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COMMON FIXED POINT THEOREMS VIA IMPLICIT RELATIONS

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Abstract. We prove common fixed point theorems for four mappings satisfying implicit relations without decreasing assumption which improve theorems of [1, 13, 14]. We also prove common fixed point theorems for four mappings satisfying implicit relations which generalize theorems of [3, 4].

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1. Introduction

Let S and T be self-mappings of a metric space (X,d). S and T are *commuting* if $STx = TSx$ for all $x \in X$. Sessa [15] defined S and T to be *weakly commuting* if, for all $x \in X$,

$$d(STx, TSx) \le d(Tx, Sx). \tag{1.1}$$

Jungck [7] defined S and T to be *compatible*, as a generalization of the weakly commuting property, if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0 \tag{1.2}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

It is easy to show that "commuting" implies "weakly commuting," which, in turn, implies "compatible," and there are examples in the literature justifying that the inclusions are proper (see [7, 15]).

Jungck et al. [6] defined S and T to be *compatible* mappings *of type (A)* if

$$\lim_{n\to\infty} d(STx_n, T^2x_n) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} d(TSx_n, S^2x_n) = 0 \tag{1.3}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Clearly, "weakly commuting" implies "compatible of type (A)". By [6], the converse is not true, and the two concepts of compatibility are independent.

Recently, Pathak and Khan [12] defined S and T to be *compatible* mappings *of type (B)*, as a generalization of compatible mappings of type (A), if

$$\begin{aligned} \lim_{n\to\infty} d(TSx_n, S^2x_n) &\le \frac{1}{2}\Big[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, T^2x_n)\Big], \\ \lim_{n\to\infty} d(STx_n, T^2x_n) &\le \frac{1}{2}\Big[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, S^2x_n)\Big] \end{aligned} \tag{1.4}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Clearly, compatible mappings of type (A) are compatible mappings of type (B), but the converse is not true [9]. However, the notions of compatibility, compatibility of type (A), and compatibility of type (B) are equivalent to one another if S and T are continuous [12].

Pathak et al. [10] defined S and T to be *compatible* mappings *of type (P)* if

$$\lim_{n\to\infty} d(S^2x_n, T^2x_n) = 0 \tag{1.5}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

The notions of compatibility, compatibility of type (A), and compatibility of type (P) are mutually equivalent if S and T are continuous [10].

Pathak et al. [11] defined S and T to be *compatible* mappings *of type (C)* as a generalization of compatible mappings of type (A) if

$$\begin{aligned} \lim_{n\to\infty} d(TSx_n, S^2x_n) &\le \frac{1}{3}\Big(\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, S^2x_n) \\ &\qquad + \lim_{n\to\infty} d(Tt, T^2x_n)\Big), \\ \lim_{n\to\infty} d(STx_n, T^2x_n) &\le \frac{1}{3}\Big(\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, T^2x_n) \\ &\qquad + \lim_{n\to\infty} d(St, S^2x_n)\Big) \end{aligned} \tag{1.6}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Clearly, compatible mappings of type (A) are also compatible mappings of type (C), but the converse implication is not true [11]. However, the properties of compatibility of type (A) and compatibility of type (C) are mutually equivalent if S and T are continuous (see [11]).

2. Preliminaries

Definition 1 ([5]). S and T are said to be *weakly compatible* if they commute at their coincidence points, i. e., the equality $Su = Tu$ for some $u \in X$ implies that $STu = TSu$.

Lemma 1 ([6, 7, 10–12])**.** *If S and T are compatible, or compatible of type* (A)*, or compatible of type* (P)*, or compatible of type* (B)*, or compatible of type* (C)*, then they are weakly compatible.*

The converse is not true in general, see [2].

Definition 2 ([8])**.** S and T are said to be *R-weakly commuting* at a point $x \in X$ if for some $R > 0$

$$d(STx, TSx) \leq Rd(Tx, Sx). \tag{2.1}$$

Definition 3 ([9])**.** S and T are said to be *pointwise R-weakly commuting* on X if, given an $x \in X$, there exists an $R > 0$ such that (2.1) holds.

It was proved in [9] that the R-weak commutativity is equivalent to commutativity at coincidence points, i. e., S and T are pointwise R-weakly commuting if and only if they are weakly compatible.

Let $\mathbb{R}_+$ be the set of all non-negative reals numbers and K_6 the family of all continuous mappings $K(t_1, t_2, t_3, t_4, t_5, t_6)\colon \mathbb{R}_+^6 \to \mathbb{R}$ with $t_3 + t_4 \neq 0$ satisfying the following conditions:

(K_1) K is decreasing in variables $t_5$and t_6.

(K_2) there exists $0 \leq h < 1$ such that for all $u, v \geq 0$ with

(K_a) $K(u, v, v, u, u+v, 0) \leq 0$ or

(K_b) $K(u, v, u, v, 0, u+v) \leq 0$

we have $u \leq hv$.

The following theorem was proved in [13].

Theorem 1. *Let S, T, I and J be self-mappings of a metric space (X, d) satisfying*

(a) *$S(X) \subset J(X)$ and $T(X) \subset I(X)$,*

(b) *the inequality*

$$F\big(d(Sx, Ty), d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Sx, Jy)\big) \leq 0$$

holds for all $x, y \in X$ and $F \in K_6$ satisfies conditions (K_1) and (K_2) if $d(Ix, Sx) + d(Jy, Ty) \neq 0$, or

$$d(Sx, Ty) = 0 \quad \text{if} \quad (Ix, Sx) + d(Jy, Ty) = 0,$$

(c) *the pairs (S, I) and (T, J) are weakly compatible.*

If one of $S(X)$, $T(X)$, $I(X)$ and $J(X)$ is a complete subspace of X, then, S, T, I and J have a unique common fixed point z in X. Further, z is the unique common fixed point of S and I and T and J.

It is our purpose in this paper to prove common fixed point theorems for weakly compatible mappings satisfying implicit relations without condition (K_1), which improve results of Ali and Imdad [1] and Popa [13, 14]. We also prove a common fixed point theorem for weakly compatible mappings satisfying implicit relations which generalizes results of Jeong and Rhoades [4] and Jeong [3].

3. IMPLICIT RELATIONS

Let F_6 be the family of all continuous mappings $F(t_1,t_2,t_3,t_4,t_5,t_6)\colon \mathbb{R}_+^6 \to \mathbb{R}$ with $t_3+t_4 \neq 0$ satisfying the following condition:

(F_1) there exists $0 \leq h < 1$ such that for all $u,v,w \geq 0$ with
(F_a) $F(u,v,v,u,w,0) \leq 0$ or
(F_b) $F(u,v,u,v,0,w) \leq 0$
we have $u \leq hv$.

Example 1. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\frac{t_2t_3}{t_3+t_4} - b\frac{t_4t_5}{t_5+t_6+1}$, where $0 < a,b < 1$ and $a+b < 1$.

(F_1) : Let $u,v,w \geq 0$ and $F(u,v,v,u,w,0) = u - a\frac{v^2}{u+v} - b\frac{uw}{w+1} \leq 0$. Then, $u \leq h_1v$, where $h_1 = \frac{a}{1-b} < 1$. Similarly, if $F(u,v,u,v,0,w) \leq 0$ then $u \leq h_2v$, where $h_2 = a < 1$. We take $h = \max\{h_1,h_2\} < 1$.

Example 2. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\frac{t_2t_4}{t_3+t_4} - b\frac{t_3t_6}{t_5+t_6+1}$, where $0 < a,b < 1$ and $a+b < 1$. The condition (F_1) can be verified as in Example 1.

Let H_6 be the family of all continuous mappings $H(t_1,t_2,t_3,t_4,t_5,t_6)\colon \mathbb{R}_+^6 \to \mathbb{R}$ with $t_5+t_6 \neq 0$ satisfying the following conditions:

(H_1) there exists $0 \leq h < 1$ such that for all $u,v,w \geq 0$ with
(H_a) $H(u,v,v,u,w,0) \leq 0$ or
(H_b) $H(u,v,u,v,0,w) \leq 0$
we have $u \leq hv$.

(H_2) $H(u,u,0,0,u,u) > 0$ for all $u > 0$.

Example 3. $H(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\frac{t_3t_6+t_4t_5}{t_5+t_6} - bt_2$, where $a,b > 0$ and $a+b < 1$.

(H_1) : Let $u,v,w \geq 0$ and $H(u,v,v,u,w,0) = u - au - bv \leq 0$. Then, $u \leq hv$, where $h = \frac{b}{1-a} < 1$. Similarly, if $H(u,v,u,v,0,w) \leq 0$ then $u \leq hv$.

(H_2) : $H(u,u,0,0,u,u) = (1-b)u > 0$ for all $u > 0$.

Example 4. $H(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \frac{at_3t_5+bt_4t_6}{t_5+t_6} - ct_2$, where $a,b,c > 0$ and $a+b+c < 1$.

(H_1) : Let $u,v,w \geq 0$ and $H(u,v,v,u,w,0) = u - av - cv \leq 0$. Then, $u \leq h_1v$, where $h_1 = a+c < 1$. Similarly, if $H(u,v,u,v,0,w) \leq 0$ then $u \leq h_2v$, where $h_2 = b+c < 1$. We take $h = \max\{h_1,h_2\} < 1$.

(H_2) : $H(u,u,0,0,u,u) = (1-c)u > 0$ for all $u > 0$.

Let G_6 be the family of all continuous mappings $G(t_1,t_2,t_3,t_4,t_5,t_6)\colon \mathbb{R}_+^6 \to \mathbb{R}$ with $t_2+t_4 \neq 0$ satisfying the following conditions:

(G_1) G is decreasing in variables t_5 and t_6.
(G_2) there exists $0 \le h < 1$ such that for all $u, v \ge 0$ with
 (G_a) $G(u,v,v,u,u+v,0) \le 0$ or
 (G_b) $G(u,v,u,v,0,u+v) \le 0$
we have $u \le hv$.
(G_3) $G(u,u,0,0,u,u) > 0$ for all $u > 0$.

Example 5. $G(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a_1 \frac{t_2t_5}{t_2+t_4} - a_2(t_3+t_4) - a_3(t_5+t_6) - a_4t_2$, where $a_1, a_2, a_3, a_4 > 0$ and $a_1 + 2a_2 + 2a_3 + a_4 < 1$.

(G_1) : It is clear.

(G_2) : Let $u, v \ge 0$ and $G(u,v,v,u,u+v,0) = u - a_1 \dfrac{v(u+v)}{u+v} - a_2(u+v) - a_3(u+v) - a_4v \le 0$. Then $u \le h_1v$, where $h_1 = \frac{a_1+a_2+a_3+a_4}{1-a_2-a_3} < 1$. Similarly, if $G(u,v,u,v,0,u+v) \le 0$ then $u \le h_2v$, where $h_2 = \dfrac{a_2+a_3+a_4}{1-a_2-a_3} < 1$. We take $h = \max\{h_1,h_2\} < 1$.

(G_3) : $G(u,u,0,0,u,u) = (1-a_1-2a_3-a_4)u > 0$ for all $u > 0$.

Let C_6 be the family of all continuous mappings $C(t_1,t_2,t_3,t_4,t_5,t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ with $t_2 + t_4 \neq 0$ satisfying the following conditions:

(C_1) there exists $0 \le h < 1$ such that for all $u, v, w \ge 0$ with
 (C_a) $C(u,v,v,u,w,0) \le 0$ or
 (C_b) $C(u,v,u,v,0,w) \le 0$
we have $u \le hv$.
(C_2) $C(u,u,0,0,u,u) > 0$ for all $u > 0$.

Example 6. $C(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\frac{t_2t_4}{t_2+t_4} - b\frac{t_3t_5}{t_5+t_6+1}$, where $0 < a, b < 1$ and $a + b < 1$.

(C_1) : Let $u, v, w \ge 0$ and $C(u,v,v,u,w,0) = u - a\frac{uv}{u+v} - b\dfrac{vw}{w+1} \le 0$. Then $u \le h_1v$, where $h_1 = a + b < 1$. Similarly, if $C(u,v,u,v,0,w) \le 0$ then $u \le h_2v$, $h_2 = \frac{a}{2} < 1$. We take $h = \max\{h_1,h_2\} < 1$.

(C_2) : $C(u,u,0,0,u,u) = u > 0$ for all $u > 0$.

Example 7. $C(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\frac{t_2t_4}{t_2+t_4} - b\frac{t_3t_6}{t_5+t_6+1}$, where $0 < a, b < 1$ and $a + 2b < 2$. (C_1) and (C_2) follow as in Example 6.

4. Main results

Theorem 2. *Let f, g, S and T be self-mappings of a metric space (X,d) satisfying the following conditions:*

$$S(X) \subset g(X), \qquad T(X) \subset f(X), \tag{4.1}$$

$$F\big(d(Sx,Ty), d(fx,gy), d(fx,Sx), d(gy,Ty), d(fx,Ty), d(Sx,gy)\big) \le 0 \tag{4.2}$$

for all $x, y \in X$ *and* $F \in F_6$ *satisfies* (F_1) *if* $d(fx,Sx)+d(gy,Ty) \neq 0$, *or*

$$d(Sx,Ty)=0 \quad \text{if} \quad d(fx,Sx)+d(gy,Ty)=0. \tag{4.3}$$

Suppose that one of $S(X)$, $T(X)$, $f(X)$, *and* $g(X)$ *is a complete subspace of* X *and the pairs* (S,f) *and* (T,g) *are weakly compatible.*

Then, f, g, S, *and* T *have a unique common fixed point* z *in* X. *Further,* z *is the unique common fixed point of* S *and* f *and* T *and* g.

Proof. Let x_0 be an arbitrary point in X. By (4.1), we can define inductively a sequence $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} = gx_{2n+1},\ y_{2n+1} = fx_{2n+2} = Tx_{2n+1} \quad \text{for } n = 0,1,2,\ldots. \tag{4.4}$$

If

$$d(fx_{2n},Sx_{2n})+d(gx_{2n+1},Tx_{2n+1}) \neq 0,$$

then, using (4.2) and (4.4), we have

$$\begin{aligned} &F\big(d(Sx_{2n},Tx_{2n+1}),d(fx_{2n},gx_{2n+1}),d(fx_{2n},Sx_{2n}),d(gx_{2n+1},Tx_{2n+1}),\\ &\qquad d(fx_{2n},Tx_{2n+1}),d(Sx_{2n},gx_{2n+1})\big)\\ &= F\big(d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n}),d(y_{2n-1},y_{2n}),d(y_{2n},y_{2n+1}),\\ &\qquad d(y_{2n-1},y_{2n+1}),0\big) \leq 0 \end{aligned}$$

By (F_a), we get

$$d(y_{2n},y_{2n+1}) \leq h d(y_{2n-1},y_{2n}).$$

Similarly, if

$$d(fx_{2n+2},Sx_{2n+2})+d(gx_{2n+1},Tx_{2n+1}) \neq 0.$$

we obtain

$$d(y_{2n+1},y_{2n+2}) \leq h d(y_{2n},y_{2n+1}).$$

Therefore,

$$d(y_n,y_{n+1}) \leq h d(y_{n-1},y_n).$$

Thus, $\{y_n\}$ is a Cauchy sequence in X and, therefore, the subsequence $\{y_{2n}\} = \{gx_{2n+1}\} \subset g(X)$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, it converges to a point $z = gv$ for some $v \in X$.

Therefore, the sequence $\{y_n\}$ converges also to z and the subsequences $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$, and $\{fx_{2n+2}\}$ converge to z.

If $z \neq Tv$, using (4.2), we obtain

$$\begin{aligned} &F\big((d(Sx_{2n},Tv),d(fx_{2n},gv),d(fx_{2n},Sx_{2n}),d(gv,Tv),d(fx_{2n},Tv),\\ &\quad d(Sx_{2n},gv)\big) \leq 0 \end{aligned}$$

Letting $n \to \infty$ and using the continuity of F, we obtain

$$F(d(z,Tv),0,0,d(z,Tv),d(z,Tv),0) \leq 0.$$

By (F_a), we get $z = Tv = gv$. Since $T(X) \subset f(X)$, there exists an $u \in X$ such that $z = fu = Tv$.

If $z \neq Su$, using (4.2) we have

$$F\big(d(Su,Tv),d(fu,gv),d(fu,Su),d(gv,Tv),d(fu,Tv),d(Su,gv)\big) = F\big(d(Su,z),0,d(z,Su),0,0,d(Su,z)\big) \leq 0.$$

By (F_b), we get $z = Su = fu$. Since the pairs (S, f) and (T,g) are weakly compatible, we get $fz = Sz$ and $gz = Tz$. Since $d(fz,Sz)+d(gv,Tv) = 0$, in view of (4.3), it follows that $d(Sz,Tv) = 0$, i. e., $z = Sz = fz$.

Similarly, we can prove that $z = gz = Tz$. Hence, z is a common fixed point of f, g, S, and T.

The proof is similar if we suppose that, instead of $g(X)$, one of $S(X)$, $T(X)$, and $f(X)$ is complete. The uniqueness of z follows from (4.3). □

Remark 1. As the function F in Theorem 2 is not decreasing in variables t_5 and t_6, theorems of [1, 13, 14] are not applicable in this case.

Considering Examples 1 and 2, we obtain two corollaries of Theorem 2.

Theorem 3. *Let f, g, S, and T be self-mappings of a metric space (X,d) satisfying (4.1),*

$$H\big(d(Sx,Ty),d(fx,gy),d(fx,Sx),d(gy,Ty),d(fx,Ty),d(Sx,gy)\big) \leq 0 \tag{4.5}$$

for all $x, y \in X$ and $H \in H_6$ satisfies (H_1) and (H_2) if $d(fx,Ty)+d(Sx,gy) \neq 0$, or

$$d(Sx,Ty) = 0 \quad \text{if} \quad d(fx,Ty)+d(Sx,gy) = 0.$$

Suppose that one of $S(X)$, $T(X)$, $f(X)$, and $g(X)$ is a complete subspace of X and the pairs (S, f) and (T,g) are weakly compatible.

Then f, g, S, and T have a unique common fixed point z in X. Further, z is the unique common fixed point of S and f and T and g.

Proof. Let x_0 be an arbitrary point in X. By (4.1), we can define inductively a sequence $\{y_n\}$ in X by (4.4).

If

$$d(fx_{2n},Tx_{2n+1})+d(Sx_{2n},gx_{2n+1}) = 0$$

or

$$d(fx_{2n+2},Tx_{2n+1})+d(gx_{2n+1},Sx_{2n+2}) = 0$$

for some n, the proof is similar to [14].

If

$$d(fx_{2n},Tx_{2n+1})+d(Sx_{2n},gx_{2n+1}) \neq 0$$

and

$$d(fx_{2n+2},Tx_{2n+1})+d(gx_{2n+1},Sx_{2n+2}) \neq 0,$$

as in the proof of Theorem 2, we obtain $z = fu = Su = gv = Tv$.

Since the pairs (S, f) and (T, g) are weakly compatible, we get $fz = Sz$ and $gz = Tz$.

If $z \neq Sz$, using (4.2) we have

$$\begin{aligned} &H\big(d(Sz,Tv),d(fz,gv),d(fz,Sz),d(gv,Tv),d(fz,Tv),d(Sz,gv)\big) \\ &\qquad\qquad = H\big(d(Sz,z),d(Sz,z),0,0,d(Sz,z),d(Sz,z)\big) \leq 0. \end{aligned}$$

which is a contradiction to (H_2). Therefore, $z = Sz = fz$.

Similarly, we can prove that $z = Tz = gz$. Hence, z is a common fixed point of f, g, S, and T. The uniqueness of z follows from (4.5) and (H_2). $\square$

Theorem 3 generalizes [4, Theorem 5] and [3, Theorem 2].

Theorem 4. *Let f, g, S, and T be self-mappings of a metric space (X,d) satisfying* (4.1),

$$G\big(d(Sx,Ty),d(fx,gy),d(fx,Sx),d(gy,Ty),d(fx,Ty),d(Sx,gy)\big) \leq 0 \tag{4.6}$$

for all $x, y \in X$ and $G \in G_6$ satisfies conditions (G_1), (G_2), and (G_3) if $d(fx,gy)+d(gy,Ty) \neq 0$, or

$$d(Sx,Ty) = 0 \quad \text{if} \quad d(fx,gy)+d(gy,Ty) = 0.$$

Suppose that one of $S(X)$, $T(X)$, $f(X)$, and $g(X)$ is a complete subspace of X and the pairs (S, f) and (T, g) are weakly compatible.

Then, f, g, S, and T have a unique common fixed point z in X. Further, z is the unique common fixed point of S and f and T and g.

Proof. Let x_0 be an arbitrary point in X. By (4.1), we can define inductively a sequence $\{y_n\}$ in X by (4.4).

If

$$d(fx_{2n},Tx_{2n+1})+d(Sx_{2n},gx_{2n+1}) \neq 0$$

using (4.2) and (4.4) we have

$$\begin{aligned} &G\big(d(Sx_{2n},Tx_{2n+1}),d(fx_{2n},gx_{2n+1}),d(fx_{2n},Sx_{2n}),d(gx_{2n+1},Tx_{2n+1}), \\ &\qquad d(fx_{2n},Tx_{2n+1}),d(Sx_{2n},gx_{2n+1})\big) \\ &= G\big(d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n}),d(y_{2n-1},y_{2n}),d(y_{2n},y_{2n+1}), \\ &\qquad d(y_{2n-1},y_{2n+1}),0\big) \leq 0. \end{aligned}$$

By (G_1) we have

$$\begin{aligned} &G\big(d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n}),d(y_{2n-1},y_{2n}), \\ &\qquad d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1}),0\big) \leq 0. \end{aligned}$$

By virtue of (G_a), we get

$$d(y_{2n}, y_{2n+1}) \le h d(y_{2n-1}, y_{2n}).$$

In the same manner, if

$$d(fx_{2n+2}, Sx_{2n+2}) + d(gx_{2n+1}, Tx_{2n+1}) \neq 0$$

we obtain

$$d(y_{2n+1}, y_{2n+2}) \le h d(y_{2n}, y_{2n+1}).$$

Therefore,

$$d(y_n, y_{n+1}) \le h d(y_{n-1}, y_n).$$

As in the proofs of Theorems 2 and 3, z is a common fixed point of f, g, S, and T. The uniqueness of z follows from (4.6) and (G_3). □

Remark 2. Theorem 4 generalizes [4, Theorem 9] and [3, Theorem 5].

Similarly to Theorem 3, we can prove the following statement which improves Theorem 4.

Theorem 5. *Let f, g, S, and T be self-mappings of a metric space (X,d) satisfying (4.1),*

$$C\big(d(Sx,Ty), d(fx,gy), d(fx,Sx), d(gy,Ty), d(fx,Ty), d(Sx,gy)\big) \le 0$$

for all $x, y \in X$ and $C \in C_6$ satisfies (C_1) and (C_2) if $d(fx,gy) + d(gy,Ty) \neq 0$, or

$$d(Sx,Ty) = 0 \quad \text{if} \quad d(fx,gy) + d(gy,Ty) = 0.$$

Suppose that one of $S(X)$ $T(X)$, $f(X)$, and $g(X)$ is a complete subspace of X and the pairs (S,f) and (T,g) are weakly compatible.

Then, f, g, S, and T have a unique common fixed point z in X. Further, z is the unique common fixed point of S and f and T and g.

If $S = T$ or $f = g$ or $S = T$ and $f = g$, or $f = g = I$, where I denotes the identity mapping, or $S = T$ and $f = g = I$ in Theorems 2, 3, 4, and 5, one can obtain several corollaries.

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EXISTENCE AND APPROXIMATION OF SOLUTIONS OF SOME FIRST ORDER ITERATIVE DIFFERENTIAL EQUATIONS

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Abstract. Existence theorems for some iterative differential equations as well as convergence theorems for a fixed point iterative method designed to approximate these solutions, are proved under weaker conditions than those due to A. Buică ("Existence and continuous dependence of solutions of some functional-differential equations," *Seminar on Fixed Point Theory 3*, pp. 1–14, 1995). To this end, we basically employ the technique of non-expansive mappings.

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1. Introduction

For most of the differential and integral equations with deviating arguments that appear in recent literature, the deviation of the argument usually involves only the time itself, see for example the very recent papers [1, 2, 6, 18, 19, 22, 23]. However, another case, in which the deviating arguments depend on both the state variable x and the time t, is of importance in theory and practice. Several papers have appeared recently that are devoted to such kind of differential equations, see for example [4, 8–12, 16, 17, 29, 31–33] and references therein.

One of the first papers studying this class of functional equations is the one by Eder [8] who considered the functional differential equation

$$x'(t) = x(x(t)), \quad t \in A \subset \mathbb{R},$$

while Fečkan [12] studied a functional differential equation of the more general form

$$x'(t) = f(x(x(t)))$$

with $f \in C^1(\mathbb{R})$. For other developments on this topic, see the very recent papers [4, 8–11, 16, 17, 29, 33] and references therein. The term to designate this class of differential equations is that of *iterative differential equations*. As mentioned in [33] and the papers cited there, iterative differential equations arise in relation to infection models and are also important in the study of the motion of charged particles with retarded interaction.

In this paper, we are interested in the study of the existence of solutions of iterative differential equations of the general form

$$x'(t) = f(t, x(x(t))),$$

and, in addition to other related papers, we illustrate how one can approximate the non-unique solution of such kind of iterative differential equations by means of iterative techniques.

The main idea is to use the powerful and more reliable technique of non-expansive operators and to adapt and use several convergence theorems from the theory of iterative approximation of fixed points of non-expansive mappings (see the very recent monographs [3,5]).

As most of the papers in literature devoted to functional differential equations with deviating argument are based on fixed point techniques and are essentially tributary to the contraction mapping principle or method of successive approximations, our approach here appears to be new not only for the study of existence of solutions but especially with regard to the approximation of solutions of these equations.

The paper is organized as follows: in Section 2 we present some basic results from the fixed point theory of non-expansive operators; in Section 3 we present the main results of the paper regarding the existence and approximation of solutions of some iterative differential equations. The paper ends with Section 4 that gives some illustrative examples. Note that our results are established under weaker assumptions than the ones obtained in [4].

2. Fixed point theory of non-expansive mappings

Let (X, d) be a metric space. A mapping $T: X \to X$ is said to be an *α-contraction* if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y), \quad \forall x, y \in X. \tag{2.1}$$

A point $x \in X$ is called a *fixed point* of T if $Tx = x$. It is well known [26] that, under the strict contraction condition (2.1) in a complete metric space X, there exists a unique fixed point of T and, moreover, the Picard iteration determined by an $x_0 \in X$ and the relation

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \dots, \tag{2.2}$$

converges to that fixed point. In the case where $\alpha = 1$ in (2.1), the mapping T is said to be *non-expansive*.

As the technique of non-expansive mappings applied to functional differential equations appears to be less frequently used in literature, in the present section we present some basic concepts and results of the fixed point theory of non-expansive operators.

Let K be a non-empty subset of a real normed linear space E and let $T: K \to K$ be a mapping. In this setting, T is *non-expansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall\, x, y \in K. \tag{2.3}$$

Although the non-expansive mappings are generalizations of α-contractions, they of contractive mappings. More precisely, if K is a non-empty closed subset of a Banach space E and $T: K \to K$ is a non-expansive mapping which is not an α-contraction, then, as is shown by the following example, T may not have fixed points.

Example 1 ([13, Example 3.3, pp. 30]). In the space $c_0(\mathbb{N})$, the isometry T defined by the relation

$$T(x_1, x_2, \dots) = (1, x_1, x_2, \dots)$$

maps the unit ball into its boundary but has no fixed points.

Moreover, as is shown by the next example, even in the cases where T has a fixed point, the Picard iteration associated to T (i. e., the sequence $\{x_n\}$ defined by (2.2) for an $x_0 \in K$), may fail to converge to the fixed point.

Example 2. Let us consider the unit interval $[0,1]$ with the usual norm. The function $T: [0,1] \to [0,1]$ given by the formula $Tx = 1 - x$ for all $x \in [0,1]$ has a unique fixed point, $x^* = \frac{1}{2}$ but, except for the trivial case $x_0 = \frac{1}{2}$, the Picard iteration starting from x_0 yields an oscillatory sequence.

For this many other reasons, some richer geometrical properties of the ambient space E are needed in order to ensure the existence of a fixed point or/and the convergence of an iterative method (generally a more complex iterative method than the Picard iteration) to a fixed point of T. In the present paper, we mainly consider Banach spaces which are uniformly convex or strictly convex (for more general settings, see [5]). For the sake of completeness, let us recall some concepts and results.

One of the most important fixed point theorems for non-expansive mappings, due to Browder, Göhde, and Kirk (see, e. g., [3]), is stated as follows.

Theorem 2.1. *If K is a non-empty closed convex and bounded subset of a uniformly convex Banach space E then any non-expansive mapping $T: K \to K$ has a fixed point.*

Remark 1. Theorem 2.1 provides no information on the approximation of a fixed point of T. From Example 2, we see that the Picard iteration does not resolve this situation, in general. Due to this fact, several other fixed point iteration procedures have been considered (see [3, 5]). The most usual ones will be defined in the sequel in view of their use.

Let K be a convex subset of a normed linear space E and let $T: K \to K$ be a self-mapping. Given an $x_0 \in K$ and a real number $\lambda \in [0,1]$, the sequence $\{x_n\}$ defined by the formula

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \quad n = 0, 1, 2, \dots, \tag{2.4}$$

is usually called the *Krasnoselskij iteration*, or *Krasnoselskij–Mann iteration*. Clearly, (2.4) reduces to the Picard iteration (2.2) for $\lambda = 1$.

For an $x_0 \in K$, the sequence $\{x_n\}$ defined by the formula

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n, \quad n = 0,1,2,\dots, \tag{2.5}$$

where $\{\lambda_n\} \subset [0,1]$ is a sequence of real numbers satisfying some appropriate conditions, is called a *Mann iteration*.

It was shown by Krasnoselskij [15] in the case where $\lambda = 1/2$, and later by Schaefer [30] for an arbitrary $\lambda \in (0,1)$, that if E is a uniformly convex Banach space and K is a convex and compact subset of E (and therefore, by Theorem 2.1, containing fixed points of T), then the Krasnoselskij iteration converges to a fixed point of T.

Moreover, Edelstein [7] proved that strict convexity of E suffices for the same conclusion. The question of whether the strict convexity assumption can be removed was answered in the affirmative by Ishikawa [14] by the following result.

Theorem 2.2. *Let K be a subset of a Banach space E and let $T\colon K \to K$ be a non-expansive mapping. For an arbitrary $x_0 \in K$, consider the Mann iteration process $\{x_n\}$ given by (2.5) under the following assumptions:*

(a) $x_n \in K$ *for all positive integers n;*
(b) $0 \le \lambda_n \le b < 1$ *for all positive integers n;*
(c) $\sum_{n=0}^{\infty} \lambda_n = \infty$.

If $\{x_n\}$ is bounded, then $x_n - Tx_n \to 0$ as $n \to \infty$.

The following corollaries of Theorem 2.2 will be particularly important for the application part of our paper.

Corollary 2.1. *Let K be a convex and compact subset of a Banach space E and let $T\colon K \to K$ be a non-expansive mapping. If the Mann iteration process $\{x_n\}$ given by (2.5) satisfies assumptions (a)–(c) of Theorem 2.2, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. See Theorem 6.17 in Chidume [5]. □

Corollary 2.2. *Let E be a real normed space, K a closed bounded convex subset of E and let $T\colon K \to K$ be a non-expansive mapping. If $I - T$ maps closed bounded subsets of E into closed subsets of E and $\{x_n\}$ is the Mann iteration defined by (2.5) with $\{\lambda_n\}$ satisfying assumptions (a)–(c) of Theorem 2.2, then $\{x_n\}$ converges strongly to a fixed point of T in K.*

Proof. See Corollary 6.19 in Chidume [5]. □

3. Existence theorems and approximation of solutions of some iterative differential equations

Consider the following initial value problem

$$\begin{cases} y'(x) = f(x, y(y(x))), & x \in [a,b], \\ y(x_0) = y_0, \end{cases} \tag{3.1}$$

where $x_0, y_0 \in [a,b]$ and $f \in C([a,b]\times[a,b])$ are given. Let us put $C_x = \max\{x-a, b-x\}$, $x \in [a,b]$, and

$$\mathscr{C}_L = \big\{ y \in C([a,b],[a,b]) : |y(t_1)-y(t_2)| \le L\,|t_1-t_2|, \forall\, t_1,t_2 \in [a,b] \big\}, \tag{3.2}$$

where $L > 0$ is given. For problem (3.1), Buică [4] established existence theorems [4, Theorems 1 and 4] as well as existence and uniqueness theorems [4, Theorems 2 and 5]. We formulate the first two of them here for the sake of completeness.

Theorem 3.1 ([4]). *Assume that the following conditions are satisfied for the initial value problem* (3.1)*:*

(1) $f \in C([a,b]\times[a,b])$*;*
(2) $\exists\, L_1 > 0 : |f(s,u)-f(s,v)| \le L_1\,|u-v|$ *for all* $s,u,v \in [a,b]$*;*
(3) *If* L *is the Lipschitz constant involved in* (3.2)*, then*

$$M = \max\{|f(s,u)| : (s,u) \in [a,b]\times[a,b]\} \le L;$$

(4) *One of the following conditions holds:*
   (a) $MC_{x_0} \le C_{y_0}$*;*
   (b) $x_0 = a,\ M(b-a) \le b-y_0,\ f(s,u) \ge 0$ *for all* $s,u \in [a,b]$*;*
   (c) $x_0 = b,\ M(b-a) \le y_0-a,\ f(s,u) \ge 0$ *for all* $s,u \in [a,b]$.

Then there exists at least one solution $y^* \in \mathscr{C}_L$ *of problem* (3.1).

Basically, Theorem 3.1 shows that, for any given $L > 0$, if (1)–(4) are satisfied, then the initial value problem (3.1) has a (possibly, non-unique) solution in $\mathscr{C}_L$.

Theorem 3.2 ([4]). *Assume that all conditions of Theorem 3.1 are satisfied and, in addition, we also have*

$$L_1 C_{x_0}(L+1) < 1. \tag{3.3}$$

Then there exists a unique solution y^* *of problem* (3.1) *in* $\mathscr{C}_L$.

Under the assumptions of Theorem 3.2, it is known that the unique solution y^* of the initial value problem (3.1) can be approximated by means of the Picard iteration $\{y_n\}$ defined by $y_1 \in \mathscr{C}_L$ arbitrary and

$$y_{n+1}(t) = y_0 + \int_{x_0}^{t} f(s, y_n(y_n(s)))ds, \quad \forall\, t \in [a,b],\ n \ge 1. \tag{3.4}$$

In view of the considerations presented in Section 2, it is clear that if condition (3.3) is weakened to

$$L_1 C_{x_0}(L+1) \le 1, \tag{3.5}$$

then, firstly, the assertion on the existence of a unique solution of problem (3.1) is not true any more and, secondly, the Picard iteration (3.4) does not generally converge to the solution.

It is therefore the aim of this paper to show that if (3.3) is replaced by (3.5), then we are still able to approximate a (non-unique) solution of the initial value problem (3.1) by means of a Krasnoselski–Mann iteration procedure. The next theorem states the main result of this paper.

Theorem 3.3. *Assume that all conditions of Theorem 3.1 are satisfied and, in addition, we have*

$$L_1 C_{x_0}(L+1) \le 1.$$

Then the initial value problem (3.1) has at least one solution y^* in $\mathscr{C}_L$ which can be approximated by the Krasnoselskij iteration

$$y_{n+1}(t) = (1-\mu)y_n(t) + \mu y_0 + \mu \int_{x_0}^{t} f(s, y_n(y_n(s)))ds, \quad t \in [a,b],\ n \ge 1, \tag{3.6}$$

where $\mu \in (0,1)$ and $y_1 \in \mathscr{C}_L$ is arbitrary.

Proof. If $L_1 C_{x_0}(L+1) < 1$, then the conclusion follows similarly to [4]. Therefore, we limit ourselves to the case where $L_1 C_{x_0}(L+1) = 1$.

It follows from [4, Lemma 1] that $\mathscr{C}_L$ is a non-empty convex and compact subset of the Banach space $(C([a,b]), \|\cdot\|)$, where $\|\cdot\|$ is the usual supremum norm. Consider the integral operator $T : \mathscr{C}_L \to C([a,b])$,

$$Ty(t) = y_0 + \int_{x_0}^{t} f(s, y(y(s)))ds, \quad t \in [a,b],\ y \in \mathscr{C}_L. \tag{3.7}$$

It is clear that $y \in \mathscr{C}_L$ is a solution of the initial value problem (3.1) if and only if y is a fixed point of T, i. e.,

$$y = Ty.$$

We first prove that $\mathscr{C}_L$ is an invariant set with respect to T, i. e., we have $T(\mathscr{C}_L) \subset \mathscr{C}_L$.

If condition (4a) holds, then for any $y \in \mathscr{C}_L$ and $t \in [a,b]$ we have

$$|(Ty)(t)| \le |y_0| + \left| \int_{x_0}^{t} f(s, y(y(s)))ds \right| \le |y_0| + M|t - x_0| \le b$$

and

$$\begin{aligned} |(Ty)(t)| &\ge |y_0| - \left| \int_{x_0}^{t} f(s, y(y(s)))ds \right| \ge |y_0| - M|t - x_0| \\ &\ge |y_0| - MC_{x_0} \ge y_0 - C_{y_0} \ge a, \end{aligned}$$

which shows that, for any $y \in \mathscr{C}_L$, one has $(Ty)(t) \in [a,b]$, $t \in [a,b]$. Now, for any $t_1, t_2 \in [a,b]$, we have

$$|(Ty)(t_1) - (Ty)(t_2)| = \left| \int_{t_1}^{t_2} f(s, y(y(s)))ds \right| \le M|t_1 - t_2| \le L|t_1 - t_2|.$$

Thus, $Ty \in \mathscr{C}_L$ for all $y \in \mathscr{C}_L$. In a similar way we treat the cases (4b) and (4c). Therefore, $T : \mathscr{C}_L \to \mathscr{C}_L$ (i. e., T is a self-mapping of $\mathscr{C}_L$).

Let $y, z \in \mathscr{C}_L$ and $t \in [a,b]$. Then

$$\begin{aligned}
|(Ty)(t) - (Tz)(t)| &\le \left| \int_{x_0}^{t} |f(s, y(y(s))) - f(s, z(z(s)))| ds \right| \\
&\le \left| \int_{x_0}^{t} L_1 |y(y(s)) - z(z(s))| ds \right| \\
&\le L_1 \left| \int_{x_0}^{t} \big(|y(y(s)) - y(z(s))| + |y(z(s)) - z(z(s))| \big) ds \right| \\
&\le L_1 \left| \int_{x_0}^{t} \big(L|y(s) - z(s)| + \max_{\xi \in [a,b]} |y(\xi) - z(\xi)| \big) ds \right| \\
&= L_1 \left| \int_{x_0}^{t} \big(L|y(s) - z(s)| + \|y - z\| \big) ds \right| \\
&\le L_1 \left| \int_{x_0}^{t} (L+1)\|y - z\| \right| ds \\
&= L_1 (L+1) \|y - z\| |t - x_0| \\
&\le L_1 C_{x_0} (L+1) \|y - z\|.
\end{aligned} \tag{3.8}$$

Now, by taking the maximum in (3.8), we get

$$\|Ty - Tz\| \le L_1 C_{x_0} (L+1) \|y - z\|$$

which, in view of condition (3.5), proves that T is non-expansive and, hence, continuous.

It now remains to apply the Schauder fixed point theorem to obtain the first part of the conclusion, and Corollary 2.1 or 2.2 to get the second one. ☐

Remark 2. In practise, one can consider $\mu = \frac{1}{2}$ in (3.6).

Now we state and prove the results corresponding to Theorems 4 and 5 of [4]. To this end, for a $\lambda \in (0,1]$ fixed, we put

$$\mathscr{C}_{L,\lambda} = \big\{ y \in \mathscr{C}_L : y(x) \le \lambda x, \forall\, x \in [a,b] \big\}.$$

Theorem 3.4. *Assume that the following conditions are satisfied:*

(i) $y_0 \le \lambda x_0$;
(ii) $\exists\, L_1 > 0 : |f(s,u) - f(s,v)| \le L_1 |u - v|$ *for all* $s, u, v \in [a,b]$;

(iii) $M \le \min\{\lambda, L\}$;
(iv) *One of the following conditions holds:*
  (a) $MC_{x_0} \le C_{y_0}$;
  (b) $x_0 = a$, $M(b-a) \le b - y_0$, *and* $f(s,u) \ge 0$ *for all* $s,u \in [a,b]$;
  (c) $x_0 = b$, $M(b-a) \le y_0 - a$, *and* $f(s,u) \ge 0$ *for all* $s,u \in [a,b]$;
(v) $M(x_0 - a) \ge y_0 - \lambda a$;
(vi) *There exists a* $\overline{\tau} > 0$ *such that* $\overline{\tau} > -\frac{\ln(1-\lambda)}{\lambda(b-x_0)}$ *(if* $x_0 \ne b$ *and* $\lambda \ne 1$*) and*

$$\frac{L_1}{\overline{\tau}}\left(L + \frac{1}{\lambda}\right)\max\left\{e^{\overline{\tau}(x_0-a)} - 1, 1 - e^{\overline{\tau}(x_0-b)}\right\} \le 1.$$

Then:

(1) *The initial value problem* (3.1) *has at least one solution in* $\mathscr{C}_{L,\lambda}$;
(2) *For any* $y_1 \in \mathscr{C}_{L,\lambda}$ *the Krasnoselskij iteration*

$$y_{n+1}(t) = (1-\mu)y_n(t) + \mu y_0 + \mu \int_{x_0}^{t} f(s, y_n(y_n(s)))ds, \quad t \in [a,b],\ n \ge 1,$$

where $\mu \in (0,1)$*, converges to a solution of problem* (3.1) *as* $n \to \infty$.

Proof. In view of [4, Lemma 1], the set $\mathscr{C}_{L,\lambda}$ is a convex and compact subset of the Banach space $C[a,b]$ endowed with Bielecki's norm given by the formula

$$\|y\|_B = \max_{x \in [a,b]} |y(x)| e^{-\tau(x-x_0)}$$

for all $y \in C[a,b]$, where $x_0 \in [a,b]$ and $\tau > 0$ are fixed. Let T be defined as in the proof of Theorem 3.3. By assumptions (ii), (iii), and (iv), it follows that

$$T(\mathscr{C}_{L,\lambda}) \subset \mathscr{C}_L.$$

Let us prove that $\mathscr{C}_{L,\lambda}$ is an invariant set with respect to the operator T. Indeed, if $y \in \mathscr{C}_{L,\lambda}$ and $x \in [a,b]$, we have

$$\begin{aligned}(Ty)(x) &\le y_0 + M(x-x_0)\\ &= y_0 + M(x-a) - M(x_0-a)\\ &\le y_0 + \lambda(x-a) - (y_0 - \lambda a) = \lambda x,\end{aligned}$$

that is, $Ty \in \mathscr{C}_{L,\lambda}$, where we have used (iii) and (v).

For $y,z \in \mathscr{C}_{L,\lambda}$ and $x \in [a,b]$, we have

$$\begin{aligned}|(Ty)(x) - (Tz)(x)| &\le L_1 \left|\int_{x_0}^{x} \Big(L|y(s)-z(s)| + |y(z(s)) - z(z(s))|\Big) ds\right|\\ &\le L_1\left(\left|\int_{x_0}^{x} L e^{\tau(s-x_0)} ds\right| + \left|\int_{x_0}^{x} e^{\tau(\lambda s - x_0)} ds\right|\right)\|y-z\|_B\\ &= L_1\left(\frac{L}{\tau}\left|e^{\tau(x-x_0)} - 1\right| + \frac{1}{\tau\lambda}\left|e^{\tau(\lambda x - x_0)} - e^{\tau(\lambda x_0 - x_0)}\right|\right)\|y-z\|_B.\end{aligned}$$

This shows that

$$
\begin{aligned}
&|(Ty)(x)-(Tz)(x)|e^{-\tau(x-x_0)} \\
&\qquad \le L_1\left(\frac{L}{\tau}\left|1-e^{-\tau(x-x_0)}\right|+\frac{1}{\tau\lambda}\left|e^{\tau(\lambda x-x)}-e^{\tau(\lambda x_0-x)}\right|\right)\|y-z\|_B \\
&\qquad = \frac{L_1}{\tau}\left(L\left|1-e^{-\tau(x-x_0)}\right|+\frac{1}{\lambda}\left|e^{\tau(\lambda-1)x}-e^{\tau(\lambda x_0-x)}\right|\right)\|y-z\|_B \\
&\qquad = L_T(x)\,\|y-z\|_B\,,
\end{aligned}
\tag{3.9}
$$

where $L_T : [a,b] \to \mathbb{R}$ is a continuous function. Then there exists a constant $\widetilde{L}_T > 0$ such that

$$\max_{x\in[a,b]} L_T(x) \le \widetilde{L}_T.$$

Thus, by (3.9) we get

$$\|Ty-Tz\|_B \le \widetilde{L}_T\,\|y-z\|_B$$

which shows that T is Lipschitzian, hence continuous. By Schauder's fixed point theorem it follows that T has at least one fixed point $y^* \in \mathscr{C}_L$, which is actually a solution of the initial value problem (3.1).

To prove the second part of the theorem, we evaluate the maximum of $L_T(x)$. It is easy to prove that $g(x) = 1-e^{-\tau(x-x_0)}$ is strictly increasing on $[a,b]$ and $g(x_0) = 0$, so

$$\max_{x\in[a,b]} |g(x)| = \max\left\{e^{\tau(x_0-a)}-1,\, 1-e^{\tau(x_0-b)}\right\}.$$

Similarly, if we set $h(x) = e^{\tau(\lambda-1)x}-e^{\tau(\lambda x_0-x)}$, then

$$h'(x) = \tau e^{\tau(\lambda-1)x}\left(\lambda-1+e^{-\tau\lambda(x-x_0)}\right).$$

It is clear that the function $h_1(x) = \lambda-1+e^{-\tau\lambda(x-x_0)}$ is strictly decreasing on $[a,b]$ and, hence, $h_1(x) \ge h_1(b) = \lambda-1+e^{-\tau\lambda(b-x_0)}$.

If $x_0 = b$, then $h_1(b) = \lambda-1 \le 0$, which shows that h is decreasing on $[a,b]$. If $\lambda = 1$, then $h_1(b) > 0$ and so h is strictly increasing on $[a,b]$. Finally, if $\lambda \ne 1$ and $x_0 \ne b$, then, by assumption (vi), we can choose $\overline{\tau} > 0$ such that

$$\overline{\tau} > -\frac{\ln(1-\lambda)}{\lambda\,(b-x_0)}$$

which implies that $h_1(b) > 0$ and hence h is strictly increasing on $[a,b]$. We put $\tau = \overline{\tau}$. Then, in each of the three cases,

$$\max_{x\in[a,b]} |h(x)| = \max\left\{\left|e^{\tau(\lambda-1)a}-e^{\tau(\lambda x_0-a)}\right|,\left|e^{\tau(\lambda-1)b}-e^{\tau(\lambda x_0-b)}\right|\right\}.$$

Using the fact that $\lambda \le 1$, we have

$$\left|e^{\tau(\lambda-1)a}-e^{\tau(\lambda x_0-a)}\right| = e^{\tau(\lambda-1)a}\left|1-e^{\tau\lambda(x_0-a)}\right|$$

$$= e^{\tau(\lambda-1)a}\left(e^{\tau\lambda(x_0-a)}-1\right)$$
$$\le e^{\tau\lambda(x_0-a)}-1.$$

Similarly,

$$\begin{aligned}\left|e^{\tau(\lambda-1)b}-e^{\tau(\lambda x_0-b)}\right| &= e^{\tau(\lambda-1)b}\left|1-e^{\tau\lambda(x_0-b)}\right|\\ &= e^{\tau(\lambda-1)b}\left(1-e^{\tau\lambda(x_0-b)}\right)\\ &\le 1-e^{\tau\lambda(x_0-b)}.\end{aligned}$$

As a consequence of the results above, we have

$$L_T(x) \le \max\left\{e^{\tau(x_0-a)}-1, 1-e^{\tau(x_0-b)}\right\}\frac{L_1}{\tau}\left(L+\frac{1}{\lambda}\right)$$

for all $x \in [a,b]$, which, by (3.9), shows that T is non-expansive. Now one can use Corollaries 2.1 and 2.2 to get the second part of the theorem. □

4. Concluding remarks and examples

Remark 3. Note that if one can find a $\tau > 0$ such that

$$\max\left\{e^{\tau(x_0-a)}-1, 1-e^{\tau(x_0-b)}\right\}\frac{L_1}{\tau}\left(L+\frac{1}{\lambda}\right) < 1,$$

then, instead of considering the non-expansive mapping principle in Theorem 3.4, we can use the contraction mapping principle similarly to [4, Theorem 5].

Note also that condition (vi) of our Theorem 3.4 is slightly simpler and weaker than the corresponding one appearing in [4]:

$$\max\left\{(\lambda-1)b, (\lambda-1)a, \frac{x_0-a}{\ln 2}, \frac{\lambda(x_0-a)}{\ln 2}\right\} L_1\left(L+\frac{e}{\lambda}\right) < 1.$$

We conclude the paper by presenting two examples which illustrate the generality and efficiency of our results.

Example 3. Consider the following initial value problem associated to an iterative differential equation similar to the ones studied in [8, 31],

$$\begin{cases} y'(x) = -\frac{1}{2} + y(y(x)), \quad x \in [0,1], \\ y\left(\frac{1}{2}\right) = \frac{1}{2}, \end{cases} \tag{4.1}$$

where $y \in C^1([0,1],[0,1])$.

We are interested in the solutions $y \in C^1([0,1],[0,1])$ belonging to the set

$$\mathscr{C}_1 = \left\{y \in C([0,1],[0,1]) : |y(t_1)-y(t_2)| \le |t_1-t_2|,\ \forall\, t_1,t_2 \in [0,1]\right\},$$

which, in view of our notation, means that $L = 1$. We also have $a = 0$, $b = 1$, and $x_0 = \frac{1}{2}$, hence $C_{x_0} = \max\{x_0-a, b-x_0\} = \frac{1}{2}$.

The function $f(x,u) = -\frac{1}{2} + u$ is Lipschitzian with the Lipschitz constant $L_1 = 1$ (i. e., f is non-expansive). This shows that

$$L_1 C_{x_0}(L+1) = 1$$

so condition (3.5) is satisfied but condition (3.3) is not. Therefore, by Theorem 3.3 (but not by Theorem 3.2!) we obtain information on the existence and approximation of the solutions of the initial value problem (4.1).

Note also that the function $y(x) = \frac{1}{2}$, $x \in [0,1]$, is a solution of the initial value problem (4.1).

Example 4. For the iterative differential equation

$$y'(x) = \frac{1}{10} y(y(t)), \qquad x \in [-1,1] \tag{4.2}$$

that has been studied in [33] with respect to equivariance solutions, consider the Cauchy problem with the initial condition

$$y(0) = 1. \tag{4.3}$$

We are interested here in the solutions $y \in C^1([-1,1],[-1,1])$ belonging to the class

$$\mathscr{C}_4 = \big\{ y \in C([0,1],[0,1]) : |y(t_1) - y(t_2)| \le 4|t_1 - t_2|,\ \forall\, t_1, t_2 \in [0,1] \big\}.$$

In this case, we have $L = 4$, $a = -1$, $b = 1$, $x_0 = 0$, and hence $C_{x_0} = \max\{x_0 - a, b - x_0\} = 1$.

The function

$$f(x,u) = \frac{1}{10} u^2$$

is Lipschitzian with the Lipschitz constant $L_1 = \frac{1}{5}$. Therefore, we have

$$L_1 C_{x_0}(L+1) = 1,$$

so condition (3.5) is satisfied but condition (3.3) is not. Therefore, by Theorem 3.3, it follows that the initial value problem (4.2), (4.3) has at least one solution in $\mathscr{C}_4$ that can be approximated by means of the iterative method

$$y_{n+1}(t) = (1-\mu) y_n(t) + \mu y_0 + \frac{\mu}{10} \int_{x_0}^{t} (y_n(y_n(s)))^2 ds, \quad t \in [-1,1],\ n \ge 1,$$

where $\mu \in (0,1)$ and $y_1 \in \mathscr{C}_4$ is arbitrary.

Example 5. Consider the initial value problem defined by the iterative differential equation in Example 3 on the interval $[1/2, 1]$ and the initial condition

$$y\left(\frac{1}{2}\right) = \frac{1}{2}. \tag{4.4}$$

We are interested here to study the solutions $y \in C^1([1/2,1],[1/2,1])$ belonging to the class $\mathscr{C}_{1,1}$. In this case, we have $L = 1$, $a = 1/2$, $b = 1$, $x_0 = 1/2$, $y_0 = 1/2$ and $L_1 = 1$.

In order to have (i) satisfied, we need $\lambda = 1$ and hence, by (iii), $M = 1$. Conditions (iv) and (v) are also satisfied, while the two conditions in (vi) reduce to $\overline{\tau} > 0$ and, respectively, to $\frac{1-e^{-\frac{\overline{\tau}}{2}}}{\overline{\tau}} \leq \frac{1}{2}$, which always have solutions. This becomes obvious if we rewrite the second inequality equivalently as

$$2e^{-\frac{\overline{\tau}}{2}} + \overline{\tau} \geq 2.$$

Thus, all conditions in Theorem 3.4 are satisfied. A solution of the initial problem in $\mathscr{C}_{1,1}$ is $y(x) = \frac{1}{2}$, $x \in [\frac{1}{2}, 1]$. Note that the conditions in Theorem 3.4 are not satisfied on the whole interval $[0,1]$, as $\mathscr{C}_{1,1}$ is a proper subclass of $\mathscr{C}_1$.

Remark 4. In a similar way, by exploiting the technique of non-expansive operators presented in this paper, we can study the iterative differential equations in [9–11, 17, 25, 29, 33], or the functional differential equations with modified argument like those in [20, 21], for which the technique of Picard operators, described in detail in [24, 26–28], is basically used. For example, in the very recent paper [17], a Picard type existence and uniqueness theorem it is obtained for iterative differential equations of the form

$$y'(x) = f(x, y(h(x) + g(y(x)))),$$

a special case of which is the differential equation involved in the initial value problem (3.1) studied in the present paper. As such iterative differential equations are used to model infective disease processes, pattern formation in the plane, and are important in investigations of dynamical systems, future works will be also devoted to them.

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QUENCHING TIME FOR A SYSTEM OF SEMILINEAR HEAT EQUATIONS

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Abstract. This paper concerns the study of the quenching time of the solution of the initial-boundary value problem for a system of reaction-diffusion equations.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with the smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem for a system of reaction-diffusion equations of the form

$$u_t = \varepsilon\Delta u + f(v) \quad \text{in} \quad \Omega\times(0,T), \tag{1.1}$$

$$v_t = \varepsilon\Delta v + g(u) \quad \text{in} \quad \Omega\times(0,T), \tag{1.2}$$

$$u = 0 \quad \text{on} \quad \partial\Omega\times(0,T), \tag{1.3}$$

$$v = 0 \quad \text{on} \quad \partial\Omega\times(0,T), \tag{1.4}$$

$$u(x,0) = u_0(x) \geq 0 \quad \text{in} \quad \Omega, \tag{1.5}$$

$$v(x,0) = v_0(x) \geq 0 \quad \text{in} \quad \Omega, \tag{1.6}$$

where $f:(-\infty,b_1)\to(0,\infty)$ is C^1 convex, increasing function with $b_1 = \text{const} > 0$, $\lim_{s\to b_1} f(s) = +\infty$, $\int_0^{b_1}\frac{ds}{f(s)} < +\infty$, $g:(-\infty,b_2)\to(0,\infty)$ is C^1 convex, increasing function with $b_2 = \text{const} > 0$, $\lim_{s\to b_2} g(s) = +\infty$, $\int_0^{b_2}\frac{ds}{g(s)} < +\infty$. The initial data (u_0,v_0) is such that $u_0\in C^1(\bar{\Omega})$, $v_0\in C^1(\bar{\Omega})$, $u_0(x)\geq 0$ in Ω, $u_0(x) = 0$ on $\partial\Omega$, $v_0(x)\geq 0$ in Ω, $v_0(x) = 0$ on $\partial\Omega$, $\sup_{x\in\Omega} u_0(x) < b_2$, $\sup_{x\in\Omega} v_0(x) < b_1$. Here, $(0,T)$ is the maximal time interval of existence of the solution (u,v). The time T may be finite or infinite. When T is infinite, then we say that the solution u exists globally. When T is finite, then the solution (u,v) develops a singularity in a finite time, namely

$$\left(\lim_{t\to T}\|u(\cdot,t)\|_\infty, \lim_{t\to T}\|v(\cdot,t)\|_\infty\right) \nleq (b_2,b_1),$$

where $\|u(\cdot,t)\|_\infty = \max_{x\in\Omega}|u(x,t)|$. Here, $(a,b) \nleq (c,d)$ means that $(a,b) \leq (c,d)$ and at least one of the equalities $a = c$, $b = d$ is valid.

Solutions for systems of semilinear heat equations which quench in a finite time have been the subject of investigation of several authors (see [3–5] and the references cited therein). By standard methods based on the maximum principle, the local existence, uniqueness, global existence, and quenching have been treated. One may also find in [1, 6, 7, 9, 10] some results about quenching for semilinear heat equations. In this paper, we are interested in the asymptotic behaviour of the quenching time as ε goes to zero. Our work is motivated by the paper [8] by Friedman and Lacey, where they consider the initial-boundary value problem

$$u_t = \varepsilon\Delta u + g(u) \quad \text{in} \quad \Omega\times(0,T), \tag{1.7}$$

$$u = 0 \quad \text{on} \quad \partial\Omega\times(0,T), \tag{1.8}$$

$$u(x,0) = u_0(x) \geq 0 \quad \text{in} \quad \Omega, \tag{1.9}$$

where $g(s)$ is a positive, increasing, convex function for the nonnegative values of s and such that $\int_0^{+\infty}\frac{ds}{g(s)} < +\infty$. The initial data u_0 is a positive and continuous function in $\bar{\Omega}$. Under some additional conditions on the initial data, they showed that the solution u of (1.7)–(1.9) blows up in a finite time, and its blow-up time goes to that of the solution $\alpha(t)$ of the

$$\alpha'(t) = g(\alpha(t)), \quad t > 0, \quad \alpha(0) = M, \tag{1.10}$$

when ε goes to zero, where $M = \sup_{x\in\Omega} u_0(x)$ (we say that a solution u *blows up* in a finite time if it reaches the value infinity in a finite time). Nabongo and Boni have obtained in [10] an analogous result in the case of the phenomenon of quenching for semilinear heat equations. In this article, we get a comparable result for the system described in (1.1)–(1.6). More precisely, under some hypotheses, we show that if ε is small enough, then the solution (u,v) of (1.1)–(1.6) quenches in a finite time, and its quenching time tends to that of the solution $(\alpha(t),\beta(t))$ of the differential system defined below

$$\alpha'(t) = f(\beta(t)), \quad t > 0, \tag{1.11}$$

$$\beta'(t) = g(\alpha(t)), \quad t > 0, \tag{1.12}$$

$$\alpha(0) = M_1, \tag{1.13}$$

$$\beta(0) = M_2, \tag{1.14}$$

as ε goes to zero, where $M_1 = \sup_{x\in\Omega} u_0(x)$, $M_2 = \sup_{x\in\Omega} v_0(x)$.

Our paper is written in the following manner. In the next section, under some conditions, we prove that if ε is small enough, then the solution (u,v) of (1.1)–(1.6) quenches in a finite time, and its quenching time tends to that of the solution of the differential system defined in (1.11)–(1.14). Finally, in the last section, we give some numerical results to illustrate our analysis.

2. Quenching solutions

In this section, under some assumptions, we prove that if ε is small enough, then the solution (u,v) of (1.1)–(1.6) quenches in a finite time, and its quenching time goes to that of the solution of the differential system defined in (1.11)–(1.14) as ε goes to zero. We start by recalling an important result.

Consider the following eigenvalue problem

$$-\Delta\varphi(x) = \lambda\varphi(x) \quad \text{in} \quad \Omega, \tag{2.1}$$

$$\varphi(x) = 0 \quad \text{on} \quad \partial\Omega, \tag{2.2}$$

$$\varphi(x) > 0 \quad \text{in} \quad \Omega. \tag{2.3}$$

It is well known that the above eigenvalue problem admits a solution (φ,λ) such that $\lambda > 0$. We can normalize φ so that $\int_\Omega \varphi(x)dx = 1$. Now, let us give our first result about the quenching time.

Theorem 1. *Suppose that $u_0(x) = 0$ and $v_0(x) = 0$. Let*

$$A = \lambda \max\left\{\frac{b_2}{f(0)}, \frac{b_1}{g(0)}\right\}.$$

If $\varepsilon < \frac{1}{A}$, then the solution (u,v) of (1.1)–(1.6) quenches in a finite time, and its quenching time T obeys the following estimates

$$0 \le T - T_0 \le \varepsilon T_0 A + o(\varepsilon),$$

where T_0 is the quenching time of the solution $(\alpha(t),\beta(t))$ of the differential system defined in (1.11)–(1.14).

Proof. Since $(0,T)$ is the maximal time interval of existence of the solution (u,v), our aim is to show that T is finite and satisfies the above estimates. Introduce the functions $w(t)$ and $z(t)$ defined as follows

$$w(t) = \int_\Omega u(x,t)\varphi(x)dx, \quad z(t) = \int_\Omega v(x,t)\varphi(x)dx, \quad t \in [0,T).$$

Due to the fact that the initial data (u_0,v_0) is nonnegative in Ω, from the maximum principle, (u,v) is also nonnegative in $\Omega\times(0,T)$. Take the derivative of w in t and use (1.1) to obtain

$$w'(t) = \varepsilon\int_\Omega \varphi\Delta u dx + \int_\Omega f(v)\varphi dx \quad \text{for } t \in (0,T).$$

Applying Green's formula, we arrive at the equality

$$w'(t) = \varepsilon \int_\Omega u \Delta\varphi dx + \int_\Omega f(v)\varphi dx \quad \text{for } t \in (0,T).$$

It follows from (2.1) and Jensen's inequality that

$$w'(t) \geq -\varepsilon\lambda w(t) + f(z(t)) \quad \text{for } t \in (0,T).$$

In the same way, we also get

$$z'(t) \geq -\varepsilon\lambda z(t) + g(w(t)) \quad \text{for } t \in (0,T).$$

It is not difficult to see that $0 \leq w(t) \leq b_2$ and $0 \leq z(t) \leq b_1$. We deduce that

$$w'(t) \geq -\varepsilon\lambda b_2 + f(z(t)) \quad \text{for } t \in (0,T),$$
$$z'(t) \geq -\varepsilon\lambda b_1 + g(w(t)) \quad \text{for } t \in (0,T),$$

whence

$$w'(t) \geq f(z(t))(1 - \frac{\varepsilon\lambda b_2}{f(z(t))}) \quad \text{for } t \in (0,T),$$
$$z'(t) \geq g(w(t))(1 - \frac{\varepsilon\lambda b_1}{g(w(t))}) \quad \text{for } t \in (0,T).$$

Since $f(z(t)) \geq f(0)$ and $g(w(t)) \geq g(0)$, we arrive at the inequalities

$$w'(t) \geq f(z(t))(1 - \frac{\varepsilon\lambda b_2}{f(0)}) \quad \text{for } t \in (0,T),$$
$$z'(t) \geq g(w(t))(1 - \frac{\varepsilon\lambda b_1}{g(0)}) \quad \text{for } t \in (0,T).$$

Taking into account the fact that $\frac{\lambda b_2}{f(0)} \leq A$ and $\frac{\lambda b_1}{g(0)} \leq A$, we find that

$$w'(t) \geq f(z(t))(1 - \varepsilon A) \quad \text{for } t \in (0,T),$$
$$z'(t) \geq g(w(t))(1 - \varepsilon A) \quad \text{for } t \in (0,T).$$

Let us set

$$w_1(t) = w\left(\frac{t}{1-\varepsilon A}\right) \quad \text{for } t \in (0,(1-\varepsilon A)T),$$
$$z_1(t) = z\left(\frac{t}{1-\varepsilon A}\right) \quad \text{for } t \in (0,(1-\varepsilon A)T).$$

A straightforward computation reveals that

$$w_1'(t) \geq f(z_1(t)) \quad \text{for } t \in (0,(1-\varepsilon A)T), \qquad w_1(0) = w(0) = 0,$$
$$z_1'(t) \geq g(w_1(t)) \quad \text{for } t \in (0,(1-\varepsilon A)T), \qquad z_1(0) = z(0) = 0.$$

From the maximum principle, we get

$$w_1(t) \geq \alpha(t) \quad \text{for } t \in (0,T_*),$$

$$z_1(t) \geq \beta(t) \quad \text{for } t \in (0, T_*),$$

where $T_* = \min\{T_0, (1-\varepsilon A)T\}$, which implies that

$$T \leq \frac{T_0}{1-\varepsilon A}. \tag{2.4}$$

In fact, suppose that $T > \frac{T_0}{1-\varepsilon A} = T'$. We deduce that

$$\big(w(T'), z(T')\big) = \big(w_1(T_0), z_1(T_0)\big) \geq \big(\alpha(T_0), \beta(T_0)\big) \not\leq (b_2, b_1).$$

Since

$$\big(\|u(\cdot, T')\|_\infty, \|v(\cdot, T')\|_\infty\big) \geq \big(w(T'), z(T')\big),$$

we have a contradiction because $(0, T)$ is the maximum time interval of existence of (u, v). On the other hand, let $(u_1(x,t), v_1(x,t))$ be such that

$$u_1(x,t) = \alpha(t) \quad \text{in} \quad \bar{\Omega} \times [0, T_0),$$
$$v_1(x,t) = \beta(t) \quad \text{in} \quad \bar{\Omega} \times [0, T_0).$$

A routine computation reveals that

$$u_{1t} = \varepsilon \Delta u_1 + f(v_1) \quad \text{in} \quad \Omega \times (0, T_0),$$
$$v_{1t} = \varepsilon \Delta v_1 + g(u_1) \quad \text{in} \quad \Omega \times (0, T_0),$$

$u_1 \geq 0$ on $\partial\Omega \times (0, T_0)$, $v_1 \geq 0$ on $\partial\Omega \times (0, T_0)$, and

$$u_1(x, 0) \geq 0 \quad \text{in} \quad \Omega,$$
$$v_1(x, 0) \geq 0 \quad \text{in} \quad \Omega.$$

The maximum principle implies that

$$0 \leq u(x,t) \leq u_1(x,t) = \alpha(t) \quad \text{in} \quad \Omega \times (0, T^0),$$
$$0 \leq v(x,t) \leq v_1(x,t) = \beta(t) \quad \text{in} \quad \Omega \times (0, T^0),$$

where $T^0 = \min\{T, T_0\}$. We deduce that

$$T \geq T_0. \tag{2.5}$$

Indeed, assume that $T < T_0$. We get

$$\big(\|u(\cdot, T)\|_\infty, \|v(\cdot, T)\|_\infty\big) \leq (\alpha(T), \beta(T)) < (b_2, b_1),$$

which is a contradiction because $(0, T)$ is the maximal time interval of existence of the solution (u, v). Applying Taylor's expansion, we obtain

$$\frac{1}{1-\varepsilon A} = 1 + \varepsilon A + o(\varepsilon).$$

Using (2.4), (2.5) and the relation above, we complete the proof. □

Now, let us consider the case where the initial data is not null. In the sequel, we suppose that there exists $a \in \Omega$ such that

$$\sup_{x\in\Omega} u_0(x) = u_0(a) \quad \text{and} \quad \sup_{x\in\Omega} v_0(x) = v_0(a).$$

Consider the following eigenvalue problem

$$-\Delta\psi(x) = \lambda_\delta \psi(x) \quad \text{in} \quad B(a,\delta), \tag{2.6}$$

$$\psi(x) = 0 \quad \text{on} \quad \partial B(a,\delta), \tag{2.7}$$

$$\psi(x) > 0 \quad \text{in} \quad B(a,\delta), \tag{2.8}$$

where $\delta > 0$, such that, $B(a,\delta) = \{x \in \mathbb{R}^N; \|x-a\| < \delta\} \subset \Omega$.

It is well known that the above problem has a solution (ψ,λ_δ) such that $\lambda_\delta = \frac{\lambda_1}{\delta^2}$, where λ_1 is the eigenvalue corresponding to the above eigenvalue problem for $\delta = 1$.

We are in position to state our result in the case where the initial data is not null.

Theorem 2. *Suppose that $M_1 = \sup_{x\in\Omega} u_0(x) > 0$, $M_2 = \sup_{x\in\Omega} v_0(x) > 0$, and let K be an upper bound of the first derivatives of u_0 and v_0. Let*

$$A = \lambda_1 \max\left\{\frac{K^2 b_2}{f(0)}, \frac{K^2 b_1}{g(0)}\right\}.$$

If

$$\varepsilon < \min\left\{(M_1/2)^3, (M_2/2)^3, (2A)^{-3}, (K\,\mathrm{dist}(a,\partial\Omega))^3\right\},$$

then the solution (u,v) of (1.1)–(1.6) quenches in a finite time, and its quenching time T obeys the estimates

$$0 \le T - T_0 \le (T_0 A + C)\varepsilon^{1/3} + o\big(\varepsilon^{1/3}\big),$$

where

$$C = \frac{1}{\min\left\{\frac{1}{2} f\left(\frac{M_2}{2}\right), \frac{1}{2} g\left(\frac{M_1}{2}\right)\right\}}$$

and T_0 is the quenching time of the solution $(\alpha(t),\beta(t))$ of the differential system defined in (1.11)–(1.14).

Proof. Due to the fact that $u_0 \in C^1(\bar{\Omega})$ and $v_0 \in C^1(\bar{\Omega})$, using the mean value theorem and the triangle inequality, we get

$$u_0(x) \ge u_0(a) - \varepsilon^{1/3} \quad \text{for } x \in B(a,\delta) \subset \Omega,$$

$$v_0(x) \ge v_0(a) - \varepsilon^{1/3} \quad \text{for } x \in B(a,\delta) \subset \Omega,$$

where $\delta = \frac{\varepsilon^{1/3}}{K}$. Since the initial data (u_0,v_0) is nonnegative in Ω, from the maximum principle, (u,v) is also nonnegative in $\Omega\times(0,T)$. Introduce the functions $w(t)$ and $z(t)$ defined as follows

$$w(t) = \int_{B(a,\delta)} u(x,t)\psi(x)dx, \quad z(t) = \int_{B(a,\delta)} v(x,t)\psi(x)dx, \quad t \in [0,T).$$

Take the derivative of w in t and use (1.1) to obtain

$$w'(t) = \varepsilon \int_{B(a,\delta)} \psi \Delta u dx + \int_{B(a,\delta)} f(v)\psi dx \quad \text{for } t \in (0,T).$$

Applying Green's formula, we arrive at

$$\begin{aligned} w'(t) = &\varepsilon \int_{B(a,\delta)} u\Delta\psi dx - \varepsilon \int_{\partial B(a,\delta)} u \frac{\partial \psi}{\partial \nu} ds + \varepsilon \int_{\partial B(a,\delta)} \psi \frac{\partial u}{\partial \nu} ds \\ &+ \int_{B(a,\delta)} f(v)\psi dx \quad \text{for } t \in (0,T), \end{aligned}$$

where ν is the exterior normal unit vector on $\partial B(a,\delta)$. Taking into account (2.6), (2.7) and the fact that $\frac{\partial \psi}{\partial \nu} \leq 0$, we arrive at the relation

$$w'(t) \geq -\varepsilon\lambda_\delta w(t) + \int_{B(a,\delta)} f(v)\psi dx \quad \text{for } t \in (0,T).$$

It follows from Jensen's inequality that

$$w'(t) \geq -\varepsilon\lambda_\delta w(t) + f(z(t)) \quad \text{for } t \in (0,T).$$

In the same way, we also prove that

$$z'(t) \geq -\varepsilon\lambda_\delta z(t) + g(w(t)) \quad \text{for } t \in (0,T).$$

As in the proof of Theorem 1, we get

$$\begin{aligned} w'(t) &\geq f(z(t)) \left(1 - \frac{\varepsilon\lambda_\delta b_2}{f(0)}\right) \quad \text{for } t \in (0,T), \\ z'(t) &\geq g(w(t)) \left(1 - \frac{\varepsilon\lambda_\delta b_1}{g(0)}\right) \quad \text{for } t \in (0,T), \end{aligned}$$

which implies that

$$\begin{aligned} w'(t) &\geq f(z(t)) \left(1 - \frac{\varepsilon^{1/3}\lambda_1 K^2 b_2}{f(0)}\right) \quad \text{for } t \in (0,T), \\ z'(t) &\geq g(w(t)) \left(1 - \frac{\varepsilon^{1/3}\lambda_1 K^2 b_1}{g(0)}\right) \quad \text{for } t \in (0,T), \end{aligned}$$

because $\lambda_\delta = \frac{\lambda_1}{\delta^2} = \frac{\lambda_1 K^2}{\varepsilon^{2/3}}$. Consequently,

$$\begin{aligned} &w'(t) \geq f(z(t)) \left(1 - \varepsilon^{1/3} A\right) \quad \text{for } t \in (0,T), \\ &w(0) \geq M_1 - \varepsilon^{1/3}, \end{aligned}$$

and

$$z'(t) \geq g(w(t)) \left(1 - \varepsilon^{1/3} A\right) \quad \text{for } t \in (0,T),$$

$$z(0) \geq M_2 - \varepsilon^{1/3}.$$

We have

$$w'(t) \geq f(z(0))(1-\varepsilon^{1/3}A) \geq \frac{1}{2} f(M_2/2)$$

for $t \in (0,T)$. In the same way, we get $z'(t) \geq \frac{1}{2} g(M_1/2)$ for $t \in (0,T)$, which implies that

$$w'(t) \geq \frac{1}{C}, \quad z'(t) \geq \frac{1}{C}$$

for $t \in (0,T)$. Using the mean value theorem, we get

$$w\big(C\varepsilon^{1/3}\big) \geq M_1, \quad z\big(C\varepsilon^{1/3}\big) \geq M_2.$$

Set

$$w_1(t) = w\left(\frac{t}{1-\varepsilon^{1/3}A} + C\varepsilon^{1/3}\right)$$

for $t \in \big(0, (T - C\varepsilon^{1/3})(1-\varepsilon^{1/3}A)\big)$, and

$$z_1(t) = z\left(\frac{t}{1-\varepsilon^{1/3}A} + C\varepsilon^{1/3}\right)$$

for $t \in \big(0, (T - C\varepsilon^{1/3})(1-\varepsilon^{1/3}A)\big)$. A straightforward computation reveals that

$$\begin{aligned} &w_1'(t) \geq f(z_1(t)) \quad \text{for } t \in \big(0, (T - C\varepsilon^{1/3})(1-\varepsilon^{1/3}A)\big), \\ &w_1(0) \geq M_1 \end{aligned}$$

and

$$\begin{aligned} &z_1'(t) \geq g(w_1(t)) \quad \text{for } t \in \big(0, (T - C\varepsilon^{1/3})(1-\varepsilon^{1/3}A)\big), \\ &z_1(0) \geq M_2. \end{aligned}$$

It follows from the maximum principle that

$$\begin{aligned} w_1(t) &\geq \alpha(t) \quad \text{for } t \in (0,T^*), \\ z_1(t) &\geq \beta(t) \quad \text{for } t \in (0,T^*), \end{aligned}$$

where $T^* = \min\{T_0, (T - C\varepsilon^{1/3})(1-\varepsilon^{1/3}A)\}$. We deduce that

$$T \leq \frac{T_0}{1-\varepsilon^{1/3}A} + C\varepsilon^{1/3}. \tag{2.9}$$

Indeed, suppose that

$$T > \frac{T_0}{1-\varepsilon^{1/3}A} + C\varepsilon^{1/3} = T'.$$

We get

$$\big(w(T'), z(T')\big) = (w_1(T_0), z_1(T_0)) \geq (\alpha(T_0), \beta(T_0)) \not\leq (b_2, b_1).$$

Since $\big(\|u(\cdot,T')\|_\infty, \|v(\cdot,T')\|_\infty\big) \geq (w(T'), z(T'))$, we have a contradiction because $(0,T)$ is the maximal time interval of existence of the solution (u,v). On the other hand, setting

$$w_2(x,t) = \alpha(t) \quad \text{in} \quad \bar{\Omega}\times[0,T_0),$$
$$z_2(x,t) = \beta(t) \quad \text{in} \quad \bar{\Omega}\times[0,T_0),$$

it is not difficult to see that

$$(w_2)_t = \varepsilon\Delta w_2 + f(z_2) \quad \text{in} \quad \Omega\times(0,T_0),$$
$$(z_2)_t = \varepsilon\Delta z_2 + g(w_2) \quad \text{in} \quad \Omega\times(0,T_0),$$
$$w_2 \geq 0 \quad \text{on} \quad \partial\Omega\times(0,T_0),$$
$$z_2 \geq 0 \quad \text{on} \quad \partial\Omega\times(0,T_0),$$
$$w_2(x,0) \geq u_0(x) \quad \text{in} \quad \Omega,$$
$$z_2(x,0) \geq v_0(x) \quad \text{in} \quad \Omega.$$

It follows from the maximum principle that

$$\alpha(t) = w_2(x,t) \geq u(x,t) \quad \text{in} \quad \Omega\times(0,T_*^*),$$
$$\beta(t) = z_2(x,t) \geq v(x,t) \quad \text{in} \quad \Omega\times(0,T_*^*),$$

where $T_*^* = \min\{T_0, T\}$. We deduce that

$$T \geq T_0. \tag{2.10}$$

Indeed, suppose that $T < T_0$. We get

$$\big(\|u(\cdot,T)\|_\infty, \|v(\cdot,T)\|_\infty\big) \leq (\alpha(T),\beta(T)) < (b_2, b_1),$$

which is a contradiction because $(0,T)$ is the maximal time interval of existence of the solution (u,v). Applying Taylor's expansion, we obtain

$$\frac{1}{1-\varepsilon^{1/3}A} = 1 + \varepsilon^{1/3} + o(\varepsilon^{1/3}).$$

Using (2.9), (2.10) and the above relation, we complete the proof. □

3. Numerical experiments

In this section, we give some computational results to confirm the theory established in the previous section. We consider the radial symmetric solution of the problem (1.1)–(1.6) in the case where $\Omega = B = \{x \in \mathbb{R}^N; \|x\| < 1\}$, $\partial\Omega = S = \{x \in \mathbb{R}^N; \|x\| = 1\}$, $f(v) = (1-v)^{-p}$, $g(u) = (2-u)^{-q}$ with $p > 0$, $q > 0$. The above problem may be rewritten in the following form

$$u_t = \varepsilon\Big(u_{rr} + \frac{N-1}{r}u_r\Big) + (1-v)^{-p}, \quad r \in (0,1), \quad t \in (0,T), \tag{3.1}$$

$$v_t = \varepsilon\Big(v_{rr} + \frac{N-1}{r}v_r\Big) + (2-u)^{-q}, \quad r \in (0,1), \quad t \in (0,T), \tag{3.2}$$

$$u_r(0,t) = 0, \quad u(1,t) = 0, \quad t \in (0,T), \tag{3.3}$$

$$v_r(0,t) = 0, \quad v(1,t) = 0, \quad t \in (0,T), \tag{3.4}$$

$$u(r,0) = \varphi(r), \quad r \in (0,1), \tag{3.5}$$

$$v(r,0) = \psi(r), \quad r \in (0,1). \tag{3.6}$$

Here, we take

$$\varphi(r) = a\sin(\pi r),$$
$$\psi(r) = b\sin(\pi r),$$

with $a \in [0,2)$, $b \in [0,1)$. We begin by the construction of some adaptive schemes as follows.

Let I be a positive integer and let $h = 1/I$. Define the grid $x_i = ih$, $0 \le i \le I$, and approximate the solution (u,v) of (3.1)–(3.6) by the solution $\left(U_h^{(n)}, V_h^{(n)}\right)$ of the explicit scheme

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \varepsilon N \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} + \left(1 - V_0^{(n)}\right)^{-p},$$

$$\frac{V_0^{(n+1)} - V_0^{(n)}}{\Delta t_n} = \varepsilon N \frac{2V_1^{(n)} - 2V_0^{(n)}}{h^2} + \left(2 - U_0^{(n)}\right)^{-q},$$

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \varepsilon \left(\frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \frac{(N-1)}{ih} \frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h} \right)$$
$$+ \left(1 - V_i^{(n)}\right)^{-p}, \quad 1 \le i \le I-1,$$

$$\frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t_n} = \varepsilon \left(\frac{V_{i+1}^{(n)} - 2V_i^{(n)} + V_{i-1}^{(n)}}{h^2} + \frac{(N-1)}{ih} \frac{V_{i+1}^{(n)} - V_{i-1}^{(n)}}{2h} \right)$$
$$+ \left(2 - U_i^{(n)}\right)^{-q}, \quad 1 \le i \le I-1,$$

$$U_I^{(n)} = 0, \qquad V_I^{(n)} = 0,$$

$$U_i^{(0)} = a\sin(\pi ih), \quad 0 \le i \le I,$$

$$V_i^{(0)} = b\sin(\pi ih), \quad 0 \le i \le I,$$

where $n \ge 0$ and $U_h^{(n)} = \left(U_0^{(n)}, U_1^{(n)}, \ldots, U_I^{(n)}\right)^T$, $V_h^{(n)} = \left(V_0^{(n)}, V_1^{(n)}, \ldots, V_I^{(n)}\right)^T$.

In order to permit the discrete solution to reproduce the property of the continuous one when the time t approaches the quenching time T, we need to adapt the size of the time step so that we take

$$\Delta t_n = \min\left\{ \frac{h^2}{2N\varepsilon}, \ h^2\left(1 - \|V_h^{(n)}\|_\infty\right)^{p+1}, \ h^2\left(2 - \|U_h^{(n)}\|_\infty\right)^{q+1} \right\}.$$

We also approximate the solution (u,v) of (3.1)–(3.6) by the solution $\big(U_h^{(n)}, V_h^{(n)}\big)$ of the following implicit scheme:

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \varepsilon N \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} + \big(1 - V_0^{(n)}\big)^{-p},$$

$$\frac{V_0^{(n+1)} - V_0^{(n)}}{\Delta t_n} = \varepsilon N \frac{2V_1^{(n+1)} - 2V_0^{(n+1)}}{h^2} + \big(2 - U_0^{(n)}\big)^{-q},$$

$$\begin{aligned}\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} &= \varepsilon \left(\frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + \frac{(N-1)}{ih} \frac{U_{i+1}^{(n+1)} - U_{i-1}^{(n+1)}}{2h} \right) \\ &\quad + \big(1 - V_i^{(n)}\big)^{-p}, \quad 1 \le i \le I-1,\end{aligned}$$

$$\begin{aligned}\frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t_n} &= \varepsilon \left(\frac{V_{i+1}^{(n+1)} - 2V_i^{(n+1)} + V_{i-1}^{(n+1)}}{h^2} + \frac{(N-1)}{ih} \frac{V_{i+1}^{(n+1)} - V_{i-1}^{(n+1)}}{2h} \right) \\ &\quad + \big(2 - U_i^{(n)}\big)^{-q}, \quad 1 \le i \le I-1,\end{aligned}$$

$$U_I^{(n+1)} = 0, \qquad V_I^{(n+1)} = 0,$$

$$U_i^{(0)} = a \sin(\pi i h), \quad 0 \le i \le I,$$

$$V_i^{(0)} = b \sin(\pi i h), \quad 0 \le i \le I.$$

Here, similarly to the case of the explicit scheme, we choose

$$\Delta t_n = \min \Big\{ h^2 \big(1 - \|V_h^{(n)}\|_\infty\big)^{p+1},\ h^2 \big(2 - \|U_h^{(n)}\|_\infty\big)^{q+1} \Big\}.$$

We note that

$$\lim_{r \to 0} \frac{u_r(r,t)}{r} = u_{rr}(0,t).$$

Hence, if $t = 0$, then we have

$$u_t(0,t) = \varepsilon N u_{rr}(0,t) + (1 - v(0,t))^{-p}.$$

This observation has been used in the construction of our schemes when $i = 0$. Let us notice that in the explicit scheme, the restriction on the time step ensures the nonnegativity of the discrete solution. For the implicit scheme, the existence and nonnegativity are also guaranteed by standard methods (see, e. g., [2]).

We need the following definition.

Definition 1. We say that the discrete solution $\big(U_h^{(n)}, V_h^{(n)}\big)$ of the explicit or implicit scheme *quenches* in a finite time if

$$\left(\lim_{n \to +\infty} \|U_h^{(n)}\|_\infty, \lim_{n \to +\infty} \|V_h^{(n)}\|_\infty \right) \not\leq (2,1),$$

and the series $\sum_{n=0}^{+\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{+\infty} \Delta t_n$ is called the *numerical quenching time* of the solution $\big(U_h^{(n)}, V_h^{(n)}\big)$.

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256. We take for the numerical quenching time

$$T^n = \sum_{j=0}^{n-1} \Delta t_j,$$

which is computed at the first time when

$$|T^{n+1} - T^n| \leq 10^{-16}.$$

The order s of the method is computed according to the formula

$$s = \frac{\log\big((T_{4h} - T_{2h})/(T_{2h} - T_h)\big)}{\log(2)}.$$

3.1. *Numerical experiments for $p = 1$, $q = 1$, $a = 0$, $b = 0$, $N = 2$*

3.1.1. *First case: $\varepsilon = \frac{1}{10}$*

I	T^n	n	CPU time	s
16	1.011490	8172	–	–
32	1.010694	31009	1	–
64	1.010360	118245	4	1.26
128	1.010207	450669	30	1.13
256	1.010135	1714170	230	1.10

TABLE 1. Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU time	s
16	1.015745	9004	–	–
32	1.014494	34740	3	–
64	1.014196	134302	16	2.08
128	1.014124	518394	118	2.06
256	1.014107	1996487	954	2.09

TABLE 2. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method.

3.1.2. *Second case:* $\varepsilon = \frac{1}{100}$

I	T^n	n	CPU time	s
16	1.000142	8084	–	–
32	1.000100	30610	1	–
64	1.000088	116756	4	1.81
128	1.000022	454329	31	2.46
256	1.000006	1726227	231	2.05

TABLE 3. Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU time	s
16	1.000288	8087	–	–
32	1.000134	30620	1	–
64	1.000100	116520	14	2.18
128	1.000093	443127	101	2.28
256	1.000091	1681491	861	1.81

TABLE 4. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method.

3.2. *Numerical experiments for* $p = 1,\ q = 1,\ a = 0,\ b = 0,\ N = 2$

3.2.1. *First case:* $\varepsilon = \frac{1}{100}$

I	T^n	n	CPU time	s
16	0.393420	11468	–	–
32	0.393698	43641	1	–
64	0.393784	166383	6	1.70
128	0.393809	633406	45	1.78
256	0.393817	2405445	338	1.65

TABLE 5. Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU time	s
16	0.393440	11469	–	–
32	0.393703	43641	2	–
64	0.393785	166384	19	1.69
128	0.393810	633407	153	1.72
256	0.393817	2405445	1093	1.84

TABLE 6. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

3.2.2. *Second case:* $\varepsilon = \frac{1}{500}$

I	T^n	n	CPU time	s
16	0.375615	11613	–	–
32	0.375680	44079		–
64	0.375702	167676	6	1.57
128	0.375709	636904	45	1.66
256	0.375711	2413063	354	1.81

TABLE 7. Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU time	s
16	0.375619	11613	–	–
32	0.375682	44079	3	–
64	0.375702	167677	19	1.66
128	0.375709	636904	150	1.52
256	0.375711	2413063	1120	1.81

TABLE 8. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method.

Remark. If we consider the problem (3.1)–(3.6) in the case where the initial data is null and $p = 1$, $q = 1$, it is not difficult to see that the quenching time of the solution of the differential system defined in (1.11)–(1.14) equals one. We observe from Tables 1–4 that when ε diminishes, then the numerical quenching time tends to one. This result has been proved in Theorem 1.

When the initial data $(\varphi(r), \psi(r))$ are such that $\varphi(r) = \frac{1}{2}\sin(\pi r)$ and $\psi(r) = \frac{1}{2}\sin(\pi r)$, and $p = 1$, $q = 1$, then we see that the quenching time of the solution of the differential system defined in (1.11)–(1.14) equals 0.375. We observe from Tables 5–8 that when ε diminishes, then the numerical quenching time decays to 0.375. This result has been established in Theorem 2.

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MULTIVALENT FUNCTIONS BASED ON A LINEAR INTEGRAL OPERATOR

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Abstract. By using a linear integral operator, a subclass of multivalent functions is introduced. Some important properties of this class such as coefficient bounds are found. Moreover, applications are also introduced.

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1. Introduction

Let $T(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in \mathbb{N},\, z \in U), \tag{1.1}$$

which are analytic and p-valent (multivalent) in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Given two functions $f, g \in T(p)$, $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ and $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$ their convolution or Hadamard product $f(z) * g(z)$ is defined by

$$f(z) * g(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n, \quad (z \in U).$$

A function $f \in T(p)$ is said to be p-valent starlike of order μ, $0 \le \mu < p$ if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \mu, \quad (z \in U),$$

this class denoted by $S_p^*(\mu)$. A function $f \in T(p)$ is said to be p-valent convex of order μ, $0 \le \mu < p$ if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \mu, \quad (z \in U),$$

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this class denoted by $C_p(\mu)$.

For a function $f \in T(p)$ given by (1.1), we define the linear integral operator $I_{p,\alpha,\beta} f(z)$ by

$$\begin{aligned} I_{p,\alpha,\beta} f(z) &= (\alpha + p\beta + 1) \int_0^1 u^\alpha f(u^\beta z) du \\ &= z^p + \sum_{n=p+1}^{\infty} \left[\frac{\alpha + p\beta + 1}{\alpha + n\beta + 1} \right] a_n z^n, \quad z \in U, \end{aligned} \tag{1.2}$$

$(\alpha, \beta \in \mathbb{R},\ p \in \mathbb{N})$. Clearly, (1.2) yields

$$f \in T(p) \Rightarrow I_{p,\alpha,\beta} \in T(p).$$

Thus, by applying the operator $I_{p,\alpha,\beta}$ successively, we can obtain

$$\begin{aligned} I^k_{p,\alpha,\beta} f(z) &= \begin{cases} I_{p,\alpha,\beta}\big(I^{k-1}_{p,\alpha,\beta} f(z)\big) & \text{for } k \in \mathbb{N}, \\ f(z) & \text{for } k = 0, \end{cases} \\ &= z^p + \sum_{n=p+1}^{\infty} \left[\frac{\alpha + p\beta + 1}{\alpha + n\beta + 1} \right]^k a_n z^n, \quad (z \in U). \end{aligned} \tag{1.3}$$

Note that when $p = 1$, $\alpha = -1$ and $\beta = 1$, operator (1.3) poses an integral operator defined by Sălăgean (see [9]). Further operators of type (1.2) involving different type of integral operators were introduced by many authors (cf. [1, 3–5, 8]).

Let $S^*_p(\alpha, \beta, \mu)$ denote the class of all functions $f \in T(p)$ satisfying the condition

$$\Re \left\{ \frac{z[I^k_{p,\alpha,\beta} f(z)]'}{I^k_{p,\alpha,\beta} f(z)} \right\} > \mu, \quad (0 \le \mu < p,\ z \in U)$$

and let $C_p(\alpha, \beta, \mu)$ be the class of all functions $f \in T(p)$ such that

$$\Re \left\{ \frac{z[I^k_{p,\alpha,\beta} f(z)]''}{[I^k_{p,\alpha,\beta} f(z)]'} \right\} > \mu, \quad (0 \le \mu < p,\ z \in U).$$

In the present paper, using Jack's lemma, the authors investigate the differential inequality

$$\left| \frac{I^k_{p,\alpha,\beta} f(z)}{z^p} + \frac{z[I^k_{p,\alpha,\beta} f(z)]'}{p I^k_{p,\alpha,\beta} f(z)} - 2 \right| < \mu, \quad (0 \le \mu < p,\ z \in U). \tag{1.4}$$

Let $S^{**}_p(\alpha, \beta, \mu)$ denote the class of analytic functions $f \in T(p)$ satisfying the inequality (1.4).

To establish our results, we need the following lemma given by Jack [2] (see also [7]).

Lemma 1. *Let $w(z)$ be analytic in U with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point z_0, then*

$$z_0 w'(z_0) = c w(z_0), \tag{1.5}$$

where c is a real number and $c \geq 1$.

2. Main results

In this section, we establish some theorems involving inequalities on p-valent functions. We first prove the following theorem.

Theorem 1. *If $f(z) \in T(p)$ satisfies the inequality*

$$\Re\left\{1+\frac{z[I^k_{p,\alpha,\beta}f(z)]''}{[I^k_{p,\alpha,\beta}f(z)]'}-p\right\} < \frac{2\mu}{\mu+2}, \quad (0 \leq \mu < p,\ z \in U), \tag{2.1}$$

then $f \in S^{**}_p(\alpha,\beta,\mu)$.

Proof. Let the function $f(z) \in T(p)$ and satisfy the inequality (2.1). Define the function $w(z)$, $z \in U$, such that

$$\frac{I^k_{p,\alpha,\beta}f(z)}{z^p} = 1+\frac{\mu}{2}w(z), \quad (z \in U,\ p \in \mathbb{N}), \tag{2.2}$$

we have that $w(z)$ is analytic in U and $w(0)=0$. Differentiating both sides of (2.2), we obtain

$$\frac{[I^k_{p,\alpha,\beta}f(z)]'}{z^{p-1}} = \frac{\mu}{2}zw'(z)+p\left(1+\frac{\mu}{2}w(z)\right), \quad (z \in U,\ p \in \mathbb{N},\ p>k,\ k \in \mathbb{N}_0). \tag{2.3}$$

Then, we have from (2.2) and (2.3) that

$$\begin{aligned} F(z) &:= \frac{z[I^k_{p,\alpha,\beta}f(z)]'}{I^k_{p,\alpha,\beta}f(z)} - p \\ &= \frac{\frac{\mu}{2}zw'(z)}{1+\frac{\mu}{2}w(z)}, \quad (z \in U,\ p \in \mathbb{N},\ p>k,\ k \in \mathbb{N}_0). \end{aligned} \tag{2.4}$$

Now, suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1.$$

Then, using Lemma 1 and letting $w(z_0)=e^{i\theta}$ such that $w(z_0)\neq -1$ in the equation (2.4), yields

$$\begin{aligned}\Re\{F(z_0)\} &= \Re\left\{\frac{\frac{\mu}{2}z_0w'(z_0)}{1+\frac{\mu}{2}w(z_0)}\right\} = \Re\left\{\frac{\frac{\mu c}{2}w(z_0)}{1+\frac{\mu}{2}w(z_0)}\right\} \\ &= c\Re\left\{\frac{\frac{\mu}{2}e^{i\theta}}{1+\frac{\mu}{2}e^{i\theta}}\right\} = c\,\frac{\mu}{2+\mu} \geq \frac{\mu}{2+\mu}.\end{aligned} \tag{2.5}$$

On the other hand define the function $\overline{w}(z)$, $z\in U$, such that

$$\frac{z[I^k_{p,\alpha,\beta}f(z)]'}{pI^k_{p,\alpha,\beta}f(z)} = 1+\frac{\mu}{2}\,\overline{w}(z), \quad (z\in U,\ p\in \mathbb{N}), \tag{2.6}$$

we find that $\overline{w}(z)$ is analytic in U and $\overline{w}(0)=0$. Differentiating both sides of (2.6), yields

$$\begin{aligned}&\frac{z}{p}\left\{\frac{z[I^k_{p,\alpha,\beta}f(z)]''}{I^k_{p,\alpha,\beta}f(z)} - z\left(\frac{[I^k_{p,\alpha,\beta}f(z)]'}{I^k_{p,\alpha,\beta}f(z)}\right)^2\right\} \\ &\quad= \frac{\mu}{2}z\overline{w}'(z) - \left(1+\frac{\mu}{2}\overline{w}(z)\right), \quad (z\in U,\ p\in\mathbb{N},\ p>k,\ k\in\mathbb{N}_0).\end{aligned} \tag{2.7}$$

Then, we have from (2.6) and (2.7)

$$\begin{aligned}\bar{F}(z) &:= 1+\frac{z[I^k_{p,\alpha,\beta}f(z)]''}{[I^k_{p,\alpha,\beta}f(z)]'} - \frac{z[I^k_{p,\alpha,\beta}f(z)]'}{I^k_{p,\alpha,\beta}f(z)} \\ &= \frac{\frac{\mu}{2}z\overline{w}'(z)}{1+\frac{\mu}{2}\overline{w}(z)}, \quad (z\in U,\ p\in\mathbb{N},\ p>k,\ k\in\mathbb{N}_0).\end{aligned} \tag{2.8}$$

Now, suppose that there exists a point $\zeta\in U$ such that

$$\max_{|z|\leq|\zeta|} |\overline{w}(z)| = |\overline{w}(\zeta)| = 1.$$

Again in view of Lemma 1 and letting $\overline{w}(\zeta)=e^{i\vartheta}$ such that $\overline{w}(\zeta)\neq -1$ in the equation (2.8), we receive

$$\begin{aligned}\Re\{\bar{F}(\zeta)\} &= \Re\left\{\frac{\frac{\mu}{2}\zeta\overline{w}'(\zeta)}{1+\frac{\mu}{2}\overline{w}(\zeta)}\right\} = \Re\left\{\frac{\frac{\mu c}{2}\overline{w}(\zeta)}{1+\frac{\mu}{2}\overline{w}(\zeta)}\right\} \\ &= c\Re\left\{\frac{\frac{\mu}{2}e^{i\vartheta}}{1+\frac{\mu}{2}e^{i\vartheta}}\right\} = c\,\frac{\mu}{2+\mu} \geq \frac{\mu}{2+\mu}.\end{aligned} \tag{2.9}$$

Combining (2.5) and (2.9), we obtain

$$\Re\left\{1+\frac{z[I^k_{p,\alpha,\beta}f(z)]''}{[I^k_{p,\alpha,\beta}f(z)]'}-p\right\}\geq\frac{2\mu}{2+\mu},\quad(0\leq\mu<p,\ z\in U),$$

which contradicts the hypothesis (2.1). Therefore, we conclude that $|w(z)|<1$ and $|\overline{w}(z)|<1$ for all $z\in U$, and it yields the inequality

$$\begin{aligned}\left|\frac{I^k_{p,\alpha,\beta}f(z)}{z^p}+\frac{z[I^k_{p,\alpha,\beta}f(z)]'}{pI^k_{p,\alpha,\beta}f(z)}-2\right|&\leq\left|\frac{\mu}{2}w(z)\right|+\left|\frac{\mu}{2}\overline{w}(z)\right|\\&=\frac{\mu}{2}\big(|w(z)|+|\overline{w}(z)|\big)\\&<\mu,\quad(0\leq\mu<p,\ z\in U),\end{aligned}$$

that is, $f(z)\in S^{**}_p(\alpha,\beta,\mu)$. Thus, the proof is completed. □

Corollary 1. *Let $f(z)\in T(p)$ and satisfy the inequality*

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}-p\right\}<\frac{2\mu}{\mu+2},\quad(0\leq\mu<p,\ z\in U).$$

Then

$$\left|\frac{f(z)}{z^p}+\frac{zf'(z)}{pf(z)}-2\right|<\mu,\quad(0\leq\mu<p,\ z\in U).$$

Proof. It is sufficient to put $k=0$. □

Corollary 2. *Let $f(z)\in T(p)$ and satisfy the inequality*

$$\Re\left\{\frac{zf''(z)}{f'(z)}\right\}<\frac{2\mu}{\mu+2},\quad(0\leq\mu<1,\ z\in U).$$

Then

$$\left|\frac{f(z)}{z}+\frac{zf'(z)}{f(z)}-2\right|<\mu,\quad(0\leq\mu<1,\ z\in U).$$

Proof. It proceeds by setting $p=1$ and $k=0$. □

As an application of Theorem 1, we have the following results which determine the bound estimates for functions $f\in T(p)$ to be in the class $S^{**}_p(\alpha,\beta,\mu)$.

Theorem 2. *Let $f(z)\in T(p)$ and satisfy*

$$\begin{aligned}\sum_{n=p+1}^{\infty}n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k|a_n|\\<2p|2(\mu+1)-p(\mu+2)|,\quad(0\leq\mu<p,\ z\in U).\end{aligned}\tag{2.10}$$

Then $f\in S^{**}_p(\alpha,\beta,\mu)$.

Proof. Let the function $f(z) \in T(p)$ and satisfy the inequality (2.10). Thus we obtain

$$\left|1+\frac{z[I^k_{p,\alpha,\beta}f(z)]''}{[I^k_{p,\alpha,\beta}f(z)]'}-p\right| \le \frac{2p(p-1)+\sum_{n=p+1}^{\infty} n(n-p)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |a_n|}{p-\sum_{n=p+1}^{\infty} n\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |a_n|}$$

the last inequality is less than $\frac{\mu}{\mu+2}$ if the assertion (2.10) holds. Now by using the fact that $\Re\{z\} \le |z|$ then in view of Theorem 1 yields $f \in S^{**}_p(\alpha,\beta,\mu)$. □

Theorem 3. *Let*

$$f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n \quad \textit{and} \quad g(z)=z^p+\sum_{n=p+1}^{\infty} b_n z^n$$

satisfy the inequality (2.10). *Then the weighted mean of f and g defined by*

$$W_i(z)=\frac{1}{2}\big[(1-i)f(z)+(1+i)g(z)\big], \quad (0<i<1)$$

is in $S^{**}_p(\alpha,\beta,\mu)$.

Proof. By the hypotheses of the theorem then in view of Theorem 2, we have

$$\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |a_n| < 2p|2(\mu+1)-p(\mu+2)|$$

and

$$\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |b_n| < 2p|2(\mu+1)-p(\mu+2)|.$$

After a simple calculation we obtain

$$W_i(z)=z^p+\sum_{n=p+1}^{\infty}\left[\frac{(1-i)}{2}a_n+\frac{(1+i)}{2}b_n\right]z^n.$$

However,

$$\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k \left|\frac{(1-i)}{2}a_n+\frac{(1+i)}{2}b_n\right|$$
$$\le \sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k \frac{1-i}{2}|a_n|$$

$$
\begin{aligned}
&+\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k \frac{1+i}{2}\,|b_n| \\
&=\frac{1-i}{2}\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |a_n| \\
&+\frac{1+i}{2}\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |b_n| \\
&\le 2p|2(\mu+1)-p(\mu+2)|\frac{1-i}{2}+2p|2(\mu+1)-p(\mu+2)|\frac{1+i}{2} \\
&=2p|2(\mu+1)-p(\mu+2)|.
\end{aligned}
$$

Consequently, by Theorem 2, $W_i(z)\in S_p^{**}(\alpha,\beta,\mu)$. □

Theorem 4. *Let*

$$f_j(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n, \quad j=1,2,3,\dots,m$$

satisfy the inequality (2.10)*. Then the arithmetic mean of* $f_j(z)$ *defined by*

$$F(z)=\frac{1}{m}\sum_{j=1}^{m} f_j(z), \quad (z\in U)$$

is in $S_p^{**}(\alpha,\beta,\mu)$.

Proof. Since f_j satisfies (2.10) then, in view of Theorem 2, we have

$$
\begin{aligned}
\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |a_{n,j}| \\
< 2p|2(\mu+1)-p(\mu+2)|.
\end{aligned}
$$

A computation gives

$$
\begin{aligned}
F(z)&=\frac{1}{m}\sum_{j=1}^{m} f_j(z) \\
&=\frac{1}{m}\sum_{j=1}^{m}\left[z^p+\sum_{n=p+1}^{\infty} a_{n,j}z^n\right] \\
&=z^p+\sum_{n=p+1}^{\infty}\left[\frac{1}{m}\sum_{j=1}^{m} a_{n,j}\right] z^n.
\end{aligned}
$$

Hence

$$\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k \left|\frac{1}{m}\sum_{j=1}^{m} a_{n,j}\right|$$
$$\le \sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k \frac{1}{m}\sum_{j=1}^{m}|a_{n,j}|$$
$$= \frac{1}{m}\sum_{j=1}^{m}\left[\sum_{n=p+1}^{\infty} n = \big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |a_{n,j}|\right]$$
$$< \frac{1}{m}\sum_{j=1}^{m} 2p|2(\mu+1)-p(\mu+2)|$$
$$= 2p|2(\mu+1)-p(\mu+2)|.$$

This implies that $F(z)\in S_p^{**}(\alpha,\beta,\mu)$. □

Theorem 5. *Let*

$$f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n$$

be in $T(p)$ such that $f(U)$ is convex. Assume that f satisfies the inequality (2.10). Then, for the Cesàro operator [6, 10–12] of f defined by the relation

$$\sigma_n(z)=\sum_{n=p+1}^{\infty}\frac{1}{n+1}\left(\sum_{i=0}^{n} a_{n,i}\right)z^n,\quad (p=1,2,\dots,\ z\in U)$$

with $\sigma_0(z)=0$, $\sigma_1(z)=z^p$, we have $\sigma_n(z)\in S_p^{**}(\alpha,\beta,\mu)$.

Proof. Since f_j satisfies (2.10) then in view of Theorem 2, we have

$$\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |a_{n,j}|$$
$$< 2p|2(\mu+1)-p(\mu+2)|.$$

For all $n\in\mathbb{N}_0$ we have

$$\sigma_n(z)=0+z^p+\sum_{n=p+1}^{\infty}\frac{1}{n+1}\left(\sum_{i=0}^{n} a_{n,i}\right)z^n,\quad (z\in U).$$

Hence

$$\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k \left|\frac{1}{n+1}\left(\sum_{i=0}^{n} a_{n,i}\right)\right|$$

$$
\begin{aligned}
&\le \sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k \frac{1}{n+1}\left(\sum_{i=0}^{n}|a_{n,i}|\right)\\
&= \frac{1}{n+1}\sum_{i=0}^{n}\left(\sum_{n=p+1}^{\infty} n\big((n-p)(\mu+2)+2\mu\big)\left[\frac{\alpha+p\beta+1}{\alpha+n\beta+1}\right]^k |a_{n,j}|\right)\\
&< \frac{1}{n+1}\sum_{i=0}^{n} 2p|2(\mu+1)-p(\mu+2)|\\
&= 2p|2(\mu+1)-p(\mu+2)|.
\end{aligned}
$$

This implies that $F(z) \in S_p^{**}(\alpha,\beta,\mu)$. □

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ON MULTIPLE MATHIEU $(\boldsymbol{a},\boldsymbol{\lambda})$-SERIES

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Abstract. In this paper we introduce the multiple Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series. We obtain two integral representations for multiple Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series applying Ivanov's and then Pogány's variant of multiple Euler-Maclaurin summation formula. Then, a bilateral bounding inequality is derived by virtue of the achieved integral expressions. Finally, the special case of multiple Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series, the multiple Mathieu $\boldsymbol{a}$-series has been investigated.

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1. INTRODUCTION

The generalization of the classical Mathieu series [3]

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^2} \qquad (r > 0) \tag{1.1}$$

has been introduced by Cerone and Lenard [1]:

$$\mathcal{S}(r,\mu,\alpha,\beta,\boldsymbol{a}) = \sum_{n=1}^{\infty} \frac{a_n^{\alpha}}{(a_n^{\beta}+r^2)^{\mu}} \qquad (r,\mu,\alpha,\beta,\boldsymbol{a}=(a_n)>0). \tag{1.2}$$

The series (1.2) is in the focus of interest by numerous authors, such as Pogány [5, 6], Qi [11, 12], Srivastava and Tomovski [13, 14]. However, according to our best knowledge, the only work on the multidimensional generalization of the series (1.2) is the paper [10] by Pogány and Tomovski, where they introduce a generalized multiple Mathieu series of the form

$$S_p^r(\boldsymbol{s},\boldsymbol{q},\rho) = \sum_{\boldsymbol{n}\in\mathbb{N}^r} \frac{2\boldsymbol{n}^{|\boldsymbol{s}|}}{(\langle \boldsymbol{n}^{\boldsymbol{q}},\boldsymbol{n}^{\boldsymbol{q}}\rangle+\rho)^{p+1}},$$

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where $\boldsymbol{n}^{\boldsymbol{q}} := (n_1^{q_1}, \ldots, n_r^{q_r})$, $\boldsymbol{n}^{\boldsymbol{\alpha}|\boldsymbol{s}|} := n_1^{\alpha_1 s_1} \cdots n_r^{\alpha_r s_r}$, $\boldsymbol{s}, \boldsymbol{q}$ have positive coordinates, i. e., $s_l, q_l > 0$, $l = 1, \ldots, r$, while $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ stands for inner product in $\mathbb{R}^r$. For $r = 1$, the above series, obviously, reduces to the classical Mathieu series (1.1).

Pogány and Tomovski found two integral representations of the multiple series $\mathcal{S}_p^r(\boldsymbol{s}, \boldsymbol{q}, \rho)$ (see Theorem 1, Eq. (9) and Theorem 3, Eq. (19) in [10]). They also derived a bilateral bounding inequality [10, Theorem 2] and established two other bounds [10, Theorems 4 and 5].

2. Summation formula for finite multiple sums

The well-known Euler–Maclaurin summation formula has the form

$$\begin{aligned} \sum_{n=k}^{l} f(n) &= \int_k^l f(x)\,\mathrm{d}x + \frac{1}{2}\big(f(k) + f(l)\big) \\ &\quad + \sum_{j=1}^{m} \frac{B_{2j}}{(2j)!}\Big(f^{(2j-1)}(l) - f^{(2j-1)}(k)\Big) \\ &\quad - \int_k^l \frac{B_{2m}(x)}{(2m)!} f^{(2m)}(x)\,\mathrm{d}x \qquad (m \in \mathbb{N}), \end{aligned} \tag{2.1}$$

where $B_p(x) = (x + B)^p$, $0 \le x < 1$, stands for the Bernoulli polynomial of order $p \in \mathbb{N}$ and $B^k = B_k$ are the Bernoulli numbers. One can rewrite it in a condensed form

$$\sum_{n=k+1}^{l} a_n = \int_k^l \big(a(x) + \{x\} a'(x)\big)\,\mathrm{d}x \equiv \int_k^l \mathfrak{d} a(x)\,\mathrm{d}x, \tag{2.2}$$

for $a \in C^1[k, l]$, $a_n = a(n)$, $k, l \in \mathbb{Z}$, $k < l$, where

$$\mathfrak{d} := 1 + \{x\} \frac{\partial}{\partial x}$$

and $\{x\}$ stands for the fractional part of a real number x (see (3) in [9] and (6.5) in [8]).

For the multidimensional bounded summation domain D, we use the summation formulas derived by Müller [4], Ivanov [2], and another type of formula due to Pogány [7].

Now the role of the Bernoulli polynomials $B_p(x)$ in (2.1) is played by the so-called *basic functions* ("Grundfunktion," see [4]) $G(x_1, \ldots, x_r)$, which satisfy the following conditions:

(1) G is a 1-periodic function in all variables;
(2) On the lattice $\mathbb{Z}^r$, the function G satisfies the equation

$$\Delta G + \lambda G = \sum_{\boldsymbol{k} :\, 4\pi^2 \boldsymbol{k}^2 = \lambda} \mathrm{e}^{2\pi i \langle \boldsymbol{k}, x \rangle}, \tag{2.3}$$

where Δ is the Laplace operator and the summation in (2.3) is carried out over all $\boldsymbol{k} = (k_1,\dots,k_r) \in \mathbb{Z}^r$ such that $4\pi^2\boldsymbol{k}^2 = 4\pi^2\langle \boldsymbol{k},\boldsymbol{k}\rangle = \lambda$.

Let us introduce the following notation: $\boldsymbol{n} = (n_1,\dots,n_r) \in \mathbb{N}_0^r$, where $\mathbb{N}_0 = \{0,1,2,\dots\}$; $\boldsymbol{x} = (x_1,\dots,x_r) \in \mathbb{R}^r$, $\mathrm{d}\boldsymbol{x} = \mathrm{d}x_1\cdots\mathrm{d}x_r$, $L = L\big(\frac{\partial}{\partial x}\big)$ is a linear differential operator with real constant coefficients, which is a polynomial in $\frac{\partial}{\partial x_j}$.*

Let us put

$$H(\boldsymbol{x}) = \sum_{\boldsymbol{n}:\, L(2\pi i\boldsymbol{n})=0} \mathrm{e}^{2\pi i\langle \boldsymbol{n},\boldsymbol{x}\rangle}. \tag{2.4}$$

Let G be a basic function of the operator L if it has period 1 with respect to each variable and satisfies the equality

$$LG = \sum_{\boldsymbol{n}\in\mathbb{Z}^r} \mathrm{e}^{2\pi i\langle \boldsymbol{n},\boldsymbol{x}\rangle}.$$

It follows that

$$G(\boldsymbol{x}) = {\sum_{\boldsymbol{n}\in\mathbb{Z}^r}}' \frac{\mathrm{e}^{2\pi i\langle \boldsymbol{n},\boldsymbol{x}\rangle}}{L(2\pi i\boldsymbol{n})}, \tag{2.5}$$

where $'$ denotes the absence of the term with zero denominator in the sum.

Let the boundary ∂D of D be smooth. Then, by Green's formula, we have

$$\int_{\bar{D}} (uLv - vMu)\,\mathrm{d}\boldsymbol{x} = \int_{\partial D} P(u,v)\,\mathrm{d}\boldsymbol{s}, \tag{2.6}$$

where $\bar{D} = D + \partial D$, M is a conjugate of L, and $P(u,v)$ is a polynomial with respect to u and v and their partial derivatives.

Theorem 1 ([2, Theorem 1])**.** *Let us assume that the boundary ∂D of D is smooth and does not contain integer points. Let L be a linear differential operator of order p with constant coefficients and $f \in C^p(\bar{D})$, where $\bar{D} = D + \partial D$. Then*

$$\sum_{\boldsymbol{n}\in D} f(\boldsymbol{n}) = \int_D f(\boldsymbol{x})H(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{\partial D} P(f,G)\,\mathrm{d}\boldsymbol{s} + \int_D GM(f)\,\mathrm{d}\boldsymbol{x}, \tag{2.7}$$

where M is a conjugate of L, whereas H and G are defined by (2.4) *and* (2.5)*.*

The following result of [7] is a multidimensional generalization of (2.2).

Theorem 2 ([7])**.** *Assume that $a : \mathbb{R}_+^r \to \mathbb{R}_+$ is a function satisfying the condition of differentiability*

$$\frac{\partial^r a}{\partial x_1\cdots\partial x_r} \in C\bigg(\prod_{j=1}^r [0,n_j]\bigg),$$

and, for any $\boldsymbol{j} = (j_1,\dots,j_r)$ with $0 \le j_l \le n_l$, $l = 1,2,\dots,r$, put $a_{\boldsymbol{j}} := a(j_1,\dots,j_r)$.

*In what follows, the expression $L(2\pi i\boldsymbol{n})$ stands for the value of that polynomial where, instead of $\frac{\partial}{\partial x_j}$, one puts $2\pi i n_j$.

Then

$$\begin{aligned}\sum_{l=1}^{r}\sum_{j_l=0}^{n_l} a_{\boldsymbol{j}} &= a_{\boldsymbol{0}} + \sum_{l=1}^{r}\int_0^{n_l} \eth_l\, a(x_l)\,\mathrm{d}x_l \\ &\quad + \sum_{1\le j<k\le r}\int_0^{n_j}\int_0^{n_k} \eth_j\eth_k\, a(x_j,x_k)\,\mathrm{d}x_j\,\mathrm{d}x_k + \cdots \\ &\quad + \int_0^{n_1}\int_0^{n_2}\cdots\int_0^{n_r} \eth_1\eth_2\cdots\eth_r\, a(x_1,x_2,\dots,x_r)\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_r,\end{aligned} \tag{2.8}$$

where $a_{\boldsymbol{0}} \equiv a(0,\dots,0)$, $a(x_{j_1},\dots,x_{j_k}) = a(\boldsymbol{x})|_{x_m=0,\, m\in\{1,\dots,r\}\setminus\{j_1,\dots,j_k\}}$, *and*

$$\eth_j := 1 + \{x_j\}\frac{\partial}{\partial x_j}.$$

3. Multiple Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series

Let $a,\lambda\colon \mathbb{R}_+^r \to \mathbb{R}_+$, where $r>1$, be certain functions, $\boldsymbol{a}$ and $\boldsymbol{\lambda}$ be their restrictions to $\mathbb{N}_0^r$, i.e., $a|_{\mathbb{N}_0^r} = (a_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{N}_0^r} = (a(n_1,\dots,n_r))_{\boldsymbol{n}\in\mathbb{N}_0^r}$, $\lambda|_{\mathbb{N}_0^r} = (\lambda_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{N}_0^r} = (\lambda(n_1,\dots,n_r))_{\boldsymbol{n}\in\mathbb{N}_0^r}$, and $\mu>0$ and $\rho>0$ be parameters. Then the series

$$\mathcal{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho) := \sum_{\boldsymbol{n}\in\mathbb{N}_0^r} \frac{a_{\boldsymbol{n}}}{(\lambda_{\boldsymbol{n}}+\rho)^\mu} \tag{3.1}$$

is called a *multiple Mathieu* $(\boldsymbol{a},\boldsymbol{\lambda})$*-series*.

We are interested in deriving an integral representation of the series (3.1). Let us begin with transforming $\mathcal{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho)$ via the Gamma function. We have

$$\begin{aligned}\mathcal{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho) &= \frac{1}{\Gamma(\mu)}\sum_{\boldsymbol{n}\in\mathbb{N}_0^r} a_{\boldsymbol{n}}\int_0^\infty \mathrm{e}^{-(\lambda_{\boldsymbol{n}}+\rho)x}x^{\mu-1}\,\mathrm{d}x \\ &= \frac{1}{\Gamma(\mu)}\int_0^\infty \mathrm{e}^{-\rho x}x^{\mu-1}\left(\sum_{\boldsymbol{n}\in\mathbb{N}_0^r} a_{\boldsymbol{n}}\mathrm{e}^{-\lambda_{\boldsymbol{n}}x}\right)\mathrm{d}x.\end{aligned}$$

The inner r-fold Dirichlet series

$$\mathcal{D}_{\boldsymbol{a},\boldsymbol{\lambda}}(x) = \sum_{\boldsymbol{n}\in\mathbb{N}_0^r} a_{\boldsymbol{n}}\mathrm{e}^{-\lambda_{\boldsymbol{n}}x},$$

by virtue of the Laplace integral formula, can be rewritten in the form

$$\mathcal{D}_{\boldsymbol{a},\boldsymbol{\lambda}}(x) = x\int_0^\infty \mathrm{e}^{-xt}\left(\sum_{\boldsymbol{n}:\lambda_{\boldsymbol{n}}\le t} a_{\boldsymbol{n}}(t)\right)\mathrm{d}t = x\int_0^\infty \mathrm{e}^{-xt}A_{\boldsymbol{n}}(t)\,\mathrm{d}t.$$

Putting all back into the expression for $\mathcal{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho)$, we obtain

$$\begin{aligned}\mathcal{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho) &= \frac{1}{\Gamma(\mu)}\int_0^\infty \mathrm{e}^{-\rho x}x^\mu \int_0^\infty \mathrm{e}^{-xt}A_{\boldsymbol{n}}(t)\,\mathrm{d}t\\ &= \frac{1}{\Gamma(\mu)}\int_0^\infty \left(\int_0^\infty \mathrm{e}^{-(\rho+t)x}x^\mu\,\mathrm{d}x\right)A_{\boldsymbol{n}}(t)\,\mathrm{d}t\\ &= \mu\int_0^\infty \frac{A_{\boldsymbol{n}}(t)}{(\rho+t)^{\mu+1}}\,\mathrm{d}t.\end{aligned}$$

Now we have to calculate the sum $A_{\boldsymbol{n}}(t)$. Let $\mathfrak{D}=\{\boldsymbol{n}:\lambda_{\boldsymbol{n}}\le t,\ t>0\}$ and let $\widehat{\lambda}_1 = \widehat{\lambda}_1(t),\dots,\widehat{\lambda}_r=\widehat{\lambda}_r(t)\in\mathbb{N}_0$ be such that

$$\lambda_{\boldsymbol{n}} = \lambda(\widehat{\lambda}_1,\dots,\widehat{\lambda}_r)\le t. \tag{3.2}$$

The summation domain is the r-dimensional parallelepiped $\mathfrak{D}=\prod_{j=1}^r[0,\widehat{\lambda}_j]$ and, obviously, $\dim\partial\mathfrak{D}=r-1$. Let us set $\mathrm{d}\boldsymbol{s}_{r-1}=\mathrm{d}s_1\cdots\mathrm{d}s_{r-1}$. Let L be an elliptic differential operator, M be its conjugate, P be the polynomial defined by (2.6), and H and G be given by (2.4) and (2.5), respectively. By Ivanov's formula (2.7), we obtain

$$A_{\boldsymbol{n}}(t)=\int_{\mathfrak{D}} a(\boldsymbol{x})H(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}+\int_{\partial\mathfrak{D}}P(a,G)\,\mathrm{d}\boldsymbol{s}_{r-1}+\int_{\mathfrak{D}}GM(a)\,\mathrm{d}\boldsymbol{x}.$$

Putting this expression for $A_{\boldsymbol{n}}(t)$ back to $\mathcal{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho)$, we immediately arrive at the following statement.

Theorem 3. *Let $a,\lambda\colon\mathbb{R}_+^r\to\mathbb{R}_+$, $a_{\boldsymbol{n}}=a(n_1,\dots,n_r)$, $\lambda_{\boldsymbol{n}}=\lambda(n_1,\dots,n_r)$, $\boldsymbol{n}\in\mathbb{N}_0^r$, $\mu,\rho>0$, and $r\in\mathbb{N}_2$. Let $t>0$ and let $\widehat{\lambda}_1=\widehat{\lambda}_1(t),\dots,\widehat{\lambda}_r=\widehat{\lambda}_r(t)\in\mathbb{N}_0$ be such that (3.2) holds. Then the multiple Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series admits the integral representation*

$$\mathcal{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho)=\mu\int_0^\infty\frac{\int_{\mathfrak{D}} a(\boldsymbol{x})H(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}+\int_{\partial\mathfrak{D}}P(a,G)\,\mathrm{d}\boldsymbol{s}_{r-1}+\int_{\mathfrak{D}}GM(a)\,\mathrm{d}\boldsymbol{x}}{(\rho+t)^{\mu+1}}\,\mathrm{d}t,$$

where $\mathfrak{D}=\prod_{j=1}^r[0,\widehat{\lambda}_j]$.

If we use formula (2.8) to calculate $A_{\boldsymbol{n}}(t)$ in $\mathcal{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho)$, we deduce

Theorem 4. *Let $a,\lambda\colon\mathbb{R}_+^r\to\mathbb{R}_+$, $a_{\boldsymbol{n}}=a(n_1,\dots,n_r)$, $\lambda_{\boldsymbol{n}}=\lambda(n_1,\dots,n_r)$, $\boldsymbol{n}\in\mathbb{N}_0^r$, $\mu>0,\rho>0$, $r>0$, and $\widehat{\lambda}_1=\widehat{\lambda}_1(t),\dots,\widehat{\lambda}_r=\widehat{\lambda}_r(t)\in\mathbb{N}_0$ such that*

$$\lambda_{\boldsymbol{n}}=\lambda(\widehat{\lambda}_1,\dots,\widehat{\lambda}_r)\le t.$$

Then multiple Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series has the following integral representation

$$\mathcal{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};\rho)=\rho^{-\mu}a_{\boldsymbol{0}}+\mu\int_0^\infty\Bigg(\sum_{l=1}^r\int_0^{\widehat{\lambda}_l}\mathfrak{d}_l\,a(x_l)\,\mathrm{d}x_l$$

$$+\sum_{1\le j<k\le r}\int_0^{\widehat{\lambda}_j}\int_0^{\widehat{\lambda}_k}\mathfrak{d}_j\mathfrak{d}_k\,a(x_j,x_k)\,\mathrm{d}x_j\,\mathrm{d}x_k+\cdots$$
$$\left.+\int_0^{\widehat{\lambda}_1}\int_0^{\widehat{\lambda}_2}\cdots\int_0^{\widehat{\lambda}_r}\mathfrak{d}_1\cdots\mathfrak{d}_r\,a(x_1,\ldots,x_r)\,\mathrm{d}x_1\cdots\mathrm{d}x_r\right)\frac{\mathrm{d}t}{(\rho+t)^{\mu+1}}\,.$$

4. Bilateral bounding inequality

We estimate $A_{\boldsymbol{n}}(t)$ given by Pogány's formula (2.8). We will focus here on the case where $r=2$, since the higher order terms are far more complicated and the procedure is identical to that applicable for $r=2$.

Let $a\colon\mathbb{R}_+^2\to\mathbb{R}$ be monotone in both variables. Since, for $j=1,2$,

$$\mathfrak{d}_j a\in\left[a,a+\frac{\partial a}{\partial x_j}\right)\qquad(\text{if } a\uparrow\text{ in } x_j); \tag{4.1}$$
$$\mathfrak{d}_j a\in\left(a+\frac{\partial a}{\partial x_j},a\right]\qquad(\text{if } a\downarrow\text{ in } x_j),$$

successively applying the operators $\mathfrak{d}_1$ and $\mathfrak{d}_2$ in (2.8), we estimate

$$\begin{aligned}\boldsymbol{A}_2&=\sum_{j=0}^{n_1}\sum_{k=0}^{n_2}a_{jk}\\&=a_{\boldsymbol{0}}+\sum_{j=1}^{2}\int_0^{n_j}\mathfrak{d}_j\,a(x_j)\,\mathrm{d}x_j+\int_0^{n_1}\int_0^{n_2}\mathfrak{d}_j\mathfrak{d}_k\,a(x_j,x_k)\,\mathrm{d}x_j\,\mathrm{d}x_k\,.\end{aligned}$$

Assume that $a\uparrow$. Then

$$\begin{aligned}A_{\boldsymbol{2}}(t)\le a_{\boldsymbol{0}}&+\sum_{j=1}^{2}\int_0^{n_j}\big(a(x_j)+a'(x_j)\big)\,\mathrm{d}x_j\\&+\int_0^{n_1}\int_0^{n_2}\Big(a(x_1,x_2)+\frac{\partial a(x_1,x_2)}{\partial x_1}\\&\qquad+\frac{\partial a(x_1,x_2)}{\partial x_2}+\frac{\partial 2a(x_1,x_2)}{\partial x_1\partial x_2}\Big)\,\mathrm{d}x_1\,\mathrm{d}x_2=:\boldsymbol{R}_2.\end{aligned}$$

Now we have

$$
\begin{aligned}
\boldsymbol{R}_2 &= a_{\mathbf{0}} + \sum_{j=1}^{2}\left(\int_0^{n_j} a(x_j)\,\mathrm{d}x_j + a(n_j) - a_{\mathbf{0}}\right) + \int_0^{n_1}\int_0^{n_2} a(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&\quad + \int_0^{n_1}\big(a(x_1,n_2)-a(x_1,0)\big)\,\mathrm{d}x_1 + \int_0^{n_2}\big(a(n_1,x_2)-a(0,x_2)\big)\,\mathrm{d}x_2\\
&\quad + a(n_1,n_2) - a(n_1,0) - a(0,n_2) + a_{\mathbf{0}}\\
&= a_{\mathbf{0}} + \sum_{j=1}^{2}\left(\int_0^{n_j} a(x_j)\,\mathrm{d}x_j + a(n_j) - a_{\mathbf{0}}\right) + \int_0^{n_1}\int_0^{n_2} a(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&\quad + \int_0^{n_1}\big(a(x_1,n_2)-a(x_1)\big)\,\mathrm{d}x_1 + \int_0^{n_2}\big(a(n_1,x_2)-a(x_2)\big)\,\mathrm{d}x_2\\
&\quad + a(n_1,n_2) - a(n_1) - a(n_2) + a_{\mathbf{0}}\\
&= a(n_1,n_2) + \int_0^{n_1} a(x_1,n_2)\,\mathrm{d}x_1 + \int_0^{n_2} a(n_1,x_2)\,\mathrm{d}x_2\\
&\quad + \int_0^{n_1}\int_0^{n_2} a(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,.
\end{aligned} \tag{4.2}
$$

Similarly, by virtue of (4.1), we have

$$
\begin{aligned}
\boldsymbol{A}_2 \ge a_{\mathbf{0}} &+ \int_0^{n_1} a(x_1)\,\mathrm{d}x_1 + \int_0^{n_2} a(x_2)\,\mathrm{d}x_2 +\\
&+ \int_0^{n_1}\int_0^{n_2} a(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 =: \boldsymbol{L}_2\,.
\end{aligned} \tag{4.3}
$$

This proves the following result.

Theorem 5. *Let $a\colon \mathbb{R}_+^2 \to \mathbb{R}$ and $a \in C^2\big([0,n_1]\times[0,n_2]\big)$, monotone in both variables. Then for the two-dimensional Euler–Maclaurin summation formula (2.8) the following bilateral bounds are valid*

$$
\begin{aligned}
\boldsymbol{L}_2 &< \sum_{j=0}^{n_1}\sum_{k=0}^{n_2} a_{jk} \le \boldsymbol{R}_2\,, \quad (a\uparrow),\\
\boldsymbol{R}_2 &\le \sum_{j=0}^{n_1}\sum_{k=0}^{n_2} a_{jk} < \boldsymbol{L}_2\,, \quad (a\downarrow),
\end{aligned}
$$

where $a(\boldsymbol{x})|_{\mathbb{N}_0^2} = \boldsymbol{a} = (a_{jk})_{\mathbb{N}_0^2}$ and bounds $\boldsymbol{L}_2$, $\boldsymbol{R}_2$ are given in (4.3), (4.2).

If we apply the above theorem to integral representation of multiple Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series given in Theorem 4, for $r=2$ with simple exchange $\widehat{\lambda}_1 \to n_1$, $\widehat{\lambda}_2 \to n_2$ we obtain

Theorem 6. *Let* $a, \lambda\colon \mathbb{R}_+^2 \to \mathbb{R}$, $a, \lambda \in C^2([0, \widehat{\lambda}_1] \times [0, \widehat{\lambda}_2])$, *where* $\widehat{\lambda}_j = \widehat{\lambda}_j(t) \in \mathbb{N}_0$, $j = 1, 2$, *for some fixed* $t > 0$, *such that* $\lambda(\widehat{\lambda}_1, \widehat{\lambda}_2) \le t$ *and* a *be monotone in both variables. Then the following bilateral inequalities hold*

$$\mu \int_0^\infty \frac{\widehat{\boldsymbol{L}}_2(t)}{(\rho+t)^{\mu+1}}\,\mathrm{d}t < \mathscr{S}_\mu(\boldsymbol{a}, \boldsymbol{\lambda}; \rho) \le \mu \int_0^\infty \frac{\widehat{\boldsymbol{R}}_2(t)}{(\rho+t)^{\mu+1}}\,\mathrm{d}t, \quad (a \uparrow),$$

$$\mu \int_0^\infty \frac{\widehat{\boldsymbol{R}}_2(t)}{(\rho+t)^{\mu+1}}\,\mathrm{d}t \le \mathscr{S}_\mu(\boldsymbol{a}, \boldsymbol{\lambda}; \rho) < \mu \int_0^\infty \frac{\widehat{\boldsymbol{L}}_2(t)}{(\rho+t)^{\mu+1}}\,\mathrm{d}t, \quad (a \downarrow),$$

where $\widehat{\boldsymbol{X}}_2(t) = \boldsymbol{X}_2\big(\widehat{\lambda}_1, \widehat{\lambda}_2\big)$, $\boldsymbol{X} \in \{\boldsymbol{R}, \boldsymbol{L}\}$, *that is*

$$\begin{aligned}
\widehat{\boldsymbol{R}}_2(t) &= a\big(\widehat{\lambda}_1(t), \widehat{\lambda}_2(t)\big) + \int_0^{\widehat{\lambda}_1(t)} a\big(x_1, \widehat{\lambda}_2(t)\big)\,\mathrm{d}x_1 \\
&\quad + \int_0^{\widehat{\lambda}_2(t)} a\big(\widehat{\lambda}_1(t), x_2\big)\,\mathrm{d}x_2 + \int_0^{\widehat{\lambda}_1(t)} \int_0^{\widehat{\lambda}_2(t)} a(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
\widehat{\boldsymbol{L}}_2(t) &= a_{\boldsymbol{0}} + \int_0^{\widehat{\lambda}_1(t)} a(x_1)\,\mathrm{d}x_1 + \\
&\quad + \int_0^{\widehat{\lambda}_2(t)} a(x_2)\,\mathrm{d}x_2 + \int_0^{\widehat{\lambda}_1(t)} \int_0^{\widehat{\lambda}_2(t)} a(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2.
\end{aligned}$$

Remark 1. Since $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ are solutions of the Diophantine equation $\lambda(\widehat{\lambda}_1, \widehat{\lambda}_2) \le t$, they are not uniquely determined, nor are the functions $\widehat{\boldsymbol{L}}_2$, $\widehat{\boldsymbol{R}}_2$.

5. MULTIPLE MATHIEU $\boldsymbol{a}$-SERIES

If we take $\lambda \equiv a$, it is straightforward to see that from multiple Mathieu $(\boldsymbol{a}, \boldsymbol{\lambda})$-series we easily obtain the series

$$\mathscr{S}_\mu(\boldsymbol{a}; \rho) := \sum_{\boldsymbol{n} \in \mathbb{N}_0^r} \frac{a_{\boldsymbol{n}}}{(a_{\boldsymbol{n}} + \rho)^\mu}$$

which we call *multiple Mathieu* $\boldsymbol{a}$*-series*. Now from integral representations for multiple Mathieu $(\boldsymbol{a}, \boldsymbol{\lambda})$-series we directly get the following results.

Corollary 1 (of Theorem 3)**.** *Let* $a\colon \mathbb{R}_+^r \to \mathbb{R}_+$, $a_{\boldsymbol{n}} = a(n_1, \ldots, n_r)$, $\boldsymbol{n} \in \mathbb{N}_0^r$, $\mu, \rho > 0$, *and* $r \in \mathbb{N}_2$. *Let* $t > 0$ *and* $\widehat{a}_1 = \widehat{a}_1(t), \ldots, \widehat{a}_r = \widehat{a}_r(t) \in \mathbb{N}_0$ *such that*

$$a_{\boldsymbol{n}} = a(\widehat{a}_1, \ldots, \widehat{a}_r) \le t.$$

If $\mathfrak{D} = \prod_{j=1}^r [0, \widehat{a}_j]$ *then multiple Mathieu* a*-series have the following integral representation*

$$\mathscr{S}_\mu(a; \rho) = \mu \int_0^\infty \frac{\int_{\mathfrak{D}} a(\boldsymbol{x}) H(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{\partial\mathfrak{D}} P(a, G)\,\mathrm{d}\boldsymbol{s}_{r-1} + \int_{\mathfrak{D}} GM(a)\,\mathrm{d}\boldsymbol{x}}{(\rho+t)^{\mu+1}}\,\mathrm{d}t,$$

with L, H and G keeping the heretofore meaning.

Corollary 2 (of Theorem 4)**.** *Let* $a\colon \mathbb{R}_+^r \to \mathbb{R}_+$, $a_{\boldsymbol{n}} = a(n_1,\dots,n_r)$, $\boldsymbol{n} \in \mathbb{N}_0^r$, $\mu > 0$, $\rho > 0$, $r > 0$, *and* $\widehat{a}_1 = \widehat{a}_1(t),\dots,\widehat{a}_r = \widehat{a}_r(t) \in \mathbb{N}_0$ *such that*

$$a_{\boldsymbol{n}} = a(\widehat{a}_1,\dots,\widehat{a}_r) \le t.$$

Then multiple Mathieu $\boldsymbol{a}$*-series possesses the integral representation*

$$\begin{aligned}\mathcal{S}_\mu(\boldsymbol{a};\rho) = \rho^{-\mu} a_{\boldsymbol{0}} + \mu \int_0^\infty \Bigg(& \sum_{l=1}^r \int_0^{\widehat{a}_l} \partial_l\, a(x_l)\,\mathrm{d}x_l \\ & + \sum_{1\le j<k\le r} \int_0^{\widehat{a}_j}\int_0^{\widehat{a}_k} \partial_j \partial_k\, a(x_j,x_k)\,\mathrm{d}x_j\,\mathrm{d}x_k + \cdots \\ & + \int_0^{\widehat{a}_1} \cdots \int_0^{\widehat{a}_r} \partial_1 \cdots \partial_r\, a(x_1,\dots,x_r)\,\mathrm{d}x_1\cdots \mathrm{d}x_r \Bigg) \frac{1}{(\rho+t)^{\mu+1}}\,\mathrm{d}t.\end{aligned}$$

Corollary 3 (of Theorem 5)**.** *Let* $a\colon \mathbb{R}_+^2 \to \mathbb{R}$, $a \in C^2\big([0,\widehat{a}_1]\times[0,\widehat{a}_2]\big)$, $\mu > 0$, $\rho > 0$, *and* $\widehat{a}_1, \widehat{a}_2 \in \mathbb{N}_0$ *such that*

$$a_{\boldsymbol{n}} = a(\widehat{a}_1,\widehat{a}_2) \le t.$$

Then the following bilateral inequalities hold

$$\begin{aligned}\mu\int_0^\infty \frac{\boldsymbol{L}_2^a(t)}{(\rho+t)^{\mu+1}}\,\mathrm{d}t < \mathcal{S}_\mu(\boldsymbol{a};\rho) \le \mu\int_0^\infty \frac{\boldsymbol{R}_2^a(t)}{(\rho+t)^{\mu+1}}\,\mathrm{d}t, \quad (a\uparrow),\\ \mu\int_0^\infty \frac{\boldsymbol{R}_2^a(t)}{(\rho+t)^{\mu+1}}\,\mathrm{d}t \le \mathcal{S}_\mu(\boldsymbol{a};\rho) < \mu\int_0^\infty \frac{\boldsymbol{L}_2^a(t)}{(\rho+t)^{\mu+1}}\,\mathrm{d}t, \quad (a\downarrow),\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{L}_2^a(t) &= a_{\boldsymbol{0}} + \sum_{j=1}^2 \int_0^{\hat{a}_j} a(x_j)\,\mathrm{d}x_j + + \int_0^{\widehat{a}_1}\int_0^{\widehat{a}_2} a(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2,\\ \boldsymbol{R}_2^a(t) &= a(\widehat{a}_1,\widehat{a}_2) + \int_0^{\widehat{a}_1} a(x_1,\widehat{a}_2)\,\mathrm{d}x_1 + \int_0^{\widehat{a}_2} a(\widehat{a}_1,x_2)\,\mathrm{d}x_2\\ &\quad + \int_0^{\widehat{a}_1}\int_0^{\widehat{a}_2} a(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2.\end{aligned}$$

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AN OPTIMAL CONDITION FOR THE UNIQUENESS OF A PERIODIC SOLUTION FOR SYSTEMS OF HIGHER ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Unimprovable efficient conditions are established for the unique solvability of the periodic problem

$$u_i^{(p_i)}(t) = \sum_{j=2}^{i+1} \ell_{i,j}(u_j)(t) + q_i(t) \qquad \text{for} \quad 1 \le i \le n-1,$$
$$u_n^{(p_n)}(t) = \sum_{j=1}^{n} \ell_{n,j}(u_j)(t) + q_n(t),$$
$$u_j^{(p)}(0) = u_j^{(p)}(\omega) \qquad \text{for} \quad 1 \le j \le n,\ 0 \le p \le p_j - 1,$$

where $\omega > 0$, $n \ge 2$, $p_j \in \mathbb{N}$, $\ell_{ij} : C([0,\omega]) \to L([0,\omega])$ are linear bounded operators, and $q_i \in L([0,\omega])$.

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1. Statement of problem and formulation of main results

On $[0,\omega]$, consider the system

$$\begin{aligned} u_i^{(p_i)}(t) &= \sum_{j=2}^{i+1} \ell_{i,j}(u_j)(t) + q_i(t) \qquad \text{for} \quad 1 \le i \le n-1, \\ u_n^{(p_n)}(t) &= \sum_{j=1}^{n} \ell_{n,j}(u_j)(t) + q_n(t), \end{aligned} \tag{1.1}$$

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with the periodic boundary conditions

$$u_j^{(p)}(0) = u_j^{(p)}(\omega) \qquad \text{for} \quad 1 \le j \le n, \; 0 \le p \le p_j - 1, \tag{1.2}$$

where $\omega > 0$, $n \ge 2$, $p_j \in \mathbb{N}$, $\ell_{ij} : C([0,\omega]) \to L([0,\omega])$ are the linear bounded operators, and $q_i \in L([0,\omega])$.

By a solution of the problem (1.1), (1.2) we understand a vector function $u = (u_i)_{i=1}^n$ with $u_i \in \widetilde{C}^{p_i-1}([0,\omega])$ $(i = \overline{1,n})$ which satisfies system (1.1) almost everywhere on $[0,\omega]$ and satisfies conditions (1.2).

Much work had been carried out on the investigation of the existence and uniqueness of the solution for a periodic boundary value problem for systems of ordinary differential equations and many interesting results have been obtained (see, for instance, [1,2,5,6,13,14] and the references therein). However, an analogous problem for functional differential equations remains investigated in significantly less detail even for linear equations. In the present paper, we study problem (1.1) (1.2) under the assumption that $\ell_{n,1}, \ell_{i,i+1}$ $(i = \overline{1,n-1})$ are monotone linear operators. We establish new unimprovable integral conditions sufficient for unique solvability of the problem (1.1), (1.2) which generalize the well-known results of A. Lasota and Z. Opial (see Remark 1.2) obtained for ordinary differential equations in [7] and, on the other hand, extend our results obtained for linear functional differential equations in [8,9,11]. These results are also new if (1.1) is the system of ordinary differential equations of the form

$$\begin{aligned} u_i^{(p_i)}(t) &= \sum_{j=2}^{i+1} l_{i,j}(t) u_j(t) + q_i(t) \qquad \text{for} \quad 1 \le i \le n-1, \\ u_n^{(p_n)}(t) &= \sum_{j=1}^{n} l_{n,j}(t) u_j(t) + q_n(t), \end{aligned} \tag{1.3}$$

where $q_i, l_{i,j} \in L([0,\omega])$. The method used for the investigation of the problem considered is based on that developed in our previous papers [8–11] for functional differential equations.

The following notation is used throughout the paper: $\mathbb{N}$ (resp., $\mathbb{R}$) is the set of all the natural (resp., real) numbers; $\mathbb{R}^n$ is the space of n-dimensional column vectors $x = (x_i)_{i=1}^n$ with elements $x_i \in \mathbb{R}$ $(i = \overline{1,n})$; $\mathbb{R}_+ = [0,+\infty[$; $C([0,\omega])$ is the Banach space of continuous functions $u : [0,\omega] \to \mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)| : 0 \le t \le \omega\}$; $\widetilde{C}^p([0,\omega])$ $(p \in \mathbb{N})$ is the set of functions $u : [0,\omega] \to \mathbb{R}$ which are absolutely continuous together with their derivatives up to order k inclusive. $L([0,\omega])$ is the Banach space of the Lebesgue integrable functions $p : [0,\omega] \to \mathbb{R}$ with the norm $\|p\|_L = \int_0^\omega |p(s)| ds$. If $\ell : C([0,\omega]) \to L([0,\omega])$ is a linear operator, then $\|\ell\| = \sup_{0 < \|x\|_C \le 1} \|\ell(x)\|_L$.

Furthermore, define the functional $\Delta : C([0,\omega]) \to \mathbb{R}_+$ by setting

$$\Delta(x) = \max\{x(t) : t \in [0,\omega]\} - \min\{x(t) : t \in [0,\omega]\}.$$

Definition 1.1. We will say that a linear operator $\ell : C([0,\omega]) \to L([0,\omega])$ is *nonnegative* (resp., *nonpositive*), if for any nonnegative $x \in C([0,\omega])$ the inequality $\ell(x)(t) \geq 0$ (resp., $\ell(x)(t) \leq 0$) for is satisfied $0 \leq t \leq \omega$. We will say that an operator ℓ is *monotone* if it is either nonnegative or nonpositive.

The following notation is used in the sequel:

$$B_0 = 1, \quad B_1 = \frac{1}{15}, \quad B_j = B_1 \sum_{m_1=1}^{2} \sum_{m_2=1}^{m_1+1} \cdots \sum_{m_{j-1}=1}^{m_{j-2}+1} \frac{1}{\eta(m_1)\ldots\eta(m_{j-1})},$$

$$C_1 = \frac{1}{8}, \quad C_j = B_1 \sum_{m_1=1}^{2} \sum_{m_2=1}^{m_1+1} \cdots \sum_{m_{j-1}=1}^{m_{j-2}+1} \frac{1}{\eta(m_1)\ldots\eta(m_{j-1})} \prod_{i=1}^{m_{j-1}+1} \left(1 + \frac{1}{2i}\right),$$

for $j \geq 2$, where

$$\eta(t) = (2t+1)(2t+3).$$

Let

$$d_0 = 1, \qquad d_1 = 4, \qquad d_2 = 32, \qquad d_3 = 192, \tag{1.4}$$

and for $p \in \mathbb{N}$ put

$$\begin{aligned} d_{2p+2} &= \frac{1}{\max\left\{\left(h_p(t)h_p(1-t)\right)^{1/2} : 0 \leq t \leq 1\right\}}, \\ d_{2p+3} &= \frac{1}{\max\left\{\left(f_p(s,t)\,f_p(1-s,1-t)\right)^{1/2} : 0 \leq s \leq 1,\ 0 \leq t \leq 1\right\}}, \end{aligned} \tag{1.5}$$

where the functions $f_p : [0,1] \times [0,1] \to \mathbb{R}_+$ and $h_p : [0,1] \to \mathbb{R}_+$ are defined as follows:

$$f_p(s,t) = \sum_{j=0}^{p-1} \alpha_{pj} t^{2(j+1)} + \alpha_{pp} t^{2p+3} s, \qquad h_p(t) = \sum_{j=0}^{p} \beta_{pj} t^{2(j+1)},$$

and

$$\alpha_{pj} = \frac{B_j}{3 \cdot 4^{j+1} d_{2(p-j)+1}}, \qquad \beta_{pj} = \frac{B_j}{3 \cdot 4^{j+1} d_{2(p-j)}} \qquad (j = \overline{0, p-1}),$$

$$\alpha_{pp} = \frac{B_p}{3 \cdot 4^{p+1}}, \qquad \beta_{pp} = \frac{C_p}{3 \cdot 4^{p+1}}.$$

Now we formulate a result from [4] in a suitable for us form.

Theorem 1.1. *Let $p \in N$, $x \in \widetilde{C}^p([0,\omega])$, $x^{(i)}(0) = x^{(i)}(\omega)$ $(i = 0,\dots,p)$, and let d_p be given by the equalities (1.4) and (1.5). Assume, moreover, that x is not equal identically to a constant. Then*

$$\Delta(x) < \frac{\omega^p}{d_p}\Delta\big(x^{(p)}\big). \tag{1.6}$$

Remark 1.1. It was shown in [4] that

$$d_4 = \frac{2^{11}\cdot 3}{5}, \quad d_5 = 2^9\cdot 3\cdot 5, \quad d_6 = \frac{2^{16}\cdot 3^2\cdot 5}{61}, \quad d_7 = \frac{2^{14}\cdot 3^2\cdot 5\cdot 7}{17},$$

and the numbers d_k, $k = \overline{1,7}$, are optimal in the sense that (1.6) cannot be replaced by the inequality

$$\Delta(x) < \frac{\omega^p}{d_p+\varepsilon}\Delta\big(x^{(p)}\big)$$

no matter how small $\varepsilon \in]0,1]$ would be.

Definition 1.2. With system (1.1), we associate the matrix $A_1 = (a^{(1)}_{i,j})^n_{i,j=1}$ defined by the equalities

$$\begin{gathered} a^{(1)}_{1,1} = -1, \quad a^{(1)}_{n,1} = \frac{\|\ell_{n,1}\|}{4}, \quad a^{(1)}_{i,1} = 0 \ \text{for} \ 2 \le i \le n-1, \\ a^{(1)}_{i+1,i+1} = \|\ell_{i+1,i+1}\| - \frac{d_{p_{i+1}-1}}{\omega^{p_{i+1}-1}}, \quad a^{(1)}_{i,i+1} = \frac{\|\ell_{n,1}\|}{4} \ \text{for} \ 1 \le i \le n-1, \\ a^{(1)}_{i,j} = 0 \ \text{for} \ i+2 \le j \le n, \quad a^{(1)}_{i,j} = \|\ell_{i,j}\| \ \text{for} \ 3 \le j+1 \le i \le n, \end{gathered} \tag{1.7}$$

and the matrices $A_k = (a^{(k)}_{i,j})^n_{i,j=1}$ $(k = \overline{2,n})$ given by the recurrence relations

$$A_2 = A_1, \tag{1.8}$$

$$a^{(k+1)}_{i,j} = a^{(k)}_{i,j} \quad \text{for} \ \ i \le k \ \ \text{or} \ \ j \notin \{k,k+1\}, \tag{1.9}$$

$$a^{(k+1)}_{i,j} = a^{(k)}_{i,j} + \frac{a^{(k)}_{k,j}}{|a^{(k)}_{k,k}|}a^{(k)}_{i,k} \quad \text{for} \ \ k+1 \le i \le n, \ \ k \le j \le k+1. \tag{1.10}$$

Theorem 1.2. *Let $\ell_{n,1}, \ell_{i,i+1} : C([0,\omega]) \to L([0,\omega])$ $(i = \overline{1,n-1})$ be linear monotone operators,*

$$\int_0^\omega \ell_{n,1}(1)(s)ds \ne 0, \quad \int_0^\omega \ell_{i,i+1}(1)(s)ds \ne 0 \quad \textit{for} \quad 1 \le i \le n-1, \tag{1.11}$$

and

$$a^{(k)}_{k,k} < 0 \quad \textit{for} \quad 2 \le k \le n, \tag{1.12}$$

where the matrices A_k are defined by relations (1.7)–(1.10). Let, moreover,

$$\int_0^\omega |\ell_{n,1}(1)(s)|ds \prod_{j=1}^{n-1} \int_0^\omega |\ell_{j,j+1}(1)(s)|ds < 4^n \prod_{j=2}^{n} |a_{j,j}^{(j)}|. \tag{1.13}$$

Then problem (1.1), (1.2) has a unique solution.

Definition 1.3. For the system (1.3) we define the matrix $A_1 = (a_{i,j}^{(1)})_{i,j=1}^n$ by the equalities (1.7)–(1.10) with

$$\ell_{i,j}(x)(t) = h_{i,j}(t)x(t) \quad \text{for} \quad i,j \in \overline{1,n}, \quad x \in C([0,\omega]). \tag{1.14}$$

Corollary 1.1. *Let*

$$0 \le \sigma_n h_{n,1}(t) \not\equiv 0, \;\; 0 \le \sigma_i h_{i,i+1}(t) \not\equiv 0 \qquad \textit{for} \quad 1 \le i \le n-1 \tag{1.15}$$

where $\sigma_i \in \{-1,1\}$ $(i = \overline{1,n})$, the matrices A_k are defined by the relations (1.8)-(1.10), (1.14) and

$$a_{k,k}^{(k)} < 0 \qquad \textit{for} \quad 2 \le k \le n. \tag{1.16}$$

Let, moreover,

$$\int_0^\omega |h_{n,1}(s)|ds \prod_{j=1}^{n-1} \int_0^\omega |h_{j,j+1}(s)|ds < 4^n \prod_{j=2}^{n} |a_{j,j}^{(j)}|. \tag{1.17}$$

Then problem (1.3), (1.2) has a unique solution.

Now, assume that

$$\begin{gathered} \ell_{1,j} \equiv 0 \text{ for } j \ne 2, \;\; \ell_{i,j} \equiv 0 \text{ for } j \notin \{i,i+1\}, \;\; i = \overline{2,n-1}, \\ \ell_{n,j} = 0 \text{ for } j = \overline{2,n-1}. \end{gathered} \tag{1.18}$$

Then system (1.1) is of the following type:

$$\begin{aligned} &u_1^{(p_1)}(t) = \ell_{1,2}(u_2)(t) + q_1(t), \\ &u_i^{(p_i)}(t) = \ell_{i,i}(u_i)(t) + \ell_{i,i+1}(u_{i+1})(t) + q_i(t) \;\; \text{for } 2 \le i \le n-1, \\ &u_n^{(p_n)}(t) = \ell_{n,1}(u_1)(t) + \ell_{n,n}(u_n)(t) + q_n(t), \end{aligned} \tag{1.19}$$

and from Theorem 1.2 we obtain

Corollary 1.2. *Let $\ell_{n,1}, \ell_{i,i+1}$ $(i = \overline{1,n-1})$ be linear monotone operators, the conditions (1.11) hold and, moreover,*

$$\int_0^\omega |\ell_{k,k}(1)(s)|ds < \frac{d_{p_k-1}}{\omega^{p_k-1}} \quad \textit{for} \quad 2 \le k \le n. \tag{1.20}$$

Let, moreover,

$$\int_0^\omega |\ell_{n,1}(1)(s)|ds \prod_{j=1}^{n-1} \int_0^\omega |\ell_{j,j+1}(1)(s)|ds < 4^n \prod_{j=2}^{n} \left(\frac{d_{p_j-1}}{\omega^{p_j-1}} - \int_0^\omega |\ell_{j,j}(1)(s)|ds \right). \tag{1.21}$$

Then problem (1.19), (1.2) *has a unique solution.*

For the cyclic feedback system

$$\begin{aligned} u_i^{(p_1)}(t) &= \ell_i(u_{i+1})(t) + q_i(t) \qquad \text{for} \quad 1 \le i \le n-1, \\ u_n^{(p_n)}(t) &= \ell_n(u_1)(t) + q_n(t), \end{aligned} \tag{1.22}$$

Corollary 1.2 yields

Corollary 1.3. *Let $\ell_i : C([0,\omega]) \to L([0,\omega])$ $(i = \overline{1,n})$ be linear monotone operators,*

$$\|\ell_i\| \neq 0 \quad \text{for} \quad i = \overline{1,n}, \tag{1.23}$$

and

$$\prod_{i=1}^{n} \|\ell_i\| < 4^n \prod_{j=2}^{n} \frac{d_{p_j-1}}{\omega^{p_j-1}}. \tag{1.24}$$

Then problem (1.22), (1.2) *has a unique solution.*

Remark 1.2. The problem

$$u''(t) = p(t)u(t) + q(t), \;\; u(0) = u(\omega), \; u'(0) = u'(\omega), \tag{1.25}$$

is equivalent to the problem (1.22), (1.2) with $n = 2$, $n_1 = n_2 = 1$ $\ell_1(x)(t) = x(t)$, $\ell_2(x)(t) = p(t)x(t)$, $q_1 \equiv 0$ and $q_2 \equiv q$.

Then if $p, q \in L([0,\omega])$, $p(t) \le 0$, and $\int_0^\omega p(s)ds \neq 0$, it follows from Corollary 1.3 that problem (1.22), (1.2) and, therefore, problem (1.25), has a unique solution provided that the condition

$$\int_0^\omega |p(s)|ds < \frac{16}{\omega}$$

is fulfilled. This follows from a well-known result of A. Lasota and Z. Opial (see [7]).

Remark 1.3. Rewrite the problem

$$u^{(4)}(t) = \ell(u)(t) + q(t), \qquad u^{(i)}(0) = u^{(i)}(\omega) \quad (i = 0,1,2,3) \tag{1.26}$$

as the systems

$$
\begin{aligned}
u_i'(t) &= u_{i+1}(t) \;\; (i=1,2,3),\\
u_4'(t) &= \ell(u_1)(t)+q(t),\\
u_i(0) &= u_i(\omega) \;\; (i=1,2,3,4)
\end{aligned}
\tag{1.27}
$$

and

$$
\begin{gathered}
u_1'(t) = u_2(t),\\
u_2^{(3)}(t) = \ell(u_1)(t)+q(t),\\
u_1(0)=u_1(\omega), \;\; u_2^{(j)}(0)=u_2^{(j)}(\omega) \;\; (j=0,1,2).
\end{gathered}
\tag{1.28}
$$

It follows from [12, Theorem 3] that problem (1.26), i. e., problems (1.27) and (1.28), are uniquely solvable if $\|\ell\| \le \frac{768}{\omega^3}$, and the constant 768 is optimal.

On the other hand, it follows from Corollary 1.3 that system (1.27) (resp., (1.28)) is uniquely solvable if $\|\ell\| \le \frac{256}{\omega^3}$ (resp., $\|\ell\| \le \frac{512}{\omega^3}$). Consequently, if we rewrite every equation of (1.1) as a system of first order equations, we debase the quality of uniqueness condition.

Example 1.1. The example below shows that condition (1.24) in Corollary 1.3 is optimal and cannot be replaced by the condition

$$
\prod_{i=1}^{n} \|\ell_i\| \le 4^n \prod_{j=2}^{n} \frac{d_{p_j-1}}{\omega^{p_j-1}}. \tag{1.24$_1$}
$$

Define the function $u_0 \in \widetilde{C}([0,1])$ on $[0,1/2]$, and extend it to $[1/2,1]$ by the equalities

$$
u_0(t) = \begin{cases} 1 & \text{for } \; 0 \le t \le 1/8,\\ \sin\pi(1-4t) & \text{for } \; 1/8 < t \le 3/8,\\ -1 & \text{for } \; 3/8 < t \le 1/2 \end{cases}
$$

and

$$
u_0(t) = u_0(1-t) \quad \text{for} \quad 1/2 < t \le 1.
$$

Now let measurable functions $\tau_i : [0,1] \to [0,1]$ and the linear nonnegative operators $\ell_i : C([0,1]) \to L([0,1]) (i=\overline{1,n})$ be given by the equalities

$$
\tau_i(t) = \begin{cases} 1/8i & \text{for } \; 0 \le u_0'(t)\\ 1/2 - 1/8i & \text{for } \; 0 > u_0'(t) \end{cases}, \quad \ell_i(x)(t) = |u_0'(t)| x(\tau_i(t)).
$$

Then it is clear that $u_0(0)=u_0(1)$, $\ell_i \ne \ell_j$ if $i \ne j$, and $\|\ell_i\| = \int_0^1 |\ell_i(1)(s)|ds = 16\pi \int_{1/8}^{1/4} \cos\pi(1-4s)ds = 4$ for $i=\overline{1,n}$. Thus, all the assumptions of Corollary 1.3 are satisfied except (1.24), instead of which condition (1.24$_1$) is fulfilled with $\omega = 1$. On the other hand, from the relations $u_0'(t) = |u_0'(t)| u_0(\tau_i(t)) = \ell_i(u_0)(t) \;\; (i = \overline{1,n})$, it follows that the vector function $(u_i(t))_{i=1}^n$ if $u_i(t) \equiv u_0(t) \;\; (i=\overline{1,n})$, is a nontrivial solution of problem (1.22), (1.2) with $\omega=1$, $q(t)\equiv 0$, $p_j=1(j=\overline{1,n})$, which contradicts the conclusion of Corollary 1.3.

2. Auxiliary propositions

Lemma 2.1. *Let the matrices A_k $(k=\overline{1,n})$ be defined by equalities (1.7)-(1.10). Then the following relations hold:*

$$a_{i,j}^{(m)}\ge 0 \qquad \textit{for} \quad i\ne j,\ m=\overline{1,n}, \tag{2.1$_m$}$$

$$a_{n,1}^{(1)}=a_{n,1}^{(n)} \tag{2.2$_0$}$$

$$a_{i,j}^{(\lambda)}\le a_{i,j}^{(m)} \qquad \textit{for} \quad i\ge m\ge 2,\ j\ge m,\ \lambda\le m. \tag{2.2$_m$}$$

Proof. It follows immediately from the definition of A_1 and A_2 that inequalities (2.1$_1$) and (2.2$_2$) are true. Assume now that (2.1$_m$) holds for $m=3,\dots,m_0$ $(m_0<n)$ and prove (2.1$_{m_0+1}$). If $i\le m_0$ or $j\notin\{m_0,\ m_0+1\}$, relation (1.9) implies inequality (2.1$_{m_0+1}$), and if $i\ge m_0+1$, $j\in\{m_0,\ m_0+1\}$, then (2.1$_{m_0+1}$) follows from (1.10).

Now we prove inequality (2.2$_m$). First assume that $j\ge m+1$. Then from (1.9) with $k=m-1$ it is clear that

$$a_{i,j}^{(\lambda)}=a_{i,j}^{(\lambda+1)}=\cdots=a_{i,j}^{(m)} \qquad \text{for} \quad j\ge m+1,\ i\ge m,\ \lambda\le m. \tag{2.3}$$

Now, let $j=m$. Then from (1.9) we get $a_{i,m}^{(\lambda)}=a_{i,m}^{(\lambda+1)}=\cdots=a_{i,m}^{(m-1)}$ for $i\ge m$, $\lambda\le m-1$. By the last equalities and (2.1$_m$), from (1.10) it follows that

$$a_{i,m}^{(m)}=a_{i,m}^{(m-1)}+\frac{a_{m-1,m}^{(m-1)}}{|a_{m-1,m-1}^{(m-1)}|}\,a_{i,m-1}^{(m-1)}\ge a_{i,m}^{(m-1)}=a_{i,m}^{(\lambda)} \quad \text{for} \quad i\ge m,\ \lambda\le m.$$

From this inequality and (2.3) we conclude that (2.2$_m$) is fulfilled for all $j\ge m$ and $i\ge m$. Equality (2.2$_0$) follows immediately from (1.8) and (1.9). □

Also we need the following simple lemma proved in the paper [11].

Lemma 2.2. *Let $\sigma\in\{-1,1\}$ and $\sigma\ell: C([0,\omega])\to L([0,\omega])$ be a nonnegative linear operator. Then*

$$-m|\ell(1)(t)|\le\sigma\ell(x)(t)\le M|\ell(1)(t)| \quad \textit{for} \quad 0\le t\le\omega,\quad x\in C([0,\omega]),$$

where $m=-\min_{0\le t\le\omega}\{x(t)\}$ and $M=\max_{0\le t\le\omega}\{x(t)\}$.

Now, consider on $[0,\omega]$ the homogeneous problem

$$v_i^{(p_i)}(t)=\sum_{j=2}^{i+1}\ell_{i,j}(v_j)(t) \qquad \text{for} \quad 1\le i\le n, \tag{2.4$_i$}$$

$$v_j^{(p)}(0)=v_j^{(p)}(\omega) \qquad \text{for} \quad 1\le j\le n,\ 0\le p\le p_j-1, \tag{2.5}$$

where the operator $\ell_{n,n+1}$ and function v_{n+1} are defined by the equalities $\ell_{n,n+1}\equiv\ell_{n,1}$ and $v_{n+1}\equiv v_1$. Then it is clear that $\Delta(v_{n+1})\equiv\Delta(v_1)$.

Lemma 2.3. *Let $\ell_{i,i+1} : C([0,\omega]) \to L([0,\omega])$ $(i = \overline{1,n})$ be linear monotone operators,*

$$\int_0^{\omega} \ell_{i,i+1}(1)(s)ds \neq 0 \qquad \text{for} \quad 1 \le i \le n, \tag{2.6}$$

the matrices A_k be defined by the equalities (1.7)–(1.10) and

$$a_{k,k}^{(k)} < 0 \qquad \text{for} \quad 2 \le k \le n. \tag{2.7}$$

Let, moreover $v = (v_i)_{i=1}^n$ be a nontrivial solution of the problem $((2.4_i))_{i=1}^n$, (2.5) for which there exists a $k_1 \in \{2,\dots,n\}$ such that $v_{k_1} \not\equiv 0$. Then if

$$k_0 = \min\{k \in \{2,\dots,n\} : v_k \not\equiv 0\}, \tag{2.8}$$

the inequalities

$$0 < \|v_k\|_C \le \Delta(v_k) \qquad \text{for} \quad k = 1,\ k_0 \le k \le n, \tag{2.9$_k$}$$

$$0 \le \frac{\omega^{p_k-1}}{d_{p_k-1}} a_{k,k}^{(k)} \Delta(v_k^{(p_k-1)}) + \frac{\omega^{p_{k+1}-1}}{d_{p_{k+1}-1}} a_{k,k+1}^{(k)} \Delta(v_{k+1}^{(p_{k+1}-1)}) \tag{2.10$_k$}$$

for $k_0 \le k \le n$ hold, where $a_{n,n+1}^{(1)} = a_{n,1}^{(1)}$.

Proof. Define the numbers $M_{k,i}, m_{k,i} \in \mathbb{R}$, $t'_{k,i}, t''_{k,i} \in [0,\omega]$ $(k = \overline{1,n},\ i = 1, p_k - 1)$ by the relations

$$\begin{aligned} M_{k,i} &= v_k^{(i)}(t'_{k,i}) = \max_{0\le t\le\omega}\{v_k^{(i)}(t)\}, \\ -m_{k,i} &= v_k^{(i)}(t''_{k,i}) = \min_{0\le t\le\omega}\{v_k^{(i)}(t)\}, \end{aligned} \tag{2.11$_k$}$$

and introduce the sets $I_k^{(1)} = [t'_{k,p_k-1}, t''_{k,p_k-1}]$, $I_k^{(2)} = I \setminus I_k^{(1)}$ for $t'_{k,p_k-1} < t''_{k,p_k-1}$. It is clear from (2.8) that

$$v_{k_0} \not\equiv 0. \tag{2.12}$$

On the other hand, from (2.4_{k_0-1}) by (2.8) we obtain

$$\int_0^{\omega} \ell_{k_0-1,k_0}(v_{k_0})(s)ds = 0. \tag{2.13}$$

Equality (2.13), in view of (2.6) and Lemma 2.2 guarantees the existence of a $t_0 \in [0,\ \omega]$ such that $v_{k_0}(t_0) = 0$. Then from (2.12) we get (2.9_{k_0}).

Let the numbers $M_{k_0,p_{k_0}-1}$, $m_{k_0,p_{k_0}-1}$, $M_{k_0+1,1}$, $m_{k_0+1,1} \in \mathbb{R}$, $t'_{k_0}, t''_{k_0} \in [0,\omega]$ be defined by the relations (2.11_k) and $t'_{k_0,p_{k_0}-1} < t''_{k_0,p_{k_0}-1}$ (the case $t''_{k_0,p_{k_0}-1} < t'_{k_0,p_{k_0}-1}$ can be considered analogously). The integration of (2.4_{k_0}) on $I_{k_0}^{(r)}$, by virtue of (2.5) and (2.8) results in

$$\Delta(v_{k_0}^{(p_{k_0}-1)}) = (-1)^r \Bigg(\int_{I_{k_0}^{(r)}} \ell_{k_0,k_0}(v_{k_0})(s)ds + \int_{I_{k_0}^{(r)}} \ell_{k_0,k_0+1}(v_{k_0+1})(s)ds \Bigg) \tag{2.14}$$

for $r = 1,2$. From the last equality, by virtue of (1.7), (2.7), (2.9$_{k_0}$) and (2.2$_{k_0}$) with $\lambda = 1$, $i = j = k_0$ we get

$$0 < -\frac{\omega^{p_{k_0}-1}}{d_{p_{k_0}-1}} a_{k_0,k_0}^{(k_0)} \Delta(v_{k_0}^{(p_{k_0}-1)}) \le (-1)^r \int_{I_{k_0}^{(r)}} \ell_{k_0,k_0+1}(v_{k_0+1})(s)ds \tag{2.15$_r$}$$

for $r = 1,2$. Assume that v_{k_0+1} is a constant sign function. Then, in view of the fact that the operator ℓ_{k_0,k_0+1} is monotone, we get a contradiction with (2.15$_1$) or (2.15$_2$), i. e., v_{k_0+1} changes its sign. Then

$$M_{k_0+1,1} > 0, \;\; m_{k_0+1,1} > 0, \tag{2.16}$$

and the inequality (2.9$_{k_0+1}$) holds ((2.9$_1$) if $k_0 = n$). If ℓ_{k_0,k_0+1} is a non-negative operator, from (2.15$_r$) ($r = 1,2$) in view of (2.16) by Lemma 2.2, we get

$$0 < -\frac{\omega^{p_{k_0}-1}}{d_{p_{k_0}-1}} a_{k_0,k_0}^{(k_0)} \Delta(v_{k_0}^{(p_{k_0}-1)}) \le m_{k_0+1} \int_{I_{k_0}^{(1)}} |\ell_{k_0,k_0+1}(1)(s)|ds,$$

$$0 < -\frac{\omega^{p_{k_0}-1}}{d_{p_{k_0}-1}} a_{k_0,k_0}^{(k_0)} \Delta(v_{k_0}^{(p_{k_0}-1)}) \le M_{k_0+1} \int_{I_{k_0}^{(2)}} |\ell_{k_0,k_0+1}(1)(s)|ds.$$

By multiplying these estimates and applying the numerical inequality $4AB \le (A+B)^2$, in view of (1.6) and notation (1.7), we obtain

$$\begin{aligned} 0 \le\ & \frac{\omega^{p_{k_0}-1}}{d_{p_{k_0}-1}} a_{k_0,k_0}^{(k_0)} \Delta\big(v_{k_0}^{(p_{k_0}-1)}\big) \\ & + \frac{1}{4}(M_{k_0+1,1} + m_{k_0+1,1}) \Bigg(\int_{I_{k_0}^{(1)}} |\ell_{k_0,k_0+1}(1)(s)|ds + \int_{I_{k_0}^{(2)}} |\ell_{k_0,k_0+1}(1)(s)|ds \Bigg) \\ \le\ & \frac{\omega^{p_{k_0}-1}}{d_{p_{k_0}-1}} a_{k_0,k_0}^{(k_0)} \Delta(v_{k_0}^{(p_{k_0}-1)}) + \frac{\omega^{p_{k_0+1}-1}}{d_{p_{k_0+1}-1}} a_{k_0,k_0+1}^{(1)} \Delta\big(v_{k_0+1}^{(p_{k_0+1}-1)}\big), \end{aligned}$$

($0 \le \frac{\omega^{p_n-1}}{d_{p_n-1}} a_{n,n}^{(n)} \Delta\big(v_n^{(p_n-1)}\big) + \frac{\omega^{p_1-1}}{d_{p_1-1}} a_{n,1}^{(1)} \Delta\big(v_1^{(p_1-1)}\big)$ if $k_0 = n$), whence, by virtue of (2.2$_0$) if $k_0 = n$ and (2.2$_{k_0}$) with $\lambda = 1$, $i = k_0$, $j = k_0 + 1$ if $k_0 < n$, the relation (2.10$_{k_0}$) follows. Analogously, from (2.15$_r$) we get (2.10$_{k_0}$) in the case where the operator ℓ_{k_0,k_0+1} is nonpositive.

We have thus proved the proposition:

(i) Let $2 \le k_0 \le n$, then the inequalities (2.9_{k_0}), (2.9_{k_0+1}) ((2.9_1) if $k_0 = n$) and (2.10_{k_0}) hold.

Now, we shall prove the following proposition:

(ii) Let $k_1 \in \{k_0,\dots,n-1\}$ be such that the inequalities (2.9_k),(2.10_k) for $(k = \overline{k_0,k_1})$, and (2.9_{k_1+1}) hold. Then the inequalities (2.9_{k_1+2}) if $k_1 \le n-2$, (2.9_1) if $k_1 = n-1$ and (2.10_{k_1+1}) hold too.

Define the numbers

$$M_{k_1+1,p_{k_1+1}-1},\ m_{k_1+1,p_{k_1+1}-1} \in \mathbb{R} \ \text{ and } \ t'_{k_1+1,p_{k_1+1}-1},\ t''_{k_1+1,p_{k_1+1}-1} \in [0,\omega],$$

by the relations (2.11_{k_1+1}) and let $t'_{k_1+1,p_{k_1+1}-1} < t''_{k_1+1,p_{k_1+1}-1}$ (the case where $t''_{k_1+1,p_{k_1+1}-1} < t'_{k_1+1,p_{k_1+1}-1}$ can be considered analogously). The integration of (2.4_{k_1+1}) on $I^{(r)}_{k_1+1}$, by virtue of (2.5) and (2.8) results in

$$\Delta(v_{k_1+1}^{(p_{k_1+1}-1)}) = (-1)^r \sum_{j=k_0}^{k_1+2} \int_{I^{(r)}_{k_1+1}} \ell_{k_1+1,j}(v_j)(s)ds \tag{2.17}$$

for $r = 1,2$. From this equality, by the conditions (1.6), (1.7), (2.7), (2.9_k) with $k = k_0,\dots,k_1+1$, and (2.2_{k_0}) with $\lambda = 1$, $i = k_1+1$, $j = k_0,\dots,k_1+1$ we get

$$0 \le \sum_{j=k_0}^{k_1+1} \frac{\omega^{p_j-1}}{d_{p_j-1}} a^{(k_0)}_{k_1+1,j} \Delta(v_j^{(p_j-1)}) + (-1)^r \int_{I^{(r)}_{k_1+1}} \ell_{k_1+1,k_1+2}(v_{k_1+2})(s)ds \tag{2.18}$$

for $r = 1,2$. By multiplying (2.10_k) with $a^{(k)}_{k_1+1,k}/|a^{(k)}_{k,k}|$ for $k \in \{k_0,\dots,k_1\}$ in view of the inequalities (2.7) we obtain

$$0 \le -\frac{\omega^{p_k-1}}{d_{p_k-1}} a^{(k)}_{k_1+1,k} \Delta(v_k^{(p_k-1)}) + \frac{\omega^{p_{k+1}-1}}{d_{p_{k+1}-1}} \frac{a^{(k)}_{k,k+1}}{|a^{(k)}_{k,k}|} a^{(k)}_{k_1+1,k} \Delta\big(v_{k+1}^{(p_{k+1}-1)}\big). \tag{2.19_k}$$

Now, summing (2.18) and (2.19_{k_0}), by virtue of (1.10) with $k = k_0$, $i = k_1 + 1$, $j = k_0 + 1$, we get the estimate

$$0 \le \frac{\omega^{p_{k_0+1}-1}}{d_{p_{k_0+1}-1}} a^{(k_0+1)}_{k_1+1,k_0+1} \Delta\big(v^{(p_{k_0+1}-1)}_{k_0+1}\big) + \sum_{j=k_0+2}^{k_1+1} \frac{\omega^{p_j-1}}{d_{p_j-1}} a^{(k_0)}_{k_1+1,j} \Delta\big(v^{(p_j-1)}_j\big) \\ + (-1)^r \int_{I^{(r)}_{k_1+1}} \ell_{k_1+1,k_1+2}(v_{k_1+2})(s)\,ds,$$

from which, by (2.2_{k_0+1}) with $i = k_1 + 1$, $j \ge k_0 + 2$, $\lambda = k_0$, we obtain

$$0 \le \sum_{j=k_0+1}^{k_1+1} \frac{\omega^{p_j-1}}{d_{p_j-1}} a^{(k_0+1)}_{k_1+1,j} \Delta(v^{(p_j-1)}_j) \\ + (-1)^r \int_{I^{(r)}_{k_1+1}} \ell_{k_1+1,k_1+2}(v_{k_1+2})(s)\,ds \tag{2.20}$$

for $r = 1,2$. Analogously, by summing (2.20) and the inequalities (2.19_k) for all $k = k_0 + 1, \dots, k_1$ we get

$$0 < -\frac{\omega^{p_{k_1+1}-1}}{d_{p_{k_1+1}-1}} a^{(k_1+1)}_{k_1+1,k_1+1} \Delta(v^{(p_{k_1+1}-1)}_{k_1+1}) \\ \le (-1)^r \int_{I^{(r)}_{k_1+1}} \ell_{k_1+1,k_1+2}(v_{k_1+2})(s)\,ds \tag{2.21}$$

for $r = 1,2$. In the same way as the inequality (2.9_{k_0+1}) and (2.10_{k_0}) follow from (2.15_r), the inequalities (2.9_{k_1+2}) ((2.9_1) if $k_0 = n-1$) and (2.10_{k_1+1}) follow from (2.21).

From the propositions (i) and (ii), by the method of mathematical induction, we obtain that the inequalities (2.9_1), (2.9_k) and (2.10_k) ($k = \overline{k_0, n}$) hold. □

3. Proofs

Proof of Theorem 1.2. It is known from the general theory of boundary value problems for functional differential equations that if $\ell_{i,j}$ ($i, j = \overline{1,n}$) are bounded linear operators, then problem (1.1), (1.2) has the Fredholm property (see [3]). Thus, problem (1.1), (1.2) is uniquely solvable if and only if the homogeneous problem $(2.4_i)_{i=1}^n$, (2.5) has only the trivial solution.

Assume that, on the contrary, the problem $(2.4_i)_{i=1}^n$, (2.5) has a nontrivial solution $v = (v_i)_{i=1}^n$. Let

$$v_1 \not\equiv 0, \quad v_i \equiv 0 \quad \text{for} \quad 2 \le i \le n. \tag{3.1}$$

Thus, from (2.4_1) and (2.4_n), it follows that

$$v_1^{(p_1)} \equiv 0, \tag{3.2}$$

and $\ell_{n,1}(v_1)(t) \equiv 0$. Then from (1.11) we obtain that $v_1 \not\equiv$ const, whence, due to conditions (2.5), it follows $v_1' \not\equiv$ const. Also it is clear that from the conditions $v_1^{(i)} \not\equiv$ const and (2.5) it follows $v_1^{(i+1)} \not\equiv$ const. Thus, by the method of mathematical induction, we obtain that $v_1^{(p_1)} \not\equiv$ const, which contradicts (3.2) and, therefore, contradicts (3.1). Consequently, there exists $k_0 \in \{2,\dots,n\}$ such that $v_{k_0} \not\equiv 0$. Then all the conditions of Lemma 2.3 are satisfied, from which we conclude that $0 < \|v_1\|_C \le \Delta(v_1)$, i. e., $v_1 \not\equiv$ const and in view of the condition (2.5) the function $v_1^{(p_1)}$ changes its sign. Thus, from (2.4$_1$), by the monotonicity of the operator $\ell_{1,2}$, we get that v_2 also changes its sign. Consequently, if $M_{2,1}$ and $m_{2,1}$ are the numbers defined by the equalities (2.11$_2$), then

$$M_{2,1} > 0, \qquad m_{2,1} > 0, \tag{3.3}$$

and if k_0 is the number defined by the equality (2.8), then $k_0 = 2$. Thus, it follows from Lemma 2.3 that the inequalities (2.9$_1$), (2.9$_k$) and (2.10$_k$) ($k = \overline{2,n}$) hold.

Now, assume that the numbers M_{1,p_1-1}, m_{1,p_1-1}, and t'_{1,p_1-1}, $t''_{1,p_1-1} \in [0,\omega[$ are defined by the equalities (2.11$_1$) and $t'_{1,p_1-1} < t''_{1,p_1-1}$ (the case where $t''_{1,p_1-1} < t'_{1,p_1-1}$ is considered analogously). By integration of (2.4$_1$) on the set $I_1^{(r)}$ and by using the inequality (2.9$_1$), we obtain

$$0 < \Delta(v_1^{(p_1-1)}) = (-1)^r \int_{I_1^{(r)}} \ell_{1,2}(v_2)(s)ds \tag{3.4}$$

for $r = 1,2$. Let us first assume that the operator $\ell_{1,2}$ is nonnegative (the case of a nonpositive $\ell_{1,2}$ can be considered analogously), then from (3.4) by (3.3) and Lemma 2.2 we obtain

$$0 < \Delta(v_1^{(p_1-1)}) \le m_2 \int_{I_1^{(1)}} |\ell_{1,2}(1)(s)|ds,$$

$$0 < \Delta(v_1^{(p_1-1)}) \le M_2 \int_{I_1^{(2)}} |\ell_{1,2}(1)(s)|ds.$$

By multiplying these estimates and applying the numerical equality $4AB \le (A+B)^2$ and the equalities (1.7), we get

$$\begin{aligned} 0 &\le a_{1,1}^{(1)}\Delta(v_1^{(p_1-1)}) + \frac{1}{4}(m_2+M_2)\left(\int_{I_1^{(1)}} |\ell_{1,2}(1)(s)|ds + \int_{I_1^{(2)}} |\ell_{1,2}(1)(s)|ds\right) \\ &= a_{1,1}^{(1)}\Delta(v_1^{(p_1-1)}) + a_{1,2}^{(1)}\Delta(v_2^{(p_2-1)}), \end{aligned}$$

i. e., all the inequalities (2.10$_k$) ($k = \overline{1,n}$) are satisfied.

On the other hand from (1.7)–(1.9) and Lemma 2.1 it is clear that

$$a_{1,1}^{(1)} = -1, \quad a_{n,1}^{(n)} = a_{n,1}^{(1)}, \quad a_{k,k+1}^{(k)} = a_{k,k+1}^{(1)} = \frac{1}{4}\|\ell_{k,k+1}\| \tag{3.5}$$

for $1 \le k \le n-1$. By multiplying all the estimates (2.10$_k$) ($k=\overline{1,n}$) and applying (3.5), we get the contradiction with condition (1.13). Thus, our assumption fails, and hence $v_i \equiv 0$ ($i=\overline{1,n}$). □

Proof of Corollary 1.1. From (1.14) and (1.15) it is clear that $\ell_{n,1}$ and $\ell_{i,i+1}$ are monotone operators and (1.11) holds. Also, from (1.16) and (1.17), the conditions (1.12) and (1.13) follow. Consequently, all the conditions of Theorem 1.1 are fulfilled for system (1.3). □

Proof of Corollary 1.2. From (1.7), (1.9), and (1.18) we obtain

$$a_{k,k}^{(k-1)} = a_{k,k}^{(k-2)} = \cdots = a_{k,k}^{(1)} = \|\ell_{k,k}\| - \frac{d_{p_k-1}}{\omega^{p_k-1}} \quad \text{for} \quad 2 \le k \le n, \tag{3.6}$$

and

$$\begin{gathered} a_{k,k-i}^{(k-i-1)} = a_{k,k-i}^{(k-i-2)} = \cdots = a_{k,k-i}^{(1)} = \|\ell_{k,k-i}\| = 0 \ \text{ for } \ 3 \le k-i \le n, \\ a_{2,1}^{(1)} = 0. \end{gathered} \tag{3.7}$$

From (1.10), (1.18) and the first equality of (3.7) we get

$$\begin{aligned} a_{k,k-1}^{(k-1)} &= a_{k,k-1}^{(k-2)} + \frac{a_{k-2,k-1}^{(k-2)}}{|a_{k-2,k-2}^{(k-2)}|} a_{k,k-2}^{(k-2)} = \frac{a_{k-2,k-1}^{(k-2)}}{|a_{k-2,k-2}^{(k-2)}|} a_{k,k-2}^{(k-2)} \\ &= \frac{a_{k-2,k-1}^{(k-2)}}{|a_{k-2,k-2}^{(k-2)}|} \frac{a_{k-3,k-2}^{(k-3)}}{|a_{k-3,k-3}^{(k-3)}|} a_{k,k-3}^{(k-3)} = \cdots = a_{k,2}^{(2)} \prod_{j=2}^{k-2} \frac{a_{j,j+1}^{(j)}}{|a_{j,j}^{(j)}|} = 0 \end{aligned} \tag{3.8}$$

for $k \ge 3$. From (3.8) and the second equality of (3.7) it is clear that

$$a_{k,k-1}^{(k-1)} = 0 \qquad \text{for} \qquad 2 \le k \le n. \tag{3.9}$$

Then from (1.10) by (3.6) and (3.9) we obtain

$$a_{k,k}^{(k)} = a_{k,k}^{(k-1)} + a_{k-1,k}^{(k-1)} a_{k,k-1}^{(k-1)} / |a_{k-1,k-1}^{(k-1)}| = \|\ell_{k,k}\| - \frac{d_{p_k-1}}{\omega^{p_k-1}}.$$

Thus, it follows from the conditions (1.20) and (1.21) that (1.12) and (1.13) hold. Therefore, all the conditions of Theorem 1.1 are fulfilled for the system (1.19). □

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NEW SHARP INEQUALITIES FOR APPROXIMATING THE FACTORIAL FUNCTION AND THE DIGAMMA FUNCTION

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Abstract. The aim of this paper is to establish new sharp upper and lower bounds for the gamma and digamma functions, starting from the Stirling's formula.

2000 *Mathematics Subject Classification:* 30E15, 26D07, 41A60

Keywords: factorial function, gamma function, digamma and polygamma functions, completely monotonic function, inequalities, Euler-Mascheroni constant

1. Introduction

Stirling's formula

$$n! \approx \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$$

is probably the most widely known and used result for approximation of the factorial function. It was discovered by the French mathematician Abraham de Moivre (1667–1754) as

$$n! \approx \text{constant} \cdot n^{n+\frac{1}{2}} e^{-n},$$

while the Scottish mathematician James Stirling (1692–1770) discovered the constant $\sqrt{2\pi}$ (see, e. g., [2, 21, 25, 27] for the proofs and further details)

Although such an approximation is satisfactory for the needs of the probability theory, in pure mathematics, more precise estimates are necessary. The following refined estimate

$$\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n}} \tag{1.1}$$

was first established by Robbins [23] and it can also be found, e. g., in [3, 4, 6, 8, 24]. Successively better results about the gamma and polygamma functions were obtained in [5, 9, 12–18, 20, 26]. In fact, (1.1) is the initial form of the more accurate expression

$$n! = \sqrt{2\pi} n^{n+1/2} e^{-n} \exp\left(\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \dots\right),$$

see, e. g., [8, 19, 24].

In this paper, we study the complete monotonicity of the functions

$$f(x)=\ln\Gamma(x+1)-\ln\sqrt{2\pi}-\left(x+\frac{1}{2}\right)\ln x+x-\frac{1}{12x}$$

and

$$g(x)=\ln\Gamma(x+1)-\ln\sqrt{2\pi}-\left(x+\frac{1}{2}\right)\ln x+x-\frac{1}{12x+1}$$

associated with the approximations (1.1). More precisely, we show that $-f$ and g are completely monotonic. As direct consequences, we establish the following double inequalities for $x\geq 1$:

$$\omega\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x}}\leq\Gamma(x+1)<\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x}},$$

where $\omega=\frac{1}{\sqrt{2\pi}}e^{\frac{11}{12}}=0.99773\ldots$ is the best possible and

$$\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x+1}}<\Gamma(x+1)\leq\eta\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x+1}},$$

where $\eta=\frac{1}{\sqrt{2\pi}}e^{\frac{12}{13}}=1.004146965\ldots$ is the best possible.

With this occasion, we state the following double inequalities for $x\geq 1$:

$$\ln x-\frac{1}{2x}-\frac{1}{12x^2}<\psi(x)\leq\ln x-\frac{1}{2x}-\frac{1}{12x^2}+\tau, \tag{1.2}$$

where $\tau=-\gamma+\frac{7}{12}=0.0061177\ldots$ is the best possible and

$$\ln x-\frac{1}{2x}-\frac{1}{12\left(x+\frac{1}{12}\right)^2}-\upsilon\leq\psi(x)<\ln x-\frac{1}{2x}-\frac{1}{12\left(x+\frac{1}{12}\right)^2}, \tag{1.3}$$

where $\upsilon=\frac{193}{338}-\gamma=0.0062097\ldots$ is the best possible. Estimates (1.2) and (1.3) refine other known results [1,7,10,11,22] of the form

$$\ln x-\frac{1}{x}<\psi(x)<\ln x-\frac{1}{2x},\quad x>1.$$

2. THE RESULTS

The gamma Γ and digamma ψ functions are defined by the equalities

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt,\quad \psi(x)=\frac{d}{dx}\left(\ln\Gamma(x)\right)=\frac{\Gamma'(x)}{\Gamma(x)},$$

for an arbitrary positive real x. We also have the recurrence relation

$$\psi(x+1)=\psi(x)+\frac{1}{x}$$

valid for all $x > 0$. The gamma function is an extension of the factorial function because, as is well-known, $\Gamma(n+1) = n!$ for $n = 0,1,2,3,\ldots$. The derivatives ψ', ψ'', ..., known as polygamma functions, admit the integral representations

$$\psi^{(n)}(x) = (-1)^{n-1}\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}dt \tag{2.1}$$

for $n = 1,2,3,\ldots$ (see, e. g., [2] for the proofs and other details). We also use the following integral representation

$$\frac{1}{x^n} = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-xt}dt, \quad n \geq 1. \tag{2.2}$$

Recall that a function f is (strictly) completely monotonic in an interval I if f has derivatives of all orders in I such that $(-1)^n f^{(n)}(x) \geq 0$ (resp., $(-1)^n f^{(n)}(x) > 0$) for all $x \in I$ and $n = 0,1,2,3,\ldots$.

Completely monotonic functions involving $\ln\Gamma(x)$ are important because they produce bounds for the polygamma functions. The famous Hausdorff–Bernstein–Widder theorem states that f is completely monotonic on $[0,\infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt}d\mu(t),$$

where μ is a non-negative measure on $[0,\infty)$ such that the integral converges for all $x > 0$, see [27].

We are now in position to give the following

Theorem 1. *Let the function $f : (0,\infty) \to \mathbb{R}$ be given by the equality*

$$f(x) = \ln\Gamma(x+1) - \ln\sqrt{2\pi} - \left(x+\frac{1}{2}\right)\ln x + x - \frac{1}{12x}.$$

Then $-f$ is completely monotonic.

Proof. We have

$$f'(x) = \psi(x) + \frac{1}{2x} - \ln x + \frac{1}{12x^2}$$

and

$$f''(x) = \psi'(x) - \frac{1}{2x^2} - \frac{1}{x} - \frac{1}{6x^3}.$$

Now, using (2.1) and (2.2), we have

$$f''(x) = \int_0^\infty \frac{te^{-tx}}{1-e^{-t}}dt - \int_0^\infty \frac{1}{2}te^{-xt}dt - \int_0^\infty e^{-xt}dt - \int_0^\infty \frac{1}{12}t^2e^{-xt}dt,$$

or

$$f''(x) = \int_0^\infty \frac{e^{-xt}}{1-e^{-t}}\varphi(t)\,dt,$$

where

$$\varphi(t) = t - \frac{1}{2}t\left(1-e^{-t}\right) - 1 + e^{-t} - \frac{1}{12}t^2\left(1-e^{-t}\right).$$

We have $\varphi'''(t) = -\frac{1}{12}t^2e^{-t} < 0$, for every $t > 0$, so φ'' is strictly decreasing. But $\varphi''(0) = 0$, thus $\varphi'' < 0$ on $(0,\infty)$. Now, φ' is strictly decreasing, with $\varphi'(0) = 0$, so $\varphi' < 0$. Finally, φ is strictly decreasing, with $\varphi(0) = 0$, so $\varphi < 0$ on $(0,\infty)$.

In consequence, $-f''$ is completely monotonic. Furthermore, f' is strictly decreasing, since $f'' < 0$. But we have $\lim_{x\to\infty} f'(x) = 0$, so $f'(x) > 0$ and consequently, f is strictly increasing. Using the fact that $\lim_{x\to\infty} f(x) = 0$, we deduce that $f < 0$. Finally, $-f$ is completely monotonic. □

Corollary 1. *The following assertions hold:*

(i) *For all $x \geq 1$, we have*

$$\omega\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x}} \leq \Gamma(x+1) < \sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x}},$$

where $\omega = \frac{1}{\sqrt{2\pi}}e^{\frac{11}{12}} = 0.99773\ldots$ is the best possible.

(ii) *For all $x \geq 1$, we have*

$$\ln x - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x) \leq \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \tau,$$

where $\tau = -\gamma + \frac{7}{12} = 0.0061177\ldots$ is the best possible.

Proof. (i) The function f is strictly increasing, so for $x \geq 1$, we have

$$f(1) \leq f(x) < \lim_{x\to\infty} f(x) = 0.$$

As $f(1) = \frac{11}{12} - \ln\sqrt{2\pi}$, we get, by exponentiating

$$\frac{1}{\sqrt{2\pi}}e^{\frac{11}{12}} \leq \frac{\Gamma(x+1)}{\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x}}} < 1,$$

or

$$\omega\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x}} \leq \Gamma(x+1) < \sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x}},$$

where $\omega = \frac{1}{\sqrt{2\pi}}e^{\frac{11}{12}} = 0.99773\ldots$ is the best possible.

(ii) The function f' is strictly decreasing, so for $x \geq 1$, we have

$$0 = \lim_{x\to\infty} f'(x) < f'(x) \leq f'(1).$$

As $f'(1) = -\gamma + \frac{7}{12}$, we get

$$\ln x - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x) \leq \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \tau,$$

where $\tau = -\gamma + \frac{7}{12} = 0.0061177\ldots$ is the best possible. □

In order to prove the next result, we need the following

Lemma 1. *Let*

$$a_n = (6n-13)\,13^{n-1} + 6n + 1 - 12^{n-1} n\,(n-1)\,, \quad n \geq 4.$$

Then $a_n > 0$ *for every* $n \geq 4$.

Proof. Since we have $a_4 = 3456$, $a_5 = 70848$, $a_6 = 1074816$, $a_7 = 14566176$, $a_8 = 189616896$, $a_9 = 2486277504$, $a_{10} = 34031238912$, $a_{11} = 495590003424$, $a_{12} = 7660358317440$, it suffices to show that $a_n > 0$, for every $n \geq 13$. We prove moreover that

$$(6n-13)\,13^{n-1} > 12^{n-1} n\,(n-1)\,, \quad n \geq 13.$$

This follows immediately from the inequalities

$$\left(\frac{13}{12}\right)^{n-1} > \frac{n}{5} > \frac{n\,(n-1)}{6n-13}\,,$$

because the first inequality can be easily proved by induction, with respect to $n \geq 13$, using

$$\frac{13}{12}\frac{n}{5} - \frac{n+1}{5} = \frac{n-12}{60} > 0,$$

and for the second inequality, we have

$$\frac{n}{5} - \frac{n\,(n-1)}{6n-13} = \frac{n\,(n-8)}{5\,(6n-13)} > 0.$$

□

Theorem 2. *Let the function* $g\colon (0,\infty) \to \mathbb{R}$ *be given by the equality*

$$g(x) = \ln \Gamma(x+1) - \ln\sqrt{2\pi} - \left(x + \frac{1}{2}\right)\ln x + x - \frac{1}{12x+1}\,.$$

Then g *is completely monotonic.*

Proof. We have

$$g'(x) = \psi(x) + \frac{1}{2x} - \ln x + \frac{1}{12}\frac{1}{\left(x+\frac{1}{12}\right)^2}$$

and

$$g''(x) = \psi'(x) - \frac{1}{2x^2} - \frac{1}{x} - \frac{1}{6}\frac{1}{\left(x+\frac{1}{12}\right)^3}.$$

Now, using (2.1) and (2.2), we have

$$g''(x) = \int_0^\infty \frac{te^{-tx}}{1-e^{-t}}\,dt - \int_0^\infty \frac{1}{2}te^{-xt}\,dt - \int_0^\infty e^{-xt}\,dt - \int_0^\infty \frac{1}{12}t^2 e^{-xt}\,dt,$$

or

$$g''(x) = \int_0^\infty \frac{e^{-xt}}{1-e^{-t}}\,\nu(t)\,dt,$$

where
$$\nu(t)=t-\frac{1}{2}t\left(1-e^{-t}\right)-1+e^{-t}-\frac{1}{12}t^2e^{-\frac{1}{12}t}\left(1-e^{-t}\right).$$
Now, $\nu>0$ on $(0,\infty)$, since straightforward computations lead us to the following power series expansion:
$$\nu(t)=\sum_{n=4}^{\infty}\frac{a_n}{n!}t^n>0,$$
where the sequence $(a_n)_{n\geq 4}$ is defined in Lemma 1.

As a consequence, g'' is completely monotonic. Furthermore, g' is strictly increasing because $g''>0$. However, we have $\lim_{x\to\infty}g'(x)=0$, so $g'(x)<0$ and, consequently, g is strictly decreasing. Using the fact that $\lim_{x\to\infty}g(x)=0$, we deduce that $g>0$. Finally, g is completely monotonic. □

Corollary 2. *The following assertions hold:*

(i) *For all $x\geq 1$, we have*
$$\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x+1}}<\Gamma(x+1)\leq\eta\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x+1}},$$
where $\eta=\frac{1}{\sqrt{2\pi}}e^{\frac{12}{13}}=1.004146965\ldots$ is the best possible value.

(ii) *For all $x\geq 1$, we have*
$$\ln x-\frac{1}{2x}-\frac{1}{12\left(x+\frac{1}{12}\right)^2}-\upsilon\leq\psi(x)<\ln x-\frac{1}{2x}-\frac{1}{12\left(x+\frac{1}{12}\right)^2},$$
where $\upsilon=\frac{193}{338}-\gamma=0.0062097\ldots$ is the best possible value.

Proof. (i) The function g is strictly decreasing, so for $x\geq 1$, we have
$$0=\lim_{x\to\infty}g(x)<g(x)\leq g(1).$$
As $g(1)=\frac{12}{13}-\ln\sqrt{2\pi}=0.0041384\ldots$, we get, by exponentiating
$$1<\frac{\Gamma(x+1)}{\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x+1}}}\leq\frac{1}{\sqrt{2\pi}}e^{\frac{12}{13}},$$
or
$$\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x+1}}\leq\Gamma(x+1)<\eta\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12x+1}},$$
where $\eta=\frac{1}{\sqrt{2\pi}}e^{\frac{12}{13}}=1.004146965\ldots$ is the best possible.

(ii) The function g' is strictly increasing, so for $x\geq 1$, we have
$$g'(1)\leq g'(x)<\lim_{x\to\infty}g'(x)=0.$$
As $g'(1)=-\gamma+\frac{193}{338}=-0.0062097\ldots$, we get
$$\ln x-\frac{1}{2x}-\frac{1}{12\left(x+\frac{1}{12}\right)^2}-\upsilon\leq\psi(x)<\ln x-\frac{1}{2x}-\frac{1}{12\left(x+\frac{1}{12}\right)^2},$$

where $\upsilon = \frac{193}{338} - \gamma = 0.0062097\ldots$ is the best possible. $\square$

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LOCAL APPROXIMATION BEHAVIOR OF MODIFIED SMK OPERATORS

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Abstract. In this paper, for a general modification of the classical Szász–Mirakjan–Kantorovich operators, we obtain many local approximation results including the classical cases. In particular, we obtain a Korovkin theorem, a Voronovskaya theorem, and some local estimates for these operators.

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Keywords: Szász–Mirakjan–Kantorovich, modulus of continuity, second modulus of smoothness, Peetre's K-functional, Korovkin theorem, Voronovskaya theorem

1. INTRODUCTION

As usual, let $C[0,\infty)$ denote the space of all continuous functions on $[0,\infty)$. The classical Szász–Mirakjan–Kantorovich (SMK) operators [4] are given by the relation

$$K_n(f;x) := ne^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\int_{I_{n,k}} f(t)dt, \tag{1.1}$$

where $I_{n,k} = \left[\frac{k}{n}, \frac{k+1}{n}\right]$ and f belongs to an appropriate subspace of $C[0,\infty)$ for which the above series is convergent. Among of these subspaces, we can take the space $C_B[0,\infty)$ of all bounded and continuous functions on $[0,\infty)$, or, the weighted space $C_\gamma[0,+\infty)$, $\gamma > 0$, defined by the equality

$$C_\gamma[0,+\infty) := \left\{f \in C[0,+\infty) : |f(t)| \le M(1+t)^\gamma \text{ for some } M > 0\right\}.$$

Assume now that (u_n) is a sequence of functions on $[0,\infty)$ such that, for a fixed $a \ge 0$,

$$0 \le u_n(x) \le x \quad \text{for every } x \in [a,\infty) \text{ and } n \in \mathbb{N}. \tag{1.2}$$

Then, we consider the following modification of SMK operators:

$$L_n(f;x) := \sum_{k=0}^{\infty} p_{k,n}(x)\int_{I_{n,k}} f(t)dt, \quad n \in \mathbb{N},\ x \in [a,\infty), \tag{1.3}$$

where

$$p_{k,n}(x) := ne^{-nu_n(x)} \frac{(nu_n(x))^k}{k!}, \quad k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \tag{1.4}$$

Throughout the paper we use the following test functions

$$e_i(y) = y^i, \quad i = 0,1,2,3,4,$$

and the moment function

$$\psi_x(y) = y - x.$$

So, using the fundamental properties of the classical SMK operators, one can get the following lemmas.

Lemma 1. *For the operators L_n, we have*

(i) $L_n(e_0;x) = 1$,

(ii) $L_n(e_1;x) = u_n(x) + \frac{1}{2n}$,

(iii) $L_n(e_2;x) = u_n^2(x) + \frac{2u_n(x)}{n} + \frac{1}{3n^2}$,

(iv) $L_n(e_3;x) = u_n^3(x) + \frac{9u_n^2(x)}{2n} + \frac{7u_n(x)}{2n^2} + \frac{1}{4n^3}$,

(v) $L_n(e_4;x) = u_n^4(x) + \frac{8u_n^3(x)}{n} + \frac{15u_n^2(x)}{n^2} + \frac{6u_n(x)}{n^3} + \frac{1}{5n^4}$.

Lemma 2. *For the operators L_n, we have*

(i) $L_n(\psi_x;x) = u_n(x) - x + \frac{1}{2n}$,

(ii) $L_n(\psi_x^2;x) = (u_n(x)-x)^2 + \frac{2u_n(x)-x}{n} + \frac{1}{3n^2}$,

(iii) $L_n(\psi_x^3;x) = (u_n(x)-x)^3 + \frac{3(3u_n(x)-x)(u_n(x)-x)}{2n} + \frac{7u_n(x)-2x}{2n^2} + \frac{1}{4n^3}$,

(iv) $L_n(\psi_x^4;x) = (u_n(x)-x)^4 + \frac{2(4u_n(x)-x)(x-u_n(x))^2}{n} + \frac{15u_n^2(x)-14xu_n(x)+2x^2}{n^2} + \frac{6u_n(x)-x^3}{n^3} + \frac{1}{5n^4}$.

Then, we see from Lemma 1 that, with some suitable choices of u_n, our operators L_n may preserve the linear functions or the test function e_2. For example, taking $a = \frac{1}{2}$, if we choose $u_n(x) = x - \frac{1}{2n}$ for $x \in \left[\frac{1}{2},\infty\right)$ and $n \in \mathbb{N}$, then the corresponding operators L_n preserve the linear functions, i. e., they preserve the test functions e_0 and e_1 (see [8]). In this case, we know from [8] that the operators L_n have a better error estimation than the classical SMK operators on $\left[\frac{1}{2},\infty\right)$. Also, taking $a = \frac{1}{3}$ and

$$u_n(x) := \frac{\sqrt{3n^2x^2+2}-\sqrt{3}}{n\sqrt{3}} \quad \text{for } x \in \left[\frac{1}{\sqrt{3}},\infty\right) \text{ and } n \in \mathbb{N},$$

we see that the corresponding operators L_n preserve the test functions e_0 and e_2. Finally, for $a = 0$ and

$$u_n(x) := \frac{-1+\sqrt{4n^2x^2+1}}{2n}, \quad x \geq 0 \text{ and } n \in \mathbb{N},$$

the corresponding operators L_n becomes the Kantorovich variant of the modified Szász–Mirakjan operators (see [6, 11]).

The first study regarding the preservation of e_0 and e_2 for the linear positive operators in order to get better error estimation, was first presented by King. In [9], King introduced a modification of the classical Bernstein polynomials and had a better error estimation than the classical ones on the interval $\left[0, \frac{1}{3}\right]$. Later, similar problems were accomplished for Szász–Mirakjan operators [6], Szász–Mirakjan–Beta operators [7], Meyer–König and Zeller operators [11], Bernstein–Chlodovsky operators [1], q-Bernstein operators [10], Baskakov operators and Stancu operators [12], and some other kinds of summation-type positive linear operators [2].

However, in the present paper, for a general sequence (u_n) satisfying (1.2), we study the local approximation behavior of the operators L_n defined by (1.3) and (1.4). First of all, we get the following Korovkin-type approximation theorem for these operators.

Theorem 1. *Let (u_n) be a sequence of functions on $[0,\infty)$ satisfying (1.2) for a fixed $a \geq 0$. If*

$$\lim_{n\to\infty} u_n(x) = x \tag{1.5}$$

uniformly with respect to $x \in [a,b]$ with $b > a$, then, for all $f \in C_\gamma[0,+\infty)$ with $\gamma \geq 2$, we have

$$\lim_{n\to\infty} L_n(f;x) = f(x)$$

uniformly with respect to $x \in [a,b]$.

Proof. For a fixed $b > 0$, consider the lattice homomorphism $H_b : C[0,+\infty) \to C[a,b]$ defined by $H_b(f) := f|_{[a,b]}$ for every $f \in C[0,+\infty)$. In this case, from (1.5), we see that, for each $i = 0,1,2$,

$$\lim_{n\to\infty} H_b(T_n(e_i)) = H_b(e_i)$$

uniformly on $[a,b]$. Hence, using the universal Korovkin-type property with respect to monotone operators (see Theorem 4.1.4(vi) of [3, p. 199]), we obtain that, for all $f \in C_\gamma[0,+\infty)$, $\gamma \geq 2$,

$$\lim_{n\to\infty} L_n(f;x) = f(x)$$

uniformly with respect to $x \in [a,b]$. $\square$

Finally, using Proposition 4.2.5(2) of [3], we can state the following approximation result in the space L_p:

Corollary 1. *Let $1 \leq p < \infty$. Then for all $f \in L_p$, we have*

$$\lim_{n\to\infty} L_n(f;x) = f(x)$$

uniformly with respect to $x \in [a,\infty)$ with $a \geq 0$.

2. A VORONOVSKAYA-TYPE THEOREM

In order to get a Voronovskaya-type theorem for the operators L_n given by (1.3) and (1.4), we need the following lemma.

Lemma 3. *Let (u_n) be a sequence of functions on $[0,\infty)$ satisfying (1.2) for a fixed $a \geq 0$. If*

$$\lim_{n\to\infty} \sqrt{n}\,(x - u_n(x)) = 0 \tag{2.1}$$

uniformly with respect to $x \in [a,b]$, $b > a$, then we have

$$\lim_{n\to\infty} n^2 L_n(\psi_x^4;x) = 3x^2 \tag{2.2}$$

uniformly with respect to $x \in [a,b]$.

Proof. Let $x \in [a,b]$, $b > a$, be fixed. Then, by (1.2), since

$$0 \leq x - u_n(x) \leq \sqrt{n}(x - u_n(x)) \quad \text{for every } n \in \mathbb{N},$$

it follows from (2.1) that

$$\lim_{n\to\infty} u_n(x) = x \tag{2.3}$$

uniformly with respect to $x \in [a,b]$. Also, since

$$0 \leq \frac{u_n(x)}{n} = \frac{u_n(x) - x}{n} + \frac{x}{n} \leq x - u_n(x) + \frac{x}{n},$$

we obtain from (2.1) that

$$\lim_{n\to\infty} \frac{u_n(x)}{n} = 0 \tag{2.4}$$

uniformly with respect to $x \in [a,b]$. Observe now that, by Lemma 2(iv),

$$\begin{aligned} n^2 L_n(\psi_x^4;x) &= \left\{\sqrt{n}(x - u_n(x))\right\}^4 \\ &\quad + 2\left\{\sqrt{n}(x - u_n(x))\right\}^2 (4u_n(x) - x) \\ &\quad + \left\{15u_n^2(x) - 14xu_n(x) + 2x^2\right\} + \frac{6}{n}u_n(x) - \frac{x}{n} + \frac{1}{5n^2}. \end{aligned}$$

Taking limit as $n \to \infty$ in the both sides of the last equality and also using (2.1), (2.3), (2.4), we immediately see that

$$\lim_{n\to\infty} n^2 L_n(\psi_x^4;x) = 3x^2$$

uniformly with respect to $x \in [a,b]$. The proof is complete. □

We now get the following result.

Theorem 2. *Let (u_n) be a sequence of functions on $[0,\infty)$ satisfying (1.2) and (2.1) for a fixed $a \geq 0$. Assume further that there exists a function ξ defined on $[a,\infty)$ such that*

$$\lim_{n\to\infty} n\,(x-u_n(x)) = \xi(x) \tag{2.5}$$

uniformly with respect to $x \in [a,b]$, $b > a$. Then, for every $f \in C_\gamma[0,+\infty)$, $\gamma \geq 4$, such that $f', f'' \in C_\gamma[0,+\infty)$, we have

$$\lim_{n\to\infty} n\{L_n(f;x)-f(x)\} = \frac{1}{2}xf''(x) + \left(\frac{1}{2}-\xi(x)\right)f'(x)$$

uniformly with respect to $x \in [a,b]$.

Proof. Let $f, f', f'' \in C_\gamma[0,+\infty)$ with $\gamma \geq 4$. Define

$$\Psi(y,x) = \begin{cases} \frac{f(y)-f(x)-(y-x)f'(x)-\frac{1}{2}(y-x)^2 f''(x)}{(y-x)^2} & \text{for } y \neq x, \\ 0 & \text{for } y = x. \end{cases}$$

Then, it is clear that $\Psi(x,x) = 0$ and that the function $\Psi(\cdot,x)$ belongs to $C_\gamma[0,+\infty)$. Hence, it follows from the Taylor theorem that

$$f(y)-f(x) = \psi_x(y)f'(x) + \frac{1}{2}\psi_x^2(y)f''(x) + \psi_x^2(y)\Psi(y,x).$$

Now, by Lemma 2(ii) and (iii), we get

$$\begin{aligned} n\{L_n(f;x)-f(x)\} &= nf'(x)L_n(\psi_x;x) + \frac{n}{2}f''(x)L_n\left(\psi_x^2;x\right) \\ &\quad + nL_n(\psi_x^2(y)\Psi(y,x);x), \end{aligned}$$

which gives

$$\begin{aligned} n\{L_n(f;x)-f(x)\} &= f'(x)\left\{n\,(u_n(x)-x)+\frac{1}{2}\right\} \\ &\quad + \frac{f''(x)}{2}\left\{\left(\sqrt{n}\,(u_n(x)-x)\right)^2 + 2u_n(x)-x\right\} \\ &\quad + nL_n(\psi_x^2(y)\Psi(y,x);x). \end{aligned} \tag{2.6}$$

If we apply the Cauchy–Schwarz inequality for the last term on the right-hand side of (2.6), then we conclude that

$$n\left|L_n\left(\psi_x^2(y)\Psi(y,x);x\right)\right| \leq \left(n^2L_n(\psi_x^4(y);x)\right)^{1/2}\left(L_n(\Psi^2(y,x);x)\right)^{1/2}. \tag{2.7}$$

Let $\eta(y,x) := \Psi^2(y,x)$. In this case, observe that $\eta(x,x) = 0$ and $\eta(\cdot,x) \in C_\gamma[0,+\infty)$. Then it follows from Theorem 1 that

$$\lim_{n\to\infty} L_n\left(\Psi^2(y,x);x\right) = \lim_{n\to\infty} L_n\left(\eta(y,x);x\right) = \eta(x,x) = 0 \tag{2.8}$$

uniformly with respect to $x \in [a,b]$, $b > a$. So, considering (2.5), (2.7) and (2.8), and also using Lemma 3, we immediately see that

$$\lim_{n\to\infty} nL_n\left(\psi_x^2(y)\Psi(y,x);x\right) = 0 \tag{2.9}$$

uniformly with respect to $x \in [a,b]$. Taking limit as $n \to \infty$ in (2.6) and also using (2.1), (2.3), (2.5), (2.9) we have

$$\lim_{n\to\infty} n\{L_n(f;x) - f(x)\} = \frac{1}{2}xf''(x) + \left(\frac{1}{2} - \xi(x)\right)f'(x)$$

uniformly with respect to $x \in [a,b]$. The proof is complete. □

We should note that one can find a sequence (u_n) satisfying all assumptions (1.2), (2.1) and (2.5) in Theorem 2. For example, if we take $a = 0$ and $u_n(x) = x$, then our operators in (1.3) turn out to be the classical SMK operators K_n defined by (1.1). In this case, we have $\xi(x) = 0$. Hence, we obtain the following result.

Corollary 2. *For the operators (1.1), if $f \in C_\gamma[0,+\infty)$, $\gamma \geq 4$, such that $f', f'' \in C_\gamma[0,+\infty)$, then we have*

$$\lim_{n\to\infty} n\{K_n(f;x) - f(x)\} = \frac{1}{2}xf''(x) + \frac{1}{2}f'(x)$$

uniformly with respect to $x \in [0,b]$, $b > 0$.

Now, if take $a = 0$ and

$$u_n(x) := u_n^{[1]}(x) = \frac{-1+\sqrt{4n^2x^2+1}}{2n}, \quad x \in [0,\infty),\ n \in \mathbb{N}, \tag{2.10}$$

then our operators L_n in (1.3) turn out to be

$$L_n^{[1]}(f;x) := ne^{-\left(-1+\sqrt{4n^2x^2+1}\right)/2} \sum_{k=0}^{\infty} \frac{\left(-1+\sqrt{4n^2x^2+1}\right)^k}{2^k k!} \int_{I_{n,k}} f(t)dt. \tag{2.11}$$

In this case, observe that

$$\xi(x) = \lim_{n\to\infty} n\left(x - u_n^{[1]}(x)\right) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{2} & \text{if } x > 0. \end{cases}$$

So, the next result immediately follows from Theorem 2.

Corollary 3. *For the operators (2.11), if $f \in C_\gamma[0,+\infty)$, $\gamma \geq 4$, such that $f', f'' \in C_\gamma[0,+\infty)$, then we have*

$$\lim_{n\to\infty} n\{L_n^{[1]}(f;x) - f(x)\} = \begin{cases} f'(0)/2 & \textit{if } x = 0, \\ xf''(x)/2 & \textit{if } x > 0. \end{cases}$$

Furthermore, if we choose $a = \frac{1}{2}$ and

$$u_n(x) := u_n^{[2]}(x) = x - \frac{1}{2n}, \quad x \in \left[\frac{1}{2}, \infty\right), \; n \in \mathbb{N},$$

then the operators in (1.3) reduce to the following operators (see [8]):

$$L_n^{[2]}(f;x) := ne^{\frac{1-2nx}{2}} \sum_{k=0}^{\infty} \frac{(2nx-1)^k}{2^k k!} \int_{I_{n,k}} f(t)dt. \tag{2.12}$$

Then, we observe that

$$\xi(x) = \lim_{n\to\infty} n\left(x - u_n^{[2]}(x)\right) = \frac{1}{2}.$$

Therefore, we get the next result at once.

Corollary 4 ([8]). *For the operators* (2.12), *if* $f \in C_\gamma[0,+\infty)$, $\gamma \geq 4$, *such that* $f', f'' \in C_\gamma[0,+\infty)$, *then we have*

$$\lim_{n\to\infty} n\{L_n^{[2]}(f;x) - f(x)\} = \frac{1}{2} x f''(x)$$

uniformly with respect to $x \in [1/2, b]$, $b > 1/2$.

Finally, taking $a = \frac{1}{\sqrt{3}}$ and

$$u_n(x) := u_n^{[3]}(x) = \frac{\sqrt{3n^2x^2+2} - \sqrt{3}}{n\sqrt{3}}, \quad x \in \left[\frac{1}{\sqrt{3}}, \infty\right), \; n \in \mathbb{N}, \tag{2.13}$$

we get the following positive linear operators:

$$L_n^{[3]}(f;x) := ne^{\frac{\sqrt{3}-\sqrt{3n^2x^2+2}}{\sqrt{3}}} \sum_{k=0}^{\infty} \frac{\left(\sqrt{3n^2x^2+2} - \sqrt{3}\right)^k}{3^{k/2} k!} \int_{I_{n,k}} f(t)dt. \tag{2.14}$$

In this case, we find that

$$\xi(x) = \lim_{n\to\infty} n\left(x - u_n^{[3]}(x)\right) = 1.$$

Then, for the corresponding operators, we have the following

Corollary 5. *For the operators* (2.14), *if* $f \in C_\gamma[0,+\infty)$, $\gamma \geq 4$, *such that* $f', f'' \in C_\gamma[0,+\infty)$, *then we have*

$$\lim_{n\to\infty} n\{L_n^{[3]}(f;x) - f(x)\} = \frac{1}{2} x f''(x) - \frac{1}{2} f'(x)$$

uniformly with respect to $x \in \left[\frac{1}{\sqrt{3}}, b\right]$, $b > \frac{1}{\sqrt{3}}$.

3. Local approximation results for the operators L_n

In order to study various local approximation properties of the operators L_n we mainly use the (usual) modulus of continuity, the second modulus of smoothness, and Peetre's K-functional.

By $C_B^2[0,\infty)$ we denote the space of all functions $f \in C_B[0,\infty)$ such that $f', f'' \in C_B[0,\infty)$. Let $\|f\|$ denote the usual supremum norm of a bounded function f. Then, the classical Peetre's K-functional and the second modulus of smoothness of a function $f \in C_B[0,\infty)$ are defined respectively by

$$K(f,\delta) := \inf_{g \in C_B^2[0,\infty)} \left\{ \|f-g\| + \delta \|g''\| \right\}$$

and

$$\omega_2(f,\delta) := \sup_{0<h\le\delta,\ x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|,$$

where $\delta > 0$. Then, by Theorem 2.4 of [5, p. 177], there exists a constant $C > 0$ such that

$$K(f,\delta) \le C\omega_2\big(f, \sqrt{\delta}\big). \tag{3.1}$$

Also, as usual, by $\omega(f,\delta)$, $\delta > 0$, we denote the usual modulus of continuity of f.

Then, we first get the following local approximation result.

Theorem 3. *Let (u_n) be a sequence of functions on $[0,\infty)$ satisfying (1.2) for a fixed $a \ge 0$. For any $f \in C_B[0,\infty)$ and for every $x \in [a,\infty)$, $n \in \mathbb{N}$, we have*

$$|L_n(f;x) - f(x)| \le C\omega_2\left(f, \sqrt{\delta_n(x)}\right) + \omega\left(f, \left|u_n(x) - x + \frac{1}{2n}\right|\right)$$

for some constant $C > 0$, where

$$\delta_n(x) := (u_n(x) - x)^2 + \frac{2u_n(x) - x}{n} + \frac{1}{3n^2}. \tag{3.2}$$

Proof. Define an operator $\Omega_n : C_B[0,\infty) \to C_B[0,\infty)$ by

$$\Omega_n(f;x) := L_n(f;x) - f\left(u_n(x) + \frac{1}{2n}\right) + f(x). \tag{3.3}$$

So, by Lemma 2(ii), we get

$$\Omega_n(\psi_x;x) = L_n(\psi_x;x) - u_n(x) - \frac{1}{2n} + x = 0. \tag{3.4}$$

Let $g \in C_B^2[0,\infty)$, the space of all functions having the second continuous derivative on $[0,\infty)$, and let $x \in [0,\infty)$. Then, it follows from the well-known Taylor formula that

$$g(y) - g(x) = \psi_x(y) g'(x) + \int_x^y \psi_t(y) g''(t)\,dt, \quad y \in [0,\infty).$$

By (3.4), we get

$$\begin{aligned}|\Omega_n(g;x)-g(x)| &= |\Omega_n(g(y)-g(x);x)| \\ &= \left|\Omega_n\left(\int_x^y \psi_t(y)g''(t)dt;x\right)\right| \\ &= \left|L_n\left(\int_x^y \psi_t(y)g''(t)dt;x\right)\right. \\ &\quad \left. -\int_x^{u_n(x)+\frac{1}{2n}} \psi_t\left(u_n(x)+\frac{1}{2n}\right)g''(t)dt\right|.\end{aligned}$$

Using (3.3) and Lemma 2(ii), we obtain that

$$\begin{aligned}|\Omega_n(g;x)-g(x)| &\le \frac{\|g''\|}{2}L_n\left(\psi_x^2;x\right)+\frac{\|g''\|}{2}\psi_x^2\left(u_n(x)+\frac{1}{2n}\right) \\ &= \frac{\|g''\|}{2}\left\{\left((u_n(x)-x)^2+\frac{2u_n(x)-x}{n}+\frac{1}{3n^2}\right)\right. \\ &\quad \left.+\left(u_n(x)-x+\frac{1}{2n}\right)^2\right\},\end{aligned}$$

which implies that

$$|\Omega_n(g;x)-g(x)| \le \|g''\|\delta_n(x), \tag{3.5}$$

where $\delta_n(x)$ is given by (3.2). Then, for any $f \in C_B[0,\infty)$, it follows from (3.5) that

$$\begin{aligned}|L_n(f;x)-f(x)| &\le |\Omega_n(f-g;x)-(f-g)(x)| \\ &\quad +|\Omega_n(g;x)-g(x)|+\left|f\left(u_n(x)+\frac{1}{2n}\right)-f(x)\right| \\ &\le 2\|f-g\|+\delta_n(x)\|g''\|+\left|f\left(u_n(x)+\frac{1}{2n}\right)-f(x)\right| \\ &\le 2\{\|f-g\|+\delta_n(x)\|g''\|\}+\left|f\left(u_n(x)+\frac{1}{2n}\right)-f(x)\right|.\end{aligned}$$

Hence, by (3.1), we deduce that

$$\begin{aligned}|L_n(f;x)-f(x)| &\le 2\{\|f-g\|+\delta_n(x)\|g''\|\}+\omega\left(f,\left|u_n(x)-x+\frac{1}{2n}\right|\right) \\ &\le 2K\left(f,\delta_n(x)\right)+\omega\left(f,\left|u_n(x)-x+\frac{1}{2n}\right|\right) \\ &\le C\omega_2\left(f,\sqrt{\delta_n(x)}\right)+\omega\left(f,\left|u_n(x)-x+\frac{1}{2n}\right|\right)\end{aligned}$$

which completes the proof. □

Using Theorem 3, one can get the following special cases.

Corollary 6. *For the classical SMK operators* (1.1)*, we have, for any* $x \geq 0$, $n \in \mathbb{N}$ *and* $f \in C_B[0,\infty)$,

$$\left|K_n(f;x) - f(x)\right| \leq C\omega_2\left(f, \sqrt{\frac{x}{n} + \frac{1}{3n^2}}\right) + \omega\left(f, \frac{1}{2n}\right).$$

Corollary 7. *For the operators* (2.11)*, we have, for any* $f \in C_B[0,\infty)$, $x \geq 0$ *and* $n \in \mathbb{N}$,

$$\left|L_n^{[1]}(f;x) - f(x)\right| \leq C\omega_2\left(f, \sqrt{\delta_n^{[1]}(x)}\right) + \omega\left(f, \frac{\sqrt{4n^2x^2+1} - 2nx}{2n}\right),$$

where

$$\delta_n^{[1]}(x) := 2x^2 - \frac{1}{6n^2} + \frac{(1-2nx)\sqrt{4n^2x^2+1}}{2n^2}.$$

Corollary 8. *For the operators* (2.12)*, we have, for any* $f \in C_B[0,\infty)$, $x \geq \frac{1}{2}$ *and* $n \in \mathbb{N}$,

$$\left|L_n^{[2]}(f;x) - f(x)\right| \leq C\omega_2\left(f, \sqrt{\delta_n^{[2]}(x)}\right),$$

where

$$\delta_n^{[2]}(x) := \frac{x}{n} - \frac{5}{12n^2}.$$

Corollary 9. *For the operators* (2.14) *we have, for any* $f \in C_B[0,\infty)$, $x \geq \frac{1}{\sqrt{3}}$ *and* $n \in \mathbb{N}$,

$$\begin{aligned}\left|L_n^{[3]}(f;x) - f(x)\right| &\leq C\omega_2\left(f, \sqrt{\delta_n^{[3]}(x)}\right)\\ &\quad + \omega\left(f, \frac{2\sqrt{3}\sqrt{3n^2x^2+2} - 6(nx+1) + 3}{6n}\right),\end{aligned}$$

where

$$\delta_n^{[3]}(x) := 2x^2 + \frac{x\left(3 - 2\sqrt{3}\sqrt{3n^2x^2+2}\right)}{3n}.$$

4. Estimates for Lipschitz-type functions

In this section, for a fixed $a \geq 0$, we consider the following Lipschitz-type space

$$\mathrm{Lip}_M^*(r) := \left\{ f \in C_B[0,\infty) : |f(t) - f(x)| \leq M\frac{|t-x|^r}{(t+x)^{r/2}};\ x,t \in (a,\infty) \right\},$$

where M is any positive constant and $0 < r \leq 1$.

In order to give an estimation in approximating the functions in $\mathrm{Lip}_M^*(r)$ we need the next lemma.

Lemma 4. *Let (u_n) be a sequence of functions on $[0,\infty)$ satisfying (1.2) for a fixed $a \geq 0$. For every $x > a$ and $n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x|\,dt \leq \sqrt{\frac{\delta_n(x)}{n}}, \tag{4.1}$$

where $p_{n,k}(x)$ and $\delta_n(x)$ are given by (1.4) and (3.2), respectively.

Proof. Applying the Cauchy–Schwarz inequality to the series in the left hand side of (4.1), we get

$$\sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x|\,dt \leq \left\{ \sum_{k=0}^{\infty} p_{n,k}(x) \left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x|\,dt \right)^2 \right\}^{1/2}.$$

If we again apply the Cauchy–Schwarz inequality to the integral in the right-hand side of the last inequality and also use Lemma 2(ii), then we see that

$$\begin{aligned}
\sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x|\,dt &\leq \frac{1}{\sqrt{n}} \left\{ \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} (t-x)^2\,dt \right\}^{1/2} \\
&= \frac{1}{\sqrt{n}} \sqrt{L_n\left(\psi_x^2; x\right)} \\
&= \sqrt{\frac{\delta_n(x)}{n}},
\end{aligned}$$

whence the result follows. □

Now we are in position to give our result.

Theorem 4. *Let (u_n) be a sequence of functions on $[0,\infty)$ satisfying (1.2) for a fixed $a \geq 0$. Then, for any $f \in \mathrm{Lip}_M^*(r)$, $r \in (0,1]$, and for every $n \in \mathbb{N}$ and $x \in (a,\infty)$, we have*

$$|L_n(f;x) - f(x)| \leq \frac{M\delta_n^{r/2}(x)}{n^{1-r+r/2}x^{r/2}}, \tag{4.2}$$

where $\delta_n(x)$ is given by (3.2).

Proof. We first assume that $r = 1$. So, let $f \in \mathrm{Lip}_M^*(1)$ and $x \in (a,\infty)$. Then, we get

$$\begin{aligned}
|L_n(f;x) - f(x)| &\leq \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t) - f(x)|\,dt \\
&\leq M \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \frac{|t-x|}{\sqrt{t+x}}\,dt.
\end{aligned}$$

Since $\frac{1}{\sqrt{t+x}} \leq \frac{1}{\sqrt{x}}$, we may write that

$$|L_n(f;x) - f(x)| \leq \frac{M}{\sqrt{x}} \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x|\, dt.$$

Now, by Lemma 4, we conclude that

$$|L_n(f;x) - f(x)| \leq M \sqrt{\frac{\delta_n(x)}{nx}}, \tag{4.3}$$

which gives the desired result for $r = 1$. Assume now that $r \in (0,1)$. Then, taking $p = \frac{1}{r}$ and $q = \frac{1}{1-r}$, for any $f \in \mathrm{Lip}_M^a(r)$, if we apply the Hölder inequality two times, then we obtain that

$$\begin{aligned} |L_n(f;x) - f(x)| &\leq \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t) - f(x)|\, dt \\ &\leq \left\{ \sum_{k=0}^{\infty} p_{n,k}(x) \left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t) - f(x)|\, dt \right)^{1/r} \right\}^r \\ &\leq \frac{1}{n^{1-r}} \left\{ \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t) - f(x)|^{1/r}\, dt \right\}^r. \end{aligned}$$

Using the definition of the space $\mathrm{Lip}_M^*(r)$ and also considering Lemma 4, we get

$$\begin{aligned} |L_n(f;x) - f(x)| &\leq \frac{M}{n^{1-r}} \left\{ \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \frac{|t-x|}{\sqrt{t+x}} dt \right\}^r \\ &\leq \frac{M}{n^{1-r} x^{r/2}} \left\{ \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x|\, dt \right\}^r \\ &\leq \frac{M \delta_n^{r/2}(x)}{n^{1-r+r/2} x^{r/2}}. \end{aligned}$$

Thus, the proof is complete. □

Finally, it should be noted that our Theorem 4 includes many special cases as in the previous sections. However, we omit the details.

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