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## CONTENTS

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# SECTIONAL SWITCHING MAPPINGS IN SEMILATTICES

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ABSTRACT. Consider a $\vee$-lattice $\mathcal{S} = (S, \vee)$ with a greatest element 1. An interval $[a, 1]$ for $a \in S$ is called a section. A mapping $f$ of $[a, 1]$ onto itself is called a switching mapping if $f(a) = 1$, $f(1) = a$ and for $x \in [a, 1]$, $a \neq x \neq 1$ we have $a \neq f(x) \neq 1$. We study $\vee$-lattices with switching mappings on all the sections. If for $p, q \in S$, $p \leq q$ the mapping on the section $[q, 1]$ is determined by that of $[p, 1]$, we say that the compatibility condition is satisfied. We will get conditions for antitony of switching mappings and a connection with complementation in sections will be shown.

*Mathematics Subject Classification:* 06A12, 06C15, 06F99, 08B05.

*Keywords:* semilattice, section, sectional mapping, switching mapping, compatibility condition

## 1. BASIC CONCEPTS

A mapping $f : A \to A$ is called an *involution* if $f(f(x)) = x$ for each $x \in A$. Let $(A, \leq)$ be an ordered set. A mapping $f : A \to A$ is *antitone* if $x \leq y$ implies $f(y) \leq f(x)$ for all $x, y \in A$. Let $\mathcal{S} = (S, \vee, 1)$ be a join-semilattice with the greatest element 1. For $a \in S$, the interval $[a, 1]$ will be called a *section* (of $\mathcal{S}$). We will study semilattices with 1 where for each $a \in S$ there is a mapping on the section $[a, 1]$; such a structure will be called a *semilattice with sectional mappings*. To distinguish among these mappings, we introduce the following notation:

for each $a \in S$ and $x \in [a, 1]$, denote by $x^a$ the image of $x$ in this sectional mapping on $[a, 1]$. Thus $x \mapsto x^a$ is a symbol for the corresponding sectional mapping on the section $[a, 1]$.

Hence, semilattices with sectional mappings can be considered as algebras with partial unary operations $x \mapsto x^a$ whose number is equal to the cardinality of $S$. To avoid the difficulty with the types of these partical algebras and to transform them into total algebras, let us introduce the following:

Let $\mathcal{S} = (S, \vee, 1)$ be a semilattice with sectional mappings. Define the so-called *induced operation* on $S$ by the rule $x \cdot y = (x \vee y)^y$.

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Evidently, "$\cdot$" is everywhere defined binary operation on $S$ since $x \vee y \in [y, 1]$ for any $x, y \in S$. Also conversely, if "$\cdot$" is induced on $S$ then for each $a \in S$ and $x \in [a, 1]$ we have $x \cdot a = (x \vee a)^a = x^a$.

Hence, the induced operation determines all the sectional mappings. Due to this, semilattices with sectional mappings will be considered as algebras of type $(2, 2, 0)$ in the signature $\langle \vee, \cdot, 1 \rangle$. For certain properties of sectional mappings we will specify the corresponding properties of the induced operation and vice versa.

**Lemma 1.** *Let $\mathcal{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectional involutions. The following conditions are equivalent for $a \in S$:*

(a) *$x \mapsto x^a$ is antitone,*

(b) *the section $[a, 1]$ is a lattice where*

$$x \wedge_a y = (x^a \vee y^a)^a \qquad \textit{(De Morgan Law).}$$

PROOF. (a) $\Rightarrow$ (b): Since the sectional mapping on $[a, 1]$ is an antitone involution, it is a bijection and $x, y \leq x \vee y$ implies $x^a, y^a \geq (x \vee y)^a$ and the existence of supremum for $x, y \in [a, 1]$ yields existence of the infimum $x \wedge_a y$. Hence, $x^a \wedge_a y^a \geq (x \vee y)^a$. However, $x^a, y^a \geq x^a \wedge_a y^a$ thus, due to $x = x^{aa}$, $y = y^{aa}$, we obtain $x, y \leq (x^a \wedge_a y^a)^a$ whence $x \vee y \leq (x^a \wedge_a y^a)^a$, i.e. $(x \vee y)^a \geq x^a \wedge_a y^a$. Altogether, we obtain (b).

(b) $\Rightarrow$ (a): Let $x, y \in [a, 1]$ and suppose $x \leq y$. Then $x \vee y = y$ and, by (b), $y^a = (x \vee y)^a = x^a \wedge_a y^a$ thus $y^a \leq x^a$, i.e. the sectional mapping on $[a, 1]$ is antitone. $\square$

## 2. SWITCHING MAPPINGS

We say that a mapping $x \mapsto x^a$ on the section $[a, 1]$ is *weakly switching* if $a^a = 1$ and $1^a = a$. In other words, a weakly switching mapping "switches" the bound elements of the section.

**Lemma 2.** *Let $\mathcal{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectional mappings.*

(a) *If for each $p \in S$ the sectional mapping $x \mapsto x^p$ is an involution, then the induced operation satisfies the identity*

$$(x \cdot y) \cdot y = (y \cdot x) \cdot x = x \vee y. \tag{I1}$$

(b) *If for each $p \in S$ the sectional mapping $x \mapsto x^p$ is weakly switching and the induced operation satisfies* (I1)*, then every sectional mapping is an involution.*

PROOF. (a) Since $x \vee y \in [y, 1]$ we have $x \cdot y = (x \vee y)^y \geq y$. Thus, if the sectional mapping is an involution, we infer

$$(x \cdot y) \cdot y = ((x \vee y)^y \vee y)^y = (x \vee y)^{yy} = x \vee y,$$

whence (I1) is evident.

(b) Let each sectional mapping be weakly switching, let $p \in S$ and $x \in [p, 1]$. Then $x \vee p = x$ and, by (I1),

$$x^{pp} = (x \cdot p) \cdot p = (p \cdot x) \cdot x = ((p \vee x)^x \vee x)^x = (x^x \vee x)^x = (1 \vee x)^x = 1^x = x$$

and thus $x \mapsto x^p$ is an involution. □

*Remark* 1. Identity (I1) is called *quasi-commutativity* in [1, 2].

A weakly switching mapping $x \mapsto x^p$ will be called a *switching mapping* if $a \neq x^a \neq 1$ for each $x \in [a, 1]$ with $a \neq x \neq 1$.

*Remark* 2. Every join-semilattice $\mathcal{S} = (S, \vee, 1)$ with a greatest element can be considered as a semilattice with sectional switching mappings. One can take for each $a \in S$ and every $x \in [a, 1]$ $a^a = 1$, $1^a = a$ and $x^a = x$ for $a \neq x \neq 1$. Hence, our concept is really universal and very natural for semilattices.

**Lemma 3.** *Let $\mathcal{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectional switching mappings, let $\leq$ be its induced order. Then $x \leq y$ if and only if $x \cdot y = 1$.*

PROOF. If $x \leq y$, then $x \cdot y = (x \vee y)^y = y^y = 1$. Conversely, if $x \cdot y = 1$, then $(x \vee y)^y = 1$ thus, since it is a switching mapping, $x \vee y = y$, whence $x \leq y$. □

The following lemma is almost evident.

**Lemma 4.** *Let $\mathcal{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectional weakly switching mappings. Then $\mathcal{S}$ satisfies the identities*

$$x \cdot x = 1, \qquad 1 \cdot x = x, \qquad x \cdot 1 = 1. \tag{I2}$$

**Theorem 1.** *The class of all semilattices with sectional switching mappings is congruence distributive and weakly regular.*[*]

PROOF. Consider the binary terms $r_1(x, y) = x \cdot y$ and $r_2(x, y) = y \cdot x$. By Lemma 4, $r_1(x, x) = r_2(x, x) = 1$. Converesely, let $r_1(x, y) = r_2(x, y) = 1$. By Lemma 3, it yields $x \leq y$ and $y \leq x$ thus $x = y$. Applying Theorem 6.4.3 of [4] (the Csákány Theorem), we conclude that the class $\mathcal{W}$ of all semilattices with sectional switching mappings is weakly regular.

Moreover, for $t(x, y) = y \cdot x$ we have $t(x, x) = 1$ and $t(1, x) = 1$ (by Lemma 4); thus, by Theorem 8.3.2 of [4], the class $\mathcal{W}$ is arithmetical in 1 and hence distributive in 1, i. e., $[1]_{\Theta \wedge (\Phi \vee \Psi)} = [1]_{(\Theta \wedge \Phi) \vee (\Theta \wedge \Psi)}$ for any $\mathcal{A} \in \mathcal{W}$ and $\Theta, \Phi, \Psi \in \operatorname{Con} \mathcal{A}$. Since $\mathcal{W}$ is weakly regular, this yields the congruence distributivity of $\mathcal{W}$. □

We are interesting in the question when sectional switching mappings are antitone. For this, we use the identity involved in [3] (see also [5, 6]).

**Theorem 2.** *Let $\mathcal{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectional switching mappings.*

[*]That is, if $\Theta, \Phi \in \operatorname{Con} \mathcal{S}$ and $[1]_\Theta = [1]_\Phi$ then $\Theta = \Phi$.

(a) *If $\mathcal{S}$ satisfies the identity*

$$(((x \cdot y) \cdot y) \cdot z) \cdot (x \cdot z) = 1 \tag{I3}$$

*then every switching mapping on $\mathcal{S}$ is antitone.*

(b) *If every sectional switching mapping on $\mathcal{S}$ is an involution then it is antitone if and only if $\mathcal{S}$ satisfies* (I3).

PROOF. (a) Suppose $z \in S$, $x, y \in [z, 1]$ and $x \leq y$. By Lemma 3 we have $x \cdot y = 1$ and, by Lemma 4 and (I3) we infer

$$(y \cdot z) \cdot (x \cdot z) = ((1 \cdot y) \cdot z) \cdot (x \cdot z) = (((x \cdot y) \cdot y) \cdot z) \cdot (x \cdot z) = 1.$$

By Lemma 3 we have $y \cdot z \leq x \cdot z$ and thus $y^z = y \cdot z \leq x \cdot z = x^z$.

(b) Let the sectional switching mappings on $\mathcal{S}$ are antitone involutions. By Lemma 2 we have $(x \cdot y) \cdot y = x \vee y$.

Since $x \vee y \vee z \geq x \vee z$ and $x \vee y \vee z, x \vee z \in [z, 1]$, we obtain $((x \cdot y) \cdot y) \cdot z = (x \vee y \vee z)^z \leq (x \vee z)^z = x \cdot z$. By Lemma 3 we infer $(((x \cdot y) \cdot y) \cdot z) \cdot (x \cdot z) = 1$. The converse is given by (a). □

## 3. THE COMPATIBILITY CONDITION

We will consider a semilattice with sectional mappings where the mapping in a smaller section is determined by that of a bigger one. More precisely, we say that $\mathcal{S} = (S, \vee, \cdot, 1)$ satisfies the *compatibility condition* if

$$p \leq q \leq x \text{ implies that } x^q = x^p \vee q. \tag{CC}$$

It is easy to verify that (CC) can be equivalently expressed as the following identity

$$(y \vee z) \cdot (x \vee y) = ((y \vee z) \cdot x) \vee (x \vee y) \tag{CCI}$$

since $x \leq x \vee y \leq x \vee y \vee z$ and $(y \vee z) \cdot (x \vee y) = (x \vee y \vee z)^{(x \vee y)}$, $(y \vee z) \cdot x = (x \vee y \vee z)^x$.

Let us also note that the compatibility condition is satisfied for complementation in any Boolean semilattice (see [1]) and in any orthomodular semilattice [2], its modification holds also for semilattices with sectionally antitone involutions which are implication algebras for MV-algebras [6].

We are going to show that (CC) does not imply either antitone or involuton of switching mappings.

*Example* 1. Let $\mathcal{S} = (\{p, x, y, z, 1\}, \vee, \cdot, 1)$ be a semilattice depicted on Figure 1, where the sectional mappings are given as follows:

$$\begin{aligned}
&\text{on } [p, 1]: \quad x^p = y^p = z,\ z^p = y,\ p^p = 1,\ 1^p = p, \\
&\text{on } [x, 1]: \quad x^x = 1,\ 1^x = x, \\
&\text{on } [y, 1]: \quad y^y = 1,\ 1^y = y, \\
&\text{on } [z, 1]: \quad z^z = 1,\ 1^z = z,
\end{aligned}$$

on $[1,1]:\quad 1^1 = 1.$

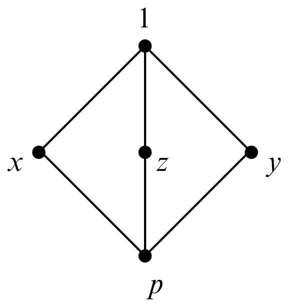


FIGURE 1.

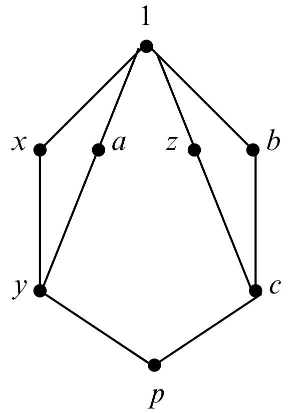


FIGURE 2.

One can easily verify that (CC) is satisfied by $\mathcal{S}$ and all the sectional mappings are antitone switching mappings. However, $a \mapsto a^p$ is not a bijection since $x \neq y$ but $x^p = y^p$.

*Example* 2. Let $\mathcal{S} = (\{p, a, b, c, x, y, z, 1\}, \vee, \cdot, 1)$ be the semilattice shown on Figure 2.

The sectional mappings are defined as follows:

on $[p,1]:\quad x^p = a,\ y^p = c,\ z^p = b, a^p = x,\ c^p = y,\ b^p = z,\ p^p = 1,$
$1^p = p,$

on $[y,1]:\quad x^y = a,\ a^y = x,\ y^y = 1,\ 1^y = y,$

on $[c,1]:\quad z^c = b,\ b^c = z,\ c^c = 1,\ 1^c = c,$

on $[x,1]:\quad x^x = 1,\ 1^x = x,$

on $[a,1]:\quad a^a = 1,\ 1^a = a,$

on $[z,1]:\quad z^z = 1,\ 1^z = z,$

on $[b,1]:\quad b^b = 1,\ 1^b = b,$

on $[1,1]:\quad 1^1 = 1.$

It is easy to verify that all of them are switching mappings satisfying (CC) and, moreover, they are involutions. However, the mapping $v \mapsto v^p$ is not antitone, since $y \leq x$ but $x^p = a$, $y^p = c$ are incomparable.

**Lemma 5.** *Let $\mathcal{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectional mappings satisfying the compatibility condition. Then*

(a) $x \vee x^p = 1$ *for each* $p \in S$ *and each* $x \in [p, 1]$*;*

(b) *If* $z \mapsto z^p$ *is a switching mapping for* $p \neq 1$, *then* $x^p \neq x$ *and if* $x < y$ *then* $x^p \neq y^p$ *for each* $x, y \in [p, 1]$*;*

(c) *If all the sectional mappings are switching, then no section of* $\mathcal{S}$ *can be a chain with more than two elements.*

PROOF. (a) Since $p \leq x \leq x$, we infer directly by (CC) $1 = x^x = x^p \vee x$.

(b) If $z \mapsto z^p$ is a switching mapping on $[p, 1]$ and $x, y \in [p, 1]$, then if $x^p = x$, by (a), we obtain $1 = x^p \vee x = x$ and, hence, $1 = x^p = 1^p = p$, a contradiction.

If $x < y$ and $x^p = y^p$, then, by (CC) and (a), $y^x = y^p \vee x = x^p \vee x = 1$. Since the sectional mapping is switching, it yields $y = x$, a contradiction.

(c) Suppose that $[p, 1]$ is a chain with more than two elements. Then there exists $x \in [p, 1]$, $p \neq x \neq 1$. We have $x^p \neq p$, $x^p \neq 1$ and, by (a), $1 = x^p \vee x = \max(x, x^p)$, a contradiction. □

Let us recall that a join semilattice $\mathcal{S} = (S, \vee, 1)$ with 1 where for $p \in S$ the section $[p, 1]$ is a lattice $([p, 1], \vee, \wedge_p)$ is called a *nearlattice* (the concept was introduced by M. Scholander [9] in 1950s).

We are interested in the case where the sectional switching mappings are sectional complementations.

**Theorem 3.** *Let* $\mathcal{S} = (S, \vee, \cdot, 1)$ *be a nearlattice with sectional switching mappings satisfying the compatibility condition. If* $x \mapsto x^p$ *is antitone on* $[p, 1]$*, then* $x^p$ *is a complement of* $x$ *for each* $x \in [p, 1]$*.*

PROOF. Assume that the sectional switching mapping on $[p, 1]$ is antitone. By Lemma 5, we have $x \vee x^p = 1$ and $x^p \vee x^{pp} = 1$ for each $x \in [p, 1]$. Take $z = x \wedge_p x^p$. Then $z \leq x$, $z \leq x^p$ and, due to the antitone property of this mapping, also $z^p \geq x^p$, $z^p \geq x^{pp}$. Thus, $z^p \geq x^p \vee x^{pp} = 1$. Therefore, it follows that $z^p = 1$, i. e., $z = p$ and $x^p$ is a complement of $x$ in the lattice $([p, 1], \vee, \wedge_p)$. □

*Remark* 3. The complement $x^p$ of $x$ in $[p, 1]$ need not be an orthocomplement although the mapping is antitone. We can see in Example 1 that this mapping need not be an involution: we have $x^{pp} = z^p = y \neq x$.

If $\mathcal{S} = (S, \vee, \cdot, 1)$ is a semilattice with sectionally antitone involutions, then we can apply De Morgan law (see Lemma 1) in each section $[p, 1]$ to prove that for $x, y \in [p, 1]$, $(x^p \vee y^p)^p$ is their infimum, i. e., every $[p, 1]$ becomes a lattice where $x \wedge_p y = (x^p \vee y^p)^p$. Hence, $\mathcal{S}$ is in fact a nearlattice. Moreover, if these sectionally antitone involutions satisfy the compatibility condition, we can prove the following.

**Theorem 4.** *Let* $\mathcal{S} = (S, \vee, \cdot, 1)$ *be a semilattice with sectionally antitone involutions satisfying the compatibility condition. Then for each* $p \in S$ *the section* $[p, 1]$ *is an orthomodular lattice where* $x^p$ *is an orthocomplement of* $x \in [p, 1]$*.*

PROOF. Naturally, sectionally antitone involutions are switching mappings, thus, by Lemma 1 and Theorem 3, $[p, 1]$ is a lattice and $x^p$ is a complement of $x \in [p, 1]$.

Since this sectional mapping is an involution, we have $x^{pp} = x$ and, due to antitony, $x \leq y$ implies $y^p \leq x^p$ for $x, y \in [p, 1]$ thus $x^p$ is an orthocomplement of $x$ in $[p, 1]$. By using the compatibility condition, $p \leq x \leq y$ implies $y^x = y^p \vee x$ and hence

$$y \wedge_p (x \vee y^p) = y \wedge_p y^x = y \wedge_x y^x = x$$

which is the orthomodular law in the lattice $([p, 1], \vee, \wedge_p)$. □

In the remaining part, we will check whether the complement $x^p$ of $x$ in $[p, 1]$ is unique. We will establish a new condition which need not be the compatibility condition.

**Theorem 5.** *Let $\mathcal{S} = (S, \vee, \cdot, 1)$ be a semilattice with sectionally antitone involutions. If for $p \in S$ and each $x, y \in [p, 1]$ the relation*

$$(x^p \vee y)^p \vee x^p = (y^p \vee x)^p \vee y^p \tag{$*$}$$

*holds, then $([p, 1], \vee, \wedge_p)$ is a Boolean algebra.*

PROOF. Due to Lemma 1, $([p, 1], \vee, \wedge_p)$ is a lattice and we can use De Morgan law for each section. Let $a \in [p, 1]$. Using of the identity $(*)$, we obtain

$$a \vee a^p = a^{pp} \vee a^p = (a^p \vee p)^p \vee a^p = (p^p \vee a)^p \vee p^p = (1 \vee a)^p \vee 1 = 1.$$

Due to the De Morgan law, we have

$$a \wedge_p a^p = a^{pp} \wedge_p a^p = (a^p \vee a)^p = 1^p = p.$$

Hence, $a^p$ is a complement of $a$ in $[p, 1]$.

Suppose now that $u \in [p, 1]$ is a complement of $a$ in $[p, 1]$, i.e. $a \vee u = 1$ and $a \wedge_p u = p$. Using the identity $(*)$ and the De Morgan law again, we derive $a = p \vee a = (a \vee u)^p \vee a = (a^{pp} \vee u)^p \vee a^{pp} = (u^p \vee a^p)^p \vee u^p = (u \wedge_p a) \vee u^p = p \vee u^p = u^p$. Thus, $a^p = u^{pp} = u$, and the complement is unique.

Since the involution is an antitone unique complementation, then, according to [8], $([p, 1], \vee, \wedge_p)$ is distributive. □

*Remark* 4. Identity $(*)$ is in fact equivalent to the assertion that $([p, 1], \vee, \wedge_p)$ is a Boolean algebra where $x^p$ is a complement of an $x \in [p, 1]$. Indeed, if $([p, 1], \vee, \wedge_p)$ is distributive, then $(x^p \vee y)^p \vee x^p = x^p \vee (x^{pp} \wedge_p y^p) = x^p \vee (x \wedge_p y^p) = (x^p \vee x) \wedge_p (x^p \vee y^p) = 1 \wedge_p (x^p \vee y^p) = x^p \vee y^p$, thus also $(y^p \vee x)^p \vee y^p = y^p \vee x^p = x^p \vee y^p$. Therefore, $(*)$ is satisfied.

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# MINIMAL PERIODS OF PERIODIC SOLUTIONS

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ABSTRACT. We derive minimal periods of non-constant periodic solutions for semilinear damped wave equations on Hilbert spaces. Similar estimates are obtained for symmetric nonconstant periodic solutions of $\mathbb{Z}_p$-symmetric autonomous ordinary differential equations.

*Mathematics Subject Classification:* 34C14, 34C25, 35B10

*Keywords:* minimal periods, weak solutions, symmetries

## 1. INTRODUCTION

In this note, we present some estimates, lower bounds, concerning periods of nontrivial periodic orbits for certain differential equations. First, in Section 2, we study damped semilinear wave equations on Hilbert spaces. Here we are inspired by [14], where semilinear parabolic equations are studied. We apply our method to the equation of a damped buckled beam [9] and also to the equation of a buckled elastic panel excited by a fluid flow over its upper surface [3, 10].

Then, in Sections 3 and 4, we investigate $\mathbb{Z}_p$-symmetric autonomous ordinary differential equations which are generalizations of odd systems and related antiperiodic solutions (see [1]).

We note that recent results on minimal periods are also derived in [5, 12, 13, 15], where discrete, continuous and delay dynamical systems are considered. Minimal periods for ordinary differential equations on one-dimensional lattices are studied in [6, 7]. Finally, we refer the reader to [14, 15] for the history of these topics.

## 2. SEMILINEAR DAMPED WAVE EQUATIONS

Let $H$ be a Hilbert space with a norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Let $A$ be an unbounded linear self-adjoint operator on $H$ with an orthonormal basis of eigenvectors $\{w_j\}_{j\geq 1}$ on $H$ and with corresponding eigenvalues $\lambda_j$, $Aw_j = \lambda_j w_j$ such that $\lambda_j \to +\infty$ as $j \to \infty$ (see [16]).

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Let $0 \leq \alpha \leq 1$. We put

$$H^\alpha := \left\{ x = \sum_{j \geq 1} x_j w_j \mid \sum_{j \geq 1} (\lambda_j^{2\alpha} + 1)\, x_j^2 < \infty \right\}$$

with an inner product $(\cdot,\cdot)_\alpha$ and corresponding norm $\|\cdot\|_\alpha$ defined by the formulae

$$(x, y)_\alpha := \sum_{j \geq 1} (\lambda_j^{2\alpha} + 1)\, x_j y_j, \quad \|x\|_\alpha := \sqrt{(x, x)_\alpha}$$

for any $x = \sum_{j \geq 1} x_j w_j$ and $y = \sum_{j \geq 1} y_j w_j$. Here, we consider $\lambda_j^{2\alpha} = (\lambda_j^2)^\alpha = |\lambda_j|^{2\alpha}$.

Let $f : H^\alpha \to H$ be globally Lipschitz continuous, i. e., $\exists L > 0$ such that

$$\|f(u_1) - f(u_2)\| \leq L \|u_1 - u_2\|_\alpha \quad \forall u_{1,2} \in H^\alpha.$$

We consider a damped abstract wave equation

$$\ddot{u} + \delta \dot{u} + Au + f(u) = 0. \tag{2.1}$$

We put

$$L_T^2(\mathbb{R}, H) = \left\{ h \in L_{\mathrm{loc}}^2(\mathbb{R}, H) : h \text{ is } T\text{-periodic} \right\},$$

$$L_{T,0}^2(\mathbb{R}, H) = \left\{ h \in L_T^2(\mathbb{R}, H) : \int_0^\infty h(t)\, dt = 0 \right\}.$$

Similarly, we define Hilbert spaces $L_T^2(\mathbb{R}, H^\alpha)$ and $L_{T,0}^2(\mathbb{R}, H^\alpha)$. For $\alpha > 0$, the usual integral norm on $L_T^2(\mathbb{R}, H^\alpha)$ is denoted by $\|\cdot\|_{\alpha,2}$, while for $\alpha = 0$, we take the standard integral norm $\|\cdot\|_2$ on $L_T^2(\mathbb{R}, H)$.

**Definition 2.1.** By a weak $T$-periodic solution of (2.1), we mean any function $u \in L_T^2(\mathbb{R}, H^\alpha)$ satisfying the relation

$$\int_0^T \{(u(t), \ddot{v}(t)) - \delta(u(t), \dot{v}(t)) + (u(t), Av(t)) + (f(u(t)), v(t))\}\, dt = 0$$

for all $v \in C_T^2(\mathbb{R}, H^1)$, where $C_T^2(\mathbb{R}, H^1)$ is defined as above and $H^1 = D(A)$.

First we study the linear equation

$$\ddot{u} + \delta \dot{u} + Au = h \tag{2.2}$$

for $h \in L_T^2(\mathbb{R}, H)$.

**Lemma 2.2.** *Let $0 \leq \alpha \leq \frac{1}{2}$. For any $h \in L_{T,0}^2(\mathbb{R}, H)$, equation* (2.2) *has a unique weak solution $u \in L_{T,0}^2(\mathbb{R}, H^\alpha)$ satisfying the estimate*

$$\|u\|_{\alpha,2} \leq \Psi_{\delta,\alpha}(T) \|h\|_2,$$

*for a function* $\Psi_{\delta,\alpha} \in C([0,\infty),\ (0,\infty))$ *with*

$$\begin{gathered}\lim_{T\to 0_+} \Psi_{\delta,\alpha}(T)/T^{1-2\alpha} = c_{\alpha,0},\\ \lim_{T\to\infty} \Psi_{\delta,\alpha}(T)/T = c_{\alpha,\infty},\end{gathered} \tag{2.3}$$

*where* $c_{\alpha,0}$ *and* $c_{\alpha,\infty}$ *are suitable positive constants.*

PROOF. We take $h(t) = \sum_{j\ge 1} h_j(t)w_j$, $u(t) = \sum_{j\ge 1} u_j(t)w_j$,

$$h_j(t) = \frac{1}{\sqrt{T}} \sum_{k\in\mathbb{Z}\setminus\{0\}} h_{j,k} e^{2\pi k t\imath/T}, \quad \bar{h}_{j,k} = h_{j,-k}$$

$$u_j(t) = \frac{1}{\sqrt{T}} \sum_{k\in\mathbb{Z}\setminus\{0\}} u_{j,k} e^{2\pi k t\imath/T}, \quad \bar{u}_{j,k} = u_{j,-k}.$$

Then

$$\|h\|_2^2 = \sum_{j\ge 1} \|h_j\|_2^2 = 2\sum_{j,k\ge 1} |h_{j,k}|^2,$$

$$\|u\|_{\alpha,2}^2 = \sum_{j\ge 1} \|u_j\|_2^2(\lambda_j^{2\alpha}+1) = 2\sum_{j,k\ge 1} |u_{j,k}|^2(\lambda_j^{2\alpha}+1).$$

So (2.2) gives

$$u_{j,k} = \frac{h_{j,k}}{\lambda_j - \frac{4\pi^2k^2}{T^2} + \delta\frac{2\pi k}{T}\imath}.$$

Hence

$$\|u\|_{\alpha,2}^2 \le 2\sum_{j,k\ge 1} \frac{\lambda_j^{2\alpha}+1}{\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2}} |h_{j,k}|^2.$$

We evaluate

$$\frac{\lambda_j^{2\alpha}+1}{\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2}} \le \frac{T^2}{4\pi^2\delta^2} + \frac{\lambda_j^{2\alpha}}{\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2}}.$$

Let us put $|\lambda_0| = \max\{|\lambda_i| \mid \lambda_i \le 0\}$. Next, if $\lambda_j - \frac{\delta^2}{2} \ge \frac{4\pi^2}{T^2}$, then

$$\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2} \ge \delta^2\left(\lambda_j - \frac{\delta^2}{4}\right).$$

Hence, we have

$$\frac{\lambda_j^{2\alpha}}{\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2}} \le \frac{\lambda_0^{2\alpha}T^2}{4\pi^2\delta^2}$$

for $\lambda_j \le 0$,

$$\frac{\lambda_j^{2\alpha}}{\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2}} \le \left(\frac{\delta^2}{2} + \frac{4\pi^2}{T^2}\right)^{2\alpha}\frac{T^2}{4\pi^2\delta^2}$$

for $0 \le \lambda_j \le \frac{\delta^2}{2} + \frac{4\pi^2}{T^2}$, and

$$\frac{\lambda_j^{2\alpha}}{\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2}} \le \frac{\lambda_i^{2\alpha}}{\delta^2\left(\lambda_i - \frac{\delta^2}{4}\right)^2} \le \frac{\left(\frac{\delta^2}{2} + \frac{4\pi^2}{T^2}\right)^{2\alpha}}{\delta^2\left(\frac{\delta^2}{4} + \frac{4\pi^2}{T^2}\right)}$$

for $\lambda_j \ge \frac{\delta^2}{2} + \frac{4\pi^2}{T^2}$. Summarising, we get

$$\frac{\lambda_j^{2\alpha}}{\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2}} \le \Phi_{\delta,\alpha}(T),$$

where

$$\Phi_{\delta,\alpha}(T) := \max\left\{\frac{\lambda_0^{2\alpha}T^2}{4\pi^2\delta^2}, \left(\frac{\delta^2}{2} + \frac{4\pi^2}{T^2}\right)^{2\alpha}\frac{T^2}{4\pi^2\delta^2}, \frac{\left(\frac{\delta^2}{2} + \frac{4\pi^2}{T^2}\right)^{2\alpha}}{\delta^2\left(\frac{\delta^2}{4} + \frac{4\pi^2}{T^2}\right)}\right\}.$$

We see that $\Psi_{\delta,\alpha}$ defined by the equality

$$\Psi_{\delta,\alpha}(T) := \sqrt{\Phi_{\delta,\alpha}(T) + \frac{T^2}{4\pi^2\delta^2}}$$

satisfies the conditions of this lemma. So we obtain

$$\|u\|_{\alpha,2}^2 \le 2\sum_{j,k\ge 1}\Psi_{\delta,\alpha}^2(T)|h_{j,k}|^2 = \Psi_{\delta,\alpha}^2(T)\|h\|_2^2.$$

Consequently, we arrive at the estimate

$$\|u\|_{\alpha,2} \le \Psi_{\delta,\alpha}(T)\|h\|_2.$$

The proof is complete. □

Now we return to (2.1) by splitting any $u \in L_T^2(\mathbb{R}, H^\alpha)$ as

$$u = u_1 + u_0$$

for $u_1 := \frac{1}{T}\int_0^T u(t)\,dt$ and $u_0 = u - u_1 \in L^2_{T,0}(\mathbb{R}, H^\alpha)$. Hence (2.1) has the form

$$\ddot{u}_0 + \delta\dot{u}_0 + Au_0 + f(u_1 + u_0) - \frac{1}{T}\int_0^T f(u_1 + u_0(t))\,dt = 0, \tag{2.4}$$

$$Au_1 + \frac{1}{T}\int_0^T f(u_1 + u_0(t))\,dt = 0. \tag{2.5}$$

We note that the linear operator $P : L^2_T(\mathbb{R}, H) \to L^2_T(\mathbb{R}, H)$ given by

$$Pu := u - \frac{1}{T}\int_0^T u(t)\,dt$$

is orthogonal and the Nemytskii operator $N : L^2_T(\mathbb{R}, H^\alpha) \to L^2_T(\mathbb{R}, H)$ given by the equality

$$N(u)(t) := f(u(t))$$

is globally Lipschitz continuous with a constant $L$. Then (2.4) gives

$$\begin{aligned}\|u_0\|_{\alpha,2} &\le \Psi_{\delta,\alpha}(T)\|PN(u_1 + u_0)\|_2\\ &= \Psi_{\delta,\alpha}(T)\|P[N(u_1 + u_0) - N(u_1)]\|_2\\ &\le \Psi_{\delta,\alpha}(T)L\|u_0\|_{\alpha,2}.\end{aligned}$$

Consequently, if

$$\Psi_{\delta,\alpha}(T)L < 1, \tag{2.6}$$

then $u_0 = 0$ and (2.5) becomes

$$Au_1 + f(u_1) = 0.$$

Summarising, we have the following result.

**Theorem 2.3.** *If* $0 \le \alpha \le \frac{1}{2}$ *and* (2.6) *holds, then any* $T$*-periodic weak solution of* (2.1) *is constant.*

Function $\Psi_{\delta,\alpha}(T)$ is depending also on $\lambda_0$. To avoid this, for $\lambda_j \le 0$ we compute

$$\frac{\lambda_j^{2\alpha}}{\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2}} \le \frac{\lambda_j^{2\alpha}}{\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2} \le \left(\frac{1-\alpha}{4\pi^2}\right)^{2-2\alpha}\alpha^{2\alpha}T^{4(1-\alpha)}.$$

Then $\Phi_{\delta,\alpha}(T)$ is replaced by

$$\widetilde{\Phi}_{\delta,\alpha}(T) := \max\left\{\left(\frac{1-\alpha}{4\pi^2}\right)^{2-2\alpha}\alpha^{2\alpha}T^{4(1-\alpha)},\right.$$
$$\left.\left(\frac{\delta^2}{2} + \frac{4\pi^2}{T^2}\right)^{2\alpha}\frac{T^2}{4\pi^2\delta^2}, \frac{\left(\frac{\delta^2}{2} + \frac{4\pi^2}{T^2}\right)^{2\alpha}}{\delta^2\left(\frac{\delta^2}{4} + \frac{4\pi^2}{T^2}\right)}\right\}$$

and

$$\tilde{\Psi}_{\delta,\alpha}(T) := \sqrt{\tilde{\Phi}_{\delta,\alpha}(T) + \frac{T^2}{4\pi^2\delta^2}}.$$

So (2.3) is replaced by

$$\begin{aligned} &\lim_{T\to 0_+} \tilde{\Psi}_{\delta,\alpha}(T)/T^{1-2\alpha} = \tilde{c}_{\alpha,0}, \\ &\lim_{T\to\infty} \tilde{\Psi}_{\delta,\alpha}(T)/T^{2(1-\alpha)} = \tilde{c}_{\alpha,\infty} \end{aligned} \tag{2.7}$$

for suitable positive constants $\tilde{c}_{\alpha,0}$ and $\tilde{c}_{\alpha,\infty}$. Thus we have the following

**Theorem 2.4.** *If* $0 \le \alpha \le \frac{1}{2}$ *and the inequality*

$$\tilde{\Psi}_{\delta,\alpha}(T)L < 1$$

*holds, then any* $T$*-periodic weak solution of* (2.1) *is constant.*

*Remark* 2.5. Semilinear parabolic differential equations on Hilbert spaces are studied in [7, 12, 14].

*Remark* 2.6. We see that $\lim_{T\to 0_+} \Psi_{\delta,1/2}(T) \neq 0$ and $\lim_{T\to 0_+} \tilde{\Psi}_{\delta,1/2}(T) \neq 0$. So for $\alpha = 1/2$ we do not get a result on the minimal periods. But we think that it is not a handicap for our above estimates. Indeed, for $\alpha = 1/2$ and $\lambda_j \ge 0$, we consider the function

$$x \longmapsto \frac{x+1}{\left(x - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2}}, \tag{2.8}$$

which has a global maximum over $[0,\infty)$:

$$\frac{4\pi^2k^2 + T^2 + \sqrt{16\pi^4k^4 + 4(2+\delta^2)\pi^2k^2T^2 + T^4}}{8\delta^2\pi^2k^2}. \tag{2.9}$$

We can check that (2.9) is decreasing with respect to $k \in \mathbb{N}$. So for $\lambda_j \ge 0$, we get

$$\frac{\lambda_j + 1}{\left(\lambda_j - \frac{4\pi^2k^2}{T^2}\right)^2 + \delta^2\frac{4\pi^2k^2}{T^2}} \le \frac{4\pi^2 + T^2 + \sqrt{16\pi^4 + 4(2+\delta^2)\pi^2T^2 + T^4}}{8\delta^2\pi^2}.$$

Consequently, when $\alpha = 1/2$ and $\lambda_j \ge 0\ \forall j \in \mathbb{N}$, the best estimate for $\Psi_{\delta,1/2}(T)$ seems to be

$$\Psi_{\delta,1/2}(T) = \sqrt{\frac{4\pi^2 + T^2 + \sqrt{16\pi^4 + 4(2+\delta^2)\pi^2T^2 + T^4}}{8\delta^2\pi^2}}. \tag{2.10}$$

Clearly, (2.10) is nonzero at $T = 0$.

Next, assuming $\lambda_j > 0\ \forall j \in \mathbb{N}$, we cannot improve much the above estimates. Now we can take

$$(x,y)_\alpha := \sum_{j\ge 1} \lambda_j^{2\alpha} x_j y_j$$

for $x = \sum_{j \geq 1} x_j w_j$ and $y = \sum_{j \geq 1} y_j w_j$. Then instead of the function (2.8), we consider the function

$$x \longmapsto \frac{x}{\left(x - \frac{4\pi^2 k^2}{T^2}\right)^2 + \delta^2 \frac{4\pi^2 k^2}{T^2}}. \tag{2.11}$$

Making the above analysis for (2.11), we obtain

$$\Psi_{\delta,1/2}(T) = \sqrt{\frac{2\pi + \sqrt{4\pi^2 + \delta^2 T^2}}{4\delta^2 \pi}}. \tag{2.12}$$

Clearly (2.12) is again nonzero at $T = 0$. But (2.12) is simpler than (2.10). Then the lower bound estimate is

$$\frac{2\pi + \sqrt{4\pi^2 + \delta^2 T^2}}{4\delta^2 \pi} \geq L^{-2}. \tag{2.13}$$

Analysing (2.13), we see that if $L \geq \delta$, then (2.13) holds for any $T > 0$, while for $\delta > L$, relation (2.13) yields

$$T \geq L^{-2} 4\pi \sqrt{\delta^2 - L^2}.$$

Consequently, we are able to estimate $T$ from below only for small $L > 0$ with respect to $\delta$.

Finally, the following semilinear parabolic equation is studied in [14]

$$\dot{u} + Au + f(u) = 0, \tag{2.14}$$

where $A$, $f$ satisfy our assumptions with $\lambda_j > 0 \; \forall j \geq 1$. Then we have

$$\frac{\lambda_j^{2\alpha}}{\frac{4\pi^2 k^2}{T^2} + \lambda_j^2} \leq \frac{\lambda_j^{2\alpha}}{\frac{4\pi^2}{T^2} + \lambda_j^2}.$$

Analysing the function

$$x \longmapsto \frac{x^\alpha}{\frac{4\pi^2}{T^2} + x}$$

on $[0, \infty)$, we get its maximum $\alpha^\alpha (1-\alpha)^{1-\alpha} (2\pi)^{2(\alpha-1)} T^{2(1-\alpha)}$. So the minimal period estimate for (2.14) is

$$T \geq K_\alpha L^{-\frac{1}{1-\alpha}} \tag{2.15}$$

for any $0 \leq \alpha < 1$ and

$$K_\alpha = \frac{2\pi}{\sqrt{1-\alpha}\,\alpha^{\frac{\alpha}{2(1-\alpha)}}}.$$

Inequality (2.15) is consistent with a similar one in [14], but here we allow $0 \leq \alpha < 1$. We note that $K_0 = 2\pi$ and $\lim_{\alpha \to 1-} K_\alpha = \infty$.

*Remark* 2.7. Theorem 2.4 could help when the shifts

$$A \longleftrightarrow A - \lambda I, \quad f \longleftrightarrow f + \lambda I$$

can improve an estimate of the Lipschitz constant $L$.

We also note that it follows from the proof of Lemma 2.2 that $\dot{u} \in L^2_T(\mathbb{R}, H)$ and

$$\|\dot{u}\|_2 \le \frac{1}{\delta}\|h\|_2$$

in Lemma 2.2. Then of course any $T$-periodic weak solution $u$ of (2.1) satisfies $\dot{u} \in L^2_T(\mathbb{R}, H)$. Hence we get

$$\|u\|_{\alpha,2} + \|\dot{u}\|_2 \le \left(\Psi_{\delta,\alpha}(T) + \frac{1}{\delta}\right)\|h\|_2. \tag{2.16}$$

*Example* 2.8. Let us consider the system

$$\begin{aligned} \ddot{u} + \delta\dot{u} - u'' + \sin(u - v) &= 0, \\ \ddot{v} + \delta\dot{v} - v'' + \sin(u + v) &= 0, \\ u(0, \cdot) = v(0, \cdot) = u(1, \cdot) = v(1, \cdot) &= 0. \end{aligned} \tag{2.17}$$

Now $H = L^2(0,1)^2$, $\alpha = 0$ and

$$A(u, v) = -(u''(x), v''(x)), \quad f(u, v) = (\sin(u - v), \sin(u + v))\,.$$

We see that $\lambda_j > 0\ \forall j \in \mathbb{N}$ and $L = 2$. So we can choose a better function $\Psi_{\delta,0}(T) = \frac{T}{2\pi\delta}$ than in the proof of Lemma 2.2. Then for (2.17), the minimal period estimate is $T \ge \pi\delta$.

Now we present the following simple result which seems to be known but we prove it here for the reader's convenience.

**Theorem 2.9.** *Let us suppose that*

(1) $f : H^\alpha \to H$ *is locally Lipschitz continuous, i. e.* $\forall k > 0\ \exists L_k > 0$ *such that*

$$\|f(u_1) - f(u_2)\| \le L_k\|u_1 - u_2\|_\alpha$$

*for all* $u_{1,2} \in H^\alpha$, $\|u_{1,2}\|_\alpha \le k$.

(2) $\exists F \in C^1(H^\alpha, \mathbb{R}) : (f(v), w) = DF(v)w\ \forall v, w \in H^\alpha$.

*Then any* $T$*-periodic weak solution* $u \in L^\infty_T(\mathbb{R}, H^\alpha)$ *of* (2.1) *is constant for all* $T > 0$.

PROOF. Let $\Gamma := \|u\|_{\alpha,\infty} + 1$ where $\|\cdot\|_{\alpha,\infty}$ is the usual sup-norm on $L^\infty_T(\mathbb{R}, H^\alpha)$. Assumption (1) gives

$$\|f(u_1) - f(u_2)\| \le L_\Gamma\|u_1 - u_2\|_\alpha, \quad \forall u_{1,2} \in H^\alpha, \quad \|u_{1,2}\|_\alpha \le \Gamma. \tag{2.18}$$

So we get

$$\|f(u(t))\| \le L_\Gamma\|u(t)\|_\alpha + \|f(0)\| \le \Gamma L_\Gamma + \|f(0)\|,$$

which implies $f(u) \in L_T^\infty(\mathbb{R}, H)$. Then the argument of Remark 2.7 shows that $\dot{u} \in L_T^2(\mathbb{R}, H)$. Moreover, (2.18) yields

$$\|f(u_1) - f(u_2)\|_2 \le L_\Gamma \|u_1 - u_2\|_{\alpha,2} \qquad \forall u_{1,2} \in L_T^\infty(\mathbb{R}, H^\alpha), \quad \|u_{1,2}\|_{\alpha,\infty} \le \Gamma. \tag{2.19}$$

Next using the notation of the proof of Lemma 2.2, from (2.1) we get the system

$$\ddot{u}_j + \delta \dot{u}_j + \lambda_j u_j + (f(u), w_j) = 0, \quad \forall j \in \mathbb{N}.$$

Since

$$\left\| \sum_{j=1}^n u_j(t) w_j \right\|_\alpha \le \|u(t)\|_\alpha \le \|u\|_{\alpha,\infty} \le \Gamma, \tag{2.20}$$

we obtain that $f\left(\sum_{j=1}^n u_j w_j\right) \in L_T^\infty(\mathbb{R}, H^\alpha)$ for any $n \in \mathbb{N}$. Then

$$\begin{aligned}
\sum_{j=1}^n \delta \|\dot{u}_j\|_2^2 &= -\int_0^T \left( f(u(t)), \sum_{j=1}^n \dot{u}_j(t) w_j \right) dt \\
&= -\int_0^T \left( f\left(\sum_{j=1}^n u_j(t) w_j\right), \sum_{j=1}^n \dot{u}_j(t) w_j \right) dt \\
&\quad + \int_0^T \left( f\left(\sum_{j=1}^n u_j(t) w_j\right) - f(u(t)), \sum_{j=1}^n \dot{u}_j(t) w_j \right) dt \\
&\le \left( f\left(\sum_{j=1}^n u_j w_j\right) - f(u) \right)_2 \sqrt{\sum_{j=1}^n \|\dot{u}_j\|_2^2}.
\end{aligned}$$

Hence (2.19) and (2.20) give

$$\delta^2 \sum_{j=1}^n \|\dot{u}_j\|_2^2 \le \left| f\left(\sum_{j=1}^n u_j w_j\right) - f(u) \right|_2^2 \le L_\Gamma^2 \left| \sum_{j=1}^n u_j w_j - u \right|_{\alpha,2}^2. \tag{2.21}$$

Since $\sum_{j=1}^n u_j w_j \to u$ in $L_T^2(\mathbb{R}, H^\alpha)$ and $\sum_{j=1}^n \dot{u}_j w_j \to \dot{u}$ in $L_T^2(\mathbb{R}, H)$ as $n \to \infty$, relation (2.21) implies that

$$\|\dot{u}\|_2^2 = \sum_{j=1}^\infty \|\dot{u}_j\|_2^2 = 0.$$

Hence, $\dot{u} = 0$. The proof is complete. □

*Example* 2.10. A standard p.d.e. which fits into the framework of Theorem 2.9 is an equation of a damped buckled beam [9]

$$\begin{gathered}\ddot{u} + \delta\dot{u} + u'''' + \left[\varkappa_1 - \varkappa_2\left(\int_0^1 u'^2(\xi,\cdot)\,d\xi\right)\right]u'' = 0,\\ u(0,\cdot) = u(1,\cdot) = u''(0,\cdot) = u''(1,\cdot) = 0,\end{gathered}\tag{2.22}$$

where $\varkappa_1$ and $\varkappa_2$ are constants. Now we take $H = L^2(0,1)$, $\alpha = 1/2$, so $H^{1/2} = H_0^2(0,1)$, and

$$\begin{gathered}Au = u''''(x), \quad f(u) = \left[\varkappa_1 - \varkappa_2\left(\int_0^1 u'^2(\xi,\cdot)\,d\xi\right)\right]u'',\\ F(u) = -\frac{1}{2}\int_0^1 \varkappa_1 u'^2(\xi)\,d\xi + \varkappa_2\frac{1}{4}\left(\int_0^1 u'^2(\xi)\,d\xi\right)^2.\end{gathered}$$

*Example* 2.11. We finish this section with another partial differential equation different from (2.17) and (2.22) of the form

$$\begin{gathered}\ddot{u} + \delta\dot{u} + u'''' + \left[\varkappa_1 - \varkappa_2\left(\int_0^1 u'^2(\xi,\cdot)\,d\xi\right)\right]u'' + \eta u' = 0,\\ u(0,\cdot) = u(1,\cdot) = u''(0,\cdot) = u''(1,\cdot) = 0,\end{gathered}\tag{2.23}$$

where $\varkappa_1, \varkappa_2 > 0$, $\eta > 0$ are constants. Problem (2.23) models the oscillation of a buckled elastic panel excited by a fluid flow over its upper surface [3, 10]. Now we again take $H = L^2(0,1)$ with the usual integral norm $\|\cdot\|$, $H^{1/2} = H_0^2(0,1)$ and $\|u\|_{1/2} = \|u''\|$. Then any weak $T$-periodic solution $u \in L_T^\infty(\mathbb{R}, H^{1/2})$ of (2.23) satisfies the relations

$$\begin{gathered}\delta\|\dot{u}\|_2^2 + \eta(u',\dot{u})_2 = 0,\\ \|\dot{u}\|_2^2 = \|u''\|_2^2 - \varkappa_1\|u'\|_2^2 + \varkappa_2\int_0^T \|u'(\cdot,t)\|^4\,dt.\end{gathered}\tag{2.24}$$

Since

$$\|u'\|_2^4 \le T\int_0^T \|u'(\cdot,t)\|^4\,dt, \quad \pi\|u'\|_2 \le \|u''\|_2,$$

from (2.24) we derive

$$\begin{aligned}\pi^2\|u'\|_2^2 + \frac{\varkappa_2}{T}\|u'\|_2^4 &\le \|u''\|_2^2 + \varkappa_2\int_0^T \|u'(\cdot,t)\|^4\,dt\\ &= \varkappa_1\|u'\|_2^2 + \|\dot{u}\|_2^2 \le \varkappa_1\|u'\|_2^2 + \frac{\eta^2}{\delta^2}\|u'\|_2^2.\end{aligned}\tag{2.25}$$

Hence

$$\|u'\|_2^2 \leq \frac{T}{\varkappa_2}\left(\varkappa_1 + \frac{\eta^2}{\delta^2} - \pi^2\right) = TK_1.$$

So we get

**Theorem 2.12.** *If* $\eta^2 \leq \delta^2(\pi^2 - \varkappa_1)$, *then the only periodic weak solution* $u \in L^\infty_T(\mathbb{R}, H^{1/2})$ *of* (2.23) *is the zero one,* $u = 0$.

Note that Theorem 2.12 is consistent with Proposition 2.1 of [10]. Now, let

$$\eta^2 > \delta^2(\pi^2 - \varkappa_1). \tag{2.26}$$

Then (2.25) implies

$$\|u''\|_2^2 \leq \left(\varkappa_1 + \frac{\eta^2}{\delta^2}\right)\|u'\|_2^2 \leq T\left(\varkappa_1 + \frac{\eta^2}{\delta^2}\right)K_1 = TK_2,$$
$$\|\dot{u}\|_2^2 \leq \frac{\eta^2}{\delta^2}\|u'\|_2^2 \leq T\frac{\eta^2}{\delta^2}K_1 = TK_3.$$

Hence

$$\|u''\|_2^2 \leq TK_2, \quad \|\dot{u}\|_2^2 \leq TK_3. \tag{2.27}$$

Since $u \in L^2_T(\mathbb{R}, H^2_0(0,1))$ and $\dot{u} \in L^2_T(\mathbb{R}, L^2(0,1))$, Theorem 4 from [4, p. 288] gives $u \in C^0_T(\mathbb{R}, H^1_0(0,1))$ along with

$$\max_{t\in\mathbb{R}} \|u'(\cdot,t)\|^2 \leq \frac{1}{T}\|u'\|_2^2 + \|u''\|_2^2 + \|\dot{u}\|_2^2 \leq K_1 + TK_2 + TK_3 = K_4. \tag{2.28}$$

Let us put

$$X := \left\{u \mid u \in L^2_T(\mathbb{R}, H^2_0(0,1)),\ \dot{u} \in L^2_T(\mathbb{R}, L^2(0,1))\right\}.$$

Then $X$ is a Hilbert space with the norm

$$|u|_X := \|u''\|_2 + \|\dot{u}\|_2.$$

Again from Theorem 4 of [4, p. 288], we have $X \subset C^0_T(\mathbb{R}, H^1_0(0,1))$ along with

$$\max_{\mathbb{R}} \|v'(\cdot,t)\|^2 \leq \left(\frac{1}{\pi^2 T} + 1\right)\|v''\|_2^2 + \|\dot{v}\|_2^2 \leq \left(\frac{1}{\pi^2 T} + 1\right)\|v\|_X^2 \tag{2.29}$$

for any $v \in X$.

We put

$$Au = u''''(x) + \varkappa_1 u'', \quad f(u) = \varkappa_2 g(u) + \eta Bu,$$
$$g(u) := -\|u'\|^2 u'', \quad Bu := u'.$$

For any $v, w \in X$ satisfying (2.27), we derive

$$\|Bv - Bw\|_2 = \|v' - w'\|_2 \leq \|v'' - w''\|_2/\pi \leq |v - u|_X/\pi,$$

and by (2.28) and (2.29)

$$
\begin{aligned}
\|g(v)-g(w)\|_2 &= \|\|v'\|^2v''-\|w'\|^2w''\|_2 \\
&\leq \|\|v'\|^2(v''-w'')\|_2+\|w''(\|v'\|^2-\|w'\|^2)\|_2 \\
&= \left(\int_0^T\int_0^1\left[\int_0^1 v'^2(\xi,t)\,d\xi\right]^2(v''(x,t)-w''(x,t))^2\,dx\,dt\right)^{1/2} \\
&\quad+\left(\int_0^T\int_0^1 w''^2(x,t)\left[\int_0^1(v'^2(\xi,t)-w'^2(\xi,t))\,d\xi\right]^2\,dx\,dt\right)^{1/2} \\
&\leq K_4\|v''-w''\|_2 \\
&\quad+\left(\int_0^T\int_0^1 w''^2(x,t)\int_0^1(v'(\xi,t)+w'(\xi,t))^2\,d\xi\right. \\
&\quad\left.\times\int_0^1(v'(\xi,t)-w'(\xi,t))^2\,d\xi\,dx\,dt\right)^{1/2} \\
&\leq K_4|v-w|_X \\
&\quad+2\sqrt{K_4\left(\frac{1}{\pi^2}+T\right)}|v-w|_X\left(\frac{1}{T}\int_0^T\int_0^1 w''^2(x,t)\,dx\,dt\right)^{1/2} \\
&\leq \left(K_4+2\sqrt{K_2K_4\left(\frac{1}{\pi^2}+T\right)}\right)|v-w|_X = K_5|v-w|_X.
\end{aligned}
$$

So $f: X\to L_T^2(\mathbb{R},H)$ has the Lipschitz constant

$$L=\varkappa_2K_5+\frac{\eta}{\pi}$$

on the subset $\mathcal{K}\subset X$ of all $u\in X$ satisfying (2.27).

It is clear that $\mathcal{K}$ is a closed and convex subset of $X$. Then it is well-known (see Lemma 3.5 in [2]) that there existsa retraction $R: X\to\mathcal{K}\subset X$ with a global Lipschitz constant 1. Using $R$, we modify $f$ outside of $\mathcal{K}$ as follows

$$\tilde{f}(u):=f(Ru).$$

Clearly $\tilde{f}: X\to L_T^2(\mathbb{R},H)$ is a globally Lipschitzian continuous with the constant $L$. According to the above construction, each $T$-periodic weak solution $u\in L_T^\infty(\mathbb{R},H^{1/2})$ of (2.23) is also a $T$-periodic weak solution of the modified one with $\tilde{f}$ in place of $f$. Furthermore, now $\lambda_j=-j^2\pi^2(\varkappa_1-j^2\pi^2)$ and let us assume for simplicity that $\varkappa_1<\pi^2$ (see (2.26)). Then we can apply Remarks 2.6 and 2.7 (see

(2.12) and (2.16)). So if $u$ is nonstationary, then

$$\left(\varkappa_2 K_5 + \frac{\eta}{\pi}\right)\left(\sqrt{\frac{2\pi + \sqrt{4\pi^2 + \delta^2 T^2}}{4\delta^2 \pi}} + \frac{1}{\delta}\right) \geq 1. \tag{2.30}$$

Clearly, relation (2.30) makes sense if

$$\left(\varkappa_2 \left(K_1 + \frac{2}{\pi}\sqrt{K_1 K_2}\right) + \frac{\eta}{\pi}\right)\frac{2}{\delta} < 1. \tag{2.31}$$

We do not express $K_5$ in terms of parameters $\varkappa_1, \varkappa_2, \eta$, $\delta$ and period $T$, since it is an awkward formula. But we note that for $\eta = \delta\sqrt{\pi^2 - \varkappa_1}$ (see (2.26)), we get $K_1 = K_2 = K_3 = K_4 = K_5 = 0$, and (2.31) becomes

$$3\pi^2/4 < \varkappa_1. \tag{2.32}$$

Hence if (2.32) holds, then there is a unique

$$\eta_0 = \eta(\varkappa_1, \varkappa_2, \delta) > \delta\sqrt{\pi^2 - \varkappa_1}$$

solving equation (see (2.31))

$$\left(\varkappa_2 \left(K_1 + \frac{2}{\pi}\sqrt{K_1 K_2}\right) + \frac{\eta_0}{\pi}\right)\frac{2}{\delta} = 1. \tag{2.33}$$

We note that the left-hand side of (2.33) is increasing with respect to $\eta_0$. Then by fixing $\eta$ such that $\delta\sqrt{\pi^2 - \varkappa_1} < \eta < \eta_0$, there is a unique $T_0 > 0$ solving the equation (see (2.30))

$$\left(\varkappa_2 K_5 + \frac{\eta}{\pi}\right)\left(\sqrt{\frac{2\pi + \sqrt{4\pi^2 + \delta^2 T_0^2}}{4\delta^2 \pi}} + \frac{1}{\delta}\right) = 1. \tag{2.34}$$

We note that the left-hand side of (2.34) is increasing with respect to $T_0$.

Summarising, we have the following result.

**Theorem 2.13.** *If $3\pi^2/4 < \varkappa_1 < \pi^2$ and $\delta\sqrt{\pi^2 - \varkappa_1} < \eta < \eta_0$, then the period $T$ of any weak nonstationary $T$-periodic solution $u \in L^\infty_T(\mathbb{R}, H^{1/2})$ of* (2.23) *satisfies inequality $T \geq T_0$. Here, $\eta_0$ and $T_0$ are the unique positive solutions of* (2.33) *and* (2.34)*, respectively.*

For instance if $\varkappa_1 = 8$, $\varkappa_2 = 5$ and $\delta = 100$, then equation (2.33) gives $\eta_0 = 152.667$. So we get $136.733 < \eta < 152.667$. Taking $\eta = 138$, we find from (2.34) that $T_0 = 0.122$. Consequently, we obtain $T \geq 0.122$ in Theorem 2.13.

## 3. Nonresonant $\mathbb{Z}_p$-symmetric autonomous ordinary differential equation

Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be an orthonormal matrix, i. e., $S^* = S^{-1}$, such that $S^p = I$ for some $p \in \mathbb{N}$. In this section, we deal with the ordinary differential equation

$$\dot{x} = g(x) \tag{3.1}$$

under the following assumptions:

(H1) $g : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz continuous, i. e., there is a constant $L > 0$ such that $\|g(x) - g(y)\| \leq L\|x - y\|\ \forall x, y \in \mathbb{R}^n$.
(H2) $g$ is $S$-symmetric, i. e., $g(Sx) = Sg(x)\ \forall x \in \mathbb{R}^n$.
(H3) $1 \notin \sigma(S)$, the spectrum of $S$.

We call (3.1) under conditions (H1)-(H3) a nonresonant $\mathbb{Z}_p$-symmetric autonomous ordinary differential equation.

*Remark* 3.1. Assumptions (H1) and (H2) are reasonable, i. e., there are many $g$ satisfying both (H1) and (H2). Indeed, let $\mathrm{Lip}\,(\mathbb{R}^n)$ be the space of all globally Lipschitz continuous mappings $h : \mathbb{R}^n \to \mathbb{R}^n$ endowed with the norm

$$\|h\|_{\mathrm{Lip}} := \inf\{L > 0 \mid L \text{ is the Lipschitz constant of } h\} + \|h(0)\|.$$

Then $\mathrm{Lip}\,(\mathbb{R}^n)$ becomes a Banach space. Moreover, we define a linear mapping $\mathcal{S} : \mathrm{Lip}\,(\mathbb{R}^n) \to \mathrm{Lip}\,(\mathbb{R}^n)$ by

$$(\mathcal{S}h)(x) := \frac{1}{p}\sum_{i=0}^{p-1} S^{-i}h(S^i x).$$

Clearly, $\mathcal{S}$ is a continuous projection of $\mathrm{Lip}\,(\mathbb{R}^n)$ onto $\mathrm{Lip}_S\,(\mathbb{R}^n)$, the space of all $S$-symmetric elements of $\mathrm{Lip}\,(\mathbb{R}^n)$. Furthermore, if $h$ has a Lipschitz constant $L > 0$, then $\mathcal{S}h$ has also a Lipschitz constant $L$. Finally we note

$$a_0 I + a_1 S + \cdots + a_{p-1}S^{p-1} \in \mathrm{Lip}_S\,(\mathbb{R}^n)$$

for any $a_j \in \mathbb{R}$, $0 \leq j \leq p-1$. Hence $\mathrm{Lip}_S\,(\mathbb{R}^n) \neq \emptyset$.

**Definition 3.2.** Let $T > 0$. We call any $x \in C^1(\mathbb{R}, \mathbb{R}^n)$ satisfying (3.1) and

$$x(t+T) = Sx(t) \quad \forall t \in \mathbb{R} \tag{3.2}$$

the $T$-$S$-symmetric solution of (3.1).

We note that any $x(t)$ satisfying (3.2) is also $pT$-periodic. Moreover, (H2) and (H3) imply $g(0) = Sg(0)$, so $g(0) = 0$. Thus $x(t) = 0$ is a trivial $T$-$S$-symmetric solution of (3.1) for any $T > 0$, which is the only constant function satisfying (3.2).

We put

$$C^r_{T,S} := \left\{x \in C^r(\mathbb{R}, \mathbb{R}^n) \mid x(t) \text{ satisfies (3.2)}\right\}$$

with the usual maximum norm $\|x\|_r$ on the interval $[0, T]$. We note that (3.2) implies

$$\|x^{(r)}(t+T)\| = \|Sx^{(r)}(t)\| = \|x^{(r)}(t)\|.$$

So norms $\|\cdot\|_r$ are well-defined.

First we solve the linear boundary value problem

$$\dot{x} = h, \quad x \in C^1_{T,S}, \quad h \in C^0_{T,S}. \tag{3.3}$$

It is easy to find its solution

$$x(t) = (S - I)^{-1} \int_0^T h(s)\, ds + \int_0^t h(s)\, ds.$$

So we arrive at the following.

**Lemma 3.3.** *For any $h \in C^0_{T,S}$, problem* (3.3) *has a unique solution $x \in C^1_{T,S}$ with an estimate*

$$\|x\|_0 \le [\|(I - S)^{-1}\| + 1]\, \|h\|_0 T.$$

The Banach fixed point theorem together with Lemma 3.3 give

**Theorem 3.4.** *If $[\|(I-S)^{-1}\|+1]\, TL < 1$, then $x = 0$ is the only $T$-$S$-symmetric solution of* (3.1).

When $S = -I$, the $T$-$S$-symmetric solutions are called $T$-antiperiodic ones [1]. Then condition (H3) holds and $g$ is just an odd mapping. Hence Theorem 3.4 implies

**Theorem 3.5.** *Let* (H1) *hold and $g$ be odd. If $x$ is a nonzero $T$-antiperiodic solution of* (3.1)*, i. e., $x(t + T) = -x(t)$ $\forall t \in \mathbb{R}$, then $T \ge \frac{2}{3L}$.*

*Example* 3.6. Now let $n = 2$, $\mathbb{R}^2 \simeq \mathbb{C}$, and $Sz = e^{\imath 2\pi/p} z$, $z \in \mathbb{C}$, $p \in \mathbb{N}$, $p \ge 3$. So $S : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation by the angle $2\pi/p$. Then $|z - Sz| = 2 \sin \frac{\pi}{p} |z|$, and

$$\|(I - S)^{-1}\| = \frac{1}{2 \sin(\pi/p)}.$$

Theorem 3.4 implies

**Theorem 3.7.** *Let $Sz = e^{\imath 2\pi/p} z$, $z \in \mathbb{C}$, $p \in \mathbb{N}$, $p \ge 3$, be a rotation in the plane by the angle $2\pi/p$. If $x$ is a nonzero $T$-$S$-symmetric solution of* (3.1) *for $g$ satisfying* (H1), (H2)*, then*

$$T \ge \frac{2L^{-1} \sin(\pi/p)}{1 + 2\sin(\pi/p)}.$$

Such result as in Theorem 3.7 should hold for a general $S$, since from $S^p = I$ and (H3) it follows that $\sigma(S)$ consists of the unit roots, i. e.,

$$\sigma(S) \subset \left\{ e^{2\pi k \imath/p} \mid k = 1, 2, \dots, p - 1 \right\}.$$

Indeed, it is not difficult to show [11] that conditions $S^* = S^{-1}$, $S^p = I$, $S \ne -I$ and $1 \notin \sigma(S)$ imply both $p \ge 3$ and the existence of an orthonormal basis

$$\{e_{11}, e_{12}, e_{21}, e_{22}, \dots, e_{k1}, e_{k2}, e_{k+1}, \dots, e_n\}$$

of $\mathbb{R}^n$ such that on each invariant subspace $V_j := [e_{j1}, e_{j2}]$, $1 \leq j \leq k$, matrix $S$, $S : V_j \to V_j$ acts as a rotation by the angle $2\pi r_j/p$, $1 \leq r_j < p$, $r_j \neq p/2$. While $Se_j = -e_j$, $k+1 \leq j \leq n$, naturally, only for even $p$. We can suppose that

$$\sin(\pi r_1/p) = \min\left\{\sin(\pi r_j/p) \mid j = 1, 2, \ldots, k\right\}.$$

Then

$$\|(I - S)^{-1}\| = \frac{1}{2\sin(\pi r_1/p)}.$$

Summarising, we get

**Theorem 3.8.** *When assumptions* (H1)-(H3) *and $S \neq -I$ are satisfied, then*

$$T \geq \frac{2L^{-1}\sin(\pi r_1/p)}{1 + 2\sin(\pi r_1/p)}$$

*for any nonzero $T$-$S$-symmetric solution of* (3.1).

We note that any nonzero $T$-$S$-symmetric solution of (3.1) is nonconstant.

*Remark* 3.9. The existence and nonexistence of forced symmetric oscillations like (3.1) are studied in [8].

*Remark* 3.10. We note that if $S : \mathbb{R}^n \to \mathbb{R}^n$ is a matrix with $S^p = I$ for some $p \in \mathbb{N}$. Then there is a scalar product $(\cdot,\cdot)$ for which $S$ is unitary, i. e., $(Su, Sv) = (u, v)$ $\forall u, v \in \mathbb{R}^n$. Indeed, let $\langle\cdot,\cdot\rangle$ be any scalar product on $\mathbb{R}^n$. Then we take [11]

$$(u, v) = \frac{1}{p}\sum_{j=0}^{p-1}\langle S^i u, S^i v\rangle.$$

## 4. Resonant $\mathbb{Z}_p$-symmetric autonomous ordinary differential equation

In this section, we proceed to study (3.1) only under assumptions (H1) and (H2), and we call (3.1) a resonant $\mathbb{Z}_p$-symmetric autonomous ordinary differential equation, since $1 \in \sigma(S)$. Then there is an orthogonal decomposition

$$\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$$

such that $S : \mathbb{R}^{n_{1,2}} \to \mathbb{R}^{n_{1,2}}$ and $S/\mathbb{R}^{n_1} = I$, $1 \notin \sigma(\tilde{S})$ for $\tilde{S} := S/\mathbb{R}^{n_2}$.

We split (3.1) as

$$\begin{aligned} \dot{x}_1 &= g_1(x_1 + x_2)\,, \\ \dot{x}_2 &= g_2(x_1 + x_2)\,, \end{aligned} \tag{4.1}$$

for $g_i(x) = P_i g(x)$, where $P_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ are orthogonal projections, and $x_i \in \mathbb{R}^{n_i}$, $i = 1, 2$. Clearly $g_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ are globally Lipschitz continuous with a constant

$L$. Assumption (H2) implies

$$g_1(x_1 + x_2) = g_1(x_1 + \tilde{S}x_2)\,,$$
$$\tilde{S}g_2(x_1 + x_2) = g_2(x_1 + \tilde{S}x_2).$$

Hence $g_2(x_1) = 0$, $\forall x_1 \in \mathbb{R}^{n_1}$. Now $T$-$S$-symmetric solutions of (4.1) are given by conditions

$$x_1(t+T) = x_1(t), \quad x_2(t+T) = \tilde{S}x_2(t). \tag{4.2}$$

We see that any constant functions satisfying (4.2) are $x_1(t) = \text{const}$ and $x_2(t) = 0$.

As in Section 2, we further split

$$x_1(t) = u(t) + c, \quad u \in L^2_{T,0}(\mathbb{R}, \mathbb{R}^{n_1}), \quad c \in \mathbb{R}^{n_1},$$

and decompose (4.1) as

$$\begin{aligned} \dot{u}(t) &= [g_1(u(t) + c + x_2(t)) - g_1(c)] \\ &\quad - \frac{1}{T}\int_0^T [g_1(u(s) + c + x_2(s)) - g_1(c)]\, ds, \\ \dot{x}_2(t) &= g_2(u(t) + c + x_2(t)), \end{aligned} \tag{4.3}$$

and

$$0 = \int_0^T g_1(u(s) + c + x_2(s))\, ds. \tag{4.4}$$

Let us put

$$L^2_{T,\tilde{S}}(\mathbb{R}, \mathbb{R}^{n_2}) := \left\{h \in L^2_{\text{loc}}(\mathbb{R}, \mathbb{R}^{n_2}) \mid h(t+T) = \tilde{S}h(t)\right\},$$
$$W^{1,2}_{T,\tilde{S}}(\mathbb{R}, \mathbb{R}^{n_2}) := W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^{n_2}) \cap L^2_{T,\tilde{S}}(\mathbb{R}, \mathbb{R}^{n_2}).$$

We need the following result similarly to Lemmas 2.2 and 3.3:

**Lemma 4.1.** (a) *For any $h \in L^2_{T,\tilde{S}}(\mathbb{R}, \mathbb{R}^{n_2})$ there is a unique function $x_2 \in W^{1,2}_{T,\tilde{S}}(\mathbb{R}, \mathbb{R}^{n_2})$ satisfying the equation $\dot{x}_2 = h$ and the estimate*

$$\|x_2\|_2 \le (\|(\tilde{S} - I)^{-1}\| + 1)T\|h\|_2,$$

*where $\|\cdot\|_2$ is the usual integral norm in $L^2((0,T), \mathbb{R}^{n_2})$.*

(b) *For any $h \in L^2_{T,0}(\mathbb{R}, \mathbb{R}^{n_1})$ there is a unique $u \in W^{1,2}_{T,0}(\mathbb{R}, \mathbb{R}^{n_1})$ satisfying $\dot{u} = h$ along with the estimate*

$$\|u\|_2 \le \frac{T}{2\pi}\|h\|_2.$$

Then from (4.3) we obtain

$$\|u\|_2^2 + \|x_2\|_2^2 \le \left(\frac{1}{4\pi^2} + (\|(\tilde{S} - I)^{-1}\| + 1)^2\right) T^2 L^2 \left(\|u\|_2^2 + \|x_2\|^2\right).$$

So if

$$\sqrt{\frac{1}{4\pi^2} + (\|(\tilde{S} - I)^{-1}\| + 1)^2} TL < 1,$$

then $u = 0$, $x_2 = 0$. Using results of Section 3, we arrive at the following

**Theorem 4.2.** *The number/period $T$ of any nonconstant solution of* (4.1)*) satisfying* (4.2) *fulfils the inequality*

$$T \geq L^{-1} \frac{2\pi}{\sqrt{9\pi^2 + 1}}$$

*for $\tilde{S} = -I$, and*

$$T \geq L^{-1} \frac{2\pi \sin(\pi \tilde{r}_1 / p)}{\sqrt{\sin^2(\pi \tilde{r}_1 / p) + \pi^2 (1 + 2 \sin(\pi \tilde{r}_1 / p))^2}}$$

*for $\tilde{S} \neq -I$, where $\tilde{r}_1$ is defined for $\tilde{S}$ as in Theorem 3.8 in place of $S$.*

*Remark* 4.3. Theorems 3.5, 3.8, 4.2 and Remark 3.10 completely solve the minimal period problem for $\mathbb{Z}_p$-symmetric autonomous ordinary differential equations, i. e., lower bounds for $T$ of $T$-$S$-symmetric solutions are derived for (3.1) satisfying assumptions (H1) and (H2) for a matrix $S : \mathbb{R}^n \to \mathbb{R}^n$ with $S^p = I$ for some $p \in \mathbb{N}$.

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# SUBDIRECTLY IRREDUCIBLE DISTRIBUTIVE NEARLATTICES

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ABSTRACT. A nearlattice means a join-semilattice having the property that every principal filter (or a section) is a lattice with respect to a semilattice order. The aim of the paper is to characterize in a simple way subdirectly irreducible distributive nearlattices.

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*Keywords:* nearlattice, semilattice, distributivity, subdirect irreducibility

## 1. INTRODUCTION

Algebraic structures being join-semilattices with respect to a naturally induced order relation appear very frequently in algebraic logic. For example, implication algebras, introduced by Abbott in the 1960s, describe algebraic properties of the logical connective implication in a classical propositional logic. Implication algebras have a very nice structure: with respect to the induced order, they are join-semilattices whose principal filters are Boolean algebras.

For various logics of quantum mechanics the corresponding algebraic structures have a semilattice structure with principal filters being special lattices, see, e. g., [1–5, 7].

This fact motivated I. Chajda and M. Kolařík to describe all $\vee$-semilattices where every principal filter is a lattice not satisfying any additional condition. More precisely, they studied the following structures:

**Definition 1.** A semilattice $\mathcal{S} = (S, \vee)$, where for each $a \in S$ the principal filter $[a) = \{x \in S \mid a \leq x\}$ is a lattice with respect to the induced order $\leq$ of $\mathcal{S}$, is called a *nearlattice.*

It has been shown [6] that nearlattices can be considered as algebras with one ternary operation. Moreover, nearlattices considered as algebras of type (3) form an equational class: indeed, if $x, y, z \in S$ for a nearlattice $\mathcal{S}$, the element $(x \vee z) \wedge (y \vee z)$ is correctly defined since both $x \vee z, y \vee z \in [z)$ and $[z)$ are lattices. The following proposition has been obtained in [6]:

**Proposition 1** ([6]). *Let $\mathcal{S} = (S, \vee)$ be a nearlattice. Define a ternary operation $m(x, y, z) = (x \vee z) \wedge (y \vee z)$ on $S$. Then $m(x, y, z)$ is an everywhere defined operation and the following identities are satisfied:*

(P1) $m(x, y, x) = x$,
(P2) $m(x, x, y) = m(y, y, x)$,
(P3) $m(m(x, x, y), m(x, x, y), z) = m(x, x, m(y, y, z))$,
(P4) $m(x, y, p) = m(y, x, p)$,
(P5) $m(m(x, y, p), z, p) = m(x, m(y, z, p), p)$,
(P6) $m(x, m(y, y, x), p) = m(x, x, p)$,
(P7) $m(m(x, x, p), m(x, x, p), m(y, x, p)) = m(x, x, p)$.

*Conversely, let $(S, m)$ be an algebra of type* (3) *satisfying the identities* (P1)–(P7). *If we define $x \vee y = m(x, x, y)$, then $(S, \vee)$ is a join-semilattice and for each $p \in S$, $([p), \leq)$ is a lattice, where for $x, y \in [p)$ their infimum is $x \wedge y = m(x, y, p)$. Hence $(S, \vee)$ is a nearlattice.*

Thus nearlattices similarly to lattices have two faces and we shall alternate in our investigations between them depending on which one is more convenient.

**Proposition 2** ([6]). *The variety of nearlattices is congruence distributive.*

The following notions of distributivity for nearlattices have been introduced in [6]:

**Definition 2.** Let $\mathcal{S} = (S, m)$ be an algebra of type (3). We call $\mathcal{S}$ *distributive* if it satisfies the identity

(D1) $m(x, m(y, y, z), p) = m(m(x, y, p), m(x, y, p), m(x, z, p))$.

If $(S, m)$ satisfies the identity

(D2) $m(x, x, m(y, z, p)) = m(m(x, x, y), m(x, x, z), p)$,

then it is called *dually distributive.*

It is expected that both notions are in case of nearlattices related. Indeed, one can prove the following statement:

**Proposition 3** ([6]). *Let $\mathcal{S} = (S, m)$ be an algebra of type* (3) *satisfying* (P1)–(P7). *Then the following conditions are equivalent:*

(1) *$\mathcal{S}$ is distributive,*
(2) *$\mathcal{S}$ is dually distributive,*
(3) *in the associated semilattice, every principal filter is a distributive lattice.*

In spite of the previous description of distributivity for nearlattices, we are able to prove by using very simple arguments that there is only one subdirectly irreducible nearlattice of the variety $\mathcal{D}N$ of distributive nearlattices, namely, the two element lattice.

## 2. Subdirectly irreducible distributive nearlattices

**Definition 3.** A subset $\varnothing \neq I \subseteq S$ of a nearlattice $\mathcal{S} = (S, \vee)$ is called an *ideal* if

(I1) $m(x, x, y) = x \vee y \in I$ for all $x, y \in I$,
(I2) $m(x, y, p) \in I$ for all $x, p \in I$ and $y \in S$.

Note that $I$ is an ideal of $\mathcal{S}$ iff $I$ is closed under suprema and $I$ is a downset of $(S, \leq)$.

Secondly, by a *filter* of $\mathcal{S} = (S, \vee)$ is meant a subset $\varnothing \neq F \subseteq S$ closed under imfima which is an upset of $(S, \leq)$.

An ideal $I$ of $(S, \vee)$ is called *prime* if for all $x, y \in S$, the existence of $x \wedge y$ and $x \wedge y \in I$ yield $x \in I$ or $y \in I$.

Let us mention that the primality of $I$ is equivalent to the following condition:

($*$) $\forall x, y, p \in S : (p \in I \text{ and } (x \vee p) \wedge (y \vee p) \in I) \Rightarrow x \vee p \in I$ or $y \vee p \in I$.

Indeed, if $I$ is prime, then evidently ($*$) holds. Conversely, assume that ($*$) holds and let $x \wedge y \in I$ exists. Then putting $p := x \wedge y$ in ($*$), we get $x = x \vee p$, $y = y \vee p$ and $(x \vee p) \wedge (y \vee p) = x \wedge y \in I$. But ($*$) yields $x = x \vee p \in I$ or $y = y \vee p \in I$ and $I$ is prime.

Clearly, for each $a \in S$, $(a] = \{x \in S \mid x \leq a\}$ is an ideal of $\mathcal{S} = (S, \vee)$ as well as $[a) = \{x \in S \mid a \leq x\}$ is a filter of $\mathcal{S} = (S, \vee)$. The set $\mathrm{Id}(\mathcal{S})$ of all ideals of $\mathcal{S}$ forms with respect to $\subseteq$ a complete lattice and for each $I \in \mathrm{Id}(\mathcal{S})$ and $a \in S$, the join $I \vee (a])$ in $\mathrm{Id}(\mathcal{S})$ is

$$I \vee (a] = \{z \in S \mid z \leq i \vee a \text{ for some } i \in I\}.$$

The following statement can be considered as a prime ideal theorem for nearlattices:

**Theorem 1.** *Let $\mathcal{S}$ be a distributive nearlattice, $I$ be an ideal of $\mathcal{S}$ and $D$ be a filter of $\mathcal{S}$. If $I \cap D = \varnothing$, then there is a prime ideal $P \supseteq I$ with $P \cap D = \varnothing$.*

Proof. It is evident that the uniton of a chain of ideals of $\mathcal{S}$ containing $I$ and having an empty intersection with $D$ is again an ideal of this kind. Hence, applying Zorn's lemma, there are maximal ideals $P \supseteq I$ with $P \cap D = \varnothing$.

Let us prove that $P$ is prime. If not, by ($*$) there exist $x, y, p \in S$ such that $x \vee p \notin P$, $y \vee p \notin P$ and $(x \vee p) \wedge (y \vee p) \in P$ for some $p \in P$.

Further, due the to maximality of $P$,

$$(P \vee (x \vee p]) \cap D \neq \varnothing, \quad (P \vee (y \vee p]) \cap D \neq \varnothing.$$

Hence there are $a, b \in D$ such that $a \leq r \vee x \vee p$ and $b \leq r \vee y \vee p$ for some $r, s \in P$.

Since $D$ is a filter of $\mathcal{S}$ and $a, b \in D$, also $r \vee x \vee p \in D$ and $s \vee y \vee p \in D$. Moreover, $r \vee x \vee p, s \vee y \vee p \in [p)$, thus also $(r \vee x \vee p) \wedge (s \vee y \vee p) \in D$. This gives also

$$\alpha = ((r \vee p) \vee (x \vee p)) \wedge ((s \vee p) \vee (y \vee p)) \in D.$$

Since all the elements mentioned are above $p$, in view of the distributivity of $[p)$ we derive

$$\alpha = ((r\vee p)\wedge(s\vee p))\vee((r\vee p)\wedge(y\vee p))\vee((x\vee p)\wedge(s\vee p))\vee((x\vee p)\wedge(y\vee p)) \in D.$$

Further, $r, s, p \in P$ yield $r \vee p, s \vee p \in P$ and all the elements in brackets of $\alpha$ belong to $P$. This gives $\alpha \in P \cap D$, which is a contradiction proving that $P$ is prime. □

**Corollary 1.** *Let $\mathcal{S} = (S, \vee)$ be a distributive nearlattice, $a, b \in S, a \neq b$. Then there is a prime ideal $P$ of $\mathcal{S}$ with $a \in P$ and $b \notin P$.*

PROOF. For $a \neq b$, we have either $(a] \cap [b) = \varnothing$ or $(b] \cap [a) = \varnothing$. The rest is a corollary of Theorem 1. □

Similarly as in distributive lattices, a typical example of a distributive nearlattice is a set-nearlattice:

**Definition 4.** For a set $S$, define on $\operatorname{Exp}(S)$ a binary operation by

$$m(X, Y, Z) := (X \cup Z) \cap (Y \cup Z).$$

Then $(\operatorname{Exp}(S), m)$ is a distributive nearlattice, called a *set-nearlattice* on $S$.

Let us show that any distributive nearlattice can be embedded into a set-nearlattice.

**Theorem 2.** *Let $\mathcal{S} = (S, \vee)$ be a distributive nearlattice, let $\mathcal{P}(S)$ be the set of all prime ideals of $\mathcal{S}$. Then the mapping $r : S \to \operatorname{Exp}(\mathcal{P}(S))$ defined by the formula*

$$r(a) := \{P \in \mathcal{P}(S) \mid a \notin P\}$$

*is an embedding of $\mathcal{S}$ into* $(\operatorname{Exp}(\mathcal{P}(S)), m)$.

PROOF. Due to the previous corollary, $r$ is an injection. So we have to prove that for all $a, b, c \in S$, the relation $r(m(a, b, c)) = m(r(a), r(b), r(c))$ is true, or

$$r((a \vee c) \wedge (b \vee c)) = (r(a) \cup r(c)) \cap (r(b) \cup r(c)). \tag{2.1}$$

Let us first show that for all $x, y \in S$,

$$r(x \wedge y) = r(x) \cap r(y). \tag{2.2}$$

Indeed, (2.2) is equivalent to the statement that for each $P \in \mathcal{P}(S)$, $x \wedge y \notin P$ if and only if $x \notin P$ and $y \notin P$. Then (2.1) can be rewritten into

$$r(a \vee c) \cap r(b \vee c) = (r(a) \cup r(c)) \cap (r(b) \cup r(c)). \tag{2.3}$$

Evidently, the inclusions $r(a \vee c) \cap r(b \vee c) \supseteq r(c)$ and $r(a \vee c) \cap r(b \vee c) \supseteq r(a) \cap r(b)$ are valid. Thus,

$$r(a \vee c) \cap r(b \vee c) \supseteq r(c) \cup (r(a) \cap r(b)) = (r(a) \cup r(c)) \cap (r(b) \cup r(c)).$$

Let us show the converse inclusion. For this, assume $P \in r(a \vee c) \cap r(b \vee c)$, i. e., $a \vee c \notin P$ and $b \vee c \notin P$. Then $P \notin r(a) \cup r(c)$ would lead to $a, c \in P$ and $a \vee c \in P$, which is a contradiction. Thus $P \in r(a) \cup r(c)$ and, analogously, $P \in r(b) \cup r(c)$, which altogether gives

$$r(a \vee c) \cap r(b \vee c) \subseteq (r(a) \cup r(c)) \cap (r(b) \cup r(c))$$

and completes the proof. □

**Corollary 2.** *Let $\mathcal{S} = (S, m)$ be a distributive nearlattice, $|S| \geq 2$. Then the identity $p \approx q$ holds in $\mathcal{S}$ iff $p \approx q$ holds in a 2-element nearlattice $\mathbf{C}_2$.*

PROOF. If $|S| \geq 2$, then $(S, \leq)$ contains a 2-element chain $M$. Moreover, $\mathcal{M} = (M, m)$ is a subalgebra of $\mathcal{S}$, hence if $p \approx q$ holds in $\mathcal{S}$, it holds also in $\mathcal{M}$.

Conversely, every set-nearlattice $(\mathrm{Exp}\,(B), m)$ is isomorphic to a direct product $\mathcal{M}^{|B|}$. Hence if $p \approx q$ holds in $\mathbf{C}_2 = \mathcal{M}$, it holds also in $(\mathrm{Exp}\,(B), m)$ and since $\mathcal{S}$ can be embedded into a set-nearlattice, $p \approx q$ holds in $\mathcal{S}$. □

It is clear that given a nearlattice $\mathcal{S} = (S, m)$ and $c \in S$, $[c)$ is a subalgebra of $\mathcal{S}$. Let us show that for an equivalence $\theta$ on $[c)$, $\theta \in \mathrm{Con}\,([c), m)$ iff $\theta \in \mathrm{Con}\,([c), \wedge, \vee)$.

Since $\wedge$ and $\vee$ are both term operations of $\mathcal{S}$, the implication $(\Rightarrow)$ is immediate. Conversely, if $\theta \in \mathrm{Con}\,([c), \wedge, \vee)$ and $(a, b) \in \theta$, $(x, y) \in \theta$ and $(u, v) \in \theta$ for some $a, b, x, y, u, v \geq c$, then

$$m(a, x, u) = (a \vee u) \wedge (x \vee u) \equiv_\theta (b \vee v) \wedge (y \vee v) = m(b, y, v),$$

verifying $\theta \in \mathrm{Con}\,([c), m)$.

Now we are ready to describe subdirectly irreducible members of the variety $\mathcal{D}N$ of all distributive nearlattices. Denote by $\theta(a, b)$ the principal congruence generated by $(a, b)$.

**Theorem 3.** *The variety $\mathcal{D}N$ has only one subdirectly irreducible member, namely* $\mathbf{C}_2$.

PROOF. Let $\mathcal{S} = (S, m)$ be a subdirectly irreducible distributive nearlattice, and assume that $\mu = \theta(a, b)$ is its monolith. We may assume $a \neq b$. Then $(a, b) \in \mu$ yields $(a, a \vee b) \in \mu, a \neq a \vee b$. Thus $\theta(a, a \vee b) \neq \omega$ and $\theta(a, b) \subseteq \theta(a, a \vee b)$.

Conversely, $(a, a \vee b) \in \mu$ gives $\theta(a, a \vee b) \subseteq \mu$ and altogether $\theta(a, a \vee b) = \theta(a, b) = \mu$.

Let now $c \in S$ be given, and denote $\mu_c = y \cap [c)^2$ the congruence induced on a subalgebra $[c)$. If $y_c \neq \omega$, then $y_c$ is the least nonzero congruence on the distributive lattice $([c), \wedge, \vee)$. Indeed, if $\theta \in \mathrm{Con}\,([c), \wedge, \vee)$ and $\theta \neq \omega$, then $\theta \in \mathrm{Con}\,([c), m)$. Since the variety $\mathcal{D}N$ is congruence distributive, $\mathcal{D}N$ has the congruence extension property. Thus there is $\psi \in \mathrm{Con}\,\mathcal{S}$ with $\theta = \psi \cap [c)^2$. Moreover, $\theta \neq \omega$ yields $\psi \neq \omega$, hence $\mu \subseteq \psi$ and $\mu_c = \mu \cap [c)^2 \subseteq \psi \cap [c)^2 = \theta$, proving that $\mu_c$ is the least nonzero congruence on $([c), m)$, and hence the monolith on $([c), \wedge, \vee)$. Now, since $\mu_a \neq \omega$, $\mu_a$ is the monolith on the distributive lattice $([a), \wedge, \vee)$, and hence

$|[a)| = 2$, i.e. $[a) = \{a, a \vee b\}$. This yields $a \vee b = 1$, the greatest element of $(S, \leq)$ (otherwise $[a) \neq \{a, a \vee b\}$).

Analogously, for every $c \leq a$, $\mu_c \neq \omega$ is the monolith on $[c)$, hence for $|[c)| = 2$ we obtain $c = a$. Then the nearlattice $\mathcal{S}$ is of the form shown on Figure 1.

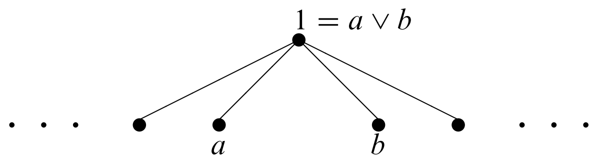


FIGURE 1.

It is easily seen that $\theta(1, a) = \{(1, a), (a, 1)\} \cup \omega$, so $\mu = \theta(1, a)$ forces $S = \{1, a\}$, which completes the proof.

$\square$

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# A CENTRAL LIMIT THEOREM FOR MIXING RANDOM FIELDS

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ABSTRACT. Detailed proofs of central limit theorems are given for mixing random fields. The results cover the case when the locations of the observations become dense in an increasing sequence of domains.

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*Keywords:* central limit theorem, $\alpha$-mixing, random field, asymptotic normality of estimators, infill asymptotics, increasing domain asymptotic

## 1. INTRODUCTION

Ibragimov and Linnik [6] proved a central limit theorem (CLT) for stationary sequences satisfying certain $\alpha$-mixing conditions. Bolthausen [1] and Guyon [11] extended it to $\alpha$-mixing random fields. Fazekas [10] and Fazekas and Kukush [8] presented the so-called infill-increasing versions of Guyon's result for the bounded and the uniformly integrable cases, respectively. These papers do not contain the proofs of the theorems mentioned (a sketch of the proof can be found in [9]). The aim of our paper is to give proofs for the above mentioned theorems. The importance of the detailed proof is the following. It turns out that the original proof by Ibragimov and Linnik [6] and Guyon [11] can be applied if the random field satisfies a certain uniform integrability condition. Guyon [11] does not assume the uniform integrability but he does not describe the step from the bounded case to the general case. Therefore we do not know if his result is valid in the general case. We mention that Ibragimov and Linnik [6] assumed stationarity, so their proof is complete.

Our Theorem 1 contains the bounded case (it is a version of the result of [10]) while Theorem 2 is the general central limit theorem (it is contained in [7]). Theorems 3 and 4 are the $p$-dimensional extensions of Theorems 1 and 2, respectively.

There is a vast literature of mixing random processes. Bradley [2] gives a recent survey of mixing conditions. Recent advances in the central limit theorems for random sequences are considered by Merlevède, Peligrad and Utev in [13].

In [4] a general CLT is proved for stationary random fields. As is pointed out in [4], certain mixing conditions imply the assumptions of the general CLT.

Note that our mixing conditions are not comparable with the ones in [4]. Moreover, in that paper stationary field and fixed designs are studied, while in our paper we do not assume stationarity and consider an infill-increasing setup.

## 2. Notation and preliminary remarks

The following notation is used. $\mathbb{Z}$ is the set of all integers, $\mathbb{Z}^d$ is the set of $d$-dimensional lattice points where $d$ is a fixed positive integer. $\mathbb{R}$ is the real line, $\mathbb{R}^d$ is the $d$-dimensional space with the usual Euclidean norm $\|\boldsymbol{x}\|$. In $\mathbb{R}^d$ we shall also consider the distance corresponding to the maximum norm

$$\varrho(\boldsymbol{x}, \boldsymbol{y}) = \max_{1 \le i \le d} |x_i - y_i|$$

where $\boldsymbol{x} = (x_1, \ldots, x_d)$, $\boldsymbol{y} = (y_1, \ldots, y_d)$. The distance of two sets in $\mathbb{R}^d$ corresponding to the maximum norm is also denoted by $\varrho : \varrho(A, B) = \inf\{\varrho(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{x} \in A, \ \boldsymbol{y} \in B\}$.

For real valued sequences $\{a_n\}$ and $\{b_n\}$, $a_n = \mathrm{o}\,(b_n)$ (resp., $a_n = \mathrm{O}\,(b_n)$) means that the sequence $a_n/b_n$ converges to 0 (resp., is bounded). We shall denote different constants with the same letter $c$. $\mathrm{I}\{A\}$ denotes the indicator function of the set $A$. $|\mathcal{D}|$ denotes the cardinality of the finite set $\mathcal{D}$.

We shall suppose the existence of an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The $\sigma$-algebra generated by a set of events or by a set of random variables will be denoted by $\sigma\{\cdot\}$. The sign $\mathbb{E}$ stands for the expectation. The variance and covariance are denoted by $\mathrm{var}\,(\cdot)$ and $\mathrm{cov}\,(\cdot,\cdot)$, respectively. The $L_p$-norm of a random (vector) variable $\eta$ is defined as

$$\|\eta\|_p = \left\{\mathbb{E}\|\eta\|^p\right\}^{1/p}, \ 1 \le p < \infty.$$

The sign "$\Rightarrow$" denotes the convergence in distribution. $\mathcal{N}(m, \Sigma)$ stands for the (vector) normal distribution with the mean (vector) $m$ and covariance (matrix) $\Sigma$.

The scheme of observations is the following. Let $T_1, T_2, \ldots$, and $T_\infty$ be domains in $\mathbb{R}^d$. Suppose that $T_1 \subset T_2 \subset T_3 \subset \ldots$, $\bigcup_{i=1}^{\infty} T_i = T_\infty$. Assume that $T_i$ is compact for each $i$, $T_\infty$ is of infinite Lebesgue measure. Let $\{\varepsilon(\boldsymbol{x}), \ \boldsymbol{x} \in T_\infty\}$ be a random field. The $n$-th set of observations consists of values of the random field $\varepsilon(\boldsymbol{x})$ taken at points $\boldsymbol{x}_{\boldsymbol{k}} \in T_n$, where $\boldsymbol{k} \in \mathcal{D}_n \subset \mathbb{Z}^d$. The choice of points $\boldsymbol{x}_{\boldsymbol{k}}$ is the following. Divide $\mathbb{R}^d$ into hyperrectangles

$$\Delta_n(\boldsymbol{k}) = \prod_{j=1}^{d} \left( \frac{k_j}{N_{jn}}, \frac{k_j + 1}{N_{jn}} \right],$$

where $\boldsymbol{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ is a $d$-dimensional integer lattice point and $\{N_{jn}\}$ is an increasing and unbounded sequence of positive integers for each $j = 1, \ldots, d$. Now, select the $n$ -th data sites $\boldsymbol{x}_{\boldsymbol{k}}$, $\boldsymbol{k} \in \mathcal{D}_n$, by choosing an arbitrary point $\boldsymbol{x}_{\boldsymbol{k}}$ from each $\Delta_n(\boldsymbol{k}) \cap T_n$ which is non-empty. Actually, each $\boldsymbol{x}_{\boldsymbol{k}} = \boldsymbol{x}_{\boldsymbol{k}}^{(n)}$ depends on

$n$ but to avoid complicated notation we often omit superscript $(n)$. We suppose that $\lim_{n\to\infty} |\mathcal{D}_n| = \infty$.

As the locations of the observations become more and more dense in an increasing sequence of domains, we call our setup infill-increasing (see [3, 12] for infill asymptotics).

Define the discrete parameter random field $Y_n(\boldsymbol{k})$ as follows. For arbitrary $n = 1, 2, \ldots$ and $\boldsymbol{k} \in \mathcal{D}_n$, let $Y_n(\boldsymbol{k})$ be a Borel measurable function of $\varepsilon(\boldsymbol{x}_{\boldsymbol{k}}^{(n)})$.

We need the notion of $\alpha$-mixing (see, e. g., [5, 11]). Let $\mathcal{A}$ and $\mathcal{B}$ be two $\sigma$-algebras in $\mathcal{F}$. The $\alpha$-mixing coefficient of $\mathcal{A}$ and $\mathcal{B}$ is

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup\{|\mathbb{P}(A)\mathbb{P}(B) - \mathbb{P}(AB)| : A \in \mathcal{A}, B \in \mathcal{B}\}.$$

The $\alpha$-mixing coefficient of $\{\varepsilon(\boldsymbol{x}) : \boldsymbol{x} \in T_\infty\}$ is

$$\alpha(r, u, v) = \sup\{\alpha(\mathcal{F}_{I_1}, \mathcal{F}_{I_2}) : \varrho(I_1, I_2) \geq r, |I_1| \leq u, |I_2| \leq v\}$$

where $I_1$ and $I_2$ are finite subsets in $T_\infty$, $\mathcal{F}_{I_i} = \sigma\{\varepsilon(\boldsymbol{x}) : \boldsymbol{x} \in I_i\}$, $i = 1, 2$.

We say that the random field $\{\varepsilon(\boldsymbol{x})\}$ is $\alpha$ -mixing if its mixing coefficients satisfy some conditions. All of these conditions mean a weak defence of the field, that is $\alpha(r, u, v)$ is small if $r$ is large.

We list the conditions that will be used in our theorems.

$$\int_0^\infty s^{d-1}\alpha^{\frac{\tau}{2+\tau}}(s, 1, 1)ds < \infty \text{ for some } 0 < \tau < 1; \tag{2.1}$$

$$\int_0^\infty s^{d-1}\alpha(s, i, j)ds < \infty \text{ for } i + j \leq 4; \tag{2.2}$$

$$\alpha(s, 1, \infty) = \mathrm{o}\,(s^{-d}) \text{ as } s \to \infty; \tag{2.3}$$

$$\Lambda_n = \mathrm{O}\,(\lambda_n) \text{ as } n \to \infty, \tag{2.4}$$

where

$$\Lambda_n = \max_{1\leq j\leq d} N_{jn}, \qquad \lambda_n = \min_{1\leq j\leq d} N_{jn}. \tag{2.5}$$

Actually (2.3) means that $\lim_{n\to\infty} \alpha(s_n, 1, k_n)s_n^d = 0$ if $s_n \to \infty$ and $k_n \to \infty$.

Let $\alpha_n(r, i, j)$ denote the $\alpha$-mixing coefficients of $Y_n(\boldsymbol{k})$. As $\boldsymbol{x}_{\boldsymbol{k}} \in \Delta_n(\boldsymbol{k})$ and $\boldsymbol{x}_{\boldsymbol{\ell}} \in \Delta_n(\boldsymbol{\ell})$ (where $\boldsymbol{\ell} = (l_1, \ldots, l_d)$), we have

$$\varrho(\boldsymbol{x}_{\boldsymbol{k}}, \boldsymbol{x}_{\boldsymbol{\ell}}) \geq \max\left\{\frac{|k_1 - l_1| - 1}{N_{1n}}, \ldots, \frac{|k_d - l_d| - 1}{N_{dn}}\right\} \geq \frac{\varrho(\boldsymbol{k}, \boldsymbol{\ell}) - 1}{\Lambda_n} \tag{2.6}$$

and

$$\varrho(\boldsymbol{x}_{\boldsymbol{k}}, \boldsymbol{x}_{\boldsymbol{\ell}}) \leq \frac{\varrho(\boldsymbol{k}, \boldsymbol{\ell}) + 1}{\lambda_n} \tag{2.7}$$

where $\Lambda_n$, $\lambda_n$ are given in (2.5). Therefore the $\alpha$-mixing coefficients $\alpha_n(r, i, j)$ of $Y_n(\boldsymbol{k})$ satisfy

$$\alpha\left(\frac{r+1}{\lambda_n}, i, j\right) \leq \alpha_n(r, i, j) \leq \alpha\left(\frac{r-1}{\Lambda_n}, i, j\right), \quad r = 1, 2, \ldots.$$

**Lemma 1.** *For $\gamma > 0$ and positive integers $i,\ j,\ n$, the relation*

$$\sum_{r=1}^{\infty} r^{d-1}\alpha_n^{\gamma}(r,i,j) \le c\left(1 + \Lambda_n^d \int_0^{\infty} r^{d-1}\alpha^{\gamma}(r,i,j)dr\right) \tag{2.8}$$

*holds, where the constant $c$ depends only on $d$.*

PROOF. We have

$$\begin{aligned}\sum_{r=1}^{\infty} r^{d-1}\alpha_n^{\gamma}(r,i,j) &\le c\left(1 + \sum_{r=1}^{\infty} r^{d-1}\alpha^{\gamma}\left(\frac{r}{\Lambda_n},i,j\right)\right)\\ &\le c\left(1 + \sum_{r=2}^{\infty}\int_{r-1}^{r} s^{d-1}\alpha^{\gamma}\left(\frac{s}{\Lambda_n},i,j\right)ds\right)\\ &\le c\left(1 + \int_0^{\infty} s^{d-1}\alpha^{\gamma}\left(\frac{s}{\Lambda_n},i,j\right)ds\right)\\ &\le c\left(1 + \Lambda_n^d\int_0^{\infty} s^{d-1}\alpha^{\gamma}(s,i,j)ds\right),\end{aligned}$$

which leads us to (2.8). □

*Remark* 1 (Davydov's inequality [5, p. 9]). The following well-known covariance inequalities are basic tools for mixing fields.

$$|\operatorname{cov}(X,Y)| \le 8\,[\alpha(\sigma(X),\sigma(Y))]^{\frac{1}{r}}\ \|X\|_p\|Y\|_q \tag{2.9}$$

for $r,p,q \ge 1$, $r^{-1}+p^{-1}+q^{-1} = 1$. In the special case where $X$ and $Y$ belong to $L_{\infty}$, we have

$$|\operatorname{cov}(X,Y)| \le 4\,[\alpha(\sigma(X),\sigma(Y))]\,\|X\|_{\infty}\|Y\|_{\infty}. \tag{2.10}$$

The following inequality is a special case of the Rosenthal inequality. For the proof, see [5].

**Lemma 2.** *Let $\xi_{\boldsymbol{k}}$, $\boldsymbol{k} \in \mathbb{Z}^d$, be centered random variables with $\mathbb{E}|\xi_{\boldsymbol{k}}|^{l+\tau} < \infty$, $\boldsymbol{k} \in \mathbb{Z}^d$. Introduce the notation*

$$L(h,\tau,\mathcal{D}) = \sum_{\boldsymbol{k}\in\mathcal{D}}\left(\mathbb{E}|\xi_{\boldsymbol{k}}|^{h+\tau}\right)^{\frac{h}{h+\tau}}$$

*if $1 < h \le 2$, $\tau \ge 0$ and $\mathcal{D}$ is a finite set in $\mathbb{Z}^d$. Let*

$$c_{1,1}^{(\tau)} = 1 + \sum_{s=1}^{\infty} s^{d-1}[\alpha_{\xi}(s,1,1)]^{\frac{\tau}{2+\tau}}$$

*where* $\alpha_\xi(s,1,1)$ *is the* $\alpha$*-mixing coefficient of the field* $\{\xi_{\boldsymbol{k}}\}$. *Now, let* $1 < l \le 2$ *and* $\tau > 0$. *Assume that* $c_{1,1}^{(\tau)} < \infty$. *Then there is a constant* $c$ *such that*

$$\mathbb{E}\left|\sum_{\boldsymbol{k}\in\mathcal{D}} \xi_{\boldsymbol{k}}\right|^{l} \le c \cdot c_{1,1}^{(\tau)} L(l,\tau,\mathcal{D}) \tag{2.11}$$

*for any finite subset* $\mathcal{D}$ *of* $\mathbb{Z}^d$.

We remark that the proof of the Rosenthal inequality (2.11) follows from (2.12) below, by using the so-called interpolation lemma. Details and the general form of the Rosenthal inequality can be found, e. g., in [8].

*Remark* 2. Using the notation of Lemma 2, let $\tau > 0$, and assume that $c_{1,1}^{(\tau)} < \infty$. Then there is a constant $c$ such that

$$\sum_{\boldsymbol{k},\boldsymbol{\ell}\in\mathcal{D}} |\operatorname{cov}(\xi_{\boldsymbol{k}},\xi_{\boldsymbol{\ell}})| \le c \cdot c_{1,1}^{(\tau)} L(2,\tau,\mathcal{D}) \tag{2.12}$$

for any finite subset $\mathcal{D}$ of $\mathbb{Z}^d$.

PROOF. For the sake of completeness we prove (2.12). By (2.9),

$$\sum_{\boldsymbol{k},\boldsymbol{\ell}\in\mathcal{D}} |\operatorname{cov}(\xi_{\boldsymbol{k}},\xi_{\boldsymbol{\ell}})| \le$$
$$\le \sum_{\boldsymbol{k}\in\mathcal{D}} \|\xi_{\boldsymbol{k}}\|_2^2 + \sum_{\substack{\boldsymbol{k},\boldsymbol{\ell}\in\mathcal{D}\\ \boldsymbol{k}\neq\boldsymbol{\ell}}} 8\left[\alpha_\xi(\|\boldsymbol{k}-\boldsymbol{\ell}\|,1,1)\right]^{\frac{\tau}{2+\tau}} \|\xi_{\boldsymbol{k}}\|_{2+\tau}\|\xi_{\boldsymbol{\ell}}\|_{2+\tau}.$$

By the inequality between the geometric and arithmetic means, the above expression is majorized by

$$\sum_{\boldsymbol{k}\in\mathcal{D}} \|\xi_{\boldsymbol{k}}\|_{2+\tau}^2 + \sum_{\substack{\boldsymbol{k},\boldsymbol{\ell}\in\mathcal{D}\\ \boldsymbol{k}\neq\boldsymbol{\ell}}} 8\left[\alpha_\xi(\|\boldsymbol{k}-\boldsymbol{\ell}\|,1,1)\right]^{\frac{\tau}{2+\tau}} \|\xi_{\boldsymbol{k}}\|_{2+\tau}^2 \le$$
$$\le \sum_{\boldsymbol{k}\in\mathcal{D}} \|\xi_{\boldsymbol{k}}\|_{2+\tau}^2 + \sum_{\boldsymbol{k}\in\mathcal{D}} c \sum_{s=1}^{\infty} s^{d-1}[\alpha_\xi(s,1,1)]^{\frac{\tau}{2+\tau}} \|\xi_{\boldsymbol{k}}\|_{2+\tau}^2.$$

This gives (2.12). □

## 3. MAIN THEOREMS

**Theorem 1.** *Let* $\varepsilon(\boldsymbol{x})$ *be a random field and let* $Y_n(\boldsymbol{k})$ *be a Borel measurable function of* $\varepsilon(\boldsymbol{x}_{\boldsymbol{k}}^{(n)})$, $\boldsymbol{k} \in \mathcal{D}_n$. *Suppose that* $\mathbb{E}Y_n(\boldsymbol{k}) = 0$ *and* $|Y_n(\boldsymbol{k})|$ *are uniformly* *bounded for* $\boldsymbol{k} \in \mathcal{D}_n$, $n = 1, 2, \ldots$. *Let* $S_n = \sum_{\boldsymbol{k}\in\mathcal{D}_n} Y_n(\boldsymbol{k})$, $n = 1, 2, \ldots$, $\sigma_n^2 = \operatorname{var}(S_n)$. *Let conditions* (2.2), (2.3), *and* (2.4) *be satisfied. Assume that*

$$\liminf_{n\to\infty} \frac{\sigma_n^2}{\Lambda_n^d |\mathcal{D}_n|} > 0. \tag{3.1}$$

*Then* $\sigma_n^{-1} S_n \Rightarrow \mathcal{N}(0,1)$ *as* $n \to \infty$.

PROOF. We follow ideas of [11]. Throughout the proof we shall suppose that the random variables $Y_n(\boldsymbol{k})$ are uniformly bounded with the bound 1: $|Y_n(\boldsymbol{k})| \le 1$, $\boldsymbol{k} \in \mathcal{D}_n$, $n = 1, 2, \ldots$.

Choose a sequence $\{m_n\}$ of positive integers such that $\lim_{n\to\infty} m_n = \infty$,

$$\lim_{n\to\infty} \alpha(m_n, 1, \infty) |\mathcal{D}_n|^{\frac{1}{2}} \Lambda_n^{-\frac{d}{2}} = 0 \tag{3.2}$$

and

$$\lim_{n\to\infty} m_n^{-d} |\mathcal{D}_n|^{\frac{1}{2}} \Lambda_n^{-\frac{d}{2}} = \infty. \tag{3.3}$$

To this end, let $x_n = \alpha(n, 1, \infty)$, $y_n = n^{-d}$ and $z_n = |\mathcal{D}_n|^{\frac{1}{2}} \Lambda_n^{-\frac{d}{2}}$ in Lemma 3 below. Then, by (2.3), $x_n/y_n \to 0$. Moreover $z_n \to \infty$ because $|\mathcal{D}_n| \Lambda_n^{-d} \ge c\mu(T_n) \to \infty$. Here, $\mu(T_n)$ is the Lebesgue measure of $T_n$.

Let $S_n(\boldsymbol{k}) = \sum_{\boldsymbol{\ell}\in\mathcal{D}_n,\, \varrho(\boldsymbol{x}_{\boldsymbol{k}}, \boldsymbol{x}_{\boldsymbol{\ell}}) \le m_n} Y_n(\boldsymbol{\ell})$ and $S_n^*(\boldsymbol{k}) = S_n - S_n(\boldsymbol{k})$ for any $\boldsymbol{k} \in \mathcal{D}_n$. Let $a_n = \sum_{\boldsymbol{\ell}\in\mathcal{D}_n} \mathbb{E}\,(Y_n(\boldsymbol{\ell}) S_n(\boldsymbol{\ell}))$, $\overline{S}_n = a_n^{-\frac{1}{2}} S_n$, and $\overline{S}_n(\boldsymbol{k}) = a_n^{-\frac{1}{2}} S_n(\boldsymbol{k})$. We have

$$\sigma_n^2 = \operatorname{var}(S_n) = a_n + \sum_{\boldsymbol{\ell}\in\mathcal{D}_n} \mathbb{E}\left(Y_n(\boldsymbol{\ell}) S_n^*(\boldsymbol{\ell})\right).$$

Using (2.10) and the same argument as in Lemma 1, we obtain

$$\begin{aligned}
\sigma_n^2 - a_n &= \left| \sum_{\boldsymbol{\ell}\in\mathcal{D}_n} \mathbb{E}\left(Y_n(\boldsymbol{\ell}) S_n^*(\boldsymbol{\ell})\right) \right| \\
&\le \sum_{\boldsymbol{k},\boldsymbol{\ell}\in\mathcal{D}_n,\ \varrho(\boldsymbol{x}_{\boldsymbol{k}}, \boldsymbol{x}_{\boldsymbol{\ell}}) \ge m_n} |\operatorname{cov}(Y_n(\boldsymbol{k}), Y_n(\boldsymbol{\ell}))| \\
&\le c|\mathcal{D}_n| \sum_{s=m_n\lambda_n-1}^{\infty} s^{d-1} \alpha\left(\frac{s-1}{\Lambda_n}, 1, 1\right) \\
&\le c|\mathcal{D}_n| \int_{m_n\lambda_n-2}^{\infty} s^{d-1} \alpha\left(\frac{s-1}{\Lambda_n}, 1, 1\right) ds \\
&\le c|\mathcal{D}_n| \Lambda_n^d \int_{\frac{m_n\lambda_n-3}{\Lambda_n}}^{\infty} s^{d-1} \alpha(s, 1, 1)\, ds \le c|\mathcal{D}_n| \Lambda_n^d \cdot \mathrm{o}(1) \le \sigma_n^2 \cdot \mathrm{o}(1).
\end{aligned}$$

In the last steps we used (2.2) and (3.1). Hence

$$\lim_{n\to\infty} \frac{a_n}{\sigma_n^2} = 1. \tag{3.4}$$

Therefore it is sufficient to prove the asymptotic normality of $\overline{S}_n$.

Since $\sup_n \mathbb{E}\overline{S}_n^2 < \infty$, by Stein's lemma (see Remark 3) it is sufficient to prove that

$$\lim_{n\to\infty} \mathbb{E}\left((it - \overline{S}_n)\, e^{it\overline{S}_n}\right) = 0 \quad \text{for every} \quad t \in \mathbb{R}. \tag{3.5}$$

Consider the decomposition

$$(it - \overline{S}_n)\, e^{it\overline{S}_n} = A_1 - A_2 - A_3,$$

where

$$A_1 = ite^{it\overline{S}_n}\left(1 - \frac{1}{a_n}\sum_{\boldsymbol{\ell}\in\mathcal{D}_n} Y_n(\boldsymbol{\ell}) S_n(\boldsymbol{\ell})\right),$$

$$A_2 = a_n^{-\frac{1}{2}} e^{it\overline{S}_n} \sum_{\boldsymbol{\ell}\in\mathcal{D}_n} Y_n(\boldsymbol{\ell})\left(1 - it\overline{S}_n(\boldsymbol{\ell}) - e^{-it\overline{S}_n(\boldsymbol{\ell})}\right),$$

$$A_3 = a_n^{-\frac{1}{2}} \sum_{\boldsymbol{\ell}\in\mathcal{D}_n} Y_n(\boldsymbol{\ell})\, e^{it(\overline{S}_n - \overline{S}_n(\boldsymbol{\ell}))}.$$

First we prove that $\lim_{n\to\infty} \mathbb{E}|A_1|^2 = 0$. We have

$$\mathbb{E}|A_1|^2 = t^2 a_n^{-2} \operatorname{var}\left(\sum_{\boldsymbol{\ell}\in\mathcal{D}_n} Y_n(\boldsymbol{\ell}) S_n(\boldsymbol{\ell})\right) =$$

$$= t^2 a_n^{-2} \sum_{\substack{\boldsymbol{j},\, \boldsymbol{j}',\, \boldsymbol{\ell},\, \boldsymbol{\ell}'\in\mathcal{D}_n,\\ \varrho(\boldsymbol{x}_{\boldsymbol{j}}, \boldsymbol{x}_{\boldsymbol{\ell}})\le m_n,\\ \varrho(\boldsymbol{x}_{\boldsymbol{j}'}, \boldsymbol{x}_{\boldsymbol{\ell}'})\le m_n}} \operatorname{cov}\left(Y_n(\boldsymbol{j})Y_n(\boldsymbol{\ell}), Y_n(\boldsymbol{j}')Y_n(\boldsymbol{\ell}')\right). \tag{3.6}$$

Now we distinguish two cases. First suppose that $\varrho(\boldsymbol{x}_{\boldsymbol{j}}, \boldsymbol{x}_{\boldsymbol{j}'}) = k \ge \frac{3m_n\Lambda_n}{\lambda_n}$. Then $\varrho(\{\boldsymbol{x}_{\boldsymbol{j}}, \boldsymbol{x}_{\boldsymbol{\ell}}\}, \{\boldsymbol{x}_{\boldsymbol{j}'}, \boldsymbol{x}_{\boldsymbol{\ell}'}\}) \ge k - 2m_n$. Then, by the covariance inequality (2.10),

$$\left|\operatorname{cov}\left(Y_n(\boldsymbol{j})Y_n(\boldsymbol{\ell}), Y_n(\boldsymbol{j}')Y_n(\boldsymbol{\ell}')\right)\right| \le 4\alpha(k - 2m_n, 2, 2),$$

because $|Y_n(\boldsymbol{\ell})| \le 1$ for each $\boldsymbol{\ell}$ and $n$. Now, we can choose $\boldsymbol{j}$ in $|\mathcal{D}_n|$ ways, then $\boldsymbol{\ell}$ at most in $m_n^d \Lambda_n^d$ ways, and when $\boldsymbol{j}'$ is chosen, then we can choose $\boldsymbol{\ell}'$ at most in

$m_n^d \Lambda_n^d$ ways. So the expression in (3.6) is less than or equal to

$$ct^2 a_n^{-2} |\mathcal{D}_n| m_n^{2d} \Lambda_n^{2d} \sup_{\boldsymbol{j} \in \mathcal{D}_n} \sum_{\{\boldsymbol{j}' \,:\, \varrho(\boldsymbol{x}_{\boldsymbol{j}}, \boldsymbol{x}_{\boldsymbol{j}'}) = k \geq \frac{3 m_n \Lambda_n}{\lambda_n}\}} \alpha(\varrho(\boldsymbol{x}_{\boldsymbol{j}}, \boldsymbol{x}_{\boldsymbol{j}'}) - 2m_n, 2, 2)$$
$$\leq ct^2 a_n^{-2} |\mathcal{D}_n| m_n^{2d} \Lambda_n^{2d} \sum_{s = 3 m_n \Lambda_n - 1}^{\infty} s^{d-1} \alpha\left(\frac{s-1}{\Lambda_n} - 2m_n, 2, 2\right).$$

In the last step we used (2.6) and (2.7). We can majorize the above sum by the following integral:

$$ct^2 a_n^{-2} |\mathcal{D}_n| m_n^{2d} \Lambda_n^{2d} \int_{3 m_n \Lambda_n - 2}^{\infty} s^{d-1} \alpha\left(\frac{s-1}{\Lambda_n} - 2m_n, 2, 2\right) ds$$
$$\leq ct^2 a_n^{-2} |\mathcal{D}_n| m_n^{2d} \Lambda_n^{2d+1} \int_{m_n + \mathrm{o}(1)}^{\infty} (\Lambda_n (s + 2m_n) + 1)^{d-1} \alpha(s, 2, 2) ds$$
$$\leq ct^2 a_n^{-2} |\mathcal{D}_n| m_n^{2d} \Lambda_n^{3d} \int_0^{\infty} s^{d-1} \alpha(s, 2, 2) ds.$$

Now, we can use (3.1) and (3.4) to show that $a_n^{-1} \leq c|\mathcal{D}_n|^{-1} \Lambda_n^{-d}$. Therefore the above expression is majorized by $c|\mathcal{D}_n|^{-1} m_n^{2d} \Lambda_n^d$, which converges to 0, as $n \to \infty$, by the choice of $m_n$ (see (3.2), (3.3)).

In the second case we suppose that $\varrho(\boldsymbol{x}_{\boldsymbol{j}}, \boldsymbol{x}_{\boldsymbol{j}'}) = k < \frac{3 m_n \Lambda_n}{\lambda_n}$. Let

$$h = \inf\{\varrho(\boldsymbol{x}_{\boldsymbol{j}}, \boldsymbol{x}_{\boldsymbol{j}'}), \varrho(\boldsymbol{x}_{\boldsymbol{j}}, \boldsymbol{x}_{\boldsymbol{\ell}}), \varrho(\boldsymbol{x}_{\boldsymbol{j}}, \boldsymbol{x}_{\boldsymbol{\ell}'})\}.$$

Then, by the covariance inequality (2.10),

$$\left|\mathrm{cov}\left(Y_n(\boldsymbol{j}) Y_n(\boldsymbol{\ell}), Y_n(\boldsymbol{j}') Y_n(\boldsymbol{\ell}')\right)\right|$$
$$\leq \left|\mathbb{E}\left(Y_n(\boldsymbol{j}) Y_n(\boldsymbol{\ell}) Y_n(\boldsymbol{j}') Y_n(\boldsymbol{\ell}')\right)\right| + \left|\mathbb{E}\left(Y_n(\boldsymbol{j}) Y_n(\boldsymbol{\ell})\right)\right| \left|\mathbb{E}\left(Y_n(\boldsymbol{j}') Y_n(\boldsymbol{\ell}')\right)\right|$$
$$\leq 4\alpha(h, 1, 3) + 4\alpha(h, 1, 1) \leq 8\alpha(h, 1, 3),$$

because $|Y_n(\boldsymbol{\ell})| \leq 1$ for each $\boldsymbol{\ell}$ and $n$. Suppose that $h = \varrho(\boldsymbol{x}_{\boldsymbol{j}}, \boldsymbol{x}_{\boldsymbol{\ell}})$ (the other two cases can be studied similarly). Then we can choose $\boldsymbol{j}$ in $|\mathcal{D}_n|$ ways, $\boldsymbol{j}'$ at most in $\Lambda_n^d \left(3 m_n \Lambda_n (\lambda_n)^{-1}\right)^d$ ways, and $\boldsymbol{\ell}'$ at most in $\Lambda_n^d m_n^d$ ways. So the expression in

(3.6) is less than or equal to

$$
\begin{aligned}
&ct^2 a_n^{-2} |\mathcal{D}_n| \Lambda_n^d \left(\frac{3m_n\Lambda_n}{\lambda_n}\right)^d \Lambda_n^d m_n^d \sum_{\{\boldsymbol{\ell}\,:\,\varrho(\boldsymbol{x_j},\boldsymbol{x_\ell})=h\le m_n\}} \alpha(h,1,3) \\
&\quad\le ct^2 a_n^{-2} |\mathcal{D}_n| \Lambda_n^{3d} m_n^{2d} \lambda_n^{-d} \sum_{\{\boldsymbol{\ell}\,:\,\varrho(\boldsymbol{j},\boldsymbol{\ell})=k\le \Lambda_n m_n+1\}} \alpha\left(\frac{k-1}{\Lambda_n},1,3\right) \\
&\quad\le ct^2 a_n^{-2} |\mathcal{D}_n| \Lambda_n^{2d} m_n^{2d} \left(1+\sum_{k=1}^{\Lambda_n m_n+1} k^{d-1}\alpha\left(\frac{k-1}{\Lambda_n},1,3\right)\right) \\
&\quad\le ct^2 a_n^{-2} |\mathcal{D}_n| \Lambda_n^{2d} m_n^{2d} \left(1+\int_0^{\Lambda_n m_n+1} s^{d-1}\alpha\left(\frac{s-1}{\Lambda_n},1,3\right) ds\right) \\
&\quad\le ct^2 a_n^{-2} |\mathcal{D}_n| \Lambda_n^{3d} m_n^{2d} \to 0, \quad n\to\infty,
\end{aligned}
$$

as it was shown above. Therefore both parts of the sum in (3.6) converge to 0, so $\lim_{n\to\infty} \mathbb{E}|A_1|^2 = 0$.

Now, we turn to $A_2$. By Taylor's expansion, we have

$$|1 - it\overline{S}_n(\boldsymbol{\ell}) - e^{-it\overline{S}_n(\boldsymbol{\ell})}| \le ct^2\,\overline{S}_n^2(\boldsymbol{\ell})$$

and, therefore,

$$
\begin{aligned}
\mathbb{E}|A_2| &\le a_n^{-\frac{1}{2}} \sum_{\boldsymbol{\ell}\in\mathcal{D}_n} \mathbb{E}\left|1 - it\overline{S}_n(\boldsymbol{\ell}) - e^{-it\overline{S}_n(\boldsymbol{\ell})}\right| \\
&\le a_n^{-\frac{1}{2}} |\mathcal{D}_n| \sup_{\boldsymbol{\ell}\in\mathcal{D}_n} \mathbb{E}\left(ct^2\overline{S}_n^2(\boldsymbol{\ell})\right) \\
&\le c a_n^{-\frac{3}{2}} |\mathcal{D}_n| \sup_{\boldsymbol{\ell}} \sum_{\varrho(\boldsymbol{\ell},:\boldsymbol{j})\le\Lambda_n m_n,\ \varrho(\boldsymbol{\ell},:\boldsymbol{j}')\le\Lambda_n m_n} \left|\operatorname{cov}\left(Y_n(\boldsymbol{j}), Y_n(\boldsymbol{j}')\right)\right| \\
&\le c a_n^{-\frac{3}{2}} |\mathcal{D}_n| m_n^d \Lambda_n^{2d} \to 0
\end{aligned}
$$

as $n\to\infty$, because of the choice of $m_n$, the relation between $a_n$ and $\sigma_n^2$, (3.1) and (3.2). In the above calculation, we have applied the covariance inequality and the majorization of the sum by an integral as we did before. Hence, $\lim_{n\to\infty} \mathbb{E}|A_2| = 0$.

Now, we shall prove that $\lim_{n\to\infty} \mathbb{E}A_3 = 0$.

$$
\begin{aligned}
|\mathbb{E}A_3| &\le a_n^{-\frac{1}{2}} \sum_{\boldsymbol{\ell}\in\mathcal{D}_n} \left|\operatorname{cov}\left(Y_n(\boldsymbol{\ell}), e^{it(\overline{S}_n-\overline{S}_n(\boldsymbol{\ell}))}\right)\right| \le \\
&\qquad \le a_n^{-\frac{1}{2}} |\mathcal{D}_n| \alpha(m_n,1,\infty) \le c|\Lambda_n|^{-\frac{d}{2}} |\mathcal{D}_n|^{\frac{1}{2}} \alpha(m_n,1,\infty) \to 0,
\end{aligned}
$$

by the choice of $m_n$.

So (3.5) is proved and, therefore, the theorem is proved. $\square$

**Theorem 2.** *Let $\varepsilon(\boldsymbol{x})$ be a random field and let $Y_n(\boldsymbol{k})$ be a Borel measurable function of $\varepsilon(\boldsymbol{x}_{\boldsymbol{k}}^{(n)})$, $\boldsymbol{k} \in \mathcal{D}_n$. Suppose that $\mathbb{E}Y_n(\boldsymbol{k}) = 0$ for $\boldsymbol{k} \in \mathcal{D}_n$, $n = 1, 2, \ldots$. Let $S_n = \sum_{\boldsymbol{k}\in\mathcal{D}_n} Y_n(\boldsymbol{k})$, $n = 1, 2, \ldots$ , $\sigma_n^2 = \operatorname{var}(S_n)$. Suppose that there exists a $\tau > 0$ such that* (2.1)*is satisfied and*

$$\{|Y_n(\boldsymbol{k})|^{2+\tau} : \boldsymbol{k} \in \mathcal{D}_n,\ n = 1, 2, \ldots\} \text{ are uniformly integrable.} \tag{3.7}$$

*Then*

$$\limsup_{n\to\infty} \frac{1}{\Lambda_n^d |\mathcal{D}_n|} \sum_{\boldsymbol{k},\boldsymbol{\ell}\in\mathcal{D}_n} |\operatorname{cov}(Y_n(\boldsymbol{k}), Y_n(\boldsymbol{\ell}))| < \infty. \tag{3.8}$$

*If, additionally, conditions* (2.2), (2.3), (2.4), *and* (3.1) *are satisfied, then $\sigma_n^{-1} S_n \Rightarrow \mathcal{N}(0,1)$ as $n \to \infty$.*

PROOF. First we prove (3.8). By the Rosenthal inequality (2.11),

$$\sum_{\boldsymbol{k},\boldsymbol{\ell}\in\mathcal{D}_n} |\operatorname{cov}(Y_n(\boldsymbol{k}), Y_n(\boldsymbol{\ell}))| \le$$

$$\le c\left(1 + \sum_{s=1}^{\infty} [\alpha_n(s,1,1)]^{\frac{\tau}{2+\tau}} s^{d-1}\right) \sum_{\boldsymbol{k}\in\mathcal{D}} \left(\mathbb{E}|Y_{\boldsymbol{k}}|^{2+\tau}\right)^{\frac{2}{2+\tau}}.$$

Due to Lemma 1, this expression is majorized by

$$c \cdot \left(1 + \Lambda_n^d \int_0^\infty s^{d-1} \alpha^{\frac{\tau}{2+\tau}}(s,1,1)\,ds\right) \times$$

$$\times |\mathcal{D}_n| \sup\left\{\|Y_n(\boldsymbol{k})\|_{2+\tau}^2 \ :\ \boldsymbol{k} \in \mathcal{D}_n,\ n = 1, 2, \ldots\right\}.$$

Therefore, (3.7) and (2.1) imply (3.8).

Now, we show that it is sufficient to prove the theorem for uniformly bounded random variables $\{Y_n(\boldsymbol{k}) : \boldsymbol{k} \in \mathcal{D}_n,\ n = 1, 2, \ldots\}$. So Theorem 1 will imply the result. We follow the ideas of Ibragimov and Linnik [6]. Let $L > 0$ and define the truncated variables by the superscript $(L)$ and the remainder by breve and superscript $(L)$: $X^{(L)} = X \cdot \mathrm{I}\{X \in [-L, L]\}$, $\breve{X}^{(L)} = X - X^{(L)}$. Let $Z_n = S_n/\sigma_n$ be the standardized sum,

$$Z_n^{(L)} = \frac{1}{\sigma_n} \sum_{\boldsymbol{k}\in\mathcal{D}_n} \left(Y_n^{(L)}(\boldsymbol{k}) - \mathbb{E}Y_n^{(L)}(\boldsymbol{k})\right)$$

be the normalized sum of the truncated variables, and

$$\breve{Z}_n^{(L)} = \frac{1}{\sigma_n} \sum_{\boldsymbol{k}\in\mathcal{D}_n} \left(\breve{Y}_n^{(L)}(\boldsymbol{k}) - \mathbb{E}\breve{Y}_n^{(L)}(\boldsymbol{k})\right)$$

be the normalized sum of the remainders. Then $Z_n = Z_n^{(L)} + \breve{Z}_n^{(L)}$ and $\mathbb{E}Z_n^2 = 1$.

By the Rosenthal inequality, (2.8), and (2.1),

$$\mathbb{E}(\breve{Z}_n^{(L)})^2 = \mathbb{E}\left|\frac{1}{\sigma_n}\sum_{\boldsymbol{k}\in\mathcal{D}_n}\left(\breve{Y}_n^{(L)}(\boldsymbol{k}) - \mathbb{E}\breve{Y}_n^{(L)}(\boldsymbol{k})\right)\right|^2 \leq$$

$$\leq c\frac{\Lambda_n^d|\mathcal{D}_n|}{\sigma_n^2}\sup_{\boldsymbol{k}\in\mathcal{D}_n}\|\breve{Y}_n^{(L)}(\boldsymbol{k})\|_{2+\tau}^2 \to 0 \quad (3.9)$$

as $L\to\infty$. We remark that this convergence is uniform in $n$. In the last step we used (3.1) and (3.7).

Let $\sigma_n^2(L) = \operatorname{var}\big(\sum_{\boldsymbol{k}\in\mathcal{D}_n} Y_n^{(L)}(\boldsymbol{k})\big)$ be the variance of the sum of the truncated variables. Now

$$\frac{\sigma_n^2(L)}{\sigma_n^2} - 1 = \mathbb{E}(Z_n^{(L)})^2 - \mathbb{E}(Z_n)^2 =$$

$$= \mathbb{E}(Z_n - \breve{Z}_n^{(L)})^2 - \mathbb{E}(Z_n)^2 = \mathbb{E}(\breve{Z}_n^{(L)})^2 - 2\mathbb{E}(Z_n\breve{Z}_n^{(L)}).$$

By (3.9), and using the Cauchy inequality for the second term, the above expression converges to 0, as $L\to\infty$, uniformly in $n$. Therefore,

$$\lim_{L\to\infty}\sup_{n\geq 1}\left|\frac{\sigma_n^2(L)}{\sigma_n^2} - 1\right| = 0. \quad (3.10)$$

Now,

$$\left|\mathbb{E}e^{itZ_n} - e^{-t^2/2}\right| \leq$$

$$\leq \mathbb{E}\left|e^{it\breve{Z}_n^{(L)}} - 1\right| + \left|\mathbb{E}e^{itZ_n^{(L)}} - e^{-\frac{\sigma_n^2(L)}{\sigma_n^2}\frac{t^2}{2}}\right| + \left|e^{-\frac{\sigma_n^2(L)}{\sigma_n^2}\frac{t^2}{2}} - e^{-\frac{t^2}{2}}\right| \leq$$

$$\leq |t|\mathbb{E}|\breve{Z}_n^{(L)}| + \sup_{v\in[1-\delta_L,1+\delta_L]}\left|\mathbb{E}e^{itvU_n} - e^{-\frac{(tv)^2}{2}}\right| + \frac{t^2}{2}\delta_L \quad (3.11)$$

where $\delta_L = \sup_{n\geq 1}\left|\frac{\sigma_n^2(L)}{\sigma_n^2} - 1\right|$ and $U_n = \frac{1}{\sigma_n(L)}\sum_{\boldsymbol{k}\in\mathcal{D}_n}\left(Y_n^{(L)}(\boldsymbol{k}) - \mathbb{E}Y_n^{(L)}(\boldsymbol{k})\right)$.

By (3.10), $\lim_{L\to\infty}\delta_L = 0$. If the theorem is valid for bounded random variables, then $U_n$ is asymptotically standard normal, therefore (3.11) implies that

$$\limsup_{n\to\infty}\left|\mathbb{E}e^{itZ_n} - e^{-t^2/2}\right| \leq |t|\sqrt{\sup_{n\geq 1}\mathbb{E}\big(\breve{Z}_n^{(L)}\big)^2} + \frac{t^2}{2}\delta_L.$$

However, using (3.9), the last expression converges to 0, as $L\to\infty$. Therefore, the theorem is valid. □

In the proof of Theorem 1 we used the existence of the following subsequence $m_n$. For the sake of completeness we give a proof of the existence.

**Lemma 3.** *Let $x_n \downarrow 0$, $y_n \downarrow 0$, $z_n \to \infty$ be real sequences such that $y_n = n^{-d}$ with $d > 0$ and $x_n/y_n \to 0$. Then there exists a sequence $m_n$ of positive integers such that $m_n \to \infty$, $x_{m_n} z_n \to 0$, and $y_{m_n} z_n \to \infty$.*

PROOF. As $\tau_n = x_n/y_n \to 0$ we can find an increasing unbounded sequence $u_k$ of positive numbers such that $\tau_n \le k^{-2}$ if $n \ge u_k^{1/d}$ . As $z_n \to \infty$, we can find an increasing unbounded sequence $\{n(k)\}$ of positive integers such that $z_n \ge k u_k$ if $n \ge n(k)$. Now let $\alpha_m = 1/k$ if $n(k) \le m < n(k+1)$ for each positive integer $k$. Therefore, $\alpha_m \to 0$ and for each positive integer $k$ we have $z_m \alpha_m \ge k u_k / k = u_k$ if $n(k) \le m < n(k+1)$.

Now let $m_n = [(z_n \alpha_n)^{1/d}] + 1$ for each $n$, where $[\cdot]$ denotes the integer part of a number. It is easy to see that $m_n \uparrow \infty$. Using the above considerations, for each positive integer $k$ we have

$$\frac{\tau_{m_n}}{\alpha_n} \le \frac{k^{-2}}{k^{-1}} = k^{-1} \quad \text{if } n(k) \le n < n(k+1). \tag{3.12}$$

Therefore, by the definition of $\tau_n$, the definition of $m_n$ and (3.12), we get

$$x_{m_n} z_n = \tau_{m_n} y_{m_n} z_n \le \tau_{m_n} (z_n \alpha_n)^{-1} z_n = \tau_{m_n} \alpha_n^{-1} \to 0$$

if $n \to \infty$. Moreover, by the definition of $\alpha_n$, we obtain

$$y_{m_n} z_n = \frac{z_n}{m_n^d} = \frac{z_n}{z_n \alpha_n} \left\{ \frac{(z_n \alpha_n)^{1/d}}{\left[(z_n \alpha_n)^{1/d}\right] + 1} \right\}^d \to \infty$$

as $n \to \infty$. □

*Remark* 3 (Stein's lemma [11, 14]). Let $\{\nu_n\}$ be a sequence of probabilities on $\mathbb{R}$ such that $\sup_n \int_{\mathbb{R}} x^2 \nu_n(dx) < \infty$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}} (it - x) e^{itx} \nu_n(dx) = 0$$

for every $t \in \mathbb{R}$. Then $\nu_n \Rightarrow \mathcal{N}(0,1)$ as $n \to \infty$.

## 4. EXTENSIONS OF MAIN THEOREMS

**Corollary 1.** *In Theorems 1 and 2, instead of* (3.1), *assume that*

$$\lim_{n\to\infty} \Lambda_n^{-d} |\mathcal{D}_n|^{-1} \sigma_n^2 = \sigma^2.$$

*Then $(\Lambda_n^d |\mathcal{D}_n|)^{-\frac{1}{2}} S_n \Rightarrow \mathcal{N}(0, \sigma^2)$ as $n \to \infty$.*

PROOF. First assume that $\sigma^2 > 0$. Then (3.1) is satisfied and therefore $\sigma_n^{-1} S_n \Rightarrow \mathcal{N}(0,1)$. So

$$(\Lambda_n^d |\mathcal{D}_n|)^{-\frac{1}{2}} S_n = \left( \frac{\sigma_n^2}{\Lambda_n^d |\mathcal{D}_n|} \right)^{\frac{1}{2}} \sigma_n^{-1} S_n \Rightarrow \sigma \mathcal{N}(0,1) = \mathcal{N}(0, \sigma^2).$$

If $\sigma^2 = 0$, then $\operatorname{var}[(\Lambda_n^d|\mathcal{D}_n|)^{-\frac{1}{2}} S_n] = (\Lambda_n^d|\mathcal{D}_n|)^{-1}\sigma_n^2 \to 0$ as $n \to \infty$. Therefore, $(\Lambda_n^d|\mathcal{D}_n|)^{-\frac{1}{2}} S_n$ converges to 0 in $L^2$, so it converges to the degenerate normal law $\mathcal{N}(0, 0)$ in distribution. $\square$

Now we turn to $p$-dimensional extensions of Theorems 1 and 2. Our Theorem 3 is the same as Theorem 3.1 of [10].

**Theorem 3.** *Let $\varepsilon(\boldsymbol{x})$ be a random field and let the $p$-dimensional random vector $Y_n(\boldsymbol{k})$ be a Borel measurable function of $\varepsilon(\boldsymbol{x}_{\boldsymbol{k}}^{(n)})$, $\boldsymbol{k} \in \mathcal{D}_n$. Suppose that $\mathbb{E}Y_n(\boldsymbol{k}) = 0$ and $\|Y_n(\boldsymbol{k})\|$ are uniformly bounded for $\boldsymbol{k} \in \mathcal{D}_n$, $n = 1, 2, \ldots$. Let $S_n = \sum_{\boldsymbol{k}\in\mathcal{D}_n} Y_n(\boldsymbol{k})$, $n = 1, 2, \ldots$, $\Sigma_n = \operatorname{var}(S_n)$. Suppose that conditions* (2.2), (2.3), *and* (2.4) *are satisfied. Assume that the limit*

$$\lim_{n\to\infty} (\Lambda_n^{-d}|\mathcal{D}_n|^{-1}\Sigma_n) = \Sigma$$

*exists. Then $(\Lambda_n^d|\mathcal{D}_n|)^{-\frac{1}{2}} S_n \Rightarrow \mathcal{N}(0, \Sigma)$ as $n \to \infty$.*

PROOF. Consider the one dimensional random field $\{a^\top Y_n(\boldsymbol{k})\}$, where $a \in \mathbb{R}^p$ is arbitrary ($a^\top$ is the transpose of $a$). Apply Theorem 1 and Corollary 1 to the field $\{a^\top Y_n(\boldsymbol{k})\}$. $\square$

Our Theorem 4 is the same as Remark 4.3 in [7].

**Theorem 4.** *Let $\varepsilon(\boldsymbol{x})$ be a random field. For each $n = 1, 2, \ldots$, and for each $\boldsymbol{k} \in \mathcal{D}_n$, let $Y_n(\boldsymbol{k})$ be a centered $p$-dimensional random vector that is $\varepsilon(\boldsymbol{x}_{\boldsymbol{k}}^{(n)})$-measurable. Let $S_n = \sum_{\boldsymbol{k}\in\mathcal{D}_n} Y_n(\boldsymbol{k})$, $n = 1, 2, \ldots$, $\Sigma_n = \operatorname{var}(S_n)$. Assume that conditions* (2.2), (2.3), *and* (2.4) *are satisfied. Moreover, assume that there exists a $\tau > 0$ such that* (2.1) *is satisfied, and*

$$\{\|Y_n(\boldsymbol{k})\|^{2+\tau} : \boldsymbol{k} \in \mathcal{D}_n,\ n = 1, 2, \ldots\} \text{ are uniformly integrable.}$$

*Assume that*

$$\liminf_{n\to\infty} \lambda_{\min}\left(\Lambda_n^{-d}|\,\mathcal{D}_n|^{-1}\Sigma_n\right) > 0.$$

*Then $\Sigma_n^{-\frac{1}{2}} S_n \Rightarrow \mathcal{N}(0, I_p)$ as $n \to \infty$.*

Here, $\lambda_{\min}(A)$ denotes the minimal eigenvalue of the matrix $A$ and $I_p$ is the $p \times p$ type unit matrix.

PROOF. Apply Theorem 2 to the field $\{a^\top Y_n(k)\}$. $\square$

Several other versions of the above theorems can be obtained. For example, the uniform integrability condition can be substituted by a strong stationarity condition and an integrability condition.

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# THE KANTOROVICH FORM OF STANCU OPERATORS

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ABSTRACT. In this paper we study the Kantorovich form of Stancu operators. As particular cases, we shall obtain similar properties of the Kantorovich form for Bernstein, Schurer and Schurer–Stancu operators.

*Mathematics Subject Classification:* 41A10, 41A25, 41A36

*Keywords:* linear positive operators, Bernstein operators, Schurer operators, Stancu operators, Kantorovich operators

## 1. INTRODUCTION

In this section, we recall some notions and results which we will use in this article (see [5]).

We define the natural number $m_0$ by

$$m_0 = \begin{cases} \max\{1, -[\beta]\}, & \text{iff} \quad \beta \in \mathbb{R} \setminus \mathbb{Z}, \\ \max\{1, 1-\beta\}, & \text{iff} \quad \beta \in \mathbb{Z}. \end{cases} \tag{1.1}$$

For the real number $p$, we have

$$m + \beta \geq \gamma_\beta = m_0 + \beta \tag{1.2}$$

for any natural number $m$, $m \geq m_0$, where

$$\gamma_\beta = \begin{cases} \max\left\{1+\beta, \{\beta\}\right\}, & \text{iff} \quad \beta \in \mathbb{R} \setminus \mathbb{Z}, \\ \max\{1+\beta, 1\}, & \text{iff} \quad \beta \in \mathbb{Z}. \end{cases} \tag{1.3}$$

For the real numbers $\alpha, \beta, \alpha \geq 0$, we set

$$\mu^{(\alpha,\beta)} = \begin{cases} 1, & \text{iff } \alpha \leq \beta, \\ 1 + \dfrac{\alpha - \beta}{\gamma_\beta}, & \text{iff } \alpha > \beta. \end{cases} \tag{1.4}$$

**Lemma 1.** *For the real numbers $\alpha$ and $\beta$, $\alpha \geq 0$, we have*

$$0 \leq \frac{k+\alpha}{m+\beta} \leq \mu^{(\alpha,\beta)} \tag{1.5}$$

*for any natural number $m$, $m \geq m_0$ and for any $k \in \{0, 1, \dots, m\}$.*

For the real numbers $\alpha$ and $\beta$, $\alpha \geq 0$, where $m_0$ and $\mu^{(\alpha,\beta)}$ are defined by (1.1)–(1.4), let the operators $P_m^{(\alpha,\beta)} : C\big([0, \mu^{(\alpha,\beta)}]\big) \to C\big([0, 1]\big)$ be defined for any function $f \in C\big([0, \mu^{(\alpha,\beta)}]\big)$ by

$$\big(P_m^{(\alpha,\beta)} f\big)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right), \tag{1.6}$$

for any natural number $m, m \geq m_0$ and for any $x \in [0, 1]$, where $p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}$ are the fundamental Bernstein polynomials. These operators are named Bernstein-Stancu operators, introduced and studied in 1969 by D. D. Stancu in the paper [7]. In [7] the domain of definition for the Bernstein–Stancu operators is $C([0, 1])$ and $0 \leq \alpha \leq \beta$.

*Remark* 1. Because there is no restriction on the real parameter $\beta$ in our construction, in the following remarks we will explain how to obtain the Bernstein, Schurer and Schurer–Stancu operators from the Stancu operators, through particularization.

*Remark* 2. If $\alpha = \beta = 0$, then $m_0 = 1$, $\mu^{(0,0)} = 1$, we obtain $P_m^{(0,0)} = B_m$, $m \geq 1$, the Bernstein operators, $B_m : C([0, 1]) \to C([0, 1])$ defined by

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.7}$$

for any function $f \in C([0, 1])$ and any $x \in [0, 1]$.

*Remark* 3. If $p$ is a natural number, $\alpha = 0$ and $\beta = -p$, then $m_0 = 1 + p$, $\mu^{(0,-\beta)} = 1 + p$. Changing $m$ with $m + p$, we obtain $P_{m+p}^{(0,-\beta)} = \widetilde{B}_{m,p}$, $m \geq 1$, the Schurer operators, $\widetilde{B}_{m,p} : C([0, 1 + p]) \to C([0, 1])$ defined by

$$\big(\widetilde{B}_{m,p} f\big)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.8}$$

for any function $f \in C([0, 1 + p])$ and any $x \in [0, 1]$, where

$$\widetilde{p}_{m,k}(x) = p_{m+p,k}(x)\binom{m+p}{k} x^k (1-x)^{m+p-k}$$

are the fundamental Schurer polynomials.

*Remark* 4. If $0 \leq \alpha$, $p$ is a natural number, substituting $m$ with $m + p$ and $\beta$ with $\beta - p$, we obtain $P_{m+p}^{(\alpha,\beta-p)} = \widetilde{S}_{m,p}^{(\alpha,\beta)}$, $m \geq m_0$, where $m_0$ is defined in (1.1) for $\beta - p$, the Schurer–Stancu operators, $\widetilde{S}_{m,p}^{(\alpha,\beta)} : C\big([0, \mu^{(\alpha,\beta-p)}]\big) \to C([0, 1])$ defined by

$$\big(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\big)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) \left(\frac{k+\alpha}{m+\beta}\right), \tag{1.9}$$

for any function $f \in C\big([0, \mu^{(\alpha,\beta-p)}]\big)$ and any $x \in [0, 1]$ (see [2], where the domain of definition for the Schurer–Stancu operators is $C([0, 1 + p])$ and the parameters $\alpha$ and $\beta$ verify $0 \le \alpha \le \beta$).

**Proposition 1.** *The operators* $\big(P_m^{(\alpha,\beta)}\big)_{m \ge m_0}$ *satisfy the relations*

$$\big(P_m^{(\alpha,\beta)} e_0\big)(x) = 1, \tag{1.10}$$

$$\big(P_m^{(\alpha,\beta)} e_1\big)(x) = x + \frac{\alpha - \beta x}{m + \beta} \tag{1.11}$$

*and*

$$\big(P_m^{(\alpha,\beta)} e_2\big)(x) = x^2 + \frac{mx(1-x) + (\alpha - \beta x)(2mx + \beta x + \alpha)}{(m+\beta)^2} \tag{1.12}$$

*for any natural number* $m$, $m \ge m_0$, *for any* $x \in [0, 1]$.

PROOF. The proof can be found in [7, 8]. □

## 2. PRELIMINARIES

For a nonzero natural number $m$, let the operator $K_m : L_1\big([0, 1]\big) \to C([0, 1])$ be defined for any function $f \in L_1\big([0, 1]\big)$ by

$$(K_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)dt, \tag{2.1}$$

for any $x \in [0, 1]$.

The operators $K_m$, where $m$ is a nonzero natural number, are named Kantorovich operators, introduced and studied in 1930 by L. V. Kantorovich (see [8]).

In the following, we consider the real numbers $\alpha$ and $\beta$, $\alpha \ge 0$, where $m_0$ and $\mu^{(\alpha,\beta)}$ are defined by (1.1)–(1.4).

**Lemma 2.** *For a natural number* $m$, $m \ge m_0$, *we have*

$$0 \le \frac{k+\alpha}{m+\beta+1} \le \frac{k+\alpha+1}{m+\beta+1} \le \mu^{(\alpha,\beta)} \tag{2.2}$$

*for any* $k \in \{0, 1, \ldots, m\}$.

PROOF. This results from (1.5). □

For a natural number $m$, $m \ge m_0$, let the operator $K_m^{(\alpha,\beta)} : L_1\big([0, \mu^{(\alpha,\beta)}]\big) \to C([0, 1])$ be defined for any function $f \in L_1\big([0, \mu^{(\alpha,\beta)}]\big)$ by

$$\big(K_m^{(\alpha,\beta)} f\big)(x) = (m+\beta+1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha+1}{m+\beta+1}} f(t)dt, \tag{2.3}$$

for any $x \in [0, 1]$. These operators are named the Kantorovich–Stancu type operators.

**Lemma 3.** *The operators* $\left(K_m^{(\alpha,\beta)}\right)_{m\geq m_0}$ *satisfy the relations*

$$\left(K_m^{(\alpha,\beta)}e_0\right)(x) = 1, \tag{2.4}$$

$$\left(K_m^{(\alpha,\beta)}e_1\right)(x) = \frac{m}{m+\beta+1}\,x + \frac{2\alpha+1}{2(m+\beta+1)}, \tag{2.5}$$

*and*

$$\left(K_m^{(\alpha,\beta)}e_2\right)(x) = \frac{m(m-1)}{(m+\beta+1)^2}\,x^2 + \frac{2m(\alpha+1)}{(m+\beta+1)^2}\,x + \frac{3\alpha^2+3\alpha+1}{3(m+\beta+1)^2} \tag{2.6}$$

*for any natural number* $m$, $m \geq m_0$, *and for any* $x \in [0,1]$.

PROOF. Using the definition of the operator $K_m^{(\alpha,\beta)}$ and applying Proposition 1.1, the conclusion follows. □

**Lemma 4.** *The operators* $\left(K_m^{(\alpha,\beta)}\right)_{m\geq m_0}$ *satisfy the relation*

$$\left(K_m^{(\alpha,\beta)}\varphi_x^2\right)(x) = \frac{-m+(\beta+1)^2}{(m+\beta+1)^2}\,x^2 + \frac{m-(2\alpha+1)(\beta+1)}{(m+\beta+1)^2}\,x + \frac{3\alpha^2+3\alpha+1}{3(m+\beta+1)^2} \tag{2.7}$$

*for any natural number* $m$, $m \geq m_0$, *for any* $x \in [0,1]$.

PROOF. We have

$$\left(K_m^{(\alpha,\beta)}\varphi_x^2\right)(x) = \left(K_m^{(\alpha,\beta)}e_2\right)(x) - 2x\left(K_m^{(\alpha,\beta)}e_1\right)(x) + x^2\left(K_m^{(\alpha,\beta)}e_0\right)(x),$$

and applying Lemma 2.2 we get the conclusion. □

**Lemma 5.** *The operators* $\left(K_m^{(\alpha,\beta)}\right)_{m\geq m_0}$ *are linear and positive.*

PROOF. The conclusion follows immediately. □

## 3. MAIN RESULTS

Let us recall that if $I \subset \mathbb{R}$ is a given interval, $f \in C_B(I)$, where $B(I) = \{f \mid f : I \to \mathbb{R},\ f \text{ bounded on } I\}$, $C(I) = \{f \mid f : I \to \mathbb{R},\ f \text{ is continuous on } I\}$, and $C_B(I) = B(I) \cap C(I)$. The first order modulus of smoothness is the function $\omega_1 : [0,\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by the formula

$$\omega_1(f;\delta) = \sup\left\{|f(x') - f(x'')| : x', x'' \in I,\ |x'-x''| \leq \delta\right\}. \tag{3.1}$$

In the sequel, we will use the following result established by O. Shisha and B. Mond (see [1, 6, 8]).

**Theorem 1.** *Let $L : C(I) \to B(I)$ be a linear and positive operator with the properties $Le_0 = e_0$.*
(i) *If $f \in C_b(I)$, then*

$$\left|(Lf)(x) - f(x)\right| \leq \left[1 + \delta^{-1}\sqrt{\left(L\varphi_x^2\right)(x)}\right]\omega_1(f;\delta) \tag{3.2}$$

*and*

$$\left|(Lf)(x) - f(x)\right| \leq \left[1 + \delta^{-2}\left(L\varphi_x^2\right)(x)\right]\omega_1(f;\delta) \tag{3.3}$$

*for any $x \in I$, for any $\delta > 0$;*
(ii) *If $f$ is a derivable function on $I$ and $f' \in C_B(I)$, then*

$$\begin{aligned}&\left|(Lf)(x) - f(x)\right| \leq \\ &\leq \left|f'(x)\right|\left|(Le_1)(x) - x\right| + \sqrt{\left(L\varphi_x^2\right)(x)}\left[1 + \delta^{-1}\sqrt{\left(L\varphi_x^2\right)(x)}\right]\omega_1(f';\delta)\end{aligned} \tag{3.4}$$

*for any $x \in I$, for any $\delta > 0$.*

**Theorem 2.** *The sequence $\left(K_m^{(\alpha,\beta)} f\right)_{m \geq m_0}$ converges uniformly on $[0,1]$ to $f$, for any $f \in C\left([0, \mu^{(\alpha,\beta)}]\right)$.*

PROOF. Applying Lemma 2.3 we get $\lim_{m\to\infty}\left(K_m^{(\alpha,\beta)}\varphi_x^2\right)(x) = 0$ uniformly on $[0,1]$. Since $K_m^{(\alpha,\beta)} e_0 = e_0$, using then the well-known Bohman–Korovkin theorem [1, 8], we obtain the result. □

**Theorem 3.** (i) *If $f \in C\left(\left[0, \mu^{(\alpha,\beta)}\right]\right)$, then*

$$\left|\left(K_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq \left(1 + \delta^{-1}\sqrt{\left(K_m^{(\alpha,\beta)}\varphi_x^2\right)(x)}\right)\omega_1(f;\delta) \tag{3.5}$$

*and*

$$\left|\left(K_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq \left[1 + \delta^{-2}\left(K_m^{(\alpha,\beta)}\varphi_x^2\right)(x)\right]\omega_1(f;\delta) \tag{3.6}$$

*for any $x \in [0,1]$, for any $\delta > 0$ and $m \in \mathbb{N}$, $m \geq m_0$.*
(ii) *If $f$ is a differentiable function on $\left[0, \mu^{(\alpha,\beta)}\right]$ and $f' \in C\left(\left[0, \mu^{(\alpha,\beta)}\right]\right)$, then*

$$\begin{aligned}\left|\left(K_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| &\leq \left|f'(x)\right|\left|-\frac{\beta+1}{m+\beta+1}x + \frac{2\alpha+1}{2(m+\beta+1)}\right| \\ &+ \sqrt{\left(K_m^{(\alpha,\beta)}\varphi_x^2\right)(x)}\left(1 + \delta^{-1}\sqrt{\left(K_m^{(\alpha,\beta)}\varphi_x^2\right)(x)}\right)\omega_1(f';\delta)\end{aligned} \tag{3.7}$$

*for any $x \in [0,1]$, for any $\delta > 0$ and $m \in \mathbb{N}$, $m \geq m_0$.*

PROOF. Applying the Theorem 3.1, we obtain the results. □

**Theorem 4.** *Let* $\delta_m^{(\alpha,\beta)}(x) = \sqrt{\left(K_m^{(\alpha,\beta)}\varphi_x^2\right)(x)}$, *where* $x \in [0,1]$ *and* $m$ *is any natural number,* $m \geq m_0$. *Then*

(1) *If* $f \in C\left(\left[0, \mu^{(\alpha,\beta)}\right]\right)$, *then*

$$\left|\left(K_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq 2\omega_1\left(f; \delta_m^{(\alpha,\beta)}(x)\right) \tag{3.8}$$

*for any* $x \in [0,1]$ *and for any natural number* $m$, $m \geq m_0$.

(2) *If* $f$ *is a derivable function on* $\left[0, \mu^{(\alpha,\beta)}\right]$ *and* $f' \in C\left(\left[0, \mu^{(\alpha,\beta)}\right]\right)$, *then*

$$\begin{aligned}\left|\left(K_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq |f'(x)| \Bigg| &-\frac{\beta+1}{m+\beta+1}x \\ &+ \frac{2\alpha+1}{2(m+\beta+1)}\Bigg| + 2\delta_m^{(\alpha,\beta)}(x)\omega_1\left(f', \delta_m^{(\alpha,\beta)}(x)\right)\end{aligned} \tag{3.9}$$

*for any* $x \in [0,1]$ *and for any natural number* $m$, $m \geq m_0$.

PROOF. Choosing $\delta = \delta_m^{(\alpha,\beta)}(x)$ in Theorem 3.3, we obtain Theorem 3.4. □

For a natural number $m, m \geq m_1$, let $f_m : [0,1] \to \mathbb{R}$ be a function of second degree defined by

$$f_m(x) = \frac{-m+(\beta+1)^2}{(m+\beta+1)^2}x^2 + \frac{m-(2\alpha+1)(\beta+1)}{(m+\beta+1)^2}x + \frac{3\alpha^2+3\alpha+1}{3(m+\beta+1)^2} \tag{3.10}$$

for any $x \in [0,1]$, where $m_1$ is the smallest natural number so that

$$m_1 \geq \max\left\{m_0, (2\alpha+1)(\beta+1), (\beta+1)^2+1, (\beta+1)(2\beta-2\alpha+1)\right\}. \tag{3.11}$$

**Lemma 6.** *The function* $f_m$ *has a maximum value*

$$M_m^{(\alpha,\beta)} = \frac{3m^2 - 2m(6\alpha\beta+3\beta+1-6\alpha^2) - (\beta+1)^2}{12\left[m-(\beta+1)^2\right](m+\beta+1)^2} > 0 \tag{3.12}$$

*at the point* $x_M = \frac{m-(2\alpha+1)(\beta+1)}{2(m-(\beta+1)^2)}$, *where* $m$ *is a natural number,* $m \geq m_1$.

PROOF. Let $a = \frac{-m+(\beta+1)^2}{(m+\beta+1)^2}$, $b = \frac{m-(2\alpha+1)(\beta+1)}{(m+\beta+1)^2}$, and $c = \frac{3\alpha^2+3\alpha+1}{3(m+\beta+1)^2}$. Then $f_m = ax^2+bx+c$. Because $m \geq m_1$, $a < 0$, the function $f_m$ has a maximum value $M_m^{(\alpha,\beta)} = -\frac{\Delta}{4a}$ at the point $x_M = -\frac{b}{2a}$. It follows immediately that $0 \leq x_M \leq 1$, since $m > (\beta+1)^2$, $m \geq (2\alpha+1)(\beta+1)$ and $m \geq (\beta+1)(2\beta-2\alpha+1)$. We have $f_m(0) = \frac{3\alpha^2+3\alpha+1}{3(m+\beta+1)^2} > 0$ and from calculations we obtain relation (3.12). □

**Lemma 7.** *We have*

$$\delta_m^{(\alpha,\beta)}(x) \leq \delta_m^{(\alpha,\beta)} \tag{3.13}$$

*for any* $x \in [0,1]$ *and for any natural number* $m$, $m \geq m_1$, *where* $\delta_m^{(\alpha,\beta)} = \sqrt{M_m^{(\alpha,\beta)}}$.

PROOF. Taking (2.7) into account, the definition of $\delta_m^{(\alpha,\beta)}(x)$ and Lemma 3.1. □

For a natural number $m$, $m \geq m_0$, let $g_m : [0,1] \to \mathbb{R}$ be a function defined by $g_m(x) = -\frac{\beta+1}{m+\beta+1}x + \frac{2\alpha+1}{2(m+\beta+1)}$, for any $x \in [0,1]$. Because the function $g_m$ is linear, then the extremal value of $g_m$ is $g_m(0)$ and $g_m(1)$. Then

$$\left|g_m(x)\right| \leq \eta_m^{(\alpha,\beta)} \tag{3.14}$$

for any $x \in [0,1]$, for any natural number $m$, $m \geq m_0$, where

$$\begin{aligned}\eta_m^{(\alpha,\beta)} &= \max\left\{\left|g_m(0)\right|, \left|g_m(1)\right|\right\} = \\ &= \max\left\{\frac{2\alpha+1}{2(m+\beta+1)}, \frac{\left|-2\beta+2\alpha-1\right|}{2(m+\beta+1)}\right\}.\end{aligned} \tag{3.15}$$

**Corollary 1.** *The following assertions are true:*

(1) *If* $f \in C\left(\left[0, \mu^{(\alpha,\beta)}\right]\right)$, *then*

$$\left|\left(K_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq 2\omega_1\left(f; \delta_m^{(\alpha,\beta)}\right) \tag{3.16}$$

*for any* $x \in [0,1]$ *and for any natural number* $m$, $m \geq m_1$.

(2) *If* $f$ *is a derivable function on* $\left[0, \mu^{(\alpha,\beta)}\right]$ *and* $f' \in C\left(\left[0, \mu^{(\alpha,\beta)}\right]\right)$, *then*

$$\left|\left(K_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq M_1 \eta_m^{(\alpha,\beta)} + 2\delta_m^{(\alpha,\beta)} \omega_1\left(f; \delta_m^{(\alpha,\beta)}\right) \tag{3.17}$$

*for any* $x \in [0,1]$ *and for any natural number* $m$, $m \geq m_1$, *where* $M_1 = \max_{x\in[0,1]} |f'(x)|$.

PROOF. It results from Theorem 3.4, Lemma 3.2 and relation (3.14). □

*Remark* 5. Through particularization, in the following applications we obtain known operators which verify the general results proved for the Stancu operators.

*Application* 1. If $\alpha = \beta = 0$ we obtain the Kantorovich operators.

*Application* 2. If $p$ is a natural number, $\alpha = \beta = 0$, substituting $m$ with $m+p$, we obtain the Kantorovich form of Schurer type operators (see [4]).

*Application* 3. If $p$ is a natural number, $0 \leq \alpha \leq \beta$, substituting $m$ with $m+p$, we obtain the Kantorovich form of Schurer–Stancu operators (see [3]).

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# THE THIRD INTERNATIONAL WORKSHOP "CONSTRUCTIVE METHODS FOR NON-LINEAR BOUNDARY VALUE PROBLEMS"

MIKLÓS RONTÓ

[Received: June 20, 2006]

In the period from 7th to 10th June, 2006, the Third International Workshop

**"CONSTRUCTIVE METHODS FOR NON-LINEAR BOUNDARY VALUE PROBLEMS"**

took place in Sárospatak, Hungary. The workshop had been organised by the Institute of Mathematics of the University of Miskolc in cooperation with the Regional Committee of the Hungarian Academy of Sciences in Miskolc and the Comenius Teacher Training College of the University of Miskolc.

Participants of the Third International Workshop "Constructive Methods for Non-Linear Boundary Value Problems" (7–10 June, 2006, Sárospatak, Hungary)

The scope of the conference covered various topics in the modern theory of boundary value problems in a broad sense, including initial value problems for equations with argument deviations, periodic solutions of functional differential equations, boundary value problems for differential equations with homogeneous non-linearities, the method of lower and upper functions, bifurcation theory, as well as certain aspects of the theory of generalised differential equations, studies of population dynamics, and applications. The workshop was attended by more than 30 researchers from Chile, the Czech Republic, Hungary, Latvia, Slovakia, and the Ukraine, who held 20 and 30-minute lectures.

The Organising Committee of the workshop consisting of G. Bognár, A. Galántai, P. Körtesi, A. Muha Miklósné, A. Samoilenko, A. Rontó, and the author of this note (chairman) gratefully acknowledges the financial support provided by the following funds and institutions:

- Institute of Mathematics, University of Miskolc;
- Faculty of Mechanical Engineering, University of Miskolc;
- Faculty of Economics, University of Miskolc;
- Tudomány Támogatásáért Észak-Magyarországon Alapítvány.

This meeting is a continuation of the two previous workshops held in Miskolc in 2000 and 2006.

We hope that the fruitful tradition of this kind of “compact” international meetings oriented at exchange of ideas of researchers working in closely related fields will continue in the future.

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# A NOTE ON TWO-DIMENSIONAL SYSTEMS OF LINEAR DIFFERENTIAL INEQUALITIES WITH ARGUMENT DEVIATIONS

JIŘÍ ŠREMR

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ABSTRACT. In this paper, the question on the positivity of the Cauchy operator of two-dimensional systems of differential equations with argument deviations is studied. Some results of [11] are refined for two-dimensional systems of differential inequalities.

*Mathematics Subject Classification:* 34K06, 34K10

*Keywords:* System of diferential equations with argument deviations, initial value problem, sign-constant solutions, theorem on differential inequalities

## 1. NOTATION AND INTRODUCTION

On the interval $[a, b]$, we consider two-dimensional differential system

$$u_i'(t) = p_{i1}(t)u_1\big(\tau_{i1}(t)\big) + p_{i2}(t)u_2\big(\tau_{i2}(t)\big) + q_i(t) \qquad (i = 1, 2) \tag{1.1}$$

with the initial conditions

$$u_1(a) = c_1, \qquad u_2(a) = c_2, \tag{1.2}$$

where $p_{ik}, q_i : [a, b] \to \mathbb{R}$ $(i, k = 1, 2)$ are Lebesgue integrable functions, $\tau_{ik} : [a, b] \to [a, b]$ $(i, k = 1, 2)$ are measurable functions, and $c_1, c_2 \in \mathbb{R}$. By a solution of the problem (1.1), (1.2), we understand an absolutely continuous vector function $u = (u_1, u_2)^T : [a, b] \to \mathbb{R}^2$ satisfying the equation (1.1) almost everywhere on $[a, b]$ and verifying the initial conditions (1.2). The following notation is used throughout the paper:

$\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$;

$\mathbb{R}^2$ is the space two-dimensional column vectors $x = (x_i)_{i=1}^2$ with the elements $x_1, x_2 \in \mathbb{R}$ and the norm

$$\|x\| = |x_1| + |x_2|;$$

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If $x, y \in \mathbb{R}^2$ then

$$x \le y \quad \text{if and only if} \quad x_1 \le y_1,\ x_2 \le y_2\,;$$

$\mathbb{R}^{2\times 2}$ is the space of the $2\times 2$-matrices $X = (x_{ik})_{i,k=1}^2$ with the elements $x_{ik} \in \mathbb{R}$ $(i, k = 1, 2)$;

$r(X)$ is the spectral radius of the matrix $X \in \mathbb{R}^{2\times 2}$;

$X^T$ is the transposed matrix to $n \times m$-matrix $X$;

If $x_1, x_2 \in \mathbb{R}$ then

$$\operatorname{diag}(x_1, x_2) = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix};$$

$C([a,b]; \mathbb{R}^2)$ is the Banach space of continuous vector functions $u : [a,b] \to \mathbb{R}^2$ equipped with the norm

$$\|u\|_C = \max\bigl\{\|u(t)\| : t \in [a,b]\bigr\};$$

$\widetilde{C}([a,b]; \mathbb{R}^2)$ is the set of absolutely continuous vector functions $u : [a,b] \to \mathbb{R}^2$;

$L([a,b]; \mathbb{R}^2)$ is the Banach space of Lebesgue integrable vector functions $h : [a,b] \to \mathbb{R}^2$ equipped with the norm

$$\|h\|_L = \int_a^b \|h(s)\| ds;$$

The equalities and inequalities with integrable functions are understood to hold almost everywhere.

The following proposition is well-known.

**Proposition 1.1.** *Let*

$$p_{i\,3-i}(t) \ge 0 \quad \textit{for } t \in [a,b],\ i = 1, 2. \tag{1.3}$$

*Then, for every vector function* $(\gamma_1, \gamma_2)^T \in \widetilde{C}([a,b]; \mathbb{R}^2)$ *satisfying*

$$\gamma_i'(t) \le \sum_{k=1}^{2} p_{ik}(t)\gamma_k(t) + q_i(t) \quad \textit{for } t \in [a,b],\ i = 1, 2,$$

$$\gamma_1(a) \le c_1, \qquad \gamma_2(a) \le c_2\,,$$

*the relation*

$$\gamma_i(t) \le u_i(t) \quad \textit{for } t \in [a,b],\ i = 1, 2$$

*holds, where* $(u_1, u_2)^T$ *is a solution of the system*

$$u_i' = \sum_{k=1}^{2} p_{ik}(t)u_k + q_i(t) \qquad (i = 1, 2) \tag{1.4}$$

*satisfying the initial conditions* (1.2).

In the other words, if the condition (1.3) is satisfied then so-called theorem on differential inequalities holds for the system (1.4) or, equivalently, the Cauchy matrix function of the homogeneous system corresponding to (1.4) is non-negative. It is very easy to show that, under the assumption (1.3), an analogous assertion for the system (1.1) does not hold in general. Consequently, some stronger assumptions have to be required for differential systems with argument deviations. There are a lot of results concerning various types of theorems on functional differential inequalities. Let us mention, among other, papers [1–4, 6, 8, 9]. We have studied this question in [11].

Following [11], we introduce a definition.

**Definition 1.1.** Let $\sigma_1, \sigma_2 \in \{-1, 1\}$. We say that a linear bounded operator $\ell : C([a,b];\mathbb{R}^2) \to L([a,b];\mathbb{R}^2)$ belongs to the set $\mathcal{S}_{ab}^{2,(\sigma_1,\sigma_2)}(a)$ if every vector function $u \in \widetilde{C}([a,b];\mathbb{R}^2)$ such that

$$\operatorname{diag}(\sigma_1,\sigma_2)\big[u'(t) - \ell(u)(t)\big] \geq 0 \quad \text{for } t \in [a,b], \tag{1.5}$$

$$\operatorname{diag}(\sigma_1,\sigma_2)\, u(a) \geq 0 \tag{1.6}$$

satisfies the conditions

$$\sigma_i u_i(t) \geq 0 \quad \text{for } t \in [a,b],\ i = 1,2. \tag{1.7}$$

*Remark* 1.1. Let the operator $\ell : C([a,b];\mathbb{R}^2) \to L([a,b];\mathbb{R}^2)$ be defined by the relation

$$\ell(v)(t) \stackrel{\text{def}}{=} \begin{pmatrix} p_{11}(t)v_1\big(\tau_{11}(t)\big) + p_{12}(t)v_2\big(\tau_{12}(t)\big) \\ p_{21}(t)v_1\big(\tau_{21}(t)\big) + p_{22}(t)v_2\big(\tau_{22}(t)\big) \end{pmatrix}$$
$$\text{for } t \in [a,b],\ v = (v_1, v_2)^T \in C([a,b];\mathbb{R}^2) \tag{1.8}$$

and let $\ell \in \mathcal{S}_{ab}^{2,(\sigma_1,\sigma_2)}(a)$. Then it is easy to verify that the homogeneous problem corresponding to (1.1), (1.2) has only the trivial solution. Therefore, according to the Fredholm property of the linear boundary value problems for functional differential systems (see, e. g., [5–7, 10]), the problem (1.1), (1.2) has a unique solution for any $(q_1, q_2)^T \in L([a,b];\mathbb{R}^2)$ and $c_1, c_2 \in \mathbb{R}$. However, the inclusion $\ell \in \mathcal{S}_{ab}^{2,(\sigma_1,\sigma_2)}(a)$ guarantees, in addition, that the solution $(u_1, u_2)^T$ of the problem indicated satisfies

$$\sigma_i u_i(t) \geq 0 \quad \text{for } t \in [a,b],\ i = 1,2$$

if the relations

$$\sigma_i q_i(t) \geq 0 \quad \text{for } t \in [a,b], \quad \sigma_i c_i \geq 0 \quad (i = 1,2)$$

are true.

In the sequel, we set

$$\tau^* = \max\Big\{\operatorname{ess\,sup}\big\{\tau_{ik}(t) : t \in [a,b]\big\} : i,k = 1,2\Big\}. \tag{1.9}$$

The following proposition is given in [11].

**Proposition 1.2** ([11, Theorem 4.1]). *Let $\sigma_1, \sigma_2 \in \{-1, 1\}$ and let the functions $p_{jk}$ $(j, k = 1, 2)$ satisfy the conditions*

$$\sigma_j \sigma_k p_{jk}(t) \geq 0 \quad \text{for } t \in [a, b], \ j, k = 1, 2. \tag{1.10}$$

*Let, moreover, there exist numbers $\delta_i > 0$ $(i = 1, 2)$ such that*

$$\max \left\{ \frac{1}{\delta_i} \sum_{k=1}^{2} \delta_k \int_a^{\tau^*} |p_{ik}(s)| ds : i = 1, 2 \right\} = 1 \tag{1.11}$$

*and $r(A) < 1$, where the matrix $A = (a_{ik})_{i,k=1}^2$ is given by the equality*

$$a_{ik} = \frac{\delta_k}{\delta_i} \sum_{j=1}^{2} \int_a^{\tau^*} |p_{ij}(s)| \left( \int_a^{\tau_{ij}(s)} |p_{jk}(\xi)| d\xi \right) ds \quad \text{for } i, k = 1, 2. \tag{1.12}$$

*Then the operator $\ell$ defined by* (1.8) *belongs to the set $\mathcal{S}_{ab}^{2,(\sigma_1,\sigma_2)}(a)$.*

We have also shown in [11] (see Example 5.3) that the assumption $r(A) < 1$ in the last proposition is optimal and cannot be weakened. On the other hand, the following example shows that the assumption indicated is not necessary for the assertion of Proposition 1.2.

*Example* 1.1. Let $p_{ii} \equiv 0$ $(i = 1, 2)$ and let $p_{12}, p_{21} : [a, b] \to \mathbb{R}_+$ be integrable functions such that

$$\int_a^b p_{12}(s) ds = 1, \qquad \int_a^b p_{21}(s) ds = 1. \tag{1.13}$$

It is clear that there exists $t_0 \in [a, b]$ satisfying

$$\int_a^{t_0} p_{12}(s) ds < 1. \tag{1.14}$$

Put $\delta_1 = 1$, $\delta_2 = 1$, $\tau_{ii} \equiv a$ $(i = 1, 2)$, $\tau_{12} \equiv b$, and $\tau_{21} \equiv t_0$. Then, the matrix $A = (a_{ik})_{i,k=1}^2$ given by (1.12) has the form

$$A = \begin{pmatrix} \int_a^b p_{12}(s) ds \int_a^b p_{21}(s) ds & 0 \\ 0 & \int_a^b p_{21}(s) ds \int_a^{t_0} p_{12}(s) ds \end{pmatrix}.$$

It is clear that, by virtue of (1.13) and (1.14), we have $r(A) = 1$.

According to (1.13), the condition (1.11) is satisfied. Therefore, it follows from Proposition 3.3 of [11] that the operator $\ell$ given by (1.8) belongs to the set $\mathcal{S}_{ab}^{2,(1,1)}(a)$ if and only if the homogeneous problem corresponding to (1.1), (1.2) has only the trivial solution. Let $u = (u_1, u_2)^T$ be a solution of the problem indicated. Then

$$u_1'(t) = p_{12}(t) u_2(b), \quad u_2'(t) = p_{21}(t) u_1(t_0) \quad \text{for } t \in [a, b], \tag{1.15}$$

$$u_1(a) = 0, \qquad u_2(a) = 0. \tag{1.16}$$

The integration of the second equality in (1.15) from $a$ to $b$, in view of (1.13) and (1.16), results in

$$u_2(b) = u_1(t_0) \int_a^b p_{21}(s)ds = u_1(t_0).$$

On the other hand, the integration of the first equality in (1.15) from $a$ to $t_0$, on account of (1.16), implies

$$u_1(t_0) = u_2(b) \int_a^{t_0} p_{12}(s)ds = u_1(t_0) \int_a^{t_0} p_{12}(s)ds.$$

Hence, using (1.14) in the last relations, we get $u_1(t_0) = 0$ and thus $u_2(b) = 0$, as well. Consequently, (1.15) and (1.16) yield $u_1 \equiv 0$ and $u_2 \equiv 0$.

Therefore, the operator $\ell$ defined by (1.8) belongs to the set $\mathcal{S}_{ab}^{2,(1,1)}(a)$ even if $r(A) = 1$.

In Section 3, we shall give efficient conditions which are not only sufficient but also necessary for the inclusion $\ell \in \mathcal{S}_{ab}^{2,(\sigma_1,\sigma_2)}(a)$ with $\ell$ given by (1.8) provided that there exist numbers $\delta_i > 0$ $(i = 1, 2)$ such that (1.10) and (1.11) are satisfied.

## 2. Auxiliary statements

In addition to (1.9), we put

$$\tau_{ii}^* = \operatorname{ess\,sup}\left\{\tau_{ii}(t) : t \in [a, b]\right\} \quad \text{for } i = 1, 2. \tag{2.1}$$

To prove the main results (see Section 3) we need some auxiliary statements. Following [4], we introduce a definition.

**Definition 2.1.** Let $C([a, b]; \mathbb{R})$ and $L([a, b]; \mathbb{R})$ denote the Banach spaces of continuous and Lebesgue integrable functions $z : [a, b] \to \mathbb{R}$, respectively, equipped with the standard norms. We say that a linear bounded operator $\varphi : C([a, b]; \mathbb{R}) \to L([a, b]; \mathbb{R})$ belongs to the set $\mathcal{S}_{ab}(a)$ if every absolutely continuous function $z : [a, b] \to \mathbb{R}$ such that

$$z'(t) \geq \varphi(z)(t) \quad \text{for } t \in [a, b], \qquad z(a) \geq 0$$

is non-negative on $[a, b]$.

For $i = 1, 2$, we put

$$\ell_{ii}(z)(t) \stackrel{\text{def}}{=} p_{ii}(t)(\tau_{ii}(t)) \quad \text{for } t \in [a, b],\ z \in C([a, b]; \mathbb{R}). \tag{2.2}$$

**Lemma 2.1** ([11, Proposition 3.1]). *Let $\sigma_1, \sigma_2 \in \{-1, 1\}$, the condition (1.10) be satisfied, and let*

$$\int_a^{\tau^*} p_{12}(s)ds \int_a^{\tau^*} p_{21}(s)ds = 0. \tag{2.3}$$

*Then the operator $\ell$ given by* (1.8) *belongs to the set* $\mathcal{S}_{ab}^{2,(\sigma_1,\sigma_2)}(a)$ *if and only if both operators* $\ell_{11}$ *and* $\ell_{22}$ *defined by* (2.2) *belong to the set* $\mathcal{S}_{ab}(a)$.

**Lemma 2.2.** *Let* $i \in \{1, 2\}$ *and*

$$p_{ii}(t) \geq 0 \quad \textit{for } t \in [a, b]. \tag{2.4}$$

*Then the following assertions are true:*

(a) *If*

$$\int_a^{\tau_{ii}^*} p_{ii}(s)ds < 1 \tag{2.5}$$

*then the operator* $\ell_{ii}$ *defined by* (2.2) *belongs to the set* $\mathcal{S}_{ab}(a)$.

(b) *Let*

$$\int_a^{\tau_{ii}^*} p_{ii}(s)ds = 1. \tag{2.6}$$

*Then the operator* $\ell_{ii}$ *defined by* (2.2) *belongs to the set* $\mathcal{S}_{ab}(a)$ *if and only if*

$$\int_a^{\tau_{ii}^*} p_{ii}(s)\left(\int_a^{\tau_{ii}(s)} p_{ii}(\xi)d\xi\right)ds < 1. \tag{2.7}$$

The results of the last lemma are partly contained in [4]. For the sake of completeness, we give the proof here.

PROOF OF LEMMA 2.2. Let $C\big([a, \tau_{ii}^*]; \mathbb{R}\big)$ be the Banach space of the continuous functions $z : [a, \tau_{ii}^*] \to \mathbb{R}$ equipped with the standard norm. Let the operator $\ell_{ii}$ be given by (2.2) and let

$$\ell_{ii}^*(z)(t) \stackrel{\text{def}}{=} p_{ii}(t)z\big(\tau_{ii}(t)\big) \quad \text{for } t \in [a, \tau_{ii}^*],\ z \in C\big([a, \tau_{ii}^*]; \mathbb{R}\big).$$

In the other words, $\ell_{ii}^*$ is the restriction of $\ell_{ii}$ to the space $C\big([a, \tau_{ii}^*]; \mathbb{R}\big)$. Since the condition (2.4) holds and

$$\tau_{ii}(t) \leq \tau_{ii}^* \quad \text{for } t \in [a, b],$$

it is clear that $\ell_{ii} \in \mathcal{S}_{ab}(a)$ if and only if $\ell_{ii}^* \in \mathcal{S}_{a\tau_{ii}^*}(a)$.

*Case (a).* Let the condition (2.5) be satisfied. By virtue of Remark 1.1 in [4], we find $\ell_{ii}^* \in \mathcal{S}_{a\tau_{ii}^*}(a)$, and thus $\ell_{ii} \in \mathcal{S}_{ab}(a)$.

*Case (b).* Let the condition (2.6) be fulfilled. According to Remark 1.1 in [4], $\ell_{ii}^* \in \mathcal{S}_{a\tau_{ii}^*}(a)$ if and only if the homogeneous problem

$$z'(t) = p_{ii}(t)z(\tau_{ii}(t)) \qquad (t \in [a, \tau_{ii}^*]), \tag{2.8}$$
$$z(a) = 0 \tag{2.9}$$

has only the trivial solution*. Consequently, to prove the lemma it is sufficient to show that the homogeneous problem (2.8), (2.9) has only the trivial solution if and only if the condition (2.7) is satisfied.

Let $z$ be a solution of the problem (2.8), (2.9). Put

$$M = \max\{z(t) : t \in [a, \tau_{ii}^*]\}, \qquad m = \min\{z(t) : t \in [a, \tau_{ii}^*]\} \tag{2.10}$$

and choose $t_M, t_m \in [a, \tau_{ii}^*]$ such that

$$z(t_M) = M, \qquad z(t_m) = m. \tag{2.11}$$

Obviously, (2.9) and (2.10) imply

$$M \geq 0. \tag{2.12}$$

We can assume without loss of generality that $t_m \leq t_M$. The integration of (2.8) from $t_m$ to $t_M$, in view of (2.4), (2.6), and (2.10)–(2.12), yields

$$M - m = \int_{t_m}^{t_M} p_{ii}(s) z\big(\tau_{ii}(s)\big) ds \leq M \int_{t_m}^{t_M} p_{ii}(s) ds \leq M.$$

Hence we get $m \geq 0$, i. e.,

$$z(t) \geq 0 \quad \text{for } t \in [a, \tau_{ii}^*]. \tag{2.13}$$

From (2.4), (2.8), and (2.13) we obtain

$$z(t) \leq z\big(\tau_{ii}^*\big) \quad \text{for } t \in [a, \tau_{ii}^*]. \tag{2.14}$$

Put

$$f(t) = \int_a^t p_{ii}(s) ds \quad \text{for } t \in [a, \tau_{ii}^*]. \tag{2.15}$$

The integration of (2.8) from $t$ to $\tau_{ii}^*$, on account of (2.4) and (2.14), yields

$$z\big(\tau_{ii}^*\big) - z(t) = \int_t^{\tau_{ii}^*} p_{ii}(s) z\big(\tau_{ii}(s)\big) ds \leq z\big(\tau_{ii}^*\big) \int_t^{\tau_{ii}^*} p_{ii}(s) ds \quad \text{for } t \in [a, \tau_{ii}^*].$$

Using (2.6), (2.15) and the last relations, we get

$$z\big(\tau_{ii}^*\big) f(t) = z(\tau_{ii}^*) \left(1 - \int_t^{\tau_{ii}^*} p_{ii}(s) ds\right) \leq z(t) \quad \text{for } t \in [a, \tau_{ii}^*]. \tag{2.16}$$

On the other hand, the integration of (2.8) from $a$ to $t$, on account of (2.4), (2.9), (2.14), and (2.15), results in

$$z(t) = \int_a^t p_{ii}(s) z\big(\tau_{ii}(s)\big) ds \leq z\big(\tau_{ii}^*\big) \int_a^t p_{ii}(s) ds = z\big(\tau_{ii}^*\big) f(t) \quad \text{for } t \in [a, \tau_{ii}^*].$$

Now, from the last relations and (2.16) we obtain

$$z(t) = z\big(\tau_{ii}^*\big) f(t) \quad \text{for } t \in [a, \tau_{ii}^*]. \tag{2.17}$$

---

*By a solution of the problem (2.8), (2.9), we mean an absolutely continuous function $z : [a, \tau_{ii}^*] \to \mathbb{R}$ satisfying the equation (2.8) almost everywhere on $[a, \tau_{ii}^*]$ and verifying also the initial condition (2.9).

Finally, the integration of (2.8) from $a$ to $\tau_{ii}^*$, with respect to (2.9) and (2.17), implies

$$z(\tau_{ii}^*) = \int_a^{\tau_{ii}^*} p_{ii}(s) z\big(\tau_{ii}(s)\big) ds = z(\tau_{ii}^*) \int_a^{\tau_{ii}^*} p_{ii}(s) f\big(\tau_{ii}(s)\big) ds,$$

whence we get

$$z(\tau_{ii}^*) \left[ 1 - \int_a^{\tau_{ii}^*} p_{ii}(s) \left( \int_a^{\tau_{ii}(s)} p_{ii}(\xi) d\xi \right) ds \right] = 0. \tag{2.18}$$

We have proved that every solution $z$ of the problem (2.8), (2.9) admits the representation (2.17), where $z(\tau_{ii}^*)$ satisfies (2.18). Consequently, if (2.7) holds then the homogeneous problem (2.8), (2.9) has only the trivial solution.

It remains to show that if (2.7) is not satisfied, i. e.,

$$\int_a^{\tau_{ii}^*} p_{ii}(s) f\big(\tau_{ii}(s)\big) ds = 1, \tag{2.19}$$

then the homogeneous problem (2.8), (2.9) has a non-trivial solution. Indeed, in view of (2.4) and (2.6), (2.15) yields

$$f(t) \le f(\tau_{ii}^*) = 1 \quad \text{for } t \in [a, \tau_{ii}^*].$$

Therefore, using (2.4) and (2.19), it is easy to verify that

$$0 \le \int_a^t p_{ii}(s) \Big[ 1 - f\big(\tau_{ii}(s)\big) \Big] ds \le \int_a^{\tau_{ii}^*} p_{ii}(s) \Big[ 1 - f\big(\tau_{ii}(s)\big) \Big] ds =$$
$$= 1 - \int_a^{\tau_{ii}^*} p_{ii}(s) f\big(\tau_{ii}(s)\big) ds = 0 \quad \text{for } t \in [a, \tau_{ii}^*],$$

whence we get

$$f(t) = \int_a^t p_{ii}(s) f\big(\tau_{ii}(s)\big) ds \quad \text{for } t \in [a, \tau_{ii}^*].$$

Consequently, $f$ is a non-trivial solution of the problem (2.8), (2.9). □

**Lemma 2.3.** *Let $\sigma_1, \sigma_2 \in \{-1, 1\}$, the condition* (1.10) *hold, and let there exist numbers $\delta_j > 0$ $(j = 1, 2)$ such that the relation*

$$\delta_1 \int_a^{\tau^*} |p_{i1}(s)| ds + \delta_2 \int_a^{\tau^*} |p_{i2}(s)| ds = \delta_i \tag{2.20$_i$}$$

*is satisfied for $i = 1, 2$. Let, moreover, $u = (u_1, u_2)^T$ be a solution of the homogeneous problem*

$$u_i'(t) = p_{i1}(t) u_1\big(\tau_{i1}(t)\big) + p_{i2}(t) u_2\big(\tau_{i2}(t)\big) \qquad (t \in [a, \tau^*],\ i = 1, 2) \tag{2.21}$$
$$u_1(a) = 0, \qquad u_2(a) = 0. \tag{2.22}$$

*Then both functions $u_1$ and $u_2$ do not change their signs on $[a, \tau^*]$. If, in addition,*

$$\int_a^{\tau^*} |p_{12}(s)|ds + \int_a^{\tau^*} |p_{21}(s)|ds > 0 \tag{2.23}$$

*then the relation*

$$\sigma_1\sigma_2 u_1(t)u_2(t) \geq 0 \quad \textit{for } t \in [a, \tau^*] \tag{2.24}$$

*is satisfied.*

PROOF. For $i = 1, 2$, we put

$$M_i = \max\big\{\sigma_i u_i(t) : t \in [a, \tau^*]\big\}, \quad m_i = -\min\big\{\sigma_i u_i(t) : t \in [a, \tau^*]\big\}. \tag{2.25}$$

Choose $t_i, T_i \in [a, \tau^*]$ $(i = 1, 2)$ such that

$$\sigma_i u_i(T_i) = M_i\,, \qquad \sigma_i u_i(t_i) = -m_i \quad \text{for } i = 1, 2, \tag{2.26}$$

and put

$$\widetilde{p}_{ik} = \int_a^{\tau^*} |p_{ik}(s)|ds \quad \text{for } i, k = 1, 2. \tag{2.27}$$

Let us first suppose that both functions $u_1$ and $u_2$ change their signs on $[a, \tau^*]$. Then we have

$$M_i > 0, \qquad m_i > 0 \tag{2.28$_i$}$$

for $i = 1, 2$. We can assume without loss of generality that $T_1 < t_1$. The integration of (2.21) with $i = 1$ from $T_1$ to $t_1$, in view of (1.10) and (2.25)–(2.27), yields

$$\begin{aligned} M_1 + m_1 &= -\sigma_1 \int_{T_1}^{t_1} p_{11}(s)u_1\big(\tau_{11}(s)\big)ds - \sigma_1 \int_{T_1}^{t_1} p_{12}(s)u_2\big(\tau_{12}(s)\big)ds \leq \\ &\leq m_1 \int_{T_1}^{t_1} |p_{11}(s)|ds + m_2 \int_{T_1}^{t_1} |p_{12}(s)|ds \leq m_1\widetilde{p}_{11} + m_2\widetilde{p}_{12}\,. \end{aligned} \tag{2.29}$$

It is clear that either $T_2 < t_2$ or $T_2 > t_2$ is satisfied.

*Case 1: $T_2 < t_2$ holds.* The integration of (2.21) with $i = 2$ from $T_2$ to $t_2$, on account of (1.10) and (2.25)–(2.27), implies

$$\begin{aligned} M_2 + m_2 &= -\sigma_2 \int_{T_2}^{t_2} p_{21}(s)u_1\big(\tau_{21}(s)\big)ds - \sigma_2 \int_{T_2}^{t_2} p_{22}(s)u_2\big(\tau_{22}(s)\big)ds \leq \\ &\leq m_1 \int_{T_2}^{t_2} |p_{21}(s)|ds + m_2 \int_{T_2}^{t_2} |p_{22}(s)|ds \leq m_1\widetilde{p}_{21} + m_2\widetilde{p}_{22}\,. \end{aligned} \tag{2.30}$$

If $\delta_1 m_2 \leq \delta_2 m_1$ then from (2.20$_1$) and (2.29) we get

$$M_1 + m_1 \leq m_1\widetilde{p}_{11} + \frac{\delta_2}{\delta_1} m_1\widetilde{p}_{12} \leq m_1\,, \tag{2.31}$$

which contradicts (2.28$_1$).

If $\delta_1 m_2 > \delta_2 m_1$ then $(2.20_2)$ and (2.30) result in

$$M_2 + m_2 \leq \frac{\delta_1}{\delta_2} m_2 \widetilde{p}_{21} + m_2 \widetilde{p}_{22} \leq m_2 \,,$$

which contradicts $(2.28_2)$.

*Case 2: $T_2 > t_2$ holds.* The integrations of (2.21) with $i = 2$ from $a$ to $t_2$ and from $t_2$ to $T_2$, on account of (1.10), (2.22), and (2.25)–(2.27), yield

$$\begin{aligned} m_2 = -\sigma_2 \int_a^{t_2} p_{21}(s) u_1\big(\tau_{21}(s)\big) ds - \sigma_2 \int_a^{t_2} p_{22}(s) u_2\big(\tau_{22}(s)\big) ds \leq \\ \leq m_1 \int_a^{t_2} |p_{21}(s)| ds + m_2 \int_a^{t_2} |p_{22}(s)| ds \leq m_1 \widetilde{p}_{21} + m_2 \widetilde{p}_{22} \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} M_2 + m_2 = \sigma_2 \int_{t_2}^{T_2} p_{21}(s) u_1\big(\tau_{21}(s)\big) ds + \sigma_2 \int_{t_2}^{T_2} p_{22}(s) u_2\big(\tau_{22}(s)\big) ds \leq \\ \leq M_1 \int_{t_2}^{T_2} |p_{21}(s)| ds + M_2 \int_{t_2}^{T_2} |p_{22}(s)| ds \leq M_1 \widetilde{p}_{21} + M_2 \widetilde{p}_{22} \,. \end{aligned} \tag{2.33}$$

If $\delta_1 m_2 \leq \delta_2 m_1$ then from $(2.20_1)$ and (2.29) we get (2.31), which contradicts $(2.28_1)$.

If $\delta_1 m_2 > \delta_2 m_1$ and $\widetilde{p}_{21} > 0$ then $(2.20_2)$ and (2.32) imply

$$m_2 < \frac{\delta_1}{\delta_2} m_2 \widetilde{p}_{21} + m_2 \widetilde{p}_{22} \leq m_2 \,,$$

which is a contradiction.

If $\delta_1 m_2 > \delta_2 m_1$ and $\widetilde{p}_{21} = 0$ then $(2.20_2)$ and (2.33) results in

$$M_2 + m_2 \leq M_2 \widetilde{p}_{22} \leq M_2 \,, \tag{2.34}$$

which contradicts $(2.28_2)$.

The contradictions obtained prove that at least one of the functions $u_1$ and $u_2$ does not change its sign on $[a, \tau^*]$. We can assume without loss of generality that

$$\sigma_1 u_1(t) \geq 0 \quad \text{for } t \in [a, \tau^*]. \tag{2.35}$$

Suppose that, on the contrary, $u_2$ changes its sign. Then $(2.28_2)$ is satisfied and either $T_2 < t_2$ or $T_2 > t_2$ is true.

*Case 1: $T_2 < t_2$ holds.* The integration of (2.21) with $i = 2$ from $T_2$ to $t_2$, in view of (1.10), $(2.20_2)$, (2.25)–(2.27), and (2.35), implies

$$M_2 + m_2 = -\sigma_2 \int_{T_2}^{t_2} p_{21}(s) u_1\big(\tau_{21}(s)\big) ds - \sigma_2 \int_{T_2}^{t_2} p_{22}(s) u_2\big(\tau_{22}(s)\big) ds \leq$$

$$\leq m_2 \int_{T_2}^{t_2} |p_{22}(s)| ds \leq m_2 \,,$$

which contradicts $(2.28_2)$.

*Case 2: $T_2 > t_2$ holds.* The integrations of (2.21) with $i = 2$ from $t_2$ to $T_2$ and from $a$ to $t_2$, with respect to (1.10), (2.25)–(2.27), and (2.35), result in (2.33) and

$$m_2 = -\sigma_2 \int_a^{t_2} p_{21}(s) u_1\big(\tau_{21}(s)\big) ds - \sigma_2 \int_a^{t_2} p_{22}(s) u_2\big(\tau_{22}(s)\big) ds \leq \\ \leq m_2 \int_a^{t_2} |p_{22}(s)| ds \leq m_2 \widetilde{p}_{22} \,. \quad (2.36)$$

If $\widetilde{p}_{21} = 0$ then from $(2.20_2)$ and (2.33) we get (2.34), which contradicts $(2.28_2)$. If $\widetilde{p}_{21} > 0$ then $(2.20_2)$ guarantees that $\widetilde{p}_{22} < 1$. Consequently, (2.36) implies $m_2 \leq 0$, which contradicts $(2.28_2)$.

We have proved that both functions $u_1$ and $u_2$ do not change their signs on $[a, \tau^*]$. Let, in addition, (2.23) holds. We will show that (2.24) is satisfied. We can assume without loss of generality that $\widetilde{p}_{12} > 0$ and the condition (2.35) is fulfilled. Suppose that, on the contrary, (2.24) is not true. Then

$$M_1 > 0 \quad (2.37)$$

and

$$\sigma_2 u_2(t) \leq 0 \quad \text{for } t \in [a, \tau^*]. \quad (2.38)$$

Obviously, $(2.20_1)$ implies that

$$\widetilde{p}_{11} < 1. \quad (2.39)$$

The integration of (2.21) with $i = 1$ from $a$ to $T_1$, in view of (1.10), (2.22), (2.25)–(2.27), and (2.38), results in

$$M_1 = \sigma_1 \int_a^{T_1} p_{11}(s) u_1\big(\tau_{11}(s)\big) ds + \sigma_1 \int_a^{T_1} p_{12}(s) u_2\big(\tau_{12}(s)\big) ds \leq \\ \leq M_1 \int_a^{T_1} |p_{11}(s)| ds \leq M_1 \widetilde{p}_{11} \,.$$

Using (2.39) in the last relations, we get $M_1 \leq 0$, which contradicts (2.37). The contradiction obtained proves that the condition (2.24) holds provided that (2.23) is satisfied. □

## 3. Main results

Recall that the numbers $\tau^*$ and $\tau_{ii}^*$ $(i = 1, 2)$ are given by (1.9) and (2.1), respectively.

**Theorem 3.1.** *Let $\sigma_1, \sigma_2 \in \{-1, 1\}$, $i \in \{1, 2\}$, and let the condition* (1.10) *hold. Let, moreover, there exist $\delta_j > 0$ $(j = 1, 2)$ such that*

$$\delta_1 \int_a^{\tau^*} |p_{i1}(s)|ds + \delta_2 \int_a^{\tau^*} |p_{i2}(s)|ds = \delta_i \tag{3.1}$$

*and*

$$\delta_1 \int_a^{\tau^*} |p_{3-i\,1}(s)|ds + \delta_2 \int_a^{\tau^*} |p_{3-i\,2}(s)|ds < \delta_{3-i}\,. \tag{3.2}$$

*Then the following assertions are true:*

(a) *If the condition* (2.5) *is satisfied then the operator $\ell$ given by* (1.8) *belongs to the set $\mathcal{S}_{ab}^{2,(\sigma_1,\sigma_2)}(a)$.*

(b) *Let the condition* (2.6) *be satisfied. Then the operator $\ell$ given by* (1.8) *belongs to the set $\mathcal{S}_{ab}^{2,(\sigma_1,\sigma_2)}(a)$ if and only if the condition* (2.7) *is true.*

PROOF. We can assume without loss of generality that $i = 1$. Let the operators $\ell$ and $\ell_{11}, \ell_{22}$ are defined by (1.8) and (2.2), respectively. We first note that (1.10) and (3.2) imply

$$\int_a^{\tau_{22}^*} p_{22}(s)ds < 1.$$

Hence, Lemma 2.2 (a) guarantees

$$\ell_{22} \in \mathcal{S}_{ab}(a). \tag{3.3}$$

*Case (a).* If $p_{12} \not\equiv 0$ then the assertion of the theorem follows from Proposition 3.2 in [11]. Therefore, suppose that $p_{12} \equiv 0$. Then, by virtue of (3.3) and Lemma 2.1, it is sufficient to show that $\ell_{11} \in \mathcal{S}_{ab}(a)$. However, using (2.5) and Lemma 2.2 (a), we see that the inclusion $\ell_{11} \in \mathcal{S}_{ab}(a)$ is true.

*Case (b).* According to (3.1) and (2.6), we get $p_{12} \equiv 0$. By virtue of (3.3) and Lemma 2.1, the operator $\ell$ belongs to the set $\mathcal{S}_{ab}^{2,(\sigma_1,\sigma_2)}(a)$ if and only if $\ell_{11} \in \mathcal{S}_{ab}(a)$. However, in view of (2.6) and Lemma 2.2 (b), $\ell_{11} \in \mathcal{S}_{ab}(a)$ if and only if the condition (2.7) is satisfied. □

**Theorem 3.2.** *Let $\sigma_1, \sigma_2 \in \{-1, 1\}$, the condition* (1.10) *hold, and let there exist $\delta_j > 0$ $(j = 1, 2)$ such that the relation* (3.1) *is satisfied for $i = 1, 2$. Then the following assertions are true:*

(a) *Let the inequality* (2.3) *be fulfilled and let the condition* (2.5) *hold for $i = 1, 2$. Then the operator $\ell$ given by* (1.8) *belongs to the set $\mathcal{S}_{ab}^{2,(\sigma_1,\sigma_2)}(a)$.*

(b) *Let the condition* (2.3) *hold and let there exist $i \in \{1, 2\}$ such that the condition* (2.6) *is satisfied. Then the operator $\ell$ given by* (1.8) *belongs to the set $\mathcal{S}_{ab}^{2,(\sigma_1,\sigma_2)}(a)$ if and only if the condition* (2.7) *holds for every $i \in \{1, 2\}$ such that the condition* (2.6) *is true.*

(c) *Let*

$$\int_a^{\tau^*} p_{12}(s)ds \int_a^{\tau^*} p_{21}(s)ds \neq 0. \tag{3.4}$$

*Then the operator* $\ell$ *given by* (1.8) *belongs to the set* $\mathcal{S}_{ab}^{2,(\sigma_1,\sigma_2)}(a)$ *if and only if there exist* $i \in \{1,2\}$ *such that*

$$\sum_{j=1}^{2} \int_a^{\tau^*} |p_{ij}(s)| \left( \sum_{k=1}^{2} \delta_k \int_a^{\tau_{ij}(s)} |p_{jk}(\xi)| d\xi \right) ds < \delta_i \,. \tag{3.5}$$

PROOF. Let the operators $\ell$ and $\ell_{11}, \ell_{22}$ are defined by (1.8) and (2.2), respectively.

*Case (a).* According to (2.5) and Lemma 2.2 (a), we get

$$\ell_{11} \in \mathcal{S}_{ab}(a), \qquad \ell_{22} \in \mathcal{S}_{ab}(a).$$

Hence, by virtue of (2.3) and Lemma 2.1, it is clear that $\ell \in \mathcal{S}_{ab}^{2,(\sigma_1,\sigma_2)}(a)$.

*Case (b).* It is easy to see from (1.10) and (3.1) that, for $i = 1,2$, either (2.5) or (2.6) is satisfied. Therefore, in view of (2.3), the assertion of the theorem follows immediately from Lemmas 2.1 and 2.2.

*Case (c).* Let the operator $\ell^* : C([a,\tau^*];\mathbb{R}^2) \to L([a,\tau^*];\mathbb{R}^2)$ be defined by the formula

$$\ell^*(v)(t) \stackrel{\text{def}}{=} \begin{pmatrix} p_{11}(t)v_1\big(\tau_{11}(t)\big) + p_{12}(t)v_2\big(\tau_{12}(t)\big) \\ p_{21}(t)v_1\big(\tau_{21}(t)\big) + p_{22}(t)v_2\big(\tau_{22}(t)\big) \end{pmatrix}$$

for $t \in [a,\tau^*]$. In the other words, $\ell^*$ is the restriction of $\ell$ into the space $C([a,\tau^*];\mathbb{R}^2)$. Since (1.10) holds and

$$\tau_{ik}(t) \le \tau^* \quad \text{for } t \in [a,b],\ i,k = 1,2,$$

it is clear that $\ell \in \mathcal{S}_{ab}^{2,(\sigma_1,\sigma_2)}(a)$ if and only if $\ell^* \in \mathcal{S}_{a\tau^*}^{2,(\sigma_1,\sigma_2)}(a)$. However, according to (3.1) and Proposition 3.3 in [11], $\ell^* \in \mathcal{S}_{a\tau^*}^{2,(\sigma_1,\sigma_2)}(a)$ if and only if the homogeneous problem (2.21), (2.22) has only the trivial solution. Consequently, to prove the theorem it is sufficient to show that the homogeneous problem (2.21), (2.22) has only the trivial solution if and only if there exists $i \in \{1,2\}$ such that (3.5) is satisfied.

Let $u = (u_1,u_2)^T$ be a solution the problem (2.21), (2.22). According to (3.4) and Lemma 2.3, we can assume that

$$\sigma_i u_i(t) \ge 0 \quad \text{for } t \in [a,\tau^*],\ i = 1,2. \tag{3.6}$$

Therefore, in view of (1.10) and (3.6), from (2.21) we get

$$\sigma_i u_i(t) \le \sigma_i u_i(\tau^*) \quad \text{for } t \in [a,\tau^*],\ i = 1,2. \tag{3.7}$$

Put

$$u_i^* = \frac{\sigma_i}{\delta_i}\, u_i(\tau^*) \quad \text{for } i = 1, 2, \tag{3.8}$$

$$f_i(t) = \sigma_i \sum_{k=1}^{2} \delta_k \int_a^t |p_{ik}(s)|ds \quad \text{for } t \in [a, \tau^*],\ i = 1, 2. \tag{3.9}$$

The integration of (2.21) from $t$ to $\tau^*$, on account of (1.10) and (3.7), implies

$$\sigma_i u_i(\tau^*) - \sigma_i u_i(t) = \int_t^{\tau^*} |p_{i1}(s)|\sigma_1 u_1\big(\tau_{i1}(s)\big)ds + \int_t^{\tau^*} |p_{i2}(s)|\sigma_2 u_2\big(\tau_{i2}(s)\big)ds$$
$$\leq \sigma_1 u_1(\tau^*) \int_t^{\tau^*} |p_{i1}(s)|ds + \sigma_2 u_2(\tau^*) \int_t^{\tau^*} |p_{i2}(s)|ds$$

for $t \in [a, \tau^*]$, $i = 1, 2$. Using the notation (3.8), we get

$$\delta_i u_i^* + \sum_{k=1}^{2} \delta_k u_k^* \int_a^t |p_{ik}(s)|ds \leq$$
$$\leq \sigma_i u_i(t) + \sum_{k=1}^{2} \delta_k u_k^* \int_a^{\tau^*} |p_{ik}(s)|ds \quad \text{for } t \in [a, \tau^*],\ i = 1, 2. \tag{3.10}$$

On the other hand, the integration of (2.21) from $a$ to $t$, in view of (1.10), (2.22), (3.7), and (3.8) yields

$$\sigma_i u_i(t) = \int_a^t |p_{i1}(s)|\sigma_1 u_1\big(\tau_{i1}(s)\big)ds + \int_a^t |p_{i2}(s)|\sigma_2 u_2\big(\tau_{i2}(s)\big)ds \leq$$
$$\leq \delta_1 u_1^* \int_a^t |p_{i1}(s)|ds + \delta_2 u_2^* \int_a^t |p_{i2}(s)|ds \tag{3.11}$$

for $t \in [a, \tau^*]$, $i = 1, 2$. Now, from (3.10) and (3.11) we obtain

$$\delta_i u_i^* \leq \delta_1 u_1^* \int_a^{\tau^*} |p_{i1}(s)|ds + \delta_2 u_2^* \int_a^{\tau^*} |p_{i2}(s)|ds \quad \text{for } i = 1, 2,$$

whence we get

$$u_i^* \left( \delta_i - \delta_i \int_a^{\tau^*} |p_{ii}(s)|ds \right) \leq u_{3-i}^* \delta_{3-i} \int_a^{\tau^*} |p_{i3-i}(s)|ds \quad (i = 1, 2). \tag{3.12}$$

By virtue of (3.1) and (3.4), (3.12) yields $u_i^* \leq u_{3-i}^*$ for $i = 1, 2$ and thus

$$u_1^* = u_2^* \big( := u^* \big). \tag{3.13}$$

Now (3.10), in view of (3.1) and (3.9), yields

$$\sigma_i u_i(t) \geq u^* \sum_{k=1}^{2} \delta_k \int_a^t |p_{ik}(s)|ds + u^* \left( \delta_i - \sum_{k=1}^{2} \delta_k \int_a^{\tau^*} |p_{ik}(s)|ds \right)$$
$$= u^* \sigma_i f_i(t) \quad \text{for } t \in [a, \tau^*],\ i = 1, 2. \quad (3.14)$$

On the other hand, using (3.1), (3.9), and (3.13), we can rewrite (3.11) as

$$\sigma_i u_i(t) \leq u^* \sum_{k=1}^{2} \delta_k \int_a^t |p_{ik}(s)|ds = u^* \sigma_i f_i(t)$$
$$\text{for } t \in [a, \tau^*],\ i = 1, 2. \quad (3.15)$$

Hence, (3.14) and (3.15) result in

$$u_i(t) = u^* f_i(t) \quad \text{for } t \in [a, \tau^*],\ i = 1, 2. \quad (3.16)$$

Finally, the integration of (2.21) from $a$ to $\tau^*$, in view of (1.10), (2.22), and (3.16), yields

$$\sigma_i u_i(\tau^*) = \int_a^{\tau^*} |p_{i1}(s)| \sigma_1 u_1\big(\tau_{i1}(s)\big) ds + \int_a^{\tau^*} |p_{i2}(s)| \sigma_2 u_2\big(\tau_{i2}(s)\big) ds =$$
$$= u^* \sum_{j=1}^{2} \int_a^{\tau^*} \big|p_{ij}(s)\big| \sigma_j f_j\big(\tau_{ij}(s)\big) ds$$

for $t \in [a, \tau^*]$, $i = 1, 2$, whence we get

$$u^* \left[ \delta_i - \sum_{j=1}^{2} \int_a^{\tau^*} |p_{ij}(s)| \left( \sum_{k=1}^{2} \delta_k \int_a^{\tau_{ij}(s)} |p_{jk}(\xi)| d\xi \right) ds \right] = 0 \quad \big(i = 1, 2\big) \quad (3.17)$$

because of the notations (3.8), (3.9), and (3.13).

We have proved that every solution $u$ of the problem (2.21), (2.22) admits the representation

$$u(t) = u^* f(t) \quad \text{for } t \in [a, \tau^*],$$

where $f = (f_1, f_2)^T$ and $u^*$ satisfies (3.17). Consequently, if there exists $i \in \{1, 2\}$ such that (3.5) is true then the homogeneous problem (2.21), (2.22) has only the trivial solution.

It remains to show that if the condition (3.5) is not satisfied for any $i \in \{1, 2\}$, i. e.,

$$\sum_{j=1}^{2} \int_a^{\tau^*} \big|p_{ij}(s)\big| \sigma_j f_j\big(\tau_{ij}(s)\big) ds = \delta_i \quad \text{for } i = 1, 2, \quad (3.18)$$

then the problem (2.21), (2.22) has a non-trivial solution. Indeed, (3.1) and (3.9) yield

$$\sigma_i f_i(t) \leq \sigma_i f_i(\tau^*) = \delta_i \quad \text{for } t \in [a, \tau^*],\ i = 1, 2.$$

Therefore, using (3.1) and (3.18), it is easy to verify that

$$0 \le \sum_{k=1}^{2} \int_a^t |p_{ik}(s)| \Big[\delta_k - \sigma_k f_k\big(\tau_{ik}(s)\big)\Big] ds \le$$
$$\le \sum_{k=1}^{2} \int_a^{\tau^*} |p_{ik}(s)| \Big[\delta_k - \sigma_k f_k\big(\tau_{ik}(s)\big)\Big] ds =$$
$$= \delta_i - \sum_{k=1}^{2} \int_a^{\tau^*} |p_{ik}(s)|\, \sigma_k f_k\big(\tau_{ik}(s)\big) ds = 0$$

for $t \in [a, \tau^*]$ and $i = 1, 2$. Hence we get

$$f_i(t) = \sigma_i \sum_{k=1}^{2} \int_a^t |p_{ik}(s)|\, \sigma_k f_k\big(\tau_{ik}(s)\big) ds = \sum_{k=1}^{2} \int_a^t p_{ik}(s) f_k\big(\tau_{ik}(s)\big) ds$$

for $t \in [a, \tau^*]$, $i = 1, 2$. Consequently, $f = (f_1, f_2)^T$ is a non-trivial solution of problem (2.21), (2.22). □

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# COMPARISONS AND COMPOSITIONS OF GALOIS-TYPE CONNECTIONS

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ABSTRACT. In a former paper, motivated by a recent theory of relators (families of relations), the first author has investigated increasingly regular and normal functions of one preordered set into another instead of Galois connections and residuated mappings of partially ordered sets.

A function $f$ of one preordered set $X$ into another $Y$ has been called

(1) increasingly $g$-normal, for some function $g$ of $Y$ into $X$ if for any $x \in X$ and $y \in Y$ we have $f(x) \leq y$ if and only if $x \leq g(y)$;

(2) increasingly $\varphi$-regular, for some function $\varphi$ of $X$ into itself if for any $x_1, x_2 \in X$ we have $x_1 \leq \varphi(x_2)$ if and only if $f(x_1) \leq f(x_2)$.

In the present paper, for instance, we shall show that if $\varphi$ is an increasingly $\psi$-regular function of $X$ into itself, then $\varphi \leq \psi$ if and only if $\varphi \circ \psi \leq \psi$, and if $f_i$ is an increasingly $g_i$-normal function of $X$ into $Y$ for each $i = 1, 2$, then $f_1 \leq f_2$ if and only if $g_2 \leq g_1$.

Moreover, for instance, we shall show that if $f$ is an increasingly $\varphi_i$-regular function of $X$ into $Y$ for each $i = 1, 2$, then $f$ is increasingly $\varphi_1 \circ \varphi_2$-regular, and if $f$ is an increasingly $g$-normal function of $X$ into $Y$ and $h$ is an increasingly $k$-normal function of $Y$ into $Z$, then $h \circ f$ is $g \circ k$-normal.

*Mathematics Subject Classification:* Primary 06A06, 06A15; Secondary 04A05, 54E15

*Keywords:* preordered sets, Galois connections, residuated mappings, closure and interior operations

## INTRODUCTION

In a former paper [13], motivated by a recent theory of relators (see [9] and [7]), the first author has investigated increasingly regular and normal functions of one preordered set into another instead of Galois connections [3, p. 55] and residuated mappings [2, p. 11] of partially ordered sets.

A function $f$ of one preordered set $X$ into another $Y$ has been called

(1) increasingly $g$-normal, for some function $g$ of $Y$ into $X$ if for any $x \in X$ and $y \in Y$ we have $f(x) \leq y$ if and only if $x \leq g(y)$;

(2) increasingly $\varphi$-regular, for some function $\varphi$ of $X$ into itself if for any $x_1, x_2 \in X$ we have $x_1 \leq \varphi(x_2)$ if and only if $f(x_1) \leq f(x_2)$.

In the present paper, for instance, we shall show that on the one hand if $\varphi$ is an increasingly $\psi$-regular function of $X$ into itself, then $\varphi \leq \psi \iff \varphi \circ \psi \leq \psi \iff \psi \circ \varphi \leq \psi$, and the other hand if $f_i$ is an increasingly $g_i$-normal function of $X$ into $Y$ for each $i = 1, 2$, then $f_1 \leq f_2 \iff f_1 \circ g_2 \leq \Delta_Y \iff g_2 \leq g_1$.

Moreover, for instance, we shall show that if $f$ is an increasingly $\varphi_i$-regular function of $X$ into $Y$ for each $i = 1, 2$, then $f$ is increasingly $\varphi_1 \circ \varphi_2$-regular. While, if $f$ is an increasingly $g$-normal function of $X$ into $Y$ and $h$ is an increasingly $k$-normal function of $Y$ into $Z$, then $h \circ f$ is $g \circ k$-normal.

The results obtained naturally extend and supplement some former results not only on Galois-connections and residuated mappings, but also on closure and interior operations. Namely, for instance, we shall show that an increasingly $\psi$-normal function $\varphi$ of $X$ into itself is a closure operation if and only if it is an interior operation.

## 1. A few basic facts on relations

A subset $F$ of a product set $X \times Y$ is called a relation on $X$ to $Y$. If in particular $F \subset X^2$, then we may simply say that $F$ is a relation on $X$. Thus, $\Delta_X = \{(x, x) : x \in X\}$ is a relation on $X$.

If $F$ is a relation on $X$ to $Y$, then for any $x \in X$ the set $F(x) = \{y \in Y : (x, y) \in F\}$ is called the image of $x$ under $F$. And the set $D_F = \{x \in X : F(x) \neq \emptyset\}$ is called the domain of $F$.

In particular, a relation $F$ on $X$ to $Y$ is called a function if for each $x \in D_F$ there exists $y \in Y$ such that $F(x) = \{y\}$. In this case, by identifying singletons with their elements, we may usually write $F(x) = y$ in place of $F(x) = \{y\}$.

More generally, if $F$ is a relation on $X$ to $Y$, then for any $A \subset X$ the set $F[A] = \bigcup_{x \in A} F(x)$ is called the image of $A$ under $F$. And the set $R_F = F[D_F]$ is called the range of $F$.

If $F$ is a relation on $X$ to $Y$ such that $D_F = X$, then we say that $F$ is a relation of $X$ to $Y$. While, if $F$ is a relation on $X$ to $Y$ such that $R_F = Y$, then we say that $F$ is a relation on $X$ onto $Y$.

If $F$ is a relation on $X$ to $Y$, then the values $F(x)$, where $x \in X$ uniquely determine $F$ since we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the inverse $F^{-1}$ of $F$ can be defined such that $F^{-1}(y) = \{x \in X : \ y \in F(x)\}$ for all $y \in Y$.

Moreover, if $F$ is a relation on $X$ to $Y$ and $G$ is a relation on $Y$ to $Z$, then the composition $G \circ F$ of $G$ and $F$ can be defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G[F[A]]$ for all $A \subset X$.

A relation $R$ on $X$ is called reflexive, antisymmetric, and transitive if $\Delta_X \subset R$, $R \cap R^{-1} \subset \Delta_X$, and $R \circ R \subset R$, respectively. Moreover, a reflexive and transitive relation is called a preorder. And an antisymmetric preorder is called a partial order.

## 2. A FEW BASIC FACTS ON ORDERED SETS

If $\leq$ is a relation on a nonvoid set $X$, then having in mind the terminology of Birkhoff [1, p. 2] the ordered pair $X(\leq) = (X, \leq)$ is called a goset (generalized ordered set). And we usually write $X$ in place of $X(\leq)$.

If $X(\leq)$ is a goset, then by taking $X^* = X$ and $\leq^* = \leq^{-1}$ we can form another goset $X^*(\leq^*)$. This is called the dual of $X(\leq)$. And we usually write $\geq$ in place of $\leq^*$.

The goset $X$ is called reflexive, transitive, and antisymmetric if the inequality relation in it has the corresponding property. Moreover, for instance, $X$ is called preordered if it is reflexive and transitive.

In particular, a preordered set will be called a proset, and a partially ordered set will be called a poset. The usual definitions on posets can be naturally extended to gosets [10, 11].

For instance, for any subset $A$ of a goset $X$, the members of the families

$$\mathrm{lb}\,(A) = \big\{x \in X : \forall\, a \in A :\ x \leq a\big\}$$

and

$$\mathrm{ub}\,(A) = \big\{x \in X : \forall\, a \in A :\ a \leq x\big\}$$

are called the lower and upper bounds of $A$ in $X$, respectively.

Moreover, the members of the families

$$\min(A) = A \cap \mathrm{lb}\,(A) \qquad\qquad \max(A) = A \cap \mathrm{ub}\,(A)$$
$$\inf(A) = \max(\mathrm{lb}\,(A)) \qquad\qquad \sup(A) = \min(\mathrm{ub}\,(A))$$

are called the minima, maxima, infima and suprema of $A$ in $X$, respectively.

Thus, for any $A, B \subset X$, we have $A \subset \mathrm{lb}\,(B)$ if and only if $B \subset \mathrm{ub}\,(A)$. Moreover, a reflexive goset $X$ is antisymmetric if and only if $\mathrm{card}\,(\max(A)) \leq 1$ (resp., $\mathrm{card}\,(\sup(A)) \leq 1$) for all $A \subset X$; see [12].

## 3. CLOSURE OPERATIONS AND REGULAR STRUCTURES

**Definition 3.1.** A function $\varphi$ of a proset $X$ into itself is called an operation on $X$. More generally, a function $f$ of $X$ into another proset $Y$ is called a structure on $X$.

*Remark* 3.2. The latter terminology has been mainly motivated by the various structures derived from relators (see [8, 10]).

**Definition 3.3.** An operation $\varphi$ on $X$ is called

(1) expansive if $\Delta_X \leq \varphi$;
(2) quasi-idempotent if $\varphi^2 \leq \varphi$.

Moreover, a structure $f$ on $X$ is called increasing if for any $x_1, x_2 \in X$, with $x_1 \leq x_2$, we have $f(x_1) \leq f(x_2)$.

*Remark* 3.4. Note that if (1) holds, then $\varphi(x) = \Delta_X(\varphi(x)) \leq \varphi(\varphi(x)) = \varphi^2(x)$ for all $x \in X$, and thus $\varphi \leq \varphi^2$. Therefore, if both (1) and (2) hold and $X$ is a poset, then $\varphi$ is actually idempotent in the sense that $\varphi^2 = \varphi$.

Now, as a natural extension of the corresponding definition of [1, p. 111], we may also have the following.

**Definition 3.5.** An increasing and expansive operation is called a preclosure operation. And a quasi-idempotent preclosure operation is called a closure operation.

Moreover, an expansive and quasi-idempotent operation is called a semiclosure operation. And an increasing and idempotent operation is called a modification operation.

*Remark* 3.6. Now, an operation $\varphi$ on $X$ may be naturally called an interior operation if it is a closure operation on $X^*$.

In [13], having in mind the ideas of [7], the first author has also introduced the following

**Definition 3.7.** A structure $f$ on $X$ is called increasingly $\varphi$-regular, for some operation $\varphi$ on $X$, if for any $x_1, x_2 \in X$ we have

$$x_1 \leq \varphi(x_2) \iff f(x_1) \leq f(x_2).$$

*Remark* 3.8. Now, a structure $f$ on $X$ to $Y$ may be naturally called decreasingly $\varphi$-regular if it is an increasingly $\varphi$-regular structure on $X$ to $Y^*$.

The above definition closely resembles a recent definition of the Galois connections [3, p. 155]. However, instead of the Galois connections, it has been more convenient to use residuated mappings [2, p. 11] in the following relevant form.

**Definition 3.9.** A structure $f$ on $X$ to $Y$ is called increasingly $g$-normal, for some structure $g$ on $Y$ to $X$, if for any $x \in X$ and $y \in Y$ we have

$$f(x) \leq y \iff x \leq g(y).$$

*Remark* 3.10. Now, a structure $f$ on $X$ to $Y$ may be naturally decreasingly $g$-normal if it is an increasingly $g$-normal structure on $X$ to $Y^*$.

For an easy illustration of the latter definition, we can note here

*Example* 3.11. Consider the family $\mathcal{P} = \mathcal{P}(X)$ of all subsets of a generalized ordered set $X$ to be partially ordered by inclusion. Moreover, define $F(A) = \operatorname{ub}(A)$ and $G(A) = \operatorname{lb}(A)$ for all $A \subset X$.

Then, by the corresponding definitions, it is clear that $F$ is a decreasingly $G$-normal structure on $\mathcal{P}$. Hence, by defining $\Phi = G \circ F$ and using the duals of the forthcoming Theorems 4.5 and 4.3, we can easily see that $F$ is a decreasingly $\Phi$-regular structure and $\Phi$ is a closure operation on $\mathcal{P}$.

To appreciate the importance of this example, note that if in particular $X$ is a poset, then by [3, p. 166] the poset $\Phi[\mathcal{P}]$ is just the Dedekind–MacNeille completion of $X$ by the cuts $\Phi(A)$.

## 4. Relationships between closure operations and regular structures

By using the above definitions, in [13], the first author has proved the following theorems.

**Theorem 4.1.** *If $f$ is an increasingly $\varphi$-regular structure on $X$, then*

(1) *$\varphi$ is expansive;*
(2) *$f$ is increasing;*
(3) *$f \leq f \circ \varphi \leq f$.*

**Corollary 4.2.** *If $f$ is an increasingly $\varphi$-regular structure on $X$ to a poset $Y$, then $f = f \circ \varphi$.*

**Theorem 4.3.** *If $\varphi$ is an operation on $X$, then the following assertions are equivalent:*

(1) *$\varphi$ is a closure operation;*
(2) *$\varphi$ is increasingly $\varphi$-regular;*
(3) *there exists an increasingly $\varphi$-regular structure $f$ on $X$.*

**Corollary 4.4.** *If $f$ is a structure and $\varphi$ is an operation on $X$, then $f$ is increasingly $\varphi$-regular if and only if $\varphi$ is a closure operation and for any $x_1, x_2 \in X$ we have $\varphi(x_1) \leq \varphi(x_2)$ if and only if $f(x_1) \leq f(x_2)$.*

**Theorem 4.5.** *If $f$ is an increasingly $g$-normal structure on $X$ to $Y$ and $\varphi$ is an operation on $X$ such that $\varphi \leq g \circ f \leq \varphi$, then $f$ is increasingly $\varphi$-regular.*

**Theorem 4.6.** *If $f$ is an increasingly $\varphi$-regular structure on $X$ onto $Y$ and $g$ is a structure on $Y$ to $X$ such $\varphi \leq g \circ f \leq \varphi$, then $f$ is increasingly $g$-normal.*

**Theorem 4.7.** *$f$ is an increasingly $g$-normal structure on $X$ to $Y$ if and only if $g$ is an increasingly $f$-normal structure on $Y^*$ to $X^*$.*

**Theorem 4.8.** *If $f$ is an increasingly $g$-normal structure on $X$ to $Y$, then $f$ and $g$ are increasing. Moreover, $\varphi = g \circ f$ is a closure operation on $X$ and $\psi = f \circ g$ is an interior operation on $Y$.*

## 5. Characterization of increasingly normal structures

**Definition 5.1.** For a structure $f$ on $X$ to $Y$, we define two relations $\Gamma_f$ and $g_f$ on $Y$ to $X$ such that

$$\Gamma_f(y) = \{x \in X : f(x) \leq y\}$$

and $g_f(y) = \max(\Gamma_f(y))$ for all $y \in Y$.

*Remark* 5.2. Note that $\Gamma_f(y) = f^{-1}[\mathrm{lb}(y)] = (f^{-1} \circ \mathrm{lb})(y)$ for all $y \in Y$.

Moreover, note that if in particular $X$ is a poset, then $g_f$ is already a function of a subset of $Y$ into $X$.

Concerning the relation $g_f$, in [13], the first author has, for instance, proved the following.

**Theorem 5.3.** *For any structures $f$ on $X$ to $Y$ and $g$ on $Y$ to $X$, the following assertions are equivalent:*

(1) *$f$ is increasingly $g$-normal;*
(2) *$f$ is increasing and $g \subset g_f$.*

*Remark* 5.4. Hence, by using that for any increasing structure $f$ on $X$ to $Y$ and $y \in Y$ we have $g_f(y) = \{x \in X : \operatorname{lb}(x) = \Gamma_f(y)\}$, we can easily get a further useful equivalent of (1).

**Definition 5.5.** For a structure $f$ on $X$ to $Y$, we define

$$\mathcal{Q}_f = \big\{ g \in X^Y : f \text{ is increasingly } g\text{-normal} \big\}.$$

Moreover, if in particular $\mathcal{Q}_f \neq \emptyset$, then we say that $f$ is increasingly normal.

Concerning increasingly normal structures, in [13], the first author has, for instance, proved the following theorems.

**Theorem 5.6.** *If $f$ is a structure on $X$ to $Y$, then the following assertions are equivalent:*

(1) *$f$ is increasingly normal;*
(2) *$f$ is increasing and $X = D_{g_f}$.*

**Theorem 5.7.** *If $f$ is an increasingly normal structure on $X$ to $Y$, then $g_f = \bigcup \mathcal{Q}_f$.*

**Theorem 5.8.** *If $f$ is an increasingly normal structure on a poset $X$ to $Y$, then $g_f$ is an increasing structure on $Y$ to $X$ and $\mathcal{Q}_f = \{g_f\}$.*

## 6. Characterization of increasingly regular structures

**Definition 6.1.** For a structure $f$ on $X$, we define two relations $\Lambda_f$ and $\varphi_f$ on $X$ such that

$$\Lambda_f(x) = \{u \in X : f(u) \leq f(x)\}$$

and $\varphi_f(x) = \max(\Lambda_f(x))$ for all $x \in X$.

*Remark* 6.2. Note that, in contrast to Remark 5.2, now we simply have $\Lambda_f = f^{-1} \circ \operatorname{lb} \circ f$.

Moreover, note also that if in particular $X$ is a poset, then $\varphi_f$ is already a function of a subset of $X$ into $X$.

Concerning the above relations, in [13], the first author has, for instance, proved the following theorems.

**Theorem 6.3.** *If $f$ is a structure on $X$, then $\Lambda_f$ is a preorder relation on $X$. Moreover, $\Lambda_f = \Gamma_f \circ f$ and $\varphi_f = g_f \circ f$.*

**Theorem 6.4.** *If $\varphi$ is an operation and $f$ is a structure on $X$, then the following assertions are equivalent:*

(1) *$f$ is increasingly $\varphi$-regular;*
(2) *$f$ is increasing and $\varphi \subset \varphi_f$.*

*Remark* 6.5. Hence, by using that for any increasing structure on $X$ and $x \in X$ we have $\varphi_f(x) = \{u \in X : \operatorname{lb}(u) = \Lambda_f(x)\}$, we can easily get a further useful equivalent of (1).

**Definition 6.6.** For a structure $f$ on $X$, we define

$$\mathcal{O}_f = \{\varphi \in X^X : f \text{ is increasingly } \varphi\text{-regular}\}.$$

Moreover, if in particular $\mathcal{O}_f \neq \varnothing$, then we say that $f$ is increasingly regular.

Concerning increasingly regular structures, in [13], the first author has, for instance, proved the following theorems.

**Theorem 6.7.** *If $f$ is an increasingly normal structure on $X$ to $Y$, then $f$ is, in particular, increasingly regular.*

**Theorem 6.8.** *If $f$ is a structure on $X$, then the following assertions are equivalent:*

(1) *$f$ is increasingly regular;*
(2) *$f$ is increasing and $X = D_{\varphi_f}$.*

**Theorem 6.9.** *If $f$ is an increasingly regular structure on $X$, then $\varphi_f = \bigcup \mathcal{O}_f$.*

**Theorem 6.10.** *If $f$ is an increasingly regular structure on $X$ onto $Y$, then $f$ is increasingly normal.*

**Theorem 6.11.** *If $f$ is an increasingly regular structure on a poset $X$, then $\varphi_f$ is a closure operation on $X$ and $\mathcal{O}_f = \{\varphi_f\}$.*

## 7. Comparison of closure operations

**Definition 7.1.** For any operation $\varphi$ on $X$, we set

$$A_\varphi = \{x \in X : \varphi(x) \leq x\}.$$

*Remark* 7.2. Note that if in particular $\varphi$ is expansive, then $x \leq \varphi(x)$ for all $x \in X$. Therefore, if $\varphi$ is an expansive operation on a poset $X$, then $A_\varphi$ is just the family of all fixed points of $\varphi$.

Moreover, note that the operation $\varphi$ is quasi-idempotent if and only if $\varphi(x) \in A_\varphi$ for all $x \in X$. Therefore, if in particular $\varphi$ is a semiclosure operation on a poset $X$, then $A_\varphi$ is just the range of $\varphi$.

Now, as a natural extension of [6, Theorem 2] and [2, Theorem 4.4, p. 30], we can also prove the following

**Theorem 7.3.** *If $\varphi$ is increasing and $\psi$ is a closure operation on $X$, then the following assertions are equivalent:*

(1) $\varphi \leq \psi$;
(2) $A_{\psi} \subset A_{\varphi}$;
(3) $\varphi \circ \psi \leq \psi$;
(4) $\psi \circ \varphi \leq \psi$.

PROOF. If $x \in A_{\psi}$, then $\psi(x) \leq x$. Moreover, if (1) holds, then we also have $\varphi(x) \leq \psi(x)$. Hence, by the transitivity of $X$, it follows that $\varphi(x) \leq x$. Therefore, $x \in A_{\varphi}$, and thus (2) also holds.

Moreover, if $x \in X$, then by the quasi-idempotency of $\psi$ we have $\psi(\psi(x)) \leq \psi(x)$, and thus $\psi(x) \in A_{\psi}$. Hence, if (2) holds, we can infer that $\psi(x) \in A_{\varphi}$. Therefore, $(\varphi \circ \psi)(x) = \varphi(\psi(x)) \leq \psi(x)$, and thus (3) also holds.

Furthermore, if $x \in X$, then by the expansivity of $\psi$ we have $x \leq \psi(x)$. Hence, by the increasingness of $\varphi$, it follows that $\varphi(x) \leq \varphi(\psi(x))$. Moreover, if (3) holds, then we also have $\varphi(\psi(x)) \leq \psi(x)$. Hence, by the transitivity of $X$, it follows that $\varphi(x) \leq \psi(x)$. Therefore, (1) also holds.

Now, it remains only to prove that (1) and (4) are also equivalent. For this, note that if $x \in X$ and (1) holds, then $\varphi(x) \leq \psi(x)$. Hence, by using the increasingness of $\psi$, we can infer that $(\psi \circ \varphi)(x) = \psi(\varphi(x)) \leq \psi(\psi(x)) = \psi^2(x)$. Moreover, by the quasi-idempotency of $\psi$, we also have $\psi^2(x) \leq \psi(x)$. Hence, by the transitivity of $X$, it follows that $(\psi \circ \varphi)(x) \leq \psi(x)$. Therefore, (4) also holds.

On the other hand, if $x \in X$, then by the expansivity of $\psi$, we also have $\varphi(x) \leq \psi(\varphi(x))$. Moreover, if (4) holds, then we also have $\psi(\varphi(x)) \leq \psi(x)$. Hence, by the transitivity of $X$, it follows that $\varphi(x) \leq \psi(x)$. Therefore, (1) also holds. $\square$

Now, as an immediate consequence of the above theorem, we can also state.

**Corollary 7.4.** *If $\varphi$ and $\psi$ are closure operations on a poset $X$ such that $A_{\varphi} = A_{\psi}$, then $\varphi = \psi$.*

Moreover, from Theorem 7.3, we can also easily get the following.

**Theorem 7.5.** *If $\varphi$ is a preclosure and $\psi$ is a closure operation on a poset $X$, then the following assertions are equivalent:*

(1) $\varphi \leq \psi$;
(2) $\psi = \varphi \circ \psi$;
(3) $\psi = \psi \circ \varphi$.

PROOF. If $x \in X$, then by the expansivity of $\varphi$, we have $\psi(x) \leq \varphi(\psi(x)) = (\varphi \circ \psi)(x)$. Moreover, if (1) holds, then by Theorem 7.3, we also have $(\varphi \circ \psi)(x) \leq \psi(x)$. Hence, by the antisymmetry of $X$, it follows that $\psi(x) = (\varphi \circ \psi)(x)$. Therefore, (2) also holds.

On the other hand, if $x \in X$, then by the expansivity of $\varphi$ we also have $x \leq \varphi(x)$. Hence, by the increasingness of $\psi$, it follows that $\psi(x) \leq \psi(\varphi(x)) = (\psi \circ \varphi)(x)$.

Moreover, if (1) holds, then by Theorem 7.3 we also have $(\psi \circ \varphi)(x) \leq \psi(x)$. Hence, by the antisymmetry of $X$, it follows that $\psi(x) = \varphi(\psi(x))$. Therefore, (3) also holds.

Now, since the converse implications (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (1) are immediate from Theorem 7.3, the proof is complete. □

Now, to include a further part of [5, Proposition 4], we can also prove the following.

**Theorem 7.6.** *If $\varphi$ and $\psi$ are closure operations on a poset $X$, then the following assertions are equivalent:*

(1) $\varphi \leq \psi$;
(2) $\varphi^{-1} \circ \varphi \subset \psi^{-1} \circ \psi$.

PROOF. If $(x, y) \in \varphi^{-1} \circ \varphi$, then $y \in (\varphi^{-1} \circ \varphi)(x) = \varphi^{-1}(\varphi(x))$, and thus $\varphi(y) = \varphi(x)$. Moreover, if (1) holds, then by Theorem 7.5, we have $\psi = \psi \circ \varphi$. Now, we can already see that $\psi(y) = (\psi \circ \varphi)(y) = \psi(\varphi(y)) = \psi(\varphi(x)) = (\psi \circ \varphi)(x) = \psi(x)$. Hence, it follows that $y \in \psi^{-1}(\psi(x)) = (\psi^{-1} \circ \psi)(x)$, and thus $(x, y) \in \psi^{-1} \circ \psi$. Therefore, (2) also holds.

On the other hand, if $x \in X$, then under the notation $y = \varphi(x)$ we have $(\psi \circ \varphi)(x) = \psi(\varphi(x)) = \psi(y)$. Moreover, by the idempotency of $\varphi$, we also have $\varphi(y) = \varphi(\varphi(x)) = \varphi^2(x) = \varphi(x)$. Hence, quite similarly the above, we can infer that $(x, y) \in \varphi^{-1} \circ \varphi$. Now, if (2) holds, then we can also state that $(x, y) \in \psi^{-1} \circ \psi$. Hence, quite similarly the above, we can infer that $\psi(y) = \psi(x)$. Now, by the equality $(\psi \circ \varphi)(x) = \psi(y)$, we can also state that $(\psi \circ \varphi)(x) = \psi(x)$, and thus $\psi \circ \varphi = \psi$. Therefore, by Theorem 7.5, (1) also holds. □

From the above results, by using Theorems 4.1 and 4.3, we can easily derive the following theorems.

**Theorem 7.7.** *If $\varphi$ is an increasingly $\psi$-regular operation on $X$, then the following assertions are equivalent:*

(1) $\varphi \leq \psi$;
(2) $A_\psi \subset A_\varphi$;
(3) $\varphi \circ \psi \leq \psi$;
(4) $\psi \circ \varphi \leq \psi$.

PROOF. Now, by Theorems 4.1 and 4.3, $\varphi$ is increasing and $\psi$ is a closure operation on $X$. Therefore, Theorem 7.3 can be applied. □

**Theorem 7.8.** *If $\varphi$ is a semiclosure and an increasingly $\psi$-regular operation on a poset $X$ such that $A_\varphi = A_\psi$, then $\varphi = \psi$.*

PROOF. Now, by Theorems 4.1 and 4.3, $\varphi$ and $\psi$ are closure operations on $X$. Therefore, Corollary 7.4 can be applied. □

**Theorem 7.9.** *If $\varphi$ is an expansive increasingly $\psi$-regular operation on a poset $X$ such that $\varphi \leq \psi$, then $\varphi = \psi$.*

PROOF. Now, by Theorems 4.1 and 4.3, $\varphi$ is a preclosure and $\psi$ is a closure operation on $X$. Therefore, by Theorem 7.5 and Corollary 4.2, $\psi = \varphi \circ \psi = \varphi$. $\square$

## 8. COMPARISON OF INCREASINGLY NORMAL STRUCTURES

By using the corresponding definitions, we can easily prove the following.

**Theorem 8.1.** *If $f_i$ is a structure on $X$ to $Y$ for each $i = 1, 2$, then the following assertions are equivalent:*

(1) $f_1 \leq f_2$;
(2) $\Gamma_{f_2} \subset \Gamma_{f_1}$.

PROOF. Suppose that (1) holds and $(y, x) \in \Gamma_{f_2}$. Then, $x \in X$, and thus $f_1(x) \leq f_2(x)$. Moreover, $x \in \Gamma_{f_2}(y)$, and thus $f_2(x) \leq y$. Hence, by the transitivity of $X$, it follows that $f_1(x) \leq y$, and thus $x \in \Gamma_{f_1}(y)$. Therefore, $(y, x) \in \Gamma_{f_1}$, and thus (2) also holds.

Suppose now that (2) holds and $x \in X$. Then, by the reflexivity of $Y$, we have $f_2(x) \leq f_2(x)$. Therefore, $x \in \Gamma_{f_2}(f_2(x))$, and thus $(f_2(x), x) \in \Gamma_{f_2}$. Hence, by using (2), we can infer that $(f_2(x), x) \in \Gamma_{f_1}$, and thus $x \in \Gamma_{f_1}(f_2(x))$. Therefore, $f_1(x) \leq f_2(x)$, and thus (1) also holds. $\square$

Moreover, as a straightforward extension of [3, Exercise 7.16, p. 172], we can prove the following.

**Theorem 8.2.** *If $f_i$ is an increasingly $g_i$-normal structure on $X$ to $Y$ for each $i = 1, 2$, then the following assertions are equivalent:*

(1) $f_1 \leq f_2$;
(2) $f_1 \circ g_2 \leq \Delta_Y$;
(3) $g_2 \leq g_1$.

PROOF. If (1) holds, then we also have $f_1 \circ g_2 \leq f_2 \circ g_2$. Moreover, since $f_2$ is increasingly $g_2$-normal, by Theorem 4.8 we also have $f_2 \circ g_2 \leq \Delta_Y$. Hence, by the transitivity of $Y$, it is clear that (2) also holds.

If (2) holds, then for any $y \in Y$ we have $f_1(g_2(y)) = (f_1 \circ g_2)(y) \leq \Delta_Y(y) = y$. Hence, by using that $f_1$ is increasingly $g_1$-normal, we can already infer that $g_2(y) \leq g_1(y)$. Therefore, (3) also holds.

Now, to prove the remaining implication (3) $\Longrightarrow$ (1), it is enough to note only that, by Theorem 4.7, $g_i$ is an increasingly $f_i$-normal structure on $Y^*$ to $X^*$ for each $i = 1, 2$. Therefore, by the corresponding definitions and the implication (1) $\Longrightarrow$ (3), we can state that $g_2 \leq g_1 \implies g_1 \leq^* g_2 \implies f_2 \leq^* f_1 \implies f_1 \leq f_2$. $\square$

Now, as an immediate consequence of the above theorem, we can also state the following reformulation of [5, Theorem 1].

**Corollary 8.3.** *If $f_i$ is an increasingly $g_i$-normal structure on poset $X$ to another $Y$ for each $i = 1, 2$, then $f_1 = f_2$ if and only if $g_1 = g_2$.*

Moreover, by using Theorem 8.2, we can also prove the following.

**Theorem 8.4.** *If $f_i$ is an increasingly normal structure on $X$ to $Y$ for each $i = 1, 2$, then the following assertions are equivalent:*

(1) $f_1 \leq f_2$;
(2) $g_{f_2}(y) \subset \operatorname{lb}(g_{f_1}(y))$ *for all* $y \in Y$.

PROOF. Suppose that (1) holds, and moreover $y \in Y$ and $x_i \in g_{f_i}(y)$ for each $i = 1, 2$. Then, by Theorem 5.7, for each $i = 1, 2$ there exists $g_i \in \mathcal{Q}_{f_i}$ such that $x_i = g_i(y)$. Moreover, by Theorem 8.2, we have $x_2 = g_2(y) \leq g_1(y) = x_1$. Hence, it is clear that $x_2 \in \operatorname{lb}(g_{f_1}(y))$, and thus (2) also holds.

To prove the converse implication, note that now, for each $i = 1, 2$, there exists $g_i \in \mathcal{Q}_{f_i}$. Moreover, by Theorem 5.3, we have $g_i(y) \in g_{f_i}(y)$ for all $y \in Y$. Hence, if (2) holds, we can infer that $g_2(y) \leq g_1(y)$ for all $y \in Y$. Now, by Theorem 8.2, it is clear that (1) also holds. □

## 9. COMPOSITIONS OF INCREASINGLY REGULAR STRUCTURES

**Theorem 9.1.** *If $f$ is an increasingly $\varphi$-regular structure on $X$, then*

$$\Lambda_f = \Lambda_f \circ \varphi \qquad \text{and} \qquad \varphi_f = \varphi_f \circ \varphi.$$

PROOF. If $x \in X$, then by the corresponding definitions, Theorem 4.1 and the transitivity of $R_f$, it is clear that

$$u \in (\Lambda_f \circ \varphi)(x) \iff u \in \Lambda_f(\varphi(x)) \iff$$
$$\iff f(u) \leq f(\varphi(x)) \iff f(u) \leq f(x) \iff u \in \Lambda_f(x).$$

Therefore, $(\Lambda_f \circ \varphi)(x) = \Lambda_f(x)$. Moreover, now we can also easily see that

$$(\varphi_f \circ \varphi)(x) = \varphi_f(\varphi(x)) =$$
$$= \max(\Lambda_f(\varphi(x))) = \max((\Lambda_f \circ \varphi)(x)) = \max(\Lambda_f(x)) = \varphi_f(x).$$

Therefore, the required equalities are also true. □

From the above theorem, by using Theorem 6.9, we can easily derive

**Corollary 9.2.** *If $f$ is an increasingly regular structure on $X$, then $\Lambda_f = \Lambda_f \circ \varphi_f$ and $\varphi_f = \varphi_f \circ \varphi_f$.*

PROOF. If $x \in X$, then by Theorem 6.9, we have

$$\varphi_f(x) = \{\varphi(x) : \varphi \in \mathcal{O}_f\}.$$

Moreover, if $F = \Lambda_f$ or $\varphi_f$, then by Theorem 9.1 we have $F \circ \varphi = F$ for all $\varphi \in \mathcal{O}_f$. Hence, it is clear that

$$(F \circ \varphi_f)(x) = F[\varphi_f(x)] = F[\{\varphi(x)\}_{\varphi \in \mathcal{O}_f}] = \\ = \bigcup_{\varphi \in \mathcal{O}_f} F(\varphi(x)) = \bigcup_{\varphi \in \mathcal{O}_f} (F \circ \varphi)(x) = \bigcup_{\varphi \in \mathcal{O}_f} F(x) = F(x).$$

Therefore, $F \circ \varphi_f = F$, and thus the required equalities are also true. □

**Theorem 9.3.** *If $f$ is an increasingly regular structure on $X$, then $\mathcal{O}_f$ is closed under composition.*

PROOF. Suppose that $\varphi, \psi \in \mathcal{O}_f$, and define $\chi = \psi \circ \varphi$. Then, by the corresponding definitions, Theorem 4.1 and the transitivity of $R_f$, it is clear that for any $x_1, x_2 \in X$ we have

$$x_1 \leq \chi(x_2) \iff x_1 \leq (\psi \circ \varphi)(x_2) \iff x_1 \leq \psi\big(\varphi(x_2)\big) \iff \\ \iff f(x_1) \leq f\big(\varphi(x_2)\big) \iff f(x_1) \leq f(x_2).$$

Therefore, $\chi \in \mathcal{O}_f$ also holds. □

*Remark* 9.4. Note that if in particular $f$ is an increasingly regular structure on a poset $X$, then by Theorem 6.11 we only have $\mathcal{O}_f = \{\varphi_f\}$.

As an extension of [2, Theorem 2.8, p. 14], we also have the following.

**Theorem 9.5.** *If $f$ is an increasingly $g$-normal structure on $X$ to $Y$ and $h$ is an increasingly $k$-normal structure on $Y$ to $Z$, then $h \circ f$ is a $g \circ k$-normal structure on $X$ to $Z$.*

PROOF. For any $x \in X$ and $z \in Z$, we have

$$(h \circ f)(x) \leq z \iff h(f(x)) \leq z \iff \\ \iff f(x) \leq k(z) \iff x \leq g(k(z)) \iff x \leq (g \circ k)(z).$$

Therefore, the required assertion is true. □

Now, as an immediate consequence of the above theorem, we can also state

**Corollary 9.6.** *If $f$ is a increasingly normal structure on $X$ to $Y$ and $h$ is an increasingly normal structure on $Y$ to $Z$, then $h \circ f$ is an increasingly normal structure on $X$ to $Z$.*

Hence, by Theorem 6.10, it is clear that in particular we also have

**Corollary 9.7.** *If $f$ is an increasingly regular structure on $X$ onto $Y$ and $h$ is an increasingly regular structure on $Y$ onto $Z$, then $h \circ f$ is an increasingly regular structure on $X$ onto $Z$.*

## 10. Some further results on closure operations

**Theorem 10.1.** *If $\varphi$ is an increasingly $\psi$-normal structure on $X$ to itself, then*

(1) $\Delta_X \leq \varphi \iff \psi \leq \Delta_X$;
(2) $\varphi^2 \leq \varphi \iff \psi \leq \psi^2$;
(3) $\varphi \leq \Delta_X \iff \Delta_X \leq \psi$;
(4) $\varphi \leq \varphi^2 \iff \psi^2 \leq \psi$.

PROOF. Clearly, $\Delta_X$ is an increasingly $\Delta_X$-normal structure on $X$ to itself. Hence, by Theorem 8.2, it is clear that (1) and (3) are true.

Moreover, from Theorem 9.5, we can see that $\varphi^2$ is an increasingly $\psi^2$-normal structure on $X$ to itself. Hence, by Theorem 8.2, it is clear that (2) and (4) are also true. □

Now, as an immediate consequence of the above theorem, we can also state

**Corollary 10.2.** *If $\varphi$ is an increasingly $\psi$-normal structure on a poset $X$ to itself, then*

(1) $\varphi = \Delta_X \iff \psi = \Delta_X$;
(2) $\varphi = \varphi^2 \iff \psi = \psi^2$.

Moreover, as a natural extension of the first part of [2, Theorem 2.10, p. 15], we can also prove

**Theorem 10.3.** *If $\varphi$ is an increasingly $\psi$-normal structure on $X$ to itself, then the following assertions are equivalent:*

(1) *$\varphi$ is a closure operation;*
(2) $\varphi \leq \psi \circ \varphi \leq \varphi$;
(3) *$\psi$ is an interior operation;*
(4) $\psi \leq \varphi \circ \psi \leq \psi$.

PROOF. From Theorem 4.8, we can see that both $\varphi$ and $\psi$ are increasing. Hence, by Theorem 10.1 and the corresponding definitions, it is clear that (1) and (3) are equivalent.

Moreover, from Theorem 4.8, we can see that $\psi \circ \varphi$ is expansive. Furthermore, if (1) holds, then $\varphi$ is quasi-idempotent. Hence, it is clear that

$$\varphi = \Delta_X \circ \varphi \leq (\psi \circ \varphi) \circ \varphi = \psi \circ \varphi^2 \leq \psi \circ \varphi.$$

On the other hand, from Theorem 4.5 we can see that $\varphi$ is increasingly $\psi \circ \varphi$-regular. Moreover, if (1) holds, then $\varphi$ is expansive. Hence, by Theorem 4.1, it is clear that

$$\psi \circ \varphi = \Delta_X \circ (\psi \circ \varphi) \leq \varphi \circ (\psi \circ \varphi) \leq \varphi.$$

Now, by the transitivity of $X$, it is clear that (2) also holds.

Conversely, if (2) holds, then from Theorem 4.5 we can see that $\varphi$ is increasingly $\varphi$-regular. Therefore, by Theorem 4.3, (1) also holds.

Now, to prove the equivalence of (1) and (4), it is enough to note only that, by Theorem 4.7, $\psi$ is an increasingly $\varphi$-normal structure on $X^*$ to itself. Therefore, by the equivalences (1) $\iff$ (3) and (1) $\iff$ (2), we can state that

$$\varphi \text{ is a closure on } X \iff \psi \text{ is an interior on } X \iff$$
$$\iff \psi \text{ is a closure on } X^* \iff \psi \leq^* \varphi \circ \psi \leq^* \psi \iff \psi \leq \varphi \circ \psi \leq \psi,$$

as required. □

Now, as an immediate consequence of Theorems 4.8 and 10.3, we can also state

**Corollary 10.4.** *If $\varphi$ is a semiclosure operation and an increasingly $\psi$-normal structure on a poset $X$ to itself, then $\varphi = \psi \circ \varphi$ and $\psi = \varphi \circ \psi$.*

Hence, it is clear in particular we also have

**Corollary 10.5.** *If $\varphi$ is a semiclosure operation and an increasingly $\psi$-normal structure on a poset $X$ to itself, then $R_\varphi = R_\psi$.*

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