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STABILITY OF MAXIMUM PRESERVING FUNCTIONAL EQUATIONS ON BANACH LATTICES

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Abstract. Analogous functional equations are studied by substituting in the Cauchy (linear) functional equation the addition with the maximum operation, to be called *maximum preserving functional equations*. The Hyers-Ulam stability problem thus is posed. We propose some sufficient conditions to solve it.

2000 *Mathematics Subject Classification:* Primary 39B05,39B22; Secondary 46B22, 20K30

Keywords: Banach spaces, maximum preserving functional equation, semi-homogeneity, continuity from below

1. INTRODUCTION

The most famous functional equation by Cauchy and known as *linear functional equation* reads:

$$f(x+y) = f(x) + f(y). \tag{1.1}$$

We would point out that equation (1.1) has been investigated for many spaces and in various perspectives such as its stability which has been intensively considered in the literature. The stability problem was first posed by M. Ulam (see [31]) in the terms: "Give conditions in order for a linear mapping near an approximately linear mapping to exist."

Given two Banach algebras E and E' a transformation $f : E \to E'$ is called δ-linear if

$$\|f(x+y) - f(x) - f(y)\| < \delta, \tag{1.2}$$

for all $x, y \in E$. The stability problem (of equation (1.1)) can be stated as follows. Does there exist for each $\varepsilon \in (0, 1)$ some $\delta > 0$ such that to each δ-linear transformation $f : E \to E'$ there corresponds a linear transformation $l : E \to E'$ satisfying the inequality $\|f(x) - l(x)\| < \varepsilon$ for all $x \in E$? This question was answered in the affirmative by Hyers [15]. Ever since various problems of stability on various spaces have come to light. We shall just list few of them: the stability of linear functional equation [4, 6], quadratic and cubic functional equations [8, 20, 23, 25, 27, 30], Jensen and Cauchy-Jensen functional equations [18], pexiderial functional equation [24, 29],

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non-Archimedian functional equation [17], functional differential equation [21], derivations and linear functions [5,22], entropy equation [9–12], functional inequalities [7, 13, 14, 16, 19, 26, 28].

Since the Cauchy linear functional equation (1.1) is an addition preserving equation, its Hyers-Ulam stability problem can be extended to the so-called *maximum preserving functional equation*, where the addition operation in (1.1) is replaced with the maximum operation on a given Banach lattice. The object of the present article is thus to formulate and solve some stability problems for this equation.

The motivation of studying the stability of maximum preserving functional equations lays in the optimal average (see [1–3], say) which, as a cone-related functional, is semi-homogeneous, maximum preserving and continuous from below (cf. [1]).

2. The main results

We would like to point out the similitude between the present communication and the result in [4].

If $\mathcal{B}$ is a Banach lattice, then $\mathcal{B}^+$ will stand for its positive cone, i.e. $\mathcal{B}^+ = \{x \in \mathcal{B} : x \geq 0\} = \{|x| : x \in \mathcal{B}\}$.

Given two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$ we say that a functional $F : \mathcal{X} \to \mathcal{Y}$ is *cone-related* if $F\left(\mathcal{X}^+\right) = \{F(|x|) : x \in \mathcal{X}\} \subset \mathcal{Y}^+$.

Some Properties. Let be given two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$ and, a cone-related functional $F : \mathcal{X} \to \mathcal{Y}$.

- **P1. Maximum Preserving Functional Equation:** $F(|x| \vee |y|) = F(|x|) \vee F(|y|)$ for all members $x, y \in \mathcal{X}$.
- **P2. Semi-homogeneity:** $F(\alpha |x|) = \alpha F(|x|)$ for all $x \in \mathcal{X}$ and every number $\alpha \in [0, \infty)$.
- **P3. Continuity From Below On The Positive Cone:** The identity $\lim_{n\to\infty} F(x_n) = F\left(\lim_{n\to\infty} x_n\right)$ holds for every increasing sequence $(x_n) \subset \mathcal{X}^+$.
- **P4.** For any increasing sequence $(x_k) \subset \mathcal{X}^+$ the inequality hereafter holds

$$\lim_{n\to\infty} \lim_{k\to\infty} \frac{F(2^n x_k)}{2^n} \leq \lim_{k\to\infty} \lim_{n\to\infty} \frac{F(2^n x_k)}{2^n}, \tag{2.1}$$

  provided that the limits exist.

We should note that the maximum preserving functional equation is known as a *join homomorphism* in Lattice Theory.

Remark 1. Given two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$ let a cone-related functional $F : \mathcal{X} \to \mathcal{Y}$ satisfy property **P1**. Then the following statements are valid.

(1) $F(|x \vee y|) \leq F(|x|) \vee F(|y|)$ for all members $x, y \in \mathcal{X}$.
(2) The semi-homogeneity implies that $F(0) = 0$.

(3) F is an increasing operator, in the sense that if $x, y \in \mathcal{X}$ are such that $|x| \leq |y|$, then $F(|x|) \leq F(|y|)$.

Theorem 1. *Let be given a continuous function $p : [0, \infty) \to (0, \infty)$ and two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$. Consider a cone-related functional $F : \mathcal{X} \to \mathcal{Y}$ for which there are numbers $\vartheta > 0$ and $\alpha \in [0, 1)$ such that*

$$\frac{\left\| F(\tau|x| \vee \eta|y|) - \frac{\tau p(\tau) F(|x|) \vee \eta p(\eta) F(|y|)}{p(\tau) \vee p(\eta)} \right\|}{\|x\|^{\alpha} + \|y\|^{\alpha}} \leq \vartheta \tag{2.2}$$

*for all $x, y \in \mathcal{X}$ and $\tau, \eta \in \mathbb{R}^{+}$. Then there is a unique cone-related mapping $T : \mathcal{X} \to \mathcal{Y}$ which satisfies properties **P1**, **P2** and inequality*

$$\frac{\|T(|x|) - F(|x|)\|}{\|x\|^{\alpha}} \leq \frac{2\vartheta}{2 - 2^{\alpha}} \tag{2.3}$$

for every $x \in \mathcal{X}$.

*Moreover, if F is continuous from below, then in order that T be continuous from below it is necessary and sufficient that F enjoy property **P4**.*

Each of the following theorem is a variation of the above result.

Theorem 2. *Let be given a continuous function $p : [0, \infty) \to (0, \infty)$ with $p(0) = 0$ and, two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$. Consider a cone-related functional $F : \mathcal{X} \to \mathcal{Y}$ for which there are numbers $\vartheta > 0$ and $\alpha \in [0, 1)$ such that*

$$\frac{\left\| F(\tau|x| \vee \eta|y|) - \frac{\tau p(\tau) F(|x|) \vee \eta p(\eta) F(|y|)}{p(\tau) + p(\eta)} \right\|}{\|x\|^{\alpha} + \|y\|^{\alpha}} \leq \vartheta \tag{2.4}$$

*for all $x, y \in \mathcal{X}$ and $\tau, \eta \in \mathbb{R}^{+}$. Then there is a unique cone-related mapping $T : \mathcal{X} \to \mathcal{Y}$ which satisfies properties **P1**, **P2** and inequality (2.3) is valid for every $x \in \mathcal{X}$. Moreover, if F is continuous from below, then in order that T be continuous from below it is necessary and sufficient that F enjoy property **P4**.*

Theorem 3. *Let be given a continuous function $p : [0, \infty) \to (0, \infty)$ and, two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$. Consider a cone-related functional $F : \mathcal{X} \to \mathcal{Y}$ for which there are numbers $\vartheta > 0$ and $\alpha \in [0, 1)$ such that*

$$\frac{\left\| F\left(\frac{\tau p(\tau)|x| \vee \eta p(\eta)|y|}{p(\tau) \vee p(\eta)}\right) - \frac{\tau p(\tau) F(|x|) \vee \eta p(\eta) F(|y|)}{p(\tau) \vee p(\eta)} \right\|}{\|x\|^{\alpha} + \|y\|^{\alpha}} \leq \vartheta \tag{2.5}$$

*for all $x, y \in \mathcal{X}$ and $\tau, \eta \in \mathbb{R}^{+}$. Then there is a unique cone-related mapping $T : \mathcal{X} \to \mathcal{Y}$ which satisfies properties **P1**, **P2** and inequality (2.3) is valid for every $x \in \mathcal{X}$. Moreover, if F is continuous from below, then in order that T be continuous from below it is necessary and sufficient that F enjoy property **P4**.*

Theorem 4. *Let be given a continuous function $p:[0,\infty)\to(0,\infty)$ with $p(0)=0$ and, two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$. Consider a cone-related functional $F:\mathcal{X}\to\mathcal{Y}$ for which there are numbers $\vartheta>0$ and $\alpha\in[0,1)$ such that*

$$\frac{\left\| F\left(\frac{\tau p(\tau)|x|\vee\eta p(\eta)|y|}{p(\tau)+p(\eta)}\right)-\frac{\tau p(\tau)F(|x|)\vee\eta p(\eta)F(|y|)}{p(\tau)+p(\eta)}\right\|}{\|x\|^{\alpha}+\|y\|^{\alpha}}\leq\vartheta \tag{2.6}$$

*for all $x,\ y\in\mathcal{X}$ and $\tau,\ \eta\in\mathbb{R}^{+}$. Then there is a unique cone-related mapping $T:\mathcal{X}\to\mathcal{Y}$ which satisfies properties **P1**, **P2** and inequality (2.3) is valid for every $x\in\mathcal{X}$. Moreover, if F is continuous from below, then in order that T be continuous from below it is necessary and sufficient that F enjoy property **P4**.*

We point out that the proof of Theorem 1 can be suitably adapted to show the validity of Theorems 2, 3, 4.

It is worth to ask the question: *Under what conditions inequalities (2.2) and (2.4)-(2.6) hold true*? The answer is formulated in the following results without proof, because of their easiness (in fact, only the triangle inequality of the norm is needed to check their validity).

Lemma 1. *Let be given a continuous function $p:[0,\infty)\to(0,\infty)$ and, two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$. Consider a cone-related functional $F:\mathcal{X}\to\mathcal{Y}$ and define the functional $F_c:\mathcal{X}\to\mathcal{Y}$ by $F_c(x)=F(x)\wedge c$, where $c\in\mathcal{Y}^{+}$. Let $\alpha\in(0,1)$ be some number, $\mathcal{Y}_0\subset\mathcal{Y}^{+}$ be some non-empty subset and consider the following 4 quantities:*

$$\sup_{c\in\mathcal{Y}_0}\sup_{\tau,\eta\in[0,\infty)}\sup_{x,y\in\mathcal{X}}\frac{\|F_c(\tau|x|\vee\eta|y|)-F(\tau|x|\vee\eta|y|)\|}{\|x\|^{\alpha}+\|y\|^{\alpha}},$$

$$\sup_{c\in\mathcal{Y}_0}\sup_{\tau,\eta\in[0,\infty)}\sup_{x,y\in\mathcal{X}}\frac{\left\|\frac{\tau p(\tau)F_c(|x|)\vee\eta p(\eta)F_c(|y|)}{p(\tau)\vee p(\eta)}-\frac{\tau p(\tau)F(|x|)\vee\eta p(\eta)F(|y|)}{p(\tau)\vee p(\eta)}\right\|}{\|x\|^{\alpha}+\|y\|^{\alpha}},$$

$$\sup_{c\in\mathcal{Y}_0}\sup_{\tau,\eta\in[0,\infty)}\sup_{x,y\in\mathcal{X}}\frac{\left\|F_c(\tau|x|\vee\eta|y|)-\frac{\tau p(\tau)F_c(|x|)\vee\eta p(\eta)F_c(|y|)}{p(\tau)\vee p(\eta)}\right\|}{\|x\|^{\alpha}+\|y\|^{\alpha}}$$

and

$$\sup_{\tau,\eta\in[0,\infty)}\sup_{x,y\in\mathcal{X}}\frac{\left\|F(\tau|x|\vee\eta|y|)-\frac{\tau p(\tau)F(|x|)\vee\eta p(\eta)F(|y|)}{p(\tau)\vee p(\eta)}\right\|}{\|x\|^{\alpha}+\|y\|^{\alpha}}.$$

If any three of them are simultaneously finite, then the fourth is also finite.

Lemma 2. *Let be given a continuous function $p:[0,\infty)\to(0,\infty)$ and, two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$. Consider a cone-related functional $F:\mathcal{X}\to\mathcal{Y}$ and define the functional $F_c:\mathcal{X}\to\mathcal{Y}$ by $F_c(x)=F(x)\wedge c$, where $c\in\mathcal{Y}^{+}$. Let $\alpha\in(0,1)$*

be some number and $\mathcal{Y}_0 \subset \mathcal{Y}^+$ some non-empty subset and consider the following 4 quantities:

$$\sup_{c\in\mathcal{Y}_0}\ \sup_{\tau,\eta\in[0,\infty)}\ \sup_{x,y\in\mathcal{X}} \frac{\|F_c(\tau|x|\vee\eta|y|)-F(\tau|x|\vee\eta|y|)\|}{\|x\|^\alpha+\|y\|^\alpha},$$

$$\sup_{c\in\mathcal{Y}_0}\ \sup_{\tau,\eta\in[0,\infty)}\ \sup_{x,y\in\mathcal{X}} \frac{\left\|\frac{\tau p(\tau)F_c(|x|)\vee\eta p(\eta)F_c(|y|)}{p(\tau)+p(\eta)}-\frac{\tau p(\tau)F(|x|)\vee\eta p(\eta)F(|y|)}{p(\tau)+p(\eta)}\right\|}{\|x\|^\alpha+\|y\|^\alpha},$$

$$\sup_{c\in\mathcal{Y}_0}\ \sup_{\tau,\eta\in[0,\infty)}\ \sup_{x,y\in\mathcal{X}} \frac{\left\|F_c(\tau|x|\vee\eta|y|)-\frac{\tau p(\tau)F_c(|x|)\vee\eta p(\eta)F_c(|y|)}{p(\tau)+p(\eta)}\right\|}{\|x\|^\alpha+\|y\|^\alpha},$$

and

$$\sup_{\tau,\eta\in[0,\infty)}\ \sup_{x,y\in\mathcal{X}} \frac{\left\|F(\tau|x|\vee\eta|y|)-\frac{\tau p(\tau)F(|x|)\vee\eta p(\eta)F(|y|)}{p(\tau)+p(\eta)}\right\|}{\|x\|^\alpha+\|y\|^\alpha}.$$

If any three of them are simultaneously finite, then the fourth is also finite.

Lemma 3. *Let be given a continuous function $p:[0,\infty)\to(0,\infty)$ and, two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$. Consider a cone-related functional $F:\mathcal{X}\to\mathcal{Y}$ and define the functional $F_c:\mathcal{X}\to\mathcal{Y}$ by $F_c(x)=F(x)\wedge c$, where $c\in\mathcal{Y}^+$. Let $\alpha\in(0,1)$ be some number and $\mathcal{Y}_0\subset\mathcal{Y}^+$ some non-empty subset and consider the following 4 quantities:*

$$\sup_{c\in\mathcal{Y}_0}\ \sup_{\tau,\eta\in[0,\infty)}\ \sup_{x,y\in\mathcal{X}} \frac{\left\|F_c\left(\frac{\tau p(\tau)|x|\vee\eta p(\eta)|y|}{p(\tau)\vee p(\eta)}\right)-F\left(\frac{\tau p(\tau)|x|\vee\eta p(\eta)|y|}{p(\tau)\vee p(\eta)}\right)\right\|}{\|x\|^\alpha+\|y\|^\alpha},$$

$$\sup_{c\in\mathcal{Y}_0}\ \sup_{\tau,\eta\in[0,\infty)}\ \sup_{x,y\in\mathcal{X}} \frac{\left\|\frac{\tau p(\tau)F_c(|x|)\vee\eta p(\eta)F_c(|y|)}{p(\tau)\vee p(\eta)}-\frac{\tau p(\tau)F(|x|)\vee\eta p(\eta)F(|y|)}{p(\tau)\vee p(\eta)}\right\|}{\|x\|^\alpha+\|y\|^\alpha},$$

$$\sup_{c\in\mathcal{Y}_0}\ \sup_{\tau,\eta\in[0,\infty)}\ \sup_{x,y\in\mathcal{X}} \frac{\left\|F_c\left(\frac{\tau p(\tau)|x|\vee\eta p(\eta)|y|}{p(\tau)\vee p(\eta)}\right)-\frac{\tau p(\tau)F_c(|x|)\vee\eta p(\eta)F_c(|y|)}{p(\tau)\vee p(\eta)}\right\|}{\|x\|^\alpha+\|y\|^\alpha}$$

and

$$\sup_{\tau,\eta\in[0,\infty)}\ \sup_{x,y\in\mathcal{X}} \frac{\left\|F\left(\frac{\tau p(\tau)|x|\vee\eta p(\eta)|y|}{p(\tau)\vee p(\eta)}\right)-\frac{\tau p(\tau)F(|x|)\vee\eta p(\eta)F(|y|)}{p(\tau)\vee p(\eta)}\right\|}{\|x\|^\alpha+\|y\|^\alpha}.$$

If any three of them are simultaneously finite, then the fourth is also finite.

Lemma 4. *Let be given a continuous function $p:[0,\infty)\to(0,\infty)$ and, two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$. Consider a cone-related functional $F:\mathcal{X}\to\mathcal{Y}$ and define the functional $F_c:\mathcal{X}\to\mathcal{Y}$ by $F_c(x)=F(x)\wedge c$, where $c\in\mathcal{Y}^+$. Let $\alpha\in(0,1)$ be some number and $\mathcal{Y}_0\subset\mathcal{Y}^+$ some non-empty subset and consider the following 4 quantities:*

$$\sup_{c\in\mathcal{Y}_0}\sup_{\tau,\eta\in[0,\infty)}\sup_{x,y\in\mathcal{X}}\frac{\left\|F_c\left(\frac{\tau p(\tau)|x|\vee\eta p(\eta)|y|}{p(\tau)+p(\eta)}\right)-F\left(\frac{\tau p(\tau)|x|\vee\eta p(\eta)|y|}{p(\tau)+p(\eta)}\right)\right\|}{\|x\|^\alpha+\|y\|^\alpha},$$

$$\sup_{c\in\mathcal{Y}_0}\sup_{\tau,\eta\in[0,\infty)}\sup_{x,y\in\mathcal{X}}\frac{\left\|\frac{\tau p(\tau)F_c(|x|)\vee\eta p(\eta)F_c(|y|)}{p(\tau)+p(\eta)}-\frac{\tau p(\tau)F(|x|)\vee\eta p(\eta)F(|y|)}{p(\tau)+p(\eta)}\right\|}{\|x\|^\alpha+\|y\|^\alpha},$$

$$\sup_{c\in\mathcal{Y}_0}\sup_{\tau,\eta\in[0,\infty)}\sup_{x,y\in\mathcal{X}}\frac{\left\|F_c\left(\frac{\tau p(\tau)|x|\vee\eta p(\eta)|y|}{p(\tau)+p(\eta)}\right)-\frac{\tau p(\tau)F_c(|x|)\vee\eta p(\eta)F_c(|y|)}{p(\tau)+p(\eta)}\right\|}{\|x\|^\alpha+\|y\|^\alpha}$$

and

$$\sup_{\tau,\eta\in[0,\infty)}\sup_{x,y\in\mathcal{X}}\frac{\left\|F\left(\frac{\tau p(\tau)|x|\vee\eta p(\eta)|y|}{p(\tau)+p(\eta)}\right)-\frac{\tau p(\tau)F(|x|)\vee\eta p(\eta)F(|y|)}{p(\tau)+p(\eta)}\right\|}{\|x\|^\alpha+\|y\|^\alpha}.$$

If any three of them are simultaneously finite, then the fourth is also finite.

3. Proof of Theorem 1

Proof. We first show by induction that for any fixed $x\in\mathcal{X}$,

$$\frac{\left\|\frac{F(2^n|x|)}{2^n}-F(|x|)\right\|}{\|x\|^\alpha}\le\vartheta\sum_{j=0}^{n-1}2^{j(\alpha-1)} \tag{3.1}$$

whenever $n\in\mathbb{N}$. In fact, for $n=1$ the statement is obvious by choosing $\tau=\eta=2$ and $x=y$ in inequality (2.2). Suppose the statement is true for $n=k$. Let us prove it for $n=k+1$. In fact, let $2x$ replace x and $n=k$ in inequality (3.1) and observe that

$$\frac{\left\|\frac{F(2^k2|x|)}{2^k}-F(2|x|)\right\|}{\|2x\|^\alpha}\le\vartheta\sum_{l=0}^{k-1}2^{l(\alpha-1)}.$$

Hence

$$\frac{\left\|\frac{F(2^{(k+1)}|x|)}{2^{(k+1)}}-\frac{1}{2}F(2|x|)\right\|}{\|x\|^\alpha}\le\vartheta2^{(\alpha-1)}\sum_{l=0}^{k-1}2^{l(\alpha-1)}\le\vartheta\sum_{l=1}^{k}2^{l(\alpha-1)}.$$

The triangle inequality yields

$$\left\| \frac{F\left(2^{(k+1)}|x|\right)}{2^{k+1}} - F(|x|) \right\| \leq \left\| \frac{F\left(2^{(k+1)}|x|\right)}{2^{k+1}} - \frac{F(2|x|)}{2} \right\| + \left\| \frac{F(2|x|)}{2} - F(|x|) \right\| \leq$$

$$\leq \left(\vartheta \sum_{j=1}^{k} 2^{j(\alpha-1)} + \vartheta \right) \|x\|^{\alpha} = \vartheta \|x\|^{\alpha} \sum_{j=0}^{k} 2^{j(\alpha-1)}.$$

We have just shown the validity of inequality (3.1) for every $n \in \mathbb{N}$. Since the geometric series $\sum_{l=0}^{\infty} 2^{l(\alpha-1)} = \frac{2}{2-2^{\alpha}}$, $(0 \leq \alpha < 1)$, we obtain that

$$\frac{\left\| \frac{F(2^n|x|)}{2^n} - F(|x|) \right\|}{\|x\|^{\alpha}} \leq \frac{2\vartheta}{2-2^{\alpha}}, \ 0 \leq \alpha < 1. \tag{3.2}$$

Next, note that for all $m > n > 0$ and making the change of variable $y = 2^n x$ we have

$$\left\| 2^{-m} F\left(2^m|x|\right) - 2^{-n} F\left(2^n|x|\right) \right\| = 2^{-n} \left\| 2^{-m+n} F\left(2^m|x|\right) - F\left(2^n|x|\right) \right\| =$$

$$= 2^{-n} \left\| 2^{-m+n} F\left(2^{m-n}|y|\right) - F(|y|) \right\| \leq 2^{-n} \frac{2\vartheta}{2-2^{\alpha}} \|y\|^{\alpha} =$$

$$= 2^{-n(1-\alpha)} \frac{2\vartheta}{2-2^{\alpha}} \|x\|^{\alpha}.$$

Consequently, passing to the limit yields,

$$\lim_{n\to\infty} \left\| 2^{-m} F\left(2^m|x|\right) - 2^{-n} F\left(2^n|x|\right) \right\| = 0.$$

Since $\mathcal{Y}$ is a Banach space, we can thus conclude that the sequence $\left(\frac{F(2^n|x|)}{2^n} \right) \subset \mathcal{Y}^+$ converges in the $\mathcal{Y}$-norm. Now, define the mapping $T : \mathcal{X} \to \mathcal{Y}$ by

$$T(|x|) := \lim_{n\to\infty} \frac{F(2^n|x|)}{2^n} \tag{3.3}$$

Clearly, T is a cone-related operator. Let us show that T is maximum preserving. In fact, letting $\tau = \eta = 2^n$ in (2.2) yields

$$\left\| F\left(2^n(|x| \vee |y|)\right) - 2^n\left(F(|x|) \vee F(|y|)\right) \right\| \leq \vartheta\left(\|x\|^{\alpha} + \|y\|^{\alpha}\right).$$

Substituting x with $2^n x$ and y with $2^n y$ in this last inequality one can get

$$\left\| F\left(4^n(|x| \vee |y|)\right) - 2^n\left(F\left(2^n|x|\right) \vee F\left(2^n|y|\right)\right) \right\| \leq 2^{n\alpha}\vartheta\left(\|x\|^{\alpha} + \|y\|^{\alpha}\right)$$

which yields

$$4^{-n} \left\| F\left(4^n(|x| \vee |y|)\right) - 2^n\left(F\left(2^n|x|\right) \vee F\left(2^n|y|\right)\right) \right\| =$$

$$= \left\| 4^{-n} F\left(4^n(|x| \vee |y|)\right) - 2^{-n}\left(F\left(2^n|x|\right) \vee F\left(2^n|y|\right)\right) \right\| \leq 2^{n(\alpha-2)}\vartheta\left(\|x\|^{\alpha} + \|y\|^{\alpha}\right).$$

Consequently, passing to the limit yields

$$\|T(|x|\vee|y|)-T(|x|)\vee T(|y|)\|=0,$$

$x,y\in\mathcal{X}$ or equivalently

$$T(|x|\vee|y|)=T(|x|)\vee T(|y|),$$

$x,y\in\mathcal{X}$, because

$$\lim_{n\to\infty}4^{-n}F\left(4^n|z|\right)=\lim_{m\to\infty}2^{-m}F\left(2^m|z|\right),\ z\in\mathcal{X}.$$

Next, we show the validity of the identity $T(\alpha|x|)=\alpha T(|x|)$ for all $x\in\mathcal{X}$ and every number $\alpha\in[0,\infty)$. In fact, in inequality (2.2) choose $\eta=\tau$, $y=0$ and substitute $2^n\tau$ for τ to observe via Remark 1/(2) that

$$\left\|F\left(2^n\tau|x|\right)-2^n\tau F(|x|)\right\|\leq\vartheta\|x\|^{\alpha}.$$

For all $x\in\mathcal{X}$ and every number $\alpha\in[0,\infty)$. This inequality can be transformed as

$$\left\|F\left(4^n\tau|x|\right)-2^n\tau F\left(2^n|x|\right)\right\|\leq\vartheta 2^{n\alpha}\|x\|^{\alpha}$$

if we replace x with 2^nx. Consequently,

$$\left\|\frac{F(4^n\tau|x|)}{4^n}-\tau\frac{F(2^n|x|)}{2^n}\right\|\leq\vartheta 2^{-n(2-\alpha)}\|x\|^{\alpha}.$$

Hence on the one hand,

$$\lim_{n\to\infty}\frac{F(4^n\tau|x|)}{4^n}=\tau\lim_{n\to\infty}\frac{F(2^n|x|)}{2^n}=\tau T(|x|)$$

and on the other hand by changing the variable $z=\tau x$

$$\lim_{n\to\infty}\frac{F(4^n\tau|x|)}{4^n}=\lim_{n\to\infty}\frac{F(4^n|z|)}{4^n}=T(|z|)=T(\tau|x|).$$

Therefore, the semi-homogeneity holds true. In the next step taking the limit in (3.2) yields (2.3). Further, let us show the unicity. In fact, assume the existence of another such cone-related functional G such that $\mathcal{S}:=\{x\in\mathcal{X}:G(|x|)\neq T(|x|)\}\neq\varnothing$. Then (2.3) implies the existence of some $\vartheta_0>0$ and $\beta\in[0,1)$ such that for each $x\in\mathcal{S}$,

$$\|G(|x|)-T(|x|)\|\leq\vartheta_0\|x\|^{\beta}. \tag{3.4}$$

One can easily deduce from the semi-homogeneity of G and T that $kx\in\mathcal{S}$ for every $k\in\mathbb{N}$ whenever $x\in\mathcal{S}$. Taking into account (3.4) one can easily deduce by the triangle inequality and the semi-homogeneity that

$$\|G(|x|)-T(|x|)\|=n^{-1}\|G(|nx|)-T(|nx|)\|\leq n^{\alpha-1}\vartheta\|x\|^{\alpha}+n^{\beta-1}\vartheta_0\|x\|^{\beta}$$

which would imply in the limit that $\|G(|x|)-T(|x|)\|=0$, or equivalently $G(|x|)=T(|x|)$. This would mean that $x\in\mathcal{S}$ implies $x\notin\mathcal{S}$, or equivalently $\mathcal{S}\cap\mathcal{S}=\varnothing$, i.e. $\mathcal{S}=\varnothing$, a contradiction, indeed. Finally, let us prove the moreover-part. In fact, assuming that F satisfies property **P4**, pick arbitrarily an increasing sequence

$(|x_k|) \subset \mathcal{X}^+$ with limit $|x| \in \mathcal{X}^+$. Then by (3.3), the monotonicity of T and the continuity from below of F we have

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{F\left(2^n|x_k|\right)}{2^n} = \lim_{k\to\infty} T\left(|x_k|\right) \leq T\left(|x|\right) = \lim_{n\to\infty}\frac{F\left(2^n|x|\right)}{2^n} = \\ = \lim_{n\to\infty}\lim_{k\to\infty}\frac{F\left(2^n|x_k|\right)}{2^n}.$$

Thus

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{F\left(2^n|x_k|\right)}{2^n} \leq T\left(|x|\right) = \lim_{n\to\infty}\lim_{k\to\infty}\frac{F\left(2^n|x_k|\right)}{2^n}. \tag{3.5}$$

By the conjunction of both inequalities (2.1) and (3.5), it follows that operator T is continuous from below. To end the proof of the moreover-part we simply note that the reverse conditional is trivial. Therefore, we can conclude on the validity of the argument. □

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THE MATRIX VERSION FOR THE MULTIVARIABLE HUMBERT POLYNOMIALS

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Abstract. In this paper, the matrix extension of the multivariable Humbert polynomials is introduced. Various families of linear, multilinear and multilateral generating matrix functions of these matrix polynomials are presented. Miscellaneous applications are also discussed.

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1. INTRODUCTION

It is well-known that special matrix functions appear in the study of many areas. Generalization of the property of orthogonality [11, 12], Rodrigues formula [6, 10], a second-order Sturm-Liouville differential equation [10], a three-term matrix recurrence [6, 7], relation between different orthogonal matrix polynomials [21] are theoretical examples. Statistics, group representation theory [17], scattering theory [15], differential equations [18, 19], Fourier series expansions [9], interpolation and quadrature [22, 23], splines [8], and medical imaging [5] are areas of application of orthogonal matrix polynomials.

Throughout this paper, for a matrix $A \in \mathbb{C}^{N\times N}$, its spectrum is denoted by $\sigma(A)$. The two-norm of A, which will be denoted by $\|A\|$, is defined by

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

where, for a vector $y \in \mathbb{C}^N$, $\|y\|_2 = \left(y^T y\right)^{1/2}$ is the Euclidean norm of y. I and θ will denote the identity matrix and the null matrix in $\mathbb{C}^{N\times N}$, respectively. We say that a matrix A in $\mathbb{C}^{N\times N}$ is a positive stable if $\Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$, where $\sigma(A)$ is the set of the eigenvalues of A. If $A_0, A_1, ..., A_n$ are elements of $\mathbb{C}^{N\times N}$ and $A_n \neq \theta$, then we call

$$P(x) = A_n x^n + A_{n-1}x^{n-1} + ... + A_1 x + A_0$$

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a matrix polynomial of degree n in x. From [20], one can see

$$(P)_n = P(P+I)(P+2I)...(P+(n-1)I);\ n \geq 1;\ (P)_0 = I. \tag{1.1}$$

For any matrix A in $\mathbb{C}^{N\times N}$, the authors exploited the following relation due to [20]

$$(1-x)^{-A} = \sum_{n=0}^{\infty} \frac{(A)_n}{n!} x^n\ , |x| < 1. \tag{1.2}$$

Gould [16] presented a systematic study of an interesting generalization of the Humbert and the Gegenbauer polynomials and several other polynomial systems, that is called generalized Humbert polynomials and defined by

$$\left(C - mxt + yt^m\right)^p = \sum_{n=0}^{\infty} P_n(m,x,y,p,C)\, t^n \tag{1.3}$$

where m is a positive integer and the other parameters are unrestricted (see also [26, p. 77, 86]).

Aktas *et al.* [1] present a systematic investigation of a multivariable extension of the Humbert polynomials generated by

$$\prod_{i=1}^{r} \left\{(C_i - m_i x_i t + y_i t^{m_i})^{-\alpha_i}\right\} = \sum_{n=0}^{\infty} P_n^{(\alpha_1,...,\alpha_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})t^n$$
$$(|m_i x_i t - y_i t^{m_i}| < |C_i|\ \ ;\ i = 1,2,...,r) \tag{1.4}$$

where $\mathbf{x} = (x_1,...,x_r)$, $\mathbf{y} = (y_1,...,y_r)$, $\mathbf{C} = (C_1,...,C_r)$, $\mathbf{m} = (m_1,...,m_r)$, $m_i = 1,2,...$ $(i = 1,2,...,r)$ and the other parameters are unrestricted.

The main objective of this paper is to construct a matrix version of the multivariable Humbert polynomials given by (1.4) and the derivation of various families of multilinear and mixed multilateral generating matrix functions for these matrix polynomials. We present some special cases of our results and also obtain several recurrence relations for these matrix polynomials.

2. Matrix extension of the multivariable Humbert polynomials

The main object of this section is to present a systematic investigation of the matrix extension of the multivariable Humbert polynomials generated by

$$\prod_{i=1}^{r} \left\{(C_i - m_i x_i t + y_i t^{m_i})^{-A_i}\right\} = \sum_{n=0}^{\infty} P_n^{(A_1,...,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})t^n$$
$$(|m_i x_i t - y_i t^{m_i}| < |C_i|\ \ ;\ i = 1,2,...,r) \tag{2.1}$$

where $A_i \in \mathbb{C}^{N\times N}$, $\mathbf{x} = (x_1,...,x_r)$, $\mathbf{y} = (y_1,...,y_r)$, $\mathbf{C} = (C_1,...,C_r)$, $\mathbf{m} = (m_1,...,m_r)$, $m_i = 1,2,...$ $(i = 1,2,...,r)$ and the other parameters are unrestricted.

(2.1) yields the following explicit representation:

$$\begin{aligned}
&P_n^{(A_1,\ldots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C}) \\
&= \sum_{m_1k_1+\ldots+m_rk_r+n_1+\ldots+n_r=n} \\
&\frac{(A_1)_{n_1+k_1}\,C_1^{-A_1-(n_1+k_1)I}\ldots(A_r)_{n_r+k_r}\,C_r^{-A_r-(n_r+k_r)I}}{n_1!\ldots n_r!k_1!\ldots k_r!} \\
&\times m_1^{n_1}\ldots m_r^{n_r}\,(-1)^{k_1+\ldots+k_r}\,x_1^{n_1}\ldots x_r^{n_r}\,y_1^{k_1}\ldots y_r^{k_r} \\
&= \sum_{m_1k_1+\ldots+m_rk_r+n_1+\ldots+n_r=n} \prod_{p=1}^{r} \\
&\left\{\frac{(A_p)_{n_p+k_p}\,C_p^{-A_p-(n_p+k_p)I}}{n_p!k_p!}\,m_p^{n_p}\,(-1)^{k_p}\,x_p^{n_p}\,y_p^{k_p}\right\}
\end{aligned} \tag{2.2}$$

where, as usual, $(A)_n$ denotes the Pochhammer symbol given by (1.1).

We notice that the case $r=1$ in (2.1) reduces to the matrix version of the generalized Humbert polynomials introduced by Gould [16]. In this case, it is generated by

$$\left(C-mxt+yt^m\right)^{-A}=\sum_{n=0}^{\infty}P_n^{(A)}(m,x,y,C)t^n \tag{2.3}$$

where $|mxt-yt^m|<|C|$, $A\in\mathbb{C}^{N\times N}$, m is a positive integer and the other parameters are unrestricted. For the special cases of (2.3), including Gegenbauer matrix polynomials, we refer [19].

It is clear that the case

$$C_i=1,\ m_i=1,\ y_i=0\ ,\ i=1,2,\ldots,r$$

of the polynomials of (2.1) reduces to matrix version of the Chan-Chyan-Srivastava multivariable polynomials, which is generated by [14]

$$\begin{aligned}
&\prod_{i=1}^{r}\left\{(1-x_it)^{-A_i}\right\}=\sum_{n=0}^{\infty}g_n^{(A_1,\ldots,A_r)}(x_1,\ldots,x_r)\,t^n \\
&\left(A_i\in\mathbb{C}^{N\times N}\ (i=1,2,\ldots,r)\ ;\ |t|<\min\left\{|x_1|^{-1},\ldots,|x_r|^{-1}\right\}\right).
\end{aligned} \tag{2.4}$$

Since $A_i=\alpha_i\in\mathbb{C}$ for $N=1$ in (2.4), we obtained the generating function of the Chan-Chyan-Srivastava multivariable polynomials [3].

On the other hand, if we choose $C_i=1$, $m_i=i$, $x_i=0$, $y_i=-x_i$, $i=1,2,\ldots,r$ in (2.1), we get a matrix version of the multivariable Lagrange-Hermite

polynomials, which is generated by [14]

$$\prod_{i=1}^{r}\left\{\left(1-x_i t^i\right)^{-A_i}\right\}=\sum_{n=0}^{\infty} h_n^{(A_1,\ldots,A_r)}(x_1,\ldots,x_r)\,t^n$$

$$\left(A_i\in\mathbb{C}^{N\times N}\ (i=1,2,\ldots,r)\ ;\ |t|<\min\left\{|x_1|^{-1},|x_2|^{-1/2},\ldots,|x_r|^{-1/r}\right\}\right). \tag{2.5}$$

Since $A_i=\alpha_i\in\mathbb{C}$ for $N=1$ in (2.5), we have the multivariable Lagrange-Hermite polynomials presented by Altın and Erkuş [2]. Furthermore, we should remark that the case $r=2$ of the polynomials corresponds to the familiar (two-variable) Lagrange-Hermite polynomials considered by Dattoli *et al.* [4].

Moreover, the special case

$$C_i=1,\ x_i=0,\ y_i=-x_i,\ \ i=1,2,\ldots,r$$

gives the matrix version of the Erkus-Srivastava multivariable polynomials generated by [14]

$$\begin{gathered}\prod_{i=1}^{r}\left\{(1-x_i t^{m_i})^{-A_i}\right\}=\sum_{n=0}^{\infty} u_n^{(A_1,\ldots,A_r)}(x_1,\ldots,x_r)\,t^n,\\ A_i\in\mathbb{C}^{N\times N}\ (i=1,2,\ldots,r),\\ |t|<\min\left\{|x_1|^{-1/m_1},|x_2|^{-1/m_2},\ldots,|x_r|^{-1/m_r}\right\}.\end{gathered} \tag{2.6}$$

Since $A_i=\alpha_i\in\mathbb{C}$ for $N=1$ in (2.6), we have the Erkus-Srivastava multivariable polynomials generated by [13].

3. An application of Srivastava's theorem on mixed generating functions

Srivastava [25] (see also the subsequent treatise on the subject by Srivastava and Manocha [26, p. 378, Theorem 12]) obtained a family of mixed generating functions for certain general multivariable and multiparameter sequences of functions. Our generating function (2.1) fits easily into the general setting of Srivastava's theorem. Thus, by applying this general result to the generating function (2.1), we obtain the

following family of mixed generating functions for matrix version of the multivariable Humbert polynomials given by (2.1)

$$\sum_{n=0}^{\infty} P_n^{(A_1+\lambda_1 nI,\ldots,A_r+\lambda_r nI)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})t^n = \frac{\prod_{i=1}^{r}\left\{(C_i - m_i x_i \xi + y_i \xi^{m_i})^{-A_i}\right\}}{1-\xi\left\{\sum_{i=1}^{r}\lambda_i m_i (C_i + y_i\xi^{m_i})^{-1}\left[-y_i\xi^{m_i-1} + x_i\frac{(C_i - m_i y_i \xi^{m_i} + y_i\xi^{m_i})}{(C_i - m_i x_i\xi + y_i\xi^{m_i})}\right]\right\}}, \tag{3.1}$$

$$\left(\xi = \xi(t) := t\prod_{i=1}^{r}\left\{\left(C_i - m_i x_i \xi + y_i \xi^{m_i}\right)^{-\lambda_i}\right\}\ ;\ \lambda_i \in \mathbb{C},\right.$$
$$\left. C_i \neq 0 \text{ and } A_i \in \mathbb{C}^{N\times N}\ \ (i = 1,\ldots,r)\right)$$

where all of the matrices commute with each other. In a special case, it is easily seen that (3.1) would at once reduce to the generating function (2.1) when $\lambda_i = 0\ (i = 1,\ldots,r)$. For the special case of $N = 1$, (3.1) gives mixed generating function for the multivariable Humbert polynomials given by [1]. Furthermore, the special case of $N = 1$ and $r = 1$ of (3.1) reduces to the mixed generating function for the generalized Humbert polynomials in [24].

4. Bilinear and bilateral generating matrix functions

In this section, we derive several families of bilinear and bilateral generating matrix functions for matrix version of the multivariable Humbert polynomials which are generated by (2.1) and given explicitly by (2.2).

We begin by stating the following theorem.

Theorem 1. *Corresponding to an identically non-vanishing function $\Omega_\mu(\mathbf{z})$ of s complex variables $z_1,\ldots,z_s$ $(s \in \mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\nu}(\mathbf{z};w) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(\mathbf{z})w^k \tag{4.1}$$

where $(a_k \neq 0,\ \mu,\nu \in \mathbb{C})$; $\mathbf{z} = (z_1,\ldots,z_s)$ and

$$\Theta_{n,p,\mu,\nu}(\mathbf{x},\mathbf{y};\mathbf{z};\zeta) := \sum_{k=0}^{[n/p]} a_k P_{n-pk}^{(A_1,\ldots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})\ \Omega_{\mu+\nu k}(\mathbf{z})\zeta^k \tag{4.2}$$

where $n,p\in\mathbb{N}$; $A_i\in\mathbb{C}^{N\times N}$; $\mathbf{x}=(x_1,...,x_r)$; $\mathbf{y}=(y_1,...,y_r)$;$\mathbf{C}=(C_1,...,C_r)$; $\mathbf{m}=(m_1,...,m_r)$, $m_i=1,2,...$ $(i=1,2,...,r)$. *Then we have*

$$\sum_{n=0}^{\infty}\Theta_{n,p,\mu,\nu}\left(\mathbf{x},\mathbf{y};\mathbf{z};\frac{\eta}{t^p}\right)t^n=\prod_{i=1}^{r}\left\{\left(C_i-m_ix_it+y_it^{m_i}\right)^{-A_i}\right\}\Lambda_{\mu,\nu}(\mathbf{z};\eta) \quad (4.3)$$

provided that each member of (4.3) exists.

Proof. Let T denote the left-hand side of the equality (4.3) of Theorem 1. Then, upon substituting the polynomials

$$\Theta_{n,p,\mu,\nu}\left(\mathbf{x},\mathbf{y};\mathbf{z};\frac{\eta}{t^p}\right)$$

from definition (4.2) into the left-hand side of (4.3), we find

$$T=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_kP_{n-pk}^{(A_1,...,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})\Omega_{\mu+\nu k}(\mathbf{z})\eta^kt^{n-pk}. \quad (4.4)$$

Replacing n by $n+pk$, we can write

$$\begin{aligned}T&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\,P_n^{(A_1,...,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})\Omega_{\mu+\nu k}(\mathbf{z})\eta^kt^n\\&=\sum_{n=0}^{\infty}P_n^{(A_1,...,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})t^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(\mathbf{z})\eta^k\\&=\prod_{i=1}^{r}\left\{\left(C_i-m_ix_it+y_it^{m_i}\right)^{-A_i}\right\}\Lambda_{\mu,\nu}(\mathbf{z};\eta),\end{aligned}$$

which completes the proof. □

In a similar manner, we can give the next result.

Theorem 2. *For a non-vanishing function* $\Omega_\mu(\mathbf{z})$ *of* s *complex variables* $z_1,\dots z_s$ $(s\in\mathbb{N})$ *and for* $p\in\mathbb{N}$, $\mu,\nu\in\mathbb{C}$, $\mathbf{z}=(z_1,...,z_s)$, $A:=(A_1,...,A_r)$, $B:=(B_1,...,B_r)$, $A_i,B_i\in\mathbb{C}^{N\times N}$ *for* $i=1,2,...,r$, *let*

$$\Xi_{\mu,\nu,\mathbf{C},\mathbf{m}}^{n,p}(\mathbf{x},\mathbf{y};\mathbf{z};w):=\sum_{k=0}^{[n/p]}a_kP_{n-pk}^{(A_1+B_1,...,A_r+B_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})\Omega_{\mu+\nu k}(\mathbf{z})w^k \quad (4.5)$$

where $a_k\neq0$; $n,k\in\mathbb{N}_0$; $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. *Then we have*

$$\begin{aligned}&\sum_{k=0}^{n}\sum_{l=0}^{[k/p]}a_lP_{n-k}^{(A_1,...,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})P_{k-pl}^{(B_1,...,B_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})\ \Omega_{\mu+\nu l}(\mathbf{z})w^l\\&=\Xi_{\mu,\nu,\mathbf{C},\mathbf{m}}^{n,p}(\mathbf{x},\mathbf{y};\mathbf{z};w)\end{aligned} \quad (4.6)$$

provided that each member of (4.6) exists where the matrices commute with each other.

5. Special cases and some further properties

It is possible to give many applications of the theorems obtained in the previous sections with the help of appropriate choices of the multivariable functions $\Omega_{\mu+\nu k}(\mathbf{z})$, $\mathbf{z}=(z_1,...,z_s)$, $k\in\mathbb{N}_0$, $s\in\mathbb{N}$. For example, if we set

$$s=r \quad \text{and} \quad \Omega_{\mu+\nu k}(\mathbf{z})=h_{\mu+\nu k}^{(B_1,...,B_r)}(\mathbf{z})$$

in Theorem 1, where the matrix version of the multivariable Lagrange-Hermite polynomials

$$h_n^{(B_1,...,B_r)}(\mathbf{x})$$

are generated by (2.5), then we obtain the following result which provides a class of bilateral generating matrix functions for the matrix version of the multivariable Lagrange-Hermite polynomials and for the matrix version of the multivariable Humbert polynomials given explicitly by (2.2).

Corollary 1. *If* $\Lambda_{\mu,\nu}(\mathbf{z};w):=\sum_{k=0}^{\infty}a_k h_{\mu+\nu k}^{(B_1,...,B_r)}(\mathbf{z})w^k$, $a_k\neq 0$, $\mu,\nu\in\mathbb{N}_0$, $\mathbf{z}=(z_1,...,z_r)$ *and*

$$\Theta_{n,p,\mu,\nu}(\mathbf{x},\mathbf{y};\mathbf{z};\zeta):=\sum_{k=0}^{[n/p]}a_k P_{n-pk}^{(A_1,...,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})h_{\mu+\nu k}^{(B_1,...,B_r)}(\mathbf{z})\zeta^k$$

where $n\in\mathbb{N}_0$; $p\in\mathbb{N}$; $A_i,B_i\in\mathbb{C}^{N\times N}$; $\mathbf{x}=(x_1,...,x_r)$; $\mathbf{y}=(y_1,...,y_r)$; $\mathbf{C}=(C_1,...,C_r)$;$\mathbf{m}=(m_1,...,m_r)$, $m_i=1,2,...$ $(i=1,2,...,r)$, *then*

$$\sum_{n=0}^{\infty}\Theta_{n,p,\mu,\nu}\left(\mathbf{x},\mathbf{y};\mathbf{z};\frac{\eta}{t^p}\right)t^n=\prod_{i=1}^{r}\left\{\left(C_i-m_i x_i t+y_i t^{m_i}\right)^{-A_i}\right\}\Lambda_{\mu,\nu}(\mathbf{z};\eta) \quad (5.1)$$

provided that each member of (5.1) exists.

Remark 1. Using the generating relation (2.5) for the matrix version of the multivariable Lagrange-Hermite polynomials and setting $a_k=1$, $\mu=0$, $\nu=1$, we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}P_{n-pk}^{(A_1,...,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})h_k^{(B_1,...,B_r)}(\mathbf{z})\,\eta^k t^{n-pk}$$
$$=\left(\prod_{i=1}^{r}\left(C_i-m_i x_i t+y_i t^{m_i}\right)^{-A_i}\right)\left(\prod_{i=1}^{r}(1-z_i\eta^i)^{-B_i}\right),$$

where

$$|\eta|<\min\left\{|z_1|^{-1},|z_2|^{-1/2},...,|z_r|^{-1/r}\right\},$$

$$|m_i x_i t - y_i t^{m_i}| < |C_i| \ ; \ i = 1,2,...,r.$$

Also, if we choose $s = 2r$ and $\Omega_{\mu+\nu k}(\mathbf{z}) = P_{\mu+\nu k}^{(E_1,...,E_r)}(\mathbf{m},\mathbf{t},\omega,\mathbf{C})$, $\mu,\nu \in \mathbb{N}_0$, $\mathbf{t} = (t_1,...,t_r)$, $\omega = (\omega_1,...,\omega_r)$ in Theorem 2, we obtain the following class of bilinear generating matrix functions for the matrix version of the multivariable Humbert polynomials given explicitly by (2.2).

Corollary 2. *If*

$$\begin{aligned}&\Lambda_{\mu,\nu,\mathbf{C},\mathbf{m}}^{n,p}(\mathbf{x},\mathbf{y};\mathbf{t},\omega;w)\\ &:= \sum_{k=0}^{[n/p]} a_k P_{n-pk}^{(A_1+B_1,...,A_r+B_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C}) P_{\mu+\nu k}^{(E_1,...,E_r)}(\mathbf{m},\mathbf{t},\omega,\mathbf{C}) w^k\end{aligned}$$

$(a_k \neq 0;\ p \in \mathbb{N}; n,k,\mu,\nu \in \mathbb{N}_0)$

where $A_i, B_i, E_i \in \mathbb{C}^{N\times N}$ *for* $i = 1,2,...,r$, *then*

$$\begin{aligned}&\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l P_{n-k}^{(A_1,...,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})\\ &P_{k-pl}^{(B_1,...,B_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C}) P_{\mu+\nu l}^{(E_1,...,E_r)}(\mathbf{m},\mathbf{t},\omega,\mathbf{C}) w^l\\ &= \Lambda_{\mu,\nu,\mathbf{C},\mathbf{m}}^{n,p}(\mathbf{x},\mathbf{y};\mathbf{t},\omega;w)\end{aligned} \tag{5.2}$$

provided that each member of (5.2) exists where $A_i B_j = B_j A_i$ *for* $i,j = 1,2,...,r$.

For example, if we set

$$s = 1 \ \text{ and } \ \Omega_{\mu+\nu k}(y) = L_{\mu+\nu k}^{(E,\lambda)}(y)$$

in Theorem 1, where the nth Laguerre matrix polynomials $L_n^{(E,\lambda)}(x)$ are defined by [18]

$$L_n^{(E,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k \lambda^k}{k!\,(n-k)!} (E+I)_n \left[(E+I)_k\right]^{-1} x^k,$$

where E is a matrix in $\mathbb{C}^{N\times N}$, $E + nI$ is invertible for every integer $n \geq 0$ and λ is a complex number with $\Re(\lambda) > 0$ and they are generated by

$$\sum_{n=0}^{\infty} L_n^{(E,\lambda)}(x) t^n = (1-t)^{-(E+I)} \exp\left(\frac{-\lambda x t}{1-t}\right), \tag{5.3}$$

$$|t| < 1,\ 0 < x < \infty,$$

then we obtain the following result which provides a class of bilateral generating matrix functions for the matrix version of the multivariable Humbert and Laguerre matrix polynomials.

Corollary 3. *If* $\Lambda_{\mu,\nu}(z;w) := \sum_{k=0}^{\infty} a_k L_{\mu+\nu k}^{(E,\lambda)}(z)w^k$ *where* $(a_k \neq 0,\ \mu,\nu \in \mathbb{N}_0)$; *and*

$$\Theta_{n,p,\mu,\nu}(\mathbf{x},\mathbf{y};z;\zeta) := \sum_{k=0}^{[n/p]} a_k P_{n-pk}^{(A_1,\ldots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C}) L_{\mu+\nu k}^{(E,\lambda)}(z)\zeta^k$$

where $n, p \in \mathbb{N}$. *Then we have*

$$\sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu}\left(\mathbf{x};\mathbf{y};z;\frac{\eta}{t^p}\right) t^n = \prod_{i=1}^{r} \left\{\left(C_i - m_i x_i t + y_i t^{m_i}\right)^{-A_i}\right\} \Lambda_{\mu,\nu}(z;\eta) \quad (5.4)$$

provided that each member of (5.4) exists.

Remark 2. Using the generating relation (5.3) for the Laguerre matrix polynomials and taking $a_k = 1$, $\mu = 0$, $\nu = 1$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} P_{n-pk}^{(A_1,\ldots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C}) L_k^{(E,\lambda)}(z)\eta^k t^{n-pk} \\
&= \prod_{i=1}^{r} \left\{\left(C_i - m_i x_i t + y_i t^{m_i}\right)^{-A_i}\right\} \\
&\quad \times (1-\eta)^{-(E+I)} \exp\left(\frac{-\lambda z\eta}{1-\eta}\right), \qquad (5.5)
\end{aligned}$$

where $|\eta| < 1,\ 0 < z < \infty$.

Remark 3. For $r = 1$ in (5.5), we have a bilateral generating matrix function of the Humbert (2.3) and Laguerre matrix polynomials:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} P_{n-pk}^{(A)}(m,x,y,C) L_k^{(E,\lambda)}(z)\eta^k t^{n-pk} \\
&= \left(C - mxt + yt^m\right)^{-A} (1-\eta)^{-(E+I)} \exp\left(\frac{-\lambda z\eta}{1-\eta}\right).
\end{aligned}$$

Remark 4. For $r = 1$ and $s = 2$ in Theorem 1, setting $\Omega_{\mu+\nu k}(z) = P_n^{(B)}(m,x,y,C)(B \in \mathbb{C}^{N\times N})$ and taking $a_k = 1$, $\mu = 0$, $\nu = 1$,we have bilinear generating matrix function for the Humbert matrix polynomials:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} P_{n-pk}^{(A)}(m,x,y,C) P_k^{(B)}(m,x,y,C)\eta^k t^{n-pk} \\
&= \left(C - mxt + yt^m\right)^{-A} \left(C - mx\eta + y\eta^m\right)^{-B}.
\end{aligned}$$

Furthermore, for every suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable function $\Omega_{\mu+\nu k}(\mathbf{y})$, $\mathbf{y} = (y_1, \ldots, y_s)$, $(s \in \mathbb{N})$, is expressed as an appropriate product of several simpler functions, then the assertions of Theorems 1 and 2 can be applied in order to derive various families of multilinear and multilateral generating matrix functions for the matrix version of the multivariable Humbert polynomials given explicitly by (2.2).

We now discuss some further properties of matrix version of the multivariable Humbert polynomials given by (2.2). First of all, the generating matrix relation (2.1) yields the following addition formula for these multivariable polynomials:

$$P_n^{(A_1+B_1,\ldots,A_r+B_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C}) = \sum_{k=0}^{n} P_{n-k}^{(A_1,\ldots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})\, P_k^{(B_1,\ldots,B_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C})$$

where $A_i, B_i \in \mathbb{C}^{N\times N}$, $A_i B_j = B_j A_i$ for $i, j = 1, 2, \ldots, r$.

On the other hand, the multivariable Humbert matrix polynomials satisfy the following equation:

$$\sum_{j=1}^{r} \left(x_j \frac{\partial}{\partial x_j} + m_j y_j \frac{\partial}{\partial y_j} \right) P_n^{(A_1,\ldots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C}) = n P_n^{(A_1,\ldots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C}). \tag{5.6}$$

If we differentiate each member of the generating function (2.1) with respect to x_j and y_j $(j = 1, 2, \ldots, r)$, we obtain the following (differential) recurrence relations for the matrix version of the multivariable Humbert polynomials:

$$\begin{aligned} &\frac{\partial}{\partial x_j} P_n^{(A_1,\ldots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C}) \\ &= \sum_{k=0}^{n-1} \sum_{l=0}^{[k/m_j]} \frac{(-1)^l (k+l-lm_j)! A_j (m_j)^{k-lm_j+1}}{(k-lm_j)! l! C_j^{k-l(m_j-1)+1}} x_j^{k-lm_j} y_j^l \\ &\quad \times P_{n-k-1}^{(A_1,\ldots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C}) \end{aligned} \tag{5.7}$$

for $n \geq 1$, and

$$\begin{aligned} &\frac{\partial}{\partial y_j} P_n^{(A_1,\ldots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C}) \\ &= -\sum_{k=0}^{n-m_j} \sum_{l=0}^{[k/m_j]} \frac{(-1)^l (k+l-lm_j)! A_j (m_j)^{k-lm_j}}{(k-lm_j)! l! C_j^{k-l(m_j-1)+1}} x_j^{k-lm_j} y_j^l \\ &\quad \times P_{n-k-m_j}^{(A_1,\ldots,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C}) \end{aligned} \tag{5.8}$$

where $n \geq m_j$ and m_j $(j = 1, 2, ..., r)$ is a positive integer and all matrices are commutative. By applying (5.6), (5.7) and (5.8), the following recurrence relation for the matrix polynomials (given explicitly by (2.2)) can be easily derived:

$$\begin{aligned}
&\sum_{j=1}^{r}\sum_{k=0}^{n-1}\sum_{l=0}^{[k/m_j]} \frac{(-1)^l \left(k+l-lm_j\right)! A_j \left(m_j\right)^{k-lm_j+1}}{\left(k-lm_j\right) l! C_j^{k-l(m_j-1)+1}} \\
&x_j^{k-lm_j+1} y_j^l P_{n-k-1}^{(A_1,...,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C}) \\
&-\sum_{j=1}^{r}\sum_{k=0}^{n-m_j}\sum_{l=0}^{[k/m_j]} \frac{(-1)^l \left(k+l-lm_j\right)! A_j \left(m_j\right)^{k-lm_j+1}}{\left(k-lm_j\right)! l! C_j^{k-l(m_j-1)+1}} \\
&x_j^{k-lm_j} y_j^{l+1} P_{n-k-m_j}^{(A_1,...,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C}) \\
&= n P_n^{(A_1,...,A_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{C}), \text{ for } n \geq m_j
\end{aligned}$$

where m_j $(j = 1, 2, ..., r)$ is a positive integer and all matrices commute.

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DIFFERENTIAL SUBORDINATION FOR A CERTAIN GENERALIZED OPERATOR

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Abstract. The authors have recently introduced a new generalized derivative operator $\mu^{n,m}_{\lambda_1,\lambda_2}$, that generalized many well-known operators. The trend of finding new differential or integral operators has attracted widespread interest. The aim of this paper is to use the relation

$$(1+n)\,\mu^{n+1,m}_{\lambda_1,\lambda_2} f(z) = \left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' + n\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)$$

to discuss some interesting results by using the technique of differential subordination. The results include both subordination and inclusion. In the case of $n = 0, \lambda_2 = 0$, we obtain the results of Oros [11].

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1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let S, $S^*(\alpha)$, $C(\alpha)$ $(0 \le \alpha < 1)$ denote the subclasses of $\mathcal{A}$ consisting of functions that are univalent, starlike of order α and convex of order α in U, respectively. In particular, the classes $S^*(0) = S^*$ and $C(0) = C$ are the familiar classes of starlike and convex functions in U, respectively. And a function $f \in C(\alpha)$ if $\mathrm{Re}(1 + \frac{zf''}{f'}) > \alpha$. Furthermore a function f analytic in U is said to be convex if it is univalent and $f(U)$ is convex.

Let $\mathcal{H}(U)$ be the class of holomorphic functions in unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Consider

$$\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots, \ (z \in U)\}, \text{ with } \mathcal{A}_1 = \mathcal{A}.$$

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For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1,2,3,\dots,\}$ we let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \dots, \quad (z \in U)\}.$$

Given two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, the Hadamard product (or convolution) $f * g$ is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k .$$

Next, we state the basic ideas on subordination. If f and g are analytic in U, then the function f is said to be subordinate to g, written as

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in U),$$

if and only if there exists the Schwarz function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)) \ (z \in U)$.

Furthermore if g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$. [see [14], p.36].

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \quad (z \in U), \tag{1.2}$$

then p is called a solution of the differential subordination.

The univalent function q is called a dominant of the solutions of the differential subordination, or simply a dominant, if $p \prec q$ for all p satisfying (1.2).

A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.2) is said to be the best dominant of (1.2). (Note that the best dominant is unique up to a rotation of U).

Now, $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_k = \begin{cases} 1 & \text{for } k = 0, x \in \mathbb{C}\backslash\{0\}, \\ x(x+1)(x+2)\dots(x+k-1) & \text{for } k \in \mathbb{N} = \{1,2,3,\dots\} \text{and } x \in \mathbb{C}. \end{cases}$$

In [1], the authors introduced and studied the generalized derivative operator $\mu^{n,m}_{\lambda_1,\lambda_2} f(z)$ given by the following definition.

Definition 1. For $f \in \mathcal{A}$ the generalized derivative operator $\mu^{n,m}_{\lambda_1,\lambda_2}$ is defined by $\mu^{n,m}_{\lambda_1,\lambda_2} : \mathcal{A} \to \mathcal{A}$,

$$\mu^{n,m}_{\lambda_1,\lambda_2} f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^m}{(1+\lambda_2(k-1))^{m-1}} c(n,k) a_k z^k, \quad (z \in U),$$

where $n, m \in \mathbb{N}_0 = \{0,1,2\dots\}$, $\lambda_2 \geq \lambda_1 \geq 0$ and $c(n,k) = \binom{n+k-1}{n} = \frac{(n+1)_{k-1}}{(1)_{k-1}}$.

Special cases of this operator includes the Ruscheweyh derivative operator in two cases when $\mu^{n,1}_{0,\lambda_2} \equiv R^n$ and $\mu^{n,0}_{\lambda_1,0} \equiv R^n$ [16], the Salagean derivative operator for $\mu^{0,m}_{1,0} \equiv S^n$ [17], the generalized Ruscheweyh derivative operator in the cases $\mu^{n,1}_{\lambda_1,\lambda_2} \equiv R^n_\lambda$ and $\mu^{n,0}_{\lambda_1,\lambda_2} \equiv R^n_\lambda$ [3], the generalized Salagean derivative operator introduced by Al-Oboudi $\mu^{0,m}_{\lambda_1,0} \equiv S^n_\beta$ [2], and the generalized Al-Shaqsi and Darus derivative operator $\mu^{n,m}_{\lambda_1,0} \equiv D^n_{\lambda,\beta}$ [5]. Now, let us recall the well known Carlson-Shaffer operator $L(a,c)$ [4] associated to the incomplete beta function $\phi(a,c;z)$, defined by

$$\begin{aligned} &L(a,c) : \mathcal{A} \to \mathcal{A}, \\ &L(a,c) f(z) := \phi(a,c;z) * f(z) \qquad (z \in U), \\ &\text{where } \phi(a,c;z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k. \end{aligned}$$

It can be easily seen that

$$\mu^{0,0}_{\lambda_1,0} f(z) = \mu^{1,0}_{0,\lambda_2} f(z) = f(z)$$

and

$$\mu^{1,0}_{\lambda_1,0} f(z) = \mu^{1,1}_{0,\lambda_2} f(z) = z f'(z).$$

Also $\mu^{a-1,0}_{\lambda_1,0} f(z) = \mu^{a-1,1}_{0,\lambda_2} f(z)$ where $a = 1,2,3,\dots$.

To prove our results, we need the following equality:

$$(1+n)\,\mu^{n+1,m}_{\lambda_1,\lambda_2} f(z) = z\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' + n\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right), \quad (z \in U) \tag{1.3}$$

where $n,m \in \mathbb{N}_0 = \{0,1,2\dots\}$ and $\lambda_2 \geq \lambda_1 \geq 0$.

In addition, we need the following lemmas to prove our main results:

Lemma 1 ([9],p.71)**.** *Let h be analytic, univalent, convex in U, with $h(0) = a$, $\gamma \neq 0$ and $Re\,\gamma \geq 0$. If $p \in \mathcal{H}[a,n]$ and*

$$p(z) + \frac{z p'(z)}{\gamma} \prec h(z), \quad (z \in U),$$

then

$$p(z) \prec q(z) \prec h(z), \quad (z \in U),$$

where $q(z) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t)\, t^{(\frac{\gamma}{n})-1} dt, \quad (z \in U)$.

The function q is convex and is the best (a,n)-dominant.

Lemma 2 ([8])**.** *Let g be a convex function in U and let*

$$h(z) = g(z) + n\alpha z g'(z),$$

where $\alpha > 0$ and n is a positive integer.

If

$$p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \dots ., \quad (z \in U),$$

is holomorphic in U and

$$p(z)+\alpha z p'(z) \prec h(z), \quad (z \in U),$$

then

$$p(z) \prec g(z)$$

and this result is sharp.

Lemma 3 ([10]). *Let $f \in \mathcal{A}$, if*

$$\operatorname{Re}\left(1+\frac{z f''(z)}{f'(z)}\right) > -\frac{1}{2},$$

then

$$\frac{2}{z}\int_0^z f(t)dt, \quad (z \in U \text{ and } z \neq 0),$$

is a convex function.

In the present paper, we shall use the method of differential subordination to derive certain properties of the generalized derivative operator $\mu^{n,m}_{\lambda_1,\lambda_2} f(z)$. Note that, differential subordination has been studied by various authors, and we follow the similar work of Oros [12] and Oros and Oros [13].

2. Main results

Before we state our first theorem, we give another definition.

Definition 2. For $n,m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$ and $0 \leq \alpha < 1$, we let $R^{n,m}_{\lambda_1,\lambda_2}(\alpha)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\operatorname{Re}\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' > \alpha, \quad (z \in U). \tag{2.1}$$

It is clear that the class $R^{0,1}_{\lambda_1,0}(\alpha) \equiv R(\lambda_1,\alpha)$ consists of functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}(\lambda_1 z f''(z) + f'(z)) > \alpha, \quad (z \in U),$$

studied by Ponnusamy [15] and many others.

Now we begin with our first result.

Theorem 1. *Let*

$$h(z) = \frac{1+(2\alpha-1)z}{1+z}, \quad (z \in U),$$

be convex in U, with h(0)=1 and $0 \leq \alpha < 1$. If $n,m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, and the differential subordination.

$$(\mu^{n+1,m}_{\lambda_1,\lambda_2} f(z))' \prec h(z), \qquad (z \in U), \tag{2.2}$$

then

$$\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' \prec q(z) = 2\alpha - 1 + \frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}},$$

where σ *is given by*

$$\sigma(x) = \int_0^z \frac{t^x}{1+t} dt, \quad (z \in U). \tag{2.3}$$

The function q *is convex and is the best dominant.*

Proof. By differentiating (1.3), with respect to z, we obtain

$$\left(\mu_{\lambda_1,\lambda_2}^{n+1,m} f(z)\right)' = \frac{(1+n)\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' + z\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)''}{1+n}. \tag{2.4}$$

Using (2.4) in (2.2), differential subordination (2.2) becomes

$$\begin{aligned} \frac{(1+n)\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' + z\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)''}{1+n} &\prec h(z) \\ &= \frac{1+(2\alpha-1)z}{1+z}. \end{aligned} \tag{2.5}$$

Let

$$\begin{aligned} p(z) = \left[\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right]' &= \left[z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^m}{(1+\lambda_2(k-1))^{m-1}} c(n,k) a_k z^k\right]' \\ &= 1 + p_1 z + p_2 z^2 + \ldots, \qquad (p \in \mathcal{H}[1,1],\ z \in U). \end{aligned} \tag{2.6}$$

Using (2.6) in (2.5), the differential subordination becomes:

$$p(z) + \frac{z p'(z)}{1+n} \prec h(z) = \frac{1+(2\alpha-1)z}{1+z}.$$

By using Lemma 1, we have

$$\begin{aligned} p(z) \prec q(z) =& \frac{(n+1)\int_0^z h(t)t^n\,dt}{z^{n+1}}, \\ =& \frac{(n+1)\int_0^z \left(\frac{1+(2\alpha-1)t}{1+t}\right)t^n\,dt}{z^{n+1}}, \\ =& \frac{(n+1)}{z^{n+1}}\left[\sigma(n)+(2\alpha-1)\int_0^z \frac{t^{n+1}}{1+t}dt\right], \\ =& 2\alpha-1+\frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}}, \end{aligned}$$

where σ is given by (2.3), so we get

$$\left[\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right]' \prec q(z) = 2\alpha-1+\frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}}.$$

The functions q is convex and is the best dominant. The proof is complete. □

Theorem 2. *If $n,m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$ and $0 \leq \alpha < 1$, then we have*

$$R_{\lambda_1,\lambda_2}^{n+1,m}(\alpha) \subset R_{\lambda_1,\lambda_2}^{n,m}(\delta)$$

where

$$\delta = 2\alpha-1+2(n+1)(1-\alpha)\sigma(n),$$

where σ is given by (2.3).

Proof. Let $f \in R_{\lambda_1,\lambda_2}^{n+1,m}(\alpha)$, then from (2.1) we have

$$\mathrm{Re}(\mu_{\lambda_1,\lambda_2}^{n+1,m} f(z))' > \alpha, \quad (z \in U),$$

which is equivalent to

$$(\mu_{\lambda_1,\lambda_2}^{n+1,m} f(z))' \prec h(z) = \frac{1+(2\alpha-1)z}{1+z}.$$

Using Theorem 1, we have

$$\left[\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right]' \prec q(z) = 2\alpha-1+\frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}}.$$

Since q is convex and $q(U)$ is symmetric with respect to the real axis, we deduce

$$\begin{aligned} \mathrm{Re}\left[\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right]' > \mathrm{Re}\, q(1) =& \delta = \delta(\alpha,\lambda_1) \\ =& 2\alpha-1+2(n+1)(1-\alpha)\sigma(n). \end{aligned}$$

From that we deduce $R^{n+1,m}_{\lambda_1,\lambda_2}(\alpha) \subset R^{n,m}_{\lambda_1,\lambda_2}(\delta)$. This completes the proof of Theorem 2. □

Theorem 3. *Let q be a convex function in U, with $q(0) = 1$ and let*

$$h(z) = q(z) + \lambda_1 z q'(z), \quad (z \in U).$$

If $n, m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $f \in \mathcal{A}$ and it satisfies the differential subordination

$$(\mu^{n+1,m}_{\lambda_1,\lambda_2} f(z))' \prec h(z), \quad (z \in U), \tag{2.7}$$

then

$$\left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right]' \prec q(z), \quad (z \in U),$$

and this result is sharp.

Proof. Let

$$p(z) = \left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)'.$$

Using (2.4), the differential subordination (2.7) becomes

$$p(z) + \frac{z p'(z)}{1+n} \prec h(z) = q(z) + \lambda_1 z q'(z), \quad (z \in U).$$

Using Lemma 2, we obtain

$$p(z) \prec q(z), \quad (z \in U).$$

Hence

$$\left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right]' \prec q(z), \quad (z \in U).$$

The result is sharp. This completes the proof of the theorem. □

We give a simple application for Theorem 3.

Example 1. For $n = 1$, $m = 0$, $\lambda_2 \geq \lambda_1 \geq 0$, $q(z) = \frac{1+z}{1-z}$, $f \in \mathcal{A}$ and $z \in U$ and applying Theorem 3, we have

$$h(z) = \frac{1+z}{1-z} + \lambda_1 z \left(\frac{1+z}{1-z}\right)' = \frac{1 + 2\lambda_1 z - z^2}{(1-z)^2}.$$

By using (2.4) we find

$$\begin{aligned}\left(\mu^{1,0}_{\lambda_1,\lambda_2}f(z)\right)' &= \left(\mu^{0,0}_{\lambda_1,\lambda_2}f(z)\right)' + z\left(\mu^{0,0}_{\lambda_1,\lambda_2}f(z)\right)'',\\ &= 1+\sum_{k=2}^{\infty}(1+\lambda_2(k-1))ka_kz^{k-1}\\ &\quad+\sum_{k=2}^{\infty}(1+\lambda_2(k-1))k(k-1)a_kz^{k-1}, \qquad (2.8)\\ &= 1+\sum_{k=2}^{\infty}(1+\lambda_2(k-1))k^2a_kz^{k-1},\\ &= \frac{f(z)*\left[z+\sum_{k=2}^{\infty}(1+\lambda_2(k-1))k^2z^k\right]}{z}.\end{aligned}$$

Similarly we compute $\left(\mu^{2,0}_{\lambda_1,\lambda_2}f(z)\right)'$. By using (2.4), we find

$$\left(\mu^{2,0}_{\lambda_1,\lambda_2}f(z)\right)' = \left(\mu^{1,0}_{\lambda_1,\lambda_2}f(z)\right)' + \frac{z}{2}\left(\mu^{1,0}_{\lambda_1,\lambda_2}f(z)\right)''. \qquad (2.9)$$

Then, by using (2.8) we have

$$\left(\mu^{1,0}_{\lambda_1,\lambda_2}f(z)\right)'' = \sum_{k=2}^{\infty}(1+\lambda_2(k-1))k^2(k-1)a_kz^{k-2}. \qquad (2.10)$$

After that, by (2.8) and (2.10), (2.9) becomes

$$\begin{aligned}\left(\mu^{2,0}_{\lambda_1,\lambda_2}f(z)\right)' &= 1+\sum_{k=2}^{\infty}(1+\lambda_2(k-1))k^2a_kz^{k-1}\\ &\quad+\frac{1}{2}\sum_{k=2}^{\infty}(1+\lambda_2(k-1))k^2(k-1)a_kz^{k-1},\\ &= 1+\sum_{k=2}^{\infty}(1+\lambda_2(k-1))k^2\frac{(k+1)}{2}a_kz^{k-1},\\ &= \frac{f(z)*\left[z+\sum_{k=2}^{\infty}\frac{1}{2}(1+\lambda_2(k-1))(1+k)k^2z^k\right]}{z}.\end{aligned}$$

From Theorem 3 we deduce that

$$\frac{f(z)*\left[z+\sum_{k=2}^{\infty}\frac{1}{2}\left(1+\lambda_2\left(k-1\right)\right)\left(1+k\right)k^2z^k\right]}{z}\prec\frac{1+2\lambda_1z-z^2}{\left(1-z\right)^2}$$

implies

$$\frac{f(z)*\left[z+\sum_{k=2}^{\infty}k^2\left(1+\lambda_2(k-1)\right)z^k\right]}{z}\prec\frac{1+z}{1-z},\quad,(z\in U).$$

Theorem 4. *Let q be a convex function in U, with $q(0)=1$ and let*

$$h(z)=q(z)+zq'(z),\quad(z\in U).$$

If $n,m\in\mathbb{N}_0$, $\lambda_2\geq\lambda_1\geq 0$, $f\in\mathcal{A}$ and satisfies the differential subordination

$$(\mu_{\lambda_1,\lambda_2}^{n,m}f(z))'\prec h(z),\tag{2.11}$$

then

$$\frac{\mu_{\lambda_1,\lambda_2}^{n,m}f(z)}{z}\prec q(z),\quad,(z\in U).$$

The result is sharp.

Proof.

$$\begin{aligned}p(z)=&\frac{\mu_{\lambda_1,\lambda_2}^{n,m}f(z)}{z},\\=&\frac{z+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1))^m}{(1+\lambda_2(k-1))^{m-1}}c(n,k)a_kz^k}{z},\\=&1+p_1z+p_2z^2+\ldots.,\qquad(p\in\mathcal{H}[1,1],\,z\in U).\end{aligned}\tag{2.12}$$

Differentiating (2.12), with respect to z, we obtain

$$\left(\mu_{\lambda_1,\lambda_2}^{n,m}f(z)\right)'=p(z)+zp'(z),\quad(z\in U).\tag{2.13}$$

Using (2.13), the differential subordination (2.11) becomes

$$p(z)+zp'(z)\prec h(z)=q(z)+zq'(z),$$

and by using Lemma 2, we deduce

$$p(z)\prec q(z),\quad(z\in U).$$

Next using (2.12), we have

$$\frac{\mu_{\lambda_1,\lambda_2}^{n,m}f(z)}{z}\prec q(z),\quad,(z\in U).$$

This proves Theorem 4. □

We give a simple application of Theorem 4.

Example 2. For $n=1, m=0, \lambda_2 \geq \lambda_1 \geq 0$, $q(z)=\frac{1}{1-z}$, $f \in \mathcal{A}$ and $z \in U$, by using Theorem 4, we obtain

$$h(z)=\frac{1}{1-z}+z\left(\frac{1}{1-z}\right)'=\frac{1}{(1-z)^2}.$$

From (1.3), we have

$$\begin{aligned}\left(\mu_{\lambda_1,\lambda_2}^{1,0} f(z)\right) &= z\left(\mu_{\lambda_1,\lambda_2}^{0,0} f(z)\right)', \\ &= z+\sum_{k=2}^{\infty}\left(1+\lambda_2(k-1)\right)k a_k z^k, \\ &= f(z)*\left[z+\sum_{k=2}^{\infty}\left(1+\lambda_2(k-1)\right)k z^k\right].\end{aligned}$$

From example 1, we have

$$\left(\mu_{\lambda_1,\lambda_2}^{1,0} f(z)\right)'=\frac{f(z)*\left[z+\sum_{k=2}^{\infty}\left(1+\lambda_2(k-1)\right)k^2 z^k\right]}{z}.$$

Now, applying Theorem 4, we deduce that

$$\frac{f(z)*\left[z+\sum_{k=2}^{\infty}\left(1+\lambda_2(k-1)\right)k^2 z^k\right]}{z} \prec \frac{1}{(1-z)^2}$$

implies

$$\frac{f(z)*\left[z+\sum_{k=2}^{\infty}\left(1+\lambda_2(k-1)\right)k z^k\right]}{z} \prec \frac{1}{1-z}.$$

Theorem 5. *Let*

$$h(z)=\frac{1+(2\alpha-1)z}{1+z}, \quad (z \in U)$$

be convex in U, with $h(0)=1$ and $0 \leq \alpha < 1$. If $n, m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $f \in \mathcal{A}$ and the differential subordination

$$(\mu_{\lambda_1,\lambda_2}^{n,m} f(z))' \prec h(z) \tag{2.14}$$

is satisfied, then

$$\frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z} \prec q(z) = 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}.$$

The function q is convex and is the best dominant.

Proof. Let

$$\begin{aligned} p(z) =& \frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z}, \qquad (2.15)\\ =& \frac{z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^m}{(1+\lambda_2(k-1))^{m-1}} c(n,k) a_k z^k}{z},\\ =& 1 + p_1 z + p_2 z^2 + \ldots., \qquad (p \in \mathcal{H}[1,1],\ z \in U). \end{aligned}$$

Differentiating (2.15), with respect to z, we obtain

$$\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' = p(z) + z p'(z), \quad (z \in U). \qquad (2.16)$$

Using (2.16), the differential subordination (2.14) becomes

$$p(z) + z p'(z) \prec h(z) = \frac{1 + (2\alpha - 1) z}{1 + z}, \quad (z \in U).$$

From Lemma 1, we deduce

$$\begin{aligned} p(z) \prec q(z) =& \frac{1}{z} \int_0^z h(t)\, dt,\\ =& \frac{1}{z} \int_0^z \left(\frac{1 + (2\alpha - 1) t}{1 + t} \right) dt,\\ =& \frac{1}{z} \left[\int_0^z \frac{1}{1+t} dt + (2\alpha - 1) \int_0^z \frac{t}{1+t} dt \right],\\ =& 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}. \end{aligned}$$

Using (2.15), we have

$$\frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z} \prec q(z) = 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}.$$

The proof is complete. □

From Theorem 5, we deduce the following Corollary:

Corollary 1. *If $f \in R^{n,m}_{\lambda_1,\lambda_2}(\alpha)$, then*

$$\operatorname{Re}\left(\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z}\right) > (2\alpha-1)+2(1-\alpha)\ln 2, \quad (z \in U).$$

Proof. Since $f \in R^{n,m}_{\lambda_1,\lambda_2}(\alpha)$, and from Definition 2 we have

$$\operatorname{Re}\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' > \alpha, \quad (z \in U),$$

which is equivalent to

$$(\mu^{n,m}_{\lambda_1,\lambda_2} f(z))' \prec h(z) = \frac{1+(2\alpha-1)z}{1+z}.$$

Using Theorem 5, we have

$$\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \prec q(z) = (2\alpha-1)+2(1-\alpha)\frac{\ln(1+z)}{z}.$$

Since q is convex and $q(U)$ is symmetric with respect to the real axis, we deduce

$$\operatorname{Re}\left(\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z}\right) > \operatorname{Re} q(1) = (2\alpha-1)+2(1-\alpha)\ln 2, \quad (z \in U).$$

□

Theorem 6. *Let $h \in \mathcal{H}(U)$, with $h(0)=1$, $h'(0) \neq 0$ and assume that it satisfies the inequality*

$$\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}, \quad (z \in U).$$

If $n,m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $f \in \mathcal{A}$ and it satisfies the differential subordination

$$\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' \prec h(z), \ , \quad (z \in U), \tag{2.17}$$

then

$$\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \prec q(z) = \frac{1}{z}\int_0^z h(t)dt.$$

Proof. Let

$$\begin{aligned} p(z) =& \frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z}, \\ =& \frac{z+\sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^m}{(1+\lambda_2(k-1))^{m-1}} c(n,k) a_k z^k}{z}, \\ =& 1+p_1 z+p_2 z^2+\ldots., \qquad (p \in \mathcal{H}[1,1],\ z \in U). \end{aligned} \tag{2.18}$$

Differentiating (2.18), with respect to z, we have

$$\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' = p(z) + z p'(z), \quad (z \in U). \tag{2.19}$$

Using (2.19), the differential subordination (2.17) becomes

$$p(z) + z p'(z) \prec h(z), \quad (z \in U).$$

From Lemma 1, we deduce

$$p(z) \prec q(z) = \frac{1}{z} \int_0^z h(t)\, dt.$$

With (2.18), we obtain

$$\frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z} \prec q(z) = \frac{1}{z} \int_0^z h(t) dt.$$

From Lemma 3, we obtain that the function q is convex, and from Lemma 1, q is the best dominant for the subordination (2.17). This completes the proof of Theorem 6. □

3. Conclusion

We remark that several subclasses of analytic univalent functions can be derived using the operator $\mu_{\lambda_1,\lambda_2}^{n,m}$. Several of their properties can be studied with this method, for example properties related to the ones that were studied in [7] and [6].

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SPLIT QUATERNION MATRICES

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Abstract. In this paper, we consider split quaternions and split quaternion matrices. Firstly, we give some properties of split quaternions. After that we investigate split quaternion matrices using properties of complex matrices. Then we define the complex adjoint matrix of split quaternion matrices and we describe some of their properties. Furthermore, we give the definition of q-determinant of split quaternion matrices.

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1. INTRODUCTION

Sir William Rowan Hamilton introduced the set of quaternions in 1843, which can be represented as

$$\mathbf{H} = \{a = a_0 + a_1 i + a_2 j + a_3 k : \quad a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

where $i^2 = j^2 = k^2 = -1$ and $ijk = -1$. One can see easily that $ij = -ji = k$. Thus the set of quaternions is a skew-field. Since quaternions are non-commutative, they differs from complex numbers ($\mathbb{C}$) and real numbers ($\mathbb{R}$). For example, for matrices of quaternions the left eigenvalues and the right eigenvalues must be not the same. Because of this, various authors have been studied quaternion matrices, for example [1], [4], [5] and [6]. After the work of Hamilton, in 1849, James Cockle introduced the set of split quaternions, which can be represented as

$$\mathbf{H_s} = \{q = q_0 + q_1 i + q_2 j + q_3 k : \quad q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

where $i^2 = -1$, $j^2 = k^2 = 1$ and $ijk = 1$. This set is 4-dimensional like the set of quaternions. $\mathbf{H_s}$ is noncommutative, too. But $\mathbf{H_s}$ contains zero-divisors, nilpotent elements and nontrivial idempotents. For more properties of split quatenions the reader is referred to [2] and [3].

In this paper, we will investigate split quaternion matrices. Firstly in Section 2, we will give some properties of split quaternions which will be used in the following sections. After that, in Section 3, we will introduce split quaternion matrices and give some properties of them. Finally in Section 4, we define the complex adjoint matrix

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of split quaternion matrices and we give some properties of these matrices. Then we will define the determinant of a split quaternion matrices using its adjoint matrix.

2. SPLIT QUATERNIONS

A split quaternion q is a vector of the form

$$q = a_0 + a_1 i + a_2 j + a_3 k$$

where a_0, a_1, a_2, a_3 are real numbers. Split quaternion algebra is an associative, non-commutative, non-division ring with four basis elements $\{1, i, j, k\}$ satisfying the equalities

$$i^2 = -1,\ j^2 = k^2 = 1,$$
$$ij = -ji = k,\ jk = -kj = -i,\ ki = -ik = j.$$

Let us denote the algebra of split quaternions by $\mathbf{H}_S$. For any $q = a_0 + a_1 i + a_2 j + a_3 k \in \mathbf{H}_S$ we define $\mathrm{Re}q = a_0$, the real part of q; $\mathrm{Im}q = a_1 i + a_2 j + a_3 k$, the imaginary part of q. The conjugate of a split quaternion is denoted by $q^* = \bar{q}$ and it is

$$q^* = \overline{q} = a_0 - a_1 i - a_2 j - a_3 k.$$

The norm of a split quaternion is defined as

$$\|q\| = \sqrt{|qq^*|} = \sqrt{|I_q|}$$

where $I_q = qq^* = a_0^2 + a_1^2 - a_2^2 - a_3^2$.If $\|q\| = 1$ then q is called unit split quaternion. Note that $p = \frac{q}{\|q\|}$ is a unit split quaternion for $q \in \mathbf{H}_S$ with $\|q\| \neq 0$. The inverse of the split quaternion q is

$$q^{-1} = \frac{q^*}{I_q}, \quad \text{if } I_q \neq 0.$$

Theorem 1. *Split quaternions satisfy the following properties for the elements p, q and $r \in \mathbf{H}_S$.*
(i) $q^* q = q q^*$,
(ii) $jc = \overline{c} j$ or $jcj^* = \overline{c}$ for any complex number c,
(iii) $q^2 = |\mathrm{Re}q|^2 - \|\mathrm{Im}q\|^2 + 2\mathrm{Re}q\,\mathrm{Im}q$,
(iv) $(qp)^* = p^* q^*$,
(v) $(pq)r = p(qr)$,
(vi) $pq \neq qp$ in general,
(vii) $q^* = q$ if and only if q is a real number,
$(viii)$ $\frac{q^*}{I_q}$ is the inverse of q if $a_0^2 + a_1^2 \neq a_2^2 + a_3^2$,
(ix) Every split quaternion q can be uniquely expressed as $q = c_1 + c_2 j$, where c_1 and c_2 are complex numbers.

Theorem 2. *Every split quaternion can be represented by a 2×2 complex matrix.*

Proof. Let $q \in \mathbf{H}_S$, then there exist complex numbers a_1 and a_2 such that $q = a_1 + a_2 j$ by Theorem 1(ix). The linear map $f_q : \mathbf{H}_S \to \mathbf{H}_S$ is defined by $f_q(p) = pq$ for all $p \in \mathbf{H}_S$. This map is bijective and

$$f_q(1) = 1(a_1 + a_2 j) = a_1 + a_2 j$$
$$f_q(j) = j(a_1 + a_2 j) = \overline{a_2} + \overline{a_1} j.$$

With this transformation split quaternions are defined as subset of the matrix ring $M_2(\mathbb{C})$, the set of 2×2 complex matrices:

$$\mathbf{H}_S' = \left\{ \begin{pmatrix} a_1 & a_2 \\ \overline{a_2} & \overline{a_1} \end{pmatrix};\ a_1, a_2 \in \mathbb{C} \right\}.$$

$\mathbf{H}_S$ and $\mathbf{H}_S'$ are essentially the same. □

Note that,

$$\mathcal{M} : q = a_1 + a_2 j \in \mathbf{H}_S \to q' = \begin{pmatrix} a_1 & a_2 \\ \overline{a_2} & \overline{a_1} \end{pmatrix} \in \mathbf{H}_S'$$

is bijective and preserves the operations. Furthermore $\|q\|^2 = \det q'$.

3. Split quaternion matrices

The set $M_{m\times n}(\mathbf{H}_S)$ denotes all $m \times n$ type matrices with split quaternion entries. If $m = n$, then the set of split quaternion matrices is denoted by $M_n(\mathbf{H}_S)$. The ordinary matrix addition and multiplication are defined. With these operations $M_n(\mathbf{H}_S)$ is a ring with a unit. The left (right) scalar multiplication is defined as, for $A = (a_{st}) \in M_{m\times n}(\mathbf{H}_S)$, $q \in \mathbf{H}_S$,

$$qA = (qa_{st}) \text{ and } Aq = (a_{st}q).$$

Here for $B \in M_{n\times o}(\mathbf{H}_S)$, $p \in \mathbf{H}_S$ we obtain

$$q(AB) = (qA)B,$$
$$(Aq)B = A(qB),$$
$$(pq)A = p(qA).$$

$M_{m\times n}(\mathbf{H}_S)$ is a left (right) vector space over $\mathbf{H}_S$.

For $A = (a_{st}) \in M_{m\times n}(\mathbf{H}_S)$, $\overline{A} = (\overline{a_{st}}) = \left(a_{st}^*\right) \in M_{m\times n}(\mathbf{H}_S)$ is the conjugate of A; $A^T = (a_{ts}) \in M_{n\times m}(\mathbf{H}_S)$ is the transpose of A; $A^* = \left(\overline{A}\right)^T \in M_{n\times m}(\mathbf{H}_S)$ is the conjugate transpose of A.

For a square matrix $A \in M_n(\mathbf{H}_S)$ if $AA^* = A^*A$ then A is called a *normal* matrix; if $A = A^*$ then A is called a *Hermitian* matrix; if $AA^* = I$ then A is called a *unitary* matrix; for $B \in M_n(\mathbf{H}_S)$ if $AB = BA = I$ then A is an invertible matrix.

Example 1. Let $A = \begin{pmatrix} i & j \\ 0 & k \end{pmatrix}$, $B = \begin{pmatrix} i & k \\ 0 & j \end{pmatrix} \in M_2(\mathbf{H}_S)$. Then,

(i) $A^{-1} = \begin{pmatrix} -i & 1 \\ 0 & k \end{pmatrix}$ and $\overline{A} = \begin{pmatrix} -i & -j \\ 0 & -k \end{pmatrix}$

(ii) $\left(\overline{A}\right)^{-1} = \begin{pmatrix} i & -1 \\ 0 & -k \end{pmatrix} \neq \begin{pmatrix} i & 1 \\ 0 & -k \end{pmatrix} = \overline{\left(A^{-1}\right)}$

(iii) $\left(A^T\right)^{-1} = \begin{pmatrix} -i & 0 \\ -1 & k \end{pmatrix} \neq \begin{pmatrix} -i & 0 \\ 1 & k \end{pmatrix} = \left(A^{-1}\right)^T$

(iv) $\overline{AB} = \begin{pmatrix} -1 & j+1 \\ 0 & -i \end{pmatrix} \neq \begin{pmatrix} -1 & -j+1 \\ 0 & i \end{pmatrix} = \overline{A}\,\overline{B}$

(v) $(AB)^T = \begin{pmatrix} -1 & 0 \\ -j+1 & i \end{pmatrix} \neq \begin{pmatrix} -1 & 0 \\ j+1 & -i \end{pmatrix} = B^T A^T$

With these examples we get the following Corollary for split quaternion matrices.

Corollary 1. *Let $A \in M_{m\times n}(\mathbf{H}_S)$ and $B \in M_{n\times o}(\mathbf{H}_S)$. Then the followings are satisfied:*

(i) $\left(\overline{A}\right)^{-1} \neq \overline{\left(A^{-1}\right)}$ *in general;*

(ii) $\left(A^T\right)^{-1} \neq \left(A^{-1}\right)^T$ *in general;*

(iii) $\overline{AB} \neq \overline{A}\,\overline{B}$ *in general;*

(iv) $(AB)^T \neq B^T A^T$ *in general.*

Theorem 3. *Let $A \in M_{m\times n}(\mathbf{H}_S)$ and $B \in M_{n\times o}(\mathbf{H}_S)$. Then,*

(i) $\left(\overline{A}\right)^T = \overline{\left(A^T\right)}$,

(ii) $(AB)^* = B^* A^*$,

(iii) $(AB)^{-1} = B^{-1}A^{-1}$ *if A and B are invertible,*

(iv) $(A^*)^{-1} = \left(A^{-1}\right)^*$ *if A is invertible.*

Proof. (i),(iii) and (iv) can be easily shown. Now we will prove (ii).

(ii) Let $A = A_1 + A_2 j$ and $B = B_1 + B_2 j$, where A_1, A_2, B_1 and B_2 are complex matrices. Then

$$\begin{aligned}(AB)^* &= \left[\left(A_1B_1 + A_2\overline{B_2}\right) + \left(A_1B_2 + A_2\overline{B_1}\right)j\right]^* \\ &= \left[\left(\overline{A_1B_1} + \overline{A_2}B_2\right) - \left(A_1B_2 + A_2\overline{B_1}\right)j\right]^T \\ &= \left(\overline{A_1B_1} + \overline{A_2}B_2\right)^T - \left(A_1B_2 + A_2\overline{B_1}\right)^T j \\ &= \left(\overline{B_1}\right)^T \left(\overline{A_1}\right)^T + (B_2)^T \left(\overline{A_2}\right)^T - (B_2)^T (A_1)^T j - \left(\overline{B_1}\right)^T (A_2)^T j.\end{aligned}$$

On the other hand, for

$$B^* = \left(\overline{B_1}\right)^T - (B_2)^T j \text{ and } A^* = \left(\overline{A_1}\right)^T - (A_2)^T j$$

we can obtain

$$B^*A^* = \left(\left(\overline{B_1}\right)^T - (B_2)^T j\right)\left(\left(\overline{A_1}\right)^T - (A_2)^T j\right)$$
$$= \left(\overline{B_1}\right)^T \left(\overline{A_1}\right)^T + (B_2)^T \left(\overline{A_2}\right)^T - (B_2)^T (A_1)^T j - \left(\overline{B_1}\right)^T (A_2)^T j.$$

Thus we get

$$(AB)^* = B^*A^*.$$

□

Proposition 1. *Let $A, B \in M_n(\mathbf{H}_S)$, if $AB = I$ then $BA = I$.*

Proof. It is well known that this proposition is true for complex matrices. Now let $A = A_1 + A_2 j$ and $B = B_1 + B_2 j$, where A_1, A_2, B_1 and B_2 are $n \times n$ complex matrices. Then

$$AB = \left(A_1 B_1 + A_2 \overline{B_2}\right) + \left(A_1 B_2 + A_2 \overline{B_1}\right) j = I.$$

From this we get

$$(A_1, A_2)\begin{pmatrix} B_1 & B_2 \\ \overline{B_2} & \overline{B_1} \end{pmatrix} = (I, 0),$$

so

$$\begin{pmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{pmatrix}\begin{pmatrix} B_1 & B_2 \\ \overline{B_2} & \overline{B_1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Since $\begin{pmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{pmatrix}$ and $\begin{pmatrix} B_1 & B_2 \\ \overline{B_2} & \overline{B_1} \end{pmatrix}$ are $2n \times 2n$ complex matrices, we get

$$\begin{pmatrix} B_1 & B_2 \\ \overline{B_2} & \overline{B_1} \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Then we can write

$$B_1 A_1 + B_2 \overline{A_2} = I, \quad B_1 A_2 + B_2 \overline{A_1} = 0,$$

so

$$\left(B_1 A_1 + B_2 \overline{A_2}\right) + \left(B_1 A_2 + B_2 \overline{A_1}\right) j = I.$$

Thus we obtai,

$$BA = I.$$

□

4. THE COMPLEX ADJOINT MATRIX OF THE SPLIT QUATERNION MATRICES

In this section we will define the complex adjoint matrix of a split quaternion matrix. After that we will give some relations between split quaternion matrices and their complex adjoint matrices.

Definition 1. Let $A = A_1 + A_2 j \in M_n(\mathbf{H}_S)$ where A_1 and A_2 are complex matrices. We define the $2n \times 2n$ complex matrices χ_A as

$$\begin{pmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{pmatrix}.$$

This matrix χ_A is called *the complex adjoint* (or *adjoint) matrix of the split quaternion matrix* A.

Example 2. Let $A = \begin{pmatrix} 1 & i+j+k \\ 0 & 1 \end{pmatrix}$ be a split quaternion matrix. Then the complex adjoint matrix of A is

$$\chi_A = \begin{pmatrix} 1 & i & 0 & 1+i \\ 0 & 1 & 0 & 0 \\ 0 & 1-i & 1 & -i \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Furthermore, $A^* = \begin{pmatrix} 1 & 0 \\ -i-j-k & 1 \end{pmatrix}$ and the complex adjoint matrix of A^* is

$$\chi_{(A^*)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -i & 1 & -1-i & 0 \\ 0 & 0 & 1 & 0 \\ -1+i & 0 & i & 1 \end{pmatrix}.$$

But

$$(\chi_A)^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -i & 1 & 1+i & 0 \\ 0 & 0 & 1 & 0 \\ 1-i & 0 & i & 1 \end{pmatrix}.$$

Thus, we get $\chi_{(A^*)} \neq (\chi_A)^*$.

Theorem 4. *Let* $A, B \in M_n(\mathbf{H}_S)$. *Then the followings hold*
(i) $\chi_{I_n} = I_{2n}$;
(ii) $\chi_{A+B} = \chi_A + \chi_B$;
(iii) $\chi_{AB} = \chi_A \chi_B$;
(iv) $\chi_{A^{-1}} = (\chi_A)^{-1}$ *if* A^{-1} *exist;*
(v) $\chi_{A^*} \neq (\chi_A)^*$ *in general.*

Proof. (i),(ii) and (iv) can be easily shown. For (v), we have given Example 2. Now we will prove (iii).
(iii) Let $A, B \in M_n(\mathbf{H}_S)$ where $A = A_1 + A_2 j$ and $B = B_1 + B_2 j$ for the complex matrices A_1, A_2, B_1 and B_2. Then the complex adjoint matrices of A and B are

$$\chi_A = \begin{pmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{pmatrix}, \quad \chi_B = \begin{pmatrix} B_1 & B_2 \\ \overline{B_2} & \overline{B_1} \end{pmatrix}.$$

If we calculate AB for $A = A_1 + A_2 j$ and $B = B_1 + B_2 j$, then we have

$$AB = \left(A_1 B_1 + A_2 \overline{B_2}\right) + \left(A_1 B_2 + A_2 \overline{B_1}\right) j.$$

So for the complex adjoint matrix of AB we can write

$$\chi_{AB} = \begin{pmatrix} A_1 B_1 + A_2 \overline{B_2} & A_1 B_2 + A_2 \overline{B_1} \\ \overline{A_1 B_2} + \overline{A_2} B_1 & \overline{A_1 B_1} + \overline{A_2} B_2 \end{pmatrix}.$$

On the other hand,

$$\chi_A \chi_B = \begin{pmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{pmatrix} \begin{pmatrix} B_1 & B_2 \\ \overline{B_2} & \overline{B_1} \end{pmatrix} = \begin{pmatrix} A_1 B_1 + A_2 \overline{B_2} & A_1 B_2 + A_2 \overline{B_1} \\ \overline{A_1 B_2} + \overline{A_2} B_1 & \overline{A_1 B_1} + \overline{A_2} B_2 \end{pmatrix}.$$

Thus we get

$$\chi_{AB} = \chi_A \chi_B.$$

□

Definition 2. Let $A \in M_n(\mathbf{H}_S)$ and $\lambda \in \mathbf{H}_S$. If λ satisfies the equation of $Ax = \lambda x$ $(Ax = x\lambda)$ then λ is called the left (right) eigenvalue of A. The set of the left eigenvalue is defined as $\sigma_l(A) = \{\lambda \in \mathbf{H}_S : Ax = \lambda x, \text{ for some } x \neq 0\}$. This set is called the left spectrum of A. Similarly, we define the right spectrum of A as $\sigma_r(A) = \{\lambda \in \mathbf{H}_S : Ax = x\lambda, \text{ for some } x \neq 0\}$.

Example 3. For the split quaternion matrix,

$$A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

some of the left eigenvalues of A are 1 and -1. Some of the right eigenvalues of A are $\{\pm 1, \pm j, \pm k\}$.

Example 4. For the split quaternion matrix,

$$A = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}$$

some of the left eigenvalues of A are $\{\pm k\}$ but none of them is a right eigenvalue. Similarly, some of the right eigenvalues of A are $\{\pm i\}$.

Theorem 5. *Let $A \in M_n(\mathbf{H}_S)$. The followings are equivalent:*
(i) A is invertible,
(ii) $Ax = 0$ has a unique solution 0,
(iii) $|\chi_A| \neq 0$, i.e., χ_A is an invertible,
(iv) A has no zero eigenvalue. More precisely if $Ax = \lambda x$ or $Ax = x\lambda$ for some split quaternion λ and some split quaternion vector $x \neq 0$, then $\lambda \neq 0$.

Proof. $(i) \Rightarrow (ii)$ This part is trivial.
$(ii) \Rightarrow (iii)$ Let $A = A_1 + A_2 j$, $x = x_1 + x_2 j$ where A_1, A_2 are complex matrices and x_1, x_2 are complex column vectors. Then

$$Ax = (A_1 x_1 + A_2 \overline{x_2}) + (A_1 x_2 + A_2 \overline{x_1})\, j\,.$$

From $Ax = 0$ we can write

$$A_1 x_1 + A_2 \overline{x_2} = 0$$

and

$$A_1 x_2 + A_2 \overline{x_1} = 0.$$

These equations give us the equations

$$A_1 x_1 + A_2 \overline{x_2} = 0$$

and

$$\overline{A_2} x_1 + \overline{A_1}\, \overline{x_2} = 0.$$

So we get that

$$Ax = 0 \text{ if and only if } \begin{pmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{pmatrix} \begin{pmatrix} x_1 \\ \overline{x_2} \end{pmatrix} = 0.$$

That is $\chi_A (x_1, \overline{x_2})^T = 0$. Since $Ax = 0$ has an unique solution, $\chi_A (x_1, \overline{x_2})^T = 0$ has an unique solution. Thus, since χ_A is a complex matrix, χ_A is invertible.
$(ii) \Leftrightarrow (iv)$ Let $Ax = 0$ has a unique solution 0 for $A \in M_n(\mathbf{H}_S)$. Suppose that A has a zero eigenvalue. Then for some split quaternion vector $x \neq 0$ the equation $Ax = \lambda x$ has zero eigenvalue. Thus $Ax = 0$, and so by our assumption $x = 0$, this is a contradiction. Now suppose that A has no zero eigenvalue. If we have $Ax = 0 = \lambda x$, then by our assumption $x = 0$.
$(iii) \Rightarrow (i)$ If χ_A is invertible, then for $A = A_1 + A_2 j$ there exist a complex matrix $\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$ such that

$$\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Here we get

$$B_1 A_1 + B_2 \overline{A_2} = I$$

and

$$B_1 A_2 + B_2 \overline{A_1} = 0.$$

Using this equations we can write

$$\left(B_1A_1 + B_2\overline{A_2}\right) + \left(B_1A_2 + B_2\overline{A_1}\right)j = I.$$

That is

$$BA = I$$

for $B = B_1 + B_2 j$. So A is an invertible split quaternion matrix by Proposition 1. □

Definition 3. Let $A \in M_n(\mathbf{H}_S)$ and χ_A be the complex adjoint matrix of A. We define the q-determinant of A by $|A|_q = |\chi_A|$. Here $|\chi_A|$ is the usual determinant of χ_A.

Theorem 6. *Let $A, B \in M_n(\mathbf{H}_S)$, then*
(i) A is invertible $\Leftrightarrow |A|_q \neq 0$.
(ii) $|AB|_q = |A|_q |B|_q$, consequently $\left|A^{-1}\right|_q = |A|_q^{-1}$ if A^{-1} is exist.
(iii) $|PAQ|_q = |A|_q$, for any elementary matrices P and Q.
(iv) Cayley- Hamilton Theorem: Let A be a square matrix of split quaternions, and let $F_A(\lambda) = |\lambda I_{2n} - \chi_A|$, which is called the characteristic polynomial of A,where λ is a complex indeterminant. Then $F_A(A) = 0$, and $F_A(\lambda_0) = 0$ if and only if λ_0 is an eigenvalue.

Proof. (i) It is clear from Theorem 5.
(ii) From Theorem 4 we have $\chi_{AB} = \chi_A\chi_B$. And so $|AB|_q = |\chi_{AB}| = |\chi_A\chi_B| = |\chi_A||\chi_B| = |A|_q|B|_q$. Here, if we let $B = A^{-1}$ then we can find easily $\left|A^{-1}\right|_q = |A|_q^{-1}$.
(iii) It is sufficient to observe that $\left|\chi_p\right| = 1$ where P is an elementary split quaternion matrix.
(iv) $F_A(\lambda)$ is a real coefficient polynomial by [5, Corollary 5.1]. For any real coefficient polynomial f, $\chi_{f(A)} = F_A(\chi_A)$. Since χ_A is a complex matrix, by the Cayley- Hamilton theorem for complex matrices we have $F_A(\chi_A) = 0$. Thus we get $\chi_{f(A)} = F_A(\chi_A) = 0$, hence $F_A(A) = 0$, as desired.

The rest of this part follows, since every split quaternion is similar to a complex number, and $Ax = x\lambda$ is equivalent to $\chi_A(x_1, -\overline{x_2})^T = \lambda(x_1, -\overline{x_2})^T$, where $x = x_1 + x_2 j$ and λ is a complex number. □

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ON COMPANION OF OSTROWSKI INEQUALITY FOR MAPPINGS WHOSE FIRST DERIVATIVES ABSOLUTE VALUE ARE CONVEX WITH APPLICATIONS

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Abstract. Several inequalities for a companion of Ostrowski inequality for absolutely continuous mappings whose first derivatives absolute value are convex (resp. concave) are established. Applications to a composite quadrature rule, to p.d.f.'s, and to special means are provided.

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1. INTRODUCTION

In 1938, Ostrowski established a very interesting inequality for differentiable mappings with bounded derivatives, as follows [8]:

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, interior of the interval I, such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the following inequality,*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]$$

holds for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

For recent results concerning Ostrowski inequality see [1], [2] and [3]. Also, the reader may be refer to the monograph [8] where various inequalities of Ostrowski type are discussed.

In [9], Guessab and Schmeisser have proved among others, the following companion of Ostrowski inequality:

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Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ satisfy the Lipschitz condition, i.e., $|f(t)-f(s)|\le M|t-s|$. Then for each $x\in[a,\frac{a+b}{2}]$, we have the inequality,*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)M. \tag{1.1}$$

The constant $1/8$ is the best possible in the sense that it cannot be replaced by a smaller constant.

We may also note that the best inequality in (1.1) is obtained for $x=\frac{3a+b}{4}$, giving the trapezoid type inequality,

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{(b-a)}{8}M \tag{1.2}$$

The constant $1/8$ is sharp in (1.2) in the sense mentioned above.

Companions of Ostrowski integral inequality for absolutely continuous functions was considered by Dragomir in [6], pp.228, as follows :

Theorem 3. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then we have the inequalities,*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.3}$$

$$\le\begin{cases}\left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|f'\|_\infty, & f'\in L_\infty[a,b]\\[2ex] \left(\frac{2}{q+1}\right)^{1/q}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\|f'\|_{[a,b],p}, & \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ and\ f'\in L_p[a,b]\\[2ex] \left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\|f'\|_{[a,b],1} & \end{cases}$$

for all $x\in[a,\frac{a+b}{2}]$.

In [7], the following theorem which was obtained by Dragomir and Agarwal contains the Hermite-Hadamard type integral inequality:

Theorem 4. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \frac{(b-a)\left(|f'(a)|+|f'(b)|\right)}{8}. \tag{1.4}$$

In [5], S.S. Dragomir established some inequalities for this companion for mappings of bounded variation. Also, Z. Liu introduced some companions of an Ostrowski type integral inequality for functions whose derivatives are absolutely continuous in [10]. Recently, N.S. Barnett et al. have proved some companions for the Ostrowski inequality and the generalized trapezoid inequality in [4].

The aim of this paper is to study a companion of Ostrowski inequality Theorem 2 for the class of functions whose derivatives in absolute value are convex (concave) functions.

2. Results

In order to prove our results, we need the following lemma (see [6]):

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I°, where $a,b \in I$ with $a < b$, such that $f' \in L_1[a,b]$. Then, the following equality holds*

$$\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b p(x,t)\,f'(t)\,dt,$$

where

$$p(x,t) = \begin{cases} t-a, & t \in [a,x] \\ t-\frac{a+b}{2}, & t \in (x,a+b-x] \\ t-b, & t \in (a+b-x,b] \end{cases},$$

for all $x \in [a, \frac{a+b}{2}]$.

A simple proof of the equality can be done by performing integration by parts. The details are left to the interested reader (see [6]).

Let us begin with the following result:

Theorem 5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I°, where $a,b \in I$ with $a < b$, such that $f' \in L_1[a,b]$. If $|f'|$ is convex on $[a,b]$, then we have the following inequality:*

$$\left| \frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{2.1}$$

$$\leq \frac{(x-a)^2}{6(b-a)}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)$$
$$+\frac{8(x-a)^2+3(a+b-2x)^2}{24(b-a)}\left(\left|f'(x)\right|+\left|f'(a+b-x)\right|\right)$$

for all $x\in[a,\frac{a+b}{2}]$.

Proof. Using Lemma 1 and the modulus, we have

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \frac{1}{b-a}\int_a^b |p(x,t)|\left|f'(t)\right|dt$$
$$=\frac{1}{b-a}\left[\int_a^x |p(x,t)|\left|f'(t)\right|dt+\int_x^{a+b-x}|p(x,t)|\left|f'(t)\right|dt\right.$$
$$\left.+\int_{a+b-x}^b |p(x,t)|\left|f'(t)\right|dt\right]$$

Since $|f'|$ is convex on $[a,b]=[a,x]\cup(x,a+b-x]\cup(a+b-x,b]$, therefore we have

$$\left|f'(t)\right|\leq\frac{t-a}{x-a}\left|f'(x)\right|+\frac{x-t}{x-a}\left|f'(a)\right|,\quad t\in[a,x];$$

$$\left|f'(t)\right|\leq\frac{t-x}{a+b-2x}\left|f'(a+b-x)\right|+\frac{a+b-x-t}{a+b-2x}\left|f'(x)\right|,\quad t\in(x,a+b-x];$$

and

$$\left|f'(t)\right|\leq\frac{t-a-b+x}{x-a}\left|f'(b)\right|+\frac{b-t}{x-a}\left|f'(a+b-x)\right|,\quad t\in(a+b-x,b];$$

which follows that,

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq\frac{1}{b-a}\int_a^x|t-a|\left[\frac{t-a}{x-a}\left|f'(x)\right|+\frac{x-t}{x-a}\left|f'(a)\right|\right]dt$$
$$+\frac{1}{b-a}\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left[\frac{t-x}{a+b-2x}\left|f'(a+b-x)\right|\right.$$
$$\left.+\frac{a+b-x-t}{a+b-2x}\left|f'(x)\right|\right]dt$$
$$+\frac{1}{b-a}\int_{a+b-x}^b|t-b|\left[\frac{t-a-b+x}{x-a}\left|f'(b)\right|+\frac{b-t}{x-a}\left|f'(a+b-x)\right|\right]dt$$

$$
\begin{aligned}
&= \frac{1}{(b-a)(x-a)}\left[\left|f'(x)\right|\frac{(x-a)^3}{3}+\left|f'(a)\right|\frac{(x-a)^3}{6}\right] \\
&\quad+\frac{1}{(b-a)(a+b-2x)}\frac{(a+b-2x)^3}{8}\left[\left|f'(a+b-x)\right|+\left|f'(x)\right|\right] \\
&\quad+\frac{1}{(b-a)(x-a)}\left[\left|f'(b)\right|\frac{(x-a)^3}{6}+\left|f'(a+b-x)\right|\frac{(x-a)^3}{3}\right] \\
&= \frac{(x-a)^2}{6(b-a)}\left[\left|f'(a)\right|+2\left|f'(x)\right|+2\left|f'(a+b-x)\right|+\left|f'(b)\right|\right] \\
&\quad+\frac{(a+b-2x)^2}{8(b-a)}\left[\left|f'(a+b-x)\right|+\left|f'(x)\right|\right] \\
&= \frac{(x-a)^2}{6(b-a)}\left(\left|f'(a)\right|+\left|f'(b)\right|\right) \\
&\quad+\frac{8(x-a)^2+3(a+b-2x)^2}{24(b-a)}\left(\left|f'(x)\right|+\left|f'(a+b-x)\right|\right),
\end{aligned}
$$

where

$$
\int_x^{a+b-x}(t-x)\left|t-\tfrac{a+b}{2}\right|dt=\int_x^{a+b-x}(a+b-x-t)\left|t-\tfrac{a+b}{2}\right|dt=\tfrac{(a+b-2x)^3}{8},
$$

which completes the proof. □

An Ostrowski type inequality may be deduced as follows:

Corollary 1. *Let f as in Theorem 5. Additionally, if f is symmetric about the x-axis, i.e., $f(a+b-x)=f(x)$, we have*

$$
\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq\frac{(x-a)^2}{6(b-a)}\left(\left|f'(a)\right|+\left|f'(b)\right|\right) \\
&\quad+\frac{8(x-a)^2+3(a+b-2x)^2}{24(b-a)}\left(\left|f'(x)\right|+\left|f'(a+b-x)\right|\right)
\end{aligned}
$$

for all $x\in[a,\frac{a+b}{2}]$.

Remark 1. In Theorem 5, if we choose $x=a$, then we get

$$
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{b-a}{8}\left(\left|f'(a)\right|+\left|f'(b)\right|\right).
$$

which is the inequality in (1.4).

Corollary 2. *In Theorem 5, if we choose*

(1) $x = \frac{3a+b}{4}$, then we get

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.2}$$
$$\leq\frac{(b-a)}{96}\left[|f'(a)|+5\left|f'\left(\frac{3a+b}{4}\right)\right|+5\left|f'\left(\frac{a+3b}{4}\right)\right|+|f'(b)|\right]$$

(2) $x = \frac{a+b}{2}$, then we get

$$\left|f\left(\tfrac{a+b}{2}\right)-\tfrac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \tfrac{(b-a)}{24}\left[|f'(a)|+4\left|f'\left(\tfrac{a+b}{2}\right)\right|+|f'(b)|\right]. \tag{2.3}$$

Another result may be considered using the Hölder inequality, as follows:

Theorem 6. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I°, where $a,b \in I$ with $a < b$, such that $f' \in L_1[a,b]$. If $|f'|^q$, $q > 1$ is convex on $[a,b]$, then we have the following inequality:*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.4}$$
$$\leq \frac{1}{2^{1/q}(b-a)(p+1)^{1/p}}\left\{(x-a)^2\left[|f'(a)|^q+|f'(x)|^q\right]^{1/q}\right.$$
$$+\frac{(a+b-2x)^2}{2}\left[|f'(x)|^q+|f'(a+b-x)|^q\right]^{1/q}$$
$$\left.+(x-a)^2\left[|f'(a+b-x)|^q+|f'(b)|^q\right]^{1/q}\right\},$$

for all $x \in [a, \frac{a+b}{2}]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1 and the Hölder inequality, we have

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.5}$$
$$\leq \frac{1}{b-a}\int_a^b |p(x,t)|\,|f'(t)|\,dt$$
$$=\frac{1}{b-a}\left[\int_a^x |t-a|\,|f'(t)|\,dt+\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\,|f'(t)|\,dt\right.$$
$$\left.+\int_{a+b-x}^b |t-b|\,|f'(t)|\,dt\right]$$

$$\leq \frac{1}{b-a}\left[\left(\int_a^x |t-a|^p\,dt\right)^{1/p}\left(\int_a^x \left|f'(t)\right|^q dt\right)^{1/q}\right.$$
$$+\left(\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|^p dt\right)^{1/p}\left(\int_x^{a+b-x}\left|f'(t)\right|^q dt\right)^{1/q}$$
$$\left.+\left(\int_{a+b-x}^b |t-b|^p\,dt\right)^{1/p}\left(\int_{a+b-x}^b \left|f'(t)\right|^q dt\right)^{1/q}\right]$$

Since $|f'|^q$ is convex on $[a,b]=[a,x]\cup(x,a+b-x]\cup(a+b-x,b]$, therefore we have

$$\left|f'(t)\right|^q \leq \frac{t-a}{x-a}\left|f'(x)\right|^q + \frac{x-t}{x-a}\left|f'(a)\right|^q,\ t\in[a,x];$$

$$\left|f'(t)\right|^q \leq \frac{t-x}{a+b-2x}\left|f'(a+b-x)\right|^q + \frac{a+b-x-t}{a+b-2x}\left|f'(x)\right|^q,\ t\in(x,a+b-x];$$

and

$$\left|f'(t)\right|^q \leq \frac{t-a-b+x}{x-a}\left|f'(b)\right|^q + \frac{b-t}{x-a}\left|f'(a+b-x)\right|^q,\ t\in(a+b-x,b];$$

which follows that,

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \frac{1}{b-a}\left[\left(\int_a^x |t-a|^p\,dt\right)^{1/p}\left(\int_a^x\left[\frac{t-a}{x-a}\left|f'(x)\right|^q+\frac{x-t}{x-a}\left|f'(a)\right|^q\right]dt\right)^{1/q}\right.$$
$$+\left(\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|^p dt\right)^{1/p}$$
$$\times\left(\int_x^{a+b-x}\left[\frac{t-x}{a+b-2x}\left|f'(a+b-x)\right|^q+\frac{a+b-x-t}{a+b-2x}\left|f'(x)\right|^q\right]dt\right)^{1/q}$$
$$+\left(\int_{a+b-x}^b |t-b|^p\,dt\right)^{1/p}$$
$$\left.\times\left(\int_{a+b-x}^b\left[\frac{t-a-b+x}{x-a}\left|f'(b)\right|^q+\frac{b-t}{x-a}\left|f'(a+b-x)\right|^q\right]dt\right)^{1/q}\right]$$

$$=\frac{1}{b-a}\left[\left(\frac{(x-a)^{p+1}}{(p+1)}\right)^{1/p}\left(\frac{x-a}{2}\right)^{1/q}\left(\left|f'(a)\right|^q+\left|f'(x)\right|^q\right)^{1/q}\right.$$

$$
\begin{aligned}
&+\left(\frac{2}{(p+1)}\left(\frac{a+b}{2}-x\right)^{p+1}\right)^{1/p}\cdot \\
&\qquad\cdot\left(\frac{a+b}{2}-x\right)^{1/q}\left(\left|f'(x)\right|^q+\left|f'(a+b-x)\right|^q\right)^{1/q} \\
&+\left(\frac{(x-a)^{p+1}}{(p+1)}\right)^{1/p}\left(\frac{x-a}{2}\right)^{1/q}\left(\left|f'(a+b-x)\right|^q+\left|f'(b)\right|^q\right)^{1/q}\Bigg] \\
&=\frac{1}{2^{1/q}(b-a)(p+1)^{1/p}}\Bigg[(x-a)^2\left(\left|f'(a)\right|^q+\left|f'(x)\right|^q\right)^{1/q} \\
&+\frac{(a+b-2x)^2}{2}\left(\left|f'(x)\right|^q+\left|f'(a+b-x)\right|^q\right)^{1/q} \\
&+(x-a)^2\left(\left|f'(a+b-x)\right|^q+\left|f'(b)\right|^q\right)^{1/q}\Bigg],
\end{aligned}
$$

since $\frac{1}{p}+\frac{1}{q}=1$, $q>1$, which completes the proof. □

Corollary 3. *Let f as in Theorem 6. Additionally, if f is symmetric about the x-axis, i.e., $f(a+b-x)=f(x)$, we have*

$$
\begin{aligned}
\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| &\le \frac{1}{2^{1/q}(b-a)(p+1)^{1/p}} \\
&\times\Bigg\{(x-a)^2\left[\left|f'(a)\right|^q+\left|f'(x)\right|^q\right]^{1/q} \\
&+\frac{(a+b-2x)^2}{2}\left[\left|f'(x)\right|^q+\left|f'(a+b-x)\right|^q\right]^{1/q} \\
&+(x-a)^2\left[\left|f'(a+b-x)\right|^q+\left|f'(b)\right|^q\right]^{1/q}\Bigg\}
\end{aligned}
$$

for all $x\in[a,\frac{a+b}{2}]$.

Corollary 4. *In Theorem 6, if we choose*

(1) *$x=a$, then we get*

$$
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{(b-a)}{2^{1+\frac{1}{q}}(1+p)^{1/p}}\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{1/q}. \tag{2.6}
$$

(2) *$x=\frac{3a+b}{4}$, then we get*

$$
\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.7}
$$

$$\leq \frac{(b-a)}{2^{4+\frac{1}{q}}(1+p)^{1/p}}\left\{\left(\left|f'(a)\right|^q+\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)^{1/q}\right.$$
$$+2\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)^{1/q}$$
$$\left.+\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\left|f'(b)\right|^q\right)^{1/q}\right\}.$$

(3) *$x=\frac{a+b}{2}$, then we get*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \frac{(b-a)}{2^{2+\frac{1}{q}}(1+p)^{1/p}}\left[\left(\left|f'(a)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right.$$
$$\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q\right)^{1/q}\right].$$

The following result holds for concave mappings.

Theorem 7. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous mapping on I°, where $a,b\in I$ with $a<b$, such that $f'\in L_1[a,b]$. If $|f'|^q$, $q>1$ is concave on $[a,b]$, then we have the following inequality:*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.8}$$
$$\leq \frac{1}{(b-a)(1+p)^{1/p}}$$
$$\times\left[(x-a)^2\left(\left|f'\left(\frac{a+x}{2}\right)\right|+\left|f'\left(\frac{2b+a-x}{2}\right)\right|\right)+\frac{(a+b-2x)^2}{2}\left|f'\left(\frac{a+b}{2}\right)\right|\right]$$

for all $x\in[a,\frac{a+b}{2}]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1, and by (2.5), we have

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \frac{1}{b-a}\left[\left(\int_a^x|t-a|^p\,dt\right)^{1/p}\left(\int_a^x\left|f'(t)\right|^q dt\right)^{1/q}\right.$$

$$+\left(\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|^p dt\right)^{1/p}\left(\int_x^{a+b-x}\left|f'(t)\right|^q dt\right)^{1/q}$$
$$+\left(\int_{a+b-x}^{b}|t-b|^p dt\right)^{1/p}\left(\int_{a+b-x}^{b}\left|f'(t)\right|^q dt\right)^{1/q}\Bigg].$$

Now, let us write,

$$\int_a^x\left|f'(t)\right|^q dt=(x-a)\int_0^1\left|f'(\lambda x+(1-\lambda)a)\right|^q d\lambda,$$

$$\int_x^{a+b-x}\left|f'(t)\right|^q dt=(a+b-2x)\int_0^1\left|f'(\lambda(a+b-x)+(1-\lambda)x)\right|^q d\lambda,$$

and

$$\int_{a+b-x}^{b}\left|f'(t)\right|^q dt=(x-a)\int_0^1\left|f'(\lambda b+(1-\lambda)(a+b-x))\right|^q d\lambda.$$

Since $|f'|^q$, $q>1$ is concave on $[a,b]=[a,x]\cup(x,a+b-x]\cup(a+b-x,b]$, we can use the Jensen integral inequality to obtain

$$\begin{aligned}
&(x-a)\int_0^1\left|f'(\lambda x+(1-\lambda)a)\right|^q d\lambda\\
&=(x-a)\int_0^1\lambda^0\left|f'(\lambda x+(1-\lambda)a)\right|^q d\lambda\\
&\le(x-a)\left(\int_0^1\lambda^0 d\lambda\right)\left|f'\left(\frac{1}{\int_0^1\lambda^0 d\lambda}\int_0^1(\lambda x+(1-\lambda)a)\,d\lambda\right)\right|^q\\
&=(x-a)\left|f'\left(\frac{x+a}{2}\right)\right|^q
\end{aligned}$$

and analogously

$$(a+b-2x)\int_0^1\left|f'(\lambda(a+b-x)+(1-\lambda)x)\right|^q d\lambda\le(a+b-2x)\left|f'\left(\frac{a+b}{2}\right)\right|^q,$$

$$(x-a)\int_0^1\left|f'(\lambda b+(1-\lambda)(a+b-x))\right|^q d\lambda\le(x-a)\left|f'\left(\frac{2b+a-x}{2}\right)\right|^q.$$

Combining all above inequalities, we get

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\leq \frac{1}{b-a}\left[\left(\frac{(x-a)^{p+1}}{(p+1)}\right)^{1/p}(x-a)^{1/q}\left|f'\left(\frac{x+a}{2}\right)\right|\right]$$
$$+\left(\frac{2}{(p+1)}\left(\frac{a+b}{2}-x\right)^{p+1}\right)^{1/p}(a+b-2x)^{1/q}\left|f'\left(\frac{a+b}{2}\right)\right|$$
$$+\left(\frac{(x-a)^{p+1}}{(p+1)}\right)^{1/p}(x-a)^{1/q}\left|f'\left(\frac{2b+a-x}{2}\right)\right|$$
$$=\frac{1}{(b-a)(1+p)^{1/p}}$$
$$\times\left[(x-a)^2\left(\left|f'\left(\frac{a+x}{2}\right)\right|+\left|f'\left(\frac{2b+a-x}{2}\right)\right|\right)+\frac{(a+b-2x)^2}{2}\left|f'\left(\frac{a+b}{2}\right)\right|\right],$$

for all $x\in[a,\frac{a+b}{2}]$, where $\frac{1}{p}+\frac{1}{q}=1$, $q>1$, which is required. □

Therefore, we may state the following Ostrowski type inequality:

Corollary 5. *Let f as in Theorem 7. Additionally, if f is symmetric about the x-axis, i.e., $f(a+b-x)=f(x)$, we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq\frac{1}{(b-a)(1+p)^{1/p}}$$
$$\times\left[(x-a)^2\left(\left|f'\left(\frac{a+x}{2}\right)\right|+\left|f'\left(\frac{2b+a-x}{2}\right)\right|\right)+\frac{(a+b-2x)^2}{2}\left|f'\left(\frac{a+b}{2}\right)\right|\right],$$

for all $x\in[a,\frac{a+b}{2}]$.

Corollary 6. *In Theorem 7, if we choose*

(1) *$x=a$, then we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{(b-a)}{2(1+p)^{1/p}}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{2.9}$$

(2) *$x=\frac{3a+b}{4}$, then we get*

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\leq \frac{(b-a)}{16(1+p)^{1/p}}\left[\left|f'\left(\frac{7a+b}{8}\right)\right|+2\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'\left(\frac{a+7b}{8}\right)\right|\right]. \quad (2.10)$$

(3) $x=\frac{a+b}{2}$, *then we get*

$$\left|f\left(\tfrac{a+b}{2}\right)-\tfrac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \tfrac{(b-a)}{4(1+p)^{1/p}}\left[\left|f'\left(\tfrac{3a+b}{4}\right)\right|+\left|f'\left(\tfrac{a+3b}{4}\right)\right|\right].$$

3. A COMPOSITE QUADRATURE FORMULA

Let $I_n : a=x_0<x_1<\cdots<x_n=b$ be a division of the interval $[a,b]$ and $h_i=x_{i+1}-x_i$, $(i=0,1,2,\cdots,n-1)$.

Consider the general quadrature formula

$$Q_n(I_n,f):=\frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i. \quad (3.1)$$

The following result holds.

Theorem 8. *Let $f : I\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous mapping on I°, where $a,b\in I$ with $a<b$, such that $f'\in L_1[a,b]$. If $|f'|$ is convex on $[a,b]$. Then, we have*

$$\int_a^b f(t)\,dt = Q_n(I_n,f)+R_n(I_n,f). \quad (3.2)$$

where, $Q_n(I_n,f)$ is defined by formula (3.1), and the remainder term $R_n(I_n,f)$ satisfies the error estimates

$$|R_n(I_n,f)| \leq \frac{1}{96}\sum_{i=0}^{n-1}h_i^2\left[|f'(x_i)|+5\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|+5\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|+|f'(x_{i+1})|\right].$$

Proof. Applying inequality (3.1) and (3.2) on the intervals $[x_i,x_{i+1}]$, we may state that

$$R_i(I_i,f)=\int_{x_i}^{x_{i+1}}f(t)\,dt-\frac{1}{2}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i.$$

Summing the above inequality over i from 0 to $n-1$, we get

$$R_n(I_n,f)=\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}f(t)\,dt-\frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i$$

$$=\int_a^b f(t)\,dt-\frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i,$$

which follows from (2.2), that

$$|R_n(I_n,f)| = \left|\int_a^b f(t)\,dt - \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i\right|$$
$$\leq \frac{1}{96}\sum_{i=0}^{n-1}h_i^2\left[\left|f'(x_i)\right|+5\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|\right.$$
$$\left.+5\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|+\left|f'(x_{i+1})\right|\right].$$

which completes the proof. □

Remark 2. One may state more inequalities, using (2.7) and (2.10). We shall omit the details.

4. Applications for P.D.F.'s

Let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f:[a,b]\to[0,1]$ with the cumulative distribution function $F(x)=Pr(X\leq x)=\int_a^b f(t)dt$.

Theorem 9. *With the assumptions of Theorem 5, we have the inequality*

$$\left|\frac{1}{2}[F(x)+F(a+b-x)]-\frac{b-E(X)}{b-a}\right|$$
$$\leq \frac{(x-a)^2}{6(b-a)}\left(\left|F'(a)\right|+\left|F'(b)\right|\right)$$
$$+\frac{8(x-a)^2+3(a+b-2x)^2}{24(b-a)}\left(\left|F'(x)\right|+\left|F'(a+b-x)\right|\right)$$

for all $x\in[a,\frac{a+b}{2}]$, *where* $E(X)$ *is the expectation of* X.

Proof. In the proof of Theorem 5 , let $f=F$, and taking into account that

$$E(X)=\int_a^b t\,dF(t)=b-\int_a^b F(t)\,dt.$$

We left the details to the interested reader. □

Corollary 7. *In Theorem 9, if we choose* $x=\frac{3a+b}{4}$, *then we get*

$$\left|\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]-\frac{b-E(X)}{b-a}\right|$$
$$\leq \frac{(b-a)}{96}\left[\left|F'(a)\right|+5\left|F'\left(\frac{3a+b}{4}\right)\right|+5\left|F'\left(\frac{a+3b}{4}\right)\right|+\left|F'(b)\right|\right].$$

Corollary 8. *In Theorem 9, if F is symmetric about the x-axis, i.e., $F(a+b-x)=F(x)$, we have*

$$\begin{aligned}&\left|F(x)-\frac{b-E(X)}{b-a}\right|\\&\leq\frac{(x-a)^2}{6(b-a)}\left(\left|F'(a)\right|+\left|F'(b)\right|\right)\\&\quad+\frac{8(x-a)^2+3(a+b-2x)^2}{24(b-a)}\left(\left|F'(x)\right|+\left|F'(a+b-x)\right|\right)\end{aligned}$$

for all $x\in[a,\frac{a+b}{2}]$.

Remark 3. One may state more inequalities, using Theorem 6 and Theorem 7. We shall omit the details.

5. Applications for special means

Recall the following means which could be considered extensions of arithmetic, logarithmic and generalized logarithmic for positive real numbers.

(1) The arithmetic mean:

$$A=A(a,b)=\frac{a+b}{2};\ a,b\in\mathbb{R}^+$$

(2) The logarithmic mean:

$$L(a,b)=\frac{b-a}{\ln|b|-\ln|a|};\ |a|\neq|b|,\ a,b\in\mathbb{R}^+$$

(3) The generalized logarithmic mean:

$$L_n(a,b)=\left[\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\right]^{\frac{1}{n}};\ n\in\mathbb{Z}\setminus\{-1,0\},\ a,b\in\mathbb{R}^+,\ a\neq b$$

(4) The identric mean:

$$I(a,b)=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & ,a\neq b\\ a & ,a=b\end{cases}\qquad a,b\in\mathbb{R}^+.$$

Now using our results, we give some applications to special means for positive real numbers.

Proposition 1. *Let $a,b\in\mathbb{R}^+$, $a<b$. Then, we have*

$$\left|A\left(\tfrac{4}{3a+b},\tfrac{4}{a+3b}\right)-L^{-1}(a,b)\right|\leq\tfrac{(b-a)}{96}\left[\tfrac{a^2+b^2}{a^2b^2}+80\left(\tfrac{1}{(3a+b)^2}+\tfrac{1}{(a+3b)^2}\right)\right].$$

Proof. The assertion was obtained by the inequality in (2.2) applied to the convex mapping $f:[a,b]\to\mathbb{R}$, $f(x)=\frac{1}{x}$. □

Proposition 2. *Let $a,b \in \mathbb{R}^+$, $a<b$, and $p \in \mathbb{Z}$, $|p| \geq 2$.Then,*

$$\left|A^p(a,b) - L_p^p(a,b)\right| \leq \frac{p(b-a)}{24}\left[a^{p-1} + 4A^{p-1}(a,b) + b^{p-1}\right].$$

Proof. The assertion was obtained by the inequality in (2.3) applied to the convex mapping $f:[a,b]\to\mathbb{R}$, $f(x)=x^p$. □

Proposition 3. *Let $a,b \in \mathbb{R}^+$, $a<b$. Then, we have*

$$|\ln I - A(\ln a, \ln b)| \leq \frac{(b-a)}{2^{1+\frac{1}{q}}(1+p)^{\frac{1}{p}}}\left(\frac{1}{a^q}+\frac{1}{b^q}\right)^{\frac{1}{q}}.$$

Proof. The assertion was obtained by the inequality in (2.6) applied to the convex mapping $f:[a,b]\to[0,\infty)$, $f(x)=-\ln x$. □

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ASYMPTOTIC REPRESENTATIONS OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $y^{(n)} = \alpha_0 p(t) \prod_{i=0}^{n-1} \varphi_i \left(y^{(i)}\right)$

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Abstract. The asymptotic representations for solutions of differential equations of the type $y^{(n)} = \alpha_0 p(t) \prod_{i=0}^{n-1} \varphi_i(y^{(i)})$, where φ_i are regularly varying in some one-sided neighborhoods of critical points, are established.

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1. Introduction

Consider the differential equation

$$y^{(n)} = \alpha_0 p(t) \prod_{i=0}^{n-1} \varphi_i(y^{(i)}), \tag{1.1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \leq +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ $(i = 0, ..., n)$ are continuous functions, $Y_i \in \{0, \pm\infty\}$ and Δ_{Y_i} is either the interval $[y_i^0, Y_i[$[2] or the interval $]Y_i, y_i^0]$.

We also suppose that every $\varphi_i(z)$ is regularly varying as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of index σ_i and $\sum_{i=0}^{n-1} \sigma_i \neq 1$.

We say that the measurable function $\varphi : \Delta_Y \to]0, +\infty[$ is regularly varying as $z \to Y$ of index σ if for every $\lambda > 0$ we have

$$\lim_{\substack{z \to Y \\ z \in \Delta_Y}} \frac{\varphi(\lambda z)}{\varphi(z)} = \lambda^\sigma. \tag{1.2}$$

[1] If $\omega > 0$ we will take $a > 0$.

[2] If $Y_i = +\infty (Y_i = -\infty)$ we take $y_i^0 > 0$ $(y_i^0 < 0)$ correspondingly.

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Here, $Y \in \{0, \pm\infty\}$, Δ_Y is some one-sided neighbourhood of Y. If $\sigma = 0$, such function is called slowly varying.

It follows from the results of the monograph [11] that regularly varying functions have the next properties.

M_1: The function $\varphi(z)$ is regularly varying of index σ as $z \to Y$ if and only if the next representation takes place

$$\varphi(z) = z^{\sigma}\theta(z),$$

where $\theta(z)$ is a slowly varying function as $z \to Y$.

M_2: If the function $L : \Delta_{Y0} \to]0, +\infty[$ is slowly varying as $z \to Y_0$, the function $\varphi : \Delta_Y \to \Delta_{Y0}$ is regularly varying as $z \to Y$, then the function $L(\varphi) : \Delta_Y \to]0, +\infty[$ is slowly varying as $z \to Y$.

M_3: If the function $\varphi : \Delta_Y \to]0, +\infty[$ satisfies the condition

$$\lim_{\substack{z \to Y \\ z \in \Delta}} \frac{z\varphi'(z)}{\varphi(z)} = \sigma \in \mathbb{R},$$

then φ is regularly varying as $z \to Y$ of index σ.

M_4: For every regularly varying function φ as $z \to Y$ the property (1.2) takes place uniformly as $\lambda \in [c, d]$ for every segment $[c, d] \subset]0, +\infty[$.

According to M_1, it is clear that for every solution y of the equation (1.1) defined on $[t_0, \omega[\subset [a, \omega[$ such that

$$y^{(i)} : [t_0, \omega[\longrightarrow \Delta_{Y_i}, \qquad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \quad (i = 0, ..., n-1), \tag{1.3}$$

the representations $\varphi_i(y^{(i)}(t)) = |y^{(i)}(t)|^{\sigma_i + o(1)}$ take place as $t \uparrow \omega$. Therefore the equation (1.1) is in some sense similar to the well known differential equation

$$y^{(n)} = \alpha_0 p(t) \prod_{i=0}^{n-1} |y^{(i)}|^{\sigma_i}. \tag{1.4}$$

We call the solution y of the equation (1.1), that satisfies (1.3), the $P_\omega(\lambda^0_{n-1})$-solution ($-\infty \le \lambda^0_{n-1} \le +\infty$) if the next condition takes place

$$\lim_{t \uparrow \omega} \frac{(y^{(n-1)}(t))^2}{y^{(n)}(t)\, y^{(n-2)}(t)} = \lambda^0_{n-1}. \tag{1.5}$$

If p is regularly varying as $t \uparrow \omega$ it is easy to show, using Proposition 9 in [10, p. 116], that every regularly varying as $t \uparrow \omega$ of index $\gamma \in R \setminus \{0, 1, ..., n-1\}$ solution of Equation (1.1), that satisfies (1.3), is the $P_\omega\left(\frac{\gamma - n + 2}{\gamma - n + 1}\right)$-solution of (1.1). By the other side, by the investigation of $P_\omega(\lambda_0)$-solutions of Equation (1.1) it will be clear,

that (1.1) has the $P_\omega\left(\frac{\gamma-n+2}{\gamma-n+1}\right)$-solutions, where $\gamma \in R \setminus \{0,1,...,n-1\}$, only if the function p is regularly varying as $t \uparrow \omega$ of index $\gamma - n + 1 - \sum_{j=0}^{n-1}(\gamma - j)\sigma_j$.

It follows from the definition of $P_\omega(\lambda_0)$-solution, that every $P_\omega(1)$-solution of Equation (1.1) is rapidly varying as $t \uparrow \omega$. By the other side, by the investigation of $P_\omega(\lambda_0)$-solutions of Equation (1.1) it will be clear, that (1.1) has the $P_\omega(1)$-solutions only if the function p is rapidly varying as $t \uparrow \omega$.

The first results about the asymptotic properties and the existence of $P_\omega(\lambda_{n-1}^0)$-solutions of Equation (1.4) are found in [4]. All $P_\omega(\lambda_{n-1}^0)$-solutions of Equation (1.4) were investigated in [5,6]. In case $n = 2$ for all $P_\omega(\lambda_{n-1}^0)$-solutions of Equation (1.1) the necessary and sufficient conditions of existence and asymptotic representations as $t \uparrow \omega$ were found later in [1–3, 8].

The aim of the work is to establish the necessary and sufficient conditions of the existence $P_\omega(\lambda_{n-1}^0)$-solutions of Equation (1.1) in general case $n \geq 2$ for $\lambda_{n-1}^0 \in \mathbb{R}\setminus\left\{0, \frac{1}{2}, \frac{2}{3}, ..., \frac{n-2}{n-1}\right\}$.

2. Results

Let us introduce subsidiary notations

$$\gamma_0 = 1 - \sum_{j=0}^{n-1}\sigma_j, \quad \mu_n = \sum_{j=0}^{n-1}(n-j-1)\sigma_j, \quad \pi_\omega(t) = \begin{cases} t & \text{if } \omega = +\infty, \\ t-\omega & \text{if } \omega < +\infty, \end{cases}$$

$$\theta_i(z) = \varphi_i(z)|z|^{-\sigma_i}, \quad a_{0i} = (n-i)\lambda_{n-1}^0 - (n-i-1) \quad (i = 1,...,n),$$

$$C = \alpha_0|\lambda_{n-1}^0 - 1|^{\mu_n}\prod_{k=0}^{n-2}\left|\prod_{j=k+1}^{n-1} a_{0j}\right|^{-\sigma_k} sign\, y_{n-1}^0,$$

$$I_0(t) = \int_{A_\omega^0}^{t} Cp(\tau)|\pi_\omega(\tau)|^{\mu_n}\,d\tau, \quad I_1(t) = \int_{A_\omega^1}^{t}\alpha_0 p(\tau)\,d\tau,$$

$$A_\omega^0 = \begin{cases} a, & \text{if } \int_a^\omega p(\tau)|\pi_\omega(\tau)|^{\gamma_0}\,d\tau = +\infty, \\ \omega, & \text{if } \int_a^\omega p(\tau)|\pi_\omega(\tau)|^{\gamma_0}\,d\tau < +\infty, \end{cases} \quad A_\omega^1 = \begin{cases} a, & \text{if } \int_a^\omega p(\tau)\,d\tau = +\infty, \\ \omega, & \text{if } \int_a^\omega p(\tau)\,d\tau < +\infty, \end{cases}$$

$$J(t) = \int_{B_\omega}^{t}|\gamma_0 I_1(\tau)|^{\frac{1}{\gamma_0}}\,d\tau, \quad B_\omega = \begin{cases} a, & \text{if } \int_a^\omega |I_1(\tau)|^{\frac{1}{\gamma_0}}\,d\tau = +\infty, \\ \omega, & \text{if } \int_a^\omega |I_1(\tau)|^{\frac{1}{\gamma_0}}\,d\tau < +\infty. \end{cases}$$

The following conclusions take place for Equation (1.1).

Theorem 1. *The next conditions are necessary for the existence of $P_\omega(\lambda^0_{n-1})$-solutions ($\lambda^0_{n-1} \in \mathbb{R}\setminus\{0,1,\frac{1}{2},\frac{2}{3},\dots,\frac{n-2}{n-1}\}$) of Equation (1.1)*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)I_0'(t)}{I_0(t)}=\frac{\gamma_0}{\lambda^0_{n-1}-1},\ \lim_{t\uparrow\omega}y_i^0|\pi_\omega(t)|^{\frac{a_{0i+1}}{\lambda^0_{n-1}-1}}=Y_i, \tag{2.1}$$

$$y_i^0y_{i+1}^0a_{0i+1}(\lambda^0_{n-1}-1)\pi_\omega(t)>0\ \text{ as } t\in[a,\omega[, \tag{2.2}$$

where $y_n^0=\alpha_0$, $i=0,\dots,n-1$.

If Equation

$$\sum_{k=0}^{n-1}\sigma_k\prod_{i=k+1}^{n-1}a_{0i}\prod_{i=1}^{k}(a_{0i}+\lambda)=(1+\lambda)\prod_{i=1}^{n-1}(a_{0i}+\lambda) \tag{2.3}$$

has no roots with zero real part, then Conditions (2.1) and (2.2) are sufficient for the existence of $P_\omega(\lambda^0_{n-1})$-solutions of Equation (1.1). For any such solution the next asymptotic representations as $t\uparrow\omega$

$$\begin{aligned}&\frac{|y^{(n-1)}(t)|^{\gamma_0}}{\prod_{j=0}^{n-1}\theta_j(y^{(j)}(t))}=\gamma_0I_0(t)[1+o(1)],\\ &\frac{y^{(i)}(t)}{y^{(n-1)}(t)}=\frac{\left[(\lambda^0_{n-1}-1)\,\pi_\omega(t)\right]^{n-i-1}}{\prod_{j=i+1}^{n-1}a_{0j}}[1+o(1)],\end{aligned} \tag{2.4}$$

where $i=0,\dots,n-2$, take place.

Theorem 2. *The next conditions are necessary for the existence of $P_\omega(1)$-solutions of Equation (1.1)*

$$\lim_{t\uparrow\omega}\frac{I_1'(t)J(t)}{I_1(t)J'(t)}=\gamma_0,\ \lim_{t\uparrow\omega}y_i^0\,|I_1(t)|^{\frac{1}{\gamma_0}}=Y_i, \tag{2.5}$$

$$\alpha_0y_{n-2}^0>0,\quad y_i^0y_{i+1}^0J(t)>0\ \text{as } t\in[a,\omega[, \tag{2.6}$$

where $i=0,\dots,n-1$.

If Equation

$$\sum_{k=0}^{n-1}\sigma_k(1+\lambda)^k=(1+\lambda)^n \tag{2.7}$$

has no roots with zero real part, then Conditions (2.5) and (2.6) are sufficient for the existence of $P_\omega(1)$-solutions of Equation (1.1). For any such solution the asymptotic

representations as $t \uparrow \omega$

$$\frac{|y^{(n-1)}(t)|^{\gamma_0}}{\prod\limits_{j=0}^{n-1}\theta_j(y^{(j)}(t))} = \gamma_0 I_1(t)\left|\frac{J(t)}{J'(t)}\right|^{\mu_n} sign\, y_{n-1}^0[1+o(1)], \tag{2.8}$$

$$\frac{y^{(i)}(t)}{y^{(n-1)}(t)} = \left(\frac{J(t)}{J'(t)}\right)^{n-i-1}[1+o(1)],$$

where $i = 0,\dots,n-2$, *take place.*

Remark 1. If

$$\sum_{i=0}^{n-2}|\sigma_i| < |1-\sigma_{n-1}|,$$

Conditions (2.1) and (2.2) are necessary and sufficient for the existence of $P_\omega(\lambda_{n-1}^0)$-solutions ($\lambda_{n-1}^0 \in \mathbb{R}\setminus\{0,1,\frac{1}{2},\frac{2}{3},\dots,\frac{n-2}{n-1}\}$) of Equation (1.1) and Conditions (2.5) and (2.6) are necessary and sufficient for the existence of $P_\omega(1)$-solutions of (1.1).

By additional conditions on the functions φ_0, φ_1, ..., φ_{n-1} the asymptotic representations as $t \uparrow \omega$ of $P_\omega(\lambda_{n-1}^0)$-solutions and their derivatives from the first to $(n-1)$-th order are found in the explicit form.

In order to formulate our following results we present the next definition.

Definition 1. We call the slowly varying function θ as $z \to Y$ $(z \in \Delta)$ satisfies the condition S if for every continuously differentiable function $L : \Delta \to]0;+\infty[$ such that

$$\lim_{\substack{z\to Y\\ z\in\Delta}} \frac{zL'(z)}{L(z)} = 0,$$

the representation

$$\theta(zL(z)) = \theta(z)[1+o(1)] \quad \text{as } z \to Y \quad (z \in \Delta)$$

takes place.

The next conclusions follow from Theorems 1 and 2.

Corollary 1. *Let the functions* θ_0, ..., θ_{n-1} *satisfy the condition* S. *Then for any* $P_\omega(\lambda_{n-1}^0)$*-solution* ($\lambda_{n-1}^0 \in \mathbb{R}\setminus\{0,1,\frac{1}{2},\frac{2}{3},\dots,\frac{n-2}{n-1}\}$) *of Equation* (1.1) *the next asymptotic representations as* $t \uparrow \omega$

$$y^{(n-1)}(t) = \left|\gamma_0 I_0(t)\prod_{j=0}^{n-1}\theta_j\left(y_j^0|\pi_\omega(t)|^{\frac{a_{0j}+1}{\lambda_{n-1}^0-1}}\right)\right|^{\frac{1}{\gamma_0}} sign\, y_{n-1}^0[1+o(1)],$$

$$y^{(i)}(t) = y^{(n-1)}(t)\frac{\left[(\lambda^0_{n-1}-1)\,\pi_\omega(t)\right]^{n-i-1}}{\prod\limits_{j=i+1}^{n-1} a_{0j}}[1+o(1)], \quad i=0,\dots,n-2$$

take place.

Corollary 2. *Let the functions θ_0, ..., θ_{n-1} satisfy the condition S. Then for any $P_\omega(1)$-solution of Equation (1.1) the next asymptotic representations as $t \uparrow \omega$*

$$y^{(n-1)}(t) = \left|\gamma_0 I_1(t)\left|\frac{J'(t)}{J(t)}\right|^{\mu_n}\prod_{j=0}^{n-1}\theta_j\left(y_j^0|J(t)|\right)\right|^{\frac{1}{\gamma_0}} sign y^0_{n-1}[1+o(1)],$$

$$y^{(i)}(t) = y^{(n-1)}(t)\left(\frac{J(t)}{J'(t)}\right)^{n-i-1}[1+o(1)], \quad i=0,\dots,n-2$$

take place.

2.1. *Preliminary considerations*

The following lemma is true.

Lemma 1. *Let $y:[t_0,\omega[\to \Delta_{Y_0}$ be a $P_\omega(\lambda^0_{n-1})$-solution of Equation (1.1), where $\lambda^0_{n-1}\in\mathbb{R}\setminus\{0,\frac{1}{2},\frac{2}{3},\dots,\frac{n-2}{n-1}\}$. Then for every function $y^{(i)}$ ($i\in\{0,\dots,n-1\}$), the inverse function $\left(y^{(i)}\right)^{-1}$ exists. Moreover, the function $y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)$ is regularly varying as $z\to Y_i$ ($z\in\Delta_{Y_i}$) of index $\frac{a_{0j+1}}{a_{0i+1}}$ for every $j\in\{0,\dots,n-1\}$.*

Proof. Let $y:[t_0,\omega[\to\Delta_{Y_0}$ be a $P_\omega(\lambda^0_{n-1})$-solution of Equation (1.1), where $\lambda^0_{n-1}\in\mathbb{R}\setminus\{0,\frac{1}{2},\frac{2}{3},\dots,\frac{n-2}{n-1}\}$. By (1.3) and (1.1) the function $y^{(i)}(t)$ is strongly monotone on $[t_0,\omega[$ for every $i\in\{0,\dots,n-1\}$, because its derivative $y^{(i+1)}$ has a fixed sign on $[t_0,\omega[$. Therefore $y^{(i)}$ has an inverse function $\left(y^{(i)}\right)^{-1}$. Due to (1.5) and Lemmas 10.1 and 10.2 from [7] we would have the asymptotic representations as $t\uparrow\omega$

$$\frac{\pi_\omega(t)y^{(k+1)}(t)}{y^{(k)}(t)} = \frac{a_{0k+1}}{(\lambda^0_{n-1}-1)}[1+o(1)], \quad (k=0,..,n-1) \tag{2.9}$$

in case $\lambda^0_{n-1}\in\mathbb{R}\setminus\{0,1,\frac{1}{2},\frac{2}{3},\dots,\frac{n-2}{n-1}\}$. Then, for every $j\in\{0,\dots,n-1\}$, according to the equality

$$\frac{z\left(y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)\right)'}{y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)} = \frac{y^{(j+1)}\left(\left(y^{(i)}\right)^{-1}(z)\right)}{y^{(i+1)}\left(\left(y^{(i)}\right)^{-1}(z)\right)}\cdot\frac{y^{(i)}\left(\left(y^{(i)}\right)^{-1}(z)\right)}{y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)} \tag{2.10}$$

and (2.9), we have

$$\lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}} \frac{z\left(y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)\right)'}{y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)} =$$

$$= \lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}} \frac{\pi_\omega\left(\left(y^{(i)}\right)^{-1}(z)\right) y^{(j+1)}\left(\left(y^{(i)}\right)^{-1}(z)\right)}{y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)}\times$$

$$\times\frac{y^{(i)}\left(\left(y^{(i)}\right)^{-1}(z)\right)}{\pi_\omega\left(\left(y^{(i)}\right)^{-1}(z)\right) y^{(i+1)}\left(\left(y^{(i)}\right)^{-1}(z)\right)} = \frac{a_{0j+1}}{a_{0i+1}}.$$

Therefore by M_3 the function $y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)$ is regularly varying as $z\to Y_i$ ($z\in\Delta_{Y_i}$) of index $\frac{a_{0j+1}}{a_{0i+1}}$ in case $\lambda^0_{n-1}\in\mathbb{R}\setminus\left\{0,1,\frac{1}{2},\frac{2}{3},...,\frac{n-2}{n-1}\right\}$.

By Lemma 10.1 from [7] and (1.5) the asymptotic representations as $t\uparrow\omega$

$$\frac{y^{(k+1)}(t)}{y^{(k)}(t)} = \frac{y^{(l+1)}(t)}{y^{(l)}(t)}[1+o(1)] \quad (l,k=0,..,n-1). \tag{2.11}$$

take place in case $\lambda^0_{n-1}=1$. Then, according to (2.10), in this case we have

$$\lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}} \frac{z\left(y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)\right)'}{y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)} = \lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}} \frac{y^{(j+1)}\left(\left(y^{(i)}\right)^{-1}(z)\right)}{y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)}$$

$$\times\frac{y^{(i)}\left(\left(y^{(i)}\right)^{-1}(z)\right)}{y^{(i+1)}\left(\left(y^{(i)}\right)^{-1}(z)\right)} = 1 = \frac{(n-j-1)-(n-j-2)}{(n-i-1)-(n-i-2)} = \frac{a_{0j+1}}{a_{0i+1}}.$$

Therefore by M_3 the function $y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)$ is regularly varying as $z\to Y_i$ ($z\in\Delta_{Y_i}$) of index $\frac{a_{0j+1}}{a_{0i+1}}$ in case $\lambda^0_{n-1}=1$. □

2.2. *Proof of the main results.*

Proof of Theorems 1 and 2. Necessity. Let $y:[t_0,\omega[\to\Delta_{Y_0}$ be a $P_\omega(\lambda^0_{n-1})$-solution of Equation (1.1), where $\lambda^0_{n-1}\in\mathbb{R}\setminus\left\{0,\frac{1}{2},\frac{2}{3},...,\frac{n-2}{n-1}\right\}$. Then by Lemmas 10.1 and 10.2 from [7], according to (1.3) and (1.5), the asymptotic representations (2.9) take place as $\lambda^0_{n-1}\neq 1$. It follows from (2.9), that (2.2) and the second of conditions (2.1) take place in this case. We get (2.11) if $\lambda^0_{n-1}=1$ like in the proof

of Lemma 1. Putting in (2.11) $k = n-1$, $l = n-2$, we get the first of inequalities (2.6). By virtue of (1.5), corollary 10.1 and note 10.1 from [7] we have as $t \uparrow \omega$ for $i = 0, .., n-2$

$$y^{(i)}(t) = \begin{cases} \dfrac{\left[(\lambda^0_{n-1}-1)\,\pi_\omega(t)\right]^{n-i-1}}{\prod\limits_{j=i+1}^{n-1} a_{0j}}\, y^{(n-1)}(t)[1+o(1)], \text{ if } \lambda^0_{n-1} \neq 1, \\ \\ y^{(n-1)}(t)\left(\dfrac{y^{(n-1)}(t)}{y^{(n)}(t)}\right)^{n-i-1}[1+o(1)], \text{ if } \lambda^0_{n-1} = 1. \end{cases} \tag{2.12}$$

This means that the second of representations (2.4) takes place.

By Lemma 1 the inverse function $\left(y^{(n-1)}\right)^{-1}$ exists for the function $y^{(n-1)}$ on $[t_0, \omega[$. So, we have

$$\theta_i(y^{(i)}(t)) = \theta_i\left(y^{(i)}\left(\left(y^{(n-1)}\right)^{-1}\left(y^{(n-1)}(t)\right)\right)\right) \quad \forall i \in \{0, ..., n-2\}.$$

From M_2 and Lemma 1 it follows, that every function $\theta_i\left(y^{(i)}\left(\left(y^{(n-1)}\right)^{-1}(z)\right)\right)$ $(i \in \{0, ..., n-2\})$ is slowly varying as $z \to Y_{n-1}$ $(z \in \Delta_{Y_{n-1}})$. In case $\lambda^0_{n-1} = 1$ for all $i \in \{1, ..., n\}$ $a_{0i} = 1$. Therefore by Lemma 1 in this case every function $y^{(i)}\left(\left(y^{(n-1)}\right)^{-1}(z)\right)$ $(i \in \{0, ..., n-2\})$ is regularly varying of index 1 as $z \to Y_{n-1}$ $(z \in \Delta_{Y_{n-1}})$. Then the function

$$\psi(z) = \begin{cases} \dfrac{|z|^{\gamma_0}}{\prod\limits_{i=0}^{n-1}\theta_i\left(y^{(i)}\left(\left(y^{(n-1)}\right)^{-1}(z)\right)\right)} \text{ if } \lambda^0_{n-1} \neq 1, \\ \\ \dfrac{z\prod\limits_{i=0}^{n-1}\left|y^{(i)}\left(\left(y^{(n-1)}\right)^{-1}(z)\right)\right|^{-\sigma_i}}{\prod\limits_{i=0}^{n-1}\theta_i\left(y^{(i)}\left(\left(y^{(n-1)}\right)^{-1}(z)\right)\right)} \text{ if } \lambda^0_{n-1} = 1, \end{cases}$$

is regularly varying as $z \to Y_{n-1}$ $(z \in \Delta_{Y_{n-1}})$ of index $\gamma_0 \neq 0$.

For all $i = 0, ..., n-2$ using (2.12) and (1.1), we get as $t \uparrow \omega$

$$\frac{y^{(n)}(t)}{y^{(n-1)}(t)}\psi\left(y^{(n-1)}(t)\right) = I'_k(t)[1+o(1)], \quad k = \begin{cases} 0, \text{ if } \lambda^0_{n-1} \neq 1, \\ 1, \text{ if } \lambda^0_{n-1} = 1. \end{cases} \tag{2.13}$$

By (2.13), Theorem 2.1 from [11] and the definition of the functions $I_k(t)$, we have as $t \uparrow \omega$

$$\psi\left(y^{(n-1)}(t)\right) = \gamma_0 I_k(t)[1+o(1)], \quad k = \begin{cases} 0, \text{ if } \lambda^0_{n-1} \neq 1, \\ 1, \text{ if } \lambda^0_{n-1} = 1. \end{cases} \tag{2.14}$$

If $\lambda_0 \neq 1$ $(k = 0)$ the first of representations (2.4) follows from this.

Using (2.14) and (2.13) we have as $t \uparrow \omega$

$$\frac{y^{(n)}(t)}{y^{(n-1)}(t)} = \frac{I_j'(t)}{\gamma_0 I_j(t)}[1+o(1)], \quad j = \begin{cases} 0, \text{ if } \lambda^0_{n-1} \neq 1, \\ 1, \text{ if } \lambda^0_{n-1} = 1. \end{cases} \tag{2.15}$$

Therefore the first of conditions (2.1) follows from (2.15), where $j = 0$ and (2.9), where $k = n-1$. The second of conditions (2.5) follows from (2.15), (2.11) and (1.3) in case $\lambda^0_{n-1} = 1$.

In view of M_2 and Lemma 1 every function $\theta_i\left(y^{(i)}\left(\left(y^{(n-2)}\right)^{-1}(z)\right)\right)$ $(i \in \{0,\dots,n-1\})$ is slowly varying as $z \to Y_{n-1}$ $(z \in \Delta_{Y_{n-1}})$. Then by Lemma 1 the function

$$\psi_1(z) = \frac{z}{y^{(n-1)}\left(\left(y^{(n-2)}\right)^{-1}(z)\right)}$$

$$\times \left| y^{(n-1)}\left(\left(y^{(n-2)}\right)^{-1}(z)\right) \prod_{i=0}^{n-1} \frac{\left|y^{(i)}\left(\left(y^{(n-2)}\right)^{-1}(z)\right)\right|^{-\sigma_i}}{\theta_i\left(y^{(i)}\left(\left(y^{(n-2)}\right)^{-1}(z)\right)\right)} \right|^{\frac{1}{\gamma_0}}$$

is regularly varying as $z \to Y_{n-1}$ $(z \in \Delta_{Y_{n-1}})$ of index 1 in case $\lambda^0_{n-1} = 1$. By rewriting (2.14), where $k = 1$ as

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} \psi_1\left(y^{(n-2)}(t)\right) = J'(t)[1+o(1)] \quad \text{as } t \uparrow \omega, \tag{2.16}$$

in case $\lambda^0_{n-1} = 1$ due to the Theorem 2.1 from [11] we get

$$\psi_1\left(y^{(n-2)}(t)\right) = J(t)[1+o(1)] \quad \text{as } t \uparrow \omega. \tag{2.17}$$

By (2.11) this representation leads to the second of inequalities (2.6). From (2.17) and (2.16) we have

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} = \frac{J'(t)}{J(t)}[1+o(1)] \quad \text{as } t \uparrow \omega. \tag{2.18}$$

Using (2.11), we get from this representation the second of representations (2.8). By putting the second of representations (2.8) in (2.14), where $k = 1$, we get the first of representations (2.8). From (2.11), (2.15) and (2.18) the first of conditions (2.5) follows.

Sufficiency. Let $\lambda^0_{n-1} \in \mathbb{R}\setminus\{0, \frac{1}{2}, \frac{2}{3}, ..., \frac{n-2}{n-1}\}$. Let also Conditions (6 – 8) take place as $\lambda^0_{n-1} \neq 1$ and Conditions (10 – 12) take place as $\lambda^0_{n-1} = 1$.

Let us consider the function

$$F(s_0, s_1, ..., s_{n-1}) = \begin{pmatrix} \prod_{i=0}^{n-1} \Phi_i(s_i) \\ \frac{s_1}{s_0} \\ \frac{s_2}{s_1} \\ ... \\ \\ \\ \frac{s_{n-1}}{s_{n-2}} \end{pmatrix},$$

where

$$\Phi_i(z) = \int\limits_{Y_i^*}^{z} \frac{d\tau}{\varphi_i(\tau)|\tau|^{1-c_i-\sigma_i}}, \tag{2.19}$$

$$c_i = \begin{cases} \frac{\gamma_0(1+n-m)}{m}, & \text{if } (n-i-1)\lambda^0_{n-1} > n-i-2, \\ \\ -\gamma_0, & \text{if } (n-i-1)\lambda^0_{n-1} < n-i-2, \end{cases} \tag{2.20}$$

m is the quantity of numbers $i \in \{0, ..., n-1\}$, for which $(n-i-1)\lambda^0_{n-1} > n-i-2$,

$$Y_i^* = \begin{cases} y_i^0, & \text{if } \left| \int\limits_{y_i^0}^{Y_i} \frac{dz}{\varphi_i(z)|z|^{1-c_i-\sigma_i}} \right| = +\infty, \\ \\ Y_i, & \text{if } \left| \int\limits_{y_i^0}^{Y_i} \frac{dz}{\varphi_i(z)|z|^{1-c_i-\sigma_i}} \right| < +\infty \end{cases} \qquad (i = 0, ..., n-1)$$

on the set $\Delta = \Delta^1_{Y_0} \times ... \times \Delta^1_{Y_{n-1}}$. For every $i \in \{0, ..., n-1\}$ the number $y_i^1 \in \Delta_{Y_i}$ is chosen in such a way, that for $z_i \in \Delta^1_{Y_i}$, where

$$\Delta^1_{Y_i} = \begin{cases} [y_i^1, Y_i[, & \text{if } \Delta_{Y_i} = [y_i^0, Y_i[, \\ \\ ]Y_i, y_i^1], & \text{if } \Delta_{Y_i} =]Y_i, y_i^0], \end{cases}$$

the next inequality holds

$$\left|\frac{z_i\Phi_i'(z_i)}{\Phi_i(z_i)}-c_i\right|<\frac{\left|1-\sum\limits_{i=0}^{n-1}\sigma_i\right|}{8mn}. \tag{2.21}$$

It is easy to see that the function Φ_i is monotone on Δ_{Y_i} for every $i\in\{0,...,n-1\}$. In view of the second of conditions (2.1) and (2.20) the following equality holds

$$\prod_{i=0}^{n-1}\lim_{\substack{z_i\to Y_i\\ z\in\Delta_{Y_i}}}\Phi_i(z_i)=\Phi_{0n}, \tag{2.22}$$

where

$$\Phi_{0n}=\begin{cases}\infty, & \text{if } \pi_\omega(t)(\lambda_{n-1}^0-1)\gamma_0>0, \text{ or } \alpha_0 I_1(t)>0 \text{ if } \lambda_{n-1}^0=1,\\ 0, & \text{if } \pi_\omega(t)(\lambda_{n-1}^0-1)\gamma_0<0, \text{ or } \alpha_0 I_1(t)<0 \text{ if } \lambda_{n-1}^0=1.\end{cases} \tag{2.23}$$

We have also

$$\lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}}\frac{z_i\Phi_i'(z_i)}{\Phi_i(z_i)}=c_i\quad(i=0,...,n-1),\qquad\sum_{i=0}^{n-1}c_i=\gamma_0. \tag{2.24}$$

We will show that F is the one to one correspondence between the set Δ and the set

$$F(\Delta)=\begin{cases}\left[\prod\limits_{i=0}^{n-1}\Phi_i(y_i^1);\Phi_{0n}\right)\times\Delta_0\times...\times\Delta_{n-2}, & \text{if } \prod\limits_{i=0}^{n-1}\Phi_i(y_i^1)<\Phi_{0n},\\ \left(\Phi_{0n};\prod\limits_{i=0}^{n-1}\Phi_i(y_i^1)\right]\times\Delta_0\times...\times\Delta_{n-2}, & \text{if } \prod\limits_{i=0}^{n-1}\Phi_i(y_i^1)>\Phi_{0n},\end{cases}$$

where

$$\Delta_i=\begin{cases}(0;+\infty), & \text{if } a_{0i+2}a_{0i+1}>0,\ y_i^0y_{i+1}^0>0,\\ (-\infty;0), & \text{if } a_{0i+2}a_{0i+1}>0,\ y_i^0y_{i+1}^0<0,\\ \left[\frac{y_{i+1}^1}{y_i^1};Y_i^0\right), & \text{if } a_{0i+2}a_{0i+1}<0,\ \frac{y_{i+1}^1}{y_i^1}<Y_i^0,\\ \left(Y_i^0;\frac{y_{i+1}^1}{y_i^1}\right], & \text{if } a_{0i+2}a_{0i+1}<0,\ \frac{y_{i+1}^1}{y_i^1}>Y_i^0,\end{cases}$$

$$Y_i^0=\begin{cases}Y_i, & \text{if } Y_{i+1}=0,\\ -\infty, & \text{if } Y_{i+1}=\infty,\end{cases}\qquad i=0,...,n-2.$$

Let us suppose that F is not a one to one mapping. Then

$$\exists\,(p_0,...,p_{n-1}),(q_0,...,q_{n-1})\in\Delta,\ (p_0,...,p_{n-1})\neq(q_0,...,q_{n-1})$$

such that

$$F(p_0,...,p_{n-1})=F(q_0,...,q_{n-1}).$$

By definition of the set Δ the last equality means that

$$\prod_{i=0}^{n-1}\Phi_i(p_i)=\prod_{i=0}^{n-1}\Phi_i(q_i),\quad \frac{p_{j+1}}{p_j}=\frac{q_{j+1}}{q_j}=k_j\in R\setminus\{0\},\quad j=1,...,n-1. \tag{2.25}$$

Then the points $(p_0,...,p_{n-1})$ and $(q_0,...,q_{n-1})$ lie on the one line

$$\frac{s_0}{1}=\frac{s_1}{k_1}=\frac{s_2}{k_1k_2}=...=\frac{s_{n-1}}{k_1...k_{n-1}}.$$

On this line $\prod_{i=0}^{n-1}\Phi_i(s_i)=\Phi_0(s_0)\prod_{i=1}^{n-1}\Phi_i(k_1...k_is_0)$. We obtain also

$$\left(\Phi_0(s_0)\prod_{i=1}^{n-1}\Phi_i(k_1...k_is_0)\right)'_{s_0}=$$

$$=\frac{\Phi_0(s_0)\prod_{i=1}^{n-1}\Phi_i(k_1...k_is_0)}{s_0}\left(\frac{s_0\Phi_0'(s_0)}{\Phi_0(s_0)}+\sum_{i=1}^{n-1}\frac{k_1...k_is_0\Phi_i'(k_1...k_is_0)}{\Phi_i(k_1...k_is_0)}\right).$$

By (2.22), (2.21) and the definition of the set Δ this means, that

$$sign\left(\left(\Phi_0(s_0)\prod_{i=1}^{n-1}\Phi_i(k_1...k_is_0)\right)'_{s_0}\right)=sign\left(y_0^0\Phi^0\left(1-\sum_{i=0}^{n-1}\sigma_i\right)\right),$$

where

$$\Phi^0=\prod_{i=0}^{n-1}sign\Phi_i(z_i),\quad (z_i\in\Delta^1_{Y_i}). \tag{2.26}$$

Therefore the function $\Phi_0(s_0)\prod_{i=1}^{n-1}\Phi_i(k_1...k_is_0)$ is strongly monotone on this line. But then (2.25) is impossible. So, there exists the inverse function $F^{-1}:\ F(\Delta)\to\Delta$. Looking on the function F, we obtain

$$F^{-1}(w_0,...,w_{n-1})=$$

$$= \begin{pmatrix} F_0^{-1}(w_0,...,w_{n-1}) \\ F_1^{-1}(w_0,...,w_{n-1}) \\ ... \\ F_{n-1}^{-1}(w_0,...,w_{n-1}) \end{pmatrix} = \begin{pmatrix} F_0^{-1}(w_0,...,w_{n-1}) \\ w_1 F_0^{-1}(w_0,...,w_{n-1}) \\ ... \\ w_1...w_{n-1} F_0^{-1}(w_0,...,w_{n-1}) \end{pmatrix}.$$

Let us show that we can choose the number $t_0 \in [a,\omega[$ in such a way, that

$$(H_0(t),...,H_{n-1}(t)) \in F(\Delta) \text{ if } t \in [t_0,\omega[, \tag{2.27}$$

where

$$H_0(t) = \gamma_0 BT(t) \left| \frac{L(t)}{L'(t)} \right|^{\sum_{i=0}^{n-1}(n-i-1)c_i},$$

$$H_i(t) = \begin{cases} \dfrac{a_{0i}}{(\lambda_{n-1}^0 - 1)\pi_\omega(t)} \text{ if } \lambda_{n-1}^0 \neq 1, \\ \\ \dfrac{J'(t)}{J(t)} \text{ if } \lambda_{n-1}^0 = 1, \end{cases} \qquad i = 1,...,n-1,$$

$$T(t) = \begin{cases} I_0(t) \text{ if } \lambda_{n-1}^0 \neq 1, \\ \\ I_1(t) \left| \dfrac{J(t)}{J'(t)} \right|^{\mu_n} \text{ if } \lambda_{n-1}^0 = 1, \end{cases} \qquad L(t) = \begin{cases} \pi_\omega(t) \text{ if } \lambda_{n-1}^0 \neq 1, \\ \\ J(t) \text{ if } \lambda_{n-1}^0 = 1, \end{cases}$$

$$B = \begin{cases} \prod_{i=0}^{n-1} \left(\dfrac{signy_i^0}{c_i} |\lambda_{n-1}^0 - 1|^{(n-i-1)c_i} \prod_{j=i+1}^{n-1} |a_{0j}|^{-c_i} \right) \text{ if } \lambda_{n-1}^0 \neq 1, \\ \\ \prod_{i=0}^{n-1} \dfrac{signy_i^0}{c_i} signy_{n-1}^0 \text{ if } \lambda_{n-1}^0 = 1. \end{cases}$$

By the type of the set $F(\Delta)$it is sufficient to prove that

$$sign H_0(t) = \Phi^0 \text{ if } t \in [a,\omega[, \tag{2.28}$$

for all $i = 1,...,n-1$

$$sign H_i(t) = signy_i^0 y_{i-1}^0 \text{ if } t \in [a,\omega[, \tag{2.29}$$

$$\lim_{t \uparrow \omega} H_0(t) = \Phi_{0n}, \tag{2.30}$$

and for all $i = 1,...,n-1$ if $a_{0i+1}a_{0i} < 0$

$$\lim_{t \uparrow \omega} H_i(t) = Y_{i-1}^0. \tag{2.31}$$

By (2.24)

$$\Phi_i(z) = \frac{|z|^{c_i}}{c_i\theta_i(z)} sign z[1+o(1)] \text{ as } z \to Y_i \ \left(z \in \Delta^1_{Y_i}\right). \tag{2.32}$$

Therefore $sign\Phi_i(z) = sign(y_i^0 c_i)$. Then by (2.26)

$$\Phi^0 = sign \prod_{i=0}^{n-1} y_i^0 c_i \tag{2.33}$$

Using the first of conditions (2.1), we obtain in case $\lambda^0_{n-1} \neq 1$

$$\pi_\omega(t)(\lambda^0_{n-1}-1)y^0_{n-1} > 0 \text{ if } t \in [a,\omega[.$$

From this inequality and the condition (2.2), where $i = n-1$, it follows that $\gamma_0 I_0(t) > 0$ for $t \in [a,\omega[$ in case $\lambda^0_{n-1} \neq 1$. Therefore, in this case from (2.33) we have (2.28).

Using the first of conditions (2.5), in case $\lambda^0_{n-1} = 1$ we obtain

$$\gamma_0\alpha_0 I_1(t)J(t) > 0 \text{ if } t \in [a,\omega[.$$

From this inequality and the condition (2.6), where $i = n-1$, it follows that $\alpha_0 y^0_{n-1} J(t) > 0$ for $t \in [a,\omega[$ in case $\lambda^0_{n-1} = 1$. Therefore, $\gamma_0 y^0_{n-1} I_1(t) > 0$ for $t \in [a,\omega[$ and from (2.33) we have (2.28) in this case.

The equality (2.29) follows from (2.2) in case $\lambda^0_{n-1} \neq 1$. If $\lambda^0_{n-1} = 1$ we get (2.29) due to the second of conditions (2.6).

It is easy to see, that if $i \in \{1,...,n-1\}$ and

$$\frac{a_{0i}}{a_{0i+1}} = \frac{(n-i)\lambda^0_{n-1}-(n-i-1)}{(n-i-1)\lambda^0_{n-1}-(n-i-2)} = 1 + \frac{\lambda^0_{n-1}-1}{(n-i-1)\lambda^0_{n-1}-(n-i-2)} < 0.$$

the next inequality

$$\frac{\lambda^0_{n-1}-1}{(n-i-1)\lambda^0_{n-1}-(n-i-2)} < 0 \tag{2.34}$$

takes place. By (2.34), the second of conditions (2.1), the definition of the function $\pi_\omega(t)$ and the definition of Y_1^0 we have (2.31).

Let us consider the equality

$$\frac{L(t)H_0'(t)}{L'(t)H_0(t)} = \frac{L(t)T'(t)}{L'(t)T(t)} + \left(\sum_{i=0}^{n-1}(n-i-1)c_i\right)\left(1-\frac{L(t)L''(t)}{(L'(t))^2}\right).$$

Here

$$\frac{L(t)T'(t)}{L'(t)T(t)} = \begin{cases} \dfrac{\pi_\omega(t)I_0'(t)}{I_0(t)} & \text{if } \lambda^0_{n-1} \neq 1, \\ \dfrac{J(t)I_1'(t)}{J'(t)I_1(t)} + \mu_n\left(1-\dfrac{J(t)J''(t)}{(J'(t))^2}\right) & \text{if } \lambda^0_{n-1} = 1, \end{cases}$$

$$\frac{L(t)L''(t)}{(L'(t))^2} = \begin{cases} 0 \text{ if } \lambda_{n-1}^0 \neq 1, \\ \\ \dfrac{J(t)J''(t)}{(J'(t))^2} \text{ if } \lambda_{n-1}^0 = 1. \end{cases}$$

In case $\lambda_{n-1}^0 = 1$

$$\frac{J(t)J''(t)}{(J'(t))^2} = \frac{J(t)I_1'(t)}{\gamma_0 J'(t)I_1(t)}. \tag{2.35}$$

So, using the first of conditions (2.1) and the first of conditions (2.5), we have as $t \uparrow \omega$

$$\frac{L(t)H_0'(t)}{L'(t)H_0(t)} = \begin{cases} \sum\limits_{i=0}^{n-1} \dfrac{a_{0i+1}c_i}{\lambda_{n-1}^0 - 1}[1 + o(1)] \text{ if } \lambda_{n-1}^0 \neq 1, \\ \\ \gamma_0[1 + o(1)] \text{ if } \lambda_{n-1}^0 = 1. \end{cases} \tag{2.36}$$

The next representation

$$\lim_{t\uparrow\omega} H_0(t) = \begin{cases} \infty, \quad \text{if } \pi_\omega(t)(\lambda_{n-1}^0 - 1)\left(\sum\limits_{i=0}^{n-1} a_{0i+1}c_i\right) > 0 \text{ or } \gamma_0 J(t) > 0, \\ \\ 0, \quad \text{if } \pi_\omega(t)(\lambda_{n-1}^0 - 1)\left(\sum\limits_{i=0}^{n-1} a_{0i+1}c_i\right) < 0 \text{ or } \gamma_0 J(t) < 0. \end{cases} \tag{2.37}$$

follows from (2.36) by the type of functions $L(t)$ and $H_0(t)$.

By definition of numbers c_i, the inequality $\gamma_0 a_{0i+1}c_i > 0$ takes place for all $i \in \{0, ..., n-1\}$. Therefore, if $\lambda_{n-1}^0 \neq 1$ we obtain (2.30) using (2.37) and (2.2). The existence of such $b_0 \in [a, \omega[$ that for any $t \in [b_0, \omega[$ the inequality $\alpha_0 I(t)\gamma_0 J(t) > 0$ takes place, follows from the first of representations (2.5) in case $\lambda_{n-1}^0 = 1$. This leads to (2.30) by (2.23).

So, we have (2.27).

Let us introduce the next notations for all $i \in \{0, ..., n-1\}$

$$Y^{[i]}(t) = F_i^{-1}(H_0(t), ..., H_{n-1}(t)).$$

From the properties of the function F and the functions H_0,...,H_{n-1} it is easy to see, that for all $i \in \{0, ..., n-1\}$ $Y^{[i]}$ is continuously differentiable on $[t_0, \omega[$.

Using (2.1), (2.2), (2.5), (2.6) and the transformation of Equation (1.1)

$$y^{(i)}(t) = Y^{[i]}(t)[1 + z_i(x)], \tag{2.38}$$

where

$$x = \beta \ln |L(t)|, \quad \beta = \begin{cases} 1 & \text{if} \quad \lim\limits_{t\uparrow\omega} L(t) = \infty, \\ -1 & \text{if} \quad \lim\limits_{t\uparrow\omega} L(t) = 0, \end{cases} \tag{2.39}$$

we get the system of differential equations

$$\begin{cases} z_i' = \dfrac{\beta L(t(x))}{L'(t(x))Y^{[i]}(t(x))}\Big[-(Y^{[i]})'(t(x))[1+z_i]+ \\ \qquad\qquad +Y^{[i+1]}(t(x))[1+z_{i+1}]\Big],\ (i=0,...,n-2) \\ z_{n-1}' = \dfrac{\beta L(t(x))}{L'(t(x))}\Bigg[G(t(x))\prod_{j=0}^{n-1}|1+z_j|^{\sigma_j}Q_j(x,z_j)- \\ \qquad\qquad -\dfrac{\left(Y^{[n-1]}\right)'(t(x))}{Y^{[n-1]}(t(x))}[1+z_{n-1}]\Bigg], \end{cases} \tag{2.40}$$

where

$$G(t)=\frac{\alpha_0 p(t)\prod_{j=0}^{n-1}\varphi_j\left(Y^{[j]}(t)\right)}{Y^{[n-1]}(t)},\quad Q_j(x,z_j)=\frac{\theta_j\left(Y^{[j]}(t(x))[1+z_j]\right)}{\theta_j\left(Y^{[j]}(t(x))\right)},$$

$j=0,...,n-1$, $t(x)$ is the inverse function for the function $x=\beta\ln|L(t)|$.

Then we consider the system (2.40) on the set

$$\Omega=[x_0,+\infty[\times D,\qquad x_0=\beta\ln|\pi_\omega(t_0)|,$$
$$D=\{(z_1,z_2):|z_i|\le\varepsilon,\ i=1,2\}.$$

The system (2.40) can be rewritten as the system

$$z_i'=\sum_{j=0}^{n-1}A_{ij}z_j+R_1^{[i]}(x,z_0,...,z_{n-1})+R_2^{[i]}(x,z_0,...,z_{n-1}),\quad i=0,...,n-1, \tag{2.41}$$

where

$$A_{ij}=\begin{cases} -\dfrac{\beta a_{0i+1}}{\lambda_{n-1}^0-1} & \text{as } j=i,\ \lambda_{n-1}^0\neq 1, \\ \dfrac{\beta a_{0i+1}}{\lambda_{n-1}^0-1} & \text{as } j=i+1,\ \lambda_{n-1}^0\neq 1, \\ -\beta & \text{as } j=i,\ \lambda_{n-1}^0=1, \\ \beta & \text{as } j=i+1,\ \lambda_{n-1}^0=1, \\ 0 & \text{as } j\notin\{i,i+1\}, \end{cases}$$
$$if\quad i=0,...,n-2,\ j=0,...,n-1,$$

$$A_{n-1\ j} = \begin{cases} \dfrac{\beta\sigma_j}{\lambda_{n-1}^0 - 1} & \text{as } j \neq n-1,\ \lambda_{n-1}^0 \neq 1, \\ \dfrac{\beta(\sigma_{n-1}-1)}{\lambda_{n-1}^0 - 1} & \text{as } j = n-1,\ \lambda_{n-1}^0 \neq 1, \\ \beta\sigma_j & \text{as } j \neq n-1,\ \lambda_{n-1}^0 = 1, \\ \beta(\sigma_{n-1}-1) & \text{as } j = n-1,\ \lambda_{n-1}^0 = 1, \end{cases}$$

$$if \quad j = 0, ..., n-1,$$

$$R_1^{[i]}(x, z_0, ..., z_{n-1}) =$$

$$= \begin{cases} 0 & \text{as } i = 0, ..., n-2, \\ \dfrac{\prod\limits_{j=0}^{n-1} |1+z_j|^{\sigma_j} - \sum\limits_{j=0}^{n-1} \sigma_j z_j}{\beta(\lambda_{n-1}^0 - 1)} & \text{as } i = n-1,\ \lambda_{n-1}^0 \neq 1, \\ \beta\left(\prod\limits_{j=0}^{n-1} |1+z_j|^{\sigma_j} - \sum\limits_{j=0}^{n-1} \sigma_j z_j\right) & \text{as } i = n-1,\ \lambda_{n-1}^0 = 1, \end{cases}$$

$$R_2^{[i]}(x, z_0, ..., z_{n-1}) = \beta[1+z_i]\left(G_0 a_{0i+1} - \frac{L(t(x))(Y^{[i]})'(t(x))}{L'(t(x))Y^{[i]}(t(x))}\right) +$$

$$+\beta[1+z_{i+1}]\left(\frac{L(t(x))Y^{[i+1]}(t(x))}{L'(t(x))Y^{[i]}(t(x))} - G_0 a_{0i+1}\right) \text{ as } i = 0, ..., n-2,$$

$$R_2^{[n-1]}(x, z_0, ..., z_{n-1}) =$$

$$= \beta \prod_{j=0}^{n-1} |1+z_j|^{\sigma_j} \left(\frac{L(t(x))}{L'(t(x))} G(t(x)) \left(\prod_{j=0}^{n-1} Q_j(x, z_j) - 1\right) + \right.$$

$$\left. + \frac{L(t(x))}{L'(t(x))} G(t(x)) - G_0\right) + \left(\frac{L(t(x))(Y^{[n-1]})'(t(x))}{L'(t(x))Y^{[n-1]}(t(x))} - G_0\right)(1+z_{n-1})$$

$$G_0 = \begin{cases} \dfrac{1}{\lambda_{n-1}^0 - 1} \text{ as} \lambda_{n-1}^0 \neq 1, \\ 1 \text{ if } \lambda_{n-1}^0 = 1. \end{cases}$$

It is clear that

$$\lim_{|z_0|+...+|z_{n-1}|\to 0} \frac{R_1^{[i]}(x, z_0, ..., z_{n-1})}{|z_0| + ... + |z_{n-1}|} = 0 \quad (i = 0, ..., n-1)$$

uniformly on $x \in [x_0, +\infty[$.

To prove that

$$\lim_{x\to+\infty} R_2^{[i]}(x, z_0, ..., z_{n-1}) = 0 \quad (i = 0, ..., n-1) \tag{2.42}$$

uniformly on $(z_0, ..., z_{n-1}) \in D$, we have to prove that for all $j \in \{0, ..., n-2\}$

$$\lim_{t\uparrow\omega} \frac{L(t)(Y^{[j]})'(t)}{L'(t)Y^{[j]}(t)} = G_0 a_{0j+1}, \quad \lim_{t\uparrow\omega} \frac{L(t)Y^{[j+1]}(t)}{L'(t)Y^{[j]}(t)} = G_0 a_{0j+1}, \tag{2.43}$$

$$\lim_{t\uparrow\omega} \frac{L(t)G(t)}{L'(t)} = G_0 \tag{2.44}$$

and for all $i \in \{0, ..., n-1\}$

$$\lim_{t\uparrow\omega} \frac{\theta_i\left(Y^{[i]}(t)[1+z_i]\right)}{\theta_i\left(Y^{[i]}(t)\right)} = 1, \quad \text{uniformly for} \quad |z_i| < \frac{1}{2}. \tag{2.45}$$

Let us show at first that for all $i \in \{0, ..., n-1\}$

$$\lim_{t\uparrow\omega} Y^{[i]}(t) = Y_i. \tag{2.46}$$

The inequality

$$\left| \frac{Y^{[i]}(t)\Phi_i'(Y^{[i]}(t))}{\Phi_i(Y^{[i]}(t))} - c_i \right| < \frac{|\gamma_0|}{8mn}$$

for all $t \in [t_0, \omega)$ follows from (2.27) and the definition of the set Δ. By this inequality for $t \in [t_0, \omega[$ we have

$$-\frac{|\gamma_0|}{8mn} + c_i < \frac{Y^{[i]}(t)\Phi_i'(Y^{[i]}(t))}{\Phi_i(Y^{[i]}(t))} < \frac{|\gamma_0|}{8mn} + c_i, \tag{2.47}$$

From (2.47), (2.36), (2.35), the first of conditions (2.2), the first of conditions (2.6) and the equalities

$$\frac{L(t)\left(Y^{[i]}(t)\right)'_t}{L'(t)Y^{[i]}(t)} = \sum_{k=0}^{n-1} \frac{L(t)H_k'(t)}{L'(t)H_k(t)} P_{ik}(t), \tag{2.48}$$

$$\frac{L(t)H_j'(t)}{L'(t)H_j(t)} = \begin{cases} 1 \text{ if } \lambda_{n-1}^0 \neq 1, \\ \dfrac{J''(t)J(t)}{(J'(t))^2} - 1 \text{ if } \lambda_{n-1}^0 = 1, \end{cases} \qquad (j = 1, ..., n-1)$$

where for all $i \in \{0,...,n-1\}$, $k \in \{1,...,n-1\}$

$$P_{i0}(t) = \frac{1}{\sum\limits_{j=0}^{n-1} \frac{Y^{[j]}(t)\Phi_j'(Y^{[j]}(t))}{\Phi_i(Y^{[j]}(t))}},\ P_{ik}(t) = l_k^i + P_{i0}(t)\sum_{j=0}^{k-1} \frac{Y^{[j]}(t)\Phi_j'(Y^{[j]}(t))}{\Phi_i(Y^{[j]}(t))},$$

$$l_k^i = \begin{cases} -1 \text{ if } i \le k-1, \\ \\ 0 \text{ if } i > k-1, \end{cases}$$

$$-\sum_{k=1}^{n-1}\left(l_k^i + \frac{1}{\gamma_0}\sum_{j=0}^{k-1} c_j\right) + \frac{1}{\gamma_0(\lambda_{n-1}^0 - 1)}\sum_{i=0}^{n-1} a_{0i}c_i =$$

$$= -\left(n-i-1+\frac{1}{\gamma_0}\sum_{k=1}^{n-1}\sum_{j=0}^{n-1} l_k^j c_j\right) + \frac{1}{\gamma_0(\lambda_{n-1}^0-1)}\sum_{i=0}^{n-1} a_{0i}c_i$$

$$= -\left(n-i-1+\frac{1}{\gamma_0}\sum_{j=0}^{n-1}(n-j-1)c_j\right) + \frac{\sum\limits_{i=0}^{n-1} c_i}{\gamma_0(\lambda_{n-1}^0-1)} + \tag{2.49}$$

$$+\frac{1}{\gamma_0}\sum_{i=0}^{n-1}(n-i-1)c_i = \frac{1}{\lambda_{n-1}^0-1} - (n-i-1) = \frac{a_{0i+1}}{\lambda_{n-1}^0-1},$$

it follows that for every $i \in \{0,1\}$ there exist constants $k_0^i, k_1^i \in \mathbb{R}$ such that

$$signk_0^i = signk_1^i = \begin{cases} sign(a_{0i+1}(\lambda_0 - 1)) \text{ if } \lambda_{n-1}^0 \ne 1, \\ \\ 1 \text{ if } \lambda_{n-1}^0 = 1, \end{cases}$$

and the number $t_2 \in [t_1, \omega[$ such that

$$\frac{k_0^i L'(t)}{L(t)} < \frac{\left(Y^{[i]}(t)\right)'}{Y^{[i]}(t)} < \frac{k_1^i L'(t)}{L(t)} \text{ for} t \in [t_2, \omega[.$$

By integrating this inequality on $[t_2, t]$ we will get

$$\ln\frac{|L(t)|^{k_0^i}}{|L(t_2)|^{k_0^i}} + \ln\left|Y^{[i]}(t_2)\right| < \ln\left|Y^{[i]}(t)\right| < \ln\frac{|L(t)|^{k_1^i}}{|L(t_2)|^{k_1^i}} + \ln\left|Y^{[i]}(t_2)\right|.$$

From (2.27) and one to one property of F it is clear, that $signY^{[i]}(t) = signy_i^0$ for every $i \in \{0,...,n-1\}$. So, we have (2.46).

The representation (2.45) follows from (2.46) M_3 and M_4.

By the definition of $Y^{[i]}$ $(i = 0, ..., n-2)$

$$F(Y^{[0]}(t), ..., Y^{[n-1]}(t)) = (H_0(t), ..., H_{n-1}(t)) \quad \forall t \in [t_2, \omega[.$$

Therefore

$$\frac{Y^{[i+1]}(t)}{Y^{[i]}(t)} = H_{i+1}(t) \quad \forall i = 0, ..., n-2.$$

So

$$\frac{L(t)Y^{[i+1]}(t)}{L'(t)Y^{[i]}(t)} = \begin{cases} \dfrac{a_{0i+1}}{\lambda^0_{n-1} - 1} \text{ if } \lambda^0_{n-1} \neq 1, \\ \\ 1 \text{ if } \lambda^0_{n-1} = 1 \end{cases}$$

and we have the second of representations (2.43). By (2.46), (2.48) and (2.24), we have

$$\lim_{t \uparrow \omega} \frac{L(t)(Y^{[i]}(t))'}{L'(t)Y^{[i]}(t)} = \begin{cases} -\sum\limits_{k=1}^{n-1} \left(l^i_k + \dfrac{1}{\gamma_0} \sum\limits_{j=0}^{k-1} c_j \right) + \dfrac{\sum\limits_{i=0}^{n-1} a_{0i} c_i}{\gamma_0(\lambda^0_{n-1} - 1)} \text{ if } \lambda^0_{n-1} \neq 1, \\ \\ 1 \text{ if } \lambda^0_{n-1} = 1. \end{cases}$$

We get the first of representations (2.43), using the equality (2.49).

By virtue of (2.32), we have the asymptotic representation

$$G(t) = \frac{\alpha_0 p(t) \prod\limits_{i=0}^{n-1} \left| Y^{[i]}(t) \right|^{\sigma_i + c_i} [1 + o(1)]}{Y^{[n-1]}(t) \prod\limits_{i=0}^{n-1} \Phi_i \left(Y^{[i]}(t) \right) \prod\limits_{i=0}^{n-1} c_i sign y^0_i} \quad \text{as } t \uparrow \omega.$$

From this representation and the equalities

$$\frac{\alpha_0 p(t) \prod\limits_{i=0}^{n-1} \left| Y^{[i]}(t) \right|^{\sigma_i + c_i}}{Y^{[n-1]}(t) \prod\limits_{i=0}^{n-1} \Phi_i \left(Y^{[i]}(t) \right) \prod\limits_{i=0}^{n-1} c_i sign y^0_i} = \frac{\alpha_0 p(t) \prod\limits_{i=0}^{n-1} \left| \frac{Y^{[i]}(t)}{Y^{[n-1]}(t)} \right|^{\sigma_i + c_i}}{H_0(t) \prod\limits_{i=0}^{n-1} c_i sign y^0_i sign y^0_{n-1}},$$

$$\frac{Y^{[i]}(t)}{Y^{[n-1]}(t)} = \prod_{k=i}^{n-2} \frac{Y^{[k]}(t)}{Y^{[k+1]}(t)} = \prod_{k=i}^{n-2} \frac{1}{H_{k+1}(t)}$$

it follows that as $t \uparrow \omega$

$$\frac{L(t)G(t)}{L'(t)} = \begin{cases} \dfrac{\pi_\omega(t) I'_0(t)}{\gamma_0 I_0(t)} [1 + o(1)] \text{ if } \lambda^0_{n-1} \neq 1, \\ \\ \dfrac{I'_1(t) J(t)}{\gamma_0 I_1(t) J'(t)} [1 + o(1)] \text{ if } \lambda^0_{n-1} = 1. \end{cases}$$

This leads to (2.44) by the first of conditions (2.1) and the first of conditions (2.5). So, (2.42) takes place.

Equation (2.3) is the characteristic equation for the matrix of coefficients of linear part of system (2.41) in case $\lambda^0_{n-1} \neq 1$. Equation (2.7) is the characteristic equation for this matrix in case $\lambda^0_{n-1} = 1$. By Conditions of Theorems 1 and 2 these equations have no roots with zero real part. Therefore all conditions of Theorem 2.2 from [9] are satisfied for system (2.41). By this Theorem the system (2.40) has at least one solution $\{z_i\}_{i=0}^{n-1} : [x_1, +\infty[\longrightarrow \mathbb{R}^n$ $(x_1 \geq x_0)$, that follows to zero as $x \to +\infty$. So, due to (2.38), (2.39) and the equality

$$F_i\left(Y^{[0]}(t), Y^{[1]}(t), ..., Y^{[n-1]}(t)\right) = H_i(t),$$

we get that Equation (1.1) has at least one solution y that admits the next asymptotic representations as $t \uparrow \omega$

$$\prod_{j=0}^{n-1} \Phi_j(y^{(j)}(t)) = H_0(t)[1 + o(1)], \quad \frac{y^{(i)}(t)}{y^{(i-1)}} = H_i(t)[1 + o(1)], \quad i = 0, ..., n-2. \tag{2.50}$$

By (2.19) and the second of representations (2.50), the first of representations (2.50) can be rewritten like the first of representations (2.4) in case $\lambda^0_{n-1} \neq 1$. In case $\lambda^0_{n-1} = 1$ let us rewrite the first of representations (2.50) like the first of representations (2.8). If $\lambda^0_{n-1} \neq 1$ we rewrite the second of representations (2.50) like the second of representations (2.8). By these representations and (1.1) it is clear that y is a $P_\omega(\lambda^0_{n-1})$-solution of (1.1). The Theorems 1 and 2 are proved.

Proof of Corollaries 1, 2. Let $y : [t_0, \omega[\to \Delta_{Y_0}$ be a $P_\omega(\lambda^0_{n-1})$-solution of Equation (1.1), where $\lambda^0_{n-1} \in \mathbb{R} \setminus \{0, \frac{1}{2}, \frac{2}{3}, ..., \frac{n-2}{n-1}\}$. It follows from (2.9), that in case $\lambda^0_{n-1} \neq 1$ the function $y^{(i)}(t)$ is regularly varying of the index $\frac{a_{0i+1}}{\lambda^0_{n-1} - 1}$. Then, by M_1, using the condition S, from the first of representations (2.4) we get the first of representations in Corollary 1.

In case $\lambda^0_{n-1} = 1$ it follows from (2.18), that for all $i \in \{0, ...n-1\}$

$$\lim_{t\uparrow\omega} \frac{J(t)\left(\frac{y^{(i)}(t)}{J(t)}\right)' \cdot \frac{1}{J'(t)}}{\frac{y^{(i)}(t)}{J(t)}} = \lim_{t\uparrow\omega} \frac{J(t)}{J'(t)}\left(\frac{y^{(i+1)}(t)}{y^{(i)}(t)} - \frac{J'(t)}{J(t)}\right) =$$

$$= \lim_{t\uparrow\omega} \frac{J(t)y^{(i+1)}(t)}{J'(t)y^{(i)}(t)} - 1 = 0.$$

So, the next representation

$$y^{(i)}(t) = |J(t)|L(|J(t)|y_i^0)signy_i^0,$$

where the function $L(z)$ is slowly varying as $z \to Y_i$, takes place for all $t \in [t_0, \omega[$. Then, by M_1, using the condition S, we get from the first of representations (2.8) the first of representations in Corollary 2.

The second representations in Corollary 1 are the second of representations (2.4). The second representations in Corollary 2 are the second of representations (2.8).

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AN ACCELERATION TECHNIQUE FOR SLOWLY CONVERGENT FIXED POINT ITERATIVE METHODS

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Abstract. In this paper we present a new acceleration method for accelerating some numerical sequences of successive approximations that arises in solving certain nonlinear equations. We give a convergence result and some numerical examples to illustrate the abilities of the proposed new acceleration technique.

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Keywords: fixed point, Picard iterations, Aitken's Δ^2 process, order of convergence

1. INTRODUCTION

In the last decade an important domain of numerical analysis was the development and the improvement of several extrapolation algorithms. By an extrapolation process a scalar sequence $\{S_n\}$, convergent to S, is transformed into a new sequence $\{T_n\}$, through a sequence transformation $T : \{S_n\} \to \{T_n\}$. It is said that the transformation T accelerates the convergence speed of the sequence $\{S_n\}$ if it is satisfying the following relation

$$\lim_{n\to\infty} \frac{T_n - S}{S_n - S} = 0. \tag{1.1}$$

In this case we say that $\{T_n\}$ converges faster to S, than $\{S_n\}$.
The aim of this paper is twofold: first, to present a new acceleration algorithm inspired from one of the many representations of Aitken's iterated Δ^2 process for which we give a convergence result and secondly, to perform with this technique an empirical study for the sequences arising in solving nonlinear equations for some of the test functions that can be found in literature.

2. THE NEW ACCELERATION METHOD

Consider a nonlinear equation

$$f(x) = 0 \tag{2.1}$$

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which is equivalently written as a fixed point problem

$$g(x) = x \tag{2.2}$$

where $g : [a,b] \to [a,b]$ is the iteration function.
We assume that α is the unique root of (2.1) in the interval $[a,b]$ and let $x_0 \in [a,b]$ be an initial approximation sufficiently close to α. Under appropriate assumptions on f (and therefore on g), the Picard iteration

$$x_{n+1} = g(x_n), \quad n = 0,1,\ldots \tag{2.3}$$

converges to the (unique) fixed point x^* of g, which is the (unique) solution of (2.1) in the interval $[a,b]$.
In what follows we remind the concept of linear convergence that can be found in Berinde [3] and the concept of convergence order, also called Q-order of convergence that can be found in monographs by Măruşter [17] and Ortega and Rheinboldt [19].

Definition 1. [9] Let us assume that the sequence $\{S_n\}$, which converges to some limit S, satisfies

$$\lim_{n\to\infty} \frac{S_{n+1} - S}{S_n - S} = \lambda.$$

If $\lambda \neq 1$ holds, we say that the sequence $\{S_n\}$ converges linearly, and if $\lambda = 1$ holds, we say that $\{S_n\}$ converges logarithmically.

Let $\{x_n\} \in \mathbb{R}$ be a sequence of real numbers convergent to $\alpha \in \mathbb{R}$ (which is obtained by iterating a fixed point equation).

Definition 2. [17],[19] Let $\{x_n\}$ converge to α. If there exist an integer constant p, and a real positive constant C such that

$$\lim_{n\to\infty} \left| \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} \right| = C$$

then p is called the order and C the constant of convergence.

It is well known that the convergence order of the Picard iteration is generally linear, see Berinde [3], so the method (2.3) converges slowly to the fixed point x^*. In order to improve the convergence speed of (2.3) we must use a convergence acceleration process. In the sequel we will present the most important ones. The most popular sequence acceleration method, for a sequence that converges slowly, is Aitken's Δ^2 process. The method was given by A.C. Aitken in 1926 [1], who used it to accelerate the convergence of Bernoulli's method for computing the dominant zero of a polynomial. The method is given by

$$A_1^{(n)} = S_n - \frac{(S_{n+1} - S_n)^2}{S_{n+2} - 2S_{n+1} + S_n} = S_n - \frac{[\Delta S_n]^2}{\Delta^2 S_n}, \quad n = 0,1,\ldots, \tag{2.4}$$

where $\{S_n\}$ is the sequence to be accelerated and Δ denotes the forward difference operator, $\Delta S_n = S_{n+1} - S_n$ and $\Delta^2 S_n = \Delta S_{n+1} - \Delta S_n = S_{n+2} - 2S_{n+1} + S_n$. As it

is known, Aitken's Δ^2 algorithm is based on a recursive computation, in which each term of the sequence is determined by three previous terms by a simple arithmetic, which implies that Aitken's Δ^2 process is a transformation of order $l = 2$.
Discussion about Aitken's Δ^2 process can be found in books by Lubkin [15], Shanks [20], Thuker [22], [23], Cordellier [10], Weniger [25]. Even a generalization to the vector case was discussed by MacLeod [16]. Modifications and generalizations of Aitken's Δ^2 process were suggested by Drummond [11] and Jamieson and O' Beirne [13]. The properties of Aitken's Δ^2 process were studied by Baker and Graves-Morries, Brezinski [2], [7], [8], Walz [24], Wimp [26]. Those properties are:
(i) Aitken's Δ^2 process accelerates linear convergence.
(ii) Aitken's Δ^2 process is regular but not accelerative for logarithmically convergent sequences.
The convergence power for Aitken's Δ^2 process can be increased by iterating it, yielding the following nonlinear recursive scheme

$$A_{k+1}^{(n)} = A_k^{(n)} - \frac{[\Delta A_k^{(n)}]^2}{\Delta^2 A_k^{(n)}}, \quad A_0^{(n)} = S_n \quad n,k \in \mathbb{N}_0, \tag{2.5}$$

where $\Delta A_k^{(n)} = A_k^{(n+1)} - A_k^{(n)}$ and $\Delta^2 A_k^{(n)} = \Delta A_k^{(n+1)} - \Delta A_k^{(n)} = A_k^{(n+2)} - 2A_k^{(n+1)} + A_k^{(n)}$, the forward difference operator Δ acts only upon the superscript n and not upon the subscript k. In the case of Aitken's iterated Δ^2 process every $A_k^{(n)}$ term of the sequence is determined by a recursive formula which requires $2k+1$ elements, consequently, $A_k^{(n)}$ is a transformation of order $l = 2k$.
Like Aitken's Δ^2 process, the iterated form has many different expressions, one of them being given by the next formula

$$A_{k+1}^{(n)} = A_k^{(n+1)} - \frac{[\Delta A_k^{(n)}][\Delta A_k^{(n+1)}]}{\Delta^2 A_k^{(n)}}, \; A_0^{(n)} = S_n \quad n,k \in \mathbb{N}_0. \tag{2.6}$$

Discussions about Aitken's iterated Δ^2 process were done in [21], [25], although about its theoretical properties very little seems to be known.
Some other important sequences transformations were obtain by Wynn [27] in 1956, this algorithm is called Wynn's ϵ-algorithm; by Brezinski [6] in 1971, which is an improvement of Wynn's ϵ-algorithm, also called the θ-algorithm and by Levin [14] in 1973.
Our new algorithm we introduce in this paper is similar to expression (2.6). The new proposed acceleration technique is given by

$$B_{k+1}^{(n)} = B_k^{(n+3)} - \frac{[\overline{\Delta} B_k^{(n+1)}][\Delta B_k^{(n+2)}]}{\overline{\Delta} B_k^{(n+1)} - \overline{\Delta} B_k^{(n)}}, \quad n,k \in \mathbb{N} \tag{2.7}$$

where $B_0^{(n)} = S_n$, $\{S_n\}$ is the sequence to be accelerated, Δ, the forward difference operator, has been defined before and we denote by $\overline{\Delta}$, the forward difference operator with two steps, $\overline{\Delta} B_k^{(n)} = B_k^{(n+2)} - B_k^{(n)}$. Like in the case of the Aitken's iterated Δ^2 process the difference operators, Δ and $\overline{\Delta}$, act only upon the superscript n and not upon the subscript k.

In the sequel we will give conditions on the $B_i's$ which ensure that $\lim_{n\to\infty} B_k^{(n)} = S$.

Theorem 1. *If* $\lim_{n\to\infty} S_n = S$, *and if* $\forall i, \exists b_i \neq 1$ *such that* $\lim_{n\to\infty} \frac{B_i^{(n+1)}}{B_i^{(n)}} = b_i$, *for* $\forall i \neq j,\ b_i \neq b_j$. *Then* $\forall k$ *we have* $\lim_{n\to\infty} B_k^{(n)} = S$.

Proof. For $k = 0$ we have

$$B_1^{(n)} = B_0^{(n+3)} - \frac{(B_0^{(n+3)} - B_0^{(n+1)})(B_0^{(n+3)} - B_0^{(n+2)})}{(B_0^{(n+3)} - B_0^{(n+1)}) - (B_0^{(n+2)} - B_0^{(n)})},$$

where $B_0^{(n)} = S_n$.

Multiplying with $\frac{B_0^{(n+2)}}{B_0^{(n+2)}}$ we get

$$B_1^{(n)} = B_0^{(n+3)} - \frac{\frac{B_0^{(n+3)}}{B_0^{(n+2)}} - \frac{B_0^{(n+1)}}{B_0^{(n+2)}}}{\frac{B_0^{(n+3)}}{B_0^{(n+2)}} - \frac{B_0^{(n+1)}}{B_0^{(n+2)}} - 1 + \frac{B_0^{(n)}}{B_0^{(n+2)}}} (B_0^{(n+3)} - B_0^{(n+2)}). \tag{2.8}$$

Taking the limit when n tends to infinity we obtain

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} B_0^{(n+3)} - \frac{b_0 - \frac{1}{b_0}}{b_0 - \frac{1}{b_0} - 1 + \frac{1}{b_0^2}} \lim_{n\to\infty} (B_0^{(n+3)} - B_0^{(n+2)}) \tag{2.9}$$

from where it results

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} B_0^{(n+3)} - \frac{b_0}{b_0 - 1} \lim_{n\to\infty} (B_0^{(n+3)} - B_0^{(n+2)}), \tag{2.10}$$

where $b_0 \neq 1$ and $B_0^{(n)} = S_n$. Hence

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} S_{n+3} - \frac{b_0}{b_0 - 1} \lim_{n\to\infty} (S_{n+3} - S_{n+2}), \tag{2.11}$$

because $\lim_{n\to\infty} S_n = S$ we obtain $\lim_{n\to\infty} B_1^{(n)} = S$.

Suppose that $\lim_{n\to\infty} B_{k-1}^{(n)} = S$ we prove that $\lim_{n\to\infty} B_k^{(n)} = S$.

$$B_k^{(n)} = B_{k-1}^{(n+3)} - \frac{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})(B_{k-1}^{(n+3)} - B_{k-1}^{(n+2)})}{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})}$$

Multiplying with $\frac{B_{k-1}^{(n+2)}}{B_{k-1}^{(n+2)}}$ we obtain

$$B_k^{(n)} = B_{k-1}^{(n+3)} - \frac{\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+2)}} - \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}}}{\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+2)}} - \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}} - 1 + \frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+2)}}} (B_{k-1}^{(n+3)} - B_{k-1}^{(n+2)}). \tag{2.12}$$

Taking the limit when n tends to infinity we obtain

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n+3)} - \frac{b_{k-1} - \frac{1}{b_{k-1}}}{b_{k-1} - \frac{1}{b_{k-1}} - 1 + \frac{1}{b_{k-1}^2}} \lim_{n\to\infty} (B_0^{(n+3)} - B_0^{(n+2)}), \tag{2.13}$$

hence there

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n+3)} - \frac{b_{k-1}}{b_{k-1} - 1} \lim_{n\to\infty} (B_{k-1}^{(n+3)} - B_{k-1}^{(n+2)}) \tag{2.14}$$

where $b_{k-1} \neq 1$ and $B_{k-1}^{(n)} = S_n$ we obtain $\lim\limits_{n\to\, infty} B_k^{(n)} = S$. □

Theorem 2. *If the conditions of Theorem 1 are satisfied and if* $\lim\limits_{n\to\infty} \frac{B_{k-1}^{(n+1)} - S}{B_{k-1}^{(n)} - S} = b_k$. *Then* $\{B_k^{(n)}\}$ *converges to S faster than* $\{B_{k-1}^{(n)}\}$, *when n tends to infinity, that is*

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = 0. \tag{2.15}$$

Moreover if $b_k \neq 0$, $\{B_k^{(n)}\}$ *converges to S faster than* $\{B_{k-1}^{(n+1)}\}$.

Proof. We prove the first part of theorem.

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = \frac{B_{k-1}^{(n+3)} - S - \frac{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})(B_{k-1}^{(n+3)} - B_{k-1}^{(n+2)})}{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})}}{B_{k-1}^{(n)} - S}, \tag{2.16}$$

adding $S - S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = \frac{B_{k-1}^{(n+3)} - S}{B_{k-1}^{(n)} - S} -$$
$$- \frac{[(B_{k-1}^{(n+3)} - S) - (B_{k-1}^{(n+1)} - S)][(B_{k-1}^{(n+3)} - S) - (B_{k-1}^{(n+2)} - S)]}{\{[(B_{k-1}^{(n+3)} - S) - (B_{k-1}^{(n+1)} - S)] - [(B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n)} - S)]\}(B_{k-1}^{(n)} - S)}.$$

Multiplying with $\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+2)}-S}$ it follows that

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S} \tag{2.17}$$

$$= \frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}\cdot\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n)}-S} - \frac{\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}\right]\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-1\right]}{\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}-1+\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right]\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}}.$$

Taking the limit when n tends to infinity we get

$$\lim_{n\to\infty}\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S} = b_k^3 - \frac{\left(b_k-\frac{1}{b_k}\right)(b_k-1)}{\left(b_k-\frac{1}{b_k}-1+\frac{1}{b_k^2}\right)\frac{1}{b_k^2}}, \tag{2.18}$$

from where it follows

$$\lim_{n\to\infty}\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S} = b_k^3 - b_k^3 = 0. \tag{2.19}$$

The proof of the second part of the theorem run as follows.

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S} = \frac{B_{k-1}^{(n+3)}-S-\frac{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})(B_{k-1}^{(n+3)}-B_{k-1}^{(n+2)})}{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})-(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})}}{B_{k-1}^{(n+1)}-S}, \tag{2.20}$$

adding $S-S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S} = \frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+1)}-S} -$$

$$\frac{[(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+1)}-S)][(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+2)}-S)]}{\{[(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+1)}-S)]-[(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n+1)}-S)]\}(B_{k-1}^{(n+1)}-S)}.$$

Multiplying with $\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+2)}-S}$

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S} = \frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}\cdot\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+1)}-S} -$$

$$- \frac{\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}\right]\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-1\right]}{\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}-1+\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right]\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}}.$$

Taking the limit when n tends to infinity

$$\lim_{n\to\infty}\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=b_k^2-\frac{\left(b_k-\frac{1}{b_k}\right)(b_k-1)}{\left(b_k-\frac{1}{b_k}-1+\ frac1b_k^2\right)\frac{1}{b_k}} \tag{2.21}$$

from where it results

$$\lim_{n\to\infty}\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=b_k^2-b_k^2=0. \tag{2.22}$$

This completes the proof. □

Our aim in the next section is to perform with our new algorithm an empirical study to test the capabilities of accelerating some numerical sequences arising in solving nonlinear equations for some test functions taken from [4], [5], [12], [18].

3. NUMERICAL EXAMPLES

In this section we present some numerical examples to illustrate the efficiency and the performance of the new proposed algorithm. The new acceleration method is applied to some numerical sequences arising in solving nonlinear equations that can be found in [4], [5], [12], [18]. To see the efficiency of the new proposed acceleration method we shall compare the sequence that arises when applying the new acceleration technique with the sequence that arises when applying Aitken's iterated Δ^2 process. In the tables the numbers of exact decimals obtained at each step are displayed. All the numerical computations listed in the tables were done with Maple 13 using 39 digit floating point arithmetic.

Example 1. [4], [5] Test function: $f(x)=x^3+4x^2-10$, which has a unique root $x^*=1.365230013414096845760806828981666078331\ldots$. This equation can be rewritten into a fixed point problem by $g(x)=\frac{1}{2}\sqrt{10-x^3}$. To apply the new acceleration technique we shall take the initial guess $x_0\in\{1.2,1.3,1.4,1.5\}$. The results for Example 1 are listed in Table 1.

Table 1

Initial guess \ No. of iterations		1	2	3	4	5	6	7	8
1.2	$B_k^{(n)}$	2	5	9	13	20	26	31	38
	$A_k^{(n)}$	2	4	4	10	14	18	23	28
1.3	$B_k^{(n)}$	3	6	10	15	22	28	35	38
	$A_k^{(n)}$	2	4	8	12	16	21	26	32
1.5	$B_k^{(n)}$	2	4	9	12	20	25	32	38
	$A_k^{(n)}$	2	4	4	9	14	18	24	28
1.6	$B_k^{(n)}$	1	4	8	12	17	23	32	34
	$A_k^{(n)}$	1	3	4	8	13	16	24	25

Example 2. [5] Test function $f(x) = x - 3^{-x} = 0$, which has a unique root $x^* = 0.547808621654097446450575408151021850346 0\ldots$. This equation can be rewritten as a fixed point problem by $g(x) = 3^{-x}$. To apply the new acceleration technique we shall take the initial guess $x_0 \in \{0.3, 0.4, 0.5, 0.6\}$. The results for Example 2 are listed in Table 2.

Table 2

Initial guess \ No. of iterations		1	2	3	4	5	6	7	8
0.3	$B_k^{(n)}$	2	4	7	11	15	21	26	30
	$A_k^{(n)}$	1	3	5	8	12	16	19	23
0.4	$B_k^{(n)}$	2	4	8	11	16	23	28	32
	$A_k^{(n)}$	1	3	6	9	12	17	21	25
0.5	$B_k^{(n)}$	3	6	10	14	19	26	32	37
	$A_k^{(n)}$	2	5	8	12	16	21	25	30
0.6	$B_k^{(n)}$	3	6	10	14	19	27	31	37
	$A_k^{(n)}$	2	5	8	12	16	21	25	29

Example 3. [4] Test function $f(x) = cosx - x = 0$, which has a unique root $x^* = 0.739085133215160641655312087673873404013 4\ldots$. This equation can be rewritten into a fixed point problem by $g(x) = cosx$. To apply the new acceleration technique we shall take the initial guess $x_0 \in \{0.5, 0.7, 0.8, 0.9\}$. The results for Example refe3 are listed in Table 3.

Table 3

Initial guess \ No. of iterations		1	2	3	4	5	6	7	8
0.5	$B_k^{(n)}$	1	3	5	8	11	14	18	23
	$A_k^{(n)}$	2	2	3	6	8	11	16	17
0.7	$B_k^{(n)}$	3	5	8	12	15	19	25	30
	$A_k^{(n)}$	2	5	7	10	14	16	22	23
0.8	$B_k^{(n)}$	3	4	7	11	13	18	22	28
	$A_k^{(n)}$	2	3	7	8	11	15	19	23
0.9	$B_k^{(n)}$	1	2	5	9	11	15	19	24
	$A_k^{(n)}$	2	3	4	6	9	12	16	19

Example 4. [12] Test function $f(x) = (x+2)e^x - 1 = 0$, which has a unique root $x^* = -0.44285440100238858314132799999993368197163\ldots$. This equation can be rewritten into a fixed point problem by $g(x) = -\ln(x+2)$. To apply the new acceleration technique we shall take the initial guess $x_0 \in \{-0.2, -0.4, -0.5, -0.6\}$. The results for Example 4 are listed in Table 4.

Table 4

Initial guess \ No. of iterations		1	2	3	4	5	6	7	8
-0.2	$B_k^{(n)}$	2	4	7	8	15	19	23	32
	$A_k^{(n)}$	1	3	6	6	11	15	19	23
-0.4	$B_k^{(n)}$	3	6	10	14	19	25	31	38
	$A_k^{(n)}$	3	6	9	12	16	20	23	29
-0.5	$B_k^{(n)}$	2	6	10	13	18	23	30	37
	$A_k^{(n)}$	3	5	8	11	15	19	23	23
-0.6	$B_k^{(n)}$	2	5	8	11	15	21	23	33
	$A_k^{(n)}$	2	4	7	9	12	16	20	21

Example 5. [18] Test function $f(x) = x^3 - 2x^2 - 5 = 0$, which has a unique root $x^* = 2.690647448028613750350788882676806151802\ldots$. This equation can be rewritten into a fixed point problem by $g(x) = \sqrt[3]{2x^2+5}$. To apply the new acceleration techniqu e we shall take the initial guess $x_0 \in \{2.3, 2.5, 2.8, 3.0\}$. The results for Example 5are listed in Table 5.

Table 5

Initial guess \ No. of iterations		1	2	3	4	5	6	7	8
2.3	$B_k^{(n)}$	3	6	9	12	16	21	28	36
	$A_k^{(n)}$	1	5	8	9	11	15	18	23
2.5	$B_k^{(n)}$	4	6	9	12	18	24	30	38
	$A_k^{(n)}$	3	5	8	8	13	17	21	27
2.8	$B_k^{(n)}$	6	7	9	14	20	26	32	38
	$A_k^{(n)}$	5	5	7	13	14	20	23	29
3.0	$B_k^{(n)}$	3	6	7	11	17	24	29	36
	$A_k^{(n)}$	3	5	5	10	12	15	20	24

4. Conclusions

In this paper we presented a new acceleration method inspired from one of the derivation of Aitken's iterated Δ^2 process given by the expression (2.6). To see the efficiency of the new acceleration method we performed a comparison between the sequence that arises when applying the new acceleration technique and the sequence that arises when applying Aitken's iterated Δ^2 process in order to see which one has the best convergence speed. We have done studies on twenty test functions, we here presented only five of them. For all test functions, that we studied, the new acceleration method improves significantly the convergence speed of a scalar sequence.

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RING-LIKE ALGEBRAS ASSOCIATED WITH SYMMETRIC DIFFERENCE

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Abstract. We generalize the one-to-one correspondence between Boolean algebras and Boolean rings to so-called difference lattices and commutative strong difference ring-like algebras. Moreover, we show that difference ring-like algebras induce some sort of symmetric difference in corresponding posets.

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1. Introduction and basic definitions

The well-known one-to-one correspondence between Boolean algebras and Boolean rings was generalized in many ways from Boolean algebras to more general structures (cf. e.g. [1],[2],[3],[4],[5] and [7],[9],[8],[10],[11]). This correspondence is realized via the so-called symmetric difference. A certain modification of this approach for orthocomplemented lattices is develope d e.g. in [11]. We already generalized the concept of symmetric difference for posets with an antitone involution in [6]. The aim of this paper is to generalize this correspondence to certain bounded lattices with an antitone involution. Moreover, we generalize Boolean rings in such a way that they induce some sort of symmetric difference on a certain bounded poset with an antitone involution.

We start our investigations with some basic definitions.

Definition 1. An **antitone involution** on a poset $(P, \leq)$ is a unary operation $'$ on P satisfying

$$x \leq y \text{ implies } x' \geq y'$$

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$$(x')' = x$$

for all $x, y \in P$. A **bounded poset with an antitone involution** is an ordered quintuple $(P, \leq, ', 0, 1)$ where $(P, \leq, 0, 1)$ is a bounded poset and $'$ is an antitone involution on $(P, \leq)$. A **bounded lattice with an antitone involution** is an algebra $(L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and $'$ is an antitone involution on the poset $(L, \leq)$ corresponding to $(L, \vee, \wedge)$.

Remark 1. An algebra $(L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ is a bounded lattice with an antitone involution if and only if $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and the de Morgan laws

$$(x \vee y)' = x' \wedge y'$$
$$(x \wedge y)' = x' \vee y'$$

hold for all $x, y \in L$.

2. Difference ring-like algebras

In this section we introduce certain generalizations of Boolean rings and show that they induce posets with an antitone involution where the addition plays the role of some sort of symmetric difference.

First we define a generalized symmetric difference on a bounded poset with an antitone involution.

Definition 2. A **generalized symmetric difference** on a bounded poset $(P, \leq, ', 0, 1)$ with an antitone involution is a binary operation $+$ on P satisfying the following axioms:

$$\begin{aligned} (x + y) + z &= x + (y + z) \\ x + y &= y + x \\ x + 0 &= x \\ x + x &= 0 \\ x + x' &= 1 \end{aligned}$$

Remark 2. The operation $+$ satisfies $x + 1 = x'$ and $x + y = x' + y'$ since $x + 1 = x + (x + x') = (x + x) + x' = 0 + x' = x' + 0 = x'$ and $x + y = (x + y) + 0 = (x + y) + (1 + 1) = (x + 1) + (y + 1) = x' + y'$.

Now we define difference ring-like algebras.

Definition 3. A **difference ring-like algebra** is an algebra $(R, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ satisfying the following axioms:

$$(x + y) + z = x + (y + z)$$
$$x + y = y + x$$

$$x+0=x$$
$$x+x=0$$
$$(xy)z=x(yz)$$
$$xx=x$$
$$x1=x$$
$$1x=x$$

$$(x+1)(xy+1)=x+1 \tag{2.1}$$
$$(xy+1)(x+1)=x+1 \tag{2.2}$$

$(R,+,\cdot,0,1)$ is called a **commutative difference ring-like algebra** whenever the operation $\cdot$ is commutative.

Remark 3. There exist non-commutative difference ring-like algebras as the following example shows:

Example 1. If $R=\{0,a,b,1\}$ and $+$ and $\cdot$ are defined by

$+$	0	a	b	1
0	0	a	b	1
a	a	0	1	b
b	b	1	0	a
1	1	b	a	0

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
1	0	a	b	1

then $(R,+,\cdot,0,1)$ is a non-commutative difference ring-like algebra since $ab=a\neq b=ba$.

Now we can prove that every difference ring-like algebra induces in a natural way a generalized symmetric difference on a certain bounded poset with an antitone involution.

Theorem 1. *If $(R,+,\cdot,0,1)$ is a difference ring-like algebra then there exists a bounded poset $\mathcal{P}=(R,\leq,',0,1)$ with an antitone involution such that $+$ is a generalized symmetric difference on $\mathcal{P}$.*

Proof. Define $x\leq y$ if $xy=yx=x$ $(x,y\in R)$ and $x':=x+1$ for all $x\in R$. Then $(R,\leq)$ is a poset with greatest element 1. Now let $a,b,c\in R$. If $a\leq b$ then $ab=ba=a$, $(b+1)(a+1)=(b+1)(ba+1)=b+1$ according to (2.1) and $(a+1)(b+1)=(ba+1)(b+1)=b+1$ according to (2.2), i. e. $b+1\leq a+1$. Moreover, $(a+1)+1=a+(1+1)=a+0=a$. Hence $'$ is an antitone involution on $(R,\leq)$ and $1'=1+1=0$ its least element. We have proved that $\mathcal{P}:=(R,\leq,',0,1)$ is a bounded poset with an antitone involution. Finally, we have

$$(a+b)+c=a+(b+c),$$
$$a+b=b+a,$$

$$a+0=a,$$
$$a+a=0 \text{ and}$$
$$a+1=a' \text{ by definition.}$$

From here we infer

$$a+a'=a+(a+1)=(a+a)+1=0+1=1+0=1.$$

□

Definition 4. The bounded poset $(R,\leq,0,1)$ constructed in the proof of Theorem 1 will be called the **poset induced** by the difference ring-like algebra $(R,+,\cdot,0,1)$.

Remark 4. Every Boolean ring is a commutative difference ring-like algebra. That the converse is not true can be seen from the following example:

Example 2. If $R=\{0,a,b,1\}$ and $+$ and $\cdot$ are defined by

$+$	0	a	b	1
0	0	a	b	1
a	a	0	1	b
b	b	1	0	a
1	1	b	a	0

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

then $(R,+,\cdot,0,1)$ is a commutative difference ring-like algebra that is not a Boolean ring since

$$(a+b)b=1\cdot b=b\neq 1=a+b=ab+bb.$$

Moreover, the induced poset is the chain $0<a<b<1$ and the involution $'$ is the permutation $(01)(ab)$.

Next we show that every commutative difference ring-like algebra already induces a generalized symmetric difference on a bounded lattice with an antitone involution.

Theorem 2. *If $(R,+,\cdot,0,1)$ is a commutative difference ring-like algebra and*

$$x\vee y:=(x+1)(y+1)+1$$
$$x\wedge y:=xy \text{ and}$$
$$x':=x+1$$

for all $x,y\in R$ then $\mathcal{L}:=(R,\vee,\wedge,',0,1)$ is a bounded lattice with an antitone involution and $+$ is a generalized symmetric differenc e on $\mathcal{L}$.

Proof. The proof follows the same lines as that of Theorem 1. However, $x\leq y$ is now equivalent to $xy=x$ $(x,y\in R)$ and $(R,\cdot)$ is a meet-semilattic e the corresponding poset of which is $(R,\leq)$. Since $(R,\leq,',0,1)$ is a bounded poset with an antitone involution and $(R,\wedge)$ with $x\wedge y:=xy$ for all $x,y\in R$ the corresponding meet-semilattice, $(R,\vee,\wedge,',0,1)$ with $x\vee y;=(x+1)(y+1)+1$ for all $x,y\in R$ is a bounded lattice with an antitone involution. The rest follows from the proof of Theorem 1. □

3. Difference lattices and commutative strong difference ring-like algebras

In this section we want to derive a one-to-one correspondence between certain lattices with some kind of symmetric difference and certain ring-like algebras generalizing the well-known one-to-one correspondence between Boolean algebras and Boolean rings. We generalize the approach by Matoušek and Pták (cf. [11]).

First we define the lattice structures.

Definition 5. A **difference** on a bounded lattice $(L,\vee,\wedge,',0,1)$ with an antitone involution is a binary operation $\triangle$ on L satisfying the following axioms:

$$(x\triangle y)\triangle z = x\triangle(y\triangle z) \tag{3.1}$$

$$x\triangle 1 = x' \tag{3.2}$$

$$1\triangle x = x' \tag{3.3}$$

$$x\triangle x = 0 \tag{3.4}$$

$$x\triangle y \le x\vee y \tag{3.5}$$

A **difference lattice** is an algebra $(L,\vee,\wedge,\triangle,',0,1)$ of type $(2,2,1,0,0,\ 2)$ such that $\mathfrak{L} = (L,\vee,\wedge,',0,1)$ is a bounded lattice with an antitone involution and $\triangle$ is a difference on $\mathfrak{L}$.

Remark 5. If $(B,\vee,\wedge,',0,1)$ is a Boolean algebra and $\triangle$ denotes the usual symmetric difference given by $x\triangle y = (x'\wedge y)\vee(x\wedge y') = (x\vee y)\wedge(x'\vee y')$ then $(B,\vee,\wedge,\triangle,',0,1)$ is a differenc e lattice with an antitone involution. That the converse is not true is shown by the following example:

Example 3. If $L = \{0,a,b,c,d,e,f,1\}$ and $\vee,\wedge,'$ and $\triangle$ are defined by the tables

$\vee$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	a	1	1	1	1	1	1
b	b	1	b	1	1	1	1	1
c	c	1	1	c	1	1	1	1
d	d	1	1	1	d	1	1	1
e	e	1	1	1	1	e	1	1
f	f	1	1	1	1	1	f	1
1	1	1	1	1	1	1	1	1

$\wedge$	0	a	b	c	d	e	f	1	$'$
0	0	0	0	0	0	0	0	0	1
a	0	a	0	0	0	0	0	a	f
b	0	0	b	0	0	0	0	b	e
c	0	0	0	c	0	0	0	c	d
d	0	0	0	0	d	0	0	d	c
e	0	0	0	0	0	e	0	e	b
f	0	0	0	0	0	0	f	f	a
1	0	a	b	c	d	e	f	1	0

$\triangle$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	0	c	b	e	d	1	f
b	b	c	0	a	f	1	d	e
c	c	b	a	0	1	f	e	d
d	d	e	f	1	0	a	b	c
e	e	d	1	f	a	0	c	b
f	f	1	d	e	b	c	0	a
1	1	f	e	d	c	b	a	0

then $(L,\vee,\wedge,\triangle,',0,1)$ is a difference lattice such that $(L,\ vee,\wedge,',0,1)$ is not a Boolean algebra since

$$(a\vee b)\wedge c = 1\wedge c = c \neq 0 = 0\vee 0 = (a\wedge c)\vee(b\wedge c).$$

Next we define the ring-like counterpart.

Definition 6. A **difference ring-like algebra** $(R,+,\cdot,0,1)$ is called **strong** if it satisfies the axiom

$$(x+y)(xy+1) = x+y.$$

Remark 6. Every Boolean ring is a commutative strong difference ring-like algebra. That the converse is not true is shown by the following example:

Example 4. Let $R=\{0,a,b,c,d,e,f,1\}$ and define $+$ and $\cdot$ as $\triangle$ and $\wedge$ in Example 3, respectively. Then $(R,+,\cdot,0,1)$ is a commutati ve strong difference ring-like algebra that is not a Boolean ring since $(a+f)b = 1\cdot b = b \neq 0 = 0+0 = ab+fb$.

Now we can prove that every difference lattice induces a commutative strong difference ring-like algebra in a natural way.

Theorem 3. *If $\mathcal{L}=(L,\vee,\wedge,\triangle,',0,1)$ is a difference lattice and one defines*

$$x+y := x\triangle y \text{ and}$$
$$xy := x\wedge y$$

for all $x,y\in L$ then $\mathbf{R}(\mathcal{L}) := (L,+,\cdot,0,1)$ is a commutative strong difference ring-like algebra.

Proof. Let $\mathcal{L}=(L,\vee,\wedge,\triangle,',0,1)$ be a difference lattice, define

$$x+y := x\triangle y \text{ and}$$
$$xy := x\wedge y$$

for all $x,y\in L$ and let $a,b,c\in L$. Then

$$(a+b)+c = (a\triangle b)\triangle c = a\triangle(b\triangle c) = a+(b+c),$$

$$
\begin{aligned}
a+1 &= a\triangle 1 = a',\\
1+a &= 1\triangle a = a',\\
a+a &= a\triangle a = 0,\\
a+0 &= a+(1+1) = (a+1)+1 = (a')' = a,\\
0+a &= (1+1)+a = 1+(1+a) = (a')' = a,\\
a+b &= (a+b)+0 = (a+b)+((b+a)+(b+a)) =\\
&= ((a+(b+b))+a)+(b+a) = ((a+0)+a)+(b+a) =\\
&= (a+a)+(b+a) = 0+(b+a) = b+a,\\
(ab)c &= (a\wedge b)\wedge c = a\wedge(b\wedge c) = a(bc),\\
ab &= a\wedge b = b\wedge a = ba,\\
aa &= a\wedge a = a,\\
a1 &= a\wedge 1 = a,\\
(a+1)(ab+1) &= a'\wedge(a\wedge b)' = a' = a+1,\\
a'+b' &= (a+1)+(1+b) = (a+(1+1))+b = (a+0)+b =\\
&= a+b \text{ and}\\
(a+b)(ab+1) &= (a+b)\wedge(a\wedge b)' = (a'+b')\wedge(a'\vee b') =\\
&= (a'\triangle b')\wedge(a'\vee b') = a'\triangle b' = a'+b' = a+b \text{ according to}\\
&\quad (3.5).
\end{aligned}
$$

Hence $\mathbf{R}(\mathcal{L}) = (L,+,\cdot,0,1)$ is a commutative strong difference ring-like algebra. □

We can also prove the converse, namely that every commutative strong difference ring-like algebra induces in a natural way a difference lattice.

Theorem 4. *If $\mathcal{R} = (R,+,\cdot,0,1)$ is a commutative strong difference ring-like algebra and*

$$
\begin{aligned}
x\vee y &:= (x+1)(y+1)+1,\\
x\wedge y &:= xy,\\
x' &:= x+1 \text{ and}\\
x\triangle y &:= x+y
\end{aligned}
$$

for all $x,y\in R$ then $\mathbf{L}(\mathcal{R}) := (R,\vee,\wedge,\triangle,',0,1)$ is a difference lattice.

Proof. Let $\mathcal{R} = (R,+,\cdot,0,1)$ be a commutative strong difference ring-like algebra, define

$$x\vee y := (x+1)(y+1)+1,$$

$$x \wedge y := xy,$$
$$x' := x + 1 \text{ and}$$
$$x \triangle y := x + y$$

for all $x, y \in R$ and let $a, b, c \in R$. According to Theorem 2, $(R, \vee, \wedge, ', 0, 1)$ is a bounded lattice with an antitone involution. Moreover, we have

$$\begin{aligned}
(a \triangle b) \triangle c &= (a + b) + c = a + (b + c) = a \triangle (b \triangle c),\\
a \triangle 1 &= a + 1 = a',\\
1 \triangle a &= 1 + a = a + 1 = a',\\
a \triangle a &= a + a = 0 \text{ and}\\
a \triangle b &= a + b = (a + b) + 0 = (a + b) + (1 + 1) = (a + 1) + (b + 1) =\\
&= ((a + 1) + (b + 1))((a + 1)(b + 1) + 1) =\\
&= ((a + 1) + (b + 1)) \wedge (a \vee b) \leq a \vee b.
\end{aligned}$$

Hence $\mathbf{L}(\mathcal{R}) = (R, \vee, \wedge, \triangle, ', 0, 1)$ is a difference lattice. □

We can now show that the correspondence between difference lattices and commutative strong difference ring-like algebras described in Theorems 3 and 4 is one-to-one.

Theorem 5.

(i) $\mathbf{L}(\mathbf{R}(\mathcal{L})) = \mathcal{L}$ *for every differen ce lattice* $\mathcal{L}$.
(ii) $\mathbf{R}(\mathbf{L}(\mathcal{R})) = \mathcal{R}$ *for every commutative strong difference ring-like algebra* $\mathcal{R}$.

Proof.

(i) If $\mathcal{L} = (L, \vee, \wedge, \triangle, ', 0, 1)$ is a difference lattice, $\mathbf{R}(\mathcal{L}) = (L, +, \cdot, 0, 1)$, $\mathbf{L}(\mathbf{R}(\mathcal{L})) = (L, \cup, \cap, \circ, ^*, 0, 1)$ and $a, b \in L$ then

$$\begin{aligned}
a \cup b &= (a + 1)(b + 1) + 1 = ((a \triangle 1) \wedge (b \triangle 1)) \triangle 1 = (a' \wedge b')' = a \vee b,\\
a \cap b &= ab = a \wedge b,\\
a \circ b &= a + b = a \triangle b \text{ and}\\
a^* &= a + 1 = a \triangle 1 = a'.
\end{aligned}$$

(ii) If $\mathcal{R} = (R, +, \cdot, 0, 1)$ is a commutative strong difference ring-like algebra, $\mathbf{L}(\mathcal{R}) = (R, \vee, \wedge, \triangle, ', 0, 1)$, $bfR(\mathbf{L}(\mathcal{R})) = (R, \oplus, \odot, 0, 1)$ and $a, b \in R$ then

$$\begin{aligned}
a \oplus b &= a \triangle b = a + b \text{ and}\\
a \odot b &= a \wedge b = ab.
\end{aligned}$$

□

4. FURTHER EXAMPLES

In this section we generalize the examples given before.

Example 5. If $n \geq 2$, $R := \{0,1\}^n$, $0 := (0,\ldots,0)$, $1 := (1,\ldots,1)$, $+$ the componentwise addition modulo 2, $0x = x0 := 0$ and $x1 = 1x := x$ for all $x \in R$ and $xy := x$ for all $x, y \in R \setminus \{0,1\}$ then $(R,+,\cdot,0,1)$ is a non- commutative difference ring-like algebra since

$$(0,\ldots,0,1)(1,\ldots,1,0) = (0,\ldots,0,1) \neq (1,\ldots,1,0) = (1,\ldots,1,0)(0,\ \ldots,0,1).$$

For $n = 2$ we obtain Example 1.

Example 6. If $n \geq 2$, $R := \{0,1\}^n$, $0 := (0,\ldots,0)$, $1 := (1,\ldots,1)$, $+$ the componentwise addition modulo 2 and $\cdot$ the minimum with respect to the lexicographical order then $(R,+,\cdot,0,1)$ is a commutative difference ring- like algebra that is not a Boolean ring since

$$\begin{aligned}&((0,\ldots,0,1)+(1,\ldots,1,0))(1,\ldots,1,0) = \\ &= 1\cdot(1,\ldots,1,0) = (1,\ldots,1,0) \neq 1 = (0,\ldots,0,1)+(1,\ldots,1,0) = \\ &= (0,\ldots,0,1)(1,\ldots,1,0)+(1,\ldots,1,0)(1,\ldots,1,0).\end{aligned}$$

For $n = 2$ we obtain Example 2.

Example 7. If $n \geq 3$, $L := \{0,1\}^n$, $0 := (0,\ldots,0)$, $1 := (1,\ldots,1)$, $x \vee x = x\ vee 0 = 0 \vee x := x$ for all $x \in L$, $x \vee y := 1$ for all $x, y \in L \setminus \{0\}$ with $x \neq y$, $x \wedge x = x \wedge 1 = 1 \wedge x := x$ for all $x \in L$, $x \wedge y := 0$ for all $x, y \in L \setminus \{1\}$ with $x \neq y$, $\triangle$ denotes the componentwise addition modulo 2 and $x' := x \triangle 1$ for all $x \in L$ then $(L,\vee,\wedge,\triangle,',0,1)$ is a difference lattice such that $(L,\vee,\wedge,',0,1)$ is not a Boolean algebra since

$$\begin{aligned}&((0,\ldots,0,1)\vee(0,\ldots,0,1,0))\wedge(0,\ldots,0,1,1) = \\ &= 1\wedge(0,\ldots,0,1,1) = (0,\ldots,0,1,1) \neq 0 = 0\vee 0 = \\ &= ((0,\ldots,0,1)\wedge(0,\ldots,0,1,1))\vee((0,\ldots,0,1,0)\wedge(0,\ldots,0,1,1)).\end{aligned}$$

For $n = 3$ we obtain Example 3.

Example 8. If $n \geq 3$, $R := \{0,1\}^n$, $0 := (0,\ldots,0)$, $1 := (1,\ldots,1)$ and $+$ and $\cdot$ are defined as $\triangle$ and $\wedge$ in Example 7, respectively , then $(R,+,\cdot,0,1)$ is a commutative strong difference ring-like algebra that is not a Boolean ring since

$$\begin{aligned}&((0,\ldots,0,1)+(1,\ldots,1,0))(0,\ldots,0,1,0) = \\ &= 1\cdot(0,\ldots,0,1,0) = (0,\ldots,0,1,0) \neq 0 = 0+0 = \\ &= (0,\ldots,0,1)(0,\ldots,0,1,0)+(1,\ldots,1,0)(0,\ldots,0,1,0).\end{aligned}$$

This commutative strong difference ring-like algebra corresponds to the difference lattice defined in Example 7. For $n = 3$ we obtain Example 4.

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TWO PROBLEMS OF GEORGE ANDREWS ON GENERATING FUNCTIONS FOR PARTITIONS

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Abstract. Based on Andrews' recent work on parity in partitions, this paper will prove two partition identities proposed by Andrews (2010), simplify two generating functions into single sum expressions and extend two double series expansions of the first and second q-exponential functions.

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In a recent paper [1] on parity in partition identities, Andrews proposed fifteen problems. The two of them, numbered with 9 and 10 and soon proved by Yee [5], demand essentially to confirm the following identities:

$$\sum_{n\geq 1}(-1)^n q^{n^2} = \sum_{i,j\geq 0} \frac{(-1)^i q^{(i-j)^2+j^2+i-j}}{(-q;q)_i (q;q)_{2j-1} (q;q)_{i-2j+1}},$$

$$\sum_{n\geq 0}(-1)^n q^{n^2} = \sum_{i,j\geq 0} \frac{(-1)^i q^{(i-j)^2+j^2+j}}{(-q;q)_i (q;q)_{2j} (q;q)_{i-2j}};$$

where the summation index n in the second equation should start from 0 instead of 1. Here we follow the standard notation of shifted factorials in base q given by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x) \quad \text{for} \quad n\in\mathbb{N}$$

which admit for $|q|<1$ the two well–defined infinite product expressions

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-q^k x) \quad \text{and} \quad (x;q)_n = (x;q)_\infty \big/ \big(q^n x;q\big)_\infty .$$

The goal of this paper will be threefold. Firstly, we shall present proofs (different from Yee's ones) for both identities proposed by Andrews through combinations of the q-Gauss summation theorem and Jacobi's triple product identity. Secondly, Andrews' double sum generating functions for partitions with parity index enumerator

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will be reduced to single sum expressions by utilizing the q-binomial theorem. Finally, two double sum expansions for $(qx;q)_\infty^{\pm 1}$ will slightly be generalized with two free variables.

Throughout the paper, the q-binomial theorem (cf. [4, II-4]) will be fundamental in our derivation, which reads as

$$(x;q)_n=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{k}{2}}x^k$$

where the q-binomial coefficient is defined by

$$\begin{bmatrix}n\\k\end{bmatrix}=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}\quad\text{for}\quad n,k\in\mathbb{N}_0\quad\text{with}\quad n\geq k.$$

1. Solutions for Andrews' Problems 9 and 10

This section will be devoted to the confirmation to the two identities demanded by Andrews [1], which will be accomplished by combining limiting cases of the q-Gauss summation theorem with Jacobi's triple product identity. Our proofs are different from those given by Yee [5], who employed a partial theta series identity due to Ramanujan [2, Entry 1.6.2].

Theorem 1 (Andrews, Problem 9 in [1])**.**

$$\sum_{n\geq 1}(-1)^n q^{n^2}=\sum_{i,j\geq 0}\frac{(-1)^i q^{(i-j)^2+j^2+i-j}}{(-q;q)_i(q;q)_{2j-1}(q;q)_{i-2j+1}}.\tag{1.1}$$

Theorem 2 (Andrews, Problem 10 in [1])**.**

$$\sum_{n\geq 0}(-1)^n q^{n^2}=\sum_{i,j\geq 0}\frac{(-1)^i q^{(i-j)^2+j^2+j}}{(-q;q)_i(q;q)_{2j}(q;q)_{i-2j}}.\tag{1.2}$$

Replacing i by $2j+k-1$, we can express the double sum in (1.1) as

$$\begin{aligned}\text{RHS}(1.1)&=\sum_{j\geq 1}\sum_{i\geq 2j-1}\frac{(-1)^i q^{(i-j)^2+j^2+i-j}}{(-q;q)_i(q;q)_{2j-1}(q;q)_{i-2j+1}}\\&=\sum_{j\geq 1}\frac{-q^{2j^2-j}}{(q^2;q^2)_{2j-1}}\sum_{k\geq 0}\frac{(-1)^k q^{k^2+2kj-k}}{(q;q)_k(-q^{2j};q)_k}.\end{aligned}$$

The inner sum with respect to k can be evaluated, by the q-Gauss summation theorem (cf. [3, §8.4] and [4, II-8]) for ${}_2\phi_1$-series, as $(-q^{2j};q)_\infty^{-1}$, which leads the double sum to the following closed form

$$\text{RHS}(1.1)=\frac{-1}{(-q;q)_\infty}\sum_{j\geq 1}\frac{q^{2j^2-j}}{(q;q)_{2j-1}}=\frac{(q;q)_\infty-(-q;q)_\infty}{2(-q;q)_\infty}$$

where the second q-exponential function (cf. [3, §8.2] and [4, II-2]) has been appealed. Finally applying Jacobi's triple product identity (cf. [3, §8.6] and [4, II-28]), we can make further reformulation

$$\begin{aligned}2\mathrm{RHS}(1.1) &= \frac{(q;q)_\infty}{(-q;q)_\infty} - 1 = (q^2;q^2)_\infty (q;q^2)_\infty^2 - 1 \\ &= \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} - 1 = 2\sum_{n\geq 1} (-1)^n q^{n^2} = 2\mathrm{LHS}(1.1)\end{aligned}$$

which proves the first identity displayed in (1.1).

The identity displayed (1.2) can be confirmed analogously. In fact, performing the replacement $i \to 2j+k$, we can express the double sum in (1.2) as

$$\begin{aligned}\mathrm{RHS}(1.2) &= \sum_{j\geq 0}\sum_{i\geq 2j} \frac{(-1)^i q^{(i-j)^2+j^2+j}}{(-q;q)_i (q;q)_{2j} (q;q)_{i-2j}} \\ &= \sum_{j\geq 0} \frac{q^{2j^2+j}}{(q^2;q^2)_{2j}} \sum_{k\geq 0} \frac{(-1)^k q^{k^2+2kj}}{(q;q)_k (-q^{2j+1};q)_k}.\end{aligned}$$

By invoking again the q-Gauss summation theorem and the second q-exponential function, we can proceed with the following simplifications

$$\mathrm{RHS}(1.2) = \frac{1}{(-q;q)_\infty} \sum_{j\geq 0} \frac{q^{2j^2+j}}{(q;q)_{2j}} = \frac{(q;q)_\infty + (-q;q)_\infty}{2(-q;q)_\infty}.$$

Then the identity (1.2) follows for the same reason as that for (1.1). □

2. Single sum expressions for Corollaries 11 and 13

For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$, define its upper even (odd) index by the number of terms in the longest decreasing subsequence of parts beginning with an even (odd) part and alternating in parity. Following Andrews [1], denote by $D_e(\infty)$ (resp. $D_o(\infty)$) the generating function of the partitions into distinct parts in three variables q, x, y, which account respectively for weights, lengths and upper even (resp. odd) parity indices. Andrews finds the following double sum expressions.

Lemma 1 (Andrews, Corollary 11 in [1])**.**

$$D_o(\infty) = \sum_{i,j\geq 0} \frac{x^i y^{2j} q^{(i-j)^2+j^2+i+j}}{(-q;q)_i (q;q)_{2j} (q;q)_{i-2j}} \tag{2.1a}$$

$$+ \sum_{i,j\geq 0} \frac{x^i y^{2j-1} q^{(i-j)^2+j^2+i-j}}{(-q;q)_i (q;q)_{2j-1} (q;q)_{i-2j+1}}, \tag{2.1b}$$

$$D_e(\infty)=\sum_{i,j\geq 0}\frac{x^i y^{2j} q^{(i-j)^2+j^2+j}}{(-q;q)_i\,(q;q)_{2j}\,(q;q)_{i-2j}} \tag{2.1c}$$

$$+\sum_{i,j\geq 0}\frac{x^i y^{2j-1} q^{(i-j)^2+j^2+2i-j}}{(-q;q)_i\,(q;q)_{2j-1}\,(q;q)_{i-2j+1}}. \tag{2.1d}$$

Recalling the q-binomial theorem, we can express $D_o(\infty)$ in (2.1a-2.1b) as follows:

$$\begin{aligned}
D_o(\infty)&=\sum_{i\geq 0}\frac{x^i q^{i^2+i}}{(q^2;q^2)_i}\sum_j\begin{bmatrix} i\\ 2j\end{bmatrix} q^{\binom{2j}{2}+2j(1-i)}y^{2j}\\
&\quad+\sum_{i\geq 1}\frac{x^i q^{i^2}}{(q^2;q^2)_i}\sum_j\begin{bmatrix} i\\ 2j-1\end{bmatrix} q^{\binom{2j-1}{2}+(2j-1)(1-i)}y^{2j-1}\\
&=\sum_{i\geq 0}\frac{x^i q^{i^2+i}}{2(q^2;q^2)_i}\Big\{(-q^{1-i}y;q)_i+(q^{1-i}y;q)_i\Big\}\\
&\quad+\sum_{i\geq 1}\frac{x^i q^{i^2}}{2(q^2;q^2)_i}\Big\{(-q^{1-i}y;q)_i-(q^{1-i}y;q)_i\Big\}.
\end{aligned}$$

The generating function $D_e(\infty)$ in (2.1c-2.1d) can similarly be treated as follows:

$$\begin{aligned}
D_e(\infty)&=\sum_{i\geq 0}\frac{x^i q^{i^2}}{(q^2;q^2)_i}\sum_j\begin{bmatrix} i\\ 2j\end{bmatrix} q^{\binom{2j}{2}+2j(1-i)}y^{2j}\\
&\quad+\sum_{i\geq 1}\frac{x^i q^{i^2+i}}{(q^2;q^2)_i}\sum_j\begin{bmatrix} i\\ 2j+1\end{bmatrix} q^{\binom{2j+1}{2}+(2j+1)(1-i)}y^{2j+1}\\
&=\sum_{i\geq 0}\frac{x^i q^{i^2}}{2(q^2;q^2)_i}\Big\{(-q^{1-i}y;q)_i+(q^{1-i}y;q)_i\Big\}\\
&\quad+\sum_{i\geq 1}\frac{x^i q^{i^2+i}}{2(q^2;q^2)_i}\Big\{(-q^{1-i}y;q)_i-(q^{1-i}y;q)_i\Big\}.
\end{aligned}$$

Recombining the summation terms and then inverting the shifted factorials involving y, we may state the resulting equations as the following theorem.

Theorem 3 (Single sum generating function)**.**

$$D_o(\infty)=\sum_{i\geq 0}\frac{(1+q^i)(-1/y;q)_i}{2(q^2;q^2)_i}q^{\binom{i}{2}+i}(xy)^i \tag{2.2a}$$

$$-\sum_{i\geq 0}\frac{(1-q^i)(1/y;q)_i}{2(q^2;q^2)_i}q^{\binom{i}{2}+i}(-xy)^i, \tag{2.2b}$$

$$D_e(\infty)=\sum_{i\geq 0}\frac{(1+q^i)(-1/y;q)_i}{2(q^2;q^2)_i}q^{\binom{i}{2}+i}(xy)^i \tag{2.2c}$$

$$+\sum_{i\geq 0}\frac{(1-q^i)(1/y;q)_i}{2(q^2;q^2)_i}q^{\binom{i}{2}+i}(-xy)^i. \tag{2.2d}$$

Instead, let $F_e(\infty)$ (resp. $F_o(\infty)$) be the generating function of the unrestricted partitions in three variables q,x,y, which account, respectively, for weights, lengths and upper even (resp. odd) parity indices. Andrews obtains also the following double sum expressions.

Lemma 2 (Andrews, Corollary 13 in [1])**.**

$$F_e(\infty)=\sum_{i,j\geq 0}\frac{x^i y^{2j}q^{2j^2+i-j}}{(-q;q)_i(q;q)_{2j}(q;q)_{i-2j}} \tag{2.3a}$$

$$+\sum_{i,j\geq 0}\frac{x^i y^{2j+1}q^{2j^2+2i+j}}{(-q;q)_i(q;q)_{2j+1}(q;q)_{i-2j-1}}, \tag{2.3b}$$

$$F_o(\infty)=\sum_{i,j\geq 0}\frac{x^i y^{2j}q^{2j^2+2i-j}}{(-q;q)_i(q;q)_{2j}(q;q)_{i-2j}} \tag{2.3c}$$

$$+\sum_{i,j\geq 0}\frac{x^i y^{2j+1}q^{2j^2+i+j}}{(-q;q)_i(q;q)_{2j+1}(q;q)_{i-2j-1}}. \tag{2.3d}$$

Following exactly the same procedure, we can show that both $F_e(\infty)$ and $F_o(\infty)$ can be reduced to the following single sum expressions.

Theorem 4 (Single sum generating function)**.**

$$F_e(\infty)=\sum_{i\geq 0}\frac{(1+q^i)(-y;q)_i}{2(q^2;q^2)_i}(qx)^i+\sum_{i\geq 0}\frac{(1-q^i)(y;q)_i}{2(q^2;q^2)_i}(qx)^i, \tag{2.4a}$$

$$F_o(\infty)=\sum_{i\geq 0}\frac{(1+q^i)(-y;q)_i}{2(q^2;q^2)_i}(qx)^i-\sum_{i\geq 0}\frac{(1-q^i)(y;q)_i}{2(q^2;q^2)_i}(qx)^i. \tag{2.4b}$$

3. Two parameter extensions of equations (9.18) and (10.19)

Letting $y=1$ in Lemmas 1 and 2, Andrews deduced, in view of partition interpretations, the following two interesting identities.

Proposition 1 (Andrews, Equation 9.18 in [1])**.**

$$(-qx;q)_\infty = \sum_{i,j\geq 0} \frac{x^i q^{(i-j)^2+j^2+i+j}}{(-q;q)_i (q;q)_{2j} (q;q)_{i-2j}} \tag{3.1a}$$

$$+ \sum_{i,j\geq 0} \frac{x^i q^{(i-j)^2+j^2+i-j}}{(-q;q)_i (q;q)_{2j-1} (q;q)_{i-2j+1}} \tag{3.1b}$$

$$= \sum_{i,j\geq 0} \frac{x^i q^{(i-j)^2+j^2+j}}{(-q;q)_i (q;q)_{2j} (q;q)_{i-2j}} \tag{3.1c}$$

$$+ \sum_{i,j\geq 0} \frac{x^i q^{(i-j)^2+j^2+3j+1}}{(-q;q)_i (q;q)_{2j+1} (q;q)_{i-2j-1}}. \tag{3.1d}$$

Proposition 2 (Andrews, Equation 10.19 in [1])**.**

$$\frac{1}{(qx;q)_\infty} = \sum_{i,j\geq 0} \frac{x^i q^{2j^2+i-j}}{(-q;q)_i (q;q)_{2j} (q;q)_{i-2j}} \tag{3.2a}$$

$$+ \sum_{i,j\geq 0} \frac{x^i q^{2j^2+2i+j}}{(-q;q)_i (q;q)_{2j+1} (q;q)_{i-2j-1}} \tag{3.2b}$$

$$= \sum_{i,j\geq 0} \frac{x^i q^{2j^2+2i-j}}{(-q;q)_i (q;q)_{2j} (q;q)_{i-2j}} \tag{3.2c}$$

$$+ \sum_{i,j\geq 0} \frac{x^i q^{2j^2+i+j}}{(-q;q)_i (q;q)_{2j+1} (q;q)_{i-2j-1}}. \tag{3.2d}$$

However, if we relate Lemmas 1 and 2 respectively to Theorems 3 and 4, the last two identities follow immediately from the first and the second q-exponential functions. This observation suggests that there may exist more general identities behind these two examples. In fact, this can be realized by examining the following nonterminating q-binomial series (cf. [3, §8.1] and [4, II-3]):

$$\frac{(qx;q)_\infty}{(qy;q)_\infty} = \sum_{k\geq 0} \frac{(x/y;q)_k}{(q;q)_k} (qy)^k \quad \text{where} \quad |qy| < 1.$$

Observe the two almost trivial equations

$$1 = \frac{(-q^{1-i};q)_i + (q^{1-i};q)_i}{(-q^{1-i};q)_i} = \frac{1+q^i}{(-q;q)_i} q^{\binom{i}{2}} \sum_j \begin{bmatrix} i \\ 2j \end{bmatrix} q^{2j^2-2ij+j}, \qquad i > 0;$$

$$1 = \frac{(-q^{1-i};q)_i - (q^{1-i};q)_i}{(-q^{1-i};q)_i} = \frac{1+q^i}{(-q;q)_i} q^{\binom{i}{2}} \sum_j \begin{bmatrix} i \\ 2j-1 \end{bmatrix} q^{2j^2-2ij+i-j}, \quad i > 0.$$

We can manipulate the q-binomial sum as follows

$$\begin{aligned}
\frac{(qx;q)_\infty}{(qy;q)_\infty} &= \sum_{i\geq 0} \frac{(x/y;q)_i}{(q;q)_i} (qy)^i \frac{1+q^i}{1+q^i} \\
&= \sum_{i\geq 0} \frac{(x/y;q)_i}{(q;q)_i} \frac{(qy)^i}{1+q^i} \frac{(-q^{1-i};q)_i + (q^{1-i};q)_i}{(-q^{1-i};q)_i} \\
&\quad + \sum_{i\geq 0} \frac{(x/y;q)_i}{(q;q)_i} \frac{(q^2y)^i}{1+q^i} \frac{(-q^{1-i};q)_i - (q^{1-i};q)_i}{(-q^{1-i};q)_i} \\
&= \sum_{i\geq 0} \frac{(x/y;q)_i}{(q;q)_i} \frac{(qy)^i}{(-q;q)_i} \sum_j \begin{bmatrix} i \\ 2j \end{bmatrix} q^{\binom{i}{2}+2j^2-2ij+j} \\
&\quad + \sum_{i\geq 0} \frac{(x/y;q)_i}{(q;q)_i} \frac{(q^2y)^i}{(-q;q)_i} \sum_j \begin{bmatrix} i \\ 2j-1 \end{bmatrix} q^{\binom{i}{2}+2j^2-2ij+i-j}.
\end{aligned}$$

By switching the two terms in the linear factor $1+q^i$, we may alternatively reformulate the q-binomial sum as

$$\begin{aligned}
\frac{(qx;q)_\infty}{(qy;q)_\infty} &= \sum_{i\geq 0} \frac{(x/y;q)_i}{(q;q)_i} (qy)^i \frac{1+q^i}{1+q^i} \\
&= \sum_{i\geq 0} \frac{(x/y;q)_i}{(q;q)_i} \frac{(q^2y)^i}{1+q^i} \frac{(-q^{1-i};q)_i + (q^{1-i};q)_i}{(-q^{1-i};q)_i} \\
&\quad + \sum_{i\geq 0} \frac{(x/y;q)_i}{(q;q)_i} \frac{(qy)^i}{1+q^i} \frac{(-q^{1-i};q)_i - (q^{1-i};q)_i}{(-q^{1-i};q)_i} \\
&= \sum_{i\geq 0} \frac{(x/y;q)_i}{(q;q)_i} \frac{(q^2y)^i}{(-q;q)_i} \sum_j \begin{bmatrix} i \\ 2j \end{bmatrix} q^{\binom{i}{2}+2j^2-2ij+j} \\
&\quad + \sum_{i\geq 0} \frac{(x/y;q)_i}{(q;q)_i} \frac{(qy)^i}{(-q;q)_i} \sum_j \begin{bmatrix} i \\ 2j-1 \end{bmatrix} q^{\binom{i}{2}+2j^2-2ij+i-j}.
\end{aligned}$$

Writing the Gaussian binomial coefficients in terms of shifted factorials, we get the following two bivariate series identities.

Theorem 5 (Two parameter generalization of Proposition 1)**.**

$$\frac{(qx;q)_\infty}{(qy;q)_\infty}=\sum_{i,j\geq 0}\frac{y^i(x/y;q)_i q^{\binom{i}{2}+2j^2-2ij+i+j}}{(-q;q)_i(q;q)_{2j}(q;q)_{i-2j}} \tag{3.3a}$$

$$+\sum_{i,j\geq 0}\frac{y^i(x/y;q)_i q^{\binom{i}{2}+2j^2-2ij+3i-j}}{(-q;q)_i(q;q)_{2j-1}(q;q)_{i-2j+1}}, \tag{3.3b}$$

$$\frac{(qx;q)_\infty}{(qy;q)_\infty}=\sum_{i,j\geq 0}\frac{y^i(x/y;q)_i q^{\binom{i}{2}+2j^2-2ij+2i+j}}{(-q;q)_i(q;q)_{2j}(q;q)_{i-2j}} \tag{3.3c}$$

$$+\sum_{i,j\geq 0}\frac{y^i(x/y;q)_i q^{\binom{i}{2}+2j^2-2ij+2i-j}}{(-q;q)_i(q;q)_{2j-1}(q;q)_{i-2j+1}}. \tag{3.3d}$$

When $y\to 0$, this theorem gives the identities displayed in Proposition 1. Similarly, consider two further simple equations

$$1=\frac{(-1;q)_i+(1;q)_i}{(-1;q)_i}=\frac{1+q^i}{(-q;q)_i}\sum_j\begin{bmatrix} i\\ 2j\end{bmatrix}q^{2j^2-j},\qquad i>0;$$

$$1=\frac{(-1;q)_i-(1;q)_i}{(-1;q)_i}=\frac{1+q^i}{(-q;q)_i}\sum_j\begin{bmatrix} i\\ 2j+1\end{bmatrix}q^{2j^2+j},\quad i>0.$$

We can express the q-binomial sum as follows

$$\begin{aligned}\frac{(qx;q)_\infty}{(qy;q)_\infty}&=\sum_{i\geq 0}\frac{(x/y;q)_i}{(q;q)_i}(qy)^i\frac{1+q^i}{1+q^i}\\
&=\sum_{i\geq 0}\frac{(x/y;q)_i}{(q;q)_i}\frac{(qy)^i}{1+q^i}\frac{(-1;q)_i+(1;q)_i}{(-1;q)_i}\\
&\quad+\sum_{i\geq 0}\frac{(x/y;q)_i}{(q;q)_i}\frac{(q^2y)^i}{1+q^i}\frac{(-1;q)_i-(1;q)_i}{(-1;q)_i}\\
&=\sum_{i\geq 0}\frac{(x/y;q)_i}{(q;q)_i}\frac{(qy)^i}{(-q;q)_i}\sum_j\begin{bmatrix} i\\ 2j\end{bmatrix}q^{2j^2-j}\\
&\quad+\sum_{i\geq 0}\frac{(x/y;q)_i}{(q;q)_i}\frac{(q^2y)^i}{(-q;q)_i}\sum_j\begin{bmatrix} i\\ 2j+1\end{bmatrix}q^{2j^2+j}.\end{aligned}$$

By interchanging the two terms in the linear factor $1+q^i$, we may alternatively reformulate the q-binomial sum as

$$\begin{aligned}\frac{(qx;q)_\infty}{(qy;q)_\infty} &= \sum_{i\geq 0}\frac{(x/y;q)_i}{(q;q)_i}(qy)^i\frac{1+q^i}{1+q^i}\\ &= \sum_{i\geq 0}\frac{(x/y;q)_i}{(q;q)_i}\frac{(q^2y)^i}{1+q^i}\frac{(-1;q)_i+(1;q)_i}{(-1;q)_i}\\ &\quad+\sum_{i\geq 0}\frac{(x/y;q)_i}{(q;q)_i}\frac{(qy)^i}{1+q^i}\frac{(-1;q)_i-(1;q)_i}{(-1;q)_i}\\ &= \sum_{i\geq 0}\frac{(x/y;q)_i}{(q;q)_i}\frac{(q^2y)^i}{(-q;q)_i}\sum_j\begin{bmatrix} i\\ 2j\end{bmatrix}q^{2j^2-j}\\ &\quad+\sum_{i\geq 0}\frac{(x/y;q)_i}{(q;q)_i}\frac{(qy)^i}{(-q;q)_i}\sum_j\begin{bmatrix} i\\ 2j+1\end{bmatrix}q^{2j^2+j}.\end{aligned}$$

Writing the Gaussian binomial coefficients in terms of shifted factorials, we get the following two bivariate series identities.

Theorem 6 (Two parameter generalization of Proposition 2)**.**

$$\frac{(qx;q)_\infty}{(qy;q)_\infty} = \sum_{i,j\geq 0}\frac{y^i(x/y;q)_i q^{2j^2+i-j}}{(-q;q)_i(q;q)_{2j}(q;q)_{i-2j}} \tag{3.4a}$$

$$+\sum_{i,j\geq 0}\frac{y^i(x/y;q)_i q^{2j^2+2i+j}}{(-q;q)_i(q;q)_{2j+1}(q;q)_{i-2j-1}}, \tag{3.4b}$$

$$\frac{(qx;q)_\infty}{(qy;q)_\infty} = \sum_{i,j\geq 0}\frac{y^i(x/y;q)_i q^{2j^2+2i-j}}{(-q;q)_i(q;q)_{2j}(q;q)_{i-2j}} \tag{3.4c}$$

$$+\sum_{i,j\geq 0}\frac{y^i(x/y;q)_i q^{2j^2+i+j}}{(-q;q)_i(q;q)_{2j+1}(q;q)_{i-2j-1}}. \tag{3.4d}$$

When $x=0$, this theorem gives the identities displayed in Proposition 2.

Open Problem: It would be interesting to find combinatorial interpretations via partitions for the two variable identities displayed in Theorems 9 and 10.

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SUPERADDITIVITY OF THE JENSEN INTEGRAL INEQUALITY WITH APPLICATIONS

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Abstract. The superadditivity and monotonicity of two functionals associated to the celebrated Jensen's integral inequality for convex functions with applications for Hölder's inequality and f-divergence measures in information theory are given.

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1. INTRODUCTION

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ – algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a $\mu-$measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ – a.e. (almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega w(x)|f(x)|\, d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\, d\mu(x)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S.S. Dragomir obtained in 2002 [10] the following result:

Theorem 1. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f) f \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. (almost everywhere) on Ω with $\int_\Omega w d\mu = 1$. Then we have the inequality:*

$$\begin{aligned} 0 &\leq \int_\Omega w(\Phi \circ f)\, d\mu - \Phi\left(\int_\Omega w f d\mu\right) \\ &\leq \int_\Omega w\left(\Phi' \circ f\right) f d\mu - \int_\Omega w\left(\Phi' \circ f\right) d\mu \int_\Omega w f d\mu \end{aligned} \tag{1.1}$$

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$$\leq \frac{1}{2}\left[\Phi'(M)-\Phi'(m)\right]\int_{\Omega} w\left|f-\int_{\Omega} wfd\mu\right|d\mu.$$

For a generalization of the first inequality in (1.1) without the differentiability assumption and the derivative Φ' replaced with a selection φ from the subdifferential $\partial\Phi$, see the paper [24] by C.P. Niculescu.

Remark 1. We notice that in the discrete case, the inequality between the first and the second term in (1.1) was proved in 1994 by Dragomir & Ionescu, see [12].

Utilising a different approach than in [10], we obtained in [11] the following two results that provide other upper bounds for the Jensen's difference:

$$\int_{\Omega} w(\Phi\circ f)\,d\mu-\Phi\left(\int_{\Omega} wfd\mu\right).$$

Theorem 2. *Let $\Phi: I\to\mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m,M\in\mathbb{R}$, $m<M$ with $[m,M]\subset\mathring{I}$, $\mathring{I}$ is the interior of I. If $f:\Omega\to\mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty<m\leq f(x)\leq M<\infty \text{ for } \mu\text{-a.e. } x\in\Omega$$

and is such that $f,\Phi\circ f\in L_w(\Omega,\mu)$, then

$$\begin{aligned}0&\leq \int_{\Omega} w(\Phi\circ f)\,d\mu-\Phi\left(\bar{f}_{\Omega,w}\right)\\ &\leq \frac{\left(M-\bar{f}_{\Omega,w}\right)\left(\bar{f}_{\Omega,w}-m\right)}{M-m}\sup_{t\in(m,M)}\Psi_{\Phi}(t;m,M)\\ &\leq \left(M-\bar{f}_{\Omega,w}\right)\left(\bar{f}_{\Omega,w}-m\right)\frac{\Phi'_{-}(M)-\Phi'_{+}(m)}{M-m}\\ &\leq \frac{1}{4}(M-m)\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right],\end{aligned} \tag{1.2}$$

where $\bar{f}_{\Omega,w}:=\int_{\Omega} w(x)f(x)\,d\mu(x)\in[m,M]$ and $\Psi_{\Phi}(\cdot;m,M):(m,M)\to\mathbb{R}$ is defined by

$$\Psi_{\Phi}(t;m,M)=\frac{\Phi(M)-\Phi(t)}{M-t}-\frac{\Phi(t)-\Phi(m)}{t-m}.$$

We also have the inequality

$$\begin{aligned}0&\leq \int_{\Omega} w(\Phi\circ f)\,d\mu-\Phi\left(\bar{f}_{\Omega,w}\right)\leq \frac{1}{4}(M-m)\Psi_{\Phi}\left(\bar{f}_{\Omega,w};m,M\right)\\ &\leq \frac{1}{4}(M-m)\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right],\end{aligned} \tag{1.3}$$

provided that $\bar{f}_{\Omega,w}\in(m,M)$.

Theorem 3. *With the assumptions of Theorem 2, we have the inequalities*

$$0 \leq \int_{\Omega} w(\Phi \circ f)\, d\mu(x) - \Phi\left(\bar{f}_{\Omega,w}\right) \tag{1.4}$$
$$\leq 2\max\left\{\frac{M-\bar{f}_{\Omega,w}}{M-m}, \frac{\bar{f}_{\Omega,w}-m}{M-m}\right\}\left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right]$$
$$\leq \frac{1}{2}\max\left\{M-\bar{f}_{\Omega,w}, \bar{f}_{\Omega,w}-m\right\}\left[\Phi'_{-}(M) - \Phi'_{+}(m)\right].$$

Motivated by these results we establish in the current paper some refinements and reverses of the Jensen integral inequality by capitalizing on the superadditivity and monotonicity properties of two associated functionals. Application in connection with the Hölder inequality and for f-divergence measures in Information Theory are provided as well.

2. Superadditivity and Monotonicity Properties

For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ -a.e. $x \in \Omega$ and $\int_{\Omega} w d\mu > 0$ we consider the functional

$$J(w;\Phi,f) := \int_{\Omega} w(\Phi \circ f)\, d\mu - \Phi\left(\frac{\int_{\Omega} wf d\mu}{\int_{\Omega} w d\mu}\right)\int_{\Omega} w d\mu \geq 0, \tag{2.1}$$

where $\Phi : I \to \mathbb{R}$ is a continuous convex function on the interval of real numbers I, $f : \Omega \to \mathbb{R}$ is μ-measurable and such that $f, \Phi \circ f \in L_w(\Omega,\mu)$.

Theorem 4. *Let $w_i : \Omega \to \mathbb{R}$, with $w_i(x) \geq 0$ for μ – a.e. (almost every) $x \in \Omega$ and $\int_{\Omega} w_i d\mu > 0, i \in \{1,2\}$. If $\Phi : I \to \mathbb{R}$ is a continuous convex function on the interval of real numbers I, $f : \Omega \to \mathbb{R}$ is μ-measurable and such that $f, \Phi \circ f \in L_{w_1}(\Omega,\mu) \cap L_{w_2}(\Omega,\mu)$, then*

$$J(w_1+w_2;\Phi,f) \geq J(w_1;\Phi,f) + J(w_2;\Phi,f) \geq 0 \tag{2.2}$$

i.e., J is a superadditive functional of weights.

Moreover, if $w_2 \geq w_1 \geq 0$ μ – a.e. on Ω, then

$$J(w_2;\Phi,f) \geq J(w_1;\Phi,f) \geq 0, \tag{2.3}$$

i.e., J is a monotonic nondecreasing functional of weights.

Proof. Utilising the convexity property of Φ we have successively

$$J(w_1+w_2;\Phi,f) \tag{2.4}$$
$$= \int_{\Omega} (w_1+w_2)(\Phi \circ f)\, d\mu - \Phi\left(\frac{\int_{\Omega}(w_1+w_2) f d\mu}{\int_{\Omega}(w_1+w_2) d\mu}\right)\int_{\Omega}(w_1+w_2) d\mu$$
$$= \int_{\Omega} w_1(\Phi \circ f)\, d\mu + \int_{\Omega} w_2(\Phi \circ f)\, d\mu$$

$$
\begin{aligned}
&-\Phi\left(\frac{\int_\Omega w_1 d\mu \cdot \frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu} + \int_\Omega w_2 d\mu \cdot \frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}}{\int_\Omega (w_1+w_2)\, d\mu}\right)\int_\Omega (w_1+w_2)\, d\mu \\
&\geq \int_\Omega w_1 (\Phi\circ f)\, d\mu + \int_\Omega w_2 (\Phi\circ f)\, d\mu \\
&-\left[\frac{\int_\Omega w_1 d\mu}{\int_\Omega (w_1+w_2)\, d\mu}\Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right) + \frac{\int_\Omega w_2 d\mu}{\int_\Omega (w_1+w_2)\, d\mu}\Phi\left(\frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}\right)\right] \\
&\times \int_\Omega (w_1+w_2)\, d\mu \\
&= \int_\Omega w_1 (\Phi\circ f)\, d\mu - \Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)\int_\Omega w_1 d\mu \\
&+ \int_\Omega w_2 (\Phi\circ f)\, d\mu - \Phi\left(\frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}\right)\int_\Omega w_2 d\mu \\
&= J(w_1;\Phi,f) + J(w_2;\Phi,f)
\end{aligned}
$$

which proves the superadditivity property.

Now, if $w_2 \geq w_1 \geq 0$, then on applying the superadditivity property we have

$$
\begin{aligned}
J(w_2;\Phi,f) &= J(w_1+(w_2-w_1);\Phi,f) \geq J(w_1;\Phi,f) + J(w_2-w_1;\Phi,f) \\
&\geq J(w_1;\Phi,f)
\end{aligned}
$$

since by the Jensen's inequality for the positive weights we have $J(w_2-w_1;\Phi,f) \geq 0$. □

The above theorem has a simple however interesting consequence that provides both a refinement and a reverse for the Jensen's integral inequality:

Corollary 1. *Let $w_i : \Omega \to \mathbb{R}$, with $w_i(x) \geq 0$ for μ – a.e. $x \in \Omega$, $\int_\Omega w_i d\mu > 0, i \in \{1,2\}$ and there exists the nonnegative constants γ, Γ such that*

$$
0 \leq \gamma \leq \frac{w_2}{w_1} \leq \Gamma < \infty \ \mu\text{-a.e. on } \Omega. \tag{2.5}
$$

If $\Phi : I \to \mathbb{R}$ is a continuous convex function on the interval of real numbers I, $f : \Omega \to \mathbb{R}$ is μ-measurable and such that $f, \Phi\circ f \in L_{w_1}(\Omega,\mu) \cap L_{w_2}(\Omega,\mu)$, then

$$
\begin{aligned}
0 \leq \gamma \cdot &\left[\int_\Omega w_1 (\Phi\circ f)\, d\mu - \Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)\int_\Omega w_1 d\mu\right] \\
&\leq \int_\Omega w_2 (\Phi\circ f)\, d\mu - \Phi\left(\frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}\right)\int_\Omega w_2 d\mu \\
&\leq \Gamma \cdot \left[\int_\Omega w_1 (\Phi\circ f)\, d\mu - \Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)\int_\Omega w_1 d\mu\right]
\end{aligned} \tag{2.6}
$$

or, equivalently,

$$0 \leq \gamma \cdot \frac{\int_\Omega w_1 d\mu}{\int_\Omega w_2 d\mu}\left[\frac{\int_\Omega w_1 (\Phi \circ f)\, d\mu}{\int_\Omega w_1 d\mu} - \Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)\right] \tag{2.7}$$
$$\leq \frac{\int_\Omega w_2 (\Phi \circ f)\, d\mu}{\int_\Omega w_2 d\mu} - \Phi\left(\frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}\right)$$
$$\leq \Gamma \cdot \frac{\int_\Omega w_1 d\mu}{\int_\Omega w_2 d\mu}\left[\frac{\int_\Omega w_1 (\Phi \circ f)\, d\mu}{\int_\Omega w_1 d\mu} - \Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)\right].$$

Proof. From (2.5) we have $\gamma w_1 \leq w_2 \leq \Gamma w_1 < \infty$ μ-a.e. on Ω and by the monotonicity property (2.3) we get

$$J(\Gamma w_1; \Phi, f) \geq J(w_2; \Phi, f) \geq J(\gamma w_1; \Phi, f). \tag{2.8}$$

Since the the functional is positive homogeneous, namely $J(\alpha w; \Phi, f) = \alpha J(w; \Phi, f)$, then we get from (2.8) the desired result (2.6). □

Remark 2. Assume that $\mu(\Omega) < \infty$ and let $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ – a.e. $x \in \Omega$, $\int_\Omega w d\mu > 0$ and w is essentially bounded, i.e. ess $\inf_{x\in\Omega} w(x)$ and ess $\sup_{x\in\Omega} w(x)$ are finite.

If $\Phi : I \to \mathbb{R}$ is a continuous convex function on the interval of real numbers I, $f : \Omega \to \mathbb{R}$ is μ-measurable and such that $f, \Phi \circ f \in L_w(\Omega, \mu) \cap L(\Omega, \mu)$, then

$$0 \leq \frac{\text{ess } \inf_{x\in\Omega} w(x)}{\frac{1}{\mu(\Omega)}\int_\Omega w d\mu}\left[\frac{\int_\Omega (\Phi \circ f)\, d\mu}{\mu(\Omega)} - \Phi\left(\frac{\int_\Omega f d\mu}{\mu(\Omega)}\right)\right] \tag{2.9}$$
$$\leq \frac{\int_\Omega w (\Phi \circ f)\, d\mu}{\int_\Omega w d\mu} - \Phi\left(\frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu}\right)$$
$$\leq \frac{\text{ess } \sup_{x\in\Omega} w(x)}{\frac{1}{\mu(\Omega)}\int_\Omega w d\mu}\left[\frac{\int_\Omega (\Phi \circ f)\, d\mu}{\mu(\Omega)} - \Phi\left(\frac{\int_\Omega f d\mu}{\mu(\Omega)}\right)\right].$$

This result can be used to provide the following result related to the Hermite-Hadamard inequality for convex functions that states that

$$\frac{1}{b-a}\int_a^b \Phi(t)\, dt \geq \Phi\left(\frac{a+b}{2}\right)$$

for any convex function $\Phi : [a, b] \to \mathbb{R}$.

Indeed , if $w : [a, b] \to [0, \infty)$ is Lebesgue integrable, then we have

$$0 \leq \frac{\text{ess } \inf_{x\in[a,b]} w(x)}{\frac{1}{b-a}\int_a^b w(t)\, dt}\left[\frac{1}{b-a}\int_a^b \Phi(t)\, dt - \Phi\left(\frac{a+b}{2}\right)\right] \tag{2.10}$$
$$\leq \frac{\int_a^b w(t)\Phi(t)\, dt}{\int_a^b w(t)\, dt} - \Phi\left(\frac{\int_\Omega w(t) t\, dt}{\int_a^b w(t)\, dt}\right)$$

$$\leq \frac{\operatorname{ess\,sup}_{x\in[a,b]} w(x)}{\frac{1}{b-a}\int_a^b w(t)\,dt}\left[\frac{1}{b-a}\int_a^b \Phi(t)\,dt - \Phi\left(\frac{a+b}{2}\right)\right].$$

Now we consider another functional depending on the weights

$$K(w;\Phi,f) := \frac{J(w;\Phi,f)}{\int_\Omega w\,d\mu} = \frac{\int_\Omega w(\Phi\circ f)\,d\mu}{\int_\Omega w\,d\mu} - \Phi\left(\frac{\int_\Omega wf\,d\mu}{\int_\Omega w\,d\mu}\right) \geq 0$$

and the composite functional

$$L(w;\Phi,f) := \left(\int_\Omega w\,d\mu\right)\ln\left[K(w;\Phi,f)+1\right] \geq 0,$$

where $\Phi : I \to \mathbb{R}$ is a continuous convex function on the interval of real numbers I and $f : \Omega \to \mathbb{R}$ is μ-measurable and such that $f, \Phi\circ f \in L_w(\Omega,\mu)$.

Theorem 5. *With the assumptions of Theorem 4, L is a superadditive and monotonic nondecreasing functional of weights.*

Proof. Let $w_i : \Omega \to \mathbb{R}$, with $w_i(x) \geq 0$ for μ – a.e. $x \in \Omega$ and $\int_\Omega w_i\,d\mu > 0, i \in \{1,2\}$ such that $f, \Phi\circ f \in L_{w_1}(\Omega,\mu)\cap L_{w_2}(\Omega,\mu)$.

Utilising the superadditivity property of J we have

$$\begin{aligned}
&L(w_1+w_2;\Phi,f) \qquad (2.11)\\
&= \left(\int_\Omega (w_1+w_2)\,d\mu\right)\ln\left[K(w_1+w_2;\Phi,f)+1\right]\\
&= \left(\int_\Omega (w_1+w_2)\,d\mu\right)\ln\left[\frac{J(w_1+w_2;\Phi,f)}{\int_\Omega (w_1+w_2)\,d\mu}+1\right]\\
&\geq \left(\int_\Omega (w_1+w_2)\,d\mu\right)\ln\left[\frac{J(w_1;\Phi,f)+J(w_2;\Phi,f)}{\int_\Omega (w_1+w_2)\,d\mu}+1\right]\\
&= \left(\int_\Omega (w_1+w_2)\,d\mu\right)\\
&\times\ln\left[\frac{\int_\Omega w_1 d\mu\cdot\frac{J(w_1;\Phi,f)}{\int_\Omega w_1 d\mu}+\int_\Omega w_2 d\mu\cdot\frac{J(w_2;\Phi,f)}{\int_\Omega w_2 d\mu}}{\int_\Omega (w_1+w_2)\,d\mu}+1\right]\\
&= \left(\int_\Omega (w_1+w_2)\,d\mu\right)\\
&\times\ln\left[\frac{\int_\Omega w_1 d\mu\cdot\left(\frac{J(w_1;\Phi,f)}{\int_\Omega w_1 d\mu}+1\right)+\int_\Omega w_2 d\mu\cdot\left(\frac{J(w_2;\Phi,f)}{\int_\Omega w_2 d\mu}+1\right)}{\int_\Omega (w_1+w_2)\,d\mu}\right]\\
&:= A.
\end{aligned}$$

By the weighted arithmetic mean - geometric mean inequality we have

$$\frac{\int_\Omega w_1 d\mu \cdot \left(\frac{J(w_1;\Phi,f)}{\int_\Omega w_1 d\mu}+1\right)+\int_\Omega w_2 d\mu \cdot \left(\frac{J(w_2;\Phi,f)}{\int_\Omega w_2 d\mu}+1\right)}{\int_\Omega (w_1+w_2)\, d\mu}$$

$$\geq \left(\frac{J(w_1;\Phi,f)}{\int_\Omega w_1 d\mu}+1\right)^{\frac{\int_\Omega w_1 d\mu}{\int_\Omega (w_1+w_2) d\mu}} \left(\frac{J(w_2;\Phi,f)}{\int_\Omega w_2 d\mu}+1\right)^{\frac{\int_\Omega w_2 d\mu}{\int_\Omega (w_1+w_2) d\mu}},$$

therefore, by taking the logarithm and utilizing the definition of the functional K, we get the inequality

$$A \geq \left(\int_\Omega w_1 d\mu\right) \ln\left(K(w_1;\Phi,f)+1\right)+\left(\int_\Omega w_2 d\mu\right) \ln\left(K(w_2;\Phi,f)+1\right) \tag{2.12}$$

$$= L(w_1;\Phi,f)+L(w_2;\Phi,f).$$

Utilising (2.11) and (2.12) we deduce the superadditivity of the functional L as a function of weights.

Since $L(w;\Phi,f) \geq 0$ for any weight w and it is superadditive, by employing a similar argument to the one in the proof of Theorem 4 we conclude that it is also monotonic nondecreasing as a function of weights. □

The following result provides another refinement and reverse of the Jensen inequality:

Corollary 2. *Let $w_i : \Omega \to \mathbb{R}$ with $w_i(x) \geq 0$ for μ – a.e. $x \in \Omega$, $\int_\Omega w_i d\mu > 0, i \in \{1,2\}$ and there exists the nonnegative constants γ, Γ such that*

$$0 \leq \gamma \leq \frac{w_2}{w_1} \leq \Gamma < \infty \ \mu\text{-a.e. on } \Omega.$$

If $\Phi : I \to \mathbb{R}$ is a continuous convex function on the interval of real numbers I, $f : \Omega \to \mathbb{R}$ is μ-measurable and such that $f, \Phi \circ f \in L_{w_1}(\Omega,\mu) \cap L_{w_2}(\Omega,\mu)$, then

$$0 \leq \left[\frac{\int_\Omega w_1 (\Phi \circ f) d\mu}{\int_\Omega w_1 d\mu} - \Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)+1\right]^{\gamma \cdot \frac{(\int_\Omega w_1 d\mu)}{(\int_\Omega w_2 d\mu)}} - 1 \tag{2.13}$$

$$\leq \frac{\int_\Omega w_2 (\Phi \circ f) d\mu}{\int_\Omega w_2 d\mu} - \Phi\left(\frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}\right)$$

$$\leq \left[\frac{\int_\Omega w_1 (\Phi \circ f) d\mu}{\int_\Omega w_1 d\mu} - \Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)+1\right]^{\Gamma \cdot \frac{(\int_\Omega w_1 d\mu)}{(\int_\Omega w_2 d\mu)}} - 1.$$

Proof. Since L is monotonic nondecreasing and positive homogeneous as a function of weights, we have

$$\gamma L(w_1;\Phi,f) \leq L(w_2;\Phi,f) \leq \Gamma L(w_1;\Phi,f)$$

which is equivalent with

$$[K(w_1;\Phi,f)+1]^{\gamma\left(\int_\Omega w_1 d\mu\right)} \leq [K(w_2;\Phi,f)+1]^{\left(\int_\Omega w_2 d\mu\right)} \leq [K(w_1;\Phi,f)+1]^{\Gamma\left(\int_\Omega w_1 d\mu\right)}$$

which provides that

$$[K(w_1;\Phi,f)+1]^{\gamma\frac{\left(\int_\Omega w_1 d\mu\right)}{\left(\int_\Omega w_2 d\mu\right)}}-1 \leq K(w_2;\Phi,f) \leq [K(w_1;\Phi,f)+1]^{\Gamma\frac{\left(\int_\Omega w_1 d\mu\right)}{\left(\int_\Omega w_2 d\mu\right)}}-1.$$

□

Remark 3. Assume that $\mu(\Omega)<\infty$ and let $w:\Omega\to\mathbb{R}$, with $w(x)\geq 0$ for μ – a.e. $x\in\Omega$, $\int_\Omega w d\mu>0$ and w is essentially bounded, i.e. ess $\inf_{x\in\Omega} w(x)$ and ess $\sup_{x\in\Omega} w(x)$ are finite.

If $\Phi:I\to\mathbb{R}$ is a continuous convex function on the interval of real numbers I, $f:\Omega\to\mathbb{R}$ is μ-measurable and such that $f,\Phi\circ f\in L_w(\Omega,\mu)\cap L(\Omega,\mu)$, then

$$\begin{aligned} 0 &\leq \left[\frac{\int_\Omega(\Phi\circ f)d\mu}{\mu(\Omega)}-\Phi\left(\frac{\int_\Omega f d\mu}{\mu(\Omega)}\right)+1\right]^{\frac{ess\inf_{x\in\Omega} w(x)}{\frac{1}{\mu(\Omega)}\left(\int_\Omega w d\mu\right)}}-1 \\ &\leq \frac{\int_\Omega w(\Phi\circ f)d\mu}{\int_\Omega w d\mu}-\Phi\left(\frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu}\right) \\ &\leq \left[\frac{\int_\Omega(\Phi\circ f)d\mu}{\mu(\Omega)}-\Phi\left(\frac{\int_\Omega f d\mu}{\mu(\Omega)}\right)+1\right]^{\frac{ess\sup_{x\in\Omega} w(x)}{\frac{1}{\mu(\Omega)}\left(\int_\Omega w d\mu\right)}}-1. \end{aligned} \tag{2.14}$$

In particular, if $w:[a,b]\to[0,\infty)$ is Lebesgue integrable, then we have the following result related to the Hermite-Hadamard inequality for the convex function $\Phi:[a,b]\to\mathbb{R}$

$$\begin{aligned} 0 &\leq \left[\frac{1}{b-a}\int_a^b \Phi(t)dt-\Phi\left(\frac{a+b}{2}\right)+1\right]^{\frac{\text{ess }\inf_{x\in[a,b]} w(x)}{\frac{1}{b-a}\int_a^b w(t)dt}}-1 \\ &\leq \frac{\int_a^b w(t)\Phi(t)dt}{\int_a^b w(t)dt}-\Phi\left(\frac{\int_\Omega w(t)t dt}{\int_a^b w(t)dt}\right) \\ &\leq \left[\frac{1}{b-a}\int_a^b \Phi(t)dt-\Phi\left(\frac{a+b}{2}\right)+1\right]^{\frac{\text{ess }\sup_{x\in[a,b]} w(x)}{\frac{1}{b-a}\int_a^b w(t)dt}}-1. \end{aligned} \tag{2.15}$$

3. Applications for the Hölder Inequality

It is well known that if $f \in L_p(\Omega,\mu)$, $p>1$, where the Lebesgue space $L_p(\Omega,\mu)$ is defined by

$$L_p(\Omega,\mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)|^p \, d\mu(x) < \infty\}$$

and $g \in L_q(\Omega,\mu)$ with $\frac{1}{p}+\frac{1}{q}=1$ then $fg \in L(\Omega,\mu) := L_1(\Omega,\mu)$ and the *Hölder inequality* holds true

$$\int_\Omega |fg|\, d\mu \le \left(\int_\Omega |f|^p \, d\mu\right)^{1/p} \left(\int_\Omega |g|^p \, d\mu\right)^{1/q}.$$

Assume that $p>1$. If $h:\Omega\to\mathbb{R}$ is μ-measurable, $\mu(\Omega)<\infty$, $|h|, |h|^p \in L_w(\Omega,\mu)\cap L(\Omega,\mu)$, then by (2.9) we have the bounds

$$\begin{aligned} 0 &\le \frac{\operatorname{ess\ inf}_{x\in\Omega} w(x)}{\frac{1}{\mu(\Omega)}\int_\Omega w d\mu}\left[\frac{1}{\mu(\Omega)}\int_\Omega |h|^p\, d\mu - \left(\frac{1}{\mu(\Omega)}\int_\Omega |h|\, d\mu\right)^p\right] \qquad (3.1)\\ &\le \frac{1}{\int_\Omega w d\mu}\int_\Omega w|h|^p\, d\mu - \left(\frac{1}{\int_\Omega w d\mu}\int_\Omega w|h|\, d\mu\right)^p \\ &\le \frac{\operatorname{ess\ sup}_{x\in\Omega} w(x)}{\frac{1}{\mu(\Omega)}\int_\Omega w d\mu}\left[\frac{1}{\mu(\Omega)}\int_\Omega |h|^p\, d\mu - \left(\frac{1}{\mu(\Omega)}\int_\Omega |h|\, d\mu\right)^p\right]. \end{aligned}$$

Proposition 1. *If $f \in L_p(\Omega,\mu)$, $g \in L_q(\Omega,\mu)$ with $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\mu(\Omega)<\infty$ and there exists the constants $\delta, \Delta > 0$ and such that*

$$\delta \le |g| \le \Delta \ \mu\text{-a.e on } \Omega, \qquad (3.2)$$

then we have

$$\begin{aligned} 0 &\le \frac{\delta^q}{\frac{1}{\mu(\Omega)}\int_\Omega |g|^q\, d\mu}\left[\frac{1}{\mu(\Omega)}\int_\Omega \frac{|f|^p}{|g|^q} d\mu - \left(\frac{1}{\mu(\Omega)}\int_\Omega \frac{|f|}{|g|^{q-1}} d\mu\right)^p\right] \qquad (3.3)\\ &\le \frac{\int_\Omega |f|^p\, d\mu}{\int_\Omega |g|^q\, d\mu} - \left(\frac{\int_\Omega |fg|\, d\mu}{\int_\Omega |g|^q\, d\mu}\right)^p \\ &\le \frac{\Delta^q}{\frac{1}{\mu(\Omega)}\int_\Omega |g|^q\, d\mu}\left[\frac{1}{\mu(\Omega)}\int_\Omega \frac{|f|^p}{|g|^q} d\mu - \left(\frac{1}{\mu(\Omega)}\int_\Omega \frac{|f|}{|g|^{q-1}} d\mu\right)^p\right]. \end{aligned}$$

Proof. The inequalities (3.3) follows from (3.1) by choosing

$$h = \frac{|f|}{|g|^{q-1}} \text{ and } w = |g|^q.$$

The details are omitted. □

Remark 4. We observe that for $p=q=2$ we have from (3.3) the following reverse of the Cauchy-Bunyakovsky-Schwarz inequality

$$0 \leq \delta^2 \mu(\Omega)\left[\frac{1}{\mu(\Omega)}\int_\Omega \left|\frac{f}{g}\right|^2 d\mu - \left(\frac{1}{\mu(\Omega)}\int_\Omega \left|\frac{f}{g}\right| d\mu\right)^2\right]\int_\Omega |g|^2 d\mu \tag{3.4}$$

$$\leq \int_\Omega |g|^2 d\mu \int_\Omega |f|^2 d\mu - \left(\int_\Omega |fg| d\mu\right)^2$$

$$\leq \Delta^2 \mu(\Omega)\left[\frac{1}{\mu(\Omega)}\int_\Omega \left|\frac{f}{g}\right|^2 d\mu - \left(\frac{1}{\mu(\Omega)}\int_\Omega \left|\frac{f}{g}\right| d\mu\right)^2\right]\int_\Omega |g|^2 d\mu,$$

provided that $f, g \in L_2(\Omega, \mu)$ and g satisfies the bounds (3.2).

Similar results can be stated by utilizing the inequality (2.13), however the details are not presented here.

4. Applications for f-Divergence Measures

One of the important issues in many applications of Probability Theory is finding an appropriate measure of *distance* (or *difference* or *discrimination*) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [16], Kullback and Leibler [21], Rényi [27], Havrda and Charvat [14], Kapur [19], Sharma and Mittal [29], Burbea and Rao [4], Rao [26], Lin [22], Csiszár [7], Ali and Silvey [1], Vajda [35], Shioya and Da-te [30] and others (see for example [23] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [26], genetics [23], finance, economics, and political science [28], [32], [33], biology [25], the analysis of contingency tables [13], approximation of probability distributions [6], [20], signal processing [17], [18] and pattern recognition [2], [5]. A number of these measures of distance are specific cases of Csiszár f-divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied.

Assume that a set Ω and the $\sigma-$finite measure μ are given. Consider the set of all probability densities on μ to be $\mathcal{P} := \left\{p | p : \Omega \to \mathbb{R},\ p(x) \geq 0,\ \int_\Omega p(x)\, d\mu(x) = 1\right\}$.

Csiszár $f-$divergence is defined as follows [8]

$$I_f(p,q) := \int_\Omega p(x) f\left[\frac{q(x)}{p(x)}\right] d\mu(x), \quad p, q \in \mathcal{P}, \tag{4.1}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u=1$. By appropriately defining this convex function, various divergences are derived.

The *Kullback-Leibler divergence* [21] is well known among the information divergences. It is defined as:

$$D_{KL}(p,q) := \int_{\Omega} p(x) \ln\left[\frac{p(x)}{q(x)}\right] d\mu(x), \quad p,q \in \mathcal{P}, \tag{4.2}$$

where ln is to base e.

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: *variation distance* D_v, *Hellinger distance* D_H [15], χ^2-*divergence* D_{χ^2}, $\alpha-$*divergence* D_α, *Bhattacharyya distance* D_B [3], *Harmonic distance* D_{Ha}, *Jeffrey's distance* D_J [16], *triangular discrimination* D_Δ [34], etc... They are defined as follows:

$$D_v(p,q) := \int_{\Omega} |p(x)-q(x)| d\mu(x), \quad p,q \in \mathcal{P}; \tag{4.3}$$

$$D_H(p,q) := \int_{\Omega} \left|\sqrt{p(x)}-\sqrt{q(x)}\right| d\mu(x), \quad p,q \in \mathcal{P}; \tag{4.4}$$

$$D_{\chi^2}(p,q) := \int_{\Omega} p(x)\left[\left(\frac{q(x)}{p(x)}\right)^2 - 1\right] d\mu(x), \quad p,q \in \mathcal{P}; \tag{4.5}$$

$$D_\alpha(p,q) := \frac{4}{1-\alpha^2}\left[1-\int_{\Omega} [p(x)]^{\frac{1-\alpha}{2}} [q(x)]^{\frac{1+\alpha}{2}} d\mu(x)\right], \quad p,q \in \mathcal{P}; \tag{4.6}$$

$$D_B(p,q) := \int_{\Omega} \sqrt{p(x)q(x)} d\mu(x), \quad p,q \in \mathcal{P}; \tag{4.7}$$

$$D_{Ha}(p,q) := \int_{\Omega} \frac{2p(x)q(x)}{p(x)+q(x)} d\mu(x), \quad p,q \in \mathcal{P}; \tag{4.8}$$

$$D_J(p,q) := \int_{\Omega} [p(x)-q(x)] \ln\left[\frac{p(x)}{q(x)}\right] d\mu(x), \quad p,q \in \mathcal{P}; \tag{4.9}$$

$$D_\Delta(p,q) := \int_{\Omega} \frac{[p(x)-q(x)]^2}{p(x)+q(x)} d\mu(x), \quad p,q \in \mathcal{P}. \tag{4.10}$$

For other divergence measures, see the paper [19] by Kapur or the book on line [31] by Taneja.

Most of the above distances (4.2) – (4.10), are particular instances of Csiszár $f-$divergence. There are also many others which are not in this class (see for example [31]). For the basic properties of Csiszár $f-$divergence see [8], [9] and [35].

The following result holds:

Proposition 2. *Let $f : (0,\infty) \to \mathbb{R}$ be a convex function with the property that $f(1) = 0$. Assume that $p,q \in \mathcal{P}$ and there exists constants $0 < s < 1 < S < \infty$ such that*

$$s \leq \frac{p(x)}{q(x)} \leq S \text{ for } \mu\text{-a.e. } x \in \Omega. \tag{4.11}$$

Then we have the inequalities

$$\begin{aligned} & s\left[I_{f\left(\frac{1}{\cdot}\right)}(q,p)-f\left(D_{\chi^2}(p,q)+1\right)\right] \\ & \leq I_f(p,q) \\ & \leq S\left[I_{f\left(\frac{1}{\cdot}\right)}(q,p)-f\left(D_{\chi^2}(p,q)+1\right)\right]. \end{aligned} \tag{4.12}$$

Proof. If we use the inequality (2.6) we get

$$\begin{aligned} & s\left[\int_\Omega qf\left(\frac{q}{p}\right)d\mu-f\left(\int_\Omega \frac{q^2}{p}d\mu\right)\right] \\ & \leq \int_\Omega pf\left(\frac{q}{p}\right)d\mu \\ & \leq S\left[\int_\Omega qf\left(\frac{q}{p}\right)d\mu-f\left(\int_\Omega \frac{q^2}{p}d\mu\right)\right]. \end{aligned} \tag{4.13}$$

Since

$$\int_\Omega \frac{q^2}{p}d\mu = D_{\chi^2}(p,q)+1$$

and

$$\int_\Omega qf\left(\frac{q}{p}\right)d\mu = I_{f\left(\frac{1}{\cdot}\right)}(q,p),$$

then from (4.13) we deduce the desired result (4.12). □

We notice that $f\left(\frac{1}{\cdot}\right)$ is not always a convex function. However the definition (4.1) can be extended to any measurable function for which the integral is finite. In applications, the examples for which $f\left(\frac{1}{\cdot}\right)$ is a convex function are of interest. Such an example is provided below.

Consider the Kullback-Leibler divergence defined in (4.2). If $p,q\in\mathcal{P}$ such that there exists constants $0<s<1<S<\infty$ with

$$s\leq\frac{p(x)}{q(x)}\leq S \text{ for } \mu\text{-a.e. } x\in\Omega.. \tag{4.14}$$

then we get from (4.12) that

$$\begin{aligned} & s\left[\ln\left(D_{\chi^2}(p,q)+1\right)-D_{KL}(q,p)\right] \\ & \leq D_{KL}(p,q) \\ & \leq S\left[\ln\left(D_{\chi^2}(p,q)+1\right)-D_{KL}(q,p)\right]. \end{aligned} \tag{4.15}$$

Similar results for f-divergence measures can be stated by utilizing the inequality (2.13), however the details are not presented here.

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AN EXTENSION OF THE MIZOGUCHI–TAKAHASHI FIXED POINT THEOREM

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Abstract. Our main theorem is an extension of the well–known Mizoguchi–Takahaashi fixed point theorem [N. Mizogochi and W. Takahashi, Fixed point theorems for multi–valued mappings on complete metric spaces, *J. Math. Anal. Appl.* 141 (1989) 177–188].

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1. Introduction and statement of results

Let (X,d) be a metric space. $CB(X)$ denotes the collection of all nonempty closed bounded subsets of X. For $A,B\in CB(X)$ and $x\in X$, define

$$D(x,A):=\inf\{d(x,a);a\in A\}$$

and

$$H(A,B):=\max\{\sup_{a\in A}D(a,B),\sup_{b\in B}D(b,A)\}.$$

Notice that H is a metric on $CB(X)$. H is called the Hausdorff metric induced by d.

Definition 1. An element $x\in X$ is said to be a fixed point of a multi–valued mapping $T:X\to CB(X)$, if it is such that $x\in T(x)$.

One can show that $(CB(X),H)$ is a complete metric space, whenever (X,d) is a complete metric space (see for example Lemma 8.1.4, of [11]).

In 1969, Nadler [6] extended the Banach contraction principle [1] to set–valued mappings as follows.

Theorem 1. *Let (X,d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $r\in[0,1)$ such that $H(Tx,Ty)\leq rd(x,y)$ for all $x,y\in X$. Then there exists $z\in X$ such that $z\in T(z)$.*

Many fixed point theorems have been proved by various authors as generalizations to the Nadler's theorem. One such generalization is due to Kaneko in [4] and Nicolae

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in [7]. Another generalization was proved by Mizoguchi and Takahashi [5] which is also well known as a positive proof of a conjecture posed by Simeon Reich [10].

Nadler's theorem was generalized by Mizoguchi and Takahaashi [5] in the following way.

Theorem 2. *Let (X,d) be a complete metric space and let T be a mapping from (X,d) into $(CB(X),H)$ satisfies*

$$H(Tx,Ty) \leq \alpha(d(x,y))d(x,y)$$

for all $x,y \in X$, where α be a function from $[0,\infty)$ into $[0,1)$ such that $\limsup_{s\to t^+}\alpha(s) < 1$ for all $t \in [0,\infty)$. Then T has a fixed point.

Recently Suzuki [12] proved the Mizoguchi–Takahashi's fixed point theorem by an interesting and short proof.

On the other hand, the Banach contraction principle was generalized by Reich [8,9] as follows.

Theorem 3. *Let (X,d) be a complete metric space and let T be a mapping from (X,d) into $(CB(X),H)$ satisfies*

$$H(Tx,Ty) \leq \beta[D(x,Tx) + D(y,Ty)]$$

for all $x,y \in X$, where $\beta \in [0,\frac{1}{2})$. Then T has a fixed point.

In 1973, Hardy and Rogers [3] extended the Reich's theorem by the following way.

Theorem 4. *Let (X,d) be a complete metric space and let T be a mapping from X into X such that*

$$d(Tx,Ty) \leq \alpha d(x,y) + \beta[d(x,Tx) + d(y,Ty)] + \gamma[d(x,Ty) + d(y,Tx)]$$

for all $x,y \in X$, where $\alpha,\beta,\gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then T has a fixed point.

Recently, the authors of the present paper [2] extended Theorems 1 and 4, as follows.

Theorem 5. *Let (X,d) be a complete metric space and let T be a mapping from X into $CB(X)$ such that*

$$H(Tx,Ty) \leq \alpha d(x,y) + \beta[D(x,Tx) + D(y,Ty)] + \gamma[D(x,Ty) + D(y,Tx)]$$

for all $x,y \in X$, where $\alpha,\beta,\gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then T has a fixed point.

In this paper, we shall generalize the above results. More precisely, we prove the following theorem, which can be regarded as an extension of all Theorems 1, 2, 3, 4 and 5.

Theorem 6. *Let (X,d) be a complete metric space and let T be mapping from X into $CB(X)$ such that*

$$\begin{aligned} H(Tx,Ty) \leq{}& \alpha(d(x,y))d(x,y) + \beta(d(x,y))[D(x,Tx) + D(y,Ty)] \\ &+ \gamma(d(x,y))[D(x,Ty) + D(y,Tx)] \end{aligned} \tag{1.1}$$

for all $x, y \in X$, where α, β, γ are mappings from $[0,\infty)$ into $[0,1)$ such that $\alpha(t) + 2\beta(t) + 2\gamma(t) < 1$ and

$$\limsup_{s\to t^+} \frac{\alpha(s)+\beta(s)+\gamma(s)}{1-(\beta(s)+\gamma(s))} < 1$$

for all $t \in [0,\infty)$. Then T has a fixed point.

Moreover, we conclude the following results by using Theorem 6.

Corollary 1. *Let (X,d) be a complete metric space and let T be a mapping from (X,d) into $(CB(X),H)$ satisfies*

$$H(Tx,Ty) \le \beta(d(x,y))[D(x,Tx) + D(y,Ty)]$$

for all $x, y \in X$, where β be a function from $[0,\infty)$ into $[0,\frac{1}{2})$ and $\limsup_{s\to t^+} \beta(s) < \frac{1}{2}$ for all $t \in [0,\infty)$. Then T has a fixed point.

Corollary 2. *Let (X,d) be a complete metric space and let T be a mapping from (X,d) into $(CB(X),H)$ satisfies*

$$H(Tx,Ty) \le \alpha(d(x,y))d(x,y) + \beta(d(x,y))[D(x,Tx) + D(y,Ty)]$$

for all $x, y \in X$, where α, β are function from $[0,\infty)$ into $[0,1)$ such that $\alpha(t) + 2\beta(t) < 1$ and $\limsup_{s\to t^+} \left(\frac{\alpha(s)+\beta(s)}{1-\beta(s)}\right) < 1$ for all $t \in [0,\infty)$. Then T has a fixed point.

2. Proof of the main theorem

Proof. Define the function η from $[0,\infty)$ into $[0,1)$ by $\eta(t) = \frac{\alpha(t)+1-2\beta(t)-2\gamma(t)}{2}$ for $t \in [0,\infty)$. Then we have the following assertions:

1) $\alpha(t) < \eta(t)$ for all $t \in [0,\infty)$.

2) $\limsup_{s\to t^+} \frac{\eta(s)+\beta(s)+\gamma(s)}{1-(\beta(s)+\gamma(s))} < 1$ for all $t \in [0,\infty)$.

3) For $x, y \in X$ and $u \in Tx$, there exists $v \in Ty$ such that

$$\begin{aligned} d(v,u) \le{}& \eta(d(x,y))d(x,y) + \beta(d(x,y))[D(x,Tx) + D(y,Ty)] \\ &+ \gamma(d(x,y))[D(x,Ty) + D(y,Tx)]. \end{aligned}$$

Putting $u = y$ in (3), we obtain that:

4) For $x \in X$ and $y \in Tx$ there exists $v \in Ty$ such that

$$\begin{aligned} d(v,y) \le{}& \eta(d(x,y))d(x,y) + \beta(d(x,y))[D(x,Tx) + D(y,Ty)] \\ &+ \gamma(d(x,y))[D(x,Ty) + D(y,Tx)]. \end{aligned}$$

Hence, we can define a sequence $\{x_n\}$ such that $x_{n+1} \in Tx_n$, $x_{n+1} \neq x_n$ and

$$d(x_{n+2},x_{n+1}) \le \eta(d(x_{n+1},x_n))d(x_{n+1},x_n) + \beta(d(x_{n+1},x_n))[D(x_n,Tx_n)$$

$$+ D(x_{n+1}, Tx_{n+1})] + \gamma(d(x_{n+1}, x_n)[D(x_n, Tx_{n+1}) + D(x_{n+1}, Tx_n)]$$

for all $n \in \mathbb{N}$. It follows that

$$d(x_{n+2}, x_{n+1}) \leq \frac{\eta(d(x_{n+1}, x_n)) + \beta(d(x_{n+1}, x_n)) + \gamma(d(x_{n+1}, x_n))}{1 - (\beta(d(x_{n+1}, x_n)) + \gamma(d(x_{n+1}, x_n)))} d(x_{n+1}, x_n)$$

for all $n \in \mathbb{N}$. On the other hand, we have

$$\frac{\eta(t) + \beta(t) + \gamma(t)}{1 - (\beta(t) + \gamma(t))} < 1$$

for all $t \in [0, \infty)$, so $\{d(x_{n+1}, x_n)\}$ is a non-increasing sequence in $\mathbb{R}$. Hence, $\{d(x_{n+1}, x_n)\}$ converges to some nonnegative integer τ. By assumption,

$$\limsup_{s \to \tau^+} \frac{\eta(s) + \beta(s) + \gamma(s)}{1 - (\beta(s) + \gamma(s))} < 1,$$

so we have

$$\frac{\eta(\tau) + \beta(\tau) + \gamma(\tau)}{1 - (\beta(\tau) + \gamma(\tau))} < 1,$$

and then there exists $r \in [0, 1)$ and $\epsilon > 0$ such that

$$\frac{\eta(s) + \beta(s) + \gamma(s)}{1 - \beta(s) + \gamma(s)} < r$$

for all $s \in [\tau, \tau + \epsilon]$. We can take $\nu \in \mathbb{N}$ such that

$$\tau \leq d(x_{n+1}, x_n) \leq \tau + \epsilon$$

for all $n \in \mathbb{N}$ with $n \geq \nu$. It follows that

$$\begin{aligned} & d(x_{n+2}, x_{n+1}) \\ \leq & \frac{\eta(d(x_{n+1}, x_n)) + \beta(d(x_{n+1}, x_n)) + \gamma(d(x_{n+1}, x_n))}{1 - (\beta(d(x_{n+1}, x_n)) + \gamma(d(x_{n+1}, x_n)))} d(x_{n+1}, x_n) \\ & \leq r d(x_{n+1}, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$ with $n \geq \nu$. This implies that

$$\sum_{n=1}^{\infty} d(x_{n+2}, x_{n+1}) \leq \sum_{n=1}^{\nu} d(x_{n+1}, x_n) + \sum_{n=1}^{\infty} r^n d(x_{\nu+1}, x_\nu) < \infty.$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, $\{x_n\}$ converges to some point $x^* \in X$. Now, we have

$$\begin{aligned} D(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + D(x_{n+1}, Tx^*) \\ &\leq d(x^*, x_{n+1}) + H(Tx_n, Tx^*) \\ &\leq d(x^*, x_{n+1}) + \alpha(d(x_n, x^*)) d(x_n, x^*) \\ &\quad + \beta(d(x_n, x^*))[D(x_n, Tx_n) + D(x^*, Tx^*)] \end{aligned}$$

$$+\gamma(d(x_n,x^*))[D(x_n,Tx^*)+D(x^*,Tx_n)]$$

for all $n\in\mathbb{N}$. Therefore

$$\begin{aligned}D(x^*,Tx^*)\le{}& d(x^*,x_{n+1})+\alpha(d(x_n,x^*))d(x_n,x^*)\\&+\beta(d(x_n,x^*))[d(x_{n+1},x_n)+D(x^*,Tx^*)]\\&+\gamma(d(x_n,x^*))[D(x_n,Tx^*)+d(x_{n+1},x^*)]\end{aligned}$$

for all $n\in\mathbb{N}$. It follows that

$$\begin{aligned}&D(x^*,Tx^*)\\&\le\limsup_{n\to\infty}(\beta(d(x_n,x^*))+\gamma(d(x_n,x^*)))D(x^*,Tx^*)\\&\le\limsup_{n\to\infty}\left(\frac{\alpha(d(x_n,x^*))+\beta(d(x_n,x^*))+\gamma(d(x_n,x^*))}{1-(\beta(d(x_n,x^*))+\gamma(d(x_n,x^*)))}\right)D(x^*,Tx^*).\end{aligned}$$

On the other hand, we have

$$\limsup_{s\to0^+}\left(\frac{\alpha(s)+\beta(s)+\gamma(s)}{1-(\beta(s)+\gamma(s))}\right)<1$$

so $D(x^*,Tx^*)=0$. Since Tx^* is closed, $x^*\in Tx^*$. □

3. An Example

Recently, Suzuki, in [12], has given an example to prove that the Mizoguchi condition is not a contraction. Using the same idea, we give an example in a more general setting.

Example 1. Let l^∞ be the Banach space consisting of all bounded real sequence with supremum norm and let $\{e_n\}$ be the canonical basis of l^∞. Let $\{\tau_n\}$ be a strictly decreasing sequence of positive real numbers that convergent to τ. Put $x_n=\tau_n\,e_n$ and $X_n=\{x_n,x_{n+1},x_{n+2},\ldots\}$ for $n\in\mathbb{N}$ and $X=X_1$. It is easy to see that X is a bounded and closed subset of l^∞. Define a mapping T from X into $CB(X)$ by

$$Tx_n=\frac{1}{2}\,X_{n+1}=\{\frac{1}{2}x_{n+1},\frac{1}{2}x_{n+2},\ldots\}$$

for $n\in\mathbb{N}$ and functions α,β,γ from $[0,\infty)$ into $[0,1)$ by

$$\alpha(t)=\beta(t)=\gamma(t)=\begin{cases}\frac{\tau_{n+1}}{6\tau_n}, & t=\tau_n \text{ for some } n\in\mathbb{N},\\[2ex] 0, & \text{otherwise.}\end{cases}$$

Then the following conditions hold:

(*a*) T satisfies (1.1) for all $x,y\in X$.

(*b*) $\alpha(t)+2\beta(t)+2\gamma(t)<1$ for all $t\in[0,\infty)$.

(*c*) $\limsup_{s\to t^+}\frac{\alpha(s)+\beta(s)+\gamma(s)}{1-\beta(s)-\gamma(s)}<1$ for all $t\in[0,\infty)$.

In fact, one can show that:

(i)

$$H(Tx_m, Tx_n) = \frac{1}{2}\tau_{n+1},$$

for all $m > n$.

(ii)

$$D(x_n, Tx_n) = D(x_n, \frac{1}{2}X_{n+1}) = \tau_n,$$

$$D(x_n, Tx_m) = D(x_n, \frac{1}{2}X_{m+1}) = \tau_n,$$

for all $m, n \in \mathbb{N}$ with $m > n$.

(iii)

$$\limsup_{t\to s^+} \frac{\alpha(t)+\beta(t)+\gamma(t)}{1-\beta(t)-\gamma(t)} = 0 \ < 1,$$

for all $s \in [0,\infty)$ with $s \neq \tau$, also,

$$\limsup_{t\to \tau^+} \frac{\alpha(t)+\beta(t)+\gamma(t)}{1-\beta(t)-\gamma(t)} = \limsup_{n\to\infty} \frac{\frac{1}{2}\ \frac{\tau_{n+1}}{\tau_n}}{1-\frac{1}{3}\ \frac{\tau_{n+1}}{\tau_n}} = \frac{3}{4} < 1.$$

Hence, we have (c). Now, let $m, n \in \mathbb{N}$ with $m > n$. Then we have

$$\begin{aligned}
H(Tx_m, Tx_n) = \frac{1}{2}\ \tau_{n+1} &\leq \frac{1}{2}\ \tau_{n+1} + \frac{\tau_{n+1}}{6\tau_n}\ \tau_m \\
&= \frac{1}{2}\ \frac{\tau_{n+1}}{\tau_n}\ \tau_n + \frac{\tau_{n+1}}{6\tau_n}\ \tau_m \\
&= (\alpha(\tau_n)+\beta(\tau_n)+\gamma(\tau_n)\ )\ \tau_n + \beta(\tau_n)\ \tau_m \\
&= \alpha(\tau_n)\ \tau_n + \beta(\tau_n)\ (\tau_n+\tau_m) + \gamma(\tau_n)\ \tau_n \\
&\leq \alpha(d(x_m,x_n))\ d(x_m,x_n) \\
&+ \beta(d(x_m,x_n))(D(x_n,Tx_n) + D(x_m,Tx_m)) \\
&+ \gamma(d(x_m,x_n))(D(x_n,Tx_m) + D(x_m,Tx_n)).
\end{aligned}$$

This means that (a) holds.

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BERNSTEIN-DOETSCH TYPE RESULTS FOR (k,h)-CONVEX FUNCTIONS

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Abstract. In this paper we define the so-called (k,h)-convex function which is a natural generalization of the usual convexity, the s-convexity in the first and second sense, the h-convexity, the Godunova-Levin functions and the P-functions.

Some regularity and Bernstein-Doetsch type results are investigated for (k,h)-convex functions.

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1. INTRODUCTION

In the paper of B. Micherda and T. Rajba [22] were introduced the notions of k-convex set and the (k,h)-convex functions in the following way:

Let $k:(0,1)\to\mathbb{R}$ be a given function. Then a subset D of a real linear space X will be called *k-convex* if $k(t)x+k(1-t)y\in D$ for all $x,y\in D$ and $t\in(0,1)$.

Let $k,h:(0,1)\to\mathbb{R}$ be two given functions and suppose that $D\subset X$ is a k-convex set. Then a function $f:D\to\mathbb{R}$ is *(k,h)-convex* if, for all $x,y\in D$ and $t\in(0,1)$,

$$f(k(t)x+k(1-t)y)\le h(t)f(x)+h(1-t)f(y).$$

They prove some properties of these sets and functions and gave some example too. Now we introduce notion of *(k,h)-convexity with respect to T*.

Let X be a real or complex topological vector space, T be a nonempty set, such that the following property fulfill:

$$t\in T \text{ if and only if } 1-t\in T.$$

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For example $T=\{1/2\}$, $T=\{t_0,1-t_0\}$, or T is an interval (or an arbitrary subset of $\mathbb{R}$) which symmetric with respect to $1/2$. Furthermore, T may be a subset of $\mathbb{C}$, but it can be a subset of $\mathbb{R}^n$ or $\mathbb{C}^n$ too.

Furthermore, let $D\subset X$ be a nonempty open, k-convex set (that is, $k(t)x+k(1-t)y\in D$ whenever $x;y\in D$ and $t\in T$), and $k,h:T\to\mathbb{R}$ be given functions. We say the function $f:D\to\mathbb{R}$ is (k,h)-convex function, if

$$f(k(t)x+k(1-t)y)\le h(t)f(x)+h(1-t)f(y), \tag{1.1}$$

for all $x;y\in D$ and $t\in T$. To avoid the trivialities and the unimportant cases, we suppose that there exists an element $t_0\in T$ such that

$$k(t_0)k(1-t_0)h(t_0)h(1-t_0)\neq 0.$$

In the present paper we investigate these (k,h)-convex functions. This notion of convexity is a natural generalization of the usual convexity, the s-convexity in first and second sense, the h-convexity, the Godunova-Levin functions and the P-functions.

In the special cases when $T=\{1/2\}$, $T=\{t_0,1-t_0\}$ or $T=\mathbb{Q}\cap[0,1]$, the corresponding convex functions are said to be *Jensen-(k,h)-convex, t_0-(k,h)-convex* and *rationally-(k,h)-convex*.

The convex functions f

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y).$$

are obviously (k,h)-convex functions with $k(t)=t,h(t)=t$.

Let $h:[0,1]\to\mathbb{R}$ be a given function. In the case, when $k(t)=t$ we get the so called h-convex functions, which was introduced by Varošanec [33] and was generalized by Házy [12]. We say that $f:D\to\mathbb{R}$ is an h-convex function if, for all $x,y\in D$ and $t\in[0,1]$, we have

$$f(tx+(1-t)y)\le h(t)f(x)+h(1-t)f(y).$$

The Godunova-Levin functions were investigated in [9]. We say that $f:I\to\mathbb{R}$ (where I is a real interval) is a Godunova-Levin function, if f is nonnegative and for all $x,y\in I$ and $t\in(0,1)$ we have

$$f\left(tx+(1-t)y\right)\le\frac{f(x)}{t}+\frac{f(y)}{1-t}.$$

Some properties of this type of functions are given in [23] by Dragomir, Pečarič and Persson, [7] by Mitrinovič and Pečarič, [24] by Mitrinovič, Pečarič and Fink. Among others, it is proved that nonnegative monotone and nonnegative convex functions belong to this class of functions. The Godunova-Levin functions are (k,h)-convex functions, with $k(t)=t,h(t)=1/t$.

The concept of s-convexity in the first sense was introduced in [26] by Orlicz. A real valued function $f:D\to\mathbb{R}$ is called *Orlicz s-convex* or *s-convex in the first*

sense, if

$$f\left(t^s x+(1-t)^s y\right) \leq t f(x)+(1-t) f(y)$$

for every $x, y \in D$, $t \in]0,1]$, where $s \in [1,\infty[$ is a fixed number. The Orlicz s-convex functions are (k,h)-convex functions, with $k(t)=t^s, h(t)=t$.

The concept of s-convexity in the second sense was introduced in [2] by Breckner. A real valued function $f : D \to \mathbb{R}$ is called *Breckner s-convex* or *s-convex in the second sense*, if

$$f(tx+(1-t)y) \leq t^s f(x)+(1-t)^s f(y)$$

for every $x, y \in D$ and $t \in [0,1]$, where $s \in]0,1]$ is a fixed number. The Breckner s-convex functions are (k,h)-convex functions, with $k(t)=t, h(t)=t^s$.

In [2, 3] (by Breckner) and [4] (by Breckner and Orban) Berstein-Doetsch type results were proved on rationally s-convex functions, moreover, for the s-Hölder property of s-convex functions. In [31] Pycia gave a new proof of the latter statement, when f is defined on a nonempty, convex subset of a finite dimensional vector space. In the paper [15] the authors Hudzik and Maligranda collected some properties of s-convex functions defined on the nonnegative half line. In the paper [5] (by Burai, Házy and Juhász) there are some Berstein-Doetsch type results on the so-called (H,s)-convex functions.

The P-functions were investigated in [7] by Dragomir, Pečarič and Persson. A real valued function $f : D \to \mathbb{R}$ (where D is a convex, open, nonempty subset of a real (complex) linear space X) is called P-function, if for every $x, y \in D$ and $t \in [0,1]$ we have

$$f(tx+(1-t)y) \leq f(x)+f(y).$$

Some results about the P-functions can be found in [30] by Pearce and Rubinov, [32] by Tseng, Yang and Dragomir. The P-functions are (k,h)-convex functions, with $k(t)=t, h(t)=1$.

In [1] Bernstein and Doetsch proved that if a function $f : D \to \mathbb{R}$ (where D is a convex, open, nonempty subset of a real (complex) linear space X) is locally bounded from above at a point of D, then the Jensen-convexity of the function yields its local boundedness and continuity as well, which implies the convexity of the function f (see Kuczma [16] for further references). This result has been generalized by several authors. The first such type results are due to Nikodem and Ng [25] for the approximately Jensen-convex functions (the so-called ε-Jensen-convexity), which was extended by Páles ([28, 29]) to approximately t-convex functions. Further generalizations can be found in papers of Házy [10, 11], Házy and Páles [13, 14], Makó and Páles [17–20], Maksa and Páles [21] and M. E. Özdemir, A. Ekinci and A. O. Akdemir [27]. In the paper [8] by Gilányi, Nikodem and Páles there are some Bernstein-Doetsch type results for quasiconvex functions.

2. Main results

In the sequel let $(X, \|\cdot\|)$ is a real (complex) normed space. We recall that a function $f : D \to \mathbb{R}$ (where $D \subset X$) is referred to as locally bounded from above on D if, for each point $p \in D$, there exist $\varrho > 0$ and a neighborhood $U(p,\varrho) := \{x \in X : \|x-p\| < \varrho\}$ such that f is bounded from above on $U(p,\varrho)$. We assume that $k, h : T \to [0,\infty[$.

Proposition 1. *Let $t_0 \in T$ be fixed such that $k(t_0)+k(1-t_0) = 1$ and $f : D \to \mathbb{R}$ be a t_0-(k,h)-convex function. Then we have*

(i) if $h(t_0)+h(1-t_0) > 1$ then f is nonnegative.
(ii) if $h(t_0)+h(1-t_0) < 1$ then f is nonpositive.

Proof. Let x be an arbitrary element of D. Using (k,h)-convexity of f

$$\begin{aligned} f(x) &= f(k(t_0)x + k(1-t_0)x) \\ &\leq h(t_0)f(x) + h(1-t_0)f(x) = (h(t_0)+h(1-t_0))f(x), \end{aligned}$$

which implies

$$0 \leq (h(t_0)+h(1-t_0)-1)f(x).$$

If $h(t_0)+h(1-t_0)-1 > 0$, then we have $f(x) \geq 0$ and if $h(t_0)+h(1-t_0)-1 < 0$, then we have $f(x) \leq 0$. □

We remark, that if $h(t_0)+h(1-t_0) = 1$, then, similarly the usual convexity and Jensen-convexity, there is no such type result. An easy consequence of the previous proposition is the following:

Corollary 1. *Let $h, k : T \to \mathbb{R}$ be given functions such that $k(t)+k(1-t) = 1$ for all $t \in T$, and let $f : D \to \mathbb{R}$ be an T-(k,h)-convex function. Then*

(i) *if $h(t)+h(1-t) \geq 1$ for all $t \in T$ and there exists $t_1 \in T$ such that $h(t_1)+h(1-t_1) > 1$, then f is nonnegative.*
(ii) *if $h(t)+h(1-t) \leq 1$ for all $t \in T$ and there exists $t_1 \in T$ such that $h(t_1)+h(1-t_1) < 1$, then f is nonpositive.*
(iii) *if there exists $t_1, t_2 \in T$ such that $h(t_1)+h(1-t_1) > 1$ and $h(t_2)+h(1-t_2) < 1$ then f is constant 0.*

The proof of this result is similar to the proof of Proposition 1.

Some immediate consequences of this proposition are expressed in the following two corollaries. The second one shows that every Breckner s-convex function is nonnegative.

Corollary 2. *Let $f : D \to \mathbb{R}$ be a (t_0,h)-convex function. Then*

(i) if $h(t_0)+h(1-t_0) > 1$, then f is nonnegative.
(ii) if $h(t_0)+h(1-t_0) < 1$, then f is nonpositive.

Corollary 3. *Every Breckner (t_0,s)-convex function is nonnegative.*

In our next results, we deal with the boundedness property of (k,h)-convex functions. We assume that $h(t)$ and $h(1-t)$ are not zero simultaneously. Because, if $h(t)=h(1-t)=0$, then the (h,k)-convexity not implies the boundedness of f. In this case we get $f(x)\leq 0$ from the inequality (1.1), that is, we get only the upper boundedness of f. The proof is analogous to those in [12].

Theorem 1. *Let $k,h:T\to\mathbb{R}$ be given nonnegative functions and let $t_0\in T$ be fixed such that $k(t_0)k(1-t_0)\neq 0$ and $k(t_0)+k(1-t_0)=1$. We assume that $h(t_0)$ and $h(1-t_0)$ are not zero simultaneously. Furthermore let $D\subset X$ be an open, nonempty, k-convex set, and let $f:D\to\mathbb{R}$ be a t_0-(k,h)-convex function. Then*

(i) *if $h(t_0)+h(1-t_0)<1$ or*
(ii) *if $h(t_0)+h(1-t_0)\geq 1$ and f is locally bounded from above at a point $p\in D$*

then f is locally bounded at every point of D.

Proof. Since $k(t_0)k(1-t_0)\neq 0$ therefore we get $k(t_0),k(1-t_0)>0$. Because of the property $h(t_0)$ and $h(1-t_0)$ are not zero simultaneously, therefore, without loss generality, we may assume that $h(t_0)>0$. In the case (i) the local upper boundedness yields from Proposition 1. So we prove that f is locally bounded from above on D in the case (ii).

First we prove that f is locally bounded from above on D. Define the sequence of sets D_n by

$$D_0:=\{p\},\qquad D_{n+1}:=k(t_0)D_n+k(1-t_0)D.$$

Using induction on n, we prove that f is locally upper bounded at each point of D_n. By assumption, f is locally bounded from above at $p\in D_0$. Assume that f is locally upper bounded at each point of D_n. For $x\in D_{n+1}$, there exist $x_0\in D_n$ and $y_0\in D$ such that $x=k(t_0)x_0+k(1-t_0)y_0$. By the inductive hypothesis, there exist $r>0$ and a constant $M_0\geq 0$ with $f(x')\leq M_0$ such that $\|x_0-x'\|<r$. Then, by the t_0-(k,h)-convexity of f, for $x'\in U_0:=U(x_0,r)$ we have

$$\begin{aligned} f(k(t_0)x'+k(1-t_0)y_0) &\leq h(t_0)f(x')+h(1-t_0)f(y_0)\\ &\leq h(t_0)M_0+h(1-t_0)f(y_0)=:M. \end{aligned}$$

Therefore, for

$$\begin{aligned} y\in U &:=k(t_0)U_0+k(1-t_0)y_0=U(k(t_0)x_0+k(1-t_0)y_0,k(t_0)r)\\ &=U(x,k(t_0)r), \end{aligned}$$

we get that $f(y)\leq M$. Thus f is locally bounded from above on D_{n+1}. On the other hand, we can show that

$$D=\bigcup_{n=1}^{\infty}D_n.$$

Since the inclusion $\bigcup_{n=1}^{\infty} D_n \subseteq D$ is obvious we just check the reverse inclusion. In fact, from the definition of D_n, it follows by induction that $D_n = (k(t_0))^n p + (1-(k(t_0))^n)D$. For fixed $x \in D$, define the sequence x_n by

$$x_n := \frac{x-(k(t_0))^n p}{1-(k(t_0))^n}.$$

Then $x_n \to x$ if $n \to \infty$. As D is open, $x_n \in D$ for some n. Therefore,

$$x = (k(t_0))^n p + (1-(k(t_0))^n)x_n \in (k(t_0))^n p + (1-(k(t_0))^n)D = D_n,$$

which the converse inclusion. Thus f is locally bounded from above on D.

Now, we prove that f is locally bounded from below. Let $q \in D$ be arbitrary. Since f is locally bounded from above at the point q, there exist $\varrho > 0$ and $M > 0$ such that

$$\sup_{x \in U(q,\varrho)} f(x) \leq M.$$

Let $x \in U(q, k(1-t_0)\varrho)$ and $y := \dfrac{q-k(t_0)x}{k(1-t_0)}$. Then y is in $U(q,\varrho)$. By t_0-(k,h)-convexity,

$$f(q) \leq h(t_0) f(x) + h(1-t_0) f(y),$$

which implies

$$f(x) \geq \frac{f(q)-h(1-t_0)f(y)}{h(t_0)} \geq \frac{f(q)-h(1-t_0)M}{h(t_0)} =: M'.$$

Therefore f is locally bounded from below at any point of D. □

The consequence of this theorem is that the local upper boundedness at a point of every convex (or Jensen-convex, or Breckner s-convex) function implies its locally boundedness.

Corollary 4. *Let $f : D \to \mathbb{R}$ be a Jensen-convex or t_0-convex function. If f is locally bounded from above at a point of D, then f is locally bounded at every point of D.*

Corollary 5. *Let $f : D \to \mathbb{R}$ be a Breckner (t_0, s)-convex function. If f is locally bounded from above at a point of D, then f is locally bounded at every point of D.*

Corollary 6. *Let $f : D \to \mathbb{R}$ be a (t_0, h)-convex function such that $h(t_0)$ and $h(1-t_0)$ are not zero simultaneously. If f is locally bounded from above at a point of D, then f is locally bounded at every point of D.*

It is a well-known fact that if a Jensen-convex function f is locally bounded above at a point of its domain (see [1, 16]), then it is continuous on the domain.

The nonnegative monotone functions - which are not necessarily continuous - belong to the class of the Godunova-Levin functions (see [7, 23, 24]), which is a special

class of (h,k)-convex functions. In [5] we gave an example of function which is Jensen-s-convex, bounded and nowhere continuous.

These examples show that the t_0-(k,h)-convexity of a function and the locally upper boundedness at a point in its domain does not imply the continuity of a t_0-(k,h) convex functions.

The next theorem provides a sufficient condition for the local boundedness to imply the continuity.

Theorem 2. *Let $T=\{t_n\}_{n\in\mathbb{N}}$ such that the sequence $\{t_n\}_{n\in\mathbb{N}}\subset[0,1]$ tends to 0, as $n\to\infty$. Let $k,h:T\to[0,\infty[$ be given nonnegative continuous functions such that $h(t_n)$ and $h(1-t_n)$ are not simultaneously zero, $k(t_n)+k(1-t_n)=1$ for every $n\in\mathbb{N}$ and the limit conditions*

$$\lim_{t\to 0}h(t)=0 \qquad \text{and} \qquad \lim_{t\to 1}h(t)=1$$

are valid. Furthermore let $D\subset X$ be an open, nonempty, k-convex set. If $f:D\to\mathbb{R}$ is a T-(k,h)-convex function and locally bounded from above at a point of D, then f is continuous on D.

Proof. Since $h(t_n)$ and $h(1-t_n)$ are not zero simultaneously, therefore, without loss generality, we may assume that $h(1-t_n)>0$.

Since f is locally bounded from above at a point $x_0\in D$, there exists a neighborhood U at x_0 and a constant $K>0$ such that $f(x)\le K$ for every $x\in U$. Let ε be an arbitrary nonnegative constant. Then there exists $n_0\in\mathbb{N}$ such that if $n\ge n_0$, then

$$h(t_n)K+[h(1-t_n)-1]\,f(x_0)<\varepsilon,$$

whence

$$\frac{h(t_n)}{h(1-t_n)}K+\left[1-\frac{1}{h(1-t_n)}\right]f(x_0)<\varepsilon.$$

Let V be a neighborhood of 0 such that $x_0+V\subseteq U$, and let $U'=x_0+k(t_n)V$. We prove that

$$|f(x)-f(x_0)|<\varepsilon \qquad (x\in U').$$

For $x\in U'$ there exist $y,z\in x_0+V$ such that

$$x=k(t_n)y+k(1-t_n)x_0=k(t_n)y+(1-k(t_n))x_0,$$
$$x_0=k(t_n)z+k(1-t_n)x=k(t_n)z+(1-k(t_n))x.$$

Indeed,

$$y-x_0=\frac{1}{k(t_n)}(x-x_0)\in\frac{1}{k(t_n)}k(t_n)V=V,$$

and

$$z-x_0=\frac{1-k(t_n)}{k(t_n)}(x_0-x)\in\frac{1-k(t_n)}{k(t_n)}k(t_n)V=(1-k(t_n))V\subseteq V.$$

According to T-(k,h)-convexity of f,

$$f(x) \le h(t_n) f(y) + h(1-t_n) f(x_0) \le h(t_n) K + h(1-t_n) f(x_0),$$
$$f(x_0) \le h(t_n) f(z) + h(1-t_n) f(x) \le h(t_n) K + h(1-t_n) f(x).$$

We get

$$f(x) - f(x_0) \le h(t_n) K + [h(1-t_n) - 1] f(x_0) < \varepsilon \tag{2.1}$$

and

$$f(x) \ge \frac{f(x_0) - h(t_n) K}{h(1-t_n)},$$

which implies

$$f(x) - f(x_0) \ge \left[\frac{1}{1-h(t_n)} - 1 \right] f(x_0) - \frac{h(t_n)}{h(1-t_n)} K > -\varepsilon. \tag{2.2}$$

The inequalities (2.1) and (2.2) show that $|f(x) - f(x_0)| < \varepsilon$, that is f is continuous at x_0, which was to be proved. □

The previous limit conditions are not necessary, since in the case of Jensen-convex functions they do not fulfill. However, the result of Bernstein and Doetsch is valid for Jensen-convex functions.

3. Convexity property of rationally-(k,h)-convex functions

The following result offers a generalization of the theorem of Bernstein-Doetsch [1], Breckner [2], Burai-Házy-Juhász [5,6] and Házy [12] for rationally-(k,h)-convex functions.

Theorem 3. *Let k,h be given nonnegative, continuous functions satisfying the limit conditions*

$$\lim_{t\to 0} h(t) = 0 \qquad \text{and} \qquad \lim_{t\to 1} h(t) = 1.$$

and $k(t) + k(1-t) = 1$, and assume that $h(t)$ and $h(1-t)$ are not zero simultaneously for all $t \in \mathbb{Q} \cap [0,1]$. Furthermore let $D \subset X$ be an open, nonempty, k-convex set. If $f : D \to \mathbb{R}$ is rationally-(k,h)-convex and locally bounded from above at a point of D, then f is continuous and (k,h)-convex.

Proof. We prove that the function f is t_0-(k,h)-convex for all $t_0 \in [0,1]$. Let $t_0 \in [0,1]$ be arbitrary. Then there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \in \mathbb{Q} \cap [0,1]$ and $t_n \to t_0$ (when n tends to ∞). Applying rationally-(k,h)-convexity of f, we get

$$f\left(k(t_n)x + k(1-t_n)y\right) \le h(t_n) f(x) + h(1-t_n) f(y). \tag{3.1}$$

The local upper boundedness of f implies the continuity of f (according to Theorem 2). Therefore, taking the limit $n \to \infty$ in (3.1), we get

$$f\left(k(t_0)x + k(1-t_0)y\right) \le h(t_0) f(x) + h(1-t_0) f(y),$$

which proves the (k,h)-convexity of f. □

Corollary 7. *Let $D \subset X$ be a nonempty, convex, open set and let $h : [0,1] \to \mathbb{R}$ be a given nonnegative, continuous function satisfying the limit conditions*

$$\lim_{t\to 0} h(t) = 0 \qquad \text{and} \qquad \lim_{t\to 1} h(t) = 1.$$

and assume that $h(t_0)$ and $h(1-t_0)$ are not simultaneously zero for all $t_0 \in \mathbb{Q} \cap [0,1]$.

If $f : D \to \mathbb{R}$ is rationally-h-convex and f is locally bounded from above at a point D, then f is continuous on D and h-convex.

Corollary 8. *Let $D \subset X$ be a nonempty, convex, open set. If $f : D \to \mathbb{R}$ is rationally-Breckner s-convex and locally bounded from above at a point D, then f is continuous on D and Breckner s-convex.*

Theorem 4. *Let $T = [0,1]$, $k,h : T \to \mathbb{R}$ be given nonnegative functions such that k is continuous on T, and $h(t) + h(1-t) = 1$. Let $\gamma(t) = k(t) + k(1-t)$ and $f : \mathbb{R}_+ \to \mathbb{R}$ be a (k,h)-convex function. Then*

(i) f is nondecreasing if $\gamma(T) = [r,1]$ (where $r < 1$).
(ii) f is nonincreasing if $\gamma(T) = [1,r]$ (where $r > 1$).
(iii) f is constant if $\gamma(T) = [r_1,r_2]$ (where $r_1 < 1 < r_2$).

Proof. We have, for $x > 0$ and $t \in [0,1]$

$$f\left(k(t)x + k(1-t)x\right) \le h(t)f(x) + h(t)f(x) = f(x).$$

Since $\gamma(t) = k(t) + k(1-t)$ and k is continuous on $[0,1]$, therefore γ is continuous on $[0,1]$.

In the case (i) we get $\gamma(T) = [r,1]$, where $r < 1$. Let $u \in [r,1]$ be arbitrary. Then there exists a $t \in [0,1]$ such that $\gamma(t) = u$. This yields that

$$f(ux) \le f(x) \qquad (x \in \mathbb{R}_+, u \in [r,1]). \tag{3.2}$$

If now $u \in [r^2,1]$ then $u^{1/2} \in [r,1]$. Therefore, by the fact that (3.2) holds for all $x \in \mathbb{R}_+$, we get

$$f(ux) = f(u^{1/2}(u^{1/2}x)) \le f(u^{1/2}x) \le f(x)$$

for all $x \in \mathbb{R}_+$. By induction we then obtain that

$$f(ux) \le f(x) \qquad (x \in \mathbb{R}_+, u \in]0,1]). \tag{3.3}$$

Therefore, taking $0 < x < y$ and applying (3.3), we get

$$f(x) = f((x/y)y) \le f(y),$$

which means that f is nondecreasing on $\mathbb{R}_+$.

The proof of the cases (ii) and (iii) are similar. □

The above results do not hold, in general, in the case of convex functions, because a convex function $f : \mathbb{R}_+ \to \mathbb{R}$, need not be non-decreasing. But in the case of Orlicz s-convex function this is true.

Corollary 9. *Let $0 < s < 1$. Let $f : \mathbb{R}_+ \to \mathbb{R}$ an Orlicz s-convex function. Then f is nondecreasing.*

Remark 1. In the paper of Hudzik and Maligranda [15] was given an example showing that the Orlicz s-convex function is nondecreasing on $]0,\infty[$, but not necessarily on $[0,\infty[$. For the readers convenience we recall the example: let $a,b,c \in \mathbb{R}$ and let

$$f(x) = \begin{cases} a & \text{if } x = 0 \\ bx^s + c & \text{if } x \neq 0. \end{cases}$$

If $b > 0$ and $c < a$ then f is non-decreasing on $]0,\infty[$ but not on $[0,\infty[$.

4. Optimization

It is a very well known fact that every local minimizer of a convex function is a global one. The same is true for (k,h)-convex functions under some assumptions.

Theorem 5. *Let $D \subset X$ be a nonempty, open, k-convex set, where $k,h : [0,1] \to \mathbb{R}$ are given nonnegative, continuous functions satisfying the limit conditions*

$$\lim_{t\to 0+} k(t) = 0 \qquad \text{and} \qquad \lim_{t\to 1-} k(t) = 1.$$

and assume that $h(t) + h(1-t) = 1$ for all $t \in [0,1]$. Then every local minimizer of a (k,h)-convex function $f : D \to \mathbb{R}$ is a global one.
Furthermore if the function f is strictly (k,h)-convex (there is strict inequality in (1.1) whenever $x \neq y$), then there is at most one global minimum.

Proof. Let $x_0 \in D$ be a local minimizer of f. Then there exists a positive real number r, such that

$$f(x_0) \leq f(y), \qquad y \in U(x_0, r).$$

Assume that x_0 is not a global minimizer. Then there exists $z \in D$, such that $f(x_0) > f(z)$. Using this and the (k,h)-convexity of f, we have

$$\begin{aligned} f(k(t)z + k(1-t)x_0) &\leq h(t) f(x_0) + h(1-t) f(z) \\ &= f(x_0) + h(1-t)\,(f(z) - f(x_0)) < f(x_0). \end{aligned}$$

On the other hand, using the limit conditions, $k(t)z + k(1-t)x_0 \in U(x_0, r)$, if t is sufficiently small, which contradicts the fact that x_0 is a local minimizer.

If f is a strictly (k,h)-convex function, and $x \neq y$ are global minimizers, then

$$f(k(t)x + k(1-t)y) < h(t) f(x) + h(1-t) f(y) = f(x),$$

which is a contradiction. □

Corollary 10. *Every local minimizer of an Orlicz-convex function $f : D \to \mathbb{R}$ is a global one. If the function f is strictly Orlicz-convex, then there is at most one global minimum.*

Corollary 11. *Every local minimizer of a convex function $f : D \to \mathbb{R}$ is a global one. If the function f is strictly convex, then there is at most one global minimum.*

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GLOBAL HÖLDER ESTIMATES FOR HYPOELLIPTIC OPERATORS WITH DRIFT ON HOMOGENEOUS GROUPS

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Abstract. Let $X_0, X_1, \ldots, X_q$ be left invariant real vector fields on the homogeneous group G, satisfying Hörmander's condition on $\mathbb{R}^N$. Assume that $X_1, \ldots, X_q$ are homogeneous of degree one and X_0 is homogeneous of degree two. In this paper we consider the following hypoelliptic operator with drift

$$L = \sum_{i,j=1}^{q} a_{ij} X_i X_j + a_0 X_0,$$

where (a_{ij}) is a $q \times q$ positive constant matrix and $a_0 \neq 0$, and obtain Global Hölder estimates for L on G by establishing several estimates of singular integrals.

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1. INTRODUCTION

Let G be a homogeneous group and $X_0, X_1, ..., X_q$ be left invariant real vector fields on $\mathbb{R}^N (q < N)$. Assume that $X_1, ..., X_q$ are homogeneous of degree one and X_0 is homogeneous of degree two, satisfying Hörmander's condition

$$rank \mathcal{L}(X_0, X_1, ..., X_q)(x) = N, x \in \mathbb{R}^N,$$

where $\mathcal{L}(X_0, X_1, ..., X_q)$ denotes the Lie algebra generated by $X_0, X_1, ..., X_q$. In this paper we are interested in the following hypoelliptic operator with drift

$$L = \sum_{i,j=1}^{q} a_{ij} X_i X_j + a_0 X_0, \tag{1.1}$$

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where $a_0 \neq 0, (a_{ij})_{i,j=1}^{q}$ is a constant matrix satisfying

$$\mu^{-1}|\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij}\xi_i\xi_j \leq \mu|\xi|^2, \xi \in R^q, \tag{1.2}$$

for a constant $\mu > 0$.

Many authors paid attention to the hypoelliptic operator. The outstanding result in [8] points out that Hörmander's condition implies (actually, is equivalent to) the hypoellipticity of L in (1.1). The existence of fundamental solutions for homogeneous hypoelliptic operators on nilpotent Lie groups was investigated by Folland in [6]. Bramanti and Brandolini in [2] proved the uniqueness of homogeneous fundamental solutions for L. Let us note that L includes the classic Laplace operator and parabolic operator on Euclidean spaces. Another special case of L is

$$L_1 = \sum_{i,j=1}^{q} a_{ij}\partial^2_{x_i x_j} + \sum_{i,j=1}^{n} b_{ij}x_i\partial_{x_j} - \partial_t,$$

where $(x,t) \in \mathbb{R}^{n+1}$, $X_0 = \sum_{i,j=1}^{n} b_{ij}x_i\partial_{x_j} - \partial_t$ and $X_i = \partial_{x_i}, i = 1,2,\ldots,q$,$(a_{ij})_{i,j=1}^{q}$ is a positive matrix in R^q, $(b_{ij})$is a constant matrix with a suitable upper triangular structure. Note that L_1 belongs to a class of Kolmogorov-Fokker-Planck ultraparabolic operators. The operator L_1 appears in many research fields, for instance, in stochastic processes and kinetic models (see [3–5]), and in mathematical finance theory (see [1, 12]). After the previous study on L_1 in [9, 10], the authors of [7, 11, 13] established an invariant Harnack inequality for the non-negative solution of $L_1 u = 0$ by applying the mean value formula. With the theory of singular integral, Polidoro and Ragusa in [14] concluded some Morrey-type imbedding results and gave a local Hölder continuity of the solution.

The aim of the paper is to prove global Hölder estimates on the homogeneous group G for L by applying the properties of the fundamental solution for L and several estimates of singular integrals on the homogeneous space . The method here is inspired by that used in [14]. Our results reflect the relations between the Morrey norms of Lu and Hölder exponents for u and $X_i u$, $i = 1,2,\ldots,q$. In order to state our main results, we first introduce the definition of Morrey space.

Definition 1. For $p \in (1,\infty)$,$\lambda \in [0, Q)$, the Morrey space on homogeneous group G is defined by

$$L^{p,\lambda}(G) = \left\{g \in L^p_{loc}(G) : \|g\|_{L^{p,\lambda}(G)} < \infty\right\},$$

where

$$\|g\|_{L^{p,\lambda}(G)} = \left(\sup_{r>0, x\in G} \int_{B_r(x)} \frac{1}{r^\lambda}|g(y)|^p dy\right)^{1/p},$$

$B_r(x)$ and Q will be given in (2.1) and (2.2), respectively. Here $L^{p,0}(G)$=$L^p(G)$.

The main results of this paper are as follows. For the case $\lambda \neq 0$, we have

Theorem 1. (1) *If $1 < p < \frac{Q}{2}$, $Q - 2p < \lambda < Q - p$, then there exists a positive constant $c = c(p,\lambda)$ such that for every $u \in C_0^\infty(G)$ and any $x, z \in G$, $x \neq z$,*

$$\frac{|u(x)-u(z)|}{\left\|z^{-1}\circ x\right\|^{\theta}} \leq c\,\|Lu\|_{L^{p,\lambda}(G)}, \tag{1.3}$$

where $\theta = \frac{2p+\lambda-Q}{p}$;

(2) *If $1 < p < \frac{Q}{2}$, $Q - p < \lambda < Q$, then there exists a positive constant $c = c(p,\lambda)$ such that for every $u \in C_0^\infty(G)$ and any $x, z \in G$, $x \neq z$,*

$$\frac{|X_i u(x) - X_i u(z)|}{\left\|z^{-1}\circ x\right\|^{\theta}} \leq c\,\|Lu\|_{L^{p,\lambda}(G)}, \tag{1.4}$$

where $i = 1, \cdots, q$ and $\theta = \frac{p+\lambda-Q}{p}$.

For $\lambda = 0$, we have the following results, which restores the known result previously proved in [1].

Remark 1. (1) Assume $\frac{Q}{2} < p < Q$. Then there exists a positive constant $c = c(p)$ such that for every $u \in C_0^\infty(G)$ and any $x, z \in G$, $x \neq z$,

$$\frac{|u(x)-u(z)|}{\left\|z^{-1}\circ x\right\|^{\theta}} \leq c\,\|Lu\|_{L^{p}(G)}, \tag{1.5}$$

where $\theta = \frac{2p-Q}{p}$;

(2) Assume $p > Q$. Then there exists a positive constant $c = c(p)$ such that for every $u \in C_0^\infty(G)$ and any $x, z \in G$, $x \neq z$,

$$\frac{|X_i u(x) - X_i u(z)|}{\left\|z^{-1}\circ x\right\|^{\theta}} \leq c\,\|Lu\|_{L^{p}(G)}, \tag{1.6}$$

where $i = 1, \cdots, q$ and $\theta = \frac{p-Q}{p}$.

The plan of the paper is as follows: in Section 2 we introduce some knowledge of homogeneous group and related lemmas. Estimates of two integral operators are proved. Section 3 is devoted to the proof of the main result.

2. PRELIMINARY

Given a pair of mappings:

$$[(x,y) \mapsto x \circ y] : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}^N; [x \mapsto x^{-1}] : \mathbb{R}^N \mapsto \mathbb{R}^N,$$

which are smooth, it follows that $\mathbb{R}^N$ with these mappings forms a group, and the identity is the origin. If there exist $0 < \omega_1 \leq \omega_2 \leq \ldots \leq \omega_N$, such that the dilations

$$D(\lambda) : (x_1, \ldots, x_N) \mapsto (\lambda^{\omega_1} x_1, \ldots, \lambda^{\omega_N} x_N), \lambda > 0,$$

are group automorphisms, then the space $\mathbb{R}^N$ with this structure is called a homogeneous group and denoted by G.

Definition 2. We define a homogeneous norm $\|\cdot\|$ in G by the following way: if for any $x \in G, x \neq 0$, it holds

$$\|x\| = \rho \quad \Leftrightarrow \quad |D(1/\rho)x| = 1,$$

where $|\cdot|$ denotes the Euclidean norm ; also, let $\|0\| = 0$.

It is not difficult to derive that the homogeneous norm satisfies

(1) $\|D(\lambda)x\| = \lambda \|x\|$ for every $x \in G, \lambda > 0$;
(2) there exists $c(G) \geq 1$, such that for every $x, y \in G$,

$$\left\|x^{-1}\right\| \leq c \|x\| and \|x \circ y\| \leq c(\|x\| + \|y\|).$$

In view of the above properties, it is natural to define the quasidistance d :

$$d(x, y) = \left\|y^{-1} \circ x\right\|.$$

The ball with respect to d is denoted by

$$B(x, r) \equiv B_r(x) = \{y \in G : d(x, y) < r\}. \tag{2.1}$$

Note $B(0, r) = D(r)B(0, 1)$, therefore

$$|B(x, r)| = r^Q |B(0, 1)|, x \in G, r > 0,$$

where

$$Q = \omega_1 + \ldots + \omega_N. \tag{2.2}$$

We will call that Q is the homogeneous dimension of G. In general, $Q \geq 3$.

Definition 3. A differential operators Y on G is said homogeneous of degree $\beta (\beta > 0)$, if for every test function φ,

$$Y(\varphi(D(\lambda)x)) = \lambda^{\beta} (Y\varphi)(D(\lambda)x), \lambda > 0, x \in G;$$

A function f is called homogeneous of degree α, if

$$f((D(\lambda)x)) = \lambda^{\alpha} f(x), \lambda > 0, x \in G.$$

Remark 2. Clearly, if Y is a differential operators of homogeneous of degree β and f is a function of homogeneous of degree α, then Yf is homogeneous of degree $\alpha - \beta$.

Lemma 1. *([2]) The operator L possesses a unique fundamental solution $\Gamma(\cdot)$, such that for every test function $u \in C_0^{\infty}(G)$ and every $x \in G$, it holds*

(1) $\Gamma(\cdot) \in C^{\infty}(G \backslash \{0\})$;

(2) $\Gamma(\cdot)$ *is homogeneous of degree* $2-Q$;
(3) $u(x)=(Lu*\Gamma)(x)=\int_{\mathbb{R}^N}\Gamma(y^{-1}\circ x)Lu(y)dy$;
(4) $X_iu(x)=\int_{R^N}X_i\Gamma(y^{-1}\circ x)Lu(y)dy$.

Remark 3. If we set $\Gamma_i=X_i\Gamma, i=1,\cdots,q$, then it is obvious from Remark 2 that $\Gamma_i(\cdot)$ is homogeneous of degree $1-Q$.

Proposition 1. *([2]) Let* $f\in C^1(R^N\backslash 0)$ *is a homogeneous function of degree* $\lambda<1$. *Then there exist two constants* $c=c(G,f)>0$ *and* $M=M(G)>1$, *such that for any* x,y *satisfying* $\|x\|\geq M\|y\|$,

$$|f(x\circ y)-f(x)|+|f(y\circ x)-f(x)|\leq c\|y\|\|x\|^{\lambda-1},$$

where $c=c(G,f)\sup_{z\in\Sigma_N}|\nabla f(z)|$, Σ_N *is the unit sphere of* $\mathbb{R}^N$.

From Proposition 1, it follows

Lemma 2. *If* $K\in C^1(G\backslash\{0\})$ *is a homogeneous function of degree* $\alpha<1$ *with respect to the group* $(D(\lambda))_{\lambda>0}$, *then there exist two constants* $c>0$ *and* $M>1$, *such that if* $\|x\|\geq M\left\|x^{-1}\circ z\right\|$, *then*

$$|K(z)-K(x)|\leq\frac{c\left\|x^{-1}\circ z\right\|}{\|x\|^{1-\alpha}}.$$

By Lemma 1 and Lemma 2, we have immediately

Lemma 3. *For every* $x,y,z\in G$, *it holds*

(1) *there exists a constant* $c>0$, *such that*

$$\Gamma(y^{-1}\circ x)\leq\frac{c}{\left\|y^{-1}\circ x\right\|^{Q-2}};$$

$$\Gamma_i(y^{-1}\circ x)\leq\frac{c}{\left\|y^{-1}\circ x\right\|^{Q-1}}.$$

(2) *there exist two constants* $c>0$ *and* $M>1$, *such that if* $\left\|y^{-1}\circ x\right\|\geq M\left\|x^{-1}\circ z\right\|$, *then*

$$\left|\Gamma(y^{-1}\circ x)-\Gamma(y^{-1}\circ z)\right|\leq\frac{c\left\|x^{-1}\circ z\right\|}{\left\|y^{-1}\circ x\right\|^{Q-1}};$$

$$\left|\Gamma_i(y^{-1}\circ x)-\Gamma_i(y^{-1}\circ z)\right|\leq\frac{c\left\|x^{-1}\circ z\right\|}{\left\|y^{-1}\circ x\right\|^{Q}}.$$

Now let us introduce two integral operators. For $p\in(1,\infty)$ and $\lambda\in[0,Q)$, fixed $z\in G$ and $\sigma>0$, we define for every $g\in L^{p,\lambda}(G)$ that

$$T_\alpha g(x)=\int_{\|y^{-1}\circ x\|\geq\sigma\|z^{-1}\circ x\|}\frac{g(y)}{\left\|y^{-1}\circ x\right\|^{Q-\alpha}}dy,\alpha\in[0,Q);$$

$$T^{\beta}g(x)=\int_{\|y^{-1}\circ x\|<\sigma\|z^{-1}\circ x\|}\frac{g(y)}{\|y^{-1}\circ x\|^{Q-\beta}}dy,\beta\in(0,Q).$$

Lemma 4. *If $\lambda+p\alpha<Q$, then there exists $c=c(p,\lambda,\alpha,\sigma)>0$, such that*

$$|T_{\alpha}g(x)|\leq c\,\|g\|_{L^{p,\lambda}(G}\|z^{-1}\circ x\|^{\frac{p\alpha+\lambda-Q}{p}};\tag{2.3}$$

if $\lambda+p\beta>Q$, then there exists $c=c(p,\lambda,\beta,\sigma)>0$, such that

$$\left|T^{\beta}g(x)\right|\leq c\,\|g\|_{L^{p,\lambda}(G)}\|z^{-1}\circ x\|^{\frac{p\beta+\lambda-Q}{p}}.\tag{2.4}$$

Proof. We follow the idea of Polidoro and Ragusa in [14]. If $\lambda+p\alpha<Q$, then it obtains by decomposing the domain of integration and applying the Hölder inequality that

$$\begin{aligned}
|T_{\alpha}g(x)|&\leq\sum_{k=1}^{\infty}\int_{2^{k-1}\sigma\|z^{-1}\circ x\|\leq\|y^{-1}\circ x\|<2^{k}\sigma\|z^{-1}\circ x\|}\frac{g(y)}{\|y^{-1}\circ x\|^{Q-\alpha}}dy\\
&\leq\sum_{k=1}^{\infty}\left(\frac{1}{2^{k-1}\sigma\|z^{-1}\circ x\|}\right)^{Q-\alpha}\int_{B_{2^{k}\sigma\|z^{-1}\circ x\|}(x)}|g(y)|dy\\
&\leq\sum_{k=1}^{\infty}\left(\frac{1}{2^{k-1}\sigma\|z^{-1}\circ x\|}\right)^{Q-\alpha}\left(\int_{B_{2^{k}\sigma\|z^{-1}\circ x\|}(x)}|g(y)|^{p}dy\right)^{\frac{1}{p}}\\
&\quad\left|B_{2^{k}\sigma\|z^{-1}\circ x\|}(x)\right|^{\frac{p-1}{p}}\\
&\leq c\sum_{k=1}^{\infty}\left(\frac{1}{2^{k-1}\sigma\|z^{-1}\circ x\|}\right)^{Q-\alpha}\left(2^{k}\sigma\|z^{-1}\circ x\|\right)^{\frac{\lambda}{p}}\|g\|_{L^{p,\lambda}(G)}\\
&\quad\left(2^{k}\sigma\|z^{-1}\circ x\|\right)^{\frac{(p-1)Q}{p}}\\
&\leq c\,\|g\|_{L^{p,\lambda}(G)}\|z^{-1}\circ x\|^{\frac{p\alpha+\lambda-Q}{p}}\sum_{k=1}^{\infty}\left(2^{\frac{p\alpha+\lambda-Q}{p}}\right)^{k}.
\end{aligned}$$

So (2.3) is proved, since the above series is convergent.

Similarly, if $\lambda+p\beta>Q$, then

$$\begin{aligned}
\left|T^{\beta}g(x)\right|&\leq\sum_{k=1}^{\infty}\int_{2^{-k}\sigma\|z^{-1}\circ x\|\leq\|y^{-1}\circ x\|<2^{1-k}\sigma\|z^{-1}\circ x\|}\frac{g(y)}{\|y^{-1}\circ x\|^{Q-\beta}}dy\\
&\leq\sum_{k=1}^{\infty}\left(\frac{1}{2^{-k}\sigma\|z^{-1}\circ x\|}\right)^{Q-\beta}\int_{B_{2^{1-k}\sigma\|z^{-1}\circ x\|}(x)}|g(y)|dy
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{\infty}\left(\frac{1}{2^{-k}\sigma\left\|z^{-1}\circ x\right\|}\right)^{Q-\beta}\left(\int_{B_{2^{1-k}\sigma\left\|z^{-1}\circ x\right\|}(x)}|g(y)|^{p}dy\right)^{\frac{1}{p}}\\
&\quad\left|B_{2^{1-k}\sigma\left\|z^{-1}\circ x\right\|}(x)\right|^{\frac{p-1}{p}}\\
&\leq c\sum_{k=1}^{\infty}\left(\frac{1}{2^{-k}\sigma\left\|z^{-1}\circ x\right\|}\right)^{Q-\beta}\left(2^{1-k}\sigma\left\|z^{-1}\circ x\right\|\right)^{\frac{\lambda}{p}}\|g\|_{L^{p,\lambda}(G)}\\
&\quad\left(2^{1-k}\sigma\left\|z^{-1}\circ x\right\|\right)^{\frac{(p-1)Q}{p}}\\
&\leq c\,\|g\|_{L^{p,\lambda}(G)}\left\|z^{-1}\circ x\right\|^{\frac{p\beta+\lambda-Q}{p}}\sum_{k=1}^{\infty}\left(2^{\frac{Q-p\beta-\lambda}{p}}\right)^{k}.
\end{aligned}
$$

This proves (2.4). □

Remark 4. In particular, when $\lambda=0$, we see that if $p\alpha<Q$, then there exists a constant $c=c(p,\alpha,\sigma)>0$, such that

$$
|T_{\alpha}g(x)|\leq c\,\|g\|_{L^{p}(G)}\left\|z^{-1}\circ x\right\|^{\frac{p\alpha-Q}{p}};\tag{2.5}
$$

if $p\beta>Q$, then there exists a constant $c=c(p,\beta,\sigma)>0$, such that

$$
\left|T^{\beta}g(x)\right|\leq c\,\|g\|_{L^{p}(G)}\left\|z^{-1}\circ x\right\|^{\frac{p\beta-Q}{p}}.\tag{2.6}
$$

3. Proofs of the main results

Proof of Theorem 1. (1) With the help of (3) in Lemma 1 and Lemma 3, we know that there exist constants $c>0$ and $M>1$ such that

$$
\begin{aligned}
|u(x)-u(z)|&=\left|\int_{\mathbb{R}^{N}}\Gamma(y^{-1}\circ x)-\Gamma(y^{-1}\circ z)Lu(y)dy\right|\\
&\leq\int_{\mathbb{R}^{N}}\left|\Gamma(y^{-1}\circ x)-\Gamma(y^{-1}\circ z)\right||Lu(y)|dy\\
&\leq\int_{\|y^{-1}\circ x\|\geq M\|x^{-1}\circ z\|}\left|\Gamma(y^{-1}\circ x)-\Gamma(y^{-1}\circ z)\right||Lu(y)|dy\\
&\quad+\int_{\|y^{-1}\circ x\|<M\|x^{-1}\circ z\|}\left|\Gamma(y^{-1}\circ x)-\Gamma(y^{-1}\circ z)\right||Lu(y)|dy\\
&\leq\int_{\|y^{-1}\circ x\|\geq M\|x^{-1}\circ z\|}\left|\Gamma(y^{-1}\circ x)-\Gamma(y^{-1}\circ z)\right||Lu(y)|dy\\
&\quad+\int_{\|y^{-1}\circ x\|<M\|x^{-1}\circ z\|}\left|\Gamma(y^{-1}\circ x)\right||Lu(y)|dy
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\|y^{-1}\circ x\|<M\|x^{-1}\circ z\|}\left|\Gamma(y^{-1}\circ z)\right||Lu(y)|dy\\
\leq&\int_{\|y^{-1}\circ x\|\geq M\|x^{-1}\circ z\|}\frac{c\left\|x^{-1}\circ z\right\|}{\left\|y^{-1}\circ x\right\|^{Q-1}}|Lu(y)|dy\\
&+\int_{\|y^{-1}\circ x\|<M\|x^{-1}\circ z\|}\frac{c}{\left\|y^{-1}\circ x\right\|^{Q-2}}|Lu(y)|dy\\
&+\int_{\|y^{-1}\circ x\|<M\|x^{-1}\circ z\|}\frac{c}{\left\|y^{-1}\circ z\right\|^{Q-2}}|Lu(y)|dy.
\end{aligned}
$$

Noting that if $\left\|y^{-1}\circ x\right\|\geq M\left\|x^{-1}\circ z\right\|$, then

$$
\left\|y^{-1}\circ x\right\|\geq M\left\|x^{-1}\circ z\right\|\geq\frac{M}{c}\left\|z^{-1}\circ x\right\|;
$$

if $\left\|y^{-1}\circ x\right\|<M\left\|x^{-1}\circ z\right\|$, then

$$
\left\|y^{-1}\circ x\right\|<Mc\left\|z^{-1}\circ x\right\|
$$

and

$$
\begin{aligned}
\left\|y^{-1}\circ z\right\|&\leq c\left(\left\|y^{-1}\circ x\right\|+\left\|x^{-1}\circ z\right\|\right)<c\left(M\left\|x^{-1}\circ z\right\|+\left\|x^{-1}\circ z\right\|\right)\\
&=c\left(1+M\right)\left\|x^{-1}\circ z\right\|,
\end{aligned}
$$

it follows

$$
\begin{aligned}
|u(x)-u(z)|\leq&\int_{\|y^{-1}\circ x\|\geq\frac{M}{c}\|z^{-1}\circ x\|}\frac{c\left\|x^{-1}\circ z\right\|}{\left\|y^{-1}\circ x\right\|^{Q-1}}|Lu(y)|dy\\
&+\int_{\|y^{-1}\circ x\|<Mc\|z^{-1}\circ x\|}\frac{c}{\left\|y^{-1}\circ x\right\|^{Q-2}}|Lu(y)|dy\\
&+\int_{\|y^{-1}\circ z\|<c(1+M)\|x^{-1}\circ z\|}\frac{c}{\left\|y^{-1}\circ z\right\|^{Q-2}}|Lu(y)|dy\\
\doteq&\ I_1+I_2+I_3.
\end{aligned}
$$

Applying Lemma 4 ($\alpha=1$ and $\sigma=\frac{M}{c}$) and noting $\lambda+p<Q$, there exists a constant $c=c(p,\lambda,\sigma)>0$ such that

$$
I_1\leq c\left\|Lu\right\|_{L^{p,\lambda}(G)}\left\|z^{-1}\circ x\right\|\left\|z^{-1}\circ x\right\|^{\frac{p+\lambda-Q}{p}}=c\left\|Lu\right\|_{L^{p,\lambda}(G)}\left\|z^{-1}\circ x\right\|^{\frac{2p+\lambda-Q}{p}};
$$

from Lemma 4 ($\beta=2$ and $\sigma=Mc$; $\beta=2$ and $\sigma=c(1+M)$, respectively) and $\lambda+2p>Q$, it follows

$$
I_2\leq c\left\|Lu\right\|_{L^{p,\lambda}(G)}\left\|z^{-1}\circ x\right\|^{\frac{2p+\lambda-Q}{p}}
$$

and

$$I_3 \leq c \|Lu\|_{L^{p,\lambda}(G)} \left\|z^{-1} \circ x\right\|^{\frac{2p+\lambda-Q}{p}}.$$

In conclusion, we deduce (1.3).

(2) We know from (4) in Lemma 1 and Lemma 3 that there exist two constants $c > 0$ and $M > 1$ such that

$$\begin{aligned}
|X_i u(x) - X_i u(z)| &= \left| \int_{\mathbb{R}^N} \Gamma_i(y^{-1} \circ x) - \Gamma_i(y^{-1} \circ z) Lu(y) dy \right| \\
&\leq \int_{\mathbb{R}^N} \left| \Gamma_i(y^{-1} \circ x) - \Gamma_i(y^{-1} \circ z) \right| |Lu(y)| dy \\
&\leq \int_{\|y^{-1} \circ x\| \geq M \|x^{-1} \circ z\|} \left| \Gamma_i(y^{-1} \circ x) - \Gamma_i(y^{-1} \circ z) \right| |Lu(y)| dy \\
&\quad + \int_{\|y^{-1} \circ x\| < M \|x^{-1} \circ z\|} \left| \Gamma_i(y^{-1} \circ x) - \Gamma_i(y^{-1} \circ z) \right| |Lu(y)| dy \\
&\leq \int_{\|y^{-1} \circ x\| \geq M \|x^{-1} \circ z\|} \left| \Gamma_i(y^{-1} \circ x) - \Gamma_i(y^{-1} \circ z) \right| |Lu(y)| dy \\
&\quad + \int_{\|y^{-1} \circ x\| < M \|x^{-1} \circ z\|} \left| \Gamma_i(y^{-1} \circ x) \right| |Lu(y)| dy \\
&\quad + \int_{\|y^{-1} \circ x\| < M \|x^{-1} \circ z\|} \left| \Gamma_i(y^{-1} \circ z) \right| |Lu(y)| dy \\
&\leq \int_{\|y^{-1} \circ x\| \geq M \|x^{-1} \circ z\|} \frac{c \left\|x^{-1} \circ z\right\|}{\left\|y^{-1} \circ x\right\|^Q} |Lu(y)| dy \\
&\quad + \int_{\|y^{-1} \circ x\| < M \|x^{-1} \circ z\|} \frac{c}{\left\|y^{-1} \circ x\right\|^{Q-1}} |Lu(y)| dy \\
&\quad + \int_{\|y^{-1} \circ x\| < M \|x^{-1} \circ z\|} \frac{c}{\left\|y^{-1} \circ z\right\|^{Q-1}} |Lu(y)| dy.
\end{aligned}$$

Let us remark that if $\left\|y^{-1} \circ x\right\| \geq M \left\|x^{-1} \circ z\right\|$, then

$$\left\|y^{-1} \circ x\right\| \geq \frac{M}{c} \left\|z^{-1} \circ x\right\|;$$

if $\left\|y^{-1} \circ x\right\| < M \left\|x^{-1} \circ z\right\|$, then

$$\left\|y^{-1} \circ x\right\| < Mc \left\|z^{-1} \circ x\right\|$$

and

$$\begin{aligned}
\left\|y^{-1} \circ z\right\| &\leq c \left(\left\|y^{-1} \circ x\right\| + \left\|x^{-1} \circ z\right\|\right) < c \left(M \left\|x^{-1} \circ z\right\| + \left\|x^{-1} \circ z\right\|\right) \\
&= c (1 + M) \left\|x^{-1} \circ z\right\|.
\end{aligned}$$

It implies

$$
\begin{aligned}
|X_i u(x) - X_i u(z)| \leq & \int_{\|y^{-1}\circ x\| \geq \frac{M}{c}\|z^{-1}\circ x\|} \frac{c\left\|x^{-1}\circ z\right\|}{\left\|y^{-1}\circ x\right\|^{Q}} |Lu(y)|dy \\
& + \int_{\|y^{-1}\circ x\| < Mc\|z^{-1}\circ x\|} \frac{c}{\left\|y^{-1}\circ x\right\|^{Q-1}} |Lu(y)|dy \\
& + \int_{\|y^{-1}\circ z\| < c(1+M)\|x^{-1}\circ z\|} \frac{c}{\left\|y^{-1}\circ z\right\|^{Q-1}} |Lu(y)|dy \\
\doteq & \; I_4 + I_5 + I_6.
\end{aligned}
$$

Applying Lemma 4 ($\alpha = 0$ and $\sigma = \frac{M}{c}$) and $\lambda < Q$, there exists a constant $c = c(p,\lambda,\sigma) > 0$ such that

$$
I_4 \leq c\,\|Lu\|_{L^{p,\lambda}(G)} \left\|z^{-1}\circ x\right\| \left\|z^{-1}\circ x\right\|^{\frac{\lambda-Q}{p}} = c\,\|Lu\|_{L^{p,\lambda}(G)} \left\|z^{-1}\circ x\right\|^{\frac{p+\lambda-Q}{p}};
$$

from Lemma 4 ($\beta = 1$ and $\sigma = Mc$; $\beta = 1$ and $\sigma = c(1+M)$, respectively) and $\lambda + p > Q$, it gets

$$
I_5 \leq c\,\|Lu\|_{L^{p,\lambda}(G)} \left\|z^{-1}\circ x\right\|^{\frac{p+\lambda-Q}{p}}
$$

and

$$
I_6 \leq c\,\|Lu\|_{L^{p,\lambda}(G)} \left\|z^{-1}\circ x\right\|^{\frac{p+\lambda-Q}{p}}.
$$

In conclusion we reach to (1.4).

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PROJECTIVE DIMENSION AND SYMMETRIC ALGEBRA OF GRAPH IDEALS

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Abstract. In this note lower bounds for the projective dimension of edge ideals are determined and the integrality of the symmetric algebra of these ideals is studied for some classes of simple graphs.
Criterions about torsion freeness of the symmetric powers are also analyzed and conditions about the acyclicity of the $\mathcal{Z}$-complex of edge ideals of graphs are obtained.

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1. Introduction and preliminaries

Algebraically speaking, the edge ideals $I(\mathcal{G})$ of finite simple graphs $\mathcal{G}$ are generated by square-free monomials of degree 2 in the polynomial ring $R = k[X_1, \ldots, X_n]$, where k is a field and n the number of vertices of $\mathcal{G}$.

The present paper deals with important problems about the projective dimension and the symmetric algebra of $I(\mathcal{G})$, and the acyclicity of the $\mathcal{Z}$-complex of $I(\mathcal{G})$ is also discussed. It is structured as follows.

Section 2 analyzes the projective dimension of the edge ideals of simple graphs with respect to the number of their edges and consequences of it.

In section 3 the symmetric algebra of the edge ideals of simple graphs $S_R(I(\mathcal{G}))$ is studied, bearing also in mind essential results in [2, 11], and it is shown when such an algebra is an integral domain.

In section 4 criterions about torsion freeness of the symmetric powers of $S_R(I(\mathcal{G}))$ are considered and acyclicity conditions for the complex $\mathcal{Z}(I(\mathcal{G}))$ are examined.

Let $\mathcal{G}$ be a graph, $V(\mathcal{G})$, $E(\mathcal{G})$ the sets of its vertices and edges, resp. $\mathcal{G}$ is said to be *simple* if, for all $\{v_i, v_j\} \in E(\mathcal{G})$, $i \neq j$, it is $v_i \neq v_j$.
A *cycle* of *length* n, $C_n \subset \mathcal{G}$, is an alternating sequence of $n+1$ distinct vertices and n edges beginning and ending at the same vertex, in which each edge is incident to the two vertices immediately preceding and following it.
A *tree* is a connected graph without cycles.

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The *degree* of a vertex $v_\ell \in V(\mathcal{G})$, denoted by $\deg(v_\ell)$, is the number of edges incident with v_ℓ, i.e. $\deg(v_\ell) = \max\{r \mid \exists\, v_1, \ldots, v_r \in V(\mathcal{G})$ with $\{v_i, v_\ell\} \in E(\mathcal{G}),$ $i = 1, \ldots, r\}$. When $\deg(v_\ell) = 0$, the vertex v_ℓ is said *isolated*.
A subset A of $V(\mathcal{G})$ is said a *minimal vertex cover* for $\mathcal{G}$ if every edge of $\mathcal{G}$ is incident with one vertex in A and there is no proper subset of A having such property. If A satisfies only the incident condition, A is called a *vertex cover* for $\mathcal{G}$.
The smallest number of vertices in any minimal vertex cover for $\mathcal{G}$ is called *vertex covering number* of $\mathcal{G}$ and denoted by $\alpha_0(\mathcal{G})$.
If $V(\mathcal{G}) = \{v_1, \ldots, v_n\}$ and $R = k[X_1, \ldots, X_n]$ is the polynomial ring over a field k such that each variable X_i corresponds to the vertex v_i, the *edge ideal* $I(\mathcal{G})$ associated to $\mathcal{G}$ is the ideal $(\{X_i X_j \mid \{v_i, v_j\} \in E(\mathcal{G})\}) \subset R$.
Note that the non-zero edge ideals are those generated by square-free monomials of degree 2. This implies that $I(\mathcal{G})$ is a graded ideal of R of initial degree 2, that is $I(\mathcal{G}) = \oplus_{i \geq 2} (I(\mathcal{G})_i)$.
If $E(\mathcal{G}) = \varnothing$, i.e. $\mathcal{G}$ has isolated vertices only, then $I(\mathcal{G}) = (0)$.
So $I(\mathcal{G})$ has a graded free resolution of length at most n. The length of the (unique) minimal resolution of $I(\mathcal{G})$ is equal to $\operatorname{pdim}_R(I(\mathcal{G}))$, the *projective dimension* of $I(\mathcal{G})$.
An ideal $\wp \subset R$ generated by $A = \{X_{i_1}, ..., X_{i_r}\}$ is a minimal prime ideal of $I(\mathcal{G})$ if and only if A is a minimal vertex cover for $\mathcal{G}$.
It is well-known that $\operatorname{ht}(I(\mathcal{G})) = \alpha_0(\mathcal{G})$ ([14], Corollary 6.1.18).
$\mathcal{G}$ is said a *Cohen-Macaulay graph* over k (*C-M* for short) if $\operatorname{depth}(R/I(\mathcal{G})) = \dim(R/I(\mathcal{G}))$, where dim is the Krull dimension.
If $\mathcal{G}$ is a C-M graph, the Auslander and Buchsbaum theorem ([4], Corollary 3.3) gives $\operatorname{pdim}_R(I(\mathcal{G})) = \alpha_0(\mathcal{G}) - 1$.
The first initial Betti number in a minimal graded resolution of $I(\mathcal{G})$ has the following geometric interpretation (see [3]).

Proposition 1. *If $\mathcal{G}$ is a graph with edge ideal $I(\mathcal{G})$ and*

$$0 \longrightarrow \bigoplus_{i=1}^{c_g} R^{b_{g_i}}(-d_{g_i}) \longrightarrow \ldots \longrightarrow R^c(-4) \oplus R^b(-3) \longrightarrow$$
$$\longrightarrow R^{|E(\mathcal{G})|}(-2) \longrightarrow I(\mathcal{G}) \longrightarrow 0$$

is the minimal graded resolution of $I(\mathcal{G})$, then $b = |E(\mathcal{L}(\mathcal{G}))| - N_t$, being N_t the number of triangles in $\mathcal{G}$, and $\mathcal{L}(\mathcal{G})$ the edge graph of $\mathcal{G}$.

Remark 1. The number c in a minimal graded resolution of $I(\mathcal{G})$ is equal to the number of unordered pairs of lines f_i, f_j such that f_i and f_j are independent and cannot be joined by an edge.

Let I be an ideal of $R = k[X_1, \ldots, X_n]$. The *symmetric algebra* of I over R is

$$S_R(I) = \bigoplus_{t \geq 0} S_t(I).$$

Let $R^m \xrightarrow{\varphi} R^q \xrightarrow{\psi} I \longrightarrow 0$ be a presentation of I, where $\varphi = (a_{ij})$ is an $q \times m$ matrix with entries in R. Then

$$S_R(I) \simeq R[T_1, \ldots, T_q]/J \ ,$$

being $J = (f_1, \ldots, f_m)$ the *ideal of relations* of $S_R(I)$, $f_j = \sum_{i,j} a_{ij}\, T_i$.

For each $p = 0, \ldots, n$ let's consider the morphism

$$\partial_p : \bigwedge^p R^n \longrightarrow \bigwedge^{p-1} R^n \otimes I$$

defined by

$$\partial_p (e_{i_1} \wedge \ldots \wedge e_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1}\, e_{i_1} \wedge \ldots \wedge \widehat{e}_{i_j} \wedge \ldots \wedge e_{i_p} \otimes \psi(e_{i_j})$$

where $e_1, \ldots, e_n$ is a basis of R^n and $\widehat{e}$ means omission.
Let's denote $Z_p(I) = \ker \partial_p$.
Then one can build the complex:

$$\mathcal{Z}(I) : 0 \longrightarrow Z_n(I) \otimes S[-n] \xrightarrow{\partial'_n} Z_{n-1}(I) \otimes S[-n+1] \longrightarrow \ldots$$

$$\ldots \longrightarrow Z_1(I) \otimes S[-1] \xrightarrow{\partial'_1} Z_0(I) \otimes S \longrightarrow S_R(I) \longrightarrow 0$$

where ∂'_p is the map induced by ∂_p, and $S[-p]_t = S_{t-p}$. More information about the complexes $\mathcal{Z}(I)$ can be found in [6].

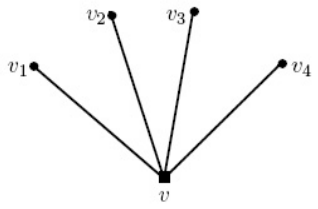
2. Bounds for projective dimension

In this section we find lower bounds for the projective dimension of the edge ideals of finite simple graphs with respect to the number of their edges.

Theorem 1. *Let $I(\mathcal{G})$ be the edge ideal of a simple graph $\mathcal{G}$ with $|E(\mathcal{G})| \geq 4$. Then* $\operatorname{pdim}_R(I(\mathcal{G})) > 1$.

Proof. It results $\operatorname{pdim}_R(I(\mathcal{G})) \geq \alpha_0(\mathcal{G}) - 1$.
So when $\alpha_0(\mathcal{G}) \geq 3$, then $\operatorname{pdim}_R(I(\mathcal{G})) > 1$ and we conclude.
We have to show only the cases $\alpha_0(\mathcal{G}) = 1, 2$.

$\alpha_0(\mathcal{G}) = 1$.
There is a unique graph $\mathcal{G}_0 = \{v, v_1, v_2, v_3, v_4\}$ with $|E(\mathcal{G}_0)| = 4$.



Every other graph $\mathcal{G}'_0$ with $|E(\mathcal{G}'_0)|>4$ must contain $\mathcal{G}_0$ and it is obtained by adding to $\mathcal{G}_0$ ℓ distinct edges of initial vertex v.
The graded minimal resolution of $I(\mathcal{G}_0)$ is

$$0 \to R(-5) \to R^4(-4) \to R^6(-3) \xrightarrow{\varphi} R^4(-2) \to I(\mathcal{G}_0) \to 0 \ ;$$

it results $\operatorname{pdim}_R(I(\mathcal{G}_0)) = 3 > 1$. The graded minimal resolution of $I(\mathcal{G}'_0)$ will be

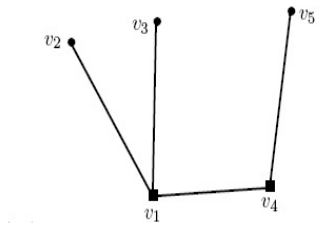
$$0 \to R(-5-\ell) \to R^{\binom{4+\ell}{4+\ell-1}}(-4-\ell) \to \dots$$

$$\dots \to R^{\binom{4+\ell}{2}}(-3) \to R^{4+\ell}(-2) \to I(\mathcal{G}'_0) \to 0 \quad .$$

Then $\operatorname{pdim}_R(I(\mathcal{G}'_0)) > \operatorname{pdim}_R(I(\mathcal{G}_0)) > 1$.

$\alpha_0(\mathcal{G}) = 2$.
We have to consider only the following simple graphs, each having 4 edges:

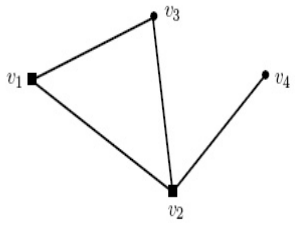
$\mathcal{G}_1$:



Resolution of $I(\mathcal{G}_1)$:

$$0 \to R(-4) \to R^4(-3) \xrightarrow{\varphi} R^4(-2) \to I(\mathcal{G}_1) \to 0.$$

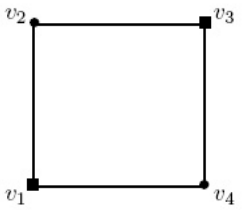
$\mathcal{G}_2$:



Resolution of $I(\mathcal{G}_2)$:

$$0 \to R(-4) \to R^4(-3) \xrightarrow{\varphi} R^4(-2) \to I(\mathcal{G}_2) \to 0.$$

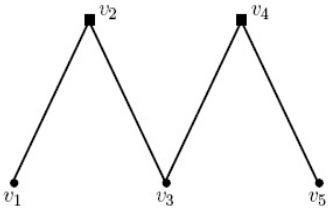
$\mathcal{G}_3$:



Resolution of $I(\mathcal{G}_3)$:

$$0 \to R(-4) \to R^4(-3) \xrightarrow{\varphi} R^4(-2) \to I(\mathcal{G}_3) \to 0\,.$$

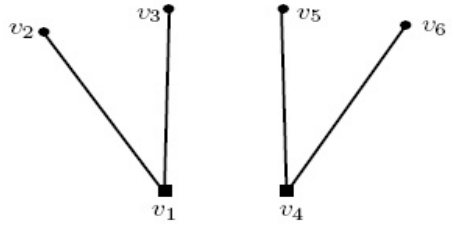
$\mathcal{G}_4$:



Resolution of $I(\mathcal{G}_4)$:

$$0 \to R(-5) \to R^3(-3) \oplus R(-4) \xrightarrow{\varphi} R^4(-2) \to I(\mathcal{G}_4) \to 0\,.$$

$\mathcal{G}_5$:



Resolution of $I(\mathcal{G}_5)$:

$$0 \to R(-6) \to R^4(-5) \to R^2(-3) \oplus R^4(-4) \xrightarrow{\varphi} R^4(-2) \to I(\mathcal{G}_5) \to 0\,.$$

It results $\operatorname{pdim}_R(I(\mathcal{G}_i)) = 2$, for $i = 1,2,3,4$; $\operatorname{pdim}_R(I(\mathcal{G}_5)) = 3$.

If we add to any of the above graphs $\mathcal{G}_i$, $i = 1,2,3,4$, a finite number of distinct edges such that in the new graphs $\mathcal{G}_i'$, $i = 1,2,3,4$, $\alpha_0(\mathcal{G}_i') = \alpha_0(\mathcal{G}_i)$, we have, by Proposition 1, that $b' + c' \geq b + c$, where b' and c' are the graded Betti numbers of the first syzygy module in the resolution of $I(\mathcal{G}_i')$.
Because $\operatorname{pdim}_R(I(\mathcal{G}_i)) = 2$, then $\operatorname{pdim}_R(I(\mathcal{G}_i')) > 1$.
Similar arguments hold for $\mathcal{G}_5$, except in the following situation:
let's $\mathcal{G}_5'$ be the graph obtained from $\mathcal{G}_5$ by adding to it the edge $\{v_1, v_4\}$. Unlike the other cases, the edge ideal of $\mathcal{G}_5'$ has projective dimension strictly less than $\operatorname{pdim}_R(I(\mathcal{G}_5))$.
In fact, the resolution of $I(\mathcal{G}_5')$ is :

$$0 \to R^2(-4) \to R^6(-3) \to R^5(-2) \to I(\mathcal{G}_5') \to 0\,.$$

However $\operatorname{pdim}_R(I(\mathcal{G}_5')) = 2 > 1$. □

It is possible to improve the above lower bound.

Proposition 2. *Let $\mathcal{G}$ be as in Theorem 1. Let $\alpha(\mathcal{G})$ be the size of the largest minimal vertex cover for $\mathcal{G}$. Then* $\operatorname{pdim}_R(I(\mathcal{G})) \geq \alpha(\mathcal{G}) - 1$.

Proof. Note that $\operatorname{pdim}_R(I(\mathcal{G})) = \operatorname{reg}(R/I(\mathcal{G})^\vee)$, the regularity of $R/I(\mathcal{G})^\vee$, where $I(\mathcal{G})^\vee$ is the Alexander dual of $I(\mathcal{G})$.
Because the Alexander dual is generated by the minimal vertex covers for $\mathcal{G}$, the degree of its largest generator is $\alpha(\mathcal{G})$.
So $\operatorname{reg}(R/I(\mathcal{G})^\vee) \geq \alpha(\mathcal{G}) - 1$. □

For example, let's consider the graph $\mathcal{G}_0$ that appears in the proof of Theorem 1. We see that any graph $\mathcal{G}$ that contains $\mathcal{G}_0$ as an induced subgraph will have at least one minimal vertex cover of at least size 4. In fact, the minimal vertex cover that does not contain v must contain v_1, v_2, v_3, v_4. So $\operatorname{pdim}_R(I(\mathcal{G})) \geq 3$.

Corollary 1. *The only simple graphs with projective dimension of their edge ideal less than or equal to 1 are the following:*
$\mathcal{P} := \{v_1, v_2\}$, *an edge.*
$\mathcal{Q} := \{\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_2, v_3\}\}\}$, *two consecutive edges.*
$\mathcal{R} := \{\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2\}, \{v_3, v_4\}\}\}$, *two non consecutive edges.*
$\mathcal{S} := \{\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\}\}$, *a triangle.*
$\mathcal{T} := \{\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\}\}$, *three consecutive edges.*

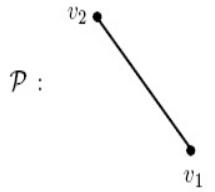
3. Integrality for symmetric algebra

In this section we examine meaningful classes of simple graphs and study the integrality of the symmetric algebra of their edge ideal.

Proposition 3. *Let $I(\mathcal{G})$ be the edge ideal of a simple graph $\mathcal{G}$ with $|E(\mathcal{G})| < 4$. Then the symmetric algebra $S_R(I(\mathcal{G}))$ is an integral domain.*

Proof. Let's consider every simple graph with at most 3 edges and construct explicitly the symmetric algebra of its edge ideal.

- Graph with 2 nodes and 1 edge

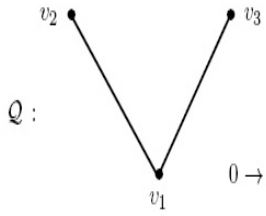


Resolution of $I(\mathcal{P})$:

$0 \to R(-2) \to I(\mathcal{P}) \to 0$

It is $S_R(I(\mathcal{P})) = R[Y]/J$, $J = (0)$; obviously, $S_R(I(\mathcal{P}))$ is integral.

- Graph with 3 nodes and 2 edges



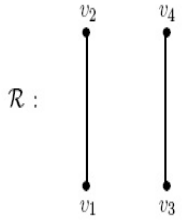
Resolution of $I(\mathcal{Q})$:

$$\varphi = \begin{pmatrix} X_1 \\ -X_3 \end{pmatrix}$$

$$0 \to R(-3) \xrightarrow{\quad} R^2(-2) \to I(\mathcal{Q}) \to 0$$

$S_R(I(\mathcal{Q})) = R[Y_1, Y_2]/J\,, \quad J = (X_1Y_1 - X_3Y_2)\,.$
$J \subset k[X_1, X_2, X_3, Y_1, Y_2]$ is prime, so $S_R(I(\mathcal{Q}))$ is integral.

- Graph with 4 nodes and 2 edges



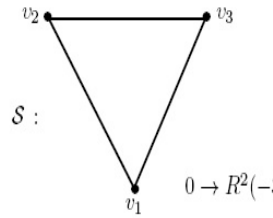
Resolution of $I(\mathcal{R})$:

$$\varphi = \begin{pmatrix} X_1X_2 \\ -X_3X_4 \end{pmatrix}$$

$$0 \to R(-4) \xrightarrow{\quad} R^2(-2) \to I(\mathcal{R}) \to 0$$

$S_R(I(\mathcal{R})) = R[Y_1, Y_2]/J\,, \quad J = (X_1X_2Y_1 - X_3X_4Y_2)\,.$
$J \subset k[X_1, X_2, X_3, X_4, Y_1, Y_2]$ is prime, so $S_R(I(\mathcal{R}))$ is integral.

- Graph with 3 nodes and 3 edges



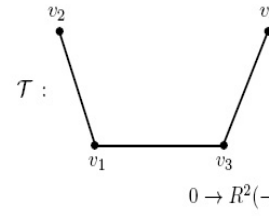
Resolution of $I(\mathcal{S})$:

$$\varphi = \begin{pmatrix} X_2 & 0 \\ -X_1 & X_1 \\ 0 & -X_3 \end{pmatrix}$$

$$0 \to R^2(-3) \xrightarrow{\quad} R^3(-2) \to I(\mathcal{S}) \to 0$$

$S_R(I(\mathcal{S})) = R[Y_1, Y_2, Y_3]/J\,, \quad J = (X_2Y_1 - X_1Y_2,\, X_1Y_2 - X_3Y_3)\,.$
$J \subset k[X_1, X_2, X_3, Y_1, Y_2, Y_3]$ is prime, so $S_R(I(\mathcal{S}))$ is integral.

- Graph (tree) with 4 nodes and 3 edges



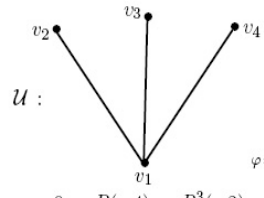
Resolution of $I(\mathcal{T})$:

$$\varphi = \begin{pmatrix} X_2 & 0 \\ -X_4 & X_1 \\ 0 & -X_3 \end{pmatrix}$$

$$0 \to R^2(-3) \xrightarrow{\quad} R^3(-2) \to I(\mathcal{T}) \to 0$$

$S_R(I(\mathcal{T})) = R[Y_1, Y_2, Y_3]/J\,, \quad J = (X_2Y_1 - X_4Y_2,\, X_1Y_2 - X_3Y_3)\,.$
$J \subset k[X_1, \ldots, X_4, Y_1, Y_2, Y_3]$ is prime, so $S_R(I(\mathcal{T}))$ is integral.

- Graph with 4 nodes and 3 edges



$\mathcal{U}$:

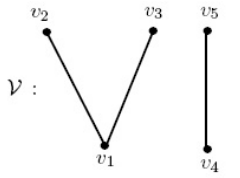
Resolution of $I(\mathcal{U})$:

$$\varphi=\begin{pmatrix} X_3 & X_2 & 0 \\ -X_4 & 0 & X_2 \\ 0 & -X_4 & -X_3 \end{pmatrix}$$

$$0 \to R(-4) \to R^3(-3) \xrightarrow{\quad} R^3(-2) \to I(\mathcal{U}) \to 0$$

$S_R(I(\mathcal{U})) = R[Y_1, Y_2, Y_3]/J$, where
$J = (X_3Y_1 - X_4Y_2,\ X_2Y_1 - X_4Y_3,\ X_2Y_2 - X_3Y_3)$.
$J \subset k[X_1, \ldots, X_4, Y_1, Y_2, Y_3]$ is prime, so $S_R(I(\mathcal{U}))$ is integral.

- Graph with 5 nodes and 3 edges



$\mathcal{V}$:

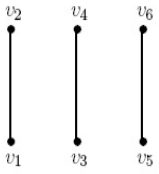
Resolution of $I(\mathcal{V})$:

$$\varphi=\begin{pmatrix} 0 & X_1X_3 & X_1X_2 \\ X_2 & -X_4X_5 & 0 \\ -X_3 & 0 & -X_4X_5 \end{pmatrix}$$

$$0 \to R(-5) \to R(-3) \oplus R^2(-4) \xrightarrow{\quad} \xrightarrow{\varphi} R^3(-2) \to I(\mathcal{V}) \to 0$$

$S_R(I(\mathcal{V})) = R[Y_1, Y_2, Y_3]/J$, where
$J = (X_2Y_2 - X_3Y_3,\ X_1X_3Y_1 - X_4X_5Y_2,\ X_1X_2Y_1 - X_4X_5Y_3)$.
$J \subset k[X_1, \ldots, X_5, Y_1, Y_2, Y_3]$ is prime, so $S_R(I(\mathcal{V}))$ is integral.

- Graph with 6 nodes and 3 edges



$\mathcal{W}$:

Resolution of $I(\mathcal{W})$:

$$\varphi=\begin{pmatrix} X_3X_4 & X_1X_2 & 0 \\ -X_5X_6 & 0 & X_1X_2 \\ 0 & -X_5X_6 & -X_3X_4 \end{pmatrix}$$

$$0 \to R(-6) \to R^3(-4) \xrightarrow{\quad} \xrightarrow{\varphi} R^3(-2) \to I(\mathcal{W}) \to 0$$

$S_R(I(\mathcal{W})) = R[Y_1, Y_2, Y_3]/J$, where
$J = (X_3X_4Y_1 - X_5X_6Y_2,\ X_1X_2Y_1 - X_5X_6Y_3,\ X_1X_2Y_2 - X_3X_4Y_3)$.
$J \subset k[X_1, \ldots, X_6, Y_1, Y_2, Y_3]$ is prime, so $S_R(I(\mathcal{W}))$ is integral. □

Corollary 2. *Let $\mathcal{G}$ be a simple graph such that* $\operatorname{pdim}_R(I(\mathcal{G})) \leqslant 1$. *Then $S_R(I(\mathcal{G}))$ is an integral domain.*

Proof. By Corollary 1, the result immediately descends. □

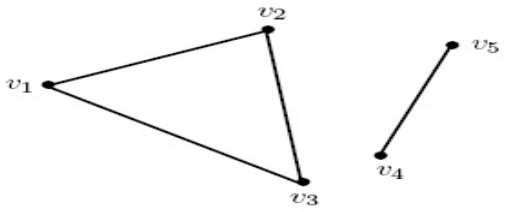
Let $\mathcal{G}$ be a Cohen-Macaulay graph with q edges. By [14], Exercise 6.2.19, if g is the height of $I(\mathcal{G})$, it results $q \leqslant (g^2+g)/2$. Hence the possible C-M graphs with $\alpha_0(\mathcal{G}) = 3$ have at most 6 edges.

Proposition 4. *Let $\mathcal{G}$ be a Cohen-Macaulay graph with no cycles of even length and* $\operatorname{pdim}_R(I(\mathcal{G})) = 2$. *Then the symmetric algebra $S_R(I(\mathcal{G}))$ is an integral domain.*

Proof. Remember that for C-M graphs it results $\operatorname{pdim}_R(I) = \alpha_0 - 1$.
By Proposition 3, the symmetric algebra of edge ideals of generic simple graphs having at most 3 edges is an integral domain.
So it remains to examine the symmetric algebra of edge ideals of C-M graphs $\mathcal{G}$ with $\operatorname{pdim}_R(I(\mathcal{G})) = 2$ and $4 \leqslant |E(\mathcal{G})| \leqslant 6$.
Computations are made using softwares `CoCoA` [13] and `Macaulay` [5].

⋆ C-M graphs with $\operatorname{pdim}_R(I) = 2$ having 4 edges are the following:

$\mathcal{H}_1$



The minimal free resolution of its edge ideal is

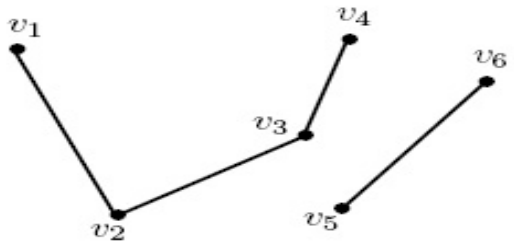
$$0 \to R^2(-5) \to R^2(-3) \oplus R^3(-4) \xrightarrow{\varphi} R^4(-2) \to I(\mathcal{H}_1) \to 0\,.$$

$S_R(I(\mathcal{H}_1)) = R[Y_1,\dots,Y_4]/J$, where

$$J = (X_1Y_3 - X_3Y_4,\ X_2Y_2 - X_3Y_4,\ X_2X_3Y_1 - X_4X_5Y_3,\ X_1X_3Y_1 - X_4X_5Y_2,\ X_1X_2Y_1 - X_4X_5Y_4)\,.$$

$J \subset k[X_1,\dots,X_5,Y_1,\dots,Y_4]$ is prime, so $S_R(I(\mathcal{H}_1))$ is integral.

$\mathcal{H}_2$



The ideal of relations of $S_R(I(\mathcal{H}_2))$ is

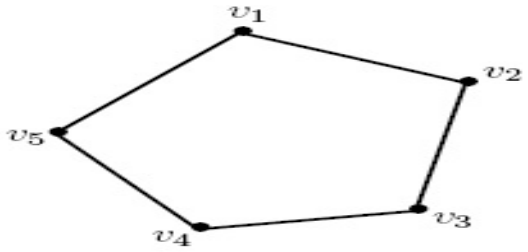
$$J = (X_2Y_2 - X_4Y_3,\ X_1Y_3 - X_3Y_4,\ X_3X_4Y_1 - X_5X_6Y_2,$$

$$X_2X_3Y_1 - X_5X_6Y_3,\, X_1X_2Y_1 - X_5X_6Y_4)\,.$$

$J \subset k[X_1,\dots,X_6,Y_1,\dots,Y_4]$ is prime, so $S_R(I(\mathcal{H}_2))$ is integral.

⋆ C-M graphs with $\operatorname{pdim}_R(I) = 2$ having 5 edges are the following:

$\mathcal{H}_3$



The minimal free resolution of its edge ideal is

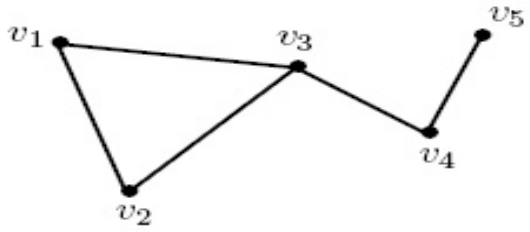
$$0 \to R(-5) \to R^5(-3) \xrightarrow{\varphi} R^5(-2) \to I(\mathcal{H}_3) \to 0\,.$$

The ideal of relations of $S_R(I(\mathcal{H}_3))$ is

$$J = (X_3Y_2 - X_5Y_3,\, X_4Y_1 - X_1Y_2,\, X_2Y_1 - X_5Y_5,\\ X_2Y_3 - X_4Y_4,\, X_1Y_4 - X_3Y_5)\,.$$

$J \subset k[X_1,\dots,X_5,Y_1,\dots,Y_5]$ is prime, so $S_R(I(\mathcal{H}_3))$ is integral.

$\mathcal{H}_4$



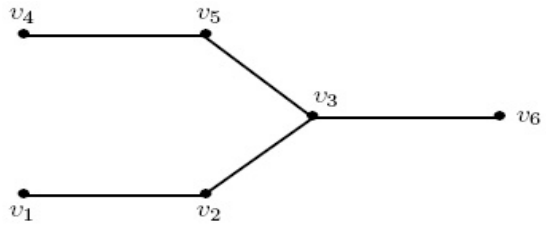
The minimal free resolution of its edge ideal is

$$0 \to R(-4) \oplus R(-5) \to R^5(-3) \oplus R(-4) \xrightarrow{\varphi} R^5(-2) \to I(\mathcal{H}_4) \to 0\,.$$

$$J = (X_3Y_1 - X_5Y_2,\, X_2Y_2 - X_4Y_4,\, X_1Y_2 - X_4Y_3,\, X_2Y_3 - X_1Y_4,\\ X_1Y_4 - X_3Y_5,\, X_1X_2Y_1 - X_4X_5Y_5) \text{ is a prime ideal.}$$

So the symmetric algebra $S_R(I(\mathcal{H}_4))$ is integral.

$\mathcal{H}_5$

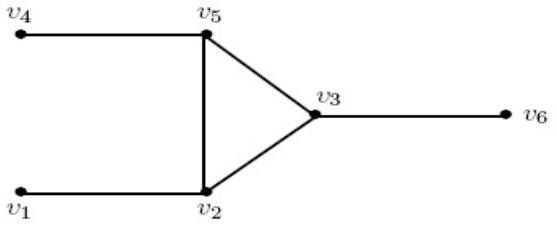


$$J = (X_6Y_1 - X_5Y_3,\ X_2Y_3 - X_6Y_4,\ X_4Y_1 - X_3Y_2,\ X_2Y_1 - X_5Y_4,\ X_1Y_4 - X_3Y_5,\ X_1X_2Y_2 - X_4X_5Y_5) \text{ is a prime ideal.}$$

So the symmetric algebra $S_R(I(\mathcal{H}_5))$ is integral.

⋆ C-M graphs with $\operatorname{pdim}_R(I) = 2$ having 6 edges are the following:

$\mathcal{H}_6$



The minimal free resolution of its edge ideal is

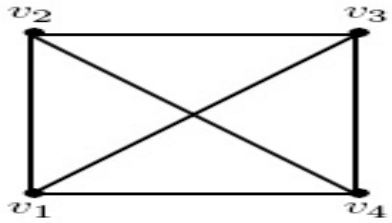
$$0 \to R^3(-4) \to R^8(-3) \xrightarrow{\varphi} R^6(-2) \to I(\mathcal{H}_6) \to 0\,.$$

The ideal of relations of $S_R(I(\mathcal{H}_6))$ is

$$J = (X_6Y_1 - X_5Y_4,\ X_2Y_4 - X_6Y_5,\ X_4Y_1 - X_3Y_3,\ X_4Y_2 - X_2Y_3,\ X_2Y_1 - X_3Y_2,\ X_3Y_2 - X_5Y_5,\ X_1Y_2 - X_5Y_6,\ X_1Y_5 - X_3Y_6)\,.$$

$J \subset k[X_1, \ldots, X_6, Y_1, \ldots, Y_6]$ is prime, so $S_R(I(\mathcal{H}_6))$ is integral.

$\mathcal{H}_7$



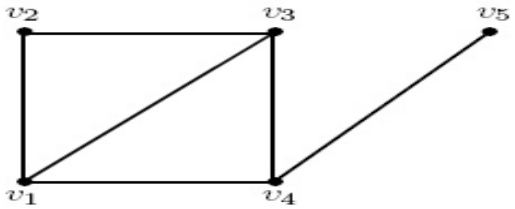
The ideal of relations of $S_R(I(\mathcal{H}_7))$ is

$$J = (X_3Y_1 - X_2Y_4,\ X_2Y_4 - X_4Y_5,\ X_4Y_2 - X_3Y_3,\ X_3Y_3 - X_1Y_4,$$

$$X_1Y_1 - X_2Y_3,\ X_2Y_3 - X_4Y_6,\ X_2Y_2 - X_1Y_5,\ X_1Y_5 - X_3Y_6).$$

J is not a prime ideal, so $S_R(I(\mathcal{H}_7))$ is not integral.

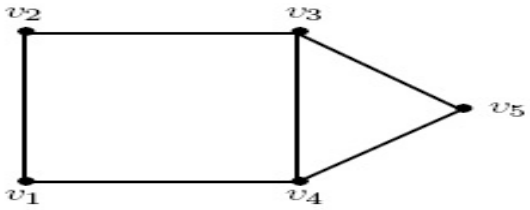
$\mathcal{H}_8$



$$J = (X_3Y_1 - X_5Y_4,\ X_1Y_1 - X_5Y_3,\ X_2Y_4 - X_4Y_5,\ X_4Y_2 - X_3Y_3,\ X_3Y_3 - X_1Y_4,\ X_2Y_3 - X_4Y_6,\ X_2Y_2 - X_1Y_5,\ X_1Y_5 - X_3Y_6).$$

J is not a prime ideal, so $S_R(I(\mathcal{H}_8))$ is not integral.

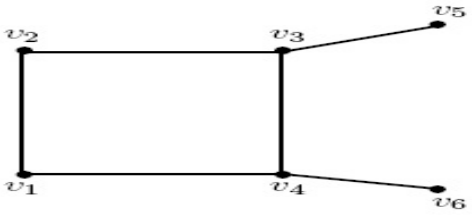
$\mathcal{H}_9$



$$J = (X_3Y_1 - X_4Y_2,\ X_4Y_2 - X_5Y_4,\ X_1Y_1 - X_5Y_3,\ X_2Y_2 - X_5Y_5,\ X_2Y_4 - X_4Y_5,\ X_3Y_3 - X_1Y_4,\ X_2Y_3 - X_4Y_6,\ X_1Y_5 - X_3Y_6).$$

J is not a prime ideal, so $S_R(I(\mathcal{H}_9))$ is not integral.

$\mathcal{H}_{10}$



$$J = (X_3Y_1 - X_6Y_4,\ X_1Y_1 - X_6Y_3,\ X_4Y_2 - X_5Y_4,\ X_2Y_2 - X_5Y_5,\ X_2Y_4 - X_4Y_5,\ X_3Y_3 - X_1Y_4,\ X_2Y_3 - X_4Y_6,\ X_1Y_5 - X_3Y_6).$$

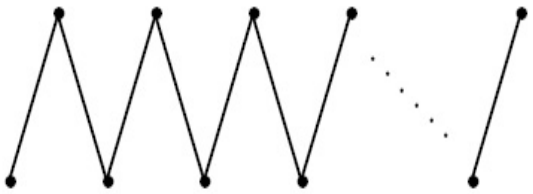
J is not a prime ideal, so $S_R(I(\mathcal{H}_{10}))$ is not integral. □

Proposition 4 can be easily extended as follows.

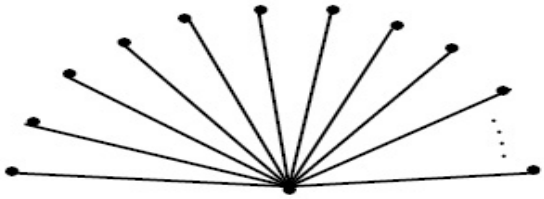
Theorem 2. *Let $\mathcal{G}$ be a simple graph such that $|E(\mathcal{G})| \leqslant 6$, with no cycles of even length, and $\alpha_0(\mathcal{G}) = 3$. Then the symmetric algebra $S(I(\mathcal{G}))$ is an integral domain.*

In the light of previous results, we are able to show that the symmetric algebra is integral for the following classes of simple graphs.

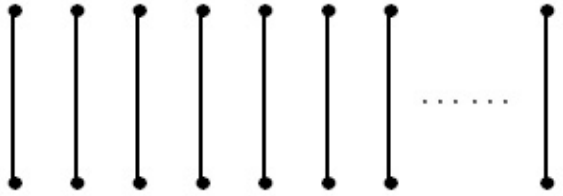
1) Connected graphs of the type



2) Connected graphs (star graphs) of the type



3) Non-connected graphs of the type



Theorem 3. *Let $\mathcal{G}$ be a simple graph as in items 1) – 3). Then the symmetric algebra $S(I(\mathcal{G}))$ is an integral domain.*

Proof. The statement is shown for star graphs (item 2)) with $n+1$ vertices and n edges.
Let $R = k[X_1, \ldots, X_{n+1}]$ be the polynomial ring whose variables are associated to the vertices of $\mathcal{G}$. The edge ideal of $\mathcal{G}$ is $I(\mathcal{G}) = (X_1X_{n+1}, \ldots, X_nX_{n+1})$.

The minimal free resolution of $I(\mathcal{G})$ is

$$0 \to R(-n-1) \to R^{\binom{n}{n-1}}(-n) \to R^{\binom{n}{n-2}}(-n+1) \to \dots$$
$$\dots \to R^{\binom{n}{3}}(-4) \to R^{\binom{n}{2}}(-3) \to R^{n}(-2) \to I(\mathcal{G}) \to 0\,.$$

The ideal of relations of $S_R(I(\mathcal{G}))$ is $J = (g_1, g_2, \dots, g_{\binom{n}{2}})$, where

$g_1 = X_{n-1}Y_1 - X_nY_2$
$g_2 = X_{n-2}Y_1 - X_nY_3$ $\quad g_n = X_{n-2}Y_2 - X_{n-1}Y_3$
$g_3 = X_{n-3}Y_1 - X_nY_4$ $\quad g_{n+1} = X_{n-3}Y_2 - X_{n-1}Y_4$ $\quad g_{2n-2} = X_{n-3}Y_3 - X_{n-2}Y_4$
$g_4 = X_{n-4}Y_1 - X_nY_5$ $\quad g_{n+2} = X_{n-4}Y_2 - X_{n-1}Y_5$ $\quad g_{2n-1} = X_{n-4}Y_3 - X_{n-2}Y_5$
$\dots\dots$ $\quad \dots\dots$ $\quad \dots\dots$
$g_{n-1} = X_1Y_1 - X_nY_n$ $\quad g_{2n-3} = X_1Y_2 - X_{n-1}Y_n$ $\quad g_{3n-6} = X_1Y_3 - X_{n-2}Y_n$

$\dots\dots\dots$ $\quad g_{\binom{n}{2}-2} = X_2Y_{n-2} - X_3Y_{n-1}$
$g_{\binom{n}{2}-1} = X_1Y_{n-2} - X_3Y_n$ $\quad g_{\binom{n}{2}} = X_1Y_{n-1} - X_2Y_n\,.$

For all $n \in \mathbb{N}$, J is a prime ideal, so $S_R(I(\mathcal{G}))$ is integral. □

4. Integrality and acyclicity

In this section we study conditions about the acyclicity of the $\mathcal{Z}$-complex of edge ideals and join them with integrality.

Remark 2. In 1964 Micali (cfr. [7]) showed, when R is an integral domain and E a finitely generated R-module, that the symmetric algebra $S_R(E)$ is an integral domain if and only if $S_t(E)$ is torsion free, $\forall\, t$.
So, for all simple graphs examined in last section, the symmetric powers $S_t(I(\mathcal{G}))$ are torsion free, $\forall\, t$.
In particular, for simple graphs such that $\operatorname{pdim}_R(I(\mathcal{G})) = 1$ (Corollaries 1 and 2), $S_t(I(\mathcal{G}))$ torsion free is equivalent to the inequality $\operatorname{grade} I_k(\varphi) \geq |E(\mathcal{G})| - 1 - k + 1$, for all $k = 1, \dots, \operatorname{rank}(\varphi)$ (see [1]).

The following joins integrality and acyclicity.

Proposition 5. *Let $\mathcal{G}$ be one of the graphs as in Corollary 1. The complex $\mathcal{Z}(S_t(I(\mathcal{G})))$ is acyclic if and only if the symmetric algebra $S_R(S_t(I(\mathcal{G})))$ is an integral domain.*

Proof. By Corollary 2, $S_R(I(\mathcal{G}))$ is an integral domain.
By Micali's theorem (cfr. Remark 2), $S_t(I(\mathcal{G}))$ is torsion-free, $\forall\, t$. Because $S_t(I(\mathcal{G}))$ is an R-module, $S_R(S_t(I(\mathcal{G})))$ integral domain is equivalent to be $\mathcal{Z}(S_t(I(\mathcal{G})))$ acyclic, according to a remark of [10] . □

Remark 3. By [12], Remark 3.8 c), if $\mathcal{G}$ is a graph such that $S_t(I(\mathcal{G}))$ is torsion free, then the complex $\mathcal{Z}(I(\mathcal{G}))$ is exact if and only if $\mathcal{Z}(I(\mathcal{G}))_{\wp}$ is exact for any $\wp \in \operatorname{Spec}(R)$ with $\operatorname{depth} R_{\wp} < |E(\mathcal{G})|$.

Proposition 6. *Let $\mathcal{G}$ be a graph with $|E(\mathcal{G})| \geqslant 4$. Then $S_R(I(\mathcal{G}))$ is not a complete intersection.*

Proof. By [12], Theorem 2.9, $S_R(I(\mathcal{G}))$ is a complete intersection if and only if $\operatorname{pdim}_R(I(\mathcal{G})) \leqslant 1$.
But when $|E(\mathcal{G})| \geqslant 4$, Proposition 1 says that $\operatorname{pdim}_R(I(\mathcal{G})) > 1$.
So $S_R(I(\mathcal{G}))$ is not a complete intersection. □

Proposition 7. *The edge ideals of graphs $\mathcal{G}$ with* $\operatorname{pdim}_R(I(\mathcal{G})) = 2$ *and* $rank\ N = rank\ Z_1(I(\mathcal{G})) = 3$ *have the $\mathcal{Z}$-complex acyclic.*

Proof. $\operatorname{rank} N = 3 \implies (\overset{3}{\bigwedge} N)^{**} \simeq R$ and the $\mathcal{Z}$-complex is

$$\mathcal{Z}(I(\mathcal{G})) \colon 0 \to R \otimes S[-3] \to (\overset{2}{\bigwedge} N)^{**} \otimes S[-2] \to$$

$$\to N \otimes S[-1] \to R^q \otimes S \to S_R(I(\mathcal{G})) \to 0$$

Because $I(\mathcal{G})$ is torsion-free, $(\overset{2}{\bigwedge} N)^{**} \simeq Z_2(I(\mathcal{G}))$ (in fact, $2 < \operatorname{rank} N$) and $Z_2(I(\mathcal{G}))$ is a reflexive module.
It follows that $(\overset{2}{\bigwedge} N)^{**}$ is a reflexive module. Such a module is a second syzygy module, so $\operatorname{depth}(\overset{2}{\bigwedge} N)^{**} \geqslant 2$ (see [4, 8]).
¿From Peskine and Szpiro's acyclicity lemma (see [9], Lemma 1.8), the complex $\mathcal{Z}(I(\mathcal{G}))$ is exact. □

Corollary 3. *The only C-M graphs with* $\operatorname{pdim}_R(I) = 2$ *and such that* $\operatorname{rank} Z_1(I) = 3$ *are the following:*

1) $\mathcal{H}_1 := \Big\{\{v_1, v_2, v_3, v_4, v_5\}, \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}, \{v_4, v_5\}\big\}\Big\}$,
a triangle + an isolated edge.
2) $\mathcal{H}_2 := \Big\{\{v_1, v_2, v_3, v_4, v_5, v_6\}, \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_5, v_6\}\big\}\Big\}$,
three consecutive edges + an isolated edge.

Proof. Looking at the minimal free resolutions of the edge ideals of $\mathcal{H}_1$ and $\mathcal{H}_2$ as in Proposition 4, it results that $\operatorname{rank} Z_1(I(\mathcal{H}_1))$ and $\operatorname{rank} Z_1(I(\mathcal{H}_2))$ are both equal to $3 = \operatorname{rank} \varphi$. □

Note that, by Proposition 7, the complexes $\mathcal{Z}(I(\mathcal{H}_1))$ and $\mathcal{Z}(I(\mathcal{H}_2))$ are acyclic.

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PARABOLIC CLASS NUMBER OF A SUBGROUP OF THE NORMALIZER OF $\Gamma_0(m)$

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Abstract. In this study, we introduce a subgroup of the normalizer of $\Gamma_0(m)$ and we calculate parabolic class number of this group.

2000 *Mathematics Subject Classification:* 20H10; 20H20

Keywords: normalizer, parabolic class number, Fuchsian group, orbit

1. INTRODUCTION

Let m be a positive integer and let $N(\Gamma_0(m))$ be the normalizer of $\Gamma_0(m)$ in $PSL(2,\mathbb{R})$. The normalizer $N(\Gamma_0(m))$ was studied for the first time by Newman in [8]. A complete description of the elements of $N(\Gamma_0(m))$ is given in [2]. Especially, a necessary and sufficient condition for $N(\Gamma_0(m))$ to act transitively on $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ was given in [1]. Moreover, the signatures and therefore the parabolic class numbers of some subgroups of $N(\Gamma_0(m))$ were calculated in [1], [5], and [6] . In this study, we introduce a subgroup of $N(\Gamma_0(m))$ and we calculate the parabolic class number of this subgroup when m is a square free positive integer.

2. PRELIMINARIES

Let Λ be a discrete subgroup of $PSL(2,\mathbb{R})$. When $x \in \mathbb{R} \cup \{\infty\}$ is a fixed point of a parabolic element of Λ, we say that x is a parabolic point of Λ. We also call a parabolic point of Λ, a cusp of Λ.

The proof the following theorem is easy and can be found in [7] or [10].

Theorem 1. *Let Λ be a discrete subgroup of $PSL(2,\mathbb{R})$ and assume that Λ^* is a subgroup of finite index in Λ. Let P be the set of the parabolic points of Λ. Then P is the set of the parabolic points of Λ^* and the parabolic class number of Λ^* is the number of orbits of Λ^* on P.*

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By a Fuchsian group Λ we will mean a finitely generated discrete subgroup of $PSL(2,\mathbb{R})$ the group of conformal homeomorphisms of the upper-half plane. The most general presentation for Λ is

Generators:

$$a_1, b_1, \cdots, a_g, b_g \ Hyperbolic$$

$$x_1, x_2, \cdots, x_r \ Elliptic$$

$$p_1, p_2, \cdots, p_s \ Parabolic$$

with the relations

$$x_1^{m_1} = x_2^{m_2} = ... = x_r^{m_r} = \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{r} x_j \prod_{k=1}^{s} p_k = 1.$$

We then say Λ has signature (see [4])

$$(g, m_1, m_2, ..., m_r; s).$$

s is the number of parabolic classes, i.e., of parabolic generators, and r is the number of elliptic generators. The m_i are the periods of Λ.

A complete description of the elements of $N(\ \Gamma_0(m))$ is given in [2]. If we represent the elements of $N(\ \Gamma_0(m))$ by the associated matrices, then the normalizer consists exactly of the matrices

$$\begin{pmatrix} ae & b/h \\ cm/h & de \end{pmatrix}$$

where $e \mid (m/h^2)$ such that $(e, m/h^2 e) = 1$ and h is the largest divisor of 24 for which $h^2 \mid m$ with the understanding that the determinant of the matrix is $e > 0$. If $e \mid n$ and $(e, n/e) = 1$, we represent this as $e \parallel n$ and we say that e is exact divisor of n. Thus we have

$$N(\Gamma_0(m)) = \left\{ \begin{pmatrix} a\sqrt{q} & b/h\sqrt{q} \\ cm/h\sqrt{q} & d\sqrt{q} \end{pmatrix} : \det = 1; 1 \le q; q \parallel \frac{m}{h^2}; a, b, c, d \in \mathbb{Z} \right\}.$$

The following theorem is given in [1] and [11].

Theorem 2. *Let m have prime power decomposition $2^{\alpha_1} 3^{\alpha_2} p_3^{\alpha_3} .... p_r^{\alpha_r}$. Then $N(\Gamma_0(m))$ acts transitively on $\hat{\mathbb{Q}}$ if and only if $\alpha_1 \le 7, \alpha_2 \le 3, \alpha_i \le 1, i = 3, 4, ..., r$.*

If m is a square free positive integer, then

$$N(\Gamma_0(m)) = \left\{ \begin{pmatrix} a\sqrt{q} & b/\sqrt{q} \\ cm/\sqrt{q} & d\sqrt{q} \end{pmatrix} : \det = 1; 1 \le q; q \parallel m; a, b, c, d \in \mathbb{Z} \right\}.$$

Let Γ be modular group. In [3], we calculated parabolic class number of the subgroup $\Gamma_0(n) \cap \Gamma^N$ such that

$$\Gamma^N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a \equiv d \equiv 0 \,(\mathrm{mod}\ N) \text{ or } b \equiv c \equiv 0 \,(\mathrm{mod}\ N) \right\}$$

and

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \,(\mathrm{mod}\ n) \right\},$$

where $(N,n) = 1$ and N is a power of a prime number. And it is shown that parabolic class number of this group is

$$N \sum_{d|n} \varphi\left(\left(d, \frac{n}{d}\right)\right).$$

Let $(m,n) = (m,N) = (n,N) = 1$ and let $\Gamma_0^*(n)$ be defined by

$$\Gamma_0^*(n) = \left\{ \begin{pmatrix} a\sqrt{q} & b/h\sqrt{q} \\ cm/h\sqrt{q} & d\sqrt{q} \end{pmatrix} \in N(\Gamma_0(m)) : c \equiv 0 (\mathrm{mod}\, n) \right\}$$

and let $\Gamma(m,N)$ be defined by

$$\Gamma(m,N) = \{A \in N(\Gamma_0(m)) : a \equiv d \equiv 0 (\mathrm{mod}\, N) \ or \ b \equiv c \equiv 0 (\mathrm{mod}\, N)\}.$$

Moreover, we define $\Gamma_0^*(m,n,N)$ by $\Gamma_0^*(m,n,N) = \Gamma(m,N) \cap \Gamma_0^*(n)$. It is clear that $\Gamma_0^*(m,n,1) = \Gamma_0^*(n)$. In this paper, we will show that the parabolic class number of $\Gamma_0^*(m,n,N)$ is

$$N \sum_{d|n} \varphi\left(\left(d, \frac{n}{d}\right)\right)$$

when $(m,n) = (m,N) = (n,N) = 1$ and N is a power of a prime number. In the next section, we give five lemmas, which are necessary for our main theorem.

3. Main Theorems

From now on, otherwise stated, we will assume that m is a square free positive integer and $(m,n) = (m,N) = (n,N) = 1$.

Lemma 1. $\left|N(\Gamma_0(m)) : \Gamma_0^*(m,n,N)\right| = \frac{|\Gamma_0(m):\Gamma_0(mnN)|}{2}$

Proof. Let $q|m$. Since m is square free, $(q, m/q) = 1$. Thus it follows that $(qN^2, (m/q)n) = 1$. Therefore there exist two integers x_1 and y_1 such that $qN^2x_1 - (mn/q)y_1 = 1$. Let

$$A_q = \begin{pmatrix} N\sqrt{q} & y_1/\sqrt{q} \\ mn/\sqrt{q} & Nx_1\sqrt{q} \end{pmatrix}.$$

Then $AA_q^{-1} \in \Gamma_0(mnN)$ for any

$$A = \begin{pmatrix} aN\sqrt{q} & b/\sqrt{q} \\ cmn/\sqrt{q} & dN\sqrt{q} \end{pmatrix} \in \Gamma_0^*(m,n,N).$$

Thus $A \in \Gamma_0(mnN)A_q$. On the other hand, since $(q, n(m/q)N^2) = 1$, there exist two integers x_2 and y_2 such that $qx_2 - n(m/q)N^2 y_2 = 1$. Let

$$B_q = \begin{pmatrix} x_2\sqrt{q} & y_2N/\sqrt{q} \\ mnN/\sqrt{q} & \sqrt{q} \end{pmatrix}.$$

Then we see that $BB_q^{-1} \in \Gamma_0(mnN)$ for any

$$B = \begin{pmatrix} a\sqrt{q} & bN/\sqrt{q} \\ cmnN/\sqrt{q} & d\sqrt{q} \end{pmatrix} \in \Gamma_0^*(m,n,N).$$

Thus it follows that $B \in \Gamma_0(mnN)B_q$. It can be seen that $\Gamma_0(mnN)A_{q_1}$ $\neq \Gamma_0(mnN)A_{q_2}$ and $\Gamma_0(mnN)B_{q_1} \neq \Gamma_0(mnN)B_{q_2}$ for $q_1 \neq q_2$. Moreover,

$$\Gamma_0(mnN)A_{q_1} \neq \Gamma_0(mnN)B_{q_2}$$

for $q_1|m$ and $q_2|m$. Thus it follows that $|\Gamma_0^*(m,n,N) : \Gamma_0(mnN)| < \infty$ and it can be shown that $|\Gamma_0^*(m,n,N) : \Gamma_0(mnN)| = 2^{t+1}$ where t is the number of the prime divisor of m. Also we have

$$\begin{aligned} |N(\Gamma_0(m)) : \Gamma_0^*(m,n,N)| &= \frac{|N(\Gamma_0(m)) : \Gamma_0(mnN)|}{|\Gamma_0^*(m,n,N) : \Gamma_0(mnN)|} \\ &= \frac{|N(\Gamma_0(m)) : \Gamma_0(m)||\Gamma_0(m) : \Gamma_0(mnN)|}{2^{t+1}} \end{aligned}$$

Since $|N(\Gamma_0(m)) : \Gamma_0(m)| = 2^t$, the proof follows. □

In view of the above lemma, it is seen that $\Gamma_0^*(m,n,N)$ is finitely generated and therefore it is a Fuchsian group.

Lemma 2. *Let $N = p^r$ where p is a prime number and let $k/s \in \hat{\mathbb{Q}}$ with $p|s$. Then there exist some $T \in \Gamma_0^*(m,n,N)$ such that $T(k/s) = k_1/s_1$ with $(s_1,N) = 1$.*

Proof. Since $(m,N) = (n,N) = 1$, we see that $(mn, N^2) = 1$ and therefore there exist two integers x and y such that $N^2x - mny = 1$. Let

$$T = \begin{pmatrix} xN & y \\ mn & N \end{pmatrix}.$$

Then $T \in \Gamma_0^*(m,n,N)$ and $T(k/s) = \frac{xNk+ys}{mnk+sN} = k_1/s_1$ with $s_1 = mnk + sN$. It can be seen easily that $(s_1, N) = 1$. □

Lemma 3. *Let $(k,s) = (s,N) = 1$. Then there exist some $T \in \Gamma_0^*(m,n,N)$ such that $T(k/s) = k_1/s_1$ with $s_1|n$.*

Proof. Let $s_1 = (s,n)$ and $q = (s,m)$. Then $s_1 = (s,kn) = (s,knN)$. Moreover $(s,m/q) = 1$, since m is square free. Therefore $(s,knN\frac{m}{q}) = s_1$ and thus there exist two integers c_1 and d_1 such that $sd_1 + kn\frac{m}{q}Nc_1 = s_1$. Since $(d_1,(knm/s_1q)N) = 1$, there exist some integer k_0 such that

$$(d_1 - \frac{knm}{s_1q}Nk_0, (m/q)nN^2) = 1.$$

Let

$$d = d_1 - \frac{knm}{s_1q}Nk_0$$

and

$$c = c_1 + \frac{s}{s_1}k_0.$$

Then $kn(m/q)Nc + sd = s_1$. On the other hand, $(dq, cN^2(m/q)n) = 1$ since $(d,c) = (d,(m/q)nN^2) = 1$. It can be seen easily that $(q,c) = 1$. Therefore $(dq, c(m/q)nN^2) = 1$. Thus there exist two integers x and y such that $xdq - c(m/q)nN^2y = 1$. Let

$$T = \begin{pmatrix} x\sqrt{q} & yN/\sqrt{q} \\ cmnN/\sqrt{q} & d\sqrt{q} \end{pmatrix}.$$

Then $T \in \Gamma_0^*(m,n,N)$ and

$$T(k/s) = \frac{qxk + sNy}{cknmN + dsq} = \frac{qxk + sNy}{q(ckn(m/q)N + ds)} = \frac{xk + N(s/q)y}{(ckn(m/q)N + ds)} = \frac{k_1}{s_1}.$$

□

Lemma 4. *Let $(a_1,d_1) = (a_2,d_1) = 1$ and $d_1|n$. Then a_1/d_1 is conjugate to a_2/d_1 under $\Gamma_0^*(m,n,N)$ if and only if $a_1 \equiv a_2 \pmod{Nt}$ where $t = (d_1, n/d_1)$.*

Proof. Suppose that a_1/d_1 is conjugate to a_2/d_1 under $\Gamma_0^*(m,n,N)$. Then there exist some $T \in \Gamma_0^*(m,n,N)$ such that $T(a_1/d_1) = a_2/d_1$. Assume that

$$T = \begin{pmatrix} a\sqrt{q} & bN/\sqrt{q} \\ cmnN/\sqrt{q} & d\sqrt{q} \end{pmatrix},$$

where $q|m$. Then we have

$$\frac{aa_1q + bNd_1}{cnmNa_1 + dd_1q} = \frac{a_2}{d_1}$$

and

$$adq - bcn(m/q)N^2 = 1.$$

Since

$$a(cmnNa_1 + dd_1q) - \frac{cmnN}{q}(aa_1q + bd_1N) = d_1,$$

$$dq(aa_1q + bd_1N) - bN(cmnNa_1 + dd_1q) = a_1q,$$

and $(d_1, a_1q) = 1$, we see that $(aa_1q + bd_1N, cmnNa_1 + dd_1q) = 1$. Therefore there exist some $u = \pm 1$ such that

$$ua_2 = aa_1q + bNd_1$$

and

$$ud_1 = cmnNa_1 + dd_1q.$$

Since $q(cn(m/q)Na_1 + dd_1) = ud_1$ and $(q, d_1) = 1$, it follows that $q = 1$. Thus

$$ua_2 = aa_1 + bNd_1,$$

$$ud_1 = cmnNa_1 + dd_1,$$

and

$$ad - bcnmN^2 = 1$$

From the above equations, we see that $ua_2 \equiv aa_1 (\mathrm{mod}\, tN)$, $u \equiv d (\mathrm{mod}\, tN)$, and $ad \equiv 1 (\mathrm{mod}\, tN)$. Therefore $a_1 \equiv u^2 a_2 (\mathrm{mod}\, tN)$, which implies that $a_1 \equiv a_2 (\mathrm{mod}\, tN)$. Now assume that

$$T = \begin{pmatrix} aN\sqrt{q} & b/\sqrt{q} \\ cmn/\sqrt{q} & dN\sqrt{q} \end{pmatrix},$$

where $adN^2q - bcn(m/q) = 1$. Then we obtain

$$\frac{aa_1Nq + bd_1}{cnma_1 + dd_1qN} = \frac{a_2}{d_1}.$$

Similarly, it is seen that $(cnma_1 + dd_1qN, aa_1Nq + bd_1) = 1$. Thus there exist some $u = \pm 1$ such that

$$ua_2 = aa_1Nq + bd_1$$

and

$$ud_1 = cmna_1 + dd_1qN.$$

Since $ud_1 = q(cn(m/q)a_1 + dd_1N)$ and $(q, d_1) = 1$, it follows that $q = 1$. Thus

$$ua_2 = aa_1N + bd_1,$$

$$ud_1 = cmna_1 + dd_1N,$$

and

$$adN^2 - bcnm = 1.$$

From the above equations, we get

$$ua_2 \equiv bd_1 (\mathrm{mod}\, N),$$

$$ud_1 \equiv cmna_1 (\mathrm{mod}\, N),$$

and

$$bcmn \equiv 1 (\mathrm{mod}\, N).$$

This implies that $a_1 \equiv a_2 (\mathrm{mod}\, N)$. In the same way we see that

$$ua_2 \equiv aa_1N (\mathrm{mod}\, t),$$

$$u \equiv dN (\mathrm{mod}\, t),$$

and

$$adN^2 \equiv 1 (\mathrm{mod}\, t).$$

Thus we have $a_1 \equiv a_2 (\mathrm{mod}\, t)$. Since $(t, N) = 1$, it follows that $a_1 \equiv a_2 (\mathrm{mod}\, tN)$.

Now suppose that $a_1 \equiv a_2 (\mathrm{mod}\, tN)$, where $t = (d_1, n/d_1)$. Let $n_1 = n/d_1$. Then $t = (d_1, n_1)$ and $(a_1 a_2, d_1) = 1$. Therefore $t = (a_1 a_2 n_1, d_1)$. This shows that $tN = (d_1 N, a_1 a_2 n_1 N)$. Since $tN | a_2 - a_1$, there exist two integers x and y such that $n_1 a_1 a_2 N x + d_1 N y = a_2 - a_1$. Thus we obtain $a_1(n_1 a_2 N x + 1) + d_1 N y = a_2$. If we take $b = y$ and $a = n_1 a_2 N x + 1$, we have $aa_1 + d_1 N b = a_2$. Let $c = n_1 d_1 x = nx$ and $d = 1 - N n_1 a_1 x$. Then $cNa_1 + dd_1 = d_1$. On the other hand, we see that

$$\begin{aligned} ad - bcN^2 &= a(1 - Nn_1 a_1 x) - bn_1 d_1 x N^2 \\ &= a - (aa_1 + d_1 N b) N n_1 x = a - n_1 a_2 N x = 1. \end{aligned}$$

Let

$$T(z) = \frac{az + bN}{cmNz + d}.$$

Then $n | c$ and therefore $T \in \Gamma_0^*(m, n, N)$. Moreover, we get

$$T(a_1/d_1) = \frac{aa_1 + bd_1 N}{cNa_1 + dd_1} = a_2/d_1.$$

□

The proof of the following lemma is easy and will be omitted.

Lemma 5. *Let $(a_1, d_1) = (a_2, d_1) = 1$, $d_1 | n$, $d_2 | n$, where d_1 and d_2 are positive integers. If a_1/d_1 is conjugate to a_2/d_2 under $\Gamma_0^*(m, n, N)$, then $d_1 = d_2$.*

In the following, we give a lemma without proof, which we use later. The lemma appears in [9], page 73, as problem.

Lemma 6. *Let m and k be positive integers. Then the number of the positive integers $\leq mk$ that are prime to m is $k\varphi(m)$.*

Theorem 3. *Let p be a prime number and $N = p^r$. Then the parabolic class number of $\Gamma_0^*(m, n, N)$ is*

$$N \sum_{d | n} \varphi\left(\left(d, \frac{n}{d}\right)\right)$$

Proof. It suffices to calculate the number of orbits of $\Gamma_0^*(m, n, N)$ on $\hat{\mathbb{Q}}$. By Lemma 4 and Lemma 6, there exist $N\varphi((d, n/d))$ different orbits for each $d \mid n$. Then from Lemma 2 to Lemma 5, it follows that the number of orbits of $\Gamma_0^*(m, n, N)$ on $\hat{\mathbb{Q}}$ is

$$\sum_{d | n} N\varphi\left(\left(d, \frac{n}{d}\right)\right) = N \sum_{d | n} \varphi\left(\left(d, \frac{n}{d}\right)\right)$$

□

Corollary 1. *Let p be a prime number. Then the parabolic class number of $\Gamma(m,p^k)$ is p^k.*

Corollary 2. *The parabolic class number of $\Gamma_0^*(n)$ is*

$$\sum_{d|n}\varphi\left(\left(d,\frac{n}{d}\right)\right).$$

Let r be a natural number and let

$$\Gamma_0^+(r)=\left\{\begin{pmatrix} a\sqrt{q} & b/\sqrt{q} \\ cr/\sqrt{q} & d\sqrt{q}\end{pmatrix} : adq-bcr/q=1; 1\le q;\ q\ \|\ r; a,b,c,d\in\mathbb{Z}\right\}.$$

Then $\Gamma_0^+(r)$ is a subgroup of the normalizer of $\Gamma_0(r)$. Moreover, any element of $\Gamma_0^+(r)$ is an Atkin-Lehner involution of $\Gamma_0(r)$. Recall that an Atkin-Lehner involution w_q of $\Gamma_0(r)$ is an element of determinant 1 of the form

$$w_q=\begin{pmatrix} a\sqrt{q} & b/\sqrt{q} \\ cr/\sqrt{q} & d\sqrt{q}\end{pmatrix}$$

for some exact divisor q of r. If $h=1$, then $\Gamma_0^+(r)$ is equal to $N(\Gamma_0(r))$, where h is the largest divisor of 24 such that $h^2|r$. The following lemmas are well known and their proofs can be found in [11].

Lemma 7. *$\Gamma_0^+(r)$ acts transitively on the set $\hat{\mathbb{Q}}=\mathbb{Q}\cup\{\infty\}$ if and only if r is a square-free positive integer.*

Lemma 8. *Let $r=m/h^2$. Then $N(\Gamma_0(m))$ acts transitively on the set $\hat{\mathbb{Q}}$ if and only if $\Gamma_0^+(r)$ acts transitively on the set $\hat{\mathbb{Q}}$.*

Let $r=m/h^2$. Then, it is well-known and easy to see that

$$N(\Gamma_0(m))=\begin{pmatrix} 1/\sqrt{h} & 0 \\ 0 & \sqrt{h}\end{pmatrix}\Gamma_0^+(r)\begin{pmatrix} 1/\sqrt{h} & 0 \\ 0 & \sqrt{h}\end{pmatrix}^{-1}.$$

Assume that $N(\Gamma_0(m))$ acts transitively on the set $\hat{\mathbb{Q}}$. Taking $r=m/h^2$ and

$$X=\begin{pmatrix} 1/\sqrt{h} & 0 \\ 0 & \sqrt{h}\end{pmatrix},$$

it is immediate that XHX^{-1} is equal to $\Gamma_0^*(m,n,N)$, provided that $H=\Gamma_0^*(r,m,N)$. Therefore, we can give the following corollary easily.

Corollary 3. *Suppose that $N(\Gamma_0(m))$ acts transitively on the set $\hat{\mathbb{Q}}$ and $N=p^k$ for some prime p. Then parabolic class number of $\Gamma_0^*(m,n,N)$ is*

$$N\sum_{d|n}\varphi\left(\left(d,\frac{n}{d}\right)\right).$$

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ON THE DIOPHANTINE EQUATION $x^2 - kxy + y^2 + 2^n = 0$

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Abstract. In this paper, we determine when the equation in the title has an infinite number of positive integer solutions x and y when $0 \leq n \leq 10$. Moreover, we give all the positive integer solutions of the same equation for $0 \leq n \leq 10$.

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1. INTRODUCTION

In [14], Yuan and Hu determined when the two equations

$$x^2 - kxy + y^2 + 2x = 0 \tag{1.1}$$

and

$$x^2 - kxy + y^2 + 4x = 0 \tag{1.2}$$

have an infinite number of positive integer solutions x and y. They showed that Eq.(1.1) has an infinite number of positive integer solutions x and y if and only if $k = 3, 4$ and Eq.(1.2) has an infinite number of positive integer solutions x and y if and only if $k = 3, 4, 6$. In the present paper, we consider the equation

$$x^2 - kxy + y^2 + 2^r x = 0, \tag{1.3}$$

where k is a positive integer and r is a nonnegative integer. Eq.(1.3) is a generalization of Eq.(1.1) and Eq.(1.2). In order to decide when Eq.(1.3) has an infinite number of positive integer solutions x and y, it is sufficient to determine when the equation

$$x^2 - kxy + y^2 + 2^n = 0 \tag{1.4}$$

has an infinite number of positive integer solutions x and y for nonnegative integer n. Let us assume that Eq.(1.3) has positive integer solutions x and y. Then it follows that $x | y^2$ and thus $y^2 = xz$ for some positive integer z. A simple computation shows that $\gcd(x, z) = 2^j$ for some nonnegative integer j. Thus $x = 2^j a^2$ and $z = 2^j b^2$

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for some positive integers a and b with $(a,b)=1$. Then it follows that $y=2^j ab$. Substituting these values of x and y into Eq.(1.3), we obtain

$$a^2-kab+b^2+2^{r-j}=0.$$

Therefore it is sufficient to know when $x^2-kxy+y^2+2^{r-j}=0$ has an infinite number of positive integer solutions for $0\le j\le r$.

Now we begin with some well known elementary properties about Pell equations. Let d be a positive integer which is not a perfect square and N be any nonzero fixed integer. Then the equation $x^2-dy^2=N$ is known as the Pell equation. For $N=\pm 1$, the equation $x^2-dy^2=\pm 1$ is known as the classical Pell equation. We use the notations (x,y), and $x+y\sqrt{d}$ interchangeably to denote solutions of the equation $x^2-dy^2=N$. Also, if x and y are both positive, we say that $x+y\sqrt{d}$ is positive solution to the equation $x^2-dy^2=N$. It is well known that the equation $x^2-dy^2=1$ always has a positive solution when $d\ge 2$. The least positive integer solution $x_1+y_1\sqrt{d}$ of the equation $x^2-dy^2=N$ is called the fundamental solution. If $x_1+y_1\sqrt{d}$ is the fundamental solution of the equation $x^2-dy^2=-1$, it is well known that $(x_1+y_1\sqrt{d})^2$ is the fundamental solution to the equation $x^2-dy^2=1$. Moreover, if $x_1+y_1\sqrt{d}$ is the fundamental solution to the equation $x^2-dy^2=1$, then all positive integer solutions to the equation $x^2-dy^2=1$ are given by

$$(x_n+y_n\sqrt{d})=(x_1+y_1\sqrt{d})^n \tag{1.5}$$

with $n\ge 1$. It can be seen that $x_n=(\alpha^n+\beta^n)/2$ and $y_n=(\alpha^n-\beta^n)/2\sqrt{d}$, where $\alpha=x_1+y_1\sqrt{d}$ and $\beta=x_1-y_1\sqrt{d}$. If $x+y\sqrt{d}$ is a solution of the equation $x^2-dy^2=N$ and $a+b\sqrt{d}$ is a solution of the equation $x^2-dy^2=1$, then $(a+b\sqrt{d})(x+y\sqrt{d})=(ax+dby)+(ay+bx)\sqrt{d}$ is also a solution of the equation $x^2-dy^2=N$. This means that if the equation $x^2-dy^2=N$ has a solution, then it has infinitely many solutions. For more information, see [10], [13], and [2].

In section 2, we determine when Eq.(1.4) has an infinite number of positive integer solutions x and y for $0\le n\le 10$. Then in section 3, we give all positive integer solutions to Eq.(1.4) for $0\le n\le 10$.

2. Main Theorems

In this section, we determine when Eq.(1.4) has an infinite number of positive integer solutions x and y for $0\le n\le 10$. Before discussing this, we give the following lemma and theorem, which will be needed in the proof of the main theorems.

Lemma 1. *Let $d>2$. If $u_1+v_1\sqrt{d}$ is the fundamental solution of the equation $u^2-dv^2=\pm 2$, then $(u_1^2+dv_1^2)/2+u_1v_1\sqrt{d}$ is the fundamental solution of the equation $x^2-dy^2=1$.*

Proof. Assume that $d>2$ and $\omega=u_1+v_1\sqrt{d}$ is the fundamental solution of the equation $u^2-dv^2=\pm 2$. Our purpose is to show that $\alpha=\omega^2/2=(u_1^2+dv_1^2)/2+$

$u_1v_1\sqrt{d}$ is the fundamental solution of the equation $x^2-dy^2=1$. On the contrary, assume that α is not the fundamental solution of the equation $x^2-dy^2=1$. Then there exists a fundamental solution $\beta=x_1+y_1\sqrt{d}$ of the equation $x^2-dy^2=1$ such that $\alpha=\beta^n$ with $n>1$. Assume that n is an even integer. Then $n=2k$ for some positive integer k. By using $\alpha=\omega^2/2$, we obtain $\omega^2=2\beta^{2k}$, i.e., $(\omega/\beta^k)^2=2$. If we write $\omega/\beta^k=a+b\sqrt{d}$, then it follows that $(\omega/\beta^k)^2=a^2+b^2d+2ab\sqrt{d}=2$. Thus $ab=0$. This shows that $a=0$ or $b=0$. If $a=0$, then $b^2d=2$, which implies that $d=2$. This contradicts with the fact that $d>2$. If $b=0$, then $a^2=2$, which is impossible. Now assume that n is an odd integer. Then $n=2t+1$ for some positive integer t. Thus it follows that $\omega^2=2\alpha=2\beta^{2t+1}$, i.e., $(\omega/\beta^t)^2=2\beta$. It is obvious that $\beta^t>1$. Writing $\omega/\beta^t=a+b\sqrt{d}$ gives $(\omega/\beta^t)^2=a^2+b^2d+2ab\sqrt{d}=2\beta=2x_1+2y_1\sqrt{d}$. Since β is the fundamental solution of the equation $x^2-dy^2=1$, it follows that $ab>0$. Assume that $a>0$ and $b>0$. Since ω is the fundamental solution of the equation $u^2-dv^2=\pm2$ and ω/β^t is a positive solution of the same equation, we get $\omega\leq\omega/\beta^t$, which implies that $\beta^t\leq1$. This is impossible since $\beta^t>1$. Assume that $a<0$ and $b<0$. Then $\omega/\beta^t=-(e+f\sqrt{d})$ for some positive integers e and f. This shows that $-\omega/\beta^t=e+f\sqrt{d}$ is a positive solution of the equation $u^2-dv^2=\pm2$. Since ω is the fundamental solution of the same equation, we get $\omega\leq-\omega/\beta^t$. From here, we find that $\beta^t\leq-1$, a contradiction. □

The following theorem is given in [10].

Theorem 1. *Let d be a positive integer which is not a perfect square. If x_1 and y_1 are natural numbers satisfying the inequality*

$$x_1>\frac{y_1^2}{2}-1$$

and if $\alpha=x_1+y_1\sqrt{d}$ is a solution of the equation $x^2-dy^2=1$, then α is the fundamental solution of this equation.

The proof of the following theorem is given in [6], and [7].

Theorem 2. *Let $k>3$. Then the equation $x^2-kxy+y^2+1=0$ has no positive integer solutions.*

Corollary 1. *The equation $x^2-kxy+y^2+1=0$ has an infinite number of positive integer solutions x and y if and only if $k=3$.*

Proof. By the above theorem, $x^2-kxy+y^2+1=0$ has no positive integer solutions when $k>3$. It is clear that the equation $x^2-kxy+y^2+1=0$ has no positive integer solutions x and y for $k=1,2$. For $k=3$, $x^2-3xy+y^2+1=0$ has an infinite number of positive integer solutions $(x,y)=(F_{2n+1},F_{2n-1})$ with $n\geq0$, where F_n is the n-th Fibonacci number defined in section 3 (see [6], Theorem 1.6). □

Theorem 3. *The equation $x^2-kxy+y^2+2=0$ has an infinite number of positive integer solutions x and y if and only if $k=4$.*

Proof. Assume that $x^2-kxy+y^2+2=0$ for some positive integers x and y. It is clear that x and y must be odd integers. Then it follows that k is even. Let $k=2t$ for some positive integer t. Then $x^2-kxy+y^2+2=0$ implies that $(x-ty)^2-(t^2-1)y^2=-2$. Let $u_1+v_1\sqrt{t^2-1}$ be the fundamental solution of the equation $u^2-\sqrt{t^2-1}v^2=-2$. Then from Lemma 1, it follows that $(u_1^2+(t^2-1)v_1^2)/2+u_1v_1\sqrt{t^2-1}$ is the fundamental solution of the equation $x^2-(t^2-1)y^2=1$. For $t>1$, since $(t,1)$ is the fundamental solution of the equation $x^2-(t^2-1)y^2=1$ by Theorem 1, we get $(u_1^2+(t^2-1)v_1^2)/2=t$ and $u_1v_1=1$. From this, it follows that $t=2$ and thus $k=4$. □

Theorem 4. *The equation $x^2-kxy+y^2+4=0$ has an infinite number of positive integer solutions x and y if and only if $k=3,6$.*

Proof. Assume that $x^2-kxy+y^2+4=0$ for some positive integers x and y. Assume that x is even. Then y is even and thus $x=2a$ and $y=2b$ for some positive integers a and b. Then it follows that $a^2-kab+b^2+1=0$, which implies that $k=3$ by Corollary 1. Now assume that x and y are odd integers. Then k is even and $4\nmid k$. Therefore $k=2t$ for some odd positive integer t. Completing the square gives $(x-ty)^2-(t^2-1)y^2=-4$. Since $8|t^2-1$, it follows that $x-ty=2m$ and thus $m^2-((t^2-1)/4)y^2=-1$. Let $d=(t^2-1)/4$ and assume that $u_1+v_1\sqrt{d}$ is the fundamental solution of the equation $u^2-dv^2=-1$. Then $(u_1+v_1\sqrt{d})^2=u_1^2+dv_1^2+2u_1v_1\sqrt{d}$ is the fundamental solution of the equation $x^2-dy^2=1$. For $t>1$, since $(t,2)$ is the fundamental solution of the equation $x^2-dy^2=1$ by Theorem 1, we get $u_1^2+dv_1^2+2u_1v_1\sqrt{d}=t+2\sqrt{d}$. Then it follows that $u_1v_1=1$ and $u_1^2+((t^2-1)/4)v_1^2=t$. From this, we see that $t=3$ and thus $k=6$. □

Theorem 5. *The equation $x^2-kxy+y^2+8=0$ has an infinite number of positive integer solutions x and y if and only if $k=4,6,10$.*

Proof. Assume that x is even. Then y is even and thus $x=2a$ and $y=2b$ for some positive integers a and b. Thus we get $a^2-kab+b^2+2=0$. By Theorem 3, it follows that $k=4$. Now assume that x and y are odd positive integers. Then k is even and $4\nmid k$. Thus $k=2t$ for some odd positive integer t. Completing the square gives $(x-ty)^2-(t^2-1)y^2=-8$, which implies that $x-ty=2m$ for some positive integer m. Thus we get $m^2-((t^2-1)/4)y^2=-2$. If $t=3$, then we get $m^2-2y^2=-2$. Since $4+3\sqrt{2}$ is a solution of the equation $m^2-2y^2=-2$, this equation has infinitely many solutions. Thus we get $k=6$. Let $d=(t^2-1)/4$ and assume that $t>3$. If $u_1+v_1\sqrt{d}$ is the fundamental solution of the equation $u^2-dv^2=-2$, then by Lemma 1, $(u_1^2+dv_1^2)/2+u_1v_1\sqrt{d}$ is the fundamental solution of the equation $x^2-dy^2=1$. For $t>1$, since $(t,2)$ is the fundamental solution of the equation

$x^2-dy^2=1$ by Theorem 1, we get $(u_1^2+dv_1^2)/2+u_1v_1\sqrt{d}=t+2\sqrt{d}$. From this, it follows that $u_1v_1=2$ and $u_1^2+((t^2-1)/4)v_1^2=2t$. Solving these equations, we see that $t=5$ and thus we get $k=10$. □

The proofs of the following theorems are similar to that of the above theorems and therefore we omit their proofs.

Theorem 6. *The equation $x^2-kxy+y^2+16=0$ has an infinite number of positive integer solutions x and y if and only if $k=3,6,18$.*

Theorem 7. *The equation $x^2-kxy+y^2+32=0$ has an infinite number of positive integer solutions x and y if and only if $k=4,6,10,14,34$.*

Now, we consider the equation

$$x^2-dy^2=N, \tag{2.1}$$

where $N\neq 0$ and d is a positive integer which is not a perfect square. If $u^2-dv^2=N$, then we say that $\alpha=u+v\sqrt{d}$ is a solution to Eq.(2.1). Let α_1 and α_2 be any two solutions to Eq.(2.1). Then α_1 and α_2 are called associated solutions if there exists a solution α to $x^2-dy^2=1$ such that

$$\alpha_1=\alpha\alpha_2.$$

The set of all solutions associated with each other forms a class of solutions to Eq.(2.1). If K is a class, then $\overline{K}=\left\{u-v\sqrt{d}\mid u+v\sqrt{d}\in K\right\}$ is also a class. We say that the class is ambiguous if $K=\overline{K}$.

Now we give the following definitions from [1].

Definition 1. Assume that $N<0$ or $N=1$. Let $u_0+v_0\sqrt{d}$ be a solution to Eq.(2.1) given in a class K such that v_0 is the least positive value of v which occurs in K. If K is not ambiguous then the number u_0 is uniquely determined. If K is ambiguous we get a uniquely determined u_0 by prescribing that $u_0\geq 0$.

Now we can give the following theorem from [10].

Theorem 8. *Let $N<0$ and $x_1+y_1\sqrt{d}$ be the fundamental solution to $x^2-dy^2=1$. If $u_0+v_0\sqrt{d}$ is the fundamental solution to the equation $u^2-dv^2=N$ in its class, then*

$$0<v_0\leq\frac{y_1\sqrt{-N}}{\sqrt{2(x_1-1)}} \text{ and } 0\leq|u_0|\leq\sqrt{\frac{-1}{2}(x_1-1)N}$$

Now we can give the following theorems.

Theorem 9. *The equation $x^2-kxy+y^2+64=0$ has an infinite number of positive integer solutions x and y if and only if $k=3,6,18,66$.*

Proof. Assume that $x^2-kxy+y^2+64=0$ for some positive integers x and y. If x is even, then y is even and thus $x=2a$ and $y=2b$ for some positive integers a and b. Substituting these values of x and y into the equation $x^2-kxy+y^2+64=0$, we get $a^2-kab+b^2+16=0$, which implies that $k=3,6,18$ by Theorem 6. Now assume that x and y are odd integers. Then k is even and $4\nmid k$. Thus $k=2t$ for some positive odd integer t. Completing the square gives $(x-ty)^2-(t^2-1)y^2=-64$. Since $8|t^2-1$, it follows that $x-ty=2n$ and $t^2-1=8s$ for some positive integers n and s. So we get $n^2-2sy^2=-16$. It is seen that n is even. Then $n=2m$ and thus $2m^2-sy^2=-8$. Since y is odd, it is seen that s is even and thus we get $m^2-((t^2-1)/16)y^2=-4$. Now we consider the equation

$$u^2-\left(\frac{t^2-1}{16}\right)v^2=-4. \tag{2.2}$$

Let $u_0+v_0\sqrt{d}$ be the fundamental solution to Eq.(2.2) in a given class K. If (m,y) is a solution in the class K, then it is seen that v_0 is odd. Since $(t,4)$ is the fundamental solution to the equation $x^2-((t^2-1)/16)y^2=1$ for $t>7$ by Theorem 1, we get

$$0<v_0\leq\frac{4\sqrt{4}}{\sqrt{2(t-1)}}\leq\frac{4\sqrt{4}}{\sqrt{2(9-1)}}=2$$

by Theorem 8. Since v_0 is odd, $v_0=1$. If we substitute the value of v_0 into Eq.(2.2), we get $(t-4u)(t+4u)=65$. First assume that $t-4u=1$ and $t+4u=65$. Then we get $t=33$ and thus $k=66$. In a similar way, if $t-4u=5$ and $t+4u=13$, then we get $t=9$ and thus $k=18$. Now assume that $1<t\leq 7$. Since $(t^2-1)/16$ is not an integer for $1<t<7$, t must be 7. But if we substitute the value of t into Eq.(2.2), we get $u^2-3v^2=-4$, which has no positive integer solutions u and v. This completes the proof. □

Theorem 10. *The equation $x^2-kxy+y^2+128=0$ has an infinite number of positive integer solutions x and y if and only if $k=4,6,10,14,34,46,130$.*

Proof. Assume that $x^2-kxy+y^2+128=0$ for some positive integers x and y. If x is even, then y is even and thus $x=2a$ and $y=2b$ for some positive integers a and b. Substituting these values of x and y into the equation $x^2-kxy+y^2+128=0$, we get $a^2-kab+b^2+32=0$, which implies that $k=4,6,10,14,34$ by Theorem 7. Now assume that x and y are odd integers. Then k is even and $4\nmid k$. Thus $k=2t$ for some positive odd integer t. Completing the square gives $(x-ty)^2-(t^2-1)y^2=-128$. Since $8|t^2-1$, it is seen that $x-ty=4m$ and thus we get $m^2-((t^2-1)/16)y^2=-8$. Now we consider the equation

$$u^2-\left(\frac{t^2-1}{16}\right)v^2=-8. \tag{2.3}$$

Let $u_0+v_0\sqrt{d}$ be the fundamental solution to Eq.(2.3) in a given class K. If (m,y) is a solution in the class K, then it is seen that v_0 is odd. Since $(t,4)$ is the fundamental

solution to the equation $x^2-((t^2-1)/16)y^2=1$ for $t>7$ by Theorem 1, we get

$$0<v_0\leq\frac{4\sqrt{8}}{\sqrt{2(t-1)}}\leq\frac{4\sqrt{8}}{\sqrt{2(9-1)}}<3$$

by Theorem 8. Since v_0 is odd, $v_0=1$. Substituting this value of v_0 into Eq.(2.3), we get $(t-4u)(t+4u)=129$. A simple computation shows that $t=23$ and $t=65$. Thus we get $k=46$ and $k=130$. Now assume that $1<t\leq 7$. Since $(t^2-1)/16$ is not an integer for $1<t<7$, t must be 7. If we substitute the value of t into Eq.(2.3), we get $u^2-3v^2=-8$, which has no positive integer solutions u and v. This completes the proof. □

Since the proofs of the following theorems are similar to that of above theorems, we omit them.

Theorem 11. *The equation $x^2-kxy+y^2+256=0$ has an infinite number of positive integer solutions x and y if and only if $k=3,6,18,66,258$.*

Theorem 12. *The equation $x^2-kxy+y^2+512=0$ has an infinite number of positive integer solutions x and y if and only if $k=4,6,10,14,34,46,66,130,174,514$.*

Theorem 13. *The equation $x^2-kxy+y^2+1024=0$ has an infinite number of positive integer solutions x and y if and only if $k=3,6,18,66,210,258,1026$.*

Assume that x and y are solutions of Eq.(1.4), where $1\leq n\leq 10$. Then it can be shown that x and y have same parity. The equation $x^2-66xy+y^2+1024=0$ has positive integer solutions $(4,4)$ and $(41,1)$. The equation $x^2-46xy+y^2+512=0$ has positive integer solutions $(6,2)$ and $(19,1)$. Morever, the equation $x^2-18xy+y^2+64=0$ has positive integer solutions $(2,2)$ and $(5,1)$. It is seen from the proofs of the above theorems that all x and y solutions of Eq.(1.4) are either odd or even for $(k,n)\notin\{(66,10),(46,9),(18,6)\}$.

3. Solutions of some of the equations $x^2-kxy+y^2+2^n=0$

In this section, we will give solutions of the equation $x^2-kxy+y^2+2^n=0$ for $0\leq n\leq 10$. Solutions of the equation $x^2-kxy+y^2+2^n=0$ are related to the generalized Fibonacci and Lucas numbers. Now we briefly mention the generalized Fibonacci and Lucas sequences $(U_n(k,s))$ and $(V_n(k,s))$. Let k and s be two integers with $k^2+4s>0$. Generalized Fibonacci sequence is defined by $U_0(k,s)=0$, $U_1(k,s)=1$ and $U_{n+1}(k,s)=kU_n(k,s)+sU_{n-1}(k,s)$ for $n\geqslant 1$ and generalized Lucas sequence is defined by $V_0(k,s)=2$, $V_1(k,s)=k$ and $V_{n+1}(k,s)=kV_n(k,s)+sV_{n-1}(k,s)$ for $n\geqslant 1$, respectively. For negative subscript, U_{-n} and V_{-n} are defined by

$$U_{-n}(k,s)=\frac{-U_n(k,s)}{(-s)^n}\text{ and }V_{-n}(k,s)=\frac{V_n(k,s)}{(-s)^n}\tag{3.1}$$

for $n \geqslant 1$. We will use U_n and V_n instead of $U_n(k,1)$ and $V_n(k,1)$, respectively. For $s=-1$, we represent (U_n) and (V_n) by $(u_n)=(U_n(k,-1))$ and $(v_n)=(V_n(k,-1))$ or briefly by (u_n) and (v_n) respectively. Also, it is seen from Eq.(3.1) that

$$u_{-n} = -U_n(k,-1) \text{ and } v_{-n} = V_n(k,-1)$$

for all $n \in \mathbb{Z}$. For $k=s=1$, the sequences (U_n) and (V_n) are called Fibonacci and Lucas sequences and they are denoted as (F_n) and (L_n), respectively. For $k=2$ and $s=1$, the sequences (U_n) and (V_n) are called Pell and Pell Lucas sequences and they are denoted as (P_n) and (Q_n), respectively. Let α and β are the roots of the equation $x^2-kx-s=0$. Then it is well known that

$$U_n = \frac{\alpha^n-\beta^n}{\alpha-\beta} \text{ and } V_n = \alpha^n+\beta^n \tag{3.2}$$

where $\alpha=(k+\sqrt{k^2+4s})/2$ and $\beta=(k-\sqrt{k^2+4s})/2$. The above identities are known as Binet's formulae. Clearly $\alpha+\beta=k$, $\alpha-\beta=\sqrt{k^2+4s}$, and $\alpha\beta=-s$ for every $n \in \mathbb{Z}$. Moreover, it is well known that

$$U_n^2 - kU_nU_{n-1} - U_{n-1}^2 = (-1)^{n-1}, \tag{3.3}$$

$$v_n = u_{n+1} - u_{n-1}$$

and

$$u_n^2 - ku_nu_{n-1} + u_{n-1}^2 = 1, \tag{3.4}$$

where $U_n = U_n(k,1)$ and $u_n = U_n(k,-1)$. For more information about generalized Fibonacci and Lucas sequences, one can consult [11], [4], [12], [5], [8], and [9].

Now we give the following two theorems that help us to find solutions of some of the equations $x^2-kxy+y^2+2^n=0$. Since the proofs of these theorems can be found in [6], [5], [8], [9], and [3], we omit their proofs.

Theorem 14. *Let $k>3$. Then all nonnegative integer solutions of the equation $x^2-kxy+y^2-1=0$ are given by $(x,y)=(u_n,u_{n-1})$ with $n \geq 0$, where $u_n = U_n(k,-1)$.*

Theorem 15. *All nonnegative integer solutions of the equation $x^2-kxy-y^2+1=0$ are given by $(x,y)=(U_{2n},U_{2n-1})$ with $n \geq 0$, where $U_n = U_n(k,1)$.*

Theorem 16. *Let $r \geq 0$ be an integer. Then all positive integer solutions of the equation $x^2-(2^{2r}+2)xy+y^2+2^{2r}=0$ are given by (x,y) $=(U_{2n+1}(2^r,1),U_{2n-1}(2^r,1))$ with $n \geq 0$.*

Proof. Assume that $x^2-(2^{2r}+2)xy+y^2+2^{2r}=0$ for some positive integers x and y. It is easily seen that $2^r|x-y$. Without loss of generality, we may suppose $x \geq y$. Let $u=(x-y)/2^r$ and $v=y$. Then we get $x=2^ru+v$ and $y=v$. Substituting these values of x and y into the equation $x^2-(2^{2r}+2)xy+y^2+2^{2r}=0$, we obtain

$$(2^ru+v)^2-(2^{2r}+2)(2^ru+v)v+v^2+2^{2r}=0$$

and this implies that $u^2-2^r uv-v^2+1=0$. Therefore by Theorem 15, we get $u=U_{2n}(2^r,1)$ and $v=U_{2n-1}(2^r,1)$ with $n\geq 0$. Thus $x=2^rU_{2n}+U_{2n-1}=U_{2n+1}$ and $y=U_{2n-1}$ with $n\geq 0$. Conversely, if $(x,y)=(U_{2n+1},U_{2n-1})$, then from identity (3.3), it follows that $x^2-(2^{2r}+2)xy+y^2+2^{2r}=0$. □

Theorem 17. *Let $r\geq 1$ be an odd integer. Then all positive integer solutions of the equation $x^2-(2^r+2)xy+y^2+2^r=0$ are given by $(x,y)=(u_{n+1}-u_n,u_n-u_{n-1})$ with $n\geq 0$, where $u_n=U_n(2^r+2,-1)$.*

Proof. Assume that $x^2-(2^r+2)xy+y^2+2^r=0$ for some positive integers x and y. It is seen that x and y have the same parity. Without loss of generality, we may suppose $x\geq y$. It can be easily seen that $2^{(r+1)/2}|x-y$. Moreover, it can be shown that

$$\frac{2^r}{4}(x+y)^2-(\frac{2^r}{4}+1)(x-y)^2=2^r.$$

This implies that

$$\left(\frac{x+y}{2}\right)^2-(2^{r-1}+2)\left(\frac{x-y}{2^{(r+1)/2}}\right)^2=1.$$

Since $\alpha=\left(2^{r-1}+1+2^{(r-1)/2}\sqrt{2^{r-1}+2}\right)$ is the fundamental solution to the equation $x^2-(2^{r-1}+2)y^2=1$ by Theorem 1, it follows from (1.5) that

$$(x+y)/2=x_n \text{ and } (x-y)/2^{(r+1)/2}=y_n$$

for some $n\geq 0$, where $x_n+y_n\sqrt{2^{r-1}+2}=\alpha^n$. It is easily seen that $x_n=v_n(2^r+2,-1)/2$ and $y_n=2^{(r-1)/2}u_n(2^r+2,-1)$. Then we get $x=(v_n+2^ru_n)/2$ and $y=(v_n-2^ru_n)/2$. Since $v_n=u_{n+1}-u_{n-1}$, it follows that $x=(u_{n+1}-u_{n-1}+2^ru_n)/2=(u_{n+1}+u_{n+1}-2u_n)/2=u_{n+1}-u_n$. In a similar way, it is seen that $y=u_n-u_{n-1}$. This shows that $(x,y)=(u_{n+1}-u_n,u_n-u_{n-1})$ with $n\geq 0$. Conversely, if $(x,y)=(u_{n+1}-u_n,u_n-u_{n-1})$, then from identity (3.4), it follows that $x^2-(2^r+2)xy+y^2+2^r=0$. □

As an alternative to Theorem 16, we can give the following theorem without proof, since its proof is similar to that of Theorem 17.

Theorem 18. *Let $r\geq 1$ be an integer. Then all positive integer solutions of the equation $x^2-(2^{2r}+2)xy+y^2+2^{2r}=0$ are given by $(x,y)=(u_{n+1}-u_n,u_n-u_{n-1})$ with $n\geq 0$, where $u_n=U_n(2^{2r}+2,-1)$.*

Now we can give the following corollaries from above theorems.

Corollary 2. *All positive integer solutions of the equation $x^2-3xy+y^2+1=0$ are given by $(x,y)=(F_{2n+1},F_{2n-1})$ with $n\geq 0$.*

Corollary 3. *All positive integer solutions of the equation $x^2-4xy+y^2+2=0$ are given by $(x,y)=(u_{n+1}-u_n,u_n-u_{n-1})$ with $n\geq 0$, where $u_n=U_n(4,-1)$.*

Corollary 4. *All positive integer solutions of the equation* $x^2-3xy+y^2+4=0$ *are given by* $(x,y)=(2F_{2n+1},2F_{2n-1})$ *with* $n\geq 0$.

Corollary 5. *All positive integer solutions of the equation* $x^2-6xy+y^2+4=0$ *are given by* $(x,y)=(P_{2n+1},P_{2n-1})$ *with* $n\geq 0$.

Corollary 6. *All positive integer solutions of the equation* $x^2-4xy+y^2+8=0$ *are given by* $(x,y)=(2u_{n+1}-2u_n,2u_n-2u_{n-1})$ *with* $n\geq 0$, *where* $u_n=U_n(4,-1)$.

Theorem 19. *All positive integer solutions of the equation* $x^2-6xy+y^2+8=0$ *are given by* $(x,y)=(3u_{n+1}-u_n,3u_n-u_{n-1})$ *with* $n\geq 0$, *where* $u_n=U_n(6,-1)$.

Proof. Assume that $x^2-6xy+y^2+8=0$ for some positive integers x and y. Then by Theorem 3, it is seen that x and y are both odd integers. Also it is easily seen that $8|x-3y$. Without loss of generality, we may suppose $x\geq 3y$. Let $u=(x-3y)/8$ and $v=(3x-17y)/8$. Then we get $x=17u-3v$ and $y=3u-v$. Substituting these values of x and y into the equation $x^2-6xy+y^2+8=0$, we obtain

$$(17u-3v)^2-6(17u-3v)(3u-v)+(3u-v)^2+8=0$$

and this shows that $u^2-6uv+v^2-1=0$. From Theorem 14, we get $(u,v)=(u_n,u_{n-1})$ with $n\geq 0$. If we substitute these values of u and v into the equations $x=17u-3v$ and $y=3u-v$, then it follows that $x=17u_n-3u_{n-1}=3u_{n+1}-u_n$ and $y=3u_n-u_{n-1}$ with $n\geq 0$. Conversely, if $(x,y)=(3u_{n+1}-u_n,3u_n-u_{n-1})$, then from identity (3.4), it follows that $x^2-6xy+y^2+8=0$. This completes the proof. □

Corollary 7. *All positive integer solutions of the equation* $x^2-10xy+y^2+8=0$ *are given by* $(x,y)=(u_{n+1}-u_n,u_n-u_{n-1})$ *with* $n\geq 0$, *where* $u_n=U_n(10,-1)$.

Now we can give the following corollaries.

Corollary 8. *All positive integer solutions of the equation* $x^2-3xy+y^2+16=0$ *are given by* $(x,y)=(4F_{2n+1},4F_{2n-1})$ *with* $n\geq 0$.

Corollary 9. *All positive integer solutions of the equation* $x^2-6xy+y^2+16=0$ *are given by* $(x,y)=(2P_{2n+1},2P_{2n-1})$ *with* $n\geq 0$.

Corollary 10. *All positive integer solutions of the equation* $x^2-18xy+y^2+16=0$ *are given by* $(x,y)=(U_{2n+1},U_{2n-1})$ *with* $n\geq 0$, *where* $U_n=U_n(4,1)$.

Corollary 11. *All positive integer solutions of the equation* $x^2-4xy+y^2+32=0$ *are given by* $(x,y)=(4u_{n+1}-4u_n,4u_n-4u_{n-1})$ *with* $n\geq 0$, *where* $u_n=U_n(4,-1)$.

Corollary 12. *All positive integer solutions of the equation* $x^2-6xy+y^2+32=0$ *are given by* $(x,y)=(6u_{n+1}-2u_n,6u_n-2u_{n-1})$ *with* $n\geq 0$, *where* $u_n=U_n(6,-1)$.

Corollary 13. *All positive integer solutions of the equation* $x^2-10xy+y^2+32=0$ *are given by* $(x,y)=(2u_{n+1}-2u_n,2u_n-2u_{n-1})$ *with* $n\geq 0$, *where* $u_n=U_n(10,-1)$.

Corollary 14. *All positive integer solutions of the equation* $x^2-34xy+y^2+32=0$ *are given by* $(x,y)=(u_{n+1}-u_n,u_n-u_{n-1})$ *with* $n\geq 0$, *where* $u_n=U_n(34,-1)$.

Theorem 20. *All positive integer solutions of the equation* $x^2-14xy+y^2+32=0$ *are given by* $(x,y)=(3u_{n+1}-u_n,u_n-u_{n-1})$ *with* $n\geq 0$, *where* $u_n=U_n(4,-1)$.

Proof. Assume that $x^2-14xy+y^2+32=0$ for some positive integers x and y. Then by Theorem 5, it is seen that x and y must be odd integers. Also, it is easily seen that $8|x-3y$. Without loss of generality, we may suppose $x\geq 3y$. Let $u=(x-3y)/8$ and $v=(x-11y)/8$. Then we get $x=11u-3v$ and $y=u-v$. Substituting these values of x and y into the equation $x^2-14xy+y^2+32=0$, we obtain

$$(11u-3v)^2-14(11u-3v)(u-v)+(u-v)^2+32=0$$

and this shows that $u^2-4uv+v^2-1=0$. Therefore by Theorem 14, we get $u=U_n(4,-1)$ and $v=U_{n-1}(4,-1)$ with $n\geq 0$. Thus it follows that $x=11u_n-3u_{n-1}=3u_{n+1}-u_n$ and $y=u_n-u_{n-1}$ with $n\geq 0$. Conversely, if $(x,y)=(3u_{n+1}-u_n,u_n-u_{n-1})$, then from identity (3.4), it follows that $x^2-14xy+y^2+32=0$. □

In order to find all positive integer solutions of the equation $x^2-46xy+y^2+128=0$, we need the following theorem given in [1].

Theorem 21. *If* $u+v\sqrt{d}$ *is a solution in nonnegative integers to the Diophantine equation* $u^2-dv^2=N$, *where* $N<0$, *then there exists a nonnegative integer* m *such that*

$$u+v\sqrt{d}=(u_1+v_1\sqrt{d})(x_1+y_1\sqrt{d})^m$$

where $u_1+v_1\sqrt{d}$ *is the fundamental solution to the class of solutions of the equation* $u^2-dv^2=N$ *to which* $u+v\sqrt{d}$ *belongs and* $x_1+y_1\sqrt{d}$ *is the fundamental solution to the equation* $x^2-dy^2=1$.

Lemma 2. *All positive integer solutions of the equation* $x^2-33y^2=-8$ *are given by* $(x,y)=(|17u_n+5u_{n-1}|,3u_n-u_{n-1})$ *with* $n\in\mathbb{Z}$, *where* $u_n=U_n(46,-1)$.

Proof. Assume that $x^2-33y^2=-8$ for some positive integers x and y. It can be seen from Theorem 8 that the equation $x^2-33y^2=-8$ has two solution classes. And the fundamental solutions of these classes are $5+\sqrt{33}$ and $-5+\sqrt{33}$. By Theorem 21, all positive integer solutions of the equation $x^2-33y^2=-8$ are given by

$$a_n+b_n\sqrt{33}=(5+\sqrt{33})(x_n+y_n\sqrt{33})$$

with $n\geq 0$ or

$$c_n+d_n\sqrt{33}=(-5+\sqrt{33})(x_n+y_n\sqrt{33})$$

with $n \geq 1$, where $x_n + y_n\sqrt{33}$ is the solution of the equation $x^2 - 33y^2 = 1$. Since the fundamental solution of this equation is $\alpha = 23 + 4\sqrt{33}$, we get $x_n + y_n\sqrt{33} = \alpha^n$ and therefore $x_n = (\alpha^n + \beta^n)/2$ and $y_n = (\alpha^n - \beta^n)/2\sqrt{33}$, where $\beta = 23 - 4\sqrt{33}$. Thus we get $b_n = 5y_n + x_n$ and $d_n = -5y_n + x_n$. It is seen that $x_n = V_n(46,-1)/2$ and $y_n = 4U_n(46,-1)$. This shows that $b_n = 20u_n + v_n/2 = u_{n+1} - 3u_n$ with $n \geq 0$ and $d_n = -20u_n + v_n/2 = 3u_n - u_{n-1}$ with $n \geq 1$. Since $3u_{-n} - u_{-n-1} = u_{n+1} - 3u_n$ for $n > 0$, we can take y as $y = u_{n+1} - 3u_n$ with $n \in \mathbb{Z}$. Substituting the value of y into the equation $x^2 - 33y^2 = -8$, we get $x = |17u_n + 5u_{n-1}|$ with $n \in \mathbb{Z}$. Conversely, if $(x,y) = (|17u_n + 5u_{n-1}|, 3u_n - u_{n-1})$, then from identity (3.4), it follows that $x^2 - 33y^2 = -8$. □

Theorem 22. *All positive integer solutions of the equation* $x^2 - 46xy + y^2 + 128 = 0$ *are given by* $(x,y) = (3u_{n+1} - u_n, 3u_n - u_{n-1})$ *with* $n \in \mathbb{Z}$, *where* $u_n = U_n(46,-1)$.

Proof. Assume that $x^2 - 46xy + y^2 + 128 = 0$ for some positive integers x and y. Completing the square gives $(x - 23y)^2 - 528y^2 = -128$, which implies that $x - 23y = \pm 4m$ for some positive integer m. Rearranging the equation gives $m^2 - 33y^2 = -8$. By Lemma 2, we get $(m,y) = (|17u_n + 5u_{n-1}|, 3u_n - u_{n-1})$ with $n \in \mathbb{Z}$. Thus $x = 23y \pm 4m = 23(3u_n - u_{n-1}) \pm 4(17u_n + 5u_{n-1})$ and therefore we get $x = 3u_{n+1} - u_n$ or $x = 3u_{n-1} - u_{n-2}$ with $n \in \mathbb{Z}$. Since $3u_{(n+2)-1} - u_{(n+2)-2} = 3u_{n+1} - u_n$, we can take $(x,y) = (3u_{n+1} - u_n, 3u_n - u_{n-1})$ with $n \in \mathbb{Z}$. Conversely, if $(x,y) = (3u_{n+1} - u_n, 3u_n - u_{n-1})$, then from identity (3.4), it follows that $x^2 - 46xy + y^2 + 128 = 0$. □

The proofs of the following theorems are similar to that of Theorem 22 and therefore we omit them.

Theorem 23. *All positive integer solutions of the equation* $x^2 - 174xy + y^2 + 512 = 0$ *are given by* $(x,y) = (3u_{n+1} - u_n, 3u_n - u_{n-1})$ *with* $n \in \mathbb{Z}$, *where* $u_n = U_n(174,-1)$.

Theorem 24. *All positive integer solutions of the equation* $x^2 - 66xy + y^2 + 512 = 0$ *are given by* $(x,y) = (9u_{n+1} - u_n, 9u_n - u_{n-1})$ *with* $n \in \mathbb{Z}$, *where* $u_n = U_n(66,-1)$.

Theorem 25. *All positive integer solutions of the equation* $x^2 - 210xy + y^2 + 1024 = 0$ *are given by* $(x,y) = (5u_{n+1} - u_n, 5u_n - u_{n-1})$ *with* $n \in \mathbb{Z}$, *where* $u_n = U_n(210,-1)$.

Theorem 26. *All positive integer solutions of the equation* $x^2 - 66xy + y^2 + 1024 = 0$ *are given by* $(x,y) = (41u_{n+1} - u_n, 41u_n - u_{n-1})$ *with* $n \in \mathbb{Z}$, *where* $u_n = U_n(66,-1)$ *or* $(x,y) = (4U_{2n+1}, 4U_{2n-1})$ *with* $n \geq 0$, *where* $U_n = U_n(8,1)$.

Since all positive integer solutions of the following equations

$$x^2-kxy+y^2+64=0,\ k\in\{3,6,18,66\},$$
$$x^2-kxy+y^2+128=0,\ k\in\{4,6,10,14,34,130\},$$
$$x^2-kxy+y^2+256=0,\ k\in\{3,6,18,66,258\},$$
$$x^2-kxy+y^2+512=0,\ k\in\{4,6,10,14,34,46,130,514\},$$

and

$$x^2-kxy+y^2+1024=0,\ k\in\{3,6,18,258,1026\}$$

can be given easily by using the previous theorems and corollaries, we do not give their solutions.

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A MATRIX APPROACH FOR GENERALIZING TWO CURIOUS DIVISIBILITY PROPERTIES

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Abstract. We shall consider two curious divisibility properties due to [1,5]. Our main purpose is to generalize these properties for a general second order linear recursion. We use generating matrix approach for our purposes. By using our results, we derive a new recursive identities for two general second order linear recurrences. We give a more general result extending earlier divisibility properties on the similar subject.

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1. INTRODUCTION

The Fibonacci numbers have many interesting properties in accordance with their nature and the recursively definition. We shall be interested in their divisibility properties along the paper. We recall that

$$\gcd(F_n, F_m) = F_{\gcd(n,m)}$$

and

$$F_{kn} \text{ is a multiple of } F_n.$$

An extension of these divisibility properties was given showing Matijasevich [4] as that if $n > 2$, the Fibonacci number F_m is a multiple of F_n^2 if and only if m is multiple of nF_n. This fact and its detailed proof was also given in [2] (pp. 294) showning

$$F_{kn} \equiv kF_n F_{n+1}^{k-1} \text{ and } F_{kn+1} \equiv F_{n+1}^k \pmod{F_n^2}.$$

The above fact and the following similar results can also be found in [1,5] :

i) $F_{kn-1} - F_{n-1}^k$ is divisible by F_n^2 $(n,k = 1,2,\ldots)$,

ii) $F_{kn-2} - (-1)^{k+1} F_{n-2}^k$ is divisible by F_n^2 $(n,k = 1,2,\ldots)$.

We mainly consider these two curious properties and then we generalize them to a general second order linear recursion $\{U_n(p)\}$, or briefly U_n, defined for $n > 1$ by

$$U_n(p) = pU_{n-1}(p) + U_{n-2}(p), \tag{1.1}$$

with initials $U_0(p) = 0$ and $U_1(p) = 1$.

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The generalized Lucas sequence $\{V_n(p)\}$, or briefly V_n, is defined by

$$V_n(p) = pV_{n-1}(p) + V_{n-2}(p),$$

with initials $V_0(p) = 2,\ V_1(p) = p$.

The Binet forms of them are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n,$$

where $\alpha, \beta = \left(p \pm \sqrt{p^2+4}\right)/2$.

Throughout the paper we consider the properties

a) $U_{kn-1} - U_{n-1}^k$ is divisible by U_n^2 $(n,k = 1,2,\ldots)$,

b) $p^{k-1}U_{kn-2} - (-1)^{k+1}U_{n-2}^k$ is divisible by U_n^2 $(n,k = 1,2,\ldots)$.

Matijasevich and Cavachi proved the properties (i) and (ii) in [2, 5] but the quotients

$$\frac{F_{kn-1} - F_{n-1}^k}{F_n^2} \quad \text{and} \quad \frac{F_{kn-2} - (-1)^{k+1}F_{n-2}^k}{F_n^2}$$

were not considered. It seems that to find the quotients are not easy.

However, we shall focus on quotients arising from (a) and (b) :

$$\frac{U_{kn-1} - U_{n-1}^k}{U_n^2} \quad \text{and} \quad \frac{p^{k-1}U_{kn-2} - (-1)^{k+1}U_{n-2}^k}{U_n^2}$$

for all n and k such that $n,k \geq 1$.

Matrix methods are essential tools for solving problems arising from number theory, for more details see [3].

Our approach is to use matrix methods for deriving these quotients explicitly. We first look at values of the quotients for fixed n and increasing values of k starting from 1 and then find a generating matrix for these values for fixed n and increasing values of k. After we find the generating matrices for these quotients, we will give two explicit statements for them. Using these two explicit statements, we obtain two new recursive identities for the general second order linear recurrences. Finally by considering and extending these two recursive identities, we find new recursive identities for the general second order linear recursions. In order to guarantee the accuracy of matrix computations, all the equalities appearing in this paper have been verified through "Maple" commands.

2. General Cases

We start with the first quotient with the case $p = 1$, $U_n = F_n$ (nth Fibonacci numbers) and its values for some n and k are shown in Table 1.

We will see that the entries of each row in Table 1 have a generating matrix. For example, the entries of the third row of Table 1, {1,5,22,94,399,1691, ...}, could be generated by powers of a matrix. Continuing this process each row entries in each

TABLE 1.

$$\left(F_{kn-1}-F_{n-1}^{k}\right)/F_{n}^{2}$$

$n\backslash k$	1	2	3	4	5	6	7	...
1	0	1	1	2	3	5	8	...
2	0	1	4	12	33	88	232	...
3	0	1	5	22	94	399	1691	...

table can be obtained from the quotients for all n,k and p. We will state this fact in general by matrix methods.

Before this process, we need some auxiliary results and definitions.

Define a second order linear sequence $\{A_n(p)\}$, or briefly A_n, for $n>1$ as

$$A_n = pA_{n-1}+A_{n-2}$$

with initials $A_0=p$ and $A_1=p^2+3$.

From the definitions of sequences $\{U_n\}$ and $\{A_n\}$, we have that

$$A_n = U_{n+2}+U_n = V_n+U_{n-1}.$$

Denote the first quotient by $s(n,k,p)$, or briefly $s(k,p)$:

$$s(n,k)=\left(U_{kn-1}-U_{n-1}^{k}\right)/U_n^2.$$

Define two matrices $H(n,p)$, or briefly $H(n)$, and $G(n,k,p)$, or briefly $G(n,k)$, of order 3 as shown:

$$H(n)=\begin{bmatrix} A_{n-1} & -U_{n-1}V_n+(-1)^{n+1} & (-1)^n U_{n-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$G(n,k)=\begin{bmatrix} s(n,k+2) & t(n,k+2) & (-1)^n U_{n-1}s(n,k+1) \\ s(n,k+1) & t(n,k+1) & (-1)^n U_{n-1}s(n,k) \\ s(n,k) & t(n,k) & (-1)^n U_{n-1}s(n,k-1) \end{bmatrix},$$

where

$$t(n,k)=\frac{U_{kn}U_{n-1}^2-U_{(k-2)n}-U_{2n}U_{n-1}^k}{U_n^3}.$$

Lemma 1. *For $n\geq 1$, the eigenvalues of the matrix $H(n)$ are α^n,β^n and U_{n-1}.*

Proof. The characteristic polynomial of $H(n)$ is

$$x^3-A_{n-1}x^2+x\left(U_{n-1}V_n+(-1)^n\right)-(-1)^n U_{n-1}$$

and it is factorized as $(x-\alpha^n)(x-\beta^n)(x-U_{n-1})$, as claimed. □

Now we state one of our main two results:

Theorem 1. *For $n \geq 1$,*

$$H(n)^k = G(n,k).$$

Proof. To prove the claim, it is sufficient to verify the followings:

$$s(n,k+1)A_{n-1} + t(n,k) = s(n,k+2)$$

and

$$s(n,k+1)\left(-U_{n-1}V_n + (-1)^{n+1}\right) + (-1)^n U_{n-1}s(n,k) = t(n,k+2).$$

For proving them, it is enough to consider the definitions of $s(n,k)$ and $t(n,k)$. The above equations give us the following matrix equation:

$$G(n,k-1)H(n) = G(n,k).$$

Since $G(n,1) = H(n)$, we deduce that $G(n,k) = H(n)^k$, as claimed. □

We see that the powers of matrix $H(n)$ generate the first quotient and since the matrix has integer entries, it is clear that the first quotient are integers.

Since all eigenvalues of $H(n)$ are different from each other, we diagonalize it. $H(n)$ is a companion matrix so that it is diagonalized with the Vandermonde matrix

$$V = \begin{bmatrix} \alpha^{2n} & \beta^{2n} & U_{n-1}^2 \\ \alpha^n & \beta^n & U_{n-1} \\ 1 & 1 & 1 \end{bmatrix},$$

as $V^{-1}H(n)V = D$, where D is the diagonal matrix $D = \text{diag}(\alpha^n, \beta^n, U_{n-1})$. Thus we obtain

$$V^{-1}H(n)^k V = D^k \text{ or } V^{-1}G(n,k)V = D^k,$$

and so

$$G(n,k) = VD^kV^{-1}.$$

Consequently, after required computations, we get the following result:

Theorem 2. *For $n,k > 0$*

$$\begin{aligned}(G(n,k))_{2,1} &= s(n,k) \\ &= \frac{(-1)^n U_{n(k-1)} + U_{n-1}^k U_n - U_{kn}U_{n-1}}{U_n^3}.\end{aligned}$$

From the definition of $s(n,k)$ and the above result, we have the following result:

Corollary 1. *For $n,k > 0$*

$$U_{n-1}U_{kn} = U_n U_{kn-1} + (-1)^n U_{n(k-1)}.$$

Now we shall focus on the second quotient

$$\frac{p^{k-1}U_{kn-2}-(-1)^{k+1}U_{n-2}^{k}}{U_n^2}$$

and we will seek an explicit formula for that.

Similar to the process for the first quotient, we need some preliminary results and definition. First we define a second order recurrence $\{B_n(p)\}$, or briefly B_n, for $n>1$ as

$$B_n = pB_{n-1}+B_{n-2}$$

with initials $B_0 = p^2-1$ and $B_1 = p^3+2p$.

For shortness, we denote the second quotient by $y(n,k)$:

$$y(n,k) = \frac{p^{k-1}U_{kn-2}-(-1)^{k+1}U_{n-2}^{k}}{U_n^2}.$$

Define two matrices $T(n)$ and $Q(n,k)$ of order 3 as shown:

$$T(n) = \begin{bmatrix} B_{n-1} & C(n) & (-1)^{n+1}p^2U_{n-2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$Q(n,k) = \begin{bmatrix} y(n,k+2) & h(n,k+2) & (-1)^{n+1}p^2U_{n-2}y(n,k+1) \\ y(n,k+1) & h(n,k+1) & (-1)^{n+1}p^2U_{n-2}y(n,k) \\ y(n,k) & h(n,k) & (-1)^{n+1}p^2U_{n-2}y(n,k-1) \end{bmatrix},$$

where $C(n) = p^2\left(D_n+2(-1)^n\right)$, D_n is defined as for $n>1$

$$D_n = \left(p^2+2\right)D_{n-1}-D_{n-2}$$

with initials $D_0 = 0$ and $D_1 = 1$, and, $h(n,k)$ is given by

$$h(n,k) = \frac{p^{k-1}U_{n-2}^2U_{kn}-p^{k+1}U_{n(k-2)}-p(-1)^kU_{n-2}^kU_{2n}}{U_n^3}.$$

Lemma 2. *For $n\geq 1$, the eigenvalues of $T(n)$ are $p\alpha^n$, $p\beta^n$ and $-U_{n-2}$.*

Proof. The characteristic polynomial of $T(n)$ is $x^3-B_{n-1}x^2-C(n)x+(-1)^n p^2U_{n-2}$ and it is factorized as $(x-p\alpha^n)(x-p\beta^n)(x+U_{n-2})$, as claimed. □

We state our second main result:

Theorem 3. *For $n\geq 1$,*

$$T(n)^k = Q(n,k).$$

Proof. To prove the claim, it is sufficient to verify the followings:

$$y(n,k+1)B_{n-1}+h(n,k+1)=y(n,k+2)$$

and

$$y(n,k+1)C(n)+(-1)^{n+1}p^2U_{n-2}y(n,k)=h(n,k+2).$$

For proving them, one can consider the definitions of $y(n,k)$ and $h(n,k)$. The above two equations give us the following matrix equation:

$$Q(n,k-1)T(n)=Q(n,k).$$

Since $Q(n,1)=T(n)$, we deduce that $Q(n,k)=T(n)^k$. □

Since all the eigenvalues of $T(n)$ are different from each other, it is diagonalized as

$$V_1^{-1}T(n)V_1=D_1,$$

where

$$V_1=\begin{bmatrix} p^2\alpha^{2n} & p^2\beta^{2n} & U_{n-2}^2 \\ p\alpha^n & p\beta^n & -U_{n-2} \\ 1 & 1 & 1 \end{bmatrix}.$$

and D_1 is the diagonal matrix $D_1=\text{diag}(\alpha^n,\beta^n,-U_{n-2})$. Thus we obtain

$$V_1^{-1}T(n)^kV_1=D_1^k \text{ or } V_1^{-1}Q(n,k)V_1=D_1^k,$$

and so

$$Q(n,k)=V_1D_1^kV_1^{-1}.$$

Therefore, by considering $(Q(n,k))_{2,1}=y(n,k)$, we get the following result:

Theorem 4. *For $n,k>0$*

$$y(n,k)=\frac{p^{k+1}(-1)^nU_{kn}+(-1)^{k+1}U_{n-2}^{k+1}U_n+p^kU_{n-2}U_{n(k+1)}}{U_n^3}$$

From the definition of $y(n,k)$ and the above result, we have the following result:

Corollary 2. *For $n,k>0$*

$$U_{n-2}U_{nk}=U_{nk-2}U_n-p(-1)^nU_{kn-n}$$

3. New recursive identities

While extending two curious properties in the previous section, we found the following two new identities for general second order recursions:

$$i)\ U_{n-1}U_{kn}=U_nU_{kn-1}+(-1)^nU_{kn-n},$$
$$ii)\ U_{n-2}U_{nk}=U_nU_{kn-2}-p(-1)^nU_{kn-n}.$$

The above recursions suggest us that they could be more generalized. Thus we give our third main result:

Theorem 5. *For all integers r,k and n,*

$$U_{n-r}(p)\,U_{kn}(p) = U_n(p)\,U_{kn-r}(p) - (-1)^{r+n}\,U_r(p)\,U_{kn-n}(p)$$

and

$$U_{n-r}(p)\,V_{kn}(p) = U_n(p)\,V_{kn-r}(p) - (-1)^{r+n}\,U_r(p)\,V_{kn-n}(p)$$

Proof. For the first identity consider

$$\begin{aligned}
&U_n(p)\,U_{kn-r}(p) - (-1)^{r+n}\,U_r(p)\,U_{kn-n}(p)\\
&= \frac{\alpha^n-\beta^n}{\alpha-\beta}\,\frac{\alpha^{kn-r}-\beta^{kn-r}}{\alpha-\beta} - (-1)^{r+n}\,\frac{\alpha^r-\beta^r}{\alpha-\beta}\,\frac{\alpha^{kn-n}-\beta^{kn-n}}{\alpha-\beta}\\
&= (\alpha^{kn+n-r} - (-1)^n\,\beta^{kn-n-r} - (-1)^n\,\alpha^{kn-n-r} + \beta^{kn+n-r} - (-1)^{n-r}\,\alpha^{kn-n+r}\\
&\quad + (-1)^n\,\beta^{kn-n-r} + (-1)^n\,\alpha^{kn-n-r} - (-1)^{n-r}\,\beta^{kn-n+r})/(\alpha-\beta)^2\\
&= (\alpha^{kn+n-r} + \beta^{kn+n-r} - (-1)^{n-r}\,\alpha^{kn-n+r} - (-1)^{n-r}\,\beta^{kn-n+r})/(\alpha-\beta)^2\\
&= \left(\alpha^{n-r}-\beta^{n-r}\right)\left(\alpha^{kn}-\beta^{kn}\right))/(\alpha-\beta)^2\\
&= U_{n-r}(p)\,U_{kn}(p),
\end{aligned}$$

as claimed. By the Binet formula, the second claim could be similarly proved. □

Our last conclusion is that the last recursive identities are new for general second order linear recurrences $\{U_n\}$ and $\{V_n\}$ according to our literature knowledge.

For the Fibonacci and Lucas cases, we have that

$$\begin{aligned}
&i)\ F_{n-r}F_{kn} = F_nF_{kn-r} - (-1)^{n+r}\,F_rF_{kn-n},\\
&ii)\ F_{n-r}L_{kn} = F_nL_{kn-r} - (-1)^{n+r}\,F_rL_{kn-n}.
\end{aligned}$$

4. Generalization of the divisibility properties

We consider two divisibility properties and then generalize them for the general second order linear recurrence $\{U_n\}$. Now we are ready to present more general case of these properties. We state this general case with the following corollary without proof:

Corollary 3. *For all integers r,*

$$U_r^{k-1}U_{kn-r} - (-1)^{(r-1)(k+1)}\,U_{n-r}^k \text{ is divisible by } U_n^2 \ \ (n,k=1,2,\ldots).$$

This corollary gives us all the divisibility properties mentioned in this paper.

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ON RINGS OF QUOTIENTS OF SEMIPRIME Γ-RINGS

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Abstract. In this paper, we investigate the rings of quotients of a semiprime Γ-ring.

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1. INTRODUCTION

Let M be an abelian additive group whose elements are denoted by $a,b,c,\cdots$ and let Γ another abelian additive group whose elements are $\gamma,\beta,\alpha,\cdots$. Suppose that $a\gamma b$ is defined to be an element of M and that $\gamma a\beta$ is defined to be an element of Γ for every a,b,γ and β. If the products satisfy the following three conditions:

(i) $(a+b)\gamma c = a\gamma c + b\gamma c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\gamma(b+c) = a\gamma b + a\gamma c$,

(ii) $(a\gamma b)\beta c = a(\gamma b\beta)c = a\gamma(b\beta c)$,

(iii) $a\alpha b = 0$ for all $a,b \in M$ implies $\alpha = 0$,

then M is called a $\Gamma-$ring in the sense of Nabusawa [7].

Barnes approached Γ-ring in a different way compared to that of Nobusawa and he defined the concept of Γ-rings.

Let M and Γ be additive abelian groups. If there exits a mapping of $M\times\Gamma\times M$ to M (the image of (a,γ,b) $a,b\in M$, $\gamma\in\Gamma$ is denoted by $(a\gamma b)$) satisfying for all $a,b,c\in M, \alpha,\beta\in\Gamma$:

(i) $(a+b)\gamma c = a\gamma c + b\gamma c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\gamma(b+c) = a\gamma b + a\gamma c$,

(ii) $(a\gamma b)\beta c = a\gamma(b\beta c)$,

then M is called a $\Gamma-$ring in the sense of Barnes [2].

Throughout the present paper, the symbol $(\Gamma,M)_N$ that stands for M is the $\Gamma-$ring in the sense of Nabusawa, and the symbol $(\Gamma,M)_B$ that stands for M is the $\Gamma-$ring in the sense of Barnes. In [5], we have shown that for all $(\Gamma,M)_B$ there exists Γ' as an additive group such that $(\Gamma',M)_N$. Therefore, if $\Gamma-$ring in the sense of Barnes, then $\Gamma'-$ring in the sense of Nabusawa.

Let M be a $\Gamma-$ring in the sense of Nabusawa. A right (resp. left) ideal of M is an additive subgroup of U such that $U\Gamma M\subseteq U$ (resp. $M\Gamma U\subseteq U$). If U is both a right and left ideal of M, then we say that U is an ideal of M. An ideal P of a $\Gamma-$ring

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M is said to be prime if for any ideals A and B of M, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. An ideal P of a Γ- ring M is said to be semiprime if for any ideal U of M, $U\Gamma U \subseteq P$ implies $U \subseteq P$. A Γ-ring M is said to be semiprime if the zero ideal is semiprime. This definition is given as "A Γ-ring M is said to be prime if $a\Gamma b = 0$ with $a, b \in M$, implies $a = 0$ or $b = 0$ and semiprime $a\Gamma a = 0$ with $a \in M$, implies $a = 0$" in [4, Theorem 2.2.23].

Let (Γ_1, M_1) and (Γ_2, M_2) be two gamma rings, $\phi : \Gamma_1 \to \Gamma_2$ and $\theta : M_1 \to M_2$ be two functions. Then an ordered pair (ϕ, θ) of mappings is called a homomorphism of (Γ_1, M_1) into (Γ_2, M_2) if it satisfies the following properties:

(i) $\theta : M_1 \to M_2$ is group homomorphism,

(ii) $\phi : \Gamma_1 \to \Gamma_2$ is group homomorphism,

(iii) $\theta(x\alpha y) = \theta(x)\phi(\alpha)\theta(y)$, for all $x, y \in M$, $\alpha \in \Gamma$,

(iv) $\phi(\alpha x\beta) = \phi(\alpha)\theta(x)\phi(\beta)$, for all $x \in M$, $\alpha, \beta \in \Gamma$.

A homomorphism (ϕ, θ) of a gamma ring (Γ_1, M_1) into a gamma ring (Γ_2, M_2) is called a monomorphism if ϕ and θ are one-one.

Let M be a Γ-ring. A commutative additive group N is called a right gamma M-module (or right ΓM-module) if for all $n, n_1, n_2 \in N$, $m, m_1, m_2 \in M$ and $\alpha, \beta \in \Gamma$,

(i) $n\alpha m \in N$,

(ii) $(n_1 + n_2)\alpha m = n_1\alpha m + n_2\alpha m$,

(iii) $n(\alpha + \beta)m = n\alpha m + n\beta m$,

(iv) $n\alpha(m_1 + m_2) = n\alpha m_1 + n\alpha m_2$.

Let N_1 and N_2 be two right gamma M-modules. Then θ is called a right gamma M-module homomorphism (or right ΓM-module homomorphism) of N_1 into N_2 if it satisfies the following properties:

(i) $\theta : N_1 \to N_2$ is group homomorphism,

(ii)$\theta(x\alpha m) = \theta(x)\alpha m$, for all $x \in N_1$, $m \in M$, $\alpha \in \Gamma$.

A great deal work has been done on Γ-ring in the sense of Barnes and Nabusawa, the results can be compared to those of in the ring theory since 1964. On the other hand, it will be seen that rings of quotients play crucial role in the study of generalized identities in prime and semiprime rings. The study of two-sided rings of quotients was initiated by W. S. Martindale [6] for prime rings and extented for semiprime rings by S. A. Amitsur in [1]. The concept of centroid of a prime Γ-ring was defined and researched in [9], [8], [10] and [11]. In [4], it was defined the rings of quotients of a prime Γ-ring and researched the some properties of it. In this paper, we investigate the rings of quotients of a semiprime Γ-ring. It was first constructed by [12] for rings. We extend these results for semiprime Γ-rings.

Throughout the present paper, M will a Γ-ring in the sense of Nabusawa and the symbol (Γ, M) stands for the $(\Gamma, M)_N$.

2. RESULTS

Definition 1. Let M be a Γ-ring. If there exists $e \in M$ and $\delta \in \Gamma$ such that for all $x \in M$ $e\delta x = x$, then (δ, e) is said to be strong left identity element of (Γ, M). Similarly, if there exists $e \in M$ and $\delta \in \Gamma$ such that for all $x \in M$ $x\delta e = x$, then (δ, e) is said to be strong right identity element of (Γ, M). If (δ, e) is both a right and left strong identity element of (Γ, M), then we say that (δ, e) is an strong identity element of (Γ, M).

Definition 2. Let M be a Γ-ring. A right ideal J of M is said to be dense if given any $0 \neq m_1 \in M, m_2 \in M$ there exists $m \in M$, $\gamma \in \Gamma$ such that $m_1\gamma m \neq 0$ and $m_2\gamma m \in J$. One defines a dense left ideal in an analogous fashion. The collection of all dense right ideal of M will be denoted by $D(\Gamma, M)$.

Let M be a Γ-ring. For a subset S of M,

$$r_\Gamma(S) = \{c \in M \,|\, S\gamma c = (0),\ \forall \gamma \in \Gamma\}$$

is called the right annihilator of S. A left annihilator $l_\Gamma(S)$ can be defined similarly.

Let N be a $\Gamma M-$ module. For any submodule J of N and any subset $S \subseteq N$ we set

$$(S : J)_\Gamma = \{c \in M \,|\, S\gamma c \subseteq J,\ \forall \gamma \in \Gamma\}.$$

In particular, for any $a \in M$

$$(a : J)_\Gamma = \{c \in M \,|\, a\gamma c \in J,\ \forall \gamma \in \Gamma\}.$$

Theorem 1. *Let M be a semiprime Γ-ring, $U, V \in D(\Gamma, M)$ and $f : U \to M$ be a right $\Gamma M-$module homomorphism. Then*

i) $f^{-1}(V) = \{c \in U \,|\, f(c) \in V\} \in D(\Gamma, M)$.

ii) $(a : U)_\Gamma \in D(\Gamma, M)$ *for all* $a \in M$.

iii) $U \cap V \in D(\Gamma, M)$.

iv) If W is a right ideal of M and $U \subseteq W$, then $W \in D(\Gamma, M)$.

v) $l_\Gamma(U) = (0) = r_\Gamma(U)$.

vi) If W is a right ideal of M and $(a : W)_\Gamma \in D(\Gamma, M)$ for all $a \in M$, then $W \in D(\Gamma, M)$.

vii) $U\Gamma V \in D(\Gamma, M)$.

Proof. i) Clearly, $f^{-1}(V)$ is a right ideal of M. Let $m_1 \neq 0$, $m_2 \in M$. By the dense right ideal U of M, we get $m_1\gamma' m' \neq 0$ and $m_2\gamma' m' \in U$ for some $m' \in M$, $\gamma' \in \Gamma$. Since $f(m_2\gamma' m') \in M$ and $V \in D(\Gamma, M)$, we see that there exists $m'' \in M$, $\gamma'' \in \Gamma$ such that $(m_1\gamma' m')\gamma'' m'' \neq 0$ and $f(m_2\gamma' m')\gamma'' m'' \in V$. Using f is right $\Gamma M-$module homomorphism, we have

$$f\left(m_2\gamma' m'\right)\gamma'' m'' = f\left(\left(m_2\gamma' m'\right)\gamma'' m''\right) = f\left(m_2\gamma'\left(m'\gamma'' m''\right)\right).$$

Setting $m = m'\gamma'' m''$, we conclude that $m_1\gamma' m \neq 0$ and $f(m_2\gamma' m) \in V$. That is, $m_1\gamma' m \neq 0$ and $m_2\gamma' m \in f^{-1}(V)$ for some $m \in M$, $\gamma' \in \Gamma$. Thus, $f^{-1}(V) \in D(\Gamma, M)$.

ii) Consider the map $\lambda_{a\gamma} : M \to M$, $\lambda_{a\gamma}(m) = a\gamma m$, $\forall m \in M, \forall \gamma \in \Gamma$. One easily checks that $\lambda_{a\gamma}$ is right $\Gamma M-$module homomorphism. We have

$$\begin{aligned}\lambda_{a\gamma}^{-1}(U) &= \left\{c \in M \,\middle|\, \lambda_{a\gamma}(c) \in U, \text{ for all } \gamma \in \Gamma\right\} \\ &= \{c \in M \,|\, a\gamma c \in U, \text{ for all } \gamma \in \Gamma\} \\ &= (a : U)_\Gamma\end{aligned}$$

According to (i), we find that $(a : U)_\Gamma \in D(\Gamma, M)$.

iii) If i is the inclusion map $U \to M$, then

$$i^{-1}(V) = U \cap V.$$

Now apply (i), we conclude that $U \cap V \in D(\Gamma, M)$.

iv) Let W be a right ideal of M, $U \subseteq W$ and $m_1 \neq 0, m_2 \in M$. Since U is a dense right ideal of M, we obtain that $m_1\gamma m \neq 0$ and $m_2\gamma m \in U$ for some $m \in M$, $\gamma \in \Gamma$. Using $U \subseteq W$, we get $m_1\gamma m \neq 0$ and $m_2\gamma m \in W$. That is $W \in D(\Gamma, M)$.

v) We assume that $U\gamma c = (0)$ for some $0 \neq c \in M$, for all $\gamma \in \Gamma$. Setting $m_1 = c = m_2$, we arrive that there exists $\alpha \in \Gamma$, $m \in M$ such that $0 \neq c\alpha m \in U$. Hence

$$(c\alpha m)\,\gamma\,(c\alpha m) \in (U\gamma c)\,\alpha m = (0) \text{ for all } \gamma \in \Gamma,$$

and so

$$(c\alpha m)\,\Gamma\,(c\alpha m) = (0).$$

By the semiprimeness of (Γ, M), we have $c\alpha m = 0$. It contradicts $c\alpha m \neq 0$. Hence $r_\Gamma(U) = (0)$.

Next we suppose $l_\Gamma(U) \neq (0)$. We see that there exists $0 \neq c \in M$ such that $c\gamma U = (0)$ for all $\gamma \in \Gamma$. On the other hand, $c\alpha m \neq 0$ for some $\alpha \in \Gamma$, $m \in M$. Indeed, if $c\alpha m = 0$ for all $\alpha \in \Gamma$, $m \in M$, then $c\Gamma c = (0)$, and so $c = 0$ by the semiprimeness of (Γ, M). But we know that $c \neq 0$.

Setting $m_1 = c\alpha m \neq 0$, $m_2 = m$, we have $(c\alpha m)\,\gamma' m' \neq 0$ and $m\gamma' m' \in U$ for some $m' \in M$, $\gamma' \in \Gamma$. But

$$(c\alpha m)\,\gamma' m' = c\alpha\left(m\gamma' m'\right) \in c\Gamma U = (0)$$

and a contradiction is reached. So we must have $l_\Gamma(U) = (0)$.

vi) Let $m_1 \neq 0$, $m_2 \in M$. Using $U \in D(\Gamma, M)$, we see that there exists $m' \in M$, $\gamma' \in \Gamma$ such that $m_1\gamma' m' \neq 0$ and $m_2\gamma' m' \in U$. By the hypothesis, we have $(m_2\gamma' m' : W)_\Gamma \in D(\Gamma, M)$. By (v), we get $l_\Gamma((m_2\gamma' m' : W)_\Gamma) = (0)$. Hence $m_1\gamma' m' \neq 0$, we obtain $m_1\gamma' m' \notin l_\Gamma((m_2\gamma' m' : W)_\Gamma)$. That is $(m_1\gamma' m')\,\gamma'' m'' \neq 0$ for some $m'' \in (m_2\gamma' m' : W)_\Gamma$, $\gamma'' \in \Gamma$. Therefore,

$$\left(m_1\gamma' m'\right)\gamma'' m'' = m_1\gamma'\left(m'\gamma'' m''\right) \neq 0$$

and $(m_2\gamma' m')\,\gamma'' m'' = m_2\gamma'\,(m'\gamma'' m'') \in W$.

Setting $m = m'\gamma'' m''$, we conclude that there exists $m \in M$, $\gamma' \in \Gamma$ such that $m_1\gamma' m \neq 0$ and $m_2\gamma' m \in W$. Thus W is a dense right ideal of M.

vii) Let $m_1 \neq 0$, $m_2 \in M$. By (ii), we have $(m_2 : U)_\Gamma \in D(\Gamma, M)$. Using (v), we get $l_\Gamma((m_2 : U)_\Gamma) = (0)$. We obtain $m_1 \notin l_\Gamma((m_2 : U)_\Gamma)$. Then we have $m_1\gamma' m' \neq 0$ for some $m' \in (m_2 : U)_\Gamma$, $\gamma' \in \Gamma$. On the other hand, we see that there exists $m'' \in V$, $\gamma'' \in \Gamma$ such that $(m_1\gamma' m')\gamma'' m'' \neq 0$. Indeed, if $(m_1\gamma' m')\gamma'' m'' = 0$, for all $m'' \in V$, $\gamma'' \in \Gamma$, then $m_1\gamma' m' \in l_\Gamma(V)$. Also, we know that $l_\Gamma(V) = (0)$. Hence we get $m_1\gamma' m' = 0$. This is a contradiction.

Setting $m = m'\gamma'' m''$, we obtain that $m_1\gamma' m \neq 0$
and $m_2\gamma' m = m_2\gamma'(m'\gamma'' m'') = (m_2\gamma' m')\gamma'' m'' \in U\Gamma V$. Therefore,
$U\Gamma V \in D(\Gamma, M)$. □

Corollary 1. *Let M be a semiprime Γ-ring and U be a right ideal of M. Then $U \in D(\Gamma, M)$ if and only if $l_\Gamma((a : U)_\Gamma) = (0)$ for all $a \in M$.*

Proof. If $U \in D(\Gamma, M)$, then $(a : U)_\Gamma \in D(\Gamma, M)$ for all $a \in M$ by Theorem 1(ii). By Theorem 1(v), we have $l_\Gamma((a : U)_\Gamma) = (0)$ for all $a \in M$. Conversely, let $m_1 \neq 0$, $m_2 \in M$. Since $m_1 \neq 0$, we get $m_1 \notin l_\Gamma((m_2 : U)_\Gamma)$. Thus, $m_1\gamma m \neq 0$ for some $m \in (m_2 : U)_\Gamma$, $\gamma \in \Gamma$. That is, $m_1\gamma m \neq 0$ and $m_2\gamma m \in U$ for some $m \in M$, $\gamma \in \Gamma$. We get $U \in D(\Gamma, M)$. This completes the proof. □

We are now in a position to construct the desired gamma ring of quotients of M. Let M be a semiprime Γ-ring. Consider the set

$$\aleph = \{(f;U) \mid U \in D(\Gamma, M)\}$$

$f : U \to M$ is a right ΓM-module homomorphism

We define $(f;U) \sim (g;V)$ if there exists $W \subseteq U \cap V$ such that $W \in D(\Gamma, M)$ and $f = g$ on W. By [3], one easily checks that "$\sim$" is an equivalence relation and we let $[f;U]$ denote the equivalence class determined by $(f;U) \in \aleph$. Let Q be the set of all equivalence classes. We define addition on Q as follow:

$$[f;U] + [g;V] = [f + g; U \cap V]$$

First of all we note that by Theorem 1(iii), $U \cap V \in D(\Gamma, M)$. Using similar arguments as in [3], we can prove that Q is a abelian additive group.

In a similar fasion, let (M, Γ) be a semiprime gamma ring. The collection of all dense left ideal of (M, Γ) will be denoted by $D(M, \Gamma)$.

$$\Re = \{(\tau;\Omega) \mid \Omega \in D(M, \Gamma),\ \tau : \Omega \to \Gamma \text{ is a left } M\Gamma\text{-module homomorphism}\}$$

We will show that Γ is a left $M\Gamma$-module. Indeed, since (M, Γ) is a gamma ring, Γ is abelian additive group. Also, for all $\gamma, \gamma_1, \gamma_2 \in \Gamma$, $m, m_1, m_2 \in M$ and $\alpha, \alpha_1, \alpha_2 \in \Gamma$,

(i) $\alpha m\gamma \in \Gamma$,
(ii) $\alpha m(\gamma_1 + \gamma_2) = \alpha m\gamma_1 + \alpha m\gamma_2$,
(iii) $\alpha(m_1 + m_2)\gamma = \alpha m_1\gamma + \alpha m_2\gamma$,
(iv) $(\alpha_1 + \alpha_2)m\gamma = \alpha_1 m\gamma + \alpha_2 m\gamma$.

On the other hand, Ω is selected to be left $M\Gamma$-module.

We define $(\tau;\Omega)\approx(\sigma;\Lambda)$ if there exists $\Pi\subseteq\Omega\cap\Lambda$ such that $\Pi\in D(M,\Gamma)$ and $\tau=\sigma$ on Π. " $\approx$ " is an equivalence relation. Let $[\tau;\Omega]$ denote the equivalence class determined by $(\tau;\Omega)\in\Re$ and Δ denote the set of all equivalence classes. We define addition of equivalence classes as follow:

$$[\tau;\Omega]+[\sigma;\Lambda]=[\tau+\sigma;\Omega\cap\Lambda]$$

In similar fasion, one easily checks that Δ is abelian additive group.

Let $\tau:\Omega\to\Gamma$ be a left $M\Gamma$-module homomorphism. Define

$$\overset{\Delta}{\tau}:M\Omega M\to M \text{ by } \overset{\Delta}{\tau}(m\gamma n)=m\tau(\gamma)n \text{ for all } m,n\in M,\ \gamma\in\Omega.$$

Now we define multiplication of equivalence classes as follow:

$$[f;U][\tau;\Omega][g;V]=\left[f\overset{\Delta}{\tau}g;(\overset{\Delta}{\tau}g)^{-1}(U)\right]$$

where $[f;U],[g;V]\in Q$ and $[\tau;\Omega]\in\Delta$. We will show that multiplication is well defined. For any $[f_1;U_1],[f_2;U_2],[g_1;V_1],[g_2;V_2]\in Q$ and $[\tau_1;\Omega_1],[\tau_2;\Omega_2]\in\Delta$, we get

$$([f_1;U_1],[\tau_1;\Omega_1],[g_1;V_1])=([f_2;U_2],[\tau_2;\Omega_2],[g_2;V_2]).$$

Then

$$[f_1;U_1]=[f_2;U_2],\ [\tau_1;\Omega_1]=[\tau_2;\Omega_2],\ [g_1;V_1]=[g_2;V_2],$$

and so

$$(f_1;U_1)\sim(f_2;U_2),\ (\tau_1;\Omega_1)\approx(\tau_2;\Omega_2),\ (g_1;V_1)\sim(g_2;V_2).$$

Hence, there exists $W_1,\ W_2\in D(\Gamma,M)$ such that $W_1\subseteq U_1\cap U_2,\ W_2\subseteq V_1\cap V_2$, $f_1=f_2$ on W_1, $g_1=g_2$ on W_2 and $\Pi\in D(M,\Gamma)$ such that $\Pi\subseteq\Omega_1\cap\Omega_2$, $\tau_1=\tau_2$ on Π.

Set $W=g_1^{-1}(W_1\Pi M)\cap W_2$. Using the same techniques in Theorem 1(vii), we prove that $W_1\Pi M\in D(\Gamma,M)$. By Theorem 1(i), we have $g_1^{-1}(W_1\Pi M)\in D(\Gamma,M)$. Also $g_1^{-1}(W_1\Pi M)\cap W_2\in D(\Gamma,M)$ by Theorem 1(iii).

Now, we will show that $W\subseteq(\overset{\Delta}{\tau_1}g_1)^{-1}(U_1)\cap(\overset{\Delta}{\tau_2}g_2)^{-1}(U_2)$. Let x be any element of $W=g_1^{-1}(W_1\Pi M)\cap W_2$. That is, $g_1(x)\in W_1\Pi M$ and $x\in W_2$. Taking $g_1(x)$ by $w_1\gamma m$, where $w_1\in W_1$, $\gamma\in\Pi$, $m\in M$, we get

$$\overset{\Delta}{\tau_1}g_1(x)=\overset{\Delta}{\tau_1}(g_1(x))=\overset{\Delta}{\tau_1}(w_1\gamma m)=w_1\tau_1(\gamma)m.$$

On the other hand, $g_1(x)=g_2(x)$ for $x\in W_2$, there by obtain

$$\overset{\Delta}{\tau_2}g_2(x)=\overset{\Delta}{\tau_2}(g_2(x))=\overset{\Delta}{\tau_2}(g_1(x))=\overset{\Delta}{\tau_2}(w_1\gamma m)=w_1\tau_2(\gamma)m.$$

For $w_1\in W_1\subseteq U_1\cap U_2$, we conclude that

$$\overset{\Delta}{\tau_1}g_1(x)=w_1\tau_1(\gamma)m\in U_1 \text{ and } \overset{\Delta}{\tau_2}g_2(x)=w_1\tau_2(\gamma)m\in U_2.$$

Hence, we find that $x \in (\overset{\Delta}{\tau_1} g_1)^{-1}(U_1)$ and $x \in (\overset{\Delta}{\tau_2} g_2)^{-1}(U_2)$, i.e., $x \in (\overset{\Delta}{\tau_1} g_1)^{-1}(U_1) \cap (\overset{\Delta}{\tau_2} g_2)^{-1}(U_2)$.

Moreover, we prove that $f_1 \overset{\Delta}{\tau_1} g_1 = f_2 \overset{\Delta}{\tau_2} g_2$ on W. Let $x \in W$. We know that $g_1(x) \in W_1 \Pi M$. Replacing $g_1(x)$ by $w_1 \gamma m$, where $w_1 \in W_1$, $\gamma \in \Pi$, $m \in M$, we have

$$(f_1 \overset{\Delta}{\tau_1} g_1)(x) = f_1(\overset{\Delta}{\tau_1}(g_1(x))) = f_1(\overset{\Delta}{\tau_1}(w_1 \gamma m)) = f_1(w_1 \tau_1(\gamma) m).$$

Using $w_1 \in W_1 \subseteq U_1$ and f_1 is a right ΓM-module homomorphism, we get

$$f_1(w_1 \tau_1(\gamma) m) = f_1(w_1) \tau_1(\gamma) m.$$

Since $f_1 = f_2$ on W_1 and $\tau_1 = \tau_2$ on Π, we obtain that

$$f_1(w_1)\tau_1(\gamma)m = f_2(w_1)\tau_2(\gamma)m = f_2(w_1 \tau_2(\gamma) m) = f_2(\overset{\Delta}{\tau_2}(w_1 \gamma m)).$$

Appliying $x \in W_2$ and $g_1 = g_2$ on W_2, we conclude that

$$(f_2 \overset{\Delta}{\tau_2})(w_1 \gamma m) = f_2(\overset{\Delta}{\tau_2}(g_1(x))) = (f_2 \overset{\Delta}{\tau_2} g_2)(x).$$

Hence we find that $f_1 \overset{\Delta}{\tau_1} g_1 = f_2 \overset{\Delta}{\tau_2} g_2$ on W, and also

Therefore,

$$\left[f_1 \overset{\Delta}{\tau_1} g_1; (\overset{\Delta}{\tau_1} g_1)^{-1}(U_1) \right] = \left[f_2 \overset{\Delta}{\tau_2} g_2; (\overset{\Delta}{\tau_2} g_2)^{-1}(U_2) \right].$$

As a result multiplication is well defined. Also, for all $[f;U], [g;V] \in Q$ and $[\tau;\Omega] \in \Delta$, we get $[f;U][\tau;\Omega][g;V] \in Q$. Indeed, we defined that $[f;U][\tau;\Omega][g;V] = \left[f \overset{\Delta}{\tau} g; (\overset{\Delta}{\tau} g)^{-1}(U) \right]$ by Theorem 1(i), we have $(\overset{\Delta}{\tau} g)^{-1}(U) \in D(\Gamma, M)$. It is clear that $f \overset{\Delta}{\tau} g$ is a right ΓM-module homomorphism. Thus, $[f;U][\tau;\Omega][g;V] \in Q$ for all $[f;U], [g;V] \in Q$ and $[\tau;\Omega] \in \Delta$.

Now we will show that (Δ, Q) is a gamma ring.

a) i) For all $[f;U], [g;V], [h;K] \in Q$ and $[\tau;\Omega] \in \Delta$,

$$\begin{aligned}
([f;U] + [g;V])[\tau;\Omega][h;K] &= [f+g; U \cap V][\tau;\Omega][h;K] \\
&= \left[(f+g) \overset{\Delta}{\tau} h; (\overset{\Delta}{\tau} h)^{-1}(U \cap V) \right] \\
&= \left[f \overset{\Delta}{\tau} h + g \overset{\Delta}{\tau} h; (\overset{\Delta}{\tau} h)^{-1}(U) \cap (\overset{\Delta}{\tau} h)^{-1}(V) \right] \\
&= \left[f \overset{\Delta}{\tau} h; (\overset{\Delta}{\tau} h)^{-1}(U) \right] + \left[g \overset{\Delta}{\tau} h; (\overset{\Delta}{\tau} h)^{-1}(V) \right] \\
&= [f;U][\tau;\Omega][h;K] + [g;V][\tau;\Omega][h;K].
\end{aligned}$$

ii) For all $[f;U],[g;V]\in Q$ and $[\tau;\Omega],[\sigma;\Lambda]\in \Delta$,

$$[f;U]([\tau;\Omega]+[\sigma;\Lambda])[g;V]=[f;U][\tau+\sigma;\Omega\cap\Lambda][g;V] \tag{2.1}$$
$$=\left[f\overset{\Delta}{(\tau+\sigma)}g;(\overset{\Delta}{(\tau+\sigma)}g)^{-1}(U)\right],$$

and

$$[f;U][\tau;\Omega][g;V]+[f;U][\sigma;\Lambda][g;V] \tag{2.2}$$
$$=\left[f\overset{\Delta}{\tau}g;(\overset{\Delta}{\tau}g)^{-1}(U)\right]+\left[f\overset{\Delta}{\sigma}g;(\overset{\Delta}{\sigma}g)^{-1}(U)\right] \tag{2.3}$$
$$=\left[f\overset{\Delta}{\tau}g+f\overset{\Delta}{\sigma}g;(\overset{\Delta}{\tau}g)^{-1}(U)\cap(\overset{\Delta}{\sigma}g)^{-1}(U)\right].$$

We will show that (2.1) and (2.3) are equivalent. Let $W=(\overset{\Delta}{\tau}g)^{-1}(U)\cap(\overset{\Delta}{\sigma}g)^{-1}(U)$ and $x\in W$. Then $x\in(\overset{\Delta}{\tau}g)^{-1}(U)$ and $x\in(\overset{\Delta}{\sigma}g)^{-1}(U)$, i.e., $(\overset{\Delta}{\tau}g)(x)\in U$ and $(\overset{\Delta}{\sigma}g)(x)\in U$. Thus $(\overset{\Delta}{\tau}g)(x)+(\overset{\Delta}{\sigma}g)(x)\in U$ and so $(\overset{\Delta}{\tau}+\overset{\Delta}{\sigma})g(x)\in U$. That is, $x\in(\overset{\Delta}{(\tau+\sigma)}g)^{-1}(U)$. Hence

$$W\subseteq(\overset{\Delta}{(\tau+\sigma)}g)^{-1}(U)\cap(\overset{\Delta}{\tau}g)^{-1}(U)\cap(\overset{\Delta}{\sigma}g)^{-1}(U).$$

Moreover, for all $w\in W$

$$(f\overset{\Delta}{(\tau+\sigma)}g)(w)=(f\overset{\Delta}{\tau}g)(w)+(f\overset{\Delta}{\sigma}g)(w)=(f\overset{\Delta}{\tau}g+f\overset{\Delta}{\sigma}g)(w).$$

iii) For all $[f;U],[g;V],[h;K]\in Q$ and $[\tau;\Omega]\in\Delta$,

$$\begin{aligned}[f;U][\tau;\Omega]([g;V]+[h;K])&=[f;U][\tau;\Omega][g+h;V\cap K]\\&=\left[f\overset{\Delta}{\tau}(g+h);(\overset{\Delta}{\tau}(g+h))^{-1}(U)\right]\\&=\left[f\overset{\Delta}{\tau}g+f\overset{\Delta}{\tau}h;(\overset{\Delta}{\tau}g)^{-1}(U)\cap(\overset{\Delta}{\tau}h)^{-1}(U)\right]\\&=\left[f\overset{\Delta}{\tau}g;(\overset{\Delta}{\tau}g)^{-1}(U)\right]+\left[f\overset{\Delta}{\tau}h;(\overset{\Delta}{\tau}h)^{-1}(U)\right]\\&=[f;U][\tau;\Omega][g;V]+[f;U][\tau;\Omega][h;K].\end{aligned}$$

Selected $W=(\overset{\Delta}{\tau}g)^{-1}(U)\cap(\overset{\Delta}{\tau}h)^{-1}(U)$, above equality is proved in anagous way.

b) i) For all $[f;U],[g;V],[h;K]\in Q$ and $[\tau;\Omega],[\sigma;\Lambda]\in\Delta$,

$$([f;U][\tau;\Omega][g;V])[\sigma;\Lambda][h;K]=\left[f\overset{\Delta}{\tau}g;(\overset{\Delta}{\tau}g)^{-1}(U)\right][\sigma;\Lambda][h;K] \tag{2.4}$$

$$= \left[(f \overset{\Delta}{\tau} g) \overset{\Delta}{\sigma} h; (\overset{\Delta}{\sigma} h)^{-1} (\overset{\Delta}{\tau} g)^{-1} (U) \right].$$

$$[f;U][\tau;\Omega]([g;V][\sigma;\Lambda][h;K]) = [f;U][\tau;\Omega]\left[g \overset{\Delta}{\sigma} h; (\overset{\Delta}{\sigma} h)^{-1} (V) \right] \tag{2.5}$$
$$= \left[f \overset{\Delta}{\tau} (g \overset{\Delta}{\sigma} h); (\overset{\Delta}{\tau} (g \overset{\Delta}{\sigma} h))^{-1} (U) \right]$$

We will prove that (2.4) and (2.5) are equivalent. Let $W = (\overset{\Delta}{\sigma} h)^{-1} (\overset{\Delta}{\tau} g)^{-1} (U)$ and $x \in W$. Then $(\overset{\Delta}{\sigma} h)(x) \in (\overset{\Delta}{\tau} g)^{-1} (U)$ and so $(\overset{\Delta}{\tau} g)(\overset{\Delta}{\sigma} h)(x) \in U$. Finally, $x \in (\overset{\Delta}{\tau} (g \overset{\Delta}{\sigma} h))^{-1} (U)$. Thus $W \subseteq (\overset{\Delta}{\sigma} h)^{-1} (\overset{\Delta}{\tau} g)^{-1} (U) \cap (\overset{\Delta}{\tau} (g \overset{\Delta}{\sigma} h))^{-1} (U)$, and also

$$((f \overset{\Delta}{\tau} g) \overset{\Delta}{\sigma} h)(w) = (f \overset{\Delta}{\tau} (g \overset{\Delta}{\sigma} h))(w), \text{ for all } w \in W.$$

ii) For all $[f;U], [g;V], [h;K] \in Q$ and $[\tau;\Omega], [\sigma;\Lambda] \in \Delta$,

$$\begin{aligned}
([f;U][\tau;\Omega][g;V])[\sigma;\Lambda][h;K] &= \left[f \overset{\Delta}{\tau} g; (\overset{\Delta}{\tau} g)^{-1} (U) \right][\sigma;\Lambda][h;K] \\
&= \left[(f \overset{\Delta}{\tau} g) \overset{\Delta}{\sigma} h; (\overset{\Delta}{\sigma} h)^{-1} (\overset{\Delta}{\tau} g)^{-1} (U) \right] \\
&= \left[f (\overset{\Delta}{\tau} g \overset{\Delta}{\sigma}) h; (\overset{\Delta}{\tau} g \overset{\Delta}{\sigma} h)^{-1} (U) \right] \\
&= [f;U][\overset{\Delta}{\tau} g \overset{\Delta}{\sigma}; (g\sigma)^{-1} (\Omega)][h;K] \\
&= [f;U]([\tau;\Omega][g;V][\sigma;\Lambda])[h;K].
\end{aligned}$$

Similar to the above can be shown.

c) Let $[f;U][\tau;\Omega][g;V] = [0;M]$ for all $[f;U], [g;V] \in Q$ and $[\tau;\Omega] \in \Delta$. For $[I_M;M] \in Q$, where $I_M : M \to M$, is an identity right ΓM-module homomorphism, we get

$$[I_M;M][\tau;\Omega][I_M;M] = [0;M],$$

and so

$$\left[I_M \overset{\Delta}{\tau} I_M; (\overset{\Delta}{\tau} I_M)^{-1} (M) \right] = [0;M].$$

Let $W = M\Omega M$. We first prove that $M\Omega M \subseteq M \cap (\overset{\Delta}{\tau} I_M)^{-1} (M)$. For any $x = m\gamma n \in M\Omega M$, where $m, n \in M$, $\gamma \in \Omega$, we have

$$(\overset{\Delta}{\tau} I_M)(x) = (\overset{\Delta}{\tau} I_M)(m\gamma n) = \overset{\Delta}{\tau} (I_M (m\gamma n)) = \overset{\Delta}{\tau} (m\gamma n) = m\tau(\gamma) n.$$

Thus we get $(\overset{\Delta}{\tau} I_M)(x) \in M$, and so $x \in (\overset{\Delta}{\tau} I_M)^{-1} (M)$. On the other hand, we know that $x \in M$. Hence $x \in M \cap (\overset{\Delta}{\tau} I_M)^{-1} (M)$.

Now, for all $w = m\gamma n$, where $m,n \in M$, $\gamma \in \Omega$,

$$I_M \overset{\Delta}{\tau} I_M(w) = I_M \overset{\Delta}{\tau} I_M(m\gamma n) = 0(m\gamma n).$$

That is

$$m\tau(\gamma)n = 0, \text{ for all } m,n \in M \text{ and } \gamma \in \Omega.$$

Since (Γ, M) is gamma ring, we have $\tau(\gamma) = 0$, for all $\gamma \in \Omega$, and so, $\tau = 0$. That is, $[\tau;\Omega] = [0;\Gamma]$.

So, we see that (Δ, Q) is a gamma ring and we call it the maximal right gamma ring of quotients of (Γ, M).

Theorem 2. *Let (M,Γ) be a semiprime gamma ring. The collection of all dense left ideal of (M,Γ) will be denoted by $D(M,\Gamma)$, $\Omega, \Lambda \in D(M,\Gamma)$ and $\tau : \Omega \to \Gamma$ be a left $M\Gamma-$module homomorphism. Then*

i) $\tau^{-1}(\Lambda) = \{\gamma \in \Omega \mid \tau(\gamma) \in \Lambda\} \in D(M,\Gamma)$.

ii)$(\alpha : \Omega)_M \in D(M,\Gamma)$ *for all* $\alpha \in \Gamma$.

iii) $\Omega \cap \Lambda \in D(M,\Gamma)$.

iv) If Π *is a left ideal of* Γ *and* $\Omega \subseteq \Pi$, *then* $\Pi \in D(M,\Gamma)$.

v) $l_M(\Omega) = (0) = r_M(\Omega)$.

vi) If Π *is a left ideal of* Γ *and* $(\alpha : \Pi)_M \in D(M,\Gamma)$ *for all* $\alpha \in \Gamma$, *then* $\Pi \in D(M,\Gamma)$.

vii) $\Omega M \Lambda \in D(M,\Gamma)$.

Proof. The procedures in Theorem 1 can be exactly applied in set $D(M,\Gamma)$ and the same results are obtained. □

One can construct the maximal left gamma ring of quotient of (M,Γ) using the following operations. Now we can define multiplication of equivalence classes as follow:

$$[\tau;\Omega][f;U][\sigma;\Lambda] = \left[\tau \overset{\Delta}{f} \sigma; (\overset{\Delta}{f}\sigma)^{-1}(\Omega)\right]$$

where $\overset{\Delta}{f} : \Gamma U \Gamma \to \Gamma$, $\overset{\Delta}{f}(\gamma m \beta) = \gamma f(m)\beta$ and $[f;U] \in Q$ and $[\tau;\Omega],[\sigma;\Lambda] \in \Delta$.

We will prove that multiplication is well defined. For any $[\tau_1;\Omega_1],[\tau_2;\Omega_2],[\sigma_1;\Lambda_1],[\sigma_2;\Lambda_2] \in \Delta$ and $[f_1;U_1],[f_2;U_2] \in Q$, we get

$$([\tau_1;\Omega_1],[f_1;U_1],[\sigma_1;\Lambda_1]) = ([\tau_2;\Omega_2],[f_2;U_2],[\sigma_2;\Lambda_2]).$$

Then

$$[\tau_1;\Omega_1] = [\tau_2;\Omega_2],\ [f_1;U_1] = [f_2;U_2],\ [\sigma_1;\Lambda_1],[\sigma_2;\Lambda_2]$$

and so

$$(\tau_1;\Omega_1) \approx (\tau_2;\Omega_2),\ (f_1;U_1) \sim (f_2;U_2),\ (\sigma_1;\Lambda_1) \approx (\sigma_2;\Lambda_2).$$

Therefore, there exists Π_1, $\Pi_2 \in D(M,\Gamma)$ such that $\Pi_1 \subseteq \Omega_1 \cap \Omega_2$, $\Pi_2 \subseteq \Lambda_1 \cap \Lambda_2, \tau_1 = \tau_2$ on Π_1, $\sigma_1 = \sigma_2$ on Π_2 and $W \in D(\Gamma, M)$ such that $W \subseteq U_1 \cap U_2$, $f_1 = f_2$ on W.

Let $\Pi = \sigma_1^{-1}(\Gamma W \Pi_1) \cap \Pi_2$. Using Theorem 2(vii), we show that $\Gamma W \Pi_1 \in D(M,\Gamma)$. By Theorem 2(i), we see that $\sigma_1^{-1}(\Gamma W \Pi_1) \in D(M,\Gamma)$. Hence $\sigma_1^{-1}(\Gamma W \Pi_1) \cap \Pi_2 \in D(M,\Gamma)$ by Theorem 2(iii).

Now, we will show that $\Pi \subseteq (\overset{\Delta}{f_1}\sigma_1)^{-1}(\Omega_1) \cap (\overset{\Delta}{f_2}\sigma_2)^{-1}(\Omega_2)$. Let α be any element of $\Pi = \sigma_1^{-1}(\Gamma W \Pi_1) \cap \Pi_2$. That is $\sigma_1(\alpha) \in \Gamma W \Pi_1$ and $\alpha \in \Pi_2$. Replacing $\sigma_1(\alpha)$ by $\gamma w \beta_1$, where $\beta_1 \in \Pi_1$, $w \in W$, $\gamma \in \Gamma$, yields that

$$(\overset{\Delta}{f_1}\sigma_1)(\alpha) = \overset{\Delta}{f_1}(\sigma_1(\alpha)) = \overset{\Delta}{f_1}(\gamma w \beta_1) = \gamma f_1(w)\beta_1.$$

On the other hand, $\sigma_1(\alpha) = \sigma_2(\alpha)$ for $\alpha \in \Pi_2$, we get

$$(\overset{\Delta}{f_2}\sigma_2)(\alpha) = \overset{\Delta}{f_2}(\sigma_2(\alpha)) = \overset{\Delta}{f_2}(\sigma_1(\alpha)) = \overset{\Delta}{f_2}(\gamma w \beta_1) = \gamma f_2(w)\beta_1.$$

For $\beta_1 \in \Pi_1 \subseteq \Omega_1 \cap \Omega_2$, we see that

$$(\overset{\Delta}{f_1}\sigma_1)(\alpha) = \gamma f_1(w)\beta_1 \in \Omega_1 \text{ and } (\overset{\Delta}{f_2}\sigma_2)(\alpha) = \gamma f_2(w)\beta_1 \in \Omega_2.$$

Hence, we arrive that $\alpha \in (\overset{\Delta}{f_1}\sigma_1)^{-1}(\Omega_1)$ and $\alpha \in (\overset{\Delta}{f_2}\sigma_2)^{-1}(\Omega_2)$, i.e., $\alpha \in (\overset{\Delta}{f_1}\sigma_1)^{-1}(\Omega_1) \cap (\overset{\Delta}{f_2}\sigma_2)^{-1}(\Omega_2)$.

Moreover, we show that $\tau_1 \overset{\Delta}{f_1}\sigma_1 = \tau_2 \overset{\Delta}{f_2}\sigma_2$ on Π. Let $\alpha \in \Pi$. We obtain that $\sigma_1(\alpha) \in \Gamma W \Pi_1$. Substituting $\gamma w \beta_1$ for $\sigma_1(\alpha)$, where $\beta_1 \in \Pi_1$, $w \in W$, $\gamma \in \Gamma$, we have

$$(\tau_1 \overset{\Delta}{f_1}\sigma_1)(\alpha) = \tau_1(\overset{\Delta}{f_1}(\sigma_1(\alpha))) = \tau_1(\overset{\Delta}{f_1}(\gamma w \beta_1)) = \tau_1(\gamma f_1(w)\beta_1).$$

Using $\beta_1 \in \Pi_1 \subseteq \Omega_1$ and τ_1 is a left $M\Gamma$-module homomorphism, we arrive that

$$\tau_1(\gamma f_1(w)\beta_1) = \gamma f_1(w)\tau_1(\beta_1).$$

Since $\tau_1 = \tau_2$ on Π_1 and $f_1 = f_2$ on W, we find that

$$\gamma f_1(w)\tau_1(\beta_1) = \gamma f_2(w)\tau_2(\beta_1) = \tau_2(\gamma f_2(w)\beta_1) = \tau_2(\overset{\Delta}{f_2}(\gamma w \beta_1)).$$

Using $\alpha \in \Pi_2$ and $\sigma_1 = \sigma_2$ on Π_2, we see that

$$(\tau_2 \overset{\Delta}{f_2})(\gamma w \beta_1) = \tau_2(\overset{\Delta}{f_2}(\sigma_1(\alpha))) = (\tau_2 \overset{\Delta}{f_2}\sigma_2)(\alpha).$$

Thus, we conclude that $\tau_1 \overset{\Delta}{f_1}\sigma_1 = \tau_2 \overset{\Delta}{f_2}\sigma_2$ on Π, and so

$$\left[\tau_1 \overset{\Delta}{f_1}\sigma_1; (\overset{\Delta}{f_1}\sigma_1)^{-1}(\Omega_1)\right] = \left[\tau_2 \overset{\Delta}{f_2}\sigma_2; (\overset{\Delta}{f_2}\sigma_2)^{-1}(\Omega_2)\right].$$

Consequently, multiplication is well defined. It is clearly that $[\tau;\Omega][f;U][\sigma;\Lambda]\in\Delta$ for all $[f;U]\in Q$ and $[\tau;\Omega],[\sigma;\Lambda]\in\Delta$.

Now we will show that (Q,Δ) is a gamma ring.

a) i) For all $[f;U]\in Q$ and $[\tau;\Omega],[\sigma;\Lambda],[\delta;\Sigma]\in\Delta$,

$$\begin{aligned}([\tau;\Omega]+[\sigma;\Lambda])[f;U][\delta;\Sigma] &= [\tau+\sigma;\Omega\cap\Lambda][f;U][\delta;\Sigma]\\ &= \left[(\tau+\sigma)\overset{\Delta}{f}\delta;(\overset{\Delta}{f}\delta)^{-1}(\Omega\cap\Lambda)\right]\\ &= \left[\tau\overset{\Delta}{f}\delta+\sigma\overset{\Delta}{f}\delta;(\overset{\Delta}{f}\delta)^{-1}(\Omega)\cap(\overset{\Delta}{f}\delta)^{-1}(\Lambda)\right]\\ &= \left[\tau\overset{\Delta}{f}\delta;(\overset{\Delta}{f}\delta)^{-1}(\Omega)\right]+\left[\sigma\overset{\Delta}{f}\delta;(\overset{\Delta}{f}\delta)^{-1}(\Lambda)\right]\\ &= [\tau;\Omega][f;U][\delta;\Sigma]+[\sigma;\Lambda][f;U][\delta;\Sigma].\end{aligned}$$

ii) For all $[f;U],[g;V]\in Q$ and $[\tau;\Omega],[\sigma;\Lambda]\in\Delta$,

$$\begin{aligned}[\tau;\Omega]([f;U]+[g;V])[\sigma;\Lambda] &= [\tau;\Omega][f+g;U\cap V][\sigma;\Lambda] \qquad (2.6)\\ &= \left[\tau(f\overset{\Delta}{+}g)\sigma;((f\overset{\Delta}{+}g)\sigma)^{-1}(\Omega)\right],\end{aligned}$$

and

$$\begin{aligned}&[\tau;\Omega][f;U][\sigma;\Lambda]+[\tau;\Omega][g;V][\sigma;\Lambda] \qquad (2.7)\\ &= \left[\tau\overset{\Delta}{f}\sigma;(\overset{\Delta}{f}\sigma)^{-1}(\Omega)\right]+\left[\tau\overset{\Delta}{g}\sigma;(\overset{\Delta}{g}\sigma)^{-1}(\Omega)\right] \qquad (2.8)\\ &= \left[\tau\overset{\Delta}{f}\sigma+\tau\overset{\Delta}{g}\sigma;(\overset{\Delta}{f}\sigma)^{-1}(\Omega)\cap(\overset{\Delta}{g}\sigma)^{-1}(\Omega)\right].\end{aligned}$$

We will show that (2.6) and (2.8) are equivalent. Let $\Pi=(\overset{\Delta}{f}\sigma)^{-1}(\Omega)\cap(\overset{\Delta}{g}\sigma)^{-1}(\Omega)$ and $\alpha\in\Pi$. Then $\alpha\in(\overset{\Delta}{f}\sigma)^{-1}(\Omega)$ and $\alpha\in(\overset{\Delta}{g}\sigma)^{-1}(\Omega)$, i.e., $(\overset{\Delta}{f}\sigma)(\alpha)\in\Omega$ and $(\overset{\Delta}{g}\sigma)(\alpha)\in\Omega$. Thus $(\overset{\Delta}{f}\sigma)(\alpha)+(\overset{\Delta}{g}\sigma)(\alpha)\in\Omega$, and so $(\overset{\Delta}{f}+\overset{\Delta}{g})\sigma(\alpha)\in\Omega$. That is $\alpha\in((f\overset{\Delta}{+}g)\sigma)^{-1}(\Omega)$. Then

$$\Pi\subseteq((f\overset{\Delta}{+}g)\sigma)^{-1}(\Omega)\cap(\overset{\Delta}{f}\sigma)^{-1}(\Omega)\cap(\overset{\Delta}{g}\sigma)^{-1}(\Omega).$$

Moreover, for all $\alpha\in\Pi$

$$(\tau(f\overset{\Delta}{+}g)\sigma)(\alpha)=(\tau\overset{\Delta}{f}\sigma)(\alpha)+(\tau\overset{\Delta}{g}\sigma)(\alpha)=(\tau\overset{\Delta}{f}\sigma+\tau\overset{\Delta}{g}\sigma)(\alpha).$$

iii) For all $[f;U]\in Q$ and $[\tau;\Omega],[\sigma;\Lambda],[\delta;\Sigma]\in\Delta$,

$$[\tau;\Omega][f;U]([\sigma;\Lambda]+[\delta;\Sigma])=[\tau;\Omega][f;U][\sigma+\delta;\Lambda\cap\Sigma]$$

$$
\begin{aligned}
&= \left[\tau \overset{\Delta}{f}(\sigma+\delta); (\overset{\Delta}{f}(\sigma+\delta))^{-1}(\Omega)\right] \\
&= \left[\tau \overset{\Delta}{f}\sigma + \tau \overset{\Delta}{f}\delta; (\overset{\Delta}{f}\sigma)^{-1}(\Omega) \cap (\overset{\Delta}{f}\delta))^{-1}(\Omega)\right] \\
&= \left[\tau \overset{\Delta}{f}\sigma; (\overset{\Delta}{f}\sigma)^{-1}(\Omega)\right] + \left[\tau \overset{\Delta}{f}\delta; (\overset{\Delta}{f}\delta))^{-1}(\Omega)\right] \\
&= [\tau;\Omega][f;U][\sigma;\Lambda] + [\tau;\Omega][f;U][\delta;\Sigma].
\end{aligned}
$$

Set $\Pi = (\overset{\Delta}{f}\sigma)^{-1}(\Omega) \cap (\overset{\Delta}{f}\delta))^{-1}(\Omega)$, application of similar arguments yields the above equality.

b) i) For all $[f;U],[g;V] \in Q$ and $[\tau;\Omega],[\sigma;\Lambda],[\delta;\Sigma] \in \Delta$,

$$
\begin{aligned}
([\tau;\Omega][f;U][\sigma;\Lambda])[g;V][\delta;\Sigma] &= \left[\tau \overset{\Delta}{f}\sigma; (\overset{\Delta}{f}\sigma)^{-1}(\Omega)\right][g;V][\delta;\Sigma] \qquad (2.9)\\
&= \left[(\tau \overset{\Delta}{f}\sigma)\overset{\Delta}{g}\delta; (\overset{\Delta}{g}\delta)^{-1}(\overset{\Delta}{f}\sigma)^{-1}(\Omega)\right].
\end{aligned}
$$

$$
\begin{aligned}
[\tau;\Omega][f;U]([\sigma;\Lambda][g;V][\delta;\Sigma]) &= [\tau;\Omega][f;U]\left[\sigma \overset{\Delta}{g}\delta; (\overset{\Delta}{g}\delta)^{-1}(\Lambda)\right] \qquad (2.10)\\
&= \left[\tau \overset{\Delta}{f}(\sigma \overset{\Delta}{g}\delta); (\overset{\Delta}{f}(\sigma \overset{\Delta}{g}\delta))^{-1}(\Omega)\right]
\end{aligned}
$$

We will prove that (2.9) and (2.10) are equivalent. Let $\Pi = (\overset{\Delta}{g}\delta)^{-1}(\overset{\Delta}{f}\sigma)^{-1}(\Omega)$ and $\alpha \in \Pi$. Then $(\overset{\Delta}{g}\delta)(\alpha) \in (\overset{\Delta}{f}\sigma)^{-1}(\Omega)$, and so $(\overset{\Delta}{f}\sigma)(\overset{\Delta}{g}\delta)(\alpha) \in \Omega$. Hence, $\alpha \in (\overset{\Delta}{f}(\sigma \overset{\Delta}{g}\delta))^{-1}(\Omega)$. Thus $\Pi \subseteq (\overset{\Delta}{g}\delta)^{-1}(\overset{\Delta}{f}\sigma)^{-1}(\Omega) \cap (\overset{\Delta}{f}(\sigma \overset{\Delta}{g}\delta))^{-1}(\Omega)$, and also

$$((\tau \overset{\Delta}{f}\sigma)\overset{\Delta}{g}\delta)(\alpha) = (\tau \overset{\Delta}{f}(\sigma \overset{\Delta}{g}\delta))(\alpha), \text{ for all } \alpha \in \Pi.$$

ii) For all $[f;U],[g;V] \in Q$ and $[\tau;\Omega],[\sigma;\Lambda],[\delta;\Sigma] \in \Delta$,

$$
\begin{aligned}
([\tau;\Omega][f;U][\sigma;\Lambda])[g;V][\delta;\Sigma] &= \left[\tau \overset{\Delta}{f}\sigma; (\overset{\Delta}{f}\sigma)^{-1}(\Omega)\right][g;V][\delta;\Sigma] \\
&= \left[(\tau \overset{\Delta}{f}\sigma)\overset{\Delta}{g}\delta; (\overset{\Delta}{g}\delta)^{-1}(\overset{\Delta}{f}\sigma)^{-1}(\Omega)\right] \\
&= \left[\tau(\overset{\Delta}{f}\sigma\overset{\Delta}{g})\delta; (\overset{\Delta}{f}\sigma\overset{\Delta}{g}\delta)^{-1}(\Omega)\right] \\
&= [\tau;\Omega][\overset{\Delta}{f}\sigma\overset{\Delta}{g}; (\sigma\overset{\Delta}{g})^{-1}(U)][\delta;\Sigma] \\
&= [\tau;\Omega]([f;U][\sigma;\Lambda][g;V])[\delta;\Sigma].
\end{aligned}
$$

Using similar arguments as above, we can prove.

c) Let $[\tau;\Omega][f;U][\sigma;\Lambda]=[0;\Gamma]$ for all $[f;U]\in Q$ and $[\tau;\Omega],[\sigma;\Lambda]\in\Delta$. For $[I_\Gamma;\Gamma]\in\Delta$, where $I_\Gamma:\Gamma\to\Gamma$ is an identity left $M\Gamma$-module homomorphism, we see that

$$[I_\Gamma;\Gamma][f;U][I_\Gamma;\Gamma]=[0;\Gamma],$$

and so

$$\left[I_\Gamma \overset{\Delta}{f} I_\Gamma;(\overset{\Delta}{f} I_\Gamma)^{-1}(\Gamma)\right]=[0;\Gamma].$$

Set $\Pi=\Gamma U\Gamma$. We first prove that $\Gamma U\Gamma\subseteq\Gamma\cap(\overset{\Delta}{f} I_\Gamma)^{-1}(\Gamma)$. For any $\gamma=\alpha u\beta\in\Gamma U\Gamma$ where $u\in U$, $\alpha,\beta\in\Gamma$, we arrive at

$$(\overset{\Delta}{f} I_\Gamma)(\gamma)=(\overset{\Delta}{f} I_\Gamma)(\alpha u\beta)=\overset{\Delta}{f}(I_\Gamma(\alpha u\beta))=\overset{\Delta}{f}(\alpha u\beta)=\alpha f(u)\beta.$$

We have $(\overset{\Delta}{f} I_\Gamma)(\gamma)\in\Gamma$, and so $\gamma\in(\overset{\Delta}{f} I_\Gamma)^{-1}(\Gamma)$. On the other hand, we get $\gamma\in\Gamma$. As a result $\gamma\in\Gamma\cap(\overset{\Delta}{f} I_\Gamma)^{-1}(\Gamma)$.

Now, for all $\gamma=\alpha u\beta\in\Gamma U\Gamma$ where $u\in U$, $\alpha,\beta\in\Gamma$,

$$I_\Gamma \overset{\Delta}{f} I_\Gamma(\gamma)=I_\Gamma \overset{\Delta}{f} I_\Gamma(\alpha u\beta)=0(\alpha u\beta).$$

That is

$$\alpha f(u)\beta=0,\ \text{for all } u\in U,\alpha,\beta\in\Gamma.$$

Since (M,Γ) is gamma ring, we get $f(u)=0$, for all $u\in U$, and so $f=0$. That is $[f;U]=[0;M]$.

We arrive that (Q,Δ) is a gamma ring and we call it the maximal left gamma ring of quotient of (M,Γ).

We proceed by showing that (Δ,Q) is characterized by certain reasonable properties that any gamma ring of quotient should have.

Remark 1. Let M be a Γ-ring, $\varepsilon\in\Gamma,e\in M$. If (ε,e) is the strong right identity element of (Γ,M), then (e,ε) is the strong left identity element of (M,Γ).

Proof. Assume that (ε,e) is the strong right identity element of (Γ,M). Thus we get $x\varepsilon e=x$ for all $x\in M$. For any $\gamma\in\Gamma$, $x,y\in M$, we have

$$x(\varepsilon e\gamma-\gamma)y=x(\varepsilon e\gamma)y-x\gamma y=(x\varepsilon e)\gamma y-x\gamma y=x\gamma y-x\gamma y=0$$

and so

$$M(\varepsilon e\gamma-\gamma)M=(0).$$

Since (Γ,M) is gamma ring, we get

$$\varepsilon e\gamma=\gamma \text{ for all } \gamma\in\Gamma.$$

□

Using similar arguments as above, we can prove the followings:

Remark 2. Let M be a Γ-ring, $\varepsilon \in \Gamma, e \in M$. If (ε, e) is the strong left identity element of (Γ, M), then (e, ε) is the strong right identity element of (M, Γ).

Let (ε, e) is the strong right identity element of (Γ, M). For a fixed element a in M, consider a mapping $\lambda_{a\varepsilon} : M \to M$ defined by $\lambda_{a\varepsilon}(x) = a\varepsilon x$ for all $x \in M$. It is easy to prove that the mapping $\lambda_{a\varepsilon}$ is a right $\Gamma M-$module homomorphism. Moreover, for all $m \in M$,

$$\lambda_{(a+b)\varepsilon}(m) = (a+b)\varepsilon m = a\varepsilon m + b\varepsilon m = \lambda_{a\varepsilon}(m) + \lambda_{b\varepsilon}(m) = (\lambda_{a\varepsilon} + \lambda_{b\varepsilon})(m),$$

and so

$$\lambda_{(a+b)\varepsilon} = (\lambda_{a\varepsilon} + \lambda_{b\varepsilon}). \tag{2.11}$$

Now consider a mapping $\mu_{e\beta} : \Gamma \to \Gamma$ defined by $\mu_{e\beta}(\alpha) = \alpha e\beta$ for all $\alpha \in \Gamma$ is a left $M\Gamma-$module homomorphism. Indeed, we shown that Γ is a left $M\Gamma-$module.

i) For all $\alpha, \gamma \in \Gamma$,

$$\mu_{e\beta}(\alpha + \gamma) = (\alpha + \gamma)e\beta = \alpha e\beta + \gamma e\beta.$$

ii) For all $\alpha, \gamma \in \Gamma$ and $m \in M$,

$$\mu_{e\beta}(\alpha m\gamma) = (\alpha m\gamma)e\beta = \alpha m(\gamma e\beta) = \alpha m\mu_{e\beta}(\gamma).$$

Thus, $\mu_{e\beta}$ is a left $M\Gamma-$module homomorphism.

Using arguments as above, we can prove the followings:

$$\mu_{e(\beta+\gamma)} = \mu_{e\beta} + \mu_{e\gamma}. \tag{2.12}$$

Let

$$\wp = \{\lambda_{a\varepsilon} \,|\, a \in M\} \text{ and } \mho = \{\mu_{e\beta} \,|\, \beta \in \Gamma\}.$$

These sets are additive groups and defining the mappings

$$\wp \times \mho \times \wp \to \wp, \ (\lambda_{x\varepsilon}, \mu_{e\gamma}, \lambda_{y\varepsilon}) \mapsto \lambda_{x\varepsilon e\gamma y\varepsilon} = \lambda_{x\gamma y\varepsilon} \tag{2.13}$$

and

$$\mho \times \wp \times \mho \to \mho, \ (\mu_{e\gamma}, \lambda_{x\varepsilon}, \mu_{e\beta}) \mapsto \mu_{e\gamma x\varepsilon e\beta} = \mu_{e\gamma x\beta}. \tag{2.14}$$

It can be shown that $(\mho, \wp)$ is a gamma ring.

In the following theorem, $D(\Gamma, M)$ will denote collection of all dense ideal of M.

Theorem 3. *Let (Γ, M) be a semiprime gamma ring with strong right identity element. Then (Γ, M) is a subring of (Δ, Q).*

Proof. Consider the mappings

$$\phi_\varepsilon : M \to Q, \ a \mapsto [\lambda_{a\varepsilon}; M] \text{ and } \psi_e : \Gamma \mapsto \Delta, \ \beta \to [\mu_{e\beta}; \Gamma].$$

One readily checks that ϕ_ε is well defined. Using (2.11) equation, we have

$$\phi_\varepsilon(a+b) = [\lambda_{(a+b)\varepsilon}; M] = [\lambda_{a\varepsilon}; M] + [\lambda_{b\varepsilon}; M] = \phi_\varepsilon(a) + \phi_\varepsilon(b),$$

$$\text{for all } a, b \in M.$$

Hence, ϕ_ε is a group homomorphism. Now, we suppose that $\phi_\varepsilon(a)=\phi_\varepsilon(b)$. Then $[\lambda_{a\varepsilon};M]=[\lambda_{b\varepsilon};M]$, and so $(\lambda_{a\varepsilon};M)\sim(\lambda_{b\varepsilon};M)$. Hence there exists $W\subseteq M\cap M$ such that $W\in D(\Gamma,M)$ and $\lambda_{a\varepsilon}=\lambda_{b\varepsilon}$ on W, i.e., $a\varepsilon w=b\varepsilon w$, for all $w\in W$. That is $(a-b)\varepsilon W=(0)$. Since W is a ideal of M, we have $(a-b)\varepsilon M\Gamma W=(0)$, and so $(a-b)\varepsilon M\in l_\Gamma(W)$. By Theorem 1(v), we get $(a-b)\varepsilon M=(0)$. In particular, taking m by e, we get $(a-b)\varepsilon e=0$, and so $a=b$. Thus ϕ_ε is one-one. Hence ϕ_ε is a group monomorphism.

On the other hand, once easly checks that ψ_e is a group monomorphism.

For all $\alpha,\beta\in\Gamma$, $m,n\in M$

$$\phi_\varepsilon(m\beta n)=\left[\lambda_{(m\beta n)\varepsilon};M\right]=\left[\lambda_{m\varepsilon e\beta n\varepsilon};M\right], \tag{2.15}$$

and

$$\phi_\varepsilon(m)\psi_e(\beta)\phi_\varepsilon(n)=[\lambda_{m\varepsilon};M]\left[\mu_{e\beta};\Gamma\right][\lambda_{n\varepsilon};M] \tag{2.16}$$

$$=[\lambda_{m\varepsilon}\overset{\Delta}{\mu}_{e\beta}\lambda_{n\varepsilon};(\overset{\Delta}{\mu}_{e\beta}\lambda_{n\varepsilon})^{-1}(M)].$$

We will show that (2.15) and (2) are equivalent. Set $W=M\Gamma M$ and $w=x\gamma y\in W$, where $\gamma\in\Gamma$, $x,y\in M$.

$$(\overset{\Delta}{\mu}_{e\beta}\lambda_{n\varepsilon})(w)=(\overset{\Delta}{\mu}_{e\beta}\lambda_{n\varepsilon})(x\gamma y)=\overset{\Delta}{\mu}_{e\beta}(n\varepsilon x\gamma y)=n\varepsilon x\mu_{e\beta}(\gamma)y=n\varepsilon x\gamma e\beta y.$$

Thus $(\overset{\Delta}{\mu}_{e\beta}\lambda_{n\varepsilon})(w)\in M$, and so $w\in(\overset{\Delta}{\mu}_{e\beta}\lambda_{n\varepsilon})^{-1}(M)$. Hence we obtain that $W\subseteq(\overset{\Delta}{\mu}_{e\beta}\lambda_{n\varepsilon})^{-1}(M)\cap M$.

On the other hand, using (2.13), we conclude that

$$\lambda_{m\varepsilon}\overset{\Delta}{\mu}_{e\beta}\lambda_{n\varepsilon}=\lambda_{m\varepsilon e\beta n\varepsilon}=\lambda_{(m\beta n)\varepsilon}.$$

That is $\lambda_{m\varepsilon}\overset{\Delta}{\mu}_{e\beta}\lambda_{n\varepsilon}=\lambda_{(m\beta n)\varepsilon}$ on W. In that case, we get

$$\phi_\varepsilon(m\beta n)=\phi_\varepsilon(m)\psi_e(\beta)\phi_\varepsilon(n).$$

Using similar arguments as above, we can shown that

$$\psi_e(\alpha m\gamma)=\psi_e(\alpha)\phi_\varepsilon(m)\psi_e(\gamma).$$

Hence $(\psi_e,\phi_\varepsilon)$ is a gamma ring monomorphism, and so (Γ,M) is a subring of (Δ,Q). □

Theorem 4. *Let (Γ,M) be a semiprime gamma ring with strong left identity element. (Δ,Q) satisfies the following properties:*

i) If $U\in D(\Gamma,M)$ and $f:U\to M$ is a right $\Gamma M-$module homomorphism, then there exists an element $q\in Q$ such that $f(u)=q\varepsilon u$ for all $u\in U$.

ii) For all $q\in Q$ there exists $U\in D(\Gamma,M)$ such that $q\varepsilon U\subseteq M$.

iii) For all $q\in Q$ and $U\in D(\Gamma,M)$, $q\varepsilon U=(0)$ if and only if $q=0$.

Proof. i) Let $q \in Q$, $u \in U$, $f : U \to M$ be a right $\Gamma M-$module homomorphism and $q = [f;U]$. Since M can be embedded in Q, we can write $u = [\lambda_{u\varepsilon};M]$ such that $\lambda_{u\varepsilon} : M \to M$, $x \mapsto u\varepsilon x$ for all $u \in U$. Then we have

$$q\varepsilon u = [f;U][\mu_{e\varepsilon};\Gamma][\lambda_{u\varepsilon};M] = [f\overset{\Delta}{\mu}_{e\varepsilon}\lambda_{u\varepsilon};(\overset{\Delta}{\mu}_{e\varepsilon}\lambda_{u\varepsilon})^{-1}(U)] \tag{2.17}$$

and

$$f(u) = \left[\lambda_{f(u)\varepsilon};M\right]. \tag{2.18}$$

We will show that (2.17) and (2.18) are equivalent. Indeed, for $W = U\Gamma M$ and $x = y\gamma m \in W$, where $y \in U$, $\gamma \in \Gamma$, $m \in M$, we have

$$\begin{aligned}(\overset{\Delta}{\mu}_{e\varepsilon}\lambda_{u\varepsilon})(x) &= (\overset{\Delta}{\mu}_{e\varepsilon}\lambda_{u\varepsilon})(y\gamma m) = \overset{\Delta}{\mu}_{e\varepsilon}(u\varepsilon y\gamma m)\\ &= u\varepsilon y\mu_{e\varepsilon}(\gamma)m = u\varepsilon y\gamma e\varepsilon m = u\varepsilon y\gamma m \in U.\end{aligned}$$

In this way, we get $x = y\gamma m \in (\overset{\Delta}{\mu}_{e\varepsilon}\lambda_{u\varepsilon})^{-1}(U)$, and so $W \subseteq (\overset{\Delta}{\mu}_{e\varepsilon}\lambda_{u\varepsilon})^{-1}(U) \cap M$. Moreover, for all $x \in W$, we get

$$\begin{aligned}(f\overset{\Delta}{\mu}_{e\varepsilon}\lambda_{u\varepsilon})(x) &= (f\overset{\Delta}{\mu}_{e\varepsilon}\lambda_{u\varepsilon})(y\gamma m) = (f\overset{\Delta}{\mu}_{e\varepsilon})(u\varepsilon y\gamma m)\\ &= f(u\varepsilon y\mu_{e\varepsilon}(\gamma)m) = f(u\varepsilon y\gamma e\varepsilon m)\\ &= f(u\varepsilon y\gamma m) = f(u)\varepsilon y\gamma m\\ &= \lambda_{f(u)\varepsilon}(y\gamma m) = \lambda_{f(u)\varepsilon}(x).\end{aligned}$$

That is $f\overset{\Delta}{\mu}_{e\varepsilon}\lambda_{u\varepsilon} = \lambda_{f(u)\varepsilon}$ on W.

Therefore, there exists an element $q \in Q$ such that $f(u) = q\varepsilon u$ for all $u \in U$.

ii) Let $q \in Q$. Then there exists $U \in D(\Gamma,M)$ such that $q = [f;U]$ and $f : U \to M$ be a right $\Gamma M-$module homomorphism. According to (i), we have $f(u) = q\varepsilon u$ for all $u \in U$. Thus $q\varepsilon U \subseteq M$. Consequently, there exists $U \in D(\Gamma,M)$ such that $q\varepsilon U \subseteq M$.

iii) Let $q \in Q$ and $U \in D(\Gamma,M)$, $q\varepsilon U = (0)$. Then $q = [f;U]$. If $q\varepsilon U = (0)$, then $q\varepsilon u = 0$, for all $u \in U$. By (i), $f(u) = q\varepsilon u = 0$, for all $u \in U$. Thus $f(u) = 0$, for all $u \in U$, and so $q = [0;U] = [0;M] = 0_Q$. Conversely, let $q = 0$. Then $q = [f;U] = [0;M]$, and so $(f;U) \sim (0;M)$. That is $f(u) = 0$ for all $u \in U \subseteq M$. According to (i), $f(u) = q\varepsilon u = 0$ for all $u \in U$, so $q\varepsilon U = (0)$. This completes the proof. □

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ON THE MODIFIED RANDIĆ INDEX OF TREES, UNICYCLIC GRAPHS AND BICYCLIC GRAPHS

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Abstract. The modified Randić index of a graph G is a graph invariant closely related to the classical Randić index, defined as

$$R'(G) = \sum_{uv \in E(G)} \frac{1}{\max\{d_G(u), d_G(v)\}},$$

where $E(G)$ is the edge set of G, $d_G(u)$ is the degree of vertex u in G. We obtain best possible lower bounds for the modified Randić index of trees, unicyclic graphs and bicyclic graphs respectively when the number of vertices and matching number are fixed.

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Keywords: Randić index, modified Randić index, tree, unicyclic graph, bicyclic graph, matching number

1. INTRODUCTION

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For $u \in V(G)$, $N_G(u)$ denotes the set of neighbors of u in G and the degree of u is $d_G(u) = |N_G(u)|$. The Randić (connectivity) index of G is defined as [12]

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u) d_G(v)}}.$$

The Randić index is one of the most successful molecular descriptors in structure–property and structure–activity relationships studies [9, 12]. Its mathematical properties as well as those of its generalizations have been studied extensively as summarized in the books [7, 10].

Various variants of Randić index have been proposed in the literature, see, e.g., [2, 4, 11, 13, 15]. To investigate the relationship between Randić index and radius of a (connected) graph, Dvořák et al. [6] proposed a modified version, which we call the

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modified Randić index, defined as

$$R'(G) = \sum_{uv \in E(G)} \frac{1}{\max\{d_G(u), d_G(v)\}}.$$

Some basic mathematical properties of the modified Randić index have been established in [1, 3, 6]. Recall that a connected graph on n vertices is known as a tree, a unicyclic graph and a bicyclic graph if it possesses $n-1$, n and $n+1$ edges, respectively. Andova et al. [1] obtained best possible lower and upper bounds for the modified Randić index of n-vertex trees and unicyclic graphs, respectively, and determined the corresponding extremal graphs.

In this paper, we obtain best possible lower bounds for the modified Randić index of trees, unicyclic graphs and bicyclic graphs respectively when the number of vertices and matching number are fixed. We also obtain best possible lower and upper bounds for the modified Randić index of n-vertex bicyclic graphs, and determine the corresponding extremal graphs.

2. Preliminaries

A matching M of the graph G is a subset of $E(G)$ such that no two edges in M share a common vertex. The matching number of G is the maximum number of edges of a matching in G. An m-matching is a matching of size m.

If M is a matching of a graph G and vertex $v \in V(G)$ is incident with an edge of M, then v is said to be M-saturated, and if every vertex of G is M-saturated, then M is a perfect matching.

For a graph G with $u, v \in V(G)$, $G-u$ denotes the graph formed from G by deleting vertex u (and its incident edges), $G+uv$ denotes the graph formed from G by adding the edge uv if $uv \notin E(G)$, and $G-uv$ denotes the graph formed from G by deleting the edge uv if $uv \in E(G)$.

First we give some lemmas that will be used.

Lemma 1. *[1] Let G be a connected graph on at least 3 vertices. Let u be a pendent vertex of G and v the unique neighbor of u. Let $a = d_G(v)$ and let l be the number of neighbors of v whose degree is at least a in G. Then*

$$R'(G) - R'(G-u) = \frac{l}{a(a-1)}.$$

Lemma 2. *Let G be a connected graph on at least 5 vertices. Let u be a pendent vertex of G and v the unique neighbor of u. Suppose that $d_G(v) = 2$. Let w be the neighbor of v different from u. Let $a = d_G(w)$ and let l be the number of neighbors of w whose degree is at least a in G. Then*

$$R'(G) - R'(G-u-v) = \begin{cases} 1 & \text{if } a = 2, \\ \frac{1}{2} + \frac{l}{a(a-1)} & \text{if } a \geq 3. \end{cases}$$

Proof. Obviously, $a \geq 2$. Suppose first that $a = 2$. Let w_1 be the neighbor of w different from v. Obviously, $d_{G-u-v}(w_1) = d_G(w_1) \geq 2$. Note that only edges uv and vw affect the difference $R'(G) - R'(G-u-v)$. Thus $R'(G) - R'(G-u-v) = \frac{1}{2} + \frac{1}{2} = 1$. Now suppose that $a \geq 3$. By Lemma 1, $R'(G) - R'(G-u) = \frac{1}{2}$ and $R'(G-u) - R'(G-u-v) = \frac{l}{a(a-1)}$. The result follows easily. □

Let P_n be the path on n vertices. To attaching a path P at a vertex u of a graph G mean that we add an edge between u and an end vertex of P.

Lemma 3. *Let G be a connected graph with at least two vertices and let $u \in V(G)$. Let G_1 be the graph obtained from G by attaching two paths P_a and P_b to u, G_2 the graph obtained from G by attaching a path P_{a+b} to u, where $a \geq b \geq 1$. Then $R'(G_1) < R'(G_2)$.*

Proof. Let $d = d_G(u)$. Then $d_{G_1}(u) = d + 2$ and $d_{G_2}(u) = d + 1$. Thus

$$\begin{aligned} R'(G_2) - R'(G_1) &= \sum_{v \in N_G(u)} \left(\frac{1}{\max\{d_G(v), d+1\}} - \frac{1}{\max\{d_G(v), d+2\}} \right) \\ &\quad + \frac{1}{2} + \frac{1}{d+1} - \frac{2}{d+2} \\ &\geq \frac{1}{2} + \frac{1}{d+1} - \frac{2}{d+2} \\ &> 0, \end{aligned}$$

implying the result. □

Lemma 4. *Let G be a connected graph with $u \in V(G)$, where $d_G(u) \geq 2$. Let H be the graph obtained from G by attaching a path P_a to u. Let u_1 be a neighbor of u in G, and u_0 the pendent vertex of the attached path in H. Let $H' = H - uu_1 + u_0u_1$. Then $R'(H') > R'(H)$.*

Proof. Let $d = d_G(u)$. Then $d_H(u) = d + 1$ and $d_{H'}(u) = d$. Note that $d_H(u_0) = 1$, $d_{H'}(u_0) = 2$, and the vertices of $V(H) \setminus \{u, u_0\}$ have the same degree in H' and H. Then

$$\begin{aligned} R'(H') - R'(H) &= \sum_{v \in N_G(u) \setminus \{u_1\}} \left(\frac{1}{\max\{d_G(v), d\}} - \frac{1}{\max\{d_G(v), d+1\}} \right) \\ &\quad + \frac{1}{d} - \frac{1}{d+1} \\ &\quad + \frac{1}{\max\{d_G(u_1), 2\}} - \frac{1}{\max\{d_G(u_1), d+1\}} \\ &\geq \frac{1}{d} - \frac{1}{d+1} > 0, \end{aligned}$$

implying the result. □

A vertex $u \in V(G)$ is called a vertex of local maximal degree if $d_G(v) < d_G(u)$ for any $v \in N_G(u)$.

3. Modified Randić index of trees

Let S_n be the star on n vertices. For positive integers n and m with $n \geq 2m$, let $\mathbb{T}_{n,m}$ be the set of trees with n vertices and matching number m, and $T_{n,m}$ the graph obtained from the star S_{n-2m+2} by attaching $m-1$ paths on two vertices to its center.

Lemma 5. *[8] Let G be a n-vertex tree with a perfect matching, where $n \geq 3$. Then G has at least two pendent vertices such that they are adjacent to vertices of degree 2.*

Lemma 6. *[8] Let G be an n-vertex tree with matching number m, where $n > 2m$. Then there is an m-matching M and a pendent vertex v such that v is not M-saturated.*

Theorem 1. *Let $G \in \mathbb{T}_{n,m}$, where $n \geq 2m$. Then $R'(G) \geq \frac{m+1}{2}$ with equality if and only if $G \cong T_{n,m}$.*

Proof. We prove the result by induction on n (for fixed m).

First suppose that $n = 2m$. We prove the result by induction on m. If $m = 1, 2$, then the result follows trivially. If $m = 3$, then $G \cong P_6$ or $T_{6,3}$, and thus the result follows since $R'(T_{6,3}) = 2 = \frac{3+1}{2} < R'(P_6) = \frac{5}{2}$.

Suppose that $m \geq 4$, and the theorem holds for $m-1$. Let $G \in \mathbb{T}_{2m,m}$. Let M be a perfect matching of G. By Lemma 5, G has a pendent vertex u_1 adjacent to a vertex u_2 of degree 2. Let u_3 be the neighbor of u_2 different from u_1. Let $a = d_G(u_3)$ and l be the number of neighbors of u_3 whose degree is at least a in G. Obviously, $u_1u_2 \in M$. Let $G' = G - u_1 - u_2$. Then $G' \in \mathbb{T}_{2m-2,m-1}$ and $M \setminus \{u_1u_2\}$ is a perfect matching of G'. By Lemma 2 and the induction hypothesis, $R'(G) = R'(G') + 1 \geq \frac{(m-1)+1}{2} + 1 > \frac{m+1}{2}$ if $a = 2$, and $R'(G) = R'(G') + \frac{1}{2} + \frac{l}{a(a-1)} \geq \frac{(m-1)+1}{2} + \frac{1}{2} + \frac{l}{a(a-1)} = \frac{m+1}{2} + \frac{l}{a(a-1)} \geq \frac{m+1}{2}$ with equalities if and only if $G' \cong T_{2m-2,m-1}$ and $l = 0$, i.e., $G' \cong T_{2m-2,m-1}$ and u_3 is a vertex of local maximal degree in G if $a \geq 3$. Thus $R'(G) \geq \frac{m+1}{2}$ with equality if and only if $G \cong T_{2m,m}$.

Suppose that $n > 2m$ and the result holds for $n-1$. Let $G \in \mathbb{T}_{n,m}$. By Lemma 6, G has an m-matching M and a pendent u such that M does not saturate u. Then $G - u \in \mathbb{T}_{n-1,m}$. Let v be the unique neighbor of u in G. Let $a = d_G(v)$ and let l be the number of neighbors of v whose degree is at least a in G. By Lemma 1 and the induction hypothesis,

$$R'(G) = R'(G-u) + \frac{l}{a(a-1)} \geq \frac{m+1}{2} + \frac{l}{a(a-1)} \geq \frac{m+1}{2}$$

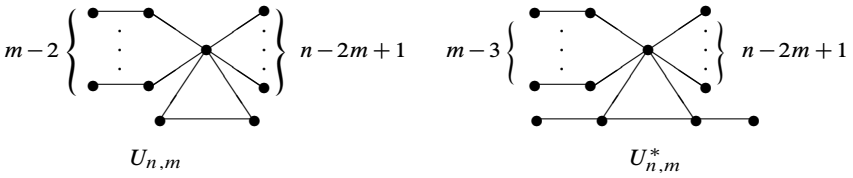

FIGURE 1. The graphs $U_{n,m}$ and $U^*_{n,m}$.

with equalities if and only if $G-u \cong T_{n-1,m}$ and $l=0$, i.e., $G-u \cong T_{n-1,m}$ and v is a vertex of local maximal degree in G. Thus $R'(G) \geq \frac{m+1}{2}$ with equality if and only if $G \cong T_{n,m}$. □

Corollary 1. *[1] Let G be a tree on $n \geq 2$ vertices. Then $R'(G) \geq 1$ with equality if and only if $G \cong S_n$.*

Proof. Let m be the matching number of G. By Theorem 1, $R'(G) \geq \frac{m+1}{2} \geq 1$ with equalities if and only if $G \cong T_{n,1} \cong S_n$. □

By Lemma 3, we have

Corollary 2. *[1] Let G be a tree on at least 2 vertices. Then $R'(G) \leq \frac{n-1}{2}$ with equality if and only if $G \cong P_n$.*

4. MODIFIED RANDIĆ INDEX OF UNICYCLIC GRAPHS

Let C_n be a cycle on n vertices. For integers n and m with $n \geq 2m$ and $m \geq 2$, let $\mathbb{U}_{n,m}$ be the set of unicyclic graph with n vertices and matching number m, and $U_{n,m}$ the graph obtained from C_3 by attaching to a vertex $n-2m+1$ pendent vertices and $m-2$ paths on two vertices. For $n \geq 2m$ and $m \geq 3$, let $U^*_{n,m}$ be the graph obtained from C_3 by attaching to a vertex $n-2m+1$ pendent vertices and $m-3$ paths on two vertices and to the other two vertices a pendent vertex each. See Figure 1 for them.

Lemma 7. *[14] Let $G \in \mathbb{U}_{n,m}$ and $G \ncong C_n$, where $n > 2m$. Then there is an m-matching M and a pendent vertex v such that M does not saturate v.*

Theorem 2. *Let $G \in \mathbb{U}_{n,m}$, where $n \geq 2m$. Then $R'(G) \geq \frac{m+1}{2}$ with equality if and only if $G \cong U_{n,m}$ or $U^*_{n,m}$.*

Proof. We prove the result by induction on n (for fixed m).
First suppose that $n = 2m$. We prove the result by induction on m.

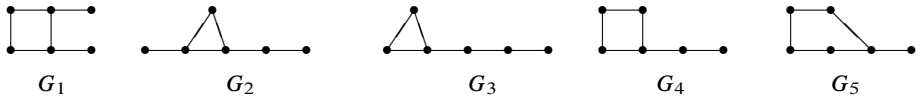

FIGURE 2. Graphs in $\mathbb{U}_{6,3} \setminus \{C_6, U_{6,3}, U^*_{6,3}\}$.

If $m = 2$, then $G \cong U_{4,2}$ or C_4, and thus the result follows since $R'(U_{4,2}) = \frac{3}{2} < R'(C_4) = 2$. If $m = 3$, then $G \cong C_6$, $U_{6,3}$, $U^*_{6,3}$ or G_i for $i = 1, 2, \ldots, 5$, see Figure 2, and thus the result follows since

$$R'(G) = \begin{cases} \frac{13}{6} & \text{if } G \cong G_i, i = 1,2 \\ 3 & \text{if } G \cong C_6, \\ \frac{5}{2} & \text{if } G \cong G_i, i = 3,4,5, \\ 2 & \text{if } G \cong U_{6,3} \text{ or } U^*_{6,3}. \end{cases}$$

Suppose that $m \geq 4$ and the result holds for graphs in $\mathbb{U}_{2m-2,m-1}$. Let $G \in \mathbb{U}_{2m,m}$. If $G \cong C_{2m}$, then $R'(G) = m > \frac{m+1}{2}$. Suppose that $G \ncong C_{2m}$. Let M be a perfect matching of G. There are two cases.

Case 1. G has a pendent vertex u whose unique neighbor v has degree two. Obviously, $uv \in M$. Let $G_1 = G - u - v$. Then $G_1 \in \mathbb{U}_{2m-2,m-1}$ and $M \setminus \{uv\}$ is a perfect matching of G_1. Let w be the neighbor of v different from u in G. Let $a = d_G(w)$ and let l be the number of neighbors of w whose degree is at least a in G. By Lemma 2 and the induction hypothesis, $R'(G) = R'(G_1) + 1 \geq \frac{(m-1)+1}{2} + 1 > \frac{m+1}{2}$ if $a = 2$, and $R'(G) = R'(G_1) + \frac{1}{2} + \frac{l}{a(a-1)} \geq \frac{(m-1)+1}{2} + \frac{1}{2} = \frac{m+1}{2}$ with equality if and only if $G_1 = U_{2m-2,m-1}$ or $U^*_{2m-2,m-1}$ and $l = 0$, i.e., $G_1 = U_{2m-2,m-1}$ or $U^*_{2m-2,m-1}$ and w is a vertex of local maximal degree in G if $a \geq 3$. Thus $R'(G) \geq \frac{m+1}{2}$ with equality if and only if $G \cong U_{2m,m}$ or $U^*_{2m,m}$.

Case 2. No neighbor of a pendent vertex has degree two in G. Let $C = u_1 u_2 \ldots u_p u_1$ be the unique cycle of G. Since G has a perfect matching, the graph obtained from G by deleting vertices of C consists of isolated vertices. Thus G is obtainable from the cycle C by attaching some pendent vertices and the maximal degree of G is three.

If all vertices on C have degree three, then $R'(G) = \frac{2m}{3} > \frac{m+1}{2}$. Suppose that there is at least one vertex of degree two on C. Since $G \ncong C_{2m}$, there are adjacent vertices on C, one of degree two and the other of degree three. Assume that $d_G(u_2) = 3$ and $d_G(u_3) = 2$. Let w_2 be the pendent neighbor of u_2 in G. Then $u_2 w_2 \in M$. Since $u_3 u_4 \in M$, we have $d_G(u_4) = 2$. Let $G_2 = G - u_2 - w_2 + u_1 u_3$. Then $G_2 \in \mathbb{U}_{2m-2,m-1}$ and $M \setminus \{u_2 w_2\}$ is a perfect matching of G_2. If $d_G(u_1) = 3$, then by the induction hypothesis,

$$R'(G) = R'(G_2) + \frac{2}{3} \geq \frac{(m-1)+1}{2} + \frac{2}{3} > \frac{m+1}{2}.$$

If $d_G(u_1) = 2$, then $|V(C)| \geq 5$, and thus $G_2 \ncong U_{2m-2,m-1}, U^*_{2m-2,m-1}$, implying that

$$R'(G) = R'(G_2) + \frac{1}{2} > \frac{(m-1)+1}{2} + \frac{1}{2} = \frac{m+1}{2}.$$

Thus $R'(G) > \frac{m+1}{2}$.

By combining Cases 1 and 2, the result follows for $n = 2m$.

Suppose that $n > 2m$ and the result holds for $n-1$. Let $G \in \mathbb{U}_{n,m}$. If $G \cong C_n$, then $n = 2m+1$, and thus $R'(G) = \frac{2m+1}{2} > \frac{m+1}{2}$. Suppose that $G \ncong C_n$. By Lemma 7, G has an m-matching M and a pendent u such that M does not saturate u. Let $G_3 = G - u$. Then $G_3 \in \mathbb{U}_{n-1,m}$. Let v be the unique neighbor of u in G. Let $a = d_G(v)$ and let l be the number of neighbors of v whose degree is at least a in G. By Lemma 1 and the induction hypothesis,

$$R'(G) = R'(G_3) + \frac{l}{a(a-1)} \geq \frac{m+1}{2} + \frac{l}{a(a-1)} \geq \frac{m+1}{2}$$

with equalities if and only if $G_3 \cong U_{n-1,m}$ or $U^*_{n-1,m}$ and $l = 0$, i.e., $G_3 \cong U_{n-1,m}$ or $U^*_{n-1,m}$, and v is a vertex of local maximal degree in G. Thus $R'(G) \geq \frac{m+1}{2}$ with equality if and only if $G \cong U_{n,m}$ or $U^*_{n,m}$. □

Corollary 3. *[1] Let G be a unicyclic graph G on $n \geq 3$ vertices. Then $R'(G) \geq \frac{3}{2}$ with equality if and only if $G \cong C_3$ or $U_{n,2}$.*

Proof. Let m be the matching number of G. If $m = 1$, then $G \cong C_3$ and $R'(G) = \frac{3}{2}$. If $m \geq 2$, then by Theorem 2, $R'(G) \geq \frac{m+1}{2} \geq \frac{3}{2}$ with equalities if and only if $G \cong U_{n,2}$. □

By Lemmas 3 and 4, we have

Corollary 4. *[1] Let G be a unicyclic graph on at least 3 vertices. Then $R'(G) \leq \frac{n}{2}$ with equality if and only if $G \cong C_n$.*

5. Modified Randić index of bicyclic graphs

In this section, we use techniques developed from [5].

For integers n and m with $n \geq 2m$ and $m \geq 2$, let $\mathbb{B}_{n,m}$ be the set of bicyclic graphs with n vertices and matching number m. Let $B^1_{n,3}$ be the graph obtained by identifying a vertex of two triangles, and attaching to this common vertex of the two triangles $n-5$ pendent vertices. Let $B^2_{n,3}$ be the graph obtained from the unique bicyclic graph on four vertices by attaching to a vertex of degree three $n-5$ pendent vertices and to a vertex of degree two a pendent vertex. For $n \geq 2m$ and $m \geq 4$, let $B_{n,m}$ be the graph obtained from the unique bicyclic graph on four vertices by attaching to a vertex of degree three $n-2m+1$ pendent vertices and $m-4$ paths on two vertices to the three other vertices a pendent vertex each, see Figure 3.

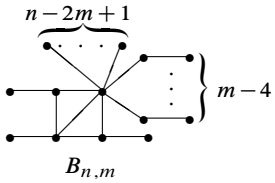

FIGURE 3. The graph $B_{n,m}$.

Lemma 8. *Let G be a connected graph with $uv \in E(G)$, where $d_G(u), d_G(v) \geq 2$, and u and v have no common neighbor in G. Let G_1 be the graph obtained from G by deleting the edge uv, identifying u and v, which is denoted by w, and adding a vertex w_1 and the edge w_1w. Then $R'(G) > R'(G_1)$.*

Proof. Note that $d_{G_1}(w) = d_G(u) + d_G(v) - 1$, and the vertices of $V(G) \setminus \{u, v\}$ have the same degree in G_1 and G. Obviously, $d_{G_1}(w) > \max\{d_G(u), d_G(v)\}$ since $d_G(u), d_G(v) \geq 2$, and for any $u' \in N_G(u) \cup N_G(v) \setminus \{u, v\}$, $u' \in N_{G_1}(w)$. Thus

$$\begin{aligned}
&R'(G_1) - R'(G)\\
&= \sum_{u' \in N_G(u)\setminus\{v\}} \left(\frac{1}{\max\{d_{G_1}(w), d_G(u')\}} - \frac{1}{\max\{d_G(u), d_G(u')\}} \right)\\
&\quad + \sum_{u' \in N_G(v)\setminus\{u\}} \left(\frac{1}{\max\{d_{G_1}(w), d_G(u')\}} - \frac{1}{\max\{d_G(v), d_G(u')\}} \right)\\
&\quad + \frac{1}{d_{G_1}(w)} - \frac{1}{\max\{d_G(u), d_G(v)\}}\\
&\leq \frac{1}{d_{G_1}(w)} - \frac{1}{\max\{d_G(u), d_G(v)\}}\\
&< 0,
\end{aligned}$$

as desired. □

Lemma 9. *Let $G \in \mathbb{B}_{6,3}$. Then $R'(G) \geq 2$ with equality if and only if $G \cong B^1_{6,3}$ or $B^2_{6,3}$.*

Proof. If G has a pendent vertex with a neighbor of degree two, then G is the graph obtained from the unique bicyclic graph on four vertices by attaching to a vertex (of degree two or three) a path on two vertices, and thus it is easily seen that $R'(G) = \frac{13}{6}$ or $\frac{5}{2}$, implying that $R'(G) > 2$. Suppose that there is no pendent vertex

with a neighbor of degree two. If there is no vertex of degree two in G, then G is obtained by attaching a pendent vertex to every vertex of degree two of the unique bicyclic graph on four vertices, and thus $R'(G) = \frac{7}{3} > 2$. Suppose that there is a vertex, say u, of degree two in G. Let v and w be the two neighbors of u in G. Then one of the two edges incident with u, say $uv \in M$, where M is a perfect matching of G. Then u lies on some cycle of G, and thus $G' = G - uw$ is a unicyclic graph with a perfect matching. Let $a = d_G(w)$ and let l be the number of neighbors of w whose degree is at least a in G. Note that $a \geq 2$. Then by Theorem 2, $R'(G) = R'(G') + \frac{1}{2} \geq \frac{3+1}{2} + \frac{1}{2} = \frac{5}{2} > 2$ if $a = 2$, and $R'(G) = R'(G') + \frac{a-l}{a} - \frac{a-l-1}{a-1} = R'(G') + \frac{l}{a(a-1)} \geq \frac{3+1}{2} = 2$ with equality if and only if $G' \cong U_{6,3}$ or $U^*_{6,3}$ and $l = 0$, i.e., $G' \cong U_{6,3}$ or $U^*_{6,3}$ and w is a vertex of local maximal degree if $a \geq 3$. The result follows. □

Lemma 10. *[16] Let $G \in \mathbb{B}_{n,m}$ with $n > 2m \geq 6$, and G has at least one pendent vertex. Then there is an m-matching M and a pendent vertex u such that u is not M-saturated.*

Let $g(m) = \frac{m+1}{2} - \frac{1}{12}$.

Let $\tilde{\mathbb{B}}(n)$ be the set of bicyclic graphs on n vertices without pendent vertices, where $n \geq 4$. Let $\mathbb{B}^1_n(t)$ be the bicyclic graph with n vertices containing two edge-disjoint cycles C_a and C_b that are joined by a path of length $t = n - a - b + 1$ (C_a and C_b has a common vertex and $a + b = n + 1$ if $t = 0$). Let $\mathbb{B}^2_n(s)$ be the set of bicyclic graphs obtained by joining two non-adjacent vertices of C_a with $4 \leq a \leq n$ by a path of length $s = n - a + 1$, where $n \geq 4$ and $s \geq 1$. Obviously, $\tilde{\mathbb{B}}(n) = \mathbb{B}^1_n(t) \cup \mathbb{B}^2_n(s)$.

Lemma 11. *Let $G \in \mathbb{B}_{2m,m}$ and no pendent vertex has neighbor of degree two, where $m \geq 4$. Then $R'(G) \geq g(m)$ with equality if and only if $m = 4$ and $G \cong B_{8,4}$.*

Proof. Since $G \in \mathbb{B}_{2m,m}$ and no pendent vertex has neighbor of degree two, G is obtainable by attaching some pendent vertices to a graph in $\tilde{\mathbb{B}}(k)$, where $m \leq k \leq 2m$, and any two pendent vertices have no common neighbor (if $k = 2m$, then no pendent vertex is attached).

Case 1. There is no vertex of degree two in G. Then either $k = m$, G is obtainable by attaching a pendent vertex to every vertex of a graph G_1 in $\tilde{\mathbb{B}}(k)$, or $k = m + 1$, G is obtainable by attaching a pendent vertex to every vertex with degree two of a graph $G_1 \in \mathbb{B}^1_{m+1}(1) \cup \mathbb{B}^2_{m+1}(1)$. By direct calculation, we have

$$R'(G) = \begin{cases} \frac{2}{3}m - \frac{1}{4} & \text{if } G_1 \in \mathbb{B}^1_m(1) \cup \mathbb{B}^2_m(1), \\ \frac{2}{3}m - \frac{1}{3} & \text{if } G_1 \in \mathbb{B}^1_m(t) \cup \mathbb{B}^1_m(0) \cup \mathbb{B}^2_m(t) \text{ with } t \geq 2, \\ \frac{2}{3}m + \frac{1}{3} & \text{if } G_1 \in \mathbb{B}^1_{m+1}(1) \cup \mathbb{B}^2_{m+1}(1). \end{cases}$$

Then $R'(G) \geq g(m)$ with equality if and only if $m = 4$ and $G \cong B_{8,4}$.

Case 2. There is a vertex, say u, of degree two in G. Denote by v and w the two neighbors of u in G. Then one of the two edges incident with u, say $uv \in M$, where

M is a perfect matching of G. Suppose that there is no vertex of degree two in any cycle of G. Since no pendent vertex has neighbor of degree two in G, u lies on the path joining the two disjoint cycles of G. For $G_1 = G - uw + vw \in \mathbb{B}_{2m,m}$, the difference of the numbers of vertices of degree two outside any cycle of G and G_1 is equal to one, and thus by Lemma 8, $R'(G_1) < R'(G)$. Repeating the operation from G to G_1, we finally get a graph $G' \in \mathbb{B}_{2m,m}$, which has no vertex of degree two, such that $R'(G) > R'(G')$, and thus the result follows from Case 1. Now suppose that u lies on some cycle of G. Consider $G' = G - uw$, which is a unicyclic graph with perfect matching. Let $a = d_G(w)$ and let k be the number of neighbors of w whose degree is at least a in G. Then by Theorem 2, $R'(G) = R'(G') + \frac{1}{2} \geq \frac{m+1}{2} + \frac{1}{2} > g(m)$ if $a = 2$, and $R'(G) = R'(G') + \frac{k}{a(a-1)} \geq \frac{m+1}{2} > g(m)$ if $a \geq 3$. □

Lemma 12. *Let $G \in \mathbb{B}_{8,4}$. Then $R'(G) \geq \frac{29}{12}$ with equality if and only if $G \cong B_{8,4}$.*

Proof. If there is no pendent vertex with neighbor of degree two in G, then by Lemma 11, $R'(G) \geq g(4) = \frac{29}{12}$ with equality if and only if $G \cong B_{8,4}$. Suppose that G has a pendent vertex u whose neighbor v is of degree two. Then $uv \in M$ and $G - u - v \in \mathbb{B}_{6,3}$. Let w be the neighbor v different from u. Let $a = d_G(w)$ and let k the number of neighbors of w whose degree is at least a. By Lemmas 2 and 9, $R'(G) = R'(G-u-v) + 1 \geq 2 + 1 = 3 > \frac{29}{12}$ if $a = 2$, and $R'(G) = R'(G-u-v) + \frac{1}{2} + \frac{k}{a(a-1)} \geq 2 + \frac{1}{2} + \frac{k}{a(a-1)} \geq \frac{5}{2} > \frac{29}{12}$ if $a \geq 3$. □

Let $B_n(a,b)$ be the graph obtained from the unique bicyclic graph on four vertices by attaching to the two vertices of degree three $a-3$ and $b-3$ pendent vertices, respectively, where $a \geq b \geq 3$, $a + b = n + 2$ and $n \geq 4$.

Theorem 3. *Among the graphs in $\mathbb{B}_{n,2}$ with $n \geq 6$,*

$$R'\left(B_n\left(\left\lceil \frac{n}{2} \right\rceil + 1, \left\lfloor \frac{n}{2} \right\rfloor + 1\right)\right) > R'\left(B_n\left(\left\lceil \frac{n}{2} \right\rceil + 2, \left\lfloor \frac{n}{2} \right\rfloor\right)\right) > \cdots > R'\left(B_n(n-1,3)\right) = \frac{5}{3},$$

and for $G \in \mathbb{B}_{n,2}$ not appearing in the above inequalities, $R'(G) = 2$.

Proof. Let $G \in \mathbb{B}_{n,2}$. Then G is of three types:
(1) $G = B_n(a,b)$ with $a \geq b \geq 3$. Then $R'(G) = 2 - \frac{1}{b}$. It follows that

$$R'\left(B_n\left(\left\lceil \frac{n}{2} \right\rceil + 1, \left\lfloor \frac{n}{2} \right\rfloor + 1\right)\right) > R'\left(B_n\left(\left\lceil \frac{n}{2} \right\rceil + 2, \left\lfloor \frac{n}{2} \right\rfloor\right)\right) > \cdots > R'\left(B_n(n-1,3)\right) = \frac{5}{3}.$$

(2) G is the graph obtained from the unique bicyclic graph on four vertices by attaching to a vertex of degree two $n-4$ pendent vertices . Then $R'(G) = 2$.

(3) G is the graph obtained from the unique bicyclic graph on five vertices in $\mathbb{B}_5^2(2)$ by attaching to one or two vertices of degree three some pendent vertices. Then $R'(G) = 2$.

Now the result follows easily. □

Theorem 4. *Let $G \in \mathbb{B}_{n,3}$. Then $R'(G) \geq 2$ with equality if and only if $G \cong B_{n,3}^1$ or $G \cong B_{n,3}^2$.*

Proof. We prove the result by induction on n. If $n = 6$, then the result follows from Lemma 9.

Suppose that $n \geq 7$ and the result holds for graphs in $\mathbb{B}_{n-1,3}$. Let $G \in \mathbb{B}_{n,3}$. If there is no pendent vertex in G, then $G \in \tilde{\mathbb{B}}(n)$, $n = 7$, and thus

$$\begin{aligned} R'(G) &= \begin{cases} 3 + \frac{1}{6} & \text{if } G \in \mathbb{B}_7^1(1) \cup \mathbb{B}_7^2(1), \\ 3 & \text{otherwise,} \end{cases} \\ &> 2. \end{aligned}$$

Suppose that there is at least one pendent vertex in G. By Lemma 10, there is a 3-matching M and a pendent vertex u of G such that u is not M-saturated. Then $G - u \in \mathbb{B}_{n-1,3}$. Let v be the unique neighbor of u. Let $a = d_G(v)$ and let l be the number of neighbors of v whose degree is at least a in G. By Lemma 1, we have

$$R'(G) = R'(G-u) + \frac{l}{a(a-1)} \geq 2 + \frac{l}{a(a-1)} \geq 2$$

with equalities if and only if $G - u \cong B_{n-1,3}^1$ or $B_{n-1,3}^2$ and $l = 0$, i.e., $G - u \cong B_{n-1,3}^1$ or $B_{n-1,3}^2$ and v is a vertex of local maximal degree. Thus $R'(G) \geq 2$ with equality if and only if $G \cong B_{n,3}^1$ or $G \cong B_{n,3}^2$ □

Theorem 5. *Let $G \in \mathbb{B}_{n,m}$, where $n \geq 2m$ and $m \geq 4$. Then $R'(G) \geq g(m)$ with equality if and only if $G \cong B_{n,m}$.*

Proof. We prove the result by induction on n (for fixed m).

First suppose that $n = 2m$. We prove the result by induction on m.

If $m = 4$, then from Lemma 12, $R'(G) \geq g(4)$ with equality if and only if $G \cong B_{8,4}$.

Suppose that $m \geq 5$ and the result holds for graphs in $\mathbb{B}_{2m-2,m-1}$. Let $G \in \mathbb{B}_{2m,m}$ with a perfect matching M. If there is no pendent vertex with neighbor of degree two in G, then by Lemma 11, $R'(G) > g(m)$. Suppose that G has a pendent vertex u whose neighbor v is of degree two. Then $uv \in M$ and $G - u - v \in \mathbb{B}_{2m-2,m-1}$. Let w be the neighbor of v different from u. Let $a = d_G(w)$ and let l be the number of neighbors of w whose degree is at least a. By Lemma 2 and the induction hypothesis, $R'(G) = R'(G-u-v) + 1 \geq g(m-1) + 1 > g(m)$ if $a = 2$, and $R'(G) = R'(G - u - v) + \frac{1}{2} + \frac{l}{a(a-1)} \geq g(m-1) + \frac{1}{2} + \frac{l}{a(a-1)} \geq g(m)$ with equalities if and only if $G - u - v \cong B_{2m-2,m-1}$ and $l = 0$, i.e., $G - u - v \cong B_{2m-2,m-1}$ and w is a vertex

of local maximal degree if $a \geq 3$. Thus $R'(G) \geq g(m)$ with equality if and only if $G \cong B_{2m,m}$.

Suppose that $n > 2m$ and the result holds for graphs in $\mathbb{B}_{n-1,m}$. Let $G \in \mathbb{B}_{n,m}$. If there is no pendent vertex in G, then $G \in \tilde{\mathbb{B}}(2m+1)$, and thus

$$\begin{aligned} R'(G) &= \begin{cases} m + \frac{1}{6} & \text{if } G_1 \in \mathbb{B}_n^1(1) \cup \mathbb{B}_n^2(1) \\ m & \text{otherwise} \end{cases} \\ &> g(m). \end{aligned}$$

Suppose that there is at least one pendent vertex in G. By Lemma 10, there is an m-matching M and a pendent vertex u of G such that u is not M-saturated. Then $G-u \in \mathbb{B}_{n-1,m}$. Let v be the unique neighbor of u. Let $a = d_G(v)$ and let l be the number of neighbors of v whose degree is at least a. By Lemma 1, we have

$$R'(G) = R'(G-u) + \frac{l}{a(a-1)} \geq g(m) + \frac{l}{a(a-1)} \geq g(m)$$

with equalities if and only if $G-u \cong B_{n-1,m}$ and $l = 0$, i.e., $G-u \cong B_{n-1,m}$ and v is a vertex of local maximal degree. Thus $R'(G) \geq g(m)$ with equality if and only if $G \cong B_{n,m}$. □

Corollary 5. *Let G be a bicyclic graph on $n \geq 6$ vertices. $R'(G) \geq \frac{5}{3}$ with equality if and only if $G \cong B_n(n-1,3)$.*

Proof. Let m be the matching number of G. If $m \geq 4$, then by Theorem 5, $R'(G) \geq g(m) \geq \frac{5}{2} - \frac{1}{12} > \frac{5}{3}$. If $m = 3$, then by Theorem 4, $R'(G) \geq 2 > \frac{5}{3}$. Thus the result follows from Theorem 3. □

Theorem 6. *Let G be a bicyclic graph on at least 5 vertices. Then $R'(G) \leq \frac{n}{2} - \frac{1}{3}$ with equality if and only if $G \in \mathbb{B}_n^1(1) \cup \mathbb{B}_n^2(1)$.*

Proof. Let G be a graph with the maximum modified Randić index among the bicyclic graphs on n vertices. By Lemmas 3 and 4, $G \in \tilde{\mathbb{B}}(n)$. By direct calculation, we have

$$R'(G) = \begin{cases} \frac{n}{2} - \frac{1}{3} & \text{if } G \in \mathbb{B}_n^1(1) \cup \mathbb{B}_n^2(1), \\ \frac{n}{2} - \frac{1}{2} & \text{if } G \in \mathbb{B}_n^1(t) \cup \mathbb{B}_n^2(t) \cup \mathbb{B}_n^1(0) \text{ with } t \geq 2. \end{cases}$$

The result follows. □

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ON THE BLOW-UP OF SOLUTIONS FOR THE UNSTABLE SIXTH ORDER PARABOLIC EQUATION

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Abstract. We study the universal blow-up of sixth-order parabolic thin film equation with the initial boundary conditions. We prove that the problem in finite time blow-up will happen, if the initial datum $u_0 \in C^{6+\alpha}(\overline{\Omega})$ with $-\int_\Omega \left(H(u_0)+\frac{1}{2}|\Delta u_0|^2\right)dx \geq 0$. And then, we get some nondegeneracy results on blow-up for this problem.

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1. INTRODUCTION

In this paper, we consider the following initial boundary problem of sixth-order equation

$$\begin{cases} u_t - \Delta(\Delta^2 u - |u|^{p-1}u) = 0, & \text{in } \Omega \times (0,T), \\ u = \Delta u = \Delta^2 u = 0, & \text{on } \partial\Omega \times [0,T), \\ u = u_0, & \text{in } \Omega \times \{0\}, \end{cases} \tag{1.1}$$

where $\Omega \subset R^N$ is a bounded smooth domain, $p > 1$.

During the past years, only a few works have been devoted to the sixth-order parabolic equation [1,4,5,7].

Recently, Evans, Galaktionov and King [4,5] considered the sixth-order thin film equation containing an unstable (backward parabolic) second-order term

$$\frac{\partial u}{\partial t} = \operatorname{div}\left[|u|^n \nabla \Delta^2 u\right] - \Delta(|u|^{p-1}u), n > 0, p > 1.$$

By a formal matched expansion technique, they show that, for the first critical exponent $p = p_0 = n + 1 + \frac{4}{N}$ for $n \in (0, \frac{5}{4})$, where N is the space dimension, the free-boundary problem with zero-height, zero-contact-angle, zero-moment, and zero-flux conditions at the interface admits a countable set of continuous branches of radially symmetric self-similar blow-up solutions $u_k(x,t) = (T-t)^{-\frac{N}{nN+6}} f_k(y)$, $y = \frac{x}{(T-t)^{\frac{1}{nN+6}}}$, where $T > 0$ is the blow-up time.

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In fact, when $n = 0$, the equation (1.1) is obtained. In this paper we study the universal blow-up and some nondegeneracy results on blow-up of the equation (1.1). Our method about universal finite time blow-up is similar to that of Elliott and Zheng [3] which treats the blow-up problem for Cahn-Hilliard equation. We can show that if the initial datum $u_0 \in C^{6+\alpha}(\overline{\Omega})$ with $-\int_\Omega \left(H(u_0) + \frac{1}{2}|\Delta u_0|^2\right) dx \geq 0$, then the solution to the above problem (1.1) should blow up in finite time.

We also establish some nondegeneracy results on the blow-up of the problem. We mainly follow the purpose of Giga and Kohn [6] and Cheng and Zheng [2]. More accurately, there is a constant $\varepsilon > 0$, depending on n, p and the constant in the estimates of the fundamental solution to $u_t - \Delta^3 u = 0$ (see (3.1) below), such that if u is a solution of the equation

$$u_t - \Delta(\Delta^2 u - |u|^{p-1}u) = 0, \qquad \text{on} \quad Q_r = B_r(a) \times [t_1 - r^6, t_1),$$

where $1 < p < 3, a \in \mathbb{R}^n, t_1 \in \mathbb{R}$ and $0 < r \leq 1$, and if

$$|u(x,t)| \leq \varepsilon (t_1 - t)^{-\frac{2}{3(p-1)}} \qquad \text{for} \quad \text{all} \quad (x,t) \in Q_r, \tag{1.2}$$

then u does not blow up at (a, t_1).

The following sections include our main results. In Section 2, we establish universal finite time blow-up. Section 3 is devoted to the nondegeneracy results on the blow-up.

2. Universal finite time blow-up

Theorem 1. *Assume $u_0 \in C^{6+\alpha}(\overline{\Omega})$ with $-\int_\Omega \left(H(u_0) + \frac{1}{2}|\Delta u_0|^2\right) dx \geq 0$. Then the solution of the problem (1.1) must blow up at a finite time, namely, for some $T > 0$*

$$\lim_{t \to T} \|u(t)\| = +\infty,$$

where $H(u) = -\frac{|u|^{p+1}}{p+1}$.

Proof. Let

$$F(t) = \int_\Omega \left(H(u) + \frac{1}{2}|\Delta u|^2\right) dx,$$

then

$$\begin{aligned} \frac{dF(t)}{dt} &= \int_\Omega \left(-|u|^{p-1}u\varphi(u)u_t + \frac{1}{2}\Delta u \Delta u_t\right) dx \\ &= \int_\Omega \left(-|u|^{p-1}u + \frac{1}{2}\Delta^2 u\right) u_t dx \\ &= -\int_\Omega |\nabla \left(-|u|^{p-1}u + \frac{1}{2}\Delta^2 u\right)|^2 dx \leq 0. \end{aligned}$$

So

$$2\int_\Omega H(u)dx - 2F(0) \leq -\|\Delta u\|^2, \tag{2.1}$$

where

$$F(0)=\int_{\Omega}\left(H(u_0)+\frac{1}{2}|\Delta u_0|^2\right)dx.$$

Let ϕ be the unique solution to

$$\begin{cases}\Delta\phi=u, & \text{in } \Omega,\\ \nabla\phi=0, & \text{on } \partial\Omega.\end{cases}$$

It is easy to get that

$$\|\nabla\phi\|^2\le C\|\Delta\phi\|_2^2\le C\|u\|^2. \tag{2.2}$$

Now multiplying (1.1) by ϕ and integrating with respect x, we obtain

$$\begin{aligned}\frac{d}{dt}\|\nabla\phi\|^2&=-2\int_{\Omega}\varphi(u)u dx-2\|\Delta u\|^2dx\\ &\ge 4\int_{\Omega}H(u)dx-4F(0)-2\int_{\Omega}\varphi(u)udx\\ &=\int_{\Omega}(2-\frac{4}{p+1})|u|^{p+1}dx-4F(0)\\ &\ge\frac{2(p-1)}{p+1}\left(\int_{\Omega}u^2dx\right)^{\frac{p+1}{2}}-4F(0).\end{aligned} \tag{2.3}$$

Combining (2.2), (2.3) and $-F(0)\ge 0$, we have

$$\frac{d}{dt}\|\nabla\phi\|^2\ge\frac{2C(p-1)}{p+1}\|\nabla\phi\|^{p+1}. \tag{2.4}$$

Let $y(t)=\|\nabla\phi\|_2^2$ with $t\in[0,T)$, then

$$y'(t)\ge\gamma\left(y(t)\right)^{\frac{p+1}{2}}, \tag{2.5}$$

where $\gamma=\frac{2C(p-1)}{p+1}$. A direct integration of (2.5) then yields

$$y^{\frac{p-1}{2}}(t)\ge\frac{1}{y^{\frac{1-p}{2}}(0)-\frac{p-1}{2}\gamma t}.$$

It turns out that the solution of the problem (1.1) will blow up in finite time. The proof of this theorem is completed. □

3. Nondegeneracy results on the blow-up

Let $\Gamma(x,t)$ be the fundamental solution to $u_t-\Delta^3u=0$. According to [8], we have the follow inequalities:

$$|D_t^{\mu}D_x^{\nu}\Gamma(x,t)|\le Ct^{-\frac{1}{6}(n+6\mu+\nu)}\exp\left\{-\omega\frac{|x|^{\frac{6}{5}}}{t^{\frac{1}{5}}}\right\},\qquad t>0, \tag{3.1}$$

where $C>0$, $\omega>0$ are constants, and μ, ν are nonnegative integers.

Our purpose in this section is to have some nondegeneracy results on the blow-up. We state that the solution $u(x,t)$ to blows up at (a,t_1) if it is not locally bounded nearby, i.e., if there is a sequence $\{(x_k,\tau_k)\}\subset\Omega\times[0,t_1)$ with $(x_k,\tau_k)\to(a,t_1)$ as $k\to\infty$ such that $|u(x_k,\tau_k)|\to\infty$.

Theorem 2. *There is a constant $\varepsilon>0$, depending on n, p and the constant in (3.1), such that if u is a solution of the equation*

$$u_t-\Delta(\Delta^2u-|u|^{p-1}u)=0,\quad on\quad Q_r=B_r(a)\times[t_1-r^6,t_1),$$

where $1<p<3, a\in\mathbb{R}^n, t_1\in\mathbb{R}$ and $0<r\le 1$, and if

$$|u(x,t)|\le\varepsilon(t_1-t)^{-\frac{2}{3(p-1)}}\qquad for\quad all\quad (x,t)\in Q_r, \tag{3.2}$$

then u does not blow up at (a,t_1).

Next, we introduce the two lemma which will be used in the article and whose proofs can be found in [2] and [6].

Lemma 1. *For $0<a<1$, $\theta>0$, and $0<h<1$, the integral*

$$I(h)=\int_h^1(s-h)^{-a}s^{-\theta}ds,$$

satisfies

$$(1)\quad I(h)\le\left(\frac{1}{1-a}+\frac{1}{a+\theta-1}\right)\quad if\quad a+\theta>1,$$

$$(2)\quad I(h)\le\frac{1}{1-a}+|\log h|\quad if\quad a+\theta=1,$$

$$(3)\quad I(h)\le\frac{1}{1-a-\theta}\quad if\quad a+\theta<1.$$

Lemma 2. *If $y(t)$, $r(t)$ and $q(t)$ are continuous functions defined on $[t_0,t_1]$, such that $y(t)\le y_0+\int_{t_0}^t y(s)r(s)ds+\int_{t_0}^t q(s)ds$, $t_0\le t\le t_1$, and $r(t)\ge 0$ on $[t_0,t_1]$, then*

$$y(t)\le exp\left\{\int_{t_0}^t r(\tau)d\tau\right\}\left[y_0+\int_{t_0}^t q(\tau)exp\left\{-\int_{t_0}^t r(\sigma)d\sigma\right\}d\tau\right].$$

Then, we began to prove the main Theorem 2.

Proof. Without loss of generality, we may assume $a=0$ and $t_1=0$. By scaling, it is sufficient to consider the case $r=1$. In the fact, if u satisfies the assumptions of the theorem with $r<1$, then $u_r(x,t)=r^{\frac{4}{p-1}}u(rx,r^6t)$ satisfies them with $r=1$ (using the same ε), and clearly u_r blow up at $(0,0)$ if u does.

Let ϕ be a smooth function supported on $B_1(0)$ such that $\phi\equiv 1$ on $B_{\frac{1}{2}}(0)$ and $0\le\phi\le 1$. Consider $\omega=\phi u$; then $\omega_t-\Delta^3\omega==g$ where

$$g = -2\nabla\Delta^2 u\nabla\phi - \Delta^2 u\Delta\phi$$
$$- \Delta(u\Delta^2\phi + 4\nabla\Delta u\nabla\phi + 6\Delta u\Delta\phi + 4\nabla u\nabla\Delta\phi) - \phi\Delta(|u|^{p-1}u)$$

The semigroup representation formula for ω gives that

$$\omega(t) = e^{(t+1)\Delta^3}\omega(-1) + \int_{-1}^{t} e^{(t-s)\Delta^3} g(s)ds \qquad \text{for} \quad -1 \le t < 0, \tag{3.3}$$

where $e^{t\Delta^3}$ is the semigroup associated with the equation $u_t - \Delta^3 u = 0$ in $\mathbb{R}^n$, i.e.,

$$(e^{t\Delta^3}h)(x) = \int_{\mathbb{R}^n} \Gamma(x-y,t)h(y)dy.$$

Notice that $\int_{\mathbb{R}^n} \Gamma(x-y,t)dy = 1$. It follows that

$$\|e^{t\Delta^3}h\| \le \|h\|_\infty. \tag{3.4}$$

The (3.1) implies that

$$|(e^{t\Delta^3}D_i h)(x)| = |\int_{\mathbb{R}^n} \Gamma(x-y,t)D_i h(y)dy|$$
$$= |\int_{\mathbb{R}^n} \frac{\partial}{\partial x_i}\Gamma(x-y,t)h(y)dy| \le Ct^{-\frac{1}{6}}\|h\|_\infty, \quad \forall i = 1,2,\ldots,n,$$

So, we get that

$$\|e^{t\Delta^3}D_i h\|_\infty \le Ct^{-\frac{1}{6}}\|h\|_\infty, \quad \|e^{t\Delta^3}D_{ij}h\|_\infty \le Ct^{-\frac{1}{3}}\|h\|_\infty,$$
$$\|e^{t\Delta^3}D_{ijk}h\|_\infty \le Ct^{-\frac{1}{2}}\|h\|_\infty, \quad \|e^{t\Delta^3}D_{ijkm}h\|_\infty \le Ct^{-\frac{2}{3}}\|h\|_\infty,$$
$$\|e^{t\Delta^3}D_{ijkmq}h\|_\infty \le Ct^{-\frac{5}{6}}\|h\|_\infty, \tag{3.5}$$

where $i,j,k,m,q \in \{1,2,\cdots,n\}$.

Now let $g = g_1 + g_2$, where $g_2 = -\phi\Delta(|u|^{p-1}u)$. As above, we estimate

$$\left|\int_{-1}^{t} e^{(t-s)\Delta^3} g_2(s)ds\right|$$
$$\le \int_{-1}^{t}\left|\int_{\mathbb{R}^n} \Delta(\phi\Gamma(x-y,t-s))(|u|^{p-1}u)(y,s)dy\right|ds$$
$$\le \int_{-1}^{t}\left|\int_{\mathbb{R}^n} \Delta\Gamma(x-y,t-s)\phi|u|^{p-1}u(y)dy\right|ds$$
$$+ \int_{-1}^{t}\left|\int_{\mathbb{R}^n} (\Gamma(x-y,t-s)\Delta\phi + 2\nabla\Gamma(x-y,t-s)\cdot\nabla\phi)|u|^{p-1}u(y)dy\right|ds$$
$$\le C\int_{-1}^{t}(t-s)^{-\frac{1}{3}}\|\phi u^p\|_\infty(s)ds + C\int_{-1}^{t}\|\Delta\phi u^p\|_\infty(s)ds$$

$$
\begin{aligned}
&+C\int_{-1}^{t}(t-s)^{-\frac{1}{6}}\|\nabla\phi u^{p}\|_{\infty}(s)ds\\
&\le C\int_{-1}^{t}(t-s)^{-\frac{1}{3}}\|u\|_{\infty}^{p-1}\|\omega\|_{\infty}(s)ds+C\int_{-1}^{t}\|u^{p}\|_{\infty}(s)ds\\
&+C\int_{-1}^{t}(t-s)^{-\frac{1}{6}}\|u\|_{\infty}^{p}(s)ds\\
&\le C\varepsilon^{p-1}\int_{-1}^{t}(t-s)^{-\frac{1}{3}}(-s)^{-\frac{2}{3}}\|\omega\|_{\infty}(s)ds+C\varepsilon^{p}\int_{-1}^{t}(-s)^{-\frac{2p}{3(p-1)}}ds\\
&+C\varepsilon^{p}\int_{-1}^{t}(t-s)^{-\frac{1}{6}}(-s)^{-\frac{2p}{3(p-1)}}ds, \qquad (3.6)
\end{aligned}
$$

due to our assumption.

On the other hand, it is found similarly that

$$
\begin{aligned}
&\left|\int_{-1}^{t}e^{(t-s)\Delta^{3}}g_{1}(s)ds\right|\\
&=\left|\int_{-1}^{t}\int_{\mathbb{R}^{n}}\Gamma(x-y,t-s)(-2\nabla\Delta^{2}u\nabla\phi-\Delta^{2}u\Delta\phi\right.\\
&\left.-\Delta(u\Delta^{2}\phi+4\nabla\Delta u\nabla\phi+6\Delta u\Delta\phi+4\nabla u\nabla\Delta\phi))dyds\right|\\
&\le C\int_{-1}^{t}(t-s)^{-\frac{5}{6}}\|u\|_{\infty}(s)ds\le C\varepsilon\int_{-1}^{t}(t-s)^{-\frac{5}{6}}(-s)^{-\frac{2}{3(p-1)}}ds. \qquad (3.7)
\end{aligned}
$$

By (3.2)-(3.4), (3.6) and (3.7), we get that for $-1\le t<0$,

$$
\begin{aligned}
\|\omega(t)\|_{\infty}&\le\varepsilon+\varepsilon^{p-1}\int_{-1}^{t}(t-s)^{-\frac{1}{3}}(-s)^{-\frac{2}{3}}\|\omega\|_{\infty}(s)ds\\
&\quad+C\varepsilon^{p}\int_{-1}^{t}(t-s)^{-\frac{1}{6}}(-s)^{-\frac{2p}{3(p-1)}}ds+C\varepsilon\int_{-1}^{t}(t-s)^{-\frac{5}{6}}(-s)^{-\frac{2}{3(p-1)}}ds\\
&\le\varepsilon+C\varepsilon^{p-1}\int_{-1}^{t}(t-s)^{-\frac{1}{3}}(-s)^{-\frac{2}{3}}\|\omega\|_{\infty}(s)ds+C\varepsilon(-t)^{\frac{1}{6}-\frac{2}{3(p-1)}},
\end{aligned}
\qquad (3.8)
$$

due to $1<p<3$ and Lemma (1).

Let $y(t)=\|\omega(t)\|_{\infty}$; therefore

$$
y(t)\le\varepsilon+C\varepsilon(-t)^{\frac{1}{6}-\frac{2}{3(p-1)}}+C\varepsilon^{p-1}\int_{-1}^{t}(t-s)^{-\frac{1}{3}}(-s)^{-\frac{2}{3}}y(s)ds. \qquad (3.9)
$$

Define $f(t)=\chi_{[-1,0]}(t)y(t)$, $\forall t<0$. We introduce a special maximal function on $(-\infty,0)$:

$$
(Mf)(t)=\sup_{r>0}\frac{1}{r}\int_{t-r}^{t}|f(s)|ds, \qquad \forall t\in(-\infty,0).
$$

Now $\forall r > 0$,

$$\begin{aligned}
&\int_{-1}^{t} (t-s)^{-\frac{1}{3}} (-s)^{-\frac{2}{3}} y(s)ds = \int_{-\infty}^{t} (t-s)^{-\frac{1}{3}} (-s)^{-\frac{2}{3}} f(s)ds \\
&= \int_{t-r}^{t} (t-s)^{-\frac{1}{3}} (-s)^{-\frac{2}{3}} f(s)ds + \int_{-\infty}^{t-r} (t-s)^{-\frac{1}{3}} (-s)^{-\frac{2}{3}} f(s)ds \\
&= I_1 + I_2.
\end{aligned}$$

We compute these two integrals, respectively.

$$\begin{aligned}
I_1 &\le (-t)^{-\frac{2}{3}} \int_{t-r}^{t} (t-s)^{-\frac{1}{3}} f(s)ds \\
&= (-t)^{-\frac{2}{3}} \sum_{k=0}^{\infty} \int_{t-\frac{r}{2^k}}^{t-\frac{r}{2^{k+1}}} (t-s)^{-\frac{1}{3}} f(s)ds \\
&\le (-t)^{-\frac{2}{3}} \sum_{k=0}^{\infty} \left(\frac{r}{2^{k+1}}\right)^{-\frac{1}{3}} \int_{t-\frac{r}{2^k}}^{t-\frac{r}{2^{k+1}}} f(s)ds \\
&\le (-t)^{-\frac{2}{3}} \sum_{k=0}^{\infty} \left(\frac{1}{2^{k+1}}\right)^{\frac{2}{3}} r^{\frac{2}{3}} (Mf)(t) \\
&= C r^{\frac{2}{3}} (-t)^{-\frac{2}{3}} (Mf)(t),
\end{aligned}$$

and

$$I_2 \le r^{-\frac{1}{3}} \int_{-\infty}^{t-r} (-s)^{-\frac{2}{3}} f(s)ds \le r^{-\frac{1}{3}} \int_{-\infty}^{t} (-s)^{-\frac{2}{3}} f(s)ds = r^{-\frac{1}{3}} \int_{-1}^{t} (-s)^{-\frac{2}{3}} f(s)ds.$$

Then,

$$f(t) \le \varepsilon + C\varepsilon(-t)^{\frac{1}{6} - \frac{2}{3(p-1)}} + C\varepsilon^{p-1} \left[r^{\frac{2}{3}} (-t)^{-\frac{2}{3}} (Mf)(t) + r^{-\frac{1}{3}} \int_{-1}^{t} (-s)^{-\frac{2}{3}} f(s)ds \right],$$

for all $r > 0$ and $t \in (-\infty, 0)$.

Let

$$r = \frac{\int_{-1}^{t} (-s)^{-\frac{2}{3}} f(s)ds}{(-t)^{-\frac{2}{3}} (Mf)(t)},$$

so we have

$$\begin{aligned}
f(t) &\le \varepsilon + C\varepsilon(-t)^{\frac{1}{6} - \frac{2}{3(p-1)}} + C\varepsilon^{p-1} \left((-t)^{-\frac{1}{3}} \int_{-1}^{t} (-s)^{-\frac{2}{3}} f(s)ds \right)^{\frac{2}{3}} ((Mf)(t))^{\frac{1}{3}} \\
&\le \varepsilon + C\varepsilon(-t)^{\frac{1}{6} - \frac{2}{3(p-1)}} + C\varepsilon^{p-1} (-t)^{-\frac{1}{3}} \int_{-1}^{t} (-s)^{-\frac{2}{3}} f(s)ds \\
&\quad + C\varepsilon^{p-1} (Mf)(t). \qquad (3.10)
\end{aligned}$$

If we define

$$g(t) = (-t)^{-\frac{1}{3}} \int_{-1}^{t} (-s)^{-\frac{2}{3}} f(s)ds,$$

then

$$g'(t) = (-t)^{-1} \left[\frac{1}{3}(-t)^{-\frac{1}{3}} \int_{-1}^{t} (-s)^{-\frac{2}{3}} f(s) + f(t) \right] \geq 0.$$

Hence $g(t)$ is increasing in $(-\infty, 0)$.

Then we get

$$\max_{-1\leq\tau\leq t} f(\tau) \leq \varepsilon + C\varepsilon(-t)^{\frac{1}{6}-\frac{2}{3(p-1)}} + C\varepsilon^{p-1} g(t) + C\varepsilon^{p-1} \max_{-1\leq\tau\leq t} (Mf)(\tau), \quad \forall t \in [-1,0), \tag{3.11}$$

where we have used $\frac{1}{6} - \frac{2}{3(p-1)} < 0$ since $1 < p < 3$.

Clearly, $\max_{-1\leq\tau\leq t}(Mf)(\tau) \leq \max_{-1\leq\tau\leq t} f(\tau)$ by our definition of the maximal function. Therefore (3.11) implies that for any $-1 \leq t < 0$,

$$\max_{-1\leq\tau\leq t} f(\tau) \leq$$

$$\frac{1}{1 - C\varepsilon^{p-1}} \left[\varepsilon + C\varepsilon(-t)^{\frac{1}{6}-\frac{2}{3(p-1)}} + C\varepsilon^{p-1}(-t)^{-\frac{1}{3}} \int_{-1}^{t} (-s)^{-\frac{2}{3}} f(s)ds \right],$$

provided that $C\varepsilon^{p-1} < 1$. Especially,

$$f(t) \leq \frac{1}{1 - C\varepsilon^{p-1}} \left[\varepsilon + C\varepsilon(-t)^{\frac{1}{6}-\frac{2}{3(p-1)}} + C\varepsilon^{p-1}(-t)^{-\frac{1}{3}} \int_{-1}^{t} (-s)^{-\frac{2}{3}} f(s)ds \right]$$
$$\forall t \in [-1,0).$$

Then for $\varepsilon > 0$ small enough, we obtain

$$(-t)^{\frac{1}{3}} f(t) \leq 2 \left[\varepsilon + C\varepsilon(-t)^{\frac{1}{2}-\frac{2}{3(p-1)}} + C\varepsilon^{p-1} \int_{-1}^{t} (-s)^{-1}(-s)^{\frac{1}{3}} f(s)ds \right]$$
$$\forall t \in [-1,0).$$

Define $h(t) = (-t)^{\frac{1}{3}} f(t)$; then

$$h(t) \leq 2\varepsilon + 2C\varepsilon(-t)^{\frac{1}{2}-\frac{2}{3(p-1)}} + 2C\varepsilon^{p-1} \int_{-1}^{t} (-s)^{-1} h(s)ds, \tag{3.12}$$

Applying Lemma(2), we have

$$h(t) \leq (-t)^{-2C\varepsilon^{p-1}} \left[2\varepsilon + C(p,\varepsilon)\varepsilon(-t)^{\frac{1}{2}-\frac{2}{3(p-1)}+2C\varepsilon^{p-1}} \right]$$
$$\leq 2\varepsilon(-t)^{-2C\varepsilon^{p-1}} + C(p,\varepsilon)\varepsilon(-t)^{\frac{1}{2}-\frac{2}{3(p-1)}}, \quad \forall t \in [-1,0).$$

Then $f(t) \leq 2\varepsilon(-t)^{-\frac{1}{3}-2C\varepsilon^{p-1}} + C(p,\varepsilon)\varepsilon(-t)^{\frac{1}{6}-\frac{2}{3(p-1)}}$, $\forall t \in [-1,0)$, or

$$y(t) \leq 2\varepsilon(-t)^{-\frac{1}{3}-2C\varepsilon^{p-1}} + C(p,\varepsilon)\varepsilon(-t)^{\frac{1}{6}-\frac{2}{3(p-1)}}, \quad \forall t \in [-1,0). \tag{3.13}$$

Choose $\varepsilon > 0$ small enough that $\frac{1}{3} + 2C\varepsilon^{p-1} < \frac{2}{3(p-1)}$ which is possible since $1 < p < 3$. Define $\alpha = \max\{\frac{1}{3} + 2C\varepsilon^{p-1}, \frac{2}{3(p-1)} - \frac{1}{6}\} \leq \frac{2}{3(p-1)}$, it is easy to find that $\alpha > \frac{1}{3}$; then (3.13) implies $y(t) \leq C(p,\varepsilon)\varepsilon(-t)^{-\alpha}$, $\forall t \in [-1,0)$. Hence

$$|u(x,t)| \leq C(p,\varepsilon)\varepsilon(-t)^{-\alpha}, \quad \forall (x,t) \in B_{\frac{1}{2}}(0) \times [-1,0). \tag{3.14}$$

Now let $\tilde{\phi}$ be a function supported on $B_{\frac{1}{2}}(o)$ with $\tilde{\phi} \equiv 1$ on $B_{\frac{1}{4}}(0)$ and $0 \leq \tilde{\phi} \leq 1$, and define $\tilde{\omega} = \tilde{\phi}u$; then we go back to (3.6)-(3.8) and we have that

$$\begin{aligned}
\|\tilde{\omega}(t)\|_\infty &\leq \varepsilon + C\int_{-1}^{t}(t-s)^{-\frac{1}{3}}\|u\|_\infty^{p-1}\|\tilde{\omega}\|_\infty ds + C\int_{-1}^{t}\|u\|_\infty^p ds \\
&\quad + C\int_{-1}^{t}(t-s)^{-\frac{1}{6}}\|u\|_\infty^p ds + C\varepsilon\int_{-1}^{t}(t-s)^{-\frac{5}{6}}\|u\|_\infty ds \\
&\leq \varepsilon + C\varepsilon^{p-1}\int_{-1}^{t}(t-s)^{-\frac{1}{3}}(-s)^{-\alpha(p-1)}(-s)^{-\alpha} ds + C\varepsilon^p\int_{-1}^{t}(-s)^{-\alpha p} ds \\
&\quad + C\varepsilon^p\int_{-1}^{t}(t-s)^{-\frac{1}{6}}(-s)^{-\alpha p} ds + C\varepsilon\int_{-1}^{t}(t-s)^{-\frac{5}{6}}(-s)^{-\alpha} ds \\
&\leq \varepsilon + C\varepsilon^{p-1}\int_{-1}^{t}(t-s)^{-\frac{1}{3}}(-s)^{-\alpha p} ds + C\varepsilon^p\int_{-1}^{t}(-s)^{-\alpha p} ds \\
&\quad + C\varepsilon^p\int_{-1}^{t}(t-s)^{-\frac{1}{6}}(-s)^{-\alpha p} ds + C\varepsilon\int_{-1}^{t}(t-s)^{-\frac{5}{6}}(-s)^{-\alpha} ds
\end{aligned} \tag{3.15}$$

due to (3.14).

Since $\frac{1}{3} < \alpha < \frac{2}{3(p-1)}$, we get

$$\frac{5}{6} - \alpha p > \frac{2}{3} - \alpha p > \frac{1}{6} - \alpha.$$

Hence by Lemma(1), we obtain

$$\|\tilde{\omega}(t)\|_\infty \leq \varepsilon + C\varepsilon^{p-1} + C\varepsilon^{p-1}(-t)^{\frac{1}{6}-\alpha} \leq (2 + C\varepsilon^{p-1})(-t)^{\frac{1}{6}-\alpha}, \quad \forall t \in [-1,0),$$

Which means, for small $\varepsilon > 0$,

$$|u(x,t)| \leq (2 + C\varepsilon^{p-1})(-t)^{\frac{1}{6}-\alpha}, \quad \forall (x,t) \in B_r(0) \times [-1,0). \tag{3.16}$$

Iterating the argument finitely many times we can get that there is a number $0 < r_0 < \frac{1}{4}$ such that

$$|u(x,t)| \leq K(-t)^{-\frac{1}{6p}}, \quad \forall (x,t) \in B_{r_0}(0) \times [-1,0), \tag{3.17}$$

where K is constant.

Next, we choose another cut-off function $\hat{\phi}$ supported on B_{r_0} such that $\hat{\phi} \equiv 1$ on $B_{\frac{r_0}{2}}$ and define $\hat{\omega} = \hat{\phi}u$. Going back to (3.15) and applying Lemma(1), we have

$$\begin{aligned}
\|\hat{\omega}(t)\|_\infty &\le \varepsilon + C\int_{-1}^{t}(t-s)^{-\frac{1}{3}}\|u\|_\infty^{p-1}\|\hat{\omega}\|_\infty ds + C\int_{-1}^{t}\|u\|_\infty^p ds \\
&\quad + C\int_{-1}^{t}(t-s)^{-\frac{1}{6}}\|u\|_\infty^p ds + C\varepsilon\int_{-1}^{t}(t-s)^{-\frac{5}{6}}\|u\|_\infty ds \\
&\le \varepsilon + CK^{p-1}\int_{-1}^{t}(t-s)^{-\frac{1}{3}}(-s)^{-\frac{1}{6}}ds + CK^p\int_{-1}^{t}(-s)^{-\frac{1}{6}}ds \\
&\quad + CK^p\int_{-1}^{t}(t-s)^{-\frac{1}{6}}(-s)^{-\frac{1}{6p}}ds + CK\int_{-1}^{t}(t-s)^{-\frac{5}{6}}(-s)^{-\frac{1}{6p}}ds \\
&\le \varepsilon + CK^{p-1}, \qquad (3.18)
\end{aligned}$$

which means that $|u(x,t)| \le C$ in $B_{\frac{r_0}{2}} \times [-1,0)$. This completes the proof of the theorem.

□

Using the same argument, we can easily draw the following conclusion.

Theorem 3. *Suppose $p \ge 3$, then for any $\delta \in (0, \frac{2}{3(p-1)})$, there is a constant $\varepsilon > 0$, depending on n, p and the constant in (3.1), such that if u is a solution of the equation*

$$u_t - \Delta(\Delta^2 u - |u|^{p-1}u) = 0, \qquad \text{on} \quad Q_r = B_r(a) \times [t_1 - r^6, t_1)$$

where $a \in \mathbb{R}^n, t_1 \in \mathbb{R}$ and $0 < r \le 1$, and if

$$|u(x,t)| \le \varepsilon(t_1 - t)^{-\frac{2}{3(p-1)}} \qquad \text{for} \quad \text{all} \quad (x,t) \in Q_r,$$

then u does not blow up at (a, t_1).

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GENERALIZATIONS OF INTEGRAL INEQUALITIES FOR FUNCTIONS WHOSE SECOND DERIVATIVES ARE CONVEX AND $m-$ CONVEX

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Abstract. In this paper, we establish some new integral inequalities for convex and $m-$convex functions by using a new kernel. The analysis used in the proofs is fairly elementary and based on the classical inequalities. We also give some comparisons and applications to special means for our results.

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1. INTRODUCTION

The following definition is well known in the literature: a function $f : I \to \mathbb{R}, \varnothing \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

holds for every $x, y \in I$ and $t \in [0,1]$.

Definition 1 (See [10])**.** A function $f : [0,b] \to \mathbb{R}$ is said to be m-convex, where $m \in [0,1]$, $b > 0$, if for every $x, y \in [0,b]$ and $t \in [0,1]$ we have :

$$f(tx+m(1-t)y) \leq tf(x)+m(1-t)f(y).$$

For recent results related to m-convex functions we refer the interest of readers to the papers [2, 3, 5, 7, 8, 10].

Many authors have been studied on integral inequalities. One of the well known of these inequalities -Simpson's inequality- is given as following:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\| f^{(4)} \right\|_\infty = \sup_{x \in (a,b)} \left| f^{(4)}(x) \right| < \infty$. Then, the following inequality*

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holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_\infty(b-a)^4.$$

In [9], Sarıkaya *et al.* proved the following lemma and established some new Simpson type inequalities for convex functions:

Lemma 1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $f''\in L_1[a,b]$, where $a,b\in I$ with $a<b$, then the following equality holds:*

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx \tag{1.1}$$

$$=(b-a)^2\int_0^1 k(t)\,f''(tb+(1-t)a)\,dt$$

where

$$k(t)=\begin{cases}\frac{t}{2}\left(\frac{1}{3}-t\right), & t\in\left[0,\frac{1}{2}\right)\\ (1-t)\left(\frac{t}{2}-\frac{1}{3}\right), & t\in\left[\frac{1}{2},1\right]\end{cases}.$$

Theorem 2 (See [9])**.** *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $f''\in L_1[a,b]$, where $a,b\in I$ with $a<b$. If $|f''|$ is a convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{1.2}$$

$$\leq\frac{(b-a)^2}{162}\left[|f''(a)|+|f''(b)|\right].$$

Theorem 3 (See [9])**.** *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $f''\in L_1[a,b]$, where $a,b\in I$ with $a<b$. If $|f''|^q$ is a convex on $[a,b]$ and $q\geq1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{1.3}$$

$$\leq(b-a)^2\left(\frac{1}{162}\right)^{1-\frac{1}{q}}\left\{\left(\frac{59}{3^5 2^7}\left|f''(b)\right|^q+\frac{133}{3^5 2^7}\left|f''(a)\right|^q\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{133}{3^5 2^7}\left|f''(b)\right|^q+\frac{59}{3^5 2^7}\left|f''(a)\right|^q\right)^{\frac{1}{q}}\Bigg\}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see the papers [1,4,6].

The main purpose of this paper is to give some generalizations of integral inequalities for convex and m-convex functions by using a more general lemma similar to Lemma 1. Some comparisons and applications to special means related to our results are also given.

2. Main Results

Throughout this paper, we will assume that $I \subset \mathbb{R}$. To prove our main theorems we need the following lemma involving a new kernel.

Lemma 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L_1[a,b]$, where $a,b \in I$ with $a<b$ and $r \in \mathbb{R}^+$ then the following equality holds:*

$$\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx \tag{2.1}$$

$$=(b-a)^2\int_0^1 k(t)\,f''(tb+(1-t)a)\,dt$$

where

$$k(t)=\begin{cases}\frac{t}{r}\left(\frac{1}{r+1}-t\right), & t\in\left[0,\frac{1}{2}\right)\\ (1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right), & t\in\left[\frac{1}{2},1\right]\end{cases}.$$

Proof. By definition of $k(t)$, we can write

$$I=\int_0^1 k(t)\,f''(tb+(1-t)a)\,dt$$

$$=\int_0^{\frac{1}{2}}\frac{t}{r}\left(\frac{1}{r+1}-t\right)f''(tb+(1-t)a)\,dt$$

$$+\int_{\frac{1}{2}}^1(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)f''(tb+(1-t)a)\,dt.$$

Integrating the right hand side of the above equality by parts twice, we have

$$\int_0^{\frac{1}{2}} \frac{t}{r}\left(\frac{1}{r+1}-t\right) f''(tb+(1-t)a)\,dt$$
$$=-\frac{r-1}{4r(r+1)(b-a)} f'\left(\frac{a+b}{2}\right)$$
$$+\frac{1}{(b-a)^2}\left[\frac{1}{r+1} f\left(\frac{a+b}{2}\right)+\frac{1}{r(r+1)} f(a)-\frac{2}{r}\int_0^{\frac{1}{2}} f\left(tb+(1-t)a\right)dt\right]$$

and

$$\int_{\frac{1}{2}}^{1} (1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right) f''(tb+(1-t)a)\,dt$$
$$=\frac{r-1}{4r(r+1)(b-a)} f'\left(\frac{a+b}{2}\right)$$
$$+\frac{1}{(b-a)^2}\left[\frac{1}{r+1} f\left(\frac{a+b}{2}\right)+\frac{1}{r(r+1)} f(b)-\frac{2}{r}\int_{\frac{1}{2}}^{1} f\left(tb+(1-t)a\right)dt\right].$$

By addition and using the change of variable $x = tb+(1-t)a$ for $t \in [0,1]$ and multiplying the both sides by $(b-a)^2$, we obtain (2.1). □

Remark 1. If we choose $r = 2$ in (2.1), we get the equality (1.1).

Theorem 4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L_1[a,b]$, where $a,b \in I$ with $a < b$ and $r \in \mathbb{R}^+$. If $|f''|$ is convex, then we have the following inequality;*

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1} f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \tag{2.2}$$
$$\leq M(b-a)^2\left[\left|f''(a)\right|+\left|f''(b)\right|\right]$$

where $M = \frac{r^3-3r+6}{24r(r+1)^3}$.

Proof. From Lemma 2 and by using convexity of $|f''(x)|$, we obtain

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1} f\left(\frac{a+b}{2}\right)-\frac{2}{r}\int_0^1 f\left(tb+(1-t)a\right)dt\right| \tag{2.3}$$

$$\leq (b-a)^2 \left\{ \int_0^{\frac{1}{2}} \left| \frac{t}{r}\left(\frac{1}{r+1}-t\right)\right| \left[t\left|f''(b)\right| + (1-t)\left|f''(a)\right|\right] dt \right.$$

$$\left. + \int_{\frac{1}{2}}^{1} \left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right| \left[t\left|f''(b)\right| + (1-t)\left|f''(a)\right|\right] dt \right\}$$

$$= (b-a)^2 \left\{ \int_0^{\frac{1}{2}} \left| \frac{t}{r}\left(\frac{1}{r+1}-t\right)\right| \left[t\left|f''(b)\right| + (1-t)\left|f''(a)\right|\right] dt \right.$$

$$\left. + \int_0^{\frac{1}{2}} \left|t\left(\frac{1-t}{r}-\frac{1}{r+1}\right)\right| \left[(1-t)\left|f''(b)\right| + t\left|f''(a)\right|\right] dt \right\}$$

$$= (b-a)^2 (J_1 + J_2)$$

where

$$J_1 = \int_0^{\frac{1}{2}} \left| \frac{t}{r}\left(\frac{1}{r+1}-t\right)\right| \left[t\left|f''(b)\right| + (1-t)\left|f''(a)\right| dt\right]$$

$$J_2 = \int_{\frac{1}{2}}^{1} \left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right| \left[t\left|f''(b)\right| + (1-t)\left|f''(a)\right| dt\right].$$

By a little hard computation, one can see that

$$J_1 = \int_0^{\frac{1}{r+1}} \left| \frac{t}{r}\left(\frac{1}{r+1}-t\right)\right| \left[t\left|f''(b)\right| + (1-t)\left|f''(a)\right| dt\right]$$

$$+ \int_{\frac{1}{r+1}}^{\frac{1}{2}} \left| \frac{t}{r}\left(t-\frac{1}{r+1}\right)\right| \left[t\left|f''(b)\right| + (1-t)\left|f''(a)\right| dt\right]$$

$$= M_1\left|f''(b)\right| + M_2\left|f''(a)\right|$$

and

$$\begin{aligned} J_2 &= \int_{\frac{1}{2}}^{\frac{r}{r+1}} \left| t\left(\frac{1-t}{r} - \frac{1}{r+1}\right)\right| \left[(1-t)\left|f''(b)\right| + t\left|f''(a)\right|\right] dt \\ &\quad + \int_{\frac{r}{r+1}}^{1} \left| t\left(\frac{1-t}{r} - \frac{1}{r+1}\right)\right| \left[(1-t)\left|f''(b)\right| + t\left|f''(a)\right|\right] dt \\ &= M_1\left|f''(a)\right| + M_2\left|f''(b)\right| \end{aligned}$$

where

$$M_1 = \frac{3r^4+4r^3-6r^2-12r+27}{192r\,(r+1)^4} \text{ and } M_2 = \frac{5r^4+4r^3-18r^2+36r+21}{192r\,(r+1)^4}.$$

By taking into account J_1, J_2, M_1 and M_2 in (2.3), we obtain

$$\begin{aligned} J_1 + J_2 &= (M_1+M_2)\left[\left|f''(a)\right| + \left|f''(b)\right|\right] \\ &= M\left[\left|f''(a)\right| + \left|f''(b)\right|\right] \end{aligned}$$

which completes the proof. □

Corollary 1. *If we take $r = 1$ in (2.2) we obtain the following inequality:*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_a^b f(x)\,dx \right| \qquad (2.4) \\ &\leq \frac{(b-a)^2}{48}\left[\left|f''(a)\right| + \left|f''(b)\right|\right]. \end{aligned}$$

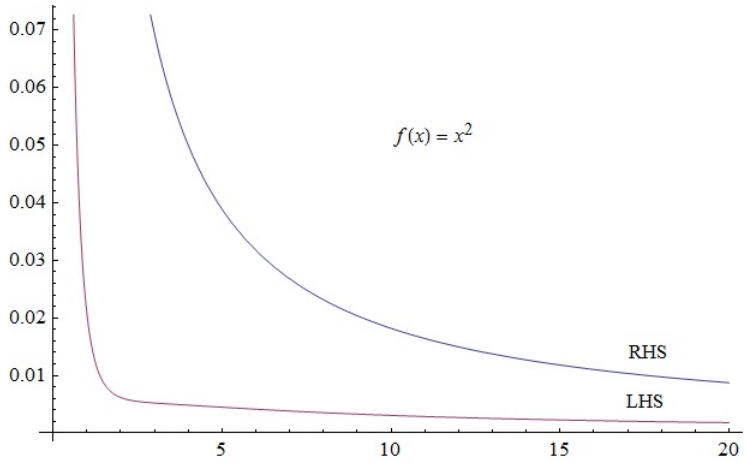
Corollary 2. *The following table shows the results in (2.2) for several values of r. In the table the left hand side of (2.2) is given by LHS and the right one is given by RHS.*

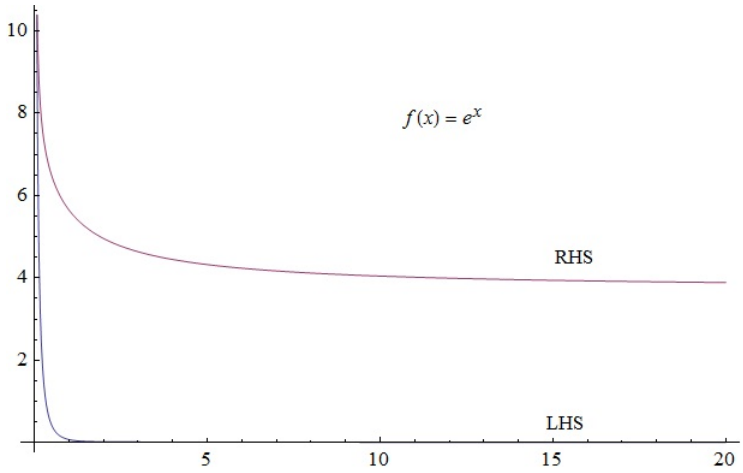
r	*LHS*	*RHS*
1	$\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{(b-a)}\int_a^b f(x)\,dx$	$\frac{(b-a)^2}{48}[\lvert f''(a)\rvert+\lvert f''(b)\rvert]$
2	$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{(b-a)}\int_a^b f(x)dx$	$\frac{(b-a)^2}{162}[\lvert f''(a)\rvert+\lvert f''(b)\rvert]$
3	$\frac{1}{12}\left[f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2}{3(b-a)}\int_a^b f(x)dx$	$\frac{(b-a)^2}{192}[\lvert f''(a)\rvert+\lvert f''(b)\rvert]$
24	$\frac{1}{600}\left[f(a)+48f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{12(b-a)}\int_a^b f(x)dx$	$\frac{(b-a)^2}{654}[\lvert f''(a)\rvert+\lvert f''(b)\rvert]$
30	$\frac{1}{930}\left[f(a)+30f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{15(b-a)}\int_a^b f(x)dx$	$\frac{(b-a)^2}{797}[\lvert f''(a)\rvert+\lvert f''(b)\rvert]$

From the table one can see that if we take $r\to\infty$ we obtain smaller upper bounds for the inequality (2.2) .

Remark 2. If we take $r=2$ in (2.2) we obtain (1.2).

Example 1. Under the assumptions of Theorem 4, if we choose $[a,b]=[0,1]$ then we can give the following graphics for some special cases of f by Mathematica programme . In these graphics x-axis shows the values of $r\in\mathbb{R}^+$:





Theorem 5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L_1[a,b]$ and if $|f''|^q$ is convex on $[a,b]$ where $a,b \in I$ with $a < b$, $r \in \mathbb{R}^+$ and $q \geq 1$, then the following inequality holds:*

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right|$$

$$\leq (b-a)^2\left(\frac{r^3-3r+6}{24r(r+1)^3}\right)^{1-\frac{1}{q}}\left\{\left[\left(M_1\left|f''(b)\right|^q+M_2\left|f''(a)\right|^q\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+\left[\left(M_2\left|f''(b)\right|^q+M_1\left|f''(a)\right|^q\right)^{\frac{1}{q}}\right]\right\}$$

where $M_1 = \frac{3r^4+4r^3-6r^2-12r+27}{192r(r+1)^4}$ *and* $M_2 = \frac{5r^4+4r^3-18r^2+36r+21}{192r(r+1)^4}$.

Proof. From Lemma 2 and by using the well known power-mean inequality for $q \geq 1$, we have

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right|$$

$$\leq (b-a)^2\int_0^1 |k(t)|\left|f''(tb+(1-t)a)\right|dt$$

$$\leq (b-a)^2\left(\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|dt\right)^{1-\frac{1}{q}}$$

$$\cdot\left(\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left|f''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|dt\right)^{1-\frac{1}{q}}$$

$$\cdot\left(\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left|f''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}.$$

On the other hand, since $|f''|^q$ is convex on $[a,b]$, we have

$$\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left|f''(tb+(1-t)a)\right|^q dt \tag{2.5}$$

$$\leq\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left[t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right]dt$$

$$=\int_0^{\frac{1}{r+1}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left[t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right]dt$$

$$+\int_{\frac{1}{r+1}}^{\frac{1}{2}}\left|\frac{t}{r}\left(t-\frac{1}{r+1}\right)\right|\left[t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right]dt$$

$$=M_1\left|f''(b)\right|^q+M_2\left|f''(a)\right|^q$$

and

$$\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left|f''(tb+(1-t)a)\right|^q dt \tag{2.6}$$

$$\leq\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left[t\left|f''(b)\right|^q+(1-t)\left|f''(a)\right|^q\right]dt$$

$$= \int_{\frac{1}{2}}^{\frac{r}{r+1}} \left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right| \left[t\left|f''(b)\right|^q + (1-t)\left|f''(a)\right|^q\right] dt$$

$$+ \int_{\frac{r}{r+1}}^{1} \left|(1-t)\left(\frac{1}{r+1}-\frac{t}{r}\right)\right| \left[t\left|f''(b)\right|^q + (1-t)\left|f''(a)\right|^q\right] dt$$

$$= M_2\left|f''(b)\right|^q + M_1\left|f''(a)\right|^q$$

where $M_1 = \frac{3r^4+4r^3-6r^2-12r+27}{192r(r+1)^4}$ and $M_2 = \frac{5r^4+4r^3-18r^2+36r+21}{192r(r+1)^4}$.

From (2.5) and (2.6) and by using the fact that

$$\int_0^{\frac{1}{2}} \left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right| dt = \int_{\frac{1}{2}}^{1} \left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right| dt = \frac{r^3-3r+6}{24r(r+1)^3}$$

we deduce

$$\left|\frac{1}{r(r+1)}\left[f(a)+f(b)\right] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)}\int_a^b f(x)\,dx\right|$$

$$\leq (b-a)^2\left(\frac{r^3-3r+6}{24r(r+1)^3}\right)^{1-\frac{1}{q}} \left\{\left[\left(M_1\left|f''(b)\right|^q + M_2\left|f''(a)\right|^q\right)^{\frac{1}{q}}\right]\right.$$

$$\left. + \left[\left(M_2\left|f''(b)\right|^q + M_1\left|f''(a)\right|^q\right)^{\frac{1}{q}}\right]\right\},$$

which completes the proof. □

Corollary 3. *i) Under the assumptions of Theorem 5, if we choose $r = 1$ we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq (b-a)^2\left(\frac{1}{48}\right)^{1-\frac{1}{q}} \left\{\left[\left(\frac{1}{192}\left|f''(b)\right|^q + \frac{3}{192}\left|f''(a)\right|^q\right)^{\frac{1}{q}}\right]\right.$$

$$\left. + \left[\left(\frac{3}{192}\left|f''(b)\right|^q + \frac{1}{192}\left|f''(a)\right|^q\right)^{\frac{1}{q}}\right]\right\}$$

ii) Let $a_1 = \frac{1}{192}|f''(b)|^q$, $b_1 = \frac{3}{192}|f''(a)|^q$, $a_2 = \frac{3}{192}|f''(b)|^q$, $b_2 = \frac{1}{192}|f''(a)|^q$. *Here* $0 < \frac{1}{q} < 1$, *for* $q > 1$, *using the fact that*

$$\sum_{k=1}^{n}(a_k+b_k)^s \le \sum_{k=1}^{n}(a_k)^s + \sum_{k=1}^{n}(b_k)^s$$

for $0 \le s \le 1$, $a_1,...,a_n \ge 0$, $b_1,...b_n \ge 0$, *we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_a^b f(x)\,dx\right|$$
$$\le (b-a)^2\left(\frac{1}{48}\right)^{1-\frac{1}{q}}\left\{\frac{1}{192^{1/q}}|f''(b)| + \left(\frac{3}{192}\right)^{1/q}|f''(a)|\right\}.$$

Now for $q \to \infty$, *we get*

$$\left|\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{24}\left(|f''(a)| + |f''(b)|\right).$$

Theorem 6. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a twice differentiable mapping on* I° *such that* $f'' \in L_1[a,b]$ *and* $|f''|$ *is* m*-convex function with* $m \in (0,1]$, *where* $a,b \in I$ *with* $a < b$ *and* $r \in \mathbb{R}^+$

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)] + \frac{2}{r+1}f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \tag{2.7}$$
$$\le (b-a)^2 M\left[m\left|f''\left(\frac{a}{m}\right)\right| + |f''(b)|\right]$$

where $M = \frac{r^3-3r+6}{24r(r+1)^3}$.

Proof. From Lemma 2 and using the property of absolute value, we can write

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)] + \frac{2}{r+1}f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)}\int_a^b f(x)\,dx\right|$$
$$\le (b-a)^2\int_0^1 \left|k(t)f''(tb+(1-t)a)\right|dt.$$

Since $|f''|$ is m-convex on $[a,b]$, we know that for any $t \in [0,1]$

$$|f''(tb+(1-t)a)| = \left|f''\left(tb+m(1-t)\frac{a}{m}\right)\right|$$

$$\leq t\left|f''(b)\right| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right|$$

hence it follows that

$$\begin{aligned}
&(b-a)^2\int_0^1 |k(t)|\left[t\left|f''(b)\right| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right|\right]dt \\
&= (b-a)^2\left\{\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left[t\left|f''(b)\right| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right|\right]dt\right. \\
&\quad \left. + \int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left[t\left|f''(b)\right| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right|\right]dt\right\} \\
&= J_1 + J_2
\end{aligned}$$

where

$$J_1 = \int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left[t\left|f''(b)\right| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right|\right]dt$$

$$J_2 = \int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left[t\left|f''(b)\right| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right|\right]dt.$$

If we compute J_1 and J_2, we have

$$\begin{aligned}
J_1 &= \int_0^{\frac{1}{r+1}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left[t\left|f''(b)\right| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right|\right]dt \\
&\quad + \int_{\frac{1}{r+1}}^{\frac{1}{2}}\left|\frac{t}{r}\left(t-\frac{1}{r+1}\right)\right|\left[t\left|f''(b)\right| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right|\right]dt \\
&= M_1\left|f''(b)\right| + M_2 m\left|f''\left(\frac{a}{m}\right)\right|
\end{aligned}$$

and

$$J_2 = \int_{\frac{1}{2}}^{\frac{r}{r+1}}\left|(1-t)\left(\frac{1}{r+1}-\frac{t}{r}\right)\right|\left[t\left|f''(b)\right| + m(1-t)\left|f''\left(\frac{a}{m}\right)\right|\right]dt$$

$$+\int_{\frac{r}{r+1}}^{1}\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left[t\left|f''(b)\right|+m(1-t)\left|f''\left(\frac{a}{m}\right)\right|\right]dt$$

$$=M_1 m\left|f''\left(\frac{a}{m}\right)\right|+M_2\left|f''(b)\right|$$

where $M_1=\frac{3r^4+4r^3-6r^2-12r+27}{192r(r+1)^4}$ and $M_2=\frac{5r^4+4r^3-18r^2+36r+21}{192r(r+1)^4}$.
By taking into account J_1 and J_2, we obtain the desired result. □

Theorem 7. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I° such that $f''\in L_1[a,b]$ and if $|f''|^q$ is m-convex on $[a,b]$ and $m\in(0,1]$ where $a,b\in I$ with $a<b$, $r\in\mathbb{R}^+$ and $q\geq 1$, then the following inequality holds;*

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \tag{2.8}$$

$$\leq(b-a)^2\left(\frac{r^3-3r+6}{24r(r+1)^3}\right)^{1-\frac{1}{q}}\left\{\left\{\left(M_1\left|f''(b)\right|^q+M_2 m\left|f''(\tfrac{a}{m})\right|^q\right)^{\frac{1}{q}}\right\}\right.$$

$$\left.+\left(M_2\left|f''(b)\right|^q+M_1 m\left|f''(\tfrac{a}{m})\right|^q\right)^{\frac{1}{q}}\right\}$$

where $M_1=\frac{3r^4+4r^3-6r^2-12r+27}{192r(r+1)^4}$ and $M_2=\frac{5r^4+4r^3-18r^2+36r+21}{192r(r+1)^4}$.

Proof. From Lemma 2 and by using the well known power-mean inequality, we have

$$\left|\frac{1}{r(r+1)}[f(a)+f(b)]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right|$$

$$\leq(b-a)^2\int_0^1|k(t)|\left|f''(tb+(1-t)a)\right|dt$$

$$\leq(b-a)^2\left(\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|dt\right)^{1-\frac{1}{q}}$$

$$\cdot\left(\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left|f''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|dt\right)^{1-\frac{1}{q}}$$

$$\cdot\left(\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left|f''(tb+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}.$$

Since $|f''|^q$ is m-convex, we have

$$\int_{0}^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left|f''(tb+(1-t)a)\right|^{q}dt \tag{2.9}$$

$$\leq\int_{0}^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left[t\left|f''(b)\right|^{q}+m(1-t)\left|f''(\frac{a}{m})\right|^{q}\right]dt$$

$$=\int_{0}^{\frac{1}{r+1}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|\left[t\left|f''(b)\right|^{q}+m(1-t)\left|f''(\frac{a}{m})\right|^{q}\right]dt$$

$$+\int_{\frac{1}{r+1}}^{\frac{1}{2}}\left|\frac{t}{r}\left(t-\frac{1}{r+1}\right)\right|\left[t\left|f''(b)\right|^{q}+m(1-t)\left|f''(\frac{a}{m})\right|^{q}\right]dt$$

$$=M_1\left|f''(b)\right|^{q}+M_2m\left|f''(\frac{a}{m})\right|^{q}$$

and

$$\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left|f''(tb+(1-t)a)\right|^{q}dt \tag{2.10}$$

$$\leq\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left[t\left|f''(b)\right|^{q}+m(1-t)\left|f''(\frac{a}{m})\right|^{q}\right]dt$$

$$=\int_{\frac{1}{2}}^{\frac{r}{r+1}}\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|\left[t\left|f''(b)\right|^{q}+m(1-t)\left|f''(\frac{a}{m})\right|^{q}\right]dt$$

$$+\int_{\frac{r}{r+1}}^{1}\left|(1-t)\left(\frac{1}{r+1}-\frac{t}{r}\right)\right|\left[t\left|f''(b)\right|^q+m(1-t)\left|f''(\frac{a}{m})\right|^q\right]dt$$

$$=M_2\left|f''(b)\right|^q+M_1m\left|f''(\frac{a}{m})\right|^q$$

where $M_1=\frac{3r^4+4r^3-6r^2-12r+27}{192r(r+1)^4}$ and $M_2=\frac{5r^4+4r^3-18r^2+36r+21}{192r(r+1)^4}$.

From (2.9) and (2.10) and by using the fact that

$$\int_0^{\frac{1}{2}}\left|\frac{t}{r}\left(\frac{1}{r+1}-t\right)\right|dt=\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{r}-\frac{1}{r+1}\right)\right|dt=\frac{r^3-3r+6}{24r(r+1)^3}$$

we obtain

$$\left|\frac{1}{r(r+1)}\left[f(a)+f(b)\right]+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{2}{r(b-a)}\int_a^b f(x)\,dx\right| \tag{2.11}$$

$$\leq(b-a)^2\left(\frac{r^3-3r+6}{24r(r+1)^3}\right)^{1-\frac{1}{q}}\left\{\left(M_1\left|f''(b)\right|^q+M_2m\left|f''(\frac{a}{m})\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(M_2\left|f''(b)\right|^q+M_1m\left|f''(\frac{a}{m})\right|^q\right)^{\frac{1}{q}}\right\},$$

which completes the proof. □

3. Applications to special means

Now, we consider some applications of our theorems to the following special means.

a) The arithmetic mean:

$$A=A(a,b):=\frac{a+b}{2},\quad a,b\geq 0,$$

b) The harmonic mean:

$$H=H(a,b):=\frac{2ab}{a+b},\quad a,b\geq 0,$$

c) The logarithmic mean:

$$L=L(a,b):=\begin{cases}a & \text{if } a=b\\ \frac{b-a}{\ln b-\ln a} & \text{if } a\neq b\end{cases},\quad a,b\geq 0,$$

d) The $p-$logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}, \quad p \in \mathbb{R}\setminus\{-1,0\};\ a,b>0.$$

Proposition 1. *Let $a,b,n \in \mathbb{R}$, then we have*

$$\left|A\left(a^n,b^n\right) + A^n(a,b) - 2L_n^n(a,b)\right| \leq \frac{(b-a)^2}{24} n(n-1) A\left(|a|^{n-2},|b|^{n-2}\right) \quad (3.1)$$

and

$$\left|\frac{1}{3}A\left(a^n,b^n\right) + \frac{2}{3}A^n(a,b) - L_n^n(a,b)\right| \leq \frac{(b-a)^2}{81} n(n-1) A\left(|a|^{n-2},|b|^{n-2}\right)$$

Proof. The assertion follows from Theorem 4 applied for $f(x) = x^n$, $x \in [a,b]$ with $r = 1$ for the first inequality and $r = 2$ for the second inequality. □

And especially for $f(x) = \frac{1}{x}$, we can get

$$\left|H(a,b) + A^{-1}(a,b) - 2L(a,b)\right| \leq \frac{(b-a)^2}{12} A\left(|a|^{-3},|b|^{-3}\right)$$

for $r = 1$ and

$$\left|\frac{1}{3}H(a,b) + \frac{2}{3}A^n(a,b) - L(a,b)\right| \leq \frac{(b-a)^2}{81} 2A\left(|a|^{-3},|b|^{-3}\right)$$

for $r = 2$.

Proposition 2. *Let $a,b,n \in \mathbb{R}$, then we have*

$$\left|A\left(a^n,b^n\right) + A^n(a,b) - 2L_n^n(a,b)\right| \leq \frac{(b-a)^2}{24} n(n-1) A\left(m|a|^{n-2},|b|^{n-2}\right) \quad (3.2)$$

and

$$\left|\frac{1}{3}A\left(a^n,b^n\right) + \frac{2}{3}A^n(a,b) - L_n^n(a,b)\right| \leq \frac{(b-a)^2}{216} n(n-1) A\left(m|a|^{n-2},|b|^{n-2}\right)$$

Proof. The assertion follows from Theorem 6 applied for $f(x) = x^n$, $x \in [a,b]$ with $r = 1$ for the first inequality and $r = 2$ for the second inequality. □

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PERIODIC SUCCESSIVE APPROXIMATIONS AND INTERVAL HALVING

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Abstract. We show how a suitable interval halving and parametrisation technique can help to essentially improve the sufficient convergence condition for the periodic successive approximations dealing with periodic solutions of nonlinear non-autonomous systems of ordinary differential equations.

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1. INTRODUCTION

It is well-known that, formally, the methods used in the the theory of boundary value problems for the investigation and approximate construction of solutions, can be characterised as analytic, functional-analytic, numerical and numerical-analytic ones (see, e. g. [1, 4, 7, 16, 20, 23] and the references therein). Among the numerical-analytic methods we share out those which are based upon some types of successive approximations.

In the theory of nonlinear oscillations, such types of numerical-analytic methods were apparently first developed by Cesari [2], Hale [6], and Samoilenko [24–26]. The latter mentioned approach had later been developed for more general types of boundary-value problems in numerous books and papers (see, e. g., [8–14, 17, 21, 22, 22, 27–29]).

Methods of the numerical-analytic type, in a sense, combine, advantages of the mentioned above approaches and are usually based upon certain iteration processes constructed explicitly in analytic form. Such an approach belongs to the few of them that offer constructive possibilities both for the investigation of the existence of solution and it approximate construction.

For a boundary value problem, the numerical-analytic approach usually replaces the problem by the Cauchy problem for a suitably perturbed system containing some artificially introduced vector parameter z, which most often has the meaning of the

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initial value of the solution and the numerical value of which is to be determined later. A solution of the Cauchy problem for the perturbed system is sought for in an analytic form by successive approximations. The functional "perturbation term," by which the modified equation differs from the original one, depends explicitly on the parameter z and generates a system of algebraic or transcendental "determining equations" from which the numerical values of z should be found. The solvability of the determining system, in turn, may be checked by studying some approximations constructed explicitly. It is clear that the complexity of the given equations and boundary conditions has essential influence on both the efficiency of construction of approximate solutions and the subsequent solvability analysis.

In order to guarantee the convergence, one usually assumes [11, 27–29] a kind of the Lipschitz condition and requires a certain smallness restriction, which most often has the form

$$r(K) \leq q_T, \tag{1.1}$$

where K is the Lipschitz matrix and q_T is a constant depending on the period T. The improvement of the condition then consists in minimising the value of q_T.

The aim of this paper is to show how a suitable interval halving and parametrization techniques can, under fairly general assumptions, weaken (1.1) to

$$r(K) \leq 2q_T \tag{1.2}$$

and, thus, essentially improve the sufficent convergence conditions established in [11, 24–27] for the periodic successive approximations related to the periodic solutions of non-linear non-autonomous systems of ordinary differential equations.

2. List of symbols

We fix an $n \in \mathbb{N}$ and a bounded set $D \subset \mathbb{R}^n$. The following symbols and conventions are used in the sequel:

(1) $|\cdot|$, $\leq$ for vectors from $\mathbb{R}^n$ are understood component-wise.
(2) $\sup\{f(z) : z \in Q\}$ for any $f = (f_i)_{i=1}^n : Q \to \mathbb{R}^n$, where $Q \subset \mathbb{R}^m$, $m \leq n$, is defined as the column vector with components $\sup\{f_i(z) : z \in Q\}$, $i = 1, 2, \ldots, n$. The symbol $\max\{f(z) : z \in Q\}$ is defined by analogy.
(3) The "vector-valued diameter" of a set $Q \subset \mathbb{R}^n$ is defined by the equality

$$\operatorname{diam} Q := \sup\{|z_1 - z_2| : \{z_1, z_2\} \subset Q\}. \tag{2.1}$$

(4) $d(Q)$: see (7.26).
(5) $r(K)$ is the maximal, in modulus, eigenvalue of a matrix K
(6) $\delta_{J,D}(f)$: see (4.2).
(7) ϱ_*: see (4.3) and (4.4).
(8) $D(r)$: see Definition 4.1, formula (4.1).
(9) $Q - Q := \{z_1 - z_2 : \{z_1, z_2\} \subset Q\}$ for any $Q \subset \mathbb{R}^n$.

3. Motivation, problem setting, and basic assumptions

In the papers [24,25], Samoilenko had suggested a method for studying T-periodic solutions of a system of non-linear ordinary differential equations

$$u'(t) = f(t,u(t)), \qquad t \in (-\infty,\infty), \tag{3.1}$$

where $f : \mathbb{R}\times D \to \mathbb{R}^n$ is a continuous function such that

$$f(t,z) = f(t+T,z) \tag{3.2}$$

for all $z \in D$ and all $t \in (-\infty,\infty)$. Here, T is a given positive number and D is the closure of a bounded and connected domain in $\mathbb{R}^n$.

We are interested in continuously differentiable solutions of system (3.1) which are periodic with period T. From now on, instead T-periodic solutions of (3.1), we shall deal with solutions of the corresponding periodic boundary value problem on the bounded interval $[0,T]$

$$u'(t) = f(t,u(t)), \qquad t \in [0,T], \tag{3.3}$$

$$u(0) = u(T). \tag{3.4}$$

The passage to problem (3.3), (3.4) is justified by assumption (3.2) and the following standard lemma (see, e. g., [11, Lemma 3.1] or [3, Lemma 2.2.1]) which is given here for completeness.

Lemma 3.1. *If a function $u : (-\infty,\infty) \to D$ is a $T-$periodic solution of a system (3.1), then its restriction to $[0,T]$ satisfies (3.3), (3.4). Conversely, if, under assumption (3.2), a function $u : [0,T] \to D$ is a solution of the boundary value problem (3.3), (3.4), then its continuous T-periodic extension to $(-\infty,\infty)$ is a T-periodic solution of system (3.1).*

Our study of the periodic boundary value problem (3.3), (3.4) is based on the assumption that the non-linearity $f : [0,T]\times D \to \mathbb{R}^n$ is Lipschitzian in the space variable. For the sake of simplicity, we assume that there exists a non-negative constant matrix $K = (k_{ij})_{i,j=1}^n$ such that

$$|f(t,x_1) - f(t,x_2)| \le K\,|x_1 - x_2| \tag{3.5}$$

for all $\{x_1,x_2\} \subset D$ and $t \in [0,T]$.

Here and below, the obvious notation $|x| = \mathrm{col}(|x_1|,|x_2|,...,|x_n|)$ is used. The inequalities and the signs "max" and "min" for vectors are understood componentwise.

The method of [24,25], originally called *numerical-analytic method for the investigation of periodic solutions*, was later referred to also as *numerical-analytic method of periodic successive approximations* [11,27–29].

The scheme of the method, which is described, in a suitable for us form, by Propositions 5.1 and 5.5 below, is extremely simple and deals with the investigation of the

parametrised equation

$$u(t) = z + \int_0^t f(s,u(s))ds - \frac{t}{T}\int_0^T f(s,u(s))ds, \qquad t \in [0,T], \tag{3.6}$$

where $z \in D$ is an parameter to be chosen later. For convenience of reference, we formulate the statements for the p-periodic problem

$$u'(t) = g(t,u(t)), \qquad t \in [t_0, t_0+p], \tag{3.7}$$

$$u(t_0) = u(t_0+p), \tag{3.8}$$

where $g : [t_0, t_0+p] \times \mathbb{R}^n \to \mathbb{R}^n$ and $t_0 \in (-\infty, \infty)$ is arbitrary but fixed.

4. Notation

The notion of a set $D(r)$ associated with D, which could have been called an *r-core* of D, will often be used in the sequel.

Definition 4.1. For any non-negative vector $r \in \mathbb{R}^n$, we put

$$D(r) := \{z \in D : B(z,r) \subset D\}, \tag{4.1}$$

where $B(z,r) := \{\xi \in \mathbb{R}^n : |\xi - z| \le r\}$.

Definition 4.2. Given a closed interval $J \subseteq \mathbb{R}$, define the vector

$$\delta_{J,D}(f) := \max_{(t,z)\in J\times D} f(t,z) - \min_{(t,z)\in J\times D} f(t,z). \tag{4.2}$$

Finally, let the positive number ϱ_* be determined by the equality

$$\varrho_*^{-1} = \inf\left\{q > 0 : q^{-1} = \int_0^{\frac{1}{2}} \exp\big(\tau(\tau-1)q\big)d\tau\right\}. \tag{4.3}$$

Note that the constant ϱ_* had already appeared in the literature, and it can be defined in various ways (see, e. g., [11, 16] for more details). One can show that

$$\varrho_* \approx 0.2927. \tag{4.4}$$

5. p-periodic successive approximations: the original version

The original, unmodified, periodic successive approximations scheme for the p-periodic problem (3.7), (3.8) is constructed as follows. With problem (3.7), (3.8), we associate the sequence of functions $u_m(\cdot,z)$, $m \ge 0$, defined according to the rule

$$u_0(t,z) := z,$$

$$u_m(t,z) := z + \int_{t_0}^t g(s,u_{m-1}(s,z))ds - \frac{t-t_0}{p}\int_{t_0}^{t_0+p} g(s,u_{m-1}(s,z))ds \tag{5.1}$$

for $t \in [t_0, t_0+p]$ and $m = 1,2,\ldots$, where the vector $z = \mathrm{col}(z_1, z_2, \ldots, z_n)$ is regarded as a parameter whose value is to be determined later. The following proposition is a consequence of [8, Theorem 1].

Proposition 5.1 ([8]). *Let the function g satisfy the Lipschitz condition of form (3.5) with a matrix K for which the inequality*

$$r(K) < \frac{1}{p\varrho_*} \tag{5.2}$$

holds, and moreover

$$D(\tfrac{p}{4}\delta_{[t_0,t_0+p],D}(g)) \neq \varnothing. \tag{5.3}$$

Then, for any fixed $z \in D(\frac{p}{4}\delta_{[t_0,t_0+p],D}(g))$, the following assertions are true:

(1) *Sequence (5.1) converges to a limit function*

$$u_\infty(t,z) = \lim_{m\to\infty} u_m(t,z) \tag{5.4}$$

uniformly in $t \in [t_0, t_0+p]$.

(2) *The limit function (5.4) satisfies p-periodic boundary conditions*

$$u_\infty(t_0,z) = u_\infty(t_0+p,z).$$

(3) *The function $u_\infty(\cdot,z)$ is the unique solution of Cauchy problem*

$$u'(t) = g(t,u(t)) - p^{-1}\Delta(z), \qquad t \in [t_0,t_0+p], \tag{5.5}$$

$$u(t_0) = z, \tag{5.6}$$

where

$$\Delta(z) := \int_{t_0}^{t_0+p} g(\tau,u_\infty(\tau,z))\,d\tau. \tag{5.7}$$

(4) *Given an arbitrarily small positive ε, one can choose a number $m_\varepsilon \geq 1$ such that the estimate*

$$|u_m(t,z) - u_\infty(t,z)| \leq \frac{\alpha_{m_\varepsilon}(t)}{2}\, K^{m_\varepsilon-1}(pc_\varepsilon K)^{m-m_\varepsilon+1}(1_n - pc_\varepsilon K)^{-1}\,\delta_{[t_0,t_0+p],D}(g)$$

holds for all $t \in [t_0,t_0+p]$ and $m \geq m_\varepsilon$, where

$$c_\varepsilon := \varrho_* + \varepsilon. \tag{5.8}$$

Remark 5.2. According to Definition 4.1, condition (5.3) means the non-emptiness of the $\frac{p}{4}\delta_{[t_0,t_0+p],D}(g)$-core of the set D, where the vector $\delta_{[t_0,t_0+p],D}(g)$ is given by formula (4.2). This correlates with the natural idea that the domain where the Lipschitz condition is assumed should be wide enough.

The proof of Proposition 5.1 is based on Lemma 5.3 formulated below, which provides an estimate for the sequence of functions α_m, $m \geq 0$, given by the formula

$$\alpha_m(t) := \left(1 - \frac{t-t_0}{p}\right)\int_{t_0}^{t} \alpha_{m-1}(s)ds + \frac{t-t_0}{p}\int_{t}^{t_0+p} \alpha_{m-1}(s)ds \tag{5.9}$$

where $\alpha_0(t) := 1$, $t \in [t_0,t_0+p]$, $m \geq 1$.

Lemma 5.3. *Given an arbitrary $\varepsilon \in (0, +\infty)$, one can specify an integer $m_\varepsilon \geq 1$ such that*

$$\alpha_{m+1}(t) \leq p\,(\varrho_* + \varepsilon)\,\alpha_m(t) \tag{5.10}$$

for all $t \in [t_0, t_0 + p]$ and $m \geq m_\varepsilon$, where $\varrho_* \approx 0.2927$ is given by (4.3).

Proof. The assertion is easily derived from [8, Lemma 3] by transforming the time interval to $[t_0, t_0 + p]$. □

Remark 5.4. It follows from [19, Lemma 4] that if $\varepsilon \geq \varepsilon_0$, where

$$\varepsilon_0 := \frac{3}{10} - \varrho_* \approx 0.00727, \tag{5.11}$$

then $m_\varepsilon = 1$ in Lemma 5.3.

Proposition 5.1 suggests that there is a relation between p-periodic solutions of the given differential equation and those of the perturbed equation (5.5), which are, at the same time, solutions of the initial value problem (5.5), (5.6). The following statement shows that, by choosing the value of z appropriately, one can use function (5.4) to construct a solution of the original periodic problem (3.3), (3.4).

Proposition 5.5. *Let the assumptions of Proposition 5.1 hold. Then:*

(1) *Given a $z \in D(\frac{p}{4}\delta_{[t_0,t_0+p],D}(g))$, the function $u_\infty(\cdot, z)$ is a solution of the p-periodic boundary value problem (3.7), (3.8) if and only if z is a root of the equation*

$$\Delta(z) = 0. \tag{5.12}$$

(2) *For any solution $u(\cdot)$ of problem (3.7), (3.8) with $u(t_0) \in D(\frac{p}{4}\delta_{[t_0,t_0+p],D}(g))$, there exists a z_0 such that $u(\cdot) = u_\infty(\cdot, z_0)$.*

The important assertion (2) means that equation (5.12), usually referred to as a *determining equation*, allows one to track *all* the solutions of the periodic boundary value problem (3.3), (3.4). The original infinite-dimensional problem is thus reduced to a system of n numerical equations.

Proof of Proposition 5.5. The statement is a consequence of [8, Corollary 1] and [15, Lemma 3.5]. □

The method thus consists of two parts, namely, the *analytic* part, when the integral equation (3.6) is dealt with by using the method of successive approximations (5.1), and the *numerical* one which consists in finding a value of the unknown parameter from equation (5.12).

We also note that it is possible to prove the existence of a solution based on the properties of a certain iteration $u_m(\cdot, z)$, constructed explicitly for a certain fixed m, by studying the *approximate determining system*

$$\Delta_m(z) = 0,$$

where $\Delta_m : D \to \mathbb{R}^n$ is defined by the formula

$$\Delta_m(z) := \int_{t_0}^{t_0+p} g(s, u_m(s,z))ds, \qquad z \in D.$$

This topic is discussed in detail, in particular, in [11], whereas a theorem of the kind specified, which corresponds to the scheme developed here, is proved in Section 10 below.

In view of (4.4), assumption (5.2), which is essential for the proof of the uniform convergence of sequence (5.1), can be rewritten in the form

$$p\,r(K) < 3.4161\ldots. \tag{5.13}$$

Inequality (5.13) can be treated either as a kind of upper bound for the Lipschitz matrix or as a smallness assumption on the period p, the latter interpretation presenting the scheme as particularly appropriate for the study of high-frequency oscillations.

Without assumption (5.13), Lemma 5.3 does not guarantee the convergence of sequence (5.1) directly. Nevertheless, it turns out that this limitation can be overcome. We are going to show that, by using a suitable parametrisation and modifying the scheme appropriately, one can always weaken the smallness condition (5.2) so that the constant on its right-hand side is *doubled*:

$$r(K) < \frac{2}{p\varrho_*}. \tag{5.14}$$

6. Interval halving and parametrisation

Our aim is to show that the approach described above can also be used in the cases where the smallness condition (5.2), which guarantees the convergence, is *not* satisfied. For this purpose, a natural trick based on the interval halving can be used.

The idea is surprisingly simple: along with the periodic boundary value problem (3.3), (3.4), consider two auxiliary problems,

$$x'(t) = f\,(t, x(t))\,, \qquad t \in [0, \tfrac{1}{2}T], \tag{6.1}$$

$$x\Big(\frac{T}{2}\Big) - x\,(0) = \lambda \tag{6.2}$$

and

$$y'(t) = f\,(t, y(t))\,, \qquad t \in [\tfrac{1}{2}T, T], \tag{6.3}$$

$$y(T) - y\Big(\frac{T}{2}\Big) = -\lambda\,, \tag{6.4}$$

where $\lambda = \operatorname{col}(\lambda_1, \ldots, \lambda_n)$ is a parameter. Our further reasoning related to problem (3.3), (3.4) uses the following simple observation.

Proposition 6.1. *Let $x : [0, \frac{1}{2}T] \to \mathbb{R}^n$ and $y : [\frac{1}{2}T, T] \to \mathbb{R}^n$ be solutions of problems (6.1), (6.2) and (6.3), (6.4), respectively, with a certain value of $\lambda \in \mathbb{R}^n$. Then the function*

$$u(t) := \chi_T(t)x(t) + (1 - \chi_T(t))\left(y(t) - y\left(\frac{T}{2}\right) + x\left(\frac{T}{2}\right)\right), \qquad t \in [0, T], \quad (6.5)$$

where $\chi_T(t) = 1$ for $0 \le t < \frac{1}{2}T$ and $\chi_T(t) = 0$ for $\frac{1}{2}T \le t \le T$, is a solution of the periodic problem boundary value problem (3.4) for the equation

$$u'(t) = f\left(t, u(t) + (1 - \chi_T(t))\left(y\left(\frac{T}{2}\right) - x\left(\frac{T}{2}\right)\right)\right), \qquad t \in [0, T]. \quad (6.6)$$

Conversely, if a certain function $u : [0, T] \to \mathbb{R}^n$ is a solution of problem (3.3), (3.4), then its restrictions $x := u|_{[0,\frac{1}{2}T]}$ and $y := u|_{[\frac{1}{2}T,T]}$ to the corresponding intervals satisfy, respectively, problems (6.1), (6.2) and (6.3), (6.4).

Proof. The idea is to rewrite the periodic boundary condition (3.4) in the form

$$u(0) - u\left(\frac{T}{2}\right) + u\left(\frac{T}{2}\right) - u(T) = 0. \quad (6.7)$$

Then it is obvious from (6.2) and (6.4) that the function u given by (6.5) satisfies (6.7) and, hence, condition (3.4) holds. Moreover, it is clear from (6.5) that $u(\frac{1}{2}T) = x(\frac{1}{2}T)$. Thus, u is continuous at the point $\frac{1}{2}T$ and, hence, $u \in C([0, T], \mathbb{R}^n)$. By direct computation, one easily verifies that function (6.5) satisfies equation (6.6) and, therefore, is a solution of problem (3.4), (6.6).

On the other hand, if a certain function $u : [0, T] \to \mathbb{R}^n$ is a solution of problem (3.3), (3.4), then its restriction $x := u|_{[0,\frac{1}{2}T]}$ to the interval $[0, \frac{1}{2}T]$ satisfies (6.2) with

$$\lambda := u\left(\frac{T}{2}\right) - u(0). \quad (6.8)$$

Put now $y := u|_{[\frac{1}{2}T,T]}$. Then, due to (3.4) and (6.8), we have

$$\begin{aligned} y(T) - y\left(\frac{T}{2}\right) &= u(0) - u\left(\frac{T}{2}\right) \\ &= -\lambda \end{aligned}$$

and, therefore, y has property (6.4) with λ given by (6.8). □

Proposition 6.1 suggests one the idea to treat the T-periodic problem (3.3), (3.4) as a kind of join of two independent two-point problems. More precisely, one may try to solve independently the auxiliary problems (6.1), (6.2) and (6.3), (6.4), considering λ as an unknown parameter, and then "glue" their solutions together by choosing the value of λ so that (6.9) holds. A rigorous formulation is contained in the following

Proposition 6.2. *Assume that* $x : [0, \frac{1}{2}T] \to \mathbb{R}^n$ *and* $y : [\frac{1}{2}T, T] \to \mathbb{R}^n$ *are solutions of problems* (6.1), (6.2) *and* (6.3), (6.4), *respectively, for a certain value of* $\lambda \in \mathbb{R}^n$. *Then the function* $u : [0, T] \to \mathbb{R}^n$ *given by formula* (6.5) *is a solution of problem* (3.3), (3.4) *if and only if the equality*

$$x\left(\frac{T}{2}\right) = y\left(\frac{T}{2}\right) \tag{6.9}$$

holds.

Conversely, if a certain $u : [0, T] \to \mathbb{R}^n$ *is a solution of problem* (3.3), (3.4), *then the functions* $x := u|_{[0,\frac{1}{2}T]}$ *and* $y := u|_{[\frac{1}{2}T,T]}$ *satisfy, respectively, problems* (6.1), (6.2) *and* (6.3), (6.4).

Proof. It is sufficient to apply Proposition 6.1 and take into account the fact that, for $x : [0, \frac{1}{2}T] \to \mathbb{R}^n$ and $y : [\frac{1}{2}T, T] \to \mathbb{R}^n$ satisfying (6.9), equality (6.6) coincides with (3.3).

The second part of the assertion is a repetition of that of Proposition 6.1. Note that condition (6.9) is satisfied automatically in that case. □

7. Successive approximations on smaller intervals

Let us now turn to the construction of the successive approximation scheme related to the T-periodic problem (3.3), (3.4). As Proposition 6.2 suggests, we shall first deal with the auxiliary problems (6.1), (6.2) and (6.3), (6.4), for which purpose appropriate iteration processes will be introduced.

More precisely, for any $\xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^n$, we set

$$x_0(t, \xi, \lambda) := \xi + \frac{2t}{T}\lambda, \qquad t \in [0, \tfrac{1}{2}T], \tag{7.1}$$

and define the recurrence sequence of functions $x_m : [0, \frac{1}{2}T] \times \mathbb{R}^{2n} \to \mathbb{R}^n$, $m = 0, 1, \ldots$, by putting

$$\begin{aligned} x_m(t, \xi, \lambda) := &\int_0^t f\left(s, x_{m-1}(s, \xi, \lambda)\right) ds - \frac{2t}{T} \int_0^{\frac{T}{2}} f\left(s, x_{m-1}(s, \xi, \lambda)\right) ds \\ &+ \xi + \frac{2t}{T}\lambda, \qquad t \in [0, \tfrac{1}{2}T], \end{aligned} \tag{7.2}$$

for all $m = 1, 2, \ldots$.

Introduce the functions $\bar{\alpha}_m : [0, \frac{1}{2}T] \to [0, +\infty)$ and $\bar{\bar{\alpha}}_m : [\frac{1}{2}T, T] \to [0, +\infty)$, $m \geq 0$, by putting $\bar{\alpha}_0 \equiv 1$, $\bar{\bar{\alpha}}_0 \equiv 1$,

$$\bar{\alpha}_{m+1}(t) := \left(1 - \frac{2t}{T}\right) \int_0^t \bar{\alpha}_m(s) ds + \frac{2t}{T} \int_t^{\frac{1}{2}T} \bar{\alpha}_m(s) ds \tag{7.3}$$

for $t \in [0, \frac{1}{2}T]$, and

$$\bar{\bar{\alpha}}_{m+1}(t) := 2\left(1-\frac{t}{T}\right)\int_{\frac{1}{2}T}^{t} \bar{\bar{\alpha}}_m(s)ds + \left(\frac{2t}{T}-1\right)\int_t^T \bar{\bar{\alpha}}_m(s)ds \tag{7.4}$$

for $t \in [\frac{1}{2}T, T]$. In particular, we have

$$\bar{\alpha}_1(t) = 2t\left(1-\frac{2t}{T}\right), \qquad t \in [0, \tfrac{1}{2}T], \tag{7.5}$$

and

$$\bar{\bar{\alpha}}_1(t) = 2\left(1-\frac{t}{T}\right)(2t-T), \qquad t \in [\tfrac{1}{2}T, T]. \tag{7.6}$$

Functions (7.3) and (7.4) are involved in the estimates of convergence given in the sequel. In the case of the iteration process (7.2), we have the following

Theorem 7.1. *Let the vector-function $f : [0,T]\times D \to \mathbb{R}^n$ satisfy the Lipschitz condition (3.5) on the set D with a matrix K such that*

$$r(K) < \frac{2}{T\varrho_*}, \tag{7.7}$$

where the constant ϱ_* is determined by equality (4.3). Moreover, assume that there exists a set $\mathscr{D} \subset D$ such that

$$D(\tfrac{T}{8}\delta_{[0,\frac{1}{2}T],D}(f) + \operatorname{diam}\mathscr{D}) \neq \varnothing. \tag{7.8}$$

Then, for all $\lambda \in \mathscr{D}-\mathscr{D}$ and $\xi \in D(\frac{T}{8}\delta_{[0,\frac{1}{2}T],D}(f) + \operatorname{diam}\mathscr{D})$:

(1) *For any $m \geq 1$, $x_m(\cdot,\xi,\lambda)$ satisfies the boundary conditions*

$$x_m(\tfrac{1}{2}T,\xi,\lambda) - x_m(0,\xi,\lambda) = \lambda. \tag{7.9}$$

(2) *The uniform, in $t \in [0, \frac{1}{2}T]$, limit*

$$\lim_{m\to\infty} x_m(t,\xi,\lambda) =: x_\infty(t,\xi,\lambda) \tag{7.10}$$

exists, and moreover,

$$x_\infty(\tfrac{1}{2}T,\xi,\lambda) - x_\infty(0,\xi,\lambda) = \lambda. \tag{7.11}$$

(3) *The function $x_\infty(\cdot,\xi,\lambda)$ is the unique solution of the Cauchy problem*

$$x'(t) = f(t,x(t)) + 2T^{-1}\Xi(\xi,\lambda), \tag{7.12}$$

$$x(0) = \xi, \tag{7.13}$$

where

$$\Xi(\xi,\lambda) := \lambda - \int_0^{\frac{T}{2}} f(\tau, x_\infty(\tau,\xi,\lambda))d\tau. \tag{7.14}$$

(4) *For an arbitrarily small positive ε, one can find a number $m_\varepsilon \geq 1$ such that the function*

$$\xi_m(\cdot,\xi,\lambda) := x_m(\cdot,\xi,\lambda) - x_\infty(\cdot,\xi,\lambda)$$

satisfies the estimate

$$|\xi_m(t,\xi,\lambda)| \leq \frac{1}{2}\bar{\alpha}_{m_\varepsilon}(t)\, K^{m_\varepsilon-1}(\tfrac{1}{2}Tc_\varepsilon K)^{m-m_\varepsilon+1}\left(1_n - \tfrac{1}{2}Tc_\varepsilon K\right)^{-1}\delta_{[0,\frac{1}{2}T],D}(f) \tag{7.15}$$

for all $t \in [0, \frac{1}{2}T]$ and $m \geq m_\varepsilon$, where c_ε is given by (5.8).

Recall that the non-negative vector diam $\mathscr{D}$ appearing in condition (7.8) is defined according to formula (2.1). By $\mathscr{D}-\mathscr{D}$, we mean the set of all the differences $z_1 - z_2$ with z_1 and z_2 both belonging to $\mathscr{D}$. The proof of Theorem 7.1 is postponed till Section 8.4.

Remark 7.2. The error estimate (7.15) may look inconvenient because it is guaranteed starting from a sufficiently large iteration number, m_ε, depending on the value of ε which can be arbitrarily small. It is, however, quite transparent when the required constant is not "too close" to ϱ_* (i. e., if ε is not "too small").

More precisely, as is seen from the proof given in Section 8, m_ε in Theorem 7.1 is, in fact, the very number appearing in Lemma 5.3. In view of Remark 5.4, $m_\varepsilon = 1$ for $\varepsilon \geq \varepsilon_0$, where

$$\varepsilon_0 \approx 0.00727$$

is given by formula (5.11). Consequently, inequality (7.15) with $\varepsilon \geq \varepsilon_0$ holds for an *arbitrary* value of $m \geq 1$.

By analogy to Theorem 7.1, let us now consider the parametrised boundary value problem (6.3), (6.4) on the interval $[\frac{1}{2}T, T]$. Put

$$y_0(t,\eta,\lambda) := \eta - \left(\frac{2t}{T} - 1\right)\lambda, \qquad t \in [\tfrac{1}{2}T, T], \tag{7.16}$$

and introduce the functions $y_m : [\frac{1}{2}T, T] \times \mathbb{R}^{2n} \to \mathbb{R}^n$, $m \geq 1$, according to the formula

$$\begin{aligned} y_m(t,\eta,\lambda) := {} & \int_{\frac{T}{2}}^{t} f\left(s, y_{m-1}(s,\eta,\lambda)\right) ds - \left(\frac{2t}{T} - 1\right)\int_{\frac{T}{2}}^{T} f\left(s, y_{m-1}(s,\eta,\lambda)\right) ds \\ & + \eta - \left(\frac{2t}{T} - 1\right)\lambda, \qquad t \in [\tfrac{1}{2}T, T], \end{aligned} \tag{7.17}$$

for all η and λ from $\mathbb{R}^n$.

Let $\mathscr{D}$ be the set appearing in Theorem 7.1. Then the following statement on sequence (7.17) holds.

Theorem 7.3. *Assume that the vector-function f satisfies conditions (3.5), (7.7), and, moreover,*

$$D(\tfrac{T}{8}\delta_{[\frac{1}{2}T,T],D}(f)+\operatorname{diam}\mathscr{D})\neq\varnothing. \tag{7.18}$$

Then, for all fixed $\lambda\in\mathscr{D}-\mathscr{D}$ and $\eta\in D(\frac{T}{8}\delta_{[\frac{1}{2}T,T],D}(f)+\operatorname{diam}\mathscr{D})$:

(1) *For any $m\geq 1$, $y_m(\cdot,\eta,\lambda)$ satisfies the boundary conditions*

$$y_m(T,\eta,\lambda)-y_m(\tfrac{1}{2}T,\eta,\lambda)=-\lambda. \tag{7.19}$$

(2) *The uniform, in $t\in[\frac{1}{2}T,T]$, limit*

$$\lim_{m\to\infty}y_m(t,\eta,\lambda)=:y_\infty(t,\eta,\lambda) \tag{7.20}$$

exists, and moreover,

$$y_\infty(T,\eta,\lambda)-y_\infty(\tfrac{1}{2}T,\eta,\lambda)=-\lambda. \tag{7.21}$$

(3) *The function $y_\infty(\cdot,\eta,\lambda)$ is the unique solution of the Cauchy problem*

$$y'(t)=f(t,y(t))+2T^{-1}\mathrm{H}(\eta,\lambda), \tag{7.22}$$

$$y(\tfrac{1}{2}T)=\eta, \tag{7.23}$$

where

$$\mathrm{H}(\eta,\lambda):=-\lambda-\int_{\frac{T}{2}}^{T}f(\tau,y_\infty(\tau,\eta,\lambda))d\tau. \tag{7.24}$$

(4) *For an arbitrarily small positive ε, one can find a number $m_\varepsilon\geq 1$ such that the function*

$$\psi_m(\cdot,\eta,\lambda):=y_m(\cdot,\eta,\lambda)-y_\infty(t,\eta,\lambda)$$

satisfies the estimate

$$|\psi_m(t,\eta,\lambda)|\leq\frac{1}{2}\bar{\bar{\alpha}}_{m_\varepsilon}(t)\,K^{m_\varepsilon-1}(\tfrac{1}{2}Tc_\varepsilon K)^{m-m_\varepsilon+1}\left(1_n-\tfrac{1}{2}Tc_\varepsilon K\right)^{-1}\delta_{[\frac{1}{2}T,T],D}(f) \tag{7.25}$$

for all $t\in[\frac{1}{2}T,T]$ and $m\geq m_\varepsilon$, where c_ε is given by (5.8).

Remark 7.4. By analogy to Remark 7.2, one can conclude that the validity of estimate (7.25) is ensured for *all* $m\geq 1$ provided that $\varepsilon\geq\varepsilon_0$ with ε_0 given by formula (5.11).

Theorems 7.1 and 7.3 suggest that sequences (7.2) and (7.17) can be used to construct the solutions of auxiliary problems (6.1), (6.2) and (6.3), (6.4), and ultimately of the original problem (3.3), (3.4).

Remark 7.5. As will be seen from the proofs (Section 8), Theorems 7.1 and 7.3 could have been formulated so that the inclusion $\lambda\in\mathscr{D}-\mathscr{D}$ would be replaced by $\lambda\in\Lambda_0$ (Theorem 7.1) and $-\lambda\in\Lambda_1$ (Theorem 7.3), where Λ_0 and Λ_1 are certain sets in

$\mathbb{R}^n$ where the unknown values of λ and $-\lambda$ are expected to belong. Assumptions (7.8) and (7.18) would then be replaced by the conditions

$$D(\tfrac{T}{8}\delta_{[0,\frac{1}{2}T],D}(f)+d(\Lambda_0))\neq\varnothing,$$
$$D(\tfrac{T}{8}\delta_{[\frac{1}{2}T,T],D}(f)+d(\Lambda_1))\neq\varnothing,$$

where

$$d(Q):=\sup\{|z|:z\in Q\} \tag{7.26}$$

for any non-empty $Q\subset\mathbb{R}^n$.

The present choice $\Lambda_0=\mathscr{D}-\mathscr{D}$, $\Lambda_1=\mathscr{D}-\mathscr{D}$, with $\mathscr{D}\subset D$, is rather natural because, as follows from the form of the auxiliary boundary conditions (6.2) and (6.4), the looked-for values of λ are, in fact, always differences of values of the solutions at some of the points 0, $\frac{1}{2}T$, T, and all those values should lie in D.

The last two statements are summarised in the following

Theorem 7.6. *Assume that the vector-function f satisfies conditions* (3.5), (7.7), *and, moreover,*

$$D(\tfrac{T}{8}\max\{\delta_{[0,\frac{1}{2}T],D}(f),\delta_{[\frac{1}{2}T,T],D}(f)\}+\operatorname{diam}\mathscr{D})\neq\varnothing, \tag{7.27}$$

Then, for any $\xi\in D(\frac{T}{8}\delta_{[0,\frac{1}{2}T],D}(f)+\operatorname{diam}\mathscr{D})$, $\eta\in D(\frac{T}{8}\delta_{[\frac{1}{2}T,T],D}(f)+\operatorname{diam}\mathscr{D})$, and $\lambda\in\mathscr{D}-\mathscr{D}$, the assertions of Theorems 7.1 and 7.3 hold.

8. Proof of convergence theorems

The proof of Theorems 7.1 and 7.3, which will be given in Section 8.4, uses the fact that, by modifying the problem suitably, the inhomogeneous terms in the boundary conditions (6.2) and (6.4) can be removed so that techniques similar to Proposition 5.1 can be applied.

8.1. *Making the boundary conditions periodic*

Let $\mathscr{D}$ be the set appearing in the formulation of Theorem 7.1. For any fixed $\lambda\in\mathscr{D}-\mathscr{D}$, consider the boundary value problems

$$\tilde{x}'(t)=f\Big(t,\tilde{x}(t)+\frac{2t}{T}\lambda\Big)-\frac{2}{T}\lambda,\qquad t\in[0,\tfrac{1}{2}T], \tag{8.1}$$
$$\tilde{x}\Big(\frac{T}{2}\Big)-\tilde{x}(0)=0 \tag{8.2}$$

and

$$\tilde{y}'(t)=f\Big(t,\tilde{y}(t)+\Big(1-\frac{2t}{T}\Big)\lambda\Big)+\frac{2}{T}\lambda,\qquad t\in[\tfrac{1}{2}T,T], \tag{8.3}$$
$$\tilde{y}(T)-\tilde{y}\Big(\frac{T}{2}\Big)=0. \tag{8.4}$$

Then, by direct computation, one easily finds that

Lemma 8.1. *The following assertions are true for an arbitrary $\lambda \in \mathbb{R}^n$:*

(1) *A function $\tilde{x} : [0, \frac{1}{2}T] \to \mathbb{R}^n$ satisfies (8.1), (8.2) if and only if the function*

$$x(t) := \tilde{x}(t) + \frac{2t}{T}\lambda, \qquad t \in [0, \tfrac{1}{2}T], \tag{8.5}$$

is a solution of problem (6.1), (6.2).

(2) *A function $\tilde{y} : [\frac{1}{2}T, T] \to \mathbb{R}^n$ satisfies (8.3), (8.4) if and only if the function*

$$y(t) := \tilde{y}(t) + \Big(1 - \frac{2t}{T}\Big)\lambda, \qquad t \in [\tfrac{1}{2}T, T], \tag{8.6}$$

is a solution of problem (6.3), (6.4).

Keeping this equivalence in mind, one can deal with problems (6.1), (6.2) and (6.3), (6.4) by using Proposition 5.1, for which purpose the corresponding iteration process should be constructed. Indeed, for any fixed $\lambda \in \mathbb{R}^n$ and $(\xi, \eta) \in D^2$, let us put

$$\begin{aligned} \tilde{x}_m(t,\xi,\lambda) := \xi &+ \int_0^t f\Big(s, \tilde{x}_{m-1}(s,\xi,\lambda) + \frac{2s}{T}\lambda\Big)ds \\ &- \frac{2t}{T}\int_0^{\frac{T}{2}} f\Big(s, \tilde{x}_{m-1}(s,\xi,\lambda) + \frac{2s}{T}\lambda\Big)ds, \qquad t \in [0, \tfrac{1}{2}T], \end{aligned} \tag{8.7}$$

and

$$\begin{aligned} \tilde{y}_m(t,\eta,\lambda) := \eta &+ \int_{\frac{T}{2}}^t f\Big(s, \tilde{y}_{m-1}(s,\eta,\lambda) + \Big(1 - \frac{2s}{T}\Big)\lambda\Big)ds \\ &- \Big(\frac{2t}{T} - 1\Big)\int_{\frac{T}{2}}^T f\Big(s, \tilde{y}_{m-1}(s,\eta,\lambda) + \Big(1 - \frac{2s}{T}\Big)\lambda\Big)ds, \quad t \in [\tfrac{1}{2}T, T], \end{aligned} \tag{8.8}$$

for all $m = 1, 2, \ldots$, where

$$\tilde{x}_0(t,\xi,\lambda) := \xi, \qquad t \in [0, \tfrac{1}{2}T], \tag{8.9}$$

$$\tilde{y}_0(t,\eta,\lambda) := \eta, \qquad t \in [\tfrac{1}{2}T, T]. \tag{8.10}$$

Similarly to Proposition 5.1, we see that functions (8.7) and (8.8) have the properties

$$\tilde{x}_m(0,\xi,\lambda) = \tilde{x}_m(\tfrac{T}{2},\xi,\lambda),$$
$$\tilde{y}_m(\tfrac{T}{2},\eta,\lambda) = \tilde{y}_m(T,\eta,\lambda)$$

for all ξ, η, and m.

8.2. *Estimates of non-linearities*

For any fixed $\lambda \in \mathscr{D} - \mathscr{D}$, put

$$D_{\lambda,i} := \left\{ z \in \mathbb{R}^n : z + (-1)^i \theta\lambda \in D \text{ for all } \theta \in [0,1] \right\} \tag{8.11}$$

for $i \in \{0,1\}$.

Lemma 8.2. *Let $\lambda \in \mathscr{D} - \mathscr{D}$ be arbitrary. Then:*

(1) $\{\xi + 2tT^{-1}\lambda : t \in [0, \frac{1}{2}T]\} \subset D$ *if and only if* $\xi \in D_{\lambda,0}$.

(2) $\{\eta + (1 - 2tT^{-1})\lambda : t \in [\frac{1}{2}T, T]\} \subset D$ *if and only if* $\eta \in D_{\lambda,1}$.

Proof. Indeed, if $\xi \in D_{\lambda,0}$, then, according to (8.11), $z + \theta\lambda \in D$ for all $\theta \in [0,1]$, and therefore $\xi + 2T^{-1}\lambda t \in D$ for any $t \in [0, \frac{1}{2}T]$. Similarly, given an $\eta \in D_{\lambda,1}$, we have $\eta - \theta\lambda \in D$ for all $\theta \in [0,1]$ and, hence, $\eta + (1 - 2tT^{-1})\lambda \in D$ for $t \in [\frac{1}{2}T, T]$. □

When estimating the integral terms in (8.7) and (8.8), the following result will be used.

Lemma 8.3 ([19])**.** *Let $u : [0,T] \to \mathbb{R}^n$ be continuous. The following estimates are true:*

$$\int_0^t \left(u(s) - \frac{2}{T} \int_0^{\frac{T}{2}} u(\sigma)\, d\sigma \right) ds \le \frac{1}{2} \bar{\alpha}_1(t) \max_{s \in [0,t],\, \sigma \in [t, \frac{1}{2}T]} |u(s) - u(\sigma)| \tag{8.12}$$

for $t \in [0, \frac{1}{2}T]$ and

$$\int_{\frac{T}{2}}^t \left(u(s) - \frac{2}{T} \int_{\frac{T}{2}}^{T} u(\sigma)\, d\sigma \right) ds \le \frac{1}{2} \bar{\bar{\alpha}}_1(t) \max_{s \in [\frac{1}{2}T, t],\, \sigma \in [t, T]} |u(s) - u(\sigma)| \tag{8.13}$$

for $t \in [\frac{1}{2}T, T]$.

Here, as before, the inequalities between vectors and the operation "max" for vector functions are understood in the component-wise sense.

Proof of Lemma 8.3. The assertion is obtained along the lines of proof of [19, Lemma 3] after rescaling the original interval to $[0, \frac{1}{2}T]$ and $[\frac{1}{2}T, T]$ respectively. The main points of the proof being the same, the details are omitted. □

Given arbitrary $i \in \{0,1\}$ and $y \in C([\frac{1}{2}iT, \frac{1}{2}(i+1)T], \mathbb{R}^n)$, put

$$(P_i v)(t) := \int_{\frac{i}{2}T}^t v(s)\, ds - \left(\frac{2t}{T} - i \right) \int_{\frac{i}{2}T}^{\frac{i+1}{2}T} v(s)\, ds \tag{8.14}$$

for all $t \in [\frac{1}{2}iT, \frac{1}{2}(i+1)T]$. It is clear that the linear mapping P_i transforms the space $C([\frac{1}{2}iT, \frac{1}{2}(i+1)T], \mathbb{R}^n)$ to itself.

Lemma 8.4. *The estimate*

$$|(P_0 v)(t)| \le \left(1 + i - \frac{2t}{T}\right) \int_{\frac{i}{2}T}^{t} |v(s)|\, ds + \left(\frac{2t}{T} - i\right) \int_{t}^{\frac{i+1}{2}T} |v(s)|\, ds \tag{8.15}$$

holds for any $i \in \{0,1\}$, $v \in C([\frac{1}{2}iT, \frac{1}{2}(i+1)T], \mathbb{R}^n)$, *and* $t \in [\frac{1}{2}iT, \frac{1}{2}(i+1)T]$.

Proof. Estimate (8.15) is established by direct computation using (8.14). □

Now define the functions $X_\lambda : [0, \frac{1}{2}T] \times D \to \mathbb{R}^n$ and $Y_\lambda : [\frac{1}{2}T, T] \times D \to \mathbb{R}^n$ by setting

$$X_\lambda(t,\xi) := f\left(t, \xi + \frac{2t}{T}\lambda\right), \qquad (t,\xi) \in [0, \tfrac{1}{2}T] \times D, \tag{8.16}$$

and

$$Y_\lambda(t,\eta) := f\left(t, \eta + \left(1 - \frac{2t}{T}\right)\lambda\right), \qquad (t,\eta) \in [\tfrac{1}{2}T, T] \times D. \tag{8.17}$$

Lemma 8.5. *Let* $\lambda \in \mathscr{D} - \mathscr{D}$, *and* $u : [0, \frac{1}{2}T] \to \mathbb{R}^n$, $v : [\frac{1}{2}T, T] \to \mathbb{R}^n$ *be arbitrary functions such that* $\{u(t) : t \in [0, \frac{1}{2}T]\} \subset D_{\lambda,0}$ *and* $\{v(t) : t \in [\frac{1}{2}T, T]\} \subset D_{\lambda,1}$. *Then:*

(1) *For* $t \in [0, \frac{1}{2}T]$,

$$\begin{aligned} |P_0 X_\lambda(\cdot, u(\cdot))|(t) &\le \frac{1}{2}\bar{\alpha}_1(t)\delta_{[0,\frac{1}{2}T],D}(f) \\ &\le \frac{T}{8}\delta_{[0,\frac{1}{2}T],D}(f) \end{aligned} \tag{8.18}$$

(2) *For* $t \in [\frac{1}{2}T, T]$,

$$\begin{aligned} |P_1 Y_\lambda(\cdot, v(\cdot))|(t) &\le \frac{1}{2}\bar{\bar{\alpha}}_1(t)\delta_{[\frac{1}{2}T,T],D}(f) \\ &\le \frac{T}{8}\delta_{[\frac{1}{2}T,T],D}(f). \end{aligned} \tag{8.19}$$

Recall that we use notation (4.2).

Proof. Let us fix a $\lambda \in \mathscr{D} - \mathscr{D}$. Let us prove, e. g., inequality (8.19). Take an $\eta \in D_{\lambda,1}$. Applying inequality (8.13) of Lemma 8.3 with $u = Y_\lambda(\cdot, \eta)$, we obtain the estimate

$$|(P_1 Y_\lambda(\cdot, v(\cdot)))(t)| \le \frac{1}{2}\bar{\bar{\alpha}}_1(t) \max_{s \in [\frac{1}{2}T, t], \sigma \in [t, T]} |Y_\lambda(s, v(s)) - Y_\lambda(\sigma, v(\sigma))| \tag{8.20}$$

valid for $t \in [\frac{1}{2}T, T]$. Since v has values in $D_{\lambda,1}$, Lemma 8.2 ensures that $v(t) + 2tT^{-1}\lambda \in D$ for all $t \in [0, \frac{1}{2}T]$ and, therefore, considering (8.17), we see that the

Lipschitz condition (3.5) assumed for f can be applied in (8.20). Then (8.17) and (8.20) yield the estimate

$$|(P_1 Y_\lambda(\cdot, v(\cdot)))(t)| \le \frac{1}{2}\bar{\bar{\alpha}}_1(t)\delta_{[\frac{1}{2}T,T],D}(f), \qquad t \in [\tfrac{1}{2}T, T]. \tag{8.21}$$

It now remains to observe that the maximal value of the function $\bar{\bar{\alpha}}_1$ given by equality (7.6) is attained at $\frac{3}{4}T$ and

$$\max_{t\in[\frac{1}{2}T,T]} \bar{\bar{\alpha}}_1(t) = \frac{T}{4}, \tag{8.22}$$

which leads us immediately to (8.13).

Inequality (8.18) is proved similarly by estimating the value of P_0 on the function $X_\lambda(\cdot, u(\cdot))$ with $u([0, \frac{1}{2}T]) \subset D_{\lambda,0}$ and using the equality

$$\max_{t\in[0,\frac{1}{2}T]} \bar{\alpha}_1(t) = \frac{T}{4} \tag{8.23}$$

which is obtained directly from formula (7.5) using the fact that $\max_{t\in[0,\frac{1}{2}T]} \bar{\alpha}_1(t) = \bar{\alpha}_1(\frac{1}{2})$. Indeed, similarly to (8.21), one shows that

$$|(P_0 X_\lambda(\cdot, u(\cdot)))(t)| \le \frac{1}{2}\bar{\alpha}_1(t)\delta_{[0,\frac{1}{2}T],D}(f), \qquad t \in [0, \tfrac{1}{2}T]. \tag{8.24}$$

The required estimate (8.18) is then a direct consequence of relations (8.23) and (8.24). Inequality (8.18) is established in a similar way. □

8.3. *Functions $\bar{\alpha}_m$ and $\bar{\bar{\alpha}}_m$, $m = 0, 1, \ldots$*

We need the following estimates for the sequences of functions (7.3) and (7.4).

Lemma 8.6. *Given any $\varepsilon \in (0, +\infty)$, one can specify an integer $m_\varepsilon \ge 1$ such that the estimates*

$$\bar{\alpha}_{m+1}(t) \le \frac{1}{2}T\,(\varrho_* + \varepsilon)\,\bar{\alpha}_m(t), \qquad t \in [0, \tfrac{1}{2}T], \tag{8.25}$$

and

$$\bar{\bar{\alpha}}_{m+1}(t) \le \frac{1}{2}T\,(\varrho_* + \varepsilon)\,\bar{\bar{\alpha}}_m(t), \qquad t \in [\tfrac{1}{2}T, T], \tag{8.26}$$

hold for all $m \ge m_\varepsilon$, where ϱ_* is defined by (4.3).

Proof. Estimate (8.25) (resp., (8.26)) is obtained directly from Lemma 5.3 by putting $t_0 := 0$ (resp., $t_0 := \frac{1}{2}T$) and $p := \frac{1}{2}T$ in (5.9). □

Remark 8.7. It follows immediately from (8.25) and (8.26) that

$$\bar{\alpha}_{m+1}(t) \le \left(\frac{T}{2}\right)^{m-m_\varepsilon+1} (\varrho_* + \varepsilon)^{m-m_\varepsilon+1}\,\bar{\alpha}_{m_\varepsilon}(t), \qquad t \in [0, \tfrac{1}{2}T], \tag{8.27}$$

and

$$\bar{\tilde{\alpha}}_{m+1}(t) \le \left(\frac{T}{2}\right)^{m-m_\varepsilon+1} (\varrho_* + \varepsilon)^{m-m_\varepsilon+1} \bar{\tilde{\alpha}}_{m_\varepsilon}(t), \qquad t \in [\tfrac{1}{2}T, T], \tag{8.28}$$

for all $m \ge m_\varepsilon$, where m_ε is the integer appearing in Lemma 8.6. Inequalities (8.27) and (8.28) improve the corresponding estimate from the work [20].

8.4. *Proofs of Theorems 7.1 and 7.3*

We shall use an approach similar to that described in Proposition 5.1 adopting it appropriately to problems (6.1), (6.2) and (6.3), (6.4).

Proof of Theorem 7.1. Let $\lambda \in \mathscr{D} - \mathscr{D}$ and

$$\xi \in D(\tfrac{T}{8}\delta_{[0,\frac{1}{2}T],D}(f) + \operatorname{diam}\mathscr{D}) \tag{8.29}$$

be fixed. It is not difficult to verify by computation that $\tilde{x}_m(\cdot,\xi,\lambda)$, $m \ge 0$, coincides with sequence (5.1) for $t_0 = 0$, $p = \frac{1}{2}T$, and $g = X_\lambda - 2T^{-1}\lambda$, where the function $X_\lambda : [0, \frac{1}{2}T] \times D \to \mathbb{R}^n$ is defined according to (8.16).

Let us show that

$$\{\tilde{x}_m(t,\xi,\lambda) : t \in [0, \tfrac{1}{2}T]\} \subset D_{\lambda,0} \tag{8.30}$$

for all $m \ge 0$. For $m = 0$, the validity of inclusion (8.30) is guaranteed directly by (8.9) and (8.29). Indeed, for all $\theta \in [0,1]$ and z such that

$$|\xi - z| \le \frac{T}{8}\delta_{[0,\frac{1}{2}T],D}(f),$$

the estimates

$$\begin{aligned} |\xi + \theta\lambda - z| &\le |\xi - z| + |\lambda| \\ &\le |\xi - z| + \operatorname{diam}\mathscr{D} \\ &\le \frac{T}{8}\delta_{[0,\frac{1}{2}T],D}(f) + \operatorname{diam}\mathscr{D} \end{aligned} \tag{8.31}$$

hold. In (8.31), we have used the assumption that λ belongs to $\mathscr{D} - \mathscr{D}$. By virtue of (8.29), inequality (8.31) guarantees that $z + \theta\lambda \in D$, i. e., $\xi \in D_{\lambda,0}$ and (8.30) holds with $m = 0$.

Assume that (8.30) holds for a certain value of m and show that the inclusion

$$\{\tilde{x}_{m+1}(t,\xi,\lambda) : t \in [0, \tfrac{1}{2}T]\} \subset D_{\lambda,0} \tag{8.32}$$

holds as well.

Indeed, considering definition (8.7) and recalling notation (8.14), we conclude that, for all m, the identity

$$\tilde{x}_{m+1}(\cdot,\xi,\lambda) - \xi = P_0 X_\lambda(\cdot, \tilde{x}_m(\cdot,\xi,\lambda)) \tag{8.33}$$

holds. Here, we have used the fact that $P_0 c = 0$ for every $c \in \mathbb{R}^n$. Therefore, taking assumption (8.30) and inequality (8.18) of Lemma 8.5 into account, we obtain the estimates

$$|\tilde{x}_{m+1}(t,\xi,\lambda) - \xi| \le \frac{T}{8}\delta_{[0,\frac{1}{2}T],D}(f)$$

valid for all $t \in [0, \frac{1}{2}T]$.

Therefore,

$$\begin{aligned} |\tilde{x}_{m+1}(t,\xi,\lambda) + \theta\lambda - \xi| &\le \frac{T}{8}\delta_{[0,\frac{1}{2}T],D}(f) + |\lambda| \\ &\le \frac{T}{8}\delta_{[0,\frac{1}{2}T],D}(f) + \operatorname{diam}\mathscr{D} \end{aligned} \tag{8.34}$$

for all $\theta \in [0,1]$ and $t \in [0, \frac{1}{2}T]$. However, in view of notation (4.1), estimate (8.34) implies that

$$\{\tilde{x}_{m+1}(t,\xi,\lambda) + \theta\lambda : \theta \in [0,1],\ t \in [0,\tfrac{1}{2}T]\} \subset D,$$

which means that (8.32) holds. Thus, inclusion (8.30) is true for all $m \ge 0$.

Now, following [8], consider the differences

$$\bar{d}_m(t,\xi,\lambda) := \tilde{x}_{m+1}(t,\xi,\lambda) - \tilde{x}_m(t,\xi,\lambda), \qquad m = 0,1,\dots. \tag{8.35}$$

By virtue of (8.33) and Lemma 8.5,

$$|\bar{d}_0(t,\xi,\lambda)| \le \frac{1}{2}\bar{\alpha}_1(t)\delta_{[0,\frac{1}{2}T],D}(f), \qquad t \in [0,\tfrac{1}{2}T]. \tag{8.36}$$

In view of (8.33),

$$\bar{d}_m(\cdot,\xi,\lambda) = P_0\left(X_\lambda(\cdot,\tilde{x}_m(\cdot,\xi,\lambda)) - X_\lambda(\cdot,\tilde{x}_{m-1}(\cdot,\xi,\lambda))\right) \tag{8.37}$$

for any m. Since (8.30) holds for all m, it follows that, for any $t \in [0, \frac{1}{2}T]$, the function $X_\lambda(t,\cdot)$ satisfies the Lipschitz condition with the same matrix K as $f(t,\cdot)$ does. Therefore, relations (8.37), (8.36), Lemma 8.4, and notation (7.3) yield

$$\begin{aligned} |\bar{d}_1(t,\xi,\lambda)| &\le \left(1 - \frac{2t}{T}\right) K \int_0^t |\bar{d}_0(s,\xi,\lambda)|\,ds + \frac{2t}{T} K \int_t^{\frac{1}{2}T} |\bar{d}_0(s,\xi,\lambda)|\,ds \\ &\le \frac{1}{2}\left(\left(1 - \frac{2t}{T}\right) K \int_0^t \bar{\alpha}_1(s)ds + \frac{2t}{T} K \int_t^{\frac{1}{2}T} \bar{\alpha}_1(s)ds\right)\delta_{[0,\frac{1}{2}T],D}(f) \\ &= \frac{1}{2}\bar{\alpha}_2(t) K \delta_{[0,\frac{1}{2}T],D}(f) \end{aligned} \tag{8.38}$$

for $t \in [0, \frac{1}{2}T]$. On the other hand, considering (8.37) and using the Lipschitz condition, we obtain the inequality

$$|\bar{d}_m(t,\xi,\lambda)| \le \left(1-\frac{2t}{T}\right) K \int_0^t |\bar{d}_{m-1}(s,\xi,\lambda)|\, ds + \frac{2t}{T} K \int_t^{\frac{1}{2}T} |\bar{d}_{m-1}(s,\xi,\lambda)|\, ds, \qquad t \in [0, \tfrac{1}{2}T], \quad (8.39)$$

for $m \ge 1$. Combining (8.36) and (8.39), recalling notation (7.3), and arguing by induction, we arrive to the estimate

$$|\bar{d}_m(t,\xi,\lambda)| \le \frac{1}{2} K^m \bar{\alpha}_{m+1}(t) \delta_{[0,\frac{1}{2}T],D}(f) \quad (8.40)$$

valid for all $t \in [0, \frac{1}{2}T]$. It follows from assumption (7.7) that, for a sufficiently small positive number ε, the inequality

$$r(K) < \frac{2}{T(\varrho_* + \varepsilon)} \quad (8.41)$$

holds. Let us fix such an ε and let $m \ge m_\varepsilon$, where m_ε is the integer appearing in Lemma 8.6. Then (8.27) and (8.40) yield

$$|\bar{d}_{m+1}(t,\xi,\lambda)| \le \frac{T^{m-m_\varepsilon+2}}{2^{m-m_\varepsilon+3}} K^{m+1} (\varrho_* + \varepsilon)^{m-m_\varepsilon+2} \bar{\alpha}_{m_\varepsilon}(t) \delta_{[0,\frac{1}{2}T],D}(f) \quad (8.42)$$

for $t \in [0, \frac{1}{2}T]$. Now note that, due to (8.41), we have

$$\lim_{m\to+\infty} 2^{-m} T^m (\varrho_* + \varepsilon)^m K^m = 0. \quad (8.43)$$

Then, arguing similarly to the proof of [8, Theorem 1], we conclude that (8.42) and (8.43) imply the convergence of the sequence $\tilde{x}_m(\cdot,\xi,\lambda)$, $m \ge 0$. Since (7.2) and (8.7) imply the identity

$$x_m(t,\xi,\lambda) = \tilde{x}_m(t,\xi,\lambda) + \frac{2t}{T}\lambda, \qquad t \in [0, \tfrac{1}{2}T], \quad (8.44)$$

the latter fact is equivalent to the convergence of $x_m(\cdot,\xi,\lambda)$, $m \ge 0$. The rest of the proof almost literally follows the lines of [8]. □

Proof of Theorem 7.3. It follows along the lines of proof of Theorem 7.1. One finds that $\tilde{y}_m(\cdot,\eta,\lambda)$, $m \ge 0$, is nothing but (5.1) with $t_0 = \frac{1}{2}T$, $p = \frac{1}{2}T$, and $g = Y_\lambda + 2T^{-1}\lambda$, where $Y_\lambda : [\frac{1}{2}T, T] \times D \to \mathbb{R}^n$ is given by (8.17). The required assertion is established by analogy, instead of (8.44), using the identity

$$y_m(t,\eta,\lambda) = \tilde{y}_m(t,\eta,\lambda) - \frac{2t}{T}\lambda, \qquad t \in [\tfrac{1}{2}T, T], \quad (8.45)$$

which follows, for any $\eta \in D(\frac{T}{8}\delta_{[\frac{1}{2}T,T],D}(f) + \operatorname{diam}\mathscr{D})$, directly from relations (7.17) and (8.8). □

9. Determining equations

Theorems 7.1 and 7.3 guarantee that, under the conditions assumed, the functions $x_\infty(\cdot,\xi,\lambda):[0,\frac{1}{2}T]\to\mathbb{R}^n$ and $y_\infty(\cdot,\eta,\lambda):[\frac{1}{2}T,T]\to\mathbb{R}^n$ are well-defined for the appropriate values of (ξ,η,λ). Therefore, one can introduce the function $u_\infty(\cdot,\xi,\eta,\lambda):[0,T]\to\mathbb{R}^n$ by putting

$$u_\infty(t,\xi,\eta,\lambda):=\chi_T(t)x_\infty(t,\xi,\lambda)\\ +(1-\chi_T(t))\big(y_\infty(t,\eta,\lambda)-y_\infty(\tfrac{1}{2}T,\eta,\lambda)+x_\infty(\tfrac{1}{2}T,\xi,\lambda)\big) \quad (9.1)$$

for all $t\in[0,T]$ and $\xi\in D(\frac{T}{8}\delta_{[0,\frac{1}{2}T],D}(f)+\operatorname{diam}\mathscr{D})$, $\eta\in D(\frac{T}{8}\delta_{[\frac{1}{2}T,T],D}(f)+\operatorname{diam}\mathscr{D})$, and $\lambda\in\mathscr{D}-\mathscr{D}$.

Theorem 9.1. *Assume that f satisfies the Lipschitz condition (3.5) with a matrix K such that (7.7) holds. Furthermore, let*

$$D(\tfrac{T}{8}\delta_{[0,T],D}(f)+\operatorname{diam}\mathscr{D})\neq\varnothing. \quad (9.2)$$

Then the following assertions hold:

(1) *The function $u_\infty(\cdot,\xi,\eta,\lambda):[0,T]\to\mathbb{R}^n$ defined by (9.1) is a solution of the boundary value problem (3.3), (3.4) if and only if the triplet (ξ,η,λ) satisfies the system of $3n$ equations*

$$\begin{aligned}\Xi(\xi,\lambda)&=0,\\ \mathrm{H}(\eta,\lambda)&=0,\\ x_\infty(\tfrac{1}{2}T,\xi,\lambda)&=\eta.\end{aligned} \quad (9.3)$$

(2) *For every solution $u(\cdot)$ of problem (3.3), (3.4) with $u(0)$ and $u(\frac{1}{2}T)$ lying in $D(\frac{T}{8}\delta_{[0,T],D}(f)+\operatorname{diam}\mathscr{D})$, there exists a triplet (ξ_0,η_0,λ_0) such that $u(\cdot)=u_\infty(\cdot,\xi_0,\eta_0,\lambda_0)$.*

Proof. It follows immediately from (4.1) that $D(\beta_0)\subseteq D(\beta_1)$ whenever $\beta_0\le\beta_1$. Therefore, using notation (4.2) and the obvious inequality

$$\max\{\delta_{[0,\frac{1}{2}T],D}(f),\delta_{[\frac{1}{2}T,T],D}(f)\}\le\delta_{[0,T],D}(f),$$

we find that that assumption (9.2) ensures the validity of relations (7.8) and (8.30) and, therefore, Theorems 7.1 and 7.3 can be applied. It then remains to use Proposition 6.1 and argue by analogy to [18, Theorem 5.2]. □

Equations (9.3) are referred to as *determining equations* [5, 11].

10. A remark on the constructive application of the scheme

Although Theorem 9.1 theoretically solves the question on the construction of a solution of the periodic problem (3.3), (3.4), its application faces with difficulties due to the fact that the explicit form of the functions $\Xi:D\times\mathbb{R}^n\to\mathbb{R}^n$ and $\mathrm{H}:D\times\mathbb{R}^n\to$

$\mathbb{R}^n$ appearing in (9.3) is usually unknown. This complication can be overcome by using the functions

$$\Xi_m(\xi,\lambda) := \lambda - \int_0^{\frac{T}{2}} f(\tau, x_m(\tau,\xi,\lambda))d\tau \tag{10.1}$$

and

$$\mathrm{H}_m(\eta,\lambda) := -\lambda - \int_{\frac{T}{2}}^{T} f(\tau, y_m(\tau,\eta,\lambda))d\tau \tag{10.2}$$

for a fixed m, which will lead one to the so-called *approximate determining equations*. More precisely, similarly to [15], one can show that, under certain natural assumptions, the exact determining system (9.3) can be replaced by its approximate analogue

$$\begin{aligned} \Xi_m(\xi,\lambda) &= 0, \\ \mathrm{H}_m(\eta,\lambda) &= 0, \\ x_m(\tfrac{1}{2}T,\xi,\lambda) &= \eta. \end{aligned} \tag{10.3}$$

It is important to note that, unlike system (9.3), the mth approximate determining system (10.3) contains only terms involving the functions x_m and y_m and, thus, known explicitly. The topic concerning the use of the approximate determining equations in solvability analysis falls out of the scope of this paper and will be developed elsewhere.

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THE FIFTH INTERNATIONAL WORKSHOP – 2012 "CONSTRUCTIVE METHODS FOR NON-LINEAR BOUNDARY VALUE PROBLEMS"

MIKLÓS RONTÓ

Received 2nd September, 2012

In the period from 28th June to 1st July, 2012, the Fifth International Workshop – 2012 *"Constructive methods for non-linear boundary-value problems"* had taken place in Tokaj, Hungary. The workshop was organised by the Institute of Mathematics of the University of Miskolc and Institute of Mathematics of the Academy of Sciences of Czech Republic in cooperation with the Regional Committee of the Hungarian Academy of Sciences in Miskolc and Junior Mathematical Society in Miskolc.

The scope of the conference covered various topics related to the modern theory of boundary value problems in a broad sense, including initial value and periodic problems for functional differential equations, certain singular Sturm–Liouville problems, the method of lower and upper functions, stability of fixed iterative methods, limit cycles, travelling waves in lattices, problems with state-dependent impulses, positivity of Green's operator for problems for functional differential equations, singular

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boundary value problems, semilinear hyperbolic differential and functional differential equations, comparison principles, invariant manifolds of extensions of dynamical systems on a torus, coupled fixed points for multivalued operators, numerical-analytic methods, variational inequalities in elasticity, delayed matrix exponents, Lambert functions, sign-constant and oscillating solutions, properties of differential equations with reflections, discontinuous differential equations with delayed or advanced arguments, and difference equations with p-Laplacians.

The workshop was attended by more than 50 researchers from Brasil, Chile, Israel, Czech Republic, Hungary, Latvia, Roumania, Spain, Slovak Republic, and Ukraine, who held 20 and 30-minute lectures.

The Organising Committee of the workshop consisting of V. Berinde, J. Diblík, P. Drábek, M. Fečkan, P. Körtesi, V. Marinec, M. Perestyuk, I. Rachůnková, A. Rontó, A. Samoilenko, F. Sadyrbaev, S. Staněk, J. Szigeti, M. Tvrdý, and the author of this note as a chairman, gratefully acknowledges the financial support provided by the following funds and institutions:

- Department of Analysis University of Miskolc
- Faculty of Mechanical Engineering and Informatics, University of Miskolc
- Tudomány Támogatásáért Észak-Magyarországon Alapítvány.

This meeting was a continuation of the four previous workshops held in Miskolc in 2000 and 2003, in Sárospatak in 2006, and in Eger in 2009 [1, 2].

We hope that the fruitful tradition of this kind of "compact-style" international meetings focusing at exchange of ideas of researchers working in closely related fields will continue in the future.

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STATISTICAL APPROXIMATION FOR MULTIVARIABLE INTEGRABLE FUNCTIONS

İLKNUR SAKAOĞLU AND MEHMET ÜNVER

Received 16 February, 2011

Abstract. Using the concept of A-statistical convergence we give Korovkin type approximation theorems for a sequence of A-statistically uniformly $bounded$ positive linear operators acting from $L_p[a,b;c,d]$ into itself.

2000 *Mathematics Subject Classification:* 41A36; 47B38

Keywords: A-statistical convergence, Korovkin type theorem, Multivariable integrable function

1. INTRODUCTION

Recall that Korovkin type approximation theory deals with the problem of approximating a function f by a sequence $\{T_n(f,x)\}$ of positive linear operators over a certain space of real valued functions (see, e.g, [1], [10]). In particular, this type of results in the space $L_p[a,b]$ of integrable functions on a compact interval may be found in [2], [3], [6], [11] and for the space $L_p[-1,1;-1,1]$ of the integrable multivariable functions on $[-1,1]\times[-1,1]$ in [12]. Also Gadjiev and Orhan [8] have given a Korovkin type approximation theorem, via statistical convergence, on L_p-spaces. Some further results concerning the statistical approximation in the space of locally integrable functions may be found in [5] and [4]. The aim of this paper is to study statistical Korovkin type results for statistically uniformly bounded sequences of positive linear operators which map the space of multivariable integrable functions into itself.

First of all, we recall some basic definitions and notations used in this paper.
Let $A=(a_{jn})$ be a nonnegative regular matrix. The A-density of $K\subseteq\mathbb{N}$ is given by

$$\delta_A(K):=\lim_j\sum_{n\in K}a_{jn}.$$

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A sequence $x = (x_n)$ is called A-statistically convergent to a number L if for every $\varepsilon > 0$,

$$\delta_A(\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}) = 0. \tag{1.1}$$

It is not difficult to see that 1.1 is equivalent to

$$\lim_{j\to\infty} \sum_{n:|x_n-L|\geq\varepsilon} a_{jn} = 0, \text{ for every } \varepsilon > 0.$$

This limit expression is denoted by $st_A - \lim_n x_n = L$. The case in which $A = C_1$, the Cesàro matrix, reduces to statistical convergence [7].

Let $A = (a_{jn})$ be a nonnegative regular matrix. Then the sequence $x = (x_n)$ is said to be strongly A-summable to L if

$$\lim_j \sum_n a_{jn} |x_n - L| = 0.$$

By $L_p[a,b;c,d]$ we denote the space of all functions f defined on $[a,b]\times[c,d]$ for which

$$\int_c^d \int_a^b |f(x,y)|^p \, dx\,dy < \infty, \; 1 \leq p < \infty.$$

In this case, the L_p norm of a function f in $L_p[a,b;c,d]$, denoted by $\|f\|_p$, is given by

$$\|f\|_p := \left(\int_c^d \int_a^b |f(x,y)|^p \, dx\,dy \right)^{1/p}.$$

If T is a positive linear operator from L_p into L_p then the operator norm $\|T\|_{L_p\to L_p}$ is given by

$$\|T\|_{L_p\to L_p} := \sup_{\|f\|_p=1} \|Tf\|_p$$

2. L_p-APPROXIMATION THEOREMS IN STATISTICAL SENSE

In [12] Zaritskaya has given the following Korovkin type theorem for a uniformly bounded sequence of positive linear operators which map the space $L_p[-1,1;-1,1]$ into itself.

Theorem 1. *Let $\{T_n\}$ be a uniformly bounded sequence of positive linear operators from $L_p[-1,1;-1,1]$ into itself. Then convergence of the sequence $\{T_n f\}$ to f in L_p norm holds for any function $f \in L_p[-1,1;-1,1]$ if and only if*

$$\lim_n \|T_n(f_i;x,y) - f_i(x,y)\|_p = 0, i = 1,2,3,4 \tag{2.1}$$

where $f_1(t,\upsilon) = 1$, $f_2(t,\upsilon) = t$, $f_3(t,\upsilon) = \upsilon$, $f_4(t,\upsilon) = t^2 + \upsilon^2$.

In this section, replacing ordinary "*limit*" operation by "$A-statistical\ limit$" operation, we give an analogues result for Theorem 1.

Theorem 2. *Let $A=(a_{jn})$ be a non-negative regular summability matrix and let $\{T_n\}$ be an $A-statistically\ uniformly\ bounded$ sequence of positive linear operators from $L_p[a,b;c,d]$ into $L_p[a,b;c,d]$, $1\le p<\infty$. Then for any function $f\in L_p[a,b;c,d]$,*

$$st_A-\lim_n \|T_n(f;x,y)-f(x,y)\|_p=0 \tag{2.2}$$

if and only if

$$st_A-\lim_n \|T_n(f_i;x,y)-f_i(x,y)\|_p=0,\ i=1,2,3,4 \tag{2.3}$$

where $f_1(t,\upsilon)=1$, $f_2(t,\upsilon)=t$, $f_3(t,\upsilon)=\upsilon$, $f_4(t,\upsilon)=t^2+\upsilon^2$.

Proof. It is obvious that 2.2 implies 2.3. To show that 2.3 implies 2.2, let $\{T_n\}$ be an $A-statistically\ uniformly\ bounded$ sequence of positive linear operators and $f\in L_p[a,b;c,d]$. Then there exist $M>0$ such that $\delta_A(K_0)=1$ where $K_0:=\left\{n\in\mathbb{N}:\|T_n\|_{L_p\to L_p}\le M\right\}$. Given $\varepsilon>0$, there exist $n_i(\varepsilon)$ and $K_i\subseteq\mathbb{N}$ of density 1 such that

$$\|T_n(f_i;x,y)-f_i(x,y)\|_p<\varepsilon,\ i=1,2,3,4 \tag{2.4}$$

for all $n\in K_i$ and $n>n_i(\varepsilon)$. Then inequality 2.4 holds for all $n\in K:=\bigcap_{i=0}^{4}K_i$ and $n>n_0:=\max\{n_i:i=1,2,3,4\}$.

Since $C[a,b;c,d]$, the set of continuous functions on $[a,b]\times[c,d]$ is dense in $L_p[a,b;c,d]$, for any $\varepsilon>0$ there exists $g\in C[a,b;c,d]$ such that

$$\|f(x,y)-g(x,y)\|_p<\varepsilon.$$

Hence for all $n\in K$ and $n>n_0$ we have

$$\begin{aligned}\|T_n(f;x,y)-f(x,y)\|_p &\le \|T_n(f-g;x,y)\|_p+\|T_n(g;x,y)-g(x,y)\|_p\\ &\quad+\|f(x,y)-g(x,y)\|_p\\ &<\varepsilon(1+M)+\|T_n(g;x,y)-g(x,y)\|_p.\end{aligned} \tag{2.5}$$

By the continuity of g on $[a,b]\times[c,d]$, for any $\varepsilon>0$ there exists $\delta>0$ such that for all (x,y), $(t,\upsilon)\in[a,b]\times[c,d]$ satisfying $(t-x)^2+(\upsilon-y)^2<\delta^2$ we have

$$|g(t,\upsilon)-g(x,y)|<\varepsilon.$$

Note that

$$\begin{aligned}|g(t,\upsilon)-g(x,y)| &\le |g(t,\upsilon)|+|g(x,y)|\\ &<\frac{2H}{\delta^2}\phi(t,\upsilon)\end{aligned}$$

for all (x,y), $(t,\upsilon) \in [a,b]\times[c,d]$ satisfying $(t-x)^2+(\upsilon-y)^2 \geq \delta^2$ where $\phi(t,\upsilon) := (t-x)^2+(\upsilon-y)^2$ and $H = \|g(x,y)\|_{C[a,b;c,d]}$. Then for any (x,y), $(t,\upsilon) \in [a,b]\times[c,d]$ we also have

$$|g(t,\upsilon)-g(x,y)| < \varepsilon + \frac{2H}{\delta^2}\phi(t,\upsilon). \tag{2.6}$$

On the other hand one can get

$$\begin{aligned}\|T_n(g;x,y)-g(x,y)\|_p \leq{}& \|T_n(|g(t,\upsilon)-g(x,y)|;x,y)\|_p \\ &+ H\,\|T_n(f_1;x,y)-f_1\|_p\,.\end{aligned} \tag{2.7}$$

Using the linearity and positivity of the operators T_n and inequality 2.6, we get for any $n \in K$ and $n > n_0$ that

$$\begin{aligned}\|T_n(|g(t,\upsilon)-g(x,y)|;x,y)\|_p &\leq \left\| T_n\left(\varepsilon+\frac{2H}{\delta^2}\phi(t,\upsilon);x,y\right)\right\|_p \\ &\leq \varepsilon\left(\|T_n(f_1;x,y)-f_1\|_p+1\right) \\ &\quad+\frac{2H}{\delta^2}\|T_n(\phi(t,\upsilon);x,y)\|_p \\ &\leq \varepsilon\left(\|T_n(f_1;x,y)-f_1\|_p+1\right) \\ &\quad+\frac{2H}{\delta^2}\left\{\|T_n(f_4;x,y)-f_4\|_p\right. \\ &\quad+\left(\alpha^2+\beta^2\right)\|T_n(f_1;x,y)-f_1\|_p \\ &\quad+2\alpha\,\|T_n(f_2;x,y)-f_2\|_p \\ &\quad\left.+2\beta\,\|T_n(f_3;x,y)-f_3\|_p\right\}\end{aligned} \tag{2.8}$$

where $\alpha = \max\{|a|,|b|\}$ and $\beta = \max\{|c|,|d|\}$. It follows from 2.5, 2.7 and 2.8, for all $n \in K$ and $n > n_0$, that

$$\begin{aligned}\|T_n(f;x,y)-f(x,y)\|_p \leq{}& \varepsilon(2+M) \\ &+\left(H+\varepsilon+\frac{2H}{\delta^2}\left(\alpha^2+\beta^2\right)\right)\|T_n(f_1;x,y)-f_1\|_p \\ &+\frac{4H\alpha}{\delta^2}\|T_n(f_2;x,y)-f_2\|_p \\ &+\frac{4H\beta}{\delta^2}\|T_n(f_3;x,y)-f_3\|_p \\ &+\frac{2H}{\delta^2}\|T_n(f_4;x,y)-f_4\|_p\,.\end{aligned} \tag{2.9}$$

Using 2.3, inequality 2.9 can be made small enough for all $n \in K$ and $n > n_0$. Hence, we have

$$st_A - \lim_n \|T_n(f;x,y)-f(x,y)\|_p = 0$$

which concludes the proof. □

Since any bounded $A-statistically$ convergent sequence is strongly $A-summable$ [9], the following result holds immediately.

Corollary 1. *Let $A=(a_{jn})$ be a nonnegative regular summability matrix and let $\{T_n\}$ be a uniformly bounded sequence of positive linear operators from $L_p[a,b;c,d]$ into $L_p[a,b;c,d]$ satisfies*

$$\lim_n \|T_n(f_1;x,y)-f_1(x,y)\|_p = 0$$

and

$$st_A-\lim_n \|T_n(f_i;x,y)-f_i(x,y)\|_p = 0,\ i=2,3,4.$$

Then for any function $f\in L_p[a,b;c,d]$

$$\lim_{j\to\infty}\sum_{n=1}^{\infty} a_{jn}\|T_n(f;x,y)-f(x,y)\|_p = 0.$$

Now using the same methods as in the proof of Theorem 2, one can get the following result easily.

Theorem 3. *Let $A=(a_{jn})$ be a non-negative regular summability matrix and let $\{T_n\}$ be an $A-statistically\ uniformly\ bounded$ sequence of positive linear operators from $L_p[J]$ into $L_p[J]$, $1\le p<\infty$. Then for any function $f\in L_p[J]$,*

$$st_A-\lim_n \|T_n f-f\|_p = 0 \tag{2.10}$$

if and only if

$$st_A-\lim_n \|T_n f_i-f_i\|_p = 0,\ i=1,2,...,m+2 \tag{2.11}$$

where $f_1(t_1,...,t_m)=1$, $f_i(t_1,...,t_m)=t_i$, $(i=2,3,...,m+1)$, $f_{m+2}(t_1,...,t_m)=\sum_{k=1}^{m} t_k^2$ *and* $J:=J_1\times J_2...\times J_m$, $J_i=[a_i,b_i]$, $i=1,2,3,...,m$.

In this theorem if we choose;
i) $m=1$ and $A=I$, the identity matrix, we get Dzyadyk's result [6].
ii) $m=1$ and $A=C_1$, the Cesàro matrix, we get Theorem 7 in [8].
iii) $m=2$ and $A=I$, we get Theorem 1.
iv) $m=2$, we get Theorem 2.

Remark 1. Let $A=(a_{jn})$ be a non-negative regular summability matrix for which $\lim_j \max_n a_{jn}=0$. Then it is well known that $A-statistical$ convergence is stronger than ordinary convergence [9]. We can choose a non-negative $A-statistically$ null but non-convergent sequence (μ_n). Let the operators U_n on $L_p[-1,1;-1,1]$ be defined by

$$U_n(f;x,y)=(1+\mu_n)T_n(f;x,y)$$

for all $f \in L_p[-1,1;-1,1]$ where

$$T_n(f;x,y) = \frac{1}{1+2^{-n}} \begin{cases} f(x,y), & 2^{-n} \le |x| \le 1 \text{ or } 2^{-n} \le |y| \le 1 \\ \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} f(x,y)dxdy, & |x| < 2^{-n} \text{ and } |y| < 2^{-n} \end{cases}.$$

To show that $\{U_n\}$ satisfies Theorem 2 but it does not satisfy Theorem 1, we will show that $\{T_n\}$ is uniformly bounded sequence satisfying condition (2.1). A simple calculation shows that for all $n \in \mathbb{N}$, $\|T_n f\|_p < 8^{1/p} \|f\|_p$ and $\|T_n\|_{L_p \to L_p} < 8^{1/p}$. Hence $\{T_n\}$ is a *uniformly bounded* sequence of positive linear operators from $L_p[-1,1;-1,1]$ into $L_p[-1,1;-1,1]$, $1 \le p < \infty$. Also it is easy to verify that

$$\|T_n(f_1;x,y) - f_1(x,y)\|_p = 4^{1/p}\left(1 - \frac{1}{1+2^{-n}}\right),$$

$$\|T_n(f_2;x,y) - f_2(x,y)\|_p = \|T_n(f_3;x,y) - f_3(x,y)\|_p$$
$$= \left\{ \frac{4}{p+1}\left[\left(1 - \frac{1}{1+2^{-n}}\right)^p \left(1-2^{-n}\right) + 2^{-n(p+2)}\right]\right\}^{1/p}$$

and

$$\|T_n(f_4;x,y) - f_4(x,y)\|_p < \left[\left(1 - \frac{1}{1+2^{-n}}\right) 2^{p+3} + 2^{p-n}16\right]^{1/p}.$$

Hence we necessarily have

$$\lim_n \|T_n(f_i;x,y) - f_i(x,y)\|_p = 0,\ i = 1,2,3,4.$$

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A SHARP THRESHOLD FOR RAINBOW CONNECTION IN SMALL-WORLD NETWORKS

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Abstract. An edge-colored graph G is rainbow connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. We prove that $p = \sqrt{\ln n/n}$ is a sharp threshold function for the property $rc(S(n,p,H)) \leq 2$ in the small-world networks. As by-products, our extension of the concept of independence in graph theory and generalized small-world network models are of independent interest.

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Keywords: rainbow connection, edge coloring, small world, networks

1. INTRODUCTION

We utilize the terminology and notation of [19] in this letter. An interesting connectivity concept of a graph was recently introduced in [3] and has attracted attention of some researchers. An edge-colored graph G is referred to as rainbow connected if any two vertices are connected by a path whose edges have distinct colors. A rainbow connected graph must be connected, and conversely, any connected graph has a trivial edge coloring that makes it rainbow connected. The rainbow connection of a connected graph G, denoted $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected.

An easy observation is that if G has n vertices then $rc(G) \leq n-1$, since one may color the edges of a given spanning tree of G with distinct colors, and color the remaining edges with one of the already used colors. It is also known that $rc(G) = 1$ if and only if G is a complete graph, and that $rc(G) = n-1$ if and only if G is a tree. Note that $rc(G) \geq diam(G)$, where $diam(G)$ denotes the diameter of G. The behavior of $rc(G)$ with respect to the minimum degree $\delta(G)$ has been dealt with in the work [2, 13], of which a primary result is $rc(G) \leq 20n/\delta(G)$. Related concepts such as rainbow path [6], rainbow tree [5] and rainbow k-connectivity [4] have also been investigated recently.

A natural and intriguing direction to explore is the random graph scenarios [12,17]. Let $G(n,p)$ be the classical random graph with n vertices and edge probability p. For

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a graph property $\mathcal{A}$, we say that $G(n,p)$ satisfies $\mathcal{A}$ almost surely if the probability that $G(n,p)$ satisfies $\mathcal{A}$ tends to 1 as n tends to infinity. A function $f(n)$ is called a sharp threshold function for the property $\mathcal{A}$ if there are two positive constants C and c such that $G(n,p)$ satisfies $\mathcal{A}$ almost surely for $p \geq Cf(n)$ and $G(n,p)$ almost surely does not satisfy $\mathcal{A}$ for $p \leq cf(n)$. A remarkable feature of random graphs is that all monotone graph properties have sharp thresholds (see e.g. [1, 10, 11]).

The parameter $rc(G)$ is monotone non-increasing in the sense that if we add an edge to G we cannot increase its rainbow connection. The authors of [2] show that $p = \sqrt{\ln n/n}$ is a sharp threshold function for the property $rc(G(n,p)) \leq 2$. In this note, we propose a generalized small-world network model and explore the threshold of rainbow connection of it. The small-world network is a model with two important characteristics: the clustering effect and the small-world phenomenon, which was originally introduced by Watts and Strogatz [18] as a model of real world complex networks. It has since been the subject of considerable research interest within the physics community, see e.g. [7, 14–16] and references therein.

The rest of the note is organized as follows. In Section 2, we present some necessary notions including the generalized small-world model and state our sharp threshold result. The proofs are given in Section 3.

2. Notions and main result

Watts-Strogatz (WS) rewiring model [18] and its variant Newman-Watts (NW) model [16] are classical small-world network models. The NW model can be regarded as the union of an Erdős-Rényi random graph $G(n,p)$ and a $2k$-regular lattice. It is known that the NW model and the WS model are, essentially, the same. A natural extension would be to use a general sparse graph to replace the low-dimensional regular lattices.

Let $S(n,p,H)$ be a small-world network that is the union of a random graph $G(n,p)$ and a graph H on n vertices. Note that $S(n,p,H)$ is not necessarily connected when H is not connected. When H is a regular lattice, we then obtain the NW model.

Next, we need to extend the classical notion of independence in graph theory to distant l-independence. A subset X of vertices in a graph G is called distant l-independent for some $l \in \mathbb{N}$, if the distance between any two vertices in V is larger than l. Thus, a distant 1-independent set is independent in the classical sense. Recall that there is another generalization of independence, called k-independence [8, 9], which requires the induced subgraph has maximum degree less than k. The relative strength relationship of these three concepts can be described as follows:

$$k\text{-independence} \;<\; \text{independence} \;<\; \text{distant } l\text{-independence}.$$

Now we are on the stage to state our main result.

Theorem 1. *Let H be a graph on n vertices, which contains a distant 2-independent set of order $\Theta(n^{\varepsilon})$ for some $\varepsilon > 0$. For the small-world network $S(n,p,H)$, $p = \sqrt{\ln n/n}$ is a sharp threshold function for the property $rc(S(n,p,H)) \leq 2$.*

Clearly, a $2k$-regular lattice with $k \ll n^{\alpha}$ for some $\alpha \in (0,1)$ serves as an eligible graph H in Theorem 1. Therefore, the above result holds for both WS and NW models.

3. Proof of Theorem 1

In this section, we will provide a proof of Theorem 1 as per the reasoning in [2]. As mentioned in Section 1, $rc(G) \geq 2$ for any non-complete graph G. The following lemma gives a sufficient condition for $rc(G) = 2$.

Lemma 1. *([2]) If G is a non-complete graph on n vertices and any two vertices of G have at least $2\ln n$ common neighbors, then $rc(G) = 2$.*

Proof of Theorem 1. For the first part of the theorem, we need to prove that for a sufficiently large constant C, the small-world network $S(n,p,H)$ with $p = C\sqrt{\ln n/n}$ almost surely has $rc(G) = 2$. Recall that $rc(G)$ is monotone non-increasing, we need only to prove this for the random graph $G(n,p)$. By Lemma 1, it suffices to show that almost surely any two vertices of $G(n,p)$ have at least $2\ln n$ common neighbors.

Fix a pair of vertices x, y, and the probability that z is a common neighbor of them is $C^2\ln n/n$. Let random variable X represents the number of common neighbors of x and y. Accordingly, we get $EX = (n-2)(C^2\ln n/n)$. By using the Chernoff bound (e.g. [12] pp.26), for large enough C, we have

$$P(X < 2\ln n) \leq P\Big(X < EX - \frac{C^2\ln n}{4}\Big) \leq e^{-\frac{C^2\ln n}{32}} = o(n^{-2}).$$

Since there are $\binom{n}{2}$ pairs of vertices in $G(n,p)$, the union bound readily yields the result.

For the other direction, it suffices to show that for a sufficiently small constant c, the small-world network $S(n,p,H)$ with $p = c\sqrt{\ln n/n}$ almost surely has $diam(S(n,p,H)) \geq 3$. By the assumption in Theorem 1, fix a distant 2-independent set X of order $\Theta(n^{\varepsilon})$ for some $\varepsilon < 1/4$ in H, and let Y be the remaining $n - \Theta(n^{\varepsilon})$ vertices. Let $\mathcal{A}$ be the event that X induces an independent set in the small-world network $S(n,p,H)$. Let $\mathcal{B}$ be the event that there exists a pair of vertices $x, y \in X$ with no common neighbor in Y. Consequently, it suffices to prove that (i) $P(\mathcal{A}) \to 1$; and (ii) $P(\mathcal{B}) \to 1$, as $n \to \infty$.

For (i): For c sufficiently small we obtain

$$\begin{aligned} P(\mathcal{A}) &= (1-p)^{\binom{\Theta(n^{\varepsilon})}{2}} = (1 - c\sqrt{\ln n/n})^{\binom{\Theta(n^{\varepsilon})}{2}} \\ &\sim e^{-\frac{c\sqrt{\ln n}}{\Theta\left(n^{\frac{1}{2}-2\varepsilon}\right)}} \to 1, \end{aligned}$$

as $n \to \infty$, since $0 < \varepsilon < 1/4$.

For (ii): For a pair $x, y \in X$, the probability that x, y have a common neighbor in Y is shown to be given by

$$1 - \left(1 - \frac{c^2 \ln n}{n}\right)^{n-\Theta(n^\varepsilon)} \sim (1 - n^{-c^2}).$$

Since the vertex set X can be divided into $\Theta(n^\varepsilon)/2 = \Theta(n^\varepsilon)$ pairs, the probability that all $\Theta(n^\varepsilon)$ pairs have a common neighbor is

$$1 - P(\mathcal{B}) = \left(1 - \left(1 - \frac{c^2 \ln n}{n}\right)^{n-\Theta(n^\varepsilon)}\right)^{\Theta(n^\varepsilon)} \sim \left(1 - n^{-c^2}\right)^{\Theta(n^\varepsilon)} \sim e^{-\frac{\Theta(n^\varepsilon)}{n^{c^2}}}. \quad (3.1)$$

For sufficiently small c, the right hand side of (3.1) tends to zero, which thus completes the proof. □

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HARMONIC NUMBERS OF ORDER TWO

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Abstract. We investigate sums of Euler type, in particular the summation, in closed form, of the product of Harmonic numbers of order two and the square of reciprocal binomial coefficients.

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Keywords: harmonic numbers, Riemann Zeta functions, binomial coefficients, series representations

1. INTRODUCTION

There are various identities in the literature for harmonic number sums in higher powers. A result that goes back to the time of Euler, see [17], or [19] is

$$\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n^p}=\frac{1}{2}\left(\zeta^2(p)+\zeta(2p)\right) \tag{1.1}$$

where $\zeta(z)=\sum\limits_{r=1}^{\infty}\frac{1}{r^z}$, $\mathbb{R}(z)>1$, denotes the familiar Riemann zeta function.

$$H_n=\gamma+\psi(n+1)=\sum_{r=1}^{n}\frac{1}{r}=\int_0^{\infty}\frac{1-t^n}{1-t}dt$$

is the nth harmonic number and γ denotes the Euler–Mascheroni constant. The *generalized* n^{th} *harmonic number* in power r, $H_n^{(r)}$, is defined for positive integers n and r as

$$H_n^{(r)}:=\sum_{m=1}^{n}\frac{1}{m^r}=\frac{(-1)^{r-1}}{(r-1)!}\Big(\quad^{(r-1)}(n+1)-\quad^{(r-1)}(1)\Big),$$

where

$$^{(m)}(z)=(-1)^{m+1}m!\sum_{r=0}^{\infty}\frac{1}{(z+r)^{m+1}}=(-1)^{m+1}m!\,\zeta(m+1,z)$$

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are the polygamma functions of order m which are defined by $\psi^{(0)}(z) \equiv \psi(z)$ and $\psi^{(m)}(z) := d^m \psi(z)/dz^m$, $m \in \mathbb{N}$ and $z \neq 0, -1, -2, \ldots$. Here $\psi(z)$ is the psi, or digamma, function, given as the logarithmic derivative of the well-known gamma function $\Gamma(z)$, *i.e.* $\psi(z) := d \log \Gamma(z)/dz$. $\zeta(\alpha, z)$ denotes the Hurwitz zeta function. There also exists the relation, which will be useful in our later analysis, between the polygamma functions and the generalized n^{th} harmonic number

$$H_n^{(r+1)} = \zeta(r+1) + \frac{(-1)^r}{r!}\psi^{(r)}(n+1) \tag{1.2}$$

and the recurrence relations

$$\psi^{(m)}(z+1) = \frac{(-1)^m m!}{z^{m+1}} + \psi^{(m)}(z), \; H_{n+1}^{(r)} = \frac{1}{(n+1)^p} + H_n^{(r)}.$$

In the paper [15] the author gave the result

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{\binom{n+k}{k}} = \frac{k}{k-1}\left(\zeta(2) - H_{k-1}^{(2)}\right). \tag{1.3}$$

and in this paper we shall extend (1.3) by providing identities for the general sums

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n\binom{n+k}{k}^2}, \; \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{\binom{n+k}{k}^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{(n+1)\binom{n+k}{k}^2}.$$

A result that goes back to Euler [8] is listed in [3] or [9] and we shall refer to in the latter part of this paper is the following:

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} = \zeta(p+q) + \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{(n+1)^q} \tag{1.4}$$

where

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{(n+1)^q} = \frac{A}{2}\left[\binom{p+q}{p} - A\right]\zeta(p+q) + B\zeta(p)\zeta(q)$$

$$-A\sum_{j=1}^{p+q}\left[\binom{2j-2}{p-1} + \binom{2j-2}{q-1}\right]\zeta(2j-1)\zeta(p+q-2j+1), \tag{1.5}$$

here

$$A = \begin{cases} 1, & \text{for } p \text{ odd}, q \text{ even} \\ -1, & \text{for } p \text{ even}, q \text{ odd} \end{cases}, \quad B = \begin{cases} 1, & \text{for } p \text{ odd}, q \text{ even} \\ 0, & \text{for } p \text{ even}, q \text{ odd} \end{cases}.$$

Many finite versions of higher order harmonic number sum identities also exist in the literature, for example, see [11], [12], and [13]. Other finite sum identities can be seen in [18]. Further work in the summation of harmonic numbers and binomial coefficients has also been done by Sofo [16]. The works of, [1], [2], [4], [6], [5], [7], [10], [14], and [20] and references therein, also investigate various representations of binomial sums and zeta functions in simpler form by the use of the Beta function and other techniques. The following results will be useful for later analysis.

Lemma 1. *Let $a > 1$ be a positive integer then*

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n\,(n+a)} = \frac{1}{a}\zeta(3) + \frac{1}{a}\zeta(2)\,H_{a-1} - \frac{1}{a}\sum_{j=1}^{a-1}\frac{H_j}{j^2} \tag{1.6}$$

and

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n\,(n+a)^2} = -\frac{3}{4a}\zeta(4) + \frac{1}{a^2}\zeta(3) + \frac{1}{a}\zeta(2)\left(\frac{H_{a-1}}{a} + 2H_{a-1}^{(2)}\right) \tag{1.7}$$

$$-\frac{1}{a}\sum_{j=1}^{a-1}\left(\frac{H_j}{aj^2} + \frac{2H_j}{j^3} + \frac{H_j^{(2)}}{j^2}\right)$$

Proof. Consider

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n\,(n+a)} = \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{1}{k^2\,n\;(n+a)}$$

since these sums are absolutely convergent, by a re-arrangement of the double sum we can write

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{1}{k^2\,n\;(n+a)} = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} \frac{1}{k^2\;(n+k)\;(n+a+k)}$$

$$= \sum_{k=1}^{\infty} \frac{1}{a\,k^2}\left[\psi(a+k) - \psi(k)\right].$$

Now

$$\sum_{k=1}^{\infty} \frac{1}{a\,k^2}\left[\psi(a+k) - \psi(k)\right] = \sum_{k=1}^{\infty} \frac{1}{a\,k^3} + \sum_{j=1}^{a-1}\sum_{k=1}^{\infty} \frac{1}{a\,k^2\,(k+j)}$$

$$= \frac{1}{a}\zeta(3) + \frac{1}{a}\sum_{j=1}^{a-1}\int_0^1 x^{j-1} Li_2(x)\,dx$$

where the Polylogarithmic function

$$Li_2(\beta) = \sum_{r\geq 1}\frac{\beta^r}{r^2}.$$

By partial fraction decomposition we have

$$\sum_{j=1}^{a-1}\sum_{k=1}^{\infty}\frac{1}{a\,k^2(k+j)} = \sum_{j=1}^{a-1}\frac{1}{a}\sum_{k=1}^{\infty}\left[\frac{1}{j\,k^2} - \frac{1}{j\,k(k+j)}\right]$$

$$= \sum_{j=1}^{a-1}\frac{1}{a}\left[\frac{\zeta(2)}{j} - \frac{H_j}{j^2}\right] = \frac{H_{a-1}\,\zeta(2)}{a} - \frac{1}{a}\sum_{j=1}^{a-1}\frac{H_j}{j^2}$$

and the result (1.6) follows. To prove (1.7) consider

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n(n+a)^2} = \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{k^2\,n\,(n+a)^2}$$

since these sums are absolutely convergent, by a re-arrangement of the double sum we can write

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{k^2\,n\,(n+a)^2} = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{k^2\,(n+k)\,(n+a+k)^2}$$

$$= \sum_{k=1}^{\infty}\frac{1}{a^2\,k^2}\left[\psi(a+k) - \psi(k) - a\psi'(a+k)\right]$$

$$= \sum_{k=1}^{\infty}\frac{1}{a^2\,k^3} + \sum_{j=1}^{a-1}\sum_{k=1}^{\infty}\frac{1}{a^2\,k^2(k+j)} + \sum_{j=1}^{a-1}\sum_{k=1}^{\infty}\frac{1}{a\,k^2(k+j)^2} - \sum_{k=1}^{\infty}\frac{\psi'(1+k)}{a\,k^2}$$

$$= \frac{\zeta(3)}{a^2} + \sum_{j=1}^{a-1}\frac{1}{a^2}\left(\frac{\zeta(2)}{j} - \frac{H_j}{j^2}\right) + \frac{1}{a}\sum_{j=1}^{a-1}\sum_{k=1}^{\infty}\begin{pmatrix}\frac{1}{k^2j^2} + \frac{1}{j^2(k+j)^2}\\ -\frac{2}{j^2k(k+j)}\end{pmatrix}$$

$$-\sum_{k=1}^{\infty}\frac{\zeta(2)-H_k^{(2)}}{a\,k^2}$$

$$=\frac{\zeta(3)}{a^2}+\sum_{j=1}^{a-1}\frac{1}{a^2}\left(\frac{\zeta(2)}{j}-\frac{H_j}{j^2}\right)$$
$$+\frac{1}{a}\sum_{j=1}^{a-1}\left(\frac{\zeta(2)}{j^2}-\frac{2H_j}{j^3}+\frac{\zeta(2)}{j^2}-\frac{H_j^{(2)}}{j^2}\right)-\frac{\zeta^2(2)}{a}$$

$$+\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{a\,k^2}.$$

From (1.1) we know that

$$\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{a\,k^2}=\frac{1}{2a}\left(\zeta^2(2)+\zeta(4)\right)=\frac{7}{4a}\zeta(4),\ \text{and also}\ \zeta^2(2)=\frac{5}{2}\zeta(4),$$

hence

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n(n+a)^2}=-\frac{3}{4a}\zeta(4)+\frac{1}{a^2}\zeta(3)+\frac{1}{a}\zeta(2)\left(\frac{H_{a-1}}{a}+2H_{a-1}^{(2)}\right)$$
$$-\frac{1}{a}\sum_{j=1}^{a-1}\left(\frac{H_j}{aj^2}+\frac{2H_j}{j^3}+\frac{H_j^{(2)}}{j^2}\right).$$

□

Some special cases are noted in the remark.

Remark 1. When $a=0$

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}=\frac{7}{4}\zeta(4),\ \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}=3\zeta(2)\zeta(3)-\frac{9}{2}\zeta(5),$$

for $a=1$ and 5 and also using (1.3) and (1.4), we have, respectively

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n\ (n+1)}=\zeta(3),\ \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n\ (n+5)}=\frac{\zeta(3)}{5}+\frac{5\zeta(2)}{12}-\frac{2953}{8640},$$

and

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n\ (n+1)^2}=\zeta(3)-\frac{3}{4}\zeta(4),$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n\ (n+5)^2}=\frac{\zeta(3)}{25}-\frac{3\zeta(4)}{20}+\frac{47\zeta(2)}{72}-\frac{51499}{57600}.$$

A related Lemma which will be useful later is the following.

Lemma 2. *Let $a>1$ be a positive integer then*

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^2(n+a)}=\frac{3\zeta(4)}{4(a-1)}-\frac{\zeta(2)H_{a-1}}{(a-1)^2}+\frac{1}{(a-1)^2}\sum_{j=1}^{a-1}\frac{H_j}{j^2} \tag{1.8}$$

and

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^2(n+a)^2}=\frac{3\zeta(4)}{2(a-1)^2}-\frac{2\zeta(2)H_{a-1}}{(a-1)^3}-\frac{2\zeta(2)H_{a-1}^{(2)}}{(a-1)^2}$$
$$+\frac{1}{(a-1)^2}\sum_{j=1}^{a-1}\left(\frac{2H_j}{(a-1)\ j^2}+\frac{2H_j}{j^3}+\frac{H_j^{(2)}}{j^2}\right) \tag{1.9}$$

Proof. Consider

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^2(n+a)}=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{k^2\ (n+1)^2\ (n+a)}$$

since these sums are absolutely convergent, by a re-arrangement of the double sum we can write

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{k^2\ (n+1)^2\ (n+a)}=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{k^2\ (n+1+k)^2\ (n+a+k)}$$

$$=\sum_{k=1}^{\infty}\frac{1}{k^2}\left[\frac{\psi(1+k)-\psi(a+k)}{(a-1)^2}+\frac{\psi'(1+k)}{(a-1)}\right].$$

Now

$$\sum_{k=1}^{\infty}\frac{1}{k^2}\left[\frac{\psi(1+k)-\psi(a+k)}{(a-1)^2}+\frac{\psi'(1+k)}{(a-1)}\right]=-\sum_{j=1}^{a-1}\sum_{k=1}^{\infty}\frac{1}{(a-1)^2k^2(k+j)}$$

$$+\sum_{k=1}^{\infty}\frac{\zeta(2)-H_k^{(2)}}{(a-1)\ k^2}$$

$$=\frac{\zeta^2(2)}{a-1}-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{(a-1)\ k^2}-\sum_{j=1}^{a-1}\sum_{k=1}^{\infty}\frac{1}{(a-1)^2k^2(k+j)}$$

$$=\frac{\zeta^2(2)}{a-1}-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{(a-1)\ k^2}-\sum_{j=1}^{a-1}\frac{1}{(a-1)^2}\left(\frac{\zeta(2)}{j}-\frac{H_j}{j^2}\right)$$

$$=\frac{\zeta^2(2)}{a-1}-\frac{1}{2(a-1)}\left(\zeta^2(2)+\zeta(4)\right)-\frac{\zeta(2)H_{a-1}}{(a-1)^2}+\sum_{j=1}^{a-1}\frac{H_j}{(a-1)^2\ j^2}$$

$$=\frac{3\zeta(4)}{4(a-1)}-\frac{\zeta(2)H_{a-1}}{(a-1)^2}+\frac{1}{(a-1)^2}\sum_{j=1}^{a-1}\frac{H_j}{j^2}.$$

To prove (1.9) consider

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^2(n+a)^2}=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{k^2\ (n+1)^2\ (n+a)^2}$$

$$=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{k^2\ (n+1+k)^2\ (n+a+k)^2}$$

$$=\sum_{k=1}^{\infty}\frac{1}{(a-1)^3\ k^2}\left[2\psi(1+k)-2\psi(a+k)+(a-1)\left(\psi'(1+k)+\psi'(a+k)\right)\right]$$

$$=\sum_{k=1}^{\infty}\frac{1}{(a-1)^3\ k^2}\left[-2\sum_{j=1}^{a-1}\frac{1}{(k+j)}+(a-1)\left(2\psi'(1+k)+\sum_{j=1}^{a-1}\frac{1}{(k+j)^2}\right)\right]$$

$$=-\frac{2}{(a-1)^3}\sum_{j=1}^{a-1}\left(\frac{\zeta(2)}{j}-\frac{H_j}{j^2}\right)+\frac{2}{(a-1)^2}\sum_{k=1}^{\infty}\frac{\zeta(2)-H_k^{(2)}}{k^2}$$

$$-\frac{1}{(a-1)^2}\sum_{j=1}^{a-1}\sum_{k=1}^{\infty}\frac{1}{(k+j)^2}$$

$$= \frac{3\zeta(4)}{2(a-1)^2} - \frac{2\zeta(2)H_{a-1}}{(a-1)^3} - \frac{2\zeta(2)H^{(2)}_{a-1}}{(a-1)^2}$$
$$+ \frac{1}{(a-1)^2}\sum_{j=1}^{a-1}\left(\frac{2H_j}{(a-1)\ j^2} + \frac{2H_j}{j^3} + \frac{H^{(2)}_j}{j^2}\right)$$

and the result (1.9) is attained. □

Remark 2. Some examples are

$$\sum_{n=1}^{\infty}\frac{H^{(2)}_n}{(n+1)^3} = 3\zeta(2)\zeta(3) - \frac{11}{2}\zeta(5),\ \sum_{n=1}^{\infty}\frac{H^{(2)}_n}{(n+1)^2(n+2)} = \frac{3\zeta(4)}{4} - \zeta(2)$$

and

$$\sum_{n=1}^{\infty}\frac{H^{(2)}_n}{(n+1)^4} = \zeta^2(3) + \frac{8}{3}\zeta(2)\zeta(4) - 6\zeta(6),\ \sum_{n=1}^{\infty}\frac{H^{(2)}_n}{(n+1)^2(n+5)^2} = \frac{3\zeta(4)}{32}$$

$$- \frac{35}{144}\zeta(2) + \frac{3827}{12288}.$$

Lemma 3. *Let $a > 1$ be a positive integer then*

$$\sum_{n=1}^{\infty}\frac{H^{(2)}_n}{(n+1)^3(n+a)} = \frac{3\zeta(2)\zeta(3)}{(a-1)} - \frac{11\zeta(5)}{2(a-1)} - \frac{3\zeta(4)}{4(a-1)^2}$$
$$+ \frac{\zeta(2)H_{a-1}}{(a-1)^3} - \frac{1}{(a-1)^3}\sum_{j=1}^{a-1}\frac{H_j}{j^2} \tag{1.10}$$

and

$$\sum_{n=1}^{\infty}\frac{H^{(2)}_n}{(n+1)^3(n+a)^2} = \frac{3\zeta(2)\zeta(3)}{(a-1)^2} - \frac{11\zeta(5)}{2(a-1)^2} - \frac{9\zeta(4)}{4(a-1)^3} + \frac{2\zeta(2)H^{(2)}_{a-1}}{(a-1)^3}$$
$$+ \frac{3\zeta(2)H_{a-1}}{(a-1)^4} - \frac{1}{(a-1)^3}\sum_{j=1}^{a-1}\left(\frac{3H_j}{(a-1)\ j^2} + \frac{2H_j}{j^3} + \frac{H^{(2)}_j}{j^2}\right) \tag{1.11}$$

Proof. Consider

$$\sum_{n=1}^{\infty}\frac{H^{(2)}_n}{(n+1)^3(n+a)} = \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{k^2\ (n+1)^3\ (n+a)}$$

since these sums are absolutely convergent, by a re-arrangement of the double sum we can write

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{k^2\ (n+1)^3\ (n+a)}=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{k^2\ (n+1+k)^3\ (n+a+k)}$$

$$=\frac{1}{2\,(a-1)^3}$$

$$\sum_{k=1}^{\infty}\frac{1}{k^2}\left[2\psi(a+k)-2\psi(1+k)-2\,(a-1)\,\psi'(1+k)-(a-1)^2\,\psi''(1+k)\right]$$

$$=\frac{1}{(a-1)^3}$$

$$\sum_{j=1}^{a-1}\sum_{k=1}^{\infty}\frac{1}{k^2\,(k+j)}-\frac{1}{(a-1)^2}\sum_{k=1}^{\infty}\frac{\psi'(1+k)}{k^2}-\frac{1}{(a-1)}\sum_{k=1}^{\infty}\frac{\psi''(1+k)}{k^2}.$$

Using the same ideas as the previous lemma together with (1.2), (1.4) and (1.5) we obtain (1.10). To prove (1.11) consider

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^3\,(n+a)^2}=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{k^2\ (n+1)^3\ (n+a)^2}$$

$$=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{k^2\ (n+1+k)^3\ (n+a+k)^2}$$

$$=\sum_{k=1}^{\infty}\frac{1}{2\,(a-1)^4\ k^2}\left[\begin{array}{c}6\psi(a+k)-6\psi(1+k)-4\,(a-1)\,\psi'(1+k)\\ \\ -2\,(a-1)\,\psi'(a+k)-(a-1)^2\,\psi''(1+k)\end{array}\right]$$

$$=\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\begin{array}{c}\frac{3}{(a-1)^4}\sum_{j=1}^{a-1}\frac{1}{(k+j)}+\frac{1}{(a-1)^3}\sum_{j=1}^{a-1}\frac{1}{(k+j)^2}+\frac{3\psi'(1+k)}{(a-1)^3}\\ \\ -\frac{\psi''(1+k)}{(a-1)^2}\end{array}\right)$$

and the result (1.11) is attained after utilizing (1.2), (1.4) and (1.5). □

Remark 3. Some examples are

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n(n+1)^3}=\zeta(3)-\frac{3\zeta(4)}{4}-3\zeta(2)\zeta(3)+\frac{11}{2}\zeta(5),$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^3(n+5)}=\frac{3}{4}\zeta(2)\zeta(3)-\frac{3\zeta(4)}{64}-\frac{11}{8}\zeta(5)+\frac{25}{768}\zeta(2)-\frac{2953}{110592}$$

and

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^3(n+2)^2}=\frac{3}{16}\zeta(2)\zeta(3)-\frac{9\zeta(4)}{256}-\frac{11}{32}\zeta(5)+\frac{635}{9216}\zeta(2)-\frac{9349}{110592}.$$

2. Three theorems

We now prove the following three theorems.

Theorem 1. *Let* $k\in\mathbb{N}$, $\mathbb{N}:=\{1,2,3,...\}$, *then*

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n\binom{n+k}{k}^2} \tag{2.1}$$

$$=\sum_{r=1}^{k}r\binom{k}{r}^2\left[\begin{array}{c}2(H_{r-1}-H_{k-r})\left(\zeta(3)+\zeta(2)H_{r-1}+\sum_{j=1}^{r-1}\frac{H_j}{j^2}\right)\\ -\frac{3}{4}\zeta(4)+\frac{1}{r}\zeta(3)+\zeta(2)\left(\frac{H_{r-1}}{r}+2H_{r-1}^{(2)}\right)\\ -\sum_{j=1}^{r-1}\left(\frac{H_j}{rj^2}+\frac{2H_j}{j^3}+\frac{H_j^{(2)}}{j^2}\right)\end{array}\right]$$

Proof. Consider the following expansion:

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n\binom{n+k}{k}^2}=\sum_{n=1}^{\infty}\frac{(k!)^2\ H_n^{(2)}}{n\prod_{r=1}^{k}(n+r)^2}=\sum_{n=1}^{\infty}\frac{(k!)^2\ H_n^{(2)}}{n\left((n+1)_{k+1}\right)^2}$$

Where $(\alpha)_r$ is Pochhammer's symbol given by
$(\alpha)_r=\alpha(\alpha+1)(\alpha+2)...(\alpha+r-1),r>0,(\alpha)_0=1$. Now

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n\binom{n+k}{k}^2}=\sum_{n=1}^{\infty}\frac{(k!)^2\ H_n^{(2)}}{n}\sum_{r=1}^{k}\left(\frac{AA_r}{(n+r)}+\frac{BB_r}{(n+r)^2}\right) \tag{2.2}$$

where

$$BB_r = \lim_{n\to(-r)} \left(\frac{(n+r)^2}{\prod_{r=1}^{k} (an+r)^2} \right) = \left(\frac{r}{k!} \left(\begin{array}{c} k \\ r \end{array} \right) \right)^2,$$

and

$$AA_r = \lim_{n\to(-r)} \frac{d}{dn} \left(\frac{(n+r)^2}{\prod_{r=1}^{k} (an+r)^2} \right) = 2\left(H_{r-1} - H_{k-r}\right) \left(\frac{r}{k!} \left(\begin{array}{c} k \\ r \end{array} \right) \right)^2.$$

Now from (2.2) and using Lemma 1

$$\sum_{n=1}^{\infty} \frac{(k!)^2 \ H_n^{(2)}}{n} \sum_{r=1}^{k} \left(\frac{AA_r}{(n+r)} + \frac{BB_r}{(n+r)^2} \right) = \sum_{r=1}^{k} \sum_{n=1}^{\infty} \left(\begin{array}{c} \frac{(k!)^2 \ H_n^{(2)} \ AA_r}{n(n+r)} \\ \\ + \frac{(k!)^2 \ H_n^{(2)} \ BB_r}{n(n+r)^2} \end{array} \right)$$

$$= \sum_{r=1}^{k} \sum_{n=1}^{\infty} \left(\frac{2\left(H_{r-1} - H_{k-r}\right) \left(r \left(\begin{array}{c} k \\ r \end{array} \right) \right)^2 \ H_n^{(2)}}{n\,(n+r)} + \frac{\left(r \left(\begin{array}{c} k \\ r \end{array} \right) \right)^2 \ H_n^{(2)}}{n\,(n+r)^2} \right)$$

and using (1.6) and (1.7) we obtain the result. □

Remark 4. For the case $k = 5$, we have the result

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n \left(\begin{array}{c} n+5 \\ 5 \end{array} \right)^2} = \zeta(3) - \frac{945}{2}\zeta(4) + 1750\zeta(2) - \frac{16370447}{6912}$$

Now we consider the following Theorem 1.

Theorem 2. *Let* $k \in \mathbb{N}\backslash\{1\}$, $\mathbb{N} := \{1, 2, 3, ...\}$, *then*

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{\left(\begin{array}{c} n+k \\ k \end{array} \right)^2} \tag{2.3}$$

$$= \sum_{r=2}^{k} \left(r \binom{k}{r} \right)^2 \left[\begin{array}{c} 2\left(H_{r-2} - H_{k-r}\right) \left(\begin{array}{c} \frac{3(r-1)\zeta(4)}{4} - \zeta(2) H_{r-1} \\ + \sum_{j=1}^{r-1} \frac{H_j}{j^2} \end{array} \right) \\ \frac{3\zeta(4)}{2} - \frac{2\zeta(2) H_{r-1}}{(r-1)} - 2\zeta(2) H_{r-1}^{(2)} \\ + \sum_{j=1}^{r-1} \left(\frac{2H_j}{(r-1)\ j^2} + \frac{2H_j}{j^3} + \frac{H_j^{(2)}}{j^2} \right) \end{array} \right]$$

Proof. Consider the following expansion:

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{\binom{n+k}{k}^2} = \sum_{n=1}^{\infty} \frac{(k!)^2\ H_n^{(2)}}{(n+1)^2 \prod_{r=2}^{k} (n+r)^2} = \sum_{n=1}^{\infty} \frac{(k!)^2\ H_n^{(2)}}{(n+1)^2 \left((n+2)_{k+1}\right)^2}$$

Now

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{\binom{n+k}{k}^2} = \sum_{n=1}^{\infty} \frac{(k!)^2\ H_n^{(2)}}{(n+1)^2} \sum_{r=2}^{k} \left(\frac{A_r}{(n+r)} + \frac{B_r}{(n+r)^2} \right) \tag{2.4}$$

where

$$B_r = \lim_{n \to (-r)} \left(\frac{(n+r)^2}{\prod_{r=2}^{k} (an+r)^2} \right) = \left(\frac{2}{k!} \binom{k}{r} \binom{r}{2} \right)^2 \tag{2.5}$$

and

$$A_r = \lim_{n \to (-r)} \frac{d}{dn} \left(\frac{(n+r)^2}{\prod_{r=2}^{k} (an+r)^2} \right) = 2\left(H_{r-2} - H_{k-r}\right) \left(\frac{2}{k!} \binom{k}{r} \binom{r}{2} \right)^2. \tag{2.6}$$

Now from (2.4) and using Lemma 2

$$\sum_{n=1}^{\infty}\frac{(k!)^2\ H_n^{(2)}}{(n+1)^2}\sum_{r=2}^{k}\left(\frac{A_r}{(n+r)}+\frac{B_r}{(n+r)^2}\right)=\sum_{r=2}^{k}\sum_{n=1}^{\infty}\left(\begin{array}{c}\frac{(k!)^2\ H_n^{(2)}\ A_r}{(n+1)^2(n+r)}\\ \\ +\frac{(k!)^2\ H_n^{(2)}\ B_r}{(n+1)^2(n+r)^2}\end{array}\right)$$

$$=\sum_{r=2}^{k}\sum_{n=1}^{\infty}\left(\frac{2\left(H_{r-2}-H_{k-r}\right)\left(2\left(\begin{array}{c}k\\r\end{array}\right)\left(\begin{array}{c}r\\2\end{array}\right)\right)^2\ H_n^{(2)}}{(n+1)^2\,(n+r)}\right.$$

$$\left.+\frac{\left(2\left(\begin{array}{c}k\\r\end{array}\right)\left(\begin{array}{c}r\\2\end{array}\right)\right)^2\ H_n^{(2)}}{(n+1)^2\,(n+r)^2}\right)$$

and using (1.8) and (1.9) we obtain the result. □

Remark 5. An example for Theorem 2 is

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{\left(\begin{array}{c}n+5\\5\end{array}\right)^2}=\frac{2625}{2}\zeta(4)-\frac{179375}{36}\zeta(2)+\frac{140498725}{20736}$$

The final theorem follows.

Theorem 3. *Let* $k\in\mathbb{N}\backslash\{1\}$, $\mathbb{N}:=\{1,2,3,...\}$, *then*

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)\left(\begin{array}{c}n+k\\k\end{array}\right)^2} \tag{2.7}$$

$$=\sum_{r=2}^{k}\left(r\left(\begin{array}{c}k\\r\end{array}\right)\right)^2$$

$$\left[\begin{array}{c} 2\left(H_{r-2}-H_{k-r}\right)\left(\begin{array}{c}3(r-1)\zeta(2)\zeta(3)-\frac{11(r-1)\zeta(5)}{2}-\frac{3\zeta(4)}{4}\\ +\frac{\zeta(2)H_{r-1}}{(r-1)}-\frac{1}{(r-1)}\sum_{j=1}^{r-1}\frac{H_j}{j^2}\end{array}\right)\\ 3\zeta(2)\zeta(3)-\frac{11\zeta(5)}{2}-\frac{9\zeta(4)}{4(r-1)}+\frac{2\zeta(2)H_{r-1}^{(2)}}{(r-1)}\\ +\frac{3\zeta(2)H_{r-1}}{(r-1)^2}-\frac{1}{(r-1)}\sum_{j=1}^{r-1}\left(\frac{3H_j}{(r-1)\,j^2}+\frac{2H_j}{j^3}+\frac{H_j^{(2)}}{j^2}\right)\end{array}\right]$$

Proof. Consider the following expansion:

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)\left(\begin{array}{c}n+k\\k\end{array}\right)^2}=\sum_{n=1}^{\infty}\frac{(k!)^2\;H_n^{(2)}}{(n+1)^3\prod_{r=2}^{k}(n+r)^2}$$

$$=\sum_{n=1}^{\infty}\frac{(k!)^2\;H_n^{(2)}}{(n+1)^3\left((n+2)_{k+1}\right)^2}.$$

Now

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)\left(\begin{array}{c}n+k\\k\end{array}\right)^2}=\sum_{n=1}^{\infty}\frac{(k!)^2\;H_n^{(2)}}{(n+1)^3}\sum_{r=2}^{k}\left(\frac{A_r}{(n+r)}+\frac{B_r}{(n+r)^2}\right) \tag{2.8}$$

where B_r and A_r are given respectively by (2.5) and (2.6). Now from (2.8) and using Lemma 3

$$\sum_{n=1}^{\infty}\frac{(k!)^2\;H_n^{(2)}}{(n+1)^3}\sum_{r=2}^{k}\left(\frac{A_r}{(n+r)}+\frac{B_r}{(n+r)^2}\right)=\sum_{r=2}^{k}\sum_{n=1}^{\infty}\left(\begin{array}{c}\frac{(k!)^2\;H_n^{(2)}\;A_r}{(n+1)^3(n+r)}\\ +\frac{(k!)^2\;H_n^{(2)}\;B_r}{(n+1)^3(n+r)^2}\end{array}\right),$$

and using (1.10) and (1.11) we obtain the result (2.7). □

Remark 6. An example for Theorem 3 is

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{(n+1)\binom{n+5}{5}^2} = 75\zeta(2)\zeta(3) - \frac{275}{2}\zeta(5) - \frac{13125}{16}\zeta(4) + \frac{5050925}{1728}\zeta(2) - \frac{40704625}{10368}.$$

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ON THE DIOPHANTINE EQUATION $x^2 + 7^\alpha \cdot 11^\beta = y^n$

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Abstract. In this paper, we give all the solutions of the Diophantine equation $x^2 + 7^\alpha \cdot 11^\beta = y^n$, for the nonnegative integers $\alpha,\ \beta,\ x,\ y,\ n \geq 3$, where x and y coprime, except when $\alpha.x$ is odd and β is even.

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Keywords: exponential equations, primitive divisors of Lucas sequences

1. INTRODUCTION

The Diophantine equation

$$x^2 + C = y^n, \qquad n \geq 3 \tag{1.1}$$

in positive integers $x,\ y,\ n$ for given a C has a rich history. In 1850, Lebesgue [25] proved that the above equation has no solutions when $C = 1$. The equation of the title is a special case of the Diophantine equation $ay^2 + by + c = dx^n$, where $a \neq 0, b,\ c$ and $d \neq 0$ are integers with $b^2 - 4ac \neq 0$, which has at most finitely many integer solutions $x,\ y,\ n \geq 3$ (see [23]). In 1993, J.H.E. Cohn [17] solved the Diophantine equation (1.1) for several values of the parameter C in the range $1 \leq C \leq 100$. The solution for the cases $C = 74, 86$ was completed by Mignotte and de Weger [31]. That had not been covered by Cohn (indeed, Cohn solved these two equations of type (1.1) except for $p = 5$, in which case difficulties occur as the class numbers of the corresponding imaginary quadratic fields are divisible by 5). In [12], Bugeaud, Mignotte and Siksek improved modular methods to solve completely (1.1) when $n \geq 3$, for C in the range $[1, 100]$. So they covered the remaining cases.

Different types of the Diophantine equation (1.1) were studied also by various mathematicians. For effectively computable upper bounds for the exponent n, we refer to [8] and [22]. However, these estimates are based on Baker's theory of lower bounds for linear forms in logarithms of algebraic numbers, so they are quite impractical. In [37], Tengely gave a method to solve the equation $x^2 + a^2 = y^n$ and applied it to $3 \leq a \leq 501$, so it includes $x^2 + 7^2 = y^n$ and $x^2 + 11^2 = y^n$. In [4], the equation $x^2 + C = 2y^n$, where C is a fixed positive integer, under the similar restrictions

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$n \geq 3$ and $\gcd(x,y) = 1$ was studied. Recently, Luca, Tengely and Togbé studied the Diophantine equation $x^2 + C = 4y^n$ for nonnegative integers $x, y, n \geq 3$ with x and y coprime for various shapes of the positive integer C in [28].

In recent years, a different form of the above equation has been considered, namely where C is a power of a fixed prime. In [6], the equation $x^2 + 2^k = y^n$ was studied under some conditions by Arif and Muriefah. A conjecture of Cohn (see [16]) was verified. It says that $x^2 + 2^k = y^n$ has no solutions with x odd and even $k > 2$ by Le [24]. In [7], Abu Muriefah and Arif, gave all the solutions of $x^2 + 3^k = y^n$ with k odd and, Luca [27], gave all the solutions with k even. Again the same equation was independently solved in 2008 by Liqun in [35] for both odd and even m. All solutions of $x^2 + 5^k = y^n$ are given with k odd in [3] and with k even in [2]. Liqun solves the same equation again in 2009, in [36]. Recently, Bérczes and Pink [9], gave all the solutions of the Diophantine equation (1.1) when $C = p^k$ and k is even, where p is any prime in the interval $[2, 100]$.

The last variant of the Diophantine equation (1.1) where C is a product of at least two prime powers were studied in some recent papers. In 2002, Luca gave complete solution of $x^2 + 2^a.3^b = y^n$ in [30]. Since then, in 2006, all the solutions of the Diophantine equation $x^2 + 2^a.5^b = y^n$ were found by Luca and Togbé in [30]. In 2008, the equations $x^2 + 5^a.13^b = y^n$ and $x^2 + 2^a 5^b.13^c = y^n$ were solved in [5] and [21]. Recently, in [14] and [13], complete solutions of the equations $x^2 + 2^a.11^b = y^n$ and $x^2 + 2^a.3^b.11^c = y^n$ were found. In [20], the complete solution (n,a,b,x,y) of the equation $x^2 + 5^a.11^b = y^n$ when $\gcd(x,y) = 1$, except for the case when xab is odd, is given. In [34], Pink gave all the *non-exceptional solutions* (in the terminology of that paper) with $C = 2^a.3^b.5^c.7^d$. Note that finding all the *exceptional solutions* of this equation seems to be a very difficult task. A more exhaustive survey on this type of problems is [32].

Here, we study the Diophantine equation

$$x^2 + 7^\alpha \cdot 11^\beta = y^n, \qquad gcd(x,y) = 1 \qquad \text{and} \qquad n \geq 3. \tag{1.2}$$

There are three papers concerned with partial solutions for equation (1.2). The known results include the following theorem:

Theorem 1. *(i) If α is even and $\beta = 0$, then the only integer solutions of the Diophantine equation*

$$x^2 + 7^{2k} = y^n$$

are

$$n = 3 \quad (x,y,k) = (524 \cdot 7^{3\lambda}, 65 \cdot 7^{2\lambda}, 1 + 3\lambda),$$
$$n = 4 \quad (x,y,k) = (24 \cdot 7^{2\lambda}, 5 \cdot 7^{\lambda}, 1 + 2\lambda) \text{ where } \lambda \geq 0 \text{ is any integer.}$$

(*ii*) *If* $\alpha=1$ *and* $\beta=0$, *then the only integer solutions* (x,y,n) *to the generalized Ramanujan–Nagell equation*

$$x^2+7=y^n$$

are

$$(1,2,3),(181,32,3),(3,2,4),(5,2,5),(181,8,5),(11,2,7),(181,2,15).$$

(*iii*) If $\alpha=0$, then the only integer solutions of the Diophantine equation

$$x^2+11^\beta=y^n$$

are

$$(x,y,\beta,n)\ =(2,5,2,3),(4,3,1,3),(58,15,1,3),(9324,443,3,3)$$

Proof. See [29], [12] and [14]. □

Our main result is the following.

Theorem 2. *The only solutions of the Diophantine equation* (1.2) *are*

$n=3:\quad (x,y,\alpha,\beta)\in\{(57,16,1,2),(797,86,1,2),(4229,284,3,4),$
$(3093,478,7,2)(4,3,0,1),(58,15,0,1),(2,5,0,2),(9324,443,0,3),$
$(1,2,1,0),(181,32,1,0),(524,65,2,0),(13,8,3,0)\};$

$n=4:\quad (x,y,\alpha,\beta)\in\{(2,3,1,1),(57,8,1,2),(8343,92,5,2),(3,2,1,0),$
$(24,5,2,0)\};$

$n=6:\quad (x,y,\alpha,\beta)=(57,4,1,2);$

$n=9:\quad (x,y,\alpha,\beta)=(13,2,3,0);$

$n=12:\quad (x,y,\alpha,\beta)=(57,2,1,2);$

When $n\geq 5, n\neq 6,9,12$, *equation* (1.2) *has no solutions* (x,y,α,β) *with at least one of* α, x *even or with* β *is odd.*

Remark 1. For $n\geq 5, n\neq 6,9,12$ the above theorem lefts out the solutions (α,β,x,y) when $\alpha.x$ is odd and β is even. These are exactly the exceptional solutions of the equation (1.2) in the terminology of [34]; see also the remark 2 at the end of this paper.

One can deduce from the Theorem 1 and Theorem 2 the following corollary.

Corollary 1. *The only integer solutions of the Diophantine equation* (1.2) *are*

$n=3:\quad (x,y,\alpha,\beta)\in\{(57,16,1,2),(797,86,1,2),(4229,284,3,4),$
$(3093,478,7,2),(4,3,0,1),(58,15,0,1),(2,5,0,2),$
$(9324,443,0,3),\ (1,2,1,0),(181,32,1,0),(524,65,2,0),(13,8,3,0)\};$

$n=4:\quad (x,y,\alpha,\beta)\in\{(2,3,1,1),(57,8,1,2),(8343,92,5,2),(3,2,1,0),$

$(24,5,2,0)\}$;

$n=5:\quad (x,y,\alpha,\beta)=(5,2,1,0),(181,8,1,0)$;

$n=6:\quad (x,y,\alpha,\beta)=(57,4,1,2)$;

$n=7:\quad (x,y,\alpha,\beta)=(11,2,1,0)$;

$n=9:\quad (x,y,\alpha,\beta)=(13,2,3,0)$;

$n=12:\quad (x,y,\alpha,\beta)=(57,2,1,2)$;

$n=15:\quad (x,y,\alpha,\beta)=(181,2,1,0)$.

2. The proof of Theorem 2

We distinguish the cases $n=3,6,9,12$, $n=4$ and $n>4$, devoting a subsection to the treatment of each case. We first treat the cases $n=3$ and $n=4$. This is achieved in Section 2.1 and Section 2.2, respectively. For the case $n=3$, we transform equation (1.2) into several elliptic equations in Weierstrass form for which we need to determine all their $\{7,11\}-$integral points. In Section 2.2, we use the same method as in Section 2.1 to determine the solutions of (1.2) for $n=4$. In the last section, we assume that $n>4$ is prime and study the equation (1.2) under this assumption. Here we use the method of primitive divisors for Lucas sequences. All the computations are done with MAGMA [11] and with Cremona's program mwrank.

2.1. *The Cases* $n=3,6,9$ *and* 12

Lemma 1. *When* $n=3$, *then only solutions to equation* (1.2) *are*

$$\begin{aligned}&(57,16,1,2),(797,86,1,2),(4229,284,3,4),(3093,478,7,2),\\&(4,3,0,1),(58,15,0,1),(2,5,0,2),(9324,443,0,3),\\&(1,2,1,0),(181,32,1,0),(524,65,2,0),(13,8,3,0);\end{aligned}\tag{2.1}$$

when $n=6$, *then only solution to equation* (1.2) *is* $(57,4,1,2)$; *when* $n=9$, *then only solution to equation* (1.2) *is* $(13,2,3,0)$; *when* $n=12$, *then only solution to equation* (1.2) *is* $(57,2,1,2)$.

Proof. Suppose $n=3$. Writing $\alpha=6k+\alpha_1$, $\beta=6l+\beta_1$ in (1.2) with $\alpha_1,\beta_1\in\{0,1,2,3,4,5\}$, we get that

$$\left(\frac{x}{7^{3k}11^{3l}},\frac{y}{7^{2k}11^{2l}}\right)$$

is an $S-$Integral point (X,Y) on the elliptic curve

$$X^2=Y^3-7^{\alpha_1}\cdot 11^{\beta_1},\tag{2.2}$$

where $S=\{7,11\}$ with the numerator of Y being coprime to 77, in view of the restriction $\gcd(x,y)=1$. Now we need to determine all the $\{7,11\}$-integral points on the above 36 elliptic curves. At this stage we note that in [33] Pethő, Zimmer, Gebel

and Herrmann developed a practical method for computing all $S-$Integral points on Weierstrass elliptic curve and their method has been implemented in MAGMA [11] as a routine under the name `SIntegralPoints`. The subroutine `SIntegralPoints` of MAGMA worked
without problems for all (α_1,β_1) except for $(\alpha_1,\beta_1)=(5,5)$. MAGMA determined the appropriate Mordell-Weil groups except this case and we deal with this exceptional case separately. By computations done for equation (2.2) when $n=3$, we obtain the following solutions for the $\{7,11\}-$integral points on the curves:

$$(1,0,0,0),(3,4,0,1),(15,58,0,1),(5,2,0,2),(11,0,0,3),(443,9324,0,3),$$
$$(2,1,1,0),(32,181,1,0),\ (478/49,3093/3431,2),(11,22,1,2),(16,57,1,2),$$
$$(1899062/117649,2338713355/40353607,1,2),(22,99,1,2),(86,797,1,2),$$
$$(88,825,1,2),(638,16115,1,2),(657547,533200074,1,2),(242,3751,1,4),$$
$$(65,524,2,0),(7,0,3,0),(8,13,3,0),(14,49,3,0),(28,147,3,0),$$
$$(154,1911,3,0),(77,0,3,3),(242,3025,3,4),(284,4229,3,4),$$
$$(1435907/49,1720637666/343,3,4).$$

We use the above points on the elliptic curves to find the corresponding solutions for equation (2.2). Identifying the coprime positive integers x and y from the above list, one obtains the solutions listed in (2.2) (note that not all of them lead to coprime values for x and y).

We give the details in case $(\alpha_1,\beta_1)=(5,5)$ of equation (2.2). Observe that if Y is even, then X is odd and $X^2+7^5 11^5\equiv 0 \pmod 8$, and hence $X^2\equiv 3 \pmod 8$, which is a contradiction. Therefore Y is always odd. We consider solutions such that X and Y are coprime.

Write $\mathbb{K}=\mathbb{Q}(i\sqrt{77})$. In this field, the primes $2,7,11$ (all primes dividing the discriminant $d_{\mathbb{K}}=4d$) ramify so there are prime ideals P_2,P_7,P_{11} such that $2\mathcal{O}_{\mathbb{K}}=P_2^2$, $7\mathcal{O}_{\mathbb{K}}=P_7^2$, $11\mathcal{O}_{\mathbb{K}}=P_{11}^2$ respectively. Now, we show that the ideals $(X+7^2 11^2\sqrt{77}i)\mathcal{O}_{\mathbb{K}}$ and $(X-7^2 11^2\sqrt{77}i)\mathcal{O}_{\mathbb{K}}$ are coprime in the ring of integers $\mathcal{O}_{\mathbb{K}}$. To show this, let us assume that the ideals $(X+7^2 11^2\sqrt{77}i)\mathcal{O}_{\mathbb{K}}$ and $(X-7^2 11^2\sqrt{77}i)\mathcal{O}_{\mathbb{K}}$ are not coprime. So, these ideals have a gcd that divides $2.7^2.11^2\sqrt{77}i$. Hence there is an ideal $P_2^a P_7^b P_{11}^c$ with $a\leq 2$, and $b,c\leq 5$. If $b>0$ then $7\mid X$. Hence $7\mid Y$, hence $7^3\mid X^2$, hence $7^2\mid X$, hence $7^4\mid Y^3$, hence $7^2\mid Y$, hence $7^5\mid X^2$, hence $7^3\mid X$. So, we have a contradiction as $7^6\mid X^2-Y^3$. Thus $b=0$. Similarly we can prove that $c=0$.

Now let $(X+7^2 11^2\sqrt{77}i)\mathcal{O}_{\mathbb{K}}=P_2^a\wp^3$ for some ideal $\wp$ not divisible by P_2, and $(X-7^2 11^2\sqrt{77}i)\mathcal{O}_{\mathbb{K}}=P_2^a\wp'^3$ (for its conjugate ideal). If we take norms, then we get that $y^3=2^a[N_{\mathbb{K}}(\wp)]^3$, where $N_{\mathbb{K}}(\wp)$ is odd. It follows that $a=0$ (as it could be at most 2). So, we showed that the ideals $(X+7^2 11^2\sqrt{77}i)\mathcal{O}_{\mathbb{K}}$ and $(X-$

$7^2 11^2\sqrt{77i})\mathcal{O}_{\mathbb{K}}$ are coprime. Equation (2.2) now implies that

$$(X + 7^2 11^2\sqrt{77i})\mathcal{O}_{\mathbb{K}} = \wp^3 \text{ and } (X - 7^2 11^2\sqrt{77i})\mathcal{O}_{\mathbb{K}} = \wp'^3$$

for the ideals $\wp$ and $\wp'$. Let $h(\mathbb{K})$ be the class number of the field $\mathbb{K}$, then $\delta^{h(\mathbb{K})}$ is principal for any ideal δ. Note that, $h(\mathbb{K}) = 8$ and so $(3, h(\mathbb{K})) = 1$. Thus since $\wp^3$ and $\wp'^3$ are principal, $\wp$ and $\wp'$ are also principal. Moreover, since the units of $\mathbb{Q}(i\sqrt{77})$ are 1 and -1, which are both cubes, we conclude that

$$(X + 7^2 11^2\sqrt{77i}) = (u + \sqrt{77i}\, v)^3 \tag{2.3}$$

$$(X - 7^2 11^2\sqrt{77i}) = (u - \sqrt{77i}\, v)^3 \tag{2.4}$$

for some integers u and v. After subtracting the conjugate equation we obtain

$$7^2 \cdot 11^2 = v(3u^2 v - 77v^2). \tag{2.5}$$

Since u and v are coprime, we have the following possibilities in equation (2.5)

$$v = \pm 1;\ v = \pm 7^2;\ v = \pm 11^2;\ v = \pm 7^2 11^2$$

All cases lead to the conclusion that no solution is obtained.

For $n = 6$, equation

$$x^2 + 7^\alpha \cdot 11^\beta = y^6$$

becomes equation

$$x^2 + 7^\alpha \cdot 11^\beta = (y^2)^3.$$

Again, here we look in the list of solutions of equation (2.1) and observe that the only solution whose y is a perfect square is $(57, 16, 1, 2)$.Therefore the only solution to equation (1.2) is $(57, 4, 1, 2)$. In the same way, one can see that the value of y above which is a perfect square is $y = 4$ for the solution $(57, 4, 1, 2)$, therefore the only solution with $n = 12$ is $(57, 2, 1, 2)$.

For $n = 9$, equation

$$x^2 + 7^\alpha \cdot 11^\beta = y^9$$

becomes equation

$$x^2 + 7^\alpha \cdot 11^\beta = (y^3)^3.$$

Again here, we look in the list of solutions of (2.1) and observe that only solution whose y is a perfect cube is $(13, 8, 3, 0)$.Therefore the only solution to equation (1.2) is $(13, 2, 3, 0)$.This completes the proof of lemma. □

If (x, y, α, β, n) is a solution of the Diophantine equation (1.2) and d is any proper divisor of n, then $(x, y^d, \alpha, \beta, n/d)$ is also a solution of the same equation. Since $n > 3$ and we have already dealt with case $n = 3$, it follows that it suffices to look at the solutions n for which $p \mid n$ for some odd prime p. In this case, we may certainly replace n by p, and thus assume for the rest of the paper that $n \in \{4, p\}$.

2.2. *The Case* $n=4$

Lemma 2. *The only solutions with* $n=4$ *of the Diophantine equation* (1.2) *are given by*

$$(x,y,\alpha,\beta)=(2,3,1,1),(57,8,1,2),(8343,92,5,2),(3,2,1,0),(24,5,2,0)$$

Proof. Suppose that $n=4$. Rewrite equation (1.2) as

$$7^\alpha\cdot 11^\beta=(y^2+x)(y^2-x). \tag{2.6}$$

From equation (2.6), we have that

$$y^2+x=7^{a_1}.11^{b_1}$$
$$y^2-x=7^{a_2}.11^{b_2}$$

where $a_1,a_2,b_1,b_2\geq 0$. Then we get that

$$2y^2=7^{a_1}.11^{b_1}+7^{a_2}.11^{b_2}$$

from the sum of two equations. We multiply the above equation by 2 and we can write the equation

$$Z^2=2.(7^{a_1}.11^{b_1}+7^{a_2}.11^{b_2}) \tag{2.7}$$

as

$$2U+2V=Z^2 \tag{2.8}$$

where $Z=2y$, $U=7^{a_1}.11^{b_1}$ and $V=7^{a_2}.11^{b_2}$.

Let $p_1,p_2,...,p_s$ $(s\geq 1)$ be fixed distinct primes. The set of $S-$Units is defined as $S=\left\{\pm p_1^{x_1}p_2^{x_2}...p_s^{x_s}\mid x_i\in\mathbb{Z},\text{ for } i=1...k\right\}$. Let $a,b\in\mathbb{Q}-\{0\}$ be fixed. In [19], B.M.M. de Weger dealt with the solutions of the Diophantine equation $ax+by=z^2$, in $a,b\in S$, $z\in\mathbb{Q}$. He showed that this equation has essentially only finitely many solutions. Moreover, he indicated how to find all the solutions of this equation for any given set of parameters a,b, $p_1,...,p_s$. The tools are the theory of p-adic linear forms in logarithms, and a computational p-adic diophantine approximation method. He actually performed all the necessary computations for solving (2.8) completely for $p_1,...,p_s=2,3,5,7$ and $a=b=1$, and reported on this elsewhere (see [18], Chapter 7). Then we can find all the solutions of the Diophantine equation (2.7). But this requires a lot of additional manual effort. To solve the equation $x^2+7^\alpha\cdot 11^\beta=y^4$ instead of this method, we prefer using MAGMA (see [11]).

Writing in (1.2) $\alpha=4k+\alpha_1$, $\beta=4l+\beta_1$ with $\alpha_1,\beta_1\in\{0,1,2,3\}$ we get that

$$\left(\frac{x}{7^{2k}11^{2l}},\frac{y}{7^{2k}11^{2l}}\right)$$

is an $S-$Integral point (X,Y) on the hyperelliptic curve

$$X^2=Y^4-7^{\alpha_1}\cdot 11^{\beta_1}, \tag{2.9}$$

where $S = \{7, 11\}$ with the numerator of Y being prime to 77, in view of the restriction $\gcd(x, y) = 1$. We use the subroutine `SIntegralLjunggrenPoints` of MAGMA to determine the $\{7, 11\}$-integral points on the above hyperelliptic curves and we only find the following solutions

$$(X, Y, \alpha_1, \beta_1) = \{(1,0,0,0), (2,3,1,0), (3,2,1,1), (8,57,1,2),$$
$$(92/7, 8343/49, 1, 2), (5,24,2,0)\}$$

With the conditions on x and y and the definition of X, Y, one can obtain the solutions listed in the statement of the lemma. □

2.3. *The Case $n > 4$ and Prime*

Lemma 3. *The Diophantine equation (1.2) has no solutions with $n > 4$ prime except possibly for α and x are odd and β even.*

Proof. Since in section 2 we have finished the study of equation $x^2 + 7^\alpha \cdot 11^\beta = y^n$ with $n = 3$, we can assume that n is a prime > 4. One can write the Diophantine equation (1.2) as $x^2 + dz^2 = y^n$, where

$$d \in \{1,\ 7,\ 11,\ 77\},\ \ z = 7^{\alpha_1} \cdot 11^{\beta_1} \tag{2.10}$$

the relation of α_1 and β_1 with α and β, respectively, is clear. If x is odd, then by z also being odd we have that y is even, so $y^n \equiv 0 \pmod 8$. As $x^2 = z^2 \equiv 1 \pmod 8$ we have $1 + d \equiv 0 \pmod 8$, so $d = 7$, implying $\alpha \equiv 1 \pmod 2$ and $\beta \equiv 0 \pmod 2$. This case is excluded in the lemma. Hence we have that x is even, and y is odd. We study in the field $\mathbb{K} = \mathbb{Q}(i\sqrt{d})$. As $\gcd(x, z) = 1$ standard argument tells us now that in $\mathbb{K}$ we have

$$(x + i\sqrt{d}z)(x - i\sqrt{d}z) = y^n, \tag{2.11}$$

where the ideals generated by $x + iz\sqrt{d}$ and $x - iz\sqrt{d}$ are coprime in $\mathbb{K}$. Hence, we obtain the ideal equation

$$\langle x + i\sqrt{d}z\rangle = \theta^n \tag{2.12}$$

Then, since the ideal class number of $\mathbb{K}$ is 1 or 8, and n is odd, we conclude that the ideal θ is principal. The cardinality of the group of units of $\mathcal{O}_\mathbb{K}$ is 2 or 4, all coprime to n. Furthermore, $\{1, i\sqrt{d}\}$ is always an integral base for $\mathcal{O}_\mathbb{K}$ except for when $d = 7$, and $d = 11$, in which cases an integral basis for $\mathcal{O}_\mathbb{K}$ is $\{1, (1 + i\sqrt{d})/2\}$. Thus, we may assume that

$$x + i\sqrt{d}z = \varphi^n,\ \varphi = \frac{u + i\sqrt{d}v}{2} \tag{2.13}$$

the relation holds with some algebraic integer $\varphi \in \mathcal{O}_\mathbb{K}$. The algebraic integers in this number field are of the form $\varphi = \frac{u+i\sqrt{d}v}{2}$, where $u, v \in \mathbb{Z}$, with u, v both even, if $d = 1, 77$ and u, v both odd if $d = 7, 11$. Note that

$$\varphi - \overline{\varphi} = vi\sqrt{d},\ \varphi + \overline{\varphi} = i\sqrt{d}v,\ \varphi\overline{\varphi} = \frac{u^2 + dv^2}{4}$$

We thus obtain

$$\frac{2\cdot 7^{\alpha_1}\cdot 11^{\beta_1}}{v}=\frac{2z}{v}=\frac{\varphi^n-\overline{\varphi}^n}{\varphi-\overline{\varphi}}\in\mathbb{Z}. \tag{2.14}$$

Let $(L_m)_{m\geq 0}$ be the sequence with general term $L_m=(\varphi^m-\overline{\varphi}^m)/(\varphi-\overline{\varphi})$ for all $m\geq 0$. This is called a *Lucas sequence*. Note that

$$L_0=0, L_1=1 \text{ and } L_m=uL_{m-1}-\frac{u^2+dv^2}{4}L_{m-2},\ m\geq 2. \tag{2.15}$$

Following the nowadays standard strategy based on the important paper [10], we distinguish two cases according as L_n has or has not primitive divisors.

Suppose first that L_n has a primitive divisor, say q. By definition, this means that the prime q divides L_n and q does not divide $(\mu-\overline{\mu})^2L_1...L_{n-1}$, hence

$$q\nmid(\varphi-\overline{\varphi})^2L_1...L_4=(dv^2).u.\frac{3u^2-dv^2}{4}.\frac{u^2-dv^2}{2}. \tag{2.16}$$

If $q=2$, then (2.16) implies that uv is odd, hence $d=11$ or 77. If $d=11$, then third factor in the right hand-most side of (2.16) is even, a contradiction. If $d=77$, then, from (2.15) we see that $L_m\equiv L_{m-1}\pmod 2$, hence L_m is odd for every $m\geq 1$, implying that 2 cannot be a primitive divisor of L_n.

If $q=7$, then (2.16) implies that $d=1,11$ and 7 does not divide $uv(3u^2-dv^2)(u^2-dv^2)$. It follows easily then that $v^2\equiv -du^2\pmod 7$, so that, by (2.15), $L_m\equiv uL_{m-1}\pmod 8$ for every $m\geq 2$. Therefore, $7\nmid L_n$, so that 7 can not be a prime divisor of L_n.

If $q=11$, then by (2.16), $d=1$ or 7. If $d=1$ then we write $u=2v_1, v=2v_1$ with $u_1,v_1\in\mathbb{Z}$, so that $\varphi=u_1+i\sqrt{d}v_1$ and (2.16) becomes $q\nmid u_1v_1(3u_1^2-dv_1^2)(u_1^2-dv_1^2)$. Moreover, $L_m=2u_1L_{m-1}-(u_1^2+dv_1^2)L_{m-2}$ for $m\geq 2$. Note that $\varphi\overline{\varphi}=u_1^2+dv_1^2\neq 0\pmod 8$; therefore, by corollary 2.2 of [10], there exists a positive integer m_{11} such that $11\mid L_{m_{11}}$ and $m_{11}\mid m$ for every m such that $11\mid L_m$. It follows then that $11\mid\gcd(L_n,L_{m_{11}})=L_{\gcd(n,m_{11})}$. Because of the minimality property of m_{11}, we conclude that $\gcd(n,m_{11})$, hence, since n is a prime, $m_{11}=n$. On the other hand, the Legendre symbol $\left(\frac{(\varphi-\overline{\varphi})^2}{11}\right)=-1$, hence by Theorem XII of [15] (or by theorem 2.2.4 (iv) of [26]), $11\mid L_{12}$. Therefore $m_{11}\mid 12$, i.e. $n\mid 12$, a contradiction, since n is a prime≥ 5. If $d=7$, then (2.16) implies $11\nmid u_1v_1(3u_1^2-dv_1^2)(u_1^2-dv_1^2)$. Moreover, $L_m=2u_1L_{m-1}-(u_1^2+dv_1^2)L_{m-2}$ for $m\geq 2$. Note that $\varphi\overline{\varphi}=u_1^2+dv_1^2\neq 0\pmod 8$; therefore, by corollary 2.2 of [10], there exists a positive integer m_{11} such that $11\mid L_{m_{11}}$ and $m_{11}\mid m$ for every m such that $11\mid L_m$.It follows then that $11\mid\gcd(L_n,L_{m_{11}})=L_{\gcd(n,m_{11})}$. Because of the minimality property of m_{11}, we conclude that $\gcd(n,m_{11})$, hence, since n is a prime, $m_{11}=n$. On the other hand, the Legendre symbol $\left(\frac{(\varphi-\overline{\varphi})^2}{11}\right)=1$, hence by Theorem XII of [15]

(or by theorem 2.2.4 (iii) of [26]), $11 \mid L_{10}$. Therefore $m_{11} \mid 10$, i.e. $n \mid 10$. Since $n \geq 5$ is a prime, we get that $n = 5$.

We conclude that 11 is primitive divisor for $d = 7$.

In particular, u and v are integers. Since 11 is coprime to $-4dv^2 = -28v^2$, we get that $v = \pm 7^{\alpha_1}$. Since $y = u^2 + 7v^2$, we get that u is even.

In the case $v = \pm 7^{\alpha_1}$, equation (2.14) becomes

$$\pm 11^{\beta_1} = 5u^4 - 70u^2v^2 + 49v^4.$$

Since u is even, it follows that the right hand side of the last equation above is congruent to 1 (mod 8). So $\pm 11^{\beta_1} \equiv 1 \pmod 8$, showing that the sign on the left hand side is positive and β_1 is odd, or the sign on the left hand side is negative and β_1 is even.

Assume first that $\beta_1 = 2\beta_0 + 1$ be odd. We get

$$11V^2 = 5U^4 - 70U^2 + 49,$$

where $(U, V) = (u/v, 11^{\beta_0}/v^2)$ is a $\{7\}$-integral point on the above elliptic curve. We get that the only such points on the above curve are $(U, V) = (\pm 7, \pm 28)$. This does not lead to solutions of our original equation.

Assume now that $\beta_1 = 2\beta_0$ is even and we get that

$$V^2 = 5U^4 - 70U^2 + 49,$$

where $(U, V) = (u/v, 11^{\beta_0}/v^2)$ is a $\{7\}$-integral point on the above elliptic curve. With MAGMA, we get that the only such point on the above curve are $(U, V) = (0, 7)$. This does not lead to solutions of our original equation.

We now recall that a particular instance of the Primitive Divisor Theorem for Lucas sequences implies that, if $n \geq 5$ is prime, then L_n always has a prime factor except for finitely many *exceptional triples* $(\varphi, \overline{\varphi}, n)$, and all of them appear in the Table 1 in [10] (see also [1]). These exceptional Lucas numbers are called *defective*.

Let us assume that we are dealing with a number L_n without primitive divisors. Then a quick look at Table 1 in [10] reveals that this is impossible. Indeed, all exceptional triples have $n = 5, 7$ or 13. The defective Lucas numbers whose roots are in $\mathbb{K} = \mathbb{Q}(i\sqrt{d})$ with $d = 7$ and $n = 5, 7$ or 13 appearing in the list (2.10) is $(\varphi,\ \overline{\varphi}) = ((1 + i\sqrt{7})/2,\ (1 - i\sqrt{7})/2)$ for which $L_7 = 7,\ L_{13} = -1$. Furthermore, with such a value for φ we get that $y = |\varphi|^2 = 2$. However, this is not convenient since for us x and y are coprime so y cannot be even. For $n = 5$ and $d = 11$,we get $L_5 = 1$ and $y = 3$ with $(\varphi,\ \overline{\varphi}) = ((1 + i\sqrt{11})/2,\ (1 - i\sqrt{11})/2)$. Therefore the equation is $x^2 + C = 3^5$, where $C = 7^{\alpha} \cdot 11^{\beta}$, with a even and b odd. Since $11^3 > 3^5$, we have $b = 1$, and next that $a = 0$. But it doesn't yield an integer value for x. The proof is completed. □

Remark 2. We mention here why the method applied for the proof of Lemma 3 does not apply when α and x are odd, β is even. In this case $d = 7$, the class

number of $\mathbb{Q}(\sqrt{7}i)$ is 1. With $\omega = \frac{1+\sqrt{7}i}{2}$ a prime dividing 2, and ω' its conjugate, let us now write $(x + z\sqrt{7}i) = \omega^b \omega^c \xi$, where ξ is an integer in $\mathbb{Q}(\sqrt{7}i)$ of odd norm, not divisible by 7 and ξ' its conjugate. As both x and z are odd and they are coprime, we may take $c = 1, b \geq 1$. Taking norms we get $y^n = 2^{b+1}\xi\xi'$, and it easily follows that $\xi = c^n$ and $b + 1 = k.n$. Now we take $\varphi = 2^{k-1}c$, $\wp = 2\omega^{n-2}$, and then we have $x + z\sqrt{7}i = \wp\varphi^n$. A way to look at the rest of argument why this case is essentially different from the primitive divisors in Lucas sequences thing: From $x + z\sqrt{7}i = \wp\varphi^n$ and its conjugate it follows that

$$z = \frac{\wp\varphi^n - \overline{\wp}\,\overline{\varphi}^n}{2\sqrt{7}i}$$

If $\wp$ is in $\mathbb{Q}$ then the right hand side is the n-th term of a Lucas sequence. As z has a very nice prime factorization $7^p 11^q$ then theory of primitive divisors will work. But in our case $\wp$ is not in $\mathbb{Q}$. Hence the right side, while it is the n-th term of a recurrence sequence, this is not a Lucas sequence, and does not have the nice divisibility properties of Lucas sequences. That's why the method of [10] fails in our case.

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ON MINIMAL SOLUTIONS OF SYSTEMS OF LINEAR EQUATIONS WITH APPLICATIONS

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Abstract. We give a thorough investigation of the structure of solution sets of both homogeneous and inhomogeneous systems of linear equations, from the viewpoint of their *minimal solutions* (which use minimal sets of columns of the coefficient matrix). We also discuss applications in chemistry (stoichiometry).

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1. INTRODUCTION

Systems of linear equations have an enormous amount of applications in many areas of science and technology. For our present research we took chemistry (stoichiometry) as a direct motivation; these applications are outlined in the last section. But beside them and beyond pure theoretical interest, it is plausible to guess that our results can be applied in further areas, too, e.g. in physics (dimensionless groups).

The main targets of our research are the *minimal solutions* of systems of linear equations. We call a solution *minimal* if it uses a minimal set of column vectors of the coefficient matrix; see Definition 1. (The difference and connections between minimal and *base* solutions are explained in Remark 3.)

The structure of the set of minimal solutions is revealed in Proposition 3 and in Propositions 7 through 8, the connection between minimal and other solutions (without any restrictions) is dealt with in Proposition 5 and Theorems 2 and 6.

The results obtained in Theorem 4 and in Theorem 8 have a crucial effect in stoichiometry: all stoichiometrical reactions can be obtained as linear combinations of *minimal* ones. This result could also simplify the research question raised in Section 5 concerning "*The second level*" in the hierarchy of equations.

Some related notions, their combinatorial and geometrical aspects are surveyed in [7].

The sections below give thorough extensions of A. Pethő's results in [3].

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2. Basic properties

Let us consider the systems of linear equations

$$A \cdot \underline{x} = \underline{b} \tag{2.1}$$

and

$$A \cdot \underline{x} = \underline{0} \tag{2.2}$$

where $A \in \mathbb{R}^{n \times m}$, $\underline{b} \in \mathbb{R}^n$ are given, and the solution vectors are $\underline{x} \in \mathbb{R}^m$. We investigate the solution sets

$$M_{A,\underline{b}} := \{\underline{x} \in \mathbb{R}^m : A \cdot \underline{x} = \underline{b}\} \tag{2.3}$$

and

$$M_{A,\underline{0}} := \{\underline{x} \in \mathbb{R}^m : A \cdot \underline{x} = \underline{0}\} \ . \tag{2.4}$$

Clearly the cases $M_{A,\underline{0}} \neq \{\underline{0}\}$ and $\left|M_{A,\underline{b}}\right| > 1$ are the interesting ones.

We denote the column vectors of A by $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_m \in \mathbb{R}^n$, that is

$$A = \left[\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_m\right] \ . \tag{2.5}$$

The assumptions below are not only for simplicity, they have real effect in stoichiometry.

Condition 1. It will be assumed that

i) A does not contain parallel column vectors,

especially

ii) A does not contain $\underline{0}$ as a column vector.

In the inhomogeneous case ($\underline{b} \neq \underline{0}$) we also assume that

iii) A does not contain a column vector parallel to $\underline{b}$.

We are interested in the structure of the *sets of column vectors* of A which effectively take part in the equality (2.1).

Definition 1. **(i)** For any vector $\underline{x} \in \mathbb{R}^m$ we write

$$\operatorname{supp}(\underline{x}) := \{i \leq m : x_i \neq 0\} \tag{2.6}$$

and call it the **support** of $\underline{x}$. (Especially $\operatorname{supp}(\underline{0}) = \varnothing$.)

(ii) For any set of vectors $M \subseteq \mathbb{R}^m$ we say that a *nonzero* vector $\underline{z} \in M$ has **minimal support** for M if the set $\operatorname{supp}(\underline{z})$ is minimal, that is *there exists no* other nonzero vector $\underline{y} \in M$ for which $\operatorname{supp}(\underline{y}) \subsetneqq \operatorname{supp}(\underline{z})$ holds.

We will call such (nonzero) vectors $\underline{z} \in M$ shortly **minimal** (for M).

(iii) For *any set* of vectors $M \subseteq \mathbb{R}^m$ we denote by $M^{\min}$ the set of all minimal (for M) vectors in M, that is

$$M^{\min} := \left\{\underline{z} \in M : \underline{z} \text{ is minimal}\right\}. \tag{2.7}$$

(iv) Clearly $M^{\min}_{A,\underline{b}}$ and $M^{\min}_{A,\underline{0}}$ stand for the sets of minimal elements in $M_{A,\underline{b}}$ and $M_{A,\underline{0}}$, respectively. We call the vectors in $M^{\min}_{A,\underline{b}}$ and $M^{\min}_{A,\underline{0}}$ the **minimal solutions** of the equalities (2.1) and (2.2).

Clearly, when solving (2.1) or (2.2), we only need those column vectors of A which correspond to $\operatorname{supp}(\underline{x})$. This is discussed in details in Definition 2 and Proposition 5 below.

We explain the difference between the popular *base solutions* and our *minimal solutions* in Remark 3.

The following observations are obvious; we extensively use them later without further mentioning.

Proposition 1. *For the three parts of Condition 1 the following equivalences hold. Considering the homogeneous equation (2.2) we have:*

i) $\iff$ $|\operatorname{supp}(\underline{x})| \geq 3$ *for all* $\underline{x} \in M_{A,\underline{0}} \setminus \{\underline{0}\}$,

ii) $\iff$ $|\operatorname{supp}(\underline{x})| \geq 2$ *for all* $\underline{x} \in M_{A,\underline{0}} \setminus \{\underline{0}\}$.

For inhomogeneous equations (2.1) ($\underline{b} \neq \underline{0}$) we have:

iii) $\iff$ $|\operatorname{supp}(\underline{x})| \geq 2$ *for all* $\underline{x} \in M_{A,\underline{b}}$.

Proposition 2. *For* any nontrivial *solution* $\underline{x} \in M_{A,\underline{0}}$ *of* (2.2) *there is a* minimal *solution* $\underline{z} \in M^{\min}_{A,\underline{0}}$ *such that*

$$\operatorname{supp}(\underline{z}) \subseteqq \operatorname{supp}(\underline{x}) \ . \tag{2.8}$$

The same is valid for (2.1) *with* $\underline{x} \in M_{A,\underline{b}}$ *and* $\underline{z} \in M^{\min}_{A,\underline{b}}$.

Proposition 3. *For any set of vectors* $M \subseteq \mathbb{R}^m$ *the sets* $\operatorname{supp}(\underline{z}) \subseteqq \{1,2,\ldots,m\}$ *for* minimal *vectors* $\underline{z} \in M^{\min}$ *form a* Sperner system, *that is for any two (distinct) minimal vectors* $\underline{z}_1, \underline{z}_2 \in M$ *we have* $\operatorname{supp}(\underline{z}_1) \subsetneqq \operatorname{supp}(\underline{z}_2)$ *and* $\operatorname{supp}(\underline{z}_2) \subsetneqq \operatorname{supp}(\underline{z}_1)$.

We want to extract and investigate the nonzero components of the solution vectors $\underline{x}$ and the corresponding column vectors of A . To simplify our later discussion we introduce one more notion.

Definition 2. For an *arbitrary* vector $\underline{x} \in \mathbb{R}^m$, matrix $A \in \mathbb{R}^{n\times m}$, and set $H \subseteq \{1,\ldots,m\}$ of indices we define the **restrictions** of $\underline{x}$ and A to the set H as

$$\underline{x}\,|_H := [x_i : i \in H] \tag{2.9}$$

and

$$A\,|_H := \left[\underline{a}_i : i \in H\right] , \tag{2.10}$$

so $\underline{x}\,|_H \in \mathbb{R}^h$ and $A\,|_H \in \mathbb{R}^{n\times h}$ where $h = |H|$.

We clearly have

Proposition 4. *For* any (fixed) *vector* $\underline{x} \in \mathbb{R}^m$ *the set-function*

$$\begin{aligned} \mu_{\underline{x}} &: \mathcal{P}\{1,\dots,m\} \longrightarrow \mathbb{R}^n \\ \mu_{\underline{x}} &: H \longmapsto (A\,|_H)\cdot(\underline{x}\,|_H) \end{aligned} \tag{2.11}$$

for $H \subseteqq \{1,\dots,m\}$ *is additive.*

The following correspondence between $\underline{x}\,|_{\mathrm{supp}(\underline{x})}$ and $\underline{x}$ is trivial.

Proposition 5. *If* $\underline{x} \in \mathbb{R}^m$ *is* any solution *of the equation* $A\cdot\underline{x} = \underline{b}$ *then* $\underline{x}\,|_{\mathrm{supp}(\underline{x})}$ *satisfies the equality*

$$\left(A\,|_{\mathrm{supp}(\underline{x})}\right)\cdot\left(\underline{x}\,|_{\mathrm{supp}(\underline{x})}\right) = \underline{b} \quad . \tag{2.12}$$

On the other hand, for any subset $H \subseteq \{1,\dots,m\}$ *and solution* $\underline{y} \in \mathbb{R}^h$ $(h = |H|)$ *of the equality*

$$(A\,|_H)\cdot\underline{y} = \underline{b} \tag{2.13}$$

there is at least one *solution* $\underline{x} \in \mathbb{R}^m$ *of the equation* $A\cdot\underline{x} = \underline{b}$ (2.1) *such that*

$$\underline{y} = \underline{x}\,|_H \quad . \tag{2.14}$$

Especially, $\mathrm{supp}\,(\underline{x}) \subseteqq H$ *can be assumed.*

Note that the solution $\underline{x}$ above is not unique in general.

Proposition 5 will be extended for homogeneous equations in Theorem 2 and for inhomogeneous ones in Theorem 6 below. It is interesting that these results are very different.

3. Homogeneous equations

We start with the extension of Proposition 5 for homogeneous equations.

Theorem 2. *Let* $\underline{z} \in M^{\min}_{A,\underline{0}}$ *be any minimal solution of* (2.2), $\underline{z} \neq \underline{0}$. *Then the equation*

$$\left(A\,|_{\mathrm{supp}(\underline{z})}\right)\cdot\underline{y} = \underline{0} \tag{3.1}$$

for $\underline{y} \in \mathbb{R}^h$ $\left(h = \left|\mathrm{supp}\left(\underline{z}\right)\right|\right)$ *has* the only *solutions*

$$\underline{y} = \lambda\cdot\underline{z}\,|_{\mathrm{supp}(\underline{z})} \tag{3.2}$$

where $\lambda \in \mathbb{R}$ *is any number.*

Proof. Clearly $\underline{y}_1 := \underline{z}\,|_{\mathrm{supp}(\underline{z})} \in \mathbb{R}^h$ is a solution of (3.1), in which no coordinate is 0 . Let further $\underline{y}_2 \in \mathbb{R}^h$ be any nontrivial solution of (3.1), *distinct* from $\lambda\cdot\underline{z}$ for all $\lambda \in \mathbb{R}$. Since $\underline{y}_2 \neq \underline{0}$, the ith coordinate of $\underline{y}_2$ is nonzero for some $i \in \mathrm{supp}\left(\underline{z}\right)$, so we can find some $\nu \in \mathbb{R}$ such that the ith coordinate of the vector

$$\underline{z}_2 := \underline{y}_1 - \nu\cdot\underline{y}_2$$

is zero.

If $\underline{z}_2 = \underline{0}$ then $\underline{y}_1$ and $\underline{y}_2$ are parallel, i.e. (3.2) holds.

If $\underline{z}_2 \neq \underline{0}$ then $\underline{z}_2 \in \mathbb{R}^h$ has $\operatorname{supp}(\underline{z}_2) \subsetneqq \operatorname{supp}(\underline{z})$ which is a contradiction since $\underline{z}$ was assumed to be minimal. □

Remark 1. We consider the solutions $\underline{x}$ and $\lambda\underline{x}$ **identical** for all $\lambda \in \mathbb{R}$, but throughout our investigation we do *not* call $\underline{0}$ a solution of (2.2). According to Theorem 2 we can say that *the solution* of (3.1), for any minimal $\underline{z} \in M^{\min}_{A,\underline{0}}$, is **unique** (i.e. up to a scalar multiplier).

Our main problem for both homogeneous and inhomogeneous equations $A \cdot \underline{x} = \underline{0}$ and $A \cdot \underline{x} = \underline{b}$ are the same. We first formulate it for the homogeneous case.

Problem 3. Can *all* solutions of the homogeneous equation $A \cdot \underline{x} = \underline{0}$ be generated from the minimal solutions? In other words: does $M^{\min}_{A,\underline{0}} \subset \mathbb{R}^m$ generate $M_{A,\underline{0}}$ with linear combinations?

The inhomogeneous version of this problem will be Problem 7.

Theorem 4. *$M^{\min}_{A,\underline{0}}$ generates $M_{A,\underline{0}}$ for all matrices $A \in \mathbb{R}^{n\times m}$.*

Proof. We have to show that each vector $\underline{x} \in M_{A,\underline{0}}$ (solution of (2.2)) is a linear combination of vectors from $M^{\min}_{A,\underline{0}}$. We proceed by induction on the *size* of $\operatorname{supp}(\underline{x})$; here $\underline{x} \neq \underline{0}$ can clearly be assumed. In the case $\underline{x} \in M^{\min}_{A,\underline{0}}$ we are done.

For $\underline{x} \notin M^{\min}_{A,\underline{0}}$ choose a minimal vector $\underline{z} \in M^{\min}_{A,\underline{0}}$ such that

$$\varnothing \neq \operatorname{supp}(\underline{z}) \subsetneqq \operatorname{supp}(\underline{x}) \ . \tag{3.3}$$

Let $k \in \operatorname{supp}(\underline{z})$ be any index and consider the vector

$$\underline{x}' := \underline{x} - \frac{x_k}{z_k} \cdot \underline{z} \ , \tag{3.4}$$

which is also a solution of (2.2).

By the definition of $\operatorname{supp}(\underline{z})$ we have $z_k \neq 0$, $x'_k = 0$ (i.e. $k \notin \operatorname{supp}(\underline{x}')$), and so

$$\operatorname{supp}(\underline{x}') \subsetneqq \operatorname{supp}(\underline{x}) \ . \tag{3.5}$$

Clearly $\underline{x}' \neq \underline{0}$ since $\underline{x}' = \underline{0}$ would imply $\underline{x} \parallel \underline{z}$ but $\underline{x} \notin M^{\min}_{A,\underline{0}}$ was assumed.

Finally, by the induction hypothesis we know that $\underline{x}'$ is a linear combination of vectors from $M^{\min}_{A,\underline{0}}$. This fact together with (3.4) and $\underline{z} \in M^{\min}_{A,\underline{0}}$ implies that $\underline{x}$ is also a linear combination of minimal vectors. □

The proof given above implies that (the supports of) *all solutions* $\underline{x} \in M_{A,\underline{0}}$ are covered by *minimal* solutions:

$$\operatorname{supp}(\underline{x}) \subseteqq \bigcup \left\{ \operatorname{supp}(\underline{z}) : \underline{z} \in M^{\min}_{A,\underline{0}} \right\} \tag{3.6}$$

what does not follow from Proposition 2.

However, not every *column* of the coefficient matrix A takes part in *any* solutions (reactions in stoichiometry); this is explained in the following assertion.

Proposition 6. *For any index $i \leq m$ the column vector $\underline{a}_i$ takes part in some solution $\underline{x} \in M_{A,\underline{0}}$ (that is, $i \in \operatorname{supp}(\underline{x})$) if and only if $\underline{a}_i$ is linearly* dependent *on the other column vectors $\{\underline{a}_1, \ldots, \underline{a}_{i-1}, \underline{a}_{i+1}, \ldots, \underline{a}_m\}$.*

Proof. For any solution vector $\underline{x}$, write the equality $A\underline{x} = \underline{0}$ as

$$x_i \cdot \underline{a}_i = -\sum_{j \neq i} x_j \cdot \underline{a}_j \ . \tag{3.7}$$

The existence of a solution $\underline{x}$ satisfying $i \in supp(\underline{x})$ is equivalent to the solvability of (3.7) with $x_i \neq 0$. This exactly means the dependency of $\underline{a}_i$ on the vectors $\{\underline{a}_1, \ldots, \underline{a}_{i-1}, \underline{a}_{i+1}, \ldots, \underline{a}_m\}$, because $\underline{a}_i \neq \underline{0}$ has been assumed. □

Nonparallel elements of $M^{\min}_{A,\underline{0}}$ can be linearly dependent. One might ask for a base of $M^{\min}_{A,\underline{0}}$; we have not investigated this question yet.

Now we proceed with the investigation of the *inner* structure of the set of minimal solutions.

Proposition 7. *Let $\underline{z} \in M^{\min}_{A,\underline{0}}$, i.e. $\underline{z}$ is a* minimal *solution of the equation $A \cdot \underline{z} = \underline{0}$. Then the set of column vectors $\underline{a}_i$ "used" by $\underline{z}$,*

$$S_{\underline{z}} := \{\underline{a}_i : i \in \operatorname{supp}(\underline{z})\} \subset \mathbb{R}^n$$

is a minimal *linearly dependent set.*

Proof. $S_{\underline{z}}$ is linearly dependent since $A \cdot \underline{z} = \underline{0}$ is in fact a linear combination of the elements of $S_{\underline{z}}$.

If some proper subset $T \subset S_{\underline{z}}$ was linearly dependent then $\underline{z}$ would not be minimal. □

This result explains the following notion.

Definition 3. A set $S \subset \mathbb{R}^n$ is called a **linear algebraic simplex** if S is minimal (linearly) dependent, that is S itself is dependent but all its proper subsets $T \subset S$ are independent.

Remark 2. The term "simplex" is also used in Euclidian and in affine geometry with different meanings. In the present paper we deal only with linear algebraic ones but always use the attribute "linear" to avoid confusion. Several notions of simplexes and their applications can be found in [7].

Remark 3. Let us now clarify the *difference* between **base solutions** and **minimal solutions** of a system of (linear) equations (2.1) and (2.2).

For *inhomogeneous* equations (2.1) the base solutions correspond to *bases* of A , i.e. r independent column vectors of A where $r = rank(A)$. The coefficients for

resulting $\underline{b}$ when using a base are always uniquely determined. So, a base solution is minimal exactly when it is *nondegenerate*.

For *homogeneous* equations (2.2) (according to [3]) the support of a base solution contains a base of A (as in the previous paragraph) plus exactly one from the remaining column vectors of A . So, in the homogeneous case, all base solutions have size exactly $r+1$, contain dependent column vectors of A , but are not necessarily minimal (a simplex). On the other hand, the set of column vectors of A corresponding to the support of a minimal solution always forms a minimal dependent set (a simplex) but may have size less than $r+1$.

The following characterization is fundamental both in theory and in applications of (linear algebraic) simplexes.

Theorem 5. *A (nonempty) set $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\} \subset \mathbb{R}^n$ of vectors is a linear algebraic simplex if and only if there exists a linear combination*

$$\gamma_1 \underline{v}_1 + \gamma_2 \underline{v}_2 + \cdots + \gamma_k \underline{v}_k = \underline{0}\,, \tag{3.8}$$

and *for* each *linear combination* (3.8) *we must have $\gamma_i \neq 0$ for* all $i \leq k$.
Moreover, the linear combination in (3.8) *is unique up to a constant factor; that is, for all linear combinations*

$$\gamma'_1 \underline{v}_1 + \gamma'_2 \underline{v}_2 + \cdots + \gamma'_k \underline{v}_k = \underline{0} \tag{3.9}$$

we must have

$$\left[\gamma'_1, \gamma'_2, \ldots, \gamma'_k\right] = \lambda \cdot [\gamma_1, \gamma_2, \ldots, \gamma_k] \tag{3.10}$$

for some $\lambda \in \mathbb{R} \setminus \{0\}$.

Proof. The linear combination (3.8) exists since S is dependent. The minimality of S implies $\gamma_i \neq 0$ for all $i \leq k$.

Suppose now that (3.10) does not hold for any $\lambda \in \mathbb{R}$ for the two sequences $[\gamma_1, \gamma_2, \ldots, \gamma_k]$ and $\left[\gamma'_1, \gamma'_2, \ldots, \gamma'_k\right]$ satisfying (3.8) and (3.9), respectively. Then the linear combination

$$\gamma'_1 \cdot (3.8) - \gamma_1 \cdot (3.9)$$

does not contain $\underline{v}_1$, contradicting the minimality of S .

On the other hand, (3.8) implies that S is linearly dependent. Clearly, S is *not* minimal if and only if there is a linear combination (3.8) where $\gamma_i = 0$ for some $i \leq k$. So, the uniqueness of (3.8) (in the sense of (3.10)) together with the assumption " $\gamma_i \neq 0$ for all $i \leq k$ " ensures that S must be minimal linearly dependent. □

Using the result above, we can sharpen Proposition 5 for *minimal* solutions $\underline{z} \in M^{\min}_{A,\underline{0}}$ as follows. (See also Remark 1.)

Corollary 1. *For any* minimal solutions $\underline{z}$, $\underline{y} \in M^{\min}_{A,\underline{0}}$ *we have*

$$\operatorname{supp}(\underline{z}) = \operatorname{supp}\left(\underline{y}\right) \iff \underline{z} \parallel \underline{y} \quad , \tag{3.11}$$

that is each minimal solution $\underline{z}$ *is* unique *(up to a scalar multiplier) on its support.*

Proof. The statement follows from Theorem 2, Proposition 7 and Theorem 5. □

In Theorem 6 we extend the result above to *inhomogeneous* equations.
Theorem 5 also implies the following property of linear algebraic simplexes.

Proposition 8. *For any two simplexes* S_1 *and* S_2 *, for which* $S_1 \cap S_2 \neq \varnothing$ *holds, and for any vector* $\underline{w} \in S_1 \cap S_2$ *there is a simplex* S_3 *contained in* $(S_1 \cap S_2) \setminus \{\underline{w}\}$ *:*

$$S_3 \subseteqq (S_1 \cap S_2) \setminus \{\underline{w}\} \ . \tag{3.12}$$

In other words, if $\underline{z}_1$, $\underline{z}_2 \in M^{\min}_{A,\underline{0}}$ *are two* minimal solutions *of the equation* $A \cdot \underline{x} = \underline{0}$ (2.2) *such that* $\operatorname{supp}(\underline{z}_1) \cap \operatorname{supp}(\underline{z}_2) \neq \varnothing$ *, then for any* $j \in \operatorname{supp}(\underline{z}_1) \cap \operatorname{supp}(\underline{z}_2)$ *one can find a* minimal solution $\underline{z}_3 \in M^{\min}_{A,\underline{0}}$ *for which*

$$j \notin \operatorname{supp}(\underline{z}_3) \subseteq \operatorname{supp}(\underline{z}_1) \cap \operatorname{supp}(\underline{z}_2) \quad . \tag{3.13}$$

Proof. Consider the linear combinations

$$\gamma_1 \underline{v}_1 + \gamma_2 \underline{v}_2 + \cdots + \gamma_k \underline{v}_k = \underline{0} \tag{3.14}$$

and

$$\delta_1 \underline{u}_1 + \delta_2 \underline{u}_2 + \cdots + \delta_\ell \underline{u}_\ell = \underline{0} \tag{3.15}$$

where $S_1 = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ and $S_2 = \{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_\ell\}$. Now, by Theorem 5 we have $\gamma_j \neq 0$ and $\delta_j \neq 0$ (where $\underline{w} = \underline{v}_j = \underline{u}_j$) and so

$$\begin{aligned} & \frac{1}{\gamma_j}\left(\gamma_1 \underline{v}_1 + \gamma_2 \underline{v}_2 + \cdots + \gamma_k \underline{v}_k\right) - \frac{1}{\delta_j}\left(\delta_1 \underline{u}_1 + \delta_2 \underline{u}_2 + \cdots + \delta_\ell \underline{u}_\ell\right) \\ = \quad & \underline{w} - \underline{w} = \underline{0} \quad . \end{aligned} \tag{3.16}$$

This means that the set $S_1 \cap S_2 \setminus \{\underline{w}\}$ is linearly dependent, hence it must contain a simplex S_3 . □

4. Inhomogeneous equations

First we extend Proposition 5 and Corollary 1 to inhomogeneous equations.

Theorem 6. *Let* $\underline{z} \in M^{\min}_{A,\underline{b}}$ *be a* minimal *solution of the* inhomogeneous *equation* $A\underline{x} = \underline{b}$ $(\underline{b} \neq \underline{0})$ *and let* $H := \operatorname{supp}(\underline{z})$ *. Then the equation*

$$(A\,|_H) \cdot \underline{y} = \underline{b} \tag{4.1}$$

has the unique *solution* $\underline{y} = \underline{z}\,|_H$ *only.*

Proof. Let $\underline{y_1} := \underline{z}\,|_H \in \mathbb{R}^h \quad (h = |H|)$, in which no coordinate is 0 . Let further $\underline{y_2} \in \mathbb{R}^h$ be any *other* solution of (4.1), assuming that the ith coordinate of $\underline{y}_2$ is nonzero but distinct from the ith coordinate of $\underline{y}_1$ (for some $i \in \operatorname{supp}(\underline{z})$).

It is well-known that for each $\alpha \in \mathbb{R}$ the vectors

$$\underline{v} = \alpha \cdot \underline{y_1} + (1-\alpha) \cdot \underline{y_2} \tag{4.2}$$

satisfy (4.1). By the assumptions above we can choose an α such that the ith coordinate of $\underline{v}$ is equal to 0 . But then $\operatorname{supp}(\underline{v}) \subsetneqq \operatorname{supp}(\underline{z})$, contradicting the minimality of $\underline{z}$. □

Second, we extend Problem 3.

Problem 7. Can *all* solutions of the *in*homogeneous equation $A \cdot \underline{x} = \underline{b}$ be generated from the minimal solutions? In other words, does $M^{\min}_{A,\underline{b}}$ generate $M_{A,\underline{b}}$?

We can prove the following result.

Theorem 8. Each *solution vector* $\underline{x} \in M_{A,\underline{b}}$ *can be written as an affine combination of some elements of* $M^{\min}_{A,\underline{b}}$ *plus a solution of the homogeneous equation*

$$\underline{x} = \sum_{i=1}^{I} \alpha_i \underline{z}_i + \underline{y} \qquad \textit{where} \qquad \sum_{i=1}^{I} \alpha_i = 1 \,, \tag{4.3}$$

all $\underline{z}_i \in M^{\min}_{A,\underline{b}}$ *are minimal solutions* ($i = 1, \ldots, I$, $\alpha_i \in \mathbb{R}$), *and* $\underline{y} \in M_{A,\underline{0}} \cup \{\underline{0}\}$.

Let us emphasize that Theorem 8 extends the well known formula $M_{A,\underline{b}} = \underline{z} + M_{A,\underline{0}}$ for *minimal* solution vectors. Moreover, together with Theorem 4 it implies that $M^{\min}_{A,\underline{b}} \cup M^{\min}_{A,\underline{0}}$ generates $M_{A,\underline{b}}$.

Proof. Let $\underline{x} \in M_{A,\underline{b}}$ be any solution vector. We proceed by induction on the *size* of $\operatorname{supp}(\underline{x})$; $\underline{x} \neq \underline{0}$ can clearly be assumed.

In the case $\underline{x} \in M^{\min}_{A,\underline{b}}$ we are done.

In the case $\underline{x} \notin M^{\min}_{A,\underline{b}}$ choose a minimal vector $\underline{z} \in M^{\min}_{A,\underline{b}}$ such that

$$\operatorname{supp}(\underline{z}) \subsetneqq \operatorname{supp}(\underline{x}) \,. \tag{4.4}$$

Such a $\underline{z}$ does exist since $\underline{x}$ is not minimal *and* $\operatorname{supp}(\underline{x}) \neq \varnothing$.

Subcase a) There exists a $\underline{z} \in M^{\min}_{A,\underline{b}}$ which has an index $k \in \operatorname{supp}(\underline{z})$ such that $z_k \neq x_k$ (and, of course $z_k \neq 0$).

Then consider the vector

$$\underline{x'} := \left(\underline{x} - \frac{x_k}{z_k} \cdot \underline{z}\right) \cdot \frac{1}{1 - \frac{x_k}{z_k}} = (\underline{x} - \beta \underline{z}) \cdot \frac{1}{1-\beta} \,. \tag{4.5}$$

It is an affine linear combination of $\underline{x}$ and $\underline{z}$, and hence also a solution of $A \cdot \underline{x} = \underline{b}$. Clearly $x'_k = 0$ (i.e. $k \notin \operatorname{supp}(\underline{x}')$), so

$$\operatorname{supp}(\underline{x}') \subsetneqq \operatorname{supp}(\underline{x}) \ , \tag{4.6}$$

moreover $\underline{x}' \neq \underline{0}$ by (4.4). Using the induction hypothesis we know that $\underline{x}'$ is an affine linear combination of vectors from $M^{\min}_{A,\underline{b}}$ plus $\underline{y}' \in M_{A,\underline{0}} \cup \{\underline{0}\}$:

$$\underline{x}' = \sum_{i=1}^{I'} \alpha'_i \underline{z}'_i + \underline{y}' \qquad \text{where} \qquad \sum_{i=1}^{I} \alpha_i = 1 \ . \tag{4.7}$$

From (4.5) and (4.7) we get

$$\underline{x} = (1-\beta)\,\underline{x}' + \beta \underline{z} = (1-\beta) \sum_{i=1}^{I'} \alpha'_i \underline{z}'_i + \beta \underline{z} + (1-\beta)\,\underline{y}' \tag{4.8}$$

which is also an affine linear combination of vectors from $M^{\min}_{A,\underline{b}}$ plus one from $M_{A,\underline{0}} \cup \{\underline{0}\}$.

Subcase b) We have

$$\underline{x}\,|_{\operatorname{supp}(\underline{z})} = \underline{z}\,|_{\operatorname{supp}(\underline{z})} \tag{4.9}$$

for all vectors $\underline{z} \in M^{\min}_{A,\underline{b}}$ satisfying (4.4).

Let $\underline{z} \in M^{\min}_{A,\underline{b}}$ be such a fixed vector. By (4.9) and Proposition 4 we have

$$A|_{\operatorname{supp}(\underline{z})} \cdot \underline{x}|_{\operatorname{supp}(\underline{z})} = A|_{\operatorname{supp}(\underline{z})} \cdot \underline{z}|_{\operatorname{supp}(\underline{z})} = \underline{b} \tag{4.10}$$

and so

$$\left(A|_{\operatorname{supp}(\underline{x}) \setminus \operatorname{supp}(\underline{z})}\right) \cdot \left(\underline{x}|_{\operatorname{supp}(\underline{x}) \setminus \operatorname{supp}(\underline{z})}\right) = \underline{0} \, . \tag{4.11}$$

Letting

$$\underline{y} := \underline{x}|_{\operatorname{supp}(\underline{x}) \setminus \operatorname{supp}(\underline{z})} \tag{4.12}$$

we have $\underline{y} \in M_{A,\underline{0}} \cup \{\underline{0}\}$, and by

$$\operatorname{supp}(\underline{z}) \cap \operatorname{supp}\left(\underline{y}\right) = \varnothing \tag{4.13}$$

we obtain $\underline{x} = \underline{z} + \underline{y}$ which clearly justifies (4.3) and proves the theorem. □

5. The second level

Recall that the set of solution vectors of the linear equation (2.2) is denoted by $M_{A,\underline{0}}$ in (2.4). Since $M_{A,\underline{0}} \subseteqq \mathbb{R}^m$, we can pick some *arbitrary* elements

$$\underline{X}_1, \ldots, \underline{X}_k \in M_{A,\underline{0}} \tag{5.1}$$

and consider the new linear equation

$$\sum_{\ell=1}^{k} m_\ell \underline{X}_\ell = \underline{0} \tag{5.2}$$

for the unknowns.

This is the second step of the **Hierarchy of equations**: the coefficients $\underline{X}_\ell$ of the equality (5.2) are, in fact, all the solutions of the previous (original) equality (2.2).

The final *connection* among the coefficients $\underline{A}_j$ of (2.2) and the solutions $M := [m_1, \ldots, m_\ell]$ of (5.2) has been raised but not investigated yet.

Considering the result of Theorem 4 we could choose $\underline{X}_1, \ldots, \underline{X}_k$ to be all the *minimal* solutions of the (original) equality (2.2).

The stoichiometrical importance of this question will be explained in the next section.

6. Chemical applications

As usual, we consider the reactions in stoichiometry as (systems of) homogeneous linear equations. For example the chemical reaction

$$NaOH + HNO_3 = NaNO_3 + H_2O \tag{6.1}$$

corresponds to the vector-equation

$$[1,0,1,1]^T + [1,1,0,3]^T = [0,1,1,3]^T + [2,0,0,1]^T \tag{6.2}$$

using the base $B = \{H, N, Na, O\}$.

In a general formulation, if the chemical compounds $A_1, A_2, \ldots, A_m$ consist of elements $E_1, E_2, \ldots, E_n$ as $A_j = \sum_{i=1}^{n} a_{i,j} E_i$ $(a_{i,j} \in \mathbb{N},\ j = 1,2,\ldots,m)$ and we write $\underline{A}_j$ for the vector $\left[a_{1,j}, \ldots, a_{n,j}\right]^T$, then there may exist a chemical reaction between the compounds $\{A_j : j \in S\}$ for any $S \subseteq \{1,2,\ldots,m\}$ if and only if the homogeneous linear equation

$$\sum_{j \in S} x_j \underline{A}_j = \underline{0} \tag{6.3}$$

has a nontrivial solution for some $x_j \in \mathbb{R}$ $(j \in S)$, that is if the vector set $\{\underline{A}_j : j \in S\}$ is *linearly dependent*. Similarly, the inhomogeneous linear equation

$$\sum_{j \in S} x_j \underline{A}_j = \underline{b} \tag{6.4}$$

corresponds to the chemical reaction resulting the compound $B = \sum_{i=1}^{n} b_i E_i$ denoted by $\underline{b} = [b_1, \ldots, b_n]^T$. (Of course the reactions obtained in the way described above are only *possibilities*, e.g. the reaction $2Au + 6HCl \to 2AuCl_3 + 3H_2$ does not occur under normal conditions.)

These ideas motivate the study of the objects (2.1) through (2.7). The question "*are there* $\underline{A}_i$ *and* $\underline{A}_j$ *parallel for some* $i \neq j$?" (Condition 1.i) means "*are the*

compounds A_i and A_j isomers or multiple doses of each other?" in the language of stoichiometry.

The *support* of a solution vector $\underline{x} \in \mathbb{R}^m$ (see Definition 1) collects the compounds which effectively take part in the reaction (6.3) or (6.4): $\operatorname{supp}(\underline{x}) \subseteqq S$ in (6.3) and in (6.4). Minimal solution vectors $\underline{x} \in M^{\min}_{A,\underline{b}}$ and $\underline{x} \in M^{\min}_{A,\underline{0}}$ clearly mean *minimal chemical reactions* in the following sense.

Definition 4. The reaction in (6.3) is called a **minimal reaction** if for no $T \subsetneqq S$ might there be any reaction among the compounds $\{A_j : j \in T\}$; that is if the vector set $\{\underline{A}_j : j \in T\}$ is *linearly independent* for any $T \subsetneqq S$.

As Proposition 7 and Definition 3 explain, minimal chemical reactions correspond to minimal linearly dependent sets of vectors, which we call (algebraic) simplexes. Simplexes are widely used e.g. in stoichiometry when finding minimal reactions and mechanisms or for finding *dimensionless groups* in dimensional analysis, see e.g. [1], [4] and [6]. Algorithmic and extremal quantitative questions of minimal reactions (simplexes) were extensively studied in several papers of the authors; we refer only to [9], [2] and [8]. Other kinds of simplexes and their several mathematical aspects are surveyed in [7].

In the present paper we focused on the *inner structure* of $M_{A,\underline{0}}$ and $M_{A,\underline{b}}$, the set of *all* reactions / solutions of the linear equations

$$\sum_{j=1}^{m} x_j \underline{A}_j = \underline{0}\,, \quad \textit{equivalently} \quad A \cdot \underline{x} = \underline{0} \tag{6.5}$$

and

$$\sum_{j=1}^{m} x_j \underline{A}_j = \underline{b}\,, \quad \textit{equivalently} \quad A \cdot \underline{x} = \underline{b} \tag{6.6}$$

i.e. of (2.2) and (2.1)). We gave thorough extensions of the results in [3].

Proposition 5 and its extensions, Theorems 2 and 6 prove the *uniqueness* of the reactions (solutions) in the sense of Remark 1, if the given set of compounds is minimal.

Theorems 4 and 8 are fundamental in our investigations, since they ensure: *All reactions can be obtained from minimal ones*.

Our further results were listed in Remark 3 through Proposition 8.

The "second level of hierarchy" corresponds to **mechanisms**: *sequences of reactions*, i.e. linear combinations of solution vectors of (6.5) (of (2.1) and (2.2)). This and other questions are planned to discuss in [5].

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CHARACTERIZATION OF UNIFORM STATISTICAL CONVERGENCE FOR DOUBLE SEQUENCES

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Abstract. In the present study we introduce uniform statistical convergence for double sequences. We present a decomposition theorem that characterizes uniform statistical convergence for double sequences.

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1. INTRODUCTION

In the summability theory uniform statistical convergence has important roles for ordinary (single) sequences. Since this type of convergence method is stronger than ordinary convergence, it is quite effective, especially when the classical limit does not exist. This idea has been introduced by Brown and Freedman [2] by using the notion of uniform density. Then the method has been discussed by many authors in various directions ([1], [4] and [5]). The concept of uniform density has recently been extended to double sequences [14].

In the present work we introduce the uniform statistical convergence by using the uniform density for double sequences. We present a decomposition theorem that characterizes uniform statistical convergence for double sequences and show that for double sequences almost convergence and statistical convergence are not compatible. Also we give a relation between uniform statistical convergence and strong uniform p-Cesàro summable double sequences. Finally by defining strongly almost convergence with respect to a modulus function we examine the new sequence space together with the space of all uniformly statistically convergent sequences.

In 1900 Pringsheim [13] introduced the concept of convergence for double sequences. A double sequence $x = (x_{ij})$ is said to be convergent in Pringsheim sense if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_{ij} - L| < \varepsilon$ whenever $i, j \geq N$. In this case L is called the Pringsheim limit of x and space of such sequences is denoted by c_2. A double sequence x is bounded if there exists a positive number M such that

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$\left|x_{ij}\right| < M$ for all i and j ,i.e.,

$$\|x\|_{(\infty,2)} = \sup_{i,j} \left|x_{ij}\right| < \infty.$$

We will denote the set of all bounded double sequences by l_∞^2. Note that in contrast to the case for single sequences, a convergent double sequence need not be bounded.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two dimensional set of positive integers and let $K(n,m)$ be the numbers of (i,j) in K such that $i \leq n$, $j \leq m$. Then the lower asymptotic density of a set $K \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{d}_2(K) = \liminf_{n,m} \frac{K(n,m)}{nm}$$

and the upper asymptotic density is defined as

$$\bar{d}_2(K) = \limsup_{n,m} \frac{K(n,m)}{nm}.$$

If $\bar{d}_2(K) = \underline{d}_2(K)$ then $d_2(K) = \bar{d}_2(K) = \underline{d}_2(K)$ is called the asymptotic density [11].

The following definition has been given in [10], [11] and [15] independently.

Definition 1. A double sequence $x = (x_{ij})$ is said to be statistically convergent if for every $\varepsilon > 0$, $d_2(A_\varepsilon) = 0$ where $A_\varepsilon = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \left|x_{ij} - L\right| \geq \varepsilon\right\}$.

We will denote the set of all statistically convergent double sequences by st^2. The lower uniform density of a set $K \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{u}_2(K) = \lim_{p,q} \frac{\min_{m,n \geq 0} \left| \{(i,j) \epsilon K : m+1 \leq i \leq m+p, n+1 \leq j \leq n+q\} \right|}{pq}$$

and the upper uniform density is defined as

$$\bar{u}_2(K) = \lim_{p,q} \frac{\max_{m,n \geq 0} \left| \{(i,j) \epsilon K : m+1 \leq i \leq m+p, n+1 \leq j \leq n+q\} \right|}{pq}.$$

If $\bar{u}_2(K) = \underline{u}_2(K)$ then $u_2(K) = \bar{u}_2(K) = \underline{u}_2(K)$ is called uniform density.

It is clear that $\underline{u}_2(K) \leq \underline{d}_2(K) \leq \bar{d}_2(K) \leq \bar{u}_2(K)$.

We now introduce the concept of uniform statistical convergence for double sequences.

Definition 2. A double sequence $x = (x_{ij})$ is said to be uniformly statistically convergent if for every $\varepsilon > 0$, $u_2(A_\varepsilon) = 0$ where
$A_\varepsilon = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \left|x_{ij} - L\right| \geq \varepsilon\right\}$.

By st_u^2 we will denote the set of all statistically convergent double sequences.

Note that the set $st_u^2 \cap l_\infty^2$ is a closed linear subspace of the normed linear space l_∞^2.

Uniform statistical convergence is closely related to statistical convergence. As in the case of single sequences, we have the following inclusions

$$c_2 \subset st_u^2 \subset st^2.$$

These inclusions are proper as the following examples show.

Example 1. Let P be the set of all primes. Define a double sequence $x = (x_{ij})$ as follows

$$x_{ij} = \begin{cases} 1 & ; \quad (i,j) \in P \times P \\ 0 & ; \quad otherwise. \end{cases}$$

We know that $u(P) = 0$ [3]. It is easy to see that $u_2(P \times P) = 0$. Hence $x = (x_{ij})$ is uniformly statistically convergent to 0 but not convergent.

Example 2. Let $A = \bigcup_{k=1}^{\infty} \left\{5^k + 1, 5^k + 2, ..., 5^k + k\right\}$. Define a double sequence $x = (x_{ij})$ as follows

$$x_{ij} = \begin{cases} 1 & ; \quad (i,j) \in A \times A \\ 0 & ; \quad otherwise. \end{cases}$$

It is known from [1] that $\underline{u}(A) = 0, \bar{u}(A) = 1, d(A) = 0$.

One can easily see that $\underline{u}_2(A \times A) = 0$, $\bar{u}_2(A \times A) = 1$, $d_2(A \times A) = 0$. Hence the double sequence x is statistically convergent to 0, but not uniformly statistically convergent.

2. Characterizations of uniform statistical convergence

This section presents some comparison results which are related to uniform statistical convergence for double sequences. Also uniform statistical convergence is characterized.

The next definition may be found in [8] and [9].

Definition 3. A double sequence $x = (x_{ij})$ is called almost convergent to L if

$$\lim_{p,q} \frac{1}{pq} \sum_{i,j=m+1,n+1}^{m+p,n+q} x_{ij} = L, \text{ (uniformly in } n \text{ and } m).$$

By ac^2, we denote the set of almost convergent double sequences.

Recall that $ac^2 \subset l_\infty^2$,[8].

We know that almost convergence and statistical convergence are not compatible for ordinary sequences [7]. This raises the question of whether this property holds for double sequences. The following examples demonstrates this fact in the affirmative.

Example 3. Let $x_{ij} = \begin{cases} ij & ; \quad i \text{ and } j \text{ square} \\ 0 & ; \quad otherwise \end{cases}$.
Since x is unbounded, it can not be almost convergent. But it is easy to see that x is statistically convergent to 0.

Example 4. Let $x_{ij} = \begin{cases} 0 & ; \quad i \text{ is even, for every } j \\ 1 & ; \quad i \text{ is odd, for every } j \end{cases}$.
One can observe that x can not be statistically convergent but it is almost convergent to $\frac{1}{2}$.

For double sequences almost convergence was introduced by Mòricz and Rhoades [8] and it was studied in some detail in [16] and [9].

The following definition is an extension of a definition given in [1] to double sequences.

Definition 4. A double sequence $x = (x_{ij})$ is said to be uniformly strong $p-$Cesàro convergent $(0 < p < \infty)$ to L if

$$\lim_{k,l} \frac{1}{kl} \sum_{i,j=m+1,n+1}^{m+k,n+l} | x_{ij} - L |^p = 0, \text{ (uniformly in } n \text{ and } m).$$

By uw_p^2, we denote the set of all uniformly strongly $p-$Cesàro convergent double sequences. It is immediate that $uw_p^2 \subset w_p^2$, where w_p^2 denotes space of all strongly $p-$Cesàro convergent double sequences studied in [11].

In case $p = 1$, x is said to be strongly almost convergent. So this implies almost convergence.

It is known that $c \subset ac$ for single (ordinary) sequences. But we have $l_\infty^2 \cap c_2 \subset ac^2$ for double sequences ([8]).

We now give our first characterization for uniform statistical convergence of double sequences.

Theorem 1. *(i) If $0 < p < \infty$ and a sequence $x = (x_{ij})$ is uniformly strongly $p-$Cesàro convergent to L, then it is uniformly statistically convergent to L.*

(ii) If $x = (x_{ij})$ is bounded and uniformly statistically convergent to L, then it is uniformly strongly $p-$Cesàro convergent to L for every p, $0 < p < \infty$.

Proof. (i) Let x be uniformly strongly $p-$Cesàro convergent to L, $0 < p < \infty$. Suppose $\varepsilon > 0$. Then for every $n, m \in \mathbb{N}$, we have

$$\sum_{i,j=1,1}^{k,l} | x_{n+i,m+j} - L |^p \geq \sum_{\substack{i,j=1,1 \\ |x_{n+i,m+j}-L| \geq \varepsilon}}^{k,l} | x_{n+i,m+j} - L |^p$$

$$\geq \varepsilon^p \sum_{\substack{i,j=1,1 \\ |x_{n+i,m+j}-L|\geq\varepsilon}}^{k,l} 1 \geq \varepsilon^p A_\varepsilon(n+1,n+k;m+1,m+l).$$

Then it follows that

$$\frac{1}{kl}\sup_{n,m}\sum_{i,j=1,1}^{k,l} |\, x_{n+i,m+j}-L\,|^p \geq \varepsilon^p \frac{1}{kl}\sup_{n,m} A_\varepsilon(n+1,n+k;m+1,m+l),$$

where

$$A_\varepsilon(n+1,n+k;m+1,m+l) =$$
$$\left\{(i,j) : n+1 \leq i \leq n+k, m+1 \leq j \leq m+l; |\, x_{ij}-L\,| \geq \varepsilon\right\}.$$

This implies $\bar{u}_2(A_\varepsilon) = 0$ and $u_2(A_\varepsilon) = 0$, so that $st_u^2 - \lim x = L$.

(ii) Suppose that x is a bounded sequence and $st_u^2 - \lim x = L$. Let $0 < p < \infty$ and $\varepsilon > 0$. According to the hypothesis we have $u_2(A_\varepsilon) = 0$.

The boundedness of $x = (x_{ij})$ implies that there exists $M > 0$ such that for every $i, j = 1, 2, ..., |\, x_{ij} - L\,| \leq M$.

Observe that for every $n, m \in \mathbb{N}$, we have

$$\frac{1}{kl}\sum_{i,j=1,1}^{k,l} |\, x_{n+i,m+j}-L\,|^p =$$
$$\frac{1}{kl}\sum_{\substack{i,j=1,1 \\ (n+i,m+j)\in A_\varepsilon}}^{k,l} |\, x_{n+i,m+j}-L\,|^p + \frac{1}{kl}\sum_{\substack{i,j=1,1 \\ (n+i,m+j)\notin A_\varepsilon}}^{k,l} |\, x_{n+i,m+j}-L\,|^p$$
$$\leq M^p \frac{1}{kl}\max_{n,m\geq 0} A_\varepsilon(n+1,n+k;m+1,m+l) + \varepsilon^p \frac{1}{kl}\sum_{i,j=1,1}^{k,l} 1$$
$$\leq M^p u_2(A_\varepsilon) + \varepsilon^p.$$

This implies that $\lim_{k,l} \frac{1}{kl} \sum_{i,j=m+1,n+1}^{m+k,n+l} |\, x_{ij}-L\,|^p = 0$ (uniformly in n and m). □

We immediately obtain the following results from Theorem 1.

Corollary 1. *Let $x = (x_{ij})$ is a bounded double sequence. If x is uniformly statistically convergent to L then x is almost convergent to L.*

But the converse of this corollary does not hold, see Example 4.

Corollary 2. *Let $x = (x_{ij})$ is a bounded double sequence. Then x is uniformly statistically convergent to L if and only if x is uniformly strongly $p-$Cesàro convergent to L for every p, $0 < p < \infty$.*

The following result characterizes uniform statistical convergence.

Theorem 2. *The following are equivalent.*

(i) The double sequence $x = (x_{kl})$ is uniformly statistically convergent to L.

(ii) There exists a sequence $y = (y_{kl})$ which is convergent and such that $u_2(\{(k,l) : x_{kl} = y_{kl}\}) = 1$.

(iii) There exists a subset $M = \{(k_i, l_j) : i, j \in \mathbb{N}\}$ such that $u_2(M) = 1$ and $(x_{k_i l_j})$ is convergent.

(iv) There exists two sequences (y_{kl}) and (z_{kl}) such that $x_{kl} = y_{kl} + z_{kl}$ for all k, l and (y_{kl}) converges to L and (z_{kl}) is uniformly statistically convergent to 0.

Proof. (i)⇒(ii): Let the double sequence (x_{kl}) be uniformly statistically convergent to L. Let the complement of the set $A_{\frac{1}{j}} = \left\{(p,q) :\mid x_{k_p l_q} - L \mid \geq \frac{1}{j}\right\}$, $j = 1, 2, ...$, be

$$K_j = \mathbb{N} \times \mathbb{N} - A_{\frac{1}{j}}.$$

Hence by the definition of uniform statistical convergence, we have $u_2(K_j) = 1$, for $j = 1, 2, ...$

By the definition of K_j we have $K_1 \supseteq K_2 \supseteq ... \supseteq K_j \supseteq K_{j+1} \supseteq ....$ Let us choose an arbitrary $s_1 = (s_1', s_1'') \in K_1$. By the definition of K_j, there exists $s_2 = (s_2', s_2'') \in K_2$ such that $s_2' > s_1'$, $s_2'' > s_1''$ and for every $p \geq s_2'$, $q \geq s_2''$

$$\frac{\min\limits_{m,n \geq 0} K_2(m+1, m+p; n+1, n+q)}{pq} > \frac{1}{2}.$$

Again by the definition of K_j there exists $s_3 = (s_3', s_3'') \in K_3$ such that $s_3' > s_2'$, $s_3'' > s_2''$ and for every $p \geq s_3'$, $q \geq s_3''$.

$$\frac{\min\limits_{m,n \geq 0} K_3(m+1, m+p; n+1, n+q)}{pq} > \frac{2}{3}.$$

Continuing in this way, we can construct increasing sequences of positive integers $s_1' < s_2' < s_3' < ... < s_j' < ...$

$s_1'' < s_2'' < s_3'' < ... < s_j'' < ...$ such that $s_j = (s_j', s_j'') \in K_j$ and for every $p \geq s_j'$, $q \geq s_j''$

$$\frac{\min\limits_{m,n \geq 0} K_j(m+1, m+p; n+1, n+q)}{pq} > 1 - \frac{1}{j}, \quad \text{for } j = 1, 2, 3, ...$$

Define K as follows: if $1 \leq k \leq s_1'$ or $1 \leq l \leq s_1''$ then $(k,l) \in K$; suppose that $j \geq 1$ and $s_j' < k \leq s_{j+1}'$ or $s_j'' < l \leq s_{j+1}''$ then $(k,l) \in K$ if and only if $(k,l) \in K_j$.

For every p,q, such that $s_j^{'} < p \leq s_{j+1}^{'}$ and $s_j^{''} < q \leq s_{j+1}^{''}$, we have

$$\frac{\min\limits_{m,n\geq 0} K(m+1,m+p;n+1,n+q)}{pq} \geq \frac{\min\limits_{m,n\geq 0} K_j(m+1,m+p;n+1,n+q)}{pq}$$
$$\geq 1-\frac{1}{j}.$$

Hence it is obvious that $u_2(K)=1$. Let $\varepsilon>0$ be given and select j such that $\frac{1}{j}<\varepsilon$. Let $p\geq s_j^{'}$, $q\geq s_j^{''}$, $(p,q)\epsilon K$. Then there exists a number $r\geq j$ such that $s_r^{'} < p \leq s_{r+1}^{'}$ and $s_r^{''} < q \leq s_{r+1}^{''}$. According to the definition of K, $(p,q)\epsilon K_r$ is obtained, we have $\mid x_{pq}-L \mid < \frac{1}{r} \leq \frac{1}{j} < \varepsilon$. Thus $\mid x_{pq}-L \mid < \varepsilon$ for every $p\geq s_j^{'}$, $q\geq s_j^{''}$, $(p,q)\epsilon K$. Hence $\lim\limits_{\substack{p,q\\(p,q)\epsilon K}} x_{pq}=L$.

Let us define $y=\left(y_{pq}\right)$ as follows

$$y_{pq}=\begin{cases} x_{pq} & ; \quad (p,q)\epsilon K \\ L & ; \quad otherwise \end{cases}.$$

So $\lim\limits_{p,q} y_{pq}=L$ and $u_2(\{(p,q):x_{pq}=y_{pq}\})=u_2(\{(p,q)\epsilon K\})=1$

(ii)$\Rightarrow$(iii): Let there exists a sequence $y=(y_{kl})$ which is convergent such that $u_2(\{(k,l):x_{kl}=y_{kl}\})=1$. Let $M=\{(k,l):x_{kl}=y_{kl}\}$ then $u_2(M)=1$. Now we can represent M as $M=\left\{(k_i,l_j):i,j\epsilon\mathbb{N}\right\}$. Clearly $\mid x_{k_il_j}-L \mid = \mid y_{k_il_j}-L \mid \to 0$ as $i,j\to\infty$

(iii)$\Rightarrow$(iv): Let (x_{kl}) be a double sequence and there exists $M=\left\{(k_i,l_j):i,j\epsilon\mathbb{N}\right\}$ such that $u_2(M)=1$ and $\lim\limits_{i,j} x_{k_il_j}=L$.

Now construct the sequences (y_{kl}) and (z_{kl}) as follows:

$$y_{kl}=\begin{cases} x_{kl} & ; \quad (k,l)\epsilon M \\ L & ; \quad otherwise \end{cases},$$

and

$$z_{kl}=\begin{cases} 0 & ; \quad (k,l)\epsilon M \\ x_{kl}-L & ; \quad otherwise \end{cases}.$$

From the above construction , it is obvious that (y_{kl}) is convergent to L and $st_u^2-\lim\limits_{k,l} z_{kl}=0$ and $x_{kl}=y_{kl}+z_{kl}$ for all $k,l\epsilon\mathbb{N}$

(iv)$\Rightarrow$(i): Suppose that there exists sequences such that $x_{kl}=y_{kl}+z_{kl}$ for all $k,l\epsilon\mathbb{N}$; (y_{kl}) converges to L and $st_u^2-\lim\limits_{k,l} z_{kl}=0$. For any $\varepsilon>0$, let $A=\left\{(k,l):\mid y_{kl}-L \mid < \frac{\varepsilon}{2}\right\}$ and $B=\left\{(k,l):\mid z_{kl} \mid < \frac{\varepsilon}{2}\right\}$. Then clearly $u_2(A)=u_2(B)=1$. This implies that

$$u_2(A\cap B)=1,\ u_2(\{(k,l):\mid x_{kl}-L \mid < \varepsilon\}) \geq u_2(A\cap B).$$

By $u_2(\{(k,l) :\mid x_{kl}-L \mid<\varepsilon\})=1$, the sequence (x_{kl}) is uniformly statistically convergent to L. □

3. A NEW SEQUENCE SPACE DEFINED BY A MODULUS FUNCTION

In the section we define a new sequence space by means of a modulus function and we relate it to uniform statistical convergence.

Recall that a modulus [6] is a function from $[0,\infty)$ to $[0,\infty)$ such that

(i) $f(x)=0$ if and only if $x=0$,

(ii) $f(x+y)\leq f(x)+f(y)$ for $x,y\geq 0$,

(iii) f is increasing,

(iv) f is continuous at the right at 0.

Because of (ii) $\mid f(x)-f(y)\mid\leq f(x-y)$ so that in view of (iv) f is continuous on $[0,\infty)$. A modulus may be bounded or unbounded. For example $f(x)=x^p(0<p\leq 1)$ is unbounded and $f(x)=\dfrac{x}{x+1}$ is bounded.

The next sequence space of double sequences is motivated from [12] where in its single version is studied:

$$\left[ac^2(f)\right]$$
$$=\left\{x=(x_{ij}):\lim_{p,q}\frac{1}{pq}\sum_{i,j=n+1,m+1}^{n+p,m+q}f(\mid x_{ij}-L\mid)=0,\ \text{uniformly in } n \text{ and } m\right\}$$

Let us give the following two theorems.

Theorem 3. *Let f be any modulus. Then $\left[ac^2(f)\right]\subset st_u^2$.*

Proof. Suppose $x\epsilon\left[ac^2(f)\right]$ and $\varepsilon>0$. Then we have for every n,m

$$\frac{1}{pq}\sum_{i,j=n+1,m+1}^{n+p,m+q}f(\mid x_{ij}-L\mid)\geq\frac{1}{pq}\sum_{\substack{i,j=n+1,m+1\\ |x_{ij}-L|\geq\varepsilon}}^{n+p,m+q}f(\mid x_{ij}-L\mid)$$
$$\geq f(\varepsilon)\frac{1}{pq}\mid E_\varepsilon\mid$$

where
$E_\varepsilon=\left\{(i,j):n+1\leq i\leq n+p,m+1\leq j\leq m+q \text{ and } \mid x_{ij}-L\mid\geq\varepsilon\right\}$. This inequality implies that $x\epsilon st_u^2$. □

Theorem 4. *$st_u^2=\left[ac^2(f)\right]$ if and only if f is bounded.*

Proof. Suppose that f is bounded and $x\epsilon st_u^2$. Since f is bounded there exists an integer K such that $f(x_{ij})<K$ for all $x_{ij}\geq 0$, $i,j=1,2,\ldots$. Then for each n,m

we have

$$\frac{1}{pq}\sum_{i,j=n+1,m+1}^{n+p,m+q} f(\mid x_{ij}-L\mid)$$

$$=\frac{1}{pq}\sum_{\substack{i,j=n+1,m+1\\ |x_{ij}-L|\geq\varepsilon}}^{n+p,m+q} f(\mid x_{ij}-L\mid)$$

$$+\frac{1}{pq}\sum_{\substack{i,j=n+1,m+1\\ |x_{ij}-L|<\varepsilon}}^{n+p,m+q} f(\mid x_{ij}-L\mid)$$

$$\leq \frac{1}{pq}K\max_{m,n\geq 0}\mid E_{\varepsilon}\mid +f(\varepsilon)$$

Conversely suppose that f is unbounded so that there is a positive sequence (u_{ij}) such that $f(u_{ij})=i^2j^2$ for $i,j=1,2,...$

Now consider the sequence $x=(x_{ij})$ defined by $x_{ij}=\begin{cases} u_{ij} & ; \quad i,j \text{ square} \\ 0 & ; \quad \text{otherwise} \end{cases}$.

Then we have

$$\frac{1}{pq}\max_{m,n\geq 0}\mid\{n+1\leq i\leq n+p, m+1\leq j\leq m+q :\mid x_{ij}-L\mid\geq\varepsilon\}\mid\leq\frac{\sqrt{p}\sqrt{q}}{pq}$$

Hence $st_u^2-\lim x=L$. But $x\notin[ac^2(f)]$. Contradicting $st_u^2=[ac^2(f)]$. This completes the proof. □

As a result of Theorem 1 and Theorem 4 we easily obtain the next corollary.

Corollary 3. *For bounded double sequences we have* $[ac^2(f)]=uw_p^2$, $(0<p<\infty)$, *if and only if* f *is bounded.*

The boundedness of the sequence can not be omitted. The next example demonstrates this fact.

Define $x=(x_{ij})$ by

$$x_{ij}=\begin{cases} ij & ; \quad i \text{ and } j \text{ square} \\ 0 & ; \quad \text{otherwise} \end{cases}$$

and take

$$f(x)=\frac{x}{x+1}.$$

It is clear that x is unbounded.

$$\frac{1}{kl}\sum_{\substack{i,j=n+1,m+1\\ i,j \text{ square}}}^{n+k,m+l} f(\mid x_{ij}\mid)\leq\frac{1}{kl}\sum_{\substack{i,j=n+1,m+1\\ i,j \text{ square}}}^{n+k,m+l}1$$

$$\leq \frac{1}{kl} \max_{m,n\geq 0} |\{n+1 \leq i \leq n+k, m+1 \leq j \leq m+l : i,j \text{ square}\}|$$

So $x = (x_{ij}) \epsilon \left[ac^2(f)\right]$.

$$\frac{1}{kl} \sum_{i,j=n+1,m+1}^{n+k,m+l} |x_{ij} - 0|^p = \frac{1}{kl} \sum_{i,j=n+1,m+1}^{n+k,m+l} {x_{ij}}^p = \frac{1}{kl} \sum_{\substack{i,j=n+1,m+1 \\ i,j \text{ square}}}^{n+k,m+l} (ij)^p$$

$$= \frac{1}{kl} \sum_{\substack{i,j=n+1,m+1 \\ i,j \text{ square}}}^{n+k,m+l} (ij)^p \qquad (*)$$

One can find p so that $(*)$ does not converge.

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STABILITY OF JUNGCK-TYPE ITERATIVE PROCEDURE FOR SOME CONTRACTIVE TYPE MAPPINGS VIA IMPLICIT RELATIONS

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Abstract. We obtain some stability results of the Jungck-type iterative procedure for common fixed points and weakly compatible mappings defined by a suitable implicit relation satisfying (E.A) property in metric spaces.

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Keywords: fixed point, stability, implicit relation, weakly compatible mappings, property (E.A)

1. INTRODUCTION

The stability of a fixed point iterative procedure was first studied by Ostrowski [35] in the case of Banach contraction mappings and this subject was later developed for certain contractive definitions by several authors (see Harder and Hicks [12], Rhoades [42], [44], Osilike [33] and [32], Berinde [6], [5], Jachymski [18], Olatinwo, Owojori and Imoru [28], [30] etc.). Osilike and Udomene [34] introduced a shorter method in order to prove stability results and this has also been applied by Imoru and Olatinwo [17], Imoru, Olatinwo and Owojori [16], [29] and some others. Moreover, Olatinwo [26] made generalizations and obtained first stability results using the concepts of pointwise convergence of sequences of operators and the fixed point iteration procedure was investigated for the case of two metrics.

Jungck [20] generalized the Banach's contraction principle, by replacing the identity map with a continuous map, thus obtaining a common fixed point theorem. Following the Jungck's contraction principle, many authors proved general common fixed points theorems and coincidence theorems (see Imdad and Ali [15], Aamri and Moutawakil [1]). Stability results of common fixed point iterative procedures and coincidence points were obtained by some authors. Singh, Bhatnagar and Mishra [46] established some stability results for Jungck and Jungck-Mann iteration procedures by employing two contractive definitions which generalized those of Osilike [32] but independent of that of Imoru and Olatinwo [17]. Moreover, Olatinwo [25], [27] obtained some stability results for nonself mappings in normed linear spaces which

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are generalizations and extensions of [5], [17], [16]. Olatinwo and Postolache [31] also studied the stability in convex metric spaces for nonself mappings satisfying certain general contractive definitions in the case of Jungck-Mann and Jungck-Ishikawa iteration procedures. Singh and Prasad [47] studied the problem of stability for coincidence points on b-metric spaces. Timiş and Berinde [49] studied the problem of weak stability of common fixed point iterative procedures for some classes of contractive type mappings and gave an illustrative example of weakly stable but not stable iterative fixed point procedure.

Using Popa's approach of implicit functions (see [40], [37], [38]) and the advantage of their versatility, several classical fixed point theorems and common fixed point theorems have been unified since they cover several contractive conditions rather than one contractive condition. Moreover, Berinde [9], [8] established stability results for fixed point iteration procedures associated to contractive mappings defined by an implicit relation.

Since a metrical common fixed point theorem generally involves conditions of commutativity, a lot of researches in this domain are aimed at weakening these conditions. The evolution of weak commutativity of Sessa [45] and compatibility of Jungck [22] developed weak conditions in order to improve common fixed points theorems.

This paper gives a general stability result for the common fixed point iteration procedure of Jungck-type in the class of weakly compatible mappings defined by means of an implicit contraction condition with five parameters.

2. Preliminaries

All common fixed point theorems are based on a commuting property. We start by recalling the main important concepts of commutativity and weak commutativity.

Let (X,d) be a metric space and $S,\ T : X \to X$ be two mappings. We say that S and T are *commuting* if

$$STx = TSx,\ \ \forall\ x \in X.$$

As a generalization of this notion, Sessa [45] defined S and T to be *weakly commuting* if

$$d(STx, TSx) \leq d(Sx, Tx),\ \ \forall\ x \in X.$$

Jungck [22] defined S and T to be *compatible*, as a generalization of weakly commuting, if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t,\ \ t \in X.$$

Jungck [22] also showed that commuting implies weakly commuting which, in turn, implies compatibility property but the converse property is not true in general, as show the following illustrative example.

Example 1 ([22]). Let the functions $f(x) = x^3$ and $g(x) = 2x^3$, with $X = \mathbb{R}$. They are compatible, since

$$|f(x) - g(x)| = |x^3| \to 0 \quad \Leftrightarrow \quad |fg(x) - gf(x)| = 6|x^9| \to 0,$$

but the pair (f, g) is not weakly commuting.

Moreover, Jungck [21] defined S and T to be *weaky compatible* if they commute at their coincidence points, i.e., if

$$Sz = Tz,\ z \in X, \ \text{ then } \ STz = TSz.$$

Jungck [22] established the inclusions between these notions, respectively that the *commuting property* implies *weakly commuting* which, in turn, implies *compatible* that implies *weakly compatible* but the reverse is not generally true.

Secondly, Aamri and Moutawakil [1] introduced a notion which is independent of the notion of weakly compatibility.

Definition 1 ([1])**.** S and T mappings satisfy (E.A) property if there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t, \ \text{ for some } \ t \in X.$$

The following example shows that a pair of mappings can satisfy the (E.A) property without being weakly compatible.

Example 2 ([3]). Let $(\mathbb{R}_+, |\cdot|)$ and define S and T by $Sx = x^2$ and $Tx = x + 2$. We have that $Sx = Tx \ \Leftrightarrow \ x = 2$. Let $\{x_n\}$ be a sequence in X, given by $x_n = 2 + \frac{1}{n}$, $n \geq 1$. Then, $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 4$, so, S and T satisfy property (E.A).

As $ST(2) = S(4) = 16$, and $TS(2) = T(4) = 6$, (S, T) is not weakly compatible.

In general, a pair satisfying (E.A) property need not follow the pattern of containment of range of one map into the range of other as it is utilized in common fixed point considerations but still it relaxes such requirements.

Example 3 ([15]). Consider $X = [-1, 1]$ with the usual metric. Define $S,\ T : X \to X$, as follows:

$$T(x) = \begin{cases} \frac{1}{2}, & x = -1, \\ \frac{x}{4}, & x \in (-1, 1), \\ \frac{3}{5}, & x = 1, \end{cases}$$

and

$$S(x) = \begin{cases} \frac{1}{2}, & x = -1, \\ \frac{x}{2}, & x \in (-1,1), \\ \frac{-1}{2}, & x = 1. \end{cases}$$

Let the sequence $\{x_n\}$ be given by $x_n = \frac{1}{n}$. Then,

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0,$$

so the pair (S,T) satisfies (E.A) property.

The mappings T and S are also weakly compatible because $T(0) = S(0) = 0$ and $ST(0) = TS(0) = 0$.

On the other hand, $T(X) = \{\frac{1}{2}, \frac{3}{5}\} \cup (\frac{-1}{4}, \frac{1}{4})$ and $S(X) = [\frac{-1}{2}, \frac{1}{2}]$. Hence, neither $T(X)$ is contained in $S(X)$ nor $S(X)$ is contained in $T(X)$.

3. Stability of common fixed point iterative procedures

For the following stability study, we shall use weakly compatible mappings satisfying the above (E.A) property.

Let the mappings $S, T : X \to X$ and $T(X) \subseteq S(X)$. For any $x_0 \in X$, consider $Sx_{n+1} = Tx_n$, $n = 0, 1, \ldots$ which is the iterative procedure introduced by Jungck [20] which becomes the Picard iterative procedure when $S = id$, the identity map on X.

Jungck showed in [20] that the mappings S and T satisfying

$$d(Tx, Ty) \leq kd(Sx, Sy), \quad 0 \leq k < 1, \quad \forall x, y \in X, \tag{3.1}$$

have a common fixed point in X, provided that S and T are commuting, $T(X) \subseteq S(X)$ and S is continuous.

The following significant improved version of this result is generally called the Jungck common fixed point principle.

Theorem 1 ([47]). *Let $S, T : X \to X$ satisfying (3.1). If $T(X) \subseteq S(X)$ and $S(X)$ or $T(X)$ is a complete subspace of X, then S and T have a coincidence. Indeed, for any $x_0 \in X$, there exists a sequence $\{x_n\}$ in X such that*

- $Sx_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$
- *$\{Sx_n\}$ converges to Sz for some z in X and $Sz = Tz = t$, that is, S and T have a coincidence at z.*

Further, if S and T commute just at z, then they have an unique common fixed point.

Definition 2 ([47]). Let (X,d) be a metric space and $S, T : X \to X$. Let z to be a coincidence point of T and S, that is, $Sz = Tz = t$. For any $x_0 \in X$, the sequence

$\{Sx_n\}$ generated by the general iterative procedure

$$Sx_{n+1} = Tx_n, \quad n = 1, 2, ..., \tag{3.2}$$

converges to $t \in X$. Let $\{Sy_n\} \subset X$ be an arbitrary sequence and set

$$\epsilon_n = d(Sy_{n+1}, Ty_n), \quad n = 0, 1, 2, \ldots .$$

Then the iterative procedure 3.2 is said to be (S,T)-stable or stable with respect to (S,T) if and only if

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \text{implies that} \quad \lim_{n\to\infty} Sy_n = t.$$

This definition reduces to that of the stability of a fixed point iterative procedure, due to Harder and Hicks [13], [12], when $S = id$.

For several examples discussing the practical aspect and theoretical importance of the stability when S is the identity mapping on X in the above definition, see Berinde [6].

In order to prove our main stability result in this paper, we shall need the next lemma.

Lemma 1 ([6]). *Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers and a constant h, $0 \le h < 1$, so that*

$$a_{n+1} \le ha_n + b_n, \quad n \ge 0.$$

- *If $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$.*
- *If $\sum_{n=0}^{\infty} b_n < \infty$, then $\sum_{n=0}^{\infty} a_n < \infty$.*

4. Implicit contractive conditions

From the class of implicit functions due to Popa [40], [37], [38], let $\mathbb{F}$ to be the set of all continuous functions $F : \mathbb{R}_+^5 \to \mathbb{R}$ satisfying the following conditions:

(1) F is continuous in each coordinate variable;
(2) If for some $u, v, w \ge 0$, we have
  (a) $F(u, v, u, v, w) \le 0$ or
  (b) $F(u, v, v, u, w) \le 0$,
  then there exists $h \in [0, 1)$, such that $u \le h \max\{v, w\}$;
(3) $F(u, u, u, u, 0) > 0$, for all $u > 0$.

In the sequel, we present some examples of contractive type mappings defined by an implicit function depending of five parameters.

Example 4 ([40]). The function $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ given by

$$F(t_1, ..., t_5) = t_1 - at_2,$$

where $a \in [0, 1)$, satisfies (1), (2a), (2b) and (3), with $h = a$.

Example 5. The function $F(t_1,t_2,t_3,t_4,t_5):\mathbb{R}_+^5\to\mathbb{R}$ given by one of the following:

(1) $F(t_1,...,t_5)=t_1-at_2$,
(2) $F(t_1,...,t_5)=t_1-bt_5$,
(3) $F(t_1,...,t_5)=t_1-c(t_3+t_4)$,

where $a,b\in[0,1)$, $c\in\left[0,\frac{1}{2}\right)$, satisfies (1), (2a), (2b) and (3), with $h=a$, b, respectively $\frac{b}{1-b}<1$.

Example 6. The function $F(t_1,t_2,t_3,t_4,t_5):\mathbb{R}_+^5\to\mathbb{R}$ given by

$$F(t_1,...,t_5)=t_1-kt_5,$$

where $k\in(0,1)$, satisfies (1), (2a), (2b) and (3), with $h=k$.

Example 7. The function $F(t_1,t_2,t_3,t_4,t_5):\mathbb{R}_+^5\to\mathbb{R}$ given by

$$F(t_1,...,t_5)=t_1-at_2-bt_5,$$

where $a,b\in(0,1)$, with $a+2b<1$, satisfies (1), (2a), (2b) and (3), with $h=a$, if $\max\{v,w\}=v$ and $h=b$, if $\max\{v,w\}=w$.

Example 8 ([40]). The function $F(t_1,t_2,t_3,t_4,t_5):\mathbb{R}_+^5\to\mathbb{R}$ given by

$$F(t_1,...,t_5)=t_1-a(t_3+t_4),$$

where $a\in\left(0,\frac{1}{2}\right)$, satisfies (1), (2a), (2b) and (3), with $h=\frac{a}{1-a}\in(0,1)$.

Example 9. The function $F(t_1,t_2,t_3,t_4,t_5):\mathbb{R}_+^5\to\mathbb{R}$ given by

$$F(t_1,...,t_5)=t_1-h\max\{t_3,t_4\},$$

where $h\in[0,1)$, satisfies (1), (2a), (2b) and (3).

Example 10. The function $F(t_1,t_2,t_3,t_4,t_5):\mathbb{R}_+^5\to\mathbb{R}$ given by

$$F(t_1,...,t_5)=t_1-at_2-bt_3-ct_4,$$

where $a,b,c\in[0,1)$, with $a+b+c<1$, satisfies (1), (2a) with $h=\frac{a+c}{1-b}\in[0,1)$, (2b) with $h=\frac{a+b}{1-c}\in[0,1)$, and (3).

Example 11. The function $F(t_1,t_2,t_3,t_4,t_5):\mathbb{R}_+^5\to\mathbb{R}$ given by

$$F(t_1,...,t_5)=t_1-at_2-bt_3-ct_4-dt_5,$$

where $a,b,c,d\in[0,1)$, with $a+b+c+2d<1$, satisfies (1), (2a) with $h=\frac{a+c}{1-b}\in[0,1)$, (2b) with $h=\frac{a+b}{1-c}\in[0,1)$, and (3), where $h=\frac{a+c}{1-b}\in[0,1)$, if $\max\{v,w\}=v$ and $h=\frac{a+b}{1-c}\in[0,1)$, if $\max\{v,w\}=w$.

Example 12 ([36]). The function $F(t_1,t_2,t_3,t_4,t_5):\mathbb{R}_+^5\to\mathbb{R}$ given by

$$F(t_1,...,t_5)=t_1-a\max\left\{t_2,\frac{t_3+t_4}{2},t_5\right\},$$

where $a\in[0,1)$, satisfies (1), (2a), (2b) and (3), respectively when $\max=t_2$ or $\max=t_5$, then $h=a$, when $\max=\frac{t_3+t_4}{2}$, then $h=\frac{\frac{a}{2}}{1-\frac{a}{2}}$.

Example 13 ([40]). The function $F(t_1,t_2,t_3,t_4,t_5):\mathbb{R}_+^5\to\mathbb{R}$ given by

$$F(t_1,...,t_5)=t_1-c\max\{t_2,t_3,t_4,t_5\},$$

where $h=c\in[0,1)$, satisfies (1), (3), when $\max=t_2$, $\max=t_4$ or $\max=t_5$ is satisfied (2a) and when $\max=t_3$ is satisfied (2b).

Example 14 ([40]). The function $F(t_1,t_2,t_3,t_4,t_5):\mathbb{R}_+^5\to\mathbb{R}$ given by

$$F(t_1,...,t_5)=t_1^2-c\max\left\{t_2t_3,t_2t_4,t_3t_4,t_5^2\right\},$$

where $c\in[0,1)$, satisfies (1), (2a) and (3), with $h=c$.

5. Main results

Using the common fixed point theorem of Imdad and Ali [15], we give the following general stability result for the common fixed point iteration procedure of Jungck-type using weakly compatible mappings defined by an implicit contraction condition satisfying (E.A) property.

Theorem 2. *Let (X,d) be a complete metric space and $S,T:X\to X$ be two mappings, such that T and S satisfy (E.A) property and $S(X)$ is a complete subspace of X.*

There exists $F\in\mathbb{F}$ such that

$$F\left(d(Tx,Ty),d(Sx,Sy),d(Sx,Ty),d(Sy,Tx),\frac{d(Sx,Tx)+d(Sy,Ty)}{2}\right)\leq 0, \tag{5.1}$$

for all $x,y\in X$. Then

(1) *if F satisfies* (2b), then the pair (T,S) has a point of coincidence;
(2) if F satisfies (3), the pair (T,S) has a common fixed point provided it is weakly compatible;
(3) if F satisfies (3), then the associated iterative procedure is (S,T)-stable.

Proof. Since T and S satisfy (E.A) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty}Tx_n=\lim_{n\to\infty}Sx_n=t,\quad t\in X.$$

Since $S(X)$ is a complete subspace of X, every convergent sequence of points of $S(X)$ has a limit in $S(X)$. Therefore,

$$\lim_{n\to\infty} Sx_n = t = Sz = \lim_{n\to\infty} Tx_n = t, \quad z \in X$$

which in turn yields that $t = Sz \in S(X)$.

Assert that $Sz = Tz$. If not, then $d(Tz, Sz) > 0$ and using (5.1), we have

$$F\left(d(Tz,Tx_n), d(Sz,Sx_n), d(Sz,Tx_n), d(Sx_n,Tz), \frac{d(Sz,Tz)+d(Sx_n,Tx_n)}{2}\right) \leq 0$$

which by letting $n \to \infty$ reduces to

$$F\left(d(Tz,t), d(Sz,t), d(Sz,t), d(t,Tz), \frac{d(Sz,Tz)+d(t,t)}{2}\right) \leq 0$$

or

$$F\left(d(Tz,Sz), 0, 0, d(Sz,Tz), \frac{d(Sz,Tz)+0}{2}\right) \leq 0$$

and according to (2b), there exists $h \in [0,1)$ such that

$$d(Tz,Sz) \leq h \max\left\{0, \frac{d(Sz,Tz)}{2}\right\} = h\frac{d(Sz,Tz)}{2} < d(Sz,Tz),$$

a contradiction.

Hence $Tz = Sz$, so z is a coincidence point of T and S.

Since S and T are weakly compatible, then

$$St = STz = TSz = Tt.$$

Now, assert that $Tt = t$. If not, then $d(Tt,t) > 0$. Again, using (5.1),

$$F\left(d(Tt,Tz), d(Sz,St), d(St,Tz), d(Sz,Tt), \frac{d(St,Tt)+d(Sz,Tz)}{2}\right) \leq 0$$

or

$$F\left(d(Tt,t), d(Tt,t), d(Tt,t), d(Tt,t), 0\right) \leq 0$$

which contradicts property (3). Hence, $Tt = t$ which shows that t is a common fixed point of T and S and the uniqueness of the common fixed point is an easy consequence of implicit relation (5.1).

In order to prove the (S,T)-stability, we take the sequence $\{Sx_n\}$ generated by the general iterative procedure $Sx_{n+1} = Tx_n$, $n = 1,2,...$, for any $x_0 \in X$, which converges to t, the common fixed point of the iterative procedure.

Let $\{Sy_n\} \subset X$ be an arbitrary sequence and set

$$\epsilon_n = d(Sy_{n+1}, Ty_n), \quad n = 0,1,2,\ldots .$$

By definition, the iterative procedure is (S,T)-stable if and only if

$$\lim_{n\to\infty} \epsilon_n = 0 \quad implies\ that \quad \lim_{n\to\infty} Sy_n = t.$$

Assume that $\lim_{n\to\infty}\epsilon_n = 0$. Then

$$d(Sy_{n+1},t) \le d(Sy_{n+1},Ty_n) + d(Ty_n,t) = \epsilon_n + d(Ty_n,t). \tag{5.2}$$

If we take $x := t$ and $y := y_n$ in (5.1), then we obtain $F(u,v,u,v,w) \le 0$, where $u := d(Ty_n,t)$, $v := d(Sy_n,t)$, $w := \frac{1}{2}d(Sy_n,Ty_n)$. Now, since F satisfies (2a), there exists $h \in [0,1)$ such that $u \le h\max\{v,w\}$, respectively $d(Ty_n,t) \le h\max\left\{d(Sy_n,t), \frac{d(Sy_n,Ty_n)}{2}\right\}$. We discuss two cases.

In the first case, when $\max\left\{d(Sy_n,t), \frac{d(Sy_n,Ty_n)}{2}\right\} = d(Sy_n,t)$, it yields that $d(Ty_n,t) \le hd(Sy_n,t)$, and then

$$d(Sy_{n+1},t) \le hd(Sy_n,t) + \epsilon_n$$

and applying Lemma 1 we get the conclusion.

For the second case, if $\max\left\{d(Sy_n,t), \frac{d(Sy_n,Ty_n)}{2}\right\} = \frac{d(Sy_n,Ty_n)}{2}$, we have

$$d(Ty_n,t) \le \frac{h}{2}d(Ty_n,Sy_n) \le \frac{h}{2}d(Ty_n,t) + \frac{h}{2}d(t,Sy_n).$$

Then,

$$(1-\frac{h}{2})d(Ty_n,t) \le \frac{h}{2}d(t,Sy_n), \text{ so,}$$

$$d(Ty_n,t) \le \frac{\frac{h}{2}}{1-\frac{h}{2}}d(Sy_n,p).$$

We denote $q := \frac{\frac{h}{2}}{1-\frac{h}{2}} \in [0,1)$, because $h \in [0,1)$ and then we get

$$d(Ty_n,t) \le qd(Sy_n,t), \text{ so,}$$

$$d(Sy_{n+1},t) \le qd(Sy_n,t) + \epsilon_n.$$

Consequently, the conclusion follow by applying Lemma 1. □

Remark 1. Theorem 2 completes Theorem 3.1 in Imdad and Ali [15] with the information about the stability of the Jungck-type iterative procedure with respect to the mappings S and T, provided that the function F satisfies a certain condition.

Corollary 1. *Let (X,d) be a complete metric space and $S,T : X \to X$ be two mappings, such that T and S satisfy (E.A) property and $S(X)$ is a complete subspace of X.*

Suppose there exists $F \in \mathbb{F}$ such that F satisfies (5.1), for all $x,y \in X$.

Then, the Jungck-type iterative procedure is (S,T)-stable.

Proof. We apply Theorem 2, with F given by Example 4 and then we obtain a stability result for Jungck's contraction principle, see [20]. □

Corollary 2. *Let (X,d) be a complete metric space and $S,T : X \to X$ be two mappings, such that T and S satisfy (E.A) property and $S(X)$ is a complete subspace of X.*

Suppose there exists $F \in \mathbb{F}$ such that F satisfies (5.1), for all $x, y \in X$.

Then, in the case of Zamfirescu's contraction conditions, the associated common fixed point iterative procedure is (S,T)-stable.

Proof. We apply Theorem 2, with F given by Example 5 and then we obtain a stability result for the Zamfirescu's fixed point theorem, see [50], corresponding to a pair of mappings with a common fixed point. □

Remark 2. **Particular cases of Theorem 2.**

(1) If F is given by Example 6, then we obtain a stability result for the Kannan's fixed point theorem, see [23], corresponding to a pair of mappings with a common fixed point;
(2) If F is given by Example 7, then we obtain a stability result for a fixed point theorem obtained by Reich (1971) and Rus (1971), see [48], corresponding to a pair of mappings with a common fixed point;
(3) If F is given by Example 8, then we obtain a stability result for the Chatterjea's fixed point theorem, see [11], corresponding to a pair of mappings with a common fixed point;
(4) If F is given by Example 11, then we obtain a stability result for the Hardy and Rogers's fixed point theorem, see [14], corresponding to a pair of mappings with a common fixed point;
(5) If F is given by Example 12, then we obtain a stability result for the Pathak and Verma's fixed point theorem, see [36], corresponding to a pair of mappings with a common fixed point in symmetric spaces.
(6) If F is given by Examples 13 and 14, then we obtain stability results for the Popa's fixed point theorem, see [40], corresponding to two pairs of mappings on two metric spaces.

6. Concluding remarks

The results obtained in this paper generalize classical fixed point theorems existing in the literature: Jungck [20], Zamfirescu [50], Kannan [23], Reich and Rus [48], Chatterjea [11], Hardy and Rogers [14] and most of their references.

The contractive conditions obtained from (5.1) with F as in Examples 1-11, imply contractive conditions used by Rhoades in [41], [42], [44], [43].

Because of the inclusions between the commutativity definitions, the weakly compatible pair of mappings is the most general type from the mentioned notions and it includes the others. The above theorem use this kind of weakly compatible mappings and it follows that it holds also for compatible, commuting and weakly commuting pair of mappings.

Olaleru [24] showed that the Jungck iteration procedure can be used to approximate the common fixed points of some weakly compatible pair of mappings defined on metric spaces. These are generalizations and extensions of the results of Berinde [7].

Recently, many authors used implicit relations in order to prove common fixed points theorems for weakly compatible mappings satisfying different contractive conditions and various properties.

Bouhadjera and Djoudi [10] proved a common fixed point theorem for four weakly compatible mappings satisfying an implicit relation without need of continuity. This theorem generalizes some results on compatible continuous mappings of Popa [38].

Aliouche [3] proved common fixed point theorems for weakly compatible mappings in metric spaces satisfying an implicit relation using (E.A) property and a common (E.A) property, which generalizes the results of Aamri and Moutawakil [1].

Aliouche [4] also proved common fixed point theorems for weakly compatible mappings satisfying implicit relations without the condition that the map to be decreasing in any variable. These theorems improve results of Ali and Imdad [2], Jeong and Rhoades [19] and Popa [39].

It is possible to obtain corresponding stability results for the above mentioned common fixed point theorems in the presence of implicit contractive conditions, a task which will be completed in forthcoming papers.

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CHARACTERIZATIONS OF Rad-SUPPLEMENTED MODULES

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Abstract. We prove that a commutative ring R is an artinian principal ideal ring if and only if the ring is semilocal and every Rad-supplemented R-module is a direct sum of w-local R-modules. Moreover, we study of extensions of Rad-supplemented modules over commutative noetherian rings, and we show that if $\frac{M}{N}$ is reduced, M is Rad-supplemented if and only if N and $\frac{M}{N}$ are Rad-supplemented. We also prove that over a dedekind domain an indecomposable, amply Rad-supplemented radical module is hollow radical.

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1. INTRODUCTION

In this note R will be an associative ring with identity. Unless otherwise mentioned, all modules will be unital left R-modules. Let R be such a ring and M be an R-module. The notation $N \subseteq M$ means that N is a submodule of M. A submodule S of M is called *small* in M, denoted by $S << M$, if $S + N \neq M$ for every proper submodule N of M. We denote by $\mathrm{Rad}(M)$ the radical of M. A non-zero module M is called *hollow* if every proper submodule of M is small in M, and it is called *local* if it is hollow and $\mathrm{Rad}(M)$ is a maximal submodule of M. Let M be a module. M is called *supplemented* if every submodule N of M has a *supplement*, that is a submodule K of M minimal with respect to $N + K = M$. Equivalently, $N + K = M$ and $N \cap K << K$ ([12]). Following [12], M is called *amply supplemented* if, for any two submodules U and V of M with $U + V = M$, V contains a supplement of U in M. Clearly, hollow modules are amply supplemented and amply supplemented modules are supplemented.

Recall from Lomp [7] that a module M is said to be *semilocal* if $\frac{M}{\mathrm{Rad}(M)}$ is semisimple, and a ring R is said to be *semilocal* if it is semilocal as a left (right) module over itself. It is shown in [7, Teorem 3.5] that a ring R is semilocal if and only if every left R-module is semilocal.

As a proper generalization of supplemented modules, the notion of Rad-supplemented modules, which has been introduced by Xue [13], has been studied recently

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(see [1, 4, 5]). Let M be a module and N be a submodule of M. A submodule K of M is called a Rad-*supplement* of N in M (according to [13], *generalized supplement*) if $N + K = M$ and $N \cap K \subseteq \mathrm{Rad}(K)$. Since $\mathrm{Rad}(K)$ is the sum of all small submodules of K, every supplement submodule is a Rad-supplement in M. A module M is called Rad-*supplemented* (according to [13], *generalized supplemented*) if every submodule N of M has a Rad-supplement K in M, and it is called *amply* Rad-*supplemented* (according to [13], *generalized amply supplemented*) if every submodule N of M has *ample* Rad-*supplements* in M, i. e., $N + L = M$ implies that N has a Rad-supplement $K \subseteq L$. In [5], the various properties of Rad-supplemented modules are extensively studied. In addition, it is shown in [1, 2.2.(2) and 2.3.(3)] that factor modules of a Rad-supplemented module and finite sums of Rad-supplemented modules are Rad-supplemented. It is of obvious interest to investigate extensions and characterizations of Rad-supplemented modules. This is the focus of our investigations in this paper.

Let Γ be a class of modules and let $0 \to N \to M \to K \to 0$ be any short exact sequence. Here M is an *extension* of N by K and Γ is called *closed under extensions* if $N, K \in \Gamma$ implies $M \in \Gamma$. It is clear that, for modules $N \subseteq M$, M is an extension of N.

In this article, we prove that a commutative ring R is an artinian principal ideal ring if and only if the ring is semilocal and every Rad-supplemented R-module is a direct sum of w-local R-modules if and only if every left R-module is a direct sum of w-local R-modules. We give a characterization of semisimple rings via Rad-supplements. We show that a semilocal ring R is left perfect if and only if every Rad-supplemented module is (generalized) semiperfect. Some examples are given in order to show that the class of Rad-supplemented modules is not generally closed under extensions. Let R be a commutative noetherian ring and M be an R-module with $N \subseteq M$. If $\frac{M}{N}$ is reduced, M is Rad-supplemented if and only if N and $\frac{M}{N}$ are Rad-supplemented. It follows that a ring R is semilocal if and only if every left R-module with Rad-supplemented radical is Rad-supplemented. Over a dedekind domain a radical module is amply Rad-supplemented and indecomposable if and only if the module is hollow radical. Every indecomposable, w-local and amply Rad-supplemented module over a dedekind domain is local.

2. Rad-SUPPLEMENTED MODULES OVER ANY RINGS

Let R be any ring and M be an R-module. A submodule N of M is called *radical* if N has no maximal submodules, i.e. $N = \mathrm{Rad}(N)$. Note that radical modules are Rad-supplemented. This fact plays a key role in our study. By $P(M)$ we denote the sum of all radical submodule of a module M. It is clear that, for any module M, $P(M)$ is the largest radical submodule and so $P(M)$ is Rad-supplemented. Using the mentioned facts, we give examples of a module, which is Rad-supplemented but not supplemented. We see, for example, the left $\mathbb{Z}$-module $M =_{\mathbb{Z}} \mathbb{Q}$.

Firstly we have the following lemma.

Lemma 1. *Let M be a module and $N \subseteq U \subseteq M$. Then U is* Rad*-supplemented if and only if $\frac{U}{P(N)}$ is* Rad*-supplemented.*

Proof. ($\Rightarrow$) Let U be Rad-supplemented. By [1, 2.2 (2)], $\frac{U}{P(N)}$ is Rad-supplemented as a factor module of U.

($\Leftarrow$) Let $U^{'}$ be any submodule of U. By the assumption, there exists a submodule $\frac{V}{P(N)}$ of $\frac{U}{P(N)}$ such that $\frac{U^{'}+P(N)}{P(N)} + \frac{V}{P(N)} = \frac{U}{P(N)}$ and

$$\Big(\frac{U^{'}+P(N)}{P(N)}\Big)\bigcap\Big(\frac{V}{P(N)}\Big) \subseteq \mathrm{Rad}\Big(\frac{V}{P(N)}\Big).$$

Then $(U^{'}+P(N))+V = U$ and hence $U^{'}+V = U$. Since $P(N) = \mathrm{Rad}(P(N)) \subseteq \mathrm{Rad}(V)$, it follows that $\frac{U^{'}\cap V+P(N)}{P(N)} = \frac{(U^{'}+P(N))\cap V}{P(N)} = (\frac{U^{'}+P(N)}{P(N)}) \cap (\frac{V}{P(N)}) \subseteq \mathrm{Rad}(\frac{V}{P(N)}) = \frac{\mathrm{Rad}(V)}{P(N)}$, which means that $U^{'} \cap V \subseteq \mathrm{Rad}(V)$. So V is a Rad-supplement of $U^{'}$ in U. Hence U is Rad-supplemented. □

Corollary 1. *Let M be a module and N be a submodule of M. M is* Rad*-supplemented if and only if $\frac{M}{P(N)}$ is* Rad*-supplemented. In particular, M is* Rad*-supplemented if and only if $\frac{M}{P(M)}$ is* Rad*-supplemented.*

Proof. It follows from Lemma 1. □

Recall from [5, Corollary 4.2] that if a submodule V of a module M is a Rad-supplement in M, then $\mathrm{Rad}(V) = V \cap \mathrm{Rad}(M)$.

Now we shall show that the rings whose modules are Rad-supplement submodules in every extension are semisimple in the following theorem.

Theorem 1. *Let R be any ring. Then the following statements are equivalent.*

(1) *R is semisimple.*
(2) *Every left R-module is a* Rad*-supplement in every extension.*
(3) *Every left R-module is a* Rad*-supplement in every injective extension .*
(4) *Every left ideal of R is a* Rad*-supplement in every injective extension.*

Proof. (1) $\Rightarrow$ (2) Let N be an R-module and M be any extension of N. By the hypothesis and [6, Corollary 8.2.2 (a)], M is semisimple, and so N is a direct summand of M. It follows that N is a Rad-supplement in M.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (1) Let I be any left ideal of R. By the hypothesis, I is a Rad-supplement in its injective hull $E(I)$. Then we have $I + J = E(I)$ and $I \cap J \subseteq \mathrm{Rad}(I)$ for some submodule $J \subseteq E(I)$. If $m \in I \cap J$, then $Rm \subseteq \mathrm{Rad}(I) \subseteq \mathrm{Rad}(E(I))$. By (4), Rm is a Rad-supplement in $E(I)$ and so $\mathrm{Rad}(Rm) = Rm \cap \mathrm{Rad}(E(I)) = Rm$.

Therefore $m = 0$. This means that $I \oplus J = E(I)$ and so I is injective, and hence a direct summand of R. By [6, Corollary 8.2.2 (a)], R is semisimple. □

A ring R is Rad-supplemented if ${}_RR$ (or R_R) is a Rad-supplemented module. It is clear that semiperfect (i.e., supplemented) rings are Rad-supplemented. Characterizations of semiperfect rings have been studied extensively by many authors recently. Now we shall give a characterization of Rad-supplemented rings. Firstly, we need the following simple lemmas.

Lemma 2. *Let R be any ring with identity. Then R is* Rad*-supplemented if and only if every cyclic R-module is* Rad*-supplemented.*

Proof. Let R be a Rad-supplemented ring. Suppose that M is any cyclic R-module. Then there exists an element m of M such that $M = Rm$. Note that $\frac{R}{\mathrm{Ann}(m)} \cong Rm$, where $\mathrm{Ann}(m)$ is the set of all elements r of R such that $rm = 0$. From [1, 2.2.(2)] the hypothesis implies that $\frac{R}{\mathrm{Ann}(m)}$ is Rad-supplemented and so Rm is Rad-supplemented. The converse is clear. □

Lemma 3. *Let M be a module with $U + V = M$ for submodules U, V of M. If V contains a* Rad*-supplement of U in M, then $U \cap V$ has a* Rad*-supplement in V.*

Proof. Suppose that a submodule K of V is a Rad-supplement of U in M. Then, we have $U + K = M$ and $U \cap K \subseteq \mathrm{Rad}(K)$. From the modular law, $U \cap V + K = V$. Since $K \subseteq V$, then $(U \cap V) \cap K = U \cap K \subseteq \mathrm{Rad}(K)$. So K is a Rad-supplement of $U \cap V$ in V. □

Theorem 2. *The following statements are equivalent for any ring R.*

(1) *R is* Rad*-supplemented.*
(2) *R has ample* Rad*-supplements in every finitely generated extension.*
(3) *Every cyclic R-module has ample* Rad*-supplements in every finitely generated extension.*

Proof. (1) $\Rightarrow$ (3) Let N be any cyclic R-module and M be any finitely generated extension of N. Since R is Rad-supplemented, by Lemma 2, every cyclic submodule of M is Rad-supplemented and so M is amply Rad-supplemented by [11, Corollary 3.6]. Therefore N has ample Rad-supplements in M.

(3) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (1) For any left ideal I of R, consider the finitely generated pushout R-module $N = \frac{R \oplus R}{K}$, where K is the set of all elements k of $R \oplus R$ such that $k = (r, -r)$ for all $r \in I$. Then there exist monomorphisms $f, g : R \to N$ such that $N = f(R) + g(R)$. The hypothesis implies that $f(R)$ has a Rad-supplement V in N with $V \subseteq g(R)$. So, by Lemma 3, V is a Rad-supplement of $f(R) \cap g(R)$ in $g(R)$. Note that $I = g^{-1}(f(R) \cap g(R))$. It follows that $R = I + g^{-1}(V)$ and $I \cap g^{-1}(V) \subseteq \mathrm{Rad}(g^{-1}(V))$. Hence R is Rad-supplemented. □

We say that a module M *w-local* if $\mathrm{Rad}(M)$ is a maximal submodule of M as in [4]. Every local module is w-local. It is well known that a commutative ring R has the property that every R-module is a direct sum of local R-modules if and only if R is an artinian principal ideal ring. Now, we prove that if R is a commutative ring and every R-module is a direct sum of w-local R-modules, then R is an artinian principal ideal ring in the following theorem.

Theorem 3. *The following are equivalent for a commutative ring R.*

(1) *Every left R-module is a direct sum of w-local R-modules.*
(2) *R is semilocal and every* Rad*-supplemented left R-module is a direct sum of w-local R-modules.*
(3) *R is an artinian principal ideal ring.*

Proof. (1) $\Rightarrow$ (2) Write $\frac{R}{\mathrm{Rad}(R)} = \bigoplus_{i\in I} N_i$, where each N_i is w-local. Since $\mathrm{Rad}(\frac{R}{\mathrm{Rad}(R)}) = 0$, for all $i \in I$, $\mathrm{Rad}(N_i) = 0$. So N_i is simple. Thus $\frac{R}{\mathrm{Rad}(R)}$ is semisimple and so R is semilocal. The rest of the proof is clear.

(2) $\Rightarrow$ (3) Let $F = R^{(\Lambda)}$any index set Λ. Suppose that $\mathrm{Rad}(\frac{F}{N}) = \frac{F}{N}$ for some submodule N of F. By the assumption, we can write $\frac{F}{N} = \bigoplus_{i\in I} M_i$ where M_i is w-local for all $i \in I$. By [12, 21.6.(5)], $\mathrm{Rad}(\frac{F}{N}) = \bigoplus_{i\in I} \mathrm{Rad}(M_i)$ and so each M_i is radical as a direct summand of $\frac{F}{N}$. Since M_i is w-local, we obtain that, for all $i \in I$, $M_i = 0$. Therefore $\frac{F}{N} = 0$. This means that $\mathrm{Rad}(F) << F$. It follows from [12, 43.9] that R is left perfect. Applying [12, 43.9] again, we deduce that every left R-module is Rad-supplemented and so every left R-module is a direct sum of w-local R-modules. If N is aw-local, then N is local because R is left perfect. Hence every left R-module is a direct sum of cyclic R-modules. By [9, Theorem 6.7], R is an artinian principal ideal ring.

(3) $\Rightarrow$ (1) is clear. □

The following corollary is an immediate consequence of Theorem 3.

Corollary 2. *Let R be a commutative semilocal ring. Then, R is an artinian principal ideal ring if and only if every* Rad*-supplemented left R-module is a direct sum of w-local R-modules.*

Let $f : P \to M$ be an epimorphism. Xue [13] calls f a *(generalized) cover* if $(\mathrm{Ker}(f) \subseteq \mathrm{Rad}(P))$ $\mathrm{Ker}(f) << P$, and calls a (generalized) cover f a *(generalized) projective cover* if P is a projective module. In the spirit of [13], a module M is said to be *(generalized) semiperfect* if every factor module of M has a (generalized) projective cover. He [13, Theorem 2.2] proved that every generalized semiperfect module is Rad-supplemented. Now, we obtain the following result.

Proposition 1. *Let R be a semilocal ring. Every* Rad*-supplemented left R-module is (generalized) semiperfect if and only if R is left perfect.*

Proof. ($\Rightarrow$) Let $M = \mathrm{Rad}(M)$. Since M is Rad-supplemented, it follows from the hypothesis that M is generalized semiperfect. Then, there exists a generalized cover $f : F \to M$ with a projective module F. Since $\mathrm{Ker}(f) \subseteq \mathrm{Rad}(F) \neq F$, it follows that $M = 0$. By [12, 43.9], R is left perfect.

($\Leftarrow$) This is immediate. □

3. Rad-SUPPLEMENTED MODULES OVER COMMUTATIVE NOETHERIAN RINGS

Throughout this section, unless otherwise stated, we shall consider commutative noetherian rings.

An R-module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M, and it is called *reduced* if every submodule of M contains a maximal submodule, that is, $P(M) = 0$. Note that $\mathrm{Rad}(M)$ is small in M for every coatomic R-module M.

Lemma 4. *The following statements are equivalent for a* Rad*-supplemented module M.*

(1) *M is coatomic.*
(2) *M is reduced.*
(3) *$\mathrm{Rad}(M)$ is small in M.*

If the module M satisfies one of the equivalent conditions, then M is supplemented.

Proof. (1) $\Rightarrow$ (2) Let M be a coatomic module. By [15, Lemma 1.1], every submodule of M is coatomic and so $P(M) = 0$, which means that M is reduced.

(2) $\Rightarrow$ (3) Suppose that $M = \mathrm{Rad}(M) + N$ for some submodule N of M. Then we can write $\mathrm{Rad}(\frac{M}{N}) = \frac{M}{N}$. Since M is Rad-supplemented, N has a Rad-supplement V in M. From (2) it follows that V has a maximal submodule K. So $\frac{K}{N\cap V}$ is a maximal submodule of $\frac{V}{N\cap V}$. Note that

$$\frac{M}{N} \cong \frac{V}{N\cap V}$$

contains a maximal submodule and thus $\frac{M}{N} = 0$. Therefore $M = N$. This proves (3).

(3) $\Rightarrow$ (1) The assumption implies that, for any proper submodule $U \subseteq M$, there exists a submodule V of M such that $U + V = M$ and $U \cap V \subseteq \mathrm{Rad}(V)$. Since $\mathrm{Rad}(M) << M$, V is not contained in a maximal submodule K of M. Then the submodule $U + V \cap K$ of M is maximal. Thus M is coatomic.

Suppose that Rad-supplemented module M satisfies one of these conditions. Then M is supplemented by [5, Proposition 7.3]. □

The following result follows from [5, Proposition 7.3]. We give this result as a consequence of Lemma 4.

Corollary 3. *For a module M, M is* Rad*-supplemented if and only if $\frac{M}{P(M)}$ is supplemented.*

A submodule of a Rad-supplemented module need not be Rad-supplemented, in general. To see this actuality, we shall consider the left $\mathbb{Z}$-module $M = {}_{\mathbb{Z}}\mathbb{Q}$. It is well known that M is Rad-supplemented. On the other hand, the submodule ${}_{\mathbb{Z}}\mathbb{Z}$ of M is not semisimple.

Now, we show that a submodule of a Rad-supplemented module is Rad-supplement ed under a certain condition.

Proposition 2. *Let M be a module and $N \subseteq M$. Suppose that $\frac{M}{N}$ is reduced. If M is* Rad*-supplemented, then N is* Rad*-supplemented.*

Proof. According to [1, 2.2.(2)], $\frac{M}{N}$ is Rad-supplemented as a factor module of M. Since $\frac{M}{N}$ is reduced, $P(\frac{M}{N}) = 0$. Therefore $\frac{M}{N}$ is supplemented by Lemma 4. Since M is Rad-supplemented, $\frac{M}{P(N)}$ is Rad-supplemented by Corollary 1. Note that

$$\frac{\frac{M}{P(N)}}{\frac{N}{P(N)}} \cong \frac{M}{N}$$

is reduced and thus $\frac{M}{P(N)}$ is reduced by [14, Lemma 1.5 (a)]. It follows from Lemma 4 that $\frac{M}{P(N)}$ is supplemented. Thus $\frac{N}{P(N)}$ is supplemented by [8, Proposition 2.6]. So $\frac{N}{P(N)}$ is Rad-supplemented. Hence N is Rad-supplemented by Lemma 1. □

Using Proposition 2, we obtain the following result.

Corollary 4. *The following statements are equivalent for any module M.*

(1) *M is* Rad*-supplemented.*
(2) *Every maximal submodule of M is* Rad*-supplemented.*
(3) *Every cofinite submodule of M is* Rad*-supplemented.*

Proof. (1) $\Rightarrow$ (3) If N is a cofinite submodule of M, then $\frac{M}{N}$ is finitely generated and so $\frac{M}{N}$ is reduced. From Proposition 2, the proof follows.

(3) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Let $M = M_1 + M_2$, where M_1 and M_2 are maximal submodules of M. Since M_1 and M_2 are Rad-supplemented modules, M is Rad-supplemented according to [1, 2.3.(3)]. If M is w-local, $\mathrm{Rad}(M)$ is maximal and so $M = \mathrm{Rad}(M) + U$ for every proper submodule U of M with $U \not\subseteq \mathrm{Rad}(M)$. By [1, 2.3.(1)], U has a Rad-supplement in M since $\mathrm{Rad}(M)$ is Rad-supplemented. Hence M is Rad-supplemented. □

The following example shows that the class of Rad-supplemented modules is not closed under extensions, in general.

Example 1. Let Λ be a collection of maximal ideals of the noetherian commutative ring $\mathbb{Z}$. Suppose that M is the left $\mathbb{Z}$-module $\prod_{\mathfrak{p}\in\Lambda}(\frac{\mathbb{Z}}{\mathfrak{p}})$. Then $\mathrm{Rad}(M) = 0$. By [3, Lemma 2.9], for some submodule N of M, we have $\frac{N}{T} \cong \mathbb{Q}$, where T is the

direct sum of simple $\mathbb{Z}$-modules $\frac{\mathbb{Z}}{p}$. Then N is an extension of T by $\mathbb{Q}$. Since T is semisimple, it is Rad-supplemented. On the other hand, the submodule N is not Rad-supplemented.

Later we shall give another example of such modules (see Example 2).

Theorem 4. *Let $0 \to N \to M \to K \to 0$ be a short exact sequence. Suppose that K is reduced. Then M is* Rad*-supplemented if and only if N and K are* Rad*-supplemented.*

Proof. ($\Rightarrow$) It follows from Proposition 2 and [1, 2.2.(2)].
($\Leftarrow$) By Lemma 4, K is supplemented. Since N is Rad-supplemented, $\frac{N}{P(N)}$ is supplemented by Corollary 3. It follows from [8, Proposition 2.6] that $\frac{M}{P(N)}$ is Rad-supplemented. Hence M is Rad-supplemented by Corollary 1. □

Corollary 5. *A module M is* Rad*-supplemented if and only if it is an extension of a* Rad*-supplemented submodule by a reduced supplemented module.*

Proof. If M has no maximal submodules, the result is obvious as $\frac{M}{P(M)} = 0$. Suppose that $M \neq P(M)$. Then this gives the existence of a reduced factor module of M. Therefore the assertion follows from Theorem 4. □

Proposition 3. *Let M be a module. M is* Rad*-supplemented if and only if M is semilocal and* Rad(M) *is* Rad*-supplemented.*

Proof. If M is Rad-supplemented, then M is semilocal. Thus $\frac{M}{\mathrm{Rad}(M)}$ is reduced. By Proposition 2, Rad(M) is Rad-supplemented . Conversely, suppose that M is semilocal and Rad(M) is Rad-supplemented. From Theorem 4 the assumption implies that M is Rad-supplemented. □

Using the above proposition we obtain the following characterization of semilocal rings.

Corollary 6. *The following conditions on a ring R is equivalent:*

(1) *R is semilocal.*
(2) *Every left R-module with* Rad*-supplemented radical is* Rad*-supplemented.*

Proof. (1) $\Rightarrow$ (2) If R is semilocal, then every left R-module is semilocal by [7, Theorem 3.5]. The result follows from Proposition 3.
(2) $\Rightarrow$ (1) Since Rad$(\frac{R}{\mathrm{Rad}(R)}) = 0$, it follows from the hypothesis that $\frac{R}{\mathrm{Rad}(R)}$ is Rad-supplemented. So $\frac{R}{\mathrm{Rad}(R)}$ is semisimple, i.e. R is semilocal. □

In [5], a module M is said to be *totally* Rad*-supplemented* if every submodule of M is Rad-supplemented. Every semisimple module is totally Rad-supplemented. It is easy to check that the class of totally Rad-supplemented modules is closed under factor modules and submodules. The following fact is a modification of Theorem 4.

Theorem 5. *Let M be a module and $\frac{M}{N}$ be reduced for some submodule N of M. Then M is totally* Rad*-supplemented if and only if N and $\frac{M}{N}$ are totally* Rad*-supplemented.*

Proof. Suppose that N and $\frac{M}{N}$ are totally Rad-supplemented. Let U be any submodule of M. By the hypothesis, $U \cap N$ and $\frac{U+N}{N}$ are Rad-supplemented. Note that

$$\frac{U+N}{N} \cong \frac{U}{U \cap N}$$

is reduced because $\frac{M}{N}$ is reduced. By Theorem 4, U is Rad-supplemented. Hence M is totally Rad-supplemented. □

Corollary 7. *Let M be a* Rad*-supplemented module. Then, M is totally* Rad*-supplemented if and only if $P(M)$ is totally* Rad*-supplemented.*

Proof. Suppose that $P(M)$ is totally Rad-supplemented. By the hypothes is and Corollary 3, $\frac{M}{P(M)}$ is supplemented. Applying [8, Proposition 2.6], we deduce that $\frac{M}{P(M)}$ is totally supplemented. Therefore M is totally Rad-supplemented by Theorem 5. □

4. Rad-SUPPLEMENTED MODULES OVER COMMUTATIVE DOMAINS

In this section a ring R will be a commutative domain. Let R be such a ring and M be an R-module. We denote by $T(M)$ the set of all elements m of M for which there exists a non-zero element r of R such that $rm = 0$, i.,e., $\mathrm{Ann}(m) \neq 0$. Then $T(M)$, which is a submodule of M, called *the torsion submodule* of M. If $M = T(M)$, then M is called a *torsion module* and M is called *torsion-free* provided $T(M) = 0$.

Proposition 4. *Let R be a non-semilocal commutative domain and M be an R-module. If M is totally* Rad*-supplemented, M is a torsion module.*

Proof. Let $0 \neq m \in M$. Suppose that $\mathrm{Ann}(m) = 0$, i.e. $R \cong Rm$. Since M is totally Rad-supplemented, the left R-submodule Rm of M is Rad-suppleme nted. So ${}_RR$ is Rad-supplemented. Therefore $\frac{R}{\mathrm{Rad}(R)}$ is semisimp le, i.e. R is semilocal. This contradicts the assumption. Hence $\mathrm{Ann}(m) \neq 0$, this implies that M is torsion. □

Corollary 8. *Let R be a non-semilocal dedekind domain and M be a totally* Rad*-supplement ed R-module. Then M is torsion.*

Let R be a dedekind domain and M be an R-module. We denote by Ω the set of all maximal (i.e., prime) ideals of R. Suppose that $\mathfrak{p}$ is any element of Ω. We denote by $T_{\mathfrak{p}}(M)$, which is a submodule of M, the set of all elements m of M for which there exists a positive integer n such that $\mathfrak{p}^n m = 0$. Then $T_{\mathfrak{p}}(M)$ is called the $\mathfrak{p}$-*primary part* of M. For a torsion module M over a dedekind domain, we have the decomposition $M = \bigoplus_{\mathfrak{p}\in\Omega} T_{\mathfrak{p}}(M)$.

Lemma 5. *Let R be a non-local dedekind domain and M be an R-module. Then M is* Rad*-supplemented if and only if $\frac{M}{P(M)}$ is torsion and every $\mathfrak{p}$-primary part of $\frac{M}{P(M)}$ is (*Rad*-)supplemented.*

Proof. According to [14, Theorem 3.1] and [5, Theorem 7.4], the proof of the lemma is clear. □

Let R be a dedekind domain and M be an R-module. By [2, Lemma 4.4], $P(M)$ is injective and so there exists a direct summand N of M such that $\frac{M}{P(M)} \cong N$. This fact and Lemma 5 give the following basic result for torsion-free modules.

Corollary 9. *Let M be a torsion-free* Rad*-supplemented module over a non-local dedekind domain. Then M is radical.*

Let M be a radical module. M is called *simply radical* if M has no proper radical submodules.

Proposition 5. *Let R be a noetherian ring and M be a simply radical R-module. If M is amply* Rad*-supplemented, M is hollow radical. In particular, every* Rad*-supplemented proper submodule of M is supplemented.*

Proof. Let U be any proper submodule of M. Suppose that $U+V=M$ for some submodule V of M. By the hypothesis, there exists a submodule V' of V such that $U+V'=M$ and $U\cap V'\subseteq \mathrm{Rad}(V')$. Since M is simply radical, it follows that $\mathrm{Rad}(V')=V'\cap \mathrm{Rad}(M)=V'\cap M=V'$. So V' is radical. Therefore $V'=M$ and so $V=M$. Then we deduce that U is small in M. Hence M is hollow radical. Suppose that a proper submodule N of M is Rad-supplemented. Since M is simply radical, every submodule of N contains a maximal submodule, i. e., $P(N)=0$. By Lemma 4, N is supplemented. □

Corollary 10. *Let R be a dedekind domain and M be a radical R-module. Then M is amply* Rad*-supplemented and indecomposable if and only if the module is hollow radical.*

Proof. Since indecomposable radical modules over dedekind domains is simply radical, M is hollow radical by Proposition 5. The converse is clear. □

Proposition 6. *Let M be a module over a Dedekind domain. Then the following statements are equivalent.*

(1) *M is indecomposable, w-local and amply* Rad*-supplemented.*
(2) *M is local.*

Proof. (1) $\Rightarrow$ (2) Let U be any proper submodule of M. Suppose that U is not contained $\mathrm{Rad}(M)$. Since M is w-local, $\mathrm{Rad}(M)$ is maximal and so $U+\mathrm{Rad}(M)=M$. By the hypothesis, there exists a submodule V of $\mathrm{Rad}(M)$ such that $U+V=M$ and $U\cap V\subseteq \mathrm{Rad}(V)$. It follows that $\mathrm{Rad}(V)=V\cap \mathrm{Rad}(M)=V$, i.e. V is radical.

Then, by [2, Lemma 4.4], V is injective and so there exists a submodule L of M such that $M = V \oplus L$. Since M is indecomposable and w-local, we get $V = 0$. Thus, $U = M$, implying that M is local.

(2) $\Rightarrow$ (1) is clear. □

Now, we give an analogous characterization of [14, Theorem 3.1] for totally Rad-supplemented modules.

Theorem 6. *Let M be a non-semilocal dedekind domain and M be an R-module. Then M is totally* Rad*-supplemented if and only if M is torsion and every $\mathfrak{p}$-primary part of M is totally* Rad*-supplemented.*

Proof. The necessity of the condition is obvious by Corollary 8. Conversely, suppose that M is torsion and every $\mathfrak{p}$-primary part of M is totally Rad-supplemented. Let $N \subseteq U \subseteq M$. Since $M = \bigoplus_{\mathfrak{p}\in\Omega} T_{\mathfrak{p}}(M)$, we have $U = \bigoplus_{\mathfrak{p}\in\Omega}(U \cap T_{\mathfrak{p}}(M))$ and $N = \bigoplus_{\mathfrak{p}\in\Omega}(N \cap T_{\mathfrak{p}}(M))$. By the hypothesis, $N \cap T_{\mathfrak{p}}(M)$ has a Rad-supplement $V_{\mathfrak{p}}$ in $U \cap T_{\mathfrak{p}}(M)$. So $U \cap T_{\mathfrak{p}}(M) = N \cap T_{\mathfrak{p}}(M) + V_{\mathfrak{p}}$ and $N \cap V_{\mathfrak{p}} \subseteq \mathrm{Rad}(V_{\mathfrak{p}})$. Let $V = \bigoplus_{\mathfrak{p}\in\Omega} V_{\mathfrak{p}}$. Then $N + V = U$. Since $N \cap V_{\mathfrak{p}} \subseteq \mathrm{Rad}(V_{\mathfrak{p}})$ for every $\mathfrak{p} \in \Omega$, by [6, Corollaries 9.1.5 (c)], $N \cap V = (\bigoplus_{\mathfrak{p}\in\Omega}(N \cap T_{\mathfrak{p}}(M))) \cap (\bigoplus_{\mathfrak{p}\in\Omega} V_{\mathfrak{p}}) \subseteq \mathrm{Rad}(V)$. Hence U is Rad-supplemented. This completes the proof. □

Finally, we give an example showing the class of (totally) Rad-supplemented modules is not closed under extensions, in general. For a module M, $\mathrm{Soc}(M)$ will indicate the sum of all simple submodules of M.

Example 2. (see [10, Example 2.3]) Consider the non-Noetherian commutative ring which is the direct product $\prod_{i\geq 1}^{\infty} F_i$, where $F_i = F$ is any field. Suppose that R is the subring of the ring consisting of all sequences $(r_n)_{n\in\mathbb{N}}$ such that there exist $r \in F, m \in \mathbb{N}$ with $r_n = r$ for all $n \geq m$. Let $M =_R R$. Then M is a regular module which is not semisimple. Therefore $\mathrm{Soc}(M)$ is a maximal submodule of M. This means that $\mathrm{Soc}(M)$ and $\frac{M}{\mathrm{Soc}(M)}$ are Rad-supplemented. On the other hand, M is not Rad-supplemented.

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APPROXIMATION PROPERTIES OF REAL AND COMPLEX POST-WIDDER OPERATORS BASED ON q-INTEGERS

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Abstract. In this paper, we study the statistical approximation properties of real and complex Post-Widder operators based on q-integers. We also obtain a Voronovskaya-type formula in statistical sense for these operators.

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1. INTRODUCTION

Positive linear operators and their q-modifications have recently become an active research area in the approximation theory (see, i.e., [1,2,6,7,10,12,13,16,18–26,28]). In this paper, we improve the classical Post-Widder operators via the techniques from the q-calculus. We obtain various statistical approximation theorems including some Korovkin-type results and Voronovskaya-type formulas in statistical and ordinary sense for these operators. Our investigations cover both real and complex cases.

Throughout the paper we employ the standard notations of q-calculus. As usual, by a q-integer and a q-factorial we mean respectively:

$$[n]_q := 1 + q + ... + q^n \text{ for } n \in \mathbb{N} \text{ with } [0]_q := 0$$

and

$$[n]_q! := [1]_q [2]_q ... [n]_q \text{ for } n \in \mathbb{N} \text{ with } [0]_q! := 1.$$

Also, a q-binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!} \text{ for } 0 \leq k \leq n.$$

Now let $0 < q < 1$. For simplicity, we use the following standard notations:

$$(1+t)_q^{\infty} := \prod_{j=0}^{\infty} \left(1 + q^j t\right).$$

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Then, we know [14, 15] that the q-Jackson definite integral of a function f is defined by

$$\int_0^a f(t)\,d_q t := a(1-q)\sum_{n=0}^{\infty} f\left(aq^n\right)q^n \text{ for } a>0.$$

Now we recall the q-analogue of the exponential function as follows (see [3, 17]):

$$E_q(x) = \sum_{n=0}^{\infty} q^{n(n-1)/2}\frac{x^n}{[n]_q!} = (1+(1-q)x)_q^{\infty}.$$

Notice that the above series converges for every x whenever $q \in (0,1)$. Then, the corresponding q-Gamma function is defined by (see [17])

$$\Gamma_q(x) := \int_0^{1/(1-q)} t^{x-1}E_q(-qt)\,d_q t.$$

As usual, let $C_B[0,\infty)$ denote the space of all bounded and continuous functions on $[0,\infty)$ endowed with the norm

$$\|f\| = \sup_{x\in[0,\infty)} |f(x)|.$$

In the present paper, we improve the classical Post-Widder operators by considering the above terminology as follows:

$$P_{n,q}(f;x) = \frac{1}{[n]_q!}\int_0^{1/(1-q)} f\left(\frac{xt}{[n]_q}\right)t^n E_q(-qt)\,d_q t, \tag{1.1}$$

where $x \in [0,\infty)$, $q \in (0,1)$ and $f \in C_B[0,\infty)$. We use the following test functions

$$e_m(y) = y^m \text{ for } m \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}.$$

Then, we first observe that these operators are positive and linear and that

$$P_{n,q}(e_0;x) = e_0(x) = 1. \tag{1.2}$$

Notice that the operators $P_{n,q}(f)$ map $C_B[0,+\infty)$ into itself. Also, it follows from (1.1) that

$$\lim_{q\to 1^-} P_{n,q}(f;x) = \frac{1}{n!}\int_0^{\infty} f\left(\frac{xt}{n}\right)t^n e^{-t}\,dt =: P_n(f;x),$$

which is the classical Post-Widder operator (see, for instance, [5, 27]).

2. Local Approximation Results for q-Post-Widder Operators

Let $f \in C_B[0,\infty)$ and $\delta > 0$. Then, we consider the first modulus of continuity, the Peetre's K-functional, and the second modulus of smoothness, which are defined respectively by:

$$\omega(f,\delta) = \sup\{|f(y)-f(x)| : |y-x| \le \delta;\ x,y \in [0,\infty)\},$$

$$K(f,\delta) = \inf\left\{\|f-g\| + \delta\left\|g''\right\| : g \in C_B^2[0,\infty)\right\},$$

$$\omega_2(f,\delta) = \sup_{0<h\le\delta}\ \sup_{x\in[0,\infty)} |f(x+2h)-2f(x+h)+f(x)|,$$

where $C_B^2[0,\infty) := \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. Then, it is well-known that (see [4])

$$K(f,\delta) \le A\omega_2(f,\sqrt{\delta})$$

for some positive constant A. With this terminology, in order to get a local approximation result for q-Post-Widder Operators we first need the following two results.

Lemma 1. *For every $m,n \in \mathbb{N}$ and $x \in [0,+\infty)$, we have*

$$P_{n,q}(e_m;x) = \frac{[n+1]_q[n+2]_q...[n+m]_q}{[n]_q^m}x^m.$$

Proof. Let $m,n \in N$ and $x \in [0,+\infty)$. Then, by (1.1) and (1.2), we get

$$\begin{aligned}
P_{n,q}(e_m;x) &= \frac{1}{[n]_q!}\int_0^{1/(1-q)}\left(\frac{xt}{[n]_q}\right)^m t^n E_q(-qt)\,d_qt \\
&= \frac{1}{[n]_q!}\frac{x^m}{[n]_q^m}\int_0^{1/(1-q)} t^{n+m}E_q(-qt)\,d_qt \\
&= \frac{1}{[n]_q!}\frac{x^m}{[n]_q^m}\Gamma_q(n+m+1) \\
&= \frac{1}{[n]_q^m}\frac{[n+m]_q!}{[n]_q!}x^m \\
&= \frac{[n+1]_q[n+2]_q...[n+m]_q}{[n]_q^m}x^m,
\end{aligned}$$

which completes the proof. □

If we use the moment function given by

$$\varphi(y) := \varphi_x(y) = y-x,$$

then the following result is an immediate consequence of Lemma 1.

Corollary 1. *For every $n \in \mathbb{N}$ and $x \in [0,+\infty)$, we have*

$$P_{n,q}(\varphi;x) = \left(\frac{[n+1]_q}{[n]_q} - 1\right)x,$$

$$P_{n,q}(\varphi^2;x) = \left(\frac{[n+1]_q[n+2]_q}{[n]_q^2} - \frac{2[n+1]_q}{[n]_q} + 1\right)x^2.$$

Now we are ready to give our approximation result.

Theorem 1. *Let $q \in (0,1)$. Then, for every $n \in \mathbb{N}$, $x \in [0,+\infty)$ and $f \in C_B[0,\infty)$, we have*

$$\left|P_{n,q}(f;x) - f(x)\right| \leq C\omega_2(f, x\delta_n) + \omega(f, x\alpha_n)$$

for some positive constant C, where

$$\delta_n := \left(\frac{[n+1]_q^2}{[n]_q^2} + \frac{[n+1]_q[n+2]_q}{[n]_q^2} - \frac{4[n+1]_q}{[n]_q} + 2\right)^{1/2} \tag{2.1}$$

and

$$\alpha_n := \frac{[n+1]_q}{[n]_q} - 1 \tag{2.2}$$

Proof. Define an auxiliary operator $P_{n,q}^* : C_B[0,\infty) \to C_B[0,\infty)$ by

$$P_{n,q}^*(f;x) := P_{n,q}(f;x) - f\left(\frac{[n+1]_q}{[n]_q}x\right) + f(x). \tag{2.3}$$

Then, by Corollary 1, we get

$$P_{n,q}^*(\varphi;x) = 0. \tag{2.4}$$

Now, for a given $g \in C_B^2[0,\infty)$, it follows from the Taylor formula that

$$g(y) - g(x) = (y-x)g'(x) + \int_x^y (y-u)g''(u)du, \quad y \in [0,\infty).$$

Taking into account (2.3) and using (2.4) we get, for every $x \in [0,\infty)$, that

$$\begin{aligned}
\left|P_{n,q}^*(g;x) - g(x)\right| &= \left|P_{n,q}^*(g(y) - g(x);x)\right| \\
&= \left|g'(x)P_{n,q}^*(\varphi;x) + P_{n,q}^*\left(\int_x^y (y-u)g''(u)du;x\right)\right| \\
&= \left|P_{n,q}^*\left(\int_x^y (y-u)g''(u)du;x\right)\right|
\end{aligned}$$

$$= \left| P_{n,q}\left(\int_x^y (y-u)g''(u)du; x \right) - \int_x^{x[n+1]_q/[n]_q} \left(\frac{[n+1]_q}{[n]_q}x - u \right) g''(u)du \right|.$$

Since

$$\left| P_{n,q}\left(\int_x^y (y-u)g''(u)du; x \right) \right| \leq \frac{\|g''\|}{2} P_{n,q}\left(\varphi^2; x\right)$$

and

$$\left| \int_x^{x[n+1]_q/[n]_q} \left(\frac{[n+1]_q}{[n]_q}x - u \right) g''(u)du \right| \leq \frac{\|g''\|}{2} \left(\frac{[n+1]_q}{[n]_q} - 1 \right)^2 x^2$$

we get

$$\left| P^*_{n,q}(g;x) - g(x) \right| \leq \frac{\|g''\|}{2} P_{n,q}\left(\varphi^2; x\right) + \frac{\|g''\|}{2} \left(\frac{[n+1]_q}{[n]_q} - 1 \right)^2 x^2.$$

Hence, Corollary 1 implies that

$$\left| P^*_{n,q}(g;x) - g(x) \right| \leq \frac{\|g''\|}{2} \left(\left(\frac{[n+1]_q[n+2]_q}{[n]_q^2} - \frac{2[n+1]_q}{[n]_q} + 1 \right) + \left(\frac{[n+1]_q}{[n]_q} - 1 \right)^2 \right) x^2,$$

which gives

$$\left| P^*_{n,q}(g;x) - g(x) \right| \leq \frac{\|g''\|}{2} \left(\frac{[n+1]_q^2}{[n]_q^2} + \frac{[n+1]_q[n+2]_q}{[n]_q^2} - \frac{4[n+1]_q}{[n]_q} + 2 \right) x^2. \tag{2.5}$$

Now, considering (2.1) and (2.2), if $f \in C_B[0,\infty)$ and $g \in C_B^2[0,\infty)$, we may write from (2.5) that

$$\begin{aligned} \left| P_{n,q}(f;x) - f(x) \right| &\leq \left| P^*_{n,q}(f-g;x) - (f-g)(x) \right| \\ &\quad + \left| P^*_{n,q}(g;x) - g(x) \right| + \left| f\left(\frac{[n+1]_q}{[n]_q}x \right) - f(x) \right| \\ &\leq 2\|f-g\| + x^2\delta_n \frac{\|g''\|}{2} + \left| f\left(\frac{[n+1]_q}{[n]_q}x \right) - f(x) \right| \\ &\leq 2\left(\|f-g\| + x^2\delta_n \left\|g''\right\| \right) + \omega\left(f, x\alpha_n\right), \end{aligned}$$

which yields that

$$\begin{aligned}\left|P_{n,q}(f;x)-f(x)\right| &\leq 2K\left(f,x^2\delta_n^2\right)+\omega\left(f,x\alpha_n\right)\\ &\leq C\omega_2\left(f,x\delta_n\right)+\omega\left(f,x\alpha_n\right).\end{aligned}$$

Therefore, the proof is completed. □

Now we consider the elements of locally Lipschitz functions. Let $0<\alpha\leq 1$ and let D be a subset of the interval $[0,\infty)$. Then, by $Lip_M(D,\alpha)$ we denote the space of all functions satisfying the condition

$$|f(y)-f(x)|\leq M\,|y-x|^{\alpha} \quad \text{for} \quad y\in\overline{D} \text{ and } x\in[0,\infty), \tag{2.6}$$

where $\overline{D}$ denotes the closure of D in $[0,\infty)$.

Then we can give our second local approximation result.

Theorem 2. *Let D be a subset of $[0,\infty)$, $q\in(0,1)$ and $\alpha\in(0,1]$. Then, for every $n\in\mathbb{N}$, $x\in[0,+\infty)$ and $f\in C_B[0,\infty)\cap Lip_M(D,\alpha)$, we have*

$$\left|P_{n,q}(f;x)-f(x)\right|\leq M\left\{\gamma_n^{\alpha}x^{\alpha}+(d(x,D))^{\alpha}\right\},$$

where M is a positive constant; $d(x,D)$ is the distance between x and E defined as $d(x,D):=\inf\{|y-x|:y\in D\}$; and γ_n is given by

$$\gamma_n:=\left(\frac{[n+1]_q[n+2]_q}{[n]_q^2}-\frac{2[n+1]_q}{[n]_q}+1\right)^{1/2}. \tag{2.7}$$

Proof. Let $\overline{D}$ denote the closure of D in $[0,\infty)$. Let $x\in[0,\infty)$ be fixed. Then, there exists a point $x_0\in\overline{D}$ such that $|x-x_0|=d(x,D)$. Using the triangle inequality, we may write that

$$|f(y)-f(x)|\leq|f(y)-f(x_0)|+|f(x)-f(x_0)|.$$

Hence, it follows from (2.6) that

$$\begin{aligned}\left|P_{n,q}(f;x)-f(x)\right| &\leq P_{n,q}(|f(y)-f(x)|;x)\\ &\leq P_{n,q}(|f(y)-f(x_0)|;x)+|f(x)-f(x_0)|\\ &\leq M\left\{P_{n,q}(|y-x_0|^{\alpha};x)+|x-x_0|^{\alpha}\right\}.\\ &= M\left\{P_{n,q}(|y-x_0|^{\alpha};x)+(d(x,D))^{\alpha}\right\}.\end{aligned}$$

Using the Cauchy-Bunyakowsky-Schwarz inequality for positive linear operators we find that

$$\left|P_{n,q}(f;x)-f(x)\right|\leq M\left\{P_{n,q}(\varphi^2;x)^{\alpha/2}+(d(x,D))^{\alpha}\right\}.$$

Thus, the proof follows from (2.7) and Corollary 1. □

3. Rates of convergence of the operators $P_{n,q}$

Consider the following weighted space

$$E := \left\{ f \in C[0,\infty) : |f(x)| \leq M(1+x^2) \text{ for some } M > 0 \right\}. \tag{3.1}$$

Now, for a given $b > 0$, by $\omega_b(f,\delta)$ we denote the usual modulus of continuity of f on the closed interval $[0,b]$, which is defined to be

$$\omega(f,\delta)_{[0,b]} := \sup\{|f(y)-f(x)| : |y-x| \leq \delta;\ x,y \in [0,b]\}.$$

The next theorem gives the rate of convergence of the operators $P_{n,q}(f)$ to f for all $f \in E$.

Theorem 3. *For every $f \in E$, $b > 0$ and $q \in (0,1)$, we have*

$$\left\| P_{n,q}(f) - f \right\|_{[0,b]} \leq C \left\{ \gamma_n^2 + \omega(f,\gamma_n)_{[0,b+1]} \right\},$$

where γ_n is given by (2.7); C is a positive constant depending on f, b; and the symbol $\|\cdot\|_{[0,b]}$ denotes the classical sup-norm on the space $C[0,b]$.

Proof. Let $x \in [0,b]$ be fixed. Assume that $f \in E$, $b > 0$ and $q \in (0,1)$. If $y \leq b+1$, then one can write, for any $\delta > 0$, that

$$|f(y)-f(x)| \leq \omega(f,|y-x|)_{[0,b+1]} \leq \left(1 + \frac{|y-x|^2}{\delta^2}\right) \omega(f,\delta)_{[0,b+1]}. \tag{3.2}$$

On the other hand, if $y > b+1$, since $y - x > 1$, we get, for some positive constant M depending on $f \in E$, that

$$\begin{aligned} |f(y)-f(x)| &\leq M\left\{2 + x^2 + y^2\right\} \\ &= M\left\{2 + x^2 + ((y-x)+x)^2\right\} \\ &= M\left\{2 + 2x^2 + 2x(y-x) + (y-x)^2\right\} \\ &\leq M\left\{2 + 2x^2 + (2x+1)(y-x)^2\right\} \\ &\leq M\left(2x^2 + 2x + 3\right)(y-x)^2, \end{aligned}$$

which implies that

$$|f(y)-f(x)| \leq N(y-x)^2, \tag{3.3}$$

where $N := M\left(2b^2 + 2b + 3\right)$. From (3.2) and (3.3), we get, for all $x \in [0,b]$ and $y \geq 0$, that

$$|f(y)-f(x)| \leq N(y-x)^2 + \left(1 + \frac{(y-x)^2}{\delta^2}\right) \omega(f,\delta)_{[0,b+1]}$$

and therefore

$$\left|P_{n,q}(f;x) - f(x)\right| \leq N P_{n,q}(\varphi^2;x) + \omega(f,\delta)_{[0,b+1]} \left(1 + \frac{1}{\delta^2} P_{n,q}(\varphi^2;x)\right).$$

Now applying Corollary 1, we obtain that

$$\begin{aligned}\left|P_{n,q}(f;x)-f(x)\right| &\leq N\gamma_n^2x^2+\omega(f,\delta)_{[0,b+1]}\left(1+\frac{\gamma_n^2x^2}{\delta^2}\right)\\ &\leq Nb^2\gamma_n^2+\omega_{b+1}(f,\delta)_{[0,b+1]}\left(1+\frac{\gamma_n^2b^2}{\delta^2}\right),\end{aligned}$$

where γ_n is given by (2.7). If we take $\delta=\gamma_n$ on the right hand side of the last inequality, then we immediately see that

$$\left|P_{n,q}(f;x)-f(x)\right| \leq C\left\{\gamma_n^2+\omega(f,\gamma_n)_{[0,b+1]}\right\},$$

where $C:=\max\{Nb^2,1+b^2\}$. Finally, taking supremum over $x\in[0,b]$, the proof is completed. □

4. Statistical approximation results

In this section, we study the statistical approximation properties of the q-Post-Widder operators. We first recall the concept of statistical convergence.

Let $A:=[a_{jn}]$, $j,n\in\mathbb{N}$, be an infinite matrix. The A-transform of a sequence $x:=(x_n)$ is given by $Ax:=((Ax)_n)$ with $(Ax)_n=\sum_{n=1}^{\infty}a_{jn}$ provided that the series converges for each $n\in\mathbb{N}$. Then, we say that a matrix A is regular if $\lim Ax=L$ whenever $\lim x=L$. Now let $A=[a_{jn}]$ be nonnegative regular summability matrix. Then, a sequence $x=(x_n)$ is said to be A-statistically convergent to L if $\lim_n\sum_{n:|x_n-L|\geq\varepsilon}a_{jn}=0$ holds for every $\varepsilon>0$. In this case, we write $st_A-\lim x=L$ (see [9]). It is well-known that if we take $A=C_1=[c_{jn}]$, the Cesáro matrix of order one defined to be $c_{jn}=1/j$ if $1\leq n\leq j$, and $c_{jn}=0$ otherwise, then we get the classical definition of statistical convergence (see [8]). In this case, we use the notation $st-\lim$ instead of $st_{C_1}-\lim$. Also, if $A=I$, the identity matrix, then it reduces to the ordinary ordinary convergence. Notice that every convergent sequence is A-statistically convergent to the same value, but the converse does not hold true. As we can see the following example, there exists an A-statistically convergent sequence but non-convergent in the usual sense.

Now let $A=[a_{jn}]$ be a nonnegative regular summability matrix. Assume that (q_n) is sequence from $(0,1)$ such that

$$st_A-\lim_n\frac{1}{[n]_{q_n}}=0,\ \ st_A-\lim_n q_n=1 \text{ and } st_A-\lim_n q_n^n=\beta\in(0,1). \tag{4.1}$$

Indeed, such a sequence (q_n) can be constructed by the following way. Take $A=C_1=[c_{jn}]$ and define the sequence (q_n) by

$$q_n:=\begin{cases}e^{-n}, & \text{if } n=m^2\ (m\in\mathbb{N})\\ \dfrac{n}{n+1}, & \text{otherwise.}\end{cases} \tag{4.2}$$

Then observe that $q_n \in (0,1)$ for each $n \in \mathbb{N}$. It is easy to check that although (q_n) is non-convergent in the usual sense, it satisfies the conditions in (4.1) with the choice of $A = C_1$ and $\beta = 1/e$. On the other hand, since

$$\frac{[n+1]_{q_n}}{[n]_{q_n}} = \frac{1-q_n^n}{1-q_n^{n-1}} \quad \text{and} \quad \frac{[n+1]_{q_n}[n+2]_{q_n}}{[n]_{q_n}^2} = \frac{(1-q_n^n)(1-q_n^{n+1})}{\left(1-q_n^{n-1}\right)^2},$$

we obtain from (4.1) that

$$st_A - \lim_n \frac{[n+1]_{q_n}}{[n]_{q_n}} = st_A - \lim_n \frac{[n+1]_{q_n}^2}{[n]_{q_n}^2} = st_A - \lim_n \frac{[n+1]_{q_n}[n+2]_{q_n}}{[n]_{q_n}^2} = 1. \quad (4.3)$$

Then it follows from (4.3) that

$$st_A - \lim_n \delta_n = st_A - \lim \alpha_n = st_A - \lim \gamma_n = 0, \quad (4.4)$$

where δ_n, α_n and γ_n are given by (2.1), (2.2) and (2.7), respectively.

Now taking a sequence (q_n) from $(0,1)$ satisfying (4.1) instead of a fixed number $q \in (0,1)$ in the definition of operators (1.1), we obtain the following (pointwise) statistical approximation result.

Theorem 4. *Let $A = [a_{jn}]$ be a nonnegative regular summability matrix, and let (q_n) be a sequence from $(0,1)$ satisfying (4.1). Then, for every $x \in [0,+\infty)$ and $f \in C_B[0,\infty)$, we have*

$$st_A - \lim_n \left| P_{n,q_n}(f;x) - f(x) \right| = 0.$$

Proof. Let $x \in [0,\infty)$ and $f \in C_B[0,\infty)$ be fixed. By the right continuity of ω and ω_2 at zero, one can get from (4.4) that

$$st_A - \lim_n \omega(f, x\alpha_n) = st_A - \lim_n \omega_2(f, x\delta_n) = 0. \quad (4.5)$$

Now, for a given $\varepsilon > 0$, define the following sets:

$$U := \left\{ n \in \mathbf{N} : \left| P_{n,q_n}(f;x) - f(x) \right| \geq \varepsilon \right\},$$
$$U_1 := \left\{ n \in \mathbf{N} : \omega(f; x\alpha_n) \geq \frac{\varepsilon}{2} \right\},$$
$$U_2 := \left\{ n \in \mathbf{N} : \omega_2(f; x\delta_n) \geq \frac{\varepsilon}{2C} \right\},$$

where C is a positive constant as in Theorem 1. Then, we may write from the inequality in Theorem 1 that

$$U \subseteq U_1 \cup U_2,$$

which gives, for every $j \in \mathbb{N}$,

$$\sum_{n \in U} a_{jn} \leq \sum_{n \in U_1} a_{jn} + \sum_{n \in U_2} a_{jn}. \quad (4.6)$$

Letting $j \to \infty$ in (4.6) and also using (4.5), we have

$$\lim_j \sum_{n \in U} a_{jn} = 0,$$

which guarantees that

$$st_A - \lim_n \left| P_{n,q_n}(f;x) - f(x) \right| = 0.$$

Thus, the proof is completed. □

If we take $A = I$, the identity matrix, by Theorem 4, we obtain the classical approximation result.

Corollary 2. *Let (q_n) be a sequence from $(0,1)$ satisfying the conditions:*

$$\lim_n \frac{1}{[n]_{q_n}} = 0, \ \lim_n q_n = 1 \text{ and } \lim_n q_n^n = \eta \in (0,1). \tag{4.7}$$

Then, for every $x \in [0,+\infty)$ and $f \in C_B[0,\infty)$, the sequence $\{P_{n,q_n}(f;x)\}$ converges (pointwise) to $f(x)$.

We should note that if a sequence (q_n) satisfies (4.7), then it also satisfies (4.1); and hence the classical (pointwise) approximation result in Corollary 2 implies the statistical one in Theorem 4. However, if we choose the sequence (q_n) as in (4.2), then we can easily see that (4.7) does not hold true while (4.1) is still valid. This example shows that our statistical approximation result in Theorem 4 is more applicable than the classical one.

Now we obtain a uniform approximation theorem in statistical sense.

Theorem 5. *Let $A = [a_{jn}]$ be a nonnegative regular summability matrix, and let (q_n) be a sequence from $(0,1)$ satisfying (4.1). Assume that D is a compact set in $[0,\infty)$. Then, for every $f \in C_B[0,\infty) \cap Lip_M(D,\alpha)$ with $M > 0$ and $\alpha \in (0,1]$, we have*

$$st_A - \lim_n \left\| P_{n,q_n}(f) - f \right\|_D = 0.$$

Proof. Let $x \in D$ be fixed. In this case, we have $d(x,D) = 0$. Using this and Theorem 2, we immediately get that, for each $x \in D$,

$$\left| P_{n,q_n}(f;x) - f(x) \right| \le M \gamma_n^{\alpha} x^{\alpha}. \tag{4.8}$$

Since D is compact, the number $L := \sup_{x \in D}\{x^{\alpha}\}$ is finite. Hence, taking supremum over $x \in D$ on the both sides of (4.8), we obtain that

$$\left\| P_{n,q_n}(f) - f \right\|_D \le LM\gamma_n^{\alpha},$$

which yields, for every $\varepsilon > 0$,

$$\left\{ n \in \mathbf{N} : \left\| P_{n,q_n}(f) - f \right\|_D \ge \varepsilon \right\} \subseteq \left\{ n \in \mathbf{N} : \gamma_n \ge \left(\frac{\varepsilon}{LM} \right)^{1/\alpha} \right\}$$

and hence

$$\sum_{n:\, \|P_{n,q_n}(f)-f\|_D \geq \varepsilon} a_{jn} \leq \sum_{n:\, \gamma_n \geq \left(\frac{\varepsilon}{LM}\right)^{1/\alpha}} a_{jn}$$

holds for every $j \in \mathbb{N}$. Now taking limit as $j \to \infty$ on both sides of the last inequality, and also considering (4.4), we deduce that

$$\lim_n \sum_{n:\, \|P_{n,q_n}(f)-f\|_D \geq \varepsilon} a_{jn} = 0,$$

which is the desired result. □

If $A = I$ in Theorem 5, then we easily get the uniform approximation result in the ordinary sense.

Corollary 3. *Let (q_n) be a sequence from $(0,1)$ satisfying (4.7), and let D be a compact set in $[0,\infty)$. Then, for every $f \in C_B[0,\infty) \cap Lip_M(D,\alpha)$ with $M > 0$ and $\alpha \in (0,1]$, the sequence $\{P_{n,q_n}(f)\}$ is uniformly convergent to f on D.*

For a uniform approximation process, we also get the next result.

Theorem 6. *Let $A = [a_{jn}]$ be a nonnegative regular summability matrix, and let (q_n) be a sequence from $(0,1)$ satisfying (4.1). Then, for every $f \in E$ and $b > 0$, we have*

$$st_A - \lim_n \|P_{n,q_n}(f) - f\|_{[0,b]} = 0.$$

Proof. Assume that $x \in [0,b]$ is fixed. Then, for a given $\varepsilon > 0$, consider the following sets:

$$V := \left\{n \in \mathbf{N} : \|P_{n,q_n}(f) - f\|_{[0,b]} \geq \varepsilon\right\},$$

$$V_1 := \left\{n \in \mathbf{N} : \gamma_n \geq \left(\frac{\varepsilon}{2C}\right)^{1/2}\right\},$$

$$V_2 := \left\{n \in \mathbf{N} : \omega(f,\gamma_n)_{[0,b+1]} \geq \frac{\varepsilon}{2C}\right\},$$

where γ_n is given by (2.7), and C is a positive constant as in Theorem 3. Then, it follows from Theorem 3 that

$$V \subseteq V_1 \cup V_2.$$

The last inclusion implies that

$$\sum_{n\in V} a_{jn} \leq \sum_{n\in V_1} a_{jn} + \sum_{n\in V_2} a_{jn}. \tag{4.9}$$

On the other hand, by the right continuity of $\omega(f,\cdot)_{[0,b+1]}$ at zero, one can get from (4.4) that

$$st_A - \lim_n \omega(f,\gamma_n)_{[0,b+1]} = 0. \tag{4.10}$$

Then, letting $j \to \infty$ in (4.9) and using (4.4), (4.10), we easily get that

$$\lim_n \sum_{n \in V} a_{jn} = 0,$$

which completes the proof. □

Of course, if we take $A = I$ in Theorem 6, we obtain the next result.

Corollary 4. *Let (q_n) be a sequence from $(0,1)$ satisfying (4.7). Then, for every $f \in E$ and $b > 0$, the sequence $\{P_{n,q_n}(f)\}$ is uniformly convergent to f on $[0,b]$.*

5. Voronovskaya-type results

In this section, we prove a Voronovskaya-type theorem for the q-Post-Widder operators. Throughout this section we consider a sequence (q_n) satisfying (4.1) instead of a fixed $q \in (0,1)$ in the definition (1.1).

We first need the following lemma.

Lemma 2. *Let $A = [a_{jn}]$ be a nonnegative regular summability matrix, and let (q_n) be a sequence from $(0,1)$ satisfying (4.1). Then, we have*

$$st_A - \lim_{n\to\infty} [n]_{q_n} P_{n,q_n}(\varphi; x) = \beta x,$$

$$st_A - \lim_{n\to\infty} [n]_{q_n} P_{n,q_n}(\varphi^2; x) = \beta x^2,$$

$$st_A - \lim_{n\to\infty} [n]_{q_n}^2 P_{n,q_n}(\varphi^4; x) = 3\beta^2(1-\beta)x^4$$

uniformly with respect to $x \in [0,b]$, $(b > 0)$, where $\beta := st_A - \lim q_n^n$ as stated in (4.1).

Proof. It follows from Corollary 1 that

$$[n]_{q_n} P_{n,q_n}(\varphi; x) = \left([n+1]_{q_n} - [n]_{q_n}\right) x$$

and

$$[n]_{q_n} P_{n,q_n}(\varphi^2; x) = \left(\frac{[n+1]_{q_n}[n+2]_{q_n}}{[n]_{q_n}} - 2[n+1]_{q_n} + [n]_{q_n} \right) x^2.$$

Then, using (4.1) we easily get

$$st_A - \lim_{n\to\infty} [n]_{q_n} P_{n,q_n}(\varphi; x) = \beta x$$

and

$$st_A - \lim_{n\to\infty} [n]_{q_n} P_{n,q_n}(\varphi^2; x) = \beta x^2$$

uniformly with respect to $x \in [0,b]$. On the other hand, by Lemma 1, we obtain that

$$P_{n,q_n}(\varphi^4; x) = P_{n,q_n}(e_4; x) - 4x P_{n,q_n}(e_3; x) + 6x^2 P_{n,q_n}(e_2; x) - 4x^3 P_{n,q_n}(e_1; x) + x^4$$

$$= \frac{[n+1]_{q_n}[n+2]_{q_n}[n+3]_{q_n}[n+4]_{q_n}}{[n]_{q_n}^4}x^4$$
$$-4\frac{[n+1]_{q_n}[n+2]_{q_n}[n+3]_{q_n}}{[n]_{q_n}^3}x^4$$
$$+6\frac{[n+1]_{q_n}[n+2]_{q_n}}{[n]_{q_n}^2}x^4 - 4\frac{[n+1]_{q_n}}{[n]_{q_n}}x^4 + x^4.$$

Then, we get

$$[n]_{q_n}^2 P_{n,q_n}(\varphi^4;x) = \left(\frac{[n+1]_{q_n}[n+2]_{q_n}[n+3]_{q_n}[n+4]_{q_n}}{[n]_{q_n}^2}\right.$$
$$-4\frac{[n+1]_{q_n}[n+2]_{q_n}[n+3]_{q_n}}{[n]_{q_n}}$$
$$\left.+6[n+1]_{q_n}[n+2]_{q_n} - 4[n]_{q_n}[n+1]_{q_n} + [n]_{q_n}^2\right)x^4.$$

Now using the fact that

$$[n]_{q_n} = \frac{1-q_n^{n-1}}{1-q_n}$$

and also considering (4.1), after some simple calculations, for each $x \in [0,\infty)$, we have

$$st_A - \lim_{n\to\infty}[n]_{q_n}^2 P_{n,q_n}(\varphi^4;x) = 3\beta^2(1-\beta)x^4,$$

uniformly with respect to $x \in [0,b]$, which is the desired result. □

Then we get our Voronovskaya-type theorem as follows.

Theorem 7. *Let $A = [a_{jn}]$ be a nonnegative regular summability matrix, and let (q_n) be a sequence from $(0,1)$ satisfying (4.1). Then, for every $f \in E$ such that f', $f'' \in E$, we have*

$$st_A - \lim_{n\to\infty}[n]_{q_n}\{P_{n,q_n}(f;x) - f(x)\} = \beta x f'(x) + \frac{\beta x^2 f''(x)}{2}$$

uniformly with respect to $x \in [0,b]$, $(b>0)$, where $\beta := st_A - \lim q_n^n$ as stated in (4.1).

Proof. Let f, f', $f'' \in E$. For each $x \geq 0$, define a function $\Psi(y) := \Psi(y,x)$ by

$$\Psi(y) = \begin{cases} \dfrac{f(y) - f(x) - (y-x)f'(x) - \dfrac{1}{2}(y-x)^2 f''(x)}{(y-x)^2}, & \text{if } y \neq x \\ 0, & \text{if } y = x. \end{cases}$$

Then by assumption we have $\Psi(x,x)=0$ and the function $\Psi(\cdot,x)$ belongs to E. Hence, by Taylor's theorem we get

$$f(y)=f(x)+(y-x)f'(x)+\frac{(y-x)^2}{2}f''(x)+(y-x)^2\Psi(y,x).$$

Hence we easily observe that

$$\begin{aligned}[n]_{q_n}\{P_{n,q_n}(f;x)-f(x)\}&=f'(x)[n]_{q_n}P_{n,q_n}(\varphi;x)\\&\quad+\frac{f''(x)}{2}[n]_{q_n}P_{n,q_n}(\varphi^2;x)\\&\quad+[n]_{q_n}P_{n,q_n}(\varphi^2\Psi;x).\end{aligned}$$

If we apply the Cauchy-Schwarz inequality for the last term on the right-hand side of the last equality, then we conclude that

$$[n]_{q_n}\left|P_{n,q_n}\left(\varphi^2\Psi;x\right)\right|\leq\left([n]_{q_n}^2P_{n,q_n}(\varphi^4;x)\right)^{1/2}\left(P_{n,q_n}(\Psi^2;x)\right)^{1/2}. \tag{5.1}$$

Let $\eta(y,x):=\Psi^2(y,x)$. In this case, observe that $\eta(x,x)=0$ and $\eta(\cdot,x)\in E$. Then, it follows from Theorem 6 that

$$st_A-\lim_{n\to\infty}P_{n,q_n}\left(\Psi^2;x\right)=st_A-\lim_{n\to\infty}P_{n,q_n}\left(\eta(y,x);x\right)=\eta(x,x)=0 \tag{5.2}$$

uniformly with respect to $x\in[0,b]$, $(b>0)$. Now, by (5.1), (5.2), and Lemma 2, we obtain that

$$st_A-\lim_{n\to\infty}[n]_{q_n}P_{n,q_n}(\varphi^2\Psi;x)=0$$

uniformly with respect to $x\in[0,b]$. Combining the above facts with Lemma 2, we observe that

$$st_A-\lim_{n\to\infty}[n]_{q_n}\left\{P_{n,q_n}(f;x)-f(x)\right\}=\beta xf'(x)+\frac{\beta x^2f''(x)}{2}$$

uniformly with respect to $x\in[0,b]$. Thus, the proof is completed. □

The next result is an immediate consequence of Theorem 7.

Corollary 5. *Let (q_n) be a sequence from $(0,1)$ satisfying (4.7). Then, for every $f\in E$ such that f', $f''\in E$, we have*

$$\lim_{n\to\infty}[n]_{q_n}\left\{P_{n,q_n}(f;x)-f(x)\right\}=\eta xf'(x)+\frac{\eta x^2f''(x)}{2}$$

uniformly with respect to $x\in[0,b]$, $(b>0)$, where $\eta=\lim q_n^n$ as stated in (4.1).

6. Approximation properties of complex q-Post-Widder operators

In this section, we will introduce the complex q-Post-Widder operators and study their approximation properties.

Now, let

$$D := \{z \in \mathbb{C} : |z| < 1\},\ \bar{D} := \{z \in \mathbb{C} : |z| \leq 1\}$$

and

$$C\left(\bar{D}\right) : \left\{f : \bar{D} \to \mathbb{C} : f \text{ is continuous on } D\right\}.$$

Then, we propose the complex q-Post-Widder operators as follows:

$$P^*_{n,q}(f;z) = \frac{1}{[n]_q!} \int_0^{1/(1-q)} f\left(\frac{ze^{it}}{[n]_q}\right) t^n E_q(-qt)\, d_q t, \tag{6.1}$$

where $z \in \bar{D}$, $n \in \mathbb{N}$, $f \in C\left(\bar{D}\right)$ and $q \in (0,1)$.

It is not hard to see that if f is a constant function on $\bar{D}$, say $f(z) \equiv c$, then we have, for every $n \in \mathbb{N}$ that $P^*_{n,q}(c;z) = c$. Hence, the operators $P^*_{n,q}$ preserve the constant functions. In order to get some geometric properties of the operators $P^*_{n,q}$ in (6.1) we first need the following concepts.

Now define the subspace of $C\left(\bar{D}\right)$ by

$$A\left(\bar{D}\right) := \left\{f \in C\left(\bar{D}\right) : f \text{ is analytic on } D \text{ with } f(0) = 0\right\}.$$

If $f \in C\left(\bar{D}\right)$, then, as in Section 3, we consider the (first) modulus of continuity of f on $\bar{D}$, denoted by $\omega(f,\delta)_{\bar{D}}$, $\delta > 0$, as follows (see [11]):

$$\omega(f;\delta)_{\bar{D}} := \sup\left\{|f(z) - f(w)| : |z - w| \leq \delta,\ z, w \in \bar{D}\right\}.$$

It is easy to see that, for any $c, \delta > 0$

$$\omega(f; c\delta)_{\bar{D}} \leq (1 + c)\omega(f;\delta)_{\bar{D}}. \tag{6.2}$$

We obtain the following result.

Theorem 8. *For each fixed $n \in \mathbb{N}$ and $q \in (0,1)$, $P^*_{n,q}\left(A\left(\bar{D}\right)\right) \subset A\left(\bar{D}\right)$.*

Proof. Let $n \in \mathbb{N}$, $f \in A\left(\bar{D}\right)$ and $q \in (0,1)$ be fixed. Since $f(0) = 0$, it follows from (6.1) that

$$P^*_{n,q}(f;0) = \frac{1}{[n]_q!} \int_0^{1/(1-q)} f(0)\, t^n E_q(-qt)\, d_q t = 0.$$

Now we show that $P_{n,q}(f)$ is continuous on $\bar{D}$. To see this assume that $z, (z_m) \in \bar{D}$ and that $\lim_m z_m = z$. Hence, we get from the definition of ω that

$$\left|P^*_{n,q}(f;z_m) - P^*_{n,q}(f;z)\right|$$

$$
\begin{aligned}
&\leq \frac{1}{[n]_q!}\int_0^{1/(1-q)}\left|f\left(\frac{z_m e^{it}}{[n]_q}\right)-f\left(\frac{ze^{it}}{[n]_q}\right)\right| t^n E_q(-qt)\,d_qt\\
&\leq \frac{\omega\left(f,\frac{|z_m-z|}{[n]_q}\right)_{\bar{D}}}{[n]_q!}\int_0^{1/(1-q)} t^n E_q(-qt)\,d_qt\\
&= \omega\left(f,\frac{|z_m-z|}{[n]_q}\right)_{\bar{D}}\\
&\leq \omega\left(f,|z_m-z|\right)_{\bar{D}}
\end{aligned}
$$

due to $[n]_q \geq 1$ for every $n \in \mathbb{N}$. Since $\lim_m z_m = z$, we may write that

$$\lim_m \omega\left(f,|z_m-z|\right)_{\bar{D}} = 0.$$

Due to the right continuity of $\omega(f,\cdot)$ at zero. Hence, we get

$$\lim_m P^*_{n,q}(f;z_m) = P_{n,q}(f;z),$$

which gives the continuity of $P^*_{n,q}(f)$ at the point $z \in \bar{D}$.

Finally, since $f \in A\left(\bar{D}\right)$, we can write $f(z) = \sum_{k=1}^{\infty} a_k z^k$ for $z \in D$. Then, we get

$$f\left(\frac{ze^{it}}{[n]_q}\right) = \sum_{k=1}^{\infty}\frac{a_k z^k e^{ikt}}{[n]_q^k}. \tag{6.3}$$

Since $\left|\frac{a_k e^{ikt}}{[n]_q^k}\right| \leq |a_k|$ for every $t \in \mathbb{R}$, the series in (6.3) is uniformly convergent with respect to $t \in \mathbb{R}$. Hence, we conclude that

$$
\begin{aligned}
P^*_{n,q}(f;z) &= \frac{1}{[n]_q!}\int_0^{1/(1-q)}\left(\sum_{k=1}^{\infty}\frac{a_k z^k e^{ikt}}{[n]_q^k}\right) t^n E_q(-qt)\,d_qt\\
&= \frac{1}{[n]_q!}\sum_{k=1}^{\infty}\frac{a_k z^k}{[n]_q^k}\left(\int_0^{1/(1-q)} e^{ikt}t^n E_q(-qt)\,d_qt\right)\\
&= \sum_{k=1}^{\infty}\ell_{n,k}z^k,
\end{aligned}
$$

where $\ell_{n,k}$ is given by

$$\ell_{n,k} := \frac{a_k}{[n]_q![n]_q^k}\int_0^{1/(1-q)} e^{ikt}t^n E_q(-qt)\,d_qt \quad (k,n \in \mathbb{N}). \tag{6.4}$$

We should remark that

$$\left|\ell_{n,k}\right| \leq |a_k| \text{ for every } k,n \in \mathbb{N}.$$

Therefore, for each $n \in \mathbb{N}$ and $f \in A\left(\bar{D}\right)$, the function $P^*_{n,q}(f)$ has a Taylor series expansion whose Taylor coefficients are $\ell_{n,k}$ given by (6.4). Combining the above facts we obtain the desired result. □

Now consider the following space:

$$B\left(\bar{D}\right) := \{f \in C(\bar{D}) : f \text{ is analytic on } D \text{ with } f(0) = 1 \text{ and } Re\,[f(z)] > 0 \text{ for every } z \in D\}.$$

Then we have the next result.

Theorem 9. *For each fixed $n \in \mathbb{N}$ and $q \in (0,1)$, $P^*_{n,q}\left(B\left(\bar{D}\right)\right) \subset B\left(\bar{D}\right)$.*

Proof. Let $n \in \mathbb{N}$, $f \in B\left(\bar{D}\right)$ and $q \in (0,1)$ be fixed. As in the proof of Theorem 8, we see that $P^*_{n,q}(f)$ is analytic on D and continuous on $\bar{D}$. Since $f(0) = 1$, we easily get that

$$P^*_{n,q}(f;0) = \frac{1}{[n]_q!} \int_0^{1/(1-q)} f(0)\, t^n E_q(-qt)\, d_q t = 1.$$

Finally, we may write that, for every $z \in D$,

$$Re\left[P^*_{n,q}(f;z)\right] = \frac{1}{[n]_q!} \int_0^{1/(1-q)} Re\left[f\left(\frac{ze^{it}}{[n]_q}\right)\right] t^n E_q(-qt)\, d_q t > 0$$

since $Re\left[f(\frac{ze^{it}}{[n]_q})\right] > 0$. Thus, the proof is completed. □

If we consider the following space of Lipschitz class functions:

$$Lip^*_M(\bar{D};\alpha) := \{f : \bar{D} \to \mathbb{C};\ |f(z) - f(w)| \leq M|z-w|^\alpha \text{ for every } z,w \in \bar{D}\},$$
$$(\alpha \in (0,1] \text{ and } M > 0),$$

we can get the next result.

Theorem 10. *For each fixed $n \in \mathbb{N}$ and $q \in (0,1)$, $P^*_{n,q}\left(Lip^*_M(\bar{D};\alpha)\right) \subset Lip^*_M(\bar{D};\alpha)$.*

Proof. Let $f \in Lip^*_M(\bar{D};\alpha)$. Then, for every $z,w \in \bar{D}$, we observe that

$$\left|P^*_{n,q_n}(f;z) - P^*_{n,q}(f;w)\right| \leq \frac{1}{[n]_q!} \int_0^{1/(1-q)} \left|f\left(\frac{ze^{it}}{[n]_q}\right) - f\left(\frac{we^{it}}{[n]_q}\right)\right| t^n E_q(-qt) d_q t$$

$$\leq \frac{M}{[n]_q!}\int_0^{1/(1-q)}\left|\frac{ze^{it}}{[n]_q}-\frac{we^{it}}{[n]_q}\right|^{\alpha} t^n E_q(-qt)d_qt$$
$$= \frac{M\,|z-w|^{\alpha}}{[n]_q^{\alpha}}$$
$$\leq M\,|z-w|^{\alpha},$$

which completes the proof. □

Using the definition of $\omega(f;\delta)_{\bar{D}}$ for $f \in C\left(\bar{D}\right)$ and $\delta > 0$, we obtain the following global smoothness preservation for the operators $P^*_{n,q}$ given by (6.1).

Theorem 11. *For each fixed $n \in \mathbb{N}$, $q \in (0,1)$ and $f \in C\left(\bar{D}\right)$, we have*

$$\omega(P^*_{n,q}(f);\delta)_{\bar{D}} \leq \omega(f;\delta)_{\bar{D}}.$$

Proof. Let $\delta > 0$, $n \in \mathbb{N}$, $q \in (0,1)$ and $f \in C\left(\bar{D}\right)$ be given. Assume that $z, w \in \bar{D}$ and $|z-w| \leq \delta$. Then, we have

$$\left|P^*_{n,q}(f;z) - P^*_{n,q}(f;w)\right| \leq \frac{1}{[n]_q!}\int_0^{1/(1-q)}\left|f\left(\frac{ze^{it}}{[n]_q}\right)-f\left(\frac{we^{it}}{[n]_q}\right)\right| t^n E_q(-qt)\,d_qt$$
$$\leq \omega\left(f;\frac{|z-w|}{[n]_q}\right)_{\bar{D}} \leq \omega\left(f;\frac{\delta}{[n]_q}\right)_{\bar{D}} \leq \omega(f;\delta)_{\bar{D}}.$$

Then, taking supremum over $|z-w| \leq \delta$, we conclude that

$$\omega(P^*_{n,q}(f);\delta)_{\bar{D}} \leq \omega(f;\delta)_{\bar{D}},$$

whence the result. □

Now we obtain an estimation with the help of $\omega(f;\delta)_{\bar{D}}$ for the operators $P^*_{n,q}$ defined by (6.1).

Theorem 12. *For each fixed $n \in \mathbb{N}$, $q \in (0,1)$ and $f \in C\left(\bar{D}\right)$, we have*

$$\left\|P^*_{n,q}(f) - f\right\|_{\bar{D}} \leq 3\omega\left(f;\frac{1}{\sqrt{[n]_q}}\right)_{\bar{D}}$$

for some (finite) positive constant M, where $\|\cdot\|_{\bar{D}}$ denotes the usual sup-norm on $\bar{D}$ as stated before.

Proof. Let $z \in \bar{D}$ and $f \in C\left(\bar{D}\right)$ be fixed. We first observe that

$$\left|P^*_{n,q}(f;z) - f(z)\right| \leq \frac{1}{[n]_q!}\int_0^{1/(1-q)}\left|f\left(\frac{ze^{it}}{[n]_q}\right)-f(z)\right| t^n E_q(-qt)\,d_qt$$

$$\leq \frac{1}{[n]_q!}\int_0^{1/(1-q)} \omega\left(f;|z|\left|\frac{e^{it}}{[n]_q}-1\right|\right)_{\bar{D}} t^n E_q(-qt)\,d_qt.$$

Since, for every $z \in D$ and $t \geq 0$,

$$|z|\left|\frac{e^{it}}{[n]_q}-1\right| \leq \sqrt{\frac{2}{[n]_q^2}-\frac{2\cos t}{[n]_q}} \leq \frac{2|\sin(t/2)|}{\sqrt{[n]_q}} \leq \frac{2}{\sqrt{[n]_q}} \tag{6.5}$$

we may write that

$$\left|P_{n,q}^*(f;z)-f(z)\right| \leq \frac{1}{[n]_q!}\omega\left(f;\frac{2}{\sqrt{[n]_q}}\right)_{\bar{D}} \int_0^{1/(1-q)} t^n E_q(-qt)\,d_qt$$

$$= \omega\left(f;\frac{2}{\sqrt{[n]_q}}\right)_{\bar{D}}$$

Then, by (6.2) we get

$$\left|P_{n,q}^*(f;z)-f(z)\right| \leq 3\omega\left(f;\frac{1}{\sqrt{[n]_q}}\right)_{\bar{D}}$$

Taking supremum over $z \in \bar{D}$ on the last inequality, the proof is completed. □

Using the Lipschitz class functions, we also obtain the next estimation result for the operators $P_{n,q}^*$.

Theorem 13. *For each fixed $n \in \mathbb{N}$, $q \in (0,1)$ and $f \in Lip_M^*(\bar{D};\alpha)$, we have*

$$\left\|P_{n,q}^*(f)-f\right\|_{\bar{D}} \leq \frac{2^\alpha M}{[n]_q^{\alpha/2}}.$$

Proof. Let $z \in \bar{D}$ and $f \in Lip_M^*(\bar{D};\alpha)$ be given. Then, one can write that

$$\left|P_{n,q}^*(f;z)-f(z)\right| \leq \frac{1}{[n]_q!}\int_0^{1/(1-q)} \left|f\left(\frac{ze^{it}}{[n]_q}\right)-f(z)\right| t^n E_q(-qt)\,d_qt$$

$$\leq \frac{M}{[n]_q!}\int_0^{1/(1-q)} |z|^\alpha \left|\frac{e^{it}}{[n]_q}-1\right|^\alpha t^n E_q(-qt)\,d_qt.$$

By (6.5), we get

$$\left|P_{n,q}^*(f;z)-f(z)\right| \leq \frac{2^\alpha M}{[n]_q^{\alpha/2}}.$$

Taking supremum over $z \in \bar{D}$ in the last inequality, the proof is completed. □

Now replacing the fixed number $q \in (0,1)$ with a sequence (q_n) from the interval $(0,1)$ which satisfies the conditions in (4.1), we can also obtain a statistical approximation result for our complex operators defined by (6.1)

Theorem 14. *Let $A := [a_{jn}]$ be a nonnegative regular summability matrix, and let (q_n) be a sequence from $(0,1)$ satisfying (4.1). Then, for every $f \in C\left(\bar{D}\right)$, we have*

$$st_A - \lim_n \left\| P^*_{n,q_n}(f) - f \right\|_{\bar{D}} = 0.$$

Proof. Let $f \in C\left(\bar{D}\right)$. Then, for a given $\varepsilon > 0$, we may write from Theorem 11 that

$$U := \left\{ n \in \mathbb{N} : \left\| P^*_{n,q_n}(f) - f \right\|_{\bar{D}} \geq \varepsilon \right\} \subseteq \left\{ n \in \mathbb{N} : \omega\left(f; \frac{1}{\sqrt{[n]_{q_n}}} \right)_{\bar{D}} \geq \frac{\varepsilon}{3} \right\} =: V.$$

Thus, for every $j \in \mathbb{N}$, we get

$$\sum_{n \in U} a_{jn} \leq \sum_{n \in V} a_{jn}. \tag{6.6}$$

On the other hand, by the right continuity of $\omega(f,\cdot)_{\bar{D}}$ at zero, one can get from (4.1) that

$$st_A - \lim_n \omega\left(f; \frac{1}{\sqrt{[n]_{q_n}}} \right)_{\bar{D}} = 0,$$

which gives

$$\lim_j \sum_{n \in V} a_{jn} = 0. \tag{6.7}$$

Now taking limit as $j \to \infty$ in the both sides of (6.6) and also considering (6.7) we obtain that

$$\lim_j \sum_{n \in U} a_{jn} = 0,$$

which means that

$$st_A - \lim_n \left\| P^*_{n,q_n}(f) - f \right\| = 0.$$

The proof is completed. □

If we replace A by the identity matrix in Theorem 14, then we get the following uniform approximation result.

Corollary 6. *Let (q_n) be a sequence from $(0,1)$ satisfying (4.7). Then, for every $f \in A\left(\bar{D}\right)$, the sequence $\left\{P^*_{n,q_n}(f)\right\}_{n \in \mathbb{N}}$ is uniformly convergent to f on $\bar{D}$.*

Finally, we give a statistical approximation result on the space $Lip^*_M(\bar{D};\alpha)$.

Theorem 15. *Let $A := [a_{jn}]$ be a nonnegative regular summability matrix, and let (q_n) be a sequence from $(0,1)$ satisfying (4.1). Then, for every $f \in Lip^*_M(\bar{D};\alpha)$, we have*

$$st_A - \lim_n \left\| P^*_{n,q_n}(f) - f \right\|_{\bar{D}} = 0.$$

Proof. Let $z \in \bar{D}$ and $f \in Lip^*_M(\bar{D};\alpha)$. By Theorem 13, we get, for every $\varepsilon > 0$, that

$$\left\{ n \in \mathbf{N} : \left\| P^*_{n,q_n}(f) - f \right\|_{\bar{D}} \geq \varepsilon \right\} \subseteq \left\{ n \in \mathbf{N} : \frac{1}{[n]_{q_n}} \geq \left(\frac{\varepsilon}{2^\alpha M} \right)^{2/\alpha} \right\}.$$

Thus, we have, for every $j \in \mathbb{N}$,

$$\sum_{n:\left\| P^*_{n,q_n}(f)-f \right\|_{\bar{D}} \geq \varepsilon} a_{jn} \leq \sum_{n:\frac{1}{[n]_{q_n}} \geq \left(\frac{\varepsilon}{2^\alpha M}\right)^{2/\alpha}} a_{jn}.$$

Now taking limit as $j \to \infty$ in and also using (4.7), we immediately obtain that

$$\lim_j \sum_{n \in Z_1} a_{jn} = 0,$$

which gives

$$st_A - \lim_n \left\| P^*_{n,q_n}(f) - f \right\|_{\bar{D}} = 0.$$

Thus, the proof is completed. □

The next result is a natural consequence of Theorem 15.

Corollary 7. *Let (q_n) be a sequence from $(0,1)$ satisfying (4.7). Then, for every $f \in Lip^*_M(\bar{D};\alpha)$, the sequence $\left\{ P^*_{n,q_n}(f) \right\}_{n \in \mathbb{N}}$ is uniformly convergent to f on $\bar{D}$.*

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