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CONTENTS

Ishak Altun

- Common fixed point theorems for weakly increasing mappings on ordered
uniform spaces 3

Csilla Bujtás and Zsolt Tuza

- Optimal combinatorial batch codes derived from dual systems 11

Stephan Foldes and Gerasimos Meletiou

- Some remarks on structural matrix rings and matrices with ideal entries . . 25

Tamás Glavosits and Csaba Kézi

- On the domain of oddness of an infimal convolution 31

Jun He and Ting-Zhu Huang

- New preconditioners for nonsymmetric and indefinite saddle point linear
systems 41

Rabha W. Ibrahim

- Global controllability of a set of fractional differential equations 51

Božidar Ivanković, Josip Pečarić, and Sanja Varošaneć

- New results on the AM-GM inequality 61

Sevda Karakuş and Kamil Demirci

- Summation process of Korovkin type approximation theorem 75

Emrah Kılıç, Neşe Ömür, and Yücel Türker Ulutaş

- Alternating sums of the powers of Fibonacci and Lucas numbers 87

Cristinel Mortici

- Refinements of some bounds related to the constant e 105

N. Rajabov, A. N. Adib, and Mohamed Elsayed

- An overdetermined system with two super-singular lines containing second
order hyperbolic equations 113



COMMON FIXED POINT THEOREMS FOR WEAKLY INCREASING MAPPINGS ON ORDERED UNIFORM SPACES

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Abstract. In the present paper, we give a common fixed point theorem for two weakly increasing mappings on ordered Hausdorff uniform spaces, using the distance p . Two multivalued versions of this result are also presented.

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1. INTRODUCTION

Recently, Aamri and El Moutawakil [2] have introduced the concept of E -distance function on uniform spaces and utilized it to improve some well known results of the existing literature involving both E -contractive or E -expansive mappings. In [3], a partial ordering on uniform spaces has been introduced utilizing E -distance function and some fixed point theorems for single valued and multivalued nondecreasing (not necessary E -contractive or E -expansive) mappings on ordered uniform spaces have been proved. In this paper, we give a common fixed point theorem for two weakly increasing single valued mappings, which may not be nondecreasing, on ordered uniform spaces. Two multivalued versions of this result are also given.

In order to have a possibly self contained presentation, we recall some relevant definitions and properties of the foundation of uniform spaces. For further informations and details, one is referred to the book by Bourbaki [5]. We call a pair (X, ϑ) to be a uniform space if it consists of a non-empty set X together with an uniformity ϑ , where we assume that the latter begins with a special kind of filter on $X \times X$ whose all elements contain the diagonal $\Delta = \{(x, x) : x \in X\}$. If $V \in \vartheta$ and $(x, y) \in V$, $(y, x) \in V$ then x and y are said to be V -close. A sequence $\{x_n\}$ in X , is said to be a Cauchy sequence w.r.t. the uniformity ϑ if for any $V \in \vartheta$, there exists $N \geq 1$ such that x_n and x_m are V -close for $m, n \geq N$. An uniformity ϑ defines a unique topology $\tau(\vartheta)$ on X for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X : (x, y) \in V\}$ when V runs over ϑ .

A uniform space (X, ϑ) is said to be Hausdorff if and only if the intersection of all the $V \in \vartheta$ reduces to the diagonal Δ of X , i.e. $(x, y) \in V$ for all $V \in \vartheta$ implies $x = y$. Notice that Hausdorffness of the topology induced by the uniformity guarantees the uniqueness of limit of a sequence in uniform spaces. An element of uniformity ϑ is said to be symmetrical if $V = V^{-1} = \{(y, x) : (x, y) \in V\}$. Since each $V \in \vartheta$ contains a symmetrical $W \in \vartheta$ and if $(x, y) \in W$ then x and y are both W and V -close, so one may assume that each $V \in \vartheta$ is symmetrical. When topological concepts are mentioned in the context of a uniform space (X, ϑ) , they are naturally interpreted with respect to the topological space $(X, \tau(\vartheta))$.

2. PRELIMINARIES

We shall use the following definitions and lemmas in the sequel.

Definition 1 ([2]). Let (X, ϑ) be a uniform space. A function $p : X \times X \rightarrow \mathbb{R}^+$ is said to be an E -distance if

- (p₁) For any $V \in \vartheta$ there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$, for some $z \in X$, imply $(x, y) \in V$,
- (p₂) $p(x, y) \leq p(x, z) + p(z, y)$, $\forall x, y, z \in X$.

One can find some examples of E -distances in [2]. Especially, one can infer from Example 1 (contained in [2]) that every metric d on a uniform space (X, ϑ) is an E -distance.

The following lemma embodies some useful properties of E -distances.

Lemma 1 ([1], [2]). Let (X, ϑ) be a Hausdorff uniform space and p be an E -distance on X . Let $\{x_n\}$ and $\{y_n\}$ be arbitrary sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x, y, z \in X$, the following holds:

- (a) If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.
- (b) If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to z .
- (c) If $p(x_n, x_m) \leq \alpha_n$ for all $m > n$, then $\{x_n\}$ is a Cauchy sequence in (X, ϑ) .

Let (X, ϑ) be a uniform space equipped with E -distance p . A sequence in X is p -Cauchy if it satisfies the usual metric condition. There are several concepts of completeness in this setting.

Definition 2 ([1], [2]). Let (X, ϑ) be a uniform space and p be an E -distance on X . Then

- (i) X said to be S -complete if for every p -Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $\lim_{n \rightarrow \infty} p(x_n, x) = 0$,
- (ii) X is said to be p -Cauchy complete if for every p -Cauchy sequence $\{x_n\}$ there exists $x \in X$ such that $\lim_{n \rightarrow \infty} x_n = x$ with respect to $\tau(\vartheta)$,

(iii) $f : X \rightarrow X$ is p -continuous if $\lim_{n \rightarrow \infty} p(x_n, x) = 0$ implies

$$\lim_{n \rightarrow \infty} p(fx_n, fx) = 0$$

and

(iv) $f : X \rightarrow X$ is $\tau(\vartheta)$ -continuous if $\lim_{n \rightarrow \infty} x_n = x$ with respect to $\tau(\vartheta)$ implies $\lim_{n \rightarrow \infty} fx_n = fx$ with respect to $\tau(\vartheta)$.

Remark 1 ([2]). Let (X, ϑ) be a Hausdorff uniform space and let $\{x_n\}$ be a p -Cauchy sequence. Suppose that X is S -complete, then there exists $x \in X$ such that $\lim_{n \rightarrow \infty} p(x_n, x) = 0$. Then Lemma 1 (b) gives that $\lim_{n \rightarrow \infty} x_n = x$ with respect to the topology $\tau(\vartheta)$. This shows that S -completeness implies p -Cauchy completeness.

Using the following lemma, whose metric version is available in [6], we can construct an order relation on a Hausdorff uniform space.

Lemma 2 ([3]). Let (X, ϑ) be a Hausdorff uniform space, p be E -distance on X and $\phi : X \rightarrow \mathbb{R}$. Define the relation $\leq$ on X as follows:

$$x \leq y \iff x = y \text{ or } p(x, y) \leq \phi(x) - \phi(y).$$

Then $\leq$ is a (partial) order on X induced by ϕ .

There are some examples in [3] for partial orders on a Hausdorff uniform space induced by ϕ .

The following theorems have been given in [3].

Theorem 1. Let (X, ϑ) be a Hausdorff uniform space and p be an E -distance on X such that X is S -complete and $\phi : X \rightarrow \mathbb{R}$ be a function which is from bounded from below. If $\leq$ is the partial order induced by ϕ and $f : X \rightarrow X$ is a p -continuous or $\tau(\vartheta)$ -continuous non-decreasing function with $x_0 \leq fx_0$ for some $x_0 \in X$, then f has a fixed point in X .

Theorem 2. Let (X, ϑ) be a Hausdorff uniform space equipped with an E -distance p such that X is S -complete and $\phi : X \rightarrow \mathbb{R}$ is bounded from below. Assume that $\leq$ is the partial order induced by ϕ . Then it follows that if $F : X \rightarrow 2^X$ is p -order closed operator with $\{x_0\} \prec_1 Fx_0$ for some $x_0 \in X$ and for all $x, y \in X$, $x \leq y \implies Fx \prec_1 Fy$ (i.e. F is non-decreasing with respect to $\prec_1$), then F has a fixed point in X .

The aim of this paper is to give some common fixed point theorems for single valued and multivalued weakly increasing mappings on ordered Hausdorff uniform spaces.

First, we recall the concept of weakly increasing mappings.

Definition 3 ([7], [8]). Let $(X, \leq)$ be a partially ordered set. Two mappings $f, g : X \rightarrow X$ are said to be weakly increasing if $fx \leq gfx$ and $gx \leq fgx$ hold for all $x \in X$.

Note that two weakly increasing mappings need not be nondecreasing.

Example 1 ([4]). Let $X = \mathbb{R}^2$ be endowed with lexicographical ordering, that is, $(x, y) \preceq (z, w) \Leftrightarrow (x < z)$ or (if $x = z$, then $y \leq w$). Let $f, g : X \rightarrow X$ be defined by

$$f(x, y) = (\max\{x, y\}, \min\{x, y\})$$

and

$$g(x, y) = (\max\{x, y\}, \frac{x+y}{2}),$$

then f and g are weakly increasing mappings. However, both f and g are not non-decreasing.

Example 2 ([4]). Let $X = [1, \infty) \times [1, \infty)$ be endowed with coordinatwise ordering, that is, $(x, y) \preceq (z, w) \Leftrightarrow x \leq z$ and $y \leq w$. Let $f, g : X \rightarrow X$ be defined by $f(x, y) = (2x, 3y)$ and $g(x, y) = (x^2, y^2)$, then $f(x, y) = (2x, 3y) \preceq gf(x, y) = g(2x, 3y) = (4x^2, 9y^2)$ and $g(x, y) = (x^2, y^2) \preceq fg(x, y) = f(x^2, y^2) = (2x^2, 3y^2)$. Thus f and g are weakly increasing mappings.

3. MAIN RESULTS

Now we present our main theorem for single valued mappings.

Theorem 3. *Let (X, ϑ) be a Hausdorff uniform space and p be an E -distance on X such that X is S -complete and let $\phi : X \rightarrow \mathbb{R}$ be a function which is bounded below. If $\preceq$ is the partial order induced by ϕ and $f, g : X \rightarrow X$ are two p -continuous or $\tau(\vartheta)$ -continuous weakly increasing functions, then f and g have a common fixed point in X .*

Proof. Let x_0 be an arbitrary point of X and define a sequence $\{x_n\}$ in X as follows:

$$x_{2n+1} = fx_{2n} \text{ and } x_{2n+2} = gx_{2n+1} \text{ for } n \in \{0, 1, \dots\}.$$

Note that, since f and g are weakly increasing, we have

$$x_1 = fx_0 \preceq gfx_0 = gx_1 = x_2,$$

$$x_2 = gx_1 \preceq fgx_1 = fx_2 = x_3$$

and by continuing this process we have

$$x_1 \preceq x_2 \preceq \dots \preceq x_n \preceq x_{n+1} \preceq \dots$$

So the sequence $\{x_n\}$ is non-decreasing. By the definition of $\preceq$ we have $\dots \phi(x_2) \leq \phi(x_1) \leq \phi(x_0)$, that is, the sequence $\{\phi(x_n)\}$ is a non-increasing sequence in $\mathbb{R}$. Since ϕ is bounded from below, $\{\phi(x_n)\}$ is convergent and hence it is Cauchy, i.e. for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$ we have $|\phi(x_m) - \phi(x_n)| < \varepsilon$. Since $x_n \preceq x_m$, we have $x_n = x_m$ or $p(x_n, x_m) \leq \phi(x_n) - \phi(x_m)$. Therefore,

$$\begin{aligned}
p(x_n, x_m) &\leq \phi(x_n) - \phi(x_m) \\
&= |\phi(x_m) - \phi(x_n)| \\
&< \varepsilon.
\end{aligned}$$

which shows that (in view of Lemma 1 (c)) $\{x_n\}$ is p -Cauchy sequence. By the S -completeness of X , there is $z \in X$ such that $\lim_{n \rightarrow \infty} p(x_n, z) = 0$. Consequently, by the p -continuity of f and g , we have $\lim_{n \rightarrow \infty} p(fx_n, fz) = 0 = \lim_{n \rightarrow \infty} p(gx_n, gz)$. Therefore $\lim_{n \rightarrow \infty} p(x_n, z) = 0 = \lim_{n \rightarrow \infty} p(fx_n, fz) = \lim_{n \rightarrow \infty} p(gx_n, gz)$ and so from Lemma 1 (a), we have $fz = gz = z$. The proof is similar when f and g are $\tau(\vartheta)$ -continuous, since S -completeness implies p -Cauchy completeness (Remark 1). $\square$

In the following we provide two multivalued versions of the preceding theorem. These results are related to the theorems in [9].

Let X be a topological space and $\leq$ be a partial order on X . Let 2^X denote the family of all nonempty subsets of X .

Definition 4 ([9]). Let A, B be two nonempty subsets of X , and assume that the relations between A and B are defined as follows:

- (r₁) If for every $a \in A$, there exists $b \in B$ such that $a \leq b$, then $A <_1 B$.
- (r₂) If for every $b \in B$, there exists $a \in A$ such that $a \leq b$, then $A <_2 B$.
- (r₃) If $A <_1 B$ and $A <_2 B$, then $A < B$.

Remark 2 ([9]). $<_1$ and $<_2$ are different relations between A and B . For example, let $X = \mathbb{R}$, $A = [\frac{1}{2}, 1]$, $B = [0, 1]$, $\leq$ be usual order on X , then $A <_1 B$ but $A \not<_2 B$; if $A = [0, 1]$, $B = [0, \frac{1}{2}]$, then $A <_2 B$ while $A \not<_1 B$.

Remark 3 ([9]). $<_1$, $<_2$ and $<$ are reflexive and transitive, but are not antisymmetric. For instance, let $X = \mathbb{R}$, $A = [0, 3]$, $B = [0, 1] \cup [2, 3]$, $\leq$ be the usual order on X , then $A < B$ and $B < A$, but $A \neq B$. Hence, they are not partial orders on 2^X .

Definition 5 ([9]). A multi-valued operator $T : X \rightarrow 2^X$ is called order closed if for monotone sequences $\{u_n\}, \{v_n\} \subset X$, $u_n \rightarrow u_0$, $v_n \rightarrow v_0$ and $v_n \in Tu_n$ imply $v_0 \in Tu_0$.

The order closed operators on uniform space using E -distance function has been characterised in [3] in the following way.

Definition 6 ([3]). Let (X, ϑ) be a Hausdorff uniform space and let p be an E -distance on X . A multi-valued operator $T : X \rightarrow 2^X$ is called p -order closed if for monotone sequences $\{u_n\}, \{v_n\} \subset X$, $\lim_{n \rightarrow \infty} p(u_n, u_0) = 0$, $\lim_{n \rightarrow \infty} p(v_n, v_0) = 0$ and $v_n \in Tu_n$ imply $v_0 \in Tu_0$.

Now, we introduce the following definition.

Definition 7. Let $(X, \preceq)$ be a partially ordered set. Two mappings $F, G : X \rightarrow 2^X$ are said to be weakly increasing with respect to $\prec_1$ if for any $x \in X$ we have $Fx \prec_1 Gy$ for all $y \in Fx$ and $Gx \prec_1 Fy$ for all $y \in Gx$. Similarly two maps $F, G : X \rightarrow 2^X$ are said to be weakly increasing with respect to $\prec_2$ if for any $x \in X$ we have $Gy \prec_2 Fx$ for all $y \in Fx$ and $Fy \prec_2 Gx$ for all $y \in Gx$.

Example 3. Let $X = [1, \infty)$ and $\leq$ be usual order on X . Consider two mappings $F, G : X \rightarrow 2^X$ defined by $Fx = [1, x^2]$ and $Gx = [1, 2x]$ for all $x \in X$. Then the pair of mappings F and G are weakly increasing with respect to $\prec_2$ but not w.r.t $\prec_1$. Indeed, since

$$Gy = [1, 2y] \prec_2 [1, x^2] = Fx \text{ for all } y \in Fx$$

and

$$Fy = [1, y^2] \prec_2 [1, 2x] = Gx \text{ for all } y \in Gx$$

so F and G are weakly increasing with respect to $\prec_2$ but $F2 = [1, 4] \not\prec_1 [1, 2] = G1$ for $1 \in F2$, so F and G are not weakly increasing with respect to $\prec_1$.

Example 4. Let $X = [0, 1]$ and $\leq$ be usual order on X . Consider two mappings $F, G : X \rightarrow 2^X$ defined by $Fx = \{0, 1\}$ and $Gx = [x, 1]$ for all $x \in X$. Then the pair of mappings F and G are weakly increasing with respect to $\prec_1$ but not w.r.t. $\prec_2$. Indeed, since

$$Fx = \{0, 1\} \prec_1 [y, 1] = Gy \text{ for all } y \in Fx$$

and

$$Gx = [x, 1] \prec_1 \{0, 1\} = Fy \text{ for all } y \in Gx$$

so F and G are weakly increasing with respect to $\prec_1$ but $G1 = \{1\} \not\prec_2 \{0, 1\} = F1$ for $1 \in F1$, so F and G are not weakly increasing with respect to $\prec_2$.

Two multivalued versions of preceding theorem are as follows:

Theorem 4. Let (X, ϑ) be a Hausdorff uniform space and p let be an E -distance on X such that X is S -complete and $\phi : X \rightarrow \mathbb{R}$ is a function which is bounded from below. If $\preceq$ is the partial order induced by ϕ and $F, G : X \rightarrow 2^X$ are two order closed and weakly increasing mappings with respect to $\prec_1$, then F and G have a common fixed point in X .

Proof. Let $x_0 \in X$ be an arbitrary point. Since $Fx_0 \neq \emptyset$, we can choose $x_1 \in Fx_0$. Now since F and G are weakly increasing with respect to $\prec_1$, we have $x_1 \in Fx_0 \prec_1 Gx_1$. Thus there exist some $x_2 \in Gx_1$ such that $x_1 \preceq x_2$. Again since F and G are weakly increasing with respect to $\prec_1$, we have $x_2 \in Gx_1 \prec_1 Fx_2$. Thus there exist some $x_3 \in Fx_2$ such that $x_2 \preceq x_3$. By continuing this process, we will get a nondecreasing sequence $\{x_n\}_{n=1}^{\infty}$ which satisfies $x_{2n+1} \in Fx_{2n}$, $x_{2n+2} \in Gx_{2n+1}$, $n = 0, 1, 2, \dots$. By the definition of $\preceq$ we have

$$\dots \phi(x_3) \leq \phi(x_2) \leq \phi(x_1),$$

that is, the sequence $\{\phi(x_n)\}$ is a non-increasing sequence in $\mathbb{R}$. Since ϕ is bounded from below, then $\{\phi(x_n)\}$ is convergent and hence it is Cauchy. That is, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$ we have $|\phi(x_m) - \phi(x_n)| < \varepsilon$. On the other hand, since $x_n \preceq x_m$, we have $x_n = x_m$ or $p(x_n, x_m) \leq \phi(x_n) - \phi(x_m)$. Therefore, for all $m > n > n_0$ we have $p(x_n, x_m) < \varepsilon$. This implies $\{x_n\}$ is p -Cauchy sequence in X . By the S -completeness of X , there is $z \in X$ such that $\lim_{n \rightarrow \infty} p(x_n, z) = 0$. Therefore, $\lim_{n \rightarrow \infty} p(x_{2n+1}, z) = 0$ and $\lim_{n \rightarrow \infty} p(x_{2n+2}, z) = 0$. Consequently, since F and G are p -order closed, $\{x_n\}_{n=1}^{\infty}$ monotone and $x_{2n+1} \in Fx_{2n}$, $x_{2n+2} \in Gx_{2n+1}$, we deduce that $z \in Fz$ and $z \in Gz$, i.e. z is a common fixed point of F and G . $\square$

Theorem 5. *Let (X, ϑ) be a Hausdorff uniform space and p be an E -distance on X such that X is S -complete and $\phi : X \rightarrow \mathbb{R}$ be a function which is bounded below. If $\preceq$ is the partial order induced by ϕ and $F, G : X \rightarrow 2^X$ are two order closed and weakly increasing mappings with respect to $\prec_2$, then F and G have a common fixed point in X .*

Proof. Let $x_0 \in X$ be arbitrary point. Since $Fx_0 \neq \emptyset$, we can choose $x_1 \in Fx_0$. Now since F and G are weakly increasing with respect to $\prec_2$, we have $Gx_1 \prec_2 Fx_0$. Thus there exist some $x_2 \in Gx_1$ such that $x_2 \preceq x_1$. Again since F and G are weakly increasing with respect to $\prec_2$ we have $Fx_2 \prec_2 Gx_1$. Thus there exist some $x_3 \in Fx_2$ such that $x_3 \preceq x_2$. If we continue this process, we will get a nonincreasing sequence $\{x_n\}_{n=1}^{\infty}$ which satisfies $x_{2n+1} \in Fx_{2n}$, $x_{2n+2} \in Gx_{2n+1}$, $n = 0, 1, 2, \dots$. By the definition $\preceq$, we have

$$\phi(x_1) \leq \phi(x_2) \leq \phi(x_3) \leq \dots$$

Since ϕ is bounded from above, then $\{\phi(x_n)\}$ is convergent and hence it is Cauchy. That is, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$ we have $|\phi(x_m) - \phi(x_n)| < \varepsilon$. On the other hand, since $x_n \preceq x_m$, we have $x_n = x_m$ or $p(x_n, x_m) \leq \phi(x_n) - \phi(x_m)$. Therefore, for all $m > n > n_0$ we have $p(x_n, x_m) < \varepsilon$. This implies that $\{x_n\}$ is p -Cauchy sequence in X . By the S -completeness of X , there is an $z \in X$ such that $\lim_{n \rightarrow \infty} p(x_n, z) = 0$. Therefore, $\lim_{n \rightarrow \infty} p(x_{2n+1}, z) = 0$ and $\lim_{n \rightarrow \infty} p(x_{2n+2}, z) = 0$. Consequently, since F and G are p -order closed, $\{x_n\}_{n=1}^{\infty}$ is monotone and $x_{2n+1} \in Fx_{2n}$, $x_{2n+2} \in Gx_{2n+1}$, we can deduce that $z \in Fz$ and $z \in Gz$, i.e. z is a common fixed point of F and G . $\square$

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OPTIMAL COMBINATORIAL BATCH CODES DERIVED FROM DUAL SYSTEMS

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Abstract. Combinatorial batch codes with parameters n , k , m , and $t = 1$ may be viewed as set systems $\mathcal{F}$ consisting of n subsets over an m -element set (repetitions allowed), satisfying the following restricted version of Hall's Condition: for every $1 \leq i \leq k$, the union of any i members of $\mathcal{F}$ has cardinality at least i . An optimization problem is to determine $N(n, k, m)$, the minimum total size $\sum_{F \in \mathcal{F}} |F|$ in such systems. Beside its theoretical interest, the problem has strong practical motivation, too, concerning distributed storage and retrieval of data in a database.

Already the case $n = m + 2$ turns out to be somewhat complicated. Here we give explicit optimal constructions and prove the following formulae: in the range $k \leq m \leq k + \sqrt{k}$

$$N(m + 2, k, m) = 2m + \left\lfloor \frac{k}{m - k + 1} \right\rfloor,$$

and if $m > k + \sqrt{k}$ then

$$N(m + 2, k, m) = N(m + 1, k, m - 1) + 1 = m + k - 2 + \lceil 2\sqrt{k + 1} \rceil$$

for all $m \geq k \geq 1$. Our method is purely combinatorial, whereas the first proof by Brualdi *et al.* [Adv. Math. Commun., 4 (2010), 419–431 & 597] used the theory of transversal matroids. We also present an optimality-preserving transformation, by which a large family of non-isomorphic optimal constructions can be derived if one is already available. Moreover, we prove a new general upper bound on $N(n, k, m)$.

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Keywords: combinatorial batch code, dual system, Hall's Condition, system of distinct representatives

1. INTRODUCTION

In this paper we study a discrete optimization problem concerning data storage and retrieval in distributed databases. In the considered model, a certain amount of data is to be stored on a given number of servers. For a fixed number k , the task is to distribute data storage on the servers in such a way that any k data can be retrieved by communicating simultaneously with k suitably chosen different servers, one server

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per data. The overall goal is to minimize storage space, or equivalently the total amount of data replication.

This problem, whose abstract formulation is given in the next subsection, was studied in detail first by Paterson *et al.* [6], as a restricted version of the batch code problem introduced by Ishai *et al.* [5]. In the latter, which is more application-oriented, a limited amount t of communication is allowed with any one server, and even an encoding/decoding phase is allowed during data distribution/retrieval. The terminology *combinatorial batch code* refers to the fact that the encoding aspect is excluded. Our results for the general case $t \geq 1$ will be announced in [4]. In the present paper we will consider only the case $t = 1$.

1.1. Several alternative formulations

The problem has several equivalent formulations, which we list in this subsection. Most of them (except the last one) are taken from [6], as purely combinatorial versions of the more complex model of [5]. Throughout the paper we shall disregard the first form, since we find the dual systems and related concepts more handy. The last condition—the complementary version of the restricted Hall Condition—is a new one introduced here and it will turn out to be useful at several points of the proofs.

Combinatorial batch code. For positive integers n, k, m , a *combinatorial batch code* $\text{CBC}(n, k, m)$ is a set system¹ $\mathcal{S}$ of $|\mathcal{S}| = m$ sets over an underlying set D of cardinality $|D| = n$, with the property that for every $1 \leq \ell \leq k$ and every ℓ -tuple $\{d_1, \dots, d_\ell\} \subseteq D$, there exists a subsystem $\{S_1, \dots, S_\ell\} \subset \mathcal{S}$ of ℓ distinct members of $\mathcal{S}$, such that $d_i \in S_i$ holds for every $1 \leq i \leq \ell$. The goal is to determine

$$N(n, k, m) := \min_{\mathcal{S}} \sum_{S \in \mathcal{S}} |S|$$

where the minimum is taken over all combinatorial batch codes $\mathcal{S}$ with parameters n, k, m .

In this setting the elements of D represent the items of the database, and the members of $\mathcal{S}$ correspond to the contents of the servers; i.e., for any $d \in D$ and $S \in \mathcal{S}$, server S stores item d if and only if $d \in S$. Hence, $N(n, k, m)$ determines the smallest possible storage space required.

In order to avoid trivialities, we shall assume that the following inequalities are valid.

- $k \geq 2$. (For $k = 1$ the only condition is that each $d \in D$ should occur in some $S \in \mathcal{S}$, hence $\mathcal{S}$ can be composed e.g. of D and $m - 1$ copies of the empty set, and consequently $N(n, 1, m) = n$ for all n, m .)

¹Throughout, ‘set system’ is meant as a ‘multisystem’. Among the members of the system repetitions are allowed; that is, distinct members of the system may correspond to the same set.

- $n > m$. (For $n \leq m$, we may take the elements of D as 1-element members (singletons) of $\mathcal{S}$, together with $m - n$ copies of $\emptyset$, hence we again have $N(n, k, m) = n$ for all $n \leq m$ and all $k \geq 1$.)
- $m \geq k$. (Assuming $n > m$ by the previous observation, the case $m < k$ would lead to the contradiction for $\ell = m + 1 \leq k$ that no ℓ -tuple can satisfy the requirement; that is, no such $\text{CBC}(n, k, m)$ exists.)

In formulating the results, however, we shall sometimes allow $n = m$ and/or $k = 1$ to hold when it fits the generality of presentation, despite that these are obvious cases.

Dual system. A combinatorial batch code as a set system can be represented with its *dual system*. We say that a set system $\mathcal{F}$ is a $\text{CBC}^*(k)$ -system if, for every $1 \leq \ell \leq k$ and every ℓ -element subsystem $\mathcal{F}' = \{F_1, \dots, F_\ell\} \subseteq \mathcal{F}$ there exist ℓ mutually different elements $x_1, \dots, x_\ell$ of the underlying set, such that $x_i \in F_i$ holds for every $1 \leq i \leq \ell$. A set system $\mathcal{F}$ over the underlying set X is called a $\text{CBC}^*(n, k, m)$ if $|\mathcal{F}| = n$, $|X| = m$, and $\mathcal{F}$ is a $\text{CBC}^*(k)$ -system.

Asterisk in the notation refers to the usage of dual systems in the sense of hypergraph theory. If $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph with vertex set $V = \{v_1, \dots, v_n\}$ and edge set $\mathcal{E} = \{E_1, \dots, E_m\}$ (i.e., $\mathcal{E}$ is a family of sets over the underlying set V , multiple sets allowed), then its *dual* is a hypergraph $\mathcal{H}^* = (V^*, \mathcal{E}^*)$ where $V^* = \{v_1^*, \dots, v_m^*\}$, $\mathcal{E}^* = \{E_1^*, \dots, E_n^*\}$, and $v_i^* \in E_j^*$ if and only if $v_j \in E_i$ ($1 \leq i \leq m$, $1 \leq j \leq n$).

The condition on ℓ -tuples of sets in a dual system clearly corresponds to the condition on ℓ -element subsets of D given in definition of $\text{CBC}(n, k, m)$. Hence, $\mathcal{S}$ is a $\text{CBC}(n, k, m)$ if and only if its dual system $\mathcal{S}^*$ is a $\text{CBC}^*(n, k, m)$. By this equivalence we have

$$N(n, k, m) = \min_{\mathcal{F}} \sum_{F \in \mathcal{F}} |F|$$

where the minimum is taken over all $\text{CBC}^*(n, k, m)$ systems.

We say that $\mathcal{F}$ is *optimal* if it is a $\text{CBC}^*(n, k, m)$ attaining equality for $N(n, k, m)$. k -Restricted Hall Condition (k -HC). The definition of $\text{CBC}^*(k)$ -system can equivalently be expressed by a Hall-type condition, in the following way. We say that a set system $\mathcal{F}$ satisfies the *k-restricted Hall Condition*, abbreviated as k -HC, if for every $\ell \leq k$ and every family $\mathcal{F}'$ of ℓ members of $\mathcal{F}$ the inequality $|\bigcup_{F \in \mathcal{F}'} F| \geq \ell$ holds. (Certainly, $|\mathcal{F}|$ -HC means precisely the classical Hall Condition for the existence of a System of Distinct Representatives.)

Restricting our attention to the range $n \geq m \geq k \geq 1$, let us introduce the notation

$$s := m - k, \quad q := n - m.$$

In this way the goal is to determine $N(k + s + q, k, k + s)$.

We will consider the definition of $\text{CBC}^*(k)$ -system in the following three equivalent formulations. From now on, each of them will be referred as k -HC.

- (k1) The set system $\mathcal{F}$ satisfies k -HC.

- (k2) For every $0 \leq j \leq k-1$, every j -element subset of the underlying set contains at most j members from the system $\mathcal{F}$.
- (k3) For every $t > s$, every t -element subset of the underlying set meets at least $t+q$ members of $\mathcal{F}$.

For the equivalence of the last formulation, we note that writing $j := k + s - t$, the condition is equivalent to the requirement (k2). Property (k3) can be considered as the complementary version of (k1) and (k2), viewing the problem ‘from the other side.’

Remark 1. For $k \geq 2$, in any $\text{CBC}^*(k)$, each singleton set $\{x\}$ may occur with multiplicity at most 1. This is the particular case $j = 1$ of the requirement (k2).

1.2. Results and brief history

There are not too many ranges of (n, k, m) for which an exact formula for $N(n, k, m)$ is known. In [6] the problem was solved for arbitrary k, m and $n \geq (k-1)\binom{m}{k-1}$. For smaller $n \geq \binom{m}{k-2}$ the optimum was determined by the present authors in [3] and independently by Bhattacharya *et al.* in [1]. These two results together yield $N(n, 3, m)$ for all n and m , as computed in [3]. The case $N(n, 2, m)$ follows already from the quoted theorem of [6]. But, in contrast, $k = 4$ which was settled in [3] is quite complex and the solution is composed of four different ranges of (n, m) .

In the paper [6], also $N(n, k, k) = k(n - k + 1)$ and $N(m + 1, k, m) = m + k$ are proven. But already the determination of $N(m + 2, k, m)$ turns out to be quite hard and requires a deep insight. This was done first by Brualdi *et al.* in [2], using the heavy machinery of transversal matroids. Our major point here is to explore the structure of batch codes with $n = m + 2$, and to develop purely combinatorial methods to obtain a transparent derivation of the formula for $N(m + 2, k, m)$, stated as Theorem 2. The main structural observations are made in Section 3, and the computation proving optimality is given in Section 4.

We also design a class of optimal batch codes with transparent structure, and describe a transformation which generates further families of codes while preserving optimality. These results are presented in Section 2 in a general setting, where a general upper bound on $N(n, k, m)$ is also proved. Some further remarks and open problems are mentioned in the concluding section.

1.3. Hypergraph preliminaries

Given a set system $\mathcal{F}$, we denote

$$S(\mathcal{F}) := \sum_{F \in \mathcal{F}} |F|.$$

If $\mathcal{F}$ is over the set X , for $x \in X$ we denote by $d(x)$ the number of sets $F \in \mathcal{F}$ containing x and call it the *degree* of x . Counting in two ways the pairs (x, F)

satisfying the incidence relation $x \in F$, we obtain $\sum_{F \in \mathcal{F}} |F| = \sum_{x \in X} d(x)$ and hence

$$S(\mathcal{F}) = \sum_{x \in X} d(x)$$

is valid. Moreover, if $d(x) \geq 2$ for all $x \in X$, we partition X into two sets:

$$X_2 = \{x \in X \mid d(x) = 2\}, \quad X_3 = \{x \in X \mid d(x) \geq 3\}.$$

Then from the degree-sum expression above we obtain

$$S(\mathcal{F}) \geq 2|X_2| + 3|X_3| = 2|X| + |X_3| = 3|X| - |X_2|. \quad (1.1)$$

The following fact can be applied in arguments proving that a construction is a $\text{CBC}^*(k)$.

Lemma 1. *If a set system $\mathcal{F}$ violates the k -restricted Hall Condition but $\mathcal{F}$ is minimal in the sense that each of its proper subsystems satisfies k -HC, then all elements of $\bigcup_{F \in \mathcal{F}} F$ are contained in at least two members of $\mathcal{F}$.*

Proof. If some $F \in \mathcal{F}$ is the unique member of $\mathcal{F}$ containing an element x , then x can be chosen to represent F in every subfamily involving F . This has no effect on k -HC in $\mathcal{F} \setminus \{F\}$. $\square$

Corollary 1. *Let $n \geq m \geq k \geq 2$, and suppose that $\mathcal{F}$ is an optimal $\text{CBC}^*(n, k, m)$. Then*

- (i) *no elements have degree 0;*
- (ii) *if $d(x) = 1$, then $\{x\} \in \mathcal{F}$.*

Proof. (i) This part was proved in [6] and [2], too; but with reference to Lemma 1 we can simply say that if x had degree 0, then from any $n \geq m$ subsets of $X \setminus \{x\}$, a non-singleton F could be replaced with the singleton $\{x\}$, contradicting the optimality of $\mathcal{F}$. (Such an $F \in \mathcal{F}$ must occur whenever x has degree 0 and $k \geq 2$, by Remark 1.)

(ii) If F is the unique member of $\mathcal{F}$ containing x , and $|F| > 1$ holds, then by Lemma 1, $\mathcal{F}$ remains a $\text{CBC}^*(k)$ -system when F is replaced with $\{x\}$. But applying this replacement we would decrease $S(\mathcal{F})$, contradicting the assumption concerning optimality. $\square$

Further structural simplification will be presented for elements of degree 2 in the next section.

2. GENERAL UPPER BOUNDS AND TRANSFORMATIONS

In this section we present constructions, general upper bounds on $N(n, k, m)$ and some transformations applicable when elements of degree 2 occur, aiming at structural simplification under which the optimality of systems remains unchanged.

2.1. Constructions and upper bounds

We present two constructions, one recursive and one explicit, providing upper bounds on $N(n, k, m)$, which will turn out to be optimal in the entire range of m if $n = m + 2$. Actually, the recursive step is very simple.

Construction A. For any three integers $k \geq 1$, $s \geq 1$ and $q \geq 0$, take a system $\mathcal{F}'$ which is a $\text{CBC}^*(k + s + q - 1, k, k + s - 1)$, and add a further vertex x^* and the singleton edge $\{x^*\}$ to it. Clearly, the system remains a $\text{CBC}^*(k)$. Hence, a $\text{CBC}^*(k + s + q, k, k + s)$ is obtained with $S(\mathcal{F}) = S(\mathcal{F}') + 1$.

Corollary 2. For every $k \geq 1$, $s \geq 1$ and $q \geq 0$,

$$N(k + s + q, k, k + s) \leq N(k + s + q - 1, k, k + s - 1) + 1.$$

Construction B. For every fixed $k \geq 1$, $s \geq 0$ and $q \geq 2$, let us write s and k in the following form, where a , b , p and r are integers:

$$s = a(q - 1) + b \quad \text{with} \quad 0 \leq b \leq q - 2, \quad \text{hence} \quad a = \left\lfloor \frac{s}{q - 1} \right\rfloor;$$

$$k = p(a + 1) + r \quad \text{with} \quad 0 \leq r \leq a, \quad \text{hence} \quad p = \left\lfloor \frac{k}{a + 1} \right\rfloor = \left\lfloor \frac{k}{\left\lfloor \frac{s}{q - 1} \right\rfloor + 1} \right\rfloor.$$

Let $X = V_0 \cup V \cup U \cup R \cup Y$, where

$$V = V_1 \cup \dots \cup V_a \quad \text{with} \quad |V_1| = \dots = |V_a| = p,$$

$$U = U_1 \cup \dots \cup U_{q-1} \quad \text{with} \quad |U_1| = \dots = |U_{q-1}| = a,$$

$$|V_0| = p, \quad |R| = r, \quad |Y| = b.$$

The sets $V_0, V_1, \dots, V_a, U_1, \dots, U_{q-1}, R$ and Y are pairwise disjoint. This means

$$|X| = p + pa + (q - 1)a + r + b = k + s.$$

Then $\mathcal{F}$ is composed of the $k + s - p$ singletons of $V \cup U \cup R \cup Y$, any p mutually disjoint $(a + 1)$ -tuples each having precisely one element in each V_i ($i = 0, 1, \dots, a$), and the following q sets: $F_j = U_j \cup V_0$ ($1 \leq j \leq q - 1$) and $F_q = R \cup V_0$. Hence, $\mathcal{F}$ has precisely $k + s + q$ members and the sum of their cardinalities is equal to

$$\begin{aligned} k + s - p + p(a + 1) + (q - 1)(a + p) + r + p &= 2k + 2s + (q - 1)p - b = \\ &= 2k + 2s + (q - 1) \left\lfloor \frac{k}{\left\lfloor \frac{s}{q - 1} \right\rfloor + 1} \right\rfloor - b. \end{aligned}$$

Depending on the fixed parameters k , s and q , some of the sets V_0, V, U, R and Y may be empty, but no member of $\mathcal{F}$ is empty. In particular, if $s < q - 1$ then $a = 0$, and hence $X = V_0 \cup Y$.

Lemma 2. *For every $k \geq 1$, $s \geq 0$ and $q \geq 2$, Construction B yields a $\text{CBC}^*(k)$ -system.*

Proof. Consider the system $\mathcal{F}$ constructed as described above with parameters k , s and q . To prove that $\mathcal{F}$ is a $\text{CBC}^*(k)$, condition k -HC will be checked in the form (k3); that is, we will prove that any $z \geq s + 1$ elements of X intersect at least $z + q$ members of $\mathcal{F}$.

Consider a subset $Z \subseteq X$ with $|Z| = z \geq s + 1$. We have two cases:

- If $Z \cap V_0 \neq \emptyset$ then the elements in $Z \cap V_0$ together intersect precisely $|Z \cap V_0| + q$ members of the system $\mathcal{F}$. Moreover, each element of $Z \setminus V_0$ is a singleton. Thus, Z meets at least $z + q$ members.
- If $Z \cap V_0 = \emptyset$, each element of Z is a singleton. Moreover, in this case $a \geq 1$ has to hold. (Indeed, otherwise $Z \subseteq Y$ would imply $z \leq b \leq s$.) Let us denote the cardinalities of $Z \cap V$, $Z \cap U$, $Z \cap R$ and $Z \cap Y$ by z_1 , z_2 , z_3 and z_4 , respectively. Since $z_4 \leq b$,

$$z_1 + z_2 + z_3 \geq z - b \geq s + 1 - b = a(q - 1) + 1$$

holds. By construction, each non-singleton member of $\mathcal{F}$ meets at most one of V , U and R , and it does not meet Y , moreover it contains at most a elements from V or U and at most $r \leq a$ elements from R . Hence, the number of sets $F \in \mathcal{F}$ intersected by Z is at least

$$z + \left\lceil \frac{z_1}{a} \right\rceil + \left\lceil \frac{z_2}{a} \right\rceil + \left\lceil \frac{z_3}{a} \right\rceil \geq z + \left\lceil \frac{z_1 + z_2 + z_3}{a} \right\rceil \geq z + \left\lceil \frac{a(q - 1) + 1}{a} \right\rceil = z + q,$$

as claimed.

Therefore, k -HC is satisfied and $\mathcal{F}$ is a $\text{CBC}^*(k)$, indeed. $\square$

Corollary 3. *Let $k \geq 1$, $s \geq 0$, $q \geq 2$, and let b denote the residue of s modulo $q - 1$. Then*

$$N(k + s + q, k, k + s) \leq 2k + 2s + (q - 1) \left\lceil \frac{k}{\left\lfloor \frac{s}{q-1} \right\rfloor + 1} \right\rceil - b.$$

If $q = 2$ then $b = 0$, and we have the following consequence:

Corollary 4. *For every $k \geq 1$ and $s \geq 0$*

$$N(k + s + 2, k, k + s) \leq 2k + 2s + \left\lceil \frac{k}{s + 1} \right\rceil.$$

2.2. Optimality-preserving transformations

By Corollary 1, the situation with elements of degree smaller than 2 is clear. Here we handle the elements of degree 2. In this subsection $n > m$ is assumed.

Transformation C.. Suppose that $\mathcal{F}$ is a $\text{CBC}^*(k)$, and some $x \in X$ has degree 2. Let $F_1, F_2 \in \mathcal{F}$ be the two members of $\mathcal{F}$ containing x . Suppose further that $|F_2| \geq 2$ and $y \in F_2 \setminus \{x\}$. Define $\mathcal{F}_{x,y}$ as the system obtained from $\mathcal{F}$ by replacing F_1 and F_2 with $F'_1 := F_1 \cup \{y\}$ and $F'_2 := F_2 \setminus \{y\}$, respectively.

Let us put some comments about the flexibility of this transformation. First, there is no condition about the degree of y . Second, it is also allowed that $y \in F_1$, in which case $S(\mathcal{F})$ decreases by 1; and otherwise $S(\mathcal{F})$ remains unchanged. Third, F_1 is allowed to be a singleton, although $|F'_1| \geq 2$ always holds because $x \neq y$.

Lemma 3. *The system $\mathcal{F}_{x,y}$ is a $\text{CBC}^*(k)$ whenever so is $\mathcal{F}$.*

Proof. Assume for a contradiction that the assertion is not valid, and let $\mathcal{F}' \subset \mathcal{F}_{x,y}$ be a minimal subsystem violating k -HC. By Lemma 1, x cannot have degree 1 in $\mathcal{F}'$; hence, either both or none of F_1, F_2 belong to $\mathcal{F}'$. But if $F_1, F_2 \notin \mathcal{F}'$ then $\mathcal{F}' \subset \mathcal{F}$ and must satisfy k -HC, whereas for $F_1, F_2 \in \mathcal{F}'$ the union $\bigcup_{F \in \mathcal{F}'} F$ is the same in $\mathcal{F}'$ as in $\mathcal{F}$. Thus, $\mathcal{F}'$ cannot violate k -HC. $\square$

Corollary 5. *Every $\text{CBC}^*(n, k, m)$ system $\mathcal{F}$ can be transformed to some $\text{CBC}^*(n, k, m)$ system $\mathcal{F}'$ such that $S(\mathcal{F}') \leq S(\mathcal{F})$, the degree of each element is at most as large in $\mathcal{F}'$ as that in $\mathcal{F}$, and every element x of degree 2 in $\mathcal{F}'$ is a singleton member $\{x\} \in \mathcal{F}'$.*

Proof. A singleton can be obtained from F_2 after $|F_2| - 1$ applications of Transformation C. This singleton will always remain in the system if we do not choose it as F_1 . Hence, the number of singleton sets can be increased as long as there exists an element of degree 2 which is not a singleton member of the system.

Since the transformation does not increase the degrees, it does not increase the value of $S(\mathcal{F})$ either. $\square$

Using the conditions and notation given for transformation C, we can also conclude:

Corollary 6. *If $\mathcal{F}$ is an optimal $\text{CBC}^*(k)$, then*

- (i) $\mathcal{F}_{x,y}$ is optimal;
- (ii) $\mathcal{F}$ can also be obtained from $\mathcal{F}_{x,y}$ by Transformation C;
- (iii) if $F', F'' \in \mathcal{F}$ share an element z of degree 2, then $F' \cap F'' = \{z\}$.

Transformation C can also be applied to derive many non-isomorphic optimal families starting from a known one.

3. LOWER BOUNDS FOR $n = m + 2$

From now on, we concentrate on the case of $q = 2$. Here we prove two lower bounds on $N(m + 2, k, m)$, which will match the constructions of the previous section for $q = 2$, and also correspond to the two ranges of m (with respect to k) in the main theorem. We again begin with the simpler one, for vertices of degree 1.

Lemma 4. *Let $k \geq 2$ and $s \geq 1$. If $\mathcal{F}$ is a $\text{CBC}^*(k+s+2, k, k+s)$ with minimum vertex degree 1, then*

$$S(\mathcal{F}) \geq N(k+s+1, k, k+s-1) + 1.$$

Proof. Consider a vertex x of degree 1 from $\mathcal{F}$. Let the only edge F incident with x be replaced by the singleton $F^* = \{x\}$. After this modification the system $\mathcal{F}^*$ is obtained. Consider any at most $(k-1)$ -element subset Y of the underlying set X . If $x \notin Y$, then Y contains exactly as many members from $\mathcal{F}^*$ as from $\mathcal{F}$, hence at most $|Y|$. In the other case, when $x \in Y$, the vertex set $Y \setminus \{x\}$ contains at most $|Y| - 1$ members from $\mathcal{F}$, hence Y contains at most $|Y|$ members from $\mathcal{F}^*$. This proves that $\mathcal{F}^*$ is a $\text{CBC}^*(k+s+2, k, k+s)$. Moreover, omitting the vertex x and the singleton $\{x\}$ from $\mathcal{F}^*$, the obtained system $\mathcal{F}^-$ still satisfies the $\text{CBC}^*(k)$ -property. This implies that $N(k+s+1, k, k+s-1) \leq S(\mathcal{F}^-) = S(\mathcal{F}^*) - 1$. By our construction, also $S(\mathcal{F}^*) \leq S(\mathcal{F})$ holds. Hence, the lemma follows. $\square$

Remark 2. The above lemma and proof are valid for any $q \geq 0$. That is, if $\mathcal{F}$ is a $\text{CBC}^*(k+s+q, k, k+s)$ with minimum vertex degree 1 where $k \geq 2$ and $s \geq 1$, then

$$S(\mathcal{F}) \geq N(k+s+q-1, k, k+s-1) + 1.$$

The crucial tool for the proof of the main theorem is the following lower bound. We will use the notation X_2 and X_3 as they were introduced in Section 1.3. Let us also recall from there that $S(\mathcal{F}) = \sum_{x \in X} d(x) \geq 2k + 2s + |X_3|$ holds by (1.1), whenever no vertices of degree less than 2 occur.

Theorem 1. *If $\mathcal{F}$ is a $\text{CBC}^*(k+s+2, k, k+s)$ with minimum vertex degree at least 2, then*

$$S(\mathcal{F}) \geq 2k + 2s + \left\lfloor \frac{k}{s+1} \right\rfloor.$$

Proof. By Corollary 5, we can assume that every $x \in X_2$ is a singleton member $\{x\} \in \mathcal{F}$. On the other hand, referring the condition k -HC in the form (k3), any $\ell \geq s+1$ elements of X_2 have to meet at least $\ell+2$ members of $\mathcal{F}$. Therefore, each $F \in \mathcal{F}$ contains at most s elements from X_2 . This yields the following lower bound on the number $|\mathcal{F}| = n = k + s + 2$ of sets:

$$n \geq |X_2| + \frac{|X_2|}{s} = |X_2| \frac{s+1}{s}.$$

This implies an upper bound on $|X_2|$, and hence a lower bound on $|X_3|$.

$$|X_3| = k + s - |X_2| \geq k + s - n \frac{s}{s+1} = \frac{k+s+2}{s+1} - 2 = \frac{k-s}{s+1}.$$

Therefore, by (1.1) we have that

$$S(\mathcal{F}) \geq 2k + 2s + \left\lceil \frac{k-s}{s+1} \right\rceil = 2k + 2s + \left\lfloor \frac{k}{s+1} \right\rfloor,$$

as it was claimed. $\square$

Remark 3. The analogous general bound

$$|X_2| \leq \frac{n \left\lfloor \frac{s}{q-1} \right\rfloor + b}{\left\lfloor \frac{s}{q-1} \right\rfloor + 1},$$

where b denotes the residue of s modulo $q - 1$, is valid for any $q \geq 2$ and any $s \geq 0$.

4. EXACT VALUES FOR $n = m + 2$

Theorem 2. For every two integers $k \geq 1$ and $s \geq 0$,

- (i) $N(k + s + 2, k, k + s) = 2k + 2s + \left\lfloor \frac{k}{s+1} \right\rfloor$, if $k \geq s^2$; and
- (ii) $N(k + s + 2, k, k + s) = 2k + s - 2 + \lceil 2\sqrt{k+1} \rceil$, if $k < s^2$.

Proof. For every $n \geq m$ and $k \geq 1$ there exists an optimal $\text{CBC}^*(k)$ without isolated vertices. Then, due to Corollaries 2 and 4, Lemma 4, and Theorem 1, for every $k \geq 1$ and $s \geq 1$

$$N(k + s + 2, k, k + s) = \min \left(N(k + s + 1, k, k + s - 1) + 1, 2k + 2s + \left\lfloor \frac{k}{s+1} \right\rfloor \right)$$

holds. Let us introduce the notation

$$a(k, s) := N(k + s + 2, k, k + s), \quad b(k, s) := 2k + 2s + \left\lfloor \frac{k}{s+1} \right\rfloor.$$

Hence, for any $k \geq 1$ and $s \geq 1$ we have

$$a(k, s) = \min(a(k, s-1) + 1, b(k, s)). \quad (4.1)$$

Before proving the statements of the theorem, let us observe that

$$b(k, s) \leq b(k, s-1) + 1 \quad \text{if} \quad k \geq s^2 \geq 1. \quad (4.2)$$

Indeed, $\frac{k}{s+1} \leq \frac{k}{s} - 1$ holds whenever $k \geq s^2 + s$. Moreover, if $s^2 \leq k < s^2 + s$, then $\left\lfloor \frac{k}{s+1} \right\rfloor = s-1$ and $\left\lfloor \frac{k}{s} \right\rfloor = s$ are valid, yielding (4.2) with equality in this range of k . On the other hand, a similar computation shows that

$$b(k, s) \geq b(k, s-1) + 1 \quad \text{if} \quad k < s^2. \quad (4.3)$$

Part (i) states that $a(k, s) = b(k, s)$ if $k \geq s^2$. For $s = 0$, our statement is the particular case $n = k + 2$ of Theorem 4 in [6]. Indeed, the result $N(n, k, k) = k(n - k + 1)$ implies $a(k, 0) = 3k = b(k, 0)$. We proceed by induction on s . Assume that $s \geq 1$, $k \geq s^2$, and that $a(k, s') = b(k, s')$ is true for all $0 \leq s' < s$ and all $k \geq s^2$. Since $k > (s-1)^2$, the induction hypothesis and (4.2) imply $a(k, s-1) + 1 = b(k, s-1) + 1 \geq b(k, s)$. Hence, $a(k, s) = b(k, s)$ follows by (4.1). This proves (i) for every $k \geq 1$ and $s \geq 0$.

Concerning Part (ii), we first prove that in case of $k < s^2$ the recursion $a(k, s) = a(k, s-1) + 1$ applies. Due to (4.1), this precisely means to verify $b(k, s) \geq a(k, s-1) + 1$. This inequality is valid indeed, because $a(k, s-1) = \min(a(k, s-2) + 1, b(k, s-1))$ and hence, applying (4.3), we obtain

$$a(k, s-1) + 1 \leq b(k, s-1) + 1 \leq b(k, s)$$

as required.

The rest of the proof for $s > \sqrt{k}$ is a purely technical matter of computation. Namely, from the ' $a(k, s-1) + 1$ ' branch of the recursion we obtain

$$\begin{aligned} N(k+s+2, k, k+s) &= b(k, \lfloor \sqrt{k} \rfloor) + (s - \lfloor \sqrt{k} \rfloor) \\ &= 2k + 2\lfloor \sqrt{k} \rfloor + \left\lfloor \frac{k}{\lfloor \sqrt{k} \rfloor + 1} \right\rfloor + (s - \lfloor \sqrt{k} \rfloor) \\ &= 2k + s + \lfloor \sqrt{k} \rfloor + \left\lfloor \frac{k}{\lfloor \sqrt{k} \rfloor + 1} \right\rfloor. \end{aligned}$$

Observing $\lfloor \sqrt{k} \rfloor = \lceil \sqrt{k+1} \rceil - 1$, the above formula yields

$$\begin{aligned} N(k+s+2, k, k+s) &= 2k + s - 1 + \lceil \sqrt{k+1} \rceil + \left\lfloor \frac{k}{\lceil \sqrt{k+1} \rceil} \right\rfloor \\ &= 2k + s - 2 + \lceil 2\sqrt{k+1} \rceil \end{aligned}$$

where the last step follows from the identity

$$\lceil \sqrt{n} \rceil + \left\lfloor \frac{n-1}{\lceil \sqrt{n} \rceil} \right\rfloor = \lceil 2\sqrt{n} \rceil - 1.$$

Indeed, for any integer $t \geq 1$, if $t^2 + 1 \leq n \leq t^2 + t$ then $\lceil \sqrt{n} \rceil = t + 1$, $\left\lfloor \frac{n-1}{\lceil \sqrt{n} \rceil} \right\rfloor = t - 1$, and $\lceil 2\sqrt{n} \rceil = 2t + 1$; and if $t^2 + t + 1 \leq n \leq t^2 + 2t + 1$ then $\lceil \sqrt{n} \rceil = t + 1$, $\left\lfloor \frac{n-1}{\lceil \sqrt{n} \rceil} \right\rfloor = t$, and $\lceil 2\sqrt{n} \rceil = 2t + 2$. $\square$

5. CONCLUDING REMARKS

We have developed some methods of combinatorial nature to design batch codes and to prove their optimality in some cases. It is very likely that the class of systems described in Constructions A and B is optimal for many ranges of (n, k, m) . We have proved this for all $n = m + 2$.

A novel contribution to methodology here is that the matroid approach of [2] is now completely replaced with a combinatorial one. By this switch we expect that some of the techniques may turn out to be applicable for larger values of $n - m$, too.

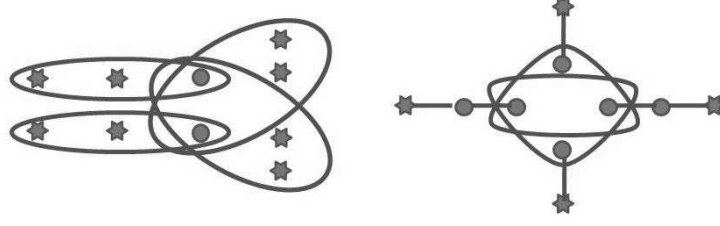


FIGURE 1. Two optimal constructions verifying $N(12, 8, 10) = 22$.
Star vertices indicate singleton members of the system.

Currently the smallest unexplored subclasses of the general problem are $n = m + 3$ and $k = 5$.

Many optimal systems. Transformation C described in Section 2.2 generates a large class of optimal systems. To illustrate this, we take the case of $(n, k, m) = (12, 8, 10)$. Figure 1 exhibits two set systems which are in some sense the two extremes. The first one is given in Construction B, where all but two of the elements are singleton members of the system. The prize of including many singletons is of course that a considerable number of larger sets must also occur. In the other system, there are only few singletons, but practically all sets are fairly small. Adding 10 new elements as singletons, we obtain the optimal $(22, 8, 20)$ system exhibited in [2] in the form of a 20×22 matrix of 0s and 1s.

One can easily verify that the two systems exhibited in Figure 1 can be obtained from each other (in either direction) via a sequence of step-by-step applications of Transformation C.

The case $m = k$. It has been proved in [6, Theorem 4] that $N(n, k, k) = k(n - k + 1)$ holds for all $n \geq k \geq 1$. An optimal system is obtained easily by taking the k elements of X as singletons, and $n - k$ copies of X for the remaining members of $\mathcal{F}$.

Here we describe an alternative optimal construction that replaces the k singletons² and two copies of X if $n - 2 \geq k \geq 3$. Let $X = \{x_1, \dots, x_k\}$, and consider any non-trivial partition $F' \cup F'' = X$. Beside F' and F'' we take the k pairs $\{x_1, x_2\}$, $\{x_2, x_3\}$, ..., $\{x_{k-1}, x_k\}$, $\{x_k, x_1\}$. The sum of cardinalities clearly remains unchanged.

Since any s pairs with $s < k$ form a union of paths in the terminology of graphs, they have at least $s + 1$ elements in their union, therefore Hall's Condition is satisfied for any at most k members of the set system obtained.

Let us note that the side condition $n - 2 \geq k \geq 3$ is natural, because otherwise (if $n \leq k + 1$ or $k = 2$) we do not obtain a new system. As a matter of fact, for $k = 2$ the unique optimal system has two (distinct) singletons and $n - 2$ identical pairs; this

²The case of $k = 3$ is slightly different from $k \geq 4$, because then one singleton will remain.

follows by Remark 1. On the other hand, for $n = k + 1$ the starting configuration of k singletons together with $\{X\}$ is 2-regular, hence the transformation described in Section 2.2 generates a bunch of non-isomorphic optimal systems. Complementing any of these systems with $n - k - 1$ copies of $\{X\}$ if $n \geq k + 2$, many further optimal configurations can be obtained.

Using parameters n, k and m . In the proofs of this paper we found simpler the usage of parameters q, k, s instead of n, k, m . It is worth, however, formulating our estimates in terms of the original parameters, too. In this way, Corollaries 2 and 3 and Theorem 2 can be transcribed as follows:

- $N(n, k, m) \leq N(n - 1, k, m - 1) + 1$ for all $n \geq m \geq k + 1 \geq 2$;
- $N(n, k, m) \leq 2m + (n - m - 1) \left\lfloor \frac{k}{\left\lfloor \frac{m-k}{n-m-1} \right\rfloor + 1} \right\rfloor - b$ for all $m \geq k \geq 1$ and $n \geq m + 2$, where b is the residue of $m - k$ modulo $n - m - 1$;
- $N(m + 2, k, m) = 2m + \left\lfloor \frac{k}{m-k+1} \right\rfloor$ for all $k \leq m \leq k + \sqrt{k}$;
- $N(m + 2, k, m) = m + k - 2 + \lceil 2\sqrt{k+1} \rceil$ for all $m > k + \sqrt{k}$.

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SOME REMARKS ON STRUCTURAL MATRIX RINGS AND MATRICES WITH IDEAL ENTRIES

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Abstract. Associating to each pre-order on the indices $1, \dots, n$ the corresponding structural matrix ring, or incidence algebra, embeds the lattice of n -element pre-orders into the lattice of $n \times n$ matrix rings. Rings within the order-convex hull of the embedding, i.e. matrix rings that contain the ring of diagonal matrices, can be viewed as incidence algebras of ideal-valued, generalized pre-order relations. Certain conjugates of the upper or lower triangular matrix rings correspond to the various linear orderings of the indices, and the incidence algebras of partial orderings arise as intersections of such conjugate matrix rings.

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Keywords: structural matrix ring, incidence algebra, pre-order, quasi-order, triangular matrix, conjugation, semiring, ideal lattice, subring lattice

1. CONJUGATE SUBRINGS

Rings are understood to be possibly non-commutative, and to have a *unit* (multiplicatively neutral) element, which is assumed to be distinct from the zero (null) element and is denoted by 1 or I or a similar symbol. *Subrings* are understood to contain the unit element. An $n \times n$ matrix is viewed as a map defined on the set $\mathbf{n}^2 = \{1, \dots, n\}^2$. (Here $n \geq 1$ is assumed.) The ring of $n \times n$ matrices over a ring R is denoted by $M_n(R)$.

A *pre-order*, also called *quasi-order*, is a reflexive and transitive binary relation $\lesssim$ on a set S , an *order* (or *partial order*) is an anti-symmetric pre-order, and a *linear* (or *total*) *order* is an order in which any two elements are comparable. Instead of the generic notation $\lesssim$, specific pre-orders may be denoted by other symbols such as θ . Given a ring R and a pre-order $\lesssim$ on $\mathbf{n}$, the *structural matrix ring* $M_n(\lesssim, R)$ over R is defined by

$$M_n(\lesssim, R) = \{ A \in M_n(R) : \forall i, j \quad A(i, j) = 0 \text{ unless } i \lesssim j \}$$

The full matrix ring $M_n(R)$, the subrings of all upper triangular (respectively lower triangular), and of all diagonal matrices, are examples of structural matrix rings.

Structural matrix rings are essentially the same as incidence algebras of finite pre-ordered sets, although the latter term is sometimes used under the assumption that the base ring R is a field [3] or that the pre-order in question is an order, possibly on an infinite set, as in [8]. Ring-theoretical properties of incidence algebras most relevant to the present context were studied in [2, 3, 7, 10, 11].

The set of all pre-orders defined on any given set constitutes a lattice whose minimum is the equality relation.

Proposition 1. *For any ring R , the map associating to each pre-order $\lesssim$ on $\mathbf{n}$ the corresponding structural matrix ring $M_n(\lesssim, R)$ provides an embedding of the lattice of pre-orders on $\mathbf{n}$ into the lattice of subrings of the matrix ring $M_n(R)$. The embedding preserves also infinite greatest lower and least upper bounds.*

Proof. Preservation of greatest lower bounds is obvious. The preservation of upper bounds is a consequence of the description of the least upper bound of a family of pre-orders as the transitive closure of the least binary relation that is implied by the pre-orders in the family. $\square$

The lattice embedding described in the proposition above is generally not surjective, and for $n \geq 2$ its range is order-convex if and only if R is a division ring. Section 2 will provide a description of the order-convex hull of the embedding's range.

Subrings S and T of $M_n(R)$ are said to be *permutation conjugates* if there is a permutation matrix P such that

$$S = \{PAP^{-1} : A \in T\}$$

Such subrings are obviously isomorphic under the automorphism $A \mapsto PAP^{-1}$ of $M_n(R)$.

All the $n!$ linear orders on the finite set $\mathbf{n}$ are isomorphic, and the isomorphisms among them are precisely the self-bijections of the underlying set $\mathbf{n}$. Consequently we have:

Proposition 2. *For any pre-order $\lesssim$ on $\mathbf{n}$ and any ring R , the following conditions are equivalent:*

- (i) $\lesssim$ is a linear order,
- (ii) $M_n(\lesssim, R)$ is a permutation conjugate of the ring of upper triangular matrices,
- (iii) $M_n(\lesssim, R)$ is a permutation conjugate of the ring of lower triangular matrices.

The well-known fact that every partial order is the intersection of its linear extensions yields:

Proposition 3. *For any pre-order $\lesssim$ on $\mathbf{n}$ and any ring R , the following conditions are equivalent:*

- (i) $\lesssim$ is an order,
- (ii) $M_n(\lesssim, R)$ is the intersection of some permutation conjugates of the ring of upper triangular matrices,

- (iii) $M_n(\lesssim R)$ is the intersection of some permutation conjugates of the ring of lower triangular matrices.

These considerations were first developed, in the context of fields, in [4]. They were related to a ring property concerning one-sided and two-sided inverses in [6], addressing questions originating in [1] and [9].

2. MATRICES WITH IDEAL ENTRIES

The order-convex hull of the embedding provided by Proposition 1 consists obviously of those matrix rings that contain the ring of diagonal matrices. We now show that these rings can be viewed as generalized incidence algebras, corresponding to generalized relations that are the analogues of pre-order relations in a reticulated semiring-valued framework.

By a *semiring* we understand a set endowed with a binary operation called *sum* (denoted additively) and a binary operation called *product* (denoted multiplicatively) such that:

- (i) the sum operation defines a commutative semigroup,
- (ii) the product operation defines a semigroup,
- (iii) both distributivity laws $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ hold for all members a, b, c of the underlying set.

The set $\mathcal{I}(R)$ of (bilateral) ideals of any ring R is a semiring under ideal sum and product of ideals. This semiring is lattice-ordered by inclusion. The set $M_n(\mathcal{I}(R))$ of $n \times n$ matrices with entries in $\mathcal{I}(R)$ is again a semiring under the obvious sum and product operations, also lattice-ordered by entry-wise inclusion: lattice join coincides with semiring sum, denoted $+$, while the lattice meet operation $\wedge$ is entry-wise intersection. The lattice $M_n(\mathcal{I}(R))$ is complete, it is isomorphic to the n^2 -th Cartesian power of the complete lattice $\mathcal{I}(R)$. The matrix $\mathbf{I}$, with the improper ideal $(1) = R$ in diagonal positions and the trivial ideal (0) in off-diagonal positions, is multiplicatively neutral in the semiring $M_n(\mathcal{I}(R))$.

For every matrix $\mathbf{U}$ with ideal entries, $\mathbf{U} \in M_n(\mathcal{I}(R))$, consider the set of matrices

$$G = \{A \in M_n(R) : \forall i, j \quad A(i, j) \in \mathbf{U}(i, j)\}$$

This set is always an additive subgroup of $M_n(R)$, and it is a subring if and only if $\mathbf{U}^2 + \mathbf{I} \leq \mathbf{U}$ in the ordered semiring of $n \times n$ matrices with ideal entries. In that case, we say that the subring G is *defined by* $\mathbf{U}$.

There is a natural lattice embedding from the lattice of all pre-orders on $\mathbf{n}$ into the lattice $M_n(\mathcal{I}(R))$, namely the map $\theta \mapsto \mathbf{U}$ where $\mathbf{U}(i, j)$ is the improper or trivial ideal of R according to whether the relation $i\theta j$ holds or not. The matrix $\mathbf{U}$ so obtained always satisfies $\mathbf{U}^2 + \mathbf{I} \leq \mathbf{U}$, and the subring of $M_n(R)$ defined by it coincides with the structural matrix ring $M_n(\theta, R)$. For this reason we denote by $M_n(R, \mathbf{U})$ the matrix ring defined by any $\mathbf{U} \in M_n(\mathcal{I}(R))$ satisfying $\mathbf{U}^2 + \mathbf{I} \leq \mathbf{U}$. Any

$\mathbf{U} \in M_n(\mathcal{I}(R))$ is viewed as an $\mathcal{I}(R)$ -valued relation on the n -element set $\mathbf{n}$, the inequality $\mathbf{U}^2 + \mathbf{I} \leq \mathbf{U}$ generalizes transitivity and reflexivity of relations, and $M_n(R, \mathbf{U})$ may then be thought of as the incidence algebra of the generalized pre-order $\mathbf{U}$.

Matrices with ideal entries $\mathbf{U} \in M_n(\mathcal{I}(R))$ satisfying the generalized transitivity condition $\mathbf{U}^2 \leq \mathbf{U}$ only were used by van Wyk [10], and recently again by Meyer, Sziget and van Wyk [7], in the description of ideals of structural matrix rings.

Any matrix with ideal entries $\mathbf{U} \in M_n(\mathcal{I}(R))$ satisfying $\mathbf{U}^2 + \mathbf{I} \leq \mathbf{U}$ is said to be *reflexive-transitive*. If R is a division ring, then the range of the map $\theta \mapsto \mathbf{U}$ described above, embedding the pre-order lattice into $M_n(\mathcal{I}(R))$, is exactly the set of reflexive-transitive matrices.

For any ring R , the set of reflexive-transitive matrices with ideal entries constitutes a complete lattice under the ordering of $M_n(\mathcal{I}(R))$ (but not a sublattice of $M_n(\mathcal{I}(R))$ in general).

Proposition 4. *For any ring R , the map $\mathbf{U} \mapsto M_n(R, \mathbf{U})$ establishes a lattice isomorphism between:*

- (i) the lattice $\{\mathbf{U} \in M_n(\mathcal{I}(R)) : \mathbf{U}^2 + \mathbf{I} \leq \mathbf{U}\}$ of $n \times n$ reflexive-transitive matrices with ideal entries,
- (ii) the lattice of subrings of $M_n(R)$ containing all diagonal matrices.

Proof. Obviously if $\mathbf{U}^2 + \mathbf{I} \leq \mathbf{U}$ then $M_n(R, \mathbf{U})$ contains all diagonal matrices, and if $\mathbf{U} \subseteq \mathbf{V}$ then $M_n(R, \mathbf{U}) \subseteq M_n(R, \mathbf{V})$.

Conversely, let $N \subseteq M_n(R)$ be any matrix ring including all diagonal matrices. For every $1 \leq i, j \leq n$ the set $U_{ij} = \{A(i, j) : A \in N\}$ is an ideal of R , the matrix $\mathbf{U} = (U_{ij})$ with ideal entries can be seen to satisfy $\mathbf{U}^2 + \mathbf{I} \leq \mathbf{U}$, and $M_n(R, \mathbf{U}) = N$. $\square$

The proof above is presented in [5] in somewhat greater detail, together with some consequences. For division rings, where there is only the trivial and the improper ideal, the structure of the lattice of subrings of $M_n(R)$ containing all diagonal matrices is independent of the choice of the particular division ring R . In the general case, it is the ideal lattice structure of R that determines the structure of this upper section of the subring lattice of $M_n(R)$.

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ON THE DOMAIN OF ODDNESS OF AN INFIMAL CONVOLUTION

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Abstract. To illustrate a recent uniqueness result on Hahn-Banach extensions, we show that if p is a sublinear functional of a vector space X , φ is a linear functional of a subspace V of X such that $\varphi(v) \leq p(v)$ for all $v \in V$, and

$$q(x) = \inf_{v \in V} (p(x-v) + \varphi(v))$$

for all $x \in X$, then the domain

$$W = \{x \in X : q(-x) = -q(x)\}$$

of the oddness of the infimal convolution q of p and φ is a subspace of X such that any of the three possible cases $V = W$, $V \neq W \neq X$ and $W = X$ can naturally occur even if X is an Euclidean space and p is a norm on X .

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Keywords: Hahn-Banach uniqueness, infimal convolution, domain of oddness

1. INTRODUCTION

Suppose that X is a vector space over $\mathbb{R}$ and p is a sublinear (positively homogeneous and subadditive) function on X to $\mathbb{R}$.

Moreover, assume that V is a subspace of X and φ is a linear function on V to $\mathbb{R}$ such that φ is dominated by p on V in the sense that $\varphi(v) \leq p(v)$ for all $v \in V$.

Now, because of the classical Hahn-Banach theorem [1, 4], a linear function f on X to $\mathbb{R}$, that extends φ and is dominated by p on X , may be called a Hahn-Banach extension of φ to X .

Moreover, in accordance with the more general definitions of Moreau [1, 3], and Strömberg [5], the function $q = p * \varphi$, defined by

$$q(x) = \inf_{v \in V} (p(x-v) + \varphi(v))$$

for all $x \in X$, can be called the infimal convolution of p and φ .

Under the above notations, the following theorem has been proved in [2] on the uniqueness of Hahn-Banach extension of φ to X .

Theorem 1. *The following assertions are equivalent :*

- (1.) *q is odd on X ;*
- (2.) *q is a Hahn-Banach extension of φ to X ;*
- (3.) *there exists a unique Hahn-Banach extension f of φ to X ;*
- (4.) *there exists at most one Hahn-Banach extension f of φ to X .*

This theorem has motivated us to investigate the domain

$$W = \{x \in X : q(-x) = -q(x)\}$$

of oddness of the function q .

By using some basic properties of q , we shall show that W is always a subspace of X containing V . Moreover, by using a particular infimal value theorem, we shall show that any of the three possible cases $V = W$, $V \neq W \neq X$, and $W = X$ can naturally occur even if X is an Euclidean space and p is a norm on X .

2. LINEARITY PROPERTIES OF q AND W

The following two basic theorems have also been established in [2].

Theorem 2. *q is a real-valued, sublinear function of X .*

Corollary 1. *$q(0) = 0$ and $\lambda q(x) \leq q(\lambda x)$ for all $\lambda \in \mathbb{R}$ and $x \in X$.*

Remark 1. Hence, one can easily derive that $q(\lambda x) = \lambda q(x)$ for all $\lambda \geq 0$ and $x \in X$.

Theorem 3. *$q(x + v) = q(x) + \varphi(v)$ for all $x \in X$ and $v \in V$.*

Corollary 2. *q is an extension of φ and $q(x + v) = q(x) + q(v)$ for all $x \in X$ and $v \in V$.*

Now, by using the above results, we can easily prove the following

Theorem 4. *W is a subspace of X containing V .*

Proof. Since φ is odd and q is an extension of φ , it is clear that $V \subset W$. Therefore, we only need to show that W is closed under the linear operations in X .

For this, note first that if $x \in W$, then $q(-x) = -q(x)$. Hence, it is clear that

$$q(-(-x)) = q(x) = -(-q(x)) = -q(-x).$$

Therefore, $-x \in W$ also holds.

Moreover, if $\lambda \geq 0$ and $x \in W$, then by Remark 2.3 and since $q(-x) = -q(x)$, we have

$$q(-\lambda x) = q(\lambda(-x)) = \lambda q(-x) = \lambda(-q(x)) = -\lambda q(x) = -q(\lambda x).$$

Therefore, $\lambda x \in W$ also holds.

While, if $\lambda < 0$ and $x \in W$, then $-\lambda > 0$ and $-x \in W$. Therefore, by the above observation, we also have

$$\lambda x = (-\lambda)(-x) \in W.$$

Finally, we note that if $x, y \in W$, then the subadditivity of q , the equalities $q(-x) = -q(x)$ and $q(-y) = -q(y)$, and the $\lambda = -1$ particular case of Corollary 2.2 altogether imply

$$\begin{aligned} q(-(x+y)) &= q(-x + (-y)) \leq q(-x) + q(-y) \\ &= -q(x) + (-q(y)) = -(q(x) + q(y)) \leq -q(x+y) \leq q(-(x+y)) \end{aligned}$$

Therefore,

$$q(-(x+y)) = -q(x+y),$$

and thus $x+y \in W$ also holds. $\square$

Remark 2. To appreciate the importance of W , note that by definition W is the largest subset of X such that q is odd on W .

Hence, by Theorems 2.6 and 1.1, we can see that W is the largest subspace of X such that φ has a unique Hahn-Banach extension ψ to W . Moreover, we necessarily have $\psi(w) = q(w)$ for all $w \in W$.

3. A PARTICULAR INFIMAL VALUE THEOREM

In the proof of our subsequent infimal value theorem, we shall use the following minimal value result.

Lemma 1. *If f is a continuous function of $\mathbb{R}$ to itself such that*

$$\lim_{t \rightarrow -\infty} f(t) = +\infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} f(t) = +\infty,$$

then there exists $\tau \in \mathbb{R}$ such that

$$f(\tau) = \min_{t \in \mathbb{R}} f(t).$$

Proof. By using the above limit conditions, we can see that there exists $r > 0$ such that

$$f(0) \leq f(t)$$

whenever either $t < -r$ or $r < t$. Now, by a basic theorem on continuous functions, we can state that there exists $\tau \in [-r, r]$ such that

$$f(\tau) \leq f(t)$$

for all $t \in [-r, r]$. Now, since $f(\tau) \leq f(0)$, it is clear that the above inequality also holds for all $t \in \mathbb{R} \setminus [-r, r]$. $\square$

Our colleague, Mihály Bessenyei has observed that the following infimal value theorem can also be proved in a more elementary way, without using the standard methods of analysis for finding local and global minima. However, our subsequent proof can be used to illustrate these methods in classrooms.

Theorem 5. *For any $\alpha, \beta \in \mathbb{R}$, with $\alpha \geq 0$ and $|\beta| \leq 1$, we have*

$$\inf_{t \in \mathbb{R}} \left(\sqrt{t^2 + \alpha} + \beta t \right) = \sqrt{\alpha(1 - \beta^2)}.$$

Proof. Define

$$f_{\alpha\beta}(t) = \sqrt{t^2 + \alpha} + \beta t$$

for all $t \in \mathbb{R}$. Then, in particular

$$f_{0\beta}(t) = \sqrt{t^2} + \beta t = |t| + \beta t = \begin{cases} (\beta + 1)t & \text{for } t \geq 0, \\ (\beta - 1)t & \text{for } t < 0. \end{cases}$$

Hence, since $\beta + 1 \geq 0$ and $\beta - 1 \leq 0$, we can see that $f_{0\beta}(t) \geq 0$ for all $t \in \mathbb{R}$. Now, since $f_{0\beta}(0) = 0$, it is clear that

$$\min_{t \in \mathbb{R}} f_{0\beta}(t) = 0 = \sqrt{0(1 - \beta^2)}.$$

This shows that the required equality is true in the particular case $\alpha = 0$. Therefore, in the sequel, we may assume that $\alpha \neq 0$, and thus $\alpha > 0$.

Now, we can also note that

$$f_{0\beta}(t) = \sqrt{t^2} + \beta t < \sqrt{t^2 + \alpha} + \beta t = f_{\alpha\beta}(t)$$

for all $t \in \mathbb{R}$. Hence, by the nonnegativity of the function $f_{0\beta}$, we can see that $f_{\alpha\beta}(t) > 0$ for all $t \in \mathbb{R}$. Moreover, we can note that

$$\begin{aligned} \lim_{t \rightarrow -\infty} f_{\alpha 1}(t) &= \lim_{t \rightarrow -\infty} \left(\sqrt{t^2 + \alpha} + t \right) \\ &= \lim_{t \rightarrow -\infty} \frac{\left(\sqrt{t^2 + \alpha} \right)^2 - t^2}{\sqrt{t^2 + \alpha} - t} = \lim_{t \rightarrow -\infty} \frac{\alpha}{\sqrt{t^2 + \alpha} - t} = 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{t \rightarrow +\infty} f_{\alpha(-1)}(t) &= \lim_{t \rightarrow +\infty} \left(\sqrt{t^2 + \alpha} - t \right) \\ &= \lim_{t \rightarrow +\infty} \frac{\left(\sqrt{t^2 + \alpha} \right)^2 - t^2}{\sqrt{t^2 + \alpha} + t} = \lim_{t \rightarrow +\infty} \frac{\alpha}{\sqrt{t^2 + \alpha} + t} = 0. \end{aligned}$$

Hence it is clear that

$$\inf_{t \in \mathbb{R}} f_{\alpha 1}(t) = 0 = \sqrt{\alpha(1 - 1^2)} \quad \text{and} \quad \inf_{t \in \mathbb{R}} f_{\alpha(-1)}(t) = 0 = \sqrt{\alpha(1 - (-1)^2)}.$$

This shows that the required equality is true in the particular case $|\beta| = 1$. Therefore, in the sequel, we may assume that $|\beta| \neq 1$, and thus $|\beta| < 1$.

Now, since $\beta - 1 < 0$ and $\beta + 1 > 0$, we can also note that

$$\lim_{t \rightarrow -\infty} f_{0\beta}(t) = +\infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} f_{0\beta}(t) = +\infty.$$

Hence, by using that $f_{0\beta} \leq f_{\alpha\beta}$, we can infer that

$$\lim_{t \rightarrow -\infty} f_{\alpha\beta}(t) = +\infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} f_{\alpha\beta}(t) = +\infty$$

also hold. Moreover, we can note that $f_{\alpha\beta}$ is continuous. Thus, by Lemma 3.1, there exists $\tau \in \mathbb{R}$ such that

$$f_{\alpha\beta}(\tau) = \min_{t \in \mathbb{R}} f_{\alpha\beta}(t).$$

Moreover, since $f_{\alpha\beta}$ is differentiable, we can also state that $f'_{\alpha\beta}(\tau) = 0$. Hence, by using that

$$f'_{\alpha\beta}(\tau) = \frac{1}{2\sqrt{\tau^2 + \alpha}} 2\tau + \beta = \frac{\tau + \beta\sqrt{\tau^2 + \alpha}}{\sqrt{\tau^2 + \alpha}},$$

we can infer that $\tau + \beta\sqrt{\tau^2 + \alpha} = 0$, and thus

$$\tau = -\beta\sqrt{\tau^2 + \alpha}.$$

This implies that $\text{sgn}(\tau) = -\text{sgn}(\beta)$. Moreover, we can also easily see that $\tau^2 = \beta^2(\tau^2 + \alpha)$, whence

$$\tau^2 = \frac{\alpha\beta^2}{1-\beta^2}, \quad \text{and thus} \quad |\tau| = \frac{\sqrt{\alpha}|\beta|}{\sqrt{1-\beta^2}}.$$

Hence, by using that

$$|\tau| = \text{sgn}(\tau)\tau \quad \text{and} \quad |\beta| = \text{sgn}(\beta)\beta = -\text{sgn}(\tau)\beta,$$

we can infer that

$$\text{sgn}(\tau)\tau = -\text{sgn}(\tau) \frac{\sqrt{\alpha}\beta}{\sqrt{1-\beta^2}}, \quad \text{and thus} \quad \tau = -\frac{\sqrt{\alpha}\beta}{\sqrt{1-\beta^2}}.$$

Namely, if $\text{sgn}(\tau) = 0$, then because of $\text{sgn}(\tau) = -\text{sgn}(\beta)$ we also have $\text{sgn}(\beta) = 0$. Therefore, not only $\tau = 0$, but also $\beta = 0$. Now, to complete the proof, it remains only to note that

$$\begin{aligned} f_{\alpha\beta}(\tau) &= \sqrt{\tau^2 + \alpha} + \beta\tau = \sqrt{\frac{\alpha\beta^2}{1-\beta^2} + \alpha} - \frac{\sqrt{\alpha}\beta^2}{\sqrt{1-\beta^2}} \\ &= \frac{\sqrt{\alpha\beta^2 + \alpha - \alpha\beta^2}}{\sqrt{1-\beta^2}} - \frac{\sqrt{\alpha}\beta^2}{\sqrt{1-\beta^2}} = \sqrt{\alpha} \frac{1-\beta^2}{\sqrt{1-\beta^2}} = \sqrt{\alpha(1-\beta^2)}. \end{aligned}$$

□

Remark 3. Note that if in particular $|\beta| \neq 1$, then we can write minimum instead of infimum in the above theorem.

4. THE FIRST ILLUSTRATING EXAMPLE

The following example is a little more difficult than that given in [2].

Example 1. Take

$$X = \mathbb{R}^2 \quad \text{and} \quad V = \mathbb{R}(1, 1).$$

Moreover, define

$$\varphi(r, r) = r \quad \text{and} \quad p(r, s) = \sqrt{r^2 + s^2}$$

for all $r, s \in \mathbb{R}$. Then, it is clear that V is a subspace of X , φ is a linear function of V , and p is a norm on X such that

$$|\varphi(r, r)| = |r| \leq \sqrt{2}|r| = \sqrt{r^2 + r^2} = p(r, r)$$

for all $r \in \mathbb{R}$. Thus, in particular, φ is dominated by p on V .

Moreover, by using Theorem 3.2, we can see that

$$\begin{aligned} q(r, s) &= \inf_{u \in \mathbb{R}} (p((r, s) - (u, u)) + \varphi(u, u)) \\ &= \inf_{u \in \mathbb{R}} (p(r - u, s - u) + \varphi(u, u)) = \inf_{u \in \mathbb{R}} \left(\sqrt{(r - u)^2 + (s - u)^2} + u \right) \\ &= \inf_{u \in \mathbb{R}} \sqrt{2} \left(\sqrt{\frac{(r - u)^2 + (s - u)^2}{2}} + \frac{u}{\sqrt{2}} \right) \\ &= \sqrt{2} \inf_{u \in \mathbb{R}} \left(\sqrt{\frac{(r - u)^2 + (s - u)^2}{2}} + \frac{u}{\sqrt{2}} \right) \\ &= \sqrt{2} \inf_{u \in \mathbb{R}} \left(\sqrt{\left(u - \frac{r+s}{2}\right)^2 + \frac{(r-s)^2}{4}} + \frac{1}{\sqrt{2}} \left(u - \frac{r+s}{2}\right) + \frac{r+s}{2\sqrt{2}} \right) \\ &= \sqrt{2} \left(\inf_{u \in \mathbb{R}} \left(\sqrt{\left(u - \frac{r+s}{2}\right)^2 + \frac{(r-s)^2}{4}} + \frac{1}{\sqrt{2}} \left(u - \frac{r+s}{2}\right) \right) + \frac{r+s}{2\sqrt{2}} \right) \\ &= \sqrt{2} \inf_{t \in \mathbb{R}} \left(\sqrt{t^2 + \frac{(r-s)^2}{4}} + \frac{1}{\sqrt{2}} t \right) + \frac{r+s}{2} \\ &= \sqrt{2} \sqrt{\frac{(r-s)^2}{4} \left(1 - \frac{1}{2}\right)} + \frac{r+s}{2} = \frac{|r-s|}{2} + \frac{r+s}{2} = \max\{r, s\} \end{aligned}$$

for all $r, s \in \mathbb{R}$.

Hence, it is clear that, for any $r, s \in \mathbb{R}$, we have

$$\begin{aligned} (r, s) \in W &\iff q(-(r, s)) = -q(r, s) \iff q(-r, -s) = -q(r, s) \\ &\iff \max\{-r, -s\} = -\max\{r, s\} \iff -\max\{-r, -s\} = \max\{r, s\} \\ &\iff \min\{r, s\} = \max\{r, s\} \iff r = s \iff (r, s) \in V. \end{aligned}$$

Therefore, in the present particular case, $W = V$ holds.

Remark 4. Note that now we have

$$(r, s) = \frac{s-r}{2}(-1, 1) + \frac{r+s}{2}(1, 1)$$

for all $r, s \in \mathbb{R}$.

Therefore, if f is a Hahn-Banach extension of φ to X , then

$$\begin{aligned} f(r, s) &= \frac{s-r}{2} f(-1, 1) + \frac{r+s}{2} f(1, 1) \\ &= \frac{s-r}{2} f(-1, 1) + \frac{r+s}{2} \varphi(1, 1) = \frac{s-r}{2} f(-1, 1) + \frac{r+s}{2} \\ &= \frac{1-f(-1, 1)}{2} r + \frac{1+f(-1, 1)}{2} s \end{aligned}$$

for all $r, s \in \mathbb{R}$. Moreover, by Theorem 1.7 of [2], we have

$$-1 = -q(1, -1) \leq f(-1, 1) \leq q(-1, 1) = 1.$$

Hence, by taking

$$\lambda = \frac{1-f(-1, 1)}{2},$$

we can see that $\lambda \in [0, 1]$ is such that

$$f(r, s) = \lambda r + (1-\lambda)s$$

for all $r, s \in \mathbb{R}$. Thus, we can obtain all Hahn-Banach extensions f of φ to X .

5. THE SECOND AND THIRD ILLUSTRATING EXAMPLES

Example 2. Let X , V and φ be as in Example 4.1. Moreover, define

$$p(r, s) = \sqrt{\frac{r^2 + s^2}{2}}$$

for all $r, s \in \mathbb{R}$. Then, it is clear that p is a norm on X such that

$$|\varphi(r, r)| = |r| = \sqrt{\frac{r^2 + r^2}{2}} = p(r, r)$$

for all $r \in \mathbb{R}$. Thus, in particular, φ is dominated by p on V .

Moreover, quite similarly as in Example 4.1, we can see that

$$\begin{aligned}
 q(r, s) &= \inf_{u \in \mathbb{R}} (p(r-u, s-u) + \varphi(u, u)) = \\
 &= \inf_{u \in \mathbb{R}} \left(\sqrt{\frac{(r-u)^2 + (s-u)^2}{2}} + u \right) \\
 &= \inf_{u \in \mathbb{R}} \left(\sqrt{\left(u - \frac{r+s}{2}\right)^2 + \frac{(r-s)^2}{4}} + \left(u - \frac{r+s}{2}\right) \right) + \frac{r+s}{2} \\
 &= \inf_{t \in \mathbb{R}} \left(\sqrt{t^2 + \frac{(r-s)^2}{4}} + t \right) + \frac{r+s}{2} \\
 &= \sqrt{\frac{(r-s)^2}{4} (1-1)} + \frac{r+s}{2} = \frac{r+s}{2}
 \end{aligned}$$

for all $r, s \in \mathbb{R}$.

Hence, it is clear that

$$q(-(r, s)) = q(-r, -s) = \frac{-r-s}{2} = -\frac{r+s}{2} = -q(r, s)$$

for all $r, s \in \mathbb{R}$, and thus q is an odd function of X . Therefore, in the present particular case, $W = X$ holds.

Remark 5. Now, by Theorem 1.1, we can state that q is the unique Hahn-Banach extension of φ to X .

Example 3. Take

$$X = \mathbb{R}^3 \quad \text{and} \quad V = \mathbb{R} (1, 1, 0).$$

Moreover, define

$$\varphi(r, r, 0) = r \quad \text{and} \quad p(r, s, t) = \sqrt{\frac{r^2 + s^2}{2}} + |t|$$

for all $r, s, t \in \mathbb{R}$. Then, it is clear that V is a subspace of X , φ is a linear function of V , and p is a norm on X such that

$$|\varphi(r, r, 0)| = |r| = \sqrt{\frac{r^2 + r^2}{2}} + |0| = p(r, r, 0)$$

for all $r \in \mathbb{R}$. Thus, in particular, φ is dominated by p on V .

Moreover, by making use of the computation given in Example 5.1, we can see that

$$\begin{aligned}
q(r, s, t) &= \inf_{u \in \mathbb{R}} (p((r, s, t) - (u, u, 0)) + \varphi(u, u, 0)) \\
&= \inf_{u \in \mathbb{R}} (p(r - u, s - u, t) + \varphi(u, u, 0)) \\
&= \inf_{u \in \mathbb{R}} \left(\sqrt{\frac{(r - u)^2 + (s - u)^2}{2}} + |t| + u \right) \\
&= \inf_{u \in \mathbb{R}} \left(\sqrt{\frac{(r - u)^2 + (s - u)^2}{2}} + u \right) + |t| = \frac{r + s}{2} + |t|
\end{aligned}$$

for all $r, s, t \in \mathbb{R}$.

Hence, it is clear that, for any $r, s, t \in \mathbb{R}$, we have

$$\begin{aligned}
(r, s, t) \in W &\iff q(-(r, s, t)) = -q(r, s, t) \\
&\iff q(-r, -s, -t) = -q(r, s, t) \\
&\iff \frac{-r - s}{2} + |-t| = -\left(\frac{r + s}{2} + |t|\right) \\
&\iff |-t| = -|t| \iff 2|t| = 0 \iff |t| = 0 \iff t = 0.
\end{aligned}$$

Therefore, in the present particular case, $W = \mathbb{R} \times \mathbb{R} \times \{0\}$, and thus $V \neq W \neq X$ holds.

Remark 6. Note that now we have

$$(r, s, t) = t(0, 0, 1) + (r, s, 0)$$

for all $r, s, t \in \mathbb{R}$.

Therefore, if f is a Hahn-Banach extension of φ to X , then by Theorem 1.1 we have

$$\begin{aligned}
f(r, s, t) &= t f(0, 0, 1) + f(r, s, 0) = t f(0, 0, 1) + q(r, s, 0) \\
&= t f(0, 0, 1) + \frac{r + s}{2} = \frac{1}{2}r + \frac{1}{2}s + f(0, 0, 1)t
\end{aligned}$$

for all $r, s, t \in \mathbb{R}$. Moreover, by Theorem 1.7 of [2], we have

$$-1 = -q(0, 0, -1) \leq f(0, 0, 1) \leq q(0, 0, 1) = 1.$$

Hence, by taking

$$\lambda = f(0, 0, 1),$$

we can see that $\lambda \in [0, 1]$ such that

$$f(r, s, t) = \frac{1}{2}r + \frac{1}{2}s + \lambda t$$

for all $r, s, t \in \mathbb{R}$. Thus, we can obtain all Hahn-Banach extensions f of φ to X .

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NEW PRECONDITIONERS FOR NONSYMMETRIC AND INDEFINITE SADDLE POINT LINEAR SYSTEMS

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Abstract. In this paper, we extend the preconditioners of Sturler and Liesen [*SIAM J. Sci. Comput.*, 26(2005): 1598-1619]. The spectral characteristics of the preconditioners are investigated and the results show that all eigenvalues of the preconditioned matrices are strongly clustered. Finally, numerical experiments are also reported for illustrating the efficiency of the presented preconditioners.

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1. INTRODUCTION

Consider the following general nonsingular saddle point linear system

$$A\xi \equiv \begin{pmatrix} G & B^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix} = f, \quad (1.1)$$

where $G \in R^{n \times n}$, $B, C \in R^{n \times m}$ have full rank, $x, b \in R^n$ and $y, q \in R^m$, and the vectors x, y are unknown. Here we assume that A is nonsingular, that will be used in the following analysis. Under these assumptions, system (1.1) has a unique solution. This system is very important and appears in many different applications of scientific computing, such as constrained optimization [1, 16], the finite element or finite difference methods of solving the Navier-Stokes equation [4–7], fluid dynamics, constrained least problems and generalized least squares problems [10–12], and the discretized time-harmonic Maxwell equations in mixed form [9].

Results for the general systems have been obtained before, for example, Murphy, Golub and Wathen [3] proposed the block-diagonal preconditioner

$$\begin{pmatrix} G^{-1} & 0 \\ 0 & (CG^{-1}B^T)^{-1} \end{pmatrix}.$$

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Then the preconditioned matrices are diagonalizable and have at most three distinct eigenvalues. Hence, a Krylov subspace method will converge in at most three steps. Because such a general system is typically large and sparse, the preconditioner is expensive. Recently, Sturler and Liesen [14] considered the following preconditioner

$$\begin{pmatrix} D^{-1} & 0 \\ 0 & (CD^{-1}B^T)^{-1} \end{pmatrix},$$

where $G = D - E$ and D can be efficiently inverted. For such a preconditioner, one should assume that A and $CD^{-1}B^T$ are invertible. But it may be expensive to compute $CD^{-1}B^T$, and the matrix $CD^{-1}B^T$ may be singular, so we introduce the following preconditioner

$$\begin{pmatrix} D^{-1} & 0 \\ 0 & (F + CD^{-1}B^T)^{-1} \end{pmatrix},$$

where D and $F + CD^{-1}B^T$ are invertible. We can get a suitable F such that $F + CD^{-1}B^T$ is invertible and we can reduce the computing cost of the Schur complement. If we choose suitable D and F , we find that all of the block diagonal preconditioners can be included. When G is singular or ill-conditioned, such preconditioners can still be applied.

This paper is organized as follows. In Section 2, we will establish new preconditioners and study their spectral analysis for the saddle point systems. The related systems are given in Section 3. In Section 4, numerical experiments are given. Finally, the conclusions are presented in Section 5.

2. BLOCK DIAGONAL PRECONDITIONER

We split G as $G = D - E$, where D is easy to solve. $CD^{-1}B^T$ is the exact Schur complement of the matrix

$$\begin{pmatrix} D & B^T \\ C & 0 \end{pmatrix}.$$

In this paper, we introduce the following preconditioner which is a generalization of the preconditioners of [14]:

$$P(D) = \begin{pmatrix} D^{-1} & 0 \\ 0 & (F + CD^{-1}B^T)^{-1} \end{pmatrix}.$$

Then we get the left preconditioned matrix:

$$B(F) = P(D)A = \begin{pmatrix} I - D^{-1}E & D^{-1}B^T \\ (F + CD^{-1}B^T)^{-1}C & 0 \end{pmatrix} = \begin{pmatrix} I - S & N \\ M & 0 \end{pmatrix}, \quad (2.1)$$

where,

$$\begin{aligned}
MN &= (F + CD^{-1}B^T)^{-1}CD^{-1}B^T \\
&= (F + CD^{-1}B^T)^{-1}(F + CD^{-1}B^T - F) \\
&= I + (-(F + CD^{-1}B^T)^{-1}F) \\
&= I + Q.
\end{aligned}$$

When $F = 0$, $MN = I$ (it is the case in [14]), and the preconditioner reduces to the preconditioner in [14]. If $B = C$, $E = C^T W^{-1}B$ and $F = W - CD^{-1}B^T$, it reduces to the block diagonal preconditioner in [13] and [8].

In the following analysis, we discuss the eigendecomposition of the matrix

$$B_0 = \begin{pmatrix} I & N \\ M & 0 \end{pmatrix}, \quad (2.2)$$

and we assume that Q is diagonalizable and $5 + 4\delta_j \neq 0$, where $Qv_j = \delta_j v_j$.

Theorem 1. *Let B_0 be of the form (2.2). Then B_0 has the following eigenvalues and eigenvectors:*

(a) $n - m$ eigenpairs of the form $(1, [u_j^T, 0]^T)$, where $u_1, \dots, u_{n-m}$ form a basis of $\text{Null}(M)$, the nullspace of M .

(b) $2m$ eigenpairs of the form $(\lambda^\pm, [u_j, (\lambda^\pm)^{-1}(Mu_j)^T]^T)$, where $\lambda^\pm = (1 \pm \sqrt{5 + 4\delta_j})/2$, and $Qv_j = \delta_j v_j$. Then the eigenvector matrix Y of B_0 is given by

$$Y = \begin{pmatrix} U_1 & U_2 & U_2 \\ 0 & (\lambda^+)^{-1}MU_2 & (\lambda^-)^{-1}MU_2 \end{pmatrix}, \quad (2.3)$$

where $U_1 = [u_1, \dots, u_{n-m}]$ and $U_2 = [u_{n-m+1}, \dots, u_n]$.

Proof. We consider the equation

$$B_0[u^T, v^T]^T = \lambda[u^T, v^T]^T,$$

which is equivalent to the following two equations

$$u + Nv = \lambda u, \quad (2.4)$$

$$Mu = \lambda v. \quad (2.5)$$

Since B_0 is nonsingular, we can assume $\lambda \neq 0$. So from (2.5), we have

$$v = \lambda^{-1}Mu.$$

First, we assume $\lambda = 1$. Substituting this into (2.4) and (2.5), we have

$$Nv = 0 \quad \text{and} \quad Mu = v. \quad (2.6)$$

Since B^T has full column rank by assumption, this implies that $v = 0$ and that B_0 has only eigenpairs of the form $(1, [u_j^T, 0]^T)$, where $u \in \text{null}(M)$.

Since C has full row rank, so does M , and B_0 has precisely $n - m$ distinct eigenpairs of this type. Next, we assume $\lambda \neq 1$. From (2.4), we can get

$$u = (\lambda - 1)^{-1} N v.$$

Substituting this into (2.5) yields

$$Q v_j = (\lambda^2 - \lambda - 1) v_j. \quad (2.7)$$

Hence, v_j must be eigenvectors of Q , we assume that $Q v_j = \delta_j v_j$. Then we solve (2.7) for λ to yield

$$\lambda^\pm = \left(1 \pm \sqrt{5 + 4\delta_j} \right) / 2.$$

So, the remaining $2m$ eigenpairs are $(\lambda^\pm, [u_j, (\lambda^\pm)^{-1}(M u_j)^T]^T)$. $\square$

We are now ready to consider the perturbation bounds on the eigenvalues of $B(F)$. Throughout this paper $\|\cdot\|$ indicates the 2-norm, and we let

$$Y = \begin{pmatrix} U_1 & U_2 & U_2 \\ 0 & (\lambda^+)^{-1} M U_2 & (\lambda^-)^{-1} M U_2 \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix},$$

$$\gamma^+ = \text{diag}((\lambda_j^- - 1)/(\lambda_j^- - \lambda_j^+)),$$

$$\gamma^- = \text{diag}((\lambda_j^+ - 1)/(\lambda_j^- - \lambda_j^+)).$$

Theorem 2. Consider matrices $B(F)$ of the form (2.1). Let Y be the eigenvector matrix of B_0 , as given by (2.2). Then for each eigenvalue λ_B of $B(F)$ there exists an eigenvalue λ of B_0 such that

$$|\lambda_B - \lambda| \leq \left\| Y^{-1} \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} Y \right\| \leq 2 \max(1, \|\gamma^+\|, \|\gamma^-\|) \|Y_{11}^{-1} S Y_{11}\|. \quad (2.8)$$

Proof. Since B_0 is diagonalizable, this follows from a classic result in perturbation theory [15]. We expand the right-hand side of (2.8) (see also [3, 14])

$$\begin{aligned} |\lambda_B - \lambda| &\leq \left\| \begin{pmatrix} \hat{I} Y_{11}^{-1} S Y_{11} & \hat{I} Y_{11}^{-1} S Y_{12} \\ -[0, \gamma^-] Y_{11}^{-1} S Y_{11} & -[0, \gamma^-] Y_{11}^{-1} S Y_{12} \end{pmatrix} \right\| \\ &\leq \sqrt{2} \max(1, \|\gamma^+\|, \|\gamma^-\|) \left\| \begin{bmatrix} Y_{11}^{-1} S Y_{11} \\ -[0, I] Y_{11}^{-1} S Y_{11} \end{bmatrix} \right\| \\ &\leq 2 \max(1, \|\gamma^+\|, \|\gamma^-\|) \|Y_{11}^{-1} S Y_{11}\|. \end{aligned}$$

$\square$

We can find that, if $\delta_j \approx -5/4$, $\gamma^\pm$ can be very large. So we should take a suitable Q such that the value of δ_j are well separated from $-5/4$. In section 4, we will present some choices.

Now we will derive a bound on $\|Y_{11}^{-1}SY_{11}\|$, following the approach in [3, 14]. Recall that $Y_{11} = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$, where $U_1^T U_1 = I$ and $U_2 = NV$ with unit columns. Let $U_2 = V_2 \Theta$, where $V_2^T V_2 = I$. Furthermore, denote $\omega_1 = \|U_1^T V_2\|$.

Theorem 3. Define Y_{11} , S , U_1 , U_2 , V_2 , Θ and ω_1 as above, and let $\kappa(\cdot)$ denote the 2-norm condition number. Then

$$\|Y_{11}^{-1}SY_{11}\| \leq \kappa(\Theta) \left(\frac{1+\omega_1}{1-\omega_1} \right)^{1/2} \|S\|.$$

The proof is similar to that of Lemma 2.3 in [14].

3. FIXED-POINT METHOD AND ITS RELATED SYSTEM

We can derive the following splitting from (2.1):

$$B(F) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} I-S & N \\ M & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \left(B_0 - \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{Bmatrix} f \\ g \end{Bmatrix}. \quad (3.1)$$

Note that

$$B_0^{-1} = \begin{pmatrix} I_n - N(MN)^{-1}M & N(MN)^{-1} \\ (MN)^{-1}M & -(MN)^{-1} \end{pmatrix}. \quad (3.2)$$

We left-multiply (3.1) by B_0^{-1} to yield the fixed-point iteration

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} (I - N(MN)^{-1}M)S & 0 \\ ((MN)^{-1}M)S & 0 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} \hat{f} \\ \hat{g} \end{pmatrix}. \quad (3.3)$$

So we can get the related system for the fixed-point iteration

$$(I - (I - N(MN)^{-1}M)S)x = \hat{f}. \quad (3.4)$$

Theorem 4. The spectral radius of the fixed point iteration matrix in (3.3) and the eigenvalues λ_R of the related system matrix in (3.4) satisfy

$$\left\{ \frac{\rho(I - (I - N(MN)^{-1}M)S)}{|1 - \lambda_R|} \right\} \leq \frac{\|S\|}{(1 - \omega_1^2)^{1/2}}, \quad (3.5)$$

where ω_1 is the largest singular value of $U_1^T U_2$.

Proof. Let $\hat{N} = N(MN)^{-1}$, $\hat{M} = M$. We have

$$(I - \hat{N}\hat{M})U_1 = U_1, \quad (I - \hat{N}\hat{M})U_2 = 0.$$

Thus

$$(I - \hat{N}\hat{M})S = (U_1, U_2) \begin{pmatrix} I_{n-m} & 0 \\ 0 & 0 \end{pmatrix} (U_1, U_2)^{-1} S,$$

Then, similary to the proof of Theorem 4.4 in [3], we have

$$\rho(I - (I - N(MN)^{-1}M)S) \Big\{ \frac{\|S\|}{|1 - \lambda_R|} \Big\} \leq \frac{\|S\|}{(1 - \omega_1^2)^{1/2}}.$$

□

Generally, this leads to better clustering and tighter bounds for the related system than for the block-diagonally preconditioned system. Because of these advantages, the related system will generally have faster convergence than the block-diagonally preconditioned system, as we will see it in section 4.

4. NUMERICAL EXAMPLES

All the numerical experiments were performed with MATLAB 7.0. In all of our runs we used a zero initial guess. The stopping criterion is $\|r^{(k)}\|_2 / \|r^{(0)}\|_2 \leq 10^{-6}$, where $r^{(k)}$ is the residual vector after the k -th iteration. The right-hand side vectors b and q are taken such that the exact solutions x and y are both vectors with all of their components equal to 1. The initial guess is chosen to be the zero vector. We will use preconditioned GMRES(10) to solve the saddle point-type systems. Our numerical experiments are similar to ones in [2]. The matrices considered are taken from [2] with a slightly altered notations. We construct the saddle point-type matrix A from a transformed matrix $\hat{A}$ of the following form

$$\hat{A} = \begin{pmatrix} F_1 & 0 & B_u^T \\ 0 & F_2 & B_v^T \\ B_u & B_v & 0 \end{pmatrix},$$

where $G = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}$ is positive real. The matrix $\hat{A}$ arises from the discretization by the maker and cell finite direrence scheme of a leaky two dimensional liddriven cavity problem in a square domain ($0 \leq x \leq 1; 0 \leq y \leq 1$). Then the matrix (B_u, B_v) is replaced by a random matrix $\hat{B}$ with the same sparsity as (B_u, B_v) , replaced by $B_1 = \hat{B}(1:m, 1:m) - \frac{3}{2}I_m$, such that B_1 is nonsingular. Denote $B_2 = \hat{B}(1:m, m+1:n)$, then we have $B = (B_1, B_2)$ with $B_1 \in R^{m,m}$ and $B_2 \in R^{m,n-m}$. Obviously, the resulting saddle point-type matrix

$$A = \begin{pmatrix} G & B^T \\ B & 0 \end{pmatrix}$$

satisfies that $\text{rank}(B^T) = \text{rank}(B) = m$.

From the matrix A we construct the following saddle point-type matrices:

$$A_1 = \begin{pmatrix} G_1 & B^T \\ B & 0 \end{pmatrix},$$

where G_1 is constructed from G by replacing its first $\frac{m}{4}$ rows and columns with zero entries. Note that G_1 is semipositive real and its nullity is $\frac{m}{4}$.

In our numerical experiments the matrix W in the augmentation block preconditioners is taken as $W = I_m$ and the mesh $h = 1/16$. During the implementation of our augmentation block preconditioners, we assume

$$P1 = \begin{pmatrix} D^{-1} & 0 \\ 0 & (CD^{-1}B^T)^{-1} \end{pmatrix},$$

where $F = 0, D = G_1 + 10B^TC$, and

$$P2 = \begin{pmatrix} D^{-1} & 0 \\ 0 & I \end{pmatrix},$$

where $F = I - CD^{-1}B^T, D = G_1 + B^TC$,

$$P3 = \begin{pmatrix} D^{-1} & 0 \\ 0 & (0.01CD^{-1}B^T)^{-1} \end{pmatrix},$$

where $F = 1.1CD^{-1}B^T, D = G_1 + 10B^TC$, respectively.

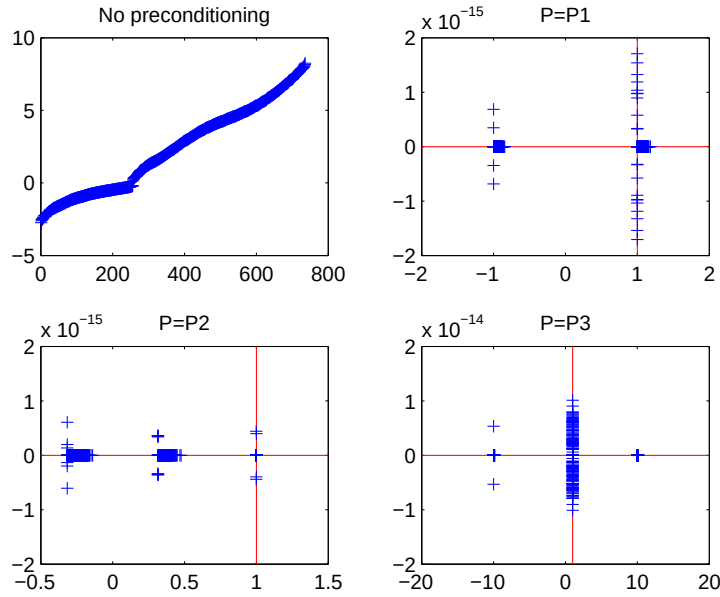


FIGURE 1. Spectrum of A and $P^{-1}A$ when $h = \frac{1}{16}(n + m = 736)$.

On Figure 1, we can observe that the eigenvalues of $P^{-1}A$ are strongly clustered, which is expected by Theorem 2.1 and Theorem 2.2. In our example,

we compared the three different preconditioners. On Figure 2, we see that the choice

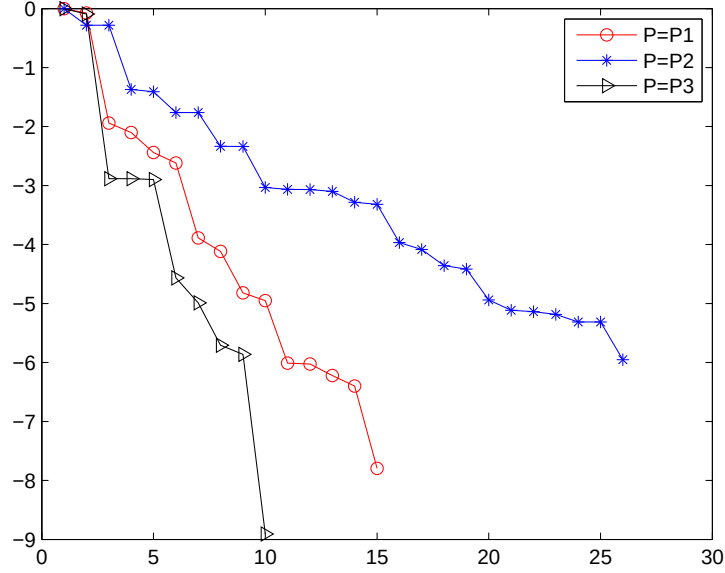


FIGURE 2. Convergence curve and total numbers of inner GMRES iterations when $h = \frac{1}{16}(n + m = 736)$.

of a suitable F improves the results.

5. CONCLUSION

In this paper, we extended the block positive definite preconditioner introduced in [3], and we proved some of its properties. Then, we gave numerical examples in order to illustrate the efficiency of our method.

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GLOBAL CONTROLLABILITY OF A SET OF FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we present the problem of global controllability (GC) of a set of fractional differential equation (SFDE).

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1. INTRODUCTION

Recently fractional differential equations have acquired many important applications in science and engineering involving different types of differential operators (see [1–3, 5, 6, 13]). In [4], the author studied the controllability of the control system

$$x(t) = \mu I^\alpha f(t, x(t), u(t)), \quad 0 < \alpha \leq 1,$$

$$u(t) \in \mu G(t, x(t), u(t)), \quad t \in J := [0, T],$$

where $\mu \in [0, 1]$, $G(t, x(t), u(t))$ denotes a set-valued function which has a nonempty closed convex set of values in $\mathbb{R}^n$ and $I^\alpha f(t)$ is the fractional integral operator defined as follows.

Definition 1 ([7, 9, 11, 12]). The fractional order integral of the function f of order $0 < \alpha \leq 1$ is defined by

$$I_a^\alpha f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau, \quad 0 < \alpha \leq 1,$$

and $I_0^\alpha f(t) = I^\alpha f(t) = f(t) * \psi_\alpha(t)$, where $(*)$ denotes the convolution product (see [9, 11]), $\psi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$ and $\psi_\alpha(t) = 0$ for $t \leq 0$ and $\psi_\alpha(t) \rightarrow \delta(t)$ (the delta function) as $\alpha \rightarrow 0$.

Definition 2 ([7, 9, 11, 12]). The fractional order derivative of the function f of order $\alpha > 1$ in the Riemann-Liouville sense is defined by

$$D_a^\alpha f(t) = \frac{d}{dt} \int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau) d\tau = \frac{d}{dt} I_a^{1-\alpha} f(t).$$

Remark 1. [9,11] The fractional order derivative and integral of a continuous function f satisfies the relation:

$$I^\alpha(D^\alpha f(t)) = D^\alpha(I^\alpha f(t)) = f(t).$$

Also we have

$$I^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \quad (\mu > -1).$$

In this work, we consider the control set of fractional differential equations (SCFDE) as follows: Let $K_c(\mathbb{R}^n)$ be the collection of all nonempty compact convex subsets of $\mathbb{R}^n$

$$D^\alpha X(t) = F(t, X(t), U(t)), \quad 0 < \alpha \leq 1, \quad (1.1)$$

where

$$(X(0) = 0, X(t) \in K_c(\mathbb{R}^n), U(t) \in K_c(\mathbb{R}^m), t \in J := [0, T]),$$

$$F(t, X(t), U(t)) : J \times K_c(\mathbb{R}^n) \times K_c(\mathbb{R}^m) \longrightarrow K_c(\mathbb{R}^n)$$

and let D^α be the fractional derivative of order $\alpha \in (0, 1]$. Thus (1.1) reduces to the system

$$\begin{pmatrix} D^\alpha x_1(t) \\ \vdots \\ D^\alpha x_n(t) \end{pmatrix} = \begin{pmatrix} f_1(t, x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)) \\ \vdots \\ f_n(t, x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)) \end{pmatrix}.$$

This paper is organized as follows: in Section 2, we recall certain basic concepts and notations which are useful in next sections. In Section 3, we present the problem of global controllability (GC) for (SCFDE), and some examples of globally controllable (GC) and non - controllable SCFDE will be given. In Section 4, we present the conclusion of this paper.

2. PRELIMINARIES

We recall some notations and concepts presented in detail in a recent series of works of V. Lakshmikantham et al. (see [8]).

Given $A, B \in K_c(\mathbb{R}^n)$, the Hausdorff distance between A and B is defined by

$$D[A, B] := \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|_{\mathbb{R}^n}, \sup_{b \in B} \inf_{a \in A} \|b - a\|_{\mathbb{R}^n} \right\}$$

where $\|\cdot\|_{\mathbb{R}^n}$ denotes the Euclidean norm in $\mathbb{R}^n$. We define the magnitude of a nonempty subset of A as

$$D[A, \theta^n] = \|A\| = \sup\{\|a\|_{\mathbb{R}^n}, a \in A\} \quad (2.1)$$

where θ^n is the zero element of $\mathbb{R}^n$ which is regarded as a one point set. The $\|A\| = D[A, \theta^n]$ norm in $K_c(\mathbb{R}^n)$ is finite when the supremum in (2.1) is attained with $A \in K_c(\mathbb{R}^n)$. The Hausdorff metric satisfies the following properties.

$$D[A + C, B + C] = D[A, B] \text{ and } D[A, B] = D[B, A],$$

$$D[\lambda A, \lambda B] = \lambda D[A, B], D[A, B] \leq D[A, C] + D[C, B],$$

$$D[A + A', B + B'] \leq D[A, B] + D[A', B']$$

for all $A, B, C \in K_c(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}_+$. If $r, s \in \mathbb{R}$ and $A, B \in K_c(\mathbb{R}^n)$, then

$$r(A + B) = rA + rB, r(sA) = (rs)A, 1.A = A.$$

Finally, let $A, B \in K_c(\mathbb{R}^n)$. The set $C \in K_c(\mathbb{R}^n)$ satisfying $A = B + C$ is called the Hausdorff difference (the geometric difference) of the sets A and B and is denoted by the symbol $A - B$.

3. MAIN RESULTS

In this section we consider different cases of the system

$$D^\alpha X(t) = F(t, X(t), U(t)), \quad 0 < \alpha \leq 1, \quad (3.1)$$

where $F(t, X(t), U(t)) \in C[J, K_c(\mathbb{R}^n), K_c(\mathbb{R}^m)]$ whose solution set is of the form

$$\begin{aligned} X(t) &= X(X_0, t, U(t)) \in K_c(\mathbb{R}^n), \quad (X(0) = X_0 = 0) \\ &= X(0, t, U(t)) \\ &:= X(t, U(t)). \end{aligned} \quad (3.2)$$

Definition 3. A state pair of the solution set $(X_0, X_1) \in K_c(\mathbb{R}^n)$ is controllable if after time t_1 we can find a control set $U(t) \in K_c(\mathbb{R}^m)$ such that

$$X(t_1) = X(X_0, t_1, U(t_1)) = X_1 \in K_c(\mathbb{R}^n). \quad (3.3)$$

Definition 4. The system (3.1) is called

- (GC): Globally controllable if every state pair of solution set $(X_0, X_1) \in K_c(\mathbb{R}^n)$ is controllable;
- (GR): Globally achievable if for every $X_1 \in K_c(\mathbb{R}^n)$ we have a state pair of solution set $(\theta^n, X_1) \in K_c(\mathbb{R}^n)$ that is controllable;
- (GNC): Globally achievable to $\theta^n \in K_c(\mathbb{R}^n)$ if for every $X_1 \in K_c(\mathbb{R}^n)$ we have state pair of solution set $(X_1, \theta^n) \in K_c(\mathbb{R}^n)$ that is controllable.
- (SGR): Globally achievable set after time t :

$$R_t(X_0) := \{X(t) \in K_c(\mathbb{R}^n) | \exists U(t) \in K_c(\mathbb{R}^m) : X(X_0, t, U(t)) = X(t)\}$$

and

$$R(X_0) := \bigcup_{t>0} R_t(X_0).$$

Definition 5. A control system (3.1) is called completely controllable (CC) if $\forall X_0, X_1 \in K_c(\mathbb{R}^n)$ there exists a continuous control $U(t) \in K_c(\mathbb{R}^m), t \in J = [0, T]$ such that $X(0) = X_0$ and $X(T) = X_1$.

In virtue of Definition 4, we have the following result:

Theorem 1. System (3.1) is

- (i) (GC) if for every $X_0 \in K_c(\mathbb{R}^n)$ we have $R(X_0) = K_c(\mathbb{R}^n)$;
- (ii) (GR) if we have $R(\theta^n) = K_c(\mathbb{R}^n)$;
- (iii) (GNC) if for every $X_0 \in K_c(\mathbb{R}^n)$ we have $\theta^n \in R(X_0)$.

Remark 2. From Definition 5, every (CC) is controllable.

In the following we study several kinds of systems (3.1) :

3.1. Stationary linear control set of a fractional differential equation

Let us define the fractional system

$$D^\alpha X(t) = AX(t) + BU(t), \quad 0 < \alpha \leq 1, \quad (3.4)$$

where $X(0) = X_0 = 0 \in K_c(\mathbb{R}^n)$, $A : K_c(\mathbb{R}^n) \rightarrow K_c(\mathbb{R}^n)$, $B : K_c(\mathbb{R}^m) \rightarrow K_c(\mathbb{R}^n)$ are operators, whose solution set is of the form

$$\begin{aligned} X(t) &= X(t, U(t)) \in K_c(\mathbb{R}^n) \\ &= \frac{B}{\Gamma(\alpha)} \int_0^t [(t-\tau)^{\alpha-1} - A]U(\tau)d\tau. \end{aligned} \quad (3.5)$$

Theorem 2. System (3.4) is (GC) if and only if B is invertible.

Proof. Sufficient condition: For $t_1 > 0$, we consider

$$L(t_1) := \int_0^{t_1} [(t-\tau)^{\alpha-1} - A]d\tau.$$

Then for all $t \in J$, the control $U(t) \in K_c(\mathbb{R}^m)$ has the form

$$U(\tau) = B^{-1}\Gamma(\alpha)L^{-1}(t_1)X_1.$$

In virtue of Remark 1, we have

$$X(t) = I^\alpha[AX(t) + BU(t)] = AI^\alpha X(t) + BI^\alpha U(t).$$

Then by using *Laplace* transform and its inverse for the fractional integral operator (see [11], p. 104), we obtain

$$\begin{aligned} \mathbb{L}\{X(t)\} &= \mathbb{L}\{AI^\alpha X(t) + BI^\alpha U(t)\} = A\mathbb{L}\{I^\alpha X(t)\} + B\mathbb{L}\{I^\alpha U(t)\} \\ \Rightarrow \mathbb{X}(s) &= A\mathbb{L}\{I^\alpha X(t)\} + B\mathbb{L}\{I^\alpha U(t)\} \Rightarrow \mathbb{X}(s) = As^{-\alpha}\mathbb{X}(s) + Bs^{-\alpha}\mathbb{U}(s) \\ &\Rightarrow \mathbb{X}(s)[1 - As^{-\alpha}] = Bs^{-\alpha}\mathbb{U}(s) \\ &\Rightarrow \mathbb{X}(s) = \frac{Bs^{-\alpha}}{1 - As^{-\alpha}}\mathbb{U}(s) = \frac{B}{s^\alpha - A}\mathbb{U}(s), \end{aligned}$$

where $\mathbb{X}(s)$ and $\mathbb{U}(s)$ are the *Laplace* transform of X and U respectively. Hence, by the inverse *Laplace* transform, we have

$$X(t) = \frac{B}{\Gamma(\alpha)} \int_0^t [(t-\tau)^{\alpha-1} - A]U(\tau)d\tau.$$

Now for $t_1 > 0$, we state that

$$\begin{aligned} X(t_1) &= \frac{B}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - \tau)^{\alpha-1} - A] U(\tau) d\tau \\ &= \frac{B}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - \tau)^{\alpha-1} - A] [B^{-1} \Gamma(\alpha) L^{-1}(t_1) X_1] d\tau \\ &= BB^{-1} \left(\int_0^{t_1} [(t_1 - \tau)^{\alpha-1} - A] d\tau \right) L^{-1}(t_1) X_1 = X_1. \end{aligned}$$

This implies that the system (3.4) is (GC).

Necessary condition: Assume that system (3.4) is (GC). If B is not invertible then the existence of the feedback control $U(t)$ implies that system (3.4) is not (GC) and this is a contradiction. $\square$

3.2. Non stationary linear control set of fractional differential equations

Let us define the linear system

$$D^\alpha X(t) = A(t)X(t) + B(t)U(t), \quad 0 < \alpha \leq 1, \quad (3.6)$$

where $X(0) = X_0 \in K_c(\mathbb{R}^n)$, $A(t) : K_c(\mathbb{R}^n) \rightarrow K_c(\mathbb{R}^n)$, $B(t) : K_c(\mathbb{R}^m) \rightarrow K_c(\mathbb{R}^n)$ are operators, whose solution set is of the form $X(t) = X(0, X_0, t, U(t)) \in K_c(\mathbb{R}^n)$.

Theorem 3. System (3.6) is (GC) if and only if $B^{-1}(t)$ exists.

Proof. Sufficient condition: For $t_1 > 0$, we consider

$$\gamma(t_1) := \int_0^{t_1} [(t - \tau)^{\alpha-1} - A(\tau)] d\tau.$$

Then for all $t \in J$, the control $U(t) \in K_c(\mathbb{R}^m)$ has the form

$$U(\tau) = B^{-1}(\tau) \Gamma(\alpha) \gamma^{-1}(t_1) X_1.$$

In view of Remark 1, we have

$$X(t) = I^\alpha [A(t)X(t) + B(t)U(t)] = I^\alpha A(t)X(t) + I^\alpha B(t)U(t).$$

By using the *Laplace* transform and its inverse for the fractional integral operator (see [11], P 104), we obtain

$$\begin{aligned} \mathbb{L}\{X(t)\} &= \mathbb{L}\{I^\alpha A(t)X(t) + I^\alpha B(t)U(t)\} \\ &\Rightarrow \mathcal{X}(s) = \mathbb{L}\{I^\alpha A(t)X(t)\} + \mathbb{L}\{I^\alpha B(t)U(t)\} \\ &\Rightarrow \mathcal{X}(s) = s^{-\alpha} \mathcal{A}(s) \mathcal{X}(s) + s^{-\alpha} \mathcal{B}(s) \mathcal{U}(s) \\ &\Rightarrow \mathcal{X}(s) [1 - \mathcal{A}(s) s^{-\alpha}] = \mathcal{B}(s) s^{-\alpha} \mathcal{U}(s) \\ &\Rightarrow \mathcal{X}(s) = \frac{\mathcal{B}(s) s^{-\alpha}}{1 - \mathcal{A}(s) s^{-\alpha}} \mathcal{U}(s) = \frac{\mathcal{B}(s)}{s^\alpha - \mathcal{A}(s)} \mathcal{U}(s), \end{aligned}$$

where $\mathcal{A}(s)$, $\mathcal{B}(s)$, $\mathcal{X}(s)$ and $\mathcal{U}(s)$ are the *Laplace* transform of A , B , X and U respectively. Hence by the inverse of *Laplace* transform, we have

$$X(t) = \frac{1}{\Gamma(\alpha)} \int_0^t [(t-\tau)^{\alpha-1} - A(\tau)] B(\tau) U(\tau) d\tau.$$

Now, for $t_1 > 0$, this yields

$$\begin{aligned} X(t_1) &= \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_1-\tau)^{\alpha-1} - A(\tau)] B(\tau) U(\tau) d\tau \\ &= \frac{B}{\Gamma(\alpha)} \int_0^{t_1} B(\tau) [(t_1-\tau)^{\alpha-1} - A(\tau)] [B^{-1}(\tau) \Gamma(\alpha) \Upsilon^{-1}(t_1) X_1] d\tau \\ &= \left(\int_0^{t_1} [(t_1-\tau)^{\alpha-1} - A(\tau)] d\tau \right) \Upsilon^{-1}(t_1) X_1 = X_1. \end{aligned}$$

This implies that system (3.6) is (GC).

Necessary condition: Assume that system (3.6) is (GC). If $B(t)$ has no inverse, then we have not the feedback control $U(t)$ implies that system (3.6) is not (GC) and this is a contradiction. $\square$

3.3. Nonlinear control set of a fractional differential equation (NLSCFDE)

Theorem 4. Let F be continuous and $\|F\| \leq M$, $0 < M < \infty$. Then the control system (3.1) is (CC).

Proof. The solution set of system (3.1) takes the form

$$X(t) = I^\alpha F(t, X(t), U(t)), \quad (t \in J).$$

By continuity of F , the solution $X(t)$ is continuous. We only need to find X_0 and X_1 . Since

$$X(0) = X_0 = 0,$$

and by Definition 1, $X(t) = \frac{t^{\alpha-1} * F(t)}{\Gamma(\alpha)}$, where $(*)$ is the convolution product, we have

$$X(T) = \frac{T^{\alpha-1} * F(T)}{\Gamma(\alpha)}, \quad (F(T) \neq 0).$$

We can choose X_0 and X_1 such that $X_0 = 0$ and $X_1 = X(T)$. Hence the system (3.1) is completely controllable on J . $\square$

Our next aim is to show that system (3.1) is (GC). For this purpose we need to the following preliminary result..

Lemma 1. Suppose that the decreasing Lyapunov-like function $V(t, X(t), U(t))$, $t \in \mathbb{R}_+$ satisfies the followings:

- (i) $V \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n) \times K_c(\mathbb{R}^m), K_c(\mathbb{R}^n)]$,
- (ii) $|V(t, X_1, U_1) - V(t, X_2, U_2)| \leq L(D[X_1, X_2] + D[U_1, U_2])$, $L > 0$

(iii) $b(\|X\|) \leq V(t, X, U) \leq a(t, \|X\|)$, for $t \in \mathbb{R}_+$, $X, X_1, X_2 \in K_c(\mathbb{R}^n)$, $U, U_1, U_2 \in K_c(\mathbb{R}^m)$, where $b(\cdot), a(t, \cdot) \in \Phi = \{\varphi \in C[\mathbb{R}_+, \mathbb{R}_+] : \varphi(\omega) \text{ is increasing in } \omega \text{ and } \varphi(\omega) \rightarrow \infty \text{ as } \omega \rightarrow \infty\}$ then for every $\mu > 0, t_0 \in \mathbb{R}_+, \exists v > 0$ such that

$$\|X_0\| < \mu \text{ implies } \|X(t)\| < v, \forall t > t_0. \quad (3.7)$$

Proof. Given $\mu > 0$, we can choose $v > 0$ such that $a(t_0, \mu) < b(v)$, $t_0 \in \mathbb{R}_+$. If (3.7) is not hold, then for $t_0 < t_1$, we obtain

$$\begin{aligned} b(v) &\leq b(\|X(t_1)\|) \leq V(t_1, X(t_1), U(t_1)) \\ &\leq V(t_0, X(t_0), U(t_0)) \leq a(t_0, \|X_0\|) < a(t_0, \mu) < b(v). \end{aligned}$$

This contradiction proves (3.7). $\square$

Theorem 5. Assume that the assumptions of Lemma 1 are satisfied. Assume that $F(t, X(t), U(t)) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n) \times K_c(\mathbb{R}^m), K_c(\mathbb{R}^n)]$ with

$$\|F(t, X(t), U(t))\| \leq \tilde{L}(\|X\| + \|U\|), \quad \tilde{L} > 0$$

such that

$$\|U\| \leq \frac{\Gamma(\alpha + 1)}{\tilde{L}T^\alpha} [-\|X_0\| - \frac{\tilde{L}T^\alpha \|X\|}{\Gamma(\alpha + 1)} + \|X_1\|].$$

In that case the system (3.1) is (GC).

Proof. The solution set of system (3.1) satisfies the following: for $t_1 \in J$

$$\begin{aligned} \|X(t_1)\| &\leq \|X_0\| + \int_0^{t_1} \frac{(t_1 - \tau)^{\alpha-1}}{\Gamma(\alpha)} \|F(\tau, X(\tau), U(\tau))\| d\tau \\ &\leq \|X_0\| + \int_0^{t_1} \frac{(t_1 - \tau)^{\alpha-1}}{\Gamma(\alpha)} \tilde{L}(\|X\| + \|U\|) d\tau \leq \|X_0\| + \frac{T^\alpha}{\Gamma(\alpha + 1)} \tilde{L}(\|X\| + \|U\|) \\ &\leq \|X_1\|. \end{aligned}$$

Our aim is to prove $\|X(t_1)\| = \|X_1\|$. If this is not true, i.e. $\|X(t_1)\| < \|X_1\|$, then in view of Lemma 1, we have

$$\begin{aligned} b(v) &\leq b(\|X(t_1)\|) \leq V(t_1, X(t_1), U(t_1)) \\ &\leq V(t_0, X_1, U(t_1)) \leq a(t_1, \|X_1\|) < a(t_1, \mu) < b(v). \end{aligned}$$

That means the state pair $(X_0, X_1) \in K_c(\mathbb{R}^n)$ is controllable. Hence the (NLSCFDE) (3.1) is (GC). $\square$

In the same manner as in Theorem 5, we can prove the following result

Theorem 6. Suppose that the assumptions of Lemma 1 hold. Assume that $F(t, X(t), U(t)) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n) \times K_c(\mathbb{R}^m), K_c(\mathbb{R}^n)]$ with

$$\|F(t, X(t), U(t))\| \leq \tilde{L}(\|X\| + \|U\|), \quad \tilde{L} > 0$$

such that $U(t) := \sigma(t)X_1$ and

$$\|\sigma\| \leq \frac{\Gamma(\alpha+1)}{\|X_1\|\widetilde{L}T^\alpha} [-\|X_0\| - \frac{\widetilde{L}T^\alpha\|X\|}{\Gamma(\alpha+1)} + \|X_1\|], \quad (3.8)$$

then system (3.1) is (GC).

Remark 3. In the case $m = n$, the control operator $\sigma(t)$ must chose as a real function such that

$$|\sigma| \leq \frac{\Gamma(\alpha+1)}{\|X_1\|\widetilde{L}T^\alpha} [-\|X_0\| - \frac{\widetilde{L}T^\alpha\|X\|}{\Gamma(\alpha+1)} + \|X_1\|],$$

then system (3.1) is (GC).

We present a few examples of these two cases:

Example 1. Global controllable case: The movement of the oil trace on the sea will be controllable. Because for every pair of state sets $(X_0, X_1) \in K_c(\mathbb{R}^n) \times K_c(\mathbb{R}^n)$, $\forall t_1 > t_0, \exists U(t) \in K_c(\mathbb{R}^m)$, such that $U(t) = \sigma(t)X_1$, (where the control operator $\sigma(t)$ satisfies (3.8)) the NLSCFDE (3.1) is (GC).

Example 2. Non controllable case: We consider the problem of transfer of an alcohol concentrations in a volatile environment. This motion of the alcohol will not be controllable, because it might move to somewhere else so we can't find any suitable feedback $U(t) = h(X(t))$.

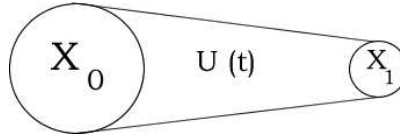


FIGURE 1. Globally controllable case

4. CONCLUSION

In this paper we used Riemann-Liouville fractional differential and integral operators for the study of the possibility of global control. We established the complete controllability of generalized systems of the form (3.1). In the case of $\alpha = 1$, the system (3.1) reduces to the form

$$D_H X(t) = F(t, X(t), U(t)), \quad (4.1)$$

where D is the Hukuhara derivative

$$(X(0) = X_0, X(t) \in K_c(\mathbb{R}^n), U(t) \in K_c(\mathbb{R}^m), t \in J := [0, T]),$$

$$F(t, X(t), U(t)) : J \times K_c(\mathbb{R}^n) \times K_c(\mathbb{R}^m) \longrightarrow K_c(\mathbb{R}^n).$$

This system has been studied by several authors (see [8, 10]).

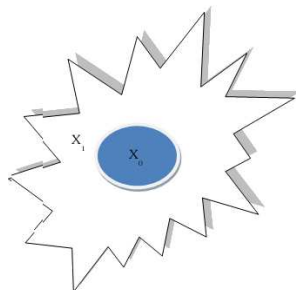


FIGURE 2. Non controllable case

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NEW RESULTS ON THE AM-GM INEQUALITY

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Abstract. There is a large number of discrete inequalities between geometric and arithmetic means. Our intention is to investigate possible generalization and integral analogues of the inequalities obtained by Redheffer, Bullen and Godunova. At the end, we will consider refinements of these inequalities by different separations of their domains.

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1. PRELIMINARES

Consider the following sequences of positive real numbers:

$(a) = (a_1, \dots, a_n)$; $(b) = (b_1, \dots, b_n)$; $(p) = (p_1, \dots, p_n)$; $(q) = (q_1, \dots, q_n)$.

We use the notations:

$$\begin{aligned} P_n &= \sum_{k=1}^n p_k, Q_n = \sum_{k=1}^n q_k, \\ A_n(a, q) &= \frac{1}{Q_n} \sum_{k=1}^n q_k a_k, G_n(a, q) = \prod_{k=1}^n a_k^{\frac{q_k}{Q_n}}, \\ \Gamma_n(a, q) &= \frac{1}{Q_n} \sum_{k=1}^n q_k G_k(a, q). \end{aligned} \quad (1.1)$$

A function $F : [a, b] \rightarrow \mathbb{R}$ is said to be convex if for all $x, y \in [a, b]$ and all $\alpha \in [0, 1]$,

$$F(\alpha x + (1 - \alpha)y) \leq \alpha F(x) + (1 - \alpha)F(y). \quad (1.2)$$

If the inequality (1.2) is reversed, then F is said to be concave.

Investigating recurrent inequalities in [5], Redheffer has obtained the following result involving the arithmetic and geometric means. The statement is given in terms of (1.1), where $G_n(a, 1) = G_n(a, q)$, $q_k = 1$, $k = 1, \dots, n$.

Theorem 1 (Redheffer). *If $t > 0$, then*

$$G_n(a, 1) < A_n(a, 1)e^{-t} + t\Gamma_n(a, 1).$$

Bullen [2] proved an inequality of Rado type generalizing Redheffer's inequality.

Theorem 2 (Bullen). *If $0 \leq tq_n \leq Q_n$, then*

$$\begin{aligned} & Q_n(A_n(a, q)e^{-t} + t\Gamma_n(a, q) - G_n(a, q)) \\ & \geq Q_{n-1}(A_{n-1}(a, q)e^{-t} + t\Gamma_{n-1}(a, q) - G_{n-1}(a, q)). \end{aligned}$$

A more general result (given below) is obtained in [3].

Theorem 3 (Godunova). *Suppose that G and F are twice continuously differentiable functions such that G is concave and $GF = (G \circ F)$ is convex.*

For $t \in \mathbb{R}$ the next inequality is true:

$$Q_n \left\{ G(A_n(F(t + a_k), q)) - t A_n \left((GF)' \left(\frac{q_k P_k}{p_k Q_k} A_k(a, p) \right), q \right) \right. \quad (1.3)$$

$$\left. - GF \left(\frac{q_n P_n}{p_n Q_n} A_n(a, p) \right) \right\} \quad (1.4)$$

$$\geq Q_{n-1} \left\{ G(A_{n-1}(F(t + a_k), q)) - t A_{n-1} \left((GF)' \left(\frac{q_k P_k}{p_k Q_k} A_k(a, p) \right), q \right) \right. \quad (1.5)$$

$$\left. - GF \left(\frac{q_n P_{n-1}}{p_n Q_{n-1}} A_{n-1}(a, p) \right) \right\}. \quad (1.6)$$

In this article we will present some generalizations and refinements of these results.

2. MAIN RESULT

Our results are based on the well-known properties of convex functions. The conditions are simpler than the ones of Theorem 3. It is obvious that Theorem 3 follows from this result.

Theorem 4. *If F is a convex differentiable function, then for all $t \in \mathbb{R}$*

$$\begin{aligned} & Q_n \left[A_n(F(t + a_k), q) - t \cdot A_n \left(F' \left(\frac{q_k P_k}{p_k Q_k} \cdot A_k(a, p) \right), q \right) \right. \quad (2.1) \\ & \quad \left. - F \left(\frac{q_n P_n}{p_n Q_n} \cdot A_n(a, p) \right) \right] \\ & \geq Q_{n-1} \left[A_{n-1}(F(t + a_k), q) - t \cdot A_{n-1} \left(F' \left(\frac{q_k P_k}{p_k Q_k} \cdot A_k(a, p) \right), q \right) \right. \\ & \quad \left. - F \left(\frac{q_n P_{n-1}}{p_n Q_{n-1}} \cdot A_{n-1}(a, p) \right) \right]. \end{aligned}$$

Proof. Obviously, we have

$$\sum_{k=1}^n q_k F(t + a_k) - \sum_{k=1}^{n-1} q_k F(t + a_k) = q_n F(t + a_n)$$

and

$$\begin{aligned} Q_n \cdot A_n \left(F' \left(\frac{q_k P_k}{p_k Q_k} \cdot A_k(a, p) \right), q \right) \\ - Q_{n-1} \cdot A_{n-1} \left(F' \left(\frac{q_k P_k}{p_k Q_k} \cdot A_k(a, p) \right), q \right) \\ = q_n \cdot F' \left(\frac{q_n P_n}{p_n Q_n} \cdot A_n(a, p) \right). \end{aligned}$$

Statement (2.1) is equivalent to

$$\begin{aligned} q_n F(t + a_n) + Q_{n-1} \cdot F \left(\frac{q_n P_{n-1}}{p_n Q_{n-1}} \cdot A_{n-1}(a, p) \right) \\ \geq Q_n F \left(\frac{q_n P_n}{p_n Q_n} \cdot A_n(a, p) \right) + t q_n F' \left(\frac{q_n P_n}{p_n Q_n} \cdot A_n(a, p) \right), \end{aligned} \quad (2.2)$$

so it remains to show (2.2). Because of $\frac{q_n}{Q_n} + \frac{Q_{n-1}}{Q_n} = 1$, the definition of convex functions implies

$$\begin{aligned} q_n \cdot F(t + a_n) + Q_{n-1} \cdot F \left(\frac{q_n P_{n-1}}{p_n Q_{n-1}} \cdot A_{n-1}(a, p) \right) \\ \geq Q_n \cdot F \left(\frac{t q_n}{Q_n} + \frac{q_n P_n}{p_n Q_n} \cdot A_n(a, p) \right). \end{aligned} \quad (2.3)$$

Presuming that the convex function F is differentiable, we have (see [4], p.5, Theorem 1.6.):

$$F(u + \alpha) \geq F(u) + \alpha \cdot F'(u). \quad (2.4)$$

Upon application of these results, the following inequality is obtained

$$\begin{aligned} Q_n \cdot F \left(\frac{q_n P_n}{p_n Q_n} \cdot A_n(a, p) + \frac{t q_n}{Q_n} \right) \\ \geq Q_n \cdot F \left(\frac{q_n P_n}{p_n Q_n} \cdot A_n(a, p) \right) + t \cdot q_n \cdot F' \left(\frac{q_n P_n}{p_n Q_n} \cdot A_n(a, p) \right). \end{aligned} \quad (2.5)$$

Now (2.3) and (2.5) imply (2.2). $\square$

Corollary 1. *If F is a strictly monotone convex function, then*

$$\begin{aligned} & Q_n \left[A_n(b, q) - F \left(\frac{q_n P_n}{p_n Q_n} A_n(F^{-1}(b), p) \right) \right] \\ & \geq Q_{n-1} \left[A_{n-1}(b, q) - F \left(\frac{q_n P_{n-1}}{p_n Q_{n-1}} A_{n-1}(F^{-1}(b), p) \right) \right]. \end{aligned}$$

Proof. It is enough to substitute $t = 0$ and $a_k = F^{-1}(b_k)$, $k = 1, 2, \dots, n$ in (2.1). $\square$

Corollary 2. *If F is a convex, differentiable function, then for every $t \in \mathbb{R}$*

$$F(A_n(a, p)) \leq A_n(F(t + a_k), p) - t A_n(F'(A_k(a, p)), p). \quad (2.6)$$

Proof. Substitute $(p) = (q)$ in (2.1) for $n = 2, 3, \dots, n-1, n$ in order to get a sequence of inequalities. Then sum these inequalities. $\square$

Inequality (2.6) turns into the inequality of Redheffer upon taking $F(u) = e^{-u}$, $a_i = \ln \frac{1}{b_i}$ and $(p) = (1, \dots, 1)$. This inequality is proved in [5] for $t \geq 0$. The same inequality is proved in [2], but only for $0 \leq t \leq 2$.

Theorem 4 ensures the statement of Theorem 2 proved in [2] only for $0 \leq tq_n \leq Q_n$.

Corollary 3. *For every $t \in \mathbb{R}$*

$$\begin{aligned} & Q_n (A_n(b, q)e^{-t} + t \Gamma_n(b, q) - G_n(b, q)) \\ & \geq Q_{n-1} (A_{n-1}(b, q)e^{-t} + t \Gamma_{n-1}(b, q) - G_{n-1}(b, q)). \end{aligned} \quad (2.7)$$

Proof. To obtain (2.7), substitute $(p) = (q)$ and $F(u) = e^{-u}$ in (2.1). Then $F(a_k) = b_k = e^{-a_k}$ and $a_k = -\ln b_k = \ln \frac{1}{b_k}$. $\square$

Remark. Theorem 3, (the first theorem of [3]), follows from Theorem 4. Substitute the composition GF instead of the convex function F in (2.2) and use the Jensen inequality for the concave function G :

$$Q_n G(A_n(F(t + a_k), q)) \geq Q_{n-1} G(A_{n-1}(F(t + a_k), q)) + q_n G(F(t + a_n)).$$

The next inequality from [1] is an immediate consequence of Theorem 3.

Corollary 4. *Suppose that F is a strictly monotone function, G is a concave function and GF is a convex function. Then*

$$\begin{aligned} & Q_n \left\{ G(A_n(b, q)) - (GF) \left(\frac{q_n P_n}{p_n Q_n} A_n(F^{-1}(b), p) \right) \right\} \\ & \geq Q_{n-1} \left\{ G(A_{n-1}(b, q)) - (GF) \left(\frac{q_n P_{n-1}}{p_n Q_{n-1}} A_{n-1}(F^{-1}(b), p) \right) \right\}. \end{aligned}$$

Taking $(p) = (q)$, the next corollary is obtained from (2.1).

Corollary 5. *If F is a differentiable convex function, then*

$$\begin{aligned} & \sum_{k=1}^n p_k F(t + a_k) - t \cdot \sum_{k=1}^n p_k F' \left(\frac{1}{P_k} \sum_{j=1}^k p_j a_j \right) \\ & \quad - \sum_{k=1}^n p_k \cdot F \left(\frac{1}{P_n} \sum_{k=1}^n p_k a_k \right) \\ & \geq \sum_{k=1}^{n-1} p_k F(t + a_k) - t \cdot \sum_{k=1}^{n-1} p_k F' \left(\frac{1}{P_k} \sum_{j=1}^k p_j a_j \right) \\ & \quad - \sum_{k=1}^{n-1} p_k \cdot F \left(\frac{1}{P_{n-1}} \sum_{k=1}^{n-1} p_k a_k \right), \end{aligned}$$

whereby $t \in \mathbb{R}$.

3. AN APPLICATION ON INTEGRAL ANALOGUE

Integral analogue is given in [3] as Theorem 2.

Theorem 5 (Godunova). *Let ψ be a convex, differentiable function and let $p(\tau)$ and $a(\tau)$ be positive functions for $\tau > 0$. Then the function*

$$\begin{aligned} \Phi(x) = & \int_0^x \psi(t + a(\tau)) p(\tau) d\tau - t \int_0^x \psi' \left(\frac{\int_0^\tau a(\xi) p(\xi) d\xi}{\int_0^\tau p(\xi) d\xi} \right) p(\tau) d\tau \\ & - \psi \left(\frac{\int_0^x a(\tau) p(\tau) d\tau}{\int_0^x p(\tau) d\tau} \right) \cdot \int_0^x p(\tau) d\tau \quad (3.1) \end{aligned}$$

is an increasing function on $(0, \infty)$.

The next corollary estimates the difference between the values, so that composition GF is calculated at the mean values on the intervals $[0, x] \subset [0, y]$.

Corollary 6. *Let GF be a convex function, let G be a concave function and suppose that a, p are positive on $(0, \infty)$. Then*

$$\begin{aligned} & GF \left(\frac{\int_0^y a(\tau) p(\tau) d\tau}{\int_0^y p(\tau) d\tau} \right) \cdot \int_0^y p(\tau) d\tau \\ & - GF \left(\frac{\int_0^x a(\tau) p(\tau) d\tau}{\int_0^x p(\tau) d\tau} \right) \cdot \int_0^x p(\tau) d\tau \end{aligned}$$

$$\begin{aligned} &\leq \int_x^y p(\tau) d\tau \cdot G \left(\frac{\int_x^y F(t+a(\tau)) p(\tau) d\tau}{\int_x^y p(\tau) d\tau} \right) \\ &\quad - t \int_x^y (GF)' \left(\frac{\int_0^\tau a(\xi) p(\xi) d\xi}{\int_0^\tau p(\xi) d\xi} \right) p(\tau) d\tau. \end{aligned}$$

Proof. Substitute $\psi = GF$ in (5). Notice that if

$$\Phi(x) = \int_0^x GF(t+a(\tau)) p(\tau) d\tau + D(x), \text{ then for } 0 \leq x \leq y \text{ we have}$$

$$\begin{aligned} &D(x) - D(y) \\ &= t \int_0^y (GF)' \left(\frac{\int_0^\tau a(\xi) p(\xi) d\xi}{\int_0^\tau p(\xi) d\xi} \right) p(\tau) d\tau \\ &\quad + GF \left(\frac{\int_0^y a(\tau) p(\tau) d\tau}{\int_0^y p(\tau) d\tau} \right) \cdot \int_0^y p(\tau) d\tau \\ &\quad - t \int_0^x (GF)' \left(\frac{\int_0^\tau a(\xi) p(\xi) d\xi}{\int_0^\tau p(\xi) d\xi} \right) p(\tau) d\tau \\ &\quad - GF \left(\frac{\int_0^x a(\tau) p(\tau) d\tau}{\int_0^x p(\tau) d\tau} \right) \cdot \int_0^x p(\tau) d\tau \\ &\leq \int_0^y GF(t+a(\tau)) p(\tau) d\tau - \int_0^x GF(t+a(\tau)) p(\tau) d\tau \\ &= \int_x^y GF(t+a(\tau)) p(\tau) d\tau \\ &\leq \int_x^y p(\tau) d\tau \cdot G \left(\frac{\int_x^y F(t+a(\tau)) p(\tau) d\tau}{\int_x^y p(\tau) d\tau} \right). \end{aligned}$$

□

In case that all calculations are well defined, Jensen's inequality for a concave function G

$$\int_0^y GF(t+a(\tau)) p(\tau) d\tau \leq P(y) \cdot G \left(\frac{1}{P(y)} \int_0^y F(t+a(\tau)) p(\tau) d\tau \right),$$

with $P(y) = \int_0^y p(\tau) d\tau$, makes it possible to define the function

$$\mathcal{F}(y) = P(y) \cdot G \left(\frac{1}{P(y)} \int_0^y F(t+a(\tau)) p(\tau) d\tau \right) - \int_0^y GF(t+a(\tau)) p(\tau) d\tau.$$

The function $\mathcal{F}$ is monotone on $(0, \infty)$.

Proposition 1. *If G is a concave function, then $0 \leq x \leq y$ ensures*

$$\mathcal{F}(x) \leq \mathcal{F}(y). \quad (3.2)$$

Proof. This statements follows from the concavity of G . There is a chain of inequalities:

$$\begin{aligned} & P(x)G\left(\frac{1}{P(x)}\int_0^x F(t+a(\tau))p(\tau)d\tau\right) - \int_0^x GF(t+a(\tau))p(\tau)d\tau \\ & \quad + \int_0^y GF(t+a(\tau))p(\tau)d\tau \\ & = P(x)G\left(\frac{1}{P(x)}\int_0^x F(t+a(\tau))p(\tau)d\tau\right) \\ & \quad + \frac{\int_x^y p(\tau)d\tau}{\int_x^y p(\tau)d\tau} \cdot \int_x^y GF(t+a(\tau))p(\tau)d\tau \\ & \leq \frac{P(y)}{P(y)} \left[P(x) \cdot G\left(\frac{1}{P(x)}\int_0^x F(t+a(\tau))p(\tau)d\tau\right) \right. \\ & \quad \left. + \int_x^y p(\tau)d\tau \cdot G\left(\frac{\int_x^y F(t+a(\tau))p(\tau)d\tau}{\int_x^y p(\tau)d\tau}\right) \right] \\ & = P(y) \left[\frac{P(x)}{P(y)} \cdot G\left(\frac{1}{P(x)}\int_0^x F(t+a(\tau))p(\tau)d\tau\right) \right. \\ & \quad \left. + \frac{\int_x^y p(\tau)d\tau}{P(y)} \cdot G\left(\frac{\int_x^y F(t+a(\tau))p(\tau)d\tau}{\int_x^y p(\tau)d\tau}\right) \right] \\ & \leq P(y) \cdot G\left(\frac{1}{P(y)}\int_0^x F(t+a(\tau))p(\tau)d\tau + \frac{1}{P(y)}\int_x^y F(t+a(\tau))p(\tau)d\tau\right) \\ & = P(y) \cdot G\left(\frac{1}{P(y)}\int_0^y F(t+a(\tau))p(\tau)d\tau\right). \end{aligned}$$

□

At this point observe certain refinements when G is concave and GF is a convex function.

Remark 1. Applying (3.2) in

$$D(x) - D(y) \leq \int_0^y GF(t+a(\tau))p(\tau)d\tau - \int_0^x GF(t+a(\tau))p(\tau)d\tau,$$

the following is obtained:

$$P(y) \cdot GF\left(\frac{\int_0^y a(\tau)p(\tau)d\tau}{\int_0^y p(\tau)d\tau}\right) - P(x) \cdot GF\left(\frac{\int_0^x a(\tau)p(\tau)d\tau}{\int_0^x p(\tau)d\tau}\right)$$

$$\begin{aligned} \leq P(y) \cdot G \left(\frac{\int_0^y F(t+a(\tau))p(\tau)d\tau}{\int_0^y p(\tau)d\tau} \right) - P(x) \cdot G \left(\frac{\int_0^x F(t+a(\tau))p(\tau)d\tau}{\int_0^x p(\tau)d\tau} \right) \\ - t \int_x^y (GF)' \left(\frac{\int_0^\tau a(\tau)p(\tau)d\tau}{\int_0^y p(\tau)d\tau} \right) p(\tau)d\tau. \end{aligned}$$

To sum up, if GF is a non-decreasing function, we can make the following observation.

Remark 2. If GF is a nondecreasing differentiable function, then $GF' \geq 0$ and for $t \geq 0$ one can obtain

$$\begin{aligned} P(y) \cdot GF \left(\frac{\int_0^y a(\tau)p(\tau)d\tau}{\int_0^y p(\tau)d\tau} \right) - P(x) \cdot GF \left(\frac{\int_0^x a(\tau)p(\tau)d\tau}{\int_0^x p(\tau)d\tau} \right) \\ \leq P(y) \cdot G \left(\frac{\int_0^y F(t+a(\tau))p(\tau)d\tau}{\int_0^y p(\tau)d\tau} \right) - P(x) \cdot G \left(\frac{\int_0^x F(t+a(\tau))p(\tau)d\tau}{\int_0^x p(\tau)d\tau} \right). \end{aligned}$$

For $t = 0$ we get ordinary monotonicity. If $[0, x] \subseteq [0, y]$, then

$$\begin{aligned} P(x) \cdot G \left(\frac{\int_0^x F(a(\tau))p(\tau)d\tau}{\int_0^x p(\tau)d\tau} \right) - P(x) \cdot GF \left(\frac{\int_0^x a(\tau)p(\tau)d\tau}{\int_0^x p(\tau)d\tau} \right) \\ \leq P(y) \cdot G \left(\frac{\int_0^y F(a(\tau))p(\tau)d\tau}{\int_0^y p(\tau)d\tau} \right) - P(y) \cdot GF \left(\frac{\int_0^y a(\tau)p(\tau)d\tau}{\int_0^y p(\tau)d\tau} \right). \end{aligned}$$

4. DECREASING PROPERTY UPON TAKING SUBSETS

In the next theorem an integral version of the statement obtained in (1.6) is given whereby $(p) = (q)$ and $t = 0$. The statement appeared as particular case of Bullen's inequality (see [4]). In [3] the author omitted the proof. We are providing it here to refresh certain basic integral inequalities.

Theorem 6. Let $\mathcal{D}$ be a measurable domain and let $\mathcal{D}_1 \subset \mathcal{D}$ be its regular measurable sub-domain. Further, suppose $a, q : \mathcal{D} \rightarrow [0, +\infty)$ is such that $Q = \int_{\mathcal{D}} q(u)dV_u \neq 0$ and $Q_1 = \int_{\mathcal{D}_1} q(u)dV_u \neq 0$. If G is concave, F is monotone and GF is a convex function, then

$$\begin{aligned} Q_1 \left[G \left(\frac{1}{Q_1} \int_{\mathcal{D}_1} F(a(u))q(u)dV_u \right) - GF \left(\frac{1}{Q_1} \int_{\mathcal{D}_1} a(u)q(u)dV_u \right) \right] \\ \leq Q \left[G \left(\frac{1}{Q} \int_{\mathcal{D}} F(a(u))q(u)dV_u \right) - GF \left(\frac{1}{Q} \int_{\mathcal{D}} a(u)q(u)dV_u \right) \right]. \quad (4.1) \end{aligned}$$

Proof. Concavity of G ensures

$$\begin{aligned} & Q \cdot G \left(\frac{1}{Q} \int_{\mathcal{D}} F(a(u))q(u)dV_u \right) \\ & \geq Q_1 \cdot G \left(\frac{1}{Q_1} \int_{\mathcal{D}_1} F(a(u))q(u)dV_u \right) \\ & \quad + Q_2 \cdot G \left(\frac{1}{Q_2} \int_{\mathcal{D}_2} F(a(u))q(u)dV_u \right), \end{aligned}$$

wherby $\mathcal{D}_2 = \mathcal{D} \setminus \mathcal{D}_1$ with $Q_2 = \int_{\mathcal{D}_2} q(u)dV_u \neq 0$. Besides, the convexity of GF implies

$$\begin{aligned} & Q \cdot GF \left(\frac{1}{Q} \int_{\mathcal{D}} a(u)q(u)dV_u \right) \\ & \leq Q_1 \cdot GF \left(\frac{1}{Q_1} \int_{\mathcal{D}_1} a(u)q(u)dV_u \right) + Q_2 \cdot GF \left(\frac{1}{Q_2} \int_{\mathcal{D}_2} a(u)q(u)dV_u \right). \end{aligned}$$

Integral version of Jensen's inequality (see [4], p. 45) for concave function G provides

$$Q_2 \cdot G \left(\frac{1}{Q_2} \int_{\mathcal{D}_2} F(a(u))q(u)dV_u \right) \geq \int_{\mathcal{D}_2} GF(a(u))q(u)dV_u.$$

Furthermore, the convexity of GF provides the statement

$$\int_{\mathcal{D}_2} GF(a(u))q(u)dV_u \geq Q_2 \cdot GF \left(\frac{1}{Q_2} \int_{\mathcal{D}_2} a(u)q(u)dV_u \right).$$

Now there is a chain

$$\begin{aligned} & Q \cdot G \left(\frac{1}{Q} \int_{\mathcal{D}} F(a(u))q(u)dV_u \right) - Q_1 \cdot G \left(\frac{1}{Q_1} \int_{\mathcal{D}_1} F(a(u))q(u)dV_u \right) \\ & = Q_2 \cdot G \left(\frac{1}{Q_2} \int_{\mathcal{D}_2} F(a(u))q(u)dV_u \right) \geq Q_2 \cdot GF \left(\frac{1}{Q_2} \int_{\mathcal{D}_2} a(u)q(u)dV_u \right) \\ & \geq Q \cdot GF \left(\frac{1}{Q} \int_{\mathcal{D}} a(u)q(u)dV_u \right) - Q_1 \cdot GF \left(\frac{1}{Q_1} \int_{\mathcal{D}_1} a(u)q(u)dV_u \right). \end{aligned}$$

The first line and the last line from this give (4.1). $\square$

Finally, we generalize (4.1).

Theorem 7. Let (Ω, Σ, μ) be a space with a positive finite measure and let $a : \Omega \rightarrow \mathbb{R}$ be a μ -measurable function. Suppose that G and F are differentiable functions such that G is concave and GF is convex. For every $t \in \mathbb{R}$, if $\Omega_1 \in \Sigma$, and $\Omega_1 \neq \Omega$, then

$$\mu(\Omega) \cdot G \left(\frac{1}{\mu(\Omega)} \int_{\Omega} F(t+a)d\mu \right) \quad (4.2)$$

$$\begin{aligned}
& -\mu(\Omega_1) \cdot G \left(\frac{1}{\mu(\Omega_1)} \int_{\Omega_1} F(t+a) d\mu \right) \\
& \geq (\mu(\Omega) - \mu(\Omega_1)) \left[GF \left(\frac{\int_{\Omega} a d\mu - \int_{\Omega_1} a d\mu}{\mu(\Omega) - \mu(\Omega_1)} \right) \right. \\
& \quad \left. + t \cdot (GF)' \left(\frac{\int_{\Omega} a d\mu - \int_{\Omega_1} a d\mu}{\mu(\Omega) - \mu(\Omega_1)} \right) \right].
\end{aligned}$$

Proof. Take $\Omega \setminus \Omega_1 = \Omega_2$. Then concavity of G ensures

$$\begin{aligned}
\mu(\Omega) \cdot G \left(\frac{1}{\mu(\Omega)} \int_{\Omega} F(t+a) d\mu \right) - \mu(\Omega_1) \cdot G \left(\frac{1}{\mu(\Omega_1)} \int_{\Omega_1} F(t+a) d\mu \right) \\
\geq \mu(\Omega_2) \cdot G \left(\frac{1}{\mu(\Omega_2)} \int_{\Omega_2} F(t+a) d\mu \right). \quad (4.3)
\end{aligned}$$

Concavity of G implies (see [4], p.51) that

$$\mu(\Omega_2) \cdot G \left(\frac{1}{\mu(\Omega_2)} \int_{\Omega_2} F(t+a) d\mu \right) \geq \int_{\Omega_2} GF(t+a) d\mu.$$

Now, convexity of GF implies

$$\int_{\Omega_2} GF(t+a) d\mu \geq \mu(\Omega_2) \cdot GF \left(\frac{1}{\mu(\Omega_2)} \int_{\Omega_2} (t+a) d\mu \right).$$

Using (2.4) and the equality $\frac{1}{\mu(\Omega_2)} \int_{\Omega_2} (t+a) d\mu = t + \frac{1}{\mu(\Omega_2)} \int_{\Omega_2} a d\mu$, we claim that

$$\begin{aligned}
GF \left(t + \frac{1}{\mu(\Omega_2)} \int_{\Omega_2} a d\mu \right) & \geq GF \left(\frac{1}{\mu(\Omega_2)} \int_{\Omega_2} a d\mu \right) \\
& + t \cdot (GF)' \left(\frac{1}{\mu(\Omega_2)} \int_{\Omega_2} a d\mu \right). \quad (4.4)
\end{aligned}$$

Finally, (4.3) and (4.4), together with the well-known properties

$$\mu(\Omega \setminus \Omega_1) = \mu(\Omega) - \mu(\Omega_1) \text{ and } \int_{\Omega \setminus \Omega_1} a d\mu = \int_{\Omega} a d\mu - \int_{\Omega_1} a d\mu \text{ give (4.2). } \quad \square$$

Theorem 7 contains (4.1) as a particular case.

Remark 3. Notice that convexity of GF , assumed in (4.4), implies

$$\begin{aligned}
\mu(\Omega) \cdot G \left(\frac{1}{\mu(\Omega)} \int_{\Omega} F(t+a) d\mu \right) - \mu(\Omega_1) \cdot G \left(\frac{1}{\mu(\Omega_1)} \int_{\Omega_1} F(t+a) d\mu \right) \\
\geq \mu(\Omega) \cdot GF \left(\frac{1}{\mu(\Omega)} \int_{\Omega} a d\mu \right) - \mu(\Omega_1) \cdot GF \left(\frac{1}{\mu(\Omega_1)} \int_{\Omega_1} a d\mu \right)
\end{aligned}$$

$$+t \cdot (GF)' \left(\frac{\int_{\Omega} a \, d\mu - \int_{\Omega_1} a \, d\mu}{\mu(\Omega) - \mu(\Omega_1)} \right) \quad (4.5)$$

and (4.1) can be obtained from (4.5) for $t = 0$.

In the similar manner as in [6], where the authors have refined Jensen's inequality, here we give an estimate after dividing the n -tuples $(a_1, \dots, a_n)$, $(p_1, \dots, p_n)$ and $(q_1, \dots, q_n)$ into groups. Firstly, we obtain the next Theorem.

Theorem 8. Suppose that G and F are differentiable functions such that G is concave and GF is convex. If $m \leq n$, then

$$\begin{aligned} & Q_n \left[G(A_n(F(t + a_k), q)) - t A_n \left((GF)' \left(\frac{q_k P_k}{p_k Q_k} A_k(a, p) \right), q \right) \right. \\ & \quad \left. - GF \left(\frac{q_n P_n}{p_n Q_n} A_n(a, p) \right) \right] \\ & - Q_m \left[G(A_m(F(t + a_k), q)) - t A_m \left((GF)' \left(\frac{q_k P_k}{p_k Q_k} A_k(a, p) \right), q \right) \right. \\ & \quad \left. - GF \left(\frac{q_m P_m}{p_m Q_m} A_m(a, p) \right) \right] \\ & \geq \sum_{k=m}^{n-1} Q_k \left[GF \left(\frac{q_k P_k}{p_k Q_k} A_k(a, p) \right) - \left(\frac{q_{k+1} P_k}{p_{k+1} Q_k} A_k(a, p) \right) \right]. \end{aligned}$$

Proof. No generality is lost if we separate each of n -tuples $(a_1, \dots, a_n)$, $(p_1, \dots, p_n)$ and $(q_1, \dots, q_n)$ by the choosing the first m members in the first group. For example: $(a_1, \dots, a_m, a_{m+1}, \dots, a_n)$.

Then the first inequality is

$$\begin{aligned} & Q_n \left[G(A_n(F(t + a_k), q)) - t A_n \left((GF)' \left(\frac{q_k P_k}{p_k Q_k} A_k(a, p) \right), q \right) \right. \\ & \quad \left. - GF \left(\frac{q_n P_n}{p_n Q_n} A_n(a, p) \right) \right] \\ & \geq Q_{n-1} \left[G(A_{n-1}(F(t + a_k), q)) - t A_{n-1} \left((GF)' \left(\frac{q_k P_k}{p_k Q_k} A_k(a, p) \right), q \right) \right. \\ & \quad \left. - GF \left(\frac{q_{n-1} P_{n-1}}{p_{n-1} Q_{n-1}} A_{n-1}(a, p) \right) \right] \\ & \quad + Q_{n-1} \left[GF \left(\frac{q_{n-1} P_{n-1}}{p_{n-1} Q_{n-1}} A_{n-1}(a, p) \right) - GF \left(\frac{q_n P_{n-1}}{p_n Q_{n-1}} A_{n-1}(a, p) \right) \right]. \end{aligned}$$

The second one is

$$Q_{n-1} \left[G(A_{n-1}(F(t + a_k), q)) - t A_{n-1} \left((GF)' \left(\frac{q_k P_k}{p_k Q_k} A_k(a, p) \right), q \right) \right]$$

$$\begin{aligned}
& -GF \left(\frac{q_{n-1} P_{n-1}}{p_{n-1} Q_{n-1}} A_{n-1}(a, p) \right) \Big] \\
& \geq Q_{n-2} \left[G(A_{n-2}(F(t + a_k), q)) - t A_{n-2} \left((GF)' \left(\frac{q_k P_k}{p_k Q_k} A_k(a, p) \right), q \right) \right. \\
& \quad \left. -GF \left(\frac{q_{n-2} P_{n-2}}{p_{n-2} Q_{n-2}} A_{n-2}(a, p) \right) \right] \\
& + Q_{n-2} \left[GF \left(\frac{q_{n-2} P_{n-2}}{p_{n-2} Q_{n-2}} A_{n-2}(a, p) \right) - GF \left(\frac{q_{n-1} P_{n-1}}{p_{n-1} Q_{n-1}} A_{n-1}(a, p) \right) \right]
\end{aligned}$$

And finally, the last one is

$$\begin{aligned}
& Q_{m+1} \left[G(A_{m+1}(F(t + a_k), q)) - t A_{m+1} \left((GF)' \left(\frac{q_k P_k}{p_k Q_k} A_k(a, p) \right), q \right) \right. \\
& \quad \left. -GF \left(\frac{q_{m+1} P_{m+1}}{p_{m+1} Q_{m+1}} A_{m+1}(a, p) \right) \right] \\
& \geq Q_m \left[G(A_m(F(t + a_k), q)) - t A_m \left((GF)' \left(\frac{q_k P_k}{p_k Q_k} A_k(a, p) \right), q \right) \right. \\
& \quad \left. -GF \left(\frac{q_m P_m}{p_m Q_m} A_m(a, p) \right) \right] \\
& + Q_m \left[GF \left(\frac{q_m P_m}{p_m Q_m} A_m(a, p) \right) - GF \left(\frac{q_{m+1} P_{m+1}}{p_{m+1} Q_{m+1}} A_{m+1}(a, p) \right) \right].
\end{aligned}$$

The desired inequality in the statement is obtained by summing these inequalities. $\square$

The sum $\sum_{k=m}^{n-1} Q_k \left[GF \left(\frac{q_k P_k}{p_k Q_k} A_k(a, p) \right) - \left(\frac{q_{k+1} P_{k+1}}{p_{k+1} Q_{k+1}} A_{k+1}(a, p) \right) \right]$ depends on the positive weights (p) and (q) . Therefore it could be positive, negative or zero. When $(p) = (q)$, the sum is zero and we obtain the following Corollary.

Corollary 7. Suppose G and F fulfill the assumptions of Theorem 8. If $n \geq m$, then

$$\begin{aligned}
& Q_n \left[G(A_n(F(t + a_k), p)) - t A_n \left((GF)'(A_k(a, p)), p \right) - GF(A_n(a, p)) \right] \\
& \geq Q_m \left[G(A_m(F(t + a_k), p)) - t A_m \left((GF)'(A_k(a, p)), p \right) - GF(A_m(a, p)) \right].
\end{aligned}$$

Finally, we state the following Corollary in the case of disjunct separation of the n -tuples as: $(a_1, \dots, a_n) = (a_1, \dots, a_m, a_{m+1}, \dots, a_n)$.

Corollary 8. Suppose that $G(u)$ and $F(u)$ satisfy conditions given in Theorem 8. If $l + m = n$ and $(b_1, \dots, b_l, c_1, \dots, c_m) = (a_1, \dots, a_n)$, then

$$Q_l \left\{ G(A_l(F(t + b_k), p)) - t A_l \left((GF)'(A_k(b, p)), p \right) \right\}$$

$$\begin{aligned}
& -GF(A_l(b, p))\} \\
& + Q_m \{G(A_m(F(t + c_k), q)) - t A_m((GF)'(A_k(c, p)), p) \\
& \quad - GF(A_m(c, p))\} \\
& \leq 2 \cdot Q_n \{G(A_n(F(t + a_k), p)) - t A_n((GF)'(A_k(a, p)), p) \\
& \quad - GF(A_n(a, p))\}.
\end{aligned}$$

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SUMMATION PROCESS OF KOROVKIN TYPE APPROXIMATION THEOREM

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Abstract. The aim of this paper is to present a Korovkin-type approximation theorem on $H_w(I^2)$, which is the subspace of all continuous and bounded real valued functions on $I^2 = [0, \infty) \times [0, \infty)$ by using $\mathcal{A}$ -summation process.

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1. INTRODUCTION

Approximation theory, which has a close relationship with other branches of mathematics, has been used in the theory of polynomial approximation and various domains of functional analysis [2], in numerical studies of differential and integral operators [15], and in the studies of the interpolation operator of Hermite-Fejér [6–8, 10] and of the partial sums of Fourier series [16]. Most of the classical approximation operators tend to converge to the value of the function being approximated. However, at points of discontinuity, they often converge to the average of the left and right limits of the function. There are, however, exceptions such as the interpolation operators of Hermite-Fejér [6]. These operators do not converge at points of simple discontinuity. In this case, the matrix summability methods of Cesàro type are strong enough to correct the lack of convergence [7]. The main purpose of using summability theory has always been to make a nonconvergent sequence converge. Some results regarding matrix summability for positive linear operators may be found in the papers [3, 4, 14, 23]. Our interest in the present paper is to obtain a Korovkin-type approximation theorem for a sequence of positive linear operators defined on $H_w(I^2)$, which is the subspace of all continuous and bounded real valued functions on $I^2 = [0, \infty) \times [0, \infty)$ by using $\mathcal{A}$ -summation process.

A double sequence $x = \{x_{m,n}\}_{m,n \in \mathbb{N}}$ is convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{m,n} - L| < \varepsilon$ whenever $m, n > N$.

In this case L is called the Pringsheim limit of x and is denoted by $P - \lim x = L$ (see [22]).

If there exists a positive number M such that $|x_{m,n}| \leq M$ for all $(m,n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, then $x = \{x_{m,n}\}$ is said to be bounded. Note that in contrast to the case for single sequences, a convergent double sequence need not to be bounded.

Let

$$A = [a_{j,k,m,n}], \quad j, k, m, n \in \mathbb{N},$$

be a four-dimensional infinite matrix. For a given double sequence $x = \{x_{m,n}\}$, the A -transform of x , denoted by $Ax := \{(Ax)_{j,k}\}$, is given by

$$(Ax)_{j,k} = \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n} x_{m,n}, \quad j, k \in \mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(j,k) \in \mathbb{N}^2$. We say that a sequence x is A -summable to l if the A -transform of x exists for all $j, k \in \mathbb{N}$ and is convergent in the Pringsheim's sense i.e.,

$$P - \lim_{p,q} \sum_{m \in \mathbb{N}}^p \sum_{n \in \mathbb{N}}^q a_{j,k,m,n} x_{m,n} = y_{j,k} \text{ and } P - \lim_{j,k} y_{j,k} = l.$$

In summability theory, a two-dimensional matrix transformation is said to be regular if it maps every convergent sequence to a convergent sequence with the same limit. The well-known characterization of regularity for two dimensional matrix transformations is known as Silverman-Toeplitz conditions (see, for instance, [13]). In 1926, Robison [24] presented a four dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a double P -convergent sequence is not necessarily bounded. The definition and the characterization of regularity for four dimensional matrices is known as Robison-Hamilton conditions, or briefly, RH -regularity. (see, [12, 24])

Recall that a four dimensional matrix $A = [a_{j,k,m,n}]$ is said to be RH -regular if it maps every bounded P -convergent sequence into a P -convergent sequence with the same P -limit. The Robison-Hamilton conditions state that a four dimensional matrix $A = [a_{j,k,m,n}]$ is RH -regular if and only if

- (i) $P - \lim_{j,k} a_{j,k,m,n} = 0$ for each $(m,n) \in \mathbb{N}^2$,
- (ii) $P - \lim_{j,k} \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n} = 1$,
- (iii) $P - \lim_{j,k} \sum_{m \in \mathbb{N}} |a_{j,k,m,n}| = 0$ for each $n \in \mathbb{N}$,
- (iv) $P - \lim_{j,k} \sum_{n \in \mathbb{N}} |a_{j,k,m,n}| = 0$ for each $m \in \mathbb{N}$,

(v) $\sum_{(m,n) \in \mathbb{N}^2} |a_{j,k,m,n}|$ is P -convergent for each $(j,k) \in \mathbb{N}^2$,

(vi) there exist finite positive integers A and B such that $\sum_{m,n > B} |a_{j,k,m,n}| < A$ holds for every $(j,k) \in \mathbb{N}^2$.

Now let $\mathcal{A} := \{A^{(i,l)}\} = \{a_{j,k,m,n}^{(i,l)}\}$ be a sequence of four-dimensional infinite matrices with non-negative real entries. For a given double sequence of real numbers, $x = \{x_{m,n}\}$ is said to be $\mathcal{A}$ -summable to l if

$$P\text{-}\lim_{j,k} \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} x_{m,n} = l$$

uniformly in i and l . If $A^{(i,l)} = A$, four-dimensional infinite matrix, then $\mathcal{A}$ -summability is the A -summability for four-dimensional infinite matrix.

Some results regarding matrix summability method for double sequences may be found in the papers [20], [21], [25].

Now let $A = [a_{j,k,m,n}]$ be a non-negative RH -regular summability matrix, and let $K \subset \mathbb{N}^2$. Then, a real double sequence $x = \{x_{m,n}\}$ is said to be A -statistically convergent to a number L if, for every $\varepsilon > 0$,

$$P\text{-}\lim_{j,k} \sum_{(m,n) \in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \{(m,n) \in \text{BeginExpansion} \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon\}.$$

In this case we write $st_A^2\text{-}\lim_{m,n} x_{m,n} = L$. Observe that, a P -convergent double sequence is A -statistically convergent to the same value but the converse does not hold true.

We should note that if we take $A = C(1,1)$, which is the double Cesàro matrix, then $C(1,1)$ -statistical convergence coincides with the notion of statistical convergence for double sequence, which was introduced in [18, 19]. Finally, if we replace the matrix A by the identity matrix for four-dimensional matrices, then A -statistical convergence reduces to the Pringsheim convergence.

2. A KOROVKIN-TYPE APPROXIMATION THEOREM

We recall that the Korovkin-type theorem on the $H_w([0, \infty))$ space was given by Gadjiev and Çakar in [11]. Similarly as in [11], let us introduce a space denoted by $H_w(I^2)$, where $I^2 := [0, \infty) \times [0, \infty)$.

Let ω be a modulus of continuity type functions such that the following conditions are satisfied:

i) ω is a non-negative increasing function on $[0, \infty)$,

- ii) $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$,
- iii) $\lim_{\delta \rightarrow 0} \omega(\delta) = 0$.

Let $H_\omega(I^2)$ be a subspace of real valued functions satisfying the following conditions: for some $M > 0$

$$|f(u, v) - f(x, y)| \leq M\omega\left(\left|\left(\frac{u}{1+u}, \frac{v}{1+v}\right) - \left(\frac{x}{1+x}, \frac{y}{1+y}\right)\right|\right) \quad (2.1)$$

where

$$\left|\left(\frac{u}{1+u}, \frac{v}{1+v}\right) - \left(\frac{x}{1+x}, \frac{y}{1+y}\right)\right| = \sqrt{\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2 + \left(\frac{v}{1+v} - \frac{y}{1+y}\right)^2}.$$

Let $C_B(I^2)$ be the space of all continuous and bounded functions on I^2 . Then $C_B(I^2)$ is a linear normed space with

$$\|f\|_{C_B(I^2)} = \sup_{x, y \geq 0} |f(x, y)|.$$

Due to (ii), we can say that $H_\omega(I^2) \subset C_B(I^2)$.

A sequence $\{L_{m,n}\}$ of positive linear operators of $H_\omega(I^2)$ into $C_B(I^2)$ is called an $\mathcal{A}$ -summation process on $H_\omega(I^2)$ if $\{L_{m,n}(f)\}$ is $\mathcal{A}$ -summable to f for every $f \in H_\omega(I^2)$, i.e.,

$$P - \lim_{j,k \rightarrow \infty} \left\| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(f) - f \right\|_{C_B(I^2)} = 0, \quad \text{uniformly in } i \text{ and } l \quad (2.2)$$

where it is assumed that the series in (2.2) converges for each $i, l, j, k \in \mathbb{N}$ and f .

Note that, the results of type (2.2) are extensions of type

$$P - \lim_{j,k \rightarrow \infty} \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} \|L_{m,n}(f) - f\|_{C_B(I^2)} = 0, \quad \text{uniformly in } i \text{ and } l$$

for all $f \in H_\omega(I^2)$.

We establish a theorem of Korovkin type with respect to the convergence behavior (2.2) for a double sequence of positive linear operators of $H_\omega(I^2)$ into $C_B(I^2)$.

Let $\{L_{m,n}\}$ be a sequence of positive linear operators of $H_\omega(I^2)$ into $C_B(I^2)$ such that

$$\sup_{i,l,j,k} \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} \|L_{m,n}(1)\|_{C_B(I^2)} < \infty. \quad (2.3)$$

Furthermore, for $i, l, j, k \in \mathbb{N}$ and $f \in H_\omega(I^2)$, let

$$B_{j,k}^{(i,l)} f = \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(f)$$

which is well defined by (2.3) and belongs to $C_B(I^2)$.

Altın, Doğru and Özarslan [1] obtained the following Korovkin theorem.

Theorem 1 ([1]). *Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $H_\omega(I^2)$ into $C_B(I^2)$. Then, for all $f \in H_\omega(I^2)$*

$$P\text{-}\lim_{m,n} \|L_{m,n}(f) - f\|_{C_B(I^2)} = 0$$

is satisfied if the following holds:

$$P\text{-}\lim_{m,n} \|L_{m,n}(f_i) - f_i\|_{C_B(I^2)} = 0 \quad (i = 0, 1, 2, 3)$$

where

$$\begin{aligned} f_0(u, v) &= 1, \quad f_1(u, v) = \frac{u}{1+u}, \quad f_2(u, v) = \frac{v}{1+v}, \\ f_3(u, v) &= \left(\frac{u}{1+u}\right)^2 + \left(\frac{v}{1+v}\right)^2. \end{aligned}$$

A -statistical analog of Theorem 1 can be given as follows.

Theorem 2 ([9]). *Let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix method. Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $H_\omega(I^2)$ into $C_B(I^2)$. Then, for all $f \in H_\omega(I^2)$*

$$st_A^2\text{-}\lim_{m,n} \|L_{m,n}(f) - f\|_{C_B(I^2)} = 0$$

is satisfied if the following holds:

$$st_A^2\text{-}\lim_{m,n} \|L_{m,n}(f_i) - f_i\|_{C_B(I^2)} = 0 \quad (i = 0, 1, 2, 3)$$

where

$$\begin{aligned} f_0(u, v) &= 1, \quad f_1(u, v) = \frac{u}{1+u}, \quad f_2(u, v) = \frac{v}{1+v}, \\ f_3(u, v) &= \left(\frac{u}{1+u}\right)^2 + \left(\frac{v}{1+v}\right)^2. \end{aligned}$$

If we replace the matrix A in Theorem 2 by the Cesáro matrix $C(1, 1)$, we immediately get the statistical Korovkin result.

Now we give the following generalization by using a $\mathcal{A}$ -summation process.

Theorem 3. *Let $\mathcal{A} = \{A^{(i,l)}\}$ be a sequence of four-dimensional infinite matrices with non-negative real entries such that*

$$\sup_{i,l,j,k} \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} < \infty. \quad (2.4)$$

Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $H_\omega(I^2)$ into $C_B(I^2)$. Assume that (2.3) holds. Then, for all $f \in H_\omega(I^2)$

$$P - \lim_{j,k} \left\| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(f) - f \right\|_{C_B(I^2)} = 0 \quad \text{uniformly in } i \text{ and } l$$

is satisfied if the following holds:

$$P - \lim_{j,k} \left\| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(f_r) - f_r \right\|_{C_B(I^2)} = 0 \quad (2.5)$$

uniformly in i and l ($r = 0, 1, 2, 3$), where

$$f_0(u, v) = 1, \quad f_1(u, v) = \frac{u}{1+u}, \quad f_2(u, v) = \frac{v}{1+v},$$

$$f_3(u, v) = \left(\frac{u}{1+u} \right)^2 + \left(\frac{v}{1+v} \right)^2.$$

Proof. If $f \in H_\omega(I^2)$, then from (2.1) we have that for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that $|f(u, v) - f(x, y)| < \varepsilon$ if

$$\sqrt{\left(\frac{u}{1+u} - \frac{x}{1+x} \right)^2 + \left(\frac{v}{1+v} - \frac{y}{1+y} \right)^2} < \delta.$$

Since f is bounded, there exists a positive constant N such that

$$|f(u, v) - f(x, y)| < \frac{2N}{\delta^2} \left[\left(\frac{u}{1+u} - \frac{x}{1+x} \right)^2 + \left(\frac{v}{1+v} - \frac{y}{1+y} \right)^2 \right]$$

if $\sqrt{\left(\frac{u}{1+u} - \frac{x}{1+x} \right)^2 + \left(\frac{v}{1+v} - \frac{y}{1+y} \right)^2} \geq \delta$. Therefore for all $(u, v), (x, y) \in I^2$ one can write

$$|f(u, v) - f(x, y)| < \varepsilon + \frac{2N}{\delta^2} \left[\left(\frac{u}{1+u} - \frac{x}{1+x} \right)^2 + \left(\frac{v}{1+v} - \frac{y}{1+y} \right)^2 \right]. \quad (2.6)$$

Now using the linearity and the positivity of the operators $L_{m,n}$ and considering the inequalities (2.6) and (2.4), for all $(x, y) \in I^2$, we obtain

$$\left| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(f; x, y) - f(x, y) \right|$$

$$\leq \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(|f(u, v) - f(x, y)|; x, y)$$

$$\begin{aligned}
& + |f(x, y)| \left| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(1; x, y) - 1 \right| \\
& \leq \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} \left(\varepsilon + \frac{2N}{\delta^2} \left[\left(\frac{u}{1+u} - \frac{x}{1+x} \right)^2 + \left(\frac{v}{1+v} - \frac{y}{1+y} \right)^2 \right]; x, y \right) \\
& \quad + N \left| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(1; x, y) - 1 \right| \\
& \leq \varepsilon + \varepsilon \left| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(1; x, y) - 1 \right| + N \left| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(1; x, y) - 1 \right| \\
& \quad + \frac{2N}{\delta^2} \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} \left(\left(\frac{u}{1+u} - \frac{x}{1+x} \right)^2; x, y \right) \\
& \quad + \frac{2N}{\delta^2} \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} \left(\left(\frac{v}{1+v} - \frac{y}{1+y} \right)^2; x, y \right) \\
& \leq \varepsilon + \left(\varepsilon + N + \frac{4N}{\delta^2} \right) \left| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(1; x, y) - 1 \right| \\
& \quad + \frac{2N}{\delta^2} \left| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} \left(\left(\frac{u}{1+u} \right)^2 + \left(\frac{v}{1+v} \right)^2; x, y \right) \right. \\
& \quad \quad \left. - \left(\left(\frac{x}{1+x} \right)^2 + \left(\frac{y}{1+y} \right)^2 \right) \right| \\
& \quad + \frac{4N}{\delta^2} \left| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} \left(\left(\frac{u}{1+u} \right); x, y \right) - \frac{x}{1+x} \right| \\
& \quad + \frac{4N}{\delta^2} \left| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} \left(\left(\frac{v}{1+v} \right); x, y \right) - \frac{y}{1+y} \right| \\
& \leq \varepsilon + B \left\{ \left| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(1; x, y) - 1 \right| \right.
\end{aligned}$$

$$\begin{aligned}
& + \left| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} \left(\left(\frac{u}{1+u} \right); x, y \right) - \frac{x}{1+x} \right| \\
& + \left| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} \left(\left(\frac{v}{1+v} \right); x, y \right) - \frac{y}{1+y} \right| \\
& + \left| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n} \left(\left(\frac{u}{1+u} \right)^2 + \left(\frac{v}{1+v} \right)^2; x, y \right) \right. \\
& \quad \left. - \left(\left(\frac{x}{1+x} \right)^2 + \left(\frac{y}{1+y} \right)^2 \right) \right| \Big\}
\end{aligned}$$

where $B := \max \left\{ \varepsilon + N + \frac{4N}{\delta^2}, \frac{2N}{\delta^2}, \frac{4N}{\delta^2} \right\}$. Then taking supremum over $(x, y) \in I^2$, we have

$$\begin{aligned}
& \left\| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(f) - f \right\|_{C_B(I^2)} \\
& \leq \varepsilon + B \left\{ \left\| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(f_0) - f_0 \right\|_{C_B(I^2)} \right. \\
& \quad + \left\| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(f_1) - f_1 \right\|_{C_B(I^2)} \\
& \quad + \left\| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(f_2) - f_2 \right\|_{C_B(I^2)} \\
& \quad \left. + \left\| \sum_{(m,n) \in \mathbb{N}^2} a_{j,k,m,n}^{(i,l)} L_{m,n}(f_3) - f_3 \right\|_{C_B(I^2)} \right\}.
\end{aligned}$$

Using (2.5) and by taking limit as $j, k \rightarrow \infty$, we obtain the desired result. $\square$

Remark 1. If we take $A^{(i,l)} = I$, I being the four-dimensional identity matrix in Theorem 3, then we immediately get Theorem 1.

Corollary 1. Now we present an example such that our new approximation result works but its classical case does not work. Let $I^2 = [0, \infty) \times [0, \infty)$ and $f : I^2 \rightarrow \mathbb{R}$. We consider the double sequence $\{L_{m,n}\}$ of positive linear operators defined by

$$L_{m,n}(f; x, y) = (1 + \alpha_{m,n}) B_{m,n}(f; x, y)$$

where $\{B_{m,n}\}$ is the Bleimann, Butzer and Hahn [5] operators defined by

$$\begin{aligned} & B_{m,n}(f; x, y) \\ &= \frac{1}{(1+x)^m} \frac{1}{(1+y)^n} \sum_{k=0}^m \sum_{l=0}^n f\left(\frac{k}{m-k+1}, \frac{l}{n-l+1}\right) \binom{m}{k} \binom{n}{l} x^k y^l \end{aligned}$$

and $(\alpha_{m,n})$ is a double sequence defined by $\alpha_{m,n} = (-1)^{m+n}$.

From [1], we have

$$\begin{aligned} B_{m,n}(f_0; x, y) &= 1, \\ B_{m,n}(f_1; x, y) &= \frac{m}{m+1} \frac{x}{1+x}, \\ B_{m,n}(f_2; x, y) &= \frac{n}{n+1} \frac{y}{1+y}, \\ B_{m,n}(f_3; x, y) &= \frac{m(m-1)}{(m+1)^2} \frac{x^2}{(1+x)^2} + \frac{m}{(m+1)^2} \frac{x}{1+x} \\ &\quad + \frac{n(n-1)}{(n+1)^2} \frac{y^2}{(1+y)^2} + \frac{n}{(n+1)^2} \frac{y}{1+y}, \end{aligned}$$

where

$$\begin{aligned} f_0(u, v) &= 1, \quad f_1(u, v) = \frac{u}{1+u}, \quad f_2(u, v) = \frac{v}{1+v}, \\ f_3(u, v) &= \left(\frac{u}{1+u}\right)^2 + \left(\frac{v}{1+v}\right)^2. \end{aligned}$$

A double sequence $x = \{x_{m,n}\}$ of real numbers is called almost convergent to a limit s if

$$P - \lim_{p,q \rightarrow \infty} \sup_{j,k \geq 0} \left| \frac{1}{pq} \sum_{m=j}^{j+p-1} \sum_{n=k}^{k+q-1} x_{m,n} - s \right| = 0,$$

that is, the average value of $\{x_{m,n}\}$ taken over any rectangle

$$\{(m, n) : j \leq m \leq j+p-1, k \leq n \leq k+q-1\}$$

tends to s as both p and q tend to ∞ , and this convergence is uniform in j and k . Now assume that $\mathcal{A} = \{A^{(i,l)}\} = \{a_{j,k,m,n}^{(i,l)}\}$ is a sequence of four-dimensional infinite matrices defined by $a_{j,k,m,n}^{(i,l)} = \frac{1}{jk}$ if $i \leq m \leq j+i-1, l \leq n \leq k+l-1$

and $a_{j,k,m,n}^{(i,l)} = 0$ otherwise. In this case $\mathcal{A}$ -summability method reduces to almost convergence of double sequences introduced by Mòricz [17]. Observe that $(\alpha_{m,n})$ is almost convergent to zero, but it is not convergent in Pringsheim's sense. Also $(\alpha_{m,n})$ is not $C(1,1)$ -statistically convergent. We conclude that for the double sequence $\{L_{m,n}\}$, since $(\alpha_{m,n})$ is almost convergent to zero, $\{L_{m,n}\}$ satisfies the conditions of Theorem 3. Also, since $(\alpha_{m,n})$ is not convergent in Pringsheim's sense and $C(1,1)$ -statistical sense, $\{L_{m,n}\}$ does not satisfy Theorem 1 and Theorem 2 (for $A = C(1,1)$).

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ALTERNATING SUMS OF THE POWERS OF FIBONACCI AND LUCAS NUMBERS

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Abstract. We shall consider alternating Melham's sums for Fibonacci and Lucas numbers of the form $\sum_{k=1}^n (-1)^k F_{2k+\varepsilon}^{2m+\varepsilon}$ and $\sum_{k=1}^n (-1)^k L_{2k+\delta}^{2m+\varepsilon}$, where $\varepsilon, \delta \in \{0, 1\}$.

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1. INTRODUCTION

The Fibonacci F_n and Lucas numbers L_n are defined by the following recursions: for $n > 0$,

$$F_{n+1} = F_n + F_{n-1} \text{ and } L_{n+1} = L_n + L_{n-1},$$

where $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$, respectively.

If the roots of the characteristic equation $x^2 - x - 1 = 0$ are α and β , then the Binet formulas are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n.$$

Wiemann and Cooper [6] raised certain conjectures for the Melham sum:

$$\sum_{k=1}^n F_{2k}^{2m+1}.$$

Ozeki [2] considered Melham's sum and he gave an explicit expansion for it as a polynomial in F_{2n+1} .

More generally, Prodinger [3] derived a formula for the sum:

$$\sum_{k=0}^n F_{2k+\delta}^{2m+\varepsilon},$$

where $\varepsilon, \delta \in \{0, 1\}$. He also evaluated the corresponding sums for the Lucas numbers.

In this paper, we consider the alternating analogs of Melham's sums. We derive explicit formulas for the sums:

$$\sum_{k=1}^n (-1)^k F_{2k+\delta}^{2m+\varepsilon} \text{ and } \sum_{k=1}^n (-1)^k L_{2k+\delta}^{2m+\varepsilon},$$

where $\varepsilon, \delta \in \{0, 1\}$.

2. ALTERNATING MELHAM'S SUMS FOR FIBONACCI NUMBERS

In this section we will start with some lemmas and then we shall derive our results about the alternating Melham's sum.

Lemma 1. *For positive integers n, m and t such that $m \geq t$,*

$$\begin{aligned} i) \quad & (-1)^{t+1} F_{(2m-2t+1)n} + F_{(2m-2t+1)(n+1)} \\ &= \sum_{j=0}^{2m-2t} (-1)^{j(t-2)} F_{(2m-2t+1)n+j+(1-3(-1)^t)/2}, \end{aligned}$$

and

$$ii) \quad F_{2(m-t)(n+1)} - (-1)^t F_{2(m-t)n} = \begin{cases} F_{2(m-t)n+m-t} L_{m-t} & \text{if } m \text{ is odd,} \\ L_{2(m-t)n+m-t} F_{m-t} & \text{if } m \text{ is even.} \end{cases}$$

Proof. i) We can write

$$\begin{aligned} & (-1)^{t+1} F_{(2m-2t+1)n} + F_{(2m-2t+1)(n+1)} \\ &= (-1)^{t+1} \left(\frac{\alpha^{(2m-2t+1)n} - \beta^{(2m-2t+1)n}}{\alpha - \beta} \right) \\ & \quad + \left(\frac{\alpha^{(2m-2t+1)(n+1)} - \beta^{(2m-2t+1)(n+1)}}{\alpha - \beta} \right) \\ &= \frac{\alpha^{(2m-2t+1)n} \left((-1)^{t+1} + \alpha^{2m-2t+1} \right)}{\alpha - \beta} \\ & \quad - \frac{\beta^{(2m-2t+1)n} \left((-1)^{t+1} + \beta^{2m-2t+1} \right)}{\alpha - \beta} \\ &= \begin{cases} \sum_{j=0}^{2m-2t} (-1)^j (F_{(2m-2t+1)n+j} + F_{(2m-2t+1)n+j+1}) & \text{if } t \text{ is odd,} \\ - \sum_{j=0}^{2m-2t} F_{(2m-2t+1)n+j} - F_{(2m-2t+1)n+j+1} & \text{if } t \text{ is even,} \end{cases} \end{aligned}$$

$$\begin{aligned}
&= \begin{cases} \sum_{j=0}^{2m-2t} (-1)^j F_{(2m-2t+1)n+j+2} & \text{if } t \text{ is odd,} \\ \sum_{j=0}^{2m-2t} F_{(2m-2t+1)n+j-1} & \text{if } t \text{ is even,} \end{cases} \\
&= \sum_{j=0}^{2m-2t} (-1)^{j(t-2)} F_{(2m-2t+1)n+j+(1-3(-1)^t)/2}.
\end{aligned}$$

ii) The proof is similar to the ones for the Binet formulas of $\{F_n\}$ and $\{L_n\}$. $\square$

From [4], we have the following result for the Gibonacci sequence $\{G_n\}$, defined by for $n > 0$

$$G_{n+1} = G_n + G_{n-1},$$

with arbitrary initial values G_0 and G_1 .

Lemma 2. Let $a, p \neq 0, q$ be arbitrary integers. Then for $n > 0$,

$$\sum_{i=a}^n G_{pi+q} = \frac{G_{p(n+1)+q} + (-1)^{p+1} G_{pn+q} + (-1)^p G_{p(a-1)+q} - G_{pa+q}}{L_p - 1 - (-1)^p},$$

and

$$\begin{aligned}
&\sum_{i=a}^n (-1)^i G_{pi+q} \\
&= \frac{(-1)^n G_{p(n+1)+q} + (-1)^{p+n} G_{pn+q} + (-1)^a G_{pa+q} + (-1)^{a+p} G_{p(a-1)+q}}{1 + (-1)^p + L_p}.
\end{aligned}$$

As a consequence of Lemma 2, for further use we state the following result:

Corollary 1. For any integer r and positive even integer t ,

$$\sum_{k=1}^n (-1)^{kr} L_{kt} = \frac{(-1)^{nr} F_{t(n+1)} + (-1)^{nr-r} F_{tn} - F_t}{F_t}$$

and

$$\sum_{k=1}^n (-1)^{kr} F_{k(t+1)} = \frac{(-1)^{(n+1)r} F_{(t+1)n} + (-1)^{nr} F_{(t+1)(n+1)} - F_{t+1}}{L_{t+1}}.$$

Proof. Clearly

$$\sum_{k=1}^n (-1)^{kr} L_{kt} = (-1)^r L_t + L_{2t} + \dots + (-1)^{nr} L_{nt}.$$

For the first claim, we consider two cases: the first case is when n is an odd integer. Here

$$\sum_{k=1}^n (-1)^{kr} L_{kt} = (-1)^r \sum_{j=1}^{(n+1)/2} L_{(2j-1)t} + \sum_{j=1}^{(n-1)/2} L_{2jt}. \quad (2.1)$$

If we take $a = 1$, $p = 2t$, $q = -t$ and $n \rightarrow \frac{n+1}{2}$ in Lemma 2, then we get

$$\sum_{j=1}^{(n+1)/2} L_{(2j-1)t} = \frac{L_{t(n+2)} - L_{tn} + L_{-t} - L_t}{L_{2t} - 2}.$$

The following identities are well known [1, 5]:

$$L_{c+t} - L_{c-t} = 5F_c F_t \quad (2.2)$$

for even t , and

$$L_{2c} - (-1)^c 2 = 5F_c^2 \text{ and } L_{-c} = (-1)^c L_c \quad (2.3)$$

for any integer c . Thus we have

$$\sum_{j=1}^{(n+1)/2} L_{(2j-1)t} = \frac{5F_{t(n+1)}F_t}{5F_t^2} = \frac{F_{t(n+1)}}{F_t}. \quad (2.4)$$

Meanwhile, if we take $a = 1$, $p = 2t$, $q = 0$ and $n \rightarrow \frac{n-1}{2}$ in Lemma 2, then we get

$$\sum_{j=1}^{(n-1)/2} L_{2jt} = \frac{L_{t(n+1)} - L_{t(n-1)} + L_0 - L_{2t}}{L_{2t} - 2}.$$

Since t is even, by (2.2) and (2.3), we rewrite the last equation as

$$\sum_{j=1}^{(n-1)/2} L_{2jt} = \frac{5F_{tn}F_t}{5F_t^2} - 1 = \frac{F_{tn}}{F_t} - 1. \quad (2.5)$$

If we substitute (2.4) and (2.5) in (2.1), then we obtain

$$\begin{aligned} \sum_{k=1}^n (-1)^{kr} L_{kt} &= (-1)^r \left(\frac{F_{t(n+1)}}{F_t} \right) + \left(\frac{F_{tn}}{F_t} - 1 \right) \\ &= \frac{(-1)^r F_{t(n+1)} + F_{tn} - F_t}{F_t}. \end{aligned} \quad (2.6)$$

For the second case, let n be an even integer, thus

$$\sum_{k=1}^n (-1)^{kr} L_{kt} = (-1)^r \sum_{j=1}^{n/2} L_{(2j-1)t} + \sum_{j=1}^{n/2} L_{2jt}. \quad (2.7)$$

By taking $a = 1$, $p = 2t$, $q = -t$ and $n \rightarrow \frac{n}{2}$ and $a = 1$, $p = 2t$, $q = 0$ and $n \rightarrow \frac{n}{2}$ in Lemma 2, respectively, we obtain the following result by (2.2) and (2.3), for even t ,

$$\sum_{j=1}^{n/2} L_{(2j-1)t} = \frac{F_{tn}}{F_t}, \quad (2.8)$$

$$\sum_{j=1}^{n/2} L_{2jt} = \frac{F_{t(n+1)}}{F_t} - 1. \quad (2.9)$$

If we put (2.8) and (2.9) in (2.7), we get

$$\begin{aligned} \sum_{k=1}^n (-1)^{kr} L_{kt} &= (-1)^r \left(\frac{F_{tn}}{F_t} \right) + \left(\frac{F_{t(n+1)}}{F_t} - 1 \right) \\ &= \frac{(-1)^r F_{tn} + F_{t(n+1)} - F_t}{F_t}. \end{aligned} \quad (2.10)$$

Combining (2.6) and (2.10), we get the final result:

$$\sum_{k=1}^n (-1)^{kr} L_{kt} = \frac{(-1)^{nr} F_{t(n+1)} + (-1)^{nr-r} F_{tn} - F_t}{F_t},$$

as claimed.

Finally by taking $a = 1$, $p = t + 1$, $q = 0$ in Lemma 2, the second claim is obtained similary to the first claim. $\square$

Theorem 1. *i) For positive odd m ,*

$$\begin{aligned} \sum_{k=1}^n (-1)^k F_k^{2m} &= \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{i(n+1)+n} \binom{2m}{i} \frac{F_{(m-i)(2n+1)}}{F_{m-i}} \\ &\quad - \frac{1}{5^m} \binom{2m-1}{m} - \frac{1}{5^m} \binom{2m}{m} n, \end{aligned}$$

and, for positive even m ,

$$\begin{aligned} &\sum_{k=1}^n (-1)^k F_k^{2m} \\ &= \begin{cases} \frac{1}{5^m} \left(\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{(m-i)(2n+1)}}{L_{m-i}} + \binom{2m-1}{m} \right) & \text{if } n \text{ is even,} \\ -\frac{1}{5^m} \left(\sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{(m-i)(2n+1)}}{L_{m-i}} + \binom{2m-1}{m-1} \right) & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

and

$$\begin{aligned}
 ii) \sum_{k=1}^n (-1)^k F_k^{2m+1} &= \frac{1}{5^m} \sum_{i=0}^m \binom{2m+1}{i} \frac{(-1)^{i(n+1)+n}}{L_{2m-2i+1}} \\
 &\quad \times \sum_{j=0}^{2(m-i)} (-1)^{j(i-2)} F_{n(2m-2i+1)+j+\frac{1-3(-1)^i}{2}} \\
 &\quad - \frac{1}{5^m} \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \frac{F_{2m-2i+1}}{L_{2m-2i+1}}.
 \end{aligned}$$

Proof. i) For odd m , consider

$$\begin{aligned}
 &\sum_{k=1}^n (-1)^k F_k^{2m} \\
 &= \sum_{k=1}^n (-1)^k \left(\frac{\alpha^k - \beta^k}{\alpha - \beta} \right)^{2m} \\
 &= \sum_{k=1}^n (-1)^k \frac{1}{(\alpha - \beta)^{2m}} \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} \alpha^{ki} \beta^{k(2m-i)} \\
 &= \sum_{k=1}^n \frac{(-1)^k}{5^m} \left(\sum_{i=0}^m (-1)^i \binom{2m}{i} (\alpha^{k(2m-i)} \beta^{ki} + \alpha^{ki} \beta^{k(2m-i)}) \right. \\
 &\quad \left. - (-1)^m \binom{2m}{m} (\alpha\beta)^{km} \right) \\
 &= \frac{1}{5^m} \left(\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{k=1}^n (-1)^{k(i+1)} L_{2k(m-i)} \right. \\
 &\quad \left. + \binom{2m}{m} \sum_{k=1}^n (-1)^{m(k+1)+k} \right).
 \end{aligned}$$

By taking $i+1 = r$ and $2(m-i) = t$ in Corollary 1, we write

$$\begin{aligned}
 &\sum_{k=1}^n (-1)^k F_k^{2m} \\
 &= \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \\
 &\quad \times \left(\frac{(-1)^{n(i+1)} F_{2(m-i)(n+1)} + (-1)^{(n+1)(i+1)} F_{2(m-i)n} - F_{2(m-i)}}{F_{2(m-i)}} \right)
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{5^m} \binom{2m}{m} \sum_{k=1}^n (-1)^{m(k+1)+k} \\
& = \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{i(n+1)+n} \binom{2m}{i} \left(\frac{F_{2(m-i)(n+1)} - (-1)^i F_{2(m-i)n}}{F_{2(m-i)}} \right) \\
& \quad - \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} + \frac{1}{5^m} \binom{2m}{m} \sum_{k=1}^n (-1)^{m(k+1)+k}.
\end{aligned}$$

Using (ii) in Lemma 1, we have the claimed result. For even m , the desired result is also obtained.

ii) Consider

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k F_k^{2m+1} \\
& = \sum_{k=1}^n (-1)^k \left(\frac{\alpha^k - \beta^k}{\alpha - \beta} \right)^{2m+1} \\
& = \sum_{k=1}^n \frac{(-1)^k}{(\alpha - \beta)^{2m+1}} \sum_{i=0}^{2m+1} \binom{2m+1}{i} (-1)^{i+1} \alpha^{ki} \beta^{k(2m+1-i)} \\
& = \sum_{k=1}^n \frac{(-1)^k}{(\alpha - \beta)^{2m+1}} \sum_{i=0}^m \binom{2m+1}{i} (-1)^i \left(\alpha^{k(2m+1-i)} \beta^{ki} - \alpha^{ki} \beta^{k(2m+1-i)} \right) \\
& = \frac{1}{5^m} \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \sum_{k=1}^n (-1)^{k(i+1)} F_{k(2m-2i+1)}.
\end{aligned}$$

By taking $r = i + 1$ and $t = 2(m - i)$ in Corollary 1, we write

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k F_k^{2m+1} \\
& = \frac{1}{5^m} \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \times \\
& \quad \frac{(-1)^{(n+1)(i+1)} F_{(2m-2i+1)n} + (-1)^{n(i+1)} F_{(2m-2i+1)(n+1)} - F_{2m-2i+1}}{L_{2m-2i+1}} \\
& = \frac{1}{5^m} \sum_{i=0}^m (-1)^{i+n(i+1)} \binom{2m+1}{i} \frac{(-1)^{i+1} F_{(2m-2i+1)n} + F_{(2m-2i+1)(n+1)}}{L_{2m-2i+1}}
\end{aligned}$$

$$-\frac{1}{5^m} \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \frac{F_{2m-2i+1}}{L_{2m-2i+1}}.$$

By Lemma 1, we obtain the claimed result. $\square$

For further use, we state a consequence of Lemma 2:

Corollary 2. *For even positive integer t and $n > 0$,*

$$\sum_{k=1}^n (-1)^k L_{2tk} = \frac{(-1)^n L_{t(2n+1)}}{L_t} - 1.$$

Proof. Substituting $a = 1$, $p = 2t$ and $q = 0$ in Lemma 2, we get

$$\begin{aligned} \sum_{k=1}^n (-1)^k L_{2tk} &= \frac{(-1)^n L_{2t(n+1)} + (-1)^n L_{2tn} - L_{2t} - 2}{2 + L_{2t}} \\ &= \frac{(-1)^n (L_{2t(n+1)} + L_{2tn}) - (L_{2t} + 2)}{2 + L_{2t}}. \end{aligned}$$

For even t , from [1, 5], we have that

$$L_{c+t} + L_{c-t} = L_c L_t \quad (2.11)$$

and for any c ,

$$L_{2c} + (-1)^c 2 = L_c^2. \quad (2.12)$$

Thus we obtain

$$\sum_{k=1}^n (-1)^k L_{2tk} = (-1)^n \frac{L_{(2n+1)t}}{L_t} - 1,$$

as claimed. $\square$

Theorem 2. *For $m > 0$,*

$$\begin{aligned} i) \quad & \sum_{k=1}^n (-1)^k F_{2k}^{2m} \\ &= \begin{cases} \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} + \frac{(-1)^m}{5^m} \binom{2m-1}{m-1} & \text{if } n \text{ is even,} \\ \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{i+1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - \frac{(-1)^m}{5^m} \binom{2m-1}{m-1} & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

and

$$ii) \quad \sum_{k=1}^n (-1)^k F_{2k}^{2m+1}$$

$$= \frac{1}{5^{m+1}} \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \frac{(-1)^n L_{(2m-2i+1)(2n+1)} - L_{2m-2i+1}}{F_{2m-2i+1}}.$$

Proof. i) We write

$$\begin{aligned} \sum_{k=1}^n (-1)^k F_{2k}^{2m} &= \sum_{k=1}^n (-1)^k \left(\frac{\alpha^{2k} - \beta^{2k}}{\alpha - \beta} \right)^{2m} \\ &= \sum_{k=1}^n (-1)^k \frac{1}{(\alpha - \beta)^{2m}} \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} \alpha^{2ki} \beta^{2k(2m-i)} \\ &= \sum_{k=1}^n \frac{(-1)^k}{5^m} \left(\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} (\alpha^{2k(2m-i)} \beta^{2ki} + \alpha^{2ki} \beta^{2k(2m-i)}) \right. \\ &\quad \left. + (-1)^m \binom{2m}{m} (\alpha\beta)^{2km} \right) \\ &= \sum_{k=1}^n \frac{(-1)^k}{5^m} \left(\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} L_{4k(m-i)} + (-1)^m \binom{2m}{m} \right) \\ &= \frac{1}{5^m} \left(\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{k=1}^n (-1)^k L_{4k(m-i)} + (-1)^m \binom{2m}{m} \sum_{k=1}^n (-1)^k \right). \end{aligned}$$

If we take $t = 2(m-i)$ in Corollary 2, we write

$$\begin{aligned} &\sum_{k=1}^n (-1)^k F_{2k}^{2m} \\ &= \frac{1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} (-1)^i \left(\frac{(-1)^n L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 1 \right) \\ &\quad + \frac{1}{5^m} \binom{2m}{m} (-1)^m \sum_{k=1}^n (-1)^k \\ &= \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{n+i} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \\ &\quad + \frac{1}{5^m} \binom{2m}{m} (-1)^m \sum_{k=1}^n (-1)^k \\ &= \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{n+i} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} \end{aligned}$$

$$+ \frac{1}{5^m} (-1)^m \left(\binom{2m-1}{m-1} + \binom{2m}{m} \left(\frac{(-1)^n - 1}{2} \right) \right).$$

According to the choice of n as an odd or as an even number, we have the conclusion.

ii) The proof is obtained similarly to the proof of (i). $\square$

For further use, we state the following result:

Corollary 3. *For positive even integer t and $n > 0$,*

$$\sum_{k=1}^n (-1)^k L_{(2k+1)t} = \frac{(-1)^n L_{2t(n+1)} - L_{2t}}{L_t}.$$

Proof. Substituting $a = 1$, $p = 2t$ and $q = t$ in Lemma 2, we get

$$\begin{aligned} \sum_{k=1}^n (-1)^k L_{(2k+1)t} &= \frac{(-1)^n L_{(2n+3)t} + (-1)^n L_{(2n+1)t} - L_{3t} - L_t}{2 + L_{2t}} \\ &= \frac{(-1)^n (L_{(2n+3)t} + L_{(2n+1)t}) - (L_{3t} + L_t)}{2 + L_{2t}}. \end{aligned}$$

By (2.11) and (2.12), we rewrite the last equation as for even t ,

$$\begin{aligned} \sum_{k=1}^n (-1)^k L_{(2k+1)t} &= (-1)^n \frac{L_t L_{2t(n+1)}}{L_t^2} - \frac{L_{3t} + L_t}{L_t^2} \\ &= (-1)^n \frac{L_{2t(n+1)}}{L_t} - \frac{L_{2t}}{L_t}. \end{aligned}$$

Thus we have the conclusion. $\square$

Theorem 3. *For $m > 0$,*

$$\begin{aligned} &i) \sum_{k=1}^n (-1)^k F_{2k+1}^{2m} \\ &= \begin{cases} \frac{-1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{4(m-i)(n+1)} + L_{4(m-i)}}{L_{2(m-i)}} - \frac{1}{5^m} \binom{2m}{m} & \text{if } n \text{ is odd} \\ \frac{1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} & \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

and

$$ii) \sum_{k=1}^n (-1)^k F_{2k+1}^{2m+1} = \frac{1}{5^{m+1}} \sum_{i=0}^m \binom{2m+1}{i} \frac{(-1)^n L_{(4m-4i+2)(n+1)} - L_{4m-4i+2}}{F_{2m-2i+1}}.$$

Proof. *i)* Consider

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k F_{2k+1}^{2m} \\
&= \sum_{k=1}^n (-1)^k \left(\frac{\alpha^{2k+1} - \beta^{2k+1}}{\alpha - \beta} \right)^{2m} \\
&= \sum_{k=1}^n \frac{(-1)^k}{(\alpha - \beta)^{2m}} \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} \alpha^{(2k+1)i} \beta^{(2k+1)(2m-i)} \\
&= \sum_{k=1}^n \frac{(-1)^k}{5^m} \left(\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \right. \\
&\quad \left(\alpha^{(2k+1)(2m-i)} \beta^{(2k+1)i} + \alpha^{(2k+1)i} \beta^{(2k+1)(2m-i)} \right) \\
&\quad \left. + \binom{2m}{m} (\alpha\beta)^{2km} \right) \\
&= \sum_{k=1}^n \frac{(-1)^k}{5^m} \left(\sum_{i=0}^{m-1} \binom{2m}{i} L_{2(2k+1)(m-i)} + \binom{2m}{m} \right) \\
&= \frac{1}{5^m} \left(\sum_{i=0}^{m-1} \binom{2m}{i} \sum_{k=1}^n (-1)^k L_{2(2k+1)(m-i)} + \binom{2m}{m} \sum_{k=1}^n (-1)^k \right)
\end{aligned}$$

If we take $t = 2(m-i)$ in Corollary 3, we write

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k F_{2k+1}^{2m} \\
&= \frac{1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{(-1)^n L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} + \frac{1}{5^m} \binom{2m}{m} \sum_{k=1}^n (-1)^k \\
&= \frac{(-1)^n}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{4(m-i)(n+1)}}{L_{2(m-i)}} \\
&\quad - \frac{1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{4(m-i)}}{L_{2(m-i)}} + \frac{1}{5^m} \binom{2m}{m} \left(\frac{(-1)^n - 1}{2} \right).
\end{aligned}$$

According to the choice of n as an odd or as an even number, we have the conclusion.

ii) The proof of (ii) is obtained similarly to the proof of (i). $\square$

3. ALTERNATING MELHAM'S SUM FOR LUCAS NUMBERS

Theorem 4. *i) For positive odd m ,*

$$\sum_{k=1}^n (-1)^k L_k^{2m} = \sum_{i=0}^{m-1} (-1)^{(i+1)n} \binom{2m}{i} \frac{F_{(m-i)(2n+1)}}{F_{m-i}} - 2^{2m-1} + \binom{2m-1}{m} + \binom{2m}{m} n,$$

and, for even m ,

$$\begin{aligned} & \sum_{k=1}^n (-1)^k L_k^{2m} \\ &= \begin{cases} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{(m-i)(2n+1)}}{L_{m-i}} - 2^{2m-1} + \binom{2m-1}{m} & \text{if } n \text{ is even,} \\ \sum_{i=0}^{m-1} (-1)^{i+1} \binom{2m}{i} \frac{L_{(m-i)(2n+1)}}{L_{m-i}} - 2^{2m-1} + \binom{2m-1}{m} - \binom{2m}{m} & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

and

$$\begin{aligned} ii) \sum_{k=1}^n (-1)^k L_k^{2m+1} &= \sum_{i=0}^m \binom{2m+1}{i} \frac{(-1)^{(i+1)n+1}}{L_{2m-2i+1}} \\ &\quad \times \sum_{j=0}^{2(m-i)} (-1)^{ji+1} L_{n(2m-2i+1)+j+\frac{1-3(-1)^i}{2}} \\ &\quad + 2 \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \frac{1}{L_{2m-2i+1}} - 2^{2m}. \end{aligned}$$

Proof. *i)* By the Binet formula of $\{L_n\}$, we have

$$\begin{aligned} & \sum_{k=1}^n (-1)^k L_k^{2m} = \sum_{k=1}^n (-1)^k (\alpha^k + \beta^k)^{2m} \\ &= \sum_{k=1}^n (-1)^k \left(\sum_{i=0}^{m-1} \binom{2m}{i} (\alpha^{k(2m-i)} \beta^{ki} + \alpha^{ki} \beta^{k(2m-i)}) + \binom{2m}{m} (\alpha\beta)^{km} \right). \end{aligned}$$

Since $\alpha\beta = -1$, we get

$$\sum_{k=1}^n (-1)^k L_k^{2m} = \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{k=1}^n (-1)^{k(i+1)} L_{2k(m-i)} + \binom{2m}{m} \sum_{k=1}^n (-1)^{k(m+1)}.$$

From Corollary 1, we write

$$\begin{aligned} & \sum_{k=1}^n (-1)^k L_k^{2m} \\ &= \sum_{i=0}^{m-1} \binom{2m}{i} \left(\frac{(-1)^{(i+1)n} F_{2(m-i)(n+1)} + (-1)^{(n+1)(i+1)} F_{2(m-i)n} - F_{2(m-i)}}{F_{2(m-i)}} \right) \\ & \quad + \binom{2m}{m} \sum_{k=1}^n (-1)^{k(m+1)} \\ &= \sum_{i=0}^{m-1} (-1)^{(i+1)n} \binom{2m}{i} \frac{F_{2(m-i)(n+1)} + (-1)^{i+1} F_{2(m-i)n}}{F_{2(m-i)}} \\ & \quad - \sum_{i=0}^{m-1} \binom{2m}{i} + \binom{2m}{m} \sum_{k=1}^n (-1)^{k(m+1)} \\ &= \begin{cases} \sum_{i=0}^{m-1} (-1)^{(i+1)n} \binom{2m}{i} \frac{F_{(m-i)(2n+1)}}{F_{m-i}} - 2^{2m-1} \\ \quad + \binom{2m-1}{m} + \binom{2m}{m} n & \text{if } m \text{ is odd,} \\ \sum_{i=0}^{m-1} (-1)^{(i+1)n} \binom{2m}{i} \frac{L_{(m-i)(2n+1)}}{L_{m-i}} - 2^{2m-1} \\ \quad + \binom{2m-1}{m} + \binom{2m}{m} \left(\frac{(-1)^n - 1}{2} \right) & \text{if } m \text{ is even.} \end{cases} \end{aligned}$$

According to the choice of n , the claim is obtained.

ii) The proof is similar to the proof of *i)*. □

Theorem 5. *i)* For $m > 0$ and even $n > 0$,

$$\sum_{k=1}^n (-1)^k L_{2k}^{2m} = \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 2^{2m-1} + \binom{2m-1}{m},$$

and, for odd $n > 0$,

$$\sum_{k=1}^n (-1)^k L_{2k}^{2m} = - \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 2^{2m-1} - \binom{2m-1}{m}.$$

i i) For positive integers m and n ,

$$\sum_{k=1}^n (-1)^k L_{2k}^{2m+1} = (-1)^n \sum_{i=0}^m \binom{2m+1}{i} \frac{F_{(2m-2i+1)(2n+1)}}{F_{2m-2i+1}} - 2^{2m}.$$

Proof. i) We write

$$\begin{aligned} & \sum_{k=1}^n (-1)^k L_{2k}^{2m} \\ &= \sum_{k=1}^n (-1)^k (\alpha^{2k} + \beta^{2k})^{2m} \\ &= \sum_{k=1}^n (-1)^k \left(\sum_{i=0}^{m-1} \binom{2m}{i} (\alpha^{2k(2m-i)} \beta^{2ki} + \alpha^{2ki} \beta^{2k(2m-i)}) + \binom{2m}{m} (\alpha\beta)^{2km} \right) \\ &= \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{k=1}^n (-1)^k L_{4k(m-i)} + \binom{2m}{m} \sum_{k=1}^n (-1)^k. \end{aligned}$$

If we take $t = 2(m-i)$ in Corollary 2, we can write

$$\begin{aligned} & \sum_{k=1}^n (-1)^k L_{2k}^{2m} \\ &= \sum_{i=0}^{m-1} \binom{2m}{i} \left(\frac{(-1)^n L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 1 \right) + \binom{2m}{m} \sum_{k=1}^n (-1)^k \\ &= (-1)^n \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - \sum_{i=0}^{m-1} \binom{2m}{i} + \binom{2m}{m} \sum_{k=1}^n (-1)^k \\ &= (-1)^n \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 2^{2m-1} + \binom{2m-1}{m-1} + \binom{2m}{m} \left(\frac{(-1)^n - 1}{2} \right). \end{aligned}$$

According to the choice of n , the proof is complete by the fact that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

ii) The proof of *(ii)* is similar to the proof of *(i)*. □

Theorem 6. For $m > 0$,

$$\begin{aligned} & i) \sum_{k=1}^n (-1)^k L_{2k+1}^{2m} \\ &= \begin{cases} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} & \text{if } n \text{ is even,} \\ (-1)^{m+1} \binom{2m}{m} - \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)(n+1)} + L_{4(m-i)}}{L_{2(m-i)}} & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

and

$$\begin{aligned} & ii) \sum_{k=1}^n (-1)^k L_{2k+1}^{2m+1} = \\ & \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \frac{(-1)^n F_{2(2m-2i+1)(n+1)} - F_{2(2m-2i+1)}}{F_{2m-2i+1}}. \end{aligned}$$

Proof. *i)* Using the Binet formula of $\{L_n\}$, we have

$$\begin{aligned} & \sum_{k=1}^n (-1)^k L_{2k+1}^{2m} \\ &= \sum_{k=1}^n (-1)^k \left(\alpha^{2k+1} + \beta^{2k+1} \right)^{2m} \\ &= \sum_{k=1}^n (-1)^k \left(\sum_{i=0}^{m-1} \binom{2m}{i} \left(\alpha^{(2k+1)(2m-i)} \beta^{(2k+1)i} + \alpha^{(2k+1)i} \beta^{(2k+1)(2m-i)} \right) \right. \\ & \quad \left. + \binom{2m}{m} (\alpha\beta)^{(2k+1)m} \right) \\ &= \sum_{k=1}^n (-1)^k \left(\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \left(\alpha^{(2k+1)(2m-2i)} + \beta^{(2k+1)(2m-2i)} \right) \right) \end{aligned}$$

$$\begin{aligned}
& + \binom{2m}{m} (-1)^m \\
& = \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{k=1}^n (-1)^k L_{2(2k+1)(m-i)} + \binom{2m}{m} (-1)^m \sum_{k=1}^n (-1)^k.
\end{aligned}$$

If we take $t = 2(m-i)$ in Corollary 3, we get

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k L_{2k+1}^{2m} \\
& = \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \left(\frac{(-1)^n L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} \right) \\
& \quad + \binom{2m}{m} (-1)^m \sum_{k=1}^n (-1)^k \\
& = \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{(-1)^n L_{4(m-i)(n+1)}}{L_{2(m-i)}} - \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)}}{L_{2(m-i)}} \\
& \quad + \binom{2m}{m} (-1)^m \sum_{k=1}^n (-1)^k \\
& = (-1)^n \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)(n+1)}}{L_{2(m-i)}} - \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)}}{L_{2(m-i)}} \\
& \quad + \binom{2m}{m} (-1)^m \left(\frac{(-1)^n - 1}{2} \right).
\end{aligned}$$

According to the choice of n , the claimed result is clear.

ii) The proof is similar to the proof of *i)*. □

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REFINEMENTS OF SOME BOUNDS RELATED TO THE CONSTANT e

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Abstract. The aim of this paper is to refine some inequalities related to the constant e . Finally, a strengthened Carleman's inequality is given.

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1. INTRODUCTION

There exists an extensive literature on inequalities involving the constant e and the exponential function in general. Recently, some authors have obtained interesting inequalities of that type and applied them to refine the Hardy and Carleman inequality [1]. It states that for every $a_n \geq 0$ with $0 < \sum_{n=1}^{\infty} a_n < \infty$,

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n.$$

One of the first contributions in this sense was achieved by Kloosterman [2, Chapter 3.8.26], who proved the following inequality

$$e - \left(1 + \frac{1}{n}\right)^n < \frac{e}{2n}, \quad n = 1, 2, 3, \dots \quad (1.1)$$

In the same direction, Yang [7] and Yang and Debnath [8] proved that for every $x \geq 1$,

$$e \left(1 - \frac{1}{2x+1}\right) < \left(1 + \frac{1}{x}\right)^x < e \left(1 - \frac{1}{2x+2}\right), \quad (1.2)$$

respectively

$$e \left(1 - \frac{1}{2x + \frac{5}{3}}\right) < \left(1 + \frac{1}{x}\right)^x < e \left(1 - \frac{1}{2x+2}\right). \quad (1.3)$$

Furthermore Zitian and Yibing [9] improved (1.2)-(1.3), showing that

$$e \left(1 - \frac{7}{14x + 12} \right) < \left(1 + \frac{1}{x} \right)^x < e \left(1 - \frac{6}{12x + 11} \right). \quad (1.4)$$

As an application of the inequalities (1.1)-(1.4), they obtained certain strengthened Carleman - Hardy's inequalities, due to the fact that

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < \sum_{n=1}^{\infty} \left(1 + \frac{1}{n} \right)^n a_n. \quad (1.5)$$

See [1, Chapter 9.12]. More precisely, Yang [7] applied (1.2)-(1.3) to (1.5) to establish the following improvement of Carleman's inequality

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \frac{1}{2n + 2} \right) a_n. \quad (1.6)$$

We show in this paper that the right-hand side of (1.4) is sharp in a sense that we explain later. The upper bound of (1.4) defines the positive function

$$f(x) = 1 + \ln \left(1 - \frac{6}{12x + 11} \right) - x \ln \left(1 + \frac{1}{x} \right), \quad x \in (0, \infty)$$

and in the first part of this paper we prove that f is completely monotonic.

As numerical computations show, for large values of x , the quantity $\left(1 + \frac{1}{x} \right)^x$ becomes closer to the right-hand side of (1.4). Motivated by this fact, we prove that the best approximation of the form

$$\left(1 + \frac{1}{x} \right)^x \approx e \left(1 - \frac{1}{ax + b} \right), \quad \text{as } x \rightarrow \infty,$$

is obtained indeed for $a = 2$ and $b = \frac{11}{6}$. In order to strengthen our assertions, we prove that for every $x \geq 1$,

$$e \left(1 - \frac{6}{12x + 11 - \frac{5}{12x}} \right) < \left(1 + \frac{1}{x} \right)^x < e \left(1 - \frac{6}{12x + 11 - \frac{5}{12x} + \frac{17}{90x^2}} \right).$$

2. A COMPLETELY MONOTONICITY RESULT

The theory of completely monotonic functions has remarkable applications in pure mathematics, and in other branches of science, such as probability, numerical and asymptotic analysis, applied statistics, or statistical physics. A collection of the most important properties of completely monotonic functions can be found in the monograph of Widder [6].

We recall that a function $z : (0, \infty) \rightarrow \mathbb{R}$ is said to be completely monotonic, if z has derivatives of all orders and satisfies $(-1)^n z^{(n)}(x) \geq 0$ for every $n = 0, 1, 2, \dots$ and $x \in (0, \infty)$.

The famous Bernstein's theorem, e.g. [6, p. 161], states that z is completely monotonic if and only if

$$z(x) = \int_0^\infty e^{-tx} d\mu(t),$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$. Completely monotonic functions z are of great importance because they provide sharp bounds for z and their derivatives.

We first prove the following

Lemma 1. *Let $a_n = 12^n - 12n(7^{n-1} - 1)$. Then $a_n > 0$, for every $n \geq 4$.*

Proof. By the binomial formula, we have

$$\begin{aligned} 12^n &= (7+5)^n > 7^n + \binom{n}{1} 5 \cdot 7^{n-1} + \binom{n}{2} 5^2 \cdot 7^{n-2} \\ &= \left(\frac{25}{14}n^2 + \frac{45}{14}n + 7 \right) 7^{n-1} > 12n \cdot 7^{n-1} > 12n(7^{n-1} - 1), \end{aligned}$$

where the penultimate inequality follows from letting $n = s + 4$, with $s \geq 0$, and

$$\left(\left(\frac{25}{14}n^2 + \frac{45}{14}n + 7 \right) - 12n \right) |_{n=s+4} = \frac{25}{14}s^2 + \frac{11}{2}s + \frac{3}{7} > 0.$$

□

Now we are in the position to state the main result of this section. We also use the following integral representations, for every $x \in (0, \infty)$,

$$\frac{1}{x} = \int_0^\infty e^{-xt} dt, \quad \frac{1}{x^2} = \int_0^\infty t e^{-xt} dt. \quad (2.1)$$

Theorem 1. *The function $f : (0, \infty) \rightarrow \mathbb{R}$ given by*

$$f(x) = 1 + \ln \left(1 - \frac{6}{12x + 11} \right) - x \ln \left(1 + \frac{1}{x} \right), \quad x \in (0, \infty)$$

is completely monotonic.

Proof. We have

$$f''(x) = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2} - \frac{1}{(x+\frac{5}{12})^2} + \frac{1}{(x+\frac{11}{12})^2}$$

and using (2.1), we obtain

$$\begin{aligned} f''(x) &= \int_0^\infty e^{-xt} dt - \int_0^\infty e^{-(x+1)t} dt - \int_0^\infty t e^{-(x+1)t} dt \\ &\quad - \int_0^\infty t e^{-(x+\frac{5}{12})t} dt + \int_0^\infty t e^{-(x+\frac{11}{12})t} dt, \end{aligned}$$

or

$$f''(x) = \int_0^\infty \varphi(t) e^{-(x+1)t} dt, \quad (2.2)$$

where $\varphi(t) = e^t - 1 - t - te^{\frac{7}{12}t} + te^{\frac{1}{12}t}$. As

$$\varphi(12t) = \sum_{n=4}^{\infty} \frac{a_n}{n!} t^n > 0,$$

it follows from (2.2) that f'' is completely monotonic. As a consequence, for every $n = 2, 3, \dots$ and $x \in (0, \infty)$,

$$(-1)^n f^{(n)}(x) \geq 0. \quad (2.3)$$

Further, f' is strictly increasing, since $f'' > 0$ and $f'(\infty) = 0$, so $f' < 0$. Now f is strictly decreasing, with $f(\infty) = 0$, so $f > 0$. Finally, (2.3) also holds for $n = 0, 1$, thus f is completely monotonic. $\square$

3. A CLASS OF APPROXIMATIONS

Motivated by the lower and upper approximations (1.4), we wonder whether there are real parameters a, b which provide the best approximation of the form

$$\left(1 + \frac{1}{x}\right)^x \approx e \left(1 - \frac{1}{ax+b}\right), \quad \text{as } x \rightarrow \infty. \quad (3.1)$$

One way to estimate the accuracy of such an approximation is to introduce the relative error sequence $w_n = w_n(a, b)$ defined by the relations

$$\left(1 + \frac{1}{n}\right)^n = e \left(1 - \frac{1}{an+b}\right) \exp w_n, \quad n \geq 1 \quad (3.2)$$

and to consider an approximation of the form (3.1) to be better, when the sequence $(w_n)_{n \geq 1}$ converges faster to zero. Further, a powerful tool for estimating the rate of convergence is the following

Lemma 2. *If $(\omega_n)_{n \geq 1}$ is convergent to zero and there exists the limit*

$$\lim_{n \rightarrow \infty} n^k (\omega_n - \omega_{n+1}) = l \in \mathbb{R}, \quad (3.3)$$

with $k > 1$, then there exists the limit:

$$\lim_{n \rightarrow \infty} n^{k-1} \omega_n = \frac{l}{k-1}.$$

We see from Lemma 2 that the rate of convergence of the sequence $(\omega_n)_{n \geq 1}$ increases together with the value k satisfying (3.3). This lemma was used by Mortici [5]-[4] for constructing asymptotic expansions, or to accelerate the convergence of certain sequences. For proof and some details, see, e.g., [5], or [3].

By (3.2), and by using a computer software, such as Maple, we get

$$w_n - w_{n+1} = \left(\frac{1}{a} - \frac{1}{2}\right) \frac{1}{n^2} + \left(\frac{7}{6} - \frac{1}{a} + \frac{1-2b}{a^2}\right) \frac{1}{n^3} + O\left(\frac{1}{n^4}\right). \quad (3.4)$$

Now we can see that the fastest sequence $w_n - w_{n+1}$, and consequently, by Lemma 2, the fastest sequence w_n are obtained when the first two coefficients of the power series (3.4) vanish, namely for $a = 2$, $b = \frac{11}{6}$. In this case of the approximation is

$$\left(1 + \frac{1}{x}\right)^x \approx e \left(1 - \frac{1}{2x + \frac{11}{6}}\right), \quad (3.5)$$

the relative error sequence $w_n(2, \frac{11}{6})$ converges to zero as n^{-3} , since

$$\lim_{n \rightarrow \infty} n^3 w_n \left(2, \frac{11}{6}\right) = -\frac{5}{288}.$$

Further we have the idea to refine the class of approximations (3.5) to

$$\left(1 + \frac{1}{x}\right)^x \approx e \left(1 - \frac{1}{2x + \frac{11}{6} + \frac{c}{x}}\right), \quad (3.6)$$

where c is real parameter. In order to find the best value of c , we consider the corresponding relative error sequence $z_n = z_n(c)$ by the relations

$$\left(1 + \frac{1}{n}\right)^n = e \left(1 - \frac{1}{2n + \frac{11}{6} + \frac{c}{n}}\right) \exp z_n, \quad n \geq 1.$$

Making appeal again to the Maple software, we get

$$z_n - z_{n+1} = \left(-\frac{3}{4}c - \frac{5}{96}\right) \frac{1}{n^4} + O\left(\frac{1}{n^5}\right).$$

Using again Lemma 2, the best value of c is obtained when the coefficient of n^{-4} vanishes, that is $c = -\frac{5}{72}$. In consequence, the best approximation of the form (3.6) is

$$\left(1 + \frac{1}{x}\right)^x \approx e \left(1 - \frac{1}{2x + \frac{11}{6} - \frac{5}{72x}}\right), \quad (3.7)$$

whose relative error sequence $z_n(-\frac{5}{72})$ converges to zero as n^{-4} , since

$$\lim_{n \rightarrow \infty} n^4 z_n \left(-\frac{5}{72}\right) = \frac{17}{2160}.$$

Now we can see that the sequence $z_n(-\frac{5}{72})$ has a better rate of convergence than $w_n(2, \frac{11}{6})$ has, and this is another evidence of the fact that the approximation (3.7) is more accurate than (3.5), for large values of x .

4. SOME SHARP BOUNDS

Our new approach presented in the previous section can be used to establish increasingly accurate approximations. Moreover, these approximations can be used to deduce some sharp bounds, as it is done in the following theorem. Remark that the expression in the right-hand side of (4.1) was obtained using the method presented above.

Theorem 2. *For every real number $x \geq 1$, we have*

$$e \left(1 - \frac{6}{12x + 11 - \frac{5}{12x}} \right) < \left(1 + \frac{1}{x} \right)^x < e \left(1 - \frac{6}{12x + 11 - \frac{5}{12x} + \frac{17}{90x^2}} \right). \quad (4.1)$$

Proof. We have to prove that $g > 0$ and $h < 0$, where

$$g(x) = x \ln \left(1 + \frac{1}{x} \right) - \ln \left(1 - \frac{6}{12x + 11 - \frac{5}{12x}} \right) - 1$$

and

$$h(x) = x \ln \left(1 + \frac{1}{x} \right) - \ln \left(1 - \frac{6}{12x + 11 - \frac{5}{12x} + \frac{17}{90x^2}} \right) - 1.$$

We have $g > 0$ and $h < 0$ on $[1, \infty)$, since

$$\begin{aligned} g''(x) &= \frac{P(x)}{x(x+1)^2(132x+144x^2-5)^2(60x+144x^2-5)^2} \\ &= -\frac{h''(x)}{Q(x)} \\ &= -\frac{Q(x)}{x(x+1)^2(900x^2-75x+2160x^3+34)^2(1980x^2-75x+2160x^3+34)^2}, \end{aligned}$$

where

$$\begin{aligned} P(x) &= 2210400x^2 - 297600x + 58821120x^3 + 123690240x^4 \\ &\quad + 67682304x^5 - 625 \\ Q(x) &= 282884760x^2 - 96687840x + 6264927720x^3 + 30001770225x^4 \\ &\quad + 284790556800x^5 + 2911276663200x^6 + 5519218176000x^7 \\ &\quad + 2875292640000x^8 + 1336336. \end{aligned}$$

(the polynomial $P(x+1)$ has all coefficients positive). Now g is strictly convex, h is strictly concave, with $g(\infty) = h(\infty) = 0$, so $g > 0$ and $h < 0$. $\square$

We further refine Yang's result (1.6) using (1.5) and Theorem 2.

Theorem 3. For every $a_n \geq 0$, $n = 1, 2, 3, \dots$, with $0 < \sum_{n=1}^{\infty} a_n < \infty$, we have

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \frac{6}{12n + 11 - \frac{5}{12n} + \frac{17}{90n^2}} \right) a_n.$$

Finally, remark that our new inequalities stated in Theorem 2 can be used to improve the variants of Hardy's inequality established by Zitian and Yibing [9].

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AN OVERDETERMINED SYSTEM WITH TWO SUPER-SINGULAR LINES CONTAINING SECOND ORDER HYPERBOLIC EQUATIONS

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Abstract. The paper treats a linear overdetermined system which contains second order hyperbolic equations with two linear super singularities. The authors obtain some new integral representations of the solutions.

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1. INTRODUCTION

We derive the general solutions of the system (2.1) that contains arbitrary constants subject to the conditions $\alpha < 1, \beta < 1, \gamma < 1$ or $\alpha > 1, \beta > 1, \gamma > 1$.

Hyperbolic differential equations with singular coefficients or singular surfaces possess importance in diverse areas of mathematical physics and mathematical engineering, including elasticity, hydrodynamics, thermodynamics and other problems. Furthermore, it is also well known that hyperbolic differential equations with one or more singular lines occur in engineering and physical processes. For example, the non-standard hyperbolic equation of second order with two singular lines is employed to describe the transformation spectrum of electric signals on long lines with variable parameters in the theory of the electric flail [3, 4, 9].

The problem of obtaining the solutions of equations of hyperbolic type with singular coefficients is described in a number of works [1, 3–5, 9]. In [9], for the equation

$$\frac{\partial^2 U}{\partial t^2} = \sum_{j=1}^n [a_j \frac{\partial^2 U}{\partial x_j^2} + \frac{\eta_j}{X_j} \frac{\partial U}{\partial x_j}] + g(x, t),$$

(here $a_j = \text{const.} > 0, \eta_j = \text{const.} > 0$) a representation of the solutions was obtained by using the solutions of the corresponding regularity equation. On this basis a solution of the Cauchy problem was received, when the boundary conditions are

given on the $t = 0$ surface. In [3], the equation

$$\frac{\partial^2 U}{\partial t^2} + \frac{\eta}{t} \frac{\partial U}{\partial t} - \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) = f(x, y, t).$$

(in the case when $\eta > 0$ in region $x > 0$, $t > 0$, $-\infty < y < \infty$) was solved

$$U(x, 0) = 0, \frac{\partial U(x, y, t)}{\partial t} \Big|_{t=0} = 0, U(0, y, t) = g(y, t).$$

Investigations of first and second order overdetermined systems with singular and super-singular coefficients are described in [6, 10]. A study of a nonlinear system of ordinary differential equations with singular and super-singular coefficient (in a case which is connected to the theory of P-Laplacians and psedo-Laplacians) is described in [8].

2. MAIN RESULTS

Let D be the following region: $D = \{0 < x < \lambda_1, 0 < y < \lambda_2\}$, which is bounded by

$$\Gamma_1 = \{0 < x < \lambda_1, y = 0\}, \quad \Gamma_2 = \{x = 0, 0 < y < \lambda_2\}.$$

In the region D we consider the following system:

$$\begin{cases} \frac{\partial u(x, y)}{\partial x \partial y} + \frac{a_1(x, y)}{y^\beta} \frac{\partial u(x, y)}{\partial x} + \frac{b_1(x, y)}{x^\alpha} \frac{\partial u(x, y)}{\partial y} + \frac{c_1(x, y)}{x^\alpha y^\beta} u(x, y) = \frac{f_1(x, y)}{x^\alpha y^\beta} \\ \frac{\partial u(x, y)}{\partial y} + \frac{b_2(x, y)}{y^\gamma} u(x, y) = \frac{f_2(x, y)}{y^\gamma} \end{cases} \quad (2.1)$$

where the coefficients $a_1(x, y)$, $b_j(x, y)$, $c_1(x, y)$, $f_j(x, y)$, $j = 1, 2$, are given functions in the domain $\overline{D}$. Moreover $\alpha = \text{const.} > 0$, $\beta = \text{const.} > 0$, $\gamma = \text{const.} > 0$.

2.1. The case of $\alpha < 1$, $\beta < 1$, $\gamma < 1$.

We first determine an integral representation of the solution of the given system in the case when the coefficients of the first equation are related together in a certain way. Then we investigate the case when the coefficients of the first equation of system (2.1) are not related to each other.

Let $C_y^1(\overline{D})$ denote the set functions which are continuous on $\overline{D}$ together with their derivatives with respect to the variable y .

We always assume that $b_1(x, y) \in C_y^1(\overline{D})$ and that the coefficients $a_1(x, y)$, $b_1(x, y)$, $b_2(x, y)$, $c_1(x, y)$ are continuous functions in the domain D .

According to [9], the first equation of system (2.1) can be written in the following form

$$\left(\frac{\partial}{\partial y} + \frac{a_1(x, y)}{y^\beta} \right) \left(\frac{\partial}{\partial x} + \frac{b_1(x, y)}{x^\alpha} \right) u(x, y) = \frac{f_1(x, y) + c_2(x, y)u(x, y)}{x^\alpha y^\beta}, \quad (2.2)$$

where

$$c_2(x, y) = y^\beta \frac{\partial b_1(x, y)}{\partial y} + a_1(x, y) \cdot b_1(x, y) - c_1(x, y).$$

2.1.1. The case of $c_2(x, y) = 0$

Suppose that the functions $a_1(x, y)$, $b_1(x, y)$ and $c_1(x, y)$ of the system (2.1) satisfy the following relation

$$c_1(x, y) = y^\beta \frac{\partial b_1(x, y)}{\partial y} + a_1(x, y) \cdot b_1(x, y)$$

for all $(x, y) \in D$.

Under this condition, the first equation of (2.1) can be written in the following form:

$$\left(\frac{\partial}{\partial y} + \frac{a_1(x, y)}{y^\beta}\right)\left(\frac{\partial}{\partial x} + \frac{b_1(x, y)}{x^\alpha}\right)u(x, y) = \frac{f_1(x, y)}{x^\alpha y^\beta}.$$

The solution of this equation is

$$\begin{aligned} u(x, y) = & \exp[-\Omega_{b_1}^\alpha(x, y)][\psi_1(y) + \int_0^x \exp[\Omega_{b_1}^\alpha(t, y) - \Omega_{a_1}^\beta(t, y)] \cdot (\varphi_1(t) + \\ & + \int_0^y \exp[\Omega_{a_1}^\beta(t, s)] \cdot t^{-\alpha} s^{-\beta} \cdot f_1(t, s) ds) dt] \equiv K_1[\varphi_1(x), \psi_1(y), f_1(x, y)], \end{aligned} \quad (2.3)$$

where $\varphi_1(x)$ and $\psi_1(y)$ are arbitrary continuous functions belonging to the class $C^1(\Gamma_1)$ and $C^1(\Gamma_2)$, respectively. In fact, these functions will be determined in the sequel,

$$\Omega_{b_1}^\alpha(x, y) = \int_0^x \frac{b_1(t, y)}{t^\alpha} dt, \quad \Omega_{a_1}^\beta(x, y) = \int_0^y \frac{a_1(x, s)}{s^\beta} ds.$$

Now, we write the second equation of (2.1) in the following form:

$$\frac{\partial}{\partial y} [\exp[\Omega_{b_2}^\gamma(x, y)] \cdot u(x, y)] = \exp[\Omega_{b_2}^\gamma(x, y)] y^{-\gamma} f_2(x, y), \quad (2.4)$$

where

$$\Omega_{b_2}^\gamma(x, y) = \int_0^y \frac{b_2(x, s)}{s^\gamma} ds.$$

By plugging $u(x, y)$ from (2.3) into (2.4) we obtain

$$\begin{aligned} \frac{\partial}{\partial y} \{ \exp[\Omega_{b_2}^\gamma(x, y) - \Omega_{b_1}^\alpha(x, y)] [\psi_1(y) + \int_0^x \exp[\Omega_{b_1}^\alpha(t, y) - \Omega_{a_1}^\beta(t, y)] \cdot (\varphi_1(t) + \\ + \int_0^y \exp[\Omega_{a_1}^\beta(t, s)] t^{-\alpha} s^{-\beta} f_1(t, s) ds) dt] \} = f_2(x, y) y^{-\gamma} \exp[\Omega_{b_2}^\gamma(x, y)]. \end{aligned} \quad (2.5)$$

Suppose that the functions $b_1(x, y)$ and $b_2(x, y)$ satisfy the following relation

$$y^\gamma \frac{\partial b_1(x, y)}{\partial y} = x^\alpha \frac{\partial b_2(x, y)}{\partial x}. \quad (2.6)$$

By differentiating and using the relation (2.6), the equality (2.5) can be expressed in the form

$$\begin{aligned} \psi_1'(y) + \frac{b_2(0, y)}{y^\gamma} \psi_1(y) &= f_2(x, y) y^{-\gamma} \cdot \exp[\Omega_{b_1}^\alpha(x, y)] - \frac{b_2(0, y)}{y^\gamma} \\ &\cdot \int_0^x \exp[\Omega_{b_1}^\alpha(t, y) - \Omega_{a_1}^\beta(t, y)] \cdot (\varphi_1(t) + \int_0^y \exp[\Omega_{a_1}^\beta(t, s)] t^{-\alpha} s^{-\beta} f_1(t, s) ds) dt - \\ &- \int_0^x \frac{\partial}{\partial y} \{ \exp[\Omega_{b_1}^\alpha(t, y) - \Omega_{a_1}^\beta(t, y)] \cdot (\varphi_1(t) + \int_0^y \exp[\Omega_{a_1}^\beta(t, s)] t^{-\alpha} s^{-\beta} f_1(t, s) ds) \} dt \end{aligned}$$

Since the left hand side does not depend on x , we have

$$\begin{aligned} \frac{\partial}{\partial x} [f_2(x, y) y^{-\gamma} \exp[\Omega_{b_1}^\alpha(x, y)]] - \frac{\partial}{\partial y} \{ \exp[\Omega_{b_1}^\alpha(x, y) - \Omega_{a_1}^\beta(x, y)] \cdot (\varphi_1(x) + \\ + \int_0^y \exp[\Omega_{a_1}^\beta(x, s)] x^{-\alpha} s^{-\beta} f_1(x, s) ds) \} &= b_2(0, y) y^{-\gamma} \exp[\Omega_{b_1}^\alpha(x, y) - \Omega_{a_1}^\beta(x, y)] \\ \cdot (\varphi_1(x) + \int_0^y \exp[\Omega_{a_1}^\beta(x, s)] x^{-\alpha} s^{-\beta} f_1(x, s) ds), \end{aligned} \quad (2.7)$$

or in a simpler form

$$\begin{aligned} x^\alpha \frac{\partial f_2(x, y)}{\partial x} + b_1(x, y) f_2(x, y) &= (b_2(x, y) - a_1(x, y) y^{\gamma-\beta}) \exp[-\Omega_{a_1}^\beta(x, y)] \cdot \\ \cdot [x^\alpha \varphi_1(x) + \int_0^y \exp[\Omega_{a_1}^\beta(x, s)] s^{-\beta} f_1(x, s) ds] &+ f_1(x, y) y^{\gamma-\beta}, \quad (x, y) \in D, \end{aligned} \quad (2.8)$$

Making use of (2.7), we can define $\psi_1(y)$ by the following ordinary differential equation

$$\psi_1'(y) + \frac{b_2(0, y)}{y^\gamma} \psi_1(y) = \frac{f_2(0, y)}{y^\gamma}.$$

According to [6, 9], the solution of this equation is represented by the following form

$$\psi_1(y) = \exp[-\Omega_{b_2}^\gamma(0, y)] [C + \int_0^y \exp[\Omega_{b_2}^\gamma(0, s)] s^{-\gamma} f_2(0, s) ds], \quad (2.9)$$

where C is an arbitrary constant.

From expression (2.8), the function $x^\alpha \varphi_1(x)$ can be expressed in the form

$$\begin{aligned} x^\alpha \varphi_1(x) &= \frac{x^\alpha \frac{\partial f_2(x, y)}{\partial x} + b_1(x, y) f_2(x, y) - f_1(x, y) y^{\gamma-\beta}}{b_2(x, y) - a_1(x, y) y^{\gamma-\beta}} \cdot \exp[\Omega_{a_1}^\beta(x, y)] - \\ &- \int_0^y \exp[\Omega_{a_1}^\beta(x, s)] s^{-\beta} f_1(x, s) ds, \quad (b_2(x, y) \neq a_1(x, y) y^{\gamma-\beta}). \end{aligned} \quad (2.10)$$

We note that the left part of this expression is independent of the variable y , so we have

$$\begin{aligned} \frac{\partial}{\partial y} \left\{ \frac{x^\alpha \frac{\partial f_2(x, y)}{\partial x} + b_1(x, y) f_2(x, y) - f_1(x, y) y^{\gamma-\beta}}{b_2(x, y) - a_1(x, y) y^{\gamma-\beta}} \cdot \exp[\Omega_{a_1}^\beta(x, y)] \right\} = \\ = \exp[\Omega_{a_1}^\beta(x, y)] y^{-\beta} f_1(x, y). \end{aligned} \quad (2.11)$$

By using (2.11) and assuming that the following limit exists,

$$\lim_{y \rightarrow 0} \left\{ \frac{x^\alpha \frac{\partial f_2(x, y)}{\partial x} + b_1(x, y) f_2(x, y) - f_1(x, y) y^{\gamma-\beta}}{b_2(x, y) - a_1(x, y) y^{\gamma-\beta}} \cdot \exp[\Omega_{a_1}^\beta(x, y)] \right\} = F_1(x), \quad (2.12)$$

the function $\varphi_1(x)$ can be determined by the following formula:

$$\varphi_1(x) = F_1(x) x^{-\alpha}, \quad \alpha < 1, \quad F_1(x) \in C(\overline{\Gamma_1}). \quad (2.13)$$

In the particular case of $\gamma = \beta$, if $y \rightarrow 0$, then

$$\varphi_1(x) = x^{-\alpha} \left[\frac{x^\alpha \frac{\partial f_2(x, 0)}{\partial x} + b_1(x, 0) f_2(x, 0) - f_1(x, 0)}{b_2(x, 0) - a_1(x, 0)} \right]$$

from equality (2.10).

After simplification, condition (2.11) can be written in the following form:

$$\begin{aligned} & x^\alpha [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}] \frac{\partial^2 f_2(x, y)}{\partial x \partial y} \\ & + \left\{ \frac{a_1(x, y)}{y^\beta} [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}] x^\alpha - \right. \\ & x^\alpha \frac{\partial}{\partial y} [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}] \left. \right\} \frac{\partial f_2(x, y)}{\partial x} + \\ & b_1(x, y) [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}] \frac{\partial f_2(x, y)}{\partial y} + \\ & + \left\{ \frac{a_1(x, y)}{y^\beta} b_1(x, y) \cdot [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}] - \right. \\ & b_1(x, y) \frac{\partial}{\partial y} [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}] + \\ & + [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}] \cdot \frac{\partial b_1(x, y)}{\partial y} \left. \right\} f_2(x, y) \\ & = y^{\gamma-\beta} [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}] \frac{\partial f_1(x, y)}{\partial y} + \\ & + \{ [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}]^2 y^{-\beta} \} \end{aligned}$$

$$\begin{aligned}
& + \frac{a_1(x, y)}{y^\beta} [b_2(x, y) - a_1(x, y)y^{\gamma-\beta}]y^{\gamma-\beta} - y^{\gamma-\beta} \frac{\partial}{\partial y} [b_2(x, y) - \\
& - a_1(x, y)y^{\gamma-\beta}] + (\gamma - \beta)y^{\gamma-\beta-1} \cdot [b_2(x, y) - a_1(x, y)y^{\gamma-\beta}] \} f_1(x, y), \quad (2.14)
\end{aligned}$$

where $(x, y) \in D$.

In what follows, $C_{xy}^2(D)$ will denote the set of all functions $u(x, y)$ that are continuous on D together with their first derivatives, and $u_{xy} \in C(D)$.

Henceforth, we have the following conclusion:

Theorem 1. *Let $\alpha < 1$, $\beta < 1$, $\gamma < 1$ in system (2.1), and assume that the coefficients and right hand sides of this system satisfy the following conditions:*

- 1) $a_1(x, y), b_2(x, y), f_2(x, y) \in C_x^1(\overline{D}), b_1(x, y), f_1(x, y) \in C_y^1(\overline{D}), f_2(x, y) \in C'_{xy}(D), \frac{\partial}{\partial y} [b_2(x, y) - a_1(x, y)y^{\gamma-\beta}] \in C(D),$
- 2) $c_1(x, y) = y^\beta \frac{\partial b_1(x, y)}{\partial y} + a_1(x, y) \cdot b_1(x, y),$
- 3) $y^\gamma \frac{\partial b_1(x, y)}{\partial y} = x^\alpha \frac{\partial b_2(x, y)}{\partial x},$
- 4) *The functions $f_1(x, y)$ and $f_2(x, y)$ satisfy conditions (2.8), (2.1.1),*
- 5) *The limit (2.12) exists.*

Then the desired solution of system (2.1) of class $C_{xy}^2(D)$ is determined by (2.3), in which $\varphi_1(x)$ which is determined by formula (2.13), where the value of $\psi_1(y)$ is defined by formula (2.9), provided that C is an arbitrary constant.

Example 1. Assume that in (2.1) $\alpha < 1, \beta < 1, \gamma < 1, \beta = \gamma$ and the coefficients are constants, moreover, suppose that the right parts are defined by the following functions:

$$\begin{aligned}
f_1(x, y) &= x^\alpha \exp[b_1\omega_\alpha(x) + a_1\omega_\beta(y)], \\
f_2(x, y) &= x \exp[b_1\omega_\alpha(x) + a_1\omega_\beta(y)]E(y).
\end{aligned}$$

where

$$E(y) = 1 - (b_2 - a_1)\omega_\beta(y), \quad \omega_\beta(y) = \frac{1}{(\beta - 1)y^{\beta-1}}, \quad \omega_\alpha(x) = \frac{1}{(\alpha - 1)x^{\alpha-1}}.$$

Then the function of the form

$$u(x, y) = c \exp[b_2\omega_\gamma(y) + b_1\omega_\alpha(x)] - x\omega_\beta(y) \exp[a_1\omega_\beta(y) + b_1\omega_\alpha(x)]$$

is the solution of system (2.1), where c is an arbitrary constant.

2.1.2. $c_2(x, y) \neq 0$

Suppose that the functions $a_1(x, y), b_1(x, y), c_1(x, y)$ satisfy the following relation

$$c_1(x, y) \neq y^\beta \frac{\partial b_1(x, y)}{\partial y} + a_1(x, y) \cdot b_1(x, y), \quad (2.15)$$

for all $(x, y) \in D$ in the system (2.1).

In this case, the first equation of system (2.1), can be represented in the form (2.2). We denote $f_3(x, y) = f_1(x, y) + c_2(x, y)u(x, y)$ where $f_3(x, y)$ is a given function. Then the solution of equation (2.2) can be written as

$$u(x, y) = K_1[\varphi_1(x), \psi_1(y), f_3(x, y)]. \quad (2.16)$$

Using these items that are dependent on $u(x, y)$, we obtain the following integral equation:

$$\begin{aligned} u(x, y) - \int_0^x dt \int_0^y [\exp[\Omega_{b_1}^\alpha(t, y) - \Omega_{b_1}^\alpha(x, y) + \Omega_{a_1}^\beta(t, s) - \Omega_{a_1}^\beta(t, y)] \cdot \\ \cdot \frac{c_2(t, s)}{t^\alpha s^\beta} u(t, s) ds = K_1[\varphi_1(x), \psi_1(y), f_1(x, y)]. \end{aligned} \quad (2.17)$$

In fact, the problem of finding the solution of the overdetermined system (2.1), in the case of $\alpha < 1$, $\beta < 1$, $\gamma < 1$, is reduced to the solution of a two dimensional Volterra integral equation with two weak singular lines in the kernel (2.17).

According to [7, 11], the solution of the integral equation (2.17) can be represented in the following form

$$\begin{aligned} u(x, y) = K_1[\varphi_1(x), \psi_1(y), f_1(x, y)] + \\ \int_0^x dt \int_0^y \Gamma(x, y; t, s) K_1[\varphi_1(t), \psi_1(s), f_1(t, s)] ds \\ \equiv T_{1,1}[\varphi_1(x), \psi_1(y), f_1(x, y)], \end{aligned} \quad (2.18)$$

where $\Gamma(x, y; t, s)$ is the resolvent of the integral equation (2.17).

We now return to the problem of finding the arbitrary functions $\varphi_1(x), \psi_1(y)$. We repeat the scheme of 2.1.1 in the case when $\gamma = \beta$, and $y \rightarrow 0$. We obtain

$$\varphi_1(x) = x^{-\alpha} \left[\frac{x^\alpha \frac{\partial f_2(x, 0)}{\partial x} + b_1(x, 0) f_2(x, 0) - f_1(x, 0) - c_2(x, 0) u(x, 0)}{b_2(x, 0) - a_1(x, 0)} \right], \quad (2.19)$$

($b_2(x, 0) \neq a_1(x, 0)$),

$$\psi_1(y) = \exp[-\Omega_{b_2}^\gamma(0, y)] [C + \int_0^y \exp[\Omega_{b_2}^\gamma(0, s)] s^{-\gamma} f_2(0, s) ds], \quad (2.20)$$

$\psi_1(0) = C$,

$$u(x, 0) = \exp[-\Omega_{b_1}^\alpha(x, 0)] [\psi_1(0) + \int_0^x \exp[\Omega_{b_1}^\alpha(t, 0)] \varphi_1(t) dt]. \quad (2.21)$$

By substituting the value of $\varphi_1(x)$ from (2.19) into (2.21), we obtain the following one dimensional Volterra type integral equation:

$$u(x, 0) + \int_0^x \exp[\Omega_{b_1}^\alpha(t, 0) - \Omega_{b_1}^\alpha(x, 0)] \cdot \frac{c_2(t, 0)u(t, 0)}{b_2(t, 0) - a_1(t, 0)} \frac{dt}{t^\alpha} = \exp[-\Omega_{b_1}^\alpha(x, 0)] \cdot \left\{ \psi_1(0) + \int_0^x \exp[\Omega_{b_1}^\alpha(t, 0)] \left[\frac{t^\alpha \frac{\partial f_2(t, 0)}{\partial t} + b_1(t, 0)f_2(t, 0) - f_1(t, 0)}{b_2(t, 0) - a_1(t, 0)} \right] \frac{dt}{t^\alpha} \right\}, \quad (2.22)$$

Take a new unknown function

$$W(x) = u(x, 0) \exp[\Omega_{b_1}^\alpha(x, 0)]. \quad (2.23)$$

Then it follows that the integral equation (2.22) can be represented in the following form

$$W(x) + \int_0^x \frac{c_2(t, 0)W(t)}{b_2(t, 0) - a_1(t, 0)} \frac{dt}{t^\alpha} = E(x), \quad \alpha < 1, \quad (2.24)$$

where

$$E(x) = \left\{ \psi_1(0) + \int_0^x \exp[\Omega_{b_1}^\alpha(t, 0)] \left[\frac{t^\alpha \frac{\partial f_2(t, 0)}{\partial t} + b_1(t, 0)f_2(t, 0) - f_1(t, 0)}{b_2(t, 0) - a_1(t, 0)} \right] \frac{dt}{t^\alpha} \right\},$$

and $\alpha < 1$. According to [2], the solution of equation (2.24) is given by the following formula

$$W(x) = E(x) - \int_0^x \exp[\Omega_{c_3}^\alpha(t, 0) - W_{c_3}^\alpha(x, 0)] \frac{c_3(t)}{t^\alpha} E(t) dt \equiv T_2[E(x)],$$

where

$$c_3(x) = \frac{c_2(x, 0)}{b_2(x, 0) - a_1(x, 0)}, \quad W_{c_3}^\alpha(x, 0) = \int_0^x \frac{c_3(t)}{t^\alpha} dt.$$

Using formula (2.23), the solution of integral equation (2.22) can be written as

$$u(x, 0) = \exp[-\Omega_{b_1}^\alpha(x, 0)] T_2[E(x)].$$

By substituting the value of $u(x, 0)$ into (2.19), we obtain the value of $\phi_1(x)$:

$$\begin{aligned} \varphi_1(x) = \\ x^{-\alpha} \left[\frac{x^\alpha \frac{\partial f_2(x, 0)}{\partial x} + b_1(x, 0)f_2(x, 0) - f_1(x, 0) - c_2(x, 0) \exp[-\Omega_{b_1}^\alpha(x, 0)] T_2[E(x)]}{b_2(x, 0) - a_1(x, 0)} \right], \end{aligned} \quad (2.25)$$

Then we substitute the values of $\varphi_1(x)$, $\psi_1(y)$ which were obtained from (2.1.2) and (2.20), respectively, in formula (2.17).

Henceforth, we deduce the following conclusion

Theorem 2. Let $\alpha < 1, \beta < 1, \gamma < 1, \gamma = \beta$ in system (2.1). Assume the followings: the coefficients and right hand sides of this system satisfies the conditions: 1, 3, 5, from theorem 1, and $c_2(x, y) \neq 0$. Moreover, suppose that the functions $f_1(x, y), f_2(x, y)$ satisfy the following conditions:

(1)

$$\begin{aligned} & x^\alpha \frac{\partial f_2(x, y)}{\partial x} + b_1(x, y) f_2(x, y) \\ &= (b_2(x, y) - a_1(x, y) y^{\gamma-\beta}) \exp[-W_{a_1}^\beta(x, y)] [x^\alpha \varphi_1(x) + \\ & \quad + \int_0^y \exp[\Omega_{a_1}^\beta(x, s)] s^{-\beta} \\ & \quad \{f_1(x, s) + c_2(x, s) T_{1,1}[\varphi_1(x), \psi_1(y), f_1(x, y)]\} ds] + \\ & \quad + y^{\gamma-\beta} [f_1(x, y) + c_2(x, y) T_{1,1}[\varphi_1(x), \psi_1(y), f_1(x, y)]], \end{aligned}$$

(2)

$$\begin{aligned} & x^\alpha [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}] \frac{\partial^2 f_2(x, y)}{\partial x \partial y} \\ & + \{ \frac{a_1(x, y)}{y^\beta} [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}] x^\alpha - x^\alpha \frac{\partial}{\partial y} [b_2(x, y) - \\ & - a_1(x, y) y^{\gamma-\beta}] \} \frac{\partial f_2(x, y)}{\partial x} + b_1(x, y) [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}] \frac{\partial f_2(x, y)}{\partial y} + \\ & \{ \frac{a_1(x, y)}{y^\beta} b_1(x, y) [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}] - b_1(x, y) \frac{\partial}{\partial y} [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}] \\ & \quad + [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}] \frac{\partial b_1(x, y)}{\partial y} \} f_2(x, y) \\ & = y^{\gamma-\beta} [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}] \\ & \quad \frac{\partial}{\partial y} [f_1(x, y) + c_2(x, y) \cdot T_{1,1}[\varphi_1(x), \psi_1(y), f_1(x, y)]] + \\ & \quad \{ [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}]^2 y^{-\beta} + \frac{a_1(x, y)}{y^\beta} \\ & \quad \cdot [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}] y^{\gamma-\beta} \\ & \quad - y^{\gamma-\beta} \frac{\partial}{\partial y} [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}] + (\gamma - \beta) y^{\gamma-\beta-1} \\ & \quad \cdot [b_2(x, y) - a_1(x, y) y^{\gamma-\beta}] \} [f_1(x, y) + c_2(x, y) T_{1,1}[\varphi_1(x), \psi_1(y), f_1(x, y)]]]. \end{aligned}$$

Then the desired solution of system (2.1) in the class $C_{xy}^2(D)$ is determined by (2.3), in which $\varphi_1(x)$ is defined by (2.1.2) and $\psi_1(y)$ is determined by formula (2.20), where C is an arbitrary constant.

2.2. The case of $b_1(x, y) \in C_y^1(\overline{D})$, $\alpha > 1$, $\beta > 1$, $\gamma > 1$.

In system (2.1) let $\alpha > 1$, $\beta > 1$, $\gamma > 1$, $c_2(x, y) = 0$. By repeating the method that was explained in section 2.1.1, we obtain the following result:

Theorem 3. Let $\alpha > 1$, $\beta > 1$, $\gamma > 1$, in system (2.1) and assume that the coefficients and right hand sides of this system satisfy the conditions 1, 2, 3 of Theorem 1. Suppose also that $a_1(0, 0) > 0$, $b_1(0, 0) > 0$, $b_2(0, 0) > 0$. Moreover, assume the existence of the following limits:

$$E_1(x) = \lim_{y \rightarrow 0} \left\{ \frac{x^\alpha \frac{\partial f_2(x, y)}{\partial x} + b_1(x, y) f_2(x, y) - f_1(x, y) y^{\gamma-\beta}}{b_2(x, y) - a_1(x, y) y^{\gamma-\beta}} \right. \\ \left. \exp[W_{a_1}^\beta(x, y) - a_1(0, 0) \omega_\beta(y)] \right\},$$

$$E_2(y) = \lim_{x \rightarrow 0} (\exp[-b_1(0, 0) \omega_\alpha(x)] f_2(x, y)),$$

where $E_1(x) \in C(\overline{\Gamma_1})$, $E_2(y) \in C(\overline{\Gamma_2})$.

Besides, suppose that the functions $f_1(x, y)$, $f_2(x, y)$ satisfy the condition (2.11) and it is also true that

$$x^\alpha \frac{\partial f_2(x, y)}{\partial x} + b_1(x, y) f_2(x, y) \\ = (b_2(x, y) - a_1(x, y) y^{\gamma-\beta}) \exp[a_1(0, 0) \omega_\beta(y) - W_{a_1}^\beta(x, y)] \\ (x^\alpha \varphi_2(x) + \int_0^y \exp[W_{a_1}^\beta(x, s) - a_1(0, 0) \omega_\beta(s)] s^{-\beta} f_1(x, s) ds) + f_1(x, y) y^{\gamma-\beta},$$

$(x, y) \in D$, $\varphi_2(x) = E_1(x) x^{-\alpha}$, $\alpha > 1$, $\beta > 1$.

Moreover, assume that the functions, $a_1(x, y)$, $b_1(x, y)$, $b_2(x, y)$ in the neighborhoods Γ_1 and Γ_2 satisfy the following conditions:

$$|b_1(x, y) - b_1(0, 0)| \leq H_1 x^{\delta_3}, \quad \delta_3 > \alpha - 1, \quad \text{at } x \rightarrow 0; \\ |a_1(x, y) - a_1(0, 0)| \leq H_2 y^{\delta_4}, \quad \delta_4 > \beta - 1, \quad \text{at } y \rightarrow 0; \\ |b_2(x, y) - b_2(0, 0)| \leq H_3 x^{\delta_5}, \quad \delta_5 > \gamma - 1, \quad \text{at } y \rightarrow 0;$$

Then the desired solution of system (2.1) in the class $C_{xy}^2(D)$ is determined by:

$$u(x, y) = \exp[-W_{b_1}^\alpha(x, y) + b_1(0, 0) \omega_\alpha(x)] \{ \psi_2(y) + \\ + \int_0^x \exp[W_{b_1}^\alpha(t, y) - b_1(0, 0) \omega_\alpha(t) + a_1(0, 0) \omega_\beta(y) - W_{a_1}^\beta(t, y)] \cdot (\varphi_2(t) + \\ + \int_0^y \exp[W_{a_1}^\beta(t, s) - a_1(0, 0) \omega_\beta(s)] \cdot t^{-\alpha} s^{-\beta} \cdot f_1(t, s) ds) dt, \} \quad (2.26)$$

where

$$\psi_2(y) = \exp[b_2(0, 0) \omega_\gamma(y) - W_{b_2}^\gamma(0, y)] [C$$

$$+ \int_0^y \exp[W_{b_2}^\gamma(0, s) - b_2(0, 0)\omega_\gamma(s)] s^{-\gamma} E_2(s) ds],$$

(here C is an arbitrary constant), and

$$W_{b_1}^\alpha(x, y) = \int_0^x \frac{b_1(t, y) - b_1(0, 0)}{t^\alpha} dt, \quad W_{a_1}^\beta(x, y) = \int_0^y \frac{a_1(x, s) - a_1(0, 0)}{s^\beta} ds,$$

$$W_{b_2}^\gamma(x, y) = \int_0^y \frac{b_2(x, s) - b_2(0, 0)}{s^\gamma} ds, \quad \omega_\gamma(y) = \frac{1}{(\gamma - 1)y^{\gamma-1}}, \quad \omega_\alpha(x) = \frac{1}{(\alpha - 1)x^{\alpha-1}}.$$

Remark 2. The integral representation (3) also satisfies the system (2.1) when $b_2(0, 0) < 0$, $a_1(0, 0) < 0$, $b_1(0, 0) < 0$, and the functions $E_2(y)$, $E_1(x)$ are such that $E_2(0) = 0$, $E_1(0) = 0$, and they have the following asymptotic behaviors:

$$E_2(y) = 0[\exp[-|b_2(0, 0)|\omega_\gamma(y)]y^{\delta_1}], \quad \delta_1 > \gamma - 1, \quad \text{as } y \rightarrow 0,$$

$$E_1(x) = 0[\exp[-|b_1(0, 0)|\omega_\alpha(x)] \cdot x^{\delta_2}], \quad \delta_2 < \alpha - 1, \quad \text{as } x \rightarrow 0.$$

Moreover, the function $f_1(x, y)$ satisfies that $f_1(x, 0) = 0$, $f_1(0, y) = 0$, and it has the following asymptotic behavior:

$$f_1(x, y) = 0[\exp[-|a_1(0, 0)|\omega_\beta(y)] \cdot y^{-\delta_4}], \quad y \rightarrow 0,$$

$$f_1(x, y) = 0[\exp[-|b_1(0, 0)|\omega_\alpha(x)] \cdot x^{-\delta_3}], \quad x \rightarrow 0.$$

Remark 1. If the coefficients of system (2.1) satisfy all of the conditions of Theorem 3, then the solution of the system (2.1) is unbounded in Γ_1 , Γ_2 , and its behavior is determined by the following asymptotic conditions:

$$u(x, y) = O[\exp(b_1(0, 0)\omega_\alpha(x))] \text{ at } x \rightarrow 0,$$

and

$$u(x, y) = O[\exp(b_2(0, 0)\omega_\gamma(y))] \text{ at } y \rightarrow 0, \quad (\gamma > \beta \quad \text{or} \quad \gamma = \beta, \quad b_2(0, 0) > a_1(0, 0)).$$

Remark 2. For all possible cases of the parameters α , β and γ , integral representations of the solutions of the system (2.1) are obtained not only in the case when $c_1(x, y) = y^\beta \frac{\partial b_1(x, y)}{\partial y} + a_1(x, y) \cdot b_1(x, y)$, but also when $c_1(x, y) \neq y^\beta \frac{\partial b_1(x, y)}{\partial y} + a_1(x, y) \cdot b_1(x, y)$.

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