HU e-ISSN 1787-2413

Miskolc Mathematical Notes

A Publication of the University of Miskolc

VOLUME 10 (2009), NUMBER 2

MISKOLC UNIVERSITY PRESS

Miskolc Mathematical Notes

VOLUME 10 (2009), NUMBER 2

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Doi: 10.18514/MMN.2009.218

Miskolc Mathematical Notes
Vol. 10 (2009), No. 2, pp. 119–127

HU e-ISSN 1787-2413

ASYMPTOTIC REPRESENTATIONS FOR SOLUTIONS OF A CLASS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Received 1 April, 2009

Abstract. The asymptotic representations for solutions of a class of nonlinear non-autonomous differential equations of the second order are established.

2000 *Mathematics Subject Classification:* 34D05

Keywords: nonlinear differential equations, nonoscillating solutions, asymptotic representations

1. STATEMENT OF THE PROBLEM AND MAIN THEOREMS

Consider the differential equation

$$y'' = \alpha_0 p(t) y |\ln|y||^{\sigma}, \tag{1.1}$$

where $\alpha_0 \in \{-1, 1\}$, $\sigma \in \mathbb{R}$, $p\colon [a,\omega[\to]0,+\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$.

This equations is a special case of the important class of equations

$$y'' = \alpha_0 p(t) y L(y),$$

where the function L is continuous in one-sided neighborhood Δ_{Y_0} of Y_0 (Y_0 is either zero, or $\pm\infty$). L is positive and slowly varying [5] as $y \to Y_0$ also.

A solution y of Eq. (1.1) defined on the interval $[t_y,\omega[\subseteq [a,\omega[$ is called a $P_\omega(\lambda_0)$-solution if it satisfies the following conditions

$$\lim_{t\uparrow\omega} y^{(k)}(t) \in \{0, \pm\infty\} \text{ for } k = 0,1 \quad \text{and} \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0. \tag{1.2}$$

In [1, 3] it was shown that, the set of all $P_\omega(\lambda_0)$-solutions of Eq. (1.1) and its asymptotic properties are divided into four classes of solutions associated with the values $\lambda_0 \in \mathbb{R}\setminus\{0,1\}$, $\lambda_0 = \pm\infty$, $\lambda_0 = 0$ and $\lambda_0 = 1$. The necessary and sufficient conditions for the existence of $P_\omega(\lambda_0)$-solutions for each class were given. Also the asymptotic representations as $t \uparrow \omega$ for $\ln|y(t)|$ are established. Moreover in [1] for $P_\omega(\lambda_0)$-solutions, associated with the values of $\lambda_0 = 0$ and $\lambda_0 = \pm\infty$, the conditions for which the exact asymptotic formulas for the solution of Eq. (1.1) was established. In this paper we obtain similar results for $P_\omega(1)$-solutions of Eq. (1.1).

We introduce the following auxiliary notation:

$$\pi_\omega(t) = \begin{cases} t & \text{if } \omega = +\infty, \\ t-\omega & \text{if } \omega < +\infty, \end{cases} \qquad I_B(t) = \int_B^t p^{\frac{1}{2}}(\tau)\,d\tau,$$

where

$$B = \begin{cases} a & \text{if } \int_a^\omega p^{\frac{1}{2}}(\tau)\,d\tau = +\infty, \\ \omega & \text{if } \int_a^\omega p^{\frac{1}{2}}(\tau)\,d\tau < +\infty. \end{cases}$$

The next theorem for the equation (1.1) was established in [1].

Theorem 1.1. *Let $\sigma \neq 2$. Then for the existence of $P_\omega(1)$-solutions of Eq. (1.1) it is necessary that*

$$\alpha_0 > 0, \qquad \lim_{t\uparrow\omega} \pi_\omega(t)\,p^{\frac{1}{2}}(t)\,|I_B(t)|^{\frac{\sigma}{2-\sigma}} = \infty. \tag{1.3}$$

If besides that function $p\colon [a,\omega[\to]0,+\infty[$ is continuously differentiable and there is the finite of equal $\pm\infty$ limit

$$\lim_{t\uparrow\omega} \frac{\left(p^{\frac{1}{2}}(t)\,|I_B(t)|^{\frac{\sigma}{2-\sigma}}\right)'}{p(t)\,|I_B(t)|^{\frac{2\sigma}{2-\sigma}}}, \tag{1.4}$$

it is considered to be sufficient. Moreover, each of these solutions admits the following asymptotic representations as $t \uparrow \omega$

$$\ln|y(t)| \sim \pm\left(\frac{2-\sigma}{2}\,I_B(t)\right)^{\frac{2}{2-\sigma}}, \qquad \frac{y'(t)}{y(t)} \sim \pm p^{\frac{1}{2}}(t)\left(\frac{2-\sigma}{2}\,I_B(t)\right)^{\frac{\sigma}{2-\sigma}}. \tag{1.5}$$

Remark 1.1. Note that by virtue of (1.1)

$$\lim_{t\uparrow\omega} |I_B(t)|^{\frac{2}{2-\sigma}} = +\infty. \tag{1.6}$$

Moreover, it was shown in [3], that if the second condition of (1.1) is satisfied, then

$$\lim_{t\uparrow\omega} \frac{\left(p^{\frac{1}{2}}(t)\,|I_B(t)|^{\frac{\sigma}{2-\sigma}}\right)'}{p(t)\,|I_B(t)|^{\frac{2\sigma}{2-\sigma}}} = 0 \tag{1.7}$$

if it exists.

From Theorem 1.1 when $\sigma = 0$, that is, for the linear differential equation

$$y'' = \alpha_0\,p(t)\,y, \tag{1.8}$$

where $\alpha_0 \in \{-1,1\}$, $p\colon [a,\omega[\to]0,+\infty[$ is a continuous function, and $-\infty < a < \omega \leq +\infty$, we obtain

Corollary 1.1. *For the existence of $P_\omega(1)$-solutions of Eq. (1.8), it is necessary that*

$$\alpha_0 > 0, \qquad \lim_{t\uparrow\omega} \pi_\omega^2(t)p(t) = +\infty. \tag{1.9}$$

If besides that function $p\colon [a,\omega[\to]0,+\infty[$ is continuously differentiable and there is the finite of equal $\pm\infty$ limit $\lim_{t\uparrow\omega} p'(t)p^{-\frac{3}{2}}(t)$ it is sufficient. Moreover, each of these solutions admits the following asymptotic representations

$$\ln|y(t)| \sim \pm I_B(t), \qquad \frac{y'(t)}{y(t)} \sim \pm p^{\frac{1}{2}}(t) \quad \text{as} \quad t\uparrow\omega.$$

Remark 1.2. In the case where the function $p\colon [a,\omega[\to]0,+\infty[$ is continuously differentiable and the limit $\lim_{t\uparrow\omega} p'(t)p^{-\frac{3}{2}}(t)$ is finite or equal to $\pm\infty$, it is easy to prove that every non-oscillatory solution y of the linear differential equation (1.8), different from solutions that admit one of the asymptotic representations $y(t)\sim c$ or $y(t)\sim c\pi_\omega(t)$ as $t\uparrow\omega$ ($c\neq 0$), is certainly a $P_\omega(\lambda_0)$-solution, where $-\infty\le\lambda_0\le+\infty$.

The aim of the present paper is to establish conditions for the existence of $P_\omega(1)$-solutions of Eq. (1.1) that admits as $t\uparrow\omega$ the asymptotic representations

$$\begin{gathered} y(t) = \big(C+o(1)\big)\exp\left(\pm\left(\frac{2-\sigma}{2}I_B(t)\right)^{\frac{2}{2-\sigma}}\right), \\ \frac{y'(t)}{y(t)} = \pm p^{\frac{1}{2}}(t)\left(\frac{2-\sigma}{2}I_B(t)\right)^{\frac{\sigma}{2-\sigma}}\big(1+o(1)\big) \end{gathered} \tag{1.10}$$

in the case $\sigma(2-\sigma)\neq 0$, and

$$y(t)\sim C\exp\big(\pm I_B(t)\big), \qquad y'(t)\sim \pm p^{\frac{1}{2}}(t)y(t)$$

in the case $\sigma=0$, where C is a nonzero real constant.

The following theorems are true for equation (1.1).

Theorem 1.2. *Let $\sigma(2-\sigma)\neq 0$, the function $p\colon [a,\omega[\to]0,+\infty[$ be continuously differentiable, the conditions (1.4) are satisfied, and*

$$\lim_{t\uparrow\omega} \frac{\left(p^{\frac{1}{2}}(t)\,|I_B(t)|^{\frac{\sigma}{2-\sigma}}\right)'}{p(t)\,|I_B(t)|^{\frac{2(\sigma-1)}{2-\sigma}}} = 0. \tag{1.11}$$

Then, for $C=\pm1$, the Eq. (1.1) has a $P_\omega(1)$-solution that admits the asymptotic representations (1.10) as $t\uparrow\omega$.

Theorem 1.3. *Let the function $p\colon [a,\omega[\to]0,+\infty[$ be continuously differentiable, the conditions (1.1) are satisfied, and*

$$\int_a^\omega \left|\frac{p'(t)}{p(t)}\right| dt < +\infty. \tag{1.12}$$

Then Eq. (1.8) *has a fundamental set* $P_\omega(1)$*-solutions, that admits the following asymptotic representations as* $t \uparrow \omega$*:*

$$y_i(t) \sim \exp\Big((-1)^{i-1} I_B(t)\Big), \quad y_i'(t) \sim (-1)^{i-1} p^{\frac{1}{2}}(t) y_i(t) \quad (i = 1,2). \tag{1.13}$$

Remark 1.3. Theorem 1.3 and Corollary 1.1 complement the known results (see [4]) on the asymptotic properties of solutions of linear differential equations.

2. Proof of main theorems

To prove the Theorems 1.2 and 1.3 presented above, we need the auxiliary statements on the existence of solutions vanishing at infinity for the system of differential equations

$$\frac{dz_i}{d\tau} = f_i(\tau) + \sum_{k=1}^{2} p_{ik}(\tau) z_k + g_i(\tau) Z_i(\tau, z_1, z_2) \quad (i = 1,2), \tag{2.1}$$

where the functions $f_i, g_i\colon [\tau_0, +\infty[\to \mathbb{R}$ $(i = 1,2)$, $p_{ik}\colon [\tau_0, +\infty[\to \mathbb{R}$ $(i,k = 1,2)$, $Z_i\colon [\tau_0, +\infty[\times \mathbb{R}_b^2 \to \mathbb{R}$ $(i = 1,2)$ are continuous, and $\mathbb{R}_b^2 = \{(z_1, z_2) \in \mathbb{R}^2 : |z_i| \le b \ (i = 1,2)\}$ for $b > 0$. For this system of equations, using Theorem 1.3 and Remark 1.4 from [2], the following Lemmas take place:

Lemma 2.1. *Suppose that*

$$p_{ii}(\tau) \neq 0 \quad \textit{for} \quad \tau \ge \tau_0, \qquad \int_{\tau_0}^{+\infty} |p_{ii}(\tau)| d\tau = +\infty \quad (i = 1,2), \tag{2.2}$$

and also assume that the following conditions are satisfied:

$$\lim_{\tau \to +\infty} \frac{f_i(\tau)}{p_{ii}(\tau)} = 0, \qquad \lim_{\tau \to +\infty} \frac{g_i(\tau)}{p_{ii}(\tau)} = const. \quad (i = 1,2), \tag{2.3}$$

$$\lim_{\tau \to +\infty} \frac{p_{12}(\tau)}{p_{11}(\tau)} = 0, \qquad \lim_{\tau \to +\infty} \frac{p_{21}(\tau)}{p_{22}(\tau)} = const., \tag{2.4}$$

and

$$\lim_{|z_1|+|z_2| \to 0} \frac{\partial Z_i(\tau, z_1, z_2)}{\partial z_k} = 0 \quad \textit{uniformly in } [\tau_0, +\infty[\quad (i,k = 1,2). \tag{2.5}$$

Then the system (2.1) *has at least one solution* $(z_1, z_2)\colon [\tau_1, +\infty[\to \mathbb{R}_b^2$ $(\tau_1 \ge \tau_0)$*, that tends to zero as* $\tau \to +\infty$*. Moreover, there exists a one-parametric family of these solutions if* $p_{11}(\tau) p_{22}(\tau) < 0$ *for* $\tau \ge \tau_0$*, and a two-parametric family of these solutions if* $p_{ii}(\tau) < 0$ *for* $\tau \ge \tau_0$*,* $i = 1,2$*.*

Lemma 2.2. *Suppose that the conditions* (2.5) *are satisfied, and*

$$\int_{\tau_0}^{+\infty} |f_i(\tau)| d\tau < +\infty \quad (i = 1,2), \qquad \int_{\tau_0}^{+\infty} |g_2(\tau)| d\tau < +\infty,$$

$$\int_{\tau_0}^{+\infty} |p_{2k}(\tau)|d\tau < +\infty \quad (k = 1,2),$$

the functions p_{1k} $(k = 1,2)$ *and* g_1 *has the form*

$$p_{1k}(\tau) = p_{1k}^0 + \delta_{1k}(\tau) \quad (k = 1,2), \qquad g_1(\tau) = g_1^0 + \gamma_1(\tau),$$

where p_{11}^0, p_{12}^0, *and* g_1^0 *are constants,* $p_{11}^0 \neq 0$, *and*

$$\int_{\tau_0}^{+\infty} |\delta_{1k}(\tau)|d\tau < +\infty \quad (k = 1,2), \qquad \int_{\tau_0}^{+\infty} |\gamma_1(\tau)|d\tau < +\infty.$$

Then the system (2.1) *has at least one solution* $(z_1, z_2)\colon [\tau_1, +\infty[\to \mathbb{R}_b^2$ $(\tau_1 \geq \tau_0)$, *that tends to zero as* $\tau \to +\infty$. *Moreover, there exists a one-parametric family of these solutions if* $p_{11}^0 < 0$.

Proof of Theorem 1.2. We choose $C \in \{-1,1\}$ and, using the transformation

$$\tau = \left(\frac{2-\sigma}{2} I_B(t)\right)^{\frac{2}{2-\sigma}},$$

$$y(t) = C\exp\left(\pm\left(\frac{2-\sigma}{2} I_B(t)\right)^{\frac{2}{2-\sigma}}\right)\big(1+v_1(\tau)\big),$$

$$y'(t) = \pm C p^{\frac{1}{2}}(t)\left(\frac{2-\sigma}{2} I_B(t)\right)^{\frac{\sigma}{2-\sigma}} \exp\left(\pm\left(\frac{2-\sigma}{2} I_B(t)\right)^{\frac{2}{2-\sigma}}\right)\big(1+v_2(\tau)\big), \tag{2.6}$$

we reduce Eq. (1.1) to the system of differential equations

$$\begin{aligned} v_1' &= \pm(v_2 - v_1), \\ v_2' &= \pm\big[(1+v_1)R(\tau,v_1) - (1+\delta_1(\tau))(1+v_2)\big], \end{aligned} \tag{2.7}$$

where

$$R(\tau,v_1) = \big|1 \pm \delta_2(\tau)\ln|1+v_1|\big|^{\sigma}, \qquad \delta_2(\tau) = \left(\frac{2-\sigma}{2} I_B(t)\right)^{-\frac{2}{2-\sigma}},$$

and

$$\delta_1(\tau) = \pm \frac{\left(p^{\frac{1}{2}}(t)\,|I_B(t)|^{\frac{\sigma}{2-\sigma}}\right)'}{\left|\frac{2-\sigma}{2}\right|^{\frac{\sigma}{2-\sigma}} p(t)\,|I_B(t)|^{\frac{2\sigma}{2-\sigma}}}.$$

Here

$$\tau(t) = \left(\frac{2-\sigma}{2} I_B(t)\right)^{\frac{2}{2-\sigma}} \to +\infty \quad \text{as} \quad t \uparrow \omega$$

and

$$\tau'(t) = p^{\frac{1}{2}}(t)\left(\frac{2-\sigma}{2} I_B(t)\right)^{\frac{\sigma}{2-\sigma}} > 0 \quad \text{for} \quad t \in]a,\omega[.$$

By the conditions (1.6) and (1.2), we get

$$\lim_{\tau\to+\infty}\delta_1(\tau)=0,\qquad \lim_{\tau\to+\infty}\delta_2(\tau)=0,\qquad \lim_{\tau\to+\infty}\frac{\delta_1(\tau)}{\delta_2(\tau)}=0. \tag{2.8}$$

Also we have

$$\delta_2'(\tau)=-\delta_1^2(\tau),$$

It follows from this, that

$$\delta_2(\tau)=\frac{1+o(1)}{\tau}\quad\text{as}\quad \tau\to+\infty. \tag{2.9}$$

Now we choose the number $\tau_0>1$ such that the inequality

$$|\delta_2(\tau)|\le\frac{1}{2}\quad\text{for}\quad \tau\ge\tau_0$$

holds and then we consider the system (2.7) on the set

$$\Omega=[\tau_0,+\infty[\times\mathbb{R}^2_{1/2},\quad\text{where}\quad \mathbb{R}^2_{1/2}=\Big\{(v_1,v_2)\in\mathbb{R}^2:|v_i|\le\frac{1}{2},\ i=1,2\Big\}.$$

On this set, the right-hand sides of system (2.7) are continuous. So, by separation of the linear part in the second equation, we rewrite the system to the form

$$v'=\pm\big[q(\tau)+(A+B(\tau))v+V(\tau,v)\big], \tag{2.10}$$

where

$$q(t)=\begin{pmatrix}0\\-\delta_1(\tau)\end{pmatrix},\qquad A=\begin{pmatrix}-1&1\\1&-1\end{pmatrix},$$

$$B(\tau)=\begin{pmatrix}0&0\\\pm\sigma\delta_2(\tau)&-\delta_1(\tau)\end{pmatrix},\qquad V(\tau,v)=\begin{pmatrix}0\\V_1(\tau,v_1)\end{pmatrix},$$

and

$$V_1(\tau,v_1)=(1+v_1)\big[(1\pm\delta_2(\tau)\ln(1+v_1))^\sigma\mp\sigma\delta_2(\tau)v_1\big]\pm\sigma\delta_2(\tau)v_1^2.$$

For the function V_1, that belongs to the nonlinear component of this system, we have

$$\frac{\partial V_1(\tau,v_1)}{\partial v_1}=\big[1\pm\delta_2(\tau)\ln(1+v_1)\big]^{\sigma-1}\big[1\pm\delta_2(\tau)\ln(1+v_1)\pm\sigma\delta_2(\tau)\big]-1\mp\sigma\delta_2(\tau).$$

From this it is clear that

$$\frac{1}{\delta_2(\tau)}\frac{\partial V_1(\tau,v_1)}{\partial v_1}\to 0\quad\text{as}\quad v_1\to 0\quad\text{uniformly in } [\tau_0,+\infty[. \tag{2.11}$$

Using an additional transformation to the system (2.10), namely

$$v=\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}, \tag{2.12}$$

we get a system of differential equations (2.1), in which

$$f_1(\tau)=\frac{\nu}{2}\delta_1(\tau),\quad f_2(\tau)=-\frac{\nu}{2}\delta_1(\tau),\quad g_1(\tau)=-\frac{\nu}{2}\delta_2(\tau),\quad g_2(\tau)=\frac{\nu}{2}\delta_2(\tau),$$

$$p_{11}(\tau)=\nu\left[-2-\frac{1}{2}\delta_1(\tau)\mp\frac{\sigma}{2}\delta_2(\tau)\right],\qquad p_{12}(\tau)=\frac{\nu}{2}\big[\delta_1(\tau)\mp\sigma\delta_2(\tau)\big],$$

$$p_{21}(\tau)=\frac{\nu}{2}\big[\delta_1(\tau)\pm\sigma\delta_2(\tau)\big],\qquad p_{22}(\tau)=-\frac{\nu}{2}\big[\delta_1(\tau)\mp\sigma\delta_2(\tau)\big],$$

and

$$Z_i(\tau,z_1,z_2)=\frac{V_1(\tau,z_1+z_2)}{\delta_2(\tau)}\quad (i=1,2),\quad \nu=\pm1.$$

By virtue of conditions (2.8) and (2.9), we have

$$p_{11}(\tau)=\nu\big(-2+o(1)\big),\quad p_{22}(\tau)=\pm\frac{\nu\sigma}{2\tau}\big(1+o(1)\big)\quad \text{as}\quad \tau\to+\infty$$

Therefore the assumptions (2.2) of Lemma 2.1 hold for some sufficiently large value of τ_0. Without loss of generality we can assume that τ_0 equals to the chosen one before. Then, taking into account (2.8), we have

$$\lim_{\tau\to+\infty}\frac{f_i(\tau)}{p_{ii}(\tau)}=0\quad (i=1,2),\qquad \lim_{\tau\to+\infty}\frac{p_{12}(\tau)}{p_{11}(\tau)}=0,\qquad \lim_{\tau\to+\infty}\frac{p_{21}(\tau)}{p_{22}(\tau)}=1,$$

and

$$\lim_{\tau\to+\infty}\frac{g_1(\tau)}{p_{11}(\tau)}=0,\qquad \lim_{\tau\to+\infty}\frac{g_2(\tau)}{p_{22}(\tau)}=\pm\frac{1}{\sigma}.$$

Therefore, the assumptions (2.3) and (2.4) of Lemma 2.1 are satisfied. Moreover, by virtue of (2.11), conditions (2.5) hold. Hence, according to Lemma 2.1, the obtained system of differential equations (2.1) has at least one solution $(z_1,z_2)\colon[\tau_1,+\infty[\to\mathbb{R}_b^2$ $(\tau_1\ge\tau_0)$, that tends to zero as $\tau\to+\infty$. By virtue of the change of variables (2.12) and (2.6), this solution is associated with a solution y of the differential equation (1.1) that satisfies the asymptotic relations (1.10) as $t\uparrow\omega$. □

Remark 2.1. Using results from [4], we can show that, if the assumptions of Theorem 1.2 are satisfied, then Eq. (1.1), for $\sigma>0$, has a one-parametric family of $P_\omega(1)$-solutions that admit the asymptotic representations

$$y(t)\sim\pm\exp\left[\left(\frac{2-\sigma}{2}I_B(t)\right)^{\frac{2}{2-\sigma}}\right],\quad \frac{y'(t)}{y(t)}\sim p^{\frac{1}{2}}(t)\left(\frac{2-\sigma}{2}I_B(t)\right)^{\frac{\sigma}{2-\sigma}}\tag{2.13}$$

as $t\uparrow\omega$, and, for $\sigma<0$, has a two-parametric family of $P_\omega(1)$-solutions admitting the asymptotic representations (2.13), and a one-parametric family of $P_\omega(1)$-solutions that admit the following asymptotic representations as $t\uparrow\omega$:

$$y(t)\sim\pm\exp\left[-\left(\frac{2-\sigma}{2}I_B(t)\right)^{\frac{2}{2-\sigma}}\right],\quad \frac{y'(t)}{y(t)}\sim -p^{\frac{1}{2}}(t)\left(\frac{2-\sigma}{2}I_B(t)\right)^{\frac{\sigma}{2-\sigma}}.$$

Proof of Theorem 1.3. Let $i \in \{1,2\}$ be arbitrary. If we use the transformation

$$\begin{aligned} \tau = I_B(t), \qquad & y(t) = \exp\Big[(-1)^{i-1} I_B(t)\Big]\big(1 + z_1(\tau) + z_2(\tau)\big), \\ y'(t) = (-1)^{i-1} p^{\frac{1}{2}}(t) & \exp\Big[(-1)^{i-1} I_B(t)\Big]\big(1 - z_1(\tau) + z_2(\tau)\big) \end{aligned} \tag{2.14}$$

in the equation (1.8), we get the system of linear differential equations

$$z_i' = f_i(\tau) + \sum_{k=1}^{2} p_{ik}(\tau) z_k \quad (i = 1,2), \tag{2.15}$$

where

$$f_1(\tau) = \frac{(-1)^{i-1}}{2}\delta(\tau), \qquad f_2(\tau) = \frac{(-1)^{i}}{2}\delta(\tau),$$

$$p_{11}(\tau) = (-1)^i \left(2 + \frac{1}{2}\delta(\tau)\right), \qquad p_{12}(\tau) = \frac{(-1)^{i-1}}{2}\delta(\tau),$$

and

$$p_{21}(\tau) = \frac{(-1)^{i-1}}{2}\delta(\tau), \quad p_{22}(\tau) = \frac{(-1)^{i-1}}{2}\delta(\tau), \quad \delta(\tau) = \frac{1}{2} p'(t) p^{-\frac{3}{2}}(t).$$

By virtue of (1.12) and substitution $\tau = I_B(t)$, we have

$$\int_{\tau_0}^{+\infty} |\delta(\tau)| d\tau = \int_a^{\omega} \left| \frac{p'(t)}{p(t)} \right| dt < +\infty,$$

where $\tau_0 = I_B(a)$. For the system (2.15), all assumptions of Lemma 2.2 are satisfied. Hence, according to this lemma, the system of differential equations (2.15) has at least one solution $(z_1, z_2)\colon [\tau_1, +\infty[\to \mathbb{R}_b^2$ $(\tau_1 \ge \tau_0)$, that tends to zero as $\tau \to +\infty$, which, by virtue of (2.14), is associated with solutions y_i of the differential equation (1.8) that admits the asymptotic representations (1.13) as $t \uparrow \omega$. Taking these representations into account, it is not difficult to see that the solution indicated is a $P_\omega(1)$-solution.

Since $i \in \{1,2\}$ was chosen arbitrarily, we have proved the existence of two $P_\omega(1)$-solutions y_1 and y_2 admitting the asymptotic representations (1.13) as $t \uparrow \omega$. It is obvious that these two solutions form a fundamental set of solutions of Eq. (1.8). □

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Doi: 10.18514/MMN.2009.182

Miskolc Mathematical Notes
Vol. 10 (2009), No. 2, pp. 129–136

HU e-ISSN 1787-2413

THE GROUP STRUCTURE OF BACHET ELLIPTIC CURVES OVER FINITE FIELDS F_p

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Received 3 December, 2007

Abstract. Bachet elliptic curves are the curves $y^2 = x^3 + a^3$ and, in this work, the group structure $E(\mathbb{F}_p)$ of these curves over finite fields $\mathbb{F}_p$ is considered. It is shown that there are two possible structures $E(\mathbb{F}_p) \cong C_{p+1}$ or $E(\mathbb{F}_p) \cong C_n \times C_{nm}$, for $m,n \in \mathbb{N}$, according to $p \equiv 5 \pmod 6$ and $p \equiv 1 \pmod 6$, respectively. A result of Washington is restated in a more specific way saying that if $E(\mathbb{F}_p) \cong Z_n \times Z_n$, then $p \equiv 7 \pmod{12}$ and $p = n^2 \mp n + 1$.

2000 *Mathematics Subject Classification:* 11G20, 14H25, 14K15, 14G99

Keywords: elliptic curves over finite fields, rational points

1. INTRODUCTION

Let p be a prime. We shall consider the elliptic curves

$$E: \ y^2 \equiv x^3 + a^3 \pmod p, \tag{1.1}$$

where a is an element of $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$. Let us denote the group of the points on E by $E\left(\mathbb{F}_p\right)$.

If $\mathbb{F}$ is a field, then an elliptic curve over $\mathbb{F}$ has, after a change of variables, the following form:

$$y^2 = x^3 + Ax + B,$$

where $A, B \in \mathbb{F}$ with $4A^3 + 27B^2 \neq 0$ in $\mathbb{F}$. Here, $D = -16\left(4A^3 + 27B^2\right)$ is called the discriminant of the curve. Elliptic curves are studied over finite and infinite fields. Here we take $\mathbb{F}$ to be a finite prime field $\mathbb{F}_p$ with characteristic $p > 3$. Then $A, B \in \mathbb{F}_p$. The set of points $(x, y) \in \mathbb{F}_p \times \mathbb{F}_p$ on E, together with a *point o at infinity*, is called the set of $\mathbb{F}_p$*-rational points* of E on $\mathbb{F}_p$ and is denoted by $E\left(\mathbb{F}_p\right)$. N_p denotes the number of rational points on this curve. It must be finite.

In fact one expects to have at most $2p + 1$ points (including o) (for every x, there exist at most two values of y). But not all elements of $\mathbb{F}_p$ have square roots. In fact

This work was supported by the research fund of Uludag University project No. F-2003/63 and F-2004/40.

only half of the elements of $\mathbb{F}_p$ have a square root. Therefore, the expected number is about $p+1$.

It is known that

$$N_p = p+1+\sum_{x=0}^{p-1}\chi\left(x^3+Ax+B\right).$$

Here we use the fact that the number of solutions of $y^2 \equiv u \pmod{p}$ is $1+\chi(u)$. The following theorem of Hasse quantifies this result.

Theorem 1.1 (Hasse, 1922)**.** *The inequality $N_p < \left(\sqrt{p}+1\right)^2$ holds.*

Now we look at the algebraic structure of $E\left(\mathbb{F}_p\right)$. Let $P\left(x_1, y_1\right)$ and $Q\left(x_2, y_2\right)$ be two points on $E : y^2 = x^3+Ax+B$. Let also

$$m = \begin{cases} \frac{y_2-y_1}{x_2-x_1} & \text{if } P \neq Q, \\ \frac{3x_1^2+A}{2y_1} & \text{if } P \neq Q, \end{cases}$$

where $y_1 \neq 0$, while when $y_1 = 0$, the point is of order 2. If we put

$$x_3 = m^2 - x_1 - x_2 \quad \text{and} \quad y_3 = m(x_1-x_3)-y_1\,,$$

then

$$P+Q = \begin{cases} \sigma & \text{if } x_1 = x_2 \text{ and } y_1+y_2 = 0, \\ Q & \text{if } P = Q, \\ (x_3, y_3) & \text{in the other cases.} \end{cases}$$

By definition $-P = (x, -y)$.

Because of the definition of addition in an arbitrary field, it takes very long to make any addition and the results are very complicated.

Here we shall deal with Bachet elliptic curves $y^2 = x^3+a^3$ modulo p. Let $N_{p,a}$ denote the number of rational points on this curve. Some results on these curves have been given in [1] and [4].

A historical problem leading to Bachet elliptic curves is that how one can write an integer as a difference of a square and a cube. In another words, for a given fixed integer c, search for the solutions of the Diophantine equation $y^2-x^3 = c$. This equation is widely called as Bachet or Mordell equation. The existence of duplication formula makes this curve interesting. This formula was found in 1621 by Bachet. When (x, y) is a solution to this equation, where $x, y \in \mathbb{Q}$, it is easy to show that

$$\left(\frac{x^4-8cx}{4y^2}, \frac{-x^6-20cx^3+8c^2}{8y^3}\right)$$

is also a solution for the same equation. Furthermore, if (x, y) is a solution such that $xy \neq 0$ and $c \neq 1, -432$, then this leads to infinitely many solutions, which could not proven by Bachet. Hence if an integer can be stated as the difference of a cube and a square, this could be done in infinitely many ways.

If $p \equiv 5 \pmod 6$, it is well known that $E\left(\mathbb{F}_p\right) \cong C_{p+1}$, the cyclic group of order $p+1$, see [2]. But when $p \equiv 1 \pmod 6$, there is no result giving the group structure of $E\left(\mathbb{F}_p\right)$. In this work, we discuss this situation. We show that this group is isomorphic to a direct product of two cyclic groups C_n and C_{nm}, i. e.,

$$E\left(F_p\right) \cong C_n \times C_{nm}$$

for $m, n \in \mathbb{N}$. If we denote the order of $E\left(\mathbb{F}_p\right)$ by $N_{p,a}$, then

$$N_{p,a} = n^2 m = p+1-b,$$

where $b > 0$ when $a \in Q_p$, and $b < 0$ otherwise. Here b is the trace of the Frobenius endomorphism.

2. Bachet elliptic curves having a group of the form $C_n \times C_{nm}$

Let E be the curve in (1.1). Then its twist is defined as the curve $y^2 \equiv x^3 + g^3 a^3$, where g is an element of Q'_p, the set of quadratic non-residues modulo p. As usual, Q_p denotes the set of quadratic residues modulo p. Here note that if $a \in Q_p$, then $ga \in Q'_p$ and when $a \in Q'_p$, then $ga \in Q_p$. It is easy to show that b of (1.1) and of its twist have different signs. Therefore

Theorem 2.1. *Let $p \equiv 1 \pmod 6$ be a prime. If (1.1) has the group isomorphic to $C_n \times C_{nm}$ with order $n^2 m = p+1-b$, then its twist is isomorphic to $C_r \times C_{rs}$ with order $r^2 s = p+1+b$.*

Let us set $t = |b|$, that is,

$$t = \left|p+1-N_{p,a}\right|.$$

We first have

Theorem 2.2. *The following assertions hold:*

(a) *Let $p \equiv 1 \pmod{12}$ be a prime. Then*

$$b \equiv 2 \pmod{12} \quad \textit{iff} \quad N_{p,a} \equiv 0 \pmod{12}$$

and

$$b \equiv 10 \pmod{12} \quad \textit{iff} \quad N_{p,a} \equiv 4 \pmod{12}.$$

(b) *Let $p \equiv 7 \pmod{12}$ be a prime. Then*

$$b \equiv 4 \pmod{12} \quad \textit{iff} \quad N_{p,a} \equiv 4 \pmod{12}$$

and

$$b \equiv 8 \pmod{12} \quad \textit{iff} \quad N_{p,a} \equiv 0 \pmod{12}.$$

Proof. (a) Let $p \equiv 1 \pmod{12}$ be a prime. Then we can write this as $p = 1+12n$, $n \in \mathbb{Z}$. Also $b \equiv 2 \pmod{12}$ can be stated as $b = 2+12m$, $m \in \mathbb{Z}$. By substituting these, we get

$$b \equiv 2 \pmod{12} \iff N_{p,a} = p+1-b$$

and hence $N_{p,a} = 1+12n+1-(2+12m) = 12(n-m)$ and this is only valid when $N_{p,a} \equiv 0 \pmod{12}$. Similarly,

$$b \equiv 10 \pmod{12} \iff N_{p,a} = p+1-b = 1+12n+1-(10+12m)$$

and therefore $N_{p,a} = -8+12(n-m)$ and this means that $N_{p,a} \equiv 4 \pmod{12}$. Part (b) is proved in a similar fashion. □

Theorem 2.3. *Let $p \equiv 1 \pmod 6$ be a prime. Then b is not divisible by 6.*

Proof. Let us consider the curve $y^2 = x^3+1$. It has a point of order 6. Therefore its reduction modulo p has also a point of order 6. Therefore

$$b \equiv p+1-N_{p,a} \equiv 2-0 \equiv 2 \pmod 6.$$

The other possibility for the curve is $y^2 = x^3+a^3$ with a is a quadratic non-residue. It is the quadratic twist of the other curve, so has $b \equiv -2 \pmod 6$. Therefore in both cases b is non-zero modulo 6. □

Corollary 2.1. *Let $p \equiv 1 \pmod 6$ be a prime. Then $N_{p,a} \equiv 0 \pmod 4$ or $N_{p,a} \equiv 4 \pmod 6$.*

Also one obtains the following result:

Corollary 2.2. *If $p \equiv 1 \pmod{12}$ is a prime, then $b \equiv \mp 2 \pmod{12}$ and if $p \equiv 7 \pmod{12}$ is a prime, then $b \equiv \mp 4 \pmod{12}$.*

We now have the following result about the number of points on curves (1.1).

Theorem 2.4. *Let $p \equiv 1 \pmod 6$ be a prime. Then:*

(a) *If $t \equiv 2 \pmod 6$, then (1.1) has $b = t$ and $N_{p,a} \equiv 0 \pmod 6$, and its twist has $b = -t$ and $N_{p,a} \equiv 4 \pmod 6$.*
(b) *If $t \equiv 4 \pmod 6$, then (1.1) has $b = t$ and $N_{p,a} \equiv 4 \pmod 6$, and its twist has $b = -t$ and $N_{p,a} \equiv 0 \pmod 6$.*

Proof. Let $p \equiv 1 \pmod 6$ be a prime. Let us put $p = 1+6n$, $n \in \mathbb{Z}$. Let $t \equiv 2 \pmod 6$. If $b = t$, then $b \equiv 2 \pmod 6$ and we put $b = 2+6m$, $m \in \mathbb{Z}$. Therefore

$$\begin{aligned} N_{p,a} &= p+1-b = 6n+1+1-2-6m \\ &= 6(n-m) \end{aligned}$$

implying that $N_{p,a} \equiv 0 \pmod 6$.
The other parts can be proven similarly. □

We then immediately have the following result concerning the elements of order 3:

Corollary 2.3. *The following assertions hold:*

(a) *Let* $p \equiv 1 \pmod{12}$ *be a prime. If* $t \equiv 2 \pmod{12}$*, then* (1.1) *has* $b = t$ *and* $N_{p,a} \equiv 0 \pmod{12}$ *and* $E(\mathbb{F}_p)$ *has elements of order* 3*. Its twist has* $b = -t$ *and* $N_{p,a} \equiv 4 \pmod{12}$ *implying that there are no elements of order* 3*.*

If $t \equiv 10 \pmod{12}$*, then* (1.1) *has* $b = t$ *and* $N_{p,a} \equiv 4 \pmod{12}$ *and* $E(\mathbb{F}_p)$ *has no elements of order* 3*, while its twist has* $b = -t$ *and* $N_{p,a} \equiv 0 \pmod{12}$ *implying that the group has elements of order* 3*.*

(b) *Let* $p \equiv 7 \pmod{12}$ *be a prime. If* $t \equiv 4 \pmod{12}$*, then* (1.1) *has* $b = t$ *and* $N_{p,a} \equiv 4 \pmod{12}$ *and therefore has no points of order* 3*, while its twist has* $b = -t$ *and* $N_{p,a} \equiv 0 \pmod{12}$ *having elements of order* 3*.*

If $t \equiv 8 \pmod{12}$*, then* (1.1) *has* $b = t$ *and* $N_{p,a} \equiv 0 \pmod{12}$ *implying that it has elements of order* 3 *while its twist has* $b = -t$ *and* $N_{p,a} \equiv 4 \pmod{12}$ *having no such elements.*

The elements of order 3 are important in the classification of these elliptic curves modulo p. We now show that their number is either 2 or 8.

Theorem 2.5. *Let* $p \equiv 1 \pmod 6$ *be a prime. If* $N_{p,a} \equiv 0 \pmod 6$*, then there are* 2 *or* 8 *points of order* 3*.*

Proof. By [3], there are at most 9 points together with the point at infinity ø, forming a subgroup which is either trivial, cyclic of order 3 or the direct product of two cyclic groups of order 3. As we want to determine the number of elements of order 3, this group cannot be trivial. Then it is C_3 or $C_3 \times C_3$ and it is well-known that it contains 2 or 8 elements of order 3, respectively. ☐

In fact, if we let $E(\mathbb{F}_p) \cong C_n \times C_{nm}$, then when 3 divides n, $E(\mathbb{F}_p)$ has 8 points of order 3, and if not, it has 2 points of order 3.

We are now going to give one of the main results in Theorem 2.8. We first need the following results:

Corollary 2.4. *Let* p *be a prime. Then for only* $x = 0$ *among all values of* x *in* $\mathbb{F}_p$*,* $x^3 + 1$ *takes the value* 1*.*

Proof. It is clear that $x = 0$ satisfies the condition. The fact that no other value of x satisfies $x^3 + 1 = 1$ is clear from the fact that p is a prime. ☐

Theorem 2.6. *Let* $p \equiv 1 \pmod 6$ *be a prime. There are* 3 *values of* x *between* 1 *and* p *so that* $x^3 + 1 \equiv 0 \pmod p$*.*

Proof. It is obvious that $x^3 \equiv a \pmod p$ has three solutions in $\mathbb{F}_p$ for every $a \neq 0$. For $a = -1$, the proof follows. ☐

Theorem 2.7. *Let* $p \equiv 1 \pmod 6$ *be a prime. Then*

$$\sum_{x \in \mathbb{F}_p} \chi\left(x^3 + 1\right) \equiv 4 \pmod 6.$$

Proof. For each $x \in \mathbb{F}_p$, calculate the p values of x^3+1. By Corollary 2.4, one of these values is 1. By Theorem 2.6, three of them are 0. The rest $p-4$ values of x^3+1 are grouped into $\frac{p-4}{3}$ triples. As $p \equiv 1 \pmod 6$, $\frac{p-4}{3}$ is odd. Indeed, let us write $p = 1+6k$, $k \in \mathbb{Z}$. Then $\frac{p-4}{3} = 2k-1$. Let us suppose that out of these triples, s triples are in Q_p and $2k-1-s$ are in Q'_p. If a triple is in Q_p, then it adds $+3$ to the sum $\sum_{x\in\mathbb{F}_p} \chi\left(x^3+1\right)$, and if it is in Q'_p, -3 is added. Therefore

$$\begin{aligned}\sum_{x\in\mathbb{F}_p} \chi\left(x^3+1\right) &= 1+3\cdot 0+s\cdot(+3)+(2k-1-s)\cdot(-3)\\ &= 6(s-k)+4\end{aligned}$$

implying the result. □

Theorem 2.8. *Let $p \equiv 1 \pmod 6$ be a prime. Then $a \in Q_p$ iff $N_{p,a} \equiv 0 \pmod 6$.*

Proof. It is well-known that

$$N_{p,a} = p+1+\sum_{x\in\mathbb{F}_p} \chi\left(x^3+a^3\right).$$

By putting $p = 1+6n$ for $n \in \mathbb{Z}$, we get $N_{p,a} = 6n+2+\sum_{x\in\mathbb{F}_p} \chi\left(x^3+a^3\right)$. Now as $\chi(a) = 1$, and as the set of the values of x^3 is the same as the set of the values of a^3x^3, we can write

$$\begin{aligned}\sum_{x\in\mathbb{F}_p} \chi\left(x^3+a^3\right) &= \sum_{x\in\mathbb{F}_p} \chi\left(a^3x^3+a^3\right)\\ &= \sum_{x\in\mathbb{F}_p} \chi\left(a^3\right)\chi\left(x^3+1\right)\\ &= \sum_{x\in\mathbb{F}_p} \chi\left(x^3+1\right),\end{aligned}$$

and by Theorem 2.7, this sum is congruent to 4 modulo 6. Hence, by putting

$$\sum_{x\in\mathbb{F}_p} \chi\left(x^3+a^3\right) = 4+6r, \quad r \in \mathbb{Z},$$

we get $N_{p,a} = 6n+2+4+6r$ implying that $N_{p,a} \equiv 0 \pmod 6$. □

Corollary 2.5. *Let $p \equiv 1 \pmod 6$ be a prime. If $N_{p,a} \equiv \pmod 6$, then $b \equiv 2 \pmod 6$.*

Proof. As $N_{p,a} = p+1-b = p+1+\sum_{x\in\mathbb{F}_p} \chi\left(x^3+a^3\right)$, we know that $b = -\sum_{x\in\mathbb{F}_p} \chi\left(x^3+a^3\right)$. By Theorem 2.7, the result follows. □

Similarly, we have

Theorem 2.9. *Let $p \equiv 1 \pmod 6$ be a prime. Then $a \in Q'_p$ iff $N \equiv 4 \pmod 6$.*

Corollary 2.6. *Let $p \equiv 1 \pmod 6$ be a prime. Let E be the curve given by* (1.1). *Then:*

(a) $a \in Q_p$ *iff* $E\left(\mathbb{F}_p\right)$ *has* 2 *or* 8 *elements of order* 3.
(b) $a \in Q'_p$ *iff* $E\left(\mathbb{F}_p\right)$ *has no elements of order* 3.

Proof. This is clear from Corollary 2.3 and Theorem 2.8. □

3. Bachet elliptic curves having a group of the form $C_n \times C_n$

Now we shall consider the case where the Bachet elliptic curves have a group isomorphic to $C_n \times C_n$ for same n. This is only possible when $p \equiv 1 \pmod 6$, as otherwise when $p \equiv 5 \pmod 6$, $E\left(\mathbb{F}_p\right)$ is isomorphic to the cyclic group C_{p+1}. We shall consider a result of Washington and refine it.

Theorem 3.1 ([5])**.** *Let E be an elliptic curve over $\mathbb{F}_q$ where q is a prime power and suppose $E\left(\mathbb{F}_q\right) \cong \mathbb{Z}_n \times \mathbb{Z}_n$ for some integer n. Then either $q = n^2+1$, $q = n^2 \mp n+1$, or $q = (n \mp 1)^2$.*

Now we give a more specific result for Bachet elliptic curves given by (1.1) over $\mathbb{F}_q$.

Theorem 3.2. *Let E be the elliptic curve in* (1.1)*. Suppose that*

$$E\left(\mathbb{F}_p\right) \cong \mathbb{Z}_n \times \mathbb{Z}_n .$$

Then $p \equiv 7 \pmod{12}$ and $p = n^2 \mp n+1$.

Proof. By Theorem 3.1, there are three possibilities $p = n^2+1$, $p = n^2 \mp n+1$, and $p = n^2 \mp 2n+1$. The latter one is immediately rules out as p cannot be a square. We need only to show that p cannot be equal to n^2+1.

If $p = n^2+1$, then $n^2 = p-1$ and hence $p-1$ is in Q_p. But it is known that $p-1$ could be in Q_p only when $p \equiv 1,5 \pmod{12}$ is a prime. Therefore the result follows. □

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Doi: 10.18514/MMN.2009.211

Miskolc Mathematical Notes
Vol. 10 (2009), No. 2, pp. 137–143

HU e-ISSN 1787-2413

ON γ-R_0 AND γ-R_1 SPACES

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Received 9 February, 2009

Abstract. In this paper, we investigate further properties of γ-R_0 and γ-R_1 spaces due to Ekici.

2000 *Mathematics Subject Classification:* 54A20, 54C10, 54D10

Keywords: γ-R_0 spaces, γ-R_1 spaces, b-open sets, b-closure and b-θ-closure operators

1. Introduction and preliminaries

The separation axioms R_0 and R_1 were introduced and studied by N. A. Shan in [22] and C. T. Yang [23]. In 1963, they were rediscovered by A. S. Davis [4]. Then, their some properties are obtained by some authors [6, 7, 15, 17]. The notion of b-open (or sp-open) sets was introduced by Andrijević [1] and Dontchev and Przemski [5] independently of each other in 1996. Since then it has been investigated in the literature (see [9, 13, 14, 18, 19]). Ekici [9, 10] introduced γ-R_0 and γ-R_1 spaces and obtained some properties of them. In this paper, we investigte further properties of γ-R_0 and γ-R_1 spaces due to Ekici [9, 10].

In this paper, by (X,τ) and (Y,φ) (or X and Y) we always mean topological spaces on which no separation axioms are assumed, unless otherwise mentioned. For a subset A of (X,τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ represent the closure of A and the interior of A with respect to τ, respectively.

Definition 1.1. Let (X,τ) be a topological space. A subset A is called

(a) b-open [1] or sp-open [5] or γ-open [14] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$,
(b) α-open [20] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$.

The family of all b-open (resp., α-open) sets in a topological space (X,τ) is denoted by $BO(X)$ (resp., $\alpha(X)$).

Definition 1.2. The complement of a b-open set is called b-closed [1].

Similarly, the family of all b-closed sets in a topological space (X,τ) is denoted by $BC(X)$.

Definition 1.3. The intersection of all b-closed sets containing A is called the b-closure [1] of A and is denoted by $\mathrm{b\,Cl}(A)$.

The union of all b-open sets contained in A is called the b-interior of A and is denoted by $\mathrm{b\,Int}(A)$. The following lemma is useful in the sequel:

Lemma 1.1 ([1]). *Let A be a subset of a space (X,τ). Then the following properties hold:*

(a) $\mathrm{b\,Cl}(A) = \mathrm{s\,Cl}(A) \cap \mathrm{p\,Cl}(A)$,
(b) $\mathrm{b\,Int}(A) = \mathrm{s\,Int}(A) \cup \mathrm{p\,Int}(A)$,
(c) $x \in \mathrm{b\,Cl}(A)$ *if and only if* $A \cap U \neq \varnothing$ *for every* $BO(X,x)$,
(d) $\mathrm{b\,Cl}(A) = A \cup \big[\mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Cl}(\mathrm{Int}(A))\big]$,
(e) $\mathrm{b\,Int}(A) = A \cap \big[\mathrm{Cl}(\mathrm{Int}(A)) \cup \mathrm{Int}(\mathrm{Cl}(A))\big]$.

The family of all b-open sets containing x of X will be denoted by $BO(X,x)$. The following basic properties of the b-closure are useful in the sequel:

Lemma 1.2 ([11]). *Let A and B be subsets of a space (X,τ). The following properties hold:*

(a) *If $A \subset B$, then* $\mathrm{b\,Cl}(A) \subset \mathrm{b\,Cl}(B)$,
(b) $\mathrm{b\,Cl}(X \setminus A) = X \setminus \mathrm{b\,Int}(A)$,
(c) $A \in BC(X,\tau)$ *if and only if* $A = \mathrm{b\,Cl}(A)$.

Lemma 1.3 ([1]). *The following assertions hold:*

(a) *The union of any family of b-open sets is a b-open set.*
(b) *The intersection of an open set and a b-open set is a b-open set.*

Definition 1.4. [10] Let X be a topological space and $S \subset X$. The γ-kernel of S, denoted by $\gamma\text{-}\ker(S)$, is defined to be the set $\gamma\text{-}\ker(S) = \cap\{U \in BO(X,\tau) \mid S \subset U\}$.

Remark 1.1. In [3], the set $\gamma\text{-}\ker(S)$ has been denoted by S^{Λ_b}.

Lemma 1.4 ([10]). *Let (X,τ) be a topological space and A be a subset of X. Then* $A^{\Lambda_b} = \{x \in X \mid \mathrm{b\,Cl}(\{x\}) \cap A \neq \varnothing\}$.

2. PRESERVATION THEOREMS

Definition 2.1. A topological space (X,τ) is called γ-R_0 [10] if its every b-open set contains the b-closure of each of its singletons.

Theorem 2.1 ([10]). *A topological space $(X,\tau$ is a γ-R_0 space if and only if for any x and y in X,* $\mathrm{b\,Cl}(\{x\}) \neq \mathrm{b\,Cl}(\{y\})$ *implies* $\mathrm{b\,Cl}(\{x\}) \cap \mathrm{b\,Cl}(\{y\}) = \varnothing$.

Definition 2.2. A function $f\colon (X,\tau) \to (Y,\varphi)$ is called

(a) b-continuous [14] (resp., α-continuous [16]) if $f^{-1}(V)$ is b-open (resp., α-open) in (X,τ) for every open set V of (Y,φ);
(b) α-open [16] if $f(U)$ is α-open in (Y,φ) for every open set U of (X,τ).

Lemma 2.1 ([14]). *Let $f\colon (X,\tau) \to (Y,\varphi)$ be an α-continuous and α-open function. Then the inverse image of each b-open set in Y is b-open in X.*

Definition 2.3. A function $f:(X,\tau) \to (Y,\varphi)$ is called strongly γ-closed [12] if $f(F)$ is γ-closed in (Y,φ) for every γ-closed set F of (X,τ).

Theorem 2.2. *If (X,τ) is a γ-R_0 space and $f:(X,\tau) \to (Y,\varphi)$ is an α-continuous, α-open and b-closed surjection, then (Y,φ) is a γ-R_0 space.*

Proof. Let V be a b-open set of Y and y any point of of V. Since f is α-continuous and α-open, $f^{-1}(V)$ is a b-open set in (X,τ), by Lemma 2.1. Since (X,τ) is γ-R_0, for a point $x \in f^{-1}(\{y\})$ by Definition 2.1, $b\mathrm{Cl}(\{x\}) \subset f^{-1}(V)$. The b-closeness of f implies that $b\mathrm{Cl}(\{y\}) = b\mathrm{Cl}(\{f(x)\}) \subset f(b\mathrm{Cl}(\{x\})) \subset V$. Therefore, (Y,φ) is γ-R_0. □

Definition 2.4. A function $f:(X,\tau) \to (Y,\varphi)$ is called quasi γ-closed [12] if $f(F)$ is closed in (Y,φ) for every γ-closed set F of (X,τ).

Theorem 2.3. *If (X,τ) is a γ-R_0 space and $f:(X,\tau) \to (Y,\varphi)$ is a strongly b-closed and b-continuous surjection, then (Y,φ) is an R_0 space.*

Proof. The proof is similar to that of Theorem 2.2 and is thus omitted. □

3. γ-R_1 SPACES

Definition 3.1. A topological space (X,τ) is called γ-R_1 [9] if for x and y in X with $b\mathrm{Cl}(\{x\}) \neq b\mathrm{Cl}(\{y\})$, there exist disjoint b-open sets U and V such that $b\mathrm{Cl}(\{x\})$ is a subset of U and $b\mathrm{Cl}(\{y\})$ is a subset of V.

Proposition 3.1. *Every γ-R_1 space is γ-R_0.*

Proof. Let U be a b-open set such that $x \in U$. If $y \notin U$, since $x \notin b\mathrm{Cl}(\{y\})$, we have $b\mathrm{Cl}(\{x\}) \neq b\mathrm{Cl}(\{y\})$. So, there exists a b-open set V such that $b\mathrm{Cl}(\{y\}) \subset V$ and $x \notin V$, which implies $y \notin b\mathrm{Cl}(\{x\})$. Hence $b\mathrm{Cl}(\{x\}) \subset U$. Therefore, (X,τ) is γ-R_0. □

Remark 3.1. The converse of Proposition 3.1 need not be true as shown in the following example.

Example 3.1. Let (X,τ) be a topological space as in [2, Example 4.16], that is, p be a fixed point of (X,τ) with τ as the cofinite topology on X. If it meaned clearly, $\tau = \{\varnothing, X, U\}$ with $U \subset (X \setminus \{p\})$ and $X \setminus U$ finite. One can obtain $BO(X) = \{\varnothing, X, U, A\}$ such that for each $A \subset U$. Since $b\mathrm{Cl}(\{x\}) = \{x\}$ for every $x \in X$, we obtain that (X,τ) is γ-R_0 by using Theorem 2.1. On the other hand, one can easily show that (X,τ) is not γ-R_1 by using Definition 3.1.

Theorem 3.1 ([9])**.** *A topological space (X,τ) is γ-R_1 if and only if for $x, y \in X$, $\{x\}^{\Lambda_b} \neq \{y\}^{\Lambda_b}$, there exist disjoint b-open sets U and V such that $b\mathrm{Cl}(\{x\}) \subset U$ and $b\mathrm{Cl}(\{y\}) \subset V$.*

Definition 3.2. A set U in a topological space (X,τ) is called b-neighborhood [11] of x if U contains a b-open set V such that $x \in V$.

Theorem 3.2. *A topological space (X,τ) is γ-R_1 if and only if the inclusion $x \in X \setminus \mathrm{bCl}(\{y\})$ implies that x and y have disjoint b-open neighborhoods.*

Proof. Necessity: Let $x \in X \setminus \mathrm{bCl}(\{y\})$. Then $\mathrm{bCl}(\{x\}) \neq \mathrm{bCl}(\{y\})$ and x, y have disjoint b-open neighborhoods.

Sufficiency: First, we show that (X,τ) is γ-R_0. Let U be a b-open set and $x \in U$. Suppose that $y \notin U$. Then, $\mathrm{bCl}(\{y\}) \cap U = \varnothing$ and $x \notin \mathrm{bCl}(\{y\})$. There exist b-open sets U_x and U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \varnothing$. Hence, $\mathrm{bCl}(\{x\}) \subset \mathrm{bCl}(U_x)$ and $\mathrm{bCl}(\{x\}) \cap U_y \subset \mathrm{bCl}(U_x) \cap U_y = \varnothing$. Therefore, $y \notin \mathrm{bCl}(\{x\})$. Consequently, $\mathrm{bCl}(\{x\}) \subset U$ and (X,τ) is γ-R_0. Next, we show that (X,τ) is γ-R_1. Suppose that $\mathrm{bCl}(\{x\}) \neq \mathrm{bCl}(\{y\})$. Then, we can assume that there exists $z \in \mathrm{bCl}(\{x\})$ such that $z \notin \mathrm{bCl}(\{y\})$. There exist b-open sets V_z and V_y such that $z \in V_z$, $y \in V_y$ and $V_z \cap V_y = \varnothing$. Since $z \in \mathrm{bCl}(\{x\})$, $x \in V_z$. Since (X,τ) is γ-R_0, we obtain $\mathrm{bCl}(\{x\}) \subset V_z$, $\mathrm{bCl}(\{y\}) \subset V_y$ and $V_z \cap V_y = \varnothing$. This shows that (X,τ) is γ-R_1. □

Corollary 3.1. *For a topological space (X,τ), the following properties are equivalent:*

(1) *(X,τ) is a γ-R_1 space,*
(2) *$\mathrm{bCl}(\{x\}) \neq \mathrm{bCl}(\{y\})$ implies that x and y have disjoint b-open neighborhoods.*

Proof. The implication (1)$\Rightarrow$(2) is obvious.

(2)$\Longrightarrow$(1): Suppose that $x \in X \setminus \mathrm{bCl}(\{y\})$. Then, $\mathrm{bCl}(\{x\}) \neq \mathrm{bCl}(\{y\})$ and by (2) x and y have disjoint b-open neighborhoods. By Theorem 3.2, (X,τ) is γ-R_1. □

The following notions are due to Park [21]. A point x of X is called a b-θ-cluster point of A if $\mathrm{bCl}(U) \cap A \neq \varnothing$ for every $U \in BO(X,x)$. The set of all b-θ-cluster points of A is called the b-θ-closure of A and is denoted by $\mathrm{bCl}_\theta(A)$. A subset A is said to be b-θ-closed if $A = \mathrm{bCl}_\theta(A)$. The complement of a b-$\theta$-closed set is said to be b-θ-open.

Lemma 3.1. *For any subset A of a topological space (X,τ), $\mathrm{bCl}(A) \subseteq \mathrm{bCl}_\theta(A)$.*

Theorem 3.3. *A topological space (X,τ) is a γ-R_1 space if and only if for each $x \in X$, $\mathrm{bCl}_\theta(\{x\}) = \mathrm{bCl}(\{x\})$.*

Proof. Necessity: Let (X,τ) be γ-R_1 and x any point of X. Assume that $y \in X \setminus \mathrm{bCl}(\{x\}))$. By Theorem 3.2, there exist disjoint b-open sets U and V such that $x \in U$ and $y \in V$. Thus, we have $\{x\} \cap \mathrm{bCl}(V) \subset U \cap \mathrm{bCl}(V) = \varnothing$ and hence $y \notin \mathrm{bCl}_\theta(\{x\})$. Therefore, we obtain $\mathrm{bCl}_\theta(\{x\}) \subseteq \mathrm{bCl}(\{x\})$. So, by using Lemma 3.1, we have $\mathrm{bCl}_\theta(\{x\}) = \mathrm{bCl}(\{x\})$.

Sufficiency: Let x, y be distinct points of X and $y \in X \setminus \mathrm{bCl}(\{x\})$. Then $y \in X \setminus \mathrm{bCl}_\theta(\{x\}))$ and hence there exists a b-open neighborhood V of y such that $\mathrm{bCl}(V) \cap \{x\} = \varnothing$. Then V and $X \setminus \mathrm{bCl}(V))$ are disjoint b-open neighborhoods of y and x, respectively. By Theorem 3.2, (X,τ) is γ-R_1. □

The following notion is due to Ekici and Caldas [13].

Definition 3.3. A topological space (X,τ) is said to be b-regular [13] if for each b-closed set F and each point $x \in X \setminus F$, there exist disjoint b-open sets U and V such that $x \in U$ and $F \subset V$.

Now, we can give a characterization of b-regular spaces.

Theorem 3.4. *For a topological space (X,τ), the following properties are equivalent:*

(1) *(X,τ) is b-regular,*
(2) *$\mathrm{bCl}_\theta(A) = \mathrm{bCl}(A)$ for every subset A of X,*
(3) *$\mathrm{bCl}_\theta(A) = A$ for every b-closed subset A of X.*

Proof. (1)⇒(2): Since $\mathrm{bCl}(A) \subseteq \mathrm{bCl}_\theta(A)$ by Lemma 3.1, we will show that $bCl_\theta(A) \subseteq \mathrm{bCl}(A)$. Assume that $x \notin \mathrm{bCl}(A)$. Since (X,τ) is b-regular, there exist disjoint b-open sets U and V such that $x \in U$ and $\mathrm{bCl}(A) \subset V$. Hence $A \cap \mathrm{bCl}(U) \subset \mathrm{bCl}(A) \cap \mathrm{bCl}(U) = \varnothing$. This implies $x \notin \mathrm{bCl}_\theta(A)$.

(2)⇒(3): Let A be a b-closed subset of X, then we have $A = \mathrm{bCl}(A)$ and $\mathrm{bCl}_\theta(A) = A$.

(3)⇒(1): Let V be a b-open set containing x in X. Hence $X \setminus V$ is b-closed. By (3), $X \setminus V = \mathrm{bCl}_\theta(X \setminus V)$. Then there exists an $U \in BO(X,x)$ such that $\mathrm{bCl}(U) \cap (X \setminus V) = \varnothing$. This implies $\mathrm{bCl}(U) \subset V$. Consequently, the proof is completed. □

Corollary 3.2. *Every b-regular space is a γ-R_1 space.*

Proof. This immediately follows from Theorems 3.3 and 3.4. □

Definition 3.4. Let ∇ be a filter base of a topological space (X,τ).

(1) The b-adherence (resp., b-θ-adherence) of a filter base ∇, denoted by $\mathrm{ad}_b \nabla$ (resp., θ-$\mathrm{ad}_b \nabla$), is defined by $\cap\{\mathrm{bCl}(V) \mid V \in \nabla\}$ (resp., $\cap\{\mathrm{bCl}_\theta(V) \mid V \in \nabla\}$),
(2) ∇ b-converges [8] (resp., ∇ b-θ-converges) to x if for each b-open set U containing x, there is $V \in \nabla$ such that $V \subset U$ (resp., $V \subset \mathrm{bCl}(U)$).

Theorem 3.5. *If a topological space (X,τ) is γ-R_1, then $\mathrm{ad}_b \nabla \subset \mathrm{bCl}(\{x\})$ for each filter base ∇ b-θ-converging to $x \in X$.*

Proof. Let (X,τ) be a γ-R_1 space, $x \in X$, and ∇ be a filter base in X b-θ-converging to x. If $\mathrm{ad}_b \nabla = \varnothing$, then the assertion of the theorem holds trivially. Assume that $y \in \mathrm{ad}_b \nabla$ and $y \in X \setminus \mathrm{bCl}_\theta(\{x\})$. Then there exists a b-open set U containing y such that $x \in X \setminus \mathrm{bCl}(U)$. Since ∇ b-θ-converging to x, there exists $V \in \nabla$ such that $V \subset \mathrm{bCl}(X \setminus \mathrm{bCl}(U))$. This implies that $V \cap U = \varnothing$ which contradicts the assumption that $y \in \mathrm{ad}_b \nabla$. Thus $y \in \mathrm{bCl}_\theta(\{x\})$ and $\mathrm{ad}_b \nabla \subset \mathrm{bCl}_\theta(\{x\})$. Since (X,τ) is γ-R_1, by Theorem 3.3 we obtain $\mathrm{ad}_b \nabla \subset \mathrm{bCl}(\{x\})$. □

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Doi: 10.18514/MMN.2009.210

Miskolc Mathematical Notes
Vol. 10 (2009), No. 2, pp. 145–153

HU e-ISSN 1787-2413

ON A SUMMATION FORMULA FOR THE CLAUSEN'S SERIES $_3F_2$ WITH APPLICATIONS

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Received 25 February, 2009

Abstract. The aim of this research paper is to establish the following summation formula for the Clausen's series $_3F_2$:

$$ {}_3F_2\left[\begin{matrix}-n, b-a-1, f+1 \\ b, f\end{matrix}; 1\right] = \frac{(a)_n (c+1)_n}{(b)_n (c)_n}, $$

where $f = c(1+a-b)/(a-c)$, in *three different ways*. For $c = \frac{1}{2}a$, we have

$$ {}_3F_2\left[\begin{matrix}-n, b-a-1, 2+a-b \\ b, 1+a-b\end{matrix}; 1\right] = \frac{(a)_n \left(1+\frac{1}{2}a\right)_n}{(b)_n \left(\frac{1}{2}a\right)_n}, $$

which is already available in the literarute. Our formula is then applied to obtain two general results, one is the Euler's transformation for the series $_2F_2$ and another is the Kummer–type first transformation for the series $_2F_2$ established recently by Paris by following a different method. The results obtained generalize the related results by Exton.

2000 *Mathematics Subject Classification:* 33C20, 33B15

Keywords: Clausen's series $_3F_2$, Euler's transformation for $_2F_2$, Kampé de Fériet function, Kummer type I transformation for $_2F_2$, summation formula

1. Introduction and results required

In 1997 Exton [3] derived four reduction formulæ for the Kampé de Fériet function and from one of the results, he deduced the following two results:

$$ (1-x)^{-h} {}_3F_2\left[\begin{matrix}h, a, 1+\frac{1}{2}a \\ b, \frac{1}{2}a\end{matrix}; -\frac{x}{1-x}\right] = {}_3F_2\left[\begin{matrix}h, b-a-1, 2+a-b \\ b, 1+a-b\end{matrix}; x\right] \tag{1} $$

and

$$ e^{-x} {}_2F_2\left[\begin{matrix}a, 1+\frac{1}{2}a \\ \frac{1}{2}a\end{matrix}; x\right] = {}_2F_2\left[\begin{matrix}b-a-1, 2+a-b \\ b, 1+a-b\end{matrix}; -x\right]. \tag{2} $$

The first author was supported in part by the Wonkwang University, Iksan 2008.

The second author was supported in part by the Ministry of Sciences, Education and Sports of Croatia under Research Project No. 112-2352818-2814.

It is of interest to compare these results with

$$(1-x)^{-\alpha}{}_2F_1\left[\begin{matrix}\alpha,\gamma-\beta\\ \gamma\end{matrix};-\frac{x}{1-x}\right]={}_2F_1\left[\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};x\right] \tag{3}$$

known as Euler's first transformation [8] and the confluent

$$e^{-x}{}_1F_1\left[\begin{matrix}a\\ b\end{matrix};x\right]={}_1F_1\left[\begin{matrix}b-a\\ b\end{matrix};-x\right] \tag{4}$$

known as Kummer's first theorem [8].

We remark in passing that the result (1) is also recorded in [10] where it is obtained by other method. The result (2) has been obtained by Miller [4] by following two different and transparent ways. On the other hand it should be remarked here that whenever generalized hypergeometric functions reduce to gamma functions, the results are very important from the applications point of view.

It is also well known that the generalized hypergeometric functions ${}_pF_q$ appear ubiquitously as solutions of plethora of problems in mathematics, statistics and mathematical physics. Thus, the results established in this paper should be eventually useful in a wide range of applications.

The aim of this research article is to establish first, a summation formula for the Clausen's series ${}_3F_2$ and then, as an application we derive two results which generalize (1) and (2).

In our present investigations we will require:

(i) Euler's second transformation formula [1]

$${}_2F_1\left[\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};x\right]=(1-x)^{\gamma-\alpha-\beta}{}_2F_1\left[\begin{matrix}\gamma-\alpha,\gamma-\beta\\ \gamma\end{matrix};x\right]; \tag{5}$$

(ii) the Beta-integral transform [2]

$$\int_0^1 x^{c-1}(1-x)^{e-c-1}{}_2F_1\left[\begin{matrix}a,b\\ d\end{matrix};x\right]\mathrm{d}x=\frac{\Gamma(c)\Gamma(e-c)}{\Gamma(e)}{}_3F_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix};1\right], \tag{6}$$

provided $\Re\{c\}>0$, $\Re\{e-c\}>0$, $\Re\{d+e-a-b-c\}>0$;

(iii) Bailey's transform [10, Eq. (2.4.10), p. 60] in summing double series

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}A(m,n-m). \tag{7}$$

2. Main summation formula

Theorem 1. *Let n be a non-negative integer and let $a,b,c\in\mathbb{C}$ such that*

$$f=\frac{c(1+a-b)}{a-c}\notin\mathbb{Z}_0^-:=\{0,-1,-2,\dots\}.$$

Then

$$ {}_3F_2\left[\begin{matrix}-n, b-a-1, f+1\\ b, f\end{matrix};1\right] = \frac{(a)_n(c+1)_n}{(b)_n(c)_n}, \tag{8}$$

where $(a)_n = \Gamma(a+n)/\Gamma(n) = a(a+1)\cdots(a+n-1)$ *stands for the Pochhammer symbol.*

First proof. Denoting by S the left hand side of (8), expressing ${}_3F_2$ as a series, we get

$$S = \sum_{r=0}^{n} \frac{(-n)_r(b-a-1)_r}{(b)_r\, r!} \left\{\frac{(f+1)_r}{(f)_r}\right\} = \sum_{r=0}^{n} \frac{(-n)_r(b-a-1)_r}{(b)_r\, r!} \left\{1+\frac{r}{f}\right\}.$$

Separate now S into two series, and after little adjusment in the second series, summing up the both expression, we arrive at

$$S = {}_1F_2\left[\begin{matrix}-n, b-a\\ b+1\end{matrix};1\right].$$

Applying Vandermonde's theorem in both hypergeometric series ${}_2F_1$, we have

$$S = \frac{(a+1)_n}{(b)_n} + \frac{n(a-c)}{bc}\,\frac{(a+1)_{n-1}}{(b+1)_{n-1}}$$

which, upon a short algebraic simplification yields

$$S = \frac{(a)_n(c+1)_n}{(b)_n(c)_n}.$$

The proof is complete. $\square$

Second proof. In a two-term transformation formula for the Clausen's series ${}_3F_2$, popularly known as *Kummer–Thomae–Whipple formula* [1], which evidently originates back to Kummer [1] and states that

$$ {}_3F_2\left[\begin{matrix}A, H, C\\ D, E\end{matrix};1\right] = \frac{\mathrm{B}(E, D+E-A-H-C)}{\mathrm{B}(E-C, D+E-A-H)}\, {}_3F_2\left[\begin{matrix}C, D-A, D-H\\ D, D+E-A-H\end{matrix};1\right] \tag{9}$$

provided $\Re\{E\} > 0$, $\Re\{D+E-A-H-C\} > 0$. Here

$$\mathrm{B}(s,r) = \int_0^1 x^{s-1}(1-x)^{r-1}\mathrm{d}x \qquad \left(\min\{\Re\{s\},\Re\{r\}\} > 0\right) \tag{10}$$

stands for the Euler Beta function (Its connection to the familiar Euler Gamma function $\Gamma(\cdot)$ one realizes by $\mathrm{B}(s,r) = \Gamma(s)\Gamma(r)/\Gamma(s+r)$).

In order to derive our summation formula (8), put $A = b-a-1$, $H = f+1$, $C = -n$, $D = f$, and $E = b$ in (9). So, we get after some reduction

$$S = \frac{(a)_n}{(b)_n}\, {}_3F_2\left[\begin{matrix}-n, f-b+a+1, -1\\ f, a\end{matrix};1\right] = \frac{(a)_n}{(b)_n}\left\{1+\frac{n(f-b+a+1)}{af}\right\}.$$

Using now $f = c(1+a-b)/(a-c)$, the last expression transforms into

$$S = \frac{(a)_n}{(b)_n}\left\{1+\frac{n}{c}\right\}.$$

Finally, recalling that $c+n = c(c+1)_n/(c)_n$, we conclude

$$S = \frac{(a)_n(c+1)_n}{(b)_n(c)_n},$$

such that completes the proof. □

Third proof. Consider the Beta-integral transform (6) in the slightly changed form:

$${}_3F_2\left[\begin{matrix}\alpha,\beta,s\\ \gamma,s+r\end{matrix};1\right] = \frac{1}{\mathrm{B}(\rho,\sigma)}\int_0^1 x^{s-1}(1-x)^{r-1}{}_2F_1\left[\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};x\right]\mathrm{d}x$$

provided $\Re\{s\}>0$, $\Re\{r\}>0$, and $\Re\{r+\gamma-\alpha-\beta\}>0$. In this, if we take $\alpha=-n$, $\beta = f+1$, $s=b-a-1$, $\gamma = f$, and $s+r=b$, that is, $r=a+1$, by (5) we get

$$\begin{aligned} S &= \frac{1}{\mathrm{B}(b-a-1,a+1)}\int_0^1 x^{b-a-2}(1-x)^a{}_2F_1\left[\begin{matrix}-n, f+1\\ f\end{matrix};x\right]\mathrm{d}x\\ &= \frac{1}{\mathrm{B}(b-a-1,a+1)}\int_0^1 x^{b-a-2}(1-x)^{a+n-1}{}_2F_1\left[\begin{matrix}-1, f+n\\ f\end{matrix};x\right]\mathrm{d}x\\ &= \frac{1}{\mathrm{B}(b-a-1,a+1)}\int_0^1 x^{b-a-2}(1-x)^{a+n-1}\left\{1-\frac{f+n}{f}x\right\}\mathrm{d}x. \end{aligned}$$

Separating the last integral into two parts and calculating the values of both, we get

$$S = \frac{1}{\mathrm{B}(b-a-1,a+1)}\left\{\mathrm{B}(b-a-1,a+n)-\left(1+\frac{n}{f}\right)\mathrm{B}(b-a,a+n)\right\}.$$

Now, straightforward calculations lead the asserted result (8). □

3. Applications

In this section, as applications of our summation formula (8), we derive the following two general formulæ such that generalize Exton's results (1), (2).

Theorem 2. *Let f be the same as before. Then*

$$(1-x)^{-h}{}_3F_2\left[\begin{matrix}h, b-a-1, f+1\\ b, f\end{matrix};-\frac{x}{1-x}\right] = {}_3F_2\left[\begin{matrix}h,a,c+1\\ b,c\end{matrix};x\right] \tag{11}$$

and

$$e^x{}_2F_2\left[\begin{matrix}b-a-1, f+1\\ b, f\end{matrix};-x\right] = {}_2F_2\left[\begin{matrix}a,c+1\\ b,c\end{matrix};x\right]. \tag{12}$$

Proof. Let us denote T the left-hand side of (11). First, expressing ${}_3F_2$ as a series, we conclude

$$T=\sum_{n=0}^{\infty}\frac{(-1)^n(h)_n(b-a-1)_n(f+1)_n}{(b)_n(f)_n\,n!}\,x^n(1-x)^{-h-n}.$$

Using the binomial expansion of $(1-x)^{-h}$ and applying the identity $(h)_n(h+n)_n=(h)_{n+m}$, we arrive at

$$T=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n(h)_{n+m}(b-a-1)_n(f+1)_n}{(b)_n(f)_n\,n!m!}\,x^{n+m}.$$

The consequence of the Bailey's transform (7) of the double-indexed summand will be

$$T=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n\,(h)_m(b-a-1)_n(f+1)_n}{(b)_n(f)_n\,n!(m-n)!}\,x^m.$$

Now, because of

$$(m-n)!=(-1)^n\,\frac{m!}{(-m)_n}\,, \tag{13}$$

we get

$$T=\sum_{m=0}^{\infty}\frac{(h)_m}{m!}\,x^m\sum_{n=0}^{m}\frac{(b-a-1)_n(f+1)_n(-m)_n}{(b)_n(f)_n\,n!}\,;$$

summing up the inner-most series, we have

$$T=\sum_{m=0}^{\infty}\frac{(h)_m}{m!}\,{}_3F_2\left[\begin{matrix}-m,b-a-1,f+1\\ b,f\end{matrix};1\right]x^m.$$

By the Theorem 1, we deduce

$$T=\sum_{m=0}^{\infty}\frac{(h)_m}{m!}\,x^m\,\frac{(a)_m(c+1)_m}{(b)_m(c)_m}={}_3F_2\left[\begin{matrix}h,a,c+1\\ b,c\end{matrix};x\right]$$

such that is the right-hand side expression in (11).

Denote U the expression on the left in the relation (12) and express both the functions involved, in series. After some little simplification we get

$$U=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m(b-a-1)_m(f+1)_m}{(b)_m(f)_m m!n!}\,x^{m+n}.$$

Making use of the Bailey's transform and the indetity (13), we have

$$U=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m(b-a-1)_m(f+1)_m}{(b)_m(f)_m\,m!(n-m)!}\,x^n$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{m=0}^{n} \frac{(-n)_m (b-a-1)_m (f+1)_m}{(b)_m (f)_m\, m!}.$$

Now, it remains to summing up the inner-most series getting

$$U = \sum_{n=0}^{\infty} \frac{x^n}{n!}\, {}_3F_2\left[\begin{matrix} -n, b-a-1, f+1 \\ b, f \end{matrix}; 1\right]$$

such that, by means of Theorem 1, clearly becomes

$$U = \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{(a)_n (c+1)_n}{(b)_n (c)_n} = {}_2F_2\left[\begin{matrix} a, c+1 \\ b, c \end{matrix}; x\right]$$

that is, Theorem 2 is proved. □

Another proof of (11). Having in the mind the earlier convention that T denotes the left side expression of (11), we conclude

$$\begin{aligned} T &= (1-x)^{-h}\, {}_3F_2\left[\begin{matrix} h, b-a-1, f+1 \\ b, f \end{matrix}; -\frac{x}{1-x}\right] \\ &= (1-x)^{-h} \sum_{n=0}^{\infty} \frac{(h)_n\, (b-a-1)_n}{(b)_n\, n!} \frac{(f+1)_n}{(f)_n} \left(-\frac{x}{1-x}\right)^n \\ &= (1-x)^{-h} \sum_{n=0}^{\infty} \frac{(h)_n\, (b-a-1)_n}{(b)_n\, n!} \left(1+\frac{n}{f}\right) \left(-\frac{x}{1-x}\right)^n. \end{aligned}$$

Separating the sum into two parts and summing the first series, we get

$$\begin{aligned} T &= (1-x)^{-h}\, {}_2F_1\left[\begin{matrix} h, b-a-1 \\ b \end{matrix}; -\frac{x}{1-x}\right] \\ &\quad + \frac{1}{f}(1-x)^{-h} \sum_{n=1}^{\infty} \frac{(h)_n (b-a-1)_n}{(b)_n (n-1)!} \left(-\frac{x}{1-x}\right)^n. \end{aligned}$$

Now, changing n into $n+1$ the remaining series T_2, say, becomes

$$\begin{aligned} T_2 &= \frac{1}{f}(1-x)^{-h} \sum_{n=0}^{\infty} \frac{(h)_{n+1} (b-a-1)_{n+1}}{(b)_{n+1}\, n!} \left(-\frac{x}{1-x}\right)^{n+1} \\ &= -\frac{h(b-a-1)x}{fb(1-x)^{h+1}} \sum_{n=0}^{\infty} \frac{(h+1)_n (b-a)_n}{(b+1)_n\, n!} \left(-\frac{x}{1-x}\right)^n, \end{aligned}$$

that is

$$T = (1-x)^{-h}\, {}_2F_1\left[\begin{matrix} h, b-a-1 \\ b \end{matrix}; -\frac{x}{1-x}\right]$$

$$-\frac{h(b-a-1)x}{fb(1-x)^{h+1}}\,{}_2F_1\left[\begin{matrix}h+1,b-a\\b+1\end{matrix};-\frac{x}{1-x}\right].$$

Using (3) and mentioning that $f=\frac{c(1+a-b)}{a-c}$, we deduce

$$\begin{aligned}T&={}_2F_1\left[\begin{matrix}h,a+1\\b\end{matrix};x\right]-\frac{h(b-a-1)x}{fb}\,{}_2F_1\left[\begin{matrix}h+1,a+1\\b+1\end{matrix};x\right]\\&={}_2F_1\left[\begin{matrix}h,a+1\\b\end{matrix};x\right]+\frac{h(a-c)x}{bc}\,{}_2F_1\left[\begin{matrix}h+1,a+1\\b+1\end{matrix};x\right]\\&=\left\{{}_2F_1\left[\begin{matrix}h,a+1\\b\end{matrix};x\right]-\frac{hx}{b}\,{}_2F_1\left[\begin{matrix}h+1,a+1\\b+1\end{matrix};x\right]\right\}\\&\quad+\frac{hax}{bc}\,{}_2F_1\left[\begin{matrix}h+1,a+1\\b+1\end{matrix};x\right].\end{aligned}$$

But it can be easily seen that

$$\left\{{}_2F_1\left[\begin{matrix}h,a+1\\b\end{matrix};x\right]-\frac{hx}{b}\,{}_2F_1\left[\begin{matrix}h+1,a+1\\b+1\end{matrix};x\right]\right\}={}_2F_1\left[\begin{matrix}h,a\\b\end{matrix};x\right].$$

Hence,

$$T={}_2F_1\left[\begin{matrix}h,a\\b\end{matrix};x\right]+\frac{hax}{bc}\,{}_2F_1\left[\begin{matrix}h+1,a+1\\b+1\end{matrix};x\right]={}_3F_2\left[\begin{matrix}h,a,c+1\\b,c\end{matrix};x\right]$$

such that coincides with the right hand expression in (11). □

Remark 1. Since the result (12) is a confluent limiting case of the (11), so it can be easily deduced by (11).

Remark 2. In (11), if we take $c=\frac{1}{2}a$, we get

$$(1-x)^{-h}\,{}_3F_2\left[\begin{matrix}h,b-a-1,2+a-b\\b,1+a-b\end{matrix};-\frac{x}{1-x}\right]={}_3F_2\left[\begin{matrix}h,a,1+\frac{1}{2}a\\b,\frac{1}{2}a\end{matrix};x\right]$$

which reduces to (1) by replacing x by $-x/(1-x)$.

Remark 3. In (12), taking $c=\frac{1}{2}a$, we get (2).

4. Conclusion

We mention below a result similar to (8) given by Miller [5]

$${}_3F_2\left[\begin{matrix}-n,c+1\\b,c\end{matrix};1\right]=\frac{(b-a-1)_n(f+1)_n}{(b)_n(f)_n},\tag{14}$$

f being the same as before.

For $c = \frac{1}{2}a$, this reduces to

$$ {}_3F_2\left[\begin{matrix}-n, a, 1+\frac{1}{2}a \\ b, \frac{1}{2}a\end{matrix}; 1\right] = \frac{(b-a-1)_n(2+a-b)_n}{(b)_n(1+a-b)_n}, $$

which is a known result in the literature [7].

The result quoted in Remark 2 belongs to Paris [6] who obtained this relation by a different method. Actually, the results mentioned in Remarks 2 and 3 have been also obtained very recently by Rathie and Paris [9] by utilising the summation formula (14) due to Miller [5].

We conclude the paper by remarking that the result (8) and Miller's result (14) are special cases of the following general result given already in the literature [7]

$$ {}_3F_2\left[\begin{matrix}a, b, d+1 \\ c, d\end{matrix}; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left\{c-a-b-1+\frac{ab}{d}\right\} $$

such that can be easily transformed into Beta function expression, reads as follows:

$$ {}_3F_2\left[\begin{matrix}a, b, d+1 \\ c, d\end{matrix}; 1\right] = \frac{\mathrm{B}(c, c-a-b-1)}{\mathrm{B}(c-a, c-b-1)}\left\{1+\frac{a(b-d)}{d(c-b-1)}\right\}. $$

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Doi: 10.18514/MMN.2009.206

Miskolc Mathematical Notes
Vol. 10 (2009), No. 2, pp. 155–162

HU e-ISSN 1787-2413

A NOTE ON INVOLUTION RINGS

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Received 29 January, 2009

Abstract. Classifications of subdirectly irreducible involution rings via some properties of trace elements are given. As an application, von Neumann regular involution rings with central idempotent norm elements are described. The structure of certain involution rings in which the norms are multiplicatively generated by nilpotents is also determined.

2000 *Mathematics Subject Classification:* 16N60, 16W10

Keywords: involution ring, subdirectly irreducible ring, von Neumann regular ring, norm, trace

1. Introduction

All rings considered are associative. Let us recall that an involution ring R is a ring with an additional unary operation $*$, called involution, such that $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$, and $(a^*)^* = a$ for all $a, b \in R$. An element of an involution ring R which is either symmetric or skew-symmetric shall be called a $*$-element and a $*$-ideal I of R is an ideal of R which is closed under involution, that is, $I^* = \{a^* \in R : a \in I\} \subseteq I$. A subring B of R such that $BAB \subseteq B$ and which is closed under involution, is called a $*$-biideal of R.

Motivated by the well-known fact that the algebraic structure of an involution ring is, to a large extent, determined by the algebraic nature of its symmetric or skew-symmetric elements, we study $*$-subdirectly irreducible rings with identity and in which norm elements are central and non-nilpotent. Such involution rings are shown to be rings of the following types: a division ring, a 2×2 matrix ring over a field, $F \oplus F$ where F is a field, a subdirectly irreducible ring with identity and square-zero heart. Abu-Khuzam [1] studied the structure of certain rings which are multiplicatively generated by their nilpotents. In this note, we shall extend this result to involution rings by conditioning only the norm of the ring. Indeed, we prove that an involution ring R in which the norm is multiplicatively generated by nilpotents is nil if it satisfies the descending chain condition (d.c.c.) on $*$-biideals or if the norm is central and R satisfies the polynomial identity $x^m = x^{m+1} f(x)$, where m is a positive integer and $f(x)$ is a polynomial with integer coefficients.

We recall that an involution ring R is called $*$-subdirectly irreducible if the intersection of all its non-zero $*$-ideals (called the $*$-heart) is non-zero. A routine application of Birkhoff's theorem yields that every involution ring R is a subdirect product of $*$-subdirectly irreducible rings, whence they serve as one source of basic building blocks of all involution rings. Hence the study of $*$-subdirectly irreducible rings is desirable.

In the sequel, for an involution ring R, we put $S = \{a \in R : a^* = a\}$, the symmetric elements in R, $K = \{a \in R : a^* = -a\}$, the skew-symmetric elements in R, $T = \{a + a^* : a \in R\}$, the trace of R, $N = \{aa^* : a \in R\}$, the norm of R. The center of R is denoted by Z. Moreover, following Lim [13], we call R a central trace involution ring if T is contained in the center of R.

2. Results

The following lemma shall be useful in the proof of the next theorem.

Lemma 1. *If R is a $*$-subdirectly irreducible ring with a non-zero central idempotent, then R has identity.*

Proof. Let $e \neq 0$ be a central idempotent in R. If e is a symmetric element, then e is the identity of R. Indeed, the $*$-ideals Re and $\{r - re : r \in R\}$ have zero intersection and $Re \neq 0$ and so $\{r - re : r \in R\} = 0$, which implies that e is the identity of R. Now suppose e is not symmetric. Then $ee^* = 0$, for, otherwise, ee^* would be the identity of R and then $ee^* = e$, contradicting the fact that e is not a symmetric. Likewise it follows that $e^*e = 0$. Clearly, $e + e^* \neq 0$ is an idempotent symmetric element and, arguing as above, we conclude that R has identity $e + e^*$. □

Theorem 1. *Let R be a $*$-subdirectly irreducible ring with a non-zero idempotent. If non-zero trace elements in R are central and non-nilpotent, then R is one of the following types:*

(i) *a field;*
(ii) *a division ring with involution, 4-dimensional over its centre and $N \subseteq Z$;*
(iii) *$F \oplus F$, where F is a field, endowed with the exchange involution;*
(iv) *the 2×2 matrix ring over a field F, with the symplectic involution, i. e.,*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(v) *a subdirectly irreducible ring with identity and heart $H = H^* \subseteq K$, $H^2 = 0$ and the additive group of H is either an elementary Abelian p-group or a torsion-free divisible group.*

Proof. Let e be an idempotent in R. Since the trace elements in R are central, the norm elements in R are also central [13]. If e is symmetric, then it is a norm element and therefore central. If e is not symmetric, then $e + e^* \neq 0$ is an idempotent trace element and therefore central. It follows from Lemma 1 that R has identity.

Now either all elements in the $*$-heart H are square zero and skew-symmetric, or all elements in the $*$-heart are skew-symmetric and $hh^* = -h^2 \neq 0$ for some $h \in H$, or there exists an element $h + h^* \neq 0$ for some $h \in H$.

In the first two cases, by [5, Proposition 1.1] and our assumptions on R, we see that the $*$-heart H is a minimal ideal of R. By [8, Proposition 6.2] the additive group of H is either an elementary Abelian p-group or a torsion-free divisible group. Moreover, we claim that R is a subdirectly irreducible ring. Indeed, if $H \cap J = 0$ for some non-zero ideal J of R then it is clear that $J \cap J^* = 0$ and $H \subseteq J \oplus J^*$. Hence, any $h \in H$ can be represented uniquely in the form $h = a + b^*$ for some $a, b \in J$. Now $h^* = a^* + b = -a - b^*$ and so $a + b = -b^* - a^* \in J \cap J^*$. This implies $b = -a$ and thus $h = a - a^*$. Therefore $ha = a^2 - a^*a = a^2$. But $(ha)^* = a^*h^* = -ha \in J \cap J^*$ so that $a^2 = 0$. Thus $(a + a^*)^2 = 0$ and, by our assumption on R, we must have $a^* = -a$, which implies $h = 2a \in J$, a contradiction.

In the first case, R is a subdirectly irreducible ring with identity and heart $H = H^* \subseteq K$, $H^2 = 0$.

In the second case, let $0 \neq a = hh^*$, $h \in H$. Then $aR = Ra = H$. Since H is a prime ring, $a^2R = a(aR) = aH \neq 0$. Moreover, since a^2 is central, $H = a^2R$ and $a = a^2b$ for some $b \in R$. Now $ab \neq 0$, $(ab)^2 = (a^2b)b = ab$ and $(ab)^* = -ab = ab$, whence $ab \in H$ is the identity of R. Therefore $H = R$ and R is a primitive ring. Since, for each $a \in R$, $a^* = -a$, we have $a^2 \in S$ and $T = 0$. If there exists $a \in R$ such that $2a \neq 0$, then $(2a)^2 = 2a^2 + 2a^2 \in T$ and so $(2a)^2$. But then $2aR$ is a nil right ideal of R, a contradiction with the fact that R is Jacobson-semisimple. Moreover, for any $a, b \in R$, $(ab)^* = -ab$, that is, $b^*a^* = ba = -ab = ab$. Hence R is a field of characteristic 2.

We study the third case. If $a = h + h^* \neq 0$ for some $h \in H$, then $a^2R = H$. Hence $a = a^2b$ for some $b \in R$ and it follows that $0 \neq ab$ is an idempotent. Thus the identity of R belongs to H, whence $H = R$ and R is $*$-simple. Either R is a simple prime ring or $R = I \oplus I^*$ where I and I^* are simple prime rings and the only proper non-zero ideals of R are I and I^* [5]. If R is a simple prime ring then, by [9, Theorem 7] and [12, Theorem 1], it is either a division ring or a 2×2 matrix ring over a field, endowed with the symplectic involution. If R is a division ring, then it follows from [6, Theorem 2] that R is either a field or 4-dimensional over its centre. Finally, we show that if $R = I \oplus I^*$, then I is a field. In this case, the identity of R is of the form $e + e^*$ with $e \in I$. Since R has no non-zero nilpotent trace elements, it is clear that I cannot have non-zero nilpotent elements. Now, for any $x \in R$, $ex - exe$ and $xe - exe$ are nilpotent elements in I and hence $ex - exe = xe - exe = 0$, whence $ex = xe$ and e is a central element and so is e^*. Then $R = R(e + e^*) = Re \oplus Re^*$, $I = Re$, and e is the identity of I. In addition, I is commutative. Indeed, if $a, b \in I$, then $(a + a^*)(b + b^*) = (b + b^*)(a + a^*)$, that is, $ab + a^*b^* = ba + b^*a^*$. Thus $ab - ba = b^*a^* - a^*b^* \in I \cap I^* = 0$ and I is commutative. Moreover, any $0 \neq c \in I$ is invertible since $cI = I = Ic$. □

The following is an example of an involution ring of type (v) in the above theorem.

Example 1. Let R be the set of all $\alpha x + \beta e$ where $\alpha, \beta \in GF(2)$ and $x^2 = 0$, $e^2 = e$, $ex = xe = x$. Then R is clearly a commutative ring with identity element and with the identity involution. The only non-zero proper ideal of R is $H = \{\alpha x : \alpha \in GF(2)\} = H^*$. Hence R is subdirectly irreducible with identity and heart H, where $H^2 = 0$. The trace of R is 0.

Corollary 1. *R is a semiprime $*$-subdirectly irreducible ring with non-zero idempotent and non-zero trace elements in R are central and non-nilpotent if and only if R is one of the following:*

(i) *a field;*
(ii) *a division ring with involution, 4-dimensional over its centre and $N \subseteq Z$;*
(iii) *$F \oplus F$ where F is a field, endowed with the exchange involution;*
(iv) *the 2×2 matrix ring over a field F, with the symplectic involution.*

Corollary 2. *Let R be a $*$-subdirectly irreducible ring with identity and with central idempotent norm and trace elements. Then R is one of the following types:*

(i) *$GF(2)$ with the identity involution;*
(ii) *$GF(4)$ with the inverse involution;*
(iii) *$GF(2) \oplus GF(2)$, endowed with the exchange involution;*
(iv) *the 2×2 matrix ring over the field $GF(2)$, with the symplectic involution;*
(v) *a subdirectly irreducible involution ring with identity, of characteristic 2 and heart H in which the elements are symmetric and $H^2 = 0$.*

Proof. A division ring R with involution in which every norm element is idempotent has the inverse involution. In fact, for any $0 \neq a \in R$, $0 \neq aa^*$ and $(aa^*)^2 = aa^*$ implies $aa^* = 1$ (the identity of R) and, similarly, $a^*a = 1$, so that $a^* = a^{-1}$. Moreover, since the trace elements are idempotent, the characteristic of R must be 2. In fact, $(1+1)^2 = 1+1$ implies $1+1 = 0$. By [7], we know that such a $*$-subdirectly irreducible ring is either $GF(2)$ or $GF(4)$. In $F \oplus F$ (F a field), $T = N = \{(a,a) : a \in F\}$ and hence, if every element in N is idempotent, it is clear that $F \cong GF(2)$. Similarly, in a 2×2 matrix ring over a field F, with the symplectic involution, $T = N = \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) : a \in F\right\}$ and the idempotence of each element in T implies $F \cong GF(2)$. $\square$

Proposition 1. *If R is a von Neumann regular involution ring with central idempotent norm elements, then R is isomorphic to a subdirect sum of involution rings R_i ($i \in \Lambda$), where each R_i is one of the following:*

(i) *$GF(2)$ with the identity involution;*
(ii) *$GF(3)$ with the identity involution;*
(iii) *$GF(4)$ with the inverse involution;*
(iv) *$GF(2) \oplus GF(2)$, endowed with the exchange involution;*
(v) *a 2×2 matrix ring over $GF(2)$, with the symplectic involution.*

Proof. Any nonzero homomorphic image of R is semiprime and has a nonzero central idempotent element. In fact, if $a+I$ is a nonzero element in the factor ring R/I, then $ax+I$, where $axa=a$, is a nonzero idempotent in R/I. Let $e+I=ax+I$. If $ee^* \notin I$, then ee^*+I is a nonzero central idempotent in R/I. On the other hand, if $ee^* \in I$, then $e+e^*+I$ is a nonzero central idempotent in R/I. Thus, by Lemma 1, R/I has identity and by [13], the norm elements are central if and only if the trace elements are central. Hence, since R/I is von Neumann regular, R/I cannot have nonzero nilpotent trace elements. By Corollary 1, R/I is either a division central trace ring, or $F \oplus F$ where F is a field, endowed with the exchange involution, or a 2×2 matrix ring over a field F, with the symplectic involution. If R/I is a division central trace ring, then the characteristic of R/I is either 2 or 3. Indeed, since the norm elements in R are idempotent, $(1+1)^4+I=(1+1)^2+I$ (where $1+I$ is the identity of R/I), whence either $(1+1)+I=I$ or $(1+1)^2+I=1+I$, that is, the characteristic of R/I is either 2 or 3. By [7], R/I is of type $GF(2)$, $GF(3)$ or $GF(4)$. □

Proposition 2. *R is a von Neumann regular involution ring with zero trace and idempotent norm elements if and only if R is a Boolean ring with identity involution.*

Proof. We prove the direct implication. For any elements $a,b,c \in R$, $a^*=-a$, $ab=-ba$ and $2abc=0$, whence $(2R)^3=0$ and so $2R=0$. Hence R has characteristic 2, R is commutative and $a^*=a$. For any $a \in R$, $a^4=a^2$ since the norm elements are idempotent. There exists $y \in R$ such that $(a^2-a)y(a^2-a)=a^2-a$, that is, $(a^2-a)^2y=a^2-a$, which implies $a^2=a$. □

For an involution ring R, we let N' be the set of symmetric nilpotent elements in R and E the set of symmetric central idempotents in R. The proof of the following proposition is as that of [11, Lemma 1], but we include it for the sake of completeness.

Proposition 3. *Let R be an involution ring of characteristic 2 having a non-zero central idempotent. Then the following conditions are equivalent:*

(i) *the symmetric elements in R can be represented as a product of elements in $E \cup N'$.*

(ii) *the symmetric elements in R are central idempotent.*

Proof. (i) implies (ii). Suppose symmetric elements in R can be represented as a product of elements in $E \cup N'$. Clearly, R has a non-zero central idempotent symmetric element e. First, we show that $eN'=0$. The $*$-subring $A=eR$ has identity e and its symmetric elements can be represented as a product of elements in $E \cup N'$. If a is any element in N', then $e+ea=x_1x_2\cdots x_n$ for some $x_i \in eE \cup eN'$. Since $e+ea$ is invertible in A, x_1 cannot be nilpotent, so that $x_1 \in eE$. Hence $e+ea=x_1(e+ea)$, which implies $x_1=e$ and $e+ea=x_2\cdots x_n$. Repeating this procedure, we eventually obtain $e+ea=e$, implying $ea=0$. This shows that $eN'=0$. Suppose now there exists a symmetric element r in R which cannot be expressed as a product of symmetric central idempotents. Let $e+r = y_1 y_2 \cdots y_k$ for certain $y_1, y_2, \ldots, y_k \in E \cup N'$. Since $eN' = 0$, we have $e = e(e+r) = (ey_1)(ey_2)\cdots(ey_k)$ and the elements $y_1, y_2, \ldots, y_k$ are central idempotent. Hence $y_i + e$ is central idempotent. Moreover, for each $i = 1, 2, \ldots, n$, $ey_i = e$. So, $r = y_1 y_2 \cdots y_k + e = (y_1+e)(y_2+e)\cdots(y_k+e)$, which is a contradiction. This shows that every symmetric element in R can be written as a product of central idempotents and hence is central idempotent. The converse is trivial. □

Abu-Khuzam [1] studied the structure of certain rings which, as semi-groups, are multiplicatively generated by their nilpotents. We now consider certain involution rings in which the norms are central and multiplicatively generated by nilpotents (that is, the norm elements can be written as a product of nilpotents). We call such an involution ring a $*$-N-ring. If R is a $*$-N-ring with identity, then $R = 0$ since the identity of R is a norm element. Moreover, any $*$-homomorphic image of a $*$-N-ring is also such a ring.

Theorem 2. *If R is a $*$-N-ring with d.c.c. on $*$-biideals, then R is nilpotent.*

Proof. Let J denote the Jacobson radical of R and suppose $J \neq R$. It is well-known that $J^* = J$. Since R/J is a semiprime involution ring with d.c.c. on $*$-biideals, R/J has an identity $e + J$ [2] and, clearly, $e + J = e^* + J = ee^* + J$. Since R/J is a $*$-N-ring, by the considerations preceding the theorem, $R/J = 0$, that is, $R = J$, a contradiction. Hence $R = J$ and R is nilpotent [3]. □

Let us recall (see [4]) that a $*$-ideal P of an involution ring R is said to be $*$-prime if $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$, where A and B are $*$-ideals of R and R is is $*$-prime if the zero ideal is $*$-prime. Birkenmeier and Groenewald showed that if P is a $*$-prime $*$-ideal which is not a prime ideal, and X is minimal among prime ideals containing P, then P is a prime ideal of X. In addition, they showed that the intersection of all the $*$-prime $*$-ideals of R coincides with the prime radical of R. Hence, if the prime radical of R is 0, then R is a subdirect product of $*$-prime rings. We shall use these facts in the proof of the next theorem and we shall also require the following lemmas.

Lemma 2 ([1])**.** *If a ring R satisfies the polynomial identity $x^m = x^{m+1}f(x)$, then the Jacobson radical of R is nil.*

Lemma 3 ([10])**.** *Let I be any non-zero ideal of a ring R. Then there exists a homomorphic image R' of R containing an isomorphic copy I' of I such that I' is essential in R'.*

The proof of the following theorem is an adaptation of the corresponding one for rings without involution in [1].

Theorem 3. *Let R be a $*$-N-ring which satisfies the polynomial identity $x^m = x^{m+1}f(x)$ (m is a positive integer and $f(x)$ is a polynomial with integer coefficients). Then R is nil.*

Proof. The Jacobson radical $J = J^*$ of R is nil by Lemma 2. Since R/J is a semiprime involution ring its prime radical is 0 and hence, by the considerations preceding the lemmas, the intersection of all $*$-prime $*$-ideals of R is 0. Hence R is a subdirect product of $*$-prime rings R_i ($i \in \Lambda$). If R_i is a prime ring, then as in [1, Theorem 2], it follows that R_i has identity and since it is a $*$-N-ring, $R_i = 0$. Now suppose R_i is not a prime ring. We have that $R_i = R/K_i$ for some $*$-prime $*$-ideal K_i of R which is not a prime ideal. Now if X_i is minimal among prime ideals containing K_i, then K_i is a prime ideal of X_i, that is X_i/K_i is a prime ring and satisfies the polynomial identity $x^m = x^{m+1} f(x)$. Hence, as in [1], it follows that X_i/K_i has an identity. According to Lemma 3, there exists a homomorphic image R_i' of R_i containing an isomorphic copy X_i' of X_i/K_i such that X_i' is essential in R_i'. Then, since X_i' has identity, $X_i' = R_i'$. Now $R_i' \cong R/Q_i$ for some ideal Q_i of R. Moreover, if $e + Q_i$ is the identity of R/Q_i, then $e^* + Q_i = ee^* + Q_i = n_1 n_2 \cdots n_k + Q_i$ for certain nilpotent elements $n_1, n_2, \ldots, n_k$ of R. Suppose n_1 has degree of nilpotence t. Then $n_1^{t-1} ee^* + Q_i = n_1^{t-1} n_1 n_2 \cdots n_k + Q_i = Q_i$ which implies $n_1^{t-1} e^* \in Q_i$. Therefore $n_1^{t-1} e^* e n_2 \cdots n_k = n_1^{t-2}(e^*e)^2 \in Q_i$. Continuing with this process, we eventually obtain $n_1^{t-(t-1)}(e^*e)^{t-1} \in Q_i$, that is $n_1(e^*e)^{t-1} \in Q_i$. Therefore $n_1(e^*e)^{t-1} n_2 \ldots n_k \in Q_i$ and $(e^*e)^t \in Q_i$. So, if t is even, $(e^*e)^t R = (e^*e)^{\frac{t}{2}} R (e^*e)^{\frac{t}{2}} \subseteq Q_i$ and since Q_i is a prime ideal of R, $(e^*e)^{\frac{t}{2}} \in Q_i$. On the other hand, if t is odd, $(e^*e)^{t+1} R \subseteq Q_i$ implies that $(e^*e)^{\frac{t+1}{2}} \in Q_i$. Continuing in this way, we eventually obtain $e^*e \in Q_i$ and hence $e^* \in Q_i$. Then $e + e^* + Q_i = e + Q_i$ and arguing as above, with $e + e^* + Q_i$ instead of $e + Q_i$, we obtain $e + e^* \in Q_i$. This implies $R/Q_i = 0$, which is a contradiction. Thus R_i is a prime ring and $R_i = 0$. It follows that $R/J = 0$ and $R = J$ is nil. □

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Doi: 10.18514/MMN.2009.194

Miskolc Mathematical Notes
Vol. 10 (2009), No. 2, pp. 163–171

HU e-ISSN 1787-2413

DECOMPOSITION OF $(1,2)*$-CONTINUITY AND $(1,2)*$-α-CONTINUITY

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Received 18 August, 2008

Abstract. In this paper, we introduce new types of sets called $(1,2)*$-D(α,p) sets, $(1,2)*$-D(α,s) sets, $(1,2)*$-D(c,α) sets, $(1,2)*$-D(c,s) sets and $(1,2)*$-D(c,p) sets and new classes of mappings called $(1,2)*$-D(α,p) continuous, $(1,2)*$-D(α,s) continuous, $(1,2)*$-D(c,α) continuous, $(1,2)*$-D(c,s) continuous and $(1,2)*$-D(c,p) continuous mappings. We obtain several characterizations of these classes, study their bitopological properties, and investigate their relation with other bitopological sets and mappings.

2000 *Mathematics Subject Classification:* 54E55

Keywords: $(1,2)*$-D(α,p) set, $(1,2)*$-D(α,s) set, $(1,2)*$-D(c,α) set, $(1,2)*$-D(c,s) set, $(1,2)*$-D(c,p) set, $(1,2)*$-D(α,p) continuous mapping, $(1,2)*$-D(α,s) continuous mapping, $(1,2)*$-D(c,α) continuous mapping, $(1,2)*$-D(c,s) continuous mapping, $(1,2)*$-D(c,p) continuous mapping

1. Introduction

Njastad [7] intiated the study of the notion of a nearly open set in a topological space. Following it, a number of research papers were written by Tong [11, 12], Hatir and Noiri [3–5], Przemski [8], Dontchev and Przemski [1] and Ganster [2] where decompositions of continuity in topological spaces are considered. Here, following these lines, we study the "decomposition of $(1,2)*$- continuity and $(1,2)*$-α-continuity" and deal with new types of sets such as $(1,2)*$-D(α,p) sets, $(1,2)*$-D(α,s) sets, $(1,2)*$-D(c,α) sets, $(1,2)*$-D(c,s) sets and $(1,2)*$-D(c,p) sets, as well as with new classes of mappings such as $(1,2)*$-D(α,p) continuous, $(1,2)*$-D(α,s) continuous, $(1,2)*$-D(c,α) continuous, $(1,2)*$-D(c,s) continuous and $(1,2)*$-D(c,p) continuous mappins. In this paper, we obtain some important results in bitopological spaces. In most of the occasions, our ideas are illustrated and substantiated by suitable examples.

2. Preliminaries

Let (X,τ_1,τ_2) and (Y,σ_1,σ_2), briefly X and Y, be bitopological spaces.

Definition 1 ([6]). Let S be a subset of X. Then S is called $\tau_{1,2}$-open if $S = A \cup B$, where $A \in \tau_1$ and $B \in \tau_2$. The complement of $\tau_{1,2}$-open set is called $\tau_{1,2}$-closed.

Definition 2 ([6]). Let S be a subset of X. Then:

(i) the $\tau_1\tau_2$-closure of S, denoted by $\tau_1\tau_2\text{-Cl}\, S$, is defined by

$$\cap\{F \mid S \subseteq F \text{ and } F \text{ is } \tau_{1,2}\text{-closed}\};$$

(ii) the $\tau_1\tau_2$-interior of S, denoted by $\tau_1\tau_2\text{-Int}\, S$, is defined by

$$\cup\{F \mid F \subseteq S \text{ and } F \text{ is } \tau_{1,2}\text{-open}\}.$$

Note 1 ([6]). Notice that $\tau_{1,2}$-open sets need not necessarily form a topology.

Definition 3. A subset S of X is called:

(i) $(1,2)*$-α-open [6] if $S \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(S)))$;
(ii) $(1,2)*$-semi-open [6] if $S \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(S))$;
(iii) $(1,2)*$-preopen [6] if $S \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(S))$;
(iv) $(1,2)*$-α-closed [10] if $\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(S))) \subseteq S$;
(v) $(1,2)*$-semi-closed [10] if $\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(S)) \subseteq S$;
(vi) $(1,2)*$-preclosed [9] if $\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-}\int(S)) \subseteq S$.

The family of $(1,2)*$-α-open sets (resp. $(1,2)*$-semi-open sets, $(1,2)*$-preopen sets) of X is denoted by $(1,2)*\text{-}\alpha O(X)$ (resp. $(1,2)*\text{-}SO(X)$, $(1,2)*\text{-}PO(X)$).

The complement of $(1,2)*$-α-open (resp. $(1,2)*$-semi-open, $(1,2)*$-preopen) set is $(1,2)*$-α-closed (resp. $(1,2)*$-semi-closed, $(1,2)*$-preclosed).

The intersection of all $(1,2)*$-α-closed (resp. $(1,2)*$-semi-closed, $(1,2)*$-preclosed) sets containing A is called the $(1,2)*$-α-closure (resp. $(1,2)*$-semi-closure, $(1,2)*$-preclosure) of A and is denoted by $(1,2)*\text{-}\alpha\,\text{Cl}(A)$ [resp. $(1,2)*\text{-s}\,\text{Cl}(A)$, $(1,2)*\text{-p}\,\text{Cl}(A)$).

The union of all $(1,2)*$-α-open (resp. $(1,2)*$-semi-open, $(1,2)*$-preopen) sets contained in A is called the $(1,2)*$-α-interior (resp. $(1,2)*$-semi-interior, $(1,2)*$-preinterior) of A and is denoted by $(1,2)*\text{-}\alpha\,\text{Int}(A)$ (resp. $(1,2)*\text{-s}\,\text{Int}(A)$, $(1,2)*\text{-p}\,\text{Int}(A)$).

Result 1 ([9]). *Let S be a subset of X. Then:*

(i) $(1,2)*\text{-s}\,\text{Cl}(S) = S \cup \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(S))$;
(ii) $(1,2)*\text{-s}\,\text{Int}(S) = S \cap \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-}\int(S))$.

3. A NEW TYPE OF SETS

Proposition 1. *For any subset A of a bitopological space X, the followings hold:*

(i) $(1,2)*\text{-}\alpha\,\text{Cl}(A) = A \cup \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$;
(ii) $(1,2)*\text{-}\alpha\,\text{Int}(A) = A \cap \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$;
(iii) $(1,2)*\text{-p}\,\text{Cl}(A) = A \cup \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$;
(iv) $(1,2)*\text{-p}\,\text{Int}(A) = A \cap \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$.

Proof. (i) Since $(1,2)*$-$\alpha\,\mathrm{Cl}(A)$ is $(1,2)*$-α-closed set, $A \subseteq (1,2)*$-$\alpha\,\mathrm{Cl}(A)$. we have $\tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(\tau_1\tau_2$-$\mathrm{Cl}((1,2)*$-$\alpha\,\mathrm{Cl}(A)))) \subseteq (1,2)*$-$\alpha\,\mathrm{Cl}(A)$ and hence $A \cup \tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(\tau_1\tau_2$-$\mathrm{Cl}(A))) \subseteq (1,2)*$-$\alpha\,\mathrm{Cl}(A)$. On the other hand, we observe that

$$\begin{aligned}
&\tau_1\tau_2\text{-}\mathrm{Cl}(\tau_1\tau_2\text{-}\mathrm{Int}(\tau_1\tau_2\text{-}\mathrm{Cl}(A \cup \tau_1\tau_2\text{-}\mathrm{Cl}(\tau_1\tau_2\text{-}\mathrm{Int}(\tau_1\tau_2\text{-}\mathrm{Cl}(A))))))\\
&\subseteq \tau_1\tau_2\text{-}\mathrm{Cl}(\tau_1\tau_2\text{-}\mathrm{Int}(\tau_1\tau_2\text{-}\mathrm{Cl}(A)) \cup \tau_1\tau_2\text{-}\mathrm{Int}(\tau_1\tau_2\text{-}\mathrm{Cl}(\tau_1\tau_2\text{-}\mathrm{Int}(\tau_1\tau_2\text{-}\mathrm{Cl}(A)))))\\
&= \tau_1\tau_2\text{-}\mathrm{Cl}(\tau_1\tau_2\text{-}\mathrm{Int}(\tau_1\tau_2\text{-}\mathrm{Cl}(A)) \cup \tau_1\tau_2\text{-}\mathrm{Int}(\tau_1\tau_2\text{-}\mathrm{Cl}(A)))\\
&= \tau_1\tau_2\text{-}\mathrm{Cl}(\tau_1\tau_2\text{-}\mathrm{Int}(\tau_1\tau_2\text{-}\mathrm{Cl}(A)))\\
&\subseteq A \cup \tau_1\tau_2\text{-}\mathrm{Cl}(\tau_1\tau_2\text{-}\mathrm{Int}(\tau_1\tau_2\text{-}\mathrm{Cl}(A))).
\end{aligned}$$

This shows that $A \cup \tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(\tau_1\tau_2$-$\mathrm{Cl}(A)))$ is $(1,2)*$-α-closed set and so, $(1,2)*$-$\alpha\,\mathrm{Cl}(A) \subseteq A \cup \tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(\tau_1\tau_2$-$\mathrm{Cl}(A)))$. Thus, $(1,2)*$-$\alpha\,\mathrm{Cl}(A) = A \cup \tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(\tau_1\tau_2$-$\mathrm{Cl}(A)))$.

(ii) $(1,2)*$-$\alpha\,\mathrm{Cl}(X \setminus A) = (X \setminus A) \cup \tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(\tau_1\tau_2$-$\mathrm{Cl}(X \setminus A)))$. Then $X \setminus (1,2)*$-$\alpha\,\mathrm{Cl}(X \setminus A) = (X \setminus (X \setminus A)) \cap (X \setminus \tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(\tau_1\tau_2$-$\mathrm{Cl}(X \setminus A))))$. Therefore $(1,2)*$-$\alpha\,\mathrm{Int}(A) = A \cap \tau_1\tau_2$-$\mathrm{Int}(\tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(A)))$.

(iii) We observe that

$$\begin{aligned}
&\tau_1\tau_2\text{-}\mathrm{Cl}(\tau_1\tau_2\text{-}\mathrm{Int}(A \cup \tau_1\tau_2\text{-}\mathrm{Cl}(\tau_1\tau_2\text{-}\mathrm{Int}(A))))\\
&\subseteq \tau_1\tau_2\text{-}\mathrm{Cl}(\tau_1\tau_2\text{-}\mathrm{Int}(A)) \cup \tau_1\tau_2\text{-}\mathrm{Cl}(\tau_1\tau_2\text{-}\mathrm{Int}(A))\\
&= \tau_1\tau_2\text{-}\mathrm{Cl}(\tau_1\tau_2\text{-}\mathrm{Int}(A)) \subseteq A \cup \tau_1\tau_2\text{-}\mathrm{Cl}(\tau_1\tau_2\text{-}\mathrm{Int}(A)).
\end{aligned}$$

This shows that the set $A \cup \tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(A))$ is $(1,2)*$-preclosed and thus $(1,2)*$-$\mathrm{pCl}(A) \subseteq A \cup \tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(A))$. On the other hand, since $(1,2)*$-$\mathrm{pCl}(A)$ is $(1,2)*$-preclosed, we obtain that $\tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(A)) \subseteq \tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}((1,2)*$-$\mathrm{pCl}(A))) \subseteq (1,2)*$-$\mathrm{pCl}(A)$ and hence $A \cup \tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(A)) \subseteq (1,2)*$-$\mathrm{pCl}(A)$. Thus, $(1,2)*$-$\mathrm{pCl}(A) = A \cup \tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(A))$.

(iv) $(1,2)*$-$\mathrm{pCl}(X \setminus A) = (X \setminus A) \cup \tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(X \setminus A))$. Then $X \setminus (1,2)*$-$\mathrm{pCl}(X \setminus A) = X \setminus (X \setminus A) \cap (X \setminus \tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(X \setminus A)))$. Hence we have $(1,2)*$-$\mathrm{pInt}(A) = A \cap \tau_1\tau_2$-$\mathrm{Int}(\tau_1\tau_2$-$\mathrm{Cl}(A))$. □

We introduce a new type of sets as follows.

Definition 4. For a bitopological space (X, τ_1, τ_2), we define

(i) $(1,2)*$-$\mathrm{D}(\alpha,\mathrm{p}) = \{S \subseteq X : (1,2)*$-$\alpha\,\mathrm{Int}(S) = (1,2)*$-$\mathrm{pInt}(S)\}$;

(ii) $(1,2)*$-$\mathrm{D}(\alpha,\mathrm{s}) = \{S \subseteq X : (1,2)*$-$\alpha\,\mathrm{Int}(S) = (1,2)*$-$\mathrm{sInt}(S)\}$.

Proposition 2. *S is a $(1,2)*$-semi-open set if and only if $\tau_1\tau_2$-$\mathrm{Cl}(S) = \tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(S))$.*

Proof. Since $S \subseteq \tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(S))$, $\tau_1\tau_2$-$\mathrm{Cl}(S) \subseteq \tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(S))$. On the other hand, since $\tau_1\tau_2$-$\mathrm{Int}(S) \subseteq S$, we have $\tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(S)) \subseteq \tau_1\tau_2$-$\mathrm{Cl}(S)$.

Thus $\tau_1\tau_2$-Cl$(S) = \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(S))$. Conversely, since $S \subseteq \tau_1\tau_2$-Cl$(S) = \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(S))$, S is a $(1,2)*$-semi-open. □

4. Comparisons

Remark 1. We have the following diagram:

$$\begin{array}{ccccc}
(1,2)*\text{-preopen set} & \Leftarrow & (1,2)*\text{-}\alpha\text{-open set} & \Rightarrow & (1,2)*\text{-semi-open set} \\
\not\Uparrow & & & & \Downarrow \\
(1,2)*\text{-D}(\alpha,\text{s})\text{ set} & & & & (1,2)*\text{-D}(\alpha,\text{p})\text{ set}
\end{array}$$

The following statements enable us to realize that none of the implications in the above diagram is reversible.

Example 1. Let$X = \{a,b,c\}$, $\tau_1 = \{\varnothing, X, \{a,b\}\}$ and $\tau_2 = \{\varnothing, X, \{b,c\}\}$. Then the sets in $\{\varnothing, X, \{a,b\}, \{b,c\}\}$ are $\tau_{1,2}$-open and the sets in $\{\varnothing, X, \{c\}, \{a\}\}$ are $\tau_{1,2}$-closed.

Example 2. Let $X = \{a,b,c\}$, $\tau_1 = \{\varnothing, X, \{a\}\}$ and $\tau_2 = \{\varnothing, X, \{b\}\}$. Then the sets in $\{\varnothing, X, \{a\}, \{b\}, \{a,b\}\}$ are $\tau_{1,2}$-open and the sets in $\{\varnothing, X, \{c\}, \{b,c\}, \{a,c\}\}$ are $\tau_{1,2}$-closed.

Example 3. Let $X = \{a,b,c,d\}$, $\tau_1 = \{\varnothing, X, \{c,d\}\}$ and $\tau_2 = \{\varnothing, X, \{a\}\}$. Then the elements of $\{\varnothing, X, \{a\}, \{c,d\}, \{a,c,d\}\}$ are $\tau_{1,2}$-open sets and the elements of $\{\varnothing, X, \{b\}, \{a,b\}, \{b,c,d\}\}$ are $\tau_{1,2}$-closed sets.

Remark 2. $(1,2)*$-preopen and $(1,2)*$-D$(\alpha,$p$)$ sets are independent each other. Indeed, in Example 1, clearly $\{b\}$ is $(1,2)*$-preopen but it is not $(1,2)*$-D$(\alpha,$p$)$ set. Moreover, $\{a\}$ is $(1,2)*$-D$(\alpha,$p$)$ set but it is not $(1,2)*$-preopen.

Remark 3. $(1,2)*$-semi-open and $(1,2)*$-D$(\alpha,$s$)$ sets are independent each other. Indeed, in Example 2, clearly $\{a,c\}$ is $(1,2)*$-semi-open set but it is not $(1,2)*$-D$(\alpha,$s$)$ set. Moreover, $\{c\}$ is $(1,2)*$-D$(\alpha,$s$)$ set but it is not $(1,2)*$-semi-open.

Remark 4. $(1,2)*$-semi-open and $(1,2)*$-preopen sets are independent each other. Indeed, in Example 1, clearly $\{b\}$ is $(1,2)*$-preopen set but it is not $(1,2)*$-semi-open set. In Example 2, $\{a,c\}$ is $(1,2)*$-semi-open set but it is not $(1,2)*$-preopen.

Theorem 1. *Let A be a subset of X. Then A is $(1,2)*$-α-open set in X if and only if A is $(1,2)*$-semi-open and $(1,2)*$-preopen in X.*

Proof. Let $A \in (1,2)*$-$\alpha O(X)$. By the definition of $(1,2)*$-α-open set, we have $A \subset \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ and $A \subset \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$. Therefore $A \in (1,2)*$-$PO(X)$ and $A \in (1,2)*$-$SO(X)$. Hence $A \in (1,2)*$-$PO(X) \cap (1,2)*$-$SO(X)$.

Conversely, let $A \in (1,2)*$-$SO(X)$. Then, by Proposition 2, $\tau_1\tau_2$-Cl$(A) = \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$. Then $\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$. Let $A \in (1,2)*$-$PO(X)$. Then $A \subset \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$. We have $A \subset \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$. Thus $A \in (1,2)*$-$\alpha O(X)$. □

Example 4. A $(1,2)*$-semi-open set need not be $(1,2)*$-α-open. In Example 2, $\{a,c\}$ is $(1,2)*$-semi-open set but it is not $(1,2)*$-α-open.

Example 5. A $(1,2)*$-preopen set need not be $(1,2)*$-α-open. In Example 1, $\{b\}$ is $(1,2)*$-preopen set but it is not $(1,2)*$-α-open.

Proposition 3. *For a bitopological space* (X,τ_1,τ_2)*, the relation* $(1,2)*$-$SO(X) \subseteq (1,2)*$-$\mathrm{D}(\alpha,\mathrm{p})$ *holds.*

Proof. Let $A \in (1,2)*$-$SO(X)$. Then, by virtue of Proposition 2, $\tau_1\tau_2$-$\mathrm{Cl}(A) = \tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(A))$. Then $\tau_1\tau_2$-$\mathrm{Int}(\tau_1\tau_2$-$\mathrm{Cl}(A)) = \tau_1\tau_2$-$\mathrm{Int}(\tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(A)))$. We have $A \cap \tau_1\tau_2$-$\mathrm{Int}(\tau_1\tau_2$-$\mathrm{Cl}(A)) = A \cap \tau_1\tau_2$-$\mathrm{Int}(\tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(A)))$. Hence, $(1,2)*$-$\mathrm{p}\,\mathrm{Int}(A) = (1,2)*$-$\alpha\,\mathrm{Int}(A)$ by Proposition 1. Therefore A is a $(1,2)*$-$\mathrm{D}(\alpha,\mathrm{p})$ set. ☐

Example 6. A $(1,2)*$-$\mathrm{D}(\alpha,\mathrm{p})$ set need not be $(1,2)*$-semi-open. In Example 2, $\{a\}$ is $(1,2)*$-$\mathrm{D}(\alpha,\mathrm{p})$ set but it is not $(1,2)*$-semi-open.

Remark 5. $(1,2)*$-$\mathrm{D}(\alpha,\mathrm{s})$ and $(1,2)*$-preopen are independent each other. Indeed, in Example 3, clearly $\{a,b,c\}$ is $(1,2)*$-preopen but it is not $(1,2)*$-$\mathrm{D}(\alpha,\mathrm{s})$. Moreover, $\{b\}$ is $(1,2)*$-$\mathrm{D}(\alpha,\mathrm{s})$ but it is not $(1,2)*$-preopen.

Theorem 2. *For a bitopological space* (X,τ_1,τ_2)*, the equality* $(1,2)*$-$\alpha O(X) = (1,2)*$-$PO(X) \cap (1,2)*$-$\mathrm{D}(\alpha,\mathrm{p})$ *holds.*

Proof. Let $A \in (1,2)*$-$PO(X) \cap (1,2)*$-$\mathrm{D}(\alpha,\mathrm{p})$. Then $A = (1,2)*$-$\mathrm{p}\,\mathrm{Int}(A)$ and $(1,2)*$-$\alpha\,\mathrm{Int}(A) = (1,2)*$-$\mathrm{p}\,\mathrm{Int}(A)$. Hence $A = (1,2)*$-$\alpha\,\mathrm{Int}(A)$. Therefore $A \in (1,2)*$-$\alpha O(X)$.

Conversely, let $A \in (1,2)*$-$\alpha O(X)$. Then $A = (1,2)*$-$\alpha\,\mathrm{Int}(A) \subseteq (1,2)*$-$\mathrm{p}\,\mathrm{Int}(A)$. But $(1,2)*$-$\mathrm{p}\,\mathrm{Int}(A) \subseteq A$. Therefore $A \in (1,2)*$-$PO(X)$ and thus we have $A = (1,2)*$-$\alpha\,\mathrm{Int}(A) = (1,2)*$-$\mathrm{p}\,\mathrm{Int}(A)$. Hence A is a $(1,2)*$-$\mathrm{D}(\alpha,\mathrm{p})$ set. ☐

Theorem 3. *For a bitopological space* (X,τ_1,τ_2)*, the equality* $(1,2)*$-$\alpha O(X) = (1,2)*$-$SO(X) \cap (1,2)*$-$\mathrm{D}(\alpha,\mathrm{s})$ *holds.*

Proof. Let $A \in (1,2)*$-$\alpha O(X)$. Then $A = (1,2)*$-$\alpha\,\mathrm{Int}(A) \subseteq (1,2)*$-$\mathrm{s}\,\mathrm{Int}(A)$. But $(1,2)*$-$\mathrm{s}\,\mathrm{Int}(A) \subseteq A$. Thus $A \in (1,2)*$-$SO(X)$. Also $A = (1,2)*$-$\alpha\,\mathrm{Int}(A) = (1,2)*$-$\mathrm{s}\,\mathrm{Int}(A)$. Thus $A \in (1,2)*$-$\mathrm{D}(\alpha,\mathrm{s})$.

Conversely, let $A \in (1,2)*$-$SO(X) \cap (1,2)*$-$\mathrm{D}(\alpha,\mathrm{s})$. Then $A = (1,2)*$-$\mathrm{s}\,\mathrm{Int}(A)$ and $(1,2)*$-$\alpha\,\mathrm{Int}(A) = (1,2)*$-$\mathrm{s}\,\mathrm{Int}(A)$. Now we have $A = (1,2)*$-$\alpha\,\mathrm{Int}(A)$. Therefore $A \in (1,2)*$-$\alpha O(X)$. ☐

5. Another new type of sets

Definition 5. For a bitopological space (X,τ_1,τ_2), we define

(i) $(1,2)*$-$\mathrm{D}(\mathrm{c},\alpha) = \big\{S \subseteq X : \tau_1\tau_2$-$\mathrm{Int}(S) = (1,2)*$-$\alpha\,\mathrm{Int}(S)\big\}$;

(ii) $(1,2)*$-$\mathrm{D}(\mathrm{c},\mathrm{s}) = \big\{S \subseteq X : \tau_1\tau_2$-$\mathrm{Int}(S) = (1,2)*$-$\mathrm{s}\,\mathrm{Int}(S)\big\}$;

(iii) $(1,2)*$-D(c, p) $= \{S \subseteq X : \tau_1\tau_2\text{-Int}(S) = (1,2)*\text{-p}\operatorname{Int}(S)\}$.

Proposition 4. *For a subset A of (X, τ_1, τ_2), the following statements are equivalent:*

(i) *A is a $(1,2)*$-D(c, s) set.*
(ii) $\tau_1\tau_2\text{-Int}(A) = A \cap \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$.

Proof. (i)⇒(ii): Let A be a $(1,2)*$-D(c, s) set. Then we have $\tau_1\tau_2\text{-Int}(A) = (1,2)*\text{-s}\operatorname{Int}(A) = A \cap \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$ by Result 1.

(ii)⇒(i): Let $\tau_1\tau_2\text{-Int}(A) = A \cap \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$. Then, by Result 1, $\tau_1\tau_2\text{-Int}(A) = (1,2)*\text{-s}\operatorname{Int}(A)$. Therefore, A is a $(1,2)*$-D(c, s) set. □

Proposition 5. *Every singleton $\{x\}$ of (X, τ_1, τ_2) is a $(1,2)*$-D(c, s) set.*

Proof. First suppose that $\{x\}$ is $\tau_{1,2}$-open. Then $\tau_1\tau_2\text{-Int}(\{x\}) = \{x\} = \{x\} \cap \tau_1\tau_2\text{-Cl}(\{x\}) = \{x\} \cap \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\{x\}))$. Therefore, by Proposition 4, $\{x\}$ is a $(1,2)*$-D(c, s) set.

Now, let $\{x\}$ be not $\tau_{1,2}$-open. Then $\tau_1\tau_2\text{-Int}(\{x\}) = \varnothing = \{x\} \cap \varnothing = \{x\} cap \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\{x\}))$. Therefore, by Proposition 4, $\{x\}$ is a $(1,2)*$-D(c, s) set. □

Proposition 6. *Every $\tau_{1,2}$-open set is a $(1,2)*$-D(c, s) set.*

Proof. $\tau_1\tau_2\text{-Int}(A) = A = A \cap \tau_1\tau_2\text{-Cl}(A) = A \cap \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)) = (1,2)*\text{-s}\operatorname{Int}(A)$ by Result 1. □

Proposition 7. *Every $\tau_{1,2}$-open set is a $(1,2)*$-D(c, α) set.*

Proof. By Proposition 1, $\tau_1\tau_2\text{-Int}(A) = A = \tau_1\tau_2\text{-Int}(A \cap \tau_1\tau_2\text{-Cl}(A)) = A \cap \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)) = A \cap \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))) = (1,2)*\text{-}\alpha\operatorname{Int}(A)$. □

Proposition 8. *For a bitopological space (X, τ_1, τ_2), the following assertions hold:*

(i) $(1,2)*$-D(c, s) $\subseteq$ $(1,2)*$-D(c, α)*;*
(ii) $(1,2)*$-D(c, p) $\subseteq$ $(1,2)*$-D(c, α).

Proof. (i) Let $A \in (1,2)*$-D(c, s). Then $\tau_1\tau_2\text{-Int}(A) = (1,2)*\text{-s}\operatorname{Int}(A) \supseteq (1,2)*\text{-}\alpha\operatorname{Int}(A)$. But $\tau_1\tau_2\text{-Int}(A) \subseteq (1,2)*\text{-}\alpha\operatorname{Int}(A)$ and thus we have $\tau_1\tau_2\text{-Int}(A) = (1,2)*\text{-}\alpha\operatorname{Int}(A)$. Therefore, $A \in (1,2)*$-D(c, α).

(ii) Let $A \in (1,2)*$-D(c, p). Then we get $\tau_1\tau_2\text{-Int}(A) = (1,2)*\text{-p}\operatorname{Int}(A) \supseteq (1,2)*\text{-}\alpha\operatorname{Int}(A)$. But $\tau_1\tau_2\text{-Int}(A) \subseteq (1,2)*\text{-}\alpha\operatorname{Int}(A)$ and thus $\tau_1\tau_2\text{-Int}(A) = (1,2)*\text{-}\alpha\operatorname{Int}(A)$. Therefore $A \in (1,2)*$-D(c, α). □

The converses of Propositions 7 and 8 are not true as can be seen from the following example.

Example 7. In Example 1, $\{b\}$ is $(1,2)*$-D(c, α) set but it is not $(1,2)*$-D(c, p). In Example 2, $\{b, c\}$ is $(1,2)*$-D(c, α) set but it is not $(1,2)*$-D(c, s). Finally, in Example 1, $\{b\}$ is $(1,2)*$-D(c, α) set but it is not $\tau_{1,2}$-open.

Remark 6. $(1,2)*$-semi-open and $(1,2)*$-D(c, s) sets are independent each other. Indeed, in Example 2, clearly $\{a,c\}$ is $(1,2)*$-semi-open but it is not $(1,2)*$-D(c, s). Moreover, $\{c\}$ is $(1,2)*$-D(c, s) set but it is not $(1,2)*$-semi-open.

Remark 7. $(1,2)*$-preopen and $(1,2)*$-D(c, p) sets are independent each other. In Example 1, clearly $\{b\}$ is $(1,2)*$-preopen but it is not $(1,2)*$-D(c, p). Moreover, $\{a\}$ is $(1,2)*$-D(c, p) set but it is not $(1,2)*$-preopen.

Remark 8. $(1,2)*$-α-open and $(1,2)*$-D(c, α) sets are independent each other. Indeed, let $X=\{a,b,c\}$ with $\tau_1=\{\varnothing,X,\{c\}\}$ and $\tau_2=\{\varnothing,X,\{b,c\}\}$. Then the sets in $\{\varnothing,X,\{c\},\{b,c\}\}$ are $\tau_{1,2}$-open and the sets in $\{\varnothing,X,\{a\},\{a,b\}\}$ are $\tau_{1,2}$-closed. Clearly $\{a,c\}$ is $(1,2)*$-α-open but it is not $(1,2)*$-D(c, α) set. Moreover, $\{a\}$ is $(1,2)*$-D(c, α) set but it is not $(1,2)*$-α-open.

Remark 9. We have the following diagram from the subsets we defined above and the Remarks given above. Also, None of the implications in the diagram is reversible.

$$\begin{array}{ccccc}
(1,2)*\text{-D(c, s)} & & \not\twoheadrightarrow & & (1,2)*\text{-semi-open} \\
\downarrow & & & & \uparrow \\
(1,2)*\text{-D(c},\alpha) & \leftarrow & \tau_{1,2}\text{-open} & \rightarrow & (1,2)*\text{-}\alpha\text{-open} \\
\uparrow & & & & \downarrow \\
(1,2)*\text{-D(c, p)} & & \not\twoheadrightarrow & & (1,2)*\text{-preopen}
\end{array}$$

Theorem 4. *For a subset A of (X,τ_1,τ_2), the following assertions are equivalent:*

(i) *A is $\tau_{1,2}$-open.*
(ii) *A is $(1,2)*$-α-open set and a $(1,2)*$-D(c, α) set.*
(iii) *A is $(1,2)*$-preopen set and a $(1,2)*$-D(c, p) set.*

Proof. The implication (i)$\Rightarrow$(ii) is obvious.

(ii)$\Rightarrow$(iii): Let A be $(1,2)*$-α-open set and a $(1,2)*$-D(c, α) set. Then $A\in(1,2)*$-$PO(X)\cap(1,2)*$-$SO(X)$ and $(1,2)*$-α-Int$(A)=\tau_1\tau_2$-Int(A). By Proposition 1, we have $A\cap\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))=\tau_1\tau_2$-Int(A) and subsequently $A\cap\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))=\tau_1\tau_2$-Int(A). Now by Proposition 1, we have $(1,2)*$-pInt$(A)=\tau_1\tau_2$-Int(A). Therefore A is a $(1,2)*$-D(c, p) set.

(iii)$\Rightarrow$(i): Let A be $(1,2)*$-preopen set and a $(1,2)*$-D(c, p) set. Then by definition $A=(1,2)*$-pInt(A) and $\tau_1\tau_2$-Int$(A)=(1,2)*$-pInt(A). Hence $A=\tau_1\tau_2$-Int(A). Therefore A is $\tau_{1,2}$-open set. $\square$

Remark 10. $(1,2)*$-α-open and $(1,2)*$-D(c, s) sets are independent each other. Indeed, in the example given in Remark 8, clearly $\{a,c\}$ is $(1,2)*$-α-open set but it is not $(1,2)*$-D(c, s) set. Moreover, $\{a\}$ is $(1,2)*$-D(c, s) set but it is not $(1,2)*$-α-open.

Theorem 5. *For a subset A of (X,τ_1,τ_2), the following assertions are equivalent:*

(i) *A is $\tau_{1,2}$-open.*
(ii) *A is $(1,2)*$-α-open set and a $(1,2)*$-D(c, s) set.*

(iii) *A is $(1,2)*$-semi-open set and a $(1,2)*$-$\mathrm{D(c,s)}$ set.*

Proof. The implications (i)⇒(ii) and (ii)⇒(iii) are obvious.
(iii)⇒(i): Let A be $(1,2)*$-semi-open set and a $(1,2)*$-$\mathrm{D(c,s)}$ set. Then $\tau_1\tau_2$-$\mathrm{Int}(A) = (1,2)*$-$\mathrm{s\,Int}(A) = A \cap \tau_1\tau_2$-$\mathrm{Cl}(\tau_1\tau_2$-$\mathrm{Int}(A)) = A \cap \tau_1\tau_2$-$\mathrm{Cl}(A)$ by the definition, Proposition 2, and Result 1. Now we have $\tau_1\tau_2$-$\mathrm{Int}(A) = A$. Therefore A is $\tau_{1,2}$-open. □

6. Decomposition of $(1,2)*$-continuity and $(1,2)*$-α-continuity

Definition 6 ([6])**.** A mapping $f : X \to Y$ is called

(i) $(1,2)*$-α-continuous if $f^{-1}(V) \in (1,2)*$-$\alpha O(X)$ for each $\sigma_{1,2}$-open set V of Y;
(ii) $(1,2)*$-semi-continuous if $f^{-1}(V) \in (1,2)*$-$SO(X)$ for each $\sigma_{1,2}$-open set V of Y;
(iii) $(1,2)*$-precontinuous if $f^{-1}(V) \in (1,2)*$-$PO(X)$ for each $\sigma_{1,2}$-open set V of Y.

We introduce new classes of mappings as follows:

Definition 7. A mapping $f : X \to Y$ is called

(i) $(1,2)*$-$\mathrm{D}(\alpha,\mathrm{p})$ continuous if $f^{-1}(V) \in (1,2)*$-$\mathrm{D}(\alpha,\mathrm{p})$ for each $\sigma_{1,2}$-open set v of Y;
(ii) $(1,2)*$-$\mathrm{D}(\alpha,\mathrm{s})$ continuous if $f^{-1}(V) \in (1,2)*$-$\mathrm{D}(\alpha,\mathrm{s})$ for each $\sigma_{1,2}$-open set v of Y;
(iii) $(1,2)*$-$\mathrm{D}(\mathrm{c},\alpha)$ continuous if $f^{-1}(V) \in (1,2)*$-$\mathrm{D}(\mathrm{c},\alpha)$ for each $\sigma_{1,2}$-open set v of Y;
(iv) $(1,2)*$-$\mathrm{D(c,s)}$ continuous if $f^{-1}(V) \in (1,2)*$-$\mathrm{D(c,s)}$ for each $\sigma_{1,2}$-open set v of Y;
(v) $(1,2)*$-$\mathrm{D(c,p)}$ continuous if $f^{-1}(V) \in (1,2)*$-$\mathrm{D(c,p)}$ for each $\sigma_{1,2}$-open set v of Y.

Theorem 6. *A mapping $f : X \to Y$ is $(1,2)*$-continuous if and only if*

(i) *it is $(1,2)*$-α-continuous and $(1,2)*$-$\mathrm{D}(\mathrm{c},\alpha)$ continuous.*
(ii) *it is $(1,2)*$-precontinuous and $(1,2)*$-$\mathrm{D(c,p)}$ continuous.*
(iii) *it is $(1,2)*$-α-continuous and $(1,2)*$-$\mathrm{D(c,s)}$ continuous.*
(iv) *it is $(1,2)*$-semi-continuous and $(1,2)*$-$\mathrm{D(c,s)}$ continuous.*

Proof. It is a decomposition of $(1,2)*$-continuity from Theorems 4 and 5. □

Theorem 7. *A mapping $f : X \to Y$ is $(1,2)*$-α-continuous if and only if*

(i) *it is $(1,2)*$-semi-continuous and $(1,2)*$-precontinuous.*
(ii) *it is $(1,2)*$-precontinuous and $(1,2)*$-$\mathrm{D}(\alpha,\mathrm{p})$ continuous.*
(iii) *it is $(1,2)*$-semi-continuous and $(1,2)*$-$\mathrm{D}(\alpha,\mathrm{s})$ continuous.*

Proof. It is a decomposition of $(1,2)*$-α-continuity from Theorems 1, 2 and 3. □

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Doi: 10.18514/MMN.2009.249

Miskolc Mathematical Notes
Vol. 10 (2009), No. 2, pp. 173–205

HU e-ISSN 1787-2413

ON A CAUCHY–NICOLETTI TYPE THREE-POINT BOUNDARY VALUE PROBLEM FOR LINEAR DIFFERENTIAL EQUATIONS WITH ARGUMENT DEVIATIONS

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Received 5th November, 2009

Abstract. We obtain some results concerning the investigation of the solutions of three-point Cauchy–Nicoletti type boundary value problems for a certain class of linear functional differential equations. We show that it is useful to reduce the given problem to the parametrized two-point boundary value problem for a suitably perturbed system containing some artificially introduced parameters both in the constructed inhomogeneous two-point boundary conditions and in the modified functional differential equations.

To study the transformed parametrized two-point problem, we use a method which is based upon special type of successive approximations constructed in an analytic form. We prove the uniform convergence of these approximations to the parametrized limit function. Our technique leads to a certain system of algebraic equations with respect to the introduced parameters whose solutions provide those numerical values of the parameters that correspond to the solutions of the given three-point boundary value problem.

2000 *Mathematics Subject Classification:* 34B15

Keywords: Functional differential equation, special deviation of argument, linear three-point boundary value problem, successive approximations

1. INTRODUCTION

In studies of solutions of various boundary value problems for ordinary and functional differential equations, it is often useful to possess appropriate techniques based upon some types of successive approximations constructed in an analytic form. To this class of methods belongs, in particular, the approach suggested originally in [22–24] for the investigation of the periodic boundary value problems for non-autonomous systems of ordinary differential equations of the form

$$x'(t) = f(t, x(t)), \qquad t \in [0, T],$$
$$x(0) = x(T).$$

The research was supported in part by AS CR, Institutional Research Plan No. AV0Z10190503 (A. Rontó) and the Hungarian Scientific Research Fund through Grant No. K68311 (M. Rontó).

Appropriate versions of the method have been obtained by various authors and can be applied in many situations for handling boundary value problems for non-linear systems of first or second order ordinary differential equations, integro-differential equations, equations with retarded argument and more general functional differential equations, as well as boundary value problems containing parameters. The given equation can be considered together with various types of boundary conditions such as the two-point conditions (both linear and non-linear), the Cauchy–Nicoletti conditions of the form

$$x_i(t_i) = d_i, \qquad i = 1, 2, \ldots, n,$$

where $0 \le t_1 \le t_2 \le \cdots \le t_n = T$, the boundary conditions of the interpolation type

$$x_j(t_i) = d_i, \qquad i = 1, 2, \ldots, n,$$

with $j \in \{1, 2, \ldots, n\}$ and $0 \le t_1 < t_2 < \cdots < t_n = T$, as well as with other kinds of multipoint conditions or more general functional boundary conditions.

It is clear that both the form and complexity of the given equations and boundary conditions have essential influence both on the possibility of efficient construction of approximate solutions and the solvability analysis of the given boundary value problem. We refer, e. g., to the books [14, 25, 26], the papers [6, 7, 9–12], and the survey [15–21] for some related references.

According to the basic idea of the method under consideration, the given boundary value problem is replaced by some "perturbed" boundary value problem containing an unknown vector-parameter $z \in \mathbb{R}^n$, whose value is to be determined later and which usually has the meaning of an initial value of the solution at a certain point. The solution of the modified problem is sought for in an analytic form by a suitable iteration process. The presence of a "perturbation term," which, of course, depends on the original equations and the boundary conditions, yields a system of algebraic or transcendental "determining equations," whose solutions give the numerical values of the parameter $z \in \mathbb{R}^n$ that correspond to the solutions of the given boundary value problem. By studying the solvability of these determining equations, one can establish existence results for the original problem.

Some modifications of the numerical-analytic approach based upon successive approximations were obtained, in particular, in [1,3–5] for the two-point boundary value problem for the system of non-linear neutral type functional differential equations

$$\begin{gathered} x'(t) = f\left(t, x(\alpha(t)), x'(\beta(t))\right), \\ Ax(0) + Bx(T) = d, \end{gathered}$$

where $\alpha, \beta : [0, T] \to [0, T]$ are given continuous functions, A and B are $(n \times n)$-matrices, $\det B \neq 0$, $d \in \mathbb{R}^n$, and the right-hand side function f defined on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ is continuous and satisfies the Lipschitz condition

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \le K|u_1 - u_2| + L|v_1 - v_2|$$

with some non-negative constant matrices K and L. In [8], we refined certain estimates related to the convergence analysis of successive approximations in the case of two-point linear boundary value problems for functional differential equations where the argument deviations possess certain special properties.

In this paper, our aim is to extend the techniques used in [8] to investigate solutions of a system of linear functional differential equation with the Cauchy–Nicoletti three-point boundary conditions. The difficulties related to this type of boundary conditions are due to the singularity of the matrices that determine them. To avoid dealing with singular matrices in the boundary conditions and simplify the construction of the solution in an analytic form, we use some parametrization technique on two levels. The first level allows one to replace the three-point boundary conditions by a family of parametrized two-point inhomogeneous conditions. The second level of parametrization is used for the construction and investigation of the above-mentioned perturbed system. Finally, the study of certain algebraic or transcendental determining equations gives one a possibility to obtain the numerical values of the parameters which correspond to the solutions of the given three-point boundary value problem.

2. Notation

The following notation is used in the sequel:

$C([0,T],\mathbb{R}^n)$ is the Banach space of the continuous functions $[0,T]\to\mathbb{R}^n$ with the standard uniform norm;

$L_1([0,T],\mathbb{R}^n)$ is the usual Banach space of the vector functions $[0,T]\to\mathbb{R}^n$ with Lebesgue integrable components;

$\mathcal{L}(\mathbb{R}^n)$ is the algebra of all the square matrices of dimension n with real elements;

$r(Q)$ is the maximal in module eigenvalue of the matrix $Q\in\mathcal{L}(\mathbb{R}^n)$;

$\mathbb{1}_m$ is the unit matrix of dimension $m\le n$;

$\mathbb{0}_{m,k}$ is the zero matrix of dimension $m\times k$;

$\mathbb{0}_m=\mathbb{0}_{m,m}$.

3. Problem setting

We consider the system of n linear functional differential equations of the form

$$x'(t)=P_0(t)x(t)+P_1(t)x(\beta(t))+f(t),\qquad t\in[0,T],\tag{3.1}$$

subjected to the inhomogeneous three-point Cauchy–Nicoletti boundary conditions

$$\begin{aligned} x_1(0)&=x_{10}, &&\ldots, & x_p(0)&=x_{p0},\\ x_{p+1}(\xi)&=d_{p+1}, &&\ldots, & x_{p+q}(\xi)&=d_{p+q},\\ x_{p+q+1}(T)&=d_{p+q+1}, &&\ldots, & x_n(T)&=d_n. \end{aligned}\tag{3.2}$$

For the sake of simplicity, we restrict ourselves to the case where only one argument deviation is present. Here, we suppose that $T\in(0,+\infty)$, the elements of the matrix-valued functions $P_j:[0,T]\to\mathcal{L}(\mathbb{R}^n)$, $j=0,1$ are Lebesgue integrable, $f\in$

$L_1([0,T],\mathbb{R}^n)$, and $\beta : [0,T] \to [0,T]$ is a Lebesgue measurable function, $x_{10} \in \mathbb{R}$, $\dots$, $x_{p0} \in \mathbb{R}$, $d_i \in \mathbb{R}$, $i = p+1, p+2, \dots, n$.

Thus, in (3.2), one has p conditions prescribed at the point 0, q conditions at a point ξ, $0 < \xi < T$, and $n-(p+q)$ conditions at the point T. Using the diagonal matrices A, B, and C of the form

$$A := \begin{pmatrix} \mathbb{1}_p & \mathbb{0}_{p,q} & \mathbb{0}_{n-(p+q),p} \\ \mathbb{0}_{q,p} & \mathbb{0}_q & \mathbb{0}_{n-(p+q),q} \\ \mathbb{0}_{n-(p+q),p} & \mathbb{0}_{n-(p+q),q} & \mathbb{0}_{n-(p+q)} \end{pmatrix}, \tag{3.3}$$

$$B := \begin{pmatrix} \mathbb{0}_p & \mathbb{0}_{p,q} & \mathbb{0}_{n-(p+q),p} \\ \mathbb{0}_{q,p} & \mathbb{1}_q & \mathbb{0}_{n-(p+q),q} \\ \mathbb{0}_{n-(p+q),p} & \mathbb{0}_{n-(p+q),q} & \mathbb{0}_{n-(p+q)} \end{pmatrix}, \tag{3.4}$$

$$C := \begin{pmatrix} \mathbb{0}_p & \mathbb{0}_{p,q} & \mathbb{0}_{n-(p+q),p} \\ \mathbb{0}_{q,p} & \mathbb{0}_q & \mathbb{0}_{n-(p+q),q} \\ \mathbb{0}_{n-(p+q),p} & \mathbb{0}_{n-(p+q),q} & \mathbb{1}_{n-(p+q)} \end{pmatrix}, \tag{3.5}$$

one can rewrite the boundary conditions (3.2) in the usual matrix-vector form

$$Ax(0) + Bx(\xi) + Cx(T) = \bar{d}, \tag{3.6}$$

where

$$\bar{d} = \operatorname{col}(x_{10}, x_{20}, \dots, x_{p0}, d_{p+1}, d_{p+2}, \dots, d_{p+q}, d_{p+q+1}, \dots, d_n). \tag{3.7}$$

It is obvious that each of the matrices (3.3), (3.4), and (3.5) appearing in condition (3.6) is singular, which causes some difficulties for the construction of suitable successive approximations.

4. Parametrization of the three-point boundary conditions

Besides the three-point boundary condition (3.6), we introduce into consideration the auxiliary two-point condition

$$Ax(0) + Cx(T) = \bar{\bar{d}}, \tag{4.1}$$

where $\bar{\bar{d}} = \operatorname{col}(x_{10}, x_{20}, \dots, x_{p0}, 0, 0, \dots, 0, d_{p+q+1}, \dots, d_n)$. Compared to (3.6), condition (4.1) may be regarded as a result of “freezeing” of the value of the function at the point ξ.

To avoid dealing with the singular matrix C in (4.1), we carry out the following parametrization:

$$\begin{array}{cccc} x_1(T) = \lambda_1, & x_2(T) = \lambda_2, & \dots, & x_p(T) = \lambda_p, \\ x_{p+1}(T) = \lambda_{p+1}, & x_{p+2}(T) = \lambda_{p+2}, & \dots, & x_{p+q}(T) = \lambda_{p+q} \end{array} \tag{4.2}$$

and, instead of (4.1), we use the two-point boundary conditions

$$Ax(0) + x(T) = d(\lambda), \tag{4.3}$$

where $\lambda = \operatorname{col}(\lambda_1, \dots, \lambda_p, \lambda_{p+1}, \dots, \lambda_{p+q}) \in \mathbb{R}^{p+q}$ and

$$d(\lambda) := \operatorname{col}(x_{10} + \lambda_1, x_{20} + \lambda_2, \dots, x_{p0} + \lambda_p, \lambda_{p+1}, \lambda_{p+2}, \dots, \lambda_{p+q}, d_{p+q+1}, \dots, d_n). \tag{4.4}$$

Instead of the three-point boundary value problem (3.1), (3.6), we first consider the two-point boundary value problem (3.1), (4.3).

5. Subsidiary statements

In the sequel, we need several auxuliary statements, many of which are related to properties of the sequence of functions $\{\alpha_m\}_{m=0}^{\infty} \subset C([0,T], \mathbb{R})$ defined by the recurrence relation

$$\alpha_{m+1}(t) = \left(1 - \frac{t}{T}\right) \int_0^t \alpha_m(s)\, ds + \frac{t}{T} \int_t^T \alpha_m(s)\, ds, \qquad m = 0, 1, 2, \dots, \tag{5.1}$$

where $\alpha_0(t) := 1$, $t \in [0,T]$. In particular, we have

$$\alpha_1(t) = 2t\left(1 - \frac{t}{T}\right), \qquad t \in [0,T]. \tag{5.2}$$

Lemma 1. *Let the sequence of functions $\{\alpha_m\}_{m=0}^{\infty} \subset C([0,T], \mathbb{R})$ be given by formula (5.1). Then:*

(1) *The function α_m is symmetric with respect to the point $\frac{T}{2}$ for all $m \geq 0$, i. e.,*

$$\alpha_m(t) = \alpha_m(T - t), \qquad t \in [0,T], \tag{5.3}$$

$$\alpha_m\left(\frac{T}{2} - t\right) = \alpha_m\left(\frac{T}{2} + t\right), \qquad t \in [0, T/2]. \tag{5.4}$$

(2) *Sequence (5.1) can be represented alternatively as*

$$\begin{aligned} \alpha_{m+1}(t) &= \int_0^t \alpha_m(s)\, ds + \frac{t}{T} \int_t^{T-t} \alpha_m(s)\, ds \\ &= \frac{t}{T} \int_t^{T-t} \alpha_m(s)\, ds + \left(1 - \frac{t}{T}\right) \int_{T-t}^t \alpha_m(s)\, ds \\ &= \int_0^{T-t} \alpha_m(s)\, ds + \left(1 - \frac{t}{T}\right) \int_{T-t}^t \alpha_m(s)\, ds, \qquad t \in [0,T]. \end{aligned} \tag{5.5}$$

(3) *For any $m \geq 1$, $\alpha_m(0) = \alpha_m(T) = 0$ and $\alpha_m(t) > 0$ for all $t \in (0,T)$.*

(4) *The maximal value of every $\alpha_m(t)$, $m \geq 0$, is achieved at the point $T/2$, namely,*

$$\max_{t \in [0,T]} \alpha_m(t) = \alpha_m\left(\frac{T}{2}\right). \tag{5.6}$$

(5) *For every $m \geq 1$,*

$$\alpha_m'(t) \cdot \operatorname{sign}\left(t - \frac{T}{2}\right) \leq 0, \qquad t \in [0, T], \tag{5.7}$$

that is, the function α_m is increasing on $(0, T/2)$ and decreasing on $(T/2, T)$.

Proof. For the proof, see [6, Lemma 1] and [8, Lemma 1]. □

Lemma 2. *For an arbitrary essentially bounded function $u : [0, T] \to \mathbb{R}$, the estimate*

$$\left| \int_0^t \left(u(\tau) - \frac{1}{T} \int_0^T u(s)\, ds \right) d\tau \right| \leq \frac{\alpha_1(t)}{2} \left(\operatorname*{ess\,sup}_{s \in [0,T]} u(s) - \operatorname*{ess\,inf}_{s \in [0,T]} u(s) \right) \tag{5.8}$$

is true for a. e. $t \in [0, T]$, where α_1 is the function defined by equality (5.2).

Proof. Inequality (5.8) is established similarly to [13, Lemma 3] (see also [14, Lemma 2.3]). □

Lemma 3. *The sequence of functions α_m, $m \geq 1$, given by relation (5.1) satisfies the inequalities*

$$\begin{aligned} \alpha_{m+1}(t) &\leq \frac{3T}{10} \alpha_m(t), \qquad m \geq 2; \\ \alpha_{m+1}(t) &\leq \frac{10}{9} \left(\frac{3T}{10} \right)^m \alpha_1(t), \qquad m \geq 0. \end{aligned} \tag{5.9}$$

Proof. Inequalities (5.9) are established by analogy to [13, Lemma 4] (see also [14, Lemma 2.4]). □

Lemma 4 ([8, Lemma 3])**.** *Let $\beta : [0, T] \to [0, T]$ be a measurable function satisfying the condition*

$$\operatorname*{ess\,inf}_{t \in [0,T]} (\beta(t) - t) \operatorname{sign}\left(t - \frac{T}{2}\right) \geq 0. \tag{5.10}$$

Then the members of the function sequence (5.1) satisfy the pointwise estimates

$$\alpha_m(\beta(t)) \leq \alpha_m(t), \qquad t \in [0, T], \; m = 1, 2, \ldots. \tag{5.11}$$

Remark 1. If $\beta : [0, T] \to [0, T]$ is a continuous function satisfying condition (5.10) then necessarily $\beta(0) = 0$, $\beta(T/2) = T/2$, and $\beta(T) = T$.

Lemma 5 ([8, Lemma 4])**.** *If a measurable function $\beta : [0, T] \to [0, T]$ satisfies the condition*

$$k_\beta := \operatorname*{ess\,sup}_{t \in [0,T]} \frac{\beta(t)(T - \beta(t))}{t(T - t)} < +\infty, \tag{5.12}$$

then

$$\alpha_1(\beta(t)) \leq k_\beta \alpha_1(t), \qquad t \in [0, T]. \tag{5.13}$$

For example, the function $\beta(t) = t^2$, $t \in [0,1]$, satisfies condition (5.12) on the interval $[0,1]$, whereas the functions $\beta(t) = \frac{1}{2}t^2$, $\beta(t) = \frac{1}{2}t$, $\beta(t) = \sin t$, $t \in [0,1]$, do not satisfy it.

6. Convergence of successive approximations for the case of a general type of argument deviation

To study the solution of the auxiliary two-point boundary value problem (3.1), (4.3) let us introduce the sequence of functions

$$
\begin{aligned}
x_{m+1}(t,z,\lambda) := z &+ \int_0^t \left(P_0(s)\, x_m(s,z,\lambda) + P_1(s)\, x_m(\beta(s),z,\lambda) + f(s)\right) ds \\
&- \frac{t}{T}\int_0^T \left(P_0(s)\, x_m(s,z,\lambda) + P_1(s)\, x_m(\beta(s),z,\lambda) + f(s)\right) ds \\
&+ \frac{t}{T}\left(d(\lambda) - (A + \mathbb{1}_n)\, z\right),
\end{aligned}
\tag{6.1}
$$

where $m \geq 0$, $x_0(t,z) = z$, $t \in [0,T]$,

$$
z = \operatorname{col}(x_{10}, x_{20}, \ldots, x_{p0}, z_{p+1}, \ldots, z_n) \in \mathbb{R}^n, \tag{6.2}
$$

$\lambda = \operatorname{col}(\lambda_1, \ldots, \lambda_p, \lambda_{p+1}, \ldots, \lambda_{p+q}) \in \mathbb{R}^{p+q}$ is a vector parameter, and $d(\lambda)$ is given by (4.4).

Remark 2. We emphasize that the first p components of the vector z are fixed and coincide with the initial values appearing in the boundary conditions (3.2), while the other its components z_k, $p < k \leq n$, are considered as free parameters. Thus, the expression "for all z," which is often used in what follows, actually means "for all $z_{p+1}, \ldots, z_n$." We hope that no confusion will arise.

Let us establish the convergence of the sequence (6.1) for arbitrary deviation function $\beta : [0,T] \to [0,T]$.

Theorem 1. *Let the elements of matrix-valued functions $P_i : [0,T] \to \mathcal{L}(\mathbb{R}^n)$, $i = 0,1$, be Lebesgue integrable, $f \in L_1([0,T],\mathbb{R}^n)$ and $\beta : [0,T] \to [0,T]$ be a Lebesgue measurable function. Moreover, assume that*

$$
r(K_0 + K_1) < \frac{2}{T}, \tag{6.3}
$$

where

$$
K_i := \operatorname*{ess\,sup}_{s\in[0,T]} |P_i(s)|, \qquad i = 0,1. \tag{6.4}
$$

Then:

(1) *All the members of sequence (6.1) are absolutely continuous functions satisfying the two-point boundary conditions*

$$
A x_m(0,z,\lambda) + x_m(T,z,\lambda) = d(\lambda), \qquad m = 1,2,\ldots, \tag{6.5}
$$

for all $\lambda \in \mathbb{R}^{p+q}$ and z of form (6.2).

(2) *The sequence of functions* (6.1) *converges to a limit function* $x^*(\cdot,z,\lambda)$,

$$x^*(t,z,\lambda) = \lim_{m\to\infty} x_m(t,z,\lambda) \tag{6.6}$$

uniformly in $t \in [0,T]$ *for all fixed* z *of form* (6.2) *and* $\lambda \in \mathbb{R}^{p+q}$.

(3) *The limit function* (6.6) *satisfies the initial condition*

$$x^*(0,z,\lambda) = z \tag{6.7}$$

and the boundary condition (4.3)

$$Ax^*(0,z,\lambda) + x^*(T,z,\lambda) = d(\lambda) \tag{6.8}$$

for all z *of form* (6.2) *and* $\lambda \in \mathbb{R}^{p+q}$.

(4) *For all fixed* z *of form* (6.2) *and* $\lambda \in \mathbb{R}^{p+q}$, *the limit function* (6.6) *is a unique absolutely continuous solution of the integro-functional equation*

$$\begin{aligned} x(t) = z &+ \int_0^t [P_0(s)x(s) + P_1(s)x(\beta(s)) + f(s)]\,ds \\ &- \frac{t}{T}\int_0^T [P_0(s)x(s) + P_1(s)x(\beta(s)) + f(s)]\,ds \\ &+ \frac{t}{T}[d(\lambda) - (A + \mathbb{1}_n)z], \qquad t \in [0,T]. \end{aligned} \tag{6.9}$$

(5) *The following estimate holds for all fixed* z *of form* (6.2) *and* $\lambda \in \mathbb{R}^{p+q}$:

$$\begin{aligned} &\left|x^*(t,z,\lambda) - x_m(t,z,\lambda)\right| \le \\ &\quad \le \left(\frac{T}{2}\right)^{m-1} (K_0+K_1)^m \left(\frac{2}{T}\mathbb{1}_n - K_0 - K_1\right)^{-1} \gamma(z,\lambda), \end{aligned} \tag{6.10}$$

where

$$\gamma(z,\lambda) := \frac{T}{2}\delta(z) + |d(\lambda) - (A+\mathbb{1}_n)z| \tag{6.11}$$

and

$$\begin{aligned} \delta(z) := \frac{1}{2}\Bigg(&\operatorname*{ess\,sup}_{s\in[0,T]} (P_0(s)z + P_1(s)z + f(s)) \\ &- \operatorname*{ess\,inf}_{s\in[0,T]} (P_0(s) + P_1(s) + f(s))\Bigg). \end{aligned} \tag{6.12}$$

In (6.10) and similar relations, below the signs $|\cdot|$, $\le$, $\ge$, ess sup, and ess inf are understood componentwise.

Proof. The validity of assertion 1 is verified by direct computation. To obtain the other required properties, let us show that, under the conditions assumed, sequence (6.1) is a Cauchy sequence in the Banach space $C([0,T],\mathbb{R}^n)$ equipped with the standard uniform norm. Indeed, due to estimate (5.8) of Lemma 2, it follows from (6.1) that, for $m = 0$ and arbitrary fixed $z_{p+1}, \ldots, z_n$ and $\lambda \in \mathbb{R}^{p+q}$,

$$\begin{aligned} |x_1(t,z,\lambda) - z| = \bigg| \int_0^t \bigg(& (P_0(s)z + P_1(s)z + f(s)) \\ & - \frac{1}{T} \int_0^T (P_0(\tau)z + P_1(\tau)z + f(\tau))\,d\tau \bigg) ds \\ & + \frac{t}{T} [d(\lambda) - (A + \mathbb{1}_n)z] \bigg| \le \alpha_1(t)\delta(z) + \tilde{\delta}(z,\lambda), \end{aligned} \tag{6.13}$$

where the vector z has form (6.2), $\tilde{\delta}$ is defined by the formula

$$\tilde{\delta}(z,\lambda) := |d(\lambda) - (A + \mathbb{1}_n)z|, \tag{6.14}$$

and α_1 is the function given by (5.2).

Let us put

$$r_{m+1}(t,z,\lambda) := x_{m+1}(t,z,\lambda) - x_m(t,z,\lambda). \tag{6.15}$$

Then, by virtue of formulae (6.1), for all $t \in [0,T]$, $n \ge 1$, $\lambda \in \mathbb{R}^{p+q}$, and z, we have

$$\begin{aligned} r_{m+1}(t,z,\lambda) &= \int_0^t [P_0(s)r_m(s,z,\lambda) + P_1(s)r_m(\beta(s),z,\lambda)]\,ds \\ &\quad - \frac{t}{T} \int_0^T [P_0(s)r_m(s,z,\lambda) + P_1(s)r_m(\beta(s),z,\lambda)]\,ds \\ &= \left(1 - \frac{t}{T}\right) \int_0^t [P_0(s)r_m(s,z,\lambda) + P_1(s)r_m(\beta(s),z,\lambda)]\,ds \\ &\quad - \frac{t}{T} \int_t^T [P_0(s)r_m(s,z,\lambda) + P_1(s)r_m(\beta(s),z,\lambda)]\,ds. \end{aligned} \tag{6.16}$$

Equalities (6.16) imply that, for all $t \in [0,T]$, $m = 1,2,..$, λ, and z,

$$\begin{aligned} |r_{m+1}(t,z,\lambda)| &\le K_0 \left(\left(1 - \frac{t}{T}\right) \int_0^t |r_m(s,z,\lambda)|\,ds + \frac{t}{T} \int_t^T |r_m(s,z,\lambda)|\,ds \right) \\ &\quad + K_1 \left(\left(1 - \frac{t}{T}\right) \int_0^t |r_m(\beta(s),z,\lambda)|\,ds + \frac{t}{T} \int_t^T r_m(\beta(s),z,\lambda)\,ds \right), \end{aligned} \tag{6.17}$$

where K_0 and K_1 are the non-negative matrices given by formula (6.4). Relation (6.13) yields

$$|r_1(t,z,\lambda)| \le \alpha_1(t)\delta(z) + \tilde{\delta}(z,\lambda), \qquad t \in [0,T], \tag{6.18}$$

where δ and $\tilde{\delta}$ are given by (6.12) and (6.14). In view of property (5.6) of Lemma 1, estimate (6.18) gives

$$|r_1(t,z,\lambda)| \le \frac{T}{2}\delta(z) + \tilde{\delta}(z,\lambda) = \gamma(z,\lambda), \qquad t \in [0,T]. \tag{6.19}$$

Let us now estimate $|r_2(t,z,\lambda)|$ using (6.17) and (6.19). We obtain

$$
\begin{aligned}
|r_2(t,z,\lambda)| \le K_0 &\left(\left(1-\frac{t}{T}\right)\int_0^t |r_1(s,z,\lambda)|\,ds + \frac{t}{T}\int_t^T |r_1(s,z,\lambda)|\,ds\right)\\
&+ K_1\left(\left(1-\frac{t}{T}\right)\int_0^t |r_1(\beta(s),z,\lambda)|\,ds + \frac{t}{T}\int_t^T |r_1(\beta(s),z,\lambda)|\,ds\right)\\
\le K_0 &\left(\left(1-\frac{t}{T}\right)\int_0^t \gamma(z,\lambda)ds + \frac{t}{T}\int_t^T \gamma(z,\lambda)ds\right)\\
&+ K_1\left(\left(1-\frac{t}{T}\right)\int_0^t \gamma(z,\lambda)ds + \frac{t}{T}\int_t^T \gamma(z,\lambda)ds\right) \qquad (6.20)
\end{aligned}
$$

for all $t \in [0,T]$. Taking relations (5.1), (5.2) into account and using property (5.6), from (6.20) we get

$$\max_{t\in[0,T]} |r_2(t,z,\lambda)| \le (K_0+K_1)\,\gamma(z,\lambda) \max_{t\in[0,T]} \alpha_1(t) = \frac{T}{2}(K_0+K_1)\,\gamma(z,\lambda).$$

Arguing by induction, we then obtain that, for all $t \in [0,T]$ and $m = 1,2,\dots,$

$$|r_m(t,z,\lambda)| \le \left(\frac{T}{2}\right)^{m-1}(K_0+K_1)^{m-1}\,\gamma(z,\lambda) = G^{m-1}\gamma(z,\lambda), \qquad (6.21)$$

where

$$G := \frac{T}{2}(K_0+K_1). \qquad (6.22)$$

Due to equality (6.15), estimate (6.21), assumption (6.3), and notation (6.22) give

$$
\begin{aligned}
\left|x_{m+j}(t,z,\lambda) - x_m(t,z,\lambda)\right| &\le \sum_{i=1}^{j} |r_{m+i}(t,z,\lambda)| \le G^m \sum_{i=0}^{j-1} G^i \gamma(z,\lambda)\\
&\le G^m \sum_{i=0}^{\infty} G^i \gamma(z,\lambda)\\
&= G^m(\mathbb{1}_n - G)^{-1}\gamma(z,\lambda) \qquad (6.23)
\end{aligned}
$$

for all $t \in [0,T], m = 1,2,\dots.$

Since, due to (6.3), $\lim_{m\to\infty} G^m = \mathbb{0}_n$, it is clear from (6.23) that (6.1) is a Cauchy sequence in the Banach space $C([0,T],\mathbb{R}^n)$ and, consequently, it converges uniformly in $t \in [0,T]$ for all fixed z of form (6.2) and $\lambda \in \mathbb{R}^{p+1}$, i.e., assertion 2 holds. Assertions 2–5 are obtained by passing to the limit.

Passing to the limit as $m \to \infty$ in (6.1) and (6.5), we show that function (6.6) is a solution of equation (6.9) and possesses property (6.8). Passing to the limit as $j \to \infty$

in (6.23), we obtain the estimate

$$|x^*(t,z,\lambda) - x_m(t,z,\lambda)| \le G^m (\mathbb{1}_n - G)^{-1} \gamma(z,\lambda)$$

for all $t \in [0,T]$, $m = 1,2,\ldots$, $\lambda \in \mathbb{R}^{p+q}$, and z of form (6.2), i. e., assertion 5 holds. This completes the proof of Theorem 1. □

Remark 3. A similar scheme can be obtained if the recurrence formula (6.1) is replaced (cf. [9]) by the relation

$$\begin{aligned} x_{m+1}(t,z,\lambda) := z + \int_0^t [P_0(s)\, x_m(s,z,\lambda) + P_1(s)\, x_m(\beta(s),z,\lambda) + f(s)]\, ds \\ - \frac{\omega(t)}{T} \int_0^T [P_0(s)\, x_m(s,z,\lambda) + P_1(s)\, x_m(\beta(s),z,\lambda) + f(s)]\, ds \\ + \frac{\omega(t)}{T} [d(\lambda) - (A + \mathbb{1}_n) z], \end{aligned}$$

where $\omega : [0,T] \to [0,T]$ is an arbitrary continuous function with the properties $\omega(0) = 0$ and $\omega(T) = T$.

Proposition 1. *If, under assumptions of Theorem 1, the function $x^*(\cdot,z,\lambda)$ satisfies the condition*

$$d(\lambda) - (A + \mathbb{1}_n) z = \int_0^T \big(P_0(s)\, x^*(s,z,\lambda) + P_1(s)\, x^*(\beta(s),z,\lambda)\big)\, ds + \int_0^T f(s)\, ds \tag{6.24}$$

for certain values of z and λ, then, for these z and λ, it is also a solution of the boundary value problem (3.1), (4.3).

The *proof* of the last statement is a straightforward application of the above theorem.

7. PROPERTIES OF THE LIMIT FUNCTION

Let us first establish a relation between the limit function of sequence (6.1) and the solution of the auxiliary two-point parametrized boundary value problem (3.1), (4.3). Along with system (3.1), we also consider the system with the additive perturbation of the right-hand side

$$x'(t) = P_0(t)x(t) + P_1(t)x(\beta(t)) + f(t) + \mu, \qquad t \in [0,T], \tag{7.1}$$

with the initial condition

$$x(0) = z, \tag{7.2}$$

where $\mu = \operatorname{col}(\mu_1,\ldots,\mu_n)$ is a control parameter. We shall see that, for any z, the vector-parameter μ can always be chosen so that the solution $x(\cdot,z,\mu)$ of the initial value problem (7.1), (7.2) is, at the same time, a solution of the two-point boundary value problem (7.1), (4.3).

Proposition 2. *Assume that the system of differential equations* (3.1) *satisfies the conditions of Theorem 1. Then, for arbitrary z of form* (6.2) *and any λ,*

$$\mu = \frac{1}{T}\left[d(\lambda) - (A + \mathbb{1}_n)z\right] - \frac{1}{T}\int_0^T \big(P_0(s)x^*(s,z,\lambda) + P_1(s)x^*(\beta(s),z,\lambda) + f(s)\big)\,ds \tag{7.3}$$

is the unique value of the vector parameter μ for which the solution $x(\cdot,z,\mu)$ of the initial value problem (7.1), (7.2) *with μ given by* (7.3) *is also a solution of the boundary value problem* (7.1), (4.3). *Moreover, with this values of μ*

$$x(t,z,\mu) = x^*(t,z,\lambda) = \lim_{m\to\infty} x_m(t,z,\lambda), \tag{7.4}$$

where $\{x_m(\cdot,z,\lambda)\}_{m=1}^{\infty}$ is the sequence of functions defined according to (6.1).

Proof. The assertion of Proposition 2 is obtained by analogy to the proof of [11, Theorem 4.2] □

Definition 1. *For any $k = 1,2,\ldots,n$, let us define the n-dimensional row vector e_k by putting*

$$e_k := (\underbrace{0,0,\ldots,0,0}_{k-1},1,0,\ldots,0). \tag{7.5}$$

Let us consider the function $\Delta : \mathbb{R}^{n-p}\times\mathbb{R}^{p+q} \to \mathbb{R}^n$ given by the formula

$$\Delta(z,\lambda) := \frac{1}{T}\left[d(\lambda) - (A + \mathbb{1}_n)z\right] - \frac{1}{T}\int_0^T \left[P_0(s)x^*(s,z,\lambda) + P_1(s)x^*(\beta(s),z,\lambda) + f(s)\right]ds \tag{7.6}$$

for z of form (6.2) with arbitrary $z_{p+1},\ldots,z_n$ and $\lambda \in \mathbb{R}^{p+q}$. Formula (7.6) makes sense provided that the limit function $x^*(\cdot,z,\lambda)$ of sequence (6.1) exists.

Proposition 3. *Assume the conditions of Theorem 1. Then the function $x^*(\cdot,z,\lambda)$ is a solution of the three-point Cauchy–Nicoletti boundary value problem* (3.1), (3.6) *if and only if the pair (z,λ) satisfies the system of $n+q$ algebraic equations**

$$\Delta(z,\lambda) = 0, \tag{7.7}$$

$$e_{p+1}x^*(\xi,z,\lambda) = d_{p+1}, \quad \ldots, \quad e_{p+q}x^*(\xi,z,\lambda) = d_{p+q}. \tag{7.8}$$

*Recall that the first p components of z are known, see Remark 2.

Proof. It is sufficient to apply Proposition 2 and notice that the differential equation obtained by the differentition of equation (6.9) coincides with (3.1) if and only if the pair (z, λ) satisfies (7.7). On the other hand, equations (7.8) bring us from the auxiliary boundary conditions (4.1) back to the three-point Cauchy–Nicoletti conditions (3.6). □

Proposition 4. *Let us define the matrix R by putting*

$$R := \sup_{t\in[0,T]} |\mathbb{1}_n - tT^{-1}(\mathbb{1}_n + A)|. \tag{7.9}$$

Under the conditions of Theorem 1, the estimate

$$|x^*(t, z^0, \lambda) - x^*(t, z^1, \lambda)| \leq \frac{2}{T}\left(\frac{2}{T}\mathbb{1}_n - K_0 - K_1\right)^{-1} R\,|z^0 - z^1|, \tag{7.10}$$

where

$$z^j = \operatorname{col}(x_{10}, x_{20}, \ldots, x_{p0}, z^j_{p+1}, z^j_{p+2}, \ldots, z^j_n), \quad j = 0, 1, \tag{7.11}$$

holds for arbitrary z^j_k, $k = p+1, p+2, \ldots, n$, $j = 0, 1$, $t \in [0, T]$, and $\lambda \in \mathbb{R}^{p+q}$.

Proof. Consider the sequence of vectors c_m, $m \geq 1$, defined by the recurrence formula

$$c_m := R|z^0 - z^1| + \frac{T}{2}(K_0 + K_1)c_{m-1}, \qquad m \geq 1,$$

with $c_0 := |z^0 - z^1|$ and the matrix R of form (7.9). Let us show that the functions

$$u_m(t) := x_m(t, z^0, \lambda) - x_m(t, z^1, \lambda), \qquad t \in [0, T],\ m \geq 1, \tag{7.12}$$

satisfy the estimate

$$|u_m(t)| \leq c_m, \qquad t \in [0, T],\ m \geq 1. \tag{7.13}$$

Indeed, for $m = 0$ relation (7.13) is satisfied in the form of an equality. Assume that (7.13) is satisfied for a given $m \geq 1$. It follows immediately from (6.1) that

$$\begin{aligned} u_{m+1}(t) &= z^0 - z^1 - \frac{t}{T}(\mathbb{1}_n + A)(z^0 - z^1) \\ &\quad + \int_0^t (P_0(s)u_m(s) + P_1(s)u_m(\beta(s)))\,ds \\ &\quad - \frac{t}{T}\int_0^T (P_0(s)u_m(s) + P_1(s)u_m(\beta(s)))\,ds \\ &= z^0 - z^1 - \frac{t}{T}(\mathbb{1}_n + A)(z^0 - z^1) \\ &\quad + \left(1 - \frac{t}{T}\right)\int_0^t (P_0(s)u_m(s) + P_1(s)u_m(\beta(s)))\,ds \end{aligned}$$

$$-\frac{t}{T}\int_t^T (P_0(s)u_m(s)+P_1(s)u_m(\beta(s)))\,ds, \tag{7.14}$$

whence, in view of (7.9),

$$\begin{aligned}|u_{m+1}(t)| \le R|z^0-z^1| &\\ &+\left(1-\frac{t}{T}\right)\int_0^t (|P_0(s)||u_m(s)|+|P_1(s)||u_m(\beta(s))|)\,ds\\ &+\frac{t}{T}\int_t^T (|P_0(s)||u_m(s)|+|P_1(s)||u_m(\beta(s))|)\,ds\end{aligned} \tag{7.15}$$

for all $t\in[0,T]$ and $m\ge 1$. Recalling formulae (6.4), (5.1) and using assumption (7.13) and equality (5.6), we obtain

$$\begin{aligned}|u_{m+1}(t)| &\le R|z^0-z^1|+\left(1-\frac{t}{T}\right)\int_0^t (K_0|u_m(s)|+K_1|u_m(\beta(s))|)\,ds\\ &\quad+\frac{t}{T}\int_t^T (K_0|u_m(s)|+K_1|u_m(\beta(s))|)\,ds\\ &\le R|z^0-z^1|+\left(\left(1-\frac{t}{T}\right)\int_0^t dt+\int_t^T dt\right)(K_0+K_1)c_m\\ &\le R|z^0-z^1|+\alpha_1\left(\frac{T}{2}\right)(K_0+K_1)c_m\\ &= R|z^0-z^1|+\frac{T}{2}(K_0+K_1)c_m=c_{m+1},\end{aligned} \tag{7.16}$$

that is, estimate (7.13) holds at the step $m+1$, and, hence, it is satisfied on every step of iteration. Considering now inequality (7.13) and iterating backwards, we obtain

$$\begin{aligned}|u_m(t)| &\le R|z^0-z^1|+\frac{T}{2}(K_0+K_1)c_{m-1}\\ &= R|z^0-z^1|+\frac{T}{2}(K_0+K_1)\left(R|z^0-z^1|+\frac{T}{2}(K_0+K_1)c_{m-2}\right)\\ &= \left(\mathbb{1}_n+\frac{T}{2}(K_0+K_1)\right)R|z^0-z^1|+\left(\frac{T}{2}\right)^2(K_0+K_1)^2c_{m-2}\end{aligned}$$

and so on, which leads us the inequality

$$|u_m(t)|\le\sum_{i=0}^{m-1}\left(\frac{T}{2}\right)^i(K_0+K_1)^i R|z^0-z^1|+\left(\frac{T}{2}\right)^m(K_0+K_1)^m|z^0-z^1|$$

valid for all $m = 1, 2, \ldots$ and $t \in [0, T]$. Passing to the limit as $m \to \infty$, using assumption (6.3), and recalling notation (7.12), we obtain the inequality

$$|x^*(t, z^0, \lambda) - x^*(t, z^1, \lambda)| \le \sum_{i=0}^{\infty} \left(\frac{T}{2}\right)^i (K_0 + K_1)^i \, R|z^0 - z^1|, \qquad t \in [0, T],$$

whence the required estimate (7.10) follows. □

Let us put

$$\mathcal{S}_\theta := (K_0 + K_1)\,(\theta \mathbb{1}_n - K_0 - K_1)^{-1} \tag{7.17}$$

for all those θ for which the inverse matrix exists.

Proposition 5. *Under the conditions of Theorem 1, formula (7.6) determines a well-defined function $\Delta : \mathbb{R}^{n-p} \times \mathbb{R}^{p+q} \to \mathbb{R}^n$, which satisfies the estimate*

$$\left|\Delta(z^0, \lambda) - \Delta(z^1, \lambda)\right| \le \left(\frac{1}{T}|A + \mathbb{1}_n| + \frac{2}{T}\mathcal{S}_{\frac{2}{T}} R\right)|z^0 - z^1|, \tag{7.18}$$

where the matrices $\mathcal{S}_{2T^{-1}}$ and R are given by (7.17) and (7.9).

For the *proof* of the last statement, it is sufficient to recall formula (7.6) and use Proposition 3.

8. An existence theorem for the Cauchy–Nicoletti problem

Theorem 1 and Proposition 3 give the following numerical-analytic algorithm for the construction of a solution of the three-point Cauchy–Nicoletti boundary value problem (3.1), (3.6).

(1) For any vector z of form (6.2), according to (6.1), we analytically construct the sequence of functions $x_m(\cdot, z, \lambda)$, depending on the parameters $(z_{p+1}, \ldots, z_n) \in \mathbb{R}^{n-p}$ and $\lambda = \operatorname{col}(\lambda_1, \ldots, \lambda_p, \lambda_{p+1}, \ldots, \lambda_{p+q}) \in \mathbb{R}^{p+q}$ and satisfying the auxiliary two-point boundary conditions (4.3).
(2) We find the limit $x^*(\cdot, z, \lambda)$ of the sequence $x_m(\cdot, z, \lambda)$ satisfying to (4.3).
(3) We construct the algebraic determining system of the form (7.7), (7.8) with respect to the $n + q$ scalar parameters $\lambda = \operatorname{col}(\lambda_1, \ldots, \lambda_p, \lambda_{p+1}, \ldots, \lambda_{p+q})$ and $(z_{p+1}, \ldots, z_n) \in \mathbb{R}^{n-p}$.
(4) Using a suitable method for the numerical solution of system (7.7), (7.8), we (approximately) find the solution

$$\begin{gathered} \operatorname{col}(z^*_{p+1}, \ldots, z^*_n) \in \mathbb{R}^{n-p}, \\ \lambda^* = \operatorname{col}(\lambda^*_1, \ldots, \lambda^*_p, \lambda^*_{p+1}, \ldots, \lambda^*_{p+q}) \in \mathbb{R}^{p+q} \end{gathered} \tag{8.1}$$

of the determining system (7.7), (7.8).
(5) Substituting values (8.1) into $x^*(\cdot, z, \lambda)$, we get the solution of the three-point Cauchy–Nicoletti boundary value problem (3.1), (3.6) in the form

$$x = x^*(\cdot, z^*, \lambda^*), \tag{8.2}$$

where $z^* = \operatorname{col}(x_{10}, x_{20}, \ldots, x_{p0}, z^*_{p+1}, \ldots, z^*_n)$. The solution (8.2) can also be obtained by solving the Cauchy problem

$$x(0) = z^* \tag{8.3}$$

for equation (3.1).

A fundamental difficulty in the realization of this approach is related to the analytic construction of the limit function $x^*(\cdot, z, \lambda)$. However, in a number of cases, this problem can be avoided because, as can be shown, it is possible to prove the existence of a solution of the three-point Cauchy–Nicoletti boundary value problem (3.1), (3.6) based on properties of a certain approximation $x_m(\cdot, z, \lambda)$ known in the analytic form.

Given some $m \geq 1$, define the function $\Delta_m : \mathbb{R}^n \times \mathbb{R}^{p+q} \to \mathbb{R}^n$ according to the formula

$$\begin{aligned} \Delta_m(z,\lambda) := {} & \frac{1}{T}\left[d(\lambda) - (A + \mathbb{1}_n) z\right] \\ & - \frac{1}{T}\int_0^T \left[P_0(s)\, x_m(s,z,\lambda) + P_1(s)\, x_m(\beta(s), z, \lambda) + f(s)\right] ds \end{aligned} \tag{8.4}$$

for z of form (6.2) with arbitrary $z_{p+1}, \ldots, z_n$, and $\lambda \in \mathbb{R}^{p+q}$. To investigate the solvability of the three-point Cauchy–Nicoletti boundary value problem (3.1), (3.6), in addition to determining system (7.7), (7.8), we introduce the mth approximate determining system

$$\begin{gathered} \Delta_m(z,\lambda) = 0, \\ e_{p+1} x_m(t,z,\lambda)(\xi) = d_{p+1}, \quad \ldots, \quad e_{p+q} x_m(t,z,\lambda)(\xi) = d_{p+q}, \end{gathered} \tag{8.5}$$

where e_i, $i = 1, 2, \ldots, n$, are the vectors given by (7.5) and the vector function $x_m(\cdot, z, \lambda)$ is defined by formula (6.1). It is natural to expect that, under suitable conditions, systems (7.7) and (8.5) are "close enough" to one another for m sufficiently large.

Lemma 6. *Assume the conditions of Theorem 1. Then, for arbitrary $m \geq 1$, $\lambda \in \mathbb{R}^{p+q}$, and all z of form (6.2), the estimate*

$$|\Delta(z,\lambda) - \Delta_m(z,\lambda)| \leq \left(\frac{T}{2}\right)^{m-1} (K_0 + K_1)^m\, \mathcal{S}_{\frac{2}{T}}\, \gamma(z,\lambda), \tag{8.6}$$

holds, where $\gamma(z,\lambda)$ is given by (6.11).

Proof. Indeed, let us fix arbitrary z and λ and put

$$\psi_m(t) := x^*(t,z,\lambda) - x_m(t,z,\lambda), \qquad t \in [0,T],\ m \geq 1.$$

By virtue of estimate (6.10) and notation (7.17), we have

$$|\psi_m(t)| \leq \left(\frac{T}{2}\right)^{m-1} (K_0 + K_1)^m \left(\frac{2}{T}\mathbb{1}_n - K_0 - K_1\right)^{-1} \gamma(z,\lambda)$$

$$= \left(\frac{T}{2}\right)^{m-1} (K_0 + K_1)^{m-1} \mathcal{S}_{\frac{2}{T}} \gamma(z, \lambda) \tag{8.7}$$

Therefore, according to (7.6) and (8.4),

$$\begin{aligned} |\Delta(z,\lambda) - \Delta_m(z,\lambda)| &= \frac{1}{T} \left| \int_0^T [P_0(s)\,\psi_m(s) + P_1(s)\,\psi_m(\beta(s))]\,ds \right| \\ &\le \frac{1}{T} \int_0^T (|P_0(s)||\psi_m(s)| + |P_1(s)||\psi_m(\beta(s))|)\,ds \\ &\le (K_0 + K_1) \left(\frac{T}{2}\right)^{m-1} (K_0 + K_1)^{m-1} \mathcal{S}_{\frac{2}{T}} \gamma(z,\lambda) \\ &= \left(\frac{T}{2}\right)^{m-1} (K_0 + K_1)^m \mathcal{S}_{\frac{2}{T}} \gamma(z,\lambda), \end{aligned}$$

which leads us to (8.6). □

Let us formulate a statement that gives sufficient conditions for the solvability of the three-point Cauchy–Nicoletti boundary value problem (3.1), (3.6).

Definition 2. *For any indices i_1 and i_2 between 1 and n, $i_2 \ge i_1$, define the $(i_2 - i_1) \times n$ matrix J_{i_1,i_2} by putting*

$$J_{i_1,i_2} := \left(\mathbb{0}_{i_2-i_1+1,i_1-1} \quad \mathbb{1}_{i_2-i_1+1} \quad \mathbb{0}_{i_2-i_1+1,n-i_2} \right), \tag{8.8}$$

so that the left multiplication of a vector by the matrix J_{i_1,i_2} is equivalent to the selection of its components with numbers from i_1 to i_2.

Introduce the mapping $\Phi_m : \mathbb{R}^{n-p} \times \mathbb{R}^{p+q} \to \mathbb{R}^{n+q}$ by setting

$$\Phi_m(z,\lambda) := \begin{pmatrix} \Delta_m(z,\lambda) \\ J_{p+1,p+q}\, x_m(\xi, z, \lambda) - J_{p+1,p+q}\bar{d} \end{pmatrix} \tag{8.9}$$

for all z of form (6.2), $\lambda \in \mathbb{R}^{p+q}$, and $m \ge 0$, where $\bar{d}$ is the vector given by equality (3.7). Recall that the first p components of the vector z are fixed and, thus, the actual number of variables on which Φ_m depends is $n + q$ (see Remark 2).

Definition 3. *Let $H \subset \mathbb{R}^{n+q}$ be a an arbitrary non-empty set. For any pair of functions $f_j = (f_{j,i})_{i=1}^{n+q} : H \to \mathbb{R}^{n+q}$, $j = 1, 2$, we write $f_1 \rhd_H f_2$ if and only if there exists a function $k : H \to \{1, 2, \ldots, n + q\}$ such that*

$$f_{1,k(x)}(x) > f_{2,k(x)}(x)$$

for all $x \in H$.

Remark 4. The relation "$\rhd_H$" has properties similar to those of the usual inequality. In particular, if $f_1 \ge f_2$ on H pointwise and componentwise and $f_2 \rhd_H f_3$, then f_1 and f_3 satisfy the relation $f_1 \rhd_H f_3$.

Let $D \subset \mathbb{R}^{n-p}$, $\varLambda \subset \mathbb{R}^{p+q}$, and $\varOmega \subset D \times \varLambda$ be the closure of a bounded domain. The following theorem holds.

Theorem 2. *Let us suppose that, in addition to the assumptions of Theorem 1, the set $\varOmega$ and a number $m \in \mathbb{N}$ can be chosen so that the approximate determining function $\varDelta_m$ constructed according to equation (8.5) satisfies the following conditions:*

(1) *The relation*

$$|\varPhi_m| \rhd_{\partial\varOmega} \left(\frac{T}{2}\right)^{m-1} \begin{pmatrix} (K_0 + K_1)^m \, \mathcal{S}_{\frac{2}{T}} \gamma \\ J_{p+1,\,p+q} \, (K_0 + K_1)^{m-1} \, \mathcal{S}_{\frac{2}{T}} \gamma \end{pmatrix} \tag{8.10}$$

holds, where γ is the function defined by formula (6.11).†

(2) *The Brouwer degree of $\varPhi_m$ over $\varOmega$ with respect to 0 satisfies the inequality*

$$\deg(\varPhi_m, \varOmega, 0) \neq 0. \tag{8.11}$$

Then the three-point Cauchy–Nicoletti boundary value problem (3.1), (3.6) has a solution x with $(x_{p+1}(0), x_{p+2}(0), \ldots, x_n(0))$ belonging to D.

Proof. Let us define the mapping $\varPhi : \mathbb{R}^{n-p} \times \mathbb{R}^{p+q} \to \mathbb{R}^n$ by setting

$$\varPhi(z,\lambda) := \begin{pmatrix} \varDelta(z,\lambda) \\ J_{p+1,p+q} \, x^*(\xi, z, \lambda) - J_{p+1,p+q} \bar{d} \end{pmatrix} \tag{8.12}$$

for all z of form (6.2) and $\lambda \in \mathbb{R}^{p+q}$, where x^* is the limit function (6.6) of sequence (6.1) and $\bar{d}$ is the vector (3.7). It is clear from from Proposition 5 that the mappings $\varPhi$ and $\varPhi_m$ are continuous.

Let us prove that the fields $\varPhi$ and $\varPhi_m$ are homotopic. For this purpose, we consider the linear deformation

$$Q_\theta(\lambda) := \varPhi_m(z,\lambda) + \theta\,[\varPhi(z,\lambda) - \varPhi_m(z,\lambda)], \qquad (z,\lambda) \in \partial\varOmega, \tag{8.13}$$

where $\theta \in [0,1]$. Obviously, Q_θ is continuous mapping on $\partial\varOmega$ for every $\theta \in [0,1]$ and, furthermore,

$$Q_0(z,\lambda) = \varPhi_m(z,\lambda), \qquad\qquad Q_1(z,\lambda) = \varPhi(z,\lambda) \tag{8.14}$$

for all $(z,\lambda) \in \partial\varOmega$.

For arbitrary $(z,\lambda) \in \partial\varOmega$ and $\theta \in [0,1]$, in view of (8.10), (8.14), and (6.10), we have

$$\begin{aligned} |Q_\theta(z,\lambda)| &= |\varPhi_m(z,\lambda) + \theta\,[\varPhi(z,\lambda) - \varPhi_m(z,\lambda)]| \\ &\geq |\varPhi_m(z,\lambda)| - |\varPhi(z,\lambda) - \varPhi_m(z,\lambda)|. \end{aligned} \tag{8.15}$$

† Recall that $\mathcal{S}_{2T^{-1}}$ is the matrix (7.17), the vector function γ is defined by (6.11), and the matrices in (8.10) are constructed according to (8.8).

On the other hand, recalling equalities (7.6), (8.4), (7.17), (8.9), and using estimates (6.10) of Theorem 1 and (8.6) of Lemma 6, we otain the componentwise inequalities

$$|\Phi(z,\lambda)-\Phi_m(z,\lambda)| \le \begin{pmatrix} \left(\frac{T}{2}\right)^{m-1}(K_0+K_1)^{m-1}\mathcal{S}_{\frac{2}{T}}\gamma(z,\lambda) \\ \left(\frac{T}{2}\right)^{m-1} J_{p+1,p+q}(K_0+K_1)^m \mathcal{S}_{\frac{2}{T}}\gamma(z,\lambda) \end{pmatrix}$$

whence, in view of (8.15), it follows that

$$|Q_\theta| \rhd_{\partial\Omega} 0, \qquad \theta \in [0,1].$$

The last relation implies, in particular, that Q_θ does not vanish on $\partial\Omega$ for any value of $\theta \in [0,1]$, i. e., deformation (8.13) is non-degenerate and, thus, Φ_m is homotopic to Φ. Using assumption (8.11) and the property of invariance of degree under homotopy, we conclude that

$$\deg(\Phi,\Omega,0) = \deg(\Phi_m,\Omega,0) \neq 0. \tag{8.16}$$

The classical topological result (see, e. g., [2, Theorem A2.5]) then guarantees the existence of vectors $(z^*,\lambda^*) \in \Omega$ such that $\Phi(z^*,\lambda^*) = 0$, which, according to (8.12), means that

$$\Delta(z^*,\lambda^*) = 0$$

and, moreover, q components of the vector $x^*(\xi, z^*, \lambda^*)$, starting from the $(p+1)$th, coincide with the corresponding components of $\bar{d}$. Thus, the pair (z^*,λ^*) satisfies the system of equations (7.7), (7.8). Applying now Proposition 3, we find that function (8.2) is a solution of the three-point Cauchy–Nicoletti boundary value problem (3.1), (3.6). □

9. Convergence of successive approximations for the special deviation functions

If the deviation function $\beta : [0,T] \to [0,T]$ satisfies the condition (5.10), then the convergence condition (5.11) can be improved.

Theorem 3. *Assume that the condition*

$$r(K_0+K_1) < \frac{10}{3T}. \tag{9.1}$$

is satisfied and, moreover, the deviation function $\beta : [0,T] \to [0,T]$ has property (5.10). Then assertions 1–4 of Theorem 1 hold. Moreover, the estimate

$$\left|x^*(t,z,\lambda) - x_m(t,z,\lambda)\right| \le$$
$$\le \frac{20}{9}\left(\frac{3T}{10}\right)^{m-2} t\left(1-\frac{t}{T}\right)(K_0+K_1)^{m-1}\mathcal{S}_{\frac{10}{3T}}\gamma(z,\lambda)$$

holds for any $m \ge 1$, $\lambda \in \mathbb{R}^{p+q}$, $t \in [0,T]$, and all fixed z of form (6.2), where $\gamma(z,\lambda)$ and $\mathcal{S}_{\frac{10}{3T}}$ are given by (6.11) and (7.17).

Proof. By virtue of Lemma 4, it follows from assumption (5.10) that inequalities (5.11) are true and, in particular,

$$\alpha_1(\beta(t)) \leq \alpha_1(t), \qquad t \in [0,T]. \tag{9.2}$$

Estimating $|r_2(t,z,\lambda)|$ by using (6.13), (5.1), (9.2), and (5.9), we get

$$\begin{aligned}|r_2(t,z,\lambda)| &\leq K_0\Bigg(\left(\left(1-\frac{t}{T}\right)\int_0^t \left(\frac{T}{2}\delta(z)+\tilde{\delta}(z,\lambda)\right)ds\right.\\ &\quad \left.+\frac{t}{T}\int_t^T \left(\frac{T}{2}\delta(z)+\tilde{\delta}(z,\lambda)\right)ds\right)\\ &\quad +K_1\left(\left(1-\frac{t}{T}\right)\int_0^t \left(\frac{T}{2}\delta(z)+\tilde{\delta}(z,\lambda)\right)ds\right.\\ &\quad \left.+\frac{t}{T}\int_t^T \left(\frac{T}{2}\delta(z)+\tilde{\delta}(z,\lambda)\right)ds\right)\\ &\leq (K_0+K_1)\,\gamma(z,\lambda)\,\alpha_1(t), \qquad t\in[0,T],\end{aligned} \tag{9.3}$$

where $\tilde{\delta}(z,\lambda)$, $\delta(z)$, and $\gamma(z,\lambda)$ are given by (6.14), (6.12), and (6.11), respectively. Relation (9.3), due to (9.2), yields

$$\begin{aligned}|r_2(\beta(t),z,\lambda)| &\leq (K_0+K_1)\,\gamma(z,\lambda)\,\alpha_1(\beta(t))\\ &\leq (K_0+K_1)\,\gamma(z,\lambda)\,\alpha_1(t), \qquad t\in[0,T].\end{aligned} \tag{9.4}$$

Arguing by induction, we find that all the functions (6.15) admit the estimates

$$|r_{m+1}(t,z,\lambda)| \leq (K_0+K_1)^m\,\gamma(z,\lambda)\,\alpha_m(t), \tag{9.5}$$

$$|r_{m+1}(\beta(t),z,\lambda)| \leq (K_0+K_1)^m\,\gamma(z,\lambda)\,\alpha_m(t), \tag{9.6}$$

for all $t\in[0,T]$ and $m\geq 1$, where the function α_m is given by (5.1).

Due to estimate (5.9) of Lemma 3, relations (9.5), (9.6) yield

$$|r_{m+1}(t,z,\lambda)| \leq \frac{10}{9}\left(\frac{3T}{10}\right)^{m-1}(K_0+K_1)^m\,\gamma(z,\lambda)\,\alpha_1(t),$$

$$|r_{m+1}(\beta(t),z,\lambda)| \leq \frac{10}{9}\left(\frac{3T}{10}\right)^{m-1}(K_0+K_1)^m\,\gamma(z,\lambda)\,\alpha_1(t),$$

and it remains to repeat, with obvious modifications, the reasoning shown at the end of the proof of Theorem 1. □

A similar improvement of the convergence condition is possible in the case where β satisfies condition (5.12).

Theorem 4. *Assume that the deviation function* $\beta : [0,T] \to [0,T]$ *has property* (5.12) *and, moreover,*

$$r(K_0 + K_1) < \frac{3}{T\eta}, \tag{9.7}$$

where

$$\eta := \max\{k_\beta, 1\} \tag{9.8}$$

and k_β *is the constant appearing in* (5.12)*. Then assertions 1–4 of Theorem 1 are true. Moreover, the estimate*

$$\left|x^*(t,z,\lambda) - x_m(t,z,\lambda)\right| \le$$
$$\le \frac{6}{T} t\left(1 - \frac{t}{T}\right)\left(\frac{T\eta}{3}\right)^{m-1} (K_0 + K_1)^{m-1} \mathcal{S}_{\frac{3}{T\eta}} \gamma(z,\lambda)$$

holds for any $m \ge 1$, $\lambda \in \mathbb{R}^{p+q}$, $t \in [0,T]$, *and all fixed* z *of form* (6.2).

Proof. It is clear from (9.8) that $\eta \ge 1$ and, hence, according to (9.3),

$$\begin{aligned}|r_2(t,z,\lambda)| &\le (K_0 + K_1)\gamma(z,\lambda)\alpha_1(t)\\ &\le \eta(K_0 + K_1)\gamma(z,\lambda)\alpha_1(t), \qquad t \in [0,T].\end{aligned} \tag{9.9}$$

Due to (5.13), we have

$$\begin{aligned}|r_2(\beta(t),z,\lambda)| &\le (K_0 + K_1)\gamma(z,\lambda)\alpha_1(\beta(t))\\ &\le k_\beta(K_0 + K_1)\gamma(z,\lambda)\alpha_1(t)\\ &\le \eta(K_0 + K_1)\gamma(z,\lambda)\alpha_1(t), \qquad t \in [0,T].\end{aligned} \tag{9.10}$$

Using (6.17), (9.9), (9.10) and (5.9) and carrying out calculations, we obtain

$$\begin{aligned}|r_3(t,z,\lambda)| &\le \eta K_0(K_0 + K_1)\gamma(z,\lambda)\alpha_2(t) + \eta K_1(K_0 + K_1)\gamma(z,\lambda)\alpha_2(t)\\ &\le \eta(K_0 + K_1)^2\gamma(z,\lambda)\alpha_2(t)\\ &\le \eta\frac{T\eta}{3}(K_0 + K_1)^2\gamma(z,\lambda)\alpha_1(t).\end{aligned} \tag{9.11}$$

Therefore, in view of (9.11), (5.12), (5.13),

$$\begin{aligned}|r_3(\beta(t),z,\lambda)| &\le \eta(K_0 + K_1)^2\gamma(z,\lambda)\alpha_2(\beta(t))\\ &\le \eta(K_0 + K_1)^2\gamma(z,\lambda)\frac{T}{3}\alpha_1(\beta(t))\\ &\le \eta\left(\frac{T\eta}{3}\right)(K_0 + K_1)^2\gamma(z,\lambda)\alpha_1(t).\end{aligned} \tag{9.12}$$

Further on, according to (6.17), (9.11), and (9.12), we find

$$|r_4(t,z,\lambda)| \le K_0\eta\left(\frac{T\eta}{3}\right)(K_0 + K_1)^2\gamma(z,\lambda)\alpha_2(t)$$

$$
\begin{aligned}
&+K_1\eta\left(\frac{T\eta}{3}\right)(K_0+K_1)^2\gamma(z,\lambda)\alpha_2(t)\\
&=\eta\left(\frac{T\eta}{3}\right)(K_0+K_1)^3\gamma(z,\lambda)\alpha_2(t)\\
&\le\left(\frac{T\eta}{3}\right)^2(K_0+K_1)^3\gamma(z,\lambda)\alpha_1(t) \qquad (9.13)\\
&\le\eta\left(\frac{T\eta}{3}\right)^2(K_0+K_1)^3\gamma(z,\lambda)\alpha_1(t).
\end{aligned}
$$

According to (9.13) and (5.13),

$$
\begin{aligned}
|r_4(\beta(t),z,\lambda)| &\le \left(\frac{T\eta}{3}\right)^2(K_0+K_1)^3\gamma(z,\lambda)\alpha_1(\beta(t))\\
&\le\eta\left(\frac{T\eta}{3}\right)^2(K_0+K_1)^3\gamma(z,\lambda)\alpha_1(t) \qquad (9.14)
\end{aligned}
$$

for a. e. $t\in[0,T]$. Arguing by induction, for any $t\in[0,T]$, we arrive at the estimates

$$
\begin{aligned}
|r_m(t,z,\lambda)| &\le \left(\frac{T\eta}{3}\right)^{m-2}(K_0+K_1)^{m-1}\gamma(z,\lambda)\alpha_1(t)\\
&=(K_0+K_1)\,\bar{\bar{G}}^{m-2}\gamma(z,\lambda)\alpha_1(t) \qquad (9.15)\\
&\le\eta(K_0+K_1)\,\bar{\bar{G}}^{m-2}\gamma(z,\lambda)\alpha_1(t),
\end{aligned}
$$

where

$$
\bar{\bar{G}} := \frac{\eta T}{3}(K_0+K_1). \qquad (9.16)
$$

Using (9.15), we also get

$$
\begin{aligned}
|r_m(\beta(t),z,\lambda)| &\le k_\beta\left(\frac{T\eta}{3}\right)^{m-2}(K_0+K_1)^{m-1}\gamma(z,\lambda)\alpha_1(t)\\
&\le\eta(K_0+K_1)\,\bar{\bar{G}}^{m-2}\gamma(z,\lambda)\alpha_1(t) \qquad (9.17)
\end{aligned}
$$

for a. e. $t\in[0,T]$. This yields

$$
\begin{aligned}
\left|x_{m+j}(t,z,\lambda)-x_m(t,z,\lambda)\right| &\le \sum_{i=1}^{j}|r_{m+i}(t,z,\lambda)|\\
&\le\eta(K_0+K_1)\,\bar{\bar{G}}^{m-1}\sum_{i=0}^{j-1}\bar{\bar{G}}^{i}\gamma(z,\lambda)\alpha_1(t)\\
&\le\eta(K_0+K_1)\,\bar{\bar{G}}^{m-1}(\mathbb{1}_n-\bar{\bar{G}})^{-1}\gamma(z,\lambda)\alpha_1(t)
\end{aligned}
$$

$$= \frac{3}{T} \bar{\bar{G}}^m (\mathbb{1}_n - \bar{\bar{G}})^{-1} \gamma(z, \lambda) \alpha_1(t) \tag{9.18}$$

for any $m \geq 1$, $j \geq 1$, $t \in [0, T]$, and arbitrary λ and z. The required assertion is now obtained from (9.18) again by analogy to the proof of Theorem 1. □

Remark 5. It is obvious that if $\eta < 3/2$, then the convergence condition (9.7) is sharper than inequality (6.3) used in the general case.

Remark 6. Estimates of Theorems 3 and 4 allow one to state analogues of the existence Theorem 2. The formulations are straightforward, and we omit them.

10. A NUMERICAL EXAMPLE

We apply techniques based on the statements of the preceding sections to the following Cauchy–Nicoletti boundary value problem

$$x_1(0) = -\frac{1}{16}, \quad x_2\left(\frac{1}{2}\right) = \frac{1}{8}, \quad x_3(1) = \frac{1}{4} \tag{10.1}$$

for the system of three equations

$$\left.\begin{aligned} x_1'(t) &= x_2(t^2) + \frac{t}{4} - \frac{t^2}{4}, \\ x_2'(t) &= t x_3(t) + \frac{1}{4} - \frac{t}{4}, \\ x_3'(t) &= t x_1(t) - \frac{t^2}{2} x_2(t) + \frac{t}{16}, \end{aligned}\right\} \tag{10.2}$$

considered in the closed domain determined by the inequalities

$$|x_1| \leq \frac{1}{2}, \qquad |x_2| \leq \frac{1}{2}, \qquad |x_3| \leq \frac{1}{3}. \tag{10.3}$$

System (10.2), of course, is a particular case of (3.1) with $T = 1$, $f_1(t) = t(1 - t)/4$, $f_2(t) = (1-t)/4$, $f_3(t) = t/16$ for $t \in [0, 1]$,

$$P_0(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & t \\ t & -\frac{t^2}{2} & 0 \end{pmatrix} \qquad P_1(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{t}{4} \end{pmatrix}, \tag{10.4}$$

and the argument transformation

$$\beta(t) = t^2, \qquad t \in [0, 1], \tag{10.5}$$

whereas (10.1) has form (3.6) with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} -1/16 \\ 1/8 \\ 1/4 \end{pmatrix}.$$

The two-point parametrized boundary conditions (4.3) have the form

$$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}x(0)+\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}x(1)=\begin{pmatrix}-\frac{1}{16}+\lambda_1\\ \lambda_2\\ \frac{1}{4}\end{pmatrix}. \tag{10.6}$$

The deviation function (10.5) satisfies condition (5.13) because, as one can verify, k_β is equal to 2 in this case and, according to (9.8), we have $\eta = 2$. Furthermore, as follows immediately from (10.4), matrices (6.4) are determined by the equalities

$$K_0=\begin{pmatrix}0&0&0\\0&0&1\\1&\frac{1}{2}&0\end{pmatrix},\qquad K_1=\begin{pmatrix}0&1&0\\0&0&0\\0&0&\frac{1}{4}\end{pmatrix}. \tag{10.7}$$

Therefore, $r(K_0+K_1)\approx 1.267<\frac{5}{3}$ and, hence, condition (9.7) is satisfied. By virtue of Theorem 4, the method of successive approximations is applicable to the two-point parametrized problem (10.2), (10.6).

One can verify directly that the triplet of functions

$$\begin{aligned} x_1^*(t)&=\frac{t^2}{8}-\frac{1}{16},\\ x_2^*(t)&=\frac{t}{4},\\ x_3^*(t)&=\frac{1}{4} \end{aligned} \tag{10.8}$$

is a solution of the boundary value problem (10.2), (10.1). Let us see how the approximation scheme based on Theorem 4 works in this case.

We take the starting approximation $x_0=(x_{i0})_{i=1}^3$ of the form

$$x_0(t)=\begin{pmatrix}-\frac{1}{16}\\ z_2\\ z_3\end{pmatrix}$$

with $z_1=-1/16$, and construct the corresponding functions of the recurrence sequence (6.1). We obtain:

$$x_{m+1}(t,z,\lambda)=\begin{pmatrix}-\frac{1}{16}\\ z_2\\ z_3\end{pmatrix}$$
$$+\begin{pmatrix}\int_0^t\left(x_{2m}(s^2)+\frac{s}{4}-\frac{s^2}{4}\right)ds-t\int_0^1\left(x_{2m}(s^2)+\frac{s}{4}-\frac{s^2}{4}\right)ds\\ \int_0^t\left(sx_{3m}(s)+\frac{1}{4}-\frac{s}{4}\right)ds-\frac{t}{T}\int_0^1\left(sx_{3m}(s)+\frac{1}{4}-\frac{s}{4}\right)ds\\ \int_0^t\left(sx_{1m}(s)-\frac{s^2}{2}x_{2m}(s)+\frac{s}{16}\right)ds-t\int_0^1\left(sx_{1m}(s)-\frac{s^2}{2}x_{2m}(s)+\frac{s}{16}\right)ds\end{pmatrix}$$

$$+t\left(\begin{pmatrix}-\frac{1}{16}+\lambda_1\\ \lambda_2\\ \frac{1}{4}\end{pmatrix}-\begin{pmatrix}2&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}-\frac{1}{16}\\ z_2\\ z_3\end{pmatrix}\right),\qquad m=0,1,2,\dots. \tag{10.9}$$

For $m=0$, formula (10.9) gives

$$\begin{aligned}
x_{11}(t,z,\lambda)&=-\frac{1}{16}+\frac{t^2}{8}-\frac{t^3}{12}+\frac{t}{48}+t\lambda_1,\\
x_{12}(t,z,\lambda)&=z_2+\frac{1}{8}t+\frac{1}{2}t^2z_3-\frac{1}{8}t^2-\frac{1}{2}tz_3+t\lambda_2-z_2t,\\
x_{13}(t,z,\lambda)&=z_3-\frac{1}{6}z_2t^3+\frac{1}{6}z_2t+\frac{1}{4}t-tz_3,
\end{aligned}\tag{10.10}$$

and the first approximate determining system

$$\begin{aligned}
\Delta_1(z,\lambda)&=0,\\
x_{12}\left(\frac{1}{2}\right)&=\frac{1}{8}
\end{aligned}$$

has the roots

$$\begin{aligned}
&\lambda_1\approx-0.020833333, &&\lambda_2\approx 0.2500000001,\\
&z_2\approx 0.13\cdot 10^{-9}, &&z_3\approx 0.25.
\end{aligned}\tag{10.11}$$

Substituting (10.11) into (10.10), we obtain the first approximation of the solution of the given three-point Cauchy–Nicoletti boundary value problem

$$\begin{aligned}
x_{11}(t,z,\lambda)&=-0.0625+\frac{t^2}{8}-\frac{t^3}{12}+0.13\cdot 10^{-9}t,\\
x_{12}(t,z,\lambda)&=0.13\cdot 10^{-9}+0.25t,\\
x_{13}(t,z,\lambda)&=0.25-0.2166666667\cdot 10^{-10}t^3.
\end{aligned}\tag{10.12}$$

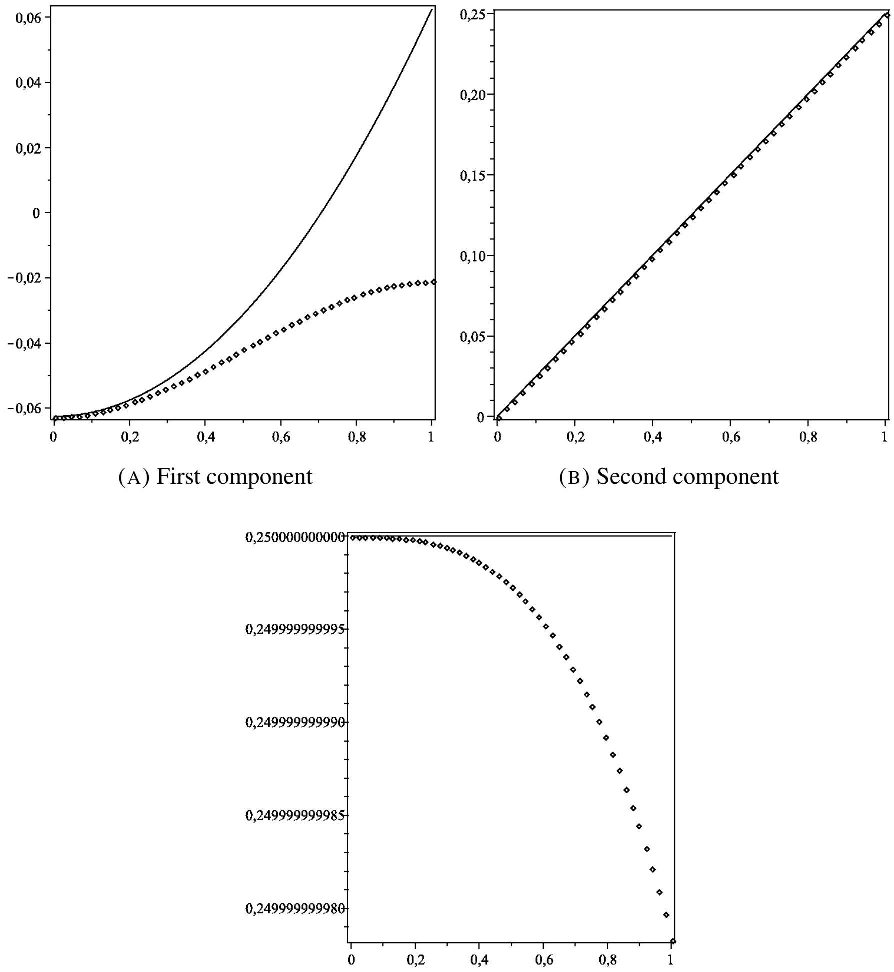
The result of computation of the second iteration is

$$\begin{aligned}
x_{21}(t,z,\lambda)&=-\frac{1}{16}+\frac{1}{10}t^5z_3-\frac{1}{40}t^5-\frac{1}{24}t^3\\
&\quad-\frac{1}{3}t^3z_2-\frac{1}{6}t^3z_3+\frac{1}{3}t^3\lambda_2+\frac{1}{8}t^2+\frac{1}{3}z_2t+\frac{1}{15}tz_3+\frac{1}{240}t-\frac{1}{3}t\lambda_2+t\lambda_1,
\end{aligned}$$

$$\begin{aligned}
x_{22}(t,z,\lambda)&=z_2+\frac{1}{24}t-\frac{1}{30}z_2t^5+\frac{1}{18}t^3z_2+\frac{1}{12}t^3-\frac{1}{3}t^3z_3+\frac{1}{2}t^2z_3-\frac{1}{8}t^2\\
&\quad-\frac{46}{45}z_2t-\frac{1}{6}tz_3+t\lambda_2,
\end{aligned}$$

and

$$x_{23}(t,z,\lambda)=z_3-\frac{1}{20}t^5z_3-\frac{1}{240}t^5+\frac{1}{64}t^4+\frac{1}{8}t^4z_2+\frac{1}{16}t^4z_3-\frac{1}{8}t^4\lambda_2+\frac{1}{144}t^3$$



(A) First component

(B) Second component

(C) Third component

FIGURE 1. The the exact solution (solid line) and its first approximation (dots).

$$+\frac{1}{3}t^3\lambda_1-\frac{1}{16}t^3z_2-\frac{81}{80}tz_3+\frac{667}{2880}t+\frac{1}{24}z_2+\frac{1}{8}t\lambda_2-\frac{1}{3}t\lambda_1.$$

The approximate solution of the second approximate determining equation has the form

$$\begin{aligned}\lambda_1 &\approx 0.06357861637, \qquad \lambda_2 \approx 0.2490649542,\\ z_2 &\approx 0.94488842\cdot 10^{-3}, \quad z_3 \approx 0.2385944095.\end{aligned} \tag{10.13}$$

Inserting values (10.13) into the expressions above, we obtain the components of the second approximation which have the form

$$
\begin{aligned}
x_{21}(t) &= -0.0625 - 0.001140559t^5 + 0.001274287t^3 + \frac{1}{8}t^2 + 0.94488842 \cdot 10^{-3}t,\\
x_{22}(t) &= 0.94488842 \cdot 10^{-3} + 0.2500000001t - 0.3149628067 \cdot 10^{-4}t^5\\
&\quad + 0.0038543573t^3 - 0.57027952 \cdot 10^{-2}t^2,\\
x_{23}(t) &= 0.2385944095 - 0.01609638715t^5 - 0.4778576375 \cdot 10^{-3}t^4\\
&\quad + 0.02797983516t^3 + 0.2 \cdot 10^{-9}t.
\end{aligned}
$$

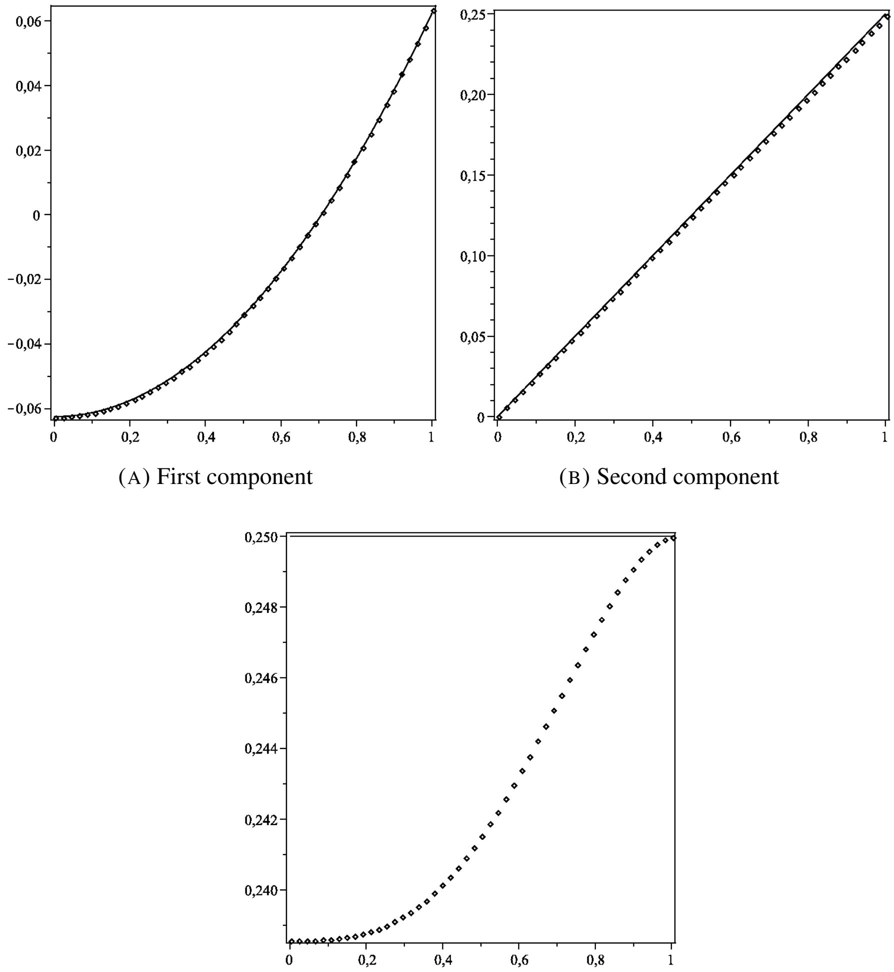
Proceeding analogously, we find that the components of the fourth iteration have the form

$$
\begin{aligned}
x_{41}(t,z,\lambda) &= -\frac{1}{63}t^7\lambda_1 - \frac{2167}{6552}t\lambda_2 + \frac{1}{624}t^{13}z_2 - \frac{1}{16} - \frac{1}{624}t^{13}\lambda_2 + \frac{1031}{327600}tz_3\\
&\quad + \frac{360259}{1081080}z_2t + \frac{10343}{10395}t\lambda_1 + \frac{1}{1248}t^{13}z_3 + \frac{1}{165}t^{11}\lambda_1 - \frac{1}{330}t^{11}z_2\\
&\quad - \frac{27}{560}t^7z_3 + \frac{1}{504}t^7z_2 + \frac{1}{168}t^7\lambda_2 + \frac{1}{10}t^5z_3 - \frac{721}{2160}t^3z_2\\
&\quad + \frac{2}{135}t^3\lambda_1 - \frac{557}{10080}t^3z_3 - \frac{24869}{362880}t^3 + \frac{1}{8}t^2 - \frac{1}{2100}t^{15}z_3\\
&\quad + \frac{2557873}{129729600}t + \frac{47}{144}t^3\lambda_2 - \frac{1}{25200}t^{15} + \frac{1}{4992}t^{13} + \frac{1}{7920}t^{11}\\
&\quad + \frac{667}{60480}t^7 - \frac{1}{40}t^5,
\end{aligned}
$$

$$
\begin{aligned}
x_{42}(t,z,\lambda) &= \frac{14773}{15120}t\lambda_2 + z_2 - \frac{1}{1728}t^8z_2 - \frac{1}{48}t^6\lambda_2 + \frac{23}{1080}t^6z_2 + \frac{1}{288}t^6z_3\\
&\quad - \frac{1}{45}t^5\lambda_2 - \frac{1}{90}t^5z_2 + \frac{1}{15}t^5\lambda_1 + \frac{1}{2}t^2z_3 - \frac{117}{700}tz_3 - \frac{453277}{453600}z_2t\\
&\quad - \frac{1}{2520}t^9 - \frac{1}{1152}t^8 + \frac{5}{1152}t^6 + \frac{1}{4800}z_2t^{10} + \frac{2}{45}t\lambda_1 - \frac{1}{84}t^7z_3\\
&\quad - \frac{1}{105}t^7z_2 + \frac{1}{105}t^7\lambda_2 + \frac{1}{225}t^5z_3 - \frac{13}{12960}t^3z_2 - \frac{1}{9}t^3\lambda_1 - \frac{187}{560}t^3z_3\\
&\quad + \frac{923}{12096}t^3 - \frac{1}{8}t2 + \frac{1933}{43200}t + \frac{61}{1080}t^3\lambda_2 + \frac{1}{1680}t^7 + \frac{1}{3600}t^5\\
&\quad + \frac{1}{288}t^8z_3 + \frac{1}{630}t^9z_3
\end{aligned}
$$

and

$$
x_{43}(t,z,\lambda) = \frac{2923}{17280}t\lambda_2 - \frac{1}{4290}t^{13}z_2 - \frac{1}{240}t^8\lambda_1 + \frac{721}{5760}t^4z_2 - \frac{1}{180}t^4\lambda_1
$$



(A) First component

(B) Second component

(C) Third component

FIGURE 2. The exact solution (solid line) and its second approximation (dots).

$$
\begin{aligned}
&+\frac{557}{26880}t^4 z_3+\frac{1}{864}t^9\lambda_2+z_3+\frac{1}{480}t^8 z_2-\frac{1}{288}t^6\lambda_2-\frac{1}{864}t^6 z_2\\
&+\frac{9}{320}t^6 z_3+\frac{1}{15}t^5\lambda_2-\frac{46}{675}t^5 z_2-\frac{5}{18144}t^9 z_2-\frac{47}{384}t^4\lambda_2\\
&-\frac{44691}{44800}t z_3-\frac{705961}{259459200}z_2 t+\frac{1}{2800}t^{10}z_3+\frac{1}{108}t^6\lambda_1
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{33600}t^{10}+\frac{19}{16128}t^{9}-\frac{1}{11520}t^{8}-\frac{667}{103680}t^{6}-\frac{719}{2160}t\lambda_1 \\
&+\frac{1}{70}t^{7}z_3-\frac{11}{180}t^{5}z_3-\frac{3413}{62370}t^{3}z_2+\frac{1}{3}t^{3}\lambda_1+\frac{1}{945}t^{3}z_3 \\
&+\frac{101}{15120}t^{3}+\frac{472559}{2073600}t-\frac{1}{9}t^{3}\lambda_2-\frac{1}{280}t^{7}-\frac{1}{72}t^{5}+\frac{24869}{967680}t^{4} \\
&-\frac{71}{12096}t^{9}z_3.
\end{aligned}
$$

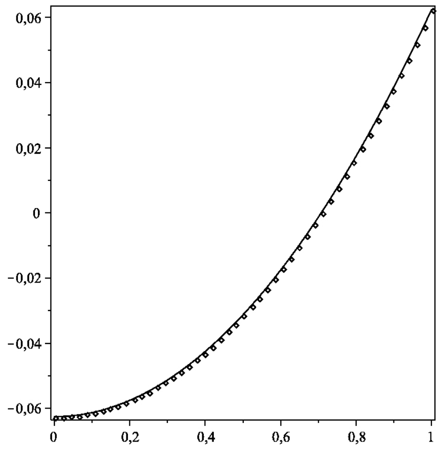
The fourth approximate determining equation has the solution

$$
\begin{aligned}
&\lambda_1 \approx 0.06249396077, \quad && \lambda_2 \approx 0.2499991352, \\
&z_2 \approx 0.9764\cdot 10^{-6}, && z_3 \approx 0.2499875012.
\end{aligned}
\tag{10.14}
$$

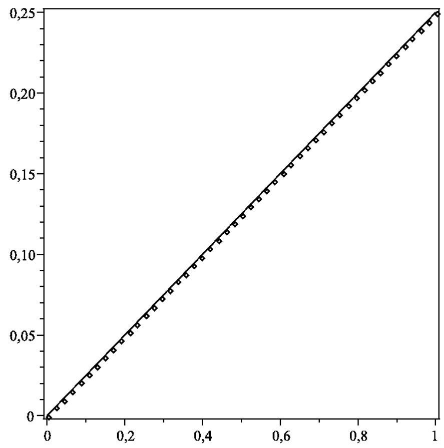
Inserting (10.14) into the formulae above, we obtain the first, second, and third components of the fourth approximation:

$$
\begin{aligned}
x_{41}(t) = &-\frac{1}{16}+0.17635985\cdot 10^{-3}t^{3}+\frac{1}{8}t^{2}+0.9763918464\cdot 10^{-6}t \\
&-0.1587242069\cdot 10^{-3}t^{15}-0.7064456410\cdot 10^{-8}t^{13} \\
&+0.5050109449\cdot 10^{-3}t^{11}-0.5284052627\cdot 10^{-3}t^{7} \\
&-0.124988\cdot 10^{-5}t^{5}, \\
x_{42}(t) = &\ 0.9764\cdot 10^{-6}+0.2034166667\cdot 10^{-9}t^{10}-0.198393\cdot 10^{-7}t^{9} \\
&-0.439637\cdot 10^{-7}t^{8}-0.45884\cdot 10^{-8}t^{6}+0.479491\cdot 10^{-5}t^{3} \\
&-0.62494\cdot 10^{-5}t^{2}+0.2500000001t+0.13126\cdot 10^{-6}t^{7} \\
&-0.449797\cdot 10^{-6}t^{5}, \\
x_{43}(t) = &\ 0.2499875012+0.1190431552\cdot 10^{-3}t^{10}+0.72095\cdot 10^{-7}t^{9} \\
&-0.3471950246\cdot 10^{-3}t^{8}+ \\
&+0.3082364001\cdot 10^{-3}t^{6}-0.198364\cdot 10^{-3}t^{3}-0.7\cdot 10^{-10}t \\
&-0.2275990676\cdot 10^{-9}t^{13}-0.178554\cdot 10^{-6}t^{7}+0.639621\cdot 10^{-6}t^{5} \\
&-0.6613494\cdot 10^{-4}t^{4}.
\end{aligned}
$$

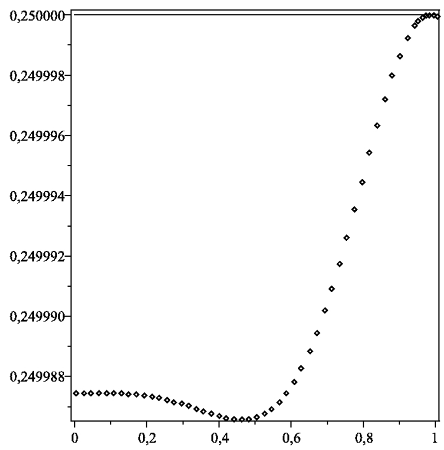
As is seen from Figures 1–3, the graph of the exact solution almost coincides with those of its approximations (especially with the graph of the fourth approximation).



(A) First component



(B) Second component

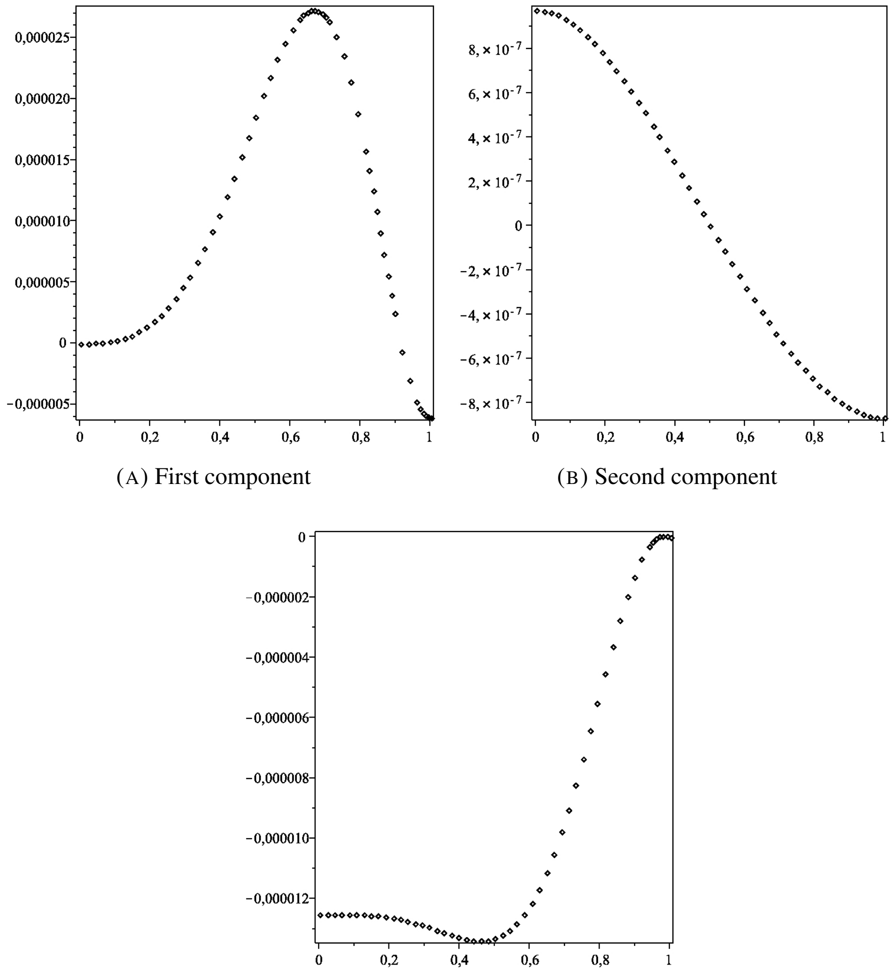


(C) Third component

FIGURE 3. The the exact solution (solid line) and its fourth approximation (dots).

For example, the error of the first approximation (i. e., the uniform deviation of the first approximation from the exact solution) admits the estimates

$$|x_1^*(t) - x_{11}(t)| \le 0.8 \cdot 10^{-1},$$
$$|x_2^*(t) - x_{12}(t)| \le 0.14 \cdot 10^{-9},$$
$$|x_3^*(t) - x_{13}(t)| \le 0.25 \cdot 10^{-10},$$



(A) First component

(B) Second component

(C) Third component

FIGURE 4. The error of the fourth approximation.

and the errors of the second and fourth approximation are

$$
\begin{aligned}
|x_1^*(t) - x_{21}(t)| &\le 0.12 \cdot 10^{-2},\\
|x_2^*(t) - x_{22}(t)| &\le 0.8 \cdot 10^{-3},\\
|x_3^*(t) - x_{23}(t)| &\le 0.12 \cdot 10^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
|x_1^*(t) - x_{41}(t)| &\le 0.25 \cdot 10^{-4}, \\
|x_2^*(t) - x_{42}(t)| &\le 0.1 \cdot 10^{-5}, \\
|x_3^*(t) - x_{43}(t)| &\le 0.12 \cdot 10^{-4}.
\end{aligned}
$$

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Doi: 10.18514/MMN.2009.145

Miskolc Mathematical Notes
Vol. 10 (2009), No. 2, pp. 207–221

HU e-ISSN 1787-2413

MAXIMUM PRINCIPLE AND EXISTENCE OF SOLUTIONS FOR NON-NECESSARILY COOPERATIVE SYSTEMS INVOLVING SCHRÖDINGER OPERATORS

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Received 13 February, 2006

Abstract. In this paper, we obtain the necessary and sufficient conditions for having the maximum principle and existence of positive solutions for some cooperative systems involving Schrödinger operators defined on unbounded domains. Then, we deduce the existence of solutions for semi-linear systems. Finally we discuss the generalized maximum principle (ϕ_q- positivity) for non-cooperative systems.

2000 *Mathematics Subject Classification:* 35P30, 35J70

Keywords: maximum principle, generalized maximum principle, positive solution, cooperative and non-cooperative semilinear systems, Schrödinger operators

1. INTRODUCTION

We consider the following semilinear system

$$\begin{cases} LY + QY = Ag(x)Y + F(x,Y) \quad \text{in} \quad \Omega, \\ Y \to 0 \quad \text{as} \quad |x| \to \infty, \\ Y = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{S}$$

where Ω is an unbounded domain of R^N, L is an $n \times n$ diagonal matrix of Laplace operators, Q is an $n \times n$ diagonal matrix of potential functions q_i $(1 \le i \le n)$, $g(x)$ is a weight function tending to zero at infinity, F is a given n-vector function and $A = (a_{ij})$ is a constant $n \times n$ cooperative matrix such that

$$a_{ij} \ge 0 \quad \text{for all} \quad i \ne j. \tag{1}$$

It is well known that the maximum principle plays an important role in the theory of partial differential equations. An analogous theory has been appeared for semilinear systems in [1,5–9,11,14,22]. In [7,9], the authors studied system (S) with $q_i = 0$ and $g(x) = 1$, defined on bounded domains with Dirichlet conditions. The problem with $q_i = 0$ defined on the whole space R^N has been established in [12, 13]. The system with equal potentials defined on R^N has been considered in [3,4].

Some applications concerning the optimal control of systems like (S) have been introduced in [15–21].

Here, we extend these results to system (S). In section two, we obtain necessary or sufficient conditions for having the maximum principle and existence of positive solutions for cooperative linear systems. Then, we study semilinear systems in section three; we adapt the method of sub and super solutions for proving the existence of nonnegative solutions. Finally, in section four, we study the generalized maximum principle (φ_q-positivity) for non-cooperative systems.

To prove our theorems, we make use of earlier results by Djellit and Yechoui [10] who proved that, for $N > 2$ and $q > 0$, if there exist $\alpha > 0$, $\beta \geq 1$, $\alpha > \beta$, and $k, c > 0$ such that

$$0 < g(x) \leq \frac{k}{(1+|x|^2)^{\alpha}}, \qquad 0 < q(x) \leq \frac{c}{(1+|x|^2)^{\beta}}, \tag{2}$$

then the eigenvalue problem

$$\begin{cases} (-\Delta + q)y = \lambda g(x) y \quad \text{in} \quad \Omega, \\ y \to 0 \quad \text{as} \quad |x| \to \infty, \\ y = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{E}$$

has simple principal eigenvalue (λ_q^+) which is associated with positive eigenfunction φ_q on V.

Moreover (λ_q^+) is characterized by

$$\lambda_q^+ \int_\Omega g(x)|y|^2 dx \leq |\nabla y|^2 + q|y|^2, \tag{3}$$

where

$$V(\Omega) = \left\{ y \in D'(\Omega) \mid \left|(1+|x|^2\right)^{-\frac{1}{2}} y \in L^2(\Omega),\ \nabla y \in L^2(\Omega) \right\}$$

is a Hilbert space with an inner product $(y, \psi)_V = \int_\Omega \left(\nabla y . \nabla \psi + \frac{1}{1+|x|^2} y \psi \right) dx$ and a norm

$$\|y\|_V = \left(\int_\Omega \left(|\nabla y|^2 + \frac{1}{1+|x|^2} |y|^2 \right) dx \right)^{\frac{1}{2}}$$

which is equivalent to

$$\|y\|_q = \left(\int_\Omega \left(|\nabla y|^2 + q|y|^2 \right) dx \right)^{\frac{1}{2}}.$$

We also introduce the Hilbert space

$$\mathcal{H} = \left\{ y : \Omega \to R \mid \int_\Omega g y^2 dx \leq \infty \right\} = L_g^2(\Omega)$$

with an inner product

$$(y,\psi)_g = \int_\Omega g y \psi \, dx.$$

2. Cooperative linear systems

In this section, we study the maximum principle and existence of positive solutions for system (S) when the right-hand side is linear.

Definition 1. We say that the maximum principle holds for system (S) if $F \geq 0$ and Y is a solution of (S), then $Y \geq 0$.

Definition 2. A non-singular square matrix $B = (b_{ij})$ is said to be an M-matrix if $b_{ij} \leq 0$ for $i \neq j$, $b_{ii} > 0$ for $i = 1, \dots, n$, and if all the principal minors extracted from B are positive.

The ith equation of system (S) can be written as

$$\begin{cases} (-\Delta + q_i) y_i = g(x) \sum_{j=1}^{n} a_{ij} y_j + f_i & \text{in} \quad \Omega, \\ y_i \to 0 \quad \text{as} \quad |x| \to \infty, \\ y_i = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{S_i}$$

Theorem 1. *Assume that* (1) *and* (2) *with* $q = q_i$ *hold, and* $f_i \geq 0$. *System* (S) *satisfies the maximum principle if*

the matrix $(\Lambda_Q^+ - A)$ *is a non-singular* M*-matrix, where* (4)

$$\Lambda_Q^+ = \begin{pmatrix} \lambda_{q_1}^+ & 0 & \dots & 0 \\ 0 & \lambda_{q_2}^+ & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{q_n}^+ \end{pmatrix}. \tag{5}$$

Moreover, if the maximum principle holds for system (S), *then*

the matrix $(\Lambda_q^+ - A)$ *is a non-singular* M*-matrix, where* (6)

$$q = \max\{q_i : 1 \leq i \leq n\}, \qquad \Lambda_q^+ = \begin{pmatrix} \lambda_q^+ & 0 & \dots & 0 \\ 0 & \lambda_q^+ & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_q^+ \end{pmatrix}.$$

Proof. First assume that $f_i \geq 0$ and $(y_i)_{i=1}^n \in \prod_{i=1}^n V_{q_i}$ is a solution of (S). Multiplying (S_i) by $y_i^- = \max\{-y_i, 0\}$ and integrating over Ω, we obtain by Green's formula that

$$\int_\Omega \nabla y_i.\nabla y_i^- dx + \int_\Omega q_i y_i y_i^- dx = \sum_{j=1}^n a_{ij} \int_\Omega g(x) y_j y_i^- dx + \int_\Omega f_i y_i^- dx,$$

i. e.,

$$\int_\Omega |\nabla y_i^-|^2 dx + \int_\Omega q_i |y_i^-|^2 dx$$
$$= a_{ii} \int_\Omega g(x)|y_i^-|^2 dx - \sum_{j\neq i}^n a_{ij} \int_\Omega g(x) y_j y_i^- dx - \int_\Omega f_i y_i^- dx,$$

and thus, by (3), we get

$$\lambda_{q_i}^+ \int_\Omega g(x)|y_i^-|^2 dx$$
$$\leq a_{ii} \int_\Omega g(x)|y_i^-|^2 dx - \sum_{j\neq i}^n a_{ij} \int_\Omega g(x) y_j y_i^- dx - \int_\Omega f_i y_i^- dx.$$

Therefore

$$(\lambda_{q_i}^+ - a_{ii}) \int_\Omega |\sqrt{g} y_i^-|^2 dx \leq \sum_{j\neq i}^n a_{ij} \int_\Omega g(x) y_j^- y_i^- dx$$

and, by the Cauchy-Schwartz inequality, we have

$$(\lambda_{q_i}^+ - a_{ii}) \left(\int_\Omega |\sqrt{g} y_i^-|^2 dx\right)^{\frac{1}{2}} - \sum_{j\neq i}^n a_{ij} \left(\int_\Omega |\sqrt{g} y_j^-|^2 dx\right)^{\frac{1}{2}} \leq 0,$$

which can be rewritten to the form

$$\begin{pmatrix} \lambda_{q_1}^+ - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda_{q_2}^+ - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda_{q_n}^+ - a_{nn} \end{pmatrix} \begin{pmatrix} \left(\int_\Omega |\sqrt{g} y_1^-|^2 dx\right)^{1/2} \\ \left(\int_\Omega |\sqrt{g} y_2^-|^2 dx\right)^{1/2} \\ \vdots \\ \left(\int_\Omega |\sqrt{g} y_n^-|^2 dx\right)^{1/2} \end{pmatrix} \leq 0.$$

Now, (4), $y_1^- = y_2^- = \cdots = y_n^- = 0$ and hence $y_1, y_2, \ldots, y_n \geq 0$.

Assume now that $0 \leq f_i \in L^2_{1/g}(\Omega)$ and that the maximum principle holds for system (S). We rewrite (S_i) as follows:

$$\begin{cases} (-\Delta + q)y_i = g(x)\sum_{j=1}^{n} a_{ij}y_j + H_i & \text{in} \quad \Omega \\ y_i \to 0 \quad \text{as} \quad |x| \to \infty \\ y_i = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $0 \leq H_i = (q - q_i)y_i + f_i \in L^2_{1/g}(\Omega)$.

Multiplying by φ_q (the eigenfunction corresponding λ_q^+) and integrating over Ω, we get

$$\int_\Omega (-\Delta + q)y_i\varphi_q dx = \sum_{j=1}^{n} a_{ij}\int_\Omega g(x)y_j\varphi_q dx + \int_\Omega H_i\varphi_q dx$$

and, by using Green's formula and (3), we obtain

$$(\lambda_q^+ - a_{ii})\int_\Omega g(x)y_i\varphi_q dx - \sum_{j\neq i} a_{ij}\int_\Omega g(x)y_j\varphi_q dx = \int_\Omega H_i\varphi_q dx \tag{7}$$

which is a Cramer system in $X_i = \int_\Omega g(x)y_i\varphi_q dx$, $1 \leq i \leq n$. Since the right-hand side is non-negative as well as X_i, we obtain

$$\begin{vmatrix} \lambda_q^+ - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda_q^+ - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda_q^+ - a_{nn} \end{vmatrix} = \left|\Lambda_q^+ - A\right| > 0.$$

The inequality $\lambda_q^+ - a_{ii} > 0$ is satisfied from the scalar case (see [13]), because the functions g, y_i, H_i and the coefficients a_{ij} for $i \neq j$ are non-negative. $\square$

Remark 1. If $q_i = q$ for $1 \leq i \leq n$, then condition (6) is the necessary and sufficient condition for having the maximum principle for system (S).

Theorem 2. *Let* (1) *and* (2) *with* $q = q_i$ *hold. Then, for* $F \geq 0$, *system* (S) *has a unique positive solution if condition* (4) *is satisfied.*

Proof. We consider the bilinear form $a\colon \prod_{i=1}^{n} V_{q_i} \times \prod_{i=1}^{n} V_{q_i} \to R$ defined by

$$\begin{aligned} a(Y,\Psi) &= a((y_1, y_2, \ldots, y_n), (\psi_1, \psi_2, \ldots, \psi_n)) \\ &= \sum_{i=1}^{n}\int_\Omega \nabla y_i.\nabla\psi_i dx + \sum_{i=1}^{n}\int_\Omega q_i y_i\psi_i dx \end{aligned}$$

$$-\sum_{j\neq i}^{n}\int_{\Omega} g(xa_{ij}y_j\psi_i\,dx-\sum_{i=1}^{n}\int_{\Omega} g(x)a_{ii}y_i\psi_i\,dx$$

We choose $m\geq 0$ such that $m+a_{ii}>0$. Then, we have

$$\begin{aligned}
a(Y,Y)&=\sum_{i=1}^{n}\int_{\Omega}\left[|\nabla y_i|^2+q_i y_i^2\right]dx\\
&\quad-\sum_{j\neq i}^{n}\int_{\Omega} g(x)a_{ij}y_iy_j\,dx-\sum_{i=1}^{n}\int_{\Omega} g(x)a_{ii}y_i^2\,dx\\
&=\sum_{i=1}^{n}\int_{\Omega}\left[|\nabla y_i|^2+(q_i+mg)y_i^2\right]dx\\
&\quad-\sum_{j\neq i}^{n}\int_{\Omega} g(x)a_{ij}y_iy_j\,dx-\sum_{i=1}^{n}(m+a_{ii})\int_{\Omega} g(x)y_i^2\,dx
\end{aligned}$$

and, by the Cauchy-Schwartz inequality and (3), we get

$$\begin{aligned}
a(Y,Y)&\geq\sum_{i=1}^{n}\int_{\Omega}\left[|\nabla y_i|^2+(q_i+mg)y_i^2\right]dx\\
&\quad-\sum_{j\neq i}^{n}a_{ij}\left(\int_{\Omega}(\sqrt{g}y_i)^2dx\right)^{\frac{1}{2}}\left(\int_{\Omega}(\sqrt{g}y_j)^2dx\right)^{\frac{1}{2}}\\
&\quad-\sum_{i=1}^{n}(m+a_{ii})\int_{\Omega} g(x)y_i^2\,dx\\
&\geq\sum_{i=1}^{n}\left(1-\frac{m+a_{ii}}{m+\lambda_{q_i}^{+}}\right)\int_{\Omega}\left[|\nabla y_i|^2+(q_i+mg)y_i^2\right]dx\\
&\quad-\sum_{j\neq i}^{n}\frac{a_{ij}}{\sqrt{(m+\lambda_{q_i}^{+})(m+\lambda_{q_j}^{+})}}\left(\int_{\Omega}\left[|\nabla y_i|^2+(q_i+mg)y_i^2\right]dx\right)^{\frac{1}{2}}\\
&\quad\times\left(\int_{\Omega}\left[|\nabla y_j|^2+(q_j+mg)y_j^2\right]dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, using (4), it follows that

$$a(Y,Y)\geq C\sum_{i=1}^{n}\int_{\Omega}\left[|\nabla y_i|^2+(q_i+mg)(y_i)^2\right]dx$$

$$\geq C\sum_{i=1}^{n}\int_{\Omega}\left[|\nabla y_i|^2+q_i(y_i)^2\right]dx = C\sum_{i=1}^{n}\|y_i\|_{q_i}^2,$$

where $C > 0$. Then, by the Lax Milgram lemma, since a is a continuous coercive bilinear form, there exists a unique solution $Y = (y_i)_{i=1}^n \in \prod_{i=1}^n V_{q_i}$. This solution is non-negative by the maximum principle. □

3. Semilinear systems

In this section, we adapt the method of sub and super solutions, to prove the existence of solutions for semilinear cooperative system (S). The proof here is analogous to that of [3, 13].

We assume that $f_i(x,Y) = f_i(x,y_1,y_2,\dots,y_n)$ is a Carathéodory function such that

$$0 \leq F(x,Y) = (f_1(x,Y), f_2(x,Y),\dots, f_n(x,Y)) \leq Ng(x)Y + h' \tag{8}$$

for all $Y \geq 0$ and $x \in \Omega$, where N is a positive constant and $0 < h' = (h,h,\dots,h)$ is a bounded vector-function in $\left(L^2_{1/g}(\Omega)\right)^n$.

Theorem 3. *Let* (1), (2) *with* $q = q_i$, *and* (8) *be satisfied. Then there exists a positive solution of system* (S) *if*

$$\text{the matrix } \left(\Lambda_Q^+ - (A+NI)\right) \text{ is a non-singular } M\text{-matrix,} \tag{9}$$

where Λ_Q^+ *is defined by the relation* (5) *and* I *denotes the identity matrix.*

Proof. We divide the proof into several steps.
Step (i): *Construction of sub and super solutions.* It is clear that

$$Y^0 = (y_1^0, y_2^0,\dots, y_n^0) = (0,0,\dots,0)$$

is a sub solution of (S), because

$$LY^0 + QY^0 - g(x)AY^0 - F(x,Y^0) \leq 0.$$

Consider now the system

$$\begin{cases} LY + QY = g(x)(A+NI)Y + h' & \text{in} \quad \Omega. \\ Y \to 0 \quad \text{as} \quad |x| \to \infty, \\ Y = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{10}$$

It follows from Theorem 2 that, under condition (9), system (10) has a unique positive solution

$$Y^* = (y_1^*, y_2^*,\dots, y_n^*).$$

By (8), we have

$$0 \leq (L+Q)Y^* - g(x)AY^* - F(x,Y^*), \tag{11}$$

i. e., $Y^* = (y_1^*, y_2^*,\dots, y_n^*)$ is a super solution of (S).

Step (ii): *Definition of a compact operator.* We introduce

$$T : \mathcal{H}^n \ni Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \geq 0 \mapsto \Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} = TY \in \prod_{i=1}^{n} V_{q_i},$$

where Ψ is a unique solution of the following system:

$$\begin{cases} (L+Q+mg(x)I)\Psi = (mI+A)g(x)Y + F(x,Y) & \text{in} \quad \Omega, \\ \Psi \to 0 \quad \text{as} \quad |x| \to \infty, \\ \Psi = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{12}$$

where $m \geq 0$ is such that $m + a_{ii} > 0$. Since the equation in (12) can be rewritten as

$$(L+Q)\Psi = -mg(x)\Psi + \bar{F}, \quad \bar{F} = (mI+A)g(x)Y + F(x,Y) \geq 0,$$

system (12) has a unique solution $\Psi \in \prod_{i=1}^{n} V_{q_i}$ and therefore T is well-defined.

Step (iii): *$K = [Y^0, Y^*]$ is invariant by T, i. e., $T(K) \subseteq K$.* For every $Y \in \prod_{i=1}^{n} V_{q_i}$, $Y \geq 0$, we have $T(Y) = \Psi \geq 0$. We show now that if $Y \leq Y^*$, then $\Psi \leq Y^*$. From (10) and (12), we obtain

$$(L+Q+mg(x)I)(Y^* - \Psi) = \tau,$$

where $\tau := (mI+A)g(x)(Y^* - Y) + Ng(x)Y^* - F(x,Y) \geq 0$. Therefore

$$(L+Q)(Y^* - \Psi) = -mg(x)(Y^* - \Psi) + \tau$$

and thus $Y^* - \Psi$ is non-negative, i. e., $\Psi \leq Y^*$.

Step (iv): *T is a continuous operator.* Let $Y_k \to Y$ in $\mathcal{H}^n$. Then we get

$$F(x, Y_k) \to F(x,Y) \quad \text{in} \quad \left(\mathcal{H}'\right)^n.$$

If we denote $\Psi_k = T(Y_k)$, by (12) it follows that

$$(L+Q+mg(x)I)(\Psi - \Psi_k) = (mI+A)g(x)(Y - Y_k) + F(x,Y) - F(x,Y_k).$$

Multiplying by $(\Psi - \Psi_k)$ and integrating over Ω, we get

$$\begin{aligned} \int_\Omega (L+Q+mg(x)I)(\Psi - \Psi_k).(\Psi - \Psi_k)dx \\ = (mI+A)\int_\Omega g(x)(Y - Y_k).(\Psi - \Psi_k)dx \\ + \int_\Omega \big(F(x,Y) - F(x,Y_k)\big).(\Psi - \Psi_k)dx, \end{aligned}$$

which, by virtue of the Green formula, yields that

$$\int_\Omega \left|\nabla(\Psi - \Psi_k)\right|^2 dx + \int_\Omega Q|\Psi - \Psi_k|^2 dx + m\int_\Omega g(x)|\Psi - \Psi_k|^2 dx$$

$$= (mI + A)\int_{\Omega} g(x)(Y - Y_k).(\Psi - \Psi_k)dx \\ + \int_{\Omega} \big(F(x,Y) - F(x,Y_k)|\big).(\Psi - \Psi_k)dx.$$

Now, using the Cauchy-Schwarz inequality, we get

$$\int_{\Omega} \big|\nabla(\Psi - \Psi_k)\big|^2 dx + \int_{\Omega} Q|\Psi - \Psi_k|^2 dx + m\int_{\Omega} g(x)|\Psi - \Psi_k|^2 dx \\ \leq (mI + A)\|Y - Y_k\|_{\mathcal{H}^n}\|\Psi - \Psi_k\|_{\mathcal{H}^n} \\ + \|F(x,Y) - F(x,Y_k)\|_{(\mathcal{H}')^n}\|\Psi - \Psi_k\|_{\mathcal{H}^n}$$

and thus

$$\|\Psi - \Psi_k\|_Q^2 \leq (mI + A)\|Y - Y_k\|_{\mathcal{H}^n}\|\Psi - \Psi_k\|_{\mathcal{H}^n} \\ + \|F(x,Y) - F(x,Y_k)\|_{(\mathcal{H}')^n}\|\Psi - \Psi_k\|_{\mathcal{H}^n},$$

where $\|\cdot\|_Q$ denotes the norm in $\prod_{i=1}^n V_{q_i}$. Since $\|Y - Y_k\|_{\mathcal{H}} \to 0$, we have $\|\Psi - \Psi_k\|_Q \to 0$.

Step (v): *T is a compact operator.* First note the following: Let $Y \geq 0$ and $\Psi = T(Y)$. Multiplying (12) by Ψ and integrating over Ω, we get

$$\int_{\Omega} \big|\nabla\Psi\big|^2 dx + \int_{\Omega} Q|\Psi|^2 dx + m\int_{\Omega} g(x)|\Psi|^2 dx \\ = (mI + A)\int_{\Omega} g(x)Y\Psi dx + \int_{\Omega} F(x,Y)\Psi dx,$$

which, in view of (8), yields that

$$\int_{\Omega} \big|\nabla\Psi\big|^2 dx + \int_{\Omega} Q|\Psi|^2 dx + m\int_{\Omega} g(x)|\Psi|^2 dx \\ \leq (mI + A)\int_{\Omega} g(x)Y\Psi dx + N\int_{\Omega} g(x)Y\Psi dx + \int_{\Omega} \sqrt{g}\Psi\frac{h}{\sqrt{g}}dx \\ = (mI + A + NI)\int_{\Omega} g(x)Y\Psi dx + \int_{\Omega} \sqrt{g}\Psi\frac{h}{\sqrt{g}}dx.$$

By the Cauchy-Schwarz inequality, we obtain

$$\|\psi\|_Q \leq C\left(\|Y\|_{(\mathcal{H})^n} + \|h\|_{\mathcal{H}'}\right) \leq C\left(\|Y\|_Q + \|h\|_{\mathcal{H}'}\right)$$

Therefore if $\{Y_k\}_{k\in N}$ is a bounded sequence in $\prod_{i=1}^n V_{q_i}$, the associated sequence $\{\Psi_k\}_{k\in N}$ is bounded in $\prod_{i=1}^n V_{q_i}$. We show now that $\{\Psi_k\}_{k\in N}$ is a Cauchy sequence in $\prod_{i=1}^n V_{q_i}$.

Suppose that $\|Y_k\|_Q^2 \le E$ for $k \in N$, E is a constant. In view of (2), we can choose R large enough such that

$$(1+|x|^2)g(x) < \frac{\epsilon}{8\gamma E} \quad \text{for} \quad |x| \ge R, \tag{13}$$

where $\epsilon > 0$ is fixed and γ is given by

$$\int_{\Omega} (1+|x|^2)^{-1} y^2 dx \le \gamma \int_{\Omega} |\nabla y|^2 dx. \tag{14}$$

Let $B = \{x \in \Omega \mid |x| < R\}$ and $B' = \{x \in \Omega \mid |x| \ge R\}$. Since $\{Y_k\}_{k\in N}$ is bounded in $\prod_{i=1}^n V_{q_i}$, Y_k is bounded in $(\mathcal{H}(B))^n$. However B is bounded and therefore the embedding $(\mathcal{H}(B))^n$ into $(L^2(B))^n$ is compact. Hence there exists a convergent subsequence still denoted by $\{Y_k\}_{k\in N}$, which is a Cauchy sequence and thus, for every j and k large enough, we have

$$\int_B g(x)|Y_k - Y_j|^2 dx \le \int_B |Y_k - Y_j|^2 dx < \frac{\epsilon}{2}.$$

Moreover using (13) and (14), we obtain

$$\begin{aligned}
\int_{B'} g(x)|Y_k - Y_j|^2 dx &= \int_{B'} (1+|x|^2)g(x)\frac{1}{1+|x|^2}|Y_k - Y_j|^2 dx \\
&\le \frac{\varepsilon}{8\gamma E}\int_{B'} \frac{1}{1+|x|^2}|Y_k - Y_j|^2 dx \\
&\le \frac{\varepsilon}{8\gamma E}\gamma\|Y_k - Y_j\|_Q^2
\end{aligned}$$

so $\int_{B'} g(x)|Y_k - Y_j|^2 dx < \frac{\epsilon}{2}$.

Then $\{\Psi_k\}_{k\in N}$ is a Cauchy sequence in $\prod_{i=1}^n V_{q_i}$. Hence it converges to Ψ and therefore T is compact in $\prod_{i=1}^n V_{q_i}$. By Schauder fixed point theorem, there exists at least one positive solution $Y = (y_i)_{i=1}^n \in \prod_{i=1}^n V_{q_i}$ of system (S) satisfying $Y^0 \le Y \le Y^*$. □

4. NON-COOPERATIVE SYSTEMS

In this section, we study the generalized maximum principle (φ_q-positivity) for $n \times n$ non-cooperative systems.

Definition 3. System (S) (with $q_i = q$) satisfies the generalized maximum principle if $F \ge 0$ and $Y = (y_1, y_2, \ldots, y_n)$ is a solution of (S), then there exists $C > 0$ such that

$$y_i \ge C\varphi_q \quad \text{in} \quad \Omega \quad \text{for all} \quad i = 1, 2, \ldots, n.$$

We start with 2×2 non cooperative systems:

$$\begin{cases} L_q y_1 = a_{11}g(x)y_1 + a_{12}g(x)y_2 + f_1 & \text{in} \quad \Omega, \\ L_q y_2 = a_{21}g(x)y_1 + a_{22}g(x)y_2 + f_2 & \text{in} \quad \Omega, \\ y_1, y_2 \to 0 \quad \text{as} \quad |x| \to \infty, \\ y_1 = y_2 = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{S_2}$$

As in [2], we can prove

Theorem 4. *Assume that $a_{12} < 0$, $a_{21} > 0$, $a_{11} > a_{22}$, $f_1 - \xi_2 f_2 \geq 0$, and $(a_{11} - a_{22})^2 + 4a_{12}a_{21} \geq 0$ are satisfied. The generalized maximum principle holds for system (S_2) if*

$$a_{11} - \lambda_q < r < \lambda_q - a_{22}.$$

Now, let us consider the following 3×3 non cooperative system

$$\begin{cases} L_q y_1 = a_{11}g(x)y_1 + a_{12}g(x)y_2 + a_{13}g(x)y_3 + f_1 & \text{in} \quad \Omega \\ L_q y_2 = a_{21}g(x)y_1 + a_{22}g(x)y_2 + a_{23}g(x)y_3 + f_2 & \text{in} \quad \Omega \\ L_q y_3 = a_{31}g(x)y_1 + a_{32}g(x)y_2 + a_{33}g(x)y_3 + f_3 & \text{in} \quad \Omega \\ y_1, y_2, y_3 \to 0 \quad \text{as} \quad |x| \to \infty \\ y_1 = y_2 = y_3 = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{S_3}$$

Assume that

$$a_{12}, a_{13} < 0, \quad a_{21}, a_{23}, a_{31}, a_{32} \geq 0 \tag{15}$$

We insert our 3×3 non-cooperative system into a 4×4 cooperative one. We introduce the following fourth equation of a new variable $y_4 = y_1 - \xi_2 y_2 - \xi_3 y_3$:

$$\begin{aligned} L_q y_4 = (a_{11} - \xi_2 a_{21} - \xi_3 a_{31} - s)g(x)y_1 \\ + (a_{12} - \xi_2 a_{22} - \xi_3 a_{32} + s\xi_2)g(x)y_2 \\ + (a_{13} - \xi_2 a_{23} - \xi_3 a_{33} + s\xi_3)g(x)y_3 + sg(x)y_4 + f_4, \end{aligned}$$

where ξ_2 and ξ_3 are positive numbers and $f_4 = f_1 - \xi_2 f_2 - \xi_3 f_3$. Then system ($S_3$) can be changed into system of the form

$$\begin{cases} L_q Y = \mathcal{N} g(x)Y + F & \text{in} \quad \Omega \\ Y \to 0 \quad \text{as} \quad |x| \to \infty \\ Y = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{S_3'}$$

where

$$L_q = \begin{pmatrix} -\Delta + q & 0 & 0 & 0 \\ 0 & -\Delta + q & 0 & 0 \\ 0 & 0 & -\Delta + q & 0 \\ 0 & 0 & 0 & -\Delta + q \end{pmatrix},$$

$Y = (y_1, y_2, y_2, y_4)$, $F = (f_1, f_2, f_3, f_4)$, and

$$\mathcal{N} = \begin{pmatrix} a_{11}-r & a_{12}+r\xi_2 & a_{13}+r\xi_3 & r \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ b_1 & b_2 & b_3 & s \end{pmatrix}$$

with

$$\begin{gathered} b_1 = a_{11} - \xi_2 a_{21} - \xi_3 a_{31} - s, \quad b_2 = a_{12} - \xi_2 a_{22} - \xi_3 a_{32} + s\xi_2, \\ b_3 = a_{13} - \xi_2 a_{23} - \xi_3 a_{33} + s\xi_3. \end{gathered} \tag{16}$$

For the cooperativeness of system (S_3'), we can choose r, s, ξ_2, and ξ_3 such that

$$r > 0, \qquad a_{12} + r\xi_2 = 0, \qquad a_{13} + r\xi_3 = 0, \tag{17}$$

$$a_{11} - \xi_2 a_{21} - \xi_3 a_{31} - s = 0, \tag{18}$$

$$a_{12} - \xi_2 a_{22} - \xi_3 a_{32} + s\xi_2 = 0, \tag{19}$$

$$a_{13} - \xi_2 a_{23} - \xi_3 a_{33} + s\xi_3 = 0, \tag{20}$$

and then, using (15), we get $\xi_3 = -\frac{a_{13}}{r} > 0$ and $\xi_2 = -\frac{a_{12}}{r} > 0$. Now, system (S_3') satisfies the maximum principle if the matrix

$$\begin{pmatrix} \lambda_q - a_{11} + r & -a_{12} - r\xi_2 & -a_{13} - r\xi_3 & -r \\ -a_{21} & \lambda_q - a_{22} & -a_{23} & 0 \\ -a_{31} & -a_{32} & \lambda_q - a_{33} & 0 \\ -b_1 & -b_2 & -b_3 & \lambda_q - s \end{pmatrix}$$

is a non-singular M-matrix, which means that

$$\lambda_q - a_{11} + r > 0,$$

$$\begin{vmatrix} \lambda_q - a_{11} + r & -a_{12} - r\xi_2 \\ -a_{21} & \lambda_q - a_{22} \end{vmatrix} > 0,$$

$$\begin{vmatrix} \lambda_q - a_{11} + r & a_{12} + r\xi_2 & -a_{13} - r\xi_3 \\ -a_{21} & \lambda_q - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda_q - a_{33} \end{vmatrix} > 0,$$

and

$$\begin{vmatrix} \lambda_q - a_{11} - r & -a_{12} - r\xi_2 & -a_{13} - r\xi_3 & -r \\ -a_{21} & \lambda_q - a_{22} & -a_{23} & 0 \\ -a_{31} & -a_{32} & \lambda_q - a_{33} & 0 \\ -b_1 & -b_2 & -b_3 & \lambda_q - s \end{vmatrix} > 0.$$

Therefore, using (17)–(20), we obtain

$$\begin{gathered} \lambda_q - a_{11} + r > 0, \quad (\lambda_q - a_{11} + r)(\lambda_q - a_{22}) > 0, \\ (\lambda_q - a_{11} + r)\big[(\lambda_q - a_{22})(\lambda_q - a_{33}) - a_{23}a_{32}\big] > 0, \end{gathered}$$

and

$$(\lambda_q - a_{11} + r)\big((\lambda_q - a_{22})(\lambda_q - a_{33})(\lambda_q - s) - a_{23}a_{32}(\lambda_q - s)\big) > 0,$$

which implies that

$$\lambda_q - a_{11} + r > 0, \quad \lambda_q - a_{22} > 0, \quad (\lambda_q - a_{22})(\lambda_q - a_{33}) > a_{23}a_{32}, \quad \lambda_q - s > 0.$$

From (15), we obtain $(\lambda_q - a_{22})(\lambda_q - a_{33}) > 0$ and, thus,

$$a_{11} - r < \lambda_q, \quad a_{22} < \lambda_q, \quad a_{33} < \lambda_q, \quad s < \lambda_q.$$

It follows from (17) and (19) that

$$a_{22} + r + \frac{\xi_3}{\xi_2} a_{32} = s, \tag{21}$$

whereas (17) and (20) yield that

$$a_{33} + r + \frac{\xi_2}{\xi_3} a_{23} = s \tag{22}$$

Since $s < \lambda_q$, from (21) and (22) we obtain

$$\lambda_q > a_{22} + r + \frac{\xi_3}{\xi_2} a_{32}, \qquad \lambda_q > a_{33} + r + \frac{\xi_2}{\xi_3} a_{23},$$

which, in view of (15), guarantee that

$$\lambda_q > a_{22} + r \qquad \lambda_q > a_{33} + r,$$

i. e.,

$$r < \lambda_q - a_{22}, \quad r < \lambda_q - a_{33}, \quad \text{and} \quad r > a_{11} - \lambda_q.$$

Consequently, we have

Theorem 5. *Assume that (15) holds. Then for $0 \le \xi_3 f_3 \le \xi_2 f_2 \le f_1$, system ($S_3$) satisfies the generalized maximum principle if*

$$a_{11} - \lambda_q < r < \lambda_q - a_{22}, \qquad a_{11} - \lambda_q < r < \lambda_q - a_{33}.$$

For non-cooperative system (S), we set the following conditions:

$$\begin{aligned} a_{1j} &< 0 \quad \text{for} \quad j = 2, 3, \ldots, n, \\ a_{ij} &\ge 0 \quad \text{for} \quad i = 2, 3, \ldots, n,\ j = 1, 2, \ldots, n,\ i \ne j. \end{aligned} \tag{23}$$

Similarly, to construct $(n+1) \times (n+1)$ cooperative system, we introduce the following equation of a new variable $y_{n+1} = y_1 - \sum_{i=2}^{n} \xi_i y_i$, where $\xi_2, \ldots, \xi_n$ are positive numbers and thus we have

Theorem 6. *Assume that (23) holds. Then for $0 \le \xi_n f_n \le \cdots \le \xi_2 f_2 \le f_1$, system (S) satisfies the generalized maximum principle if*

$$a_{11} - \lambda_q < r < \lambda_q - a_{ii} \quad \textit{for all} \quad i = 2, 3, \ldots, n.$$

ACKNOWLEDGEMENT

The authors would like to express their gratitude to Professor Jacqueline Fleckinger (QREMAQ – Université Toulouse 1, France) for her constructive suggestions.

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Doi: 10.18514/MMN.2009.233

Miskolc Mathematical Notes
Vol. 10 (2009), No. 2, pp. 223–224

HU e-ISSN 1787-2413

THE FOURTH INTERNATIONAL WORKSHOP-2009 "CONSTRUCTIVE METHODS FOR NON-LINEAR BOUNDARY-VALUE PROBLEMS"

MIKLÓS RONTÓ

Received 18 September, 2009

In the period from 1th to 4th July, 2009, the Fourth International Workshop-2009

"*Constructive methods for non-linear boundary-value problems*"

had taken place in Eger, Hungary. The workshop was organized by the Institute of Mathematics of the University of Miskolc, Institute of Mathematics and Informatics of the Eszterházy Károly College in Eger, and Institute of Mathematics of the Academy of Sciences of the Czech Republic in cooperation with the Regional Committee of the Hungarian Academy of Sciences in Miskolc.

The scope of the conference covered various topics related to the modern theory of boundary value problems in a broad sence, including initial value and periodic problems for functional differential equations, certain boundary value problems for differential equations with homogeneous non-linearities, the method of lower and upper functions, bifurcation theory, singular differential equations, limit cycles, difference equations with p-Laplacians, the Fučík spectrum, invariant manifolds, stochastic differential equations, impulsive boundary value problems, optimal control problems as well as certain aspects of the theory of generalized differential equations and applications. The workshop was attented by more than 60 researchers from Chile, Israel, Czech Republic, Hungary, Latvia, Romania, Russia, Spain, Slovak Republic, and Ukraine, who held 20 and 30-minute lectures.

The Organizing Committee of the workshop consisting of M. Fečkan, P. Körtesi, K. Liptai, F. Mátyás, V. Marinec, M. Perestyuk, A. Rontó, A. Samoilenko, J. Szigeti, M. Tvrdý, and the author of this note (chairman) gratefully acknowledges the financial support provided by the following funds and institutions:

- Department of Analysis University of Miskolc,
- Faculty of Mechanical Engineering and Informatics, University of Miskolc,
- Tudomány Támogatásáért Észak-Magyarországon Alapítvány.

This meeting was a continuation of the three previous workshops held in Miskolc in 2000 and 2003 and in Sárospatak in 2006.

We hope that the fruitful tradition of this kind of “compact” international meetings oriented at exchange of ideas of researchers working in closely related fields will continue in future.

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Miskolc Mathematical Notes

Abstracted in *Zentralblatt für Mathematik*, *Mathematical Reviews*, and Реферативный журнал „Математика”

Responsible for publication: Rector of the University of Miskolc
Published by the Miskolc University Press under the leadership of Dr. József Péter
Responsible for duplication: Works manager Tibor Kovács
Number of copies printed: 200
Put to the Press on December 15, 2009
Number of permission: TU 2009-500-ME **HU e-ISSN 1787-2413**