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SOME COMMON FIXED POINT THEOREMS FOR A CLASS OF FUZZY CONTRACTIVE MAPPINGS

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Abstract. The purpose of this paper is to state and prove a new lemma generalizing Lemma 3.1 of Arora and Sharma [1] and Proposition 3.2 of Lee and Cho [10]. Some common fixed point theorems for a type of fuzzy contractive mappings are also established. These theorems extend and generalize several previous results [3, 14, 21, 22].

1991 *Mathematics Subject Classification:* 47H10, 54H25

Keywords: fuzzy sets, fuzzy map, fuzzy contractive mappings, common fixed points.

1. Introduction

Common fixed point theorems have been applied to diverse problems during the last few decades. These theorems provide techniques for solving a variety of applied problems in mathematical science and in dynamic programming (see, e. g., [4, 15, 16]). Extensions of the Banach contraction principle to multivalued mappings were initiated independently by Markin [11] and Nadler [13]. Therefore, results on fixed points of contractive type multivalued mappings have been carried out by many authors (see, for example, [2, 17, 21]).

The theory of fuzzy sets was investigated by Zadeh [24] in 1965. Some applications on results in this theory are discussed (see [9, 23]). In 1981, Heilpern [7] first introduced the concept of fuzzy contractive mappings and proved a fixed point theorem for these mappings in metric linear spaces. His result is a generalization of the fixed point theorem for point-to-set maps of Nadler [13]. Later, several fixed point theorems for types of fuzzy contractive mappings appeared (see, for instance, [1, 18–20]).

In this paper, we state and prove a new lemma generalizing Lemma 3.1 of Arora and Sharma [1] and Proposition 3.2 of Lee and Cho [10]. Two common fixed point theorems of a type of fuzzy contractive mappings are established. These theorems generalize and extend results in [3, 14, 21, 22]. Finally, we state a conclusion containing a brief of our results and future research.

2. Basic preliminaries

The definitions and terminologies for further discussions are taken from Heilpern [7]. Let (X,d) be a metric linear space. A *fuzzy set* in X is a function with domain X and values in $[0,1]$. If A is a fuzzy set and $x \in X$, then the function-value $A(x)$ is called the *grade of membership* of x in A. The collection of all fuzzy sets in X is denoted by $\mathfrak{F}(X)$.

Let $A \in \mathfrak{F}(X)$ and $\alpha \in [0,1]$. The *α-level set* of A, denoted by A_α, is defined by the formula

$$A_\alpha = \begin{cases} \{x : A(x) \geq \alpha\} & \text{if } \alpha \in (0,1], \\ \overline{\{x : A(x) > 0\}} & \text{if } \alpha = 0. \end{cases} \tag{2.1}$$

where $\overline{B}$ is the closure of a (nonfuzzy) set B.

Definition 1. A fuzzy set A in X is an *approximate quantity* if and only if its α-level set is a nonempty compact convex (nonfuzzy) subset of X for each $\alpha \in [0,1]$ and $\sup_{x\in X} A(x) = 1$.

The set of all approximate quantities, denoted $W(X)$, is a subcollection of $\mathfrak{F}(X)$.

Definition 2. Let $A, B \in W(X)$, $\alpha \in [0,1]$ and $CP(X)$ be the set of all nonempty compact subsets of X. Then one puts $p_\alpha(A,B) = \inf_{x\in A_\alpha, y\in B_\alpha} d(x,y)$, $\delta_\alpha(A,B) = \sup_{x\in A_\alpha, y\in B_\alpha} d(x,y)$, and $D_\alpha(A,B) = H(A_\alpha, B_\alpha)$, where H is the *Hausdorff metric* between two sets in the collection $CP(X)$.

We define the functions $p(A,B) = \sup_\alpha p_\alpha(A,B)$, $\delta(A,B) = \sup_\alpha \delta_\alpha(A,B)$, and $D(A,B) = \sup_\alpha D_\alpha(A,B)$.

Note that p_α is nondecreasing function of α.

Definition 3. Let $A, B \in W(X)$. Then A is said to be *more accurate* than B (or B includes A), denoted by $A \subset B$, if and only if $A(x) \leq B(x)$ for each $x \in X$.

The relation $\subset$ induces a partial ordering on $W(X)$.

Definition 4. Let X be an arbitrary set and Y be a metric linear space. F is said to be a *fuzzy mapping* if and only if F is a mapping from the set X into $W(Y)$, i. e., $F(x) \in W(Y)$ for each $x \in X$.

The following lemma and proposition are used in the sequel.

Lemma 1 ([12])**.** *Suppose that $\gamma : [0,\infty) \to [0,\infty)$ is a right continuous function such that $\gamma(t) < t$ for all $t > 0$. Then for every $t > 0$, $\lim_{n\to\infty} \gamma^n(t) = 0$, where γ^n is the nth iterate of γ, $n \in \mathbb{N} \cup \{0\}$.**

Proposition 1 ([13])**.** *If $A, B \in CP(X)$ and $a \in A$, then there exists $b \in B$ such that $d(a,b) \leq H(A,B)$.*

*$\mathbb{N}$ is the set of all positive integers

We consider the set Φ of all functions $\phi : [0,\infty)^5 \to [0,\infty)$ with the following properties:

(i) ϕ is nondecreasing with respect to each variable,
(ii) ϕ is right continuous with respect to each variable,
(iii) for each $t > 0$, $\Psi(t) = \max\{\phi(t,t,t,t,t), \phi(t,t,t,2t,0), \phi(t,t,t,0,2t)\} < t$.

3. Main results

Throughout this paper, let (X,d) be a metric space. We consider a subcollection of $\mathfrak{F}(X)$ denoted by $W^*(X)$. Each fuzzy set $A \in W^*(x)$, its α-level set is a nonempty compact (nonfuzzy) subset of X for each $\alpha \in [0,1]$. It is obvious that each element $A \in W(X)$ leads one to $A \in W^*(X)$ but the converse is not true. Now, we introduce the improvements of the lemmas in Heilpern [7] as follows.

Lemma 2. *If $\{x_0\} \subset A$ for each $A \in W^*(X)$ and $x_0 \in X$, then $p_\alpha(x_0,B) \le D_\alpha(A,B)$ for each $B \in W^*(X)$.*

Lemma 3. *$p_\alpha(x,A) \le d(x,y) + p_\alpha(y,A)$ for all $x,y \in X$ and $A \in W^*(X)$.*

Lemma 4. *Let $x \in X, A \in W^*(X)$ and $\{x\}$ be a fuzzy set with membership function equal to a characteristic function of the set $\{x\}$. Then $\{x\} \subset A$ if and only if $p_\alpha(x,A) = 0$ for each $\alpha \in [0,1]$.*

Proof. If $\{x\} \subset A$, then $x \in A_\alpha$ for each $\alpha \in [0,1]$. This implies that $p_\alpha(x,A) = \inf_{y \in A_\alpha} d(x,y) = 0$ for any $\alpha \in [0,1]$. Conversely, if $p_\alpha(x,A) = 0$, then we have $\inf_{y \in A_\alpha} d(x,y) = 0$. It follows that $x \in \overline{A_\alpha} = A_\alpha$ for an arbitrary $\alpha \in [0,1]$. Then $\{x\} \subset A$. □

Also, we state and prove a new lemma in the following way.

Lemma 5. *Let (X,d) be a complete metric space, $F : X \to W^*(X)$ be a fuzzy map and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.*

Proof. For $n \in \mathbb{N}$, $((F(x_0))_{n/(n+1)})$ is a decreasing sequence of nonempty compact subsets of X. Thus we have from Proposition 11.4 and Remark 11.5 of [25, pp. 495–496] that $\bigcap_{n=1}^{\infty}(F(x_0))_{n/(n+1)}$ is nonempty and compact.

Let $x_1 \in \bigcap_{n=1}^{\infty}(F(x_0))_{n/(n+1)}$. Then $\frac{n}{n+1} \le (F(x_0))(x_1) \le 1$. As $n \to \infty$, we get that $(F(x_0))(x_1) = 1$. This implies that $\{x_1\} \subset F(x_0)$. □

Remark 1. It is clear that Lemma 5 is a generalization of Lemma 3.1 of Arora and Sharma [1] and Proposition 3.2 of Lee and Cho [10].

Now, we are ready to prove our main theorems.

Theorem 1. *Let (X,d) be a complete metric space and F_1, F_2 be fuzzy mappings from X into $W^*(X)$. If there is a $\phi \in \Phi$ such that for all $x,y \in X$,*

$$D(F_1(x),F_2(y)) \leq \phi(d(x,y),p(x,F_1(x)), p(y,F_2(y)),p(x,F_2(y)),p(y,F_1(x))), \quad (3.1)$$

then there exists $z \in X$ such that $\{z\} \subset F_1(z)$ and $\{z\} \subset F_2(z)$.

Proof. Let $x_0 \in X$. Then by Lemma 5, there exists $x_1 \in X$ such that $\{x_1\} \subset F_1(x_0)$. For $x_1 \in X$, the set $(F_2(x_1))_1$ is nonempty compact subset of X. Since $(F_1(x_0))_1$ and $(F_2(x_1))_1$ belong to $CP(X)$ and $x_1 \in (F_1(x_0))_1$, Proposition 1 asserts that there exists $x_2 \in (F_2(x_1))_1$ such that $d(x_1,x_2) \leq D_1(F_1(x_0),F_2(x_1))$. So, we have from Lemma 4 and the property (i) of ϕ that

$$\begin{aligned} d(x_1,x_2) &\leq D_1(F_1(x_0),F_2(x_1)) \leq D(F_1(x_0),F_2(x_1)) \\ &\leq \phi(d(x_0,x_1), \\ &\qquad p(x_0,F_1(x_0)),p(x_1,F_2(x_1)),p(x_0,F_2(x_1)),p(x_1,F_1(x_0))) \\ &\leq \phi(d(x_0,x_1),d(x_0,x_1),d(x_1,x_2),d(x_0,x_1)+d(x_1,x_2),0). \end{aligned}$$

If $d(x_1,x_2) > d(x_0,x_1)$, then

$$d(x_1,x_2) \leq \phi(d(x_1,x_2),d(x_1,x_2),d(x_1,x_2),2d(x_1,x_2),0) < d(x_1,x_2).$$

This contradiction demands that

$$d(x_1,x_2) \leq \phi(d(x_0,x_1),d(x_0,x_1),d(x_0,x_1),2d(x_0,x_1),0).$$

Similarly, one can deduce that

$$d(x_2,x_3) \leq \phi(d(x_1,x_2),d(x_1,x_2),d(x_1,x_2),0,2d(x_1,x_2)).$$

By induction, we have a sequence (x_n) of points in X such that, for all $n \in \mathbb{N} \cup \{0\}$,

$$\{x_{2n+1}\} \subset F_1(x_{2n}), \quad \{x_{2n+2}\} \subset F_2(x_{2n+1}).$$

It follows by induction that $d(x_n,x_{n+1}) \leq \Psi^n(d(x_0,x_1))$, where Ψ is defined in the property (iii) of ϕ. Then, Lemma 1 gives that $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$. Since

$$d(x_n,x_m) \leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + ... + d(x_{m-1},x_m),$$

then $\lim_{n,m\to\infty} d(x_n,x_m) = 0$. Therefore, (x_n) is a Cauchy sequence. Since X is a complete metric space, then there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Next, we show that $\{z\} \subset F_i(z), i = 1,2$. Now, we get from Lemma 2 and Lemma 3 that

$$\begin{aligned} p_\alpha(z,F_2(z)) &\leq d(z,x_{2n+1}) + p_\alpha(x_{2n+1},F_2(z)) \\ &\leq d(z,x_{2n+1}) + D_\alpha(F_1(x_{2n}),F_2(z)), \end{aligned}$$

for each $\alpha \in [0,1]$. Taking the supremum on α in the last inequality, we obtain from the property (i) of ϕ that

$$\begin{aligned} p(z, F_2(z)) &\leq d(z, x_{2n+1}) + D(F_1(x_{2n}), F_2(z)) \\ &\leq d(z, x_{2n+1}) + \phi(d(x_{2n}, z), p(x_{2n}, F_1(x_{2n})), p(z, F_2(z)), \\ &\qquad p(x_{2n}, F_2(z)), p(z, F_1(x_{2n}))) \\ &\leq d(z, x_{2n+1}) + \phi(d(x_{2n}, z), d(x_{2n}, x_{2n+1}), p(z, F_2(z)), \\ &\qquad p(x_{2n}, F_2(z)), d(z, x_{2n+1})). \end{aligned}$$

As $n \to \infty$, we have from the properties (i), (ii) and (iii) of ϕ with $p(z, F_2(z)) \neq 0$ that

$$\begin{aligned} p(z, F_2(z)) &\leq \phi(0, 0, p(z, F_2(z)), p(z, F_2(z)), 0) \\ &\leq \phi(p(z, F_2(z)), p(z, F_2(z)), p(z, F_2(z)), p(z, F_2(z)), p(z, F_2(z))) \\ &< p(z, F_2(z)). \end{aligned}$$

This contradiction yields $p(z, F_2(z)) = 0$. We then get from Lemma 4 that $\{z\} \subset F_2(z)$. Similarly, one can show that $\{z\} \subset F_1(z)$. □

Example 1. Let $X = [0,1]$ endowed with the metric d defined by $d(x,y) = |x-y|$. It is clear that (X,d) is a complete metric space. Assume that $\phi(t_1,t_2,t_3,t_4,t_5) = \frac{3}{4}t_1$ for arbitrary $t_i \in [0,\infty)$, $i = \overline{1,5}$. It is obvious that $\Psi(t) < t$ for all $t > 0$. Let $F_1 = F_2 = F$. Define a fuzzy mapping F on X such that for all $x \in X$, $F(x)$ is the characteristic function for $\{\frac{3}{4}x\}$. For each $x, y \in X$,

$$\begin{aligned} D(F(x), F(y)) &= \frac{3}{4} d(x,y) \\ &= \phi(d(x,y), p(x, F(x)), p(y, F(y)), p(x, F(y)), p(y, F(x))). \end{aligned} \tag{3.2}$$

The characteristic function for $\{0\}$ is the fixed point of F.

As corollaries of Theorem 1, we get the following statements.

Corollary 1. *Let (X,d) be a complete metric space and F_1, F_2 be fuzzy mappings from X into $W^*(X)$ satisfying the following conditions: for any x, y in X,*

$$\begin{aligned} D(F_1(x), F_2(y)) \leq a_1 p(x, F_1(x)) + a_2 p(y, F_2(y)) + a_3 p(y, F_1(x)) \\ + a_4 p(y, F_1(x)) + a_5 d(x,y), \end{aligned} \tag{3.3}$$

where a_1, a_2, a_3, a_4, and a_5 are non-negative real numbers, $\sum_{j=1}^{5} a_i < 1$ and $a_1 = a_2$ or $a_3 = a_5$. Then there exists $z \in X$ such that $\{z\} \subset F_1(z)$ and $\{z\} \subset F_2(z)$.

Proof. We consider the function $\phi : [0,\infty)^5 \to [0,\infty)$ defined by the formula

$$\phi(x_1, x_2, x_3, x_4, x_5) = a_1 x_2 + a_2 x_3 + a_3 x_5 + a_4 x_4 + a_5 x_1, \tag{3.4}$$

where $\sum_{i=1}^{i=5} a_i < 1$ such that $a_1 = a_2$ or $a_3 = a_4$. Since $\phi \in \Phi$, we have from Theorem 1 that there exists $z \in X$ such that $\{z\} \subset F_1(z)$ and $\{z\} \subset F_2(z)$. □

The following corollary is a fuzzy version of the fixed point theorem of Singh and Whitfield [21] for multivalued mappings.

Corollary 2. *Let (X,d) be a complete metric space and F_1, F_2 be fuzzy mappings from X into $W^*(X)$. If there is a constant α, $0 \le \alpha < 1$, such that, for each $x, y \in X$,*

$$D(F_1(x), F_2(y)) \le \alpha \max\Big\{ d(x,y), \frac{1}{2}[p(x,F_1(x)) + p(y,F_2(y))], \frac{1}{2}[p(x,F_2(y)) + p(y,F_1(x))]\Big\}, \quad (3.5)$$

then there exists $z \in X$ such that $\{z\} \subset F_1(z)$ and $\{z\} \subset F_2(z)$.

Proof. We consider the function $\phi : [0,\infty)^5 \to [0,\infty)$ defined by

$$\phi(x_1,x_2,x_3,x_4,x_5) = \alpha \max\left\{ x_1, \frac{1}{2}[x_2+x_3], \frac{1}{2}[x_4+x_5]\right\}. \quad (3.6)$$

Since $\phi \in \Phi$, we get from Theorem 1 that there exists $z \in X$ such that $\{z\} \subset F_1(z)$ and $\{z\} \subset F_2(z)$. $\square$

Remark 2. (1) If there is a $\phi \in \Phi$ such that, for each $x, y \in X$,

$$\delta(F_1(x), F_2(y)) \le \phi(d(x,y), p(x,F_1(x)), p(y,F_2(y)), p(x,F_2(y)), p(y,F_1(x))), \quad (3.7)$$

then the conclusion of Theorem 1 remains valid. This result is considered as a special case of Theorem 1 because $D(F_1(x),F_2(y)) \le \delta(F_1(x),F_2(y))$ [8, p. 414]. Moreover, this result generalizes Theorem 3.3 of Park and Jeong [14].

(2) Corollary 1 is [22, Theorem 3.1] without condition (a), where condition (a) reads as follows: "for each $x \in X$, there exists $\alpha(x) \in (0,1]$ such that $(F_1(x))_{\alpha(x)}$ and $(F_2(x))_{\alpha(x)}$ are nonempty closed bounded subsets of $\mathfrak{F}(X)$." Also, Corollary 1 generalizes [3, Theorem 3.1].

(3) Theorems 3.1 and 3.4 of Park and Jeong [14] are special cases of Theorem 1.

The following theorem generalizes Theorem 1 to a sequence of fuzzy contractive mappings.

Theorem 2. *Let $(F_n : n \in \mathbb{N} \cup \{0\})$ be a sequence of fuzzy mappings from a complete metric space (X,d) into $W^*(X)$. If there is a $\phi \in \Phi$ such that, for all $x, y \in X$,*

$$D(F_0(x), F_n(y)) \le \phi(d(x,y), p(x,F_0(x)), p(y,F_n(y)), p(x,F_n(y)), p(y,F_0(x))) \qquad \forall (n \in \mathbb{N}), \quad (3.8)$$

then there exists a common fixed point of the family $(F_n : n \in \mathbb{N} \cup \{0\})$.

Proof. Putting $F_1 = F_0$ and $F_2 = F_n$ for all $n \in \mathbb{N}$ in Theorem 1. Then there exists a common fixed point of the family $(F_n : n \in \mathbb{N} \cup \{0\})$. $\square$

Remark 3. If there is a $\phi \in \Phi$ such that, for all $x, y \in X$,

$$\delta(F_0(x), F_n(y)) \leq \phi(d(x,y), p(x, F_0(x)), p(y, F_n(y)), p(x, F_n(y)), p(y, F_0(x))) \qquad (\forall n \in \mathbb{N}), \quad (3.9)$$

then the conclusion of Theorem 2 remains valid. This result is considered as a special case of Theorem 2 for the same reason as in Remark 2 (1).

4. Conclusion

This paper presents an improvement of some results in [1, 7, 10]. Also, it presents two common fixed point theorems for a type of fuzzy contractive mappings. These theorems generalize and extend results in [3, 14, 22] and [21], respectively. A fixed point theorem for fuzzy contractive mappings is stated generalizing [1, Theorem 3.5]. Many applications of our main theorems are possible, e. g., for differential and integral equations. In view of the references [5,6], some future research can be done, for example:

(1) I believe that our results can be hold for $FC(X)$, where $FC(X) = \{A \in \mathfrak{F}(X) :$ A_α is a nonempty closed (nonfuzzy) subset of X for each $\alpha \in [0,1]\}$,

(2) it is also possible to generalize our results to quasi-metric spaces.

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ASYMPTOTIC EQUIVALENCE OF A QUASILINEAR IMPULSIVE DIFFERENTIAL EQUATION AND A LINEAR ORDINARY DIFFERENTIAL EQUATION

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Abstract. The main goal of the paper is to obtain new sufficient conditions for the asymptotic equivalence of the quasilinear impulsive system of differential equations and the linear system of ordinary differential equations.

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Keywords: asymptotic equivalence, quasilinear systems, impulsive differential equations

1. Introduction and preliminaries

As it is well known, one of the most efficient ways to clarify the structure of a complex set is to arrange an equivalence with another set which is then carefully analysed. Apparently, the first results concerning the asymptotic behaviour of systems on the basis of one-to-one correspondence between sets of solutions were obtained in [4, 5, 8, 9].

In this paper, we obtain new sufficient conditions for the asymptotic equivalence of linear and quasilinear systems of differential equations with piecewise constant arguments. Homeomorphism, which is the most important version of the correspondence in a theoretical sense, is considered. The method of the paper was also used in the works [1, 2].

Let $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{R}$ be, respectively, the sets of all integer, natural, and real numbers. Denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^n$, where $n \in \mathbb{N}$. Fix a sequence θ_i, $i \in \mathbb{Z}$, such that $\theta_i < \theta_{i+1}$ for all $i \in \mathbb{Z}$ and $|\theta_i| \to \infty$ as $|i| \to \infty$.

We consider the system of impulsive differential equations

$$\begin{aligned} z'(t) &= Cz(t) + f(t, z(t)), \\ \Delta z|_{t=\theta_i} &= J_i(z), \end{aligned} \tag{1}$$

and the linear system of ordinary differential equations

$$x'(t) = Cx(t), \tag{2}$$

where $x, z \in \mathbb{R}^n$, $t \in \mathbb{R}$.

The following assumptions will be needed throughout the paper:

(c1) *C is a constant $n \times n$ real valued matrix;*

(c2) *$f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$;*

(c3) *The estimate*

$$\|f(t,z_1) - f(t,z_2)\| + \|J_i(z_1) - J_i(z_2)\| \le \eta(t)\|z_1 - z_2\|$$

holds for all $z_1, z_2 \in \mathbb{R}^n$, $t \in \mathbb{R}_+$, and $i \in \mathbb{Z}$ with some nonnegative function $\eta(t)$ defined on $\mathbb{R}_+$;

(c4) *The following equalities are true*

$$\begin{aligned} f(t,0) &= 0 \quad \text{for all } t \in \mathbb{R}; \\ J_i(0) &= 0 \quad \text{for all } i \in \mathbb{Z}. \end{aligned} \tag{3}$$

To investigate the asymptotic properties of the solutions, the following definition can be efficiently applied.

Definition ([3, 6])**.** *A homeomorphism H between the sets of solutions $x(t)$ and $z(t)$ is called an asymptotic equivalence if $z(t) = H(x(t))$ implies that $x(t) - z(t) \to 0$ as $t \to \infty$.*

2. Main result

In this section we consider the theorem about the asymptotic equivalence of systems (1) and (2). The theorem is a development of V. Yakubovich's result [6, 9] for impulsive differential equations. Let $\alpha = \min_j \operatorname{Re}\lambda_j$ and $\beta = \max_j \operatorname{Re}\lambda_j$, where $\operatorname{Re}\lambda_j$ denotes the real part of the eigenvalue λ_j of the matrix C. Let m_α and m_β be the maximum of the degrees of elementary divisors of C corresponding to eigenvalues with the real parts equal to α and β, respectively. Clearly, there exist constants κ_1 and κ_2 such that

$$\|e^{Ct}\| \le \kappa_1 t^{m_\beta - 1} e^{\beta t} \text{ and } \|e^{-Ct}\| \le \kappa_2 t^{m_\alpha - 1} e^{-\alpha t}$$

for all $t \in \mathbb{R}_+ = [0, \infty)$. We shall also assume that

(c5) *The relation*

$$l_0 := \int_0^\infty t^{m_\beta + m_\alpha - 2} e^{(\beta - \alpha)t} \eta(t)\, dt + \sum_{0 \le \theta_i < \infty} \theta_i^{m_\beta + m_\alpha - 2} e^{(\beta - \alpha)\theta_i} \eta(\theta_i) < +\infty$$

is satisified.

The following lemma can be easily established by direct substitution.

Lemma 1. *If $z(t)$ is a solution of (1), then there is a solution $u(t)$ of the equation*

$$\begin{aligned} u' &= e^{-Ct} f(t, e^{Ct} u), \\ \Delta u|_{t=\theta_i} &= e^{-C\theta_i} J_i(e^{C\theta_i} u), \end{aligned} \tag{4}$$

such that

$$z(t) = e^{Ct}u(t). \tag{5}$$

Conversely, if $u(t)$ is a solution of (4), then $y(t)$ in (5) is a solution of (2).

Lemma 2. *If conditions (c1)–(c5) are valid, then every solution of (4) is bounded on $\mathbb{R}_+$ and for each solution u of (4) there exists a constant vector $c_u \in \mathbb{R}^n$ such that $u(t) \to c_u$ as $t \to \infty$.*

Proof. Let $u(t) = u(t, t_0, u_0)$ denote a solution of (4) satisfying $u(t_0) = u_0$, $t_0 \geq 0$. It is known [7] that

$$u(t) = u_0 + \int_{t_0}^{t} e^{-Cs} f(s, e^{Cs}u(s))\,ds + \sum_{t_0 \leq \theta_i < t} e^{-C\theta_i} J_i(e^{C\theta_i}u)$$

for all $t \geq t_0$. Using (c4), we obtain

$$\begin{aligned}\|u(t)\| \leq \|u_0\| + k_1 \int_{t_0}^{t} s^{m_\beta + m_\alpha - 2} e^{(\beta s - \alpha)s} \eta(s)\|u(s)\|\,ds \\ + k_2 \sum_{t_0 \leq \theta_i < t} \theta_i^{m_\beta + m_\alpha - 2} e^{(\beta - \alpha)\theta_i} \eta(\theta_i)\|u(\theta_i)\|, \quad t \geq t_0,\end{aligned}$$

with some positive k_1 and k_2. By applying the Gronwall–Bellman Lemma for discontinuous functions [7] we obtain that $|u(t)| \leq M$ for all $t \in \mathbb{R}_+$, with some positive M. To prove the second part of the theorem, we first note that

$$\begin{aligned}\left| \int_{t_0}^{t} e^{-Cs} f(s, e^{Cs}u(s))\,ds + \sum_{t_0 \leq \theta_i < t} e^{-C\theta_i} J_i(e^{C\theta_i}u(\theta_i)) \right| \\ \leq Mk_1 \int_0^\infty t^{m_\beta + m_\alpha - 2} e^{(\beta - \alpha)t} \eta(t)dt \\ + Mk_2 \sum_{0 \leq \theta_i < \infty} \theta_i^{m_\beta + m_\alpha - 2} e^{(\beta - \alpha)\theta_i} \eta(\theta_i) < \infty.\end{aligned}$$

So we may define

$$c_u = u_0 + \int_{t_0}^{\infty} e^{-Cs} f(s, e^{Cs}u(s))\,ds + \sum_{t_0 \leq \theta_i < \infty} e^{-C\theta_i} J_i(e^{C\theta_i}u(\theta_i)).$$

It follows that

$$u(t) = c_u - \int_t^\infty e^{-Cs} f(s, e^{Cs}u(s))ds - \sum_{t \leq \theta_i < \infty} e^{-C\theta_i} J_i(e^{C\theta_i}u(\theta_i)),$$

which completes the proof. □

Theorem 1. *If conditions* (c1)–(c5) *are satisfied, then every solution* $y(t)$ *of* (1) *possesses an asymptotic representation of the form*

$$z(t) = e^{Ct}[c + o(1)],$$

where $c \in \mathbb{R}^n$ *is a constant vector and for a solution* $u(t)$ *of* (4),

$$o(1) = -\int_t^{\infty} e^{-Cs} f(s, e^{Cs}u(s))\,ds - \sum_{t \le \theta_i < \infty} e^{-C\theta_i} J_i(e^{C\theta_i}u(\theta_i)).$$

The proof follows from Lemma 1 and Lemma 2.

Theorem 2. *Assume that conditions* (c1)–(c5) *are fulfilled, and, moreover,*

$$\lim_{t\to\infty}\Bigg(\int_t^{\infty}(s-t)^{m_\alpha - 1}s^{m_\beta - 1}e^{\alpha(t-s)}e^{\beta s}\eta(s)ds + \sum_{t\le\theta_i<\infty}(\theta_i - t)^{m_\alpha-1}\theta_i^{m_\beta-1}e^{\alpha(t-\theta_i)}e^{\beta\theta_i}\eta(\theta_i)\Bigg) = 0. \quad (6)$$

Then systems (1) *and* (2) *are asymptotically equivalent.*

Proof. In view of Lemma 2, we see that

$$\begin{aligned} z(t) &= e^{Ct}\Bigg(c_u - \int_t^{\infty} e^{-Cs} f(s, e^{Cs}u(s))\,ds - \sum_{t\le\theta_i<\infty} e^{-C\theta_i}J_i(e^{C\theta_i}u(\theta_i))\Bigg) \\ &= x(t) - \int_t^{\infty} e^{-C(s-t)} f(s, e^{Cs}u(s))\,ds - \sum_{t\le\theta_i<\infty} e^{-C(\theta_i - t)}J_i(e^{C\theta_i}u(\theta_i)), \end{aligned}$$

where $x(t) = e^{Ct}c_u$ is a solution of (2) and $u(t) = u(t, t_0, u_0)$ is a solution of (4). It is clear that a given u_0 results in a homeomorphism between $x(t)$ and $y(t)$. Thus, we find that there exists a bicontinuous and one-to-one correspondence between u_0 and c_u. In view of (6), we also see that $x(t) - z(t) \to 0$ as $t \to \infty$, which completes the proof of the theorem. □

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FINE LIMITS OF GENERALIZED POTENTIAL-TYPE INTEGRAL OPERATORS WITH NON-ISOTROPIC KERNEL

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Abstract. This paper deals with the fine limits of generalized potential-type operators with non-isotropic kernels defined for functions on $\mathbb{R}^n$ satisfying appropriate conditions.

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1. INTRODUCTION

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be positive numbers with $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ and $\|x\|_\lambda = (|x_1|^{\frac{1}{\lambda_1}} + \ldots + |x_n|^{\frac{1}{\lambda_n}})^{\frac{|\lambda|}{n}}$, $x \in \mathbb{R}^n$. The expression $\|x - y\|_\lambda$, where $x,\ y \in \mathbb{R}^n$, is called the λ-distance or non-isotropic distance between x and y. This distance is an important concept in the theory of partial differential equations and imbedding theorems. Some problems with the λ-distance were examined in [6, 7].

It can be seen that λ-distance becomes the ordinary Euclidean distance $|x - y|$ for $\lambda_j = \frac{1}{2}$, $j = 1, 2, \dots, n$. The λ-distance has the following properties.

Using the inequality $(a + b)^m \le 2^m (a^m + b^m)$, $m > 1$, we obtain

$$\|x - y\|_\lambda \le M_\lambda (\|x\|_\lambda + \|y\|_\lambda), \tag{1.1}$$

where $M_\lambda = 2^{\left(1+\frac{1}{\lambda_{\min}}\right)\frac{|\lambda|}{n}}$ and $\lambda_{\min} = \min(\lambda_1, \lambda_2, \ldots \lambda_n)$.

Several authors have investigated the properties of classical Riesz potentials and their generalizations. For example, taking some appropriate conditions on the kernel depending on Euclidean distance type of $K(|x - y|)$, Gadjiev [3] proved a variant of the Hardy–Littlewood–Sobolev theorem. He also gave the properties of convergence almost everywhere. In [1], a theorem similar to results of [3] was proved for potential-type integrals with kernel depending on the λ-distance.

Some results on potential-type integral operators and Riesz potentials given by generalized shift operators can be found in [2, 4, 5]. Various generalizations of the Riesz potentials are given in [10].

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A potential-type integral operator depending on the λ-distance and defined for non-negative measurable functions f on $\mathbb{R}^n$ is given by the equality

$$(Lf)(x) = \int_{\mathbb{R}^n} K(\|x-y\|_\lambda) f(y)dy,$$

where K is the kernel function satisfying the following conditions (see [1]):

(K_1) *K is a non-negative continuous and decreasing function on semiaxis* $[0,\infty)$ *and* $\lim_{t\to 0} K(t) = \infty$*;*

(K_2) $L(r) = \int_r^a K\left(2\beta M_\lambda t^{\frac{2|\lambda|}{n}}\right) t^{2|\lambda|-\delta-1} dt < \infty$ *for* $0 < \delta < 2|\lambda|$, $\beta \in (0,1)$ *and* $0 \leq r < a$.

We know that $(Lf)(x) \neq \infty$ if and only if

$$\int_{\mathbb{R}^n} K(\beta(1+\|y\|_\lambda)) f(y)dy, \tag{1.2}$$

where $\beta \in (0,1)$. Hence it is seen that $(Lf)(x) \neq \infty$ when f is integrable on $\mathbb{R}^n$. Note that (1) is equivalent to

$$\int_{\mathbb{R}^n - B_\lambda(x,1)} K(\beta\|x-y\|_\lambda) f(y)dy$$

for every $x \in \mathbb{R}^n$, and $\beta \in (0,1)$, where $B_\lambda(x,1)$ is λ-ball centered at x with radius 1. That is $B_\lambda(x,1) = \{y \in \mathbb{R}^n : \|x-y\|_\lambda < 1\}$.

In what follows, we investigate the fine limits of generalized potential-type integral operators with non-isotropic kernels Lf at $x_0 \in \mathbb{R}^n$. Our results are generalizations of the corresponding results for classical Riesz potentials given in [9, 11].

To obtain a general result, we assume the condition

$$\int_{\mathbb{R}^n} \phi_p(f(y))\, w\left(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right) dy < \infty. \tag{1.3}$$

where $x_0 \in \mathbb{R}^n$ and $\phi_p(r)$ is positive monotone function on interval $(0,\infty)$ having the following properties:

(φ_1) *$\phi_p(r)$ is of the form $r^p\varphi(r)$, where $1 \leq p < \infty$ and φ is a positive non-decreasing function on interval* $(0,\infty)$.

(φ_2) *There exists A_1 such that $\varphi(2r) \leq A_1\varphi(r)$ whenever $r > 0$.*

Throughout this paper, let $w(r)$ be a positive non-increasing function on $(0,\infty)$ satisfying the condition:

(w_1) *There exists $A_2 > 0$ such that $A_2^{-1}w(r) \leq w(2r) \leq A_2 w(r)$ whenever $r > 0$.*

In this paper we will use some ideas from [9, 11]. By the symbol M, we denote a positive constant whose value may change depending on the context.

2. Preliminary lemmas

First we collect properties which follow from conditions (φ_1) and (φ_2).

Lemma 2.1. *The function φ satisfies the doubling condition, that is, there exists $A_3 > 1$ such that*

$$\varphi(r) \leq \varphi(2r) \leq A_3\varphi(r) \quad \text{for } r > 0.$$

Lemma 2.2. *For any $\gamma > 0$, there exists $A_4(\gamma) > 1$ such that*

$$A_4^{-1}(\gamma)\varphi(r) \leq \varphi(r^\gamma) \leq A_4(\gamma)\varphi(r), \text{ whenever } r > 0.$$

3. The estimate of Lf

We write $(Lf)(x) = L_1(x) + L_2(x) + L_3(x)$ for $x \in \mathbb{R}^n - \{x_0\}$, where

$$L_1(x) = \int_{\mathbb{R}^n - B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda)} K(\|x-y\|_\lambda) f(y)dy,$$
$$L_2(x) = \int_{B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda) - B_\lambda(x, \|x-x_0\|_\lambda/2M_\lambda)} K(\|x-y\|_\lambda) f(y)dy,$$
$$L_3(x) = \int_{B_\lambda(x, \|x-x_0\|_\lambda/2M_\lambda)} K(\|x-y\|_\lambda) f(y)dy.$$

Using (1.1), then for any $x, y \in \mathbb{R}^n$

$$\|x-y\|_\lambda \geq \frac{1}{M_\lambda}\|y-x_0\|_\lambda - \|x-x_0\|_\lambda.$$

It is obvious that, if $y \in \mathbb{R}^n - B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda)$, then $\|x-y\|_\lambda \geq \frac{1}{2M_\lambda}\|y-x_0\|_\lambda$. Taking into account $L_1(x)$, we have the inequality

$$L_1(x) \leq M\int_{\mathbb{R}^n - B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda)} K(\beta\|y-x_0\|_\lambda) f(y)dy \tag{3.1}$$

for any $\beta = \frac{1}{2M_\lambda} \in (0,1)$. For $y \in B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda) - B_\lambda(x, \|x-x_0\|_\lambda/2M_\lambda)$, since $\|y-x\|_\lambda \geq \frac{1}{2M_\lambda}\|x-x_0\|_\lambda$, we have similarly

$$L_2(x) \leq K(\beta\|x-x_0\|_\lambda)\int_{B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda) - B_\lambda(x, \|x-x_0\|_\lambda/2M_\lambda)} f(y)dy \tag{3.2}$$

for any $\beta = \frac{1}{2M_\lambda} \in (0,1)$.

Let us begin with the Hölder type inequality.

Lemma 3.1. *Let $p>1$, $\delta>0$, and f be a non-negative measurable function on $\mathbb{R}^n$. If $0\le 2M_\lambda\|x-x_0\|_\lambda<2M_\lambda a^{\frac{2|\lambda|}{n}}<1$, then*

$$\begin{aligned}
&\int_{\mathbb{R}^n-B_\lambda(x_0,2M_\lambda\|x-x_0\|_\lambda)} K(\beta\|y-x_0\|_\lambda)\,f(y)dy\\
&\quad\le \int_{\mathbb{R}^n-B_\lambda(x_0,2M_\lambda a^{\frac{2|\lambda|}{n}})} K(\beta\|y-x_0\|_\lambda)\,f(y)dy + ML\left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right)\\
&\quad+MR_1\left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right)\left(\int_{B_\lambda(x_0,2M_\lambda a^{\frac{2|\lambda|}{n}})}\phi_p(f(y))w\left(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right)dy\right)^{\frac{1}{p}},
\end{aligned}$$

where $R_1(r)=\left(\int_r^a K^{p'}\left(2\beta M_\lambda t^{\frac{2|\lambda|}{n}}\right)\left[\varphi(t^{-1})w(t)\right]^{\frac{p'}{p}}t^{2|\lambda|-1}dt\right)^{\frac{1}{p'}}$ if $0<2M_\lambda r^{\frac{2|\lambda|}{n}}<1$ and $R_1(r)=R_1\left((2M_\lambda)^{-\frac{n}{2|\lambda|}}\right)$ in the other cases.

Proof. Without loss of generality we assume that $f=0$ outside of $B_\lambda(x_0,2M_\lambda a^{\frac{2|\lambda|}{n}})$. We have

$$\begin{aligned}
&\int_{\mathbb{R}^n-B_\lambda(x_0,2M_\lambda\|x-x_0\|_\lambda)} K(\beta\|y-x_0\|_\lambda)\,f(y)dy=\int_{A(y)}K(\beta\|y-x_0\|_\lambda)\,f(y)dy\\
&\qquad\le\int_{\left\{y\in A(y);\,f(y)>\|y-x_0\|_\lambda^{-\frac{\delta n}{2|\lambda|}}\right\}}K(\beta\|y-x_0\|_\lambda)\,f(y)dy\\
&\qquad+\int_{\left\{y\in A(y);\,f(y)\le\|y-x_0\|_\lambda^{-\frac{\delta n}{2|\lambda|}}\right\}}K(\beta\|y-x_0\|_\lambda)\,f(y)dy=:L_{11}+L_{12},
\end{aligned}$$

where $A(y)=B_\lambda(x_0,2M_\lambda a^{\frac{2|\lambda|}{n}})-B_\lambda(x_0,2M_\lambda\|x-x_0\|_\lambda)$. Consider the integral L_{11}. From Hölder's inequality, we obtain

$$\begin{aligned}
L_{11}(x)&\le\left(\int_{U(y)}f^p(y)\varphi(f(y))w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})dy\right)^{\frac{1}{p}}\\
&\times\left(\int_{U(y)}K(\beta\|y-x_0\|_\lambda)^{p'}\left[\varphi(f(y))w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})\right]^{-\frac{P'}{p}}dy\right)^{\frac{1}{p'}},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p'}=1$ and $U(y)=\left\{y\in A(y);\ f(y)>\|y-x_0\|_\lambda^{-\frac{\delta n}{2|\lambda|}}\right\}$.

Since φ is a non-decreasing function, we have $\varphi(f(y)) \geq \varphi(\|y-x_0\|_\lambda^{-\frac{\delta n}{2|\lambda|}})$ and therefore, Lemma 2.2 implies $\varphi(\|y-x_0\|_\lambda^{-\frac{\delta n}{2|\lambda|}}) \geq M\varphi(\|y-x_0\|_\lambda^{-\frac{n}{2|\lambda|}})$. Thus,

$$L_{11}(x) \leq M\left(\int_{U(y)} \phi_p(f(y))w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})dy\right)^{\frac{1}{p}}$$
$$\times\left(\int_{U(y)} K(\beta\|y-x_0\|_\lambda)^{p'}\left[\varphi(\|y-x_0\|_\lambda^{-\frac{n}{2|\lambda|}})w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})\right]^{-\frac{P'}{p}}dy\right)^{\frac{1}{p'}}. \tag{3.3}$$

The right hand side integral with respect to y may be easily calculated. Namely, passing to generalized spherical coordinates by transformation

$$\begin{aligned}
y_1 &= x_{01} + (t\cos\theta_1)^{2\lambda_1},\\
y_2 &= x_{02} + (t\sin\theta_1\cos\theta_2)^{2\lambda_2},\\
&\vdots\\
y_n &= x_{0n} + (t\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-1})^{2\lambda_n},
\end{aligned}$$

where θ_j, $j=1,2,\ldots,n$, are the coordinates of the point θ on unit sphere. We can see that the Jacobian of this transformation $t^{2|\lambda|-1}\Omega_\lambda(\theta)$, where $\Omega_\lambda(\theta)$ depends on angles $\theta_1,\theta_2,\ldots,\theta_{n-1}$ only $0\leq\theta_1,\ldots,\theta_{n-2}\leq\pi$, $0\leq\theta_{n-1}\leq 2\pi$ and

$$\Omega_\lambda(\theta) = 2^n\prod_{j=1}^{n-1}\left(\cos\theta_j\right)^{2\lambda_j-1}\left(\sin\theta_j\right)^{2|\lambda|-\sum_{k=1}^{j}\lambda_k-1}.$$

Here the integral $\int_{S^{n-1}}\Omega_\lambda(\theta)d\theta$ is finite, where S^{n-1} is the unit ball in $\mathbb{R}^n$. Consequently, from (3.3) we have

$$L_{11}(x) \leq M\left(\int_{(2M_\lambda)^{\frac{n}{2|\lambda|}}\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}}}^{(2M_\lambda)^{\frac{n}{2|\lambda|}}a} K^{p'}\left(\beta t^{\frac{2|\lambda|}{n}}\right)\left[\varphi(t^{-1})w(t)\right]^{-\frac{p'}{p}}t^{2|\lambda|-1}dt\right)^{\frac{1}{p'}}$$
$$\times\left(\int_{U(y)}\phi_p(f(y))w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})dy\right)^{\frac{1}{p}}. \tag{3.4}$$

Let us now consider the integral L_{12}. By passing to generalized spherical coordinates, we get

$$L_{12}(x) \leq \int_{A(y)} K\left(\|y-x_0\|_\lambda\right) \|y-x_0\|_\lambda^{-\frac{n}{2|\lambda|}\delta} dy$$
$$= ML\left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right), \tag{3.5}$$

where $L(r)$ is defined in the condition (K_2). Relations (3.4) and (3.5) give the desired conclusion. □

Lemma 3.2. *Let f be a non-negative measurable function on $\mathbb{R}^n$. If $0 < 2M_\lambda \|x-x_0\|_\lambda < 1$ and $0 < \delta < 2|\lambda|$, then there exists a positive M such that*

$$L_2(x) \leq MR_2(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}}) \left(\int_{B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda)} \phi_p(f(y)) w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}}) dy\right)^{\frac{1}{p}}$$
$$+ M\|x-x_0\|_\lambda^{2|\lambda|-\delta},$$

where

$$R_2(r) = K\left(\beta t^{\frac{2|\lambda|}{n}}\right) r^{\frac{2|\lambda|}{n}\frac{2|\lambda|}{p'}} \left[\varphi\left(r^{-\frac{2|\lambda|}{n}}\right) w(r)\right]^{-\frac{1}{p}}.$$

Proof. It follows from (3.2) that

$$L_2(x) \leq K\left(\beta\|x-x_0\|_\lambda\right) \int_{B(x)} f(y) dy$$
$$\leq K\left(\beta\|x-x_0\|_\lambda\right) \left\{ \int_{\{y\in B(x);\, f(y)>\|x-x_0\|_\lambda^{-\delta}\}} f(y) dy \right.$$
$$\left. + \int_{\{y\in B(x);\, f(y)\leq\|x-x_0\|_\lambda^{-\delta}\}} f(y) dy \right\} =: L_{21}(x) + L_{22}(x),$$

where $B(x) = B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda) - B_\lambda(x, \|x-x_0\|_\lambda/2M_\lambda)$.

Let us first consider L_{21}. Since φ is a non-decreasing function, by Lemma 3.1, we get

$$L_{21}(x) \leq M\left[\varphi\left(\|x-x_0\|_\lambda^{-1}\right)\right]^{-\frac{1}{p}} K\left(\beta\|x-x_0\|_\lambda\right) \int_{B(x)} f(y) \left[\varphi(f(y))\right]^{\frac{1}{p}} dy.$$

From Hölder's inequality, we obtain

$$L_{21}(x) \leq M\left[\varphi\left(\|x-x_0\|_\lambda^{-1}\right)\right]^{-\frac{1}{p}} K\left(\beta\|x-x_0\|_\lambda\right)$$
$$\times \left(\int_{B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda)} dy\right)^{\frac{1}{p'}} \left(\int_{B(x)} f(y)^p \varphi(f(y)) dy\right)^{\frac{1}{p}},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Therefore, because w is a non-increasing function, it follows that

$$L_{21}(x) \leq M\left[\varphi\left(\|x-x_0\|_\lambda^{-1}\right)\right]^{-\frac{1}{p}} K\left(\beta\|x-x_0\|_\lambda\right)\|x-x_0\|_\lambda^{\frac{2|\lambda|}{P'}} \tag{3.6}$$

$$\times\left(\int_{B(x)}\phi_p(f(y))dy\right)^{\frac{1}{p}}$$

$$\leq M\left(\varphi\left(\|x-x_0\|_\lambda^{-1}\right)w(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}})\right)^{-\frac{1}{p}} K\left(\beta\|x-x_0\|_\lambda\right)\|x-x_0\|_\lambda^{\frac{2|\lambda|}{P'}}$$

$$\times\left(\int_{B_\lambda(x_0,2M_\lambda\|x-x_0\|_\lambda)}\phi_p(f(y))w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})dy\right)^{\frac{1}{p}}. \tag{3.7}$$

On the other hand, we have

$$L_{22} \leq K\left(\beta\|x-x_0\|_\lambda\right)\int_{B_\lambda(x_0,2M_\lambda\|x-x_0\|_\lambda)}\|x-x_0\|_\lambda^{-\delta}dy$$

$$\leq MK\left(\beta\|x-x_0\|_\lambda\right)\|x-x_0\|_\lambda^{2|\lambda|-\delta}. \tag{3.8}$$

We have the desired conclusion from (3.7) and (3.8). ☐

Lemma 3.3. *Let f be a non-negative measurable function on $\mathbb{R}^n$. If $\delta > 0$, then there exists a positive M such that*

$$L_3(x) \leq MR_3\left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right)\left(\int_{B_\lambda(x,\|x-x_0\|_\lambda/2M_\lambda)}\phi_p(f(y))w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})dy\right)^{\frac{1}{p}}$$

$$+M\int_0^{(\|x-x_0\|_\lambda/2M_\lambda)^{\frac{n}{2|\lambda|}}} K^{p'}\left(2\beta M_\lambda t^{\frac{2|\lambda|}{n}}\right)t^{2|\lambda|-\delta-1}dt$$

where $R_3(r)=\varphi^*(r)\omega(r)^{-\frac{1}{p}}$ *and* $\varphi^*(r)=\left(\int_0^r K^{p'}(t^{\frac{2|\lambda|}{n}})\left[\varphi(t^{-1})\right]^{-\frac{p'}{p}}t^{2|\lambda|-1}dt\right)^{\frac{1}{p'}}$.

Proof. By change of variable, we have

$$L_3(x)=\int_{B_\lambda(0,\|x-x_0\|_\lambda/2M_\lambda)}K\left(\|y\|_\lambda\right)f(x+y)dy.$$

In a way similar to the proof of Lemmas 3.1 and 3.2, we obtain

$$
\begin{aligned}
L_3(x) \le M & \left(\int_0^{(\|x-x_0\|_\lambda/2M_\lambda)^{\frac{n}{2|\lambda|}}} K^{p'}\left(t^{\frac{2|\lambda|}{n}}\right) \left[\varphi(t^{-1})\right]^{-\frac{p'}{p}} t^{2|\lambda|-1} dt \right)^{\frac{1}{p'}} \\
& \times \left(\int_{B_\lambda(0,\|x-x_0\|_\lambda/2M_\lambda)} \phi_p(f(x+y))dy \right)^{\frac{1}{p}} \\
& + M \int_0^{(\|x-x_0\|_\lambda/2M_\lambda)^{\frac{n}{2|\lambda|}}} K\left(\beta 2M_\lambda t^{\frac{2|\lambda|}{n}}\right) t^{2|\lambda|-\delta-1} dt \\
\le M\varphi^* & \left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}} \right) w\left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}} \right)^{-\frac{1}{p}} \\
& \times \left(\int_{B_\lambda(x,\|x-x_0\|_\lambda/2M_\lambda)} \phi_p(f(y)) w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}}) dy \right)^{\frac{1}{p}} \\
& + \int_0^{(\|x-x_0\|_\lambda/2M_\lambda)^{\frac{n}{2|\lambda|}}} K\left(\beta 2M_\lambda t^{\frac{2|\lambda|}{n}}\right) t^{2|\lambda|-\delta-1} dt,
\end{aligned}
$$

as required. □

4. Fine limit of $R_\alpha f$

We consider the function

$$
\begin{aligned}
R(r) &= R_1(r) + R_2(r) + R_3(r) \\
&= R_1(r) + K\left(\beta t^{\frac{2|\lambda|}{n}}\right) r^{\frac{2|\lambda|}{n}\frac{2|\lambda|}{p'}} \left(w(r)\varphi(r^{-\frac{2|\lambda|}{n}})\right)^{-\frac{1}{p}} + \varphi^*(r) w(r)^{-\frac{1}{p}}.
\end{aligned}
$$

Theorem 4.1. *Let $p > 1$ and f be a non-negative measurable function on $\mathbb{R}^n$ satisfying conditions (1.2) and (1.3). If $\varphi^*(1) < \infty$ and $\lim_{r\to 0} R(r) = \infty$, then*

$$
\lim_{x \to x_0} \left[R\left(\|x-x_0\|_\lambda\right)\right]^{-1} (Lf)(x) = 0.
$$

If $R(r)$ is bounded, then $(Lf)(x_0)$ is finite and $(Lf)(x)$ tends to $(Lf)(x_0)$ as $x \to x_0$.

Proof. By condition (1.2), the integral

$$
\int_{\mathbb{R}^n - B_\lambda(x_0, 2M_\lambda a^{\frac{2|\lambda|}{n}})} K\left(\beta \|y-x_0\|_\lambda\right) f(y) dy
$$

is finite. It follows from (3.1), the condition K_2 and Lemma 3.1 that

$$\limsup_{x\to x_0}\left(R\left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right)\right)^{-1}L_1(x)$$
$$\le M\left(\int_{B_\lambda(x_0,2M_\lambda a^{\frac{2|\lambda|}{n}})}\phi_p(f(y))w\left(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right)dy\right)^{\frac{1}{p}}.$$

Since a is arbitrary, we see that the integral in the left-hand side of the last estimate is equal to zero.

In view of Lemmas 3.2 and 3.3 and condition (1.3), we have

$$\lim_{x\to x_0}[R(\|x-x_0\|_\lambda)]^{-1}(L_2(x)+L_3(x))=0.$$

If we combine these results, we have

$$\lim_{x\to x_0}[R(\|x-x_0\|_\lambda)]^{-1}(Lf)(x)=0.$$

If $R(r)$ is bounded, then Lemmas 3.2 and 3.3 imply that $L_2(x)+L_3(x)$ tends to zero at $x\to x_0$. Furthermore, in view of Lemma 3.1, we have $\limsup_{x\to x_0}L_1(x)<\infty$. Thus it follows that $(Lf)(x_0)$ is finite. Hence

$$L_1(x)+L_2(x)=\int_{\mathbb{R}^n-B_\lambda(x,\|x-x_0\|_\lambda/2M_\lambda)}K(\|x-y\|_\lambda)\,f(y)dy.$$

Since $\|y-x_0\|_\lambda\le 2M_\lambda^2\|y-x\|_\lambda$ for $y\in\mathbb{R}^n-B_\lambda(x,\|x-x_0\|_\lambda/2M_\lambda)$, we have by Lebesgue's dominated convergence theorem

$$\lim_{x\to x_0}(L_1(x)+L_2(x))=(Lf)(x_0).$$

However, we also know that $\lim_{x\to x_0}L_3(x)=0$. The proof of Theorem 4.1 is thus complete. □

Corollary 4.1 ([8,9])**.** *Let $p=\frac{n}{\alpha}$ and $\varphi^*(1)<\infty$. If f is a non-negative measurable function on $\mathbb{R}^n$ satisfying* (1.2) *and the condition*

$$\int_{\mathbb{R}^n}\phi_p(f(y))dy<\infty,$$

then $L_\alpha f$ is continuous on $\mathbb{R}^n$ with $K(t)=t^{\alpha-n}$, $0<\alpha<n$, and $\lambda_k=\frac{1}{2}$, $k=1,2,\ldots,n$.

Corollary 4.2 ([1])**.** *Let f be a non-negative measurable function satisfying conditions* (1.2) *and the condition*

$$\int_{\mathbb{R}^n}\phi_p(f(y))dy<\infty,$$

then $L_\alpha f$ is continuous on $\mathbb{R}^n$.

Proposition 4.1. *Let $ap = n$, $\varphi^*(1) < \infty$, $x_0 = 0$, $K(t) = t^{\alpha - n}$, and*

$$\lim_{r\to 0} r^{\frac{2|\lambda|}{p'}} (w(r))^{-\frac{1}{p}} \left(\varphi(r^{-1})\right)^{-\frac{1}{p}} = 0.$$

Then for any positive non-decreasing function $a(r)$ on $(0,\infty)$ such that

$$\lim_{r\to 0} a(r) = \infty,$$

there exists a non-negative measurable function f satisfying (1.2) and (1.3) such that

$$\limsup_{x\to x_0} a\left(\|x\|_\lambda^{\frac{n}{2|\lambda|}}\right)\left(w\left(\|x\|_\lambda^{\frac{n}{2|\lambda|}}\right)\varphi\left(\|x\|_\lambda^{-\frac{n}{2|\lambda|}}\right)\right)^{-\frac{1}{p}} R_\alpha f(x) = \infty,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Let (j_i) be a sequence of positive integers such that $j_i + 2 < J_{i+1}$ and $\sum_i a_i^{-\frac{1}{p}} < \infty$, where $a_i(r_j) = a_i$ and $r_j = 2^{-j_i}$. We set

$$f(y) = a_i^{-\frac{1}{p}} \left(\varphi(r_j^{-1})\right)^{\frac{1}{p'}} (w(r_j))^{-\frac{1}{p}} \|x_i - y\|_\lambda^{-\alpha} \left[\varphi(\|x_i - y\|_\lambda^{-1})\right]^{-1}$$

if $y \in \bigcup_{i=1}^{\infty} B_\lambda(x_i, (2r_j)^{\frac{2|\lambda|}{n}}) - B_\lambda(x_i, (r_j)^{\frac{2|\lambda|}{n}})$, otherwise $f(y) = 0$, where $x_i = (r_j, 0, \ldots, 0) \in \mathbb{R}^n$.

Let us now show that f meets all the conditions in the proposition. If we use Lemmas 2.1 and 2.2, then we have

$$\begin{aligned}
\int f(y)dy &= \sum_i a_i^{-\frac{1}{p}} \left(\varphi(r_j^{-1})\right)^{\frac{1}{p'}} (w(r_j))^{-\frac{1}{p}} \\
&\quad \times \int_{B_\lambda(x_i,(2r_j)^{\frac{2|\lambda|}{n}}) - B_\lambda(x_i,(r_j)^{\frac{2|\lambda|}{n}})} \|x_i - y\|_\lambda^{-\alpha} \left[\varphi(\|x_i - y\|_\lambda^{-1})\right]^{-1} dy \\
&\le M \sum_i a_i^{-\frac{1}{p}} \left(\varphi(r_j^{-1})\right)^{\frac{1}{p'}} (w(r_j))^{-\frac{1}{p}} \int_{r_j}^{2r_j} t^{-\frac{2|\lambda|}{n}\alpha} \left(\varphi(t^{-\frac{2|\lambda|}{n}})\right)^{-1} t^{2|\lambda|-1} dt \\
&\le M \sum_i a_i^{-\frac{1}{p}} \left\{ r_j^{\frac{2|\lambda|}{p'}} \left(\varphi(r_j^{-1})\right)^{\frac{1}{p}} (w(r_j))^{-\frac{1}{p}} \right\} \\
&\le M \sum_i a_i^{-\frac{1}{p}} < \infty.
\end{aligned}$$

Consequently f satisfies (1.2). On the other hand, since the values $\left(a_i^{-\frac{1}{p}}\right)$ and $\left(r_j^{\frac{2|\lambda|}{p'}}\left(\varphi(r_j^{-1})\right)^{-\frac{1}{p}}\left(w(r_j)\right)^{-\frac{1}{p}}\right)$ are bounded, we have

$$\begin{aligned} f(y) &\le M\left(\varphi(r_j^{-1})\right)^{\frac{1}{p'}}\left(w(r_j)\right)^{-\frac{1}{p}}\|x_i-y\|_\lambda^{-\alpha}\left[\varphi(\|x_i-y\|_\lambda)\right]^{-1} \\ &\le M\left(\varphi(r_j^{-1})\right)^{-\frac{1}{p}}\left\{r_j^{\frac{2|\lambda|}{p'}}\varphi(r_j^{-1})^{-\frac{1}{p}}\|x_i-y\|_\lambda^{-\alpha}\right\}^{-1} \\ &\le M\|x_i-y\|_\lambda^{-\alpha-\frac{n}{p'}}. \end{aligned}$$

Thus, the inequality $\varphi(f(y)) \le \varphi(\|x_i-y\|_\lambda^{-1})$ holds.

Now we show that f satisfies (1.3). Using condition (w_1), we get

$$\begin{aligned} &\int_{\mathbb{R}^n}\phi_p(f(y))w\left(\|y\|_\lambda^{\frac{n}{2|\lambda|}}\right)dy \\ &\le \sum_i a_i^{-1}\left(\varphi(r_j^{-1})\right)^{\frac{p}{p'}}\int_{B_\lambda(x_i,(2r_j)^{\frac{2|\lambda|}{n}})-B_\lambda(x_i,(r_j)^{\frac{2|\lambda|}{n}})}\|x_i-y\|_\lambda^{-\alpha p} \\ &\qquad\times\left[\varphi(\|x_i-y\|_\lambda^{-1})\right]^{-\frac{p}{p'}}dy \\ &\le M\sum_i a_i^{-1}\left(\varphi(r_j^{-1})\right)^{\frac{p}{p'}}\int_{r_j}^{2r_j}t^{-\frac{2|\lambda|}{n}\alpha p}\left(\varphi(t^{-\frac{2|\lambda|}{n}})\right)^{-\frac{p}{p'}}t^{2|\lambda|-1}dt \\ &\le M\sum_i a_i^{-1}\left(\varphi(r_j^{-1})\right)^{\frac{p}{p'}}\int_{r_j}^{2r_j}\left(\varphi(t^{-1})\right)^{-\frac{p}{p'}}t^{-1}dt \\ &\le M\sum_i a_i^{-1}<\infty. \end{aligned}$$

Finally,

$$\begin{aligned} R_\alpha f(x_i) &\ge a_i^{-\frac{1}{p}}\left(\varphi(r_j^{-1})\right)^{\frac{1}{p'}}\left(w(r_j)\right)^{-\frac{1}{p}} \\ &\qquad\times\int_{B_\lambda(x_i,(2r_j)^{\frac{2|\lambda|}{n}})-B_\lambda(x_i,(r_j)^{\frac{2|\lambda|}{n}})}\|x_i-y\|_\lambda^{-n}\left[\varphi(\|x_i-y\|_\lambda^{-1})\right]^{-1}dy \\ &\ge Ma_i^{-\frac{1}{p}}\left(\varphi(r_j^{-1})\right)^{\frac{1}{p'}}\left(w(r_j)\right)^{-\frac{1}{p}}\int_{r_j}^{2r_j}\left(\varphi(t^{-1})\right)^{-1}t^{-1}dt \\ &\ge Ma_i^{-\frac{1}{p}}\left(\varphi(r_j^{-1})\right)^{-\frac{1}{p}}\left(w(r_j)\right)^{-\frac{1}{p}}. \end{aligned}$$

Thus we have

$$a_i\left(\varphi(r_j^{-1})\right)^{\frac{1}{p}}\left(w(r_j)\right)^{\frac{1}{p}}R_\alpha f(x^{(i)}) \ge Ma_i^{\frac{1}{p}}.$$

This proves the proposition for $j\to\infty$. $\square$

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ON DIRECTIONAL DERIVATIVE SETS OF MAX-MIN SET-VALUED MAPS

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Abstract. In this paper, necessary statements are given for a minimum point and a maximum point of the max-min function. Moreover estimations for the directional lower and upper derivative sets of the max-min set-valued map which are used to state a characterization of the directional derivative of the max-min functions are given. Furthermore, a sufficient condition ensuring the existence of the directional derivative of the max-min function is obtained by using the lower differentiability of the max-min set-valued maps.

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1. Introduction

The directional derivatives of max-min functions were studied in [7]. It is well-known that max-min functions are considered and occur in control theory problems, parametric optimization problems and differential game theory problems [4]. Moreover marginal functions are max-min functions (see [3–6, 10–14]).

In this paper, necessary statements are given for a minimum point and a maximum point of the max-min function. It is well-known that the directional lower and upper derivative sets of max-min set valued maps are used to state a characterization of the directional derivative of max-min function [7]. In this paper, estimations for the directional lower and upper derivative sets of max-min set valued maps are given. Furthermore, a sufficient condition ensuring the existence of the directional derivative of max-min function is obtained by using the lower differentiability of max-min set-valued maps.

In this study, $\mathrm{cl}\,(\mathbb{R}^m)$ $(\mathrm{comp}\,(\mathbb{R}^m))$ denotes the set of all nonempty closed (compact) subsets in $\mathbb{R}^m$. Let $a(\cdot) : \mathbb{R}^n \to \mathrm{cl}\,(\mathbb{R}^m)$ be an upper semi-continuous set-valued map. For $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ and vector $f \in \mathbb{R}^n$, let us consider the following sets:

$$Da(x,y) \mid (f) = \left\{d \in \mathbb{R}^m : \liminf_{\delta \to +0} \delta^{-1} \operatorname{dist}(y + \delta d, a(x + \delta f)) = 0\right\},$$

$$D^*a(x,y) \mid (f) = \left\{v \in \mathbb{R}^m : \lim_{\delta \to +0} \delta^{-1} \operatorname{dist}(y + \delta d, a(x + \delta f)) = 0\right\},$$

where $x \in \mathbb{R}^n$, $D \subset \mathbb{R}^n$, $\operatorname{dist}(x, D) = \inf_{d \in D} \|x - d\|$. $Da(x, y) \mid (f)$ ($D^*a(x, y) \mid (f)$) is called the upper (lower) derivative set of the set-valued map $a(\cdot)$ at (x, y) in direction f. Note that the directional upper (lower) derivative set of the set-valued map $a(\cdot)$ is closed and there is a connection between the upper (lower) derivative set of a set-valued map and the upper (lower) contingent cone which is used to investigate several problems in nonsmooth analysis [1,2,8]. It is obvious that $D^*a(x, y) \mid (f) \subset Da(x, y) \mid (f)$. The symbol

$$A = \operatorname{gr} a(\cdot) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in a(x)\}$$

denotes the graph of the set-valued map $a(\cdot)$. Since $a(\cdot)$ is upper semicontinuous, A is a closed set. It is possible to show that $Da(x, y) \mid (f) = D^*a(x, y) \mid (f) = \varnothing$ if $(x, y) \notin A$, $Da(x, y) \mid (f) = D^*a(x, y) \mid (f) = \mathbb{R}^m$ if $(x, y) \in \operatorname{int} A$, where $\operatorname{int} A$ denotes the interior of A.

Let $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be a function. The lower and upper derivative of $f(\cdot)$ at the point x in direction v are denoted by the symbols $\frac{\partial^- f(x)}{\partial v}$ and $\frac{\partial^+ f(x)}{\partial v}$ respectively, and defined by the formulas

$$\frac{\partial^- f(x)}{\partial v} = \liminf_{\delta \to +0} [f(x + \delta v) - f(x)]\,\delta^{-1},$$

and

$$\frac{\partial^+ f(x)}{\partial v} = \limsup_{\delta \to +0} [f(x + \delta v) - f(x)]\,\delta^{-1}.$$

If

$$\frac{\partial f(x)}{\partial v} = \lim_{\delta \to +0} [f(x + \delta v) - f(x)]\,\delta^{-1}$$

exists and is finite, then $f(\cdot)$ is said to be differentiable at the point x in direction v and $\frac{\partial f(x)}{\partial v}$ denotes the derivative of $f(\cdot)$ at the point x in direction v.

Let $a(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m)$, $b(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^k)$ be set-valued maps and $\sigma(\cdot, \cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ be a continuous function on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$. The max-min function is denoted by $m(\cdot)$ and is defined by

$$m(x) = \max_{y \in a(x)} \min_{z \in b(x)} \sigma(x, y, z). \tag{1.1}$$

In this paper, we will assume that $a(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m)$, $b(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^k)$ are continuous set-valued maps and $\sigma(\cdot, \cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ is a continuous function on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$ and locally Lipschitz on $\mathbb{R}^m \times \mathbb{R}^k$, i. e., for every bounded $D \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$, there exists $L(D) > 0$ such that

$$|\sigma(x, y_1, z_1) - \sigma(x, y_2, z_2)| \leq L(D)\,\|(y_1 - y_2, z_1 - z_2)\|$$

for any (x, y_1, z_1), $(x, y_2, z_2) \in D$. Under these conditions $m(\cdot)$ is a continuous function (see [1]). Let

$$Y_*(x) = \{(y_*, z_*) \in a(x) \times b(x) : \\ m(x) = \max_{y \in a(x)} \min_{z \in b(x)} \sigma(x, y, z) = \sigma(x, y_*, z_*)\}. \quad (1.2)$$

The map $x \mapsto Y_*(x)$ is an upper semicontinuous set-valued map and it is called a max-min set-valued map.

2. Minimization and maximization problems of max-min function

Now we give necessary statements for a minimum point and a maximum point of max-min function.

Theorem 1. *Let the function $m(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be in the form (1.1). Suppose that $x_* \in \mathbb{R}^n$ is a minimum point of max-min function $m(\cdot)$. Then*

$$\inf_{f \in \mathbb{R}^n} \inf_{(y,z) \in Y_*(x_*)} \inf_{(d,n) \in DY_*(x_*,y,z)|(f)} \frac{\partial^+ \sigma(x_*, y, z)}{\partial(f, d, n)} \geq 0.$$

Proof. Let $f \in \mathbb{R}^n$. Since $x_* \in \mathbb{R}^n$ is a minimum point of max-min function $m(\cdot)$, then it follows that $m(x_* + \delta f) \geq m(x_*)$ for all $\delta > 0$. In that case, it follows from here that

$$\frac{\partial^- m(x_*)}{\partial f} = \liminf_{\delta \to +0} \frac{1}{\delta} [m(x_* + \delta f) - m(x_*)] \geq 0.$$

Hence, from here and of [7, Proposition 7], we have

$$0 \leq \frac{\partial^- m(x_*)}{\partial f} \leq \inf_{(y,z) \in Y_*(x_*)} \inf_{(d,n) \in DY_*(x_*,y,z)|(f)} \frac{\partial^+ \sigma(x_*, y, z)}{\partial(f, d, n)}$$

and hence we obtain the inequality. □

Theorem 2. *Let the function $m(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be in the form (1.1). Suppose that $x_* \in \mathbb{R}^n$ is a maximum point of max-min function $m(\cdot)$ and there exists a $(y, z) \in Y_*(x_*)$ such that $DY_*(x_*, y, z) \mid (f) \neq \varnothing$ for all $f \in \mathbb{R}^n$. Then*

$$\sup_{f \in \mathbb{R}^n} \inf_{(y,z) \in Y_*(x_*)} \inf_{(d,n) \in DY_*(x_*,y,z)|(f)} \frac{\partial^- \sigma(x_*, y, z)}{\partial(f, d, n)} \leq 0.$$

Proof. Let $f \in \mathbb{R}^n$. Since $x_* \in \mathbb{R}^n$ is a maximum point of max-min function $m(\cdot)$, then it follows that $m(x_* + \delta f) \leq m(x_*)$ for all $\delta > 0$. In that case, it follows from here that

$$\frac{\partial^+ m(x_*)}{\partial f} = \limsup_{\delta \to +0} \frac{1}{\delta} [m(x_* + \delta f) - m(x_*)] \leq 0.$$

Hence, from here and [7, Proposition 8], we have

$$0 \geq \frac{\partial^+ m(x_*)}{\partial f} \geq \inf_{(y,z) \in Y_*(x_*)} \inf_{(d,n) \in DY_*(x_*,y,z)|(f)} \frac{\partial^- \sigma(x_*, y, z)}{\partial(f, d, n)}$$

and hence we obtain the inequality. □

3. Directional derivative sets of max-min set-valued maps

Now we give the estimations for the directional lower and upper derivative sets of max-min set-valued map which are used to state a characterization of the directional derivative of the max-min functions in [7].

Let us take the max-min function $m(\cdot)$ such as:

$$m(x) = \min_{y \in a(x)} \sigma(x,y) \tag{3.1}$$

where $a(\cdot) : \mathbb{R}^n \to \mathrm{comp}\,(\mathbb{R}^m)$ is a continuous set-valued map and $\sigma(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a continuous function on $\mathbb{R}^n \times \mathbb{R}^m$ and locally Lipschitz on $\mathbb{R}^m$, i. e., for every bounded $D \subset \mathbb{R}^n \times \mathbb{R}^m$, there exists $L(D) > 0$ such that

$$|\sigma(x,y_1) - \sigma(x,y_2)| \leq L(D)\,\|y_1 - y_2\|$$

for any (x,y_1), $(x,y_2) \in D$. Under these conditions $m(\cdot)$ is a continuous function (see [1]). Then we take

$$Y_*(x) = \Big\{ y_* \in a(x) : m(x) = \min_{y \in a(x)} \sigma(x,y) = \sigma(x,y_*) \Big\}. \tag{3.2}$$

Theorem 3. *Let the set-valued map $Y_*(\cdot) : \mathbb{R}^n \to \mathrm{comp}\,(\mathbb{R}^m)$ be in the form (3.2). Suppose that there exists a $y_* \in Y_*(x_*)$ such that $\sigma(\cdot,\cdot)$ is a derivable function at the point (x_*,y_*) in direction (f,d) for all $d \in \mathbb{R}^m$. Then*

$$D^*Y_*(x_*,y_*) \mid (f) \subset \left\{ p \in \mathbb{R}^m : \inf_{d \in Da(x_*,y_*)|(f)} \frac{\partial \sigma(x_*,y_*)}{\partial (f,d)} = \frac{\partial \sigma(x_*,y_*)}{\partial (f,p)} \right\}. \tag{3.3}$$

Proof. Let $y_* \in Y_*(x_*)$ such that $\sigma(\cdot,\cdot)$ is a derivable function at the point (x_*,y_*) in direction (f,d) for all $d \in \mathbb{R}^m$.

Let $D^*Y_*(x_*,y_*) \mid (f) = \varnothing$. Then the statement (3.3) holds.

Let $Da(x_*,y_*) \mid (f) = \varnothing$. Then

$$\inf_{d \in Da(x_*,y_*)|(f)} \frac{\partial \sigma(x_*,y_*)}{\partial (f,d)} = +\infty.$$

Since $D^*Y_*(x_*,y_*) \mid (f) \subset D^*a(x_*,y_*) \mid (f) \subset Da(x_*,y_*) \mid (f)$, then it follows that statement (3.3) holds.

Now $D^*Y_*(x_*,y_*) \mid (f) \neq \varnothing$. Take $p \in D^*Y_*(x_*,y_*) \mid (f)$. Then from the definition of $D^*Y_*(x_*,y_*) \mid (f)$, there exists a $\delta_* > 0$ such that for all $\delta \in [0,\delta_*]$,

$$y_*(\delta) = y_* + \delta p + o_1(\delta) \in Y_*(x_* + \delta f)$$

where $\|o_1(\delta)\|\,\delta^{-1} \to 0$ as $\delta \to +0$. Since

$$Y_*(x) = \{ y_* \in a(x) : \sigma(x,y) \geq \sigma(x,y_*), \forall\, y \in a(x) \},$$

then it follows that for any $y \in a(x_* + \delta f)$,

$$\sigma(x_* + \delta f, y) \geq \sigma(x_* + \delta f, y_* + \delta p + o_1(\delta)). \tag{3.4}$$

Choose any $d \in Da(x_*, y_*) \mid (f)$. Then from the definition of $Da(x_*, y_*) \mid (f)$, there exists a sequence $y_k \in a(x_* + \delta_k f)$ where $\delta_k > 0$ and $\delta_k \to +0$ as $k \to \infty$, such that

$$y_k = y_* + \delta_k d + o_2(\delta_k)$$

where $\|o_2(\delta_k)\| \delta_k^{-1} \to 0$ as $k \to \infty$. In that case from (3.4) since $\delta_k \to +0$, then it follows that there exists a $k_0 \in \mathbb{N}$ such that $\delta_k \in [0, \delta_*]$ for all $k \geq k_0$ and

$$\begin{aligned}\sigma(x_* + \delta_k f, y_* + \delta_k d + o_2(\delta_k)) - \sigma(x_*, y_*) \\ \geq \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) - \sigma(x_*, y_*).\end{aligned}$$

Since the function $\sigma(\cdot, \cdot)$ is locally Lipschitzian on $\mathbb{R}^m$, then it follows that for $k = 1, 2, \ldots,$ there exists $L_1 > 0$ and $L_2 > 0$ such that

$$\begin{aligned}&| \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) - \sigma(x_* + \delta_k f, y_* + \delta_k p) | \leq L_1 \|o_1(\delta_k)\|, \\ &| \sigma(x_* + \delta_k f, y_* + \delta_k d + o_2(\delta_k)) - \sigma(x_* + \delta_k f, y_* + \delta_k d) | \leq L_2 \|o_2(\delta_k)\|.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\frac{\partial \sigma(x_*, y_*)}{\partial(f, p)} &= \lim_{\delta \to +0} [\sigma(x_* + \delta f, y_* + \delta p) - \sigma(x_*, y_*)] \delta^{-1} \\ &= \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k p) - \sigma(x_*, y_*)] \delta_k^{-1} \\ &= \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k p) - \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) \\ &\quad + \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) - \sigma(x_*, y_*)] \delta_k^{-1} \\ &\leq \lim_{k \to \infty} [L_1 \|o_1(\delta_k)\| + \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) - \sigma(x_*, y_*)] \delta_k^{-1} \\ &= \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) - \sigma(x_*, y_*)] \delta_k^{-1} \\ &\leq \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k d + o_2(\delta_k)) - \sigma(x_*, y_*)] \delta_k^{-1} \\ &= \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k d + o_2(\delta_k)) - \sigma(x_* + \delta_k f, y_* + \delta_k d) \\ &\quad + \sigma(x_* + \delta_k f, y_* + \delta_k d) - \sigma(x_*, y_*)] \delta_k^{-1} \\ &\leq \lim_{k \to \infty} L_2 \|o_2(\delta_k)\| \delta_k^{-1} + \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k d) - \sigma(x_*, y_*)] \delta_k^{-1} \\ &= \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k d) - \sigma(x_*, y_*)] \delta_k^{-1} \\ &= \lim_{\delta \to +0} [\sigma(x_* + \delta f, y_* + \delta d) - \sigma(x_*, y_*)] \delta^{-1} \\ &= \frac{\partial \sigma(x_*, y_*)}{\partial(f, d)}.\end{aligned}$$

Thus $\frac{\partial\sigma(x_*,y_*)}{\partial(f,p)} \leq \frac{\partial\sigma(x_*,y_*)}{\partial(f,d)}$ for any $d \in Da(x_*,y_*) \mid (f)$. It follows from here that

$$\inf_{d\in Da(x_*,y_*)|(f)} \frac{\partial\sigma(x_*,y_*)}{\partial(f,d)} \geq \frac{\partial\sigma(x_*,y_*)}{\partial(f,p)}.$$

Since

$$D^*Y_*(x_*,y_*) \mid (f) \subset DY_*(x_*,y_*) \mid (f) \subset Da(x_*,y_*) \mid (f),$$

then for any $p \in D^*Y_*(x_*,y_*) \mid (f)$, $p \in Da(x_*,y_*) \mid (f)$ and it follows that the relation

$$\inf_{d\in Da(x_*,y_*)|(f)} \frac{\partial\sigma(x_*,y_*)}{\partial(f,d)} \leq \frac{\partial\sigma(x_*,y_*)}{\partial(f,p)}$$

is satisfied. In that case, it follows from the above two inequalities that the statement holds. □

Theorem 4. *Let the set-valued map $Y_*(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m)$ be in the form (3.2). Suppose that there exists a $y_* \in Y_*(x_*)$ such that $\sigma(\cdot,\cdot)$ is a derivable function at the point (x_*,y_*) in the direction (f,d) for all $d \in \mathbb{R}^m$. Then*

$$DY_*(x_*,y_*) \mid (f) \subset \left\{p \in \mathbb{R}^m : \inf_{d\in D^*a(x_*,y_*)|(f)} \frac{\partial\sigma(x_*,y_*)}{\partial(f,d)} \geq \frac{\partial\sigma(x_*,y_*)}{\partial(f,p)}\right\}. \tag{3.5}$$

Proof. Let $y_* \in Y_*(x_*)$ such that $\sigma(\cdot,\cdot)$ is a derivable function at the point (x_*,y_*) in direction (f,d) for all $d \in \mathbb{R}^m$.

Let $DY_*(x_*,y_*) \mid (f) = \varnothing$. Then statement (3.5) holds.

Let $D^*a(x_*,y_*) \mid (f) = \varnothing$. Then

$$+\infty = \inf_{d\in D^*a(x_*,y_*)|(f)} \frac{\partial\sigma(x_*,y_*)}{\partial(f,d)} > \frac{\partial\sigma(x_*,y_*)}{\partial(f,p)}$$

and it follows that the statement (3.5) holds.

Now let $DY_*(x_*,y_*) \mid (f) \neq \varnothing$ and $D^*a(x_*,y_*) \mid (f) \neq \varnothing$. Let us take $p \in DY_*(x_*,y_*) \mid (f)$. Then from the definition of $DY_*(x_*,y_*) \mid (f)$, there exists a sequence $y_k \in Y_*(x_* + \delta_k f)$, where $\delta_k > 0$ and $\delta_k \to +0$ as $k \to \infty$ such that

$$y_k = y_* + \delta_k p + o_1(\delta_k)$$

where $\|o_1(\delta_k)\|/\delta_k \to 0$ as $k \to \infty$. Since

$$Y_*(x) = \{y_* \in a(x) : \ \sigma(x,y) \geq \sigma(x,y_*), \forall y \in a(x)\},$$

then it follows that for any $y \in a(x_* + \delta_k f)$,

$$\sigma(x_* + \delta_k f, y) \geq \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)). \tag{3.6}$$

Choose any $d \in D^*a(x_*,y_*) \mid (f)$. Then from the definition of $D^*a(x_*,y_*) \mid (f)$, there exists a $\delta_* > 0$ such that for all $\delta \in [0,\delta_*]$,

$$y(\delta) = y_* + \delta d + o_2(\delta) \in a(x_* + \delta f)$$

where $\|o_2(\delta)\|\,\delta^{-1} \to 0$ as $\delta \to +0$. In that case from (3.6) since $\delta_k \to +0$, then it follows that there exists a $k_0 \in \mathbb{N}$ such that $\delta_k \in [0, \delta_*]$ for all $k \geq k_0$ and

$$\sigma(x_* + \delta_k f, y_* + \delta_k d + o_2(\delta_k)) - \sigma(x_*, y_*) \\ \geq \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) - \sigma(x_*, y_*).$$

Since the function $\sigma(\cdot,\cdot)$ is locally Lipschitz on $\mathbb{R}^m$, then it follows that for $k = 1, 2, \ldots,$ there exist $L_1 > 0$ and $L_2 > 0$ such that

$$|\,\sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k) - \sigma(x_* + \delta_k f, y_* + \delta_k p)\,| \leq L_1 \,\|o_1(\delta_k)\|\,,$$
$$|\,\sigma(x_* + \delta_k f, y_* + \delta_k d + o_2(\delta_k)) - \sigma(x_* + \delta_k f, y_* + \delta_k d)\,| \leq L_2 \,\|o_2(\delta_k)\|\,.$$

Consequently, we have

$$\begin{aligned}
\frac{\partial \sigma(x_*, y_*)}{\partial (f, p)} &= \lim_{\delta \to +0} [\sigma(x_* + \delta f, y_* + \delta p) - \sigma(x_*, y_*)]\,\delta^{-1} \\
&= \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k p) - \sigma(x_*, y_*)]\,\delta_k^{-1} \\
&= \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k p) - \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) \\
&\qquad + \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) - \sigma(x_*, y_*)]\delta_k^{-1} \\
&\leq \lim_{k \to \infty} [L_1 \,\|o_1(\delta_k)\| + \sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) - \sigma(x_*, y_*)]\delta_k^{-1} \\
&= \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k p + o_1(\delta_k)) - \sigma(x_*, y_*)]\,\delta_k^{-1} \\
&\leq \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k d + o_2(\delta_k)) - \sigma(x_*, y_*)]\,\delta_k^{-1} \\
&= \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k d + o_2(\delta_k)) - \sigma(x_* + \delta_k f, y_* + \delta_k d) \\
&\qquad + \sigma(x_* + \delta_k f, y_* + \delta_k d) - \sigma(x_*, y_*)]\delta_k^{-1} \\
&\leq \lim_{k \to \infty} L_2 \,\|o_2(\delta_k)\|\,\delta_k^{-1} + \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k d) - \sigma(x_*, y_*)]\delta_k^{-1} \\
&= \lim_{k \to \infty} [\sigma(x_* + \delta_k f, y_* + \delta_k d) - \sigma(x_*, y_*)]\delta_k^{-1} \\
&= \lim_{\delta \to +0} [\sigma(x_* + \delta f, y_* + \delta d) - \sigma(x_*, y_*)]\,\delta^{-1} \\
&= \frac{\partial \sigma(x_*, y_*)}{\partial (f, d)}.
\end{aligned}$$

Thus, $\frac{\partial \sigma(x_*, y_*)}{\partial (f, p)} \leq \frac{\partial \sigma(x_*, y_*)}{\partial (f, d)}$ for any $d \in D^* a(x_*, y_*) \,|\, (f)$. It follows from here that

$$\inf_{d \in D^* a(x_*, y_*) | (f)} \frac{\partial \sigma(x_*, y_*)}{\partial (f, d)} \geq \frac{\partial \sigma(x_*, y_*)}{\partial (f, p)}$$

and hence the statement holds. $\square$

Corollary 1. *Let the set-valued map $Y_*(\cdot):\mathbb{R}^n\to \mathrm{comp}\,(\mathbb{R}^m)$ be in the form (3.2). Suppose that there exists a $y_*\in Y_*(x_*)$ such that $\sigma(\cdot,\cdot)$ is a differentiable function at the point (x_*,y_*). Then*

$$D^*Y_*(x_*,y_*)\mid(f)\subset\left\{p\in\mathbb{R}^m:\inf_{d\in Da(x_*,y_*)|(f)}\left\langle\frac{\partial\sigma(x_*,y_*)}{\partial y},d\right\rangle=\left\langle\frac{\partial\sigma(x_*,y_*)}{\partial y},p\right\rangle\right\},$$

where the symbol $\langle\cdot,\cdot\rangle$ denotes the inner product.

Proof. Since $\sigma(\cdot,\cdot)$ is a differentiable function at the point (x_*,y_*), then it is a derivable function at the point (x_*,y_*) in any direction $(f,d)\in\mathbb{R}^n\times\mathbb{R}^m$ and

$$\frac{\partial\sigma(x_*,y_*)}{\partial(f,d)}=\left\langle\frac{\partial\sigma(x_*,y_*)}{\partial x},f\right\rangle=\left\langle\frac{\partial\sigma(x_*,y_*)}{\partial y},d\right\rangle.$$

Then from Theorem 3, we obtain the corollary. □

Corollary 2. *Let the set-valued map $Y_*(\cdot):\mathbb{R}^n\to \mathrm{comp}\,(\mathbb{R}^m)$ be in the form (3.2). Suppose that there exists a $y_*\in Y_*(x_*)$ such that $\sigma(\cdot,\cdot)$ is a differentiable function at the point (x_*,y_*). Then*

$$DY_*(x_*,y_*)\mid(f)\subset\left\{p\in\mathbb{R}^m:\inf_{d\in D^*a(x_*,y_*)|(f)}\left\langle\frac{\partial\sigma(x_*,y_*)}{\partial y},d\right\rangle\geq\left\langle\frac{\partial\sigma(x_*,y_*)}{\partial y},p\right\rangle\right\},$$

where the symbol $\langle\cdot,\cdot\rangle$ denotes the inner product.

Proof. It is similar to that of Theorem 4. □

Now, we give an example for the above theorems.

Example 1. Take a constant vector $l\in\mathbb{R}^m$. $\sigma(\cdot,\cdot):\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ is defined by

$$(x,y)\to\sigma(x,y)=\langle l,y\rangle$$

and $a(\cdot)$ is defined by

$$x\to a(x)=\{y\in\mathbb{R}^m: b(x,y)\geq 0\}$$

where $b(\cdot,\cdot):\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ is continuous differentiable such that $a(x)$ is bounded for all $x\in\mathbb{R}^m$ and $\frac{\partial b(x,y)}{\partial y}\neq 0$ for any $y\in L(x)$ where

$$L(x)=\{y\in\mathbb{R}^m: b(x,y)=0\}.$$

Then $a(x):\mathbb{R}^m\to\mathrm{comp}\,(\mathbb{R}^m)$ and

$$Y_*(x)\subset\partial a(x)\subset L(x)$$

where $\partial a(x)$ is a boundary of $a(x)$. It is shown that

$$Y_*(x)\subset K(x)\tag{3.7}$$

where

$$K(x)=\{y_0\in a(x):\langle l,d\rangle\geq 0 \text{ for all } d\in T_{a(x)}(y_0)\},\tag{3.8}$$

and

$$T_{a(x)}(y_0) = \Big\{ d \in \mathbb{R}^m : \exists \delta_k > 0 \ (\delta_k \to +0 \text{ as } k \to \infty), \\ \exists y_k \in a(x) \ni \ d = \lim_{k\to\infty} \frac{y_k - y_0}{\delta_k} \Big\},$$

where $T_{a(x)}(y_0)$ is an upper contingent cone of $a(x)$ at y_0.

Let $y_* \in Y_*(x)$. Then $y_* \in \partial a(x)$ and $b(x, y_*) = 0$. Since $\frac{\partial b(x,y_*)}{\partial y} \neq 0$, we get

$$T_{a(x)}(y_0) = \left\{ d \in \mathbb{R}^m : \left\langle \frac{\partial b(x, y_*)}{\partial y}, d \right\rangle \geq 0 \right\} \tag{3.9}$$

and from [7, Corollary 2],

$$\begin{aligned} Da(x, y_*) \mid (f) &= D^* a(x, y_*) \mid (f) \\ &= \left\{ d \in \mathbb{R}^m : \left\langle \frac{\partial b(x, y_*)}{\partial x}, f \right\rangle + \left\langle \frac{\partial b(x, y_*)}{\partial y}, d \right\rangle \geq 0 \right\}. \end{aligned} \tag{3.10}$$

Since $y_* \in Y_*(x)$, then it follows from (3.7) that $y_* \in K(x)$ and since $y_* \in K(x)$, then from (3.8), $\langle l, d \rangle \geq 0$ for all $d \in T_{a(x)}(y_*)$. Then from (3.9) we get

$$\langle l, d \rangle \geq 0 \text{ for all } d \in \mathbb{R}^m \text{ such that } \left\langle \frac{\partial b(x, y_*)}{\partial y}, d \right\rangle \geq 0.$$

This yields $l = \alpha(x) \frac{\partial b(x,y_*)}{\partial y}$ with $\alpha(x) > 0$, and follows that

$$Y_*(x) \subset \left\{ y_* \in \partial a(x) : l = \alpha(x) \frac{\partial b(x, y_*)}{\partial y}, \ \alpha(x) > 0 \right\}. \tag{3.11}$$

Therefore, Theorems 3 and 4 and statements (3.10), (3.11) yield

$$\begin{aligned} D^* Y_*(x, y_*) \mid (f) &\subset \left\{ p \in \mathbb{R}^m : \min_{d \in Da(x,y_*)|(f)} \langle l, d \rangle = \langle l, p \rangle \right\} \\ &= \left\{ p \in \mathbb{R}^m : \langle l, p \rangle = -\alpha(x) \left\langle \frac{\partial b(x, y_*)}{\partial x}, f \right\rangle \right\}, \\ DY_*(x, y_*) \mid (f) &\subset \left\{ p \in \mathbb{R}^m : \min_{d \in D^* a(x,y_*)|(f)} \langle l, d \rangle \geq \langle l, p \rangle \right\} \\ &= \left\{ p \in \mathbb{R}^m : \langle l, p \rangle \leq -\alpha(x) \left\langle \frac{\partial b(x, y_*)}{\partial x}, f \right\rangle \right\}. \end{aligned}$$

4. Directional differentiability of max-min function

In [7], the directional lower and upper derivatives of max-min function are investigated by using the directional lower and upper derivative sets of max-min set-valued map.

Now, in this paper, a sufficient condition ensuring the existence of the directional derivative of the max-min function is obtained by using the lower differentiability of max-min set-valued maps. Hence by using the lower different, a characterization of the upper and lower directional derivatives of max-min functions is obtained. For $A \subset \mathbb{R}^n$ and $C \subset \mathbb{R}^n$, we put $\beta(A,C) = \sup_{a\in A} d(a,C)$, $d_H(A,C) = \max\{\beta(A,C),\beta(C,A)\}$. It is known that $(\mathrm{comp}(\mathbb{R}^n), d_H(\cdot,\cdot))$ is a metric space.

Definition 1. The set-valued map $a(\cdot) : \mathbb{R}^n \to \mathrm{comp}(\mathbb{R}^m)$ is said to be lower differentiable at the point $x \in \mathbb{R}^n$ in direction $f \in \mathbb{R}^n$, if there exist $G^-(x,f) \in \mathrm{comp}(\mathbb{R}^m)$, $G^+(x,f) \in \mathrm{comp}(\mathbb{R}^m)$ such that

$$\lim_{\delta\to+0} \frac{1}{\delta}\beta(a(x)+\delta G^+(x,f), a(x+\delta f)+\delta G^-(x,f)) = 0.$$

In that case the pair $(G^-(x,f), G^+(x,f))$ is said to be lower differential of the set-valued map $a(\cdot) : \mathbb{R}^n \to \mathrm{comp}(\mathbb{R}^m)$ at the point $x \in \mathbb{R}^n$ in the direction $f \in \mathbb{R}^n$ [9].

In [9], the following two propositions are given.

Proposition 1. *Let the set-valued map $a(\cdot) : \mathbb{R}^n \to \mathrm{comp}(\mathbb{R}^m)$ be lower differentiable at the point $x \in \mathbb{R}^n$ in direction $f \in \mathbb{R}^n$. Then $Da(x,y) \mid (f) \neq \varnothing$ for every $y \in a(x)$.*

Proposition 2. *Let the set-valued map $a(\cdot) : \mathbb{R}^n \to \mathrm{comp}(\mathbb{R}^m)$ be lower differentiable at the point $x \in \mathbb{R}^n$ in direction $f \in \mathbb{R}^n$ and the pair $(0, G^+(x,f))$ be its lower differential. Then $D^*a(x,y) \mid (f) \neq \varnothing$ for every $y \in a(x)$.*

In [7], the following two propositions characterizing upper and lower directional derivatives of max-min function $m(\cdot)$ are proved.

Proposition 3. *For all $x \in \mathbb{R}^n$ and $f \in \mathbb{R}^n$*

$$\frac{\partial^- m(x)}{\partial f} \le \inf_{(y,z)\in Y_*(x)} \ \inf_{(d,n)\in DY_*(x,y,z)|(f)} \frac{\partial^+\sigma(x,y,z)}{\partial(f,d,n)},$$

$$\frac{\partial^+ m(x)}{\partial f} \le \inf_{(y,z)\in Y_*(x)} \ \inf_{(d,n)\in D^*Y_*(x,y,z)|(f)} \frac{\partial^+\sigma(x,y,z)}{\partial(f,d,n)}.$$

Proposition 4. *Let $x \in \mathbb{R}^n$, $f \in \mathbb{R}^n$ and there exists $(y_*, z_*) \in Y_*(x)$ such that $DY_*(x,y_*,z_*) \mid (f) \neq \varnothing$. Then*

$$\frac{\partial^+ m(x)}{\partial f} \ge \inf_{(y,z)\in Y_*(x)} \ \inf_{(d,n)\in DY_*(x,y,z)|(f)} \frac{\partial^-\sigma(x,y,z)}{\partial(f,d,n)}.$$

Moreover if there exists $(y^*, z^*) \in Y_*(x)$ such that $D^*Y_*(x,y^*,z^*) \mid (f) \neq \varnothing$ then

$$\frac{\partial^- m(x)}{\partial f} \ge \inf_{(y,z)\in Y_*(x)} \ \inf_{(d,n)\in D^*Y_*(x,y,z)|(f)} \frac{\partial^-\sigma(x,y,z)}{\partial(f,d,n)}.$$

Now, by using the lower differentiability of max-min set-valued maps, we give a characterization of the upper and lower directional derivatives of $m(\cdot)$ which follows from Propositions 1 – 4.

Theorem 5. *Suppose that the set-valued map $Y_*(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m \times \mathbb{R}^k)$ is lower differentiable at the point $x \in \mathbb{R}^n$ in the direction $f \in \mathbb{R}^n$. Then*

$$\frac{\partial^+ m(x)}{\partial f} \leq \inf_{(y,z)\in Y_*(x)} \inf_{(d,n)\in D^* Y_*(x,y,z)|(f)} \frac{\partial^+ \sigma(x,y,z)}{\partial(f,d,n)},$$

$$\frac{\partial^+ m(x)}{\partial f} \geq \inf_{(y,z)\in Y_*(x)} \inf_{(d,n)\in D Y_*(x,y,z)|(f)} \frac{\partial^- \sigma(x,y,z)}{\partial(f,d,n)}.$$

Theorem 6. *Suppose that the set-valued map $Y_*(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m \times \mathbb{R}^k)$ is lower differentiable at the point $x \in \mathbb{R}^n$ in direction $f \in \mathbb{R}^n$ and $(0, G^+(x,f))$ is its lower differential. Let the function $\sigma(\cdot,\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ be directional differentiable at the point (x,y,z) in direction $(f,d,n) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$ for any $(y,z) \in Y_*(x)$ and $(d,n) \in \mathbb{R}^m \times \mathbb{R}^k$. Then*

$$\frac{\partial m(x)}{\partial f} = \inf_{(y,z)\in Y_*(x)} \inf_{(d,n)\in D Y_*(x,y,z)|(f)} \frac{\partial \sigma(x,y,z)}{\partial(f,d,n)}.$$

5. Conclusions

Necessary statements are given for a minimum point and a maximum point of the max-min function. The estimations for the directional lower and upper derivative sets of max-min set-valued map which are used to state a characterization of the directional derivative of max-min function are given. Moreover, by using the lower differentiability of max-min set-valued maps, sufficient condition ensuring the existence of the directional derivative of the max-min function is obtained.

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ZIP BIFURCATION IN AN AMPLE CLASS OF COMPETITIVE SYSTEMS

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Abstract. In this research we study the occurrence of *zip bifurcation* in an ample class of $(n+1)$-dimensional prey–predator system modeling the competition among n species of predators for one species of prey. A similar study was made first by Farkas [3] for a three dimensional prey–predator system where he studied the competition between two species of predators for one species of prey.

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1. INTRODUCTION

The phenomenon of a *zip bifurcation* had been discovered by Farkas [3] in 1984 for a three-dimensional prey-predator system. The model was not a structurally stable one, however, it illustrated the intuitively evident fact that at low values of the carrying capacity K both predators could survive but as K grew, only one of them survived. Recently, theoretical results related to this phenomenon have been generalized first to a four-dimensional system [1, 4] and later to an n-dimensional ordinary differential system [5]. The first model [1] arose in economy and politology, whereas the other one [4, 5] appeared in a specific prey-predator model.

The purpose of this paper is to study the phenomenon of *zip bifurcation* in an ample class of ordinary differential systems modeling the competition among n species of predators for a single prey. The model under consideration is described by the system of ordinary differential equations

$$\begin{aligned} \dot{S} &= \gamma g(S,K)S - \sum_{i=1}^{n} p(S,a_i)x_i, \\ \dot{x_i} &= p(S,a_i)x_i - d_i x_i, \quad i = 1,\dots,n. \end{aligned} \tag{1.1}$$

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The parameters involved in system (1.1) are all non-negative and have the following meaning:

S	quantity of prey
x_i	quantity of predator i
$\gamma g(S,K)$	per capita growth rate of prey in absence of predators*
K	carrying capacity of the environment with respect to the prey
d_i	death rate of the ith predator
$p(S,a_i)$	per capita birth rate of the ith predator, where a_i is a constant

The function $g(S,K)$ possesses the following properties:

$$g \in C^2(]0,\infty[\times]0,\infty[), \qquad g \in C^0([0,\infty[\times]0,\infty[),$$
$$g(0,K) = 1, \quad g'_S(S,K) < 0 < g''_{SK}(S,K), \quad S \geq 0,\ K > 0. \tag{1.2}$$

Condition (1.2) means that the highest specific growth rate of prey is achieved at $S = 0$, $x_1 = 0$ $x_2 = 0$, and it is $\gamma > 0$; the growth rate decreases if the quantity of prey increases, and the rate of decrease in growth rate $g'_S(S,K)$ is negative and an increasing function of the carrying capacity K, i. e., the effect of the increase in prey diminishes with an increase in K.

$$\lim_{K\to\infty} g'_S(S,K) = 0 \tag{1.3}$$

uniformly in $[\delta, S_0[$ for any $0 < \delta < S_0$, and the improper integral $\int_0^{S_0} g'_S(S,K)dS = 0$ is uniformly convergent in $[K_0,\infty[$ for any $K_0 > 0$. Relation (1.3) means that for very high values of K changes in the quantity of prey have a negligible effect on the growth rate. It is easy to see that conditions (1.2)–(1.3) imply that

$$\lim_{K\to\infty} g(S,K) = 1, \quad S \geq 0 \tag{1.4}$$

and

$$(K-S)g(S,K) > 0, \quad S \geq 0,\ K > 0,\ S \neq K. \tag{1.5}$$

Inequality (1.4) means that (in absence of predator) the growth rate of prey $g(S,K) > 0$ if $S < K$ and $g(S,K) < 0$ if $S > K$ therefore, by continuity of function g we have $g(K,K) = 0$.

The function $p(S,a)$ possesses the following properties:

$$p \in C^2(]0,\infty[\times]0,\infty[), \qquad p \in C^0([0,\infty[\times]0,\infty[),$$
$$p(0,a) = 0, \quad p'_S(S,a) > 0, \quad S > 0,\ a > 0 \tag{1.6}$$

Conditions (1.6) mean that the per capita birth rate p of the predators (also called the “predation rate” or the “functional response”) is zero in the absence of prey and is an increasing function of the quantity of prey.

$$p'_S(S,a) < \frac{p(S,a)}{S}, \quad S > 0,\ a > 0. \tag{1.7}$$

Condition (1.7) is a "weak concavity" condition, sometimes called Krasonselskij's condition. If p is a strictly concave function of S (for any $a > 0$), (1.7) is implied with the possible exception of isolated points where it holds with an equality sign.

$$p'_a(S,a) < 0, \quad S > 0,\ a > 0. \tag{1.8}$$

Inequality (1.8) throws light on the role of the parameter a; the birth rate of the predator is a decreasing function of a, i. e., the higher the value of a is the more food is needed to maintain the same birth rate of the specific predator. In the original model of Hsu, Hubbell, and Waltman [7] a is the "half-saturation constant". The conditions imply that $p(S,a)$, S and a are all greater than zero. In the case where p is a bounded function for fixed $a > 0$, $m_i = \sup_{S>0} p(S,a_i)$ is the "maximal birth rate" of the ith predator. Clearly,

$$\lim_{S\to\infty} p(S,a_i) = \begin{cases} m_i & \text{if } p \text{ is bounded,} \\ 0 & \text{if } p \text{ is not bounded.} \end{cases} \tag{1.9}$$

We shall assume that $a_1 > a_2 > \cdots > a_n$, i. e.,

$$p(S,a_i) < p(S,a_{i+1}) \quad \text{for all} \quad S > 0 \tag{1.10}$$

according to condition (1.8). We shall also assume from now on that $d_1 < d_2 < \cdots < d_n$, as a consequence, (1.10) does not imply that the net growth rate of predator $i+1$ also exceeds that of predator i.

Another important characteristic of the respective predator species is the prey *threshold quantity* $S = \lambda_i$, above which their growth rate is positive, i. e., $p(\lambda_i, a_i) = d_i$, $i = 1,2,\ldots,n$. However, we shall assume that $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, i. e., the n species have equal prey thresholds although they achieve this by different means. Thus, our assumption will be that there exists a $\lambda > 0$ such that

$$p(\lambda,a_i) = d_i, \quad i = 1,2,\ldots,n. \tag{1.11}$$

We note that, because of condition (1.6), equation (1.11) has one and only one solution λ if and only if either p or m_i is greater than d_i. The real content of (1.11) is that the n solutions for $i = 1,2,\ldots,n$ coincide.

In the next section we study the equilibrium points for the system and we prove its dissipativeness. In Section 3 we establish the conditions under which zip bifurcation occurs in the model.

2. Equilibrium points

We first show that system (1.1) is dissipative before studying its equilibrium points.

Proposition 1. *Any solution of system* (1.1) *with initial values* $S^0 > 0$, $x_i^0 > 0$, $i = 1,2,\ldots,n$, *is bounded in* $[0,\infty[$.

Proof. We first observe that any solution of (1.1) whose initial value has positive components remains with positive components, as long as the solution exists. We will prove that the solution exists for all time $t \geq 0$ and there exists a bounded J in $\mathbb{R}_+^{n+1}$ which attracts the solutions starting on any bounded set in $\mathbb{R}_+^{n+1}$.

Let $d_0 = \min\{d_1, \ldots, d_n\}$ and

$$V(S, x_1, \ldots, x_n) = S + x_1 + \cdots + x_n.$$

If $z(t) = (S(t), x_1(t), \ldots, x_n(t))$ is a solution of (1.1), then as long as it exists, we have

$$\frac{d}{dt} V(z(t)) = \gamma g(S, K) S - \sum_{i=1}^{n} d_i x_i.$$

Since $\gamma g(S, K) S < -dS + 2\gamma K$, where $0 < d < \gamma$ for all $S \in \mathbb{R}_+$, according to conditions (1.2) and (1.5), we have

$$\frac{d}{dt} V(z(t)) \leq -dS - \sum_{i=1}^{n} d_i x_i + 2\gamma K.$$

Letting $\alpha = \min\{d_0, d\}$, we have

$$\frac{d}{dt} V(z(t)) \leq -\alpha V(z(t)) + 2\gamma K$$

and therefore

$$V(z(t)) \leq V(z(0)) e^{-\alpha t} + \frac{2\gamma K}{\alpha}$$

as long as the solution exists. If B is a bounded set contained in $\mathbb{R}_+^{n+1}$, then there exists $R > 0$ such that $V(z(0)) \leq R$. Let $t_0 = \frac{1}{\alpha} \log \frac{2\gamma K}{\alpha R}$ and $z(0) \in B$. For $t \geq t_0$, we have

$$V(z(t)) \leq R \frac{2\gamma K}{\alpha R} + \frac{2\gamma K}{\alpha} \leq \frac{4\gamma K}{\alpha}.$$

This implies that any solution is defined for $t \geq 0$ and the compact set

$$J = \Big\{ (S, x_1, \ldots, x_n) : S \geq 0, x_1 \geq 0, \ldots, x_n \geq 0, \\ \text{and } S + x_1 + \cdots + x_n \leq \frac{\gamma K (1 + d_0)^2}{2\alpha} \Big\}$$

attracts all bounded sets B. Therefore, the system is dissipative and its global attractor is contained in J. $\square$

Let us now consoder the equilibrium points of (1.1), that is, the solutions of the system

$$\begin{aligned} \gamma g(S,K)S-\sum_{i=1}^{n} p(S,a_i)x_i &= 0, \\ \big(p(S,a_i)-d_i\big)x_i &= 0, \quad i=1,2,\ldots,n. \end{aligned} \tag{2.1}$$

For $i=1,\ldots,n$, let λ_i be the prey *quantity threshold* for species i. Then, apart from the obvious two solutions $(S,x_1,\ldots,x_n)=(0,0,\ldots,0)$ and $(S,x_1,\ldots,x_n)=(K,0,\ldots 0)$, equation (2.1) has biologically interesting solutions only if $m_i>d_i$ and $\lambda_1=\cdots=\lambda_n$. Henceforth, we assume that $m_i>d_i$ and $\lambda_1=\cdots=\lambda_n=\lambda$, i. e., the n species have equal prey thresholds although they achieve this by different means. In order to have the predators' survival it is necessary that $m_i>d_i$, $i=1,2,\ldots,n$. Predator i begins to grow only if the right-hand side of the ith equation in system (1.1) is positive, i. e., if

$$(p(S,a_i)-d_i)x_i>0,$$

and it means by condition (1.6) that $S>\lambda$, where $0<\lambda<K$.

The equilibrim points of system (1.1) are the origin $(S,x_1,\ldots,x_n)=(0,0,\ldots,0)$, the point $(S,x_1,\ldots,x_n)=(K,0,\ldots,0)$ and the points of the $(n-1)$-dimensional hyperplane

$$\begin{aligned} H_K=\big\{&(S,x_1,\ldots,x_n)\in\mathbb{R}^{n+1}: S=\lambda,\ x_i\geq 0,\ i=1,\ldots,n, \\ &p(\lambda,a_1)x_1+p(\lambda,a_2)x_2+\cdots+p(\lambda,a_n)x_n=\gamma\lambda g(\lambda,K)\big\}. \end{aligned} \tag{2.2}$$

To study the stability of these equilibrium points, observe that the Jacobian matrix $J(S,x_1,\ldots,x_n)$ of system (1.1) is

$$\begin{pmatrix} \begin{smallmatrix}\gamma g(S,K)+\gamma g_S'(S,K)S\\ -\sum_{i=1}^n p_S'(S,a_i)x_i\end{smallmatrix} & -p(S,a_1) & -p(S,a_2) & \ldots & -p(S,a_n) \\ p_S'(S,a_1)x_1 & p(S,a_1)-d_1 & 0 & \ldots & 0 \\ p_S'(S,a_2)x_2 & 0 & p(S,a_2)-d_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_S'(S,a_n)x_n & 0 & 0 & \ldots & p(S,a_n)-d_n \end{pmatrix}$$

Now, we have

$$J(0,0,\ldots,0)=\begin{pmatrix} \gamma & 0 & 0 & \ldots & 0 \\ 0 & -d_1 & 0 & \ldots & 0 \\ 0 & 0 & -d_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -d_n \end{pmatrix}$$

so $(S,x_1,\dots,x_n)=(0,0,\dots,0)$ is unstable, with a n-dimensional stable manifold and a 1-dimensional unstable manifold. On the other hand,

$$J(K,0,\dots,0) = \begin{pmatrix} \gamma K g'_S(K,K) & p(K,a_2) & -p(K,a_2) & \dots & -p(K,a_n) \\ 0 & p(K,a_1)-d_1 & 0 & \dots & 0 \\ 0 & 0 & p(K,a_2)-d_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & p(K,a_n)-d_n \end{pmatrix}$$

by conditions (1.6) and (1.11), $(S,x_1,\dots,x_n)=(K,0,\dots,0)$ is asymptotically stable if $K<\lambda$ and unstable if $K>\lambda$ with a 1-dimensional stable manifold and a n-dimensional unstable manifold. It is well-know (see [6] and [2]) that

$$K>\lambda \tag{2.3}$$

is a necessary condition for the survival of each predator. Therefore, (2.3) will also be assumed from now on. Note that, by (1.4), if $K<\lambda$, then H_K is empty, and if $K=\lambda$, then its only point is the origin. In the next section we fix our attention on the study of stability of the equilibrium points belonging to H_K.

3. Coexistence and extinction by zip bifurcation

In this section we shall study the stability of the hyperplane H_K. The elements of H_K are denoted by $(\lambda,\xi_1,\dots,\xi_n)$, i. e., $(\lambda,\xi_1,\dots,\xi_n)\in H_K$.

Evaluating the Jacobian matrix of (1.1) at an arbitrary point of H_K we get that $J=J(\lambda,\xi_1,\dots,\xi_n)$ is given by the formula

$$\begin{pmatrix} \begin{smallmatrix}\gamma g(\lambda,K)+\gamma\lambda g'_S(\lambda,K) \\ -\sum_{i=1}^n p'_S(\lambda,a_i)\xi_i\end{smallmatrix} & -p(\lambda,a_1) & -p(\lambda,a_2) & \dots & -p(\lambda,a_n) \\ p'_S(\lambda,a_1)\xi_1 & 0 & 0 & \dots & 0 \\ p'_S(\lambda,a_2)\xi_2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p'_S(\lambda,a_n)\xi_n & 0 & 0 & \dots & 0 \end{pmatrix}$$

since by condition (1.11) we have $p(\lambda,a_i)-d_i=0$, $i=1,2,\dots,n$. So, the characteristic polynomial is

$$P(\mu)=\left(\mu-\gamma g(\lambda,K)-\gamma\lambda g'_S(\lambda,K)+\sum_{i=1}^n p'_S(\lambda,a_i)\xi_i\right)\mu^n \\ +p(\lambda,a_1)\Delta_{12}+p(\lambda,a_2)\Delta_{13}+\dots+p(\lambda,a_n)\Delta_{1n+1}\,.$$

where $\Delta_{1j} = (-1)^{j+1} \det\big(\mu - J\big)_{1j}$, and $\det(\mu - J)_{1j}$ is the determinant of the submatrix $(\mu - J)_{1j}$ obtained from $\mu - J$ by eliminating line 1 and column j, $j = 2,3,\ldots,n+1$, i. e., of the matrix

$$\begin{array}{cccccccccc} & & & & & \mathrm{col}(j-1) & \mathrm{col}(j+1) & & & \end{array}$$
$$\begin{pmatrix} -p'_S(\lambda,a_1)\xi_1 & \mu & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ -p'_S(\lambda,a_2)\xi_2 & 0 & \mu & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ -p'_S(\lambda,a_3)\xi_3 & 0 & 0 & \mu & \ldots & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ -p'_S(\lambda,a_{n-1})\xi_{n-1} & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & \mu & 0 \\ -p'_S(\lambda,a_n)\xi_n & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & \mu \end{pmatrix}.$$

So we have

$$\begin{aligned} \Delta_{12} &= (-1)^3\Big((-1)^2\big(-p'_S(\lambda,a_1)\big)\mu^{n-1}\Big) = \xi_1 p'_S(\lambda,a_1)\mu^{n-1}, \\ \Delta_{13} &= \xi_2 p'_S(\lambda,a_2)\mu^{n-1}, \\ &\vdots \\ \Delta_{1n} &= \xi_n p'_S(\lambda,a_n)\mu^{n-1}. \end{aligned}$$

Thus, the characteristic polynomial associated to $J(\lambda,\xi_1,\ldots,\xi_n)$ is

$$\begin{aligned} P(\mu) =& \Big(\mu - \gamma g(\lambda,K) - \gamma\lambda g'_S(\lambda,K) + \sum_{i=1}^{n} p'_S(\lambda,a_i)\xi_i\Big)\mu^n \\ &+ \xi_1 p(\lambda,a_1)p'_S(\lambda,a_1)\mu^{n-1} + \xi_2 p(\lambda,a_2)p'_S(\lambda,a_2)\mu^{n-1} \\ &+ \cdots + \xi_n p(\lambda,a_n)p'_S(\lambda,a_n)\mu^{n-1} \\ =& \mu^{n-1}\Big[\mu^2 + \Big(\sum_{i=1}^{n} p'_S(\lambda,a_i)\xi_i - \gamma g(\lambda,K) - \gamma\lambda g'_S(\lambda,K)\Big)\mu \\ &+ \sum_{i=1}^{n}\xi_i p(\lambda,a_i)p'_S(\lambda,a_i)\Big]. \end{aligned} \tag{3.1}$$

This means that each equilibrium point in H has 0 as an eigenvalue with multiplicity $n-2$ and two eigenvalues with real part negative if the quadratic polynomial in square brackets is stable, respectively positive if the polynomial is unstable. Now, the quadratic polynomial in square brackets is stable if and only if

$$\sum_{i=1}^{n} p'_S(\lambda,a_i)\xi_i > \gamma g(\lambda,K) + \gamma\lambda g'_S(\lambda,K).$$

We can rewrite the above inequality as follows

$$\gamma\lambda g(\lambda,K)+\gamma\lambda^2 g'_S(\lambda,K)<\sum_{i=1}^{n}\big(\lambda p'_S(\lambda,a_i)-p(\lambda,a_i)\big)\,\xi_i+\sum_{i=1}^{n}p(\lambda,a_i)\xi_i.$$

Now, $(\xi_1,\xi_2,\dots,\xi_n)$ satisfy (2.2), i. e., $\sum_{i=1}^{n}p(\lambda,a_i)\xi_i=\gamma\lambda g(\lambda,K)$, hence we obtain

$$\sum_{i=1}^{n}\big(p(\lambda,a_i)-\lambda p'_S(\lambda,a_i)\big)\,\xi_i<-\gamma\lambda^2 g'_S(\lambda,K). \tag{3.2}$$

By condition (1.7) the left-hand side of (3.2) is positive for all $(\xi_1,\xi_2,\dots,\xi_n)\in H_K$. In view of (1.2) and (1.3) the right-hand side is positive, decreases and tends to zero for $K\to\infty$. Let us consider the hyperplane B_k given by

$$B_K=\Bigg\{(S,x_1,\dots,x_n)\in\mathbb{R}^{n+1}:S=\lambda,\ \xi_i\geq 0,\ i=1,\dots,n,\\ \sum_{i=1}^{n}\big[p(\lambda,a_i)-\lambda p'_S(\lambda,a_i)\big]\xi_i=-\gamma\lambda^2 g'_S(\lambda,K)\Bigg\}. \tag{3.3}$$

Let us denote the ith intersection coordinate of H_K with the coordinate axes by $(0,\dots,0,x_i^{H_K},0,\dots,0)$ and of B_k by $(0,\dots,0,x_i^{B_K},0,\dots,0)$, where

$$\begin{aligned} x_i^{H_K}&=\frac{\gamma\lambda g(\lambda,K)}{p(\lambda,a_i)},\\ x_i^{B_K}&=-\frac{\gamma\lambda^2 g'_S(\lambda,K)}{p(\lambda,a_i)-\lambda p'_S(\lambda,a_i)},\qquad i=1,\dots,n.\end{aligned} \tag{3.4}$$

Observe that by conditions (1.6) and (1.9) for $K=\lambda$ we have

$$x_i^{H_\lambda}=0\quad\text{and}\quad x_i^{B_\lambda}>0\quad\text{for all}\quad i=1,\dots,n. \tag{3.5}$$

On the other hand, $\lim_{K\to\infty}g'_S(S,K)=0$ and $\lim_{K\to\infty}g(S,K)=1$, so by continuity there exist $K=K_i>\lambda$ such that $x_i^{H_{K_i}}=x_i^{B_{K_i}}$, i. e.,

$$\frac{\gamma\lambda g(\lambda,K)}{p(\lambda,a_i)}=-\frac{\gamma\lambda^2 g'_S(\lambda,K)}{p(\lambda,a_i)-\lambda p'_S(\lambda,a_i)},\quad i=1,\dots,n. \tag{3.6}$$

Let us assume that

$$g'_K(S,K)\geq 0,\quad K>\lambda,\ S>0\quad\text{and}\quad p''_{Sa}(S,a)>0\quad S>0,\ a>0. \tag{3.7}$$

Note that from (3.6) we have $\frac{g(\lambda,K)}{p(\lambda,a_i)}+\frac{\lambda g'_S(\lambda,K)}{p(\lambda,a_i)-\lambda p'_S(\lambda,a_i)}=0$ so, define

$$F(a,K)=\frac{g(\lambda,K)}{p(\lambda,a)}+\frac{\lambda g'_S(\lambda,K)}{p(\lambda,a)-\lambda p'_S(\lambda,a)}$$

and observe that

$$\frac{\partial F}{\partial K}(a,K) = \frac{g'_K(\lambda,K)}{p(\lambda,a)} + \frac{\lambda g''_{SK}(\lambda,K)}{p(\lambda,a)-\lambda p'_S(\lambda,a)} > 0 \tag{3.8}$$

in agreement with conditions (1.2) and (3.7). Thus, by the Implicit Function Theorem we can write

$$K = K(x_i,a_i) \quad \text{and} \quad \frac{\partial K}{\partial a_i} = -\frac{\frac{\partial F_i(x_i,a_i,K)}{\partial a_i}}{\frac{\partial F_i(x_i,a_i,K)}{\partial K}},$$

that is,

$$\frac{\partial K}{\partial a_i} = -\frac{g(\lambda,K)p'_a(\lambda,a)}{p(\lambda,a)^2} - \frac{\lambda g'_S(\lambda,K)(p'_a(\lambda,a)-\lambda p''_{Sa}(\lambda,a))}{(p(\lambda,a)-\lambda p'_S(\lambda,a))^2} > 0 \tag{3.9}$$

in accordance with conditions (1.2), (1.5), (1.8), and (3.7). Hence, by properties (3.8) and (3.9), K is an increasing funtion of a. Therefore, we have

$$\lambda < K_n < K_{n-1} < \cdots < K_1 \tag{3.10}$$

furthermore, by conditions (1.3), (1.4), (3.5) and the continuity of $x_i^{H_K}$ and $x_i^{B_K}$, $i = 1,\dots,n$, for $\lambda < K < K_n$ the hyperplane (3.3) is above H_K and reaches H_K at, $x_n^{H_K} = x_n^{B_K}$ when $K = K_n$ (as K is increased both planes are displaced parallel). As K is increased further the hyperplane B_K cuts into H_K, reaches $x_{n-1}^{H_K} = x_{n-1}^{B_K}$ at $K = K_2$ and so on, reaches $x_1^{H_K} = x_1^{B_K}$ at $K = K_1$. After that, B_K cuts H_K such that now H_K is above B_K and condition (3.2) holds with an inverted inequality sign. In the process in that part of H_K which is already "above" B_K condition (3.2) holds with an inverted inequality sign. This means that the equilibria on this part of the plane have a two dimensional unstable manifold which fills a neighbourhood of this part of H_K. The points left behind by the intersection between H_K and B_K become destabilized. Farkas [3] called this phenomenon a *zip bifurcation*. We have thus arrived at

Theorem 1. *Suppose that conditions* (1.2) – (1.9)*,* (1.11)*, and* (3.7) *hold and assume that $a_1 > a_2 > \cdots > a_n$.*

(1) *If $\lambda < K < K_n$, then each equilibrium in H_K is stable in the Liapunov sense and H_K is an attractor of system* (1.1)*;*
(2) *If $K_1 < K < \infty$, then all these equilibria are unstable and H_K is repellor;*
(3) *If K is increased from one extreme to the other one of the interval (K_n, K_1), then the hyperplane intersection of H_K and B_K is traveling through H_K from the vertex on the axis x_n to the vertex x_1 and the equilibria left behind get destabilized. For $K_n < K < K_1$ this hyperplane intersection divides H_K in two parts, "the upper one" is a repellor and "the lower one" is an attractor of the system, i. e., the system undergoes a zip bifurcation.*

Proof. The assertion follows from the calculations carried out above. □

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ON THE ACTION OF THE STEENROD SQUARES ON POLYNOMIAL ALGEBRA

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Abstract. In this work we gather and formulize some useful tools for handling the action of the Steenrod squares on monomials. In particular, we introduce some matrices and call them Sq-matrices, which are sufficient tools in algorithmic calculations with the Steenrod squares on polynomials.

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1. Introduction

In this section we introduce briefly the hit problem for the polynomial algebra $\mathbf{P}(n) = \mathbb{F}_2[x_1, x_2, \ldots, x_n] = \bigoplus_{d\geq 0} \mathbf{P}^d(n)$, viewed as a graded module over the Steenrod algebra $\mathcal{A}$ at prime 2. The grading is by the homogeneous polynomials $\mathbf{P}^d(n)$ of degree d in the variables $x_1, x_2, \ldots, x_n$ of grading 1. We refer to [1, 6] in cohomology operations, to [6, 7] in the Steenrod algebra, and to [3] and the comprehensive reference [8] for the hit problem.

The Steenrod algebra $\mathcal{A}$ is defined to be the graded algebra over the field $\mathbb{F}_2$, generated by the Steenrod squares Sq^k, in grading $k \geq 0$, subject to the Adem relations [3, 8].

From a topological point of view, the Steenrod algebra is the algebra of stable cohomology operations for ordinary cohomology $\mathbf{H}^*$ over $\mathbb{F}_2$. The polynomial algebra $\mathbf{P}(n)$ realizes the cohomology of products of n copies of infinite real projective spaces.

For the present purpose we only need to know that the Steenrod algebra acts by composition of linear operators on $\mathbf{P}(n)$ and the action of the Steenrod squares $Sq^k : \mathbf{P}^d(n) \to \mathbf{P}^{d+k}(n)$ is determined by the following rules [8].

Proposition 1.1. *For homogeneous elements f, g in $\mathbf{P}(n)$ we have*

(i) *Sq^0 is the identity homomorphism;*
(ii) *$Sq^k(f) = f^2$ if $\deg(f) = k$ and $Sq^k(f) = 0$ if $\deg(f) < k$;*

(iii) *The Cartan formula* $Sq^k(fg) = \sum_{0\le r\le k} Sq^r(f)Sq^{k-r}(g)$.

The Cartan formula can be expressed in a more concise form by defining the *total* Steenrod square by $Sq = Sq^0 + Sq^1 + \cdots$. This acts on $\mathbf{P}(n)$ since by property (ii) in the above proposition, only a finite number of Sq^k's can be nonzero on a given polynomial. The Cartan formula then says that $Sq(fg) = Sq(f)Sq(g)$, so Sq is a ring homomorphism $Sq : \mathbf{P}(n) \to \mathbf{P}(n)$. Now, we can use Sq to compute the operator Sq^k via the following lemmas [7].

Lemma 1.2. *If* $\deg(x) = 1$, *then* $Sq^k(x^\alpha) = \binom{\alpha}{k}x^{k+\alpha}$, *for any non-negative integer* α.

Proof. Properties (i), (ii) in Proposition 1.1 give $Sq(x) = x + x^2 = x(1+x)$, so

$$Sq(x^\alpha) = Sq(x)^\alpha = x^\alpha(1+x)^\alpha = \sum_k \binom{\alpha}{k} x^{k+\alpha}$$

and hence $Sq^k(x^\alpha) = \binom{\alpha}{k}x^{k+\alpha}$. □

The following lemma is now immediate.

Lemma 1.3. *If* $\deg(x) = 1$, *then*

$$Sq^k(x^{2^\tau}) = \begin{cases} x^{2^\tau} & \text{if } \; k = 0, \\ 0 & \text{if } \; 0 < k < 2^\tau, \\ x^{2^{\tau+1}} & \text{if } \; k = 2^\tau. \end{cases}$$

Remark. It is clear by Proposition 1.1 that $Sq^k(x^{2^\tau}) = 0$ if $k > 2^\tau$.

2. Some results in the n variables

Our main goal in this section is to extend Lemmas 1.2 and 1.3 and get some useful tools for handling the Steenrod operations. In particular, we show that given $\tau \ge 1$, the $Sq^k(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n})$, where $1 \le k < 2^\tau$ and $1 \le \alpha_i \le 2^\tau$, determine all $Sq^\ell(x_1^{\beta_1}x_2^{\beta_2}\cdots x_n^{\beta_n})$ for any $\beta_i \ge 1$ and any ℓ. On the other hand if we change the places of α_i and α_j in $Sq^k(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n})$, the results will be a permutation of x_i and x_j. So, to handle the Sq's it is sufficient to know only

$$Sq^k(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}) \quad \text{with } \alpha_1 \le \alpha_2 \cdots \le \alpha_n \le 2^\tau \text{ and } 1 \le k < 2^\tau,$$

for some $\tau > 0$.

Throughout the paper we shall adopt the following notations for any positive integer τ.

$$\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}, \tag{1}$$

$$\boldsymbol{x}^{\boldsymbol{m}(2^\tau)} = x_1^{m_1(2^\tau)}x_2^{m_2(2^\tau)}\cdots x_n^{m_n(2^\tau)}, \tag{2}$$

where, α_i and m_i are non negative integers for $1 \leq i \leq n$.

The following lemma is an extension of Lemma 1.3.

Lemma 2.1.

$$Sq^k(\boldsymbol{x}^{2^\tau}) = \begin{cases} \boldsymbol{x}^{2^\tau} & \text{if} \quad k = 0, & \text{(i)} \\ 0 & \text{if} \quad 0 < k < 2^\tau, & \text{(ii)} \\ \boldsymbol{x}^{2^\tau} \sum_{j=1}^{n} x_j^{2^\tau} & \text{if} \quad k = 2^\tau. & \text{(iii)} \end{cases}$$

Proof. Item (i) is trivial. For (ii) and (iii) use induction on n, noting the fact that the one variable case is consistent with Lemma 1.3 in each case. □

We prove even more general results.

Lemma 2.2.

$$Sq^k(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}) = \begin{cases} \boldsymbol{x}^{\boldsymbol{m}(2^\tau)} & \textit{if } k = 0, & \text{(i)} \\ 0 & \textit{if } 0 < k < 2^\tau, & \text{(ii)} \\ 0 & \textit{if } k = 2^\tau \textit{ and all the } m_i \textit{ even}, & \text{(iii)} \\ \boldsymbol{x}^{\boldsymbol{m}(2^\tau)} \sum_{j=1}^{h} x_j^{2^\tau} & \textit{if } k = 2^\tau,\ m_1, m_2, \ldots m_h \textit{ are odd, and the other } m_i\textit{'s are even.} & \text{(iv)} \end{cases}$$

Proof. Item (i) is trivial. To prove (ii), expand notation (2) as

$$\boldsymbol{x}^{\boldsymbol{m}(2^\tau)} = \big(\overbrace{x_1^{2^\tau} \ldots x_1^{2^\tau}}^{m_1 \text{ times}} \overbrace{x_2^{2^\tau} \ldots x_2^{2^\tau}}^{m_2 \text{ times}} \ldots \overbrace{x_n^{2^\tau} \ldots x_n^{2^\tau}}^{m_n \text{ times}}\big).$$

Now, the result follows from Lemma 2.1(ii) taking $n = m_1 + m_2 + \cdots + m_n$.

We prove (iii) by induction on n, the number of variables. In the one variable case, put $m_1 = 2n_1$. Then, using (ii) we get $Sq^k(x_1^{m_1(2^\tau)}) = 0$. Assume now the result for smaller variables than n. Then

$$\begin{aligned} Sq^k(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}) =& Sq^0\big(x_1^{m_1(2^\tau)} x_2^{m_2(2^\tau)} \ldots x_{n-1}^{m_{n-1}(2^\tau)}\big) Sq^{2^\tau}\big(x_n^{m_n(2^\tau)}\big) \\ &+ Sq^{2^\tau}\big(x_1^{m_1(2^\tau)} x_2^{m_2(2^\tau)} \ldots x_{n-1}^{m_{n-1}(2^\tau)}\big) Sq^0\big(x_n^{m_n(2^\tau)}\big) \quad \text{(by (ii))} \\ =& 0, \end{aligned}$$

by assumption.

Finally, we prove (iv) by induction, this time, on h. Let m_1 be odd and the other m_i's even. Then, from (ii) and (iii) it follows that

$$Sq^k(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}) = \boldsymbol{x}^{\boldsymbol{m}(2^\tau)} x_1^{2^\tau}.$$

Now assume the result is true for smaller values than h. Then

$$\begin{aligned}
Sq^k(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}) =& Sq^0\left(x_h^{m_h(2^\tau)}\right)Sq^{2^\tau}\left(\prod_{h\neq i=1}^{n} x_i^{m_i(2^\tau)}\right)\\
&+ Sq^{2^\tau}(x_h^{m_h(2^\tau)})Sq^0\left(\prod_{h\neq i=1}^{n} x_i^{m_i(2^\tau)}\right) \quad \text{(by (ii))}\\
=& x_h^{m_h(2^\tau)}\left(\prod_{h\neq i=1}^{n} x_i^{m_i(2^\tau)}\right)\left(\sum_{j=1}^{h-1} x_j^{2^\tau}\right)\\
&+ x_h^{m_h(2^\tau)+2^\tau}\prod_{h\neq i=1}^{n} x_i^{m_i(2^\tau)} \quad \text{(by assumption)}\\
=& \prod_{i=1}^{n} x_i^{m_i(2^\tau)}\left(\sum_{j=1}^{h-1} x_j^{2^\tau} + x_h^{2^\tau}\right)\\
=& \boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\sum_{j=1}^{h} x_j^{2^\tau}.
\end{aligned}$$

The proof is complete. $\square$

Corollary 2.3. *In our earlier notations* (1) *and* (2)*, where* $\alpha_i, m_i \geq 0$ *and* $\tau > 0$*, assume in addition that* $1 \leq \alpha_i < 2^\tau$ *for* $1 \leq i \leq n$*. Let also* $0 \leq k < 2^\tau$*. Then*

$$Sq^k(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\boldsymbol{x}^{\boldsymbol{\alpha}}) = \boldsymbol{x}^{\boldsymbol{m}(2^\tau)}Sq^k(\boldsymbol{x}^{\boldsymbol{\alpha}}). \tag{3}$$

If, in addition, for $1 \leq h \leq n$*, we assume* $m_1, m_2, \ldots, m_h$ *are odd and other* m_i*'s even, then*

$$Sq^{2^\tau}(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\boldsymbol{x}^{\boldsymbol{\alpha}}) = \boldsymbol{x}^{\boldsymbol{m}(2^\tau)}Sq^{2^\tau}(\boldsymbol{x}^{\boldsymbol{\alpha}}) + \boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\boldsymbol{x}^{\boldsymbol{\alpha}}\sum_{j=1}^{h} x_j^{2^\tau} \tag{4}$$

and

$$Sq^{k+2^\tau}(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\boldsymbol{x}^{\boldsymbol{\alpha}}) = \boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\sum_{j=1}^{h} x_j^{2^\tau}\,Sq^k(\boldsymbol{x}^{\boldsymbol{\alpha}}). \tag{5}$$

In particular, if all m_i*'s are even, then*

$$Sq^{2^\tau}(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\boldsymbol{x}^{\boldsymbol{\alpha}}) = \boldsymbol{x}^{\boldsymbol{m}(2^\tau)}Sq^{2^\tau}(\boldsymbol{x}^{\boldsymbol{\alpha}}) \tag{6}$$

and

$$Sq^{k+2^\tau}(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\boldsymbol{x}^{\boldsymbol{\alpha}}) = 0. \tag{7}$$

Proof. For $k = 0$ the relation (3) is trivial. Let $1 \le k < 2^\tau$. Then by the Cartan formula

$$Sq^k(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\boldsymbol{x}^{\boldsymbol{\alpha}}) = \boldsymbol{x}^{\boldsymbol{m}(2^\tau)} Sq^k(\boldsymbol{x}^{\boldsymbol{\alpha}}) + \sum_{r=1}^{k} Sq^r(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}) Sq^{k-r}(\boldsymbol{x}^{\boldsymbol{\alpha}}).$$

But if $1 \le r \le k$, then $0 < r < 2^\tau$, and hence by Lemma 2.2 (ii), we have $Sq^r(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}) = 0$. This proves the relation (3).

To prove the relation (4), by Lemma 2.2 (ii) and the Cartan formula we have

$$Sq^{2^\tau}(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}\boldsymbol{x}^{\boldsymbol{\alpha}}) = Sq^0(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}) Sq^{2^\tau}(\boldsymbol{x}^{\boldsymbol{\alpha}}) + Sq^{2^\tau}(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}) Sq^0(\boldsymbol{x}^{\boldsymbol{\alpha}}).$$

Now the result follows from Lemma 2.2 (iv).

Finally, to prove the relation (5) expand the left hand side of (5) using the Cartan formula. Now, by Lemma 2.2 (ii) and Lemma 1.1 (ii), we see that all the terms in the expansion are zero except $Sq^{2^\tau}(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}) Sq^k(\boldsymbol{x}^{\boldsymbol{\alpha}})$, which is the right hand side of (5) by Lemma 2.2.

The previous corollary shows clearly the main object stated at the beginning of this section. In the following example we illustrate all the cases in Corollary 2.3, i. e., relations (3)–(7). □

Example 2.4.

$$\begin{aligned}
Sq^3(x^{14}y^{11}) &= Sq^3(x^{3\cdot 2^2}y^{2\cdot 2^2}x^2y^3) = x^{3\cdot 2^2}y^{2\cdot 2^2}Sq^3(x^2y^3)\\
&= x^{12}y^8(x^2y^6 + x^4y^4) = x^{14}y^{14} + x^{16}y^{12},\\
Sq^4(x^{14}y^{11}) &= Sq^{2^2}(x^{3\cdot 2^2}y^{2\cdot 2^2}x^2y^3) = x^{3\cdot 2^2}y^{2\cdot 2^2}(Sq^{2^2}(x^2y^3) + x^2y^3\cdot x^{2^2})\\
&= x^{12}y^8(x^2y^7 + x^4y^5 + x^6y^3) = x^{14}y^{15} + x^{16}y^{13} + x^{18}y^{11},\\
Sq^7(x^{14}y^{11}) &= Sq^{3+2^2}(x^{3\cdot 2^2}y^{2\cdot 2^2}x^2y^3) = x^{3\cdot 2^2}y^{2\cdot 2^2}\cdot x^{2^2}Sq^3(x^2y^3)\\
&= x^{16}y^8(x^2y^6 + x^4y^4) = x^{18}y^{14} + x^{20}y^{12},\\
Sq^4(x^{18}y^{11}) &= Sq^{2^2}(x^{4\cdot 2^2}y^{2\cdot 2^2}x^2y^3) = x^{4\cdot 2^2}y^{2\cdot 2^2}Sq^{2^2}(x^2y^3)\\
&= x^{16}y^8(x^2y^7 + x^4y^5) = x^{18}y^{15} + x^{20}y^{13},\\
Sq^7(x^{18}y^{11}) &= Sq^{3+2^2}(x^{4\cdot 2^2}y^{2\cdot 2^2}x^2y^3) = 0.
\end{aligned}$$

Since every n-variable polynomial over $\mathbb{F}_2$ is the sum of n-variable monomials, the following result is concluded directly from equation (3) in Corollary 2.3.

Corollary 2.5. *Let $\tau \ge 1$ and $0 \le k < 2^\tau$. Then given integers $m_i \ge 0$ for $1 \le i \le n$, and any n-variable polynomial f,*

$$Sq^k(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)} f) = \boldsymbol{x}^{\boldsymbol{m}(2^\tau)} Sq^k(f).$$

In particular, if $\deg(f) < k$, *then*

$$Sq^k(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)} f) = 0.$$

3. Application 1

A homogeneous element f of grading d in a graded module $\mathbf{M}$ over $\mathcal{A}$ is said to be *hit* if it can be written as

$$f = \sum_{k>0} Sq^k(f_k),$$

where the pre-image elements f_k have a degree less than d. The hit problem is to discover criteria for elements of $\mathbf{M}$ to be hit and find minimal generating sets for $\mathbf{M}$ as an $\mathcal{A}$-module. However, we shall not go deeply into the hit problem. The following result is a direct consequence of Corollary 2.5.

Proposition 3.1. *Let* $\tau \geq 1$, *and let* f, g *be* n*-variable polynomials. Then* f *is hit via*

$$f = \sum_{0<k<2^\tau} Sq^k(f_k),$$

if and only if $g = \boldsymbol{x}^{\boldsymbol{m}(2^\tau)} f$ *is hit via*

$$g = \sum_{0<k<2^\tau} Sq^k(\boldsymbol{x}^{\boldsymbol{m}(2^\tau)} f_k).$$

Example 3.2. Consider the hit polynomial

$$f = xy^5 = Sq^1(xy^4) + Sq^2(xy^3).$$

Then, by Proposition 3.1, $g_1 = x^{2^2} y^{2^3} f = x^5 y^{13}$ is hit and

$$x^5 y^{13} = Sq^1(x^5 y^{12}) + Sq^2(x^5 y^{11}).$$

But $g_2 = x^{2^1} f = x^3 y^5$ is not hit. Here $k = 2 = 2^1 = 2^\tau$ and we cannot use the lemma to conclude that $x^3 y^5 = Sq^1(x^3 y^4) + Sq^2(x^3 y^3)$.

Let the monomial $\boldsymbol{x}^{\boldsymbol{\alpha}'}$ be a permutation of the monomial $\boldsymbol{x}^{\boldsymbol{\alpha}}$. Then by equation (3) in Corollary 2.3 we have

$$Sq^k[\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}(\boldsymbol{x}^{\boldsymbol{\alpha}} + \boldsymbol{x}^{\boldsymbol{\alpha}'})] = \boldsymbol{x}^{\boldsymbol{m}(2^\tau)} Sq^k(\boldsymbol{x}^{\boldsymbol{\alpha}} + \boldsymbol{x}^{\boldsymbol{\alpha}'}).$$

Using this fact we can state the same results as Corollary 2.5 and Proposition 3.1 for symmetric polynomials. In particular, in Proposition 3.1 both f and g may be chosen symmetric if we take $\boldsymbol{x}^{\boldsymbol{m}(2^\tau)}$ symmetric n-variable, i. e.,

$$\boldsymbol{x}^{\boldsymbol{m}(2^\tau)} = x_1^{m_1(2^\tau)} x_2^{m_2(2^\tau)} \cdots x_n^{m_n(2^\tau)},$$

where $m_1(2^\tau) = \cdots = m_n(2^\tau) > 0$. If this is the case, f_k will be symmetric as well. For the symmetric hit problem we refer to [4, 5].

4. Application 2

In this section we get some tools for handling the Steenrod squares in the 2-variable case and, since higher variables are determined recursively from two variables, these tools apply for general n.

Proposition 4.1. *Given $\tau \geq 1$, let*

(i) $0 \leq \alpha < 2^\tau$,
(ii) $2^\tau \leq \beta < 2^{\tau+1} - 1$,
(iii) $2^\tau < \alpha + \beta < 2^{\tau+1}$.

Then

$$Sq^{2^\tau}(x^\alpha y^\beta) = x^\alpha y^{\beta+2^\tau}.$$

Proof. Put $\beta = \beta' + 2^\tau$, where $0 \leq \beta' \leq 2^\tau - 1$. By the Cartan formula and Lemma 1.2 we have

$$\begin{aligned} Sq^{2^\tau}(x^\alpha y^\beta) &= Sq^0(x^\alpha) Sq^{2^\tau}(y^\beta) + \sum_{r=1}^{2^\tau} Sq^r(x^\alpha) Sq^{2^\tau - r}(y^\beta) \\ &= x^\alpha y^{\beta+2^\tau} + y^{2^\tau} \sum_{r=1}^{2^\tau} Sq^r(x^\alpha) Sq^{2^\tau - r}(y^{\beta'}). \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{r=1}^{2^\tau} Sq^r(x^\alpha) Sq^{2^\tau - r}(y^{\beta'}) = &\sum_{1 \leq r \leq \alpha} Sq^r(x^\alpha) Sq^{2^\tau - r}(y^{\beta'}) \\ &+ \sum_{\alpha < r \leq 2^\tau} Sq^r(x^\alpha) Sq^{2^\tau - r}(y^{\beta'}). \end{aligned}$$

If $r \leq \alpha$, then $2^\tau - r > \beta'$ and hence $Sq^{2^\tau - r}(y^{\beta'}) = 0$, and if $r > \alpha$, then $Sq^r(x^\alpha) = 0$. Therefore,

$$\sum_{r=1}^{2^\tau} Sq^r(x^\alpha) Sq^{2^\tau - r}(y^{\beta'}) = 0,$$

and the proof is completed. $\square$

Proposition 4.2. *Let $m \geq n + 2$, $n \geq 1$. Let*

(i) $2^{m-2} \leq \alpha \leq 2^{m-2} + 2^{n-1} - 1$,
(ii) $2^{m-2} + 2^{n-1} - 1 \leq \beta \leq 2^{m-2} + 2^n - 2$,
(iii) $\alpha + \beta = 2^{m-1} + 2^n - 2$.

Then

$$Sq^{2^{m-1}}(x^\alpha y^\beta) = \left(x^{\alpha+2^{m-2}} y^{\beta+2^{m-2}}\right).$$

Proof. By the Cartan formula we have

$$Sq^{2^{m-1}}(x^\alpha y^\beta) = \sum_{0\le r<2^{m-1}-\beta} Sq^r(x^\alpha)Sq^{2^{m-1}-r}(y^\beta).$$

If $0 \le r < 2^{m-1}-\beta$, then $2^{m-1}-r > \beta$ and $Sq^{2^{m-1}-r}(y^\beta) = 0$.
If $2^{m-1}-\beta \le r < 2^{m-2}$, then $r > \alpha - 2^{m-2}$. Hence, by Corollary 2.5,

$$Sq^r(x^\alpha) = Sq^r\big(x^{2^{m-2}}x^{\alpha-2^{m-2}}\big) = 0.$$

If $2^{m-2} < r \le \alpha$, then $2^{m-1}-r > \beta - 2^{m-2}$. Once again,

$$Sq^{2^{m-1}-r}(y^\beta) = Sq^{2^{m-1}-r}\big(y^{2^{m-2}}y^{\beta-2^{m-2}}\big) = 0.$$

Finally, if $\alpha < r \le 2^{m-1}$, then $Sq^r(x^\alpha) = 0$. So, by splitting the summation, one sees that only the middle term is non-zero. Thus,

$$Sq^{2^{m-1}}(x^\alpha y^\beta) = Sq^{2^{m-2}}(x^\alpha)Sq^{2^{m-2}}(y^\beta) = \big(x^{\alpha+2^{m-2}}y^{\beta+2^{m-2}}\big).$$

The proof is complete. □

5. Application 3

The subject of this section is to introduce some particular matrices, which we call *Sq-matrices*, and apply them to simplify the action of the Steenrod squares. To do this, we need some preliminaries.

Definition 5.1. Let M be an $m\times n$ matrix. By a *reverse transpose of* M, denoted M^{rt}, we mean an $n\times m$ matrix obtained by reversing the order of rows of M^t, the transpose of M. Therefore,

$$M^{rt}_{ji} = M^t_{(n+1-j)i} = M_{i(n+1-j)},$$

for $1 \le i \le m, 1 \le j \le n$.

The following result follows directly from the definition.

Proposition 5.2. *Given any $m\times n$ matrix M, the product MM^{rt} is a symmetric $m\times m$ matrix.*

The next lemma in [2] describes how binomial coefficients can be computed modulo a prime.

Lemma 5.3. *If p is a prime, then $\binom{m}{n} = \prod_i \binom{m_i}{n_i}$ mod p, where $m = \sum_i m_i p^i$ and $n = \sum_i n_i p^i$, with $0 \le m_i < p$ and $0 \le n_i < p$, are the p-adic expansions of m and n.*

When $n = 2$, for example, the extreme cases of a dyadic expansion consisting of a single 1 or all 1's give

$$Sq\big(x^{2^k}\big) = x^{2^k} + x^{2^{k+1}}$$

and

$$Sq\big(x^{2^k-1}\big) = x^{2^k-1} + x^{2^k} + x^{2^k+1} + \cdots + x^{2^{k+1}-2}$$

for all x with degree 1. More generally, the coefficient of $Sq(x^n)$ can be read from the $(n+1)^{th}$ row of the mod 2 Pascal triangle, a portion of which is shown in Figure 1, where dots denote zeros [2].

Definition 5.4. Let m be a positive integer. For $1 \leq k \leq 2^m - 1$ the Sq*-matrix* $\mathcal{S}_k$ is a $2^m \times (k+1)$ matrix defined by

$$(\mathcal{S}_k)_{ij} = Sq^{j-1}(x^i).$$

In other words, the terms of $Sq(x^n)$ can be read from the nth row of $\mathcal{S}_k$.

Without any confusion, if convenient, in expression of Sq-matrices each non zero entry may be denoted only by the power of x in it. Figure 2 shows the Sq-matrix $\mathcal{S}_{31}$ where, as in Figure 1, dots denote zeros. Note that it contains the Sq-matrices $\mathcal{S}_{15}, \mathcal{S}_7, \mathcal{S}_3$, and $\mathcal{S}_1$ as sub-blocks.

As seen comparing Figures 1 and 2, if we remove the top row of Figure 1 then, up to arrangement, the position of zeros in both figures are the same.

In the following algorithm, given a positive integer m, we construct the Sq-matrix $\mathcal{S}_{2^m-1}$ using Corollary 2.3. For $1 \leq k \leq 2^m - 1$ the Sq-matrix $\mathcal{S}_k$ can be obtained from $\mathcal{S}_{2^m-1}$ by choosing the first k columns.

Algorithm 5.5.

1) Define

$$\mathcal{S}_1 = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix};$$

2) For $t = 1$ to $m - 1$ do
   2.1) For $i, j = 1$ to 2^t do
      2.1.1) Define $\mathcal{T}_{ij} = (\mathcal{S}_{2^t-1})_{ij} + 2^t$;
      2.1.2) Define $\mathcal{U}_{ij} = (\mathcal{S}_{2^t-1})_{ij} + 2^{t+1}$;
      2.1.3) Define $\mathbf{0}$ to be the $2^t \times 2^t$ zero matrix;
   2.2) Define the $2^{t+1} \times 2^{t+1}$ matrix

$$\mathcal{S} = \left(\begin{array}{c|c} \mathcal{S}_{2^t-1} & \mathbf{0} \\ \hline \mathcal{T} & \mathcal{U} \end{array}\right);$$

   2.3) Define $\mathcal{S}_{2^{t+1}-1}$ to be the matrix obtained from $\mathcal{S}$ by substituting $\mathbf{0}_{2^t1}$ by 2^t, and $\mathcal{U}_{2^t1}$ by 0.

```
                  1
                 1 1
                1 . 1
               1 1 1 1
              1 . . . 1
             1 1 . . 1 1
            1 . 1 . 1 . 1
           1 1 1 1 1 1 1 1
          1 . . . . . . . 1
         1 1 . . . . . . 1 1
        1 . 1 . . . . . 1 . 1
       1 1 1 1 . . . . 1 1 1 1
      1 . . . 1 . . . 1 . . . 1
     1 1 . . 1 1 . . 1 1 . . 1 1
    1 . 1 . 1 . 1 . 1 . 1 . 1 . 1
   1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
  1 . . . . . . . . . . . . . . . 1
```

FIGURE 1. Mod 2 Pascal triangle

```
 1 2 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
 2 . 4 . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
 3 4 5 6 . . . . . . . . . . . . . . . . . . . . . . . . . . . .
 4 . . . 8 . . . . . . . . . . . . . . . . . . . . . . . . . . .
 5 6 . . 910 . . . . . . . . . . . . . . . . . . . . . . . . . .
 6 . 8 .10 .12 . . . . . . . . . . . . . . . . . . . . . . . . .
 7 8 91011121314 . . . . . . . . . . . . . . . . . . . . . . . .
 8 . . . . . . .16 . . . . . . . . . . . . . . . . . . . . . . .
 910 . . . . . .1718 . . . . . . . . . . . . . . . . . . . . . .
10 .12 . . . . .18 .20 . . . . . . . . . . . . . . . . . . . . .
11121314 . . . .19202122 . . . . . . . . . . . . . . . . . . . .
12 . . .16 . . .20 . . .24 . . . . . . . . . . . . . . . . . . .
1314 . .1718 . .2122 . .2526 . . . . . . . . . . . . . . . . . .
14 .16 .18 .20 .22 .24 .26 .28 . . . . . . . . . . . . . . . . .
15161718192021222324252627282930 . . . . . . . . . . . . . . . .
16 . . . . . . . . . . . . . . .32 . . . . . . . . . . . . . . .
1718 . . . . . . . . . . . . . .3334 . . . . . . . . . . . . . .
18 .20 . . . . . . . . . . . . .34 .36 . . . . . . . . . . . . .
19202122 . . . . . . . . . . . .35363738 . . . . . . . . . . . .
20 . . .24 . . . . . . . . . . .36 . . .40 . . . . . . . . . . .
2122 . .2526 . . . . . . . . . .3738 . .4142 . . . . . . . . . .
22 .24 .26 .28 . . . . . . . . .38 .4. .42 .44 . . . . . . . . .
2324252627282930 . . . . . . . .3940414243444546 . . . . . . . .
24 . . . . . . .32 . . . . . . .40 . . . . . . .48 . . . . . . .
2526 . . . . . .3334 . . . . . .4142 . . . . . .495. . . . . . .
26 .28 . . . . .34 .36 . . . . .42 .44 . . . . .50 .52 . . . . .
27282930 . . . .35363738 . . . .43444546 . . . .51525354 . . . .
28 . . .32 . . .36 . . .40 . . .44 . . .48 . . .52 . . .56 . . .
2930 . .3334 . .3738 . .4142 . .4546 . .4950 . .5354 . .5758 . .
30 .32 .34 .36 .38 .40 .42 .44 .46 .48 .50 .52 .54 .56 .58 .60 .
3132333435363738394041424344454647484950515253545556575859606162
32 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
```

FIGURE 2. The Sq-matrix $\mathcal{S}_{31}$

The following observation follows directly from Definition 5.4 and the Cartan formula, where in $\mathcal{S}_k^{rt}$ the symbol y is used instead of x.

Proposition 5.6. *Let m be a positive integer and $1 \le k \le 2^m - 1$. Then*

$$(\mathcal{S}_k \mathcal{S}_k^{rt})_{ij} = Sq^k(x^i y^j), \quad 1 \le i, j \le 2^m.$$

For example

$$\mathcal{S}_1 = \begin{pmatrix} x^1 & x^2 \\ x^2 & 0 \end{pmatrix}, \qquad \mathcal{S}_1^{rt} = \begin{pmatrix} y^2 & 0 \\ y^1 & y^2 \end{pmatrix}$$

$$\mathcal{S}_1 \mathcal{S}_1^{rt} = \begin{pmatrix} Sq^1(x^1 y^1) & Sq^1(x^1 y^2) \\ Sq^1(x^2 y^1) & Sq^1(x^2 y^2) \end{pmatrix} = \begin{pmatrix} x^1 y^2 + x^2 y^1 & x^2 y^2 \\ x^2 y^2 & 0 \end{pmatrix}.$$

We extend the argument above. To do this, suppose that X, Y are monomials in positive grading with distinct variables. Let x, y be distinct variables different from those in X, Y. Given $m > 0$ and $1 \le k \le 2^m - 1$, define the $2^m \times (k+1)$ Sq-matrices

$$(\mathcal{X}_k)_{ij} = Sq^{j-1}(X x^i), \qquad (\mathcal{Y}_k)_{ij} = Sq^{j-1}(Y y^i).$$

The following observation is analogous to Proposition 5.6.

Proposition 5.7. *Let m be a positive integer and $1 \le k \le 2^m - 1$. Then*

$$(\mathcal{X}_k \mathcal{Y}_k^{rt})_{ij} = Sq^k(X x^i Y y^j), \quad 1 \le i, j \le 2^m.$$

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TWO IMPROVED ZONE METHODS

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Abstract. The first section gives a very simple introduction to zone functions. Using this function, the next two sections present algorithms for computing integral values and global maxima with error bounds. These algorithms are simpler and more effective than the previous ones published by the author. The last section describes some experience about the C++ and Fortran codes used.

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1. Introduction

Computational methods very often reduce the solving of a complicated task to a set of elementary problems. Since the finding of solution boxes of a system of inequalities, the computation of integral values with error bounds, the approximation of global maxima and others can be reduced to computing solution boxes of one inequality, therefore the creation and the handling by computer code of these solution boxes (as new tools of computational methods) are useful issues. First consider a particular case. Let

$$g : D \subset \mathbb{R}^2 \to \mathbb{R}, \quad g(x_1, x_2) = \sin x_1 + \frac{x_1^2}{\ln x_2}, \text{ where } D = ((0,\pi),(1,10)).$$

Try to give a solution box to the inequality $g(x_1, x_2) > \alpha$ around the point $c = (1,2) \in D$ for $\alpha = 1$. Since $g(c) > 1$ and g is continuous on D, there are solution boxes in D around c. It comes from the structure of g that

$$\sin x_1 > 0.5, \quad \frac{x_1^2}{\ln x_2} > 0.5$$

is a sufficient condition for satisfying $g(x_1, x_2) > 1$. Since $x_1^2 > 0.5$ and $\ln x_2 < 1$ is a sufficient condition for satisfying $x_1^2/\ln x_2 > 0.5$, therefore the three elementary inequalities

$$\sin x_1 > 0.5, \quad x_1^2 > 0.5, \quad \ln x_2 < 1$$

determine the solution box (zone)

$$Z_g(c,\alpha) = \Big(\big(\max(\pi/6, \sqrt{0.5}), \min(5\pi/6, \pi)\big), (1,e)\Big) \approx ((0.71, 2.62), (1, 2.72)).$$

Fortunately, $c \in Z_g(c,\alpha)$ is satisfied, i. e., our casual choices in producing the three elementary inequalities were suitable. Notice that the inequality $g(x_1,x_2) > \alpha$ is equivalent to $g(x_1,x_2) \neq \alpha$ because of the continuity, hence the cases $<, >$ can be discussed at the same time. Now let

$$g : D \subset \mathbb{R}^m \to \mathbb{R}, \text{ where } D = \big((\underline{x}_1, \overline{x}_1), (\underline{x}_2, \overline{x}_2), \ldots, (\underline{x}_m, \overline{x}_m)\big)$$

be such a continuous (multivariate real) function on the open box D that is built of the well-known (univariate real) elementary functions

$$x^a, a^x, \log_a x, |x|, \sin x, \ldots, \arcsin x, \ldots, \sinh x, \ldots, \sinh^{-1} x, \ldots$$

by function operations $+$, $*$, $\circ$ (the last symbol being used for composite functions). The zone function

$$(c,\alpha) \mapsto Z_g(c,\alpha), \text{ where } c \in D, \quad g(c) \neq \alpha,$$

assigns a nonempty open box (interval, zone) $Z_g(c,\alpha) \subset D$ around point c to every pair (c,α), in which the function value $g(x)$ is not equal to α anywhere. If $g(c) < \alpha$, then, because of the continuity of g on $Z_g(c,\alpha)$,it is also true that $x \in Z_g(c,\alpha)$ implies $-\infty < g(x) < \alpha$. Consequently, the zone function Z_g assigns the box $Z_g(c,\alpha)$ of the domain to the interval $(-\infty,\alpha)$ of function values. Similarly, if $g(c) > \alpha$, then the zone function Z_g assigns the box $Z_g(c,\alpha)$ of the domain to the interval (α,∞) of function values. This property was used intensively in finding solution boxes of systems of inequalities [5], in computing integral values [4], and in approximating global minima [3]. Here we note that interval extension functions used in interval methods (see, e. g., [1, 6]) are inverse type functions, they assign intervals of function values to boxes of domain. The handling of zone functions and interval extension functions requires a highly different computational background. The creation of zone functions is based on two facts:

- zone functions to the above mentioned univariate elementary functions can be created easily,
- the creation of a zone function of a multivariate function g can be reduced to the previous case, by rules belonging to the function operations $+$, $*$, $\circ$ (by suitable choices in every step of the reduction).

Naturally a zone function is not given by a formula, but its values (boxes, zones) are defined uniquely by reduction rules. A variety of rules can be seen in [2]. A reduction seen in our particular example can be easily followed by a Maple code, because Maple can recognize the operands of an expression. Unfortunately, the Maple programs are slow, therefore the author uses a simple numerically coded form of g in

C++ and Fortran programs nowadays. The experience with Maple V Release 5, Visual C++ version 6.0 and Lahey Fortran 90 version 4.5 codes used in [2] can be summarized as follows:

- Maple requires only the conventional (convenient) form of g, C++ and Fortran require a numerically coded form of g,
- the computation efforts (the evaluation times) belonging to $Z_g(c,\alpha)$ and $g(c)$ can be characterized by the formula: $\mathrm{effort}\big(Z_g(c,\alpha)\big) \approx 10 * \mathrm{effort}(g(c))$, with respect to all three programming languages,
- the speeds belonging to the evaluation of $Z_g(c,\alpha)$ satisfy the relations: speed (C++) $\approx 200 *$ speed (Maple), speed (Fortran) $\approx 300 *$ speed (Maple).

At the end of this section, let us emphasize some facts about zone functions:

(i) The reduction process on the expression $g(x)$, the determination of the elementary inequalities, uses only the structure of the expression $g(x)$ and supposes only the continuity of g on D.
(ii) The zone $Z_g(c,\alpha)$ is not a symmetrical box around c. Often it has a large volume, although $g(c) \neq \alpha$ is only just satisfied.
(iii) The preparation of a numerically coded form of the expression $g(x)$ is easy (see in [2]), nevertheless the author also has a short Maple program for this work.

2. Algorithm for computing integrals

Let the definite integral

$$\int \cdots \int_V f(x_1,x_2,\ldots,x_{m-1})dx_1dx_2\ldots dx_{m-1}$$

be given, where the $m-1$ dimensional point set V is described by the system of inequalities

$$f_i(x_1,x_2,\ldots,x_{m-1}) \geq 0, \quad i = 1,2,\ldots,n-1,\ m \geq 2,\ n \geq 1,$$

the multivariate real functions $f, f_1, f_2, \ldots, f_{n-1}$ are continuous on the closed box $D \supset V$ and are built from the above univariate real elementary functions by using the usual function operations. Let us assume that we know (rough) lower and upper bounds $\underline{x}_m \leq 0$, $\overline{x}_m \geq 0$ so that

$$\underline{x}_m \leq f(x_1,x_2,\ldots,x_{m-1}) \leq \overline{x}_m, \quad \forall (x_1,x_2,\ldots,x_{m-1}) \in D.$$

According to [4] the computation of the integral value is equivalent (consider the geometrical meaning of definite integrals) to the computation of the volumes of the solution sets of the two systems of inequalities

$$\begin{aligned} f_i(x_1,x_2,\ldots,x_{m-1}) &\geq 0, \quad i = 1,2,\ldots,n-1, \\ f(x_1,x_2,\ldots,x_{m-1}) - x_m &\geq 0, \end{aligned} \tag{2.1}$$

where $(x_1,\dots,x_m) \in I_+ = D \times [0,\overline{x}_m] = \big([\underline{x}_1,\overline{x}_1],\dots,[\underline{x}_{m-1},\overline{x}_{m-1}],[0,\overline{x}_m]\big)$, and

$$\begin{aligned} f_i(x_1,x_2,\dots,x_{m-1}) &\ge 0, \quad i = 1,2,\dots,n-1 \\ -f(x_1,x_2,\dots,x_{m-1}) - x_m &\ge 0, \end{aligned} \tag{2.2}$$

where $(x_1,\dots,x_m) \in I_- = D \times [0,-\underline{x}_m] = \big([\underline{x}_1,\overline{x}_1],\dots,[\underline{x}_{m-1},\overline{x}_{m-1}],[0,-\underline{x}_m]\big)$. The integral value is the difference of the first and second volumes. Since problems (2.1) and (2.2) are very similar, it is sufficient to make an algorithm for the first problem. If $f(x_1,x_2,\dots,x_{m-1}) \ge 0$ on D, then $\underline{x}_m = 0$ and only the system of inequalities (2.1) is used indeed. Our algorithm is made for the case $\underline{x}_m = 0$ and the codes of the algorithm run on the adequate segment once (with the data of (2.1)) or twice (with the data of (2.1) and (2.2)). The notation

$$f_n(x_1,x_2,\dots,x_{m-1},x_m) := f(x_1,x_2,\dots,x_{m-1}) - x_m$$

is also used from now on. The volume of the box $I = I_+$ is known (denoted by $\operatorname{vol}(I)$), furthermore boxes to the solution set S of (2.1) and boxes to the complementary set $\overline{S} = I \setminus S$ can be computed by using zone functions. If the approximating value to the integral value is denoted by $\mu(S)$ and its error bound by ε, then $\mu(S) = 0$, $\varepsilon = \operatorname{vol}(I)$ are the initial values. If the computed box Z fulfils $Z \subset S$, then $\mu(S) = \mu(S) + \operatorname{vol}(Z)$, $\varepsilon = \varepsilon - \operatorname{vol}(Z)$ and if $Z \subset \overline{S}$, then $\varepsilon = \varepsilon - \operatorname{vol}(Z)$. Hence the error bound is improved in every step and the approximating value of the integral is improved in cases of $Z \subset S$. A detailed description of the algorithm is as follows.

(a) Define the first element of a box (interval) sequence $\{I_k\}$ by $I_1 = I$. Let $n_b = 1$, $e_b = 0$, $\mu(S) = 0$, $\varepsilon = \operatorname{vol}(I)$, where n_b, e_b, $\mu(S)$, ε denote the number of boxes in the sequence, the number of the examined boxes, the approximating value of $\operatorname{vol}(S)$, and the error bound, respectively.

(b) Let $e_b = e_b + 1$. Compute $\min f_i(c)$, where $1 \le i \le n$ and c is the centre of I_{n_b}.

  (1) If $\min f_i(c) < 0$, then compute the box (zone) $Z = Z_{f_i^*}(c,0) \subset \overline{S}$, where the function f_i^* is the first among $f_1, f_2, \dots, f_n$, having the smallest function value at c. (The "worst inequality" is used here.) Let $\varepsilon = \varepsilon - \operatorname{vol}(Z)$.

  (2) If $\min f_i(c) \ge 0$, then $Z := I_{n_b}$ and $Z := Z \cap Z_{f_i}(c,0)$, $i = 1,2,\dots,n$. Let $\mu(S) = \mu(S) + \operatorname{vol}(Z)$, $\varepsilon = \varepsilon - \operatorname{vol}(Z)$.

(c) Divide the set $I_{n_b} \setminus Z$ in L boxes (if the set is empty, then $L := 0$). Filter the "unimportant" (too small) boxes by the simple condition $\operatorname{vol}(\text{box}) > \kappa$, where κ is a (small) given value. Place the $L^* \le L$ new boxes into the box sequence as n_bth, (n_b+1)th, $\dots$, (n_b+L^*-1)th elements and let $n_b = n_b + L^* - 1$. If $n_b = 0$, then give $\mu(S)$, ε, e_b and stop, otherwise go to (b).

The function and volume values used several times are computed once and are stored. The partitioning of the set $I_{n_b} \setminus Z$ comes from a special case of the box complementation algorithm (see [4,6]). The box sequence $\{I_k\}$ always contains only the boxes which are waiting for examination (the new boxes are indexed from n_b). There are two essential differences between this algorithm and the algorithm of [4]:

(i) Before step (b), the (first) maximum volume element of the box sequence $\{I_k\}$ is selected (the "most promising box" is used at the beginning of every new loop) in [4].

(ii) The too small boxes are filtered by the simple condition $\mathrm{vol}\,(\mathrm{box}) > \kappa$ here, and by the sophisticated condition $\mathrm{vol}\,(\mathrm{box}) * (E_b - e_b) > \varepsilon$, where E_b is a given upper bound to e_b, in [4].

Now survey the general case ($\underline{x}_m < 0$) and the final results more exactly. If (temporarily) $\mu(S)^+$, ε^+, e_b^+ and $\mu(S)^-$, ε^-, e_b^- denote the output values of the first and second running of the adequate segment, respectively, then $\mu(S)^+ - (\mu(S)^- + \varepsilon^-)$ is lower bound and $(\mu(S)^+ + \varepsilon^+) - \mu(S)^-$ is upper bound for the exact integral value (because $\mu(S)^+$, $\mu(S)^-$ are lower approximations). Hence the mean value $\mu(S) = \mu(S)^+ + \varepsilon^+/2 - (\mu(S)^- + \varepsilon^-/2)$ is an improved value for the integral, the mean value $\varepsilon = \pm(\varepsilon^+/2 + \varepsilon^-/2)$ is an improved value for the error and the value $e_b = e_b^+ + e_b^-$ is the full number of the boxes examined, i. e., these $\mu(S)$, ε, e_b are printed as final results (if $\underline{x}_m = 0$, then $\mu(S)^- = \varepsilon^- = e_b^- = 0$). The improved $\mu(S)$ is obtained by an "error equalization", therefore it often has much less error than ε.* Our numerical examples illustrate very convincingly that simplicity is useful in this integral algorithm, i. e., the selection mentioned in [4] can be too expensive. Table 1 contains the essential data for the following three integrals (where $I_\pm = I_+ \cup I_-$):

$$\int_V \arctan\left(\cos x - 3^x\right) dx$$

with $I_\pm = [[-1,3],[-10,10]]$;

$$\iint_V \frac{2+\cos(x-3)\cos(y+2)}{1+|x|+4|y|}\,dxdy, \qquad \begin{cases} 16 - x^2 - 4y^2 \geq 0, \\ x^2 - y^2 - 4 \geq 0, \end{cases}$$

with $I_\pm = [[-5,5],[-5,5],[0,3]]$;

$$\iiint_V |x|\,\sqrt{y^2+|z|}\,dxdydz, \qquad 4 - 2x^2 - 3y^2 - 4z^2 \geq 0,$$

with $I_\pm = [[-2,2],[0,2],[-2,0],[0,6]]$.

The programs worked on a PC Pentium 4 with a 3.2 GHz processor and running times belong to our Visual C++ version 6.0 code.

The experience is as follows. (1) Naturally, other numerical methods (e. g., the quadrature methods for integrals with one variable) could compute approximating

*See the "likely errors" coming from the comparison of different results at the following integrals.

Definite integral	$\mu(S) \pm \varepsilon$, e_b, time, for $\kappa = 10^{-5}$	$\mu(S) \pm \varepsilon$, e_b, time, for $\kappa = 10^{-6}$	$\mu(S) \pm \varepsilon$, e_b, time, with code of [4]
$\int_V \cdots$	-3.2988±0.0067 2953, 0.02 sec	-3.2987±0.0021 9392, 0.06 sec	-3.2987±0.0021 9813, 0.15 sec
$\iint_V \cdots$	2.4566±0.1706 50052, 0.8 sec	2.4590±0.0778 222029, 4 sec	2.4590±0.0772 228740, 149 sec
$\iiint_V \cdots$	0.6579±0.1329 31486, 0.4 sec	0.6524±0.0703 166939, 2 sec	0.6522±0.0706 169198, 91 sec

TABLE 1. Approximating integral values with error bounds

integral values more quickly, without guaranteed error bounds. The author does not know alternative computer codes for solving the problem stated here.

(2) In the 2nd and 3rd examples $I_\pm = I_+$ and we use rough enough upper bounds for the integrands. In the first example $I_\pm = I_+ \cup I_-$ and we use rough lower and upper bounds (-10 and 10 instead of $-\pi/2$ and $\pi/2$) for the integrand. The experience is that the accuracy essentially does not depend on the circumstances mentioned. For example, if $-\pi/2$ and $\pi/2$ are used in place of -10 and 10, respectively, the result is practically the same.

(3) By the geometrical meaning, the solution set S was covered with rectangles in the 1st example, with cuboids in the 2nd example, with four dimensional (abstract) boxes in the third integral. In general, the computational effort to achieve a good covering is greater and greater as the number of dimensions increases. (The second example requires prominent computational effort because the integrand is "wavy" over a large box of 100 units in volume, and the set V contains two disjunct parts.)

(4) The real strength of our algorithm compared to [4] is illustrated in the 2nd and 3rd columns of Table 1. The results of the second column come by the algorithm of [4] after 2.5, 37.25, 45.5 times more time, respectively. Hence the selection step of [4] could be too expensive indeed.

3. ALGORITHM FOR FINDING GLOBAL MAXIMA

Consider the problem

$$\begin{aligned} &\text{maximize} && f(x_1, x_2, \ldots, x_{m-1}) \\ &\text{subject to} && f_i(x_1, x_2, \ldots, x_{m-1}) \geq 0, \quad i = 1, 2, \ldots, n-1,\ m \geq 2,\ n \geq 1; \\ & && (x_1, \ldots, x_{m-1}) \in D = \big([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_{m-1}, \overline{x}_{m-1}]\big) \subset \mathbb{R}^{m-1}, \end{aligned}$$

where the multivariate real functions f_i, $i = 1, \ldots, n-1$ and f are continuous on the box D and built from the elementary functions mentioned. Let us assume that we know (rough) lower and upper bounds $\underline{x}_m$, $\overline{x}_m$ that

$$\underline{x}_m \le f(x_1, x_2, \ldots, x_{m-1}) \le \overline{x}_m, \quad \forall (x_1, x_2, \ldots, x_{m-1}) \in D.$$

Define the system of inequalities

$$\begin{aligned} f_i(x_1, x_2, \ldots, x_{m-1}) &\ge 0, \quad i = 1, 2, \ldots, n-1 \\ f(x_1, x_2, \ldots, x_{m-1}) - x_m &\ge 0, \end{aligned}$$

where $(x_1, x_2, \ldots, x_m) \in I = \big([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_{m-1}, \overline{x}_{m-1}], [\underline{x}_m, \overline{x}_m]\big)$, or briefly

$$f_i(x_1, x_2, \ldots, x_m) \ge 0, \quad i = 1, 2, \ldots, n, \ (x_1, x_2, \ldots, x_m) \in I, \tag{3.1}$$

where $f_n(x_1, x_2, \ldots, x_m) = f(x_1, x_2, \ldots, x_{m-1}) - x_m$. The solution of our problem is a point of the solution set S of (3.1) with the largest mth coordinate. By a fine scanning of S seen in the integral algorithm, a good approximation value ω can be obtained for the maximum function value (belonging to the set of feasible points) and henceforth it is proved that the value $\omega + \varepsilon$ (where ε is a supposed error bound) is an upper bound to the maximum value. More exactly the aim is to do a fine scanning only around the solution point, therefore a second filter is used besides the simple one seen in the integral algorithm. A detailed description of the algorithm is as follows (it is supposed that the set of feasible points is not empty).

(a) Define the first element of a box (interval) sequence $\{I_k\}$ by $I_1 = I$. Let $n_b = 1$, $e_b = 0$, $\omega = -\infty$, where n_b, e_b, ω denote the number of boxes in the sequence, the number of the examined boxes, and the approximating value to the maximum function value, respectively.

(b) Let $e_b = e_b + 1$. Compute $\min f_i(c)$, where $1 \le i \le n$ and c is the centre of I_{n_b}. If $\min f_i(c) \ge 0$ (c is a feasible point) and $f(c) > \omega$, then $p := c$, $\omega := f(c)$.
 (1) If $\min f_i(c) < 0$, then compute the box (zone) $Z = Z_{f_i^*}(c, 0) \subset \overline{S}$, where the function f_i^* is the first among $f_1, f_2, \ldots, f_n$ having the smallest function value at c. (The "worst inequality" is used here.)
 (2) If $\min f_i(c) \ge 0$, then $Z := I_{n_b}$ and $Z := Z \cap Z_{f_i}(c, 0)$, $i = 1, 2, \ldots, n$.

(c) Divide the set $I_{n_b} \setminus Z$ in L boxes (if the set is empty, then $L := 0$). Filter the 'unimportant' boxes by the conditions $\mathrm{vol}(\text{box}) > \kappa$ (where κ is a small given value) and $\overline{x}_m^* > \omega$ (where $\overline{x}_m^*$ is the maximum value of the mth coordinate in the box). Place the $L^* \le L$ new boxes into the box sequence as n_bth, $(n_b + 1)$th, $\ldots$, $(n_b + L^* - 1)$th elements and let $n_b = n_b + L^* - 1$. If $n_b = 0$ or $e_b \ge E_b$ (E_b is an upper bound to the number of the examined boxes), then give p, ω, e_b and go to (d), otherwise go to (b).

(d) Modify I and κ. Let $I = \big([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_{m-1}, \overline{x}_{m-1}], [\omega + \varepsilon, \overline{x}_m]\big)$ and $\kappa = 0$. Run (a)-(c) with the new initial values again. If the second return to (d) is realized by $n_b = 0$ (in such a case $\omega = -\infty$), then give $\omega + \varepsilon$ as an upper bound to the maximum function value and e_b as the number of the examined boxes in the second running. If the second return to (d) is realized by $e_b \geq E_b$, then give e_b. Stop.

The function and volume values used several times are computed once and are stored. The partitioning of the set $I_{n_b} \setminus Z$ comes from a special case of the box complementation algorithm (see [4, 6]). The box sequence $\{I_k\}$ always contains only the boxes which are waiting for examination (the new boxes are indexed from n_b). There are three essential differences between this algorithm and the algorithm of [3]:

(i) The latter one is a strongly heuristic (with hindsight, also a little confused) method for our problem without using solution boxes of nonlinear systems of inequalities.
(ii) A selection step is used at the beginning of every new loop in [3].
(iii) The algorithm of [3] does not prove upper bound to the maximum function value (this is why there are no comparisons between the two algorithms in the next table).

Global optimum problem	$\omega+\varepsilon$, e_b+e_b, time, for $\kappa=10^{-5}$	$\omega+\varepsilon$, e_b+e_b, time, for $\kappa=10^{-6}$
$\arctan(\cos x - 3^x)$ ...	0.2988+0.01 265+64, 0.002sec	0.2988+0.01 542+64, 0.004sec
$\dfrac{2+\cos(x-3)\cos(y+2)}{1+\lvert x\rvert+4\lvert y\rvert}$ ...	0.6206+0.01 3160+497, 0.05sec	0.6256+0.01 6481+419, 0.11sec
$\lvert x\rvert\sqrt{y^2+\lvert z\rvert}$ ...	0.9222+0.01 3104+578715, 6sec	0.9312+0.01 7643+45165, 0.5sec

TABLE 2. Approximations of global maxima with error bounds

Table 2 contains the essential data for the three problems (the functions of the integral problems are used again):

$$\arctan\left(\cos x - 3^x\right) \to \max$$

with $I = [[-1,3],[-10,10]]$;

$$\frac{2+\cos(x-3)\cos(y+2)}{1+|x|+4|y|} \to \max, \qquad \begin{cases} 16-x^2-4y^2 \geq 0, \\ x^2-y^2-4 \geq 0, \end{cases}$$

with $I = [[-5,5],[-5,5],[0,3]]$;

$$|x|\sqrt{y^2+|z|} \to \max, \qquad 4-2x^2-3y^2-4z^2 \geq 0,$$

with $I = [[-2,2],[0,2],[-2,0],[0,6]]$.

The programs worked on a PC Pentium 4 with a 3.2 GHz processor and running times belong to our Visual C++ version 6.0 code.

The experience is as follows.

(1) Comparing the first member of $e_b + e_b$ to the value e_b seen in the integral problem (265, 3160, 3104 instead of 2953, 50052, 31486 for $\kappa = 10^{-5}$, respectively) the effect of the second filter $\overline{x}_m^* > \omega$ is illustrated. Without this filter the examination in the first step would be similar to the examination of the integral algorithm, the filter $\overline{x}_m^* > \omega$ can utilize the intermediate ω values very well. (A good second filter to integral algorithms could be applied only with difficulty.)

(2) A very sharp or faulty value $\omega + \varepsilon$ as upper bound to the maximum function value belonging to the set of feasible points could cause huge computational efforts. The stop criterion $e_b \geq E_b$ can prevent the needless work. In the above 6 cases the value $E_b = 10^6$ was used, and it allows the entire computation with the sharp upper bound of the 5th problem ($e_b = 578715$ shows the computational effort in proving the solution set is empty in a system of inequalities which very nearly has some solutions).

(3) A faulty result never happened to be obtained because of the effect of rounding errors. The boxes are computed by lower estimates, see the proofs of rules in [2], and our C++ program uses double type for real variables. (The circumstances are the same for the integral algorithm.)

4. Codes for the two algorithms

Our Visual C++ version 6.0 and Lahey Fortran 90 version 4.5 programs have 5 segments for both algorithms. The first 3 segments are the same in the integral and global maximum algorithms. The function (subroutine) segment fval computes the function values from the numerically coded form, i. e., it handles the function

$$\text{fval} : (G, c) \mapsto g(c),$$

where G is a numerically coded form of the multivariate real function g, and c is a point of the domain. For computing the zone function values (boxes) the function (subroutine) segment zone is used, which handles the function

$$\text{zone} : (D, G, c, \alpha) \mapsto Z_g(c, \alpha),$$

where D is the domain box of the multivariate real function g, G is a numerically coded form of g, $c \in D$ and $\alpha \in \mathbb{R}$. To divide the difference of a closed m-dimensional box U and an open m-dimensional box T into closed boxes $B_1, B_2, \ldots, B_L$ which do not contain common interior points, our function (subroutine) segment ints handles the function

$$\text{ints} : (U, T) \mapsto (L, \{B_1, B_2, \ldots, B_L\}).$$

In the integral algorithm, the function (subroutine) segment appr, for computing approximating values to problem (2.1), handles the function

$$\text{appr} : (F, I, \kappa) \mapsto (\mu(S), \varepsilon, e_b),$$

where F is the numerically coded form of

$$(x_1,\ldots,x_m) \mapsto (f_1(x_1,\ldots,x_m),\ldots,f_n(x_1,\ldots,x_m))$$

and I, κ, $\mu(S)$, ε, e_b come from our algorithm description. The main (program) segment produces data for calling the segment appr. These data are m, n, F, $I_\pm = I_+ \cup I_-$ and κ. The segment appr is called only once if $I_\pm = I_+$ and it is called twice if vol $(I_-) \neq 0$. At the global maximum algorithm, the function (subroutine) segment glob, for computing approximating values to problem (3.1), handles the function

$$\text{glob} : (F, I, \kappa, E_b) \mapsto (p, \omega, e_b),$$

where F is the numerically coded form of

$$(x_1,\ldots,x_m) \mapsto (f_1(x_1,\ldots,x_m),\ldots,f_n(x_1,\ldots,x_m))$$

and I, κ, E_b, p, ω, e_b come from our algorithm description. The main (program) segment produces data for calling the segment glob. These data are m, n, F, I, κ and E_b. First the segment glob is called for finding p and ω, and the aim of the second call is to prove that the value $\omega+\varepsilon$ is upper bound to the exact solution. Naturally, these segments have some other (auxiliary) parameters and sometimes their notations cannot follow names in this paper perfectly. Finally, some remarks and comparisons between C++ and Fortran programs are given.

(1) The organization of the C++ and Fortran programs differs from one another essentially in two cases only. The Fortran subroutine segments can handle all output parameters and large arrays in natural mode (as local variables), the C++ function segments use global variables for these data. The lengths of numerical coded forms of multivariate functions are stored as zero indexed elements of the arrays in question in C++ programs and they are stored in an independent array in Fortran programs. Naturally, the first 4 segments of both C++ and Fortran programs are unaltered in the 3 different problems, only the calling segment has to be changed.

(2) Our C++ programs use double type for real variables, and double precision variables are in the Fortran programs. The results of Table 1 and Table 2 (obtained by C++ programs) are the same essentially as those by Fortran programs. The Fortran codes are hardly faster than the C++ ones (only 0-10 percent are the differences in running time). This fact is somewhat surprising, because in the former methods (with selections) or in the zone function tests (with simple real variables) differences of 30-50 percent often appeared.

(3) The author would gladly send the Maple, C++, and Fortran codes mentioned to interested readers in e-mail as an attached file.

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