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MULTIPLE SOLUTIONS FOR IMPULSIVE HIGHER ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The Leggett-Williams fixed point theorem is applied to obtain triple positive solutions for impulsive nth order functional differential equations.

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1. INTRODUCTION

In this paper, we are concerned with the existence of three nonnegative solutions for initial value problems for nth order functional differential equations with impulsive effects

$$y^{(n)}(t) = f(t, y_t), \quad t \in J = [0,T],\ t \neq t_k,\ k = 1,\dots,m, \tag{1.1}$$

$$\Delta y^{(i-1)}|_{t=t_k} = I_k^i(y(t_k^-)), \quad i = 1,\dots,n,\ k = 1,\dots,m, \tag{1.2}$$

$$y(t) = \phi(t), \quad t \in [-r,0], \tag{1.3}$$

$$y^{(i-1)}(0) = \eta_i, \quad i = 2,\dots,n, \tag{1.4}$$

where $f: J \times D \to \mathbb{R}$ is a given function, $D = \{\psi: [-r,0] \to \mathbb{R}_+ \mid \psi$ is continuous everywhere except for a finite number of points s at which $\psi(s)$ and the right limit (s^+) exist and $\psi(s^-) = \psi(s)\}$, $\phi \in D$, $0 < r < \infty$, $\eta_i \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k^i \in C(\mathbb{R}, \mathbb{R}_+)$ $(i = 1,\dots,n,\ k = 1,2,\dots,m)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, and $\mathbb{R}$ is a real Banach space with the norm $|\cdot|$.

For any function y defined on $[-r,T]$ and any $t \in J$, we denote by y_t the function defined by

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r,0].$$

Impulsive differential equations can be used to model and describe real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. That is why, in recent years, they have become an object of investigation. We refer to the monographs of Bainov and Simeonov [5], Benchohra, Henderson and Ntouyas [6], Lakshmikantham et al [14], and Samoilenko

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and Perestyuk [16] where numerous properties of their solutions are studied, and a detailed bibliography is given.

Recently extensions to functional differential equations with impulsive effects have been done by Benchohra et al. [7], Dong [9], and Franco et al. [11].

The existence of multiple solutions for boundary value problems for impulsive differential equations was studied by Agarwal and O'Regan [1], Erbe et al. [10], and Guo and Liu [13]. Notice that when the impulses are absent (i. e., $I_k = 0$, $k = 1,\dots,m$) the existence of three solutions and multiple solutions for ordinary differential equations was studied in [2–4].

The main results here make use of the Leggett–Williams fixed point theorem [15] in obtaining multiple positive solutions for the problem (1.1)–(1.4). This paper can be viewed as a generalization of the recent paper [8].

2. Preliminaries

In this section, we provide notations, definitions, and preliminary facts which are used throughout the paper.

By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_J := \sup\{|y(t)| : t \in J\}.$$

$L^1(J,\mathbb{R})$ denotes the Banach space of measurable functions $y\colon J \to \mathbb{R}$ which are Lebesgue integrable, with

$$\|y\|_{L^1} = \int_0^T |y(t)|dt.$$

Let $(E,\|\cdot\|)$ be a Banach space and C be a cone in E. By a concave nonnegative continuous functional ψ on C we mean a continuous mapping $\psi\colon C \to [0,\infty]$ with

$$\psi(\lambda x + (1-\lambda)y) \geq \lambda\psi(x) + (1-\lambda)\psi(y) \quad \text{for all } x,y \in C \text{ and } \lambda \in [0,1].$$

For $k = 0,\dots,m$, let $J_k = (t_k, t_{k+1}]$ and for $y\colon [-r,T] \to \mathbb{R}$, define y_k to be the restriction of y to J_k. In order to define the solution of (1.1)–(1.4), we shall consider the space

$$\Omega = \big\{y\colon [-r,T] \to \mathbb{R} \mid y_k \in C(J_k,\mathbb{R}),\ k = 0,\dots,m, \text{ and there exists } y(t_k^+),\ k = 1,\dots,m\big\},$$

which is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y_k\|_{J_k} : k = 0,\dots,m\}.$$

Definition 1. A map $f\colon J \times D \to \mathbb{R}$ is said to be L^1-Carathéodory if

(i) $t \mapsto f(t,u)$ is measurable for each $u \in D$;

(ii) $u \mapsto f(t,u)$ is continuous for almost all $t \in J$;

(iii) for each $q > 0$, there exists $h_q \in L^1(J, \mathbb{R}_+)$ such that

$$|f(t,u)| \le h_q(t) \quad \text{for all } \|u\| \le q \text{ and for almost all } t \in J.$$

Our consideration is based on the following fixed point theorem by Leggett and Williams [15] (see also Guo and Lakshmikantham [12]).

Theorem 2.1. *Let E be a Banach space, $C \subset E$ a cone of E and $R > 0$ a constant. Let $C_R = \{y \in C : \|y\| < R\}$. Suppose a concave nonnegative continuous functional ψ exists on the cone C with $\psi(y) \le \|y\|$ for $y \in \bar{C}_R$, and let $N : \bar{C}_R \to \bar{C}_R$ be a completely continuous operator. Assume there are numbers ϱ, L and K with $0 < \varrho < L < K \le R$ such that*

(A1) *$\{y \in C(\psi, L, K) : \psi(y) > L\} \ne \varnothing$ and $\psi(N(y)) > L$ for all $y \in C(\psi, L, K)$;*
(A2) *$\|N(y)\| < \varrho$ for all $y \in \bar{C}_\varrho$;*
(A3) *$\psi(N(y)) > L$ for all $y \in C(\psi, L, R)$ with $\|N(y)\| > K$, where $C(\psi, L, K) = \{y \in C : \psi(y) \ge L \text{ and } \|y\| \le K\}$.*

Then N has at least three fixed points y_1, y_2, y_3 in $\bar{C}_R$. Furthermore, we have

$$y_1 \in C_\varrho, \quad y_2 \in \{y \in C(\psi, L, R) : \psi(y) > L\},$$

and

$$y_3 \in \bar{C}_R \setminus \{C(\psi, L, R) \cup \bar{C}_\varrho\}.$$

3. Triple solutions of impulsive functional differential equations

In what follows we will assume that f is an L^1-Carathéodory function. Let us start by defining what we mean by a solution of problem (1.1)–(1.4).

Definition 2. A function $y \in \Omega$ such that $y \in \Omega \cap AC^{(n-1)}((t_k, t_{k+1}), \mathbb{R})$, for $k = 0, \dots, m$, is said to be a solution of (1.1)–(1.4) if y satisfies the equation $y^{(n-1)}(t) = f(t, y_t)$ a. e. on $J \setminus \{t_1, \dots, t_m\}$, the conditions $\Delta y^{(i-1)}|_{t=t_k} = I_k^i(y(t_k^-))$, $i = 1, \dots, n$, $k = 1, \dots, m$, $y(t) = \phi(t)$, $t \in [-r, 0]$, and $y^{(i-1)}(0) = \eta_i$, $i = 2, \dots, n$.

Here, $AC^{(n-1)}((t_k, t_{k+1}), \mathbb{R}) = \{y : (t_k, t_{k+1}) \to \mathbb{R} \mid y^{(i)} \text{ is absolutely continuous}, i = 0, \dots, n-1\}$.

In our application of Theorem 2.1 to obtain triple positive solutions of (1.1)–(1.4), we will draw upon the following list of hypotheses.

(H1) There exist constants c_k^i, $i = 1, \dots, n$, $k = 1, \dots, m$, such that

$$I_k^i(y) \le c_k^i, \quad i = 1, \dots, n, \ k = 1, \dots, m, \text{ for each } y \in \mathbb{R};$$

(H2) There exists a function $h \in L^1(J, \mathbb{R}_+)$ and $\varrho > 0$ and $0 < M < 1$ such that

$$f(t,u) \le M h(t) \quad \text{for a. e. } t \in J \text{ and each } u \in D$$

and

$$\|\phi\| + \sum_{i=2}^{n} |\eta_i| \frac{T^{i-1}}{(i-1)!} + M \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} h(s)ds + \sum_{k=1}^{m} \sum_{i=1}^{n} \frac{(T-t_k)^{i-1}}{(i-1)!} c_k^i < \varrho;$$

(H3) There exist $L > \varrho$ and $M \leq M_1 < 1$ and an interval $[a,b] \subset (0,T)$ such that, for each $y \in \Omega$ with $\min_{t\in[a,b]} y(t) \geq L$,

$$\begin{aligned}
&\min_{t\in[a,b]} \left(\phi(0) + \sum_{i=2}^{n} \eta_i \frac{t^{i-1}}{(i-1)!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,y_s)ds \right. \\
&\qquad \left. + \sum_{0<t_k<t} \sum_{i=1}^{n} \frac{(t-t_k)^{i-1}}{(i-1)!} I_k^i(y(t_k^-)) \right) \\
&\geq M_1 \left(\phi(0) + \sum_{i=2}^{n} \eta_i \frac{T^{i-1}}{(i-1)!} + \int_0^T \frac{T^{n-1}}{(n-1)!} f(s,y_s)ds \right. \\
&\qquad \left. + \sum_{k=1}^{m} \sum_{i=1}^{n} \frac{(T-t_k)^{i-1}}{(i-1)!} I_k^i(y(t_k^-)) \right) \\
&> L;
\end{aligned}$$

(H4) There exist $R > L$ and $M_1 \leq M_2 < 1$ such that

$$\|\phi\| + \sum_{i=2}^{n} \eta_i \frac{T^{i-1}}{(i-1)!} + M_2 \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} h(s)ds + \sum_{k=1}^{m} \sum_{i=1}^{n} \frac{(T-t_k)^{i-1}}{(i-1)!} c_k^i \leq R.$$

Theorem 3.1. *Suppose that hypotheses* (H1)–(H4) *are satisfied. Then problem* (1.1)–(1.4) *has three nonnegative solutions* y_1, y_2, *and* y_3 *with*

$$\|y_1\|_\Omega < \varrho, \quad y_2(t) > L \quad \text{for } t \in [0,T],$$

and

$$\|y_3\|_\Omega > \varrho \quad \text{with} \quad \min_{t\in[a,b]} y_3(t) < L.$$

Proof. In making application of Theorem 2.1, we naturally transform problem (1.1)–(1.4) into a fixed point problem. Define an operator $N:\Omega \to \Omega$ by

$$N(y)(t) = \begin{cases} \phi(t) & \text{if } t \in [-r,0], \\ \phi(0) + \sum_{i=2}^{n} \eta_i \dfrac{t^{i-1}}{(i-1)!} + \displaystyle\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,y_s)ds & \\ \qquad + \displaystyle\sum_{0<t_k<t} \sum_{i=1}^{n} \frac{(t-t_k)^{i-1}}{(i-1)!} I_k^i(y(t_k^-)) & \text{if } t \in [0,T]. \end{cases}$$

It is fairly standard that N is completely continuous. Rather than repeat the usual complete continuity arguments, we refer the reader to [7, 8] for details. We content ourselves in verifying that the other hypotheses of Theorem 2.1 are satisfied. Let

$$C = \{y \in \Omega : y(t) \geq 0 \text{ for } t \in [-r,T]\}$$

be a cone in Ω. Since f and I_k^i are all positive functions, it is immediate that $N(C) \subset C$, and moreover, $N|_{\bar{C}_R}$ is completely continuous. In fact, if $y \in \bar{C}_R$, then by (H4), $\|N(y)\|_\Omega \leq R$. In particular, $N:\bar{C}_R \to \bar{C}_R$ is completely continuous.

Next, let $\psi:C \to [0,\infty)$ be defined by

$$\psi(y) = \min_{t\in[a,b]} y(t).$$

Then, it is clear that ψ is a nonnegative concave continuous functional and

$$\psi(y) \leq \|y\|_\Omega \quad \text{for } y \in \bar{C}_R.$$

Now it remains to show that the hypotheses (A1) and (A2) of Theorem 2.1 are satisfied. First, for $y \in \bar{C}_\varrho$, we have from (H1) and (H2),

$$\begin{aligned} |N(y)(t)| \leq & |\phi(0)| + \sum_{i=2}^{n} |\eta_i| \frac{T^{i-1}}{(i-1)!} + \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} f(s,y_s)ds \\ & + \sum_{0<t_k<t} \sum_{i=1}^{n} + \frac{(T-t_k)^{i-1}}{(i-1)!} I_k(y(t_k^-)) \\ \leq & \|\phi\| + \sum_{i=2}^{n} |\eta_i| \frac{T^{i-1}}{(i-1)!} + M \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} h(s)ds \\ & + \sum_{k=1}^{m} \sum_{i=1}^{n} \frac{(T-t_k)^{i-1}}{(i-1)!} c_k^i \\ < & \varrho. \end{aligned}$$

As a consequence, (A2) of Theorem 2.1 holds. Next, let K be such that $\frac{L}{M} \leq K \leq R$ and let $z(t) = \frac{1}{2}(L+K)$ for $t \in [-r,T]$. By the definition of $C(\psi,L,K)$, the function

z belongs to $C(\psi, L, K)$. Moreover, $\psi(z) = \frac{1}{2}(L+K) > L$, so that

$$\{y \in C(\psi, L, K) : \psi(y) > L\} \neq \emptyset.$$

So, if we choose $y \in C(\psi, L, K)$, then from (H3), we have

$$\begin{aligned}
\psi(N(y)) = \min_{t\in[a,b]} \Bigg(& \phi(0) + \sum_{i=2}^{n} \eta_i \frac{t^{i-1}}{(i-1)!} + \sum_{0<t_k<t} \sum_{i=1}^{n} \frac{(t-t_k)^{i-1}}{(i-1)!} I_k^i(y(t_k)) \\
& + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, y_s) ds \Bigg) \\
\geq & M_1 \Bigg(\phi(0) + \sum_{i=2}^{n} \eta_i \frac{T^{i-1}}{(i-1)!} + \int_0^T \frac{T^{n-1}}{(n-1)!} f(s, y_s) ds \\
& + \sum_{k=1}^{m} \sum_{i=1}^{n} \frac{(T-t_k)^{i-1}}{(i-1)!} I_k^i(y(t_k^-)) \Bigg) \\
> & L.
\end{aligned}$$

As a consequence, (A1) of Theorem 2.1 is fulfilled.

Finally, to see that (A3) of Theorem 2.1 holds, let $y \in C(\psi, L, R)$ with $\|N(y)\|_{\Omega} > K$. Again from (H3), we have

$$\begin{aligned}
\psi(N(y)) = \min_{t\in[a,b]} \Bigg(& \phi(0) + \sum_{i=2}^{n} \eta_i \frac{t^{i-1}}{(i-1)!} + \sum_{0<t_k<t} \sum_{i=1}^{n} \frac{(t-t_k)^{i-1}}{(i-1)!} I_k^i(y(t_k)) \Bigg) \\
& + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, y_s) ds) \\
\geq & M_1 \Bigg(\phi(0) + \sum_{i=2}^{n} \eta_i \frac{T^{i-1}}{(i-1)!} + \int_0^T \frac{T^{n-1}}{(n-1)!} f(s, y_s) ds \\
& + \sum_{k=1}^{m} \sum_{i=1}^{n} \frac{(T-t_k)^{i-1}}{(i-1)!} I_k^i(y(t_k^-)) \Bigg) \\
\geq & M_1 ||N(y)||_{\Omega} > M_1 K \geq L.
\end{aligned}$$

Thus, Theorem 2.1 implies that N has at least three fixed points y_1, y_2, and y_3 which are positive solutions of problem (1.1)–(1.4). Furthermore, we have $y_1 \in C_{\varrho}$,

$$y_2 \in \{y \in C(\psi, L, R) : \psi(y) > L\},$$

and $y_3 \in C_R \setminus (C(\psi, L, R) \cup C_{\varrho})$.The proof is complete. □

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NOTES ON THE REPRESENTATION OF DISTRIBUTIVE MODAL ALGEBRAS

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Abstract. In this work we give some results on the representation by means of relational structures of distributive lattices endowed with two modal operators $\Box$ and $\Diamond$, and two weak forms of negation Δ and ∇.

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Keywords: distributive modal algebras, relational structures, representation, canonical varieties

1. INTRODUCTION

In this paper we give some new results on the representation of distributive modal algebras by means of ordered sets with binary relations. Distributive modal algebras are bounded distributive lattices endowed with two modal operators $\Box$ and $\Diamond$, and two weak forms of negation Δ and ∇. These classes of algebras are the algebraic interpretation of certain free-negations modal logics, i. e., logics with modal operators but without classical negation (see, for instance, [2, 4]).

In [1], a relational semantics for Positive Modal Logic is introduced. In this semantics the relational structures, or frames, are triples $\langle X, \leq, R\rangle$ where X is a set, R is a binary relation and $\leq$ is a quasiorder of X, that is, a reflexive and transitive relation on X, such that $\leq \circ R \subseteq R \circ \leq$ and $\leq^{-1} \circ R \subseteq R \circ \leq^{-1}$. This semantics is inherited from a suitable intuitionistic modal logic from which Positive Modal Logic is the positive fragment. Reasons in favour of this semantics are given in [1, 2].

In this paper we study the representation for more extensive classes of distributive modal algebras by means of relational structures of type $\langle X, \leq, R_\Box, R_\Diamond, R_\Delta, R_\nabla\rangle$, where $\leq$ is an ordering of X, and $R_\Box$, $R_\Diamond$, R_Δ, R_∇ are binary relations defined on X.

In Section 2, we introduce the definitions and necessary notions to develop this paper, and we develop the representation by means of relational structures for some subvarieties of the variety of distributive modal algebras $\mathcal{DMA}$. In Section 3we prove that the varieties considered are closed under canonical extensions. Let us

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recall that the notion of canonical variety is the algebraic interpretation of the notion of canonical modal logic (see [6]).

2. Distributive modal algebras

A *distributive modal algebra* is an algebra $\boldsymbol{A} = \langle A, \vee, \wedge, \Box, \Diamond, \Delta, \nabla, 0, 1\rangle$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and $\Box$, $\Diamond$, Δ, and ∇ are unary operations defined on A and satisfying the identities

(M1) $\Box(a \wedge b) = \Box a \wedge \Box b$ and $\Box 1 = 1$;
(M2) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ and $\Diamond 0 = 0$;
(M3) $\Delta(a \vee b) = \Delta a \wedge \Delta b$ and $\Delta 0 = 1$;
(M4) $\nabla(a \wedge b) = \nabla a \vee \nabla b$ and $\nabla 1 = 0$.

The variety of distributive modal algebras shall be denoted by $\mathcal{DMA}$. A $\Box$-*algebra* is an algebra $\boldsymbol{A} = \langle A, \vee, \wedge, \Box, 0, 1\rangle$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice with an operator $\Box$ satisfying (M1). A $\Diamond$-*algebra* is an algebra $\boldsymbol{A} = \langle A, \vee, \wedge, \Diamond, 0, 1\rangle$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice with an operator $\Diamond$ satisfying (M2). A $\Box\Diamond$-*algebra* is an algebra $\boldsymbol{A} = \langle A, \vee, \wedge, \Box, \Diamond, 0, 1\rangle$ such that it is a $\Box$-algebra and a $\Diamond$-algebra. Similarly, a Δ-*algebra* (resp., ∇-*algebra*) is an algebra $\boldsymbol{A} = \langle A, \vee, \wedge, \Delta, 0, 1\rangle$ (resp., $\boldsymbol{A} = \langle A, \vee, \wedge, \nabla, 0, 1\rangle$) such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice with an operator Δ (resp., ∇) satisfying (M3) (resp., (M4)). A $\Delta\nabla$-*algebra* is an algebra $\boldsymbol{A} = \langle A, \vee, \wedge, \Delta, \nabla, 0, 1\rangle$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice, and it is an Δ-algebra and a ∇-algebra.

A $\Box\Diamond$-*algebra* $\boldsymbol{A}$ is a *Positive Modal algebra*, or PM-algebra, if the following conditions are satisfied:

(P1) $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$;
(P2) $\Box(a \vee b) \leq \Box a \vee \Diamond b$.

A $\Delta\nabla$-*algebra* $\boldsymbol{A}$ is a *Negative Modal algebra*, or NM-algebra, if the following conditions are satisfied:

(N1) $\Delta(a \wedge b) \leq \Delta a \vee \nabla b$;
(N2) $\Delta a \wedge \nabla b \leq \nabla(a \vee b)$.

Let us consider a poset $\langle X, \leq\rangle$, i. e., X is a set and $\leq$ is a reflexive, antisymmetric and transitive binary relation on X. A subset $U \subseteq X$ is said to be *increasing* if for all $x, y \in X$ such that $x \in U$ and $x \leq y$, we have $y \in U$. The set of all increasing subsets of X is denoted by $\mathcal{P}_i(X)$. It is clear that $\langle \mathcal{P}_i(X), \cup, \cap, \varnothing, X\rangle$ is a bounded distributive lattice. For each $Y \subseteq X$, the increasing set (resp., decreasing set) generated by Y is $[Y) = \{x \in X : \exists y \in Y : y \leq x\}$ (resp., $(Y] = \{x \in X : \exists y \in Y : x \leq y\}$). Let Y be a subset of a set X. The theoretical complement of Y is denoted by $Y^c = X - Y$.

Let S and R be binary relations defined on a set X. The composition of R with S is denoted by $R \circ S$. The image of $x \in X$ by means of the relation R is $R(x) = \{y \in X : (x, y) \in R\}$.

Let X be a set and let R be a binary relation on X. We define four operators on $\mathcal{P}(X)$ as follows:

$$\Box_R(U) = \{x \in X : R(x) \subseteq U\}, \qquad \Diamond_R(U) = \{x \in X : R(x) \cap U \neq \varnothing\},$$
$$\Delta_R(U) = \{x \in X : R(x) \cap U = \varnothing\}, \quad \nabla_R(U) = \{x \in X : R(x) \not\subseteq U\}.$$

We note that, in $\mathcal{P}(X)$, all operators can be defined from $\Box_R$, i. e., $\Diamond_R(U) = \Box_R(U^c)^c$, $\Delta_R(U) = \Box_R(U^c)$, and $\nabla_R(U) = \Box_R(U)^c$, but in $\mathcal{P}_i(X)$ this fact is not valid.

Proposition 1. *Let $\langle X, \leq\rangle$ be a poset and let R be a binary relation on X. Then $\Box_R(U) = \Box_{R\circ\leq}(U)$, $\Diamond_R(U) = \Diamond_{R\circ\leq^{-1}}(U)$, $\Delta_R(U) = \Delta_{R\circ\leq^{-1}}(U)$, and $\nabla_R(U) = \nabla_{R\circ\leq}(U)$ for all $U \in \mathcal{P}_i(X)$.*

Proof. Let $U \in \mathcal{P}_i(X)$. We prove that $\Box_R(U) = \Box_{R\circ\leq}(U)$. Let $x, y \in X$ such that $x \in \Box_R(U)$, and $(x, y) \in R \circ \leq$. Then there exists $z \in X$ such that $(x, z) \in R$ and $z \leq y$. Since $R(x) \subseteq U$, $z \in U$, and as U is increasing, $y \in U$. Thus, $x \in \Box_{R\circ\leq}(U)$.

Since $\leq$ is reflexive, $R \subseteq R \circ \leq$. So, $\Box_{R\circ\leq}(U) \subseteq \Box_R(U)$.

Let $U \in \mathcal{P}_i(X)$. We prove that $\Delta_R(U) = \Delta_{R\circ\leq^{-1}}(U)$. Let $x \in \Delta_R(U)$. Suppose that $(R \circ \leq^{-1})(x) \cap U \neq \varnothing$. Then there exist $y, z \in X$ such that $(x, y) \in R$, $z \leq y$, and $z \in U$. So, $y \in U \cap R(x)$, which is a contradiction. Thus, $x \in \Delta_{R\circ\leq^{-1}}(U)$. The other inclusion it follows by the fact that $R \subseteq R \circ \leq^{-1}$.

The proofs of the identities $\Diamond_R(U) = \Diamond_{R\circ\leq^{-1}}(U)$ and $\nabla_R(U) = \nabla_{R\circ\leq}(U)$ are similar. □

In general, the bounded distributive lattice $\mathcal{P}_i(X)$ is not closed under the modal operators or the weak negations. In the next result we shall give conditions on the relation R for that $\mathcal{P}_i(X)$ be closed under these operations.

Proposition 2. *Let $\langle X, \leq\rangle$ be a poset and let R be a binary relation on X. Then*

(1) $\leq \circ R \subseteq R \circ \leq$ *if and only if* $\Box_R(U) \in \mathcal{P}_i(X)$, *for all* $U \in \mathcal{P}_i(X)$.
(2) $\leq^{-1} \circ R \subseteq R \circ \leq^{-1}$ *if and only if* $\Diamond_R(U) \in \mathcal{P}_i(X)$, *for all* $U \in \mathcal{P}_i(X)$.
(3) $\leq \circ R \subseteq R \circ \leq^{-1}$ *if and only if* $\Delta_R(U) \in \mathcal{P}_i(X)$, *for all* $U \in \mathcal{P}_i(X)$.
(4) $\leq^{-1} \circ R \subseteq R \circ \leq$ *if and only if* $\nabla_R(U) \in \mathcal{P}_i(X)$, *for all* $U \in \mathcal{P}_i(X)$.

Proof. We prove only (1), (3), and (4).

(1) Let us show that if $U \in \mathcal{P}_i(X)$, then $\Box_R(U) \in \mathcal{P}_i(X)$. Let $x \leq y$ and $R(x) \subseteq U$. Let $z \in X$ such that $(y, z) \in R$. Then there exists $k \in X$ such that $(x, k) \in R$ and $k \leq z$. As $R(x) \subseteq U$, $k \in U$, and since U is increasing, $z \in U$.

Assume that $\Box_R(U) \in \mathcal{P}_i(X)$, for all $U \in \mathcal{P}_i(X)$. Let $x, y, z \in X$ such that $x \leq y$ and $(y, z) \in R$. Let us consider the increasing set $[R(x)) = U_x$. As $R(x) \subseteq U_x$, $x \in \Box_R(U_x)$. Then $y \in \Box_R(U_x)$. Thus, $z \in U_x$, i. e. there exists $k \in X$ such that $(x, k) \in R$ and $k \leq z$, i. e., $(x, z) \in R \circ \leq$.

(3) Assume that $\Delta_R(U) \in \mathcal{P}_i(X)$, for all $U \in \mathcal{P}_i(X)$. Let $x, y, z \in X$ such that $x \leq y$ and $(y, z) \in R$. Suppose that for every $k \in R(x)$, $k \notin [z)$, i. e., $R(x) \cap [z) = \varnothing$.

Then $x \in \Delta_R([z))$. So $y \in \Delta_R([z))$. Thus, $R(y) \cap [z) = \varnothing$, which is a contradiction. Therefore, there exists $k \in X$ such that $(x,k) \in R$ and $z \leq k$, i. e., $(x,z) \in R \circ \leq^{-1}$.

(4) Assume that $\nabla_R(U) \in \mathcal{P}_i(X)$, for all $U \in \mathcal{P}_i(X)$. Let $x, y, z \in X$ such that $z \leq x$ and $(z,y) \in R$. Then $R(z) \cap (y] \neq \varnothing$. Since $U_y = (y]^c \in \mathcal{P}_i(X)$, $z \in \nabla_R(U_y)$, and as $\nabla_R(U_y)$ is increasing, $x \in \nabla_R(U_y)$. Then, $R(x) \cap (y] \neq \varnothing$. Thus, there exists $k \in X$ such that $(x,k) \in R$ and $k \leq y$, i. e., $(x,y) \in R \circ \leq$. The proof of the converse implication is straightforward. □

Theorem 3. *Let $\langle X, \leq \rangle$ be a poset and let $R_\Box$, $R_\Diamond$, R_Δ and R_∇ be binary relations on X. Then:*

(1) *The following conditions are equivalent:*
   (a) $R_\Box \subseteq ((R_\Box \circ \leq) \cap (R_\Diamond \circ \leq^{-1})) \circ \leq$;
   (b) $\Box_{R_\Box}(U \cup V) \subseteq \Box_{R_\Box}(U) \cup \Diamond_{R_\Diamond}(V)$, *for all* $U, V \in \mathcal{P}_i(X)$.

(2) *The following conditions are equivalent:*
   (a) $R_\Diamond \subseteq ((R_\Box \circ \leq) \cap (R_\Diamond \circ \leq^{-1})) \circ \leq$;
   (b) $\Box_{R_\Box}(U) \cap \Diamond_{R_\Diamond}(V) \subseteq \Diamond_{R_\Diamond}(U \cap V)$, *for all* $U, V \in \mathcal{P}_i(X)$.

(3) *The following conditions are equivalent:*
   (a) $R_\Delta \subseteq ((R_\Delta \circ \leq^{-1}) \cap (R_\nabla \circ \leq)) \circ \leq^{-1}$;
   (b) $\Delta_{R_\Delta}(U \cap V) \subseteq \Delta_{R_\Delta}(U) \cup \nabla_{R_\nabla}(V)$, *for all* $U, V \in \mathcal{P}_i(X)$.

(4) *The following conditions are equivalent:*
   (a) $R_\nabla \subseteq ((R_\Delta \circ \leq^{-1}) \cap (R_\nabla \circ \leq)) \circ \leq$;
   (b) $\Delta_{R_\Delta}(U) \cap \nabla_{R_\nabla}(V) \subseteq \nabla_{R_\nabla}(U \cup V)$, *for all* $U, V \in \mathcal{P}_i(X)$.

Proof. (1) (a) ⇒ (b) Let $x \in X$ and $U, V \in \mathcal{P}_i(X)$ such that $x \in \Box_{R_\Box}(U \cup V)$. Assume that $x \notin \Box_{R_\Box}(U)$. Then there exists $y \in X$ such that $(x,y) \in R_\Box$ and $y \notin U$. By assumption, there exists $z \in X$ such that $(x,z) \in R_\Box \circ \leq$, $(x,z) \in R_\Diamond \circ \leq^{-1}$, and $z \leq y$. As $y \notin U$, $z \notin U$, and since $x \in \Box_{R_\Box}(U \cup V) = \Box_{R_\Box \circ \leq}(U \cup V)$, $z \in V$. Then, $z \in (R_\Diamond \circ \leq^{-1})(x) \cap V$, and thus $x \in \Diamond_{R_\Diamond \circ \leq^{-1}}(V) = \Diamond_{R_\Diamond}(V)$.

(b) ⇒ (a) Let $x, y \in X$ such that $(x,y) \in R_\Box$. We shall prove that

$$(R_\Box \circ \leq)(x) \cap (R_\Diamond \circ \leq^{-1})(x) \cap (y] \neq \varnothing.$$

Let us consider the sets

$$U = X \setminus (R_\Diamond \circ \leq^{-1})(x) = (R_\Diamond \circ \leq^{-1})(x)^c \quad \text{and} \quad V = X \setminus (y] = (y]^c.$$

It is clear that $V \in \mathcal{P}_i(X)$. We prove that $U \in \mathcal{P}_i(X)$. Let $a \leq b$ and $a \in U$. If $b \notin U$, there exists $c \in X$ such that $(x,c) \in R_\Diamond$ and $b \leq c$. As $a \leq b \leq c$, $(x,a) \in (R_\Diamond \circ \leq^{-1})$, which is a contradiction. Then, $U \in \mathcal{P}_i(X)$.

Moreover, since $y \in R_\Box(x)$, $R_\Box(x) \cap (y] \neq \varnothing$. So, $x \notin \Box_{R_\Box}(V)$. Also, $x \notin \Diamond_{R_\Diamond}(U)$, because $(R_\Diamond \circ \leq^{-1})(x) \cap (R_\Diamond \circ \leq^{-1})(x)^c = \varnothing$. Thus, $x \notin \Box_{R_\Box}(V) \cup \Diamond_{R_\Diamond}(U)$, and consequently $x \notin \Box_{R_\Box}(U \cup V)$, i. e.,

$$(R_\Box \circ \leq)(x) \not\subseteq U \cup V = ((R_\Diamond \circ \leq^{-1})(x) \cap (y])^c.$$

(3) (a) $\Rightarrow$ (b) Let $x \in X$ and $U, V \in \mathcal{P}_i(X)$ such that $x \in \Delta_{R_\Delta}(U \cap V)$. Then $R_\Delta(x) \cap U \cap V = \varnothing$. Assume that $x \notin \Delta_{R_\Delta}(U)$. Then there exists $y \in X$ such that $(x, y) \in R_\Delta$ and $y \in U$. By assumption, there exists $z \in X$ such that $(x, z) \in (R_\Delta \circ \leq^{-1}) \cap (R_\nabla \circ \leq)$ and $y \leq z$. Then $z \in U$, and since $x \in \Delta_{R_\Delta}(U \cap V) = x \in \Delta_{R_\Delta \circ \leq^{-1}}(U \cap V)$, $z \notin V$. As $z \in (R_\nabla \circ \leq)(x)$, $x \in \nabla_{R_\nabla \circ \leq}(V) = \nabla_{R_\nabla}(V)$.

(b) $\Rightarrow$ (a) Let $x, y \in X$ such that $(x, y) \in R_\Delta$. We shall prove that

$$(R_\Delta \circ \leq^{-1})(x) \cap (R_\nabla \circ \leq)(x) \cap [y) \neq \varnothing.$$

Let us consider the sets

$$U = (R_\nabla \circ \leq)(x) \quad \text{and} \quad V = [y).$$

It is easy to see that $(R_\nabla \circ \leq)(x) \in \mathcal{P}_i(X)$. Moreover, $x \notin \nabla_{R_\nabla}(U)$, because $R_\nabla(x) \subseteq (R_\nabla \circ \leq)(x)$. Also, $x \notin \Delta_{R_\Delta}(V)$, because $y \in R_\Delta(x) \cap [y)$. Thus, $x \notin \Delta_{R_\Delta}(U) \cup \nabla_{R_\nabla}(V)$, and consequently $x \notin \Delta_{R_\Delta}(U \cap V) = \Delta_{R_\Delta \circ \leq^{-1}}(U \cap V)$. Therefore,

$$(R_\Delta \circ \leq^{-1})(x) \cap U \cap V = (R_\Delta \circ \leq^{-1})(x) \cap (R_\nabla \circ \leq)(x) \cap [y) \neq \varnothing.$$

The proofs of the assertions (2) and (4) are similar. □

Theorem 4. *Let $\langle X, \leq \rangle$ be a poset. The following conditions are equivalent:*

(1) *There exists a binary relation R on X such that*
   (a) $\leq \circ R \subseteq R \circ \leq$,
   (b) $\leq^{-1} \circ R \subseteq R \circ \leq^{-1}$.

(2) *There exist two binary relations $R_\Box$ and $R_\Diamond$ on X such that*
   (a) $R_\Box \subseteq (R_\Box \circ \leq) \cap (R_\Diamond \circ \leq^{-1}) \circ \leq$,
   (b) $R_\Diamond \subseteq (R_\Box \circ \leq) \cap (R_\Diamond \circ \leq^{-1}) \circ \leq^{-1}$,
   (c) $\leq \circ R_\Box \subseteq R_\Box \circ \leq$,
   (d) $\leq^{-1} \circ R_\Diamond \subseteq R_\Diamond \circ \leq^{-1}$.

(3) *The $\Box\Diamond$-algebra $\langle \mathcal{P}_i(X), \cup, \cap, \Box_R, \Diamond_R, \varnothing, X \rangle$ is a PM-algebra.*

(4) *The $\Box\Diamond$-algebra $\langle \mathcal{P}_i(X), \cup, \cap, \Box_{R_\Box}, \Diamond_{R_\Diamond}, \varnothing, X \rangle$ is a PM-algebra.*

Proof. (1) $\Rightarrow$ (2) Define the binary relations $R_\Box = R \circ \leq$ and $R_\Diamond = R \circ \leq^{-1}$. Since $\leq$ is reflexive,

$$R \subseteq R_\Box \cap R_\Diamond \subseteq (R_\Box \circ \leq) \cap (R_\Diamond \circ \leq^{-1}).$$

So,

$$R_\Box = (R \circ \leq) \subseteq (R_\Box \circ \leq) \cap (R_\Diamond \circ \leq^{-1}) \circ \leq,$$

and

$$\leq \circ R_\Box = (\leq \circ R \circ \leq) \subseteq R \circ \leq \circ \leq = R_\Box \circ \leq.$$

The proof for the relation $R_\Diamond$ is similar.

(2) $\Rightarrow$ (1) Given the relations $R_\Box$ and $R_\Diamond$, we define the relation

$$R = (R_\Box \circ \leq) \cap (R_\Diamond \circ \leq^{-1}).$$

Let $x \leq y$ and $(y,z) \in R = (R_\Box \circ \leq) \cap (R_\Diamond \circ \leq^{-1})$. Then there exists $k \in X$ such that $(y,k) \in R_\Box$ and $k \leq z$. As $(x,k) \in \leq \circ R_\Box \subseteq R_\Box \circ \leq$, there exists $w \in X$ such that $(x,w) \in R_\Box$ and $w \leq k$. Since $R_\Box \subseteq (R_\Box \circ \leq) \cap (R_\Diamond \circ \leq^{-1}) \circ \leq$, there exists $d \in X$ such that $(x,d) \in R$ and $d \leq w$. Thus $(x,z) \in R \circ \leq$.

The proof of the inclusion $\leq^{-1} \circ R \subseteq R \circ \leq^{-1}$ is similar. The equivalence of (1) and (3) follows from Proposition 2. The equivalence of (2) and (4) follows from Theorem 3 and Proposition 2. □

Theorem 5. *Let $\langle X, \leq \rangle$ be a poset. The following conditions are equivalent:*

(1) *There exists a binary relation R on X such that*
   (a) $\leq \circ R \subseteq R \circ \leq^{-1}$,
   (b) $\leq^{-1} \circ R \subseteq R \circ \leq^{-1}$.

(2) *There exist two binary relations R_Δ and R_∇ on X such that*
   (a) $R_\Delta \subseteq ((R_\Delta \circ \leq^{-1}) \cap (R_\nabla \circ \leq)) \circ \leq^{-1}$,
   (b) $R_\nabla \subseteq ((R_\Delta \circ \leq^{-1}) \cap (R_\nabla \circ \leq)) \circ \leq$,
   (c) $\leq \circ R_\Delta \subseteq R_\Delta \circ \leq^{-1}$,
   (d) $\leq^{-1} \circ R_\nabla \subseteq R_\nabla \circ \leq$.

(3) *There exists a binary relation R on X such that the $\Box\Diamond$-algebra*

$$\langle \mathcal{P}_i(X), \cup, \cap, \Delta_R, \nabla_R, \varnothing, X \rangle$$

is a NM-algebra.

(4) *There exist two binary relations R_Δ and R_∇ on X such that the $\Box\Diamond$-algebra $\langle \mathcal{P}_i(X), \cup, \cap, \Delta_{R_\Delta}, \nabla_{R_\nabla}, \varnothing, X \rangle$ is a NM-algebra.*

Proof. The proof of the direction (1) ⇒ (2) is similar to the proof of the previous theorem, but taking $R_\Delta = R \circ \leq^{-1}$ and $R_\nabla = R \circ \leq$. For the direction (2) ⇒ (1) we need define the relation R as $R = (R_\Delta \circ \leq^{-1}) \cap (R_\nabla \circ \leq)$. □

3. Canonical varieties

Let $\boldsymbol{A}$ be a bounded distributive lattice. By $X(\boldsymbol{A})$ we shall denote the set of all prime filters of $\boldsymbol{A}$. The filter (resp., ideal) generated by a subset $H \subseteq A$ will be denoted by $[H)$ (resp., $(H]$). The lattice of all filters (resp., ideals) of $\boldsymbol{A}$ is denoted by $\mathrm{Fi}(\boldsymbol{A})$ (resp., $\mathrm{Id}(\boldsymbol{A})$).

Let $\boldsymbol{A}$ be a distributive modal algebra. We define binary relations $R^{\boldsymbol{A}}_\Box$, $R^{\boldsymbol{A}}_\Diamond$, $R^{\boldsymbol{A}}_\Delta$, and $R^{\boldsymbol{A}}_\nabla$ on $X(\boldsymbol{A})$ as follows:

(1) $(P,Q) \in R^{\boldsymbol{A}}_\Box \Leftrightarrow \Box^{-1}(P) \subseteq Q$,
(2) $(P,Q) \in R^{\boldsymbol{A}}_\Diamond \Leftrightarrow Q \subseteq \Diamond^{-1}(P)$,
(3) $(P,Q) \in R^{\boldsymbol{A}}_\Delta \Leftrightarrow Q \subseteq (\Delta^{-1}(P))^c$,
(4) $(P,Q) \in R^{\boldsymbol{A}}_\nabla \Leftrightarrow (\nabla^{-1}(P))^c \subseteq Q$

with $P, Q \in X(\boldsymbol{A})$.

Lemma 6. *Let $\boldsymbol{A}$ be a distributive modal algebra. Then, for each $P \in X(\boldsymbol{A})$ and for each $a \in A$, the following assertions hold:*

(1) $\Box a \in P$ *if and only if for every* $Q \in X(\boldsymbol{A})$ *such that* $(P, Q) \in R^{\boldsymbol{A}}_{\Box}$ *it holds* $a \in Q$.
(2) $\Diamond a \in P$ *if and only if there exists* $Q \in X(A)$ *such that* $(P, Q) \in R^{\boldsymbol{A}}_{\Diamond}$ *and* $a \in Q$.
(3) $\Delta a \in P$ *if and only if for every* $Q \in X(\boldsymbol{A})$ *such that* $(P, Q) \in R^{\boldsymbol{A}}_{\Delta}$ *it holds* $a \notin Q$.
(4) $\nabla a \in P$ *if and only if there exists* $Q \in X(\boldsymbol{A})$ *such that* $(P, Q) \in R^{\boldsymbol{A}}_{\nabla}$ *and* $a \notin Q$.

Proof. For the proof for the operators $\Box$ and $\Diamond$, see [1, 5]. For the weak negation Δ, see [3]. Let $\nabla a \in P$, and let us consider the filter $(\nabla^{-1}(P))^c$. As $a \notin (\nabla^{-1}(P))^c$, there exists $Q \in X(\boldsymbol{A})$ such that $(\nabla^{-1}(P))^c \subseteq Q$ and $a \notin Q$. The converse implication is immediate. □

Let us consider the relational structure $\mathcal{F}(\boldsymbol{A}) = \langle X(\boldsymbol{A}), R^{\boldsymbol{A}}_{\Box}, R^{\boldsymbol{A}}_{\Diamond}, R^{\boldsymbol{A}}_{\Delta}, R^{\boldsymbol{A}}_{\nabla}\rangle$. It is easy to see that

$$\subseteq \circ R^{\boldsymbol{A}}_{\Box} \subseteq R^{\boldsymbol{A}}_{\Box} \circ \subseteq, \qquad \subseteq^{-1} \circ R^{\boldsymbol{A}}_{\Diamond} \subseteq R^{\boldsymbol{A}}_{\Diamond} \circ \subseteq^{-1},$$

$$\subseteq \circ R^{\boldsymbol{A}}_{\Delta} \subseteq R^{\boldsymbol{A}}_{\Delta} \circ \subseteq^{-1}, \qquad \subseteq^{-1} \circ R^{\boldsymbol{A}}_{\nabla} \subseteq R^{\boldsymbol{A}}_{\nabla} \circ \subseteq .$$

Thus, by Proposition 2, we get that the algebra

$$\mathrm{Ec}(\boldsymbol{A}) = \langle \mathcal{P}_i(X(\boldsymbol{A})), \cup, \cap, \Box, \Diamond, \Delta, \nabla, \varnothing, X(\boldsymbol{A})\rangle$$

is a distributive modal algebra, and the map $\beta_{\boldsymbol{A}} : A \to \mathcal{P}_i(X(\boldsymbol{A}))$ defined by the equality $\beta_{\boldsymbol{A}}(a) = \{P \in X(\boldsymbol{A}) : a \in P\}$ is an injective homomorphism of distributive modal algebras. The algebra $\mathrm{Ec}(\boldsymbol{A})$ is called the *canonical extension* of $\boldsymbol{A}$.

Definition 1. A variety $\mathcal{V}$ of distributive modal algebras is canonical if for any $\boldsymbol{A} \in \mathcal{V}$, $\mathrm{Ec}(\boldsymbol{A}) \in \mathcal{V}$.

The notion of canonical varieties is the algebraic interpretation of canonical modal logics (see [6]). Let us recall that a modal logic is canonical if their canonical frame is a frame of the logic.

Now, we shall see that if $\boldsymbol{A}$ is a PM-algebra (resp., NM-algebra), then $\mathrm{Ec}(\boldsymbol{A})$ is a PM-algebra (resp., NM-algebra).

Theorem 7. *Let $\boldsymbol{A}$ be distributive modal algebra. Then:*

(1) $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$ *is valid in* $\boldsymbol{A}$ *if and only if* $R^{\boldsymbol{A}}_{\Diamond} \subseteq (R^{\boldsymbol{A}}_{\Box} \cap R^{\boldsymbol{A}}_{\Diamond}) \circ \subseteq^{-1}$.
(2) $\Box(a \vee b) \leq \Box a \vee \Diamond b$ *is valid in* $\boldsymbol{A}$ *if and only if* $R^{\boldsymbol{A}}_{\Box} \subseteq (R^{\boldsymbol{A}}_{\Box} \cap R^{\boldsymbol{A}}_{\Diamond}) \circ \subseteq$.
(3) $\Delta(a \wedge b) \leq \Delta a \vee \nabla b$ *is valid in* $\boldsymbol{A}$ *if and only if* $R^{\boldsymbol{A}}_{\Delta} \subseteq (R^{\boldsymbol{A}}_{\Delta} \cap R^{\boldsymbol{A}}_{\nabla}) \circ \subseteq^{-1}$.
(4) $\Delta a \wedge \nabla b \leq \nabla(a \vee b)$ *is valid in* $\boldsymbol{A}$ *if and only if* $R^{\boldsymbol{A}}_{\nabla} \subseteq (R^{\boldsymbol{A}}_{\Delta} \cap R^{\boldsymbol{A}}_{\nabla}) \circ \subseteq$.

Proof. We prove only (3) and (4).

For (3), we assume that $\Delta(a \wedge b) \leq \Delta a \vee \nabla b$ is valid in $\boldsymbol{A}$. Let $P, Q \in X(\boldsymbol{A})$ such that $Q \subseteq \Delta^{-1}(P)^c$. Let us consider the filter $F(\nabla^{-1}(P)^c \cup Q)$. We prove that

$$F(\nabla^{-1}(P)^c \cup Q) \cap \Delta^{-1}(P) = \varnothing.$$

Suppose the opposite. Then there are $a \in \nabla^{-1}(P)^c$, $b \in Q$ and $c \in \Delta^{-1}(P)$ such that $a \wedge b \leq c$. So,

$$\Delta c \leq \Delta(a \wedge b),$$

and by the hypothesis we obtain

$$\Delta c \leq \Delta a \vee \nabla b.$$

Since $\Delta c \in P$, $\Delta a \vee \nabla b \in P$, and as P is a prime filter, $\Delta a \in P$ or $\nabla b \in P$, which is a contradiction. Thus, there exists $D \in X(\boldsymbol{A})$ such that

$$\nabla^{-1}(P)^c \subseteq D, \quad Q \subseteq D, \quad \text{and} \quad D \cap \Delta^{-1}(P) = \varnothing,$$

i. e., $(P, Q) \in (R_{\Delta}^{\boldsymbol{A}} \cap R_{\nabla}^{\boldsymbol{A}}) \circ \subseteq^{-1}$.

Assume that $R_{\Delta}^{\boldsymbol{A}} \subseteq (R_{\Delta}^{\boldsymbol{A}} \cap R_{\nabla}^{\boldsymbol{A}}) \circ \subseteq^{-1}$. Let $a, b \in A$ and suppose that $\Delta(a \wedge b) \not\leq \Delta a \vee \nabla b$. Then there exits $P \in X(\boldsymbol{A})$ such that $\Delta(a \wedge b) \in P$, and $\Delta a, \nabla b \notin P$. So there exists $Q \in X(\boldsymbol{A})$ such that $(P, Q) \in R_{\Delta}^{\boldsymbol{A}}$ and $a \in Q$. By assumption, there exist $D \in X(\boldsymbol{A})$ such that $(P, D) \in R_{\Delta}^{\boldsymbol{A}} \cap R_{\nabla}^{\boldsymbol{A}}$ and $Q \subseteq D$. Since $\Delta(a \wedge b) \in P$ and $(P, D) \in R_{\Delta}^{\boldsymbol{A}}$, $a \wedge b \notin D$, but as $(P, D) \in R_{\nabla}^{\boldsymbol{A}}$, and $\nabla b \notin P$, $b \in D$. Also, $c \in D$, because $c \in Q \subseteq D$. Then, $b \wedge c \in D$, which is a contradiction. Thus, $\Delta(a \wedge b) \leq \Delta a \vee \nabla b$ for all $a, b \in A$.

(4) Suppose that $\Delta a \wedge \nabla b \leq \nabla(a \vee b)$ is valid in $\boldsymbol{A}$. Let $P, Q \in X(\boldsymbol{A})$ such that $\nabla^{-1}(P)^c \subseteq Q$. Let us consider the ideal $I = I(\Delta^{-1}(P) \cup Q^c)$. We prove that

$$\nabla^{-1}(P)^c \cap I = \varnothing.$$

Suppose the contrary. Then, there exists $c \in \nabla^{-1}(P)^c$, $a \notin Q$, and $b \in \Delta^{-1}(P)$ such that $c \leq a \vee b$. So, $\nabla(a \vee b) \leq \nabla c$. By hypothesis we have

$$\Delta b \wedge \nabla a \leq \nabla c.$$

Since $\Delta b \in P$ and $a \in \nabla^{-1}(P)$, $\nabla c \in P$, which is a contradiction. Thus, there exists $D \in X(\boldsymbol{A})$ such that $(P, D) \in R_{\Delta}^{\boldsymbol{A}} \cap R_{\nabla}^{\boldsymbol{A}}$ and $D \subseteq Q$, i. e., $R_{\nabla}^{\boldsymbol{A}} \subseteq (R_{\Delta}^{\boldsymbol{A}} \cap R_{\nabla}^{\boldsymbol{A}}) \circ \subseteq$. The proof of the other implication is omitted. □

Let $\boldsymbol{A}$ be a PM-algebra. Let us define the relation $R_A = R_{\Box}^A \cap R_{\Diamond}^A$. Then it is easy to see that $R_{\Box}^A = R_A \circ \subseteq$, $R_{\Diamond}^A = R_A \circ \subseteq^{-1}$, and $\subseteq \circ R_A \subseteq R_A \circ \subseteq$ and $\subseteq^{-1} \circ R_A \subseteq R_A \circ \subseteq^{-1}$. Consequently, by Theorem 7 we can deduce that the axioms that relate the operators $\Box$ and $\Diamond$ allow us to define the single relation $R_A = R_{\Box}^A \cap R_{\Diamond}^A$.

Let $\boldsymbol{A}$ be a NM-algebra. Let us define the relation $R_A = R_{\Delta}^A \cap R_{\nabla}^A$. Then it is easy to see that $R_{\Delta}^A = R_A \circ \subseteq^{-1}$, $R_{\nabla}^A = R_A \circ \subseteq$, and $\subseteq \circ R_A \subseteq R_A \circ \subseteq$ and

$\subseteq^{-1} \circ R_A \subseteq R_A \circ \subseteq^{-1}$. So, by Theorem 7 we can deduce that the axioms that relate the operators Δ and ∇ allow us to define the single relation $R_A = R^A_\Delta \cap R^A_\nabla$.

Let $\mathcal{V}$ be a variety and let Φ be a set of identities in the algebraic language of $\mathcal{V}$. We denote by $\mathcal{V} + \Phi$ the variety generated by $\mathcal{V}$ and by Φ. We note that the previous results imply that, whenever $\boldsymbol{A}$ belongs to some of the varieties $\mathcal{DML} + \Phi$, where $\Phi \subseteq \{(P1), (P2), (N1), (N2)\}$, then the relational structure $\langle X(\boldsymbol{A}), R^{\boldsymbol{A}}_\Box, R^{\boldsymbol{A}}_\Diamond, R^{\boldsymbol{A}}_\Delta, R^{\boldsymbol{A}}_\nabla \rangle$ satisfies the appropriate first-order conditions that correspond to the identities of Φ.

Theorem 8. *Let $\mathcal{DMA}$ be the variety of distributive modal algebras. The variety $\mathcal{DML}$ and any extension of it that is obtained by adding any subset of the set of identities $\{(P1), (P2), (N1), (N2)\}$ is canonical.*

Proof. It is clear that if $\boldsymbol{A} \in \mathcal{DMA}$, then $\mathrm{Ec}(\boldsymbol{A}) \in \mathcal{DMA}$. Thus, $\mathcal{DMA}$ is canonical. If $\boldsymbol{A}$ is an algebra in some of the varieties $\mathcal{DMA} + \Phi$, where Φ is any subset of $\{(P1), (P2), (N1), (N2)\}$, then we should verify that the canonical extension $\mathrm{Ec}(\boldsymbol{A})$ belong to $\mathcal{DMA} + \Phi$. But this it follows from Theorem 7 and the results of the previous section. □

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APPROXIMATION OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH DELAYED ARGUMENTS

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Abstract. We discuss boundary value problems for first-order functional differential equations and investigate the approximation of extremal solutions of such problems. We use a monotone iterative method.

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1. INTRODUCTION

Let us consider the problem

$$\begin{cases} x'(t) = f(t, x(t), x(\alpha(t))), & t \in J := [0, T], \\ x(0) = rx(T), & r \in (0, 1], \end{cases} \tag{1.1}$$

where $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and the function $\alpha \in C(J, J)$ is such that $\alpha(t) \leq t$ for all $t \in J$.

In this paper we extend some results of paper [3] concerning the case where $r = 1$ and f satisfies a left-sided Lipshitz condition with suitable constants. We show the applicability of the monotone iterative method in obtaining monotone sequences approximating the extremal solutions of (1.1). We refer, e. g., to [2] for details about the monotone iterative method. Note that we use more general definition of lower and upper solution than the classical definition used, e. g., in [1].

We start by proving a comparison theorem used in the sequel. In Section 3, we prove the existence of a solution of the linear problem associated to (1.1). Finally, we prove the existence of monotone sequences approximating the extremal solutions of problem (1.1).

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2. Comparison results

Lemma 1 ([1]). *Suppose that $\alpha \in C(J,J)$, $\alpha(t) \le t$ on J, $M \in C(J,\mathbb{R})$, $q \in C^1(J,\mathbb{R})$ is such that*

$$q'(t) \le -M(t)q(t) - N(t)q(\alpha(t)), \quad t \in J, \qquad q(0) \le 0,$$

and

$$\int_0^T N(t)e^{\int_{\alpha(t)}^t M(s)ds}dt \le 1,$$

where N is a nonnegative function integrable on J. Then $q(t) \le 0$ on J.

Theorem 1. *Assume that $r \in (0,1]$ and*

(1) *$x \in C^1(J,\mathbb{R})$, $\alpha \in C(J,J)$, $\alpha(t) \le t$, $t \in J$, $M \in C(J,\mathbb{R})$, N is integrable on J, $M(t) > 0$, $t \in J$ and $N(t) \ge 0$, $t \in J$,*
(2) *$x'(t) + M(t)x(t) + N(t)x(\alpha(t)) \le 0$, $t \in J$, if $x(0) \le rx(T)$,*
(3) *$x'(t) + M(t)x(t) + N(t)x(\alpha(t)) + \frac{M(t)t + N(t)\alpha(t)+1}{rT}[x(0) - rx(T)] \le 0$, $t \in J$, if $x(0) > rx(T)$,*
(4) *$\int_0^T N(t)e^{\int_{\alpha(t)}^t M(s)ds}dt \le 1$.*

Then $x(t) \le 0$, $t \in J$.

Proof. In case (1), we assume that $x \ge 0$ on J and $x \not\equiv 0$. Then $x'(t) \le 0$, $t \in J$, so x is nonincreasing and $x(t) \le x(0)$ $t \in J$ so $x(0) \ge x(T)$. On the other hand, we have $x(0) \le rx(T) \le x(T)$. Thus $x(0) = x(T)$ and x is a constant function, $x(t) = C > 0$. So $x' \equiv 0$ and $[M(t) + N(t)]C \le 0$. Hence $C = 0$ and finally $x(t) = 0$, $t \in J$. It is a contradiction.

Thus we can consider that x has some negative value. Note that if $x(0) \le 0$, then $x(t) \le 0$, $t \in J$, by Lemma 1. Assume that $x(0) > 0$, then $x(T) > 0$. Let us consider the function v defined by

$$v(t) = e^{\int_0^t M(s)ds}x(t), \quad t \in J.$$

We have $v(0) = x(0) > 0$ and $v(T) = e^{\int_0^T M(s)ds}x(T) > 0$. Since x has some negative value, there exists $t_* \in (0,T)$ such that

$$v(t_*) = \min_{t\in[0,T]} v(t) < 0.$$

Note that

$$v'(t) = e^{\int_0^t M(s)ds}M(t)x(t) + e^{\int_0^t M(s)ds}x'(t) \le -N(t)e^{\int_{\alpha(t)}^t M(s)ds}v(\alpha(t)).$$

The integration of this from t_* to T yields

$$-v(t_*) < v(T) - v(t_*) \le -\int_{t_*}^T N(t)e^{\int_{\alpha(t)}^t M(s)ds}v(\alpha(t))dt$$

$$\leq -v(t_*)\int_{t_*}^{T} N(t)e^{\int_{\alpha(t)}^{t} M(s)\,ds}dt \leq -v(t_*).$$

So $-v(t_*) < -v(t_*)$, which is a contradiction. Thus $x(0) \leq 0$ and, in view of Lemma 1, we get $x(t) \leq 0$, $t \in J$.

In case (3), let us consider the function

$$w(t) = x(t) + \frac{t}{rT}[x(0) - rx(T)], \quad t \in J.$$

It yields that $w(0) = rw(T)$. Moreover,

$$w'(t) + M(t)w(t) + N(t)w(\alpha(t)) = x'(t) + M(t)x(t) + N(t)x(\alpha(t)) + \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[x(0) - rx(T)] \leq 0.$$

This is included in case (2). Thus $w(t) \leq 0$, $t \in J$ and finally $x(t) \leq 0$, $t \in J$. □

3. Linear problem

Now we are going to prove the existence of solution of linear problem associated to (1.1). We will need that result in succeeding section.

Theorem 2. *Let $\sigma \in C(J,\mathbb{R})$, $\alpha \in C(J,J)$, $\alpha(t) \leq t$, $t \in J$, $M, N \in C(J,\mathbb{R})$, $M(t) > 0$, $t \in J$, $N(t) \geq 0$, $t \in J$ and*

$$\begin{cases} x'(t) + M(t)x(t) + N(t)x(\alpha(t)) = \sigma(t), & t \in J, \\ x(0) = rx(T), & r \in (0,1]. \end{cases} \tag{3.1}$$

Moreover, assume that there exist functions $y_0, z_0 \in C^1(J,\mathbb{R})$ such that

(1) *$y_0 \leq z_0$ on J,*

(2)

$$y_0'(t) + M(t)y_0(t) + N(t)y_0(\alpha(t)) \leq \sigma(t) - a(t), \quad t \in J,$$
$$z_0'(t) + M(t)z_0(t) + N(t)z_0(\alpha(t)) \geq \sigma(t) - b(t), \quad t \in J,$$

where

$$a(t) = \begin{cases} 0 & \text{if } y_0(0) \leq ry_0(T), \\ \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[y_0(0) - ry_0(T)] & \text{if } y_0(0) > ry_0(T), \end{cases}$$

and

$$b(t) = \begin{cases} 0 & \text{if } z_0(0) \geq rz_0(T), \\ \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[z_0(0) - rz_0(T)] & \text{if } z_0(0) < rz_0(T). \end{cases}$$

(3) $\int_0^T N(t)e^{\int_{\alpha(t)}^{t} M(s)ds}dt \leq 1$.

Then there exists, in sector $[y_0, z_0]_* = \{w \in C^1(J,\mathbb{R}) : y_0(t) \leq w(t) \leq z_0(t),\ t \in J\}$, a unique solution of (3.1).

Proof. First we prove the uniqueness of a solution. Let x_1, x_2 be solutions of (3.1). Put $v_1 = x_1 - x_2$ and $v_2 = x_2 - x_1$. Then

$$v_1(0) = x_1(0) - x_2(0) = r[x_1(T) - x_2(T)] = rv_1(T),$$
$$v_1'(t) + M(t)v_1(t) + N(t)v_1(\alpha(t)) = \sigma(t) - \sigma(t) = 0, \quad t \in J,$$

and

$$v_2(0) = rv_2(T),$$
$$v_2'(t) + M(t)v_2(t) + N(t)v_2(\alpha(t)) = 0, \quad t \in J.$$

In view of Theorem 1, we have $v_1 \le 0$ and $v_2 \le 0$. Hence $x_1 = x_2$.

Now we show that if x is a solution of (3.1), then $y_0 \le x \le z_0$. Put $w_1 = y_0 - x$ and $w_2 = x - z_0$. Then we have

$$w_1'(t) + M(t)w_1(t) + N(t)w_1(\alpha(t)) \le \sigma(t) - \sigma(t) = 0 \quad \text{if } w_1(0) \le rw_1(T),$$

$$\begin{aligned} w_1'(t) + M(t)w_1(t) + N(t)w_1(\alpha(t)) + \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[w_1(0) - rw_1(T)] \\ \le \sigma(t) - \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[y_0(0) - ry_0(T)] \\ -\sigma(t) + \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[y_0(0) - ry_0(T)] \\ = 0 \quad \text{if } w_1(0) > rw_1(T), \end{aligned}$$

and

$$w_2'(t) + M(t)w_2(t) + N(t)w_2(\alpha(t)) \le 0 \quad \text{if } w_2(0) \le rw_2(T)$$

$$\begin{aligned} w_2'(t) + M(t)w_2(t) + N(t)w_2(\alpha(t)) \\ + \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[w_2(0) - rw_2(T)] \le 0 \quad \text{if } w_2(0) > rw_2(T). \end{aligned}$$

In view of Theorem 1, $w_1 \le 0, w_2 \le 0$ on J. It shows that $y_0(t) \le x(t) \le z_0(t)$, $t \in J$.

Finally, we prove that problem (3.1) has the solution x. Let us consider two functions

$$\bar{y}_0(t) = \begin{cases} y_0(t) & \text{if } y_0(0) \le ry_0(T), \\ y_0(t) + \frac{t}{rT}[y_0(0) - ry_0(T)] & \text{if } y_0(0) > ry_0(T), \end{cases}$$

and

$$\bar{z}_0(t) = \begin{cases} z_0(t) & \text{if } z_0(0) \ge rz_0(T), \\ z_0(t) + \frac{t}{rT}[z_0(0) - rz_0(T)] & \text{if } z_0(0) < rz_0(T). \end{cases}$$

We have $y_0(t) \le \bar{y}_0(t)$ and $\bar{z}_0(t) \le z_0(t)$, $t \in J$. Moreover, $\bar{y}_0(0) \le r\bar{y}_0(T)$ and $\bar{z}_0(0) \ge r\bar{z}_0(T)$. Note that if $y_0(0) > ry_0(T)$, then $\bar{y}_0(0) = r\bar{y}_0(T)$ and if $z_0(0) < rz_0(T)$, then $\bar{z}_0(0) = r\bar{z}_0(T)$.

We show that $\overline{y}_0$ and $\overline{z}_0$ are classical lower and upper solutions, respectively, of (3.1) and that $\overline{y}_0 \le \overline{z}_0$. We have

$$\begin{aligned}\overline{y}_0'(t) + M(t)\overline{y}_0(t) + N(t)\overline{y}_0(\alpha(t)) &= y_0'(t) + M(t)y_0(t) + N(t)y_0(\alpha(t)) \\ &\le \sigma(t),\ t \in J, \quad \text{if } y_0(0) \le r y_0(T)\end{aligned}$$

and

$$\begin{aligned}\overline{y}_0'(t) + M(t)\overline{y}_0(t) + N(t)\overline{y}_0(\alpha(t)) &= y_0'(t) + M(t)y_0(t) + N(t)y_0(\alpha(t)) \\ &\quad + \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[y_0(0) - r y_0(T)] \\ &\le \sigma(t) - \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[y_0(0) - r y_0(T)] \\ &\quad + \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[y_0(0) - r y_0(T)] \\ &= \sigma(t),\ t \in J, \quad \text{if } y_0(0) > r y_0(T).\end{aligned}$$

Similarly,

$$\overline{z}_0'(t) + M(t)\overline{z}_0(t) + N(t)\overline{z}_0(\alpha(t)) \ge \sigma(t), \quad t \in J.$$

Thus $\overline{y}_0$ is a classical lower and $\overline{z}_0$ a classical upper solution of (3.1).

Now consider the function $w = \overline{y}_0 - \overline{z}_0 \in C^1(J, \mathbb{R})$. We have

$$w'(t) + M(t)w(t) + N(t)w(\alpha(t)) \le 0, \quad t \in J$$

and $w(0) \le 0$. Lemma 1 yields $w \le 0$ on J, hence $\overline{y}_0 \le \overline{z}_0$.

Setting

$$\begin{cases} \overline{y}_{n+1}'(t) = \sigma(t) - M(t)\overline{y}_{n+1}(t) - N(t)\overline{y}_{n+1}(\alpha(t)), & t \in J, \\ \overline{y}_{n+1}(0) = r\overline{y}_n(T), \end{cases}$$

and

$$\begin{cases} \overline{z}_{n+1}'(t) = \sigma(t) - M(t)\overline{z}_{n+1}(t) - N(t)\overline{z}_{n+1}(\alpha(t)), & t \in J, \\ \overline{z}_{n+1}(0) = r\overline{z}_n(T), \end{cases}$$

and arguing similarly to the proof of [1, Theorem 3.1], we show that there exists a solution of (3.1). □

4. Approximation of extremal solutions of (1.1)

In this section we develop monotone iterative technique for (1.1).

Theorem 3. *Let $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\alpha \in C(J, J)$, $\alpha(t) \le t$, $t \in J$, $M, N \in C(J, \mathbb{R})$, $M(t) > 0$, $t \in J$, $N(t) \ge 0$, $t \in J$. Moreover, assume that there exist functions $y_0, z_0 \in C^1(J, \mathbb{R})$ such that*

(1) *$y_0 \le z_0$ on J,*

(2) y_0, z_0 *are lower and upper solutions of* (1.1)*, respectively, i. e.,*

$$y_0'(t) \le f(t, y_0(t), y_0(\alpha(t))) - a(t), \quad t \in J,$$
$$z_0'(t) \ge f(t, z_0(t), z_0(\alpha(t))) - b(t), \quad t \in J,$$

where

$$a(t) = \begin{cases} 0 & \text{if } y_0(0) \le r y_0(T), \\ \frac{M(t)t + N(t)\alpha(t) + 1}{rT} [y_0(0) - r y_0(T)] & \text{if } y_0(0) > r y_0(T), \end{cases}$$

and

$$b(t) = \begin{cases} 0 & \text{if } z_0(0) \ge r z_0(T), \\ \frac{M(t)t + N(t)\alpha(t) + 1}{rT} [z_0(0) - r z_0(T)] & \text{if } z_0(0) < r z_0(T), \end{cases}$$

(3) $f(t,u,v) - f(t,\overline{u},\overline{v}) \le M(t)[\overline{u} - u] + N(t)[\overline{v} - v]$, *if* $y_0(t) \le u \le \overline{u} \le z_0(t)$, $y_0(\alpha(t)) \le u \le \overline{u} \le z_0(\alpha(t))$,

(4) $\int_0^T N(t) e^{\int_{\alpha(t)}^t M(s)ds} dt \le 1$.

Then there exist monotone sequences $\{y_n\} \uparrow y$ *and* $\{z_n\} \downarrow z$ *uniformly on* J *with* $y_0 \le y_n \le z_n \le z_0$ *for every* $n \in \mathbb{N}$ *and* $y, z \in C^1(J, \mathbb{R})$. *Functions* y *and* z *are extremal solutions of* (1.1).

Proof. Let us consider the problem

$$\begin{cases} x'(t) + M(t)x(t) + N(t)x(\alpha(t)) = \sigma_u(t), \quad t \in J, \\ x(0) = r x(T), \quad r \in (0,1], \end{cases} \tag{4.1}$$

where $\sigma_u(t) = f(t, u(t), u(\alpha(t))) + M(t)u(t) + N(t)u(\alpha(t))$ for $u \in C(J, \mathbb{R})$, $y_0 \le u \le z_0$ on J. Note that, in view of Theorem 2, this problem has exactly one solution. Define operator $A\colon [y_0, z_0] \to [y_0, z_0]$ as $u \mapsto v$, where $Au = v$ is the unique solution of (4.1).

First we show that A is well defined. Put $w = y_0 - v$. If $y_0(0) \le r y_0(T)$ then we have

$$w(0) = y_0(0) - v(0) \le r y_0(T) - r v(T) = r w(T)$$

and

$$\begin{aligned} w'(t) + M(t)w(t) + N(t)w(\alpha(t)) &= y_0'(t) + M(t)y_0(t) + N(t)y_0(\alpha(t)) \\ &\quad - v'(t) - M(t)v(t) + N(t)v(\alpha(t)) \\ &\le f(t, y_0(y), y_0(\alpha(t))) - f(t, u(t), u(\alpha(t))) \\ &\quad + M(t)[y_0(t) - u(t)] + N(t)[y_0(\alpha(t)) - u(\alpha(t))] \\ &\le M(t)[u(t) - y_0(t)] + N(t)[u(\alpha(t)) - y_0(\alpha(t))] \\ &\quad + M(t)[y_0(t) - u(t)] + N(t)[y_0(\alpha(t)) - u(\alpha(t))] \\ &= 0. \end{aligned}$$

By Theorem 1, we have $w(t) \le 0$, $t \in J$. Similarly if $y_0(0) > r y_0(T)$ then

$$w(0) > r w(T)$$

and

$$w'(t) + M(t)w(t) + N(t)w(\alpha(t)) + \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[w(0) - r w(T)] \le 0.$$

Hence $w \le 0$ on J. Analogously, we can show that $v \le z_0$. Thus A is well defined.

Now we prove that A is monotone increasing. Put $w = v_1 - v_2$, where $v_1 = Au_1$, $v_2 = Au_2$ and $u_1 \le u_2$. We have $w(0) = r w(T)$ and

$$\begin{aligned} w'(t) + M(t)w(t) + N(t)w(\alpha(t)) & \\ &= f(t, u_1(t), u_1(\alpha(t))) + M(t)u_1(t) + N(t)u_1(\alpha(t)) \\ &\quad - f(t, u_2(t), u_2(\alpha(t))) - M(t)u_2(t) - N(t)u_2(\alpha(t)) \\ &\le M(t)[u_2(t) - u_1(t)] + N(t)[u_2(\alpha(t)) - u_1(\alpha(t))] \\ &\quad + M(t)[u_1(t) - u_2(t)] + N(t)[u_1(\alpha(t)) - u_2(\alpha(t))] \\ &= 0. \end{aligned}$$

In view of Theorem 1, $v_1 \le v_2$. Since v_1 and v_2 were arbitrary, A is monotone increasing.

Define the sequences $\{y_n\}$ and $\{z_n\}$ as follows:

$$y_{n+1} = Ay_n, \qquad z_{n+1} = Az_n, \quad n \ge 0.$$

Using the mathematical induction we can show that these sequences have the properties

$$y_0 \le y_1 \le \cdots \le y_n \le z_n \le \cdots \le z_1 \le z_0, \quad n \ge 0,$$

because A is monotone increasing. Thus the sequence $\{y_n\}$ is increasing and $y_n \le z_0$, $n \ge 0$. Hence, there exists $\lim_{n\to\infty} y_n(t) = y(t)$ for $t \in J$. The convergence is uniform since $\{y_n\}$ is bounded in $C^1(J, \mathbb{R})$. Similarly $\{z_n\} \downarrow z$ uniformly on J. It is easy to see that y and z are extremal solutions of (1.1). □

Example 1. Let us consider the problem

$$\begin{cases} x'(t) = e^{-x(t)} - tx\left(\dfrac{1}{3}t\right) - \dfrac{1}{2}, & t \in [0,1], \\ x(0) = \dfrac{1}{3}x(1). \end{cases}$$

Put $y_0 = 0$ and $z_0 = 1$. All assumptions of Theorem 3 are satisfied with $M(t) = 1$ and $N(t) = t$. Thus there exist monotone sequences converging uniformly to the extremal solutions of above problem in the sector $[0,1]_*$.

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NUMERICAL-ANALYTIC METHOD FOR INVESTIGATING BOUNDARY VALUE PROBLEMS FOR IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. The new numerical-analytic method for investigation of the nonlinear periodical impulsive systems of differential equations is substantiated. The problems of existing and approximate construction of the solutions are studied and the error estimates are obtained.

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1. INTRODUCTION

There are numerous methods for investigating and approximate constructing of the solutions of the boundary value problems for ordinary differential equations and, in particular, for differential equations with impulses (see, e. g., [1, 2, 5, 6] and the references therein). From the large variety of them, we can distinguish the numerical-analytic method of successive approximations developed in [5,6] where the problem on the existence and approximate construction of periodic solutions of the impulsive differential system of the form

$$x'(t) = f(t,x), \quad t \neq \tau_i, \qquad x(\tau_i + 0) - x(\tau_i) = I_i(x).$$

is studied on the assumption that f is a the time-periodic function. Another important assumption guaranteeing the applicability of the approach of the works mentioned requires that f and I_i should satisfy the Lipschitz conditions with sufficiently small constants.

In this paper we attempt to generalise the numerical-analytic method of successive approximations [5,6] to nonlinear systems of differential equations with impulse action of the form

$$x'(t) = A(t)x + f(t,x), \quad t \neq \tau_i, \qquad x(\tau_i + 0) - x(\tau_i) = B_i x + I_i(x),$$

in the cases where there exist a nontrivial solutions of the corresponding linear boundary value problem. The Lipschitz conditions concerns not all right-hand side term but only the non-linearity.

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2. Regularisation of linear impulsive boundary value problems

Let us first consider a linear inhomogeneous system of differential equations with impulse action

$$\begin{aligned} x'(t) &= A(t)x + h(t), \quad t \in [a,b] \setminus \{\tau_1, \tau_2, \dots, \tau_p\}, \\ x(\tau_i + 0) - x(\tau_i) &= B_i x(\tau_i) + a_i, \quad i = 1, \dots, p, \end{aligned} \tag{2.1}$$

under the linear functional boundary conditions

$$lx = \alpha, \tag{2.2}$$

where $A(\cdot)$ and $h(\cdot)$ are, respectively, $(n \times n)$-matrix and $(n \times 1)$-vector functions that arepiecewise continuous, with discontinuity of the first kind at $t = \tau_i$, $i = 1, 2, \dots, p$. The second relation in (2.1) determines the jumps of the solutions. Here B_i, $i = 1, 2, \dots, p$, are $(n \times n)$-constant matrices such that

$$\det(\mathbb{1}_n + B_i) \neq 0, \qquad i = 1, 2, \dots, p,$$

$\{a_1, a_2, \dots, a_p, \alpha\} \subset \mathbb{R}^n$ are constant column vectors, and $a \leq \tau_1 < \tau_2 < \cdots < \tau_p \leq b$. Equality (2.2), where l is an n-dimensional bounded linear vector functional, determines the boundary conditions.

Let $PC([a,b] \setminus \{\tau_i\}_{i=1}^p, \mathbb{R}^n)$ stand for the space of functions $x : [a,b] \to \mathbb{R}^n$ which are continuous from the left with discontinuity of the first kind at the points τ_i, $i = 1, \dots, p$ endowed with the norm $\|x\| = \sup\{|x(t)| : t \in [a,b]\}$. Moreover, let $PC^1_{\text{loc}}([a,b] \setminus \{\tau_i\}_{i=1}^p, \mathbb{R}^n)$ denote the set of functions $x \in PC([a,b] \setminus \{\tau_i\}_{i=1}^p, \mathbb{R}^n)$ which are continuously differentiable on every open interval contained in $[a,b] \setminus \{\tau_i\}_{i=1}^p$. A function $x \in PC^1_{\text{loc}}([a,b] \setminus \{\tau_i\}_{i=1}^p, \mathbb{R}^n)$ is said to be a solution of the problem (2.1), (2.2) if it satisfies the equalities (2.1) and verifies also the boundary condition (2.2).

According to the Riesz theorem [3], for an arbitrary linear bounded vector functional $l : PC([a,b] \setminus \{\tau_i\}_{i=1}^p, \mathbb{R}^n) \to \mathbb{R}^n$, there exist a matrix function of bounded variation $C(\cdot)$ such that by means of Riemann–Stieltjes integral we can represent the boundary condition (2.2) in the form

$$\int_a^b [dC(t)]x(t) = \alpha.$$

It is known [5] that solution $x(t, x_0)$ of impulsive differential system (2.1) having the initial value $x(a, x_0) = x_0$ is of the form

$$x(t, x_0) = X(t)x_0 + \int_a^t X(t,s)h(s)ds + \sum_{a \leq \tau_i < t} X(t, \tau_i + 0)a_i, \tag{2.3}$$

where $X(\cdot)$ is a fundamental matrix for the linear homogeneous impulsive system

$$x'(t) = A(t)x, \quad t \notin \{\tau_i\}_{i=1}^p, \qquad x(\tau_i + 0) - x(\tau_i) = B_i x, \quad i = 1, 2, \dots, p. \tag{2.4}$$

Recall that the matrix function X has the properties* $X(a) = \mathbb{1}_n$, $X(t) = X(t,a)$, where $X(t,s) = X(t)X^{-1}(s)$, and $X(t,\tau_i+0) = X(t,\tau_i)(\mathbb{1}_n + B_i)^{-1}$, $i = 1,2,\dots,p$.

While substituting (2.3) into the boundary conditions (2.2) we can see that the initial value x_0 of the solution $x(t,x_0)$ of the boundary value problem (2.1), (2.2) must satisfy the equation

$$Gx_0 = \alpha - \int_a^b Z(s)h(s)ds - \sum_{i=1}^{p} Z(\tau_i+0)a_i. \tag{2.5}$$

Here $Z(s) = \int_s^b [dC(t)]X(t,s)$ and $G = Z(a) = lX = \int_a^b [dC(t)]X(t)$.

In a noncritical case [1] (when a linear homogeneous impulsive system (2.4) does not have a nontrivial solutions), the algebraic system (2.5) has a unique solution

$$x_0 = G^{-1}\Big(\alpha - \int_a^b Z(s)h(s)ds - \sum_{i=1}^{p} Z(\tau_i+0)a_i\Big),$$

which is the initial value of a unique solution of the problem (2.1), (2.2)

$$x(t) = X(t)G^{-1}\alpha + \int_a^t X(t,s)h(s)ds - X(t)G^{-1}\int_a^b Z(s)h(s)ds + \sum_{a\le\tau_i<t} X(t,\tau_i+0)a_i - X(t)G^{-1}\sum_{i=1}^{p} Z(\tau_i+0)a_i.$$

Further we will consider a critical case [1], i. e., the case where

(A) *There exist k, $1 \le k \le n$, linearly independent solutions of the linear homogeneous impulsive boundary value problem*

$$\begin{gathered} x'(t) = A(t)x, \quad t \notin \{\tau_i\}_{i=1}^{p}, \\ x(\tau_i+0) - x(\tau_i) = B_i x(\tau_i), \quad i = 1,2,\dots,p; \\ lx = 0. \end{gathered} \tag{2.6}$$

Lemma 1. *Assume that there exist k linearly independent solutions of the linear homogeneous boundary value problem with impulse action (2.6). Then for an arbitrary function $h(t)$ there exists a function $H(t)$ such that inhomogeneous impulsive differential system*

$$\begin{aligned} x'(t) &= A(t)x + h(t) + H(t), \quad t \notin \{\tau_1,\tau_2,\dots,\tau_p\}, \\ x(\tau_i+0) - x(\tau_i) &= B_i x + a_i, \qquad i = 1,2,\dots,p, \end{aligned} \tag{2.7}$$

possesses a k-parametric family of solutions which satisfy the boundary condition (2.2).

*Here and below, $\mathbb{1}_n$ stands for the unit matrix of dimension n.

Proof. It is known [1] that the boundary value problem (2.7), (2.2) has a solution if and only if the condition

$$P_{G^*}\left(\alpha-\int_a^b Z(s)h(s)ds-\int_a^b Z(s)H(s)ds-\sum_{i=1}^{p}Z(\tau_i+0)a_i\right)=0 \qquad (2.8)$$

is fulfilled,[†] where $P_{G^*}\colon \mathbb{R}^n\to\ker G^*$ (resp., $P_G\colon \mathbb{R}^n\to\ker G$) is an orthoprojector to the null space $\ker G^*=\{z: z\in\mathbb{R}^n,\ zG^*=0\}$ (resp., $\ker G=\{y: y\in\mathbb{R}^n,\ Gy=0\}$) of G^* (resp., G).

We denote by the symbol $P_{G;k}$ the $(n\times k)$-matrix whose columns are the linearly independent columns of P_G constituting a basis in $\ker G$. Similarly, by $P_{G^*;k}$ we denote the $(k\times n)$-matrix whose rows are the linearly independent rows of the matrix P_{G^*} constituting a basis in $\ker G^*$. Thus,

$$\operatorname{rank}P_G=\operatorname{rank}P_{G;k}=\operatorname{rank}P_{G^*}=\operatorname{rank}P_{G^*;k}=k.$$

Let us set

$$H(t):=Z^*(t)(P_{G_k^*})^*R_1^{-1}P_{G_k^*}\left(\alpha-\int_a^b Z(s)h(s)ds-\sum_{a\le\tau_i<b}Z(\tau_i+0)a_i\right) \qquad (2.9)$$

for all $t\in[a,b]$, where $R_1:=P_{G^*;k}R_2(P_{G_k^*})^*$ and

$$R_2:=\int_a^b Z(\tau)Z^*(\tau)d\tau.$$

One can see that condition (2.8) is fulfilled if $H(t)$ is of the form (2.9). The solution of the impulsive differential system (2.7) is

$$x(t)=X(t)x_0+\int_a^t X(t,s)h(s)ds+\sum_{a\le\tau_i<t}X(t,\tau_i+0)a_i$$
$$+\int_a^t X(t,s)Z^*(s)ds(P_{G_k^*})^*R_1^{-1}P_{G^*;k}\left(\alpha-\int_a^b Z(s)h(s)ds-\sum_{i=1}^{p}Z(\tau_i+0)a_i\right). \qquad (2.10)$$

Substituting (2.10) into (2.2) we can see that $x(t)$ satisfy the boundary condition if and only if the initial value x_0 is a solution of the algebraic system

$$Gx_0=\left(\mathbb{1}_n-R_2P_{G_k^*}^*R_1^{-1}P_{G_k^*}\right)\left(\alpha-\int_a^b Z(s)h(s)ds-\sum_{i=1}^{p}Z(\tau_i+0)a_i\right). \qquad (2.11)$$

Since $P_{G;k}\left(\mathbb{1}_n-R_2P_{G_k^*}^*R_1^{-1}P_{G^*;k}\right)=0$, the system (2.11) is solvable and its general solution is of the form [1, 7]

[†]Here and below, we identify projectors with the corresponding matrices.

$$x_0 = P_{G;k}\xi + G^+\Big(\mathbb{1}_n - R_2 P^*_{G^*_k} R_1^{-1} P_{G^*;k}\Big)\Big(\alpha - \int_a^b Z(s)h(s)ds - \sum_{i=1}^{p} Z(\tau_i+0)a_i\Big), \quad (2.12)$$

where ξ is an arbitrary k-dimensional column, G^+ is a unique Moore–Penrose generalised inverse $(n\times n)$-matrix [4, 7]. If substitute the initial value x_0 given by (2.12) in representation (2.10) we obtain the general solution of the boundary value problem boundary value problem (2.2), (2.7):

$$\begin{aligned} x(t) &= X(t)P_{G;k}\xi \\ &+ X(t)G^+\Big(\mathbb{1}_n - R_2 P^*_{G^*_k} R_1^{-1} P_{G^*_k}\Big)\Big(\alpha - \int_a^b Z(\tau)h(\tau)d\tau - \sum_{i=1}^{p} Z(\tau_i+0)a_i\Big) \\ &+ \int_a^t X(t,s)h(s)ds + \sum_{a\le\tau_i<t} X(t,\tau_i+0)a_i \\ &+ \int_a^t X(t,s)Z^*(s)ds P^*_{G^*_k} R_1^{-1} P_{G^*;k}\Big(\alpha - \int_a^b Z(\tau)h(\tau)d\tau - \sum_{i=1}^{p} Z(\tau_i+0)a_i\Big). \end{aligned}$$

Finally we can rewrite it in the form

$$\begin{aligned} x(t,\xi) = X(t)P_{G;k}\xi + X(t)\Big(G^+ + (R(t) - G^+R_2)P^*_{G^*_k}R_1^{-1}P_{G^*;k}\Big)\alpha \\ + \int_a^b L(t,s)h(s)ds + \sum_{i=1}^{p} L(t,\tau_i+0)a_i, \quad (2.13) \end{aligned}$$

where

$$R(t) = \int_a^t X(s)^{-1}Z^*(s)ds$$

and

$$L(t,s) = \begin{cases} X(t,s) - X(t)\Big(G^+ + \big(R(t) - G^+R_2\big)P^*_{G^*_k}R_1^{-1}P_{G^*_k}\Big)Z(s), & 0\le s\le t\le b, \\ -X(t)\Big(G^+ + \big(R(t) - G^+R_2\big)P^*_{G^*_k}R_1^{-1}P_{G^*_k}\Big)Z(s), & 0\le t<s\le b. \end{cases}$$

Considering (2.13), we conclude that the lemma is proved. □

3. The boundary value problem for nonlinear impulsive differential equations in a critical case

Now we consider the nonlinear system of differential equations with impulse action

$$\begin{aligned} x'(t) &= A(t)x + f(t,x), \quad t\notin\{\tau_i\}_{i=1}^{p}, \\ x(\tau_i+0) - x(\tau_i) &= B_i x(\tau_i) + I_i(x(\tau_i)), \end{aligned} \quad (3.1)$$

under the boundary condition (2.2). We will investigate a critical case, i. e., the case, when the condition (A) is fulfilled.

We suppose that, for $t \in [a,b]$, $x \in D \subset \mathbb{R}^n$, where D is a closed and bounded domain, the following conditions hold:

(B) $A(t)$, $f(t,x)$ are continuous or piecewise continuous for t with discontinuity of the first kind for $t = \tau_i$ respectively $(n \times n)$-matrix function and $(n \times 1)$-vector function, B_i are $(n \times n)$-constant matrices such that $\det(\mathbb{1}_n + B_i) \neq 0$, a_i, α are the n-dimensional column vectors of constants, $\tau_0 < a \leq \tau_1 < \tau_2 < \cdots < \tau_p \leq b$, l is n-dimensional bounded linear vector functional, and the following inequalities hold:

$$\begin{gathered} |f(t,x)| \leq M(t), \qquad |I_i(x)| \leq m_i, \\ |f(t,x') - f(t,x'')| \leq K(t)|x' - x''|, \\ |I_i(x') - I_i(x'')| \leq K_i|x' - x''|, \end{gathered} \tag{3.2}$$

where m_i and K_i are constant vectors and constant matrices with nonnegative components respectively, $M(t)$ and $K(t)$ are piecewise continuous vector function and matrix function with nonnegative integrable components, respectively. Here the notation $|f(t,x)| = \operatorname{col}(|f_1(t,x)|, \ldots, |f_n(t,x)|)$ is used and all the inequalities are meant componentwise;

(C) the domain $D_\beta \equiv \{\xi \in \mathbb{R}^k \mid B(x_0(t,\xi), \beta) \subseteq D,\ t \in [a,b]\}$ is non-empty, where $x_0(t,\xi) = X(t)P_{G;k}\xi$,

$$\beta = \max_{t\in[a,b]} \left(\left| X(t)\left(G^+ + (R(t) - G^+R_2)P^*_{G_k^*} R_1^{-1} P_{G^*;k}\right)\alpha \right| + \int_a^b |L(t,s)|\, M(s)ds + \sum_{i=1}^{p} |L(t,\tau_i+0)\, m_i| \right)$$

and $B(y,\varrho) = \{x \in \mathbb{R}^n : |x - y| \leq \varrho\}$ for all $y, \varrho \in \mathbb{R}^n$;

(D) the maximum eigenvalue of the following matrix Q is less then one:

$$Q = \sup_{t\in[a,b]} \left\{ \int_a^b |L(t,s)|K(s)ds + \sum_{i=1}^{p} |L(t,\tau_i+0)|K_i \right\}.$$

We also consider k-parametric family of mappings $\mathcal{L}_\xi : PC([a,b] \setminus \{\tau_i\}_{i=1}^p, \mathbb{R}^n) \to PC([a,b] \setminus \{\tau_i\}_{i=1}^p, \mathbb{R}^n)$ and vector functional $\mu : PC([a,b] \setminus \{\tau_i\}_{i=1}^p, \mathbb{R}^n) \to \mathbb{R}^k$ defined by the relations

$$\begin{aligned} (\mathcal{L}_\xi x)(t) \stackrel{\text{def}}{=} {} & X(t)P_{G;k}\xi + X(t)\left(G^+ + \left(R(t) - G^+R_2\right)P^*_{G_k^*} R_1^{-1} P_{G^*;k}\right)\alpha \\ & + \int_a^b L(t,s)f(s,x(s))ds + \sum_{i=1}^{p} L(t,\tau_i+0)I_i(x(\tau_i)) \end{aligned}$$

and

$$\mu(x) \stackrel{\text{def}}{=} P_{G^*;k}\left(\alpha - \int_a^b Z(s)f(s,x(s))ds - \sum_{i=1}^{p} Z(\tau_i+0)I_i(x(\tau_i))\right).$$

Lemma 2. *Let the linear homogeneous boundary value problem (2.2), (2.4) has k, $1 \le k \le n$, linearly independent solutions. Then:*

(1) *If $\varphi \in PC^1_{\text{loc}}([a,b]\setminus\{\tau_i\}_{i=1}^p, \mathbb{R}^n)$ is a solution of the boundary value problem (2.2), (3.1) then there exists $\xi \in \mathbb{R}^k$ such that φ is a solution of the system of equations*

$$x = \mathcal{L}_\xi x, \tag{3.3}$$

$$\mu(x) = 0. \tag{3.4}$$

Moreover, the initial value of the solution is

$$\varphi(a) = P_{G_k}\xi + G^+\left(\alpha - \int_a^b Z(s)f(s,\varphi(s))ds + \sum_{i=1}^{p} Z(\tau_i+0)I_i(\varphi(\tau_i))\right). \tag{3.5}$$

(2) *If $\varphi \in PC([a,b]\setminus\{\tau_i\}_{i=1}^p, \mathbb{R}^n)$ is a solution of the system of equations (3.3), (3.4) with some $\xi \in \mathbb{R}^k$ then $\varphi \in PC^1_{\text{loc}}([a,b]\setminus\{\tau_i\}_{i=1}^p, \mathbb{R}^n)$ and φ is a solution of the boundary value problem (2.2), (3.1).*

Proof. Necessity. Let φ be a solution of the problem (2.2), (3.1). Then

$$\varphi(t) \equiv X(t)\varphi(a) + \int_a^t X(t,s)f(s,\varphi(s))ds + \sum_{a\le\tau_i<t} X(t,\tau_i+0)I_i(\varphi(\tau_i)). \tag{3.6}$$

When substituting (3.6) into the boundary condition (2.2) we can see that $\varphi(a)$ is a solution of algebraic system

$$G\varphi(a) = \alpha - \int_a^b Z(s)f(s,\varphi(s))ds - \sum_{i=1}^{p} Z(\tau_i+0)I_i(\varphi(\tau_i)), \tag{3.7}$$

and therefore, there exists $\xi \in \mathbb{R}^k$ such that (3.5) holds. Consequently, $\mu(\varphi) = 0$. Considering this equality, by direct computation it can be verified that φ is also a solution of the equation (3.3).

Sufficiency. Let ξ and φ be such that equations (3.3) and (3.4) are satisfied. Therefore, $\varphi \in PC^1_{\text{loc}}([a,b]\setminus\{\tau_i\}_{i=1}^p, \mathbb{R}^n)$, (3.6) fulfils and $\varphi(a)$ satisfies (3.5). From this follows that φ is a solution of problem (2.2), (3.1). □

For the investigation of the problem of the existence and approximate constructing of the solutions of problem (2.2), (3.1) we consider the k-parametric family of sequences of functions given by the formulas $x_0(t,\xi) = X(t)P_{G;k}\xi$ and

$$x_m(t,\xi) = x_0(t,\xi) + X(t)\Big(G^+ + \big(R(t) - G^+R_2\big)P^*_{G_k^*}R_1^{-1}P_{G^*;k}\Big)\alpha$$

$$+\int_a^b L(t,s)f(s,x_{m-1}(s,\xi)ds+\sum_{i=1}^{p}L(t,\tau_i+0)I_i(x_{m-1}(\tau_i,\xi) \quad (3.8)$$

for $m=1,2,\dots$ and $\xi\in\mathbb{R}^k$. Each of these functions satisfies the boundary condition (2.2).

Theorem 1. *Assume that for the boundary value problem (2.2), (3.1) the conditions (A)–(D) hold. Then:*

(1) *for all $\xi\in D_\beta\subset\mathbb{R}^k$, the operator $\mathcal{L}_\xi$ has a fixed point $x^*(\cdot,\xi)$ in the set $PC([a,b]\setminus\{\tau_i\}_{i=1}^p,\mathbb{R}^n)$, which coincides with the limit function $x^*(t,\xi)=\lim_{m\to\infty}x_m(t,\xi)$ of the sequence (3.8) and the following error estimation holds:*

$$|x^*(t,\xi)-x_m(t,\xi)|\le(\mathbb{1}_n-Q)^{-1}Q^m\beta; \quad (3.9)$$

(2) *the limit function $x^*(t,\xi)$ satisfies the boundary condition (2.2) for arbitrary $\xi\in\mathbb{R}^k$ and the initial value is*

$$x^*(a,\xi)=P_{G_k}\xi+G^+(\mathbb{1}_n-R_2P^*_{G_k^*}R_1^{-1}P_{G^*;k})\times\Big(\alpha-\int_a^b Z(s)f(s,x^*(s,\xi))ds-\sum_{i=1}^{p}Z(\tau_i+0)I_i(x^*(\tau_i,\xi))\Big); \quad (3.10)$$

(3) *the limit function $x^*(t)=x^*(t,\xi^*)$ is a solution of the boundary value problem (2.2), (3.1) if and only if ξ^* is a solution of the determining equation $\Delta(\xi)=0$, where*

$$\Delta(\xi)\stackrel{\text{def}}{=}\mu(x^*(\cdot,\xi))\stackrel{\text{def}}{=}P_{G^*;k}\Big(\alpha-\int_a^b Z(s)f(s,x^*(s,\xi))ds-\sum_{i=1}^{p}Z(\tau_i+0)I_i(x^*(\tau_i,\xi))\Big). \quad (3.11)$$

Proof. Since

$$|(\mathcal{L}_\xi x)(t)-x_0(t,\xi)|\le\Big|X(t)\Big(G^+ +(R(t)-G^+R_2)P^*_{G_k^*}R_1^{-1}P_{G_k^*}\Big)\alpha\Big| +\int_a^b|L(t,s)f(s,x(s))|ds+\sum_{i=1}^{p}|L(t,\tau_i+0)I_i(x(\tau_i))|\le\beta,$$

from the condition (C) it follows that $(\mathcal{L}_\xi x)(t)\in D$ for all $\xi\in D_\beta$, $x\in PC([a,b]\setminus\{\tau_i\}_{i=1}^p,\mathbb{R}^n)$.

Let $\mathcal{Q}$ be the operator in $PC([a,b]\setminus\{\tau_i\}_{i=1}^p,\mathbb{R}^n)$ acting according to the rule

$$(\mathcal{Q}x)(t)=\int_a^b|L(t,s)|K(s)x(s)ds+\sum_{i=1}^{p}|L(t,\tau_i+0)|K_ix(\tau_i).$$

Then, from (3.8) and the Lipschitz conditions (3.2) we get

$$\begin{aligned}
|x_{m+1}(t,\xi)-x_m(t,\xi)| &= \left|\left(\mathcal{L}_\xi(x_m(\cdot,\xi)-x_{m-1}(\cdot,\xi))\right)(t)\right| \\
&\le \left(\mathcal{Q}|x_m(\cdot,\xi)-x_{m-1}(\cdot,\xi)|\right)(t) \\
&\le \left(\mathcal{Q}^2|x_{m-1}(\cdot,\xi)-x_{m-2}(\cdot,\xi)|\right)(t) \\
&\le \cdots \\
&\le \left(\mathcal{Q}^m|x_1(\cdot,\xi)-x_0(\cdot,\xi)|\right)(t) \\
&\le (\mathcal{Q}^m\beta)(t),
\end{aligned}$$

and thus

$$\begin{aligned}
|x_{m+j}(t,\xi)-x_m(t,\xi)| &\le \sum_{i=0}^{j-1}|x_{m+i+1}(t,\xi)-x_{m+i}(t,\xi)| \\
&\le \sum_{i=0}^{j-1}(\mathcal{Q}^{m+i}\beta)(t) \qquad (3.12)\\
&\le \sum_{i=0}^{j-1}Q^{m+i}\beta.
\end{aligned}$$

It follows from the condition (D) that $x_m(t,\xi)$ is a Cauchy sequence. Therefore it uniformly converges to a continuous function $x^*(t,\xi)$. Passing to the limit as $j\to\infty$ in (3.12), we prove the error estimation (3.9). Since all functions satisfy the boundary condition (2.2), we conclude that so does the limit function $x^*(t,\xi)$.

Taking the limit as $m\to\infty$ we get that $x^*(t,\xi)$ satisfies the equation

$$\begin{aligned}
x(t) = X(t)P_{G;k}\xi + X(t)\left(G^+ + \left(R(t)-G^+R_2\right)P^*_{G^*_k}R_1^{-1}P_{G^*_k}\right)\alpha \\
+\int_a^b L(t,s)f(s,x(s)ds) + \sum_{i=1}^{p}L(t,\tau_i+0)I_i(x(\tau_i)). \qquad (3.13)
\end{aligned}$$

According to the Lemma 2, it means that the limit function $x^*(t,\xi)$ is a solution of the boundary value problem (2.2), (3.1) if and only if the condition $\Delta(\xi)=0$ is fulfilled. Further, putting $t=a$ to (3.13) it is easy to see that (3.10) is the initial value of the solution. □

The next statement gives us the satisfactory conditions for the existence of the solution of the boundary value problem (2.2), (3.1). This conditions are based upon the properties of the approximations $x_m(t,\xi)$ and not upon those of the limit function $x^*(t,\xi)$.

Theorem 2. *Let us suppose that conditions* (A)–(D) *hold and furthermore:*

(1) *there exists closed convex subset* $D' \subset D_\beta \subset \mathbb{R}^k$ *such that, for some fixed* $m \in \mathbb{N}$, *the approximating equation*

$$\Delta_m(\xi) \stackrel{\text{def}}{=} \mu(x_m(\cdot,\xi))$$
$$\stackrel{\text{def}}{=} P_{G_k^*}\Big(\alpha - \int_a^b Z(s) f\,(s, x_m(s,\xi))\,ds - \sum_{i=1}^{p} Z(\tau_i+0) I_i(x_m(\tau_i,\xi))\Big) = 0 \quad (3.14)$$

possesses only one isolated solution $\xi = \xi_{0m}$ *of non-zero index;*

(2) *on the boundary* $\partial D'$ *of the set* D', *the condition*

$$\inf_{\xi\in\partial D'} |\Delta_m(\xi)| > Q_1(\mathbb{1}_n - Q)^{-1} Q^m \beta \quad (3.15)$$

is satisfied, where

$$Q_1 = \int_a^b |P_{G_{k^*}^*} Z(s)| K(s) ds + \sum_{i=1}^{p} |P_{G_{k^*}^*} Z(\tau_i+0)| K_i.$$

Then there exist a solution $x^*(t) = x^*(t,\xi^*)$ *of the boundary value problem* (2.2), (3.1).

Proof. We introduce the continuous for $\xi \in \partial D'$ and $\theta \in [a,b]$ vector field family

$$\Delta(\theta,\xi) = \Delta_m(\xi) + \theta(\Delta(\xi) - \Delta_m(\xi)), \quad 0 \le \theta \le 1,$$

connecting the vector fields $\Delta(0,\xi) = \Delta_m(\xi)$ and $\Delta(1,\xi) = \Delta(\xi)$. Let us assume that there exist $\theta_0 \in [0,1]$ such that $\Delta(\theta_0,\xi) = 0$. Then

$$\Delta_m(\xi) = -\theta_0\big(\Delta(\xi) - \Delta_m(\xi)\big). \quad (3.16)$$

From (3.9), (3.11), (3.14) and the Lipschitz condition (3.2) we have

$$\begin{aligned}
&|\Delta(\xi) - \Delta_m(\xi)| \\
&\quad \le \int_a^b |P_{G^*;k} Z(s)|\,|f(s,x^*(s,\xi)) - f(s,x_m(s,\xi))|ds \\
&\qquad + \sum_{i=1}^{p} |P_{G^*;k} Z(\tau_i+0)|\,|I_i(x^*(\tau_i,\xi)) - I_i(x_m(\tau_i,\xi))| \\
&\quad \le \int_a^b |P_{G^*;k} Z(s)| K(s) |x^*(s,\xi) - x_m(s,\xi)|ds \\
&\qquad + \sum_{i=1}^{p} |P_{G^*;k} Z(\tau_i+0)|\,|x^*(\tau_i,\xi) - x_m(\tau_i,\xi)| \\
&\quad \le Q_1(\mathbb{1}_n - Q)^{-1} Q^m \beta.
\end{aligned}$$

But in this case from (3.16) we obtain the inequality

$$|\Delta_m(\xi)| \le |\Delta(\xi) - \Delta_m(\xi)| \le Q_1(\mathbb{1}_n - Q)^{-1} Q^m \beta,$$

which contradict the condition (3.15). It means that the field family $\Delta(\theta,\xi)$ does not assume the value zero on $\partial D'$, therefore vector fields $\Delta(\xi)$ and $\Delta_m(\xi)$ are homotopic. It means that the rotation of vector field $\Delta(\xi)$ on the boundary $\partial D'$ is also non-zero and consequently $\Delta(\xi)$ assumes the value zero at least in one point $\xi^* \in D'$. □

4. Noncritical case

In a noncritical case, i. e., when there is only trivial solution of the linear homogeneous problem (2.6), we obtain $\det G \neq 0$, $P_{G_k} = 0$ and $\Delta(\xi) \equiv 0$. This means that is not necessary to solve the determining equation.

Theorem 3. *Let us suppose that the linear homogeneous boundary value problem (2.6) has only trivial solution, the condition (B) holds, and*

(C1) $B(\tilde{x}_0(t),\tilde{\beta}) \subseteq D$ *for all* $t \in [a,b]$, *where*

$$\tilde{x}_0(t) := X(t)G^{-1}\alpha, \qquad t \in [a,b],$$

$B(y,\varrho) = \{x \in \mathbb{R}^n : |x-y| \leq \varrho\}$ *for all* $y,\varrho \in \mathbb{R}^n$, *the vector* $\tilde{\beta}$ *is given by the formula*

$$\tilde{\beta} = \max_{t\in[a,b]} \left(\int_a^b |\tilde{L}(t,s)|M(s)ds + \sum_{i=1}^{p} |\tilde{L}(t,\tau_i+0)|m_i \right),$$

and

$$\tilde{L}(t,s) = \begin{cases} X(t,s) - X(t)G^{-1}Z(s), & 0 \leq s \leq t \leq b, \\ -X(t)G^{-1}Z(s), & 0 \leq t < s \leq b; \end{cases}$$

(D1) *the greatest eigenvalue of the following matrix* $\tilde{Q}$ *is less then one:*

$$\tilde{Q} := \sup_{t\in[a,b]} \left\{ \int_a^b |\tilde{L}(t,s)|K(s)ds + \sum_{i=1}^{p} |\tilde{L}(t,\tau_i+0)|K_i \right\}.$$

Then there exist a unique solution $x^*(\cdot)$ *of the boundary value problem (2.2), (3.1), which is the limit function of the sequence*

$$\tilde{x}_m(t) = X(t)G^{-1}\alpha + \int_a^b \tilde{L}(t,s)f(s,\tilde{x}_{m-1}(s))ds + \sum_{i=1}^{p} \tilde{L}(t,\tau_i+0)I(\tilde{x}_{m-1}(\tau_i)), \quad m = 1,2,\ldots.$$

Moreover, the initial value of $x^*(\cdot)$ *satisfies the relation*

$$x^*(a) = G^{-1}\left(\alpha - \int_a^b Z(\tau)f(\tau,x^*(\tau))d\tau - \sum_{i=1}^{p} Z(\tau_i+0)I_i(x^*(\tau_i))\right).$$

Proof. It can be carried out in the same way as that of Theorem 1. □

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ON THE TOPOLOGICAL PROPERTIES OF THE GENERALIZED CLARKE SUBDIFFERENTIAL

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Abstract. We give some topological properties of the generalized Clarke subdifferential which is an extension of the notion of the Clarke subdifferentiability [5]. Potential advantages of generalized Clarke subdifferential over other concepts of subdifferentiability might be related to the fact that it offers a rich calculus and applications. It allows us, for example, to present a convenient test for the weak metric regularity of mappings non-necessarily strictly Fréchet differentiable.

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1. An introduction to generalized Clarke subdifferentiability

We shall be working in binormed space $(E, \|\cdot\|_1, \|\cdot\|_2)$ such that $(E, \|\cdot\|_2)$ is a Banach space and, for some $c > 0$, the condition $\|\cdot\|_1 \leq c\|\cdot\|_2$ holds.

Let U be an open set of $(E, \|\cdot\|_1)$, and let h_2 be a $\|\cdot\|_2$ locally Lipschitz real function around $\overline{x} \in U$.

Let v any vector in E. The Clarke generalized directional derivative of h_2 at $\overline{x}$ in the direction v with respect to the norm $\|\cdot\|_2$, denoted $h_2^{0,2}(\overline{x}, v)$, is defined as follows:

$$h_2^{0,2}(\overline{x}, v) = \limsup_{x' \to_{\|\cdot\|_2} \overline{x},\, t \downarrow 0} \frac{h_2(x' + tv) - h_2(x')}{t}.$$

The Clarke subdifferential of h_2 at $\overline{x}$ with respect to the norm $\|\cdot\|_2$, denoted $\partial_0^2 h_2(\overline{x})$, is the subset of $(E, \|\cdot\|_2)'$ given by

$$\partial_0^2 h_2(\overline{x}) = \{\xi \in (E, \|\cdot\|_2)' : h_2^{0,2}(\overline{x}, v) \geq \langle \xi, v \rangle \ \forall v \in E\}.$$

According to [5], the generalized Clarke subdifferential of h_2 at $\overline{x}$ with respect to the pair of norms $(\|\cdot\|_1, \|\cdot\|_2)$, denoted $\partial_0^{1,2} h_2(\overline{x})$, is the subset of $(E, \|\cdot\|_2)'$ given by

$$\partial_0^{1,2} h_2(\overline{x}) = \{\xi \in (E, \|\cdot\|_2)' : h_2^{0,1}(\overline{x}, v) \geq \langle \xi, v \rangle \ \forall v \in E\}.$$

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Notice that the Clarke subdifferential $\partial_0^2 h_2(\overline{x})$ is smaller than the generalized Clarke subdifferential $\partial_0^{1,2} h_2(\overline{x})$. More precisely, we have $\partial_0^2 h_2(\overline{x}) \subset \partial_0^{1,2} h_2(\overline{x})$ and also $\partial_0^1 h_2(\overline{x}) \subset \partial_0^{1,2} h_2(\overline{x})$.

If moreover, h_2 is locally Lipschitz around $\overline{x}$ with respect to the norm $\|\cdot\|_1$, then $\partial_0^2 h_2(\overline{x}) = \partial_0^1 h_2(\overline{x}) = \partial_0^{1,2} h_2(\overline{x})$.

In particular, if we take $\|\cdot\|_1 = \|\cdot\|_2$, then $\partial_0^2 h_2(\overline{x}) = \partial_0^1 h_2(\overline{x}) = \partial_0^{1,2} h_2(\overline{x})$.

Thus, the generalized Clarke subdifferential is a generalized version of the Clarke subdifferential.

Let us note that $h_2^{0,1}(\overline{x},\cdot)$ is the support function of the generalized Clarke subdifferential $\partial_0^{1,2} h_2(\overline{x})$ viewed as subset of $(E,\|\cdot\|_2)'$. Finally, note that even if $h_2^{0,1}(\overline{x},.)$ is the support function of the Clarke subdifferential $\partial_0^1 h_2(\overline{x})$, the generalized Clarke subdifferential $\partial_0^{1,2} h_2(\overline{x})$ does not necessarily coincide with $\partial_0^1 h_2(\overline{x})$, because $\partial_0^1 h_2(\overline{x})$ is not, in general, weak*-closed in $((E,\|\cdot\|_2)', *\sigma((E,\|\cdot\|_2)', E))$.

We refer to [1–6] for more details on the notions mentioned.

2. Basic properties of generalized Clarke subdifferential

Assume that all the hypothesis of the above paragraph are satisfied.

2.1. *Topological properties of generalized Clarke subdifferential*

Our goals in this paragraph is to give some topological properties of generalized Clarke subdifferential. For this, we introduce the notion of locally Lipschitzian mappings with respect to the pair of norms.

We say that h_2 is $(\|\cdot\|_1, \|\cdot\|_2)$ locally Lipschitzian around $\overline{x}$ if for some nonnegative scalar K, one has

$$|h_2(y) - h_2(y')| \leq K\|y - y'\|_2$$

for all y, y' close to $\overline{x}$ with respect to the norm $\|\cdot\|_1$.

The following summarizes some basic properties of the generalized Clarke subdifferential.

Proposition 1. *Let $\partial_0^{1,2} h_2(\overline{x})$ be a nonempty, convex, closed subset of the space $((E,\|.\|_2)', *\sigma((E,\|.\|_2)', E))$. Moreover, if h_2 is $(\|\cdot\|_1, \|\cdot\|_2)$ locally Lipschitzian around $\overline{x}$, then $\partial_0^{1,2} h_2(\overline{x})$ is weakly*-compact in $((E,\|.\|_2)', *\sigma((E,\|.\|_2)', E))$.*

Proof. The set $\partial_0^{1,2} h_2(\overline{x})$ is nonempty, because $\partial_0^2 h_2(\overline{x}) \subset \partial_0^{1,2} h_2(\overline{x})$. The convexity and the closeness of $\partial_0^{1,2} h_2(\overline{x})$ are trivially fulfilled.

Assume now that h_2 is $(\|\cdot\|_1, \|\cdot\|_2)$ locally Lipschitzian around $\overline{x}$. Then, $\partial_0^{1,2} h_2(\overline{x})$ is bounded in $((E,\|\cdot\|_2)', \|\cdot\|_{(E,\|\cdot\|_2)'})$. Therefore, using the Alaoglu theorem, we deduce the weak*-compactness of $\partial_0^{1,2} h_2(\overline{x})$.

The assertion below reiterates that the pair of multifunctions $(\partial_0^2 h_2, \partial_0^{1,2} h_2)$ is closed from $(U,\|\cdot\|_1)$ to $((E,\|\cdot\|_2)', *\sigma((E,\|\cdot\|_2)', E))$ in the following sense: if

x_i and ξ_i are sequences in U and $(E, \|\cdot\|_2)'$ such that $\xi_i \in \partial_0^2 h_2(x_i)$, x_i converges to x with respect to the norm $\|\cdot\|_1$, and ξ_i converges to ξ in the weak topology $*\sigma((E, \|\cdot\|_2)', E)$, then $\xi \in \partial_0^{1,2} h_2(x)$. □

Proposition 2. *The pair of multifunctions* $(\partial_0^2 h_2, \partial_0^{1,2} h_2)$ *is closed from* $(U, \|.\|_1)$ *to* $((E, \|.\|_2)', *\sigma((E, \|.\|_2)', E))$.

Proof. Let x_i and ξ_i be sequences in U and $(E, \|\cdot\|_2)'$ such that $\xi_i \in \partial_0^2 h_2(x_i)$, x_i converges to x with respect to the norm $\|\cdot\|_1$, and ξ_i converges to ξ in the weak topology $*\,\sigma((E, \|\cdot\|_2)', E)$.

Let any $v \in E$ be given. Then $\langle \xi_i, v\rangle$ converges to $\langle \xi, v\rangle$.One has $h_2^{0,2}(x_i, v) \geq \langle \xi_i, v\rangle$, which implies that $h_2^{0,1}(x_i, v) \geq \langle \xi_i, v\rangle$.

By the upper semicontinuity of $h_2^{0,1}$ with respect to the norm $\|\cdot\|_1$, we deduce that $h_2^{0,1}(x, v) \geq \langle \xi, v\rangle$. Since v is arbitrary, ξ belongs to $\partial_0^{1,2} h_2(x)$. □

Remark 1. If h_2 is $(\|\cdot\|_1, \|\cdot\|_2)$ locally Lipschitzian around $\bar{x}$. Then, $\zeta \in \partial_0^{1,2} h_2(\bar{x})$ if and only if $h_2^{0,1}(\bar{x}, v) \geq \langle \zeta, v\rangle \ \forall v \in E$.

2.2. *Relation to Taylor derivatives and generalized subderivatives*

The main result of this section will be that if h_2 is $(\|\cdot\|_1, \|\cdot\|_2)$ locally Lipschitzian around $\bar{x}$, then $\partial_0^{1,2} h_2(\bar{x})$ is reduced to a singleton if and only if h_2 is $(\|\cdot\|_1, \|\cdot\|_2)$ strictly Taylor differentiable at $\bar{x}$.

By B_1, respectively, B_2, we denote the unit ball in $(E, \|\cdot\|_1)$, respectively, the unit ball in $(E, \|\cdot\|_2)$.

Let h be a map acting from U into a Banach space $(Y, \|\cdot\|)$, and let $x \in U$. We shall say that h is $(\|\cdot\|_1, \|\cdot\|_2)$ strictly Taylor differentiable at a point x if there exists a continuous linear operator from $(E, \|\cdot\|_2)$ to $(Y, \|\cdot\|)$ denoted $\nabla h(x)$ such that for each v, the condition

$$\lim_{x' \to_{\|\cdot\|_1} x,\, t \downarrow 0} \frac{h(x' + tv) - h(x')}{t} = \nabla h(x)v$$

holds, where the convergence is uniform for v in compact sets in $(E, \|.\|_2)$. Note that ours is a Hadamard-type strict derivative.

Let us remark that if h is a $(\|\cdot\|_1, \|\cdot\|_2)$ strictly Taylor differentiable at x, then h is not necessarily $\|\cdot\|_1$ strictly differentiable at x. But if $\|\cdot\|_1$ is equivalent to the norm $\|\cdot\|_2$, then h is necessarily $\|\cdot\|_1$-strictly differentiable at x.

So the notion of strict differentiability in terms of Taylor strengthens and generalizes the elegant notion of Hadamard-strict differentiability.

Proposition 3. *Let* L *be a continuous linear operator from* $(E, \|\cdot\|_2)$ *to* $(Y, \|\cdot\|)$, *and let* $x \in U$. *The following assertions are equivalent:*

(i) h *is* $(\|\cdot\|_1, \|\cdot\|_2)$ *strictly Taylor differentiable at* x *and* $\nabla h(x) = L$;

(ii) *h is $(\|\cdot\|_1, \|\cdot\|_2)$ locally Lipschitzian around x, and for each v in E one has*

$$\lim_{x' \to_{\|\cdot\|_1} x,\, t \downarrow 0} \frac{h(x'+tv)-h(x')}{t} = Lv.$$

Proof. Assume (i). The equality in (ii) holds by assumption, so to prove (ii) it suffices to show that h is $(\|\cdot\|_1, \|\cdot\|_2)$ locally Lipschitzian around x. If this is not the case, there exist sequences $\{x_n\}$ and $\{x'_n\}$ converging to x with respect to the norm $\|\cdot\|_1$ such that x_n and x'_n lie in $x + \frac{1}{n}B_1$ and

$$\|h(x_n) - h(x'_n)\| > n\|x_n - x'_n\|_2.$$

Let us define $t_n > 0$ and v_n via $x'_n = x_n + t_n v_n$ and $\|v_n\|_2 = n^{-\frac{1}{2}}$. It follows that $t_n \to 0$.

Let V consists of the points in the sequence $\{v_n\}$ together with 0. Note that V is compact in $(E, \|\cdot\|_2)$, so that by definition of $\nabla h(x)$ for any ε there exists n_0 such that, for all $n \geq n_0$, for all $v \in V$, one has

$$\left\| \frac{h(x_n + t_n v) - h(x_n)}{t_n} - \nabla h(x)v \right\| < \varepsilon.$$

But this is impossible since when $v = v_n$, the term ${t_n}^{-1}(h(x_n + t_n v) - h(x_n))$ has norm exceeding $n^{\frac{1}{2}}$ by construction. Thus, (ii) holds.

We now posit (ii). Let V any compact subset of $(E, \|\cdot\|_2)$ and ε any positive number. In view of (ii), there exists for each v in V a number $\delta(v)$ such that

$$\left\| \frac{h(x'+tv)-h(x')}{t} - \nabla h(x)v \right\| < \varepsilon$$

for all $x' \in x + \delta(v)B_1$ and $t \in]0, \delta[$,. Since the norm of

$$\frac{h(x'+tv')-h(x')}{t} - \frac{h(x'+tv)-h(x')}{t}$$

is bounded above by $k\|v - v'\|_2$ (where K is a $(\|\cdot\|_1, \|\cdot\|_2)$ Lipschitz constant for h and where x' is sufficiently $\|\cdot\|_1$ near x and t is sufficiently near 0), we deduce from the last inequality that for a suitable redefinition of $\delta(v)$, one has

$$\left\| \frac{h(x'+tv')-h(x')}{t} - Lv' \right\| < 2\varepsilon$$

for all x' in $x + \delta(v)B_1$, v' in $v + \delta(v)B_2$, and t in $]0, \delta[$. A finite number of the $\|\cdot\|_2$-open sets $\{v + \delta(v)B_2 : v \in V\}$ will cover V, say, those that correspond to v_1,...,v_n. If we set $\delta' = \min_{1 \leq i \leq n} \delta(v_i)$, it follows then that

$$\left\| \frac{h(x'+tv')-h(x')}{t} - Lv' \right\| < 2\varepsilon$$

for any $v \in V$, for all $x' \in x + \delta' B_1$ and $t \in]0, \delta'[$. Thus, h is $(\|\cdot\|_1, \|\cdot\|_2)$ strictly Taylor differentiable at x and $\nabla h(x) = L$, and the proof is complete. □

Now we can prove the main result of this section.

Theorem 1. *Assume that h_2 is $(\|\cdot\|_1, \|\cdot\|_2)$ locally Lipschitzian around $\bar{x}$. Then $\partial_0^{1,2} h_2(\bar{x})$ is reduced to a singleton $\{\zeta\}$ if and only if h_2 is $(\|\cdot\|_1, \|\cdot\|_2)$ strictly Taylor differentiable at $\bar{x}$. In this case, $\zeta = \nabla h_2(\bar{x})$.*

In particular, if $\|\cdot\|_1 = \|\cdot\|_2$ then $\partial_0^1 h_2(\bar{x})$ is reduced to a singleton $\{\zeta\}$ if and only if h_2 is $\|\cdot\|_1$ strictly differentiable at $\bar{x}$.

Proof. Suppose first that h_2 is $(\|\cdot\|_1, \|\cdot\|_2)$ strictly Taylor differentiable at $\bar{x}$, then $h_2^{0,1}(\bar{x}, v) = \nabla h_2(\bar{x})v$. Therefore, $\partial_0^{1,2} h_2(\bar{x})$ is reduced to a singleton $\{\nabla h_2(\bar{x})\}$.

To prove the converse, it suffices to show that the condition of Proposition 3 holds if $\partial_0^{1,2} h_2(\bar{x})$ is reduced to a singleton $\{\zeta\}$. We begin by showing that $h_2^{0,1}(\bar{x}, v) = \langle \zeta, v \rangle$ for each v.

Let $v \in E$ (note that $h_2^{0,1}(\bar{x}, v) \geq \langle \zeta, v \rangle$). By the Hahn-Banach theorem there exists $\zeta' \in (E, \|\cdot\|_1)'$ majorized by $h_2^{0,1}(\bar{x}, \cdot)$ and agreeing with $h_2^{0,1}(\bar{x}, \cdot)$ at v. Since h_2 is $(\|\cdot\|_1, \|\cdot\|_2)$ locally Lipschitzian around $\bar{x}$, then $\zeta' \in (E, \|\cdot\|_2)'$. It follows then that $\zeta' \in \partial_0^{1,2} h_2(\bar{x})$. Consequently, $\zeta = \zeta'$. Therefore, $h_2^{0,1}(\bar{x}, v) = \langle \zeta, v \rangle$.

We now calculate

$$
\begin{aligned}
\liminf_{x' \to_{\|\cdot\|_1} \bar{x},\, t \downarrow 0} \frac{h_2(x' + tv) - h_2(x')}{t} &= - \limsup_{x' \to_{\|\cdot\|_1} \bar{x},\, t \downarrow 0} \frac{h_2(x') - h_2(x' + tv)}{t} \\
&= - \limsup_{x' \to_{\|\cdot\|_1} \bar{x},\, t \downarrow 0} \frac{h_2(x' + tv - tv) - h_2(x' + tv)}{t} \\
&= -h_2^{0,1}(\bar{x}, -v) \\
&= -\langle \zeta, -v \rangle \\
&= \langle \zeta, v \rangle \\
&= h_2^{0,1}(\bar{x}, v) \\
&= \limsup_{x' \to_{\|\cdot\|_1} \bar{x},\, t \downarrow 0} \frac{h_2(x' + tv) - h_2(x')}{t}.
\end{aligned}
$$

This establishes the limit condition of Proposition 3 and completes the proof. □

Example 1. Let Ω a bounded domain in R^2, $E = W_0^{1,2}(\Omega)$ the Sobolev space with the usual norm $\|\cdot\|_2 = \|\cdot\|_{W_0^{1,2}(\Omega)}$. Let also p and ε such that $0 < \varepsilon < 1$, $\varepsilon + 2 < p < \infty$. Set $\|\cdot\|_1 = \|\cdot\|_{L^p(\Omega)}$. Note that $(E, \|\cdot\|_2)$ is a separable Banach space.

Since $W_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$, it follows that $(E, \|\cdot\|_1, \|\cdot\|_2)$ is a binormed space such that $\|\cdot\|_1 \leq c\|\cdot\|_2$ for some $c > 0$.

Set $g(u) = |u|^{\varepsilon+2}$ and consider the functional G defined on E by

$$G(x) = \int_\Omega g(x(s))ds.$$

Then G is $\|\cdot\|_2$ twice Fréchet differentiable at every $x \in E$ and

$$G^{(1)}(x)h = \int_\Omega g'(x(s))h(s)ds,$$
$$G^{(2)}(x)(h_1,h_2) = \int_\Omega g''(x(s))h_1(s)h_2(s)ds.$$

We assert that G is $(\|\cdot\|_1, \|\cdot\|_2)$ strictly differentiable at every $x \in E$. Indeed, suppose by the contrary that there exist $x \in E$, $\varepsilon > 0$, $x'_m \in E$, and $h_m \in E$ such that $x'_m \to x$ in $(E, \|\cdot\|_1)$, $h_m \to 0$ in $(E, \|\cdot\|_1)$ and $|r(x'_m, h_m)| > \varepsilon\|h_m\|_2$, where

$$r(x', h) = G(x'+h) - G(x') - G^{(1)}(x)h.$$

Then,

$$|r(x'_m, h_m)| = \left| \int_\Omega \int_0^1 \big(g'(x'_m(s) + t h_m(s)) - g'(x(s))\big) h_m(s) dt ds \right|.$$

Let $p' > 1$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. Since $p > \varepsilon + 2$ then $p > p'(\varepsilon+1)$. Thus, $x'_m \to x$ in $(E, \|\cdot\|_{L^{p'(\varepsilon+1)}(\Omega)})$ and $h_m \to 0$ in $(E, \|\cdot\|_{L^{p'(\varepsilon+1)}(\Omega)})$. Without loss of generality we can suppose that there exists $Z_1 \in L^{p'(\varepsilon+1)}(\Omega)$ such that

$$|x'_m(s)| + |h_m(s)| \le Z_1(s), \quad h_m(s) \to 0, \quad x'_m(s) \to x(s)$$

almost everywhere on Ω. Using the Hölder inequality, we obtain

$$|r(x'_m, h_m)| \le \left(\int_\Omega \int_0^1 |g'(x'_m(s) + t h_m(s)) - g'(x(s))|^{p'} dt ds \right)^{\frac{1}{p'}} \|h_m\|_{L^p(\Omega)}.$$

Consequently,

$$|r(x'_m, h_m)| \le c \left(\int_\Omega \int_0^1 |g'(x'_m(s) + t h_m(s)) - g'(x(s))|^{p'} dt ds \right)^{\frac{1}{p'}} \|h_m\|_2.$$

But this contradicts the fact that $|r(x'_m, h_m)| > \varepsilon\|h_m\|_2$, because

$$\int_\Omega \int_0^1 |g'(x'_m(s) + t h_m(s)) - g'(x(s))|^{p'} dt ds \to 0,$$

by the dominated convergence theorem.

Let us remark that G is not $\|\cdot\|_1$ Fréchet differentiable at any point $x \in E$. Indeed, let $\alpha_m \to \infty$ and $d_m \to \infty$ such that $|g(d_m)| \ge \alpha_m |d_m|^p$. By the countable additivity of the Lebesgue measure there exist $C > 0$ and $\Omega' \subset \Omega$ such that $\mu(\Omega') > 0$, $\operatorname{dist}(\Omega', \partial\Omega) > 0$, and $|x(s)| \le C$ for all $s \in \Omega'$.

In this case, put $D = \max\{g(u) : |u| \leq C\} < \infty$. Choose $\Omega_m \subset \Omega'$ such that $\mu(\Omega_m) = |d_m|^{-p}|\alpha_m|^{-\frac{1}{2}}$ for large m.

Let h_m be defined by the relation

$$h_m(s) = \begin{cases} d_m - x(s) & \text{for} \quad s \in \Omega_m, \\ 0 & \text{for} \quad s \in \Omega \setminus \Omega_m. \end{cases}$$

It then follows that $\|h_m\|_1 \to 0$ and

$$|G(x+h_m) - G(x)| \geq \alpha_m |d_m|^p \mu(\Omega_m) - D\mu(\Omega_m) = |\alpha_m|^{\frac{1}{2}} - D\mu(\Omega_m) \to \infty.$$

Let $k_m \in C_0^\infty(\Omega)$ such that $\|k_m - h_m\|_1 \to 0$. Then $\|k_m\|_1 \to 0$, but $G(x+k_m) - G(x) \to \infty$, because the Lebesgue integral is absolutely continuous. Therefore, G is not $\|\cdot\|_1$ Fréchet differentiable at x and consequently, G is not $\|\cdot\|_1$ strictly Fréchet differentiable at x.

Thus, applying Theorem 1, we conclude that $\partial_0^{1,2} G(x) = \partial_0^2 G(x) = \{\nabla G(x)\}$.

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ON SOLVABILITY OF SECOND-ORDER EVOLUTION INCLUSIONS WITH VOLTERRA TYPE OPERATORS

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This paper is dedicated to the memory of the Corresponding Member of the National Academy of Sciences of Ukraine, Professor Valeriy S. Mel'nik.

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Abstract. We consider second-order differential-operators inclusions with Volterra type operators. The problem of the existence of solutions of the Cauchy problem for the given inclusions is investigated. Important *a priori* estimates are obtained. An example illustrating the approach is given.

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Keywords: second-order evolution inclusion, Volterra type operator, pseudomonotone map

1. INTRODUCTION

The progress in the investigation of non-linear boundary problems for partial differential equations became possible thanks to the intense development of the methods of non-linear analysis which had found their application in various parts of mathematics. It has recently become natural to reduce these problems to the study of non-linear operator and differential-operator equations and inclusions in functional spaces. Within such an approach, the results for concrete systems are obtained as rather simple consequences of operator theorems [2, 10].

The evolution differential equations and inclusions are studied rather actively. To prove the properties of the resolving operator (non-emptiness, compactness, connectedness), the method of monotony, method of compactness, and their combinations are often used.

In the present work, we study the solvability of the evolution inclusion with multivalued non-coercive maps

$$y'' + A(y') + B(y) \ni f,$$

which is important for applications.

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Recent related investigations concern a class of problems with a strongly monotone operator A and multi-valued operator B that can be presented as the sum of a single-valued linear self-conjugated monotone operator and a multi-valued demiclosed bounded operator. These problems are coercive. They were considered, e. g., by Papageorgiou and Yannakakis [13, 14]. More particular cases of evolution inclusions were studied by Ahmed and Kerbal [1], Gasiński and Smołka [3], Kartsatos and Markov [4], Migórski [12], and other authors.

Our goal here is to extend the approach indicated to a wider class of problems, namely, to problems with a multi-valued non-coercive non-monotone operator A and a multi-valued operator B satisfying similar conditions.

The idea of passing to subsequences in the classical definition of a single-valued pseudomonotone operator was suggested by Skrypnik [15]. It was developed for the first order differential-operator equations and inclusions in infinite-dimensional spaces with $+$-coercive W_{λ_0}-pseudomonotone maps by Mel'nik, Zgurovskii, and Novikov [11, 18, 19] and Kas'yanov [5–8]. This gave one the possibility to investigate a substantially wider class of problems arising in applications. In particular, this methodology, combined with the non-coercive theory [2, 9, 18], which we apply to the second-order evolution inclusions, allows one to sufficiently extend the class of problems with multi-valued maps for which we can obtain the solvability. Since the operators are multi-valued, such extension faced with considerable difficulties which are not typical for the differential-operator equations. Here, the proof of the solvability is based on the method of singular perturbations [9, 10] and allows us to obtain important a priori estimates for solutions. It makes possible to study properties for the obtained solutions (e. g., dynamics). As an example illustrating the suggested approach, we consider a class of problems with non-linear operators. The obtained results are new for both inclusions and equations.

We note that the solvability of second-order differential-operator equations was investigated by the authors in [16, 17].

2. Problem setting

Let H be a real Hilbert space with the inner product $(\cdot,\cdot)$, and let $(V_1, \|\cdot\|_{V_1})$ and $(V_2, \|\cdot\|_{V_2})$ be some real reflexive separable Banach spaces continuously embedded into H and such that

$$V := V_1 \cap V_2$$

is dense in the spaces V_1, V_2, and H. We assume that one of the embeddings $V_i \subset H$, $i = 1, 2$, is compact. In what follows, the space topologically conjugate to H (with respect to the bilinear form $(\cdot,\cdot)$) is identified with H. Then we have

$$V_i \subset H \subset V_i^* \qquad (i = 1, 2)$$

with continuous and dense embeddings, where $(V_i^*, \|\cdot\|_{V_i^*})$, $i = 1,2$, is the space topologically conjugate to V_i, $i = 1,2$, with respect to the canonical bilinear form

$$\langle\cdot,\cdot\rangle_{V_i}: V_i^* \times V_i \to \mathbb{R} \qquad (i = 1,2)$$

that coincides on $H \times V$ with the inner product $(\cdot,\cdot)$ in H. Let us consider the reflexive function spaces $Y = L_2(S;H)$ and

$$X_i := L_{r_i}(S;H) \cap L_{p_i}(S;V_i) \qquad (i = 1,2)$$

with

$$\|y\|_{X_i} := \|y\|_{L_{p_i}(S;V_i)} + \|y\|_{L_{r_i}(S;H)} \qquad (i = 1,2),$$

where $S := [0,T]$, $1 < p_i \le r_i < +\infty$, $i = 1,2$, and $\max\{r_1; r_2\} \ge 2$.

Let us consider the reflexive (it follows from [2, Chapter 1]) Banach space $X := X_1 \cap X_2$ with the norm $\|y\|_X := \|y\|_{X_1} + \|y\|_{X_2}$. We note that the space X is continuously and densely embedded in Y.

We identify $L_{q_i}(S;V_i^*) + L_{r_i'}(S;H)$ with X_i^*. Similarly, $Y^* \equiv Y$ and

$$X^* = X_1^* + X_2^* \equiv L_{q_1}(S;V_1^*) + L_{q_2}(S;V_2^*) + L_{r_1'}(S;H) + L_{r_2'}(S;H),$$

where ${r_i}^{-1} + {r_i'}^{-1} = {p_i}^{-1} + {q_i}^{-1} = 1$.

Let $A, B: X \rightrightarrows X^*$ be strict multi-valued maps. We consider the Cauchy problem for the differential-operator inclusion with non-coercive multi-valued maps of W_{λ_0}-pseudomonotone type

$$\begin{cases} y'' + Ay' + By \ni f, \\ y(0) = a_0,\ y'(0) = \overline{0},\ y \in C(S;V),\ y' \in C(S;H), \end{cases} \tag{2.1}$$

where $a_0 \in V$ and $f \in X^*$ are fixed.

On $X^* \times X$ we consider the pairing

$$\begin{aligned} \langle f, y\rangle &= \int_S (f_{11}(\tau), y(\tau))_H \, d\tau + \int_S (f_{12}(\tau), y(\tau))_H \, d\tau \\ &\quad + \int_S \langle f_{21}(\tau), y(\tau)\rangle_{V_1} \, d\tau + \int_S \langle f_{22}(\tau), y(\tau)\rangle_{V_2} \, d\tau \\ &= \int_S (f(\tau), y(\tau)) \, d\tau, \end{aligned}$$

where $f = f_{11} + f_{12} + f_{21} + f_{22}$, $f_{1i} \in L_{r_i'}(S;H)$, and $f_{2i} \in L_{q_i}(S;V_i^*)$. Note that, for any $f \in X^*$,

$$\|f\|_{X^*} = \inf_{\substack{f = f_{11}+f_{12}+f_{21}+f_{22}: \\ f_{1i} \in L_{r_i'}(S;H),\ f_{2i} \in L_{q_i}(S;V_i^*)\ (i=1,2)}} \varphi(f_{11}, f_{12}, f_{21}, f_{22}),$$

where

$$\varphi(f_{11}, f_{12}, f_{21}, f_{22}) = \max\left\{\|f_{11}\|_{L_{r_1'}(S;H)}, \|f_{12}\|_{L_{r_2'}(S;H)}, \|f_{21}\|_{L_{q_1}(S;V_1^*)}, \|f_{22}\|_{L_{q_2}(S;V_2^*)}\right\}.$$

Moreover, let

$$W = \{y \in X \mid y' \in X^*\}$$

and $\|y\|_W = \|y\|_X + \|y'\|_{X^*}$ for all $y \in W$, where the derivative y' of the element $y \in X$ is considered in the sense of scalar distribution space $\mathcal{D}^*(S;V^*) = \mathcal{L}(\mathcal{D}(S);V_w^*)$ with $V = V_1 \cap V_2$ and $V_w^* = (V^*, \sigma(V^*, V))$ [2]. We note that W is a reflexive Banach space with a compact embedding $W \subset Y$ [10].

3. Classes of maps

Let Y be a reflexive Banach space, Y^* be its topologically conjugated space, $\langle\cdot,\cdot\rangle_Y : Y^* \times Y \to \mathbb{R}$ be the pairing, and $A: Y \rightrightarrows Y^*$ be a strict multi-valued map. Let us define its upper support function $[A(y), w]_+ := \sup_{d \in A(y)} \langle d, w\rangle_Y$ and lower support function $[A(y), w]_- := \inf_{d \in A(y)} \langle d, w\rangle_Y$, where $y, w \in Y$, and its upper norm $\|A(y)\|_+ := \sup_{d \in A(y)} \|d\|_{Y^*}$ and lower norm $\|A(y)\|_- := \inf_{d \in A(y)} \|d\|_{Y^*}$. Consider the associated maps $\mathrm{co}A: Y \rightrightarrows Y^*$ and $\overline{\mathrm{co}}A: Y \rightrightarrows Y^*$ defined by the relations $(\mathrm{co}A)(y) = \mathrm{co}(A(y))$ and $(\overline{\mathrm{co}}A(y)) = \overline{\mathrm{co}}(A(y))$ respectively, where $(\overline{\mathrm{co}}^* A(y))$ is the weak closure of $\mathrm{co}(A(y))$ in Y^* and $\mathrm{co}(A(y))$ is the convex hull of $A(y) \subset Y^*$.

Proposition 1 ([18]). *Let $A, B : Y \rightrightarrows Y^*$. Then*

(1) *for all $y, v_1, v_2 \in Y$ the relations*

$$[A(y), v_1 + v_2]_+ \le [A(y), v_1]_+ + [A(y), v_2]_+,$$
$$[A(y), v_1 + v_2]_- \ge [A(y), v_1]_- + [A(y), v_2]_-,$$
$$[A(y), v_1 + v_2]_+ \ge [A(y), v_1]_+ + [A(y), v_2]_-,$$
$$[A(y), v_1 + v_2]_- \le [A(y), v_1]_+ + [A(y), v_2]_-$$

are satisfied;

(2) *the equalities*

$$[A(y), v]_+ = -[A(y), -v]_-,$$
$$[A(y) + B(y), v]_{+(-)} = [A(y), v]_{+(-)} + [B(y), v]_{+(-)}$$

hold for all $y, v \in Y$;

(3) $[A(y), v]_{+(-)} = [\overline{\mathrm{co}}^* A(y), v]_{+(-)}$ *for all $y, v \in Y$;*

(4) *for all $y, v \in Y$ the relations*

$$[A(y), v]_{+(-)} \le \|A(y)\|_{+(-)} \|v\|_Y,$$
$$d_H\big(A(y), B(y)\big) \ge \big|\|A(y)\|_{+(-)} - \|B(y)\|_{+(-)}\big|,$$
$$\|A(y) - B(y)\|_+ \ge \big|\|A(y)\|_+ - \|B(y)\|_-\big|$$

are fulfilled, where $d_H(\cdot,\cdot)$ is the Hausdorff metric.

Proposition 2 ([18]). *The inclusion $d \in \overline{\mathrm{co}}^* A(y)$ is true if and only if*

$$[A(y), v]_+ \geq \langle d, v\rangle_Y \quad \textit{for all } v \in Y.$$

Proposition 3 ([18]). *Let $D \subset Y$ and $a(\cdot,\cdot)\colon D \times Y \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. For every $y \in D$, the functional $Y \ni w \mapsto a(y,w)$ is positively homogeneous, convex, and lower semi-continuous if and only if there exists a multi-valued map $A\colon Y \rightrightarrows Y^*$ such that $D(A) = D$ and*

$$a(y,w) = [A(y), w]_+ \quad \textit{for all } y \in D(A),\ w \in Y.$$

Remark 1. In what follows, $y_n \rightharpoonup y$ in Y means that y_n weakly converges to y in a reflexive Banach space Y.

Definition 1. Let us denote the family of all non-empty closed convex bounded subsets of the space Y by $C_v(Y)$.

Definition 2. An operator $A\colon X \rightrightarrows X^*$ is called a *Volterra type operator* if, for any $t \in S$, from the equality $u(s) = v(s)$ for a. e. $s \in [0,t]$ $(u, v \in X)$ it follows that $(\overline{\mathrm{co}}A(u))(s) = (\overline{\mathrm{co}}A(v))(s)$ for a. e. $s \in [0,t]$, i. e., $[A(u), \xi_t]_+ = [A(v), \xi_t]_+$ for all $\xi_t \in X$ such that $\xi_t(s) = 0$ for a. e. $s \in S \setminus [0,t]$.

Definition 3. A strict multi-valued map $A\colon Y \rightrightarrows Y^*$ is called:

(1) $+(-)$-*coercive* if there exists a lower bounded, on bounded in $\mathbb{R}_+$ sets, real function $\gamma\colon \mathbb{R}_+ \to \mathbb{R}$ such that $\gamma(s) \to +\infty$ as $s \to +\infty$ and

$$[A(y), y]_{+(-)} \geq \gamma(\|y\|_Y)\|y\|_Y \quad \text{for all } y \in Y;$$

(2) *bounded* if for any $L > 0$ there is $l > 0$, such that

$$\|A(y)\|_+ \leq l \quad \text{for all } y \in Y,\ \|y\|_Y \leq L;$$

(3) *locally bounded* if for all $y \in Y$ there exist $m > 0$ and $M > 0$ such that

$$\|A(\xi)\|_+ \leq M \quad \text{for all } \xi \in Y,\ \|y - \xi\|_Y \leq m;$$

(4) *finite-dimension locally bounded* if $A|_F$ is locally bounded on $(F, \|\cdot\|_Y)$ for any finite-dimensional subspace $F \subset Y$.

Definition 4. We say that a multi-valued map $A\colon X \rightrightarrows X^*$ possesses the *property* (Π) if the following implication holds: If for some non-empty bounded subset $B \subset Y$, constant $k > 0$, and selector d of A, the relation

$$\langle d(y), y\rangle_Y \leq k \quad \text{for all } y \in B$$

holds, then there is a $K > 0$ such that

$$\|d(y)\|_{Y^*} \leq K \quad \text{for all } y \in B.$$

Definition 5. We say that a function $C: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *belongs to the class* Φ if $C(r_1;\cdot) : \mathbb{R}_+ \to \mathbb{R}$ is continuous for any $r_1 \geq 0$ and

$$\lim_{\tau \to 0+} \tau^{-1} C(r_1; \tau r_2) = 0$$

for all $r_1, r_2 \geq 0$.

Now let W be some normed space with the norm $\|\cdot\|_W$. We suppose that $W \subset Y$ with a continuous embedding. Let also $\|\cdot\|'_W$ be a (semi-)norm on Y which is compact with respect to $\|\cdot\|_W$ on W and continuous with respect to $\|\cdot\|_Y$ on Y. Moreover, let $C \in \Phi$.

Definition 6. A strict multi-valued map $A: Y \rightrightarrows Y^*$ is called:

(1) *radially lower semi-continuous* (or, shortly, RLSC) if

$$\liminf_{t \to 0+} [A(y + t\xi), \xi]_+ \geq [A(y), \xi]_-$$

for all $y, \xi \in Y$;

(2) *radially upper semi-continuous* (or RUSC) if, for all $y, \xi \in Y$, the real function $t \mapsto [A(y + t\xi), \xi]_+$ is upper semi-continuous from the right at the point $t = 0$;

(3) *operator with semi-bounded variation on* W (or (Y, W)-SBV) if, for all $R \geq 0$ and $y_1, y_2 \in Y$ such that $\|y_1\|_Y \leq R$ and $\|y_2\|_Y \leq R$, the inequality

$$[A(y_1), y_1 - y_2]_- \geq [A(y_2), y_1 - y_2]_+ - C(R; \|y_1 - y_2\|'_W)$$

is satisfied;

(4) *operator with* N*-semi-bounded variation on* W (or N-SBV *on* W) if, for all $R \geq 0$ and every $y_1, y_2 \in Y$ such that $\|y_1\|_Y \leq R$ and $\|y_2\|_Y \leq R$, the condition

$$[A(y_1), y_1 - y_2]_- \geq [A(y_2), y_1 - y_2]_- - C(R; \|y_1 - y_2\|'_W)$$

holds;

(5) λ_0*-pseudomonotone on* W (or W_{λ_0}*-pseudomonotone*) if, for arbitrary sequences $\{y_n\}_{n \geq 0} \subset W$ and $\{d_n\}_{n \geq 1}$ such that $d_n \in \overline{\text{co}} A(y_n)$ for all $n \geq 1$, $y_n \rightharpoonup y_0$ in W, and $d_n \rightharpoonup d_0$ in Y^*, from the inequality

$$\limsup_{n \to \infty} \langle d_n, y_n - y_0 \rangle_Y \leq 0$$

it follows that there exist subsequences $\{y_{n_k}\}_{k \geq 1} \subset \{y_n\}_{n \geq 1}$ and $\{d_{n_k}\}_{k \geq 1} \subset \{d_n\}_{n \geq 1}$ for which the inequality

$$\liminf_{k \to \infty} \langle d_{n_k}, y_{n_k} - w \rangle_Y \geq [A(y_0), y_0 - w]_-$$

holds for all $w \in Y$.

Remark 2. The idea on passing to a subsequence in the definition of a single-valued pseudomonotone operator was proposed by Skrypnik [15].

Lemma 1 ([18]). *Any strict multi-valued operator $A: Y \rightrightarrows Y^*$ with $(Y; W)$-*SBV *is bounded-valued, locally bounded, and satisfies property (Π). Furthermore, if A is* RLSC*, then it is also λ_0-pseudomonotone on W.*

Let $Y := Y_1 \cap Y_2$, where $(Y_1, \|\cdot\|_{Y_1})$ and $(Y_2, \|\cdot\|_{Y_2})$ are some reflexive Banach spaces.

Definition 7. A pair $(A; B)$ of maps $A: Y_1 \to C_v(Y_1^*)$ and $B: Y_2 \to C_v(Y_2^*)$ is called *s-mutually bounded* if, for any constant $M > 0$, bounded set $D \subset Y$, and selectors d_A of A and d_B of B, there exists a $K > 0$ such that the relations $y \in D$ and

$$\langle d_A(y), y\rangle_{Y_1} + \langle d_B(y), y\rangle_{Y_2} \le M$$

imply that $\|d_A(y)\|_{Y_1^*} \le K$ or $\|d_B(y)\|_{Y_2^*} \le K$.

Remark 3. A bounded strict multi-valued map $A: Y \rightrightarrows Y^*$ satisfies condition (Π).

If one operator of the pair $(A; B)$ is bounded, then the pair $(A; B)$ is s-mutually bounded. Moreover, if both operators from $(A; B)$ satisfy condition (Π), then their sum also satisfies condition (Π) and the pair $(A; B)$ is s-mutually bounded.

Let now $W := W_1 \cap W_2$, where $(W_1, \|\cdot\|_{W_1})$ and $(W_2, \|\cdot\|_{W_2})$ are Banach spaces such that $W_i \subset Y_i$, $i = 1, 2$, with a continuous embedding.

Lemma 2 ([7]). *Let $A : Y_1 \to C_v(Y_1^*)$ and $B : Y_2 \to C_v(Y_2^*)$ be multi-valued maps which are λ_0-pseudomonotone on W_1 and W_2, respectively, and such that the pair $(A; B)$ is s-mutually bounded. Then the map $C := A + B : Y \to C_v(Y^*)$ is λ_0-pseudomonotone on W.*

Definition 8. We say that a multi-valued map $A: X \rightrightarrows X^*$ satisfies *condition (H)* if, for any $y \in X$, $n \ge 1$, $\{d_i\}_{i=1}^n \subset A(y)$ and measurable $E_j \subset S$ $(j = 1, \dots, n)$ such that $\bigcup_{j=1}^n E_j = S$ and $E_i \cap E_j = \varnothing$ for all $i, j = 1, \dots, n$, $i \ne j$, the inclusion $d \in \overline{\mathrm{co}} A(y)$ holds, where $d = \sum_{j=1}^n d_j \chi_{E_j}$ and

$$\chi_{E_j}(\tau) := \begin{cases} 1 & \text{for } \tau \in E_j, \\ 0 & \text{for } \tau \in S \setminus E_j. \end{cases}$$

4. Main result

Theorem 1. *Let $\lambda_A \ge 0$ be fixed, $p_0 := \min\{p_1, p_2\}$, the space V be compactly embedded in some Banach space V_0, and the embedding $V_0 \subset V^*$ be continuous. Moreover, let the map*[*] *$A + \lambda_A I: X \to C_v(X^*)$ be $+$-coercive and* RLSC *multi-valued map of the Volterra type with $(X; W)$-*SBV *$(\|\cdot\|'_W = \|\cdot\|_{L_{p_0}(S;V_0)})$ satisfying condition (H). Let $B: Y \to C_v(Y^*)$ be a multi-valued operator of the Volterra type which fulfils condition (H), the growth condition*

$$\|By\|_+ \le c_1 \|y\|_Y + c_2 \quad \text{for all } y \in Y \tag{4.1}$$

[*]Here, $I: X \to X^*$ is the identical motion.

with some $c_1, c_2 \geq 0$, and the continuity condition

$$d_H(B(z), B(z_0)) \to 0 \quad \text{as } z \to z_0. \tag{4.2}$$

Then for any $a_0 \in V$ and $f \in X^*$ there exist at least one solution u of problem (2.1) with $u' \in W$.

Here, $d_H(\cdot,\cdot)$ is the Hausdorff metric on $C_v(Y^*)$, i. e.,

$$d_H(C, D) := \max\{\operatorname{dist}(C; D), \operatorname{dist}(D, C)\}$$

with $\operatorname{dist}(C; D) := \sup_{c \in C} \inf_{d \in D} \|c - d\|_{Y^*}$ for $C, D \in C_v(Y^*)$.

Proof. Let us reduce the evolution inclusion (2.1) to a first-order inclusion. Let $R: X \to X$ (resp., $R: Y \to Y$) be the Volterra type operator defined by the relation

$$(Rv)(t) = a_0 + \int_0^t v(s)ds \quad \text{for all } t \in S \text{ and every } v \in X \text{ (resp., } v \in Y).$$

It is clear that R is a Lipschitz continuous operator from X into X (resp., from Y into Y). Consider the problem

$$\begin{cases} v' + A(v) + B(Rv) \ni f, \\ v(0) = \overline{0},\ v \in W. \end{cases} \tag{4.3}$$

If $v \in W$ is a solution of problem (4.3), then $u = Rv \in X$ is a solution of problem (2.1) such that $u' \in W \subset X$.

Let us set $\mathcal{A} := A + B \circ R: X \to C_v(X^*)$ and $\lambda = \lambda_A + \lambda_B$, where $\lambda_B = 1 + c_1 c_3$ and c_3 is the Lipschitz constant for the operator $R: Y \to Y$. For an arbitrary $y \in X$ and a. e. $t \in S$, we set

$$y_\lambda(t) = e^{-\lambda t} y(t), \qquad \hat{y}_\lambda(t) = e^{\lambda t} y(t), \tag{4.4}$$

and

$$(A_\lambda y)(t) = e^{-\lambda t}(\mathcal{A}\hat{y}_\lambda)(t) + \lambda y(t).$$

Then $g \in A_\lambda(y_\lambda) \iff \langle g, w\rangle_X \leq [A(y) + \lambda y, w_\lambda]_+$ for all $w \in X$. The set $A_\lambda(y_\lambda)$ is non-empty because every g defined by the relation

$$g(t) = e^{-\lambda t} d(t) + \lambda y_\lambda(t) \quad \text{for a. e. } t \in S \text{ and all } d \in \mathcal{A}(y)$$

belongs to $A_\lambda(y_\lambda)$.

We note that $A_\lambda: X \to C_v(X^*)$ and $v \in W$ is a solution of problem (4.3) if and only if $v_\lambda \in W$ is such that

$$v_\lambda' + A_\lambda v_\lambda \ni f_\lambda, \quad v_\lambda(0) = \overline{0}, \tag{4.5}$$

where $f_\lambda(t) = e^{-\lambda t} f(t)$. It turns out that $A_\lambda: X \to C_v(X^*)$ possesses the following properties:

(α_1) A_λ is $+$-coercive on X,

(α_2) A_λ is λ_0-pseudomonotone on W,
(α_3) A_λ is locally bounded on X,
(α_4) A_λ satisfies condition (Π) on X.

Let us prove the assertion above.

PROPERTY (α_1). Let us fix $y \in X$, $\|y\|_X \neq 0$. As $\|y_\lambda\|_X \leq \|y\|_X$, then

$$\|y_\lambda\|_X^{-1}[A_\lambda y_\lambda, y_\lambda]_+ \geq \|y\|_X^{-1} \sup_{\zeta(y)\in A(y)} \int_S e^{-2\lambda t}\big(\zeta(y)(t)+\lambda_A y(t), y(t)\big)dt$$
$$+ \|y\|_X^{-1} \inf_{\zeta(y)\in B(Ry)} \int_S e^{-2\lambda t}\big(\zeta(y)(t)+\lambda_B y(t), y(t)\big)dt. \quad (4.6)$$

We first estimate the first term. We remark that

$$[(A+\lambda_A I)y, y]_+ \geq \widehat{\gamma}(\|y\|_X)\|y\|_X \quad \text{for all } y \in X,$$

where $\widehat{\gamma} : \mathbb{R}_+ \to \mathbb{R}$ can be chosen as a non-decreasing function lower bounded on bounded, in $\mathbb{R}_+$, sets such that $\widehat{\gamma}(r) \to +\infty$ as $r \to \infty$.

Since A is a Volterra type operator, we see that, for any $u \in X$,

$$\sup_{\zeta(u)\in A(u)} \int_0^t \big(\zeta(u)(\tau)+\lambda_A u(\tau), u(\tau)\big)d\tau \geq \widehat{\gamma}(\|u\|_{X_t})\|u\|_{X_t} \quad \text{for all } t \in S,$$

where $\|u\|_{X_t} = \|u_t\|_X$. Let us set

$$g_{\zeta(y)}(\tau) = \big(\zeta(y)(\tau)+\lambda_A y(\tau), y(\tau)\big), \quad \zeta(y) \in A(y),\ \tau \in S,$$

and $h(t) = \widehat{\gamma}(\|y\|_{X_t})\|y\|_{X_t}$ for $t \in S$. Then $h(t) \geq \min\{\widehat{\gamma}(0), 0\}\|y\|_X$ and

$$\sup_{\zeta(y)\in A(y)} \int_0^t g_{\zeta(y)}(\tau)d\tau \geq h(t)$$

for all $t \in S$. Similarly to the definition of A_λ, for any $u \in X$ and a. e. $t \in S$, we put

$$(A_1 u)(t) = \big(e^{-2\lambda t} - e^{-2\lambda T}\big)\big((Au)(t)+\lambda_A u(t)\big),$$
$$(A_2 u)(t) = e^{-2\lambda T}\big((Au)(t)+\lambda_A u(t)\big),$$

and

$$(\widehat{A}u)(t) = e^{-2\lambda t}\big((Au)(t)+\lambda_A u(t)\big).$$

Then, due to Proposition 1, we get

$$[\widehat{A}y, y]_+ = [A_1 y, y]_+ + [A_2 y, y]_+$$
$$\geq e^{-2\lambda T}h(T) + 2\lambda T \sup_{\zeta(y)\in A(y)} \inf_{s\in S} e^{-2\lambda s} \int_0^s \big(\zeta(y)(\tau)+\lambda_A y(\tau), y(\tau)\big)d\tau.$$

Further, using condition (H) for the operator A, we prove that

$$\sup_{\zeta(y)\in A(y)} \inf_{s\in S} e^{-2\lambda s} \int_0^s \big(\zeta(y)(\tau)+\lambda_A y(\tau), y(\tau)\big)d\tau \ge -C_1\|y\|_X,$$

where $C_1 = \max\{-\hat{\gamma}(0), 0\} \ge 0$ does not depend on y. Consequently, we obtain

$$\|y\|_X^{-1} \sup_{\zeta(y)\in A(y)} \int_0^T e^{-2\lambda\tau}\big(\zeta(y)(\tau)+\lambda_A y(\tau), y(\tau)\big)d\tau \ge e^{-2\lambda T}\hat{\gamma}(\|y\|_X) - 2\lambda C_1 T. \quad (4.7)$$

Let us estimate the second term. Analogously to the previous case, using the Volterra property of the operator $B\circ R$, we obtain that, for all $t\in S$,

$$\inf_{\zeta(y)\in B(Ry)} \int_0^t \big(\zeta(y)(\tau)+\lambda_B y(\tau), y(\tau)\big)d\tau \ge -(c_2+c_1\|R\bar{0}\|_Y)c_4\|y\|_X > -\infty,$$

where $c_4 > 0$ is such that $\|\cdot\|_Y \le c_4\|\cdot\|_X$. Then

$$\inf_{\zeta(y)\in B(Ry)} \int_0^T e^{-2\lambda\tau}\big(\zeta(y)(\tau)+\lambda_B y(\tau), y(\tau)\big)d\tau \ge -e^{-2\lambda T}(c_2+c_1\|R\bar{0}\|_Y)c_4\|y\|_X + 2\lambda\int_0^T e^{-2\lambda s} \inf_{\zeta(y)\in B(Ry)} \int_0^s \big(\zeta(y)(\tau)+\lambda_B y(\tau), y(\tau)\big)d\tau ds \ge -(c_2+c_1\|R\bar{0}\|_Y)c_4\|y\|_X.$$

Therefore, in view of (4.7), it follows from (4.6) that

$$\|y_\lambda\|_X^{-1}[A_\lambda y_\lambda, y_\lambda]_+ \ge e^{-2\lambda T}\hat{\gamma}(\|y_\lambda\|_X) - 2\lambda C_1 T - (c_2+c_1\|R\bar{0}\|_Y)c_4,$$

because $\|y\|_X \ge \|y_\lambda\|_X$ and the function $\hat{\gamma}$ is non-decreasing. Since y is arbitrary, we have proved that $A_\lambda: X\to C_v(X^*)$ is $+$-coercive.

PROPERTY (α_2). For any $y\in X$ and a. e. $t\in S$ we set

$$(A_\lambda^1 y)(t) = e^{-\lambda t}(A\hat{y}_\lambda)(t)+\lambda_A y(t), \quad (A_\lambda^2 y)(t) = e^{-\lambda t}(B(R\hat{y}_\lambda))(t)+\lambda_B y(t),$$

where $\hat{y}_\lambda$ is given by (4.4). Let us note that $A_\lambda^1 + A_\lambda^2 = A_\lambda$.

At first we show that A_λ^1 is an RLSC operator with $(X;W)$-SBV. Let us prove the semi-boundedness of the variation. By virtue of the assumptions of the theorem, for all $R>0$ and $y,\xi\in X$ such that $\|y\|_X\le R$ and $\|\xi\|_X\le R$, we have

$$[A(y)-A(\xi)+\lambda_A y-\lambda_A\xi, y-\xi]_- + C_A(R;\|y-\xi\|'_W) \ge 0.$$

Let us set $\widehat{C}_A(R;t) := \max_{\tau\in[0,t]} C_A(R;\tau)$ for all $R,t\ge 0$ (note that $\widehat{C}_A\in\Phi$) and

$$z_t(\tau) := \begin{cases} z(\tau) & \text{for } 0\le\tau\le t,\\ \bar{0} & \text{for } t<\tau\le T \end{cases}$$

for a. e. $t \in S$ and all $z \in X$. Let ζ and η be fixed selectors of A. Since A is a Volterra type operator, for all $y, \xi \in X$ and $t \in S$, we have

$$\int_0^t \big(\zeta(y)(\tau) + \lambda_A y(\tau) - \eta(\xi)(\tau) - \lambda_A \xi(\tau), y(\tau) - \xi(\tau)\big) d\tau + \widehat{C}_A(R; \|y - \xi\|'_{W_t})$$
$$\geq [(A + \lambda_A I)(y_t) - (A + \lambda_A I)(\xi_t), y_t - \xi_t]_- + \widehat{C}_A(R; \|y_t - \xi_t\|'_W) \geq 0$$

because $\|y_t\|_X \leq \|y\|_X$ and $\|y_t - \xi_t\|'_W \leq \|y - \xi\|'_W$. Here, $\|\cdot\|'_{W_t} = \|\cdot\|_{L_{p_0}([0,t];V_0)}$.

Let us fix $y, \xi \in X$ and set

$$g(\tau) = \big(\zeta(y)(\tau) + \lambda_A y(\tau) - \eta(\xi)(\tau) - \lambda_A \xi(\tau), y(\tau) - \xi(\tau)\big), \quad \tau \in S,$$

and $h(t) = \widehat{C}_A(R; \|y - \xi\|'_{W_t})$, $t \in S$. We have proved that

$$\int_0^t g(\tau) d\tau \geq -h(t) \quad \text{for all } t \in S.$$

The function $S \ni t \mapsto h(t)$ is non-decreasing and, thus,

$$\int_0^T e^{-2\lambda\tau} g(\tau) d\tau = e^{-2\lambda T} \int_0^T g(\tau) d\tau + 2\lambda \int_0^T e^{-2\lambda\tau} \int_0^\tau g(s) ds d\tau \geq -h(T).$$

Consequently,

$$[A^1_\lambda y_\lambda, y_\lambda - \xi_\lambda]_-$$
$$\geq [A^1_\lambda \xi_\lambda, y_\lambda - \xi_\lambda]_+ - \widehat{C}_A(R; \|y - \xi\|_{L_{p_0}(S;V_0)}) \quad \text{for all } y, \xi \in X. \quad (4.8)$$

Now we consider the space $L_{p_0,\lambda}(S; V_0)$ that consists of measurable functions $g: S \to V_0$ for which the integral $\int_S e^{\lambda t p_0} \|g(t)\|^{p_0}_{V_0} dt$ is finite. Then

$$\|y - \xi\|_{L_{p_0}(S;V_0)} = \left(\int_S e^{\lambda t p_0} \|y_\lambda(t) - \xi_\lambda(t)\|^{p_0}_{V_0} dt\right)^{1/p_0} = \|y_\lambda - \xi_\lambda\|_{L_{p_0,\lambda}(S;V_0)}.$$

Therefore, from (4.8) we obtain

$$[A^1_\lambda y_\lambda, y_\lambda - \xi_\lambda]_- \geq [A^1_\lambda \xi_\lambda, y_\lambda - \xi_\lambda]_+ - \widehat{C}_A(R; \|y_\lambda - \xi_\lambda\|_{L_{p_0,\lambda}(S;V_0)}).$$

The proof of the fact that the mapping $A^1_\lambda : X \to C_v(X^*)$ has $(X; W)$-SBV is concluded by taking into account the compactness of the embedding $W \subset L_{p_0,\lambda}(S; V_0)$ [10, Theorem 1.5.1]. The RLSC for A^1_λ is clear.

Since an arbitrary RLSC multi-valued operator with $(X; W)$-SBV is λ_0-pseudomonotone on W [8], we have proved that A^1_λ is λ_0-pseudomonotone on W.

Let us now consider A^2_λ. We first show that A^2_λ is an operator with N-SBV on W (the radial upper semi-continuity is clear).

We first prove that $(B\circ R): X\to C_v(X^*)$ is the operator with N-SBV on W with $\|\cdot\|'_W=\|\cdot\|_Y$. For all $R\ge 0$ and $t\ge 0$, we set

$$C_B(R,t)=t\sup\big\{d_H(B(Rz_1),B(Rz_2))\mid z_1,z_2\in X:\ \|z_1-z_2\|_Y\le t \\ \text{and } \|z_i\|_X\le R,\ i=1,2\big\}.$$

Similarly to [17], we show that $C_B\in\Phi$. Then for any $R>0$ and $y,\xi\in X$ such that $\|y\|_X\le R$, $\|\xi\|_X\le R$ it follows that

$$[B(Ry)+\lambda_B y,y-\xi]_- -[B(R\xi)+\lambda_B\xi,y-\xi]_- +C_B(R;\|y-\xi\|_Y)\ge 0.$$

Let us set $\widehat{C}_B(R;\cdot)=\max_{\tau\in[0,t]}C_B(R;\tau)$ for all $R,t\ge 0$ (note that $\widehat{C}_B\in\Phi$).

Since $B\circ R$ is the Volterra type operator, for all $y,\xi\in X$ and $t\in S$, we have

$$\inf_{\zeta(y)\in B(Ry)}\int_0^t\big(\zeta(y)(\tau)+\lambda_B y(\tau),y(\tau)-\xi(\tau)\big)d\tau \\ -\inf_{\eta(\xi)\in B(R\xi)}\int_0^t\big(\eta(\xi)(\tau)+\lambda_B\xi(\tau),y(\tau)-\xi(\tau)\big)d\tau \\ +\widehat{C}_B(R;\|y-\xi\|_{Y_t})\ge 0,$$

where $\|\cdot\|_{Y_t}=\|\cdot\|_{L_2([0,t];H)}$. Let us fix some $y,\xi\in X$ and set

$$g_\beta(\tau)=\big(\beta(\tau),y(\tau)-\xi(\tau)\big),\quad \beta\in X^*,\ \tau\in S,$$

and $h(t)=\widehat{C}_B(R;\|y-\xi\|_{Y_t})$, $t\in S$. We have thus proved that

$$\inf_{\zeta\in B(Ry)+\lambda_B y}\int_0^t g_\zeta(\tau)d\tau-\inf_{\eta\in B(R\xi)+\lambda_B\xi}\int_0^t g_\eta(\tau)d\tau\ge -h(t)\quad\text{for all } t\in S.$$

The function $S\ni t\mapsto h(t)$ is non-decreasing and thus, for an arbitrary $\zeta\in B(Ry)+\lambda_B y$, we get

$$\int_0^T e^{-2\lambda\tau}\big(\zeta(\tau),y(\tau)-\xi(\tau)\big)d\tau-\inf_{\eta\in B(R\xi)+\lambda_B\xi}\int_0^T e^{-2\lambda\tau}\big(\eta(\tau),y(\tau)-\xi(\tau)\big)d\tau \\ =\sup_{\eta\in B(R\xi)+\lambda_B\xi}\int_0^T e^{-2\lambda\tau}\big(\zeta(\tau)-\eta(\tau),y(\tau)-\xi(\tau)\big)d\tau \\ \ge e^{-2\lambda T}\inf_{\eta\in B(R\xi)+\lambda_B\xi}\int_0^T\big(\zeta(\tau)-\eta(\tau),y(\tau)-\xi(\tau)\big)d\tau \\ +\sup_{\eta\in B(R\xi)+\lambda_B\xi}\int_0^T\Big(e^{-2\lambda\tau}-e^{-2\lambda T}\Big)\big(\zeta(\tau)-\eta(\tau),y(\tau)-\xi(\tau)\big)d\tau \\ \ge -e^{-2\lambda T}h(T)+2\lambda T\sup_{\eta\in B(R\xi)+\lambda_B\xi}\inf_{s\in S}e^{-2\lambda s}\int_0^s\big(\zeta(\tau)-\eta(\tau),y(\tau)-\xi(\tau)\big)d\tau.$$

Using property (H) of the operator B we can prove that, for an arbitrary $\zeta \in B(Ry) + \lambda_B y$, the relation

$$\sup_{\eta\in B(R\xi)+\lambda_B\xi} \inf_{s\in S} e^{-2\lambda s} \int_0^s \big(\zeta(\tau)-\eta(\tau), y(\tau)-\xi(\tau)\big)d\tau \geq -h(T)$$

holds. Consequently,

$$\begin{aligned}\inf_{\zeta\in B(Ry)+\lambda_B y} &\int_0^T e^{-2\lambda\tau}\big(\zeta(\tau), y(\tau)-\xi(\tau)\big)d\tau \\ &- \inf_{\eta\in B(R\xi)+\lambda_B\xi}\int_0^T e^{-2\lambda\tau}\big(\eta(\tau), y(\tau)-\xi(\tau)\big)d\tau \\ &\geq -\big(e^{-2\lambda T}+2\lambda T\big)\hat{C}_B(R;\|y-\xi\|_Y).\end{aligned}$$

Let us set $\widetilde{C}_B(R;t) = \big(e^{-2\lambda T}+2\lambda T\big)\hat{C}_B(R;t)$ for $R,t\geq 0$ (note that $\widetilde{C}_B \in \Phi$). Then

$$\begin{aligned}\big[A_\lambda^2 y_\lambda, y_\lambda-\xi_\lambda\big]_- - \big[A_\lambda^2\xi_\lambda, y_\lambda-\xi_\lambda\big]_- &\geq -\widetilde{C}_B(R;\|y-\xi\|_Y) \\ &= -\widetilde{C}_B(R;\|y-\xi\|_{L_2(S;H)}).\end{aligned} \tag{4.9}$$

Now we consider the space $L_{2,\lambda}(S;H)$ consisting of the measurable functions $g\colon S\to H$ for which the integral $\int_S e^{2\lambda t}\|g(t)\|_H^2\,dt$ is finite. Then

$$\|y-\xi\|_{L_2(S;H)} = \left(\int_S e^{2\lambda t}\|y_\lambda(t)-\xi_\lambda(t)\|_H^2\,dt\right)^{1/2} = \|y_\lambda-\xi_\lambda\|_{L_{2,\lambda}(S;H)}.$$

Therefore, from (4.9), we obtain

$$\big[A_\lambda^2 y_\lambda, y_\lambda-\xi_\lambda\big]_- \geq \big[A_\lambda^2\xi_\lambda, y_\lambda-\xi_\lambda\big]_- - \widetilde{C}_B(R;\|y_\lambda-\xi_\lambda\|_{L_{2,\lambda}(S;H)}).$$

To prove that $A_\lambda^2\colon X\to C_v(X^*)$ is N-SBV, it is sufficient to note that embedding $W\subset L_{2,\lambda}(S;H)$ is compact. This is a direct sequence of the compactness of the embedding $W\subset Y$.

Let us now check the λ_0-pseudomonotony of A_λ^2 on W. Let $y_{\lambda,n}\to y_\lambda$ weakly in W (therefore $y_{\lambda,n}\to y_\lambda$ in $Y_\lambda := L_{2,\lambda}(S;H)$), $A_\lambda^2(y_{\lambda,n})\ni d_{\lambda,n}\to d_\lambda\in X^*$ weakly in X^*, and

$$\limsup_{n\to\infty}\langle d_{\lambda,n}, y_{\lambda,n}-y_\lambda\rangle \leq 0.$$

Since A_λ^2 is an operator with N-SBV on W, we conclude that for every $v\in X$

$$\begin{aligned}\liminf_{n\to\infty}\langle d_{\lambda,n}, y_{\lambda,n}-v_\lambda\rangle &\geq \liminf_{n\to\infty}[A_\lambda^2(y_{\lambda,n}), y_{\lambda,n}-v_\lambda]_- \\ &\geq \liminf_{n\to\infty}[A_\lambda^2(v_\lambda), y_{\lambda,n}-v_\lambda]_- - \widetilde{C}_B(R;\|y_\lambda-v_\lambda\|_{Y_\lambda}).\end{aligned} \tag{4.10}$$

At first we estimate the first term at the right-hand side of (4.10). It is easy to show that the function $Y_\lambda\ni h\mapsto [A_\lambda^2(v_\lambda),h]_-$ is continuous for all $v\in X$.

Therefore, from (4.10) we obtain that $\langle d_{\lambda,n}, y_{\lambda,n} - y_\lambda \rangle \to 0$ and

$$\liminf_{n\to\infty} \langle d_{\lambda,n}, y_\lambda - v_\lambda \rangle \geq [A^2_\lambda(v_\lambda), y_\lambda - v_\lambda]_- - \widetilde{C}_B(R; \|y_\lambda - v_\lambda\|_{Y_\lambda})$$

for all $v \in X$. Substituting $tw_\lambda + (1-t)y_\lambda$, where $w \in X$, $t \in [0,1]$, for v_λ in the last inequality, dividing the result by t, and passing to the limit as $t \to 0+$, due to the RUSC for A^2_λ, we obtain

$$\liminf_{n\to\infty} \langle d_{\lambda,n}, y_{\lambda,n} - w_\lambda \rangle \geq [A^2_\lambda(y_\lambda), y_\lambda - w_\lambda]_-$$

for all $w \in X$. Therefore, the λ_0-pseudomonotony A^2_λ on W is proved.

In order to prove the λ_0-pseudomonotony of A_λ on W, we use Lemma 2. Let us note that the pair $(A^1_\lambda, A^2_\lambda)$ is s-mutually bounded because A^2_λ is bounded as a consequence of (4.1) and the boundedness of the identity map.

PROPERTIES (α_3) AND (α_4). These properties follow from $(X;W)$-SBV of A^1_λ, N-SBV of A^2_λ, and Lemma 1.

In order to prove the solvability for problem (4.3) we use [8, Theorem 3.1]. Let $L : W_{\overline{0}} \subset X \to X^*$ be a densely defined linear operator, $Lz = z'$, $D(L) = W_{\overline{0}} = \{z \in W \mid z(0) = \overline{0}\}$, and $A_\lambda : X \to C_v(X^*)$ be a multi-valued map. Let us consider the problem

$$Lz + A_\lambda z \ni f_\lambda, \quad z \in W_{\overline{0}}. \tag{4.11}$$

Let us note that $D(L) = W_{\overline{0}}$ is a reflexive Banach space with respect to the graph norm of the derivative. Conditions (α_1)–(α_4) guarantee that there exists at least one solution z of problem (4.3) in $W_{\overline{0}}$. The function $\hat{z}_\lambda$ is then a solution of problem (4.3) and $R\hat{z}_\lambda$ is a solution of the original problem. □

Corollary 1. *Assume that $\lambda_A \geq 0$ is fixed, $p_2 \geq 2$, $p_0 = \min\{p_1, p_2\}$, the space V is compactly embedded in a Banach space V_0, and the embedding $V_0 \subset V^*$ is continuous. Moreover, let $A + \lambda_A I : X \to C_v(X^*)$ be a $+$-coercive and* RLSC *multi-valued operator of the Volterra type with $(X;W)$-*SBV *($\|\cdot\|'_W = \|\cdot\|_{L_{p_0}(S;V_0)}$) satisfying condition (H), and $B : Y \to C_v(Y^*)$ be a multi-valued operator of the Volterra type satisfying condition (H), the growth condition (4.1), and the continuity condition (4.2)[†], and $C : X \to X^*$ be an operator with the property*

$$(Cu)(t) = C_0(u(t)) \quad \textit{for all } u \in X,\ t \in S,$$

where $C_0 : V_2 \to V_2^*$ is a linear, bounded, self-conjugate, and monotone operator.

Then for arbitrary $a_0 \in V$ and $f \in X^*$ there exists at least one solution of the problem

$$\begin{cases} y'' + Ay' + By + Cy \ni f, \\ y(0) = a_0,\ y'(0) = \overline{0},\ y \in C(S;V),\ y' \in C(S;H). \end{cases} \tag{4.12}$$

[†]We recall that $d_H(\cdot,\cdot)$ is the Hausdorff metric on $C_v(Y^*)$

5. AN EXAMPLE

Let $\Omega \subset \mathbb{R}^n$ be a bounded region with the regular boundary $\partial\Omega$, $S = [0,T]$, $Q = \Omega \times S$, $\Gamma_T = \partial\Omega \times S$, $1 < p = p_1 = p_2$, and $\Phi\colon \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the "growth condition"

$$|\Phi(t)| \le c_1|t| + c_2 \quad \text{for all } t \in \mathbb{R}, \tag{5.1}$$

where $c_1, c_2 \in \mathbb{R}$, and the "sign condition"

$$(\Phi(t) - \Phi(s))(t-s) \ge -c_3(s-t)^2 \quad \text{for all } t,s \in \mathbb{R} \tag{5.2}$$

with some $c_3 > 0$. Moreover, let $S \times \mathbb{R} \ni (t,y) \mapsto \theta_i(t,y) \in \mathbb{R}_+$, $i = 1,2$, be single-valued continuous functions such that

$$-c_2(1+|x|) \le \theta_1(t,x) \le \theta_2(t,x) \le c_1(1+|x|) \quad \text{for all } t \in S,\ x \in \mathbb{R}, \tag{5.3}$$

where $c_1, c_2 \ge 0$. For an any $f \in X^* = L_2(S;L_2(\Omega)) + L_q(S;W^{-1,q}(\Omega))$, we consider the problem

$$\begin{aligned} &\frac{\partial^2 y(x,t)}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial^2 y(x,t)}{\partial x_i\, \partial t}\right|^{p-2} \frac{\partial^2 y(x,t)}{\partial x_i\, \partial t}\right) + \left|\frac{\partial y(x,t)}{\partial t}\right|^{p-2} \frac{\partial y(x,t)}{\partial t} \\ &\qquad + \Phi\left(\frac{\partial y(x,t)}{\partial t}\right) - \Delta y(x,t) \\ &\qquad + [\theta_1(t,y(x,t)); \theta_2(t,y(x,t))] \ni f(x,t) \quad \text{a. e. on } Q, \\ &\qquad y(x,0) = 0, \quad \left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = 0 \qquad \text{a.e. on } \Omega, \\ &\qquad\qquad y(x,t) = 0 \quad \text{a. e. on } \partial\Omega. \end{aligned} \tag{5.4}$$

For the operator $A\colon L_p(S;W_0^{1,p}(\Omega)) \to L_q(S;W^{-1,q}(\Omega))$, we take $(Au)(t) = A(u(t))$, $t \in S$ [16, 17], where $A(\varphi) = A_1(\varphi) + A_2(\varphi)$,

$$A_1(\varphi) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial \varphi}{\partial x_i}\right|^{p-2} \frac{\partial \varphi}{\partial x_i}\right) + |\varphi|^{p-2}\varphi,$$

and $A_2(\varphi) = \Phi(\varphi)$ for all $\varphi \in C_0^2(\bar{\Omega})$. For the map $B\colon L_2(Q) \to L_2(Q)$, we take

$$B(u) = \{v \in L_2(Q) \mid \theta_1(t,u(x,t)) \le v(x,t) \le \theta_2(t,u(x,t)) \text{ for a. e. } (x,t) \in Q\},$$

and let the operator $C\colon L_2(S;H_0^1(\Omega)) \to L_2(S;H^{-1}(\Omega))$ be defined by the relation $(Cu)(t) = C_0(u(t))$, $t \in S$, where $C_0(v) = -\Delta v$ for $v \in H_0^1(\Omega)$. Moreover, let $H = L_2(\Omega)$, $V_1 = W_0^{1,p}(\Omega)$, $V_2 = H_0^1(\Omega)$, and let $Y = L_2(S;H) = L_2(Q)$,

$$X = L_p(S;V_1) \cap L_2(S;H) \cap L_2(S;V_2), \qquad X^* = L_q(S;V_1^*) + L_2(S;V_2^*).$$

Then, according to Corollary 1, problem (5.4) has a solution $y \in C(S;V)$ such that $y' \in C(S;H)$ and $y'' \in X^*$.

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A MODIFIED TWO-SIDED APPROXIMATION METHOD FOR A FOUR-POINT VALLÉE–POUSSIN TYPE PROBLEM

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Abstract. We develop a modified two-sided approximation method for a four-point boundary value problem of the Vallée–Poussin type for a system of non-linear differential equations of fourth order with argument deviations.

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1. INTRODUCTION

There are many works dealing with constructive methods for approximate integration of boundary value problems for ordinary differential equations, which allow one to obtain a direct algorithm to error estimation (see, e. g., [4, 10, 11] and references therein). These methods include the two-sided methods, which give provide a possibility to construct approximate solutions and, on every step of iteration, obtain *a posteriori* error estimates of the successive approximations. Numerous research papers are devoted to the construction of new modifications of two-sided methods aimed at the study of various boundary value problems for ordinary differential equations (see, e. g., [1–3, 9].

This paper is devoted to the investigation of a four-point boundary-value problem of the Vallée–Poussin type for a system of non-linear differential equations with argument deviation by using a suitable version of the two-sided method generalising the works [5, 6].

2. PROBLEM SETTINGS, DEFINITIONS AND NOTATIONS

Let us consider the following problem of Vallée-Poussin's type: to find a solution $Y = (y_i)_{i=1}^n$ of the system of differential equations

$$Y^{(4)}(x) = F\left(x, Y(x), (\mathcal{J}_\Lambda Y)(x), (\mathcal{J}_\Theta Y)(x)\right), \quad x \in [0, \ell], \tag{2.1}$$

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which satisfies the conditions

$$Y(0) = A_1, \quad Y(\ell/3) = A_2, \quad Y(2\ell/3) = A_3, \quad Y(\ell) = A_4, \tag{2.2}$$

and

$$Y(x) = \begin{cases} \Phi(x) & \text{if } x \in [\lambda_0, 0], \\ \Psi(x) & \text{if } x \in [\ell, \theta_0], \end{cases} \tag{2.3}$$

where $F\colon [0,\ell]\times\mathbb{R}^{3n} \to \mathbb{R}^n$, the vector-functions $\Lambda = (\lambda_i)_{i=1}^n$ and $\Theta = (\theta_i)_{i=1}^n$ from $C([0,\ell],\mathbb{R}^n)$* are such that $\lambda_i(x) \le x$, $\theta_i(x) \ge x$ for all $x \in [0,\ell]$, $i = \overline{1,n}$,

$$\lambda_0 := \min\left\{\lambda_i(x) \mid x \in [0,\ell],\ i = \overline{1,n}\right\}, \quad \theta_0 := \max\left\{\theta_i(x) \mid x \in [0,\ell],\ i = \overline{1,n}\right\},$$

and $A_s = (a_{is})_{i=1}^n \in \mathbb{R}^n$ for $s = \overline{1,4}$, and $\Phi \in C([\lambda_0, 0], \mathbb{R}^n)$, $\Psi \in C([\ell, \theta_0], \mathbb{R}^n)$ are given initial vector-functions satisfying the conditions

$$\Phi(0) = A_1, \quad \Psi(\ell) = A_4. \tag{2.4}$$

The operator $\mathcal{J}_\Gamma\colon C([\lambda_0,\theta_0],\mathbb{R}^n) \to C([0,\ell],\mathbb{R}^n)$ appearing in (2.1) is defined by the formula

$$(\mathcal{J}_\Gamma Y)(x) := \big(y_i(\gamma_i(x))\big)_{i=1}^n, \quad x \in [0,\ell],$$

for any $\Gamma = (\gamma_i)_{i=1}^n \in C([0,\ell],\mathbb{R}^n)$ and $Y = (y_i)_{i=1}^n \in C([\lambda_0,\theta_0],\mathbb{R}^n)$.

3. Assumptions

In the sequel, let us suppose that the right-hand side $F\colon [0,\ell]\times\mathcal{D}^3 \to \mathbb{R}^n$, $\mathcal{D} \subseteq \mathbb{R}^n$, of the equation (2.1) belongs to the class $\mathcal{M}_{\mathcal{D}}([0,\ell])$, where $\mathcal{M}_{\mathcal{D}}([0,\ell])$ denotes the set of the vector-functions F satisfying the following conditions:

(1) $F \in C([0,\ell]\times\mathcal{D}^3, \mathbb{R}^n)$;
(2) there exists a vector-function $H \in C([0,\ell]\times\mathcal{D}^6, \mathbb{R}^n)$ such that:
   (a) the equality
   $$H(x,U,U) = F(x,U)$$
   holds for all $x \in [0,\ell]$ and $U \in \mathcal{D}^3$;
   (b) the inequality

$$\begin{aligned} &H\big(x, P_1(x), (\mathcal{J}_\Lambda P_1)(x), (\mathcal{J}_\Theta P_1)(x), Q_2(x), (\mathcal{J}_\Lambda Q_2)(x), (\mathcal{J}_\Theta Q_2)(x)\big) \\ &\ge H\big(x, Q_1(x), (\mathcal{J}_\Lambda Q_1)(x), (\mathcal{J}_\Theta Q_1)(x), P_2(x), (\mathcal{J}_\Lambda P_2)(x), (\mathcal{J}_\Theta P_2)(x)\big) \end{aligned} \tag{3.1}$$

   is satisfied for all $x \in [0,\ell]$ and every vector-functions $P_k, Q_k\colon [\tau_0,\theta_0] \to \mathbb{R}^n$, $k = 1,2$, whose restrictions on $[0,\ell]$ belong to $C^4([0,\ell],\mathbb{R}^n)$, such that $P_k(x), Q_k(x) \in \mathcal{D}$ for all $x \in [\lambda_0,\theta_0]$, $k = 1,2$, and

$$\begin{aligned} P_k(x) &\le Q_k(x) && \text{for } x \in [0,\ell/3] \cup [2\ell/3,\ell],\ k = 1,2, \\ P_k(x) &\ge Q_k(x) && \text{for } x \in [\ell/3, 2\ell/3],\ k = 1,2, \\ P_k^{(4)}(x) &\ge Q_k^{(4)}(x) && \text{for } x \in [0,\ell],\ k = 1,2. \end{aligned}$$

*$C([0,\ell],\mathbb{R}^n)$ is the usual Banach space of continuous vector-functions from $[0,\ell]$ to $\mathbb{R}^n$.

(c) the vector-function H satisfies the Lipschitz condition with a non-negative matrix $K = (k_{ij})_{i,j=1}^{n}$, i. e.,

$$|H(x, P_{10}, P_{11}, P_{12}, Q_{10}, Q_{11}, Q_{12}) - H(x, P_{00}, P_{01}, P_{02}, Q_{00}, Q_{01}, Q_{02})| \le K\left(\sum_{s=0}^{2}(|P_{1s} - P_{0s}| + |Q_{1s} - Q_{0s}|)\right), \quad (3.2)$$

for all $P_{s0}, P_{s1}, P_{s2}, Q_{s0}, Q_{s1}, Q_{s2}$ from $\mathcal{D}$, $s = 0, 1$, and all $x \in [0, \ell]$.

In (3.1), (3.2), and all similar relations below, the inequalities between vectors and the absolute value sign are understood component-wise.

4. Preliminary considerations

Due to the fact that the corresponding linearised homogeneous boundary value problem has only the trivial solution on $[0, \ell]$, the solution Y of problem (2.1)–(2.3) can be represented in the form

$$Y(x) = \begin{cases} \Phi(x) & \text{for } x \in [\lambda_0, 0], \\ \Omega(x) - (\mathcal{T}F(\cdot, Y(\cdot), (\mathcal{J}_A Y)(\cdot), (\mathcal{J}_\Theta Y)(\cdot)))(x) & \text{for } x \in [0, \ell], \\ \Psi(x) & \text{for } x \in [\ell, \theta_0], \end{cases} \quad (4.1)$$

where the vector-function $\Omega(x) = (\omega_i(x))_{i=1}^{n}$ has the components

$$\omega_i(x) = a_{i1} + \frac{243}{4\ell^6}\begin{vmatrix} x & 0 & x^2 & x^3 \\ \frac{\ell}{3} & a_{i2} - a_{i1} & \frac{\ell^2}{9} & \frac{\ell^3}{27} \\ \frac{2\ell}{3} & a_{i3} - a_{i1} & \frac{4\ell^2}{9} & \frac{8\ell^3}{27} \\ \ell & a_{i4} - a_{i1} & \ell^2 & \ell^3 \end{vmatrix}, \quad x \in [0, \ell],$$

the operator $\mathcal{T}: C([0, \ell], \mathbb{R}^n) \to C([0, \ell], \mathbb{R}^n)$ for any $Z \in C([0, \ell], \mathbb{R}^n)$ is defined by the formula

$$(\mathcal{T}Z)(x) := \frac{81}{8\ell^6}\int_0^\ell \mathcal{G}(x, \xi) Z(\xi) d\xi, \quad x \in [0, \ell],$$

and $\mathcal{G}$ is the Green function [7, 8] of the problem given by the relations

$$\mathcal{G}(x, \xi) = \begin{cases} \mathcal{G}_1(x, \xi), & 0 \le x \le \frac{\ell}{3}, \\ \mathcal{G}_2(x, \xi), & \frac{\ell}{3} \le x \le \frac{2\ell}{3}, \\ \mathcal{G}_3(x, \xi), & \frac{2\ell}{3} \le x \le \ell, \end{cases} \qquad \mathcal{G}_1(x, \xi) = \begin{cases} R_{11}(x, \xi), & 0 \le \xi \le x, \\ R_{12}(x, \xi), & x \le \xi \le \frac{\ell}{3}, \\ R_{13}(x, \xi), & \frac{\ell}{3} \le \xi \le \frac{2\ell}{3}, \\ R_{14}(x, \xi), & \frac{2\ell}{3} \le \xi \le \ell, \end{cases}$$

$$\mathcal{G}_2(x, \xi) = \begin{cases} R_{21}(x, \xi), & 0 \le \xi \le \frac{\ell}{3}, \\ R_{22}(x, \xi), & \frac{\ell}{3} \le \xi \le x, \\ R_{23}(x, \xi), & x \le \xi \le \frac{2\ell}{3}, \\ R_{24}(x, \xi), & \frac{2\ell}{3} \le \xi \le \ell, \end{cases} \qquad \mathcal{G}_3(x, \xi) = \begin{cases} R_{31}(x, \xi), & 0 \le \xi \le \frac{\ell}{3}, \\ R_{32}(x, \xi), & \frac{\ell}{3} \le \xi \le \frac{2\ell}{3}, \\ R_{33}(x, \xi), & \frac{2\ell}{3} \le \xi \le x, \\ R_{34}(x, \xi), & x \le \xi \le \ell, \end{cases}$$

$$R_{k1}(x,\xi)=\begin{vmatrix} x & (x-\xi)^3 & x^2 & x^3\\ \frac{\ell}{3} & (\frac{\ell}{3}-\xi)^3 & \frac{\ell^2}{9} & \frac{\ell^3}{27}\\ \frac{2\ell}{3} & (\frac{2\ell}{3}-\xi)^3 & \frac{4\ell^2}{9} & \frac{8\ell^3}{27}\\ \ell & (\ell-\xi)^3 & \ell^2 & \ell^3 \end{vmatrix},\ R_{k4}(x,\xi)=\begin{vmatrix} x & 0 & x^2 & x^3\\ \frac{\ell}{3} & 0 & \frac{\ell^2}{9} & \frac{\ell^3}{27}\\ \frac{2\ell}{3} & 0 & \frac{4\ell^2}{9} & \frac{8\ell^3}{27}\\ \ell & (\ell-\xi)^3 & \ell^2 & \ell^3 \end{vmatrix}$$

for $k=\overline{1,3}$,

$$R_{12}(x,\xi)=\begin{vmatrix} x & 0 & x^2 & x^3\\ \frac{\ell}{3} & (\frac{\ell}{3}-\xi)^3 & \frac{\ell^2}{9} & \frac{\ell^3}{27}\\ \frac{2\ell}{3} & (\frac{2\ell}{3}-\xi)^3 & \frac{4\ell^2}{9} & \frac{8\ell^3}{27}\\ \ell & (\ell-\xi)^3 & \ell^2 & \ell^3 \end{vmatrix},\ R_{33}(x,\xi)=\begin{vmatrix} x & (x-\xi)^3 & x^2 & x^3\\ \frac{\ell}{3} & 0 & \frac{\ell^2}{9} & \frac{\ell^3}{27}\\ \frac{2\ell}{3} & 0 & \frac{4\ell^2}{9} & \frac{8\ell^3}{27}\\ \ell & (\ell-\xi)^3 & \ell^2 & \ell^3 \end{vmatrix},$$

and

$$R_{22}(x,\xi)=R_{32}(x,\xi)=\begin{vmatrix} x & (x-\xi)^3 & x^2 & x^3\\ \frac{\ell}{3} & 0 & \frac{\ell^2}{9} & \frac{\ell^3}{27}\\ \frac{2\ell}{3} & (\frac{2\ell}{3}-\xi)^3 & \frac{4\ell^2}{9} & \frac{8\ell^3}{27}\\ \ell & (\ell-\xi)^3 & \ell^2 & \ell^3 \end{vmatrix},$$

$$R_{13}(x,\xi)=R_{23}(x,\xi)=\begin{vmatrix} x & 0 & x^2 & x^3\\ \frac{\ell}{3} & 0 & \frac{\ell^2}{9} & \frac{\ell^3}{27}\\ \frac{2\ell}{3} & (\frac{2\ell}{3}-\xi)^3 & \frac{4\ell^2}{9} & \frac{8\ell^3}{27}\\ \ell & (\ell-\xi)^3 & \ell^2 & \ell^3 \end{vmatrix}.$$

It is easy to see that

$$\mathcal{G}_1(x,\xi)\ge 0,\quad \mathcal{G}_2(x,\xi)\le 0,\quad \mathcal{G}_3(x,\xi)\ge 0\quad \text{for } (x,\xi)\in[0,\ell]\times[0,\ell]. \tag{4.2}$$

Definition. Vector-functions $Z_0, V_0\colon[\lambda_0,\theta_0]\to\mathcal{D}$ whose restrictions on $[0,\ell]$ belong to the space $C^4([0,\ell],\mathbb{R}^n)$ are called *comparison functions of problem* (2.1)–(2.3) if they satisfy the boundary conditions (2.2), the initial condition (2.3), and the inequalities

$$\begin{aligned} &Z_0(x)\le V_0(x) \quad \text{for } x\in[0,\ell/3]\cup[2\ell/3,\ell],\\ &Z_0(x)\ge V_0(x) \quad \text{for } x\in[\ell/3,2\ell/3]. \end{aligned} \tag{4.3}$$

Notation. For any vector-functions $P, Q\colon[\lambda_0,\theta_0]\to\mathbb{R}^n$ we set

$$\langle P,Q\rangle=\big\{u\in\mathbb{R}^n \mid \min\{P(x),Q(x)\}\le u\le\max\{P(x),Q(x)\} \text{ for some } x\in[\lambda_0,\theta_0]\big\},$$

where the operations "min" and "max" for vectors are understood component-wise.

5. Construction of the alternative two-sided method for problem (2.1)–(2.3)

Let us construct the successive approximations $\{Z_p\}_{p=1}^{\infty}$ and $\{V_p\}_{p=1}^{\infty}$ of a solution of problem (2.1)–(2.3) according to the formulae

$$Z_{p+1}(x)=\begin{cases}\Phi(x) & \text{for } x\in[\lambda_0,0],\\ \Omega(x)-(\mathcal{T}F_p)(x) & \text{for } x\in[0,\ell],\\ \Psi(x) & \text{for } x\in[\ell,\theta_0],\end{cases}$$
$$V_{p+1}(x)=\begin{cases}\Phi(x) & \text{for } x\in[\lambda_0,0],\\ \Omega(x)-(\mathcal{T}F^p)(x) & \text{for } x\in[0,\ell],\\ \Psi(x) & \text{for } x\in[\ell,\theta_0],\end{cases} \tag{5.1}$$

where

$$F^p(x)=H\big(x,Z_p(x),(\mathcal{J}_\Lambda Z_p)(x),(\mathcal{J}_\Theta Z_p)(x),V_p(x),(\mathcal{J}_\Lambda V_p)(x),(\mathcal{J}_\Theta V_p)(x)\big),$$
$$F_p(x)=H\big(x,V_p(x),(\mathcal{J}_\Lambda V_p)(x),(\mathcal{J}_\Theta V_p)(x),Z_p(x),(\mathcal{J}_\Lambda Z_p)(x),(\mathcal{J}_\Theta Z_p)(x)\big)$$

for all $x\in[0,\ell]$, and the zero approximations Z_0 and V_0 are comparison functions of problem (2.1)–(2.3) satisfying the conditions

$$\begin{aligned}\alpha_0(x)&:=Z_0^{(4)}(x)-F_0(x)\ge 0,\\ \beta_0(x)&:=V_0^{(4)}(x)-F^0(x)\le 0\end{aligned} \tag{5.2}$$

for all $x\in[0,\ell]$.

The iteration process (5.1) can be represented in the form

$$Z_{p+1}(x)-Z_p(x)=(\mathcal{T}\alpha_p)(x),\quad V_{p+1}(x)-V_p(x)=(\mathcal{T}\beta_p)(x),\quad x\in[0,\ell], \tag{5.3}$$

where

$$\alpha_p(x):=Z_p^{(4)}(x)-F_p(x),\quad \beta_p(x):=V_p^{(4)}(x)-F^p(x),\quad x\in[0,\ell],\ p\in\mathbb{N}. \tag{5.4}$$

Hence, from (5.3) and (5.4), for any $p\in\mathbb{N}\cup\{0\}$, we obtain

$$\alpha_{p+1}(x)=F_p(x)-F_{p+1}(x),\quad \beta_{p+1}(x)=F^p(x)-F^{p+1}(x),\quad x\in[0,\ell], \tag{5.5}$$

$$\begin{aligned}Z_p(x)-Z_{p+2}(x)&=-\mathcal{T}(\alpha_p+\alpha_{p+1})(x),\quad x\in[0,\ell],\\ V_p(x)-V_{p+2}(x)&=-\mathcal{T}(\beta_p+\beta_{p+1})(x),\quad x\in[0,\ell],\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}\alpha_{p+1}(x)+\alpha_{p+2}(x)&=F_p(x)-F_{p+2}(x),\quad x\in[0,\ell],\\ \beta_{p+1}(x)+\beta_{p+2}(x)&=F^p(x)-F^{p+2}(x),\quad x\in[0,\ell].\end{aligned} \tag{5.7}$$

Taking into account conditions (4.2), (5.2), and (5.3) with $p=0$, we can see that

$$\begin{aligned}&Z_1(x)-Z_0(x)\ge 0,\quad V_1(x)-V_0(x)\le 0,\quad x\in[0,\ell/3]\cup[2\ell/3,\ell],\\ &Z_1(x)-Z_0(x)\le 0,\quad V_1(x)-V_0(x)\ge 0,\quad x\in[\ell/3,2\ell/3].\end{aligned} \tag{5.8}$$

Thus, if $Z_1(x), V_1(x) \in \mathcal{D}$ for all $x \in [\lambda_0, \theta_0]$, then from (5.5) with $p = 0$, by virtue of (5.2), (5.8), and (3.1), we obtain $\alpha_1(x) \le 0$, $\beta_1(x) \ge 0$ for all $x \in [0, \ell]$. Therefore, from (4.2) and (5.3) with $p = 1$ we get

$$\begin{aligned} &Z_2(x) - Z_1(x) \le 0, \quad V_2(x) - V_1(x) \ge 0, \qquad x \in [0, \ell/3] \cup [2\ell/3, \ell], \\ &Z_2(x) - Z_1(x) \ge 0, \quad V_2(x) - V_1(x) \le 0, \qquad x \in [\ell/3, 2\ell/3]. \end{aligned} \tag{5.9}$$

Assume, in addition, that

$$\alpha_0(x) + \alpha_1(x) \ge 0, \quad \beta_0(x) + \beta_1(x) \le 0, \quad x \in [0, \ell]. \tag{5.10}$$

Then from (5.6) with $p = 0$ we obtain

$$\begin{aligned} &Z_0(x) - Z_2(x) \le 0, \quad V_0(x) - V_2(x) \ge 0, \quad x \in [0, \ell/3] \cup [2\ell/3, \ell], \\ &Z_0(x) - Z_2(x) \ge 0, \quad V_0(x) - V_2(x) \le 0, \quad x \in [\ell/3, 2\ell/3], \end{aligned} \tag{5.11}$$

and thus (5.9) and (5.11) result in

$$Z_0(x) \le Z_2(x) \le Z_1(x), \quad V_1(x) \le V_2(x) \le V_0(x), \qquad \text{for } x \in [0, \ell/3] \cup [2\ell/3, \ell], \tag{5.12}$$

and

$$Z_1(x) \le Z_2(x) \le Z_0(x), \quad V_0(x) \le V_2(x) \le V_1(x) \quad \text{for } x \in [\ell/3, 2\ell/3]. \tag{5.13}$$

Therefore, we have proved that if $\langle Z_0, Z_1 \rangle \subseteq \mathcal{D}$, $\langle V_1, V_0 \rangle \subseteq \mathcal{D}$, and conditions (5.10) hold, then the values $Z_2(x)$ and $V_2(x)$ of the next approximations which are obtained according to (5.1) also belong to the set $\mathcal{D}$.

From (3.1), (5.10), (5.12), (5.13), and (5.3), (5.5), (5.7) with $p = 2, 1, 0$, we get

$$\begin{aligned} &\alpha_2(x) \ge 0, \quad \beta_2(x) \le 0, \quad x \in [0, \ell], \\ &Z_3(x) - Z_2(x) \ge 0, \quad V_3(x) - V_2(x) \le 0, \quad x \in [0, \ell/3] \cup [2\ell/3, \ell], \\ &Z_3(x) - Z_2(x) \le 0, \quad V_3(x) - V_2(x) \ge 0, \quad x \in [\ell/3, 2\ell/3], \end{aligned}$$

and

$$\alpha_1(x) + \alpha_2(x) \le 0, \quad \beta_1(x) + \beta_2(x) \ge 0, \quad x \in [0, \ell].$$

Hence, from (5.6) with $p = 1$ we obtain

$$\begin{aligned} &Z_1(x) - Z_3(x) \ge 0, \quad V_1(x) - V_3(x) \le 0, \quad x \in [0, \ell/3] \cup [2\ell/3, \ell], \\ &Z_1(x) - Z_3(x) \le 0, \quad V_1(x) - V_3(x) \ge 0, \quad x \in [\ell/3, 2\ell/3]. \end{aligned}$$

Consequently,

$$\begin{aligned} &Z_0(x) \le Z_2(x) \le Z_3(x) \le Z_1(x), \quad V_1(x) \le V_3(x) \le V_2(x) \le V_0(x), \\ &\qquad\qquad x \in [0, \ell/3] \cup [2\ell/3, \ell], \\ &Z_1(x) \le Z_3(x) \le Z_2(x) \le Z_0(x), \quad V_0(x) \le V_2(x) \le V_3(x) \le V_1(x), \\ &\qquad\qquad x \in [\ell/3, 2\ell/3], \end{aligned}$$

and thus $Z_3(x), V_3(x) \in \mathcal{D}$ for all $x \in [\lambda_0, \theta_0]$.

Using the method of the mathematical induction we can show that if $\langle Z_0, Z_1\rangle \subseteq \mathcal{D}$, $\langle V_1, V_0\rangle \subseteq \mathcal{D}$, and conditions (5.10) hold, then the sequences $\{Z_p\}_{p=1}^{\infty}$ and $\{V_p\}_{p=1}^{\infty}$, which are constructed according to (5.1), satisfy the inequalities

$$Z_{2p}(x) \le Z_{2p+2}(x) \le Z_{2p+3}(x) \le Z_{2p+1}(x),$$
$$V_{2p+1}(x) \le V_{2p+3}(x) \le V_{2p+2}(x) \le V_{2p}(x)$$

for $x \in [0, \ell/3] \cup [2\ell/3, \ell]$, $p = 0, 1, 2, \ldots$, and

$$Z_{2p+1}(x) \le Z_{2p+3}(x) \le Z_{2p+2}(x) \le Z_{2p}(x),$$
$$V_{2p}(x) \le V_{2p+2}(x) \le V_{2p+3}(x) \le V_{2p+1}(x)$$

for $x \in [\ell/3, 2\ell/3]$, $p = 0, 1, 2, \ldots$.

Let us now find a sufficient condition for the uniform, on $[\lambda_0, \theta_0]$, convergence of the sequences $\{Z_p\}_{p=1}^{\infty}$ and $\{V_p\}_{p=1}^{\infty}$ to the unique solution of the boundary value problem (2.1)–(2.3).

For any vector $P = (p_i)_{i=1}^{n} \in \mathbb{R}^n$, we set

$$\|P\| := \max_{i=\overline{1,n}} |p_i|.$$

Let us also put

$$W_p(x) := Z_p(x) - V_p(x), \quad x \in [\lambda_0, \theta_0],\ p = 0, 1, 2 \ldots,$$

$$\epsilon := \max_{x\in[0,\ell]} \left\{ \|Z_0(x) - Z_1(x)\|, \|V_0(x) - V_1(x)\|, \|W_0(x)\| \right\},$$

and

$$d := \max_{x\in[0,\ell]} \int_0^{\ell} |\mathcal{G}(x,\xi)| d\xi = \frac{4\ell^{10}}{3^7}.$$

Then using (5.3), (5.5), we can prove by induction the error estimate

$$\max_{x\in[0,\ell]} \left\{ \|Z_{p+1}(x) - Z_p(x)\|, \|V_{p+1}(x) - V_p(x)\| \right\}$$
$$\le \epsilon \left(\frac{81}{8\ell^6} d 6 \|K\| \right)^p = \epsilon \left(\frac{\ell^4}{9} \|K\| \right)^p \quad (5.14)$$

valid for all $p \in \mathbb{N}$, where K is the matrix appearing in the Lipschitz condition (3.2) and $\|K\| = \max_{i=\overline{1,n}} \{\sum_{j=1}^{n} k_{ij}\}$.

If $\|K\|$ satisfies the inequality

$$\|K\| < \frac{9}{\ell^4}, \quad (5.15)$$

then it follows from estimate (5.14) that the approximations $\{Z_p\}_{p=1}^{\infty}$ and $\{V_p\}_{p=1}^{\infty}$ converge, respectively, to certain limits Y_* and Y^* uniformly on $[\lambda_0, \theta_0]$.

Let us show that $Y_*(x) \equiv Y^*(x)$. From (5.1) we have

$$W_{p+1}(x) = \begin{cases} 0 & \text{for } x \in [\lambda_0, 0], \\ \big(\mathcal{T}(F^p - F_p)\big)(x) & \text{for } x \in [0, \ell], \\ 0 & \text{for } x \in [\ell, \theta_0]. \end{cases}$$

It is easy to show that the estimate

$$\max_{x\in[0,\ell]} \big\|W_p(x)\big\| \le \epsilon \left(\frac{81}{8\ell^6}\, d\, 6\|K\|\right)^p = \epsilon \left(\frac{\ell^4}{9}\,\|K\|\right)^p \tag{5.16}$$

is true for $p \in \mathbb{N}$. If condition (5.15) holds, then $\lim_{p\to\infty} W_p(x) = 0$ uniformly on $[0, \ell]$, and thus

$$Y_*(x) = Y^*(x) =: Y(x), \quad x \in [\lambda_0, \theta_0].$$

Passing in equalities (5.1) to the limit as $p \to \infty$, we obtain the equality

$$Y(x) = \begin{cases} \Phi(x) & \text{for } x \in [\lambda_0, 0], \\ \Omega(x) - \big(\mathcal{T}\tilde{H}\big)(x) & \text{for } x \in [0, \ell], \\ \Psi(x) & \text{for } x \in [\ell, \theta_0], \end{cases}$$

where

$$\begin{aligned} \tilde{H}(x) &:= H\big(x, Y(x), (\mathcal{J}_\Lambda Y)(x), (\mathcal{J}_\Theta Y)(x), Y(x), (\mathcal{J}_\Lambda Y)(x), (\mathcal{J}_\Theta Y)(x)\big) \\ &= F\big(x, Y(x), (\mathcal{J}_\Lambda Y)(x), (\mathcal{J}_\Theta Y)(x)\big), \quad x \in [0, \ell], \end{aligned}$$

i. e., Y is a solution of problem (2.1)–(2.3).

The uniqueness of the solution Y under the condition (5.15) can be easily proved by using the Lipschitz condition (3.2).

Consequently, we have proved the following

Theorem. *Let $F \in \mathcal{M}_{\mathcal{D}}([0, \ell])$ and Z_0, V_0 be comparison functions of problem (2.1)–(2.3) satisfying conditions (5.2). In addition, let the first approximations Z_1 and V_1 constructed according to formulae (5.1) be such that $\langle Z_0, Z_1\rangle \subseteq \mathcal{D}$, $\langle V_1, V_0\rangle \subseteq \mathcal{D}$, and conditions (5.10) hold. Assume also that condition (5.15) is satisfied.*

Then the sequences of approximations $\{Z_p\}_{p=1}^{\infty}$ and $\{V_p\}_{p=1}^{\infty}$ constructed according to (5.1) converge uniformly on $[\lambda_0, \theta_0]$ to the unique solution Y of problem (2.1)–(2.3) and, moreover,

$$\begin{aligned} Z_{2p}(x) \le Z_{2p+2}(x) \le Y(x) \le Z_{2p+3}(x) \le Z_{2p+1}(x), \\ V_{2p+1}(x) \le V_{2p+3}(x) \le Y(x) \le V_{2p+2}(x) \le V_{2p}(x) \end{aligned}$$

for $x \in [0, \ell/3] \cup [2\ell/3, \ell]$, $p = 0, 1, 2, \ldots$, and

$$Z_{2p+1}(x) \le Z_{2p+3}(x) \le Y(x) \le Z_{2p+2}(x) \le Z_{2p}(x),$$

$$V_{2p}(x) \le V_{2p+2}(x) \le Y(x) \le V_{2p+3}(x) \le V_{2p+1}(x)$$

for $x \in [\ell/3, 2\ell/3]$, $p = 0, 1, 2, \ldots$.

Remark. If the domain $\mathcal{D}$ is "large" enough, then there exist comparison functions Z_0, V_0 of problem (2.1)–(2.3) satisfying conditions (5.2).

Indeed, let $U : [\lambda_0, \theta_0] \to \mathbb{R}^n$ be an arbitrary vector-function which satisfies the boundary conditions (2.2) and the initial condition (2.3) and is such that $U|_{[0,\ell]} \in C^4([0,\ell], \mathbb{R}^n)$ and $U(x) \in \mathcal{D}$ for all $x \in [\lambda_0, \theta_0]$. Then we set

$$\alpha(x) := U^{(4)}(x) - F\left(x, U(x), (\mathcal{J}_\Lambda U)(x), (\mathcal{J}_\Theta U)(x)\right), \quad x \in [0,\ell]. \tag{5.17}$$

It is clear that the problems

$$\eta^{(4)} = |\alpha(x)|,$$
$$\eta(0) = 0, \quad \eta(\ell/3) = 0, \quad \eta(2\ell/3) = 0, \quad \eta(\ell) = 0$$

and

$$q^{(4)} = -|\alpha(x)|,$$
$$q(0) = 0, \quad q(\ell/3) = 0, \quad q(2\ell/3) = 0, \quad q(\ell) = 0$$

have unique solutions η and q, respectively. Relations (4.1) and (4.2) yield

$$\begin{aligned} &\eta(x) \le 0, \quad q(x) \ge 0, \quad x \in [0,\ell/3] \cup [2\ell/3, \ell], \\ &\eta(x) \ge 0, \quad q(x) \le 0, \quad x \in [\ell/3, 2\ell/3]. \end{aligned} \tag{5.18}$$

Now we put

$$\begin{aligned} &Z_0(x) = U(x) + \eta(x), \quad V_0(x) = U(x) + q(x), \quad x \in [0,\ell], \\ &Z_0(x) = U(x), \qquad\qquad V_0(x) = U(x), \qquad\qquad x \in [\lambda_0, 0] \cup [\ell, \theta_0]. \end{aligned}$$

It is easy to see that Z_0 and V_0 satisfy the boundary conditions (2.2), the initial condition (2.3), and inequalities (4.3). If $Z_0(x), V_0(x) \in \mathcal{D}$ for all $x \in [\lambda_0, \theta_0]$, then Z_0, V_0 are comparison functions of problem (2.1)–(2.3) and, using (5.17), (5.18) and assumptions (2a) and (2b) of Section 3, we get

$$\begin{aligned} Z_0^{(4)}(x) - F_0(x) &= U^{(4)}(x) + |\alpha(x)| - F_0(x) = \\ &= \alpha(x) + |\alpha(x)| + F\left(x, U(x), (\mathcal{J}_\Lambda U)(x), (\mathcal{J}_\Theta U)(x)\right) - F_0(x) \ge 0 \end{aligned}$$

and

$$\begin{aligned} V_0^{(4)}(x) - F^0(x) &= U^{(4)}(x) - |\alpha(x)| - F^0(x) = \\ &= \alpha(x) - |\alpha(x)| + F\left(x, U(x), (\mathcal{J}_\Lambda U)(x), (\mathcal{J}_\Theta U)(x)\right) - F^0(x) \le 0 \end{aligned}$$

for all $x \in [0,\ell]$. Consequently, Z_0 and V_0 also satisfy conditions (5.2).

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BOOK REVIEW: "TOPOLOGICAL DEGREE APPROACH TO BIFURCATION PROBLEMS" BY MICHAL FEČKAN

A. RONTÓ

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Topological bifurcation theory is one of the most essential topics in mathematics. This book contains original bifurcation results for the existence of oscillations and chaotic behaviour of differential equations and discrete dynamical systems under variation of involved parameters. Using topological degree theory and a perturbation approach in dynamical systems, a broad variety of non-linear problems are studied, including: non-smooth mechanical systems with dry frictions; weakly coupled oscillators; systems with relay hysteresis; differential equations on infinite lattices of Frenkel–Kontorova and discretized Klein–Gordon types; blue sky catastrophes for reversible dynamical systems; buckling of beams; and discontinuous wave equations.

Precise and complete proofs, together with concrete applications with many stimulating and illustrating examples, make this book valuable to both the applied sciences and mathematical fields, ensuring the book should not only be of interest to mathematicians but to physicists and theoretically inclined engineers interested in bifurcation theory and its applications to dynamical systems and non-linear analysis.

(Publisher's description)

The theory of bifurcations is one of the most important and actively studied topics in non-linear analysis [1, 5–7, 10, 12–14, 17]. It has numerous applications in physics, biology, chemistry, and economics, where phenomena are often modelled by using differential or difference equations depending on parameters and it is necessary to find out which qualitative changes of the system may be caused by small changes of the parameters involved. The basic approaches here are the topological, analytical, and variational ones. When the non-linearities appearing in the equations under investigation are not smooth enough, the classical approaches relying on the differentiability fail, whereas the topological tools prove to be rather efficient. It is this wide class of bifurcation problems that is investigated in Michal Fečkan's recent monograph [11] published by Springer-Verlag in 2008.

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The book [11] is devoted to the study of various non-smooth bifurcation problems ranging from non-smooth mechanical systems [3] through ordinary differential equations on infinite non-linear lattices [21] up to undamped abstract wave equations in Hilbert spaces with applications to non-linear beam and string partial differential equations [14]. It is organised as follows.

In the introductory Chapter 1, an illustrative example is presented on the bifurcation of chaotic solutions in a differential model describing the apparatus consisting of a slender beam clamped to a rigid framework which supports two magnets. The apparatus is periodically forced using electromagnetic vibration generator. Mathematically, this leads one to the well-known periodically forced and damped Duffing equation. A Melnikov bifurcation function is obtained which determines wedge-shaped regions in the parametric space where the perturbed Duffing equation is chaotic. This example is used to outline main ideas of the approach of the book, which is based on the perturbation method from the theory of dynamical systems, the Lyapunov–Schmidt decomposition method [17, 20], and the Brouwer and Leray–Schauder theories of topological degree [15, 17].

Chapter 2 of [11] contains a brief review of some known mathematical results from functional analysis, differential topology, theory of multivalued mappings, and theory of dynamical systems, which are systematically applied in the proofs of the main results of the monograph.

The main original contribution of the book starts from Chapter 3, where bifurcations of periodic and subharmonic solutions from either periodic or homoclinic trajectories are studied for several types of differential systems, namely, for differential equations with dry frictions, weakly coupled non-linear oscillators, and forced systems with relay hysteresis. As a preparatory step to discover chaos in discontinuous systems, the existence of infinitely many subharmonics with periods tending to infinity is established for weakly discontinuous differential equations. Systems of ordinary differential equations and inclusions containing small parameters multiplying the derivatives are studied (see, e. g., [19, 24]). Among the rest, classical results on the saddle-node and Poincaré–Andronov bifurcations of periodic solutions [12, 18] are extended to non-smooth differential equations. Many concrete examples illustrate the theory.

In Chapter 4, the existence of chaotic solutions of discontinuous differential equations is studied by extending the approach of Chapter 3. The unperturbed system is supposed to have either one homoclinic solution or a manifold filled with such solutions. Melnikov type conditions are established for proving the existence of chaos for perturbed systems. Almost-periodically and quasi-periodically forced differential inclusions are studied as well. The chapter ends with a review of recent results on the homoclinic bifurcation for other types of discontinuous differential equations.

Chapter 5 deals with the study of chaos for diffeomorphisms in the cases where the intersection of the stable and unstable manifolds of a hyperbolic fixed point is

only topologically transversal. Topological degree methods here again prove to be an important tool. An extension of the classical Smale–Birkhoff homoclinic theorem [12] is obtained for the existence of topologically transversal homoclinic and heteroclinic points of diffeomorphisms. Bifurcations of such points are also studied. The rest of the chapter deals with the accumulation of periodic points of reversible diffeomorphisms near homoclinic points, with extensions of this phenomenon, known as the *blue sky catastrophe* [8–10], to chains of time-reversible oscillators.

In Chapter 6, the author studies equations on infinite non-linear lattices obtained as a result of a spatial discretisation of partial differential equations [22]. The persistence of kink travelling waves of partial differential equations under discretisation is investigated. The idea here is to consider a travelling wave equation of the lattice equation as an evolution equation in a suitable Banach space. Then the centre manifold method [12,13,18] is applied to lower the dimension of this infinite dimensional evolution equation. A result on the bifurcation of periodic solutions of certain singularly perturbed ordinary differential equations is obtained, where the small parameter determining the perturbation is the discretisation step size. Statements on the bifurcation of periodic solutions are then used when analysing travelling wave solutions of the sine-Gordon and Klein–Gordon discretised lattice equations [22]. The chapter also contains a review of results on the existence of travelling waves for differential equations on two-dimensional lattices.

Chapter 7 is devoted to the study of the existence of periodic and subharmonic solutions of undamped abstract wave equations. Here, the analysis is complicated by the presence of infinitely many resonant terms, which effect is known as the *problem of small divisors* [16]. To overcome this difficulty, certain Diophantine-type inequalities are introduced. Then using analytical and topological arguments, the bifurcations of subharmonic solutions from homoclinic ones and the bifurcations of periodic solutions from periodic ones are investigated, with applications to several types of periodically forced non-linear beam equations. Weakly non-linear wave equations are also studied.

The final Chapter 8 concerns discontinuous wave equations with infinitely many resonances. The approach here uses that of [2], which is extended to the case of a multi-valued monotone mapping. For this purpose, the notion of a topological degree for monotone multi-valued mappings is constructed by combining the Browder–Skrypnik topological degree theory [4, 23] with Mawhin's coincidence index theory [17]. This topological degree is then used to extend the Krasnoselskii bifurcation theorem and theorems on bifurcations from infinity [1,6,17] to multi-valued operator equations in Hilbert spaces. The resulting abstract bifurcation theorems are applied to study bifurcations of weak forced large-amplitude periodic solutions of discontinuous undamped semilinear wave equations. Finally, the author discusses the possibilities to apply the methods developed in the monograph for detecting the presence of chaos for weakly discontinuous and periodically forced semilinear wave equations.

The book presents to the reader many sophisticated original theorems on a wide range of problems of bifurcation theory. The exposition is based on the author's recently published results that have not yet been included in any other monograph.

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