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INFINITE ORDER SOLUTIONS OF COMPLEX LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper we investigate the growth of solutions of the differential equation $f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0$, where $A_0(z), \ldots, A_{k-1}(z)$ are entire functions with $0 < \sigma(A_0) \leq 1/2$. We will show that if there exists a real constant $\beta < \sigma(A_0)$ and a set $E_\beta \subset (1, +\infty)$ with $\underline{\log \operatorname{dens}}\, E_\beta = 1$, such that for all $r \in E_\beta$, we have $\min_{|z|=r} |A_j(z)| \leq \exp(r^\beta)$ $(j = 1, 2, \ldots, k-1)$, then every solution $f \not\equiv 0$ of the above differential equation is of infinite order with hyper-order $\sigma_2(f) \geq \sigma(A_0)$. The paper extends previous results by the author and Hamani.

1991 *Mathematics Subject Classification:* 34M10, 30D35

Keywords: linear differential equations, growth of entire function, hyper-order

1. INTRODUCTION AND STATEMENT OF MAIN RESULT

For $k \geq 2$ we consider the linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1.1}$$

where $A_0(z), \ldots, A_{k-1}(z)$ are entire functions with $A_0(z) \not\equiv 0$. It is well-known that all solutions of equation (1.1) are entire functions and if some of the coefficients of (1.1) are transcendental, then (1.1) has at least one solution with order $\sigma(f) = +\infty$.

Recently, the growth theory of complex differential equations has been an active research area, and the growth problems of linear differential equations have an important aspect in this area. By defining the hyper-order [9, 11], the growth of infinite order solutions of differential equations has been more precisely estimated (see [3, 4, 9]). Our starting point for this paper is a result by Kwon in [9]:

Theorem A ([9, p. 489])**.** *Let $A(z)$ and $B(z)$ be entire functions where $0 < \sigma(B) < 1/2$, and let there exist a real constant $\beta < \sigma(B)$ and a set $E_\beta \subset [0, +\infty)$ with $\overline{\operatorname{dens}}\, E_\beta = 1$ such that for all $r \in E_\beta$, we have*

$$\min_{|z|=r} |A(z)| \leq \exp\left(r^\beta\right). \tag{1.2}$$

Then every solution $f \not\equiv 0$ *of the linear differential equation*

$$f'' + A(z) f' + B(z) f = 0 \tag{1.3}$$

is of infinite order with hyper-order

$$\sigma_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log T(r,f)}{\log r} \geq \sigma(B). \tag{1.4}$$

Here the order $\sigma(f)$ of an entire function f is defined by

$$\sigma(f) = \overline{\lim_{r\to+\infty}} \frac{\log T(r,f)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log\log M(r, f)}{\log r}, \tag{1.5}$$

where $T(r,f)$ is the Nevanlinna characteristic function of f (see [7]), and $M(r,f) = \max_{|z|=r} |f(z)|$.

This result has been recently generalized to a higher order case (1.1) by the author and Hamani as follows (see [4]):

Theorem B ([4]). *Let* $A_0(z), \ldots, A_{k-1}(z)$ *be entire functions with* $0 < \sigma(A_0) < 1/2$, *and let there exist a real constant* $\beta < \sigma(A_0)$ *and a set* $E_\beta \subset [0,+\infty)$ *with* $\underline{\text{dens}}\, E_\beta = 1$ *such that for all* $r \in E_\beta$, *we have*

$$\min_{|z|=r} |A_j(z)| \leq \exp(r^\beta) \qquad (j = 1,2,\ldots,k-1). \tag{1.6}$$

Then every solution $f \not\equiv 0$ *of* (1.1) *is of infinite order with hyper-order*

$$\sigma_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log T(r,f)}{\log r} \geq \sigma(A_0). \tag{1.7}$$

In this paper, we extend Theorem B by proving:

Theorem 1.1. *Let* $A_0(z), \ldots, A_{k-1}(z)$ *be entire functions where* $0 < \sigma(A_0) \leq 1/2$, *and let there exist a real constant* $\beta < \sigma(A_0)$ *and a set* $E_\beta \subset (1,+\infty)$ *with* $\underline{\log \text{dens}}\, E_\beta = 1$ *such that for all* $r \in E_\beta$, *we have*

$$\min_{|z|=r} |A_j(z)| \leq \exp(r^\beta) \qquad (j = 1,2,\ldots,k-1). \tag{1.8}$$

Then every solution $f \not\equiv 0$ *of* (1.1) *is of infinite order with hyper-order*

$$\sigma_2(f) = \lim_{r\to+\infty} \frac{\log\log T(r,f)}{\log r} \geq \sigma(A_0). \tag{1.9}$$

Thus our contribution is to treat the case $\sigma(A_0) = \frac{1}{2}$. The method used in this paper will be quite different from that in the proof of Theorem B (see [4]). The main ingredient in the proof is Lemma 2.2, based on a special case of a theorem of Drasin and Shea [5, Theorem 8.1].

Remark 1.1. The conclusions of Theorem 1.1 are also held under the hypothesis $0 < \mu(A_0) \le 1/2$, where $\mu(f)$ is the lower order of f defined by

$$\mu(f) = \varliminf_{r\to+\infty} \frac{\log T(r,f)}{\log r} = \varliminf_{r\to+\infty} \frac{\log\log M(r,f)}{\log r}. \tag{1.10}$$

To ease the exposition we treat in detail only Theorem 1.1 and in Section 2 we confine ourselves to highlighting the modifications of our proof necessary to obtain the lower order result.

2. Preliminary lemmas

Our proofs depend mainly upon the following lemmas. Before starting these Lemmas, we recall the concepts of density and logarithmic density of subsets of $(1,+\infty)$. For $E \subset (1,+\infty)$, we define the linear measure of a set E by $\mathrm{m}(E) = \int_1^{+\infty} \chi_E(t)\,dt$, where χ_E is the characteristic function of E, and define the logarithmic measure of a set E by $\mathrm{lm}(E) = \int_1^{+\infty} \frac{\chi_E(t)}{t}\,dt$. The upper density and the upper logarithmic density of E are defined by the formulae

$$\overline{\mathrm{dens}}\, E = \varlimsup_{r\to+\infty} \frac{\mathrm{m}(E\cap[1,r])}{r-1}, \qquad \overline{\log\mathrm{dens}}\, E = \varlimsup_{r\to+\infty} \frac{\mathrm{lm}(E\cap[1,r])}{\log r}. \tag{2.1}$$

The lower density and the lower logarithmic density, $\underline{\mathrm{dens}}\, E$ and $\underline{\log\mathrm{dens}}\, E$, are defined similarly with the limit superior replaced by the limit inferior. It is easy to verify [10, p. 121] that

$$0 \le \underline{\mathrm{dens}}\, E \le \underline{\log\mathrm{dens}}\, E \le \overline{\log\mathrm{dens}} E \le \overline{\mathrm{dens}}\, E \le 1 \tag{2.2}$$

for any $E \subset (1,+\infty)$.

For entire f, let $L(r,f) = \min_{|z|=r} |f(z)|$. The method of Denjoy–Kjellberg [8, pp. 193–196] shows for entire f with $f(0)=1$ and for $0<\lambda<1$ that there exist positive constants $k_1(\lambda)$ and $k_2(\lambda)$ such that

$$\begin{aligned}\int_r^R (\log L(t,f) - (\cos\pi\lambda)\log M(t,f))\,\frac{dt}{t^{1+\lambda}}& \\ \ge k_1(\lambda)\frac{\log M(r,f)}{r^\lambda} - k_2(\lambda)\frac{\log M(4R,f)}{R^\lambda}&\end{aligned} \tag{2.3}$$

valid for $0<r<R$.

If $\mu(f) < \sigma(f) \le 1/2$, let β and α satisfy $\mu(f) < \beta < \alpha < \lambda < \sigma(f) \le 1/2$. Let us choose $r_m \to +\infty$ with $R_m > r_m$ satisfying $r_m^\lambda = o(\log M(r_m,f))$ and $\log M(4R_m,f) = o\big(R_m^\lambda\big)$ to conclude that

$$\int_{r_m}^{R_m} \frac{\log L(t,f)}{t^{1+\lambda}}\,dt \to \infty$$

as $m \to \infty$. Thus there exists $s_m \to \infty$ with

$$\log L(s_m, f) > s_m^{\alpha}. \tag{2.4}$$

If $\mu(f) = \sigma(f) < 1/2$ and $\beta < \sigma(f)$, the classical $\cos \pi\sigma$ theorem (contained in (2.3)) yields $s_m \to \infty$ satisfying (2.4). In both of the above cases results of Barry (see [1, p. 294] and [2, Theorem 4]) also imply the existence of unbounded s_m satisfying (2.4).

If $\beta < \mu(f) = \sigma(f) = \frac{1}{2}$, then either there exists $s_m \to \infty$ satisfying

$$\log L(s_m, f) > \varepsilon \log M(s_m, f) \tag{2.5}$$

for some $\varepsilon > 0$ and hence also satisfying (2.4), or

$$\log L(r, f) = o(\log M(r, f)). \tag{2.6}$$

Lemma 2.1 ([6])**.** *Let $f(z)$ be a nonconstant entire function, and let $\alpha > 1$ and $\varepsilon > 0$ be given constants. Then the following two statements hold:*

(i) *There exist a constant $c > 0$ and a set $E_1 \subset [0, \infty)$ having a finite linear measure such that for all z satisfying $|z| = r \notin E_1$, we have*

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq c \left[T(\alpha r, f) r^{\varepsilon} \log T(\alpha r, f) \right]^k \quad (k \in \mathbb{N}). \tag{2.7}$$

(ii) *There exist a constant $c > 0$ and a set $E_2 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) \setminus E_2$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that for all z satisfying $\arg z = \psi_0$ and $|z| = r \geq R_0$, we have*

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq c \left[T(\alpha r, f) \log T(\alpha r, f) \right]^k \quad (k \in \mathbb{N}). \tag{2.8}$$

Lemma 2.2 ([5,7])**.** *Suppose that $f(z)$ is entire with $\sigma(f) \leq 1/2$ and $\rho < \sigma(f)$. Then either there exists $\{r_m\}$ such that $r_m \to \infty$ and*

$$\min_{|z| = r_m} \log |f(z)| > r_m^{\rho}, \tag{2.9}$$

or, if

$$K_r(\rho) = K_r = \left\{ \theta \in [0, 2\pi] : \log \left| f(re^{i\theta}) \right| < r^{\rho} \right\}, \tag{2.10}$$

there exists a set $E(1) \subset [1, +\infty)$ with $\underline{\log \operatorname{dens}} E(1) = 1$ such that for all $r \in E(1)$, K_r satisfies

$$\mathrm{m}(K_r) \to 0 \qquad \textit{as } r_m \to \infty. \tag{2.11}$$

3. Proof and example

Proof of Theorem 1.1. Let $f \not\equiv 0$ be a solution of (1.1). It follows from (1.1) that

$$|A_0(z)| \le \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)| \left|\frac{f'}{f}\right|. \tag{3.1}$$

Let $\beta < \sigma(A_0)$ and suppose that $\beta < \alpha < \sigma(A_0)$ and that there is a set $E_\beta \subset (1, +\infty)$ of lower logarithmic density 1 satisfying (1.8). Set

$$E = \Big\{ z : |z| = r \in E_\beta \\ \text{and } |A_j(z)| = \min_{|z|=r} |A_j(z)| \quad (j = 1, 2, \ldots, k-1) \Big\}. \tag{3.2}$$

Then $\underline{\log \operatorname{dens}} \{|z| : z \in E\} = 1$ and

$$|A_j(z)| \le \exp\left(r^\beta\right) (\, j = 1, 2, \ldots, k-1) \tag{3.3}$$

for all $z \in E$.

The proof is divided into two cases depending on the behavior of the minimum modulus of $A_0(z)$ on $|z| = r$ by Lemma 2.2. First, we assume that there exists r_m such that $r_m \to \infty$ and

$$\log\left|A_0\left(r_m e^{i\theta}\right)\right| > r_m^\alpha \tag{3.4}$$

for all $\theta \in [0, 2\pi]$, and for any positive real number $\alpha < \sigma(A_0)$. Furthermore, by Lemma 2.1 (ii), there exists $\theta_0 \in [\beta_1, \beta_2]$ such that if $z = re^{i\theta_0}$,

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le [T(2r, f)]^{k+1} \qquad (j = 1, \ldots, k) \tag{3.5}$$

for all sufficiently large r. Hence from (3.1), (3.3), (3.4) and (3.5), we have

$$\exp\left(r_m^\alpha\right) \le \left(1 + (k-1)\exp\left(r_m^\beta\right)\right) [T(2r_m, f)]^{k+1}, \tag{3.6}$$

for all sufficiently large r_m. Therefore f has infinite order and

$$\sigma_2(f) = \overline{\lim_{r_m \to +\infty}} \frac{\log\log T(r_m, f)}{\log r_m} \ge \alpha. \tag{3.7}$$

Thus the result of Theorem 1.1 follows since α is arbitrary. Theorem 1.1 is proved in the first case.

Now, we assume that there is a set $E(1)$ with $\underline{\log \operatorname{dens}}\, E(1) = 1$ such that for all $r \in E(1)$, we have

$$\mathrm{m}(K_r) \to 0, \text{ as } r \to \infty, \tag{3.8}$$

where

$$K_r = K_r(\alpha) = \left\{\theta : \log\left|A_0\left(re^{i\theta}\right)\right| < r^\alpha\right\}. \tag{3.9}$$

By Lemma 2.1 (i) and (3.8), we have a set $E_1 \subset E(1)$ with $\underline{\log\operatorname{dens}} E_1 = 1$ such that if $z = re^{i\theta}$

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le r\,[T(2r,f)]^{k+1} \qquad (j = 1,\ldots,k) \tag{3.10}$$

holds for all $r \in E_1$, and for all $\theta \in [0,2\pi]$. Therefore it follows from (3.1), (3.3), (3.9) and (3.10) that

$$\exp(r^{\alpha}) \le \left(1 + (k-1)\exp(r^{\beta})\right) r\,[T(2r,f)]^{k+1}, \tag{3.11}$$

for all $r \in E_1 \cap E$, and for all $\theta \in [0,2\pi] \setminus K_r$. Since $[0,2\pi] \setminus K_r$ is nonempty for all sufficiently large $r \in E_1 \cap E$ by (3.8), we conclude that f has infinite order and

$$\sigma_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log T(r,f)}{\log r} \ge \alpha. \tag{3.12}$$

Thus the result of Theorem 1.1 follows since α is arbitrary. Hence the proof of Theorem 1.1 is now complete. □

Next, we give an example which illustrates Theorem 1.1.

Example. Let $P_1(z),\ldots,P_{k-1}(z)$ be nonconstant polynomials, and let $h_1(z)$, $\ldots, h_{k-1}(z)$ be entire functions satisfying $\sigma(h_j) < \deg P_j$ $(j = 1,\ldots,k-1)$. Then, by Theorem 1.1, every solution $f \not\equiv 0$ of the equation

$$f^{(k)} + h_{k-1}(z)e^{P_{k-1}(z)} f^{(k-1)} + \cdots + h_1(z)e^{P_1(z)} f' + \frac{\sin\sqrt{z}}{\sqrt{z}} f = 0 \tag{3.13}$$

is of infinite order with $\sigma_2(f) \ge \sigma\left(\frac{\sin\sqrt{z}}{\sqrt{z}}\right) = \frac{1}{2}$ since

$$\min_{|z|=r} \left|h_j(z)e^{P_j(z)}\right| \to 0 \quad (j = 1,\ldots,k-1)$$

as $r \to +\infty$.

Remark 3.1. The conclusions of our theorem are also held under the hypothesis $0 < \mu(A_0) \le 1/2$. For $\beta < \mu(A_0)$, we first establish as before that either (2.4) holds for some sequence $s_m \to \infty$ or (2.11) holds for a set $E(1)$ of lower logarithmic density 1. If $\mu(A_0) < 1/2$, we conclude from Kjellberg's lower order extension [8] of the $\cos\pi\sigma$ theorem (essentially (2.3)) that for each $\beta < \mu(A_0)$ there exists $s_m \to \infty$ satisfying (2.4). Thus we suppose $\mu(A_0) = 1/2$. Either (2.5) holds for some $s_m \to \infty$ and hence (2.4) also holds, or alternatively (2.6) holds. By the remarks following Theorem 8.1 in [5, p. 283], (2.6) implies the set K_r defined in (2.10) satisfies (2.11) for all r in some set $E(1)$ of lower logarithmic density 1.

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ALGEBRAIC STRUCTURES DERIVED FROM BCK-ALGEBRAS

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Abstract. Commutative BCK-algebras can be viewed as semilattices whose sections have antitone involutions and it is known that bounded commutative BCK-algebras are equivalent to MV-algebras. In the first part of this paper we assign to an arbitrary BCK-algebra a semilattice-like structure every section of which possesses a certain antitone mapping. The remaining part is devoted to algebras of the MV-language $\{\oplus, \neg, 0\}$ which are defined on bounded BCK-algebras in the same way as MV-algebras.

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Keywords: BCK-algebra, MV-algebra, semilattice, antitone mapping

1.

A *BCK-algebra* is an algebra $\mathcal{A} = (A; \to, 1)$ of type $(2,0)$ satisfying the following quasi-identities:

(I) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,
(II) $x \to ((x \to y) \to y) = 1$,
(III) $x \to x = 1$,
(IV) $x \to 1 = 1$,
(V) $(x \to y = 1 \;\&\; y \to x = 1) \;\Rightarrow\; x = y$.

BCK-algebras were introduced by Y. Imai and K. Iséki [5–7] and form an algebraic semantics for C. A. Meredith's logic.

The relation $\leq$ on A given by

$$x \leq y \quad \Leftrightarrow \quad x \to y = 1 \tag{1}$$

is a partial order on A with 1 as the top element, but the poset $(A; \leq)$ has no particular properties because any poset $(P; \leq)$ with 1 can be made a BCK-algebra $(P; \to, 1)$ by setting $x \to y := 1$ for $x \leq y$, and $x \to y := y$ otherwise.

By a *bounded BCK-algebra* we mean an algebra $\mathcal{A} = (A; \to, 0, 1)$, where $(A; \to, 1)$ is a BCK-algebra with the bottom element 0.

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In every BCK-algebra $(A;\to,1)$, the following hold for all $x,y,z\in A$:

$$x\le y \quad\Rightarrow\quad y\to z\le x\to z, \tag{2}$$

$$x\le y \quad\Rightarrow\quad z\to x\le z\to y, \tag{3}$$

$$x\to(y\to z)=y\to(x\to z), \qquad \text{(exchange)} \tag{4}$$

$$y\le x\to y, \tag{5}$$

$$1\to x=x, \tag{6}$$

$$x\to y\le(z\to x)\to(z\to y), \tag{7}$$

$$((x\to y)\to y)\to y=x\to y. \tag{8}$$

A *commutative BCK-algebra* is a BCK-algebra that satisfies the identity

$$(x\to y)\to y=(y\to x)\to x. \tag{9}$$

In this case, the underlying poset is a join-semilattice in which $x\vee y=(x\to y)\to y$. Commutative BCK-algebras form a variety that is axiomatized by the identities (9), (4), (III) and (6).

We have proved in [1,2] that commutative BCK-algebras (named here *weak implication algebras* and defined in a slightly different way) can be characterized as join-semilattices whose sections (= principal order filters) posses antitone involutions.

Recall that by a *semilattice with sectionally antitone involutions* we mean a structure $\mathcal{S}=(S;\vee,(^a)_{a\in S},1)$, where $(S;\vee)$ is a join-semilattice with the greatest element 1, and for every $a\in S$, the mapping $x\mapsto x^a$ is an antitone involution on the section $[a,1]=\{x\in S: a\le x\}$. If $\mathcal{A}=(A;\to,1)$ is a commutative BCK-algebra, then $\mathcal{S}(\mathcal{A})=(A;\vee,(^a)_{a\in A},1)$, where $x\vee y=(x\to y)\to y$ and $x^a=x\to a$ ($x\in[a,1]$), is a semilattice with sectionally antitone involutions in which we have $x\to y=(x\vee y)^y$. On the other hand, given a structure $\mathcal{S}=(S;\vee,(^a)_{a\in S},1)$, we define a new algebra $\mathcal{A}(\mathcal{S})=(S;\to,1)$ via $x\to y=(x\vee y)^y$. Then $\mathcal{A}(\mathcal{S})$ is a BCK-algebra if and only if it satisfies identity (4), and if this is the case, then $\mathcal{A}(\mathcal{S})$ is a commutative BCK-algebra.

Our first objective is to give a similar description for general BCK-algebras, i. e. to an arbitrary BCK-algebra we assign a semilattice-like structure the sections of which have certain antitone mappings, and also conversely, we describe the reverse passage from such structures to BCK-algebras.

Theorem 1. *Let $\mathcal{A}=(A;\to,1)$ be a BCK-algebra. Define a binary term operation $\sqcup$ on A by*

$$x\sqcup y=(x\to y)\to y,$$

and for every $a\in A$, a unary operation ^{a} on the section $[a,1]=\{x\in A: a\le x\}$ by

$$x^a=x\to a.$$

Then the structure $\mathcal{S}(\mathcal{A})=(A;\sqcup,(^a)_{a\in A},1)$ satisfies the following quasi-identities:

(i) $x\sqcup x=x$,

(ii) $(x \sqcup y = y \ \&\ y \sqcup x = x) \ \Rightarrow\ x = y$,
(iii) $x \sqcup y = (x \sqcup y) \sqcup y = x \sqcup (x \sqcup y) = y \sqcup (x \sqcup y)$,
(iv) $(x \sqcup z) \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$,
(v) $x \sqcup 1 = 1$,
(vi) $x^x = 1$, $1^x = x$,
(vii) $x \sqcup y = (x \sqcup y)^{yy} = ((x \sqcup y)^y \sqcup y)^y$,
(viii) $(x \sqcup y)^y \sqcup ((x \sqcup z) \sqcup (y \sqcup z))^{y \sqcup z} = ((x \sqcup z) \sqcup (y \sqcup z))^{y \sqcup z}$,
(ix) $((x \sqcup z)^z \sqcup (y \sqcup z))^{y \sqcup z} = ((y \sqcup z)^z \sqcup (x \sqcup z))^{x \sqcup z}$,
(x) $((x \sqcup y) \sqcup x)^x = (x \sqcup y)^x$.

Proof. First note that $x \sqcup y \in [y, 1]$ by (5), hence using (8) we have

$$(x \sqcup y)^y = ((x \to y) \to y) \to y = x \to y. \tag{10}$$

Further

$$x \leq y \quad \Leftrightarrow \quad x \to y = 1 \quad \Leftrightarrow \quad x \sqcup y = y. \tag{11}$$

Indeed, $x \to y = 1$ implies $x \sqcup y = (x \to y) \to y = 1 \to y = y$, and conversely, if $y = x \sqcup y = (x \to y) \to y$, then $1 = y \to y = ((x \to y) \to y) \to y = x \to y$.

Now, we can verify the properties (i)–(x) by direct computations:

(i) $x \sqcup x = (x \to x) \to x = 1 \to x = x$.

(ii) If $x \sqcup y = y$ and $y \sqcup x = x$, then $x \to y = 1$ and $y \to x = 1$ by (11), so that $x = y$ by axiom (V).

(iii) We have $(x \sqcup y) \sqcup y = (((x \to y) \to y) \to y) \to y = (x \to y) \to y = x \sqcup y$, and the equalities $x \sqcup (x \sqcup y) = y \sqcup (x \sqcup y) = x \sqcup y$ follow from (11) as $x, y \leq x \sqcup y$.

(iv) Since $x \leq x \sqcup y$ implies $x \sqcup z = (x \to z) \to z \leq ((x \sqcup y) \to z) \to z = (x \sqcup y) \sqcup z$, we have $(x \sqcup z) \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$ by (11).

(v) $x \sqcup 1 = (x \to 1) \to 1 = 1$.

(vi) $x^x = x \to x = 1$ and $1^x = 1 \to x = x$.

(vii) According to (10), $(x \sqcup y)^{yy} = (x \to y) \to y = x \sqcup y$ and $((x \sqcup y)^y \sqcup y)^y = ((x \to y) \sqcup y)^y = (x \to y) \to y = x \sqcup y$.

(viii) Due to (10), we have

$$\begin{aligned} ((x \sqcup z) \sqcup (y \sqcup z))^{y \sqcup z} &= (x \sqcup z) \to (y \sqcup z) \\ &= ((x \to z) \to z) \to ((y \to z) \to z) \\ &= (y \to z) \to (((x \to z) \to z) \to z) \\ &= (y \to z) \to (x \to z), \end{aligned}$$

whence

$$\begin{aligned} (x \sqcup y)^y \sqcup ((x \sqcup z) \sqcup (y \sqcup z))^{y \sqcup z} &= (x \to y) \sqcup ((y \to z) \to (x \to z)) \\ &= (y \to z) \to (x \to z) \\ &= ((x \sqcup z) \sqcup (y \sqcup z))^{y \sqcup z} \end{aligned}$$

by (I) and (11).

(ix) Again, in view of (10),

$$\begin{aligned}((x \sqcup z)^z \sqcup (y \sqcup z))^{y \sqcup z} &= (x \to z) \to (y \sqcup z)\\ &= (x \to z) \to ((y \to z) \to z)\\ &= (y \to z) \to ((x \to z) \to z)\\ &= (y \to z) \to (x \sqcup z)\\ &= ((y \sqcup z)^z \sqcup (x \sqcup z))^{x \sqcup z}.\end{aligned}$$

(x) We have $((x \sqcup y) \sqcup x)^x = (((x \sqcup y) \to x) \to x) \to x = (x \sqcup y) \to x = (x \sqcup y)^x$. □

Lemma 2. *Let* $(A; \sqcup)$ *be a groupoid satisfying the quasi-identities* (i)–(iv) *of Theorem 1. Then the binary relation defined by*

$$x \leq y \quad \Leftrightarrow \quad x \sqcup y = y \tag{12}$$

is a partial order on A *such that, for every* $x, y \in A$, $x \sqcup y$ *is a common upper bound of* x, y.

Proof. By (i) and (ii), $\leq$ is reflexive and antisymmetric. For transitivity, assume that $x \sqcup y = y$ and $y \sqcup z = z$. Then $x \sqcup z = (x \sqcup z) \sqcup z = (x \sqcup z) \sqcup (y \sqcup z) = (x \sqcup z) \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z = y \sqcup z = z$ by (iii) and (iv). Thus $\leq$ is a partial order on A. Moreover, from (iii) we conclude that $x, y \leq x \sqcup y$. □

Therefore, any BCK-algebra induces a semilattice-like structure with a join-like operation $\sqcup$. Another kind of generalizations of join-semilattices was introduced by J. Ježek and R. Quackenbush [8]:

A *directoid* is a groupoid $(A; \sqcup)$ satisfying the identities

(a) $x \sqcup x = x$,
(b) $(x \sqcup y) \sqcup x = x \sqcup y$,
(c) $y \sqcup (x \sqcup y) = x \sqcup y$,
(d) $x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$.

The relation $\leq$ given by (12) is a partial order. The binary operation $\sqcup$ assigns to a pair (x, y) a common upper bound of $\{x, y\}$ in such a way that $x \sqcup y = y \sqcup x = y$ provided $x \leq y$.

Observe that this is the point where directoids differ from our semilattice-like structures since in our case $x \leq y$ does not imply $y \sqcup x = y$.

Lemma 3. *Let* $\mathcal{A} = (A; \to, 1)$ *be a BCK-algebra and* $\sqcup$ *be the binary operation defined in Theorem 1. Then the following conditions are equivalent:*

(a) $(A; \sqcup)$ *is a directoid;*
(b) $\mathcal{A}$ *is a commutative BCK-algebra;*
(c) $(A; \sqcup)$ *is a join-semilattice.*

Proof. (a) $\Rightarrow$ (b). Assume that $(A;\sqcup)$ is a directoid. Remember that $x \leq y$ iff $x \to y = 1$ iff $x \sqcup y = y$. Since $(A;\sqcup)$ is a directoid, $x \leq y$ entails $x \sqcup y = y \sqcup x = y$, so $\mathcal{A}$ satisfies the quasi-identity

$$x \leq y \quad \Rightarrow \quad y = (y \to x) \to x.$$

Hence $x \leq (x \to y) \to y$ yields $(x \to y) \to y = (((x \to y) \to y) \to x) \to x$. But from $y \leq (x \to y) \to y$ it follows $(y \to x) \to x \leq (((x \to y) \to y) \to x) \to x$, and hence $(y \to x) \to x \leq (x \to y) \to y$. The converse inequality is obtained by interchanging x and y, thus $\mathcal{A}$ is a commutative BCK-algebra.

(b) $\Rightarrow$ (c). As we already know, if $\mathcal{A}$ is a commutative BCK-algebra, then $x \sqcup y = (x \to y) \to y$ is the supremum of $\{x, y\}$, hence $(A;\sqcup)$ is a join-semilattice.

(c) $\Rightarrow$ (a). Clearly, every join-semilattice is a directoid. □

Theorem 4. *Let $\mathcal{S} = (S;\sqcup,(^a)_{a\in S},1)$ be a structure—where $\sqcup$ is a binary operation on S and for each $a \in S$, $^a : x \mapsto x^a$ is unary operation on $\{x \in S : a \sqcup x = x\}$, and 1 is a distinguished element of S—satisfying the quasi-identities* (i)–(ix) *from Theorem 1. Define a new binary operation $\to$ on S by*

$$x \to y = (x \sqcup y)^y.$$

Then $\mathcal{A}(\mathcal{S}) = (S;\to,1)$ is a BCK-algebra.

Proof. The definition of $\to$ is correct since $y \sqcup (x \sqcup y) = x \sqcup y$ by (iii). Furthermore, we note that

$$x \sqcup y = y \quad \Leftrightarrow \quad x \to y = 1. \tag{13}$$

Indeed, if $x \sqcup y = y$, then $x \to y = (x \sqcup y)^y = y^y = 1$, and conversely, $1 = x \to y = (x \sqcup y)^y$ implies $y = 1^y = (x \sqcup y)^{yy} = x \sqcup y$.

Now, we verify the axioms of BCK-algebras:

(I) By (viii) we have

$$\begin{aligned}(x \to y) \sqcup ((x \sqcup z) \to (y \sqcup z)) &= (x \sqcup y)^y \sqcup ((x \sqcup z) \sqcup (y \sqcup z))^{y\sqcup z} \\ &= ((x \sqcup z) \sqcup (y \sqcup z))^{y\sqcup z} \\ &= (x \sqcup z) \to (y \sqcup z),\end{aligned}$$

so $(x \to y) \to ((x \sqcup z) \to (y \sqcup z)) = 1$. Further, using (vii) and (ix), we obtain

$$\begin{aligned}(x \to z) \to ((y \to z) \to z) &= (x \to z) \to ((y \sqcup z)^z \sqcup z)^z \\ &= (x \to z) \to (y \sqcup z) \\ &= ((x \sqcup z)^z \sqcup (y \sqcup z))^{y\sqcup z} \\ &= ((y \sqcup z)^z \sqcup (x \sqcup z))^{x\sqcup z} \\ &= (y \to z) \to (x \sqcup z) \\ &= (y \to z) \to ((x \to z) \to z).\end{aligned}$$

When we replace x by $x \to z$, we get

$$\begin{aligned}(x \sqcup z) \to (y \sqcup z) &= ((x \to z) \to z) \to ((y \to z) \to z)\\ &= (y \to z) \to (((x \to z) \to z) \to z)\\ &= (y \to z) \to (x \to z)\end{aligned}$$

since $((x \to z) \to z) \to z = (((x \sqcup z)^z \sqcup z)^z \sqcup z)^z = ((x \sqcup z) \sqcup z)^z = (x \sqcup z)^z = x \to z$. Altogether, we have proved

$$(x \to y) \to ((y \to z) \to (x \to z)) = (x \to y) \to ((x \sqcup z) \to (y \sqcup z)) = 1.$$

(II) We have $(x \to y) \to y = ((x \sqcup y)^y \sqcup y)^y = x \sqcup y$, hence $x \sqcup ((x \to y) \to y) = x \sqcup (x \sqcup y) = x \sqcup y = (x \to y) \to y$ and by (13) we obtain $x \to ((x \to y) \to y) = 1$.

(III) Due to (13), from $x \sqcup x = x$ it follows $x \to x = 1$.

(IV) Analogously, $x \sqcup 1 = 1$ gives $x \to 1 = 1$.

(V) If $x \to y = 1$ and $y \to x = 1$, then $x \sqcup y = y$ and $y \sqcup x = x$ which imply $x = y$. □

Remark 5. Observe that in Theorem 4 we did not employ the identity (x) of Theorem 1. Actually, this identity is useful in order to establish a one-to-one correspondence between BCK-algebras and semilattice-like structures with sectionally antitone mappings:

Theorem 6. *Let $\mathcal{A} = (A; \to, 1)$ be a BCK-algebra and let $\mathcal{S} = (S; \sqcup, (^a)_{a \in S}, 1)$ be an algebra as in Theorem 4 satisfying* (i)–(x) *of Theorem 1. Then $\mathcal{A}(\mathcal{S}(\mathcal{A})) = \mathcal{A}$ and $\mathcal{S}(\mathcal{A}(\mathcal{S})) = \mathcal{S}$.*

Proof. Let $\mathcal{S}(\mathcal{A}) = (A; \sqcup, (^a)_{a \in A}, 1)$ be the structure satisfying (i)–(x) which is assigned to a given BCK-algebra $\mathcal{A}$ by Theorem 1. Then in $\mathcal{A}(\mathcal{S}(\mathcal{A})) = (A; \rightsquigarrow, 1)$ we have $x \rightsquigarrow y = (x \sqcup y)^y = ((x \to y) \to y) \to y = x \to y$, so that $\mathcal{A}(\mathcal{S}(\mathcal{A})) = \mathcal{A}$.

Conversely, let $\mathcal{S} = (S; \sqcup, (^a)_{a \in S}, 1)$ be a structure that satisfies (i)–(x) of Theorem 1, $\mathcal{A}(\mathcal{S}) = (S; \to, 1)$ its corresponding BCK-algebra (cf. Theorem 4) and $\mathcal{S}(\mathcal{A}(\mathcal{S})) = (S; \Cup, (r_a)_{a \in S}, 1)$. Then $x \Cup y = (x \to y) \to y = ((x \sqcup y)^y \sqcup y)^y = x \sqcup y$, and for $x \in [a, 1]$, $r_a(x) = x \to a = (x \sqcup a)^a = ((a \sqcup x) \sqcup a)^a = (a \sqcup x)^a = x^a$ in view of (x), and hence $\mathcal{S}(\mathcal{A}(\mathcal{S})) = \mathcal{S}$. □

Corollary 7. *Let $\mathcal{S} = (S; \sqcup, (^a)_{a \in S}, 1)$ be an algebra satisfying* (i)–(x) *of Theorem 1. Then the relation defined by* (12) *is a partial order on S, 1 is the greatest element of S and for every $x, y \in S$, $x, y \le x \sqcup y$. Moreover, for each $a \in S$, $x \mapsto x^a$ is an antitone mapping on $[a, 1] = \{x \in S : a \le x\}$.*

2.

The MV-algebras were introduced by C. C. Chang [3] as an algebraic counterpart of the Łukasiewicz many-valued propositional logic. Here we use the present simplest definition from [4]:

An MV-algebra is an algebra $\mathcal{M} = (M; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the identities

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(MV2) $x \oplus y = y \oplus x$,
(MV3) $x \oplus 0 = x$,
(MV4) $x \oplus \neg 0 = \neg 0$,
(MV5) $\neg\neg x = x$,
(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

MV-algebras are known to be termwise equivalent to bounded commutative BCK-algebras (see [4]):

(a) Let $\mathcal{M} = (M; \oplus, \neg, 0)$ be an MV-algebra and define $x \to y = \neg x \oplus y$ and $1 = \neg 0$. Then $\mathcal{A}(\mathcal{M}) = (M; \to, 0, 1)$ is a bounded commutative BCK-algebra in which $x \oplus y = (x \to 0) \to y = (y \to 0) \to x$ and $\neg x = x \to 0$.

(b) Let $\mathcal{A} = (A; \to, 0, 1)$ be a bounded commutative BCK-algebra. Define $x \oplus y = (x \to 0) \to y$ and $\neg x = x \to 0$. Then $\mathcal{M}(\mathcal{A}) = (A; \oplus, \neg, 0)$ is an MV-algebra in which $x \to y = \neg x \oplus y$.

In what follows, we are concerned with algebras in the language $\{\oplus, \neg, 0\}$ which arise from bounded (non-commutative) BCK-algebras in the same manner as MV-algebras.

Let $\mathcal{A} = (A; \to, 0, 1)$ be a bounded BCK-algebra and define binary operation $\oplus$ and a unary operation $\neg$ on A by

$$x \oplus y = (x \to 0) \to y, \tag{14}$$

$$\neg x = x \to 0. \tag{15}$$

We refer to $\mathcal{M}(\mathcal{A}) = (A; \oplus, \neg, 0)$ as the *induced algebra* of a BCK-algebra $\mathcal{A}$. We also introduce a supplementary binary operation $\odot$ by

$$x \odot y = (x \to (y \to 0)) \to 0. \tag{16}$$

Lemma 8. *Given a bounded BCK-algebra $\mathcal{A}$, its induced algebra $\mathcal{M}(\mathcal{A})$ satisfies the following identities:*

(1) $0 \oplus x = x$, $x \oplus 0 = \neg\neg x$,
(2) $x \oplus \neg\neg y = y \oplus \neg\neg x$,
(3) $x \oplus 1 = 1 \oplus x = 1$,
(4) $\neg\neg\neg x = \neg x$,
(5) $\neg x \oplus 0 = \neg x$,
(6) $\neg\neg x \oplus y = x \oplus y$,
(7) $x \oplus \neg y = \neg y \oplus \neg\neg x$,
(8) $x \oplus (y \oplus z) = y \oplus (x \oplus z)$.

In addition, we have

(9) $x \odot 1 = 1 \odot x = \neg\neg x$,
(10) $x \odot y = y \odot x = \neg\neg x \odot y = \neg(\neg x \oplus \neg y)$,

(11) $x \odot 0 = 0$,
(12) $\neg x \odot 1 = \neg x$,
(13) $\neg(\neg x \odot \neg y) = \neg\neg(x \oplus \neg\neg y)$.

Proof. (1) $0 \oplus x = (0 \to 0) \to x = 1 \to x = x$ and $x \oplus 0 = (x \to 0) \to 0 = \neg\neg x$.

(2) $x \oplus \neg\neg y = (x \to 0) \to ((y \to 0) \to 0) = (y \to 0) \to ((x \to 0) \to 0) = y \oplus \neg\neg x$.

(3) $x \oplus 1 = (x \to 0) \to 1 = 1$ and $1 \oplus x = (1 \to 0) \to x = 0 \to x = 1$.

(4) $\neg\neg\neg x = ((x \to 0) \to 0) \to 0 = x \to 0 = \neg x$.

(5) This follows from (1) and (4).

(6) $\neg\neg x \oplus y = (((x \to 0) \to 0) \to 0) \to y = (x \to 0) \to y = x \oplus y$.

(7) This is a consequence of (2) and (4).

(8) $x \oplus (y \oplus z) = (x \to 0) \to ((y \to 0) \to z) = (y \to 0) \to ((x \to 0) \to z) = y \oplus (x \oplus z)$.

(9) $x \odot 1 = (x \to (1 \to 0)) \to 0 = (x \to 0) \to 0 = \neg\neg x$ and analogously $1 \odot x = (1 \to (x \to 0)) \to 0 = (x \to 0) \to 0 = \neg\neg x$.

(10) We have $x \odot y = (x \to (y \to 0)) \to 0 = (y \to (x \to 0)) \to 0 = y \odot x$ and $\neg\neg x \odot y = (((x \to 0) \to 0) \to (y \to 0)) \to 0 = (y \to (((x \to 0) \to 0) \to 0)) \to 0 = (y \to (x \to 0)) \to 0 = y \odot x$, and finally $\neg\neg x \odot y = (((x \to 0) \to 0) \to (y \to 0)) \to 0 = \neg(\neg x \oplus \neg y)$.

(11) $x \odot 0 = (x \to (0 \to 0)) \to 0 = (x \to 1) \to 0 = 1 \to 0 = 0$.

(12) This is a consequence of (9) and (4).

(13) By (10) and (7) we have $\neg(\neg x \odot \neg y) = \neg(\neg y \odot \neg x) = \neg\neg(\neg\neg y \oplus \neg\neg x) = \neg\neg(x \oplus \neg\neg y)$. □

Lemma 9. *Let $\mathcal{M}(\mathcal{A})$ be the induced algebra of a BCK-algebra $\mathcal{A}$. Then*

(a) *$x \le y$ implies $x \oplus z \le y \oplus z$ and $z \oplus x \le z \oplus y$, and the same is true for $\odot$,*
(b) *$x \le y \to \neg z$ iff $x \odot y \le \neg z$,*

where $\le$ is the partial order of $\mathcal{A}$ defined by (1).

Proof. (a) From $x \le y$ it follows $x \to 0 \ge y \to 0$ and hence $x \oplus z = (x \to 0) \to z \le (y \to 0) \to z = y \oplus z$. Similarly, $x \le y$ yields $z \oplus x = (z \to 0) \to x \le (z \to 0) \to y = z \oplus y$.

Now, let $x \le y$. Then $\neg x \ge \neg y$ which implies $\neg x \oplus \neg z \ge \neg y \oplus \neg z$ and hence $x \odot z = \neg(\neg x \oplus \neg y) \le \neg(\neg y \oplus \neg z) = y \odot z$.

(b) If $x \le y \to \neg z$, then $x \odot y \le (y \to \neg z) \odot y = ((y \to (z \to 0)) \to (y \to 0)) \to 0 \le ((z \to 0) \to 0) \to 0 = z \to 0 = \neg z$ since $(z \to 0) \to 0 \le (y \to (z \to 0)) \to (y \to 0)$. Conversely, if $x \odot y \le \neg z$, then $y \to \neg z \ge y \to (x \odot y) = y \to ((x \to (y \to 0)) \to 0) = (x \to (y \to 0)) \to (y \to 0) = (y \to (x \to 0)) \to (y \to 0) \ge (x \to 0) \to 0 \ge x$. □

From now on, we will assume that a given bounded BCK-algebra $\mathcal{A}$ satisfies the *law of double negation*

$$x = \neg\neg x. \tag{17}$$

It should be underlined that (17) is not enough for a BCK-algebra $\mathcal{A}$ to be commutative. Indeed, for instance, the following bounded BCK-algebra obeys the law of double negation but it is not commutative as $(a \to b) \to b = b \neq 1 = (b \to a) \to a$:

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	a	1	1
1	0	a	b	1

Recall that a structure $(P;\leq,\cdot,e)$ is a *partially ordered monoid* or briefly a *po-monoid* if $(P;\cdot,e)$ is a monoid, $(P;\leq)$ is a poset and, for all $x,y,z \in P$, $x \leq y$ implies $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$.

A *partially ordered commutative residuated integral monoid*, a *pocrim* for short, is a structure $(P;\leq,\cdot,\to,1)$ such that $(P;\leq,\cdot,1)$ is a commutative po-monoid with the greatest element 1, and for all $x,y,z \in P$,

$$x \cdot y \leq z \quad \Leftrightarrow \quad x \leq y \to z. \tag{18}$$

Lemma 10. *Let $\mathcal{A} = (A;\to,0,1)$ be a bounded BCK-algebra satisfying the law of double negation (17). Then both $(A;\leq,\oplus,0)$ and $(A;\leq,\odot,1)$ are commutative po-monoids. Moreover, $(A;\leq,\odot,\to,1)$ is a pocrim.*

Proof. First, we show that $\oplus$ is commutative:

$$\begin{aligned} x \oplus y &= (x \to 0) \to y \\ &= (x \to 0) \to ((y \to 0) \to 0) \\ &= (y \to 0) \to ((x \to 0) \to 0) \\ &= (y \to 0) \to x \\ &= y \oplus x. \end{aligned}$$

Further, $\oplus$ is associative:

$$\begin{aligned} (x \oplus y) \oplus z &= (((x \to 0) \to y) \to 0) \to z \\ &= (((x \to 0) \to y) \to 0) \to ((z \to 0) \to 0) \\ &= (z \to 0) \to ((((x \to 0) \to y) \to 0) \to 0) \\ &= (z \to 0) \to ((x \to 0) \to y) \\ &= (x \to 0) \to ((z \to 0) \to y) \\ &= x \oplus (z \oplus y) \\ &= x \oplus (y \oplus z). \end{aligned}$$

Commutativity and associativity of the operation $\odot$ is a direct consequence. Now, by Lemma 8 (1), (9) and Lemma 9 (a), $(A;\leq,\oplus,1)$ as well as $(A;\leq,\odot,1)$ is a commutative po-monoid.

In order to prove the latter statement, it suffices to note that by (b) of Lemma 9 we have $x\leq y\to\neg\neg z=y\to z$ if and only if $x\odot y\leq\neg\neg z=z$ verifying (18). $\square$

In the next theorem we characterize "MV-like" algebras arising from bounded BCK-algebras satisfying the law of double negation:

Theorem 11. *Given a bounded BCK-algebra $\mathcal{A}=(A;\to,0,1)$ satisfying the law of double negation* (17)*, the induced algebra $\mathcal{M}(\mathcal{A})=(A;\oplus,\neg,0)$ fulfils the identities* (MV1)–(MV5) *and the axioms*

(A1) $\neg(\neg x\oplus y)\oplus\neg(\neg y\oplus z)\oplus\neg x\oplus z=1$,
(A2) $(\neg x\oplus y=1\ \&\ \neg y\oplus x=1)\ \Rightarrow\ x=y$;

moreover, we have $x\to y=\neg x\oplus y$.

Conversely, let $\mathcal{M}=(M;\oplus,\neg,0)$ be an algebra of type $(2,1,0)$ and put $1=\neg 0$. If $\mathcal{M}$ satisfies (MV1)–(MV5), (A1) *and* (A2)*, then upon defining $x\to y=\neg x\oplus y$, the algebra $\mathcal{A}(\mathcal{M})=(M;\to,0,1)$ is a bounded BCK-algebra satisfying* (17)*. In addition, $x\oplus y=(x\to 0)\to y$ and $\neg x=x\to 0$.*

Proof. Let $\mathcal{A}$ be a bounded BCK-algebra with (17). Due to Lemma 8 and the identity (17) $\mathcal{M}(\mathcal{A})$ satisfies (MV1)–(MV5). Further, $\neg x\oplus y=((x\to 0)\to 0)\to y=x\to y$, and hence (A1) can be written as

$$(x\to y)\to((y\to z)\to(x\to z))=1$$

which is just axiom (I). Similarly, (A2) can be rewritten in the form

$$(x\to y=1\ \&\ y\to x=1)\ \Rightarrow\ x=y$$

which is (V), the fifth axiom of BCK-algebras.

Conversely, let $\mathcal{M}=(M;\oplus,\neg,0)$ be an algebra having the required properties. By (A1) we have $(x\to y)\to((y\to z)\to(x\to z))=\neg(\neg x\oplus y)\oplus(\neg(\neg y\oplus z)\oplus(\neg x\oplus z))=1$ proving (I). When we put $y=z=0$ in (A1), we obtain $x\to x=x\oplus\neg x=\neg\neg x\oplus\neg x=\neg(\neg x\oplus 0)\oplus\neg(\neg 0\oplus 0)\oplus\neg x\oplus 0=1$, i. e., $\mathcal{A}(\mathcal{M})$ fulfils (III). Now, $x\to((x\to y)\to y)=\neg x\oplus\neg(\neg x\oplus y)\oplus y=1$ which is (II). The axiom (IV) is clear: $x\to 1=\neg x\oplus 1=1$. Finally, according to (A2), $x\to y=1$ and $y\to x=1$ imply $x=y$. Altogether, $\mathcal{A}(\mathcal{M})$ is a BCK-algebra. Obviously, it is bounded and obeys (17).

In addition, $\neg x=\neg x\oplus 0=x\to 0$ and $x\oplus y=\neg(\neg x\oplus 0)\oplus y=(x\to 0)\to y$. $\square$

Corollary 12. *Bounded BCK-algebras satisfying the law of double negation are termwise equivalent to the class of all algebras $\mathcal{M}=(M;\oplus,\neg,0)$ that satisfy* (MV1)–(MV5), (A1) *and* (A2).

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SOME NEW INVERSE-TYPE HILBERT-PACHPATTE INEQUALITIES

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Abstract. Inverses of some new inequalities similar to the Hilbert–Pachpatte inequality are established.

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1. Introduction

In recent years several authors [2,3,5–10] have given considerable attention to the Hilbert inequalities and Hilbert type inequalities and their various generalizations. In particular, in 1988, B. G. Pachpatte [6] proved some new inequalities similar to Hilbert's inequality [4, p. 226], The main purpose of this paper is to establish their inverses.

2. Main results

Our main results are given in the following theorems.

Theorem 1. *Let* $0 < p \leq 1, 0 < q \leq 1$ *and* $\{a_m\}$ *and* $\{b_n\}$ *be two nonnegative sequences of real numbers defined for* $m = 1, 2, \ldots, k$ *and* $n = 1, 2, \ldots, r$*, where* k *and* r *are the natural numbers and define* $A_m = \sum_{s=1}^{m} a_s$ *and* $B_n = \sum_{t=1}^{n} b_t$*. Then*

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for $3n-2m>0$

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^pB_n^q}{3n-2m}\geq pqk^{-2}r^3\left(\sum_{m=1}^{k}(k-m+1)\left(a_mA_m^p\right)^{-\frac{11}{3}}\right)^3 \times\left(\sum_{n=1}^{r}(r-n+1)\left(b_nB_n^{q-1}\right)^{-\frac{1}{2}}\right)^{-2}. \quad (1)$$

Proof. By using the following inequality (see [4, p. 39])

$$rx^{r-1}(x-y)\leq x^r-y^r, \qquad 0<r\leq 1,$$

where x and y are positive real numbers, we obtain that

$$\sum_{m=0}^{k-1}pA_{m+1}^{p-1}\left(A_{m+1}-A_m\right)\leq\sum_{m=0}^{k-1}\left(A_{m+1}^p-A_m^p\right),$$

that is

$$A_k^p\geq p\sum_{m=1}^{k}a_mA_m^{p-1}. \quad (2)$$

From (2) and in view of Hölder's inequality [4, p. 24], we have

$$\frac{A_m^pB_n^q}{3n-2m}\geq pq\,(3n-2m)^{-1}\left(\sum_{s=1}^{m}a_sA_s^{p-1}\right)\left(\sum_{t=1}^{n}b_tB_t^{q-1}\right)$$
$$\geq pq(3n-2m)^{-1}n^3m^{-2}\left(\sum_{s=1}^{m}\left(a_sA_s^{p-1}\right)^{\frac{1}{3}}\right)^3 \times\left(\sum_{t=1}^{n}\left(b_tB_t^{q-1}\right)^{-\frac{1}{2}}\right)^{-2}. \quad (3)$$

Using the following inequality [1, p. 15]

$$m^{\frac{1}{h}}n^{\frac{1}{l}}\geq\frac{m}{h}+\frac{n}{l}, \quad (4)$$

where $m>0$, $n>0$, $h^{-1}+l^{-1}=1$, and $h<1$, we easily get that

$$n^3m^{-2}\geq 3n-2m. \quad (5)$$

Taking the sum on both sides of (3) over n from 1 to r first and then the sum over m from 1 to k and in view of inequality (5) and using again Hölder's inequality, we obtain

$$
\begin{aligned}
\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q}{3n-2m} \geq pq &\left(\sum_{m=1}^{k}\left(\sum_{s=1}^{m}\left(a_s A_s^{p-1}\right)^{\frac{1}{3}}\right)^3\right) \\
&\times\left(\sum_{n=1}^{r}\left(\sum_{t=1}^{n}\left(b_t B_t^{q-1}\right)^{-\frac{1}{2}}\right)^{-2}\right) \\
\geq pqk^{-2}r^3 &\left(\sum_{m=1}^{k}\left(\sum_{s=1}^{m}\left(a_s A_s^{p-1}\right)^{\frac{1}{3}}\right)\right)^3\left(\sum_{n=1}^{r}\left(\sum_{t=1}^{n}\left(b_t B_t^{q-1}\right)^{-\frac{1}{2}}\right)\right)^{-2} \\
= pqk^{-2}r^3 &\left(\sum_{s=1}^{k}\left(a_s A_s^{p-1}\right)^{\frac{1}{3}}\left(\sum_{m=s}^{k}1\right)\right)^3\left(\sum_{t=1}^{r}\left(b_t B_t^{q-1}\right)^{-\frac{1}{2}}\left(\sum_{n=t}^{r}1\right)\right)^{-2} \\
= pqk^{-2}r^3 &\left(\sum_{s=1}^{k}(k-s+1)\left(a_s A_s^{p-1}\right)^{\frac{1}{3}}\right)^3 \\
&\times\left(\sum_{t=1}^{r}(r-t+1)\left(b_t B_t^{q-1}\right)^{-\frac{1}{2}}\right)^{-2} \\
= pqk^{-2}r^3 &\left(\sum_{m=1}^{k}(k-m+1)\left(a_m A_m^{p-1}\right)^{\frac{1}{3}}\right)^3 \\
&\times\left(\sum_{n=1}^{r}(r-n+1)\left(b_n B_n^{q-1}\right)^{-\frac{1}{2}}\right)^{-2},
\end{aligned}
$$

which completes the proof. □

Remark 1. Inequality (1) is just an inverse of the following Inequality A, which was proved by Pachpatte [6]:

Inequality A.

$$
\begin{aligned}
\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q}{m+n} \leq \frac{1}{2}pq\,(kr)^{\frac{1}{2}} &\left(\sum_{m=1}^{k}(k-m+1)\left(a_m A_m^{p-1}\right)^2\right)^{\frac{1}{2}} \\
&\times\left(\sum_{n=1}^{r}(r-n+1)\left(b_n B_n^{q-1}\right)^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Theorem 2. *Let $\{a_m\},\{b_n\},A_m,B_n$ be as defined in Theorem 1. Let $\{p_m\}$ and $\{q_n\}$ be two positive sequences for $m=1,2,\dots,k$ and $n=1,2,\dots,r$, where k and r are natural numbers and put $P_m=\sum_{s=1}^m p_s$, $Q_n=\sum_{t=1}^n q_t$. Let ϕ and ψ be two real-valued nonnegative, concave, and supermultiplicative functions** defined on $\mathbb{R}_+$. Then for $3n-2m>0$*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(A_m)\psi(B_n)}{3n-2m}\geq M'(k,r)\left(\sum_{m=1}^{k}\left(p_m\phi\left(\frac{a_m}{p_m}\right)\right)^{\frac{1}{3}}(k-m+1)\right)^{3} \times\left(\sum_{n=1}^{r}\left(q_n\psi\left(\frac{b_n}{q_n}\right)\right)^{-\frac{1}{2}}(r-n+1)\right)^{-2}, \quad (6)$$

where

$$M'(k,r)=\left(\sum_{m=1}^{k}\left(\frac{\phi(P_m)}{P_m}\right)^{-\frac{1}{2}}\right)^{-2}\left(\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\right)^{\frac{1}{3}}\right)^{3}.$$

Proof. From the hypotheses and by Jensen's inequality and Hölder's inequality, we obtain

$$\begin{aligned}\frac{\phi(A_m)\psi(B_n)}{3n-2m}&\geq(3n-2m)^{-1}\\&\quad\times\phi(P_m)\psi(Q_n)\phi\left(\frac{\sum_{s=1}^{m}p_s a_s/p_s}{\sum_{s=1}^{m}p_s}\right)\psi\left(\frac{\sum_{t=1}^{n}q_t b_t/q_t}{\sum_{t=1}^{n}q_t}\right)\\&\geq(3n-2m)^{-1}\frac{\phi(P_m)}{P_m}\frac{\psi(Q_n)}{Q_n}\sum_{s=1}^{m}p_s\phi\left(\frac{a_s}{p_s}\right)\sum_{t=1}^{n}q_t\phi\left(\frac{b_t}{q_t}\right)\\&\geq(3n-2m)^{-1}n^3m^{-2}\frac{\phi(P_m)}{P_m}\frac{\psi(Q_n)}{Q_n}\left(\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^{\frac{1}{3}}\right)^{3}\\&\quad\times\left(\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^{-\frac{1}{2}}\right)^{-2}\end{aligned}$$

---

* f is said to be a supermultiplicative function if $f(xy)\geq f(x)f(y)$ for $x,y\in\mathbb{R}_+:=[0,+\infty)$.

Taking the sum over n from 1 to r first and then the sum over m from 1 to k and using (5), in view of Hölder's inequality, we have

$$\begin{aligned}
\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(A_m)\psi(B_n)}{3n-2m} &\geq \sum_{m=1}^{k}\left(\frac{\phi(P_m)}{P_m}\left(\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^{\frac{1}{3}}\right)^{3}\right) \\
&\quad \times\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\left(\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^{-\frac{1}{2}}\right)^{-2}\right) \\
&\geq \left(\sum_{m=1}^{k}\left(\frac{\phi((P_m))}{P_m}\right)^{-\frac{1}{2}}\right)^{-2}\left(\sum_{m=1}^{k}\sum_{s=1}^{m}p_s\phi\left(\frac{a_s}{p_s}\right)^{\frac{1}{3}}\right)^{3} \\
&\quad \times\left(\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\right)^{\frac{1}{3}}\right)^{3}\left(\sum_{n=1}^{r}\sum_{t=1}^{n}q_t\psi\left(\frac{b_t}{q_t}\right)^{-\frac{1}{2}}\right)^{-2} \\
&= M'(k,r)\left(\sum_{s=1}^{k}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^{\frac{1}{3}}\left(\sum_{m=s}^{k}1\right)\right)^{3} \\
&\quad \times\left(\sum_{t=1}^{r}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^{-\frac{1}{2}}\left(\sum_{n=t}^{r}1\right)\right)^{-2} \\
&= M'(k,r)\left(\sum_{s=1}^{k}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^{\frac{1}{3}}(k-s+1)\right)^{3} \\
&\quad \times\left(\sum_{t=1}^{r}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^{-\frac{1}{2}}(r-t+1)\right)^{-2} \\
&= M'(k,r)\left(\sum_{m=1}^{k}\left(p_m\phi\left(\frac{a_m}{p_m}\right)\right)^{\frac{1}{3}}(k-m+1)\right)^{3} \\
&\quad \times\left(\sum_{n=1}^{r}\left(q_n\psi\left(\frac{b_n}{q_n}\right)\right)^{-\frac{1}{2}}(r-n+1)\right)^{-2}.
\end{aligned}$$

The proof is complete. □

Remark 2. Inequality (6) is just an inverse of the following Inequality B, which was proved by Pachpatte [6]:

Inequality B.

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(A_m)\psi(B_n)}{m+n}\le M(k,r)\left(\sum_{m=1}^{k}(k-m+1)\left(p_m\phi\left(\frac{a_m}{p_m}\right)\right)^2\right)^{\frac{1}{2}}$$
$$\times\left(\sum_{n=1}^{r}(r-n+1)\left(q_n\psi\left(\frac{b_n}{q_n}\right)\right)^2\right)^{\frac{1}{2}},$$

where

$$M(k,r)=\frac{1}{2}\left(\sum_{m=1}^{k}\left(\frac{\phi(P_m)}{P_m}\right)^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\right)^2\right)^{\frac{1}{2}}.$$

Similarly, the following two theorems can also be established.

Theorem 3. *Let $\{a_m\}$ and $\{b_n\}$ be as in Theorem 1 and set $A_m=(1/m)\sum_{s=1}^{m}a_s$ and $B_n=(1/n)\sum_{t=1}^{n}b_t$, for $m=1,2,\ldots,k$ and $n=1,2,\ldots,r$, where k and r are natural numbers. Let ϕ and ψ be two real-valued, nonnegative, and concave functions defined on $\mathbb{R}_+$. Then for $3n-2m>0$,*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{mn}{3n-2m}\phi(A_m)\psi(B_n)\ge k^{-2}r^3\left(\sum_{m=1}^{k}(k-m+1)(\phi(a_m))^{\frac{1}{3}}\right)^3$$
$$\times\left(\sum_{n=1}^{r}(r-n+1)(\psi(b_n))^{-\frac{1}{2}}\right)^{-2}. \quad (7)$$

Remark 3. Inequality (7) is just an inverse of the following Inequality C, which was proved by Pachpatte [6]:

Inequality C.

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{mn}{m+n}\phi(A_m)\psi(B_n)\le\frac{1}{2}(kr)^{\frac{1}{2}}\left(\sum_{m=1}^{k}(k-m+1)(\phi(a_m))^2\right)^{\frac{1}{2}}$$
$$\times\left(\sum_{n=1}^{r}(r-n+1)(\psi(b_n))^2\right)^{\frac{1}{2}}.$$

Theorem 4. *Let $\{a_m\},\{b_n\},\{p_m\},\{q_n\},P_m,Q_n$ be as in Theorem 2 and put $A_m=(1/p_m)\sum_{s=1}^{m}p_sa_s$, $B_n=(1/Q_n)\sum_{t=1}^{n}q_tb_t$, for $m=1,2,\ldots k,n=1,2,\ldots,r$, where k and r are natural numbers. Let ϕ and ψ be as defined in Theorem 3. Then* *for* $3n-2m>0$,

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{P_mQ_n\phi(A_m)\psi(B_n)}{3n-2m}\geq k^{-2}r^{3}\left(\sum_{m=1}^{k}(k-m+1)(p_m\phi(a_m))^{\frac{1}{3}}\right)^{3}\times\left(\sum_{n=1}^{r}(r-n+1)(q_n\psi(b_n))^{-\frac{1}{2}}\right)^{-2}. \quad (8)$$

Remark 4. Inequality (8) is just an inverse of the following Inequality D, which was proved by Pachpatte [6]:

Inequality D.

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{P_mQ_n\phi(A_m)\psi(B_n)}{m+n}\leq\frac{\sqrt{kr}}{2}\left(\sum_{m=1}^{k}(k-m+1)(p_m\phi(a_m))^{2}\right)^{\frac{1}{2}}\times\left(\sum_{n=1}^{r}(r-n+1)(q_n\psi(b_n))^{2}\right)^{\frac{1}{2}}.$$

The proofs of Theorems 3 and 4 can be completed by following the same steps as in the proof of Theorem 2 with suitable changes. Here, we omit the details.

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NOTES ON GENERALIZED (α, β)-DERIVATIONS IN PRIME RINGS

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Abstract. Let R be a prime ring with characteristics different from two, $\alpha, \beta \in \mathrm{Aut}(R)$, and I a nonzero ideal of R. Let $d : R \to R$ be a (α, β)-derivation and $f : R \to R$ be generalized (α, β)-derivation associated with d. In this case: (i) if $af(x) = 0$ ($f(x)a = 0$) for all $x \in I$, then $a = 0$; (ii) if $\beta d = d\beta$ and $[f(x), a] = 0$ for all $x \in I$, then $a \in Z$ or $d(a) = 0$; (iii) if f acts as an endomorphism or anti-homomorphism on I, then $d = 0$; (iv) if $\beta d = d\beta$ and $f(xy) = f(yx)$ for all $x, y \in I$ then R is a commutative ring; (v) if f satisfies $f(x^2) = f(x)\alpha(x) + \beta(x)d(x)$ for all $x \in I$, then $f(xy) = f(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in I$.

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Keywords: generalized derivation, (α, β)-derivation, Jordan derivation

1. Introduction

The notion of a generalized derivation of a prime ring R was introduced by M. Brešar [6] and B. Hvala [10]. An additive map f of an associative ring R is called a generalized derivation if there is a derivation d of R such that

$$f(xy) = f(x)y + xd(y) \text{ for all } x, y \in R.$$

A similar notion was used by A. Nakajima [12] and N. Argaç and E. Albaş [1], who established a series of categorical properties.

As a well-known result proved by I. N. Herstein in [9, Theorem] states that if R is a prime ring of characteristics different from two and if d is a nonzero derivation of R such that $[d(x), a] = 0$ for all $x \in R$, then either R is commutative or $a \in Z$. Later on, the authors extended this result to (σ, τ)-derivation in [3]. In [8], Jui-Chi Chang showed that several results hold for any generalized (α, β)-derivation of a prime ring. The purpose of this note is to generalize these theorems for a nonzero ideal of prime ring and generalized (α, β)-derivation of R.

In [4], H. E. Bell and L. C. Kappe proved that d is a derivation of R which is either an endomorphism or anti-endomorphism in semi-prime ring R or nonzero right ideal of R, then $d = 0$. In view of this result, it is natural to question whether this result is true for generalized (α, β)-derivations of a prime ring. Our second aim in this paper is to show that the above mentioned result holds for generalized (α, β)-derivation of a prime ring.

In [7], M. Brešar and J. Vukman proved that every (α, β)-Jordan derivation on a prime ring with characteristics different from two is a (α, β)-derivation. We showed that this result holds for generalized (α, β)-derivations on a nonzero ideal of prime ring R.

Throughout this paper, R will be a prime ring with char $R \neq 2$, I a nonzero ideal of R, Z the center of R. Let α and β be two automorphisms of R into itself. For any two elements x, y of R, we denote $[x, y] = xy - yx$ and make extensive use of the basic commutator identity $[xy, z] = x[y, z] + [x, z]y$.

2. Results

Definition 1 ([8, Definition 1])**.** Let R be a ring, α and β automorphisms of R and d a (α, β)-derivation of R. An additive mapping $f : R \to R$ is said to be right generalized (α, β)-derivation associated with d if

$$f(xy) = f(x)\alpha(y) + \beta(x)d(y) \text{ for all } x, y \in R. \tag{2.1}$$

and f is said to be left generalized (α, β)-derivation associated with d if

$$f(xy) = d(x)\alpha(y) + \beta(x)f(y) \text{ for all } x, y \in R. \tag{2.2}$$

f is said to be a generalized (α, β)-derivation associated with d if it is both a left and right generalized (α, β)-derivation associated with d.

Lemma 1 ([8, Lemma 4])**.** *Let $f \neq 0$ be a generalized (α, β)-derivation of prime ring R associated with d and $a \in R$.*

(i) *If $af(x) = 0$ for all $x \in R$, then $a = 0$.*
(ii) *If $f(x)a = 0$ for all $x \in R$, then $a = 0$.*

Lemma 2. *Let R be a prime ring with characteristics not equal to two, I a nonzero ideal of R and $f : R \to R$ be a generalized (α, β)-derivation of prime ring R associated with d. If $f(x) = 0$ for all $x \in I$, then $f = 0$.*

Proof. For all $x, y \in I$

$$0 = f(xy) = f(x)\alpha(y) + \beta(x)d(y)$$

and so,

$$I\beta^{-1}(d(y)) = 0 \text{ for all } y \in I.$$

Since R is a prime ring, we get $d(y) = 0$, for all $y \in I$. Hence, by the hypothesis, $0 = f(xy) = f(x)\alpha(y)$ for all $x \in I, r \in R$. This yields that $f = 0$. □

Lemma 3. *Let $f \neq 0$ be a generalized (α, β)-derivation of prime ring R associated with nonzero (α, β)-derivation d, I a nonzero ideal of R and $a \in R$.*

(i) *If $af(x) = 0$ for all $x \in I$, then $a = 0$.*
(ii) *If $f(x)a = 0$ for all $x \in I$, then $a = 0$.*

Proof. (i) For any $x\in I, r\in R$,

$$0=af(xr)=af(x)\alpha(R)+a\beta(x)d(R)$$

and so,

$$\beta^{-1}(a)I\beta^{-1}(d(R))=0.$$

Since I is a nonzero ideal of prime ring R and $d\neq 0$, we get $a=0$.

(ii) Similarly. □

Theorem 1. *Let R be a prime ring with characteristics not two, I a nonzero ideal of R and $f: R\to R$ be a generalized (α,β)-derivation of prime ring R associated with nonzero (α,β)-derivation d. If $f(x)\in Z$ for all $x\in I$, then R is a commutative ring.*

Proof. By the hypothesis for all $x,y\in I$,

$$\begin{aligned}0&=[\alpha(y),f(xy)]=[\alpha(y),f(x)\alpha(y)+\beta(x)d(y)]\\&=\beta(x)[\alpha(y),d(y)]+[\alpha(y),\beta(x)]d(y)\\&=\beta(x)\alpha(y)d(y)-\beta(x)d(y)\alpha(y)+\alpha(y)\beta(x)d(y)-\beta(x)\alpha(y)d(y)\end{aligned}$$

and so,

$$\alpha(y)\beta(x)d(y)-\beta(x)d(y)\alpha(y)=0 \text{ for all } x,y\in I. \tag{2.3}$$

Writing $xz, z\in I$ by x in (2.3) and using this equation, we have

$$\begin{aligned}0&=\alpha(y)\beta(x)\beta(z)d(y)-\beta(x)\beta(z)d(y)\alpha(y)\\&=\alpha(y)\beta(x)\beta(z)d(y)-\beta(x)\alpha(y)\beta(z)d(y)\\&=[\alpha(y),\beta(x)]\beta(z)d(y).\end{aligned}$$

Using I a nonzero ideal of prime ring R and $d\neq 0$, one can obtain that R is a commutative ring. □

Theorem 2. *Let R be a prime ring with characteristics not two, I a nonzero ideal of R, d be a nonzero (α,β)-derivation such that $d\beta=\beta d$ and $f: R\to R$ be a generalized (α,β)-derivation of prime ring R associated with d and $a\in R$. If $[f(x),a]=0$ for all $x\in I$, then $a\in Z$ or $d(a)=0$.*

Proof. By the hypothesis for all $x\in R, y\in I$, we get

$$\begin{aligned}0&=[f(xy),a]=[d(x)\alpha(y)+\beta(x)f(y),a]\\&=d(x)[\alpha(y),a]+[d(x),a]\alpha(y)+[\beta(x),a]f(y)\\&=d(x)\alpha(y)a-d(x)a\alpha(y)+d(x)a\alpha(y)-ad(x)\alpha(y)\\&\quad+\beta(x)af(y)-a\beta(x)f(y)\end{aligned}$$

and so,

$$d(x)\alpha(y)a-ad(x)\alpha(y)+\beta(x)af(y)-a\beta(x)f(y)=0 \tag{2.4}$$

for all $x \in R, y \in I$. Writing $yz, z \in R$ by y in (2.4) and using (2.4), we get

$$\begin{aligned}
0 &= d(x)\alpha(y)\alpha(z)a - ad(x)\alpha(y)\alpha(z) \\
&\quad + \beta(x)af(y)\alpha(z) + \beta(x)a\beta(y)d(z) - \\
&\quad - a\beta(x)f(y)\alpha(z) - a\beta(x)\beta(y)d(z) \\
&= d(x)\alpha(y)\alpha(z)a - ad(x)\alpha(y)\alpha(z) + (\beta(x)af(y) - \\
&\quad - a\beta(x)f(y))\alpha(z) + \beta(x)a\beta(y)d(z) - a\beta(x)\beta(y)d(z) \\
&= d(x)\alpha(y)\alpha(z)a - d(x)\alpha(y)a\alpha(z) + \beta(x)a\beta(y)d(z) \\
&\quad - a\beta(x)\beta(y)d(z)
\end{aligned}$$

and so,

$$d(x)\alpha(y)[\alpha(z), a] + [\beta(x), a]\beta(y)d(z) = 0 \tag{2.5}$$

for all $x, z \in R, y \in I$. Replacing x by $\beta^{-1}(a)$ in (2.5), we have

$$d(\beta^{-1}(a))\alpha(y)[\alpha(z), a] = 0, \text{ for all } z \in R, y \in I.$$

Since I is a nonzero ideal of prime ring R and $d\beta = \beta d$, we get $d(a) = 0$ or $a \in Z$. □

Theorem 3. *Let R be a prime ring with characteristics not two, I a nonzero ideal of R, d be a nonzero (α, β)-derivation such that $d\beta = \beta d$ and $f : R \to R$ be a generalized (α, β)-derivation of prime ring R associated with d. If $[f(x), f(y)] = 0$ for all $x, y \in I$, then R is a commutative ring or $df = 0$.*

Proof. By Theorem 2, we have $f(I) \subset Z$ or $df = 0$. The proof is completed by Theorem 1. □

Theorem 4. *Let R be a prime ring with characteristics not two, I a nonzero ideal of R and $f : R \to R$ be a generalized (α, β)-derivation of prime ring R associated with d. If $f(xy) = f(x)f(y)$ for all $x, y \in I$, then $d = 0$.*

Proof. For any $x, y \in I$,

$$f(xy) = f(x)\alpha(y) + \beta(x)d(y) = f(x)f(y), \text{ for all } x, y \in I. \tag{2.6}$$

Writing $xw, w \in I$ for x in (2.6), we have

$$f(xw)\alpha(y) + \beta(xw)d(y) = f(xw)f(y)$$

Since d acts as a homomorphism on I, we get

$$\begin{aligned}
f(x)f(w)\alpha(y) + \beta(xw)d(y) &= f(x)f(w)f(y) = f(x)f(wy) \\
&= f(x)f(w)\alpha(y) + f(x)\beta(w)d(y)
\end{aligned}$$

and so, we have $\beta(x)\beta(w)d(y) = f(x)\beta(w)d(y)$, i. e.,

$$(\beta(x) - f(x))\beta(w)d(y) = 0, \text{ for all } x, y \in I.$$

Since I a nonzero ideal of prime ring R and β an automorphism of R, we conclude that

$$f(x)=\beta(x), \text{for all } x\in I \text{ or } d(y)=0, \text{ for all } y\in I.$$

Let us assume that $f(x)=\beta(x)$, for all $x\in I$. Replacing x by $xy, y\in I$, we have $f(xy)=\beta(xy)$ and $d(x)\alpha(y)+\beta(x)f(y)=\beta(x)\beta(y)=\beta(x)f(y)$.

Thus, we obtain that $d(x)=0$ for all $x\in I$ for any cases. That is $d=0$. This completes the proof. □

Theorem 5. *Let R be a prime ring with characteristics not two, I a nonzero ideal of R and $f:R\to R$ be a generalized (α,β)-derivation of prime ring R associated with d. If $f(xy)=f(y)f(x)$ for all $x,y\in I$, then $d=0$.*

Proof. Since d acts as an anti-homorphism on I, we get

$$f(xy)=d(x)\alpha(y)+\beta(x)f(y)=f(y)f(x), \text{ for all } x,y\in I. \tag{2.7}$$

Replacing y by $xy, y\in I$ in (2.7), we obtain $d(x)\alpha(xy)+\beta(x)f(xy)=f(xy)f(x)$,

$$d(x)\alpha(xy)+\beta(x)f(y)f(x)=d(x)\alpha(y)f(x)+\beta(x)f(y)f(x)$$

and so,

$$d(x)\alpha(x)\alpha(y)=d(x)\alpha(y)f(x), \text{ for } x,y\in I. \tag{2.8}$$

Substituting $yw, w\in I$ for y in this equation and using (2.8), we get

$$\begin{aligned}0&=d(x)\alpha(x)\alpha(y)\alpha(w)-d(x)\alpha(y)\alpha(w)f(x)\\&=d(x)\alpha(y)f(x)\alpha(w)-d(x)\alpha(y)\alpha(w)f(x)\\&=d(x)\alpha(y)[f(x),\alpha(w)].\end{aligned}$$

Since α is an automorphism and I a nonzero ideal of prime ring R, we obtain

$$d(x)=0 \text{ or } [f(x),\alpha(w)]=0, \text{ for all } x,w\in I.$$

Now let us define $A=\{x\in I \mid d(x)=0\}$ and $B=\{x\in I \mid [f(x),\alpha(w)]=0$, for all $w\in I\}$. Clearly each A and B is an additive subgroup of I. Moreover, I is the set theoretic union of A and B. But a group cannot be the set theoretic union of two proper subgroups, hence $A=I$ or $B=I$. In the former case, we have $d(R)=0$, which completes the proof. If $B=I$, then we get

$$0=[f(x),\alpha(wr)]=\alpha(w)[f(x),\alpha(R)]$$

for all $x,w\in I, r\in R$.

Thus, we obtain that $f(I)\subset Z$. Since f acts as an endomorphism of R, it follows that $d=0$, via Theorem 4. □

Theorem 6. *Let R be a prime ring with characteristics not two, I a nonzero ideal of R and $f:R\to R$ be a generalized (α,β)-derivation of prime ring R associated with nonzero (α,β)-derivation d. If $\beta d=d\beta$ and $f(xy)=f(yx)$ for all $x,y\in I$, then R is a commutative ring.*

Proof. Let $c \in I$ be a constant element such that $f(c) = 0$ (i. e., $[x, y]$) and z be an arbitrary element of I. The condition that $f(cz) = f(zc)$ yields that $f(c)\alpha(z) + \beta(c)d(z) = d(z)\alpha(c) + \beta(z)f(c)$ and so,

$$\beta(c)d(z) = d(z)\alpha(c), \text{ for } z \in I.$$

Thus $[d(z), c]_{\alpha,\beta} = 0$, for all $z \in I$. Replacing z by wz, $w \in I$ and using this equation, we have

$$\begin{aligned} 0 &= [d(wz), c]_{\alpha,\beta} = [d(w)\alpha(z) + \beta(w)d(z), c]_{\alpha,\beta} \\ &= d(w)\alpha([z, c]) + [d(w), c]_{\alpha,\beta}\alpha(z) + \beta(w)[d(z), c]_{\alpha,\beta} + \beta([w, c])d(z) \end{aligned}$$

and so,

$$d(w)\alpha([z, c]) + \beta([w, c])d(z) = 0, \text{ for all } w, z \in I. \tag{2.9}$$

Writing c (since $c \in I$) by z in (2.9), we have $\beta([w, c])d(c) = 0$.This gives us $[w, c]\beta^{-1}(d(c)) = 0$, for all $w \in I$, all $c \in I$ such that $f(c) = 0$. If we take $rw, r \in R, w \in I$ instead of w in this equation, we have

$$[r, c]w\beta^{-1}(d(c)) = 0, \text{ for all } r \in R.$$

Since I is a nonzero ideal of prime ring R, we obtain

$$c \in Z \text{ or } d(c) = 0, \text{ for all } c \in I \text{ such that } f(c) = 0.$$

So, we obtain

$$[x, y] \in Z \text{ or } d([x, y]) = 0, \text{ for all } x, y \in I.$$

because of $f([x, y]) = 0$, for all $x, y \in I$. If $d([x, y]) = 0$, for all $y \in I\}$.

This shows that additive group I is the union of two of its additive subgroups $A = \{x \in I \mid d([x, y]) = 0, \text{for all } y \in I\}$ and $B = \{x \in I \mid [x, y] \in Z, \text{for all } y \in I\}$.

If $I = A$, then R is a commutative ring (i. e., $[x, y] \in Z$, for all $x, y \in I$) by [2, Theorem 3]. So, we can take $I = B$ and $[x, y] \in Z$, for all $x, y \in I$. Let us define an inner derivation $I_x : R \to R, I_x^2(y) = 0$, for all $y \in I$. This shows $I \subset Z$ and so R is a commutative ring by [5, Theorem 1]. □

Definition 2. Let R be a ring, β and α automorphisms of R and d a (α, β)-derivation of R. An additive mapping $f : R \to R$ is said to be right generalized (α, β)-Jordan derivation associated with d if

$$f(x^2) = f(x)\alpha(x) + \beta(x)d(x) \text{ for all } x \in R. \tag{2.10}$$

and f is said to be left generalized (α, β)-Jordan derivation associated with d if

$$f(x^2) = d(x)\alpha(x) + \beta(x)f(x) \text{for all } x \in R. \tag{2.11}$$

f is said to be a generalized (α, β)-Jordan derivation associated with d if it is both a left and right generalized (α, β)-Jordan derivation associated with d.

Lemma 4. *Let $f : R \to R$ be a right generalized (α,β)-Jordan derivation on I. For all $x,y,z \in I$,*

$$f(xy+yx) = f(x)\alpha(y)+\beta(x)d(y)+f(y)\alpha(x)+\beta(y)d(x); \tag{1}$$

$$f(xyx) = f(x)\alpha(yx)+\beta(x)d(y)\alpha(x)+\beta(xy)d(x); \tag{2}$$

$$\begin{aligned} f(xyz+zyx) &= f(x)\alpha(yz)+\beta(x)d(y)\alpha(z)+\beta(xy)d(z) \\ &\quad + f(z)\alpha(yx)+\beta(z)d(y)\alpha(x)+\beta(zy)d(x) \\ &\quad + f(z)\alpha(yx)+\beta(z)d(y)\alpha(x)+\beta(zy)d(x). \end{aligned} \tag{3}$$

Proof. Assertion (1). Linearizing (2.10), we get

$$\begin{aligned} f((x+y)^2) &= f((x+y)(x+y) = f(x^2+xy+yx+y^2) \\ &= f(x^2)+f(xy+yx)+f(y^2) \\ &= f(x)\alpha(x)+\beta(x)d(x) \\ &\quad + f(xy+yx)+f(y)\alpha(y)+\beta(y)d(y) \end{aligned} \tag{2.12}$$

for all $x,y \in I$. On the other hand,

$$\begin{aligned} f((x+y)^2) &= f(x+y)\alpha(x+y)+\beta(x+y)d(x+y) \\ &= f(x)\alpha(x)+f(y)\alpha(x)+f(x)\alpha(y)+f(y)\alpha(y) \\ &+\beta(x)d(x)++\beta(x)d(y)+\beta(y)d(x)+\beta(y)d(y) \end{aligned} \tag{2.13}$$

for all $x,y \in I$. Comparing (2.12) and (2.13), we have

$$f(xy+yx) = f(x)\alpha(y)+\beta(x)d(y)+f(y)\alpha(x)+\beta(y)d(x)$$

for all $x,y \in I$.

Assertion (2). Replacing y by $xy+yx$ in (1), we get

$$\begin{aligned} f(x(xy+yx)+(xy+yx)x) &= f(x^2y+xyx+xyx+yx^2) \\ &= f(x^2y+yx^2)+2f(xyx) \\ &= f(x^2)\alpha(y)+\beta(x^2)d(y)+f(y)\alpha(x^2) \\ &\quad +\beta(y)d(x^2)+2f(xyx) \\ &= f(x)\alpha(xy)+\beta(x)d(x)\alpha(y)+\beta(x^2)d(y) \\ &\quad + f(y)\alpha(x^2)+\beta(y)d(x)\alpha(x) \\ &\quad +\beta(yx)d(x)+2f(xyx) \end{aligned} \tag{2.14}$$

for all $x, y \in I$. On the other hand, we have

$$\begin{aligned} f(x(xy+yx)+(xy+yx)x) = {} & f(x)\alpha(xy+yx)+\beta(x)d(xy+yx) \\ & + f(xy+yx)\alpha(x)+\beta(xy+yx)d(x) \\ = {} & f(x)\alpha(xy)+f(x)\alpha(yx)+\beta(x)d(x)\alpha(y)+\beta(x^2)d(y) \\ & +\beta(x)d(y)\alpha(x)+\beta(xy)d(x)+f(x)\alpha(yx) \\ & +\beta(x)d(y)\alpha(x)+f(y)\alpha(x^2)+\beta(y)d(x)\alpha(x) \\ & +\beta(xy)d(x)+\beta(yx)d(x) \end{aligned} \tag{2.15}$$

for all $x, y \in I$. Comparing (2.14) and (2.15) and using char $R \neq 2$, we get the required result.

Assertion (3). Linearizing (2) on x, we get

$$\begin{aligned} f((x+z)y(x+y)) = {} & f(xyx+xyz+zyx+zyz) \\ = {} & f(xyx)+f(xyz+zyx)+f(zyz) \\ = {} & f(x)\alpha(yx)+\beta(x)d(y)\alpha(x)+\beta(xy)d(x) \\ & + f(xyz+zyx)+f(z)\alpha(yz) \\ & +\beta(z)d(y)\alpha(z)+\beta(zy)d(z) \end{aligned} \tag{2.16}$$

for all $x, y, z \in I$. Computing now $f((x+z)y(x+z))$ in another way, we get

$$\begin{aligned} f((x+z)y(x+z)) = {} & f(x+z)\alpha(yx+yz)+\beta(x+z)d(y)\alpha(x+z) \\ & +\beta(xy+zy)d(x+z) \\ = {} & f(x)\alpha(yx)+f(x)\alpha(yz)+f(z)\alpha(yx)+f(z)\alpha(yz) \\ & +\beta(x)d(y)\alpha(x)+\beta(x)d(y)\alpha(z) \\ & +\beta(z)d(y)\alpha(x)+\beta(z)d(y)\alpha(z) \\ & +\beta(xy)d(x)+\beta(xy)d(z)+\beta(zy)d(x) \\ & +\beta(zy)d(z) \end{aligned} \tag{2.17}$$

for all $x, y, z \in I$. Comparing (2.16) and (2.17), we have

$$\begin{aligned} f(xyz+zyx) = {} & f(x)\alpha(yz)+\beta(x)d(y)\alpha(z)+\beta(xy)d(z)+f(z)\alpha(yx) \\ & +\beta(z)d(y)\alpha(x)+\beta(zy)d(x) \end{aligned}$$

for all $x, y, z \in I$. □

We introduce the abbreviation

$$x^y = f(xy) - f(x)\alpha(y) - \beta(x)d(y)$$

for all $x, y \in I$.

Observe that, by Lemma 4 (1), we have $f(xy+yx) = f(x)\alpha(y) + \beta(x)d(y) + f(y)\alpha(x) + \beta(y)d(x)$ and so, $f(xy) - f(x)\alpha(y) - \beta(x)d(y) = -(f(yx) - f(y)\alpha(x) - \beta(y)d(x))$. That is,

$$x^y = -y^x \text{ for all } x, y \in I. \tag{2.18}$$

Lemma 5. *Let $f : R \to R$ be a right generalized (α,β)-Jordan derivation on I. For all $x, y \in I$,*

$$x^y[\alpha(x), \alpha(y)] = 0.$$

Proof. Replacing z by xy in Lemma 4 (3), we get

$$\begin{aligned} f((xy)^2 + xy^2x) = f((xy)^2) + f(xy^2x) &= f(xy)\alpha(xy) + \beta(xy)d(xy) \\ &\quad + f(x)\alpha(x^2y) + \beta(x)d(y^2)\alpha(x) + \beta(xy^2)d(x) \\ &= f(xy)\alpha(xy) + \beta(xy)d(x)\alpha(y) + \beta(xyx)d(y) \\ &\quad + f(x)\alpha(x^2y) + \beta(x)d(y)\alpha(yx) \\ &\quad + \beta(xy)d(y)\alpha(x) + \beta(xy^2)d(x) \end{aligned} \tag{2.19}$$

for all $x, y \in I$. On the other hand, we get

$$\begin{aligned} f(xy(xy) + (xy)yx) &= f(x)\alpha(yxy) + \beta(x)d(y)\alpha(xy) + \beta(xy)d(xy) \\ &\quad + f(xy)\alpha(yx) + \beta(xy)d(y)\alpha(x) + \beta(xy^2)d(x) \\ &= f(x)\alpha(yxy) + \beta(x)d(y)\alpha(xy) + \beta(xy)d(x)\alpha(y) \\ &\quad + \beta(xyx)d(y) + f(xy)\alpha(yx) + \beta(xy)d(y)\alpha(x) \\ &\quad + \beta(xy^2)d(x) \end{aligned} \tag{2.20}$$

for all $x, y \in I$. Comparing these two equations, we have

$$\begin{aligned} &f(xy)\alpha(xy) + f(x)\alpha(y^2x) + \beta(x)d(y)\alpha(yx) \\ &\qquad = f(x)\alpha(yxy) + \beta(x)d(y)\alpha(xy) + f(xy)\alpha(yx). \end{aligned}$$

That is, $x^y[\alpha(x), \alpha(y)] = 0$ for all $x, y \in I$. □

Theorem 7. *Let R be a non-commutative prime ring with characteristics not two, I a nonzero ideal of R. If f be a right generalized (α,β)-Jordan derivation on I, then f is generalized (α,β)-derivation on I.*

Proof. From Lemma 4 (3), we have

$$\begin{aligned} f(xwy + ywx) &= f(x)\alpha(wy) + \beta(x)d(w)\alpha(y) + \beta(xw)d(y) \\ &\quad + f(y)\alpha(wx) + \beta(y)d(w)\alpha(x) + \beta(yw)d(x) \end{aligned} \tag{2.21}$$

for all $x, y, w \in I$. Replacing x by xy and y by yx in (2.21), we get

$$\begin{aligned} f((xy)w(yx)+(yx)w(xy)) &= f(xy)\alpha(wyx)+\beta(xy)d(w)\alpha(yx) \\ &\quad +\beta(xyw)d(yx)+f(yx)\alpha(wxy) \\ &\quad +\beta(yx)d(w)\alpha(xy)+\beta(yxw)d(xy) \\ &= f(xy)\alpha(wyx)+\beta(xy)d(w)\alpha(yx)+\beta(xyw)d(y)\alpha(x) \\ &\quad +\beta(xywy)d(x)+f(yx)\alpha(wxy)+\beta(yx)d(w)\alpha(xy) \\ &\quad +\beta(yxw)d(x)\alpha(y)+\beta(yxwx)d(y) \end{aligned} \tag{2.22}$$

for all $x, y, w \in I$. On the other hand, we have

$$\begin{aligned} f((xy)w(yx)+(yx)w(xy)) &= f(x(ywy)x)+f(y(xwx)y) \\ &= f(x)\alpha(ywyx)+\beta(x)d(ywy)\alpha(x)+\beta(xywy)d(x) \\ &\quad +f(y)\alpha(xwxy)+\beta(y)d(xwx)\alpha(y)+\beta(yxwx)d(y) \\ &= f(x)\alpha(ywyx)+\beta(x)d(y)\alpha(wyx)+\beta(xy)d(w)\alpha(yx) \\ &\quad +\beta(xyw)d(y)\alpha(x)+\beta(xywy)d(x)+f(y)\alpha(xwxy) \\ &\quad +\beta(y)d(x)\alpha(wxy)+\beta(yx)d(w)\alpha(xy) \\ &\quad +\beta(yxw)d(x)\alpha(y)+\beta(yxwx)d(y) \end{aligned} \tag{2.23}$$

for all $x, y, w \in I$. Comparing equations (2.22) and (2.23), we obtain

$$\begin{aligned} &\{f(yx)-f(y)\alpha(x)-\beta(y)d(x)\}\alpha(w)\alpha(xy) \\ &\qquad +\{f(xy)-f(x)\alpha(y)-\beta(x)d(y)\}\alpha(w)\alpha(yx)=0 \end{aligned}$$

and hence

$$y^x\alpha(w)\alpha(xy)+x^y\alpha(w)\alpha(yx)=0$$

for all $x, y, w \in I$. Using (2.18), we get $x^y\alpha(I)\alpha([y,x])=0$ for all $x, y \in I$. Since I is a nonzero ideal of prime ring R, we obtain for each pair $x, y \in I$ either $x^y=0$ or $[x,y]=0$. Notice that the mappings $(x,y)\to x^y$ and $(x,y)\to[x,y]$ satisfy the requirements of [5, Lemma 4]. Hence $x^y=0$ for all $x, y \in I$ or $[x,y]^2=0$ for all $x, y \in I$. If $[x,y]^2=0$ for all $x, y \in I$, then for each $x \in I$, $I_x(y)^2=0$, for all $y \in I$, where I_x is the inner derivation. This yields that R is commutative, a contradiction by [11, Theorem 3]. Thus, we have $x^y=0$ for all $x, y \in I$. This completes the proof. □

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GENERALIZED HYBRID INTEGRAL MELER–FOK TRANSFORM OF THE SECOND TYPE AND ITS APPLICATIONS

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Abstract. By the method of delta-like sequence, a generalized integral transformation of Meler–Fok type one the segment $[R_0, R]$ with n contact points is obtained. We consider main examples of application of the transform for the solution of singular boundary value problems of mathematical physics of non-isotopic solids.

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1. Introduction

In solving linear boundary and mixed problems of mathematical physics of homogeneous environments in the spherical system of coordinates by the method of separation of variables, the equations with the differential Legendres operator

$$\Lambda_m = \frac{d^2}{dr^2} + \operatorname{cth} r \frac{d}{dr} + \frac{1}{4} - \frac{m^2}{\operatorname{sh}^2 r}, \qquad m \geq 0 \tag{1.1}$$

arise. The direct

$$F_0[f(r)] = \int_0^\infty f(r) P_{-\frac{1}{2}+i\lambda}(\operatorname{ch} r) \operatorname{sh} r \, dr \equiv \tilde{f}(\lambda) \tag{1.2}$$

and inverse

$$F_0^{-1}[\tilde{f}(\lambda)] = \int_0^\infty \tilde{f}(\lambda) P_{-\frac{1}{2}+i\lambda}(\operatorname{ch} r) \lambda \operatorname{th}(\pi\lambda) \, d\lambda \equiv f(r) \tag{1.3}$$

integral transforms generated in the polar axis $r \geq 0$ by the Legendres differential operator

$$\Lambda_0 = \frac{d^2}{dr^2} + \operatorname{cth} r \frac{d}{dr} + \frac{1}{4}, \tag{1.4}$$

were first obtained in 1861 by F. G. Meler and strictly substantiated by V. A. Fok [4] and M. M. Lebedev [12]. These transforms are efficiently used for solving axis-symmetrical problems of the theory of potential in the domains formed by two spheres that intersect and in the domains limited by the surfaces of hyperboloids of rotation and toroidal surfaces.

In the case of absence of axial symmetry the generalized integral transforms of Meler–Fok are used [14]:

$$F_m[f(r)] = \int_0^\infty f(r)\, P^m_{-\frac{1}{2}+i\lambda}(\operatorname{ch} r) \operatorname{sh} r dr \equiv \tilde{f}(\lambda), \tag{1.5}$$

$$F_m^{-1}[\tilde{f}(\lambda)] = (-1)^m \int_0^\infty \tilde{f}(\lambda)\, P^{-m}_{-\frac{1}{2}+i\lambda}(\operatorname{ch} r)\, \lambda \operatorname{th}(\pi\lambda)\, d\lambda \equiv f(r); \tag{1.6}$$

they are generated by a differential operator Λ_m $(m = 1, 2, 3, \dots)$ on the polar axis $r \geq 0$. The integral transforms of Meler–Fok on the polar axis $r \geq R_0 > 0$ were obtained in the papers [1, 15, 16].

A natural generalization of the differential operator (1.1) is [3, 17]

$$\Lambda_{(\mu)} = \frac{d^2}{dr^2} + \operatorname{cth} r \frac{d}{dr} + \frac{1}{4} + \frac{1}{2}\left(\frac{\mu_1^2}{1-\operatorname{ch} r} + \frac{\mu_2^2}{1+\operatorname{ch} r}\right), \tag{1.7}$$

where $(\mu) = (\mu_1; \mu_2)$; $\mu_1 \geq \mu_2 > 0$. Operator (1.7) will be called *the generalized Legendres differential operator*. It is obvious that at $\mu_1 = \mu_2 = m$ an operator (1.7) coincides with operator (1.1).

The integral transforms of Meler–Fok type generated on the polar axis $r \geq 0$, $r \geq R_0 > 0$ and the polar segments $[0, R]$ and $[R_0, R]$ by the generalized Legendres differential operator (1.7) are obtained in [8–10]. Proper hybrid integral transforms, generated in these axes with one, two and n contact points by a hybrid Legendres differential operator are considered in [5–7]. In this paper, by using the method of a delta-like sequence [13], limited hybrid integral transforms of Meler–Fok type are constructed on the segment $[R_0, R]$ with n contact points. The transforms obtained are applied to the solving of some singular boundary problems of mathematical physics.

2. Basic part

Let us construct the limited integral transform generated in the set

$$I_n = \left\{ r : r \in \bigcup_{k=1}^{n+1} (R_{k-1}, R_k);\ R_0 > 0,\ R_{n+1} \equiv R < \infty \right\}$$

by the generalized hybrid Legendres differential operator

$$L_{(\mu)} = \sum_{k}^{n+1} \theta(r - R_{k-1}\theta)(R_k - r)\, a_k^2 \Lambda_{(\mu)_k}, \tag{2.1}$$

where $(\mu) = ((\mu)_1, (\mu)_2, \dots, (\mu)_{n+1})$, $(\mu)_k = (\mu_{1k}, \mu_{2k})$, and $\theta(x)$ is the Heavyside function.

For the domain of definition of the operator $L_{(\mu)}$ we take the set G of vector-functions $g = (g_1, g_2, \ldots, g_n, g_{n+1})$ which have the following properties:

(1) The vector-function

$$f(r) = \left(\Lambda_{(\mu)_1}\left[g_1(r)\right]; \Lambda_{(\mu)_2}\left[g_2(r)\right]; \ldots; \Lambda_{(\mu)_{n+1}}\left[g_{n+1}(r)\right] \right)$$

is continuous on the set I_n;

(2) The components $g_j(r)$ of the vector-function $g(r)$ satisfy the conjugate conditions

$$\left[\left(\alpha_{j1}^k \frac{d}{dr} + \beta_{j1}^k \right) g_k(r) - \left(\alpha_{j2}^k \frac{d}{dr} + \beta_{j2}^k \right) g_{k+1}(r) \right] \Big|_{r=R_k} = 0,$$

$j = 1, 2,\ k = \overline{1,n}$;

(3) The boundary conditions

$$\left(\alpha_{11}^0 \frac{d}{dr} + \beta_{11}^0 \right) g_1(r) \Big|_{r=R_0} = 0 \quad (\text{or } g_0 \neq 0),$$

$$\left(\alpha_{22}^{n+1} \frac{d}{dr} + \beta_{22}^{n+1} \right) g_{n+1}(r) \Big|_{r=R_{n+1}} = 0 \quad (\text{or } g_R \neq 0).$$

are satisfied.

We suppose that $c_{1k}c_{2k} > 0$, $\alpha_{11}^0 \leq 0$, $\beta_{11}^0 \geq 0$, $\alpha_{22}^{n+1} \geq 0$, $\beta_{22}^{n+1} \geq 0$,

$$\left|\alpha_{11}^0\right| + \beta_{11}^0 \neq 0, \quad \alpha_{22}^{n+1} + \beta_{22}^{n+1} \neq 0, \quad c_{jk} = \alpha_{2j}^k \beta_{1j}^k - \alpha_{1j}^k \beta_{2j}^k.$$

Let us define the numbers

$$\sigma_k = \frac{1}{a_k^2} \prod_{j=k}^{n} \frac{c_{1j}}{c_{2j}}, \quad k = \overline{1,n}, \quad \sigma_n = \frac{1}{a_n^2} \frac{c_{1n}}{c_{2n}}, \quad \sigma_{n+1} = \frac{1}{a_{n+1}^2},$$

and the gravimetric function

$$\sigma(r) = \left(\sum_{k=1}^{n+1} \theta(r - R_{k-1})\, \theta(R_k - r)\, \sigma_k \right) \operatorname{sh} r.$$

The scalar product of the elements of set G will be defined by a formula

$$(u, v) = \int_{R_0}^{R} u(r)\, v(r)\, \sigma(r)\, dr \equiv \sum_{k=1}^{n+1} \int_{R_{k-1}}^{R_k} u_k(r)\, v_k(r)\, \sigma_k \operatorname{sh} r\, dr,$$

where $u \in G$, $v \in G$.

Theorem 1. *The components of the vector-functions u and v from the domain of definition of operator $L_{(\mu)}$ satisfy the basic identity*

$$u'_k(R_k)\,v_k(R_k)-u_k(R_k)\,v'_k(R_k) = \frac{c_{2k}}{c_{1k}}\left(u'_{k+1}(R_k)\,v_{k+1}(R_k)-u_{k+1}(R_k)\,v'_{k+1}(R_k)\right). \quad (2.2)$$

Proof. Let us define the numbers

$$c^k_{11}=\alpha^k_{11}\alpha^k_{22}-\alpha^k_{21}\alpha^k_{12}, \qquad c^k_{12}=\alpha^k_{11}\beta^k_{22}-\alpha^k_{21}\beta^k_{12},$$
$$c^k_{21}=\beta^k_{11}\alpha^k_{22}-\beta^k_{21}\alpha^k_{12}, \qquad c^k_{22}=\beta^k_{11}\beta^k_{22}-\beta^k_{21}\beta^k_{12}$$

for $k=\overline{1,n}$. From the conjugate conditions

$$\alpha^k_{11}u'_k(R_k)+\beta^k_{11}u_k(R_k)=\alpha^k_{12}u'_{k+1}(R_k)+\beta^k_{12}u_{k+1}(R_k),$$
$$\alpha^k_{21}u'_k(R_k)+\beta^k_{21}u_k(R_k)=\alpha^k_{22}u'_{k+1}(R_k)+\beta^k_{22}u_{k+1}(R_k),$$

by Cramer rule [11] we find the relations

$$\left.\begin{aligned} u'_k(R_k)&=c^{-1}_{1k}[a^k_{21}u'_{k+1}(R_k)+a^k_{22}u_{k+1}(R_k)],\\ u_k(R_k)&=-c^{-1}_{1k}[a^k_{11}u'_{k+1}(R_k)+a^k_{12}u_{k+1}(R_k)].\end{aligned}\right\} \quad (2.3)$$

Like for $v(r)\in G$, we have

$$\left.\begin{aligned} v'_k(R_k)&=c^{-1}_{1k}[a^k_{21}v'_{k+1}(R_k)+a^k_{22}v_{k+1}(R_k)],\\ v_k(R_k)&=-c^{-1}_{1k}[a^k_{11}v'_{k+1}(R_k)+a^k_{12}v_{k+1}(R_k)].\end{aligned}\right\} \quad (2.4)$$

On the basis of equalities (2.3), (2.4) we find

$$u'_k(R_k)\,v_k(R_k)-u_k(R_k)\,v'_k(R_k)=\frac{1}{c^2_{1k}}\left(a^k_{11}a^k_{22}-a^k_{12}a^k_{21}\right) \times\left(u'_{k+1}(R_k)\,v_{k+1}(R_k)-u_{k+1}(R_k)\,v'_{k+1}(R_k)\right).$$

Taking into account the inequality $a^k_{11}a^k_{22}-a^k_{12}a^k_{21}=c_{1k}c_{2k}>0$, we obtain the basic identity (2.2). The theorem is proved. $\square$

Theorem 2. *The generalized Legendres differential operator $L_{(\mu)}$ defined by equality (2.1) is selfconjugate.*

Proof. For any $u(r)\in G$ and $v(r)\in G$ we have directly

$$\left(L_{(\mu)}[u],v\right)=\sum_{k=1}^{n+1}\int_{R_{k-1}}^{R_k} a^2_k\Lambda_{(\mu)_k}[u_k]\,v_k(r)\,\sigma_k\,\mathrm{sh}\,r\,dr.$$

Let us integrate by parts twice. We get

$$\begin{aligned}
&\sum_{k=1}^{n+1}\int_{R_{k-1}}^{R_k} a_k^2 \Lambda_{(\mu)_k}[u_k]\, v_k(r)\,\sigma_k \operatorname{sh} r\, dr \\
&\quad= \sum_{k=1}^{n+1} a_k^2\Big\{\big[u_k'(r)\, v_k(r) - u_k(r)\, v_k'(r)\big]\sigma_k \operatorname{sh} r\Big|_{R_{k-1}}^{R_k} \\
&\qquad+ \int_{R_{k-1}}^{R_k} a_k^2 u_k(r)\, \Lambda_{(\mu)_k}[v_k(r)]\,\sigma_k \operatorname{sh} r dr\Big\} \\
&\quad= \big(u(r), \Lambda_{(\mu)}[v(r)]\big) \\
&\qquad+ \sum_{k=1}^{n}\big\{\sigma_k \operatorname{sh} R_k \big[u_k'(R_k)\, v_k(R_k) - u_k(R_k)\, v_k'(R_k)\big] a_k^2 \\
&\qquad- a_{k+1}^2\sigma_{k+1} \operatorname{sh} R_k \big[u_{k+1}'(R_k)\, v_{k+1}(R_k) - u_{k+1}(R_k)\, v_{k+1}'(R_k)\big]\big\} \\
&\qquad+ \sigma_{n+1} a_{n+1}^2 \operatorname{sh} R_{k+1}[u_{n+1}'(r)\, v_{n+1}(r) - u_{n+1}(r)\, v_{n+1}'(r)] \\
&\qquad- \sigma_1 a_1^2 \operatorname{sh} R_0 \big[u_1'(R_0) v_1(R_0) - u_1(R_0) v_1'(R_0)\big].
\end{aligned} \tag{2.5}$$

Due to the basic identity (2.2) and the structure of σ_k we have

$$\begin{aligned}
&a_k^2\sigma_k \operatorname{sh} R_k \big[u_k'(R_k)\, v_k(R_k) - u_k(R_k)\, v_k'(R_k)\big] \\
&\qquad= a_k^2\sigma_k \operatorname{sh} R_k \frac{c_{2k}}{c_{1k}}\big[u_{k+1}'(R_k)\, v_{k+1}(R_k) - u_{k+1}(R_k)\, v_{k+1}'(R_k)\big] \\
&\qquad= a_{k+1}^2\sigma_{k+1} \operatorname{sh} R_k \big[u_{k+1}'(R_k)\, v_{k+1}(R_k) - u_{k+1}(R_k)\, v_{k+1}'(R_k)\big].
\end{aligned}$$

Consequently, expression in the curly braces equals to zero. If $\alpha_{11}^0 \neq 0$, as a result of the boundary condition in point $r = R_0$, we have

$$\begin{aligned}
&\left(\frac{du_1}{dr} v_1 - u_1\frac{dv_1}{dr}\right)\Big|_{r=R_0} \\
&\quad= \left[\frac{1}{\alpha_{11}^0}\left(\alpha_{11}^0\frac{du_1}{dr} + \beta_{11}^0 u_1\right) - \frac{\beta_{11}^0}{\alpha_{11}^0} u_1 v_1 - u_1\frac{dv_1}{dr}\right]\Big|_{r=R_0} \\
&\quad= \frac{v_1(R_0)}{\alpha_{11}^0}\left(\alpha_{11}^0\frac{du_1}{dr} + \beta_{11}^0 u_1\right)\Big|_{r=R_0} - \frac{u_1(R_0)}{\alpha_{11}^0}\left(\alpha_{11}^0\frac{dv_1}{dr} + \beta_{11}^0 v_1\right)\Big|_{r=R_0} \\
&\quad= \left(\alpha_{11}^0\right)^{-1}[v_1(R_0)\cdot 0 - u_1(R_0)\cdot 0] \equiv 0.
\end{aligned}$$

If $\alpha_{22}^{n+1} \neq 0$, in view of the boundary condition at the point $r = R_{n+1}$, we get

$$\left(\frac{du_{n+1}}{dr} v_{n+1} - u_{n+1} \frac{dv_{n+1}}{dr}\right)\Bigg|_{r=R_{n+1}}$$
$$= \alpha_{22}^{n+1} \left[v_{n+1}(R_{n+1}) \left(\alpha_{22}^{n+1} \frac{d}{dr} + \beta_{22}^{n+1}\right) u_{n+1}(r)\Big|_{r=R} \right.$$
$$\left. u_{n+1}(R_{n+1}) \left(\alpha_{22}^{n+1} \frac{d}{dr} + \beta_{22}^{n+1}\right) v_{n+1}(r)\Big|_{r=R_{n+1}} \right]$$
$$= \frac{v_{n+1}(R_{n+1})}{\alpha_{22}^{n+1}} \cdot 0 - \frac{u_{n+1}(R_{n+1})}{\alpha_{22}^{n+1}} \cdot 0 \equiv 0.$$

Consequently, as a result of the boundary conditions, expressions in points $r = R_0$ and $r = R \equiv R_{n+1}$ equal a zero. Equality (2.5) gives

$$\left(L_{(\mu)}[u], v\right) = \sum_{k=1}^{n+1} a_k^2 \int_{R_{k-1}}^{R_k} \Lambda_{(\mu)_k}[u_k]\, v_k(r)\, \sigma_k \operatorname{sh} r\, dr = (u, L_{(\mu)}[v]).$$

The last equality means that the operator $L_{(\mu)}$ is selfconjugate. The theorem is proved. □

As an operator $L_{(\mu)}$ in the set I_n has not the special points and is selfconjugate, its spectrum is discrete and real [2].

Let us find eigenvalues and vector eigenfunctions of the GDLO $L_{(\mu)}$ defined by equality (2.1) as a solution of spectral Schturm–Liouville problem: to construct non-trivial limited in the set I_n solution of the separate system of the generalized Legendres differential equations

$$\left(\Lambda_{(\mu)_j} + b_j^2\right) V_{(\mu);j}(r, \beta) = 0, \qquad r \in \left(R_{j-1}, R_j\right), \qquad j = \overline{1, n+1} \tag{2.6}$$

under the boundary conditions

$$\begin{aligned} \left(\alpha_{11}^0 \frac{d}{dr} + \beta_{11}^0\right) V_{(\mu);1}(r, \beta)\Big|_{r=R_0} &= 0, \\ \left(\alpha_{22}^{n+1} \frac{d}{dr} + \beta_{22}^{n+1}\right) V_{(\mu);n+1}(r, \beta)\Big|_{r=R_{n+1}} &= 0 \end{aligned} \tag{2.7}$$

and the conjugate conditions

$$\left[\left(\alpha_{j1}^k \frac{d}{dr} + \beta_{j1}^k\right) V_{(\mu);k}(r, \beta) - \left(\alpha_{j2}^k \frac{d}{dr} + \beta_{j2}^k\right) V_{(\mu);k+1}(r, \beta)\right]\Bigg|_{r=R_k} = 0, \qquad j = 1, 2,\ k = \overline{1, n}, \tag{2.8}$$

where $b_j = a_j^{-1}(\beta^2+\gamma_j^2)^{1/2}$, $a_j>0$, $\gamma_j^2\geq 0$, $j=\overline{1,n+1}$, and β is a spectral parameter.

The presence of the fundamental system of solutions [17] allows us to search for the solution of homogeneous boundary problem (2.6), (2.7), (2.8) according to the rule

$$V_{(\mu);j}(r,\beta) = C_j A^{(\mu)_j}_{-1/2+ib_j}(\operatorname{ch} r) + D_j B^{(\mu)_j}_{-1/2+ib_j}(\operatorname{ch} r), \qquad j=\overline{1,n+1}.$$

The boundary conditions (2.7) and conjugate conditions (2.8) for the determination of the values C_j and D_j $(j=\overline{1,n+1})$ give the following system of $(2n+2)=2(n+1)$ algebraic equations:

$$\left.\begin{aligned}
&Y^{(\mu)_1,01}_{-1/2+ib_1;11}(\operatorname{ch} R_0)\,C_1 + Y^{(\mu)_1,02}_{-1/2+ib_1;11}(\operatorname{ch} R_0)\,D_1 = 0,\\
&Y^{(\mu)_j,j1}_{-1/2+ib_j;m1}\big(\operatorname{ch} R_j\big)\,C_j + Y^{(\mu)_j,j2}_{-1/2+ib_j;m1}\big(\operatorname{ch} R_j\big)\,D_j\\
&\qquad\qquad -Y^{(\mu)_{j+1},j_1}_{-1/2+ib_{j+1};m_2}\big(\operatorname{ch} R_j\big)\,C_{j+1}\\
&\qquad\qquad -Y^{(\mu)_{j+1},j_2}_{-1/2+ib_{j+1};m_2}\big(\operatorname{ch} R_j\big)\,D_{j+1} = 0, \quad j=\overline{1,n},\\
&Y^{(\mu)_{n+1};n+1,1}_{-1/2+ib_{n+1};22}(\operatorname{ch} R_{n+1})\,C_{n+1}\\
&\qquad\qquad +Y^{(\mu)_{n+1};n+1,2}_{-1/2+ib_{n+1};22}(\operatorname{ch} R_{n+1})\,D_{n+1} = 0, \quad m=1,2.
\end{aligned}\right\} \tag{2.9}$$

Let us define the functions

$$\omega^{(0)}_{(\mu)_1;1}(\beta) = Y^{(\mu)_1;01}_{-1/2+ib_1;11}(\operatorname{ch} R_0),$$
$$\omega^{(0)}_{(\mu)_1;2}(\beta) = Y^{(\mu)_1;02}_{-1/2+ib_1;11}(\operatorname{ch} R_0)$$

and put

$$\begin{aligned}
&\psi^k_{((\mu)_k;(\mu)_{k+1});mj}(\beta,\operatorname{ch} R_k,\operatorname{ch} R_{k+1})\\
&\qquad = Y^{(\mu)_k;km}_{-1/2+ib_k;11}(\operatorname{ch} R_k)\,Y^{(\mu)_{k+1};kj}_{-1/2+ib_{k+1};22}(\operatorname{ch} R_{k+1})\\
&\qquad\quad - Y^{(\mu)_k;km}_{-1/2+ib_k;21}(\operatorname{ch} R_k)\,Y^{(\mu)_{k+1};kj}_{-1/2+ib_{k+1};12}(\operatorname{ch} R_{k+1}), \quad k=\overline{1,n},
\end{aligned}$$

and

$$\begin{aligned}
\omega^{(k)}_{(\mu)_{k+1};j}(\beta) &= \omega^{(k-1)}_{(\mu)_k;2}(\beta)\,\psi^k_{((\mu)_k;(\mu)_{k+1});1j}(\beta,\operatorname{ch} R_k,\operatorname{ch} R_{k+1})\\
&\quad -\omega^{(k-1)}_{(\mu)_k;1}(\beta)\,\psi^k_{((\mu)_k;(\mu)_{k+1});2j}(\beta,\operatorname{ch} R_k,\operatorname{ch} R_{k+1}), \quad j=1,2,\ k=\overline{1,n},
\end{aligned}$$

where $(\mu)_k = ((\mu)_1,(\mu)_2,\ldots,(\mu)_k)$, $(\mu)=((\mu)_1,(\mu)_2,\ldots,(\mu)_n,(\mu)_{n+1})$.

In order that the algebraic system (2.9) have a non-trivial solution, it is necessary and sufficient that its determinant be equal zero [11]:

$$\delta_{(\mu);n}(\beta) \equiv Y^{(\mu)_{n+1};n+1,1}_{-1/2+ib_{n+1};22}(\operatorname{ch} r)\,\omega^{(n)}_{(\mu)_{n+1};2}(\beta) \\ - Y^{(\mu)_{n+1};n+1,2}_{-1/2+ib_{n+1};22}(\operatorname{ch} r)\,\omega^{(n)}_{(\mu)_{n+1};1}(\beta) = 0.$$

The transcendent equation (2.10) is the equation for determination of eigenvalues of the operator $L_{(\mu)}$.

Theorem 3 (On the discrete spectrum)**.** *The solutions β_m of the transcendent equation* (2.10) *form a discrete spectrum: they are real, distinct, located symmetrically with respect to $\beta = 0$, their moduli make a droningly growing number sequence with a unique maximum point $\beta = \infty$.*

We will put in the system (2.10) $\beta = \beta_m(b_j(\beta_m) \equiv b_{jm} = a_j^{-1}(\beta_m^2 + \gamma_j^2)^{1/2})$ and we will cast aside the last equation as a result of linear dependence. If

$$V_{(\mu);1}(r,\beta_m) = A_0\Big(\omega^{(0)}_{(\mu)_1;2}(\beta_m)\,A^{(\mu)_1}_{-1/2+ib_{1m}}(\operatorname{ch} r) \\ - \omega^{(0)}_{(\mu)_1;1}(\beta_m)\,B^{(\mu)_1}_{-1/2+ib_{1m}}(\operatorname{ch} r)\Big), \quad (2.10)$$

where $A_0 = A_0(\beta_m)$ is subject to determination, then equations of the system, except the first and last, form n recurrent systems on two equations in each. The first equation of the system is satisfied. At

$$A_0 \equiv \Delta_{(\mu);n}(\beta_m) = \prod_{k=1}^{n} \frac{c_{2k}}{\operatorname{sh} R_k}\,\frac{1}{S_{(\mu)_k}(b_{km})},$$

as a result of solving the recurrent systems, we get the structure of the components of vector eigenfunction

$$V_{(\mu);1}(r,\beta_m) = \Delta_{(\mu);n}(\beta_m)\Big(\omega^{(0)}_{(\mu)_1;2}(\beta_m)\,A^{(\mu)_1}_{-1/2+ib_{1m}}(\operatorname{ch} r) \\ - \omega^{(0)}_{(\mu)_1;1}(\beta_m)\,B^{(\mu)_1}_{-1/2+ib_{1m}}(\operatorname{ch} r)\Big),$$

$$V_{(\mu);k}(r,\beta_m) = \left(\prod_{s=k}^{n} \frac{c_{2s}}{\operatorname{sh} R_s}\,\frac{1}{S_{(\mu)_s}(b_{sm})}\right)\Big[\omega^{(k-1)}_{(\mu)_k;2}(\beta_m)\,A^{(\mu)_k}_{-1/2+ib_{km}}(\operatorname{ch} r) \\ - \omega^{(k-1)}_{(\mu)_k;1}(\beta_m)\,B^{(\mu)_k}_{-1/2+ib_{km}}(\operatorname{ch} r)\Big]; \quad k = \overline{2,n};$$

$$V_{(\mu);n+1}(r,\beta_m) = \omega^{(n)}_{(\mu)_{n+1};2}(\beta_m)\,A^{(\mu)_{n+1}}_{-1/2+ib_{n+1,m}}(\operatorname{ch} r) \\ - \omega^{(n)}_{(\mu)_{n+1};1}(\beta_m)\,B^{(\mu)_{n+1}}_{-1/2+ib_{n+1,m}}(\operatorname{ch} r).$$

To an eigenvalue β_m, one (spectral) vector eigenfunction corresponds,

$$V_{(\mu)}(r,\beta_m) = \sum_{k=1}^{n+1} \theta(r-R_{k-1})\theta(R_k-r)V_{(\mu);k}(r,\beta_m)$$

with the square of the norm equal to

$$\begin{aligned}\left\| V_{(\mu)}(r,\beta_m)\right\|^2 &= \left(V_{(\mu)}(r,\beta_m), V_{(\mu)}(r,\beta_m)\right)\\ &\equiv \sum_{k=1}^{n+1}\int_{R_{k-1}}^{R_k}\left[V_{(\mu);k}(r,\beta_m)\right]^2 \sigma_k \operatorname{sh} r\, dr.\end{aligned}$$

Theorem 4 (On a discrete function)**.** *The system of the vector eigenfunctions* $\left\{V_{(\mu)}(r,\beta_m)\right\}_{m=1}^{\infty}$ *is orthogonal and complete in the set* I_n.

Theorem 5 (A Steklov-type theorem)**.** *Any vector-function* $g(r) \in G$ *is represented by absolutely and uniformly convergent in every compact set* $I_n^* \subset I_n$ *by a Fourier series for the system* $\{V_{(\mu)}(r,\beta_m)\}_{m=1}^{\infty}$ *of the vector eigenfunctions of the operator* $L_{(\mu)}$*:*

$$g(r) = \sum_{m=1}^{\infty}\int_{R_0}^{R_{n+1}} g(\rho)V_{(\mu)}(\rho,\beta_m)\sigma(\rho)d\rho \frac{V_{(\mu)}(r,\beta_m)}{\left\|V_{(\mu)}(r,\beta_m)\right\|^2}. \tag{2.11}$$

The Fourier series (2.11) determines the direct $M_{(\mu);4n}$ and inverse $M_{(\mu);4n}^{-1}$ limited generalized integral Meler–Fok transform of the second type generated in the set I_n by the GDLO $L_{(\mu)}$:

$$\begin{aligned}M_{(\mu);4n}[g(r)] &= \int_{R_0}^{R} g(r)V_{(\mu)}(r,\beta_m)\sigma(r)\,dr \equiv \tilde{g}_m = \sum_{k=1}^{n+1}\tilde{g}_{km}\\ &= \sum_{k=1}^{n+1}\int_{R_{k-1}}^{R_k} g_k(r)V_{(\mu);k}(r,\beta_m)\sigma_k \operatorname{sh} r\, dr.\end{aligned} \tag{2.12}$$

and

$$M_{(\mu);4n}^{-1}[\tilde{g}_m] = \sum_{m=1}^{\infty}\tilde{g}_m \frac{V_{(\mu)}(r,\beta_m)}{\left\|V_{(\mu)}(r,\beta_m)\right\|^2} \equiv g(r). \tag{2.13}$$

Theorem 6 (On the basic identity)**.** *If the vector-function* $g \in G$*, and the components* g_k *of the vector-function* g *satisfy innhomogeneous boundary conditions and the innhomogeneous conjugate conditions, then the basic identity for the integral* *transform of the GDLO $L_{(\mu)}$ determined by equality (2.1) is true:*

$$
\begin{aligned}
M_{(\mu);4n}\left[L_{(\mu)}(g)\right] = -\beta_m^2 \tilde{g}_m + \sigma_{n+1} a_{n+1}^2 \\
+ (\alpha_{22}^{n+1})^{-1} \operatorname{sh} R V_{(\mu);n+1}(R_{n+1}, \beta_m) g_R \\
- a_1^2 \sigma_1 \left(\alpha_{11}^0\right)^{-1} \operatorname{sh} R_0 V_{(\mu);1}(R_0, \beta_m) g_0 \\
- \sum_{k=1}^{n+1} \gamma_k^2 \tilde{g}_{km} \\
+ \sum_{k=1}^{n} a_k^2 \sigma_k (c_{1k})^{-1} \operatorname{sh} R_k \big(Z_{(\mu);12}^k(\beta_m) \omega_{2k} \\
- Z_{(\mu);22}^k(\beta_m) \omega_{1k} \big). \qquad (2.14)
\end{aligned}
$$

The proof of Theorems 2 – 6 repeats the logical scheme of proof of the respective theorems in [7].

3. Applications

The presence of the basic identity (2.14) allows us to apply the integral transforms inculcated by formulas (2.12) and (2.13) for the construction of the exact analytical solution of the proper singular problems of the mathematical physics of inhomogeneous structures.

3.1. *An example: a statics problem*

Let us consider the problem to construct, in the domain $D_n = \{(r,z) : r \in I_n;\ z \in (-\infty; +\infty)\}$, the bounded solution of the separate elliptic system of equation with the generalized Legendres operator

$$
\left(\frac{\partial^2}{\partial z^2} + a_j^2 \Lambda_{(\mu)j} - \chi_j^2 \right) u_j(r,z) = -f_j(r,z), \qquad j = \overline{1, n+1} \qquad (3.1)
$$

under the boundary conditions

$$
\begin{aligned}
\left(\alpha_{11}^0 \frac{\partial}{\partial r} + \beta_{11}^0 \right) u_1 \bigg|_{r=R_0} &= g_0(z), \\
\left(\alpha_{22}^{n+1} \frac{\partial}{\partial r} + \beta_{22}^{n+1} \right) u_{n+1} \bigg|_{r=R_{n+1}} &= g_R(z)
\end{aligned} \qquad (3.2)
$$

and the conjugate conditions

$$\left(\left(\alpha_{j1}^{0}\frac{\partial}{\partial r}+\beta_{j1}^{k}\right)u_{k}\left(r,z\right) - \left(\alpha_{j2}^{n+1}\frac{\partial}{\partial r}+\beta_{j2}^{n+1}\right)u_{k+1}\left(r,z\right)\right)\Bigg|_{r=R_k}=\omega_{jk}\left(z\right), \quad (3.3)$$

where $j=1,2$, $k=\overline{1,n}$.

We rewrite system (3.1) in the matrix form

$$\begin{pmatrix} \left(\frac{\partial^2}{\partial z^2}+a_1^2\Lambda_{(\mu)_1}-\chi_1^2\right)u_1\left(r,z\right) \\ \left(\frac{\partial^2}{\partial z^2}+a_2^2\Lambda_{(\mu)_2}-\chi_2^2\right)u_2\left(r,z\right) \\ \vdots \\ \left(\frac{\partial^2}{\partial z^2}+a_{n+1}^2\Lambda_{(\mu)_{n+1}}-\chi_{n+1}^2\right)u_{n+1}\left(r,z\right) \end{pmatrix} = -\begin{pmatrix} f_1\left(r,z\right) \\ f_2\left(r,z\right) \\ \vdots \\ f_{n+1}\left(r,z\right) \end{pmatrix}.$$

The integral operator $M_{(\mu);4n}$, in accordance with (2.12), we represent as the operator matrix-row

$$\begin{aligned} M_{(\mu);4n}\left[\cdots\right]=\Bigg(&\int_{R_0}^{R_1}\cdots V_{(\mu);1}\left(r,\beta_m\right)\sigma_1 \operatorname{sh} r\,dr, \\ &\int_{R_1}^{R_2}\cdots V_{(\mu);2}\left(r,\beta_m\right)\sigma_2 \operatorname{sh} r\,dr, \\ &\ldots\ldots\ldots \\ &\int_{R_{n-1}}^{R_n}\cdots V_{(\mu);n}\left(r,\beta_m\right)\sigma_n \operatorname{sh} r\,dr \\ &\int_{R_n}^{R_{n+1}}\cdots V_{(\mu);n+1}\left(r,\beta_m\right)\sigma_{n+1} \operatorname{sh} r\,dr\Bigg). \end{aligned} \quad (3.4)$$

Let us assume that $\chi_1^2=\max_{1\le j\le n+1}\{\chi_j^2\}$ (in the contrary case, it is possible to renumber these values). Let us put $\gamma_j^2=\chi_1^2-\chi_j^2\ge 0$ for $j=\overline{1,\,n+1}$ and apply, by the rule of multiplication of matrices, the operator matrix-tow (3.4) to system (3.1). As a result of identity (2.14), we get a boundary problem on the structure of bounded solutions of the second-order ordinary differential equation with constant coefficients in the set $I_1=\{z:\,z\in(-\infty,+\infty)\}$

$$\left(\frac{d^2}{dz^2}-q_m^2\right)\tilde{u}_m\left(z\right)=-\tilde{F}_m\left(z\right), \quad (3.5)$$

where $q_m = \left(\beta_m^2 + \chi_1^2\right)^{1/2}$ and

$$\begin{aligned}\tilde{F}_m(z) = \tilde{f}_m(z) &+ \left(a_{22}^{n+1}\right)^{-1} \operatorname{sh} R_{n+1}\, V_{(\mu);n+1}(R,\beta_m)\, g_R(z)\\ &- \left(\alpha_{11}^{0}\right)^{-1} \operatorname{sh} R_0\, a_1^2\sigma_1 V_{(\mu);1}(R_0,\beta_m)\, g_0(z)\\ &+ \sum_{k=1}^{n} a_k^2\sigma_k \frac{\operatorname{sh} R_k}{c_{1k}} \left(Z_{(\mu);12}^{k}(\beta_m)\,\omega_{2k}(z) - Z_{(\mu);22}^{k}\left(\beta_{(\mu)}\right)\omega_{1k}(z)\right).\end{aligned}$$

It is checked directly that the following function is the unique bounded solution of equation (3.5):

$$\tilde{u}_m(z) = \int_{-\infty}^{\infty} \frac{1}{2q_m} e^{-q_m|z-\zeta|}\, \tilde{F}_m(\zeta)\, d\zeta. \tag{3.6}$$

The operator $M_{(\mu);4n}^{-1}$, in accordance with the rule (2.13), as inverse to (2.12), we will represent as the operator matrix-column

$$M_{(\mu);4n}^{-1}[\cdots] = \begin{pmatrix} \sum_{m=1}^{\infty}\cdots \frac{V_{(\mu);1}(r,\beta_m)}{\left\|V_{(\mu)}(r,\beta_m)\right\|^2} \\ \sum_{m=1}^{\infty}\cdots \frac{V_{(\mu);2}(r,\beta_m)}{\left\|V_{(\mu)}(r,\beta_m)\right\|^2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{m=1}^{\infty}\cdots \frac{V_{(\mu);n+1}(r,\beta_m)}{\left\|V_{(\mu)}(r,\beta_m)\right\|^2} \end{pmatrix}. \tag{3.7}$$

We will apply by rule of multiplication of matrices the operator matrix-column to the matrix-element $[\tilde{u}_m(z)]$, where the function $\tilde{u}_m(z)$ is defined by formula (3.6). As a result of elementary transformations, we get the unique bounded solution of the elliptic problem (3.1), (3.2), (3.3):

$$\begin{aligned}u_j(r,z) = \sum_{k=1}^{n+1} &\int_{-\infty}^{\infty}\int_{R_{k-1}}^{R_k} E_{(\mu);jk}(r,\rho,z,\zeta)\, f_k(\rho,\zeta)\,\sigma_k \operatorname{sh}\rho\, d\rho\, d\zeta\\ &+ \int_{-\infty}^{\infty} \left(W_{(\mu);1j}(r,z,\zeta)\, g_0(\zeta) + W_{(\mu);n+1,j}(r,z,\zeta)\, g_R(\zeta)\right) d\zeta\\ &+ \sum_{k=1}^{n} \int_{-\infty}^{\infty} \left(R_{(\mu);12}^{jk}(r,z,\zeta)\,\omega_{2k}(\zeta) - R_{(\mu);22}^{jk}(r,z,\zeta)\,\omega_{1k}(\zeta)\right) d\zeta\end{aligned}$$

for $j = \overline{1,\, n+1}$.

Here, the principal solutions appear: (1) the influence functions generated by the inhomogeneity of system (3.1):

$$E_{(\mu);jk}(r,\rho,z,\zeta) = \sum_{m=1}^{\infty} \frac{1}{2q_m} e^{-q_m|z-\zeta|} \frac{V_{(\mu);j}(r,\beta_m)\, V_{(\mu);k}(\rho,\beta_m)}{\left\|V_{(\mu)}(r,\beta_m)\right\|^2},$$

where $j,k = \overline{1,n+1}$;

(2) Green's functions generated by boundary condition at $r = R_0$:

$$W_{(\mu);1k}(r,z,\zeta) = -a_1^2\sigma_1 \operatorname{sh} R_0 \sum_{m=1}^{\infty} \frac{e^{-q_m|z-\zeta|}}{2q_m} \frac{V_{(\mu);1}(R_0,\beta_m)\,V_{(\mu);j}(R_0,\beta_m)}{\alpha_{11}^0 \left\| V_{(\mu)}(r,\beta_m)\right\|^2};$$

(3) Green's functions generated by boundary condition at $r = R_{n+1} \equiv R$:

$$W_{(\mu);n+1,j}(r,z,\zeta) = \frac{\operatorname{sh} R}{\alpha_{22}^{n+1}} \sum_{m=1}^{\infty} \frac{e^{-q_m|z-\zeta|}}{2q_m} \frac{V_{(\mu);j}(r,\beta_m)\,V_{(\mu);n+1}(R,\beta_m)}{\left\| V_{(\mu)}(r,\beta_m)\right\|^2},$$

where $j = \overline{1,\, n+1}$;

(4) Green's functions generated by inhomogeneity of the conjugate conditions:

$$R_{(\mu);s2}^{jk}(r,z,\zeta) = \sum_{m=1}^{\infty} \frac{e^{-q_m|z-\zeta|}}{2q_m} a_k^2\sigma_k \frac{\operatorname{sh} R_k}{c_{1k}} Z_{(\mu);s2}^{k}(\beta_m) \frac{V_{(\mu);j}(r,\beta_m)}{\left\| V_{(\mu)}(r,\beta_m)\right\|^2},$$

where $s = 1,2$, $k = \overline{1,n}$.

The vector-function $u(r,z) = \{u_1(r,z), u_2(r,z), \ldots, u_n(r,z), u_{n+1}(r,z)\}$ determines the integral image of the unique analytical solution of the given elliptic boundary problem.

3.2. *An example: a quasi-statics problem*

Let us construct in the domain $D_n^+ = \{(t,r) : t \in (0,\infty),\ r \in I_n\}$ the bounded solution of the separate system of equation of parabolic type with the generalized Legendres operator

$$\frac{\partial u}{\partial t} + \chi_j^2 u_j - a_j^2 \Lambda_{(\mu)j}(u_j) = f_j(t,r), \qquad j = \overline{1,\, n+1} \tag{3.8}$$

with the initial conditions

$$u_j(t,r)\big|_{t=0} = g_j(r),\ r \in \left(R_{j-1}, R_j\right), \qquad j = \overline{1,n+1}, \tag{3.9}$$

the boundary conditions

$$\left.\begin{aligned} \left(\alpha_{11}^0 \frac{\partial}{\partial r} + \beta_{11}^0\right) u_1\Big|_{r=R_0} &= g_0(t), \\ \left(\alpha_{22}^{n+1} \frac{\partial}{\partial r} + \beta_{22}^{n+1}\right) u_{n+1}\Big|_{r=R_{n+1}} &= g_R(t), \end{aligned}\right\} \tag{3.10}$$

and the conjugate conditions

$$\left(\left(\alpha_{j1}^k \frac{\partial}{\partial r} + \beta_{j1}^k\right) u_k - \left(\alpha_{j2}^k \frac{\partial}{\partial r} + \beta_{j2}^k\right) u_{k+1}\right)\Bigg|_{r=R_k} = \omega_{jk}(t),$$

$$j = 1,2,\ \ k = \overline{1,n}. \tag{3.11}$$

We assume that the compatibility conditions hold:

$$\left(\alpha_{11}^{0}\frac{d}{dr}+\beta_{11}^{0}\right)g_1\Big|_{r=R_0}=g_0\left(0\right),$$

$$\left(\alpha_{22}^{n+1}\frac{d}{dr}+\beta_{22}^{n+1}\right)g_{n+1}\Big|_{r=R_{n+1}}=g_R\left(0\right),$$

and

$$\left[\left(\alpha_{j1}^{k}\frac{d}{dr}+\beta_{j1}^{k}\right)g_k-\left(\alpha_{j2}^{k}\frac{d}{dr}+\beta_{j2}^{k}\right)g_{k+1}\right]\Big|_{r=R_k}=\omega_{jk}\left(0\right)$$

for $j=1,2$, $k=\overline{1,n}$.

We rewrite system (3.8) in the matrix form

$$\begin{pmatrix}\left(\frac{\partial}{\partial t}+\chi_1^2-a_1^2\Lambda_{(\mu)_1}\right)u_1\left(t,r\right)\\ \left(\frac{\partial}{\partial t}+\chi_2^2-a_2^2\Lambda_{(\mu)_2}\right)u_2\left(t,r\right)\\ \vdots\\ \left(\frac{\partial}{\partial t}+\chi_{n+1}^2-a_{n+1}^2\Lambda_{(\mu)_{n+1}}\right)u_{n+1}\end{pmatrix}=-\begin{pmatrix}f_1\left(t,r\right)\\ f_2\left(t,r\right)\\ \vdots\\ f_{n+1}\left(t,r\right)\end{pmatrix}\tag{3.12}$$

and, similarly, the initial conditions (3.9) in the form

$$\begin{pmatrix}u_1\left(t,r\right)\\ u_2\left(t,r\right)\\ \vdots\\ u_{n+1}\left(t,r\right)\end{pmatrix}\Bigg|_{t=0}=\begin{pmatrix}g_1\left(r\right)\\ g_2\left(r\right)\\ \vdots\\ g_{n+1}\left(r\right)\end{pmatrix}.\tag{3.13}$$

Supposing that $\chi_1^2=\max\left\{\chi_1^2,\chi_2^2,\ldots,\chi_n^2,\chi_{n+1}^2\right\}$, we will apply to problem (3.12), (3.13), by the rule of multiplication of matrices, the operator matrix-row (3.4). As a result of identity (2.14), we get the Cauchy problem

$$\left(\frac{d}{dt}+q_m^2\right)\tilde{u}_m\left(t\right)=\tilde{F}_m\left(t\right),\qquad \tilde{u}_m|_{t=0}=g_m;\tag{3.14}$$

where $\tilde{F}_m\left(t\right)$ is as above.

The unique solution of the Cauchy problem (3.14) is the function

$$\tilde{u}_m\left(t\right)=e^{-q_m^2t}g_m+\int_0^t e^{-q_m^2(t-\tau)}\tilde{F}_m\left(\tau\right)d\tau.\tag{3.15}$$

Let us determine the principal solutions of the given parabolic problem: (1) the influence functions generated by the presence of the thermal sources

$$H_{(\mu);jk}\left(t,r,\rho\right)=\sum_{m=1}^{\infty}e^{-q_m^2t}\frac{V_{(\mu);j}\left(r,\beta_m\right)V_{(\mu);k}\left(\rho,\beta_m\right)}{\left\|V_{(\mu)}\left(r,\beta_m\right)\right\|^2},\qquad j,k=\overline{1,\,n+1};$$

(2) Green's functions generated by the boundary condition at $r = R_0$:

$$W_{(\mu);1j}(t,r) = -a_1^2\sigma_1 \operatorname{sh} R_0 \sum_{m=1}^{\infty} e^{-q_m^2 t} \frac{V_{(\mu);1}(R_0,\beta_m)\, V_{(\mu);j}(r,\beta_m)}{\alpha_{11}^0 \left\| V_{(\mu)}(r,\beta_m)\right\|^2},$$

$j = \overline{1,n+1}$;

(3) Green's functions generated by the boundary condition at the point $r = R_{n+1}$:

$$W_{(\mu);n+1,j}(t,r) = \sum_{m=1}^{\infty} e^{-q_m^2 t} \frac{\operatorname{sh} R}{\alpha_{22}^{n+1}} \frac{V_{(\mu);j}(r,\beta_m)\, V_{(\mu);n+1}(R,\beta_m)}{\left\| V_{(\mu)}(r,\beta_m)\right\|^2},$$

$j = \overline{1,n+1}$;

(4) Green's functions generated by the inhomogeneity of the conjugate conditions:

$$R_{(\mu);i2}^{jk}(t,r) = a_k^2\sigma_k \frac{\operatorname{sh} R_k}{c_{1k}} \sum_{m=1}^{\infty} e^{-q_m^2 t} Z_{(\mu);i2}^{k}(\beta_m) \frac{V_{(\mu);j}(r,\beta_m)}{\left\| V_{(\mu)}(r,\beta_m)\right\|^2},$$

$i = 1,2$, $k = \overline{1,n}$.

As a result of application to the matrix-element $[\tilde{u}_m(t)]$, where a function $\tilde{u}_m(t)$ is defined by a formula (3.15), by rule of multiplication of matrices of operator matrix-column (3.7), we get the unique bounded solution of parabolic initial-boundary problem (3.8)–(3.11):

$$\begin{aligned} u_j(t,r) = {} & \sum_{k=1}^{n+1} \int_0^t \int_{R_{k-1}}^{R_k} H_{(\mu);jk}(t-\tau,r,\rho)\,[f_k(\tau,\rho) \\ & \quad + \delta_+(\tau)\, g_k(\rho)]\,\sigma_k \operatorname{sh}\rho \,d\rho\, d\tau \\ & + \int_{-0}^{t} \left(W_{(\mu);1j}(t-\tau,r)\, g_0(\tau) + W_{(\mu);n+1,j}(t-\tau,r)\, g_R(\tau)\right) d\tau \\ & + \sum_{k=1}^{n} \int_0^t \Big(R_{(\mu);12}^{jk}(t-\tau,r)\,\omega_{2k}(\tau) \\ & \quad - R_{(\mu);22}^{jk}(t-\tau,r)\,\omega_{1k}(\tau)\Big)\, d\tau, \end{aligned} \tag{3.16}$$

where $j = \overline{1,n+1}$, and $\delta_+(\tau)$ is the Dirac measure concentrated at 0^+. The vector-function $u(t,r) = \{u_1(t,r), u_2(t,r), u_3(t,r), \ldots, u_n(t,r), u_{n+1}(t,r)\}$, the components $u_i(t,r)$ of which are determined by formulas (3.16), fully describes the exact analytical solution of the given parabolic problem.

3.3. *An example: a dynamics problem*

Consider the problem to construct in the domain D_n^+ the bounded solution of the separate hyperbolic system of equations with the generalized Legendres operator

$$\left(\frac{\partial^2}{\partial t^2} + \chi_j^2 u_j - a_j^2 \Lambda_{(\mu)j}\right) u_j(t,r) = f_j(t,r), \quad j = \overline{1,n+1} \tag{38}$$

under the initial conditions

$$u_j(t,r)\big|_{t=0} = \varphi_j(r), \quad \left.\frac{\partial u_j}{\partial t}\right|_{t=0} = \psi_j(r), \quad j = \overline{1,\, n+1}, \tag{39}$$

and the boundary conditions (3.10) and the conjugate conditions (3.11).

We assume that the compatibility conditions are satisfied:

$$\left.\left(\alpha_{11}^0 \frac{d}{dr} + \beta_{11}^0\right)\varphi_1\right|_{r=R_0} = g_0(0),$$

$$\left.\left(\alpha_{22}^{n+1} \frac{d}{dr} + \beta_{22}^{n+1}\right)\varphi_{n+1}\right|_{r=R_{n+1}} = g_R(0);$$

$$\left.\left[\left(\alpha_{j1}^k \frac{d}{dr} + \beta_{j1}^k\right)\varphi_k - \left(\alpha_{j2}^k \frac{d}{dr} + \beta_{j2}^k\right)\varphi_{k+1}\right]\right|_{r=R_k} = \omega_{jk}(0)$$

for $j = 1, 2$, $k = \overline{1,n}$;

$$\left.\left(\alpha_{11}^0 \frac{d}{dr} + \beta_{11}^0\right)\psi_1\right|_{r=R_0} = g_0'(0),$$

$$\left.\left(\alpha_{22}^{n+1} \frac{d}{dr} + \beta_{22}^{n+1}\right)\psi_{n+1}\right|_{r=R_{n+1}} = g_R'(0),$$

and

$$\left.\left[\left(\alpha_{j1}^k \frac{d}{dr} + \beta_{j1}^k\right)\psi_k - \left(\alpha_{j2}^k \frac{d}{dr} + \beta_{j2}^k\right)\psi_{k+1}\right]\right|_{r=R_k} = \omega_{jk}'(0)$$

for $j = 1, 2$, $k = \overline{1,n}$.

Constructed according to the logical scheme of solving parabolic problem (3.8)–(3.11), the unique bounded solution of the given hyperbolic problem has the form

$$\begin{aligned} u_j(t,r) = {} & \sum_{k=1}^{n+1} \int_0^t \int_{R_{k-1}}^{R_k} H_{(\mu);jk}(t-\tau,r,\rho)\big(f_k(\tau,\rho) \\ & \qquad + \delta_+(\tau)\,\psi_k(\rho)\big)\,\sigma_k \operatorname{sh}\rho \, d\rho \, d\tau \\ & + \frac{\partial}{\partial t} \sum_{k=1}^{n+1} \int_{R_{k-1}}^{R_k} H_{(\mu);jk}(t,r,\rho)\,\varphi_k(\rho)\,\sigma_k \operatorname{sh}\rho \, d\rho \\ & + \int_0^t \big(W_{(\mu);1j}(t-\tau,r)\; g_0(\tau) + W_{(\mu);n+1,j}(t-\tau,r)\, g_R(\tau)\big)\, d\tau \\ & + \sum_{k=1}^{n} \int_0^t \Big(R^{jk}_{(\mu);12}(t-\tau,r)\,\omega_{2k}(\tau) \\ & \qquad - R^{jk}_{(\mu);22}(t-\tau,r)\,\omega_{1k}(\tau)\Big)\, d\tau, \qquad j = \overline{1,n+1}. \end{aligned} \tag{3.17}$$

The principal solutions appearing in formulas (3.17) are determined by the formulas given above, in which the function $\exp(-q_m^2 t)$ is replaced by $q_m^{-1} \sin q_m t$. Note that the constructed solutions of statics, quasi-statics and dynamics problems are of algorithmic character and continuously rely on parameters and data of the problem under consideration. These solutions can be used in both theoretical research and engineering calculations.

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PERIODIC OSCILLATION IN THE DELAYED CF-SYSTEM

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Abstract. This paper deals with the stability analysis of an autonomous system of differential equations and a nonlinear system of reaction diffusion equations with discrete time delay which models the behavior of a predator population whose dynamics depends on the past history of the prey population on which it is acting. This delay is regarded as the lag due to gestation of the predator. Basic mathematical properties of the model are discussed. The most important observation is that as the delay is increased, the originally asymptotic stable interior equilibrium loses its stability, furthermore at a certain critical value a Hopf bifurcation takes place: a small amplitude periodic solution arises. The possibility of a Turing–Hopf, i. e., the occurrence of time-periodic pattern is also studied.

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1. INTRODUCTION

The system of equations constructed by M. Cavani and M. Farkas (CF-system), which describes the predator-prey interaction with Holling's type functional response of the predator is governed by

$$\left.\begin{aligned} \dot{N} &= \varepsilon N\,(1-N/K) - \beta NP/(\beta+N), \\ \dot{P} &= -P(\gamma+\delta P)/(1+P) + \beta NP/(\beta+N), \end{aligned}\right\} \tag{1.1}$$

where dot means differentiation with respect to time t; $N(t)$ and $P(t)$ are the prey and predator densities at time t, $\varepsilon > 0$, $\beta > 0$, $K > 0$ and $\kappa > 0$ are the specific growth rate of prey, the conversion rate and the carrying capacity with respect to the prey, $\gamma > 0$ and $\delta > 0$ are the minimal mortality and the limiting mortality of the predator, respectively (see [5]). This system has the advantage over the other Gause-type predator-prey systems (cf. [11]) that here predator mortality is neither a constant nor an unbounded function (as it is, e. g., in [19] and, respectively, in [16]), still, it is increasing with quantity. To have more realism the authors of the above system have introduced an infinite delay into the second equation of the system for prey density and have shown that under some conditions the increase of the delay destabilizes the originally stable equilibrium by Hopf bifurcation. Delays play an important role in the dynamics of populations. In many processes of the real word, especially, in a great many biological phenomena, the present dynamics, the present rate of change of the state variables depends not only on the present state of the processes but also on the history of the phenomenon, on past values of the state variables. As in physical sciences, in biology there are two traditions on which one can call for insight into delay problems (see [15]). In population biology one finds many systems in which the emphasis is on the distribute lag (as in [5]) but there is also a long tradition of using discrete delays to account for individual development (see [17] and the references therein).

In this paper we incorporate a discrete lag into the CF-system and into its reaction diffusion version and study the existence of periodic solutions. This delay will be considered as the lag due to the gestation of the predator.

The organization of the paper is as follows: the Section 2 is about the basic mathematical properties of the delayed CF-system such as steady states, nonnegativity and existence of solutions, respectively. In Section 3 we will discuss the stability of the trivial solution and the existence of Hopf bifurcation. In Section 4 the CF-system will be modified by assuming that the prey and predator are diffusing according to Ficks law in a spatial habitat with hexagonal boundary and the effect of the delay on the stability of the positive equilibrium will be studied.

2. The model

Let us consider the delayed version of (1.1)

$$\left.\begin{aligned} \dot{N} &= \varepsilon N\,(1-N/K) - \beta NP/(\beta+N), \\ \dot{P} &= -P(\gamma+\delta P)/(1+P) + \beta N(\cdot-\tau)P/(\beta+N(\cdot-\tau)) \end{aligned}\right\} \tag{2.1}$$

with the initial conditions $\varphi = (\phi_1, P)$ in the Banach space

$$\left\{\varphi \in C\left([-\tau,0], \mathbb{R}^2_+\right) \;\middle|\; \phi_1(\theta) = N(\theta)\right\}$$

where $\varphi_1(\theta) \geq 0$ $(\theta \in [-\tau, 0))$.

In [5] the authors have shown that the following conditions are reasonable and natural:

$$\gamma < \beta \leq \delta, \tag{2.2}$$

$$\beta < K, \tag{2.3}$$

$$\gamma < \beta K/(\beta+K)\,; \tag{2.4}$$

and under these conditions (1.1) has at least three equilibria: $(0,0)$ and $(K,0)$ which are unstable and at least one equilibrium with positive coordinates $(\overline{N}, \overline{P})$ as intersection of the prey null-cline

$$P = H_1(N) := (K-N)(\beta+N)\varepsilon/(\beta K)$$

and the predator null-cline

$$P = H_2(N) := ((\beta-\gamma)N - \beta\gamma)\,/\,((\delta-\beta)N + \beta\delta)$$

which is asymptotically stable if

$$0 < (K-\beta)/2 \le \bar{N} \tag{2.5}$$

holds. If $0 < \bar{N} < (K-\beta)/2$, i. e., $\bar{N}$ lies in the Allée-effec zone (in the zone where the increase of the prey density is favorable to its growth rate), then it may or may not be stable.

Example 1.

(1) For $\beta := 0.5000$, $\gamma := 0.1000$, $\delta := 0.5000$, $\varepsilon := 0.1000$ and $K := 5.0000$ the equilibrium $(\bar{N}, \bar{P}) = (0.2100, 0.1360)$ lies in the Allée-effect zone $(0.2100 < 4.5000/2)$ and is asymptotically stable (see Figure 1).

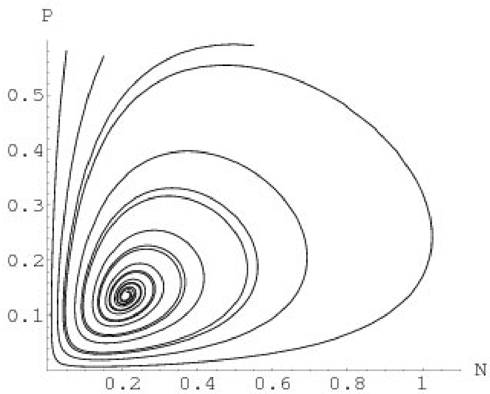

FIGURE 1. The interior equilibrium of the system (1.1) showing its asymptotic stability (MATHEMATICA®)

(2) For $\beta := 0.1065$, $\gamma := 0.0085$, $\delta := 0.1065$, $\varepsilon := 1.6000$, and $K := 35.3500$, the equilibrium $(\bar{N}, \bar{P}) = (15.1736, 131.0240)$ lies in the Allée-effect zone $(15.1736 < 35.2435/2)$ and is unstable.

In what follows we assume that inequalities (2.2)–(2.4) hold.

System (2.1) describes an animal population, therefore it is very important to prove that there are positive initial data for which both predator and prey quantities remain positive, we will even prove that all positive initial data have this property.

Clearly, system (2.1) is of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x}(\cdot - \tau)),$$

with Lipschitz continuous $\boldsymbol{f}$, continuous $\boldsymbol{g}$ and $\boldsymbol{g}(\boldsymbol{s}) \ge 0$ for $\boldsymbol{s} \in (\mathbb{R}^+)^2$. Thus, applying Theorem 1.2 from [1] and the majorant method described, e. g., in [10, p. 101], we see the validity of the following

Theorem 1. *Solutions of* (2.1) *with nonnegative initial conditions are defined on* $[0,+\infty)$ *and remain nonnegative for all* $t \geq 0$.

3. Stability of the equilibria

Clearly the equilibria of (1.1) are steady states of (2.1), also. Now, we are going to determine the stability of equilibria for system (2.1). The variational system of (2.1) with respect to the solution $(\widetilde{N},\widetilde{P})$ (see [9, p. 478]) takes the form

$$\begin{pmatrix}\dot V_N\\ \dot V_P\end{pmatrix}=\begin{pmatrix}\varepsilon-\frac{2\varepsilon\widetilde{N}}{K}-\frac{\beta^2\widetilde{P}}{(\beta+\widetilde{N})^2} & -\frac{\beta\widetilde{N}}{\beta+\widetilde{N}}\\ 0 & \frac{\beta\widetilde{N}}{\beta+\widetilde{N}}-\frac{\gamma(1+\widetilde{P})-\delta\widetilde{P}(2+\widetilde{P})}{(1+\widetilde{P})^2}\end{pmatrix} \times\begin{pmatrix}V_N\\ V_P\end{pmatrix}+\begin{pmatrix}0 & 0\\ \frac{\beta^2\widetilde{P}}{(\beta+\widetilde{N})^2} & 0\end{pmatrix}\begin{pmatrix}V_N(\cdot-\tau)\\ V_P(\cdot-\tau)\end{pmatrix}.$$

Thus, the linearized system: 1. At $(0,0)$ has the form

$$\begin{pmatrix}\dot V_N\\ \dot V_P\end{pmatrix}=\begin{pmatrix}\varepsilon & 0\\ 0 & -\gamma\end{pmatrix}\begin{pmatrix}V_N\\ V_P\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}V_N(\cdot-\tau)\\ V_P(\cdot-\tau)\end{pmatrix}$$

which leads one to the characteristic equation

$$\Delta_{(0,0)}(z,\tau):=(\varepsilon-z)(-\gamma-z)=0;$$

2. At $(K,0)$ is

$$\begin{pmatrix}\dot V_N\\ \dot V_P\end{pmatrix}=\begin{pmatrix}-\varepsilon & -\frac{\beta K}{\beta+K}\\ 0 & \frac{\beta K}{\beta+K}-\gamma\end{pmatrix}\begin{pmatrix}V_N\\ V_P\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}V_N(\cdot-\tau)\\ V_P(\cdot-\tau)\end{pmatrix}$$

whose (asymptotical) stability is determined by the real part of the roots of the characteristic equation

$$\Delta_{(K,0)}(z,\tau):=(-\varepsilon-z)\left(\frac{\beta K}{\beta+K}-\gamma-z\right)=0;$$

3. At $(\bar N,\bar P)$ has the form

$$\begin{pmatrix}\dot V_N\\ \dot V_P\end{pmatrix}=\begin{pmatrix}\eta\Theta_1\Theta_2 & -\Theta_1\\ 0 & -\eta\Theta_3\Theta_4\end{pmatrix}\begin{pmatrix}V_N\\ V_P\end{pmatrix}+\begin{pmatrix}0 & 0\\ \beta^2\eta\Theta_3 & 0\end{pmatrix}\begin{pmatrix}V_N(\cdot-\tau)\\ V_P(\cdot-\tau)\end{pmatrix},$$

where $\eta:=\varepsilon(K\beta)^{-1}$ and

$$\left.\begin{aligned}\Theta_1 &:= \frac{\beta\bar N}{\beta+\bar N}, & \Theta_2 &:= K-\beta-2\bar N,\\ \Theta_3 &:= \frac{K-\bar N}{\beta+\bar N}, & \Theta_4 &:= \frac{\left((\delta-\beta)\bar N+\beta\delta\right)^2}{\delta-\gamma}\end{aligned}\right\}$$

(see [6]). The corresponding characteristic equation is

$$\begin{aligned}\Delta_{(\bar N,\bar P)}(z,\tau) &:= \det\begin{pmatrix}\eta\Theta_1\Theta_2 - z & -\Theta_1\\ \beta^2\eta\Theta_3\exp(-z\tau) & -\eta\Theta_3\Theta_4 - z\end{pmatrix}\\ &= z^2+\eta(\Theta_3\Theta_4-\Theta_1\Theta_2)z-\eta^2\Theta_1\Theta_2\Theta_3\Theta_4+\\ &\quad+\beta^2\eta\Theta_1\Theta_3\exp(-z\tau)=0.\end{aligned}$$

Since ε is a positive root of $\Delta_{(0,0)}$, it follows that $(0,0)$ is an unstable equilibrium of (2.1) for all $\tau \geq 0$. Because of (2.4), $\beta K/(\beta+K)-\gamma$ is a positive root of $\Delta_{(K,0)}$, and, therefore, $(K,0)$ is an unstable equilibrium of (2.1) for all $\tau \geq 0$.

Clearly, $\Delta_{(\bar N,\bar P)}$ has the form

$$\Delta_{(\bar N,\bar P)}(z,\tau) \equiv p(z)+q(z)\exp(-z\tau)$$

where $p(z) := z^2+\eta(\Theta_3\Theta_4-\Theta_1\Theta_2)z-\eta^2\Theta_1\Theta_2\Theta_3\Theta_4$ and $q(z) := \beta^2\eta\Theta_1\Theta_3$ are trivially analytic functions in a right half-plane $\operatorname{Re} z > -c$ $(c>0)$ that satisfy the following conditions:

(1) p and q have no common imaginary root;
(2) $\overline{p(-iy)} = p(iy)$, $\overline{q(-iy)} = q(iy)$ $(y \in \mathbb{R})$;
(3) $p(0)+q(0) \neq 0$;
(4) $\limsup_{|z|\to\infty,\operatorname{Re} z\geq 0}\left|\dfrac{q(z)}{p(z)}\right| < 1.$

To obtain stability switch one needs to have an imaginary root of $\Delta_{(\bar N,\bar P)}$. Let $z = iy$. Then $\Delta_{(\bar N,\bar P)}(z,\tau)=0 \Leftrightarrow |p(z)|=|q(z)|$ and

$$\begin{aligned}\sin(y\tau) &= \frac{p^I(y)q^R(y)-p^R(y)q^I(y)}{|q(iy)|^2},\\ \cos(y\tau) &= -\frac{p^R(y)q^R(y)+p^I(y)q^I(y)}{|q(iy)|^2},\end{aligned}$$

where $p(iy) =: p^R(y)+p^I(y)i$ and $q(iy) =: q^R(y)+q^I(y)i$. In this case, one has $|q(iy)|^2 = \beta^4\eta^2\Theta_1^2\Theta_3^2 > 0$.

Let us define the auxiliary function $F(y) := |p(iy)|^2-|q(iy)|^2$ $(y\in\mathbb{R})$. Hence

$$F(y) = y^4+\eta^2\left[\Theta_1^2\Theta_2^2+\Theta_3^2\Theta_4^2\right]y^2+\eta^2\Theta_1^2\Theta_3^2\left[\eta^2\Theta_2^2\Theta_4^2-\beta^4\right] \quad (y\in\mathbb{R}).$$

The necessary condition for the change in stability is the existence of $\psi_0>0$ such that $F(\psi_0)=0$. Thus, applying the theorem of [2,7] (see also [14]) we can prove the following

Theorem 2. *If*

(1) $\beta^2 < |\eta\Theta_2\Theta_4|$, *then the equilibrium* $(\bar N,\bar P)$ *of* (2.1) *remains stable for all* $\tau \geq 0$*;*

(2) $\beta^2 > |\eta\Theta_2\Theta_4|$, *then as* τ *increases and passes through* τ_0,

$$\tau_0 := \frac{1}{\psi_0}\sin^{-1}\left(\frac{(\Theta_3\Theta_4-\Theta_1\Theta_2)\psi_0}{\beta^2\Theta_1\Theta_3}\right),$$

where

$$\psi_0 := \Big\{\Big(-\eta^2\left[\Theta_1^2\Theta_2^2+\Theta_3^2\Theta_4^2\right] + \sqrt{\eta^2\left[\Theta_1^2\Theta_2^2-\Theta_3^2\Theta_4^2\right]^2+4\beta^4\eta^2\Theta_1^2\Theta_3^2}\Big)/2\Big\}^{1/2},$$

the equilibrium $(\bar{N},\bar{P})$ *undergoes a Poincaré–Andronov–Hopf bifurcation, i. e., there occurs a small amplitude periodic solution with a period approximately equal to* $2\pi/\psi_0$.

Proof. Step 1. If $\beta^2 < |\eta\Theta_2\Theta_4|$, then F has no positive root, therefore no stability switch may occur, i. e., the stability of $(\bar{N},\bar{P})$ does not change as τ is increased from zero to infinity.

Step 2. If $\beta^2 > |\eta\Theta_2\Theta_4|$, then as F has one positive root ψ_0, and there may be only the critical value τ_0 for which

$$\sin\psi_0\tau_0 = \frac{\eta\,(\Theta_3\Theta_4-\Theta_1\Theta_2)\,\psi_0}{\beta^2\eta\Theta_1\Theta_3}$$

holds and the the equilibrium $(\bar{N},\bar{P})$ loses its stability. Therefore we only need to check the sign of $\varphi'(\tau_0)$ where $z=\varphi+\psi i$ and $\varphi(\tau_0)=0$. Using the implicit function theorem we are going to determine the derivative of the implicit function z_0 at τ_0 (where z_0 denotes the root of (3.3) that assumes the value $i\psi_0$ at τ_0)

$$\begin{aligned} z_0'(\tau_0) &= -\frac{\partial_\tau\Delta_{(\bar{N},\bar{P})}(i\psi_0,\tau_0)}{\partial_z\Delta_{(\bar{N},\bar{P})}(i\psi_0,\tau_0)} \\ &= \frac{i\psi_0\beta^2\eta\Theta_1\Theta_3}{[2i\psi_0+\eta(\Theta_3\Theta_4-\Theta_1\Theta_2)]\exp(i\psi_0\tau_0)-\tau_0\beta^2\eta\Theta_1\Theta_3}. \end{aligned}$$

Hence we have

$$\begin{aligned} \varphi'(\tau_0) &= \frac{d}{d\tau}\operatorname{Re} z_0(\tau)|_{\tau=\tau_0} = \operatorname{Re}\left.\frac{dz_0(\tau)}{d\tau}\right|_{\tau=\tau_0} \\ &= \psi_0^2\beta^4\eta^2\Theta_1^2\Theta_3^2\left[2\psi_0^2+\eta^2\left(\Theta_1^2\Theta_2^2+\Theta_3^2\Theta_4^2\right)\right] \\ &\quad\times\Big\{\big[\eta(\Theta_3\Theta_4-\Theta_1\Theta_2)\psi_0^2-\eta^3(\Theta_3\Theta_4-\Theta_1\Theta_2) \\ &\quad\times\ \Theta_1\Theta_2\Theta_3\Theta_4+\tau_0\beta^4\eta^2\Theta_1^2\Theta_3^2\big]^2 \\ &\quad+\left[2\psi_0^3+\eta^2\psi_0\left(\Theta_1^2\Theta_2^2+\Theta_3^2\Theta_4^2\right)\right]^2\Big\}^{-1} > 0. \end{aligned}$$

Thus, by the Hopf bifurcation theorem [8, 12] the theorem is proved. □

Example 2. In the case where $\beta := 0.1000$, $\gamma := 0.0100$, $\delta := 0.1055$, $\varepsilon := 1.0000$, and $K := 1.0000$m the equilibrium $(\overline{N}, \overline{P}) = (0.4486, 3.0250)$ lies in the Allée-effect zone $(0.4486 < 0.9000/2)$ and is asymptotically stable for $\tau = 0$ and for $\tau_0 := 1.9090$ there is a switching in stability (see Figure 2).

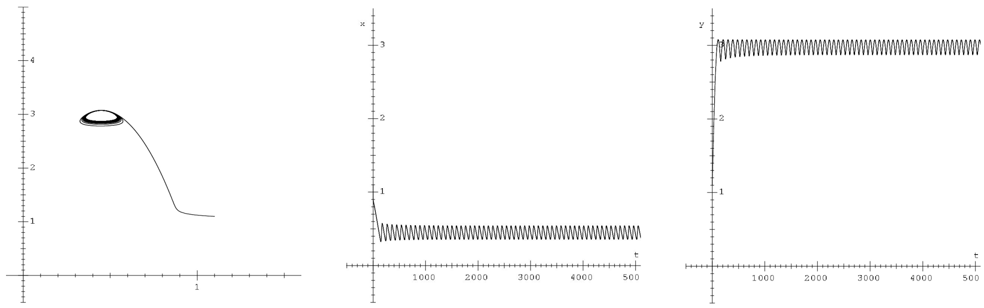

FIGURE 2. Time evolution of system (2.1) for $\beta := 0.1000$, $\gamma := 0.0100$, $\delta := 0.1055$, $\varepsilon := 1.1000$, the delay $\tau_0 := 2.2$ and time $t = 5100.0000$ with the notation $x := N$ and $y := P$ (DifEqu®).

4. THE MODEL WITH DIFFUSION

Let us modify system (2.1) assuming that the prey and predator are diffusing to Fick's law in the spatial habitat

$$\varOmega_H := \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, |x| < \frac{H\sqrt{3}}{2},\ \left| y + \frac{x}{\sqrt{3}} \right| < H \right\} \quad (H > 0)$$

i. e., consider the reaction-diffusion system

$$\left.\begin{aligned} \partial_t N &= \Delta_{\boldsymbol{r}} N + \varepsilon N(1 - N/K) - \beta NP/(\beta + N), \\ \partial_t P &= d\Delta_{\boldsymbol{r}} P - P(\gamma + \delta P)/(1 + P) + \beta N(\cdot - \tau)P/(\beta + N(\cdot - \tau)), \end{aligned}\right\} \tag{4.1}$$

where d is the diffusion coefficient about which we assume only that it has a positive sign,

$$(\boldsymbol{n} \cdot \nabla_{\boldsymbol{r}})\, \mathrm{col}\,[N, P] = \mathbf{0} \quad \text{in} \quad \partial\varOmega_H \times [-\tau, +\infty) \tag{4.2}$$

and

$$\mathrm{col}\,[N, P] = \boldsymbol{\Phi} \geq \mathbf{0} \quad \text{on} \quad \overline{\varOmega}_H \times [-\tau, 0]. \tag{4.3}$$

Clearly, a spatially constant solution $\mathrm{col}\,[N(\cdot), P(\cdot)]$ of system (4.1) satisfies boundary conditions (4.2) and system (2.1). The equilibria of system (2.1) are constant solutions of (4.1), (4.2) at the same time.

The linearized equation: 1. At $(0,0)$ has the form

$$\left.\begin{aligned}\partial_t V_N &= \Delta_{\boldsymbol{r}} V_N + \varepsilon V_N\,,\\ \partial_t V_P &= d\Delta_{\boldsymbol{r}} V_P - \gamma V_P\,,\end{aligned}\right\} \tag{4.4}$$

which is asymptotically stable if for all $n \in \mathbb{N}$ the polynomial

$$\begin{aligned}\widetilde{\Delta}_{(0,0)}(z,\tau) &:= \det\begin{pmatrix}\varepsilon-\lambda_n-z & 0\\ 0 & -\gamma-\lambda_n d-z\end{pmatrix}\\ &= z^2-(\varepsilon-\lambda_n-\gamma-\lambda_n d)z+(\lambda_n-\varepsilon)(\gamma+\lambda_n d)\end{aligned}$$

has roots with negative real part [3, 4], where $\lambda_n := 16\pi^2/(9H^2)n^2$ $(n \in \mathbb{N})$ is the nth eigenvalue of the minus Laplacian on Ω_H with no-flux boundary conditions.

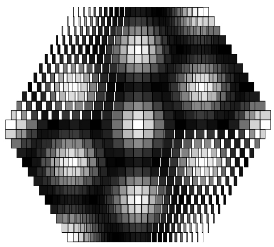

FIGURE 3. Eigenfunction of the minus Laplacian corresponding to the eigenvalue $\lambda_3(H)$ with $H = 5$ (Maple®).

2. At $(K,0)$ is

$$\left.\begin{aligned}\partial_t V_N &= \Delta_{\boldsymbol{r}} V_N - \varepsilon V_N - \frac{\beta K}{\beta+K}V_P\,,\\ \partial_t V_P &= d\Delta_{\boldsymbol{r}} V_P + \left(\frac{\beta K}{\beta+K}-\gamma\right)V_P\,,\end{aligned}\right\} \tag{4.5}$$

whose (asymptotical) stability is determined by the real part of the roots of the polynomial

$$\begin{aligned}\widetilde{\Delta}_{(K,0)}(z,\tau) &:= \det\begin{pmatrix}-\varepsilon-\lambda_n-z & -\frac{\beta K}{\beta+K}\\ 0 & \frac{\beta K}{\beta+K}-\gamma-\lambda_n d-z\end{pmatrix}\\ &= z^2+\left(\varepsilon+\lambda_n-\frac{\beta K}{\beta+K}+\gamma+\lambda_n d\right)z+(\varepsilon+\lambda_n)\\ &\quad\times\left(\gamma+\lambda_n d-\frac{\beta K}{\beta+K}\right).\end{aligned}$$

3. At $(\bar{N}, \bar{P})$ is

$$\left.\begin{aligned} \partial_t V_N &= \Delta_{\boldsymbol{r}} V_N + \eta\Theta_1\Theta_2 V_N - \Theta_1 V_P \\ \partial_t V_P &= d\Delta_{\boldsymbol{r}} V_P - \eta\Theta_3\Theta_4 V_P + \beta^2\eta\Theta_3 V_N(\cdot - \tau) \end{aligned}\right\} \tag{4.6}$$

which is asymptotically stable if for all $n \in \mathbb{N}$ the quasi-polynomial

$$\begin{aligned} \widetilde{\Delta}_{(\bar{N},\bar{P})}(z,\tau) &:= \det\begin{pmatrix} \eta\Theta_1\Theta_2 - \lambda_n - z & -\Theta_1 \\ \beta^2\eta\Theta_3\exp(-z\tau) & -\eta\Theta_3\Theta_4 - \lambda_n d - z \end{pmatrix} \\ &= z^2 - [\eta(\Theta_1\Theta_2 - \Theta_3\Theta_4) - \lambda_n(d+1)]z \\ &\qquad - [\eta\Theta_1\Theta_2 - \lambda_n][\eta\Theta_3\Theta_4 + \lambda_n d] + +\beta^2\eta\Theta_1\Theta_3\exp(-z\tau) \end{aligned}$$

has roots with negative real part (see [18]).

Since for $n = 0$ the constant term of $p_{(0,0)}$ is negative, therefore $(0,0)$ remains an unstable equilibrium of (4.1)–(4.3) for all $\tau > 0$.

Because of (2.4) there exist $n \in \mathbb{N}$ such that the constant term $p_{(K,0)}$ is negative, therefore $(K,0)$ remains an unstable equilibrium of (4.1)–(4.3) for all $\tau > 0$.

The stability of $p_{(\bar{N},\bar{P})}$ depends on several parameters. In the remainder of this section we assume that $d > 1$. In the case without delay,

$$\begin{aligned} \widetilde{\Delta}_{(\bar{N},\bar{P})}(z,0) \equiv z^2 &- [\eta(\Theta_1\Theta_2 - \Theta_3\Theta_4) - \lambda_n(d+1)]z \\ &- [\eta\Theta_1\Theta_2 - \lambda_n][\eta\Theta_3\Theta_4 + \lambda_n d] + \beta^2\eta\Theta_1\Theta_3, \end{aligned}$$

which is stable (see [6, 13]) if

(1) $\Theta_2 \leq 0$, i. e., the equilibrium $(\bar{N}, \bar{P})$ lies outside the Allée-effect zone (on the descending branch of the prey null-cline) or
(2) the parameters ε, β, K, γ, δ are Turing admissible which implies $\Theta_2 > 0$, and

$$\frac{\eta\Theta_1\Theta_2}{\lambda_1} \leq 1 \quad \text{or} \quad \Theta_2 \leq \frac{16\pi^2}{9H^2\eta\Theta_1}. \tag{4.7}$$

With respect to the parameter τ, the quasi-polynomial for $p_{(\bar{N},\bar{P})}$ has a similar structure as $\Delta_{(\bar{N},\bar{P})}$:

$$\widetilde{\Delta}_{(\bar{N},\bar{P})}(z,\tau) \equiv \widetilde{p}(z) + \widetilde{q}(z)\exp(-z\tau) \tag{4.8}$$

where

$$\widetilde{p}(z) := z^2 - [\eta(\Theta_1\Theta_2 - \Theta_3\Theta_4) - \lambda_n(d+1)]z - (\eta\Theta_1\Theta_2 - \lambda_n)(\eta\Theta_3\Theta_4 + \lambda_n d)$$

and $\widetilde{q}(z) := \beta^2\eta\Theta_1\Theta_3$ are trivially analytic functions in the right half-plane $\operatorname{Re} z > -c$ $(c > 0)$ and satisfy the following conditions:

(1) $\widetilde{p}$ and $\widetilde{q}$ have no common imaginary root;
(2) $\overline{\widetilde{p}(-iy)} = \widetilde{p}(iy)$, $\overline{\widetilde{q}(-iy)} = \widetilde{q}(iy)$ $(y \in \mathbb{R})$;
(3) $\widetilde{p}(0) + \widetilde{q}(0) \neq 0$;

(4) $\limsup_{|z|\to\infty,\,\mathrm{Re}\,z\geq 0}\left|\dfrac{\widetilde{q}(z)}{\widetilde{p}(z)}\right| < 1.$

To obtain stability switch one has to examine the existence of real roots of the function (see [2, 7])

$$\begin{aligned}\widetilde{F}(y) &:= |\widetilde{p}(iy)|^2 - |\widetilde{q}(iy)|^2 = \left|-y^2 + Aiy + B\right|^2 - |C|^2 \\ &= \left(B - y^2\right)^2 + A^2y^2 + C^2 = y^4 + (A^2 - 2B)y^2 + B^2 - C^2 \quad (y \in \mathbb{R}),\end{aligned}$$

where

$$\begin{aligned}A &:= \lambda_n(d+1) - \eta\,(\Theta_1\Theta_2 - \Theta_3\Theta_4)\,,\\ B &:= -\,(\eta\Theta_1\Theta_2 - \lambda_n)\,(\eta\Theta_3\Theta_4 + \lambda_n d)\,,\end{aligned}$$

and $C := \beta^2\eta\Theta_1\Theta_3$. A stability switch may occur only if $\widetilde{F}$ has a positive root $\widetilde{\psi}_0$. Viéte's formula implies that it may happen in the following three cases:

$$B^2 - C^2 > 0, \qquad A^2 - 2B < 0, \qquad (A^2 - 2B)^2 = 4(B^2 - C^2); \tag{4.9a}$$

$$B^2 - C^2 \in \mathbb{R}, \qquad A^2 - 2B < 0, \qquad (A^2 - 2B)^2 > 4(B^2 - C^2); \tag{4.9b}$$

$$B^2 - C^2 < 0, \qquad A^2 - 2B \in \mathbb{R}, \qquad (A^2 - 2B)^2 > 4(B^2 - C^2), \tag{4.9c}$$

where case (4.9c) reduces to $B^2 - C^2 < 0$ because this inequality implies the condition $(A^2 - 2B)^2 > 4(B^2 - C^2)$. Because $(A^2 - 2B)^2 - 4(B^2 - C^2) = A^2(A^2 - 4B) + 4C^2$ and $A^2 - 4B = [\lambda_n(d-1) + \eta(\Theta_1\Theta_2 + \Theta_3\Theta_4)]^2 > 0$ we deal with one of the last two cases. If $B^2 - C^2 < 0$, then we have one positive solution

$$\widetilde{\psi}_0 := \frac{1}{\sqrt{2}}\sqrt{2B - A^2 + \sqrt{A^2(A^2 - 4B) + 4C^2}}$$

at which switching can occur. The condition $B^2 - C^2 < 0$ is equivalent to

$$\begin{aligned}&[\eta\Theta_1\Theta_2 - \lambda_n]^2\,[\eta\Theta_3\Theta_4 + \lambda_n d]^2 - \beta^4\eta^2\Theta_1^2\Theta_3^2\\ &\qquad = \left[\eta^2\Theta_1\Theta_2\Theta_3\Theta_4 + \eta\Theta_1\Theta_2\lambda_n d - \eta\Theta_3\Theta_4\lambda_n - \lambda_n^2 d\right]^2 - \left[\beta^2\eta\Theta_1\Theta_3\right]^2\\ &\qquad = \left[\Theta_1\Theta_3\eta\left(\eta\Theta_2\Theta_4 + \beta^2\right) + \lambda_n(\eta\Theta_1\Theta_2 d - \eta\Theta_3\Theta_4 - \lambda_n d)\right]\\ &\qquad\quad \times\left[\Theta_1\Theta_3\eta\left(\eta\Theta_2\Theta_4 - \beta^2\right) + \lambda_n(\eta\Theta_1\Theta_2 d - \eta\Theta_3\Theta_4 - \lambda_n d)\right] < 0.\end{aligned}$$

Thus, we conclude that if $\beta^2 > |\eta\Theta_1\Theta_2|$ and

(1) $\Theta_2 < 0$, i. e., the equilibrium $(\bar{N}, \bar{P})$ lies outside the Allée-effect zone, or
(2) $\Theta_2 > 0$ and $\frac{\eta\Theta_1\Theta_2}{\lambda_1} \leq 1$, i. e., the equilibrium $(\bar{N}, \bar{P})$ lies in Allée-effect zone and is asymptotically stable if the diffusive system is a system without delay,

then for every integer n such that

$$d\lambda_n^2 + \eta(\Theta_3\Theta_4 - \Theta_1\Theta_2 d)\lambda_n < \Theta_1\Theta_3\eta\left(\eta\Theta_2\Theta_4 + \beta^2\right) \tag{4.10}$$

there may be a switching in stability at the point

$$\tau_0(n) = \frac{1}{\widetilde{\psi}_0} \sin^{-1} \left(\frac{[\lambda_n(d+1) - \eta(\Theta_1\Theta_2 - \Theta_3\Theta_4)]\,\widetilde{\psi}_0}{\beta^2\eta\Theta_1\Theta_3} \right).$$

Numerical calculations show that in the following cases there is a switch of stability indeed.

Example 3. For $\beta := 0.1000$, $\gamma := 0.0100$, $\delta := 0.1055$, $\varepsilon := 1.0000$ and $K := 1.0000$ the asymptotically stable equilibrium $(\bar{N}, \bar{P}) = (0.4486, 3.0250)$ of (1.1) lies inside the Allée-effect zone $(0.4486 < 0.9000/2)$, for $d = 1.1$, $H = 55$ it is diffusionally stable and (4.9) assumes the form

$$0.0030 \cdot n^4 + 0.0008 \cdot n^2 < 0.0083$$

which is fulfilled for $n \in \{0; 1\}$ and

(1) for $n = 0$ there is a switching in stability at $\tau_0(0) = 18.9969$ because for a root $z = \varphi + \psi i$ of (4.8) at which $\varphi(\tau_0(0)) = 0$ holds $\operatorname{sign} \varphi'(\tau_0(0)) = 1$;
(2) for $n = 1$ there is a switching in stability at $\tau_0(1) = 1508.1225$ because for a root $z = \varphi + \psi i$ of (4.8) at which $\varphi(\tau_0(0)) = 0$ holds $\operatorname{sign} \varphi'(\tau_0(0)) = 1$.

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HARDY–SOBOLEV TYPE INEQUALITIES FOR GENERALIZED BAOUENDI–GRUSHIN OPERATORS

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Abstract. In this paper we establish a class of Hardy–Sobolev type inequalities related to generalized Baouendi–Grushin operators. Our results contain the well-known Hardy type inequality and Sobolev type inequality for the class of operators. Furthermore, some new inequalities are obtained.

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1. INTRODUCTION

In [6], a Hardy type inequality for the generalized Baouendi–Grushin vector fields

$$Z_i = \frac{\partial}{\partial x_i}, \quad Z_{n+j} = |x|^\alpha \frac{\partial}{\partial y_j}, \quad (1 \le i \le n, 1 \le j \le m)$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2 \dots, y_m) \in \mathbb{R}^m$, $\alpha > 0$, is given by

$$\int_{\mathbb{R}^{n+m}} \frac{u^2}{d^2} \psi_{2\alpha} dxdy \le \left(\frac{2}{Q-2}\right)^2 \int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy, \tag{1.1}$$

for $u \in L^2\left(\mathbb{R}^{n+m}, \psi_{2\alpha} dxdy\right)$ and $|\nabla_L u| \in L^2(\mathbb{R}^{n+m})$, with $\nabla_L = (Z_1, \dots, Z_n, Z_{n+1} \dots, Z_{n+m})$. Using (1.1), a unique continuation for the generalized Baouendi-Grushin operator

$$\mathfrak{L}_\alpha = \Delta_x + |x|^{2\alpha} \Delta_y = \sum_{i=1}^{n+m} Z_i = \nabla_L \cdot \nabla_L$$

was proved. The proof of (1.1) used representation formulae of functions by the fundamental solution of $\mathfrak{L}_\alpha$ at the origin.

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The general version of (1.1) was established with a different approach in [10]. For any open subset $\Omega \subset \mathbb{R}^{n+m}$, D'Ambrosio obtained also the following Hardy type inequalities in [3]:

Let $p > 1, n, m \geq 1$, and $\beta, \gamma \in \mathbb{R}$ such that $n + (1+\alpha)m > \gamma - \beta - p$ and $n > \alpha p - \beta$. Then, for every $u \in D^{1,p}\left(\Omega, |x|^{\beta-\alpha p} d^{(1+\alpha)p-\gamma}\right)$ it follows

$$c^p_{Q,p,\beta,\gamma} \int_\Omega |u|^p \frac{|x|^\beta}{d^\gamma} dxdy \leq \int_\Omega |\nabla_L u|^p |x|^{\beta-\alpha p} d^{(1+\alpha)p-\gamma} dxdy, \tag{1.2}$$

where $c_{Q,p,\beta,\gamma} = \frac{Q+\beta-\gamma}{p}$. If $(0,0) \in \Omega$, then the constant (1.2) $c^p_{Q,p,\beta,\gamma}$ is sharp.

Recently, a Sobolev type inequality for the vector fields $Z_1, \ldots, Z_{n+m}$ (see [7]) states

$$\left(\int_{\mathbb{R}^{n+m}} |u|^{2^*} dxdy\right)^{\frac{1}{2^*}} \leq S \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{1}{2}}, \tag{1.3}$$

where S is a positive constant, $2^* = \frac{2Q}{Q-2}$, $Q = n + (\alpha+1)m$ is the homogeneous dimension with respect to the dilations

$$\delta_\lambda(x,y) = \left(\lambda x, \lambda^{\alpha+1} y\right), \qquad \lambda > 0,\ (x,y) \in \mathbb{R}^{n+m}, \tag{1.4}$$

which is induced by $\mathfrak{L}_\alpha$. Inequality (1.3) contains the result for $\alpha = 1$ in [2].

We define the following distance from the origin on $\mathbb{R}^{n+m}$

$$d(x,y) = \left(|x|^{2(\alpha+1)} + (\alpha+1)^2 |y|^2\right)^{\frac{1}{2(\alpha+1)}}.$$

It is easy to check that

$$\begin{aligned} \nabla_L d &= \frac{|x|^\alpha}{d^{2\alpha+1}} (|x|^\alpha x_1, |x|^\alpha x_2, \ldots, |x|^\alpha x_n, \\ &\qquad (\alpha+1) y_1, (\alpha+1) y_2, \ldots, (\alpha+1) y_m), \\ |\nabla_L d|^2 &= \frac{|x|^{2\alpha}}{d^{2\alpha}} = \psi_{2\alpha}. \end{aligned}$$

Let $C^k_0(\Omega)$ be the set of functions with compact in $C^k(\Omega)$ and $1 < p < \infty$. We denote by $D^{1,p}(\Omega)$ the closure of $C^\infty_0(\Omega)$ under the norm $\left(\int_\Omega |\nabla_L u|^p d\xi\right)^{1/p}$.

In this paper we will establish a class of Hardy-Sobolev type inequalities related to generalized Baouendi-Grushin operators. Our results contain the well-known Hardy type inequality and Sobolev type inequality for the class of operators.

This paper is organized as follows. In the next section, we prove a Hardy-Sobolev type inequality for $\mathfrak{L}_\alpha$. This generalizes the inequalities in the Euclidean space in [1]. In Section 3, we give some new Hardy type inequalities on bounded domains. Our results include those of [3].

2. Hardy-Sobolev inequalities

In this section, set $(x,y) = (x',y',x'',y'') = (z_1,z_2) \in \mathbb{R}^{n+m}$ with $z_1 = (x',y') \in \mathbb{R}^{k+l}$ $z_2 = (x'',y'') \in \mathbb{R}^{n-k+(m-l)}$, $1 \le k \le n, 1 \le l \le m$, and

$$d_1 = d\left(x',y'\right) = \left(\left|x'\right|^{2(\alpha+1)} + (\alpha+1)^2 \left|y'\right|^2\right)^{\frac{1}{2(\alpha+1)}}.$$

We denote by $B'_R(R) = \{(x',y') \in \mathbb{R}^{k+l} \mid d_1 < R\}$ the open ball centered at $(0,0)$ with radius $R > 0$, and put $\nabla_{L'} = (Z_1,\dots,Z_k,Z_{n+1},\dots,Z_{n+l})$.

We note that the polar coordinate transformation defined in [5, 8] implies

$$dz_1 = dx'dy' = \rho^{k+(\alpha+1)l-1} d\rho d\sigma_1, \tag{2.1}$$

where $d\sigma_1 = \left(\frac{1}{\alpha+1}\right)^l |\sin\theta|^{\frac{k}{\alpha+1}-1} |\cos\theta|^{l-1} d\theta d\omega_k d\omega_l$, and ω_k and ω_l are the Lebesgue measures of the unitary Euclidean spheres in $\mathbb{R}^k$ and $\mathbb{R}^l$, respectively.

The main inequalities in this section are the following.

Theorem 1 (Hardy-Sobolev type inequalities)**.** *Let us assume that s satisfies the relations $0 \le s \le 2 < k + (\alpha+1)l \le Q$, and put*

$$2_*(s) = \frac{2(Q-s)}{Q-2}.$$

Then there exists a positive constant $C(s,\alpha,k,l)$ such that for every $u \in D^{1,2}(\mathbb{R}^{n+m})$

$$\int_{\mathbb{R}^{n+m}} \frac{|x'|^{s\alpha}}{d_1^{s\alpha}} \frac{|u|^{2_*(s)}}{d_1^s} dxdy \le C\left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{Q-s}{Q-2}}, \tag{2.2}$$

where $D^{1,2}(\mathbb{R}^{n+m})$ is the completion of $C_0^\infty(\mathbb{R}^{n+m})$ under the norm

$$\|u\| = \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{1}{2}}.$$

Remark 1. If $s=0, k=n, l=m$, then (2.2) is (1.3); if $s=2, k=n, l=m$, then (2.2) is (1.1).

We prove first a lemma, which gives a representation formula of functions only depending on vector fields $(Z_1,\dots,Z_{n+m})$ and dilation of $\mathcal{L}_\alpha$. It establishes the connection between the function u and its generalized gradient $\nabla_L u$.

Lemma 1. *For any $u(x,y) \in C_0^\infty(\mathbb{R}^{n+m})$, we have*

$$u(x,y) = -\int_1^\infty \left[\frac{1}{\lambda |x|^{2\alpha}} \left\langle \nabla_L u, \nabla_L\left(\frac{d^{2(\alpha+1)}}{2(\alpha+1)}\right)\right\rangle\right] \circ \delta_\lambda d\lambda. \tag{2.3}$$

Proof. Clearly,

$$\begin{aligned}\nabla_L\left(d^{2(\alpha+1)}\right) &= 2(\alpha+1) d^{2\alpha+1} \nabla_L d = 2(\alpha+1)|x|^\alpha \\ &\quad \left(|x|^\alpha x_1, |x|^\alpha x_2, \dots, |x|^\alpha x_n, (\alpha+1)y_1, (\alpha+1)y_2, \dots, (\alpha+1)y_m\right).\end{aligned} \tag{2.4}$$

By (1.4) and (2.4), one has

$$
\begin{aligned}
\frac{d}{d\lambda} u \circ \delta_\lambda &= \frac{du\left(\delta_\lambda(x,y)\right)}{d\lambda} \\
&= \left[\sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} \circ \delta_\lambda(x,y) + (\alpha+1) \sum_{j=1}^{m} \lambda^{\alpha} y_j \frac{\partial u}{\partial y_j} \circ \delta_\lambda(x,y)\right] \\
&= \left[\sum_{i=1}^{n} \frac{x_i}{\lambda} \frac{\partial u}{\partial x_i} + (\alpha+1) \sum_{j=1}^{m} \frac{y_j}{\lambda} \frac{\partial u}{\partial y_j}\right] \circ \delta_\lambda(x,y) \\
&= \left[\frac{1}{\lambda |x|^{2\alpha}} \left(\sum_{i=1}^{n} |x|^{2\alpha} x_i \frac{\partial u}{\partial x_i} + (\alpha+1) \sum_{j=1}^{m} |x|^{2\alpha} y_j \frac{\partial u}{\partial y_j}\right)\right] \circ \delta_\lambda(x,y) \\
&= \left[\frac{1}{\lambda |x|^{2\alpha}} \left\langle \nabla_L u, \nabla_L \left(\frac{d^{2(\alpha+1)}}{2(\alpha+1)}\right)\right\rangle\right] \circ \delta_\lambda(x,y). \qquad (2.5)
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
u(x,y) &= -\int_1^{\infty} \frac{d}{d\lambda} \left(u\left(\delta_\lambda(x,y)\right)\right) d\lambda \\
&= -\int_1^{\infty} \left[\frac{1}{\lambda |x|^{2\alpha}} \left\langle \nabla_L u, \nabla_L \left(\frac{d^{2(\alpha+1)}}{2(\alpha+1)}\right)\right\rangle\right] \circ \delta_\lambda d\lambda,
\end{aligned}
$$

as required. □

Remark 2. We note that $\nabla_L\left(d^{2(\alpha+1)}\right) = 2(\alpha+1) d^{2\alpha+1} \nabla_L d$ in the proof above, and so

$$
\begin{aligned}
\left|\nabla_L\left(d^{2(\alpha+1)}\right)\right|^2 &= 2\left[(\alpha+1) d^{2\alpha+1}\right]^2 |\nabla_L d|^2 \\
&= [2(\alpha+1)]^2 |x|^{2\alpha} d^{2(\alpha+1)}. \qquad (2.6)
\end{aligned}
$$

Proof of Theorem 1. We need only to consider the cases where $u \geq 0$ and $u \in C_0^{\infty}(\mathbb{R}^{n+m})$. Introduce the notation $S_1 = \partial B_1' = \left\{(x',y') \in \mathbb{R}^{k+l}\ (x',y') = 1\right\}$ and $\vartheta = (\tau_1, \tau_2) = (\tau_{11}, \ldots, \tau_{1k}, \tau_{21}, \ldots, \tau_{2k}) \in S_1$. We introduce the transformation

$$
z_1 = \left(x', y'\right) = (\rho, \vartheta),
$$

where $\rho = d_1$, $\vartheta = (\tau_1, \tau_2) = \delta_{\frac{1}{\rho}}(x', y')$. By Lemma 1, we get

$$\begin{aligned}
&\int_{\mathbb{R}^{n+m}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha}} \frac{|u|^{2_*}}{\rho^s} dx dy = \int_{\mathbb{R}^{n-k+(m-l)}} dz_2 \int_{\mathbb{R}^{k+l}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha}} \frac{|u|^{2_*}}{\rho^s} dz_1 \\
&\quad = -\int_1^\infty d\lambda \int_{\mathbb{R}^{n-k+(m-l)}} dz_2 \\
&\quad\quad \times \int_{\mathbb{R}^{k+l}} \left[\frac{|x'|^{s\alpha}}{\rho^{s\alpha+s}} \frac{1}{\lambda |x'|^{2\alpha}} \left\langle \nabla_{L'} u^{2_*}, \nabla_{L'} \left(\frac{\rho^{2(\alpha+1)}}{2(\alpha+1)} \right) \right\rangle \right] \circ \delta_\lambda (x', y')\, dz_1. \quad (2.7)
\end{aligned}$$

Putting

$$F = \frac{1}{|x'|^{2\alpha}} \left\langle \nabla_{L'} u^{2_*}, \nabla_{L'} \left(\frac{\rho^{2(\alpha+1)}}{2(\alpha+1)} \right) \right\rangle,$$

we obtain from (2.1) that

$$\begin{aligned}
&\int_{\mathbb{R}^{n+m}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha}} \frac{|u|^{2_*}}{\rho^s} dx dy \quad (2.8) \\
&\quad = -\int_1^\infty d\lambda \int_{\mathbb{R}^{n-k+(m-l)}} dz_2 \int_{\mathbb{R}^{k+l}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha+s}} \frac{F}{\lambda} \circ \delta_\lambda (x', y') dz_1 \\
&\quad = -\int_1^\infty d\lambda \int_{\mathbb{R}^{n-k+(m-l)}} dz_2 \int_{S_1} d\sigma_1 \int_0^\infty \frac{\rho^{s\alpha} |\tau_1|^{s\alpha}}{\rho^{s\alpha+s}} \frac{F}{\lambda} \circ \delta_{\lambda\rho}(\vartheta)\, \rho^{k+(\alpha+1)l-1} d\rho \\
&\quad = -\int_1^\infty \lambda^{-(k+(\alpha+1)l-1-s)-2} d\lambda \int_{\mathbb{R}^{n-k+(m-l)}} dz_2 \\
&\quad\quad \times \int_{S_1} d\sigma_1 \int_0^\infty |\tau_1|^{s\alpha} F \circ \delta_r(\vartheta)\, r^{k+(\alpha+1)l-1-s} dr \quad (\lambda\rho = r) \\
&\quad = -\frac{1}{k+(\alpha+1)l-s} \int_{\mathbb{R}^{n-k+(m-l)}} dz_2 \int_{\mathbb{R}^{k+l}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha+s}} F dz_1 \\
&\quad = -\frac{1}{k+(\alpha+1)l-s} \int_{\mathbb{R}^{n-k+(m-l)}} dz_2 \\
&\quad\quad \times \int_{\mathbb{R}^{k+l}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha+s}} \left[\frac{1}{|x'|^{2\alpha}} \left\langle \nabla_{L'} u^{2_*}, \nabla_{L'} \left(\frac{\rho^{2(\alpha+1)}}{2(\alpha+1)} \right) \right\rangle \right] dz_1. \quad (2.9)
\end{aligned}$$

Now we consider the cases $1 \le s \le 2$ and $0 < s < 1$, respectively.

Case 1: $1 \le s \le 2$. If $1 < s < 2$, then (2.6), (2.9) and Hölder's inequality yield

$$
\begin{aligned}
&\int_{\mathbb{R}^{n+m}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha}} \frac{|u|^{2_*}}{\rho^s} dxdy \\
&\le \frac{1}{k+(\alpha+1)l-s} \int_{\mathbb{R}^{n-k+(m-l)}} dz_2 \int_{\mathbb{R}^{k+l}} \frac{|x'|^{s\alpha-2\alpha}}{\rho^{s\alpha+s}} |\nabla_{L'} u^{2_*}| \frac{\left|\nabla_{L'}\left(\rho^{2(\alpha+1)}\right)\right|}{2(\alpha+1)} dz_1 \\
&= \frac{2_*}{k+(\alpha+1)l-s} \int_{\mathbb{R}^{n+m}} \frac{|x'|^{s\alpha-2\alpha}}{\rho^{s\alpha+s}} u^{2_*-1} |\nabla_{L'} u| \left|x'\right|^{\alpha} \rho^{\alpha+1} dxdy \\
&= \frac{2_*}{k+(\alpha+1)l-s} \int_{\mathbb{R}^{n+m}} \frac{|x'|^{\alpha(s-1)}}{\rho^{\alpha(s-1)+s-1}} u^{2_*-1} |\nabla_{L'} u| dxdy \\
&\le \frac{2_*}{k+(\alpha+1)l-s} \int_{\mathbb{R}^{n+m}} \frac{|x'|^{\alpha(s-1)}}{\rho^{\alpha(s-1)}} \frac{u^{2_*\frac{s-1}{s}}}{\rho^{s-1}} u^{2_*\frac{1}{s}-1} |\nabla_L u| dxdy \\
&\le \frac{2_*}{k+(\alpha+1)l-s} \left(\int_{\mathbb{R}^{n+m}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha}} \frac{u^{2_*}}{\rho^s} dxdy\right)^{\frac{s-1}{s}} \left(\int_{\mathbb{R}^{n+m}} u^{2^*} dxdy\right)^{\frac{2-s}{2s}} \\
&\quad \times \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{1}{2}}. \qquad (2.10)
\end{aligned}
$$

By the Sobolev type inequality (1.3) we have

$$
\begin{aligned}
&\int_{\mathbb{R}^{n+m}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha}} \frac{|u|^{2_*}}{\rho^s} dxdy \\
&\qquad \le \left(\frac{2_*}{k+(\alpha+1)l-s}\right)^s \left(\int_{\mathbb{R}^{n+m}} u^{2^*} dxdy\right)^{\frac{2-s}{2}} \left(\int_{R^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{s}{2}} \\
&\qquad \le \left(\frac{2_*}{k+(\alpha+1)l-s}\right)^s S^{\frac{2^*(2-s)}{2}} \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{s}{2}+\frac{2^*}{2}\cdot\frac{2-s}{2}} \\
&\qquad = \left(\frac{2_*}{k+(\alpha+1)l-s}\right)^s S^{\frac{2^*(2-s)}{2}} \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{Q-s}{Q-2}}. \qquad (2.11)
\end{aligned}
$$

For $s = 1$, relation (2.10) leads one to

$$
\int_{\mathbb{R}^{n+m}} \frac{|x'|^{\alpha}}{\rho^{\alpha}} \frac{|u|^{2_*(1)}}{\rho} dxdy \le \frac{2_*(1)}{k+(\alpha+1)l-1} S^{\frac{2^*}{2}} \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{Q-1}{Q-2}}, \qquad (2.12)
$$

where $2_*(1) = \frac{2(Q-1)}{Q-2}$.

If $s=2$, then $2_*=2$ and by (2.10), it follows

$$\int_{\mathbb{R}^{n+m}} \frac{|x'|^{2\alpha}}{\rho^{2\alpha}} \frac{|u|^2}{\rho^2} dxdy \le \frac{2}{k+(\alpha+1)l-2} \int_{\mathbb{R}^{n+m}} \frac{|x'|^{\alpha}}{\rho^{\alpha+1}} u|\nabla_L u| dxdy$$

$$\le \frac{2}{k+(\alpha+1)l-2} \left(\int_{\mathbb{R}^{n+m}} \left(\frac{|x'|^{\alpha} u}{\rho^{\alpha+1}}\right)^2 dxdy\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{1}{2}},$$

namely,

$$\int_{\mathbb{R}^{n+m}} \frac{|x'|^{2\alpha}}{\rho^{2\alpha}} \frac{|u|^2}{\rho^2} dxdy \le \left(\frac{2}{k+(\alpha+1)l-2}\right)^2 \int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy.$$

Case 2: $0<s<1$. Using

$$2_*(s) = \frac{2(Q-s)}{Q-2} = (1-s)2^* + s2_*(1)$$

and (2.12), we have

$$\begin{aligned}
\int_{\mathbb{R}^{n+m}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha}} \frac{|u|^{2_*}}{\rho^s} dxdy &= \int_{\mathbb{R}^{n+m}} |u|^{(1-s)2^*} \left(\frac{|x'|^{\alpha} u^{2_*(1)}}{\rho^{\alpha+1}}\right)^s dxdy \\
&\le \left(\int_{\mathbb{R}^{n+m}} |u|^{2^*} dxdy\right)^{1-s} \left(\int_{\mathbb{R}^{n+m}} \frac{|x'|^{\alpha}}{\rho^{\alpha}} \frac{|u|^{2_*(1)}}{\rho} dxdy\right)^s \\
&\le S^{2^*(1-s)} \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{2^*(1-s)}{2}} \left(\frac{2_*(1)}{k+(\alpha+1)l-1} S^{\frac{2^*}{2}}\right)^s \\
&\quad \times \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{(Q-1)s}{Q-2}} \\
&= S^{\frac{2^*(2-s)}{2}} \left(\frac{2_*(1)}{k+(\alpha+1)l-1}\right)^s \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{Q-s}{Q-2}},
\end{aligned}$$

and the proof is complete.

$\square$

3. Hardy type inequalities on bounded domains

In this section, let $\Omega \subset \mathbb{R}^{n+m}$ denote any bounded domain, $(x,y)=(z_1,z_2)\in\Omega$, $d_1 = d(x',y')$, $B'_R(R)$, $\nabla_{L'}$ as before. Define

$$r_\epsilon := \left(\epsilon^2 + |x'|^2\right)^{\frac{1}{2}},$$

$$\sigma_\epsilon := \begin{pmatrix} I_k & 0 \\ 0 & r_\epsilon^{\alpha} I_l \end{pmatrix},$$

and set $\sigma_\epsilon \nabla = \nabla^\epsilon_{L'}$. For any vector field $h \in C^1(\Omega_1, \mathbb{R}^{k+l})$, we shall write $\mathrm{div}^\epsilon_{L'} = \mathrm{div}\,(\sigma_\epsilon h)$.

The following is the main result of this section.

Theorem 2. *Let $p > 1, k, l \geq 1$, and $\beta, \gamma \in \mathbb{R}$, be such that $k + (\alpha+1)\,l > \gamma - \beta - p$ and $k > \alpha p - \beta$. Then, for any $u \in D^{1,p}\big(\Omega, |x'|^{\beta-\alpha p}\, d_1^{(\alpha+1)p-\gamma}\big)$, we get*

$$c^p_{k,l,p,\beta,\gamma} \int_\Omega |u|^p \frac{|x'|^\beta}{d_1^\gamma} dxdy \leq \int_\Omega |\nabla_L u|^p \left|x'\right|^{\beta-\alpha p} d_1^{(\alpha+1)p-\gamma} dxdy, \tag{3.1}$$

where $c_{k,l,p,\beta,\gamma} = \frac{k+(\alpha+1)l+\beta-\gamma}{p}$.

If $(0,0) \in \Omega$, then the constant $c^p_{k,l,p,\beta,\gamma}$ in (3.1) is sharp.

Corollary 1. *If $1 < p < k + (\alpha+1)\,l$, then*

$$c^p_{k,l,p} \int_\Omega \frac{|x'|^{\alpha p}}{d_1^{\alpha p}} \frac{|u|^p}{d_1^p} dxdy \leq \int_\Omega |\nabla_L u|^p \, dxdy \tag{3.2}$$

for $u \in D^{1,p}\,(\Omega)$;

$$c^p_{k,l,p} \int_\Omega \frac{|u|^p}{d_1^p} dxdy \leq \int_\Omega |\nabla_L u|^p \frac{|x'|^{\alpha p}}{d_1^{\alpha p}} dxdy,$$

$$\text{for } k > \alpha p,\ u \in D^{1,p}\left(\Omega, \frac{|x'|^{\alpha p}}{d_1^{\alpha p}}\right); \tag{3.3}$$

and

$$c^p_{k,l,p} \int_\Omega \frac{|u|^p}{d_1^p} dxdy \leq \int_\Omega |\nabla_L u|^p \frac{|x'|^{(\alpha+1)p}}{d_1^{(\alpha+1)p}} dxdy \tag{3.4}$$

$$\text{for } k > (\alpha+1)\,p,\ u \in D^{1,p}\left(\Omega, \frac{|x'|^{(\alpha+1)p}}{d_1^{(\alpha+1)p}}\right),$$

where $c_{k,l,p} = \frac{k+(\alpha+1)l-p}{p}$.

In order to prove our results, we will use the following statement in [3].

Proposition 1. *Let $\epsilon > 0$, and $h \in C^1\big(\Omega_1, \mathbb{R}^{k+l}\big)$ such that $\mathrm{div}^\epsilon_{L'} h > 0$. Then for any $p > 1$ and $u \in C^1_0\left(\Omega, \mathbb{R}^{n+m}\right)$, one has*

$$\int_{\Omega_1} |u|^p \,\mathrm{div}^\epsilon_{L'} h \, dz_1 \leq p^p \int_{\Omega_1} |h|^p |\,\mathrm{div}^\epsilon_{L'} h|^{-(p-1)} |\nabla^\epsilon_{L'} u| dz_1. \tag{3.5}$$

The following proposition is also useful. For a similar description in the Euclidean space see [9].

Proposition 2. *If $\omega \in C_0^\infty\left(\mathbb{R}^{n+m}\right)$, then*

$$\inf_{\omega\in C_0^\infty(\mathbb{R}^{n+m}),\,\omega\neq 0} \frac{\int_{\mathbb{R}^{n+m}} |\nabla_L \omega|^p \,dxdy}{\int_{\mathbb{R}^{n+m}} |\omega|^p \,dxdy} = 0. \tag{3.6}$$

Proof. Set $\varphi \in C_0^\infty(\mathbb{R}^{n+m})$ to be a nonnegative radial function and satisfy

$$\varphi(d) = \begin{cases} 1 & \text{if } d \leq \frac{1}{2}; \\ 0 & \text{if } d \geq 1, \end{cases}$$

with $\int_{\mathbb{R}^{n+m}} \varphi^p \,dxdy = 1$. Letting $\epsilon < 1$ and putting $\varphi_\epsilon(d) = \epsilon^{-\frac{Q}{p}} \varphi\left(\frac{d}{\epsilon}\right)$, we get $\int_{\mathbb{R}^{n+m}} \varphi_\epsilon(d)^p \,dxdy = 1$. Note that $\nabla_L \varphi_\epsilon(d) = \epsilon^{-\frac{Q}{p}-1} \varphi'\left(\frac{d}{\epsilon}\right) \nabla_L d$. Setting $d = \epsilon\rho$ and using (2.1) in $\mathbb{R}^{n+m}$, we get

$$\begin{aligned} \int_{\mathbb{R}^{n+m}} |\nabla_L \varphi_\epsilon(d)|^p \,dxdy &= \epsilon^{-Q-p} \int_{\mathbb{R}^{n+m}} \psi_{p\alpha} \left|\varphi'\left(\frac{d}{\epsilon}\right)\right|^p dxdy \\ &= \epsilon^{-p} s_{n,m} \int_{\{\rho<1\}} \rho^{Q-1} \left|\varphi'(\rho)\right|^p d\rho \longrightarrow 0 \quad (\epsilon \to \infty) \end{aligned}$$

where

$$s_{n,m} = \left(\frac{1}{\alpha+1}\right)^m \omega_n \omega_m \int_{a_1}^{a_2} |\sin\theta|^{\frac{n+p\alpha}{\alpha+1}-1} |\cos\theta|^{m-1} \,d\theta.$$

This completes the proof. ∎

Proof of Theorem 2. Without loss of generality, we shall consider a smooth function $u \in C_0^\infty(\Omega)$. The general case will follow by the density argument. For $\epsilon > 0$, define

$$d_{1,\epsilon} = \left(r_\epsilon^{2(\alpha+1)} + (\alpha+1)^2 |y'|^2\right)^{\frac{1}{2(\alpha+1)}}$$

and

$$h_\epsilon^1 = \frac{1}{d_{1,\epsilon}^\gamma} \begin{pmatrix} x' r_\epsilon^\beta \\ (\alpha+1)|x'|^2 y' r_\epsilon^{\beta-\alpha-2} \end{pmatrix}.$$

A simple computation shows that

$$\begin{aligned} \operatorname{div}_{L'}^\epsilon h_\epsilon^1 &= \operatorname{div} \frac{1}{d_{1,\epsilon}^\gamma} \begin{pmatrix} x' r_\epsilon^\beta \\ (\alpha+1)|x'|^2 y' r_\epsilon^{\beta-2} \end{pmatrix} \\ &= \frac{r_\epsilon^\beta}{d_{1,\epsilon}^\gamma} \left(k + ((\alpha+1)l + \beta - \gamma) \frac{|x'|^2}{r_\epsilon^2}\right) \end{aligned} \tag{3.7}$$

and

$$|h_\epsilon^1| = \frac{r_\epsilon^{\beta-\alpha-2}|x'|^2}{d_{1,\epsilon}^\gamma}\left(\frac{r_\epsilon^{2\alpha+4}}{|x'|^2}+(\alpha+1)^2|y'|^2\right)^{\frac{1}{2}}. \tag{3.8}$$

Putting $f_\epsilon(s) = k + ((\alpha+1)l+\beta-\gamma)\frac{|s|^2}{\epsilon^2+s^2}$, $s \geq 0$, it is easy to see that

$$f_\epsilon(s) \geq \begin{cases} k & \text{if } (\alpha+1)l+\beta-\gamma \geq 0, \\ k+(\alpha+1)l+\beta-\gamma & \text{if } (\alpha+1)l+\beta-\gamma < 0, \end{cases} \tag{3.9}$$

for every $\epsilon > 0$ and $s \geq 0$. Since $r_\epsilon \geq \epsilon$, if $k+(\alpha+1)l > \gamma-\beta$, then $\operatorname{div}_{L'}^\epsilon h_\epsilon^1 > 0$.

Now we choose $h = h_\epsilon^1$ in (3.5) and obtain

$$\begin{aligned}
&\int_\Omega |u|^p \frac{r_\epsilon^\beta}{d_{1,\epsilon}^\gamma} f_\epsilon\left(|x'|\right)dxdy = \int_{\Omega_2} dz_2 \int_{\Omega_1} |u|^p \frac{r_\epsilon^\beta}{d_{1,\epsilon}^\gamma} f_\epsilon\left(|x'|\right)dz_1 \\
&\leq p^p \int_{\Omega_2} dz_2 \int_{\Omega_1} |h_\epsilon^1|^p \left|\frac{r_\epsilon^\beta}{d_{1,\epsilon}^\gamma} f_\epsilon\left(|x'|\right)^{-(p-1)}\right| |\nabla_{L'}^\epsilon u|^p dz_1 \\
&= p^p \int_{\Omega_2} dz_2 \int_{\Omega_1} |\nabla_{L'}^\epsilon u|^p \frac{r_\epsilon^{-\beta(p-1)}\left(r_\epsilon^{\beta-\alpha-2}|x'|^2\right)^p \left(\frac{r_\epsilon^{2\alpha+4}}{|x'|^2}+(\alpha+1)^2|y'|^2\right)^{\frac{p}{2}}}{d_{1,\epsilon}^{\gamma p} d_{1,\epsilon}^{-\gamma(p-1)} f_\epsilon\left(|x'|\right)^{p-1}} dz_1 \\
&= p^p \int_{\Omega_2} dz_2 \int_{\Omega_1} |\nabla_{L'}^\epsilon u|^p \frac{r_\epsilon^{\beta-(2+\alpha)p}|x'|^{2p}\left(\frac{r_\epsilon^{2\alpha+4}}{|x'|^2}+(\alpha+1)^2|y'|^2\right)^{\frac{p}{2}}}{d_{1,\epsilon}^\gamma f_\epsilon\left(|x'|\right)^{p-1}} dz_1.
\end{aligned} \tag{3.10}$$

Let $m_1 = \min\{k, k+(\alpha+1)l+\beta-\gamma\} > 0$. By $r < r_\epsilon$ and (3.9) the right-hand side of (3.10) can be estimated as follows:

$$\begin{aligned}
&|\nabla_{L'}^\epsilon u|^p \frac{r_\epsilon^{\beta-(2+\alpha)p}|x'|^{2p}\left(\frac{r_\epsilon^{2\alpha+4}}{|x'|^2}+(\alpha+1)^2|y'|^2\right)^{\frac{p}{2}}}{d_{1,\epsilon}^\gamma f_\epsilon\left(|x'|\right)^{p-1}} \\
&\qquad\qquad \leq \frac{|\nabla_{L'}^\epsilon u|^p}{m_1^{p-1}} r_\epsilon^{\beta-\alpha p} d_{1,\epsilon}^{(\alpha+1)p-\gamma}.
\end{aligned}$$

Therefore, following the assumptions $k+(\alpha+1)l>\gamma-\beta-p$ and $k>\alpha p-\beta$, and applying the Lebesgue dominated convergence theorem to (3.10), we have

$$\begin{aligned}\int_\Omega |u|^p \frac{|x'|^\beta}{d_1^\gamma} f\left(|x'|\right) dxdy &= \lim_{\epsilon\to 0}\int_\Omega |u|^p \frac{r_\epsilon^\beta}{d_{1,\epsilon}^\gamma} f_\epsilon\left(|x'|\right) dxdy \\ &\le \lim_{\epsilon\to 0} p^p \int_{\Omega_2} dz_2 \int_{\Omega_1} \frac{|\nabla_{L'}^\epsilon u|^p}{m_1^{p-1}} r_\epsilon^{\beta-\alpha p} d_{1,\epsilon}^{(\alpha+1)p-\gamma} dz_1 \\ &\le p^p \int_\Omega \frac{|\nabla_L u|^p}{m_1^{p-1}} \left|x'\right|^{\beta-\alpha p} d_1^{(\alpha+1)p-\gamma} dxdy.\end{aligned}$$

This shows the claim of (3.1).

Choosing $(\beta,\gamma)=(\alpha p,(\alpha+1)p)$, $(\beta,\gamma)=(0,p)$ and $(\beta,\gamma)=(-p,0)$ in (3.1), implies the inequalities (3.2), (3.3) and (3.4), respectively.

Next, if $(0,0)\in\Omega$, we prove the constant $c_{k,l,p,\beta,\gamma}^p$ is sharp. The approach here comes from that in [9].

Let $C(\Omega)$ be the best constant in (3.1), that is

$$C(\Omega)=\inf_{\phi\in C_0^\infty(\Omega),\phi\neq 0}\frac{\int_\Omega |\nabla_L\phi|^p |x'|^{\beta-p\alpha} d_1^{(\alpha+1)p-\gamma} dxdy}{\int_\Omega |\phi|^p \frac{|x'|^\beta}{d_1^\gamma} dxdy}.$$

From (3.1), we easily get $C(\Omega)\ge c_{k,l,p,\beta,\gamma}^p$. We shall prove the equality sign holds.

Put $\phi\in C_0^\infty(\Omega)$ and $\phi=\upsilon(z_1)\omega(z_2)$, where $\upsilon(z_1)\in C_0^\infty(\mathbb{R}^{k+l}\setminus\{(0,0)\})$, $\omega(z_2)\in C_0^\infty(\mathbb{R}^{n-k+(m-l)})$. By the convexity of the function $(a^2+b^2)^{\frac{p}{2}}$ for $a,b\ge 0$, we have

$$\left(a^2+b^2\right)^{\frac{p}{2}} \le (1-\lambda)^{1-p} a^p + \lambda^{1-p} b^p,$$

where $p>1, 0<\lambda<1$. Therefore, we find

$$\begin{aligned}|\nabla_L\phi|^p &= |\left(\nabla_L\left(\upsilon(z_1)\omega(z_2)\right)\right)^2|^{\frac{p}{2}} \\ &= \left(|\nabla_{L'}\upsilon|^2\omega^2+\upsilon^2|\nabla_{L''}\omega|^2\right)^{\frac{p}{2}} \\ &\le (1-\lambda)^{1-p}|\nabla_{L'}\upsilon|^p\omega^p+\lambda^{1-p}\upsilon^p|\nabla_{L''}\omega|^p,\end{aligned}\tag{3.11}$$

where $\nabla_{L''} := (Z_{k+1}, \ldots, Z_n, Z_{n+l+1}, \ldots, Z_m)$. By (3.11), this leads one to

$$\begin{aligned}
C(\Omega) &\le \frac{\int_\Omega |\nabla_L \phi|^p |x'|^{\beta - p\alpha} d_1^{(\alpha+1)p-\gamma} dxdy}{\int_\Omega |\phi|^p \frac{|x'|^\beta}{d_1^\gamma} dxdy} \\
&\le (1-\lambda)^{1-p} \frac{\int_\Omega |\nabla_{L'} \upsilon|^p |\omega|^p |x'|^{\beta - p\alpha} d_1^{(\alpha+1)p-\gamma} dxdy}{\int_\Omega |\upsilon\omega|^p \frac{|x'|^\beta}{d_1^\gamma} dxdy} \\
&\quad + \lambda^{1-p} \frac{\int_\Omega |\upsilon|^p |\nabla_{L''}\omega|^p |x'|^{\beta - p\alpha} d_1^{(\alpha+1)p-\gamma} dxdy}{\int_\Omega |\upsilon\omega|^p \frac{|x'|^\beta}{d_1^\gamma} dxdy} \\
&\le (1-\lambda)^{1-p} \frac{\int_{\mathbb{R}^{n-k+(m-l)}} |\omega|^p dz_2 \int_{\mathbb{R}^{k+l}} |\nabla_{L'}\upsilon|^p |x'|^{\beta - p\alpha} d_1^{(\alpha+1)p-\gamma} dz_1}{\int_{\mathbb{R}^{n-k+(m-l)}} |\omega|^p dz_2 \int_{\mathbb{R}^{k+l}} |\upsilon|^p \frac{|x'|^\beta}{d_1^\gamma} dz_1} \\
&\quad + \lambda^{1-p} \frac{\int_{\mathbb{R}^{n-k+(m-l)}} |\nabla_{L''}\omega|^p dz_2 \int_{\mathbb{R}^{k+l}} |\upsilon|^p |x'|^{\beta - p\alpha} d_1^{(\alpha+1)p-\gamma} dz_1}{\int_{\mathbb{R}^{n-k+(m-l)}} |\omega|^p dz_2 \int_{\mathbb{R}^{k+l}} |\upsilon|^p \frac{|x'|^\beta}{d_1^\gamma} dz_1} \\
&= (1-\lambda)^{1-p} \frac{\int_{\mathbb{R}^{k+l}} |\nabla_{L'}\upsilon|^p |x'|^{\beta - p\alpha} d_1^{(\alpha+1)p-\gamma} dz_1}{\int_{\mathbb{R}^{k+l}} |\upsilon|^p \frac{|x'|^\beta}{d_1^\gamma} dz_1} \\
&\quad + \lambda^{1-p} \frac{\int_{\mathbb{R}^{n-k+(m-l)}} |\nabla_{L''}\omega|^p dz_2}{\int_{\mathbb{R}^{n-k+(m-l)}} |\omega|^p dz_2} \cdot \frac{\int_{\mathbb{R}^{k+l}} |\upsilon|^p |x'|^{\beta - p\alpha} d_1^{(\alpha+1)p-\gamma} dz_1}{\int_{\mathbb{R}^{k+l}} |\upsilon|^p \frac{|x'|^\beta}{d_1^\gamma} dz_1}.
\end{aligned}$$

From (3.6), we get

$$\inf_{\omega \in C_0^\infty(\mathbb{R}^{n-k+(m-l)}),\ \omega \ne 0} \frac{\int_{\mathbb{R}^{n-k+(m-l)}} |\nabla_{L''}\omega|^p dz_2}{\int_{\mathbb{R}^{n-k+(m-l)}} |\omega|^p dz_2} = 0.$$

For $0 < \lambda < 1$, we have

$$\begin{aligned}
C(\Omega) &\le \inf_{\phi \in C_0^\infty(\Omega), \phi \ne 0} \frac{\int_\Omega |\nabla_L \phi|^p |x'|^{\beta - p\alpha} d_1^{(\alpha+1)p-\gamma} dxdy}{\int_\Omega |\phi|^p \frac{|x'|^\beta}{d_1^\gamma} dxdy} \\
&\le (1-\lambda)^{1-p} \inf_{\upsilon \in C_0^\infty(\mathbb{R}^{k+l}), \upsilon \ne 0} \frac{\int_{\mathbb{R}^{k+l}} |\nabla_{L'}\upsilon|^p |x'|^{\beta - p\alpha} d_1^{(\alpha+1)p-\gamma} dz_1}{\int_{\mathbb{R}^{k+l}} |\upsilon|^p \frac{|x'|^\beta}{d_1^\gamma} dz_1}.
\end{aligned} \tag{3.12}$$

We choose $Q = k + (\alpha+1)l$ in the best constant $c^p_{Q,p,\beta,\gamma}$ of (1.2), and let $\lambda \to 0$ in (3.12), hence

$$C(\Omega) \le c^p_{k,l,p,\beta,\gamma}.$$

The theorem is proved. □

Proof of Corollary 1. Choosing $(\beta,\gamma) = (\alpha p, (\alpha+1)\,p)$, $(\beta,\gamma) = (0,p)$ and taking $(\beta,\gamma) = (-p,0)$ in (3.1), we get inequalities (3.2), (3.3) and (3.4) respectively. The proof of the corollary is complete. □

Remark 3. We can also select $h = d_{1,\epsilon}^{1-p} \left|\nabla_{L'} d_{1,\epsilon}\right|^{p-2} \nabla_{L'} d_{1,\epsilon}$ to prove (3.2), as in the proof of Theorem 2.

Remark 4. In (3.1), by choosing $\beta = \alpha p$, $\gamma = \alpha p - \lambda$, $\lambda \in \mathbb{R}$, we obtain

$$c_{k,l,p,\lambda}^{p} \int_{\Omega} |u|^{p} \left(\frac{|x'|}{d_1}\right)^{\alpha p} d_1^{\lambda}\, dxdy \le \int_{\Omega} |\nabla_L u|^{p}\, d_1^{p+\lambda} dxdy,$$

where $c_{k,l,p,\lambda} = \frac{|k+(\alpha+1)l+\lambda|}{p}$. If $p = k + (\alpha+1)\,l$, and $\lambda < -1$, we let $\tilde{c}_{k,l,p,\lambda} = \frac{|1+\lambda|}{p}$, $\Omega = \left\{(x',y',x'',y'') \in \mathbb{R}^{k+l} \times \mathbb{R}^{n-k+(m-l)} \,|\, d_1 < R\right\}$, and then have

$$\tilde{c}_{k,l,p,\lambda}^{p} \int_{\Omega} |u|^{p} \left(\frac{|x'|}{d_1}\right)^{\alpha p} \left(\ln\left(\frac{R}{d_1}\right)\right)^{\lambda} dxdy \le \int_{\Omega} |\nabla_L u|^{p} \left(\ln\left(\frac{R}{d_1}\right)\right)^{p+\lambda} dxdy.$$

In particular

$$\left(\frac{p-1}{p}\right)^{p} \int_{\Omega} \frac{|u|^{p}}{\left(d_1 \ln\left(\frac{R}{d_1}\right)\right)^{p}} \left(\frac{|x'|}{d_1}\right)^{\alpha p} dxdy \le \int_{\Omega} |\nabla_L u|^{p}\, dxdy.$$

This result is more general than D'Ambrosio's in [4].

Remark 5. We note that if $k = n$ and $l = m$, then (3.1) coincides with (1.2).

Theorem 3. *Let* $1 < p < k$. *Then, for every* $u \in D^{1,p}(\Omega)$, *there exists a constant* $b_{k,p} = \frac{k-p}{p}$ *such that the following inequalities hold:*

$$b_{k,p}^{p} \int_{\Omega} \frac{|u|^{p}}{|x'|^{p}} dxdy \le \int_{\Omega} |\nabla_L u|^{p}\, dxdy, \tag{3.13}$$

$$b_{k,p}^{p} \int_{\Omega} \frac{|u|^{p}}{|d_1|^{p}} dxdy \le \int_{\Omega} |\nabla_L u|^{p}\, dxdy. \tag{3.14}$$

In particular, if $p = 2$ *and* $k \ge 3$, *we have*

$$\left(\frac{k-2}{2}\right)^{2} \int_{\Omega} \frac{u^2}{d_1^2} dxdy \le \left(\frac{k-2}{2}\right)^{2} \int_{\Omega} \frac{u^2}{|x'|^2} dxdy \le \int_{\Omega} |\nabla_L u|^{2}\, dxdy.$$

Proof. Let us first set

$$h_\epsilon^2 = \frac{1}{r_\epsilon^p} \begin{pmatrix} x' \\ 0 \end{pmatrix}.$$

A simple calculation shows that

$$\operatorname{div}_{L'}^{\epsilon} h_\epsilon^2 = \frac{1}{r_\epsilon^p} \left(k - p\frac{|x'|^2}{r_\epsilon^2}\right)$$

and

$$|h_\epsilon^2| = \frac{|x'|}{r_\epsilon^p}.$$

Since $p < k$, we get $\mathrm{div}_{L'}^{\epsilon} h_\epsilon^2 > 0$. From (3.5) we have

$$\begin{aligned}
\int_\Omega \frac{|u|^p}{r_\epsilon^p}\left(k - p\frac{|x'|^2}{r_\epsilon^2}\right) dxdy &= \int_{\Omega_2} dz_2 \int_{\Omega_1} \frac{|u|^p}{r_\epsilon^p}\left(k - p\frac{|x'|^2}{r_\epsilon^2}\right) dz_1 \\
&\leq \int_{\Omega_2} dz_2 \int_{\Omega_1} \left(\frac{|x'|}{r_\epsilon^p}\right)^p \left[\frac{1}{r_\epsilon^p}\left(k - p\frac{|x'|^2}{r_\epsilon^2}\right)\right]^{-(p-1)} |\nabla_{L'} u|^p \, dz_1 \\
&\leq \int_\Omega \frac{|x'|^p}{r_\epsilon^p\left(k - p\frac{|x'|^2}{r_\epsilon^2}\right)^{(p-1)}} |\nabla_L u|^p \, dz_1.
\end{aligned}$$

Using the Lebesgue dominated convergence theorem and putting $\epsilon \to 0$ in the estimate above, we obtain (3.13). Using $|x'| \leq d_1$ and (3.13), we get

$$\int_\Omega \frac{|u|^p}{d_1^p} dxdy \leq \int_\Omega \frac{|u|^p}{|x'|^p} dxdy.$$

Hence (3.14) is obtained. □

Remark 6. From the result above, it follows that the best constants in (3.13) and (3.14) lie in the interval $[(k-p)^p p^{-p},\ (k+(\alpha+1)l-p)^p p^{-p}]$.

As a consequence of (3.5), we give a Poincaré inequality for the vector fields on domains Ω contained in a slab. More precisely, we have

Corollary 2. *Let $\Omega \subset \mathbb{R}^{n+m}$ be an open subset. Suppose that there exist $R > 0, s \in \mathbb{R}$ and an integer $j : 1 \leq j \leq n$, such that for any $(x, y) \in \Omega$, it follows that $|x_j - s| \leq R$. Then, for every $u \in C_0^1(\Omega)$, we have*

$$c \int_\Omega |u|^p \, dxdy \leq \int_\Omega |\nabla_L u|^p \, dxdy.$$

where $c = (pR)^{-p}$.

The proof of the corollary follows from Theorem 3 by using the vector field defined by the formula

$$h := \begin{pmatrix} 0 \\ x_i - s \\ 0 \end{pmatrix}.$$

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BOUNDS FOR SINGULAR VALUES OF A BLOCK COMPANION MATRIX

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Abstract. Upper bounds for the squares of the singular values of a block companion matrix are derived. The principal tools applied are matricial and matrix norms and inequalities concerning the spectral radius of nonnegative matrices.

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1. Introduction

Singular value decomposition (SVD) was established for real matrices in 1870 by E. Beltrami and C. Jordan, and for complex squares matrices by Autonne. Later, in 1939, Eckart and Yong computed the SVD of any rectangular matrices. The study of singular values has been receiving a lot of attention since they constitute one of the most useful tools in handling, reliably, various problems which arise in a wide variety of application areas, such as linear algebra [3], statistical analysis, signal and image processing [4], electrical network theory [4], control and systems theory [3] and biomedical engineering [2]. On the other hand, we must remember that scalar companion matrices appear frequently in dynamic system theory: their special structure allows converting higher-order scalar difference and differential equations to state space form as well as simplifies theoretical considerations as for example feedback analysis.

Besides, note that any completely observable single-output system can be transformed into a companion form system [1]. Hence, the motivation for this work lies in the fact that our research is mainly concerned with system theory and we verify that these two concepts (singular values and companion matrices) assume an important role in the area referred to. The paper [9], where the authors, making use of the spectral radius of a matrix, matricial and matrix norms, derived bounds for the singular values of a block companion matrix, attracted our attention. Here, taking into account that the singular values different from one, of the right block companion matrix

$$C_R = \begin{pmatrix} 0 & \dots & 0 & -A_0 \\ I & \dots & 0 & -A_1 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & I & -A_{m-1} \end{pmatrix}$$

associated to the matrix polynomial

$$M(\lambda) = A_m \lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_2\lambda^2 + A_1\lambda + A_0$$

are the positive square roots of the latent roots of the λ-matrix

$$Q(\lambda) = I\lambda^2 - (\mathcal{A} + I)\lambda + A_0^T A_0,$$

we simplify the upper bound for the squares of the singular values of the C_R presented in [9].

2. Preliminaries

For the materialization of our purpose we need to remember the following definitions and results.

Definition 1 ([8])**.** Let $A_0, A_1, \ldots, A_{m-1}$ and $X \in Q_n(\mathbb{R})$. The polynomial

$$M(X) = A_m X^m + A_{m-1}X^{m-1} + \cdots + A_2X^2 + A_1X + A_0 \tag{2.1}$$

is called a matrix polynomial of degree m.

Definition 2 ([8])**.** If in (2.1) X is a scalar matrix, i. e., $X = \lambda I$ $(\lambda \in \mathbb{C})$, then

$$M(X) = A_m \lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_2\lambda^2 + A_1\lambda + A_0$$

is designated by lambda-matrix.

Definition 3 ([8])**.** Given a matrix polynomial

$$M(X) = X^m + A_{m-1}X^{m-1} + \cdots + A_2X^2 + A_1X + A_0,$$

the right block companion matrix associated with $M(X)$ is a matrix $C_R \in M_m(Q_n)$ defined by

$$C_R = \begin{pmatrix} 0 & \dots & 0 & -A_0 \\ I & \dots & 0 & -A_1 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & I & -A_{m-1} \end{pmatrix}$$

Theorem 1 ([3])**.** *Let $A \in M_m(Q_n)$ of rank r. Then there exist orthogonal matrices $U \in Q_m(\mathbb{R})$ and $V \in Q_n(\mathbb{R})$ such that*

$$U^T A V^T = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} = \Sigma,$$

where $\Sigma_1 = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$ and $\sigma_i > 0$, $i = 1, 2, \ldots, r$. The factorization

$$A = U\Sigma V^T$$

is called the singular value decomposition of the matrix A and the principal elements of Σ_1 are the singular values of A.

Remark 1. Recall that $\sigma_1^2, \sigma_2^2, \ldots, \sigma_r^2$ are the eigenvalues different from zero of $A^T A$ and $A A^T$.

Definition 4 ([8])**.** The latent roots of the lambda-matrix

$$M(\lambda) = A_m \lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_2\lambda^2 + A_1\lambda + A_0$$

are the zeros of det $M(\lambda) = 0$.

Definition 5 ([10])**.** Let $A = (a_{ij}) \in Q_n(\mathbb{C})$. The maximum column sum matrix norm of A is given by the formula

$$\|A\|_1 := \max_j \sum_i |a_{ij}|.$$

Finally, we need to bear in mind the following result which plays an important role in order to attain our aim:

Lemma 1 ([5])**.** *The singular values of the block companion matrix C_R are the square roots of the eigenvalues of the quadratic λ-matrix*

$$I\lambda^2 - (\mathcal{A} + I)\lambda + A_0^T A_0, \quad \mathcal{A} = \sum_{j=0}^{m-1} A_j^T A_j.$$

3. RESULTS

In this section, we are going to establish upper bounds for the squares of the singular values of the right block companion matrix C_R.

Proposition 1. *The singular values σ_j^2, $j = 1, \ldots, 2n$, different from one, of the right block companion matrix $C_R \in M_m(Q_n)$ associated with the matrix polynomial*

$$M(\lambda) = A_m \lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_2\lambda^2 + A_1\lambda + A_0,$$

satisfy the following inequality:

$$\sigma_j^2 \leq \left(1 + \max\left\{\|\mathcal{A} + I\|_1^2, \left\|\left(A_0^T A_0\right)^2\right\|_1 + \left\|(\mathcal{A} + I)^2\right\|_1\right\}\right)^{\frac{1}{2}}. \tag{3.1}$$

Proof. First, we consider the block companion matrix $C_R \in M_2(Q_n)$ defined by

$$C = \begin{pmatrix} 0 & -A_0^T A_0 \\ I & \mathcal{A} + I \end{pmatrix}$$

associated with the matrix polynomial

$$Q(\lambda) = I\lambda^2 - (\mathcal{A} + I)\lambda + A_0^T A_0,$$

where

$$\mathcal{A} = \sum_{j=0}^{m-1} A_j^T A_j .$$

Consequently, since $(\mathcal{A}+I)^T = \mathcal{A}+I$, we get

$$C^T C = \begin{pmatrix} I & \mathcal{A}+I \\ \mathcal{A}+I & \left(A_0^T A_0\right)^2 + (\mathcal{A}+I)^2 \end{pmatrix}.$$

Further, we form the matrix $D = \operatorname{diag}\left[\|\mathcal{A}+I\|_1 I, I\right]$ and take the norm $\|\cdot\|_1$ of each block of the matrix

$$D^{-1} C^T C D = \begin{pmatrix} I & \|\mathcal{A}+I\|_1^{-1} (\mathcal{A}+I) \\ \|\mathcal{A}+I\|_1 (\mathcal{A}+I) & \left(A_0^T A_0\right)^2 + (\mathcal{A}+I)^2 \end{pmatrix},$$

obtaining the matricial norm of $D^{-1}C^TCD$ denoted by $\Phi_1\left(D^{-1}C^TCD\right)$.

We have

$$\Phi_1\left(D^{-1}C^TCD\right) = \begin{pmatrix} 1 & 1 \\ \|\mathcal{A}+I\|_1^2 & \left\|\left(A_0^T A_0\right)^2 + (\mathcal{A}+I)^2\right\|_1 \end{pmatrix}.$$

Taking into account the following relations concerning the spectral radius ρ of a matrix, matricial and matrix norms [9, 11]

$$\rho(A) \le \|A\|_i, \qquad i = 1, \infty, \tag{3.2}$$

$$\rho(A) \le \left(\rho\left(A^T A\right)\right)^{\frac{1}{2}} \le \left(\rho\left(\Phi_i\left(A^T A\right)\right)\right)^{\frac{1}{2}} \le \left\|\Phi_i\left(A^T A\right)\right\|_j^{\frac{1}{2}}, \quad i, j = 1, \infty, \tag{3.3}$$

we can write

$$\begin{aligned} \rho\left(C^T C\right) &= \rho\left(D^{-1}C^TCD\right) \\ &\le \sqrt{\rho\begin{pmatrix} 1 & 1 \\ \|\mathcal{A}+I\|_1^2 & \|\left(A_0^T A_0\right)^2 + (\mathcal{A}+I)^2\|_1 \end{pmatrix}}. \end{aligned} \tag{3.4}$$

Thus

$$|\lambda| \le \left\| \begin{pmatrix} 1 & 1 \\ \|\mathcal{A}+I\|_1^2 & \left\|\left(A_0^T A_0\right)^2 + (\mathcal{A}+I)^2\right\|_1 \end{pmatrix} \right\|_1^{\frac{1}{2}},$$

or equivalently,

$$|\lambda| \le \left[1 + \max\left\{\|\mathcal{A}+I\|_1^2, \left\|\left(A_0^T A_0\right)^2\right\|_1 + \left\|(\mathcal{A}+I)^2\right\|_1\right\}\right]^{\frac{1}{2}},$$

where λ denotes a latent root of $Q(\lambda)$. $\square$

Remark 2. Note that the matrix on the right member of (3.4) is a square matrix of order two. So, evaluating directly the eigenvalues, we obtain

$$|\lambda| \leq \left(\frac{p+\sqrt{p^2-4\times(q-r)}}{2}\right)^{\frac{1}{2}},$$

where $r = \|\mathcal{A}+I\|_1^2$, $p = 1+\left\|\left(A_0^T A_0\right)^2+(\mathcal{A}+I)^2\right\|_1$, and

$$q = \left\|\left(A_0^T A_0\right)^2+(\mathcal{A}+I)^2\right\|_1,$$

which provides a sharper bound but is not so practical for computing.

Finding bounds for the eigenvalues of a matrix polynomial is a problem to which many researchers have given contributions. In [5] a variety of lower and upper bounds were derived from block companion matrix, singular values, numerical radius, norms of the coefficient matrices of $M(\lambda)$, characteristic polynomial, fractional powers of norms and maximization over unit circle. We are going to apply some of these results to the quadratic λ-matrix

$$Q(\lambda) = I\lambda^2-(\mathcal{A}+I)\lambda+A_0^T A_0,$$

where

$$\mathcal{A} = \sum_{j=0}^{m-1} A_j^T A_j,$$

and to compare them with the estimate obtained in Proposition 1. Let us consider $Q_U(\lambda) = Q(\lambda)$ and

$$Q_L(\lambda) = I\lambda^2-\left(A_0^T A_0\right)^{-1}(\mathcal{A}+I)\lambda+\left(A_0^T A_0\right)^{-1},$$

the two λ-matrices associated with $Q(\lambda)$, whose block companion matrices are defined by the formulas

$$C_U = \begin{pmatrix} 0 & I \\ -A_0^T A_0 & \mathcal{A}+I \end{pmatrix}$$

and

$$C_L = \begin{pmatrix} 0 & I \\ -\left(A_0^T A_0\right)^{-1} & \left(A_0^T A_0\right)^{-1}(\mathcal{A}+I) \end{pmatrix},$$

respectively. Using the norms of C_U and C_L [5], we obtain the estimates

$$\max\left\{\left\|\left(A_0^T A_0\right)^{-1}\right\|_1, 1+\left\|\left(A_0^T A_0\right)^{-1}(\mathcal{A}+I)\right\|_1\right\}^{-1} \leq \sigma_j^2 \leq \max\left\{1+\|\mathcal{A}+I\|_1, \|A_0^T A_0\|_1\right\}, \quad (3.5)$$

$$\max\left\{1, \left\|\left(A_0^T A_0\right)^{-1} - \left(A_0^T A_0\right)^{-1}(\mathcal{A}+I)\right\|_\infty\right\}^{-1} \leq \sigma_j^2$$
$$\leq \max\left\{1, \left\|\left(A_0^T A_0\right) - (\mathcal{A}+I)\right\|_\infty\right\}$$

and

$$\left\|I + \left(A_0^T A_0\right)^{-2} + \left(A_0^T A_0\right)^{-1} + (\mathcal{A}+I)^2\left(A_0^T A_0\right)^{-1}\right\|_2^{-\frac{1}{2}} \leq \sigma_j^2$$
$$\leq \left\|I + \left(A_0^T A_0\right)^{-2} + (\mathcal{A}+I)^2\right\|_2^{\frac{1}{2}}.$$

The numerical radius of a matrix provides another upper bound for the eigenvalues of $Q(\lambda)$ as it has been shown in [5]. Invoking Lemma 2.11 we have

$$\sigma_j^2 \leq \frac{1}{2}\left(1 + \|\mathcal{A}+I\|_2 + \left\|\left(A_0^T A_0\right) - (\mathcal{A}+I)\right\|_2\right)$$

In the same article, N. J. Higham and F. Tisseur by using norms of the coefficient matrices of $M(\lambda)$ obtained a generalization of the Cauchy bound. In our case, we get

$$\sigma_j^2 \leq \frac{1}{2}\left(\|\mathcal{A}+I\| + \left(\|\mathcal{A}+I\|^2 + 4\|A_0^T A_0\|\right)^{\frac{1}{2}}\right)$$

and

$$\sigma_j^2 \geq \frac{1}{2}\left(-\|\mathcal{A}+I\| + \left(\|\mathcal{A}+I\|^2 + 4\|\left(A_0^T A_0\right)^{-1}\|^{-1}\right)^{\frac{1}{2}}\right).$$

It should be mentioned that the same upper bound, based in matrix inequalities, spectral radius and spectral norm can be found in [7]. Other methods involving fractional powers of norms and maximization over unit circle yield bounds for the eigenvalues of $M(\lambda)$. For example, the application of Lemma 4.1 from [5] leads one to the following bounds for the squares of the singular values of C_R:

$$\left(1 + \|\left(A_0^T A_0\right)^{-1}\|\right)^{-1} \min\left\{\|\mathcal{A}+I\|^{-1}, 1\right\} \leq \sigma_j^2$$
$$\leq 2\max\left\{\|\mathcal{A}+I\|, \|A_0^T A_0\|^{\frac{1}{2}}\right\} \quad (3.6)$$

Any of these upper bounds require the computation of matrix norms subordinate to vector norms. From the similarities presented by the above majorization relations, we think that there is not an upper bound that outperforms all the others. The accuracy of a bound depends of the range of the elements of the coefficient matrices of $M(\lambda)$. Assume that the maximum in (3.1), (3.5) and (3.6) is attained by the first term and $\|\mathcal{A}+I\|_1$ is of a large order of magnitude. Thus, the first bound provides the same estimate for σ^2 as the one obtained by (3.5) and by the Cauchy bound. However, it is sharpper than the easily expressed bound given by (3.6). The following example makes it clear what we have referred to.

Example 1. Let us consider the 5×5 matrix polynomial

$$M(\lambda) = A_m \lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_2\lambda^2 + A_1\lambda + A_0$$

of degree $m = 9$ whose coefficient matrices are of the form

$$A_i = 10^{i-2}\,\text{randn}, \quad i = 0, 1, \ldots, 8; \qquad \text{and } A_9 = I,$$

where "randn" denotes a random matrix with elements removed from a Normal distribution $N(0, 1)$. In our experiment, performed by using Matlab 5.3, the maximal value of the eigenvalues of the λ-matrix

$$Q(\lambda) = I\lambda^2 - (\mathcal{A} + I)\lambda + A_0^T A_0$$

is 1.3909×10^{13}. The upper bounds presented in (3.1), (3.5) as well as the Cauchy bound yield an estimate of 1.5458×10^{13}. On the other hand, using (3.6), we verify that $\sigma_j^2 \leq 3.09168 \times 10^{13}$.

Example 2. Observe that some robustness measures for state space models in control and system theory such as nearness to uncontrollability and nearness to instability require the computation of singular values [6]. More concretely, singular values provide bounds on the nearness to instability of closed loop systems of the form

$$x(k+1) = (A + BF)x(k),$$

where

$$A = \begin{pmatrix} 0 & I_{n-1} \\ a_1 & \mathcal{A} \end{pmatrix}$$

with $\mathcal{A} = \begin{pmatrix} a_2 & \ldots & a_n \end{pmatrix}$, $B = \begin{pmatrix} 0_{n-1} \\ 1 \end{pmatrix}$, and F the feedback matrix. In this case, considering the set of all stable matrices

$$P_S = \left\{ A \in \mathbb{R}^{n \times n} : \text{Re}(\lambda) < 0 \text{ for all eigenvalues of } A \right\},$$

Kenny and Laub [6] pointed out that the distance from $A + BF$ to the complement of P_S, $d\left(A + BF, P_S^C\right)$, provides a means of measuring nearness to instability and that

$$d\left(A + BF, P_S^C\right) = \min_{\text{Re}\,\lambda \geq 0} \sigma_{\min}(A + BF - \lambda I) \leq \sigma_{\min}(A + BF),$$

where $\sigma_{\min}(\cdot)$ is the smallest singular value of a matrix. So, information about the location of singular values is useful. Let us consider the discrete-time linear invariant system

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$A = \left(\begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline -1 & 0 & -1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 & 0 & -2 \end{array}\right), \quad B = \left(\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 1 \end{array}\right),$$

and $F = \left(\begin{array}{cc|cc|cc} -2 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$. So,

$$A + BF = \left(\begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline -1 & 0 & -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & 1 & 0 & -2 \end{array}\right).$$

In order to apply the result obtained, we need to find the matrices $\left(A_0^T A_0\right)^2$ and $\mathcal{A} + I$. We verify that $\left(A_0^T A_0\right)^2 = \left(\begin{smallmatrix} 5 & -3 \\ -3 & 2 \end{smallmatrix}\right)$, and $\mathcal{A} + I = \left(\begin{smallmatrix} 5 & -2 \\ -2 & 8 \end{smallmatrix}\right)$, $\mathcal{A} = \sum_{j=0}^{2} A_j^T A_j$, which allows us to obtain $\|\mathcal{A} + I\|_1^2 = 100$, $\|(A_0^T A_0)^2\|_1 = 8$, and $\|(\mathcal{A} + I)^2\|_1 = 94$. In this way, by Proposition 1, we conclude that

$$\sigma_j^2 \leq \sqrt{103}, \quad j = 1, 2, 3, 4.$$

Observe that the maximal square singular value, $\sigma_{\max}^2(A + BF) = 9.5809$.

4. Concluding remarks

In this paper, from the relationship between singular values of a block companion matrix and the latent roots of a lambda matrix and using inequalities (3.2) and (3.3), we obtain upper bounds for the squares of the singular values of the right block companion matrix C_R. We would like to emphasize that the simplicity of inequality (3.1) is a consequence of the application of Lemma 1 that enables us to solve our problem beginning with a quadratic lambda-matrix, instead of one of order m. Finally, we hope that this work could be useful in the study of the controllability and stability of systems in scalar companion form.

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A GENERALIZATION OF GENERALIZED BANACH FIXED-POINT THEOREM IN A WEAK LEFT SMALL SLF-DISTANCE SPACE

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Abstract. In this paper we generalize the multivalued contraction theorem of S. B. Nadler [8]. As corollaries we obtain fixed point theorems for a multivalued function in complete dislocated metric spaces and complete partial metric space.

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1. Introduction

The generalized Banach fixed point Theorem was introduced in [8]. Hitzler and Seda (2000) introduced the dislocated metrics space in [3, 4] as a generalization of metrics where self-distances need not be zero. Also dislocated metrics were studied under the name of metric domains in the context of domain theory in [1]. The notion of dislocated metric is useful in the context of electronic engineering (see [3]). In 1985, S. G. Matthews [6] introduced a generalization of Banach contraction principle. In 2000, Hitzler and Seda [4] introduced an alternative proof of this result in dislocated metric spaces. The plan of this paper is as follows. In Section 2, some preliminaries are presented. In Section 3, we define a weak left small self distance space and give a new topology on a distance space different from the distance topology of P. Waszkiewicz [9, 10]. In Section 4, we generalize the multivalued contraction theorem by S. B. Nadler [8]. As corollaries of this result we obtain fixed point theorems for a multivalued function in a complete dislocated metric space and a complete partial metric space.

2. Preliminaries

In this section we give some preliminaries.

Definition 1 ([9, 10])**.** A distance on a set X is a map $d : X \times X \to [0, \infty)$. A pair (X, d) is called a distance space.

Let $\varepsilon > 0$ and $x \in X$. Then

$$B_d(x,\varepsilon) = \{y \in X \mid d(x,y) < d(x,x) + \varepsilon\}$$

and $N_x = \{A \subseteq X \mid \exists \varepsilon > 0,\ B_d(x,\varepsilon) \subseteq A\}$. The family $\tau_d = \{A \subseteq X | \forall x \in A,\ A \in N_x\}$ is a topology on X and called the distance, the topology τ_d is called the distance topology. If $d(x,y)$ is replaced by $d(y,x)$ it is called the dual distance topology.

We need the following conditions for a distance function:

(d1) $\forall x \in X, d(x,x) = 0$,
(d2) $\forall x,y \in X, d(x,y) = d(y,x) = 0 \Rightarrow x = y$,
(d3) $\forall x,y \in X, d(x,y) = d(y,x)$,
(d4) $\forall x,y,z \in X,\ d(x,y) \le d(x,z) + d(z,y)$,
(d5) $\forall\ x,y \in X,\ d(x,x) \le \min\{d(x,y), d(y,x)\}$,
(d6) $\forall x,y,z \in X, d(x,y) \le d(x,z) + d(z,y) - d(z,z)$.

If d satisfies conditions (d1)–(d4), then (X,d) is called a metric space. If it satisfies conditions (d2)–(d4), then (X,d) is called a dislocated metric space (d-metric space for short). If d satisfies conditions (d2), (d3), (d5) and (d6) then (X,d) is called a partial metric space [7].

One can deduce that every partial metric space is a dislocated metric space.

Definition 2. Let (X,d) be a distance space. A multivalued function f on $A \subseteq X$ is a function $f : A \to 2^A - \{\phi\}$.

Theorem 1 ([3,6]). *Let (X,d) be a complete d-metric space and let $f : X \to X$ be a Banach contraction function. Then f has a unique fixed point.*

Theorem 2 ([8]). *Let (X,d) be a complete metric space and $A \subseteq X$. Suppose that f is a multivalued function on A, A is closed, $f(x)$ is closed for all $x \in A$, D is the Hausdorff metric, and multivalued k-contraction condition*

$$D(f(x), f(y)) \le kd(x,y)$$

is satisfied for all $x,y \in A$ and fixed $k \in [0,1)$. Then f has a fixed point, i. e., $\exists x \in A$ such that $x \in f(x)$.

Theorem 3 ([2,5]). *Let (X,d) be a metric space. A mapping T from X to itself is said to be asymptotically regular if and only if*

$$\lim_{n\to\infty} d\left(T^n(x), T^{n+1}(x)\right) = 0,\ \forall x \in X.$$

3. On distance spaces

Definition 3. Let (X,d) be a distance space and let $A \subseteq X$.

(1) The left d-closure of A is denoted and defined by

$$d_l - \mathrm{cl}(A) = \left\{x \in X \mid \exists (x_n) \subset X,\ \lim_{n\to\infty} d(x_n, A) = \lim_{n\to\infty} d(x_n, x) = 0\right\}$$

if $A \ne \phi$, and $d_l - \mathrm{cl}(\phi) = \phi$.

(2) A is called left-d-closed (d_l-closed for short) iff $d_l - \mathrm{cl}(A) \subseteq A$,
(3) For $a \in X$ and $B \subseteq X$, we put $d(a,B) = \inf_{b \in B} d(a,b)$ and $d(B,a) = \inf_{b \in B} d(b,a)$.
(4) For $A \subseteq X$, $B \subseteq X$, we put $d(A,B) = \inf\{d(a,b) \mid a \in A, b \in B\}$,
(5) The Hausdorff distance D on X is defined by

$$D(A,B) = \max\left\{\sup_{a \in A} d(a,B), \sup_{b \in B} d(A,b)\right\},$$

where $A \subseteq X$, $B \subseteq X$.

In [8, Theorem 2.2], above the definition of Hausdorff metric coincides with the definition of Hausdorff distance iff d is a metric.

Definition 4. A sequence (x_n) in a distance space (X,d) is said to be left convergent to an $x \in X$ (respectively, asymptotically regular) iff $\lim_{n\to\infty} d(x_n,x) = 0$ (respectively, $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$). If $\lim_{n\to\infty} d(x_n,x) = 0$, then x is called a left limit of (x_n).

If (X,d) is a metric space and T is a mapping from X to itself, then T is asymptotically regular iff $\forall\, x \in X$, the sequence $T^n(x)$ is left asymptotically regular.

It is clear that every left Cauchy sequence (see Definition 4.2 below) is left asymptotically regular. The converse is not true even in metric spaces, for instance let $X = \mathbb{R}$ and $d(x,y) = |x-y|\ \forall\, x,y \in X$, then (x_n) in X defined by $x_n = \sum_{k=1}^{n} k^{-1}$ is asymptotically regular but not Cauchy.

Definition 5. A distance space is called complete left asymptotically regular distance iff every left asymptotically regular sequence (x_n) in X, left converges to some point in X.

Theorem 4. *Let (X,d) be a distance space. Then*

$$\tau_{d_l} = \left\{A \mid A \in 2^X \text{ and } A^c \text{ is } d_l\text{-closed}\right\}$$

is a topology on X, where A^c is the complement of A.

Proof. (i) Since $d_l - \mathrm{cl}(\phi) = \phi \subseteq \phi$, then ϕ is d_l-closed and so $X \in \tau_{d_l}$. Since $d_l - \mathrm{cl}(x) \subseteq X$, then X is d_l-closed and so $\phi \in \tau_{d_l}$.

(ii) Let $A, B \in \tau_{d_l}$, $x \in d_l - \mathrm{cl}(A^c \cup B^c)$ then, there is a sequence (x_n) such that

$$\lim_{n\to\infty} d\left(x_n, A^c \cup B^c\right) = \lim_{n\to\infty} d(x_n,x) = 0,$$

this implies that either

$$\lim_{n\to\infty} d\left(x_n, A^c\right) = \lim_{n\to\infty} d(x_n,x) = 0$$

or $\lim_{n\to\infty} d(x_n,B^c) = \lim_{n\to\infty} d(x_n,x) = 0 \Rightarrow x \in d_l - \mathrm{cl}(A^c)$ or $x \in d_l - \mathrm{cl}(B^c) \Rightarrow x \in A^c \cup B^c = (A \cap B)^c$.

(iii) Let $\{A_j \mid j \in J\} \subseteq \tau_{d_l}$ and $x \in d_l - \operatorname{cl}\left(\bigcap_{j\in J} A_j^c\right)$. Then there exists a sequence (x_n) such that $\lim_{n\to\infty} d\left(x_n, \bigcap_{j\in J} A_j^c\right) = \lim_{n\to\infty} d\,(x_n, x) = 0$. Therefore, $\lim_{n\to\infty} d(x_n, A_j^c) = \lim_{n\to\infty} d\,(x_n, x) = 0$ for all $j \in J \Rightarrow x \in d_l - \operatorname{cl}(A_j^c)\ \forall j \in J$ $\Rightarrow x \in \bigcap_{j\in J} A_j^c = \left(\bigcup_{j\in J} A_j\right)^c$. □

Definition 6 ([9, 10])**.** A distance space (X, d) is called a small self-distance iff d satisfies the condition (d5) above.

Definition 7. A distance space (X, d) is called a weak left small self-distance iff d satisfies the condition $d\,(x, y) = 0 \Rightarrow d\,(x, x) = 0,\ \forall x, y \in X$.

It is clear that any small self distance space is weak left small self-distance. The following counterexamples illustrate that τ_{d_l} and the distance topology τ_d by P. Wasztiewiecz are different concepts even in a weak left small self distance space.

Counterexample 1. Let $X = \{a, b, c\}$ and $d : X \times X \to [0, \infty)$ is defined by the relation $d\,(x, y) = 1$ for all $x, y \in X$. It is clear that (X, d) is a weak left small self distance space. Since $\{a\} \notin \tau_d$ and $\tau_{d_l} = 2^X$, it follows that $\tau_{d_l} \nsubseteq \tau_d$.

Counterexample 2. Let $X = \{a, b, c\}$ and $d : X \times X \to [0, \infty)$ is defined as follows: $d\,(b, a) = d\,(b, b) = 0$, $d\,(a, a) = 2^{-1}$ and $d\,(x, y) = 2$ for all $(x, y) \in X \times X - \{(b, a), (b, b), (a, a)\}$. Then (X, d) is a weak left small self distance space. Since $\{a\} \in \tau_d$ and $\{a\} \notin \tau_{d_l}$, it follows that $\tau_d \nsubseteq \tau_{d_l}$.

4. Main results

The following counterexample illustrates that there exists a sequence in a weak left small self distance satisfying (d4) which is left convergent but not left asymptotically regular.

Counterexample 3. Let $X = \{a, b, c\}$ and d is a distance function defined by the relations $d\,(a, b) = d\,(b, a) = d\,(c, b) = d\,(c, a) = d\,(c, c) = 1$, $d\,(b, c) = d\,(a, c) = d\,(b, b) = d\,(a, a) = 0$. The desired sequence is (x_n) given by the formula

$$x_n = \begin{cases} a & \text{if } n \text{ is odd,} \\ b & \text{if } n \text{ is even.} \end{cases}$$

Definition 8. Let (X, d) be a distance space and $A \subseteq X$. A is called left-asymptotically regular-closed iff if (x_n) is a left asymptotically regular sequence left convergent to $x \in X$, then $x \in A$.

Theorem 5. *Let (X, d) be a complete left asymptotically regular weak left small self distance space and $A \subseteq X$. If f is a multivalued function on A, A is left-asymptotically regular-closed, $f\,(x)$ is left-d-closed $\forall x \in A$ and f satisfies the following condition: $D\,(f\,(x), f\,(y)) \le k d\,(x, y)\ \forall x, y \in A$ for fixed $k \in (0, 1)$, then f has a fixed point.*

Proof. (I) Choose a $k_0 \in (k,1)$ and an $x_0 \in A$. If there is not an element $x_1 \in f(x_0)$such that $x_1 \neq x_0$, then since $f(x_0) \neq \phi$ we have $x_0 \in f(x_0)$. Thus x_0 is a fixed point of f. If there exists $x \in f(x_0)$ and $x \neq x_0$such that $d(x_0,x) = 0$, then $d(x_0, f(x_0)) = 0$. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ where $x_n = x_0$ $\forall n \in \mathbb{N}$. Then $\lim_{n\to\infty} d(x_n, f(x_0)) = \lim_{n\to\infty} d(x_n,x_0) = 0$, because (X,d) is a weak left small self distance space, i. e., since $d(x_0,x) = 0$ we have $d(x_0,x_0) = 0$. Thus $x_0 \in d_l - \mathrm{cl}(f(x_0)) \subseteq f(x_0)$. Hence x_0 is a fixed point. For the other case choose $x_1 \in f(x_0)$such that $x_1 \neq x_0$ and $d(x_0,x_1) > 0$. Now,

$$d(x_1, f(x_1)) \leq D(f(x_0), f(x_1)) < k_0 d(x_0,x_1).$$

Then there exists $x_2 \in f(x_1)$such that $d(x_1,x_2) < k_0 d(x_0,x_1)$. By repeating this process one can construct a sequence (x_n)such that $x_{n+1} \in f(x_n)$, $d(x_n,x_{n+1}) < k_0^n d(x_0,x_1)$ for all $n \in \mathbb{N}$. Thus $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$ and so (x_n) is a left asymptotically regular sequence. Since (X,d) is a complete left asymptotically regular distance space, then exists $x \in X$such that $\lim_{n\to\infty} d(x_n,x) = 0$. Since A is left-asymptotically regular-closed, then $x \in A$.

(II) Since $d(x_{n+1}, f(x)) \leq D(f(x_n), f(x)) \leq k d(x_n,x)$ $\forall n \in \mathbb{N}$, then

$$\lim_{n\to\infty} d(x_{n+1}, f(x)) \leq k \lim_{n\to\infty} d(x_n,x) = 0,$$

i. e., $\lim_{n\to\infty} d(x_{n+1}, f(x)) = 0$. Now,

$$\lim_{n\to\infty} d(x_{n+1}, f(x)) = \lim_{n\to\infty} d(x_{n+1},x) = 0 \Rightarrow x \in d_l - \mathrm{cl}(f(x)) \subseteq f(x)$$

then $x \in f(x)$. □

Proposition 1. *Let (X,d) be a distance space and $A \subseteq X$. Let f be a multivalued function on A such that*

(1) $D(f(x), f(y)) = 0$ $\forall x,y \in A$,
(2) $\exists\, x \in A$ *such that*
  (a) $x \in f(\ell)$ *for some* $\ell \in A$,
  (b) $d(x, f(x)) = 0 \Rightarrow x \in f(x)$.

Then f has a fixed point.

Definition 9. Let (X,d) be a distance space. A sequence (x_n) in X is called left Cauchy iff $\lim_{n\to\infty} d(x_n,x_m) = 0$ $\forall\, n,m \in \mathbb{N}$such that $m > n$.

Definition 10. A distance space (X,d) is called left complete iff every left Cauchy sequence in X, left converges.

Definition 11. Let (X,d) be a distance space and $A \subseteq X$. A is called left-Cauchy-closed iff if (x_n) is a left Cauchy sequence left convergent to $x \in X$, then $x \in A$.

One can prove that the sequence (x_n) defined in (I) in the proof of Theorem 5 is left Cauchy if the triangular inequality holds. So one can have the following theorem.

Theorem 6. *Let (X,d) be a left complete left small self distance spacesuch that d satisfying (d4) and let $A \subseteq X$. If f is a multivalued function on A, A is left-Cauchy-closed, $f(x)$ is left-d-closed $\forall\ x \in A$ and f satisfies the following condition: $D(f(x), f(y)) \leq kd(x,y)\ \forall\ x,y \in A$ for fixed $k \in (0,1)$, then f has a fixed point.*

Remarks. (1) If d satisfies (d3) the term "left" in each concept is omitted.

(2) Let (X,d) be a distance space satisfies (d3). If A is closed; i. e., if every convergent sequence in A converges in A; then A is left-asymptotically regular-closed.

(3) If (X,d) satisfies (d3) and (d2), then (X,d) is a weak small self distance.

From Theorem 6 and the remarks above one can have the following corollaries.

Corollary 1. *Let (X,d) be a complete d-metric space and $A \subset X$. If f is a multivalued function on Asuch that $\forall x \in A$, A is closed, $f(x)$ is d-closed and f satisfies the condition mentioned in Theorem 6, then f has a fixed point.*

Corollary 2. *If (X,d) is a complete partial metric space and $A \subseteq X$. If f is a multivalued function on Asuch that $\forall\ x \in A$, A is closed, $f(x)$ is d-closed and f satisfies the condition mentioned in Theorem 5, then f has a fixed point.*

The following corollary illustrates that Theorem 6 and Proposition 1 generalize the generalized k-contraction fixed point Theorem by S. B. Nadler.

Corollary 3 ([8])**.** *Let (X,d) be a complete metric space and $A \subseteq X$. Suppose that $f : A \to 2^A - \{\phi\}$ is a multivalued map such that A is closed, $f(x)$ is closed for all $x \in A$, and multivalued k-contraction condition*

$$D(f(x), f(y)) \leq kd(x,y)$$

is satisfied for all $x,y \in A$ and fixed $k \in [0,1)$. Then f has a fixed point.

The proof follows from the above notes and the following fact: In metric spaces, the topology τ_{d_l} coincides with the usual metric topology.

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