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CONTENTS

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ON UNIQUE SOLVABILITY OF QUADRATIC INTEGRAL EQUATIONS WITH LINEAR MODIFICATION OF THE ARGUMENT

MOUFFAK BENCHOHRA AND MOHAMED ABDALLA DARWISH

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Abstract. In this paper, we investigate the existence of a unique solution on a semi-infinite interval for quadratic integral equations with linear modification of the argument in Fréchet spaces using a nonlinear alternative of Leray-Schauder type for contraction maps in Fréchet spaces.

2000 *Mathematics Subject Classification:* 45G10, 45M99, 47H09, 47H10

Keywords: quadratic integral equations, linear modification of the argument, existence and uniqueness, fixed-point, contraction maps, nonlinear alternative of Leray-Schauder type, Fréchet space

1. Introduction

In this paper, we establish the existence of the unique solution, defined on a semi-infinite interval $J := [0, +\infty)$ for the following quadratic integral equations with a linear modification of the argument

$$x(t) = f(t) + (Ax)(t) \int_0^T u(t, s, x(s), x(\alpha s))ds, \quad t \in J, \tag{1.1}$$

and

$$x(t) = f(t) + g(t, x(t)) \int_0^T u(t, s, x(s), x(\alpha s))ds, \quad t \in J, \tag{1.2}$$

where $f: J \to \mathbb{R}$, $g: J \times \mathbb{R} \to \mathbb{R}$, $u: J \times J_T \times \mathbb{R}^2 \to \mathbb{R}$ are given functions, $0 < \alpha < 1$, $J_T := [0, T]$, and $A: C(J; \mathbb{R}) \to C(J; \mathbb{R})$ is an appropriate operator. Here $C(J; \mathbb{R})$ denotes the space of continuous functions $x: J \to \mathbb{R}$.

Integral equations arise naturally in many applications in describing numerous real world problems (see, for instance, the books [1, 2, 8, 10, 20] and references therein). Also quadratic integral equations have many useful applications in describing numerous events and problems of the real world. For example, quadratic integral equations are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. Especially, the so-called quadratic integral equation of Chandrasekher type can be very often encountered in many applications; see for instance the book by Chandrasekher [7] and the research papers by Banas *et al* [3], Benchohra and Darwish [4], Darwish [9], Hu *et al* [14], Kelley [15], Leggett [17] and Stuart [21] and references therein.

The study of differential equations with a modified argument is relatively new, it was initiated only in the past thirty years or so. These equations arise in the modelling of problems from the natural and social sciences such as biology, economics and physics. A special class is represented by the differential equations with affine modification of the argument which can be differential equations with linear modification of the argument or differential equation with delay. For more information and results concerning these equations, see [6, 11, 13, 16, 18, 19].

More recently, Caballero *et al* [5] investigated the so-called quadratic integral equation of the Volterra type with linear modification of the argument, Volterra counter part of equation (1.1), and proved the existence of monotonic solutions in $C([0,1],\mathbb{R})$. There equation can be considered with connection to the following Cauchy problem [5, 19]:

$$\begin{aligned} &x'(t) = u(t,s,x(t),x(\lambda t)), \quad t \in [0,1],\ 0<\lambda<1,\\ &x(0)=u_0. \end{aligned}$$

In this paper, we investigate the question of unique solvability of equation (1.1) and (1.2). Motivated by the previous papers considered for integral equations on a bounded interval, we extend here these results to semi-infinite intervals for a class of quadratic integral equations. The method we are going to use is to reduce the existence of the unique solution for the quadratic integral equation (1.1) to the search for the existence of the unique fixed-point of an appropriate operator on the Fréchet space $C(J;\mathbb{R})$ by applying a nonlinear alternative of Leray–Schauder type for contraction maps due to Frigon and Granas [12].

2. Preliminaries

We introduce notations, definitions, and theorems which are used throughout this paper.

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. Let $Y \subset X$, we say that Y is bounded if for every $n\in\mathbb{N}$, there exists $M_n>0$ such that

$$\|y\|_n \le M_n \quad \text{for every } y \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n,\|\cdot\|_n)\}$ as follows: For every $n\in\mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by the formula

$$x \sim_n y \quad \text{if and only if} \quad \|x-y\|_n = 0 \qquad \text{for all } x,y\in X.$$

We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y\subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $x\in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n = \{[x]_n : x \in Y\}$. We denote $\overline{Y^n}$, $\mathrm{Int}_n(Y^n)$, and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|$ in X^n. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies

$$\|x\|_1 \le \|x\|_2 \le \|x\|_3 \le \cdots \quad \text{for every } x \in X.$$

Definition 1 ([12]). A function $f\colon X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in (0,1)$ such that

$$\|f(x) - f(y)\|_n \le k_n \|x - y\|_n \quad \text{for all } x, y \in X.$$

Theorem 2.1 ([12]). *Suppose that U is an open subset of a Fréchet space X, $0 \in U$, and $F\colon \overline{U} \to X$ is a contraction such that $F(\overline{U})$ is bounded. Then either,*

(C1) *F has a unique fixed point in $\overline{U}$; or*
(C2) *there exist $\lambda \in (0,1)$ and $u \in \partial U$ with the property $u = \lambda F(u)$.*

3. Main theorem

In this section, we will study equation (1.1) assuming that the following assumptions are satisfied:

(a_1) $f\colon J \to \mathbb{R}$ is a continuous function.
(a_2) For each $n \in \mathbb{N}$ there exists $L_n > 0$ such that

$$|(Ax)(t) - (A\overline{x})(t)| \le L_n |x(t) - \overline{x}(t)| \quad \text{for each } x, \overline{x} \in C(J;\mathbb{R}) \text{ and } t \in [0,n].$$

(a_3) There exist nonnegative constants a and b such that

$$|(Ax)(t)| \le a + b|x(t)|$$

holds for each $x \in C(J;\mathbb{R})$ and $t \in J$.
(a_4) $u\colon J \times J_T \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and, for each $n \in \mathbb{N}$, there exists a constant $L_n^* > 0$ such that

$$|u(t,s,x,y) - u(t,s,\overline{x},\overline{y})| \le L_n^*(|x - \overline{x}| + |y - \overline{y}|)$$

holds for all $t \in [0,n]$, $s \in J_T$, and $x, y, \overline{x}, \overline{y} \in \mathbb{R}$.
(a_5) There exists a continuous nondecreasing function $\psi\colon J^2 \to (0,\infty)$ and $p \in C(J;\mathbb{R}_+)$ such that

$$|u(t,s,x,y)| \le p(s)\psi(|x|,|y|)$$

for each $(t,s) \in J \times J_T$ and $x, y \in \mathbb{R}$, and, moreover, there exist constants $M_n \in J$, $n \in \mathbb{N}$, such that

$$\frac{M_n}{\|f\|_n + T(a + bM_n)\psi(M_n, M_n)p^*} > 1 \tag{3.1}$$

holds for all $n \in \mathbb{N}$, where $p^* = \sup\{p(s) : s \in J_T\}$.

Theorem 3.1. *Let the assumptions* (a_1)–(a_5) *be satisfied. If, in addition, the inequality*

$$2\,(a+b\,M_n)\,L_n^*\,T+TL_n\,\psi(M_n,M_n)p^*<1 \tag{3.2}$$

holds for all $n\in\mathbb{N}$*, then the equation* (1.1) *has a unique solution.*

Proof. For every $n\in\mathbb{N}$, we define in $C(J;\mathbb{R})$ the semi-norms by the formula

$$\|y\|_n:=\sup\{|y(t)|:t\in[0,n]\}.$$

Then $C(J;\mathbb{R})$ is a Fréchet space with the family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$.

Transform the problem (1.1) into a fixed-point problem. Consider the operator $\mathcal{F}:C(J;\mathbb{R})\to C(J;\mathbb{R})$ defined by the relation

$$(\mathcal{F}y)(t)=f(t)+(Ay)(t)\int_0^T u(t,s,y(s),y(\alpha s))ds,\quad t\in J.$$

Let y be a possible solution of the problem (1.1). For given $n\in\mathbb{N}$ and $t\le n$, in view of (a_1), (a_2), and (a_5), we have

$$\begin{aligned}|y(t)|&\le|f(t)|+|(Ay)(t)|\int_0^T|u(t,s,y(s),y(\alpha s))|ds\\&\le|f(t)|+(a+b|y(t)|)\int_0^T p(s)\psi(|y(s)|,|y(\alpha s)|)ds\\&\le\|f\|_n+T(a+b\|y\|_n)\psi(\|y\|_n,\|y\|_n)p^*\end{aligned}$$

and thus

$$\frac{\|y\|_n}{\|f\|_n+T(a+b\|y\|_n)\psi(\|y\|_n,\|y\|_n)p^*}\le1.$$

From (3.1) it follows that $\|y\|_n\neq M_n$ for each $n\in\mathbb{N}$. Now, set

$$\Omega=\{y\in(J;\mathbb{R}):\|y\|_n<M_n\text{ for every }n\in\mathbb{N}\}.$$

Clearly, Ω is an open subset of $C(J;\mathbb{R})$. We shall show that $\mathcal{F}:\bar{\Omega}\to C(J;\mathbb{R})$ is a contraction operator. Indeed, consider $y,\bar{y}\in C(J;\mathbb{R})$, for each $t\in[0,n]$ and $n\in\mathbb{N}$, from (a_2)–(a_5) we get

$$\begin{aligned}&|(\mathcal{F}y)(t)-(\mathcal{F}\bar{y})(t)|\\&\le\left|(Ay)(t)\int_0^T u(t,s,y(s),y(\alpha s))ds-(A\bar{y})(t)\int_0^T u(t,s,\bar{y}(s),\bar{y}(\alpha s))ds\right|\\&\le\left|(Ay)(t)\int_0^T u(t,s,y(s),y(\alpha s))ds-(Ay)(t)\int_0^T u(t,s,\bar{y}(s),\bar{y}(\alpha s))ds\right|\\&\quad+\left|(Ay)(t)\int_0^T u(t,s,\bar{y}(s),\bar{y}(\alpha s))ds-(A\bar{y})(t)\int_0^T u(t,s,\bar{y}(s),\bar{y}(\alpha s))ds\right|\end{aligned}$$

$$
\begin{aligned}
&\le |(Ay)(t)| \int_0^T |u(t,s,y(s),y(\alpha s)) - u(t,s,\overline{y}(s),\overline{y}(\alpha s))|ds \\
&\quad + |(Ay)(t)-(A\overline{y})(t)| \int_0^T |u(t,s,\overline{y}(s),\overline{y}(\alpha s))|ds \\
&\le (a+b|y(t)|)\, L_n^* \int_0^T (|y(s)-\overline{y}(s)| + |y(\alpha s)-\overline{y}(\alpha s)|)ds \\
&\quad + L_n|y(t)-\overline{y}(t)| \int_0^T p(s)\psi(|\overline{y}(s)|,|\overline{y}(\alpha s)|)ds \\
&\le \left[2(a+bM_n)L_n^*T + TL_n\psi(M_n,M_n)p^*\right] \|y-\overline{y}\|_n.
\end{aligned}
$$

Therefore,

$$\|\mathcal{F}y - \mathcal{F}\overline{y}\|_n \le \left[2(a+bM_n)L_n^*T + TL_n\psi(M_n,M_n)p^*\right] \|y-\overline{y}\|_n.$$

So by (3.2) the operator $\mathcal{F}$ is a contraction for all $n \in \mathbb{N}$. From the choice of Ω there is no $y \in \partial\Omega$ such that $y = \lambda\mathcal{F}(y)$ for some $\lambda \in (0,1)$. Then the statement (C2) in Theorem 2.1 does not hold. A consequence of the Leray–Schauder type nonlinear alternative from [12] yields that (C1) holds and thus we deduce that the operator $\mathcal{F}$ has a unique fixed-point y in $\overline{\Omega}$, which is a solution to Equation (1.1). This completes the proof. □

Theorem 3.2. *Let the following assumptions be satisfied:*

($\hat{a}_1$) $f\colon J \to \mathbb{R}$ *is a continuous function.*

($\hat{a}_2$) $g\colon J \times \mathbb{R} \to \mathbb{R}$ *is continuous and, for each* $n \in \mathbb{N}$, *there exists* $L_n > 0$ *such that*

$$|g(t,x) - g(t,\overline{x})| \le L_n|x-\overline{x}| \quad \text{for all } x,\overline{x} \in \mathbb{R} \text{ and } t \in [0,n].$$

($\hat{a}_3$) $u\colon J \times J_T \times \mathbb{R}^2 \to \mathbb{R}$ *is a continuous function and, for each* $n \in \mathbb{N}$, *there exists a constant* $L_n^* > 0$ *such that*

$$|u(t,s,x,y) - u(t,s,\overline{x},\overline{y})| \le L_n^*(|x-\overline{x}| + |y-\overline{y}|)$$

holds for all $(t,s) \in [0,n] \times J_T$ *and* $x,y,\overline{x},\overline{y} \in \mathbb{R}$.

($\hat{a}_4$) *There exists a continuous nondecreasing function* $\psi\colon J^2 \to (0,\infty)$ *and* $p \in C(J;\mathbb{R}_+)$ *such that*

$$|u(t,s,x,y)| \le p(s)\psi(|x|,|y|)$$

for each $(t,s) \in J \times J_T$ *and* $x,y \in \mathbb{R}$ *and, moreover, there exists constants* $M_n \in J$, $n \in \mathbb{N}$, *such that*

$$\frac{M_n}{\|f\|_n + T(L_nM_n + m_n)\psi(M_n,M_n)p^*} > 1$$

holds for all $n \in \mathbb{N}$, *where* $p^* = \sup\{p(s) : s \in J_T\}$ *and* $m_n = \sup\{g(t,0) : t \in [0,n]\}$.

If, in addition, the inequality

$$2(L_n M_n + m_n)TL_n^* + TL_n \psi(M_n, M_n) p^* < 1 \tag{3.3}$$

holds, then Equation (1.2) has a unique solution.

Proof. The proof is similar to those of Theorem 3.1. □

Example 1. Consider the quadratic integral equation of the Urysohn type

$$x(t) = 1 + \frac{|x(t)|}{1+|x(t)|} \int_0^T \frac{ts}{t^3+1} \left(x(s) + x\left(\frac{s}{2}\right)\right) ds, \quad t \in J := [0, +\infty). \tag{3.4}$$

Set $f(t) = 1$ for each $t \in J$, $\psi(x,y) = x + y$ for all $x, y \geq 0$,

$$(Ax)(t) = \frac{|x(t)|}{1+|x(t)|}, \quad t \in J \text{ and } x \in C(J;\mathbb{R}),$$

and

$$u(t,s,x,y) = \frac{ts}{t^3+1}(x+y)$$

for all $(t,s) \in J \times J_T$ and $x, y \in \mathbb{R}$. It is clear that (3.4) is a particular case of equation (1.1). Let us show that conditions (a_1)–(a_5) hold.

For each $n \in \mathbb{N}$, $t \in [0,n]$, and $x, \overline{x} \in C(J;\mathbb{R}_+)$, we have

$$\begin{aligned} |(Ax)(t) - (A\overline{x})(t)| &= \left| \frac{|x(t)|}{1+|x(t)|} - \frac{|\overline{x}(t)|}{1+|\overline{x}(t)|} \right| = \left| \frac{|x(t)| - |\overline{x}(t)|}{(1+|x(t)|)(1+|\overline{x}(t)|)} \right| \\ &\leq \frac{|x(t) - \overline{x}(t)|}{(1+|x(t)|)(1+|\overline{x}(t)|)} \leq |x(t) - \overline{x}(t)|. \end{aligned}$$

Hence (a_2) is satisfied with $L_n = 1$.

For each $t \in J$ and $x \in C(J;\mathbb{R})$, we have

$$|(Ax)(t)| = \left| \frac{|x(t)|}{1+|x(t)|} \right| \leq |x(t)|.$$

Hence (a_3) holds with $a = 0$ and $b = 1$.

For each $n \in \mathbb{N}$, $(t,s) \in [0,n] \times J_T$, and $x, y, \overline{x}, \overline{y} \in \mathbb{R}$, we have

$$\begin{aligned} \left| u(t,s,x,\overline{x}) - u(t,s,y,\overline{y}) \right| &= \left| \frac{ts}{t^3+1} [(x+\overline{x}) - (y+\overline{y})] \right| \\ &\leq \left| \frac{ts}{t^3+1} [(x-y) + (\overline{x}-\overline{y})] \right| \\ &\leq \frac{nT}{n^3+1} [|x-y| + |\overline{x}-\overline{y}|] \\ &\leq T\,[|x-y| + |\overline{x}-\overline{y}|]. \end{aligned}$$

Hence (a_4) is satisfied with $L_n^* = T$.

For each $n \in \mathbb{N}$, $(t,s) \in [0,n] \times J_T$, and $x, y \in \mathbb{R}$, we have

$$\begin{aligned}|u(t,s,x,y)| &= \left|\frac{ts}{t^3+1}(x+y)\right| \\ &\le s(|x|+|y|) = s\psi(|x|,|y|).\end{aligned}$$

To conclude that (a_5) holds we shall show that (3.1) is satisfied. Indeed

$$\begin{aligned}\frac{M_n}{\|f\|_n + T(a+bM_n)\psi(M_n,M_n)p^*} > 1 &\iff \frac{M_n}{1+2T^2M_n^2} > 1 \\ &\iff 2T^2M_n^2 - M_n + 1 < 0.\end{aligned}$$

Notice that the last inequality holds for T such that $1-8T^2 > 0$, i. e.,

$$T < \frac{1}{2\sqrt{2}}. \tag{3.5}$$

Hence for $T > 0$ satisfying (3.5), there exists $M_n > 0$ satisfying (3.1).

Finally let us show that (3.2) is satisfied.

$$\begin{aligned}2(a+bM_n)L_n^*T + TL_n\psi(M_n,M_n)p^* - 1 &= 2M_nT^2 + 2T^2M_n - 1 \\ &= 4M_nT^2 - 1.\end{aligned}$$

Hence (3.2) is satisfied for T or M_n satisfying $4M_nT^2 - 1 < 0$, i. e., for

$$0 < T < \frac{1}{2\sqrt{M_n}}$$

or $0 < M_n < \frac{1}{4}T^{-2}$. Consequently, if T satisfies the inequalities

$$0 < T < \min\left\{\frac{1}{2\sqrt{M_n}}, \frac{1}{2\sqrt{2}}\right\},$$

then it follows from Theorem 3.1 that equation (3.4) has a unique solution.

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ON THE UNIQUENESS, STABILITY AND HYPERBOLICITY OF SYMMETRIC AND PERIODIC SOLUTIONS OF WEAKLY NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We show the existence of unique periodic and symmetric solutions of weakly nonlinear ordinary differential equations. Conditions of stability and hyperbolicity of these solutions are established as well. Concrete examples are presented to show advantages of our method to the classical averaging theory.

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Keywords: periodic solution, symmetric systems, stability of solution, k-hyperbolicity

1. Introduction

In this paper we consider systems of ordinary differential equations under periodic and symmetric assumptions. More concretely, we mostly consider a weakly nonlinear ordinary differential equation of the form

$$x'(t) = \varepsilon f(x(t), t), \qquad t \in \mathbb{R}, \tag{1.1}$$

with a small parameter $\varepsilon \in \mathbb{R}$, where $x\colon \mathbb{R} \to \mathbb{R}^n$. We suppose that the function $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ in (1.1) is sufficiently smooth, pT-periodic in the second variable, and symmetric in the first one, i. e.,

$$Af(x,t) = f(Ax, t+T), \qquad t \in \mathbb{R}, \tag{1.2}$$

where $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear map such that $A^p = \mathbb{1}$ for some $p \in \mathbb{N}$. Due to this property, we have the following useful construction. Let $\langle \cdot, \cdot \rangle$ be a scalar product on $\mathbb{R}^n$. By setting

$$(x, y) := \frac{1}{p} \sum_{i=0}^{p-1} \langle A^i x, A^i y \rangle, \qquad \{x, y\} \subset \mathbb{R}^n,$$

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as a new scalar product on $\mathbb{R}^n$, we obtain $(Ax,Ay) = (x,y)$ and $|Ax| = |x|$ for $|x| := \sqrt{(x,x)}$, $x \in \mathbb{R}^n$, and thus A is unitary with respect to $(\cdot,\cdot)$. We mostly suppose $1 \notin \sigma(A)$, where $\sigma(A)$ is the spectrum of A. The norm* on $\mathcal{L}(\mathbb{R}^n)$ generated by $|\cdot|$ is denoted by $\|\cdot\|$.

Remark 1.1. We study (1.1) always in an arbitrary but fixed open bounded subset of $\mathbb{R}^n$. So we do not study bifurcations from infinity of (1.1) for ε small.

Next, for $T > 0$ we introduce the Banach space

$$X := \left\{x \in C^0(\mathbb{R},\mathbb{R}^n) \mid x(t+T) = Ax(t),\ t \in \mathbb{R}\right\}. \tag{1.3}$$

Note $|x(\cdot)|$ is a T-periodic function because $|x(t+T)| = |Ax(t)| = |x(t)|$ for every t. So we take the norm $\|x\| = \max_{t\in[0,T]}|x(t)| = \max_{t\in\mathbb{R}}|x(t)|$ on X. It is easy to see that if a function x belongs to the space X then it is a pT-periodic function.

Definition 1.1. By a *symmetric and periodic solution* x of equation (1.1) we understand a solution $x \in X$, where X is defined by (1.3).

The main goal of this paper is to find a unique symmetric and periodic solution (see Sections 2 and 3) for (1.1) and to establish conditions, under which this solution is either stable (see Section 4) or hyperbolic (see Sections 5 and 6). We also present examples to illustrate the theory in Sections 7 and 8. We show that our symmetric averaging method could help when the classical averaging theory fails [18, 27].

The results presented here are generalizations of achievements for anti-periodic problems with $A = -\mathbb{1}$ [1,2], and continuations of [7]. Doubly symmetric solutions of reversible systems are studied in [20]. Symmetric properties of periodic solutions of nonlinear nonautonomous ordinary differential equations are also studied in [23, 24].

2. Uniqueness for general nonlinearities

In spite of the fact that we are interested in (1.1), we start with its general form with $\varepsilon = 1$, i. e., we first consider the equation

$$x'(t) = f(x(t),t), \qquad t \in \mathbb{R}. \tag{2.1}$$

Looking for a symmetric and periodic solution x of equation (2.1) is equivalent to solve (2.1) on $t \in [0,T]$ under the boundary condition

$$x(T) = Ax(0). \tag{2.2}$$

The following lemma guarantees the existence and uniqueness of a solution of problem (2.1), (2.2).

Lemma 2.1. *Assume that* $1 \notin \sigma(A)$ *and, moreover, there exists a constant* $K > 0$ *such that*

$$|f(x_1,t) - f(x_2,t)| \le K|x_1 - x_2| \tag{2.3}$$

*As usual, the symbol $\mathcal{L}(\mathbb{R}^n)$ stands for the algebra of square matrices of dimension n.

for all x_1, x_2 from $\mathbb{R}^n$. If

$$\left(\|(A-\mathbb{1})^{-1}\|+1\right)KT<1, \tag{2.4}$$

then equation (2.1) has a unique solution $x\in X$.

Proof. Let us first solve the boundary value problem

$$x'=\widetilde{h}, \qquad x(T)=Ax(0), \tag{2.5}$$

where $\widetilde{h}\in X$. Obviously, $x(t)=x(0)+\int_0^t\widetilde{h}(s)ds$ and $Ax(0)=x(T)=x(0)+\int_0^T\widetilde{h}(s)ds$, whence

$$x(0)=(A-\mathbb{1})^{-1}\int_0^T\widetilde{h}(s)ds.$$

and, therefore, the solution of (2.5) has the form

$$x(t)=(A-\mathbb{1})^{-1}\int_0^T\widetilde{h}(s)ds+\int_0^t\widetilde{h}(s)ds, \quad t\in[0,T].$$

Let us put

$$B(x)(t):=(A-\mathbb{1})^{-1}\int_0^T f(x(s),s)ds+\int_0^t f(x(s),s)ds, \quad t\in[0,T],$$

for an arbitrary x from X. For any $x_1,x_2\in X$, using the Lagrange theorem and (2.3), we get

$$\begin{aligned}\|B(x_1)-B(x_2)\| &= \Bigg\|(A-\mathbb{1})^{-1}\int_0^T(f(x_1(s),s)-f(x_2(s),s))ds\\ &\qquad+\int_0^t(f(x_1(s),s)-f(x_2(s),s))ds\Bigg\|\\ &\le\left(\|(A-\mathbb{1})^{-1}\|+1\right)KT\|x_1(t)-x_2(t)\|.\end{aligned}$$

Taking condition (2.4) into account and applying the Banach fixed point theorem we get that problem (2.1), (2.2) has a unique solution $x\in X$. The proof is complete. □

Now we replace Lipschitz condition (2.3) as follows. Let $|\cdot|_0$ be a norm on $\mathbb{R}^n$. Then for any $x,y\in\mathbb{R}^n$ the limit [3,4]

$$N(x,y):=\lim_{\lambda\to+0}\frac{|x+\lambda y|_0-|x|_0}{\lambda}$$

exists and has the following elementary properties:

$$\left.\begin{aligned} N(cx,cy) &= cN(x,y)\,,\\ |N(x,y_1)-N(x,y_1)| &\le |y_1-y_2|_0\,,\\ N\left(x,\frac{1}{m}\sum_{k=1}^{m} y_k\right) &\le \frac{1}{m}\sum_{k=1}^{m} N(x,y_k) \end{aligned}\right\} \tag{2.6}$$

for any $c>0$ and x, y, y_1, y_2, $\ldots$, y_m from $\mathbb{R}^n$. So the function $y\mapsto N(x,y)$ is globally Lipschitz continuous and convex. Next, for any $B\in\mathcal{L}(\mathbb{R}^n)$, there is its Lozinskii logarithmic norm [16]

$$\mu(B) := \lim_{\lambda\to+0}\frac{\|\mathbb{1}+\lambda B\|_0-1}{\lambda}\,,$$

where $\|B\|_0 := \max_{|x|_0=1}|Bx|_0$. Note that

$$\sup_{|x|_0=1} N(x,Bx) = \mu(B). \tag{2.7}$$

Now we are ready to prove the following results motivated by [6, 14, 15] and [30, Theorem 9.6].

Lemma 2.2. *Let $f\in C^1(\mathbb{R}^{n+1},\mathbb{R}^n)$. If there is a constant $\alpha>0$ such that*

$$\mu\left(D_x f(x,t)\right) \le -\alpha \quad \forall (x,t)\in\mathbb{R}^{n+1}$$

then (2.1) possesses a unique symmetric and periodic solution which is globally asymptotically stable.

Proof. Let $x(t)$ be a solution of (2.1) defined on $[0,\delta)$ for some $\delta>0$. Then by (2.6), (2.7) and the Jensen inequality [25], for almost each $t\in[0,\delta)$, we derive

$$\begin{aligned} \frac{d}{dt}|x(t)|_0 &= N(x(t),x'(t)) = N(x(t),f(x(t),t))\\ &\le N(x(t),f(x(t),t)-f(0,t)) + |f(0,t)|_0\\ &\le N\left(x(t),\int_0^1 D_x f(\theta x(t),t)x(t)d\theta\right) + \|f(0,\cdot)\|_0\\ &\le \int_0^1 N\left(x(t),D_x f(\theta x(t),t)x(t)\right)d\theta + \|f(0,\cdot)\|_0\\ &\le \int_0^1 \mu\left(D_x f(\theta x(t),t)\right)d\theta|x(t)|_0 + \|f(0,\cdot)\|_0\\ &\le -\alpha|x(t)|_0 + \|f(0,\cdot)\|_0\,, \end{aligned} \tag{2.8}$$

where $\|f(0,\cdot)\|_0 = \max_{t\in\mathbb{R}} |f(0,t)$. Note $|x(t)|_0$ is locally absolute continuous. The Gronwall inequality implies

$$|x(t)|_0 \le \mathrm{e}^{-\alpha t}|x(0)|_0 + \frac{\|f(0,\cdot)\|_0}{\alpha}\left(1-\mathrm{e}^{-\alpha t}\right),$$

so $x(t)$ is defined for any $t \ge 0$ and it terminates in the ball $B(0, 2\|f(0,\cdot)\|_0/\alpha)$ centered at 0 with the radius $2\|f(0,\cdot)\|_0/\alpha$, i. e., (2.1) is dissipative possessing a pT-periodic solution $x_0(t)$ inside $B(0, 2\|f(0,\cdot)\|_0/\alpha)$ (see [4]).

Next, let $x_1(t)$ and $x_2(t)$ be two solutions of (2.1) on $\mathbb{R}_+$. Then similarly to the consideration above, for almost every $t \in \mathbb{R}_+$, we obtain

$$\begin{aligned}
\frac{d}{dt}|x_1(t)-x_2(t)|_0 &= N\left(x_1(t)-x_2(t), \frac{d}{dt}(x_1(t)-x_2(t))\right)\\
&= N(x_1(t)-x_2(t), f(x_1(t),t)-f(x_2(t),t))\\
&= N\left(x_1(t)-x_2(t), \int_0^1 D_x f(\theta x_1(t)+(1-\theta)x_2(t),t)(x_1(t)-x_2(t))d\theta\right)\\
&\le \int_0^1 N\left(x_1(t)-x_2(t), D_x f(\theta x_1(t)+(1-\theta)x_2(t),t)(x_1(t)-x_2(t))\right)d\theta\\
&\le \int_0^1 \mu\left(D_x f(\theta x_1(t)+(1-\theta)x_2(t),t)\right)d\theta |x_1(t)-x_2(t)|_0\\
&\le -\alpha|x_1(t)-x_2(t)|_0,
\end{aligned}$$

i. e., the Gronwall inequality again implies

$$|x_1(t)-x_2(t)|_0 \le \mathrm{e}^{-\alpha t}|x_1(0)-x_2(0)|_0. \tag{2.9}$$

So (2.9) gives the uniqueness of $x_0(t)$ and it is globally asymptotically stable. It is easy to verify that $x(t) := A^{-1}x_0(t+T)$ is also a pT-periodic solution of (2.1). The uniqueness gives $x(t) = x_0(t)$, i. e., $x_0 \in X$. The proof is finished. □

Lemma 2.2 is extended in Corollary 4.1 of Section 4 in the context of averaging theory. The above result can be slightly modified as follows.

Lemma 2.3. *Let $f \in C^1(\mathbb{R}^{n+1}, \mathbb{R}^n)$. Assume that*

$$\mu(D_x f(x,t)) \le 0 \quad \forall (x,t) \in \mathbb{R}^{n+1}$$

and, moreover, that (2.1) possesses a bounded uniformly asymptotically stable solution x_0 on $\mathbb{R}_+$. Then x_0 is symmetric, periodic, and globally asymptotically stable.

Proof. Let $x(t)$ be a solution of (2.1) defined on $[0,\delta)$ for some $\delta > 0$. Then for $\alpha = 0$, (2.8) implies that $|x(t)|_0 \le |x(0)|_0 + \|f(0,\cdot)\|_0 t$ holds for all $t \ge 0$. So $x(t)$ is defined on $[0,\infty)$.

Next, let $x_1(t)$ and $x_2(t)$ be two solutions of (2.1) on $\mathbb{R}_+$. Then for $\alpha = 0$, (2.9) implies

$$|x_1(t)-x_2(t)|_0 \le |x_1(0)-x_2(0)|_0 \quad \forall t \ge 0. \tag{2.10}$$

Furthermore, from the uniform asymptotic stability of $x_0(t)$ on $\mathbb{R}_+$, there is an $\delta_0 > 0$ such that

$$\{x \in \mathbb{R}^n \mid \operatorname{dist}(x, \{x_0(t) \mid t \in \mathbb{R}_+\}) < \delta_0\} \subset B_A,$$

where B_A is the domain of attraction of $x_0(t)$. Then (2.10) implies that $B(z, \delta_0/2) \subset B_A$ holds for any $z \in B_A$. Consequently, $B_A = \mathbb{R}^n$. Since $x_0(t)$ is bounded on $\mathbb{R}_+$ by assumption, all solutions of (2.1) terminates in a large ball. So (2.1) has a pT-periodic solution which now must be $x_0(t)$. Then like above it is unique and symmetric. The proof is finished. □

We recall the following well-known results:

(1) If $|x|_0 = \sqrt{\sum_{k=1}^m x_k^2}$ then $\mu(B) = \Lambda\left(\frac{1}{2}(B + B^*)\right)$, where Λ means the largest eigenvalue.

(2) If $|x|_0 = \max_{k=1,2,\dots,m} |x_k|$ then

$$\mu(B) = \max_{k=1,2,\dots,m} \left(b_{kk} + \sum_{j \neq k} |b_{kj}| \right),$$

where b_{kj}, $k, j = 1, 2, \dots, m$, mean the entries of matrix B.

(3) If $|x|_0 = \sum_{k=1}^m |x_k|$ then $\mu(B) = \max_{k=1,2,\dots,m} \left(b_{kk} + \sum_{j \neq k} |b_{jk}|\right)$.

Applying Lemmas 2.2 and 2.3, we obtain the following particular results used in the sequel [6].

Lemma 2.4. *Let us put*

$$J(x,t) := \frac{1}{2}\left(D_x f(x,t) + D_x f(x,t)^*\right) \tag{2.11}$$

and let $\Lambda(x,t)$ be the largest eigenvalue of $J(x,t)$. If there is a constant $\alpha > 0$ such that

$$\Lambda(x,t) \leq -\alpha \quad \forall (x,t) \in \mathbb{R}^{n+1} \tag{2.12}$$

then (2.1) possesses a unique symmetric and periodic solution which is globally asymptotically stable.

Lemma 2.5. *Assume*

$$\Lambda(x,t) \leq 0 \quad \forall (x,t) \in \mathbb{R}^{n+1} \tag{2.13}$$

and (2.1) possesses a bounded, uniformly asymptotically stable solution $x_0(t)$ on $\mathbb{R}_+$. Then $x_0(t)$ is symmetric, periodic and globally asymptotically stable.

Remark 2.1. We do not need to suppose that $1 \notin \sigma(A)$ in Lemmas 2.2–2.5.

Remark 2.2. By reversing the time $t \leftrightarrow -t$ in (2.1), we get similar results like above ensuring that $x_0(t)$ is a global repeller, namely, (2.12) and (2.13) are replaced by the inequalities $\lambda(x,t) \geq \alpha$ and $\lambda(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^{n+1}$, respectively, where $\lambda(x,t)$ is the smallest eigenvalue of $J(x,t)$.

3. Uniqueness for weak nonlinearities

First, we consider the Cauchy problem

$$x'(t) = \varepsilon f(x(t), t), \quad t \in [0, T], \tag{3.1}$$

$$x(0) = x. \tag{3.2}$$

Let us denote by $\varphi_\varepsilon(x, t)$ the unique solution of problem (3.1), (3.2). Note that

$$\varphi_\varepsilon(x, 0) = x. \tag{3.3}$$

Obviously, the equation

$$\varphi_\varepsilon(x, pT) = x \tag{3.4}$$

determines pT-periodic solutions of (1.1). For this reason, we consider the diffeomorphism given by the formula

$$P_\varepsilon(x) := \varphi_\varepsilon(x, pT). \tag{3.5}$$

Taking (3.4) and (3.5) into account, we get

$$P_\varepsilon(x) = x. \tag{3.6}$$

On the other hand, the T-periodic and symmetric solutions of (1.1), i. e., the solutions belonging to X, are given by the equation

$$Ax = \varphi_\varepsilon(x, T).$$

In this connection, we introduce the new mapping

$$g_\varepsilon(x) := A^{-1}\varphi_\varepsilon(x, T). \tag{3.7}$$

Summarizing the considerations above, we arrive at the following simple lemma.

Lemma 3.1. *A fixed point x of P_ε, respectively of g_ε, determines a pT-periodic, respectively a periodic and symmetric, solution of equation (1.1) satisfying (3.2).*

The basic relationship between mappings P_ε and g_ε is expressed by the following lemma.

Lemma 3.2. *The relation*

$$P_\varepsilon(x) = [g_\varepsilon(x)]^p \tag{3.8}$$

holds, where P_ε and g_ε are defined, respectively, by (3.5) and (3.7).

In (3.8), $[g_\varepsilon(x)]^p$ is defined as

$$[g_\varepsilon(x)]^p = \underbrace{g_\varepsilon(g_\varepsilon(g_\varepsilon(g_\varepsilon \dots (x)))))}_{p}, \quad x \in \mathbb{R}^n.$$

Proof of Lemma 3.2. It is easy to see that

$$\varphi_\varepsilon(x,t+T) = A\varphi_\varepsilon(A^{-1}\varphi_\varepsilon(x,T),t),$$

which implies

$$\varphi_\varepsilon(x,2T) = A\varphi_\varepsilon(A^{-1}\varphi_\varepsilon(x,T),T) = AAA^{-1}\varphi_\varepsilon(g_\varepsilon(x),T).$$

Taking into account (3.7), the last relation is equal to

$$\varphi_\varepsilon(x,2T) = A^2[g_\varepsilon(x)]^2.$$

Let us apply inductive method. Assume that relation

$$\varphi_\varepsilon(x,iT) = A^i[g_\varepsilon(x)]^i$$

is true. Now we show that this relation is true also for $i+1$:

$$\begin{aligned}\varphi_\varepsilon(x,(i+1)T) &= \varphi_\varepsilon(x,iT+T) = A\varphi_\varepsilon(A^{-1}\varphi_\varepsilon(x,T),iT)\\ &= A\varphi_\varepsilon(g_\varepsilon(x),iT) = A\cdot A^i g_\varepsilon[g_\varepsilon(x)]^i = A^{i+1}[g_\varepsilon(x)]^{i+1}.\end{aligned}$$

So, finally we obtain

$$P_\varepsilon(x) = \varphi_\varepsilon(x,pT) = A^p[g_\varepsilon(x)]^p = [g_\varepsilon(x)]^p.$$

The proof is finished. □

In the sequel we assume for simplicity that f is C^∞-smooth. Asymptotic expansion can be obtained by the methods of averaging and multiple scales. So there exist the Taylor series of P_ε and g_ε with respect to the small parameter ε:

$$P_\varepsilon(x) = P_0(x) + \varepsilon P_1(x) + \dots \tag{3.9}$$

and

$$g_\varepsilon(x) = g_0(x) + \varepsilon g_1(x) + \dots.$$

Remark 3.1. It follows from the definition of the mapping P_ε given by formula (3.5) that $P_0 = \mathbb{1}$.

The next lemma follows from Lemma 3.2.

Lemma 3.3. *If x is a solution of equation*

$$g_\varepsilon(x) = x, \tag{3.10}$$

then it is also a solution of equation (3.6).

Let us solve equation (3.10), which, by (3.7), has the form

$$H(x,\varepsilon) := A^{-1}\varphi_\varepsilon(x,T) - x = 0.$$

Since

$$g_0(x) = A^{-1}\varphi_0(x,T) = A^{-1}x, \tag{3.11}$$

we obtain that $H(0,0) = 0$ and $D_xH(0,0) = A^{-1} - \mathbb{1}$. Using the assumption $1 \notin \sigma(A)$ and applying the Implicit Function Theorem, we get the following result.

Theorem 3.1. *If $1 \notin \sigma(A)$ then equation (3.10) has a unique C^∞-smooth solution $x_\varepsilon = O(\varepsilon)$ for $\varepsilon \neq 0$ small. Note that*

$$P_\varepsilon(x_\varepsilon) = x_\varepsilon .$$

Theorem 3.1 implies the following one.

Theorem 3.2. *If $1 \notin \sigma(A)$ and $\varepsilon \neq 0$ is small, then equation (1.1) has a unique pT-periodic solution $x_\varepsilon(t) \in X$ such that $x_\varepsilon(0) = x_\varepsilon$ and $x_\varepsilon(t) = O(\varepsilon)$.*

Proof. Lemmas 3.1, 3.2 and Theorem 3.1 immediately give the result. □

4. Stability of periodic and symmetric solutions

To study stability of the T-periodic and symmetric solution $x_\varepsilon(\cdot) \in X$ of equation (1.1), we consider the linearization

$$M(\varepsilon) := D_x g_\varepsilon(x_\varepsilon) \tag{4.1}$$

of mapping (3.7) at the fixed point $x_\varepsilon = O(\varepsilon)$. Note that, by (3.8), we have

$$[D_x g_\varepsilon(x_\varepsilon)]^p = D_x P_\varepsilon(x_\varepsilon),$$

which, by the Dunford spectral mapping theorem [11], yields

$$\sigma\left(D_x P_\varepsilon(x_\varepsilon)\right) = \left[\sigma\left(D_x g_\varepsilon(x_\varepsilon)\right)\right]^p . \tag{4.2}$$

Taking (3.7) into account, we get

$$D_x g_\varepsilon(x) = A^{-1} D_x \varphi_\varepsilon(x, T).$$

So, using (3.11), we obtain

$$D_x g_0(x) = A^{-1} D_x \varphi_0(x, T) = A^{-1}. \tag{4.3}$$

From (4.1), (4.3) we have that

$$M(0) = A^{-1}, \tag{4.4}$$

but A is a unitary linear mapping with all eigenvalues on the unit circle.

Let us consider the decomposition of the mapping $M(\varepsilon)$ into the Taylor series on a small parameter ε using (4.4):

$$M(\varepsilon) = A^{-1} + \varepsilon M_1 + \dots . \tag{4.5}$$

By definition,

$$M_1 = \frac{d}{d\varepsilon} M(0),$$

it is easy to see, taking into account (4.1), $x_0 = 0$ and $D_x^2 g_0(0) = 0$, that

$$\begin{aligned} M_1 = \frac{d}{d\varepsilon}\left[D_x g_\varepsilon(x_\varepsilon)\right]\Big|_{\varepsilon=0} &= D_x^2 g_0(0)\frac{d}{d\varepsilon}x_0 + \frac{d}{d\varepsilon} D_x g_0(x_0) \\ &= A^{-1}\frac{d}{d\varepsilon} D_x \varphi_0(0, T). \end{aligned} \tag{4.6}$$

Let us consider problem (3.1), (3.2). By (3.3), we get that

$$\dot{\varphi}_\varepsilon(x,t) = \varepsilon f(\varphi_\varepsilon(x,t),t), \tag{4.7}$$

$$\varphi_\varepsilon(x,0) = x. \tag{4.8}$$

We differentiate (4.7), (4.8) with respect to x and get

$$\frac{d}{dt}(D_x\varphi_\varepsilon(x,t)) = \varepsilon D_x f(\varphi_\varepsilon(x,t),t) D_x\varphi_\varepsilon(x,t), \tag{4.9}$$

$$D_x\varphi_\varepsilon(x,0) = \mathbb{1}. \tag{4.10}$$

Now differentiating (4.9), (4.10) with respect to ε and putting $\varepsilon = 0$, we get

$$\frac{d}{dt}\Big(\frac{d}{d\varepsilon} D_x\varphi_0(x,t)\Big) = D_x f(x,t) D_x\varphi_0(x,t),$$

$$\frac{d}{d\varepsilon} D_x\varphi_0(x,0) = 0.$$

Thus,

$$\frac{d}{d\varepsilon} D_x\varphi_0(x,T) = \int_0^T D_x f(x,s)ds. \tag{4.11}$$

Using (4.6) and (4.11) we arrive at the equality

$$M_1 = A^{-1}\int_0^T D_x f(0,s)ds.$$

Now we return to the Taylor series (4.5). Considering previous calculations and using that $x_\varepsilon = O(\varepsilon)$ from Theorem 3.1, we derive

$$\begin{aligned} M(\varepsilon) &= A^{-1} + \varepsilon A^{-1}\int_0^T D_x f(0,s)ds + \cdots \\ &= A^{-1}\Big(\mathbb{1} + \varepsilon\int_0^T D_x f(0,s)ds + \cdots\Big) = A^{-1}G(\varepsilon), \end{aligned} \tag{4.12}$$

where $G(\varepsilon) = AM(\varepsilon) = \mathbb{1} + \varepsilon\int_0^T D_x f(0,s)ds + \cdots$. Since A^{-1} is unitary, we get that $M(\varepsilon)$ is stable if $G(\varepsilon)$ is stable. Let us now prove a generalization of a classical result [5,6,22].

Theorem 4.1. *Let $M(\varepsilon) \in \mathcal{L}(\mathbb{R}^n)$ be given by (4.12). If*

$$\operatorname{Re}\left\{\sigma\left(\int_0^T D_x f(0,s)ds\right)\right\} \subset (-\infty,0), \tag{4.13}$$

then there exists an $\varepsilon_0 > 0$ such that $r(M(\varepsilon)) < 1$ for any $0 < \varepsilon \le \varepsilon_0$, where $r(M(\varepsilon))$ is the spectral radius of $M(\varepsilon)$.

In order to prove Theorem 4.1, we recall the following well-known results [10,11,17].

Theorem 4.2. *Let $K \in \mathcal{L}(\mathbb{R}^n)$. Then $r(K) < 1$ if and only if there exists a norm on $\mathbb{R}^n$ such that $\|K\| < 1$.*

Theorem 4.3. *Assume that (4.13) is fulfilled then there exists a scalar product $\langle\cdot,\cdot\rangle_1$ and constant α, $\alpha > 0$ such that*

$$\left\langle \left(\int_0^T D_x f(0,s)ds \right) x, x \right\rangle_1 \le -\alpha \|x\|_1^2 \quad \text{for any } x \in \mathbb{R}^n, \tag{4.14}$$

where $\|x\|_1 = \sqrt{\langle x,x\rangle_1}$, $x \in \mathbb{R}^n$.

Proof of Theorem 4.1. Consider the expression $\|G(\varepsilon)x\|_1^2 = \langle G(\varepsilon)x, G(\varepsilon)x\rangle_1$. Differentiating $\|G(\varepsilon)\|_1^2$ with respect to ε, we get

$$\frac{d}{d\varepsilon}\|G(\varepsilon)x\|_1^2 = 2\langle G'(\varepsilon)x, G(\varepsilon)x\rangle_1 .$$

Using the Lagrange theorem, we obtain

$$\begin{aligned} \|G(\varepsilon)x\|_1^2 &= \|G(0)x\|_1^2 + \frac{d}{d\varepsilon}\|G(\theta)x\|_1^2\varepsilon \\ &= \|x\|_1^2 + 2\langle G'(\theta)x, G(\theta)x\rangle_1\varepsilon, \quad 0 \le \theta \le \varepsilon. \end{aligned} \tag{4.15}$$

By virtue of (4.14), we arrive at the equality

$$\langle G'(0)x, G(0)x\rangle_1 = \left\langle \left(\int_0^T D_x f(0,s)ds \right) x, x \right\rangle_1 \le -\alpha\|x\|_1^2 .$$

Thus,

$$2\langle G'(\theta)x, G(\theta)x\rangle_1 \le -\alpha\|x\|_1^2$$

for $0 < \varepsilon \le \varepsilon_0$ with ε_0 small enough. Returning to (4.15), we obtain

$$\|G(\varepsilon)x\|_1^2 \le \|x\|_1^2 - \alpha\|x\|_1^2\varepsilon = (1-\alpha\varepsilon)\|x\|_1^2 .$$

Supposing that $\varepsilon_0\alpha < 1$, we get

$$\|G(\varepsilon)x\|_1 \le \sqrt{1-\alpha\varepsilon}\|x\|_1 \quad \forall x \in \mathbb{R}^n.$$

Consequently, we have that $\|G(\varepsilon)\|_1 \le \sqrt{1-\alpha\varepsilon} < 1$. Then Theorem 4.2 implies that $r(G(\varepsilon)) < 1$. Finally, since A is unitary with respect to the norm $|\cdot|$, we have [11]

$$r(M(\varepsilon)) = \lim_{k\to\infty} \sqrt[k]{\|M(\varepsilon)^k\|} = \lim_{k\to\infty} \sqrt[k]{\|G(\varepsilon)^k\|} = r(G(\varepsilon)) < 1,$$

which completes the proof of Theorem 4.1. □

Summarizing, we obtain the following result.

Theorem 4.4. *Suppose $1 \notin \sigma(A)$. Then for any $\varepsilon > 0$ small, the unique symmetric and T-periodic solution $x(t) = O(\varepsilon)$ of equation (1.1) is asymptotically stable if condition (4.13) is fulfilled.*

Proof. We know from Theorem 4.1 that $r\left(D_x g_\varepsilon(x_\varepsilon)\right) < 1$. Then, using (4.2), we have

$$r\left(D_x P_\varepsilon(x_\varepsilon)\right) = r\left(D_x g_\varepsilon(x_\varepsilon)\right)^p < 1.$$

The proof is complete. □

Finally, we generalize Lemma 2.2 as follows to get a practical global stability of symmetric and periodic solutions of (1.1) [12]. By using the change of variables [8]

$$x \longleftrightarrow x + \varepsilon\left(\int_0^t f(x,s)\,ds - \overline{f}(x)t\right),$$

equation (1.1) is transformed to the form

$$x'(t) = \varepsilon \overline{f}(x(t)) + \varepsilon^2 g(x(t),t,\varepsilon) \tag{4.16}$$

with a C^∞-smooth and pT-periodic g. Here $\overline{f}(x) := \frac{1}{pT}\int_0^{pT} f(x,s)ds$. Note that $A\overline{f}(x) = \overline{f}(Ax)$. Therefore, $A\overline{f}(0) = \overline{f}(0)$ and, thus, $\overline{f}(0) = 0$ because $1 \notin \sigma(A)$. Suppose the existence of a norm $|\cdot|_0$ on $\mathbb{R}^n$ and a constant $\alpha > 0$ such that

$$\mu(D\overline{f}(x)) < -\alpha \quad \text{for any } x \in \mathbb{R}^n, \tag{4.17}$$

where μ is the corresponding Lozinskii logarithmic norm. Then from the proof of Lemma 2.2 we see that equilibrium $z = 0$ is a global attractor for the system

$$\dot{z} = \varepsilon \overline{f}(z). \tag{4.18}$$

Now we are ready to prove the following results [28].

Theorem 4.5. *Suppose that (4.17) holds for a norm $|\cdot|_0$ on $\mathbb{R}^n$ and a constant $\alpha > 0$ with the corresponding Lozinskii logarithmic norm μ. Take $R_0 > R_1 > 0$ and put*

$$K_0 := \max_{|x|_0 \le R_0,\, |\varepsilon| \le 1,\, t \in \mathbb{R}} |g(x,t,\varepsilon)|_0,$$

and

$$K_1 := \max_{|x|_0 \le R_0,\, |\varepsilon| \le 1,\, t \in \mathbb{R}} \|D_x g(x,t,\varepsilon)\|_0.$$

If

$$0 < \varepsilon < \min\left\{1, \frac{R_0 - R_1}{K_0}\alpha, \frac{R_0\alpha}{3K_0}, \frac{K_1}{\alpha}\right\}, \tag{4.19}$$

then any solution of (4.16) starting from the ball $B(0,R_1)$ remains in $B(0,R_0)$ and tends asymptotically to a unique pT-periodic solution of (4.16).

Moreover, if $x(\cdot)$ and $z(\cdot)$ are solutions of (4.16) and (4.18) on $\mathbb{R}_+$ such that $x(0) = z(0)$ and $|x(0)| \le R_1$, then

$$|x(t) - z(t)|_0 \le \frac{K_0}{\alpha}\varepsilon \tag{4.20}$$

for all $t \in \mathbb{R}_+$.

Proof. We take ε satisfying (4.19). Let $x(t)$ be a solution of (4.16) such that $|x(0)| \le R_1$ and $|x(t)| \le R_0$ for all $t \in [0,\delta)$. Then from the proof of Lemma 2.2, we obtain

$$|x(t)|_0 \le \mathrm{e}^{-\alpha\varepsilon t} R_1 + \frac{\varepsilon K_0}{\alpha} \le R_1 + \frac{\varepsilon K_0}{\alpha} \le R_0$$

for any $t \in [0,\delta)$. Thus, in fact, $\delta = +\infty$. Since

$$\frac{\varepsilon K_0}{\alpha} \le \frac{R_0}{3} < \frac{R_0}{2},$$

all solutions starting from $B(0,R_1)$ terminate in $B(0,R_0/2)$ as $t \to \infty$. Next, again from the proof of Lemma 2.2, we obtain

$$|x_1(t)-x_2(t)|_0 \le \mathrm{e}^{-\varepsilon(\alpha-\varepsilon K_1)t}|x_1(t)-x_2(t)|_0 \le 2\mathrm{e}^{-\varepsilon(\alpha-\varepsilon K_1)t} R_1 \quad \forall t \ge 0$$

for any two solutions $x_1(t)$ and $x_2(t)$ of (4.16) starting in $B(0,R_1)$. Consequently, (4.16) has a unique pT-periodic solution in $B(0,R_1)$ and it is attracting all solutions starting in this ball. Finally, like above we see that any solution of (4.18) starting in ball $B(0,R_1)$ remains in $B(0,R_1)$ and it asymptotically tends to an equilibrium $z = 0$. Estimate (4.20) is derived similarly for any solutions $x(\cdot)$ and $z(\cdot)$ of (4.16) and (4.18) on $\mathbb{R}_+$ with $x(0) = z(0)$ and $|x(0)| \le R_1$. The proof is complete. □

We get immediately from Theorem 4.5 the following result.

Corollary 4.1. *Assume that (4.17) holds for a norm $|\cdot|_0$ on $\mathbb{R}^n$ and a constant $\alpha > 0$ with the corresponding Lozinskii logarithmic norm μ. Then for any bounded subset $\mathcal{S} \subset \mathbb{R}^n$ there are constants $\widetilde{K} > 0$ and $\widetilde{\varepsilon} > 0$ such that for any $0 < \varepsilon < \widetilde{\varepsilon}$, if $x(t)$ and $z(t)$ are solutions of (1.1) and (4.18) on $\mathbb{R}_+$ with $x(0) = z(0) \in \mathcal{S}$, then*

$$|x(t)-z(t)|_0 \le \widetilde{K}_0\varepsilon \tag{4.21}$$

for all $t \in \mathbb{R}_+$. Moreover, $x(t)$ asymptotically tends to a unique periodic and symmetric solution of (1.1).

Remark 4.1. Results similar to the estimates (4.20) and (4.21) above are obtained in the paper [27].

5. Averaging theory

In this section we survey classical results [5, 9, 18, 21, 22, 26, 27] on the local uniqueness and stability of pT-periodic solutions of equation (1.1). We already know that pT-periodic solutions are described by mapping (3.5). So, now we recall some known results for this mapping. Taking into account (3.9), we have

$$P_\varepsilon(x) = \varphi_0(x,pT) + \varepsilon\varphi_1(x,pT) + \cdots \tag{5.1}$$

From (3.6) we have

$$P_0(x) + \varepsilon P_1(x) + \cdots = x,$$

which, in view of the relation $P_0(x) = \varphi_0(x, pT) = x$, becomes $\varepsilon P_1(x) + \cdots = 0$. It is easy to see that $P_1(x) = \frac{d}{d\varepsilon} P_\varepsilon(x)|_{\varepsilon=0}$, and using (5.1) we get

$$P_1(x) = \frac{d}{d\varepsilon}\varphi_0(x, pT). \tag{5.2}$$

Similar as it is done in Section 4 we solve the problem (4.7), (4.8) and considering (5.2), one can get that

$$P_1(x) = \frac{d}{d\varepsilon} D\varphi_0(0, pT) = \int_0^{pT} f(x,s)ds \tag{5.3}$$

and

$$D_x P_1(x) = \int_0^{pT} D_x f(x,s)ds. \tag{5.4}$$

Now repeating the procedure of the previous sections, we are ready to state the following classical result.

Theorem 5.1. *If there exists $x_0 \in \mathbb{R}^n$ such that*

$$\int_0^{pT} f(x_0,s)ds = 0$$

and

$$\det \int_0^{pT} D_x f(x_0,s)ds \neq 0,$$

then, for any $\varepsilon \neq 0$ small, equation (1.1) has a unique pT-periodic solution $x(t) = x_0 + O(\varepsilon)$. Moreover, for $\varepsilon > 0$ small, the following statements are true:

(1) *If* $\operatorname{Re}\left\{\sigma\left(\int_0^{pT} D_x f(x_0,s)ds\right)\right\} \subset (-\infty, 0)$ *then $x(t)$ is asymptotically stable.*
(2) *If* $\operatorname{Re}\left\{\sigma\left(\int_0^{pT} D_x f(x_0,s)ds\right)\right\} \cap (0,\infty) \neq \varnothing$ *then $x(t)$ is unstable.*
(3) *If* $\operatorname{Re}\left\{\sigma\left(\int_0^{pT} D_x f(x_0,s)ds\right)\right\} \subset (0,\infty)$ *then $x(t)$ is a repeller.*
(4) *If* $\operatorname{Re}\left\{\sigma\left(\int_0^{pT} D_x f(x_0,s)ds\right)\right\} \cap \{0\} = \varnothing$ *then $x(t)$ is hyperbolic with the same hyperbolicity type as $\int_0^{pT} D_x f(x_0,s)ds$.*

Remark 5.1. By (1.2), (5.3), and (5.4), we have

$$AP_1(x) = P_1(Ax), \quad AD_x P_1(x) = D_x P_1(Ax)A \quad \forall x \in \mathbb{R}^n. \tag{5.5}$$

So if $P_1(x_0) = 0$ then $P_1(x_j) = 0$ and $\sigma(D_x P_1(x_j)) = \sigma(D_x P_1(x_0))$ for $x_j := A^j x_0$ and $j = 0, 1, \ldots, p-1$. Assuming $x_0 \neq 0$ from $1 \notin \sigma(A)$, there is the maximal $p_0 > 1$, $p_0|p$ such that x_j are different for $j = 0, 1, \ldots, p_0 - 1$. Then if

$$\det D_x P_1(x_0) \neq 0,$$

then, by Theorem 5.1, we have p_0 distinct pT-periodic solutions $x_j(t) = x_j + O(\varepsilon)$ for $j = 0, 1, \dots, p_0 - 1$. By (1.2), the function $\widetilde{x}_j(t) := A^j x_0(t - jT) = A^j x_0 + O(\varepsilon) = x_j + O(\varepsilon)$ is also a solution of (1.1). The uniqueness from Theorem 5.1 yields $\widetilde{x}_j(t) = x_j(t)$. Consequently, $x_j(t)$, $j = 0, 1, \dots, p_0 - 1$ are distinct solutions of (1.1), but geometrically they are A-symmetric to one other with same local asymptotic properties.

6. k-Hyperbolicity

To study hyperbolicity of periodic solutions of equation (1.1) we need the following results from [22].

Definition 6.1. A continuous matrix function $L_\varepsilon \colon \mathbb{R}^n \to \mathbb{R}^n$ of $\varepsilon \geq 0$ is *k-hyperbolic* if, for every matrix function N_ε defined for $\varepsilon \geq 0$ satisfying $N_\varepsilon = o(\varepsilon^k)$, there exists an interval $0 < \varepsilon < \varepsilon_1$ in which $L_\varepsilon + N_\varepsilon$ is hyperbolic of the same type (i. e., with the same number of eigenvalues on each side of the unit circle).

Definition 6.2. A continuous matrix function $L_\varepsilon \colon \mathbb{R}^n \to \mathbb{R}^n$ of $\varepsilon \geq 0$ is *strongly k-hyperbolic* if there exists a continuous real matrix C_ε defined in an interval $0 \leq \varepsilon < \varepsilon_0$ such that C_ε is regular (even for $\varepsilon = 0$) and such that

$$C^{-1} L_\varepsilon C_\varepsilon = \begin{pmatrix} A_\varepsilon & 0 \\ 0 & B_\varepsilon \end{pmatrix}$$

for $0 < \varepsilon < \varepsilon_0$, where A_ε and B_ε are $r \times r$ and $s \times s$ blocks, respectively, and $\|A_\varepsilon\| < 1 - c\varepsilon^k$, $\|B_\varepsilon^{-1}\| < 1 - c\varepsilon^k$ for some $c > 0$.

Now repeating proofs of Theorems 2.2 and 2.4 of [22], we arrive at the following generalizations.

Theorem 6.1. *Strong k-hyperbolicity implies k-hyperbolicity.*

Theorem 6.2. *If $L_\varepsilon = L_0 + \varepsilon L_1 + \dots + \varepsilon^k L_k$, if the eigenvalues of L_0 are distinct numbers on the unit circle, and if the eigenvalues $\lambda_i(\varepsilon)$ of L_ε suitably numbered satisfy $|\lambda_i(\varepsilon)| < 1 - c\varepsilon^k$ for $i = 1, \dots r$, $|\lambda_i(\varepsilon)| > 1 - c\varepsilon^k$ for $i = r + 1, \dots, n$, for some constant $c > 0$ and $\varepsilon > 0$ small, then L_ε is strongly k-hyperbolic.*

Remark 6.1. A difference between our Theorem 6.2 and [22, Theorem 2.2] is that it is supposed $L_\varepsilon = \mathbb{1} + \varepsilon L_1 + \dots + \varepsilon^k L_k$ and the eigenvalues of L_1 are distinct in [22, Theorem 2.2].

Now we can improve Theorem 4.4 as follows.

Theorem 6.3. *Suppose $1 \notin \sigma(A)$ and all eigenvalues of A are distinct complex numbers on the unit circle. Let $x_\varepsilon = O(\varepsilon)$ be the unique solution of (3.10). If*

$$D_x g_\varepsilon(x_\varepsilon) = G_{\varepsilon,k} + o(\varepsilon^k) := A^{-1} + \varepsilon G_1 + \dots + \varepsilon^k G_k + o(\varepsilon^k)$$

with a k-hyperbolic matrix function $G_{\varepsilon,k}$ then $D_x g_\varepsilon(x_\varepsilon)$ is hyperbolic of the same type as $G_{\varepsilon,k}$. In particular, $G_{\varepsilon,k}$ is k-hyperbolic if it fulfills assumptions of Theorem 6.2.

According to (4.2), from Theorem 6.3, we obtain the following consequence.

Theorem 6.4. *Suppose $1 \notin \sigma(A)$ and all eigenvalues of A are distinct complex numbers on the unit circle. If the eigenvalues $\lambda_i(\varepsilon)$ of $A^{-1} + \varepsilon G_1 + \cdots + \varepsilon^k G_k$ suitably numbered satisfy $|\lambda_i(\varepsilon)| < 1 - c\varepsilon^k$ for $i = 1, \dots r$, $|\lambda_i(\varepsilon)| > 1 - c\varepsilon^k$ for $i = r+1, \dots, n$, for some constant $c > 0$ and $\varepsilon > 0$ small.*

Then the unique symmetric and T-periodic solution of (1.1) is hyperbolic for any $\varepsilon > 0$ small.

7. Applications to weakly nonlinear problems

In this section, we present two concrete weakly nonlinear ordinary differential equations to illustrate our theory.

7.1. *A scalar problem*

Let us consider a scalar differential equation

$$x' = \varepsilon(x - x^3 + \sin t), \quad x \in \mathbb{R}, \tag{7.1}$$

with $T = \pi$, $p = 2$ and $f(x,t) = x - x^3 + \sin t$, $Ax = -x$. Indeed, we verify condition (1.2):

$$\begin{aligned} Af(x,t) &= -f(x,t) = -(x - x^3 + \sin t) = \\ &= (-x) - (-x^3) + \sin(t + \pi) = f(-x, t+\pi) = f(Ax, t+\pi). \end{aligned}$$

Let us establish the stability of the pT-periodic solution of the symmetric problem (1.1), (2.2). Using (5.3) we get

$$P_1(x) = \int_0^{2\pi} (x - x^3 + \sin t)dt = 2\pi(x - x^3).$$

Solving the equation $P_1(x) = 0$ we find that

$$x = 0, \qquad x = -1, \qquad x = 1.$$

It is easy to see, using (5.4), that

$$P_1'(x) = 2\pi(1 - 3x^2),$$

$$P_1'(-1) < 0, \quad P_1'(0) > 0, \quad P_1'(1) < 0.$$

From Theorem 5.1 we know that $x_0(t) = O(\varepsilon)$, $x_\pm(t) = \pm 1 + O(\varepsilon)$ are the only 2π-periodic solutions of the problem (7.1) for $\varepsilon > 0$ small, while $x_0(t)$ is unstable (a repeller) and $x_\pm(t)$ are asymptotically stable. Note by Remark 5.1, curves $x_\pm(t)$

are symmetric with respect to the origin to each others. Furthermore, from Theorem 3.2 we know that $x_0(t)$ is the only π-antiperiodic solution of (7.1). Now using Theorem 4.4 together with a calculation

$$\int_0^{\pi} f'(0,s)ds = \pi > 0,$$

we reverify its instability.

7.2. *A planar problem*

Let us consider a planar differential system

$$\begin{aligned} x_1' &= \varepsilon\left(f_1(x_1,x_2)+h_1(t)\right), \\ x_2' &= \varepsilon\left(f_2(x_1,x_2)+h_2(t)\right) \end{aligned} \tag{7.2}$$

with smooth functions $f_{1,2}$, $h_{1,2}$, $T=\pi/2$, $p=4$, and with

$$A=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{7.3}$$

Then symmetry condition (1.2) implies

$$f_1(x_1,x_2)=f_2(-x_2,x_1), \qquad f_2(x_1,x_2)=-f_1(-x_2,x_1), \tag{7.4}$$

and

$$h_1\left(t+\frac{\pi}{2}\right)=-h_2(t), \qquad h_2\left(t+\frac{\pi}{2}\right)=h_1(t). \tag{7.5}$$

Note that (7.5) gives $h_{1,2}(t+\pi)=-h_{1,2}(t)$ and, thus,

$$\int_0^{2\pi} h_{1,2}(t)dt=0.$$

Consequently, (5.3) now has the form

$$P_1(x)=2\pi\left(f_1(x_1,x_2),f_2(x_1,x_2)\right), \tag{7.6}$$

where $x=(x_1,x_2)$. Symmetry conditions (7.4) are satisfied, e. g., for the polynomials

$$\begin{aligned} f_1(x_1,x_2) &= a_0x_1+b_0x_2+\sum_{\substack{j,k=0,\,j+k\geq 3\\ 2|j+k+1}}^{m}\left(a_{jk}x_1^jx_2^k+b_{kj}x_1^kx_2^j\right), \\ f_2(x_1,x_2) &= -b_0x_1+a_0x_2 \\ &\quad -\sum_{\substack{j,k=0,\,j+k\geq 3\\ 2|j+k+1}}^{m}\left((-1)^kb_{kj}x_1^jx_2^k+(-1)^ja_{jk}x_1^kx_2^j\right). \end{aligned} \tag{7.7}$$

Hence $D_x P_1(0) = \left(\begin{smallmatrix} a_0 & b_0 \\ b_0 & a_0 \end{smallmatrix}\right)$. Then from Theorem 5.1 we know that $x_0(t) = O(\varepsilon)$ is the only symmetric and $\pi/2$-periodic solution of (7.2) which is in addition asymptotically stable (a repeller) if $a_0 < 0$ ($a_0 > 0$), respectively, for $\varepsilon > 0$ small. We need to compute higher-order terms for $a_0 = 0$, but it is not carried out here. Since in general polynomials (7.7) are difficult to handle, we consider the following particular cases.

7.2.1. *Degenerate cases*

If $a_0 = b_0 = 0$ then $D_x P_1(0) = 0$ and so Theorem 5.1 is not applicable. But Theorem 3.2 still guarantees the existence and uniqueness of $x_0(t)$. This is one of advantages of our results. For instance, consider

$$\begin{aligned} x_1' &= \varepsilon\left(a_j x_1^j + b_j x_2^j + h_1(t)\right), \\ x_2' &= \varepsilon\left(-b_j x_1^j + a_j x_2^j + h_2(t)\right), \end{aligned} \tag{7.8}$$

where $a_j, b_j \in \mathbb{R}\setminus\{0\}$, $j \in \mathbb{N}$, $j \geq 3$ is odd and $h_{1,2} \neq 0$ satisfy (7.5). The linearization of (7.8) along $x_0(t) = \big(x_{1,0}(t), x_{2,0}(t)\big)$ has the form

$$\begin{aligned} x_1' &= \varepsilon\left(ja_j x_{1,0}(t)^{j-1}x_1 + jb_j x_{2,0}(t)^{j-1}x_2\right), \\ x_2' &= \varepsilon\left(-jb_j x_{1,0}(t)^{j-1}x_1 + ja_j x_{2,0}(t)^{j-1}x_2\right). \end{aligned} \tag{7.9}$$

Let $X_\varepsilon(t)$ be the fundamental matrix solution of (7.9). Then in notations of Sections 4 and 5, we have $D_x g_\varepsilon(x_\varepsilon) = A^{-1}X_\varepsilon(\pi/2)$ and $D_x P_\varepsilon(x_\varepsilon) = X_\varepsilon(2\pi)$ for $x_\varepsilon = x_0(0)$. Since

$$\sigma\left(A^{-1}X_0(\pi/2)\right) = \left\{\mathrm{e}^{\pm\pi\imath/2}\right\},$$

it follows that $\sigma\left(A^{-1}X_\varepsilon(\pi/2)\right) = \left\{\lambda_\varepsilon, \overline{\lambda}_\varepsilon\right\}$ is close to $\left\{\mathrm{e}^{\pm\pi\imath/2}\right\}$. Let $\sigma\left(X_\varepsilon(2\pi)\right) = \{\lambda_{\varepsilon,1}, \lambda_{\varepsilon,2}\}$. Then, due to (4.2), we have $\lambda_{\varepsilon,1} = \lambda_\varepsilon^4$ and $\lambda_{\varepsilon,2} = (\overline{\lambda}_\varepsilon)^4 = \overline{\lambda_\varepsilon^4} = \overline{\lambda}_{\varepsilon,1}$. On the other hand, by the Liouville theorem [11], we know that

$$\lambda_{\varepsilon,1}\lambda_{\varepsilon,2} = \det X_\varepsilon(2\pi) = \mathrm{e}^{\displaystyle\int_0^{2\pi} ja_j\left(x_{1,0}(t)^{j-1} + x_{2,0}(t)^{j-1}\right)dt}.$$

Therefore, since $\int_0^{2\pi}\left(x_{1,0}(t)^{j-1} + x_{2,0}(t)^{j-1}\right)dt > 0$, we see that $|\lambda_{\varepsilon,1}| = |\lambda_{\varepsilon,2}| < 1$ for $a_j < 0$ and $|\lambda_{\varepsilon,1}| = |\lambda_{\varepsilon,2}| > 1$ for $a_j > 0$. Furthermore, the relation $P_1(x) = 0$ has the form

$$a_j x_1^j + b_j x_2^j = 0, \quad -b_j x_1^j + a_j x_2^j = 0, \tag{7.10}$$

which gives $(x_1x_2)^j(a_j^2 + b_j^2) = 0$. Thus, (7.10) has only zero solution $x_1 = x_2 = 0$. Summarizing, we have the following result.

Theorem 7.1. *Weakly nonlinear system (7.8) has the only 2π-periodic solution $x_0(t) = O(\varepsilon)$ for $\varepsilon > 0$ small, which is moreover $-\pi/2$-symmetric, i. e., has the property*

$$x_0(t+\pi/2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x_0(t) \quad \forall t \in \mathbb{R}. \tag{7.11}$$

If $a_j < 0$ then $x_0(t)$ is globally asymptotically stable, and if $a_j > 0$ then $x_0(t)$ is a global repeller.

Proof. We concentrate on the case $a_j < 0$, since the case $a_j > 0$ can be handled similarly. We already know that (7.8) possesses a solution $x_0(t)$ which is asymptotically stable. Now we apply Lemma 2.5. By (2.11), we have

$$J(x,t) = ja_j\varepsilon \begin{pmatrix} x_1^{j-1} & 0 \\ 0 & x_2^{j-1} \end{pmatrix},$$

which clearly satisfies (2.13). The proof is finished. □

7.2.2. *Nondegenerate cases*

First we consider a linear perturbation of (7.8) of the form

$$\begin{aligned} x_1' &= \varepsilon\left(a_0x_1 + b_0x_2 + a_jx_1^j + h_1(t)\right) \\ x_2' &= \varepsilon\left(-b_0x_1 + a_0x_2 + a_jx_2^j + h_2(t)\right), \end{aligned} \tag{7.12}$$

where $a_0, b_0, a_j \in \mathbb{R}$, $j \in \mathbb{N}$, $j \geq 3$ is odd and $h_{1,2}$ satisfy (7.5). Then

$$J(x,t) = \varepsilon \begin{pmatrix} a_0 + ja_jx_1^{j-1} & 0 \\ 0 & a_0 + ja_jx_2^{j-1} \end{pmatrix}$$

which, clearly, satisfies relation (2.12) for $a_0 < 0$, $a_j \leq 0$ and any $\varepsilon > 0$. Thus, applying Lemma 2.4, we obtain the following result.

Theorem 7.2. *Assume $a_0 < 0$, $a_j \leq 0$. Then nonlinear system (7.12) has the only 2π-periodic solution $x_0(t) = O(\varepsilon)$ for any $\varepsilon > 0$, which is moreover $-\pi/2$-symmetric, i. e., it satisfies (7.11) and it is globally asymptotically stable.*

Remark 7.1. Except of Theorem 7.2 in this paper, we do not know anything about dynamics of (1.1) as ε is increasing.

Finally, to get further interesting results, we consider the following simple form of (7.7):

$$\begin{aligned} f_1(x_1,x_2) &= x_1 + x_2 + ax_1^2x_2 + bx_1x_2^2, \\ f_2(x_1,x_2) &= -x_1 + x_2 + bx_1^2x_2 - ax_1x_2^2, \end{aligned} \tag{7.13}$$

where $a, b \in \mathbb{R}$ are constants with $(a,b) \neq (0,0)$. Hence $x_0(t)$ is a repeller for $\varepsilon > 0$ small. We intend to find more 2π-periodic solutions of (7.2) with (7.13). For this reason, we solve

$$\begin{aligned} x_1 + x_2 + a x_1^2 x_2 + b x_1 x_2^2 &= 0, \\ -x_1 + x_2 + b x_1^2 x_2 - a x_1 x_2^2 &= 0, \end{aligned} \tag{7.14}$$

which gives

$$x_2\left(2 + (a+b)x_1^2 - (a-b)x_1\right) = 0. \tag{7.15}$$

For $x_2 = 0$ from (7.14) we derive $x_1 = 0$ which gives $x_0(t)$. If $x_2 \neq 0$ and $a = b \geq 0$ then (7.15) has no more solutions. So we suppose $a \neq b$ and then (7.15) gives $x_1 \neq 0$ along with

$$x_2 = \frac{(a+b)x_2^2 + 2}{(a-b)x_1}. \tag{7.16}$$

Inserting (7.16) into the first equation of (7.14), after some computation, we arrive at

$$2(a+b) + 4(a^2+b^2)x_1^2 + (a+b)(a^2+b^2)x_1^4 = 0. \tag{7.17}$$

If $a + b = 0$ then (7.17) gives $x_1 = 0$ and so we again obtain $x_2 = 0$. Hence we suppose $a + b \neq 0$, and then (7.17) gives

$$\begin{aligned} x_{1,+}^2 &= \frac{\sqrt{2}(a-b) - 2\sqrt{a^2+b^2}}{(a+b)\sqrt{a^2+b^2}}, \\ x_{1,-}^2 &= -\frac{\sqrt{2}(a-b) + 2\sqrt{a^2+b^2}}{(a+b)\sqrt{a^2+b^2}}. \end{aligned} \tag{7.18}$$

It is elementary to observe

$$\sqrt{2}(a-b) - 2\sqrt{a^2+b^2} < 0, \quad \sqrt{2}(a-b) + 2\sqrt{a^2+b^2} > 0$$

whenever $a + b \neq 0$. So (7.18) has a real solution if and only if $a + b < 0$, which is from now supposed. Then by (7.16), we get four solutions of (7.14). But since according to (7.4), system of equations (7.14) is equivariant with respect to the symmetry (7.3), these solutions are rotationally symmetric (see also Remark 5.1). So we concentrate on the first solution given by the formulae

$$\begin{aligned} x_{1,+} &= \sqrt{\frac{\sqrt{2}(a-b) - 2\sqrt{a^2+b^2}}{(a+b)\sqrt{a^2+b^2}}}, \\ x_{2,+} &= \sqrt{\frac{2(a+b)}{\sqrt{a^2+b^2}\left(\sqrt{2}(a-b) - 2\sqrt{a^2+b^2}\right)}}. \end{aligned} \tag{7.19}$$

Then by (7.6) and (7.13) we get

$$D_x P_1(x_{1,+}, x_{2,+}) = \frac{2\pi}{(a+b)\sqrt{a^2+b^2}} \begin{pmatrix} u & v \\ w & w \end{pmatrix},$$

where

$$u = (a-b)\sqrt{a^2+b^2} + \sqrt{2}\left(2a^2+ab+b^2\right),$$
$$v = (b-a)\sqrt{a^2+b^2} + \sqrt{2}\left(a^2+ab+2b^2\right),$$

and

$$w = (a-b)\sqrt{a^2+b^2} + \sqrt{2}\left(a^2+ab+2b^2\right).$$

It is interesting to note $\det D_x P_1(x_{1,+}, x_{2,+}) = -16\pi$. The eigenvalues of the matrix $D_x P_1(x_{1,+}, x_{2,+})$ are given by the formulae

$$\lambda_\pm = 2\pi \frac{a-b \mp \sqrt{9a^2+14ab+9b^2}}{a+b}.$$

Since $a+b<0$, we see that $\lambda_+ > 0$ and $\lambda_- < 0$. Summarizing, we obtain the following result.

Theorem 7.3. *Consider the system*

$$\begin{aligned} x_1' &= \varepsilon\left(x_1+x_2+ax_1^2x_2+bx_1x_2^2+h_1(t)\right), \\ x_2' &= \varepsilon\left(-x_1+x_2+bx_1^2x_2-ax_1x_2^2+h_2(t)\right), \end{aligned} \tag{7.20}$$

where $a,b \in \mathbb{R}$ are constants with $(a,b) \neq (0,0)$, $h_{1,2}$ are continuous functions satisfying (7.5) and $\varepsilon > 0$ is a small parameter. Then:

(1) *If $a+b \geq 0$ then (7.20) has the only 2π-periodic solution $x_0(t) = O(\varepsilon)$ which is in addition a repeller and $-\pi/2$-symmetric, i. e., it satisfies (7.11).*
(2) *If $a+b<0$ then (7.20) has in addition precisely four more 2π-periodic solutions $x_j(t)$, $j=1,2,3,4$ which are hyperbolic with the same hyperbolicity type and orbitally $-\pi/2$-symmetric to each others.*

Proof. The proof follows immediately from the above computations together with application of Theorem 5.1 and Remark 5.1. □

8. Applications to planar weakly linear problems

We consider

$$x' = \varepsilon B(t)x, \tag{8.1}$$

where $B(t) = \left(B_{ik}(t)\right)_{i,k=1}^2$ is a smooth 2×2-matrix function, 2π-periodic in $t \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$ is a small real parameter.

We first study (8.1) without symmetries using standard approach [8, 9, 13, 19, 21, 22, 29]. If $X_\varepsilon(t)x = \varphi_\varepsilon(x,t)$ is the Cauchy solution of (8.1), where $X_\varepsilon(t)$ is its fundamental matrix solution, then we expand $X_\varepsilon(t)$ in the form

$$X_\varepsilon(t) = \sum_{i\geq 0} Y_i(t)\varepsilon^i, \quad Y_0(0) = \mathbb{1}, \quad Y_i(0) = 0 \quad \forall i \geq 1 \tag{8.2}$$

to derive from (8.1) the recurrence formulas

$$Y_{i+1}(t) = \int_0^t B(s)Y_i(s)ds, \quad Y_0(t) = \mathbb{1}.$$

Thus, we have

$$X_\varepsilon(2\pi) = \mathbb{1} + \varepsilon \int_0^{2\pi} B(t)dt + \varepsilon^2 \int_0^{2\pi} \int_0^t B(t)B(s)dsdt + O(\varepsilon^3). \tag{8.3}$$

If $\int_0^{2\pi} B(t)dt \neq 0$ then we can apply results of Section 6 (see Remark 6.1). Here we concentrate on the degenerate cases by supposing (see also [9])

$$\int_0^{2\pi} B(t)dt = 0. \tag{8.4}$$

Then (8.3) becomes

$$X_\varepsilon(2\pi) = \mathbb{1} + \varepsilon^2 \int_0^{2\pi} \int_0^t B(t)B(s)dsdt + O(\varepsilon^3). \tag{8.5}$$

The results of [19, 21, 22] are difficult to apply to (8.5) (see Remark 6.1). We show that if we consider some symmetries to (8.1), then our theory helps to study such degenerate cases. Now $f(x,t) = B(t)x$, so symmetry condition (1.2) gives

$$AB(t) = B(t+T)A, \tag{8.6}$$

where $A^p = \mathbb{1}$ for some $p \in \mathbb{N}$, $p \geq 2$, and $T = 2\pi/p$. Then we have $X_\varepsilon(iT) = A^i \left(A^{-1}X_\varepsilon(T)\right)^i$ for all $i \geq 1$, which implies

$$\left(A^{-1}X_\varepsilon\left(\frac{2\pi}{p}\right)\right)^p = X_\varepsilon(2\pi). \tag{8.7}$$

We already know (8.7) from Lemma 3.2. In spite of the fact that $Y_1(2\pi) = 0$, in many cases we could have $Y_1(T) \neq 0$. Thus, using (8.7), we could still study the existence and stability of the zero solution $x = 0$ of (8.1) also in the degenerate case (8.4).

Finally, it is better to pass in computations to the complex variable $z = x_1 + \imath x_2 \in \mathbb{C}$ for $x = (x_1, x_2) \in \mathbb{R}^2$. Then (8.1) has the form

$$\dot{z} = \varepsilon(a(t)z + b(t)\bar{z}) \tag{8.8}$$

in which

$$\begin{aligned} a(t) &= \frac{B_{11}(t) + B_{22}(t)}{2} + \imath \frac{B_{21}(t) - B_{12}(t)}{2}, \\ b(t) &= \frac{B_{11}(t) - B_{22}(t)}{2} + \imath \frac{B_{12}(t) + B_{21}(t)}{2}, \end{aligned} \tag{8.9}$$

and $\bar{z} = x_1 - \imath x_2$. Next we again expand

$$z_\varepsilon(t) = \sum_{i \geq 0} w_i(t)\varepsilon^i, \quad w_0(0) = w, \quad w_i(0) = 0 \quad \forall i \geq 1 \tag{8.10}$$

to derive from (8.8) the recurrence formulas

$$w_{i+1}(t)=\int_0^t \left(a(s)w_i(s)+b(s)\bar{w}_i(s)\right)ds,\quad w_0(t)=w. \tag{8.11}$$

In the next sections, we demonstrate the above approach by considering two different classes of symmetries A of (8.1).

8.1. *Rotation symmetries*

We consider the rotation $Az=\mathrm{e}^{\frac{2\pi}{n}\imath}z$, so $p=n$ and $T=\frac{2\pi}{n}$. Note $n\in\mathbb{N}$ and $n\geq 3$. Now $f(z,t)=a(t)z+b(t)\bar{z}$, so the symmetry condition (1.2) (see also (8.6)) implies

$$a(t+T)=a(t),\quad b(t+T)=\mathrm{e}^{\frac{4\pi}{n}\imath}b(t). \tag{8.12}$$

Note that (8.4) gives

$$\int_0^{2\pi} a(t)dt=0,\quad \int_0^{2\pi} b(t)dt=0. \tag{8.13}$$

So both (8.12) and (8.13) imply $\int_0^T a(t)dt=0$. But we need either $\int_0^T a(t)dt\neq 0$ or $\int_0^T b(t)dt\neq 0$. For this reason, we further consider $f(z,t)=b(t)\bar{z}$. Next, expanding $b(t)=\sum_{k\in\mathbb{Z}\setminus\{0\}} b_k\,\mathrm{e}^{k\imath t}$ and using (8.12) we derive

$$b(t)=\sum_{m\in\mathbb{Z}} b_m\,\mathrm{e}^{(2+mn)\imath t}. \tag{8.14}$$

Then

$$\widetilde{b}=\int_0^T b(t)dt=\int_0^{2\pi/n} b(t)dt=\left(\mathrm{e}^{\frac{4\pi}{n}\imath}-1\right)\sum_{m\in\mathbb{Z}}\frac{b_m}{(2+mn)\imath}.$$

In general $\widetilde{b}\neq 0$, so by (4.12) we consider the linear map $z\mapsto \widetilde{b}\bar{z}$, which for $\widetilde{b}=\widetilde{b}_1+\imath\widetilde{b}_2$ and $z=x_1+\imath x_2$ has the form

$$(x_1,x_2)\mapsto \begin{pmatrix}\widetilde{b}_1 & \widetilde{b}_2\\ \widetilde{b}_2 & -\widetilde{b}_1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}.$$

However, the matrix $\begin{pmatrix}\widetilde{b}_1 & \widetilde{b}_2\\ \widetilde{b}_2 & -\widetilde{b}_1\end{pmatrix}\neq 0$ is always hyperbolic and not asymptotically stable. So we can not apply Theorem 4.1. Consequently, we get now only that the equation

$$\dot{z}=\varepsilon b(t)\bar{z} \tag{8.15}$$

has zero solution the only symmetric one for $\varepsilon \neq 0$ small, when $b(t)$ is given by (8.14). Furthermore, by (8.2) we obtain

$$A^{-1}X_\varepsilon\left(\frac{2\pi}{n}\right) = A^{-1}\left(\mathbb{1}+\varepsilon Y_1\left(\frac{2\pi}{n}\right)\right)+O(\varepsilon^2)$$
$$= \begin{pmatrix} \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} \\ -\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix}\begin{pmatrix} 1+\varepsilon\widetilde{b}_1 & \varepsilon\widetilde{b}_2 \\ \varepsilon\widetilde{b}_2 & 1-\varepsilon\widetilde{b}_1 \end{pmatrix}+O(\varepsilon^2).$$

The eigenvalues of $A^{-1}\left(\mathbb{1}+\varepsilon Y_1\left(\frac{2\pi}{n}\right)\right)$ are given by

$$\lambda_\pm = \cos\frac{2\pi}{n} \pm \frac{\imath}{\sqrt{2}}\sqrt{1-\cos\frac{4\pi}{n}-2|\widetilde{b}|^2\varepsilon^2}$$

for ε small with

$$|\lambda_\pm| = \sqrt{1-|\widetilde{b}|^2\varepsilon^2} = 1-\frac{|\widetilde{b}|^2}{2}\varepsilon^2+O(\varepsilon^3).$$

So the matrix $A^{-1}\left(\mathbb{1}+\varepsilon Y_1\left(\frac{2\pi}{n}\right)\right)$ is by Theorem 6.2 strongly 2-hyperbolic, but we should have its strongly 1-hyperbolicity in order to apply results of Section 6 concerning the hyperbolicity of the zero solution. We should need to compute higher order terms in (8.2).

Next, by (8.11) and (8.13), we get

$$z_\varepsilon(2\pi) = \left(1+\varepsilon^2\int_0^{2\pi}\int_0^t b(t)\overline{b}(s)dsdt\right)w+O\left(\varepsilon^3\right),$$

and using (8.14), we derive

$$z_\varepsilon(2\pi) = \left(1+\varepsilon^2\vartheta\imath\right)w+O\left(\varepsilon^3\right)$$

for

$$\vartheta = 2\pi\sum_{m\in\mathbb{Z}}\frac{|b_m|^2}{2+mn},$$

which is a nonzero constant for $b(t)\neq 0$. In the variable x, we get

$$X_\varepsilon(2\pi) = B_2(\varepsilon)+O(\varepsilon^3) \tag{8.16}$$

for

$$B_2(\varepsilon) = \begin{pmatrix} 1 & -\varepsilon^2\vartheta \\ \varepsilon^2\vartheta & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\varepsilon^2\begin{pmatrix} 0 & -\vartheta \\ \vartheta & 0 \end{pmatrix}.$$

Since

$$1\notin\sigma\left(\begin{pmatrix} 0 & -\vartheta \\ \vartheta & 0 \end{pmatrix}\right),$$

equation (8.15) has the only zero 2π-periodic solution. Note the eigenvalues of $B_2(\varepsilon)$ are $1 \pm \varepsilon^2 \vartheta\imath$. Furthermore, we see that

$$\left\| B_2(\varepsilon)^{-1} \right\| = \frac{1}{\sqrt{1+\varepsilon^4\vartheta^2}} \leq 1 - \frac{\vartheta^2}{3}\varepsilon^4$$

for $\varepsilon \neq 0$ small, so by Definition 6.2, $B_2(\varepsilon)$ is strongly 4-hyperbolic, but we should have its strongly 2-hyperbolicity in order to apply results of Section 6 concerning the hyperbolicity of the zero solution. Consequently, the expansion approach (8.2) seems to be difficult for application to (8.15), and hence our approach is as follows. Since $\sigma\left(A^{-1}X_0\left(\frac{2\pi}{n}\right)\right) = \left\{\mathrm{e}^{\pm 2\pi\imath/n}\right\}$, so $\sigma\left(A^{-1}X_\varepsilon\left(\frac{2\pi}{n}\right)\right) = \left\{\lambda_\varepsilon, \overline{\lambda}_\varepsilon\right\}$ is close to $\left\{\mathrm{e}^{\pm 2\pi\imath/n}\right\}$ for ε small. Next, (8.15) has the form

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \varepsilon \begin{pmatrix} b_1(t) & b_2(t) \\ b_2(t) & -b_1(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Hence $\sigma\left(X_\varepsilon(2\pi)\right) = \{\lambda_{\varepsilon,1}, \lambda_{\varepsilon,2}\}$ with $\lambda_{\varepsilon,1}\lambda_{\varepsilon,2} = 1$. But (8.7) gives $\lambda_{\varepsilon,1} = \lambda_\varepsilon^p$ and $\lambda_{\varepsilon,2} = \overline{\lambda}_\varepsilon^p = \overline{\lambda}_{\varepsilon,1}$. This implies that $|\lambda_{\varepsilon,1}| = |\lambda_{\varepsilon,2}| = 1$. Summarizing, we obtain the following result.

Theorem 8.1. *Assume* (8.13) *and* $b(t) \neq 0$ *is given by* (8.14)*. Then the zero solution is the only symmetric one of equation* (8.15) *for* $\varepsilon \neq 0$ *small. It is stable but not asymptotically stable. Moreover, it is the only* 2π*-periodic solution of* (8.15).

We see that simple arguments using symmetries of (8.15) provide the stability of this equation, while expansion method seems to be rather awkward.

8.2. *Reflection symmetries*

Now we consider $Az = \mathrm{e}^{\frac{\pi}{n}\imath}\overline{z}$ with $p = 2n$ and $T = \frac{\pi}{n}$. Note that A has the form

$$A = \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} \\ -\sin\frac{\pi}{n} & -\cos\frac{\pi}{n} \end{pmatrix},$$

satisfies the relation $A^2 = \mathbb{1}$, and has the eigenvalues 1 and -1 and the corresponding orthogonal eigenvectors $\left(\sin\frac{\pi}{n}, \cos\frac{\pi}{n} - 1\right)$ and $\left(\sin\frac{\pi}{n}, 1 + \cos\frac{\pi}{n}\right)$, respectively. Thus, A is the reflection with respect to the line $x_2 = -\tan\left(\frac{\pi}{2n}\right)x_1$.

Next, the symmetry condition (1.2) (see also (8.6)) gives

$$a(t+T) = \overline{a}(t), \quad b(t+T) = \mathrm{e}^{\frac{2\pi}{n}\imath}\overline{b}(t). \tag{8.17}$$

Using the above approach, we derive

$$a(t) = \sum_{k\in\mathbb{N}} \left(a_k + \overline{a}_k\,\mathrm{e}^{k\frac{\pi}{n}\imath}\right)\mathrm{e}^{kt\imath} \tag{8.18}$$

and

$$b(t) = \sum_{k\in\mathbb{N}} \left(b_k + \overline{b}_k\,\mathrm{e}^{(2+k)\frac{\pi}{n}\imath}\right)\mathrm{e}^{kt\imath}. \tag{8.19}$$

Then

$$\hat{a} = \int_0^T a(t)dt = \int_0^{\pi/n} a(t)dt = \sum_{k\in\mathbb{N}} \left(a_k + \bar{a}_k \mathrm{e}^{k\frac{\pi}{n}\imath}\right) \frac{\mathrm{e}^{k\frac{\pi}{n}\imath}-1}{k\imath} \tag{8.20}$$

and

$$\hat{b} = \int_0^T b(t)dt = \int_0^{\pi/n} b(t)dt = \sum_{k\in\mathbb{N}} \left(b_k + \bar{b}_k \mathrm{e}^{(2+k)\frac{\pi}{n}\imath}\right) \frac{\mathrm{e}^{k\frac{\pi}{n}\imath}-1}{k\imath}. \tag{8.21}$$

In general $\hat{a} \neq 0$ and $\hat{b} \neq 0$, so by (4.12) we consider the linear map $z \mapsto \hat{a}z + \hat{b}\bar{z}$.

Now we want to show that for any given $\hat{a}, \hat{b} \in \mathbb{C}$, we can find $a(t), b(t)$ satisfying (8.20) and (8.21). For this reason, we first derive

$$\begin{aligned}\left(a_k + \bar{a}_k \mathrm{e}^{k\frac{\pi}{n}\imath}\right) \frac{\mathrm{e}^{k\frac{\pi}{n}\imath}-1}{k\imath} &= \left(a_k \mathrm{e}^{-k\frac{\pi}{2n}\imath} + \bar{a}_k \mathrm{e}^{k\frac{\pi}{2n}\imath}\right) \mathrm{e}^{k\frac{\pi}{n}\imath} \frac{\mathrm{e}^{k\frac{\pi}{2n}\imath} - \mathrm{e}^{-k\frac{\pi}{2n}\imath}}{k\imath} \\ &= \left(4\,\mathrm{Re}\left\{a_k \mathrm{e}^{-k\frac{\pi}{2n}\imath}\right\} \frac{\sin k\frac{\pi}{2n}}{k}\right) \mathrm{e}^{k\frac{\pi}{n}\imath}.\end{aligned}$$

So the range of the mapping

$$\mathbb{C} \ni a_k \mapsto \left(a_k + \bar{a}_k \mathrm{e}^{k\frac{\pi}{n}\imath}\right) \frac{\mathrm{e}^{k\frac{\pi}{n}\imath}-1}{k\imath} \in \mathbb{C} \sim \mathbb{R}^2$$

in the plain $\mathbb{R}^2$ is the line $x_2 = \tan k\frac{\pi}{n} x_1$. Consequently, the mapping

$$\mathbb{C}\times\mathbb{C} \ni (a_1,a_2) \mapsto \left(a_1 + \bar{a}_1 \mathrm{e}^{\frac{\pi}{n}\imath}\right) \frac{\mathrm{e}^{\frac{\pi}{n}\imath}-1}{\imath} + \left(a_2 + \bar{a}_2 \mathrm{e}^{2\frac{\pi}{n}\imath}\right) \frac{\mathrm{e}^{2\frac{\pi}{n}\imath}-1}{2\imath} \in \mathbb{C}$$

is onto (surjective). Hence, for any $\hat{a} \in \mathbb{C}$, there are $a_1, a_2 \in \mathbb{C}$ such that (8.20) holds with

$$a(t) = \left(a_1 + \bar{a}_1 \mathrm{e}^{\frac{\pi}{n}\imath}\right) \mathrm{e}^{t\imath} + \left(a_2 + \bar{a}_2 \mathrm{e}^{2\frac{\pi}{n}\imath}\right) \mathrm{e}^{2t\imath}. \tag{8.22}$$

Similarly we have

$$\left(b_k + \bar{b}_k \mathrm{e}^{(2+k)\frac{\pi}{n}\imath}\right) \frac{\mathrm{e}^{k\frac{\pi}{n}\imath}-1}{k\imath} = \left(4\,\mathrm{Re}\left\{b_k \mathrm{e}^{-(2+k)\frac{\pi}{2n}\imath}\right\} \frac{\sin k\frac{\pi}{2n}}{k}\right) \mathrm{e}^{(1+k)\frac{\pi}{n}\imath}.$$

So the range of the mapping

$$\mathbb{C} \ni b_k \mapsto \left(b_k + \bar{b}_k \mathrm{e}^{(2+k)\frac{\pi}{n}\imath}\right) \frac{\mathrm{e}^{k\frac{\pi}{n}\imath}-1}{k\imath} \in \mathbb{C} \sim \mathbb{R}^2$$

in the plain $\mathbb{R}^2$ is the line $x_2 = \tan(1+k)\frac{\pi}{n} x_1$. Consequently, the mapping

$$\mathbb{C}\times\mathbb{C} \ni (b_1,b_2) \mapsto \left(b_1 + \bar{b}_1 \mathrm{e}^{3\frac{\pi}{n}\imath}\right) \frac{\mathrm{e}^{\frac{\pi}{n}\imath}-1}{\imath} + \left(b_2 + \bar{b}_2 \mathrm{e}^{4\frac{\pi}{n}\imath}\right) \frac{\mathrm{e}^{2\frac{\pi}{n}\imath}-1}{2\imath} \in \mathbb{C}$$

is onto (surjective). Hence, for any $\widehat{b} \in \mathbb{C}$, there are $b_1, b_2 \in \mathbb{C}$ such that (8.21) holds with

$$b(t) = \left(b_1 + \overline{b}_1 \mathrm{e}^{3\frac{\pi}{n}\imath}\right) \mathrm{e}^{t\imath} + \left(b_2 + \overline{b}_2 \mathrm{e}^{4\frac{\pi}{n}\imath}\right) \mathrm{e}^{2t\imath}. \tag{8.23}$$

Summarizing we see that, for any given $a,b,c,d \in \mathbb{R}$, by adjusting suitable $a_1,a_2,b_1,b_2 \in \mathbb{C}$, we have $Y_1\left(\frac{\pi}{n}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Next we derive

$$\begin{aligned} A^{-1} X_\varepsilon\left(\frac{\pi}{n}\right) &= A^{-1}\left(\mathbb{1} + \varepsilon Y_1\left(\frac{\pi}{n}\right)\right) + O(\varepsilon^2) \\ &= \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} \\ -\sin\frac{\pi}{n} & -\cos\frac{\pi}{n} \end{pmatrix} \begin{pmatrix} 1+\varepsilon a & \varepsilon b \\ \varepsilon c & 1+\varepsilon d \end{pmatrix} + O(\varepsilon^2). \end{aligned}$$

The eigenvalues $\widetilde{\lambda}_\pm$ of $A^{-1}\left(\mathbb{1} + \varepsilon Y_1\left(\frac{\pi}{n}\right)\right)$ are given by the equalities

$$\begin{aligned} \widetilde{\lambda}_\pm = \frac{1}{2}\Bigg(&\varepsilon(a-d)\cos\frac{\pi}{n} - \varepsilon(b+c)\sin\frac{\pi}{n} \\ &\pm\sqrt{4 + 4\varepsilon(a+d) + \varepsilon^2\left(ad - bc + \left((d-a)\cos\frac{\pi}{n} + (b+c)\sin\frac{\pi}{n}\right)^2\right)}\Bigg) \end{aligned}$$

for ε small, which yield

$$\left.\begin{aligned} \widetilde{\lambda}_+ &= 1 + \varepsilon\frac{\sin\frac{\pi}{n}}{2}\left(a\cot\frac{\pi}{2n} + d\tan\frac{\pi}{2n} - (b+c)\right) + O(\varepsilon^2), \\ \widetilde{\lambda}_- &= -1 - \varepsilon\frac{\sin\frac{\pi}{n}}{2}\left(a\tan\frac{\pi}{2n} + d\cot\frac{\pi}{2n} + (b+c)\right) + O(\varepsilon^2). \end{aligned}\right\} \tag{8.24}$$

The two planes

$$\begin{aligned} x_1\cot\frac{\pi}{2n} + x_2\tan\frac{\pi}{2n} - x_3 &= 0, \\ x_1\tan\frac{\pi}{2n} + x_2\cot\frac{\pi}{2n} + x_3 &= 0 \end{aligned}$$

decompose the space $\mathbb{R}^3$ into four subspaces where, by (8.24), we get four different types of a strong 1-hyperbolicity of $A^{-1}\left(\mathbb{1} + \varepsilon Y_1\left(\frac{\pi}{n}\right)\right)$ with the corresponding hyperbolicities of $A^{-1}X_\varepsilon\left(\frac{\pi}{n}\right)$. For instance, $A^{-1}X_\varepsilon\left(\frac{\pi}{n}\right)$ is asymptotically stable for $\varepsilon > 0$ small if

$$a\cot\frac{\pi}{2n} + d\tan\frac{\pi}{2n} < b + c < -a\tan\frac{\pi}{2n} - d\cot\frac{\pi}{2n}. \tag{8.25}$$

Note that (8.25) yields $a + d < 0$. On the other hand, the real parts of the eigenvalues of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ lie in $\subset (-\infty, 0)$ if and only if

$$a + d < 0, \quad ad - bc > 0, \tag{8.26}$$

which condition follows from the Routh–Hurwitz criterion [6]. The relationship between constants $a,b,c,d \in \mathbb{R}$ and $a_1,a_2,b_1,b_2 \in \mathbb{C}$ are given by (8.20), (8.21) and (see (8.9)) also by the equalities

$$a = \operatorname{Re}\widehat{a} + \operatorname{Re}\widehat{b}, \quad b = \operatorname{Im}\widehat{b} - \operatorname{Im}\widehat{a}, \quad c = \operatorname{Im}\widehat{a} + \operatorname{Im}\widehat{b}, \quad d = \operatorname{Re}\widehat{a} - \operatorname{Re}\widehat{b}, \tag{8.27}$$

which are obtained by solving the equality

$$\widehat{a}z + \widehat{b}\overline{z} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where $z = x_1 + \imath x_2$, for any x_1 and x_2 from $\mathbb{R}$. Inserting (8.27) into (8.25) and (8.26), summarizing, we obtain the following result.

Theorem 8.2. *Assume* (8.13)*, and $a(t)$ and $b(t)$ are given by* (8.22) *and* (8.23)*, respectively. Then the equation*

$$\dot{z} = \varepsilon(a(t)z + b(t)\overline{z}) \tag{8.28}$$

has zero solution the only symmetric one for $\varepsilon \neq 0$ small which can be either hyperbolic, asymptotically stable or unstable, respectively.† *For instance, it is asymptotically stable for $\varepsilon > 0$ small if $\operatorname{Re}\widehat{a} < 0$ and either $|\widehat{a}| > |\widehat{b}|$ or the inequalities*

$$\operatorname{Re}\widehat{a}\sec\frac{\pi}{n} + \operatorname{Re}\widehat{b}\cot\frac{\pi}{n} < \operatorname{Im}\widehat{b} < -\operatorname{Re}\widehat{a}\sec\frac{\pi}{n} + \operatorname{Re}\widehat{b}\cot\frac{\pi}{n}$$

are satisfied. Constants $\widehat{a}$ and $\widehat{b}$ are given by (8.20) *and* (8.21)*, respectively.*

We do not compute $Y_2(2\pi)$ in (8.2) for (8.28) since it is an awkward formula, but we again see that in general, for instance in the hyperbolic cases, the zero solution is the only 2π-periodic solution of (8.28).

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†Of course, depending on the choice of $a_1, a_2, b_1, b_2 \in \mathbb{C}$.

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LATTICE OF QUASIORDERS OF MONOUNARY ALGEBRAS

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Abstract. For an algebra $\mathcal{A}$, the lattice Quord $\mathcal{A}$ of all quasiorders of $\mathcal{A}$, i. e., of all reflexive and transitive relations compatible with all fundamental operations of $\mathcal{A}$, is dealt with. In the present paper we prove that if $\mathcal{A}$ is a monounary algebra, then Quord $\mathcal{A}$ is distributive if and only if it is modular and we find necessary and sufficient conditions for $\mathcal{A}$ under which Quord $\mathcal{A}$ is distributive.

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1. INTRODUCTION

A reflexive and transitive relation on an algebra $\mathcal{A}$, which is compatible with all fundamental operations of $\mathcal{A}$, is said to be a *quasiorder of* $\mathcal{A}$. Quasiorders of an algebra can be considered as a common generalization of its congruences and compatible partial orders.

The lattices of all quasiorders on a set have several special properties (e. g., they are atomistic, dually atomistic and complemented [4]). By [2, 7], every algebraic lattice is isomorphic to the quasiorder lattice of a suitable algebra. Only few properties of the lattice of quasiorders are known in the case of concrete classes of algebras. For example, for the majority algebras it is known that their quasiorder lattice is always distributive [3, 8]). In [9] the question how endomorphisms of quasiorders behave, in particular, under which conditions $\operatorname{End} q \subseteq \operatorname{End} q'$ for quasiorders q, q' on a set A ($\operatorname{End} q$ is the set of all mappings preserving q). They describe the quasiorder lattice of the algebra $(A, \operatorname{End} q)$.

We will deal with the lattice $\operatorname{Quord}(A, f)$ of all quasiorders of (A, f), where (A, f) is a monounary algebra.

Let (A, f) be a monounary algebra. Clearly, if $|A| = 1$, then $\operatorname{Quord}(A, f)$ is a 1-element lattice and if (A, f) is a 2-element cycle, then $\operatorname{Quord}(A, f)$ is a 2-element lattice. Also, we show that if $|A| = 2$ and (A, f) is not a cycle, then $\operatorname{Quord}(A, f) \cong M_2$.

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The aim of this paper is to find necessary and sufficient conditions under which the lattice $\operatorname{Quord}(A, f)$ is modular or distributive, respectively.

2. Preliminaries

First we recall some basic notions. By a monounary algebra we understand a pair (A, f) where A is a nonempty set and $f: A \to A$ is a mapping.

A monounary algebra (A, f) is called *connected* if, for arbitrary $x, y \in A$, there are non-negative integers n, m such that $f^n(x) = f^m(y)$. A maximal connected subalgebra of a monounary algebra is called a *connected component*.

An element $x \in A$ is referred to as *cyclic*, if there exists a positive integer n such that $f^n(x) = x$. In this case the set $\{x, f^1(x), f^2(x), \dots, f^{n-1}(x)\}$ is said to be a cycle.

A *quasiorder* of an algebra $\mathcal{A} = (A, F)$ is a reflexive and transitive binary relation on A, which is compatible with all operations $f \in F$. A quasiorder is a congruence of $\mathcal{A}$, if it is symmetric. We will denote by $\operatorname{Quord}\mathcal{A}$ and $\operatorname{Con}\mathcal{A}$ the lattice of all quasiorders and of all congruences of $\mathcal{A}$, respectively, ordered by inclusion.

For an algebra $\mathcal{A} = (A, F)$, $a, b \in A$, let $\alpha(a, b)$ and $\theta(a, b)$ be the smallest quasiorder and the smallest congruence, respectively, such that $(a, b) \in \alpha(a, b)$, $(a, b) \in \theta(a, b)$.

Trivial relations $I = \{(a, a) : a \in A\}$ and A^2 are congruences. For a monounary algebra (A, f), so called "natural ordering" ν defined for $x, y \in A$ by the formula $(x, y) \in \nu$ if $y = f^n(x)$ for some non-negative integer was studied, e. g., in [6]. It is easy to see that $\nu \in \operatorname{Quord}(A, f)$. Thus, e. g., if some connected component of (A, f) has at least 3 elements, then ν is a nontrivial quasiorder.

If $\alpha \in \operatorname{Quord}(A, f)$, $x \in A$, then we denote

$$[x]_\alpha = \{y \in A : (x, y) \in \alpha,\ (y, x) \in \alpha\}.$$

The symbols $\mathbb{N}$ and $\mathbb{Z}$ are used for the sets of all positive integers and of all integers. Next, if $n \in \mathbb{N}$, then $\mathbb{Z}_n$ is the system of all integers modulo n.

The *pentagon* N_5 and the *diamond* M_3 are used as in [5]; also, for $n \in \mathbb{N}$, M_n is a lattice with $n+2$ elements $0, 1, a_1, a_2, \dots, a_n$ such that $0 < a_i < 1$, $a_i \not\leq a_j$ for any $i, j \in \{1, \dots, n\}$, $i \neq j$.

Lemma 2.1. *Let us assume that (A, f) is a monounary algebra with $A = \{0, 1\}$, $f(0) = f(1) = 0$. Put $\alpha = I \cup \{(0, 1)\}$ and $\beta = I \cup \{(1, 0)\}$. Then* $\operatorname{Quord}(A, f) = \{I, A^2, \alpha, \beta\} \cong M_2$.

Proof. If $(0, 1) \in \gamma$ for some $\gamma \in \operatorname{Quord}(A, f)$, then $(f(0), f(1)) \in \gamma$ must be valid, i. e., $(0, 1) \in \gamma$ implies $(0, 0) \in \gamma$, which is trivially satisfied. Hence the smallest element γ belonging to $\operatorname{Quord}(A, f)$ and such that $(0, 1) \in \gamma$ is equal to α. Similarly, β is the smallest quasiorder δ with the property that $(1, 0) \in \delta$. The remaining members of $\operatorname{Quord}(A, f)$ are I and A^2. Hence $\operatorname{Quord}(A, f) \cong M_2$. $\square$

Lemma 2.2. *If* (A, f) *is a monounary algebra containing at least three 1-element cycles, then* Quord (A, f) *contains a pentagon.*

Proof. Let $a, b, c \in A$ be distinct elements of A which form one-element cycles. Put

$$\alpha_1 = I \cup \{(b,a)\}, \qquad \alpha_2 = I \cup \{(b,a),(c,a)\},$$

and

$$\beta_1 = I \cup \{(c,b)\}, \qquad \beta_2 = I \cup \{(c,b),(b,a),(c,a)\}.$$

Then $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \operatorname{Quord}(A, f)$. Next, $\alpha_1 < \alpha_2 < \beta_2$, $\beta_1 < \beta_2$ and $\alpha_1 \wedge \beta_1 = \alpha_2 \wedge \beta_1 = I$. Next, $\alpha_1 \leq \beta_2$, $\beta_1 \leq \beta_2$, thus $\alpha_1 \vee \beta_1 \leq \beta_2$. If $(x, y) \in \beta_2 - (\alpha_1 \cup \alpha_2)$, then $(x, y) = (c, a)$. Since $(c, b) \in \beta_1$, $(b, a) \in \alpha_1$, we get by transitivity that $(c, a) \in \beta_1 \vee \alpha_1$. Hence $\alpha_1 \vee \beta_1 = \beta_2$. Therefore, $\alpha_2 \vee \beta_1 = \beta_2$. Thus we have a pentagon consisting of the quasiorders $\{I, \alpha_1, \alpha_2, \beta_1, \beta_2\}$. □

Corollary 2.1. *Let* (A, f) *be a monounary algebra,* $|A| \geq 3$, $f(x) = x$ *for each* $x \in A$. *The lattice* Quord (A, f) *contains a pentagon.*

Let us remark that the statement of 2.1 can be proved in a shorter way: In this case the lattice Quord (A, f) coincides with the lattice of all quasiorders of a set A. It is known that if $|A| \geq 3$, then this lattice is not modular.

3. Connected monounary algebra

From the paper of Berman [1] concerning congruences it follows that if $n \in \mathbb{N}$, then θ is a congruence relation of an n-element cycle (C, f) if and only if there is $d \in \mathbb{N}$ such that d divides n and for each $x \in C$,

$$[x]_\theta = \left\{x, f^d(x), \dots, f^{(\frac{n}{d}-1)d}(x)\right\}.$$

The congruence with this property will be denoted θ_d. It is easy to verify that for each $x \in C$, θ_d is the smallest congruence containing the pair $(x, f^d(x))$.

Lemma 3.1. *Let* (A, f) *be an* n*-element cycle,* $n \in \mathbb{N}$. *Then* Quord $(A, f) =$ Con $(A, f) = \{\theta_d : d/n\}$.

Proof. Without loss of generality we can assume that $A = \mathbb{Z}_n$ and that $f(x) \equiv x + 1 \pmod{n}$ for $x \in \mathbb{Z}_n$. Since each congruence is a quasiorder of (A, f), we have to prove that Quord $(A, f) \subseteq \{\theta_d : d/n\}$.

Let $\alpha \in \operatorname{Quord}(A, f)$. If $\alpha = I$, then $\alpha = \theta_n$. Suppose that $\alpha \neq I$, i. e., $(i_1, i_2) \in \alpha$ for some $i_1, i_2 \in \mathbb{Z}_n$, $i_1 \neq i_2$. This implies that $(f^{n-i_1}(i_1), f^{n-i_1}(i_2)) \in \alpha$. We have

$$f^{n-i_1}(i_1) = f^{n-i_1}\big(f^{i_1}(0)\big) = f^n(0) = 0,$$

and thus there exists the smallest d with $(0, d) \in \alpha$. Denote by e the greatest common divisor of d and n. Then the numbers $n_1 = \frac{n}{e}$ and $d_1 = \frac{d}{e}$ are mutually prime. If $jd \equiv j'd \mod n$ for some $0 \leq j \leq j' < n_1$, then $n/jd - j'd$, $n_1/(j - j')d_1$, thus

$n_1/j - j'$ which yields that $j = j'$. Therefore, the elements $0, d, 2d, \dots, (n_1 - 1)d$ are distinct. Further,

$$(0,d) \in \alpha \ \Rightarrow\ (d,2d) = \big(f^d(0), f^d(d)\big) \in \alpha$$
$$\Rightarrow\ (2d,3d) \in \alpha\ \Rightarrow\ \cdots\ \Rightarrow\ \big((n_1-1)d, n_1 d\big) \in \alpha.$$

Since $n_1 d = \frac{n}{e} d = n d_1 \equiv 0 \mod n$, we have $((n_1-1)d, 0) \in \alpha$, therefore we obtain

$$[0]_\alpha \supseteq \big\{0, d, 2d, \dots, (n_1-1)d\big\}. \tag{3.1}$$

Assume that $(0, f^m(0)) \in \alpha$ for some $m \in \mathbb{N}$. There are $q \in \mathbb{N}$, $z \in \mathbb{N} \cup \{0\}$ such that $m = qd + z$, $0 \le z < d$. We have $(qd, 0) \in \alpha$ by (3.1), thus $(f^z(qd), f^z(0)) \in \alpha$, $(qd+z, z) \in \alpha$, $(m,z) \in \alpha$. Hence

$$(0,m) \in \alpha,\ (m,z) \in \alpha\ \Rightarrow\ (0,z) \in \alpha, \tag{3.2}$$

which, together with the assumption that d was the smallest positive integer with $(0,d) \in \alpha$, implies that $z = 0$. Then by (3.2), $(m,0) \in \alpha$, $m \in [0]_\alpha$. In view of (3.1) then

$$[0]_\alpha = \big\{0, d, 2d, \dots, (n_1-1)d\big\} = \Big\{0, d, 2d, \dots, \Big(\frac{n}{d} - 1\Big)d\Big\}. \tag{3.3}$$

This implies

$$[1]_\alpha = \Big\{1, 1+d, 1+2d, \dots, 1 + \Big(\frac{n}{d} - 1\Big)d\Big\},$$
$$\vdots$$
$$[d-1]_\alpha = \Big\{d-1, d-1+d, d-1+2d, \dots, d-1 + \Big(\frac{n}{d} - 1\Big)d\Big\}.$$

Therefore, d/n and $\alpha = \theta_d$. □

Corollary 3.1. *Let (A, f) be an n-element cycle for some $n \in N$. Then the lattice* Quord (A, f) *is distributive.*

Proof. By Lemma 3.1, the lattice Quord (A, f) is isomorphic to the lattice D_n of all divisors of n, since $\theta_{d_1} \subseteq \theta_{d_2}$ if and only if d_2/d_1. □

Lemma 3.2. *Let (A, f) be a connected monounary algebra containing an n-element cycle $C, |A| = n + 1 \ge 3$. Then*

$$\mathrm{Quord}\,(A, f) \cong \mathrm{Quord}(C, f) \times M_2 = \mathrm{Con}(C, f) \times M_2$$

and Quord (A, f) *is distributive.*

Proof. We can suppose that $A = \mathbb{Z}_n \cup \{a\}$, $f(i) \equiv i + 1 \mod n$ for each $i \in \mathbb{Z}_n$, $f(a) = 0$. By Lemma 3.1, $\mathrm{Quord}(\mathbb{Z}_n, f) = \{\theta_d : d/n\}$. Let $\alpha \in \mathrm{Quord}\,(A, f)$. Then $\alpha' = \alpha \cap \mathbb{Z}_n^2$ is a quasiorder of $(\mathbb{Z}_n, f)$ and there is d dividing n with $\alpha' = \theta_d$. One of the following conditions is satisfied:

(1) $\alpha = \alpha' \cup \{(a,a)\}$,

(2) there is $y \in \mathbb{Z}_n$ with $(a, y) \in \alpha$,
(3) there is $z \in \mathbb{Z}_n$ with $(z, a) \in \alpha$.

Let (3.2) hold. Then $(f(a), f(y)) \in \alpha$, thus $(0, y+1) \in \alpha$. Hence $(0, y+1) \in \theta_d$, $d/y+1$ and then

$$[y+1]_{\theta_d} = \left\{0, d, 2d, \dots, \left(\frac{n}{d}-1\right)d\right\}.$$

This implies

$$[y]_{\theta_d} = \left\{d-1, d-1+d, \dots, d-1+\left(\frac{n}{d}-1\right)d\right\}.$$

Then $\{(a,t) : t \in [y]_{\theta_d}\} \subseteq \alpha$. If there is $y' \neq y$ with $y' \in \mathbb{Z}_n$, $(a, y') \in \alpha$, then $(y, y') \in \alpha'$ and $[y]_{\theta_d} = [y']_{\theta_d}$. Thus $(a,t) \in \alpha$ if and only if $t \in \{a, d-1, d-1+d, \dots, d-1+(\frac{n}{d}-1)d\}$. Analogously, if (3.3) is valid, then $(t,a) \in \alpha$ if and only if $t \in \{a, d-1, d-1+d, \dots, d-1+(\frac{n}{d}-1)d\}$. Therefore, if (3.1) fails to holds, then exactly one of the following conditions is satisfied:

$$\alpha = \alpha' \cup \left\{(a,t) : t \in \left\{a, d-1, d-1+d, \dots, d-1+\left(\frac{n}{d}-1\right)d\right\}\right\}, \tag{3.4}$$

$$\alpha = \alpha' \cup \left\{(t,a) : t \in \left\{a, d-1, d-1+d, \dots, d-1+\left(\frac{n}{d}-1\right)d\right\}\right\}, \tag{3.5}$$

$$\alpha = \alpha' \cup \left\{(t,a),(a,t) : t \in \left\{a, d-1, d-1+d, \dots, d-1+\left(\frac{n}{d}-1\right)d\right\}\right\}. \tag{3.6}$$

Let us define a mapping $\varphi\colon \mathrm{Quord}(A, f) \to \mathrm{Quord}(\mathbb{Z}_n, f) \times M_2$; the elements of M_2 are denoted by $0,1,2,3$ (0-the least, 1-the greatest element of M_2). If α satisfies (3.1), then we put $\varphi(\alpha) = (\alpha', 0)$. If α satisfies (3.4), ((3.5), (3.6), respectively), then we put $\varphi(\alpha) = (\alpha', 2)$ ($\varphi(\alpha) = (\alpha', 3)$, $\varphi(\alpha) = (\alpha', 1)$, respectively). It is a routine calculation to verify that φ is a lattice isomorphism, which yields the required assertion. □

Lemma 3.3. *If (A, f) is a connected monounary algebra fulfilling no of the assumptions of 3.1 and 3.2, $|A| > 2$, then* $\mathrm{Quord}(A, f)$ *contains a pentagon, thus it fails to be modular.*

Proof. By the assumption, there exist distinct elements $0,1,2 \in A$ such that if we denote $C = \{f^k(0) : k \in \mathbb{N} \cup \{0\}\}$, then $1,2 \notin C$, $f(1) = 0$ and $f(2) \in C \cup \{1\}$.

Consider the following relations on the set A:

$$\alpha_0 = I \cup C^2 \cup \{1\} \times C,$$
$$\alpha_1 = I \cup C^2 \cup \{1,2\} \times C,$$
$$\alpha_2 = I \cup C^2 \cup \{1,2\} \times C \cup \{(2,1)\}$$

and

$$\beta_1 = I \cup (C \cup \{1\})^2,$$
$$\beta_2 = I \cup (C \cup \{1\})^2 \cup \{2\} \times (C \cup \{1\}).$$

The above assumptions imply that these relations are quasiorders of (A, f). Further,

(1) $\alpha_0 < \alpha_1 < \alpha_2 < \beta_2$,

(2) $\alpha_0 < \beta_1 < \beta_2$.

Let $(x, y) \in \alpha_2 \wedge \beta_1$, $x \neq y$. Then either $(x, y) \in C \times C$ or $x = 1$, $y \in C$. Thus

(3) $\alpha_2 \wedge \beta_1 = \alpha_0$.

Since $\alpha_1 < \alpha_2$, then (3) yields

(4) $\alpha_1 \wedge \beta_1 = \alpha_0$.

Let us count $\alpha_1 \vee \beta_1$. We have $\alpha_1 < \beta_2$, $\beta_1 < \beta_2$, thus $\alpha_1 \vee \beta_1 \leq \beta_2$. Let $(x, y) \in \beta_2$, $(x, y) \notin \alpha_1$, $(x, y) \notin \beta_1$. Then $x = 2$, $y = 1$. We have $(2, 0) \in \alpha_1$, $(0, 1) \in \beta_1$, which implies that $(2, 1) \in \alpha_1 \vee \beta_1$, i. e., $(x, y) \in \alpha_1 \vee \beta_1$. Therefore,

(5) $\alpha_1 \vee \beta_1 = \beta_2$.

Next, $\alpha_2 > \alpha_1$ yields in view of (5) that

(6) $\alpha_2 \vee \beta_1 = \beta_2$.

We have shown that Quord(A, f) contains a pentagon, and, therefore, it is not modular. □

4. General case

Assume that $|A| > 2$ and that (A, f) is non-connected. Then either $f(x) = x$ for each $x \in A$ or there is a connected component B of (A, f) such that $|B| \geq 2$. In the first case it was shown in Lemma 2.2 that Quord(A, f) contains a pentagon.

In Lemmas 4.1 and 4.2 , and Corollary 4.1, let B, C be distinct connected components of (A, f) such that $|B| \geq 2$. Then $B \cap C = \varnothing$. Let us put

$$\alpha_1 = I \cup B \times C, \qquad \beta_1 = I \cup C \times B$$

and

$$\alpha_2 = I \cup B^2 \cup B \times C, \qquad \beta_2 = I \cup (B \cup C)^2$$

Since $|B| \geq 2$, these relations are mutually distinct.

Lemma 4.1. *The relations $\alpha_1, \alpha_2, \beta_1, \beta_2$ belong to* Quord(A, f).

Proof. It is easy to see that each of the relations is reflexive and transitive. Let $x, y \in A$, $x \neq y$. If $(x, y) \in \alpha_1$, then $x \in B$, $y \in C$, thus $f(x) \in B$, $f(y) \in C$ and $(f(x), f(y)) \in \alpha_1$. Hence $\alpha_1 \in$ Quord(A, f). The remaining assertions can be proved analogously. □

Lemma 4.2. *The following relations assertions hold:*

(i) $\alpha_1 < \alpha_2 < \beta_2$, $\beta_1 < \beta_2$,
(ii) $\alpha_2 \wedge \beta_1 = \alpha_1 \wedge \beta_1 = I$,
(iii) $\alpha_1 \vee \beta_1 = \alpha_2 \vee \beta_1 = \beta_2$.

Proof. The validity of (i) and (ii) is obvious. We have $\alpha_1 < \beta_2$, $\beta_1 < \beta_2$, thus $\alpha_1 \vee \beta_1 \leq \beta_2$. Let $(u,v) \in \beta_2$. If $u = v$, then $(u,v) \in \alpha_1 \vee \beta_1$. If $(u,v) \in B \times C$ or $(u,v) \in C \times B$, then $(u,v) \in \alpha_1$ or $(u,v) \in \beta_1$, thus $(u,v) \in \alpha_1 \vee \beta_1$. If $(u,v) \in B \times B$, then take an arbitrary $w \in C$; we have

$$(u,w) \in B \times C \leq \alpha_1, \quad (w,v) \in C \times B \leq \beta_1$$

and thus

$$(u,v) \in \alpha_1 \vee \beta_1 .$$

The case $(u,v) \in C \times C$ is similar. Therefore $\beta_2 \leq \alpha_1 \vee \beta_1$.

We have shown that $\alpha_1 \vee \beta_1 = \beta_2$. Next, since $\alpha_1 \leq \alpha_2 \leq \beta_2$, this implies that also $\alpha_2 \vee \beta_1 = \beta_2$. ☐

Corollary 4.1. Quord (A, f) *contains a pentagon.*

Let us summarize the obtained results.

Theorem 4.1. *Let (A, f) be a monounary algebra. The following conditions are equivalent:*

(i) *The lattice* Quord (A, f) *is modular.*
(ii) *The lattice* Quord (A, f) *is distributive.*
(iii) *Either $|A| \leq 2$ or (A, f) is connected and there exists a cycle C of (A, f) such that $|A| \leq |C| + 1$.*

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CONVENIENT TEST FOR THE WEAK METRIC REGULARITY OF A NON STRICTLY FRÉCHET DIFFERENTIABLE MAPPINGS

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Abstract. Using a recent extension of Fréchet differentiability (approach of Taylor mappings, see [1]), the notion of Clarke subdifferential in binormed spaces, and the notion of closed pair of multifunctions, we present a convenient test for the weak metric regularity of a non-strictly Fréchet differentiable functions in terms of the surjectivity of its Taylor strict derivative. As an application, we give an example of a non-strictly Fréchet differentiable function for which the given test works.

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Keywords: weak metric regularity, Clarke subdifferential in binormed spaces, strictly Taylor differentiable mappings, strictly Fréchet differentiable mappings, closed pair of multifunctions

1. AN INTRODUCTION TO METRIC REGULARITY

Throughout this section, we consider an open set U in a normed space $(E, \|\cdot\|_1)$, a closed set $S \subset U$, a Euclidean space $(Y, \|\cdot\|)$, and a continuous map $h: U \to Y$.

Let $x \in S$, $T_S^1(x)$ the Clarke tangent cone to S at x with respect to the norm $\|\cdot\|_1$, d_S^1 the distance function to the set S with respect to the norm $\|\cdot\|_1$.

The calculation of $K^1_{h^{-1}(h(x))}$ contingent cone to $h^{-1}(h(x))$ at x with respect to the norm $\|\cdot\|_1$ requires first to bound the distance of a point z to the set $h^{-1}(h(x))$ in terms of the function value $h(z)$. This leads us to the notion of metric regularity.

We say h is $\|\cdot\|_1$ weakly metrically regular on S at the point x if there is a real constant K such that

$$d^1_{S\cap h^{-1}(h(x))}(z) \leq K\|h(z) - h(x)\|$$

for all z close to x with respect to the norm $\|\cdot\|_1$. If h is $\|\cdot\|_1$ strictly Fréchet differentiable at x, we can present a convenient condition for weak metric regularity as follows:

Theorem 1. *If h is $\|\cdot\|_1$ strictly Fréchet differentiable at the point x in S and $\nabla h(x)(T_S^1(x)) = Y$ then h is $\|\cdot\|_1$ weakly metrically regular on S at x.*

A pretty application of the above theorem is the classical Liusternik theorem.

Theorem 2. *If h is $\|\cdot\|_1$ strictly Fréchet differentiable at the point x and $\nabla h(x)$ is surjective then the set $h^{-1}(h(x))$ is tangentially regular at x and $K^1_{h^{-1}(h(x))} = N(\nabla h(x))$, where $N(\cdot)$ denote the null space.*

Using a recent extension of Fréchet differentiability (approach of Taylor mappings, see [1]) and the notion of Clarke subdifferential in binormed spaces, we want to establish the same conclusion of the theorem 1 for non $\|\cdot\|_1$ strictly Fréchet differentiable mappings.

For this, we pause to recall some terminology before proving the main result.

2. Strictly Taylor differentiable mappings

Let $\|\cdot\|_1$, $\|\cdot\|_2$ two norms defined on a linear space E such that $(E, \|\cdot\|_2)$ is a Banach space and, for some $c > 0$, the condition $\|\cdot\|_1 \leq c\,\|\cdot\|_2$ holds.

Let U an open subset of $(E, \|\cdot\|_1)$, S a closed subset of $(E, \|\cdot\|_1)$ such that $S \subset U$, $(Y, \|\cdot\|)$ is a Euclidean space, $h\colon U \to Y$.

According to [1], we say that h is a $(\|\cdot\|_1, \|\cdot\|_2)$ strictly Taylor differentiable mapping at a point $x \in U$ if there exists a continuous linear operator from $(E, \|\cdot\|_2)$ to $(Y, \|\cdot\|)$ such that

$$h(x' + s) - h(x') - Ls = o(\|s\|_2)$$

in $(Y, \|\cdot\|)$ as $x' \to_{\|\cdot\|_1} x$, $s \to_{\|\cdot\|_1} 0$. Let us first note that such L is unique since h is $\|\cdot\|_2$ Fréchet differentiable at x. If h is $(\|\cdot\|_1, \|\cdot\|_2)$ strictly differentiable at $x \in U$, then we set $\nabla h(x) = L$.

Even if h is a $(\|\cdot\|_1, \|\cdot\|_2)$ strictly Taylor differentiable mapping at a point x, h is not necessarily $\|\cdot\|_1$ Fréchet differentiable at x. But if $\|\cdot\|_1$ is equivalent to the norm $\|\cdot\|_2$, then h is necessarily strictly $\|\cdot\|_1$ Fréchet differentiable at x.

So the notion of strict differentiability in terms of Taylor strengthens and generalizes the elegant notion of strict Fréchet differentiability.

In order to generalize the classical Theorem 1, we extend the notion of Clarke subdifferentiability in a normed spaces to binormed spaces.

3. Clarke subdifferential in binormed spaces

Throughout this section we consider a binormed space $(E, \|\cdot\|_1, \|\cdot\|_2)$ such that $(E, \|\cdot\|_2)$ is a Banach space and, for some $c > 0$, the inequality $\|\cdot\|_1 \leq c\,\|\cdot\|_2$ holds, an open set U of a normed space $(E, \|\cdot\|_1)$, a closed subset S of $(E, \|\cdot\|_1)$ such that $S \subset U$, a Euclidean space $(Y, \|\cdot\|)$, a locally Lipschitzian around $\overline{x} \in U$ with respect to the norm $\|\cdot\|_2$ mapping $h\colon U \to (Y, \|\cdot\|)$, and a $\|\cdot\|_2$ locally Lipschitzian real function h_2 around $\overline{x} \in U$.

Let v any vector in E. The Clarke generalized directional derivative of h_2 at $\overline{x}$ in the direction v with respect to the norm $\|\cdot\|_2$, denoted $h_2^{0,2}(\overline{x}, v)$, is defined as follows:

$$h_2^{0,2}(\overline{x},v) = \limsup_{x' \to_{\|\cdot\|_2} \overline{x}, t \downarrow 0} \frac{h_2(x'+tv) - h_2(x')}{t}.$$

The Clarke subdifferential of h_2 at $\overline{x}$ with respect to the norm $\|\cdot\|_2$, denoted $\partial_0^2 h_2(\overline{x})$, is the subset of $(E, \|\cdot\|_2)'$ given by

$$\partial_0^2 h_2(\overline{x}) = \{\xi \in (E, \|\cdot\|_2)' : \ h_2^{0,2}(\overline{x}, v) \geq \langle \xi, v \rangle \ \forall v \in E\}.$$

Analogously, the Clarke subdifferential of h_2 at $\overline{x}$ with respect to the pair of norms $(\|\cdot\|_1, \|\cdot\|_2)$, denoted $\partial_0^{1,2} h_2(\overline{x})$, is the subset of $(E, \|\cdot\|_2)'$ given by

$$\partial_0^{1,2} h_2(\overline{x}) = \{\xi \in (E, \|\cdot\|_2)' : \ h_2^{0,1}(\overline{x}, v) \geq \langle \xi, v \rangle \ \forall v \in E\}.$$

Notice that the Clarke subdifferentials $\partial_0^2 h_2(\overline{x})$ and $\partial_0^1 h_2(\overline{x})$ are smaller than the generalized Clarke subdifferential $\partial_0^{1,2} h_2(\overline{x})$. For a point x in S, the generalized Clarke tangent cone to S at x with respect to the norms $(\|\cdot\|_1, \|\cdot\|_2)$, denoted $T_S^{1,2}(x)$, is the subset of E given by

$$T_S^{1,2}(x) = (\mathrm{cl}\,(R^+ \partial_0^{1,2} d_S^2(x))^-,$$

where cl denotes the $*\sigma((E, \|\cdot\|_2)', E)$ closure.

Analogously, the generalized Clarke normal cone to S at x with respect to the norms $(\|\cdot\|_1, \|\cdot\|_2)$, denoted $N_S^{1,2}(x)$, is the subset of $(E, \|\cdot\|_2)'$ given by

$$N_S^{1,2}(x) = \mathrm{cl}\,(R^+ \partial_0^{1,2} d_S^2(x)),$$

where cl denotes $*\sigma((E, \|\cdot\|_2)', E)$ closure.

From the above definitions, we deduce immediately that the generalized tangent cone $T_S^{1,2}(x)$ is smaller than the tangent cones $T_S^2(x)$ and $T_S^1(x)$.

The first assertion below reiterates that if h_2 is a $\|\cdot\|_2$ locally Lipschitzian real function on U, then the pair of multifunctions $(\partial_0^2 h_2, \partial_0^{1,2} h_2)$ is closed from $(U, \|\cdot\|_1)$ to $((E, \|\cdot\|_2)', *\sigma((E, \|\cdot\|_2)', E))$ in the following sense: if x_i and ξ_i are sequences in U and $(E, \|\cdot\|_2)'$ such that $\xi_i \in \partial_0^2 h_2(x_i)$, x_i converges to x with respect to the norm $\|\cdot\|_1$, and ξ_i converges to ξ for the weak topology $*\sigma((E, \|\cdot\|_2)', E)$, then $\xi \in \partial_0^{1,2} h_2(x)$.

Proposition 1. *Assume that h_2 is a $\|\cdot\|_2$ locally Lipschitzian real function on U. Then the pair of multifunctions $(\partial_0^2 h_2, \partial_0^{1,2} h_2)$ is closed from $(U, \|\cdot\|_1)$ to $((E, \|\cdot\|_2)'$, $*\sigma((E, \|\cdot\|_2)', E))$.*

Proof. Let x_i and ξ_i be sequences in U and $(E, \|\cdot\|_2)'$ such that $\xi_i \in \partial_0^2 h_2(x_i)$, x_i converges to x with respect to the norm $\|\cdot\|_1$, and ξ_i converges to ξ for the weak topology $*\sigma((E, \|\cdot\|_2)', E)$.

Let any $v \in E$ be given. Then $\langle \xi_i, v \rangle$ converges to $\langle \xi, v \rangle$.One has $h_2^{0,2}(x_i, v) \geq \langle \xi_i, v \rangle$, which implies that $h_2^{0,1}(x_i, v) \geq \langle \xi_i, v \rangle$.

By the upper semicontinuity of $h_2^{0,1}$ with respect to the norm $\|\cdot\|_1$, we deduce that $h_2^{0,1}(x,v) \geq \langle \xi, v\rangle$. Since v is arbitrary, ξ belongs to $\partial_0^{1,2}h_2(x)$. □

Now we can give the main result.

4. A CONVENIENT TEST FOR WEAK METRIC REGULARITY OF $(\|\cdot\|_1, \|\cdot\|_2)$ STRICTLY TAYLOR DIFFERENTIABLE MAPPINGS

Assume that all the hypothesis of Section 3 are satisfied and suppose also that $(E, \|\cdot\|_2)$ is a separable Banach space. We can now present a convenient condition for weak metric regularity of $(\|\cdot\|_1, \|\cdot\|_2)$ strictly Taylor differentiable functions.

Theorem 3 (Surjectivity and metric regularity)**.** *If h is $\|\cdot\|_2$ strictly differentiable at the point x in S and $\nabla h(x)(T_S^{1,2}(x)) = Y$ then h is $\|\cdot\|_1$ weakly metrically regular on S at x.*

Let us remark that in our theorem, to give a convenient test for the $\|\cdot\|_1$ weak metric regularity of a function at a point in terms of the surjectivity of its $\|\cdot\|_2$ strict derivative there, we do not require that h is $\|\cdot\|_1$ strictly differentiable at the point x as in the classical Theorem 1.

Proof. Without loss of generality, suppose that $x = 0$ and $h(0) = 0$. If h is not $\|\cdot\|_1$ weakly metrically regular on S at the point 0 then there is a sequence $v_r \to 0$ in $(S, \|\cdot\|_1)$ such that $h(v_r) \neq 0$ for all r, and a real sequence $\delta_r \downarrow 0$ such that the function $\|h(\cdot)\| + \delta_r\|\cdot - v_r\|_1$ is minimized on S at v_r. Denoting the local Lipschitz constant by L, we deduce from the nonsmooth calculus and the Exact penalization theorem the condition

$$0 \in \partial_0^2(\|h\|)(v_r) + \delta_r B^* + L\partial_0^2 d_S^2(v_r),$$

Where B^* denote the unit ball in $(E, \|\cdot\|_2)'$.

Hence there are elements u_r of $\partial_0^2(\|h\|)(v_r)$ and w_r of $L\partial_0^2 d_S^2(v_r)$ such that $u_r + w_r$ approaches zero with respect to the dual norm $\|\cdot\|_2'$.

By choosing a subsequence we can assume $\|h(v_r)\|^{-1}h(v_r) \to y \neq 0$. Consequently, $u_r \to (\nabla h(0))^* y$ for the weak topology $*\sigma((E, \|\cdot\|_2)', E)$. Since the pair of multifunctions $(\partial_0^2 d_S^2, \partial_0^{1,2} d_S^2)$ is closed at 0, we deduce $-(\nabla h(0))^* y \in L\partial_0^{1,2} d_S^2(0) \subset N_S^{1,2}(0)$. However, by assumption there is a nonzero element p of $T_S^{1,2}(0)$ such that $\nabla h(0)p = -y$, so we arrive at the contradiction

$$0 \geq \langle p, -(\nabla h(0))^* y\rangle = \langle \nabla h(0)p, -y\rangle = \|y\| > 0,$$

which completes the proof. □

Corollary 1. *If h is $(\|\cdot\|_1, \|\cdot\|_2)$ strictly differentiable at the point x in S and $\nabla h(x)(T_S^{1,2}(x)) = Y$ then h is $\|\cdot\|_1$ weakly metrically regular on S at x.*

Proof. Since h is $(\|\cdot\|_1, \|\cdot\|_2)$ strictly differentiable at the point x then h is $\|\cdot\|_2$ strictly differentiable at the point x. Consequently, using Theorem 3, we obtain the result. □

An immediate consequence of Corollary 1 is the generalized Liusternik theorem.

Theorem 4 (Generalized Liusternik theorem)**.** *If h is $(\|\cdot\|_1, \|\cdot\|_2)$ strictly differentiable at the point x in S and $\nabla h(x)$ is surjective then the set $h^{-1}(h(x))$ is tangentially regular at x with respect to the norm $\|\cdot\|_1$ and $K^1_{h^{-1}(h(x))}(x) = N(\nabla h(x))$.*

Proof. Since h is $\|\cdot\|_2$ strictly differentiable at x then by the classical Liusternik theorem, we deduce that $K^1_{h^{-1}(h(x))}(x) = K^2_{h^{-1}(h(x))}(x) = N(\nabla h(x))$.

Let us now prove that $h^{-1}(h(x))$ is tangentially regular at x with respect to the norm $\|\cdot\|_1$. Assume without loss of generality that $x = 0$ and $h(0) = 0$. Its clear that $K^1_{h^{-1}(0)}(0) \subset N(\nabla h(x))$. So it suffices to prove $N(\nabla h(0)) \subset T^1_{h^{-1}(0)}(0)$.

Fix any element p of $N(\nabla h(0))$ and consider a sequence $x_r \to 0$ in $h^{-1}(0)$ with respect to the norm $\|\cdot\|_1$ and $t_r \downarrow 0$, $t_r > 0$. The previous result shows h is $\|\cdot\|_1$ weakly metrically regular at zero, so there is a constant k such that the inequality

$$d^1_{h^{-1}(0)}(x_r + t_r p) \leq K\|h(x_r + t_r p)\|$$

holds for all large r, and hence there are points $z_r \in h^{-1}(0)$ satisfying the estimate

$$\|x_r + t_r p - z_r\|_1 \leq K\|h(x_r + t_r p)\|.$$

If we define directions $p_r = t_r^{-1}(z_r - x_r)$ then clearly the points $x_r + t_r p_r$ lie in $h^{-1}(0)$ for large r, and since

$$\|p_r - p\|_1 = \frac{\|x_r + t_r p - z_r\|_1}{t_r} \leq K\frac{\|h(x_r + t_r p) - h(x_r)\|}{t_r} \to k\|\nabla h(0)p\| = 0,$$

we obtain $p \in T^1_{h^{-1}(0)}(0)$. □

Let us give now an example of a non-strictly Fréchet differentiable function for which the test given in Theorem 3 works.

Example 1. Let Ω be a bounded domain in $\mathbb{R}^2$, $E = W_0^{1,2}(\Omega)$ the Sobolev space with the usual norm $\|\cdot\|_2 = \|\cdot\|_{W_0^{1,2}(\Omega)}$. Let also p and ε such that $0 < \varepsilon < 1$, $\varepsilon + 2 < p < \infty$. Set $\|\cdot\|_1 = \|\cdot\|_{L^p(\Omega)}$. Note that $(E, \|\cdot\|_2)$ is a separable Banach space. Since $W_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$, it follows that $(E, \|\cdot\|_1, \|\cdot\|_2)$ is a binormed space such that for some $c > 0$, the inequality $\|\cdot\|_1 \leq c\,\|\cdot\|_2$ holds.

Set $g(u) = |u|^{\varepsilon+2}$ and consider the functional G defined on E by the formula

$$G(x) = \int_\Omega g(x(s))ds.$$

Then G is $\|\cdot\|_2$ twice Fréchet differentiable at every $x \in E$ and

$$G^{(1)}(x)h = \int_{\Omega} g'(x(s))h(s)ds,$$

$$G^{(2)}(x)(h_1, h_2) = \int_{\Omega} g''(x(s))h_1(s)h_2(s)ds.$$

Let us note that G is not $\|\cdot\|_1$ Fréchet differentiable at any point $x \in E$. Indeed, let $\alpha_m \to \infty$ and $d_m \to \infty$ such that $|g(d_m)| \geq \alpha_m |d_m|^p$. By the countable additivity of the Lebesgue measure there exist $C > 0$ and $\Omega' \subset \Omega$ such that $\mu(\Omega') > 0$, $\operatorname{dist}(\Omega', \partial\Omega) > 0$, and $|x(s)| \leq C$ for all $s \in \Omega'$. In this case, put $D = \max\{g(u) : |u| \leq C\} < \infty$. Choose $\Omega_m \subset \Omega'$ such that $\mu(\Omega_m) = |d_m|^{-p} |\alpha_m|^{-\frac{1}{2}}$ for large m.

Let h_m be defined by the formula

$$h_m(s) = \begin{cases} d_m - x(s) & \text{for } s \in \Omega_m, \\ 0 & \text{for } s \in \Omega \setminus \Omega_m. \end{cases}$$

It follows then that $\|h_m\|_1 \to 0$ and

$$\begin{aligned} |G(x + h_m) - G(x)| &\geq \alpha_m |d_m|^p \mu(\Omega_m) - D\mu(\Omega_m) \\ &= |\alpha_m|^{\frac{1}{2}} - D\mu(\Omega_m) \to +\infty. \end{aligned}$$

Let $k_m \in C_0^\infty(\Omega)$ such that $\|k_m - h_m\|_1 \to 0$. Then $\|k_m\|_1 \to 0$, but $G(x + k_m) - G(x) \to \infty$ since the Lebesgue integral is absolutely continuous. Therefore, G is not $\|\cdot\|_1$ Fréchet differentiable at x and consequently, G is not $\|\cdot\|_1$ strictly Fréchet differentiable at x.

So to test the weak metric regularity of the functional G on a closed set S of $(E, \|\cdot\|_1)$, the classical Theorem 1 cannot be used, but we can apply the Theorem 3 since G is $\|\cdot\|_2$ strictly differentiable at x. Hence, if S is a closed set in $(E, \|\cdot\|_1)$ such that $\nabla G(x)(T_S^{1,2}(x)) = \mathbb{R}$ then G is $\|\cdot\|_1$ weakly metrically regular on S at x.

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LAWS AND IDENTITIES FOR SOME UPPER TRIANGULAR MATRICES

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Abstract. J. C. Robson has investigated the ideal I_n of all polynomials in the free associative algebra $R\langle x\rangle$ over a non-commutative ring R generated by x and the n^2 entries of an $n\times n$ matrix $\alpha=(a_{ij})$, which are satisfied by α. He found four cubics generating the ideal for $n=2$ and proved its finite generation for any n. Ts. Rashkova has considered the ideal I_2 for matrix algebras with involution over a noncommutative ring and over a field of characteristic zero. In the paper the ideal I_3 is described for some special upper triangular matrices over a field of characteristic 0. The T-ideal $T(U_2(G))$ is investigated as well for G denoting the infinite dimensional Grassmann algebra.

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1. INTRODUCTION

The Cayley–Hamilton theorem and the corresponding trace identity play a fundamental role in proving classical results about the polynomial and trace identities of the $n\times n$ matrix algebra $M_n(K)$ over a field K of characteristic zero.

The structure theory of semisimple rings and quantum matrices for example show the importance of matrices over non-commutative rings in the theory of PI-algebras and other branches of algebra as well.

J. C. Robson investigated in [12–14] the ideal I_n of all polynomials (including nonmonics) in the free associative algebra $R\langle x\rangle$ over a non-commutative ring R generated by x and the n^2 entries of an $n\times n$ matrix $\alpha=(a_{ij})$, which are satisfied by α. Those polynomials we call the laws over R of a non-commutative $n\times n$ matrix α. These are not polynomial identities since the entries of α are allowed as coefficients in the laws and they vary with the choice of α.

Robson showed that I_n is an insertive ideal (meaning that its homogeneous elements are closed under inner multiplication by a constant a_{ij} in some fixed position) and as such is finitely generated (see [13] and [12, Theorem 2.3]).

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The minimal degree of polynomials in I_n remains unknown. However, for the case $n=2$, Robson [12, Proposition 3.2] found four polynomials of degree 3 (least possible) in I_2 and Pearson showed in [10, Corollary] that these four Robson cubics do indeed generate I_2 as an insertive ideal.

2. Results

The first study of the non-commutative case for $n=3$ was done in [9]. The results there provide further evidence of the tantalizing complexity of even these small matrices. Each of the four found laws of degree 7 has 1156 terms. Thus, special cases of 3×3 matrices even over a field are of interest.

In [11] we study the ideal I_3 for some special 3×3 upper triangular matrices considered in [7]. These algebras could be endowed with involution $*$ and their $*$-codimensions have important properties.

Here we give a complete answer for the existing laws in these algebras and investigate the T-ideal $T(U_2(G))$ of the 2×2 upper triangular matrices over G, where G stands for the infinite dimensional Grassmann algebra.

2.1. *Laws for upper triangular matrices over a field*

2.1.1. *Special upper triangular matrices*

In [7] D. La Mattina and P. Misso study some associative algebras with involution generating $*$-varieties of algebras with linear or linearly bounded sequences of $*$-codimensions. Questions concerning laws for them were discussed in [11]. Here we give the complete answers.

Let K be a field of characteristic zero and

$$M_2(K)=\left\{x=\begin{pmatrix} a & b & c\\ 0 & a & b\\ 0 & 0 & a\end{pmatrix} : a,b,c\in K\right\}.$$

All coefficients in the polynomials below mean the corresponding scalar matrix, i. e., $a=aE$ for example.

Theorem 1. *The only Robson cubic for a matrix from $M_2(K)$ in the general case is $r(x)=(x-a)^3$. For $b=0$, $c\neq 0$ a law of minimal degree is $(x-a)^2$ and for $b=c=0$ it is $x-a$.*

Proof. For a matrix x from the considered algebra we prove directly that $(x-a)^3=0$. Due to [8, Lemma 2, p.432] considering the subring T of the diagonal elements of a generic upper triangular matrix A and $w(x)\in T\langle x\rangle$, then $w(A)=0$ iff $w(x)\in\langle x-a_{11}\rangle\langle x-a_{22}\rangle\cdots\langle x-a_{nn}\rangle$, where $\langle f\rangle$ is the ideal generated by f. In the cited Lemma T is a subring of a non-commutative ring. In Theorem 1 the elements of the matrix are elements of a field and then $T\langle x\rangle=K\langle x\rangle$ and the possible laws of degree less or equal to 3 could be given with the linear combination

$$A = \alpha_1 + \alpha_2 x + \alpha_3(x-a) + \alpha_4 x^2 + \alpha_5(x-a)^2 + \alpha_6 x(x-a).$$

Equating to zero the corresponding entries we get the system

$$\begin{aligned} \alpha_1 + a\alpha_2 + a^2\alpha_4 &= 0, \\ b\alpha_2 + b\alpha_3 + 2ab\alpha_4 + ab\alpha_6 &= 0, \\ c\alpha_2 + c\alpha_3 + (2ac+b^2)\alpha_4 + b^2\alpha_5 + (ac+b^2)\alpha_6 &= 0. \end{aligned}$$

For $b \neq 0$ the system has a solution

$$\begin{aligned} \alpha_5 &= -\alpha_4 - \alpha_6, \\ \alpha_1 &= a\alpha_3 + a^2\alpha_4 + a^2\alpha_6, \\ \alpha_2 &= -\alpha_3 - 2a\alpha_4 - a\alpha_6. \end{aligned}$$

In this case A, is identically equal to zero. Considering $b = 0$, we get the rest of the statement. □

Let us put

$$M_3(K) = \left\{ x = \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in K \right\}.$$

Theorem 2. *The only Robson cubic for a matrix from $M_3(K)$ in the general case is $r(x) = (x-a)^2 x$. For the three cases $b \neq 0$, $c = d = 0$; $d \neq 0$, $b = c = 0$, and $b = c = d = 0$ a law of minimal degree is $(x-a)x$.*

Proof. Directly we calculate that for a matrix x from the considered algebra we have $(x-a)^2 x = 0$. Then, due to Lemma 2 from [8, p. 432], considerations analogous to the proof of Theorem 1 lead us to the linear combination

$$A = \alpha_1 + \alpha_2 x + \alpha_3(x-a) + \alpha_4 x^2 + \alpha_5(x-a)^2 + \alpha_6 x(x-a)$$

of all possible laws of degree ≤ 3. Equating to zero the corresponding entries, we obtain the system

$$\begin{aligned} \alpha_1 + a\alpha_2 + a^2\alpha_4 &= 0, \\ b\alpha_2 + b\alpha_3 + ab\alpha_4 - ab\alpha_5 &= 0, \\ c\alpha_2 + c\alpha_3 + (2ac+bd)\alpha_4 + bd\alpha_5 + (ac+bd)\alpha_6 &= 0, \\ \alpha_1 - a\alpha_3 + a^2\alpha_5 &= 0, \\ d\alpha_2 + d\alpha_3 + ad\alpha_4 - ad\alpha_5 &= 0. \end{aligned}$$

For $b \neq 0$, $d \neq 0$ the system has a solution

$$\begin{aligned}
\alpha_6 &= -\alpha_4 - \alpha_5, \\
\alpha_1 &= a\alpha_3 - a^2\alpha_5, \\
\alpha_2 &= -\alpha_3 - a\alpha_4 + a\alpha_5.
\end{aligned}$$

In this case A is identically equal to zero.

The three cases $b \neq 0$, $c = d = 0$; $d \neq 0$, $b = c = 0$, and $b = c = d = 0$ give the law $(x-a)x$. □

Let us put

$$M_4(K) = \left\{ x = \begin{pmatrix} 0 & b & c \\ 0 & a & d \\ 0 & 0 & 0 \end{pmatrix} : a,b,c,d \in K \right\}.$$

Theorem 3. *The only Robson cubic for a matrix from $M4(K)$ in the general case is $r(x) = (x-a)x^2$. For $c = d = 0$ or $b = c = 0$ a law of minimal degree is $(x-a)$.*

Proof. It follows the above pattern of proof. Directly we get $(x-a)x^2 = 0$. The corresponding system is

$$\begin{aligned}
\alpha_1 + a\alpha_2 + a^2\alpha_4 &= 0, \\
b\alpha_2 + b\alpha_3 + ab\alpha_4 - ab\alpha_5 &= 0, \\
c\alpha_2 + c\alpha_3 + bd\alpha_4 + (bd-2ac)\alpha_5 + (bd-ac)\alpha_6 &= 0, \\
\alpha_1 - a\alpha_3 + a^2\alpha_5 &= 0, \\
d\alpha_2 + d\alpha_3 + ad\alpha_4 - ad\alpha_5 &= 0.
\end{aligned}$$

For $b \neq 0$, $d \neq 0$ the system has the same solution as in the previous theorem and A is identically equal to zero.

The two cases $c = d = 0$ and $b = c = 0$ give the law $(x-a)x$. □

2.1.2. *The general upper triangular case*

Now we consider the general case, namely, the algebra $U_3(K,*)$ of the upper triangular matrices of order 3 with the involution $*$ reflection along the second diagonal. We could find the analogues of the Robson cubics for the $*$-symmetric matrices and for the $*$-skew-symmetric matrices as well.

Theorem 4. *The only law of minimal degree for a symmetric matrix $x \in U_3^+(K,*)$, where*

$$U_3^+(K,*) = \left\{ x = \begin{pmatrix} a & b & d \\ 0 & c & b \\ 0 & 0 & a \end{pmatrix} : a,b,c \in K \right\},$$

is $(x-a)^2(x-c)$. For $b = d = 0$ it is $(x-a)(x-c)$.

Proof. Directly we calculate that $(x-a)^2(x-c) = 0$. Then as explained in the above proofs we have to form the linear combination

$$A = \alpha_1 + \alpha_2(x-a) + \alpha_3(x-a)^2 + \alpha_4(x-c) + \alpha_5(x-c)^2 + \alpha_6(x-a)(x-c)$$

of all possible laws of degree ≤ 3. Equating to zero the corresponding entries we get the system

$$\begin{aligned}
\alpha_1 + (a-c)\alpha_4 + (a-c)^2\alpha_5 &= 0,\\
b\alpha_2 + b(c-a)\alpha_3 + b\alpha_4 + b(a-c)\alpha_5 &= 0,\\
d\alpha_2 + b^2\alpha_3 + d\alpha_4 + (2d(a-c)+b^2)\alpha_5 + (d(a-c)+b^2)\alpha_6 &= 0,\\
\alpha_1 + (c-a)\alpha_2 + (c-a)^2\alpha_3 &= 0,\\
b\alpha_2 + b(c-a)\alpha_3 + b\alpha_4 + b(a-c)\alpha_5 &= 0,\\
\alpha_1 + (a-c)\alpha_4 + (a-c)^2\alpha_5 &= 0.
\end{aligned}$$

For $b \neq 0$, $d \neq 0$ the system has a solution

$$\begin{aligned}
\alpha_6 &= -\alpha_3 - \alpha_5,\\
\alpha_1 &= -(a-c)\alpha_4 - (a-c)^2\alpha_5,\\
\alpha_2 &= -(-a+c)\alpha_3 - \alpha_4 - (a-c)\alpha_5.
\end{aligned}$$

In this case A, is identically equal to zero. For $b = d = 0$, we obtain the quadratic law $(x-a)(x-c)$. $\square$

We point that in the general case $(x-a)^2(x-c) = 0$ is in fact the Cayley–Hamilton theorem in a factor form, i. e., $x^3 - (2a+c)x^3 + a(a+2c)x - a^2c = 0$.

Theorem 5. *The only law of minimal degree for a skew-symmetric matrix $x \in U_3^-(K,*)$, where*

$$U_3^-(K,*) = \left\{ x = \begin{pmatrix} a & b & 0 \\ 0 & c & -b \\ 0 & 0 & -a \end{pmatrix} : a,b,c \in K \right\},$$

is $(x-a)(x-c)(x+a)$. If $a = b = 0$, then a law is $x(x-c)$.

Proof. The law $(x-a)(x-c)(x+a) = 0$ is checked directly. Then we form the linear combination

$$\begin{aligned}
A = \alpha_1 &+ \alpha_2(x-a) + \alpha_3(x-a)^2 + \alpha_4(x-c)\\
&+ \alpha_5(x-c)^2 + \alpha_6(x+a) + \alpha_7(x+a)^2\\
&+ \alpha_8(x-a)(x-c) + \alpha_9(x-a)(x+a) + \alpha_{10}(x-c)(x+a)
\end{aligned}$$

of all possible laws of degree ≤ 3. Equating to zero the corresponding entries, we obtain the system

$$\begin{aligned}
\alpha_1+(a-c)\alpha_4+(a-c)^2\alpha_5+2a\alpha_6+4a^2\alpha_7+2a(a-c)\alpha_{10}&=0,\\
b\alpha_2+b(c-a)\alpha_3+b\alpha_4+b(a-c)\alpha_5+b\alpha_6&\\
+b(c+3a)\alpha_7+b(c+a)\alpha_9+2ab\alpha_{10}&=0,\\
-b^2\alpha_3-b^2\alpha_5-b^2\alpha_7-b^2\alpha_8-b^2\alpha_9-b^2\alpha_{10}&=0,\\
\alpha_1+(c-a)\alpha_2+(c-a)^2\alpha_3+(c+a)\alpha_6+(c+a)^2\alpha_7+(c-a)(c+a)\alpha_9&=0,\\
-b\alpha_2-b(c-3a)\alpha_3-b\alpha_4+b(a+c)\alpha_5-b\alpha_6&\\
-b(c+a)\alpha_7+2ab\alpha_8-b(c-a)\alpha_9&=0,\\
\alpha_1-2a\alpha_2+4a^2\alpha_3-(a+c)\alpha_4+(a+c)^2\alpha_5+2a(a+c)\alpha_8&=0.
\end{aligned}$$

Its solution is

$$\begin{aligned}
&\alpha_1=-2a(a-c)\alpha_{10}-(a-c)\alpha_4-(a-c)^2\alpha_5-2a\alpha_6-4a^2\alpha_7,\\
&\alpha_8=-\alpha_{10}-\alpha_3-\alpha_5-\alpha_7-\alpha_9,\\
&\alpha_2=-2a\alpha_{10}-(-a+c)\alpha_3-\alpha_4-(a-c)\alpha_5-\alpha_6-(3a+c)\alpha_7-(a+c)\alpha_9.
\end{aligned}$$

In this case A, is identically equal to zero. The case $a=b=0$ leads one directly to the validity of the law $x(x-c)$. □

The law in the general case illustrates the Cayley–Hamilton theorem in a factor form, namely $x^3-cx^2-a^2x+a^2b=0$. All the computations are made using the computer algebra system *Mathematica*.

2.2. *Laws and identities for upper triangular matrices over the Grassmann algebra*

2.2.1. *Preliminaries*

We consider the matrix algebra of the 2×2 upper triangular matrices $U_2(G)$ over the Grassmann algebra G.

We recall the definition of the infinite dimensional Grassmann algebra G, namely,

$$G=G(V)=K\langle v_1,v_2,\ldots \mid v_iv_j+v_jv_i=0,\ i,j=1,2,\ldots\rangle.$$

The algebra G' (without 1) has a basis $v_{i_1}v_{i_2}\ldots v_{i_k}$, where $1\leq i_1<i_2<\ldots<i_k$. The elements v_i are called generators of G' while the elements $v_{i_1}v_{i_2}\ldots v_{i_k}$ for $1\leq i_1<i_2<\ldots<i_k$ are called basic monomials of G'. For $G=G'\cup 1$, a generator is 1 as well. The algebras G and G' are PI-equivalent (they satisfy one and the same identities). It is easy to be seen that $G'=J(G)$ for $J(G)$ being the Jacobson radical of the algebra.

The algebra G is in the mainstream of recent research in PI theory. Its importance is connected mainly with the structure theory for the T-ideals of identities of associative algebras developed by Kemer in [5]. Kemer proved [5, Theorem 1.2] that any T-prime T-ideal can be obtained as the T-ideal of identities of one of the following algebras: $M_n(K)$, $M_n(G)$ and $M_{n,u}(G)$, the latter being the algebra of $n\times n$ supermatrices over $G = G_0 \oplus G_1$ with G_0 blocks (with entries of even degree) of sizes $u\times u$ and $(n-u)\times(n-u)$ and with G_1 blocks (with entries of odd degree) of sizes $u\times(n-u)$ and $(n-u)\times u$.

Well known facts concerning the algebra G are the following:

Proposition 1 ([6, Corollary, p. 437])**.** *The T-ideal $T(G)$ is generated by the identity* $[x_1,x_2,x_3] = 0$.

Proposition 2 ([4, Exercise 5.3])**.** *For $G_k = G(V_k)$ over k-dimensional vector space V_k all identities follow from the identity $[x_1,x_2,x_3] = 0$ and the standard identity $S_{2p}(x_1,\dots,x_{2p}) = \sum_{\sigma\in\mathrm{Sym}(2p)}(-1)^{\sigma}x_{\sigma(1)}\dots x_{\sigma(2p)} = 0$, where p is the minimal integer with $2p > k$.*

Remark 1. In the monograph [4, Exercise 5.3] one could see that the identity $[x_1,x_2]\dots[x_{2p+1},x_{2p+2}] = 0$ on G_{2p+1} is equivalent to the standard identity of degree $2p+2$.

Remark 2. It could be seen [4, Exercise 5.8] that the T-ideal of the algebra $M_2(K)$ from Theorem 1 is generated by the identities $[x_1,x_2,x_3] = 0$ and $S_4(x_1,\dots,x_4) = 0$. Nevertheless, the algebra $M_2(K)$ is not isomorphic to the Grassmann algebra G_2 of the two-dimensional vector space.

For the rest of the paper we will use capital letters for the matrices with entries from the Grassmann algebra.

2.2.2. *The T-ideal $T(U_2(G))$*

Theorem 6. *The identity $[X_1,X_2,X_3][X_4,X_5,X_6] = 0$ holds on the algebra $U_2(G)$.*

Proof. As the considered polynomial is multilinear we could rely on [16, Remark 3.1] stating that it is an identity on $M_2(G)$ if and only if for every choice of the matrix units e_{a_i,b_i} and either $v_i^* = v_i$ or $v_i^* = 1$, the substitution $x_i \to e_{a_i,b_i}v_i^*$ in the polynomial gives zero.

We take the matrices $X_i = \begin{pmatrix} a_{1i} & b_{1i} \\ 0 & c_{1i}\end{pmatrix}$ for $i = 1,2,3$ belonging to $U_2(G)$ with entries being generators of G. It is easy to see that

$$[X_1,X_2,X_3] = \begin{pmatrix} [a_{11},a_{12},a_{13}] & * \\ 0 & [c_{11},c_{12},c_{13}]\end{pmatrix} = \begin{pmatrix} 0 & * \\ 0 & 0\end{pmatrix}.$$

The same form has the matrix $[X_4,X_5,X_6]$. As the only possibly non-zero entry of the considered matrices is the (1,2)-entry, the multiplication gives 0. $\square$

Remark 3. Theorem 6 could be reformulated as follows. Let $f(x,y,z)=[x,y,z]$. The 2×2 upper triangular matrices over the Grassmann algebra satisfy the identity $f^2=0$ while their entries satisfy the identity $f=0$.

In [3, Theorem 3.1], Domokos gave a compact form of a theorem of Szigeti from [15], namely,

Proposition 3. *For any 2×2 matrix X over a K-algebra S satisfying the identity $[x_1,x_2,x_3]=0$ we have that*

$$X^4-2X^3(\operatorname{tr}X)+X^2(2\operatorname{tr}^2X-\operatorname{tr}X^2)+X\Big(\frac{1}{2}\operatorname{tr}X\circ\operatorname{tr}X^2-\operatorname{tr}^3X\Big)$$
$$+\frac{1}{4}\Big(\operatorname{tr}^4X+\operatorname{tr}^2X^2+\frac{1}{2}\operatorname{tr}^2X\operatorname{tr}X^2-\frac{5}{2}\operatorname{tr}X^2\operatorname{tr}^2X+2[\operatorname{tr}X^3,\operatorname{tr}X]\Big)E$$

and

$$X^4-2(\operatorname{tr}X)X^3+(2\operatorname{tr}^2X-\operatorname{tr}X^2)X^2+\Big(\frac{1}{2}\operatorname{tr}X\circ\operatorname{tr}X^2-\operatorname{tr}^3X\Big)X$$
$$+\frac{1}{4}\Big(\operatorname{tr}^4X+\operatorname{tr}^2X^2-\frac{5}{2}\operatorname{tr}^2X\operatorname{tr}X^2+\frac{1}{2}\operatorname{tr}X^2\operatorname{tr}^2X-2[\operatorname{tr}X^3,\operatorname{tr}X]\Big)E$$

are equal to zero in $S^{2\times 2}$.

In [15], Szigeti developed a new theory of determinants of $n\times n$ matrices over rings satisfying the polynomial identity of m-Lie nilpotency

$$[[[\cdots[x_1,x_2],x_3],\cdots],x_m],x_{m+1}]=0.$$

As the Grassmann algebra is 2-Lie nilpotent the defined in [15] right m-adjoint of a matrix, the right m-determinant of a matrix rd_m and the right m-characteristic polynomial $p(x)$ of a matrix and their properties could be interpreted for the matrix algebra $U_2(G)$.

Proposition 4 ([15, Theorem 4.2])**.** *If $p(x)=\lambda_0+\lambda_1x+\cdots+\lambda_dx^d$ is the right m-characteristic polynomial of a $n\times n$ matrix $A\in M_n(R)$ over a m-Lie nilpotent ring R then the left substitution of A into $p(x)$ is zero: $(A)p=E\lambda_0+A\lambda_1+\cdots+A^d\lambda_d=0$.*

Again in [15], Szigeti pointed out the identity of "algebraicity" for matrices over the Grassmann algebra.

Proposition 5 ([15, Theorem 5.1])**.** *The polynomial identity*

$$S_{2n^2}\big([Y^{2n^2},Z],[Y^{2n^2-1},Z],\dots,[Y^2,Z],[Y,Z]\big)=0$$

holds on $M_n(G)$ for any two matrices Y and Z.

Now we give some laws and identities for the upper triangular matrices over the Grassmann algebra G.

Theorem 7. *Let the matrix $X = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ belong to $U_2(J(G))$ for a, b, c being basic monomials of G'.*

(I) *The following laws are valid for X:*

$$(X-a)(X-c) = 0,$$

$$X^3(\operatorname{tr} X) = 0,$$

$$(\operatorname{tr} X)X^3 = 0.$$

(II) *Two identities hold for any matrices X and Y of the considered type, namely $X^2Y^2 = 0$ and $(X^2Y)^2 = 0$.*

Thus any matrix X is nilpotent of index 4. A matrix X with $\operatorname{tr} X = 0$ is nilpotent of index 3.

Proof. (I) Direct calculations give the validity of the three stated laws for a matrix X.

(II) Again applying direct calculations we get that the only non-zero entries in X^2, Y^2 and X^2Y are the $(1,2)$ entries and the corresponding multiplication gives zero. □

We see that both Proposition 3 and Proposition 4 for $n = 2$ are compatible with Theorem 7 as for such a matrix $A \in U_2(G)$ we have $\operatorname{rdet}_2 A = 0$ and $p(x) = \operatorname{rdet}_2(A - Ex) = x^4 - 2\operatorname{rdet} A x^3$. Thus $A^4 - 2(\operatorname{tr} A)A^3 = 0$.

Corollary 1. *An identity of degree 9 holds for any two matrices Y and Z from $U_2(G)$ with entries being basic monomials, namely, $S_3([Y^3, Z], [Y^2, Z], [Y, Z]) = 0$.*

Proof. Applying Proposition 5 and the index of nilpotency of the matrix Y. □

Remark 4. If we consider the Grassmann algebra over a finite dimensional space, the corresponding identities have much smaller degrees.

Theorem 8. *The following two assertions hold:*

(I) *On the algebra $U_2(J(G_2))$ we get the identity $XYZ = 0$ (respectively, $X^3 = 0$).*

(II) *Any matrix $X = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ from $U_2(J(G_2))$ satisfies the law $(\operatorname{tr} X)X^2 = 0$, respectively, $X^2(\operatorname{tr} X) = 0$.*

Proof. In the considered algebra the square of every element is zero. The identity and the law are proved directly. □

Multiplying two linear combinations of the basic elements e_1, e_2 and e_1e_2 we get only αe_1e_2. Its product with any other linear combination of e_1, e_2 and e_1e_2 gives zero.

Analogous considerations are valid for the linear combinations of the basic elements of any finite dimensional Grassmann algebra (over a n-dimensional vector space). The multiplication of n linear combinations will result in $\alpha e_1 e_2 \dots e_n$ and the result of the next multiplication will be zero. Thus, we come to the following

Corollary 2. *The matrix algebra $U_k(J(G_n))$ (respectively, $M_k(J(G_n))$) is nilpotent of class $\leq n+1$.*

Corollary 3. *The polynomial identity $S_{n-1}([Y^{n-1},Z],[Y^{n-2},Z],\dots,[Y,Z]) = 0$ holds on $M_k(J(G_n))$ for $n \geq 3$ and $k \geq 2$.*

As the algebra $U_2(G)$ is a subalgebra of $M_2(G)$ we turn to the Hall identity, the four degree standard identity for $M_2(K)$ and the product commutator identity for $U_2(K)$ considering the Grassmann algebra G instead of the field K. We get

Theorem 9. *The polynomials $[[x_1,x_2]^2,x_1]$, $S_4(x_1,x_2,x_3,x_4)$, and $[x_1,x_2][x_3,x_4]$ are not identities for the algebra $U_2(G)$.*

Proof. A counter example for the validity of the first and the third identity gives the matrices

$$X_1 = \begin{pmatrix} e_1 & e_2 \\ 0 & e_3 \end{pmatrix}$$

and

$$X_2 = \begin{pmatrix} e_4e_5 + e_6 & 1 \\ 0 & e_1 + e_3 \end{pmatrix}.$$

We get that the $(1,2)$-entry of $[x_1,x_2]^2$ is nonzero, namely,

$$2e_1e_3e_6 + 2e_1e_2e_4e_5e_6 - 2e_1e_2e_3e_4e_5 + 4e_1e_2e_3e_6.$$

The $(1,2)$-entry of $[[x_1,x_2]^2,x_1]$ is $-2e_1e_2e_3e_4e_5e_6$. For the second statement, we rely on the general case considered later. $\square$

In [1, 2], a connection is given between the identities on $M_n(K)$ and those on $M_n(G)$.

Proposition 6 ([2, Proposition 2.1])**.** *Let $f_1,\dots,f_d \in K\langle x_1,\dots,x_m\rangle$ be elements of the T-ideal of identities of $M_n(K)$. If $d > \frac{1}{2}n^2m$, then $f_1f_2\cdots f_d = 0$ is an identity on $M_n(G)$.*

Remark 5. Applying the result to $M_2(G)$ and the standard polynomial S_4 we get that $S_4^9 = 0$ is an identity on $M_2(G)$. However, this is not the best possible result. Really on $M_2(G)$ we get the identities $S_4^5 = 0$ and $[[x,y]^2,x]^5 = 0$. Thus we get two identities of degree 20 and 25, respectively, for $U_2(G)$.

The above Proposition 6 has an analogue for the upper triangular matrices U_n.

Theorem 10. *Let $f_1,\dots,f_d \in K\langle x_1,\dots,x_m\rangle$ be elements of the T-ideal of identities of $U_n(K)$. If $d > \frac{1}{4}n(n+1)m$, then $f_1f_2\cdots f_d = 0$ is an identity on $U_n(G)$.*

Proof. It follows the proof of Proposition 6 taking into account the dimension of $U_n(K)$ and the fact that the relatively free algebra $F(U_n(K))$ has as a basis the monomials

$$x_1^{a_1}\cdots x_m^{a_m}[x_{i_{11}},x_{i_{21}},\ldots,x_{i_{p_1 1}}]\cdots[x_{i_{1r}},x_{i_{2r}},\ldots,x_{i_{p_r r}}],$$

where the number r of participating commutators is $\leq n-1$ and the indices in each commutator $[x_{i_{1s}},x_{i_{2s}},\ldots,x_{i_{p_s s}}]$ satisfy the relations $i_{1s}>i_{2s}\leq\cdots<i_{p_s s}$. □

Proposition 7 ([1, Lemma, p. 1509])**.** *The algebra $M_n(G)$ satisfies the identity S_{2n}^k for some $k>1$ but satisfies neither S_{2n} nor identities of the form S_m^k for any k when $m<2n$.*

Theorem 11. *The algebra $U_n(G)$ does not satisfy the identity $S_{2n}=0$.*

Proof. We have

$$S_{2n}(e_{11},e_{12},e_{22},e_{23},\ldots,e_{n-1,n-1},e_{n-1,n},ee_{nn},fe_{nn})=2efe_{1n}\neq 0$$

for $e,f\in G$ such that $ef=-fe\neq 0$. □

In [16, Proposition 4.1 and Corollary 6.1], Vishne described an efficient way to use the S_n-module structure in the computation of the multilinear identities of degree n of a given algebra. He used the method to show that the minimal degree of an identity for $M_2(G)$ is 8 and gave explicit identities of degree 8. He described a class of identities for $M_2(G)$, namely,

Proposition 8 ([16, Corollary 4.3])**.** *Let f be a multilinear polynomial of degree 8. If $\operatorname{tr} f(x_{\sigma(1)},\ldots,x_{\sigma(8)})=0$ for every $x_1,\ldots,x_8\in M_2(G)$, then f is an identity of $M_2(G)$.*

We use the notation $A_n=\sum_{\sigma\in\mathrm{Sym}(n)}x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}$. By G_0' we denote the even part of G' and by G_1' its odd part.

Proposition 9. *The following identities hold:*

(1) *On $U_2(G_1')$ we have $A_k^2=0$ for every integer k.*
(2) *On $U_2(G_0')$ we have $S_k^2=0$ for every integer k.*

Proof. For $x_1,x_2\in U_2(G_1')$ and $A_2=(a_{ij})$ we get $a_{11}=a_{21}=a_{22}=0$. Thus $A_2^2=0$. Then we use induction. Let $A_{k-1}(x_1,\ldots,x_{k-1})$ have only one nonzero entry, namely the $(1,2)$-entry. We have

$$\begin{aligned}A_k(x_1,\ldots,x_k)&=A_{k-1}(x_1,\ldots,x_{k-1})x_k+A_{k-1}(x_1,\ldots,x_{k-2},x_k)x_{k-1}\\&\quad+A_{k-1}(x_1,\ldots,x_{k-3},x_{k-1},x_k)x_{k-2}+\cdots\\&\quad+A_{k-1}(x_1,x_3,\ldots,x_k)x_2+A_{k-1}(x_2,\ldots,x_k)x_1.\end{aligned}$$

The multiplication by x_i keeps the three zero entries in every summand. So for $A_k=(b_{ij})$ we have $b_{11}=b_{21}=b_{22}=0$ and thus $A_k^2=0$.

The arguments for $S_2^2=0$ are similar as for $x_1,x_2\in U_2(G_0')$ and for $S_2=(c_{ij})$ we have $c_{11}=c_{21}=c_{22}=0$. The recursive formulas

$$S_k(x_1,\dots,x_k)=\sum_{i=1}^{k}x_i(-1)^{k-1}S_{k-1}(x_{i+1},\dots,x_k,x_1,\dots,x_{i-1})$$

for k even and

$$S_k(x_1,\dots,x_k)=\sum_{i=1}^{k}x_iS_{k-1}(x_{i+1},\dots,x_k,x_1,\dots,x_{i-1})$$

for k odd show that for $S_k=(d_{ij})$ we have $d_{11}=d_{21}=d_{22}=0$ and, therefore, $S_k^2=0$. $\square$

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SUCCESSIVE APPROXIMATION METHOD FOR SOME LINEAR BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH A SPECIAL TYPE OF ARGUMENT DEVIATION

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Abstract. We discuss successive approximation techniques for the investigation of solutions of some linear two-point boundary value problems for differential equations with special types of argument deviation.

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1. INTRODUCTION

In studies of various boundary value problems, it is often important to possess techniques based upon some types of successive approximations constructed explicitly in an analytic form. To this class of methods belongs, in particular, the approach suggested in [19–22] and originally aimed at the investigation of the periodic boundary value problem

$$x'(t) = f(t, x(t)), \qquad t \in [0, T], \tag{1.1}$$

$$x(0) = x(T), \tag{1.2}$$

which corresponds to the problem on periodic solutions of systems of first order ordinary differential equations with non-linear non-autonomous time-periodic right-hand sides [22]. Appropriate versions of the method can be applied in many situations to systems of first or second order ordinary differential equations, integro-differential equations, equations with retarded argument, and other more general functional-differential equations under various boundary conditions (periodic, Cauchy–Nicolletti, multipoint, etc.; see, e. g., [10, 23] for a bibliography). A survey of many papers devoted to the issues mentioned and published during the last three decades can be found in [11–17]. The approach indicated suggests one to replace the given boundary value problem by a family of Cauchy problems for a suitably perturbed differential system containing some artificially introduced parameter $z \in \mathbb{R}^n$ whose value is to be determined later. This idea is most transparent for the periodic boundary value problem (1.1), (1.2) considered in [19–22], in which case the parameter has the meaning of the initial value of the solution at zero and the corresponding Cauchy problem has the form

$$\begin{aligned} x'(t) &= f(t, x(t)) + \Delta(x), \qquad t \in [0, T], \\ x(0) &= z. \end{aligned} \tag{1.3}$$

where the functional given by the formula $\Delta(x) := -T^{-1} \int_0^T f(s, x(s))\, ds$ plays the role of a "perturbation term." The choice of such perturbation terms is, of course, not unique. Instead of problem (1.3), or, equivalently, of the integral functional equation

$$x(t) = z + \int_0^t f(s, x(s))\, ds - \frac{t}{T} \int_0^T f(s, x(s))\, ds, \qquad t \in [0, T], \tag{1.4}$$

one can also consider the system

$$x(t) = z + \int_0^t f(s, x(s))\, ds - \omega(t) \int_0^T f(s, x(s))\, ds, \qquad t \in [0, T],$$

where $\omega : [0, T] \to [0, T]$ is a continuous function with the properties $\omega(0) = 0$ and $\omega(T) = 1$, the arbitrariness of which in some cases [8] allows one to drop the smallness restriction of the Lipschitz constants involved.

A similar procedure also applies in other situtations. For example, in the case of the two-point boundary value problem

$$Ax(0) + Bx(T) = d \tag{1.5}$$

for equation (1.1), where $\{A, B\} \subset \mathcal{L}(\mathbb{R}^n)$, $d \in \mathbb{R}^n$, $\det B \neq 0$, the corresponding parametrized family of auxiliary differential equations can be chosen [23] in the form (1.3) with

$$\Delta(x) := -\frac{1}{T} \int_0^T f(s, x(s, z))\, ds + \frac{1}{T} \left(B^{-1} d - \left(B^{-1} A + \mathbb{1}_n \right) z \right), \tag{1.6}$$

or, equivalently, as the equation

$$\begin{aligned} x(t) = z + \int_0^t f(s, x(s))\, ds - \frac{t}{T} \int_0^T f(s, x(s))\, ds \\ + \frac{t}{T} \left(B^{-1} d - \left(B^{-1} A + \mathbb{1}_n \right) z \right), \qquad t \in [0, T]. \end{aligned} \tag{1.7}$$

The solution of problem (1.3) in each case is sought for in an analytic form by the method of successive approximations similar to the Picard iterations. All the iterations $x_m(\cdot, z)$, $m = 1, 2, \ldots$, depend upon the parameter z and, for arbitrary its values, satisfy the given boundary conditions. For the periodic boundary value problem (1.1), (1.2), according to (1.4), the successive approximations have the form [19–22]

$$x_m(t,z) := z + \int_0^t f(s, x_{m-1}(s,z))\,ds - \frac{t}{T}\int_0^T f(s, x_{m-1}(s,z))\,ds, \qquad m \geq 1, \tag{1.8}$$

where $x_0(t,z) := z$, $t \in [0,T]$, and have the properties

$$x_m(0,z) = x_m(T,z) = z, \qquad m = 1,2,\ldots.$$

Sequence (1.8) is a particular case of the iteration process

$$\begin{aligned} x_m(t,z) := z + \int_0^t \left(f(s, x_{m-1}(s,z)) - \frac{1}{T}\int_0^T f(s, x_{m-1}(s,z))\,ds \right) ds \\ + \frac{t}{T}\left(B^{-1}d - \left(B^{-1}A + \mathbb{1}_n\right)z\right), \qquad m = 1,2,\ldots, \end{aligned} \tag{1.9}$$

used in [23] in the case of the two-point boundary value problem (1.1), (1.5). Here, $x_0(t,z) := z$, $t \in [0,T]$, and

$$x_m(0,z) = z, \qquad Ax_m(0,z) + Bx_m(T,z) = d \tag{1.10}$$

for all $m = 1,2,\ldots$.

The procedure of passing from the original differential system (1.1) to its "perturbed" counterpart (1.3) and the investigation of the latter by using successive approximations leads one to a certain system of algebraic or transcendental "determining equations," which give those numerical values $z = z^*$ of the parameter that correspond to the solutions of the given boundary value problem. For example, the determining system corresponding to the successive approximations above has the form

$$\int_0^T f\left(s, x^*(s,z)\right) ds = B^{-1}d - \left(B^{-1}A + \mathbb{1}_n\right)z, \tag{1.11}$$

for the two-point boundary value problem (1.1), (1.5) and

$$\int_0^T f\left(s, x^*(s,z)\right) ds = 0, \tag{1.12}$$

in the case of the periodic boundary value problem (1.1), (1.2), where $x^*(\cdot,z)$ is the limit function of sequences (1.8) and (1.9), respectively. The idea to introduce a parameter of the kind indicated resembles the Lyapunov–Schmidt approach (see, e. g., [2,18]), with the important difference that the resulting approximations automatically satisfy the given boundary conditions.

In some related works auxiliary equations are constructed in a different way. For instance, after the change of variable

$$x(t) = z + \int_0^t y(s,z)ds - \frac{t}{T}\int_0^T y(s,z)ds,$$

instead of the integral functional equation (1.4), one arrives at the equation

$$y(t,z) = f\left(t, z + \int_0^t y(s,z)ds - \frac{t}{T}\int_0^T y(s,z)ds\right), \qquad t \in [0,T],$$

which should be considered together with the determining equation

$$\int_0^T y(s,z)ds = 0.$$

Analogously, the substitution

$$x(t,z) = z + \int_0^t y(s,z)ds - \frac{t}{T}\int_0^T y(s,z)ds + \frac{t}{T}\left(B^{-1}d - \left(B^{-1}A + I\right)z\right) \tag{1.13}$$

brings equation (1.7) to the form

$$y(t,z) = f\left(t, z + \int_0^t y(s,z)ds - \frac{t}{T}\int_0^T y(s,z)ds \right.$$
$$\left. + \frac{t}{T}\left(B^{-1}d - \left(B^{-1}A + I\right)z\right)\right) \tag{1.14}$$

with the determining equations

$$\int_0^T y(s,z)ds = B^{-1}d - \left(B^{-1}A + I\right)z. \tag{1.15}$$

Equations of type (1.14) was used in [1, 4–6] in studies of the two-point boundary value problem (1.5) for the neutral type functional differential equation

$$x'(t) = f\left(t, x(\alpha(t)), x'(\beta(t))\right), \qquad t \in [0,T], \tag{1.16}$$

where $\alpha, \beta : [0,T] \to [0,T]$ are given continuous functions, $\{A, B\} \subset \mathcal{L}(\mathbb{R}^n)$, $\det B \neq 0$, $d \in \mathbb{R}^n$, and $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in the first argument and for all u, v satisfies the Lipschitz condition

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq K|u_1 - u_2| + L|v_1 - v_2|$$

with certain non-negative constant matrices $\{K, L\} \subset \mathcal{L}(\mathbb{R}^n)$. The approach to problem (1.16), (1.5) developed in [1] is based upon finding a solution $y = y^*(\cdot, z)$ of the equation

$$y(t,z) = f\left(t, z + \frac{\alpha(t)}{T}B^{-1}[d - (A+B)z] + \frac{T - \alpha(t)}{T}\int_0^{\alpha(t)} y(s,z)ds \right.$$
$$\left. - \frac{\alpha(T)}{T}\int_{\alpha(t)}^T y(s,z)ds,\ y(\beta(t),z)\right), \qquad t \in [0,T],$$

as the uniform limit of successive approximations, after which the value $z = z^*$ of the vector parameter z is determined from equations (1.15). Of course, certain assumptions are necessary to ensure the applicability of such a scheme. The successive approximations constructed in [1] for problem (1.16), (1.5) converge if

$$r\left(2 \sup_{t\in[0,T]} \left(1 - T^{-1}\alpha(t)\right)\alpha(t) K + L\right) < 1. \tag{1.17}$$

Note that $\max_{t\in[0,T]} \left(1 - T^{-1}\alpha(t)\right)\alpha(t) \le \max_{t\in[0,T]} \left(1 - T^{-1}t\right)t = \frac{T}{4}$ and, therefore, if the range of the function α in (1.16) contains values greater than or equal to $\frac{T}{2}$, then inequality (1.17) takes the form

$$r\left(\tfrac{T}{2}K + L\right) < 1. \tag{1.18}$$

The applicability of the approach based on a substitution of type (1.13) to the more general system of functional differential equations

$$\begin{aligned} x'(t) = f\,\big(t, x(\alpha_1(t)), x(\alpha_2(t)), \dots, x\left(\alpha_p(t)\right),& \\ x'(\beta_1(t)), x'(\beta_2(t)), \dots, x'\left(\beta_q(t)\right)\big)& \end{aligned} \tag{1.19}$$

with the boundary conditions (1.5) is guaranteed in [1] by rather restrictive conditions (see [1, Assumptions H_1' and H_2']).

In the present paper, we suggest a refinement of certain estimates related to the analysis of convergence of successive approximations of type (1.9) associated with a two-point boundary value problem for a class of systems of linear functional differential equations involving argument deviations that possess certain special properties.

2. Notation

The following notation is used in the sequel.

(1) $C([0,T], \mathbb{R}^n)$ is the Banach space of the continuous functions $[0,T] \to \mathbb{R}^n$ with the standard uniform norm.
(2) $L_1([0,T], \mathbb{R}^n)$ is the usual Banach space of the vector functions $[0,T] \to \mathbb{R}^n$ with Lebesgue integrable components.
(3) $\mathcal{L}(\mathbb{R}^n)$ is the algebra of square matrices of dimension n with real elements.
(4) $r(A)$ is the maximal in modulus eigenvalue of a matrix $A \in \mathcal{L}(\mathbb{R}^n)$.
(5) $\mathbb{1}_n$ is the unit matrix of dimension n.

3. Problem setting

We consider the system of linear differential equations with argument deviations

$$x'(t) = P_0(t)x(t) + P_1(t)x(\beta(t)) + f(t), \qquad t \in [0,T], \tag{3.1}$$

subjected to the inhomogeneous two-point boundary conditions of the form

$$Ax(0) + Bx(T) = d. \tag{3.2}$$

For the sake of simplicity, we restrict ourselves to the case where only one argument deviation is present. Here, $T \in (0, +\infty)$, the elements of the matrix-valued functions $P_i : [0,T] \to \mathcal{L}(\mathbb{R}^n)$, $i = 0,1$, are Lebesgue integrable, $f \in L_1([0,T], \mathbb{R}^n)$, $\{A, B\} \subset \mathcal{L}(\mathbb{R}^n)$, $d \in \mathbb{R}^n$, and $\beta : [0,T] \to [0,T]$ is a Lebesgue measurable function. In the continuously differentiable case, problem (3.1), (3.2) is a particular case of the nonlinear boundary value problem (1.19), (1.5) considered in [1] for

$$\alpha_1(t) := t, \qquad \alpha_2(t) := \beta(t), \qquad t \in [0,T]. \tag{3.3}$$

The aim of this paper is to suggest a scheme for investigation of the two-point problem (3.1), (3.2) based upon equations of type (1.7), (1.11), which are constructed without application of substitution (1.13) and equation (1.14). In particular, it will be shown that, for certain classes of argument deviations β in the differential system (3.1), the corresponding iteration process for problem (3.1), (3.2) is convergent provided that

$$r(K_1 + K_2) < \frac{10}{3T}, \tag{3.4}$$

where $K_i := \operatorname{ess\,sup}_{t\in[0,T]} |P_i(t)|$, $i = 0,1$.

Note that, in the case of problem (3.1), (3.2), the assumptions of [1], in particular, imply that the inequalities

$$r(d_1 K_1 + d_2 K_2) < 1, \tag{3.5}$$

and

$$r\left(T\left|(A+B)^{-1}B\right|C^{-1}\sum_{i=1}^{2} K_i \sup_{t\in[0,T]}\left|\mathbb{1}_n - \frac{\alpha_i(t)}{T}B^{-1}(A+B)\right|\right) < 1 \tag{3.6}$$

are satisfied, where $C := \mathbb{1}_n - \sum_{i=1}^{2} d_i K_i$, the functions α_i, $i = 1,2$, are given by (3.3), and

$$d_i := \frac{2}{T}\sup_{t\in[0,T]}(T - \alpha_i(t))\alpha_i(t), \qquad i = 1,2. \tag{3.7}$$

These conditions are usually stronger than condition (3.4). To notice this, it sufficient to compare (3.4) with (3.5) and notice, e. g., that, in the case indicated, $d_1 = \frac{T}{2}$ and, therefore, condition (3.5) has the form

$$r\left(K_1 + \frac{4}{T^2}\sup_{t\in[0,T]}(T - \beta(t))\beta(t)K_2\right) < \frac{2}{T}.$$

4. Formulae for the iteration process

With the boundary value problem (3.1), (3.2), we associate the sequence of continuous functions defined by the recurrence relation

$$x_{m+1}(t,z) := z + \int_0^t \left[P_0(s)\, x_m(s,z) + P_1(s)\, x_m(\beta(s),z) + f(s)\right] ds -$$
$$- \frac{t}{T}\int_0^T \left[P_0(s)\, x_m(s,z) + P_1(s)\, x_m(\beta(s),z) + f(s)\right] ds$$
$$+ \frac{t}{T}\left(B^{-1}d - \left(B^{-1}A + \mathbb{1}_n\right) z\right), \quad (4.1)$$

where $m = 0,1,2,\ldots$ and $x_0(\cdot,z) = z$ for all $z \in \mathbb{R}^n$.

Proposition 1. *All the members of sequence (4.1), starting with the first one, are absolutely continuous functions satisfying the boundary conditions (3.2) for an arbitrary $z \in \mathbb{R}^n$:*

$$Ax_m(0,z) + Bx_m(T,z) = d, \qquad m = 1,2,\ldots,\ z \in \mathbb{R}^n, \quad (4.2)$$

an the initial condition

$$x_m(0,z) = z, \qquad m = 1,2,\ldots,\ z \in \mathbb{R}^n. \quad (4.3)$$

Proof. The statement indicated is an immediate consequence of formula (4.1) and the form of function x_0. □

Let $G \in \mathcal{L}(\mathbb{R}^n)$ and $g : [0,T] \times \mathbb{R}^n \to [0,+\infty)$.

Definition 1. We say that the successive approximation method (4.1) for problem (3.1), (3.2) *is applicable with an estimate of the type* (G,g) if

(1) $r(G) < 1$;
(2) $\sup_{t\in[0,T]} g(t,z) < +\infty$ for all $z \in \mathbb{R}^n$;
(3) For any $z \in \mathbb{R}^n$ there exists a continuous function $x^*(\cdot,z) : [0,T] \to \mathbb{R}^n$ such that the pointwise and componentwise estimates

$$|x^*(t,z) - x_m(t,z)| \leq G^m(\mathbb{1}_n - G)^{-1} g(t,z), \qquad t \in [0,T],\ m = 1,2,\ldots, \quad (4.4)$$

are true.

Proposition 2. *Let for certain $G \in \mathcal{L}(\mathbb{R}^n)$ and $g : [0,T] \times \mathbb{R}^n \to [0,+\infty)$ the successive approximation method (4.1) be applicable to problem (3.1), (3.2) with an estimate of the type (G,g). Then:*

(1) *For any fixed $z \in \mathbb{R}^n$, the function $x^* : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ appearing in (4.4) coincides with the uniform limit of sequence (4.1):*

$$\lim_{m\to\infty} \max_{t\in[0,T]} |x^*(t,z) - x_m(t,z)| = 0.$$

(2) *The limit function of sequence* (4.1),

$$x^*(t,z) = \lim_{m\to\infty} x_m(t,z), \tag{4.5}$$

for an arbitrary $z \in \mathbb{R}^n$ *satisfies the initial condition*

$$x^*(0,z) = z \tag{4.6}$$

and the boundary conditions (3.2)*:*

$$Ax^*(0,z) + Bx^*(T,z) = d, \qquad z \in \mathbb{R}^n.$$

(3) *Function* (4.5) *is an absolutely continuous solution of the integro-functional equation*

$$\begin{aligned} x(t) = z + \int_0^t &[P_0(s)x(s) + P_1(s)x(\beta(s)) + f(s)]\,ds \\ &- \frac{t}{T}\int_0^T [P_0(s)x(s) + P_1(s)x(\beta(s)) + f(s)]\,ds \\ &+ \frac{t}{T}\left[B^{-1}d - \left(B^{-1}A + \mathbb{1}_n\right)z\right], \quad t \in [0,T], \end{aligned} \tag{4.7}$$

or, which is equivalent, of the Cauchy problem (4.6) *for the functional differential equation*

$$x'(t) = P_0(t)x(t) + P_1(t)x(\beta(t)) + f(t) + \Delta(x;z), \qquad t \in [0,T], \tag{4.8}$$

where

$$\begin{aligned} \Delta(x;z) := &\frac{1}{T}\left[B^{-1}d - \left(B^{-1}A + \mathbb{1}_n\right)z\right] \\ &- \frac{1}{T}\int_0^T [P_0(s)x(s) + P_1(s)x(\beta(s)) + f(s)]\,ds, \qquad z \in \mathbb{R}^n. \end{aligned} \tag{4.9}$$

Proof. This assertion is easily obtained by taking Definition 1 and Proposition 1 into account. □

Let us formulate a general statement which establishes a relation between the limit function x^* of the sequence (4.1) to the solutions of the two-point problem (3.1), (3.2). For this purpose, consider the Cauchy problem

$$x'(t) = P_0(t)x(t) + P_1(t)x(\beta(t)) + f(t) + \mu, \qquad t \in [0,T], \tag{4.10}$$

$$x(0) = z, \tag{4.11}$$

where $\mu \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$ are certain parameters.

Proposition 3. *Let us assume that the successive approximation method* (4.1) *for problem* (3.1), (3.2) *is applicable with an estimate of the type* (G,g) *for certain* $G \in$

$\mathcal{L}(\mathbb{R}^n)$ and $g : [0,T] \times \mathbb{R}^n \to [0,+\infty)$. Then the solution $x(\cdot)$ of the initial value problem (4.10)*,* (4.11) *satisfies the two-point boundary condition* (3.2) *if, and only if*

$$\mu = \frac{1}{T}\left[B^{-1}d - \left(B^{-1}A + \mathbb{1}_n\right)z\right] - \frac{1}{T}\int_0^T \left[P_0(s)\,x^*(s,z) + P_1(s)\,x^*(\beta(s),z) + f(s)\right]ds, \tag{4.12}$$

where $x^*(\cdot,z) : [0,T] \to \mathbb{R}^n$ is the function given by (4.5)*.*

Proof. By virtue of Proposition 2, for an arbitrary $z \in \mathbb{R}^n$, the function $x^*(\cdot,z)$ defined by formula (4.5) satisfies the integro-functional equation (4.7). Differentiating (4.7), we find that $x = x^*(\cdot,z)$ is a solution of the Cauchy problem (4.10), (4.11) with μ given by equality (4.12). □

Proposition 3 implies immediately the following

Proposition 4. *Let the successive approximation method* (4.1) *be applicable to problem* (3.1)*,* (3.2) *with an estimate of the type (G,g) for certain $G \in \mathcal{L}(\mathbb{R}^n)$ and $g : [0,T] \times \mathbb{R}^n \to [0,+\infty)$. Then the limit function $x^*(\cdot,z)$ of the recurrence sequence* (4.1) *is a solution of the boundary value problem* (3.1)*,* (3.2) *if, and only if the value of the vector parameter $z \in \mathbb{R}^n$ in* (4.1) *satisfies the system of equations*

$$\left(B^{-1}A + \mathbb{1}_n\right)z = B^{-1}d - \int_0^T \left[P_0(s)\,x^*(s,z) + P_1(s)\,x^*(\beta(s),z) + f(s)\right]ds. \tag{4.13}$$

Proof. It suffices to apply Proposition 3 and notice that equation (4.8) coincides with equation (3.1) if and only if relation (4.13) holds. □

Equations of type (4.13) in Proposition 4 are often called *determining equations* (see, e. g., [3]), because it is from them one has to determine the actual values of the parameters involved in the iteration process.

Remark 1. In practice, it is convenient to fix some m and consider the "approximate determining equation"

$$B^{-1}d - \left(B^{-1}A + \mathbb{1}_n\right)z = \int_0^T \left[A(s)\,x_m(s,z) + B(s)\,x_m(\beta(s),z) + f(s)\right]ds. \tag{4.14}$$

If equation (4.14) has an isolated solution $z = z_m$ in a certain open domain $D \subset \mathbb{R}^n$, under additional assumptions one can show that the corresponding *exact* system of determining equations (4.13) is also solvable and, therefore, by virtue of Proposition 4, the boundary value problem (3.1), (3.2) has a solution (see, e. g., Theorem 3.1 in [10] or Theorem 7.1 in [22]). In this case, the function

$$[0,T] \ni t \longmapsto X_m(t) := x_m(t,z_m) \tag{4.15}$$

can be regarded as the *mth approximation* to a solution of problem (3.1), (3.2).

5. Auxiliary statements

Let us define the sequence of functions $\{\alpha_m\}_{m=0}^{\infty} \subset C([0,T],\mathbb{R})$ by the recurrence relation

$$\alpha_m(t) = \left(1-\frac{t}{T}\right)\int_0^t \alpha_{m-1}(s)\,ds + \frac{t}{T}\int_t^T \alpha_{m-1}(s)\,ds, \qquad t \in [0,T], \tag{5.1}$$

for $m = 1,2,\ldots$, where $\alpha_0(t) := 1$, $t \in [0,T]$. It is obvious that, in particular,

$$\alpha_1(t) = 2t\left(1-\frac{t}{T}\right), \qquad t \in [0,T], \tag{5.2}$$

and

$$\max_{t\in[0,T]} \alpha_1(t) = \frac{T}{2}. \tag{5.3}$$

Lemma 1. *Each of functions* (5.1) *is non-negative, takes zero values at the points* 0 *and* T, *and possesses the property*

$$\alpha_m\left(\frac{T}{2}-t\right) = \alpha_m\left(\frac{T}{2}+t\right), \qquad m \geq 1,\ t \in [0,T/2]. \tag{5.4}$$

Moreover,

$$\alpha_m'(t)\,\mathrm{sign}\left(t-\frac{T}{2}\right) \leq 0, \qquad t \in [0,T],\ m = 1,2,\ldots,n. \tag{5.5}$$

Proof. The first assertion is an immediate consequence of [7, Lemma 1]. Let us verify that α_m is non-decreasing on $[0,T/2]$ for any $m \geq 1$. Indeed, it follows from (5.1) and the non-negativity of α_m that

$$\begin{aligned}
\alpha_m'(t) &= \frac{1}{T}\left(-\int_0^t \alpha_{m-1}(s)\,ds + \int_t^T \alpha_{m-1}(s)\,ds\right) + \left(1-\frac{2t}{T}\right)\alpha_{m-1}(t)\\
&= \frac{1}{T}\left(\int_0^T \alpha_{m-1}(s)\,ds - 2\int_0^t \alpha_{m-1}(s)\,ds\right) + \left(1-\frac{2t}{T}\right)\alpha_{m-1}(t).\\
&\geq \frac{1}{T}\left(\int_0^T \alpha_{m-1}(s)\,ds - 2\int_0^{\frac{T}{2}} \alpha_{m-1}(s)\,ds\right) + \left(1-\frac{2t}{T}\right)\alpha_{m-1}(t)
\end{aligned} \tag{5.6}$$

for any $t \in [0,T/2]$ and $m \geq 1$. On the other hand, relation (5.4) of Lemma 1 implies that

$$\int_0^{\frac{T}{2}} \alpha_{m-1}(s)\,ds = \int_{\frac{T}{2}}^T \alpha_{m-1}(s)\,ds, \qquad m = 1,2,\ldots,$$

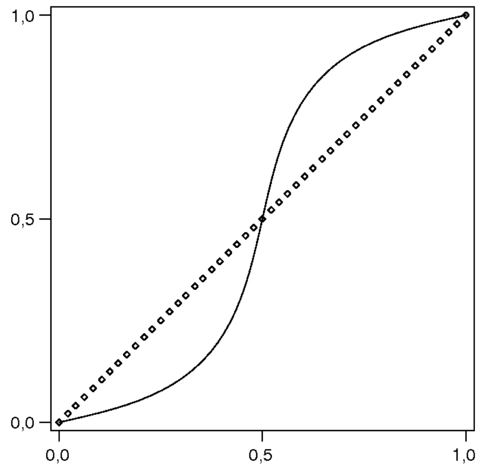

FIGURE 1. An example of an argument transformation β satisfying condition (5.8) for $T = 1$

and, therefore, (5.6) yields the estimate

$$\alpha_m'(t) \geq \frac{1}{T}\left(-\int_0^{\frac{T}{2}} \alpha_{m-1}(s)\,ds + \int_{\frac{T}{2}}^{T} \alpha_{m-1}(s)\,ds\right) + \left(1 - \frac{2t}{T}\right)\alpha_{m-1}(t) \geq 0$$

for $t \in [0, T/2]$ and $m \geq 1$. The required relation (5.5) now follows from the symmetry property (5.2). □

Lemma 2. *For an arbitrary essentially bounded function $u : [0,T] \to \mathbb{R}$, the estimate*

$$\left|\int_0^t \left(u(\tau) - \frac{1}{T}\int_0^T u(s)\,ds\right) d\tau\right| \leq \frac{1}{2}\alpha_1(t)\left(\operatorname*{ess\,sup}_{s\in[0,T]} u(s) - \operatorname*{ess\,inf}_{s\in[0,T]} u(s)\right) \tag{5.7}$$

is true for all $t \in [0,T]$, where α_1 is the function defined by equality (5.2).

Inequality (5.7) is established similarly to the Lemma from [9], and the explicit proof is therefore omitted.

Lemma 3. *Let $\beta : [0,T] \to [0,T]$ be a measurable function satisfying the condition*

$$\operatorname*{ess\,inf}_{t\in[0,T]} (\beta(t) - t)\operatorname{sign}\left(t - \frac{T}{2}\right) \geq 0. \tag{5.8}$$

Then the members of the function sequence (5.1) satisfy the pointwise estimates

$$\alpha_m(\beta(t)) \leq \alpha_m(t), \qquad m = 1, 2, \ldots, \; t \in [0,T]. \tag{5.9}$$

Proof. We shall use properties of functions (5.1). Indeed, it follows from Lemma 1 that (5.5) holds for any $t \in [0,T]$ and $m \geq 1$, i. e., each α_m, $m = 1,2,\ldots$, is non-decreasing on the interval $[0,T/2]$ and non-increasing on $[T/2,T]$. Combining (5.5) and (5.8), we easily obtain the required estimate (5.9). □

Note that if $\beta : [0,T] \to [0,T]$ is a continuous function satisfying condition (5.8) then necessarily $\beta(0) = 0$, $\beta\left(\frac{T}{2}\right) = \frac{T}{2}$, and $\beta(T) = T$.

Lemma 4. *If a measurable function $\beta : [0,T] \to [0,T]$ satisfies the condition*

$$k_T(\beta) := \operatorname*{ess\,sup}_{t\in(0,T)} \frac{\beta(t)(T-\beta(t))}{t(T-t)} < +\infty, \tag{5.10}$$

then

$$\alpha_1(\beta(t)) \leq k_T(\beta)\alpha_1(t), \qquad t \in [0,T]. \tag{5.11}$$

Proof. The assertion of Lemma 4 follows immediately from (5.10) and properties of function (5.2). □

It should be noted that (5.10) does not imply the condition $k_T(\lambda\beta) < +\infty$, where $\lambda \in \mathbb{R}$.

Example 1. The function $\beta(t) = t^2$, $t \in [0,1]$, satisfies condition (5.10) on the interval $[0,1]$. However, the condition mentioned is not satisfied for none of the functions $\beta(t) = \frac{1}{2}t^2$, $\beta(t) = \frac{t}{2}$, and $\beta(t) = \sin t$, $t \in [0,1]$. This fact is a consequence of the elementary relations

$$\sup_{t\in(0,T)} \frac{t^2\left(1-t^2\right)}{t(1-t)} = 2, \qquad \lim_{t\to 1^-} \frac{\frac{t^2}{2}\left(1-\frac{t^2}{2}\right)}{t(1-t)} = +\infty$$

and

$$\lim_{t\to 1^-} \frac{\frac{t}{2}\left(1-\frac{t}{2}\right)}{t(1-t)} = +\infty, \qquad \lim_{t\to 1^-} \frac{\sin t\,(1-\sin t)}{1-t} = +\infty.$$

6. Convergence of successive approximations for the case of a general type of argument deviation

Theorem 1. *Let $P_i : [0,T] \to \mathcal{L}(\mathbb{R}^n)$, $i = 0,1$, be matrix-valued functions with essentially bounded elements, the function $\beta : [0,T] \to [0,T]$ be measurable, and the matrix B be non-singular. Let, moreover,*

$$r\left(\widetilde{P}_0 + \widetilde{P}_1\right) < \frac{2}{T}, \tag{6.1}$$

where

$$\widetilde{P}_i := \operatorname*{ess\,sup}_{t\in[0,T]} |P_i(t)|, \qquad i = 0,1. \tag{6.2}$$

Then the successive approximation method (4.1) *is applicable to problem* (3.1), (3.2) *with an estimate of the type* $\left(T(\widetilde{P}_0+\widetilde{P}_1)/2,\gamma\right)$*, where*

$$\gamma(z) := \frac{T}{2}\delta(z) + \left|B^{-1}d - \left(B^{-1}A + 1\!\!1_n\right)z\right| \tag{6.3}$$

and

$$\delta(z) := \frac{1}{2}\left(\operatorname*{ess\,sup}_{t\in[0,T]}(P_0(t)z + P_1(t)z + f(t)) - \operatorname*{ess\,inf}_{t\in[0,T]}(P_0(t)z + P_1(t)z + f(t))\right) \tag{6.4}$$

for all $z \in \mathbb{R}^n$.

The symbols ess inf, ess sup, $|\cdot|$, and $\leq$, in (6.2) and similar relations for vector and matrix-valued functions are understood in the componentwise sense (e. g., if $P = (p_{ij})_{i,j=1}^n : [0,T] \to \mathcal{L}(\mathbb{R}^n)$, then $\operatorname{ess\,sup}_{t\in[0,T]}|P(t)|$ is the matrix with the components $\operatorname{ess\,sup}_{t\in[0,T]}|p_{ij}(t)|$, $i, j = 1,2,\dots,n$, and so on).

Proof. Let us show that, under the conditions assumed, sequence (4.1) is a Cauchy sequence in $C([0,T],\mathbb{R}^n)$. Due to estimate (5.7) of Lemma 2, it follows from (4.1) that for $m = 0$ and an arbitrary fixed $z \in \mathbb{R}^n$

$$|x_1(t,z) - z| = \left|\int_0^t \left(Q(s)z + f(s) - \frac{1}{T}\int_0^T (Q(\tau)z + f(\tau))\,d\tau\right)ds + \frac{t}{T}\left[B^{-1}d - \left(B^{-1}A + 1\!\!1_n\right)z\right]\right| \leq \alpha_1(t)\,\delta(z) + \delta_1(z), \tag{6.5}$$

where $Q := P_0 + P_1$, α_1 is the function given by (5.2), and

$$\delta_1(z) := \left|B^{-1}d - \left(B^{-1}A + 1\!\!1_n\right)z\right|, \qquad z \in \mathbb{R}^n. \tag{6.6}$$

According to formulae (4.1), for all $t \in [0,T]$, $z \in \mathbb{R}^n$ and $m = 1,2,\dots$ we have

$$\begin{aligned}
r_{m+1}(t,z) &= \int_0^t [P_0(s)\,r_m(s,z) + P_1(s)\,r_m(\beta(s),z)]\,ds - \\
&\quad - \frac{t}{T}\int_0^T [P_0(s)\,r_m(s,z) + P_1(s)r_m(\beta(s),z)]\,ds = \\
&= \left(1 - \frac{t}{T}\right)\int_0^t [P_0(s)\,r_m(s,z) + P_1(s)\,r_m(\beta(s),z)]\,ds - \\
&\quad - \frac{t}{T}\int_t^T [P_0(s)\,r_m(s,z) + P_1(s)\,r_m(\beta(s),z)]\,ds,
\end{aligned} \tag{6.7}$$

where

$$r_m(t,z) := x_m(t,z) - x_{m-1}(t,z), \qquad t \in [0,T],\ m = 1,2,\dots, \tag{6.8}$$

Equalities (6.7) imply that, for all $m = 1,2,\dots$, $z \in \mathbb{R}^n$ and all $t \in [0,T]$,

$$\begin{aligned} |r_{m+1}(t,z)| \le \widetilde{P}_0 &\left[\left(1-\frac{t}{T}\right)\int_0^t |r_m(s,z)|\,ds + \frac{t}{T}\int_t^T |r_m(s,z)|\,ds\right] + \\ &+ \widetilde{P}_1\left[\left(1-\frac{t}{T}\right)\int_0^t |r_m(\beta(s),z)|\,ds + \frac{t}{T}\int_t^T |r_m(\beta(s),z)|\,ds\right], \end{aligned} \tag{6.9}$$

where $\widetilde{P}_0$ and $\widetilde{P}_1$ are the non-negative matrices defined by formula (6.2). Relation (6.5) yields

$$|r_1(t,z)| = |x_1(t,z) - z| \le \alpha_1(t)\,\delta(z) + \delta_1(z), \qquad t \in [0,T], \tag{6.10}$$

where δ_1 is given by (6.6). In view of property (5.3) of the function α_1, estimate (6.10) gives

$$|r_1(t,z)| \le \frac{T}{2}\delta(z) + \delta_1(z) = \gamma(z), \qquad t \in [0,T]. \tag{6.11}$$

Let us now estimate $r_2(t,z)$ using (6.9) and (6.11):

$$\begin{aligned} |r_2(t,z)| \le \widetilde{P}_0 &\left(\left(1-\frac{t}{T}\right)\int_0^t |r_1(s,z)|\,ds + \frac{t}{T}\int_t^T |r_1(s,z)|\,ds\right) \\ &+ \widetilde{P}_1\left(\left(1-\frac{t}{T}\right)\int_0^t |r_1(\beta(s),z)|\,ds + \frac{t}{T}\int_t^T |r_1(\beta(s),z)|\,ds\right) \\ &\le \widetilde{P}_0\left(\left(1-\frac{t}{T}\right)\int_0^t \gamma(z)\,ds + \frac{t}{T}\int_t^T \gamma(z)\,ds\right) \\ &+ \widetilde{P}_1\left(\left(1-\frac{t}{T}\right)\int_0^t \gamma(z)\,ds + \frac{t}{T}\int_t^T \gamma(z)\,ds\right), \qquad t \in [0,T]. \end{aligned} \tag{6.12}$$

Taking the definitions (6.3) and (6.4) into account and using equality (5.3), from (6.12) we get

$$|r_2(t,z)| \le \left(\widetilde{P}_0 + \widetilde{P}_1\right)\gamma(z)\,\alpha_1(t) \le \frac{T}{2}\left(\widetilde{P}_0 + \widetilde{P}_1\right)\gamma(z), \qquad t \in [0,T].$$

Arguing by induction, we then obtain that, for all $t \in [0,T]$ and $m = 1,2,\dots$, the estimates

$$|r_m(t,z)| \le \left(\widetilde{P}_0 + \widetilde{P}_1\right)^{m-1}\left(\frac{T}{2}\right)^{m-1}\gamma(z) = G^{m-1}\gamma(z), \tag{6.13}$$

where

$$G := \frac{T}{2}\big(\widetilde{P}_0 + \widetilde{P}_1\big), \tag{6.14}$$

are true. Estimate (6.13), according to definition (6.8), yields

$$\left|x_{m+j}(t,z) - x_m(t,z)\right| \le \sum_{i=1}^{j} |r_{m+i}(t,z)| \le G^m \sum_{i=0}^{j-1} G^i \gamma(z)$$

for $t \in [0,T]$, $m = 1,2,\dots$, whence, by virtue of assumption (6.1) and notation (6.14), it follows that

$$\left|x_{m+j}(t,z) - x_m(t,z)\right| \le G^m \sum_{i=0}^{\infty} G^i \gamma(z) = G^m (\mathbb{1}_n - G)^{-1} \gamma(z) \tag{6.15}$$

for all $t \in [0,T]$ and $m = 1,2,\dots$. Since, due to (6.1), $G^m \to 0$ for $m \to \infty$, it is clear from (6.15) that (4.1) is a Cauchy sequence in $C([0,T],\mathbb{R}^n)$ and, therefore,

$$\left|x^*(t,z) - x_m(t,z)\right| \le G^m (\mathbb{1}_n - G)^{-1} \gamma(z)$$

for all $t \in [0,T]$, $z \in \mathbb{R}^n$ and $m = 1,2,\dots$. According to Definition 1, this leads us to the assertion of Theorem 1. □

Remark 2. In the case where the function β is continuous and the set of its values contains numbers greater than or equal to $\frac{T}{2}$, condition (6.1), which guarantees the applicability of the successive approximation method (4.1), coincides with condition (1.7) from [1] .

7. Convergence of successive approximations for special kinds of argument deviations

In the cases where the function $\beta : [0,T] \to [0,T]$ satisfies one of conditions (5.8) and (5.10), the assumption (6.1) of Theorem 1 can be weakened.

Theorem 2. *If the function $\beta : [0,T] \to [0,T]$ possesses property (5.8) and, moreover, the inequality*

$$r\big(\widetilde{P}_0 + \widetilde{P}_1\big) < \frac{10}{3T},$$

is satisfied, then the successive approximation method (4.1) is applicable to problem (3.1), (3.2) with an estimate of the type $\big(3T(\widetilde{P}_0 + \widetilde{P}_1)/10, g\big)$, where

$$g(t,z) := \frac{100}{27T}\alpha_1(t)\gamma(z), \qquad t \in [0,T],\ z \in \mathbb{R}^n,$$

and the function γ is defined by formula (6.3).

Proof. It follows from Lemma 3 that condition (5.8) ensures the validity of estimate (5.9). Therefore,

$$\alpha_1(\beta(t)) \le \alpha_1(t), \qquad t \in [0,T], \tag{7.1}$$

where α_1 is the function (5.2). Let us estimate the value $r_2(t,z)$. Using (6.9), (6.11), and (7.1), we have

$$\begin{aligned}
|r_2(t,z)| &\le \widetilde{P}_0\left(\left(1-\frac{t}{T}\right)\int_0^t\left(\frac{T}{2}\delta(z)+\delta_1(z)\right)ds+\frac{t}{T}\int_t^T\left(\frac{T}{2}\delta(z)+\delta_1(z)\right)ds\right)\\
&\quad+\widetilde{P}_1\left(\left(1-\frac{t}{T}\right)\int_0^t\left(\frac{T}{2}\delta(z)+\delta_1(z)\right)ds+\frac{t}{T}\int_t^T\left(\frac{T}{2}\delta(z)+\delta_1(z)\right)ds\right)\\
&\le \left(\widetilde{P}_0+\widetilde{P}_1\right)\gamma(z)\alpha_1(t)
\end{aligned} \tag{7.2}$$

for all $t \in [0,T]$, where δ_1 and γ are defined by relations (6.6) and (6.3). From estimate (7.2), due to (7.1), we obtain

$$|r_2(\beta(t),z)| \le \left(\widetilde{P}_0+\widetilde{P}_1\right)\gamma(z)\alpha_1(\beta(t)) \le \left(\widetilde{P}_0+\widetilde{P}_1\right)\gamma(z)\alpha_1(t), \qquad t \in [0,T].$$

Arguing by induction, we arrive at the pointwise and coordinatewise estimates

$$|r_{m+1}(t,z)| \le \left(\widetilde{P}_0+\widetilde{P}_1\right)^m\gamma(z)\alpha_m(t), \qquad t \in [0,T],\ m=1,2,\ldots, \tag{7.3}$$

and

$$|r_{m+1}(\beta(t),z)| \le \left(\widetilde{P}_0+\widetilde{P}_1\right)^m\gamma(z)\alpha_m(t), \qquad t \in [0,T],\ m=1,2,\ldots, \tag{7.4}$$

where α_m, $m=1,2,\ldots$, are the functions of sequence (5.1). By virtue of Lemma 2.4 from [10], the estimates

$$\alpha_{m+1}(t) \le \frac{3T}{10}\alpha_m(t),\ m \ge 2, \qquad \alpha_{m+1}(t) \le \left(\frac{3T}{10}\right)^m\bar{\alpha}_1(t),\ m \ge 0, \tag{7.5}$$

are true for all $t \in [0,T]$ and $m \ge 1$, where

$$\bar{\alpha}_1(t) := \frac{10}{9}\alpha_1(t), \qquad t \in [0,T].$$

Taking (7.5) into account and using (7.3) and (7.4), arrive at the inequalities

$$|r_{m+1}(t,z)| \le \left(\widetilde{P}_0+\widetilde{P}_1\right)\left(\frac{3T}{10}\left(\widetilde{P}_0+\widetilde{P}_1\right)\right)^{m-1}\gamma(z)\bar{\alpha}_1(t), \quad t \in [0,T],\ m \ge 1,$$

and

$$|r_{m+1}(\beta(t),z)| \le \left(\widetilde{P}_0+\widetilde{P}_1\right)\left(\frac{3T}{10}\left(\widetilde{P}_0+\widetilde{P}_1\right)\right)^{m-1}\gamma(z)\bar{\alpha}_1(t), \quad t \in [0,T],\ m \ge 1.$$

The rest of the argument is very similar to that of the proof of Theorem 1 and, therefore, is omitted. $\square$

The next statement deals with the case where the argument deviation β in equation (3.1) satisfies condition (5.10).

Theorem 3. *Let us assume that the function $\beta : [0,T] \to [0,T]$ possesses property (5.10) and, moreover, the inequality*

$$r\big(\widetilde{P}_0 + \widetilde{P}_1\big) < \frac{3}{T\max\{1, k_T(\beta)\}}, \tag{7.6}$$

holds, where $k_T(\beta)$ is the positive constant appearing in relation (5.10).

Then the successive approximation method (4.1) is applicable to problem (3.1), (3.2) with an estimate of the type $\big(T\max\{1,k_T(\beta)\}(\widetilde{P}_0 + \widetilde{P}_1)/3, g\big)$, where

$$g(t,z) := \frac{3\alpha_1(t)}{T}\gamma(z), \qquad t \in [0,T],\ z \in \mathbb{R}^n, \tag{7.7}$$

and γ is given by formula (6.3).

Proof. Let us put $\eta := \max\{1, k_T(\beta)\}$. According to (6.12), we have

$$|r_2(t,z)| \le \big(\widetilde{P}_0 + \widetilde{P}_1\big)\gamma(z)\alpha_1(t) \le \eta\big(\widetilde{P}_0 + \widetilde{P}_1\big)\gamma(z)\alpha_1(t), \qquad t \in [0,T], \tag{7.8}$$

and, hence, in view of (5.11),

$$\begin{aligned} |r_2(\beta(t),z)| &\le \big(\widetilde{P}_0 + \widetilde{P}_1\big)\gamma(z)\alpha_1(\beta(t)) \le k_T(\beta)\big(\widetilde{P}_0 + \widetilde{P}_1\big)\gamma(z)\alpha_1(t) \\ &\le \eta\big(\widetilde{P}_0 + \widetilde{P}_1\big)\gamma(z)\alpha_1(t), \qquad t \in [0,T]. \end{aligned} \tag{7.9}$$

Using (6.9), (7.8), (7.9), (5.1), and (7.5) and carrying out calculations, we obtain

$$\begin{aligned} |r_3(t,z)| &\le \eta\widetilde{P}_0\big(\widetilde{P}_0 + \widetilde{P}_1\big)\gamma(z)\alpha_2(t) + \eta\widetilde{P}_1\big(\widetilde{P}_0 + \widetilde{P}_1\big)\gamma(z)\alpha_2(t) \\ &\le \eta\big(\widetilde{P}_0 + \widetilde{P}_1\big)^2\gamma(z)\alpha_2(t) \le \frac{T\eta}{3}\big(\widetilde{P}_0 + \widetilde{P}_1\big)^2\gamma(z)\alpha_1(t) \\ &\le \eta\frac{T\eta}{3}\big(\widetilde{P}_0 + \widetilde{P}_1\big)^2\gamma(z)\alpha_1(t), \qquad t \in [0,T]. \end{aligned} \tag{7.10}$$

Therefore, by virtue of (7.10), (7.5), we have

$$\begin{aligned} |r_3(\beta(t),z)| &\le \eta\big(\widetilde{P}_0 + \widetilde{P}_1\big)^2\gamma(z)\alpha_2(\beta(t)) \\ &\le \eta\big(\widetilde{P}_0 + \widetilde{P}_1\big)^2\gamma(z)\frac{T}{3}\alpha_1(\beta(t))) \le k_T(\beta)\frac{T\eta}{3}\big(\widetilde{P}_0 + \widetilde{P}_1\big)^2\gamma(z)\alpha_1(t) \\ &\le \eta\frac{T\eta}{3}\big(\widetilde{P}_0 + \widetilde{P}_1\big)^2\gamma(z)\alpha_1(t), \qquad t \in [0,T]. \end{aligned} \tag{7.11}$$

Estimates (6.9), (7.10), and (7.11) yield

$$
\begin{aligned}
|r_4(t,z)| &\le \widetilde{P}_0 \eta \frac{T\eta}{3} (\widetilde{P}_0 + \widetilde{P}_1)^2 \gamma(z) \alpha_2(t) + \widetilde{P}_1 \eta \frac{T\eta}{3} (\widetilde{P}_0 + \widetilde{P}_1)^2 \gamma(z) \alpha_2(t) \\
&= \eta \frac{T\eta}{3} (\widetilde{P}_0 + \widetilde{P}_1)^3 \gamma(z) \alpha_2(t) \le \left(\frac{T\eta}{3}\right)^2 \left(\widetilde{P}_0 + \widetilde{P}_1\right)^3 \gamma(z) \alpha_1(t) \\
&\le \eta \left(\frac{T\eta}{3}\right)^2 \left(\widetilde{P}_0 + \widetilde{P}_1\right)^3 \gamma(z) \alpha_1(t), \qquad t \in [0,T], \quad (7.12)
\end{aligned}
$$

and, therefore, according to (7.12),

$$
\begin{aligned}
|r_4(\beta(t),z)| &\le \eta \frac{\eta T}{3} (\widetilde{P}_0 + \widetilde{P}_1)^3 \gamma(z) \alpha_2(\beta(t)) \\
&\le \eta \frac{\eta T}{3} (\widetilde{P}_0 + \widetilde{P}_1)^3 \gamma(z) \frac{T}{3} \alpha_1(\beta(t)) \\
&\le \eta \frac{\eta T}{3} (\widetilde{P}_0 + \widetilde{P}_1)^3 \gamma(z) \frac{T}{3} k_T(\beta) \alpha_1(t) \\
&\le \eta \left(\frac{\eta T}{3}\right)^2 \left(\widetilde{P}_0 + \widetilde{P}_1\right)^3 \gamma(z) \alpha_1(t) \qquad (7.13)
\end{aligned}
$$

for a. e. $t \in [0,T]$. Arguing by induction, we find that, for all $m \ge 1$, the estimates

$$
\begin{aligned}
|r_m(t,z)| &\le \left(\frac{\eta T}{3}\right)^{m-2} \left(\widetilde{P}_0 + \widetilde{P}_1\right)^{m-1} \gamma(z) \alpha_1(t) \\
&\le \left(\widetilde{P}_0 + \widetilde{P}_1\right) G_2^{m-2} \gamma(z) \alpha_1(t) \\
&\le \eta \left(\widetilde{P}_0 + \widetilde{P}_1\right) G_2^{m-2} \gamma(z) \alpha_1(t), \qquad t \in [0,T],
\end{aligned}
$$

and

$$
\begin{aligned}
|r_m(\beta(t),z)| &\le \eta \left(\frac{\eta T}{3}\right)^{m-2} \left(\widetilde{P}_0 + \widetilde{P}_1\right)^{m-1} \gamma(z) \alpha_1(t) \\
&\le \eta \left(\widetilde{P}_0 + \widetilde{P}_1\right) G_2^{m-2} \gamma(z) \alpha_1(t), \quad t \in [0,T],
\end{aligned}
$$

are true, where

$$
G_2 := \frac{\eta T}{3} \left(\widetilde{P}_0 + \widetilde{P}_1\right).
$$

This, as in the proof of Theorem 1, yields

$$
\begin{aligned}
\left|x_{m+j}(t,z)-x_m(t,z)\right| &\le \sum_{i=1}^{j}\left|r_{m+i}(t,z)\right| \\
&\le \eta\big(\widetilde{P}_0+\widetilde{P}_1\big)\left(G_2^{m-1}+G_2^{m}+\cdots+G^{m+j-2}\right)\gamma(z)\alpha_1(t) \\
&= \eta\big(\widetilde{P}_0+\widetilde{P}_1\big)\,G_2^{m-1}\sum_{i=0}^{j-1}G_2^{i}\gamma(z)\alpha_1(t) \\
&\le \eta\big(\widetilde{P}_0+\widetilde{P}_1\big)G_2^{m-1}(I-G_2)^{-1}\gamma(z)\alpha_1(t) \qquad (7.14)
\end{aligned}
$$

for $t\in[0,T]$, $m\ge 1$, $j\ge 1$. The last inequality proves that the members of sequence (4.1) satisfy estimates (4.4), where $G=G_2$ and the function $g:[0,T]\times\mathbb{R}^n\to\mathbb{R}^n$ is given by formula (7.7). By virtue of condition (7.6) and Definition 1, this means that method (4.1) for problem (3.1), (3.2) is applicable with an estimate of the type $\big(T\eta(\widetilde{P}_0+\widetilde{P}_1)/3, g\big)$, as required. □

Remark 3. In the cases where the value $k_T(\beta)$ in the assumption (5.10) of Lemma 4 satisfies the inequality

$$1\le k_T(\beta)<\frac{3}{2},$$

condition (7.6) of Theorem 3 is less restrictive than condition (6.1) of Theorem 1.

8. Examples

The conditions of the theorems presented above are easily verified in concrete cases.

Example 2. Let us consider the system of two differential equations with argument deviation

$$x_1'(t)=\frac{t}{2}x_1\left(t^2\right)+x_2(t)-\frac{t}{8}\left(t^4+1\right), \tag{8.1}$$

$$x_2'(t)=\frac{1}{2}\,x_1(t)-\frac{1}{2}x_2\left(t^2\right)+\frac{3}{8}+\frac{t^2}{8},\qquad t\in[0,1], \tag{8.2}$$

under the inhomogeneous two-point boundary conditions

$$\frac{1}{2}x_1(0)+x_2(0)-\frac{1}{4}x_2(1)=0, \tag{8.3}$$

$$x_1(0)+x_1(1)-x_2(1)=\frac{1}{4}. \tag{8.4}$$

One can verify directly that the pair of functions

$$x_1(t)=\frac{1}{4}\left(t^2+1\right),\qquad x_2(t)=\frac{t}{2},\qquad t\in[0,1], \tag{8.5}$$

is a solution of problem (8.1), (8.2), (8.3).

Problem (8.1), (8.2), (8.3) is, of course, a particular case of problem (3.1), (3.2) when $T = 1$,

$$f_1(t) = -\frac{t}{8}\left(t^4 + 1\right), \qquad f_2(t) = \frac{3}{8} + \frac{t^2}{8}, \qquad t \in [0, 1],$$

the function determining the deviation of argument has the form

$$\beta(t) = t^2, \qquad t \in [0, 1], \tag{8.6}$$

the matrix-valued functions P_0 and P_1 are given by the equalities

$$P_0(t) = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad P_1(t) = \begin{pmatrix} \frac{t}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad t \in [0, 1], \tag{8.7}$$

and $f(t) = \operatorname{col}\left(f_1(t), f_2(t)\right)$, $t \in [0, 1]$. It is also clear that the boundary conditions (8.3), (8.4) can be rewritten in form (3.2) with

$$A = \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\frac{1}{4} \\ 1 & -1 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}. \tag{8.8}$$

The argument deviation (8.6) satisfies (5.10), and, therefore, by Lemma 4, the value $k_T(\beta)$ is finite and estimate (5.11) is true. It is easy to see that, in this case, $k_T(\beta) = 2$ because, in view of (5.2) and (8.6), $\alpha_1(\beta(t))(\alpha_1(t))^{-1} = 2t^2(1-t^2)(\alpha_1(t))^{-1} = t(1+t)$ for $t \in [0, 1]$.

It is obvious from (6.2) and (8.7) that, in this case,

$$\widetilde{P}_0 = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad \widetilde{P}_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Therefore, condition (7.6) is satisfied because the greatest eigenvalue of the matrix $\widetilde{P}_0 + \widetilde{P}_1 = \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is equal to $(1 + \sqrt{2})/2 \approx 1.207$, and, hence, is less than $3T^{-1} \max\{1, k_T(\beta)\} = \frac{3}{2}$. By virtue of Theorem 3, the method of successive approximations (4.1) is applicable for problem (8.1), (8.2), (8.3) with an estimate of the type

$$\left(\frac{2}{3}\begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, 3\alpha_1\gamma\right), \tag{8.9}$$

where α_1 and γ are functions given by formulae (5.2) and (6.3), respectively. Therefore, one can use Theorem 3 and construct the successive approximations defined by formula (4.1). After solving the approximate determining equations (4.14) for some m, this, according to Remark 1, will allow one to obtain approximations (4.15) to the solutions in question.

Let us put $x_{0i}(t) := z_i$, $i = 1,2$, $t \in [0,1]$, and construct the corresponding functions $x_m = \binom{x_{m1}}{x_{m2}} : [0,1] \times \mathbb{R} \to \mathbb{R}^2$ for some m. For $m = 1$, formula (4.1) gives

$$x_{11}(t,z) = -\frac{t^6}{48} + \frac{1}{4}\left(z_1 - \frac{1}{4}\right)t^2 + \left(4z_2 - \frac{z_1}{4} + \frac{1}{3}\right)t + z_1, \tag{8.10}$$

$$x_{12}(t,z) = \frac{t^3}{24} + \left(2z_1 + 3z_2 - \frac{1}{24}\right)t + z_2, \qquad t \in [0,1], \tag{8.11}$$

and the corresponding equation (4.14) has the root

$$z_1 \approx 0.2541165999, \qquad z_2 \approx 0.006962987687. \tag{8.12}$$

Substituting the values (8.12) into formulae (8.10) and (8.11), we obtain the first approximation (4.15) of the form

$$X_{11}(t) = 0.2541165999 - \frac{t^6}{48} + 0.001029150000\,t^2 + 0.2976561342\,t,$$
$$X_{12}(t) = 0.006962987687 + 0.4874554962\,t + \frac{t^3}{24}, \qquad t \in [0,1].$$

We see from Figure 2 that the function X_{11} still differs significantly from the first component of the exact solution (8.5).

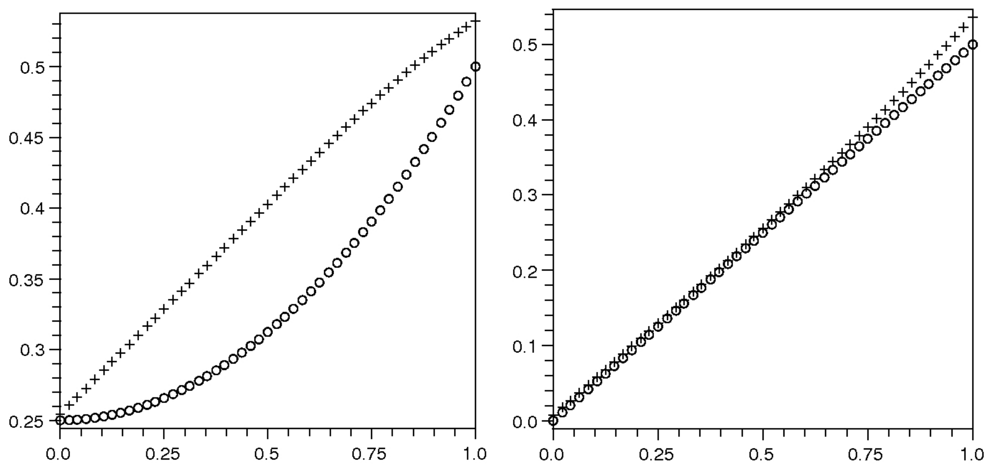

FIGURE 2. The first approximation (crosses) and the exact solution (circles) of the boundary value problem (8.1), (8.2), (8.3), (8.4)

Using expressions (8.10) and (8.11) for x_{1i}, $i = 1,2$, and computing function (4.1) for $m = 2$, we obtain

$$x_{21}(t,z) = -\frac{t^{14}}{1344} + \frac{1}{48}\left(z_1 - \frac{5}{4}\right)t^6 + \frac{1}{32}\left(\frac{5}{3} + 16z_2 - z_1\right)$$
$$+\frac{1}{4}\left(6z_2 - \frac{1}{3} + 5z_1\right)t^2 + \frac{1}{32}\left(64z_2 + \frac{69}{7} - \frac{119z_1}{3}\right)t, \tag{8.13}$$
$$x_{22}(t,z) = -\frac{1}{224}t^7 + \frac{1}{2}\left(\frac{11}{144} - \frac{7z_1}{12} - z_2\right)t^3 + \left(z_2 + \frac{1}{12} - \frac{z_1}{16}\right)t^2$$
$$+\frac{1}{2}\left(\frac{113z_1}{24} + 5z_2 - \frac{59}{252}\right)t + z_2, \qquad t \in [0,1], \tag{8.14}$$

and the corresponding system of approximate determining equations (4.14) has the root

$$z_1 \approx 0.2506508168, \qquad z_2 \approx -0.0005320453803. \tag{8.15}$$

Substituting (8.15) into (8.13) and (8.14), we obtain the second approximation, the graphs of whose components are presented on Figure 3:

$$X_{21}(t) = 0.2506508168 - \frac{t^{14}}{1344} - 0.02081977465t^6 + 0.04398447260t^4$$
$$+0.2291821196t^2, \tag{8.16}$$
$$X_{22}(t) = -0.0005320453803 - \frac{t^7}{224} - 0.03464602110t^3 + 0.06713561190t^2$$
$$+0.4716801924t, \qquad t \in [0,1]. \tag{8.17}$$

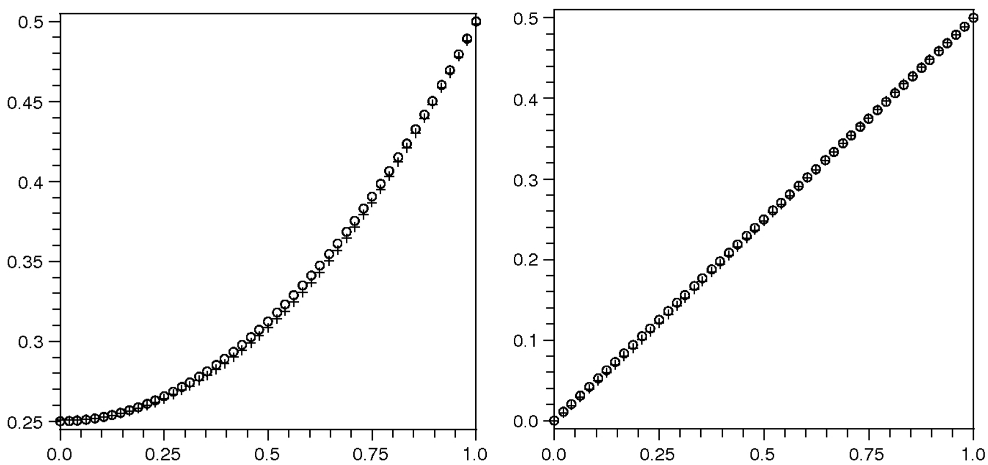

FIGURE 3. The second approximation (crosses) and the exact solution (circles) of the boundary value problem (8.1), (8.2), (8.3), (8.4)

Higher approximations are constructed by analogy. For example, using *Maple*,one obtains the roots of the fourth approximate determining equation

$$z_1 \approx 0.2499876647, \qquad z_2 \approx 0.000007202426452 \tag{8.18}$$

and the corresponding fourth approximation

$$\begin{aligned} X_{41}(t) = {} & 0.2499876647 - \frac{t^{62}}{9999360} - 0.00001240094662\,t^{30} \\ & + 0.00005031229545\,t^{22} - \frac{t^{18}}{64512} + \frac{t^{16}}{129024} - 0.00006201770500\,t^{14} \\ & - 0.0004454679680\,t^{10} + 0.001534762062\,t^{8} - 0.001579817533\,t^{6} \\ & + 0.0005047247500\,t^{4} - 0.0001525352967\,t^{3} + 0.2501969927\,t^{2} \\ & + 0.0000021079\,t, \end{aligned} \tag{8.19}$$

$$\begin{aligned} X_{42}(t) = {} & 0.000007202426452 - \frac{11t^{31}}{4999680} - 0.00005787007260\,t^{15} \\ & + 0.0002071768032\,t^{11} - \frac{t^{9}}{32256} - 0.0002156414800\,t^{7} \\ & - 0.0008451753460\,t^{5} + 0.002821512780\,t^{4} - 0.002444944367\,t^{3} \\ & + 0.0003682070000\,t^{2} + 0.5001968735\,t, \qquad t \in [0,1]. \end{aligned} \tag{8.20}$$

The results of computation clearly show that functions (8.19) and (8.20) provide a good approximation of solution (8.5). In particular, numbers (8.18) are rather accurate approximations of the initial values $\frac{1}{4}$ and 0 of the exact solution at the point 0 (see also Table 1). In fact, already starting from $m = 2$, the graph of the mth approximation constructed according to the method under consideration practically coincides with that of the exact solution.

i	$X_{i1}(0)$	$X_{i2}(0)$
1	0.2541165999	0.006962987687
2	0.2506508168	−0.0005320453803
3	0.2499480368	−0.00003329847918
4	0.2499876647	0.000007202426452

TABLE 1. Initial values of the approximate solutions of problem (8.1), (8.2), (8.3)

Along with condition (7.6), inequality (6.1) for the problem considered in Example 2 is satisfied as well. It should be noted, however, that condition (3.6), which follows from the assumptions of the work [1], is not fulfilled.

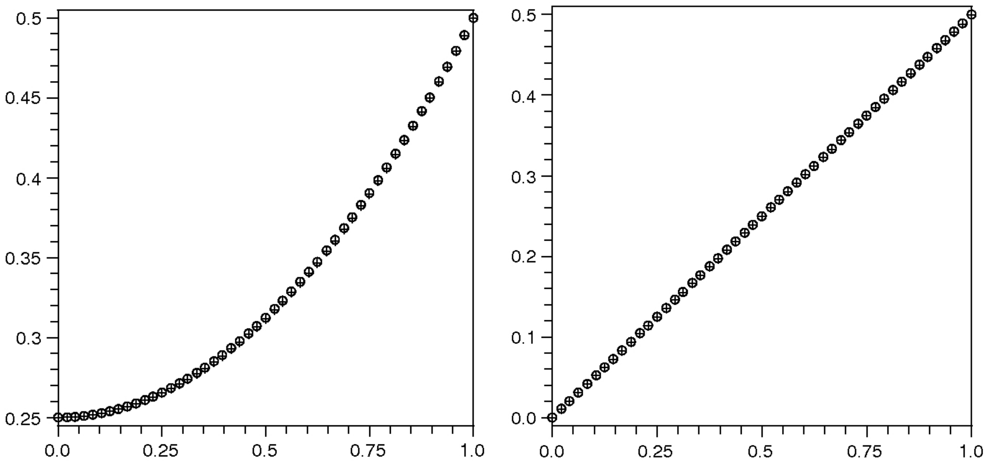

FIGURE 4. The third approximation (crosses) and the exact solution (circles) of the boundary value problem (8.1), (8.2), (8.3), (8.4)

Example 3. Consider the boundary value problem (8.3), (8.4) for the system of equations

$$x_1'(t) = \frac{t}{2}x_1\left(1-\tfrac{3t}{4}(1-t)\right) + x_2(t) - \frac{t}{8}\left(1+\left(1-\frac{3t}{4}(1-t)\right)^2\right), \tag{8.21}$$

$$x_2'(t) = \frac{1}{2}\,x_1(t) - \frac{1}{2}x_2\left(1-\tfrac{3t}{4}(1-t)\right) + \frac{1}{16}\left(10-3t+t^2\right), \qquad t \in [0,1], \tag{8.22}$$

Clearly, problem (8.21), (8.22), (8.3) is a particular case of problem (3.1), (3.2), where $T = 1$,

$$f_1(t) = -\frac{t}{8}\left(1+\left(1-\frac{3t}{4}(1-t)\right)^2\right), \qquad f_2(t) = \frac{1}{16}\left(10-3t+t^2\right), \qquad t \in [0,1],$$

the argument deviation β has the form

$$\beta(t) = 1 - \frac{3t}{4}(1-t), \qquad t \in [0,1], \tag{8.23}$$

and P_0, P_1, A, B, and d are given by formulae (8.7), (8.8). Function (8.23), as is easy to verify, transforms the interval $[0,1]$ into itself and satisfies condition (5.10), and the value of $k_T(\beta)$ in (5.11) is equal to $\frac{3}{4}$. Therefore, by Theorem 3, the method of successive approximations for problem (8.21), (8.22), (8.3) is applicable with an estimate of the type

$$\left(\frac{1}{3}\begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, 3\alpha_1\gamma\right), \tag{8.24}$$

where α_1 and γ are functions given by formulae (5.2) and (6.3), respectively. It is worth pointing out that estimate (8.24) is sharper than estimate (8.9) appearing in Example 2, which differs from Example 3 only by the forcing terms and the type of argument deviation. Note that the same function (8.5) is an exact solution of the problem indicated.

i	$X_{i1}(0)$	$X_{i2}(0)$
0	0.5429142442	−0.07493943798
1	0.2407327110	0.01169161945
2	0.2497717670	−0.0001512180660
3	0.2500928841	−0.00009129853295
4	0.2500037453	0.000002253878265
5	0.2499986308	0.000001057944723

TABLE 2. Initial values of the approximate solutions of problem (8.21), (8.22), (8.3)

Carrying out the computation according to formulae (4.1) and solving the approximate determining equations (4.14), we have constructed five approximations X_{i1}, X_{i2}, $i = 1, \ldots, 5$, to the respective components of a solution of problem (8.21), (8.22), (8.3). The results of computation (see, e. g., Table 2) show rather clearly that the approximations constructed according the scheme suggested are very accurate already on the first few steps of iteration.

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NONTRIVIAL SYMMETRIC SOLUTION OF A NONLINEAR SECOND-ORDER THREE-POINT BOUNDARY VALUE PROBLEM

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Abstract. In this paper, we study the existence of nontrivial symmetric solution for the second-order three-point boundary value problem for a function $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ which is continuous and $f(t,\cdot)$ is symmetric on $[0,1]$. We shall formulate conditions on f which guarantee the existence of nontrivial symmetric solution. As an application, we also give some examples to demonstrate our results.

2000 *Mathematics Subject Classification:* 34B10, 34B15

Keywords: three-point boundary value problem, nonlinear alternative of Leray-Schauder, nontrivial symmetric solution

1. Introduction

In this paper, we study the existence of nontrivial symmetric solution for the three-point boundary value problem

$$u''(t)+f(t,u(t))=0,\quad 0<t<1, \tag{1.1}$$

$$u(0)=u(1)=\alpha u(\eta), \tag{1.2}$$

where $\eta\in(0,1)$, $\alpha\in\mathbb{R}$, $f\in C([0,1]\times\mathbb{R},\mathbb{R})$, $f(\cdot,x)$ is symmetric on $[0,1]$ for every $x\in\mathbb{R}$, and $\mathbb{R}=(-\infty,+\infty)$.

The three-point boundary value problems for ordinary differential equations arise in a variety of different areas of applied mathematics and physics. For example, the vibrations of a guy wire of a uniform cross-section and composed of three parts of different densities can be set up as a three-point boundary value problem; also many problems in the theory of elastic stability can be handled as a multi-point problem. Many authors studied nonlinear three-point or multi-point boundary value problems and many excellent results have been established. For detail, we refer the reader to [2–9, 11–17, 19] for some recent results of nonlinear three-point boundary value problems. But the problems of the existence of symmetric positive solutions to the

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second-order three-point boundary value problems has received very little attention in the literature to the best of the author's knowledge. Very recently, Kosmatov [10] studied the existence of triple symmetric solutions for a multi-point boundary value problem by Leggett–Williams fixed point theorem. In [18], we studied the existence of symmetric positive for boundary value problem (1.1), (1.2) by Schauder's fixed point theorem. Motivated by the above mentioned papers, in this paper, we shall study the existence of nontrivial symmetric solution for the boundary value problem (1.1), (1.2) and establish some simple criteria of this problem.

The paper is organized as follows. In Section 2, we obtain some existence results for nontrivial symmetric solution of the boundary value problem (1.1), (1.2). In Section 3, as an application, we give some examples to illustrate the results we obtained. The key tool in our approach is the following Leray–Schauder nonlinear alternative, see for example [1].

Theorem 1.1. *Let E be Banach space and Ω be a bounded open subset of E, $0 \in \Omega$, $T\colon \overline{\Omega} \to E$ be a completely continuous operator. Then, either there exist $x \in \partial\Omega$ and $\lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

2. Main results

Let $E = C([0,1])$ be a Banach space equipped with norm $\|y\| = \sup_{t\in[0,1]} |y(t)|$ for any $y \in E$. A solution $u(t)$ of the boundary value problem (1.1),(1.2) is called a nontrivial symmetric solution if $u(t) \not\equiv 0$ and $u(t) = u(1-t)$ for all $t \in [0,1]$. To state and prove the main results of this paper, we first give some lemmas.

Lemma 2.1. *Let $h \in C([0,1]), \alpha \neq 1$, $\eta \in (0,1)$, then the three-point boundary value problem*

$$u'' + h(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = u(1) = \alpha u(\eta) \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)h(s)ds + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)ds, \tag{2.3}$$

where

$$G(x,y) = \begin{cases} x(1-y) & \textit{for } 0 \le x \le y \le 1, \\ y(1-x) & \textit{for } 0 \le y \le x \le 1. \end{cases}$$

Proof. From (2.1) we have $u''(t) = -h(t)$. For $t \in [0,1]$, integrating from 0 to t, we get

$$u'(t) = -\int_0^t h(s)ds + B.$$

For $t \in [0,1]$, integrating from 0 to t yields $u(t) = -\int_0^t \left(\int_0^x h(s)ds\right) dx + Bt + A$, i. e.,

$$u(t) = -\int_0^t (t-s)h(s)ds + Bt + A.$$

Thus, $u(0) = A$ and

$$u(1) = -\int_0^1 (1-s)h(s)ds + B + A,$$
$$u(\eta) = -\int_0^\eta (\eta-s)h(s)ds + \eta B + A.$$

Combining with (2.2) we conclude that

$$B = \int_0^1 (1-s)h(s)ds,$$
$$A = \frac{\alpha\eta}{1-\alpha}\int_0^1 (1-s)h(s)ds - \frac{\alpha}{1-\alpha}\int_0^\eta (\eta-s)h(s)ds.$$

Therefore, the three-point boundary value problem (2.1), (2.2) has a unique solution

$$\begin{aligned} u(t) = &-\int_0^t (t-s)h(s)ds + t\int_0^1 (1-s)h(s)ds \\ &+ \frac{\alpha\eta}{1-\alpha}\int_0^1 (1-s)h(s)ds - \frac{\alpha}{1-\alpha}\int_0^\eta (\eta-s)h(s)ds \\ = &\int_0^1 G(t,s)h(s)ds + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)ds. \end{aligned}$$

This completes the proof. □

The following lemma is obvious.

Lemma 2.2. *For any $x, y \in [0,1]$, we have*

(1) $G(x,y) = G(1-x, 1-y)$,
(2) $G(x,y) \le G(y,y) = y(1-y) \le \frac{1}{4}$.

Lemma 2.3. *Let $\alpha \neq 1$, $\eta \in (0,1)$, and $h \in C([0,1])$ be symmetric on $[0,1]$. Then the unique solution $u(t)$ of the boundary value problem (2.1), (2.2) is symmetric on $[0,1]$.*

Proof. From (2.3) we have

$$u(t) = \int_0^1 G(t,s)h(s)ds + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)h(s)ds.$$

Therefore, for any $t \in [0,1]$, by Lemma 2.2 we get

$$
\begin{aligned}
u(1-t) &= \int_0^1 G(1-t,s)h(s)ds + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)h(s)ds \\
&= \int_1^0 G(1-t,1-s)h(1-s)d(1-s) + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)h(s)ds \\
&= \int_0^1 G(t,s)h(s)ds + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)h(s)ds \\
&= u(t).
\end{aligned}
$$

i. e., the solution $u(t)$ is symmetric on [0,1]. □

Define an integral operator $T: E \to E$ by the formula

$$
Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)f(s,u(s))ds. \tag{2.4}
$$

By Lemma 2.1, the boundary value problem (2.1), (2.2) has a solution $u = u(t)$ if and only if u is a fixed point of the operator T defined by (2.4). So we only need to seek a fixed point of T in E. By the Ascoli–Arzela theorem, we can prove that T is a completely continuous operator.

Now we present and prove our main results.

Theorem 2.1. *Suppose that $f \in C([0,1]\times\mathbb{R},\mathbb{R})$, $f(t,0) \not\equiv 0$ $\alpha \neq 1$, and there exist two non-negative symmetric functions $a, b \in L^1[0,1]$ such that*

$$
|f(t,x)| \le a(t)|x| + b(t) \quad \text{for a. e. } t \in [0,1] \text{ and all } x \in \mathbb{R}
$$

and

$$
\int_0^1 G(s,s)a(s)ds + \frac{|\alpha|}{|1-\alpha|}\int_0^1 G(\eta,s)a(s)ds < 1.
$$

Then the boundary value problem (1.1), (1.2) has at least one nontrivial symmetric solution $u^* \in C([0,1])$.

Proof. Let

$$
A = \int_0^1 G(s,s)a(s)ds + \frac{|\alpha|}{|1-\alpha|}\int_0^1 G(\eta,s)a(s)ds \tag{2.5}
$$

and

$$
B = \int_0^1 G(s,s)b(s)ds + \frac{|\alpha|}{|1-\alpha|}\int_0^1 G(\eta,s)b(s)ds.
$$

Then we have $A < 1$. Since $f(t,0) \not\equiv 0$, there exists a subinterval $[\sigma,\tau] \subset [0,1]$ such that

$$
\min_{\sigma \le t \le \tau} |f(t,0)| > 0.
$$

On the other hand, the inequality $b(t) \geq |f(t,0)|$ holds for a. e. $t \in [0,1]$ and thus we know that $B > 0$. Put $r = B(1-A)^{-1}$ and

$$\Omega = \{u \in C([0,1]) : u \text{ is symmetric on } [0,1] \text{ and } \|u\| < r\}.$$

Suppose that $u \in \partial\Omega$ and $\lambda > 1$ are such that $Tu = \lambda u$. Then

$$\begin{aligned}
\lambda r = \lambda\|u\| = \|Tu\| &= \max_{0\leq t\leq 1} |(Tu)(t)| \\
&\leq \int_0^1 G(s,s)|f(s,u(s))|ds + \frac{|\alpha|}{|1-\alpha|}\int_0^1 G(\eta,s)|f(s,u(s))|ds \\
&\leq \int_0^1 G(s,s)(a(s)|u(s)|+b(s))ds \\
&\quad + \frac{|\alpha|}{|1-\alpha|}\int_0^1 G(\eta,s)(a(s)|u(s)|+b(s))ds \\
&\leq \left(\int_0^1 G(s,s)a(s)ds + \frac{|\alpha|}{|1-\alpha|}\int_0^1 G(\eta,s)a(s)ds\right)\|u\| \\
&\quad + \int_0^1 G(s,s)b(s)ds + \frac{|\alpha|}{|1-\alpha|}\int_0^1 G(\eta,s)b(s)ds \\
&= A\|u\| + B = Ar + B.
\end{aligned}$$

Therefore,

$$\lambda \leq A + \frac{B}{r} = A + \frac{B}{B(1-A)^{-1}} = A + (1-A) = 1,$$

which contradicts the inequality $\lambda > 1$. By Theorem 1.1, T has a fixed point $u^* \in \bar{\Omega}$. By $f(t,0) \not\equiv 0$, the boundary value problem (1.1), (1.2) has a nontrivial symmetric solution $u^* \in C([0,1])$. This completes the proof. □

Theorem 2.2. *Suppose that $f \in C([0,1]\times\mathbb{R},\mathbb{R})$, $f(t,0) \not\equiv 0$, $\alpha \neq 1$, and there exist two non-negative symmetric functions $a,b \in L^1[0,1]$ such that*

$$|f(t,x)| \leq a(t)|x| + b(t) \quad \textit{for a. e. } t \in [0,1] \textit{ and all } x \in \mathbb{R}$$

Assume that one of the following hypotheses is satisfied:

(1) *There exists a $p > 1$ such that $a \in L^p[0,1]$ and*

$$\int_0^1 a^p(s)ds < \left(\frac{4|1-\alpha|(1+q)^{1/q}}{|1-\alpha|(1+q)^{1/q} + 4|\alpha|\eta(1-\eta)}\right)^p,$$

where $1/p + 1/q = 1$.

(2) *There exists a constant $\mu > -1$ such that*

$$a(s) \leq \frac{4^{1+\mu}|1-\alpha|}{|1-\alpha|+|\alpha|}[s(1-s)]^{\mu} \quad \textit{for a. e. } s \in [0,1].$$

(3) *The function a satisfies*

$$a(s) \le \frac{6|1-\alpha|}{|1-\alpha|+3|\alpha|\eta(1-\eta)} \quad \text{for a. e. } s \in [0,1]$$

and

$$\operatorname{mes}\left\{s \in [0,1]:\ a(s) < \frac{6|1-\alpha|}{|1-\alpha|+3|\alpha|\eta(1-\eta)}\right\} > 0.$$

Then the boundary value problem (1.1)*,* (1.2) *has at least one nontrivial solution* $u^* \in C([0,1])$.

Proof. Let A be given by formula (2.5). In order to apply Theorem 2.1, we only need to prove that $A < 1$.

(1) By using the Hölder inequality, we have

$$\begin{aligned}
A &\le \left(\int_0^1 a^p(s)ds\right)^{1/p}\left\{\left(\int_0^1 [G(s,s)]^q ds\right)^{1/q} + \frac{|\alpha|}{|1-\alpha|}\left(\int_0^1 [G(\eta,s)]^q ds\right)^{1/q}\right\} \\
&\le \left(\int_0^1 a^p(s)ds\right)^{1/p}\left(\frac{1}{4} + \frac{|\alpha|}{|1-\alpha|}\,\frac{\eta(1-\eta)}{(1+q)^{1/q}}\right) \\
&< \frac{4|1-\alpha|(1+q)^{1/q}}{|1-\alpha|(1+q)^{1/q}+4|\alpha|\eta(1-\eta)}\,\frac{|1-\alpha|(1+q)^{1/q}+4|\alpha|\eta(1-\eta)}{4|1-\alpha|(1+q)^{1/q}} = 1.
\end{aligned}$$

(2) In this case, we have

$$\begin{aligned}
A &\le \frac{4^{1+\mu}|1-\alpha|}{|1-\alpha|+|\alpha|}\left(\int_0^1 G(s,s)[s(1-s)]^\mu ds + \frac{|\alpha|}{|1-\alpha|}\int_0^1 G(\eta,s)[s(1-s)]^\mu ds\right) \\
&\le \frac{4^{1+\mu}|1-\alpha|}{|1-\alpha|+|\alpha|}\left(\int_0^1 [s(1-s)]^{1+\mu} ds + \frac{|\alpha|}{|1-\alpha|}\int_0^1 G(s,s)[s(1-s)]^\mu ds\right) \\
&< \frac{4^{1+\mu}|1-\alpha|}{|1-\alpha|+|\alpha|}\left(\frac{1}{4^{1+\mu}} + \frac{|\alpha|}{|1-\alpha|}\,\frac{1}{4^{1+\mu}}\right) \\
&= \frac{4^{1+\mu}|1-\alpha|}{|1-\alpha|+|\alpha|}\,\frac{|1-\alpha|+|\alpha|}{4^{1+\mu}|1-\alpha|} = 1.
\end{aligned}$$

(3) In this case, we have

$$\begin{aligned}
A &< \frac{6|1-\alpha|}{|1-\alpha|+3|\alpha|\eta(1-\eta)}\left(\int_0^1 G(s,s)ds + \frac{|\alpha|}{|1-\alpha|}\int_0^1 G(\eta,s)ds\right) \\
&= \frac{6|1-\alpha|}{|1-\alpha|+3|\alpha|\eta(1-\eta)}\left(\frac{1}{6} + \frac{|\alpha|\eta(1-\eta)}{2|1-\alpha|}\right) = 1.
\end{aligned}$$

The proof is complete. $\square$

Corollary 2.1. *Suppose that* $f \in C([0,1]\times\mathbb{R},\mathbb{R})$*,* $f(t,0)\not\equiv 0$*,* $\alpha \neq 1$*, and*

$$\Delta := \limsup_{|x|\to\infty} \max_{t\in[0,1]} \left|\frac{f(t,x)}{x}\right| < \frac{6|1-\alpha|}{|1-\alpha|+3|\alpha|\eta(1-\eta)}\,.$$

Then problem (1.1)*,* (1.2) *has at least one nontrivial solution* $u^* \in C([0,1])$*.*

Proof. Let us put $\varepsilon = \frac{1}{2}\left(\frac{6|1-\alpha|}{|1-\alpha|+3|\alpha|\eta(1-\eta)} - \Delta\right)$. Then there exists $c > 0$ such that

$$|f(t,x)| \le \left(\frac{6|1-\alpha|}{|1-\alpha|+3|\alpha|\eta(1-\eta)} - \varepsilon\right)|x|$$

for $(t,x)\in[0,1]\times\mathbb{R}\setminus(-c,c)$. Set $M = \max\{|f(t,x)| : (t,x)\in[0,1]\times[-c,c]\}$. Then

$$|f(t,x)| \le \left(\frac{6|1-\alpha|}{|1-\alpha|+3|\alpha|\eta(1-\eta)} - \varepsilon\right)|x| + M$$

for $(t,x)\in[0,1]\times\mathbb{R}$. Consequently, the assumption of Theorem 2.2(3) is satisfied, where

$$a(s) = \frac{6|1-\alpha|}{|1-\alpha|+3|\alpha|\eta(1-\eta)} - \varepsilon$$

and $b(s) = M$ for $s\in[0,1]$. □

Corollary 2.2. *Suppose that* $f \in C([0,1]\times\mathbb{R},\mathbb{R})$*,* $f(t,0)\not\equiv 0$*,* $\alpha\in(0,1)$*, and there exist two non-negative symmetric functions* $a,b\in L^1[0,1]$ *such that*

$$|f(t,x)| \le a(t)|x| + b(t) \quad \text{for a. e. } t\in[0,1] \text{ and all } x\in\mathbb{R}.$$

Let one of following assumptions hold:

(1) *There exists constant* $p > 1$ *such that* $a\in L^p[0,1]$ *and*

$$\int_0^1 a^p(s)ds < (4(1-\alpha))^p\,.$$

(2) *There exists constant* $\mu > -1$ *such that*

$$a(s) \le 4^{1+\mu}(1-\alpha)\,(s(1-s))^{\mu} \quad \text{for a. e. } s\in[0,1].$$

(3) *The function* a *satisfies the estimate*

$$a(s) \le 6\,(1-\alpha) \quad \text{for a. e. } s\in[0,1].$$

Then the boundary value problem (1.1)*,* (1.2) *has at least one nontrivial solution* $u^*\in C([0,1])$*.*

Proof. The validity of the corollary follows immediately from Theorem 2.2 by using the inequality $\eta(1-\eta)\le\frac{1}{4}$. □

Corollary 2.3. *Suppose that $f \in C([0,1]\times\mathbb{R},\mathbb{R})$, $f(t,0)\not\equiv 0$, $\alpha\in(0,1)$, and*

$$\Lambda := \limsup_{|x|\to\infty}\max_{t\in[0,1]}\left|\frac{f(t,x)}{x}\right| \le 6(1-\alpha).$$

Then the boundary value problem (1.1), (1.2) has at least one nontrivial solution $u^*\in C([0,1])$.

Proof. The validity of the corollary follows immediately from Corollary 2.1 by using the inequality $\eta(1-\eta)\le\frac{1}{4}$. □

3. Examples

In this section, in order to illustrate the results obtained, we consider some examples.

Example 3.1. Consider the three-point boundary value problem

$$\begin{gathered} u''+\min\{t,1-t\}u(5\sin u-7)+\sin(t(1-t))=0, \quad 0<t<1, \\ u(0)=u(1)=-u(1/2). \end{gathered} \tag{3.1}$$

Set $\alpha=-1$, $\eta=\frac{1}{2}$, $f(t,x)=\min\{t,1-t\}x(5\sin x-7)+\sin[t(1-t)]$ for $(t,x)\in[0,1]\times\mathbb{R}$, and $a(t)=12\min\{t,1-t\}$ and $b(t)=\sin[t(1-t)]$ for $t\in[0,1]$. It is easy to see that $a,b\in L^1[0,1]$ are non-negative, symmetric, and the inequality $|f(t,x)|\le a(t)|x|+b(t)$ holds for all $(t,x)\in[0,1]\times\mathbb{R}$. Moreover, we have

$$A=\int_0^1 G(s,s)a(s)ds+\frac{|\alpha|}{|1-\alpha|}\int_0^1 G(\eta,s)a(s)ds=\frac{5}{8}+\frac{1}{4}<1.$$

Hence, by Theorem 2.1, the boundary value problem (3.1) has at least one nontrivial symmetric solution.

Example 3.2. Consider the three-point boundary value problem

$$\begin{gathered} u''+\frac{2\sqrt{6t(1-t)}}{1+u^4}u^5e^{-u^2}+5e^{t(1-t)}=0, \quad 0<t<1, \\ u(0)=u(1)=2u(1/4). \end{gathered} \tag{3.2}$$

Set $\alpha=2$, $\eta=\frac{1}{4}$, $f(t,x)=\frac{2\sqrt{6t(1-t)}}{1+x^4}x^5e^{-x^2}+5e^{t(1-t)}$ for $(t,x)\in[0,1]\times\mathbb{R}$, and $a(t)=2\sqrt{6t(1-t)}$ and $b(t)=5e^{t(1-t)}$ for $t\in[0,1]$. It is easy to prove that $a,b\in L^1[0,1]$ are non-negative, symmetric, and the inequality $|f(t,x)|\le a(t)|x|+b(t)$ holds for all $(t,x)\in[0,1]\times\mathbb{R}$. If we put $p=q=2$, then

$$\int_0^1 a^p(s)ds=\int_0^1 24s(1-s)ds=4,$$

and

$$\left(\frac{4|1-\alpha|(1+q)^{1/q}}{|1-\alpha|(1+q)^{1/q}+4|\alpha|\eta(1-\eta)}\right)^p=64(7-4\sqrt{3}).$$

Therefore

$$\int_0^1 a^p(s)ds < \left(\frac{4|1-\alpha|(1+q)^{1/q}}{|1-\alpha|(1+q)^{1/q}+4|\alpha|\eta(1-\eta)}\right)^p.$$

Hence, by Theorem 2.2(1), the boundary value problem (3.2) has at least one nontrivial symmetric solution.

Example 3.3. Consider the three-point boundary value problem

$$\begin{gathered} u''+\frac{4u^3\sqrt{t(1-t)}}{5(1+u^2)}+4t(1-t)u-\max\{t,1-t\}=0, \quad 0<t<1, \\ u(0)=u(1)=-2u(\eta). \end{gathered} \tag{3.3}$$

Set $\alpha=-2$, $\eta\in(0,1)$, $f(t,x)=\frac{4x^3\sqrt{t(1-t)}}{5(1+x^2)}+4t(1-t)x-\max\{t,1-t\}$ for $(t,x)\in[0,1]\times\mathbb{R}$, and $a(t)=\frac{4}{5}\sqrt{t(1-t)}+4t(1-t)$ and $b(t)=\max\{t,1-t\}$ for $t\in[0,1]$. It is easy to prove that $a,b\in L^1[0,1]$ are non-negative, symmetric, and the inequality $|f(t,x)|\le a(t)|x|+b(t)$ holds for all $(t,x)\in[0,1]\times\mathbb{R}$. If we put $\mu=\frac{1}{2}$, then

$$\begin{aligned} a(s)=&\frac{4}{5}\sqrt{s(1-s)}+4s(1-s)\le\frac{24}{5}\sqrt{s(1-s)} \\ =&\frac{4^{1+\mu}|1-\alpha|}{|1-\alpha|+|\alpha|}[s(1-s)]^\mu \end{aligned}$$

for $s\in[0,1]$. Hence, by Theorem 2.2(2), the boundary value problem (3.3) has at least one nontrivial symmetric solution.

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AN INTERESTING MATRIX EQUATION

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Abstract. We study the solutions of a special matrix equation, particularly, their eigenvalues. These matrix solutions have an interesting relationship to unitary matrices.

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1. INTRODUCTION

In this note, we study the matrix equation

$$AEA^* = E, \tag{1.1}$$

where

$$E = ((-1)^r \delta_{r,n+1-j})^n_{r,j=1}, \quad n = 2k,\ 1 \le k < +\infty, \tag{1.2}$$

$\delta_{r,j}$ is the Kronecker delta, and A is an $n \times n$ matrix over the complex numbers $\mathbb{C}$. This study is motivated by the theory of differential operators where this equation plays a critical role in the characterization of self-adjoint domains [1–4].

2. EIGENVALUES

In this section we characterize the eigenvalues of matrix solutions of equation (1.1). Let $\mathbb{1}$ denote the identity matrix.

Lemma 1. *We have*

$$EE = -\mathbb{1}, \quad E^* = E^{-1} = -E, \quad \det(E) = 1. \tag{2.1}$$

If A is a solution of (1.1)*, then*

$$\det(A) = \pm 1. \tag{2.2}$$

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Proof. The properties of E follow from a direct computation. For (2.2) note that $\det(A) = \det(A^*)$ and thus

$$\det(AEA^*) = \det(A)\det(E)\det(A^*) = [\det(A)]^2 = 1. \tag{2.3}$$

□

Remark 1. The simple examples $A = \mathbb{1}$ and $A = i\,\mathbb{1}$ show that both signs can occur in (2.2).

Lemma 2. *If A is a solution of* (1.1)*, then*[*]

$$\sigma(A^{-1}) = \sigma(A^*). \tag{2.4}$$

Proof. From (1.1) and (2.1) we get $AEA^*E = EE = -\mathbb{1}$. Hence $(AE)^{-1} = -A^*E$ and $A^* = -EA^{-1}E$, $A^{-1} = -EA^*E$. Therefore,

$$\begin{aligned}\det(A^{-1} - \lambda\mathbb{1}) &= \det(-EA^*E - \lambda E(-E)\mathbb{1})\\ &= \det(-E)\det(A^* - \lambda\mathbb{1})\det(E)\\ &= \det(A^* - \lambda\mathbb{1}),\end{aligned}$$

and (2.4) follows. □

Remark 2. Of course (2.4) holds for any unitary matrix U, because $U^* = U^{-1}$. All eigenvalues of a unitary matrix lie on the unit circle of the complex plane. Below we derive the corresponding result for matrix solutions A of (1.1). We will see that the eigenvalues of A need not lie on the unit circle but those which do are related to each other in interesting ways and also those which are not on the unit circle have interesting relationships to each other.

Let the spectrum of A have the form

$$\sigma(A) = \{\lambda_j = r_j e^{i\theta_j} : j = 1, \ldots, n\}, \tag{2.5}$$

where $r_j > 0$ and $-\pi < \theta_j \leq \pi$ for $j = 1, \ldots, n$. We can now give our main result.

Theorem 1. *Let A be a solution of* (1.1)*. Then the following properties hold:*

(1) *If, for some j, an eigenvalue λ_j of A is not on the unit circle, i. e., $r_j \neq 1$, then $\frac{1}{r_j} e^{i\theta_j}$ is also an eigenvalue of A.*
(2) *Let l be the number of pairs of eigenvalues of A not on the unit circle, given by* (1) *and counting multiplicity, then l can be any one of the numbers $0, 1, 2, \ldots, k$.*
(3) *Assume that there exist l $(0 \leq l \leq k)$ pairs of eigenvalues*

$$r_1 e^{i\theta_1}, \frac{1}{r_1} e^{i\theta_1}, \ldots, r_l e^{i\theta_l}, \frac{1}{r_l} e^{i\theta_l}$$

[*]As usual, the symbol $\sigma(A)$ stands for the spectrum of A.

of A which are not on the unit circle, where $r_j > 0$, $r_j \neq 1$, $j = 0, 1, \ldots, l$ ($j = 0$ means that there are no such eigenvalues, i. e., all eigenvalues are on the unit circle), then there exist $d = 2(k-l)$ eigenvalues of A,

$$e^{i\theta_{l+1}}, \cdots, e^{i\theta_{l+d}}$$

on the unit circle and θ_j satisfy the relations

$$\sum_{j=1}^{l} 2\theta_j + \sum_{j=1}^{d} \theta_{l+j} \in \{j\pi : j = 0, \pm 2, \ldots, \pm 2(k-1), 2k\} \tag{2.6}$$

if $\det(A) = 1$, *and the relations*

$$\sum_{j=1}^{l} 2\theta_j + \sum_{j=1}^{d} \theta_{l+j} \in \{j\pi : j = \pm 1, \pm 3, \pm 5, \ldots, \pm(n-1)\} \tag{2.7}$$

if $\det(A) = -1$.

Proof. By Lemma 2, we have $\sigma(A^{-1}) = \sigma(A^*)$, i. e.,

$$\left\{\frac{1}{\lambda_j} : j = 1, \ldots, n\right\} = \{\overline{\lambda}_h : h = 1, \ldots, n\}.$$

Therefore, for any $\overline{\lambda}_h \in \sigma(A^*)$, there exist $\frac{1}{\lambda_j}$ such that $\overline{\lambda}_h = \frac{1}{\lambda_j}$, i. e.,

$$r_h e^{-i\theta_h} r_j e^{i\theta_j} = 1.$$

Hence, $r_h r_j = 1$ and $\theta_j = \theta_h$ (or $\theta_j - \theta_h = 2\pi$). Therefore, if for some j, $r_j e^{i\theta_j}$ with $r_j > 0$, $-\pi < \theta_j \leq \pi$, and $r_j \neq 1$ is an eigenvalue of A, then $\frac{1}{r_j} e^{i\theta_j}$ is also an eigenvalue of A.

By (1), the eigenvalues not on the unit circle occur in pairs where each pair has the same angle and the conclusion follows.

Conditions (2.6) and (2.7) follow from the fact that $\det(A) = \prod_{j=1}^{n} \lambda_j = \pm 1$. □

Remark 3. Let A satisfy (1.1). Then A may have all its eigenvalues on the unit circle or there may be none on the unit circle. In all other cases there are an even number of eigenvalues not on the unit circle and these can be paired so that any one pair has the same angle; the angles of the eigenvalues are related to each other according to (2.6) and (2.7). The examples in the next section will illustrate these points further.

3. Examples

In this section we give some examples to illustrate Theorem 1.

Example 1. Let $n = 2$. In the second order case either both eigenvalues are on the unit circle or neither one is.

CASE 1. Both eigenvalues are on the unit circle, i. e., $\sigma(A) = \left\{e^{i\theta_1}, e^{i\theta_2}\right\}$.

(1) When $\det(A) = 1$, we have $e^{i(\theta_1+\theta_2)} = 1$ and $\theta_2 = -\theta_1$ or $\theta_1 + \theta_2 = 2\pi$. Therefore, $\sigma(A) = \{e^{i\theta}, e^{-i\theta}\}$. Note that the eigenvalues of A are symmetric with respect to the real axis. When $\lambda = 1$ or $\lambda = -1$ is an eigenvalue, then it must have multiplicity 2.

(2) If $\det(A) = -1$, then we have $\theta_1 + \theta_2 = \pi$ and the eigenvalues of A are symmetric about the imaginary axis. When $\lambda = i$ or $\lambda = -i$ is an eigenvalue, then it must be a double eigenvalue.

CASE 2. The eigenvalues of A are not on the unit circle, i. e., $\sigma(A) = \left\{re^{i\theta}, \frac{1}{r}\, e^{i\theta}\right\}$, where $r > 0$ and $r \neq 1$.

(1) When $\det(A) = 1$, we have $e^{i(2\theta)} = 1$ and $\theta = 0$ or π. Then the eigenvalues of A are r and $\frac{1}{r}$, or $-r$ and $-\frac{1}{r}$.

(2) If $\det(A) = -1$, then we have $e^{i(2\theta)} = -1$ and $\theta = \frac{\pi}{2}$ or $-\frac{\pi}{2}$. Then the eigenvalues of A are ri and $\frac{1}{r}\, i$, or $-ri$ and $-\frac{1}{r}\, i$.

Example 2. Assume that $n = 4$. Then exactly one of the following three cases must hold:

(1) All the eigenvalues are on the unit circle, i. e., $\sigma(A) = \left\{e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}\right\}$, where the numbers θ_i are such that $\sum_{j=1}^{4} \theta_j = 0, \pm 2\pi, 4\pi$ if $\det(A) = 1$, and $\sum_{j=1}^{4} \theta_j = \pm\pi, \pm 3\pi$ if $\det(A) = -1$.

(2) One pair of eigenvalues is not on the unit circle and the other two eigenvalues are on the unit circle, i. e.,

$$\sigma(A) = \left\{e^{i\theta_1}, e^{i\theta_2}, re^{i\theta}, \frac{1}{r}\, e^{i\theta}\right\},$$

where $r > 0$ and $r \neq 1$. In this case θ_1, θ_2, and θ satisfy $\theta_1 + \theta_2 + 2\theta = 0, \pm 2\pi, 4\pi$ when $\det(A) = 1$, and $\theta_1 + \theta_2 + 2\theta = \pm\pi, \pm 3\pi$ when $\det(A) = -1$.

(3) No eigenvalue of A is on the unit circle. In this case there are two pairs of eigenvalues each pair having the same angle, i. e.,

$$\sigma(A) = \left\{r_1 e^{i\theta_1}, \frac{1}{r_1}\, e^{i\theta_1}, r_2 e^{i\theta_2}, \frac{1}{r_2}\, e^{i\theta_2}\right\},$$

where $r_j > 0$ and $r_j \neq 1$, and the numbers θ_1, θ_2 satisfy $\theta_1 + \theta_2 = 0, \pm\pi, 2\pi$ when $\det(A) = 1$, and $\theta_1 + \theta_2 = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$ when $\det(A) = -1$. Here θ_1 and θ_2 are not necessarily distinct.

Example 3. Assume that $n = 6$. There are exactly four subcases:

(1) All eigenvalues are on the unit circle, i. e.,

$$\sigma(A) = \left\{e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}, e^{i\theta_5} e^{i\theta_6}\right\},$$

where the numbers θ_j satisfy $\sum_{j=1}^{6}\theta_j = 0, \pm 2\pi, \pm 4\pi, 6\pi$ if $\det(A) = 1$, and $\sum_{j=1}^{6}\theta_j = \pm\pi, \pm 3\pi, \pm 5\pi$, if $\det(A) = -1$.

(2) One pair is not on the unit circle and the other four eigenvalues are on the unit circle , i. e.,

$$\sigma(A) = \left\{e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}, re^{i\theta}, \frac{1}{r}\, e^{i\theta}\right\},$$

where $r > 0$, $r \neq 1$, and the numbers θ_j and θ satisfy

$$\sum_{j=1}^{4}\theta_j + 2\theta = 0, \pm 2\pi, \pm 4\pi, 6\pi$$

when $\det(A) = 1$, and

$$\sum_{j=1}^{4}\theta_j + 2\theta = \pm\pi, \pm 3\pi, \pm 5\pi$$

when $\det(A) = -1$.

(3) Two pairs are not on the unit circle and the other two eigenvalues are on the unit circle, i. e.,

$$\sigma(A) = \left\{e^{i\theta_1}, e^{i\theta_2}, r_1e^{i\theta_3}, \frac{1}{r_1}\, e^{i\theta_3}, r_2e^{i\theta_4}, \frac{1}{r_2}\, e^{i\theta_4}\right\},$$

where $r_j > 0$ and $r_j \neq 1$, and the numbers θ_j satisfy

$$\theta_1 + \theta_2 + 2\theta_3 + 2\theta_4 = 0, \pm 2\pi, \pm 4\pi, 6\pi,$$

when $\det(A) = 1$, and

$$\theta_1 + \theta_2 + 2\theta_3 + 2\theta_4 = \pm\pi, \pm 3\pi, \pm 5\pi.$$

when $\det(A) = -1$.

(4) No eigenvalue is on the unit circle, i. e.,

$$\sigma(A) = \left\{r_1e^{i\theta_1}, \frac{1}{r_1}\, e^{i\theta_1}, r_2e^{i\theta_2}, \frac{1}{r_2}\, e^{i\theta_2}, r_3e^{i\theta_3}, \frac{1}{r_3}\, e^{i\theta_3}\right\},$$

where $r_j > 0$ and $r_j \neq 1$, and the numbers θ_j satisfy

$$\sum_{j=1}^{3}\theta_j = 0, \pm\pi, \pm 2\pi, 3\pi \tag{3.1}$$

when $\det(A) = 1$, and

$$\sum_{j=1}^{3}\theta_j = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2} \tag{3.2}$$

when $\det(A) = -1$.

Example 4. Let

$$A = \begin{pmatrix} r_1 e^{i\theta_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & r_2 e^{i\theta_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & r_3 e^{i\theta_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r_3} e^{i\theta_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r_2} e^{i\theta_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{r_1} e^{i\theta_1} \end{pmatrix} \tag{3.3}$$

Then A satisfies (1.1) for any $r_j > 0$ and $\theta_j \in (-\pi, \pi]$. Choosing $r_j = 1$ for $j = 1, 2, 3$ all six eigenvalues are on the unit circle and their angles satisfy the equations (3.1) or (3.2). By choosing three distinct θ_j we get three distinct eigenvalues each with multiplicity 2, by choosing $\theta_1 = \theta_2 \neq \theta_3$ we get two distinct eigenvalues, one having multiplicity 2, the other 4. Finally by choosing $\theta_1 = \theta_2 = \theta_3$ we get one eigenvalue with multiplicity 6. Clearly this example extends to $n = 2k$ for $k > 3$ and can be used to show that the eigenvalues on the unit circle can have multiplicity m for any m, $m = 2, 4, 6, \ldots, 2k$.

Remark 4. Just as in the above examples, for $n = 2k$ there are exactly $k + 1$ cases ranging from exactly j pairs not on the unit circle for $j = 1, 2, \ldots, k$ to all eigenvalues on the unit circle.

Remark 5. It is interesting to compare matrices satisfying (1.1) with unitary matrices. Clearly equation (1.1) is equivalent with

$$(AE)(EA)^* = \mathbb{1} \tag{3.4}$$

which can be compared to

$$UU^* = \mathbb{1},$$

but as Theorem 1 shows there are major differences, as well as similarities, in the behavior of the eigenvalues of the matrices satisfying these two equations. Lemma 1 holds for unitary matrices since $U^{-1} = U^*$ but all the eigenvalues of a unitary matrix lie on the unit circle. The fact that one E in (3.4) is “on the wrong side” is significant.

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