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A CLASS OF CONCAVE YOUNG FUNCTIONS POSSESSING A POSITIVE FIXED POINT

N. K. AGBEKO

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Abstract. We obtained the class of all concave Young functions which possess a positive fixed point.

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Keywords: concave Young functions, degree of contraction, fixed points

1. INTRODUCTION

Let $\varphi : (0,\infty) \to (0,\infty)$ be a right-continuous and decreasing function such that it is integrable on every finite interval $(0,x)$. It is easily seen that the function $\Phi : [0,\infty) \to [0,\infty)$, defined by the equality

$$\Phi(x) = \int_0^x \varphi(t)\,dt, \tag{1.1}$$

is a nonnegative, increasing and concave function with $\Phi(0) = 0$. We further assume that $\Phi(\infty) = \infty$. Function Φ is thus referred to as a *concave Young function* in the literature, and the set of all such functions will be denoted by $\mathcal{Y}_{\text{conc}}$. For more about these functions see, e. g, [1–3, 5].

In [3], we obtained the following results.

Proposition 1.1. *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ and $s \in (0,\infty)$ be arbitrary. Then*

$$|\Phi(x) - \Phi(y)| \le \varphi(s)\,|x - y|$$

for all numbers $x, y \in (s,\infty)$.

We sought for all those positive numbers that can be a fixed point for a given concave Young function.

Theorem 1.1. *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ and c^* be any positive number. In order that the equality $\Phi(c^*) = c^*$ hold, it is necessary and sufficient that the range of the function $\Phi|_{[c^*,\infty)} : [c^*,\infty) \to [0,\infty)$, defined by $\Phi|_{[c^*,\infty)}(x) = \Phi(x)$, should equal the interval $[c^*,\infty)$.*

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Let $\Phi \in \mathcal{Y}_{\text{conc}}$ with

$$\int_1^{\infty} \frac{\varphi(t)}{t} dt < \infty. \tag{1.2}$$

In [3], a number $c \in (0,\infty)$ was called the degree of contraction of Φ if

$$\int_c^{\infty} \frac{\varphi(t)}{t} dt = 1$$

and

$$\int_c^{bc} \frac{\varphi(t)}{t} dt = \varphi(c)$$

for some $b \in (1,\infty)$. We intend to extend this notion to other concave Young functions which do not possess property (1.2).

2. Main result

Theorem 2.1. *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ be arbitrary with φ denoting its derivative. In order that there be a constant $s > 0$ for which $\varphi(s) < 1$, it is necessary and sufficient that Φ admit a positive fixed point, i. e., $\Phi(x) = x$ for some number $x > 0$.*

Proof. To prove the sufficiency, assume that there is a number $s > 0$ such that $\varphi(s) < 1$. Then by recalling Proposition 1.1 one can easily observe that Φ is a contraction in the interval (s,∞). Consequently, the Contraction Principle [6] yields $\Phi(x) = x$ for some $x \geq s$. Next, let us show the necessity. Assume that there exists some $x_0 > 0$ for which $\Phi(x_0) = x_0$, but in the contrary $\varphi(t) \geq 1$ for all $t > 0$. Then it is easy to check that $\Phi(x) \geq x$ for all $x > 0$. Since Φ is a strictly concave and increasing function, the graph of Φ must lie below that of the line $y = x$ on the interval (x_0,∞). This fact, however, contradicts the inequality $\Phi(x) \geq x$ for all $x > 0$. □

Proposition 2.1. *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ be arbitrary with φ denoting its derivative. If $x_0 \in (0,\infty)$ is such that $\Phi(x_0) = x_0$, then $\varphi(x_0) < 1$.*

Proof. It is not difficult to see that $\Phi(t) \geq t\varphi(t)$ whenever $t \in (0,\infty)$. Assume the existence of some $x_0 \in (0,\infty)$ for which $\Phi(x_0) = x_0$. Then, as noted above,

$$x_0 = \Phi(x_0) \geq x_0 \varphi(x_0),$$

and hence $\varphi(x_0) \leq 1$. Now, suppose that $\varphi(x_0) = 1$. Since φ is a decreasing function on $(0,\infty)$, there must be some $\varepsilon \in (0,1)$ such that $\varphi(x_0 + \varepsilon) < 1$, making Φ be a contraction on $(x_0 + \varepsilon,\infty)$, via Proposition 1.1. But then it would mean that there must be some $x^* \in (x_0 + \varepsilon,\infty)$ with $\Phi(x^*) = x^*$. Necessarily, it would ensue that Φ is not a concave function on the interval $(x_0, x^*]$, a contradiction. Therefore, $\varphi(x_0) < 1$. □

Now, we are in a position to reformulate the definition of the degree of contraction to cover a broader class of concave Young functions.

Definition 2.1. A number $s > 0$ is called the degree of contraction of a function $\Phi \in \mathcal{Y}_{\text{conc}}$ if $\varphi(s) = 1$, where φ is the derivative of Φ.

We note in this case that $\varphi(s+\delta) < 1$ for any positive number δ, which makes Φ be a contraction on the interval $(s+\delta, \infty)$ for some suitable δ.

Example 1. The degree of contraction of $\Phi(x) = 4\sqrt{x+1} - 4$, $x \in [0, \infty)$, equals 3.

Example 2. For any fixed number $p \in (0,1)$, the degree of contraction of the function $\Phi_p(x) = x^p$, $x \in [0, \infty)$ is equal to $p^{1/(1-p)}$.

Example 3. The function $\Phi(x) = \log(x+1)$, $x \in [0, \infty)$, has no degree of contraction.

Example 4. The degree of contraction of function $\Phi(x) = 2\log(x+1)$ exists and equals 1.

Example 5. The concave Young function Φ defined by $\Phi(x) = \dfrac{x}{2} + \sqrt{x}$ does not meet condition (1.2). Yet its degree of contraction exists and equals 1.

An algorithm for finding positive fixed points for concave Young functions:

Step 1: Input $\Phi(x)$ a concave Young function, c_0 a positive number.
Step 2: Compute the derivative $\varphi(x)$ of $\Phi(x)$.
Step 3: Starting from c_0 find an approximate root of the equation $\varphi(x) - 1 = 0$ and put the result into c.
Step 4: If $c = 0$ then STOP else GOTO *Step 5*.
Step 5: Starting from c apply the Fixed Point algorithm, i. e.,
$x_0 := c$; $x_{k+1} := \Phi(x_k)$; $k = k+1$.

3. Concluding remarks

In dynamic models, stationary equilibrium is typically described as a solution of the equation $x = f(x)$, where f is a mapping which determines the current state as a function of the previous state, or as a function of the expected future state. In many cases x is a finite dimensional vector, and in general positive solutions (i. e., fixed points of f) are rather sought for. Problems of this kind have been investigated for decades, and often for concave functions. Alfred Tarski in [7] obtained, in particular, the following result.

Theorem 3.1 (Tarski)**.** *Suppose f is an increasing function from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $f(a) > a$ for some positive vector a, and $f(b) < b$ for some vector $b > a$. Then f has a positive fixed point.*

For the proof we refer the reader, e. g., to [4]. In [4], J. Kennan obtained the result stated below by using Tarski's theorem and [4, Theorem 3.1]. He observed that it gave simple sufficient conditions for the existence and uniqueness of a positive fixed point.

Theorem 3.2 ([4, Theorem 3.3])**.** *Suppose that f is an increasing and stricly concave function from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $f(0) \geq 0$, $f(a) > a$ for some positive vector a, and $f(b) < b$ for some vector $b > a$. Then f has a unique positive fixed point.*

We note that the concavity and increasing property of f mean that every component f_k $(k = 1, \ldots, n)$ of f, considered as a function from $\mathbb{R}^n$ to $\mathbb{R}$, is increasing and strictly concave in every argument $x_j \in \mathbb{R}$, $j = 1, \ldots, n$.

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HIGHER-ORDER BOUNDARY VALUE PROBLEMS FOR CARATHÉODORY DIFFERENTIAL INCLUSIONS

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Abstract. In this paper we prove existence results for boundary value problems for higher-order differential inclusion $x^{(n)}(t) \in F(t, x(t))$ with nonlocal boundary conditions, where F is a compact convex L^1-Carathéodory multifunction.

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Keywords: boundary value problems, Carathéodory multifunction, fixed point

1. INTRODUCTION

The aim of this paper is to establish the existence of solutions of the higher-order boundary value problems

$$\begin{cases} x^{(n)}(t) \in F(t, x(t)) \quad \text{a. e. on } [0,1], \\ x^{(i)}(0) = 0, \quad 0 \le i \le n-2, \\ x(\eta) = x(1) \end{cases} \tag{1.1}$$

with $n \ge 2$,

$$\begin{cases} x^{(n)}(t) \in F(t, x(t)) \quad \text{a. e. on } [0,1], \\ x(0) = x'(\eta), \\ x(1) = x(\tau) \end{cases} \tag{1.2}$$

with $n \ge 2$,

$$\begin{cases} x^{(n)}(t) \in F(t, x(t)) \quad \text{a. e. on } [0,1], \\ x^{(i)}(0) = x^{(i+1)}(\eta), \quad 2 \le i \le n-2, \\ x(0) = x'(\eta), \\ x(1) = x(\tau) \end{cases} \tag{1.3}$$

with $n \ge 4$, and

$$\begin{cases} x^{(n)}(t) \in F(t, x(t)) \quad \text{a. e. on } [0,1], \\ x^{(i)}(0) = x^{(i+1)}(\eta), \quad 0 \le i \le n-2 \end{cases} \tag{1.4}$$

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with $n \geq 2$, where $F:[0,1]\times\mathbb{R}\to 2^{\mathbb{R}}$ is a compact convex L^1-Carathéodory multi-valued map and $(\eta,\tau)\in]0,1[^2$.

Three and four-point boundary value problems for second-order differential inclusions was initiated by Benchohra and Ntouyas [1, 2]. The authors proved the existence of solutions on compact intervals for the problems (1.1) and (1.2) in the particular case $n = 2$. In order to obtain solutions of (1.1) and (1.2), the authors reduce the existence of solutions to the search for fixed points of a suitable multi-valued map on the Banach space $C([0,1],\mathbb{R})$. Indeed, they used the fixed point theorem for condensing maps due to Martelli [5].

In this paper, we extend the results of Benchohra and Ntouyas [1,2] to the nth order boundary value problems and prove the existence of solutions of (1.3) and (1.4). We shall adopt the techniques used by Benchohra and Ntouyas in the previous papers.

2. Preliminaries

Let $(E,\|\cdot\|)$ be a Banach space. We denote by $C([0,1],E)$ the Banach space of continuous functions from $[0,1]$ to E equipped with the norm $\|x\|_\infty := \sup\{\|x(t)\| : t\in[0,1]\}$. A multifunction is said to be measurable if its graph is measurable. For more details on measurability theory, we refer the reader to the book of Castaing and Valadier [3].

Definition 1. A multi-valued map $F:[0,1]\times\mathbb{R}\to 2^{\mathbb{R}}$ is said to be an L^1-Carathéodory if

(i) $t\mapsto F(t,x)$ is measurable for all $x\in\mathbb{R}$;
(ii) $x\mapsto F(t,x)$ is upper semi-continuous for almost all $t\in[0,1]$;
(iii) For each $k>0$, there exists $h_k\in L^1([0,1];\mathbb{R}^+)$ such that

$$\|F(t,x)\| := \sup\{\|y\| : y\in F(t,x)\} \leq h_k(t)$$

for all $\|x\|\leq k$ and for almost all $t\in[0,1]$.

Definition 2. Let E be a separable Banach space, X a nonempty subset of E, and $G:X\to 2^E$ a multi-valued map. We say that:

(1) G is upper semi-continuous on X if for each $x\in X$ the set $G(x)$ is a nonempty closed subset of E and if, for each open set B of E containing $G(x)$, there exists an open neighborhood V of x such that $G(V)\subset B$.
(2) G has a fixed point if there is $x\in X$ such that $x\in G(x)$.
(3) G is said to be completely continuous if $G(B)$ is relatively compact for every B bounded set of X.
(4) If G is upper semi-continuous, it is said to be condensing map if, for any subset $B\subset X$ with $\alpha(B)\neq 0$, we have

$$\alpha(G(B)) < \alpha(B),$$

where α denotes the Kuratowski measure of noncompactness.*

*Note that a completely continuous multivalued map is the easiest example of a condensing map.

It is known that if the multi-valued map G is completely continuous with nonempty compact values, the G is upper semi-continuous if and only if G has a closed graph.

Definition 3. A function $x:[0,1]\to\mathbb{R}$ is said to be solution of (1.1) (resp., (1.2), (1.3), (1.4)) if x is $(n-1)$-times differentiable, $x^{(n-1)}$ is absolutely continuous and x satisfies the conditions of (1.1) (resp., (1.2), (1.3), (1.4)).

Let $\eta\in\mathbb{R}$ and $n\in\mathbb{N}\setminus\{0,1\}$. For the techniques reasoning, we will need, in the sequel, the sequence of functions $(\varphi_p)_{2\le p\le n}$ defined by the following formulas.

For all $t\in[0,1]$, we put

$$\begin{aligned}\varphi_2(t)&=1,\\ \varphi_3(t)&=t+\varphi_2(\eta),\\ \varphi_p(t)&=\frac{t^{p-2}}{(p-2)!}+\sum_{k=3}^{p-1}\varphi_{k-1}(\eta)\frac{t^{p-k}}{(p-k)!}+\varphi_{p-1}(\eta).\end{aligned}$$

Remark 1. The following assertions hold:

(a) For all $t\in[0,1]$ and $k\in\{0,\dots,n-2\}$, $\varphi_n^{(k)}(t)=\varphi_{n-k}(t)$;
(b) For all $k\in\{0,\dots,n-3\}$, $\varphi_{n-k}(0)=\varphi_{n-k-1}(\eta)$;
(c) For all $k\in\{0,\dots,n-2\}$, the function $\varphi_n^{(k)}$ is increasing.

3. Main results

Assume that the following hypotheses hold:

(H_1) $F:[0,1]\times\mathbb{R}\to 2^{\mathbb{R}}$ is an L^1-Carathéodory set-valued map with nonempty compact convex values;

(H_2) There exists a function $m\in L^1([0,1],\mathbb{R}^+)$ such that

$$\|F(t,x)\|\le m(t)$$

for almost all $t\in[0,1]$ and all $x\in\mathbb{R}$.

We shall prove the following main results.

Theorem 1. *If assumptions* (H_1) *and* (H_2) *are satisfied, then problem* (1.1) *has at least one solution on* $[0,1]$.

Theorem 2. *If assumptions* (H_1) *and* (H_2) *are satisfied, then problems* (1.2) *and* (1.3) *have at least one solution on* $[0,1]$.

Theorem 3. *If assumptions* (H_1) *and* (H_2) *are satisfied, then problem* (1.4) *has at least one solution on* $[0,1]$.

4. Proof of the main results

For $y \in C([0,1],\mathbb{R})$, we set

$$S_F(y) := \{g \in L^1([0,1],\mathbb{R}) : g(t) \in F(t,y(t)) \text{ for a. e. } t \in [0,1]\}.$$

In the sequel, we will use the following important lemmas. They will play a crucial role in the proof of the main results.

Lemma 1 ([4]). *If $\dim(E) < \infty$ and $F:[0,1]\times E \to 2^E$ is compact and convex then $S_F(y) \neq \varnothing$ for all $y \in E$.*

Lemma 2 ([4]). *If F satisfies (H_1) and $S_F \neq \varnothing$ then, for any linear continuous mapping $\Gamma: L^1([0,1],E) \to C([0,1],E)$, the convex compact multi-function $\Gamma \circ S_F : C([0,1],E) \to 2^{C([0,1],E)}$ has a closed graph.*

Lemma 3 ([5]). *Let $T: E \to 2^E$ be a convex compact condensing multi-valued mapping. If the set*

$$\Omega := \{y \in E : \lambda y \in T(y) \text{ for some } \lambda > 1\}$$

is bounded, then T has a fixed point.

Proof of Theorem 1. By Lemma 1, for $y \in C([0,1],\mathbb{R})$, $S_F(y)$ is nonempty. Let us transform the problem into a fixed point problem. Consider the multi-valued map $T : C([0,1],\mathbb{R}) \to 2^{C([0,1],\mathbb{R})}$ defined as follows: for $y \in C([0,1],\mathbb{R})$, $T(y)$ is the set of all $z \in C([0,1],\mathbb{R})$ such that

$$z(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} g(s)ds + \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} g(s)ds - \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g(s)ds,$$

where $g \in S_F(y)$. We shall show that T satisfies the assumptions of Lemma 3. The proof will be given in several steps:

Step 1: *$T(y)$ is convex for each $y \in C([0,1],\mathbb{R})$.* Let $h_1, h_2 \in T(y)$, then

$$h_i(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} g_i(s)ds + \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} g_i(s)ds - \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g_i(s)ds,$$

where $g_i \in S_F(y)$ and $i = 1,2$. Let $0 \le \alpha \le 1$. For all $t \in [0,1]$ we have

$$(\alpha h_1 + (1-\alpha)h_2)(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} (\alpha g_1(s) + (1-\alpha)g_2(s))ds + \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} (\alpha g_1(s) + (1-\alpha)g_2(s))ds$$

$$-\frac{t^{n-1}}{1-\eta^{n-1}}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}(\alpha g_1(s)+(1-\alpha)g_2(s))ds.$$

The set $S_F(y)$ is convex because F is convex. Hence $\alpha h_1+(1-\alpha)h_2 \in T(y)$.

STEP 2: *T is bounded on bounded sets of $C([0,1],\mathbb{R})$.* Indeed, it is sufficient to show that $T(B_r)$ is bounded for all $r \geq 0$, where $B_r = \{y \in C([0,1],\mathbb{R}) : \|y\|_\infty \leq r\}$. Let $h \in T(B_r)$. For all $t \in [0,1]$ we have

$$h(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}g(s)ds + \frac{t^{n-1}}{1-\eta^{n-1}}\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!}g(s)ds - \frac{t^{n-1}}{1-\eta^{n-1}}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}g(s)ds,$$

where $y \in B_r$ and $g \in S_F(y)$. Thus, by (H_2),

$$|h(t)| \leq \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}m(s)ds + \frac{t^{n-1}}{1-\eta^{n-1}}\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!}m(s)ds + \frac{t^{n-1}}{1-\eta^{n-1}}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}m(s)ds.$$

Then

$$\|h\|_\infty \leq \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}m(s)ds + \frac{1}{1-\eta^{n-1}}\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!}m(s)ds + \frac{1}{1-\eta^{n-1}}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}m(s)ds.$$

Hence $T(B_r) \subset B_\delta$, where δ is the right-hand term in the above inequality.

STEP 3: *T sends bounded sets of $C([0,1],\mathbb{R})$ into equicontinuous sets.* Indeed, let $h \in T(B_r)$. For all $t \in [0,1]$, we have

$$h(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}g(s)ds + \frac{t^{n-1}}{1-\eta^{n-1}}\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!}g(s)ds - \frac{t^{n-1}}{1-\eta^{n-1}}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}g(s)ds,$$

where $y \in B_r$ and $g \in S_F(y)$. Let $t,s \in [0,1]$ be such that $t < s$. We have

$$\begin{aligned}|h(s)-h(t)| &\leq \int_t^s \frac{(s-\tau)^{n-1}}{(n-1)!}|g(\tau)|d\tau + \int_0^t \frac{(s-\tau)^{n-1}-(t-\tau)^{n-1}}{(n-1)!}|g(\tau)|d\tau \\ &\quad + \frac{s^{n-1}-t^{n-1}}{1-\eta^{n-1}}\left(\int_0^\eta \frac{(\eta-\tau)^{n-1}}{(n-1)!}|g(\tau)|d\tau + \int_0^1 \frac{(1-\tau)^{n-1}}{(n-1)!}|g(\tau)|d\tau\right)\end{aligned}$$

$$\leq \int_t^s \frac{(1-\tau)^{n-1}}{(n-1)!} m(\tau) d\tau + \int_0^1 \frac{(s-\tau)^{n-1} - (t-\tau)^{n-1}}{(n-1)!} m(\tau) d\tau + \frac{s^{n-1} - t^{n-1}}{1-\eta^{n-1}} \left(\int_0^\eta \frac{(\eta-\tau)^{n-1}}{(n-1)!} m(\tau) d\tau + \int_0^1 \frac{(1-\tau)^{n-1}}{(n-1)!} m(\tau) d\tau \right).$$

The right-hand side of the above inequality converges to 0 as s tends to t. Now, by Steps 1, 2, and 3 combined with the Arzelà–Ascoli theorem, we conclude that T is completely continuous.

STEP 4: *T has a closed graph.* Let $(y_p)_p$ a sequence converging to y and consider a sequence $(h_p)_p$ such that $h_p \in T(y_p)$ and $(h_p)_p$ converges to h. We shall prove that $h \in T(y)$. We have

$$h_p(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} g_p(s) ds + \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} g_p(s) ds - \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g_p(s) ds,$$

where $g_p \in S_F(y_p)$.

Now, we consider the linear continuous operator $\Gamma : L^1([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ defined by

$$\Gamma(g)(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} g(s) ds + \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} g(s) ds - \frac{t^{n-1}}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g(s) ds.$$

We have $h_p \in \Gamma \circ S_F(y_p)$. From Lemma 2, $\Gamma \circ S_F$ has a closed graph, then $h \in \Gamma \circ S_F(y)$. Thus, there exists a $g \in S_F(y)$ such that

$$h(t) = \Gamma(g)(t), \qquad t \in [0,1],$$

which implies that $h \in T(y)$. Consequently, T is upper semi-continuous.

STEP 5: *The following set is bounded:*

$$\Omega = \{ y \in C([0,1], \mathbb{R}) : \lambda y \in T(y) \text{ for some } \lambda > 1 \}.$$

Indeed, let $y \in \Omega$. Then

$$y(t) = \lambda^{-1} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} g(s) ds + \frac{\lambda^{-1} t^{n-1}}{1-\eta^{n-1}} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} g(s) ds - \frac{\lambda^{-1} t^{n-1}}{1-\eta^{n-1}} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} g(s) ds.$$

where $g \in S_F(y)$. So, we conclude that

$$\|y\|_\infty \le \lambda^{-1}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}m(s)ds + \frac{\lambda^{-1}}{1-\eta^{n-1}}\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!}m(s)ds + \frac{\lambda^{-1}}{1-\eta^{n-1}}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}m(s)ds.$$

This shows that Ω is bounded. Thus, T satisfies all the conditions of Lemma 3. Therefore, T has a fixed point which is a solution of (1.1). □

Proof of Theorem 2. Transform the problem into a fixed point problem. Set for all $t \in [0,1]$

$$\psi_n^g(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}g(s)ds + \sum_{k=0}^{n-2}\varphi_n^{(k)}(t)\int_0^\eta \frac{(\eta-s)^k}{k!}g(s)ds,$$

where $g \in S_F(y)$. Consider the multi-valued map $T : C([0,1],\mathbb{R}) \to 2^{C([0,1],\mathbb{R})}$ defined as follows: for $y \in C([0,1],\mathbb{R})$, one puts

$$T(y) := \left\{ z \in C([0,1],\mathbb{R}) : z(t) = \psi_n^g(t) + \frac{1+t}{1-\tau}\big(\psi_n^g(\tau) - \psi_n^g(1)\big) \text{ for } t \in [0,1] \right\}.$$

Following the steps of the proof of Theorem 1, we can easily show that T has a fixed point y.

Now we shall show that y is a solution of (1.2) and (1.3). We have

$$y(t) = \psi_n^g(t) + \frac{1+t}{1-\tau}\big(\psi_n^g(\tau) - \psi_n^g(1)\big),$$

where $g \in S_F(y)$. Then $y(1) = y(\tau)$. On the other hand, for $0 \le i \le n-2$ and $t \in [0,1]$, we have

$$\begin{aligned}
[\psi_n^g]^{(i)}(t) &= \int_0^t \frac{(t-s)^{n-i-1}}{(n-i-1)!}g(s)ds + \sum_{k=0}^{n-2}\varphi_n^{(k+i)}(t)\int_0^\eta \frac{(\eta-s)^k}{k!}g(s)ds \\
&= \int_0^t \frac{(t-s)^{n-i-1}}{(n-i-1)!}g(s)ds + \sum_{l=i}^{n+i-2}\varphi_n^{(l)}(t)\int_0^\eta \frac{(\eta-s)^{l-i}}{(l-i)!}g(s)ds \\
&= \int_0^t \frac{(t-s)^{n-i-1}}{(n-i-1)!}g(s)ds + \sum_{l=i}^{n-2}\varphi_n^{(l)}(t)\int_0^\eta \frac{(\eta-s)^{l-i}}{(l-i)!}g(s)ds.
\end{aligned}$$

Then, by Remark 1(a) and (b),

$$\begin{aligned}
[\psi_n^g]^{(i)}(0) &= \int_0^\eta \frac{(\eta-s)^{n-i-2}}{(n-i-2)!}g(s)ds + \sum_{l=i}^{n-3}\varphi_n^{(l)}(0)\int_0^\eta \frac{(\eta-s)^{l-i}}{(l-i)!}g(s)ds \\
&= \int_0^\eta \frac{(\eta-s)^{n-i-2}}{(n-i-2)!}g(s)ds + \sum_{l=i}^{n-3}\varphi_{n-l-1}(\eta)\int_0^\eta \frac{(\eta-s)^{l-i}}{(l-i)!}g(s)ds,
\end{aligned}$$

and, by Remark 1(a),

$$[\psi_n^g]^{(i+1)}(\eta) = \int_0^\eta \frac{(\eta-s)^{n-i-2}}{(n-i-2)!}\, g(s)ds + \sum_{l=i}^{n-3} \varphi_n^{(l+1)}(\eta) \int_0^\eta \frac{(\eta-s)^{l-i}}{(l-i)!}\, g(s)ds$$
$$= \int_0^\eta \frac{(\eta-s)^{n-i-2}}{(n-i-2)!}\, g(s)ds + \sum_{l=i}^{n-3} \varphi_{n-l-1}(\eta) \int_0^\eta \frac{(\eta-s)^{l-i}}{(l-i)!}\, g(s)ds.$$

Consequently,

$$[\psi_n^g]^{(i+1)}(\eta) = [\psi_n^g]^{(i)}(0), \tag{4.1}$$

which implies that $y(0) = y'(\eta)$ and

$$y^{(i)}(0) = y^{(i+1)}(\eta), \qquad 2 \le i \le n-2, \tag{4.2}$$

whenever $n \ge 4$. Finally, it is clear that

$$y^{(n)}(t) = g(t), \qquad t \in [0,1], \tag{4.3}$$

and, hence,

$$y^{(n)}(t) \in F(t, y(t)), \qquad t \in [0,1], \tag{4.4}$$

as required. □

Proof of Theorem 3. Consider the multi-valued map $T : C([0,1],\mathbb{R}) \to 2^{C([0,1],\mathbb{R})}$ defined as follows: for $y \in C([0,1],\mathbb{R})$, one sets

$$T(y) := \{z \in C([0,1],\mathbb{R}) : z(t) = \psi_n^g(t) \text{ for } t \in [0,1]\}.$$

Following the steps of the proof of Theorem 1, we show that T has a fixed point y. Let us show that y is a solution of (1.4). Indeed, by (4.1), we have (4.2). In view of (4.3), it follows that (4.4) also holds. □

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ON THE COMPACTNESS OF THE ATTAINABLE SETS OF CONTROL SYSTEMS WITH p-INTEGRABLE CONTROLS

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Abstract. Quasi-linear systems governed by p-integrable controls, for $1 < p < \infty$ with the constraint $\|u\|_p \leq \mu_0$ are studied. It is proved that the attainable sets of the system at the instant of time t are compact subsets of $\mathbb{R}^n$.

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1. INTRODUCTION

In this paper quasi-linear control systems which are nonlinear with respect to phase state vector, linear with respect to control vector and where control inputs are constrained by an integral inequality are studied. It is well known that attainable sets play an important role in control theory. Many problems of optimization, dynamics, game theory can be stated and solved in terms of attainable sets. There is already much research on the topological and geometric properties of attainable sets. Linear systems are well studied and most properties of attainable sets are proved [4,5]. For nonlinear systems, attainable sets are still an object of active study (see, e. g., [1–3,6–8]).

Consider a control system whose behavior is described by the differential equation

$$x'(t) = f(t,x(t)) + B(t,x(t))u(t), \quad x(t_0) \in X_0, \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the n-dimensional phase state vector of the system, $u \in \mathbb{R}^r$ is the r-dimensional control vector, $t \in [t_0,T]$ $(t_0 < T < \infty)$ is the time, $f(t,x)$ is an n-dimensional vector function, $B(t,x)$ is an $(n \times r)$-dimensional matrix function, and $X_0 \subset \mathbb{R}^n$.

It is assumed that the realizations $u(t)$, $t \in [t_0,T]$, of the control u are restricted by the constraint

$$\int_{t_0}^{T} \|u(t)\|^p dt \leq \mu_0^p, \quad \mu_0 > 0,\ 1 < p < \infty, \tag{1.2}$$

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where $\|\cdot\|$ denotes the Euclidean norm. It is also assumed that the functions $(t,x)\mapsto f(t,x)$, $(t,x)\mapsto B(t,x)$ and the set X_0 satisfy the following conditions:

(A) The set $X_0\subset\mathbb{R}^n$ is compact;

(B) The functions $(t,x)\mapsto f(t,x)$ and $(t,x)\mapsto B(t,x)$ are continuous with respect to (t,x) and locally Lipschitz with respect to x, that is, for any bounded set $D\subset[t_0,T]\times\mathbb{R}^n$, there exist Lipschitz constants $L_i=L_i(D)\in(0,\infty)$, $i=1,2$, such that

$$\|f(t,x^*)-f(t,x_*)\|\leq L_1\|x^*-x_*\|,$$
$$\|B(t,x^*)-B(t,x_*)\|\leq L_2\|x^*-x_*\|$$

for any (t,x^*) and (t,x_*) from D;

(C) There exist constants $\gamma_i\in(0,\infty)$, $i=1,2$, such that

$$\|f(t,x)\|\leq\gamma_1(1+\|x\|),$$
$$\|B(t,x)\|\leq\gamma_2(1+\|x\|)$$

for every $(t,x)\in[t_0,T]\times\mathbb{R}^n$.

Every function $u\in L_p([t_0,T],\mathbb{R}^r)$, $1<p<\infty$, satisfying inequality (1.2) is said to be an *admissible control*, where $L_p([t_0,T],\mathbb{R}^r)$ denotes the space of p-power integrable functions. By the symbol $\mathcal{U}$ we denote the set of all admissible control functions u.

Let $u_*\in\mathcal{U}$. An absolutely continuous function $x_*:[t_0,T]\to\mathbb{R}^n$ which satisfies the equation

$$x_*'(t)=f(t,x_*(t))+B(t,x_*(t))u_*(t)$$

a. e. on $[t_0,T]$ and has the property

$$x_*(t_0)=x_0\in X_0 \tag{1.3}$$

is said to be a *solution* of system (1.1) with the initial condition (1.3) generated by the admissible control function u_*. By the symbol $X(t_0,x_0)$, we denote the set of all solutions of system (1.1) with initial condition $x(t_0)=x_0$, generated by all admissible control functions $u\in\mathcal{U}$. We also put

$$X(t_0,X_0)=\{x\in X(t_0,x_0):x_0\in X_0\},$$
$$X(t;t_0,X_0)=\{x(t)\in\mathbb{R}^n:x\in X(t_0,X_0)\}.$$

The set $X(t;t_0,X_0)$ is called the *attainable set* of the system (1.1) with constraint (1.2) at the instant of time t. It is obvious that the set $X(t;t_0,X_0)$ consists of all $x\in\mathbb{R}^n$, at which the solutions of the system (1.1) which are generated by all possible controls $u\in\mathcal{U}$ arrive at the instant of time $t\in[t_0,T]$.

In this paper we prove that the attainable set $X(t;t_0,X_0)$ defined above is a compact subset of $\mathbb{R}^n$ for all $t\in[t_0,T]$.

2. PRELIMINARIES

First, let us give a useful inequality which will be used frequently in the following sections:

$$\int_{t_0}^{t} (K_1 + K_2\|u(\tau)\|)\,d\tau \le K_1(T-t_0) + K_2(T-t_0)^{\frac{p-1}{p}}\mu_0 \tag{2.1}$$

for every $u \in \mathcal{U}$ and all $t \in [t_0, T]$, where K_1 and K_2 are positive constants. This inequality can be easily obtained from the Hölder integral inequality [9, p. 122].

The following proposition states that the graphs of the all solutions of system (1.1) with constraint (1.2) is bounded.

Proposition 1. *The inequality*

$$\|x(t)\| \le r$$

holds for all $x \in X(t_0, X_0)$ and $t \in [t_0, T]$, where

$$q = \gamma_1(T-t_0) + \gamma_2\mu_0(T-t_0)^{\frac{p-1}{p}}, \tag{2.2}$$

$$d_* = \max\{\|v\| : v \in X_0\}, \tag{2.3}$$

and

$$r = (d_* + q)\exp(q). \tag{2.4}$$

Proof. Let $x \in X(t_0, X_0)$ be any solution of system (1.1). Then there exist $x_0 \in X_0$ and $u \in \mathcal{U}$ such that

$$x(t) = x_0 + \int_{t_0}^{t} \big(f(\tau, x(\tau)) + B(\tau, x(\tau))u(\tau)\big)d\tau, \quad t \in [t_0, T].$$

Calculating the norm of both sides, using condition (C), and recalling that d_* is given by (2.3), we obtain,

$$\|x(t)\| \le d_* + \gamma_1(T-t_0) + \gamma_1\int_{t_0}^{t}\|x(\tau)\|d\tau + \gamma_2\int_{t_0}^{t}\|u(\tau)\|d\tau + \gamma_2\int_{t_0}^{t}\|x(\tau)\|\,\|u(\tau)\|d\tau. \tag{2.5}$$

Since $u \in \mathcal{U}$, using Hölder's integral inequality, we find

$$\int_{t_0}^{t}\|u(\tau)\|d\tau \le \left(\int_{t_0}^{t} 1^{\frac{p}{p-1}}d\tau\right)^{\frac{p-1}{p}}\left(\int_{t_0}^{t}\|u(\tau)\|^p d\tau\right)^{\frac{1}{p}} \le \mu_0(T-t_0)^{\frac{p-1}{p}}. \tag{2.6}$$

By virtue of (2.6) and since $q = \gamma_1(T-t_0) + \gamma_2\mu_0(T-t_0)^{\frac{p-1}{p}}$, we obtain

$$\|x(t)\| \le d_* + q + \int_{t_0}^{t}\big(\gamma_1 + \gamma_2\|u(\tau)\|\big)\|x(\tau)\|d\tau.$$

Using Gronwall's inequality [9, p. 189], we get

$$\|x(t)\| \leq (d_* + q) \exp\left(\int_{t_0}^{t} \Big(\gamma_1 + \gamma_2 \|u(\tau)\| \Big) d\tau \right).$$

In view of inequality (2.1), we find

$$\|x(t)\| \leq (d_* + q) \exp(q).$$

The right hand side of this last inequality is exactly the number r appearing in (2.4), so it is clear that the inequality

$$\|x(t)\| \leq r$$

holds for all $x \in X(t_0, X_0)$ and all $t \in [t_0, T]$. □

Definition. The set

$$Z(t_0, X_0) = \big\{ (t, x(t)) \in [t_0, T] \times \mathbb{R}^n : x \in X(t_0, X_0) \big\}$$

is called the *integral funnel* of system (1.1) with constraint (1.2).

Proposition 1 yields the following corollaries.

Corollary 1. *Graphs of all solutions of system* (1.1) *are bounded by the cylinder*

$$D = \big\{ (t, x) \in [t_0, T] \times \mathbb{R}^n : \|x\| \leq r \big\}. \tag{2.7}$$

In other words, the inclusion $Z(t_0, X_0) \subset D$ *holds.*

Here, $r > 0$ is defined by the formula (2.4). From now on, D will denote the cylinder (2.7).

Corollary 2. *The set of all solutions* $X(t_0, X_0)$ *of system* (1.1) *is a uniformly bounded subset of* $C([t_0, T], \mathbb{R}^n)$. *Furthermore, the attainable sets at the instant of time* t *are bounded subsets of* $\mathbb{R}^n$ *for all* $t \in [t_0, T]$.

Here, $C([t_0, T], \mathbb{R}^n)$ denotes the space of continuous functions $x : [t_0, T] \to \mathbb{R}^n$ with the norm $\|x\|_C = \max_{t \in [t_0, T]} \|x(t)\|$.

3. Compactness of attainable sets

In this section, by means of Ascoli's theorem [9, p. 109], it is proved that the attainable set of the system (1.1) with constraint (1.2) is a compact subset of $\mathbb{R}^n$ at the instant of time $t \in [t_0, T]$. Since we have already shown that the set of all solutions is uniformly bounded, we only need to prove its equicontinuity and relative compactness.

The following proposition states that set of all solutions $X(t_0, X_0)$ is an equicontinuous family of functions in $C([t_0, T], \mathbb{R}^n)$.

Proposition 2. *The set of all solutions* $X(t_0, X_0)$ *of system* (1.1) *with constraint* (1.2) *is an equicontinuous family in* $C([t_0, T], \mathbb{R}^n)$.

Proof. Let $\varepsilon > 0$ and $x \in X(t_0, X_0)$ be any solution of system (1.1). Then there exist $x_0 \in X_0$ and $u \in \mathcal{U}$ such that the equality

$$x(t) = x_0 + \int_{t_0}^{t} \big(f(\tau, x(\tau)) + B(\tau, x(\tau))u(\tau)\big)d\tau$$

holds for all $t \in [t_0, T]$.

Let us choose arbitrary $t_1, t_2 \in [t_0, T]$. Without loss of generality we can suppose that $t_1 \le t_2$. Therefore,

$$\begin{aligned}\|x(t_1) - x(t_2)\| &= \left\| \int_{t_1}^{t_2} [f(\tau, x(\tau)) + B(\tau, x(\tau))u(\tau)]\, d\tau \right\| \\ &\le \int_{t_1}^{t_2} \|f(\tau, x(\tau))\| d\tau + \int_{t_1}^{t_2} \|B(\tau, x(\tau))\|\, \|u(\tau)\| d\tau.\end{aligned}$$

Since the functions f and B are continuous and the set D is compact, we can set

$$\begin{aligned}K_1 &= \max\big\{\|f(t,x)\| : (t,x) \in D\big\}, \\ K_2 &= \max\big\{\|B(t,x)\| : (t,x) \in D\big\}.\end{aligned}$$

Then we obtain

$$\|x(t_1) - x(t_2)\| \le K_1|t_2 - t_1| + K_2 \int_{t_1}^{t_2} \|u(\tau)\| d\tau.$$

Using Hölder's integral inequality we get

$$\begin{aligned}\|x(t_1) - x(t_2)\| &\le K_1|t_2 - t_1| + K_2 \left(\int_{t_1}^{t_2} 1^{\frac{p}{p-1}} d\tau \right)^{\frac{p-1}{p}} \left(\int_{t_1}^{t_2} \|u(\tau)\|^p d\tau \right)^{\frac{1}{p}} \\ &\le K_1|t_2 - t_1| + K_2\mu_0|t_2 - t_1|^{\frac{p-1}{p}}\end{aligned}$$

for all $t_1, t_2 \in [t_0, T]$. Setting $K = \max\{K_1, \mu_0 K_2\}$, we get

$$\begin{aligned}\|x(t_1) - x(t_2)\| &\le K|t_2 - t_1| + K|t_2 - t_1|^{\frac{p-1}{p}} \\ &= K|t_2 - t_1|^{\frac{p-1}{p}} \left(1 + |t_2 - t_1|^{\frac{1}{p}}\right) \le K\left(1 + (T - t_0)^{\frac{1}{p}}\right)|t_2 - t_1|^{\frac{p-1}{p}}\end{aligned}$$

for all $t_1, t_2 \in [t_0, T]$. Setting $K_* = K\left(1 + (T - t_0)^{\frac{1}{p}}\right)$, we find

$$\|x(t_1) - x(t_2)\| \le K_*|t_2 - t_1|^{\frac{p-1}{p}}.$$

Therefore, if t_1 and t_2 are such that $|t_1 - t_2| < \delta(\varepsilon)$ with $\delta(\varepsilon) = (\varepsilon/K_*)^{\frac{p}{p-1}}$, then the inequality

$$\|x(t_1) - x(t_2)\| < \varepsilon$$

holds. Since the element $x \in X(t_0, X_0)$ is arbitrary, this completes the proof of our proposition. □

Combining Corollary 2, Proposition 2, and Ascoli's theorem, we obtain the following corollary.

Corollary 3. *The set of all solutions $X(t_0, X_0)$ of system (1.1) with constraint (1.2) is a relatively compact subset of $C([t_0, T], \mathbb{R}^n)$.*[*]

Proposition 3. *The set of all solutions $X(t_0, X_0)$ of system (1.1) with constraint (1.2) is a closed subset of $C([t_0, T], \mathbb{R}^n)$.*

Proof. Let x_k be any sequence in $X(t_0, X_0)$ and $x_k \to x_*$ as $k \to \infty$. It is necessary to prove that $x_* \in X(t_0, X_0)$.

Since $x_k \in X(t_0, X_0)$ for all $k = 1, 2, \ldots$, there exist $c_k \in X_0$ and $u_k \in \mathcal{U}$ such that

$$x_k(t) = c_k + \int_{t_0}^{t} \big(f(\tau, x_k(\tau)) + B(\tau, x_k(\tau))u_k(\tau)\big)d\tau$$

holds for all $t \in [t_0, T]$. Since $X_0 \subset \mathbb{R}^n$ is compact, the sequence $\{c_k\}_{k=1}^{\infty}$ contains a convergent subsequence. Assume without loss of generality that $c_k \to c_0 \in X_0$ as $k \to \infty$.

In view of the fact that the set of admissible control functions

$$\mathcal{U} = \Big\{u \in L_p([t_0, T], \mathbb{R}^r) : \int_{t_0}^{T} \|u(\tau)\|^p d\tau \le \mu_0^p\Big\},$$

where $\mu_0 > 0$ and $p \in (1, \infty)$, is a weakly compact subset of $L_p([t_0, T], \mathbb{R}^r)$ and $u_k \in \mathcal{U}$ for all $k = 1, 2, \ldots$, the sequence $\{u_k\}_{k=1}^{\infty}$ has weakly convergent subsequence. Suppose without loss of generality that $u_k \overset{w}{\to} u_0 \in \mathcal{U}$ as $k \to \infty$.

By the definition of weak convergence, we have

$$\int_{t_0}^{T} v(\tau)u_k(\tau)d\tau \to \int_{t_0}^{T} v(\tau)u_0(\tau)d\tau$$

as $k \to \infty$ for all $v \in L_q([t_0, T], \mathbb{R}^r)$, where $p^{-1} + q^{-1} = 1$.

Let us denote by x_0 the solution of system (1.1) generated by admissible control $u_0 \in \mathcal{U}$ which starts from the initial point (t_0, c_0). Then, for all $t \in [t_0, T]$, the value $x_0(t)$ can be represented in the form

$$x_0(t) = c_0 + \int_{t_0}^{t} \big(f(\tau, x_0(\tau)) + B(\tau, x_0(\tau))u_0(\tau)\big)d\tau.$$

[*]That is, $X(t_0, X_0)$ is a subset of $C([t_0, T], \mathbb{R}^n)$ with compact closure.

From here and condition (B) we obtain

$$\begin{aligned}\|x_k(t)-x_0(t)\| \le & \|c_k-c_0\| + \int_{t_0}^{t} \|f(\tau,x_k(\tau))-f(\tau,x_0(\tau))\|d\tau \\ & + \left\| \int_{t_0}^{t} [B(\tau,x_k(\tau))u_k(\tau) - B(\tau,x_0(\tau))u_0(\tau)]\,d\tau \right\| \\ \le & \|c_k-c_0\| + L_1 \int_{t_0}^{t} \|x_k(\tau)-x_0(\tau)\|d\tau + \\ & + L_2 \int_{t_0}^{t} \|x_k(\tau)-x_0(\tau)\|\,\|u_k(\tau)\|d\tau + \\ & + \left\| \int_{t_0}^{t} B(\tau,x_0(\tau))\Big(u_k(\tau)-u_0(\tau)\Big)d\tau \right\| \end{aligned} \tag{3.1}$$

for all $k = 1,2,\dots$ and $t \in [t_0,T]$. Since the function $B(\cdot,x_0(\cdot)):[t_0,T] \to \mathbb{R}^{n\times r}$ is continuous and $c_k \to c_0$, $u_k \xrightarrow{w} u_0$ as $k \to \infty$, it follows that, for all $\varepsilon > 0$, there exists a $K(\varepsilon) > 0$ such that the inequality

$$\|c_k-c_0\| + \left\| \int_{t_0}^{t} B(\tau,x_0(\tau))\Big(u_k(\tau)-u_0(\tau)\Big)d\tau \right\| \le \varepsilon, \quad t \in [t_0,T]$$

is valid for all $k \ge K(\varepsilon)$. Using (3.1), we get

$$\|x_k(t)-x_0(t)\| \le \int_{t_0}^{t} \Big(L_1 + L_2\|u_k(\tau)\|\Big)\|x_k(\tau)-x_0(\tau)\|d\tau + \varepsilon$$

for all $k \ge K(\varepsilon)$ and $t \in [t_0,T]$. Applying Gronwall's inequality, we find

$$\|x_k(t)-x_0(t)\| \le \varepsilon \exp\left(\int_{t_0}^{t} \Big(L_1 + L_2\|u_k(\tau)\|\Big)d\tau \right)$$

for all $k \ge K(\varepsilon)$ and $t \in [t_0,T]$.

Using (2.1), we obtain

$$\|x_k(t)-x_0(t)\| \le \varepsilon \exp\Big(L_1(T-t_0) + L_2\mu_0(T-t_0)^{\frac{p-1}{p}} \Big)$$

for all $k \ge K(\varepsilon)$ and $t \in [t_0,T]$. This means that $x_k \to x_0$ as $k \to \infty$.

Since the limit is unique, wesee that the solution x_* must be equal to x_0. Therefore, $x_* \in X(t_0,X_0)$, which implies that the set of all solutions $X(t_0,X_0)$ of system (1.1) is closed. □

Using the closedness of $X(t_0,X_0)$, we also obtain

Corollary 4. *The attainable set $X(t;t_0,X_0)$ of system (1.1) with constraint (1.2) at the instant of time t is closed subset of $\mathbb{R}^n$ for all $t \in [t_0,T]$.*

Combining all the results stated above, we arrive at the following main theorem.

Theorem. *The set of all solutions $X(t_0, X_0)$ and the attainable set $X(t; t_0, X_0)$, at the instant of time t, of system (1.1) with constraint (1.2) are compact subsets of $C([t_0, T], \mathbb{R}^n)$ and $\mathbb{R}^n$, respectively.*

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MORE ON CONTRA δ-PRECONTINUOUS FUNCTIONS

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Abstract. In [4], Dontchev introduced and investigated a new notion of continuity called contra-continuity. Recently, Jafari and Noiri [8–10] introduced new generalizations of contra-continuity called contra-α-continuity, contra-super-continuity and contra-precontinuity. Recently, Ekici and Noiri [6] have introduced a new class of continuity called contra δ-precontinuity as a generalization of contra-continuity. In this paper, we obtain some more properties of contra δ-precontinuous functions.

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1. INTRODUCTION AND PRELIMINARIES

Recently, Jafari and Noiri have introduced and investigated the notions of contra-precontinuity [10], contra-α-continuity [9] and contra-super-continuity [8] as a continuation of research done by Dontchev [4] and Dontchev and Noiri [5] on the interesting notions of contra-continuity and contra-semi-continuity, respectively. Caldas and Jafari [3] introduced and investigated the notion of contra-β-continuous functions in topological spaces. Raychaudhuri and Mukherjee [15] introduced the notions of δ-preopen sets and δ-almost continuity in topological spaces. The class of δ-preopen sets is larger than one of preopen sets. Recently, by using δ-preopen sets, Ekici and Noiri [6] have introduced the notion of contra δ-precontinuity as a generalization of contra-precontinuity.

In this paper, we obtain the further properties of contra δ-precontinuous functions. Throughout this paper, all spaces (X,τ) and (Y,σ) (or X and Y) are topological spaces. A subset A of X is said to be regular open (resp., regular closed) if $A = \mathrm{Int}(\mathrm{Cl}(A))$ (resp., $A = \mathrm{Cl}(\mathrm{Int}(A))$) where $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure and interior of A. A subset A of a space X is called preopen [12] (resp., semi-open [11], α-open [14], β-open [1]) if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$ (resp., $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))))$. The complement of a preopen (resp., semi-open, α-open, β-open) set is said to be preclosed (resp., semi-closed, α-closed, β-closed). The collection of all closed (resp., semi-open, clopen) subsets of X will

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be denoted by $C(X)$ (resp., $SO(X)$, $CO(X)$). We set $C(X,x)=\{V\in C(X)\mid x\in V\}$ for $x\in X$. We define $CO(X,x)$ in a similar way.

The notion of the δ-closure of A which is denoted by $\delta\mathrm{Cl}(A)$ was introduced by Veličko [19] and is widely investigated in the literature. The δ-closure of A is the set $\{x\in X\mid \mathrm{Int}(\mathrm{Cl}(U))\cap A\neq\varnothing$ for every open set U containing $x\}$. If $\delta\mathrm{Cl}(A)=A$, then A is said to be δ-closed [19]. The complement of a δ-closed set is said to be δ-open. The union of all δ-open sets contained in A is called the δ-interior of A and is denoted by $\delta\mathrm{Int}(A)$. A subset A of a topological space X is said to be δ-preopen [15] if $A\subset\mathrm{Int}(\delta\mathrm{Cl}(A))$. The complement of a δ-preopen se is said to be δ-preclosed. The intersection (union) of all δ-preclosed (δ-preopen) sets containing (contained in) A in X is called the δ-preclosure (δ-preinterior) of A and is denoted by $\delta\mathrm{Cl}_p(A)$ (resp., $\delta\mathrm{Int}_p(A)$). By $\delta PO(X)$ (resp., $\delta PC(X)$), we denote the collection of all δ-preopen (resp., δ-preclosed) sets of X.

Lemma 1 ([2,15,17])**.** *The following properties holds for the δ-preclosure of a set in a space X:*

(1) *Arbitrary union (intersection) of δ-preopen (δ-preclosed) sets in X is δ-preopen (resp., δ-preclosed).*
(2) *A is δ-preclosed in X iff $A=\delta\mathrm{Cl}_p(A)$.*
(3) *$\delta\mathrm{Cl}_p(A)\subset\delta\mathrm{Cl}_p(B)$ whenever $A\subset B(\subset X)$.*
(4) *$\delta\mathrm{Cl}_p(A)$ is δ-preclosed in X.*
(5) *$\delta\mathrm{Cl}_p(\delta\mathrm{Cl}_p(A))=\delta\mathrm{Cl}_p(A)$.*
(6) *$\delta\mathrm{Cl}_p(A)=\{x\in X\mid U\cap A\neq\varnothing$ for every δ-preopen set U containing $x\}$.*
(7) *$\delta\mathrm{Cl}_p(A)=A\cup\mathrm{Cl}(\delta\mathrm{Int}(A))$.*
(8) *If A is δ-open, then $\delta\mathrm{Cl}_p(A)=\mathrm{Cl}(A)$.*
(9) *If $Y\subset X$ is δ-open and $U\in\delta PO(Y)$, then $U\in\delta PO(X)$.*
(10) *$U\cap V\in\delta PO(U)$ if U is δ-open and $V\in\delta PO(X)$.*

Definition 1. A function $f:X\to Y$ is said to be contra δ-precontinuous [6] (resp., δ-almost continuous [15]) if $f^{-1}(V)$ is δ-preclosed (resp., δ-preopen) in X for each open set V of Y.

Definition 2. Let A be a subset of a space (X,τ). The set $\cap\{U\in\tau\mid A\subset U\}$ is called the kernel of A [13] and is denoted by $\ker(A)$.

Lemma 2 (Jafari and Noiri [8])**.** *The following properties hold for subsets A, B of a space X:*

(1) *$x\in\ker(A)$ if and only if $A\cap F\neq\varnothing$ for any $F\in C(X,x)$.*
(2) *$A\subset\ker(A)$ and $A=\ker(A)$ if A is open in X.*
(3) *If $A\subset B$, then $\ker(A)\subset\ker(B)$.*

2. Contra δ-precontinuous functions

Theorem 1. *The following assertions are equivalent for a function $f:X\to Y$:*

(1) *f is contra δ-precontinuous.*
(2) *For every closed subset F of Y, $f^{-1}(F) \in \delta PO(X)$.*
(3) *For each $x \in X$ and each $F \in C(Y, f(x))$, there exists $U \in \delta PO(X, x)$ such that $f(U) \subset F$.*
(4) *$f(\delta\mathrm{Cl}_p(A)) \subset \ker(f(A))$ for every subset A of X .*
(5) *$\delta\mathrm{Cl}_p(f^{-1}(B)) \subset f^{-1}(\ker(B))$ for every subset B of Y.*

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4) : Let A be any subset of X. Suppose that $y \notin \ker(f(A))$. Then, by Lemma 2, there exists $F \in C(Y, y)$ such that $f(A) \cap F = \varnothing$. For any $x \in f^{-1}(F)$, by (3) there exists $U_x \in \delta PO(X, x)$ such that $f(U_x) \subset F$. Hence $f(A \cap U_x) \subset f(A) \cap f(U_x) \subset f(A) \cap F = \varnothing$ and $A \cup U_x = \varnothing$. This shows that $x \notin \delta\mathrm{Cl}_p(A)$ for any $x \in f^{-1}(F)$. Therefore, $f^{-1}(F) \cap \delta\mathrm{Cl}_p(A) = \varnothing$ and hence $F \cap f(\delta\mathrm{Cl}_p(A)) = \varnothing$. Thus, $y \notin f(\delta\mathrm{Cl}_p(A))$. Consequently, we obtain $f(\delta\mathrm{Cl}_p(A)) \subset \ker(f(A))$.

(4) $\Rightarrow$ (5) : Let B be any subset of Y. Them, by (4) and Lemma 2, we have $f(\delta\mathrm{Cl}_p(f^{-1}(B))) \subset \ker(f(f^{-1}(B))) \subset \ker(B)$ and therefore $\delta\mathrm{Cl}_p(f^{-1}(B)) \subset f^{-1}(\ker(B))$.

(5) $\Rightarrow$ (1) : Let V be any open set of Y. Then, by virtue of Lemma 2, we have $\delta\mathrm{Cl}_p(f^{-1}(V)) \subset f^{-1}(\ker(V)) = f^{-1}(V)$ and $\delta\mathrm{Cl}_p(f^{-1}(V)) = f^{-1}(V)$. This shows that $f^{-1}(V)$ is δ-preclosed in X. □

The following two examples show that δ-almost continuous and contra δ-precontinuous are independent concepts.

Example 1. The identity function on the real line (with the usual topology) is continuous and hence δ-almost continuous but not contra δ-precontinuous, since the preimage of each singleton fails to be δ-preopen.

Example 2. Let $X = \{a, b\}$ be the Sierpinski space endowed with the topology $\tau = \{\varnothing, \{a\}, X\}$. Let $f : X \to X$ be defined by $f(a) = b$ and $f(b) = a$. Since the inverse image of every open set is δ-preclosed, then f is contra δ-precontinuous, but $f^{-1}(\{a\})$ is not δ-preopen in (X, τ). Therefore f is not δ-almost continuous.

Definition 3. A function $f : X \to Y$ is said to be contra-continuous [4] (resp., contra-α-continuous [9], contra-precontinuous [10], contra-semi-continuous [5], contra-β-continuous [3]) if, for each open set V of Y, $f^{-1}(V)$ is closed (resp., α-closed, preclosed, semi-closed, β-closed) in X.

For the functions defined above, we have the following implications:

$$\begin{array}{ccccccc} A & \Longrightarrow & B & \Longrightarrow & C & \Longrightarrow & D \\ & & \Downarrow & & \Downarrow & & \\ & & C & \Longrightarrow & F & & \end{array}$$

The meaning of symbols here is as follows: A = contra-continuity, B = contra-α-continuity, C = contra-precontinuity, D = contra δ-precontinuity, E = contra-semi-continuity, and F = contra-β-continuity.

It should be mentioned that none of these implications is reversible as shown by the examples stated below.

Example 3 (Jafari and Noiri [9]). Let $X = \{a,b,c\}$. Put $\tau = \{\varnothing, \{a\}, X\}$ and $\sigma = \{\varnothing, \{b\}, \{c\}, \{b,c\}, X\}$. Then the identity function $f:(X,\tau) \to (X,\sigma)$ is contra-α-continuous but not contra-continuous.

Lemma 3 (Caldas et al. [7])**.** *Let A be a subset of (X,τ). Then the following properties hold:*

(1) *If A is preopen in (X,τ), then it is δ-preopen in (X,τ).*
(2) *A is δ-preopen in (X,τ) if and only if it is preopen in (X,τ_s).*
(3) *A is δ-preclosed in (X,τ) if and only if it is preclosed in (X,τ_s) .*

Since $\mathrm{Cl}(A) \subset \delta\,\mathrm{Cl}(A)$ for any subset A of X, therefore, every contra-precontinuous is contra-δ-precontinuous but not conversely as following example shows.

Example 4 ([5]). A contra-semi-continuous function need not be contra-precontinuous. Let $f:R \to R$ be the function $f(x) = [x]$, where $[x]$ is the Gaussian symbol. If V is a closed subset of the real line, its preimage $U = f^{-1}(V)$ is the union of the intervals of the form $[n, n+1]$, $n \in Z$; hence U is semi-open being union of semi-open sets. But f is not contra-precontinuous, because $f^{-1}(0.5, 1.5) = [1,2)$ is not preclosed in R.

Example 5 ([5]). A contra-precontinuous function need not be contra-semi-continuous. Let $X = \{a,b\}$, $\tau = \{\varnothing, X\}$ and $\sigma = \{\varnothing, \{a\}, X\}$. The identity function $f:(X,\tau) \to (Y,\sigma)$ is contra-precontinuous as only the trivial subsets of X are open in (X,τ). However, $f^{-1}(\{a\}) = \{a\}$ is not semi-closed in (X,τ); hence f is not contra-semi-continuous.

Example 6 ([6]). Let R be the set of real numbers, τ be the countable extension topology on R, i. e., the topology with subbase $\tau_1 \cup \tau_2$, where τ_1 is the Euclidean topology of R and τ_2 is the topology of countable complements of R, and σ be the discrete topology of R. Define a function $f:(R,\tau) \to (R,\sigma)$ as follows: $f(x) = 1$ if x is rational, and $f(x) = 2$ if x is irrational. Then f is contra δ-precontinuous but not contra-β-continuous, because $\{1\}$ is closed in (R,σ) and $f^{-1}(\{1\}) = Q$, where Q is the set of rationals, is not β-open in (R,τ).

Example 7 ([3]). Let $X = \{a,b,c\}$, $\tau = \{\varnothing, \{a\}, \{b\}, \{a,b\}, X\}$ and $Y = \{p,q\}$, $\sigma = \{\varnothing, \{p\}, Y\}$. Let $f:(X,\tau) \to (Y,\sigma)$ be defined by $f(a) = p$ and $f(b) = f(c) = q$. Then f is contra-β-continuous but not contra δ-precontinuous since $f^{-1}(\{q\}) = \{b,c\}$ is β-open but not δ-preopen.

Theorem 2. *If a function $f:X \to Y$ is contra δ-precontinuous and Y is regular, then f is δ-almost continuous.*

Proof. Let x be an arbitrary point of X and V an open set of Y containing $f(x)$. Since Y is regular, there exists an open set W in Y containing $f(x)$ such that $\mathrm{Cl}(W) \subset V$. Since f is contra δ-precontinuous, so by Theorem 1 there exists $U \in \delta PO(X,x)$ such that $f(U) \subset \mathrm{Cl}(W)$. Then $f(U) \subset \mathrm{Cl}(W) \subset V$. Hence, f is δ-almost continuous. □

The converse of Theorem 2 is not true. Example 1 shows that δ-almost continuity does not necessarily imply contra δ-precontinuity even if the range is regular.

Definition 4. A function $f: X \to Y$ is said to be:

(1) (δ, s)-preopen if $f(U) \in SO(Y)$ for every δ-preopen set of X.
(2) contra-$I(\delta, p)$-continuous if for each $x \in X$ and each $F \in C(Y, f(x))$, there exists $U \in \delta PO(X,x)$ such that $\mathrm{Int}(f(U)) \subset F$.

Theorem 3. *If a function $f: X \to Y$ is contra-$I(\delta, p)$-continuous and (δ, s)-preopen, then f is contra δ-precontinuous.*

Proof. Suppose that $x \in X$ and $F \in C(Y, f(x))$. Since f is contra-$I(\delta, p)$-continuous, there exists $U \in \delta PO(X,x)$ such that $\mathrm{Int}(f(U)) \subset F$. By hypothesis f is (δ, s)-preopen, therefore $f(U) \in SO(Y)$ and $f(U) \subset \mathrm{Cl}(\mathrm{Int}(f(U)) \subset F$. This shows that f is contra δ-precontinuous. □

Definition 5. A space (X, τ) is said to be:

(1) locally (δ, p)-indiscrete if every δ-preopen set of X is closed in X.
(2) δp-space if every δ-preopen set of X is open in X.
(3) δS-space if and only if every δ-preopen subset of X is semi-open.

The following theorem follows immediately from Definition 5.

Theorem 4. *If a function $f: X \to Y$ is contra δ-precontinuous and X is a δS-space (resp., δp-space, locally (δ, p)-indiscrete), then f is contra-semi-continuous (resp., contra-continuous, continuous).*

Recall that a topological space is said to be:

(1) (δ, p)-T_2 ([16]) if for each pair of distinct points x and y in X there exist $U \in \delta PO(X,x)$ and $V \in \delta PO(X,y)$ such that $U \cap V = \varnothing$.
(2) Ultra Hausdorff [18] if for each pair of distinct points x and y in X there exist $U \in CO(X,x)$ and $V \in CO(X,y)$ such that $U \cap V = \varnothing$.

Theorem 5. *If X is a topological space and for each pair of distinct points x_1 and x_2 in X there exists a map f of X into a Urysohn topological space Y such that $f(x_1) \neq f(x_2)$ and f is contra δ-precontinuous at x_1 and x_2, then X is (δ, p)-T_2.*

Proof. Let x_1 and x_2 be any distinct points in X. Then by hypothesis, there is a Urysohn space Y and a function $f: X \to Y$, which satisfies the conditions of the theorem. Let $y_i = f(x_i)$ for $i = 1, 2$. Then $y_1 \neq y_2$. Since Y is Urysohn, there

exist open neighbourhoods U_{y_1} and U_{y_2} of y_1 and y_2 respectively in Y such that $\mathrm{Cl}(U_{y_1}) \cap \mathrm{Cl}(U_{y_2}) = \varnothing$. Since f is contra δ-precontinuous at x_i, there exists a δ-preopen neighbourhood W_{x_i} of x_i in X such that $f(W_{x_i}) \subset \mathrm{Cl}(U_{y_i})$ for $i = 1,2$. Hence we get $W_{x_1} \cap W_{x_2} = \varnothing$ because $\mathrm{Cl}(U_{y_1}) \cap \mathrm{Cl}(U_{y_2}) = \varnothing$. Then X is (δ, p)-T_2. □

Corollary 1. *If f is a contra δ-precontinuous injection of a topological space X into a Urysohn space Y, then X is (δ, p)-T_2.*

Proof. For each pair of distinct points x_1 and x_2 in X, f is a contra δ-precontinuous function of X into a Urysohn space Y such that $f(x_1) \neq f(x_2)$ because f is injective. Hence by Theorem 5, X is (δ, p)-T_2 . □

Corollary 2. *If f is a contra δ-precontinuous injection of a topological space X into a Ultra Hausdorff space Y, then X is (δ, p)-T_2.*

Proof. Let x_1 and x_2 be any distinct points in X. Then since f is injective and Y is Ultra Hausdorff $f(x_1) \neq f(x_2)$, and there exist $V_1, V_2 \in CO(Y)$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $V_1 \cap V_2 = \varnothing$. Then $x_i \in f^{-1}(V_i) \in \delta PO(X)$ for $i = 1,2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \varnothing$. Thus X is (δ, p)-T_2. □

Lemma 4 ([15])**.** *If A_i is a δ-preopen set in a topological space X_i for $i = 1,2,\ldots,n$, then $A_1 \times \cdots \times A_n$ is also δ-preopen in the product space $X_1 \times \cdots \times X_n$.*

Theorem 6. *Let $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ be two functions, where*

(1) *Y is a Urysohn space,*
(2) *f_1 and f_2 are contra δ-precontinuous.*

Then the set

$$\{(x_1, x_2) \mid f_1(x_1) = f_2(x_2)\}$$

is δ-preclosed in the product space $X_1 \times X_2$.

Proof. Let A denote the set $\{(x_1, x_2) \mid f_1(x_1) = f_2(x_2)\}$. In order to show that A is δ-preclosed, we show that $(X_1 \times X_2) \setminus A$ is δ-preopen. Let $(x_1, x_2) \notin A$. Then $f_1(x_1) \neq f_2(x_2)$. Since Y is Urysohn, there exist open V_1 and V_2 of $f_1(x_1)$ and $f_2(x_2)$ such that $C(V_1) \cap C(V_2) = \varnothing$. Since f_i $(i = 1,2)$ is contra δ-precontinuous, $f_i^{-1}(C(V_i))$ is a δ-preopen set containing x_i in X_i $(i = 1,2)$. Hence, by virtue of Lemma 4, $f_1^{-1}(C(V_1)) \times f_2^{-1}(C(V_2))$ is δ-preopen. Further $(x_1, x_2) \in f_1^{-1}C(V_1) \times f_2^{-1}C(V_2) \subset (X_1 \times X_2) \setminus A$. It follows that $(X_1 \times X_2) \setminus A$ is δ-preopen. Thus A is δ-preclosed in the product space $X_1 \times X_2$. □

Corollary 3. *If $f : X \to Y$ is contra δ-precontinuous and Y is a Urysohn space, then*

$$A = \{(x_1, x_2) \mid f(x_1) = f(x_2)\}$$

is δ-preclosed in the product space $X_1 \times X_2$.

Definition 6. A topological space X is said to be:

(1) (δ, p)-normal if each pair of non-empty disjoint closed sets can be separated by disjoint δ-preopen sets.
(2) Ultra normal [18] if each pair of non-empty disjoint closed sets can be separated by disjoint clopen sets.

Theorem 7. *If $f: X \to Y$ is a contra δ-precontinuous, closed injection and Y is ultra normal, then X is (δ, p)-normal.*

Proof. Let F_1 and F_2 be disjoint closed subsets of X. Since f is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of Y. Since Y is ultra normal $f(F_1)$ and $f(F_2)$ are separated by disjoint clopen sets V_1 and V_2, respectively. Hence $F_i \subset f^{-1}(V_i)$, $f^{-1}(V_i) \in \delta PO(X, x)$ for $i = 1, 2$, and

$$f^{-1}(V_1) \cap f^{-1}(V_2) = \varnothing.$$

Thus, X is (δ, p)-normal. □

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ARCWISE CONNECTEDNESS OF THE SOLUTION SET OF A NONCLOSED NONCONVEX INTEGRAL INCLUSION

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Abstract. We consider a class of nonconvex nonclosed integral inclusions and we prove the arcwise connectedness of the solution set of this problem.

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Keywords: integral inclusion, set-valued contraction, fixed point

1. INTRODUCTION

This note is concerned with the following Fredholm type integral inclusion

$$x(t) = \lambda(t) + \int_0^1 f(t,s,u(s))ds, \tag{1.1}$$

$$u(t) \in F(t,x(t),G(t,x(t))) \quad \text{a. e. in } I, \tag{1.2}$$

where X is a separable Banach space, $I := [0,1]$, $F: I \times X \times X \to \mathcal{P}(X)$, $G: I \times X \to \mathcal{P}(X)$, $\lambda: I \to X$, and $f: I \times I \times X \to X$.

When F does not depend on the last variable, (1.2) reduces to

$$u(t) \in F(t,x(t)). \tag{1.3}$$

In this case, existence results for problem (1.1), (1.3) and the stability of the solution set with respect to small perturbations of the free term are obtained via fixed point techniques in Theorem 2.8.9 in [6] (see also references therein). In [6], the set-valued map F is assumed to be at least closed-valued. Such an assumption is quite natural in order to obtain good properties of the solution set, but it is also interesting to investigate the problem when the right-hand side of (1.3) may have nonclosed values.

Following the approach in [4, 5], we consider the problem (1.1), (1.2), where F and G are closed-valued set-valued maps, Lipschitzian with respect to the second variable and F is contractive in the third variable. Obviously, the right-hand side of the inclusion in (1.2) is in general neither convex nor closed. We prove the arcwise connectedness of the solution set to (1.1), (1.2). The main tool is a result [4, 5]

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concerning the arcwise connectedness of the fixed point set of a class of nonconvex nonclosed set-valued contractions.

This idea was already used for similar results for other classes of differential inclusions [1–4]. We note that a similar result for Volterra type integral inclusions is obtained in [1], but it is well known that, in general, the results for Fredholm integral equations cannot be obtained from the corresponding results for Volterra integral equations.

The paper is organized as follows: in Section 2, we recall some preliminary results that we use in the sequel and, in Section 3, we prove our main result.

2. Preliminaries

Let Z be a metric space with the distance d_Z and let $\mathcal{P}(Z)$ be the family of all nonempty closed subsets of Z. For $a \in Z$ and $A, B \in \mathcal{P}(Z)$, set $d_Z(a,B) = \inf_{b\in B} d_Z(a,b)$ and $d_Z^*(A,B) = \sup_{a\in A} d_Z(a,B)$. Denote by D_Z the Pompeiu–Hausdorff generalized metric on $\mathcal{P}(Z)$ defined by

$$D_Z(A,B) = \max\left\{d_Z^*(A,B), d_Z^*(B,A)\right\}, \quad A,B \in \mathcal{P}(Z).$$

In what follows, when the product $Z = Z_1 \times Z_2$ of metric spaces Z_i, $i = 1,2$, is considered, it is assumed that Z is equipped with the distance

$$d_Z\big((z_1,z_2),(z_1',z_2')\big) = \sum_{i=1}^{2} d_{Z_i}(z_i,z_i').$$

Let X be a nonempty set and let $F\colon X \to \mathcal{P}(Z)$ be a set-valued map from X to Z. The range of F is the set $F(X) = \bigcup_{x\in X} F(x)$. Let $(X,\mathcal{F})$ be a measurable space. The set-valued map $F\colon X \to \mathcal{P}(Z)$ is called measurable if $F^{-1}(\Omega) \in \mathcal{F}$ for any open set $\Omega \subset Z$, where $F^{-1}(\Omega) = \left\{x \in X; F(x) \cap \Omega \neq \varnothing\right\}$. Let (X,d_X) be a metric space. The set-valued map F is called Hausdorff continuous if, for any $x_0 \in X$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$, $d_X(x,x_0) < \delta$ implies $D_Z(F(x),F(x_0)) < \varepsilon$.

Let $(T,\mathcal{F},\mu)$ be a finite, positive, nonatomic measure space and let $(X,|\cdot|_X)$ be a Banach space. We denote by $L^1(T,X)$ the Banach space of all (equivalent classes of) Bochner integrable functions $u\colon T \to X$ endowed with the norm

$$\|u\|_1 = \int_T |u(t)|_X \, d\mu.$$

A nonempty set $K \subset L^1(T,X)$ is called decomposable if, for every $u,v \in K$ and every $A \in \mathcal{F}$, one has

$$\chi_A \cdot u + \chi_{T\setminus A} \cdot v \in K$$

where χ_B indicates the characteristic function of $B \in \mathcal{F}$.

A metric space Z is called an absolute retract if, for any metric space X and any nonempty closed set $X_0 \subset X$, every continuous function $g\colon X_0 \to Z$ has a continuous

extension $g: X \to Z$ over X. It is obvious that every continuous image of an absolute retract is an arcwise connected space.

In what follows we recall some preliminary results that are the main tools in the proof of our result.

Let $(T, \mathcal{F}, \mu)$ be a finite, positive, nonatomic measure space, S a separable Banach space and let $(X, |\cdot|_X)$ be a real Banach space. To simplify the notation we write E in place of $L^1(T, X)$.

Lemma 1 ([4]). *Assume that $\varphi: S \times E \to \mathcal{P}(E)$ and $\psi: S \times E \times E \to \mathcal{P}(E)$ are Hausdorff continuous set-valued maps with nonempty, closed, decomposable values, satisfying the following conditions:*

(a) *There exists $L \in [0,1)$ such that, for every $s \in S$ and every $u, u' \in E$,*

$$D_E(\varphi(s,u), \varphi(s,u')) \leq L\|u - u'\|_E.$$

(b) *There exists $M \in [0,1)$ such that $L + M < 1$ and for every $s \in S$ and every $(u,v), (u',v') \in E \times E$,*

$$D_E(\psi(s,u,v), \psi(s,u',v')) \leq M(\|u - u'\|_E + \|v - v'\|_E).$$

Set $\operatorname{Fix}\big(\Gamma(s,\cdot)\big) = \{u \in E; u \in \Gamma(s,u)\}$, *where* $\Gamma(s,u) = \psi(s,u,\varphi(s,u))$, $(s,u) \in S \times E$. *Then*

(1) *For every $s \in S$ the set* $\operatorname{Fix}\big(\Gamma(s,\cdot)\big)$ *is nonempty and arcwise connected.*

(2) *For any $s_i \in S$ and any $u_i \in \operatorname{Fix}\big(\Gamma(s,\cdot)\big)$, $i = 1, \ldots, p$, there exists a continuous function $\gamma: S \to E$ such that $\gamma(s) \in \operatorname{Fix}\big(\Gamma(s,\cdot)\big)$ for all $s \in S$ and $\gamma(s_i) = u_i$, $i = 1, \ldots, p$.*

Lemma 2 ([4]). *Let $U: T \to \mathcal{P}(X)$ and $V: T \times X \to \mathcal{P}(X)$ be two nonempty closed-valued set-valued maps satisfying the following conditions:*

(a) *U is measurable and there exists $r \in L^1(T, \mathbb{R})$ such that $D_X(U(t), \{0\}) \leq r(t)$ for almost all $t \in T$.*

(b) *The set-valued map $t \mapsto V(t,x)$ is measurable for every $x \in X$.*

(c) *The set-valued map $x \mapsto V(t,x)$ is Hausdorff continuous for all $t \in T$.*

Let $v: T \to X$ be a measurable selection from $t \mapsto V(t, U(t))$.

Then there exists a selection $u \in L^1(T, X)$ such that $v(t) \in V(t, u(t))$, $t \in T$.

Let X be a separable Banach space, $C(I, X)$ the Banach space of continuous functions defined on I with values in X endowed with the norm $\|x\|_C = \sup_{t \in I} |x(t)|$. We consider the problem (1.1), (1.2) defined by the mappings $F: I \times X \times X \to \mathcal{P}(X)$, $G: I \times X \to \mathcal{P}(X)$, $\lambda: I \to X$, and $f: I \times I \times X \to X$. A function $x \in C(I, X)$ is said to be a solution of (1.1), (1.2) if there exists a function $u \in L^1(I, X)$ such that (1.1), (1.2) are satisfied.

In order to study problem (1.1), (1.2) we introduce the following hypothesis.

Hypothesis 1. *Let $\lambda \in C(I,X)$ and $F: I \times X \times X \to \mathcal{P}(X)$, $G: I \times X \to \mathcal{P}(X)$ be two set-valued maps with nonempty closed values, satisfying the following assumptions:*

(i) *The set-valued maps $t \mapsto F(t,u,v)$ and $t \mapsto G(t,u)$ are measurable for all $u, v \in X$.*

(ii) *There exist $l \in L^1(I,\mathbb{R})$ such that, for arbitrary $u, u' \in X$,*

$$D\big(G(t,u), G(t,u')\big) \leq l(t)|u-u'| \quad \text{a. e. in } I.$$

(iii) *There exist $m \in L^1(I,\mathbb{R})$ and $\theta \in [0,1)$ such that, for arbitrary $u, v, u', v' \in X$,*

$$D\big(F(t,u,v), F(t,u',v')\big) \leq m(t)|u-u'| + \theta|v-v'| \quad \text{a. e. in } I.$$

(iv) *There exist $g_1, g_2 \in L^1(I,\mathbb{R})$ such that*

$$d\big(\{0\}, F(t,\{0\},\{0\})\big) \leq g_1(t), \quad d\big(\{0\}, G(t,\{0\})\big) \leq g_2(t) \quad \text{a. e. in } I.$$

(v) *$f: I \times I \times X \to X$ is a continuous and bounded function and there exists a constant $M > 0$ such that*

$$|f(t,s,u_1) - f(t,s,u_2)| \leq M|u_1 - u_2| \quad \forall u_1, u_2 \in X.$$

For $\lambda \in C(I,X)$, we denote by $S(\lambda)$ the solution set of (1.1), (1.2). In what follows, $N(t) = \max\{l(t), m(t)\}$, $t \in I$, and $N^* = \int_0^1 N(s)ds$.

3. The main result

Even if the set-valued map from the right-hand side of (1.1), (1.2) has, in general, nonclosed nonconvex values, the solution set $S(\lambda)$ has some meaningful properties, stated in Theorem 1 below.

Theorem 1. *Assume that Hypothesis 1 is satisfied and $2N^*M + \theta < 1$. Then:*

(1) *For every $\lambda \in C(I,X)$, the solution set $S(\lambda)$ of (1.1), (1.2) is nonempty and arcwise connected in the space $C(I,X)$.*

(2) *For any $\lambda_i \in C(I,X)$ and any $x_i \in S(\lambda_i)$, $i = 1, \ldots, p$, there exists a continuous function $s: C(I,X) \to C(I,X)$ such that $s(\lambda) \in S(\lambda)$ for any $\lambda \in C(I,X)$ and $s(\lambda_i) = x_i$, $i = 1, \ldots, p$.*

(3) *The set $S = \cup_{\lambda \in C(I,X)} S(\lambda)$ is arcwise connected in $C(I,X)$.*

Proof. (1) For $\lambda \in C(I,X)$ and $u \in L^1(I,X)$, set

$$x_\lambda(t) = \lambda(t) + \int_0^1 f(t,s,u(s))ds, \quad t \in I.$$

We shall prove that the set-valued maps $\varphi: C(I,X) \times L^1(I,X) \to \mathcal{P}(L^1(I,X))$ and $\psi: C(I,X) \times L^1(I,X) \times L^1(I,X) \to \mathcal{P}(L^1(I,X))$, given by the formulas

$$\varphi(\lambda,u) = \big\{v \in L^1(I,X);\ v(t) \in G(t, x_\lambda(t)) \text{ a. e. in } I\big\}$$

and

$$\psi(\lambda,u,v)=\{w\in L^1(I,X);\ w(t)\in F(t,x_\lambda(t),v(t))\ \text{a. e. in } I\}$$

for $\lambda\in C(I,X)$ and $u,v\in L^1(I,X)$, satisfy the hypotheses of Lemma 1.

Since x_λ is measurable and G satisfies Hypothesis 1(i) and (ii), the set-valued map $t\mapsto G(t,x_\lambda(t))$ is measurable and nonempty closed-valued, it has a measurable selection. Therefore, due to Hypothesis 1(iv), the set $\varphi(\lambda,u)$ is nonempty. The fact that the set $\varphi(\lambda,u)$ is closed and decomposable follows by simple computation. In the same way we obtain that $\psi(\lambda,u,v)$ is a nonempty closed decomposable set.

Pick $(\lambda,u),(\lambda_1,u_1)\in C(I,X)\times L^1(I,X)$ and choose $v\in\varphi(\lambda,u)$. For each $\varepsilon>0$ there exists $v_1\in\varphi(\lambda_1,u_1)$ such that, for every $t\in I$, one has

$$\begin{aligned}|v(t)-v_1(t)|&\le D\big(G(t,x_\lambda(t)),G(t,u_{\lambda_1}(t))\big)+\varepsilon\\&\le N(t)\left(|\lambda(t)-\lambda_1(t)|+\int_0^1 M|u(s)-u_1(s)|ds\right)+\varepsilon.\end{aligned}$$

Hence

$$\|v-v_1\|_1\le N^*\|\lambda-\lambda_1\|_C+N^*M\|u-u_1\|_1+\varepsilon$$

for any $\varepsilon>0$.

This implies

$$d_{L^1(I,X)}\big(v,\varphi(\lambda_1,u_1)\big)\le N^*\|\lambda-\lambda_1\|_C+N^*M\|u-u_1\|_1$$

for all $v\in\varphi(\lambda,u)$. Therefore,

$$d^*_{L^1(I,X)}\big(\varphi(\lambda,u),\varphi(\lambda_1,u_1)\big)\le N^*\|\lambda-\lambda_1\|_C+N^*M\|u-u_1\|_1.$$

Consequently,

$$D_{L^1(I,X)}\big(\varphi(\lambda,u),\varphi(\lambda_1,u_1)\big)\le N^*\|\lambda-\lambda_1\|_C+N^*M\|u-u_1\|_1,$$

which shows that φ is Hausdorff continuous and satisfies the assumptions of Lemma 1.

Let us pick $(\lambda,u,v),(\lambda_1,u_1,v_1)\in C(I,X)\times L^1(I,X)\times L^1(I,X)$ and choose $w\in\psi(\lambda,u,v)$. Then, as before, for each $\varepsilon>0$ there exists $w_1\in\psi(\lambda_1,u_1,v_1)$ such that

$$\begin{aligned}|w(t)-w_1(t)|&\le D\big(F(t,x_\lambda(t),v(t)),F(t,x_{\lambda_1}(t),v_1(t))big\big)+\varepsilon\\&\le N(t)\left(|\lambda(t)-\lambda_1(t)|+M\int_0^1|u(s)-u_1(s)|ds\right)+\theta|v(t)-v_1(t)|+\varepsilon\end{aligned}$$

for every $t\in I$. Hence

$$\begin{aligned}\|w-w_1\|_1&\le N^*\|\lambda-\lambda_1\|_C+N^*M\|u-u_1\|_1+\theta\|v-v_1\|_1+\varepsilon\\&\le N^*\|\lambda-\lambda_1\|_C+(N^*M+\theta)(\|u-u_1\|_1+\|v-v_1\|_1)+\varepsilon\\&\le N^*\|\lambda-\lambda_1\|_C+(N^*M+\theta)d_{L^1(I,X)\times L^1(I,X)}\big((u,v),(u_1,v_1)\big)+\varepsilon.\end{aligned}$$

As above, we deduce that

$$D_{L^1(I,X)}\big(\psi(\lambda,u,v),\psi(\lambda_1,u_1,v_1)\big) \leq N^*\|\lambda-\lambda_1\|_C+(N^*M+\theta)d_{L^1(I,X)\times L^1(I,X)}\big((u,v),(u_1,v_1)\big),$$

namely, the set-valued map ψ is Hausdorff continuous and satisfies the hypothesis of Lemma 1.

Let us put $\Gamma(\lambda,u)=\psi(\lambda,u,\varphi(\lambda,u))$, $(\lambda,u)\in C(I,X)\times L^1(I,X)$. According to Lemma 1, the set $\mathrm{Fix}(\Gamma(s,\cdot))=\{u\in E;u\in\Gamma(s,u)\}$ is nonempty and arcwise connected in $L^1(I,X)$. Moreover, for fixed $\lambda_i\in C(I,X)$ and $u_i\in\mathrm{Fix}(\Gamma(\lambda_i,\cdot))$, $i=1,\dots,p$, there exists a continuous function $\gamma:C(I,X)\to L^1(I,X)$ such that

$$\gamma(\lambda)\in\mathrm{Fix}(\Gamma(\lambda,\cdot)),\quad \forall\lambda\in C(I,X) \tag{3.1}$$

and

$$\gamma(\lambda_i)=u_i,\quad i=1,\dots,p. \tag{3.2}$$

We shall prove that

$$\mathrm{Fix}(\Gamma(\lambda,\cdot))=\big\{u\in L^1(I,X);\ u(t)\in F(t,x_\lambda(t),G(t,x_\lambda(t)))\ \text{a.e. in } I\big\}. \tag{3.3}$$

Denote by $A(\lambda)$ the right-hand side of (3.3). If $u\in\mathrm{Fix}(\Gamma(\lambda,\cdot))$, then there is $v\in\varphi(\lambda,v)$ such that $u\in\psi(\lambda,u,v)$. Therefore, $v(t)\in G(t,x_\lambda(t))$ and

$$u(t)\in F(t,x_\lambda(t),v(t))\subset F(t,u_\lambda(t),G(t,u_\lambda(t)))\quad \text{a.e. in } I,$$

so that $\mathrm{Fix}(\Gamma(\lambda,\cdot))\subset A(\lambda)$.

Let now $u\in A(\lambda)$. By Lemma 2, there exists a selection $v\in L^1(I,X)$ of the set-valued map $t\to G(t,x_\lambda(t)))$ satisfying

$$u(t)\in F(t,x_\lambda(t),v(t))\quad \text{a.e. in } I.$$

Hence, $v\in\varphi(\lambda,v)$, $u\in\psi(\lambda,u,v)$ and thus $u\in\Gamma(\lambda,u)$, which completes the proof of (3.3).

We next note that, taking into account Hypothesis 1(v), the function $T:L^1(I,X)\to C(I,X)$,

$$T(u)(t):=\int_0^1 f(t,s,u(s))ds,\quad t\in I,$$

is continuous and one has

$$S(\lambda)=\lambda+T(\mathrm{Fix}(\Gamma(\lambda,\cdot))),\quad \lambda\in C(I,X). \tag{3.4}$$

Since $\mathrm{Fix}(\Gamma(\lambda,\cdot))$ is nonempty and arcwise connected in $L^1(I,X)$, the set $S(\lambda)$ has the same properties in $C(I,X)$.

(2) Let $\lambda_i\in C(I,X)$ and let $x_i\in S(\lambda_i)$, $i=1,\dots,p$ be fixed. By (3.4), there exists $v_i\in\mathrm{Fix}(\Gamma(\lambda_i,\cdot))$ such that

$$x_i=\lambda_i+T(v_i),\quad i=1,\dots,p.$$

If $\gamma: C(I,X) \to L^1(I,X)$ is a continuous function satisfying (3.1) and (3.2), we define, for every $\lambda \in C(I,X)$,

$$s(\lambda) = \lambda + T(\gamma(\lambda)).$$

Obviously, the function $s: C(I,X) \to C(I,X)$ is continuous, $s(\lambda) \in S(\lambda)$ for all $\lambda \in C(I,X)$, and

$$s(\lambda_i) = \lambda_i + T(\gamma(\lambda_i)) = \lambda_i + T(v_i) = x_i, \quad i = 1, \dots, p.$$

(3) Let $x_1, x_2 \in S = \cup_{\lambda \in C(I,X)} S(\lambda)$ and choose $\lambda_i \in C(I,X)$, $i = 1,2$ such that $x_i \in S(\lambda_i)$, $i = 1,2$. From the conclusion of (2) we deduce the existence of a continuous function $s: C(I,X) \to C(I,X)$ satisfying $s(\lambda_i) = x_i$, $i = 1,2$ and $s(\lambda) \in S(\lambda)$, $\lambda \in C(I,X)$. Let $h: [0,1] \to C(I,X)$ be a continuous mapping such that $h(0) = \lambda_1$ and $h(1) = \lambda_2$. Then the function $s \circ h : [0,1] \to C(I,X)$ is continuous and satisfies the relations

$$s \circ h(0) = u_1, \qquad s \circ h(1) = u_2,$$

and

$$s \circ h(\tau) \in S(h(\tau)) \subset S, \quad \tau \in [0,1].$$

The proof is complete. □

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ON ALMOST AND WEAKLY δ-SEMICONTINUOUS MULTIFUNCTIONS

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Abstract. The aim of this paper is to introduce two new classes of multifunctions called almost δ-semicontinuous multifunctions and weakly δ-semicontinuous multifunctions which are generalization of δ-semicontinuous multifunctions. In this paper we obtain several properties of such multifunctions.

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Keywords: δ-semicontinuous, almost continuity, weakly continuity, multifunction

1. INTRODUCTION

Weaker and stronger forms of open sets play an important role in topologically spaces. By using these sets many authors introduced and studied various types of generalizations of continuity. In this paper, we introduce the notions of almost and weakly δ-semicontinuous multifunctions and investigate the relationships among δ-semicontinuity, almost δ-semicontinuity and weakly δ-semicontinuity for multifunctions.

In this paper, the spaces X and Y mean topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of X, $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ represent the closure of A and the interior of A, respectively.

A subset A of a space X is said to be regular open if $A = \mathrm{int}(\mathrm{cl}(A))$ [15]. The complement of a regular open set is said to be regular closed. The δ-interior [16] of a subset A of (X, τ) is the union of all regular open sets of X contained in A is denoted by $\delta\text{-int}(A)$. A subset A is called δ-open [16] if $A = \delta\text{-int}(A)$. The complement of δ-open set is called δ-closed. The δ-closure of a subset A is defined by $\{x \in X : A \cap \mathrm{int}(\mathrm{cl}(U)) \neq \varnothing, U \in \tau \text{ and } x \in U\}$ and is denoted by $\delta\text{-cl}(A)$.

A subset A of a space X is said to be δ-semiopen [10] (resp., semi-open [4], preopen [6], β-open [1]) if $A \subset \mathrm{cl}(\delta\text{-int}(A))$ (resp., $A \subset \mathrm{cl}(\mathrm{int}(A))$, $A \subset \mathrm{int}(\mathrm{cl}(A))$, $A \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))))$. The family of all δ-semiopen (resp., semi-open, preopen, β-open, δ-open, δ-semiclosed) sets of X is denoted by $\delta SO(X)$ (resp., $SO(X)$, $PO(X)$, $\beta O(X)$, $\delta O(X)$, $\delta SC(X)$). The family of all δ-semiopen (resp., semi-open, δ-semiclosed) sets

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of X containing a point $x \in X$ is denoted by $\delta SO(X,x)$ (resp., $SO(X,x)$, $\delta SC(X,x)$). The complement of a δ-semiopen set is said to be δ-semiclosed. The intersection of all δ-semiclosed sets (resp., semi-closed) of X containing A is called the δ-semiclosure [10] (resp., semi-closure [2]) of A and is denoted by δ-scl(A) (resp., scl(A)). The union of all δ-semiopen (resp., semi-open) sets of X contained in A is called δ-semi-interior (resp., semi-interior) of A and is denoted by δ-sint(A) (resp., sint(A)).

Definition 1. A multifunction $F: X \to Y$ is said to be:

(1) upper quasi-continuous [12] if for each $x \in X$ and each open set V of Y containing $F(x)$, there exists $U \in SO(X,x)$ such that $F(U) \subset V$,
(2) lower quasi-continuous [12] if for each $x \in X$ and each open set V of Y such that $F(x) \cap V \neq \varnothing$, there exists $U \in SO(X,x)$ such that $F(u) \cap V \neq \varnothing$ for every $u \in U$.

Definition 2. For a multifunction $F: X \to Y$, the graph multifunction $G_F: X \to X \times Y$ is defined in the following way: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$, and the subset $\big\{\{x\} \times F(x) : x \in X\big\} \subset X \times Y$ is called the multigraph of F and is denoted by $G(F)$ [14].

Definition 3. A multifunction $F: X \to Y$ is said to be:

(1) lower δ-semicontinuous [3] if for each $x \in X$ and each open set V such that $x \in F^-(V)$, there exists a δ-semiopen set U containing x such that $U \subset F^-(V)$,
(2) upper δ-semicontinuous [3] if for each $x \in X$ and each open set V such that $x \in F^+(V)$, there exists a δ-semiopen set U containing x such that $U \subset F^+(V)$.

Lemma 1 ([7,9])**.** *For a multifunction $F: X \to Y$, the following assertions hold:*

(1) $G_F^+(A \times B) = A \cap F^+(B)$,
(2) $G_F^-(A \times B) = A \cap F^-(B)$ *for any subsets $A \subset X$ and $B \subset Y$.*

Lemma 2 ([7])**.** $\mathrm{scl}(V) = \mathrm{int}(\mathrm{cl}(V))$ *for every preopen set V of a space X.*

2. Almost and weakly δ-semicontinuous multifunctions

In this section, characterizations of almost and weakly δ-semicontinuous multifunctions and properties of them are introduced.

Definition 4. A multifunction $F: X \to Y$ is said to be:

(1) lower almost δ-semicontinuous if for each $x \in X$ and each open set V such that $x \in F^-(V)$, there exists a δ-semiopen set U containing x such that $U \subset F^-(\mathrm{int}(\mathrm{cl}(V)))$,
(2) upper almost δ-semicontinuous if for each $x \in X$ and each open set V such that $x \in F^+(V)$, there exists a δ-semiopen set U containing x such that $U \subset F^+(\mathrm{int}(\mathrm{cl}(V)))$.

Theorem 1. *Let $F: X \to Y$ be a multifunction. Then the following statements are equivalent:*

(1) *F is upper almost δ-semicontinuous multifunction,*
(2) *$F^+(\mathrm{int}(\mathrm{cl}(V))) \in \delta SO(X)$ for any open set $V \subset Y$,*
(3) *$F^-(\mathrm{cl}(\mathrm{int}(K))) \in \delta SC(X)$ for any closed set $K \subset Y$,*
(4) *$F^+(G) \in \delta SO(X)$ for any regular open set G of Y,*
(5) *$F^-(K) \in \delta SC(X)$ for any regular closed set K of Y,*
(6) *for each $x \in X$ and each open set V containing $F(x)$, there exists a δ-semiopen set U containing x such that $F(U) \subset \mathrm{scl}(V)$,*
(7) *$F^+(V) \subset \delta\text{-sint}(F^+(\mathrm{scl}(V)))$ for every open set V in Y,*
(8) *$\delta\text{-scl}(F^-(\mathrm{sint}(K))) \subset F^-(K)$ for every closed set K of Y,*
(9) *$\delta\text{-scl}(F^-(\mathrm{cl}(\mathrm{int}(K)))) \subset F^-(K)$ for every closed set K of Y,*
(10) *$\delta\text{-scl}(F^-(V)) \subset F^-(\mathrm{cl}(V))$ for every $V \in \beta O(Y)$,*
(11) *$\delta\text{-scl}(F^-(V)) \subset F^-(\mathrm{cl}(V))$ for every $V \in SO(Y)$,*
(12) *$F^+(V) \subset \delta\text{-sint}(F^+(\mathrm{int}(\mathrm{cl}(V))))$ for every $V \in PO(Y)$.*

Proof. (1)$\Rightarrow$(2): Let V be any open set of Y and $x \in F^+(\mathrm{int}(\mathrm{cl}(V)))$. By (1), there exists $U_x \in \delta SO(X,x)$ such that $U_x \subset F^+(\mathrm{int}(\mathrm{cl}(V)))$. Hence, $F^+(\mathrm{int}(\mathrm{cl}(V))) \in \delta SO(X)$.

(2)$\Rightarrow$(1): Let V be any open set of Y and $x \in F^+(V)$. By (2), $F^+(\mathrm{int}(\mathrm{cl}(V))) \in \delta SO(X)$. Take $U = F^+(\mathrm{int}(\mathrm{cl}(V)))$. Then, $F(U) \subset \mathrm{int}(\mathrm{cl}(V))$. Hence, F is upper almost δ-semicontinuous.

(2)$\Leftrightarrow$(3): Let K be any closed set of Y. Then, $Y \setminus K$ is an open set of Y. By (2), $F^+(\mathrm{int}(\mathrm{cl}(Y \setminus K))) \in \delta SO(X)$. Since $\mathrm{int}(\mathrm{cl}(Y \setminus K)) = Y \setminus \mathrm{cl}(\mathrm{int}(K))$, it follows that $F^+(\mathrm{int}(\mathrm{cl}(Y \setminus K))) = X \setminus F^-(\mathrm{cl}(\mathrm{int}(K)))$. We obtain that $F^-(\mathrm{cl}(\mathrm{int}(K)))$ is δ-semiclosed in X. The converse implication is proved by analogy.

(2)$\Leftrightarrow$(4), (3)$\Leftrightarrow$(5): Obvious.

(1)$\Leftrightarrow$(6): Let $x \in X$ and V be an open set containing $F(x)$. By (1) and Lemma 2, there exists a δ-semiopen set U containing x such that $U \subset F^+(\mathrm{scl}(V))$. The converse implication is obtained similarly.

(5)$\Rightarrow$(7): Let V be an open set in Y and $x \in F^+(V)$. Then we have $F(x) \subset V \subset \mathrm{scl}(V)$ and hence $x \in F^+(\mathrm{scl}(V)) = X \setminus F^-(Y \setminus \mathrm{scl}(V))$. Since $Y \setminus \mathrm{scl}(V)$ is regular closed, then $F^-(Y \setminus \mathrm{scl}(V))$ is δ-semiclosed in Y. Therefore, $F^+(\mathrm{scl}(V))$ is δ-semiopen and hence $x \in \delta\text{-sint}(F^+(\mathrm{scl}(V)))$. Consequently, we obtain $F^+(V) \subset \delta\text{-sint}(F^+(\mathrm{scl}(V)))$.

(7)$\Rightarrow$(8): Let K be any closed set of Y. Then, since $Y \setminus K$ is open, we obtain $X \setminus F^-(K) = F^+(Y \setminus K) \subset \delta\text{-sint}(F^+(\mathrm{scl}(Y \setminus K))) = \delta\text{-sint}(F^+(Y \setminus \mathrm{sint}(K))) = \delta\text{-sint}(X \setminus F^-(\mathrm{sint}(K))) = X \setminus \delta\text{-scl}(F^-(\mathrm{sint}(K)))$. We thus obtain the inclusion $\delta\text{-scl}(F^-(\mathrm{sint}(K))) \subset F^-(K)$.

(8)$\Rightarrow$(9): Since $\mathrm{sint}(K) = \mathrm{cl}(\mathrm{int}(K))$ for every closed set K, the proof is obvious.

(9)$\Rightarrow$(5): Let K be any regular closed set in Y. Then $\delta\text{-scl}(F^-(\mathrm{cl}(\mathrm{int}(K)))) = \delta\text{-scl}(F^-(K)) \subset F^-(K)$. We obtain that F is upper almost δ-semicontinuous.

(5)⇒(10): Let V be any β-open set in Y. Since $\mathrm{cl}(V)$ is regular closed, $F^-(\mathrm{cl}(V))$ is δ-semiclosed and $F^-(V) \subset F^-(\mathrm{cl}(V))$. Hence, $\delta\text{-scl}(F^-(V)) \subset F^-(\mathrm{cl}(V))$.

(10)⇒(11): Obvious.

(11)⇒(12): Let V be a preopen set. Then we have the inclusions $V \subset \mathrm{int}(\mathrm{cl}(V))$ and $\mathrm{cl}(\mathrm{int}(Y \setminus V)) \subset Y \setminus V$. Moreover, since the set $\mathrm{cl}(\mathrm{int}(Y \setminus V))$ is semi-open, it follows that $X \setminus \delta\text{-sint}(F^+(\mathrm{int}(\mathrm{cl}(V)))) = \delta\text{-scl}(X \setminus F^+(\mathrm{int}(\mathrm{cl}(V)))) = \delta\text{-scl}(F^-(Y \setminus \mathrm{int}(\mathrm{cl}(V)))) = \delta\text{-scl}(F^-(\mathrm{cl}(\mathrm{int}(Y \setminus V)))) \subset F^-(\mathrm{cl}(\mathrm{int}(Y \setminus V))) \subset F^-(Y \setminus V) \subset X \setminus F^+(V)$. Hence, we obtain $F^+(V) \subset \delta\text{-sint}(F^+(\mathrm{int}(\mathrm{cl}(V))))$.

(12)⇒(4): Let V be a regular open set. Since V is preopen, we get $F^+(V) \subset \delta\text{-sint}(F^+(\mathrm{int}(\mathrm{cl}(V)))) = \delta\text{-sint}(F^+(V))$. Hence, $F^+(V)$ is δ-semiopen. □

Theorem 2. *Let $F: X \to Y$ be a multifunction. Then the following statements are equivalent:*

(1) *F is lower almost δ-semicontinuous multifunction,*
(2) *$F^-(\mathrm{int}(\mathrm{cl}(V))) \in \delta SO(X)$ for any open set $V \subset Y$,*
(3) *$F^+(\mathrm{cl}(\mathrm{int}(K))) \in \delta SC(X)$ for any closed set $K \subset Y$,*
(4) *$F^-(G) \in \delta SO(X)$ for any regular open set G of Y,*
(5) *$F^+(K) \in \delta SC(X)$ for any regular closed set K of Y,*
(6) *for each $x \in X$ and each open set V such that $F(x) \cap V \neq \varnothing$, there exists a δ-semiopen set U containing x such that $F(u) \cap \mathrm{scl}(V) \neq \varnothing$ for every $u \in U$,*
(7) *$F^-(V) \subset \delta\text{-sint}(F^-(\mathrm{scl}(V)))$ for every open set V in Y,*
(8) *$\delta\text{-scl}(F^+(\mathrm{sint}(K))) \subset F^+(K)$ for every closed set K of Y,*
(9) *$\delta\text{-scl}(F^+(\mathrm{cl}(\mathrm{int}(K)))) \subset F^+(K)$ for every closed set K of Y,*
(10) *$\delta\text{-scl}(F^+(V)) \subset F^+(\mathrm{cl}(V))$ for every $V \in \beta O(Y)$,*
(11) *$\delta\text{-scl}(F^+(V)) \subset F^+(\mathrm{cl}(V))$ for every $V \in SO(Y)$,*
(12) *$F^-(V) \subset \delta\text{-sint}(F^-(\mathrm{int}(\mathrm{cl}(V))))$ for every $V \in PO(Y)$.*

Proof. It is similar to that of the previous theorem. □

Definition 5. A multifunction $F: X \to Y$ is said to be:

(1) lower weakly δ-semicontinuous if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists a δ-semiopen set U containing x such that $U \subset F^-(\mathrm{cl}(V))$,
(2) upper weakly δ-semicontinuous if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists a δ-semiopen set U containing x such that $U \subset F^+(\mathrm{cl}(V))$.

Theorem 3. *Let $F: X \to Y$ be a multifunction. Then the following statements are equivalent:*

(1) *F is upper weakly δ-semicontinuous multifunction,*
(2) *for each $x \in X$ and each open set V containing $F(x)$, there exists a δ-semiopen set U containing x such that $F(U) \subset \mathrm{cl}(V)$,*

(3) $F^+(V) \subset \mathrm{cl}(\delta\text{-int}(F^+(\mathrm{cl}(V))))$ *for any open set* V,
(4) $\mathrm{int}(\delta\text{-cl}(F^-(V))) \subset F^-(\mathrm{cl}(V))$ *for any open set* V,
(5) $\mathrm{int}(\delta\text{-cl}(F^-(\mathrm{int}(K)))) \subset F^-(K)$ *for any closed set* K,
(6) $\delta\text{-scl}(F^-(\mathrm{int}(K))) \subset F^-(K)$ *for any closed set* K,
(7) $\delta\text{-scl}(F^-(\mathrm{int}(\mathrm{cl}(B)))) \subset F^-(\mathrm{cl}(B))$ *for any subset* B,
(8) $F^+(\mathrm{int}(B)) \subset \delta\text{-sint}(F^+(\mathrm{cl}(\mathrm{int}(B))))$ *for any subset* B,
(9) $F^+(V) \subset \delta\text{-sint}(F^+(\mathrm{cl}(V)))$ *for any open set* V,
(10) $\delta\text{-scl}(F^-(V)) \subset F^-(\mathrm{cl}(V))$ *for any open set* V.

Proof. (1)$\Leftrightarrow$(2): Obvious.

(1)$\Rightarrow$(3): Let V be any open set and $x \in F^+(V)$. Then $F(x) \subset V$. By (1), there exists a δ-semiopen set U containing x such that $U \subset F^+(\mathrm{cl}(V))$. Since U is δ-semiopen, we have $x \in U \subset \mathrm{cl}(\delta\text{-int}(U)) \subset \mathrm{cl}(\delta\text{-int}(F^+(\mathrm{cl}(V))))$.

(3)$\Rightarrow$(4): Let V be any open set. Then we get

$$\begin{aligned} X \setminus F^-(\mathrm{cl}(V)) &= F^+(Y \setminus \mathrm{cl}(V)) \subset \mathrm{cl}(\delta\text{-int}(F^+(\mathrm{cl}(Y \setminus \mathrm{cl}(V))))) \\ &= \mathrm{cl}(\delta\text{-int}(F^+(Y \setminus \mathrm{int}(\mathrm{cl}(V))))) \subset \mathrm{cl}(\delta\text{-int}(F^+(Y \setminus V))) \\ &= \mathrm{cl}(\delta\text{-int}(X \setminus F^-(V))) = X \setminus \mathrm{int}(\delta\text{-cl}(F^-(V))). \end{aligned}$$

Hence, we obtain $\mathrm{int}(\delta\text{-cl}(F^-(V))) \subset F^-(\mathrm{cl}(V))$.

(4)$\Rightarrow$(5): Let K be any closed set. Since $\mathrm{int}(K)$ is open, $\mathrm{int}(\delta\text{-cl}(F^-(\mathrm{int}(K)))) \subset F^-(\mathrm{cl}(\mathrm{int}(K))) \subset F^-(K)$.

(5)$\Rightarrow$(6): Let K be any closed set. Then we have $\mathrm{int}(\delta\text{-cl}(F^-(\mathrm{int}(K)))) \subset F^-(K)$ and $F^-(\mathrm{int}(K)) \subset F^-(K)$. Since the equality $\delta\text{-scl}(F^-(\mathrm{int}(K))) = F^-(\mathrm{int}(K)) \cup \mathrm{int}(\delta\text{-cl}(F^-(\mathrm{int}(K))))$ holds, we obtain $\delta\text{-scl}(F^-(\mathrm{int}(K))) \subset F^-(K)$.

(6)$\Rightarrow$(7): Let B be any subset of Y, then $\mathrm{cl}(B)$ is closed in Y. Thus we obtain $\delta\text{-scl}(F^-(\mathrm{int}(\mathrm{cl}(B)))) \subset F^-(\mathrm{cl}(B))$.

(7)$\Rightarrow$(8): Let B be any subset of Y. Then, we obtain $F^+(\mathrm{int}(B)) = X \setminus F^-(\mathrm{cl}(Y \setminus B)) \subset X \setminus \delta\text{-scl}(F^-(\mathrm{int}(\mathrm{cl}(Y \setminus B)))) = \delta\text{-sint}(F^+(\mathrm{cl}(\mathrm{int}(B))))$.

(8)$\Rightarrow$(9): Obvious.

(9)$\Rightarrow$(10): Let V be any open set. By (9), we get

$$\begin{aligned} \delta\text{-scl}(F^-(V)) &\subset \delta\text{-scl}(F^-(\mathrm{int}(\mathrm{cl}(V)))) = \delta\text{-scl}(X \setminus F^+(Y \setminus \mathrm{int}(\mathrm{cl}(V)))) \\ &= X \setminus \delta\text{-sint}(F^+(Y \setminus \mathrm{int}(\mathrm{cl}(V)))) = X \setminus \delta\text{-sint}(F^+(\mathrm{cl}(Y \setminus \mathrm{cl}(V)))) \\ &\subset X \setminus F^+(Y \setminus \mathrm{cl}(V)) = F^-(\mathrm{cl}(V)). \end{aligned}$$

Hence, $\delta\text{-scl}(F^-(V)) \subset F^-(\mathrm{cl}(V))$.

(10)$\Rightarrow$(1): Let V be any open set and $F(x) \subset V$. Set $U = \delta\text{-sint}(F^+(\mathrm{cl}(V)))$. Then we have $U = \delta\text{-sint}(F^+(\mathrm{cl}(V))) = X \setminus \delta\text{-scl}(F^-(Y \setminus \mathrm{cl}(V))) \supset X \setminus F^-(\mathrm{cl}(Y \setminus \mathrm{cl}(V))) = F^+(\mathrm{int}(\mathrm{cl}(V))) \supset F^+(V)$ and $x \in F^+(V)$. Hence, $U \in \delta SO(X, x)$ and $F(U) \subset \mathrm{cl}(V)$. This shows that F is upper weakly δ-semicontinuous. □

Remark 1. For a multifunction $F: X \to Y$, the following implications hold:

$$\begin{array}{c} \text{upper weakly } \delta\text{-semicontinuous} \\ \Uparrow \\ \text{upper almost } \delta\text{-semicontinuous} \\ \Uparrow \\ \text{upper } \delta\text{-semicontinuous} \\ \Downarrow \\ \text{upper quasi-continuous} \end{array}$$

The following examples show that these implications are not reversible.

Example 1. Let $X = \{a, b, c\}$. Let τ be a topology on X given by the relation $\tau = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$. Define the multifunction $F: X \to X$ by $F(a) = \{a\}$, $F(b) = \{b\}$ and $F(c) = \{c\}$. Then F is upper almost δ-semicontinuous but not upper δ-semicontinuous.

Example 2. Let $X = \{a, b, c, d\}$. Let τ be a topology on X given by the relation $\tau = \{\varnothing, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Define the multifunction $F: X \to X$ by $F(a) = \{a, b\}$, $F(b) = \{d\}$, $F(c) = \{a, c, d\}$ and $F(d) = \{a, c, d\}$. Then F is upper weakly δ-semicontinuous but not upper almost δ-semicontinuous.

We refer the reader to [3] for an example concerning the remaining implication.

Theorem 4. *Let $F: X \to Y$ be a multifunction. Then the following statements are equivalent:*

(1) *F is lower weakly δ-semicontinuous multifunction,*
(2) *for each $x \in X$ and each open set V such that $F(x) \cap V \neq \varnothing$, there exists a δ-semiopen set U containing x such that if $y \in U$, then $F(y) \cap \mathrm{cl}(V) \neq \varnothing$,*
(3) *$F^-(V) \subset \mathrm{cl}(\delta\text{-int}(F^-(\mathrm{cl}(V))))$ for any open set V,*
(4) *$\mathrm{int}(\delta\text{-cl}(F^+(V))) \subset F^+(\mathrm{cl}(V))$ for any open set V,*
(5) *$\mathrm{int}(\delta\text{-cl}(F^+(\mathrm{int}(K)))) \subset F^+(K)$ for any closed set K,*
(6) *$\delta\text{-scl}(F^+(\mathrm{int}(K))) \subset F^+(K)$ for any closed set K,*
(7) *$\delta\text{-scl}(F^+(\mathrm{int}(\mathrm{cl}(B)))) \subset F^+(\mathrm{cl}(B))$ for any subset B,*
(8) *$F^-(\mathrm{int}(B)) \subset \delta\text{-sint}(F^-(\mathrm{cl}(\mathrm{int}(B))))$ for any subset B,*
(9) *$F^-(V) \subset \delta\text{-sint}(F^-(\mathrm{cl}(V)))$ for any open set V,*
(10) *$\delta\text{-scl}(F^+(V)) \subset F^+(\mathrm{cl}(V))$ for any open set V.*

Proof. The proof is similar to that of the previous theorem. □

Theorem 5. *Let $F: X \to \prod_{i \in I} X_i$ be a multifunction from a topological space X to the product space $\prod_{i \in I} X_i$ and let $p_i: \prod_{i \in I} X_i \to X_i$ be the projection for each $i \in I$. If F is upper (lower) weakly δ-semicontinuous multifunction, then $p_i \circ F$ is upper (resp., lower) weakly δ-semicontinuous multifunction for each $i \in I$.*

Proof. We shall prove this only for the upper continuity.

Let V_{i_0} be an open set in X_{i_0}. We have $(p_{i_0} \circ F)^+(V_{i_0}) = F^+(p_{i_0}^+(V_{i_0})) = F^+\big(V_{i_0} \times \prod_{i \in I} X_i\big)$. Take $x \in (p_{i_0} \circ F)^+(V_{i_0})$. Since F is upper weakly δ-semicontinuous and $V_{i_0} \times \prod_{i \in I} X_i$ is an open set, there exists a δ-semiopen set U containing x such that $U \subset F^+(\mathrm{cl}(V_{i_0} \times \prod_{i \in I} X_i))$. Since $F^+(\mathrm{cl}(V_{i_0} \times \prod_{i \in I} X_i)) = F^+(\mathrm{cl}(V_{i_0}) \times \prod_{i \in I} X_i) = \big(p_{i_0} \circ F\big)^+(\mathrm{cl}(V_{i_0}))$, then $p_{i_0} \circ F$ is upper weakly δ-semicontinuous multifunction. □

Theorem 6. *Let $F: X \to Y$ be a multifunction and let S be a δ-open set in X. If F is a lower (upper) weakly δ-semicontinuous, then the restriction multifunction $F|_S: S \to Y$ is a lower (resp., upper) weakly δ-semicontinuous.*

Proof. Suppose that V is an open set in Y. Let $x \in S$ and let $x \in (F|_S)^-(V)$. Since F is lower weakly δ-semicontinuous multifunction, it follows that there exists aδ-semiopen set G such that $x \in G \subset F^-(\mathrm{cl}(V))$. We obtain $x \in G \cap S \in \delta SO(S)$ and $G \cap S \subset (F|_S)^-(\mathrm{cl}(V))$. Thus, we show that the restriction multifunction $F|_S$ is a lower weakly δ-semicontinuous.

The proof of the upper weakly δ-semicontinuity of $F|_S$ is similar to that given above. □

Theorem 7. *Let $F: X \to Y$ be a multifunction. If the graph multifunction of F is upper weakly δ-semicontinuous, then F is upper weakly δ-semicontinuous.*

Proof. Suppose that $G_F: X \to X \times Y$ is upper weakly δ-semicontinuous. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since $X \times V$ is open in $X \times Y$ and $G_F(x) \subset X \times V$, there exists $U \in \delta SO(X, x)$ such that $G_F(U) \subset \mathrm{cl}(X \times V) = X \times \mathrm{cl}(V)$. By Lemma 1, we have $U \subset G_F^+(X \times \mathrm{cl}(V)) = F^+(\mathrm{cl}(V))$ and $F(U) \subset \mathrm{cl}(V)$. This shows that F is upper weakly δ-semicontinuous. □

Theorem 8. *A multifunction $F: X \to Y$ is lower weakly δ-semicontinuous provided that $G_F: X \to X \times Y$ is lower weakly δ-semicontinuous.*

Proof. Suppose that G_F is lower weakly δ-semicontinuous. Let $x \in X$ and V be any open set of Y such that $x \in F^-(V)$. Then $X \times V$ is open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \varnothing$. Since G_F is lower weakly δ-semicontinuous, there exists a δ-semiopen set U containing x such that $U \subset G_F^-(\mathrm{cl}(X \times V))$. Since $G_F^-(\mathrm{cl}(X \times V)) = G_F^-(X \times \mathrm{cl}(V))$, by Lemma 1, we have $U \subset F^-(\mathrm{cl}(V))$. This shows that F is lower weakly δ-semicontinuous. □

Theorem 9. *Suppose that $F_1: X \to Y$, $F_2: X \to Z$ are multifunctions. Let $F_1 \times F_2: X \to Y \times Z$ be a multifunction which is defined by $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$ for each $x \in X$. If $F_1 \times F_2$ is upper (lower) weakly δ-semicontinuous multifunction, then F_1 and F_2 are upper (resp., lower) weakly δ-semicontinuous multifunctions.*

Proof. Let $x \in X$ and let $K \subset Y$, $H \subset Z$ be open sets such that $x \in F_1^+(K)$ and $x \in F_2^+(H)$. Then we obtain that $F_1(x) \subset K$ and $F_2(x) \subset H$ and so $F_1(x) \times$

$F_2(x) = (F_1 \times F_2)(x) \subset K \times H$. We have $x \in (F_1 \times F_2)^+(K \times H)$. Since $F_1 \times F_2$ is upper weakly δ-semicontinuous multifunction, it follows that there exists a δ-semiopen set U containing x such that $U \subset (F_1 \times F_2)^+(\mathrm{cl}(K \times H))$. We obtain that $U \subset F_1^+(\mathrm{cl}(K))$ and $U \subset F_2^+(\mathrm{cl}(H))$. Thus, we obtain that F_1 and F_2 are upper weakly δ-semicontinuous multifunctions.

The proof of the lower weakly γ-continuity of F_1 and F_2 is similar to that presented above. □

Definition 6. Let $F: X \to Y$ be a multifunction. The multigraph $G(F)$ is said to be δ_s-graph in $X \times Y$ if for each $(x, y) \notin G(F)$, there exist δ-semiopen set U and open set V containing x and y, respectively, such that $(U \times V) \cap G(F) = \varnothing$.

Definition 7. A subset A of a topological space X is said to be α-paracompact [17] if every cover of A by open sets of X is refined by a cover of A which consists of open sets of X and is locally finite in X .

Furthermore, a multifunction $F: (X, \tau) \to (Y, \sigma)$ is called punctually α-paracompact [13] if $F(x)$ is α-paracompact for each point $x \in X$.

Theorem 10. *Let $F: (X, \tau) \to (Y, \sigma)$ be an upper weakly δ-semicontinuous and punctually α-paracompact multifunction into a Hausdorff space (Y, σ). Then the multigraph $G(F)$ of F is a δ_s-graph in $X \times Y$.*

Proof. Suppose that $(x_0, y_0) \notin G(F)$. Then $y_0 \notin F(x_0)$. Since (Y, σ) is a Hausdorff space, for each $y \in F(x_0)$, there exist open sets $V(y)$ and $W(y)$ containing y and y_0, respectively, such that $V(y) \cap W(y) = \varnothing$. The family $\{V(y) : y \in F(x_0)\}$ is an open cover of $F(x_0)$ which is α-paracompact. Thus, it has a locally finite open refinement $\Phi = \{U_\beta : \beta \in I\}$ which covers $F(x_0)$. Let W_0 be an open neighborhood of y_0 such that W_0 intersects only finitely many members U_{β_1}, U_{β_2}, …, U_{β_n} of Φ. Choose y_1, y_2, …, y_n in $F(x_0)$ such that $U_{\beta_i} \subset V(y_i)$ for each $i = 1, 2, \ldots, n$ and set $W = W_0 \cap (\cap_{i=1}^n W(y_i))$. Then W is an open neighborhood of y_0 with $W \cap \left(\cup_{\beta \in I} U_\beta\right) = \varnothing$, which implies that $W \cap \mathrm{cl}(\cup_{\beta \in I} U_\beta) = \varnothing$. By the upper weakly δ-semicontinuity of F, there exists $U \in \delta SO(X, x_0)$ such that $F(U) \subset \mathrm{cl}(\cup_{\beta \in I} U_\beta)$. It follows that $(U \times W) \cap G(F) = \varnothing$. Hence, the graph $G(F)$ is a δ_s-graph in $X \times Y$. □

Definition 8. A space X is said to be semi-connected [11] if it cannot be expressed as the union of two non-empty disjoint semiopen sets.

Lemma 3 ([8])**.** *For a space X, the following properties are equivalent:*

(1) *X is semi-connected,*
(2) *X cannot be expressed as the union of two nonempty disjoint δ-semiopen sets.*

Theorem 11. *Let F be a multifunction from a semi-connected topological space X onto a topological space Y such that F is point connected. If F is upper almost δ-semicontinuous multifunction, then Y is a connected space.*

Proof. Let $F: X \to Y$ be a upper almost δ-semicontinuous multifunction from a semi-connected topological space X onto a topological space Y. Suppose that Y is not connected and let $Y = H \cup K$ be a partition of Y. Then both H and K are open and closed subsets of Y. Since F is upper almost δ-semicontinuous multifunction, $F^+(H)$ and $F^+(K)$ are δ-semiopen subsets of X. In view of the fact that $F^+(H)$, $F^+(K)$ are disjoint and F is point connected, $X = F^+(H) \cup F^+(K)$ is a partition of X. This is contrary to the semi-connectedness of X. Hence, it is obtained that Y is a connected space. □

Definition 9. A topological space (X, τ) is said to be semi-T_2 [5] if, for each distinct points $x, y \in X$, there exist semiopen sets U and V containing x and y, respectively, such that $U \cap V = \varnothing$.

Lemma 4 ([8])**.** *For a space X, the following properties are equivalent:*

(1) *X is semi-T_2,*
(2) *for each distinct points $x, y \in X$, there exist $U, V \in \delta SO(X)$ containing x and y, respectively, such that $U \cap V = \varnothing$.*

Theorem 12. *Let $F: X \to Y$ be an upper almost δ-semicontinuous multifunction and point closed from a topological space X to a normal topological space Y and let $F(x) \cap F(y) = \varnothing$ for each distinct pair $x, y \in X$. Then X is a semi-T_2.*

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \varnothing$. Since Y is a normal space, it follows that there exists disjoints open sets U and V containing $F(x)$ and $F(y)$, respectively. Thus $F^+(\mathrm{int}(\mathrm{cl}(U)))$ and $F^+(\mathrm{int}(\mathrm{cl}(V)))$ are disjoint δ-semiopen sets containing x and y, respectively. Thus, it is obtained that X is semi-T_2. □

Recall that the δ-semifrontier of a subset A of a space X, denoted by δ-sfr(A), is defined by δ-sfr$(A) = \delta$-scl$(A) \cap \delta$-scl$(X \setminus A) = \delta$-scl$(A) \setminus \delta$-sint(A).

Theorem 13. *The set all points of X at which a multifunction $F: X \to Y$ is not upper almost δ-semicontinuous (lower almost δ-semicontinuous) is identical with the union of the δ-semifrontier of the upper (lower) inverse images of regular open sets containing (meeting) $F(x)$.*

Proof. Let $x \in X$ at which F is not upper almost δ-semicontinuous. Then there exists a regular open set V of Y containing $F(x)$ such that $U \cap (X \setminus F^+(V)) \neq \varnothing$ for every $U \in \delta SO(X, x)$. Therefore, we have $x \in \delta$-scl$(X \setminus F^+(V)) = X \setminus \delta$-sint$(F^+(V))$ and $x \in F^+(V)$. Thus, we obtain $x \in \delta$-sfr$(F^+(V))$.

Conversely, suppose that V is a regular open set of Y containing $F(x)$ such that $x \in \delta$-sfr$(F^+(V))$. If F is upper almost δ-semicontinuous at x, then there exists $U \in \delta SO(X, x)$ such that $U \subset F^+(V)$; hence $x \in \delta$-sint$(F^+(V))$. This is a contradiction and hence F is not upper almost δ-semicontinuous at x.

The case of the lower almost δ-semicontinuity can be considered similarly. □

3. Relation between types of δ-semicontinuity

In this section, the relationships among δ-semicontinuous multifunctions, almost δ-semicontinuous multifunctions and weakly δ-semicontinuous multifunctions are investigated.

Definition 10. A multifunction $F: X \to Y$ is said to be almost δ-semiopen if $F(U) \subset \text{int}(\text{cl}(F(U)))$ for every δ-semiopen set U in X.

Theorem 14. *If a multifunction $F: X \to Y$ is upper weakly δ-semicontinuous and almost δ-semiopen, then F is upper almost δ-semicontinuous.*

Proof. Let V be an arbitrary open set containing $F(x)$. There exists a δ-semiopen set U containing x such that $F(U) \subset \text{cl}(V)$. Since F is almost δ-semiopen, we have $F(U) \subset \text{int}(\text{cl}(F(U))) \subset \text{int}(\text{cl}(V))$. It then follows that F is upper almost δ-semicontinuous. □

Theorem 15. *Let $F: X \to Y$ be a multifunction such that $F(x)$ is an open set of Y for each $x \in X$. Then the following assertions are equivalent:*

(1) *F is lower δ-semicontinuous,*
(2) *F is lower almost δ-semicontinuous,*
(3) *F is lower weakly δ-semicontinuous.*

Proof. (1)⇒(2), (2)⇒(3): Obvious.
(3)⇒(1): Let $x \in X$ and V be an open set of Y such that $F(x) \cap V \neq \varnothing$. Then there exists a δ-semiopen set U containing x such that $F(u) \cap \text{cl}(V) \neq \varnothing$ for each $u \in U$. Since $F(u)$ is open, $F(u) \cap V \neq \varnothing$ for each $u \in U$ and hence F is lower δ-semicontinuous. □

Theorem 16. *If $F: X \to Y$ is lower weakly δ-semicontinuous and there exists an open basis $\Psi = \{V_i : i \in I\}$ of the topology for Y such that $F^-(\text{cl}(V_i)) \subset F^-(V_i)$ for every $i \in I$, then F is lower δ-semicontinuous.*

Proof. Let $\Psi = \{V_i : i \in I\}$ be an open basis of the topology for Y such that $F^-(\text{cl}(V_i)) \subset F^-(V_i)$ for every $i \in I$. For any open set V, there exists a subset Ψ_0 of Ψ such that $V = \cup_{i \in \Psi_0} V_i$. Therefore, by Theorem 4, we obtain

$$\begin{aligned} F^-(V) = F^-(\cup_{i\in\Psi_0} V_i) = \cup_{i\in\Psi_0} F^-(V_i) &\subset \cup_{i\in\Psi_0} \delta\text{-sint}(F^-(\text{cl}(V_i))) \\ &\subset \cup_{i\in\Psi_0} \delta\text{-sint}(F^-(V_i)) \subset \delta\text{-sint}(\cup_{i\in\Psi_0} F^-(V_i)) \\ &= \delta\text{-sint}(F^-(\cup_{i\in\Psi_0} V_i)) = \delta\text{-sint}(F^-(V)). \end{aligned}$$

This shows that F is lower δ-semicontinuous. □

Theorem 17. *If $F: X \to Y$ is upper weakly δ-semicontinuous and satisfies the condition $F^+(\text{cl}(V)) \subset F^+(V)$ for every open set V of Y, then F is upper δ-semicontinuous.*

Proof. Let V be any open set. Since F upper weakly δ-semicontinuous, we get the inclusion $F^+(V) \subset \delta\text{-sint}(F^+(\text{cl}(V)))$ and hence $F^+(V) \subset \delta\text{-sint}(F^+(\text{cl}(V))) \subset \delta\text{-sint}(F^+(V))$. Thus, $F^+(V)$ is δ-semiopen and F is upper δ-semicontinuous. □

Theorem 18. *Let $F: X \to Y$ be a multifunction such that $F(x)$ is closed in Y for each $x \in X$ and Y is normal. Then the following assertions are equivalent:*

(1) *F is upper weakly δ-semicontinuous,*
(2) *F is upper almost δ-semicontinuous,*
(3) *F is upper δ-semicontinuous.*

Proof. (3)⇒(2), (2)⇒(1): Obvious.

(1)⇒(3): Suppose that F is upper weakly δ-semicontinuous. Let $x \in X$ and G be an open set containing $F(x)$. Since $F(x)$ is closed in Y, by the normality of Y, there exists an open set V such that $F(x) \subset V \subset \text{cl}(V) \subset G$. Since F is upper weakly δ-semicontinuous, there exists a δ-semiopen set U containing x such that $F(U) \subset \text{cl}(V) \subset G$. This shows that F is upper δ-semicontinuous. □

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GBS OPERATORS OF DURRMEYER–STANCU TYPE

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Abstract. Considering four given real parameters α_1, β_1, α_2, and β_2, which satisfy the conditions $0 \leq \alpha_1 \leq \beta_1$ and $0 \leq \alpha_2 \leq \beta_2$, we construct the bivariate Durrmeyer–Stancu operators $D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}$ defined by relation (3.1). Next, we consider the associated GBS (Generalized Boolean Sum) operators and establish some approximation properties of these operators.

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1. INTRODUCTION

In 1967, J. L. Durrmeyer [3] introduced the operators $D_m: L_1([0,1]) \to C([0,1])$ defined by the formula

$$(D_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_0^1 p_{m,k}(t) f(t) dt, \tag{1.1}$$

for any $f \in L_1([0,1])$, any $x \in [0,1]$, and any positive integer m, where

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \qquad x \in [0,1],$$

are the fundamental Bernstein polynomials. These operators are called the *Durrmeyer operators*.

Let α and β be two given real parameters satisfying the conditions $0 \leq \alpha \leq \beta$. In the paper [5], D. D. Stancu introduced and studied the linear positive operators $P_m^{(\alpha,\beta)}: C([0,1]) \to C([0,1])$ defined for any function $f \in C([0,1])$ and any positive integer m by the formula

$$\left(P_m^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right) \tag{1.2}$$

for all $x \in [0,1]$. These operators are called the *Stancu operators*.

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In the paper [4], the authors introduced and studied the Durrmeyer–Stancu type operators $\left(D_m^{(\alpha,\beta)}\right)_{m\geq 1}$ for two given real parameters α and β with $0\leq\alpha\leq\beta$. The linear operators $D_m^{(\alpha,\beta)}: L_1([0,1])\to C([0,1])$ are defined, for any function $f\in L_1([0,1])$ and any positive integer m, by the relation

$$\left(D_m^{(\alpha,\beta)}f\right)(x)=(m+1)\sum_{k=0}^{m}p_{m,k}(x)\int_0^1 p_{m,k}(t)f\left(\frac{mt+\alpha}{m+\beta}\right)dt \tag{1.3}$$

for any $x\in[0,1]$.

2. Preliminaries

In the following, we recall some results which we will use in this paper (see [1, 2, 4]). Let the real numbers a,b,c,d so that $a<b$, $c<d$,

$$C_b([a,b]\times[c,d])=\left\{f:[a,b]\times[c,d]\to\mathbb{R}: f\ B\text{-continuous on } [a,b]\times[c,d]\right\},$$
$$M([a,b]\times[c,d])=\left\{f:[a,b]\times[c,d]\to\mathbb{R}: f \text{ bounded on } [a,b]\times[c,d]\right\},$$

and $e_{ij}(x,y)=x^i y^j$, $i,j\in\mathbb{N}$, $0\leq i,\ j\leq 2$ are the test functions.

If $f:[a,b]\times[c,d]\to\mathbb{R}$, f is B-bounded function on $[a,b]\times[c,d]$, then the function $\omega_{\text{mixed}}:[0,b-a]\times[0,d-c]\to\mathbb{R}$, defined by the formula

$$\omega_{\text{mixed}}(f;\delta_1,\delta_2)=\sup\left\{|(\Delta f)((x,y),(s,t))|:|s-x|\leq\delta_1,\ |t-y|\leq\delta_2\right\}$$

for any $(\delta_1,\delta_2)\in[0,b-a]\times[0,d-c]$, is called the *mixed modulus of smoothness* of function f.

Theorem 2.1. *Let $(L_{m,n})_{m,n\in\mathbb{N}}$ be a sequence of linear positive bivariate operators, $L_{m,n}:C_b([a,b]\times[c,d])\to M([a,b]\times[c,d])$, $m,n\in\mathbb{N}$. If*

(i) $(L_{m,n}e_{00})(x,y)=1$,
(ii) $(L_{m,n}e_{10})(x,y)=e_{10}(x,y)+u_{m,n}(x,y)$,
(iii) $(L_{m,n}e_{01})(x,y)=e_{01}(x,y)+v_{m,n}(x,y)$,
(iv) $(L_{m,n}e_{22})(x,y)=e_{22}(x,y)+w_{m,n}(x,y)$,
(v) $\lim_{m,n\to\infty}u_{m,n}(x,y)=\lim_{m,n\to\infty}v_{m,n}(x,y)=\lim_{m,n\to\infty}w_{m,n}(x,y)=0$ *uniformly on the rectangle* $[a,b]\times[c,d]$,

then, for any $f\in C_b([a,b]\times[c,d])$, the sequence $(U_{m,n}f)_{m,n\in\mathbb{N}}$ converges uniformly to f on the rectangle $[a,b]\times[c,d]$, where, for any $m,n\in\mathbb{N}$, the operator $U_{m,n}:\mathbb{R}^{[a,b]\times[c,d]}\to\mathbb{R}^{[a,b]\times[c,d]}$ is defined by

$$(U_{m,n}f)(x,y)=\left(L_{m,n}(f(\cdot,y)+f(x,*)-f(\cdot,*))\right)(x,y) \tag{2.1}$$

for any $f\in\mathbb{R}^{[a,b]\times[c,d]}$, any $(x,y)\in[a,b]\times[c,d]$, "$\cdot$" and "$*$" stand for the first and second variable.

Remark 2.1. For the positive integers m and n, the operator $U_{m,n}$ is called GBS operator associated to $L_{m,n}$ operator. The term "GBS operator" (Generalized Boolean Sum operator) was introduced by C. Badea and C. Cottin in the paper [2].

Remark 2.2. Theorem 2.1 is a Korovkin type theorem for B-continuous functions and is due to C. Badea, I. Badea, and H. H. Gonska (see [1]).

Theorem 2.2. *Let $(L_{m,n})_{m,n\in\mathbb{N}}$ be an arbitrary sequence of linear positive operators $L_{m,n}: C_b([a,b]\times[c,d]) \to M([a,b]\times[c,d])$ and $(U_{m,n})_{m,n\in\mathbb{N}}$ the sequence of associated GBS operators. Then, for any $f \in C_b([a,b]\times[c,d])$, any $(x,y) \in [a,b]\times[c,d]$, any $(\delta_1,\delta_2) \in [0,b-a]\times[0,d-c]$ and any positive integers m,n, the following*

$$\begin{aligned}\left|(U_{m,n}f)(x,y)-f(x,y)\right| &\le |f(x,y)|\,|1-(L_{m,n}e_{00})(x,y)| \\ &+\Bigg((L_{m,n}e_{00})(x,y)+\delta_1^{-1}\sqrt{(L_{m,n}(\cdot-x)^2)(x,y)} \\ &+\delta_2^{-1}\sqrt{(L_{m,n}(*-y)^2)(x,y)} \\ &+\delta_1^{-1}\delta_2^{-1}\sqrt{(L_{m,n}(\cdot-x)^2(*-y)^2)(x,y)}\Bigg)\omega_{\text{mixed}}(f;\delta_1,\delta_2)\end{aligned} \tag{2.2}$$

holds.

Remark 2.3. Theorem 2.2 is a Shisha–Mond type theorem for B-continuous functions; it is due to C. Badea and C. Cottin (see [2]).

Theorem 2.3. *Let two given real parameters α and β with $0 \le \alpha \le \beta$. The oprators $\left(D_m^{(\alpha,\beta)}\right)_{m\ge1}$ verify the following*

$$\left(D_m^{(\alpha,\beta)}e_0\right)(x) = 1, \tag{2.3}$$

$$\left(D_m^{(\alpha,\beta)}e_1\right)(x) = \frac{m^2}{(m+\beta)(m+2)}\,x + \frac{(\alpha+1)m+2\alpha}{(m+\beta)(m+2)}, \tag{2.4}$$

$$\begin{aligned}\left(D_m^{(\alpha,\beta)}e_2\right)(x) &= \frac{m^3(m-1)}{(m+\beta)^2(m+2)(m+3)}\,x^2 \\ &+\frac{4m^3+2\alpha m^2(m+3)}{(m+\beta)^2(m+2)(m+3)}\,x \\ &+\frac{2m^2+2\alpha m(m+3)+\alpha^2(m+2)(m+3)}{(m+\beta)^2(m+2)(m+3)},\end{aligned} \tag{2.5}$$

$$\begin{aligned}\left(D_m^{(\alpha,\beta)}\varphi_x^2\right)(x) &= \left(\frac{m}{m+\beta}\right)^2\frac{2(m-3)x(1-x)+2}{(m+2)(m+3)} \\ &+\frac{[\beta^2(m+2)+4\beta m]x^2-2x[\alpha\beta(m+2)+\beta m+2\alpha m]x+\alpha^2(m+2)+2\alpha m}{(m+\beta)^2(m+2)},\end{aligned} \tag{2.6}$$

and

$$\delta_m^{(\alpha,\beta)}(x) \leq \delta_m^{(\alpha,\beta)}, \tag{2.7}$$

where

$$\delta_m^{(\alpha,\beta)}(x) = \sqrt{\left(D_m^{(\alpha,\beta)}\varphi_x^2\right)(x)} \tag{2.8}$$

and

$$\begin{aligned}\delta_m^{(\alpha,\beta)} =& \left(\left(\frac{m}{m+\beta}\right)^2 \frac{m+1}{2(m+2)(m+3)}\right. \\ &\left. + \max\left\{\frac{\alpha^2(m+2)+2\alpha m}{(m+\beta)^2(m+2)}, \frac{(\alpha-\beta)^2(m+2)-2(\alpha-\beta)m}{(m+\beta)^2(m+2)}\right\}\right)^{\frac{1}{2}}\end{aligned} \tag{2.9}$$

for any positive integer m, and any $x \in [0,1]$.

Proof. For the proof, see [4]. □

3. Main results

In this section, let the given real parameters α_1, α_2, β_1, and β_2 satisfying the conditions $0 \leq \alpha_1 \leq \beta_1$ and $0 \leq \alpha_2 \leq \beta_2$.

We construct the bivariate Durrmeyer–Stancu operators

$$D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}: L_1([0,1]\times[0,1]) \to C([0,1]\times[0,1]),$$

defined for any positive integers m,n and any $f \in L_1([0,1]\times[0,1])$ by

$$\begin{aligned}\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} f\right)(x,y) &= (m+1)(n+1)\sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x)p_{n,j}(y) \\ &\times \int_0^1\int_0^1 p_{m,k}(s)p_{n,j}(t) f\left(\frac{ms+\alpha_1}{m+\beta_1}, \frac{nt+\alpha_2}{n+\beta_2}\right) ds\,dt\end{aligned} \tag{3.1}$$

for any $(x,y) \in [0,1]\times[0,1]$.

Let ${}^xD_m^{(\alpha_1,\beta_1)}$, ${}^yD_n^{(\alpha_2,\beta_2)}$ be the parametric extensions of the operator (1.3), where $x,y \in [0,1]$.

If $x \in [0,1]$, then ${}^xD_m^{(\alpha_1,\beta_1)}: L_1([0,1]\times[0,1]) \to C([0,1]\times[0,1])$ is defined for any positive integer m and any $f \in L_1([0,1]\times[0,1])$ by the formula

$$\left({}^xD_m^{(\alpha_1,\beta_1)} f\right)(x,y) = (m+1)\sum_{k=0}^{m} p_{m,k}(x)\int_0^1 p_{m,k}(s) f\left(\frac{ms+\alpha_1}{m+\beta_1}, y\right) ds \tag{3.2}$$

for any $y \in [0,1]$ and if $y \in [0,1]$, then ${}^{y}D_n^{(\alpha_2,\beta_2)}\colon L_1([0,1]\times[0,1]) \to C([0,1]\times[0,1])$ is defined for any positive integer n and any $f \in L_1([0,1]\times[0,1])$ by the formula

$$\left({}^{y}D_n^{(\alpha_1,\beta_2)} f\right)(x,y) = (n+1)\sum_{j=0}^{n} p_{n,j}(y)\int_0^1 p_{n,j}(t) f\left(x, \frac{nt+\alpha_2}{n+\beta_2}\right) dt \tag{3.3}$$

for any $x \in [0,1]$.

The GBS operator of Durrmeyer–Stancu is the "boolean sum" of the parametric extensions of the operators ${}^{x}D_m^{(\alpha_1,\beta_1)}$ and ${}^{y}D_n^{(\alpha_2,\beta_2)}$, so

$$U_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}\colon L_1([0,1]\times[0,1]) \to C([0,1]\times[0,1])$$

is defined for any positive integers m and n by the relation

$$\begin{aligned} U_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} &= {}^{x}D_m^{(\alpha_1,\beta_1)} \oplus {}^{y}D_n^{(\alpha_2,\beta_2)} = \\ &= {}^{x}D_m^{(\alpha_1,\beta_1)} + {}^{y}D_n^{(\alpha_2,\beta_2)} - {}^{x}D_m^{(\alpha_1,\beta_1)} \circ {}^{y}D_n^{(\alpha_2,\beta_2)}, \end{aligned} \tag{3.4}$$

where

$${}^{x}D_m^{(\alpha_1,\beta_1)} \circ {}^{y}D_n^{(\alpha_2,\beta_2)} = D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}.$$

Lemma 3.1. *The Durrmeyer–Stancu type GBS operator is defined, for any $f \in L_1([0,1]\times[0,1])$ and any positive integers m and n, by the relation*

$$\begin{aligned} \left(U_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} f\right)(x,y) &= (m+1)(n+1)\sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x)p_{n,j}(y) \\ &\times \int_0^1\int_0^1 p_{m,k}(s)p_{n,j}(t)\left(f\left(\frac{ms+\alpha_1}{m+\beta_1}, y\right) + f\left(x, \frac{nt+\alpha_2}{n+\beta_2}\right)\right. \\ &\left. - f\left(\frac{ms+\alpha_1}{m+\beta_1}, \frac{nt+\alpha_2}{n+\beta_2}\right)\right) ds\,dt \end{aligned} \tag{3.5}$$

for all $(x,y) \in [0,1]\times[0,1]$.

Proof. The assertion follows from definitions (3.1)–(3.4). □

Lemma 3.2. *For any positive integers m,n and any $(x,y) \in [0,1]\times[0,1]$, the parametric extensions ${}^{x}D_m^{(\alpha_1,\beta_1)}$ and ${}^{y}D_n^{(\alpha_2,\beta_2)}$ are linear and positive operators.*

Proof. The assertion is obtained by direct computation. □

Lemma 3.3. *operators Let m and n be given positive integers. The operator $D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_1)}$ is linear and positive.*

Proof. The proof is immediate. □

The operator $D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_1)}$ is called in what follows the *bivariate Durrmeyer–Stancu operator*.

Lemma 3.4. *For any nonzero natural numbers m, n and arbitrary $(x,y)\in[0,1]\times[0,1]$, we have*

$$\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}e_{00}\right)(x,y)=1, \tag{3.6}$$

$$\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}e_{10}\right)(x,y)=\frac{m^2}{(m+\beta_1)(m+2)}x+\frac{(\alpha_1+1)m+2\alpha_1}{(m+\beta_1)(m+2)}, \tag{3.7}$$

$$\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}e_{01}\right)(x,y)=\frac{n^2}{(n+\beta_2)(n+2)}y+\frac{(\alpha_2+1)n+2\alpha_2}{(n+\beta_2)(n+2)}, \tag{3.8}$$

$$\begin{aligned}\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}e_{22}\right)(x,y)&=\frac{m^3(m-1)}{(m+\beta_1)^2(m+2)(m+3)}x^2\\&+\frac{4m^3+2\alpha_1m^2(m+3)}{(m+\beta_1)^2(m+2)(m+3)}x+\frac{2m^2+2\alpha_1m(m+3)+\alpha_1^2(m+2)(m+3)}{(m+\beta_1)^2(m+2)(m+3)}\\&+\frac{n^3(n-1)}{(n+\beta_2)^2(n+2)(n+3)}y^2\frac{4n^3+2\alpha_2n^2(n+3)}{(n+\beta_2)^2(n+2)(n+3)}y\\&+\frac{2n^2+2\alpha_2n(n+3)+\alpha_2^2(n+2)(n+3)}{(n+\beta_2)^2(n+2)(n+3)},\end{aligned} \tag{3.9}$$

$$\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}(\cdot-x)^2\right)(x,y)=\left(D_m^{(\alpha_1,\beta_1)}\varphi_x^2\right)(x)=\left(\delta_m^{(\alpha_1,\beta_1)}(x)\right)^2, \tag{3.10}$$

$$\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}(*-y)^2\right)(x,y)=\left(D_n^{(\alpha_2,\beta_2)}\varphi_y^2\right)(y)=\left(\delta_n^{(\alpha_2,\beta_2)}(y)\right)^2, \tag{3.11}$$

$$\delta_m^{(\alpha_1,\beta_1)}(x)\le\delta_m^{(\alpha_1,\beta_1)} \tag{3.12}$$

and

$$\delta_n^{(\alpha_2,\beta_2)}(y)\le\delta_n^{(\alpha_2,\beta_2)}, \tag{3.13}$$

where the values $\delta_m^{(\alpha_1,\beta_1)}(x)$, $\delta_m^{(\alpha_1,\beta_1)}$, $\delta_n^{(\alpha_2,\beta_2)}(y)$, and $\delta_n^{(\alpha_2,\beta_2)}$ are defined by relations (2.7)–(2.9).

Proof. The assertion follows from Theorem 2.3. □

Lemma 3.5. *The bivariate Durrmeyer–Stancu operators satisfy the relations*

$$\lim_{m,n\to\infty}\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}e_{10}\right)(x,y)=e_{10}(x,y), \tag{3.14}$$

$$\lim_{m,n\to\infty}\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}e_{01}\right)(x,y)=e_{01}(x,y), \tag{3.15}$$

and

$$\lim_{m,n\to\infty}\left(D_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}e_{22}\right)(x,y)=e_{22}(x,y) \tag{3.16}$$

uniformly on $[0,1]\times[0,1]$.

Proof. Relations (3.14), (3.15), and (3.16) are consequences of Lemma 3.4. □

Theorem 3.1. *For any* $f \in C_b([0,1]\times[0,1]) \cap L_1([0,1]\times[0,1])$*, the sequence*

$$\left(U_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} f\right)_{m,n\geq 1}$$

converges to f *uniformly on* $[0,1]\times[0,1]$.

Proof. It is sufficient to apply Theorem 2.1 and Lemma 3.5. □

Theorem 3.2. *For any positive integers* m *and* n*, any function* $f \in C_b([0,1]\times[0,1]) \cap L_1([0,1]\times[0,1])$ *and any* $(x,y) \in [0,1]\times[0,1]$*, the operator* $U_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)}$ *has the properties*

$$\begin{aligned}\left|\left(U_{m,n}{}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} f\right)(x,y) - f(x,y)\right| \leq \Bigg(1 + \frac{1}{\delta_1}\delta_m^{(\alpha_1,\beta_1)}(x) \\ + \frac{1}{\delta_2}\delta_n^{(\alpha_2,\beta_2)}(y) + \frac{1}{\delta_1\delta_2}\delta_m^{(\alpha_1,\beta_1)}(x)\delta_n^{(\alpha_2,\beta_2)}(y)\Bigg)\omega_{\text{mixed}}(f;\delta_1,\delta_2)\end{aligned} \tag{3.17}$$

for any $\delta_1, \delta_2 \in (0,1]$ *and*

$$\begin{aligned}&\left|\left(U_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} f\right)(x,y) - f(x,y)\right| \\ &\qquad \leq 4\omega_{\text{mixed}}\left(f;\delta_m^{(\alpha_1,\beta_1)}(x),\delta_n^{(\alpha_2,\beta_2)}(y)\right) \\ &\qquad \leq 4\omega_{\text{mixed}}\left(f;\delta_m^{(\alpha_1,\beta_1)},\delta_n^{(\alpha_2,\beta_2)}\right).\end{aligned} \tag{3.18}$$

Proof. For the first inequality (3.17), we apply Theorem 2.2, Lemma 3.4, and relations (3.10) and (3.11). For the inequality (3.18), in the inequality (3.17) we choose

$$\begin{aligned}\delta_1 &= \delta_m^{(\alpha_1,\beta_1)}(x), \\ \delta_2 &= \delta_n^{(\alpha_2,\beta_2)}(y),\end{aligned}$$

and afterwards we take the relations (3.12) and (3.13) into account. □

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STATISTICAL APPROXIMATION BY SOME POSITIVE LINEAR OPERATORS OF DISCRETE TYPE

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Abstract. The aim of this paper is to present a class of linear positive operators of discrete type and its statistical approximation properties obtained by using a Bohman–Korovkin type theorem.

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1. INTRODUCTION

The study of the statistical convergence for sequences of positive linear operators was attempted in the year 2002 by A. G. Gadjiev and C. Orhan [8]. The research orientation was proved to be extremely fertile, many researchers approaching this subject recently [2–4]. Motivated by this research direction, we construct a general class of positive linear operators of discrete type and study its statistical approximation properties.

In order to construct the operators, we need some notation on A-statistical convergence. Let $A := (a_{kn})_{k,n\in\mathbb{N}}$ be a non-negative regular summability matrix. For a given sequence of real numbers, $x := (x_n)_{n\in\mathbb{N}}$, the sequence $Ax := ((Ax)_k)$ defined by the formula

$$(Ax)_k := \sum_{n=1}^{\infty} a_{kn}x_n$$

is called the A-transform of x whenever the series converges for each $k \in \mathbb{N}$. A sequence x is said to be A-statistically convergent to a real number L if for every $\varepsilon > 0$, one has

$$\lim_k \sum_{n:|x_n-L|\geq\varepsilon} a_{kn} = 0.$$

We denote this limit by $\mathrm{st}_A - \lim x = L$ (see [6]).

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2. Definition of operators

We set $\mathbb{R}_+ := [0,\infty)$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Let D be a given interval of the real line. We denote by $C(D)$ the space of all real-valued continuous functions on D. For each $n \in \mathbb{N}$ we consider a set of indices I_n and a net on D called $(x_{n,j})_{j\in I_n}$. We set $e_i(x) = x^i$, $i \geq 0$, $x \in D$. Following [1], let $(l_n)_{n\geq 1}$ be a sequence of positive linear operators of discrete type, defined by the equality

$$(l_n f)(x) = \sum_{j\in I_n} u_{n,j}(x) f(x_{n,j}), \quad x \in D,\ f \in C(D), \tag{2.1}$$

where $(u_{n,j})_{j\in I_n}$ is a family of continuous functions on D satisfying the following conditions

$$u_{n,j}(x) \geq 0, \quad x \in D, \tag{2.2}$$

$$\sum_{j\in I_n} u_{n,j}(x) = e_0(x), \quad x \in D, \tag{2.3}$$

$$\sum_{j\in I_n} u_{n,j}(x)\, x_{n,j} = e_1(x), \quad x \in D, \tag{2.4}$$

$$\sum_{j\in I_n} u_{n,j}(x)\, x_{n,j}^2 = e_2(x) + \psi_n(x), \quad x \in D, \tag{2.5}$$

where $\psi_n \in C(D)$.

Under this assumptions the sequence $(l_n)_{n\geq 1}$ can be indicated by the following system

$$l_n : \left\langle D, I_n, x_{n,j}, u_{n,j}(x); \psi_n \right\rangle, \quad (n,j) \in \mathbb{N}\times I_n,\ x \in D. \tag{2.6}$$

We denote by $C_B(D)$ the space of all continuous functions on D and bounded on the entire line, i. e.,

$$|f(x)| \leq M_f \quad \text{for all } x \in \mathbb{R},$$

where M_f is a constant depending on f. $C_B(D)$ is a Banach space with respect to the supremal norm $\|\cdot\|$.

In [1], compounding two sequences of operators given by (2.6), the author constructed a sequence of positive linear operators $(L_{n,\lambda})_{n\geq 1}$ acting on $C(\mathbb{R}_+)$.

In what follows, we will replace the conditions (2.3)–(2.5) imposed on the sequence $(u_{n,j})_{j\in I_n}$ by the following ones:

$$\mathrm{st}_A - \lim_n \left\| \sum_{j\in I_n} u_{n,j} - e_0 \right\| = 0, \tag{2.7}$$

$$\mathrm{st}_A - \lim_n \left\| \sum_{j\in I_n} u_{n,j}\, x_{n,j} - e_1 \right\| = 0, \tag{2.8}$$

$$\operatorname{st}_A - \lim_n \left\| \sum_{j \in I_n} u_{n,j} x_{n,j}^2 - e_2 \right\| = 0. \tag{2.9}$$

A sequence of positive linear operators of the form (2.1) which satisfies the conditions (2.2) and (2.7)–(2.9) will be denoted by

$$\widetilde{l}_n : \langle D, I_n, x_{n,j}, u_{n,j}(x) \rangle, \quad (n,j) \in \mathbb{N} \times I_n,\ x \in D. \tag{2.10}$$

Further on we will consider two sequences of operators of the type (2.10) and (2.6), respectively,

$$\widetilde{l}_n : \langle [0,1], I_n, x_{n,j}, u_{n,j}(x) \rangle, \quad (n,j) \in \mathbb{N} \times I_n,\ x \in [0,1],$$

and

$$l_n : \langle [0,b], J_n, y_{n,j}, v_{n,j}(x); \psi_n \rangle, \quad (n,j) \in \mathbb{N} \times J_n,\ x \in [0,b],$$

such that $1 \leq b$ and, for any $n \in \mathbb{N}$, there is a function $\widetilde{\psi}_n \in C([0,b])$ with the property

$$x_{n,j} \psi_j(x) = \widetilde{\psi}_n(x), \quad j \in I_n,\ x \in [0,b]. \tag{2.11}$$

Let us consider a continuous function $\lambda : [0,b] \to [0,1]$. Now we are ready to introduce the operator $\widetilde{L}_{n,\lambda}$ by putting

$$\left(\widetilde{L}_{n,\lambda} f\right)(x) = \sum_{j \in I_n} \sum_{s \in J_j} u_{n,j}\left(\lambda(x)\right) v_{j,s}(x) f\left(x_{n,j} y_{j,s} + (1 - x_{n,j})x\right) \tag{2.12}$$

for all $x \in [0,b]$, $f \in C_B([0,b])$, and $n \in \mathbb{N}$. We observe that these operators are positive and linear.

3. A Bohman–Korovkin type theorem

In [8], Gadjiev and Orhan proved the following Bohman–Korovkin type statistical approximation theorem.

Theorem A. *If a sequence of positive linear operators $A_n : C_B([a,b]) \to B([a,b])$ satisfies the conditions*

$$\operatorname{st} - \lim_n \| A_n e_i - e_i \| = 0 \quad \text{for } i = 0,1,2,$$

then, for any function $f \in C_B([a,b])$, we have

$$\operatorname{st} - \lim_n \| A_n f - f \| = 0,$$

where $B([a,b])$ is the space of all real-valued functions bounded on $[a,b]$.

We note that the above theorem is given for statistical convergence, but it also stands for A-statistical convergence. To obtain our main result we need the next lemma.

Lemma 1. *Let $A := (a_{kn})_{k,n\in\mathbb{N}}$ be a non-negative regular summability matrix and let the operators $\widetilde{L}_{n,\lambda}$ be defined by (2.12) such that the following conditions hold*

(1) *the sequence $\left(\|\widetilde{\psi}_n\|\right)_{n\in\mathbb{N}}$ is bounded,*
(2) $\mathrm{st}_A - \lim_n \|\widetilde{\psi}_n\| = 0$.

Then the following identities hold:

$$\mathrm{st}_A - \lim_n \left\|\widetilde{L}_{n,\lambda}e_i - e_i\right\| = 0 \quad \text{for } i = 0,1,2. \tag{3.1}$$

Proof. From (2.12) and (2.3) it follows that

$$\left(\widetilde{L}_{n,\lambda}e_0\right)(x) = \sum_{j\in I_n} u_{n,j}\left(\lambda(x)\right), \quad x \in [0,b].$$

Since $e_0\left(\lambda(x)\right) = e_0(x) = 1$ for all $x \in [0,b]$, we obtain

$$\left|\left(\widetilde{L}_{n,\lambda}e_0\right)(x) - e_0(x)\right| = \left|\sum_{j\in I_n} u_{n,j}\left(\lambda(x)\right) - e_0(x)\right| \le \left\|\sum_{j\in I_n} u_{n,j} - e_0\right\|.$$

Now using the above relation and (2.7), we get (3.1) for $i = 0$.

By the definition (2.12) of the operator $\widetilde{L}_{n,\lambda}$ we have

$$\begin{aligned}\left(\widetilde{L}_{n,\lambda}e_1\right)(x) &= \sum_{j\in I_n} u_{n,j}\left(\lambda(x)\right)x_{n,j}\sum_{s\in J_j} v_{j,s}(x)y_{j,s} \\ &\quad + x\sum_{j\in I_n} u_{n,j}\left(\lambda(x)\right)\sum_{s\in J_j} v_{j,s}(x) \\ &\quad - x\sum_{j\in I_n} u_{n,j}\left(\lambda(x)\right)x_{n,j}\sum_{s\in J_j} v_{j,s}(x).\end{aligned}$$

Using (2.3) and (2.4) we obtain

$$\left(\widetilde{L}_{n,\lambda}e_1\right)(x) = x\sum_{j\in I_n} u_{n,j}\left(\lambda(x)\right), \quad x \in [0,b].$$

Hence, we get

$$\begin{aligned}\left|\left(\widetilde{L}_{n,\lambda}e_1\right)(x) - e_1(x)\right| &= \left|x\sum_{j\in I_n} u_{n,j}\left(\lambda(x)\right) - e_1(x)\right| \\ &= |x|\left|\sum_{j\in I_n} u_{n,j}\left(\lambda(x)\right) - e_0(x)\right| \le b\left\|\sum_{j\in I_n} u_{n,j} - e_0\right\|.\end{aligned}$$

Since for a given $\varepsilon > 0$ we have

$$T_1 := \left\{n \in \mathbb{N} : \left\|\widetilde{L}_{n,\lambda}e_1 - e_1\right\| \ge \varepsilon\right\} \subseteq \left\{n \in \mathbb{N} : \left\|\sum_{j\in I_n} u_{n,j} - e_0\right\| \ge \frac{\varepsilon}{b}\right\} := T_2,$$

we get

$$\sum_{T_1} a_{kn} \le \sum_{T_2} a_{kn}.$$

Taking $k \to \infty$, we obtain (3.1) for $i = 1$.

By (2.3)–(2.5), (2.11), and an elementary calculus it follows that

$$\begin{aligned}\big(\widetilde{L}_{n,\lambda}e_2\big)(x) &= \sum_{j\in I_n} u_{n,j}\big(\lambda(x)\big)x_{n,j}^2\psi_j(x) + x^2\sum_{j\in I_n} u_{n,j}\big(\lambda(x)\big)\\ &= \sum_{j\in I_n} u_{n,j}\big(\lambda(x)\big)x_{n,j}\widetilde{\psi}_n(x) + x^2\sum_{j\in I_n} u_{n,j}\big(\lambda(x)\big).\end{aligned}$$

Let $M := \sup_{n\in\mathbb{N}}\{\|\widetilde{\psi}_n\|\}$. Then

$$\begin{aligned}\Big|\big(\widetilde{L}_{n,\lambda}e_2\big)(x) - e_2(x)\Big| &\le \Big|\widetilde{\psi}_n(x)\Big(\sum_{j\in I_n} u_{n,j}\big(\lambda(x)\big)x_{n,j} - e_1\big(\lambda(x)\big)\Big)\Big|\\ &\quad + \big|\lambda(x)\widetilde{\psi}_n(x)\big| + \Big|x^2\Big(\sum_{j\in I_n} u_{n,j}\big(\lambda(x)\big) - e_0(x)\Big)\Big|\\ &\le M\Big\|\sum_{j\in I_n} u_{n,j}x_{n,j} - e_1\Big\| + b\|\widetilde{\psi}_n\| + b^2\Big\|\sum_{j\in I_n} u_{n,j} - e_0\Big\|.\end{aligned}$$

Let us set $K := \max\{M, b^2\}$. Then

$$\|\widetilde{L}_{n,\lambda}e_2 - e_2\| \le K\left(\Big\|\sum_{j\in I_n} u_{n,j}x_{n,j} - e_1\Big\| + \|\widetilde{\psi}_n\| + \Big\|\sum_{j\in I_n} u_{n,j} - e_0\Big\|\right). \tag{3.2}$$

For a given $\varepsilon > 0$, we put

$$\begin{aligned}U &:= \Big\{n\in\mathbb{N} : \Big\|\sum_{j\in I_n} u_{n,j}x_{n,j} - e_1\Big\| + \|\widetilde{\psi}_n\| + \Big\|\sum_{j\in I_n} u_{n,j} - e_0\Big\| \ge \frac{\varepsilon}{K}\Big\},\\ U_1 &:= \Big\{n\in\mathbb{N} : \Big\|\sum_{j\in I_n} u_{n,j}x_{n,j} - e_1\Big\| \ge \frac{\varepsilon}{3K}\Big\},\\ U_2 &:= \Big\{n\in\mathbb{N} : \|\widetilde{\psi}_n\| \ge \frac{\varepsilon}{3K}\Big\},\end{aligned}$$

and

$$U_3 := \Big\{n\in\mathbb{N} : \Big\|\sum_{j\in I_n} u_{n,j} - e_0\Big\| \ge \frac{\varepsilon}{3K}\Big\}.$$

It is obvious that $U \subset U_1 \cup U_2 \cup U_3$. Inequality (3.2) implies that

$$\sum_{n:\|\widetilde{L}_{n,\lambda}e_2 - e_2\|\ge\varepsilon} a_{kn} \le \sum_{n\in U} a_{kn} \le \sum_{n\in U_1} a_{kn} + \sum_{n\in U_2} a_{kn} + \sum_{n\in U_3} a_{kn}.$$

Passing to the limit as $k \to \infty$, we complete the proof. □

Remark 1. If the non-negative regular summability matrix A is the Cesáro matrix of order one, then the A-statistical convergence reduces to the statistical convergence [5, 7]. Consequently, Lemma 1 also stands for statistical convergence.

Using the above lemma and Theorem A, we obtain the following result.

Theorem 1. *Let $A := (a_{kn})_{k,n\in\mathbb{N}}$ be a non-negative regular summability matrix and let the operators $\widetilde{L}_{n,\lambda}$ be defined by (2.12). If the conditions of Lemma 1 hold, then for any function $f \in C_B([0,b])$ we have*

$$\mathrm{st}_A - \lim_n \left\| \widetilde{L}_{n,\lambda} f - f \right\| = 0.$$

4. Example

In what follows we present a particular sequence of positive linear operators of the form (2.12) which is statistically convergent with respect to the sup-norm to the approximated function but it is not uniformly convergent.

Let

$$\widetilde{l}_n : \left\langle [0,1], \{0,\dots,n\}, \frac{j}{n}, u_{n,j}(x) \right\rangle, \quad (n,j) \in \mathbb{N}\times\{0,\dots,n\},\ x \in [0,1]$$

be such that

$$u_{n,j}(x) = \mathrm{C}_n^j \left(\phi_n(x)\right)^j \left(1-\phi_n(x)\right)^{n-j}, \quad x \in [0,1],$$

and

$$\phi_n = \begin{cases} e_1 & \text{if } \lg n \notin \mathbb{N}_0, \\ n e_1 & \text{if } \lg n \in \mathbb{N}_0 \end{cases}$$

for all $(n,j) \in \mathbb{N}\times\{0,\dots,n\}$. It is clear that taking $\phi_n = e_1$ for all $n \in \mathbb{N}$, $\widetilde{l}_n$ becomes the nth Bernstein polynomial.

We must should now check that $(u_{n,j})$ satisfies conditions (2.2), (2.7)–(2.9). It is obvious that (2.2) and (2.7) are fulfilled. By using an elementary calculus, we obtain

$$\left| \sum_{j=0}^{n} u_{n,j}(x) \frac{j}{n} - e_1(x) \right| = \left| \phi_n(x) - e_1(x) \right| = 0$$

for all n with the property $\lg n \notin \mathbb{N}_0$. Letting $\varepsilon > 0$ we obtain

$$\left\{ n \in \mathbb{N} : \left\| \sum_{j=0}^{n} u_{n,j} \frac{j}{n} - e_1 \right\| \geq \varepsilon \right\} = \left\{ n \in \mathbb{N} : \lg n \in \mathbb{N}_0 \right\},$$

and (2.8) is thus satisfied.

It remains only to verify (2.9). Indeed,

$$\left|\sum_{j=0}^{n} u_{n,j}(x)\left(\frac{j}{n}\right)^2 - e_2(x)\right| = \left|\frac{\phi_n(x)}{n} + \frac{n-1}{n}\left(\phi_n(x)\right)^2 - e_2(x)\right|$$

$$= \left|\frac{x}{n} + \frac{n-1}{n}x^2 - x^2\right| \leq \frac{x}{n} \leq \frac{1}{n}$$

for all n with the property $\lg n \notin \mathbb{N}_0$. Letting $\varepsilon > 0$ we get

$$\left\{n \in \mathbb{N} : \left\|\sum_{j=0}^{n} u_{n,j}\left(\frac{j}{n}\right)^2 - e_2\right\| \geq \varepsilon\right\} \subseteq \left\{n \in \mathbb{N} : \lg n \in \mathbb{N}_0 \text{ and } \frac{1}{n} \geq \varepsilon\right\}$$

and, therefore, (2.9) is also satisfied.

Choosing

$$l_n = s_n : \left\langle [0,\infty), \mathbb{N}_0, \frac{j}{n}, e^{-nx}\frac{(nx)^j}{j!}; \frac{e_1}{n}\right\rangle, \quad (n,j) \in \mathbb{N}\times\mathbb{N}_0,\ x \in [0,\infty)$$

(the Favard–Szász–Mirakjan operators), we are able to define the operators $\widetilde{L}_{n,\lambda}$ by putting

$$\left(\widetilde{L}_{n,\lambda} f\right)(x) = \sum_{j=0}^{n}\sum_{s=0}^{\infty} C_n^j \left(\phi_n\left(\lambda(x)\right)\right)^j \left(1-\phi_n\left(\lambda(x)\right)\right)^{n-j}$$

$$\times e^{-jx}\frac{(jx)^s}{s!} f\left(\frac{s}{n} + \left(1-\frac{j}{n}\right)x\right) \quad (4.1)$$

for all $x \geq 0$, $f \in C_B(\mathbb{R}_+)$, and $n \in \mathbb{N}$, where $\lambda : [0,\infty) \to [0,1]$ is a continuous function.

On the basis of Lemma 1 and Theorem 1 we deduce that the sequence of operators $(\widetilde{L}_{n,\lambda} f)$ defined by (4.1) is A-statistically convergent to f for any function $f \in C_B(D)$, where D is a compact interval on the positive semiaxis.

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