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ON FUZZY α-CONTINUOUS MULTIFUNCTIONS

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Abstract. In this paper we use fuzzy α-sets in order to obtain certain characterizations and properties of upper (or lower) fuzzy α-continuous multifunctions.

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Keywords: fuzzy α-open, fuzzy α-continuous, fuzzy multifunction

1. INTRODUCTION

In 1968 Chang [3] introduced fuzzy topological spaces by using fuzzy sets [12]. Since then several workers have contributed to this area: various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good many of them have been extended to the setting of multifunctions.

In 1988 Neubrunn [6] and others [9]] introduced the concept of α-continuous multifunctions. Njasted [7] and Mashhour [4] introduced α-open (α-closed) sets, respectively. Bin Shahna in [2] defined these concepts in the fuzzy setting. In this paper our purpose is to define upper (lower) fuzzy α-continuous multifunctions and to obtain several characterizations of upper (lower) fuzzy α-continuous multifunctions.

Fuzzy sets on a universe X will be denoted by μ, ρ, η, etc. Fuzzy points will be denoted by x_ϵ, y_ν, etc. For any fuzzy points x_ϵ and any fuzzy set μ, we write $x_\epsilon \in \mu$ iff $\epsilon \leq \mu(x)$. A fuzzy point x_ϵ is called quasi-coincident with a fuzzy set ρ, denoted by x_ϵ q ρ, iff $\epsilon + \rho(x) > 1$.

A fuzzy set μ is called quasi-coincident with a fuzzy set ρ, denoted by μ q ρ, iff there exists a $x \in X$ such that $\mu(x) + \rho(x) > 1$. [10, 11]

In this paper we use the concept of fuzzy topological space as introduced in [3]. By int (μ) and cl (μ), we mean the interior of μ and the closure of μ, respectively.

Let (X, τ) be a topological space in the classical sense and (Y, ν) be a fuzzy topological space. $F : X \to Y$ is called a fuzzy multifunction iff for each $x \in X$, $F(x)$ is a fuzzy set in Y. [8]

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Let $F : X \to Y$ be a fuzzy multifunction from a fuzzy topological space X to a fuzzy topological space Y. For any fuzzy set $\mu \leq X$, $F^+(\mu)$ and $F^-(\mu)$ are defined by $F^+(\mu) = \{x \in X : F(x) \leq \mu\}$, $F^-(\mu) = \{x \in X : F(x)q\mu\}$. [5]

2. Fuzzy α-continuous multifunction

Definition 1. Let (X, τ) be a fuzzy topological space and let $\mu \leq X$ be a fuzzy set. Then it is said that:

(i) μ is fuzzy α-open set [2] if $\mu \leq$ int cl int μ.
(ii) μ is fuzzy α-closed set [2] if $\mu \geq$ cl int cl μ.
(iii) μ is fuzzy semiopen set [1] if $\mu \leq$ cl int μ.
(iv) μ is fuzzy preopen set [2] if $\mu \leq$ int cl μ.

Definition 2. Let $F : X \to Y$ be a fuzzy multifunction from a fuzzy topological space (X, τ) to a fuzzy topological space (Y, ν). Then it is said that F is :

(1) Upper fuzzy α-continuous at $x_\epsilon \in X$ iff for each fuzzy open set μ of Y containing $F(x_\epsilon)$, there exists a fuzzy α-open set ρ containing x_ϵ such that $\rho \leq F^+(\mu)$.
(2) Lower fuzzy α-continuous at $x_\epsilon \in X$ iff for each fuzzy open set μ of Y such that $x_\epsilon \in F^-(\mu)$ there exists a fuzzy α-open set ρ containing x_ϵ such that $\rho \leq F^-(\mu)$.
(3) Upper (lower) fuzzy α-continuous iff it has this property at each point of X.

We know that a net $(x^\alpha_{\epsilon_\alpha})$ in a fuzzy topological space (X, τ) is said to be eventually in the fuzzy set $\rho \leq X$ if there exists an index $\alpha_0 \in J$ such that $x^\alpha_{\epsilon_\alpha} \in \rho$ for all $\alpha \geq \alpha_0$.

The following theorem states some characterizations of upper fuzzy α-continuous multifunction.

Definition 3. A sequence (x_{ϵ_n}) is said to α-converge to a point X if for every fuzzy α-open set μ containing x_ϵ there exists an index n_0 such that for $n \geq n_0$, $x_{\epsilon_n} \in \mu$. This is denoted by $x_{\epsilon_n} \to_\alpha x_\epsilon$.

Theorem 1. *Let $F : X \to Y$ be a fuzzy multifunction from a fuzzy topological (X, τ) to a fuzzy topological space (Y, ν). Then the following statements are equivalent:*

(i) *F is upper fuzzy α-continuous.*
(ii) *For each $x_\epsilon \in X$ and for each fuzzy open set μ such that $x_\epsilon \in F^+(\mu)$ there exists a fuzzy α -open set ρ containing x_ϵ such that $\rho \leq F^+(\mu)$.*
(iii) *$F^+(\mu)$ is a fuzzy α-open set for any fuzzy open set $\mu \leq Y$.*
(iv) *$F^-(\mu)$ is a fuzzy α-closed set for any fuzzy open set $\mu \leq Y$.*
(v) *For each $x_\epsilon \in X$ and for each net $(x^\alpha_{\epsilon_\alpha})$ which α-converges to x_ϵ in X and for each fuzzy open set $\mu \leq Y$ such that $x_\epsilon \in F^+(\mu)$, the net $(x^\alpha_{\epsilon_\alpha})$ is eventually in $F^+(\mu)$.*

Proof. (i)$\Leftrightarrow$(ii) this statement is obvious.

(i)$\Leftrightarrow$(iii). Let $x_\epsilon \in F^+(\mu)$ and let μ be a fuzzy open set. It follows from (i) that there exists a fuzzy α-open set ρ_{x_ϵ} containing x_ϵ such that $\rho_{x_\epsilon} \leq F^+(\mu)$. It follows that $F^+(\mu) = \bigvee_{x_\epsilon \in F^+(\mu)} \rho_{x_\epsilon}$ and hence $F^+(\mu)$ is fuzzy α-open.

The converse can be shown easily.

(iii)$\Rightarrow$(iv) Let $\mu \leq Y$ be a fuzzy open set. We have that $Y \setminus \mu$ is a fuzzy open set. From (iii), $F^+(Y \setminus \mu) = X \setminus F^-(\mu)$ is a fuzzy α-open set. Then it is obtained that $F^-(\mu)$ is a fuzzy α-closed set.

(i)$\Rightarrow$(v). Let $(x^\alpha_{\epsilon_\alpha})$ be a net which α -converges to x_ϵ in X and let $\mu \leq Y$ be any fuzzy open set such that $x_\epsilon \in F^+(\mu)$. Since F is an upper fuzzy α-continuous multifunction, it follows that there exists a fuzzy α-open set $\rho \leq X$ containing x_ϵ such that $\rho \leq F^+(\mu)$. Since $(x^\alpha_{\epsilon_\alpha})$ α-converges to x_ϵ, it follows that there exists an index $\alpha_o \in J$ such that $(x^\alpha_{\epsilon_\alpha}) \in \rho$ for all $\alpha \geq \alpha_o$ from here, we obtain that $x^\alpha_{\epsilon_\alpha} \in \rho \leq F^+(\mu)$ for all $\alpha \geq \alpha_o$. Thus the net $(x^\alpha_{\epsilon_\alpha})$ is eventually in $F^+(\mu)$.

(v)$\Rightarrow$(i). Suppose that is not true. There exists a point x_ϵ and a fuzzy open set μ with $x_\epsilon \in F^+(\mu)$ such that $\rho \not\leq F^+(\mu)$ for each fuzzy α-open set $\rho \leq X$ containing x_ϵ. Let $x_{\epsilon_\rho} \in \rho$ and $x_\epsilon \notin F^+(\mu)$ for each fuzzy α-open set $\rho \leq X$ containing x_ϵ. Then for the α-neighborhood net (x_{ϵ_ρ}), $x_{\epsilon_\rho} \to_\alpha x_\epsilon$, but (x_{ϵ_ρ}) is not eventually in $F^+(\mu)$. This is a contradiction. Thus, F is an upper fuzzy α-continuous multifunction. □

Remark 1. For a fuzzy multifunction $F : X \to Y$ from a fuzzy topological (X, τ) to a fuzzy topological space (Y, ν), the following implication holds:
Upper fuzzy continuous $\Longrightarrow$ Upper fuzzy α-continuous.

The following example show that the reverse need not be true.

Example 1. Let $X = \{x, y\}$ with topologies $\tau = \{X, \phi, \mu\}$ and $\nu = \{X, \phi, \rho\}$, where the fuzzy sets μ, ρ are defined as:

$$\mu(x) = 0.3, \quad \mu(y) = 0.6$$
$$\rho(x) = 0.7, \quad \rho(y) = 0.4$$

A fuzzy multifunction $F : (X, \tau) \to (Y, \nu)$ given by $x_\epsilon \to F(x_\epsilon) = \{x_\epsilon\}$ is upper α-continuous, but it is not upper continuous.

The following theorem states some characterizations of a lower fuzzy α-continuous multifunction.

Theorem 2. *Let $F : X \to Y$ be a fuzzy multifunction from a fuzzy topological (X, τ) to a fuzzy topological space (Y, ν). Then the following statements are equivalent.*

(i) *F is lower fuzzy α-continuous.*
(ii) *For each $x_\epsilon \in X$ and for each fuzzy open set μ such that $x_\epsilon \in F^-(\mu)$ there exists a fuzzy α-open set ρ containing x_ϵ such that $\rho \leq F^-(\mu)$.*
(iii) *$F^-(\mu)$ is a fuzzy α-open set for any fuzzy open set $\mu \leq Y$,*

(iv) *$F^+(\mu)$ is a fuzzy α-closed set for any fuzzy open set $\mu \leq Y$,*
(v) *For each $x_\epsilon \in X$ and for each net $(x^\alpha_{\epsilon_\alpha})$ which α-converges to x_ϵ in X and for each fuzzy open set $\mu \leq Y$ such that $x_\epsilon \in F^-(\mu)$, the net $(x^\alpha_{\epsilon_\alpha})$ is eventually in $F^-(\mu)$.*

Proof. It can be obtained similarly as Theorem 1. □

Theorem 3. *Let $F : X \to Y$ be a fuzzy multifunction from a fuzzy topological (X, τ) to a fuzzy topological space (Y, ν) and let $F(X)$ be endowed with subspace fuzzy topology. If F is an upper fuzzy α-continuous multifunction, then $F : X \to F(X)$ is an upper fuzzy α-continuous multifunction.*

Proof. Since F is an upper fuzzy α-continuous, $F(X \wedge F(X)) = F^+(\mu) \wedge \wedge F^+(F(X)) = F^+(\mu)$ is fuzzy α-open for each fuzzy open subset μ of Y. Hence $F : X \to F(X)$ is an upper fuzzy α-continuous multifunction. □

Definition 4. Suppose that $(X, \tau), (Y, \nu)$ and (Z, ω) are fuzzy topological spaces. It is known that if $F_1 : X \to Y$ and $F_2 : Y \to Z$ are fuzzy multifunctions, then the fuzzy multifunction $F_1 \text{o } F_2 : X \to Z$ is defined by $(F_1 \text{ o } F_2)(x_\epsilon) = F_2(F_1(x_\epsilon))$ for each $x_\epsilon \in X$.

Theorem 4. *Let $(X, \tau), (Y, \nu)$ and (Z, ω) be fuzzy topological space and let $F : X \to Y$ and $G : Y \to Z$ be fuzzy multifunction. If $F : X \to Y$ is an upper (lower) fuzzy continuous multifunction and $G : Y \to Z$ is an upper (lower) fuzzy α-continuous multifunction. Then G o $F : X \to Z$ is an upper (lower) fuzzy α-continuous multifunction.*

Proof. Let $\lambda \leq Z$ be any fuzzy open set. From the definition of G o F, we have $(G \text{ o } F)^+(\lambda) = F^+(G^+(\lambda))((G \text{ o } F)^-(\lambda) = F^-(G^-(\lambda)))$, since G is an upper (lower) fuzzy α-continuous, it follows that $G^+(\lambda)(G^-(\lambda))$ is a fuzzy open set. Since F is an upper (lower) fuzzy continuous, it follows that $F^+(G^+(\lambda))(F^-(G^-(\lambda)))$ is a fuzzy α-open set, this shows that G o F is an upper (lower) fuzzy α-continuous. □

Theorem 5. *Let $F : X \to Y$ be a fuzzy multifunction from a fuzzy topological (X, τ) to a fuzzy topological space (Y, ν). If F is a lower(upper) fuzzy α-continuous multifunction and $\mu \leq X$ is a fuzzy set, then the restriction multifunction $F|_\mu : \mu \to Y$ is an lower (upper) fuzzy α-continuous multifunction.*

Proof. Suppose that $\beta \leq Y$ is a fuzzy open set. Let $x_\epsilon \in \mu$ and let $x_\epsilon \in F^-|_\mu(\beta)$. Since F is a lower fuzzy α-continuous multifunction, if follows that there exists a fuzzy open set $x_\epsilon \in \rho$ such that $\rho \leq F^-(\beta)$. From here we obtain that $x_\epsilon \in \rho \wedge \mu$ and $\rho \wedge \mu \leq F|_\mu(\beta)$. Thus, we show that the restriction multifunction $F|_\mu$ is lower fuzzy α-continuous multifunction. □

The proof for the case of the upper fuzzy α-continuity of the multifunction $F|_\mu$ is similar to the above.

Theorem 6. *Let $F : X \to Y$ be a fuzzy multifunction from a fuzzy topological (X, τ) to a fuzzy topological space (Y, ν), let $\{\lambda_\gamma : \gamma \in \Phi\}$ be a fuzzy open cover of X. If the restriction multifunction $F_\gamma = F_{\lambda_\gamma}$ is lower (upper) fuzzy α-continuous multifunction for each $\gamma \in \Phi$, then F is lower (upper) fuzzy α-continuous multifunction.*

Proof. Let $\mu \leq Y$ be any fuzzy open set. Since F_γ is lower fuzzy α-continuous for each γ, we know that $F_\gamma^-(\mu) \leq int_{\lambda_\gamma}(F_\gamma^-(\mu))$ and from here $F^-(\mu) \wedge \lambda_\gamma \leq int_{\lambda_\gamma}(F^-(\mu) \wedge \lambda_\gamma)$ and $F^-(\mu) \wedge \lambda_\gamma \leq int(F^-(\mu)) \wedge \lambda_\gamma$. Since $\{\lambda_\gamma : \gamma \in \Phi\}$ is a fuzzy open cover of X. It follows that $F^-(\mu) \leq int(F^-(\mu))$. Thus, we obtain that F is lower(upper) fuzzy α-continuous multifunction. □

The proof of the upper fuzzy α-continuity of F is similar to the above.

Definition 5. Suppose that $F : X \to Y$ is a fuzzy multifunction from a fuzzy topological space X to a fuzzy topological space Y. The fuzzy graph multifunction $G_F : X \to X \times Y$ of F is defined as $G_F(x_\epsilon) = \{x_\epsilon\} \times F(x_\epsilon)$.

Theorem 7. *Let $F : X \to Y$ be a fuzzy multifunction from a fuzzy topological (X, τ) to a fuzzy topological space (Y, ν). If the graph function of F is lower(upper) fuzzy α-continuous multifunction, then F is lower(upper) fuzzy α-continuous multifunction.*

Proof. For the fuzzy sets $\beta \leq X, \eta \leq Y$, we take

$$(\beta \times \eta)(z, y) = \begin{cases} 0 & \text{if } z \notin \beta \\ \eta(y) & \text{if } z \in \beta \end{cases}$$

Let $x_\epsilon \in X$ and let $\mu \in Y$ be a fuzzy open set such that $x_\epsilon \in F^-(\mu)$. We obtain that $x_\epsilon \in G_F^-(X \times \mu)$ and $X \times \mu$ is a fuzzy open set. Since fuzzy graph multifunction G_F is lower fuzzy α-continuous, it follows that there exists a fuzzy α-open set $\rho \leq X$ containing x_ϵ such that $\rho \leq G_F^-(X \times \mu)$. From here, we obtain that $\rho \leq F^-(\mu)$. Thus, F is lower fuzzy α-continuous multifunction. □

The proof of the upper fuzzy α-continuity of F is similar to the above.

Theorem 8. *Suppose that (X, τ) and (X_α, τ_α) are fuzzy topological space where $\alpha \in J$. Let $F : X \to \prod_{\alpha \in J} X_\alpha$ be a fuzzy multifunction from X to the product space $\prod_{\alpha \in J} X_\alpha$ and let $P_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ be the projection multifunction for each $\alpha \in J$ which is defined by $P_\alpha((x_\alpha)) = \{x_\alpha\}$. If F is an upper (lower) fuzzy α-continuous multifunction, then P_α o F is an upper (lower) fuzzy α-continuous multifunction for each $\alpha \in J$.*

Proof. Take any $\alpha_o \in J$. Let μ_{α_o} be a fuzzy open set in (X_α, τ_α). Then $(P_{\alpha_o}$ o $F)^+(\mu_{\alpha_o}) = F^+(P_{\alpha_o}^+(\mu_{\alpha_o})) = F^+(\mu_{\alpha_o} \times \prod_{\alpha \neq \alpha_o} X_\alpha)$ (resp.,$(P_{\alpha_o}$ o $F)^-(\mu_{\alpha_o}) = F^-(P_{\alpha_o}^-(\mu_{\alpha_o})) = F^-(\mu_{\alpha_o} \times \prod_{\alpha \neq \alpha_o} X_\alpha)$).

Since F is upper (lower) fuzzy α-continuous multifunction and since $\mu_{\alpha_o} \times \prod_{\alpha \neq \alpha_o} X_\alpha$ is a fuzzy open set, it follows that $F^+(\mu_{\alpha_o} \times \prod_{\alpha \neq \alpha_o} X_\alpha)$ (resp., $F^-(\mu_{\alpha_o} \times \prod_{\alpha \neq \alpha_o} X_\alpha)$) is fuzzy α- open in (X, τ). It shows that P_{α_o} o F is upper (lower) fuzzy α-continuous multifunction.

Hence, we obtain that P_α o F is an upper (lower) fuzzy α-continuous multi function for each $\alpha \in J$. □

Theorem 9. *Suppose that for each $\alpha \in J$, (X_α, τ_α) and (Y_α, ν_α) are fuzzy topological spaces. Let $F_\alpha : X_\alpha \to Y_\alpha$ be a fuzzy multifunction for each $\alpha \in J$ and let $F : \prod_{\alpha \in J} X_\alpha \to \prod_{\alpha \in J} Y_\alpha$ be defined by $F((x_\alpha)) = \prod_{\alpha \in J} F_\alpha(x_\alpha)$ from the product space $\prod_{\alpha \in J} X_\alpha$ to product space $\prod_{\alpha \in J} Y_\alpha$. If F is an upper (lower) fuzzy α-continuous multifunction, then each F_α is an upper (lower) fuzzy α-continuous multifunction for each $\alpha \in J$.*

Proof. Let $\mu_\alpha \leq Y_\alpha$ be a fuzzy open set. Then $\mu_\alpha \times \prod_{\alpha \neq \beta} Y_\beta$ is a fuzzy open set. Since F is an upper (lower) fuzzy α-continuous multifunction, it follows that $F^+(\mu_\alpha \times \prod_{\alpha \neq \beta} Y_\beta) = F^+(\mu_\alpha) \times \prod_{\alpha \neq \beta} X_\beta$, $(F^-(\mu_\alpha \times \prod_{\alpha \neq \beta} Y_\beta) = F^-(\mu_\alpha) \times \prod_{\alpha \neq \beta} X_\beta)$ is a fuzzy α-open set. Consequently, we obtain that $F^+(\mu_\alpha)$ $(F^-(\mu_\alpha))$ is a fuzzy α-open set. Thus, we show that F_α is an upper (lower) fuzzy α-continuous multifunction. □

Theorem 10. *Suppose that (X_1, τ_1), (X_2, τ_2), (Y_1, ν_1) and (Y_2, ν_2) are fuzzy topological spaces and $F_1 : X_1 \to Y_1$, $F_2 : X_2 \to Y_2$ are fuzzy multifunctions and suppose that if $\eta \times \beta$ is fuzzy α-open set then η and β are fuzzy α-open sets for any fuzzy sets $\eta \leq Y_1$, $\beta \leq Y_2$. Let $F_1 \times F_2 : X_1 \times X_2 \to Y_1 \times Y_2$ be a fuzzy multifunction which is defined by $(F_1 \times F_2)(x_\epsilon, y_\nu) = F_1(x_\epsilon) \times F_2(y_\nu)$. If $F_1 \times F_2$ is an upper (lower) fuzzy α-continuous multifunction, then F_1 and F_2 are upper (lower) fuzzy α-continuous multifunctions.*

Proof. We know that $(\mu^* \times \beta^*)(x_\epsilon, y_\nu) = min\{\mu^*(x), \beta^*(y)\}$ for any fuzzy sets μ^*, β^* and for any fuzzy point x_ϵ, y_ν.

Let $\mu \times \beta \leq Y_1 \times Y_2$ be a fuzzy open set. It known that $(F_1 \times F_2)^+(\mu \times \beta) = F_1^+(\mu) \times F_2^+(\beta)$. Since $F_1 \times F_2$ is an upper fuzzy α-continuous multifunction, it follows that $F_1^+(\mu) \times F_2^+(\beta)$ is a fuzzy α-open set. From here, $F_1^+(\mu)$ and $F_2^+(\beta)$ are fuzzy α-open sets. Hence, it is obtain that F_1 and F_2 are upper fuzzy α-continuous multifunctions. □

The proof of the lower fuzzy α-continuity of the multifunctions F_1 and F_2 is similar to the above.

Theorem 11. *Suppose that (X, τ), (Y, ν) and (Z, ω) are fuzzy topological spaces and $F_1 : X \to Y$, $F_2 : X \to Z$ are fuzzy multifunction and suppose that if $\eta \times \beta$ is a fuzzy α-open set, then η and β are fuzzy α-open sets for any fuzzy sets $\eta \leq Y$, $\beta \leq Z$. Let $F_1 \times F_2 : X \to Y \times Z$ be a fuzzy multifunction which is defined by*

$(F_1 \times F_2)(x_\epsilon) = F_1(x_\epsilon) \times F_2(x_\epsilon)$. *If $F_1 \times F_2$ is an upper (lower) fuzzy α-continuous multifunction, then F_1 and F_2 are upper (lower) fuzzy α-continuous multifunctions.*

Proof. Let $x_\epsilon \in X$ and let $\mu \leq Y$, $\beta \leq Z$ be fuzzy α-open sets such that $x_\epsilon \in F_1^+(\mu)$ and $x_\epsilon \in F_2^+(\beta)$. Then we obtain that $F_1(x_\epsilon) \leq \mu$ and $F_2(x_\epsilon) \leq \beta$ and from here, $F_1(x_\epsilon) \times F_2(x_\epsilon) = (F_1 \times F_2)(x_\epsilon) \leq \mu \times \beta$. We have $x_\epsilon \in (F_1 \times F_2)^+(\mu \times \beta)$. Since $F_1 \times F_2$ is an upper fuzzy α-continuous multifunction, it follows that there exist a fuzzy α-open set ρ containing x_ϵ such that $\rho \leq (F_1 \times F_2)^+(\mu \times \beta)$. We obtain that $\rho \leq F_1^+(\mu)$ and $\rho \leq F_2^+(\beta)$. Thus we obtain that F_1 and F_2 are fuzzy α-continuous multifunctions. □

The proof of the lower fuzzy α-continuity of the multifunctions F_1 and F_2 is similar to the above.

Lemma 1 ([2])**.** *A fuzzy set in fuzzy topological space X is a fuzzy α-open set if and only if it is fuzzy semiopen and fuzzy preopen.*

Theorem 12. *Let $F : X \to Y$ be a fuzzy multifunction from a fuzzy topological (X, τ) to a fuzzy topological space (Y, ν). Then F is an upper fuzzy α-continuous if and only if it is an upper fuzzy semicontinuous and upper fuzzy precontinuous.*

Proof. Let F be upper fuzzy semicontinuous and upper fuzzy precontinuous, and let μ be a fuzzy open set in Y. Then $F^+(\mu)$ is fuzzy semiopen and fuzzy preopen, it follows from lemma 1 that $F^+(\mu)$is a fuzzy α-open set, and hence F is an upper fuzzy α-continuous multifunction. The converse is immediate. □

Theorem 13. *Let $F : X \to Y$ be a fuzzy multifunction from a fuzzy topological space (X, τ) to a fuzzy topological space (Y, ν). Then F is a lower fuzzy α-continuous if and only if it is lower fuzzy semicontinuous and lower fuzzy precontinuous.*

Proof. Similar to that of Theorem 12 and is omitted. □

REFERENCES

[1] K. K. Azad, "On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity," *J. Math. Anal. Appl.*, vol. 82, no. 1, pp. 14–32, 1981.

[2] A. S. Bin Shahna, "On fuzzy strong semicontinuity and fuzzy precontinuity," *Fuzzy Sets and Systems*, vol. 44, no. 2, pp. 303–308, 1991.

[3] C. L. Chang, "Fuzzy topological spaces," *J. Math. Anal. Appl.*, vol. 24, pp. 182–190, 1968.

[4] A. S. Mashhour, M. E. Abd El-Monsef, and S. N. El-Deep, "On precontinuous and weak precontinuous mappings," *Proc. Math. Phys. Soc. Egypt*, vol. 53, pp. 47–53, 1982.

[5] M. N. Mukherjee and S. Malakar, "On almost continuous and weakly continuous fuzzy multifunctions," *Fuzzy Sets and Systems*, vol. 41, no. 1, pp. 113–125, 1991.

[6] T. Neubrunn, *Strongly quasi-continuous multivalued mappings*, ser. Research and Exposition in Mathematics. Berlin: Heldermann, 1988, vol. 16.

[7] O. Njåstad, "On some classes of nearly open sets," *Pacific J. Math.*, vol. 15, pp. 961–970, 1965.

[8] N. S. Papageorgiou, "Fuzzy topology and fuzzy multifunctions," *J. Math. Anal. Appl.*, vol. 109, no. 2, pp. 397–425, 1985.

[9] V. Popa and T. Noiri, "On upper and lower α-continuous multifunctions," *Math. Slovaca*, vol. 43, no. 4, pp. 477–491, 1993.
[10] P. M. Pu and Y. M. Liu, "Fuzzy topology. i. neighborhood structure of a fuzzy point and moore-smith convergence," *J. Math. Anal. Appl.*, vol. 76, no. 2, pp. 571–599, 1980.
[11] P.-M. Pu and Y. M. Liu, "Fuzzy topology. ii. product and quotient spaces," *J. Math. Anal. Appl.*, vol. 77, no. 1, pp. 20–37, 1980.
[12] L. A. Zadeh, "Fuzzy sets," *Inform. and Control*, vol. 8, pp. 338–353, 1965.

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FIRST ORDER NONHOMOGENEOUS LINEAR DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper, we investigate the complex oscillation of the nonhomogeneous linear differential polynomial $g_f = d_1 f' + d_0 f + b$. Here $d_0(z)$, $d_1(z)$, $b(z)$ are meromorphic functions such that at least one of $d_0(z)$ and $d_1(z)$ does not vanish identically with $\rho_p(d_j) < \infty$ $(j = 0, 1)$, $\rho_p(b) < \infty$, and f is a solution of the differential equation $f'' + A(z) f = 0$, where $A(z)$ is a transcendental meromorphic function with finite iterated order $\rho_p(A) = \rho > 0$.

2000 *Mathematics Subject Classification:* 34M10, 30D35

Keywords: differential polynomials, linear differential equations, iterated order, iterated exponent of convergence of the sequence of distinct zeros, iterated exponent of convergence of the sequence of distinct fixed points

1. Introduction and main result

For the definition of the iterated order of a meromorphic function, we use the same definition as in [3]([2], p. 317), ([4], p. 129). For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N}$. We also define for all sufficiently large r the functions $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1 ([3,4])**.** Let f be a meromorphic function. Then the iterated $p-$order $\rho_p(f)$ of f is defined by

$$\rho_p(f) = \overline{\lim_{r \to +\infty}} \frac{\log_p T(r, f)}{\log r} \quad (p \geq 1 \text{ is an integer}), \tag{1.1}$$

where $T(r, f)$ is the Nevanlinna characteristic function of f. For $p = 1$, this notation is called order and for $p = 2$ hyper-order (see [7]).

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Definition 2 ([3]). The finiteness degree of the order of a meromorphic function f is defined by

$$i(f)=\begin{cases}0, & \text{if } f \text{ is rational,}\\ \min\left\{j\in\mathbb{N}:\rho_j(f)<+\infty\right\}, & \text{if } f \text{ is transcendental such that}\\ & \text{some } j\in\mathbb{N} \text{ with } \rho_j(f)<+\infty \text{ exists,}\\ +\infty, & \text{for } f \text{ with } \rho_j(f)=+\infty \text{ for all } j\in\mathbb{N}.\end{cases}$$

Definition 3 ([3,5]). Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f)=\overline{\lim_{r\to+\infty}}\frac{\log_p \overline{N}\left(r,\frac{1}{f}\right)}{\log r} \quad (p\geq 1 \text{ is an integer}), \tag{1.2}$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z:|z|<r\}$. For $p=1$, this notation is called the exponent of convergence of the sequence of distinct zeros and for $p=2$ the hyper-exponent of convergence of the sequence of distinct zeros (see [6]).

Definition 4 ([5]). Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\overline{\tau}_p(f)=\overline{\lambda}_p(f-z)=\overline{\lim_{r\to+\infty}}\frac{\log_p \overline{N}\left(r,\frac{1}{f-z}\right)}{\log r} \quad (p\geq 1 \text{ is an integer}). \tag{1.3}$$

For $p=1$, this notation is called the exponent of convergence of the sequence of distinct fixed points and for $p=2$ the hyper-exponent of convergence of the sequence of distinct fixed points (see [6]). Thus $\overline{\tau}_p(f)=\overline{\lambda}_p(f-z)$ is an indication of oscillation of distinct fixed points of $f(z)$.

Let $\mathcal{L}(\mathbf{G})$ denote a differential subfield of the field $\mathcal{M}(\mathbf{G})$ of meromorphic functions in a domain $\mathbf{G}\subset\mathbb{C}$. If $\mathbf{G}=\mathbb{C}$, we simply use $\mathcal{L}$ instead of $\mathcal{L}(\mathbb{C})$. A special case of such differential subfields is

$$\mathcal{L}_{p+1,\rho}=\left\{g \text{ meromorphic: } \rho_{p+1}(g)<\rho\right\}, \tag{1.4}$$

where ρ is a positive constant.

Consider the linear differential equation

$$f''+A(z)f=0, \tag{1.5}$$

where $A(z)$ is a transcendental meromorphic function with finite iterated order $\rho_p(A)=\rho>0$. Since the beginning of the last four decades, a substantial number of research articles have been written to describe the fixed points of general transcendental meromorphic functions (see [8]). However, there are few studies on the fixed points of solutions of general differential equations. In [5] Laine and Rieppo have

investigated the fixed points and iterated order of equation (1.5) and have obtained the following result:

Theorem 1 ([5]). *Let $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \lim_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$, and let f be a transcendental meromorphic solution of equation (1.5). Suppose, moreover, that either:*

(i) *all poles of f are of uniformly bounded multiplicity or*
(ii) $\delta(\infty, f) > 0$.

Then $\rho_{p+1}(f) = \rho_p(A) = \rho$. Moreover, let

$$P[f] = P\left(f, f', \ldots, f^{(m)}\right) = \sum_{j=0}^{m} p_j f^{(j)} \tag{1.6}$$

be a linear differential polynomial with coefficients $p_j \in \mathcal{L}_{p+1,\rho}$, assuming that at least one of the coefficients p_j does not vanish identically. Then for the fixed points of $P[f]$, we have $\overline{\tau}_{p+1}(P[f]) = \rho$, provided that neither $P[f]$ nor $P[f] - z$ vanishes identically.

Remark 1 ([5], p. 904). In Theorem 1, in order to study $P[f]$, the authors consider $m \leq 1$. Indeed, if $m \geq 2$, we obtain, by repeated differentiation of (1.5), that $f^{(k)} = q_{k,0} f + q_{k,1} f'$, $q_{k,0}, q_{k,1} \in \mathcal{L}_{p+1,\rho}$ for $k = 2, \ldots, m$. Substitution into (1.6) yields the required reduction.

The main purpose of this paper is to investigate the fixed points of the nonhomogeneous linear differential polynomial $g_f = d_1 f' + d_0 f + b$, where $d_0(z)$, $d_1(z)$, $b(z)$ are meromorphic functions generated by solutions of equation (1.5). Instead of looking at the zeros of $g_f - z$, we proceed to a slight generalization by considering zeros of $g_f - \varphi(z)$, where φ is a meromorphic function of finite iterated $p-$order, while the solution of respective differential equation is of infinite iterated $p-$order.

Let us denote by

$$\alpha_0 = d_0' - d_1 A, \quad \alpha_1 = d_0 + d_1', \tag{1.7}$$

$$h = d_1 \alpha_0 - d_0 \alpha_1, \tag{1.8}$$

where A, d_j $(j = 0, 1)$ are meromorphic functions. We obtain:

Theorem 2. *Let $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$, let $d_0(z)$, $d_1(z)$, $b(z)$ be meromorphic functions such that at least one of $d_0(z)$, $d_1(z)$ does not vanish identically with $\rho_p(d_j) < \infty$ $(j = 0, 1)$, $\rho_p(b) < \infty$ such that $h \not\equiv 0$. Let $\varphi(z)$ $(\not\equiv 0)$ be a meromorphic function of finite iterated $p-$ order such that $d_1\left(\varphi' - b'\right) - \alpha_1(\varphi - b) \not\equiv 0$. Suppose, moreover, that either:*

(i) *all poles of* f *are of uniformly bounded multiplicity or*
(ii) $\delta(\infty, f) > 0$.

If $f(z) \not\equiv 0$ *is a meromorphic solution of* (1.5), *then the differential polynomial* $g_f = d_1 f' + d_0 f + b$ *satisfies* $\overline{\lambda}_p(g_f - \varphi) = \rho_p(f) = +\infty$ *and* $\overline{\lambda}_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho_p(A) = \rho$.

Setting $p = 1$ and $\varphi(z) = z$ in Theorem 2, we obtain the following corollary:

Corollary 1. *Let* $A(z)$ *be a transcendental meromorphic function of finite order* $\rho(A) = \rho > 0$ *such that* $\delta(\infty, A) = \delta > 0$, *let* $d_0(z), d_1(z), b(z)$ *be meromorphic functions such that at least one of* $d_0(z), d_1(z)$ *does not vanish identically with* $\rho(d_j) < \infty$ $(j = 0, 1), \rho(b) < \infty$ *such that* $h \not\equiv 0$ *and* $d_1\left(1 - b'\right) - \alpha_1(z - b) \not\equiv 0$. *Suppose, moreover, that either:*

(i) *all poles of* f *are of uniformly bounded multiplicity or*
(ii) $\delta(\infty, f) > 0$.

If $f(z) \not\equiv 0$ *is a meromorphic solution of* (1.5), *then the differential polynomial* $g_f = d_1 f' + d_0 f + b$ *has infinitely many fixed points and satisfies* $\overline{\tau}(g_f) = \rho(f) = +\infty$, $\overline{\tau}_2(g_f) = \rho_2(f) = \rho(A) = \rho$.

2. Some Lemmas

We need the following lemmas in the proofs of our theorem.

Lemma 1 (see Remark 1.3 of [3]). *If* f *is a meromorphic function with* $i(f) = p \geq 1$, *then* $\rho_p(f) = \rho_p\left(f'\right)$.

Lemma 2 ([5]). *If* f *is a meromorphic function with* $0 < \rho_p(f) < \rho$ $(p \geq 1)$, *then* $\rho_{p+1}(f) = 0$.

Lemma 3 ([1]). *Let* $k \geq 2$ *and* $A(z)$ *be a transcendental meromorphic function of finite iterated order* $\rho_p(A) = \rho > 0$ *such that* $\delta(\infty, A) = \delta > 0$. *Suppose, moreover, that either:*

(i) *all poles of* f *are of uniformly bounded multiplicity or*
(ii) $\delta(\infty, f) > 0$.

Then every meromorphic solution $f(z) \not\equiv 0$ *of*

$$f^{(k)} + A(z) f = 0, \tag{2.1}$$

satisfies $i(f) = p + 1$, $\rho_p(f) = +\infty$ *and* $\rho_{p+1}(f) = \rho_p(A) = \rho$.

Lemma 4 ([1]). *Let* $A_0, A_1, \ldots, A_{k-1}$, $F \not\equiv 0$ *be finite iterated* p*-order meromorphic functions. If* f *is a meromorphic solution with* $\rho_p(f) = +\infty$ *and* $\rho_{p+1}(f) = \rho < +\infty$ *of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F, \tag{2.2}$$

then $\overline{\lambda}_p(f) = \rho_p(f) = +\infty$ *and* $\overline{\lambda}_{p+1}(f) = \rho_{p+1}(f) = \rho$.

Lemma 5. *Let* $A(z)$ *be a transcendental meromorphic function with finite iterated order* $\rho_p(A) = \rho > 0$ *such that* $\delta(\infty, A) = \delta > 0$. *Let* $d_0(z)$, $d_1(z)$, $b(z)$ *be meromorphic functions such that at least one of* $d_0(z)$, $d_1(z)$ *does not vanish identically with* $\rho_p(d_j) < \infty$ $(j = 0, 1)$, $\rho_p(b) < \infty$ *such that* $h \not\equiv 0$. *Suppose, moreover, that either:*

(i) *all poles of* f *are of uniformly bounded multiplicity or*
(ii) $\delta(\infty, f) > 0$.

If $f(z) \not\equiv 0$ *is a meromorphic solution of* (1.5), *then the differential polynomial*

$$g_f = d_1 f' + d_0 f + b \tag{2.3}$$

satisfies $i(g_f) = p + 1$, $\rho_p(g_f) = \rho_p(f) = +\infty$ *and* $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A) = \rho$.

Proof. Suppose that f $(\not\equiv 0)$ is a meromorphic solution of equation (1.5). Then by Lemma 3, we have $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho_p(A) = \rho$. Differentiating both sides of equation (2.3) and replacing f'' with $f'' = -Af$, we obtain

$$g_f' - b' = (d_0 + d_1')f' + (d_0' - d_1 A)f. \tag{2.4}$$

Then by (1.7), (2.3) and (2.4), we have

$$d_1 f' + d_0 f = g_f - b, \tag{2.5}$$

$$\alpha_1 f' + \alpha_0 f = g_f' - b'. \tag{2.6}$$

Set

$$h = d_1\alpha_0 - \alpha_1 d_0 = d_1\left(d_0' - d_1 A\right) - d_0(d_0 + d_1'). \tag{2.7}$$

By the condition $h \not\equiv 0$ and (2.5) – (2.7), we get

$$f = \frac{d_1\left(g_f' - b'\right) - \alpha_1\left(g_f - b\right)}{h}. \tag{2.8}$$

If $\rho_p(g_f) < +\infty$, then by (2.8) and Lemma 1 we obtain $\rho_p(f) < +\infty$, and this is a contradiction. Hence $\rho_p(g_f) = \rho_p(f) = +\infty$.

Now, we prove that $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho$. By (2.3), Lemma 1 and Lemma 2, we get $\rho_{p+1}(g_f) \le \rho_{p+1}(f)$ and by (2.8) we have $\rho_{p+1}(f) \le \rho_{p+1}(g_f)$. This yield $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho$. □

Remark 2. In Lemma 5, if we don't have the condition $h \not\equiv 0$, then the differential polynomial can be of finite iterated p-order. For example, if $d_0' - d_1 A \equiv 0$ and $d_1' + d_0 \equiv 0$, then $h \equiv 0$ and $g_f' - b' \equiv 0$. It follows that $\rho_p(g_f) = \rho_p\left(g_f'\right) = \rho_p\left(b'\right) = \rho_p(b) < +\infty$. Hence, the condition $h \not\equiv 0$ is necessary in Theorem 2.

3. Proof of Theorem 2

Suppose that $f(z) \not\equiv 0$ is a meromorphic solution of (1.5). Then by Lemma 3, we have $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho_p(A) = \rho$. By Lemma 5, it is clear that $g_f \not\equiv 0$ and $g_f \not\equiv \varphi$. Set $w(z) = d_1 f' + d_0 f + b - \varphi$, since $\rho_p(\varphi) < +\infty$, then by Lemma 5 we have $\rho_p(w) = \rho_p(g_f) = \rho_p(f) = +\infty$ and $\rho_{p+1}(w) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A) = \rho$. In order to prove $\overline{\lambda}_p(g_f - \varphi) = +\infty$ and $\overline{\lambda}_{p+1}(g_f - \varphi) = \rho_p(A) = \rho$, we need to prove only $\overline{\lambda}_p(w) = +\infty$ and $\overline{\lambda}_{p+1}(w) = \rho_p(A) = \rho$. Substituting $g_f = w + \varphi$ into (2.8)

$$f = \frac{d_1 w' - \alpha_1 w}{h} + \psi, \tag{3.1}$$

where

$$\psi = \frac{d_1\left(\varphi' - b'\right) - \alpha_1(\varphi - b)}{h}. \tag{3.2}$$

Substituting (3.1) into equation (1.5), we obtain

$$\frac{d_1}{h} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w = -\left(\psi'' + A(z)\psi\right) = W, \tag{3.3}$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions with $\rho_p(\phi_j) < \infty$ $(j = 0, 1, 2)$. By $\rho_p(\psi) < +\infty$ and the condition $\psi \not\equiv 0$, it follows by Lemma 3 that $W \not\equiv 0$. By Lemma 4, we obtain $\overline{\lambda}_p(w) = \rho_p(w) = +\infty$ and $\overline{\lambda}_{p+1}(w) = \rho_{p+1}(w) = \rho$, i.e., $\overline{\lambda}_p(g_f - \varphi) = \rho_p(f) = +\infty$ and $\overline{\lambda}_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho_p(A) = \rho$.

Remark 3. From the proof of Theorem 2, we see that the condition $d_1\left(\varphi' - b'\right) - \alpha_1(\varphi - b) \not\equiv 0$ is necessary because if $d_1\left(\varphi' - b'\right) - \alpha_1(\varphi - b) \equiv 0$, then $\psi \equiv 0$ and $W \equiv 0$.

References

[1] B. Belaïdi, "Oscillation of fixed points of solutions of some linear differential equations," *Acta Math. Univ. Comenian. (N.S.)*, vol. 77, no. 2, pp. 263–269, 2008.

[2] L. G. Bernal, "On growth k-order of solutions of a complex homogeneous linear differential equation," *Proc. Amer. Math. Soc.*, vol. 101, no. 2, pp. 317–322, 1987.

[3] L. Kinnunen, "Linear differential equations with solutions of finite iterated order," *Southeast Asian Bull. Math.*, vol. 22, no. 4, pp. 385–405, 1998.

[4] I. Laine, *Nevanlinna theory and complex differential equations*, ser. de Gruyter Studies in Mathematics. Berlin: Walter de Gruyter & Co., 1993, vol. 15.

[5] I. Laine and J. Rieppo, "Differential polynomials generated by linear differential equations," *Complex Var. Theory Appl.*, vol. 49, no. 12, pp. 897–911, 2004.

[6] M.-S. Liu and X.-M. Zhang, "Fixed points of meromorphic solutions of higher order linear differential equations," *Ann. Acad. Sci. Fenn. Math.*, vol. 31, no. 1, pp. 191–211, 2006.

[7] C.-C. Yang and H.-X. Yi, *Uniqueness theory of meromorphic functions*, ser. Mathematics and its Applications. Dordrecht: Kluwer Academic Publishers Group, 2003, vol. 557.

[8] Q. T. Zhang and C. C. Yang, *The Fixed Points and Resolution Theory of Meromorphic Functions*. Beijing: Beijing University Press, 1988.

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p-MINKOWSKI INEQUALITIES FOR MIXED INTERSECTION BODIES

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Abstract. In this paper, we establish p-Minkowski inequality for mixed intersection bodies, which is a general form of the Minkowski inequality for mixed intersection bodies.

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1. INTRODUCTION

One might say the history of intersection bodies began with the paper [4] of Busemann. Intersection bodies were first explicitly defined and named by Lutwak[13]. It was here that the duality between intersection bodies and projection bodies was first made clear. Despite the considerable ingenuity of earlier attacks on the Busemann-Petty problem, it seems fair to say that the work [13] of Lutwak represents the beginning of its eventual solution. In [13], Lutwak also showed that if a convex body is sufficiently smooth and not an intersection body, then there exists a centered star body such that the conditions of Busemann-Petty problem holds, but the in the result the inequality's direction is reversed. Following Lutwak, the intersection body of order i of a star body is introduced by Zhang[21]. It follows from this definition that every intersection body of order i of a star body is an intersection body of a star body, and vice versa. As Zhang observes, the new definition of intersection body allows a more appealing formulation, namely: The Busemann-Petty problem has a positive answer in n-dimensional Euclidean space if and only if each centered convex body is an intersection body. The intersection body plays an essential role in Busemann's theory[5] of area in Minkowski spaces. The intersection body also plays an important role in the Brunn-Minkowski theory.

In recent years some author including Ball[1, 2], Bourgain[3], Gardner[6–8], Schneider[19] and Lutwak[11, 12, 14–18], et al. have given considerable attention to the Brunn-Minkowski theory and their various generalizations. The purpose of

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this paper is firstly to establish the Minkowski inequality for the dual Quermassintegral sum, which is a generalization of the Minkowski inequality for mixed intersection bodies. Then, the Brunn-Minkowski inequality and the Aleksandrov-Fenchel inequality for mixed intersection bodies are proved and some related results are also given. In this work we shall derive, for intersection bodies, all the analogous inequalities for Lutwak's mixed projection body inequalities [17]. Thus, this work may be seen as additional evidence for the natural duality between intersection and projection bodies.

2. Notation and Preliminaries

The setting of this paper is the n-dimensional Euclidean space $\mathbb{R}^n (n > 2)$. Let $\mathbb{C}^n$ denote the set of non-empty convex figures (compact, convex subsets) and let $\mathcal{K}^n$ denote the subset of $\mathbb{C}^n$ consisting of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}. For $u \in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u. We will use K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u. We use $V(K)$ for the n-dimensional volume of the convex body K. The support function of $K \in \mathcal{K}^n$, $h(K,\cdot)$, is defined on $\mathbb{R}^n$ by $h(K,\cdot) = Max\{x \cdot y : y \in K\}$. Let δ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$, $\delta(K,L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, its radial function $\rho(K,\cdot) : S^{n-1} \to \mathbb{R}$, is defined (for $u \in S^{n-1}$) by $\rho(K,u) = Max\{\lambda \geq 0 : \lambda u \in K\}$. If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Let φ^n denote the set of star bodies in $\mathbb{R}^n$.

3. Dual mixed volumes

If $K_1, \ldots, K_r \in \varphi^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, then the radial Minkowski linear combination, $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$, is defined by

$$\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r : x_i \in K_i\}.$$

The following property will be used later. If $K, L \in \varphi^n$ and $\lambda, \mu \geq 0$

$$\rho(\lambda K \tilde{+} \mu L, \cdot) = \lambda \rho(K,\cdot) + \mu \rho(L,\cdot). \tag{3.1}$$

For $K_1, \ldots, K_r \in \varphi^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, the volume of the radial Minkowski linear combination $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$ is a homogeneous nth-degree polynomial in the λ_i [19],

$$V(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r) = \sum \tilde{V}_{i_1,\ldots,i_n} \lambda_{i_1} \cdots \lambda_{i_n} \tag{3.2}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ whose entries are positive integers not exceeding r. If we require the coefficients of the polynomial in (3.2)

to be symmetric in their arguments, then they are uniquely determined. The coefficient $\tilde{V}_{i_1,\ldots,i_n}$ is nonnegative and depends only on the bodies $K_{i_1},\ldots,K_{i_n}$. It is written as $\tilde{V}(K_{i_1},\ldots,K_{i_n})$ and is called the *dual mixed volume* of $K_{i_1},\ldots,K_{i_n}$. If $K_1=\cdots=K_{n-i}=K$, $K_{n-i+1}=\cdots=K_n=L$, the dual mixed volumes is written as $\tilde{V}_i(K,L)$. The dual mixed volumes $\tilde{V}_i(K,B)$ is written as $\tilde{W}_i(K)$.

Dual Quermassintegral is a special case of the p-th dual volume:

$$\tilde{V}_p(K)=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^p dS(u),\ -\infty<p<\infty. \tag{3.3}$$

Taking $p=n-i$ in $\tilde{V}_p(K)$, $\tilde{V}_p(K)$ changes to the well know dual Quermassintegral $\tilde{W}_i(K)$. Taking $p=n$ in $\tilde{V}_p(K)$, $\tilde{V}_p(K)$ changes to the well know general volume $V_i(K)$.

If $K_i\in\varphi^n(i=1,2,\ldots,n-1)$, then the dual mixed volume of $K_i\cap E_u(i=1,2,\ldots,n-1)$ will be denoted by $\tilde{v}(K_1\cap E_u,\ldots,K_{n-1}\cap E_u)$. If $K_1=\ldots=K_{n-1-i}=K$ and $K_{n-i}=\ldots=K_{n-1}=L$, then $\tilde{v}(K_1\cap E_u,\ldots,K_{n-1}\cap E_u)$ is written as $\tilde{v}_i(K\cap E_u,L\cap E_u)$. If $L=B$, then $\tilde{v}_i(K\cap E_u,B\cap E_u)$ is written as $\tilde{w}_i(K\cap E_u)$.

4. Intersection bodies

For $K\in\varphi^n$, there is a unique star body IK whose radial function satisfies (for $u\in S^{n-1}$)

$$\rho(IK,u)=v(K\cap E_u). \tag{4.1}$$

It is called the *intersection bodies* of K. From a result of Busemann, it follows that IK is convex if K is convex and centrally symmetric with respect to the origin. Clearly any intersection body is centered.

The volume of an intersection body is given by

$$V(IK)=\frac{1}{n}\int_{S^{n-1}}v(K\cap E_u)^n dS(u).$$

The mixed intersection bodies of $K_1,\ldots,K_{n-1}\in\varphi^n$, is $I(K_1,\ldots,K_{n-1})$, whose radial function is defined by

$$\rho(I(K_1,\ldots,K_{n-1}),u)=\tilde{v}(K_1\cap E_u,\ldots,K_{n-1}\cap E_u), \tag{4.2}$$

where $\tilde{v}$ is $(n-1)$-dimensional dual mixed volume. If $K\in\varphi^n$ with $\rho(K,u)\in C(S^{n-1})$, and $i\in\mathbb{R}$ is positive, the *intersection body of order i* of K is the centered star body I_iK such that [21]

$$\rho(I_iK)=\frac{1}{n-1}\int_{S^{n-1}}\rho(K,u)^{n-i-1}dS(u),\ for u\in S^{n-1},$$

where $I_iK=I(\underbrace{K,\ldots,K}_{n-i-1},\underbrace{B,\ldots,B}_{i})$.

If $K_1=\cdots=K_{n-i-1}=K,K_{n-i}=\cdots=K_{n-1}=L$, then $I(K_1,\ldots,K_{n-1})$ is written as $I_i(K,L)$. If $L=B$, then $I_i(K,L)$ is written as I_iK and is called the ith

intersection body of K. For I_0K simply write IK. The terminology was introduced by Zhang[21].

5. Main results

Lemma A. *If $K, L \in \varphi^n$, $-\infty < p < \infty$ and $0 < j < n-1$ then*

$$\tilde{V}_p(IK) = \frac{1}{n}\int_{S^{n-1}} v(K\cap E_u)^p dS(u),$$

$$\tilde{V}_p(I_jK) = \frac{1}{n}\int_{S^{n-1}} \tilde{w}_j(K\cap E_u)^p dS(u),$$

$$\tilde{V}_p(I_j(K,L)) = \frac{1}{n}\int_{S^{n-1}} \tilde{v}_j(K\cap E_u, L\cap E_u)^p dS(u).$$

To prove this we use (3.3) in conjunction with the fact (4.2).

Lemma B ([11])**.** *If $K_1,\dots,K_n \in \varphi^n$, then*

$$\tilde{V}(K_1,\dots,K_n)^r \le \prod_{j=1}^{r} \tilde{V}(\underbrace{K_j,\dots,K_j}_{r}, K_{r+1},\dots,K_n),$$

with equality if and only if $K_1,\dots,K_n$ are all dilations of each other. We shall need the following trivial elementary inequality:

Lemma C. *If $a,b \ge 0$ and $c,d > 0$ then for $0 < p < 1$*

$$(a+b)^p(c+d)^{1-p} \ge a^p c^{p-1} + b^p d^{p-1},$$

with equality if and only if $ad = bc$.

6. The Minkowski inequality for dual Quermassintegral sum of mixed intersection bodies

In [10], Leng introduced the concept of *i-Quermassintegral difference function* of convex bodies as follows. If $K, D \in \mathcal{K}^n$ and $D \subset K$, then the i- Quermassintegral difference function of convex bodies K and D, $D_{w_i}(K,D)$, was defined by

$$D_{w_i}(K,D) = W_i(K) - W_i(D) \quad (0 \le i \le n-1).$$

In the section, we first introduce a new concept, *dual Quermassintegral sum function*, as follows.

If $K, D \in \varphi^n$, then the p-dual volume sum function of star bodies K and D, $S_{\tilde{v}_p}(K,D)$, is defined by

$$S_{\tilde{v}_p}(K,D) = \tilde{V}_p(K) + \tilde{V}_p(D).$$

When $p = n$, we have $S_v(K,D) = V(K) + V(D)$, which is called the *dual volume sum function* of star bodies K and L.

The following Minkowski inequality for mixed intersection bodies will be established: If $K, L \in \varphi^n$, and $0 \le i < n$ and $0 < j < n-1$, then

$$\tilde{W}_i(I_j(K,L))^{n-1} \le \tilde{W}_i(IK)^{n-j-1}\tilde{W}_i(IL)^j, \tag{6.1}$$

with equality if and only if K and L are dilates.

This is just the special $D = D' = \varnothing$ case of:

Theorem 1. *If $K, L, D, D' \in \varphi^n$. Let D' is a dilates copy of D, and $-\infty < p < \infty$ and $0 < j < n-1$, then*

$$S_{\tilde{v}_p}(I_j(K,L), I_j(D,D'))^{n-1} \le S_{\tilde{v}_p}(IK, ID)^{n-j-1} S_{\tilde{v}_p}(IL, ID')^j, \tag{6.2}$$

with equality if and only if K and L are dilates.

Proof. In view of the special case of Lemma B, we obtain that

$$\tilde{v}_j(K \cap E_u, L \cap E_u) \le v(K \cap E_u)^{\frac{(n-j-1)}{n-1}} v(L \cap E_u)^{\frac{j}{n-1}}. \tag{6.3}$$

with equality if and only if $K \cap E_u$ and $L \cap E_u$ are dilates, this happens if and only if K and L are dilates [20].

From Lemma A, (6.3) and in view of Minkowski inequality for integral [9], we have $j < n-1$

$$\begin{aligned} n\tilde{V}_p(I_j(K,L)) &= \left(\|\tilde{v}_j(K \cap E_u, L \cap E_u)\|_p\right)^p \\ &\le \left(\|v(K \cap E_u)^{\frac{n-j-1}{n-1}} v(L \cap E_u)^{\frac{j}{n-1}}\|_p\right)^p \\ &\le \left(\|v(K \cap E_u\|_p\right)^{\frac{p(n-j-1)}{n-1}} \left(\|v(K \cap E_u\|_p\right)^{\frac{jp}{n-1}} \\ &= (n\tilde{V}_p(IK))^{\frac{(n-j-1)}{n-1}} (n\tilde{V}_p(IL))^{\frac{j}{n-1}} \\ &= n\tilde{V}_p(IK)^{\frac{(n-j-1)}{n-1}} \tilde{V}_p(IL)^{\frac{j}{n-1}}. \end{aligned} \tag{6.4}$$

In view of the conditions of (6.3) and the Minkowski inequality for the integral, it follows that the equality holds if and only if K and L are dilates.

Moreover, we consider the case of $j = n-1$ of the inequality (6.4). If $j = n-1$, inequality (6.4) reduces to

$$\tilde{W}_{n-1}(I_j(K,L))^{n-1} \le \tilde{W}_{n-1}(IK)^{n-j-1}\tilde{W}_{n-1}(IL)^j. \tag{6.5}$$

From Lemma A, (6.5) changes to

$$\begin{aligned} &\left(\int_{S^{n-1}} \tilde{v}_j(K \cap E_u, L \cap E_u) dS(u)\right)^{n-1} \\ &\le \left(\int_{S^{n-1}} v(K \cap E_u) dS(u)\right)^{n-j-1} \left(\int_{S^{n-1}} v(L \cap E_u) dS(u)\right)^j. \end{aligned} \tag{6.6}$$

On the other hand, integrating two sides of (6.3) and in view of Hölder inequality for integral, we obtain that

$$\begin{aligned}&\int_{S^{n-1}} \tilde{v}_j(K\cap E_u, L\cap E_u)dS(u)\\ &\le \int_{S^{n-1}} v(K\cap E_u)^{\frac{n-j-1}{n-1}} v(L\cap E_u)^{\frac{j}{n-1}} dS(u)\\ &\le \left(\int_{S^{n-1}} v(K\cap E_u)dS(u)\right)^{\frac{n-j-1}{n-1}} \left(\int_{S^{n-1}} v(L\cap E_u)dS(u)\right)^{\frac{j}{n-1}}.\end{aligned}$$

Moreover, from the inequality (6.4), we obtain that

$$\tilde{V}_p(I_j(K,L))^{n-1} \le \tilde{V}_p(IK)^{n-j-1}\tilde{V}_p(IL)^j,$$

with equality if and only if K and L are dilates, and

$$\tilde{V}_p(I_j(D,D'))^{n-1} = \tilde{V}_p(ID)^{n-j-1}\tilde{V}_p(ID')^j.$$

Hence, from the inequality in Lemma C, we have

$$\begin{aligned}&S_{\tilde{v}_p}(I_j(K,L), I_j(D,D'))\\ &\le \tilde{V}_p(IK)^{(n-j-1)/(n-1)}\tilde{V}_p(IL)^{j/(n-1)} + \tilde{V}_p(ID)^{(n-j-1)/(n-1)}\tilde{V}_p(ID')^{j/(n-1)}\\ &\le S_{\tilde{v}_p}(IK,ID)^{n-j-1}S_{\tilde{v}_p}(IL,ID')^j.\end{aligned}$$

The proof of Theorem 1 is complete. □

Remark 1. Let D and D' be single points and take $p = n-i$ in (6.2), then (6.2) changes to inequality (6.1).

$$\tilde{W}_i(I_1(K,L))^{n-1} \le \tilde{W}_i(IK)^{n-2}\tilde{W}_i(IL),$$

with equality if and only if K and L are dilates.

This is just a dual form of the following inequality which was given by Lutwak[17].

Minkowski inequality for mixed projection bodies If $K, L \in \mathcal{K}^n$, and $0 \le i < n$, then

$$W_i(\Pi_1(K,L))^{n-1} \ge W_i(\Pi K)^{n-2}W_i(\Pi L),$$

with equality if and only if K and L are homothetic.

Taking $p = n-i$ in (6.2), it changes to

$$S_{\tilde{w}_i}(I_j(K,L), I_j(D,D'))^{n-1} \le S_{\tilde{w}_i}(IK,ID)^{n-j-1}S_{\tilde{w}_i}(IL,ID')^j,$$

with equality if and only if K and L are dilates.

This is just a rederivation of a recent result of Zhao and Leng[22].

Let D and D' be single points and take $p = n$ in (6.2), then it changes to the well know Minkowski inequality for mixed projection bodies.

REFERENCES

[1] K. Ball, "Shadows of convex bodies," *Trans. Amer. Math. Soc.*, vol. 327, no. 2, pp. 891–901, 1991.

[2] K. Ball, "Volume ratios and a reverse isoperimetric inequality," *J. London Math. Soc. (2)*, vol. 44, no. 2, pp. 351–359, 1991.

[3] J. Bourgain, "On the busemann-petty problem for perturbations of the ball," *Geom. Funct. Anal.*, vol. 1, no. 1, pp. 1–13, 1991.

[4] H. Busemann, "A theorem on convex bodies of the brunn-minkowski type," *Proc. Nat. Acad. Sci. U. S. A.*, vol. 35, pp. 27–31, 1949.

[5] H. Busemann, "The foundations of minkowskian geometry," *Comment. Math. Helv.*, vol. 24, pp. 156–187, 1950.

[6] R. J. Gardner, "Intersection bodies and the busemann-petty problem," *Trans. Amer. Math. Soc.*, vol. 342, no. 1, pp. 435–445, 1994.

[7] R. J. Gardner, "A positive answer to the busemann-petty problem in three dimensions," *Ann. of Math. (2)*, vol. 140, no. 2, pp. 435–447, 1994.

[8] R. J. Gardner, *Geometric tomography*, ser. Encyclopedia of Mathematics and its Applications. Cambridge: Cambridge University Press, 1995, vol. 58.

[9] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*. Cambridge: Cambridge University Press, 1934.

[10] G.-S. Leng, "The brunn-minkowski inequality for volume differences," *Adv. in Appl. Math.*, vol. 32, no. 3, pp. 615–624, 2004.

[11] E. Lutwak, "Dual mixed volumes," *Pacific J. Math.*, vol. 58, no. 2, pp. 531–538, 1975.

[12] E. Lutwak, "Mixed projection inequalities," *Trans. Amer. Math. Soc.*, vol. 287, no. 1, pp. 91–105, 1985.

[13] E. Lutwak, "Intersection bodies and dual mixed volumes," *Adv. in Math.*, vol. 71, no. 2, pp. 232–261, 1988.

[14] E. Lutwak, "Centroid bodies and dual mixed volumes," *Proc. London Math. Soc. (3)*, vol. 60, no. 2, pp. 365–391, 1990.

[15] E. Lutwak, "On quermassintegrals of mixed projection bodies," *Geom. Dedicata*, vol. 33, no. 1, pp. 51–58, 1990.

[16] E. Lutwak, "The brunn-minkowski-firey theory. i. mixed volumes and the minkowski problem," *J. Differential Geom.*, vol. 38, no. 1, pp. 131–150, 1993.

[17] E. Lutwak, "Inequalities for mixed projection bodies," *Trans. Amer. Math. Soc.*, vol. 339, no. 2, pp. 901–916, 1993.

[18] E. Lutwak, "The brunn-minkowski-firey theory. ii. affine and geominimal surface areas," *Adv. Math.*, vol. 118, no. 2, pp. 244–294, 1996.

[19] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, ser. Encyclopedia of Mathematics and its Applications. Cambridge: Cambridge University Press, 1993, vol. 44.

[20] W. Süss, "Zusammensetzung von eikörpern und homothetische eiflächen," *Tohoku Math. J.*, vol. 35, pp. 47–50, 1932.

[21] G. Y. Zhang, "Centered bodies and dual mixed volumes," *Trans. Amer. Math. Soc.*, vol. 345, no. 2, pp. 777–801, 1994.

[22] C.-J. Zhao and G.-S. Leng, "Inequalities for dual quermassintegrals of mixed intersection bodies," *Proc. Indian Acad. Sci. Math. Sci.*, vol. 115, no. 1, pp. 79–91, 2005.

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EXTREMAL PROPERTIES OF ZAGREB COINDICES AND DEGREE DISTANCE OF GRAPHS

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Abstract. The degree distance, Zagreb coindices and reverse degree distance of a connected graph have been studied in mathematical chemistry. In this paper some new extremal values of these topological invariants over some special classes of graphs are determined.

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1. INTRODUCTION

All graphs in this paper are finite and simple. A graph invariant is a function on a graph that does not depend on the labeling of its vertices. Such quantities are also called topological indices. Two of these graph invariants are known under various names, the most commonly used ones are the first and second Zagreb indices. Due to their chemical relevance they have been subject of numerous papers in the chemical literature [9,12,13,16,20], while the first Zagreb index attracted a significant attention of mathematicians.

The problem of determining extremal values and the corresponding extremal graphs of some graph invariants is the topic of several papers [2, 4, 10, 13–15, 17, 20]. The aim of this paper is to investigate similar problems for a recently introduced generalization of Zagreb indices.

Let G be a connected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. For every vertex $u \in V(G)$, the edge connecting u and v is denoted by uv and $\deg_G(u)$ denotes the degree of u in G. The distance $d_G(u,v)$ is defined as the length of a shortest path connecting u and v and the diameter of G, $\mathrm{diam}_G(G)$, is the maximum possible distance between any two vertices in the graph. We will omit the subscript G when the graph is clear from the context.

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The first and second Zagreb indices were originally defined as

$$M_1(G) = \sum_{u \in V(G)} \deg(u)^2$$

and

$$M_2(G) = \sum_{uv \in E(G)} \deg(u)\deg(v),$$

respectively. The first Zagreb index can be also expressed as a sum over edges of G, $M_1(G) = \sum_{uv \in E(G)}[\deg(u) + \deg(v)]$. We refer the reader to [16] for the proof of this fact. The readers interested in more information on Zagreb indices can be referred to [4, 9, 10, 12, 14, 16, 17] and to the references therein.

The Zagreb indices can be viewed as the contributions of pairs of adjacent vertices to certain degree-weighted generalizations of Wiener polynomials [7]. It turned out that computing such polynomials for certain composite graphs depends on such contributions from pairs of non-adjacent vertices.

The first and second Zagreb coindices were first introduced by Došlić. They are defined as follows:

$$\bar{M}_1(G) = \sum_{uv \notin E(G)} [\deg(u) + \deg(v)],$$

$$\bar{M}_2(G) = \sum_{uv \notin E(G)} [\deg(u)\deg(v)].$$

Dobrynin and Kochetova [6] and Gutman [8] introduced a new graph invariant, "degree distance", defined as follows: the degree distance of a vertex x, denoted by $D'(x)$, is defined as $D'(x) = D(x)\deg(x)$, where $D(x) = \sum_{y \in V(G)} d(x, y)$ and the degree distance of G, denoted by $D'(G)$, is

$$\begin{aligned} D'(G) &= \sum_{x \in V(G)} D'(x) = \sum_{x \in V(G)} D(x)\deg(x) \\ &= \frac{1}{2} \sum_{x,y \in V(G)} d(x, y)[\deg(x) + \deg(y)]. \end{aligned}$$

If G is $n-$vertex graph then the reverse Wiener matrix is an $n \times n$ matrix $RW(G) = [RW_{ij}]$ such that $RW_{ij} = \mathrm{diam}(G) - d(v_i, v_j)$, if $i \neq j$ and 0 otherwise. The reverse degree distance of G is defined as ${}^r D'(G) = \sum_{i=1}^{n} \deg(v_i) \sum_{j=1}^{n} RW_{ij}$ [21].

The girth of G is the length of a shortest cycle contained in G. A Moore graph is a graph of diameter k with girth $2k + 1$. Those graphs have the minimum number of vertices possible for a regular graph with given diameter and maximum degree. For a real number k, let $\sum_k(G)$ denote the sum of the $k-$th powers of the degrees of G. We denote by $\sigma_2(n, m)$ the maximum value of $\sum_2(G)$ when G is a graph (not necessarily connected) with n vertices and m edges. Also, let $\sum_2(n, m) = max_{G \in \wp_{n,m}} \sum_2(G)$ where $\wp_{n,m}$ denotes the family of connected graphs on n vertices and m edges.

For the sake of completeness we state here some results from the references [1, 3, 10, 11, 14, 17, 21] which will be useful throughout the paper.

Lemma A (Ashrafi, Došlić and Hamzeh [1]). *Suppose G is a connected graph with exactly n vertices and m edges. Then we have:*

(a) $\bar{M}_1(G) = 2m(n-1) - M_1(G)$,
(b) $\bar{M}_2(G) = 2m^2 - M_2(G) - \frac{1}{2}M_1(G)$.

The subdivision graph $S(G)$ of a graph G is obtained by inserting a new vertex of degree two on each edge of G. If G has n vertices and m edges, then $S(G)$ has $n+m$ vertices and $2m$ edges.

Lemma B (Ilić and Stevanović [10]). *Suppose G is a graph with exactly n vertices and m edges. Then*

(a) *It holds that $M_1(G) \geq \frac{4m^2}{n}$. The equality is attained if and only if G is regular.*
(b) *It holds that $M_2(G) \geq \frac{4m^3}{n^2}$. The equality is attained if and only if G is regular.*
(c) *Let Δ be the maximum vertex degree in G. Then $\frac{M_1(G)}{n} \leq \frac{\Delta M_1(G)}{2m}$ and $\frac{M_2(G)}{m} \leq \frac{\Delta M_1(G)}{2m}$. Equality is attained simultaneously in both inequalities if and only if G is regular.*
(d) *$M_1(S(G)) = M_1(G) + 4m$ and $M_2(S(G)) = 2M_1(G)$.*

Lemma C (Liu and Liu [14]). *Let G be a connected graph with n vertices and m edges. Then*

(a) *If $\Delta \geq \frac{2m-\delta(n-1)}{2}$, then $M_1(G) \leq m(\Delta + \delta + 1)$.*
(b) *$M_1(G) \leq \frac{(\Delta+\delta)^2}{n\Delta\delta}m^2$ with equality if and only if G is regular.*

Lemma D (Sun and Wei [17]). *If G is a connected bicyclic graph with exactly n vertices, m edges and without pendant vertices, then $\frac{M_1(G)}{n} \leq \frac{M_2(G)}{m}$, with equality if and only if $G = K_{2,3}$.*

Lemma E (Zhou and Trinajstić [21]). *Let G be a connected graph with n vertices, m edges and diameter l. Then*

(a) ${}^rD'(G) = 2m(n-1)l - D'(G)$.
(b) *If $n \geq 2$ then $(l-1)M_1(G) \leq {}^rD'(G) \leq 2m(l-2)(n-1) + M_1(G)$ with either equality if and only if $l \leq 2$.*
(c) *Let G be a connected triangle-free and quadrangle-free graph with $n \geq 3$ vertices, minimal degree δ and maximal degree Δ. Then $[l-1+\delta(l-2)]M_1(G) - 2m(l-2)\,\delta \leq {}^rD'(G) \leq (\Delta+2)M_1(G) + 2m[(l-3)(n-1) - \Delta]$ with right equality if and only if G is a Moore graph of diameter 2, or a regular graph of diameter 3 (and girth 5, 6 or 7) and with left equality if and only if $G = S_n$, or G is a Moore graph of diameter 2, or a regular graph of diameter 3 (and girth 5, 6 or 7).*

(d) $M_1(G) \leq n(n-1)$ *with equality if and only if* G *is the star or a Moore graph of diameter 2.*

(e) *Let* G *be a connected graph with minimal degree* δ *and maximal degree* Δ. *Then* $2l(n-1)m-2\Delta W(G) \leq {}^{r}D'(G) \leq 2l(n-1)m-2\delta W(G)$ *with either equality if and only if* G *is a regular graph.*

(f) *Let* G *be a tree with* n *vertices and diameter* $\mathrm{diam}(G)$. *Then* ${}^{r}D'(G) = 2(n-1)^2\mathrm{diam}(G)-4W(G)+n(n-1)$.

The join $G+H$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph union $G \cup H$ together with all the edges joining $V(G)$ and $V(H)$.

Lemma F (Bucicovschi and Cioabă [3])**.** *If* $1 \leq n-1 \leq m$ *and* G *is a connected graph with* n *vertices and* m *edges, then* $D'(G) \geq 4(n-1)m-\sum_2(n,m) = 4(n-2)m-n^2+5n-4-\sigma_2(n-1,m-n+1)$. *Equality happens if and only if* G *is a join of* K_1 *and a graph* G' *on* $n-1$ *vertices and* $m-n+1$ *edges with* $\sum_2(G') = \sigma_2(n-1,m-n+1)$.

Let $\wp_n^1$ and $\wp_n^2$ denote the classes of connected unicyclic and bicyclic graphs, respectively. Note that any graph in $\wp_n^1$ contains a unique cycle and it has n edges and every graph in $\wp_n^2$ contains two linearly independent cycles, cycles without common edges, having $n+1$ edges.

Lemma G (Tomescu [11])**.** *Suppose* G *is a connected graph with* n *vertices,* m *edges. Then we have:*

(a) *For every* $n \geq 3$ *we have* $min_{G\in\wp_n^1} D'(G) = 3n^2-3n-6$ *and the unique extremal graph is* $K_{1,n-1}+e$.

(b) *For every* $n \geq 4$ *we have* $min_{G\in\wp_n^2} D'(G) = 3n^2+n-18$. *The extremal graph is unique and may be obtained from* $K_{1,n-1}$ *by adding two edges having a common extremity.*

Throughout this paper our notation is standard. For terms and concepts not defined here we refer the reader to any of several standard monographs such as, e.g., [5] or [18].

2. Results

In this section some new extremal values of Zagreb coindices, degree distance and reverse degree distance over some special classes of graphs are determined. We begin by Zagreb coindices of graphs.

2.1. *Zagreb coindices*

The aim of this subsection is to obtain new lower and upper bounds for the first and second Zagreb coindices of graphs.

Proposition 1. *Let* G *be a simple graph with* n *vertices and* m *edges. Then* $\bar{M}_1(G) \leq \frac{-4m^2}{n}+2m(n-1)$ *with equality if and only if* G *is regular.*

Proof. Apply Lemma A(a) and Lemma B(a). □

Proposition 2. *Let G be a simple graph with n vertices, m edges. Then $2m^2 - \frac{M_1(G)}{2}(\Delta+1) \leq \bar{M}_2(G) \leq 2m^2(1 - \frac{2m}{n^2} - \frac{1}{n})$. The right hand (left hand) side of this inequality is satisfied if and only if G is regular.*

Proof. The right hand side is a consequence of Lemma A(b) and Lemma B(b). To prove the left hand side of inequality, we notice that by Lemma A and Lemma B(c), we have:

$$\begin{aligned}\frac{\bar{M}_2(G)}{m} &= 2m - \frac{M_2(G)}{m} - \frac{1}{2m}M_1(G)\\ &\geq 2m - \frac{\Delta M_1(G)}{2m} - \frac{1}{2m}M_1(G)\\ &= 2m^2 - \frac{M_1(G)}{2}(\Delta+1),\end{aligned}$$

which completes our argument. By Lemma A and Lemma B(c) the equality holds if and only if G is regular. □

Proposition 3. *Let G be a simple graph with n vertices, m edges.*

(a) $\bar{M}_1(S(G)) \leq 4m(n-2) + 4m^2(1 - \frac{1}{n})$, *the equality holds if and only if G is regular.*

(b) $\bar{M}_2(S(G)) \leq 8m^2 - 2m - \frac{10m^2}{n}$, *the equality holds if and only if G is regular.*

(c) $\bar{M}_2(S(G)) = -2m^2 + 18m - 10mn + \frac{5}{2}\bar{M}_1(S(G))$.

Proof. The parts (a) and (b) are obtained from Lemma A and Lemma B(a) and the following inequalities:

$$\begin{aligned}\bar{M}_1(S(G)) &= 4m(n+m-1) - M_1(S(G))\\ &= 4m(n+m-1) - M_1(G) - 4m\\ &= 4m(n+m-2) - M_1(G)\\ &\leq 4mn + 4m^2 - 8m - \frac{4m^2}{n}\\ &= 4m(n-2) + 4m^2(1 - \frac{1}{n}),\end{aligned}$$

and,

$$\begin{aligned}\bar{M}_2(S(G)) &= 8m^2 - M_2(S(G)) - \frac{1}{2}M_1(S(G))\\ &= 8m^2 - 2m - \frac{5}{2}M_1(G)\\ &\leq 8m^2 - 2m - \frac{10m^2}{n}.\end{aligned}$$

The equalities in parts (a) and (b) are also obtained from Lemma A and Lemma B(a). For part (c), we apply Lemma A, Lemma B(d) and the following equality:

$$\begin{aligned}\bar{M}_2(S(G)) &= 8m^2 - 2m + \frac{5}{2}(-4mn - 4m^2 + 8m + \bar{M}_1(S(G))) \\ &= -2m^2 + 18m - 10mn + \frac{5}{2}\bar{M}_1(S(G)).\end{aligned}$$

This completes our proof. □

Proposition 4. *Suppose G is a graph with n vertices and m edges. Then* $\bar{M}_1(G) \geq 2m(n-1) - m(\Delta + \delta + 1)$.

Proof. Apply Lemma A(a) and Lemma C(a). □

Proposition 5. *Suppose G is a connected graph with n vertices, m edges.* $\bar{M}_1(G) \geq 2m(n-1) - \frac{(\Delta+\delta)^2}{n\Delta\delta}m^2$. *The equality holds if and only if G is regular.*

Proof. Apply Lemma A(a) and Lemma C(b). □

Proposition 6. *Let G be a connected bicyclic graph with n vertices and m edges without pendant vertices, then* $\frac{\bar{M}_1(G)}{n} \geq \frac{2m(n-1)}{n} + \frac{\bar{M}_2(G)}{m} - 2m + \frac{M_1(G)}{2m}$ *with equality if and only if G is isomorphic to $K_{2,3}$.*

Proof. By Lemmas A and D $\frac{\bar{M}_1(G)}{n} = \frac{2m(n-1)}{n} - \frac{M_1(G)}{n} \geq \frac{2m(n-1)}{n} - \frac{M_2(G)}{m}$ and $\frac{\bar{M}_2(G)}{m} = 2m - \frac{M_2(G)}{m} - \frac{M_1(G)}{2m}$, as desired. □

Proposition 7. *Let G be a graph of order n containing m edges, then* $2(m^2 + \bar{m}^2) - \binom{n}{2}(n-1)^2 - \frac{n(n-1)^2}{2} \leq \bar{M}_2(G) + \bar{M}_2(\bar{G}) \leq 2(m^2 + \bar{m}^2) - \binom{n}{2}(\frac{n-1}{2})^2 - \frac{n(n-1)^2}{2}$. *The equality in right hand side is satisfied if and only* $n \cong 1 \pmod 4$ *and G is* $\frac{n-1}{2}$*-regular. The equality in left hand side is satisfied if and only G is isomorphic to complete graph K_n.*

Proof. Apply [19, Theorems 2.2 and 3.1]. □

2.2. *Degree Distance of Graphs*

The degree distance of a graph is a degree analogue of the Wiener index of the graph. So, it is useful to find the lower and upper bounds of this topological index on some classes of graphs as trees, triangle- and quadrangle-free graphs and so on.

Proposition 8. *Let G be a connected graph with n vertices, m edges and diameter* $\mathrm{diam}(G)$. *Then* $4(n-1)m - M_1(G) \leq D'(G) \leq -(\mathrm{diam}(G) - 1)M_1(G) + 2(n-1)m\mathrm{diam}(G)$ *with either equality if and only if* $\mathrm{diam}(G) \leq 2$.

Proof. Apply Lemma E(1) and Lemma E(2). □

Proposition 9. *Let G be a connected triangle- and quadrangle-free graph with $n \geq 3$ vertices, minimal degree δ, maximal degree Δ, m edges and diameter l. Then*

$$\begin{aligned}-(\Delta+2)M_1(G)+2m[3(n-1)+\Delta] &\leq D'(G)\\ &\leq -[l-1+\delta(l-2)]M_1(G)+2(l-2)m\delta\\ &\quad +2(n-1)ml\end{aligned}$$

with right equality if and only if G is a Moore graph of diameter 2, or a regular graph of diameter 3 (and girth 5, 6 or 7) and with left equality if and only if $G = S_n$, or G is a Moore graph of diameter 2, or a regular graph of diameter 3 (and girth 5, 6 or 7).

Proof. Apply Lemma E(1) and Lemma E(3). □

Proposition 10. *Let G be a connected graph with n vertices, maximal degree Δ and m edges. Then $-(\Delta+2)n(n-1)+2m[3(n-1)+\Delta] \leq D'(G)$ with equality if and only if G is the star or a Moore graph of diameter 2.*

Proof. Apply Lemma E(1) and Lemma E(4). □

Proposition 11. *Let G be a connected graph with minimal degree δ and maximal degree Δ. Then $2\delta W(G) \leq D'(G) \leq 2\Delta W(G)$ with either equality if and only if G is a regular graph.*

Proof. Apply Lemma E(1) and Lemma E(5). □

Proposition 12. *Let G be a tree with n vertices and diameter* $\mathrm{diam}(G)$. *Then $D'(G) = (n-1)[\mathrm{diam}(G)(2m-2n+2)-n]+4W(G)$.*

Proof. Apply Lemma E(1) and Lemma E(6). □

2.3. *Reverse Degree Distance of Graphs*

The reverse degree distance of graphs is a new topological index proposed by Zhou and Trinajstić [21]. In the next two propositions, some extremal properties of this topological index are investigated.

Proposition 13. *If $1 \leq n-1 \leq m$ and G is a connected graph with n vertices and m edges, then ${}^rD'(G) \leq -4(n-1)m+\sum_2(n,m)+2(n-1)m\mathrm{diam}(G)$. Equality happens if and only if G is a join of K_1 and a graph G' on $n-1$ vertices and $m-n+1$ edges with $\sum_2(G') = \sigma_2(n-1,m-n+1)$.*

Proof. Apply Lemmas E(1) and F. □

Proposition 14. *Suppose G is a connected graph with n vertices and m edges. Then for every $n \geq 3$ we have ${}^rD'(G) \leq -3n^2+3n+6+2m(n-1)\mathrm{diam}(G)$ and the unique extremal graph is $K_{1,n-1}+e$.*

Proof. Apply Lemmas E(1) and G(a). □

Proposition 15. *Suppose G is a connected graph with n vertices and m edges. Then for every $n \geq 4$ we have ${}^{r}D'(G) \leq -3n^2 - n + 18 + 2m(n-1)\mathrm{diam}(G)$ and the extremal graph is unique and may be obtained from $K_{1,n-1}$ by adding two edges having a common extremity.*

Proof. Apply Lemmas E(1) and G(b). □

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REFERENCES

[1] A. R. Ashrafi, T. Došlić, and A. Hamzeh, "The zagreb coindices of graph operations," *Discrete Appl. Math.*, vol. 158, no. 15, pp. 1571–1578, 2010.
[2] A. R. Ashrafi, T. Došlić, and A. Hamzeh, "Extremal graphs with respect to the zagreb coindices," *MATCH Commun. Math. Comput. Chem.*, vol. 65, no. 1, pp. 85–92, 2011.
[3] O. Bucicovschi and S. M. Cioabă, "The minimum degree distance of graphs of given order and size," *Discrete Appl. Math.*, vol. 156, no. 18, pp. 3518–3521, 2008.
[4] K. C. Das and I. Gutman, "Some properties of the second zagreb index," *MATCH Commun. Math. Comput. Chem.*, vol. 52, pp. 103–112, 2004.
[5] M. V. Diudea, I. Gutman, and L. Jäntschi, *Molecular Topology*, ser. Huntington. New York: Nova Science, 2001.
[6] A. A. Dobrynin and A. A. Kochetova, "Degree distance of agraph: A degree analogue of the wiener index," *J. Chem. Inf. Comput. Sci.*, vol. 34, no. 5, pp. 1082–1086, 1994.
[7] T. Došlić, "Vertex-weighted wiener polynomials for composite graphs," *Ars Math. Contemp.*, vol. 1, no. 1, pp. 66–80, 2008.
[8] I. Gutman, "Selected properties of the schultz molecular topological index," *J. Chem. Inf. Comput. Sci.*, vol. 34, no. 5, pp. 1087–1089, 1994.
[9] I. Gutman and K. C. Das, "The first zagreb index 30 years after," *MATCH Commun. Math. Comput. Chem.*, vol. 50, pp. 83–92, 2004.
[10] A. Ilić and D. Stevanović, "On comparing zagreb indices," *MATCH Commun. Math. Comput. Chem.*, vol. 62, no. 3, pp. 681–687, 2009.
[11] A. Ioan Tomescu, "Unicyclic and bicyclic graphs having minimum degree distance," *Discrete Appl. Math.*, vol. 156, no. 1, pp. 125–130, 2008.
[12] M. H. Khalifeh, H. Yousefi-Azari, and A. R. Ashrafi, "The first and second zagreb indices of some graph operations," *Discrete Appl. Math.*, vol. 157, no. 4, pp. 804–811, 2009.
[13] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, and S. G. Wagner, "Some new results on distance-based graph invariants," *European J. Combin.*, vol. 30, no. 5, pp. 1149–1163, 2009.
[14] M. Liu and B. Liu, "New sharp upper bounds for the first zagreb index," *MATCH Commun. Math. Comput. Chem.*, vol. 62, no. 3, pp. 689–698, 2009.
[15] M. J. Nadjafi-Arani, G. H. Fath-Tabar, and A. R. Ashrafi, "Extremal graphs with respect to the vertex pi index," *Appl. Math. Lett.*, vol. 22, no. 12, pp. 1838–1840, 2009.
[16] S. Nikolić, G. Kovačević, A. Miličević, and N. Trinajstić, "The zagreb indices 30 years after," *Croat. Chem. Acta*, vol. 76, no. 2, pp. 113–124, 2003.
[17] L. Sun and S. Wei, "Comparing the zagreb indices for connected bicyclic graphs," *MATCH Commun. Math. Comput. Chem.*, vol. 62, no. 3, pp. 699–714, 2009.
[18] D. B. West, *Introduction to graph theory*. Upper Saddle River: Prentice Hall, 1996.

[19] L. Zhang and B. Wu, "The nordhaus-goddum-type inequalities for some chemical indices," *MATCH Commun. Math. Comput. Chem.*, vol. 54, no. 1, pp. 189–194, 2005.
[20] B. Zhou and I. Gutman, "Relations between wiener, hyper-wiener and zagreb indices," *Chem. Phys. Lett.*, vol. 394, no. 1-3, pp. 93–95, 2004.
[21] B. Zhou and N. Trinajstić, "On reverse degree distance," *J. Math. Chem.*, vol. 47, no. 1, pp. 268–275, 2010.

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EXISTENCE OF THE SOLUTION OF FRACTIONAL INTEGRAL INCLUSION WITH TIME DELAY

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Abstract. In this article, we establish the existence of solutions for integral inclusion of fractional order. The study holds in the case when the set-valued (multi-valued) function has convex and non-convex values. Also we illustrate our results with examples to which our abstract theory applies.

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1. INTRODUCTION

The class of fractional operator equations of various types plays very important role not only in mathematics but also in physics, control systems, dynamical systems and engineering. Naturally, such equations required to be solved. There are numerous books focused in this direction, that is concerning the linear and nonlinear problems involving different types of fractional derivatives as well as integral see [20, 22, 23, 27, 28]. In [13, 14, 17], El-Sayed and A. G. Ibrahim gave the concept of the definite integral of fractional order for set-valued function. Integral and integro-differential (linear and nonlinear) in abstract spaces has been studied by several authors see [3, 18, 24, 25]. Applications of this type of problem arise in the study of control systems (see [1, 2, 10]). This paper concerned with the following fractional integral inclusion.

$$u(t)-\sum_{i=1}^{n} b_i(t)u(t-\tau_i) \in I^{\alpha}F(t,u(t));\ 0<\alpha\leq 1,\ t\in J:=[0,T], \tag{1.1}$$

where $\tau_i \leq t \in J, \forall i = 1,...,n$, $F : J\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a set-valued function with nonempty values in $\mathbb{R}$, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $I^{\alpha}F(t,u(t))$ are the definite integral for the set-valued functions F of order α which is defined as

$$I^{\alpha}F(t,u(t)) = \left\{ \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau,u(\tau))d\tau : f(t,u)\in S_F(u)\right\},$$

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where

$$S_F(u) = \left\{ f \in L^1(J,\mathbb{R}) : f(t) \in F(t,u(t)) \ a.e. \ t \in J \right\}$$

denotes the set of selections of F and $b_i : J \to \mathbb{R}$ is a continuous function such that $B := \max_{1\le i\le n} \{sup_{t\in J} \{|b_i(t)|\}\}$.

This paper will be organized as follows: In Section 2, we recall briefly some basic definitions and preliminary facts from set-valued analysis which will be used later. In Section 3, we establish the existence of the solution for the single-valued problem

$$u(t) = \sum_{i=1}^{n} b_i(t)u(t-\tau_i) + I^{\alpha} f(t,u(t)); \ 0 < \alpha \le 1, \ t \in J, \ f \in F(t,u(t)), \quad (1.2)$$

by using the Schauder fixed point theorem, (see [6]) and the uniqueness of the solution by using the Banach fixed point theorem (see [29]). In Section 4, we shall study the existence of solution for the set-valued problem 1.1 when F has convex values, as well when it has with non-convex values, via the single-valued problem as well as fixed point theorems of the set-valued function. In the first case (convex) a fixed point theorem due to Bohnenblust and Karlin (see [8, 31]) is used. A fixed point theorem for contraction set-valued functions due to Covitz and Nadler [11] is applied in the second one (non-convex). Also we illustrate our results with examples.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from set-valued analysis which are used throughout this paper. For further background and details pertaining to this section we refer the reader to [4, 7, 15, 16, 19, 26, 30].

$\mathcal{B} := C[J,\mathbb{R}]$ is the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|u\| = sup\{|u(t)| : t \in J\}$$

for each $u \in \mathcal{B}$. $\mathcal{L} := L^1[J,\mathbb{R}]$ denotes the Banach space of measurable functions $u : J \to \mathbb{R}$ which are Lebesgue integrable normed by

$$\|u\|_{L^1} = \int_0^T |u(t)|dt,$$

for each $u \in \mathcal{L}$. Let $(X,|.|)$ be a normed space, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, $\mathcal{P}_c(X) = \{Y \in \mathcal{P}(X) : Y \text{ is convex}\}$, $\mathcal{P}_{cl,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed and convex}\}$, $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A set-valued function $F : X \to \mathcal{P}(X)$ is called *convex* (*closed*) valued if $F(x)$ is *convex* (*closed*) for all $x \in X$. F is called bounded valued on bounded set B if $F(B) = \bigcup_{x\in B} F(x)$ is bounded in X for all $B \in \mathcal{P}_b(X)$ i.e. $sup_{x\in B}\{sup\{|u| : u \in F(x)\}\} < \infty$. F is called upper semi-continuous (u.s.c) on X if for each $x_0 \in X$ the set $F(x_0)$ is nonempty closed subset of X and if for each open set N of X containing $F(x_0)$, there exists an open neighborhood N_0 of x_0 such that $F(N_0) \subseteq N$. F is *u.s.c* if

the set $F^{-1}(A) = \{x \in X : Fx \subset A\}$ is open in X for every open set A in X. F is called lower semi-continuous (l.s.c) on X if A is any open subset of X then $F^{-1}(A) = \{x \in X : Fx \bigcap A \neq \varnothing\}$ is open in X. F is called $continuous$ if it is lower as well as upper semi-continuous on X. F is called compact if for every M bounded subset of X, $F(M)$ is relatively compact. Finally F is called completely continuous if it is upper semi-continuous and compact on X.

The following definitions are used in the sequel.

Definition 1. A mapping $p : J \times \mathbb{R} \to \mathbb{R}$ is said to be Carathéodory if

(i) $t \to p(t,u)$ is measurable for each $u \in \mathbb{R}$,
(ii) $u \to p(t,u)$ is continuous a.e. for $t \in J$.

A *Carathéodory* function $p(t,u)$ is called $L^1(J,\mathbb{R})$-*Carathéodory* if

(iii) for each number $r > 0$ there exists a function $h_r \in L^1(J,\mathbb{R})$ such that $|p(t,u)| \leq h_r(t)$ a.e $t \in J$ for all $u \in \mathbb{R}$ with $|u| \leq r$.

A *Carathéodory* function $p(t,u)$ is called $L^1_X(J,\mathbb{R})$-*Carathéodory* if

(iv) there exist a function $h \in L^1(J,\mathbb{R})$ such that $|p(t,u)| \leq h(t)$ a.e $t \in J$ for all $u \in \mathbb{R}$ where h is called the bounded function of p.

Definition 2. A set-valued function $F : J \to \mathcal{P}(\mathbb{R})$ is said to be measurable if for any $x \in X$, the function $t \mapsto d(x, F(t)) = inf\{|x-u| : u \in F(t)\}$ is measurable.

Definition 3. A set-valued function $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called Carathéodory if

(i) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$, and
(ii) $x \mapsto F(t,x)$ is u.s.c. for almost $t \in J$.

Definition 4. A set-valued function $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called L^1-Carathéodory if

(i) F is Carathéodory and
(ii) For each $r > 0$, there exists $h_r \in L^1(J,\mathbb{R})$ such that $\|F(t,u)\| = sup\{|f| : f \in F(t,u)\} \leq h_r(t)$ for all $|u| \leq r$ and for a.e. $t \in J$.

Definition 5 ([12])**.** A set-valued function $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called $\mathbf{L}^1_X$-Carathéodory if there exists a function $h \in L^1(J,\mathbb{R})$ such that

$$\|F(t,u)\| = sup\{|f| : f \in F(t,u)\} \leq h(t),\ a.e.\ t \in J$$

for all $x \in \mathbb{R}$, and the function h is called a growth function of F on $J \times \mathbb{R}$.

Let $A, B \in \mathcal{P}_{cl}(X)$ and let $a \in A$. Then by

$$D(a,B) = inf\{\|a-b\| : b \in B\}\ and\ \ \rho(A,B) = sup\{D(a,B) : a \in A\}$$

the function $H : \mathcal{P}_{cl}(X) \times \mathcal{P}_{cl,b}(X) \to \mathbb{R}^+$ defined by

$$H(A,B) = max\{\rho(A,B), \rho(B,A)\}$$

is metric and is called Hausdorff metric on X. Moreover $(\mathcal{P}_{cl,b}(X), H)$ is a metric space and $(\mathcal{P}_{cl}(X), H)$ is a complete metric space(see [21]). It is clear that

$$H(0, C) = sup\{\|c\| : c \in C; C \in \mathcal{P}_b(X)\}.$$

Definition 6. A set-valued function $F : \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ is called

(i) γ-*Lipschitz* if and only if there exist $\gamma > 0$ such that

$$H(F(x), F(y)) \leq \gamma\|x - y\|, \, for \, each \, x, y \in X$$

the constant γ is called a Lipschitz constant.

(ii) a *contraction* if and only if it is γ-*Lipschitz* with $\gamma < 1$.

Definition 7. A set-valued function $F : J \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ is called

(i) $\gamma(t)$-Lipschitz if and only if there exists $\gamma \in L^1(J, \mathbb{R}^+)$ such that

$$H(F(t, x), F(t, y)) \leq \gamma(t)\|x - y\|, \, for \, each \, x, y \in X.$$

(ii) a contraction if and only if it is $\gamma(t)$-Lipschitz with $\|\gamma\| < 1$

The following remark and lemmas are used in the sequel.

Remark 1 ([5]). Let $M \subset X$. If $F : M \to \mathcal{P}(X)$ is closed and $F(M)$ is relatively compact then F is u.s.c. on M. And if $F : X \to \mathcal{P}(X)$ is closed and compact operator then F is u.s.c. on X.

Lemma 1 (Bohnenblust-Karlin)**.** *Let M be a closed and convex subset of the Banach space X and $F : M \to \mathcal{P}(M)$ a set-valued function. Suppose that:*

(i) *the set $F(M)$ is relatively compact;*
(ii) *the set-value function F is u.s.c on M;*
(iii) *the set $F(x)$ is nonempty closed and convex for all $x \in M$.*

Then there is $x \in M$ such that $x \in F(x)$.

Lemma 2 (Covitz-Nadler)**.** *Let (X, d) be a complete metric space. If $G : X \to \mathcal{P}_{cl}(X)$ is a contraction, then G has a fixed point.*

3. Single-valued problem

In this section we prove that the integral equation 1.2 has a solution $u(t)$ on J. Let us formulate the following assumption:

(H1) The function f is $L^1_X(J, \mathbb{R})$-Carathéodory with bound function $h \in L^1(J, \mathbb{R})$ such that $|f(t, u)| \leq h(t)$ a.e. $t \in J$ for all $u \in \mathbb{R}$.

Theorem 1. *Let $(H1)$ hold. If $1 - nB > 0$ then equation 1.2 has at least one solution $u(t)$ on J.*

Proof. Define an operator P by

$$(Pu)(t) := \sum_{i=1}^{n} b_i(t)u(t-\tau_i) + I^{\alpha} f(t,u(t)). \tag{3.1}$$

Set $r := \frac{T^{\alpha}\|h\|}{\Gamma(\alpha+1)(1-nB)}$, then by the assumption of the theorem and the properties of the fractional calculus we obtain that

$$\begin{aligned}|(Pu)(t)| &\leq \sum_{i=1}^{n} |b_i(t)||u(t-\tau_i)| + I^{\alpha}|f(t,u(t))| \\ &\leq nB\|u\| + I^{\alpha}|h(t)| \leq nB\|u\| + \|h\|\frac{T^{\alpha}}{\Gamma(\alpha+1)}.\end{aligned}$$

So we obtain that

$$\|Pu\| \leq \frac{T^{\alpha}\|h\|}{\Gamma(\alpha+1)(1-nB)}.$$

That is $P : B_r \to B_r$. Then P maps B_r into itself. In fact, P maps the convex closure of $P[B_r]$ into itself. Since f is bounded on B_r, thus $P[B_r]$ is equicontinuous and the Schauder fixed point theorem shows that P has at least one fixed point $u \in \mathcal{B} = C[J,\mathbb{R}]$ such that $Pu = u$, which is corresponding to the solution of 1.2. □

For the uniqueness let us consider the following assumption

(H2) The function f satisfies that there exists a function $\ell(t) \in L^1(J,\mathbb{R}^+)$ such that for each $u, v \in C[J,\mathbb{R}]$ we have

$$|f(t,u) - f(t,v)| \leq \ell(t)\|u-v\|.$$

Theorem 2. *Let (H2) hold. If* $\frac{nB\Gamma(\alpha+1)+T^{\alpha}\|\ell\|}{\Gamma(\alpha+1)} < 1$, *then 1.2 has a unique solution* $u(t)$ *on* J.

Proof. Assume the operator P defined in equation 3.1 then we have

$$\begin{aligned}|(Pu)(t)-(Pv)(t)| &\leq \sum_{i=1}^{n} |b_i(t)||u(t-\tau_i)-v(t-\tau_i)| + I^{\alpha}|f(t,u(t))-f(t,v(t))| \\ &\leq nB\|u-v\| + I^{\alpha}\ell(t)\|u-v\| \\ &\leq nB\|u-v\| + \frac{\|\ell\|T^{\alpha}}{\Gamma(\alpha+1)}\|u-v\| \\ &= (nB + \frac{\|\ell\|T^{\alpha}}{\Gamma(\alpha+1)})\|u-v\|.\end{aligned}$$

□

Hence by the assumption of the theorem we have that P is a contraction mapping. Then in view of Banach fixed point Theorem, P has a unique fixed point which is corresponding to the solution of equation 1.2.

4. Set-valued problem

In this section we establish the existence results for the integral inclusion 1.1 when the right hand side is convex as well as when it is non-convex valued. The study will be taken in view of the single-valued theorems as well as fixed point theorems of set-valued function. Let us illustrate the following assumption

(H3) The set-valued function $F : J \times \mathbb{R} \to \mathcal{P}_{cl,c}(\mathbb{R})$ is $L^1_X(J,\mathbb{R})$-Carathéodory with a growth function $h \in L^1(J,\mathbb{R})$ such that $\|F(t,u)\| \leq h(t)$ a.e $t \in J$ for all $u \in \mathbb{R}$.

Theorem 3. *Let* $(H3)$ *hold. If* F *is lower semi-continuous (l.s.c) and* $1-nB > 0$ *then the integral inclusion 1.1 has at least one solution* $u(t)$ *on* J.

Proof. The proof here is depending on the single-valued problem. For each $u(t)$ in $\mathbb{R}$ the set $S_F(u)$ is nonempty, since by $(H3)$, F has a non-empty measurable selection (see [9]). Thus there exists a function $f(t,u) \in F$ where f is a $L^1_X(J,\mathbb{R})$-Carathéodory function with a bounded function $h \in L^1(J,\mathbb{R})$ such that $\|f(t,u)\| \leq h(t)$ a.e $t \in J$ for all $u \in \mathbb{R}$. Hence the assumptions of Theorem 2 are satisfied so the integral inclusion 1.1 has a solution. □

Theorem 4. *(Convex case) Let* $(H3)$ *hold. If* $1-nB > 0$ *then the integral inclusion 1.1 has at least one solution* $u(t)$ *on* J.

Proof. Now we shall show that the assumptions of Lemma 1 are satisfied. Consider the following space $\Omega := \{u : J \to \mathbb{R} | u(t) \in C[J,\mathbb{R}]\}$ which is a Banach space endow with the sup. norm. Consider the set-valued operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by

$$(Nu)(t) := \{u \in \Omega : u(t) = \sum_{i=1}^{n} b_i(t)u(t-\tau_i) + I^{\alpha} f(t,u(t)); f \in S_F(u)\}. \quad (4.1)$$

We shall show that N satisfies the assumptions of Lemma 1. The proof will be given by several steps. Define a set $K := \left\{u \in \Omega : \|u\| \leq r := \frac{T^{\alpha}\|h\|}{\Gamma(\alpha+1)(1-nB)}\right\}$ such that $sup\{|u(t)| : u \in \Omega\} < \infty$. It is clear that K is nonempty, bounded, closed and convex subset of Ω. Note that by (H3) $S_F(u) \neq \varnothing$. □

Step 1: $N(K) \subset K$. Let $u \in K$, we must show that $N(u) \in K$. For all $h(t) \in N(u)$, we have

$$|h(t)| \leq \sum_{i=1}^{n} |b_i(t)||u(t-\tau_i)| + I^{\alpha}|f(t,u(t))|$$

$$\leq nB\|u\| + I^{\alpha}|h(t)| \leq nB\|u\| + \frac{\|h\|_{L^1}T^{\alpha}}{\Gamma(\alpha+1)}$$
$$\leq nrB + (1-nB)r = r.$$

Step 2: $N(K)$ is relatively compact set. We must show that N is a compact operator. Since K is bounded closed and convex set and $N(K) \subset K$, it follows that $N(K)$ is bounded closed and convex set. Moreover, for $0 \leq t_1 \leq t_2 \leq T$ such that $|t_2 - t_1| < \delta$, $\delta > 0$ and $u \in K$ then for all $h(t) \in N(u)$,

$$|h(t_1) - h(t_2)|$$
$$= |\sum_{i=1}^{n} b_i(t_1)u(t_1-\tau_i) + I^{\alpha}f(t,u(t_1)) - \sum_{i=1}^{n} b_i(t_2)u(t_2-\tau_i) - I^{\alpha}f(t,u(t_2))|$$
$$\leq 2nB\|u\| + |I^{\alpha}f(t,u(t_1)) - I^{\alpha}f(t,u(t_2))|$$
$$\leq 2nB\|u\| + [\frac{\|f\|}{\Gamma(\alpha+1)}]|(t_1^{\alpha} - t_2^{\alpha} + 2(t_2-t_1)^{\alpha})|$$
$$\leq 2nB\|u\| + [\frac{2\|f\|}{\Gamma(\alpha+1)}]|(t_2-t_1)^{\alpha}|$$
$$\leq 2nB\|u\| + [\frac{2\|f\|}{\Gamma(\alpha+1)}]|(t_2-t_1)|^{\alpha}$$
$$\leq 2[nB\|u\| + \frac{2\delta^{\alpha}\|f\|}{\Gamma(\alpha+1)},$$

which is independent of u, and that implies that $N(K)$ is equicontinuous. An application of Arzela-Ascoli Theorem yields that N maps K into a compact set in Ω, that is $N; K \to \mathcal{P}(\Omega)$ is a compact operator. Thus $N(K)$ is relatively compact set.

Step 3: N is upper semi-continuous on K. Since $N : K \to K$ is closed and compact and that $N(K)$ is relatively compact set then in view of Remark 1, we obtain that N is u.s.c on K.

Step 4: $N(u)$ is a convex. Let $h_1(t), h_2(t) \in N(u)$ then there exist two functions $f_1, f_2 \in S_F(u)$ such that

$$h_j(t) = \sum_{i=1}^{n} b_i(t)u(t-\tau_i) + I^{\alpha}f_j(t,u(t)); \ j = 1,2.$$

Let $0 \leq \delta \leq 1$, then for each $t \in J$ we have

$$[\delta h_1 + (1-\delta)h_2](t) = \delta[\sum_{i=1}^{n} b_i(t)u(t-\tau_i) + I^{\alpha}f_1(t,u(t))]$$
$$+(1-\delta)[\sum_{i=1}^{n} b_i(t)u(t-\tau_i) + I^{\alpha}f_2(t,u(t))]$$

$$= \sum_{i=1}^{n} b_i(t)[\delta u(t-\tau_i) + (1-\delta)u(t-\tau_i)]$$
$$+[\delta I^{\alpha} f_1(t,u(t)) + (1-\delta) I^{\alpha} f_2(t,u(t))]$$
$$= \sum_{i=1}^{n} b_i(t)[\delta u(t-\tau_i) + (1-\delta)u(t-\tau_i)] + I^{\alpha}[\delta f_1(t,u(t)) + (1-\delta) f_2(t,u(t))]$$
$$= \sum_{i=1}^{n} b_i(t)u(t-\tau_i) + I^{\alpha}[\delta f_1(t,u(t)) + (1-\delta) f_2(t,u(t))]$$

by the convexity of $S_F(u)$, and $F(t,u)$ implies that $[\delta h_1 + (1-\delta)h_2](t) \in N(u)$. Hence $N(u)$ is convex. As a consequence of Lemma 1, we deduce that N has a fixed point which is a solution for the integral inclusion 1.1. Hence the proof is complete.

In order to study the existence for the problem 1.1 in non-convex case, we introduce the following assumptions.

(H4) $F : J \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$, $(t,.) \mapsto F(t,u)$ is measurable for each $u \in \mathbb{R}$.

(H5) $F : J \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ is $\ell(t)$-*Lipschitz*

$$H(F(t,u), F(t,v)) \leq \ell(t)\|u-v\|.$$

Theorem 5 (Non-convex case)**.** *Let* $(H4)$ *and* $(H5)$ *hold. If*

$$\frac{nB\Gamma(\alpha+1) + T^{\alpha}\|\ell\|_{L^1}}{\Gamma(\alpha+1)} < 1,$$

then the integral inclusion 1.1 has at least one solution $u(t)$ *on* J.

Proof. For each $u(t)$ in $\mathbb{R}$ the set $S_F(u)$ is nonempty since by $(H4)$, F has a nonempty measurable selection (see [9]). Then there exists a function $f(t,u) \in F$ such that f is $\ell(t)$-*Lipschitz*. Thus by the assumption of the theorem and by Theorem 2, the inclusion 1.1 has a solution. Hence the proof is done in view of the single-valued problem. □

In order to study the existence for problem 1.1 by using Lemma 2, we must show that $N : \Omega \to \mathcal{P}(\Omega)$ defined in 4.1 is a contraction set-valued mapping. The proof will be given in two steps.

Step 1: $N(u) \in \mathcal{P}_{cl}(\Omega)$ for each $u \in \Omega$. Let $\{u_m\}_{m\geq 0} \in N(u)$ such that $u_m \to \widetilde{u}$ in Ω. Then $\widetilde{u} \in \Omega$ and there exists $f_m \in S_F(u)$ such that for $t \in J$

$$u_m(t) = \sum_{i=1}^{n} b_i(t)u(t-\tau_i) + I^{\alpha} f_m(t,u(t)).$$

Using the fact that F has closed values, we get that f_m converges to f in $L^1(J,\mathbb{R})$ and hence $f \in S_F(u)$. Then for each $t \in J$

$$u_m(t) \to \widetilde{u}(t) = \sum_{i=1}^{n} b_i(t)u(t-\tau_i) + I^{\alpha} f(t,u(t)).$$

So $\widetilde{u} \in N(u)$.

Step 2: There exists $\gamma < 1$ such that

$$H(N(u),N(v)) \leq \gamma \|u-v\|_{\Omega},\ for\ each\ u,v \in \Omega.$$

Let $u,v \in \Omega$. Then by $(H5)$ there exists $f \in F$ that satisfies

$$|f(t,u)-f(t,v)| \leq \ell(t)\|u-v\|_{\Omega}.$$

Then $h_1(t) \in N(u)$ where

$$h_1(t) = \sum_{i=1}^{n} b_i(t)u(t-\tau_i) + I^{\alpha} f(t,u(t)).$$

Then for $h_2(t) \in N(v)$, where

$$h_2(t) = \sum_{i=1}^{n} b_i(t)v(t-\tau_i) + I^{\alpha} f(t,v(t)),$$

we have

$$\begin{aligned}
|h_1(t)-h_2(t)| &\leq \sum_{i=1}^{n} b_i(t)|u(t-\tau_i)-v(t-\tau_i)| + |I^{\alpha} f(t,u(t)) - I^{\alpha} f(t,v(t))| \\
&\leq \sum_{i=1}^{n} b_i(t)|u(t-\tau_i)-v(t-\tau_i)| + I^{\alpha}|f(t,u(t))-f(t,v(t))| \\
&\leq nB\|u-v\|_{\Omega} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}|f(t,u(t))-f(t,v(t))| \\
&\leq nB\|u-v\|_{\Omega} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\ell(t)\|u-v\|_{\Omega} \\
&\leq nB\|u-v\|_{\Omega} + \frac{T^{\alpha}\|\ell\|_{L^1}}{\Gamma(\alpha+1)}\|u-v\|_{\Omega} \\
&= [nB + \frac{T^{\alpha}\|\ell\|_{L^1}}{\Gamma(\alpha+1)}]\|u-v\|_{\Omega}.
\end{aligned}$$

Denote

$$\gamma := [\frac{nB\Gamma(\alpha+1) + T^{\alpha}\|\ell\|_{L^1}}{\Gamma(\alpha+1)}].$$

It follows that

$$H(N(u),N(v)) \leq \gamma \|u-v\|_{\Omega},\ for\ each\ u,v \in \Omega,$$

where $\gamma < 1$. It follows that N is a contraction set-valued mapping. Then, in view of Lemma 2, N has a fixed point which is corresponding to a solution of inclusion 1.1. This completes the proof.

Example 1. Let $J = [0,1]$ denote a closed and bounded interval in $\mathbb{R}$. Consider the problem

$$u(t) \in \frac{t^3}{8}u(t-\tau_1) + \frac{t^4}{12}u(t-\tau_2) + \frac{1}{4}u(t-\tau_3) + I^{\alpha}F(t,u);\ t \in [0,1] \tag{4.2}$$

where $n = 3, \alpha = 0.5, B = 0.25$ and

$$F(t,u) = \begin{cases} h(t), & \text{if } u < 0; \\ [h(t)\exp^{\frac{-u(t)}{2}}, h(t)], & \text{if } u \geq 0. \end{cases}$$

It is clear that F is $L^1_X(J,\mathbb{R})-Carathéodory$ with a growth function $h \in L^1(J,\mathbb{R}^+)$ such that $\|F(t,u)\| \leq h(t)$ a.e $t \in J$ for all $u \in \mathbb{R}$. Therefore if $\|h\| \leq \frac{\sqrt{\pi}}{8}$ then in view of Theorem 4, the problem 4.2 has a solution.

Example 2. Let $J = [0,1]$ denote a closed and bounded interval in $\mathbb{R}$. Consider the problem

$$u(t) \in \frac{t}{5}u(t-\tau_1) + \frac{t^2}{10}u(t-\tau_2) + I^{\alpha}F(t,u);\ t \in [0,1] \tag{4.3}$$

where $n = 2, \alpha = 0.5, B = \frac{1}{5}$ and

$$F(t,u) = \begin{cases} 0 & \text{if } u < 0; \\ [0, \frac{u(t)}{4}] & \text{if } u \geq 0. \end{cases}$$

It is clear that F is $\ell(t)$-Lipschitzian set-valued function with $\|\ell\| = \frac{1}{4}$. Therefore, for $\gamma = 0.6 < 1$, we have that F is a contraction set-valued mapping. Thus, in view of Theorem 5, the problem 4.3 has a solution.

REFERENCES

[1] N. U. Ahmed and X. Xiang, "Existence of solutions for a class of nonlinear evolution equations with nonmonotone perturbations." *Nonlinear Anal.*, vol. 22, no. 1, pp. 81–89, 1994.

[2] N. U. Ahmed and K. L. Teo, *Optimal control of distributed parameter systems*. New York-Amsterdam: North-Holland Publishing Co., 1981.

[3] S. Aizicovici and V. Staicu, "Continuous selections of solution sets to volterra integral inclusions in banach spaces," *Electron. J. Differential Equations*, vol. 2006, no. 1, pp. 1–11, 2006.

[4] J.-P. Aubin and A. Cellina, *Differential inclusions. Set-valued maps and viability theory*, ser. Fundamental Principles of Mathematical Sciences. Berlin: Springer-Verlag, 1984, vol. 264.

[5] C. Avramescu, "A fixed point theorem for multivalued mappings," *Electron. J. Qual. Theory Differ. Equ.*, no. 17, pp. 1–10, 2004.

[6] K. Balachandar and J. Dauer, *Elements of control theory*. Narosa Publishing House, 1999.

[7] V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*. Leiden: Noordhoff International Publishing, 1976.

[8] H. F. Bohnenblust and S. Karlin, *On a theorem of Ville*, ser. Annals of Mathematics Studies. Princeton: Princeton University Press, 1950, vol. 24.

[9] C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, ser. Lecture Notes in Mathematics. Berlin-New York: Springer-Verlag, 1977, vol. 580.

[10] A. Cernea, "Existence for nonconvex integral inclusions via fixed points," *Arch. Math. (Brno)*, vol. 39, no. 4, pp. 293–298, 2003.

[11] K. Deimling, *Multivalued differential equations*, ser. de Gruyter Series in Nonlinear Analysis and Applications. Berlin: Walter de Gruyter & Co., 1992, vol. 1.

[12] B. C. Dhage, "Multi-valued operators and fixed point theorems in banach algebras. i." *Taiwanese J. Math.*, vol. 10, no. 4, pp. 1025–1045, 2006.

[13] A. M. A. El-Sayed and A. G. Ibrahim, "Multivalued fractional differential equations," *Appl. Math. Comput.*, vol. 68, no. 1, pp. 15–25, 1995.

[14] A. M. A. El-Sayed and A. G. Ibrahim, "Set-valued integral equations of fractional-orders," *Appl. Math. Comput.*, vol. 118, no. 1, pp. 113–121, 2001.

[15] S. Hu and N. S. Papageorgiou, *Handbook of multivalued analysis. Vol. I.*, ser. Mathematics and its Applications. Dordrecht: Kluwer Academic Publishers, 1997, vol. 419.

[16] S. Hu and N. S. Papageorgiou, *Handbook of multivalued analysis. Vol. II.*, ser. Mathematics and its Applications. Dordrecht: Kluwer Academic Publishers, 2000, vol. 500.

[17] A. G. Ibrahim and A. M. A. El-Sayed, "Definite integral of fractional order for set-valued functions," *J. Fract. Calc.*, vol. 11, pp. 81–87, 1997.

[18] M. Kanakaraj and K. Balachandran, "Existence of solutions of sobolev-type semilinear mixed integrodifferential inclusions in banach spaces," *J. Appl. Math. Stochastic Anal.*, vol. 16, no. 2, pp. 163–170, 2003.

[19] A. G. Kartsatos and K.-Y. Shin, "Solvability of functional evolutions via compactness methods in general banach spaces," *Nonlinear Anal.*, vol. 21, no. 7, pp. 517–535, 1993.

[20] V. Kiryakova, *Generalized fractional calculus and applications*, ser. Pitman Research Notes in Mathematics Series. New York: Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., 1994, vol. 301.

[21] M. Kisielewicz, *Differential inclusions and optimal control*, ser. Mathematics and its Applications (East European Series). Dordrecht: Kluwer Academic Publishers Group, 1991, vol. 44.

[22] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, ser. A Wiley-Interscience Publication. New York: John Wiley & Sons, Inc., 1993.

[23] K. B. Oldham and J. Spanier, *The fractional calculus*, ser. Mathematics in Science and Engineering. New York-London: Academic Press, 1974, vol. 111.

[24] N. S. Papageorgiou, "On integral inclusions of volterra type in banach spaces," *Czechoslovak Math. J.*, vol. 42(117), no. 4, pp. 693–714, 1992.

[25] N. S. Papageorgiou, "On nonconvex valued volterra integral inclusions in banach spaces," *Czechoslovak Math. J.*, vol. 44(119), no. 4, pp. 631–648, 1994.

[26] N. H. Pavel, *Nonlinear evolution operators and semigroups*, ser. Lecture Notes in Mathematics. Berlin: Springer-Verlag, 1987, vol. 1260.

[27] I. Podlubny, *Fractional differential equations*, ser. Mathematics in Science and Engineering. San Diego: Academic Press, Inc., 1999, vol. 198.

[28] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional integrals and derivatives*. Yverdon: Gordon and Breach Science Publishers, 1993.

[29] D. R. Smart, *Existence for nonconvex integral inclusions via fixed points*, ser. Fixed PointTheorems. Cambridge: Cambridge University Press, 1980.

[30] I. I. Vrabie, *Compactness methods for nonlinear evolutions*, ser. Pitman Monographs and Surveys in Pure and Applied Mathematics. New York: Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., 1987, vol. 32.

[31] E. Zeidler, *Nonlinear functional analysis and its applications. I.*, ser. Fixed-point theorems. New York: Springer-Verlag, 1986.

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SECOND ORDER LINEAR RECURSIONS WHOSE SUBSCRIPTS ARE A POWER

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Abstract. We consider two kinds of second order linear recurrences whose subscripts are powers and present certain new identities including these recurrences. Furthermore, we derive first-order nonlinear homogeneous recurrence relations for these recurrences. Our results generalize earlier results as well as they provide new solutions for certain uncompleted cases of the literature.

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1. INTRODUCTION

Let p be a nonzero integer such that $\Delta = p^2 + 4 \neq 0$. Define the generalized Fibonacci type $\{u_n\}$ and Lucas type $\{v_n\}$ sequences as follows:

$$u_n = pu_{n-1} + u_{n-2}$$

and

$$v_n = pv_{n-1} + v_{n-2},$$

where $u_0 = 0, u_1 = 1$ and $v_0 = 2, v_1 = p$, respectively.

If the roots of $x^2 - px - 1 = 0$ are α and β, then the Binet forms of $\{u_n\}$ and $\{v_n\}$ are

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } v_n = \alpha^n + \beta^n.$$

If $p = 1$, then $u_n = F_n$ (nth Fibonacci number) and $v_n = L_n$ (nth Lucas number) respectively.

Usiskin [7, 8] suggested the following problems: For $n > 0$, show that

$$F_{3^n} = \prod_{k=0}^{n-1} L_{2.3^k} - 1 \tag{1.1}$$

and

$$L_{3^n} = \prod_{k=0}^{n-1} L_{2.3^k} + 1. \tag{1.2}$$

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In [4], the author asked for the solution of the first order cubic recurrence relation:

$$a_{n+1} = 5a_n^3 - 3a_n \tag{1.3}$$

with $a_0 = 1$.

Then in [6], the solution is given as $a_n = F_{3^n}$. The same problem appeared as Problem 1809 in Crux Mathematicorum 20 (1994): 19-20.

In the same issue, there was a proposal to solve the recurrence

$$P_{n+1} = 25P_n^5 - 25P_n^3 + 5P_n, \quad P_0 = 1.$$

The solution was given as $P_n = 5^n$. Also the following recurrences and their solutions were commented by Murray S. Klamkin as an addendum to the solution of the problem given in [6]:

$$\begin{aligned} A_{n+1} &= A_n^2 - 2, \quad A_1 = 3, \\ B_{n+1} &= B_n^4 - 4B_n^2 + 2, \quad B_1 = 7, \\ C_{n+1} &= C_n^6 - 6C_n^4 + 9C_n^2 - 2, \; C_1 = 18, \end{aligned}$$

where $A_n = L_{2^n}$, $B_n = L_{4^n}$ and $C_n = L_{6^n}$.

In [1], the author presented some identities involving Fibonacci numbers of the form F_{k^n} for positive odd k and gave a first-order nonlinear homogeneous recurrence relation for F_{k^n}, which generalized (1.3), (1.1) and (1.2).

Recently in [2], *Helmut Prodinger* gave a general expansion formula for a sum of powers of Fibonacci numbers, as considered by Melham, as well as some extensions.

In this paper, we consider two generalized second order recursion sequences and then generalize the results of [1] for the odd k case as well as derive a new first-order nonlinear homogeneous recurrence relation for the sequence $\{u_{k^n}\}$ for possible even k. Further we present that the generalized Lucas numbers $v_{k^{n+1}}$ is a polynomial of generalized Fibonacci numbers u_{k^n} of degree k for even k.

2. Main Results

In this section, in order to derive a recurrence relation for both even and odd subscripted terms u_{k^n}, we start with the following result.

Proposition 1. *For $n \geq 1$ and even k,*

$$u_{k^n} = u_k \prod_{i=1}^{n-1} \left(\sum_{j=1}^{k/2} v_{(2j-1)k^i} \right).$$

Proof. Consider

$$u_{k^n} = u_k \frac{u_{k^2}}{u_k} \frac{u_{k^3}}{u_{k^2}} \cdots \frac{u_{k^n}}{u_{k^{n-1}}} = u_k \prod_{i=1}^{n-1} \frac{u_{k^{i+1}}}{u_{k^i}}. \tag{2.1}$$

From [5], we have that for even $k \geq 2$,

$$\frac{m^k - n^k}{m-k} = \sum_{j=0}^{k/2-1} (mn)^i \left(m^{k-2i-1} + n^{k-2i-1}\right) \tag{2.2}$$

When $m = \alpha^n, n = \beta^n$ in (2.2), we get

$$\frac{u_{kn}}{u_n} = \sum_{j=0}^{k/2-1} (\alpha\beta)^{jn} \left(\alpha^{(k-2j-1)n} + \beta^{(k-2j-1)n}\right) = \sum_{j=0}^{k/2-1} (-1)^{jn} v_{(k-2j-1)n}$$

$$\frac{u_{k^{i+1}}}{u_{k^i}} = \sum_{j=0}^{k/2-1} (-1)^{jk^i} v_{(k-2j-1)k^i} = \sum_{j=1}^{k/2} v_{(2j-1)k^i} \tag{2.3}$$

By (2.3), equation (2.1) equals to

$$u_{k^n} = u_k \prod_{i=1}^{n-1} \frac{u_{k^{i+1}}}{u_{k^i}} = u_k \prod_{i=1}^{n-1} \left(\sum_{j=1}^{k/2} v_{(2j-1)k^i}\right).$$

Thus the proof is complete. □

For the Fibonacci and Lucas case when $p = 1$, we also refer to [9].
For later use, we give the following two identities:

$$\begin{aligned} \sum_{i=1}^{r} (-1)^i v_{ai} &= \sum_{i=1}^{r} (-1)^i \alpha^{ai} + \sum_{i=1}^{r} (-1)^i \beta^{ai} \\ &= \frac{-\alpha^a + (-1)^r (\alpha^a)^{r+1}}{1+\alpha^a} + \frac{-\beta^a + (-1)^r (\beta^a)^{r+1}}{1+\beta^a} \\ &= \frac{(-1)^{a+r} v_{ar} + (-1)^r v_{a(r+1)} - v_a - 2(-1)^a}{v_a + 1 + (-1)^a} \end{aligned} \tag{2.4}$$

and

$$v_{a+b} - (-1)^b v_{a-b} = \Delta u_a u_b. \tag{2.5}$$

Proposition 2. *For odd k and $n \geq 1$,*

$$u_{k^n} = (-1)^{(n-1)(k-1)/2} u_k \prod_{i=1}^{n-1} \left(1 + \sum_{j=1}^{(k-1)/2} (-1)^j v_{2k^i j}\right).$$

Proof. If $(k-1)/2$ is even, we write by (2.4) and (2.5)

$$u_{k^n} = u_k \prod_{i=1}^{n-1} \frac{u_{k^{i+1}}}{u_{k^i}} = u_k \prod_{i=1}^{n-1} \left(1 + \frac{\Delta u_{k^{i+1}} u_{k^i}}{\Delta u_{k^i} u_{k^i}} - 1\right)$$

$$
\begin{aligned}
&= u_k \prod_{i=1}^{n-1} \left(1 + \frac{v_{k^{i+1}+k^i} - (-1)^{k^i} v_{k^{i+1}-k^i}}{v_{k^i+k^i} - (-1)^{k^i} v_{k^i-k^i}} - 1 \right) \\
&= u_k \prod_{i=1}^{n-1} \left(1 + \frac{v_{2k^i \frac{k+1}{2}} + v_{2k^i \frac{k-1}{2}} - v_{2k^i} - 2}{v_{2k^i} + 2} \right) \\
&= u_k \prod_{i=1}^{n-1} \left(1 + \sum_{j=1}^{(k-1)/2} (-1)^j v_{2k^i j} \right) \\
&= (-1)^{(n-1)(k-1)/2} u_k \prod_{i=1}^{n-1} \left(1 + \sum_{j=1}^{(k-1)/2} (-1)^j v_{2k^i j} \right).
\end{aligned}
$$

If $(k-1)/2$ is odd, then we write

$$
\begin{aligned}
u_{k^n} &= (-1)^{n-1} u_k \prod_{i=1}^{n-1} \frac{u_{k^{i+1}}}{u_{k^i}} = (-1)^{n-1} u_k \prod_{i=1}^{n-1} \left(1 - \frac{\Delta u_{k^{i+1}} u_{k^i}}{\Delta u_{k^i}} - 1 \right) \\
&= (-1)^{n-1} u_k \prod_{i=1}^{n-1} \left(1 + \frac{-v_{2k^i \frac{k+1}{2}} - v_{2k^i \frac{k-1}{2}} - v_{2k^i} - 2}{v_{2k^i} + 2} \right) \\
&= (-1)^{n-1} u_k \prod_{i=1}^{n-1} \left(1 + \sum_{k=1}^{(k-1)/2} (-1)^j v_{2k^i j} \right).
\end{aligned}
$$

So we have proved the conclusion for all cases. □

3. Recurrence Relation for $\{u_{k^n}\}$

We shall derive recurrence relations for $\{u_{k^n}\}$ or $\{v_{k^n}\}$ for odd or even k. Thus we need the following result:

Lemma 1. *For $n, q \geq 0$,*

$$
\begin{aligned}
u_{(2q+1)n} &= u_n \sum_{k=0}^{q} (-1)^{n(q+k)} \frac{2q+1}{q+k+1} \left(p^2+4\right)^k \binom{q+k+1}{2k+1} u_n^{2k}, \\
v_{2qn} &= \sum_{k=0}^{q} (-1)^{n(q+k)} \frac{2q}{q+k} \binom{q+k}{2k} \left(p^2+4\right)^k u_n^{2k}, \\
v_{(2q+1)n} &= v_n \sum_{k=0}^{q} (-1)^{(n+1)(q+k)} \frac{2q+1}{q+k+1} \binom{q+k+1}{2k+1} v_n^{2k},
\end{aligned}
$$

$$v_{2qn} = \sum_{k=0}^{q} (-1)^{(n+1)(q+k)} \frac{2q}{q+k} \binom{q+k}{2k} v_n^{2k}.$$

Proof. The proof can be easily obtained from [5] by considering the classical binomial expansions for $a^n - b^n$ and $a^n + b^n$ where a and b are real numbers. □

We give a recurrence relation for the sequence $\{u_{k^n}\}$ for odd k.

Proposition 3. *For odd $k > 0$ and $n \geq 0$,*

$$u_{k^{n+1}} = \triangle^{(k-1)/2} u_{k^n}^k - \sum_{i=0}^{(k-3)/2} \triangle^i C_{i,k} u_{k^n}^{2i+1}$$

where the coefficients $C_{i,k}$ are given by

$$C_{i,k} = (-1)^{(k+1)/2+i} \frac{2k}{k+2i+1} \binom{(k+1)/2+i}{2i+1}$$

for $0 \leq i \leq (k-3)/2$

Proof. Consider

$$u_{k^n}^k = \frac{1}{\triangle^{k/2}} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \beta^{jk^n} \alpha^{(k-j)k^n} \tag{3.1}$$

$$= \frac{1}{\triangle^{(k-1)/2}} \left(u_{k^{n+1}} + \sum_{j=1}^{(k-1)/2} \binom{k}{j} u_{(k-2j)k^n} \right),$$

where Δ is defined as before. By (3.1), we obtain for odd k,

$$u_{k^{n+1}} = \triangle^{(k-1)/2} u_{k^n}^k - \sum_{j=1}^{(k-1)/2} \binom{k}{j} u_{(k-2j)k^n}. \tag{3.2}$$

Then by (3.2) and Lemma 1, we write

$$u_{k^{n+1}} = \triangle^{(k-1)/2} u_{k^n}^k - \sum_{j=1}^{\frac{k-1}{2}} \sum_{i=0}^{\frac{k-1}{2}-j} (-1)^{\frac{k-1}{2}+i-j} \triangle^i \binom{k}{j} \binom{\frac{k+1}{2}+i-j}{2i+1} \frac{k-2j}{\frac{k+1}{2}+i-j} u_{k^n}^{2i+1}$$

which, after reversing the order of summation, can be rewritten as

$$u_{k^{n+1}} = \triangle^{(k-1)/2} u_{k^n}^k - \sum_{i=0}^{(k-1)/2} \triangle^i A_{i,k} u_{k^n}^{2i+1}, \tag{3.3}$$

where

$$A_{i,k}=\sum_{j=1}^{(k-1)/2-i}(-1)^{(k-1)/2+i-j}\binom{k}{j}\binom{(k+1)/2+i-j}{2i+1}\frac{k-2j}{(k+1)/2+i-j}.$$

Since $A_{(k-1)/2,k}=0$, the equality (3.3) becomes

$$u_{k^{n}+1}=\triangle^{(k-1)/2}u_{k^{n}}^{k}-\sum_{i=0}^{(k-3)/2}\triangle^{i}A_{i,k}u_{k^{n}}^{2i+1}.$$

From (pp. 58, [3]), we have the combinatorial identity: for $1\leq m\leq(k-3)/2$

$$\sum_{j=1}^{m}(-1)^{j}\frac{k-2j}{k-m-j}\binom{k}{j}\binom{k-m-j}{m-j}=(-1)^{m}\frac{k}{k-m}\binom{k-m}{m}. \tag{3.4}$$

If we replace m by $\frac{k-1}{2}-i$ in (3.4), then we obtain $C_{i,k}=A_{i,k}$. Thus the proof is complete. □

In a similar manner, we may give the following result:

Proposition 4. *For $n>0$ and odd $k>1$,*

$$u_{k^{n}+1}=\Delta^{\frac{k-1}{2}}u_{k^{n}}^{k}-\sum_{i=0}^{\frac{k-3}{2}}(-1)^{\frac{k+1}{2}+i}\frac{2k}{k-2i-1}\binom{\frac{k-1}{2}+i}{2i+1}\Delta^{i}u_{k^{n}}^{2i+1}.$$

Proof. For odd k, we get

$$\begin{aligned}u_{k^{n}}^{k}&=\frac{1}{\Delta^{k}}\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}\beta^{jk^{n}}\alpha^{(k-j)k^{n}}\\&=\frac{1}{\Delta^{\frac{k-1}{2}}}\left(u_{k^{n}+1}+\sum_{j=1}^{\frac{k-1}{2}}\binom{k}{j}u_{(k-2j)k^{n}}\right).\end{aligned}$$

So we write

$$u_{k^{n}+1}=\Delta^{\frac{k-1}{2}}u_{k^{n}}^{k}-\sum_{j=1}^{\frac{k-1}{2}}\binom{k}{j}u_{(k-2j)k^{n}}.$$

Using Lemma 1 and reversing the order of summation, we write

$$\begin{aligned}u_{k^{n}+1}=\Delta^{\frac{k-1}{2}}u_{k^{n}}^{k}-u_{k^{n}}&\sum_{j=1}^{\frac{k-1}{2}}\sum_{i=0}^{\frac{k-1}{2}-j}(-1)^{\frac{k-1}{2}+i-j}\\&\times\frac{2(k-2j)}{k-2j+2i+1}\binom{k}{j}\binom{\frac{k-1}{2}+i-j+1}{2i+1}\Delta^{i}u_{k^{n}}^{2i}\end{aligned}$$

$$= \Delta^{\frac{k-1}{2}} u_{k^n}^k - u_{k^n} \sum_{i=0}^{\frac{k-3}{2}} \sum_{j=1}^{\frac{k-1}{2}-i} (-1)^{\frac{k-1}{2}+i-j} \times \frac{2(k-2j)}{k-2j+2i+1} \binom{k}{j} \binom{\frac{k-1}{2}+i-j+1}{2i+1} \Delta^i u_{k^n}^{2i}.$$

By simplifying, we derive

$$u_{k^{n+1}} = \Delta^{\frac{k-1}{2}} u_{k^n}^k - u_{k^n} \sum_{i=0}^{\frac{k-3}{2}} (-1)^{\frac{k+1}{2}+i} \frac{2k}{k-2i-1} \binom{\frac{k-1}{2}+i}{2i+1} \Delta^i u_{k^n}^{2i}.$$

□

We give a recurrence relation for the sequence $\{v_{k^n}\}$ for odd k.

Proposition 5. *For $n > 0$ and odd $k > 1$,*

$$v_{k^{n+1}} = v_{k^n}^k - \sum_{i=0}^{\frac{k-1}{2}} E_{i,k} v_{k^n}^{2i+1},$$

where

$$E_{i,k} = \frac{2k}{-k+2i+1} \binom{\frac{k-1}{2}+i}{2i+1}$$

for $0 \le i < (k-1)/2$.

Proof. By the Binet formula of $\{v_n\}$ and the binomial expansion, we write

$$v_{k^n}^k = \sum_{j=0}^{k} \binom{k}{j} \beta^{jk^n} \alpha^{(k-j)k^n}$$

$$= v_{k^{n+1}} + \sum_{j=1}^{\frac{k-1}{2}} (-1)^j \binom{k}{j} v_{(k-2j)k^n}.$$

By Lemma 1, we write

$$v_{k^{n+1}} = v_{k^n}^k - \sum_{j=1}^{\frac{k-1}{2}} \sum_{i=0}^{\frac{k-1}{2}-j} (-1)^j \binom{k}{j} \binom{\frac{k+1}{2}+i-j}{2i+1} \frac{k-2j}{\frac{k+1}{2}+i-j} v_{k^n}^{2i+1}$$

and by reversing the order of summation, we get

$$v_{k^{n+1}} = v_{k^n}^k - \sum_{i=0}^{\frac{k-1}{2}} \sum_{j=1}^{\frac{k-1}{2}-i} (-1)^j \binom{k}{j} \binom{\frac{k+1}{2}+i-j}{2i+1} \frac{k-2j}{\frac{k+1}{2}+i-j} v_{k^n}^{2i+1}$$

which, by the definition of $C_{i,k}$, gives

$$v_{k^{n+1}} = v_{k^n}^k - \sum_{i=0}^{\frac{k-3}{2}} \sum_{j=1}^{\frac{k-1}{2}-i} (-1)^j \binom{k}{j} \binom{\frac{k+1}{2}+i-j}{2i+1} \frac{k-2j}{\frac{k+1}{2}+i-j} v_{k^n}^{2i+1}.$$

If we take $m = \frac{k-1}{2} - i$ in (3.4), we get

$$v_{k^{n+1}} = v_{k^n}^k - \sum_{i=0}^{\frac{k-3}{2}} E_{i,k} v_{k^n}^{2i+1},$$

where

$$E_{i,k} = \frac{2k}{-k+2i+1} \binom{\frac{k-1}{2}+i}{2i+1}.$$

□

For example, when $k = 5$,

$$v_{5^{n+1}} = v_{5^n}^5 + 5v_{5^n}^3 + 5v_{5^n}^1. \tag{3.5}$$

We give a recurrence relation for the sequence $\{v_{k^n}\}$ for even k.

Proposition 6. *For n and even $k > 0$,*

$$v_{k^{n+1}} = v_{k^n}^k - \sum_{i=0}^{\frac{k-2}{2}} H_{i,k} v_{k^n}^{2i},$$

where for $1 \le m \le k/2$

$$H_{i,k} = (-1)^{(\frac{k}{2}+i)} \binom{\frac{k}{2}+i-1}{2i} \frac{2k}{-k+2i}.$$

Proof. It is easy to see that

$$\begin{aligned} v_{k^n}^k &= \sum_{j=0}^{k} \binom{k}{j} \beta^{jk^n} \alpha^{(k-j)k^n} \\ &= v_{k^{n+1}} - \binom{k}{\frac{k}{2}} + \sum_{j=1}^{\frac{k}{2}} \binom{k}{j} v_{(k-2j)k^n}, \end{aligned}$$

Then by Lemma 1 and reversing the order of summation, we get

$$v_{k^n}^k = v_{k^{n+1}} - \binom{k}{\frac{k}{2}} + \sum_{j=1}^{\frac{k}{2}} \binom{k}{j} v_{(k-2j)k^n}$$

$$= v_{k^n+1} + \binom{k}{\frac{k}{2}} + \sum_{j=1}^{\frac{k}{2}} \sum_{i=0}^{\frac{k}{2}-j} (-1)^{(\frac{k}{2}-j+i)} \binom{k}{j} \binom{\frac{k}{2}-j+i}{2i} \frac{2\left(\frac{k}{2}-j\right)}{\frac{k}{2}-j+i} v_{k^n}^{2i}$$

$$= v_{k^n+1} + \sum_{i=0}^{\frac{k-2}{2}} \sum_{j=1}^{\frac{k}{2}-i} (-1)^{(\frac{k}{2}-j+i)} \binom{k}{j} \binom{\frac{k}{2}-j+i}{2i} \frac{2\left(\frac{k}{2}-j\right)}{\frac{k}{2}-j+i} v_{k^n}^{2i}$$

$$= v_{k^n+1} + \sum_{i=0}^{\frac{k-2}{2}} (-1)^{(\frac{k}{2}+i)} \binom{\frac{k}{2}+i-1}{2i} \frac{2k}{-k+2i} v_{k^n}^{2i}.$$

Thus the proof is complete. □

When $k = 6$, we get

$$v_{6^n+1} = v_{6^n}^6 - 6v_{6^n}^4 + 9v_{6^n}^2 - 2v_{6^n}^0. \tag{3.6}$$

Here we note that the coefficients of the formulas in (3.5) and (3.6) with adjusted sings appears to be the terms of the sequence *A034807* in the OEIS.

4. A Polynomial Representations for v_{k^n+1}

In this section, we show that the generalized Lucas numbers v_{k^n+1} are polynomials of the generalized Fibonacci numbers u_{k^n} of degree k for even k.

Proposition 7. *For even $k > 0$ and $n \geq 0$,*

$$v_{k^n+1} = \sum_{i=0}^{\frac{k-2}{2}} D_{i,k} \triangle^i u_{k^n}^{2i},$$

where $D_{i,k}$ is given by

$$D_{i,k} = \frac{2k}{k+2i} \binom{i+\frac{k}{2}}{2i}$$

for $1 \leq m < k/2$.

Proof. Consider

$$u_{k^n}^k = \frac{1}{\triangle^{\frac{k}{2}}} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \beta^{jk^n} \alpha^{(k-j)k^n}$$

$$= \frac{1}{\Delta^{\frac{k}{2}}} \left((-1)^{\frac{k}{2}} \binom{k}{k/2} + v_{k^n+1} - (-1)^{\frac{k}{2}} \binom{k}{k/2} v_0 \right.$$

$$\left. + \sum_{j=1}^{\frac{k}{2}} (-1)^j \binom{k}{j} v_{(k-2j)k^n} \right)$$

$$= \frac{1}{\triangle^{\frac{k}{2}}}\left(v_{k^{n+1}} - (-1)^{\frac{k}{2}}\binom{k}{k/2} + \sum_{j=1}^{\frac{k}{2}}(-1)^j\binom{k}{j}v_{(k-2j)k^n}\right)$$

After using Lemma 1 and reversing the order of summation, we get for even k,

$$u_{k^n}^k = \frac{1}{\triangle^{\frac{k}{2}}}\left(v_{k^{n+1}} + (-1)^{\frac{k}{2}}\binom{k}{k/2} + \sum_{j=1}^{\frac{k-2}{2}}\sum_{i=0}^{\frac{k}{2}-j}\left((-1)^j\frac{2\left(\frac{k}{2}-j\right)}{\left(\frac{k}{2}-j\right)+i}\binom{k}{j}\binom{\left(\frac{k}{2}-j\right)+i}{2i}\triangle^i u_{k^n}^{2i}\right)\right)$$

which becomes

$$u_{k^n}^k = \frac{1}{\triangle^{\frac{k}{2}}}\left(v_{k^{n+1}} + \sum_{i=0}^{\frac{k-2}{2}}\sum_{j=1}^{\frac{k}{2}-i}\left((-1)^j\frac{2\left(\frac{k}{2}-j\right)}{\left(\frac{k}{2}-j\right)+i}\binom{k}{j}\binom{\left(\frac{k}{2}-j\right)+i}{2i}\triangle^i u_{k^n}^{2i}\right)\right).$$

If we take $m = \frac{k}{2} - i$ in (3.4) for $1 \le m \le k/2$, the last equation takes the form:

$$u_{k^n}^k = \frac{1}{\triangle^{\frac{k}{2}}}\left(v_{k^{n+1}} + \sum_{i=0}^{\frac{k-2}{2}} D_{i,k}\triangle^i u_{k^n}^{2i}\right)$$

where $D_{i,k}$ is the right hand side of (3.4) for $m = \frac{k}{2} - i$, that is,

$$D_{i,k} = \left(\frac{-2k}{k+2i}\right)\binom{i+\frac{1}{2}k}{2i}.$$

Thus we have proved the conclusion. □

When $k = 6$, we have that

$$v_{6^{n+1}} = 125u_{6^n}^6 + 150u_{6^n}^4 + 45u_{6^n}^2 + 2. \tag{4.1}$$

Note that the coefficients of the formula in (4.1) with adjusted sings appears to be the terms of the sequence *A104064* in the OEIS.

REFERENCES

[1] P. Filipponi, "On the fibonacci numbers whose subscript is a power," *Fibonacci Quart.*, vol. 34, no. 3, pp. 271–276, 1996.

[2] H. Prodinger, "On a sum of melham and its variants," *Fibonacci Quart.*, vol. 46/47, no. 3, pp. 207–215, 2008/09.

[3] J. Riordan, *Combinatorial identities*. Huntington, N.Y.: Robert E. Krieger Publishing Co., 1979.

[4] A. Stinchcomb, "Problem b-769: First order cubic recurrence relation," *Fibonacci Quart.*, vol. 32, no. 4, p. 373, 1994.

[5] M. N. S. Swamy, "On certain identities involving fibonacci and lucas numbers," *Fibonacci Quart.*, vol. 35, no. 3, pp. 230–232, 1997.

[6] D. C. Terr, "Solution b-769: The recurrence of f_{3^n}," *Fibonacci Quart.*, vol. 33, no. 5, p. 468, 1995.
[7] Z. Usiskin, "Problem b-265: Fibonacci numbers from lucas products," *Fibonacci Quart.*, vol. 11, no. 3, p. 333, 1973.
[8] Z. Usiskin, "Problem b-266: Lucas numbers from lucas products," *Fibonacci Quart.*, vol. 11, no. 3, p. 333, 1973.
[9] S. Vajda, *Fibonacci & Lucas numbers, and the golden section*, ser. Ellis Horwood Series: Mathematics and its Applications. New York: Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], 1989.

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ON GENERALIZED LUCAS POLYNOMIALS AND EULER NUMBERS

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Abstract. In this paper we study the relationship between the generalized Lucas polynomials and the Euler numbers and give several interesting identities involving them.

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1. INTRODUCTION

As usual, the Fibonacci polynomials $\{F_n(x)\}$ are defined by the second-order linear recurrence

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x) \tag{1.1}$$

for $n \geq 0$ and $F_0(x) = 0$, $F_1(x) = 1$.

The Lucas polynomials $\{L_n(x)\}$ are defined by the second-order linear recurrence

$$L_{n+2}(x) = xL_{n+1}(x) + L_n(x) \tag{1.2}$$

for $n \geq 0$ and $L_0(x) = 2, L_1(x) = x$.

These polynomials are of great importance in the study of many subjects such as algebra, geometry and number theory itself. Obviously (1.1) and (1.2) have a deep relationship with Fibonacci numbers $\{F_n\}_{n\geq 0}$ and Lucas numbers $\{L_n\}_{n\geq 0}$ respectively.

Wenpeng Zhang [7] and Fengzhen Zhao [8] obtained some identities involving the Fibonacci numbers. Yuan Yi and Wenpeng Zhang [5] studied the calculation on the summation involving the Fibonacci polynomials and obtained the following

$$\sum_{a_1+\cdots+a_k=n} F_{a_1+1}(x)\,F_{a_2+1}(x)\ldots F_{a_k+1}(x) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+k-1-m}{m}\binom{n+k-1-2m}{k-1}x^{n-2m} \tag{1.3}$$

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where k, n are positive integers and the summation is over all $k-$dimension nonnegative integer coordinates $(a_1, a_2, \ldots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$ where k, n are any positive integers.

Tianping Zhang and Yuankui Ma [6] studied the relationship between the generalized Fibonacci polynomials and the famous Bernoulli numbers and gave several interesting identities involving them. They [4] also studied the relationships between the Chebyshev polynomials of the first kind and the Euler numbers.

In this paper we study the relationship between the generalized Lucas polynomials and the Euler numbers and give several interesting identities involving them.

2. Main Results

Definition 1. Let $\{V_n(x)\}$ be the polynomial sequence. The generalized Lucas polynomials are defined by $V_0(x) = 2$, $V_1(x) = P(x)$ and

$$V_{n+2}(x) = P(x)V_{n+1}(x) + Q(x)V_n(x) \tag{2.1}$$

where $P(x)$ and $Q(x)$ are polynomials of an indeterminate x with real coefficients.

Definition 2 ([2])**.** The ordinary Euler numbers are defined by their generating function as follows

$$F(t) = \frac{2}{e^t + 1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \ |t| < \pi \tag{2.2}$$

where our notation replaced E^m by $E_m (m \geq 0)$, symbolically.

The first few ones are $1, -\frac{1}{2}, 0, \frac{1}{4}, \cdots$, and $E_{2k} = 0$ for $k = 1, 2, 3, \cdots$. These numbers arise in the series expansions of trigonometric functions and are extremely important in number theory and analysis.

Theorem 1. *For all positive integers k and n with $k \leq n$, we have the formula*

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n} \frac{V_{a_1}(x)}{a_1!} \cdots \frac{V_{a_k}(x)}{a_k!} \frac{E_{b_1}}{b_1!} \cdots \frac{E_{b_k}}{b_k!} \left(\sqrt{\Delta(x)}\right)^{b_1+\cdots+b_k} \tag{2.3}$$

$$= 2^k \frac{(k\beta(x))^n}{n!}, \tag{2.4}$$

where $\beta(x) = \frac{P(x)-\sqrt{\Delta(x)}}{2}$ and $\Delta(x) = P^2(x) + 4Q(x)$.

Proof. Let $\alpha(x) = \frac{P(x)+\sqrt{\Delta(x)}}{2}$ and $\beta(x) = \frac{P(x)-\sqrt{\Delta(x)}}{2}$ denote the roots of characteristic polynomial $\lambda^2 - P(x)\lambda - Q(x)$ of the generalized Lucas polynomial sequence $\{V_n(x)\}$. Then the terms of the sequence $\{V_n(x)\}$ can be expressed as [1, 3]

$$V_n(x) = \left(\frac{P(x) + \sqrt{\Delta(x)}}{2}\right)^n + \left(\frac{P(x) - \sqrt{\Delta(x)}}{2}\right)^n. \tag{2.5}$$

Then we easily deduce that the generating function of $V(t,x)$ is

$$V(t,x)=\sum_{n=0}^{\infty}V_n(x)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\alpha(x)^n+\beta(x)^n\right)\frac{t^n}{n!}. \tag{2.6}$$

That is,

$$V(t,x)=e^{\alpha(x)t}+e^{\beta(x)t}=e^{\beta(x)t}(1+e^{t\sqrt{\Delta(x)}}).$$

Therefore, we have

$$\begin{aligned}e^{\beta(x)t}&=\frac{V(t,x)}{1+e^{t\sqrt{\Delta(x)}}}\\&=\frac{V(t,x)}{2}\frac{2}{1+e^{t\sqrt{\Delta(x)}}}.\end{aligned}$$

Then from (2.2) and (2.5) we have

$$e^{\beta(x)t}=\frac{1}{2}\sum_{m=0}^{\infty}V_m(x)\frac{t^m}{m!}\sum_{n=0}^{\infty}E_n\frac{\left(t\sqrt{\Delta(x)}\right)^n}{n!}. \tag{2.7}$$

Note that for two absolutely convergent power series $\sum_{n=0}^{\infty}a_nt^n$ and $\sum_{n=0}^{\infty}b_nt^n$, we have

$$\left(\sum_{n=0}^{\infty}a_nt^n\right)\left(\sum_{n=0}^{\infty}b_nt^n\right)=\sum_{n=0}^{\infty}\left(\sum_{u+v=n}^{\infty}a_ub_v\right)t^n, \tag{2.8}$$

so for the k^{th} power of both sides of formula (2.6) we have

$$LHS=\left(e^{\beta(x)t}\right)^k=e^{k\beta(x)t}=\sum_{n=0}^{\infty}\frac{(k\beta(x))^n}{n!}t^n,$$

$$\begin{aligned}RHS=\sum_{n=0}^{\infty}\Bigg(&\frac{1}{2^k}\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n}\frac{V_{a_1}(x)}{a_1!}\\&\cdots\frac{V_{a_k}(x)}{a_k!}\frac{E_{b_1}}{b_1!}\cdots\frac{E_{b_k}}{b_k!}\left(\sqrt{\Delta(x)}\right)^{b_1+\cdots+b_k}\Bigg)t^n.\end{aligned}$$

Comparing the coefficients of t^n, we obtain the following identity

$$\begin{aligned}\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n}&\frac{V_{a_1}(x)}{a_1!}\cdots\frac{V_{a_k}(x)}{a_k!}\frac{E_{b_1}}{b_1!}\cdots\frac{E_{b_k}}{b_k!}\left(\sqrt{\Delta(x)}\right)^{b_1+\ldots+b_k}\\&=2^k\frac{(k\beta(x))^n}{n!}.\end{aligned}$$

□

If we take $P(x)=2x$ and $Q(x)=1$ in Theorem 1, then we have

Corollary 1. *For all positive integers k and n with $k \leq n$, we have*

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n} \frac{P_{a_1}(x)}{a_1!}\cdots\frac{P_{a_k}(x)}{a_k!}\frac{E_{b_1}}{b_1!}\cdots\frac{E_{b_k}}{b_k!}\left(2\sqrt{x^2+1}\right)^{b_1+\cdots+b_k} = 2^k\frac{(k\beta(x))^n}{n!},$$

where $\{P_n(x)\}$ is the Pell-Lucas Polynomial sequence, $\beta(x) = x - \sqrt{x^2+1}$.

If we take $P(x) = x$ and $Q(x) = 1$ in Theorem 1, then we have

Corollary 2. *For all positive integers k and n with $k \leq n$, we have*

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n} \frac{L_{a_1}(x)}{a_1!}\cdots\frac{L_{a_k}(x)}{a_k!}\frac{E_{b_1}}{b_1!}\cdots\frac{E_{b_k}}{b_k!}\left(\sqrt{x^2+4}\right)^{b_1+\cdots+b_k} = 2^k\frac{(k\beta^*(x))^n}{n!},$$

where $\{L_n(x)\}$ is Lucas Polynomial sequence, $\beta^*(x) = \frac{x-\sqrt{x^2+4}}{2}$.

If $P(x)$ and $Q(x)$ in Theorem 1 are integers with $P^2+4Q > 0$ and $V_{n+2} = PV_{n+1} + QV_n$, where $V_0 = 2$, $V_1 = P$, then we obtain the following

Corollary 3. *For all positive integers k and n with $k \leq n$, we have the identity*

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n} \frac{V_{a_1}}{a_1!}\cdots\frac{V_{a_k}}{a_k!}\frac{E_{b_1}}{b_1!}\cdots\frac{E_{b_k}}{b_k!}\left(\sqrt{P^2+4Q}\right)^{b_1+\cdots+b_k} = 2^k\frac{(k\beta^\bullet(x))^n}{n!},$$

where $\beta^\bullet(x) = \frac{P-\sqrt{P^2+4Q}}{2}$.

Taking $x = 1$ in Corollary 2 or $P = Q = 1$ in Corollary 3, we obtain the identity

Corollary 4 ([4])**.** *For all positive integers k and n with $k \leq n$, we have*

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n} \frac{L_{a_1}}{a_1!}\cdots\frac{L_{a_k}}{a_k!}\frac{E_{b_1}}{b_1!}\cdots\frac{E_{b_k}}{b_k!}\left(\sqrt{5}\right)^{b_1+\ldots+b_k} = 2^k\frac{\left(k\beta'(1)\right)^n}{n!},$$

where $\beta'(1) = \frac{1-\sqrt{5}}{2}$.

In particular, taking $k = 1, 2, 3$ in Corollary 4, we easily get

Corollary 5. *[4] For all positive integers n, we have*

$$\sum_{a_1+b_1=n} \frac{L_{a_1}}{a_1!}\frac{E_{b_1}}{b_1!}\left(\sqrt{5}\right)^{b_1} = 2\frac{\left(\beta'(1)\right)^n}{n!}.$$

Corollary 6. *[4] For all positive integers $n \geq 2$, we have*

$$\sum_{a_1+a_2+b_1.+b_2=n} \frac{L_{a_1}}{a_1!}\frac{L_{a_2}}{a_2!}\frac{E_{b_1}}{b_1!}\frac{E_{b_2}}{b_2!}\left(\sqrt{5}\right)^{b_1+b_2} = 2^2\frac{\left(2\beta'(1)\right)^n}{n!}.$$

Corollary 7 ([4])**.** *For all positive integers $n \geq 3$, we have*

$$\sum_{a_1+a_2+a_3+b_1+b_2+b_3=n} \frac{L_{a_1}}{a_1!}\frac{L_{a_2}}{a_2!}\frac{L_{a_3}}{a_3!}\frac{E_{b_1}}{b_1!}\frac{E_{b_2}}{b_2!}\frac{E_{b_3}}{b_3!}\left(\sqrt{5}\right)^{b_1+b_2+b_3} = 2^3\frac{\left(3\beta'(1)\right)^n}{n!}.$$

REFERENCES

[1] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*. Oxford: Oxford University Press, 1962.

[2] T. Kim, "On the q-extension of euler and genocchi numbers." *J. Math. Anal. Appl.*, vol. 326, no. 2, pp. 1458–1465, 2007.

[3] T. Koshy, *Fibonacci and Lucas numbers with applications*, ser. Pure and Applied Mathematics (New York). New York: Wiley-Interscience, 2001.

[4] Y. K. Ma and T. P. Zhang, "An identity involving the first-kind chebyshev polynomials and the euler numbers," *J. Ningxia Univ. Nat. Sci. Ed.*, vol. 27, no. 1, pp. 13–14,17, 2006.

[5] Y. Yuan and W. Zhang, "Some identities involving the fibonacci polynomials," *Fibonacci Quart.*, vol. 40, no. 4, pp. 314–318, 2002.

[6] T. Zhang and Y. Ma, "On generalized fibonacci polynomials and bernoulli numbers," *J. Integer Seq.*, vol. 8, no. 5, p. 6, 2005.

[7] W. Zhang, "Some identities involving the fibonacci numbers," *Fibonacci Quart.*, vol. 35, no. 3, pp. 225–229, 1997.

[8] F. Zhao and T. Wang, "Generalizations of some identities involving the fibonacci numbers," *Fibonacci Quart.*, vol. 39, no. 2, pp. 165–167, 2001.

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SOME RESULTS ON SYMMETRIC BI-(σ,τ)-DERIVATIONS IN NEAR-RINGS

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Abstract. The aim of this paper is to investigate certain results on 3-prime near-rings and generalize these results on near-rings to semi-group ideals of 3-prime near-rings.

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Keywords: near-ring, prime near-ring, semigroup ideal, derivation, symmetric bi-derivation, symmetric bi-(σ,τ)-derivation

1. INTRODUCTION

For preliminary definitions and results related to near-rings, we refer for Pilz [5]. The concept of a symmetric bi-derivation has been introduced by Maksa [3]. In recent years, many mathematicians studied the commutativity of prime and semiprime rings admitting suitably-constrained symmetric bi-derivations. In [4], Öztürk (together with Jun) introduced the notion of a symmetric bi-derivations in near-rings and proved some results. Moreover the concept of symmetric bi-(σ,τ)-derivation of near-ring has been introduced by Çeven (together with Öztürk) [2]. The aim of this paper is to investigate some results on 3-prime near-rings and generalize the results on near-rings to semi-group ideals of 3-prime near-rings.

2. PRELIMINARIES

Throughout this paper N will be a zero-symmetric left near-ring, and usually N will be 3-prime, that is, it will have the property that $xNy = \{0\}$ implies $x = 0$ or $y = 0$. The symbol Z will denote the multiplicative center of N. A nonempty subset U of N will be called semigroup right ideal (resp. semigroup left ideal) if $UN \subset U$ (resp. $NU \subset U$) and U is both a semigroup right ideal and a semigroup left ideal, it is called a semigroup ideal. For $x, y \in N$, the symbol $[x, y]$ will denote the commutator $xy - yx$, while the symbol (x, y) will denote the additive-group commutator $x + y - x - y$. A mapping $D : N \times N \to N$ is said to be symmetric if $D(x, y) = D(y, x)$ for all $x, y \in N$. A mapping $d : N \to N$ defined by $d(x) = D(x, x)$ is called the trace of D where $D : N \times N \to N$ is a symmetric mapping. It

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is obvious that, if $D : N \times N \to N$ is a symmetric mapping which is also bi-additive (i.e., additive in both arguments), then the trace of D satisfies the relation $d(x+y) = d(x) + 2D(x,y) + d(y)$ for all $x, y \in N$. A symmetric bi-additive mapping $D : N \times N \to N$ is called a symmetric bi-derivation if $D(xy,z) = D(x,z)\, y + xD(y,z)$ is fulfilled for all $x, y, z \in N$. Then, for any $y \in N$, a mapping $x \mapsto D(x,y)$ is a derivation. A symmetric bi-additive mapping $D : N \times N \to N$ is called symmetric bi-(σ,τ)-derivation if there exist automorphisms $\sigma, \tau : N \to N$ such that $D(xy,z) = D(x,z)\,\sigma(y) + \tau(x)\, D(y,z)$ for all $x, y, z \in N$. Note that if $\sigma = 1$ and $\tau = 1$ then D is a symmetric bi-derivation.

3. Results

The following lemmas and theorems are necessary for the paper.

Lemma 1 ([3, Lemma 3]). *Let N be a 3-prime near-ring.*

(i) *If $z \in Z - \{0\}$, then z is not a zero divisor.*
(ii) *If $Z - \{0\}$ contains an element z for which $z + z \in Z$, then $(N, +)$ is abelian.*

Lemma 2 ([1, Lemma 1.3]). *Let N be 3-prime, and let d be a nonzero derivation on N.*

(i) *If U is a nonzero semi-group right ideal (resp. semi-group left ideal) and x is an element of N such that $Ux = \{0\}$ (resp. $xU = \{0\}$), then $x = 0$.*
(ii) *If U is nonzero semi-group right ideal or semi-group left ideal, then $d(U) \neq \{0\}$.*
(iii) *If U is a nonzero semi-group right ideal and x is an element of N which centralizes U, then $x \in Z$.*

Lemma 3 ([1, Lemma 1.4]). *Let N be 3-prime, and U a nonzero semi-group ideal of N. Let d be a nonzero derivation on N.*

(i) *If $x, y \in N$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$.*
(ii) *If $x \in N$ and $d(U)\, x = \{0\}$, then $x = 0$.*
(iii) *If $x \in N$ and $xd(U) = \{0\}$, then $x = 0$.*

Lemma 4 ([1, Lemma 1.5]). *If N is 3-prime and Z contains a nonzero semigroup left ideal or semigroup right ideal, N is a commutative ring.*

Lemma 5 ([2, Lemma 3]). *Let N be a 2-torsion free 3-prime near-ring, D a symmetric bi-(σ,τ)-derivation of N and d the trace of D. If $xd(N) = \{0\}$ for all $x \in N$, then $x = 0$ or $D = 0$.*

Lemma 6 ([2, Lemma 4]). *Let N be a near-ring. D is a symmetric bi-(σ,τ)-derivation of N if and only if $D(xy,z) = \tau(x)\, D(y,z) + D(x,z)\,\sigma(y)$ for all $x, y, z \in N$.*

Lemma 7 ([2, Lemma 5]). *Let N be a near-ring, D a symmetric bi-(σ,τ)-derivation of N. Then, for all $x,y,z,w \in N$,*

(i) $[D(x,z)\sigma(y)+\tau(x)D(y,z)]w = D(x,z)\sigma(y)w+\tau(x)D(y,z)w$,
(ii) $[\tau(x)D(y,z)+D(x,z)\sigma(y)]w = \tau(x)D(y,z)w+D(x,z)\sigma(y)w$.

Theorem 1 ([2, Theorem 1]). *Let N be a 3-prime near-ring, D a nonzero symmetric bi-(σ,τ)-derivation of N. If N is 2-torsion free and $D(N,N)\subset Z$, then N is commutative ring.*

Lemma 8. *Let N be a 2-torsion free 3-prime near-ring, U a nonzero semigroup ideal of N and D symmetric bi-(σ,τ)-derivation of N. If $D(U,U)=\{0\}$, then $D=0$.*

Proof. Suppose $D(x,y)=0$ for all $x,y\in U$. Then taking xz, $z\in N$ instead of x, we have

$$\begin{aligned} 0 = D(xz,y) &= D(x,y)\sigma(z)+\tau(x)D(z,y) \\ &= \tau(x)D(y,z) \end{aligned}$$

Replacing yw by y, $w\in N$ in last relation, we get

$$\begin{aligned} 0 &= \tau(x)D(y,z)\sigma(w)+\tau(x)\tau(y)D(w,z) \\ &= \tau(x)\tau(y)D(w,z) \end{aligned}$$

for all $x,y\in U$, $w,z\in N$. Since τ is an automorphism of N and, $U\neq\{0\}$, we get $D(N,N)=\{0\}$ by Lemma 3. That is, $D=0$. □

Lemma 9. *Let N be a 2-torsion free 3-prime near-ring, U a nonzero semigroup ideal of N, D a nonzero symmetric bi-(σ,τ)-derivation of N and $a\in N$. If $D(U,U)a=\{0\}$, then $a=0$. (Similarly, if $D(U,U)a=0$, then $a=0$.)*

Proof. Suppose $D(x,y)a=0$ for all $x,y\in U$. Taking wy, $w\in N$ instead of y and using Lemma 7(i), we get

$$\begin{aligned} 0 = D(x,wy)a &= [D(w,x)\sigma(y)+\tau(w)D(y,x)]a \\ &= D(w,x)\sigma(y)a+\tau(w)D(y,x)a \\ &= D(x,w)\sigma(y)a \end{aligned}$$

Since σ is an automorphism of N and $U\neq\{0\}$, we get $D(N,U)=\{0\}$ or $a=0$ by Lemma 3(i). Assume $D(N,U)=\{0\}$. Hence $D(U,U)=\{0\}$, and so $D=0$ by Lemma 8. This completes the proof.

The other case ($aD(U,U)=\{0\}$) can be treated similarly. □

Theorem 2. *Let N be a 3-prime near-ring, U a nonzero semigroup ideal of N and D a nonzero symmetric bi-(σ,τ)-derivation of N. If N is 2-torsion free and $D(U,U)\subset Z$, then N is commutative ring.*

Proof. Since $D(x,y) \in Z$ for all $x,y \in U$, we have $D(x,y)z = zD(x,y)$, for all $x,y \in U$ and $z \in N$. Relacing xw by w, $w \in U$, we get

$$\begin{aligned} &D(xw,y)z = zD(xw,y) \\ &D(x,y)\sigma(w)z + \tau(x)D(w,y)z \\ &= zD(x,y)\sigma(w) + z\tau(x)D(w,y) \end{aligned} \tag{3.1}$$

Taking $z = \sigma(w)$ and using $D(U,U) \subset Z$, we obtain that

$$\begin{aligned} &D(x,y)\sigma(w)\sigma(w) + \tau(x)D(w,y)\sigma(w) \\ &= \sigma(w)D(x,y)\sigma(w) + \sigma(w)\tau(x)D(w,y) \end{aligned}$$

and so

$$D(w,y)[\sigma(w),\tau(x)] = 0, \text{ for all } x,y,w \in U$$

Since Z contains no nonzero divisors of zero, we see that for each $w \in U$, either $D(w,y) = 0$ for all $y \in U$ or $[\tau(x),\sigma(w)] = 0$, for all $x \in U$. In the first case, equation (3.1) yields

$$D(x,y)[\sigma(w),z] = 0, \qquad \text{for all } x,y,w \in U, z \in N.$$

By the same argument as above, we have, we have $[\sigma(w),z] = 0$, for all $z \in N$. In particular, we obtain for all $x \in U$, $[\tau(x),\sigma(w)] = 0$, in each case. That is, $U \subset Z$ by Lemma 2(iii). Hence N is a commutative ring by Lemma 4. This completes the proof. □

Theorem 3. *Let N be a 2-torsion free 3-prime near-ring. Let D_1 and D_2 be nonzero symmetric bi-(σ,τ)-derivations of N and d_1 and let d_2 be nonzero trace of D_1 and D_2, respectively. If $d_2(y)$, $d_2(y) + d_2(y) \in C\ (D_1(x,z))$ for all $x,y,z \in N$, then $(N,+)$ is abelian and $d_2(N) \subseteq Z$.*

Proof. Since $d_2(y)$, $d_2(y) + d_2(y) \in C\ (D_1(x,z))$ for all $x,y,z \in N$, if both w and $w+w$ commute elementwise with $D_1(x,z)$ for all $s \in N$, then

$$[D_1(x,z) + D_1(s,z)](w+w) = [D_1(x,z) + D_1(x,z) + D_1(s,z) + D_1(s,z)]w$$

On the other hand,

$$[D_1(x,z) + D_1(s,z)](w+w) = [D_1(x,z) + D_1(s,z) + D_1(x,z) + D_1(s,z)]w$$

Comparing these expressions we get, for all $x,s \in N$,

$$D_1((x,s),z)w = 0 \tag{3.2}$$

Thus, let $w = d_2(y)$ in (3.2), we get $D_1((x,s),z)d_2(y) = 0$ for all $x,z,s,y \in N$, so $D_1((x,s),z) = 0$. Since (x,s) is also an additive commutator for any $z \in N$, we have

$$\begin{aligned} 0 = D_1((x,s)z,z) &= D_1((x,s),z)\sigma(z) + \tau((x,s))d_1(z) \\ &= \tau((x,s))d_1(z) \end{aligned}$$

Since τ is an automorphism, we get $(x,s)=0$. That is, $(N,+)$ is abelian.

Now, replacing xw by x in $D_1(x,z)d_2(y)=d_2(y)D_1(x,z)$, we get

$$\begin{aligned}0&=D_1(xw,z)d_2(y)-d_2(y)D_1(xw,z)\\&=D_1(x,z)\sigma(w)d_2(y)+\tau(x)D_1(w,z)d_2(y)\\&\quad-d_2(y)\tau(x)D_1(w,z)-d_2(y)D_1(x,z)\sigma(w)\end{aligned}$$

Taking $d_1(t)$ for $\sigma(w)$ in the last relation, we get

$$\tau(x)D_1(w,z)d_2(y)=d_2(y)\tau(x)D_1(w,z) \tag{3.3}$$

Replacing wu by w in (3.3) and using (3.3), we have

$$\begin{aligned}\tau(x)D_1(w,z)\sigma(u)d_2(y)&=d_2(y)\tau(x)D_1(w,z)\sigma(u)\\&=\tau(x)D_1(w,z)d_2(y)\sigma(u)\\\tau(x)D_1(w,z)[\sigma(u),d_2(y)]&=0\end{aligned}$$

Since U is a nonzero semi-group ideal and D_1 is nonzero, we get $[\sigma(u),d_2(y)]=0$. Since σ is an automorphism, we have $d_2(N)\subseteq Z$. □

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REFERENCES

[1] H. E. Bell, *On derivations in near-rings. II.*, ser. Mathematics and its Applications. Dordrecht: Kluwer Acad. Publ., 1997, vol. 426.

[2] Y. Çeven and M. A. Öztürk, "Some properties of symmetric bi-(σ,τ)-derivations in near-rings," *Commun. Korean Math. Soc.*, vol. 22, no. 4, pp. 487–491, 2007.

[3] G. Maksa, "On the trace of symmetric bi-derivations," *C. R. Math. Rep. Acad. Sci. Canada*, vol. 9, no. 6, pp. 303–307, 1987.

[4] M. A. Öztürk and Y. B. Jun, "On the trace of symmetric bi-derivations in near-rings," *Inter. J. Pure and Appl. Math.*, vol. 17, no. 1, pp. 9–102, 2004.

[5] G. Pilz, *Near-rings*, ser. North-Holland Mathematics Studies. Amsterdam: North-Holland Publishing Co., 1983, vol. 23.

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THE LATTICE OF REGULAR COEQUALITIES

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Abstract. For a coequality q we say that it is regular coequality on set X ordered under the anti-order α if there exists an anti-order θ on X/q such that the natural mapping $\pi : X \longrightarrow X/q$ is a reverse isotone surjection of anti-ordered sets. The lattice of regular coequalities is described.

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Keywords: constructive mathematics, set with apartness, coequality relation, anti-ordered set, quasi-antiorder relation, regular coequality

1. INTRODUCTION AND PRELIMINARIES

This short investigation, in the framework of Bishop's constructive mathematics ([1–3] and [9]), is continuation of the author's previous papers [6–8]. Bishop's constructive mathematics is developed on Constructive Logic ([9]) - logic without the Law of Excluded Middle $P \vee \neg P$. Let us note that in Constructive Logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold, but the implication $P \Longrightarrow \neg\neg P$ holds even in Minimal Logic. We have to note that 'the crazy axiom' $\neg P \Longrightarrow (P \Longrightarrow Q)$ is included in the Constructive Logic. In Constructive Logic the 'Weak Law of Excluded Middle' $\neg P \vee \neg\neg P$ does not hold, too. It is interesting, that in Constructive Logic the following deduction principle

$$A \vee B, \neg A \vdash B$$

holds, but this is impossible to prove without 'the crazy axiom'. Bishop's Constructive Mathematics is consistent with Classical Mathematics.

A relational structure $(X, =, \neq)$, where the relation "$\neq$" is a binary relation on X, which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq z \Longrightarrow x \neq y \vee y \neq z,\ x \neq y \wedge y = z \Longrightarrow x \neq z$$

will be called a set. Following Heyting, the relation $\neq$ is called *apartness*. A relation q on X is a *coequality relation* on X if and only if it is consistent, symmetric and

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cotransitive ([4,5]):

$$q \subseteq \neq,\ q^{-1} = q,\ q \subseteq q * q,$$

where "$*$ " is the *filled product* between relations (see [4]). Let β be a consitent relation on X. We put ${}^1\beta = \beta$ and ${}^n\beta = \beta * ... * \beta$ (n factors, $n \in N$). Then ([4]) the relation $c(\beta) = \bigcap_{n \in N} {}^n\beta$, the *cotransitive fulfillment* of β, is the maximal consistent and cotransitive relation on the set X under β.

A relation α on X is an *antiorder* ([6]) on X if and only if

$$\alpha \subseteq \neq,\ \alpha \subseteq \alpha * \alpha,\ \neq \subseteq \alpha \cup \alpha^{-1} (linearity).$$

A relation σ on X is a *quasi-antiorder* ([6]) on X if

$$\sigma \subseteq \neq,\ \sigma \subseteq \sigma * \sigma.$$

Let x be an element of X and let A be a subset of X. We use the notation $x \bowtie A$ if and only if $(\forall a \in A)(x \neq a)$, and $A^C = \{x \in X : x \bowtie A\}$. If σ is a quasi-antiorder on X, then the relation $q = \sigma \cup \sigma^{-1}$ is a coequality on X. Firstly, the relation $q^C = \{(x,y) \in X \times X : (x,y) \bowtie q\}$ is a equality relation on X compatible with q, in the following sense $(\forall a,b,c \in X)((a,b) \in q^C \wedge (b,c) \in q \Longrightarrow (a,c) \in q)$. We can construct the factor-set $X/(q^C,q) = \{aq^C : a \in X\}$ with

$$aq^C =_1 bq^C \Longleftrightarrow (a,b) \bowtie q,\ aq^C \neq_1 bq^C \Longleftrightarrow (a,b) \in q.$$

We can also construct the factor-set $X/q = \{aq : aX\}$ with

$$aq =_1 bq \Longleftrightarrow (a,b) \bowtie q,\ aq \neq_1 bq \Longleftrightarrow (a,b) \in q.$$

It is easy to check that $X/(q^C,q) \cong X/q$. The mapping $\pi : X \longrightarrow X/q$, defined by $\pi(a) = aq$ for any $a \in X$, is a strongly extensional surjection.

Secondly, note that the relation α^C is an order relation on set $(X, \neg \neq, \neq)$. If the relation $\neg\alpha$ is an order relation on $(X,=,\neq)$, when the apartness is tight, $\neg \neq \subseteq =$ ([5]), then the relation α is called *excise relation* on X. (The notion of anti-order relation is more general then notion of excise relation.)

For a given anti-ordered set $(X,=,\neq,\alpha)$ it is essential to know if there exists a coequality relation q on X such that X/q is an anti-ordered set. This plays an important role in the investigation of anti-ordered sets. The following question is natural: If $(X,=,\neq,\alpha)$ is an anti-ordered set and q a coequality on X, is the set X/q an anti-ordered set? A possible anti-order on X/q could be the relation Θ on X/q defined by the anti-order α on X, where $\Theta = \{(xq, yq) \in X/q \times X/q : (x,y) \in \alpha\}$. But it is not an anti-order, in general. The following question arises: Is there a coequality q on X for which X/q is anti-ordered set? The concept of quasi-antiorder relation was introduced in [6]. According to [6], if $(X,=,\neq,\alpha)$ is an anti-ordered set and σ a quasi-antiorder on X, then the relation q on X, defined by $q = \sigma \cup \sigma^{-1}$

is a coequality relation on X and the set X/q is an ordered set under anti-order Θ defined by $(xq, yq) \in \Theta \Longleftrightarrow (x, y) \in \sigma$. So, according to the results in [6], each quasi-antiorder σ on an ordered set X under anti-order α induces a coequality relation $q = \sigma \cup \sigma^{-1}$ on X such that X/q is an ordered set under antiorder Θ. In [7] we prove that the converse of this statement also holds. If $(X, =, \neq, \alpha)$ is an anti-ordered set and q a coequality relation on X and if there exists an antiorder relation θ_1 on X/q such that $(X/q, =_1, \neq_1, \theta_1)$ is an ordered set under antiorder θ_1, then there exists a quasi-antiorder σ on X such that $q = \sigma \cup \sigma^{-1}$ and $\theta_1 = \Theta$. So, each coequality relation q on a set $(X, =, \neq, \alpha)$ such that X/q is an anti-ordered set induces a quasi-antiorder on X. This was the motivation of a new notion. For that we need the following notion: Let f be a strongly extensional mapping of anti-ordered sets from $(X, =, \neq, \alpha)$ into $(Y, =, \neq, \beta)$. For f we say that it is *reverse isotone* if

$$(\forall a, b \in X)((f(a), f(b)) \in \beta \Longrightarrow (a, b) \in \alpha)$$

holds. A coequality relation q on X is called *regular* if there is an antiorder "θ_1" on X/q satisfying the following conditions:

(1) $(X/q, =_1, \neq_1, \theta_1)$ is an anti-ordered set;
(2) The mapping $\pi : X \ni a \longmapsto aq \in X/q$ is an anti-order reverse isotone surjection.

We call the antiorder "θ_1" on X/q a *regular antiorder with respect* to a regular coequality q on X and the anti-order α.

It is obviously that the regular antiorder on X/q with respect to a regular coequality q and to the antiorder α on X is in general not unique. The following questions now naturally arise: Does there exist the maximal regular antiorder on X/q with respect to a regular coequality q on X? Are all coequalities on anti-ordered sets regular? Trying to find an answers for the above questions, in this note we give a description of the family of regular coequalities. In Theorem 1 and Theorem 2 we give necessary and sufficient conditions such that coequality on an anti-ordered set is regular. In Theorem 3 we give a construction of the maximal quasi-antiorder on the anti-ordered set X induced by a regular coequality q on X. The section "The lattice of regular coequalities" contains the main results of this paper. We prove that the family of all regular coequalities with respect to the one anti-order relation on ordered set is a complete lattice and describe that lattice.

For the necessary undefined notions, the reader is referred to books [1–3, 9] and to papers [4–8].

2. Regular anticongruences

In the following lemma we describe classes of a quasi-antiorder relation:

Lemma 1 ([7, Lemma 0]). *Let σ be a quasi-antiorder on set X. Then $x\sigma$ (σx) is a strongly extensional subset of X, such that $x \bowtie x\sigma$ $(x \bowtie \sigma x)$, for each $x \in X$.*

In order to obtain the relationship between regular anticongruence and quasi-antiorder on X, the following theorem is essential.

Theorem 1 ([7, Theorem 1]). *Let $(X,=,\neq,\alpha)$ be an anti-ordered set, let q be a coequality on X. The following are equivalent:*

(1) *q is regular.*
(2) *there exists a quasi-antiorder σ on X, such that $q = \sigma \cup \sigma^1$.*

Theorem 2 ([7, Corollary 2]). *Let $(X,=,\neq,\alpha)$ be an anti-ordered set and let q be a coequality on X. The following are equivalent:*

(1) *q is regular;*
(2) *there exists an anti-ordered set $(T,=,\neq,\theta)$ and a strongly extensional reverse isotone mapping $\varphi : S \longrightarrow T$ such that $q = \{(a,b) \in X \times X : \varphi(a) \neq \varphi(b)\}$.*

Recall that, by Lemma 1, any class aq of coequality relation q, generated by the element $a \in X$, is strongly extensional subset of X. Besides, we have the following assertion, which is crucial for the characterization of regular coequality on an anti-ordered set $(X,=,\neq,\alpha)$: If q is a regular coequality relation on an anti-ordered set X, then for every q- class aq in X we have

$$((x,y) \bowtie \alpha \wedge (y,z) \bowtie \alpha \wedge x,z \bowtie aq) \Longrightarrow y \bowtie aq$$

for any $x,y,z,a \in X$. If q is a regular coequality on a set X, then there exists an anti-order relation θ on X/q such that the natural mapping $\pi : X \longrightarrow X/q$ is a strongly extensive reverse isotone surjection. Besides, there exists a quasi-antiorder σ under α, defined by $(x,y) \in \sigma \Longleftrightarrow (xq,yq) \in \theta$ such that $\sigma \cup \sigma^{-1} = q$. Let t be an arbitrary element of aq. Then $(a,t) \in q = \sigma \cup \sigma^{-1}$. Thus $(a,t) \in \sigma$ or $(t,a) \in \sigma$. Hence, we have

$(a,t) \in \sigma \Longrightarrow ((a,x) \in \sigma \subseteq q \vee (x,y) \in \sigma \subseteq \alpha \vee (y,t) \in \sigma \subseteq q \subseteq \neq)$
$\Longrightarrow t \neq y;$
$(t,a) \in \sigma \Longrightarrow ((t,y) \in \sigma \subseteq \neq \vee (y,z) \in \sigma \subseteq \alpha \vee (z,a) \in \sigma \subseteq q$
$\Longrightarrow t \neq y.$

So, in both cases, we have that $t \in aq \Longrightarrow t \neq y$. Therefore, $y \bowtie aq$.

We also have

$$((x,y) \bowtie \alpha \wedge (y,z) \bowtie \alpha \wedge y \in aq) \Longrightarrow x \in aq \vee z \in aq$$

for any $x,y,a \in X$. Indeed, if $x,y,z,a \in X$ such that $(x,y) \bowtie \alpha$ and $(y,z) \bowtie \alpha$ and $x \in aq$, then $(a,y) \in q = \sigma \cup \sigma^{-1} \Longrightarrow ((a,y) \in \sigma \vee (y,a) \in \sigma)$. Thus, we have
$((a,y) \in \sigma \vee (y,a) \in \sigma) \Longrightarrow$
$((a,x) \in \sigma \subseteq q \vee (x,y) \in \sigma \subseteq \alpha) \vee ((y,z) \in \sigma \subseteq \alpha \vee (z,a) \in \sigma \subseteq q) \Longrightarrow$
$x \in aq \vee z \in aq.$

Let q be a regular coequality relation on the anti-ordered set $(X,=,\neq,\alpha)$. Then there exists anti-order θ on X/q such that the natural mapping $\pi : X \longrightarrow X/q$ is reverse isotone. Hence, by [7], there exists a quasi-antiorder σ under α such that $q = \sigma \cup \sigma^{-1}$ and $\theta \subseteq \{(aq,bq) \in X/q \times X/q : (a,b) \in \sigma\}$. In the following theorem we show that there exists such maximal quasi-antiorder τ under α and we prove that there exists such construction of that relation.

Theorem 3 ([7, Theorem 3])**.** *Let q be a regular coequality relation on anti-ordered set $(X,=,\neq,\alpha)$. Then there exists the maximal quasi-antiorder relation τ under α such that $q = \tau \cup \tau^{-1}$ and $\theta \subseteq \{(aq,bq) \in X/q \times X/q : (a,b) \in \tau\}$. That relation is exactly the following relation $c(q \cap \alpha) = \bigcap_{n\in N} {}^{n}(q \cap \alpha)$.*

At the end of this consideration, we give the following assertion:

Theorem 4 ([7, Corollary 4])**.** *Let q be a regular coequality on anti-ordered set $(X,=,\neq,\alpha)$. Then there exists the maximal antiorder relation on X/q. That relation is exactly the following relation $\{(aq,bq) \in X/q \times X/q : (a,b) \in c(q \cap \alpha)\}$.*

3. The lattice of regular coequalities

Let $(X,=,\neq,\alpha)$ be anti-ordered set. We denote by $\Re(X,\alpha)$ the family of all regular coequality relations on X with respect to α and $\Im(X,\alpha)$ denotes the family of all quasi-antiorder relation on X included in α .

Theorem 5. *Let X be an anti-ordered set. Then $\Re(X,\alpha)$ is a complete lattice.*

Proof. Let $\{q_k\}_{k\in K}$ be a family of regular coequality relations on X.
(1) Then $\bigcup_k q_k$ is a regular coequality relation on X. If fact, if θ_k is an anti-order relation on X/q_k with respect to q_k and α, then $\bigcup_k \theta_k$ is an anti-order relation on $X/(\bigcup_k q_k)$ with respect to $\bigcup_k q_k$ and α.
(2) For each k there exists a quasi-antiorder relation σ_k on X under α such that $q_k = \sigma_k \cup (\sigma_k)^{-1}$. Then $\tau = c(\bigcap_k \sigma_k)$ is the maximal quasi-antiorder relation under $(\bigcap_k \sigma_k \subseteq)\alpha$. Thus, the relation $q = \tau \cup \tau^{-1}$ is a coequality relation on X and the relation $\theta = \pi \circ \tau \circ \pi^{-1}$ is an anti-order relation on X/q . So, the relation q is a regular coequality relation on X with respect to and α. □

Let us note that family $\Im(X,\alpha)$ is a completely lattice. Indeed, in the following theorem we give prove this fact:

Theorem 6. *If $\{\tau_k\}_{k\in J}$ is a family of quasi-antiorders on a set $(X,=,\neq,\alpha)$, then $\bigcup_{k\in J} \tau_k$ and $c(\bigcap_{k\in J} \tau_k)$ are quasi-antiorders in X. So, the family $\Im(X,\alpha)$ is a completely lattice.*

Proof. Let $\{\tau_k\}_{k\in J}$ be a family of quasi-antiorders on a set $(X,=,\neq)$ under α and let (x,z) be an arbitrary elements of X such that $(x,z) \in \bigcup_{k\in J} \tau_k$. Then, there exists k in J such that $(x,z) \in \tau_k$. Hence, for every $y \in X$ we have $(x,y) \in \tau_k (y,z) \in \tau_k$.

So, $(x,y) \in \bigcup_{k\in J} \tau_k \vee (y,z) \in \bigcup_{k\in J} \tau_k$. On the other hand, for every k in J holds $\tau_k \subseteq \alpha$. From this we have $\bigcup_{k\in J} \tau_k \subseteq \alpha$.
It is clear that the relation $c(\bigcap_{k\in J} \tau_k)$ is the maximal quasi-antiorder relation under $\bigcap_{k\in J} \tau_k (\subseteq \alpha)$. □

As end of this consideration we establish connection between lattices $\Re(X,\alpha)$ and $\Im(X,\alpha)$

Theorem 7. *The mapping*

$$\varphi : \Im(X,\alpha) \longrightarrow \Re(X,\alpha),$$

defined by $\varphi(\tau) = \tau \cup \tau^{-1}$ *, is a strongly extensional surjective function. Relations* $\varepsilon = Ker\varphi$ *and* $\omega = Antiker\varphi = \{(\tau,\sigma) \in \Im(X,\alpha) \times \Im(X,\alpha) : \tau \cup \tau^{-1} \neq \sigma \cup \sigma^{-1}\}$ *are compatible equality and diversity relation on* $\Im(X,\alpha)$*, and the following isomorphism* $\Im(X,\alpha)/(\varepsilon,\omega) \cong \Re(X,\alpha)$ *exists.*

Proof. (1) The mapping φ is a well-defined strongly extensional function: If τ is a quasi-antiorder relation on X, then $q(\tau) = \tau \cup \tau^{-1}$ is a coequality relation on X. Then, there exists an anti-order relation θ on X/q defined by $(aq,bq) \in \theta \Longleftrightarrow (a,b) \in \tau$ and the natural mapping $\pi : X \longrightarrow X/q(\tau)$ is reverse isotone. This means that $\varphi(\tau) = \tau \cup \tau^{-1} = q \in \Re(X,\alpha)$. Let σ and τ be elements of $\Im(X,\alpha)$ such that $\tau\varepsilon = \sigma\varepsilon$. Then $(\tau,\sigma) \in \varepsilon$ and $\varphi(\tau) = \tau \cup \tau^{-1} = \sigma \cup \sigma^{-1} = \varphi(\sigma)$. Suppose that $\varphi(\tau) = \tau \cup \tau^{-1} \neq \sigma \cup \sigma^{-1} = \varphi(\sigma)$ for some $\sigma,\tau \in \Im(X,\alpha)$. Then there exists an element $(x,y) \in X \times X$ such that $((x,y) \in \tau \cup \tau^{-1}$ and $(x,y) \bowtie \sigma \cup \sigma^{-1})$ or $((x,y) \in \sigma \cup \sigma^{-1}$ and $(x,y) \bowtie \tau \cup \tau^{-1})$. In the first case, we have:
$((x,y) \in \tau \vee (x,y) \in \tau^{-1}) \wedge (x,y) \bowtie \sigma \wedge (x,y) \bowtie \sigma^{-1} \Longrightarrow$
$((x,y) \in \tau \wedge (x,y) \bowtie \sigma) \vee ((x,y) \in \tau^{-1} \wedge (x,y) \bowtie \sigma^{-1}) \Longleftrightarrow$
$((x,y) \in \tau \wedge (x,y) \bowtie \sigma) \vee ((y,x) \in \tau \wedge (y,x) \bowtie \sigma) \Longrightarrow \tau \neq \sigma.$

In the second case we derive similar implication analogously.
(2) φ is an injective function. In fact: let τ and σ be elements of $\Im(X,\alpha)$ such that $\varphi(\tau) = \tau \cup \tau^{-1} = \sigma \cup \sigma^{-1} = \varphi(\sigma)$. Then, $(\tau,\sigma) \in \varepsilon$ and $\tau\varepsilon = \sigma\varepsilon$.
(3) φ is an embedding. Indeed, let τ and σ be elements of $\Im(X,\alpha)$ such that $\tau\varepsilon \neq \sigma\varepsilon$, i.e. such that $(\tau,\sigma) \in \omega$. It means $\varphi(\tau) = \tau \cup \tau^{-1} \neq \sigma \cup \sigma^{-1} = \varphi(\sigma)$.
(4) φ is a surjective function: Let q be a regular coequality relation on X with respect to α , i.e. let q be a coequality relation on X such that there exists an anti-order θ on X/q and the natural mapping $\pi : X \longrightarrow X/q$ is reverse isotone. Then, there exists a quasi-antiorder $\sigma (\subseteq \alpha)$ on X such that $\sigma \cup \sigma^{-1} = q$. Thus, $\sigma \in \Im(X,\alpha)$ and $\varphi(\sigma\varepsilon) = \sigma \cup \sigma^{-1} = q$. □

REFERENCES

[1] E. Bishop, *Foundations of constructive analysis*. New York-Toronto: McGraw-Hill Book Co., 1967.

[2] D. Bridges and F. Richman, *Varieties of constructive mathematics*, ser. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 1987, vol. 97.
[3] R. Mines, F. Richman, and W. Ruitenburg, *A course in constructive algebra*, ser. Universitext. New York: Springer-Verlag, 1988.
[4] D. A. Romano, "On construction of maximal coequality relation and its applications," *Proceedings of the VIII international conference on logic and computer science: theoretical foundations of computer science, Lira '97*, pp. 225–230, 1997.
[5] D. A. Romano, "Some relations and subsets generated by principal consistent subset of semigroup with apartness," *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.*, vol. 13, pp. 7–25, 2002.
[6] D. A. Romano, "A note on the quasi-antiorder in a semigroup," *Novi Sad J. Math.*, vol. 37, no. 1, pp. 3–8, 2007.
[7] D. A. Romano, "On regular anti-congruence in anti-ordered semigroups," *Publ. Inst. Math. (Beograd) (N.S.)*, vol. 81, no. 95, pp. 95–102, 2007.
[8] D. A. Romano, "A remark on coequality relation in anti-ordered set," *Int. J. Contemp. Math. Sci.*, vol. 3, no. 1-4, pp. 43–48, 2008.
[9] A. S. Troelstra and D. van Dalen, *Constructivism in mathematics. Vol. I.*, ser. Studies in Logic and the Foundations of Mathematics. Amsterdam: North-Holland Publishing Co., 1988, vol. 121.

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NECESSARY CONDITIONS FOR DOUBLE SUMMABILITY FACTOR THEOREM

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Abstract. We obtain necessary conditions for the series $\sum\sum c_{mn}$, which is absolutely summable of order k by a doubly triangular matrix method A, to be such that $\sum\sum c_{mn}\lambda_{mn}$ is absolutely summable of order k by a doubly triangular matrix B.

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1. INTRODUCTION

A doubly infinite matrix $A = (a_{mnij})$ is said to be doubly triangular if $a_{mnij} = 0$ for $i > m$ and $j > n$. The mn-th terms of the A-transform of a double sequence $\{s_{mn}\}$ is defined by

$$T_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{mnij}s_{ij}.$$

A series $\sum\sum c_{mn}$, with partial sums s_{mn} is said to be absolutely A-summable, of order $k \geq 1$, if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|\Delta_{11}T_{m-1,n-1}|^k < \infty, \tag{1.1}$$

where for any double sequence $\{u_{mn}\}$, and for any four-fold sequence $\{a_{mnij}\}$, we define

$$\begin{aligned}
\Delta_{11}u_{mn} &= u_{mn} - u_{m+1,n} - u_{m,n+1} + u_{m+1,n+1},\\
\Delta_{11}a_{mnij} &= a_{mnij} - a_{m+1,n,i,j} - a_{m,n+1,i,j} + a_{m+1,n+1,i,j},\\
\Delta_{ij}a_{mnij} &= a_{mnij} - a_{m,n,i+1,j} - a_{m,n,i,j+1} + a_{m,n,i+1,j+1},\\
\Delta_{i0}a_{mnij} &= a_{mnij} - a_{m,n,i+1,j}, \quad \text{and}\\
\Delta_{0j}a_{mnij} &= a_{mnij} - a_{m,n,i,j+1}.
\end{aligned}$$

The one-dimensional version of (1.1) appears in [1].

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Associated with A there are two matrices $\overline{A}$ and $\hat{A}$ defined by

$$\bar{a}_{mnij} = \sum_{\mu=i}^{m}\sum_{\nu=j}^{n} a_{mn\mu\nu}, 0 \le i \le m, 0 \le j \le n, m,n = 0,1,\ldots,$$

and

$$\hat{a}_{mnij} = \Delta_{11}\bar{a}_{m-1,n-1,i,j}, 0 \le i \le m, 0 \le j \le n, m,n = 1,2,\ldots.$$

It is easily verified that $\hat{a}_{0000} = \bar{a}_{0000} = a_{0000}$. In [3] it is shown that

$$\hat{a}_{mnij} = \sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j-1} \Delta_{11} a_{m-1,n-1,\mu,\nu}.$$

Thus $\hat{a}_{mni0} = \hat{a}_{mn0j} = 0$.

Let x_{mn} denote the mn-th term of the A-transform of the sequence of partial sums $\{s_{mn}\}$ of the series $\sum\sum c_{mn}$.

Then

$$\begin{aligned} x_{mn} &= \sum_{i=0}^{m}\sum_{j=0}^{n} a_{mnij}s_{ij} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{mnij}\sum_{\mu=0}^{i}\sum_{\nu=0}^{j} c_{\mu\nu}\lambda_{\mu\nu} \\ &= \sum_{\mu=0}^{m}\sum_{\nu=0}^{n}\sum_{i=\mu}^{m}\sum_{j=\nu}^{n} a_{mnuv}c_{\mu\nu}\lambda_{\mu\nu} \\ &= \sum_{\mu=0}^{m}\sum_{\nu=0}^{n} \bar{a}_{mn\mu\nu}c_{\mu\nu}\lambda_{\mu\nu}, \end{aligned}$$

and a direct calculation verifies that

$$X_{mn} := \Delta_{11}x_{m-1,n-1} = \sum_{\nu=1}^{m}\sum_{\mu=1}^{n} \hat{a}_{mn\mu\nu}c_{\mu\nu}\lambda_{\mu\nu},$$

since

$$\bar{a}_{m-1,n-1,m,\nu} = a_{m-1,n-1,\mu,n} = \hat{a}_{m,n-1,\mu,n} = \hat{a}_{m-1,n,m,n} = 0.$$

In a recent paper Savas and Rhoades[2] obtained sufficient conditions for the series $\sum\sum c_{mn}$, which is absolutely summable of order k by a doubly triangular matrix method A, to be such that $\sum\sum c_{mn}\lambda_{mn}$ is absolutely summable of order k by a doubly triangular matrix B.

In this paper we obtain necessary conditions for the series $\sum\sum c_{mn}$, which is absolutely summable of order k by a doubly triangular matrix method A, to be such that $\sum\sum c_{mn}\lambda_{mn}$ is absolutely summable of order k by a doubly triangular matrix B.

2. Main Theorem

Theorem 1. *Let A and B be doubly triangular matrices with A satisfying*

$$\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\Delta_{uv}\hat{a}_{mnuv}|^k = O(M^k(\hat{a}_{uvuv})), \tag{2.1}$$

where

$$M(\hat{a}_{uvuv}) := \max\{|\hat{a}_{uvuv}|, |\Delta_{u0}\hat{a}_{u+1v,u,v}|, |\Delta_{0v}\hat{a}_{uv+1,u,v}|\}.$$

Then the necessary conditions of the fact that the $|A|_k$ summability of $\sum\sum c_{mn}$ implies the $|B|_k$ summability of $\sum\sum c_{mn}\lambda_{mn}$ are the following items:

(i) $|\hat{b}_{uvuv}\lambda_{uv}| = O(M(\hat{a}_{uvuv}))$,

(ii) $|\Delta_{u0}\hat{b}_{u+1,v,u,v}\lambda_{uv}| = O(M(\hat{a}_{uvuv}))$,

(iii) $|\Delta_{0v}\hat{b}_{u,v+1,u,v}\lambda_{uv}| = O(M(\hat{a}_{uvuv}))$,

(iv) $\displaystyle\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\Delta_{uv}\hat{b}_{mnuv}\lambda_{uv}|^k = O((uv)^{k-1}M^k(\hat{a}_{uvuv}))$,

(v) $\displaystyle\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\hat{b}_{m,n,u+1,v+1}\lambda_{u+1,v+1}|^k =$
$O\Big(\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\hat{a}_{m,n,u+1,v+1}|^k\Big)$.

Proof. For $k \geq 1$ define

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|Y_{mn}|^k < \infty, \tag{2.2}$$

whenever

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|X_{mn}|^k < \infty, \tag{2.3}$$

where

$$Y_{mn} = \Delta_{11}y_{m-1,n-1},$$

$$y_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n}\bar{b}_{mnij}c_{ij}\lambda_{ij}.$$

The space of sequences satisfying (2.3) is a Banach space if it is normed by

$$\|X\| = \Big(|X_{00}|^k + |X_{01}|^k + |X_{10}|^k + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|X_{mn}|^k\Big)^{1/k}. \tag{2.4}$$

We shall also consider the space of sequences $\{Y_{mn}\}$ that satisfy (2.2). This space is also a BK-space with respect to the norm

$$\|Y\| = \left(|Y_{00}|^k + |Y_{01}|^k + |Y_{10}|^k + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|Y_{mn}|^k\right)^{1/k}. \tag{2.5}$$

The transformation $\sum_{i=0}^{m}\sum_{j=0}^{n}\bar{b}_{mnij}c_{ij}\lambda_{ij}$ maps sequences satisfying (2.3) into sequences satisfying (2.2). By the Banach-Steinhaus Theorem there exists a constant $K > 0$ such that

$$\|Y\| \leq K\|X\|. \tag{2.6}$$

For fixed u, v, the sequence $\{c_{ij}\}$ is defined by $c_{uv} = c_{u+1,v+1} = 1, c_{u+1,v} = c_{u,v+1} = -1, c_{ij} = 0$, otherwise,

$$X_{mn} = \begin{cases} 0, & m \leq u, n < v, \\ 0, & m < u, n \leq v, \\ \hat{a}_{mnuv}, & m = u, n = v, \\ \Delta_{u0}\hat{a}_{mnuv}, & m = u+1, n = v, \\ \Delta_{0v}\hat{a}_{mnuv}, & m = u, n = v+1, \\ \Delta_{uv}\hat{a}_{mnuv}, & m > u, n > v, \end{cases}$$

and

$$Y_{mn} = \begin{cases} 0, & m \leq u, n < v, \\ 0, & m < u, n \leq v, \\ \hat{b}_{mnuv}\lambda_{uv}, & m = u, n = v, \\ \Delta_{u0}\hat{b}_{mnuv}\lambda_{uv}, & m = u+1, n = v, \\ \Delta_{0v}\hat{b}_{mnuv}\lambda_{uv}, & m = u, n = v+1, \\ \Delta_{uv}\hat{b}_{mnuv}\lambda_{uv}, & m > u, n > v. \end{cases}$$

From (2.4) and (2.5) it follows that

$$\begin{aligned} \|X\| = \Big\{&(uv)^{k-1}|\hat{a}_{uvuv}|^k + ((u+1)v)^{k-1}|\Delta_{u0}a_{u+1,v,u,v}|^k \\ &+ (u(v+1))^{k-1}|\Delta_{0v}a_{u,v+1u,v}|^k \\ &+ \sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\Delta_{uv}\hat{a}_{mnuv}|^k\Big\}^{1/k}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \|Y\| = \Big\{&(uv)^{k-1}\hat{b}_{uvuv}\lambda_{uv}|^k + ((u+1)v)^{k-1}|\Delta_{u0}\hat{b}_{u+1,v,u,v}\lambda_{uv}|^k \\ &+ (u(v+1))^{k-1}|\Delta_{0v}\hat{b}_{u,v+1u,v}\lambda_{uv}|^k \end{aligned} \tag{2.8}$$

$$+\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\Delta_{uv}\hat{a}_{mnuv}\lambda_{uv}|^k\Big\}^{1/k}.$$

Substituting (2.7) and (2.8) into (2.6), along with (2.1), gives

$$\begin{aligned}(uv)^{k-1}|\hat{b}_{uvuv}\lambda_{uv}|^k+((u+1)v)^{k-1}|\Delta_{u0}\hat{b}_{u+1v,u,v}\lambda_{uv}|^k\\ +(u(v+1))^{k-1}|\Delta_{0v}\hat{b}_{u,v+1u,v}\lambda_{uv}|^k\\ +\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\Delta_{uv}\hat{b}_{mnuv}\lambda_{uv}|^k\le K^k\Big\{(uv)^{k-1}|\hat{a}_{uvuv}|^k\\ ((u+1)v)^{k-1}|\Delta_{u0}\hat{a}_{u+1v,u,v}|^k+(u(v+1))^{k-1}\Delta_{u0}\hat{a}_{u,v+1u,v}|^k\\ +\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\Delta_{uv}\hat{a}_{mnuv}|^k\Big\}\\ =K^k\{O(1)(uv)^{k-1}M^k(\hat{a}_{uvuv})\}.\end{aligned}$$

The above inequality implies that each term of the left hand side is $O(\{(uv)^{k-1}M^k(\hat{a}_{uvuv})\})$.

Using the first term one obtains

$$(uv)^{k-1}|\hat{b}_{uvuv}\lambda_{uv}|^k=O(\{(uv)^{k-1}M^k(\hat{a}_{uvuv})\}).$$

Thus

$$|\hat{b}_{uvuv}\lambda_{uv}|=O(M(\hat{a}_{uvuv})),$$

which is condition (i).

In a similar manner one obtains conditions (ii) - (iv).

Using the sequence, defined by $c_{u+1,v+1}=1$, and $c_{ij}=0$ otherwise, yields

$$X_{mn}=\begin{cases}0, & m\le u+1,n\le v,\\ 0, & m\le u,n\le v+1,\\ \hat{a}_{m,n,u+1,v+1}, & m\ge u+1,n\ge v+1\end{cases}$$

and

$$Y_{mn}=\begin{cases}0. & m\le u+1,n\le v,\\ 0, & m\le u,n\le v+1,\\ \hat{b}_{m,n,u+1,v+1}\lambda_{u+1,v+1}, & m\ge u+1,n\ge v+1.\end{cases}$$

The corresponding norms are

$$\|X\|=\Big\{\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\hat{a}_{m,n,u+1,v+1}|^k\Big\}^{1/k}$$

and

$$\|Y\| = \Big\{ \sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\hat{b}_{m,n,u+1,v+1} \lambda_{u+1,v+1}|^k \Big\}^{1/k}.$$

Applying (2.6), we have

$$\sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\hat{b}_{m,n,u+1,v+1} \lambda_{u+1,v+1}|^k$$
$$\leq K^k \Big\{ \sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\hat{a}_{m,n,u+1,v+1}|^k \Big\},$$

which implies (v). □

Every summability factor theorem becomes an inclusion theorem by setting each $\lambda_{mn} = 1$.

Corollary 1. *Let A and B two doubly triangular matrices, A satisfying (2.1). Then necessary conditions of the fact that the $|A|_k$ summability of $\sum\sum c_{mn}$ implies the $|B|_k$ summability of $\sum\sum c_{mn}$ are the following items:*

(i) $|\hat{b}_{uvuv}| = O(M(\hat{a}_{uvuv}))$,
(ii) $|\Delta_{u0}\hat{b}_{u+1,v,u,v}| = O(M(\hat{a}_{uvuv}))$,
(iii) $|\Delta_{0v}\hat{b}_{u,v+1,u,v}| = O(M(\hat{a}_{uvuv}))$,
(iv) $\sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\Delta_{uv}\hat{b}_{mnuv}|^k = O((uv)^{k-1} M^k(\hat{a}_{uvuv}))$, *and*
(v) $\sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\hat{b}_{m,n,u+1,v+1}|^k$
$= O\Big(\Big\{ \sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\hat{a}_{m,n,u+1,v+1}|^k \Big\}^{1/k} \Big)$.

Proof. To prove Corollary 1 simply put $\lambda_{mn} = 1$ in Theorem 1. □

We shall call a doubly infinite matrix a product matrix if it can be written as the termwise product of two singly infinite matrices F and G; i.e., $a_{mnij} = f_{mi} g_{nj}$ for each i, j, m, n.

A doubly infinite weighted mean matrix P has nonzero entries p_{ij}/P_{mn}, where p_{00} is positive and all of the other p_{ij} are nonnegative, and $P_{mn} := \sum_{i=0}^{m} \sum_{j=0}^{n} p_{ij}$. If P is a product matrix then the nonzero entries are $p_i q_j / P_m Q_n$, where $p_0 > 0$, $p_i > 0$ for $i > 0, q_0 > 0, q_i \geq 0$ for $j > 0$ and $P_m := \sum_{i=0}^{m} p_i, Q_n := \sum_{j=0}^{n} q_j$. Now we have the following corollary

Corollary 2. *Let B be a doubly triangular matrix, P a product weighted mean matrix satisfying*

$$\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}\left|\Delta_{uv}\left(\frac{p_m q_n P_{u-1}Q_{v-1}}{P_m P_{m-1}Q_n Q_{n-1}}\right)\right|^k = O\left(\frac{p_u q_v}{P_u Q_v}\right).$$

Then necessary conditions for $\sum\sum c_{mn}$ summable $|P|_k$ to imply that $\sum\sum c_{mn}\lambda_{mn}$ is summable $|B|_k$ are

(i) $|\hat{b}_{uvuv}\lambda_{uv}| = O\left(\frac{p_u q_v}{P_u Q_v}\right)$,

(ii) $|\Delta_{u0}\hat{b}_{u+1,v,u,v}\lambda_{uv}| = O\left(\frac{p_u q_v}{P_u Q_v}\right)$,

(iii) $|\Delta_{0v}\hat{b}_{u,v+1,u,v}\lambda_{uv}| = O\left(\frac{p_u q_v}{P_u Q_v}\right)$,

(iv) $\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\Delta_{uv}\hat{b}_{uvuv}\lambda_{uv}|^k = O\left((uv)^{k-1}\left(\frac{p_u q_v}{Pu Q_v}\right)^k\right)$, *and*

(v) $\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\hat{b}_{m,n,u+1,v+1}\lambda_{u+1,v+1}|^k = O(1)$.

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REFERENCES

[1] B. E. Rhoades and E. Savaş, "General summability factor theorems and applications," *Sarajevo J. Math.*, vol. 1(13), no. 1, pp. 59–73, 2005.

[2] E. Savaş and B. E. Rhoades, "Double summability factor theorems and applications," *Math. Inequal. Appl.*, vol. 10, no. 1, pp. 125–149, 2007.

[3] E. Savaş and B. E. Rhoades, "Double absolute summability factor theorems and applications," *Nonlinear Anal.*, vol. 69, no. 1, pp. 189–200, 2008.

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ON THE INSTABILITY OF THE SOLUTIONS OF SOME NONLINEAR VECTOR DIFFERENTIAL EQUATIONS OF FOURTH ORDER

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Abstract. The purpose of this paper is to obtain some new sufficient conditions which guarantee the instability of the trivial solution of some certain nonlinear vector differential equations of fourth order. Our results improve and include some well known instability results, which were established on the instability of the trivial solution of certain scalar differential equations of fourth order, to certain nonlinear vector differential equations of fourth order.

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1. INTRODUCTION

Consider the following nonlinear vector differential equations

$$x^{(4)} + A\ddot{x} + G(x,\dot{x},\ddot{x},\dddot{x})\ddot{x} + H(x)\dot{x} + F(x,\dot{x},\ddot{x},\dddot{x})x = 0 \tag{1.1}$$

and

$$x^{(4)} + \Phi(\dddot{x}) + \Psi(x,\dot{x},\ddot{x},\dddot{x})\ddot{x} + C\dot{x} + Dx = 0, \tag{1.2}$$

where A , C and D are constant symmetric $n \times n$ - matrices; G, H, F and Ψ are continuous (symmetric $n \times n$ -) matrix valued functions; $\Phi : R^n \to R^n$, $\Phi(0) = 0$ and Φ is continuous. Throughout this paper, instead of equations (1.1) and (1.2), we consider the system of differential equations:

$$\begin{aligned} &\dot{x} = y, \dot{y} = z, \dot{z} = w, \\ &\dot{w} = -Aw - G(x,y,z,w)z - H(x)y - F(x,y,z,w)x \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} &\dot{x} = y, \dot{y} = z, \dot{z} = w, \\ &\dot{w} = -\Phi(w) - \Psi(x,y,z,w)z - Cy - Dx, \end{aligned} \tag{1.4}$$

which were obtained as usual by setting $\dot{x} = y$, $\ddot{x} = z$, $\dddot{x} = w$ from equations (1.1) and (1.2), respectively.

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Now, let $J(H(x)x\,|x\,)$ and $J_{\Phi}(w)$, respectively, denote the linear operators from the matrix $H(x)$ and gradient vector field $\Phi(w)$ to the matrices

$$J(H(x)x\,|x\,)=\left(\frac{\partial}{\partial x_j}\sum_{k=1}^{n}h_{ik}x_k\right)=H(x)+\left(\sum_{k=1}^{n}\frac{\partial h_{ik}}{\partial x_j}x_k\right)$$

and

$$J_{\Phi}(w)=\left(\frac{\partial \phi_i}{\partial w_j}\right),(i,\ j=1,\ 2,\ldots,n),$$

where $(x_1,x_2,\ldots,x_n)$, $(w_1,w_2,\ldots,w_n)$, (h_{ik}) and $(\phi_1,\phi_2,\ldots,\phi_n)$ are components of x, w, H and Φ, respectively. Further, it is assumed that the matrices $J(H(x)x\,|x\,)$ and $J_{\Phi}(w)$ exist and are symmetric and continuous.

The unstable behavior of solutions for various certain scalar and vector linear and nonlinear differential equations of fourth order has been discussed in the literature over the past three decades, and they are still being investigated. We refer the reader to the papers of Ezeilo [2–4], Sadek [7], Skrapek [8], Tunç [9–11] and the references thereof for some publications performed on the subject.

It should be noted that, in 2000, Ezeilo [3] considered the following fourth order scalar nonlinear differential equations

$$x^{(4)}+a_1\dddot{x}+g(x,\dot{x},\ddot{x},\dddot{x})\ddot{x}+h(x)\dot{x}+f(x,\dot{x},\ddot{x},\dddot{x})=0 \tag{1.5}$$

and

$$x^{(4)}+p(\dddot{x},\ddot{x})+q(x,\dot{x},\ddot{x},\dddot{x})\ddot{x}+a_3\dot{x}+a_4x=0, \tag{1.6}$$

respectively, where a_1 , a_3 and a_4 are constants. He proved two theorems on the instability of the trivial solution, $x=0$, of equations (1.5) and (1.6). To the best of our knowledge, so far, neither the instability of solutions of nonlinear vector differential equations of the form (1.1) and (1.2) has been investigated in the literature nor papers have been published which were based on the results of Ezeilo [3]. When $f(x,\dot{x},\ddot{x},\dddot{x})=F(x,\dot{x},\ddot{x},\dddot{x})x$ in (1.5) and $p(\dddot{x},\ddot{x})=\Phi(\dddot{x})$ in (1.6), our problems, which will be related to equations (1.1) and (1.2), improve and include the problems discussed by Ezeilo [3] from the scalar cases to the vector cases, respectively. It is also worth mentioning that throughout the aforementioned papers, based on Krasovskii's criterion (see Krasovskii [5]), the Lyapunov's [6] second (or direct) method has been used as a basic tool to prove the results established thereof. In this paper, we utilize the same method to verify our main results. It is also well known that, by now, the Lyapunov's [6] second (or direct) method is the most effective tool to study the stability and instability of solutions of nonlinear differential equations of higher order, when there is no analytical expression for solutions. The motivation for the present work has been inspired basically by the paper of Ezeilo [3], Tunç [11] and the papers mentioned above. Finally, all the papers aforementioned have been published without including an example on the topic, except that of Sadek [7] and Tunç [11]. In this paper, we also give an example for the illustration of the subject.

We now consider the constant-coefficient fourth order scalar differential equation

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + a_4 x = 0.$$

We know from the qualitative theory of ordinary differential equations that the trivial solution of the above equation is unstable if and only if the associated (auxiliary) equation

$$\psi(r) := r^4 + a_1 r^3 + a_2 r^2 + a_3 r + a_4 = 0$$

has at least one root with a positive real part. Naturally, the existence of such a root depends on (though not always all of) the coefficients a_1, a_2, a_3 and a_4 of ψ. The corresponding auxiliary equation $\psi(r) = 0$ of the above equation has no purely imaginary roots $r = i\lambda$, ($\lambda \neq 0$) if

$$\chi_1 := \lambda^4 - a_2\lambda^2 + a_4 \neq 0$$

or if

$$\chi_2 := a_1\lambda^2 - a_3 \neq 0.$$

Note that χ_1 may be rearranged in the form

$$\chi_1 = \left(\lambda^2 - \frac{1}{2}a_2\right)^2 + a_4 - \frac{1}{2}a_2^2$$

so that $\chi_1 \neq 0$ holds if, for example, $a_4 > \frac{1}{4}a_2^2$. It is also easy to see that $\chi_2 \neq 0$ will hold if

$$a_1 a_3 < 0, a_4 \neq 0,$$

where the condition on a_4 here is necessary in order to ensure that $\lambda \neq 0$. It should also be noted that it was shown in Ezeilo [2] that the existence of the relation $a_4 > \frac{1}{4}a_2^2$ between the coefficients a_2 and a_4 of the equation is a sufficient condition that guarantees that the solution $x = 0$ of aforementioned equation is unstable. The above preamble is merely intended to give the background to the assumptions which play a dominant role in our treatment here and to give some indication of how these assumptions stand with respect to the Routh-Hurwitz criteria.

Throughout the paper, the symbol $\langle x, y\rangle$ is used to denote the usual scalar product in R^n for given any x, y in R^n, that is, $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$, thus $\|x\|^2 = \langle x, x\rangle$. It is also well known that a real symmetric matrix $A = (a_{ij})$, $(i, j = 1, 2, \ldots, n)$ is said to be positive definite if and only if the quadratic form $Q := x^T Ax$ $(x \in R^n)$ is positive definite.

2. Main results

We first state the following algebraic result, which will be useful in the proof of the main results

Lemma 1. *Let A be a real symmetric $n \times n$ -matrix and*

$$a' \geq \lambda_i(A) \geq a > 0, (i = 1, 2, \ldots, n),$$

where a' and a are constants, resp. $\lambda_i(A)$ denotes the $i-th$ eigenvalue of A. Then

$$a' \langle x, x \rangle \geq \langle Ax, x \rangle \geq a \langle x, x \rangle$$

and

$$a'^2 \langle x, x \rangle \geq \langle Ax, Ax \rangle \geq a^2 \langle x, x \rangle .$$

Proof. See Bellman [1]. □

Our first result is the following theorem

Theorem 1. *In addition to the basic assumptions imposed on A, G, H and F that appear in equation (1.1), we suppose that there exists a constant a_1 such that the following conditions hold:*

$$\lambda_i(A) \geq a_1 > 0$$

and

$$\lambda_i(F(x, y, z, w,)) > \frac{1}{4} [\lambda_i(G(x, y, z, w))]^2, (i = 1, 2, \ldots, n),$$

for all x, y, z, $w \in R^n$.

Then the trivial solution $x = 0$ of equation (1.1) is unstable for arbitrary $H(x)$.

Remark 1. In order to prove Theorem 1, it will be sufficient to show that there exists a continuous Lyapunov function $V = V(x, y, z, w)$ which has the following Krasovskii properties (see Krasovskii [5]):

- (K_1) In every neighborhood of $(0, 0, 0, 0)$, there exists a point (ξ, η, ζ, μ) such that $V(\xi, \eta, \zeta, \mu) > 0$;
- (K_2) the time derivative $\dot{V}$ along solution paths of the system (1.3) is positive semi-definite; and
- (K_3) the only solution of the system (1.3) which satisfies $\dot{V}(t) = 0$, $(t \geq 0)$, is the trivial solution $(0, 0, 0, 0)$.

Remark 2. It should be noted that there is no restriction on the eigenvalues of the matrix H in (1.3).

Remark 3. Our first result, Theorem 1, includes the result of Ezeilo [3, Theorem 1] and Sadek [7, Theorem 1], and improves the result of Ezeilo [3, Theorem 1] from scalar differential equation to the vector differential equation, (1.1). Equation (1.1) is also completely different from those studied in the literature, (see also Ezeilo [2–4], Sadek [7], Skrapek [8] and Tunç [9–11].

Proof. For the proof of Theorem 1, we define a Lyapunov function $V_0 = V_0(x, y, z, w)$ given by:

$$V_0 = -\langle x, w\rangle + \langle y, z\rangle - \langle x, Az\rangle + \frac{1}{2}\langle Ay, y\rangle - \int_0^1 \langle H(\sigma x)x, x\rangle\, d\sigma. \tag{2.1}$$

Observe that $V_0(0,0,0,0) = 0$. Indeed, in view of the assumptions of Theorem 1, we have

$$\begin{aligned} V_0(0,\varepsilon,\varepsilon,\varepsilon) &= \langle \varepsilon, \varepsilon\rangle + \frac{1}{2}\langle A\varepsilon, \varepsilon\rangle \\ &\geq \langle \varepsilon, \varepsilon\rangle + \frac{1}{2}a_1 \langle \varepsilon, \varepsilon\rangle \\ &= \|\varepsilon\|^2 + \frac{1}{2}a_1 \|\varepsilon\|^2 > 0 \end{aligned}$$

for all arbitrary $\varepsilon \neq 0$, $\varepsilon \in R^n$, which verifies the property (K_1) of Krasovskii [5].

Let $(x, y, z, w) = (x(t), y(t), z(t), w(t))$ be an arbitrary solution of the system (1.3). A straightforward calculation from (2.1) and (1.3) yields that

$$\begin{aligned} \dot{V}_0 = \frac{d}{dt}V_0(x,y,z,w) &= \langle z, z\rangle + \langle x, G(x,y,z,w)z\rangle + \langle x, F(x,y,z,w)x\rangle \\ &\quad + \langle H(x)y, x\rangle - \frac{d}{dt}\int_0^1 \langle H(\sigma x)x, x\rangle\, d\sigma. \end{aligned} \tag{2.2}$$

Now, recall that

$$\begin{aligned} \frac{d}{dt}\int_0^1 \langle H(\sigma x)x, x\rangle\, d\sigma &= \int_0^1 \langle H(\sigma x)x, y\rangle\, d\sigma + \int_0^1 \langle \sigma J\left(H(\sigma x)x\,|\sigma x\right)y, x\rangle\, d\sigma \\ &= \int_0^1 \langle H(\sigma x)x, y\rangle\, d\sigma + \int_0^1 \sigma\langle J\left(H(\sigma x)x\,|\sigma x\right)x, y\rangle\, d\sigma \\ &= \int_0^1 \langle H(\sigma x)x, y\rangle\, d\sigma + \int_0^1 \sigma\frac{\sigma}{\partial\sigma}\langle H(\sigma x), y\rangle\, d\sigma \\ &= \sigma\langle H(\sigma x)x, y\rangle\,|_0^1 = \langle H(x)x, y\rangle. \end{aligned} \tag{2.3}$$

Substituting the estimate (2.3) into (2.2), we obtain

$$\dot{V}_0 = \langle z, z\rangle + \langle x, G(x,y,z,w)z\rangle + \langle x, F(x,y,z,w)x\rangle$$

Now, it is clear that

$$\dot{V}_0(0,y,z,w) = \langle z,z\rangle = \|z\|^2 \geq 0.$$

Next, the function $\dot{V}_0$ can be rearranged as follows

$$\dot{V}_0 \quad = \left\|z + \tfrac{1}{2}G(x,y,z,w)x\right\|^2 + \langle x, F(x,y,z,w)x\rangle$$

$$-\tfrac{1}{4}\langle G(x,y,z,w)x, G(x,y,z,w)x\rangle .$$

Thus, we conclude that

$$\dot{V}_0 \geq \langle x, F(x,y,z,w)x\rangle - \frac{1}{4}\langle G(x,y,z,w)x, G(x,y,z,w)x\rangle > 0$$

because of the assumption of Theorem 1, $\lambda_i(F(x,y,z,w)) > \frac{1}{4}[\lambda_i(G(x,y,z,w))]^2$.

Hence, the above discussion clarifies that the function $\dot{V}_0$ is positive semi-definite. This case verifies the property (K_2) of Krasovskii [5]. Next, let $(x(t), y(t), z(t), w(t))$ be an arbitrary solution of the system (1.3). We observe from the foregoing estimate of $\dot{V}_0$ that $\dot{V}_0(t) = 0$ $(t \geq 0)$ necessarily implies that $x(t) = 0$ for all $t \geq 0$. In turn, this implies that

$$y(t) = \dot{x}(t), z(t) = \dot{y}(t) = 0, w(t) = \dot{z}(t) = 0 \text{ for all } t \geq 0.$$

Hence the property of (K_3) of Krasovskii [5] holds for the function V_0.

This completes the proof of Theorem 1. □

Example 1. In the system (1.3) in case of $n = 2$ let A, G and F take the following form:

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix},$$

$$G = \begin{bmatrix} 1+\frac{1}{1+x^2+y^2} & \frac{1}{1+x^2+y^2} \\ \frac{1}{1+x^2+y^2} & 1+\frac{1}{1+x^2+y^2} \end{bmatrix}$$

and

$$F = \begin{bmatrix} 4+\frac{1}{1+x^2+y^2} & 0 \\ 0 & 4+\frac{1}{1+x^2+y^2} \end{bmatrix}.$$

It is obvious that A , G and F are symmetric matrices. Then, by an easy calculation, we obtain eigenvalues of the matrices A , $G(x,y)$ and $F(x,y)$ as follows

$$\lambda_1(A) = 3, \lambda_2(A) = 8,$$

$$\lambda_1(G) = 1, \lambda_2(G) = 1 + \frac{2}{1+x^2+y^2}$$

and

$$\lambda_1(F) = 4 + \frac{1}{1+x^2+y^2}, \lambda_2(F) = 4 + \frac{1}{1+x^2+y^2}.$$

Clearly, all the conditions of Theorem 1 hold for arbitrary $H(x)$. Indeed, this fact shows that the trivial solution of corresponding vector differential equation for the

above case $n = 2$ is unstable. It should also be noted that, when G and F reduce to the linear case, our conclusions remain valid, too.

Our last result is the following theorem

Theorem 2. *In addition to the basic assumptions imposed on C, D, Ψ and Φ that appear in equation (1.2), we suppose that there exist constants a_3 and a_4 such that the following conditions hold:*

$$\lambda_i(C) \geq a_3 > 0, \lambda_i(D) \geq a_4 > 0$$

and

$$\lambda_i(D) > \frac{1}{4}\left[\lambda_i(\Psi(x,y,z,w))\right]^2, (i = 1, 2, \ldots, n),$$

for all $x, y, z, w \in R^n$.

Then the trivial solution $x = 0$ of equation (1.2) is unstable for arbitrary Φ provided that $\Phi(0) = 0$.

Remark 4. It should be noted that there exists no restriction on the vector field Φ in the system (1.4), except $J_{\Phi}(w)$ is symmetric.

Remark 5. Our second result, Theorem 2 includes the result of Ezeilo [3, Theorem 2] for the special case $n = 1$ in the system (1.4), and improves the same result to the vector case. Equation (1.2) is also completely different from those investigated in the relevant literature, (see also Ezeilo [2–4], Sadek [7], Skrapek [8] and Tunç [9–11]. Remark 1 is also valid for Theorem 2.

Proof. For the proof of Theorem 2, we define a Lyapunov function $V_1 = V_1(x,y,z,w)$ given by:

$$\begin{aligned} V_1 = &-\langle w, Cy\rangle + \frac{1}{2}\langle Cz, z\rangle - \langle Dx, w\rangle + \langle Dy, z\rangle \\ &- \int_0^1 \langle \Phi(\sigma w), w\rangle \, d\sigma. \end{aligned} \tag{2.4}$$

We observe that $V_1(0,0,0,0) = 0$. Indeed, in view of the assumptions of Theorem 2, we also have that

$$\begin{aligned} V_1(0,0,\varepsilon,0) &= \frac{1}{2}\langle C\varepsilon, \varepsilon\rangle \\ &\geq \frac{a_3}{2}\langle \varepsilon, \varepsilon\rangle \\ &= \frac{a_3}{2}\|\varepsilon\|^2 > 0 \end{aligned}$$

for all arbitrary $\varepsilon \neq 0$, $\varepsilon \in R^n$, which verifies the property (K_1) of Krasovskii [5]. Next, let $(x,y,z,w) = (x(t), y(t), z(t), w(t))$ be an arbitrary solution of the system

(1.4). A straightforward calculation from (2.4) and (1.4) gives that

$$\dot{V}_1 = \frac{d}{dt}V_1(x,y,z,w) = -\langle \dot{w}, Cy\rangle - \langle \dot{w}, Dx\rangle + \langle Dz, z\rangle - \frac{d}{dt}\int_0^1 \langle \Phi(\sigma w), w\rangle \, d\sigma. \tag{2.5}$$

Now, recall that

$$\frac{d}{dt}\int_0^1 \langle \Phi(\sigma w), w\rangle \, d\sigma = \int_0^1 \sigma \, \langle J_\Phi(\sigma w)\dot{w}, w\rangle \, d\sigma + \int_0^1 \langle \Phi(\sigma w), \dot{w}\rangle \, d\sigma.$$

But, J_Φ being symmetric,

$$\langle J_\Phi(\sigma w)\dot{w}, w\rangle = \langle \dot{w}, J_\Phi(\sigma w)w\rangle$$

so that

$$\begin{aligned}\int_0^1 \sigma \, \langle J_\Phi(\sigma w)\dot{w}, w\rangle \, d\sigma &= \int_0^1 \sigma \, \langle \dot{w}, J_\Phi(\sigma w)w\rangle \, d\sigma = \int_0^1 \sigma \frac{d}{d\sigma}\langle \dot{w}, \Phi(\sigma w)\rangle \, d\sigma \\ &= \sigma \, \langle \dot{w}, \Phi(\sigma w)\rangle \, \big|_0^1 - \int_0^1 \langle \dot{w}, \Phi(\sigma w)\rangle \, d\sigma \\ &= \langle \dot{w}, \Phi(w)\rangle - \int_0^1 \langle \dot{w}, \Phi(\sigma w)\rangle \, d\sigma.\end{aligned}$$

Hence,

$$\frac{d}{dt}\int_0^1 \langle \Phi(\sigma w), w\rangle \, d\sigma = \langle \dot{w}, \Phi(w)\rangle. \tag{2.6}$$

Substituting the estimate (2.6) into (2.5), and then considering the system (1.4) and the assumptions of Theorem 2, we obtain

$$\begin{aligned}\dot{V}_1 &= -\langle \dot{w}, \Phi(w)\rangle - \langle \dot{w}, Cy\rangle - \langle \dot{w}, Dx\rangle + \langle Dz, z\rangle \\ &= \langle \dot{w}, \dot{w}\rangle + \langle \dot{w}, \Psi(x,y,z,w)z\rangle + \langle Dz, z\rangle \\ &= \left\| \dot{w} + \frac{1}{2}\Psi(x,y,z,w)z \right\|^2 + \langle Dz, z\rangle \\ &\quad - \frac{1}{4}\langle \Psi(x,y,z,w)z, \, \Psi(x,y,z,w)z\rangle.\end{aligned}$$

Now, using the definition of $\dot{w}$ in (1.4), we get

$$\begin{aligned}\dot{V}_1 &= \left\| \dot{w} + \frac{1}{2}\Psi(x,y,z,w)z \right\|^2 + \langle Dz, z\rangle \\ &\quad - \frac{1}{4}\langle \Psi(x,y,z,w)z,\ \Psi(x,y,z,w)z\rangle \\ &= \left\| \Phi(w) + \frac{1}{2}\Psi(x,y,z,w)z + Cy + Dx \right\|^2 + \langle Dz, z\rangle \\ &\quad - \frac{1}{4}\langle \Psi(x,y,z,w)z,\ \Psi(x,y,z,w)z\rangle .\end{aligned} \tag{2.7}$$

Clearly, it follows from (2.7) that

$$\dot{V}_1 \geq \langle Dz, z\rangle - \frac{1}{4}\langle \Psi(x,y,z,w)z,\ \Psi(x,y,z,w)z\rangle .$$

As a result, the above discussion and (2.7) provide that the function $\dot{V}_1$ is positive semi-definite because of assumption $\lambda_i(D) > \frac{1}{4}[\lambda_i(\Psi(x,y,z,w))]^2$ of Theorem 2. This case verifies the property (K_2) of Krasovskii [5].

Finally, let $(x(t), y(t), z(t), w(t))$ be an arbitrary solution of the system (1.4). We observe from the foregoing estimate of $\dot{V}_1$ that $\dot{V}_1(t) = 0$ $(t \geq 0)$ necessarily implies that $z(t) = 0$ for all $t \geq 0$. In turn, this fact implies that

$$w(t) = \dot{z}(t) = 0 \text{ for all } t \geq 0.$$

Since $\Phi(0) = 0$ we have also

$$Cy(t) + Dx(t) = 0 \text{ for all } t \geq 0,$$

that is,

$$C\dot{x}(t) + Dx(t) = 0 \text{ for all } t \geq 0. \tag{2.8}$$

Because of $\ddot{x}(t) = 0$, $\dot{x}(t) = c \in R^n$ for all $t \geq 0$. Hence by (2.8), since $D \neq 0$, we have that $x(t) = c \in R^n$ for all $t \geq 0$. But this implies that $\dot{x}(t) = 0$ and thus also, by (2.8), that $x(t) = 0$ for all $t \geq 0$.

This completes the verification of the property of (K_3) of Krasovskii [5] for the function V_1.

Theorem 2 is thereby established. □

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REFERENCES

[1] R. Bellman, *Introduction to matrix analysis*, ser. Classics in Applied Mathematics. Philadelphia: Society for Industrial and Applied Mathematics (SIAM), 1997, vol. 19.

[2] J. O. C. Ezeilo, "An instability theorem for a certain fourth order differential equation," *Bull. London Math. Soc.*, vol. 10, no. 2, pp. 184–185, 1978.

[3] J. O. C. Ezeilo, "Further instability theorems for some fourth order differential equations," *J. Nigerian Math. Soc.*, vol. 19, pp. 1–7, 2000.

[4] J. O. C. Ezeilo, "Extension of certain instability theorems for some fourth and fifth order differential equations," *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)*, vol. 66, no. 4, pp. 239–242, 1979.

[5] N. N. Krasovskiĭ, "On conditions of inversion of a. m. lyapunov's theorems on instability for stationary systems of differential equations," *Dokl. Akad. Nauk SSSR (N.S.)*, vol. 101, pp. 17–20, 1955.

[6] A. M. Liapunov, *Stability of motion*, ser. Mathematics in Science and Engineering. New York-London: Academic Press, 1966, vol. 30.

[7] A. I. Sadek, "Instability results for certain systems of fourth and fifth order differential equations," *Appl. Math. Comput.*, vol. 145, no. 2-3, pp. 541–549, 2003.

[8] W. A. Skrapek, "Instability results for fourth-order differential equations," *Proc. Roy. Soc. Edinburgh Sect. A*, vol. 85, no. 3-4, pp. 247–250, 1980.

[9] C. Tunç, "An instability theorem for a certain vector differential equation of the fourth order," *JIPAM. J. Inequal. Pure Appl. Math.*, vol. 5, no. 1, p. 5, 2004.

[10] C. Tunç, "A further instability result for a certain vector differential equation of fourth order," *Int. J. Math. Game Theory Algebra*, vol. 15, no. 5, pp. 489–495, 2006.

[11] C. Tunç, "Instability of solutions for certain nonlinear vector differential equations of fourth order," *Nonlinear Oscil. (N. Y.)*, vol. 12, no. 1, pp. 123–132, 2009.

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LIPSCHITZ ESTIMATES FOR MULTILINEAR COMMUTATORS OF SINGULAR INTEGRAL OPERATORS ON SPACES OF HOMOGENEOUS TYPE

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Abstract. In this paper we discuss the multilinear commutators of singular integral operators on spaces of homogeneous type. By using Hölder's inequalities and the (L^p, L^q) type boundedness of fractional integral operators on spaces of homogeneous type, we obtain that this kind of multilinear commutators is bounded from L^p to L^q, from L^p to Triebel-Lizorkin spaces and from $H^p_{\vec{b}}$ to L^p. We also establish their $(H\dot{K}^{\alpha,p}_{q_1,\vec{b}}, \dot{K}^{\alpha,p}_{q_2})$ type boundedness on spaces of homogeneous type.

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1. INTRODUCTION

As a development in the theory of singular integral operators, their commutators have been extensively studied in recent years. The results are plentiful and substantial. For instance, let $b \in BMO(R^n)$ and let T be a Calderón-Zygmund operator. A classical result of Coifman, Rochberb and Weiss(see [9] states that the commutator $[b, T]$ is bounded on $L^p(R^n)$ $(1 < p < \infty)$. Chanillo (see [5]) proves a similiar result when T is replaced by fractional operators (see [7, 16]). The main purpose of this paper is to consider the continuity of the multilinear commutators associated with singular integral operators and Lipschitz functions in certain Hardy and Herz-Hardy spaces defined on spaces of homogeneous type. We also obtain the boundedness of the commutator $[b, T]$ generated by a singular integral operator T and a Lipschitz function b on Hardy and Herz type Hardy spaces. Not only that, we prove that this commutator is bounded from Hardy spaces to weak Lebesgue spaces and from Herz type Hardy spaces to weak Herz spaces.

First of all, let us introduce some definitions (see [1–6, 8, 14, 15]).

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Give a set X, a function $d : X \times X \to R^+$ is called a quasi-distance on X if the following conditions are satisfied:

(i) for every x and y in X, $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$,
(ii) for every x and y in X, $d(x,y) = d(y,x)$,
(iii) there exists a constant $k \geq 1$ such that

$$d(x,y) \leq k(d(x,z) + d(z,y)) \tag{1.1}$$

for every x, y and z in X.

Let μ be a positive measure on the σ-algebra of subsets of X which contains the r-balls $B(x,r) = \{y : d(x,y) < r\}$. We assume that μ satisfies a doubling condition, that is, there exists a constant $A_1 > 1$ such that

$$0 < \mu(B(x,2r)) \leq A_1\mu(B(x,r)) < \infty \tag{1.2}$$

holds for all $x \in X$ and $r > 0$.

A structure (X,d,μ), with d and μ as above, is called a space of homogeneous type(Coifman and Weiss type). The constants k and A_1 in (1.1) and (1.2) will be called the constants of the space.

From (1.2), we can say that there exists a constant $n_0 > 1$ such that $A_1 \leq 2^{n_0}$, in other words, there exists a constant $n_0 > 1$ such that $\mu(B(x,2r)) \leq 2^{n_0}\mu(B(x,r))$.

We say that (X,d,μ) satisfies a reverse doubling condition, that is, there exists a constant $A_2 > 1$ such that

$$0 < A_2\mu(B(x,r)) \leq \mu(B(x,2r)) < \infty \tag{1.3}$$

holds for all $x \in X$ and $r > 0$. It can be proved that, under some general additional geometric assumptions on the space (X,d), (1.3) is actually a consequence of the doubling condition on μ (see [16]).

Definition 1 (see [12])**.** Let $d > 0$ and $0 < \theta \leq 1$. A space of homogeneous type $(X,\rho,\mu)_{d,\theta}$ is a set X together with a quasi-metric ρ and a nonnegative Borel regular measure μ on X with supp $\mu = X$ and there exists a constant $C_0 > 1$ such that for all $0 < r < \mathrm{diam}X$ and for all $x, x', y \in X$,

$$\mu(B(x,r)) \sim r^d$$

and

$$|\rho(x,y) - \rho(x',y)| \leq C_0\rho(x,x')^\theta[\rho(x,y) + \rho(x',y)]^{1-\theta}.$$

In this paper, we will consider this particular class of spaces of homogeneous type which satisfies the definitions of Coifman and Weiss and we will use $(X,\rho,\mu)_{d,\theta}$, too. Moreover, in [13], Macias and Segovia have proved that in the sense of equivalent topology, a space of homogeneous type in the sense of Coifman and Weiss is a space of homogeneous type in the sense of Definition 1 with $d = 1$ and $\theta \in (0,1)$.

Definition 2. For $\beta > 0$, the Lipschitz space $Lip_\beta(X)$ is the space of functions such that

$$||f||_{Lip_\beta} = \sup_{\substack{x,y\in X \\ x\neq y}} \frac{|f(x)-f(y)|}{\rho(x,y)^\beta} < \infty.$$

where ρ is the quasi-distance.

Given a positive integer m and $1 \le i \le m$, we denote by C_i^m the family of all finite subsets $\sigma = \{\sigma(1),\dots,\sigma(i)\}$ of $\{1,\dots,m\}$ with i different elements. For $\sigma \in C_i^m$, set $\sigma^c = \{1,\dots,m\} \setminus \sigma$. Let $\vec{b} = (b_1,\dots,b_m)$ and $\sigma = \{\sigma(1),\dots,\sigma(i)\} \in C_i^m$, set $\vec{b}_\sigma = (b_{\sigma(1)},\dots,b_{\sigma(i)})$ and $b_\sigma = b_{\sigma(1)}\dots b_{\sigma(i)}$. With this notation, if $\beta_{\sigma(1)} + \dots + \beta_{\sigma(i)} = \beta_\sigma$, we write

$$||\vec{b}_\sigma||_{Lip_{\beta_\sigma}} = ||b_{\sigma(1)}||_{Lip_{\beta_{\sigma(1)}}} \dots ||b_{\sigma(i)}||_{Lip_{\beta_{\sigma(i)}}}.$$

For the product of all the functions, we simply write

$$||\vec{b}_\sigma||_{Lip_\beta} = \prod_{i=1}^m ||b_i||_{Lip_{\beta_i}},$$

where $\sum_{i=1}^m \beta_i = \beta$.

Definition 3. Let T be the singular integral operator

$$T(f)(x) = \int_X K(x,y) f(y) d\mu(y),$$

where K is a locally integrable function on $X \times X \setminus \{(x,y)|\ x = y\}$ and satisfies the following properties:

(1) $|K(x,y)| \le \dfrac{C}{\mu(B(x,d(x,y)))}$,

(2) there exists a $p_0, 1 < p_0 \le \infty$, such that T is bounded on $L^{p_0}(X)$,

(3) $|K(x,y) - K(x,y')| + |K(y,x) - K(y',x)| \le C \dfrac{d(y,y')^\delta}{\mu(B(y,d(x,y)))d(x,y)^\delta}$, when $d(x,y) \ge 2d(y,y')$, with some $\delta \in (0,1]$.

Similary to the definitions of other commutators, we introduce the definition of the commutator generated by a singular integral operator of Definition 3 and by a Lipschitz function as follows.

Definition 4. Let $\vec{b} = (b_1,\dots,b_m)$ be a finite family of locally integrable functions with $b_i \in \wedge_\beta(X), 1 \le i \le m$ and lat T be the singular integral operator. then we define the multilinear commutator as follows:

$$T_{\vec{b}}(f)(x) = \int_X \prod_{i=1}^m (b_i(x) - b_i(y)) K(x,y) f(y) d\mu(y).$$

Note that when $b_1 = \cdots = b_m$, $T_{\vec{b}}$ is just the m^{th} order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1–4, 6, 8]).
We prove the boundedness ofthe commutator $T_{\vec{b}}$ on Lebesgue spaces in the following section.

2. Boundedness from $L^p(X)$ to $L^q(X)$

In this section, we obtain the following results.

Theorem 1. *Let $b_i \in Lip_{\beta_i}(X)$, $1 \le i \le m$, $\sum_{i=1}^m \beta_i = \beta$, $0 < \beta < 1$ and $1/q = 1/p - \beta$. Then $T_{\vec{b}}$ is bounded from $L^p(X)$ to $L^q(X)$.*

Theorem 2. *Let $b_i \in Lip_{\beta_i}(X)$, $1 \le i \le m$, $\sum_{i=1}^m \beta_i = \beta$, $0 < \beta < 1$ and $1/q = 1/p - \beta$. Then $T_{\vec{b}}$ is bounded from $L^1(X)$ to weak $L^{1/(1-\beta)}(X)$.*

To prove the above results, we use the following lemmas.

Lemma 1 (see [10])**.** *Let $0 < \beta < 1$, $f \in L^p(X)$ and $1/q = 1/p - \beta$. Then I_β is bounded from $L^p(X)$ to $L^q(X)$.*

Lemma 2 (see [13])**.** *Let $0 < \beta < 1$, $f \in L^p(X)$ and $1/q = 1/p - \beta$. Then I_β is bounded from $L^1(X)$ to weak $L^{1/(1-\beta)}(X)$.*

Here I_β is the fractional integral operator.

Proof of Theorem 1 and 2. By Definition 4 and Definition 2, for any $x \in X$, we have

$$\begin{aligned}
|T_{\vec{b}}(f)(x)| &\le \int_X \prod_{i=1}^m |(b_i(x) - b_i(y))||K(x,y)||f(y)|d\mu(y) \\
&\le \int_X \prod_{i=1}^m \left(\sup_{x \ne y} \frac{|b_i(x) - b_i(y)|}{d(x,y)^{\beta_i}} d(x,y)^{\beta_i} \right) |K(x,y)||f(y)|d\mu(y) \\
&\le ||\vec{b}||_{Lip_\beta} \int_X d(x,y)^\beta \frac{|f(y)|}{\mu(B(x,d(x,y)))} d\mu(y) \\
&\le ||\vec{b}||_{Lip_\beta} \int_X \frac{|f(y)|}{\mu(B(x,d(x,y)))^{1-\beta}} d\mu(y).
\end{aligned}$$

So, by Lemma 1 and Lemma 2,

$$||T_{\vec{b}}(f)||_{L^q} \le C||\vec{b}||_{Lip_\beta} ||I_\beta(|f|)||_{L^q} \le C||\vec{b}||_{Lip_\beta} ||f||_{L^p},$$

and

$$\mu\left(\left\{x \in X : |T_{\vec{b}}(f)(X)| > \lambda\right\}\right) \le \mu\left(\left\{x \in X : |I_\beta(|f|)(x)| > \frac{\lambda}{C||\vec{b}||_{Lip_\beta}}\right\}\right)$$

$$\leq C\left(||\vec{b}||_{Lip_\beta}\frac{||f||_{L^1}}{\lambda}\right)^{1/1-\beta} \leq C\left(\frac{||f||_{L^1}}{\lambda}\right)^{1/1-\beta}.$$

Thus we have finished the proof of the theorems. □

3. Boundedness from $L^p(X)$ to Triebel-Lizorkin Spaces

In this section, we are going to show that the multilinear commutator of a singular integral operator is bounded from $L^p(X)$ to Triebel-Lizorkin spaces.

Theorem 3. *Let $b_i \in Lip_{\beta_i}(X)$, $1 \leq i \leq m$, $\sum_{i=1}^m \beta_i = \beta$, $0 < \beta \leq \delta/n_0$ and $1/q = 1/p - \beta$. Then $T_{\vec{b}}$ is bounded from $L^p(X)$ to $F_p^{\beta,\infty}(X)$.*

To prove above theorems, we use some Lemmas.

Lemma 3 (see [13]). *Let $0 < \beta < 1$, $1 < p < \infty$, we have*

$$\begin{aligned}||f||_{F_p^{\beta,\infty}} &= ||\sup_B \frac{1}{\mu(B)^{1+\beta}}\int_B |f(x) - f_B|d\mu(x)||_{L^p} \\ &= ||\sup_B \inf_c \frac{1}{\mu(B)^{1+\beta}}\int_B |f(x) - c|d\mu(x)||_{L^p}\end{aligned}$$

Lemma 4 (see [13]). *Let $0 < \beta < 1$, $1 \leq p < \infty$, we have*

$$\begin{aligned}||f||_{Lip_\beta} &= \sup_B \frac{1}{\mu(B)^{1+\beta}}\int_B |f(x) - f_B|d\mu(x) \\ &= \sup_B \mu(B)^{-\beta}\left[\frac{1}{\mu(B)}\int_B |f(x) - f_B|^p d\mu(x)\right]^{1/p}.\end{aligned}$$

Lemma 5 (see [13]). *Let balls B_1, B_2 with $B_1 \subset B_2$. Then $|f_{B_1} - f_{B_2}| \leq C||f||_{Lip_\beta}\mu(B_2)^\beta$.*

Proof of Theorem 3. Fix a ball $B = B(x_0, r)$ and $\tilde{x} \in B$. Set $\vec{b}_B = ((b_1)_B, \ldots, (b_m)_B)$, where $(b_i)_B = \mu(B)^{-1}\int_B b_i(y)d\mu(y)$, $1 \leq i \leq m$. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{(2B)^c}$, then we have

$$\begin{aligned}T_{\vec{b}}(f)(x) &= \int_X \prod_{i=1}^m (b_i(x) - b_i(y))K(x,y)f(y)d\mu(y) \\ &= \int_X \prod_{i=1}^m ((b_i(x) - (b_i)_B) + ((b_i)_B - b_i(y)))K(x,y)f(y)d\mu(y) \\ &\leq C\sum_{i=0}^m \sum_{\sigma \in C_i^m} (-1)^{m-i}(\vec{b}(x) - \vec{b}_B)_\sigma \int_X (\vec{b}(y) - \vec{b}_B)_{\sigma^c} K(x,y)f(y)d\mu(y)\end{aligned}$$

$$\le C\prod_{i=1}^{m}(b_i(x)-(b_i)_B)T(f)(x)$$
$$+\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}(-1)^{m-i}(\vec{b}(x)-\vec{b}_B)_\sigma T((\vec{b}-\vec{b}_B)_{\sigma^c}f)(x)$$
$$+(-1)^m T(\prod_{i=1}^{m}(b_i-(b_i)_B)f)(x).$$

So, let $c=T(\prod_{i=1}^{m}(b_i-(b_i)_B)f_2)(x_0)$. It is easy to see now that

$$\frac{1}{\mu(B)^{1+\beta}}\int_B|T_{\vec{b}}(f)(x)-c|d\mu(x)$$
$$\le\frac{1}{\mu(B)^{1+\beta}}\int_B|T_{\vec{b}}(f)(x)-T(\prod_{i=1}^{m}(b_i-(b_i)_B)f_2)(x_0)|d\mu(x)$$
$$\le\frac{C}{\mu(B)^{1+\beta}}\int_B|\prod_{i=1}^{m}(b_i(x)-(b_i)_B)T(f)(x)|d\mu(x)$$
$$+\frac{C}{\mu(B)^{1+\beta}}\int_B|\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}(\vec{b}(x)-\vec{b}_B)_\sigma T((\vec{b}-\vec{b}_B)_{\sigma^c}f)(x)|d\mu(x)$$
$$+\frac{C}{\mu(B)^{1+\beta}}\int_B|T(\prod_{i=1}^{m}(b_i-(b_i)_B)f_1)(x)|d\mu(x)$$
$$+\frac{C}{\mu(B)^{1+\beta}}\int_B|T(\prod_{i=1}^{m}(b_i(x)-(b_i)_B)f_2)(x)-T(\prod_{i=1}^{m}(b_i-(b_i)_B)f_2)(x_0)|d\mu(x)$$
$$=I+II+III+IV.$$

For the term I, by Lemma 4 and by Hölder's inequalities with exponent $1/q_1+\cdots+1/q_m+1/q=1, q>1, q_i>1(i=1,\ldots,m)$, we have

$$I\le\frac{C}{\mu(B)^{1+\beta}}\prod_{i=1}^{m}(\int_B|b_i(x)-(b_i)_B|^{q_i}d\mu(x))^{1/q_i}(\int_B|T(f)(x)|^q d\mu(x))^{1/q}$$
$$\le C\|\vec{b}\|_{Lip_\beta}M_q(|T(f)|)(\tilde{x})$$

For the term II, by Lemma 4, by Hölder's inequalities with exponent $1/s+1/s'=1$, $1/q+1/q'=1$ and by the $L^p(1<p<\infty)$ boundedness of T, we have

$$II\le\frac{C}{\mu(B)^{1+\beta}}\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}\int_B|(\vec{b}(x)-\vec{b}_B)_\sigma T((\vec{b}-\vec{b}_B)_{\sigma^c}f)(x)|d\mu(x)$$

$$\leq \frac{C}{\mu(B)^{1+\beta}} \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} \left(\int_B |(\vec{b}(x) - \vec{b}_B)_\sigma|^{q'} d\mu(x) \right)^{1/q'}$$

$$\times \left(\int_B |T((\vec{b} - \vec{b}_B)_{\sigma^c} f)(x)|^q d\mu(x) \right)^{1/q}$$

$$\leq C \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} \mu(B)^{-\beta + \beta_{\sigma(1)} + \cdots + \beta_{\sigma(i)}} ||\vec{b}_\sigma||_{Lip_\beta}$$

$$\times \left(\frac{1}{\mu(B)} \int_B |(\vec{b}(x) - \vec{b}_B)_{\sigma^c} f(x)|^q d\mu(x) \right)^{1/q}$$

$$\leq C \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} \mu(B)^{-\beta + \beta_{\sigma(1)} + \cdots + \beta_{\sigma(i)}} ||\vec{b}_\sigma||_{Lip_\beta}$$

$$\times \left(\frac{1}{\mu(B)} \int_B |(\vec{b}(x) - \vec{b}_B)_{\sigma^c}|^{s'q} d\mu(x) \right)^{1/s'q} \left(\frac{1}{\mu(B)} \int_B |f(x)|^{sq} d\mu(x) \right)^{1/sq}$$

$$\leq C ||\vec{b}||_{Lip_\beta} M_{sq}(f)(\tilde{x}).$$

For the term III, by the $L^p (1 < p < \infty)$ boundedness of T and by Hölder's inequalities with exponent $1/s_1 + \cdots + 1/s_m + 1/s = 1$, $s_i > 1, i = 1, \ldots, m$, $s > 1$, we have

$$III \leq \frac{1}{\mu(B)^{1+\beta}} \int_B |T(\prod_{i=1}^m (b_i - (b_i)_B) f_1)(x)| d\mu(x)$$

$$\leq C\mu(B)^{-\beta} \left(\frac{1}{\mu(B)} \int_X |T(\prod_{i=1}^m (b_i - (b_i)_B) f_1)(x)|^q \chi_B d\mu(x) \right)^{1/q}$$

$$\leq C\mu(B)^{-\beta} \left(\frac{1}{\mu(B)} \int_B |\prod_{i=1}^m (b_i(x) - (b_i)_B) f(x)|^q d\mu(x) \right)^{1/q}$$

$$\leq C \prod_{i=1}^m \mu(B)^{-\beta_i} \left(\frac{1}{\mu(B)} \int_B |(b_i(x) - (b_i)_B)|^{s_i q} d\mu(x) \right)^{1/s_i q}$$

$$\times \left(\frac{1}{\mu(B)} \int_B |f(x)|^{sq} d\mu(x) \right)^{1/sq}$$

$$\leq C ||\vec{b}||_{Lip_\beta} M_{sq}(f)(\tilde{x}).$$

For the term IV, by Hölder's inequalities with exponent $1/q' + 1/q = 1$, we get

$$\int_X \prod_{i=1}^m |(b_i(y) - (b_i)_B)| |K(x, y) - K(x_0, y)| |f_2(y)| d\mu(y)$$

$$\leq \int_{(2B)^c} \prod_{i=1}^{m} |(b_i(y)-(b_i)_B)| \frac{d(x,x_0)^\delta}{d(x_0,y)^\delta \mu(B(x_0,d(x_0,y)))} |f(y)| d\mu(y)$$

$$\leq C \sum_{k=1}^{\infty} \frac{d(x,x_0)^\delta}{(2^k r)^\delta \mu(2^k B)} \int_{2^{k+1}B} \prod_{i=1}^{m} |(b_i(y)-(b_i)_B)||f(y)| d\mu(y)$$

$$\leq C \sum_{k=1}^{\infty} \frac{d(x,x_0)^\delta}{(2^k r)^\delta \mu(2^k B)} \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} |(\vec{b}_{2^{k+1}B} - \vec{b}_B)_{\sigma^c}|$$

$$\times \int_{2^{k+1}B} |(\vec{b}(y) - \vec{b}_{2^{k+1}B})_\sigma| |f(y)| d\mu(y)$$

$$\leq C \sum_{k=1}^{\infty} \frac{d(x,x_0)^\delta}{(2^k r)^\delta \mu(2^k B)} \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} |(\vec{b}_{2^{k+1}B} - \vec{b}_B)_{\sigma^c}|$$

$$\times \Big(\int_{2^{k+1}B} |(\vec{b}(y) - \vec{b}_{2^{k+1}B})_\sigma|^{q'} d\mu(y)\Big)^{1/q'} \Big(\int_{2^{k+1}B} |f(y)|^q d\mu(y)\Big)^{1/q}$$

$$\leq C \sum_{k=1}^{\infty} \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} \frac{d(x,x_0)^\delta}{(2^k r)^\delta \mu(2^k B)}$$

$$\times ||\vec{b}_{\sigma^c}||_{Lip_\beta} \mu(2^{k+1}B)^{\beta_{\sigma^c}} \mu(2^{k+1}B)^{\beta_\sigma} ||\vec{b}_\sigma||_{\wedge_\beta} M_q(|f|)(\tilde{x}) \mu(2^{k+1}B)$$

$$\leq C ||\vec{b}||_{Lip_\beta} M_q(f)(\tilde{x}) \sum_{k=1}^{\infty} \frac{d(x,x_0)^\delta}{(2^k r)^\delta \mu(2^k B)} \mu(2^{k+1}B)^{1+\beta},$$

so,

$$IV \leq C ||\vec{b}||_{Lip_\beta} M_q(f)(\tilde{x}) \frac{1}{\mu(B)^{1+\beta}} \sum_{k=1}^{\infty} \mu(2^{k+1}B)^{1+\beta} \int_B \frac{d(x,x_0)^\delta}{(2^k r)^\delta \mu(2^k B)} d\mu(x)$$

$$\leq C ||\vec{b}||_{Lip_\beta} M_q(f)(\tilde{x}) \sum_{k=1}^{\infty} 2^{-k\delta} \mu(B)^{-\beta} \mu(2^k B)^\beta$$

$$\leq C ||\vec{b}||_{Lip_\beta} M_q(f)(\tilde{x}) \sum_{k=1}^{\infty} 2^{k n_0 (\beta - \delta/n_0)}$$

$$\leq C ||\vec{b}||_{Lip_\beta} M_q(f)(\tilde{x}).$$

We put these estimates together, by using Lemma 3 and taking the supremum over all B such that $\tilde{x} \in B$, we obtain

$$||f||_{F_p^{\beta,\infty}} \leq C ||\vec{b}||_{Lip_\beta} (||M_q(T(f))||_{L^p} + ||M_{qs}(f)||_{L^p} + ||M_q(f)||_{L^p})$$

$$\leq C ||\vec{b}||_{Lip_\beta} ||f||_{L^p}$$

Thus we have finished the proof of the theorem. □

4. Boundedness on Hardy Spaces

In this section, we discuss the boundedness of the multiliner commutator $T_{\vec{b}}$ generated by the singular integral operator T and the Lipschitz function $\vec{b}$ on Hardy spaces.

First, let us review the atomic decomposition of Hardy spaces, which is important in the study the boundedness of operators.

Definition 5. Let $b_i (i = 1, \cdots, m)$ be locally integrable functions and $0 < p \leq 1$. A bounded measurable function $a(x)$ on X is called a $(p, \vec{b})$ atom, if

(1) $supp\ a \subset B = B(x_0, r)$,
(2) $||a||_{L^\infty} \leq \mu(B)^{-1/p}$,
(3) $\int_B a(y)d\mu(y) = \int_B a(y) \prod_{l \in \sigma} b_l(y)d\mu(y) = 0$ for any $\sigma \in C_j^m, 1 \leq j \leq m$.

A tempered distribution f is said to belong to the atomic space $H_{\vec{b}}^p(X)$, if, in the Schwartz distribution sense, it can be written as $f(x) = \sum_i \lambda_i a_i(x)$, where a_i are $(p, \vec{b})$ atoms, $\lambda_i \in C$ and $\sum_i |\lambda_i|^p < \infty$. Moreover, $||f||_{H_{\vec{b}}^p} = \inf(\sum_i |\lambda_i|^p)^{1/p}$, where the infimum are taken over all of the above decompositions of f.

Theorem 4. *Let $b_i \in Lip_{\beta_i}(X)$, $1 \leq i \leq m$, $\sum_{i=1}^m \beta_i = \beta$, $0 < \beta \leq 1$, $0 < p \leq 1$ and $1/q = 1/p - \beta$. Then $T_{\vec{b}}$ is bounded from $H_{\vec{b}}^p(X)$ to $L^q(X)$.*

Theorem 5. *Let $b_i \in Lip_{\beta_i}(X)$, $1 \leq i \leq m$, $\sum_{i=1}^m \beta_i = \beta$, $0 < \beta \leq \delta/n_0$ and $0 < p \leq 1$. Then $T_{\vec{b}}$ is bounded from $H_{\vec{b}}^{1/(1+\beta)}(X)$ to weak $L^1(X)$.*

Proof of Theorem 4. By a standard argument, it suffices to show that there exists a constants $C > 0$ such that $||T_{\vec{b}}(a)||_{L^q} \leq C$ for each $(p, \vec{b})$ atom a.

$$||T_{\vec{b}}(a)||_{L^q} \leq (\int_{2B} |T_{\vec{b}}(a)(x)|^q d\mu(x))^{1/q} + (\int_{(2B)^c} |T_{\vec{b}}(a)(x)|^q d\mu(x))^{1/q}$$
$$= I + II$$

For the term I, taking $q > 1$, by Hölder's inequality and by Theorem 1, we get

$$I \leq C||T_{\vec{b}}(a)||_{L^q} \leq C||\vec{b}||_{Lip_\beta}||a||_{L^p} \leq C||\vec{b}||_{Lip_\beta}||a||_{L^\infty} \cdot \mu(B)^{1/p} \leq C.$$

For the term II, for $y \in B, x \in (2B)^c$, by the vanishing of the moment of a and by Hölder's inequalities, we get

$$|T_{\vec{b}}(a)(x)| = \int_B \prod_{i=1}^m |(b_i(x) - b_i(y))||K(x,y)||a(y)|d\mu(y)$$

$$\leq \int_B \prod_{i=1}^m |(b_i(x)-b_i(x_0))+(b_i(x_0)-b_i(y))| \\ |K(x,y)-K(x,x_0)||a(y)|d\mu(y)$$

$$\leq \prod_{i=1}^m |b_i(x)-b_i(x_0)| \int_B |K(x,y)-K(x,x_0)||a(y)|d\mu(y)$$

$$+ \sum_{i=1}^{m-1} \sum_{\sigma\in C_i^m} |(\vec{b}(x)-\vec{b}(x_0))_\sigma| \\ \int_B |(\vec{b}(y)-\vec{b}(x_0))_{\sigma^c}||K(x,y)-K(x,x_0)||a(y)|d\mu(y)$$

$$+ \int_B \prod_{i=1}^m |b_i(y)-b_i(x_0)||K(x,y)-K(x,x_0)||a(y)|d\mu(y)$$

$$\leq C\frac{r^\delta}{d(x,x_0)^\delta \mu(B(x_0,d(x,x_0)))}\cdot\{||\vec{b}||_{Lip_\beta}\rho(x,x_0)^\beta||a||_{L^\infty}\mu(B)$$

$$+ \sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m} ||\vec{b}_\sigma||_{Lip_\beta}\rho(x,x_0)^{\beta_\sigma}\int_B ||\vec{b}_{\sigma_c}||_{Lip_\beta}\rho(y,x_0)^{\beta_{\sigma^c}}|a(y)|d\mu(y)$$

$$+ \int_B ||\vec{b}||_{Lip_\beta}\rho(y,x_0)^\beta|a(y)|d\mu(y)\}$$

$$\leq C\frac{r^\delta}{d(x,x_0)^\delta\mu(B(x_0,d(x,x_0)))}||\vec{b}||_{Lip_\beta}\rho(x,x_0)^\beta||a||_{L^\infty}\mu(B)$$

$$\leq C\frac{r^\delta}{d(x,x_0)^\delta\mu(B(x_0,d(x,x_0)))}||\vec{b}||_{Lip_\beta}\rho(x,x_0)^\beta\mu(B)^{1/p'},$$

so, we have

$$II^q = \int_{(2B)^c} |T_{\vec{b}}(a)(x)|^q d\mu(x)$$

$$\leq C\int_{(2B)^c}\frac{r^{\delta q}}{d(x,x_0)^{\delta q}\mu(B(x_0,d(x,x_0)))^q}||\vec{b}||^q_{Lip_\beta}\rho(x,x_0)^{\beta q}\mu(B)^{q/p'}d\mu(x)$$

$$\leq C||\vec{b}||^q_{Lip_\beta}\sum_{k=1}^\infty\int_{2^{k+1}B}\frac{r^{\delta q}}{(2^k r)^{\delta q}\mu(2^k B)^q}\mu(2^k B)^{\beta q}\mu(B)^{q/p'}d\mu(x)$$

$$\leq C||\vec{b}||^q_{Lip_\beta}\sum_{k=1}^{\infty}2^{-k\delta q}\mu(B)^{q/p'}\mu(2^kB)^{q(-1+1/p)}$$
$$\leq C||\vec{b}||^q_{Lip_\beta}.$$

Thus, we have finished the proof of the theorem. □

Proof of Theorem 5. By the atomic decomposition, we know that there exist $(1/(1+\beta),\vec{b})-$atom a_j, supp $a_j \in B_j = B(x_j,r_j)$, so, we obtain that

$$||T_{\vec{b}}(f)||_{L^1} \leq \int_{2B_j} T_{\vec{b}}(f)(x)d\mu(x) + \int_{(2B)^c} T_{\vec{b}}(f)(x)d\mu(x)$$
$$= M_1 + M_2.$$

For M_1, by the Minkowski and Hölder inequalities, by the (L^p,L^p)-boundedness of T and by Definition 1, we have

$$|M_1| = C\sum_{j=-\infty}^{\infty}|\lambda_j|\int_{2B_j}|T_{\vec{b}}(a_j)(x)|d\mu(x)$$
$$\leq C\sum_{j=-\infty}^{\infty}|\lambda_j|\int_{2B_j}\int_{B_j}\prod_{i=1}^{m}|b_i(x)-b_i(y)||K(x,y)||a_j(y)|d\mu(y)d\mu(x)$$
$$\leq C\sum_{j=-\infty}^{\infty}|\lambda_j|\int_{2B_j}\int_{B_j}||\vec{b}||_{\wedge_\beta}d(x,y)^\beta|K(x,y)||a_j(y)|d\mu(y)d\mu(x)$$
$$\leq C||\vec{b}||_{Lip_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|\int_{2B_j}d(x,x_0)^\beta|T(a_j)(x)|d\mu(x)$$
$$\leq C||\vec{b}||_{Lip_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|{r_j}^\beta||T(a_j)||_{L^p}\mu(2B_j)^{1/p'}$$
$$\leq C||\vec{b}||_{Lip_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|{r_j}^\beta||a_j||_{L^p}\mu(2B_j)^{1/p'}$$
$$\leq C||\vec{b}||_{Lip_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|.$$

For M_2, similary to case II in Theorem 4 and by Definition 5, we have

$$T_{\vec{b}}(f)(x) \leq C\sum_{j=-\infty}^{\infty}\lambda_j\prod_{i=1}^{m}|b_i(x)-b_i(x_0)|\int_{B_j}|K(x,y)-K(x,x_0)||a_j(y)|d\mu(y)$$

$$+\sum_{j=-\infty}^{\infty}|\lambda_j|\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b}(x_0))_\sigma|$$
$$\int_{B_j}|(\vec{b}(y)-\vec{b}(x_0))_{\sigma^c}||K(x,y)-K(x,x_0)||a_j(y)|d\mu(y)$$
$$+\sum_{j=-\infty}^{\infty}|\lambda_j|\prod_{i=1}^{m}\int_{B_j}|b_i(y)-b_i(x_0)||K(x,y)-K(x,x_0)||a_j(y)|d\mu(y)$$

$$\le C\sum_{j=-\infty}^{\infty}|\lambda_j||\vec{b}||_{Lip_\beta}\frac{{r_j}^\delta}{d(x,x_0)^\delta\mu(B(x_0,d(x,x_0)))}\rho(x,x_0)^\beta||a_j||_{L^\infty}\mu(B_j)$$
$$\le C\sum_{j=-\infty}^{\infty}|\lambda_j||\vec{b}||_{Lip_\beta}\frac{{r_j}^\delta}{d(x,x_0)^\delta\mu(B(x_0,d(x,x_0)))}\rho(x,x_0)^\beta\mu(B_j)^{-\beta},$$

so,

$$\begin{aligned}
M_2 &\le C\sum_{j=-\infty}^{\infty}|\lambda_j|\mu(B_j)^{-\beta}\\
&\qquad\sum_{k=1}^{\infty}\int_{2^{k+1}B_j\setminus 2^kB_j}\frac{{r_j}^\delta}{d(x,x_0)^\delta\mu(B(x_0,d(x,x_0)))}\rho(x,x_0)^\beta d\mu(x)\\
&\le C\sum_{j=-\infty}^{\infty}|\lambda_j|\mu(B_j)^{-\beta}\sum_{k=1}^{\infty}\frac{{r_j}^\delta}{(2^kr_j)^\delta\mu(2^kB_j)}\mu(2^{k+1}B_j)^{1+\beta}\\
&\le C\sum_{j=-\infty}^{\infty}|\lambda_j|\mu(B_j)^{-\beta}\sum_{k=1}^{\infty}2^{-k\delta}\mu(2^kB_j)^\beta\\
&\le C\sum_{j=-\infty}^{\infty}|\lambda_j|\sum_{k=1}^{\infty}2^{k(-\delta+n_0\beta)}\\
&\le C\sum_{j=-\infty}^{\infty}|\lambda_j|.
\end{aligned}$$

Therefore, by the Jensen inequality, we have

$$\begin{aligned}
\mu(x\in X:|T_{\vec{b}}(f)|>\lambda) &\le C\lambda^{-1}||T_{\vec{b}}(f)||_{L^1}\\
&\le C\lambda^{-1}\sum_{j=-\infty}^{\infty}|\lambda_j|
\end{aligned}$$

$$\leq C\lambda^{-1}\Big(\sum_{j=-\infty}^{\infty}|\lambda_j|^{1/(1+\beta)}\Big)^{(1+\beta)}.$$

Taking infinum over all the central atomic decomposition completes the proof of the theorem. □

5. Boundedness on Herz type Hardy Spaces

First, we introduce the definition of Herz type Hardy space and its atomic decomposition characteristic as follows.

Definition 6. Let b_i $(i = 1, \cdots, m)$ be locally integrable functions, $1 < q < \infty$, $\alpha \geq 1 - 1/q$. A function a is called a central $(\alpha, q, \vec{b})$-atom (or a central $(\alpha, q, \vec{b})$-atom of restricted type), if

(1) $\operatorname{supp} a \subset B = B(x_0, r)$(or for some $r \geq 1$),
(2) $||a||_{L^q} \leq \mu(B)^{-\alpha}$,
(3) $\int_B a(x)d\mu(x) = \int_B a(x)\prod_{i\in\sigma} b_i(x)d\mu(x) = 0$ for any $\sigma \in C_i^m$, $1 \leq i \leq m$.

Definition 7. Let $\alpha \in R$, $0 < p < \infty$ and $1 \leq q < \infty$. For $k \in \mathbf{Z}$ and $x_0 \in X$, set $B_k = \{x \in X : d(x_0, x) \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and by χ_0 the characteristic function of B_0.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(X) = \Big\{f \in L^q_{loc}(X \setminus \{x_0\}) : ||f||_{\dot{K}_q^{\alpha,p}} < \infty\Big\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left(\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}||f\chi_k||^p_{L^q}\right)^{1/p}.$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(X) = \Big\{f \in L^q_{loc}(X) : ||f||_{K_q^{\alpha,p}} < \infty\Big\},$$

where

$$||f||_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty}\mu(B_k)^{\alpha p}||f\chi_k||^p_{L^q} + ||f\chi_0||^p_{L^q}\right]^{1/p}.$$

A function f is said to belong to the homogeneous $H\dot{K}^{\alpha,p}_{q,\vec{b}}(X)$ (or to the nonhomogeneous $HK^{\alpha,p}_{q,\vec{b}}(X)$), if it can be written as $f = \sum_{j=-\infty}^{\infty}\lambda_j a_j(x)$ (or $f =$

$\sum_{j=0}^{\infty}\lambda_j a_j(x)$), where a_j is a central $(\alpha,q,\vec{b})$-atom (or a central $(\alpha,q,\vec{b})$-atom of restricted type) supported on $B(x_0,2^j)$ and $\sum_{j=-\infty}^{\infty}|\lambda_j|^p<\infty$(or $\sum_{j=0}^{\infty}|\lambda_j|^p<\infty$). Moreover,

$$||f||_{H\dot{K}^{\alpha,p}_{q,\vec{b}}}=\inf(\sum_{j=-\infty}^{\infty}|\lambda_j|^p)^{1/p}(\text{or}\inf(\sum_{j=0}^{\infty}|\lambda_j|^p)^{1/p}),$$

where the infimum are taken over all of the above decompositions of f.

Definition 8 (see [11]). Let $\alpha\in\mathbf{R}$, $0<p<\infty$ and $1\le q<\infty$(B_k and C_k are the same as Definition 6).

(1) A measure function f is said to belong to the homogeneous weak Herz space $W\dot{K}^{\alpha,p}_q(X)$, if

$$||f||_{W\dot{K}^{\alpha,p}_q}=\sup_{\lambda>0}\lambda(\sum_{k=-\infty}^{+\infty}\mu(B_k)^{\alpha p}\mu(\{x\in C_k:|f(x)|>\lambda\})^{p/q})^{1/p}<\infty;$$

(2) A measure function f is said to belong to the inhomogeneous weak Herz space $WK^{\alpha,p}_q(X)$, if

$$||f||_{WK^{\alpha,p}_q}=\sup_{\lambda>0}\lambda(\sum_{k=1}^{+\infty}\mu(B_k)^{\alpha p}\mu(\{x\in C_k:|f(x)|>\lambda\})^{p/q}+\mu(\{x\in B_0:|f(x)|>\lambda\})^{p/q})^{1/p}<\infty.$$

Now we state our theorems.

Theorem 6. *Let $b_i\in Lip_{\beta_i}(X)$, $1\le i\le m$, $\sum_{i=1}^m\beta_i=\beta$, $0<\beta\le 1$, $1/q_2=1/q_1-\beta$, $0<p<\infty$ and $1-1/q_1\le\alpha<1-1/q_1+\delta/n_0$. Then the multilinear commutator $T_{\vec{b}}$ is bounded from $H\dot{K}^{\alpha,p}_{q_1,\vec{b}}(X)$ to $\dot{K}^{\alpha,p}_{q_2}(X)$.*

Theorem 7. *Let $b_i\in Lip_{\beta_i}(X)$, $1\le i\le m$, $\sum_{i=1}^m\beta_i=\beta$, $0<\beta\le\delta/n_0$, $1/q_2=1/q_1-\beta$ and let $0<p\le 1$. Then the multilinear commutator $T_{\vec{b}}$ is bounded from $H\dot{K}^{\alpha,p}_{q_1}(X)$ to $W\dot{K}^{\alpha,p}_{q_2}(X)$.*

Proof of Theorem 6. Let $f\in H\dot{K}^{\alpha,p}_{q_1,\vec{b}}(X)$ and let $f(x)=\sum_{j=-\infty}^{\infty}\lambda_j a_j(x)$ be the atomic decomposition of f as in Definition 7 We write

$$||T_{\vec{b}}(f)(x)||_{\dot{K}^{\alpha,p}_{q_2}}=\left(\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}||T_{\vec{b}}(f)\chi_k||^p_{L^{q_2}}\right)^{1/p}$$

$$\le C\left[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|||T_{\vec{b}}(a_j)\chi_k||_{L^{q_2}}\right)^p\right]^{1/p}$$

$$\leq C\left[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}\Big(\sum_{j=-\infty}^{k-3}|\lambda_j|||T_{\vec{b}}(a_j)\chi_k||_{L^{q_2}}\Big)^p\right]^{1/p}$$
$$+C\left[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}\Big(\sum_{j=k-2}^{\infty}|\lambda_j|||T_{\vec{b}}(a_j)\chi_k||_{L^{q_2}}\Big)^p\right]^{1/p}$$
$$=J+JJ.$$

For the term JJ, by Hölder's inequality and as Theorem 1 states that $T_{\vec{b}}$ is bounded from $L^{q_1}(X)$ to $L^{q_2}(X)$, we have

$$JJ\leq C\left[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|||T_{\vec{b}}(a_j)\chi_k||_{L^{q_2}}\right)^p\right]^{1/p}$$
$$\leq C||\vec{b}||_{Lip_\beta}$$
$$\times\left[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}\Big(\sum_{j=k-2}^{\infty}|\lambda_j|||a_j\chi_k||_{L^{q_1}}\Big)^p\right]^{1/p}$$
$$\leq C||\vec{b}||_{Lip_\beta}\left[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}\Big(\sum_{j=k-2}^{\infty}|\lambda_j|\mu(B_j)^{-\alpha}\Big)^p\right]^{1/p}$$
$$\leq C||\vec{b}||_{Lip_\beta}$$
$$\times\begin{cases}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=-\infty}^{j+2}\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{\alpha p}\right]^{1/p}, & 0<p\leq 1\\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2}\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{\alpha p/2}\right)\right. & \\ \qquad\left.\left(\sum_{k=-\infty}^{j+2}\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{\alpha p'/2}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases}$$
$$\leq C||\vec{b}||_{Lip_\beta}\begin{cases}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=-\infty}^{j+2}A_2{}^{(k-j)\alpha p}\right]^{1/p}, & 0<p\leq 1\\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p(\sum_{k=-\infty}^{j+2}A_2{}^{(k-j)\alpha p/2})\right. & \\ \qquad\left.(\sum_{k=-\infty}^{j+2}A_2{}^{(k-j)\alpha p'/2})^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases}$$

$$\leq C||\vec{b}||_{Lip_\beta}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}$$
$$\leq C||f||_{H\dot{K}^{\alpha,p}_{q_1,\vec{b}}}.$$

For the term J, let $C_k = B_k\backslash B_{k-1}$, $\chi_k = \chi_{C_k}$, $b_j^i = \mu(B_j)^{-1}\int_{B_j} b_i(x)d\mu(x)$, $1 \leq i \leq m$. By Hölder's inequality, by the vanishing moment of atom $a$and by the reverse doubling condition, we get, for $x \in 2^{k+1}B \setminus 2^k B$,

$$|T_{\vec{b}}(a_j)(x)| \leq \int_{B_j}\prod_{i=1}^{m}|(b_i(x)-b_i(y))||K(x,y)||a_j(y)|d\mu(y)$$
$$\leq \int_{B_j}\prod_{i=1}^{m}(|b_i(x)-b_i(x_0))|+|(b_i(y)-b_i(x_0)|)|K(x,y)-K(x,x_0)||a_j(y)|d\mu(y)$$
$$\leq C\prod_{i=1}^{m}|(b_i(x)-b_i(x_0))|\int_{B_j}\frac{d(x_0,y)^\delta}{d(x,x_0)^\delta\mu(B(x_0,d(x,x_0)))}|a_j(y)|d\mu(y)$$
$$+\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b}(x_0))_\sigma|$$
$$\int_{B_j}|(\vec{b}(y)-\vec{b}(x_0))_{\sigma_c}|\frac{d(x_0,y)^\delta}{d(x,x_0)^\delta\mu(B(x_0,d(x,x_0)))}|a_j(y)|d\mu(y)$$
$$+\prod_{i=1}^{m}\int_{B_j}|(b_i(y)-b_i(x_0))|\frac{d(x_0,y)^\delta}{d(x,x_0)^\delta\mu(B(x_0,d(x,x_0)))}|a_j(y)|d\mu(y)$$

$$\leq C\frac{{r_j}^\delta}{d(x,x_0)^\delta\mu(B(x_0,d(x,x_0)))}\cdot\{||\vec{b}||_{Lip_\beta}\rho(x,x_0)^\beta||a_j||_{L^{q_1}}\mu(B_j)^{1/q_1'}$$
$$+\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}||\vec{b}_\sigma||_{Lip_\beta}\rho(x,x_0)^{\beta_\sigma}\int_{B_j}||\vec{b}_{\sigma_c}||_{Lip_\beta}\rho(y,x_0)^{\beta_{\sigma^c}}|a_j(y)|d\mu(y)$$
$$+\int_{B_j}||\vec{b}||_{Lip_\beta}\rho(y,x_0)^\beta|a_j(y)|d\mu(y)\}$$
$$\leq C\frac{{r_j}^\delta}{d(x,x_0)^\delta\mu(B(x_0,d(x,x_0)))}||\vec{b}||_{Lip_\beta}\rho(x,x_0)^\beta||a_j||_{L^{q_1}}\mu(B_j)^{1/q_1'}$$
$$\leq C\frac{{r_j}^\delta}{d(x,x_0)^\delta\mu(B(x_0,d(x,x_0)))}||\vec{b}||_{Lip_\beta}\rho(x,x_0)^\beta\mu(B_j)^{-\alpha+1/q_1'},$$

so

$$||T_{\vec{b}}(a_j)\chi_k||_{L^{q_2}}$$
$$\leq C||\vec{b}||_{Lip_\beta}\frac{r_j^\delta}{r_k^\delta\mu(B_k)}\mu(B_j)^{-\alpha+1/q_1'}\Big(\int_{B_k\setminus B_{k-1}}\rho(x,u)^{\beta q_2}d\mu(x)\Big)^{1/q_2}$$
$$\leq C||\vec{b}||_{Lip_\beta}2^{(j-k)\delta}\mu(B_k)^{-1/q_2'+\beta}\mu(B_j)^{-\alpha+1/q_1'}$$

and

$$J = C\Big[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}\Big(\sum_{j=-\infty}^{k-3}|\lambda_j|||T_{\vec{b}}(a_j)\chi_k||_{L^{q_2}}\Big)^p\Big]^{1/p}$$
$$\leq C||\vec{b}||_{Lip_\beta}\Big[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}$$
$$\Big(\sum_{j=-\infty}^{k-3}|\lambda_j|2^{(j-k)\delta}\mu(B_j)^{1-1/q_1-\alpha}\mu(B_k)^{-(1-1/q_2-\beta)}\Big)^p\Big]^{1/p}$$
$$\leq C||\vec{b}||_{Lip_\beta}\begin{cases}\Big[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}2^{(j-k)\delta p}2^{(j-k)n_0(1-1/q_2-\alpha-\beta)p}\Big]^{1/p}, \\ \text{if} \quad 0<p\leq 1 \\ \Big[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\Big(\sum_{k=j+3}^{\infty}2^{(j-k)\delta p/2}2^{(j-k)n_0(1-1/q_2-\alpha-\beta)p/2}\Big) \\ \quad\times\Big(\sum_{k=j+3}^{\infty}2^{(j-k)\delta p'/2}2^{(j-k)n_0(1-1/q_2-\alpha-\beta)p'/2}\Big)^{p/p'}\Big]^{1/p}, \\ \text{if} \quad 1<p<\infty\end{cases}$$
$$\leq C||\vec{b}||_{Lip_\beta}\begin{cases}\Big[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}2^{(j-k)n_0(1-1/q_1+\delta/n_0-\alpha)p}\Big]^{1/p}, \\ \text{if} \quad 0<p\leq 1 \\ \Big[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\Big(\sum_{k=j+3}^{\infty}2^{(j-k)n_0(1-1/q_1+\delta/n_0-\alpha)p/2}\Big) \\ \quad\times\Big(\sum_{k=j+3}^{\infty}2^{(j-k)n_0(1-1/q_1+\delta/n_0-\alpha)p'/2}\Big)^{p/p'}\Big]^{1/p}, \\ \text{if} \quad 1<p<\infty\end{cases}$$
$$\leq C||\vec{b}||_{Lip_\beta}\Big(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\Big)^{1/p}$$
$$\leq C||f||_{H\dot{K}^{\alpha,p}_{q_1,\vec{b}}}.$$

This completes the proof of the theorem. □

Remark 1. Similar results related to nonhomogeneous weak Herz spaces also hold. The details are omitted in order to save space.

Proof of Theorem 7. We write $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where each a_j is a central (α, q_1) atom supported on B_j and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$. Write

$$||T_{\vec{b}}(f)||^p_{W\dot{K}^{\alpha,p}_{q_2}}$$

$$\leq \sup_{\lambda>0} \lambda^p \sum_{k=-\infty}^{\infty} \mu(B_k)^{\alpha p} \mu(\{x \in C_k : |T_{\vec{b}}(\sum_{j=k-1}^{\infty} \lambda_j a_j)(x)| > \lambda/2\})^{p/q_2}$$

$$+ \sup_{\lambda>0} \lambda^p \sum_{k=-\infty}^{\infty} \mu(B_k)^{\alpha p} \mu(\{x \in C_k : |T_{\vec{b}}(\sum_{j=-\infty}^{k-2} \lambda_j a_j)(x)| > \lambda/2\})^{p/q_2}$$

$$= G_1 + G_2.$$

By the (L^{q_1}, L^{q_2}) boundedness of $T_{\vec{b}}$ and by an estimate, similar to the one for JJ in Theorem 6, we get

$$G_1 \leq C \sum_{k=-\infty}^{\infty} \mu(B_k)^{\alpha p} ||T_{\vec{b}}(\sum_{j=k-1}^{\infty} \lambda_j a_j)(x)\chi_k||^p_{L^{q_2}} \leq C||\vec{b}||^p_{\wedge_\beta} \sum_{j=-\infty}^{\infty} |\lambda_j|^p.$$

For G_2, let us use the estimate

$$|T_{\vec{b}}(a_j)(x)| \leq C||\vec{b}||_{Lip_\beta} \frac{{r_j}^\delta}{d(x,x_0)^\delta \mu(B(x_0, d(x,x_0)))} \rho(x,x_0)^\beta \mu(B_j)^{-\alpha+1/q_1'},$$

which was obtained in the estimate for J in the proof of Theorem 6. Notice that when $x \in C_k$,

$$\lambda < 2 \sum_{j=-\infty}^{k-2} |\lambda_j||T_{\vec{b}}(a_j)| \leq C||\vec{b}||_{Lip_\beta} \sum_{j=-\infty}^{k-2} |\lambda_j| \frac{{r_j}^\delta}{d(x,x_0)^\delta \mu(B(x_0, d(x,x_0)))}$$

$$\times \rho(x,x_0)^\beta \mu(B_j)^{-\alpha+1/q_1'}$$

$$\leq C||\vec{b}||_{Lip_\beta} \sum_{j=-\infty}^{k-2} |\lambda_j| 2^{(j-k)\delta} \mu(B_k)^{-1+\beta} \mu(B_j)^{-\alpha+1/q_1'}$$

$$\leq C||\vec{b}||_{Lip_\beta} \sum_{j=-\infty}^{k-2} |\lambda_j| 2^{(j-k)\delta} \mu(B_k)^{-\alpha-1/q_2}.$$

For $\lambda > 0$, choosing $k_0 \in Z$ such that B_{k_0} is the maximal set satisfying

$$\mu(B_{k_0})^{\alpha+1/q_2} \leq C||\vec{b}||_{Lip_\beta} \lambda^{-1} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p},$$

then if $k > k_0$, it is easy to see that

$$\{x \in C_k : |T_{\vec{b}}(\sum_{j=-\infty}^{k-2} \lambda_j a_j)(x)| > \lambda/2\}$$

is an empty set, thus

$$\begin{aligned} G_2 &\leq \sup_{\lambda>0} \lambda^p \sum_{k=-\infty}^{k_0} \mu(B_k)^{(\alpha+1/q_2)p} \\ &\leq C\,||\vec{b}||^p_{Lip_\beta} \sum_{j=-\infty}^{k-2} |\lambda_j|^p 2^{(j-k)\delta p} \mu(B_k)^{-(\alpha+1/q_2)p} \sum_{k=-\infty}^{k_0} \mu(B_k)^{(\alpha+1/q_2)p} \\ &\leq C\,||\vec{b}||^p_{Lip_\beta} \sum_{k=-\infty}^{k_0} \sum_{j=-\infty}^{k-2} |\lambda_j|^p 2^{(j-k)\delta p} \\ &\leq C\,||\vec{b}||^p_{Lip_\beta} \sum_{j=-\infty}^{\infty} |\lambda_j|^p. \end{aligned}$$

So, we get,

$$||T_{\vec{b}}(f)||_{W\dot{K}^{\alpha,p}_{q_2}} \leq C\,||\vec{b}||_{Lip_\beta} (\sum_{j=-\infty}^{\infty} |\lambda_j|^p)^{1/p}.$$

Taking the infinum, we have completed the proof. □

REFERENCES

[1] J. Alvarez, "Continuity properties for linear commutators of calderón-zygmund operators," *Collect. Math.*, vol. 1998, no. 1, pp. 17–31, 49.

[2] J. Alvarez, R. J. Bagby, D. S. Kurtz, and C. Pérez, "Weighted estimates for commutators of linear operators," *Studia Math.*, vol. 104, no. 2, pp. 195–209, 1993.

[3] A. Bernardis, S. Hartzstein, and G. Pradolini, "Weighted inequalities for commutators of fractional integrals on spaces of homogeneous type," *J. Math. Anal. Appl.*, vol. 322, no. 2, pp. 825–846, 2006.

[4] J. J. Betancor, "A commutator theorem for fractional integrals in spaces of homogeneous type," *Int. J. Math. Math. Sci.*, vol. 24, no. 6, pp. 403–418, 2000.

[5] M. Bramanti and M. C. Cerutti, "Commutators of singular integrals on homogeneous spaces," *Boll. Un. Mat. Ital. B (7)*, vol. 10, no. 4, pp. 843–883, 1996.

[6] S. Chanillo, "A note on commutators," *Indiana Univ. Math. J.*, vol. 31, no. 1, pp. 7–16, 1982.

[7] Q. Chen and L. Liu, "Continuity for multilinear integral operators on some hardy and herz type spaces," *Int. J. Open Probl. Comput. Sci. Math.*, vol. 2, no. 2, pp. 215–230, 2009.

[8] W. Chen and E. Sawyer, "Endpoint estimates for commutators of singular integrals on spaces of homogeneous type," *J. Math. Anal. Appl.*, vol. 282, no. 2, pp. 553–566, 2003.

[9] R. R. Coifman, R. Rochberg, and G. Weiss, "Factorization theorems for hardy spaces in several variables," *Ann. of Math. (2)*, vol. 103, no. 3, pp. 611–635, 1976.

[10] A. E. Gatto and S. Vági, *Fractional integrals on spaces of homogeneous type*, ser. Lecture Notes in Pure and Appl. Math. New York: Marcel Dekker, Inc., 1990, vol. 122.

[11] I. Genebashvili, A. Gogatishvili, V. Kokilashvili, and M. Krbec, *Weight theory for integral transforms on spaces of homogeneous type*, ser. Pitman Monographs and Surveys in Pure and Applied Mathematics. Harlow: Longman, 1998, vol. 92.

[12] Y. Han and D. Yang, "New characterizations and applications of inhomogeneous besov and triebel-lizorkin spaces on homogeneous type spaces and fractals," *Dissertationes Math. (Rozprawy Mat.)*, vol. 403, p. 102, 2002.

[13] R. A. Macías and C. Segovia, "Lipschitz functions on spaces of homogeneous type," *Adv. in Math.*, vol. 33, no. 3, pp. 257–270, 1979.

[14] G. Pradolini and O. Salinas, "Commutators of singular integrals on spaces of homogeneous type," *Czechoslovak Math. J.*, vol. 57(132), no. 1, pp. 75–93, 2007.

[15] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, ser. Princeton Mathematical Series. Princeton: Princeton University Press, 1993, vol. 43.

[16] M. Wang and L. Liu, "Lipschitz estimates for multilinear commutator of littlewood-paley operator," *Acta Math. Acad. Paedagog. Nyházi. (N. S.)*, vol. 25, no. 1, pp. 85–104, 2009.

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