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RELATIONS BETWEEN THE HYPERGEOMETRIC FUNCTION OF APPELL F_3 AND KAMPÉ DE FÉRIET FUNCTIONS

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Abstract. Carlson [Some extensions of Lardner's relations between ${}_0F_3$ and Bessel functions, *SIAM J. Math. Anal.* **1**(2) (1970), 232–242] presented certain connections between Bessel and generalized hypergeometric functions, which generalizes some earlier results. Here, by simply splitting the hypergeometric Appell series F_3 into four parts, we show how some useful and generalized relations between F_3 and Kampé de Fériet function $F_{2;1;1}^{0;4;4}$ can be obtained. These main results are shown to be specialized to yield certain relations between functions ${}_0F_1$, ${}_0F_3$, $F_{2;1;1}^{0;4;4}$, and the Humbert function Ψ_2 some of which are still reduced to produce the Carlson's relations and some other interesting relations between the exponential function, the hyperbolic functions, and modified Bessel functions. Furthermore, decomposition formulas and integral representations of Euler type with hypergeometric function in the kernel for the function $F_{2;1;1}^{0;4;4}$ are derived by means of Burchnall-Chaundy operator method which has recently been revived.

2000 *Mathematics Subject Classification:* Primary 33C20; Secondary 44A45

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1. INTRODUCTION

Lardner [14] presented certain interesting relations between Bessel functions and the confluent hypergeometric series ${}_0F_3$, for example,

$$ {}_0F_3\left(\frac{1}{2},\frac{1}{2},1;z\right)=\frac{1}{2}\left[J_0\left(4z^{\frac{1}{4}}\right)+I_0\left(4z^{\frac{1}{4}}\right)\right] $$

and

$$ \mathrm{ber}(x)={}_0F_3\left(\frac{1}{2},\frac{1}{2},1;-\frac{x^4}{256}\right),\quad \text{and}\quad \mathrm{bei}(x)=\frac{x^2}{4}{}_0F_3\left(\frac{3}{2},\frac{3}{2},1;-\frac{x^4}{256}\right), $$

where J_ν and I_ν denote a Bessel function and a modified Bessel function of order ν (see [1]; also [19]) defined by

$$ J_\nu(z)=\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}{}_0F_1\left(-;\nu+1;-\frac{z^2}{4}\right), $$

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$$I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(-;\nu+1;\frac{z^2}{4}\right),$$

and $\mathrm{ber}(x)$ and $\mathrm{bei}(x)$ (x real) denote Kelvin's functions (see [7, p. 6]) defined by

$$\mathrm{ber}(x) + i\,\mathrm{bei}(x) = J_0\left(x\,e^{i\frac{3}{4}\pi}\right) = I_0\left(x\,e^{i\frac{1}{4}\pi}\right).$$

Carlson [5] generalized these results for arbitrary parameters to yield the following results

$${}_0F_3\left(\frac{1}{2},c,c+\frac{1}{2};z\right) = \frac{1}{2}\,\Gamma(2c)\left(2z^{\frac{1}{4}}\right)^{1-2c}\left[I_{2c-1}\left(4z^{\frac{1}{4}}\right) + J_{2c-1}\left(4z^{\frac{1}{4}}\right)\right] \tag{1.1}$$

and

$${}_0F_3\left(\frac{3}{2},c,c+\frac{1}{2};z\right) = \frac{1}{2}\,\Gamma(2c)\left(2z^{\frac{1}{4}}\right)^{-2c}\left[I_{2c-2}\left(4z^{\frac{1}{4}}\right) - J_{2c-2}\left(4z^{\frac{1}{4}}\right)\right]. \tag{1.2}$$

Here, by simply splitting the hypergeometric Appell series F_3 into four parts, we can investigate relations between F_3 and the Kampé de Fériet function $F^{0;4;4}_{2;1;1}$. Our main results can be specialized to yield certain relations between functions ${}_0F_1$, ${}_0F_3$, $F^{0;4;4}_{2;1;1}$, and Ψ_2. Some of these can be reduced to Carlson's relations (1.1) and (1.2) (see Corollary 1), while some of them yield interesting relations between the exponential function, the hyperbolic functions, and the modified Bessel functions (see Corollary 2). Furthermore, decomposition formulas and integral representations of Euler type with hypergeometric function in the kernel for the function $F^{0;4;4}_{2;1;1}$ are derived by means of Burchnall-Chaundy operator method (see [3, 4, 6]), which has recently been revived (see, for example, [11–13]).

It may be remarked before passing to the next section, that certain special functions have still played important roles in theories and applications (see, for example, [9–13]). In fact, multiple hypergeometric functions arise mostly during the solution of certain differential equations. Yet, there are other important cases in which these special functions are involved. For example, many auxiliary algebraic and integral transformations in various physical models have been found to be connected with these functions. Especially, many problems in gas dynamics lead to those second-order partial differential equations which can be solvable in terms of multiple hypergeometric functions. In the investigation of the boundary value problems for these partial differential equations, it might be necessary to decompose the hypergeometric functions of several variables in terms of simpler hypergeometric functions of the Gauss, Appell, and Kampé de Fériet types, and so on (see, for example, [15, 16, 18]). More specially, the hypergeometric function F_2 determines the fundamental solutions of the generalized bi-axially symmetric Helmholtz equation [9] and of certain three-dimensional elliptic equations with singular coefficients [10].

2. Relationships between the Appell function F_3 and Kampé de Fériet function $F_{2;1;1}^{0;4;4}$

We use the notations as in [2, 8]. The hypergeometric function of Appell F_3 is defined by

$$F_3(a_1,a_2,b_1,b_2;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a_1)_m(a_2)_n(b_1)_m(b_2)_n}{(c)_{m+n}\,m!n!}x^m y^n,\ |x|<1,\ |y|<1, \tag{2.1}$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}\quad(\lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-) \tag{2.2}$$

in terms of the Gamma function Γ, $\mathbb{C}$ and $\mathbb{Z}_0^-$ are the sets of complex numbers and nonpostive integers, $n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, and $\mathbb{N}$ is the set of positive integers. Since the series F_3 converges absolutely for any $c\in\mathbb{C}\setminus\mathbb{Z}_0^-$ and any finite $x,\,y\in\mathbb{C}$, it can be rearranged into the following four series:

$$\begin{aligned}
&F_3(a_1,a_2,b_1,b_2;c;x,y)\\
&=\sum_{p,q=0}^{\infty}\frac{(a_1)_{2p}(a_2)_{2q}(b_1)_{2p}(b_2)_{2q}}{(c)_{2p+2q}(2p)!(2q)!}x^{2p}y^{2q}\\
&+xy\sum_{p,q=0}^{\infty}\frac{(a_1)_{2p+1}(a_2)_{2q+1}(b_1)_{2p+1}(b_2)_{2q+1}}{(c)_{2p+1+2q+1}(2p+1)!(2q+1)!}x^{2p}y^{2q}\\
&+y\sum_{p,q=0}^{\infty}\frac{(a_1)_{2p}(a_2)_{2q+1}(b_1)_{2p}(b_2)_{2q+1}}{(c)_{2p+2q+1}(2p)!(2q+1)!}x^{2p}y^{2q}\\
&+x\sum_{p,q=0}^{\infty}\frac{(a_1)_{2p+1}(a_2)_{2q}(b_1)_{2p+1}(b_2)_{2q}}{(c)_{2p+1+2q}(2p+1)!(2q)!}x^{2p}y^{2q}.
\end{aligned} \tag{2.3}$$

Applying some known (or easily-derivable) identities for Pochhammer symbols (see [8]) such as

$$(\alpha)_{2m+2n+2}=\alpha(\alpha+1)\,4^{m+n}\left(\frac{\alpha+2}{2}\right)_{m+n}\left(\frac{\alpha+3}{2}\right)_{m+n}\quad(m,n\in\mathbb{N}_0)$$

to (2.3) and simplifying each of the four resulting series, we can express the Appell function F_3 in terms of Kampé de Fériet function $F_{2;1;1}^{0;4;4}$ (see [17, p. 27]) as in the following theorem.

Theorem 1. *The following relationship between the Appell function F_3 and Kampé de Fériet function $F_{2;1;1}^{0;4;4}$ holds true.*

$$
\begin{aligned}
F_3(a_1,a_2,b_1,b_2;c;x,y) &= F_{2;1;1}^{0;4;4}\left[\begin{matrix} -: & \frac{a_1}{2}, & \frac{a_1+1}{2}, & \frac{b_1}{2}, & \frac{b_1+1}{2}; \\ \frac{c}{2}, & \frac{c+1}{2}: & & & \frac{1}{2}; \end{matrix}\right. \\
&\qquad \left.\begin{matrix} \frac{a_2}{2}, & \frac{a_2+1}{2}, & \frac{b_2}{2}, & \frac{b_2+1}{2}; \\ & & & \frac{1}{2}; \end{matrix}\ x^2, y^2\right] \\
&\quad + \frac{a_1a_2b_1b_2xy}{c(c+1)}\, F_{2;1;1}^{0;4;4}\left[\begin{matrix} -: & \frac{a_1+1}{2}, & \frac{a_1+2}{2}, & \frac{b_1+1}{2}, & \frac{b_1+2}{2}; \\ \frac{c+2}{2}, & \frac{c+3}{2}: & & & \frac{3}{2}; \end{matrix}\right. \\
&\qquad \left.\begin{matrix} \frac{a_2+1}{2}, & \frac{a_2+2}{2}, & \frac{b_2+1}{2}, & \frac{b_2+2}{2}; \\ & & & \frac{3}{2}; \end{matrix}\ x^2, y^2\right] \\
&\quad + \frac{a_2b_2y}{c}\, F_{2;1;1}^{0;4;4}\left[\begin{matrix} -: & \frac{a_1}{2}, & \frac{a_1+1}{2}, & \frac{b_1}{2}, & \frac{b_1+1}{2}; \\ \frac{c+1}{2}, & \frac{c+2}{2}: & & & \frac{1}{2}; \end{matrix}\right. \\
&\qquad \left.\begin{matrix} \frac{a_2+1}{2}, & \frac{a_2+2}{2}, & \frac{b_2+1}{2}, & \frac{b_2+2}{2}; \\ & & & \frac{3}{2}; \end{matrix}\ x^2, y^2\right] \\
&\quad + \frac{a_1b_1x}{c}\, F_{2;1;1}^{0;4;4}\left[\begin{matrix} -: & \frac{a_1+1}{2}, & \frac{a_1+2}{2}, & \frac{b_1+1}{2}, & \frac{b_1+2}{2}; \\ \frac{c+1}{2}, & \frac{c+2}{2}: & & & \frac{3}{2}; \end{matrix}\right. \\
&\qquad \left.\begin{matrix} \frac{a_2}{2}, & \frac{a_2+1}{2}, & \frac{b_2}{2}, & \frac{b_2+1}{2}; \\ & & & \frac{1}{2}; \end{matrix}\ x^2, y^2\right],
\end{aligned}
\tag{2.4}
$$

where the Kampé de Fériet function $F_{2;1;1}^{0;4;4}$ is defined by

$$
\begin{aligned}
&F_{2;1;1}^{0;4;4}\left[\begin{matrix} -: & b_1,b_2,b_3,b_4; & c_1,c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & \beta; & \gamma; \end{matrix}\ x, y\right] \qquad (2.5)\\
&= \sum_{m,n=0}^{\infty} \frac{(b_1)_m (b_2)_m (b_3)_m (b_4)_m (c_1)_n (c_2)_n (c_3)_n (c_4)_n}{(\alpha_1)_{m+n} (\alpha_2)_{m+n} (\beta)_m (\gamma)_n m!n!} x^m y^n \ (|x|<1; |y|<1).
\end{aligned}
$$

In view of the relation in (2.4), the four Kampé de Fériet functions in Theorem 1 can be expressed in terms of the Appell functions F_3 as in the following theorem.

Theorem 2. *Each of the following relationship between Kampé de Fériet function $F_{2;1;1}^{0;4;4}$ and the Appell function F_3 holds true.*

$$
\begin{aligned}
&4F_{2;1;1}^{0;4;4}\left[\begin{matrix} -: & \frac{a_1}{2}, \frac{a_1+1}{2}, \frac{b_1}{2}, \frac{b_1+1}{2}; & \frac{a_2}{2}, \frac{a_2+1}{2}, \frac{b_2}{2}, \frac{b_2+1}{2}; \\ \frac{c}{2}, \frac{c+1}{2}: & \frac{1}{2}; & \frac{1}{2}; \end{matrix}\ x^2, y^2\right] \\
&\quad = F_3(a_1,a_2,b_1,b_2;c;x,y) + F_3(a_1,a_2,b_1,b_2;c;-x,y) \\
&\qquad + F_3(a_1,a_2,b_1,b_2;c;x,-y) + F_3(a_1,a_2,b_1,b_2;c;-x,-y);
\end{aligned}
\tag{2.6}
$$

$$4\frac{a_1a_2b_1b_2xy}{c(c+1)}F_{2;1;1}^{0;4;4}\left[\begin{matrix} -: & \frac{a_1+1}{2}, & \frac{a_1+2}{2}, & \frac{b_1+1}{2}, & \frac{b_1+2}{2}; \\ \frac{c+2}{2}, & \frac{c+3}{2}: & & & \frac{3}{2}; \end{matrix}\right.$$
$$\left.\begin{matrix} \frac{a_2+1}{2}, & \frac{a_2+2}{2}, & \frac{b_2+1}{2}, & \frac{b_2+2}{2}; \\ & & & \frac{3}{2}; \end{matrix}\ x^2,\ y^2\right] \tag{2.7}$$
$$= F_3(a_1,a_2,b_1,b_2;c;x,y)+F_3(a_1,a_2,b_1,b_2;c;-x,y)$$
$$-F_3(a_1,a_2,b_1,b_2;c;x,-y)+F_3(a_1,a_2,b_1,b_2;c;-x,-y);$$

$$4\frac{a_2b_2y}{c}F_{2;1;1}^{0;4;4}\left[\begin{matrix} -: & \frac{a_1}{2},\frac{a_1+1}{2},\frac{b_1}{2},\frac{b_1+1}{2}; & \frac{a_2+1}{2},\frac{a_2+2}{2},\frac{b_2+1}{2},\frac{b_2+2}{2}; \\ \frac{c+1}{2},\frac{c+2}{2}: & \frac{1}{2}; & \frac{3}{2}; \end{matrix}\ x^2,\ y^2\right]$$
$$= F_3(a_1,a_2,b_1,b_2;c;x,y)+F_3(a_1,a_2,b_1,b_2;c;-x,y)$$
$$-F_3(a_1,a_2,b_1,b_2;c;x,-y)-F_3(a_1,a_2,b_1,b_2;c;-x,-y); \tag{2.8}$$

$$4\frac{a_1b_1x}{c}F_{2;1;1}^{0;4;4}\left[\begin{matrix} -: & \frac{a_1+1}{2}, & \frac{a_1+2}{2}, & \frac{b_1+1}{2}, & \frac{b_1+2}{2}; \\ \frac{c+1}{2}, & \frac{c+2}{2}: & & & \frac{3}{2}; \end{matrix}\right.$$
$$\left.\begin{matrix} \frac{a_2}{2}, & \frac{a_2+1}{2}, & \frac{b_2}{2}, & \frac{b_2+1}{2}; \\ & & & \frac{1}{2}; \end{matrix}\ x^2,\ y^2\right] \tag{2.9}$$
$$= F_3(a_1,a_2,b_1,b_2;c;x,y)-F_3(a_1,a_2,b_1,b_2;c;-x,y)$$
$$+F_3(a_1,a_2,b_1,b_2;c;x,-y)-F_3(a_1,a_2,b_1,b_2;c;-x,-y).$$

3. Integral representations of Euler type for $F_{2;1;1}^{0;4;4}$

The Kampé de Fériet hypergeometric function $F_{2;1;1}^{0;4;4}$ has the following Euler type integral representation:

$$F_{2:1;1}^{0:4;4}\left[\begin{matrix} -: & b_1,b_2,b_3,b_4; & c_1,c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & \beta; & \gamma; \end{matrix}\ x,y\right]$$
$$= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(b_1)\Gamma(c_2)\Gamma(\alpha_1-b_1)\Gamma(\alpha_2-c_2)}$$
$$\cdot\int_0^1\int_0^1 \xi^{b_1-1}\eta^{c_2-1}(1-\xi)^{\alpha_1-b_1-1}(1-\eta)^{\alpha_2-c_2-1}$$
$$\cdot{}_3F_2\left(\begin{matrix} b_2,b_3,b_4 \\ \alpha_2-c_2,\beta \end{matrix}\ x\xi(1-\eta)\right){}_3F_2\left(\begin{matrix} c_1,c_3,c_4 \\ \alpha_1-b_1,\gamma \end{matrix}\ y(1-\xi)\eta\right)d\xi d\eta$$
$$(\Re(\alpha_1)>\Re(b_1)>0,\ \Re(\alpha_2)>\Re(c_2)>0);$$

$$F_{2:1;1}^{0:4;4}\left[\begin{matrix} -: & b_1,b_2,b_3,b_4; & c_1,c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & \beta; & \gamma; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(b_1)\Gamma(c_1)\Gamma(\beta-b_1)\Gamma(\gamma-c_1)}$$
$$\cdot\int_0^1\int_0^1 \xi^{b_1-1}\eta^{c_1-1}(1-\xi)^{\beta-b_1-1}(1-\eta)^{\gamma-c_1-1}$$
$$\cdot F_{2:0;0}^{0:3;3}\left[\begin{matrix} -: & b_2,b_3,b_4; & c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & -; & -; \end{matrix} x\xi,y\eta\right]d\xi d\eta$$
$$(\Re(\beta)>\Re(b_1)>0,\ \Re(\gamma)>\Re(c_1)>0);$$

$$F_{2:1;1}^{0:4;4}\left[\begin{matrix} -: & b_1,b_2,b_3,b_4; & \alpha_1-b_1,c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & \beta; & \gamma; \end{matrix} x,y\right]=\frac{\Gamma(\alpha_1)}{\Gamma(b_1)\Gamma(\alpha_1-b_1)}$$
$$\cdot\int_0^1 \xi^{b_1-1}(1-\xi)^{\alpha_1-b_1-1}F_{1:1;1}^{0:3;3}\left[\begin{matrix} -: & b_2,b_3,b_4; & c_2,c_3,c_4; \\ \alpha_2: & \beta; & \gamma; \end{matrix} x\xi,y(1-\xi)\right]d\xi$$
$$(\Re(\alpha_1)>\ \Re(b_1)>0);$$

$$F_{2:1;1}^{0:4;4}\left[\begin{matrix} -: & b_1,b_2,b_3,b_4; & c_1,c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & \beta; & \gamma; \end{matrix} x,y\right]=\frac{\Gamma(b_1+c_1)}{\Gamma(b_1)\Gamma(c_1)}$$
$$\cdot\int_0^1 \xi^{b_1-1}(1-\xi)^{c_1-1}F_{2:1;1}^{1:3;3}\left[\begin{matrix} b_1+c_1: & b_2,b_3,b_4; & c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & \beta; & \gamma; \end{matrix} x\xi,y(1-\xi)\right]d\xi$$
$$(\Re(b_1)>0,\ \Re(c_1)>0);$$

$$F_{2:1;1}^{0:4;4}\left[\begin{matrix} -: & b_1,b_2,b_3,b_4; & c_1,c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & \beta; & \gamma; \end{matrix} x,y\right]=\frac{\Gamma(b_1+c_1)\Gamma(b_2+c_2)}{\Gamma(b_1)\Gamma(b_2)\Gamma(c_1)\Gamma(c_2)}$$
$$\cdot\int_0^1\int_0^1 \xi^{b_1-1}\eta^{b_2-1}(1-\xi)^{c_1-1}(1-\eta)^{c_2-1}$$
$$F_{2:1;1}^{2:2;2}\left[\begin{matrix} b_1+c_1,b_2+c_2: & b_3,b_4; & c_3,c_4; \\ \alpha_1,\alpha_2: & \beta; & \gamma; \end{matrix} x\xi\eta,y(1-\xi)(1-\eta)\right]d\xi d\eta$$
$$(\Re(b_i)>0,\ \Re(c_i)>0)\ (i=1,2);$$

$$F_{2:1;1}^{0:4;4}\left[\begin{matrix} -: & b_1,b_2,b_3,b_4; & c_1,c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & \beta; & \gamma; \end{matrix} x,y\right]$$
$$= \frac{\Gamma(b_1+c_1)\,\Gamma(b_2+c_2)\,\Gamma(b_3+c_3)}{\Gamma(b_1)\,\Gamma(b_2)\,\Gamma(b_3)\,\Gamma(c_1)\,\Gamma(c_2)\,\Gamma(c_3)}$$
$$\cdot\int_0^1\int_0^1\int_0^1 \xi^{b_1-1}\eta^{b_2-1}\zeta^{b_3-1}(1-\xi)^{c_1-1}(1-\eta)^{c_2-1}(1-\zeta)^{c_3-1}$$
$$\cdot F_{2:1;1}^{3:1;1}\left[\begin{matrix} b_1+c_1,b_2+c_2,b_3+c_3: & b_4; & c_4; \\ \alpha_1,\alpha_2: & \beta; & \gamma; \end{matrix} x\xi\eta\zeta, y(1-\xi)(1-\eta)(1-\zeta)\right]$$
$$d\xi\,d\eta\,d\zeta \qquad (\Re(b_i)>0,\ \Re(c_i)>0)\ (i=1,2,3);$$

$$F_{2:1;1}^{0:4;4}\left[\begin{matrix} -: & b_1,b_2,b_3,b_4; & c_1,c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & \beta; & \gamma; \end{matrix} x,y\right] = \prod_{i=1}^{4}\frac{\Gamma(b_i+c_i)}{\Gamma(b_i)\,\Gamma(c_i)}$$
$$\cdot\int_0^1\cdots\int_0^1 \xi^{b_1-1}\eta^{b_2-1}\zeta^{b_3-1}\omega^{b_4-1}(1-\xi)^{c_1-1}(1-\eta)^{c_2-1}(1-\zeta)^{c_3-1}(1-\omega)^{c_4-1}$$
$$\cdot F_{2:1;1}^{3:1;1}\left[\begin{matrix} b_1+c_1, & b_2+c_2, & b_3+c_3, & b_4+c_4: & -; & -; \\ & & \alpha_1, & \alpha_2: & \beta; & \gamma; \end{matrix}\right.$$
$$\left. x\xi\eta\zeta\omega, y(1-\xi)(1-\eta)(1-\zeta)(1-\omega)\right] d\xi\,d\eta\,d\zeta\,d\omega$$
$$(\Re(b_i)>0,\ \Re(c_i)>0)\ (i=1,2,3).$$

Proof. Note that each of the integral representations in Section 3 can be proved by expressing the series definition of the involved special function in each integrand and changing the order of the integration and the summation, and finally using the following well-known relationship between the Beta function $B(\alpha,\beta)$ and the Gamma function Γ:

$$B(\alpha,\beta) := \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\Re(\alpha)>0;\ \Re(\beta)>0) \\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-). \end{cases}$$

□

4. Relations deduced from the main results

In this section we deduce certain interesting relations from the main results in Section 2. Transforming b_2 by $1/\varepsilon$ and y by $\varepsilon\,y$ in the formulas (2.4), (2.6), (2.7),

(2.8) and (2.9), and taking the limit of each of the obtained identities as $\varepsilon \to 0$, we get some identities as in the following corollary (for a review of this method, see [2]).

Corollary 1. *Each of the following identities holds true.*

$$\begin{aligned}\Xi_1(a_1,a_2,b_1;c;x,y) &= F_{2:1;1}^{0:4;2}\left[\begin{matrix}-:\\ \frac{c}{2},\frac{c+1}{2}:\end{matrix}\begin{matrix}\frac{a_1}{2},\frac{a_1+1}{2},\frac{b_1}{2},\frac{b_1+1}{2};\\ \frac{1}{2};\end{matrix}\begin{matrix}\frac{a_2}{2},\frac{a_2+1}{2};\\ \frac{1}{2};\end{matrix}x^2,\frac{y^2}{4}\right]\\ &+\frac{a_1a_2b_1xy}{c(c+1)}F_{2:1;1}^{0:4;2}\left[\begin{matrix}-:\\ \frac{c+2}{2},\frac{c+3}{2}:\end{matrix}\begin{matrix}\frac{a_1+1}{2},\frac{a_1+2}{2},\frac{b_1+1}{2},\frac{b_1+2}{2};\\ \frac{3}{2};\end{matrix}\begin{matrix}\frac{a_2+1}{2},\frac{a_2+2}{2};\\ \frac{3}{2};\end{matrix}x^2,\frac{y^2}{4}\right]\\ &+\frac{a_2y}{c}F_{2:1;1}^{0:4;2}\left[\begin{matrix}-:\\ \frac{c+1}{2},\frac{c+2}{2}:\end{matrix}\begin{matrix}\frac{a_1}{2},\frac{a_1+1}{2},\frac{b_1}{2},\frac{b_1+1}{2};\\ \frac{1}{2};\end{matrix}\begin{matrix}\frac{a_2+1}{2},\frac{a_2+2}{2};\\ \frac{3}{2};\end{matrix}x^2,\frac{y^2}{4}\right]\\ &+\frac{a_1b_1x}{c}F_{2:1;1}^{0:4;2}\left[\begin{matrix}-:\\ \frac{c+1}{2},\frac{c+2}{2}:\end{matrix}\begin{matrix}\frac{a_1+1}{2},\frac{a_1+2}{2},\frac{b_1+1}{2},\frac{b_1+2}{2};\\ \frac{3}{2};\end{matrix}\begin{matrix}\frac{a_2}{2},\frac{a_2+1}{2};\\ \frac{1}{2};\end{matrix}x^2,\frac{y^2}{4}\right];\end{aligned} \tag{4.1}$$

$$\begin{aligned}4F_{2;1;1}^{0;4;2}\left[\begin{matrix}-:\\ \frac{c}{2},\frac{c+1}{2}:\end{matrix}\begin{matrix}\frac{a_1}{2},\frac{a_1+1}{2},\frac{b_1}{2},\frac{b_1+1}{2};\\ \frac{1}{2};\end{matrix}\begin{matrix}\frac{a_2}{2},\frac{a_2+1}{2};\\ \frac{1}{2};\end{matrix}x^2,\frac{y^2}{4}\right] &= \Xi_1(a_1,a_2,b_1;c;x,y)\\ +\Xi_1(a_1,a_2,b_1;c;-x,y)+\Xi_1(a_1,a_2,b_1;c;x,-y)&+\Xi_1(a_1,a_2,b_1;c;-x,-y);\end{aligned} \tag{4.2}$$

$$\begin{aligned}4\frac{a_1a_2b_1xy}{c(c+1)}F_{2;1;1}^{0;4;2}\left[\begin{matrix}-:\\ \frac{c+2}{2},\frac{c+3}{2}:\end{matrix}\begin{matrix}\frac{a_1+1}{2},\frac{a_1+2}{2},\frac{b_1+1}{2},\frac{b_1+2}{2};\\ \frac{3}{2};\end{matrix}\begin{matrix}\frac{a_2+1}{2},\frac{a_2+2}{2};\\ \frac{3}{2};\end{matrix}x^2,\frac{y^2}{4}\right]&\\ =\Xi_1(a_1,a_2,b_1;c;x,y)-\Xi_1(a_1,a_2,b_1;c;-x,y)&\\ -\Xi_1(a_1,a_2,b_1;c;x,-y)+\Xi_1(a_1,a_2,b_1;c;-x,-y);&\end{aligned} \tag{4.3}$$

$$\begin{aligned}4\frac{a_2y}{c}F_{2;1;1}^{0;4;2}\left[\begin{matrix}-:\\ \frac{c+1}{2},\frac{c+2}{2}:\end{matrix}\begin{matrix}\frac{a_1}{2},\frac{a_1+1}{2},\frac{b_1}{2},\frac{b_1+1}{2};\\ \frac{1}{2};\end{matrix}\begin{matrix}\frac{a_2+1}{2},\frac{a_2+2}{2};\\ \frac{3}{2};\end{matrix}x^2,\frac{y^2}{4}\right]&\\ =\Xi_1(a_1,a_2,b_1;c;x,y)+\Xi_1(a_1,a_2,b_1;c;-x,y)&\\ -\Xi_1(a_1,a_2,b_1;c;x,-y)-\Xi_1(a_1,a_2,b_1;c;-x,-y);&\end{aligned} \tag{4.4}$$

$$\begin{aligned}4\frac{a_1b_1x}{c}F_{2;1;1}^{0;4;2}\left[\begin{matrix}-:\\ \frac{c+1}{2},\frac{c+2}{2}:\end{matrix}\begin{matrix}\frac{a_1+1}{2},\frac{a_1+2}{2},\frac{b_1+1}{2},\frac{b_1+2}{2};\\ \frac{3}{2};\end{matrix}\begin{matrix}\frac{a_2}{2},\frac{a_2+1}{2};\\ \frac{1}{2};\end{matrix}x^2,y^2\right]&\\ =\Xi_1(a_1,a_2,b_1;c;x,y)-\Xi_1(a_1,a_2,b_1;c;-x,y)&\\ +\Xi_1(a_1,a_2,b_1;c;x,-y)-\Xi_1(a_1,a_2,b_1;c;-x,-y)&\end{aligned} \tag{4.5}$$

where Ξ_1 denotes the Humbert hypergeometric function (see [2, 8, 17]) defined by

$$\Xi_1(a_1,a_2,b_1;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a_1)_m(a_2)_n(b_1)_m}{(c)_{m+n}\,m!n!}x^m y^n\ (|x|<1;\ |y|<\infty). \tag{4.6}$$

Transforming a_2 by $1/\varepsilon$ and y by εy in the formulas (4.1), (4.2), (4.3), (4.4) and (4.5), and taking the limit of each of the obtained identities as $\varepsilon \to 0$, we deduce some identities as in the following corollary (again, see [2]).

Corollary 2. *Each of the following functional relations holds true.*

$$\begin{aligned}
\Xi_2(a_1,b_1;c;x,y) &= F^{0:4;0}_{2:1;1}\left[\begin{matrix} -: \\ \frac{c}{2},\frac{c+1}{2}: \end{matrix}\ \begin{matrix} \frac{a_1}{2},\frac{a_1+1}{2},\frac{b_1}{2},\frac{b_1+1}{2}; \\ \frac{1}{2}; \end{matrix}\ \begin{matrix} -; \\ \frac{1}{2}; \end{matrix}\ x^2,\frac{y^2}{16}\right] \\
&+\frac{a_1b_1xy}{c\,(c+1)}F^{0:4;0}_{2:1;1}\left[\begin{matrix} -: \\ \frac{c+2}{2},\frac{c+3}{2}: \end{matrix}\ \begin{matrix} \frac{a_1+1}{2},\frac{a_1+2}{2},\frac{b_1+1}{2},\frac{b_1+2}{2}; \\ \frac{3}{2}; \end{matrix}\ \begin{matrix} ; \\ \frac{3}{2}; \end{matrix}\ x^2,\frac{y^2}{16}\right] \\
&+\frac{y}{c}F^{0:4;0}_{2:1;1}\left[\begin{matrix} -: \\ \frac{c+1}{2},\frac{c+2}{2}: \end{matrix}\ \begin{matrix} \frac{a_1}{2},\frac{a_1+1}{2},\frac{b_1}{2},\frac{b_1+1}{2}; \\ \frac{1}{2}; \end{matrix}\ \begin{matrix} -; \\ \frac{3}{2}; \end{matrix}\ x^2,\frac{y^2}{16}\right] \\
&+\frac{a_1b_1x}{c}F^{0:4;0}_{2:1;1}\left[\begin{matrix} -: \\ \frac{c+1}{2},\frac{c+2}{2}: \end{matrix}\ \begin{matrix} \frac{a_1+1}{2},\frac{a_1+2}{2},\frac{b_1+1}{2},\frac{b_1+2}{2}; \\ \frac{3}{2}; \end{matrix}\ \begin{matrix} -; \\ \frac{1}{2}; \end{matrix}\ x^2,\frac{y^2}{16}\right];
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
&4F^{0;4;0}_{2;1;1}\left[\begin{matrix} -: \\ \frac{c}{2},\frac{c+1}{2}: \end{matrix}\ \begin{matrix} \frac{a_1}{2},\frac{a_1+1}{2},\frac{b_1}{2},\frac{b_1+1}{2}; \\ \frac{1}{2}; \end{matrix}\ \begin{matrix} -; \\ \frac{1}{2}; \end{matrix}\ x^2,\frac{y^2}{16}\right] \\
&\quad=\Xi_2(a_1,b_1;c;x,y)+\Xi_2(a_1,b_1;c;-x,y) \\
&\qquad+\Xi_2(a_1,b_1;c;x,-y)+\Xi_2(a_1,b_1;c;-x,-y);
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
&4\frac{a_1b_1xy}{c\,(c+1)}F^{0:4;0}_{2:1;1}\left[\begin{matrix} -: \\ \frac{c+2}{2},\frac{c+3}{2}: \end{matrix}\ \begin{matrix} \frac{a_1+1}{2},\frac{a_1+2}{2},\frac{b_1+1}{2},\frac{b_1+2}{2}; \\ \frac{3}{2}; \end{matrix}\ \begin{matrix} -; \\ \frac{3}{2}; \end{matrix}\ x^2,\frac{y^2}{16}\right] \\
&\quad=\Xi_2(a_1,b_1;c;x,y)-\Xi_2(a_1,b_1;c;-x,y) \\
&\qquad-\Xi_2(a_1,b_1;c;x,-y)+\Xi_2(a_1,b_1;c;-x,-y);
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&4\frac{y}{c}F^{0:4;0}_{2:1;1}\left[\begin{matrix} -: \\ \frac{c+1}{2},\frac{c+2}{2}: \end{matrix}\ \begin{matrix} \frac{a_1}{2},\frac{a_1+1}{2},\frac{b_1}{2},\frac{b_1+1}{2}; \\ \frac{1}{2}; \end{matrix}\ \begin{matrix} -; \\ \frac{3}{2}; \end{matrix}\ x^2,\frac{y^2}{16}\right] \\
&\quad=\Xi_2(a_1,b_1;c;x,y)+\Xi_2(a_1,b_1;c;-x,y) \\
&\qquad-\Xi_2(a_1,b_1;c;x,-y)-\Xi_2(a_1,b_1;c;-x,-y);
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
&4\frac{a_1b_1x}{c}F^{0:4;0}_{2:1;1}\left[\begin{matrix} -: \\ \frac{c+1}{2},\frac{c+2}{2}: \end{matrix}\ \begin{matrix} \frac{a_1+1}{2},\frac{a_1+2}{2},\frac{b_1+1}{2},\frac{b_1+2}{2}; \\ \frac{3}{2}; \end{matrix}\ \begin{matrix} -; \\ \frac{1}{2}; \end{matrix}\ x^2,\frac{y^2}{16}\right] \\
&\quad=\Xi_2(a_1,b_1;c;x,y)-\Xi_2(a_1,b_1;c;-x,y) \\
&\qquad+\Xi_2(a_1,b_1;c;x,-y)-\Xi_2(a_1,b_1;c;-x,-y),
\end{aligned} \tag{4.11}$$

where Ξ_2 denotes the Humbert hypergeometric function (see [2, 8, 17]) defined by

$$\Xi_2(a_1,b_1;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a_1)_m\,(b_1)_m}{(c)_{m+n}\,m!n!}x^m y^n\ (|x|<1;\ |y|<\infty). \tag{4.12}$$

Transforming b_1 by $1/\varepsilon$ and x by εx in the formulas (4.1), (4.2), (4.3), (4.4) and (4.5), and taking the limit of each of the resulting identities as $\varepsilon \to 0$, we obtain some identities as in the following corollary (see [2]).

Corollary 3. *We deduce the following identities.*

$$\begin{aligned}
\Phi_2(a_1,a_2;c;x,y) &= F_{2:1;1}^{0:2;2}\left[\begin{matrix} -: & \frac{a_1}{2},\frac{a_1+1}{2},-; & \frac{a_2}{2},\frac{a_2+1}{2}; \\ \frac{c}{2},\frac{c+1}{2}: & \frac{1}{2}; & \frac{1}{2}; \end{matrix}\ \frac{x^2}{4},\frac{y^2}{4}\right] \\
&+\frac{a_1a_2xy}{c(c+1)}F_{2:1;1}^{0:2;2}\left[\begin{matrix} -: & \frac{a_1+1}{2},\frac{a_1+2}{2},-; & \frac{a_2+1}{2},\frac{a_2+2}{2}; \\ \frac{c+2}{2},\frac{c+3}{2}: & \frac{3}{2}; & \frac{3}{2}; \end{matrix}\ \frac{x^2}{4},\frac{y^2}{4}\right] \\
&+\frac{a_2y}{c}F_{2:1;1}^{0:2;2}\left[\begin{matrix} -: & \frac{a_1}{2},\frac{a_1+1}{2},-; & \frac{a_2+1}{2},\frac{a_2+2}{2}; \\ \frac{c+1}{2},\frac{c+2}{2}: & \frac{1}{2}; & \frac{3}{2}; \end{matrix}\ \frac{x^2}{4},\frac{y^2}{4}\right] \\
&+\frac{a_1x}{c}F_{2:1;1}^{0:2;2}\left[\begin{matrix} -: & \frac{a_1+1}{2},\frac{a_1+2}{2},-; & \frac{a_2}{2},\frac{a_2+1}{2}; \\ \frac{c+1}{2},\frac{c+2}{2}: & \frac{3}{2}; & \frac{1}{2}; \end{matrix}\ \frac{x^2}{4},\frac{y^2}{4}\right];
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&4F_{2:1;1}^{0:2;2}\left[\begin{matrix} -: & \frac{a_1}{2},\frac{a_1+1}{2}; & \frac{a_2}{2},\frac{a_2+1}{2}; \\ \frac{c}{2},\frac{c+1}{2}: & \frac{1}{2}; & \frac{1}{2}; \end{matrix}\ \frac{x^2}{4},\frac{y^2}{4}\right] \\
&\quad=\Phi_2(a_1,a_2;c;x,y)+\Phi_2(a_1,a_2;c;-x,y) \\
&\qquad+\Phi_2(a_1,a_2;c;x,-y)+\Phi_2(a_1,a_2;c;-x,-y);
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&4\frac{a_1a_2xy}{c(c+1)}F_{2:1;1}^{0:2;2}\left[\begin{matrix} -; & \frac{a_1+1}{2},\frac{a_1+2}{2}; & \frac{a_2+1}{2},\frac{a_2+2}{2},-; \\ \frac{c+2}{2},\frac{c+3}{2}; & \frac{3}{2}; & \frac{3}{2}; \end{matrix}\ \frac{x^2}{4},\frac{y^2}{4}\right] \\
&\quad=\Phi_2(a_1,a_2;c;x,y)-\Phi_2(a_1,a_2;c;-x,y) \\
&\qquad-\Phi_2(a_1,a_2;c;x,-y)+\Phi_2(a_1,a_2;c;-x,-y);
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
&4\frac{a_2y}{c}F_{2:1;1}^{0:2;2}\left[\begin{matrix} -; & \frac{a_1}{2},\frac{a_1+1}{2}; & \frac{a_2+1}{2},\frac{a_2+2}{2},-; \\ \frac{c+1}{2},\frac{c+2}{2}; & \frac{1}{2}; & \frac{3}{2}; \end{matrix}\ \frac{x^2}{4},\frac{y^2}{4}\right] \\
&\quad=\Phi_2(a_1,a_2;c;x,y)+\Phi_2(a_1,a_2;c;-x,y) \\
&\qquad-\Phi_2(a_1,a_2;c;x,-y)-\Phi_2(a_1,a_2;c;-x,-y);
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
&4\frac{a_1x}{c}F_{2:1;1}^{0:2;2}\left[\begin{matrix} -: & \frac{a_1+1}{2},\frac{a_1+2}{2}; & \frac{a_2}{2},\frac{a_2+1}{2},-; \\ \frac{c+1}{2},\frac{c+2}{2}: & \frac{3}{2}; & \frac{1}{2}; \end{matrix}\ \frac{x^2}{4},\frac{y^2}{4}\right] \\
&\quad=\Phi_2(a_1,a_2;c;x,y)-\Phi_2(a_1,a_2;c;-x,y) \\
&\qquad+\Phi_2(a_1,a_2;c;x,-y)-\Phi_2(a_1,a_2;c;-x,-y),
\end{aligned} \tag{4.17}$$

where Φ_2 denotes the Humbert hypergeometric function (see [2, 8, 17]) defined by

$$\Phi_2(a_1,a_2;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a_1)_m(a_2)_n}{(c)_{m+n}\,m!n!}x^m y^n \quad (|x|<\infty;\ |y|<\infty). \tag{4.18}$$

Transforming a_2 by $1/\varepsilon$ and y by $\varepsilon\, y$ in the formulas (4.13), (4.14), (4.15), (4.16) and (4.17), and taking the limit of each of the resulting identities as $\varepsilon \to 0$, we obtain some identities as in the following corollary (see [2]).

Corollary 4. *We find that the following functional relations hold true.*

$$\begin{aligned}
&\Phi_3\,(a_1;c;x,y) = F^{0:2;0}_{2:1;1}\left[\begin{matrix} -: & \frac{a_1}{2},\frac{a_1+1}{2},-; & -; \\ \frac{c}{2},\frac{c+1}{2}: & \frac{1}{2}; & \frac{1}{2}; \end{matrix}\ \frac{x^2}{4},\frac{y^2}{16}\right]\\
&+\frac{a_1 xy}{c\,(c+1)}F^{0:2;0}_{2:1;1}\left[\begin{matrix} -: & \frac{a_1+1}{2},\frac{a_1+2}{2},-; & -; \\ \frac{c+2}{2},\frac{c+3}{2}: & \frac{3}{2}; & \frac{3}{2}; \end{matrix}\ \frac{x^2}{4},\frac{y^2}{16}\right]\\
&+\frac{y}{c}F^{0:2;0}_{2:1;1}\left[\begin{matrix} -: & \frac{a_1}{2},\frac{a_1+1}{2},-; & -; \\ \frac{c+1}{2},\frac{c+2}{2}: & \frac{1}{2}; & \frac{3}{2}; \end{matrix}\ \frac{x^2}{4},\frac{y^2}{16}\right]\\
&+\frac{a_1 x}{c}F^{0:2;0}_{2:1;1}\left[\begin{matrix} -: & \frac{a_1+1}{2},\frac{a_1+2}{2},-; & -; \\ \frac{c+1}{2},\frac{c+2}{2}: & \frac{3}{2}; & \frac{1}{2}; \end{matrix}\ \frac{x^2}{4},\frac{y^2}{16}\right];
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
&4F^{0:2;0}_{2:1;1}\left[\begin{matrix} -: & \frac{a_1}{2},\frac{a_1+1}{2}; & -; \\ \frac{c}{2},\frac{c+1}{2}: & \frac{1}{2}; & \frac{1}{2}; \end{matrix}\ \frac{x^2}{4},\frac{y^2}{16}\right]\\
&=\Phi_3\,(a_1;c;x,y)+\Phi_3\,(a_1;c;-x,y)+\Phi_3\,(a_1;c;x,-y)+\Phi_3\,(a_1;c;-x,-y)\,;
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
&4\frac{a_1 xy}{c\,(c+1)}F^{0:2;0}_{2:1;1}\left[\begin{matrix} -; & \frac{a_1+1}{2},\frac{a_1+2}{2}; & -; \\ \frac{c+2}{2},\frac{c+3}{2}; & \frac{3}{2}; & \frac{3}{2}; \end{matrix}\ \frac{x^2}{4},\frac{y^2}{16}\right]\\
&=\Phi_3\,(a_1;c;x,y)-\Phi_3\,(a_1;c;-x,y)-\Phi_3\,(a_1;c;x,-y)+\Phi_3\,(a_1;c;-x,-y)\,;
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
&4\frac{y}{c}F^{0:2;0}_{2:1;1}\left[\begin{matrix} -; & \frac{a_1}{2},\frac{a_1+1}{2}; & -; \\ \frac{c+1}{2},\frac{c+2}{2}; & \frac{1}{2}; & \frac{3}{2}; \end{matrix}\ \frac{x^2}{4},\frac{y^2}{16}\right]\\
&=\Phi_3\,(a_1;c;x,y)+\Phi_3\,(a_1;c;-x,y)-\Phi_3\,(a_1;c;x,-y)-\Phi_3\,(a_1;c;-x,-y)\,;
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
&4\frac{a_1 x}{c}F^{0:2;0}_{2:1;1}\left[\begin{matrix} -: & \frac{a_1+1}{2},\frac{a_1+2}{2}; & -; \\ \frac{c+1}{2},\frac{c+2}{2}: & \frac{3}{2}; & \frac{1}{2}; \end{matrix}\ \frac{x^2}{4},\frac{y^2}{16}\right]\\
&=\Phi_3\,(a_1;c;x,y)-\Phi_3\,(a_1;c;-x,y)+\Phi_3\,(a_1;c;x,-y)-\Phi_3\,(a_1;c;-x,-y)\,,
\end{aligned} \tag{4.23}$$

where Φ_3 denotes the Humbert hypergeometric function (see [2, 8, 17]) defined by

$$\Phi_3\,(a_1,c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a_1)_m}{(c)_{m+n}\,m!n!}x^m y^n\ \ (|x|<\infty;\ |y|<\infty)\,. \tag{4.24}$$

Transforming a_1 by $1/\varepsilon$ and x by $\varepsilon\, x$ in the formulas (4.19), (4.20), (4.21), (4.22) and (4.23), and taking the limit of each of the obtained identities as $\varepsilon \to 0$, we deduce some identities as in the following corollary (see [2]).

Corollary 5. *Each of the following identities holds true.*

$$
\begin{aligned}
{}_0F_1(c;x+y) = {} & F_{2:1;1}^{0:0;0}\left[\begin{matrix} -: & -; & -; \\ \frac{c}{2}, \frac{c+1}{2}: & \frac{1}{2}; & \frac{1}{2}; \end{matrix} \frac{x^2}{16}, \frac{y^2}{16}\right] \\
& + \frac{xy}{c(c+1)} F_{2:1;1}^{0:0;0}\left[\begin{matrix} -: & -; & -; \\ \frac{c+2}{2}, \frac{c+3}{2}: & \frac{3}{2}; & \frac{3}{2}; \end{matrix} \frac{x^2}{16}, \frac{y^2}{16}\right] \\
& + \frac{y}{c} F_{2:1;1}^{0:0;0}\left[\begin{matrix} -: & -; & -; \\ \frac{c+1}{2}, \frac{c+2}{2}: & \frac{1}{2}; & \frac{3}{2}; \end{matrix} \frac{x^2}{16}, \frac{y^2}{16}\right] \\
& + \frac{x}{c} F_{2:1;1}^{0:0;0}\left[\begin{matrix} -: & -; & -; \\ \frac{c+1}{2}, \frac{c+2}{2}: & \frac{3}{2}; & \frac{1}{2}; \end{matrix} \frac{x^2}{16}, \frac{y^2}{16}\right];
\end{aligned} \tag{4.25}
$$

$$
\begin{aligned}
4F_{2:1;1}^{0:0;0}\left[\begin{matrix} -: & -; & -; \\ \frac{c}{2}, \frac{c+1}{2}: & \frac{1}{2}; & \frac{1}{2}; \end{matrix} \frac{x^2}{16}, \frac{y^2}{16}\right] = {} & {}_0F_1(c;x+y) \\
& + {}_0F_1(a_1;c;-x+y) + {}_0F_1(a_1;c;x-y) + {}_0F_1(a_1;c;-x-y);
\end{aligned} \tag{4.26}
$$

$$
\begin{aligned}
\frac{4xy}{c(c+1)} F_{2:1;1}^{0:0;0}\left[\begin{matrix} -; & -; & -; \\ \frac{c+2}{2}, \frac{c+3}{2}; & \frac{3}{2}; & \frac{3}{2}; \end{matrix} \frac{x^2}{16}, \frac{y^2}{16}\right] = {} & {}_0F_1(c;x+y) \\
& - {}_0F_1(c;-x+y) - {}_0F_1(c;x-y) + {}_0F_1(c;-x-y);
\end{aligned} \tag{4.27}
$$

$$
\begin{aligned}
\frac{4y}{c} F_{2:1;1}^{0:0;0}\left[\begin{matrix} -; & -; & -; \\ \frac{c+1}{2}, \frac{c+2}{2}; & \frac{1}{2}; & \frac{3}{2}; \end{matrix} \frac{x^2}{16}, \frac{y^2}{16}\right] = {} & {}_0F_1(c;x+y) \\
& + {}_0F_1(c;-x+y) - {}_0F_1(c;x-y) - {}_0F_1(c;-x-y);
\end{aligned} \tag{4.28}
$$

$$
\begin{aligned}
\frac{4x}{c} F_{2:1;1}^{0:0;0}\left[\begin{matrix} -: & -; & -; \\ \frac{c+1}{2}, \frac{c+2}{2}: & \frac{3}{2}; & \frac{1}{2}; \end{matrix} \frac{x^2}{16}, \frac{y^2}{16}\right] = {} & {}_0F_1(c;x+y) \\
& - {}_0F_1(c;-x+y) + {}_0F_1(c;x-y) - {}_0F_1(c;-x-y),
\end{aligned} \tag{4.29}
$$

where ${}_0F_1(c;x)$ *denotes the Bessel function (see [2, 8, 17]).*

5. Decomposition formulas for the Kampé de Fériet function $F_{2;1;1}^{0;4;4}$

Now we need some decomposition formulas for the function $F_{2;1;1}^{0;4;4}$ in order to investigate properties of the Appell function F_3. For this purpose we recall the definition of the Burchnall-Chaundy operators (see [3, 4, 6])

$$
\nabla_{x,y}(h) := \sum_{k=0}^{\infty} \frac{(-\delta_1)_k (-\delta_2)_k}{(h)_k\, k!}, \tag{5.1}
$$

$$
\begin{aligned}
\Delta_{x,y}(h) &:= \sum_{k=0}^{\infty} \frac{(-\delta_1)_k (-\delta_2)_k}{(1-h-\delta_1-\delta_2)_k\, k!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (h)_{2k} (-\delta_1)_k (-\delta_2)_k}{(h+k-1)_k (h+\delta_1)_k (h+\delta_2)_k\, k!}
\end{aligned} \tag{5.2}
$$

$$\left(\delta_1 := x\frac{\partial}{\partial x};\ \delta_2 := y\frac{\partial}{\partial y}\right)$$

and its one-dimensional analogue of (5.1), (5.2)

$$H_x(\alpha,\beta) := \sum_{k=0}^{\infty} \frac{(\beta-\alpha)_k\,(-\delta_1)_k}{(\beta)_k\,k!}, \tag{5.3}$$

and

$$\bar{H}_x(\alpha,\beta) := \sum_{k=0}^{\infty} \frac{(\beta-\alpha)_k\,(-\delta_1)_k}{(1-\alpha-\delta_1)_k\,k!}. \tag{5.4}$$

By making use of the operators (5.1), (5.2), (5.3) and (5.4), we find the following operator identities:

Lemma 1. *Each of the following operator identities holds true.*

$$\begin{aligned} &F(b_1,b_2;\beta;x)\,F(c_1,c_2;\gamma;y)\\ &= \nabla_{x,y}(\alpha_1)\,\nabla_{x,y}(\alpha_2)\,F_{2:1;1}^{0:4;4}\left[\begin{matrix} -: & b_1,b_2,\alpha_1,\alpha_2; & c_1,c_2,\alpha_1,\alpha_2; \\ \alpha_1,\alpha_2: & \beta; & \gamma; \end{matrix}\ x,y\right]; \end{aligned} \tag{5.5}$$

$$\begin{aligned} &{}_3F_2(b_1,b_2,b_3;\alpha_1,\beta_1;x)\,{}_3F_2(c_1,c_2,c_3;\alpha_2,\gamma_1;y)\\ &= \nabla_{x,y}(\alpha_1)\,\nabla_{x,y}(\alpha_2)\,F_{2:1;1}^{0:4;4}\left[\begin{matrix} -: & b_1,b_2,b_3,\alpha_2; & c_1,c_2,c_3,\alpha_1; \\ \alpha_1,\alpha_2: & \beta_1; & \gamma_1; \end{matrix}\ x,y\right]; \end{aligned} \tag{5.6}$$

$$\begin{aligned} &{}_4F_3(b_1,b_2,b_3,b_4;\alpha_1,\alpha_2,\beta_1;x)\,{}_4F_3(c_1,c_2,c_3,c_4;\alpha_1,\alpha_2,\gamma_1;y)\\ &= \nabla_{x,y}(\alpha_1)\,\nabla_{x,y}(\alpha_2)\,F_{2:1;1}^{0:4;4}\left[\begin{matrix} -: & b_1,b_2,b_3,b_4; & c_1,c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & \beta_1; & \gamma_1; \end{matrix}\ x,y\right]; \end{aligned} \tag{5.7}$$

$$\begin{aligned} &F_{2:1;1}^{0:4;4}\left[\begin{matrix} -: & b_1,b_2,b_3,b_4; & c_1,c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & \beta_1; & \gamma_1; \end{matrix}\ x,y\right] = \Delta_{x,y}(\alpha_1)\,\Delta_{x,y}(\alpha_2)\\ &\quad\cdot {}_4F_3(b_1,b_2,b_3,b_4;\alpha_1,\alpha_2,\beta_1;x)\,{}_4F_3(c_1,c_2,c_3,c_4;\alpha_1,\alpha_2,\gamma_1;y); \end{aligned} \tag{5.8}$$

$$\begin{aligned} &F_{1:1;1}^{0:3;3}\left[\begin{matrix} -: & b_2,b_3,b_4; & c_2,c_3,c_4; \\ \alpha_2: & \beta; & \gamma; \end{matrix}\ x,y\right]\\ &= \nabla_{x,y}(\alpha_1)\,F_{2:1;1}^{0:4;4}\left[\begin{matrix} -: & \alpha_1,b_2,b_3,b_4; & \alpha_1,c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & \beta; & \gamma; \end{matrix}\ x,y\right]; \end{aligned} \tag{5.9}$$

$$\begin{aligned} &F_{2:1;1}^{0:4;4}\left[\begin{matrix} -: & b_1,b_2,b_3,b_4; & c_1,c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & \beta; & \gamma; \end{matrix}\ x,y\right]\\ &= H_x(b_1,\beta)\,F_{2:0;1}^{0:3;3}\left[\begin{matrix} -: & b_2,b_3,b_4; & c_1,c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & -; & \gamma; \end{matrix}\ x,y\right]; \end{aligned} \tag{5.10}$$

$$F_{2:0;1}^{0:3;3}\left[\begin{matrix} -: & b_2,b_3,b_4; & c_1,c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & -; & \gamma; \end{matrix} x,y\right]$$
$$= \bar{H}_x(b_1,\beta)\, F_{2:1;1}^{0:4;4}\left[\begin{matrix} -: & b_1,b_2,b_3,b_4; & c_1,c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & \beta; & \gamma; \end{matrix} x,y\right]; \tag{5.11}$$

$$F_{2:1;1}^{0:4;4}\left[\begin{matrix} -: & b_1,b_2,b_3,b_4; & c_1,c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & \beta; & \gamma; \end{matrix} x,y\right]$$
$$= H_x(b_1,\beta)\, H_y(c_1,\gamma)\, F_{2:0;0}^{0:3;3}\left[\begin{matrix} -: & b_2,b_3,b_4; & c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & -; & -; \end{matrix} x,y\right]; \tag{5.12}$$

$$F_{2:0;0}^{0:3;3}\left[\begin{matrix} -: & b_2,b_3,b_4; & c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & -; & -; \end{matrix} x,y\right]$$
$$= \bar{H}_x(b_1,\beta)\, \bar{H}_y(c_1,\gamma)\, F_{2:1;1}^{0:4;4}\left[\begin{matrix} -: & b_1,b_2,b_3,b_4; & c_1,c_2,c_3,c_4; \\ \alpha_1,\alpha_2: & \beta; & \gamma; \end{matrix} x,y\right]. \tag{5.13}$$

The operator identities in Lemma 1 can be proved by using the known Mellin transformations (see [14]). Furthermore, by carrying out the operator identities (5.5) to (5.13) respectively, we find some decomposition formulas as in the following theorem.

Theorem 3. *Each of the following decomposition formulas holds true.*

$$F(b_1,b_2;\beta;x)\, F(c_1,c_2;\gamma;y)$$
$$= \sum_{i,j=0}^{\infty} \frac{\left[(\alpha_1)_{i+j}\right]^2 (\alpha_2)_{i+j}\,(b_1)_{i+j}\,(b_2)_{i+j}\,(c_1)_{i+j}\,(c_2)_{i+j}}{(\alpha_1)_i\,(\alpha_1)_{2i+2j}\,(\alpha_2)_{i+2j}\,(\beta)_{i+j}\,(\gamma)_{i+j}\,i!\,j!} x^{i+j}\,y^{i+j}$$
$$\cdot F_{2:1;1}^{0:4;4}\left[\begin{matrix} -: & b_1+i+j, & b_2+i+j, & \alpha_1+i+j, \\ \alpha_1+2i+2j, & \alpha_2+i+2j: & & \end{matrix}\right.$$
$$\left.\begin{matrix} \alpha_2+i+j; & c_1+i+j, & c_2+i+j, & \alpha_1+i+j, & \alpha_2+i+j; \\ \beta+i+j; & & & & \gamma+i+j; \end{matrix} x,y\right]; \tag{5.14}$$

$${}_3F_2(b_1,b_2,b_3;\alpha_1,\beta_1;x)\, {}_3F_2(c_1,c_2,c_3;\alpha_2,\gamma_1;y)$$
$$= \sum_{i,j=0}^{\infty} \frac{(\alpha_1)_{i+j}\,(b_1)_{i+j}\,(b_2)_{i+j}\,(b_3)_{i+j}\,(c_1)_{i+j}\,(c_2)_{i+j}\,(c_3)_{i+j}}{(\alpha_1)_i\,(\alpha_1)_{2i+2j}\,(\alpha_2)_{i+2j}\,(\beta_1)_{i+j}\,(\gamma_1)_{i+j}\,i!\,j!} x^{i+j}\,y^{i+j}$$
$$\cdot F_{2:1;1}^{0:4;4}\left[\begin{matrix} -: & b_1+i+j, & b_2+i+j, & b_3+i+j, \\ \alpha_1+2i+2j, & \alpha_2+i+2j: & & \end{matrix}\right.$$
$$\left.\begin{matrix} \alpha_2+i+j; & c_1+i+j, & c_2+i+j, & c_3+i+j, & \alpha_1+i+j; \\ \beta_1+i+j; & & & & \gamma_1+i+j; \end{matrix} x,y\right]; \tag{5.15}$$

$$
\begin{aligned}
&{}_4F_3(b_1,b_2,b_3,b_4;\alpha_1,\alpha_2,\beta_1;x)\,{}_4F_3(c_1,c_2,c_3,c_4;\alpha_1,\alpha_2,\gamma_1;y)\\
&=\sum_{i,j=0}^{\infty}\frac{(b_1)_{i+j}(b_2)_{i+j}(b_3)_{i+j}(b_4)_{i+j}(c_1)_{i+j}(c_2)_{i+j}(c_3)_{i+j}(c_4)_{i+j}}{(\alpha_1)_i(\alpha_1)_{2i+2j}(\alpha_2)_{i+j}(\alpha_2)_{i+2j}(\beta_1)_{i+j}(\gamma_1)_{i+j}\,i!\,j!}x^{i+j}y^{i+j}\\
&\cdot F^{0:4;4}_{2:1;1}\left[\begin{matrix}-: & b_1+i+j, & b_2+i+j, & b_3+i+j,\\ \alpha_1+2i+2j, & \alpha_2+i+2j: & & \end{matrix}\right.\\
&\qquad\left.\begin{matrix} b_4+i+j; & c_1+i+j, & c_2+i+j, & c_3+i+j, & c_4+i+j;\\ \beta_1+i+j; & & & & \gamma_1+i+j; \end{matrix}\ x,y\right];
\end{aligned}
\tag{5.16}
$$

$$
\begin{aligned}
&F^{0:4;4}_{2:1;1}\left[\begin{matrix}-: & b_1, & b_2, & b_3, & b_4; & c_1, & c_2, & c_3, & c_4;\\ \alpha_1, & \alpha_2: & & & \beta_1; & & & & \gamma_1; \end{matrix}\ x,y\right]\\
&=\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(\alpha_1)_{2i}(\alpha_2)_{2i+2j}(b_1)_{i+j}(b_2)_{i+j}(b_3)_{i+j}}{(\alpha_1+i-1)_i(\alpha_2+2i+j-1)_j\left[(\alpha_1)_{2i+j}\right]^2}\\
&\quad\cdot\frac{(b_4)_{i+j}(c_1)_{i+j}(c_2)_{i+j}(c_3)_{i+j}(c_4)_{i+j}}{\left[(\alpha_2)_{2i+2j}\right]^2(\beta_1)_{i+j}(\gamma_1)_{i+j}\,i!\,j!}x^{i+j}y^{i+j}\\
&\quad\cdot{}_4F_3\left[\begin{matrix} b_1+i+j, & b_2+i+j, & b_3+i+j, & b_4+i+j;\\ & \alpha_1+2i+j, & \alpha_2+2i+2j, & \beta_1+i+j; \end{matrix}\ x\right]\\
&\quad\cdot{}_4F_3\left[\begin{matrix} c_1+i+j, & c_2+i+j, & c_3+i+j, & c_4+i+j;\\ & \alpha_1+2i+j, & \alpha_2+2i+2j, & \gamma_1+i+j; \end{matrix}\ y\right];
\end{aligned}
\tag{5.17}
$$

$$
\begin{aligned}
&F^{0:3;3}_{1:1;1}\left[\begin{matrix}-: & b_2, & b_3, & b_4; & c_2, & c_3, & c_4;\\ \alpha_2: & & & \beta; & & & \gamma; \end{matrix}\ x,y\right]\\
&=\sum_{i=0}^{\infty}\frac{(\alpha_1)_i(b_2)_i(b_3)_i(b_4)_i(c_2)_i(c_3)_i(c_4)_i}{(\alpha_1)_{2i}(\alpha_2)_{2i}(\beta)_i(\gamma)_i\,i!}x^i y^i\\
&\cdot F^{0:4;4}_{2:1;1}\left[\begin{matrix}-: & \alpha_1+i, & b_2+i, & b_3+i, & b_4+i;\\ \alpha_1+2i, & \alpha_2+2i: & & & \beta+i;\\ & & \alpha_1+i, & c_2+i, & c_3+i, & c_4+i;\\ & & & & & \gamma_1+i; \end{matrix}\ x,y\right];
\end{aligned}
\tag{5.18}
$$

$$
\begin{aligned}
&F^{0:4;4}_{2:1;1}\left[\begin{matrix}-: & b_1, & b_2, & b_3, & b_4; & c_1, & c_2, & c_3, & c_4;\\ \alpha_1, & \alpha_2: & & & \beta; & & & & \gamma; \end{matrix}\ x,y\right]\\
&=\sum_{i=0}^{\infty}\frac{(-1)^i(\beta-b_1)_i(b_2)_i(b_3)_i(b_4)_i}{(\alpha_1)_i(\alpha_2)_i(\beta)_i\,i!}x^i\\
&\cdot F^{0:3;4}_{2:0;1}\left[\begin{matrix}-: & b_2+i, b_3+i, b_4+i; & c_1, c_2, c_3, c_4;\\ \alpha_1+i, \alpha_2+i: & -; & \gamma; \end{matrix}\ x,y\right];
\end{aligned}
\tag{5.19}
$$

$$
\begin{aligned}
&F_{2:0;1}^{0:3;4}\left[\begin{matrix} & -: & b_2, & b_3, & b_4; & c_1, & c_2, & c_3, & c_4; \\ \alpha_1, & \alpha_2: & & -; & & & \gamma; & & \end{matrix}\; x, y\right] \\
&=\sum_{i=0}^{\infty}\frac{(\beta-b_1)_i\,(b_2)_i\,(b_3)_i\,(b_4)_i}{(\alpha_1)_i\,(\alpha_2)_i\,(\beta)_i\,i!}x^i \qquad (5.20)\\
&\cdot F_{2:1;1}^{0:4;4}\left[\begin{matrix} & -: & b_1, b_2+i, b_3+i, b_4+i; & c_1, c_2, c_3, c_4; \\ \alpha_1+i, \alpha_2+i: & & \beta+i; & \gamma; & \end{matrix}\; x, y\right];
\end{aligned}
$$

$$
\begin{aligned}
&F_{2:1;1}^{0:4;4}\left[\begin{matrix} & -: & b_1, & b_2, & b_3, & b_4; & c_1, & c_2, & c_3, & c_4; \\ \alpha_1, & \alpha_2: & & & \beta; & & & & \gamma; & \end{matrix}\; x, y\right] \\
&=\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\,(\beta-b_1)_i\,(\gamma-c_1)_j\,(b_2)_i\,(b_3)_i\,(b_4)_i\,(c_2)_j\,(c_3)_j\,(c_4)_j}{(\alpha_1)_{i+j}\,(\alpha_2)_{i+j}\,(\beta)_i\,(\gamma)_j\,i!\,j!}x^i\,y^j \\
&\cdot F_{2:0;0}^{0:3;3}\left[\begin{matrix} & & -: & b_2+i, & b_3+i, & b_4+i; \\ \alpha_1+i+j, & \alpha_2+i+j: & & & & -; \\ & & & c_2+j, & c_3+j, & c_4+j; \end{matrix}\; x, y\right]; \qquad (5.21)
\end{aligned}
$$

$$
\begin{aligned}
&F_{2:0;0}^{0:3;3}\left[\begin{matrix} & -: & b_2, & b_3, & b_4; & c_2, & c_3, & c_4; \\ \alpha_1, & \alpha_2: & & -; & & & -; & \end{matrix}\; x, y\right] \\
&=\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\,(\beta-b_1)_i\,(\gamma-c_1)_j\,(b_2)_i\,(b_3)_i\,(b_4)_i\,(c_2)_j\,(c_3)_j\,(c_4)_j}{(\alpha_1)_{i+j}\,(\alpha_2)_{i+j}\,(\beta)_i\,(\gamma)_j\,i!\,j!}x^i\,y^j \\
&\cdot F_{2:1;1}^{0:4;4}\left[\begin{matrix} & & -: & b_1, & b_2+i, & b_3+i, & b_4+i; \\ \alpha_1+i+j, & \alpha_2+i+j: & & & & & \beta+i; \\ & & & c_1, & c_2+j, & c_3+j, & c_4+j; \\ & & & & & & \gamma+j; \end{matrix}\; x, y\right]. \qquad (5.22)
\end{aligned}
$$

Note that certain decomposition formulas for multiple hypergeometric functions have been presented in [11–13].

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AUGMENTED ECCENTRIC CONNECTIVITY INDEX

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Abstract. We review basic mathematical properties of the augmented eccentric connectivity index. Explicit formulas are presented for several classes of graphs, in particular for some open and closed unbranched polymers and nanostructures. Asymptotic behavior is explored and compression ratios are computed for those polymers.

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1. INTRODUCTION

The eccentric connectivity index is one of topological invariants that have been recently used in the quantitative structure-activity relationship (QSAR) and quantitative structure-property relationship (QSPR) studies. Like many other useful invariants, it spawned several generalizations. One of them, the augmented eccentric connectivity index, turned out to be even better suited to certain tasks than the original invariant. In spite of its usefulness, its mathematical properties have never been investigated. The main goal of this paper is to review some basic mathematical properties of the augmented eccentric connectivity index and, where it is possible, to establish explicit formulas for certain classes of graphs. It turned out that those properties are very far from the properties of the original invariant and from the properties of other distance-based invariants such as, e.g., the Wiener number. We find that this is due to the combined effects of the non-local and non-linear nature of the vertex contributions.

2. DEFINITIONS AND PRELIMINARIES

All graphs in this paper are finite, simple and connected. For terms and concepts that are not defined here we refer the reader to any of the standard monographs such as, e.g., [7] or [13].

Let G be a graph on n vertices. We denote the vertex and the edge set of G by $V(G)$ and $E(G)$, respectively. For two vertices u and v of $V(G)$ we define their **distance** $d(u,v)$ as the length of a shortest path connecting u and v in G. For a given vertex u of $V(G)$ its **eccentricity** $\varepsilon(u)$ is the largest distance between u and any other

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vertex v of G. Hence, $\varepsilon(u) = \max_{v \in V(G)} d(u,v)$. The maximum eccentricity over all vertices of G is called the **diameter** of G and is denoted by $D(G)$; the minimum eccentricity among the vertices of G is called **radius** of G and is denoted by $R(G)$. The set of all vertices of minimum eccentricity is called the **center** of G.

The **eccentric connectivity index** $\xi(G)$ of a graph G is defined as

$$\xi(G) = \sum \delta_u \varepsilon(u),$$

where δ_u denotes the degree of vertex u, i. e., the number of its neighbors in G. The eccentric connectivity index was introduced by Madan *et al.* and used in a series of papers concerned with QSAR/QSPR studies [6,11,12]. The study of its mathematical properties started only recently [1–4, 9, 14]; we refer the reader to a recent paper by Ilić [8] for a comprehensive survey.

The **augmented eccentric connectivity index** ${}^A\xi(G)$ of a graph G is defined as

$${}^A\xi(G) = \sum_{u \in V(G)} \frac{M(u)}{\varepsilon(u)},$$

where $M(u)$ denotes the product of degrees of all neighbors of vertex u. It was introduced in another paper by the above mentioned group of authors [5]. We can see that the vertex contributions are both non-local (the degrees are taken over the neighborhoods and then multiplied) and non-linear in $\varepsilon(u)$. While the first fact is reflected in the name, the second one is completely ignored – it would have been better to call the new invariant something like augmented reciprocal eccentric connectivity index. The combination of those two properties makes the augmented eccentric connectivity index rather unyielding to the standard approach to distance-based invariants, as a number of difficulties arise when one tries to obtain explicit formulas or to find the extremal graphs and values.

A graph G is **vertex-transitive** if its automorphism group is transitive. For a vertex-transitive graph G its center coincides with $V(G)$. Since a vertex-transitive graph is necessarily regular, we have a particularly simple expression for the eccentric connectivity index of a vertex transitive graph.

Proposition 1. *Let G be a vertex-transitive graph on n vertices of degree δ. Then*

$${}^A\xi(G) = \frac{n\delta^\delta}{R(G)}.$$

As a consequence, we obtain explicit formulas for the eccentric connectivity indices of several familiar classes of graphs.

Corollary 1.

(1) ${}^A\xi(K_n) = n(n-1)^{n-1}$;

(2) ${}^A\xi(C_n) = \dfrac{4n}{\lfloor \frac{n}{2} \rfloor} = \begin{cases} 8, & n \quad \text{even} \\ \dfrac{8n}{n-1}, & n \quad \text{odd} \end{cases}$;

(3) $^{A}\xi(\Pi_m) = \dfrac{54m}{\left\lfloor \frac{m}{2} \right\rfloor + 1} = \begin{cases} \frac{108m}{m+2}, & m \quad \text{even} \\ \frac{108m}{m+1}, & m \quad \text{odd} \end{cases}$;

(4) $^{A}\xi(A_m) = \dfrac{512m}{\left\lfloor \frac{m+1}{2} \right\rfloor} = \begin{cases} 1024, & m \quad \text{even} \\ \frac{1024m}{m+1}, & m \quad \text{odd} \end{cases}$;

(5) $^{A}\xi(Q_m) = 2^m m^{m-1}$.

Here K_n, C_n, Π_m, A_m and Q_m denote the complete graph on n vertices, the cycle on n vertices, the m-sided prism, the m-sided antiprism, and the m-dimensional hypercube, respectively.

It is immediately seen that the complete graph has the largest augmented eccentric connectivity index among all graphs on the same number of vertices. Namely, no vertex in a simple graph G_n on n vertices can contribute more than $(n-1)^{n-1}$ to $^{A}\xi(G_n)$, and the only graph on n vertices in which each vertex contributes that much is K_n. Hence, for any graph $G_n \neq K_n$ on n vertices we have $^{A}\xi(G_n) <^{A} \xi(K_n)$. The above result also shows that the upper bound on $^{A}\xi(G_n)$ is super-factorial (i.e., it grows faster than the factorial function) in the number of vertices, in a sharp contrast to the polynomial behavior of the eccentric connectivity index. On the other hand, the result for the cycles shows that no lower bounds better than a constant can be established for general graphs. Hence, the augmented eccentric connectivity index can, at least in principle, attain its values in a very large span. This might partly explain and account for its observed low degeneracy.

The following results can be easily obtained by a straightforward computation.

Proposition 2. *Let W_n and B_n denote the graphs of the pyramid and the bipyramid with n-gonal base, $n \geq 3$. Then*

$$^{A}\xi(W_n) = 3^n + \frac{9}{2}n^2 \quad \text{and} \quad ^{A}\xi(B_n) = 4^n + 8n^3.$$

(The pyramid graph W_n is also known as the wheel graph on n spokes.)

Proposition 3. *Let $K_{m,n}$ be a complete bipartite graph on $m+n$ vertices. For $m,n \geq 2$, we have*

$$^{A}\xi(K_{m,n}) = \frac{1}{2}\left(m^{n+1} + n^{m+1}\right).$$

The case of the complete bipartite graph $K_{m,n}$ when one of the classes of bipartition is of size 1 brings us to the trees, hence it is treated separately. In order to facilitate the comparison with other trees on n vertices, we find it more convenient to state the result as follows.

Proposition 4. *Let $S_n = K_{1,n-1}$ be a star on $n \geq 3$ vertices. Then*

$$^{A}\xi(S_n) = \frac{(n-1)^2}{2} + 1.$$

3. Linear *vs.* circular polymers and the compression ratio

It is intuitively clear that a typical distance-based graph invariant (such as the eccentric connectivity index or the Wiener index) of an open linear polymer should exceed the value of the same invariant for a closed polymer of the same size and type. That idea has been quantified by introducing the concept of compression ratio in [1], where it was shown that the eccentric connectivity index reduces by a factor of $2/3$ when an open polymer is closed by joining its end units, for a large enough number of units. Let G_n^o be an open linear polymer built from n identical units joined in a certain way, and G_n^c be a closed linear polymer obtained from G_n^o by identifying the free parts of its end units. For a graph invariant $i(G)$, its **compression ratio** is the limit of the quotient $i(G_n^c)/i(G_n^o)$ when $n \longrightarrow \infty$. For example, it can be easily verified that the compression ratio for the Wiener numbers of paths and cycles is $3/4$.

We will show in this section that the augmented eccentric connectivity index behaves exactly in the opposite way: its values for open polymers tends to be smaller than for their closed analogues. Hence we obtain "compression ratios" larger than one.

The simplest linear polymers are paths and cycles, and we start with them.

3.1. *Paths* vs. *cycles*

We start by recalling that the n-th harmonic number H_n is given as the n-th partial sum of the harmonic series, $H_n = \sum_{i=1}^{n} \frac{1}{i}$. Since the eccentricities of vertices of a path increase by one from $\lceil n/2 \rceil$ to n, and the products of degrees of their neighborhoods are fairly constant, the following formula can be verified by a direct computation. Here P_n denotes the path of length n, hence on $n+1$ vertices.

Proposition 5. *Let P_n be a path of length $n \geq 3$. Then ${}^A\xi(P_n) = 8(H_n - H_{\lfloor n/2 \rfloor}) - R_n$, where*

$$R_n = \begin{cases} \frac{4}{n(n-1)}, & n \text{ even} \\ \frac{8n-4}{n(n-1)}, & n \text{ odd} \end{cases}.$$

It is well known that harmonic numbers asymptotically behave as $H_n \approx \ln n + \gamma$, where γ is the Euler–Mascheroni constant. Hence, ${}^A\xi(P_n)$ is asymptotically equal to $8 \ln 2$. As a corollary, we obtain the compression ratio of the augmented eccentric connectivity index for paths and cycles.

Corollary 2.

$$\lim_{n \longrightarrow \infty} \frac{{}^A\xi(C_n)}{{}^A\xi(P_n)} = \frac{1}{\ln 2}.$$

3.2. *Ladder graphs* vs. *prisms*

Let L_n be a ladder graph, i.e, a graph obtained as the Cartesian product of P_n and K_2. It has n squares, as shown in Fig. 1. It can be verified by direct computation that

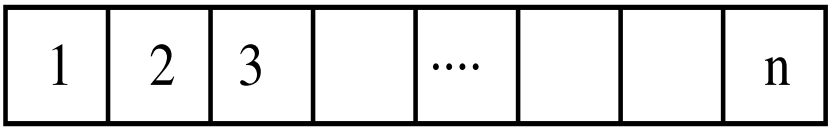

FIGURE 1. A ladder graph.

for even n the following formula is valid.

$$^{A}\xi(L_n) = 108(H_n - H_{n/2}) - \frac{36}{n} + \frac{24}{n+1} - \frac{108}{n+2}.$$

For an odd n the formula differs only in the correction terms. Both cases can be expressed by a single formula.

Proposition 6.

$$^{A}\xi(L_n) = 108(H_n - H_{\lceil n/2 \rceil}) - \frac{36}{n} + \frac{24}{n+1} - \frac{1+(-1)^n}{2}\frac{108}{n+2}.$$

Asymptotically, both cases behave like $108\ln 2$. Since the closed version of L_n is the prism Π_n, we obtain again $\frac{1}{\ln 2}$ as the compression ratio.

3.3. *Hexagonal chains* vs. *hexagonal belts*

In this subsection Z_h^o and A_h^o denote an open zigzag and an open armchair chain of h hexagons, respectively. They are shown in Figs 2 and 3. The corresponding

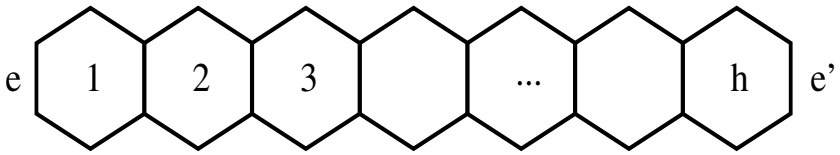

FIGURE 2. A zigzag hexagonal chain.

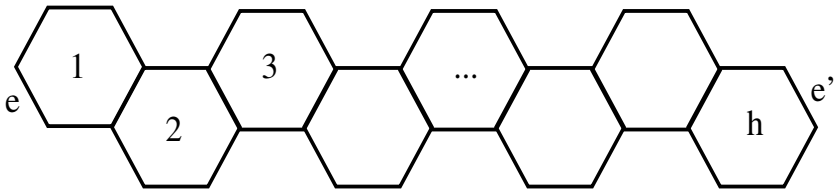

FIGURE 3. An armchair hexagonal chain.

closed hexagonal polymers are obtained in both cases by identifying the edges e and

e'. They are denoted by Z_h^c and A_h^c and called the zigzag and the armchair hexagonal belt, respectively. We start with belts, since their augmented eccentric connectivity indices are easier to compute.

In the case of zigzag belts there are $4h$ vertices; all of them have the eccentricity equal to $h+1$. For $2h$ of them the product of degrees of their neighbors is equal to 9, for another $2h$ of them it is equal to 12. By summing all contributions we readily obtain ${}^A\xi(Z_h^c) = 42\frac{h}{h+1}$.

The armchair belt A_h^c also has $4h$ vertices. For $h > 4$ (note that h must be even!) all of them have the eccentricity equal to h. By summing the contributions of two types of vertices we obtain ${}^A\xi(A_h^c) = 2h\left(\frac{6}{h} + \frac{18}{h}\right) = 48$. For the exceptional case $h = 4$ we have ${}^A\xi(A_4^c) = \frac{228}{5}$.

Now we consider open chains. The case of the armchair chain is simpler, in the sense that the formula does not depend on the parity of h. By ordering the contributions of vertices by increasing eccentricity we obtain

$$\begin{aligned} {}^A\xi(A_h^o) &= 2\left(\sum_{i=1}^{h-1}\frac{6}{h+i} + \sum_{i=2}^{h-1}\frac{18}{h+i} + \frac{4}{2h} + \frac{6}{2h} + \frac{4}{2h+1}\right) \\ &= 48H_{2h} - 48H_h - 2\frac{46h^2+35h+7}{h(h+1)(2h+1)}. \end{aligned}$$

For large values of h this behaves like $48\ln 2$, and by dividing ${}^A\xi(A_h^c) = 48$ by this value we obtain the expected compression ratio of $\frac{1}{\ln 2}$.

For the zigzag chain we must consider two cases. For an even h there are two vertices of degree 3 on the symmetry axis (and hence with the smallest eccentricity of $h+1$) and their joint contribution to ${}^A\xi(Z_h^o)$ is equal to $\frac{24}{h+1}$. All other classes of vertices with the same eccentricity have four elements, and the eccentricities increase from $h+2$ to $2h+1$. However, the resulting sum cannot be neatly expressed in terms of harmonic numbers, due to the fact that the numerators of the contributions from the vertices with an even eccentricity are not the same as of those with odd eccentricities. Instead, the digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$ appears, and we obtain

$${}^A\xi(Z_h^o) = 36(H_{2h} - H_h) + 6\left(\psi\left(h+\frac{1}{2}\right) - \psi\left(\frac{h+3}{2}\right)\right) + 2\frac{26h^2+11h-3}{h(h+1)(2h+1)}.$$

(The use of digamma function is not essential for obtaining our results, but it allows us to present them in a more compact way.) It can be checked by direct calculation that the above quantity behaves as $42\ln 2$ for large values of h, and again we obtain the expected compression ratio $\frac{1}{\ln 2}$. The case of odd h is similar and we omit the details.

3.4. *Nanotubes* vs. *nanotori*

We start by considering a hexagonal network such as the one shown in Fig. 4. Its odd columns are made of r hexagons, while the even ones have $r-1$ hexagons. By wrapping the network around its vertical symmetry axis so that the leftmost and the rightmost vertex of the same row are connected by an edge, we obtain a nanotube with r rows and $2s$ columns of hexagons. Such a nanotube is called an **armchair nanotube** and denoted by $N(r,s)$. We note that in an armchair nanotube the number

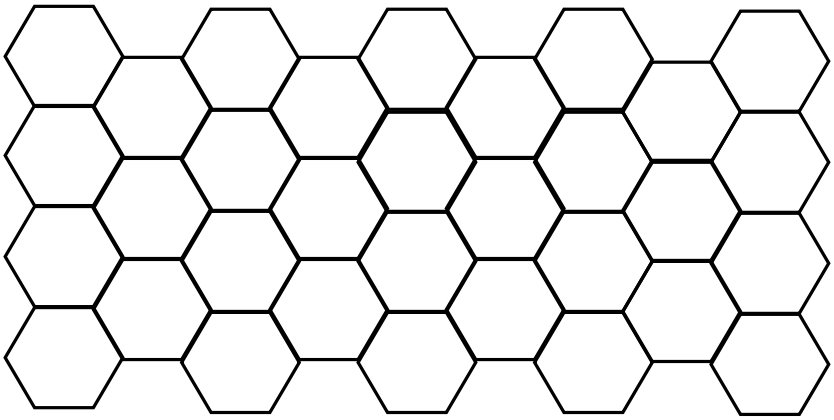

FIGURE 4. An armchair nanotube ($N(4,5)$).

of columns is always even. For the sake of simplicity we restrict our attention only to the armchair nanotubes that have a horizontal plane of symmetry.

The distances (and hence the eccentricities) relevant for our purpose were computed in a paper dealing with the eccentric connectivity index of armchair nanotubes [10]. By using those results we can derive explicit formulas for ${}^{A}\xi(N(r,s))$ in terms of r and s. The exact expressions depend on the relationship between those two parameters. For example, when r is small ($2r \leq s$), all vertices have the same eccentricity and it is equal to $2s$. There are $4s$ vertices of degree 2 whose neighbors have degrees 2 and 3. Hence their collective contribution is equal to $4s \cdot 2 \cdot 3/(2s) = 12$. Further, there are $4s$ vertices of degree 3 whose neighbors contribute $2 \cdot 3 \cdot 3$ to the numerators, and their collective contribution is equal to $4s \cdot 18/(2s) = 36$. Finally, all other $(2r+1)2s - 8s = 4rs - 6s$ vertices contribute $27/(2s)$ each, making it altogether $54r - 81$. By adding the contributions from all types of vertices we obtain that for small r the augmented eccentric connectivity index of $N(r,s)$ is given by $54r - 33$. Since the other computations involved are similarly straightforward, we omit the details.

Proposition 7. *The values of* ${}^A\xi(N(r,s))$ *are given by following formulas:*

$$\begin{array}{ll} 54r-33 & , \quad \text{for } 2r \le s; \\ 108s(H_{2r+s}-H_{2s})+27(2s-2r+1)-\frac{84s}{2r+s}-\frac{36s}{2r+s-1} & , \quad \text{for } s<2r<2s; \\ 108s(H_{2r+s}-H_{r+s-3})-\frac{54s}{r+s}-\frac{84s}{2r+s}-\frac{36s}{2r+s-1} & , \quad \text{for } r \ge s. \end{array}$$

Again, for large values of r we can observe the familiar logarithmic behavior of ${}^A\xi(N(r,s))$ with respect to r.

From an armchair nanotube we can make an **armchair nanotorus** $T(r,s)$ by identifying the uppermost and the lowermost horizontal edges in each column of r hexagons. When r is large enough ($r \ge s$) the eccentricity of all its vertices will be given by $r+s$. All vertices are now of degree 3, hence all numerators are equal to 27, and the augmented eccentric connectivity index of $T(r,s)$ is given by ${}^A\xi(T(r,s)) = (4rs+2s)\frac{27}{r+s}$. For large enough values of r it behaves asymptotically as $108s$. By considering the quotient ${}^A\xi(T(r,s))/{}^A\xi(N(r,s))$ we again recover the familiar compression ratio of $1/\ln 2$.

Similar results can be derived also for zigzag nanotubes and nanotori.

4. Concluding remarks

In this paper we have presented some basic mathematical properties of the augmented eccentric connectivity index. However, our results still leave much to be desired. For example, we did not find the minimum value of ${}^A\xi(G)$ over all graphs on a given number of vertices. We are convinced that this value is achieved for P_{n-1}, but the conjecture did not yield to the usual techniques. Further, it would be interesting to determine for which graphs G their augmented eccentric connectivity index remains bounded. We have seen that ${}^A\xi(G)$ can grow very fast with the number of vertices. As of now, we do not know for which classes of graphs the growth rate remains polynomial.

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STATISTICAL RATES IN APPROXIMATION BY POSITIVE LINEAR OPERATORS

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Abstract. This study is the continuation of our former work [O. Duman and E. Erkuş, Comput. Math. Appl. 52 (2006) 967-974] in which we obtained a statistical Korovkin-type approximation theorem for a sequence of positive linear operators defined on the space of all real-valued continuous and 2π periodic functions on the real m-dimensional space. In this paper, we compute the statistical rates of this statistical approximation.

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Keywords: statistical convergence, statistical rates, positive linear operators, periodic functions, modulus of continuity

1. INTRODUCTION

The motivation of this work are [3,4]. Let m be a positive integer, and let $C^*(\mathbb{R}^m)$ denote the space of all real-valued continuous and 2π periodic functions on the real m-dimensional space $\mathbb{R}^m$. Here, the 2π-periodicitiy of a function $f \in C^*(\mathbb{R}^m)$ is given by

$$\begin{aligned} f(u_1,u_2,...,u_m) &= f(u_1+2k\pi,u_2,....,u_m) \\ &= f(u_1,u_2+2k\pi,...,u_m) \\ &\quad ... \\ &= f(u_1,u_2,...,u_m+2k\pi) \end{aligned}$$

for every $\mathbf{u}=(u_1,u_2,...,u_m)\in\mathbb{R}^m$ and $k=0,\pm1,...$(see, for instance, [10, p. 126]). Consider the usual supremum norm on $C^*(\mathbb{R}^m)$ defined by

$$\|f\|_* := \sup_{(u_1,u_2,...,u_m)\in\mathbb{R}^m} |f(u_1,u_2,...,u_m)|, \quad f\in C^*(\mathbb{R}^m).$$

Assume that

$$A:=[a_{jn}] \quad (j,n\in\mathbb{N}:=\{1,2,...\})$$

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is a non-negative summability matrix. Recently, in [4], it has been proved that, for any sequence $\{L_n\}$ of positive linear operators mapping $C^*(\mathbb{R}^m)$ into itself,

$$st_A - \lim_n \|L_n(f) - f\|_* = 0 \quad \text{for all} \quad f \in C^*(\mathbb{R}^m) \tag{1.1}$$

if and only if

$$st_A - \lim_n \|L_n(f_i) - f_i\|_* = 0 \quad \text{for each } i = 0, 1, ..., 2m+1, \tag{1.2}$$

where

$$\begin{aligned} f_0(u_1, ..., u_m) &= 1, \\ f_i(u_1, ..., u_m) &= \cos u_i \quad \text{for} \quad i = 1, 2, ..., m, \\ f_j(u_1, ..., u_m) &= \sin u_j \quad \text{for} \quad j = m+1, m+2, ..., 2m. \end{aligned}$$

Notice that by $st_A - \lim_n x_n = L$ we mean that the sequence $x := \{x_n\}$ is A-statistically convergent to L (see [8]), i.e., for every $\varepsilon > 0$,

$$\lim_j \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0.$$

It is well-known that every convergent sequence is A-statistically convergent to the same value, however the converse is not always true. Also, taking special regular matrices, one can obtain many convergence methods from the A-statistical convergence. For example, if $A = C_1$, the Cesáro matrix, then it reduces to the concept of statistical convergence (see [7, 9]), and if $A = I$, the identity matrix, then it coincides with the ordinary convergence. Hence, with such properties, the usage of A-statistical convergence in approximation theory provides us more powerful results than the classical theory does. Observe that the above statistical approximation theorem in the space $C^*(\mathbb{R}^m)$ contains the classical uniform convergence. However, it is also possible to construct a sequence of positive linear operators satisfying (1.1) or (1.2) but not the corresponding classical case (see [4]).

In this paper, we mainly discuss the following problem: how can we compute the statistical rates of the A-statistical convergence to zero of the difference sequence $\{\|L_n(f) - f\|_*\}$ in (1.1)? Such a problem has recently been investigated for the one dimensional case by Duman [3]. In order to compute the statistical rates we use the following two definitions (see, for instance, [5, 6]):

Let $\{p_n\}$ be a positive non-increasing sequence of real numbers. For a given sequence $\{x_n\}$, we say that

- $\{x_n\}$ is A-statistically convergent to the number L with rate $o(p_n)$, denoted by $x_n - L = st_A - o(p_n)$ as $n \to \infty$, if, for every $\varepsilon > 0$,

$$\lim_j \frac{1}{p_j} \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0,$$

- $\{x_n\}$ is A-statistically convergent to the number L with rate $o_m(p_n)$, denoted by $x_n - L = st_A - o_m(p_n)$ as $n \to \infty$, if, for every $\varepsilon > 0$,

$$\lim_j \sum_{n:|x_n - L| \geq \varepsilon p_n} a_{jn} = 0.$$

Notice that the *rate of convergence* given by the little o is influenced more strongly by the summability method then by the terms of the sequence $\{x_n\}$. For instance, when one takes the identity matrix I, if $\{p_n\}$ is any positive non-increasing sequence satisfying the following inequality:

$$\frac{1}{p_n} \leq M \quad (\text{for } M > 0 \text{ and } n \in \mathbb{N}),$$

then we have $x_n - L = st_A - o(p_n)$ as $n \to \infty$ for any convergent sequence $\{x_n - L\}$ regardless of how slowly it goes to zero. To avoid such an unfortunate situation, one may borrow the concept of *convergence in measure* from the measure theory to define the rate of convergence as in the notion of the little o_m.

2. Statistical rates of the approximation

In this section, we compute the statistical rates of the statistical approximation in (1.1). To see this we first recall the concept of modulus of continuity in the space $C^*(\mathbb{R}^m)$. The modulus of continuity of a function f belonging to $C^*(\mathbb{R}^m)$ is given by, for any $\delta > 0$,

$$w(f,\delta) = \sup_{\sqrt{(u_1-x_1)^2+...+(u_m-x_m)^2} \leq \delta} |f(u_1,...,u_m) - f(x_1,...,x_m)|. \tag{2.1}$$

It is well-known that a necessary and sufficient condition for a function f to belong to $C^*(\mathbb{R}^m)$ is $\lim_{\delta \to 0} w(f,\delta) = 0$ (see [1, p. 80]). Furthermore, it follows from (2.1) that, for every $f \in C^*(\mathbb{R}^m)$ and $(u_1,...,u_m), (x_1,...,x_m) \in \mathbb{R}^m$,

$$|f(u_1,...,u_m) - f(x_1,...,x_m)| \leq w\left(f, \sqrt{(u_1-x_1)^2 + ... + (u_m-x_m)^2}\right), \tag{2.2}$$

and

$$w(f, c\delta) \leq (1+c) w(f,\delta) \quad \text{for any } c, \delta > 0. \tag{2.3}$$

Now we are ready to state our main results.

Theorem 1. *Let $A = [a_{jn}]$ be a non-negative regular summability matrix and let $\{L_n\}$ be a sequence of positive linear operators mapping $C^*(\mathbb{R}^m)$ into itself. For each $(x_1,...,x_m) \in \mathbb{R}^m$, define the function $\varphi_{x_1,...,x_m}$ by*

$$\varphi_{x_1,...,x_m}(u_1,...,u_m) = \sin^2\left(\frac{u_1 - x_1}{2}\right) + ... + \sin^2\left(\frac{u_m - x_m}{2}\right). \tag{2.4}$$

Assume that $\{p_n\}$ and $\{q_n\}$ are positive non-increasing sequences of real numbers. If, for every $f \in C^*(\mathbb{R}^m)$, the conditions

(i) $\|L_n(f_0)-f_0\|_* = st_A - o(p_n)$ *as* $n\to\infty$ *with* $f_0(u_1,...,u_m)=1$,
(ii) $w(f,\delta) = st_A - o(q_n)$ *as* $n\to\infty$ *with* $\delta_n := \sqrt{\|L_n(\varphi_{x_1,...,x_m})\|_*}$

hold, then we have

$$\|L_n(f)-f\|_* = st_A - o(r_n) \ \text{ as } \ n\to\infty$$

with $r_n := \max\{p_n,\ q_n\}$ *for each* $n\in\mathbb{N}$.

Proof. Let $f\in C^*(\mathbb{R}^m)$ and $(x_1,...,x_m)\in\mathbb{K} := [-\pi,\pi]\times...\times[-\pi,\pi]$ be fixed. Since

$$\begin{aligned} & |L_n(f;x_1,...,x_m)-f(x_1,...,x_m)| \\ \leq\ & L_n(|f(u_1,...,u_m)-f(x_1,...,x_m)|;x_1,...,x_m) \\ & +|f(x_1,...,x_m)|\,|L_n(f_0;x_1,...,x_m)-f_0(x_1,...,x_m)|, \end{aligned}$$

it follows from (2.2) that

$$\begin{aligned} & |L_n(f;x_1,...,x_m)-f(x_1,...,x_m)| \\ & \leq L_n\left(w\left(f,\sqrt{(u_1-x_1)^2+...+(u_m-x_m)^2}\right);x_1,...,x_m\right) \\ & +M\,|L_n(f_0;x_1,...,x_m)-f_0(x_1,...,x_m)|, \end{aligned} \tag{2.5}$$

where $M := \|f\|_*$. Since, for all $t\in[-\pi,\pi]$,

$$|t|\leq \pi\left|\sin\frac{t}{2}\right|$$

we have

$$\sqrt{(u_1-x_1)^2+...+(u_m-x_m)^2}\leq \pi\sqrt{\varphi_{x_1,...,x_m}(u_1,...,u_m)},$$

where $\varphi_{x_1,...,x_m}$ is given by (2.4). Combining this with (2.5), we obtain that

$$\begin{aligned} |L_n(f;x_1,...,x_m)-f(x_1,...,x_m)| &\leq L_n\left(w\left(f,\pi\sqrt{\varphi_{x_1,...,x_m}}\right);x_1,...,x_m\right) \\ &+M\,|L_n(f_0;x_1,...,x_m)-f_0(x_1,...,x_m)|. \end{aligned}$$

Also, using (2.3), we may write that, for any $\delta>0$,

$$\begin{aligned} |L_n(f;x_1,...,x_m)-f(x_1,...,x_m)| &\leq w(f,\delta)L_n\left(1+\frac{\pi}{\delta}\sqrt{\varphi_{x_1,...,x_m}};x_1,...,x_m\right) \\ &+M\,|L_n(f_0;x_1,...,x_m)-f_0(x_1,...,x_m)| \\ &\leq w(f,\delta)L_n(f_0;x_1,...,x_m) \\ &+\frac{\pi\,w(f,\delta)}{\delta}L_n\left(\sqrt{\varphi_{x_1,...,x_m}};x_1,...,x_m\right) \\ &+M\,|L_n(f_0;x_1,...,x_m)-f_0(x_1,...,x_m)|. \end{aligned}$$

From the Cauchy-Schwarz inequality for positive linear operators (see [2]), we have

$$|L_n(f;x_1,...,x_m)-f(x_1,...,x_m)|$$

$$\begin{aligned}&\leq w(f,\delta)+w(f,\delta)\left|L_n\left(f_0;x_1,...,x_m\right)-f_0\left(x_1,...,x_m\right)\right|\\&+\frac{\pi w(f,\delta)}{\delta}\sqrt{L_n\left(f_0;x_1,...,x_m\right)}\sqrt{L_n\left(\varphi_{x_1,...,x_m};x_1,...,x_m\right)}\\&+M\left|L_n\left(f_0;x_1,...,x_m\right)-f_0\left(x_1,...,x_m\right)\right|.\end{aligned}$$

Now taking supremum over $(x_1,...,x_m)\in\mathbb{K}$ and choosing $\delta:=\delta_n=\sqrt{\left\|L_n\left(\varphi_{x_1,...,x_m}\right)\right\|_*}$, we have

$$\begin{aligned}\left\|L_n(f)-f\right\|_* &\leq (1+\pi)\,w(f,\delta_n)+w(f,\delta_n)\left\|L_n(f_0)-f_0\right\|_*\\&\quad+\pi w(f,\delta_n)\sqrt{\left\|L_n(f_0)-f_0\right\|_*}\\&\quad+M\left\|L_n(f_0)-f_0\right\|_*.\end{aligned}$$

Letting $C:=\max\{1+\pi,\ M\}$, the last inequality gives that

$$\begin{aligned}\left\|L_n(f)-f\right\|_* &\leq C\left\{w(f,\delta_n)+w(f,\delta_n)\left\|L_n(f_0)-f_0\right\|_*\right.\\&\quad\left.+w(f,\delta_n)\sqrt{\left\|L_n(f_0)-f_0\right\|_*}+\left\|L_n(f_0)-f_0\right\|_*\right\}.\end{aligned}\tag{2.6}$$

Now, for a given $\varepsilon>0$, define the following sets:

$$\begin{aligned}D &:= \{n\in\mathbb{N}:\left\|L_n(f)-f\right\|_*\geq\varepsilon\},\\D_1 &:= \left\{n\in\mathbb{N}:\left\|L_n(f_0)-f_0\right\|_*\geq\frac{\varepsilon}{4C}\right\},\\D_2 &:= \left\{n\in\mathbb{N}:w(f,\delta_n)\geq\frac{\varepsilon}{4C}\right\},\\D_3 &:= \left\{n\in\mathbb{N}:w(f,\delta_n)\left\|L_n(f_0)-f_0\right\|_*\geq\frac{\varepsilon}{4C}\right\},\\D_4 &:= \left\{n\in\mathbb{N}:w(f,\delta_n)\sqrt{\left\|L_n(f_0)-f_0\right\|_*}\geq\frac{\varepsilon}{4C}\right\}.\end{aligned}$$

By (2.6) it is easy to that $D\subseteq\bigcup_{i=0}^{4}D_i$. Furthermore, consider the sets

$$D_3':=\left\{n\in\mathbb{N}:w(f,\delta_n)\geq\sqrt{\frac{\varepsilon}{4C}}\right\},$$

$$D_3'':=\left\{n\in\mathbb{N}:\left\|L_n(f_0)-f_0\right\|_*\geq\sqrt{\frac{\varepsilon}{4C}}\right\}.$$

Then observe that $D_3\subseteq D_3'\cup D_3''$ and $D_4\subseteq D_1\cup D_3''$, so we have

$$D\subseteq D_1\cup D_2\cup D_3'\cup D_3''.$$

This inclusion implies, for all $j\in\mathbb{N}$, that

$$\frac{1}{r_j}\sum_{n\in D}a_{jn}\leq\frac{1}{p_j}\sum_{n\in D_1}a_{jn}+\frac{1}{q_j}\sum_{n\in D_2}a_{jn}+\frac{1}{q_j}\sum_{n\in D_3'}a_{jn}+\frac{1}{p_j}\sum_{n\in D_3''}a_{jn}.\tag{2.7}$$

Now taking limit as $j \to \infty$ in (2.7) and using the hypotheses (i) and (ii), we conclude that

$$\lim_j \frac{1}{r_j} \sum_{n \in D} a_{jn} = 0,$$

which means

$$\| L_n(f) - f \|_* = st_A - o(r_n) \text{ as } n \to \infty,$$

so the theorem is proved. □

In a similar manner, we get the next result, which involves the statistical rate with the little o_m.

Theorem 2. *Let $A = [a_{jn}]$, $\{L_n\}$, $\{\delta_n\}$, $\{p_n\}$ and $\{q_n\}$ be the same as in Theorem 1. Assume that the conditions (i) and (ii) of Theorem 1 hold for the little o_m instead of the little o. Then, for any $f \in C^*(\mathbb{R}^m)$, we have*

$$\| L_n(f) - f \|_* = st_A - o_m(s_n) \text{ as } n \to \infty$$

with $s_n := \max\{p_n, q_n, \sqrt{p_n}, p_n q_n\}$ for each $n \in \mathbb{N}$.

Proof. Since $s_n = \max\{p_n, q_n, \sqrt{p_n}, p_n q_n\}$, from (2.6), we immediately get that

$$\begin{aligned} \frac{1}{s_n} \| L_n(f) - f \|_* \leq C \Bigg\{ & \frac{1}{q_n} w(f, \delta_n) + \frac{1}{p_n q_n} w(f, \delta_n) \| L_n(f_0) - f_0 \|_* \\ & + \frac{1}{q_n} w(f, \delta_n) \sqrt{\frac{1}{p_n} \| L_n(f_0) - f_0 \|_*} + \frac{1}{p_n} \| L_n(f_0) - f_0 \|_* \Bigg\}. \end{aligned} \tag{2.8}$$

Then, for a given $\varepsilon > 0$, consider the following sets:

$$\begin{aligned} E &:= \{ n \in \mathbb{N} : \| L_n(f) - f \|_* \geq \varepsilon s_n \}, \\ E_1 &:= \left\{ n \in \mathbb{N} : \| L_n(f_0) - f_0 \|_* \geq \frac{\varepsilon p_n}{4C} \right\}, \\ E_2 &:= \left\{ n \in \mathbb{N} : w(f, \delta_n) \geq \frac{\varepsilon q_n}{4C} \right\}, \\ E_3 &:= \left\{ n \in \mathbb{N} : \frac{w(f, \delta_n)}{q_n} \frac{\| L_n(f_0) - f_0 \|_*}{p_n} \geq \frac{\varepsilon}{4C} \right\}, \\ E_4 &:= \left\{ n \in \mathbb{N} : \frac{w(f, \delta_n)}{q_n} \sqrt{\frac{\| L_n(f_0) - f_0 \|_*}{p_n}} \geq \frac{\varepsilon}{4C} \right\}. \end{aligned}$$

In this case, it follows from (2.8) that

$$E \subseteq \bigcup_{i=1}^{4} E_i.$$

Also letting

$$E_3' := \left\{ n \in \mathbb{N} : w(f, \delta_n) \geq q_n \sqrt{\frac{\varepsilon}{4C}} \right\},$$

$$E_3'' := \left\{ n \in \mathbb{N} : \| L_n(f_0) - f_0 \|_* \geq p_n \sqrt{\frac{\varepsilon}{4C}} \right\},$$

we get $E_3 \subseteq E_3' \cup E_3''$ and $E_4 \subseteq E_1 \cup E_3''$, which implies

$$E \subseteq E_1 \cup E_2 \cup E_3' \cup E_3''.$$

Hence, the last inclusion yields that, for every $j \in \mathbb{N}$,

$$\sum_{n \in E} a_{jn} \leq \sum_{n \in E_1} a_{jn} + \sum_{n \in E_2} a_{jn} + \sum_{n \in E_3'} a_{jn} + \sum_{n \in E_3''} a_{jn}.$$

Now taking limit as $j \to \infty$ and using the hypotheses, we immediately see that

$$\lim_j \sum_{n \in E} a_{jn} = 0,$$

whence the result is proved. □

Remark 1. Specializing the sequences $\{p_n\}$ and $\{q_n\}$ as $p_n = q_n = 1$ for each $n \in \mathbb{N}$, Theorem 2.3 of [4] is a special case of our Theorem 1 or Theorem 2. So, our results give us the statistical rates of approximation to a function $f \in C^*(\mathbb{R}^m)$ by positive linear operators mapping $C^*(\mathbb{R}^m)$ into itself.

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DIFFERENTIAL OPERATOR GENERALIZED BY FRACTIONAL DERIVATIVES

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Abstract. By using the fractional derivative operator of Owa and Srivastava, we define a new linear multiplier fractional differential operator. Some generalized classes of analytic functions containing this multiplier are introduced. Basic properties of these classes are studied, such as inclusion relations and coefficient bounds. Some well known subclasses are pointed out as new special cases of our results. Moreover, the Cesáro partial sums σ_m of functions f are considered, and sharp lower bounds for the ratios of real part of f and σ_m (and also of f' and σ'_m) are determined in the unit open disk.

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1. INTRODUCTION

Let $\mathcal{H}$ be the class of analytic functions in $U := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$. Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the unit disk U.

Given two functions $f, g \in \mathcal{A}$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ their convolution or Hadamard product $f(z) * g(z)$ is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \ (z \in U).$$

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For several functions $f_1(z), ..., f_m(z) \in \mathcal{A}$,

$$f_1(z) * ... * f_m(z) = z + \sum_{n=2}^{\infty} (a_{1n}...a_{mn}) z^n, \qquad (z \in U).$$

A function $f \in \mathcal{A}$ is called starlike of order μ if it satisfies the following inequality

$$\Re\{\frac{z f'(z)}{f(z)}\} > \mu, \ (z \in U)$$

for some $0 \leq \mu < 1$. We denoted this by by class $\mathcal{S}(\mu)$. A function $f \in \mathcal{A}$ is called convex of order μ if it satisfies the following inequality

$$\Re\{\frac{z f''(z)}{f'(z)} + 1\} > \mu, \ (z \in U)$$

for some $0 \leq \mu < 1$. We denoted this class $\mathcal{C}(\mu)$. Note that $f \in \mathcal{C}(\mu)$ if and only if $z f' \in \mathcal{S}(\mu)$. The following definitions are required in our present investigation.

Definition 1. (see [8]) (Subordination Principal). For two functions f and g analytic in U, we say that the function $f(z)$ is subordinated to $g(z)$ in U and write $f(z) \prec g(z)(z \in U)$, if there exists a Schwarz function $w(z)$ analytic in U with $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = g(w(z)), z \in U$. In particular, if the function $g(z)$ is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition 2. (see [9]) (Differential subordination) Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\phi(p(z)), z p'(z)) \prec h(z)$ then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, $p \prec q$. If p and $\phi(p(z)), z p'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \phi(p(z)), z p'(z))$, then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$.

Definition 3. (see [10]) The fractional derivative of order α is defined, for a function $f(z)$, by

$$D_z^{\alpha} f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\alpha}} d\zeta; \ 0 \leq \alpha < 1,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when$(z-\zeta) > 0$.

Definition 4. [15] A function $f \in S$ (the class of univalent functions in U) is said to be in the class $SH(\mu)$ if it satisfies

$$|\frac{z f'(z)}{f(z)} - 2\mu(\sqrt{2}-1)| < \Re\{\sqrt{2}\frac{z f'(z)}{f(z)}\} + 2\mu(\sqrt{2}-1)$$

for some $\mu > 0$ and for all $z \in U$.

Remark 1. Geometric interpretation: Let $\Omega(\mu) := \{\frac{z f'(z)}{f(z)},\ f \in HS(\mu)\}$ then

$$\Omega(\mu) = \{\omega = u + iv : v^2 < 4\mu u + u^2; u > 0\}.$$

Note that $\Omega(\mu)$ is the interior of a hyperbola in the right half-plane which is symmetric with respect to the real axis and has vertex at the origin.

Define the function $\varphi_\alpha(a,c;z)$ by

$$\varphi(a,c;z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^n,\ (z \in U; a \in \mathbb{R},\ c \in \mathbb{R}\backslash\{0,-1,-2,...\}),$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n=0); \\ a(a+1)(a+2)...(a+n-1), & (n \in \mathbb{N}). \end{cases}$$

Corresponding to the function $\varphi_\alpha(a,c;z)$, Carlson and Shaffer [5] introduced a linear operator $L(a,c)$ by

$$L(a,c)f(z) := \varphi(a,c;z) * f(z),\ f \in \mathcal{A}.$$

Note that $L(a,a)$ is the identity operator.

In [11], Owa and Srivastava introduced the operator $\Phi^\alpha : \mathcal{A} \to \mathcal{A}$, that is known as an extension of fractional derivative and fractional integral, as follows

$$\begin{aligned} \Phi^\alpha f(z) :=& \Gamma(2-\alpha) z^\alpha D_z^\alpha f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} a_n z^n \\ &= \varphi(2,2-\alpha;z) * f(z) \\ &= L(2,2-\alpha;z) f(z). \end{aligned} \tag{1.2}$$

Note that $\Phi^0 f(z) = f(z)$. We define the linear multiplier fractional differential operator $D_{\beta,\lambda}^{k,\alpha}$ as follows

$$\begin{aligned} D^0 f(z) &= f(z) \\ &= z + \sum_{n=2}^{\infty} a_n z^n, \end{aligned}$$

$$\begin{aligned} D_{\beta,\lambda}^{1,\alpha} f(z) &= (\beta-\lambda)\Phi^\alpha f(z) + \lambda z (\Phi^\alpha f(z))' + (1-\beta)z \\ &= z + \sum_{n=2}^{\infty} [\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}][\lambda(n-1)+\beta] a_n z^n \\ &= D_{\beta,\lambda}^{\alpha}(f(z)), \end{aligned}$$

$$
\begin{aligned}
D^{2,\alpha}_{\beta,\lambda} f(z) &= D^{\alpha}_{\beta,\lambda}(D^{1,\alpha}_{\beta,\lambda} f(z)) \\
&= z + \sum_{n=2}^{\infty} \{[\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}][\lambda(n-1)+\beta]\}^2 a_n z^n, \\
&\vdots \\
D^{k,\alpha}_{\beta,\lambda} f(z) &= D^{\alpha}_{\beta,\lambda}\left(D^{k-1,\alpha}_{\beta,\lambda} f(z)\right) \\
&= z + \sum_{n=2}^{\infty} \{[\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}][\lambda(n-1)+\beta]\}^k a_n z^n \qquad (1.3) \\
&:= z + \sum_{n=2}^{\infty} \Psi_{n,k}(\alpha,\beta,\lambda) a_n z^n,
\end{aligned}
$$

for $0 \leq \alpha < 1, \beta \geq 1, \lambda \geq 0$ and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with $D^{k,\alpha}_{\beta,\lambda} f(0) = 0$.

Remark 2. (1) When $\alpha = 0, \beta = 1$, we have Al-Oboudi's differential operator (see [3]).

(2) When $\alpha = 0, \beta = 1$ and $\lambda = 1$, we get Sălăgean's differential operator (see [16]).

(3) When $k = 1, \beta = 1$ and $\lambda = 0$, we obtain the Owa-Srivastava fractional differential operator (see [11]).

(4) When $\beta = 1$, we get the linear multiplier fractional differential operator $D^{k,\alpha}_{\lambda}$ that introduced in [4].

Using the operator $D^{k,\alpha}_{\beta,\lambda}$, we define the following classes.

Definition 5. Let $SH_{k,\alpha,\beta,\lambda}(\mu), k \in \mathbb{N}$ be the class of all functions $f \in \mathcal{A}$ and univalent in U satisfying

$$
\left| \frac{D^{k+1,\alpha}_{\beta,\lambda} f(z)}{D^{k,\alpha}_{\beta,\lambda} f(z)} - 2\mu(\sqrt{2}-1) \right| < \Re\left\{ \sqrt{2} \frac{D^{k+1,\alpha}_{\beta,\lambda} f(z)}{D^{k,\alpha}_{\beta,\lambda} f(z)} \right\} + 2\mu(\sqrt{2}-1)
$$

for some μ, $(\mu > 0)$ and for all $z \in U$.

Remark 3. Geometric interpretation: If we denote with p_μ the analytic and univalent functions with the properties $p_\mu(0) = 1, p'_\mu(0) > 0 \, and \, p_\mu(U) = \Omega(\mu)$ (see Remark 1), then $f \in SH_{k,\alpha,\beta,\lambda}(\mu)$ if and only if

$$
\frac{D^{k+1,\alpha}_{\beta,\lambda} f(z)}{D^{k,\alpha}_{\beta,\lambda} f(z)} \prec p_\mu(z), \; z \in U,
$$

such that $p_\mu(z) = (1+2\mu)\sqrt{\frac{1+bz}{1-z}} - 2\mu$, where $b = \frac{1+4\mu-4\mu^2}{(1+2\mu)^2}$ and the branch of the square root $\sqrt{w}$ is chosen so that $Im\sqrt{w} \geq 0$.

We can observe that

$$SH_{k,0,1,1}(\mu) \equiv SH_k(\mu)$$

where the class $SH_k(\mu)$ was introduced in [1]. Also we have

$$SH_{0,0,1,1}(\mu) \equiv SH(\mu),$$

where the class $SH(\mu)$ was proposed in [15].

Definition 6. Let $0 \leq \alpha < 1, \beta \geq 0, \lambda \geq 0$, and $f \in \mathcal{A}$, $\nu \in [0,\infty)$. We define the class $(\nu,k)_{\alpha,\beta,\lambda} - \mathcal{S}$ by requiring that $f \in \mathcal{S}$ and

$$\Re\Big(\frac{D^{k,\alpha}_{\beta,\lambda} f(z)}{f(z)}\Big) > \nu\Big|\frac{D^{k,\alpha}_{\beta,\lambda} f(z)}{f(z)} - 1\Big|, \ (z \in U).$$

Note that

$$(\nu,k)_{0,1,1} - \mathcal{S} \equiv (\nu,k) - \mathcal{S},$$

where the class $(\nu,k) - \mathcal{S}$ is defined and studied in [2].

Remark 4. (Geometric interpretation): A function $f \in (\nu,k)_{\alpha,\beta,\lambda} - \mathcal{S}$ if and only if

$$\frac{D^{k,\alpha}_{\beta,\lambda} f(z)}{f(z)} \prec p_\nu(z), \ (\nu \in [0,\infty), \ z \in U)$$

where p_ν denoted the function which maps the unit disk conformally onto the region Ω_ν, such that $1 \in \Omega_\nu$ and

$$\partial\Omega_\nu = \{u + iv : u^2 = \nu^2(u-1)^2 + \nu^2 v^2\}.$$

The domain Ω_ν is elliptic for $\nu > 1$, is hyperbolic when $0 < \nu < 1$, parabolic for $\nu = 1$, and is the right half-plane when $\nu = 0$.

Definition 7. Let $0 \leq \alpha < 1, \beta \geq 0, \lambda \geq 0$, and $f \in \mathcal{A}$, $\nu \in [0,\infty)$. We define the class $(\nu,k)_{\alpha,\beta,\lambda} - \mathcal{CC}$ with respect to the function $g \in (\nu,k)_{\alpha,\beta,\lambda} - \mathcal{S}$

$$\Re\Big(\frac{D^{k,\alpha}_{\beta,\lambda} f(z)}{g(z)}\Big) > \nu\Big|\frac{D^{k,\alpha}_{\beta,\lambda} f(z)}{g(z)} - 1\Big|, \ (z \in U).$$

Note that

$$(\nu,k)_{0,1,1} - \mathcal{CC} \equiv (\nu,k) - \mathcal{CC},$$

where the class $(\nu,k) - \mathcal{CC}$ is defined and studied in [1].

Remark 5. (Geometric interpretation): A function $f \in (\nu,k)_{\alpha,\beta,\lambda} - \mathcal{CC}$ with respect to $g \in (\nu,k)_{\alpha,\beta,\lambda} - \mathcal{S}$ if and only if

$$\frac{D^{k,\alpha}_{\beta,\lambda} f(z)}{g(z)} \prec p_\nu(z), \ (\nu \in [0,\infty), \ z \in U)$$

where p_ν is defined in Remark 4. Or $\frac{D^{k,\alpha}_{\beta,\lambda} f(z)}{g(z)}$ take all values in the domain Ω_ν.

We need the following preliminaries in the sequel. The Libera-Pascu integral operator $L_a : \mathcal{A} \to \mathcal{A}$ is defined by

$$f(z) := L_a F(z) = \frac{1+a}{z^a} \int_0^z F(t) t^{a-1} dt, \ a \in \mathbb{C}, \ \Re(a) \geq 0.$$

For $a = 1$ we obtain the Libera integral operator, for $a = 0$ we obtain the Alexander integral operator and in the cases $a = 1, 2, 3, ...$ we obtain the Bernardi integral operator.

Lemma 1. *(see [7]) Let ϕ be convex in U with $\Re\{\kappa\phi(z) + \nu\} > 0$ for $\kappa, \nu \in \mathbb{C}$, also let $p(z) \in \mathcal{H}(U)$ with $p(0) = \phi(0)$ and assume that the Briot-Bouquet differential subordination*

$$p(z) + \frac{z p'(z)}{\kappa p(z) + \nu} \prec \phi(z), \ z \in U$$

is satisfied. These imply that

$$p(z) \prec \phi(z), \ z \in U.$$

Lemma 2. *(see [7]) Let q be convex in U and $\phi : U \to \mathbb{C}$ with $\Re\{\phi(z)\} > 0$. If $p(z) \in \mathcal{H}(U)$ and assume that the subordination*

$$p(z) + \phi(z).z p'(z) \prec q(z), \ z \in U$$

is satisfied, then

$$p(z) \prec q(z), \ z \in U.$$

2. The class $SH_{k,\alpha,\beta,\lambda}(\mu)$

In this section we study the inclusion property of functions in the class $SH_{k,\alpha,\beta,\lambda}(\mu)$. Further we show that if $F \in SH_{k,\alpha,\beta,\lambda}(\mu)$ implies that for the Libera integral operator $L_a F(z) \in SH_{k,\alpha,\beta,\lambda}(\mu)$.

Theorem 1. *Let $\mu > 0, 0 \leq \alpha < 1, \beta \geq 1, \lambda > 0$. We have*

$$SH_{k+1,\alpha,\beta,\lambda}(\mu) \subset SH_{k,\alpha,\beta,\lambda}(\mu).$$

Proof. Let $f \in SH_{k+1,\alpha,\beta,\lambda}(\mu)$. Denote

$$p(z) := \frac{D^{k+1,\alpha}_{\beta,\lambda} f(z)}{D^{k,\alpha}_{\beta,\lambda} f(z)}, \ p(0) = 1, \ (z \in U).$$

Then we have

$$\frac{D^{k+2,\alpha}_{\beta,\lambda} f(z)}{D^{k+1,\alpha}_{\beta,\lambda} f(z)} = \frac{D^{k+2,\alpha}_{\beta,\lambda} f(z)}{D^{k,\alpha}_{\beta,\lambda} f(z)} . \frac{D^{k,\alpha}_{\beta,\lambda} f(z)}{D^{k+1,\alpha}_{\beta,\lambda} f(z)} = \frac{1}{p(z)} \frac{D^{k+2,\alpha}_{\beta,\lambda} f(z)}{D^{k,\alpha}_{\beta,\lambda} f(z)}. \tag{2.1}$$

Now, by using the definition of the operator (1.3), we verify

$$\begin{aligned} p^2(z) + \lambda z p'(z) &= \Big[\frac{D^{k+1,\alpha}_{\beta,\lambda} f(z)}{D^{k,\alpha}_{\beta,\lambda} f(z)}\Big]^2 \\ &\quad + \lambda\Big[\frac{D^{k,\alpha}_{\beta,\lambda} f(z).z(D^{k+1,\alpha}_{\beta,\lambda} f(z))' - D^{k+1,\alpha}_{\beta,\lambda} f(z).z(D^{k,\alpha}_{\beta,\lambda} f(z))'}{(D^{k,\alpha}_{\beta,\lambda} f(z))^2}\Big] \\ &= \Big[\frac{D^{k+1,\alpha}_{\beta,\lambda} f(z)}{D^{k,\alpha}_{\beta,\lambda} f(z)}\Big]^2 \\ &\quad + \Big[\frac{D^{k,\alpha}_{\beta,\lambda} f(z).(D^{k+2,\alpha}_{\beta,\lambda} f(z) - (\beta-\lambda) D^{k+1,\alpha}_{\beta,\lambda} f(z)) - (1-\beta)z}{(D^{k,\alpha}_{\beta,\lambda} f(z))^2}\Big] \\ &\quad - \Big[\frac{D^{k+1,\alpha}_{\beta,\lambda} f(z).(D^{k+1,\alpha}_{\beta,\lambda} f(z) - (\beta-\lambda) D^{k,\alpha}_{\beta,\lambda} f(z)) - (1-\beta)z}{(D^{k,\alpha}_{\beta,\lambda} f(z))^2}\Big] \\ &= \frac{(D^{k+1,\alpha}_{\beta,\lambda} f(z))^2}{(D^{k,\alpha}_{\beta,\lambda} f(z))^2} + \frac{D^{k,\alpha}_{\beta,\lambda} f(z) D^{k+2,\alpha}_{\beta,\lambda} f(z) - \big(D^{k+1,\alpha}_{\beta,\lambda} f(z)\big)^2}{\big(D^{k,\alpha}_{\beta,\lambda} f(z)\big)^2} \\ &= \frac{D^{k+2,\alpha}_{\beta,\lambda} f(z)}{D^{k,\alpha}_{\beta,\lambda} f(z)}. \end{aligned} \tag{2.2}$$

Putting (2.2) in (2.1), we obtain

$$\frac{D^{k+2,\alpha}_{\beta,\lambda} f(z)}{D^{k+1,\alpha}_{\beta,\lambda} f(z)} = p(z) + \lambda \frac{z p'(z)}{p(z)}, \ (z \in U). \tag{2.3}$$

Since $f \in SH_{k+1,\alpha,\beta,\lambda}(\mu)$, then in view of Remark 3, we have

$$\frac{D^{k+2,\alpha}_{\beta,\lambda} f(z)}{D^{k+1,\alpha}_{\beta,\lambda} f(z)} = p(z) + \lambda \frac{z p'(z)}{p(z)} \prec p_\mu(z),$$

with $p(0) = p_\mu(0) = 1, \mu > 0,$ and $\Re(p_\mu(z)) > 0$. Hence, from Lemma 1, we obtain

$$p(z) \prec p_\mu(z), \ (z \in U),$$

or

$$\frac{D^{k+1,\alpha}_{\beta,\lambda} f(z)}{D^{k,\alpha}_{\beta,\lambda} f(z)} \prec p_\mu(z), \ (z \in U).$$

Now Remark 3 gives $f(z) \in SH_{k,\alpha,\beta,\lambda}(\mu)$. □

The next results can be found in [1, 2] respectively.

Corollary 1. *Let $\mu > 0$. We have*

$$SH(\mu) \subset \mathcal{S}.$$

Proof. By letting $k = 0$ in Theorem 1. □

Corollary 2. *Let $\mu > 0$. We have*

$$SH_{n+1,\lambda}(\mu) \subset SH_{n,\lambda}(\mu).$$

Proof. By assuming $\alpha = 0$ and $\beta = 1$ in Theorem 1. □

Theorem 2. *Let $\mu > 0, 0 \le \alpha < 1, \beta \ge 1, \lambda > 0$. If $F \in SH_{k,\alpha,\beta,\lambda}(\mu)$ with $\lambda(1+a) > \beta$, then the Libera-Pascu integral operator $f(z) := L_a F(z)$ is in $SH_{k,\alpha,\beta,\lambda}(\mu)$.*

Proof. Let $F \in SH_{k,\alpha,\beta,\lambda}(\mu)$. Denote

$$p(z) := \frac{D^{k+1,\alpha}_{\beta,\lambda} f(z)}{D^{k,\alpha}_{\beta,\lambda} f(z)}, \ p(0) = 1, \ (z \in U) \tag{2.4}$$

and assume that $0 < \Re\{p(z)\} \le 1$. From the definition of the Libera-Pascu integral operator we have

$$(1+a)F(z) = af(z) + zf'(z),$$

by using the linear operator $D^{k+1,\alpha}_{\beta,\lambda}$, we have

$$\begin{aligned}(1+a)D^{k+1,\alpha}_{\beta,\lambda} F(z) &= aD^{k+1,\alpha}_{\beta,\lambda} f(z) + D^{k+1,\alpha}_{\beta,\lambda}(zf'(z)) \\ &= aD^{k+1,\alpha}_{\beta,\lambda} f(z) + z(D^{k+1,\alpha}_{\beta,\lambda} f(z))' \\ = aD^{k+1,\alpha}_{\beta,\lambda} f(z) &+ \frac{D^{k+2,\alpha}_{\beta,\lambda} f(z) - (\beta-\lambda)D^{k+1,\alpha}_{\beta,\lambda} f(z) - (1-\beta)z}{\lambda},\end{aligned}$$

or equivalently,

$$\lambda(1+a)D^{k+1,\alpha}_{\beta,\lambda} F(z) = [\lambda(1+a)-\beta]D^{k+1,\alpha}_{\beta,\lambda} f(z) + D^{k+2,\alpha}_{\beta,\lambda} f(z) - (1-\beta)z.$$

Similarly, we obtain

$$\lambda(1+a)D^{k,\alpha}_{\beta,\lambda} F(z) = [\lambda(1+a)-\beta]D^{k,\alpha}_{\beta,\lambda} f(z) + D^{k+1,\alpha}_{\beta,\lambda} f(z) - (1-\beta)z.$$

Then by using (2.4) and (2.3) we have

$$\frac{D^{k+1,\alpha}_{\beta,\lambda}F(z)}{D^{k,\alpha}_{\beta,\lambda}F(z)} = \frac{[\lambda(1+a)-\beta]\frac{D^{k+1,\alpha}_{\beta,\lambda}f(z)}{D^{k,\alpha}_{\beta,\lambda}f(z)} + \frac{D^{k+2,\alpha}_{\beta,\lambda}f(z)}{D^{k+1,\alpha}_{\beta,\lambda}f(z)}.\frac{D^{k+1,\alpha}_{\beta,\lambda}f(z)}{D^{k,\alpha}_{\beta,\lambda}f(z)} - \frac{(1-\beta)z}{D^{k,\alpha}_{\beta,\lambda}f(z)}}{[\lambda(1+a)-\beta] + \frac{D^{k+1,\alpha}_{\beta,\lambda}f(z)}{D^{k,\alpha}_{\beta,\lambda}} - \frac{(1-\beta)z}{D^{k,\alpha}_{\beta,\lambda}f(z)}}$$

$$:= \frac{[\lambda(1+a)-\beta]p(z) + \frac{D^{k+2,\alpha}_{\beta,\lambda}f(z)}{D^{k+1,\alpha}_{\beta,\lambda}f(z)}.p(z) + \varphi(z)}{[\lambda(1+a)-\beta] + p(z) + \varphi(z)}$$

$$= \frac{[\lambda(1+a)-\beta]p(z) + p^2(z) + \lambda z p'(z) + \varphi(z)}{[\lambda(1+a)-\beta] + p(z) + \varphi(z)}$$

$$= \frac{p(z)\{[\lambda(1+a)-\beta] + p(z) + \varphi(z)\} + \lambda z p'(z) + \varphi(z)(1-p(z))}{[\lambda(1+a)-\beta] + p(z) + \varphi(z)}$$

$$= p(z) + z p'(z).\Big(\frac{\lambda + \frac{\varphi(z)(1-p(z))}{z p'(z)}}{[\lambda(1+a)-\beta] + p(z) + \varphi(z)}\Big)$$

$$:= p(z) + z p'(z).\phi(z),$$

where $\Re(\phi(z)) > 0$ as $z \to 1^+$. From $F \in HS_{k,\alpha,\beta,\lambda}(\mu)$, we have

$$p(z) + z p'(z).\phi(z) \prec p_\mu(z),$$

thus in view of Lemma 2, we obtain

$$p(z) \prec p_\mu(z), \ (z \in U)$$

or

$$\frac{D^{k+1,\alpha}_{\beta,\lambda}f(z)}{D^{k,\alpha}_{\beta,\lambda}f(z)} \prec p_\mu(z), \ p(0) = 1, \ (z \in U).$$

Implies $f \in HS_{k,\alpha,\beta,\lambda}(\mu)$. This completes the proof. □

The next results can be found in [2].

Corollary 3. *Let $\mu > 0$. If $F \in SH_{n,\lambda}(\mu)$. Then the $f \in SH_{n,\lambda}(\mu)$ where f is the Libera-Pascu integral operator.*

Proof. Assume $\alpha = 0$ and $\beta = 1$. Hence $\phi(z) = \frac{\lambda}{[\lambda(1+a)-1]+p(z)}$ in Theorem 2. □

3. The class $(\nu,k)_{\alpha,\beta,\lambda} - \mathcal{CC}$

In this section, we show that for any function $F \in \mathcal{A}$ in the classes $(\nu,k)_{\alpha,\beta,\lambda} - \mathcal{S}$ and $(\nu,k)_{\alpha,\beta,\lambda} - \mathcal{CC}$, the Libera-Pascu integral operator acting on F is also belongs to these classes. Moreover, some well known subclasses are obtained as special cases of our results.

Theorem 3. *If $F \in (\nu,k)_{\alpha,\beta,\lambda} - \mathcal{S}$, $\nu \in [0,\infty)$ and $0 \le \alpha < 1, \beta > 0, \lambda > 0$, then Libera-Pascu integral operator $f(z) = L_a F(z) \in (\nu,k)_{\alpha,\beta,\lambda} - \mathcal{S}$.*

Proof. Let $F \in (\nu,k)_{\alpha,\beta,\lambda} - \mathcal{S}$. Consider

$$p(z) := \frac{D^{k,\alpha}_{\beta,\lambda} f(z)}{f(z)}, \ p(0) = 1, \ (z \in U). \tag{3.1}$$

From the definition of the Libera-Pascu integral operator we obtain

$$(1+a)F(z) = af(z) + zf'(z). \tag{3.2}$$

By using the linear operator $D^{k,\alpha}_{\beta,\lambda}$, we have

$$\begin{aligned}
(1+a)D^{k,\alpha}_{\beta,\lambda}F(z) &= aD^{k,\alpha}_{\beta,\lambda}f(z) + D^{k,\alpha}_{\beta,\lambda}(zf'(z)) \\
&= aD^{k,\alpha}_{\beta,\lambda}f(z) + z(D^{k,\alpha}_{\beta,\lambda}f(z))' \\
&= aD^{k,\alpha}_{\beta,\lambda}f(z) + \frac{D^{k+1,\alpha}_{\beta,\lambda}f(z) - (\beta-\lambda)D^{k,\alpha}_{\beta,\lambda}f(z) - (1-\beta)z}{\lambda} \\
&= [a + \frac{\lambda-\beta}{\lambda}]D^{k,\alpha}_{\beta,\lambda}f(z) + \frac{1}{\lambda}D^{k+1,\alpha}_{\beta,\lambda}f(z) + \frac{(\beta-1)z}{\lambda} \\
&:= AD^{k,\alpha}_{\beta,\lambda}f(z) + BD^{k+1,\alpha}_{\beta,\lambda}f(z) + \varphi(z).
\end{aligned} \tag{3.3}$$

From (3.2) and (3.3) we obtain

$$\begin{aligned}
\frac{D^{k,\alpha}_{\beta,\lambda}F(z)}{F(z)} &= \frac{AD^{k,\alpha}_{\beta,\lambda}f(z) + BD^{k+1,\alpha}_{\beta,\lambda}f(z) + \varphi(z)}{af(z) + zf'(z)} \\
&= \frac{f(z)\Big[A\frac{D^{k,\alpha}_{\beta,\lambda}f(z)}{f(z)} + B\frac{D^{k+1,\alpha}_{\beta,\lambda}f(z)}{f(z)} + \frac{\varphi(z)}{f(z)}\Big]}{f(z)[a + \frac{zf'(z)}{f(z)}]} \\
&= \frac{A\frac{D^{k,\alpha}_{\beta,\lambda}f(z)}{f(z)} + B\frac{D^{k+1,\alpha}_{\beta,\lambda}f(z)}{f(z)} + \frac{\varphi(z)}{f(z)}}{a + \frac{zf'(z)}{f(z)}}.
\end{aligned} \tag{3.4}$$

From (3.1) we have

$$
\begin{aligned}
zp'(z) &= z\frac{f(z)[D^{k,\alpha}_{\beta,\lambda}f(z)]' - D^{k,\alpha}_{\beta,\lambda}f(z).f'(z)}{f^2(z)} \\
&= \frac{z[D^{k,\alpha}_{\beta,\lambda}f(z)]'}{f(z)} - \frac{D^{k,\alpha}_{\beta,\lambda}}{f(z)}.\frac{zf'(z)}{f(z)} \\
&= \frac{\frac{D^{k+1,\alpha}_{\beta,\lambda}f(z)-(\beta-\lambda)D^{k,\alpha}_{\beta,\lambda}f(z)-(1-\beta)z}{\lambda}}{f(z)} - \frac{D^{k,\alpha}_{\beta,\lambda}f(z)}{f(z)}.\frac{zf'(z)}{f(z)} \\
&:= \frac{BD^{k+1,\alpha}_{\beta,\lambda}f(z)}{f(z)} - \frac{CD^{\delta}_{\alpha,\beta,\lambda}f(z)}{f(z)} + \frac{\varphi(z)}{f(z)} - p(z).\frac{zf'(z)}{f(z)} \\
&= \frac{BD^{k+1,\alpha}_{\beta,\lambda}f(z)}{f(z)} - Cp(z) + \frac{\varphi(z)}{f(z)} - p(z).\frac{zf'(z)}{f(z)},
\end{aligned} \tag{3.5}
$$

where $C := \frac{(\beta-\lambda)}{\lambda}$. Hence

$$
\frac{BD^{k+1,\alpha}_{\beta,\lambda}f(z)}{f(z)} = zp'(z) + p(z)[C + \frac{zf'(z)}{f(z)}] - \frac{\varphi(z)}{f(z)}. \tag{3.6}
$$

Substitute (3.6) in (3.4), then we obtain

$$
\begin{aligned}
\frac{D^{k,\alpha}_{\beta,\lambda}F(z)}{F(z)} &= \frac{Ap(z) + B\frac{D^{k+1,\alpha}_{\beta,\lambda}f(z)}{f(z)} + \frac{\varphi(z)}{f(z)}}{a + \frac{zf'(z)}{f(z)}} \\
&= \frac{zp'(z) + p(z)[A + C + \frac{zf'(z)}{f(z)}]}{a + \frac{zf'(z)}{f(z)}} \\
&= \frac{zp'(z) + p(z)[a + \frac{zf'(z)}{f(z)}]}{a + \frac{zf'(z)}{f(z)}} \\
&= p(z) + \frac{1}{a + \frac{zf'(z)}{f(z)}}zp'(z).
\end{aligned} \tag{3.7}
$$

From the hypothesis, we have

$$
\frac{D^{k,\alpha}_{\beta,\lambda}F(z)}{F(z)} \prec p_\nu(z), \ (\nu \in [0,\infty), z \in U).
$$

Consequently, we have

$$
p(z) + \frac{1}{a + \frac{zf'(z)}{f(z)}}zp'(z) \prec p_\nu(z), \ (\nu \in [0,\infty), z \in U).
$$

Now in view of Lemma 2, we get

$$p(z) \prec p_\nu(z), \ (\nu \in [0,\infty), z \in U).$$

Equivalently

$$\frac{D^{k,\alpha}_{\beta,\lambda} f(z)}{f(z)} \prec p_\nu(z), \ (\nu \in [0,\infty), z \in U),$$

which implies $f(z) = L_a F(z) \in (\nu,k)_{\alpha,\beta,\lambda} - \mathcal{S}$. □

The next result can be found in [2].

Corollary 4. *If $F \in (\nu,n) - \mathcal{S}$, $\nu \in [0,\infty)$, then Libera-Pascu integral operator $f(z) = L_a F(z) \in (\nu,n) - \mathcal{S}$.*

Proof. Let $\alpha = 0, \beta = \lambda = 1$ in Theorem 3. □

Corollary 5. *If $F \in (\nu,k)_\lambda - \mathcal{S}$, $\nu \in [0,\infty)$, then Libera-Pascu integral operator $f(z) = L_a F(z) \in (\nu,k)_\lambda - \mathcal{S}$.*

Proof. Put $\alpha = 0, \beta = 1$ in Theorem 3. Note that in this case we have Al-Oboudi's differential operator. □

Theorem 4. *If $F \in (\nu,k)_{\alpha,\beta,\lambda} - \mathcal{CC}$, $\nu \in [0,\infty)$ and $0 \le \alpha < 1, \beta > 0, \lambda > 0$, with respect to the function $G(z) \in (\nu,k)_{\alpha,\beta,\lambda} - \mathcal{S}$, then the Libera-Pascu integral operator $f(z) = L_a F(z) \in (\nu,k)_{\alpha,\beta,\lambda} - \mathcal{CC}$ with respect to the function $g(z) = L_a G(z) \in (\nu,k)_{\alpha,\beta,\lambda} - \mathcal{S}$.*

Proof. Let $F \in (\nu,k)_{\alpha,\beta,\lambda} - \mathcal{CC}$ and

$$p(z) := \frac{D^{k,\alpha}_{\beta,\lambda} f(z)}{g(z)}, \ p(0) = 1, \ (z \in U). \tag{3.8}$$

Differentiating the Libera-Pascu integral operator $f(z) = L_a F(z)$ and operating by $D^{k,\alpha}_{\beta,\lambda}$, we have

$$(1+a) D^{k,\alpha}_{\beta,\lambda} F(z) = A D^{k,\alpha}_{\beta,\lambda} f(z) + B D^{k+1,\alpha}_{\beta,\lambda} f(z) + \varphi(z), \tag{3.9}$$

where A, B and $\varphi(z)$ are defined in Theorem 3. We also observe that

$$(1+a) G(z) = a g(z) + z g'(z). \tag{3.10}$$

By using (3.9) and (3.10) we get

$$\begin{aligned}\frac{D^{k,\alpha}_{\beta,\lambda}F(z)}{G(z)} &= \frac{AD^{k,\alpha}_{\beta,\lambda}f(z)+BD^{k+1,\alpha}_{\beta,\lambda}f(z)+\varphi(z)}{ag(z)+zg'(z)}\\ &= \frac{g(z)\Big[A\frac{D^{k,\alpha}_{\beta,\lambda}f(z)}{g(z)}+B\frac{D^{k+1,\alpha}_{\beta,\lambda}f(z)}{g(z)}+\frac{\varphi(z)}{g(z)}\Big]}{g(z)[a+\frac{zg'(z)}{g(z)}]}\\ &= \frac{A\frac{D^{k,\alpha}_{\beta,\lambda}f(z)}{g(z)}+B\frac{D^{k+1,\alpha}_{\beta,\lambda}f(z)}{g(z)}+\frac{\varphi(z)}{g(z)}}{a+\frac{zg'(z)}{g(z)}}.\end{aligned} \tag{3.11}$$

From (3.8) we have

$$\begin{aligned}zp'(z) &= z\frac{g(z)[D^{k,\alpha}_{\beta,\lambda}f(z)]'-D^{k,\alpha}_{\beta,\lambda}f(z).g'(z)}{g^2(z)}\\ &= \frac{z[D^{k,\alpha}_{\beta,\lambda}f(z)]'}{g(z)}-\frac{D^{k,\alpha}_{\beta,\lambda}f(z)}{g(z)}.\frac{zg'(z)}{g(z)}\\ &= \frac{D^{k+1,\alpha}_{\beta,\lambda}f(z)-(\beta-\lambda)D^{k,\alpha}_{\beta,\lambda}f(z)-(1-\beta)z}{\lambda g(z)}-\frac{D^{k,\alpha}_{\beta,\lambda}f(z)}{g(z)}.\frac{zg'(z)}{g(z)}\\ &:= \frac{BD^{k+1,\alpha}_{\beta,\lambda}f(z)}{g(z)}-\frac{CD^{k,\alpha}_{\beta,\lambda}f(z)}{g(z)}+\frac{\varphi(z)}{g(z)}-p(z).\frac{zg'(z)}{g(z)}\\ &= \frac{BD^{k+1,\alpha}_{\beta,\lambda}f(z)}{g(z)}-Cp(z)-p(z).\frac{zg'(z)}{g(z)}+\frac{\varphi(z)}{g(z)},\end{aligned} \tag{3.12}$$

where C defined in Theorem 3. Hence

$$\frac{BD^{k+1,\alpha}_{\beta,\lambda}f(z)}{g(z)} = zp'(z)+p(z)[C+\frac{zg'(z)}{g(z)}]-\frac{\varphi(z)}{g(z)}. \tag{3.13}$$

Substitute (3.13) in (3.11) we obtain

$$\begin{aligned}\frac{D^{k,\alpha}_{\beta,\lambda}F(z)}{G(z)} &= p(z)+\frac{1}{a+\frac{zg'(z)}{g(z)}}zp'(z)\\ &:= p(z)+\phi(z)zp'(z),\end{aligned} \tag{3.14}$$

where $\Re\{\phi\}>0$, $z\in U$. Thus in view of Lemma 2, we obtain

$$p(z)\prec p_\nu(z),\ (\nu\in[0,\infty), z\in U)$$

or equivalently

$$\frac{D^{k,\alpha}_{\beta,\lambda} f(z)}{g(z)} \prec p_\nu(z), \ (\nu \in [0,\infty), z \in U).$$

Hence $f(z) = L_a F(z) \in (\nu,k)_{\alpha,\beta,\lambda} - \mathcal{CC}$ with respect to the function $g(z) = L_a G(z) \in (\nu,k)_{\alpha,\beta,\lambda} - \mathcal{S}$. □

The next result can be found in [2].

Corollary 6. *If $F \in (\nu,k) - \mathcal{CC}$, $\nu \in [0,\infty)$ with respect to the function $G(z) \in (\nu,k) - \mathcal{S}$ then Libera-Pascu integral operator $f(z) = L_a F(z) \in (\nu,n) - \mathcal{CC}$ with respect to the function $g(z) = L_a G(z) \in (\nu,k) - \mathcal{S}$.*

Proof. Let $\alpha = 0, \beta = \lambda = 1$ in Theorem 4. □

Corollary 7. *If $F \in (\nu,k)_\lambda - \mathcal{CC}$, $\nu \in [0,\infty)$ and $\lambda > 0$, with respect to the function $G(z) \in (\nu,k)_\lambda - \mathcal{S}$, then Libera-Pascu integral operator $f(z) = L_a F(z) \in (\nu,k)_\lambda - \mathcal{CC}$ with respect to the function $g(z) = L_a G(z) \in (\nu,k)_\lambda - \mathcal{S}$.*

Proof. Put $\alpha = 0, \beta = 1$ in Theorem 4. Note that in this case we have Al-Oboudi's differential operator. □

4. The class $(\nu,k)_{\alpha,\beta,\lambda} - \mathcal{S}$

In this section, we determine coefficient bounds for functions of the form (1.1), with positive and negative coefficients, in the class $(\nu,k)_{\alpha,\beta,\lambda} - \mathcal{S}$. This study is required in the next section.

Theorem 5. *A sufficient condition for a function $f(z)$ of the form (1.1) to be in class $(\nu,k)_{\alpha,\beta,\lambda} - \mathcal{S}$ is*

$$\sum_{n=2}^{\infty} |a_n| [(\Psi_{n,k}(\alpha,\beta,\lambda) - 1)(1+\nu) + 1] < 1, \tag{4.1}$$

where $\nu \in [0,\infty)$ and $0 \le \alpha < 1, \beta > 0, \lambda > 0$.

Proof. It suffices to show that

$$\nu \left| \frac{D^{k,\alpha}_{\beta,\lambda} f(z)}{f(z)} - 1 \right| - \Re\left(\frac{D^{k,\alpha}_{\beta,\lambda} f(z)}{f(z)} - 1 \right) \le 1.$$

We obtain

$$\begin{aligned} \nu \left| \frac{D^{k,\alpha}_{\beta,\lambda} f(z)}{f(z)} - 1 \right| - \Re\left(\frac{D^{k,\alpha}_{\beta,\lambda} f(z)}{f(z)} - 1 \right) &\le (1+\nu) \left| \frac{D^{k,\alpha}_{\beta,\lambda} f(z)}{f(z)} - 1 \right| \\ &\le \frac{(1+\nu) \sum_{n=2}^{\infty} |a_n| (\Psi_{n,k}(\alpha,\beta,\lambda) - 1) |z|^{n-1}}{1 - \sum_{n=2}^{\infty} |a_n| |z|^{n-1}} \end{aligned}$$

$$\leq \frac{(1+\nu)\sum_{n=2}^{\infty}|a_n|(\Psi_{n,k}(\alpha,\beta,\lambda)-1)}{1-\sum_{n=2}^{\infty}|a_n|}.$$

This last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty}|a_n|[(\Psi_{n,k}(\alpha,\beta,\lambda)-1)(1+\nu)+1]<1,$$

and the proof is complete. □

Now we prove that the condition (4.1) is also necessary for f with negative coefficients. Let $\mathcal{T}$ be the class of all analytic functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n,\ (a_n\geq 0,\ z\in U).$$

Then we have the following result.

Theorem 6. *A sufficient and necessary condition for $f\in\mathcal{T}$ to be in the class $(\nu,k)_{\alpha,\beta,\lambda}-\mathcal{TS}=(\nu,k)_{\alpha,\beta,\lambda}-\mathcal{S}\cap\mathcal{T}$ (here $\nu\in[0,\infty)$ and $0\leq\alpha<1,\beta>0,\lambda>0$), is that*

$$\sum_{n=2}^{\infty}a_n[(\Psi_{n,k}(\alpha,\beta,\lambda)-1)(1+\nu)+1]<1.$$

Proof. In view of Theorem 1, we need to prove only the necessity. If $f\in(\nu,k)_{\alpha,\beta,\lambda}-\mathcal{TS}$ and z is real, then

$$\begin{aligned}1>\nu\left|\frac{D^{k,\alpha}_{\beta,\lambda}f(z)}{f(z)}-1\right|-\Re\left(\frac{D^{k,\alpha}_{\beta,\lambda}f(z)}{f(z)}-1\right)\\ \geq-(1+\nu)\Re\left(\frac{D^{k,\alpha}_{\beta,\lambda}f(z)}{f(z)}-1\right)\\ =-(1+\nu)\Re\left(\frac{-\sum_{n=2}^{\infty}a_n(\Psi_{n,k}(\alpha,\beta,\lambda)-1)z^{n-1}}{1-\sum_{n=2}^{\infty}a_n z^{n-1}}\right).\end{aligned}$$

Letting $z\to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}a_n[(\Psi_{n,k}(\alpha,\beta,\lambda)-1)(1+\nu)+1]<1.$$

□

Corollary 8. *The extreme points of $(\nu,k)_{\alpha,\beta,\lambda}-\mathcal{TS}$ are the functions given by*

$$f_1(z)=1,\ \text{and}\ f_n(z)=z-\frac{1}{[(\Psi_{n,k}(\alpha,\beta,\lambda)-1)(1+\nu)+1]}z^n,$$

where $n=2,3,\ldots,\nu\in[0,\infty)$ and $0\leq\alpha<1,\beta>0,\lambda>0$.

5. Cesáro sum

Recently, Silverman [14] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums:

$$\Re\Big\{\frac{f(z)}{f_N(z)}\Big\}, \Re\Big\{\frac{f_N(z)}{f(z)}\Big\}, \Re\Big\{\frac{f'(z)}{f'_N(z)}\Big\}, \text{ and } \Re\Big\{\frac{f'_N(z)}{f'(z)}\Big\}.$$

In this section, we consider the Cesáro sum σ_m of functions in the class $(\nu,k)_{\alpha,\beta,\lambda}-\mathcal{S}$ and obtain sharp lower bounds for the ratios of real part of $f(z)$ and σ_m, and also of $f'(z)$ and σ'_m. Let us first construct the Cesáro sum $\sigma_m(z)$ of functions $f\in\mathcal{A}$ by

$$\begin{aligned}
\sigma_m(z) &= \frac{1}{m+1}\sum_{n=0}^{m} s_n(z)\\
&= \frac{1}{m+1}\big[s_0(z)+s_1(z)+..+s_m(z)\big]\\
&= \frac{1}{m+1}\big[a_0+(a_0+a_1z)+..+(a_0+...+a_mz^m)\big]\\
&= \frac{1}{m+1}\big[(m+1)a_0+ma_1z+..+a_mz^m\big] \qquad (5.1)\\
&= \sum_{n=0}^{m}\Big(\frac{m-n+1}{m+1}\Big)a_mz^n\\
&= f(z)*\sum_{n=0}^{m}\Big(\frac{m-n+1}{m+1}\Big)z^n\\
&:= f(z)*g_m(z)
\end{aligned}$$

where $s_0(z)=0$, $s_1(z)=z$. Note that the classical Cesáro means play an important role in geometric function theory (see [6, 12, 13]). In the sequel, we will make use of the result that $\Re\Big\{\frac{(1+w(z))}{(1-w(z))}\Big\}>0$, $(z\in U)$ if and only if $w(z)=\sum_{n=1}^{\infty}c_nz^n$ satisfies the inequality $|w(z)|<|z|$.

Theorem 7. *Let the function $f(z)$ of the form (1.1) satisfies condition (4.1), then*

$$\Re\{\frac{f(z)}{\sigma_m(z)}\}\geq\frac{(\Psi_{m+1,k}(\alpha,\beta,\lambda)-1)(1+\nu)}{(\Psi_{m+1,k}(\alpha,\beta,\lambda)-1)(1+\nu)+1},\ (z\in U),$$

where $\sigma_m(z)$ defined in (5.1).

Proof. Assume that $f\in\mathcal{A}$ and satisfies (4.1). By setting

$$d_m:=[(\Psi_{m,k}(\alpha,\beta,\lambda)-1)(1+\nu)+1],\quad C(n,m):=\frac{m-n+1}{m+1}$$

and

$$w(z) = d_{m+1}\Big\{\frac{f(z)}{\sigma_m(z)} - \Big(1 - \frac{1}{d_{m+1}}\Big)\Big\}$$
$$= 1 + \frac{d_{m+1}[\sum_{n=2}^{\infty} a_n z^{n-1} - \sum_{n=2}^{m} C(n,m) a_n z^{n-1}]}{1 + \sum_{n=2}^{m} C(n,m) a_n z^{n-1}}$$

when $C(n,m) \to 1$, we find that

$$\Big|\frac{w(z)-1}{w(z)+1}\Big| \le \frac{d_{m+1}\sum_{n=m+1}^{\infty}|a_n|}{2 - 2\sum_{n=2}^{m}|a_n| - d_{m+1}\sum_{n=m+1}^{\infty}|a_n|}$$
$$\le 1, \quad (z \in U)$$

if and only if

$$2d_{m+1}\sum_{n=m+1}^{\infty}|a_n| \le 2 - 2\sum_{n=2}^{m}|a_n|$$

which is equivalent to

$$\sum_{n=2}^{m}|a_n| + d_{m+1}\sum_{n=m+1}^{\infty}|a_n| \le 1. \tag{5.2}$$

In order to see that

$$f(z) = z + \frac{z^{m+1}}{d_{m+1}}, \quad (z \in U)$$

gives sharp result, we observe that for $z = re^{\frac{\pi i}{m}}$

$$\frac{f(z)}{\sigma_m(z)} = 1 + \frac{z^m}{d_{m+1}} \to 1 - \frac{1}{d_{m+1}} \quad as \quad z \to 1^-.$$

Hence the proof is complete. □

In the same manner of Theorem 7, we can verify the following result

Theorem 8. *Let the function $f(z)$ of the form (1.1) satisfies condition (4.1), then*

$$\Re\{\frac{f'(z)}{\sigma_m'(z)}\} \ge \frac{(\Psi_{m+1,k}(\alpha,\beta,\lambda)-1)(1+\nu) - m}{(\Psi_{m+1,k}(\alpha,\beta,\lambda)-1)(1+\nu) + 1}, \quad (z \in U),$$

where $\sigma_m(z)$ defined in (5.1).

Proof. Assume that $f \in \mathcal{A}$ and satisfies (4.1). By letting

$$w(z) = d_{m+1}\Big\{\frac{f'(z)}{\sigma_m'(z)} - \Big(1 - \frac{m+1}{d_{m+1}}\Big)\Big\}$$
$$= 1 + \frac{\frac{m+1}{d_{m+1}}[\sum_{n=2}^{\infty} n a_n z^{n-1} - \sum_{n=2}^{m} C(n,m) n a_n z^{n-1}]}{1 + \sum_{n=2}^{m} C(n,m) n a_n z^{n-1}},$$

where d_m is defined in Theorem 7. □

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A NOTE ON COMMON FIXED POINT THEOREMS IN PARTIAL METRIC SPACES

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Abstract. In this manuscript, we consider the notion of generalized Sehgal contraction condition in a partial metric space. For the pair of two self mappings (S,T) which satisfies Sehgal contraction condition, we obtain a unique common fixed point.

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1. Introduction and preliminaries

Partial metric space (in short PMS), is one of the attempts to generalize the notion of the metric space that by replacing the condition $d(x,x)=0$ with the condition $d(x,x)\leq d(x,y)$ for all x,y in the definition of the metric (see e.g.[14, 15]). In these initial papers, Matthews discussed not only the general topological properties of partial metric spaces but also some properties of convergence of sequences. In [14, 15], he proved a fixed point theorem for contractive mappings of partial metric spaces: Any mapping T of a complete partial metric space X into itself that satisfies, for some $0\leq k<1$, the inequality $d(Tx,Ty)\leq kd(x,y)$, for all $x,y\in X$, has a unique fixed point. Recently, many authors (see e.g.[1, 2, 8, 10–17, 20]) focused on this subject and generalized some fixed point theorems from the class of metric spaces to the class of partial metric spaces. In this manuscript, we discuss existence and uniqueness of a common fixed point of self-mappings S,T of partial metric spaces.

A partial metric space (See e.g.[14, 15]) is a pair $(X,p:X\times X\to\mathbb{R}^+)$ (where $\mathbb{R}^+$ denotes the set of all non negative real numbers) such that

(PM1) $p(x,y)=p(y,x)$ (symmetry)
(PM2) If $0\leq p(x,x)=p(x,y)=p(y,y)$ then $x=y$ (equality)
(PM3) $p(x,x)\leq p(x,y)$ (small self-distances)
(PM4) $p(x,z)+p(y,y)\leq p(x,y)+p(y,z)$ (triangularity)

for all $x,y,z\in X$. For a partial metric p on X, the functions $d_p,d_m:X\times X\to\mathbb{R}^+$ given by

$$d_p(x,y)=2p(x,y)-p(x,x)-p(y,y) \tag{1.1}$$

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and

$$d_m(x,y) = \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\} \tag{1.2}$$

are (usual) metrics on X. It is clear that d_p and d_m are equivalent. Each partial metric p on X generates a T_0 topology τ_p on X with a base consisting of the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Example 1. A basic example of partial metric is $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$. It is clear that p is not a (usual) metric. Note that in this case $d_p(x,y) = |x-y|$ and $d_m(x,y) = \frac{1}{2}|x-y|$.

Example 2. (See [8]) Let $X = \{[a,b] : a,b, \in \mathbb{R},\ a \leq b\}$ and define $p([a,b],[c,d])$ $= \max\{b,d\} - \min\{a,c\}$. Then (X,p) is a partial metric spaces.

Example 3. (See [8]) Let $X := [0,1] \cup [2,3]$ and define $p : X \times X \to [0,\infty)$ by $p(x,y) = \begin{cases} \max\{x,y\} & \text{if } \{x,y\} \cap [2,3] \neq \varnothing, \\ |x-y| & \text{if } \{x,y\} \subset [0,1]. \end{cases}$
Then (X, p) is a complete partial metric space.

Definition 1. (See e.g. [14, 15])

- (i) A sequence $\{x_n\}$ in a PMS (X,p) converges to $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$,
- (ii) a sequence $\{x_n\}$ in a PMS (X,p) is called a Cauchy if and only if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and finite),
- (iii) A PMS (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.
- (iv) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0,\delta)) \subset B(f(x_0),\varepsilon)$.

Lemma 1. *(See e.g.[14, 15])*

- (A) *A sequence $\{x_n\}$ is Cauchy in a PMS (X,p) if and only if $\{x_n\}$ is Cauchy in a metric space (X,d_p),*
- (B) *A PMS (X,p) is complete if and only if a metric space (X,d_p) is complete. Moreover,*

$$\lim_{n\to\infty} d_p(x,x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m) \tag{1.3}$$

Remark 1. Since d_p and d_m are equivalent, we can take d_p instead of d_m in this Lemma.

Let (X,p) be a PMS and denote the closure of the set $\{p(x,y) : x,y \in X\}$ by P and $P^3 = P \times P \times P$. A function $\phi : P^3 \to \mathbb{R}^+$ is right continuous if and only if

- (S1) the sequences $\{a_n\},\{b_n\},\{c_n\}$ decrease and converge to $a,b,c \in P$, respectively,

then

$$\phi(a_n, b_n, c_n) \to \phi(a, b, c).$$

The function ϕ is called symmetric if and only if

$$\phi(a, b, c) = \phi(b, a, c), \text{ for all } (a, b, c) \in P^3.$$

In the spirit of Sehgal [19], we state the following definition for partial metric spaces.

Definition 2. Let (X, p) be a PMS and $S, T : X \to X$ be two mappings. The pair (S, T) is said to satisfy Sehgal k-condition if and only if there are maps $I_S : S \times X \to \mathbb{Z}^+$ and $I_T : T \times X \to \mathbb{Z}^+$ such that if $r(x) = I_S(S, x)$ and $q(x) = I_T(T, x)$, then

$$p(S^{r(x)}x, T^{q(y)}y) \leq k\phi(p(S^{r(x)}x, x), p(y, T^{q(y)}y), p(x, y)) \tag{1.4}$$

for all $x, y \in X$, where $k \in \mathbb{R}$ and ϕ is a symmetric right continuous. If $0 \leq k < 1$, then we say that (S, T) satisfy Sehgal contraction condition.

2. Main results

The following two lemmas are easy to prove but they will be very useful in the proof of the main theorem.

Lemma 2. *(See e.g. [1, 10]) Let (X, p) be a complete PMS. Then*

(A) *If $p(x, y) = 0$ then $x = y$,*
(B) *If $x \neq y$, then $p(x, y) > 0$.*

Lemma 3. *(See e.g. [1, 10]) Assume $x_n \to z$ as $n \to \infty$ in a PMS (X, p) such that $p(z, z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

The following theorem extends the results of [19].

Theorem 1. *Let (X, p) be a complete partial metric space. Suppose that $S, T : X \to X$ are two mappings such that the pair (S, T) satisfies Sehgal's contraction contraction.*

(A) If $\phi(a, b, c) \leq \max\{a, b, c\}$, for $(a, b, c) \in P^3$, then S and T have a unique common fixed point in X, that is, $S^{r(z)}z = T^{q(z)}z = z$.

Proof. Let $x_0 \in X$. Define the sequence $\{x_n\}_{n=1}^{\infty}$ in a way that $x_2 = T^{q(x_1)}x_1$ and $x_1 = S^{r(x_0)}x_0$ and inductively

$$x_{2n+2} = T^{q(x_{2n+1})}x_{2n+1} \quad \text{and} \quad x_{2n+1} = S^{r(x_{2n})}x_{2n} \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

If n is odd, due to (1.4), we have

$$p(x_{n+1}, x_{n+2}) = p(Tx_n, Sx_{n+1}) \leq k\phi(p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), p(x_n, x_{n+1})) \tag{2.1}$$

Regarding the assumption of (A),

$$p(x_{n+1},x_{n+2}) = p(Tx_n,Sx_{n+1}) \leq k\max\{p(x_n,x_{n+1}),p(x_{n+1},x_{n+2})\} \quad (2.2)$$

If $\max\{p(x_n,x_{n+1}),p(x_{n+1},x_{n+2})\} = p(x_{n+1},x_{n+2})$ then the expression (2.2) turns into

$$p(x_{n+1},x_{n+2}) \leq kp(x_{n+1},x_{n+2}).$$

Since $k<1$, this is impossible. Thus, we have

$$p(x_{n+1},x_{n+2}) \leq kp(x_n,x_{n+1}). \quad (2.3)$$

If n is even, analogously we observe that $p(x_{n+1},x_{n+2}) \leq kp(x_n,x_{n+1})$. Observe that $\{p(x_n,x_{n+1})\}$ is a non-negative, non-increasing sequence of reals. Regarding (2.3), one can observe that

$$p(x_n,x_{n+1}) \leq k^n p(x_0,x_1),\ \forall n = 0,1,2,\cdots \quad (2.4)$$

Letting $n\to\infty$, the right hand side of (2.4) tends to zero.

Consider now

$$\begin{aligned} d_p(x_{n+1},x_{n+2}) &= 2p(x_{n+1},x_{n+2}) - p(x_{n+1},x_{n+1}) - p(x_{n+2},x_{n+2}) \\ &\leq 2p(x_{n+1},x_{n+2}) \\ &\leq 2k^{n+1}p(x_0,x_1). \end{aligned} \quad (2.5)$$

Hence, regarding (2.4), we have $\lim_{n\to\infty} d_p(x_{n+1},x_{n+2}) = 0$. Moreover,

$$\begin{aligned} d_p(x_{n+1},x_{n+s}) &\leq d_p(x_{n+s-1},x_{n+s}) + \cdots + d_p(x_{n+1},x_{n+2}) \\ &\leq 2k^{n+s}p(x_0,x_1) + \cdots + 2k^{n+1}p(x_0,x_1) \end{aligned} \quad (2.6)$$

which implies that $\{x_n\}$ is a Cauchy sequence in (X,d_p) that is, $d_p(x_n,x_m)\to 0$. Since (X,p) is complete, by Lemma 1.3, (X,d_p) is complete and the sequence $\{x_n\}$ is convergent in (X,d_p), say to $z\in X$.

By Lemma 1.3,

$$p(z,z) = \lim_{n\to\infty} p(x_n,z) = \lim_{n,m\to\infty} p(x_n,x_m) \quad (2.7)$$

Since $\{x_n\}$ is a Cauchy sequence in (X,d_p), we have $\lim_{n,m\to\infty} d_p(x_n,x_m) = 0$. Since

$$\max\{p(x_n,x_n),p(x_{n+1},x_{n+1})\} \leq p(x_n,x_{n+1}),$$

then by (2.4), it follows that

$$\max\{p(x_n,x_n),p(x_{n+1},x_{n+1})\} \leq k^{n+1}p(x_0,x_1) \quad (2.8)$$

Thus from (2.4), (2.8) and from the definition of d_p, we have $\lim_{n,m\to\infty} p(x_n,x_m) = 0$. Therefore from (2.7) we have

$$p(z,z) = \lim_{n\to\infty} p(x_n,z) = \lim_{n,m\to\infty} p(x_n,x_m) = 0. \quad (2.9)$$

We assert that $T^{q(z)}z = z$. Assume $T^{q(z)}z \neq z$, then $p(z, T^{q(z)}z) > 0$. Let $\{x_{2n(i)}\}$ be subsequence of $\{x_{2n}\}$ and hence $\{x_n\}$. Due to (PM4), we have

$$\begin{aligned} p(Sx_{2n(i)}, T^{q(z)}z) &= p(Sx_{2n(i)}, T^{q(z)}z) \\ &\leq k\phi(p(x_{2n(i)}, x_{2n(i)+1}), p(T^{q(z)}z, z), p(x_{2n(i)}, z) \end{aligned} \tag{2.10}$$

Letting $n \to \infty$ and taking the assumption of (A) and (2.9) into account, we get that

$$p(z, T^{q(z)}z) \quad \leq k\phi(0, p(T^{q(z)}z, z), 0) \leq kp(T^{q(z)}z, z) \tag{2.11}$$

Since $k < 1$, then $p(T^{q(z)}z, z) = 0$. By Lemma 2, we get $T^{q(z)}z = z$. Analogously, if we choose a subsequence $\{x_{2n(i)+1}\}$ be subsequence of $\{x_{2n+1}\}$, we obtain $S^{r(z)}z = z$.

Assume now that there exists $w \in X$ such that $S^{r(w)} = T^{q(w)}w = w$. By (PM3)

$$p(z, z) \quad \leq p(z, w) \text{ and } \quad p(w, w) \quad \leq p(z, w) \tag{2.12}$$

Regarding that the function ϕ satisfies the condition of (A) with (2.12), we get

$$\begin{aligned} p(z, w) = p(S^{r(z)}z, T^{q(w)}w) \quad &\leq k\phi(p(z, S^{r(z)}z), p(T^{q(w)}w, w), p(z, w)) \\ &\leq k\phi(p(z, z), p(w, w), p(z, w)) \\ &\leq kp(z, w) \end{aligned}$$

Since $k < 1$, it yields a contradiction.

Thus, $p(z, w) = 0$ and by Lemma 2 we have $z = w$. □

Corollary 1. *Let (X, p) be a complete partial metric space. Suppose I_T and I_S are defined as above. $S, T : X \to X$ are two mappings such that the pair (S, T) satisfies one of the following condition:*

- (A) $p(S^{r(x)}x, T^{q(y)}y) \leq k \max\{p(S^{r(x)}x, x), p(y, T^{q(y)}y), p(x, y)\}$ *for some* $0 \leq k < 1$,
- (B) $p(S^{r(x)}x, T^{q(y)}y) \leq \alpha p(S^{r(x)}x, x) + \beta p(y, T^{q(y)}y) + \gamma p(x, y)$ *for some non-negative reals* α, β, γ *with* $\alpha + \beta + \gamma < 1$.

Then S and T have a unique common fixed point in X, that is, $S^{r(z)}z = T^{q(z)}z = z$.

Proof. For (A), we choose a function $\phi(a, b, c) = \max\{a, b, c\}$ as in Theorem 1. In case of (B), set $k = \alpha + \beta + \gamma$. Then (A) implies (B). □

Notice that this corollary generalizes also some results of ([6] -[4]).

Corollary 2. *Let (X, p) be a complete partial metric space. Let $S, T : X \to X$ be two mappings such that the pair (S, T) satisfies the following condition:*

$$p(S^r x, T^q y) \leq k\phi(p(S^r x, x), p(y, T^q y), p(x, y)) \tag{2.13}$$

for all $x, y \in X$ where $0 \leq k < 1$ and ϕ is symmetric right-continuous. If $\phi(a, b, c) \leq \max\{a, b, c\}$ then S and T have a unique common fixed point theorem.

Proof. By Theorem 1,and by taking the maps I_T, I_S as a constant, we get that S^r and T^q have a unique common fixed point, say $z \in X$. Now consider

$$S^q(Sz) = S^{q+1}z = S(S^q z) = Sz$$

which says that Sz is a fixed point of S^q. Since z is the unique fixed point of S^q, then $Sz = z$. Analogously, one can get $Tz = z$. □

Corollary 3. *Let (X, p) be a complete partial metric space. Let $S, T : X \to X$ be two mappings such that the pair (S, T) satisfies the following condition:*

$$p(S^r x, T^q y) \leq k \max\{p(S^r x, x), p(y, T^q y), p(x, y)\} \tag{2.14}$$

for all $x, y \in X$ where $0 \leq k < 1$ and ϕ is symmetric right-continuous. If $\phi(a,b,c) \leq \max\{a,b,c\}$ then S and T have a unique common fixed point theorem.

Corollary 4. *Let (X, p) be a complete partial metric space. Let $S, T : X \to X$ be two mappings such that the pair (S, T) satisfies the following condition:*

$$p(S^r x, T^q y) \leq \alpha p(S^r x, x) + \beta p(y, T^q y) + \gamma p(x, y) \tag{2.15}$$

for all $x, y \in X$, where for some non-negative reals α, β, γ with $\alpha + \beta + \gamma < 1$. $0 \leq k < 1$ and ϕ is symmetric right-continuous. If $\phi(a,b,c) \leq \max\{a,b,c\}$ then S and T have a unique common fixed point theorem.

Remark 2. Consider Corollary 4 and take $S = T$.

(1) If we set $r = q$ in (2.15) then we get Reich type fixed point theorem (See e.g. [2, 18]).
(2) If we set $r = q = 1$ and $\gamma = 0$ in (2.15) we get Kannan type fixed point theorem (See e.g. [2, 9])
(3) If we set $r = q = 1$ and $\alpha = \beta = 0$ in (2.15) we get Banach type fixed point theorem (See e.g. [2, 3, 15, 17] and [4–7])

Example 4. Let $X = [0, 1]$ and $p(x, y) = \max\{x, y\}$. It is clear that (X, p) is a partial metric spaces but not a metric. Suppose that $Sx = Tx = \frac{x}{2}$ and I_S, I_T are constant mappings, such as $r(x) = 2 = q(y)$. Take $\phi(a,b,c) = \frac{1}{3}[a + b + c]$. Let $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. For $\frac{k}{3}$ the condition of Corollary 2 is satisfied. Clearly, 0 is the common fixed point of S, T.

Example 5. Let $X = [1, 15]$ and $p(x, y) = \max\{x, y\}$. Here (X, p) is a complete metric spaces. Define the self-mappings $S, T : X \to X$ as $Tx = \frac{x^2}{1+x}$ and $Sx = \begin{cases} \frac{x}{1+x} & \text{if } 1 < x \leq 15 \\ 0 & \text{if } 0 \leq x \leq 1 \end{cases}$. Set $\phi(a,b,c) = \frac{19}{20}\max\{x, y\}$. Without loss of generality, assume $y < x$. Thus, $p(Tx, x) = x$, $p(x, y) = x$, $p(Sy, y) = y$ and $p(Tx, Sy) = \frac{x^2}{1+x}$. Clearly, $p(Tx, Sy) = \frac{x^2}{1+x} \leq \phi(x, y, x) = \frac{19}{20}x$. Hence, it satisfies the conditions of Corollary 2.14 for $r = 1$ and $q = 1$, and 0 is the unique common fixed point of S and T.

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NOTES ON THE COMMUTATIVITY OF PRIME NEAR-RINGS

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Abstract. Let N be a 3−prime right near-ring and let f be a generalized $(\theta,\theta)-$ derivation on N with associated $(\theta,\theta)-$derivation d. It is proved that N must be a commutative ring if $d \neq 0$ and one of the following conditions is satisfied for all $x, y \in N$: (i) $f([x,y]) = 0$; (ii) $f([x,y]) = \theta([x,y])$; (iii) $f(xoy) = 0$; (iv) $f(xoy) = \theta(xoy)$; (v) $f([x,y]) = \theta(xoy)$; (vi) $f(xoy) = \theta([x,y])$. We also prove theorems which assert that N is commutative, but not necessarily a ring.

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1. INTRODUCTION

An additively written group $(N,+)$ equipped with a binary operation $\cdot : N \times N \to N$, $(x,y) \to xy$ such that $(xy)z = x(yz)$ and $(x+y)z = xz + yz$ for all $x, y, z \in N$ is called a right near-ring. Recall that a near-ring N is called 3−prime if for any $x, y \in N$, $xNy = 0$ implies that $x = 0$ or $y = 0$. For $x, y \in N$ the symbol $[x,y]$ will denote $xy - yx$, while the symbol xoy will denote $xy + yx$. Z is the multiplicative center of N. An additive mapping $d : N \to N$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$, or equivalently, as noted in [12], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. Recently, in [7], Bresar defined the following concept. An additive mapping $F : N \to N$ is called a generalized derivation if there exists a derivation $d : N \to N$ such that

$$F(xy) = F(x)y + xd(y), \text{ for all } x, y \in N.$$

Basic examples are derivations and generalized inner derivations (i.e., maps of type $x \to ax + xb$ for some $a, b \in N$). One may observe that the concept of generalized derivations includes the concept of derivations and of left multipliers (i.e., $F(xy) = F(x)y$, for all $x, y \in N$).

Inspired by the definition of derivation (resp. generalized derivation), we define the notion of $(\theta,\phi)-$derivation (resp. generalized $(\theta,\phi)-$derivation) as follows: Let θ, ϕ be two near-ring automorphisms of N. An additive mapping $d : N \to N$ is called a $(\theta,\phi)-$derivation (resp. generalized $(\theta,\phi)-$derivation) if $d(xy) = \phi(x)d(y) +$

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$d(x)\theta(y)$ (resp. $f(xy) = f(x)\theta(y) + \phi(x)d(y)$, where d is a $(\theta,\phi)-$derivation) holds for all $x, y \in N$. It is noted that $d(xy) = d(x)\theta(y) + \phi(x)d(y)$, for all $x, y \in N$ in [9, Lemma 1]. Of course a $(1,1)-$derivation (resp. generalized $(1,1)-$derivation) is a derivation (resp. generalized derivation) on N, where 1 is the identity on N.

Many authors have investigated the properties of derivations of prime and semiprime rings. The study of derivations of near-rings was initiated by H. E. Bell and G. Mason in 1987 [5]. Some recent results on rings deal with commutativity on prime and semiprime rings admitting suitably constrained derivations. It is natural to look for comparable results on near-rings and this has been done in [3], [5], [6], [4], [2], [9].

In [8], Daif and Bell showed that the ideal I of a semiprime ring is contained in the center of R if

$$d([x,y]) = [x,y] \text{ for all } x, y \in I \text{ or } d([x,y]) = -[x,y] \text{ for all } x, y \in I. \tag{1.1}$$

Several authors have obtained commutativity results for prime or semiprime rings admitting derivations or generalized derivations d satisfying (1.1) or similar conditions (see [1], [11], [10]). The first purpose of this paper is to show that $3-$prime near-rings must be commutative rings if they admit appropriate generalized $(\theta,\theta)-$derivations satisfying conditions related to (1.1). The second aim is to prove some commutativity theorems for $3-$prime near-rings with $(\theta,\theta)-$derivations.

2. Results on generalized $(\theta,\theta)-$derivations

Lemma 1. *[9, Theorem 2] Let N be a $3-$prime near-ring admitting a non trivial $(\sigma,\tau)-$derivation d. If $d(N) \subset Z$, then $(N,+)$ is abelian. Moreover, if N is 2-torsion free and σ,τ commute with d, then N is a commutative ring.*

Theorem 1. *Let N be a 2-torsion free $3-$prime near-ring, (f,d) a generalized $(\theta,\theta)-$ derivation of N and $d\theta = \theta d$. If $f([x,y]) = 0$ for all $x, y \in N$ and $d \neq 0$, then N is a commutative ring.*

Proof. By the hyphothesis, we have

$$f([x,y]) = 0, \text{ for all } x, y \in N. \tag{2.1}$$

Replacing y by yx in (2.1) and using $[x,yx] = [x,y]x$, we obtain that

$$f([x,y])\theta(x) + \theta([x,y])d(x) = 0, \text{ for all } x, y \in N.$$

By (2.1), we get

$$\theta([x,y])d(x) = 0, \text{ for all } x, y \in N,$$

and so

$$\theta(x)\theta(y)d(x) = \theta(y)\theta(x)d(x), \text{ for all } x, y \in N. \tag{2.2}$$

Taking zy, $z \in N$ instead of y in (2.2) and using (2.2), we arrive at

$$\theta([x,z])\theta(y)d(x) = 0, \text{ for all } x, y, z \in N.$$

Since θ is an automorphism of N, we have

$$\theta([x,z])Nd(x)=0, \text{ for all } x,z\in N.$$

By the primeness of N, we get either $\theta([x,z])=0$ or $d(x)=0$ for each $x\in N$. Again using $\theta\in AutN$, we conclude that

$$x\in Z \text{ or } d(x)=0 \text{ for each } x\in N.$$

If $x\in Z$, then $d(x)\in Z$. Indeed, for all $y\in N$, we get

$$xy=yx,$$

and so

$$d(xy)=d(yx), \text{ for all } y\in N,$$

$$\theta(x)d(y)+d(x)\theta(y)=d(y)\theta(x)+\theta(y)d(x), \text{ for all } y\in N.$$

Using $x\in Z$ in this equation, we obtain that

$$d(x)\theta(y)=\theta(y)d(x), \text{ for all } y\in N$$

and so

$$d(x)y=yd(x), \text{ for all } y\in N.$$

Thus $d(x)\in Z$, for all $x\in N$.

By Lemma 1, we conclude that N is a commutative ring. This completes the proof. □

Theorem 2. *Let N be a 2-torsion free 3−prime near-ring, (f,d) a generalized $(\theta,\theta)-$ derivation of N and $d\theta=\theta d$. If $f([x,y])=\pm\theta([x,y])$ for all $x,y\in N$ and $d\neq 0$, then N is a commutative ring.*

Proof. Replacing y by yx in the hypothesis yields that

$$f([x,y]x)=\pm\theta([x,y]x), \text{ for all } x,y\in N,$$

and so

$$f([x,y])\theta(x)+\theta([x,y])d(x)=\pm\theta([x,y])\theta(x), \text{ for all } x,y\in N.$$

Using our hypothesis, the above relation yields that

$$\pm\theta([x,y])\theta(x)+\theta([x,y])d(x)=\pm\theta([x,y])\theta(x), \text{ for all } x,y\in N,$$

and so

$$\theta([x,y])d(x)=0, \text{ for all } x,y\in N.$$

Arguing in the similar manner as we have done in the proof of Theorem 1, we find N is a commutative ring. □

Theorem 3. *Let N be a 2-torsion free 3−prime near-ring, (f,d) a generalized $(\theta,\theta)-$ derivation of N and $d\theta=\theta d$. If $f(xoy)=0$ for all $x,y\in N$ and $d\neq 0$, then N is a commutative ring.*

Proof. Assume that

$$f(xoy) = 0, \text{ for all } x, y \in N. \tag{2.3}$$

Substituting yx for y in (2.3), we get

$$f(xoy)\theta(x) + \theta(xoy)d(x) = 0, \text{ for all } x, y \in N.$$

By (2.3), we obtain that

$$\theta(xoy)d(x) = 0, \text{ for all } x, y \in N,$$

and so

$$\theta(x)\theta(y)d(x) = -\theta(y)\theta(x)d(x), \text{ for all } x, y \in N.$$

Taking zy, $z \in N$ instead of y in this relation and using this equation, we have

$$\theta(x)\theta(z)\theta(y)d(x) = \theta(z)\theta(x)\theta(y)d(x), \text{ for all } x, y, z \in N,$$

and so

$$\theta([x,z])\theta(y)d(x) = 0, \text{ for all } x, y, z \in N.$$

Applying the same techniques in the proof of Theorem 1, we conclude that N is a commutative ring. □

Theorem 4. *Let N be a 2-torsion free 3−prime near-ring, (f,d) a generalized $(\theta,\theta)-$ derivation of N and $d\theta = \theta d$. If $f(xoy) = \pm\theta(xoy)$ for all $x, y \in N$ and $d \neq 0$, then N is commutative ring.*

Proof. We have

$$f(xoy) = \pm\theta(xoy), \text{ for all } x, y \in N. \tag{2.4}$$

Substituting yx for y in (2.4), we obtain that

$$f(xoy)\theta(x) + \theta(xoy)d(x) = \pm\theta(xoy)\theta(x), \text{ for all } x, y \in N.$$

Using (2.4), we get

$$\theta(xoy)d(x) = 0, \text{ for all } x, y \in N.$$

Replacing y by zy in the above relation, we arrive at

$$\theta([x,z])\theta(y)d(x) = 0, \text{ for all } x, y, z \in N.$$

Again using the same arguments in the proof of Theorem 1, we find the required result. □

Theorem 5. *Let N be a 2-torsion free 3−prime near-ring, (f,d) a generalized $(\theta,\theta)-$ derivation of N and $d\theta = \theta d$. If $f([x,y]) = \pm\theta(xoy)$, for all $x, y \in N$ and $d \neq 0$, then N is a commutative ring.*

Proof. Writing yx by y in the hypothesis, we have

$$f([x,y])\theta(x)+\theta([x,y])d(x)=\pm\theta(xoy)\theta(x),\ \text{for all } x,y\in N,$$

and so

$$\pm\theta(xoy)\theta(x)+\theta([x,y])d(x)=\pm\theta(xoy)\theta(x),\ \text{for all } x,y\in N.$$

That is

$$\theta([x,y])d(x)=0,\ \text{for all } x,y\in N.$$

Using the same arguments in the proof of Theorem 1, we arrive at the required result. □

Theorem 6. *Let N be a 2-torsion free 3−prime near-ring, (f,d) a generalized $(\theta,\theta)-$ derivation of N and $d\theta=\theta d$. If $f(xoy)=\pm\theta([x,y])$ for all $x,y\in N$ and $d\neq 0$, then N is a commutative ring.*

Proof. Suppose that

$$f(xoy)=\pm\theta([x,y]),\ \text{for all } x,y\in N.$$

Replacing y by yx in this equation gives that

$$f(xoy)\theta(x)+\theta(xoy)d(x)=\pm\theta([x,y])\theta(x),\ \text{for all } x,y\in N.$$

By the hypothesis, we have

$$\pm\theta([x,y])\theta(x)+\theta(xoy)d(x)=\pm\theta([x,y])\theta(x),\ \text{for all } x,y\in N,$$

and so

$$\theta(xoy)d(x)=0,\ \text{for all } x,y\in N.$$

Arguing in the similar manner as we have done in the proof of Theorem 3, we conclude that N is a commutative ring. □

Remark 1. Each of the above theorems yields on obvious result for $(\theta,\theta)-$derivations.

3. Results on $(\theta,\theta)-$derivations

Lemma 2. *Let N be a right near-ring, d a $(\theta,\theta)-$derivation of N and $a\in N$. Then*

$$a(d(x)\theta(y)+\theta(x)d(y))=ad(x)\theta(y)+a\theta(x)d(y),\ \textit{for all } x,y\in N.$$

Proof. Given $x,y\in N$, obtain

$$\begin{aligned} d(a(xy)) &= d(a)\theta(xy)+\theta(a)d(xy) \\ &= d(a)\theta(x)\theta(y)+\theta(a)(d(x)\theta(y)+\theta(x)d(y)). \end{aligned} \tag{3.1}$$

On the other hand,

$$\begin{aligned} d((ax)y) &= d(ax)\theta(y)+\theta(ax)d(y) \\ &= d(a)\theta(x)\theta(y)+\theta(a)d(x)\theta(y)+\theta(a)\theta(x)d(y). \end{aligned} \tag{3.2}$$

Comparing (3.1) and (3.2), we conclude that

$$\theta(a)(d(x)\theta(y)+\theta(x)d(y))=\theta(a)d(x)\theta(y)+\theta(a)\theta(x)d(y),$$

for all $x,y\in N$. Since θ is an automorphism of N, we can write this equation as

$$a(d(x)\theta(y)+\theta(x)d(y))=ad(x)\theta(y)+a\theta(x)d(y),\text{ for all } x,y\in N.$$

□

Theorem 7. *Let N be a 2-torsion free 3−prime near-ring, d a (θ,θ)−derivation of N. If $d(x)d(y)=\theta([x,y])$ for all $x,y\in N$, then N is commutative.*

Proof. Assume that

$$d(x)d(y)=\theta([x,y]),\text{ for all } x,y\in N. \tag{3.3}$$

Replacing y by yx in (3.3), we obtain that

$$d(x)(d(y)\theta(x)+\theta(y)d(x))=\theta([x,y])\theta(x),\text{ for all } x,y\in N.$$

By Lemma 2, we have

$$d(x)d(y)\theta(x)+d(x)\theta(y)d(x)=\theta([x,y])\theta(x),\text{ for all } x,y\in N.$$

Using equation (3.3), we find that

$$\theta([x,y])\theta(x)+d(x)\theta(y)d(x)=\theta([x,y])\theta(x),\text{ for all } x,y\in N,$$

and so

$$d(x)\theta(y)d(x)=0,\text{ for all } x,y\in N.$$

Since θ is an automorphism of N, we get

$$d(x)Nd(x)=0,\text{ for all } x\in N.$$

By the primeness of N, we arrive at $d(x)=0$, for all $x\in N$. If $d=0$, then we have $\theta([x,y])=0$ for all $x,y\in N$ by the hypothesis, and so N is commutative. □

Theorem 8. *Let N be a 2-torsion free 3−prime near-ring, d a (θ,θ)−derivation of N. If $d(x)d(y)=\theta(xoy)$ for all $x,y\in N$, then N is commutative.*

Proof. Replacing y by yx in the hypothesis, we have

$$d(x)(d(y)\theta(x)+\theta(y)d(x))=\theta(xoy)\theta(x),\text{ for all } x,y\in N.$$

By Lemma 2, we get

$$d(x)d(y)\theta(x)+d(x)\theta(y)d(x)=\theta(xoy)\theta(x),\text{ for all } x,y\in N.$$

Using the hypothesis, we obtain that

$$\theta(xoy)\theta(x)+d(x)\theta(y)d(x)=\theta(xoy)\theta(x),\text{ for all } x,y\in N.$$

and so

$$d(x)\theta(y)d(x)=0,\text{ for all } x,y\in N.$$

Since θ is an automorphism of N, we get

$$d(x) Nd(x) = 0, \text{ for all } x \in N.$$

By the primeness of N, we obtain that $d = 0$. If $d = 0$, then we have $\theta(xoy) = 0$, for all $x, y \in N$ by the hypothesis, and so $xy = -yx$, for all $x, y \in N$. Writing yz by y in this equation, we have

$$xyz = -yzx = yxz, \text{ for all } x, y, z \in N,$$

and so

$$[x, y] z = 0, \text{ for all } x, y, z \in N.$$

Since N is a 3−prime near-ring, we get $[x, y] = 0$, for all $x, y \in N$, and so, N is commutative. □

Theorem 9. *Let N be a 2-torsion free 3−prime near-ring and d, h be two $(\theta, \theta)-$ derivations. If $d(x)\theta(y) = \theta(x)h(y)$ for all $x, y \in N$, then $d = h = 0$.*

Proof. We get

$$d(x)\theta(y) = \theta(x)h(y), \text{ for all } x, y \in N. \tag{3.4}$$

Replacing y by yz, $z \in N$ in (3.4), we arrive at

$$d(x)\theta(y)\theta(z) = \theta(x)(h(y)\theta(z) + \theta(y)h(z)), \text{ for all } x, y, z \in N.$$

By Lemma 2, we have

$$d(x)\theta(y)\theta(z) = \theta(x)h(y)\theta(z) + \theta(x)\theta(y)h(z), \text{ for all } x, y, z \in N.$$

Using (3.4), we find that

$$\theta(x)h(y)\theta(z) = \theta(x)h(y)\theta(z) + \theta(x)\theta(y)h(z), \text{ for all } x, y, z \in N,$$

and so

$$\theta(x)\theta(y)h(z) = 0, \text{ for all } x, y, z \in N.$$

That is

$$\theta(x) Nh(z) = 0, \text{ for all } x, z \in N.$$

By the primeness of N gives $h = 0$. If $h = 0$, then $d(x)\theta(y) = 0$, for all $x, y \in N$ by the hypothesis. Again using the primeness of N, we get $d = 0$. This completes the proof. □

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ON WEAKLY SS-PERMUTABLE SUBGROUPS OF FINITE GROUPS

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Abstract. Suppose that G is a finite group and H is a subgroup of G. We say that: (1) H is ss-permutable in G if there is a subgroup B of G such that $G = HB$ and H permutes with every Sylow subgroup of B; (2) H is weakly ss-permutable in G if there are a subnormal subgroup T of G and an ss-permutable subgroup H_{ss} of G contained in H such that $G = HT$ and $H \cap T \leq H_{ss}$. We investigate the influence of weakly ss-permutable subgroups on the p-nilpotency and p-supersolvability of finite groups.

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1. INTRODUCTION

All groups considered in this paper are finite. A subgroup H of a group G is said to be s-permutable (or s-quasinormal) in G if H permutes with every Sylow subgroups of G [5]. In 2008, Shirong Li, etc. [7], introduced the concept of ss-permutability (or ss-quasinormality) which is a generalization of s-permutability. A subgroup H of a group G is called ss-permutable in G if there is a subgroup B of G such that $G = HB$ and H permutes with every Sylow subgroup of B. Li investigated the influence of ss-quasinormallity of some subgroups on the structure of finite groups. More recently, Xuanli He, etc. [3], introduced the following concept, which covers both ss-permutability and Skiba's weakly s-permutability [10] (A subgroup H of a group G is called weakly s-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the maximal s-permutable subgroup of G contained in H).

Definition 1. Let H be a subgroup of G. H is called weakly ss-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{ss}$, where H_{ss} is an ss-permutable subgroup of G contained in H

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In [3], Xuanli He studied the influence of weakly ss-permutable subgroups on the supersolvability of groups. In the present paper we characterize p-nilpotency and p-supersolvability of finite groups with the assumption that some maximal subgroups or 2-maximal subgroups are weakly ss-permutable.

2. PRELIMINARIES

Lemma 1 ([3], Lemma 2.2)**.** *Let U be a weakly ss-permutable subgroup of a group G and N a normal subgroup of G. Then*

(a) If $U \leq H \leq G$, then U is weakly ss-permutable in H.

(b) Suppose that U is a p-group for some prime p. If $N \leq U$, then U/N is weakly ss-permutable in G/N.

(c) Suppose U is a p-group for some prime p and N is a p'-subgroup, then UN/N is weakly ss-permutable in G/N.

(d) Suppose U is a p-group for some prime p and U is not ss-permutable in G. Then G has a normal subgroup M such that $|G : M| = p$ and $G = UM$.

(e) If $U \leq O_p(G)$ for some prime p, then U is weakly s-permutable in G.

Lemma 2. *Let p be a prime dividing the order of a group G and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and P is abelian, then G is p-nilpotent.*

Proof. Since $N_G(P)$ is p-nilpotent, $N_G(P) = P \times H$, where H is the normal p-complement of $N_G(P)$. Since P is abelian and $[P, H] = 1$, we see that $C_G(P) = P \times H = N_G(P)$. Hence G is p-nilpotent. □

Lemma 3 ([1], A, 1.2)**.** *Let U, V, and W be subgroups of a group G. Then the following statements are equivalent:*

(1) $U \cap VW = (U \cap V)(U \cap W)$.

(2) $UV \cap UW = U(V \cap W)$.

Lemma 4 ([4], VI, 4.10)**.** *Assume that A and B are two subgroups of a group G and $G \neq AB$. If $AB^g = B^g A$ holds for any $g \in G$, then either A or B is contained in a nontrivial normal subgroup of G.*

Lemma 5 ([13], Lemma 3.16)**.** *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type. Also let P be a normal p-subgroup of a group G such that $G/P \in \mathcal{F}$. If G is A_4-free and $|P| \leqslant p^2$, then $G \in \mathcal{F}$.*

Lemma 6 ([6], Lemma 2.6)**.** *Let H be a solvable normal subgroup of a group $G (H \neq 1)$. If every minimal normal subgroup of G which is contained in H is not contained in $\Phi(G)$, then the Fitting subgroup $F(H)$ of H is the direct product of minimal normal subgroups of G which are contained in H.*

Lemma 7 ([8], Lemma 2.2)**.** *If P is a s-permutable p-subgroup of G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 8 ([12], Lemma 2.8)**.** *Let M be a maximal subgroup of G and P a normal p-subgroup of G such that $G = PM$, where p is a prime. Then $P \cap M$ is a normal subgroup of G.*

3. Results

Theorem 1. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$. If $N_G(P)$ is p-nilpotent and every maximal subgroups of P is weakly ss-permutable in G, then G is p-nilpotent.*

Proof. It is easy to see that the theorem holds when $p = 2$ by [3, Theorem 3.1], so it suffices to prove the theorem for the case of odd prime. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) G is not a non-abelian simple group.

By Lemma 2, $p^3||P|$ and so there exists a non-identity maximal subgroup P_1 of P. By the hypothesis, P_1 is weakly ss-permutable in G. Then there are a subnormal subgroup T of G and an ss-permutable subgroup $(P_1)_{ss}$ of G contained in P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{ss}$. Suppose G is simple, then $T = G$ and so $P_1 = (P_1)_{ss}$ is ss-permutable in G. By [7, Lemma 2.5] and Lemma 4, G has a nontrivial normal subgroup, a contradiction.

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, we consider $G/O_{p'}(G)$. By Lemma 1, it is easy to see that every maximal subgroups of $PO_{p'}(G)/O_{p'}(G)$ is weakly ss-permutable in $G/O_{p'}(G)$. Since $N_{G/O_{p'}(G)}(PO_{p'}(G)/O_{p'}(G)) = N_G(P)O_{p'}(G)/O_{p'}(G)$ is p-nilpotent, $G/O_{p'}(G)$ satisfies all the hypotheses of our theorem. The minimality of G yields that $G/O_{p'}(G)$ is p-nilpotent, and so G is p-nilpotent, a contradiction.

(3) If M is a proper subgroup of G with $P \leq M < G$, then M is p-nilpotent.

It is clear to see $N_M(P) \leq N_G(P)$ and hence $N_M(P)$ is p-nilpotent. Applying Lemma 1, we immediately see that M satisfies the hypotheses of our theorem. Now, by the minimality of G, M is p-nilpotent.

(4) G has a unique minimal normal subgroup N such that G/N is p-nilpotent. Moreover $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G. We shall prove that G/N satisfies the hypothesis of the theorem. Since P is a Sylow p-subgroup of G, PN/N is a Sylow p-subgroup of G/N. If $|PN/N| \leqslant p^2$, then G/N is p-nilpotent by Lemma 2. So we suppose $|PN/N| \geqslant p^3$. Let M_1/N be a maximal subgroup of PN/N. Then $M_1 = N(M_1 \cap P)$. Let $P_1 = M_1 \cap P$. It follows that $P_1 \cap N = M_1 \cap P \cap N = P \cap N$ is a Sylow p-subgroup of N. Since $p = |PN/N :$

$M_1/N| = |PN : (M_1 \cap P)N| = |P : M_1 \cap P| = |P : P_1|$, P_1 is a maximal subgroup of P. By the hypothesis, P_1 is weakly ss-permutable in G, then there are a subnormal subgroup T of G and an ss-permutable subgroup $(P_1)_{ss}$ of G contained in P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{ss}$. So $G/N = P_1N/N \cdot TN/N = M_1/N \cdot TN/N$. Since $(|N : P_1 \cap N|, |N : T \cap N|) = 1$, $(P_1 \cap N)(T \cap N) = N = N \cap G = N \cap (P_1T)$. By Lemma 3, $(P_1N) \cap (TN) = (P_1 \cap T)N$. It follows that $(P_1N/N) \cap (TN/N) = (P_1N \cap TN)/N = (P_1 \cap T)N/N \leq (P_1)_{ss}N/N$. Since $(P_1)_{ss}N/N$ is ss-permutable in G/N by [7, Lemma 2.1], M_1/N is weakly ss-permutable in G/N. Since $N_{G/N}(PN/N) = N_G(P)N/N$ is p-nilpotent, we have that G/N satisfies the hypothesis of the theorem. The choice of G yields that G/N is p-nilpotent. Consequently the uniqueness of N and the fact that $\Phi(G) = 1$ are obvious.

(5) $G = PQ$ is solvable, where Q is a Sylow q-subgroup of G with $p \neq q$.

Since G is not p-nilpotent, by a result of Thompson [11, Corollary], there exists a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent. If $N_G(H) < G$, we must have $N_G(H)$ is p-nilpotent by step (3), a contradiction. We obtain $N_G(H) = G$. This leads to $O_p(G) \neq 1$. By step (4), $G/O_p(G)$ is p-nilpotent and therefore G is p-solvable. Then for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup of Q such that $G_1 = PQ$ is a subgroup of G [2, Theorem 6.3.5]. Invoking our claim (3) above, G_1 is p-nilpotent if $G_1 < G$. This leads to $Q \leq C_G(O_p(G)) \leq O_p(G)$ [9, Theorem 9.3.1], a contradiction. Thus, we have proved that $G = PQ$ is solvable.

(6) The final contradiction.

By step (4), there exists a maximal subgroup M of G such that $G = MN$ and $M \cap N = 1$. Since N is an elementary abelian p-group, $N \leq C_G(N)$ and $C_G(N) \cap M \trianglelefteq G$. By the uniqueness of N, we have $C_G(N) \cap M = 1$ and $N = C_G(N)$. But $N \leq O_p(G) \leq F(G) \leq C_G(N)$, hence $N = O_p(G) = C_G(N)$. Obviously $P = P \cap NM = N(P \cap M)$. Since $P \cap M < P$, we take a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. By our hypotheses, P_1 is weakly ss-permutable in G, then there are a subnormal subgroup T of G and an ss-permutable subgroup $(P_1)_{ss}$ of G contained in P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{ss}$. Since $|G : T|$ is a power of p and $T \lhd\lhd G$, $O^p(G) \leq T$. From the fact that N is the unique minimal normal subgroup of G, we have $N \leq O^p(G) \leq T$. Hence $N \cap P_1 = N \cap (P_1)_{ss}$. By [7, Lemma 2.5], $(P_1)_{ss}G_q = G_q(P_1)_{ss}$ for any Sylow q-subgroup G_q of G, where $q \neq p$. Since $N \cap P_1 = N \cap (P_1)_{ss} = N \cap (P_1)_{ss}G_q$, we have that $N \cap P_1$ is normalized by G_q. Obviously, $N \cap P_1 \trianglelefteq P$. Therefore, $N \cap P_1$ is normal in G. The minimality of N implies that $N \cap P_1 = 1$ or $N \cap P_1 = N$. If $N \cap P_1 = N$, then $N \leq P_1$ and so $P = NP_1 = P_1$, a contradiction. Hence we have $N \cap P_1 = 1$. Since $|N : P_1 \cap N| = |NP_1 : P_1| = |P : P_1| = p$, $P_1 \cap N$ is a maximal subgroup of N.

Therefore $|N| = p$, and so $\mathrm{Aut}(N)$ is a cyclic group of order $p-1$. If $q > p$, then NQ is p-nilpotent and therefore $Q \leq C_G(N) = N$, a contradiction. On the other hand, if $q < p$, then, since $N = C_G(N)$, we see that $M \cong G/N = N_G(N)/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$ and therefore M, and in particular Q, is cyclic. Since Q is a cyclic group and $q < p$, we know that G is q-nilpotent and therefore P is normal in G. Hence $N_G(P) = G$ is p-nilpotent, a contradiction. □

Corollary 1. *Let p be a prime dividing the order of a group G and H a normal subgroup of G such that G/H is p-nilpotent. If $N_G(P)$ is p-nilpotent and there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is weakly ss-permutable in G, then G is p-nilpotent.*

Proof. It is clear that $N_H(P)$ is p-nilpotent and that every maximal subgroup of P is weakly ss-permutable in H. By Theorem 1, H is p-nilpotent. Now let $H_{p'}$ be the normal Hall p'-subgroup of H. Then $H_{p'}$ is normal in G. If $H_{p'} \neq 1$, then we consider $G/H_{p'}$. It is easy to see that $G/H_{p'}$ satisfies all the hypotheses of our corollary for the normal subgroup $H/H_{p'}$ of $G/H_{p'}$ by Lemma 1. Now by induction, we see that $G/H_{p'}$ is p-nilpotent and so G is p-nilpotent. Hence we may assume $H_{p'} = 1$ and therefore $H = P$ is a p-group. In this case, by our hypotheses, $N_G(P) = G$ is p-nilpotent. □

Theorem 2. *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If G is A_4-free and every 2-maximal subgroup of P is weakly ss-permutable in G, then G is p-nilpotent.*

Proof. Let P_2 be a 2-maximal subgroup of P. By our hypotheses, P_2 is weakly ss-permutable in G, then there are a subnormal subgroup T of G and an ss-permutable subgroup $(P_2)_{ss}$ of G contained in P_2 such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{ss}$. If P_2 is not ss-permutable in G, then G has a normal subgroup M such that $|G : M| = p$ by Lemma 1(d). It follows that every maximal subgroup of $P \cap M$ is weakly ss-permutable in M by Lemma 1(1). Hence we have that M is p-nilpotent by [3, Theorem 3.1]. It is easy to see that G is p-nilpotent. Now we may assume that every 2-maximal subgroup of P is ss-permutable in G. By [7, Theorem 1.7], we get that G is p-nilpotent too. □

Corollary 2. *Suppose that every 2-maximal subgroup of any Sylow subgroup of a group G is weakly ss-permutable in G. If G is A_4-free, then G is a Sylow tower group of supersolvable type.*

Proof. Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. Then every 2-maximal subgroup of P is weakly ss-permutable in G. By Theorem 2, G is p-nilpotent. Let U be the normal p-complement of G. By Lemma 1, U satisfies the hypothesis of the Corollary. It follows by induction that U, and hence G is a Sylow tower group of supersolvable type. □

Corollary 3. *Let p be the smallest prime dividing the order of a group G and G is A_4-free. Assume that H is a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every 2-maximal subgroup of P is weakly ss-permutable in G, then G is p-nilpotent.*

Proof. By Lemma 1, every 2-maximal subgroup of P is weakly ss-permutable in H. By Theorem 2, H is p-nilpotent. Now, let $H_{p'}$ be the normal p-complement of H. Then $H_{p'} \trianglelefteq G$. By using the arguments as in the proof of Corollary 1, we may assume that $H_{p'} = 1$ and $H = P$ is a p-group. Since G/H is p-nilpotent, let K/H be the normal p-complement of G/H. By Schur-Zassenhaus's theorem, there exists a Hall p'-subgroup $K_{p'}$ of K such that $K = HK_{p'}$. By Theorem 2, K is p-nilpotent and so $K = H \times K_{p'}$. Hence $K_{p'}$ is a normal p-complement of G. □

Corollary 4. *Let H be a normal subgroup of a group G such that G/H is 2-nilpotent. If there exists a Sylow 2-subgroup P of H such that every 2-maximal subgroup of P is weakly ss-permutable in G and $3 \nmid |G|$, then G is 2-nilpotent.*

Corollary 5. *Let G be a group of odd order and H a normal subgroup of G such that G/H is 3-nilpotent. If there exists a Sylow 3-subgroup P of H such that every 2-maximal subgroup of P is weakly ss-permutable in G, then G is 3-nilpotent.*

Theorem 3. *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and G is A_4-free. Then $G \in \mathcal{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every 2-maximal subgroup of any Sylow subgroup of H is weakly ss-permutable in G.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order. By Lemma 1, every 2-maximal subgroup of any Sylow subgroup of H is weakly ss-permutable in H. By Corollary 2, H is a Sylow tower group of supersolvable type. Let p be the maximal prime divisor of $|H|$ and let P be a Sylow p-subgroup of H. Then P must be a normal subgroup of G and every 2-maximal subgroup of P is weakly ss-permutable in G. It is easy to see that all 2-maximal subgroups of every Sylow subgroup of H/P are weakly ss-permutable in G/P by Lemma 1 and G/P is A_4-free. By the minimality of G, we have $G/P \in \mathcal{F}$. Let N be a minimal normal subgroup of G contained in P.

(1) $P = N$.

If $N < P$, then $(G/N)/(P/N) \cong G/P \in \mathcal{F}$. We will show that $G/N \in \mathcal{F}$. If $|P/N| \leqslant p^2$, then $G/N \in \mathcal{F}$ by Lemma 5. If $|P/N| > p^2$, then every 2-maximal subgroup of P/N is weakly ss-permutable in G/N by Lemma 1. By the minimality of G, we have $G/N \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, N is the unique minimal normal subgroup of G contained in P and $N \not\leq \Phi(G)$. By Lemma 6, it follows that $P = F(P) = N$.

(2) The final contradiction.

Since $N \trianglelefteq G$, we may take a 2-maximal N_2 of N such that $N_2 \trianglelefteq G_p$, where G_p is a Sylow p-subgroup of G. By the hypothesis, N_2 is weakly ss-permutable in G. Then there are a subnormal subgroup T of G and an ss-permutable subgroup $(N_2)_{ss}$ of G contained in N_2 such that $G = N_2T$ and $N_2 \cap T \leq (N_2)_{ss}$. Thus $G = NT$ and $N = N \cap N_2T = N_2(N \cap T)$. This implies that $N \cap T \neq 1$. But since $N \cap T$ is normal in G and N is minimal normal in G, $N \cap T = N$. It follows that $T = G$ and so $N_2 = (N_2)_{ss}$ is ss-permutable in G. By [7, Lemma 2.2], N_2 is s-permutable in G. By Lemma 7, $O^p(G) \leq N_G(N_2)$. Thus $N_2 \trianglelefteq G_pO^p(G) = G$. It follows that $N_2 = 1$ and so $|N| = p^2$. By Lemma 5, $G \in \mathcal{F}$, a contradiction. □

Theorem 4. *Let p be a prime, G be a p-solvable group. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is weakly ss-permutable in G, then G is p-supersolvable.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order.

(1) G has a unique minimal normal subgroup N such that G/N is p-supersolvable.

Let N be a minimal normal subgroup of G . Since P is the Sylow p-subgroup of G, PN/N is the Sylow p-subgroup of G/N. Let M/N be a maximal subgroup of PN/N, then $M = (M \cap P)N$. Let $P_1 = M \cap P$. Obviously, P_1 is the maximal subgroup of P. Since G is p-solvable, N is an elementary abelian p-group or p'-group. If N is a p'-group, then $M/N = P_1N/N$. If N is a p-group, then $M/N = P_1/N$. By hypothesis, P_1 is weakly ss-permutable in G and so M/N is weakly ss-permutable in G/N by Lemma 1. Hence G/N satisfies all the hypotheses of our theorem. The minimal choice of G implies that G/N is p-supersolvable. Clearly, N is the unique minimal normal subgroup of G as the class of p-supersolvable group is a formation.

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $G/O_{p'}(G) \cong (G/N)/(O_{p'}(G)/N)$ is p-supersolvable by step (1) and so G is p-supersolvable, a contradiction.

(3) The final contradiction.

Since G is p-solvable, N is an elementary abelian p-group by step (2). If N is contained in all maximal subgroups of G, then $N \leq \Phi(G)$ and so G is p-supersolvable, a contradiction. Hence there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Applying Lemma 8, we have $O_p(G) \cap M \trianglelefteq G$, so that $O_p(G) \cap M = 1$ and $N = O_p(G)$. By using the arguments as in the proof of Theorem 1, we have $|N| = p$ and so G is p-supersolvable. □

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ON THE PARAMETRIZATION OF NONLINEAR BOUNDARY VALUE PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. We consider a boundary value problem with nonlinear boundary conditions. By using a suitable parametrization, we reduce the original problem to the parametrized one containing linear boundary restrictions. We construct a numerical-analytic scheme that is suitable for the study of the solutions of the transformed boundary value problem.

2000 *Mathematics Subject Classification:* 34B15, 34B08

Keywords: parametrized boundary–value problem, successive approximations, numerical–analytic method, determining system

1. INTRODUCTION

Our aim is to show that, for some types of nonlinear boundary value problems with nonlinear boundary conditions, it is useful to introduce certain parametrization techniques.

2. PROBLEM SETTING

We consider the following nonlinear two-point boundary value problem subjected to the nonlinear boundary conditions

$$\frac{dx(t)}{dt} = f(t, x(t)), \ t \in [0, T], \ x \in \mathbb{R}^n, \tag{2.1}$$

$$g(x(0), x(T)) = 0, \tag{2.2}$$

where $f : [0, T] \times D \to \mathbb{R}^n$ and $g : D \times D \to \mathbb{R}^n$ $(n \geq 2)$ are continuous, $D \subset \mathbb{R}^n$ is a closed and bounded domain.

We have to find a continuously differentiable solution of the system of differential equations (2.1) satisfying the nonlinear boundary restrictions (2.2).

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3. Construction of an equivalent problem with linear boundary conditions

To pass to the linear boundary conditions in (2.2), we replace the values of the components of the solution (2.1), (2.2) at the points $t = 0$, $t = T$ by parameters z_1, $z_2, \ldots, z_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$:

$$\begin{array}{c} x(0) = col\,(x_1(0), x_2(0), \ldots, x_n(0)) = col\,(z_1, z_2, \ldots, z_n)\,, \\ x(T) = col\,(x_1(T), x_2(T), \ldots, x_n(T)) = col\,(\lambda_1, \lambda_2, \ldots, \lambda_n)\,. \end{array} \tag{3.1}$$

Let us rewrite the boundary conditions (2.2) in the form:

$$x(T) + g\,(x(0), x(T)) = x(T). \tag{3.2}$$

Using the parametrization (3.1), the non–linear boundary restrictions (3.2) can be written as:

$$\lambda + g\,(z, \lambda) = x(T). \tag{3.3}$$

Let us put:

$$d(z, \lambda) := \lambda + g\,(z, \lambda)\,, \tag{3.4}$$

where

$$\begin{array}{c} z := x(0) = col\,(z_1, z_2, \ldots, z_n)\,, \\ \lambda := x(T) = col\,(\lambda_1, \lambda_2, \ldots, \lambda_n)\,. \end{array} \tag{3.5}$$

Taking into account (3.4), the parametrized boundary conditions (3.3) can be written in the form:

$$x(T) = d(z, \lambda), \tag{3.6}$$

where $A \equiv 0$, $C \equiv I_n$ and I_n is an initial $(n \times n)$ matrix.

So instead of the original boundary value problem of (2.1)and (2.2), we obtain an equivalent parametrized one (2.1), (3.6).

Remark 1. The set of the solutions of the non–linear boundary value problem (2.1), (2.2) coincides with the set of the solutions of the problem (2.1), (3.6) satisfying the additional conditions (3.1).

4. Construction of successive approximations

Let us introduce the vector

$$\delta_D(f) := \frac{1}{2}\left[\max_{(t,x)\in[0,T]\times D} f(t,x) - \min_{(t,x)\in[0,T]\times D} f(t,x)\right], \tag{4.1}$$

satisfying the inequality:

$$\delta_D(f) \le \max_{(t,x)\in[0,T]\times D} |f\,(t,x)|\,.$$

The given boundary value problem (2.1), (2.2) is such that the subset

$$D_\beta := \left\{ z \in D : B\left(z, \max_{t\in[0,T]} \left| z + \frac{t}{T}\left(d(z,\lambda) - z\right)\right| \subset D, \forall \lambda \in D \right)\right\}$$

is non–empty

$$D_\beta \neq \varnothing. \tag{4.2}$$

Assume that the function $f(t,x)$ is continuous in the domain $[0,T]\times D$ and satisfies a Lipschitz condition of the form

$$|f(t,u) - f(t,v)| \leq K\,|u-v|\,, \tag{4.3}$$

for all $t \in [0,T]$, $\{u,v\} \subset D$ with some non–negative constant matrix $K = \left(k_{ij}\right)_{i,j=1}^{n}$.

Moreover, we suppose that the spectral radius $r(K)$ of the matrix K satisfies the following inequality

$$r(K) < \frac{10}{3T}. \tag{4.4}$$

Let us connect with the parametrized boundary–value problem (2.1), (3.6) the sequence of functions:

$$x_m(t,z,\lambda) := z + \int_0^t f(s, x_{m-1}(s,z,\lambda))ds - \\ - \frac{t}{T}\int_0^T f(s, x_{m-1}(s,z,\lambda))ds + \frac{t}{T}\left[d\,(z,\lambda) - z\right], \tag{4.5}$$

where $m = 1,2,3,\ldots,$

$$x_0(t,z,\lambda) = z + \frac{t}{T}\left(d\,(z,\lambda) - z\right) \in\ D_\beta, \tag{4.6}$$

$\lambda = col(\lambda_1,\lambda_2,\ldots,\lambda_n) \in D$,
$x_m\,(t,z,\lambda) = col\left(x_{m,1}\,(t,z,\lambda)\,, x_{m,2}\,(t,z,\lambda)\,,\ldots, x_{m,n}\,(t,z,\lambda)\right)$ and z,λ are parameters.

It is easy to check that the functions $x_m\,(t,z,\lambda)$ satisfy linear boundary conditions (3.6) for all $m \geq 1$, $\lambda \in D$ and $z \in D_\beta$.

The following statement establishes the convergence of the sequence (4.5) and its relation to the original boundary–value problem (2.1), (2.2).

Theorem 1. *Assume that the function $f : [0,T]\times D \to \mathbb{R}^n$ in the right side of the system of differential equations (2.1) and the parametrized boundary restrictions (3.6) satisfy conditions (4.2)–(4.4).*

Then for all fixed $\lambda \in D$, $z \in D_\beta$:

(1) *The functions of the sequence (4.5) are continuously differentiable and satisfy the parametrized boundary conditions (3.6):*

$$x_m(T,z,\lambda) = d\,(z,\lambda)\,, \tag{4.7}$$

m=1,2,3,....

(2) *The sequence of functions (4.5) for $t \in [0,T]$ converges uniformly as $m \to \infty$ to the limit function*

$$x^*(t,z,\lambda) = \lim_{m\to\infty} x_m(t,z,\lambda). \tag{4.8}$$

(3) *The limit function $x^*(t,z,\lambda)$ satisfies the initial conditions*

$$x^*(0,z,\lambda) = z,$$

and the parametrized linear boundary conditions:

$$x^*(T,z,\lambda) = \lambda + g\,(z,\lambda)\,.$$

(4) *The limit function (4.8) for all $t \in [0,T]$ is a unique continuously differentiable solution of the integral equation*

$$x(t) = z + \int_0^t f(s,x(s))ds - \frac{t}{T}\int_0^T f(s,x(s))ds + \frac{t}{T}\left[d\,(z,\lambda) - z\right], \tag{4.9}$$

i. e. it is the solution of the Cauchy problem for the modified system of differential equations:

$$\frac{dx}{dt} = f(t,x) + \Delta(z,\lambda), \tag{4.10}$$

$$x(0) = z, \tag{4.11}$$

where

$$\Delta\,(z,\lambda) := \frac{1}{T}\left[d\,(z,\lambda) - z\right] - \frac{1}{T}\int_0^T f(s,x(s))ds. \tag{4.12}$$

(5) *The following error estimation holds:*

$$\left|x^*(t,z,\lambda) - x_m(t,z,\lambda)\right| \le \frac{20}{9}t\left(1-\frac{1}{T}\right)Q^m(I_n - Q)^{-1}\delta_D(f), \tag{4.13}$$

where the matrix

$$Q := \frac{3T}{10}K. \tag{4.14}$$

Proof. We will prove that the sequence of functions (4.5) is a Cauchy sequence in the Banach space $C([0,T],\mathbb{R}^n)$.

First we show that $x_m(t,z,\lambda) \in D$, for all $(t,z,\lambda) \in [0,T] \times D_\beta \times D$, $m \in \mathbb{N}$.

Indeed, using the estimation of Lemma 2.3 from [3] (see also Lemma 3 [2] and Lemma 2 [1]):

$$\left|\int_0^t\left[f(\tau)-\frac{1}{T}\int_0^T f(s)ds\right]d\tau\right|\le\frac{1}{2}\alpha_1(t)\left[\max_{t\in[0.T]}f(t)-\min_{t\in[0,T]}f(t)\right],\quad(4.15)$$

where

$$\alpha_1(t)=2t\left(1-\frac{t}{T}\right),\ |\alpha_1(t)|\le\frac{T}{2},\ t\in[0,T],\quad(4.16)$$

relation (4.5) for $m=0$ implies that:

$$|x_1(t,z,\lambda)-x_0(t,z,\lambda)|\le\left|\int_0^t\left[f(t,z)-\frac{1}{T}\int_0^T f(s,z)ds\right]dt\right|\le$$

$$\le\alpha_1(t)\delta_D(f)\le\frac{T}{2}\delta_D(f).\quad(4.17)$$

Therefore, by virtue of (4.17), we conclude that $x_1(t,z,\lambda)\in D$ whenever $(t,z,\lambda)\in[0,T]\times D_\beta\times D$.

By induction we can easily establish that all functions (4.5) are also contained in the domain D for $\forall m=1,2,3,\ldots,t\in[0,T]$, $z\in D_\beta$, $\lambda\in D$.

Now, consider the difference of functions:

$$x_{m+1}(t,z,\lambda)-x_m(t,z,\lambda)=\int_0^t[f(s,x_m(s,z,\lambda))-f(s,x_{m-1}(s,z,\lambda))]\,ds-$$

$$-\frac{t}{T}\int_0^T[\,f(s,x_m(s,z,\lambda))-f(s,x_{m-1}(s,z,\lambda))\,]\,ds,\quad(4.18)$$

for *m=1,2,3,* ... .

and introduce the notation:

$$r_m(t,z,\lambda):=|x_m(t,z,\lambda)-x_{m-1}(t,z,\lambda)|,m=1,2,3,\ldots.$$

By virtue of the estimation (4.15) and of the Lipschitz condition (4.3), we have:

$$r_{m+1}(t,z,\lambda)\le K\left[\left(1-\frac{t}{T}\right)\int_0^t r_m(s,z,\lambda)ds+\frac{t}{T}\int_t^T r_m(s,z,\lambda)ds\right],\quad(4.19)$$

for *m=0,1,2,* ... .

According to (4.17)

$$r_1(t,z,\lambda)=|x_1(t,z,\lambda)-x_0(t,z,\lambda)|\le\alpha_1(t)\delta_D(f).\quad(4.20)$$

By virtue of the statement of Lemma 3 [1] of the form

$$\alpha_{m+1}(t)\le\frac{10}{9}\left(\frac{3}{10}T\right)^m\alpha_1(t),m=0,1,2,\ldots\quad(4.21)$$

we obtain for the sequence of functions

$$\alpha_{m+1}(t)=\left(1-\frac{t}{T}\right)\int_0^t\alpha_m(s)ds+\frac{t}{T}\int_t^T\alpha_m(s)ds,\ m=0,1,2,\ldots, \tag{4.22}$$

$$\alpha_0(t)=1,\ \alpha_1(t)=2t\left(1-\frac{t}{T}\right),$$

and the statement (4.22), from (4.19) in the case of $m=1$ follows:

$$r_2(t,z,\lambda)\le K\delta_D(f)\left[\left(1-\frac{t}{T}\right)\int_0^t\alpha_1(s)ds+\frac{t}{T}\int_t^T\alpha_1(s)ds\right]\le K\alpha_2(t)\,\delta_D(f)$$

By induction we can easily obtain

$$r_{m+1}(t,z,\lambda)\le K^m\alpha_{m+1}(t)\delta_D(f), \tag{4.23}$$

$m=0,1,2,\ldots,$
where $\alpha_{m+1}(t)$, $\alpha_m(t)$ are calculated according to (4.22), and $\delta_D(f)$ is given by (4.1).

By virtue of the estimate (4.21), from (4.23) we have

$$r_{m+1}(t,z,\lambda)\le\frac{10}{9}\alpha(t)\Big[Q^m\delta_D(f)+KQ^{m-1}\big|d\,(z,\lambda)-z\big|\Big], \tag{4.24}$$

$\forall m=1,2,3,\ldots,$ where the matrix Q is given by (4.14).

Therefore, in view of (4.24)

$$\begin{aligned}\big|x_{m+j}(t,z,\lambda)-x_m(t,z,\lambda)\big|&\le\big|x_{m+j}(t,z,\lambda)-x_{m+j-1}(t,z,\lambda)\big|+\\+\big|x_{m+j-1}(t,z,\lambda)-x_{m+j-2}(t,z,\lambda)\big|&+\ldots+|x_{m+1}(t,z,\lambda)-x_m(t,z,\lambda)|=\\=\sum_{i=1}^{j}r_{m+i}(t,z,\lambda)&\le\frac{10}{9}\alpha_1(t)\sum_{i=1}^{j}Q^{m+i}\delta_D(f)=\\=\frac{10}{9}\alpha_1(t)Q^m\sum_{i=0}^{j-1}Q^i\delta_D(f).&\end{aligned} \tag{4.25}$$

Since, due to condition (4.4), the maximum eigenvalue of the matrix Q of the form (4.14) does not exceed the unity, we have

$$\sum_{i=0}^{j-1}Q^i\le(I_n-Q)^{-1},\ \lim_{m\to\infty}Q^m=[0].$$

Therefore we can conclude from (4.25) that, according to the Cauchy criterium, the sequence $\{x_m(t,z,\lambda)\}$ of the form (4.5) uniformly converges in the domain $(t,z,\lambda)\in[0,T]\times D_\beta\times D$ to the limit function $x^*(t,z,\lambda)$. Since all functions $x_m(t,z,\lambda)$ of the sequence (4.5) satisfy the boundary conditions (3.6) for all values of the artificially introduced parameters, the limit function $x^*(t,z,\lambda)$ also satisfies these conditions.

Passing to the limit as $m \to \infty$ in equality (4.5) we show that the limit function satisfies both the integral equation (4.9) and the Cauchy problem (4.10), (4.11), where $\Delta(z,\lambda)$ is given by (4.12).

□

Consider the Cauchy problem

$$\frac{dx}{dt} = f(t,x) + \mu, t \in [0,T] \tag{4.26}$$

$$x(0) = z, \tag{4.27}$$

where $\mu = col(\mu_1, \ldots, \mu_n)$ is control parameter.

Theorem 2. *Under the conditions of Theorem 1, the solution $x = x(t,z,\lambda,\mu)$ of the initial value problem (4.26), (4.27) satisfies the boundary conditions (3.6) if and only if $x = x(t,z,\lambda,\mu)$ coincides with the limit function $x^*(t,z,\lambda,\mu)$ of the sequence (4.5). Besides*

$$\mu = \mu_{z,\lambda} = \frac{1}{T}[d(z,\lambda) - z] - \frac{1}{T}\int_0^T f(s, x^*(s,z,\lambda))ds. \tag{4.28}$$

Proof. Sufficiency. Let us suppose that $\mu_{z,\lambda}$ in the right side of the system of differential equations (4.26) is given by (4.28). By virtue of Theorem 1, the limit function (4.8) of the sequence (4.5) is the unique solution of the boundary–value problem (4.26), (3.6) for fixed values of parameters z and λ when $\mu = \mu_{z,\lambda}$. Besides the limit function $x^*(t,z,\lambda,\mu)$ satisfies initial conditions (4.27), i.e. it is a solution of the Cauchy problem (4.26), (4.27) when $\mu = \mu_{z,\lambda}$.
Necessity. Let us fix an arbitrary $\bar{\mu} \in \mathbb{R}^n$ and assume that the initial value problem (4.29), (4.27):

$$\frac{dx}{dt} = f(t,x) + \bar{\mu}, t \in [0,T] \tag{4.29}$$

has a solution $\bar{x}(t)$, that satisfies the two–point boundary conditions (3.6). Then $\bar{x}(t)$ satisfies an integral equation:

$$\bar{x}(t) = z + \int_0^t f(s, \bar{x}(s))ds + \bar{\mu}t \tag{4.30}$$

for all $t \in [0,T]$.

When $t = T$ from (4.30) we get:

$$T\bar{\mu} = \bar{x}(T) - z - \int_0^T f(s, \bar{x}(s))ds. \tag{4.31}$$

Under the assumption $\bar{x}(t)$ satisfies boundary restrictions (3.6):

$$\bar{x}(T) = d(z,\lambda), \tag{4.32}$$

and the initial condition

$$\bar{x}(0) = z.$$

Substituting (4.32) in (4.31) we get:

$$\bar{\mu} = \frac{1}{T}d(z,\lambda) - \frac{1}{T}z - \frac{1}{T}\int_0^T f(s,\bar{x}(s))ds. \tag{4.33}$$

On the other hand, it is proved that the limit function $x^*(t,z,\lambda,\mu)$ is a solution of the initial value problem (4.26), (4.27) for $\mu = \mu_{z,\lambda}$ and is given by formula (4.28) and satisfies boundary conditions (3.6).

By analogy

$$x^*(t,z,\lambda,\mu) = z + \int_0^t f(s,x^*(s,z,\lambda,\mu))ds + \mu_{z,\lambda}t, \tag{4.34}$$

$$T\mu_{z,\lambda} = x^*(T,z,\lambda,\mu) - z - \int_0^T f(s,x^*(s,z,\lambda,\mu))ds, \tag{4.35}$$

$$x^*(T,z,\lambda,\mu) = d(z,\lambda), \tag{4.36}$$

$$x^*(0,z,\lambda,\mu) = z.$$

By virtue of (4.34)–(4.36) it is easy to get that

$$\mu_{z,\lambda} = \frac{1}{T}d(z,\lambda) - \frac{1}{T}z - \frac{1}{T}\int_0^T f(s,x^*(s,z,\lambda,\mu))ds. \tag{4.37}$$

Substituting (4.33) in (4.30) and (4.37) in (4.34), we get that for all $t \in [0,T]$

$$\bar{x}(t) = z + \int_0^t f(s,\bar{x}(s))ds + \frac{1}{T}[d(z,\lambda) - z] - \frac{1}{T}\int_0^T f(s,\bar{x}(s))ds, \tag{4.38}$$

$$x^*(t,z,\lambda,\mu) = z + \int_0^t f(s,x^*(s,z,\lambda,\mu))ds + \frac{1}{T}[d(z,\lambda) - z] - \\ - \frac{1}{T}\int_0^T f(s,x^*(s,z,\lambda,\mu))ds. \tag{4.39}$$

Using Theorem 1 $\bar{x}(t) \in D$ and $x^*(t,z,\lambda,\mu) \in D$. Obviously that form (4.38), (4.39) implies that

$$x^*(t,z,\lambda,\mu) - \bar{x}(t) = \int_0^t [f(s,x^*(s,z,\lambda,\mu)) - f(s,\bar{x}(s))]ds - \\ - \frac{1}{T}\int_0^T [f(s,x^*(s,z,\lambda,\mu)) - f(s,\bar{x}(s))]ds. \tag{4.40}$$

On the bases of the Lipschitz condition (4.3), from the relation (4.40) we get that the function

$$\omega(t) = |x^*(t,z,\lambda,\mu) - \bar{x}(t)|, t \in [0,T] \tag{4.41}$$

satisfies integral inequalities:

$$\omega(t) \le K\left[\int_0^t \omega(s)ds + \frac{t}{T}\int_0^T \omega(s)ds\right] \le K\alpha_1(t) \max_{s\in[0,T]} \omega(s), t \in [0,T], \tag{4.42}$$

where $\alpha_1(t)$ is given by (4.16).

Using (4.42) recursively, we come to an inequality:

$$\omega(t) \le K^m \alpha_m(t) \max_{s\in[0,T]} \omega(s), t \in [0,T], \tag{4.43}$$

where $m \in \mathbb{N}$ and functions $\alpha_m(t)$ are given by the formula (4.22).

Considering the estimations (4.21), from (4.43) for each $m \in \mathbb{N}$ we get an estimation:

$$\omega(t) \le K\alpha_1(t)\frac{10}{9}\left(\frac{3T}{10}K\right)^{m-1} \cdot \max_{s\in[0,T]} \omega(s), t \in [0,T]. \tag{4.44}$$

Letting $m \to \infty$ in the last inequality and by virtue of (4.4), we come to the conclusion that

$$\max_{s\in[0,T]} \omega(s) \le Q^m \max_{s\in[0,T]} \omega(s) \to 0.$$

It means that the function $\bar{x}(t)$ coincides with $x^*(t,z,\lambda,\mu)$. Starting with (4.33) and (4.37), we come to the conclusion that $\bar{\mu} = \mu_{z,\lambda}$.

□

Let's find out the relation of the limit function $x = x^*(t,z,\lambda)$ of the sequence of functions (4.5) to the solution of the parametrized boundary value problem (2.1), (3.6) or the equivalent problem (2.1), (2.2).

Theorem 3. *Let the conditions (4.2)–(4.4) are hold for the boundary–value problem (2.1), (2.2).*

Then the pair $(x^*(\cdot,z^*,\lambda^*),\lambda^*)$ *is the solution of the parametrized boundary value problem (2.1), (3.6) if and only if* $z^* = (z_1^*, z_2^*, \ldots, z_n^*)$, $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$ *satisfy the following determining system of algebraic or transcendental equations*

$$\Delta(z,\lambda) = \frac{1}{T}[d(z,\lambda) - z] - \frac{1}{T}\int_0^T f(s,x^*(s,z,\lambda))ds = 0, \tag{4.45}$$

$$x^*(T,z,\lambda) = \lambda. \tag{4.46}$$

Proof. It suffices to apply Theorem 2 and notice that the differential equation in (4.10) coincides with (2.1) if and only if pair (z^*, λ^*) satisfies an equation

$$\Delta(z^*, \lambda^*) = 0.$$

Taking into account (3.1) and the equivalence (2.1), (2.2) and (2.1), (3.6), it is clear that $(x^*(\cdot, z^*, \lambda^*), \lambda^*)$ coincides with the solution of the parametrized boundary value problem (2.1), (3.1), (3.6) if and only if $(x^*(\cdot, z^*, \lambda^*), \lambda^*)$ satisfies an equation

$$x^*\left(T, z, \lambda^*\right) = \lambda^*.$$

It means that the pair $(x^*(\cdot, z^*, \lambda^*), \lambda^*)$ is the solution of the parametrized boundary value problem (2.1), (3.6) if and only if (4.45), (4.46) is hold.

□

The next statement proves that the system of determining equations (4.45), (4.46) defines all possible solutions of the original boundary value problem (2.1), (2.2).

Lemma 1. *Let all conditions of Theorem 1 be satisfied. Besides there exist some vectors $z \in D_\beta$ and $\lambda \in D$ that satisfy the system of determining equations (4.45), (4.46).*

Then the non-linear boundary–value problem (2.1), (2.2) has the solution $x(\cdot)$ such that:

$$\begin{aligned} x(0) &= z, \\ x(T) &= \lambda. \end{aligned} \tag{4.47}$$

Moreover this solution is given by formula

$$x(t) = x^*(t, z, \lambda), t = [0, T], \tag{4.48}$$

where $x^*(t, z, \lambda)$ is the limit function of the sequence (4.5). And if the boundary–value problem (2.1), (2.2) has a solution $x(\cdot)$, then this solution is given by (4.48), and the system of determining equations (4.45), (4.46) is satisfied when

$$z = x(0),$$

$$\lambda = x(T).$$

Proof. We will apply Theorems 2 and 3. If there exist such $z \in D_\beta$ and $\lambda \in D$ that satisfy determining system (4.45), (4.46), then, according to Theorem 3,the function (4.48) is a solution of the original boundary–value problem (2.1), (2.2). On the other hand, if $x(\cdot)$ is the solution of the original boundary–value problem (2.1), (2.2), then this function is the solution of the Cauchy problem (4.26), (4.27) for

$$\mu = 0,$$

$$z = x(0).$$

As $x(\cdot)$ satisfies boundary restrictions (2.2) and equivalent conditions (3.6), by virtue of Theorem 2, the equality (4.48) holds. Besides,

$$\begin{array}{c} \mu = \mu_{z,\lambda} = 0, \\ z = x(0), \end{array} \tag{4.49}$$

where vector λ is defined by (3.5). But, $\mu_{z,\lambda}$ is given by formula (4.28), that's why the first equation (4.45) of the determining system is satisfied, if

$$z = x(0), \lambda = col(x_1(T), \ldots, x_n(T)) :$$

$$\Delta(x(0), \lambda) = 0. \tag{4.50}$$

From (3.6) follows that the second equation (4.46) of the determining system holds, too. So we specified such pairs $(z, \lambda) = (x(0), x(T))$ that satisfy the system of determining equations (4.45), (4.46), this proves the Lemma.

□

Remark 2. The main difficulty of the realization of this method is to find the limit function $x^*(\cdot, z, \lambda)$. But in most cases this problem can be solved using the properties of the approximate solution $x_m(\cdot, z, \lambda)$ that was built in an analytic form.

For $m \geq 1$ let us define the function $\Delta_m : D_\beta \times D \to \mathbb{R}^n$ by formula

$$\Delta_m(z, \lambda) := \frac{1}{T}[d(z, \lambda) - z] - \frac{1}{T}\int_0^T f(s, x_m(s, z, \lambda))ds, \tag{4.51}$$

where z and λ are given by the relation (3.5). To investigate the solubility of the parametrized boundary value problem (2.1), (3.6) we observe an approximate determining system of algebraic or transcendental equations of the form

$$\Delta_m(z, \lambda) = \frac{1}{T}[d(z, \lambda) - z] - \frac{1}{T}\int_0^T f(s, x_m(s, z, \lambda))ds = 0, \tag{4.52}$$

$$x_m(T, z, \lambda) = \lambda, \tag{4.53}$$

where $x_m(\cdot, z, \lambda)$ is a vector–function, that is defined by the recursive relation (4.5). By increasing m in the systems (4.45), (4.46) and (4.52), (4.53), one can achieve the needed precision of the approximate solution of the original boundary value problem (2.1), (2.2).

5. Example

Consider the system

$$\begin{cases} \frac{dx_1}{dt} = x_2 = f_1(t, x_1, x_2), \\ \frac{dx_2}{dt} = \frac{9}{32} + \frac{1}{16}t^2 + \frac{1}{8}tx_2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_1 = f_2(t, x_1, x_2), \end{cases} \tag{5.1}$$

where $t \in [0, 1]$,

with non–linear two–point boundary conditions

$$\begin{cases} g_1(x(0),x(1)) := x_1(0)+x_1(1)-x_2(1)^2-\frac{3}{16}=0, \\ g_2(x(0),x(1)) := x_2(0)+x_1(1)-x_2(1)+\frac{1}{16}=0. \end{cases} \tag{5.2}$$

It is easy to check that an exact solution of the problem (5.1), (5.2) are the functions

$$\begin{cases} x_1^* = \frac{1}{8}t^2+\frac{1}{16}, \\ x_2^* = \frac{1}{4}t. \end{cases} \tag{5.3}$$

Suppose that the boundary–value problem (5.1), (5.2) is considered in the domain

$$D = \left\{ (x_1,x_2) : |x_1| \le 1, |x_2| \le \frac{3}{4} \right\}. \tag{5.4}$$

Boundary conditions (5.2) can be rewritten in the form

$$x(1) + g(x(0),x(1)) = x(1), \tag{5.5}$$

where $g(x(0),x(1)) = col\,(g_1(x(0),x(1)), g_2(x(0),x(1)))$.

Let us replace the values of the components of the solution of the boundary–value problem (5.1), (5.2) at the points $t=0$ and $t=1$ by parameters z_1, z_2 and λ_1, λ_2 :

$$\begin{array}{l} x(0) = col\,(x_1(0),x_2(0)) = col\,(z_1,z_2), \\ x(1) = col\,(x_1(1),x_2(1)) = col\,(\lambda_1,\lambda_2). \end{array} \tag{5.6}$$

Using (5.6), the boundary restrictions (5.5) can be rewritten as

$$\lambda + g(z,\lambda) = x(1), \tag{5.7}$$

where

$$\begin{array}{l} z = col\,(z_1,z_2), \\ \lambda = col\,(\lambda_1,\lambda_2). \end{array} \tag{5.8}$$

Let us put

$$d(z,\lambda) := Az + \lambda + g(z,\lambda), \tag{5.9}$$

where z and λ are given by (5.8).

Using (5.9), the parametrized boundary conditions (5.7) can be written in the form:

$$Ax(0) + x(1) = d(z,\lambda). \tag{5.10}$$

It is easy to check that the matrix K from the Lipschitz condition (4.3) is

$$K = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{7}{8} \end{pmatrix},$$

and

$$r(K) < 1.27 < \frac{10}{3T},$$

when $T=1$.

Vector $\delta_D(f)$ can be chosen as

$$\delta_D(f) \le \begin{pmatrix} \frac{3}{4} \\ \frac{355}{512} \end{pmatrix}.$$

One can verify that, for the parametrized boundary value problem in this example, all needed conditions are fulfilled. So, we can proceed with application of the numerical-analytic scheme described above, and thus construct the sequence of approximate solutions.

The result of the first iteration is:

$$x_{11} = z_1 + 0.5t^2\lambda_2 + 0.3125t^2 + 1.5t\lambda_1 + 0.21875t - t\lambda_2^2,$$
$$x_{12} = z_2 + 0.02278645833t^3 - 0.1666666666t^3\lambda_1^2 + \\ + 0.02083333333t^3\lambda_1 - 0.5t^2z_2\lambda_1 + 0.03125t^2z_2 - 0.5t^2\lambda_1 + 0.25t^2\lambda_2^2 + \\ + 0.046875t^2 - 0.007161458333t + 0.1666666667t\lambda_1^2 + 1.479166667t\lambda_1 + \\ + 0.5tz_2\lambda_1 - 0.03125z_2t - 0.25t\lambda_2^2,$$

for all $t \in [0,1]$.

The computation shows that the approximate solutions of the approximate determining equation are

$$z_1 := z_{11} = 0.06249675051,$$
$$z_2 := z_{12} = 0.00001071252252,$$
$$\lambda_1 := \lambda_{11} = 0.1875172131,$$
$$\lambda_2 := \lambda_{12} = 0.2500279256.$$

The first and second components of the first approximation are

$$x_{11} = 0.06249675051 + 0.1250086066t^2 + 0.0000118560t,$$
$$x_{12} = 0.00001071252252 + 0.02083261607t^3 - \\ - 0.03125578527t^2 + 0.2604403824t.$$

The first approximation and the exact solution of the original boundary–value problem is shown on Figure 1.

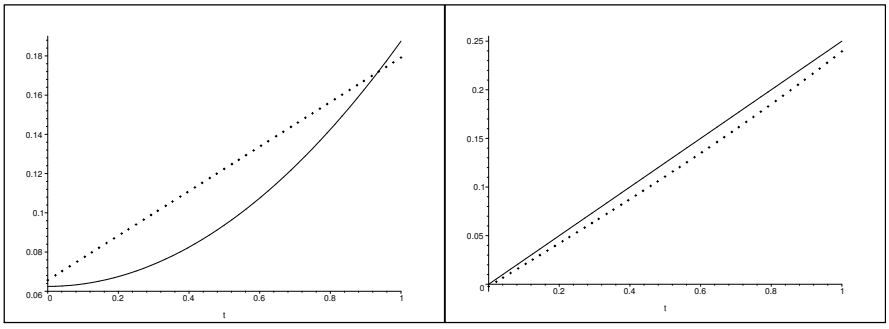

FIGURE 1. The first components of the exact solution (solid line) and its first approximation (drawn with dots)

The error of the first approximation is

$$\max_{t\in[0,1]} \left|x_1^*(t) - x_{11}(t)\right| \le 0.33 \cdot 10^{-5},$$

$$\max_{t\in[0,1]}\left|x_2^*(t)-x_{12}(t)\right|\le 0.98\cdot 10^{-3}.$$

The approximate solutions of the third approximate determining equation are:

$$\begin{gathered}z_1:=z_{31}=0.06250000284,\\ z_2:=z_{32}=0.0000001048559559,\\ \lambda_1:=\lambda_{31}=0.1875000988,\\ \lambda_2:=\lambda_{32}=0.2500002034.\end{gathered}$$

So the first and second components of the third approximation have the form:

$$\begin{gathered}x_{31}=0.00001089110198t+0.1249893435t^2-0.000003875246484t^8-\\ -0.0001130313603t^4+0.00001550100414t^7-0.0001103154996t^6+\\ +0.0002115888487t^5+0.06250000284,\end{gathered}$$

$$\begin{gathered}x_{32}=0.2500862247t-0.0008671786995t^3-0.000001635343948t^8+\\ +0.001302085198t^4+0.00001180290239t^7-0.00002203673649t^6-\\ -0.0005095320438t^5.\end{gathered}$$

The third approximation and the exact solution of the original boundary value problem is shown on Figure 2.

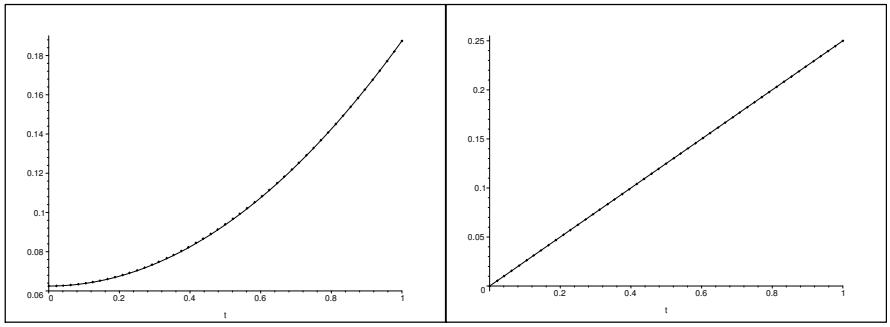

FIGURE 2. The first components of the exact solution (solid line) and its first approximation (drawn with dots)

The error of the third approximation is

$$\max_{t\in[0,1]}\left|x_1^*(t)-x_{31}(t)\right|\le 0.14\cdot 10^{-5},$$

$$\max_{t\in[0,1]}\left|x_2^*(t)-x_{32}(t)\right|\le 0.13\cdot 10^{-4}.$$

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$q-$SZÁSZ SCHURER OPERATORS

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Abstract. In this paper, we introduce $q-$Szász Schurer operators and calculate their moments. The transformation properties, Korovkin type approximation theorem and rate of convergence of the operators are studied. We further obtain global estimates for $q-$Szász Schurer operators in terms of some Lipschitz classes.

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1. INTRODUCTION

Korovkin type approximation theory has been constructed on linear positive operators and has been an active research field during the last century because of its simple applicability [2]. After the works of A. Lupaş [9] and G.M. Phillips [19], where they introduced different analogues of the q-Bernstein polynomials $B_{n,q}(f;x)$, $n = 1,2,..,0 < q < \infty$, intensive research has been conducted on linear positive operators based on q-integers (see [4, 6, 8, 15–18, 20, 21]).

In [4], [3] q-Szász Mirakjan operators were defined and their approximation properties were investigated. For $0 \leq x < \dfrac{b_n}{(1-q)[n]}$, $f \in C[0,\infty)$, and $\{b_n\}$ is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$, in [4], q-Szász Mirakjan operators were defined by

$$S_{n,q}(f)(x) := E_q\left(-[n]\frac{x}{b_n}\right)\sum_{k=0}^{\infty} f\left(\frac{[k]b_n}{[n]}\right)\frac{[n]^k x^k}{[k]!b_n^k},$$

where

$$[k] = \begin{cases} (1-q^k)/(1-q) & if\ q \neq 1 \\ k & if\ q = 1 \end{cases}, \quad [k]! = \begin{cases} [1][2]...[k] & if\ k \geq 1 \\ 1 & if\ k = 0 \end{cases},$$

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$$E_q(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{[n]!} x^n = (-(1-q)x;q)_\infty ; x \in \mathbb{R}, |q| < 1, \qquad (1.1)$$

$$(a;q)_\infty = \prod_{j=0}^{\infty} (1-aq^j) \qquad (a \in \mathbb{C}).$$

As it was pointed out in [11], it is better to refer to these operators as q-parametric Szász-Mirakjan-Chlodowsky operators because of their structures. In [11], Mahmudov introduced the following q-Szász-Mirakjan operators

$$S^*_{n,q}(f)(x) = \frac{1}{E_q([n]x)} \sum_{k=0}^{\infty} f\left(\frac{[k]}{q^{k-2}[n]}\right) q^{\frac{k(k-1)}{2}} \frac{[n]^k x^k}{[k]!},$$

where $x \in [0,\infty)$, $0 < q < 1$, $f \in C[0,\infty)$ and studied convergence properties of these operators. We note that these operators do not preserve linear functions. In [11], Mahmudov obtained inequalities for the weighted approximation error, and also for the rate of convergence in terms of weighted moduli of continuity, and a Voronoskaja-type formula for q-Szász-Mirakjan operators was derived, too.

Now let $E_q(x)$ be defined by (1.1). In the present paper we introduce $q-$Szász Schurer operators which is defined for fixed $p \in \mathbb{N}_0$ by

$$S_{n,q}(f;x;p) = \frac{1}{E([n+p]x)} \sum_{k=0}^{\infty} f\left(\frac{[k]}{q^{k-1}[n]}\right) q^{\frac{k(k-1)}{2}} \frac{[n+p]^k x^k}{[k]!} \qquad (1.2)$$

where $x \in [0,\infty)$, $0 < q < 1$, $f \in C[0,\infty)$. It is clear from (1.1) that

$$S_{n,q}(1;x;p) = 1, \qquad (1.3)$$

for each fixed $p \in \mathbb{N}_0$. Note that in the case $p = 0$, $q-$Szász Schurer operators reduces to the slightly modified form of the Mahmudov's q-Szász-Mirakjan operators. Furthermore, letting $q \to 1$ in (1.2), one can get Schurer-Szász-Mirakjan operators (see p.p. 338,[2]).

There are some recent papers related to the $q-$Szász operators. For instance, in [1] and in [10] different variants of King type $q-$Szász operators (preserving x^2 and providing better error estimation) has been considered. On the other hand, integral type $q-$Szász operators were recently investigated in [13],[14],[12] and [7]. These types of modifications of the $q-$Szász Schurer operators will be the subject of future studies.

In this paper we compute the moments of the $q-$Szász Schurer operators and investigate their transformation properties. Korovkin type approximation result and rate of convergence of the operators are obtained in Section 3. Section 4 is devoted to global estimates for $q-$Szász Schurer operators in terms of some Lipschitz classes.

2. AUXILIARY LEMMAS

In this section we calculate the moments of the $q-$Szász Schurer operators and investigate their transformation properties. We start with the following lemma:

Lemma 1. *For $m \in \mathbb{N}_0$ and fixed $p \in \mathbb{N}_0$, we have*

$$S_{n,q}(t^{m+1};x;p) = \frac{[n+p]x}{[n]} \sum_{j=0}^{m} \binom{m}{j} \frac{1}{q^j [n]^{m-j}} S_{n,q}(t^j;x;p). \tag{2.1}$$

Using (1.3) and the above lemma, we get the following results immediately.

Corollary 1. *For fixed $p \in \mathbb{N}_0$, we have*

$$\begin{aligned} S_{n,q}(t;x;p) &= \frac{[n+p]x}{[n]}, \quad S_{n,q}(t^2;x;p) = \frac{1}{q}\left(\frac{[n+p]x}{[n]}\right)^2 + \frac{[n+p]x}{[n]^2}, \\ S_{n,q}(t^3;x;p) &= \frac{1}{q^3}\left(\frac{[n+p]x}{[n]}\right)^3 + \frac{2q+1}{q^2}\frac{[n+p]^2x^2}{[n]^3} + \frac{[n+p]x}{[n]^3}. \end{aligned}$$

Note that the operators $S_{n,q}(t;x;0)$ preserve the linear functions while $S^*_{n,q}(f)(x)$ does not preserve them. Now, using linearity of the operators, we have the following:

Corollary 2. *For the second central moment, we have*

$$S_{n,q}((t-x)^2;x;p) = \left[\left(\frac{[n+p]}{q[n]} - 2\right)\frac{[n+p]}{[n]} + 1\right]x^2 + \frac{[n+p]}{[n]^2}x.$$

Lemma 2. *For every $x \geq 0$, we have*

$$\begin{aligned} &\frac{1}{E([n+p]x)} \sum_{k=0}^{\infty} \left|\frac{[k]}{q^{k-1}[n]} - x\right| q^{\frac{k(k-1)}{2}} \frac{[n+p]^k x^k}{[k]!} \\ \leq\ &\left(\left[\left(\frac{[n+p]}{q[n]} - 2\right)\frac{[n+p]}{[n]} + 1\right]x^2 + \frac{[n+p]}{[n]^2}x\right)^{1/2}. \end{aligned}$$

From (2.1), it is not difficult to obtain the following result by induction.

Lemma 3. *For the m^{th} moment ($m = 1,2,...$) we have the following:*

$$\begin{aligned} S_{n,q}(t^m;x;p) = q^{\frac{-m(m-1)}{2}} &\left(\frac{[n+p]x}{[n]}\right)^m \\ &+ \frac{b_{m,1}(q)}{[n]}\left(\frac{[n+p]x}{[n]}\right)^{m-1} + ... + \frac{b_{m,m-1}(q)}{[n]^{m-1}}\frac{[n+p]x}{[n]}, \end{aligned} \tag{2.2}$$

where the coefficients $b_{m,i}(q)$ ($i = 1,2,...,m-1$) are constants depending on m and q. Furthermore, $b_{m,m-1}(q) = 1$ ($m = 1,2,...$).

Now let $C_B[0,\infty)$ denote the space of all real valued continuous bounded functions on $[0,\infty)$ endowed with the norm

$$\|f\| = \sup_{x\in[0,\infty)} |f(x)|,\ f \in C_B[0,\infty).$$

It is obvious by (1.3) that $S_{n,q}$ maps $C_B[0,\infty)$ into itself.

Let $r \in \mathbb{N}_0 := \{0,1,...\}$ and define the weight function μ_r as follows: $\mu_0(x) := 1$ and $\mu_r(x) := (1+x^r)^{-1}$ for $x \geq 0$ and $r \in \mathbb{N}_0$. Also for $r \in \mathbb{N}_0$, let $C_r[0,\infty)$ be the space of functions $f \in C[0,\infty)$, such that $\mu_r f$ is uniformly continuous and bounded on $[0,\infty)$, endowed with the norm

$$\|f\|_r := \sup_{x\in[0,\infty)} \mu_r(x)\,|f(x)| \quad \text{for} \quad f \in C_r[0,\infty).$$

Using (2.2), we easily obtain the following transformation property.

Lemma 4. *For the operators $S_{n,q}$, there exists a constant $K_r(q) > 0$ depending on r and q, such that*

$$\mu_r(x) S_{n,q}\left(\frac{1}{\mu_r};x;p\right) \leq K_r(q). \tag{2.3}$$

Furthermore, for all $f \in C_r[0,\infty)$, we have

$$\left\|S_{n,q}(f)\right\|_r \leq K_r(q)\,\|f\|_r, \tag{2.4}$$

which guarantees that $S_{n,q}$ maps $C_r[0,\infty)$ into $C_r[0,\infty)$.

3. Korovkin type theorem

In this section we consider the space

$$E := \left\{ f \in C_2[0,\infty) : \exists \lim_{x\to\infty} \frac{|f(x)|}{1+x^2} < \infty \right\},$$

endowed with the norm

$$\|f\|_2 := \sup_{x\in[0,\infty)} \mu_2(x)\,|f(x)|\ .$$

Now let $b > 0$. The usual modulus of continuity of f on the closed interval $[0,b]$ is defined by

$$\omega_b(f,\delta) = \sup_{\substack{|t-x|\leq\delta \\ x,t\in[0,b]}} |f(t)-f(x)|. \tag{3.1}$$

It is well known that, for a function $f \in E$, we have $\lim_{\delta\to\infty} \omega_b(f,\delta) = 0$.

We start by obtaining the rate of convergence of the operators $S_{n,q}(f;x;p)$ to $f(x)$, for all $f \in E$.

Theorem 1. *Let $f \in E$ and let $\omega_{b+1}(f,\delta)$ $(b>0)$ be its modulus of continuity on the finite interval $[0,b+1] \subset [0,\infty)$. Then for fixed $q \in (0,1)$, we have*

$$\left\| S_{n,q}(f;x;p) - f(x) \right\|_{C[0,b]} \leq 2\omega_{b+1}(f,\delta(q))$$
$$+N_f\left(1+b^2\right)\left\{\left[\left(\frac{[n+p]}{q[n]}-2\right)\frac{[n+p]}{[n]}+1\right]b^2+\frac{[n+p]}{[n]^2}b\right\}$$

where $\delta=\delta(q)=\left[\left[\left(\frac{[n+p]}{q[n]}-2\right)\frac{[n+p]}{[n]}+1\right]b^2+\frac{[n+p]}{[n]^2}b\right]^{1/2}$ and N_f is a positive constant depending on f.

Proof. Let $q \in (0,1)$ be fixed. For $x \in [0,b]$ and $t \leq b+1$, we can write from (3.1) that

$$|f(t)-f(x)| \leq \omega_{b+1}(f,|t-x|) \leq \left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta) \tag{3.2}$$

where $\delta>0$. On the other hand, for $x \in [0,b]$ and $t>b+1$, using the fact that $t-x>1$, we have

$$\begin{aligned}|f(t)-f(x)| &\leq M_f(1+x^2+t^2) \leq M_f(2+3x^2+2(t-x)^2)\\ &\leq N_f\left(1+b^2\right)(t-x)^2 \end{aligned}\tag{3.3}$$

where $N_f=6M_f$. Combining (3.2) and (3.3), we get for all $x \in [0,b]$ and $t \geq 0$ that $|f(t)-f(x)| \leq N_f\left(1+b^2\right)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta)$ and therefore

$$\begin{aligned}\left|S_{n,q}(f;x;p)-f(x)\right| \quad &\leq \quad N_f\left(1+b^2\right)S_{n,q}((t-x)^2;x;p)\\ &\quad +\left(1+\frac{S_{n,q}(|t-x|;x;p)}{\delta}\right)\omega_{b+1}(f,\delta).\end{aligned}$$

By Lemma 2 and Corollary 2, we obtain

$$\begin{aligned}&\left|S_{n,q}(f;x;p)-f(x)\right|\\ &\leq N_f\left(1+b^2\right)S_{n,q}((t-x)^2;x;p)+\left(1+\frac{\left[S_{n,q}((t-x)^2;x;p)\right]^{1/2}}{\delta}\right)\omega_{b+1}(f,\delta)\\ &\leq N_f\left(1+b^2\right)\left\{\left[\left(\frac{[n+p]}{q[n]}-2\right)\frac{[n+p]}{[n]}+1\right]x^2+\frac{[n+p]}{[n]^2}x\right\}\\ &+\left(1+\frac{\left[\left[\left(\frac{[n+p]}{q[n]}-2\right)\frac{[n+p]}{[n]}+1\right]x^2+\frac{[n+p]}{[n]^2}x\right]^{1/2}}{\delta}\right)\omega_{b+1}(f,\delta)\\ &\leq N_f\left(1+b^2\right)\left\{\left[\left(\frac{[n+p]}{q[n]}-2\right)\frac{[n+p]}{[n]}+1\right]b^2+\frac{[n+p]}{[n]^2}b\right\}+2\omega_{b+1}(f,\delta(q)),\end{aligned}$$

where $\delta = \delta(q) = \left[\left[\left(\frac{[n+p]}{q[n]} - 2\right)\frac{[n+p]}{[n]} + 1\right]b^2 + \frac{[n+p]}{[n]^2}b\right]^{1/2}$. Whence the result follows. □

Corollary 3. *Let $q := q_n \in (0,1)$ and fix $p \in \mathbb{N}_0$. Then for all $f \in E$, $\{S_{n,q}(f;x;p)\}$ converges uniformly to f on $[0,b]$ if and only if $\lim_{n\to\infty} q_n = 1$.*

4. Global result

In this section we consider the following Lipschitz classes:

$$\begin{aligned}
\Delta_h^2 f(x) &: = f(x+2h) - 2f(x+h) + f(x),\\
\omega_r^2(f,\delta) &: = \sup_{h\in(0,\delta]} \left\|\Delta_h^2 f\right\|_r,\\
\omega_r^1(f,\delta) &: = \sup\{\mu_r(x)\,|f(t)-f(x)| : |t-x| \le \delta \text{ and } t,x \ge 0\}\\
Lip_r^2\alpha &: = \left\{f \in C_r[0,\infty) : \omega_r^2(f;\delta) = O(\delta^\alpha) \text{ as } \delta \to 0^+\right\},
\end{aligned}$$

where $h > 0$ and $0 < \alpha \le 2$.

Lemma 5. *Fix $r \in \mathbb{N}$, $p \in \mathbb{N}_0, q \in (0,1)$. For the sequence of operators $\{S_{n,q}(f;x;p)\}$, there exists a constant $M_r(q,p)$ such that*

$$\begin{aligned}
&\mu_r(x) S_{n,q}\left(\frac{\psi_x^2}{\mu_r};x;p\right)\\
&\le M_r(q,p)\left\{\left[\left(\frac{[n+p]}{q^{2r+1}[n]} - \frac{2}{q^r}\right)\frac{[n+p]}{[n]} + 1\right]x^2 + \frac{[n+p]}{[n]^2}x\right\}
\end{aligned}$$

where $\psi_x^2(t) = (t-x)^2$.

Proof. Now let $r = 1$. Since, for each fixed $p \in \mathbb{N}_0, q \in (0,1)$ and for all $n \in \mathbb{N}$, we have $\frac{[n+p]}{q[n]} - 2 \le \frac{[n+p]}{q^3[n]} - \frac{2}{q}$, then we can write using Corollary 1 and Corollary 2 that

$$\begin{aligned}
&\mu_1(x) S_{n,q}\left(\frac{\psi_x^2}{\mu_1};x;p\right)\\
=\ &\mu_1(x)\left\{S_{n,q}\left(\psi_x^2;x;p\right) + S_{n,q}\left(t^3;x;p\right) - 2xS_{n,q}\left(t^2;x;p\right) + x^2 S_{n,q}\left(t;x;p\right)\right\}\\
\le\ &\mu_1(x)\left\{\left[\left(\left(\frac{[n+p]}{q^3[n]} - \frac{2}{q}\right)\frac{[n+p]}{[n]} + 1\right)x^2\right]\left(1 + \frac{[n+p]}{[n]}x\right)\right.\\
&\left. + \frac{[n+p]}{[n]^2}x\left[\frac{2q+1}{q^2}\frac{[n+p]}{[n]}x - 2x + \frac{1}{[n]} + 1\right]\right\}\\
\le\ &M_r(q,p)\left\{\left[\left(\left(\frac{[n+p]}{q^3[n]} - \frac{2}{q}\right)\frac{[n+p]}{[n]} + 1\right)x^2\right] + \frac{[n+p]}{[n]^2}x\right\}.
\end{aligned}$$

Finally, assume that $r \geq 2$. Since, for each fixed $p \in \mathbb{N}_0, q \in (0,1)$ and for all $n \in \mathbb{N}$, we have $\frac{[n+p]}{q[n]} - 2 \leq \frac{[n+p]}{q^{2r+1}[n]} - \frac{2}{q^r}$, then, we get from Lemma 3 that

$$S_{n,q}\left(\frac{\psi_x^2}{\mu_r};x;p\right) = S_{n,q}\left(\psi_x^2;x;p\right) + S_{n,q}(e_{r+2};x;p) - 2xS_{n,q}(e_{r+1};x;p)$$
$$+x^2 S_{n,q}(e_r;x;p) = S_{n,q}\left(\psi_x^2;x;p\right)$$
$$+q^{\frac{-(r-1)(r+2)}{2}}\left(\frac{[n+p]x}{[n]}\right)^{r+2} + \frac{b_{r+2,1}(q)}{[n]}\left(\frac{[n+p]x}{[n]}\right)^{r+1} + \ldots + \frac{1}{[n]^{r+1}}\frac{[n+p]x}{[n]}$$
$$-2x\left\{q^{\frac{-r(r+1)}{2}}\left(\frac{[n+p]x}{[n]}\right)^{r+1} + \frac{b_{r+1,1}(q)}{[n]}\left(\frac{[n+p]x}{[n]}\right)^{r} + \ldots + \frac{1}{[n]^r}\frac{[n+p]x}{[n]}\right\}$$
$$+x^2\left\{q^{\frac{-r(r-1)}{2}}\left(\frac{[n+p]x}{[n]}\right)^{r} + \frac{b_{r,1}(q)}{[n]}\left(\frac{[n+p]x}{[n]}\right)^{r-1} + \ldots + \frac{1}{[n]^{r-1}}\frac{[n+p]x}{[n]}\right\},$$
$$= \left[\left(\frac{[n+p]}{q[n]} - 2\right)\frac{[n+p]}{[n]} + 1\right]x^2 + \frac{[n+p]}{[n]^2}x$$
$$+q^{\frac{-r(r-1)}{2}}\left[\left(\frac{[n+p]}{q^{2r+1}[n]} - \frac{2}{q^r}\right)\frac{[n+p]}{[n]} + 1\right]x^2\left(\frac{[n+p]x}{[n]}\right)^r$$
$$+\frac{[n+p]}{[n]^2}x\left[(b_{r+2}(q) - 2b_{r+1}(q) + b_r(q))\left(\frac{[n+p]x}{[n]}\right)^r + \ldots + \frac{1}{[n]^r}\right]$$

which implies that

$$\mu_r(x)S_{n,q}\left(\frac{\psi_x^2}{\mu_r};x;p\right)$$
$$\leq M_r(q,p)\left\{\left[\left(\frac{[n+p]}{q^{2r+1}[n]} - \frac{2}{q^r}\right)\frac{[n+p]}{[n]} + 1\right]x^2 + \frac{[n+p]}{[n]^2}x\right\},$$

whence the result holds. □

Now, for fixed $p \in \mathbb{N}_0$, consider the space

$$C_p^2[0,\infty) := \left\{f \in C_p[0,\infty) : f'' \in C_p[0,\infty)\right\}.$$

Then we have the following result.

Lemma 6. *Let $g \in C_p^2[0,\infty)$. For the operators $S_{n,q}^*(f;x;p) = S_{n,q}(f;x;p) - f(\frac{[n+p]x}{[n]}) + f(x)$, there exists a positive constant $M_r(q,p)$ such that, for all $x \in [0,\infty)$ and $n \in \mathbb{N}$, we have*

$$\mu_r(x)\left|S_{n,q}^*(g;x;p) - g(x)\right| \leq M_r(q,p)\left\|g''\right\|_r$$
$$\times\left\{\left[\left(1 + \frac{1}{q^{2r+1}}\right)\frac{[n+p]^2}{[n]^2} - 2\left(1 + \frac{1}{q^r}\right)\frac{[n+p]}{[n]} + 2\right]x^2 + \frac{[n+p]x}{[n]^2}\right\}.$$

Proof. Let $\psi_x(y) = y - x$. Using Taylor formula, we may write that

$$g(y) - g(x) = \psi_x(y) g'(x) + \int_x^y \psi_t(y) g''(t) dt, \quad y \in [0, \infty).$$

Since $S^*_{n,q}(\psi_x(y); x; p) = 0$, we get

$$\begin{aligned} \left| S^*_{n,q}(g; x; p) - g(x) \right| &= \left| S^*_{n,q}(g(y) - g(x); x; p) \right| \\ &= \left| S^*_{n,q}\left(\int_x^y \psi_t(y) g''(t) dt; x; p \right) \right| \\ = \left| S_{n,q}\left(\int_x^y \psi_t(y) g''(t) dt; x; p \right) - \int_x^{\frac{[n+p]x}{[n]}} \psi_t\left(\frac{[n+p]x}{[n]} \right) g''(t) dt \right|. \end{aligned}$$

Using the fact that

$$\left| \int_x^y \psi_t(y) g''(t) dt \right| \leq \frac{\|g''\|_r \psi_x^2(y)}{2} \left(\frac{1}{\mu_r(x)} + \frac{1}{\mu_r(y)} \right)$$

and

$$\left| \int_x^{\frac{[n+p]x}{[n]}} \psi_t\left(\frac{[n+p]x}{[n]} \right) g''(t) dt \right| \leq \frac{\|g''\|_r}{2\mu_r(x)} \left(\frac{[n+p]x}{[n]} - x \right)^2,$$

it follows from Lemma 5 that

$$\begin{aligned} \mu_r(x) \left| S^*_{n,q}(g; x; p) - g(x) \right| &\leq \frac{\|g''\|_r}{2} \left\{ S_{n,q}(\psi_x^2; x) + \mu_r(x) S_{n,q}\left(\frac{\psi_x^2}{\mu_r}; x \right) \right\} \\ &\quad + \frac{\|g''\|_r}{2} \left(\frac{[n+p]x}{[n]} - x \right)^2 \\ &\leq M_r(q, p) \left\| g'' \right\|_r \\ &\quad \times \left\{ \left[\left(1 + \frac{1}{q^{2r+1}} \right) \frac{[n+p]^2}{[n]^2} - 2\left(1 + \frac{1}{q^r} \right) \frac{[n+p]}{[n]} + 2 \right] x^2 + \frac{[n+p]x}{[n]^2} \right\}. \end{aligned}$$

The Lemma is proved. □

The main result of the section is the following.

Theorem 2. *For each fixed $r \in \mathbb{N}, p \in \mathbb{N}_0$ and for all $n \in \mathbb{N}$, $f \in C_r[0,\infty)$ and $x \in [0,\infty)$ and fixed $q \in (0,1)$, there exists an absolute constant $M_r(q,p) > 0$ such that*

$$\mu_r(x)\left|S_{n,q}(f;x;p) - f(x)\right| \leq \omega_r^1\left(f; \frac{[n+p]x}{[n]} - x\right) + M_r(q,p)$$
$$\times\omega_r^2\left(f, \sqrt{\left[\left(1+\frac{1}{q^{2r+1}}\right)\frac{[n+p]^2}{[n]^2} - 2\left(1+\frac{1}{q^r}\right)\frac{[n+p]}{[n]} + 2\right]x^2 + \frac{[n+p]x}{[n]^2}}\right),$$

where μ_r is the same as in Section 2. Particularly, if $f \in Lip_r^2\alpha$ for some $\alpha \in (0,2]$, then

$$\mu_r(x)\left|S_{n,q}(f;x;p) - f(x)\right| \leq \omega_r^1\left(f; \frac{[n+p]x}{[n]} - x\right) + M_r(q,p)$$
$$\times\left(\left[\left(1+\frac{1}{q^{2r+1}}\right)\frac{[n+p]^2}{[n]^2} - 2\left(1+\frac{1}{q^r}\right)\frac{[n+p]}{[n]} + 2\right]x^2 + \frac{[n+p]x}{[n]^2}\right)^{\frac{\alpha}{2}}$$

holds.

We first consider the modified Steklov means (see [5]) of a function $f \in C_r[0,\infty)$ as follows:

$$f_h(y) := \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\{2f(y+s+t) - f(y+2(s+t))\}\,ds\,dt,$$

where $h > 0$ and $y \geq 0$. In this case, it is clear that

$$f(y) - f_h(y) = \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\Delta_{s+t}^2 f(y)\,ds\,dt,$$

which guarantees that

$$\|f - f_h\|_r \leq \omega_r^2(f;h). \tag{4.1}$$

Furthermore, we have $f_h''(y) = \frac{1}{h^2}\left(8\Delta_{h/2}^2 f(y) - \Delta_h^2 f(y)\right)$, which implies

$$\left\|f_h''\right\|_r \leq \frac{9}{h^2}\omega_r^2(f;h). \tag{4.2}$$

Then, combining (4.1) with (4.2) we conclude that the Steklov means f_h corresponding to $f \in C_r[0,\infty)$ belongs to $C_r^2[0,\infty)$.

Proof. Let $r \in \mathbb{N}$, $f \in C_r[0,\infty)$ and $x \in [0,\infty)$ be fixed. Assume that, for $h > 0$, f_h denotes the Steklov means of f. For any $n \in \mathbb{N}$, the following inequality holds:

$$\left|S_{n,q}(f;x;p) - f(x)\right| \leq S_{n,q}^*\left(|f(y) - f_h(y)|;x;p\right) + |f(x) - f_h(x)|$$

$$+\left|S_{n,q}^{*}\left(f_h;x;p\right)-f_h(x)\right|+\left|f\left(\frac{[n+p]x}{[n]}\right)-f(x)\right|.$$

Since $f_h \in C_r^2[0,\infty)$, it follows from Lemma 6 that

$$\begin{aligned}
&\mu_r(x)\left|S_{n,q}(f;x;p)-f(x)\right| \leq \|f-f_h\|_r \left\{\mu_r(x)S_{n,q}^{*}\left(\frac{1}{\mu_r};x;p\right)+1\right\}\\
&+M_r(q,p)\left\|f_h''\right\|_p \left\{\left[\left(1+\frac{1}{q^{2r+1}}\right)\frac{[n+p]^2}{[n]^2}-2\left(1+\frac{1}{q^r}\right)\frac{[n+p]}{[n]}+2\right]x^2\right.\\
&\left.+\frac{[n+p]x}{[n]^2}\right\}+\mu_r(x)\left|f\left(\frac{[n+p]x}{[n]}\right)-f(x)\right|\\
\leq\ &\|f-f_h\|_p \left\{\mu_r(x)S_{n,q}\left(\frac{1}{\mu_r};x\right)+3\right\}\\
&+M_r(q,p)\left\|f_h''\right\|_r \left\{\left[\left(1+\frac{1}{q^{2r+1}}\right)\frac{[n+p]^2}{[n]^2}-2\left(1+\frac{1}{q^r}\right)\frac{[n+p]}{[n]}+2\right]x^2\right.\\
&\left.+\frac{[n+p]x}{[n]^2}\right\}+\mu_r(x)\left|f\left(\frac{[n+p]x}{[n]}\right)-f(x)\right|.
\end{aligned}$$

By (4.1) and (4.2), the last inequality yields that

$$\begin{aligned}
&\mu_r(x)\left|S_{n,q}(f;x;p)-f(x)\right| \leq M_r(q)\omega_r^2(f;h)\\
&\times\left\{1+\frac{1}{h^2}\left\{\left[\left(1+\frac{1}{q^{2r+1}}\right)\frac{[n+p]^2}{[n]^2}-2\left(1+\frac{1}{q^r}\right)\frac{[n+p]}{[n]}+2\right]x^2\right.\right.\\
&\left.\left.+\frac{[n+p]x}{[n]^2}\right\}\right\}+\omega_r^1\left(f;f\left(\frac{[n+p]x}{[n]}\right)-f(x)\right).
\end{aligned}$$

Thus, choosing $h=\sqrt{\left[\left(1+\frac{1}{q^{2r+1}}\right)\frac{[n+p]^2}{[n]^2}-2\left(1+\frac{1}{q^r}\right)\frac{[n+p]}{[n]}+2\right]x^2+\frac{[n+p]x}{[n]^2}}$, the first part of the proof is completed. The remaining part of the proof can be easily obtained from the definition of the space $Lip_p^2\alpha$. □

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INVESTIGATION OF A CLASS OF TWO-DIMENSIONAL CONJUGATE INTEGRAL EQUATION WITH FIXED SUPER-SINGULAR KERNELS

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Abstract. In this paper, two-dimensional linear conjugate Volterra integral equations containing super-singularities in the kernels are considered. The existence of a unique solution in a certain function class is established. Formulas representing the solution are given.

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Keywords: conjugate Volterra integral equation, super-singularity in kernel, two-dimensional integral equation

1. INTRODUCTION

Let D denote the rectangle $D = \{a < x < a_0,\ b_0 < y < b\}$ and introduce the sets $\Gamma_1 = \{a < x < a_0,\ y = b\}$ and $\Gamma_2 = \{x = a,\ b_0 < y < b\}$. In D, we consider the two-dimensional integral equation

$$u(x,y) + \lambda \int_a^x \frac{u(t,y)}{(t-a)^\alpha} dt - \mu \int_y^b \frac{u(x,s)}{(b-s)^\beta} ds + \delta \int_a^x \frac{dt}{(t-a)^\alpha} \int_y^b \frac{u(t,s)}{(b-s)^\beta} ds = f(x,y), \quad (1.1)$$

and the integral equation conjugate to equation (1.1)

$$T^{\alpha,\beta}_{\lambda,\mu}(v) \equiv v(x,y) + \frac{\lambda}{(x-a)^\alpha} \int_x^{a_0} v(t,y) dt - \frac{\mu}{(b-y)^\beta} \int_{b_0}^y v(x,s) ds + \frac{\delta}{(x-a)^\alpha (b-y)^\beta} \int_x^{a_0} dt \int_{b_0}^y v(t,s) ds = g(x,y), \quad (1.2)$$

where $\alpha > 1$, $\beta > 1$, $\{\lambda, \mu, \delta\} \subset \mathbb{R}$, $f(x,y)$, $g(x,y)$ are the given functions and $u(x,y)$, $v(x,y)$, are the unknown functions.

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By the study of the solutions of the integral equations (1.1) and (1.2), the problem can be reduced to the determination of the continuous solution of a hyperbolic equation with two super-singular lines and its conjugate equation in D.

We seek a solution of equation (1.1) in the class of functions $u(x,y) \in C(\overline{D})$ that vanish on the singular lines Γ_1 and Γ_2. Moreover, we will assume that the unknown function $u(x,y)$ vanishes as $x \to a$ by an order higher than $\alpha - 1$, and it vanishes as $y \to b$ by an order higher than $\beta - 1$.

We note that the one-dimensional integral equations of types (1.1) and (1.2) are studied in [2,4–6]. Two-dimensional , three-dimensional and some many-dimensional Volterra type integral equations of type (1.1) are studied in [2,3,7–9]. One-dimensional singular integral equations with Cauchy kernels are considered in [1].

2. The case where $\delta = -\lambda\mu$

In this case, the integral equation (1.2) can be represented in the following form:

$$v(x,y) - \frac{\mu}{(b-y)^\beta}\int_{b_0}^{y} v(x,s)ds + \frac{\lambda}{(x-a)^\alpha}\int_{x}^{a_0}\left[v(t,y) - \frac{\mu}{(b-y)^\beta}\int_{b_0}^{y} v(t,s)ds\right]dt = g(x,y).$$

If we introduce a new unknown function

$$W(x,y) = v(x,y) - \frac{\mu}{(b-y)^\beta}\int_{b_0}^{y} v(x,s)ds, \tag{2.1}$$

we arrive to a one-dimensional conjugate Volterra type integral equation

$$W(x,y) + \frac{\lambda}{(x-a)^\alpha}\int_{x}^{a_0} W(t,y)dt = g(x,y). \tag{2.2}$$

In the case where $a < x < a_0$, according to [3], the integral equation (2.2) has a unique solution which is given by the formula

$$W(x,y) = g(x,y) - \frac{\lambda}{(x-a)^\alpha}\int_{x}^{a_0} \exp[\lambda(\omega_a^\alpha(t) - \omega_a^\alpha(x))]g(t,y)dt, \tag{2.3}$$

where $\omega_a^\alpha(x) = [(\alpha-1)(x-a)^{\alpha-1}]^{-1}$.

Analogously, the solution of the integral equation (2.1), for $b_0 < y < b$, is given by the formula

$$v(x,y) = W(x,y) + \frac{\mu}{(b-y)^\beta}\int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y) - \omega_b^\beta(s))]W(t,y)ds, \tag{2.4}$$

where $\omega_b^\beta(y) = [(\beta-1)(b-y)^{\beta-1}]^{-1}$.

By substituting the value of $W(x,y)$ into (2.4), we obtain a general solution of the integral equation (1.2) in the form

$$v(x,y) = g(x,y) - \frac{\lambda}{(x-a)^\alpha}\int_x^{a_0} \exp[\lambda(\omega_a^\alpha(t) - \omega_a^\alpha(x))]g(t,y)dt + \frac{\mu}{(b-y)^\beta}\int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y) - \omega_b^\beta(s))]g(x,s)ds - \frac{\lambda\mu}{(x-a)^\alpha(b-y)^\beta}\int_x^{a_0} \exp[\lambda(\omega_a^\alpha(t) - \omega_a^\alpha(x))]dt \times \int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y) - \omega_b^\beta(s))]g(t,s)ds \equiv (T_{\lambda,\mu}^{\alpha,\beta})^{-1}g. \quad (2.5)$$

We thus obtain the following

Theorem 1. *Assume that in equation (1.2) the parameters are related by the equality $\delta = -\lambda\mu$, and $g(x,y) \in C(D)$. Then the non-homogeneous integral equation (1.2) has a unique solution in class $C(D)$, which is given by formula (2.5).*

3. The case where $\delta \neq -\lambda\mu$

In this case, the integral equation (1.2) can be represented in the following form:

$$T_{\lambda,\mu}^{\alpha,\beta}(v) \equiv v(x,y) + \frac{\lambda}{(x-a)^\alpha}\int_x^{a_0} v(t,y)dt - \frac{\mu}{(b-y)^\beta}\int_{b_0}^{y} v(x,s)ds - \frac{\lambda\mu}{(x-a)^\alpha(b-y)^\beta}\int_x^{a_0} dt\int_{b_0}^{y} v(t,s)ds = g(x,y) - \frac{\delta_1}{(x-a)^\alpha(b-y)^\beta}\int_x^{a_0} dt\int_{b_0}^{y} v(t,s)ds.$$

Let us introduce the function

$$g_1(x,y) = g(x,y) - \frac{\delta_1}{(x-a)^\alpha(b-y)^\beta}\int_x^{a_0} dt\int_{b_0}^{y} v(t,s)ds,$$

where $\delta_1 = \delta + \lambda\mu$. Clearly, $g_1(x,y) \in C(D)$. Then the solution of the integral equation $T_{\lambda,\mu}^{\alpha,\beta}(v) = g_1(x,y)$ is as follows:

$$v(x,y) = g_1(x,y) - \frac{\lambda}{(x-a)^\alpha}\int_x^{a_0} \exp[\lambda(\omega_a^\alpha(t) - \omega_a^\alpha(x))]g_1(t,y)dt + \frac{\mu}{(b-y)^\beta}\int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y) - \omega_b^\beta(s))]g_1(x,s)ds - \frac{\lambda\mu}{(x-a)^\alpha(b-y)^\beta}\int_x^{a_0} \exp[\lambda(\omega_a^\alpha(t) - \omega_a^\alpha(x))]dt \times \int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y) - \omega_b^\beta(s))]g_1(t,s)ds \equiv (T_{\lambda,\mu}^{\alpha,\beta})^{-1}g_1(x,y). \quad (3.1)$$

In formula (3.1), by substituting the value of $g_1(x,y)$ and rearranging the appropriate terms, we arrive to the solution of the integral equation

$$\Phi(x,y)+\frac{\delta_1}{(x-a)^{\alpha}(b-y)^{\beta}}\int_x^{a_0}dt\int_{b_0}^{y}\Phi(t,s)ds=E^{\alpha,\beta}_{\lambda,\mu}[g(x,y)], \tag{3.2}$$

where

$$\begin{aligned}E^{\alpha,\beta}_{\lambda,\mu}[g(x,y)]&=\exp[\lambda\omega_a^{\alpha}(x)-\mu\omega_b^{\beta}(y)](T^{\alpha,\beta}_{\lambda,\mu})^{-1}g(x,y)\\&=\exp[\lambda\omega_a^{\alpha}(x)-\mu\omega_b^{\beta}(y)]g(x,y)\\&-\frac{\lambda}{(x-a)^{\alpha}}\exp[-\mu\omega_b^{\beta}(y)]\int_x^{a_0}\exp[\lambda\omega_a^{\alpha}(t)]g(t,y)dt\\&+\frac{\mu}{(b-y)^{\beta}}\exp[\lambda\omega_a^{\alpha}(x)]\int_{b_0}^{y}\exp[-\mu\omega_b^{\beta}(s)]g(x,s)ds\\&-\frac{\lambda\mu}{(x-a)^{\alpha}(b-y)^{\beta}}\int_x^{a_0}\exp[\lambda\omega_a^{\alpha}(t)]dt\int_{b_0}^{y}\exp[-\mu\omega_b^{\beta}(s)]g(t,s)ds\end{aligned} \tag{3.3}$$

and

$$\Phi(x,y)=\exp[\lambda\omega_a^{\alpha}(x)-\mu\omega_a^{\alpha}(y)]v(x,y).$$

4. Representation of a solution by functional series of $\exp(-\omega_a^{\alpha}(x))$

We seek a solution for integral equation (3.2) in the class of functions that can be represented in the form

$$\Phi(x,y)=\sum_{n=1}^{\infty}(\exp(-\omega_a^{\alpha}(x)))^n\Phi_n(y)(x-a)^{-\alpha}, \tag{4.1}$$

where $\Phi_n(y)$ are unknown functions.

We assume that function $g(x,y)$ admits representation in the form

$$g(x,y)=\exp[-\lambda\omega_a^{\alpha}(x)+\mu\omega_b^{\beta}(y)]\sum_{n=1}^{\infty}[\exp(-\omega_a^{\alpha}(x))^n](x-a)^{-\alpha}g_n(y), \tag{4.2}$$

where $g_n(y)$ are known functions. Moreover, assume that the series (4.2) converges absolutely and uniformly. By substituting this value $g(x,y)$ into (3.3), we have

$$\begin{aligned}E^{\alpha,\beta}_{\lambda,\mu}[g(x,y)]&=\sum_{n=1}^{\infty}[\exp(-\omega_a^{\alpha}(x))]^n(x-a)^{-\alpha}\\&\times\left(\frac{n+\lambda}{n}\right)\left[g_n(y)+\mu(b-y)^{-\mu}\int_{b_0}^{y}g_n(s)ds\right]\\&-(x-a)^{-\alpha}\lambda\left[\sum_{n=1}^{\infty}n^{-1}\exp(-\omega_a^{\alpha}(a_0))^n\left(g_n(y)+\mu(b-y)^{-\mu}\int_{b_0}^{y}g_n(s)ds\right)\right].\end{aligned}$$

By substituting the values of $\Phi(x,y)$ and $E^{\alpha,\beta}_{\lambda,\mu}[g(x,y)]$ into the integral equation (3.2) and equating the coefficients at $[\exp(-\omega^{\alpha}_{a}(x))]^k$, $k=0,1,2,\ldots$, we obtain the following relations between the functions $\Phi_n(y)$ and $g_n(y)$, $n=0,1,2,\ldots$:

$$\delta_1(b-y)^{-\beta}\sum_{n=1}^{\infty}n^{-1}[\exp(-\omega^{\alpha}_{a}(a_0))]^n\int_{b_0}^{y}\Phi_n(s)ds$$
$$=-\sum_{n=1}^{\infty}n^{-1}[\exp(-\omega^{\alpha}_{a}(a_0))]^n\lambda[g_n(y)+(b-y)^{-\beta}\mu\int_{b_0}^{y}g_n(s)ds] \quad (4.3)$$

and

$$\Phi_n(y)-\frac{\delta_1}{n(b-y)^{\beta}}\int_{b_0}^{y}\Phi_n(s)ds=\left(\frac{n+\lambda}{n}\right)g_n(y)+\frac{\mu(n+\lambda)}{n(b-y)^{\mu}}\int_{b_0}^{y}g_n(s)ds. \quad (4.4)$$

According to [3], if the system of integral equation (4.4) has a solution, then it can be represented in the form

$$\Phi_n(y)=\frac{n+\lambda}{n}\Bigg[g_n(y)$$
$$+\left(\frac{\mu n-\delta_1}{n}\right)\frac{1}{(b-y)^{\beta}}\int_{b_0}^{y}\exp\left[\frac{\delta_1}{n}(\omega^{\beta}_{b}(s)-\omega^{\beta}_{b}(y))\right]g_n(s)ds\Bigg], \quad (4.5)$$

where $n=0,1,2,\ldots$, $\omega^{\beta}_{b}(y)=[(\beta-1)(b-y)^{\beta-1}]^{-1}$.

Furthermore, it follows from equality (4.3) that

$$\delta_1(b-y)^{-\beta}\int_{b_0}^{y}\Phi_n(s)ds=-\lambda[g_n(y)+\mu(b-y)^{-\beta}\int_{b_0}^{y}g_n(s)ds] \quad (4.6)$$

for $n=0,1,2,\ldots$. From expression (4.6), by substituting the values of $\Phi_n(s)$ according to formula (4.5), we obtain the equality

$$\left(\frac{n+\lambda}{n}\right)\Bigg[\frac{\mu n}{(b-y)^{\beta}}\int_{b_0}^{y}g_n(s)ds$$
$$-\frac{(\mu n-\delta_1)}{(b-y)^{\beta}}\int_{b_0}^{y}\exp\left(\frac{\delta_1}{n}(\omega^{\beta}_{b}(s)-\omega^{\beta}_{b}(y))\right)g_n(s)ds\Bigg]$$
$$=-\lambda\left[g_n(y)+\frac{\mu}{(b-y)^{\beta}}\int_{b_0}^{y}g_n(s)ds\right], \quad n=0,1,2,3\ldots. \quad (4.7)$$

From formula (4.1), by substituting the value $\Phi_n(y)$ from equality (4.5), where $\Phi(x,y)=v(x,y)\exp[\lambda\omega^{\alpha}_{a}(x)-\mu\omega^{\beta}_{b}(y)]$, we find

$$v(x,y)=\exp[-\lambda\omega_a^\alpha(x)+\mu\omega_b^\beta(y)]\sum_{n=1}^{\infty}\frac{\exp(-\omega_a^\alpha(x))^n}{(x-a)^\alpha}\left(\frac{n+\lambda}{n}\right)$$
$$\times\left[g_n(y)+\frac{\mu n-\delta_1}{n(b-y)^\beta}\int_{b_0}^{y}(\exp(\omega_b^\beta(s)-\omega_b^\beta(y)))^{\frac{\delta_1}{n}}g_n(s)ds\right]. \quad (4.8)$$

Thus, we arrive at the following conclusion.

Theorem 2. *Assume that in the integral equation* (1.2) $\delta\neq-\lambda\mu$, *and that the function* $g(x,y)$ *is represented by series* (4.2), *which converges absolutely and uniformly. Then the integral equation* (1.2) *has a solution in the class of functions* $v(x,y)$ *that are representable in the form*

$$v(x,y)=\exp[-\lambda\omega_a^\alpha(x)+\mu\omega_b^\beta(y)]\sum_{n=1}^{\infty}\frac{\Phi_n(y)}{(x-a)^\alpha\exp(\omega_a^\alpha(x))^n}.$$

Moreover, if the functions $g_k(y)$, $k=1,2,\ldots$, *in* (4.2) *satisfy the infinite system of solvability conditions* (4.7), *then that solution is unique and can be represented by by formula* (4.8).

Remark 1. In the case where $\delta\neq-\lambda\mu$, the solution of the integral equation (3.2) could be sought in the class of functions that are representable by a functional series of $\exp(-\omega_b^\beta(y))$, i. e.,

$$\Phi(x,y)=\sum_{n=1}^{\infty}\exp(-n\omega_b^\beta(y))(b-y)^{-\beta}W_n(x)$$

where $W_n(x)$ are unknown functions. Then one assumes that the function $g(x,y)$ is represented in the form

$$g(x,y)=\exp[-\lambda\omega_a^\alpha(x)+\mu\omega_b^\beta(y)]\sum_{n=1}^{\infty}\frac{\exp(-n\omega_b^\beta(y))}{(b-y)^\beta}g_n(x).$$

By modifying suitably the argument above, in that case, one can also obtain a statement similar to Theorem 2.

5. Remarks on a non-model integral equation

In the domain D, we consider the two-dimensional integral equation

$$u(x,y)+\int_a^x\frac{A(t)u(t,y)}{(t-a)^\alpha}dt-\int_y^b\frac{B(s)u(x,s)}{(b-s)^\beta}ds$$
$$+\int_a^x\frac{dt}{(t-a)^\alpha}\int_y^b\frac{c(t,s)u(t,s)}{(b-s)^\beta}ds=f(x,y), \quad (5.1)$$

and its conjugate equation

$$v(x,y)+\frac{A(x)}{(x-a)^{\alpha}}\int_{x}^{a_0}v(t,y)dt-\frac{B(y)}{(b-y)^{\beta}}\int_{b_0}^{y}v(x,s)ds + \frac{c(x,y)}{(x-a)^{\alpha}(b-y)^{\beta}}\int_{x}^{a_0}dt\int_{b_0}^{y}v(t,s)ds=g(x,y). \quad (5.2)$$

Integral equations of form (5.1) are studied in [8].

Remark 2. One can find a solution of the integral equation (5.2) if $c(x,y)\equiv -A(x)B(y)$. In that case, as is shown in [6], the question is reduced to finding a solution of two split systems of one-dimensional conjugate integral equations of type (5.2).

Remark 3. In the case where $c(x,y)\not\equiv -A(x)B(y)$, the problem of finding solution for integral equation (5.2) is reduced to the problem of the determination of a solution of the integral equation

$$v(x,y) + \frac{c_1(x,y)}{(x-a)^{\alpha}(b-y)^{\beta}}\int_{x}^{a_0}\exp[A(a)(\omega_{\alpha}(t)-\omega_{\alpha}(x))-W^{-}_{A,\alpha}(t)-W^{-}_{A,\alpha}(x)]dt \times\int_{b_0}^{y}\exp[B(b)(\omega^{\beta}_{b}(y)-\omega^{\beta}_{b}(s))+W^{-}_{b,\beta}(s)-W^{-}_{b,\beta}(y)]v(t,s)ds \equiv (T^{\alpha,\beta}_{A(x),B(y)})^{-1}(g(x,y)),$$

for any $(x,y)\in D$, where $(T^{\alpha,\beta}_{A(x),B(y)})^{-1}$ is a known integral operator.

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A GENERALIZATION OF ČEBYŠEV TYPE INEQUALITIES FOR FIRST DIFFERENTIABLE MAPPINGS

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Abstract. In this paper, we improve and further generalize some Čebyšev type inequalities involving functions whose derivatives belong to L_p spaces via certain integral identities.

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1. INTRODUCTION

In 1882, P. L. Čebyšev proved the following important integral inequality [13],[9, p.207]:

$$T(f,g) \leq \frac{1}{12}(b-a)^2 \left\|f'\right\|_\infty \left\|g'\right\|_\infty, \tag{1.1}$$

where $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous functions whose derivatives $f',g'\in L_\infty[a,b]$ and

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}\int_a^b f(x)dx\left(\frac{1}{b-a}\int_a^b g(x)dx\right), \tag{1.2}$$

which is called the Čebyšev functional, provided the integrals in (1.2) exist.

During the past few years many researchers have given considerable attention to the inequality of (1.1). Various generalizations, extensions and variants of this inequality have appeared in the literature, see ([1–13]) and the references cited therein.

Recently, Pachpatte [8] established a new Čebyšev type inequality involving functions whose derivatives belong to L_p spaces:

Theorem 1. *Let $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous functions whose derivatives $f',g'\in L_p[a,b]$, $p>1$. Then, we have inequalities*

$$T(f,g)\leq\frac{1}{(b-a)^3}\left\|f'\right\|_p\left\|g'\right\|_p\int_a^b (B(x))^{\frac{2}{q}}\,dx,$$

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and

$$T(f,g) \leq \frac{1}{2(b-a)^2}\int_a^b \left[g(x)\left\|f'\right\|_p + f(x)\left\|g'\right\|_p\right](B(x))^{\frac{1}{q}}\,dx, \tag{1.3}$$

where

$$B(x) = \frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1},$$

for $x \in [a,b]$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Motivated by the results of Pachpatte, in the present paper we establish some new Čebyšev type inequalities for $\frac{1}{p}+\frac{1}{q}=1$. The analysis used in the proofs is elementary and our results provide new estimates for these types of inequalities.

2. Main results

Let $[a,b] \subset \mathbb{R}$, $a<b$; and as usual for any function $h \in L_p[a,b]$, $p>1$, we define $\|h\|_p = \left(\int_a^b |h(t)|^p\,dt\right)^{\frac{1}{p}}$. We use the following notation to simplify the details of presentation. For suitable functions $f,g:[a,b]\to\mathbb{R}$ and $\lambda \in [0,1]$, we set

$$\begin{aligned} & T_\lambda(f,g) \\ = \; & \frac{1}{b-a}\left(1-\frac{\lambda}{2}\right)\int_a^b f(x)g(x)dx \\ & -\frac{\lambda}{4(b-a)^2}\left\{\int_a^b [(x-b)f(b)-(x-a)f(a)]g(x)dx\right. \\ & \qquad \left. +\int_a^b [(x-b)g(b)-(x-a)g(a)]f(x)dx\right\} \\ & -\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right). \end{aligned}$$

Now, let us state our main results.

Theorem 2. *Let* $f,g:[a,b]\to\mathbb{R}$ *be absolutely continuous functions whose derivatives* $f',g' \in L_p[a,b]$, $p>1$. *Then, the following inequality holds:*

$$|T_\lambda(f,g)| \leq \frac{1}{2(b-a)^2}\int_a^b \left[g(x)\left\|f'\right\|_p + f(x)\left\|g'\right\|_p\right](Z(x))^{\frac{1}{q}}\,dx, \tag{2.1}$$

where

$$Z(x) = \frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\left(\left(\frac{\lambda}{2}\right)^{q+1}+\left(1-\frac{\lambda}{2}\right)^{q+1}\right),$$

for $x \in [a,b]$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $\lambda \in [0,1]$.

Proof. From the hypotheses, we can write the following identities,

$$\frac{1}{b-a}\int_a^b p(x,t)f'(t)dt$$
$$=\left(1-\frac{\lambda}{2}\right)f(x)-\lambda\frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(t)dt, \tag{2.2}$$

and

$$\frac{1}{b-a}\int_a^b p(x,t)g'(t)dt$$
$$=\left(1-\frac{\lambda}{2}\right)g(x)-\lambda\frac{(x-b)g(b)-(x-a)g(a)}{2(b-a)}-\frac{1}{b-a}\int_a^b g(t)dt, \tag{2.3}$$

for $x\in[a,b]$, $\lambda\in[0,1]$, where

$$p(x,t)=\begin{cases} t-\left(a+\lambda\dfrac{x-a}{2}\right), & a\le t\le x \\ t-\left(b-\lambda\dfrac{b-x}{2}\right), & x<t\le b \end{cases}.$$

A simple proof of equalities of (2.2) and (2.3) can be done by integrating in the left hand side and using the identity of $p(x,t)$. The details are left for the interested reader.

Multiplying both sides of (2.2) and (2.3) by $g(x)$ and $f(x)$ respectively and adding the resulting identities, we get

$$\begin{aligned} &\frac{1}{b-a}g(x)\int_a^b p(x,t)f'(t)dt+\frac{1}{b-a}f(x)\int_a^b p(x,t)g'(t)dt \qquad (2.4)\\ =\;& 2\left(1-\frac{\lambda}{2}\right)f(x)g(x)-\lambda\frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)}g(x)\\ &-\lambda\frac{(x-b)g(b)-(x-a)g(a)}{2(b-a)}f(x)\\ &-\frac{1}{b-a}g(x)\int_a^b f(t)dt-\frac{1}{b-a}f(x)\int_a^b g(t)dt. \end{aligned}$$

Multiplying both sides of (2.4) by $\frac{1}{b-a}$, we get

$$\begin{aligned} &\frac{2}{b-a}\left(1-\frac{\lambda}{2}\right)f(x)g(x) \qquad (2.5)\\ &-\lambda\frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)^2}g(x)-\lambda\frac{(x-b)g(b)-(x-a)g(a)}{2(b-a)^2}f(x)\\ &-\frac{1}{(b-a)^2}g(x)\int_a^b f(t)dt-\frac{1}{(b-a)^2}f(x)\int_a^b g(t)dt \end{aligned}$$

$$= \frac{1}{(b-a)^2} g(x) \int_a^b p(x,t) f'(t)dt + \frac{1}{(b-a)^2} f(x) \int_a^b p(x,t) g'(t)dt.$$

Integrating both sides of (2.5) with respect to x over $[a,b]$, we get

$$\begin{aligned} & \frac{2}{b-a}\left(1-\frac{\lambda}{2}\right)\int_a^b f(x)g(x)dx \\ & -\frac{\lambda}{2(b-a)^2}\left\{\int_a^b [(x-b)f(b)-(x-a)f(a)]g(x)dx\right. \\ & \left.+\int_a^b [(x-b)g(b)-(x-a)g(a)]f(x)dx\right\} \\ & -2\left(\frac{1}{b-a}\int_a^b g(x)dx\right)\left(\frac{1}{b-a}\int_a^b f(x)dx\right) \\ = & \frac{1}{(b-a)^2}\int_a^b g(x)\left(\int_a^b p(x,t)f'(t)dt\right)dx \\ & +\frac{1}{(b-a)^2}\int_a^b f(x)\left(\int_a^b p(x,t)g'(t)dt\right)dx. \end{aligned} \tag{2.6}$$

After the rewriting of the equality of (2.5), we obtain

$$T_\lambda(f,g) = \frac{1}{2(b-a)^2}\int_a^b \left[g(x)\int_a^b p(x,t)f'(t)dt + f(x)\int_a^b p(x,t)g'(t)dt\right]dx. \tag{2.7}$$

From (2.7) and using the properties of modulus, Hölder's integral inequality, we have

$$\begin{aligned} & |T_\lambda(f,g)| \\ & \le \frac{1}{2(b-a)^2}\int_a^b \left[|g(x)|\int_a^b |p(x,t)|\,|f'(t)|\,dt + |f(x)|\int_a^b |p(x,t)|\,|g'(t)|\,dt\right]dx \\ & \le \frac{1}{2(b-a)^2}\int_a^b \left[|g(x)|\left(\int_a^b |p(x,t)|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^b |f'(t)|^p\,dt\right)^{\frac{1}{p}}\right. \\ & \qquad \left. + |f(x)|\left(\int_a^b |p(x,t)|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^b |g'(t)|^p\,dt\right)^{\frac{1}{p}}\right]dx \\ & = \frac{1}{2(b-a)^2}\int_a^b \left[g(x)\|f'\|_p + f(x)\|g'\|_p\right]\left(\int_a^b |p(x,t)|^q\,dt\right)^{\frac{1}{q}}dx. \end{aligned} \tag{2.8}$$

A simple calculation shows that

$$\begin{aligned}
&\int_a^b |p(x,t)|^q\,dt \qquad (2.9)\\
&= \int_a^x \left|t-\left(a+\lambda\frac{x-a}{2}\right)\right|^q dt + \int_x^b \left|t-\left(b-\lambda\frac{b-x}{2}\right)\right|^q dt\\
&= (x-a)^q\int_a^x \left|\frac{t-a}{x-a}-\frac{\lambda}{2}\right|^q dt + (b-x)^q\int_x^b \left|\frac{t-b}{b-x}-\frac{\lambda}{2}\right|^q dt\\
&= \frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\left(\left(\frac{\lambda}{2}\right)^{q+1}+\left(1-\frac{\lambda}{2}\right)^{q+1}\right)\\
&= Z(x).
\end{aligned}$$

Using (2.8) and (2.9), we get the required inequality in (2.1). The proof is complete. □

Remark 1. If we choose $\lambda = 0$ in (2.1), then we get (1.3).

Corollary 1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function whose derivative $f' \in L_p[a,b]$, $p > 1$. Then, the following inequality holds:*

$$\left|\left(1-\frac{\lambda}{2}\right)f(x)-\lambda\frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{1}{b-a}\left\|f'\right\|_p (Z(x))^{\frac{1}{q}}. \qquad (2.10)$$

Proof. From (2.2) using the properties of modulus and applying Hölder's integral inequalty, we have

$$\begin{aligned}
&\left|\left(1-\frac{\lambda}{2}\right)f(x)-\lambda\frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(t)dt\right|\\
&\le \frac{1}{b-a}\left(\int_a^b |f'(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_a^b |p(x,t)|^q\,dt\right)^{\frac{1}{q}}\\
&= \frac{1}{b-a}\|f'\|_p (Z(x))^{\frac{1}{q}}.
\end{aligned}$$

□

Remark 2. i) If we choose $\lambda = 1$ in (2.10), then we get

$$\left|\frac{1}{2}f(x)-\frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{1}{2(b-a)}\left(\frac{(x-a)^{q+1}+(b-x)^{q+1}}{(q+1)}\right)^{\frac{1}{q}}\left\|f'\right\|_p.$$

ii) For $\lambda = 0$ in (2.10), we have,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{(b-a)}\left(\frac{(x-a)^{q+1}+(b-x)^{q+1}}{(q+1)}\right)^{\frac{1}{q}}\left\|f'\right\|_p.$$

Theorem 3. *Let $f,g:[a,b]\to\mathbb{R}$ be absolutely continuous functions whose derivatives $f',g'\in L[a,b]$. If $|f'|^q$ and $|g'|^q$ is convex on $[a,b]$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$. Then, the following inequality holds:*

$$\begin{aligned}&|T(f,g)|\\ \leq\ &\frac{1}{2}\int_a^b\left\{\left[|f(x)|\left(\frac{|g'(a)|^q+|g'(b)|^q}{2}\right)^{\frac{1}{q}}\right.\right.\\ &\left.\left.+\,|g(x)|\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right]M^{\frac{1}{p}}(x)\right\}dx \qquad (2.11)\end{aligned}$$

where

$$M(x)=\frac{(b-x)^{p+1}+(x-a)^{p+1}}{(b-a)^{p+1}(p+1)}.$$

Proof. From the hypotheses, we can write the following identities,

$$f(x)=\frac{1}{b-a}\int_a^b f(x)dx+(b-a)\int_0^1 P(x,t)f'(ta+(1-t)b)dt, \qquad (2.12)$$

and

$$g(x)=\frac{1}{b-a}\int_a^b g(x)dx+(b-a)\int_0^1 P(x,t)g'(ta+(1-t)b)dt, \qquad (2.13)$$

for $x\in[a,b]$, where

$$P(x,t)=\begin{cases}t, & 0\leq t\leq \frac{b-x}{b-a}\\ t-1, & \frac{b-x}{b-a}<t\leq 1\end{cases}.$$

A simple proof of equalities of (2.12) and (2.13) can be obtained by integrating in the right hand side and using the identity of $p(x,t)$. The details are left for the interested reader.

Multiplying both sides of (2.12) and (2.13) by $g(x)$ and $f(x)$ respectively and adding the resulting identities, we get

$$2f(x)g(x) \qquad (2.14)$$

$$= \frac{g(x)}{b-a}\int_a^b f(x)dx + (b-a)g(x)\int_0^1 P(x,t)f'(ta+(1-t)b)dt$$
$$+\frac{f(x)}{b-a}\int_a^b g(x)dx + (b-a)f(x)\int_0^1 P(x,t)g'(ta+(1-t)b)dt.$$

Integrating both sides of (2.14) with respect to x over $[a,b]$ and dividing both sides of the resulting identity $(b-a)$, we get

$$\frac{2}{b-a}\int_a^b f(x)g(x)dx \tag{2.15}$$
$$= 2\left(\frac{1}{b-a}\int_a^b g(x)dx\right)\left(\frac{1}{b-a}\int_a^b f(x)dx\right)$$
$$+\int_a^b g(x)\left[\int_0^1 P(x,t)f'(ta+(1-t)b)dt\right]dx$$
$$+\int_a^b f(x)\left[\int_0^1 P(x,t)g'(ta+(1-t)b)dt\right]dx.$$

Rewriting the identity (2.15), we get

$$T(f,g) = \frac{1}{2}\int_a^b g(x)\left[\int_0^1 P(x,t)f'(ta+(1-t)b)dt\right]dx \tag{2.16}$$
$$+\frac{1}{2}\int_a^b f(x)\left[\int_0^1 P(x,t)g'(ta+(1-t)b)dt\right]dx.$$

From (2.16), using the properties of modulus and rearranging the terms, we get

$$|T(f,g)|$$
$$\leq \frac{1}{2}\int_a^b |f(x)|\left[\int_0^1 |P(x,t)|\left|g'(ta+(1-t)b)\right|dt\right]dx$$
$$+\frac{1}{2}\int_a^b |g(x)|\left[\int_0^1 |P(x,t)|\left|f'(ta+(1-t)b)\right|dt\right]dx.$$

Applying Hölder's integral inequality and the convexity of $|f'|^q$ and $|g'|^q$, we obtain

$$|T(f,g)| \leq \frac{1}{2}\int_a^b \left[|f(x)|\left(\int_0^1 |P(x,t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|g'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\,|g(x)|\left(\int_0^1 |P(x,t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right]dx$$

$$= \frac{1}{2}\int_a^b \left\{ \left[|f(x)| \left(\frac{|g'(a)|^q + |g'(b)|^q}{2} \right)^{\frac{1}{q}} \right.\right. \tag{2.17}$$
$$\left.\left. + |g(x)| \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \right] \left(\int_0^1 |P(x,t)|^p \, dt \right)^{\frac{1}{p}} \right\} dx.$$

A simple computation gives

$$\int_0^1 |P(x,t)|^p \, dt = \int_0^{\frac{b-x}{b-a}} t^p dt + \int_{\frac{b-x}{b-a}}^1 (t-1)^p \, dt \tag{2.18}$$
$$= \frac{(b-x)^{p+1} + (x-a)^{p+1}}{(b-a)^{p+1}(p+1)} = M(x).$$

Using (2.18) in (2.18), we get the required inequality in (2.11). □

Corollary 2. *If in theorem 3 we require that* $|f'(x)| \le K$, $K > 0$ *and* $|g'(x)| \le N$, $N > 0$, *then the following inequality holds*

$$|T(f,g)| \le \frac{1}{2}\left(\int_a^b [|f(x)| N + |g(x)| K] M^{\frac{1}{p}}(x) dx \right),$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

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OPTIMAL FOURTH-ORDER ITERATIVE METHODS FREE FROM DERIVATIVES

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Abstract. The construction of optimal fourth-order iterative schemes for solving univariate nonlinear equations is discussed. Per iteration, the methods consist of three evaluations of the function and they are free from any derivative calculation which property is so fruitful in engineering problems. We analytically show the fourth-order convergence. Numerical examples are considered to confirm the applicability and to justify the rapid convergence of the novel iterative algorithms.

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1. INTRODUCTION

This paper is concerned with the numerical solution of single valued nonlinear equations of the form $f(x) = 0$. The exact and analytical solutions of such equations are not always at hand. That is why the accurate iterative methods, in which the total number of evaluations is appropriate, are required. Let α be the root of such an equation. This root can fall into two categories, i.e., it can be a simple zero [8] or a multiple zero [2]. In this study, we are concerned with simple zeros. Let α be in the open interval D, let it be the simple root the nonlinear equation $f(x) = 0$, then $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. In engineering problems and/or in the real-world situations, when the calculation of the derivatives of the functions are not practical and/or cost so much time, we need root solvers that do not use derivative calculations per iteration in order to obtain an accurate approximation of the exact root. For example, consider the problem of finding the simple roots of the function $g(x) = \sin(2\cos(x)) - 1 - x^2 + e^{\sin(x^3)}$. Clearly, finding the derivatives of this function is hard and takes a great deal of time. Hence, we provide efficient fourth-order methods which are optimal and are based on the still unproved Kung-Traub hypothesis [4] concerning the "multi-point without memory" iteration schemes. According to their conjecture of optimality for multi-point iterations, a without memory iteration with n evaluation per cycle can reach the optimal order of convergence $2^{(n-1)}$ and the

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optimal efficiency index $2^{(n-1)/n}$. In fact, in 1964 Traub in the fundamental work [9] described that iterative (derivative-free or derivative-involved) methods for solving one variable nonlinear equations. He distinguished two main classes; one-point (one-step) with or without memory iterations and; multi-point (multi-step) iterations with or without memory. He also proved that the optimal order of convergence for a one-point (one-step) method, i.e. for Newton's or for Steffensen's, is two. After 10 years of that, in 1974 Kung and Traub [4] mentioned a hypothesis stating that the maximum convergence order for the multi-point without memory iterations cannot exceed $2^{(n-1)}$ where n is the whole number of evaluations per full cycle. The reason of interest in multi-point iterations is that they possess better convergence rate and efficiency in solving real-world problems. Thus, in order to obtain an efficient method of order four in this work, we are allowed to use three evaluations of the function per iteration in the sense of Kung-Traub conjecture.

We here remark that if the sequence $\{x_n\}_{n=0}^{\infty}$ for a positive λ and p satisfies the following relation

$$\lim_{n\to\infty} \frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^p} = \lambda, \tag{1.1}$$

then the iterative method which produces this sequence has local order of convergence p. We also remind that, the Ostrowski-Traub efficiency index could be provided by $p^{(1/\theta)}$, where θ is the total number of evaluations of the iterative scheme per full cycle.

The article is summarized as follows. After collecting some important derivative-free root solvers in Section 2, our optimal algorithms are developed in Section 3. In Section 4, comparisons are made between the existing methods and the new techniques to prove that our novel derivative-free techniques are effective and convenient. Finally in Section 5, our conclusion is presented.

2. Selections from the literature

For a long time the Steffensen's method, which is given by

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n+f(x_n))-f(x_n)}, \tag{2.1}$$

was the only available quadratically derivative-free scheme. This method was obtained by replacing the forward finite-difference approximation for the first derivative in the well-known Newton's method.

In 2001, Wu et al. [10] presented another optimal derivative-free second-order method as follows

$$x_{n+1} = x_n - \frac{f(x_n)^2}{bf(x_n)^2+f(x_n+f(x_n))-f(x_n)}, \tag{2.2}$$

where the parameter b should be chosen such that the denominator is non-zero; for example, $b = sign\{f(x_n+f(x_n))-f(x_n)\}$. Motivated by these methods, two-step

iterative methods have been invented to increase the order of convergence and the efficiency indices.

In 2007, a derivative-free method of order three [3], in which we have three evaluations of the function, had been presented in the following form

$$\begin{cases} y_n = x_n - \frac{f(x_n)^2}{f(x_n+f(x_n))-f(x_n)}, \\ x_{n+1} = x_n - \frac{f^3(x_n)}{[f(x_n+f(x_n))-f(x_n)][f(x_n)-f(y_n)]}. \end{cases} \tag{2.3}$$

In 2010, another third-order iterative algorithm had been developed by Dehghan and Hajarian in [1]

$$\begin{cases} y_n = x_n - \frac{f(x_n)^2}{f(x_n+f(x_n))-f(x_n)}, \\ x_{n+1} = x_n - \frac{f(x_n)[f(y_n)+f(x_n)]}{f(x_n+f(x_n))-f(x_n)}. \end{cases} \tag{2.4}$$

As we can see, this algorithm also includes three evaluations of the function per iteration and therefore is not optimal with high efficiency index.

Recently, an accurate optimal fourth-order method [5] was proposed by Liu et al. as follows

$$\begin{cases} y_n = x_n - \frac{f(x_n)^2}{f(x_n+f(x_n))-f(x_n)}, \\ x_{n+1} = y_n - \frac{f[x_n,y_n]-f[y_n,z_n]+f[x_n,z_n]}{f[x_n,y_n]^2} f(y_n), \end{cases} \tag{2.5}$$

where $z_n = x_n + f(x_n)$. This method consists of three evaluations of the function per iteration in order to obtain fourth-order convergence. In this method $f[x_n, y_n], f[y_n, z_n], f[x_n, z_n]$ are divided differences of $f(x)$. We recall that they can be defined recursively via $f[x_i] = f(x_i)$; $f[x_i, x_j] = \frac{f[x_i]-f[x_j]}{x_i-x_j}$, $x_i \neq x_j$.

In 2011, Zheng et al. in [11] extended the approach given by Liu et al. (2.5) to provide a three-parameter family of iterartions with optimal convergence rate four

$$\begin{cases} y_n = x_n - \frac{f(x_n)^2}{f(x_n+f(x_n))-f(x_n)}, \\ x_{n+1} = y_n - \frac{f[x_n,y_n]+(p-1)f[y_n,z_n]-(p-1)f[x_n,z_n]-\beta(y_n-x_n)(y_n-z_n)}{f[x_n,y_n]+pf[y_n,z_n]-pf[x_n,z_n]+a(y_n-x_n)(y_n-z_n)} \frac{f(y_n)}{f[x_n,y_n]}, \end{cases} \tag{2.6}$$

where $z_n = x_n + f(x_n)$ and β, a, p are real valued parameters.

We remark here that Kung and Traub in the fundamental paper [4] provided the following derivative-free family ($\beta \in \mathbb{R} - \{0\}$) of methods by using the Inverse Interpolation

$$\begin{cases} y_n = x_n + \beta f(x_n), \\ z_n = y_n - \beta \frac{f(x_n)f(y_n)}{f(y_n)-f(x_n)}, \\ x_{n+1} = z_n - \frac{f(x_n)f(y_n)}{f(z_n)-f(x_n)}[\frac{1}{f[y_n,x_n]} - \frac{1}{f[z_n,y_n]}], \end{cases} \tag{2.7}$$

which have the following error equation $e_{n+1}=\frac{c_2(2c_2^2-c_1c_3)(1+c_1\beta)^2}{c_1^3}e_n^4+O(e_n^5)$ where $c_j=\frac{f^{(j)}(\alpha)}{j!}$, $j\geq 1$.

For a more detailed description of this topic consult the papers [6,7] and the references therein.

3. The proposed methods

In this section, our contribution is based on the following two-step relation

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}. \end{cases} \tag{3.1}$$

Unfortunately, (3.1) includes two evaluations of the function and two evaluations of the first derivative per full cycle to the reach the convergence order 4 with $4^{1/4}\approx 1.4142$ as its index of efficiency. Hence, we must remedy two problems in (3.1). First, the derivative evaluations should disappear from this structure, and second, the number of evaluations must be decreased from four to three in such a way that the convergence rate does not decrease. Due to these requirements, the first thing that we could do is to replace $f'(y_n)$ with $f'(x_n)$ in the second sub-step. The second thing is to approximate $f'(x_n)$ as Steffensen did; see (2.1). This path reduces the number of evaluations from four to three, unfortunately the order decreases as well. Anyhow, we attain

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,A_n]}, \\ x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)}. \end{cases} \tag{3.2}$$

where $A_n=x_n+f(x_n)$ and $n=0,1,2,\cdots$. To remedy the existing problems in (3.2), first, $f'(x_n)$ should be estimated as efficiently as possible. Hopefully, we have the value of the function in the nodes x_n, A_n and y_n by now. By using the well-known method of undeterministic coefficients in the interpolating polynomial $p(t)=a_1+a_2(t-x_n)+a_3(t-x_n)^2$, we obtain a robust approximation of $f'(x_n)$ in the second step of (3.2). Note that at the second step of (3.2), we have the values of the function in three nodes, but in the first step we have the values of the function only at two nodes. That is why f at this step can be approximated by an interpolation polynomial of order two. In other words, $p'(x_n)=a_2$. Thus, by substituting the known data in the interpolation conditions $p(x_n)=f(x_n)$, $p(A_n)=f(A_n)$ and $p(y_n)=f(y_n)$; we obtain $f'(x_n)\approx\frac{x_nf[x_n,A_n]+A_nf[x_n,y_n]-x_nf[x_n,y_n]-f[x_n,A_n]y_n}{A_n-y_n}$. Therefore, we attain

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,A_n]}, \\ x_{n+1} = y_n - \frac{(A_n-y_n)f(y_n)}{(x_n-y_n)f[x_n,A_n]+(A_n-x_n)f[x_n,y_n]}, \end{cases} \tag{3.3}$$

where the total number of evaluations are three and it is free from derivative.(3.3) satisfies the error equation $e_{n+1}=\frac{2(1+c_1)c_2^2}{c_1^2}e_n^3+O(e_n^4)$, which shows that it is cubically

convergent and reaches the index of efficiency $3^{1/3} \approx 1.442$. Thus, our aim has not been achieved at this point. In order to reach the optimal order of convergence four with three evaluations of the function per iteration; we construct a weight function at the second step as follows:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n, A_n]}, \\ x_{n+1} = y_n - \frac{(A_n - y_n) f(y_n)}{(x_n - y_n) f[x_n, A_n] + (A_n - x_n) f[x_n, y_n]} [1 + \gamma \frac{f(y_n)}{f(A_n)}]. \end{cases} \tag{3.4}$$

Its error equation is $e_{n+1} = -\frac{(1+c_1)c_2^2(-2+\gamma)}{c_1^2} e_n^3 + O(e_n^4)$ and γ should be determined such that the order of convergence reaches four. That is to say, the coefficient of e_n^3 should vanish. Due to this, $\gamma = 2$ is taken into consideration. Finally, we obtain

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n, A_n]}, \\ x_{n+1} = y_n - \frac{(A_n - y_n) f(y_n)}{(x_n - y_n) f[x_n, A_n] + (A_n - x_n) f[x_n, y_n]} [1 + 2 \frac{f(y_n)}{f(A_n)}]. \end{cases} \tag{3.5}$$

The order of convergence for the preceding method is analyzed in Theorem 1.

Theorem 1. *Assume that $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is a sufficiently many times differentiable function with a simple zero $\alpha \subseteq D$, let D be an open interval, let x_0 be a guess close enough to α. In this case the new method (3.5) has fourth-order convergence.*

Proof. Let $e_n = x_n - \alpha$, $f(\alpha) = 0$ and $c_j = \frac{f^{(j)}(\alpha)}{j!}$, $j \geq 1$. Using Taylor expansion, we have

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5), \tag{3.6}$$

and $f(A_n) = f(x_n + f(x_n)) = e_n(1 + c_1 + e_n(c_2 + e_n(c_3 + c_4 e_n)))(c_1 + c_2 e_n(1 + c_1 + e_n(c_2 + e_n(c_3 + c_4 e_n))) + c_3 e_n^2(1 + c_1 + e_n(c_2 + e_n(c_3 + c_4 e_n)))^2 + c_4 e_n^3(1 + c_1 + e_n(c_2 + e_n(c_3 + c_4 e_n)))^3)$. Furthermore, we obtain

$$x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)} = \alpha + (1 + \frac{1}{c_1}) c_2 e_n^2$$

$$+ \frac{(-(2 + (-2 + c_1)c_1)c_2^2 + c_1(1 + c_1)(2 + c_1)c_3)}{c_1^2} e_n^3 + O(e_n^4). \tag{3.7}$$

Now we should expand $f(y_n)$ around the simple root by using (3.7). We have

$$f(y_n) = (1 + c_1)c_2 e_n^2 + (-\frac{(2 + c_1(2 + c_1))c_2^2}{c_1} + (1 + c_1)(2 + c_1)c_3) e_n^3$$

$$+ \frac{1}{c_1^2}((5 + c_1(7 + c_1(4 + c_1)))c_2^3 - c_1(7 + c_1(10 + c_1(7 + 2c_1)))c_2 c_3$$

$$+ c_1^2(1 + c_1)(3 + c_1(3 + c_1))c_4) e_n^4 + O(e_n^5). \tag{3.8}$$

Using (3.8) and the Taylor expansion for the approximation function which is used in the second step, of our derivative-free method gives us

$$\frac{(A_n-y_n)f(y_n)}{(x_n-y_n)f[x_n,A_n]+(A_n-x_n)f[x_n,y_n]}=(1+\frac{1}{c_1})c_2e_n^2 + \frac{(2+c_1)(-(2+c_1)c_2^2+c_1(1+c_1)c_3)}{c_1^2}e_n^3+O(e_n^4). \tag{3.9}$$

For the weight function in the last step of (3.5), we obtain

$$1+2\frac{f(y_n)}{f(A_n)}=1+\frac{2c_2}{c_1}e_n^1+\frac{(-2(3+2c_1)c_2^2+2c_1(2+c_1)c_3)}{c_1^2}e_n^2+O(e_n^3). \tag{3.10}$$

And finally, considering the above relations (3.7)-(3.10) in the last step of (3.5), we obtain

$$e_{n+1}=x_{n+1}-\alpha=\frac{(1+c_1)c_2((5+3c_1)c_2^2-c_1(1+c_1)c_3)}{c_1^3}e_n^4+O(e_n^5). \tag{3.11}$$

This completes the proof and shows that our proposed optimal derivative-free method is a fourth-order algorithm and agrees with the still un-proved conjecture of Kung-Traub on the optimality of multi-point iterations without memory. □

Remark 1. If one chooses the backward finite difference approximation in the Steffensen's method, i.e. $A_n=x_n-f(x_n)$, then another novel optimal fourth order method can be obtained as follows

$$\begin{cases} y_n=x_n-\frac{f(x_n)}{f[x_n,A_n]},\\ x_{n+1}=y_n-\frac{(A_n-y_n)f(y_n)}{(x_n-y_n)f[x_n,A_n]+(A_n-x_n)f[x_n,y_n]}[1+2\frac{f(y_n)}{f(A_n)}], \end{cases} \tag{3.12}$$

where its error equation is

$$e_{n+1}=x_{n+1}-\alpha=\frac{(-1+c_1)c_2((-5+3c_1)c_2^2-(-1+c_1)c_1c_3)}{c_1^3}e_n^4+O(e_n^5). \tag{3.13}$$

4. Numerical implementations

In this section, numerical examples are furnished to re-verify the effectiveness of the proposed derivative-free methods. The comparison among the presented methods (**PM1**) (3.5) and (**PM2**) (3.12), the third-order method of Jain (**JM**), the third-order method of Dehghan and Hajarian (**DHM**), the fourth-order method of Liu et al. (**LM**), the fourth-order method of Kung and Traub with $\beta=1$ (**KTM1**) and $\beta=3$ (**KTM3**) are given. The test functions with their roots (up to 10 decimal places when they are not integers) are:

$$
\begin{aligned}
&f_1(x) = (\sin x)^2 + x, && \alpha = 0,\\
&f_2(x) = (1+x^3)\cos(\tfrac{\pi x}{2}) + \sqrt{1-x^2} - \tfrac{2(9\sqrt{2}+7\sqrt{3})}{27}, && \alpha = 1/3,\\
&f_3(x) = (\sin x)^2 - x^2 + 1, && \alpha \approx 1.4044916482,\\
&f_4(x) = e^{-x} + \sin(x) - 1, && \alpha \approx 2.0768312745,\\
&f_5(x) = xe^{-x} - 0.1, && \alpha \approx 0.1118325591,\\
&f_6(x) = x^2 + \sin(x) + x, && \alpha = 0,\\
&f_7(x) = \sin(2\cos(x)) - 1 - x^2 + e^{\sin(x^3)}, && \alpha \approx 1.3061752018,\\
&f_8(x) = \sin(2\cos(x)) - 1 - x^2 + e^{\sin(x^3)}, && \alpha \approx -0.7848959876,\\
&f_9(x) = \cos(x) + \sin(2x)\sqrt{1-x^2} + \sin(x^2) + x^{14} + x^3 + \tfrac{1}{2x}, && \alpha \approx -0.9257722498,\\
&f_{10}(x) = \tan(\ln(x)) + x^3 + \tfrac{1}{2x}, && \alpha \approx 0.3974036712.
\end{aligned}
$$

The results are provided in Table 1 based on the same Total Number of Evaluations (TNE) (12), i.e. after 4 iterations for JM, LM, KTM1, KTM2, PM1, and PM2. In fact, the absolute values of each test function for each starting point is listed in Table 1 after 4 full iterations. Note that Div. represents that the considered iterative scheme is divergent. All calculations were done with MATLAB 7.6 using 600 digit floating point (Digits: =600) with VPA Command. In the examples considered in this paper, the stopping criteria is the $|f(x_n)| \leq \varepsilon$, where $\varepsilon = 10^{-600}$. In the implementation of the iterative methods, the appropriate choice of initial approximation value x_0 is very important since a badly chosen initial approximation produces a bad predictor and, consequently, destroys rapid convergence. Thus, we have selected the starting points close enough to the sought zero.

It could be said from these numerical experiments that the presented methods appear to be superior in compared to the other methods. Table 1 also reveals that the methods introduced in this study have similar performance compared to the other known methods of the same order for most of the numerical cases. From the results displayed in Table 1 and the number of numerical experiments, it can be concluded that the proposed multi-point derivative-free schemes possess quick convergence for good initial approximations. Moreover, after an extensive experimentation, we could mention that among all of the compared derivative-free schemes, it is impossible to select one as the best. Since, they behave similarly, i.e. for some initial guesses one is better while for other starting points another one would be appropriate.

In one word, we can infer that the convergence behavior of the considered multi-point methods strongly depends on the structure of the tested functions and the accuracy of the starting points. To compare the efficiency indices of different derivative-free methods, we provide Table 2. The computational order of convergence (namely, COC), which can be defined by

$$COC = \frac{\ln(|\, x_{n+1} - \alpha \,|/|\, x_n - \alpha \,|)}{\ln(|\, x_n - \alpha \,|/|\, x_{n-1} - \alpha \,|)}, \tag{4.1}$$

TABLE 1. Comparison of different derivative-free methods with the same TNE (12)

Fun.	Guess.	JM	DHM	LM	KTM1	KTM3	PM1	PM2
f_1	0.6	0.3e-40	0.2e-32	0.1e-90	0.7e-88	0.4e-6	0.1e-81	0.1e-204
f_1	0.8	0.6e-15	0.2e-9	0.2e-25	0.2e-22	0.5e-6	0.2e-10	0.8e-137
f_1	-0.2	0.1e-36	0.2	0.2e-83	0.3e-72	0.1e-22	0.4e-27	0.4e-318
f_2	0.5	0.4e-100	0.5e-100	0.9e-276	0.5e-344	0.5e-34	0.4e-255	0.3e-106
f_2	0.4	0.7e-137	0.9e-136	0.1e-384	0.1e-468	0.6e-200	0.6e-360	0.5e-195
f_2	0.2	0.3e-85	0.2e-80	0.2e-228	0.5e-268	0.4e-154	0.2e-190	0.3e-40
f_3	1.7	0.2e-39	0.1e-51	0.2e-172	0.1e-72	0.3e-21	0.1e-106	0.9e-84
f_3	1.2	0.1e-58	0.1e-79	0.1e-278	0.8e-168	0.2e-83	0.9e-171	0.2e-22
f_3	1.5	0.2e-82	0.4e-93	0.6e-308	0.7e-226	0.6e-94	0.7e-238	0.6e-171
f_4	1.9	0.3e-88	0.1e-80	0.7e-246	0.1e-263	0.6e-242	0.1e-209	0.1e-122
f_4	2.3	0.3e-93	0.6e-89	0.5e-265	0.1e-286	0.8e-188	0.3e-241	0.1e-162
f_4	2.1	0.1e-166	0.6e-161	0.3e-495	0.9e-515	0.1e-454	0.8e-467	0.1e-385
f_5	0.3	0.8e-43	0.2e-18	0.1e-117	0.3e-105	0.2e-53	0.1e-54	0.8e-163
f_5	0	0.9e-67	0.8e-51	0.7e-200	0.2e-187	0.6e-146	0.7e-162	0.2e-291
f_5	0.4	0.4e-26	Div.	0.1e-63	0.1e-52	0.1e-10	0.4e-3	0.4e-95
f_6	0.3	0.3e-56	0.5e-37	0.2e-146	0.9e-138	0.9e-102	0.2e-125	0.1e-183
f_6	-0.2	0.2e-53	0.5e-20	0.1e-122	0.4e-107	0.5e-8	0.1e-72	0.1e-258
f_6	0.1	0.8e-89	0.9e-67	0.1e-245	0.1e-239	0.2e-185	0.2e-218	0.3e-316
f_7	1.29	0.3e-95	0.4e-72	0.9e-218	0.2e-213	0.3e-135	0.1e-210	0.1e-90
f_7	1.33	0.1e-38	1.1	0.2e-106	0.3e-64	41.1	0.1e-60	0.1e-180
f_7	1.32	0.8e-83	0.1e-35	0.2e-179	0.3e-155	Div.	0.4e-147	0.5e-231
f_8	-0.6	0.5e-35	Div.	0.3e-58	0.4e-45	281	0.1e-18	0.1e-155
f_8	-0.9	0.3e-75	0.6e-55	0.2e-181	0.1e-176	0.3e-133	0.1e-169	0.2e-181
f_8	-0.7	0.4e-67	0.1e-33	0.2e-160	0.3e-144	0.4e-56	0.1e-117	0.2e-234
f_9	-0.91	0.6-53	0.3e-29	0.5e-249	0.4e-136	0.5e-78	0.3e-114	0.6e-59
f_9	-0.93	0.4e-97	0.2e-72	0.2e-400	0.5e-267	0.6e173	0.2e-254	0.1e-224
f_9	-0.9	0.2e-38	Div.	0.2e-170	0.2e-93	0.5e-43	0.8e-73	0.1e-9
f_{10}	0.4	0.4e-122	0.2e-90	0.9e-498	0.8e-365	0.2e-297	0.4e-326	0.2e-382
f_{10}	0.42	0.1e-49	0.3e-18	0.3e-293	0.9e-151	0.4e-147	0.3e-98	0.3e-137
f_{10}	0.36	0.1e-25	0.6e-2	0.2e-124	0.2e-50	0.4e-51	0.8e-73	0.9e-30

is very close to 4 (at least to the fifth decimal place) for the fourth-order methods and to 3 (at least to the fifth decimal place) for the third-order methods which were considered in Table 1.

TABLE 2. Comparison of efficiency indices for some famous derivative-free algorithms

Methods	SM	(2.2)	JM	DHM	LM	KTM	PM1	PM2
Efficiency Index	1.4142	1.4142	1.4421	1.4421	1.5874	1.5874	1.5874	1.5874

5. CONCLUDING REMARKS AND DISCUSSIONS ON FUTURE WORKS

In this contribution, accurate optimal fourth-order derivative-free algorithms for solving single variable nonlinear equations are developed and their advantages with respect to the other existed well-known methods are illustrated by numerical examples. The algorithms consist of three function evaluations per iteration and therefore their efficiency index is 1.5874 which is bigger than lot of the existing derivative-free methods. In addition, these derivative-free methods can be applied for finding the multiple zeros of nonlinear equations by applying a suitable transformation and converting the multiple zeros of the nonlinear function to a simple one.

To suggest future works, we direct the attention toward two directions, first to multi-point schemes with memory, and to higher-order multi-point iterations without memory. For the first case, one should consider more complicated versions of forward/backward finite difference approximations of order one, e.g. $f'(x_n) = [f(x_n + \beta f(x_n)) - f(x_n)]/\beta f(x_n) + O(\beta f(x_n)))$. In the first case, this would provide families depending on one parameter $\beta \neq 0$, and then one can estimate β by an iteration per step using only the known data of the first step. For the latter one, we should take into account the three steps cycle, in which the first two steps are PM1 or PM2, and the subsequent estimate of the newly appeared first derivatives of the function is obtained by a combination of already known values or by using a weight function.

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A NEW GENERALIZATION OF OSTROWSKI-GRÜSS TYPE INEQUALITIES INVOLVING FUNCTIONS OF TWO INDEPENDENT VARIABLES

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Abstract. The main purpose of this paper is to derive a new inequality of Ostrowski-Grüss type with a parameter involving functions of two independent variables.

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1. INTRODUCTION

In [4], Dragomir and Wang proved the following Ostrowski-Grüss type integral inequality.

Theorem 1. *Let $f:[a,b]\to R$ be continuous on $[a,b]$, differentiable on (a,b) and suppose that $\gamma\le f'(x)\le\Gamma$ for all $x\in(a,b)$. Then we have*

$$\left|f(x)-\frac{f(b)-f(a)}{b-a}(x-\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\frac{1}{4}(b-a)(\Gamma-\gamma) \quad (1.1)$$

for all $x\in[a,b]$.

In [2], Cheng not only gave a sharp version of the above inequality but also generalized it as follows.

Theorem 2. *Let the assumptions of Theorem 1 hold. Then for all $x\in[a,b]$, we have*

$$\left|\frac{1}{2}f(x)-\frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\frac{1}{2}\frac{(x-a)^2+(b-x)^2}{2(b-a)}\frac{\Gamma-\gamma}{2}. \quad (1.2)$$

In a recent paper [12], Wang et al. proved the following inequality with a parameter.

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Theorem 3. *Let the assumptions of Theorem 1 hold. Then for all $x \in [a,b]$ and $\lambda \in [0,1]$, we have*

$$\left| \left(1-\frac{\lambda}{2}\right) f(x) - \lambda \frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)} \right. \tag{1.3}$$
$$\left. -\frac{\Gamma+\gamma}{2}(1-\lambda)\left(x-\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right|$$
$$\leq \left(1-\lambda+\frac{\lambda^2}{2}\right)\frac{(x-a)^2+(b-x)^2}{2(b-a)}\frac{\Gamma-\gamma}{2}. \tag{1.4}$$

More recently, Sarikaya [10] established the following generalization of (1.2) involving functions of two independent variables.

Theorem 4. *Let $f : [a,b]\times[c,d] \to R$ be an absolutely continuous function such that the partial derivative of order 2 exist and suppose that there exist constants $\gamma, \Gamma \in R$ with $\gamma \leq \frac{\partial^2 f(t,s)}{\partial t \partial s} \leq \Gamma$ for all $(t,s) \in [a,b]\times[c,d]$. Then we have*

$$\left| \frac{1}{4}f(x,y) + \frac{1}{4}H(x,y) - \frac{1}{2(b-a)}\int_a^b f(t,y)dt - \frac{1}{2(d-c)}\int_c^d f(x,s)ds \right.$$
$$-\frac{1}{2(b-a)(d-c)}\left\{ \int_a^b [(y-c)f(t,c)+(d-y)f(t,d)]dt \right. \tag{1.5}$$
$$\left. \left. + \int_c^d [(x-a)f(a,s)+(b-x)f(b,s)]ds \right\} \right.$$
$$\left. + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dtds \right|$$
$$\leq \frac{1}{4}\frac{\left[(x-a)^2+(b-x)^2\right]\left[(y-c)^2+(d-y)^2\right]}{4(b-a)(d-c)}\frac{\Gamma-\gamma}{2} \tag{1.6}$$

for all $(x,y) \in [a,b]\times[c,d]$, where

$$H(x,y) = \frac{(x-a)[(y-c)f(a,c)+(d-y)f(a,d)]+(b-x)[(y-c)f(b,c)+(d-y)f(b,d)]}{(b-a)(d-c)}$$
$$+\frac{(x-a)f(a,y)+(b-x)f(b,y)}{b-a}+\frac{(y-c)f(x,c)+(d-y)f(x,d)}{d-c}.$$

For other related work, we refer the reader to [3, 5–9, 12].

In this paper, we shall derive a new inequality of Ostrowski-Grüss type with a parameter for absolutely continuous functions of two independent variables, which will not only provides a generalization of the inequalities (1.4) and (1.5), but also gives some other interesting inequalities as special cases.

2. Main result

Theorem 5. *Let $f:[a,b]\times[c,d]\to R$ be an absolutely continuous function such that the partial derivative of order 2 exist and suppose that there exist constants $\gamma,\Gamma\in R$ with $\gamma\le\frac{\partial^2 f(t,s)}{\partial t\partial s}\le\Gamma$ for all $(t,s)\in[a,b]\times[c,d]$. Then we have*

$$\begin{aligned}
&\left|\left(1-\frac{\lambda}{2}\right)^2 f(x,y)+\frac{\lambda}{2}\left(1-\frac{\lambda}{2}\right)H_1(x,y)+\left(\frac{\lambda}{2}\right)^2 H_2(x,y)\right.\\
&\quad-\left(1-\frac{\lambda}{2}\right)\left[\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{d-c}\int_c^d f(x,s)ds\right]\\
&\quad-\frac{\lambda}{2}\frac{1}{(b-a)(d-c)}\left\{\int_a^b[(y-c)f(t,c)+(d-y)f(t,d)]dt\right.\\
&\qquad\left.+\int_c^d[(x-a)f(a,s)+(b-x)f(b,s)]ds\right\}\\
&\left.-\frac{\Gamma+\gamma}{2}(1-\lambda)^2\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dtds\right|\\
&\le\left(1-\lambda+\frac{\lambda^2}{2}\right)^2\frac{\left[(x-a)^2+(b-x)^2\right]\left[(y-c)^2+(d-y)^2\right]}{4(b-a)(d-c)}\frac{\Gamma-\gamma}{2}
\end{aligned}\tag{2.1}$$

for all $(x,y)\in[a,b]\times[c,d]$ and $\lambda\in[0,1]$, where

$$H_1(x,y)=\frac{(x-a)f(a,y)+(b-x)f(b,y)}{b-a}+\frac{(y-c)f(x,c)+(d-y)f(x,d)}{d-c}$$

and

$$\begin{aligned}
&H_2(x,y)\\
&=\frac{(x-a)[(y-c)f(a,c)+(d-y)f(a,d)]+(b-x)[(y-c)f(b,c)+(d-y)f(b,d)]}{(b-a)(d-c)}.
\end{aligned}$$

Proof. We define the mapping: $p:[a,b]\times[a,b]\to R$, $q:[c,d]\times[c,d]\to R$ as

$$p(x,t)=\begin{cases}t-\left(a+\lambda\dfrac{x-a}{2}\right) & t\in[a,x]\\ t-\left(b-\lambda\dfrac{b-x}{2}\right) & t\in(x,b]\end{cases}$$

and

$$q(y,s)=\begin{cases}s-\left(c+\lambda\dfrac{y-c}{2}\right) & s\in[c,y]\\ s-\left(d-\lambda\dfrac{d-y}{2}\right) & s\in(y,d]\end{cases}$$

By definitions of $p(x,t)$ and $q(y,s)$, we have

$$\begin{aligned}
&\int_a^b \int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t\,\partial s}dsdt \\
&= \int_a^x \int_c^y \left[t-\left(a+\lambda\frac{x-a}{2}\right)\right]\left[s-\left(c+\lambda\frac{y-c}{2}\right)\right]\frac{\partial^2 f(t,s)}{\partial t\,\partial s}dsdt \\
&+ \int_a^x \int_y^d \left[t-\left(a+\lambda\frac{x-a}{2}\right)\right]\left[s-\left(d-\lambda\frac{d-y}{2}\right)\right]\frac{\partial^2 f(t,s)}{\partial t\,\partial s}dsdt \\
&+ \int_x^b \int_c^y \left[t-\left(b-\lambda\frac{b-x}{2}\right)\right]\left[s-\left(c+\lambda\frac{y-c}{2}\right)\right]\frac{\partial^2 f(t,s)}{\partial t\,\partial s}dsdt \\
&+ \int_x^b \int_y^d \left[t-\left(b-\lambda\frac{b-x}{2}\right)\right]\left[s-\left(d-\lambda\frac{d-y}{2}\right)\right]\frac{\partial^2 f(t,s)}{\partial t\,\partial s}dsdt \qquad (2.2)
\end{aligned}$$

Integrating by parts twice, we can state:

$$\begin{aligned}
&\int_a^x \int_c^y \left[t-\left(a+\lambda\frac{x-a}{2}\right)\right]\left[s-\left(c+\lambda\frac{y-c}{2}\right)\right]\frac{\partial^2 f(t,s)}{\partial t\,\partial s}dsdt \\
&= (x-a)(y-c)\left[\left(1-\frac{\lambda}{2}\right)^2 f(x,y)+\left(1-\frac{\lambda}{2}\right)\frac{\lambda}{2}f(a,y)\right. \\
&\qquad \left.+\left(1-\frac{\lambda}{2}\right)\frac{\lambda}{2}f(x,c)+\left(\frac{\lambda}{2}\right)^2 f(a,c)\right] \\
&\qquad -(y-c)\int_a^x \left[\left(1-\frac{\lambda}{2}\right)f(t,y)+\frac{\lambda}{2}f(t,c)\right]dt \qquad (2.3) \\
&-(x-a)\int_c^y \left[\left(1-\frac{\lambda}{2}\right)f(x,s)+\frac{\lambda}{2}f(a,s)\right]ds+\int_a^x \int_c^y f(t,s)dsdt,
\end{aligned}$$

$$\begin{aligned}
&\int_a^x \int_y^d \left[t-\left(a+\lambda\frac{x-a}{2}\right)\right]\left[s-\left(d-\lambda\frac{d-y}{2}\right)\right]\frac{\partial^2 f(t,s)}{\partial t\,\partial s}dsdt \\
&= (x-a)(d-y) \\
&\left[\left(1-\frac{\lambda}{2}\right)^2 f(x,y)+\left(1-\frac{\lambda}{2}\right)\frac{\lambda}{2}f(a,y)+\left(1-\frac{\lambda}{2}\right)\frac{\lambda}{2}f(x,d)+\left(\frac{\lambda}{2}\right)^2 f(a,d)\right] \\
&\qquad -(d-y)\int_a^x \left[\left(1-\frac{\lambda}{2}\right)f(t,y)+\frac{\lambda}{2}f(t,d)\right]dt \qquad (2.4) \\
&-(x-a)\int_y^d \left[\left(1-\frac{\lambda}{2}\right)f(x,s)+\frac{\lambda}{2}f(a,s)\right]ds+\int_a^x \int_y^d f(t,s)dsdt,
\end{aligned}$$

$$\int_x^b \int_c^y \left[t-\left(b-\lambda\frac{b-x}{2}\right)\right]\left[s-\left(c+\lambda\frac{y-c}{2}\right)\right]\frac{\partial^2 f(t,s)}{\partial t\,\partial s}dsdt$$
$$=(b-x)(y-c)$$
$$\left[\left(1-\frac{\lambda}{2}\right)^2 f(x,y)+\left(1-\frac{\lambda}{2}\right)\frac{\lambda}{2}f(b,y)+\left(1-\frac{\lambda}{2}\right)\frac{\lambda}{2}f(x,c)+\left(\frac{\lambda}{2}\right)^2 f(b,c)\right]$$
$$-(y-c)\int_x^b\left[\left(1-\frac{\lambda}{2}\right)f(t,y)+\frac{\lambda}{2}f(t,c)\right]dt$$
$$-(b-x)\int_c^y\left[\left(1-\frac{\lambda}{2}\right)f(x,s)+\frac{\lambda}{2}f(b,s)\right]ds+\int_x^b\int_c^y f(t,s)dsdt, \qquad (2.5)$$

$$\int_x^b \int_y^b \left[t-\left(b-\lambda\frac{b-x}{2}\right)\right]\left[s-\left(c-\lambda\frac{d-y}{2}\right)\right]\frac{\partial^2 f(t,s)}{\partial t\,\partial s}dsdt$$
$$=(b-x)(d-y)$$
$$\left[\left(1-\frac{\lambda}{2}\right)^2 f(x,y)+\left(1-\frac{\lambda}{2}\right)\frac{\lambda}{2}f(b,y)+\left(1-\frac{\lambda}{2}\right)\frac{\lambda}{2}f(x,d)+\left(\frac{\lambda}{2}\right)^2 f(b,d)\right]$$
$$-(d-y)\int_x^b\left[\left(1-\frac{\lambda}{2}\right)f(t,y)+\frac{\lambda}{2}f(t,d)\right]dt$$
$$-(b-x)\int_c^y\left[\left(1-\frac{\lambda}{2}\right)f(x,s)+\frac{\lambda}{2}f(b,s)\right]ds+\int_x^b\int_y^d f(t,s)dsdt. \qquad (2.6)$$

After adding (2)-(2) and rewriting we easily deduce:

$$\int_a^b\int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t\,\partial s}dsdt$$
$$=(b-a)(d-c)\left(1-\frac{\lambda}{2}\right)^2 f(x,y)$$
$$+\frac{\lambda}{2}\left(1-\frac{\lambda}{2}\right)\{(b-a)[(y-c)f(x,c)+(d-y)f(x,d)]$$
$$+(d-c)[(x-a)f(a,y)+(b-x)f(b,y)]\}$$
$$+\left(\frac{\lambda}{2}\right)^2\{(x-a)[(y-c)f(a,c)+(d-y)f(a,d)]$$
$$+(b-x)[(y-c)f(b,c)+(d-y)f(b,d)]\}$$
$$-\frac{\lambda}{2}\left\{\int_a^b[(y-c)f(t,c)+(d-y)f(t,d)]dt+\int_c^d[(x-a)f(a,s)+(b-x)f(b,s)]ds\right\}$$

$$-\left(1-\frac{\lambda}{2}\right)\left[(d-c)\int_a^b f(t,y)dt+(b-a)\int_c^d f(x,s)ds\right]+\int_a^b\int_c^d f(s,t)dsdt. \tag{2.7}$$

We also have

$$\int_a^b\int_c^d p(x,t)q(y,s)dsdt=(b-a)(d-c)(1-\lambda)^2\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right) \tag{2.8}$$

Let $M=\frac{\Gamma+\gamma}{2}$. From (2) and (2.8), it follows that

$$\begin{aligned}&\int_a^b\int_c^d p(x,t)q(y,s)\left[\frac{\partial^2 f(t,s)}{\partial t\,\partial s}-M\right]dsdt\\&=\int_a^b\int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t\,\partial s}dtds\\&-M(b-a)(d-c)(1-\lambda)^2\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right).\end{aligned} \tag{2.9}$$

On the other hand, we get

$$\begin{aligned}&\left|\int_a^b\int_c^d p(x,t)q(y,s)\left[\frac{\partial^2 f(t,s)}{\partial t\,\partial s}-M\right]dsdt\right|\\&\le\max_{(t,s)\in[a,b]\times[c,d]}\left|\frac{\partial^2 f(t,s)}{\partial t\,\partial s}-M\right|\int_a^b\int_c^d|p(x,t)q(y,s)|\,dtds.\end{aligned} \tag{2.10}$$

We also have

$$\max_{(t,s)\in[a,b]\times[c,d]}\left|\frac{\partial^2 f(t,s)}{\partial t\,\partial s}-M\right|\le\frac{\Gamma-\gamma}{2} \tag{2.11}$$

and

$$\begin{aligned}&\int_a^b\int_c^d|p(x,t)q(y,s)|\,dsdt\\&=\left(1-\lambda+\frac{\lambda^2}{2}\right)^2\frac{[(x-a)^2+(b-x)^2][(y-c)^2+(d-y)^2]}{4}\end{aligned} \tag{2.12}$$

From (2) to (2), we easily get

$$\begin{aligned}&\left|\int_a^b\int_c^d p(x,t)q(y,s)\left[\frac{\partial^2 f(t,s)}{\partial t\,\partial s}-M\right]dsdt\right|\\&\le\left(1-\lambda+\frac{\lambda^2}{2}\right)^2\frac{[(x-a)^2+(b-x)^2][(y-c)^2+(d-y)^2]}{4}\frac{\Gamma-\lambda}{2}.\end{aligned} \tag{2.13}$$

From (2), (2) and (2), we see that (5) holds. □

Corollary 1. *Under the assumptions of Theorem 5, we have*

$$\left| f(x,y) - \frac{1}{b-a}\int_a^b f(t,y)dt - \frac{1}{d-c}\int_c^d f(x,s)ds - \frac{\Gamma+\gamma}{2}\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right) + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dtds \right| \le \frac{\left[(x-a)^2+(b-x)^2\right]\left[(y-c)^2+(d-y)^2\right]}{8(b-a)(d-c)}(\Gamma-\gamma), \tag{2.14}$$

and especially

$$\left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - \frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt - \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dtds \right| \le (\Gamma-\gamma)\frac{(b-a)(d-c)}{16}. \tag{2.15}$$

Proof. We take $\lambda = 0$ in (5) to get (2.14) and $x = \frac{a+b}{2}, y = \frac{c+d}{2}$ in (2.14) to get (1). □

Remark 1. If we assume that $f(s,t) = h(s)h(t)$, $h : [a,b] \to R$, then from (5) we can get (1.4) (for $x = y$). Consequently, (5) can be also regarded as a generalization of (1.4) for double integrals. If we take $\lambda = 1$ in (5), then the inequality (1.5) is recaptured. Thus (5) may also be regarded as a generalization of (1.5) with a parameter.

Remark 2. If we take $\lambda = \frac{1}{2}$, or $\lambda = \frac{1}{3}$, or $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$, or $x = a$ and $y = c$, or $x = a$ and $y = d$, or $x = b$ and $y = c$, or $x = b$ and $y = d$ in (5), we can get other interesting inequalities for these special cases.

Remark 3. As in [1, 4, 11], we can apply the above obtained inequalities (5) et al. in numerical integration obtaining some general cubature formulae.

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ENDPOINT BOUNDEDNESS FOR MULTILINEAR COMMUTATORS OF LITTLEWOOD-PALEY OPERATORS

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Abstract. In this paper, we prove endpoint boundedness of multilinear commutators of Littlewood-Paley operators.

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1. Introduction and notations

Let $b \in BMO(R^n)$ and let T be the Calderón-Zygmund operator. The commutator $[b,T]$ of b and T is defined by

$$[b,T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

A classical result of Coifman, Rochberb and Weiss (see [3]) states that the commutator $[b,T]$ is bounded on $L^p(R^n)$, $(1 < p < \infty)$. In [2] and [5], boundedness properties of the commutators for the extreme values of p are obtained. In this paper, we will introduce the multilinear commutators of Littlewood-Paley operators and prove boundedness properties of the operators in the extreme cases.

First let us introduce some notations (see [1],[4],[8],[9],[10]). Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and for a locally integrable function f, let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x) = \sup_{x\in Q} |Q|^{-1}\int_Q |f(y) - f_Q|dy$. Moreover, f is said to belong to $BMO(R^n)$ if $f^{\#} \in L^\infty$ and define $||f||_{BMO} = ||f^{\#}||_{L^\infty}$. We also define the central BMO space by $CMO(R^n)$, which is the space of those functions $f \in L_{loc}(R^n)$ such that

$$||f||_{CMO} = \sup_{r>1} |Q(0,r)|^{-1}\int_Q |f(y) - f_Q|dy < \infty.$$

It has been known that(see [9])

$$||f||_{CMO} = \sup_{r>1} |Q(0,r)|^{-1}\int_Q |f(y) - f_Q|dy < \infty.$$

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Also, we give the concepts of the atom and H^1 space. A function a is called as a H^1 atom if there exists a cube Q such that a is supported on Q, $||a||_{L^\infty} \le |Q|^{-1}$ and $\int a(x)dx = 0$. It is well known that the Hardy space $H^1(R^n)$ can be characterized in terms of the atomic decomposition (see [4], [9]).

Definition 1. Let $0 < \delta < n$ and $1 < p < n/\delta$. We shall call $B_p^\delta(R^n)$ the space of those functions f on R^n such that

$$||f||_{B_p^\delta} = \sup_{r>1} r^{-n(1/p-\delta/n)} ||f\chi_{Q(0,r)}||_{L^p} < \infty.$$

Definition 2. Let $\varepsilon > 0$, $n > \delta > 0$ and let ψ be a fixed function that satisfies the following properties:

1) $\int \psi(x)dx = 0$,
2) $|\psi(x)| \le C(1+|x|)^{-(n+1-\delta)}$,
3) $|\psi(x+y)-\psi(x)| \le C|y|^\varepsilon (1+|x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$;

We denote $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ is written as $\chi_{\Gamma(x)}$. Let $\vec{b} = (b_1, \cdots, b_m)$ with $b_j \in BMO(R^n)$ for $1 \le j \le m$. Then the multilinear commutator of our Littlewood-Paley operator is defined by

$$S_\delta^{\vec{b}}(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^{\vec{b}}(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x,y) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(z))\psi_t(y-z)f(z)dz;$$

When $m = 1$, set

$$S_\delta^b(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^b(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^b(f)(x,y) = \int_{R^n} (b(x) - b(z))\psi_t(y-z)f(z)dz$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. Set $F_t(f)(x) = f * \psi_t(x)$, we also define

$$S_\delta(f)(x) = \left[\int\int_{\Gamma(x)} |f * \psi_t(x)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

which is the Littlewood-Paley operator (see [1][6, 7][10]).

Let H be the Hilbert space $H = \{h : ||h|| = (\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+1})^{1/2} < \infty\}$. Then for each fixed $x \in R^n$, $F_t(f)(x,y)$ may be viewed as a mapping from $[0,+\infty)$ to H. It is clear that

$$S_\delta(f)(x) = ||\chi_{\Gamma(x)}F_t(f)(y)|| \quad \text{and} \quad S_\delta^{\vec{b}}(f)(x) = ||\chi_{\Gamma(x)}F_t^{\vec{b}}(f)(y)||.$$

Given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and set
$||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$.

2. Theorems and proofs

We begin with a few preliminary lemmas.

Lemma 1. *Let $1 < r < \infty$, $b_j \in BMO(R^n)$ for $j = 1, \cdots, k$ and $k \in N$. Then we have*

$$\frac{1}{|Q|} \int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q| dy \le C \prod_{j=1}^k ||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|} \int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|^r dy \right)^{1/r} \le C \prod_{j=1}^k ||b_j||_{BMO}.$$

Lemma 2. *Let $0 < \delta < n$, $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then S_δ is bounded from $L^p(R^n)$ to $L^q(R^n)$.*

Theorem 1. *Let $0 < \delta < n$ and $\vec{b} = (b_1, \cdots, b_m)$ with $b_j \in BMO(R^n)$ for $1 \le j \le m$. Then $S_\delta^{\vec{b}}$ is bounded from $L^{n/\delta}(R^n)$ to $BMO(R^n)$.*

Proof of Theorem 1. It is necessary only to prove that there exist a constant C_Q such that

$$\frac{1}{|Q|} \int_Q |S_\delta^{\vec{b}}(f)(x) - C_Q| dx \le C ||f||_{L^{n/\delta}}.$$

Fix a cube $Q = Q(x_0, r)$, we decompose f into $f = f_1 + f_2$ with $f_1 = f\chi_Q$, $f_2 = f\chi_{(R^n \setminus Q)}$.

When $m = 1$, set $(b_1)_Q = |Q|^{-1} \int_Q b_1(y) dy$, we have

$$F_t^{b_1}(f)(x, y)$$
$$= (b_1(x) - (b_1)_Q) F_t(f)(y) - F_t((b_1 - (b_1)_Q) f_1)(y) - F_t((b_1 - (b_1)_Q) f_2)(y),$$

so

$$\begin{aligned}
&|S_\delta^{b_1}(f)(x) - S_\delta(((b_1)_Q - b_1) f_2)(x_0)| \\
&= \left| ||\chi_{\Gamma(x)} F_t^{b_1}(f)(x, y)|| - ||\chi_{\Gamma(x_0)} F_t(((b_1)_Q - b_1) f_2)(y)|| \right| \\
&\le ||\chi_{\Gamma(x)} F_t^{b_1}(f)(x, y) - \chi_{\Gamma(x_0)} F_t(((b_1)_Q - b_1) f_2)(y)|| \\
&\le ||\chi_{\Gamma(x)} (b_1(x) - (b_1)_Q) F_t(f)(y)|| + ||\chi_{\Gamma(x)} F_t(((b_1)_Q - b_1) f_1)(y)||
\end{aligned}$$

$$+||\chi_{\Gamma(x)}F_t((b_1-(b_1)_Q)f_2)(y)-\chi_{\Gamma(x_0)}F_t((b_1-(b_1)_Q)f_2)(y)||$$
$$=A(x)+B(x)+C(x).$$

For $A(x)$, set $1<p<n/\delta$, $1/q=1/p-\delta/n$ and $1/q+1/q'=1$. By Hölder's inequality and Lemma 1,2, we have

$$\begin{aligned}\frac{1}{Q}\int_Q|A(x)|dx &\le \left(\frac{1}{|Q|}\int_Q|b_1(x)-(b_1)_Q|^{q'}dx\right)^{1/q'}\\ &\quad\left(\frac{1}{|Q|}\int_{R^n}|S_\delta(f)(x)|^q\chi_Q(x)dx\right)^{1/q}\\ &\le C||b_1||_{BMO}\frac{1}{|Q|^q}\left(\int_{R^n}|f(x)|^p\chi_Q(x)dx\right)^{1/p}\\ &\le C||b_1||_{BMO}\frac{1}{|Q|^q}||f||_{L^{n/\delta}}|Q|^{(1-\delta p/n)p}\\ &\le C||b_1||_{BMO}||f||_{L^{n/\delta}}.\end{aligned}$$

For $B(x)$, take $1<r<n/\delta$ and $1/s=1/r-\delta/n$, then by Hölder's inequality we have

$$\begin{aligned}\frac{1}{|Q|}\int_Q|B(x)|dx &\le \left(\frac{1}{|Q|}\int_{R^n}(S_\delta((b_1(x)-(b_1)_Q)f_1)(x))^s dx\right)^{1/s}\\ &\le C|Q|^{-1/s}||(b_1(x)-(b_1)_Q)f\chi_Q||_{L^r}\\ &\le C\left(\frac{1}{|Q|}\int_Q|b_1(x)-(b_1)_Q|^s dx\right)^{1/s}||f||_{L^{n/\delta}}\\ &\le C||b_1||_{BMO}||f||_{L^{n/\delta}}.\end{aligned}$$

For $C(x)$, we have

$$\begin{aligned}&C(x)\\ &\le\left[\int\int_{R^{n+1}_+}\left(\int_{Q^c}|\chi_{\Gamma(x)}-\chi_{\Gamma(x_0)}||b_1(z)-(b_1)_Q||\psi_t(y-z)||f(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}\\ &\le C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)||\int\int_{|x-y|\le t}\frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2-2\delta}}\\ &\quad-\int\int_{|x_0-y|\le t}\frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2-2\delta}}|^{1/2}dz\\ &\le C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)|\end{aligned}$$

$$\times\Bigg(\int\int_{|y|\le t,|x+y-z|\le t}\left|\frac{1}{(t+|x+y-z|)^{2n+2-2\delta}}\right.$$
$$\left.-\frac{1}{(t+|x_0+y-z|)^{2n+2-2\delta}}\right|\frac{dydt}{t^{n-1}}\Bigg)^{1/2}dz$$
$$\le\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)|$$
$$\left(\int\int_{|y|\le t,|x+y-z|\le t}\frac{|x-x_0|t^{1-n}}{(t+|x+y-z|)^{2n+3-2\delta}}dydt\right)^{1/2}dz,$$

note that $2t+|x+y-z|\ge 2t+|x-z|-|y|\ge t+|x-z|$ when $|y|\le t$ and

$$\int_0^\infty\frac{t\,dt}{(t+|x-z|)^{2n+3-2\delta}}=C|x-z|^{-2n-1+2\delta}.$$

Then, for $x\in Q$,

$$C(x)$$
$$\le C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)||x-x_0|^{1/2}$$
$$\left(\int\int_{|y|\le t}\frac{t^{1-n}dydt}{(t+|x-z|)^{2n+3-2\delta}}\right)^{1/2}dz$$
$$\le C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)||x-x_0|^{1/2}$$
$$\left(\int_0^\infty\frac{t\,dt}{(t+|x-z|)^{2n+3-2\delta}}\right)^{1/2}dz$$
$$\le C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)|\frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2-\delta}}dz$$
$$\le C\sum_{k=0}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|b_1(z)-(b_1)_Q||f(z)|\frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2-\delta}}dz$$
$$\le C\sum_{k=1}^\infty 2^{-k/2}\left(\frac{1}{|2^kQ|}\int_{2^kQ}|b_1(z)-(b_1)_Q|^{n/(n-\delta)}dz\right)^{(n-\delta)/n}$$
$$\left(\int_{2^kQ}|f(z)|^{n/\delta}dz\right)^{\delta/n}$$
$$\le C\sum_{k=1}^\infty k2^{-k/2}||b_1||_{BMO}||f||_{L^{n/\delta}}$$
$$\le C||b_1||_{BMO}||f||_{L^{n/\delta}}.$$

When $m>1$, set $\vec{b}_Q=((b_1)_Q,\cdots,(b_m)_Q)\in R^n$, where $(b_j)_Q=|Q|^{-1}\int_Q b_j(y)dy,\ 1\le j\le m$. Then we have

$$\begin{aligned}
F_t^{\vec{b}}(f)(x,y)&=(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y)\\
&\quad+(-1)^m F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f)(y)\\
&\quad+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x)-\vec{b}_Q)_\sigma\int_{R^n}(\vec{b}(z)-\vec{b}_Q)_{\sigma^c}\psi_t(y-z)f(z)dz\\
&=(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y)\\
&\quad+(-1)^m F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(y)\\
&\quad+(-1)^m F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(y)\\
&\quad+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x)-\vec{b}_Q)_\sigma F_t((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x,y),
\end{aligned}$$

thus

$$\begin{aligned}
&|S_\delta^{\vec{b}}(f)(x)-S_\delta(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)|\\
&\le||\chi_{\Gamma(x)}F_t^{\vec{b}}(f)(x,y)-\chi_{\Gamma(x_0)}F_t(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(y)||\\
&\le||\chi_{\Gamma(x)}(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y)||\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\chi_{\Gamma(x)}(\vec{b}(x)-\vec{b}_Q)_\sigma F_t((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x,y)||\\
&+||\chi_{\Gamma(x)}F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(y)||\\
&+||\chi_{\Gamma(x)}F_t(\prod_{j=1}^m(b_j-(b_j)_Q)f_2)(y)-\chi_{\Gamma(x_0)}F_t(\prod_{j=1}^m(b_j-(b_j)_Q)f_2)(y)||\\
&=I_1(x)+I_2(x)+I_3(x)+I_4(x).
\end{aligned}$$

For $I_1(x)$, by taking $1<p<n/\delta$ and $1/q=1/p-\delta/n$, and by Hölder's inequality and Lemma 1,2, we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q I_1(x)dx\\
&\le\left(\frac{1}{|Q|}\int_Q|\prod_{j=1}^m(b_j(x)-(b_j)_Q)|^{q'}dx\right)^{1/q'}\left(\frac{1}{|Q|}\int_Q|S_\delta(f)(x)|^q dx\right)^{1/q}\\
&\le C||\vec{b}||_{BMO}|Q|^{-1/q}\left(\int_Q|f(x)|^p dx\right)^{1/p}\le C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}$$

For $I_2(x)$, if we take $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$, then

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q I_2(x)dx\\
&\le \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|}\int_Q |(b(\vec{x})-\vec{b}_Q)_\sigma||S_\delta((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x)|dx\\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x)-\vec{b}_Q)_\sigma|^{q'}dx\right)^{1/q'}\\
&\qquad\qquad\left(\frac{1}{|Q|}\int_Q |S_\delta((\vec{b}-\vec{b}_Q)_{\sigma^c})f)(x)|^q dx\right)^{1/q}\\
&\le C\sum_{j=1}^{m-1}||\vec{b}_\sigma||_{BMO}|Q|^{1/q}\left(\int_{R^n}|(b(\vec{x})-\vec{b}_Q)_{\sigma^c})f(x)|^p\chi_Q dx\right)^{1/q}\\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}\left(\frac{1}{|Q|}\int_Q|(\vec{b}(x)-\vec{b}_Q)_\sigma|^q dx\right)^{1/q}||f||_{L^{n/\delta}}\\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}||\vec{b}_{\sigma^c}||_{BMO}||f||_{L^{n/\delta}}\\
&\le C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}$$

For $I_3(x)$, take $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$, so we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q I_3(x)dx\\
&\le\left(\frac{1}{|Q|}\int_Q|S_\delta((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|^q dx\right)^{1/q}\\
&\le C|Q|^{-1/q}||((b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)f_1(x)||_{L^p}\\
&\le C\left(\frac{1}{|Q|}\int_Q|(b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)|^q dx\right)^{1/q}||f||_{L^{n/\delta}}\\
&\le C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}$$

For $I_4(x)$, similary as in the proof of $C(x)$ in Case $m=1$, we have

$$I_4(x) \le C\int_{Q^c} |x_0-x|^{1/2}|x_0-z|^{-(n+1/2-\delta)}\left|\prod_{j=1}^m (b_j(z)-(b_j)_Q)\right||f(z)|dz$$

$$\le C\sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ} |x_0-x|^{1/2}|x_0-z|^{-(n+1/2-\delta)}\left|\prod_{j=1}^m (b_j(z)-(b_j)_Q)\right||f(z)|dz$$

$$\le C\sum_{k=0}^{\infty} 2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}\left|\prod_{j=1}^m (b_j(z)-(b_j)_Q)\right||f(z)|dz$$

$$\le C\sum_{k=1}^{\infty} 2^{-k/2}\left(\frac{1}{|2^kQ|}\int_{2^kQ}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right|^{n/(n-\delta)}dy\right)^{(n-\delta)/n}||f||_{L^{n/\delta}}$$

$$\le C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.$$

This completes the proof of Theorem 1. □

Theorem 2. *Let $0<\delta<n$, $1<p<n/\delta$ and $\vec{b}=(b_1,\cdots,b_m)$ with $b_j\in BMO(R^n)$ for $1\le j\le m$. Then $S_\delta^{\vec{b}}$ is bounded from $B_p^\delta(R^n)$ to $CMO(R^n)$.*

Proof of Theorem 2. It suffices to prove that there exist a constant C_Q, such that

$$\frac{1}{|Q|}\int_Q |S_\delta^{\vec{b}}(f)(x)-C_Q|dx \le C||f||_{B_p^\delta}$$

holds for any cube $Q=Q(0,r)$ with $r>1$. Fix a cube $Q=Q(0,r)$ with $r>1$. Set $f_1=f\chi_Q$, $f_2=f\chi_{R^n\setminus Q}$ and $\vec{b}_Q=((b_1)_Q,\cdots,(b_m)_Q)\in R^n$, where $(b_j)_Q=|Q|^{-1}\int_Q|b_j(y)|dy$, $1\le j\le m$. Then we have

$$\begin{aligned}
&|S_\delta^{\vec{b}}(f)(x)-S_\delta(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)|\\
&\le ||\chi_{\Gamma(x)}(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y)||\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\chi_{\Gamma(x)}(\vec{b}(x)-\vec{b}_Q)_\sigma F_t((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x,y)||\\
&+||\chi_{\Gamma(x)}F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(y)||\\
&+||\chi_{\Gamma(x)}F_t(\prod_{j=1}^m(b_j-(b_j)_Q)f_2)(y)-\chi_{\Gamma(x_0)}F_t(\prod_{j=1}^m(b_j-(b_j)_Q)f_2)(y)||\\
&=H_1(x)+H_2(x)+H_3(x)+H_4(x).
\end{aligned}$$

For $H_1(x)$, we take $1 < p < n/\delta$, $1/s = 1/r - \delta/n$. By Hölder's inequality and Lemma 1,2, we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q H_1(x)dx\\
&\le \left(\frac{1}{|Q|}\int_Q |\prod_{j=1}^m (b_j(x)-(b_j)_Q)|^{q'}dx\right)^{1/q'}\left(\frac{1}{|Q|}\int_Q |S_\delta(f)(x)|^q dx\right)^{1/q}\\
&\le C||\vec{b}||_{BMO}|Q|^{-1/q}\left(\int_Q |f(x)|^p dx\right)^{1/p}\\
&\le C||\vec{b}||_{BMO}d^{-n(1/p-\delta/n)}||f\chi_Q||_{L^p}\\
&\le C||\vec{b}||_{BMO}||f||_{B_p^\delta}.
\end{aligned}$$

For $H_2(x)$, taking $1 < p < n/\delta$, $1/s = 1/r - \delta/n$, and $1/s' + 1/s = 1$, we obtain that

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q H_2(x)dx\\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x)-\vec{b}_Q)_\sigma|^{s'}dx\right)^{1/s'}\\
&\qquad\qquad\left(\frac{1}{|Q|}\int_Q |S_\delta((\vec{b}-\vec{b}_Q)_{\sigma^c})f)(x)|^s dx\right)^{1/s}\\
&\le C\sum_{j=1}^{m-1}||\vec{b}_\sigma||_{BMO}|Q|^{-1/s}\left(\int_{R^n}|(b(\vec{x})-\vec{b}_Q)_{\sigma^c})f(x)|^r\chi_Q dx\right)^{1/r}\\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|^{pr/(p-r)}dx\right)^{(p-r)/pr}\\
&\qquad\qquad|Q|^{(\delta/n-1/p)}||f\chi_Q||_{L^p}\\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}||\vec{b}_{\sigma^c}||_{BMO}d^{-n(1/p-\delta/n)}||f\chi_Q||_{L^p}\\
&\le C||\vec{b}||_{BMO}||f||_{B_p^\delta}.
\end{aligned}$$

For $H_3(x)$, taking $1 < p < n/\delta$, $1/s = 1/r - \delta/n$ and $1/s' + 1/s = 1$, we get

$$\frac{1}{|Q|}\int_Q H_3(x)dx$$

$$\le \left(\frac{1}{|Q|}\int_Q |\mu_\delta((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|^s dx\right)^{1/s}$$
$$\le C|Q|^{-1/s}||((b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)f\chi_Q||_{L^r}$$
$$\le C\left(\frac{1}{|Q|}\int_Q |(b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)|^{pr/(p-r)}dx\right)^{(p-r)/pr}$$
$$d^{-n(1/p-\delta/n)}||f\chi_Q||_{L^p}$$
$$\le C||\vec{b}||_{BMO}||f||_{B_p^\delta}.$$

For $H_4(x)$, we have

$$H_4(x)$$
$$\le \left[\int\int_{R_+^{n+1}}\left(\int_{Q^c}|\chi_{\Gamma(x)}-\chi_{\Gamma(x_0)}|\right.\right.$$
$$\left.\left.\prod_{j=1}^m |b_j(z)-(b_j)_Q||\psi_t(y-z)||f(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}$$
$$\le C\sum_{k=0}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2-\delta)}|\prod_{j=1}^m(b_j(z)-(b_j)_Q)||f(z)|dz$$
$$\le C\sum_{k=1}^\infty 2^{-k/2}\frac{1}{|2^kQ|^{1-\delta/n}}\int_{2^kQ}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right||f(y)|dy$$
$$\le C\sum_{k=1}^\infty 2^{-k/2}\frac{1}{|2^kQ|^{1-\delta/n}}\left(\int_{2^kQ}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|^{p/(p-1)}dy\right)^{(p-1)/p}$$
$$\times\left(\int_{2^kQ}|f(y)|^p dy\right)^{1/p}$$
$$\le C\sum_{k=1}^\infty 2^{-k/2}\left(\frac{1}{|2^kQ|}\int_{2^kQ}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|^{p/(p-1)}dy\right)^{(p-1)/p}$$
$$\times|2^kQ|^{-(1/p-\delta/n)}||f\chi_{2^kQ}||_{L^p}$$
$$\le C||\vec{b}||_{BMO}||f||_{B_p^\delta}.$$

This completes the proof of Theorem 2. □

Theorem 3. *Let $0<\delta<n$ and $\vec{b}=(b_1,\cdots,b_m)$ with $b_j\in BMO(R^n)$ for $1\le j\le m$. Assume that the following inequality holds for any $H^1(R^n)$−atom a supported on a certain cube Q and for $u\in Q$.*

$$\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\int_{(2Q)^c}\left[|(b(x)-b_Q)_{\sigma^c}|\right.$$

$$\left.\left(\int\int_{\Gamma(x)}\left(\int_Q|(b(z)-b_Q)_\sigma\psi_t(y-u)a(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}\right]^{n/(n-\delta)}dx$$

$$\le C$$

Then $S_\delta^{\vec{b}}$ is bounded from $H^1(R^n)$ to $L^{n/(n-\delta)}(R^n)$.

Proof of Theorem 3. Let a be an atom supported in some cube Q and $u\in Q$. We write

$$\int_{R^n}|S_\delta^{\vec{b}}(a)(x)|^{n/(n-\delta)}dx$$

$$=\int_{2Q}|S_\delta^{\vec{b}}(a)(x)|^{n/(n-\delta)}dx+\int_{(2Q)^c}|S_\delta^{\vec{b}}(a)(x)|^{n/(n-\delta)}dx=I+II.$$

For I, take $1<p<n/\delta$ and $1/q=1/p-\delta/n$. Then we have

$$I\le||\mu_\delta^{\vec{b}}(a)||_{L^q}^{n/(n-\delta)}|2Q|^{1-n/((n-\delta)q)}\le C||a||_{L^p}^{n/(n-\delta)}|Q|^{1-n/((n-\delta)q)}\le C.$$

For II, we first calculate $F_t^{\vec{b}}(a)(x)$. When $m=1$, we have

$$\begin{aligned}|F_t^{b_1}(a)(x,y)|&\le|\int_Q\psi_t(y-z)a(z)(b_1(x)-(b_1)_Q)dz|\\&\quad+|\int_Q(\psi_t(y-z)-\psi_t(y-u))a(z)(b_1(z)-(b_1)_Q)dz|\\&\quad+|\int_Q\psi_t(y-u)(b_1(z)-(b_1)_Q)a(z)dz|\\&=\nu_1'+\nu_2'+\nu_3',\end{aligned}$$

so

$$S_\delta^{b_1}(a)(x)=||\chi_{\Gamma(x)}F_t^{b_1}(a)(x,y)||$$

$$\le\left(\int\int_{\Gamma(x)}|\nu_1'|^2\frac{dydt}{t^{n+1}}\right)^{1/2}+\left(\int\int_{\Gamma(x)}|\nu_2'|^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$

$$+\left(\int\int_{\Gamma(x)}|\nu_3'|^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$

$$= A'(x) + B'(x) + C'(x).$$

For $A'(x)$, we have

$$A'(x) \le S_\delta(a)(x)|b_1(x) - (b_1)_Q|,$$

thus

$$\left(\int_{(2Q)^c} (A'(x))^{n/(n-\delta)} dx\right)^{(n-\delta)/n}$$

$$= (\int_{(2Q)^c} [|b_1(x) - (b_1)_Q|$$

$$\times \left(\int\int_{\Gamma(x)} |\int_Q (\psi_t(y-z) - \psi_t(y-u))a(z)dz|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}]^{n/(n-\delta)} dx)^{(n-\delta)/n}$$

$$\le C|Q|^{1+\varepsilon/n}||a||_{L^\infty}$$

$$\sum_{k=1}^{\infty} \left(\int_{2^{k+1}Q} \left(\frac{|2^k Q|^{\delta/n}}{|2^k Q|^{(n+\varepsilon)/n}} |b_1(x) - (b_1)_Q|\right)^{n/(n-\delta)} dx\right)^{(n-\delta)/n}$$

$$\le C \sum_{k=1}^{\infty} 2^{-k\varepsilon} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |b_1(x) - (b_1)_Q|^{n/(n-\delta)} dx\right)^{(n-\delta)/n}$$

$$\le C||b_1||_{BMO}.$$

For $B'(x)$, we have

$$B'(x)$$

$$\le C \left(\int\int_{\Gamma(x)} \left(\int_Q \frac{t|u-z|^\varepsilon}{(t+|y-u|)^{n+1+\varepsilon-\delta}} |a(z)||b(z) - (b_1)_Q| dz\right)^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$$

$$\le C \left(\int_0^\infty \frac{tdt}{(t+|x-u|)^{2(n+1+\varepsilon-\delta)}}\right)^{1/2} \int_Q |u-z|^\varepsilon |a(z)||b_1(z) - (b_1)_Q| dz$$

$$\le C||b_1||_{BMO}|x-u|^{-(n+\varepsilon-\delta)}|Q|^{1+\varepsilon/n}||a||_{L^\infty},$$

thus

$$\left(\int_{(2Q)^c} (B'(x))^{n/(n-\delta)} dx\right)^{(n-\delta)/n}$$

$$\le C||b_1||_{BMO}||a||_{L^\infty} \left[\sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} \left(\frac{|Q|^{1+\varepsilon/n}}{|x-u|^{n+\varepsilon-\delta}}\right)^{n/(n-\delta)} dx\right]^{(n-\delta)/n}$$

$$\leq C|||b_1||_{BMO}\sum_{k=1}^{\infty}2^{-k\varepsilon}$$
$$\leq C|||b_1||_{BMO}.$$

From that we know, if

$$\int_{(2Q)^c}(C'(x))^{n/(n-\delta)}dx$$
$$=\int_{(2Q)^c}\left[\left(\int\int_{\Gamma(x)}\left(\int_Q|(b_1(z)-(b_1)_Q)\right.\right.\right.$$
$$\left.\left.\left.\psi_t(y-u)a(z)|dz)^2\frac{dydt}{t^{n+1}}\right)^{1/2}\right]^{n/(n-\delta)}dx\leq C,$$

then

$$\int_{R^n}|S_\delta^{b_1}(a)(x)|^{n/(n-\delta)}dx\leq C.$$

When $m>1$, we have

$$|F_t^{\vec{b}}(a)(x,y)|\leq\left|\prod_{j=1}^m(b_j(x)-(b_j)_Q)\int_Q\psi_t(y-z)a(z)dz\right|$$
$$+\sum_{j=1}^m\sum_{\sigma\in C_j^m}\left|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\int_Q(\psi_t(y-z)-\psi_t(y-u))(\vec{b}(z)-\vec{b}_Q)_\sigma a(z)dz\right|$$
$$+\sum_{j=1}^m\sum_{\sigma\in C_j^m}\left|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\int_Q\psi_t(y-u)(\vec{b}(z)-\vec{b}_Q)_\sigma a(z)dz\right|$$
$$=\nu_1+\nu_2+\nu_3,$$

so

$$S_\delta^{\vec{b}}(a)(x)=||\chi_{\Gamma(x)}F_t^{\vec{b}}(a)(x,y)||$$
$$\leq\left(\int\int_{\Gamma(x)}|\nu_1|^2\frac{dydt}{t^{n+1}}\right)^{1/2}+\left(\int\int_{\Gamma(x)}|\nu_2|^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$
$$+\left(\int\int_{\Gamma(x)}|\nu_3|^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$
$$=A(x)+B(x)+C(x).$$

For $A(x)$, we have

$$A(x) = \left(\int\int_{\Gamma(x)} \prod_{j=1}^{m} |b_j(x) - (b_j)_Q|^2 \left| \int_Q \psi_t(y-z) a(z) dz \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2}$$
$$= \prod_{j=1}^{m} |b_j(x) - (b_j)_Q| S_\delta(a)(x),$$

thus

$$\left(\int_{(2Q)^c} (A(x))^{n/(n-\delta)} dx \right)^{(n-\delta)/n}$$
$$\leq C ||a||_{L^\infty}$$
$$\left[\sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} \left(\frac{|Q|^{1+\varepsilon/n}}{|x-u|^{n+\varepsilon-\delta}} \prod_{j=1}^{m} |b_j(x) - (b_j)_Q| \right)^{n/(n-\delta)} dx \right]^{(n-\delta)/n}$$
$$\leq C |Q|^{1+\varepsilon/n} ||a||_{L^\infty}$$
$$\sum_{k=1}^{\infty} \left(\int_{2^{k+1}Q} \left(\frac{|2^k Q|^{\delta/n}}{|2^k Q|^{(n+\varepsilon)/n}} \prod_{j=1}^{m} |b_j(x) - (b_j)_Q| \right)^{n/(n-\delta)} dx \right)^{(n-\delta)/n}$$
$$\leq C \sum_{k=1}^{\infty} 2^{-k\varepsilon} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} \left(\prod_{j=1}^{m} |b_j(x) - (b_j)_Q| \right)^{n/(n-\delta)} dx \right)^{(n-\delta)/n}$$
$$\leq C ||\vec{b}||_{BMO}.$$

For $B(x)$, we have

$$B(x) = \Big(\int\int_{\Gamma(x)} \Big| \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} (\vec{b}(x) - \vec{b}_Q)_{\sigma^c}$$
$$\times \int_Q (\psi_t(y-z) - \psi_t(y-u)) (\vec{b}(z) - \vec{b}_Q)_\sigma a(z) dz \Big|^2 \frac{dy\,dt}{t^{n+1}} \Big)^{1/2}$$
$$\leq C \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}_Q)_{\sigma^c}|$$
$$\times \left(\int\int_{\Gamma(x)} \left(\int_Q \frac{t|u-z|^\varepsilon}{(t+|y-u|)^{n+1+\varepsilon-\delta}} |(\vec{b}(z) - \vec{b}_Q)_\sigma| |a(z)| dz \right)^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2}$$

$$\leq C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-b_Q)_{\sigma^c}|\left(\int\int_{\Gamma(x)}\frac{t^{1-n}dydt}{(t+|y-u|)^{2(n+1+\varepsilon-\delta)}}\right)^{1/2}$$

$$\times\int_Q|u-z|^{\varepsilon}|a(z)||(\vec{b}(z)-\vec{b}_Q)_{\sigma}|dz$$

$$\leq C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}||x-u|^{-(n+\varepsilon)}|Q|^{1+\varepsilon/n-\delta}||a||_{L^\infty}||\vec{b}_\sigma||_{BMO},$$

thus

$$\left(\int_{(2Q)^c}(B(x))^{n/(n-\delta)}dx\right)^{(n-\delta)/n}$$

$$\leq C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\sum_{k=1}^{\infty}2^{-k\varepsilon}$$

$$\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\right|^{n/(n-\delta)}dx\right)^{(n-\delta)/n}||\vec{b}_\sigma||_{BMO}$$

$$\leq C||\vec{b}||_{BMO}.$$

So, if

$$\int_{(2Q)^c}(C(x))^{n/(n-\delta)}dx$$

$$=\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\int_{(2Q)^c}\left[|(b(x)-b_Q)_{\sigma^c}|\left(\int\int_{\Gamma(x)}\left(\int_Q|(b(z)-b_Q)_{\sigma}\right.\right.\right.$$

$$\left.\left.\left.\psi_t(y-u)a(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}\right]^{n/(n-\delta)}dx$$

$$\leq C,$$

then

$$\int_{R^n}|S_\delta^{\vec{b}}(a)(x)|^{n/(n-\delta)}dx\leq C.$$

This completes the proof of Theorem 3. □

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ON RIEMANNIAN MANIFOLDS ADMITTING W_2-CURVATURE TENSOR

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Abstract. The object of this paper is to study the properties of flat spacetimes under some conditions regarding the W_2-curvature tensor. In the first section, several results are obtained on the geometrical symmetries of this curvature tensor. It is shown that in a spacetime with W_2-curvature tensor filled with a perfect fluid, the energy momentum tensor satisfying the Einstein's equations with a cosmological constant is a quadratic conformal Killing tensor. It is also proved that a necessary and sufficient condition for the energy momentum tensor to be a quadratic Killing tensor is that the scalar curvature of this space must be constant. In a radiative perfect fluid, it is shown that the sectional curvature is constant.

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1. INTRODUCTION

In 1970, Pokhariyal and Mishra, [9] introduced a new curvature tensor in an n-dimensional manifold (M,g) denoted by W_2 and defined by

$$W_2(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}(g(X,Z)S(Y,U) - g(Y,Z)S(X,U)). \tag{1.1}$$

Some authors have studied the W_2-curvature tensor on some special manifolds before, [3,5,8,16].

A non-flat n-dimensional Riemannian manifold (M,g) is called generalized recurrent Riemannian manifold [4] if its curvature tensor satisfies the condition

$$\nabla R = A \otimes R + B \otimes G \tag{1.2}$$

where A and B non-zero 1-forms. $\otimes$ is the tensor product, ∇ denotes the Levi-Civita connection, and G is a tensor type (0,4) given by

$$G(X,Y,Z,U) = g(X,U)g(Y,Z) - g(X,Z)g(Y,U) \tag{1.3}$$

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for all $X,Y,Z,U \in \chi(M)$, where $\chi(M)$ is the Lie algebra of smooth vector fields on M. This manifold is denoted by (GK_n). These manifolds have been studied by some authors before, [1,2,10,11], etc.

In this paper, the properties of a W_2-flat spacetime are studied. Some theorems on the W_2 curvature tensor are proved.

2. Some properties of W_2-curvature tensor

Definition 1. Let (M,g) be a manifold with Levi-Civita connection ∇. A quadratic Killing tensor is a generalization of a Killing vector and is defined as a second order symmetric tensor A satisfying the condition [13,15]

$$(\nabla_X A)(Y,Z)+(\nabla_Y A)(Z,X)+(\nabla_Z A)(X,Y)=0 \tag{2.1}$$

Definition 2. Let (M,g) be a manifold with Levi-Civita connection ∇. A quadratic conformal Killing tensor is an analogous generalization of a conformal Killing vector and is defined as a second order symmetric tensor A satisfying the condition [13,15]

$$\begin{aligned}&(\nabla_X A)(Y,Z)+(\nabla_Y A)(Z,X)+(\nabla_Z A)(X,Y)=\\&k(X)g(Y,Z)+k(Y)g(Z,X)+k(Z)g(X,Y).\end{aligned} \tag{2.2}$$

Now, we have the following theorems:

Theorem 1. *If the Ricci tensor of M admitting W_2-curvature tensor is a quadratic conformal Killing tensor then $\tilde{W}_2(Y,Z)$ (which is type of (0,2)) is also quadratic conformal Killing tensor.*

Proof. By (1.1) we have

$$\tilde{W}_2(Y,Z)=\frac{n}{n-1}(S(Y,Z)-\frac{r}{n}g(Y,Z)) \tag{2.3}$$

where $S(Y,Z)$ and r are the Ricci tensor and the scalar curvature of M, respectively. If we take the covariant derivative of (2.3), we find

$$(\nabla_X \tilde{W}_2)(Y,Z)=\frac{n}{n-1}(\nabla_X S)(Y,Z)-\frac{1}{n-1}(\nabla_X r)g(Y,Z) \tag{2.4}$$

Permutating the indices X,Y,Z cyclically in (2.4) and adding the three equations, we get

$$\begin{aligned}&(\nabla_X \tilde{W}_2)(Y,Z)+(\nabla_Y \tilde{W}_2)(Z,X)+(\nabla_Z \tilde{W}_2)(X,Y)\\&=\frac{n}{n-1}((\nabla_X S)(Y,Z)+(\nabla_Y S)(Z,X)+(\nabla_Z S)(X,Y))\\&-\frac{1}{n-1}((\nabla_X r)g(Y,Z)+(\nabla_Y r)g(Z,X)+(\nabla_Z r)g(X,Y)).\end{aligned} \tag{2.5}$$

If we assume that the Ricci tensor of M is a quadratic conformal Killing tensor, from (2.2) and (2.5), we obtain that

$$\begin{aligned}&(\nabla_X \tilde{W}_2)(Y,Z)+(\nabla_Y \tilde{W}_2)(Z,X)+(\nabla_Z \tilde{W}_2)(X,Y)\\&=(\frac{n}{n-1}a(X)-\frac{1}{n-1}(\nabla_X r))g(Y,Z)\\&+(\frac{n}{n-1}a(Y)-\frac{1}{n-1}(\nabla_Y r))g(Z,X)\\&+(\frac{n}{n-1}a(Z)-\frac{1}{n-1}(\nabla_Z r))g(X,Y).\end{aligned} \tag{2.6}$$

By taking $\frac{n}{n-1}a(X)-\frac{1}{n-1}(\nabla_X r)=\alpha(X)$, it can be seen that

$$\begin{aligned}&(\nabla_X \tilde{W}_2)(Y,Z)+(\nabla_Y \tilde{W}_2)(Z,X)+(\nabla_Z \tilde{W}_2)(X,Y)\\&=\alpha(X)g(Y,Z)+\alpha(Y)g(Z,X)+\alpha(Z)g(X,Y).\end{aligned}$$

This relation completes the proof. □

Theorem 2. *Let the Ricci tensor of M admitting a W_2-curvature tensor be a quadratic Killing tensor. A necessary and sufficient condition for $\tilde{W}_2(X,Y)$ to be a quadratic Killing tensor is that the scalar curvature of M be constant.*

Proof. Let us consider that the Ricci tensor of M is a quadratic Killing tensor. From equations (2.1) and (2.5), we have

$$\begin{aligned}&(\nabla_X \tilde{W}_2)(Y,Z)+(\nabla_Y \tilde{W}_2)(Z,X)+(\nabla_Z \tilde{W}_2)(X,Y)\\&-\frac{1}{n-1}((\nabla_X r)g(Y,Z)+(\nabla_Y r)g(Z,X)+(\nabla_Z r)g(X,Y)).\end{aligned} \tag{2.7}$$

If $\tilde{W}_2(X,Y)$ is a quadratic Killing tensor then we get

$$(\nabla_X r)g(Y,Z)+(\nabla_Y r)g(Z,X)+(\nabla_Z r)g(X,Y)=0 \tag{2.8}$$

Walker's Lemma,[14] states the following. If $a(X,Y)$ and $b(X)$ are numbers that satisfy $a(X,Y)=a(Y,X)$ and

$$a(X,Y)b(Z)+a(Y,Z)b(X)+a(Z,X)b(Y)=0 \tag{2.9}$$

for all X,Y,Z, then either all the $a(X,Y)$ are zero or all the $b(X)$ are zero. Hence, by the above Lemma, we get from (2.8) and (2.9) that either $g(X,Y)=0$ or $(\nabla_X r)=0$. As $g(X,Y)\neq 0$, we get $(\nabla_X r)=0$, i.e., the scalar curvature of M is constant. Conversely, if the scalar curvature is constant, then, from (2.7), $\tilde{W}_2(X,Y)$ is a quadratic Killing tensor. Thus, the proof is completed. □

The geometrical symmetries of a Riemannian manifold are expressed through the equation

$$L_\xi A-2\Omega A=0 \tag{2.10}$$

where A represents a geometrical/physical quantity L_ξ, denotes the Lie derivative with respect to the vector field ξ and Ω is a scalar, [7].

One of the most simple and widely used example is the metric inheritance symmetry for which $A = g$ in (2.10); in this case, ξ is the Killing vector field if Ω is zero,i.e.,

$$(L_\xi g)(X,Y) = 2\Omega g(X,Y) \tag{2.11}$$

A Riemannian manifold M is said to admit a symmetry called a curvature collineation *(CC)* provided there exists a vector field ξ such that

$$(L_\xi R)(X,Y)Z = 0 \tag{2.12}$$

where $R(X,Y)Z$ is the Riemannian curvature tensor, [6].

Now, we shall investigate the role of such symmetry inheritance for the W_2-curvature tensor of a Riemannian manifold.

Theorem 3. *If a Riemannian manifold M admitting W_2-curvature tensor with a Killing vector ξ is a* (CC) *then the Lie derivative of the W_2-curvature tensor is zero.*

Proof. If ξ is a Killing vector of M then we have

$$(L_\xi g)(X,Y) = 0 \tag{2.13}$$

Since M admits a *(CC)* then we have also from (2.12)

$$(L_\xi S)(X,Y) = 0 \tag{2.14}$$

By taking the Lie derivative of (1.1) and using (2.12)-(2.14), we obtain

$$(L_\xi W)(X,Y)Z = 0$$

The proof is completed. □

Theorem 4. *If a Riemannian manifold M admitting W_2-curvature tensor with a conformal Killing vector ξ has a symmetry inheritance then the W_2-curvature tensor has also symmetry inheritance property.*

Proof. Let us assume that ξ of M is a conformal Killing vector and M has a symmetry inheritance. By taking the Lie derivative of (1.1) and using the equations (2.10) and (2.11), we find

$$\begin{aligned}(L_\xi W)(X,Y,Z,U) &= 2\Omega R(X,Y,Z,U)\\ &\quad + \frac{2\Omega}{n-1}(g(X,Z)S(Y,U) - g(Y,Z)S(X,U))\\ &= 2\Omega W(X,Y,Z,U).\end{aligned}$$

The proof is completed. □

3. W_2-FLAT SPACETIMES

Theorem. [12] A W_2-flat space is an Einstein space, i.e,

$$S(X,Y) = \frac{r}{n} g(X,Y) \tag{3.1}$$

We denote the W_2 flat spacetime as $(W_2FS)_n$ and we consider that our space is a perfect fluid. In local coordinates, a perfect fluid is a spacetime (M,g) satisfying the Einstein equations

$$S_{ij} - \frac{r}{2} g_{ij} + \lambda g_{ij} = kT_{ij} \tag{3.2}$$

where S_{ij} and r denote the Ricci tensor and the scalar curvature, respectively. λ is the cosmological constant and T_{ij} is the energy-momentum tensor. We can state the following theorem:

Theorem 5. *For a $(W_2FS)_4$, the energy-momentum tensor satisfying the Einstein's equations with a cosmological constant is locally symmetric.*

Proof. In a $(W_2FS)_4$, by (3.1), we have

$$S_{ij} = \frac{r}{4} g_{ij} \tag{3.3}$$

and then we get from (3.2) and (3.3)

$$T_{ij} = \frac{1}{k}(\lambda - \frac{r}{4}) g_{ij} \tag{3.4}$$

By taking the covariant derivative of (3.4), we get

$$\nabla_k T_{ij} = -\frac{1}{4k} (\nabla_k r) g_{ij}. \tag{3.5}$$

Since r is constant in $(W_2FS)_4$,

$$\nabla_k r = 0 \tag{3.6}$$

then (3.5) reduces to

$$\nabla_k T_{ij} = 0. \tag{3.7}$$

This completes the proof. □

Theorem 6. *For a $(W_2S)_4$, whose the Ricci tensor is quadratic Killing, a necessary and sufficient condition the energy-momentum tensor satisfying the Einstein's equations with a cosmological constant be a quadratic Killing tensor is that the scalar curvature of this space must be constant.*

Proof. If T_{ij} and R_{ij} are quadratic Killing tensors, (3.5) reduces to

$$(\nabla_k r) g_{ij} + (\nabla_i r) g_{jk} + (\nabla_j r) g_{ki} = 0 \tag{3.8}$$

By Walker's Lemma,[14], from (2.9) we get $\nabla_k r = 0$. Thus we can say that the scalar curvature of this space is constant. Conversely, if r is constant, from (3.5), it can be obtained that

$$\nabla_k T_{ij} + \nabla_i T_{jk} + \nabla_j T_{ki} = 0$$

Thus, the proof is completed. □

Now, we consider that a radiative perfect fluid in $(W_2S)_4$. Thus, we have

$$T_{ij} = \sigma a_i a_j, \qquad a_i a^i = -1 \tag{3.9}$$

In this case, by taking the covariant derivative of (3.9), we find

$$\nabla_k T_{ij} = (\nabla_k \sigma) a_i a_j + \sigma((\nabla_k a_i) a_j + (\nabla_k a_j) a_i) \tag{3.10}$$

Multiplying (3.9) by g^{ij}, we get

$$T = -\sigma \tag{3.11}$$

By taking the covariant derivative of (3.11), it can be found that

$$\nabla_k T = -\nabla_k \sigma \tag{3.12}$$

Now, we can state the following theorem:

Theorem 7. *For a radiative perfect fluid in $(W_2S)_4$, if the energy-momentum tensor satisfying the Einstein's equations with a cosmological constant is generalized recurrent then the integral curves of the vector field a_i are geodesics.*

Proof. For a radiative perfect fluid in $(W_2S)_4$, if T_{ij} is generalized recurrent then from (1.2)

$$\nabla_k T_{ij} = \lambda_k T_{ij} + \beta_k g_{ij} \tag{3.13}$$

and

$$\nabla_k T = \lambda_k T + 4\beta_k \tag{3.14}$$

Moreover, by (3.12) and (3.14), we get

$$\nabla_k \sigma = \lambda_k \sigma - 4\beta_k \tag{3.15}$$

Putting equations (3.13) and (3.15) in (3.10), we have the following relation

$$\lambda_k T_{ij} + \beta_k g_{ij} = (\lambda_k \sigma - 4\beta_k) a_i a_j + \sigma((\nabla_k a_i) a_j + (\nabla_k a_j) a_i) \tag{3.16}$$

By (3.9), (3.16) takes the form

$$\lambda_k \sigma a_i a_j + \beta_k g_{ij} = \lambda_k \sigma a_i a_j - 4\beta_k a_i a_j + \sigma((\nabla_k a_i) a_j + (\nabla_k a_j) a_i) \tag{3.17}$$

Multiplying (3.17) by g^{ij} and using (3.9), we obtain

$$2\sigma a^i (\nabla_k a_i) = 0 \tag{3.18}$$

Since, in a radiative perfect fluid, $\sigma \neq 0$ then from (3.18), we finally get

$$a^i (\nabla_k a_i) = 0 \tag{3.19}$$

From (3.19), we can say that the integral curves of the vector field a_i are geodesics. In this case, the proof is completed. □

Now, by using the above theorem, we can state the following:

Theorem 8. *For a radiative perfect fluid in $(W_2FS)_4$, the sectional curvature is constant.*

Proof. For a radiative perfect fluid in $(W_2FS)_4$, the scalar curvature of this space is constant. Thus, we can say that for a radiative perfect fluid in $(W_2FS)_4$, the scalar curvature of this space is constant.

The sectional curvature of a Riemannian manifold M is in the form

$$K(\pi) = \frac{R_{hijk}X^hY^iX^jY^k}{(g_{ij}g_{hk} - g_{ik}g_{hj})X^hY^iX^jY^k}. \tag{3.20}$$

By (1.1), if M is a W_2 flat manifold then we have

$$R_{ijkh} = \frac{1}{n-1}(g_{jk}S_{ih} - g_{ik}S_{jh}). \tag{3.21}$$

Thus, multiplying (3.21) by g^{ih} and using (3.21) again, we get

$$R_{ijkh} = \frac{r}{n(n-1)}(g_{jk}g_{ih} - g_{ik}g_{jh}). \tag{3.22}$$

Finally, from (3.20) and (3.22), it can be seen that

$$K(\pi) = \frac{r}{n(n-1)}. \tag{3.23}$$

□

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