

Miskolc Mathematical Notes

A Publication of the University of Miskolc

VOLUME 13 (2012), NUMBER 1



MISKOLC UNIVERSITY PRESS



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A VIABILITY RESULT FOR FUNCTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES

MYELKEBIR AITALIOUBRAHIM

Received 16 February, 2011

Abstract. This paper is concerned with the existence of viable or monotone solutions of a first order functional differential inclusion in a separable Banach space. We consider two cases with moving constraints.

2000 Mathematics Subject Classification: 34A60; 49J52

Keywords: multifunction, measurability, selection, functional differential inclusion

1. INTRODUCTION

Let E be a separable Banach space with norm $\|\cdot\|$. For a segment I in $\mathbb{R}$, we denote by $\mathcal{C}(I, E)$ the Banach space of continuous functions from I to E equipped with the norm $\|x(\cdot)\|_\infty := \sup\{\|x(t)\|; t \in I\}$. For a positive number a , we put $\mathcal{C}_a := \mathcal{C}([-a, 0], E)$ and for any $t \in [0, T]$, $T > 0$, we define the operator $T(t)$ from $\mathcal{C}([-a, T], E)$ to $\mathcal{C}_a$ with $(T(t)(x(\cdot)))(s) := (T(t)x)(s) := x(t+s)$, $s \in [-a, 0]$. For a given nonempty subset K of E we introduce the set $\mathcal{K}_0 := \{\varphi(\cdot) \in \mathcal{C}_a; \varphi(0) \in K\}$.

In this paper, we shall prove local existence of the solutions to the following functional differential inclusions:

$$\begin{cases} \dot{x}(t) \in F(t, T(t)x), & \text{a.e. on } [0, T]; \\ x(s) = \varphi(s), & \forall s \in [-a, 0]; \\ x(t) \in C(t), & \forall t \in [0, T]; \end{cases} \quad (1.1)$$

$$\begin{cases} \dot{x}(t) \in F(t, T(t)x), & \text{a.e. on } [0, T]; \\ x(s) = \varphi(s), & \forall s \in [-a, 0]; \\ x(t) \in P(x(s)), & \forall s \in [0, T], \quad \forall t \in [s, T]; \end{cases} \quad (1.2)$$

where F is a closed multifunction, measurable with respect to the first argument and Lipschitz continuous with respect to the second argument, C and P are two set-valued maps, and φ is a given function in $\mathcal{C}_a$.

This kind of problems was initiated by Haddad [5, 6], in the case when F is globally upper semi-continuous and takes convex compact values and C is fixed. Then

this line of research was pursued by several authors. Regarding the existence of solutions in a separable Banach space, we refer to Syam [9], where two results are given: the first one deals with a fixed convex constraint C and a globally measurable multifunction F which is upper semi-continuous in its second argument, while in the second one the constraint is convex and moving, but F is globally upper semi-continuous. See also the work of Gavioli and Malaguti [4].

Duc Ha, in [3], established the existence of viable solutions of (1.1) without memory in a Banach space when C is fixed and F is closed, integrably bounded, measurable with respect to the first argument and Lipschitz continuous with respect to the second argument. The author established a multi-valued version of Larrieu's work [7], assuming the following tangential condition:

$$\liminf_{h \rightarrow 0^+} \frac{1}{h} e \left(x + \int_t^{t+h} F(s, x) ds, C \right) = 0, \quad (1.3)$$

where $e(.,.)$ denotes the Hausdorff excess. Duc Ha's result has been extended by Lupulescu and Necula [8] to functional differential inclusions, but under the same hypotheses on F , assuming that C is always fixed. They used a suitable adapted form of the tangential condition of (1.3).

Our work extends the results of [3, 8]. Indeed, we get existence results, in a separable Banach space, for functional differential inclusions, with a not necessarily convex constraint which depends on time or on the variable $x \in E$. The right-hand side is not necessarily convex. As is known, viability problems need tangential conditions. For problems (1.1) and (1.2), we shall use a tangency condition which is weaker than the one used in [3, 8].

The paper is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel. In Section 3, we give some auxiliary results. In Section 4, we prove the existence of solutions for (1.1), while in Section 4 we establish the existence of monotone solutions for (1.2).

2. STATEMENT OF THE MAIN RESULT

For the purpose of measurability, E (resp. $\Omega \subset E$) is endowed with the σ -algebra $B(E)$ (resp. $B(\Omega)$) of Borel subsets for the strong topology and $[0, 1]$ is endowed with Lebesgue measure and the σ -algebra of Lebesgue measurable subsets. For $x \in E$ and $r > 0$, let $B(x, r) := \{y \in E; \|y - x\| < r\}$ be the open ball centered at x with radius r and let $\bar{B}(x, r)$ be its closure and put $B = B(0, 1)$. For $\varphi(.) \in \mathcal{C}_a$ let $B_a(\varphi(.), r) := \{\psi(.) \in \mathcal{C}_a; \|\varphi(.) - \psi(.)\|_\infty < r\}$ and let $\bar{B}_a(\varphi(.), r)$ be its closure. For $x \in E$ and for nonempty subsets A, B of E , we denote by $d_A(x)$ or $d(x, A)$ the real $\inf\{\|y - x\|; y \in A\}$, $e(A, B) := \sup\{d_B(x); x \in A\}$ and $H(A, B) = \max\{e(A, B), e(B, A)\}$. For a multifunction F we denote by $Gr(F)$ its graph. A multifunction is said to be measurable if its graph is measurable. For more

detail on measurability theory, we refer the reader to the book of Castaing-Valadier [2].

Let us recall the following Lemmas that will be used in the sequel. For the proofs, we refer the reader to [10].

Lemma 1. *Let Ω be a nonempty set in E . Assume that $F : [a, b] \times \Omega \rightarrow 2^E$ is a multifunction with nonempty closed values satisfying:*

- *For every $x \in \Omega$, $F(\cdot, x)$ is measurable on $[a, b]$;*
- *For every $t \in [a, b]$, $F(t, \cdot)$ is (Hausdorff) continuous on Ω .*

Then for any measurable function $x(\cdot) : [a, b] \rightarrow \Omega$, the multifunction $F(\cdot, x(\cdot))$ is measurable on $[a, b]$.

Lemma 2. *Let $G : [a, b] \rightarrow 2^E$ be a measurable multifunction and $y(\cdot) : [a, b] \rightarrow E$ a measurable function. Then for any positive measurable function $r(\cdot) : [a, b] \rightarrow \mathbb{R}^+$, there exists a measurable selection $g(\cdot)$ of G such that for almost all $t \in [a, b]$*

$$\|g(t) - y(t)\| \leq d(y(t), G(t)) + r(t).$$

Let us introduce the following hypotheses which we shall use throughout this paper.

Hypothesis (H).

- (H1) $C : [-1, 1] \rightarrow 2^E$ is a lower semicontinuous set-valued map with compact graph and $\mathcal{K} : [-1, 1] \rightarrow \mathcal{C}_a$ is a set-valued map defined by $\mathcal{K}(t) = \{\varphi \in \mathcal{C}_a, \varphi(0) \in C(t)\}$;
- (H2) $F : Gr(\mathcal{K}) \rightarrow 2^E$ is a set-valued map with nonempty closed values satisfying
- (i) $t \mapsto F(t, \psi)$ is measurable,
 - (ii) There exists a function $m(\cdot) \in L^1([0, 1], \mathbb{R}^+)$ such that for all $t \in [0, 1]$ and $\psi_1, \psi_2 \in \mathcal{K}(t)$

$$H(F(t, \psi_1), F(t, \psi_2)) \leq m(t)\|\psi_1 - \psi_2\|_\infty,$$

- (iii) There exists $g(\cdot) \in L^1([0, 1], \mathbb{R}^+)$ such that for all $t \in [0, 1]$ and $\psi \in \mathcal{K}(t)$

$$\|F(t, \psi)\| := \sup_{y \in F(t, \psi)} \|y\| \leq g(t).$$

Hypothesis (A).

- (A1) K is a nonempty compact subset in E and $P : E \rightarrow 2^K$ is a lower semicontinuous set-valued map satisfying
- (i) For all $x \in K$, $x \in P(x)$,
 - (ii) For all $x \in K$ and all $y \in P(x)$ we have $P(y) \subseteq P(x)$;

(A2) $F : [-1, 1] \times \mathcal{K}_0 \rightarrow 2^E$ is a set-valued map with nonempty closed values satisfying

- (i) For each $\psi \in \mathcal{K}_0$, $t \mapsto F(t, \psi)$ is measurable,
- (ii) There exists a function $m(\cdot) \in L^1([0, 1], \mathbb{R}^+)$ such that for all $t \in [0, 1]$ and for all $\psi_1, \psi_2 \in \mathcal{K}_0$

$$H(F(t, \psi_1), F(t, \psi_2)) \leq m(t) \|\psi_1 - \psi_2\|_\infty,$$

- (iii) There exists $g(\cdot) \in L^1([0, 1], \mathbb{R}^+)$ such that for all $t \in [0, 1]$ and for all $\psi \in \mathcal{K}_0$

$$\|F(t, \psi)\| \leq g(t).$$

In the rest of the paper, fix $\varphi \in \mathcal{C}_a$ such that $\varphi(0) \in C(0)$ for Problem (1.1) and $\varphi(0) \in K$ for Problem (1.2). For $\varepsilon > 0$ set,

$$\begin{aligned} \eta(\varepsilon) := \sup \left\{ \rho \in]0, \varepsilon] : \left| \int_{t_1}^{t_2} g(\tau) d\tau \right| < \varepsilon, \|\varphi(t_1) - \varphi(t_2)\| < \varepsilon \right. \\ \left. \text{and } \left| \int_{t_1}^{t_2} m(\tau) d\tau \right| < \varepsilon \text{ if } |t_1 - t_2| \leq \rho \right\}. \end{aligned} \quad (2.1)$$

In the sequel we need the following notation. For given measurable functions $y(\cdot) : [0, 1] \rightarrow E$ and $r(\cdot) : [0, 1] \rightarrow \mathbb{R}^+$ set

$$S_{y,r}(\psi) = \left\{ f \in L^1([-1, 1], E) : f(s) \in F(s, \psi) \text{ for all } s \in [-1, 1] \text{ and } \right. \\ \left. \|f(s) - y(s)\| \leq d(y(s), F(s, \psi)) + r(s) \text{ for all } s \in [0, 1] \right\}$$

where $\psi \in \mathcal{C}_a$. Remark that the set $S_{y,r}(\psi)$ is nonempty if we take into account that F satisfies the condition (H2) or (A2). Indeed, this fact can be trivially deduced from Lemma 1 and Lemma 2.

We are now ready to introduce the tangential conditions used in this work.

- (H3) (**Tangential condition**) For all measurable functions $y(\cdot) : [0, 1] \rightarrow E$, $r(\cdot) : [0, 1] \rightarrow \mathbb{R}^+$ and for every $t \in [0, 1]$ and $\psi \in \mathcal{K}(t)$, there exists $f \in S_{y,r}(\psi)$ such that

$$\liminf_{h \rightarrow 0^+} \frac{1}{h} d \left(\psi(0) + \int_t^{t+h} f(s) ds, C(t+h) \right) = 0.$$

- (A3) (**Tangential condition**) For all measurable functions $y(\cdot) : [0, 1] \rightarrow E$, $r(\cdot) : [0, 1] \rightarrow \mathbb{R}^+$ and for every $(t, \psi) \in [0, 1] \times \mathcal{K}_0$, there exists $f \in S_{y,r}(\psi)$ such that

$$\liminf_{h \rightarrow 0^+} \frac{1}{h} d \left(\psi(0) + \int_t^{t+h} f(s) ds, P(\psi(0)) \right) = 0.$$

These tangential conditions are weaker than the ones used in [3, 8]. It is important to notice here that, if F and C do not depend on time, (H3) and (A3) are equivalent to

$$F(\psi) \cap T_C(\psi(0)) \neq \emptyset \text{ and } F(\psi) \cap T_{P(\psi(0))}(\psi(0)) \neq \emptyset$$

respectively, where $T_K(x)$ is the contingent cone at x to K and $\varphi(\cdot) \in \mathcal{C}_a$. However, in this case the condition (1.3) is equivalent to $F(x) \subseteq T_C(x)$.

In the next sections, we shall prove the following theorems.

Theorem 1. *If assumptions (H1)-(H3) are satisfied, then for all $(z_0, \varphi) \in E \times \mathcal{C}_a$ such that $\varphi(0) = z_0 \in C(0)$, there exist $T > 0$ and a continuous function $x(\cdot) : [-a, T] \rightarrow E$ which is absolutely continuous on $[0, T]$ and it is a solution of (1.1).*

Theorem 2. *If assumptions (A1)-(A3) are satisfied, then for all $(z_0, \varphi) \in K \times \mathcal{K}_0$ such that $\varphi(0) = z_0$, there exist $T > 0$ and a continuous function $x(\cdot) : [-a, T] \rightarrow E$ which is absolutely continuous on $[0, T]$ and it is a monotone solution of (1.2).*

3. PRELIMINARY RESULTS

In this section, we shall prove some auxiliary results that will be used in the next sections. Consider first the following hypotheses which we shall use throughout this section.

Hypothesis (C).

- (C1) $C : [-1, 1] \rightarrow 2^E$ is a lower semicontinuous set-valued map with compact graph;
- (C2) $F : Gr(C) \rightarrow 2^E$ is a set-valued map with nonempty closed values satisfying
 - (i) $t \mapsto F(t, x)$ is measurable,
 - (ii) There exists a function $m(\cdot) \in L^1([0, 1], \mathbb{R}^+)$ such that for all $t \in [0, 1]$ and $x_1, x_2 \in \mathcal{C}(t)$

$$H(F(t, x_1), F(t, x_2)) \leq m(t) \|x_1 - x_2\|,$$

- (iii) There exists $g(\cdot) \in L^1([0, 1], \mathbb{R}^+)$ such that for all $t \in [0, 1]$ and $x \in \mathcal{C}(t)$

$$\|F(t, x)\| \leq g(t);$$

- (C3) (**Tangential condition**) For all measurable functions $y(\cdot) : [0, 1] \rightarrow E$, $r(\cdot) : [0, 1] \rightarrow \mathbb{R}^+$ and for every $t \in [0, 1]$ and $x \in C(t)$, there exists $f \in S_{y, r}(x)$ such that

$$\liminf_{h \rightarrow 0^+} \frac{1}{h} d \left(x + \int_t^{t+h} f(s) ds, C(t+h) \right) = 0.$$

Hypothesis (L).

- (L1) K is a nonempty compact subset in E and $P : E \rightarrow 2^K$ is a lower semicontinuous set-valued map satisfying
- (i) For all $x \in K$, $x \in P(x)$,
 - (ii) For all $x \in K$ and all $y \in P(x)$ we have $P(y) \subseteq P(x)$;
- (L2) $F : [-1, 1] \times K \rightarrow 2^E$ is a set-valued map with nonempty closed values satisfying
- (i) For each $x \in K$, $t \mapsto F(t, x)$ is measurable,
 - (ii) There exists a function $m(\cdot) \in L^1([0, 1], \mathbb{R}^+)$ such that for all $t \in [0, 1]$ and for all $x_1, x_2 \in K$

$$H(F(t, x_1), F(t, x_2)) \leq m(t)\|x_1 - x_2\|,$$

- (iii) There exists $g(\cdot) \in L^1([0, 1], \mathbb{R}^+)$ such that for all $t \in [0, 1]$ and for all $x \in K$

$$\|F(t, x)\| \leq g(t);$$

- (L3) (**Tangential condition**) For all measurable functions $y(\cdot) : [0, 1] \rightarrow E$, $r(\cdot) : [0, 1] \rightarrow \mathbb{R}^+$ and for every $(t, x) \in [0, 1] \times K$, there exists $f \in S_{y, r}(x)$ such that

$$\liminf_{h \rightarrow 0^+} \frac{1}{h} d \left(x + \int_t^{t+h} f(s) ds, P(x) \right) = 0.$$

In the following, let $b \in]0, 1[$ be fixed and, for all $t_0 \in [0, b]$ set $[t_0, b_0] = [t_0, t_0 + \frac{1}{4}\eta(\frac{\varepsilon}{4})] \cap [0, b]$.

Lemma 3. *If assumptions (C1)-(C3) are satisfied, then for all $0 < \varepsilon < a$, $t_0 \in [0, b]$ and for all measurable function $y(\cdot) : [0, b] \rightarrow E$, there exists $\eta > 0$ ($\eta < \varepsilon$) such that for all $t \in [t_0, b_0]$ and $x \in C(t)$, there exist $f \in S_{y, (2m(\cdot)+1)\varepsilon}(x)$, $h_{t,x} \in [\eta, \frac{1}{4}\eta(\frac{\varepsilon}{4})]$ and $u \in \varepsilon B$ such that*

$$\left(x + h_{t,x}u + \int_t^{t+h_{t,x}} f(s) ds \right) \in C(t + h_{t,x}).$$

Proof. Let $0 < \varepsilon < a$, $t_0 \in [0, b]$ and $y(\cdot) : [0, b] \rightarrow E$ be a measurable function. Consider $(s, z) \in M := \{(\bar{s}, \bar{z})/\bar{s} \in [0, b] \text{ and } \bar{z} \in C(\bar{s})\}$. Note that if the subset M of $Gr(C)$ is closed then it is compact. By the tangential condition, there exists $f_{s,z} \in S_{y, \frac{\varepsilon}{4}}(z)$ such that

$$\liminf_{h \rightarrow 0^+} \frac{1}{h} d \left(z + \int_s^{s+h} f_{s,z}(\tau) d\tau, C(s+h) \right) = 0.$$

Then there exists $\bar{h}_{s,z} \in]0, \min\{\frac{1}{4}\eta(\frac{\varepsilon}{4}), 1-b\}]$ satisfying

$$d\left(z + \int_s^{s+\bar{h}_{s,z}} f_{s,z}(\tau) d\tau, C(s + \bar{h}_{s,z})\right) < \bar{h}_{s,z} \frac{\varepsilon}{4}.$$

Consider the subset

$$N(s, z) = \left\{ (\bar{s}, \bar{z}) \in]-\bar{h}_{s,z}, 1 - \bar{h}_{s,z}[\times E : \right. \\ \left. d\left(\bar{z} + \int_{\bar{s}}^{\bar{s}+\bar{h}_{s,z}} f_{s,z}(\mu) d\mu, C(\bar{s} + \bar{h}_{s,z})\right) < \bar{h}_{s,z} \frac{\varepsilon}{4} \right\}.$$

Since C is lower semicontinuous, Corollary 1.2.1 in [1] shows that the function

$$(\bar{s}, \bar{z}) \mapsto d\left(\bar{z} + \int_{\bar{s}}^{\bar{s}+\bar{h}_{s,z}} f_{s,z}(\mu) d\mu, C(\bar{s} + \bar{h}_{s,z})\right)$$

is upper semicontinuous. So the set $N(s, z)$ is open. Moreover, since (s, z) belongs to $N(s, z)$, there exists $0 < \eta_{s,z} < \bar{h}_{s,z}$ such that $]s - \eta_{s,z}, s + \eta_{s,z}[\times B(z, \eta_{s,z})$ is contained in $N(s, z)$, therefore, the compact subset M can be covered by q such subsets $]s_i - \eta_{s_i, z_i}, s_i + \eta_{s_i, z_i}[\times B(z_i, \eta_{s_i, z_i})$. For simplicity, we set

$$h_i := \bar{h}_{s_i, z_i} \text{ and } \eta_i := \eta_{s_i, z_i}, i = 1, \dots, q.$$

Put $\eta = \min\{h_i/1 \leq i \leq q\}$, take $t \in [t_0, b_0]$ and $x \in C(t)$, and let $i \in \{1, \dots, q\}$ such that $(t, x) \in]s_i - \eta_i, s_i + \eta_i[\times B(z_i, \eta_i)$, hence $(t, x) \in N(s_i, z_i)$. So there exists $f_{s_i, z_i} \in S_{y, \frac{\varepsilon}{4}}(z_i)$ such that

$$d\left(x + \int_t^{t+h_i} f_{s_i, z_i}(\tau) d\tau, C(t + h_i)\right) < h_i \frac{\varepsilon}{4}.$$

On the other hand, let $f_i \in S_{f_{s_i, z_i}, \frac{\varepsilon}{4}}(x)$. By (C2) we have for all $\tau \in [t, t + h_i]$

$$\|f_i(\tau) - f_{s_i, z_i}(\tau)\| \leq d(f_{s_i, z_i}(\tau), F(\tau, x)) + \frac{\varepsilon}{4} \leq m(\tau)\|x - z_i\| + \frac{\varepsilon}{4} \leq h_i m(\tau) + \frac{\varepsilon}{4}.$$

Hence by (2.1) we get

$$\int_t^{t+h_i} \|f_i(\tau) - f_{s_i, z_i}(\tau)\| d\tau \leq h_i \int_t^{t+h_i} m(\tau) d\tau + \frac{h_i \varepsilon}{4} \leq \frac{h_i \varepsilon}{4} + \frac{h_i \varepsilon}{4} \leq \frac{h_i \varepsilon}{2}.$$

Thus

$$\begin{aligned} & d\left(x + \int_t^{t+h_i} f_i(\tau) d\tau, C(t + h_i)\right) \\ & \leq d\left(x + \int_t^{t+h_i} f_{s_i, z_i}(\tau) d\tau, C(t + h_i)\right) + \int_t^{t+h_i} \|f_{s_i, z_i}(\tau) - f_i(\tau)\| d\tau \end{aligned}$$

$$\begin{aligned} &\leq \frac{h_i \varepsilon}{4} + \frac{h_i \varepsilon}{2} \\ &\leq \frac{3h_i \varepsilon}{4}. \end{aligned}$$

Let $x_i \in C(t + h_i)$ such that

$$\frac{1}{h_i} \left\| x_i - x - \int_t^{t+h_i} f_i(\tau) d\tau \right\| \leq \frac{1}{h_i} d \left(x + \int_t^{t+h_i} f_i(\tau) d\tau, C(t + h_i) \right) + \frac{\varepsilon}{4} \leq \varepsilon.$$

Set

$$u = \frac{1}{h_i} \left(x_i - x - \int_t^{t+h_i} f_i(\tau) d\tau \right),$$

then $u \in \varepsilon B$ and

$$x_i = \left(x + h_i u + \int_t^{t+h_i} f_i(\tau) d\tau \right) \in C(t + h_i).$$

Note that $f_i \in S_{y, (2m(\cdot)+1)\varepsilon}(x)$. Indeed, one has

$$\begin{aligned} \|f_i(\tau) - y(\tau)\| &\leq \|f_i(\tau) - f_{s_i, z_i}(\tau)\| + \|f_{s_i, z_i}(\tau) - y(\tau)\| \\ &\leq m(\tau)h_i + \frac{\varepsilon}{4} + d(y(\tau), F(\tau, z_i)) + \frac{\varepsilon}{4} \\ &\leq m(\tau)h_i + \frac{2\varepsilon}{4} + d(y(\tau), F(\tau, x)) + m(\tau)\|x - z_i\| \\ &\leq m(\tau)\varepsilon + \frac{2\varepsilon}{4} + d(y(\tau), F(\tau, x)) + m(\tau)h_i \\ &\leq d(y(\tau), F(\tau, x)) + (2m(\tau) + 1)\varepsilon. \end{aligned}$$

□

The proof of the next Lemma is similar to that of Lemma 3.

Lemma 4. *If assumptions (L1)-(L3) are satisfied, then for all $0 < \varepsilon < a$, for all $t_0 \in [0, b]$ and for all measurable function $y(\cdot) : [0, b] \rightarrow E$, there exists $\eta > 0$ ($\eta < \varepsilon$) such that $\forall (t, x) \in [t_0, b_0] \times K$, there exist $f \in S_{y, (2m(\cdot)+1)\varepsilon}(x)$, $h_{t,x} \in [\eta, \frac{1}{4}\eta(\frac{\varepsilon}{4})]$ and $u \in \varepsilon B$ such that*

$$\left(x + h_{t,x} u + \int_t^{t+h_{t,x}} f(s) ds \right) \in P(x).$$

We need the following Propositions in the next sections.

Proposition 1. *If assumptions (C1)-(C3) are satisfied, then for all $0 < \varepsilon < a$, $t_0 \in [0, b]$, $z_0 \in C(t_0)$ and $y(\cdot) \in L^1([0, b], E)$, there exist a continuous mapping $z(\cdot) : [t_0, b_0] \rightarrow E$, a step function $\theta(\cdot) : [t_0, b_0] \rightarrow [t_0, b_0]$ and $f(\cdot) \in L^1([t_0, b_0], E)$ such that*

- (i) $z(t_0) = z_0$, $f(t) \in F(t, z(\theta(t)))$, $0 \leq t - \theta(t) \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$ and $z(\theta(t)) \in C(\theta(t))$, for all $t \in [t_0, b_0]$;
- (ii) $\|f(t) - y(t)\| \leq d(y(t), F(t, z(\theta(t)))) + (2m(t) + 1)\varepsilon$ for all $t \in [t_0, b_0]$;
- (iii) $\left\| z(t) - z(t_0) - \int_{t_0}^t f(\tau) d\tau \right\| \leq \varepsilon(t - t_0)$ for all $t \in [t_0, b_0]$.

Proof. Let $0 < \varepsilon < a$, $t_0 \in [0, b]$, $z_0 \in C(t_0)$ and $y(\cdot) \in L^1([0, b], E)$ be fixed. By Lemma 3, there exist $h_0 \in [\eta, \frac{1}{4}\eta(\frac{\varepsilon}{4})]$, $f_0 \in S_{y, (2m(\cdot)+1)\varepsilon}(z_0)$ and $u_0 \in \varepsilon B$ such that

$$\left(z_0 + h_0 u_0 + \int_{t_0}^{t_0+h_0} f_0(s) ds \right) \in C(t_0 + h_0).$$

Set

$$z_1 = z_0 + h_0 u_0 + \int_{t_0}^{t_0+h_0} f_0(s) ds \text{ and } t_1 = t_0 + h_0$$

We reiterate this process for constructing finite sequences $(h_i)_i \subset [\eta, \frac{1}{4}\eta(\frac{\varepsilon}{4})]$, $(f_i(\cdot))_i, (t_i)_i, (u_i)_i \subset \varepsilon B$, and $(z_i)_i$ satisfying the following assertions for $i \geq 0$:

- (a) $t_{i+1} = t_i + h_i$ and $z_{i+1} \in C(t_{i+1})$
- (b) $z_{i+1} = z_i + h_i u_i + \int_{t_i}^{t_{i+1}} f_i(s) ds$;
- (c) $f_i \in S_{y, (2m(\cdot)+1)\varepsilon}(z_i)$.

It is easy to see that for $i = 0$ the assertions (a)-(c) are fulfilled. Let now $i \geq 1$. Assume that (a)-(c) are satisfied for any $i = 1, \dots, q$. If $b_0 \leq t_{q+1}$, then we stop the process of iterations and we get that (a)-(c) are satisfied with $t_q < b_0 \leq t_{q+1}$. In the other case, for (t_{q+1}, z_{q+1}) , by Lemma 3, there exist $h_{q+1} \in [\eta, \frac{1}{4}\eta(\frac{\varepsilon}{4})]$, $f_{q+1} \in S_{y, (2m(\cdot)+1)\varepsilon}(z_{q+1})$ and $u_{q+1} \in \varepsilon B$ such that

$$\left(z_{q+1} + h_{q+1} u_{q+1} + \int_{t_{q+1}}^{t_{q+1}+h_{q+1}} f_{q+1}(s) ds \right) \in C(t_{q+1} + h_{q+1}).$$

Set

$$z_{q+2} = z_{q+1} + h_{q+1} u_{q+1} + \int_{t_{q+1}}^{t_{q+1}+h_{q+1}} f_{q+1}(s) ds \text{ and } t_{q+2} = t_{q+1} + h_{q+1}.$$

Thus the conditions (a)-(c) are satisfied for $i = q + 1$. Since $h_i \geq \eta > 0$, there exists an integer s such that $t_s < b_0 \leq t_{s+1}$. Now, we define the function $\theta(\cdot) : [t_0, b_0] \rightarrow [t_0, b_0]$ and $f(\cdot), u(\cdot) \in L^1([t_0, b_0], E)$ by setting for all $t \in [t_q, t_{q+1}] \cap [t_0, b_0]$

$$\theta(t) = t_q, f(t) = f_q(t), u(t) = u_q,$$

$$\theta(b_0) = b_0, f(b_0) = f_{s+1}(b_0), u(b_0) = u_{s+1} \text{ if } t_{s+1} = b_0.$$

Also we define the function $z(\cdot) : [t_0, b_0] \rightarrow E$ by $z(t_0) = z_0$ and

$$z(t) = z(t_0) + \int_{t_0}^t f(s)ds + \int_{t_0}^t u(s)ds, \quad \forall t \in [t_0, b_0].$$

These definitions will enable us to derive assertions (i) and (ii). Moreover, we have for all $t \in [t_0, b_0]$

$$\left\| z(t) - z(t_0) - \int_{t_0}^t f(s)ds \right\| \leq \varepsilon(t - t_0).$$

The proof of Proposition 1 is complete. $\square$

The same proof of Proposition 1 holds for the following Proposition.

Proposition 2. *If assumptions (L1)-(L3) are satisfied, then for every $\varepsilon \in]0, a[$, $t_0 \in [0, b]$, $z_0 \in K$ and $y(\cdot) \in L^1([0, b], E)$, there exist $z(\cdot) \in \mathcal{C}([t_0, +\infty[, E)$, $\theta(\cdot), \tilde{\theta}(\cdot) : [t_0, +\infty[\rightarrow [t_0, +\infty[$ and $f(\cdot) \in L^1([t_0, b_0], E)$ such that*

- (i) $z(t_0) = z_0$, $f(t) \in F(t, z(\theta(t)))$, $0 \leq t - \theta(t) \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$, $0 \leq \tilde{\theta}(t) - t \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$ and $z(\tilde{\theta}(t)) \in P(z(\theta(t)))$ for all $t \in [t_0, b_0]$;
- (ii) $\|f(t) - y(t)\| \leq d(y(t), F(t, z(\theta(t)))) + (2m(t) + 1)\varepsilon$ for all $t \in [t_0, b_0]$;
- (iii) $\left\| z(t) - z(t_0) - \int_{t_0}^t f(\tau)d\tau \right\| \leq \varepsilon(t - t_0)$ for all $t \in [t_0, b_0]$.

Remark that, in this case, $\tilde{\theta}$ and θ are defined as follows: $\tilde{\theta}(t) = t_{q+1}$ and $\theta(t) = t_q$ for all $t \in [t_q, t_{q+1}[$.

4. PROOF OF THEOREM 1

Set $\varphi(0) = z_0$ and let $T_1 > 0$ be such that

$$\int_0^{T_1} m(t)dt < 1. \quad (4.1)$$

Put $T = \inf\{T_1, b\}$.

We shall show the following Proposition. It will be used in order to obtain a sequence of approximated solutions.

Proposition 3. *If assumptions (H1)-(H3) are satisfied, then for all $0 < \varepsilon < a$ and for all $y(\cdot) \in L^1([0, b], E)$, there exist a continuous mapping $z(\cdot) : [-a, T] \rightarrow E$, step functions $\theta(\cdot), \bar{\theta}(\cdot) : [0, T] \rightarrow [0, T]$, $\Gamma(\cdot) : [0, T] \rightarrow \mathcal{C}_a$ and $f(\cdot) \in L^1([0, T], E)$ such that*

- (i) $f(t) \in F(t, \Gamma(t))$ and $z(\theta(t)) \in C(\theta(t))$, for all $t \in [0, T]$ and $z \equiv \varphi$ on $[-a, 0]$;
- (ii) $0 \leq t - \theta(t) \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$ and $0 \leq t - \bar{\theta}(t) \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$ for all $t \in [0, T]$;
- (iii) $\|f(t) - y(t)\| \leq d(y(t), F(t, \Gamma(t))) + (2m(t) + 1)\varepsilon$ for all $t \in [0, T]$;

- (iv) $\left\| z(t) - z(0) - \int_0^t f(\tau) d\tau \right\| \leq \varepsilon t$ for all $t \in [0, T]$.
 (v) For all $t \in [0, T]$

$$\Gamma(t)(s) = \begin{cases} z(\bar{\theta}(t) + \frac{1}{4}\eta(\frac{\varepsilon}{4}) + s) & -a \leq s \leq -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \\ -\frac{4s}{\eta(\frac{\varepsilon}{4})}z(\bar{\theta}(t)) + (1 + \frac{4s}{\eta(\frac{\varepsilon}{4})})z(\theta(t)) & -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \leq s \leq 0. \end{cases}$$

Proof. Let $0 < \varepsilon < a$ and $y(\cdot) \in L^1([0, b], E)$ be fixed. Set $t_0 = 0$ and put

$$z(t) = \varphi(t), \quad \forall t \in [-a, 0].$$

Consider the function $\Gamma_0 : E \rightarrow \mathcal{C}_a$ defined by

$$\Gamma_0(x)(s) = \begin{cases} z(t_0 + \frac{1}{4}\eta(\frac{\varepsilon}{4}) + s) & -a \leq s \leq -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \\ -\frac{4s}{\eta(\frac{\varepsilon}{4})}z(t_0) + (1 + \frac{4s}{\eta(\frac{\varepsilon}{4})})x & -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \leq s \leq 0, \end{cases}$$

for all $x \in E$. The set-valued map $G_0 : Gr(C) \rightarrow 2^E$ defined by $G_0(t, x) = F(t, \Gamma_0(x))$ satisfies all assumptions (C1)-(C3). Then by Proposition 1, there exist a continuous mapping $z_0(\cdot) : [t_0, t_0 + \frac{1}{4}\eta(\frac{\varepsilon}{4})] \cap [0, T] \rightarrow E$, a step function $\theta_0(\cdot) : [t_0, t_0 + \frac{1}{4}\eta(\frac{\varepsilon}{4})] \cap [0, T] \rightarrow [t_0, t_0 + \frac{1}{4}\eta(\frac{\varepsilon}{4})] \cap [0, T]$ and $f_0(\cdot) \in L^1([t_0, t_0 + \frac{1}{4}\eta(\frac{\varepsilon}{4})] \cap [0, T], E)$ such that

- (i) $z_0(t_0) = z_0$, $f_0(t) \in F(t, \Gamma_0(z_0(\theta_0(t))))$, $0 \leq t - \theta_0(t) \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$ and $z_0(\theta_0(t)) \in C(\theta_0(t))$, for all $t \in [t_0, t_0 + \frac{1}{4}\eta(\frac{\varepsilon}{4})] \cap [0, T]$;
- (ii) $\|f_0(t) - y(t)\| \leq d(y(t), F(t, \Gamma_0(z_0(\theta_0(t)))) + (2m(t) + 1)\varepsilon$ for all $t \in [t_0, t_0 + \frac{1}{4}\eta(\frac{\varepsilon}{4})] \cap [0, T]$;
- (iii) $\left\| z_0(t) - z_0(t_0) - \int_{t_0}^t f_0(\tau) d\tau \right\| \leq \varepsilon(t - t_0)$ for all $t \in [t_0, t_0 + \frac{1}{4}\eta(\frac{\varepsilon}{4})] \cap [0, T]$.

Set $t_1 = t_0 + \frac{1}{4}\eta(\frac{\varepsilon}{4})$ and $z(t) = z_0(t)$, for all $t \in [t_0, t_1] \cap [0, T]$.

We reiterate this process for constructing sequences $z_i(\cdot) : [t_i, t_{i+1}] \cap [0, T] \rightarrow E$, $\theta_i(\cdot) : [t_i, t_{i+1}] \cap [0, T] \rightarrow [t_i, t_{i+1}] \cap [0, T]$, $\Gamma_i : E \rightarrow \mathcal{C}_a$ and $f_i(\cdot) \in L^1([t_i, t_{i+1}] \cap [0, T], E)$ and a continuous function $z(\cdot) : [-a, t_{i+1}] \cap [-a, T] \rightarrow E$ satisfying the following assertions for $i \geq 0$:

- (a) $z(t_i) = z_i(t_i)$, $f_i(t) \in F(t, \Gamma_i(z_i(\theta_i(t))))$, $0 \leq t - \theta_i(t) \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$ and $z_i(\theta_i(t)) \in C(\theta_i(t))$, for all $t \in [t_i, t_{i+1}] \cap [0, T]$;
- (b) $\|f_i(t) - y(t)\| \leq d(y(t), F(t, \Gamma_i(z_i(\theta_i(t)))) + (2m(t) + 1)\varepsilon$ for all $t \in [t_i, t_{i+1}] \cap [0, T]$;
- (c) $\left\| z_i(t) - z_i(t_i) - \int_{t_i}^t f_i(\tau) d\tau \right\| \leq \varepsilon(t - t_i)$ for all $t \in [t_i, t_{i+1}] \cap [0, T]$.
- (d) For all $x \in E$

$$\Gamma_i(x)(s) = \begin{cases} z(t_i + \frac{1}{4}\eta(\frac{\varepsilon}{4}) + s) & -a \leq s \leq -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \\ -\frac{4s}{\eta(\frac{\varepsilon}{4})}z(t_i) + (1 + \frac{4s}{\eta(\frac{\varepsilon}{4})})x & -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \leq s \leq 0, \end{cases}$$

The assertions (a)-(d) are fulfilled for $i = 0$. Let now $i \geq 1$. Assume that (a)-(d) are satisfied for any $i = 1, \dots, q$. If $T \leq t_{q+1}$, then we stop this process of iterations and we get that (a)-(d) are satisfied with $t_q < T \leq t_{q+1}$. In the other case $t_{q+1} < T$, consider the function $\Gamma_{q+1} : E \rightarrow \mathcal{C}_a$ defined by

$$\Gamma_{q+1}(x)(s) = \begin{cases} z(t_{q+1} + \frac{1}{4}\eta(\frac{\varepsilon}{4}) + s) & -a \leq s \leq -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \\ -\frac{4s}{\eta(\frac{\varepsilon}{4})}z(t_{q+1}) + (1 + \frac{4s}{\eta(\frac{\varepsilon}{4})})x & -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \leq s \leq 0, \end{cases}$$

for all $x \in E$. The set-valued map $G_{q+1} : Gr(C) \rightarrow 2^E$ defined by $G_{q+1}(t, x) = F(t, \Gamma_{q+1}(x))$ satisfies all assumptions (C1)-(C3). Then by Proposition 1, there exist a continuous mapping $z_{q+1}(\cdot) : [t_{q+1}, t_{q+1} + \frac{1}{4}\eta(\frac{\varepsilon}{4})] \cap [0, T] \rightarrow E$, a step function $\theta_{q+1}(\cdot) : [t_{q+1}, t_{q+1} + \frac{1}{4}\eta(\frac{\varepsilon}{4})] \cap [0, T] \rightarrow [t_{q+1}, t_{q+1} + \frac{1}{4}\eta(\frac{\varepsilon}{4})] \cap [0, T]$ and $f_{q+1}(\cdot) \in L^1([t_{q+1}, t_{q+1} + \frac{1}{4}\eta(\frac{\varepsilon}{4})] \cap [0, T], E)$ satisfying the assertions (a)-(d) for $i = q + 1$. Set $t_{q+2} = t_{q+1} + \frac{1}{4}\eta(\frac{\varepsilon}{4})$ and $z(t) = z_{q+1}(t)$ for all $t \in [t_{q+1}, t_{q+2}] \cap [0, T]$. Thus the conditions (a)-(d) are satisfied for $i = q + 1$. Since $t_{i+1} - t_i = \frac{1}{4}\eta(\frac{\varepsilon}{4})$, there exists an integer s such that $t_s < T \leq t_{s+1}$. Further on, we define the functions $\theta(\cdot), \bar{\theta}(\cdot) : [0, T] \rightarrow [0, T]$, $\Gamma : [0, T] \rightarrow \mathcal{C}_a$ and $f(\cdot) \in L^1([0, T], E)$ by setting for all $t \in [t_q, t_{q+1}] \cap [0, T]$

$$\theta(t) = \theta_q(t), f(t) = f_q(t), \bar{\theta}(t) = t_q, \Gamma(t) = \Gamma_q(z_q(\theta_q(t)))$$

and if $t_{s+1} = T$

$$\theta(T) = \theta_{s+1}(T), f(T) = f_{s+1}(T), \bar{\theta}(T) = T, \Gamma(T) = \Gamma_{s+1}(z_{s+1}(\theta_{s+1}(T))).$$

Moreover, for all $t \in [0, T]$, there exists $0 \leq q \leq s$ such that $t \in [t_q, t_{q+1}] \cap [0, T]$. Then

$$\begin{aligned} & \left\| z(t) - z_0 - \int_0^t f(s)ds \right\| \\ & \leq \left\| z(t) - z(t_q) - \int_{t_q}^t f_q(s)ds \right\| + \left\| z(t_q) - z_0 - \int_{t_0}^{t_q} f(s)ds \right\| \\ & \leq \varepsilon(t - t_q) + \sum_{i=1}^q \left\| z(t_i) - z(t_{i-1}) - \int_{t_{i-1}}^{t_i} f_{i-1}(s)ds \right\| \\ & \leq \varepsilon(t - t_q) + \sum_{i=1}^q \varepsilon(t_i - t_{i-1}) \\ & \leq \varepsilon t. \end{aligned}$$

Hence the proof of Proposition 3 is complete. $\square$

Now we are prepared to prove our Theorem 1. Let $(\varepsilon_n)_{n \geq 1}$ be a strictly decreasing sequence of positive scalars such that $0 < \varepsilon_n < a$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \varepsilon_n <$

∞ . In view of Proposition 3, we can define inductively sequences $(f_n(\cdot))_{n \geq 1} \subset L^1([0, T], E)$, $(z_n(\cdot))_{n \geq 1} \subset \mathcal{C}([-a, T], E)$, $(\Gamma_n(\cdot))_{n \geq 1} \subset \mathcal{C}([0, T], \mathcal{C}_a)$ and $(\theta_n(\cdot))_{n \geq 1}, (\bar{\theta}_n(\cdot))_{n \geq 1} \subset S([0, T], [0, T])$ where $S([0, T], [0, T])$ denotes the space of step functions from $[0, T]$ into $[0, T]$ such that

- (1) $f_n(t) \in F(t, \Gamma_n(t))$ and $z_n(\theta_n(t)) \in C(\theta_n(t))$, for all $t \in [0, T]$ and $z_n \equiv \varphi$ on $[-a, 0]$;
- (2) $0 \leq t - \theta_n(t) \leq \frac{1}{4}\eta(\frac{\varepsilon_n}{4})$ and $0 \leq t - \bar{\theta}_n(t) \leq \frac{1}{4}\eta(\frac{\varepsilon_n}{4})$ for all $t \in [0, T]$;
- (3) $\|f_{n+1}(t) - f_n(t)\| \leq d(f_n(t), F(t, \Gamma_{n+1}(t))) + (2m(t) + 1)\varepsilon_{n+1}$ for all $t \in [0, T]$;
- (4) $\left\| z_n(t) - z_0 - \int_0^t f_n(\tau) d\tau \right\| \leq \varepsilon_n t$ for all $t \in [0, T]$.
- (5) For all $t \in [0, T]$

$$\Gamma_n(t)(s) = \begin{cases} z_n(\bar{\theta}_n(t) + \frac{1}{4}\eta(\frac{\varepsilon_n}{4}) + s) & -a \leq s \leq -\frac{1}{4}\eta(\frac{\varepsilon_n}{4}) \\ -\frac{4s}{\eta(\frac{\varepsilon_n}{4})}z_n(\theta_n(t)) + (1 + \frac{4s}{\eta(\frac{\varepsilon_n}{4})})z_n(\theta_n(t)) & -\frac{1}{4}\eta(\frac{\varepsilon_n}{4}) \leq s \leq 0. \end{cases}$$

From (1) and (3), we deduce for all $t \in [0, T]$

$$\begin{aligned} \|f_{n+1}(t) - f_n(t)\| &\leq H\left(F(t, \Gamma_n(t)), F(t, \Gamma_{n+1}(t))\right) + (2m(t) + 1)\varepsilon_{n+1} \\ &\leq m(t)\|\Gamma_n(t) - \Gamma_{n+1}(t)\|_\infty + (2m(t) + 1)\varepsilon_{n+1}. \end{aligned} \quad (4.2)$$

Claim 1. For all $t \in [0, T]$,

- $\|\Gamma_n(t) - \Gamma_{n+1}(t)\|_\infty \leq \|z_n(\cdot) - z_{n+1}(\cdot)\|_\infty + 12\varepsilon_n$.
- $\|T(t)z_n - \Gamma_n(t)\|_\infty \leq 8\varepsilon_n$.

Proof. First, remark that for all $t, \bar{t} \in [-a, T]$ and $\rho > 0$ such that $|t - \bar{t}| \leq \eta(\rho)$ we have $\|z_n(t) - z_n(\bar{t})\| \leq 2(\varepsilon_n + \rho)$. Indeed, let $t, \bar{t} \in [-a, T]$ such that $|t - \bar{t}| \leq \eta(\rho)$. If $t, \bar{t} \in [0, T]$ and $\bar{t} \leq t$, one has

$$\begin{aligned} &\|z_n(t) - z_n(\bar{t})\| \\ &\leq \left\| z_n(t) - z_0 - \int_0^t f_n(s) ds \right\| + \left\| z_n(\bar{t}) - z_0 - \int_0^{\bar{t}} f_n(s) ds \right\| \\ &\quad + \int_{\bar{t}}^t \|f_n(s)\| ds \\ &\leq \varepsilon_n t + \varepsilon_n \bar{t} + \int_{\bar{t}}^t g(s) ds \\ &\leq 2\varepsilon_n + \rho. \end{aligned}$$

If $t, \bar{t} \in [-a, 0]$, by construction, we have

$$\|z_n(t) - z_n(\bar{t})\| = \|\varphi(t) - \varphi(\bar{t})\| \leq \rho.$$

If $t \in [0, T]$ and $\bar{t} \in [-a, 0]$ one has $|t| \leq \eta(\rho)$ and $|\bar{t}| \leq \eta(\rho)$. Then

$$\begin{aligned} & \|z_n(t) - z_n(\bar{t})\| \\ & \leq \left\| z_n(t) - z_0 - \int_0^t f_n(s) ds \right\| + \|\varphi(\bar{t}) - \varphi(0)\| + \int_0^t \|f_n(s)\| ds \\ & \leq \varepsilon_n t + \rho + \rho \\ & \leq \varepsilon_n + 2\rho. \end{aligned}$$

Hence we conclude that for all $t, \bar{t} \in [-a, T]$ such that $|t - \bar{t}| \leq \eta(\rho)$, we have $\|z_n(t) - z_n(\bar{t})\| \leq 2(\varepsilon_n + \rho)$.

Now, let $t \in [0, T]$ and $-a \leq s \leq -\frac{1}{4}\eta(\frac{\varepsilon_n}{4})$. We have

$$\begin{aligned} & \left| \bar{\theta}_{n+1}(t) + \frac{1}{4}\eta\left(\frac{\varepsilon_{n+1}}{4}\right) + s - \bar{\theta}_n(t) - \frac{1}{4}\eta\left(\frac{\varepsilon_n}{4}\right) - s \right| \\ & \leq |\bar{\theta}_{n+1}(t) - t| + |\bar{\theta}_n(t) - t| + \frac{1}{4}\eta\left(\frac{\varepsilon_{n+1}}{4}\right) + \frac{1}{4}\eta\left(\frac{\varepsilon_n}{4}\right) \\ & \leq \frac{1}{4}\eta\left(\frac{\varepsilon_{n+1}}{4}\right) + \frac{1}{4}\eta\left(\frac{\varepsilon_n}{4}\right) + \frac{1}{4}\eta\left(\frac{\varepsilon_n}{4}\right) + \frac{1}{4}\eta\left(\frac{\varepsilon_n}{4}\right) \\ & \leq \eta\left(\frac{\varepsilon_n}{4}\right). \end{aligned}$$

Thus

$$\begin{aligned} & \|\Gamma_{n+1}(t)(s) - \Gamma_n(t)(s)\| \\ & = \|z_{n+1}(\bar{\theta}_{n+1}(t) + \frac{1}{4}\eta(\frac{\varepsilon_{n+1}}{4}) + s) - z_n(\bar{\theta}_n(t) + \frac{1}{4}\eta(\frac{\varepsilon_n}{4}) + s)\| \\ & \leq \|z_{n+1}(\bar{\theta}_{n+1}(t) + \frac{1}{4}\eta(\frac{\varepsilon_{n+1}}{4}) + s) - z_{n+1}(\bar{\theta}_n(t) + \frac{1}{4}\eta(\frac{\varepsilon_n}{4}) + s)\| \\ & \quad + \|z_{n+1}(\bar{\theta}_n(t) + \frac{1}{4}\eta(\frac{\varepsilon_n}{4}) + s) - z_n(\bar{\theta}_n(t) + \frac{1}{4}\eta(\frac{\varepsilon_n}{4}) + s)\| \\ & \leq 2(\varepsilon_{n+1} + \frac{\varepsilon_n}{4}) + \|z_{n+1}(\cdot) - z_n(\cdot)\|_\infty \\ & \leq 4\varepsilon_n + \|z_{n+1}(\cdot) - z_n(\cdot)\|_\infty. \end{aligned}$$

If $-\frac{1}{4}\eta(\frac{\varepsilon_n}{4}) \leq s \leq -\frac{1}{4}\eta(\frac{\varepsilon_{n+1}}{4})$ we get

$$\begin{aligned} \left| \bar{\theta}_{n+1}(t) + \frac{1}{4}\eta\left(\frac{\varepsilon_{n+1}}{4}\right) + s - \theta_n(t) \right| & \leq |\bar{\theta}_{n+1}(t) - t| + |\theta_n(t) - t| + \frac{1}{4}\eta\left(\frac{\varepsilon_{n+1}}{4}\right) + |s| \\ & \leq \frac{1}{4}\eta\left(\frac{\varepsilon_{n+1}}{4}\right) + \frac{1}{4}\eta\left(\frac{\varepsilon_n}{4}\right) + \frac{1}{4}\eta\left(\frac{\varepsilon_n}{4}\right) + \frac{1}{4}\eta\left(\frac{\varepsilon_n}{4}\right) \\ & \leq \eta\left(\frac{\varepsilon_n}{4}\right). \end{aligned}$$

Then

$$\begin{aligned}
& \|\Gamma_{n+1}(t)(s) - \Gamma_n(t)(s)\| \\
&= \left\| z_{n+1}(\bar{\theta}_{n+1}(t) + \frac{1}{4}\eta(\frac{\varepsilon_{n+1}}{4}) + s) + \frac{4s}{\eta(\frac{\varepsilon_n}{4})} z_n(\bar{\theta}_n(t)) - (1 + \frac{4s}{\eta(\frac{\varepsilon_n}{4})}) z_n(\theta_n(t)) \right\| \\
&\leq \|z_{n+1}(\bar{\theta}_{n+1}(t) + \frac{1}{4}\eta(\frac{\varepsilon_{n+1}}{4}) + s) - z_n(\theta_n(t))\| + \|z_n(\bar{\theta}_n(t)) - z_n(\theta_n(t))\| \\
&\leq \|z_{n+1}(\bar{\theta}_{n+1}(t) + \frac{1}{4}\eta(\frac{\varepsilon_{n+1}}{4}) + s) - z_n(\bar{\theta}_{n+1}(t) + \frac{1}{4}\eta(\frac{\varepsilon_{n+1}}{4}) + s)\| \\
&\quad + \|z_n(\bar{\theta}_{n+1}(t) + \frac{1}{4}\eta(\frac{\varepsilon_{n+1}}{4}) + s) - z_n(\theta_n(t))\| + \|z_n(\bar{\theta}_n(t)) - z_n(\theta_n(t))\| \\
&\leq \|z_{n+1}(\cdot) - z_n(\cdot)\|_\infty + 2(\varepsilon_n + \frac{\varepsilon_n}{4}) + 2(\varepsilon_n + \frac{\varepsilon_n}{4}) \\
&\leq \|z_{n+1}(\cdot) - z_n(\cdot)\|_\infty + 8\varepsilon_n.
\end{aligned}$$

If $-\frac{1}{4}\eta(\frac{\varepsilon_{n+1}}{4}) \leq s \leq 0$ one has

$$\begin{aligned}
& \|\Gamma_{n+1}(t)(s) - \Gamma_n(t)(s)\| \\
&\leq \frac{4|s|}{\eta(\frac{\varepsilon_{n+1}}{4})} \|z_{n+1}(\bar{\theta}_{n+1}(t)) - z_{n+1}(\theta_{n+1}(t))\| + \frac{4|s|}{\eta(\frac{\varepsilon_n}{4})} \|z_n(\bar{\theta}_n(t)) - z_n(\theta_n(t))\| \\
&\quad + \|z_{n+1}(\theta_{n+1}(t)) - z_n(\theta_n(t))\| \\
&\leq \|z_{n+1}(\bar{\theta}_{n+1}(t)) - z_{n+1}(\theta_{n+1}(t))\| + \|z_n(\bar{\theta}_n(t)) - z_n(\theta_n(t))\| \\
&\quad + \|z_{n+1}(\theta_{n+1}(t)) - z_n(\theta_n(t))\| \\
&\leq 2(\varepsilon_{n+1} + \frac{\varepsilon_n}{4}) + 2(\varepsilon_n + \frac{\varepsilon_n}{4}) \\
&\quad + \|z_{n+1}(\theta_{n+1}(t)) - z_{n+1}(\theta_n(t))\| + \|z_{n+1}(\theta_n(t)) - z_n(\theta_n(t))\| \\
&\leq 8\varepsilon_n + 2(\varepsilon_{n+1} + \frac{\varepsilon_n}{4}) + \|z_{n+1}(\cdot) - z_n(\cdot)\|_\infty \\
&\leq \|z_{n+1}(\cdot) - z_n(\cdot)\|_\infty + 12\varepsilon_n.
\end{aligned}$$

Thus we conclude that $\|\Gamma_{n+1}(t) - \Gamma_n(t)\|_\infty \leq \|z_{n+1}(\cdot) - z_n(\cdot)\|_\infty + 12\varepsilon_n$.

By the same arguments as above, for $t \in [0, T]$, if $-a \leq s \leq -\frac{1}{4}\eta(\frac{\varepsilon_n}{4})$ we have

$$\begin{aligned}
\left| \bar{\theta}_n(t) + \frac{1}{4}\eta(\frac{\varepsilon_n}{4}) + s - t - s \right| &\leq |\bar{\theta}_n(t) - t| + \frac{1}{4}\eta(\frac{\varepsilon_n}{4}) \\
&\leq \frac{1}{4}\eta(\frac{\varepsilon_n}{4}) + \frac{1}{4}\eta(\frac{\varepsilon_n}{4}) \\
&\leq \eta(\frac{\varepsilon_n}{4}).
\end{aligned}$$

Then

$$\|T(t)z_n(s) - \Gamma_n(t)(s)\|$$

$$\begin{aligned}
&= \|z_n(t+s) - z_n(\bar{\theta}_n(t) + \frac{1}{4}\eta(\frac{\varepsilon_n}{4}) + s)\| \\
&\leq 2(\varepsilon_n + \frac{\varepsilon_n}{4}) \\
&\leq 4\varepsilon_n.
\end{aligned}$$

If $-\frac{1}{4}\eta(\frac{\varepsilon_n}{4}) \leq s \leq 0$ we get

$$\begin{aligned}
|\theta_n(t) - t - s| &\leq |\theta_n(t) - t| + |s| \\
&\leq \frac{1}{4}\eta(\frac{\varepsilon_n}{4}) + \frac{1}{4}\eta(\frac{\varepsilon_n}{4}) \\
&\leq \eta(\frac{\varepsilon_n}{4}).
\end{aligned}$$

So

$$\begin{aligned}
&\|T(t)z_n(s) - \Gamma_n(t)(s)\| \\
&= \|z_n(t+s) + \frac{4s}{\eta(\frac{\varepsilon_n}{4})}z_n(\bar{\theta}_n(t)) - (1 + \frac{4s}{\eta(\frac{\varepsilon_n}{4})})z_n(\theta_n(t))\| \\
&\leq \|z_n(t+s) - z_n(\theta_n(t))\| + \|z_n(\bar{\theta}_n(t)) - z_n(\theta_n(t))\| \\
&\leq 2(\varepsilon_n + \frac{\varepsilon_n}{4}) + 2(\varepsilon_n + \frac{\varepsilon_n}{4}) \\
&\leq 8\varepsilon_n.
\end{aligned}$$

Thus we conclude that $\|T(t)z_n - \Gamma_n(t)\|_\infty \leq 8\varepsilon_n$. □

Now, Relations (4.1), (4.2) and Claim 1 yield for all $t \in [0, T]$

$$\begin{aligned}
&\|z_{n+1}(t) - z_n(t)\| \\
&\leq \left\| z_{n+1}(t) - z_0 - \int_0^t f_{n+1}(s)ds \right\| + \int_0^t \|f_{n+1}(s) - f_n(s)\|ds \\
&+ \left\| z_n(t) - z_0 - \int_0^t f_n(s)ds \right\| \\
&\leq \varepsilon_{n+1}t + \varepsilon_n t + \|z_n(\cdot) - z_{n+1}(\cdot)\|_\infty \int_0^t m(s)ds + 12\varepsilon_n \int_0^t m(s)ds \\
&+ 2\varepsilon_{n+1} \int_0^t m(s)ds + t\varepsilon_{n+1} \\
&\leq 2\varepsilon_n t + \|z_n(\cdot) - z_{n+1}(\cdot)\|_\infty \int_0^T m(s)ds + 12\varepsilon_n + 2\varepsilon_n + t\varepsilon_n \\
&\leq 17\varepsilon_n + \|z_n(\cdot) - z_{n+1}(\cdot)\|_\infty \int_0^T m(s)ds.
\end{aligned}$$

It follows that

$$\|z_n(\cdot) - z_{n+1}(\cdot)\|_\infty \leq \frac{17\varepsilon_n}{1-L}, \quad (4.3)$$

where $L = \int_0^T m(s)ds$. Therefore we have, for $n < m$

$$\|z_m(\cdot) - z_n(\cdot)\|_\infty \leq \frac{17}{1-L} \sum_{i=n}^{m-1} \varepsilon_i.$$

Thus the sequence $(z_n(\cdot))_{n \geq 1}$ converges uniformly on $[-a, T]$ to a function $x(\cdot)$. Also, since all functions $z_n(\cdot)$ agree with $\varphi(\cdot)$ on $[-a, 0]$, we have $x(\cdot) \equiv \varphi(\cdot)$ on $[-a, 0]$.

Additionally, observe that $z_n(\theta_n(t))$ converges uniformly to $x(t)$ on $[0, T]$. Indeed, for $t \in [0, T]$, we have $|t - \theta_n(t)| \leq \frac{1}{4}\eta(\frac{\varepsilon_n}{4})$ then

$$\begin{aligned} \|z_n(t) - z_n(\theta_n(t))\| &\leq 2(\varepsilon_n + \frac{\varepsilon_n}{4}) \\ &\leq 4\varepsilon_n. \end{aligned}$$

Hence $\|z_n(t) - z_n(\theta_n(t))\|$ converges to 0 as $n \rightarrow \infty$. Since

$$\|z_n(\theta_n(t)) - x(t)\| \leq \|z_n(t) - z_n(\theta_n(t))\| + \|z_n(t) - x(t)\|$$

and $(z_n(\cdot))$ converges uniformly to $x(\cdot)$, it follows $z_n(\theta_n(t))$ converges uniformly to $x(t)$ on $[0, T]$. By construction, we have $z_n(\theta_n(t)) \in C(\theta_n(t))$ for every $t \in [0, T]$ and the graph of C is closed, then $x(t) \in C(t)$ for all $t \in [0, T]$. Furthermore, the uniform convergence of $z_n(\cdot)$ to $x(\cdot)$ on $[-a, T]$ implies that $T(t)z_n$ converges to $T(t)x$ uniformly on $[-a, 0]$. From Claim 1 we deduce that

$$\Gamma_n(t) \text{ converges to } T(t)x \text{ in } \mathcal{C}_a. \quad (4.4)$$

Now, we return to Relation (4.2). By Claim 1 and Relation (4.3) we have for all $t \in [0, T]$

$$\begin{aligned} \|f_{n+1}(t) - f_n(t)\| &\leq m(t)\|\Gamma_n(t) - \Gamma_{n+1}(t)\|_\infty + (2m(t) + 1)\varepsilon_{n+1} \\ &\leq m(t)\left(\|z_n(\cdot) - z_{n+1}(\cdot)\|_\infty + 12\varepsilon_n\right) + (2m(t) + 1)\varepsilon_n \\ &\leq m(t)\left(\frac{17\varepsilon_n}{1-L} + 12\varepsilon_n\right) + (2m(t) + 1)\varepsilon_n \\ &\leq \left(m(t)\left(\frac{17}{1-L} + 14\right) + 1\right)\varepsilon_n. \end{aligned}$$

This implies (as above) that $(f_n(t))_{n \geq 1}$ is a Cauchy sequence and $(f_n(\cdot))_{n \geq 1}$ converges point-wisely to $f(\cdot)$. Further, since $\|f_n(t)\| \leq g(t)$, by (4) and by the Lebesgue dominated convergence theorem, we have

$$x(t) = \lim_{n \rightarrow \infty} z_n(t) = \lim_{n \rightarrow \infty} \left(z_0 + \int_0^t f_n(s)ds \right) = z_0 + \int_0^t f(s)ds.$$

Finally, observe that by (1),

$$\begin{aligned} d(f(t), F(t, T(t)x)) &\leq \|f(t) - f_n(t)\| + H(F(t, \Gamma_n(t)), F(t, T(t)x)) \\ &\leq \|f(t) - f_n(t)\| + m(t)\|\Gamma_n(t) - T(t)x\|_\infty. \end{aligned}$$

Since $f_n(t)$ converges to $f(t)$ and by (4.4) the last term converges to 0. So that $\dot{x}(t) = f(t) \in F(t, T(t)x)$ a.e in $[0, T]$. The proof is complete.

5. PROOF OF THEOREM 2

Set $\varphi(0) = z_0$. We emphasize that the choice of T and $\eta(\varepsilon)$ are as in the last sections.

The same proof of Proposition 3 holds for the following Proposition. We shall apply it in order to construct a sequence of approximated solutions.

Proposition 4. *If assumptions (L1)-(L3) are satisfied, then for all $0 < \varepsilon < a$ and $y(\cdot) \in L^1([0, b], E)$, there exist a continuous mapping $z(\cdot) : [-a, +\infty[\rightarrow E$, step functions $\theta(\cdot), \tilde{\theta}(\cdot), \bar{\theta}(\cdot) : [0, +\infty[\rightarrow [0, +\infty[$, $\Gamma(\cdot) : [0, T] \rightarrow \mathcal{C}_a$, and $f(\cdot) \in L^1([0, T], E)$ such that*

- (i) $f(t) \in F(t, \Gamma(t))$ and $z(\tilde{\theta}(t)) \in P(z(\theta(t)))$, for all $t \in [0, T]$ and $z \equiv \varphi$ on $[-a, 0]$;
- (ii) $0 \leq t - \theta(t) \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$, $0 \leq \tilde{\theta}(t) - t \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$ and $0 \leq t - \bar{\theta}(t) \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$ for all $t \in [0, T]$;
- (iii) $\|f(t) - y(t)\| \leq d(y(t), F(t, \Gamma(t))) + (2m(t) + 1)\varepsilon$ for all $t \in [0, T]$;
- (iv) $\left\| z(t) - z(0) - \int_0^t f(\tau) d\tau \right\| \leq \varepsilon t$ for all $t \in [0, T]$.
- (v) For all $t \in [0, T]$

$$\Gamma(t)(s) = \begin{cases} z(\bar{\theta}(t) + \frac{1}{4}\eta(\frac{\varepsilon}{4}) + s) & -a \leq s \leq -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \\ -\frac{4s}{\eta(\frac{\varepsilon}{4})}z(\tilde{\theta}(t)) + (1 + \frac{4s}{\eta(\frac{\varepsilon}{4})})z(\theta(t)) & -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \leq s \leq 0. \end{cases}$$

Now, let $(\varepsilon_n)_{n \geq 1}$ be a strictly decreasing sequence of positive scalars such that $0 < \varepsilon_n < a$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. In view of Proposition 4, we can define inductively sequences $(f_n(\cdot))_{n \geq 1} \subset L^1([0, T], E)$, $(z_n(\cdot))_{n \geq 1} \subset \mathcal{C}([-a, +\infty[, E)$, $(\Gamma_n(\cdot))_{n \geq 1} \subset \mathcal{C}([0, T], \mathcal{C}_a)$ and $(\theta_n(\cdot))_{n \geq 1}$, $(\tilde{\theta}_n(\cdot))_{n \geq 1}$, $(\bar{\theta}_n(\cdot))_{n \geq 1} \subset S([0, +\infty[, [0, +\infty[)$ such that

- (1) $f_n(t) \in F(t, \Gamma_n(t))$ and $z_n(\tilde{\theta}_n(t)) \in P(z_n(\theta_n(t)))$, for all $t \in [0, T]$ and $z_n \equiv \varphi$ on $[-a, 0]$;
- (2) $0 \leq t - \theta_n(t) \leq \frac{1}{4}\eta(\frac{\varepsilon_n}{4})$, $0 \leq \tilde{\theta}_n(t) - t \leq \frac{1}{4}\eta(\frac{\varepsilon_n}{4})$ and $0 \leq t - \bar{\theta}_n(t) \leq \frac{1}{4}\eta(\frac{\varepsilon_n}{4})$ for all $t \in [0, T]$;
- (3) $\|f_{n+1}(t) - f_n(t)\| \leq d(f_n(t), F(t, \Gamma_{n+1}(t))) + (2m(t) + 1)\varepsilon_{n+1}$ for all $t \in [0, T]$;

- (4) $\left\| z_n(t) - z_0 - \int_0^t f_n(\tau) d\tau \right\| \leq \varepsilon_n t$ for all $t \in [0, T]$.
 (5) For all $t \in [0, T]$

$$\Gamma_n(t)(s) = \begin{cases} z_n(\bar{\theta}_n(t) + \frac{1}{4}\eta(\frac{\varepsilon_n}{4}) + s) & -a \leq s \leq -\frac{1}{4}\eta(\frac{\varepsilon_n}{4}) \\ -\frac{4s}{\eta(\frac{\varepsilon_n}{4})}z_n(\bar{\theta}_n(t)) + (1 + \frac{4s}{\eta(\frac{\varepsilon_n}{4})})z_n(\theta_n(t)) & -\frac{1}{4}\eta(\frac{\varepsilon_n}{4}) \leq s \leq 0. \end{cases}$$

We can apply the same technique as above to prove that $(z_n(\cdot))_{n \geq 1}$ converges uniformly to an absolutely continuous function $x(\cdot)$ and we can deduce that, for almost all $t \in [0, T]$, $\dot{x}(t) \in F(t, T(t)x)$. To complete the proof of Theorem 2, it remains to prove that $x(t) \in P(x(t))$, for all $t \in [0, T]$ and if $t' < t$ then $x(t) \in P(x(t'))$. Since $z_n(\bar{\theta}_n(t)) \in P(z_n(\theta_n(t))) \subset K$ for all $t \in [0, T]$ and K is compact, we conclude that $x(t) \in K$ for all $t \in [0, T]$. Then by (A1), $x(t) \in P(x(t))$ for all $t \in [0, T]$. Next, let $t', t \in [0, T]$ such that $t' < t$. Then for n large enough we can find $p, q \in \{0, \dots, s\}$ such that $q = p + i$ where $1 \leq i \leq s$, $t' \in [t_p^n, t_{p+1}^n[$ and $t \in [t_q^n, t_{q+1}^n[$, where t_i^n is defined in the proof of Proposition 4. By construction, we have $z_n(t_{q-1}^n) \in P(z_n(t_{q-2}^n))$, which together with (A1) gives

$$P(z_n(t_{q-1}^n)) \subseteq P(z_n(t_{q-2}^n)).$$

Similarly,

$$P(z_n(t_{q-2}^n)) \subseteq P(z_n(t_{q-3}^n)).$$

If we continue for $i - 1$ steps, we obtain

$$P(z_n(t_{q-1}^n)) \subseteq P(z_n(t_p^n)).$$

By the fact that $z_n(t_q^n) \in P(z_n(t_{q-1}^n))$, we have $z_n(t_q^n) \in P(z_n(t_p^n))$. This means that $z_n(\theta_n(t)) \in P(z_n(\theta_n(t')))$. By letting $n \rightarrow +\infty$, we get $x(t) \in P(x(t'))$. The proof is complete.

ACKNOWLEDGEMENTS

The author would like to thank the referee for his careful and thorough reading of the paper.

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Author’s address

Myelkebir Aitalioubrahim

High school Ibn Khaldoune, BP 13100, commune Bouznika, Morocco.

E-mail address: aitalifr@hotmail.com



SOLUTION OF THE CONVEX PROGRAMMING PROBLEM VIA SECOND ORDER DIFFERENTIAL EQUATION SYSTEM

T. HAJBA AND M. KOVÁCS

Received 16 February, 2011

Abstract. In this paper we investigate a family of second order differential equations which can be considered as the continuous versions of the Fletcher-Reeves iteration for minimization applied to a regularized and penalized convex programming problem. We show that synchronizing the parameter functions of the differential equation the stationary limit point of the trajectories is the minimal norm solution of the given convex programming problem.

2000 Mathematics Subject Classification: Primary 34D05; Secondary 90C25; 65K05

Keywords: second order differential equation, minimizer trajectory, stationary point in limit, Lyapunov-type methods, convex programming, Fletcher-Reeves iteration

1. INTRODUCTION

Several types of ordinary differential equation (shortly ODE) systems are known whose trajectories minimize a given function. Some of them arise from physics and technology, others are motivated by iterative numerical methods.

The asymptotic behaviour of their trajectories is studied by lot of mathematicians, physicists and engineers. Some of them deal with either the gradient or the Newton method and model the given method by a system of first order differential equations (shortly FO-ODE) system (e.g. Antipin [3], Attouch and Cominetti [4], Bot-saris [7], Brown and Bartholomew-Biggs [9], Cominetti et al. [13], Evtushenko and Zhadan [14], Flâm [16], Hauser and Nedić [21], Branin and Hoo [8], Kovács [22,23], (Vasil'ev and Kovács [28]), Venec and Rybashov [31], Zhang et al [32] etc.).

There are some ODE models in connection with the optimization that arise from physical problems such as e.g. the heavy ball with friction, the problem of a nonlinear oscillator with dumping, evolution equations with linear dumping and convex potential. These problems generally lead to a second order differential equations (shortly SO-ODE) system. Results concerning such type of SO-ODE models can be found e.g. in the papers Alvarez [1]; Attouch [2], Attouch et al. [6], Cabot et al. [11, 12]), Goudou and Munier [17], Vasil'ev et al. [27], Vasil'ev and Nedich [29]. There are some papers discussing higher order methods (e.g. Vasiljev and Nedić [30], Vasil'ev

and Nedić [29], Nedić [24]), too. However, the mentioned papers, except Vasil'ev et al. [27], deal with such SO-ODE systems that are linear in $\dot{\mathbf{x}}$.

From the mentioned methods only a few deal with the solution of constrained optimization problem. The authors of the papers Antipin [3], Nedić [24], Vasil'ev and Nedich [29], Vasiljev and Nedić [30] use the projection gradient for handling the constraints, but this technique makes it possible to use only relatively simple constraints. In Kovács [22] the penalty function technique is used and in Vasiljev and Nedić [30], Vasil'ev and Nedich [29] and Vasil'ev et al. [27] the two techniques are used together.

In the cases when the minimum point is not unique, a suitable regularization appears in the ODE systems to control the trajectories to the direction of a well-defined solution (see e.g. in Attouch and Czarnecki [5], Cabot [10], Cominetti et al. [13], Kovács [22], Vasil'ev et al [27] etc.)

It can be seen, that the higher the order of the optimizer ODE system is and more optimization and control parameters are built into the ODE system, the more complicated the synchronization of the conditions between the parameter functions.

In this paper we deal with a SO-ODE system of four parameter functions, two of them define the structure of the SODE system, one is the penalty parameter and one is the control parameter of the regularization.

2. MATHEMATICAL PRELIMINARIES

In this section we formulate our convex programming problem (shortly CP problem) for that we try to associate SO-ODE systems whose trajectories' limit points give the solution of the CP problem. We describe those known techniques, methods and lemmas that will be used in our investigation.

2.1. CP problem

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \dots, m$ be convex continuously differentiable functions and let $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = m + 1, \dots, s$ be linear ones, and let us consider the following CP problem:

$$f(\mathbf{x}) \rightarrow \min_{\mathbf{x} \in X} \quad (2.1)$$

where

$$X = \{\mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq 0, i = 1, \dots, m, g_i(\mathbf{x}) = 0, i = m + 1, \dots, s\} \quad (2.2)$$

We assume that the problem (2.1)-(2.2) satisfies the following conditions:

- A1):** it is Slater-regular, i.e. there exists a Slater-point $\bar{\mathbf{x}}$ such that $g_i(\bar{\mathbf{x}}) < 0$, $i = 1, \dots, m$,
- A2):** the minimum is finite, i.e. $f_* = \inf f(\mathbf{x}) > -\infty$, and the given CP problem has at least one solution, i.e. $X_* = \{\mathbf{x} \in X : f(\mathbf{x}) = f_*\} \neq \emptyset$.

The constrained optimization problem may be solved by solving a sequence of unconstrained minimization problems

$$\Phi_k(\mathbf{x}) = f(\mathbf{x}) + A_k P(\mathbf{x}) \rightarrow \min_{\mathbf{x} \in \mathbb{R}^n}, \quad k = 1, 2, \dots, \quad (2.3)$$

where the *penalty function* $P(\mathbf{x})$ is defined by

$$P(\mathbf{x}) = \frac{1}{p} \sum_{i=1}^s |g_i^+(\mathbf{x})|^p, \quad |g_i^+(\mathbf{x})| = \begin{cases} \max(0, g_i(\mathbf{x})), & i = 1, \dots, m \\ |g_i(\mathbf{x})|, & i = m+1, \dots, s \end{cases} \quad (2.4)$$

and $p > 1$, moreover $A_k \rightarrow \infty$ if $k \rightarrow \infty$. If $\mathbf{x}_k^*$ is a minimum point of the k -th problem $\Phi_k(\mathbf{x}) \rightarrow \min_{\mathbf{x} \in \mathbb{R}^n}$ then there is a subsequence $\{\mathbf{x}_{k_j}\}$ of $\{\mathbf{x}_k\}$ which converges to one of the minimum point of the original problem (2.1)-(2.2) (see Fiacco and McCormick [15]).

However from the convexity of the objective and constraints functions it does not follow that the minimum point of the CP problem is unique, and hence it can not be expected that any subsequence tends to the same minimum point. Our aim is to find the so called *minimal norm solution*, i.e. we seek the point $\mathbf{x} \in X_*$ such that

$$\|\mathbf{x}_*\| = \inf_{\mathbf{x} \in X_*} \|\mathbf{x}\|. \quad (2.5)$$

Therefore we use the regularization technique of A. N. Tichonov (see Tichonov [26], Tikhonov and Vasilev [25]).

The regularization of the convex programming means that the penalized minimizing function will be approximated with a bundle of strong convex functions depending on a parameter of the regularization. By harmonizing the chosen regularization parameter with the coefficient of the penalty function, one can expect that the sequence of the unique minimum points of the auxiliary functions tends to one of the well defined minimum point of the original minimizing function independently of the starting point. The possibility of this type of regularization is based on Lemma 1. It simplifies the Theorem of (Vasil'ev and Kovács [28]) in the following sense: the used topological space is $\mathbb{R}^n$ with the norm-topology, all constraints are handled by penalty function of type (2.4), $\Omega(\mathbf{x}) = \|\mathbf{x}\|^2$. Moreover, it is also assumed that the objective and constraints functions values as well as the unconstrained minimum can be computed exactly.

Lemma 1. *Let the CP-problem (2.1)-(2.2) satisfy the conditions A1-A2 and let $\mathbf{x}_*$ denote the minimal norm solution of this problem. Let us introduce the bundle of functions*

$$T_k(\mathbf{x}) = f(\mathbf{x}) + A_k P(\mathbf{x}) + \frac{1}{2} \lambda_k \|\mathbf{x}\|^2, \quad k = 1, 2, \dots, \quad (2.6)$$

where $P(\mathbf{x})$ is defined by (2.4), $\{A_k : k = 0, 1, 2, \dots\}$ is an infinite monotone increasing and $\{\lambda_k : k = 0, 1, 2, \dots\}$ is an infinite monotone decreasing sequence satisfying the

conditions

$$\lim_{k \rightarrow \infty} A_k = \infty, \quad \lim_{k \rightarrow \infty} \lambda_k = 0, \quad \lim_{k \rightarrow \infty} A_k^{1/(p-1)} \lambda_k = \infty.$$

If $\mathbf{x}_k^*$ denotes the unique solution of the unconstrained minimization problem $T_k(\mathbf{x}) \rightarrow \min_{\mathbf{x} \in \mathbb{R}^n}$ with fixed k , then the sequence $\{\|\mathbf{x}_k^*\|, k = 0, 1, \dots\}$ is bounded and

$$\lim_{k \rightarrow \infty} f(\mathbf{x}_k^*) = f_*; \quad \lim_{k \rightarrow \infty} A_k P(\mathbf{x}_k^*) = 0; \quad \lim_{k \rightarrow \infty} \|\mathbf{x}_k^* - \mathbf{x}_*\| = 0.$$

This lemma remains valid if the parameter sequences are changed for continuous functions of time t and everywhere in the lemma the convergence is considered as $t \rightarrow \infty$.

In our investigation the continuous version of this lemma will be used.

2.2. ODE systems associated to CP problems

The most frequently used methods of unconstrained optimization are iterative methods, e.g. the gradient and Newton methods, the Fletcher-Reeves iteration etc. Any iterative method can be considered as Euler-discretization of an ODE system with stepsize 1. This ODE system is called the continuous version of the given method. So, with different numerical methods one can associate several types of differential equations with the same CP problem and one can define those conditions, under which the trajectories of these ODE systems converge to a minimum point. The associated ODE system may be first or higher order.

To speak about the limit point of the trajectories if $t \rightarrow \infty$ it is necessary that the right maximal interval will be the whole $[t_0, \infty)$. To guarantee this property we will use the idea of local Lipschitz continuity.

We will say that the vector function $\Phi : W \rightarrow \mathbb{R}^k$ defined on the open subset $W \subseteq \mathbb{R}^n$ is *local Lipschitz continuous* if it is Lipschitz continuous on every compact subset of W . If $I_0 \subset \mathbb{R}$ is open, then the function $\Phi(\mathbf{x}, t) : W \times I_0 \rightarrow \mathbb{R}^k$ is local Lipschitz continuous if it is Lipschitz continuous in $\mathbf{x}$ on any compact subset $V \subset W$, and continuous in t on any closed interval $I \subset I_0$.

If an ODE system associated with an iterative method of minimization has a unique solution on $[t_0, \infty)$ then we say that

- it is a *minimizing model* if along its trajectories $\lim_{t \rightarrow \infty} f(\mathbf{x}(t)) = f_*$;
- it is *convergent* if any trajectory of it converges in norm to some $\mathbf{x}_* \in X_*$, i.e. $\|\mathbf{x}(t) - \mathbf{x}_*\| \rightarrow 0$.

The trajectories of a convergent minimizing model are called *minimizing trajectories*.

The ODE systems associated with an unconstrained optimization problem $f(\mathbf{x}) \rightarrow \min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$ by gradient and Newton methods can be written in the forms

$$\begin{aligned} \dot{\mathbf{x}} &= -\nabla f(\mathbf{x}), & \dot{\mathbf{x}} &= -\mathbf{H}(\mathbf{x})^{-1} \nabla f(\mathbf{x}) \\ \mathbf{x}(t_0) &= \mathbf{x}_0, & \mathbf{x}(t_0) &= \mathbf{x}_0, \end{aligned}$$

respectively, where $\mathbf{H}(\mathbf{x})$ is the Hessian of $f(\mathbf{x})$. In these cases if $\mathbf{x}_*$ denotes a minimum point of the function $f(\mathbf{x})$ then $\mathbf{x}_*$ is a stationary point of both FO-ODE system.

In general, $\mathbf{x}_*$ is a *stationary point* for a FO-ODE system if its right hand side turns into a null-vector in this point for all $t \geq t_0$. If the minimum point is a stationary point of the associated ODE system then the convergence of the trajectories to $\mathbf{x}_*$ is equivalent with the asymptotic stability of $\mathbf{x}_*$, therefore the well known Lyapunov function methods are useful to prove the convergence with an appropriately chosen Lyapunov function (see e.g. Evtushenko and Zhadan [14], Flåm [16], Venec and Rybashov [31]).

In contrary, in a FO-ODE system associated with a regularized unconstrained or a regularized convex programming problem given by (2.1)-(2.2) where the constraints are handled by penalty function of type (2.4) the minimum point $\mathbf{x}_*$ is not necessary stationary point (e.g. in the case of regularized continuous gradient method).

As it was pointed out in Kovács [23], the minimization models modelled by FO-ODE systems can be divided into two classes. Those models described by a FO-ODE system for which the point $\mathbf{x}_*$ is a stationary point of the system belong to the first class. To the second class of the models belong those continuous first order models, for which the minimum point is not stationary, but the trajectories exist on the whole $[t_0, \infty)$ and along the trajectories the right hand side vector of the differential equation system tends to the null-vector if $t \rightarrow \infty$. Following Kovács [23] we say in this case, that $\mathbf{x}_*$ is *stationary in limit*.

The definition of stationary and stationary in limit points can be extended for higher order ODE systems, too. We will say, that a point is *stationary point* or *stationary in limit point* of a second or higher order ODE system if it is stationary or stationary in limit point respectively for the equivalent FO-ODE system.

As it was shown in Kovács [22, 23] the Lyapunov-type methods are also applicable to prove the convergence of the trajectories to a point stationary in limit.

The following lemma will be useful in proving the existence of the trajectories of the examined SO-SODE system on $[t_0, \infty)$.

Lemma 2. *Suppose that there exist $0 \leq T_0 \leq T_{\max} < \infty$ such that*

- (1) *the parametric scalar functions $g(t, \tau), \tau \in (T_0, T_{\max}]$ are defined and non-negative for all $t \in [T_0, \tau)$, they are continuously differentiable with respect to t ;*
- (2) *$g(T_0, \tau) \leq K$ uniformly in τ ;*
- (3) *for every fixed $\tau \in (T_0, T_{\max}]$ the function $g(t, \tau)$ satisfies the differential inequality*

$$\frac{d}{dt}g(t, \tau) \leq -\varphi(t)g(t, \tau) + \psi(t)(\tau - t)^k \quad (2.7)$$

on the interval $[T_0, \tau)$, where k is a nonnegative integer, the functions $\varphi(t) > 0$ and $\psi(t)$ are defined and continuous on $[T_0, \infty)$ and integrable on any finite interval of $[T_0, \infty)$.

Then for every fixed $\tau \in (T_0, T_{\max}]$ the function $g(t, \tau)$ is bounded on $[T_0, \tau]$ and $g(\tau, \tau) = \lim_{t \rightarrow \tau} g(t, \tau) < \infty$.

Proof. Let

$$u(t, \tau) = g(T_0, \tau) e^{-\int_{T_0}^t \varphi(v) dv} + \int_{T_0}^t \psi(\theta) (\tau - \theta)^k e^{\int_{T_0}^\theta \varphi(v) dv} d\theta \quad (2.8)$$

be the solution of the linear ODE

$$\dot{u} = -\varphi(t)u + \psi(t)(\tau - t)^k, \quad u(T_0, \tau) = g(T_0, \tau). \quad (2.9)$$

Since $\varphi(t)$ and $\psi(t)$ are continuous on $[T_0, \infty)$, $u(t, \tau)$ is the unique solution on $[T_0, \infty)$, consequently $u(t, \tau)$ is bounded on every closed finite interval of $[T_0, \infty)$.

Using the Gronwall lemma we have that for every $\tau \in (T_0, T_{\max}]$

$$g(t, \tau) \leq u(t, \tau) \quad \text{for all } T_0 \leq t < \tau,$$

from this fact and from the continuity of $g(t, \tau)$ the boundedness of $g(t, \tau)$ on $[T_0, \tau]$ immediately follows. $\square$

To examine the limit points of the trajectories when the maximal right hand interval of existence is $[t_0, \infty)$, the following lemma taken from Kovács [23] will be useful. However, we have to remark that detailed proof of this statement can only be found in the proof of the Theorem of Kovács [22], and only for the special case $k = 1$. Therefore, here we sketch its proof, too.

Lemma 3. *Let the conditions 1-3 of the Lemma 2 be satisfied when $T_{\max} = \infty$. Moreover assume that the functions $\varphi(t)$ and $\psi(t)$ are endowed with the following properties:*

$$(4) \int_{T_0}^{\infty} \varphi(t) dt = \infty, \quad \lim_{t \rightarrow \infty} \frac{\psi(t)}{\varphi^{k+1}(t)} = 0;$$

$$(5) \text{ in the case } k \geq 1 \text{ the function } \varphi(t) \text{ is differentiable and } \lim_{t \rightarrow \infty} \frac{\dot{\varphi}(t)}{\varphi^2(t)} = 0,$$

then $\lim_{\tau \rightarrow \infty} g(\tau, \tau) = 0$.

Proof. It is enough to prove that $u(\tau, \tau) \rightarrow \infty$ if $t \rightarrow \infty$ where $u(t, \tau)$ is defined by (2.8). Under the condition 3. the convergence of the first term of (2.8) to zero is trivial. By induction on k it can be proved that for all nonnegative integer k the limit $\lim_{\tau \rightarrow \infty} e^{\int_{T_0}^{\tau} \varphi(v) dv} \varphi^k(\tau) = \infty$ holds true and hence we can estimate the second term by applying $(k + 1)$ times the L'Hospital rule and finally by the additional conditions 4 and 5 of this lemma. $\square$

The function $g(t, \tau)$ plays the role of a Lyapunov-like function in stability theory or the energy function in the theory of dissipative systems.

3. MINIMIZING PROPERTY OF THE REGULARIZED FR-TYPE SO-ODE (R-FR-TYPE SO-ODE) SYSTEM ASSOCIATED WITH THE CP PROBLEM

3.1. Construction of the R-FR-type SO-ODE system associated with the CP problem

Let us now examine an ODE system associated with the CP problem (2.1)-(2.2) by the Fletcher-Reeves iteration applied to the minimization of the penalized and regularized unconstrained function (2.6). This iteration can be formulated as follows: Let be given two vectors, namely $\mathbf{x}_k \in \mathbb{R}^n$ and $\mathbf{d}_k \in \mathbb{R}^n$ which are called the k -th *state vector* and k -th *searching direction*, respectively. Let the sequence $(\mathbf{x}_k, \mathbf{d}_k) \in \mathbb{R}^n \times \mathbb{R}^n$ be defined by

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k, \quad (3.1)$$

$$\mathbf{d}_{k+1} = -\nabla_{\mathbf{x}} F(\mathbf{x}_{k+1}) + \beta_k \mathbf{d}_k, \quad k = 0, 1, 2, \dots, \quad (3.2)$$

starting with $\mathbf{x}_0 \in \mathbb{R}^n, \mathbf{d}_0 = -\nabla_{\mathbf{x}} F_k(\mathbf{x}_0)$. To obtain a convergent process we have to use well defined (we do not present here the details) changing rules for the sequences α_k, β_k, A_k and λ_k .

It is easy to see that this iteration can be considered as the Euler discretization of the non-autonomous FO-ODE system of $2n$ variables

$$\dot{\mathbf{x}} = \alpha(t) \mathbf{d} \quad (3.3)$$

$$\dot{\mathbf{d}} = -\nabla_{\mathbf{x}} F(\mathbf{x} + \alpha(t) \mathbf{d}, t) + (\beta(t) - 1) \mathbf{d} \quad (3.4)$$

were

$$F(\mathbf{x}, t) = f(\mathbf{x}) + A(t)P(\mathbf{x}) + \frac{1}{2}\lambda(t)\|\mathbf{x}\|^2, \quad (3.5)$$

corresponds to (2.6) and the rules of change of all parameters are described by continuous functions.

(3.3)-(3.4) can be transformed by the substitution $\mathbf{p}(t) = \alpha(t)\mathbf{d}(t)$ into the following ones:

$$\dot{\mathbf{x}} = \mathbf{p} \quad (3.6)$$

$$\dot{\mathbf{p}} = -\alpha(t)\nabla_{\mathbf{x}} F(\mathbf{x} + \mathbf{p}, t) - \gamma(t)\mathbf{p} \quad (3.7)$$

with the initial values

$$\mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{p}(t_0) = \mathbf{p}_0, \quad (3.8)$$

where $\gamma(t) = -\frac{\dot{\alpha}(t)}{\alpha(t)} - \beta(t) + 1$.

It is clear, that the FO-ODE system (3.6)-(3.7) with the initial values (3.8) is equivalent to the SO-ODE system

$$\ddot{\mathbf{x}} + \gamma(t)\dot{\mathbf{x}} + \alpha(t)\nabla_{\mathbf{x}} F(\mathbf{x} + \dot{\mathbf{x}}, t) = 0 \quad (3.9)$$

with initial values

$$\mathbf{x}(t_0) = \mathbf{x}_0, \quad \dot{\mathbf{x}}(t_0) = \mathbf{p}_0. \quad (3.10)$$

The ODE systems given by (3.6),(3.7) and (3.8) or by (3.9) and (3.10) will be called *regularized FR-type SO-ODE (shortly R-FR type SO-ODE) system associated with the CP problem*.

Since the Fletcher-Reeves iteration uses the new state point in the construction of the new direction, our SO-ODE system usually will not be linear in the first derivative vector $\dot{\mathbf{x}}$ in contrary to the examined SO-ODE systems mentioned in the Introduction. In connection with optimization, such type of SO-ODE systems has been investigated in Hajba [18] for the unconstrained minimization of a strong convex function. The asymptotic behaviour of the regularized FR-type SO-ODE system associated with the unconstrained minimization of convex but not necessary strong convex function has been analyzed in Hajba [20]. A similar problem for constrained optimization has been examined in Vasil'ev et al. [27], but their result is obtained under other (and more complicated) assumptions than the ones used here.

It is worth looking at the special case of the CP problem when the objective function is quadratic and the constraints are given with a linear equality system, i.e. a hyperspace. Namely, let $\mathbf{Q}$ be a symmetric positive semidefinite $n \times n$ matrix and $\mathbf{c} \in \mathbb{R}^n$. Moreover, let be given an $m \times n$ matrix $\mathbf{B}$, and a vector $\mathbf{b}_0 \in \mathbb{R}^m$. Consider the CP problem

$$\frac{1}{2} \langle \mathbf{x}, \mathbf{Q}\mathbf{x} \rangle + \langle \mathbf{c}, \mathbf{x} \rangle \quad \text{subject to} \quad \mathbf{B}\mathbf{x} = \mathbf{b}_0.$$

To define the function $F(\mathbf{x}, t)$ we use the penalty function

$$P(\mathbf{x}) = \frac{1}{2} \|\mathbf{B}\mathbf{x} - \mathbf{b}_0\|^2.$$

Then the SO-ODE system (3.9) takes the following shape

$$\ddot{\mathbf{x}} + (\gamma(t)\mathbf{E} + \mathbf{H}(t))\dot{\mathbf{x}} + \mathbf{H}(t)\mathbf{x} + \mathbf{h}(t) = \mathbf{0},$$

with matrix functions

$$\mathbf{H}(t) = \alpha(t)(\mathbf{Q} + A(t)\mathbf{B}^T\mathbf{B} + \lambda(t)\mathbf{E}), \quad \mathbf{h}(t) = \alpha(t)(\mathbf{c} + A(t)\mathbf{B}^T\mathbf{b}_0),$$

where $\mathbf{E}$ denotes the unit matrix and the upper index T denotes transposition.

Although in this special case the SO-ODE system is linear in $\dot{\mathbf{x}}$, too, but its coefficient is a time dependent matrix function. So, our SO-ODE system for this relatively simple problem also essentially differs from the SO-ODE systems analyzed in papers mentioned in the Introduction, and it has not been examined yet.

3.2. Asymptotic behaviour of the trajectories of the R-FR-type SO-ODE systems

The following proposition characterizes the R-FR-type SO-ODE systems associated with the CP problem given by (2.1)-(2.2).

Proposition 1. *Let us assume that*

- (1) *the CP-problem defined by (2.1)-(2.2) satisfies the conditions **A1** and **A2** and the functions $f(\mathbf{x})$ and $g_i(\mathbf{x})$, $i = 1, \dots, m$ are continuously differentiable convex functions on $\mathbb{R}^n$ and the gradients $\nabla f(\mathbf{x})$ and $\nabla g_i(\mathbf{x})$, $i = 1, \dots, m$ are local Lipschitz continuous;*
- (2) *the penalty function $P(\mathbf{x})$ in the construction of $F(\mathbf{x}, t)$ is defined by (2.4);*
- (3) *the parameter functions $\alpha(t)$ and $\gamma(t)$ of the systems (3.6)-(3.7) and (3.9) fulfill the following conditions:*
 - a) *$\alpha(t)$ is a positive, monotone decreasing and continuously differentiable function on $[t_0, \infty)$;*
 - b) *$\gamma(t)$ is a monotone nonincreasing, continuously differentiable function on $[t_0, \infty)$ and $\gamma(t) \geq \gamma_\infty > 1$ for all $t \geq t_0$;*
- (4) *the coefficient $A(t)$ of the penalty function in the construction of $F(\mathbf{x}, t)$ is a positive continuously differentiable concave function and satisfies the conditions $\lim_{t \rightarrow \infty} A(t) = \infty$ and $\alpha(t)A(t) \leq A_0 < \infty$;*
- (5) *for the regularization parameter $\lambda(t)$ the following assumptions hold:*
 - a) *$\lambda(t)$ is a positive continuously differentiable monotone decreasing convex function on $[t_0, \infty)$ for all $t \geq t_0$ and $\lim_{t \rightarrow \infty} \lambda(t) = \lim_{t \rightarrow \infty} \dot{\lambda}(t) = 0$;*
 - b) *$\alpha(t_0)\lambda(t_0) = 1$, while $\int_{t_0}^{\infty} \alpha(t)\lambda(t) dt = \infty$;*
 - c) *$\lim_{t \rightarrow \infty} \frac{\dot{\alpha}(t)}{\alpha^2(t)\lambda(t)} = \lim_{t \rightarrow \infty} \frac{\dot{\lambda}(t)}{\alpha(t)\lambda^2(t)} = \lim_{t \rightarrow \infty} \frac{\dot{A}(t)}{\alpha(t)\lambda^2(t)} = 0$;*
 - d) *$\lim_{t \rightarrow \infty} A(t)^{1/(p-1)}\lambda(t) = \infty$.*

Then

- (1) *the trajectories of (3.6)-(3.7), respectively of (3.9) exist and unique on the whole half-line $[t_0, \infty)$ with any initial point (3.8) resp. (3.10);*
- (2) *the trajectories converge to the minimal norm solution, i.e. if $\mathbf{x}_*$ satisfies the condition $\inf_{\mathbf{x} \in X_*} \|\mathbf{x}\| = \|\mathbf{x}_*\|$, then $\lim_{t \rightarrow \infty} \|\mathbf{x}(t) - \mathbf{x}_*\| = 0$;*
- (3) *$\lim_{t \rightarrow \infty} f(\mathbf{x}(t)) = f(\mathbf{x}_*) = \inf_{\mathbf{x} \in X} f(\mathbf{x})$, i.e. the SO-ODE system given by (3.6)-(3.7) or (3.9) is a minimizing model and its trajectories are minimizing trajectories;*
- (4) *$\lim_{t \rightarrow \infty} \|\mathbf{p}(t)\| = \lim_{t \rightarrow \infty} \|\dot{\mathbf{x}}(t)\| = 0$;*
- (5) *$\lim_{t \rightarrow \infty} \|\ddot{\mathbf{x}}(t)\| = 0$, i.e. the minimal norm solution $\mathbf{x}_*$ is a stationary in limit minimum point.*

Proof. The continuity in t of the right hand side of (3.6)-(3.7), resp. (3.9) immediately follows from the continuity conditions given for the parameter functions. From the convexity and continuity of $g_i(\mathbf{x})$, $i = 1, \dots, m$ follows the local Lipschitz

continuity of these functions which together with the assumed local Lipschitz continuity of $\nabla g_i(\mathbf{x}), i = 1, \dots, m$ calls forth the local Lipschitz continuity of $\nabla P(\mathbf{x})$. Consequently the right hand side of (3.6)-(3.7) is also local Lipschitz continuous in $\mathbf{x}$. This property guaranties the existence and uniqueness of the trajectories on a right maximal interval $[t_0, t_{\max})$ with $t_0 < t_{\max} \leq \infty$. In order to prove that $t_{\max} = \infty$, let us show that $|\dot{\mathbf{x}}(t)|$ and $|\ddot{\mathbf{x}}(t)| = |\dot{\mathbf{p}}(t)|$ are bounded.

For every fixed $t_0 < \tau < \infty$ the function $F(\mathbf{x}, \tau)$ defined by (3.5) is a strong convex function, therefore it has a unique minimum point $\mathbf{x}_\tau^*$. Let $\mathbf{x}_*$ be the minimal norm solution of the CP-problem. Under the assumption 4, 5.a) and 5.d) of the proposition the conditions of the Lemma 1 are satisfied, so there exists a constant K such that $\|\mathbf{x}_\tau^*\| \leq K$ for all $t_0 \leq \tau$.

Let us introduce the parametric function

$$g(t, \tau) = \frac{1}{2} \|\mathbf{x}(t) - \mathbf{x}_\tau^* + \mathbf{p}(t)\|^2 + \frac{1}{4} \alpha(t) \lambda(t) \|\mathbf{x}(t) - \mathbf{x}_\tau^*\|^2 + \frac{1}{2} (\gamma(t) - 1) \|\mathbf{x}(t) - \mathbf{x}_\tau^*\|^2$$

for fixed $t_{\max} \geq \tau \geq t_0$.

It follows from the conditions 3.a), 3.b) and 5.a) that $g(t, \tau) \geq 0$ for all $t_0 \leq t < \tau$, and Lemma 1 guaranties the uniformly boundedness of $g(t_0, \tau)$ in τ .

For the derivative of $g(t, \tau)$ we have

$$\begin{aligned} \frac{d}{dt} g(t, \tau) &= -\alpha(t) \left\langle \nabla_{\mathbf{x}} F(\mathbf{x}(t) + \mathbf{p}(t), t), \mathbf{x}(t) - \mathbf{x}_\tau^* + \mathbf{p}(t) \right\rangle + \\ &+ \frac{1}{4} \frac{d}{dt} (\alpha(t) \lambda(t)) \|\mathbf{x}(t) - \mathbf{x}_\tau^*\|^2 + \frac{1}{2} \alpha(t) \lambda(t) \left\langle \mathbf{x}(t) - \mathbf{x}_\tau^*, \mathbf{p}(t) \right\rangle + \\ &+ (1 - \gamma(t)) \|\mathbf{p}(t)\|^2 + \frac{1}{2} \dot{\gamma}(t) \|\mathbf{x}(t) - \mathbf{x}_\tau^*\|^2 \end{aligned}$$

for all $t_0 \leq t < \tau$. (Here and in the following $\langle \cdot, \cdot \rangle$ denotes the scalar product.)

Omitting the non-positive terms we have that

$$\begin{aligned} \frac{d}{dt} g(t, \tau) &\leq -\alpha(t) \left\langle \nabla_{\mathbf{x}} F(\mathbf{x}(t) + \mathbf{p}(t), t), \mathbf{x}(t) - \mathbf{x}_\tau^* + \mathbf{p}(t) \right\rangle + \\ &+ \frac{1}{2} \alpha(t) \lambda(t) \left\langle \mathbf{x}(t) - \mathbf{x}_\tau^*, \mathbf{p}(t) \right\rangle = \\ &= -\alpha(t) \left\langle \nabla_{\mathbf{x}} F(\mathbf{x}(t) + \mathbf{p}(t), t) - \nabla_{\mathbf{x}} F(\mathbf{x}_\tau^*, t), \mathbf{x}(t) - \mathbf{x}_\tau^* + \mathbf{p}(t) \right\rangle - \\ &- \alpha(t) \left\langle \nabla_{\mathbf{x}} F(\mathbf{x}_\tau^*, t) - \nabla_{\mathbf{x}} F(\mathbf{x}_\tau^*, \tau), \mathbf{x}(t) - \mathbf{x}_\tau^* + \mathbf{p}(t) \right\rangle + \\ &+ \frac{1}{2} \alpha(t) \lambda(t) \left\langle \mathbf{x}(t) - \mathbf{x}_\tau^*, \mathbf{p}(t) \right\rangle. \end{aligned} \tag{3.11}$$

For every fixed t the convexity modulus of the strong convex function $F(\mathbf{x}, t)$ is $\frac{1}{2}\lambda(t)$, therefore

$$\begin{aligned} -\alpha(t) \left\langle \nabla_{\mathbf{x}} F(\mathbf{x}(t) + \mathbf{p}(t), t) - \nabla_{\mathbf{x}} F(\mathbf{x}_{\tau}^*, t), \mathbf{x}(t) - \mathbf{x}_{\tau}^* + \mathbf{p}(t) \right\rangle &\leq \\ &\leq -\alpha(t) \lambda(t) \|\mathbf{x}(t) - \mathbf{x}_{\tau}^* + \mathbf{p}(t)\|^2. \end{aligned} \quad (3.12)$$

To estimate the term $-\alpha(t) \left\langle \nabla_{\mathbf{x}} F(\mathbf{x}_{\tau}^*, t) - \nabla_{\mathbf{x}} F(\mathbf{x}_{\tau}^*, \tau), \mathbf{x}(t) - \mathbf{x}_{\tau}^* + \mathbf{p}(t) \right\rangle$ we apply the inequality

$$-\langle \mathbf{a}, \mathbf{b} \rangle \leq \frac{1}{\lambda(t)} \|\mathbf{a}\|^2 + \frac{\lambda(t)}{4} \|\mathbf{b}\|^2.$$

Namely,

$$\begin{aligned} -\alpha(t) \left\langle \nabla_{\mathbf{x}} F(\mathbf{x}_{\tau}^*, t) - \nabla_{\mathbf{x}} F(\mathbf{x}_{\tau}^*, \tau), \mathbf{x}(t) - \mathbf{x}_{\tau}^* + \mathbf{p}(t) \right\rangle &= \\ &= -\alpha(t) \left\langle (A(t) - A(\tau)) \nabla P(\mathbf{x}_{\tau}^*), \mathbf{x}(t) - \mathbf{x}_{\tau}^* + \mathbf{p}(t) \right\rangle - \\ &\quad -\alpha(t) \left\langle (\lambda(t) - \lambda(\tau)) \mathbf{x}_{\tau}^*, \mathbf{x}(t) - \mathbf{x}_{\tau}^* + \mathbf{p}(t) \right\rangle \leq \\ &\leq \frac{\alpha(t)}{\lambda(t)} \left((A(t) - A(\tau))^2 \|\nabla P(\mathbf{x}_{\tau}^*)\|^2 + (\lambda(t) - \lambda(\tau))^2 \|\mathbf{x}_{\tau}^*\|^2 \right) + \\ &\quad + \frac{\alpha(t) \lambda(t)}{2} \|\mathbf{x}(t) - \mathbf{x}_{\tau}^* + \mathbf{p}(t)\|^2 \end{aligned}$$

Since the sequence $\|\mathbf{x}_{\tau}^*\|$ is bounded due to the Lemma 1 and $\|\nabla P(\mathbf{x})\|$ is continuous, therefore there exists a constant $K \geq 0$ do not depending on the parameter τ such that $\max\{\|\nabla P(\mathbf{x}_{\tau}^*), \|\mathbf{x}_{\tau}^*\|\} \leq K^{\frac{1}{2}}$. Using this fact and Conditions 4. and 5.a we have that

$$\begin{aligned} -\alpha(t) \left\langle \nabla_{\mathbf{x}} F(\mathbf{x}_{\tau}^*, t) - \nabla_{\mathbf{x}} F(\mathbf{x}_{\tau}^*, \tau), \mathbf{x}(t) - \mathbf{x}_{\tau}^* + \mathbf{p}(t) \right\rangle &\leq \\ &\leq K \frac{\alpha(t)}{\lambda(t)} \left(\dot{A}^2(t) + \dot{\lambda}^2(t) \right) (\tau - t)^2 + \frac{\alpha(t) \lambda(t)}{2} \|\mathbf{x}(t) - \mathbf{x}_{\tau}^* + \mathbf{p}(t)\|^2 = \\ &= \frac{\alpha(t) \lambda(t)}{2} \|\mathbf{x}(t) - \mathbf{x}_{\tau}^* + \mathbf{p}(t)\|^2 + \psi(t) (\tau - t)^2, \end{aligned} \quad (3.13)$$

where $\psi(t) = K \frac{\alpha(t)}{\lambda(t)} \left(\dot{A}^2(t) + \dot{\lambda}^2(t) \right)$.

Substituting the inequalities (3.12) and (3.13) into (3.11) and taking into consideration that $\gamma(t)$ is non-increasing we have that

$$\begin{aligned} \frac{d}{dt} g(t, \tau) &\leq -\frac{1}{2} \alpha(t) \lambda(t) \|\mathbf{x}(t) - \mathbf{x}_{\tau}^* + \mathbf{p}(t)\|^2 + \frac{1}{2} \alpha(t) \lambda(t) \left\langle \mathbf{x}(t) - \mathbf{x}_{\tau}^*, \mathbf{p}(t) \right\rangle + \\ &\quad + \psi(t) (\tau - t)^2 = \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{4}\alpha(t)\lambda(t)\|\mathbf{x}(t) - \mathbf{x}_\tau^* + \mathbf{p}(t)\|^2 - \frac{1}{4}\alpha(t)\lambda(t)\|\mathbf{x}(t) - \mathbf{x}_\tau^* + \mathbf{p}(t)\|^2 + \\
&\quad + \frac{1}{2}\alpha(t)\lambda(t)\langle \mathbf{x}(t) - \mathbf{x}_\tau^*, \mathbf{p}(t) \rangle + \psi(t)(\tau - t)^2 \leq \\
&\leq -\frac{1}{4}\alpha(t)\lambda(t)\|\mathbf{x}(t) + \mathbf{p}(t) - \mathbf{x}_\tau^*\|^2 - \frac{1}{8}\alpha(t)\lambda(t)\|\mathbf{x} - \mathbf{x}_\tau^*\|^2 - \\
&\quad - \frac{1}{8}\alpha(t)\lambda(t) \cdot \frac{\gamma(t) - 1}{\gamma(t_0) - 1} \|\mathbf{x} - \mathbf{x}_\tau^*\|^2 + \psi(t)(\tau - t)^2 = \\
&= -a(t)\frac{1}{2}\|\mathbf{x}(t) + \mathbf{p}(t) - \mathbf{x}_\tau^*\|^2 - b(t)\frac{1}{4}\alpha(t)\lambda(t)\|\mathbf{x} - \mathbf{x}_\tau^*\|^2 - \\
&\quad - c(t)\frac{1}{2}(\gamma(t) - 1)\|\mathbf{x} - \mathbf{x}_\tau^*\|^2 + \psi(t)(\tau - t)^2,
\end{aligned}$$

where $a(t) = \frac{1}{2}\alpha(t)\lambda(t)$, $b(t) = \frac{1}{2}$, $c(t) = \frac{1}{4(\gamma(t_0) - 1)}\alpha(t)\lambda(t) = C_1\alpha(t)\gamma(t)$.

Since $\alpha(t)\lambda(t) \leq 1$, so $a(t) \leq \frac{1}{2}$ for all $t \geq t_0$. Otherwise, $c(t) = 2C_1a(t)$, consequently, there exists $K_1 = \min\{1, 2C_1\} > 0$, depending only on $\gamma(t_0)$ such that

$$\frac{d}{dt}g(t, \tau) \leq -K_1\alpha(t)\lambda(t)g(t, \tau) + \psi(t)(\tau - t)^2$$

for all $t_0 \leq t < \tau$.

The functions $g(t, \tau)$, $\varphi(t) = K_1\alpha(t)\lambda(t)$ and $\psi(t)$ satisfy the conditions 1 and 2 of the Lemma 2, so there exists a constant K_g such that $g(t, t_{\max}) \leq K_g$ for all $t_0 \leq t < t_{\max}$.

Let us assume, that the right maximal interval of existence is finite, i.e. $t_{\max} < \infty$. Then $g(t, t_{\max}) \leq K_g$ for all $t \in [t_0, t_{\max})$. From the definition of $g(t, \tau)$ immediately follows that

$$\|\mathbf{x}(t) - \mathbf{x}_{t_{\max}}^* + \mathbf{p}(t)\| \leq S, \quad \|\mathbf{x}(t) - \mathbf{x}_{t_{\max}}^*\| \leq S$$

with some constant S for all $t \in [t_0, t_{\max})$. Using the triangle inequality, the monotonicity of the parameter functions $\alpha(t)$ and $\lambda(t)$ and the boundedness of $\mathbf{x}_\tau^*$ obtained from the Lemma 1, one can easily derive constant R such that for

$$\|\mathbf{x}(t)\| \leq \|\mathbf{x}(t) - \mathbf{x}_{t_{\max}}^*\| + \|\mathbf{x}_{t_{\max}}^*\| \leq R, \quad (3.14)$$

and

$$\|\dot{\mathbf{x}}(t)\| = \|\mathbf{p}(t)\| \leq \|\mathbf{x}(t) - \mathbf{x}_{t_{\max}}^* + \mathbf{p}(t)\| + \|\mathbf{x}(t) - \mathbf{x}_{t_{\max}}^*\| \leq R. \quad (3.15)$$

for all $t \in [t_0, t_{\max})$. Let

$$B = \{(\mathbf{x}, \mathbf{p}, t) : \|\mathbf{x}\| \leq R, \|\mathbf{p}\| \leq R, |t - t_0| \leq t_{\max}\}.$$

Using the continuity of the vector-function $\nabla_{\mathbf{x}}F(\mathbf{x}, t)$ and the upper estimate $\sup_{(\mathbf{x}, \mathbf{p}, t) \in B} \|\nabla_{\mathbf{x}}F(\mathbf{x} + \mathbf{p}, t)\|$, together with the continuity of $\alpha(t)$, $A(t)$, $\lambda(t)$ we have

$\|\ddot{\mathbf{x}}(t)\| = \|\dot{\mathbf{p}}\| \leq R$ if R is chosen large enough.

Since B is a compact set and $(\mathbf{x}(t), \mathbf{p}(t), t_{\max}) \in B$, applying again the local existence theorem with the initial value $(\mathbf{x}(t_{\max}), \mathbf{p}(t_{\max}))$, we can extend the maximal solution to a strictly larger interval, which is a clear contradiction.

Consequently, the solution $(\mathbf{x}(t), \mathbf{p}(t))$ of (3.6)-(3.7) exists on the whole halfline $[t_0, \infty)$, so we can speak about the limit points of the trajectories when $t \rightarrow \infty$.

Conditions 5.a), 5.b) and 5.c) ensure that $g(t, \tau)$ satisfies the conditions 4 and 5 of the Lemma 3 and hence $\lim_{\tau \rightarrow \infty} g(\tau, \tau) = 0$. From here and from the continuity of the function $f(\mathbf{x})$ and by using the triangle inequality as in the previous part of the proof, we immediately obtain all statements of the proposition. $\square$

3.3. Admissible parameter functions of the minimizing R-FR-type SO-ODE system

Studying the unconstrained minimization problem without regularization we have found several classes of parameter functions (see Hajba [19]) of the SO-ODE system, which satisfy the conditions given in Hajba [18] for the convergence. Since our problem examined in this paper is more complicated, here we give only two parameter classes which satisfy the conditions of the Proposition 1 assuming that

$$P(\mathbf{x}) = \sum_{i=1}^m |g_i^+(\mathbf{x})|^2.$$

It can be seen that we have the widest freedom in the choice of the function $\gamma(t)$, since it does not depend on the other parameters, only Condition 3.b) must be satisfied.

Let

$$\alpha(t) = \alpha_0(1+t)^{-\alpha}, \quad \text{and} \quad \lambda(t) = \lambda_0(1+t)^{-\lambda}.$$

Substituting these functions into the limit conditions of the Proposition 1. we can look for $A(t)$ in different forms. Namely,

Case 1. Let $A(t) = A_0(1+t)^A$. In this case the conditions of the Proposition 1. are satisfied if and only if

$$\begin{aligned} 0 < A_0, & \quad 0 < \alpha_0, & \quad 0 < \lambda_0 & \quad \alpha_0 \lambda_0 = 1, \\ 0 < \lambda < A \leq \alpha, & \quad \alpha + \lambda < 1, & \quad A \leq 1 - \alpha - 2\lambda. \end{aligned}$$

In general the obtained system of inequalities has lots of solutions, e.g. an appropriate choice of the parameters is

$$A_0 = \alpha_0 = \lambda_0 = 1, \quad A = \alpha = \frac{1}{4}, \quad \lambda = \frac{1}{8}.$$

Case 2. Let $A(t) = A_0(1+t)^A \ln(1+t)$. In this case the following inequality system must be satisfied:

$$\begin{aligned} 0 < A_0, & \quad 0 < \alpha_0, & \quad 0 < \lambda_0 & \quad \alpha_0 \lambda_0 = 1, \\ 0 < \lambda < A < \alpha, & \quad \alpha + \lambda < 1, & \quad A < \frac{1}{2}, & \quad A \leq 1 - \alpha - 2\lambda. \end{aligned}$$

This system also has solution, e.g.

$$A_0 = \alpha_0 = \lambda_0 = 1, \quad A = \frac{1}{4}, \quad \alpha = \frac{3}{8}, \quad \lambda = \frac{1}{8}$$

satisfies it.

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Authors' addresses

T. Hajba

Széchenyi István University, Faculty of Engineering Sciences, Dept. of Mathematics and Computer Science, Egyetem tér 1., H-9026 Győr, Hungary

E-mail address: hajbat@sze.hu

M. Kovács

Eötvös Loránd University, Dept. of Operations Research, Pázmány sétány 1-3, H-1117 Budapest, Hungary

E-mail address: margo@cs.elte.hu



EXISTENCE OF NONLINEAR LANE-EMDEN EQUATION OF FRACTIONAL ORDER

RABHA W. IBRAHIM

Received 19 December, 2011

Abstract. We study a Dirichlet boundary value problem for the Lane-Emden equation involving two fractional orders. Lane-Emden equation has been widely used to describe a variety of phenomena in physics and astrophysics, including aspects of stellar structure, the thermal history of a spherical cloud of gas, isothermal gas spheres, and thermionic currents. However, ordinary Lane-Emden equation does not provide a correct description of the dynamics of systems in complex media. In order to overcome this problem and describe dynamical processes in a fractal medium, numerous generalizations of Lane-Emden equation have been proposed. One such generalization replaces the ordinary derivative by a fractional derivative in the Lane-Emden equation. This gives rise to the fractional Lane-Emden equation with a single index. Recently, a new type of Lane-Emden equation with two different fractional orders has been introduced which provides a more flexible model for fractal processes as compared with the usual one characterized by a single index. The contraction mapping principle and Krasnoselskii's fixed point theorem are applied to prove the existence of solutions of the problem in a Banach space.

2000 Mathematics Subject Classification: 34G20; 26A33; 34A12

Keywords: fractional calculus, fractional differential equation, Lane-Emden equation, Riemann-Liouville fractional operators, Volterra integral equation

1. INTRODUCTION

The theory of singular boundary value problems has become an important area of investigation in the past three decades (see [2–5, 10]). One of the equations arising in this topic is the Lane-Emden equation. Lane-Emden type equations, first published by Jonathan Homer Lane in 1870 [18], and further explored in detail by Emden [11], represents such phenomena and have significant applications. It is a second- order ordinary differential equation with an arbitrary index, known as the polytropic index, that occurs in one of its terms. The Lane-Emden equation describes a variety of phenomena in physics and astrophysics, including aspects of stellar structure, the thermal history of a spherical cloud of gas, isothermal gas spheres and thermionic currents [8].

The solution of the Lane-Emden problem, as well as other various linear and non-linear singular initial value problems in quantum mechanics and astrophysics, is numerically challenging because of the singularity behavior at the origin. The approximate solutions of the Lane-Emden equation were given by homotopy perturbation method [9], variational iteration method [26], and Sinc-Collocation method [23], an implicit series solution [21]. Recently, Parand et. al [22] proposed an approximation algorithm for the solution of the nonlinear Lane-Emden type equation using Hermite functions collocation method. Moreover, Adibi and Rismani [1] introduced a modified Legendre-spectral method. Finally, Bhrawy and Alofi [6] imposed a Jacobi-Gauss collocation method for solving nonlinear Lane-Emden type equations.

Lane-Emden equations have the following form

$$u''(t) + \frac{a}{t}u'(t) + f(t, u) = g(t), \quad 0 < t \leq 1, a \geq 0 \quad (1.1)$$

with the initial condition

$$u(0) = A, \quad u'(0) = B,$$

where A, B are constants, $f(t, u)$ is a continuous real valued function and $g(t) \in C[0, 1]$.

2. FRACTIONAL CALCULUS

Fractional calculus and its applications (that is the theory of derivatives and integrals of any arbitrary real or complex order) has importance in diverse areas of mathematical physics and engineering sciences. It generalized the ideas of integer order differentiation and n -fold integration. Fractional derivatives introduce an excellent instrument for the description of general properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials, as well as in the description of properties of gases, liquids and rocks, and in many other fields (see [19, 27]).

The class of fractional differential equations of various types plays important role and is an useful tool not only in mathematics but also in physics, control systems, dynamical systems and engineering in order to create the mathematical models of many physical phenomena. Naturally, such equations are desired to be solved. Many studies on fractional calculus and fractional differential equations, involving different operators such as Riemann-Liouville operators, Erdélyi-Kober operators, Weyl-Riesz operators, Caputo operators and Grünwald-Letnikov operators, have appeared during the past three decades. The existence of positive solution and multi-positive solutions for nonlinear fractional differential equation are established and studied [7, 12, 15, 16, 20, 24]. Moreover, by using the concepts of subordination and superordination of analytic functions, the existence of analytic solutions for fractional differential equations in complex domain are suggested and posed in [13, 14].

One of the most frequently used tools in the theory of fractional calculus is the the Riemann-Liouville operator (see [24]). The Riemann-Liouville fractional derivative are well posed for the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations. Moreover, this operator possesses advantages of fast convergence, higher stability and higher accuracy to derive different types of numerical algorithms (see [17]).

Definition 1. The fractional (arbitrary) order integral of the function f of order $\alpha > 0$ is defined by

$$I_a^\alpha f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau.$$

When $a = 0$, we write $I_a^\alpha f(t) = f(t) * \phi_\alpha(t)$, where $(*)$ denotes the convolution product (see [24]), $\phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$ and $\phi_\alpha(t) = 0$, $t \leq 0$ and $\phi_\alpha \rightarrow \delta(t)$ as $\alpha \rightarrow 0$ where $\delta(t)$ is the delta function.

Definition 2. The fractional (arbitrary) order derivative of the function f of order $0 \leq \alpha < 1$ is defined by

$$D_a^\alpha f(t) = \frac{d}{dt} \int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau) d\tau = \frac{d}{dt} I_a^{1-\alpha} f(t).$$

Remark 1. From Definition 1 and Definition 2, we have

$$D^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, \quad \mu > -1; 0 < \alpha < 1$$

and

$$I^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \quad \mu > -1; \alpha > 0.$$

In this note, we consider the fractional Lane-Emden equations of the following form

$$D^\beta \left(D^\alpha + \frac{a}{t} \right) u(t) + f(t, u) = g(t), \quad (2.1)$$

$$(0 < t \leq 1, a \geq 0, 0 < \alpha, \beta \leq 1)$$

with boundary conditions

$$u(0) = u(\mu) = u(1) = 0, \quad 0 < \mu < 1,$$

where $f(t, u)$ is a continuous real valued function and $g(t) \in C[0, 1]$.

Now, we state a known result due to Krasnoselskii [25] which is needed to prove the existence of solution.

Theorem 1. Let M be a closed convex and nonempty subset of a Banach space X . Let A, B be the operators such that (i) $Ax + By \in M$ whenever $x, y \in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.

3. EXISTENCE OF SOLUTIONS

In this section, we establish the solution of Eq. 2.1. Assume that $\mathcal{B}$ is a Banach space X of all continues bounded functions endowed with the supremum norm. We introduce the following linear problem for the Lane-Emden equation involving two fractional orders:

$$D^\beta \left(D^\alpha + \frac{a}{t} \right) u(t) = g(t), \quad (3.1)$$

$$(0 < t \leq 1, a \geq 0, 0 < \alpha, \beta \leq 1)$$

with the boundary conditions

$$u(0) = u(\mu) = u(1) = 0, \quad 0 < \mu < 1.$$

Lemma 1. *A unique solution of the linear three-point boundary value problem for the Lane-Emden equation 3.1 is given by*

$$\begin{aligned} u(t) = & \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} g(s) ds - \frac{a}{\tau} u(\tau) \right) d\tau \\ & + t^\alpha \frac{(t-\mu)}{1-\mu} \left[\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} g(s) ds - \frac{a}{\tau} u(\tau) \right) d\tau \right] \\ & - \frac{t^\alpha(1-t)}{\mu^\alpha(1-\mu)} \left[\int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} g(s) ds - \frac{a}{\tau} u(\tau) \right) d\tau \right], \end{aligned} \quad (3.2)$$

where $g \in C[0, 1]$.

Proof. The general solution of

$$D^\beta \left(D^\alpha + \frac{a}{t} \right) u(t) = g(t)$$

can be written as

$$\begin{aligned} u(t) = & \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} g(s) ds - \frac{a}{\tau} u(\tau) \right) d\tau \\ & - \frac{c_1}{\Gamma(\alpha+2)} t^{\alpha+1} - \frac{c_2}{\Gamma(\alpha+1)} t^\alpha - c_3. \end{aligned} \quad (3.3)$$

Using the boundary conditions for 3.1 we obtain

$$\begin{aligned}
c_3 &= 0, \\
c_2 &= \frac{\mu(\alpha+1)}{(1-\mu)} \left[\int_0^1 (1-\tau)^{\alpha-1} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} g(s) ds - \frac{a}{\tau} u(\tau) \right) d\tau \right] \\
&\quad - \frac{1}{\mu^{\alpha+1}} \left[\int_0^\mu (\mu-\tau)^{\alpha-1} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} g(s) ds - \frac{a}{\tau} u(\tau) \right) d\tau \right] \\
c_1 &= \frac{(\alpha+2)(\alpha+1)}{1-\mu} \left[\int_0^1 (1-\tau)^{\alpha-1} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} g(s) ds - \frac{a}{\tau} u(\tau) \right) d\tau \right] \\
&\quad - \frac{1}{\mu^\alpha} \int_0^\mu (\mu-\tau)^{\alpha-1} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} g(s) ds - \frac{a}{\tau} u(\tau) \right) d\tau
\end{aligned}$$

Substituting the last assertions in 3.3, we obtain the solution given by 3.2. This completes the proof. $\square$

Theorem 2. Let $f : [0, 1] \times X \longrightarrow X$ be a jointly continuous function satisfying the condition

$$|f(t, u) - f(t, v)| \leq L|u - v|, \quad \forall t \in [0, 1], u, v \in X.$$

Moreover, assume that $\sup_{t \in [0, 1]} |g(t)| = \gamma$. Then the boundary value problem 2.1 has a unique solution provided by

$$\ell := \frac{L}{\Gamma(\alpha + \beta + 1)} (2 + \mu^{\alpha+\beta}) + a \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} (2 + \mu^{2\alpha-1}) < 1, \quad a \geq 0.$$

Proof. Define $P : X \rightarrow X$ by

$$\begin{aligned}
(Pu)(t) &= \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} (g(s) - f(s, u(s))) ds - \frac{a}{\tau} u(\tau) \right) d\tau \\
&\quad + t^\alpha \frac{(t-\mu)}{1-\mu} \\
&\quad \left[\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} (g(s) - f(s, u(s))) ds - \frac{a}{\tau} u(\tau) \right) d\tau \right] \\
&\quad - \frac{t^\alpha (1-t)}{\mu^\alpha (1-\mu)} \\
&\quad \left[\int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} (g(s) - f(s, u(s))) ds - \frac{a}{\tau} u(\tau) \right) d\tau \right] \quad (3.4)
\end{aligned}$$

Setting $\sup_{t \in [0, 1]} |f(t, u)| = M$ and we chose

$$r \geq \frac{1}{1-\epsilon} \left(\frac{\gamma + M}{\Gamma(\alpha + \beta + 1)} (2 + \mu^{\alpha+\beta}) \right), \quad (3.5)$$

where ϵ such that $\ell \leq \epsilon < 1$. Now we show that $PB_r \subset B_r$, where $B_r := \{u \in X : \|u\| \leq r\}$. For $u \in X$, we have

$$\begin{aligned}
& \|(Pu)(t)\| \\
&= \sup_{t \in [0,1]} \left| \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} (g(s) - f(s, u(s))) ds - \frac{a}{\tau} u(\tau) \right) d\tau \right. \\
&+ t^\alpha \frac{(t-\mu)}{1-\mu} \left[\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} (g(s) - f(s, u(s))) ds - \frac{a}{\tau} u(\tau) \right) d\tau \right] \\
&\quad \left. - \frac{t^\alpha(1-t)}{\mu^\alpha(1-\mu)} \right. \\
&\quad \left. \left[\int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} (g(s) - f(s, u(s))) ds - \frac{a}{\tau} u(\tau) \right) d\tau \right] \right| \\
&\leq \sup_{t \in [0,1]} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \\
&\quad \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} |g(s) - f(s, u(s)) - f(s, 0) + f(s, 0)| ds + \left| \frac{a}{\tau} u(\tau) \right| \right) d\tau \\
&\quad + \sup_{t \in [0,1]} t^\alpha \left| \frac{(t-\mu)}{1-\mu} \right| \left[\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \right. \\
&\quad \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} |g(s) - f(s, u(s)) - f(s, 0) + f(s, 0)| ds + \left| \frac{a}{\tau} u(\tau) \right| \right) d\tau \right] \\
&\quad + \sup_{t \in [0,1]} \frac{t^\alpha(1-t)}{\mu^\alpha(1-\mu)} \left[\int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)} \right. \\
&\quad \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} |g(s) - f(s, u(s)) - f(s, 0) + f(s, 0)| ds + \left| \frac{a}{\tau} u(\tau) \right| \right) d\tau \right] \\
&\leq \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} ds \right) d\tau [\gamma + M + L\|u\|] \\
&\quad + \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{\tau} d\tau a\|u\| \\
&+ \sup_{t \in [0,1]} \left(t^\alpha \left| \frac{(t-\mu)}{1-\mu} \right| \right) \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} ds \right) d\tau [\gamma + M + L\|u\|] \\
&\quad + \sup_{t \in [0,1]} \left(t^\alpha \left| \frac{(t-\mu)}{1-\mu} \right| \right) \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{\tau} d\tau a\|u\| \\
&+ \sup_{t \in [0,1]} \left(\frac{t^\alpha(1-t)}{\mu^\alpha(1-\mu)} \right) \left[\int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} ds \right) d\tau [\gamma + M + L\|u\|] \right.
\end{aligned}$$

$$\begin{aligned}
& + \sup_{t \in [0,1]} \left(\frac{t^\alpha(1-t)}{\mu^\alpha(1-\mu)} \right) \left[\int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)} \frac{a}{\tau} \|u\| d\tau \right] \\
& \leq (\gamma + M + Lr) \left[2 \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} ds \right) d\tau \right. \\
& \quad \left. + \int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} ds \right) d\tau \right] \\
& + ar \left[2 \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \tau^{\alpha-1} d\tau + \int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)} \tau^{\alpha-1} d\tau \right] \\
& = (\gamma + M + Lr) \left[\frac{2}{\Gamma(\alpha)\Gamma(\beta+1)} \int_0^1 (1-\tau)^{\alpha-1} \tau^\beta d\tau + \int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta+1)} \tau^\beta d\tau \right] \\
& + ar \left[2 \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \tau^{\alpha-1} d\tau + \int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)} \tau^{\alpha-1} d\tau \right].
\end{aligned}$$

Now by using 3.5, the relations for the Beta function

$$B(z, w) = \int_0^1 (1-\tau)^{z-1} \tau^{w-1} d\tau = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)},$$

$$B(z, w+1) = \int_0^1 (1-\tau)^{z-1} \tau^w d\tau = \frac{\Gamma(z)\Gamma(w+1)}{\Gamma(z+w+1)}$$

and

$$\int_0^\mu (\mu-\tau)^{z-1} \tau^w d\tau = \frac{\Gamma(z)\Gamma(w+1)}{\Gamma(z+w+1)} \mu^{z+w}$$

we find that

$$\begin{aligned}
& \| (Pu)(t) \| \\
& \leq (\gamma + M + Lr) \left[\frac{2}{\Gamma(\alpha)\Gamma(\beta+1)} \int_0^1 (1-\tau)^{\alpha-1} \tau^\beta d\tau + \int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta+1)} \tau^\beta d\tau \right] \\
& + ar \left[2 \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \tau^{\alpha-1} d\tau + \int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta+1)} \tau^\beta d\tau \right] \\
& = \frac{\gamma + M + Lr}{\Gamma(\alpha + \beta + 1)} (2 + \mu^{\alpha+\beta}) + ar \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} (2 + \mu^{2\alpha-1}) \\
& \leq (\ell + 1 - \epsilon)r \leq r.
\end{aligned}$$

We proceed to prove that P is a contraction mapping. For $u, v \in X$ and for all $t \in [0, 1]$ we pose

$$\begin{aligned}
& \| (Pu)(t) - (Pv)(t) \| \\
& = \sup_{t \in [0,1]} |(Pu)(t) - (Pv)(t)|
\end{aligned}$$

$$\begin{aligned}
& \leq \sup_{t \in [0,1]} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \\
& \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} |f(s, u(s)) - f(s, v(s))| ds + \frac{a}{\tau} |u(\tau) - v(\tau)| \right) d\tau \\
& + \sup_{t \in [0,1]} t^\alpha \left[\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \right. \\
& \left. \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} |f(s, u(s)) - f(s, v(s))| ds + \frac{a}{\tau} |u(\tau) - v(\tau)| \right) d\tau \right] \\
& + \sup_{t \in [0,1]} \frac{t^\alpha(1-t)}{\mu^\alpha(1-\mu)} \left[\int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)} \right. \\
& \left. \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} |f(s, u(s)) - f(s, v(s))| ds + \frac{a}{\tau} |u(\tau) - v(\tau)| \right) d\tau \right] \\
& \leq \sup_{t \in [0,1]} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} ds \right) d\tau [L\|u-v\|] \\
& + \sup_{t \in [0,1]} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{\tau} d\tau a \|u-v\| \\
& + \sup_{t \in [0,1]} t^\alpha \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} ds \right) d\tau [L\|u-v\|] \\
& + \sup_{t \in [0,1]} t^\alpha \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{\tau} d\tau a \|u-v\| \\
& + \sup_{t \in [0,1]} \frac{t^\alpha(1-t)}{\mu^\alpha(1-\mu)} \left[\int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)} \right. \\
& \left. \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} [L\|u-v\|] ds + \frac{a}{\tau} \|u-v\| \right) d\tau \right] \\
& \leq \ell \|u-v\|,
\end{aligned}$$

then P is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. This completes the proof. $\square$

Theorem 3. Let $f : [0, 1] \times X \longrightarrow X$ be a jointly continuous function and maps bounded subsets of $[0, 1] \times X$ into relatively compact subsets of X . Furthermore, assume that

$$|f(t, u) - f(t, v)| \leq L|u - v|, \quad \forall t \in [0, 1], u, v \in X,$$

$$|f(t, u)| \leq \sigma(t), \quad \sigma \in (L^1[0, 1], R^+)$$

and $\sup_{t \in [0,1]} |g(t)| = \gamma$. If

$$\left(\frac{L(1 + \mu^{\alpha+\beta})}{\Gamma(\alpha + \beta + 1)} + \frac{a\Gamma(\alpha)(1 + \mu^{2\alpha-1})}{\Gamma(2\alpha)} \right) < 1$$

then the boundary value problem 2.1 has at least one solution on $[0, 1]$.

Proof. Define two operators A and B on B_r where

$$r \geq \frac{\frac{\gamma + \|\sigma\|_{L^1}}{\Gamma(\alpha + \beta + 1)}(2 + \mu^{\alpha+\beta})}{1 - a \frac{2\Gamma(\alpha)}{\Gamma(2\alpha)}(2 + \mu^{2\alpha-1})}$$

as follows:

$$(Au)(t) := \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} (g(s) - f(s, u(s))) ds - \frac{a}{\tau} u(\tau) \right) d\tau$$

and

$$\begin{aligned} (Bu)(t) := & \\ & t^\alpha \frac{(t-\mu)}{1-\mu} \left[\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} (g(s) - f(s, u(s))) ds - \frac{a}{\tau} u(\tau) \right) d\tau \right] \\ & - \frac{t^\alpha(1-t)}{\mu^\alpha(1-\mu)} \left[\int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} (g(s) - f(s, u(s))) ds - \frac{a}{\tau} u(\tau) \right) d\tau \right]. \end{aligned}$$

For $u, v \in B_r$ we obtain

$$\begin{aligned} & \|Au + Bv\| \\ & \leq [\gamma + \|\sigma\|_{L^1}] \left[\frac{2}{\Gamma(\alpha + \beta + 1)} + \frac{1}{\Gamma(\alpha + \beta + 1)} \mu^{\alpha+\beta} \right] \\ & \quad + ar \left[\frac{2\Gamma(\alpha)}{\Gamma(2\alpha)} + \frac{1}{\Gamma(\alpha + \beta + 1)} \mu^{\alpha+\beta} \right] \\ & \leq r. \end{aligned}$$

Thus, $A, B \in B_r$. Furthermore, for $u, v \in B_r$,

$$\begin{aligned} & \|(Bu)(t) - (Bv)(t)\| \\ & \leq \sup_{t \in [0,1]} t^\alpha \frac{(t-\mu)}{1-\mu} \left[\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} |f(s, u(s)) - f(s, v(s))| ds \right. \right. \\ & \quad \left. \left. + \frac{a}{\tau} |u(\tau) - v(\tau)| \right) d\tau \right] \\ & \quad + \sup_{t \in [0,1]} \frac{t^\alpha(1-t)}{\mu^\alpha(1-\mu)} \left[\int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} |f(s, u(s)) - f(s, v(s))| ds \right. \right. \\ & \quad \left. \left. + \frac{a}{\tau} |u(\tau) - v(\tau)| \right) d\tau \right] \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{t \in [0,1]} t^\alpha \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} ds \right) d\tau [L\|u-v\|] \\
&\quad + t^\alpha \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{\tau} d\tau a \|u-v\| \\
&\quad + \sup_{t \in [0,1]} \frac{t^\alpha(1-t)}{\mu^\alpha(1-\mu)} \left[\int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)} \right. \\
&\quad \left. \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} [L\|u-v\|] ds + \frac{a}{\tau} \|u-v\| \right) d\tau \right] \\
&\leq [L\|u-v\|] \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} ds \right) d\tau \\
&\quad + a \|u-v\| \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \tau^{\alpha-1} d\tau \\
&\quad + \left[\int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} [L\|u-v\|] ds \right. \right. \\
&\quad \left. \left. + a \|u-v\| \int_0^\mu \frac{(\mu-\tau)^{\alpha-1}}{\Gamma(\alpha)} \tau^{\alpha-1} d\tau \right) d\tau \right] \\
&= \left(\frac{L(1+\mu^{\alpha+\beta})}{\Gamma(\alpha+\beta+1)} + \frac{a\Gamma(\alpha)(1+\mu^{2\alpha-1})}{\Gamma(2\alpha)} \right) \|u-v\|,
\end{aligned}$$

it follows that B is a contraction mapping in B_r . The continuity of f implies that the operator A is continuous. Also, A is uniformly bounded on B_r as

$$\begin{aligned}
&\|(Au)(t)\| \\
&= \sup_{t \in [0,1]} \left| \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} (g(s) - f(s, u(s))) ds - \frac{a}{\tau} u(\tau) \right) d\tau \right| \\
&\leq \sup_{t \in [0,1]} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} |g(s) + f(s, u(s))| ds + \left| \frac{a}{\tau} u(\tau) \right| \right) d\tau \\
&\leq \sup_{t \in [0,1]} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} ds \right) d\tau [\gamma + \|\sigma\|_{L^1}] \\
&\quad + \sup_{t \in [0,1]} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \frac{\tau^\alpha}{\tau} d\tau a \|u\| \\
&\leq [\gamma + \|\sigma\|_{L^1}] \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} ds \right) d\tau \\
&\quad + a \|u\| \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \tau^{\alpha-1} d\tau
\end{aligned}$$

$$\leq \frac{[\gamma + \|\sigma\|_{L^1}]}{\Gamma(\alpha + \beta + 1)} + \frac{\Gamma(\alpha)ar}{\Gamma(2\alpha)}.$$

Next we show that A is a compact operator in B_r . Let $t_1, t_2 \in [0, 1]$ we have

$$\begin{aligned} & \| (Au)(t_1) - (Au)(t_2) \| \\ &= \left\| \int_0^{t_1} \frac{(t_1 - \tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau - s)^{\beta-1}}{\Gamma(\beta)} (g(s) - f(s, u(s))) ds + \frac{a}{\tau} u(\tau) \right) d\tau \right. \\ & \quad \left. - \int_0^{t_2} \frac{(t_2 - \tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \frac{(\tau - s)^{\beta-1}}{\Gamma(\beta)} (g(s) - f(s, u(s))) ds + \frac{a}{\tau} u(\tau) \right) d\tau \right\| \\ & \leq \frac{(\gamma + \|f\|)}{\Gamma(\alpha + \beta + 1)} |t_1^{\alpha+\beta} - t_2^{\alpha+\beta}| + \frac{a\Gamma(\alpha)}{\Gamma(2\alpha)} |t_1^{2\alpha-1} - t_2^{2\alpha-1}| \end{aligned}$$

which is independent of u . Thus, A is equicontinuous. We used the fact that f maps bounded subsets into relatively compact subsets. Therefore, A is relatively compact on B_r . Hence, by the Arzela-Ascoli theorem, A is compact on B_r . Thus all the assumptions of Theorem 1 are satisfied and the conclusion of Theorem 1 implies that the boundary value problem 2.1 has at least one solution on $[0, 1]$. This completes the proof. $\square$

4. APPLICATIONS

In this section, we illustrate some applications of the theory established in this paper.

Example 1. Consider the following (BVP)

$$\begin{aligned} D^{1/2} \left(D^{1/2} + \frac{0.1}{t} \right) u(t) + \frac{1}{(t+2)^2} \frac{|u|}{1+|u|} &= t, \quad t \in (0, 1) \\ u(0) = u\left(\frac{1}{2}\right) = u(1) &= 0. \end{aligned} \quad (4.1)$$

It is clear that

$$|f(t, u) - f(t, v)| \leq \frac{1}{4} |u - v|,$$

and $\sup_{t \in (0,1)} g(t) = 1$. Moreover, we have

$$\ell = \frac{L}{\Gamma(\alpha + \beta + 1)} (2 + \mu^{\alpha+\beta}) + a \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} (2 + \mu^{2\alpha-1}) = 0.625 + 0.3544 = 0.9794 < 1$$

Thus, by Theorem 2, the boundary value problem 4.1 has a unique solution on $[0, 1]$.

Example 2. Consider the following (BVP)

$$\begin{aligned} D^{1/4} \left(D^{1/2} + \frac{0.1}{t} \right) u(t) + \frac{1}{(t+2)^2} \frac{|u|}{1+|u|} &= t, \quad t \in (0, 1) \\ u(0) = u\left(\frac{3}{4}\right) = u(1) &= 0. \end{aligned} \quad (4.2)$$

It is clear that $|f(t, u)| \leq \frac{1}{(t+2)^2} := \sigma(t)$,

$$|f(t, u) - f(t, v)| \leq \frac{1}{4}|u - v|,$$

and $\sup_{t \in (0,1)} g(t) = 1$. Moreover, we have

$$\begin{aligned} & \left(\frac{L(1 + \mu^{\alpha+\beta})}{\Gamma(\alpha + \beta + 1)} + \frac{a\Gamma(\alpha)(1 + \mu^{2\alpha-1})}{\Gamma(2\alpha)} \right) \\ &= \frac{1.8}{3\Gamma(\frac{3}{4})} + 0.1772 = 0.5 + 0.1772 = 0.6772 < 1. \end{aligned}$$

Thus, by Theorem 3, the boundary value problem 4.2 has at least one solution on $[0, 1]$. If $\mu = \frac{1}{3}$, we obtain

$$\left(\frac{L(1 + \mu^{\alpha+\beta})}{\Gamma(\alpha + \beta + 1)} + \frac{a\Gamma(\alpha)(1 + \mu^{2\alpha-1})}{\Gamma(2\alpha)} \right) = 1.07 > 1;$$

hence the problem has not a solution.

Note that when $a \geq 1$ the (BVPS) 4.1 and 4.2 have no solutions in $[0, 1]$:

Example 3. Consider the following (BVP)

$$D^\beta \left(D^\alpha + \frac{1}{t} \right) u(t) + \frac{1}{(t+2)^2} \frac{|u|}{1+|u|} = t, \quad t \in (0, 1) \quad (4.3)$$

$$u(0) = u(\mu) = u(1) = 0.$$

It is clear that $|f(t, u)| \leq \frac{1}{(t+2)^2} = \sigma(t)$,

$$|f(t, u) - f(t, v)| \leq \frac{1}{4}|u - v|,$$

and $\sup_{t \in (0,1)} g(t) = 1$. Moreover, since $\Gamma(\alpha) > \Gamma(2\alpha)$ for $0 < \alpha \leq 1$, we have

$$\left(\frac{L(1 + \mu^{\alpha+\beta})}{\Gamma(\alpha + \beta + 1)} + \frac{a\Gamma(\alpha)(1 + \mu^{2\alpha-1})}{\Gamma(2\alpha)} \right) > 1, \quad \forall 0 < \alpha, \beta \leq 1.$$

We have the following proposition:

Proposition 1. *If $a \geq 1$, then the solution of the boundary value problem 2.1 does not exist for all $0 < \alpha, \beta \leq 1$.*

5. CONCLUSIONS

The existence of solutions for a Dirichlet boundary value problem involving Lane-Emden equation with two different fractional orders has been discussed. We apply the concepts of fractional calculus together with fixed point theorems to establish the existence results. First of all, we find the unique solution for a linear Dirichlet boundary value problem involving Lane-Emden equation with two different fractional orders, which in fact provides the platform to prove the existence of solutions for the associated nonlinear fractional Lane-Emden equation with two different orders. Our approach is applicable to a variety of real world problems.

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Author's address

Rabha W. Ibrahim

Institute of Mathematical Sciences, University Malaya, 50603, Kuala Lumpur, Malaysia

E-mail address: rabhaibrahim@yahoo.com



FORMULAS FOR WEIGHTED BINOMIAL SUMS USING THE POWERS OF TERMS OF BINARY RECURRENCES

EMRAH KILIÇ, YÜCEL TÜRKER ULUTAŞ, AND NEŞE ÖMÜR

Received 30 March, 2011

Abstract. In this paper, we give general formulas for some weighted binomial sums, using the powers of terms of certain binary recurrences. As an applications of our results, we show that the weighted binomial sums of the generalized Fibonacci and Lucas numbers can be expressed via the second kinds of Chebyshev polynomials and Stirling numbers.

2000 Mathematics Subject Classification: 11B37; 11B39; 05A19

Keywords: binary recurrences, weighted binomial sums, Stirling numbers of the second kind, Chebyshev polynomials of the second kind

1. INTRODUCTION

For $n > 1$, define the binary recurrences $\{U_n\}$ and $\{V_n\}$ by

$$U_n = pU_{n-1} - U_{n-2} \quad \text{and} \quad V_n = pV_{n-1} - V_{n-2}$$

where $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = p$, respectively.

The Fibonacci subsequence $\{F_{2n}\}$ and the Pell subsequence $\{P_{2n}\}$ are the special cases of the sequence $\{U_n\}$ for $p = 3$ and $p = 6$, respectively. It is also known that the natural numbers are special cases of the sequence $\{U_n\}$ for $p = 2$.

The Binet formulas of $\{U_n\}$ and $\{V_n\}$ are as follows:

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n, \quad (1.1)$$

where $\alpha, \beta = \left(p \pm \sqrt{p^2 - 4}\right)/2$.

From [4], we have that for $k \geq 0$ and $n > 1$,

$$\begin{aligned} U_{kn} &= V_k U_{k(n-1)} - U_{k(n-2)}, \\ V_{kn} &= V_k V_{k(n-1)} - V_{k(n-2)}. \end{aligned}$$

Wiemann and Cooper [9] mentioned some conjectures of Melham for the sum:

$$\sum_{h=1}^n F_{2h}^{2m+1},$$

where F_n stands for the n^{th} Fibonacci number.

Ozeki [6] considered Melham's sum and gave an explicit expansion for it as a polynomial in F_{2n+1} .

In general, Prodinger [7] derived a general formula for the sum:

$$\sum_{h=0}^n F_{2h+\delta}^{2m+\varepsilon},$$

where $\varepsilon, \delta \in \{0, 1\}$, as well as for the evaluations of the corresponding sums for Lucas numbers.

In [5], we considered alternating Melham's sums for Fibonacci and Lucas numbers of the form $\sum_{h=1}^n (-1)^h F_{2h+\delta}^{2m+\varepsilon}$ and $\sum_{h=1}^n (-1)^h L_{2h+\delta}^{2m+\varepsilon}$, where $\varepsilon, \delta \in \{0, 1\}$.

Recently Khan and Kwong [3] considered the sums

$$\sum_{h=0}^n h^m \binom{n}{h} U_h \quad \text{and} \quad \sum_{h=0}^n (-1)^{n+h} h^m \binom{n}{h} U_h, \quad (1.2)$$

and expressed them in terms of two associated sequences. The special cases of $m = 2, 3$ leads to interesting binomial and Fibonacci identities.

In this paper, we shall give general formulas for the sums

$$\begin{aligned} & \sum_{h=0}^n \binom{n}{h} h^m U_{ht}^{2m+\varepsilon}, \quad \sum_{h=0}^n \binom{n}{h} h^m V_{ht}^{2m+\varepsilon}, \\ & \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m U_{ht}^{2m+\varepsilon}, \quad \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m V_{ht}^{2m+\varepsilon}, \end{aligned}$$

where t is a positive integer and $\varepsilon \in \{0, 1\}$. In order to do this, firstly we will consider general cases of the sums given by (1.2) and we shall derive similar formulas for their Lucas counterparts in the second section. After this, by using these results, we state our main results in the third section.

2. GENERALIZED WEIGHTED BINOMIAL IDENTITIES

In this section, we will give generalizations of the results of [3] by considering the sequence $\{U_{nk}\}$ instead of the sequence $\{U_n\}$. We also give similar formulas for the Lucas sequence $\{V_n\}$. While deriving these results, we follow the proof strategy of [3].

Define the sequences $\{X_{kn}\}$, $\{Y_{kn}\}$, $\{W_{kn}\}$ and $\{Z_{kn}\}$ for $n \geq 2$ as follows:

$$\begin{aligned} X_0 &= 0, \quad X_k = U_k, \quad X_{kn} = (V_k + 2)(X_{k(n-1)} - X_{k(n-2)}), \\ Y_0 &= 0, \quad Y_k = U_k, \quad Y_{kn} = (V_k - 2)(Y_{k(n-1)} + Y_{k(n-2)}), \\ W_0 &= 2, \quad W_k = V_k + 2, \quad W_{kn} = (V_k + 2)(W_{k(n-1)} - W_{k(n-2)}), \\ Z_0 &= 2, \quad Z_k = V_k - 2, \quad Z_{kn} = (V_k - 2)(Z_{k(n-1)} + Z_{k(n-2)}). \end{aligned}$$

The Binet formulas of $\{X_{kn}\}$, $\{Y_{kn}\}$, $\{W_{kn}\}$ and $\{Z_{kn}\}$ are

$$X_{kn} = \frac{(1 + \alpha^k)^n - (1 + \beta^k)^n}{\alpha - \beta}, \quad Y_{kn} = \frac{(\alpha^k - 1)^n - (\beta^k - 1)^n}{\alpha - \beta},$$

$$W_{kn} = (1 + \alpha^k)^n + (1 + \beta^k)^n, \text{ and } Z_{kn} = (\alpha^k - 1)^n + (\beta^k - 1)^n,$$

where $\alpha^k, \beta^k = (V_k \pm \sqrt{V_k^2 - 4})/2$.

Lemma 1. For $n \geq 0$, we have

$$\sum_{h=0}^n \binom{n}{h} U_{hk} = X_{nk}, \quad (2.1)$$

$$\sum_{h=0}^n \binom{n}{h} h U_{hk} = n (X_{kn} - X_{k(n-1)}), \quad (2.2)$$

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} U_{hk} = Y_{kn}, \quad (2.3)$$

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h U_{hk} = n (Y_{kn} + Y_{k(n-1)}). \quad (2.4)$$

Proof. Since

$$\sum_{h=0}^n \binom{n}{h} \alpha^{hk} = (1 + \alpha^k)^n \text{ and } \sum_{h=0}^n \binom{n}{h} \beta^{hk} = (1 + \beta^k)^n,$$

the first claim follows from the Binet formula of $\{U_{hk}\}$. Similarly by considering

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} \alpha^{hk} = (\alpha^k - 1)^n, \quad \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} \beta^{hk} = (\beta^k - 1)^n,$$

we have the third claim.

Considering

$$\begin{aligned} \sum_{h=0}^n \binom{n}{h} h \alpha^{hk} &= \frac{\alpha}{k} \cdot \frac{d}{d\alpha} \left[\sum_{h=0}^n \binom{n}{h} \alpha^{hk} \right] = \frac{\alpha}{k} \frac{d}{d\alpha} ((1 + \alpha^k)^n) \\ &= n \alpha^k (1 + \alpha^k)^{n-1} = n \left((1 + \alpha^k)^n - (1 + \alpha^k)^{n-1} \right), \end{aligned} \quad (2.5)$$

and similarly

$$\sum_{h=0}^n \binom{n}{h} h \beta^{hk} = n \left((1 + \beta^k)^n - (1 + \beta^k)^{n-1} \right), \quad (2.6)$$

one can easily obtain the rest of claimed identities. $\square$

Define the operators D_U and Δ_U on X_{kn} and Y_{kn} for $n \geq 1$, respectively, as follows:

$$\begin{aligned} D_U(X_{kn}) &= n(X_{kn} - X_{k(n-1)}), \\ \Delta_U(Y_{kn}) &= n(Y_{kn} + Y_{k(n-1)}). \end{aligned}$$

Lemma 2. For $n \geq 0$

$$\begin{aligned} \sum_{h=0}^n \binom{n}{h} V_{hk} &= W_{nk}, \\ \sum_{h=0}^n \binom{n}{h} h V_{hk} &= n(W_{kn} - W_{k(n-1)}), \\ \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} V_{hk} &= Z_{kn}, \\ \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h V_{hk} &= n(Z_{kn} + Z_{k(n-1)}). \end{aligned}$$

Proof. The proof is similar to the proof of Lemma 1. $\square$

Define the operators D_V and Δ_V on W_{kn} and Z_{kn} for $n \geq 1$, respectively, as follows:

$$\begin{aligned} D_V(W_{kn}) &= n(W_{kn} - W_{k(n-1)}), \\ \Delta_V(Z_{kn}) &= n(Z_{kn} + Z_{k(n-1)}). \end{aligned}$$

In [3], the authors stated that if $\sum_{h=0}^n h^m \binom{n}{h} U_h$ is of the form $\sum_{k \geq 0} a_k X_k$, then $\sum_{h=0}^n h^m \binom{n}{h} U_h = D(\sum_{k \geq 0} a_k X_k)$. Hence the coefficients a_k can be computed iteratively as follows.

For $m \geq 0$, define the polynomials $a_{m,r}(n)$ recursively as follows [3]:

$$a_{m,r}(n) = (n-r)a_{m-1,r}(n) - (n-r+1)a_{m-1,r-1}(n), \quad m \geq 1, \quad (2.7)$$

with the initial value $a_{0,0}(n) = 1$ and the convention that $a_{m,r}(n) = 0$ if $r < 0$ or $r > m$.

Thus, we see that a similar statement is also valid for the sum $\sum_{h=0}^n \binom{n}{h} h^m U_{hk}$, so we can give the following result.

Theorem 1. For $n \geq 0$

$$\sum_{h=0}^n \binom{n}{h} h^m U_{hk} = \sum_{r=0}^m a_{m,r}(n) X_{k(n-r)}, \quad (2.8)$$

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m U_{hk} = \sum_{r=0}^m (-1)^r a_{m,r}(n) Y_{k(n-r)}, \quad (2.9)$$

$$\sum_{h=0}^n \binom{n}{h} h^m V_{hk} = \sum_{r=0}^m a_{m,r}(n) W_{k(n-r)}, \quad (2.10)$$

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m V_{hk} = \sum_{r=0}^m (-1)^r a_{m,r}(n) Z_{k(n-r)}. \quad (2.11)$$

Proof. It is known that

$$\sum_{h=0}^n \binom{n}{h} h^m U_{hk} = D \left[\sum_{h=0}^n \binom{n}{h} h^{m-1} U_{hk} \right].$$

Thus

$$\begin{aligned} \sum_{r=0}^m a_{m,r}(n) X_{k(n-r)} &= D \left[\sum_{r=0}^{m-1} a_{m-1,r}(n) X_{k(n-r)} \right] \\ &= \sum_{r=0}^{m-1} a_{m-1,r}(n) (n-r) (X_{k(n-r)} - X_{k(n-r-1)}) \\ &= a_{m-1,0}(n) X_{kn} \\ &\quad + \sum_{r=1}^{m-1} (n-r) a_{m-1,r}(n) - (n-r+1) a_{m-1,r}(n) X_{k(n-r)} \\ &\quad - (n-m+1) a_{m-1,m-1}(n) X_{k(n-m)}. \end{aligned}$$

Since $a_{m,r}(n) = 0$ if $r < 0$ or $r > m$, we write

$$\sum_{r=0}^m a_{m,r}(n) X_{k(n-r)} = \sum_{r=0}^m (n-r) a_{m-1,r}(n) - (n-r+1) a_{m-1,r}(n) X_{k(n-r)}.$$

The recurrence for $a_{m,r}(n)$ follows directly by comparing coefficients. The rest of the claimed identities could be proved similarly. $\square$

For example, when $p = 2$ in (2.8), we have that $\alpha = \beta = 1$, so $X_n = n2^{n-1}$, which was also given in [3]. In order to get different examples of our results, suppose that $k = m = 2$, then by using the results above, we obtain

$$\sum_{h=0}^n \binom{n}{h} h^2 U_{2h} = \sum_{r=0}^2 a_{2,r}(n) X_{2(n-r)} = 2^{n-2} n^2 (n+3).$$

Now let $p = 3$, $k = 2$ and $m = 1$. Then $U_{2h} = F_{4h}$, so we have

$$Y_{2n} = \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} F_{4h} = \begin{cases} 5^{(n-1)/2} L_{2n}, & \text{if } n \text{ is odd,} \\ 5^{n/2} F_{2n}, & \text{if } n \text{ is even.} \end{cases} \quad (2.12)$$

By (2.12), we also get

$$\begin{aligned} \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h F_{4h} &= \sum_{r=0}^1 (-1)^r a_{1,r}(n) Y_{2(n-r)} \\ &= a_{1,0}(n) Y_{2n} - a_{1,1}(n) Y_{2(n-1)} \\ &= \begin{cases} n 5^{(n-1)/2} (L_{2n} + F_{2n-2}), & \text{if } n \text{ is odd,} \\ n 5^{(n-2)/2} (5F_{2n} + L_{2n-2}), & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Similar to the above examples, one can obtain various results for different values of k and p from the results above.

3. THE MAIN RESULTS

In this section, we give general formulas for the sums:

$$\begin{aligned} &\sum_{h=0}^n \binom{n}{h} h^m U_{ht}^{2m+\varepsilon}, \quad \sum_{h=0}^n \binom{n}{h} h^m V_{ht}^{2m+\varepsilon}, \\ &\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m U_{ht}^{2m+\varepsilon} \quad \text{and} \quad \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m V_{ht}^{2m+\varepsilon}, \end{aligned}$$

where t is a positive integer and $\varepsilon \in \{0, 1\}$.

We shall assume that p in the definition of the sequence $\{U_n\}$ is a positive integer different from 2 throughout in this section.

For the readers convenience and for later use, it would be convenient to recall some facts from [8]: For any real numbers m and n ,

$$(m+n)^t = \begin{cases} \sum_{i=0}^{(t-1)/2} \binom{t}{i} (mn)^i (m^{t-2i} + n^{t-2i}) & \text{if } t \text{ is odd,} \\ \sum_{i=0}^{t/2-1} \binom{t}{i} (mn)^i (m^{t-2i} + n^{t-2i}) + \binom{t}{t/2} (mn)^{t/2} & \text{if } t \text{ is even,} \end{cases} \quad (3.1)$$

and

$$(m-n)^t = \begin{cases} \sum_{i=0}^{(t-1)/2} \binom{t}{i} (mn)^i (-1)^i (m^{t-2i} - n^{t-2i}) & \text{if } t \text{ is odd,} \\ \sum_{i=0}^{t/2-1} \binom{t}{i} (mn)^i (-1)^i (m^{t-2i} + n^{t-2i}) + \binom{t}{t/2} (mn)^{t/2} (-1)^{t/2} & \text{if } t \text{ is even.} \end{cases} \quad (3.2)$$

Theorem 2. i) For $t, m > 0$,

$$\sum_{h=0}^n \binom{n}{h} h^m U_{ht}^{2m} = \frac{U_t^{2m}}{(V_t^2 - 4)^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{r=0}^m a_{m,r}(n) W_{t(2m-2i)(n-r)}$$

$$+ \frac{U_t^{2m}}{(V_t^2 - 4)^m} \binom{2m}{m} (-1)^m 2^{n-m} P(n),$$

where $P(n)$ is a monic polynomial of degree m satisfying $\sum_{h=0}^n \binom{n}{h} h^m = 2^{n-m} P(n)$.

i) For $t, m > 0$,

$$\sum_{h=0}^n \binom{n}{h} h^m U_{ht}^{2m+1} = \frac{U_t^{2m}}{(V_t^2 - 4)^m} \sum_{i=0}^m (-1)^i \binom{2m}{i} \sum_{r=0}^m a_{m,r}(n) X_{t(2m+1-2i)(n-r)}.$$

Proof. i) For $t > 0$, by the Binet formula of $\{U_n\}$, we have

$$\sum_{h=0}^n \binom{n}{h} h^m U_{ht}^{2m} = \sum_{h=0}^n \binom{n}{h} h^m \left(\frac{\alpha^{ht} - \beta^{ht}}{\alpha - \beta} \right)^{2m}.$$

Using (3.2), we write

$$\begin{aligned} & \sum_{h=0}^n \binom{n}{h} h^m U_{ht}^{2m} \\ &= \frac{1}{(\alpha - \beta)^{2m}} \sum_{h=0}^n \binom{n}{h} h^m \left[\sum_{i=0}^{m-1} \binom{2m}{i} (-1)^i (\alpha^{ht(2m-2i)} + \beta^{ht(2m-2i)}) \right. \\ & \quad \left. + \binom{2m}{m} (-1)^m \right] \\ &= \frac{1}{(p^2 - 4)^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{h=0}^n \binom{n}{h} h^m V_{ht(2m-2i)} \\ & \quad + \binom{2m}{m} (-1)^m \sum_{h=0}^n \binom{n}{h} h^m. \end{aligned}$$

By taking $k = 2t(m-i)$ in (2.10) and $\sum_{h=0}^n \binom{n}{h} h^m = 2^{n-m} P(n)$, where $P(n)$ is a monic polynomial of degree m (for the coefficients of this polynomials, see the sequence A102573 in the OEIS or see [1], p. 135), we write

$$\begin{aligned} & \sum_{h=0}^n \binom{n}{h} h^m U_{ht}^{2m} \\ &= \frac{U_t^{2m}}{(V_t^2 - 4)^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{r=0}^m a_{m,r}(n) W_{t(2m-2i)(n-r)} \end{aligned}$$

$$+ \frac{U_t^{2m}}{(V_t^2 - 4)^m} \binom{2m}{m} (-1)^m 2^{n-m} P(n).$$

ii) The proof is similar to the proof of i). □

For example, when $p = 6$, $m = 3$ and $t = 2$, we derive

$$\begin{aligned} \sum_{h=0}^n \binom{n}{h} h^3 U_{2h}^7 &= \frac{1}{32} \sum_{i=0}^3 (-1)^i \binom{6}{i} \sum_{r=0}^3 a_{3,r}(n) X_{(7-2i)(n-r)} \\ &= \frac{1}{32} \left(\sum_{r=0}^3 a_{3,r}(n) X_{7(n-r)} - 6 \sum_{r=0}^3 a_{3,r}(n) X_{5(n-r)} \right. \\ &\quad \left. + 15 \sum_{r=0}^3 a_{3,r}(n) X_{3(n-r)} - 20 \sum_{r=0}^3 a_{3,r}(n) X_{(n-r)} \right). \end{aligned}$$

For $p = 6$ in the definition of sequence $\{U_n\}$, we have $U_n = \frac{1}{2} P_{2n}$, where P_n is the n^{th} Pell number. Thus

$$X_n = \frac{1}{2} \sum_{h=0}^n \binom{n}{h} P_{2h} = U_n(\sqrt{2}), \quad (3.3)$$

where $U_n(x)$ is the Chebyshev's polynomials of the second kind. We have also another formula for X_n as shown:

$$X_n = \frac{1}{2} \sum_{h=0}^n \binom{n}{h} P_{2h} = W_{n+1}(0, 1; 8, -8),$$

where $W_n(a, b; p, q)$ is the Horadam sequence (see [2]).

Using (3.3), we get

$$\begin{aligned} &\sum_{h=0}^n \binom{n}{h} h^3 U_{2h}^7 \\ &= \frac{1}{32} \left[n^3 U_{7n}(\sqrt{2}) - n(2n^2 + 2n - 1) U_{7(n-1)}(\sqrt{2}) \right. \\ &\quad + 3n(n-1)^2 U_{7(n-2)}(\sqrt{2}) - n(n-1)(n-2) U_{7(n-3)}(\sqrt{2}) \\ &\quad - 6(n^3 U_{5n}(\sqrt{2}) - n(2n^2 + 2n - 1) U_{5(n-1)}(\sqrt{2}) \\ &\quad + 3n(n-1)^2 U_{5(n-2)}(\sqrt{2}) - n(n-1)(n-2) U_{5(n-3)}(\sqrt{2})) \\ &\quad + 15(n^3 U_{3n}(\sqrt{2}) - n(2n^2 + 2n - 1) U_{3(n-1)}(\sqrt{2}) \\ &\quad \left. + 3n(n-1)^2 U_{3(n-2)}(\sqrt{2}) - n(n-1)(n-2) U_{3(n-3)}(\sqrt{2})) \right] \end{aligned}$$

$$-20 \left(n^3 U_n(\sqrt{2}) - n(2n^2 + 2n - 1) U_{(n-1)}(\sqrt{2}) \right. \\ \left. + 3n(n-1)^2 U_{(n-2)}(\sqrt{2}) - n(n-1)(n-2) U_{(n-3)}(\sqrt{2}) \right).$$

Theorem 3. i) For $t, m > 0$,

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m U_{ht}^{2m} \\ = \frac{U_t^{2m}}{(V_t^2 - 4)^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{r=0}^m (-1)^r a_{m,r}(n) Z_{2t(m-i)(n-r)} \\ + \frac{U_t^{2m}}{(V_t^2 - 4)^m} \binom{2m}{m} (-1)^m n! S(m, n),$$

where $S(m, n)$ is the Stirling numbers of the second kind.

ii) For $t, m > 0$,

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m U_{ht}^{2m+1} \\ = \frac{U_t^{2m}}{(V_t^2 - 4)^m} \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \sum_{r=0}^m (-1)^r a_{m,r}(n) Y_{t(2m+1-2i)(n-r)}.$$

Proof. i) For $t > 0$, consider

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m U_{ht}^{2m} = \sum_{h=0}^n \binom{n}{h} (-1)^h h^m \left(\frac{\alpha^{ht} - \beta^{ht}}{\alpha - \beta} \right)^{2m}.$$

By (3.2), we write

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m U_{ht}^{2m} \\ = \frac{1}{(\alpha - \beta)^{2m}} \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m \\ \left[\sum_{i=0}^{m-1} \binom{2m}{i} (-1)^i (\alpha^{ht(2m-2i)} + \beta^{ht(2m-2i)}) + \binom{2m}{m} (-1)^m \right] \\ = \frac{1}{(\alpha - \beta)^{2m}} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m V_{ht(2m-2i)}$$

$$+ \frac{1}{(\alpha - \beta)^{2m}} \binom{2m}{m} (-1)^m \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m.$$

By taking $k = 2t(m - i)$ in (2.11) and since

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m = n! S(m, n)$$

where $S(m, n)$ is the Stirling numbers of the second kind, the claim is obtained.

ii) The proof is similar to the proof of *i)*. □

For example, when $p = 3$, we get $U_n = F_{2n}$. For $m = t = 2$, we have

$$\begin{aligned} & \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^2 U_{2h}^5 \\ &= \frac{1}{5^2} \sum_{i=0}^2 (-1)^i \binom{5}{i} \sum_{r=0}^2 (-1)^r a_{2,r}(n) Y_{2(5-2i)(n-r)} \\ &= \frac{1}{5^2} \left(\sum_{r=0}^2 (-1)^r a_{2,r}(n) Y_{10(n-r)} - 5 \sum_{r=0}^2 (-1)^r a_{2,r}(n) Y_{6(n-r)} \right. \\ & \quad \left. + 10 \sum_{r=0}^2 (-1)^r a_{2,r}(n) Y_{2(n-r)} \right). \end{aligned}$$

Since

$$Y_n = \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} F_{2h} = F_n,$$

we get

$$\begin{aligned} \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^2 U_{2h}^5 &= \frac{1}{5^2} [n F_{10n} + n(2n-1) F_{10(n-1)} + n(n-1) F_{10(n-2)} \\ & \quad - 5(n F_{6n} + n(2n-1) F_{6(n-1)} + n(n-1) F_{6(n-2)}) \\ & \quad + 10(n F_{2n} + n(2n-1) F_{2(n-1)} + n(n-1) F_{2(n-2)})]. \end{aligned}$$

Theorem 4. *i)* For $t, m > 0$,

$$\sum_{h=0}^n \binom{n}{h} h^m V_{ht}^{2m} = \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{r=0}^m a_{m,r}(n) W_{t(2m-2i)(n-r)} + \binom{2m}{m} 2^{n-m} P(n),$$

where the polynomial $P(n)$ is defined as before.

ii) For $t, m > 0$,

$$\sum_{h=0}^n \binom{n}{h} h^m V_{ht}^{2m+1} = \sum_{i=0}^m \binom{2m+1}{i} \sum_{r=0}^m a_{m,r}(n) W_{t(2m+1-2i)(n-r)}.$$

Proof. i) For $t > 0$, by the Binet formula of $\{V_n\}$, we write

$$\sum_{h=0}^n \binom{n}{h} h^m V_{ht}^{2m} = \sum_{h=0}^n \binom{n}{h} h^m (\alpha^{ht} + \beta^{ht})^{2m},$$

which, by (3.1) and since $\alpha\beta = 1$, satisfies

$$\begin{aligned} \sum_{h=0}^n \binom{n}{h} h^m V_{ht}^{2m} &= \sum_{h=0}^n \binom{n}{h} h^m \left[\sum_{i=0}^{m-1} \binom{2m}{i} (\alpha^{ht(2m-2i)} + \beta^{ht(2m-2i)}) \right. \\ &\quad \left. + \binom{2m}{m} (\alpha\beta)^{thm} \right] \\ &= \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{h=0}^n \binom{n}{h} h^m V_{ht(2m-2i)} + \binom{2m}{m} \sum_{h=0}^n \binom{n}{h} h^m. \end{aligned}$$

By taking $k = 2t(m-i)$ in (2.10) and since $\sum_{h=0}^n \binom{n}{h} h^m = 2^{n-m} P(n)$, where $P(n)$ is defined as before, we get

$$\sum_{h=0}^n \binom{n}{h} h^m V_{ht}^{2m} = \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{r=0}^m a_{m,r}(n) W_{t(2m-2i)(n-r)} + \binom{2m}{m} 2^{n-m} P(n).$$

ii) The proof is similar to the proof of i). □

Theorem 5. i) For $t, m > 0$,

$$\begin{aligned} &\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m V_{ht}^{2m} \\ &= \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{r=0}^m (-1)^r a_{m,r}(n) Z_{t(2m-2i)(n-r)} + \binom{2m}{m} n! S(m, n), \end{aligned}$$

where $S(m, n)$ is the Stirling numbers of the second kind.

ii) For $t, m > 0$,

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m V_{ht}^{2m+1}$$

$$= \sum_{i=0}^m \binom{2m+1}{i} \sum_{r=0}^m (-1)^r a_{m,r}(n) Z_{t(2m+1-2i)(n-r)}.$$

Proof. i) From the Binet formula of $\{V_n\}$, we write

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m V_{ht}^{2m} = \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m (\alpha^{ht} + \beta^{ht})^{2m},$$

which, by (3.1) and since $\alpha\beta = 1$, is equivalent to

$$\begin{aligned} & \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m V_{ht}^{2m} \\ &= \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m \left(\sum_{i=0}^{m-1} \binom{2m}{i} (\alpha^{ht(2m-2i)} + \beta^{ht(2m-2i)}) \right. \\ & \quad \left. + \binom{2m}{m} (\alpha\beta)^{thm} \right) \\ &= \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m V_{ht(2m-2i)} \\ & \quad + \binom{2m}{m} \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m. \end{aligned}$$

By taking $k = 2t(m-i)$ in (2.11) and since

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m = n! S(m, n),$$

where $S(m, n)$ is defined as before, the claim is obtained.

ii) The proof is similar to the proof of i). □

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Authors’ addresses

Emrah Kılıç

TOBB Economics and Technology University, Mathematics Department, 06560 Ankara, Turkey

E-mail address: ekilic@etu.edu.tr

Yücel Türker Ulutaş

Kocaeli University, Mathematics Department, 41380 Izmit, Turkey

E-mail address: turkery@kocaeli.edu.tr

Neşe Ömür

Kocaeli University, Mathematics Department, 41380 Izmit, Turkey

E-mail address: neseomur@kocaeli.edu.tr



EXISTENCE RESULTS FOR SOME NONLINEAR INTEGRAL EQUATIONS

MONICA LAURAN

Received 30 June, 2011

Abstract. In this paper we shall establish sufficient conditions for the existence of solutions of the integral equation of Volterra type and for its solvability in Banach space and C_L . The main tools used in our study are the nonexpansive operator technique, contraction principle and Schaefer's fixed point theorem.

2000 *Mathematics Subject Classification:* 45B05; 45D05; 47H10

Keywords: nonlinear integral equation, existence of solutions, nonexpansiv mapping

1. INTRODUCTION

The theory of integral equations has many applications in describing numerous events and problems of the real world. For example, integral equations are often applicable in engineering, mathematical physics, economics and biology. It is a well known fact that the nonlinear quadric equations are often encountered in various applications.

Fixed point theorems used in nonlinear functional analysis allows us, in general, to obtain existence theorems concerning the investigated functional-operator equations. In this paper we study the existence of solutions of nonlinear integral equation of Volterra type by using the concepts of nonexpansive operators, contraction principles and the Schaefer's fixed point theorem. The result generalize previous results of [1–3, 6, 7].

2. PRELIMINARIES

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper. By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J to $\mathbb{R}$ with norm

$$\|y\| := \sup\{|y(t)| : t \in J\}$$

We can now formulate one of the most important fixed point theorems that will used throughout the paper. These theorems can be found in papers such as [3, 5, 8].

Definition 1 ([3]). Let (X, d) be a metric space and $B \subseteq C(X, \mathbb{R})$. B is a equicontinuous means if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in B, x, a \in X$,

$$d(x, a) < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Theorem 1 (Arzelá-Ascoli, [5]). Let (X, d) be a compact metric space and let $B \subseteq C(X, \mathbb{R})$. Then B is compact if and only if B is closed, bounded and equicontinuous.

Theorem 2 (Schaefer, [5]). Let X be a normed linear space. If $A : X \rightarrow X$ is a completely continuous map, then either the subset

$$\{x \in X : x = \lambda Ax, \text{ for some } 0 \leq \lambda \leq 1\}$$

is bounded or A has a fixed point.

Theorem 3 (Schaefer, [5]). Let X be a Banach space and $K \subset X$ a nonempty, convex, compact subset of X . If $T : K \rightarrow K$ is a continuous operator, then T has at least one fixed point in K .

3. MAIN RESULTS

Consider the following nonlinear functional-integral equation

$$x(t) = g(t, x(t)) + h(t, x(t)) \int_0^T f(t, s, x(s)) ds, \quad T \geq 0. \quad (3.1)$$

Let $J = [0, T]$ and

$$C_L = \{x \in C(J, \mathbb{R}) \mid |x(t_1) - x(t_2)| \leq L \cdot |t_1 - t_2|, \\ (\forall) t_1, t_2 \in \mathbb{R}_+, L > 0\} \quad (3.2)$$

Our first result is based on the Banach fixed point theorem.

Theorem 4. Assume that

(H₁) $f : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ $g, h : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ are given continuous functions.

(H₂) There exists a constant $L_1 > 0$ such that

$$|g(t, x) - g(t, y)| \leq L_1 \cdot |x - y|, \quad (\forall) t \in J, x, y \in \mathbb{R}.$$

(H₃) There exists a constant $L_2 > 0$ such that

$$|h(t, x) - h(t, y)| \leq L_2 \cdot |x - y|, \quad (\forall) t \in J, x, y \in \mathbb{R}.$$

(H₄) Then exists a constant $L_3 > 0$ such that

$$|f(t, s, x) - f(t, s, y)| \leq L_3 \cdot |x - y|, \quad (\forall) t, s \in J, x, y \in \mathbb{R}.$$

If

$$\sup_{t \geq 0} \left\{ L_1 + L_2 \cdot \int_0^T |f(t, s, x(s))| ds + L_3 \cdot T \cdot |h(t, x(t))| \right\} \leq k < 1, \quad (\forall) t \in J(*)$$

then equation 3.1 has a unique solution on $J = [0, T]$.

Proof. Transform the equation 3.1 into a fixed point problem.
Consider the operator

$$F : C(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$$

defined by

$$F(x)(t) = g(t, x(t)) + h(t, x(t)) \int_0^T f(t, s, x(s)) ds.$$

The fixed point of the operator F are solution of equation 3.1. We shall use the Banach contraction principle to prove that F has a fixed point.

Let $x, y \in C(J, \mathbb{R})$. We have

$$\begin{aligned} |(Fx)(t) - (Fy)(t)| &\leq |g(t, x(t)) - g(t, y(t))| + \\ &\left| h(t, x(t)) \int_0^T f(t, s, x(s)) ds - h(t, y(t)) \int_0^T f(t, s, y(s)) ds + \right. \\ &\left. + h(t, x(t)) \int_0^T f(t, s, y(s)) ds - h(t, x(t)) \int_0^T f(t, s, y(s)) ds \right| \leq \\ &\leq L_1 |x(t) - y(t)| + |h(t, x(t))| \cdot \int_0^T |f(t, s, x(s)) - f(t, s, y(s))| ds + \\ &+ |h(t, x(t)) - h(t, y(t))| \cdot \int_0^T |f(t, s, y(s))| ds \leq L_1 |x(t) - y(t)| + \\ &+ L_3 \cdot |h(t, x(t))| \cdot \int_0^t |x(s) - y(s)| ds + L_2 \cdot |x(t) - y(t)| \cdot \int_0^T |f(t, s, y(s))| ds. \end{aligned}$$

Thus

$$\|Fx - Fy\| \leq \left(L_1 + L_2 \cdot \int_0^T |f(t, s, y(s))| ds + L_3 \cdot T \cdot |h(t, x(t))| \right) \cdot \|x - y\|.$$

Consequently F is a contraction. As a consequence of the Banach fixed point theorem, we deduce that F has a fixed point which is a solution of equation 3.1. $\square$

The second result is based on Schaefer's fixed point theorem.

Theorem 5. Assume the following conditions in equation 3.1:
(H₁). The functions

$$\begin{aligned} g &: \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \\ h &: \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \\ f &: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \end{aligned}$$

are continuous.

(H₂). If L is the Lipschitz constant involved in 3.2, then

$$\left| \int_0^T f(t_1, s, x(s)) ds - \int_0^T f(t_2, s, x(s)) ds \right| \leq L \cdot |t_1 - t_2|,$$

for each $(\forall) t_1, t_2, s \in J$ and $x \in \mathbb{R}$ and

$$Q = \max_{t \in J} \left\{ \left| h(t, x(t)) \int_0^T f(t, s, x(s)) ds \right| \right\}, (\forall) t, s \in J \text{ and } x \in \mathbb{R}.$$

(H₃). There exists the constants $m, M, p \in \mathbb{R}$ such that

$$\begin{aligned} |g(t, x) - g(t, y)| &\leq m \cdot |x - y| \text{ for each } x, y \in \mathbb{R} \text{ and } t \in J, \\ |g(t, x) - g(s, x)| &\leq M \cdot L \cdot |t - s| \text{ and } (\forall) t, s \in J, x \in \mathbb{R}, \\ |g(t, x(t))| &\leq p, (\forall) t, s \in J, x \in \mathbb{R}. \end{aligned}$$

(H₄). There exists the constants $n, N \in \mathbb{R}$ such that

$$\begin{aligned} |h(t, x) - h(t, y)| &\leq n \cdot |x - y| \text{ } (\forall) x, y \in \mathbb{R} \text{ and } t \in J, \\ |h(t, x) - h(s, x)| &\leq N \cdot L \cdot |t - s| \text{ } (\forall) t, s \in J \text{ and } x \in \mathbb{R} \end{aligned}$$

(H₅). There exist the constant $k \in (0, 1]$ such that

$$\sup_{t \in J} \left\{ m + M + (n + N) \int_0^T |f(t, s, x(t))| ds + |h(t, x(t))| \right\} \leq k.$$

Then the equation 3.1 has at least one solution on C_L .

Proof. It is known, (see Lemma 1 in [4]), that C_L is nonempty and convex, moreover, it is a compact subset of the Banach space $(C[a, b], \|\cdot\|)$, where $\|\cdot\|$ is the usual supremum norm.

Now we transform equation 3.1 into a fixed point problem.

Consider the integral operator

$$F : C_L \rightarrow C(J, \mathbb{R})$$

$$(Fx)(t) = g(t, x(t)) + h(t, x(t)) \int_0^T f(t, s, x(s)) ds.$$

Clearly, the fixed points of the operator F are solutions of equation 3.1. We shall use Schaefer's fixed point theorem to prove that F has a fixed point.

Step 1. F is continuous

Let $\{x_n\}$ be a sequence such that $x_n \rightarrow x$ in $C(J, \mathbb{R})$. Then for each $t \in J$

$$\begin{aligned} & |F(x_n)(t) - F(x)(t)| \\ & \leq |g(t, x_n(t)) - g(t, x(t))| + |h(t, x_n(t))| \cdot \int_0^T |f(t, s, x_n(s)) - f(t, s, x(s))| ds + \\ & \quad + |h(t, x_n(t)) - h(t, x(t))| \cdot \int_0^T |f(t, s, x(s))| ds. \end{aligned}$$

Since f, g and h are continuous functions, we have

$$\|Fx_n - Fx\| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Step 2. F maps bounded sets in C_L into bounded sets in $C(J, \mathbb{R})$.

Indeed, it is enough to show for any $\eta > 0$, there exists a positive constant l such that for each $x \in B_\eta = \{x \in C_L : \|x\| \leq \eta\}$, we have $\|Fx\| \leq l$.

By (H₃) - (H₄) we have for each $t \in J$

$$|(Fx)(t)| \leq |g(t, x(t))| + \left| h(t, x(t)) \int_0^T f(t, s, x(s)) ds \right| \leq p + Q.$$

Thus

$$\|Fx\| \leq p + Q := l.$$

Step 3. F maps bounded sets into equicontinuous sets of C_L .

$$\begin{aligned} & |(Fx)(t_1) - (Fx)(t_2)| \leq |g(t_1, x(t_1)) - g(t_2, x(t_2))| + \\ & + \left| h(t_1, x(t_1)) \int_0^T f(t_1, s, x(s)) ds - h(t_2, x(t_2)) \int_0^T f(t_2, s, x(s)) ds \right| \leq \\ & \leq |g(t_1, x(t_1)) - g(t_2, x(t_1))| + |g(t_2, x(t_1)) - g(t_2, x(t_2))| + \end{aligned}$$

$$\begin{aligned}
& + |h(t_1, x(t_2)) - h(t_2, x(t_1))| \cdot \int_0^T |f(t_1, s, x(s))| ds + \\
& + |h(t_2, x(t_1))| \cdot \left| \int_0^T f(t_1, s, x(s)) ds - \int_0^T f(t_2, s, x(s)) ds \right| \leq \\
& \leq M \cdot L \cdot |t_1 - t_2| + m \cdot |x(t_1) - x(t_2)| + \\
& + |h(t_1, x(t_2)) - h(t_1, x(t_1))| \cdot \int_0^T |f(t_1, s, x(s))| ds + \\
& + |h(t_1, x(t_1)) - h(t_2, x(t_1))| \cdot \int_0^T |f(t_1, s, x(s))| ds + \\
& + |h(t_2, x(t_1))| \cdot \left| \int_0^T f(t_1, s, x) ds - \int_0^T f(t_2, s, x) ds \right| \leq \\
& \leq M \cdot L \cdot |t_1 - t_2| + m \cdot L \cdot |t_1 - t_2| + \\
& + (N \cdot L + n \cdot L) \cdot |t_1 - t_2| \cdot \int_0^T |f(t_1, s, x)| ds + \\
& + |h(t_2, x(t_1))| \cdot L \cdot |t_1 - t_2| = \\
& = L \cdot |t_1 - t_2| \left[m + M + (n + N) \int_0^T |f(t_1, s, x)| ds + |h(t_2, x(t_1))| \right] \leq \\
& \leq L \cdot |t_1 - t_2|
\end{aligned}$$

So $Fx \in C_L$, $(\forall) x \in C_L$. Therefore $T : C_L \rightarrow C_L$.

As $t_1 \rightarrow t_2$, the right-hand side of the above inequality tends to zero. As a consequence of steps 1 to 3, by using the Arzelà-Ascoli theorem, we can conclude that $F : C_L \rightarrow C_L$ is continuous and completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set

$$\varepsilon = \{x \in C_L : x = \lambda Fx \text{ for some } 0 \leq \lambda \leq 1\}$$

is bounded.

Let $x \in \varepsilon$ then $x = \lambda Fx$ for some $0 \leq \lambda \leq 1$

$$|x(t)| \leq |g(t, x(t))| + \left| h(t, x(t)) \int_0^T f(t, s, x(s)) ds \right| \leq p + Q := R$$

As a consequence of Schaefer's fixed point theorem we deduce that F has a fixed point which is a solution of equation 3.1. $\square$

In the following theorem we shall give an existence result for equation 3.1 by means of Schauder's fixed point theorem.

Theorem 6. Assume that (H_1) , (H_3) , (H_4) in theorem 5 hold and (H'_2) There exists a constant L' such that

$$|f(t, s, u) - f(t, s, v)| \leq L' \cdot |u - v|, \text{ for each } t, s \in J, u, v \in \mathbb{R}.$$

If

$$\sup_{t \geq 0} \left\{ m + n \int_0^t |f(t, s, x(s))| ds + t \cdot L' \cdot |h(t, x(t))| \right\} \leq 1$$

or

$$\lim_{t \rightarrow \infty} t \cdot |h(t, x(t))| = 0 \text{ uniformly with respect to } x \in C_L,$$

then the equation 3.1 has at least one solution in C_L .

Proof. Using the proof of theorem 4 and theorem 5, one can prove that F is non-expansive. Applying the Schauder's fixed point theorem, we deduce that F has at least one fixed point which is a solution of equation 3.1. $\square$

4. AN EXAMPLE

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following nonlinear integral equation

$$x(t) = \frac{1}{t^2 + 9} \cdot x(t) + \cos(tx(t)) \int_0^T \frac{e^{-t}}{1 + e^t} \cdot \frac{|x(s)|}{1 + |x(s)|} ds \quad (4.1)$$

In this case we have

$$g : \mathbb{R}_+ \rightarrow \mathbb{R}, \quad g(t, x(t)) = \frac{1}{t^2 + 9} \cdot x(t), \text{ for all } t \in J,$$

$$h : \mathbb{R}_+ \rightarrow \mathbb{R}, \quad h(t, x(t)) = \cos(tx(t)), \text{ for all } t \in J,$$

and

$$f : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}, \quad f(t, s, x(s)) = \frac{e^{-t}}{1 + e^t} \cdot \frac{|x(s)|}{1 + |x(s)|}, \text{ for all } t, s \in J.$$

We are interested to study the solution $x(t)$ belonging to

$$C_1 = \{x \in C(J, J) : |x(t_1) - x(t_2)| \leq |t_1 - t_2|, (\forall) t_1, t_2 \in J\},$$

which means $L = 1$. We have $L_1 = \frac{1}{9}, L_2 = \frac{1}{2}, L_3 = \frac{1}{2}$ in Theorem 4.

If $L_1 < k < 1$ in 4, the equation 4.1 has a unique solution in $J = [0, T]$.

For T , it is especially convenient to set $m = \frac{1}{9}, M = \frac{1}{36}, Q = \frac{1}{2}, n = \frac{1}{2}, N = \frac{1}{2}$ and $L' = \frac{1}{2}$ in theorem 6. Then equation 4.1 has at least one solution in C_1 .

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Author's address

Monica Luran

North University of Baia Mare, Department of Mathematics and Computer Science, Victoriei 76, 430122 Baia Mare, Romania

E-mail address: lauranmonica@yahoo.com

**2-ABSORBING AND n -WEAKLY PRIME SUBMODULES**

S. MORADI AND A. AZIZI

Received 4 May, 2011

Abstract. Let R be a commutative ring with identity, and let $n > 1$ be an integer. A proper submodule N of an R -module M will be called 2-absorbing [resp. n -weakly prime], if $r, s \in R$ and $x \in M$ with $rsx \in N$ [resp. $rsx \in N \setminus (N : M)^{n-1}N$] implies that $rs \in (N : M)$ or $rx \in N$, or $sx \in N$. These concepts are generalizations of the notions of 2-absorbing ideals and weakly prime submodules, which have been studied in [3, 4, 6, 7]. We will study 2-absorbing and n -weakly prime submodules in this paper. Among other results, it is proved that if $(N : M)^{n-1}N \neq (N : M)^2N$, then N is 2-absorbing if and only if it is n -weakly prime.

2000 *Mathematics Subject Classification:* 13A10; 13C99; 13E05; 13F05; 13F15

Keywords: n -weakly prime submodule, prime submodule, primary submodule, 2-absorbing submodule, Weakly prime submodule

1. INTRODUCTION

Throughout this paper all rings are commutative with identity and all modules are unitary. Also we take R as a commutative ring with identity, M as an R -module, and $n > 1$ is a positive integer.

Let N be a submodule of M . The ideal $\{r \in R \mid rM \subseteq N\}$ is denoted by $(N : M)$.

It is said that a proper submodule N of M is *prime* if for $r \in R$ and $a \in M$ with $ra \in N$, either $a \in N$ or $r \in (N : M)$. If N is a prime submodule of M , then one can easily see that $P = (N : M)$ is a prime ideal of R , and we say N is a P -prime submodule. Prime submodules have been studied extensively in many papers (see, for example, [2], [4], [3]), so studying its generalization can be helpful in the amplification of this theory.

As a generalization of prime submodules, a proper submodule N of M is called *weakly prime*, if $r, s \in R$ and $x \in M$ with $rsx \in N$ implies that $rx \in N$ or $sx \in N$ (see [3, 4, 7]).

In this paper, we will introduce and study two generalizations of weakly prime submodules.

2. 2-ABSORBING SUBMODULES

According to [6] an ideal I of a ring R is called *2-absorbing*, if $abc \in I$ for $a, b, c \in I$ implies that $ab \in I$ or $bc \in I$ or $ac \in I$.

A generalization of weakly prime submodules, which is also a module version of 2-absorbing ideals, is introduced as follows:

Definition 1. A proper submodule N of M will be called 2-absorbing if for $r, s \in R$ and $x \in M$, $rsx \in N$ implies that $rs \in (N : M)$ or $rx \in N$ or $sx \in N$.

Lemma 1 (Theorem 2.1, Theorem 2.4, and Theorem 2.5 in [6]). *Let I be a 2-absorbing ideal of R with $\sqrt{I} = J$. Then*

- (1) J is a 2-absorbing ideal of R with $J^2 \subseteq I \subseteq J = \{r \in R \mid r^2 \in I\}$.
- (2) $\{(I : r)\}_{r \in J \setminus I}$ is a chain of prime ideals.
- (3) Either J is a prime ideal of R , or $J = P_1 \cap P_2$ with $P_1 P_2 \subseteq I$, where P_1, P_2 are the only distinct prime ideals of R , which are minimal over I .

For each $r \in R$ and every submodule N of M , we consider $N_r = (N :_M r) = \{x \in M \mid rx \in N\}$.

Part (ii) of the following lemma proves that 2-absorbing submodules are not too far from prime submodules.

Proposition 1. *Let N be a 2-absorbing submodule of M with $\sqrt{(N : M)} = J$. Then*

- (i) $(N : M)$ and J are 2-absorbing ideals of R . Furthermore $J^2 \subseteq (N : M) \subseteq J = \{r \in R \mid r^2 \in (N : M)\}$.
- (ii) If $(N : M) \neq J$, then for every $r \in J \setminus (N : M)$, N_r is a prime submodule containing N with $J \subseteq (N_r : M)$. Moreover $\{(N_r : M)\}_{r \in J \setminus (N : M)}$ is a chain of prime ideals.
- (iii) Either J is a prime ideal of R , or $J = P_1 \cap P_2$, where P_1, P_2 are the only distinct minimal prime ideals over $(N : M)$ and $P_1 P_2 \subseteq (N : M)$.

Proof. (i) Let $s, t, r \in R$ with $str \in (N : M)$. If $sr, tr \notin (N : M)$, then there exist $x, y \in M \setminus N$ such that $srx, try \notin N$.

Since $st(r(x + y)) \in N$ and N is 2-absorbing, $st \in (N : M)$ or $sr(x + y) \in N$ or $tr(x + y) \in N$. If $sr(x + y) \in N$, then since $srx \notin N$, we have $sry \notin N$. So as $st(ry) \in N$ and $try \notin N$, $st \in (N : M)$.

Similarly in case $tr(x + y) \in N$, we get $st \in (N : M)$.

Now since $(N : M)$ is a 2-absorbing ideal, by Lemma 1(1), J is also a 2-absorbing ideal with $J^2 \subseteq (N : M) \subseteq J = \{r \in R \mid r^2 \in (N : M)\}$.

(ii) To prove that N_r is a prime submodule, let $sx \in N_r$, where $s \in R \setminus (N_r : M)$ and $x \in M$. Then by the definition of N_r , $srx \in N$ and as N is 2-absorbing, $rs \in (N : M)$ or $rx \in N$ or $sx \in N$.

If $rs \in (N : M)$, then $srM \subseteq N$, that is $s \in (N_r : M)$, which is a contradiction. If $rx \in N$, then $x \in N_r$ by the definition of N_r , which completes the proof.

Now suppose $sx \in N$. By part (i), $r^2 \in J^2 \subseteq (N : M)$, so $rM \subseteq N_r$, particularly $rx \in N_r$. Then $(r+s)x \in N_r$, that is $r(r+s)x \in N$, and since N is 2-absorbing, $rx \in N$ or $(r+s)x \in N$ or $r(r+s) \in N$.

If $rx \in N$, then $x \in N_r$, which completes the proof. Also if $(r+s)x \in N$, then from $sx \in N$, again we get $rx \in N$ and so $x \in N_r$.

Now assume $r(r+s) \in (N : M)$. According to part (i), $r^2 \in J^2 \subseteq (N : M)$, hence $rs \in (N : M)$, and so $s \in (N_r : M)$. Whence N_r is a prime submodule of M .

One can easily see that $((N : M) : r) = (N_r : M)$. By part (i), $rJ \subseteq J^2 \subseteq (N : M)$, so $J \subseteq ((N : M) : r) = (N_r : M)$.

For the proof of the rest of this part note that by part (i), $(N : M)$ is a 2-absorbing ideal. Hence by Lemma 1(2), $\{((N : M) : r)\}_{r \in J \setminus (N : M)}$ is a chain of prime ideals and $(N_r : M) = ((N : M) : r)$.

(iii) By part (i), $(N : M)$ is a 2-absorbing ideal, so the proof is clear by Lemma 1(3). $\square$

Let S be a multiplicatively closed subset of R , and W a submodule of $S^{-1}M$ as $S^{-1}R$ -module. We consider $W^c = \{x \in M \mid \frac{x}{1} \in W\}$.

The proof of the following lemma is easy and we leave it to the reader.

Lemma 2. *Let N be an 2-absorbing submodule of M , and S a multiplicatively closed subset of R .*

- (i) *If $S^{-1}N \neq S^{-1}M$, then $S^{-1}N$ is a 2-absorbing submodule of $S^{-1}M$.*
- (ii) *If W is a 2-absorbing submodule of a $S^{-1}R$ -module $S^{-1}M$, then W^c is a 2-absorbing submodule of M .*

Lemma 3 (Proposition 1 in [9]). *Let S be a multiplicatively closed subset of R . If N is a P -prime submodule of M such that $(N : M) \cap S = \emptyset$, then $S^{-1}N$ is a prime submodule of $S^{-1}M$ as an $S^{-1}R$ -module.*

Let N be a 2-absorbing submodule of M with $(N : M) \neq \sqrt{(N : M)}$. Then evidently $(N_r : M) = ((N : M) : r)$, so

according to Proposition 1(ii), $\mathfrak{P} = \cap \{((N : M) : r) \mid r \in \sqrt{(N : M)} \setminus (N : M)\}$ is a prime ideal. In this case we say $\mathfrak{P}$ is the prime ideal related to N .

Corollary 1. *Let N be a 2-absorbing submodule of M with $(N : M) \neq \sqrt{(N : M)}$ and $\dim R < \infty$. Suppose S is a multiplicatively closed subset of R , and $\mathfrak{P}$ is the prime ideal related to N .*

- (i) *If $S \cap \mathfrak{P} = \emptyset$, then $S^{-1}N$ is a 2-absorbing submodule of $S^{-1}M$.*
- (ii) *$N_{\mathfrak{P}}$ is a 2-absorbing submodule of the $R_{\mathfrak{P}}$ -module $M_{\mathfrak{P}}$.*

Proof. (i) By Lemma 2(i), it is enough to prove that $S^{-1}M \neq S^{-1}N$. According to Proposition 1(ii), $\{(N_r : M)\}_{r \in \sqrt{(N : M)} \setminus (N : M)}$ is a chain of prime ideals, and since $\dim R < \infty$, this chain has a minimal element, say $(N_{r_0} : M)$. Now since $(N_r : M) = ((N : M) : r)$ for each $r \in \sqrt{(N : M)} \setminus (N : M)$, by our assumption we get

$S \cap (N_{r_0} : M) = S \cap \mathfrak{P} = \emptyset$. Now according to Proposition 1(ii) and Lemma 3, $S^{-1}N_{r_0}$ is a prime submodule of $S^{-1}M$ containing $S^{-1}N$. Hence $S^{-1}N \neq S^{-1}M$.

(ii) The proof is clear by part (i). $\square$

Lemma 4. *Let N be an P -primary submodule of M . Then N is 2-absorbing if and only if $P^2 \subseteq (N : M)$. In particular for every maximal submodule K of M , $(K : M)^2$ is a 2-absorbing ideal of R .*

Proof. If N is 2-absorbing, then by Proposition 1(i), $P^2 \subseteq (N : M)$.

For the converse suppose that $rsx \in N$ for some $r, s \in R$ and $x \in M$. If $rx, sx \notin N$, then since N is P -primary, $r, s \in P$ and so $rs \in P^2 \subseteq (N : M)$. Therefore N is 2-absorbing. $\square$

Example 1. Let $\mathfrak{M}$ be a maximal ideal of R .

- (a) Evidently, every weakly prime submodule is 2-absorbing. In particular if $\{P_i\}_{i \in \mathbb{N}}$ is a chain of prime ideals, then it is easy to see that for the free R -module $\oplus_{i \in \mathbb{N}} R$, the submodule $\oplus_{i \in \mathbb{N}} P_i$ is 2-absorbing.
- (b) Let F be a faithfully flat R -module. Then $\mathfrak{M}F$ and $\mathfrak{M}^2F$ are 2-absorbing submodules, particularly if F is a free module, or a projective module over an integral domain.
- (c) Let R be a Noetherian domain which is not a field. If F is a free R -module, then $\mathfrak{M}^k F$ is a primary submodule for $2 < k \in \mathbb{N}$, but it is not 2-absorbing.
- (d) Let R be a Dedekind domain which is not a field. If F is a free R -module, then $\mathfrak{M}^2 F$ is a 2-absorbing submodule but it is not weakly prime.
- (e) If R is a unique factorization domain and p is an irreducible element of R , then for the free R -module $R \oplus R$, the submodule $N = Rp \oplus Rp^2$ is 2-absorbing, but it is not weakly prime.

Proof. (a) The proof is easy, so it is omitted.

(b) Since F is faithfully flat, $\mathfrak{M}F$ and $\mathfrak{M}^2F$ are proper submodules of F . Clearly $\sqrt{(N : F)} = \mathfrak{M}$, where $N = \mathfrak{M}^k F$ for $k \in \mathbb{N}$. Then N is a primary submodule, since $\sqrt{(N : F)}$ is a maximal ideal. Evidently $\mathfrak{M}^2 \subseteq (\mathfrak{M}^2 F : F)$ and $\mathfrak{M}^2 \subseteq (\mathfrak{M}F : F)$, so by Lemma 4, the submodules $\mathfrak{M}F$ and $\mathfrak{M}^2F$ are 2-absorbing.

(c) It is easy to see that in case F is a free module, $(IF : F) = I$ for each ideal I of R . As it was proved in part (b), $\mathfrak{M}^k F$ is a primary submodule. However, if $\mathfrak{M}^k F$ is 2-absorbing, then $\mathfrak{M}^2 \subseteq (\mathfrak{M}^k F : F) = \mathfrak{M}^k \subseteq \mathfrak{M}^2$ according to Lemma 4. Thus $\mathfrak{M}^2 = \mathfrak{M}^k$. Now by Nakayama's lemma, there exists $r \in R$ such that $r\mathfrak{M}^2 = 0$ and $r - 1 \in \mathfrak{M}^{k-2}$. Then either $r = 0$, or $\mathfrak{M} = 0$, and both are impossible.

(d) Note that for every weakly prime submodule N of a module M , the ideal $(N : M)$ is prime. Although $(\mathfrak{M}^2 F : F) = \mathfrak{M}^2$ is not a prime ideal, consequently $\mathfrak{M}^2 F$ is not weakly prime.

(e) A straightforward calculation shows that N is 2-absorbing. But N is not weakly prime, because $p.p(1, 1) \in N$, however $p(1, 1) \notin N$. $\square$

Lemma 5 (Lemma 4 in [5]). *Let M be a finitely generated R -module and B a submodule of M . If $(B : M) \subseteq P$, where P is a prime ideal of R , then there exists a P -prime submodule N of M containing B .*

Let P be a prime ideal of R . For simplification, we denote the submodule $((P^2)_P M_P)^c$ of M by $P^{(2)}M$.

The following corollary supplies abundant examples of 2-absorbing submodules.

Corollary 2. *Let P be a prime ideal of R . If one of the following holds, then $P^{(2)}M$ is 2-absorbing.*

- (i) $(P^2)_P M_P \neq M_P$.
- (ii) M is finitely generated and $\text{ann}(M) \subseteq P$.

Proof. (i) Evidently $(P^2)_P \subseteq ((P^2)_P M_P : M_P)$, so $P_P \subseteq \sqrt{((P^2)_P M_P : M_P)}$, and since P_P is a maximal ideal, $\sqrt{((P^2)_P M_P : M_P)} = P_P$. Therefore $(P^2)_P M_P$ is a P_P -primary submodule of M_P . Then clearly $P^{(2)}M$ is a P -primary submodule of M . Now the proof is given by Lemma 4, as $P^2 \subseteq (P^{(2)}M : M)$.

(ii) By part (i), it is enough to prove that $(P^2)_P M_P \neq M_P$.

According to Lemma 5, there exists a P -prime submodule N of M . Then by Lemma 3, N_P is a P_P -prime submodule of M_P . Now from $P_P M_P \subseteq N_P$, we get $(P^2)_P M_P \subseteq N_P$. Consequently $(P^2)_P M_P \neq M_P$. $\square$

In the following, if

$$\mathcal{A} = \{N \mid N \text{ is a } P\text{-primary and 2-absorbing submodule of } M\} = \emptyset,$$

then we consider $\bigcap \mathcal{A} = M$.

Corollary 3. *If P is a prime ideal of R , then*

$$P^{(2)}M = \bigcap \{N \mid N \text{ is a } P\text{-primary and 2-absorbing submodule of } M\}.$$

Proof. Set $\mathcal{A} = \{N \mid N \text{ is a } P\text{-primary and 2-absorbing submodule of } M\}$.

If $P^{(2)}M = M$, then $\mathcal{A} = \emptyset$, because if N is a P -primary and 2-absorbing submodule of M , by Lemma 4, $P^2 M \subseteq N$. Therefore $M = P^{(2)}M \subseteq (N_P)^c = N$, which is impossible. Hence $\mathcal{A} = \emptyset$, and so in this case $\bigcap \mathcal{A} = M = P^{(2)}M$.

Now let $P^{(2)}M \neq M$. By Corollary 2(i), $P^{(2)}M$ is 2-absorbing. Also in the proof of Corollary 2(i), we showed that $P^{(2)}M$ is P -primary, so $P^{(2)}M \in \mathcal{A}$. Consequently $\bigcap \mathcal{A} \subseteq P^{(2)}M$.

Now suppose that N' is a P -primary and 2-absorbing submodule of M . Then Lemma 4 implies that $P^{(2)}M \subseteq (N'_P)^c = N'$. Consequently $P^{(2)}M = \bigcap \mathcal{A}$. $\square$

A prime ideal P of R is said to be a divided prime ideal if $P \subseteq Rr$ for every $r \in R \setminus P$.

We consider $T(M) = \{m \in M \mid \exists 0 \neq r \in R, rm = 0\}$. If M is a nonzero module with $T(M) = 0$, then it is easy to see that R is an integral domain, and in this case we say M is a torsion-free module.

Theorem 1. *Let M be a nonzero finitely generated module and P a divided prime ideal. If $T(M) \subseteq P^2M$, then P^2M is 2-absorbing and*

$$P^2M = \bigcap \{N \mid N \text{ is a } P\text{-primary and 2-absorbing submodule of } M\},$$

particularly if M is a torsion-free module.

Proof. First we show that P^2M is a proper submodule of M . If $P^2M = M$, then by Nakayama's lemma, there exists $a \in R$ such that $1 - a \in P^2$ and $aM = 0$. Since $1 - a \in P$, $a \notin P$ and as P is a divided prime ideal, $1 - a \in P \subseteq Ra$. Thus there exists $t \in R$ with $1 - a = ta$. Therefore $M = (1 - a)M = taM = 0$, which is impossible.

Now by Corollary 3 and Lemma 4, it suffices to show that P^2M is P -primary. Suppose that $rx = s_1t_1y_1 + \cdots + s_nt_ny_n \in P^2M$, where $s_i, t_i \in P$, $y_i, x \in M$, and $r \in R$. If $r \notin P$, then since P is a divided prime, $P \subseteq Rr$, and hence there exist $r_1, \dots, r_n \in R$ such that $s_i = rr_i \in P$, for $i = 1, \dots, n$. Thus for each i , $r_i \in P$ and $r(r_1t_1y_1 + \cdots + r_nt_ny_n) = rx \in P^2M$. Hence as $x - (r_1t_1y_1 + \cdots + r_nt_ny_n) \in T(M) \subseteq P^2M$, and $r_1t_1y_1 + \cdots + r_nt_ny_n \in P^2M$, we have $x \in P^2M$, which completes the proof. $\square$

According to [1] an ideal I of R is called an n -almost prime ideal if for $a, b \in R$ with $ab \in I \setminus I^n$, either $a \in I$ or $b \in I$. The case $n = 2$ is called an *almost prime* ideal and it is due to [8].

Theorem 2. *Let R be a Noetherian domain, which is not a field. Then the following are equivalent.*

- (i) R is Dedekind domain.
- (ii) If I is a 2-absorbing ideal of R , then I is almost prime or $I = P_1 \cap P_2$ or $I = P^2$, where P, P_1, P_2 are prime ideals of R .

Proof. (i) $\Rightarrow$ (ii) The proof is given by [6, Theorem 3.14].

(ii) $\Rightarrow$ (i) We prove that every localization of R at any nonzero prime ideal has the property introduced in (ii).

Let J be a 2-absorbing ideal of $R_{\mathfrak{P}}$, where $\mathfrak{P}$ is a nonzero prime ideal of R . By Lemma 2, J^c is a 2-absorbing ideal of R , and hence by our assumption, J^c is almost prime or $J^c = P_1 \cap P_2$ or $J^c = P^2$, for some prime ideals P, P_1, P_2 of R .

By [10, Proposition 2.10(ii)], the localization of an almost prime ideal is almost prime if it is a proper ideal. Hence if J^c is an almost prime ideal, then $(J^c)_{\mathfrak{P}} = J \neq R$, and so J is an almost prime ideal of $R_{\mathfrak{P}}$.

If $J^c = P_1 \cap P_2$, then $J = (J^c)_{\mathfrak{P}} = (P_1)_{\mathfrak{P}} \cap (P_2)_{\mathfrak{P}}$, and since J is a proper ideal, at least one of $(P_1)_{\mathfrak{P}}$ or $(P_2)_{\mathfrak{P}}$ is a prime ideal. So in this case either J is a prime ideal or the intersection of two prime ideals.

In case $J^c = P^2$, then $J = (J^c)_{\mathfrak{P}} = (P_{\mathfrak{P}})^2$, and as J is proper, the ideal $(P)_{\mathfrak{P}}$ is prime.

Therefore by considering the localization of R , we may suppose that $\mathfrak{M}$ is the only maximal ideal of R . If $\mathfrak{M} = \mathfrak{M}^2$, then by Nakayama's lemma, $\mathfrak{M} = 0$, that is R is a field. Now let $s \in \mathfrak{M} \setminus \mathfrak{M}^2$, and set $I = \mathfrak{M}^2 + Rs$.

First we prove that every ideal K with $\mathfrak{M}^2 \subset K$ is almost prime. (*)

Evidently $\sqrt{K} = \mathfrak{M}$, and so K is a primary ideal with $\mathfrak{M}^2 \subseteq K$. So by Lemma 4, K is 2-absorbing and the hypothesis in (ii) implies that K is almost prime, or $K = P_1 \cap P_2$ or $K = P^2$, where P, P_1, P_2 are prime ideals of R . If $K = P^2$, then $\mathfrak{M}^2 \subseteq K = P^2$, and so $\mathfrak{M} = P$. Thus $K = \mathfrak{M}^2$, which is impossible. If $K = P_1 \cap P_2$, then $\mathfrak{M}^2 \subseteq P_1$ and $\mathfrak{M}^2 \subseteq P_2$ and so $P_1 = P_2 = \mathfrak{M}$, that is in this case $K = \mathfrak{M}$, so evidently K is (almost) prime.

By (*) in above, I is an almost prime ideal. We will prove that $I^2 = \mathfrak{M}^2$. On the contrary let $a, b \in \mathfrak{M}$ such that $ab \notin I^2$. Thus $ab \in I \setminus I^2$, and since I is almost prime, we have $a \in I$ or $b \in I$ and not both, as $ab \notin I^2$, then suppose $a \in I$ and $b \notin I$. Note that $b^2 \in \mathfrak{M}^2 \subseteq I$. Hence $b(a+b) \in I$. If $b(a+b) \notin I^2$, then $b \in I$ or $a+b \in I$, which is impossible. Hence $b(a+b) \in I^2$, and $ab \notin I^2$, therefore $b^2 \notin I^2$. Then $b^2 \in I \setminus I^2$, and so $b \in I$, which is a contradiction.

Consequently $\mathfrak{M}^2 = I^2 = \mathfrak{M}^4 + \mathfrak{M}^2s + Rs^2 = \mathfrak{M}^2(\mathfrak{M}^2 + Rs) + Rs^2$. Hence by Nakayama's lemma $\mathfrak{M}^2 = Rs^2 \subseteq Rs$, and as $s \notin \mathfrak{M}^2$, we have $\mathfrak{M}^2 \subset Rs$. Thus again by (*), Rs is almost prime. By [8, Lemma 2.6], every principal and almost prime ideal is a prime ideal, hence Rs is a prime ideal. Now since $\mathfrak{M}^2 \subseteq Rs$, $\mathfrak{M} = Rs$, that is $\mathfrak{M}$ is a principal ideal. Therefore R is a discrete valuation domain, in case R is local.

Now for the general case, note that every localization of R is a discrete valuation domain, hence R is a Dedekind domain. $\square$

3. n -WEAKLY PRIME SUBMODULES

Another generalization of weakly prime submodules is introduced in the following. The following definition is also a generalization and a module version of n -almost prime ideals which was introduced and studied in [1].

Definition 2. Let $n > 1$ be an integer. A proper submodule N of M will be called n -weakly prime, if for $r, s \in R$ and $x \in M$, $rsx \in N \setminus (N : M)^{n-1}N$ implies that $rs \in (N : M)$ or $rx \in N$ or $sx \in N$.

If we consider R as an R -module, then evidently a proper ideal I of R is n -weakly prime if for $a, b, c \in R$, $abc \in I \setminus I^n$ implies that $ab \in I$ or $bc \in I$ or $ac \in I$.

Remark 1. For any submodule, we have the following implications:

- (1) $\text{Prime} \implies \text{weakly prime} \implies 2\text{-absorbing} \implies n\text{-weakly prime}$.
- (2) $n\text{-weakly prime} \implies (n-1)\text{-weakly prime}$, for each $n > 2$.

Evidently the zero submodule is n -weakly prime, but it is not necessarily 2-absorbing. The following example introduces non trivial n -weakly prime submodules, which are not 2-absorbing.

Example 2. Let $R = \frac{K[X_1, X_2, X_3, X_4]}{\langle \bar{X}_1^2, \bar{X}_2^2, \bar{X}_3^2, \bar{X}_4^2, \bar{X}_1\bar{X}_2\bar{X}_3, \bar{X}_1\bar{X}_2\bar{X}_4, \bar{X}_1\bar{X}_3\bar{X}_4, \bar{X}_2\bar{X}_3\bar{X}_4 \rangle}$, where K is a field of characteristic 2 and $\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4$ are independent indeterminates. Consider $M = R \oplus R$ and $I = \langle \bar{X}_1\bar{X}_2 + \bar{X}_3\bar{X}_4 \rangle$. Then the two submodules $N = \{(x, x) \mid x \in I\}$ and $N' = I \oplus I$ are n -weakly prime, but they are not 2-absorbing.

Proof. Evidently $(R, \mathfrak{M})$ is a local ring with $\mathfrak{M}^3 = 0$, where $\mathfrak{M} = \langle \bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4 \rangle$. First we prove that $\bar{X}_1\bar{X}_2 + \bar{X}_3\bar{X}_4$ is irreducible.

Suppose $fg = \bar{X}_1\bar{X}_2 + \bar{X}_3\bar{X}_4$ (*), with f, g non unit. Note that $\mathfrak{M}^3 = 0$, then we can consider $f = a_1\bar{X}_1 + a_2\bar{X}_2 + a_3\bar{X}_3 + a_4\bar{X}_4 \in \mathfrak{M}$, and $g = b_1\bar{X}_1 + b_2\bar{X}_2 + b_3\bar{X}_3 + b_4\bar{X}_4 \in \mathfrak{M}$, where $a_i, b_i \in K$. From (*) we get:

$$\begin{aligned} (1) \ a_1b_2 + a_2b_1 &= 1 & (2) \ a_1b_3 + a_3b_1 &= 0 & (3) \ a_1b_4 + a_4b_1 &= 0 \\ (4) \ a_2b_3 + a_3b_2 &= 0 & (5) \ a_2b_4 + a_4b_2 &= 0 & (6) \ a_3b_4 + a_4b_3 &= 1 \end{aligned}$$

By (2), $0 = a_1b_4(a_1b_3 + a_3b_1)$ and by (3), $0 = a_1b_3(a_1b_4 + a_4b_1)$ and so $a_1b_1(a_3b_4 - a_4b_3) = 0$. Since the characteristic of K is 2, $-a_4b_3 = a_4b_3$ and so $a_1b_1(a_3b_4 + a_4b_3) = 0$. Hence by (6), $a_1b_1 = 0$. Then $a_1 = 0$ or $b_1 = 0$. The case $a_1 = b_1 = 0$ is impossible, by (1). If $0 = a_1$ and $0 \neq b_1$, then (2) and (3) imply that $a_3 = 0 = a_4$ and this is a contradiction by (6).

In case $0 \neq a_1$ and $0 = b_1$, then by (2), (3) we get $b_3 = 0 = b_4$, which is a again impossible, according to (6). Consequently $\bar{X}_1\bar{X}_2 + \bar{X}_3\bar{X}_4$ is irreducible.

One can easily see that $(N : M) = 0$, and so $(N : M)^{n-1}N = 0$. Also it is easy to see that $I \subseteq \mathfrak{M}^2$ and $(N' : M) = I$. Then $I^2 \subseteq \mathfrak{M}^4 = 0$, and thus $(N' : M)^{n-1}N' = 0$.

To show that N is n -weakly prime, let $(\circ, \circ) \neq rs(a, b) \in N$, where $r, s \in R$ and $(a, b) \in M$. We can assume $\circ \neq rsa \in I$. Then for some $h \in R$, $\circ \neq rsa = h(\bar{X}_1\bar{X}_2 + \bar{X}_3\bar{X}_4)$. But since $I\mathfrak{M} \subseteq \mathfrak{M}^3 = 0$, $h \in R \setminus \mathfrak{M}$. Thus h is unit and so $rsah^{-1} = \bar{X}_1\bar{X}_2 + \bar{X}_3\bar{X}_4$ and it is irreducible, therefore r or sa is unit. Hence r or s is unit and so $s(a, b) \in r^{-1}N = N$ or $r(a, b) \in s^{-1}N = N$. This show that N is n -weakly prime. The same argument proves that N' is n -weakly prime.

Now if on the contrary N is a 2-absorbing submodule, then again by Proposition 1(i), $(N : M) = 0$ must be a 2-absorbing ideal and as $0 = \mathfrak{M}^3 \subseteq (N : M)$, we will have $\mathfrak{M}^2 \subseteq (N : M) = 0$, which is impossible. Thus N is not a 2-absorbing submodule.

If N' is a 2-absorbing submodule, then by Proposition 1(i), $(N' : M) = I$ is a 2-absorbing ideal of R and since $0 = \mathfrak{M}^3 \subseteq I$, then $\mathfrak{M}^2 \subseteq I$. Consequently $\bar{X}_1\bar{X}_2 \in \mathfrak{M}^2 \subseteq I$. Then for some $h' \in R$, $\circ \neq \bar{X}_1\bar{X}_2 = h'(\bar{X}_1\bar{X}_2 + \bar{X}_3\bar{X}_4)$. As $\mathfrak{M}^3 = 0$, h' is unit and since $\bar{X}_1\bar{X}_2 + \bar{X}_3\bar{X}_4$ is irreducible, $\bar{X}_1$ or $\bar{X}_2$ is unit, which is impossible. $\square$

Evidently $(N : M)^{n-1}N \subseteq (N : M)^2N$, for each submodule N of M for each $n > 2$. We now introduce a simple criteria for an n -weakly prime submodule to be 2-absorbing.

Theorem 3. *Let N be a submodule of M with $(N : M)^2 N \not\subseteq (N : M)^{n-1} N$. Then N is 2-absorbing if and only if it is n -weakly prime.*

Proof. Let N be an n -weakly prime submodule. Suppose $rsx \in N$, where $r, s \in R$ and $x \in M$. If $rx, sx \notin N$ and $rs \notin (N : M)$, then we prove that $(N : M)^2 N \subseteq (N : M)^{n-1} N$, which is impossible and so N is 2-absorbing.

First we show that the following facts hold:

- (i) $rsx \in (N : M)^{n-1} N$.
- (ii) $rsN \subseteq (N : M)^{n-1} N$.
- (iii) $r(N : M)x, s(N : M)x \subseteq (N : M)^{n-1} N$.
- (iv) $(N : M)^2 x \subseteq (N : M)^{n-1} N$.
- (v) $r(N : M)N, s(N : M)N \subseteq (N : M)^{n-1} N$.

(i) Since N is n -weakly prime and $rx, sx \notin N$ and $rs \notin (N : M)$, then $rsx \in (N : M)^{n-1} N$.

(ii) If $rsN \not\subseteq (N : M)^{n-1} N$, then for some $y \in N$ we have $rsy \notin (N : M)^{n-1} N$. So since $rsx \in (N : M)^{n-1} N$, $rs(x+y) \notin (N : M)^{n-1} N$. Hence $rs(x+y) \in N \setminus (N : M)^{n-1} N$ and then $r(x+y) \in N$ or $s(x+y) \in N$ or $rs \in (N : M)$. Thus $rx \in N$ or $sx \in N$ or $rs \in (N : M)$, which is impossible. Consequently $rsN \subseteq (N : M)^{n-1} N$.

(iii) Let $r(N : M)x \not\subseteq (N : M)^{n-1} N$. Then there exists $t \in (N : M)$ such that $rtx \in N \setminus (N : M)^{n-1} N$. Clearly $r(s+t)x \in N$. We have $r(s+t)x \notin (N : M)^{n-1} N$, otherwise since $rsx \in (N : M)^{n-1} N$, $rtx \in (N : M)^{n-1} N$, which is a contradiction. Then $r(s+t)x \in N \setminus (N : M)^{n-1} N$ and hence $rx \in N$ or $(s+t)x \in N$ or $r(s+t) \in (N : M)$, which implies $rx \in N$ or $sx \in N$ or $rs \in (N : M)$, a contradiction to our assumption. Therefore $r(N : M)x \subseteq (N : M)^{n-1} N$. Similarly $s(N : M)x \subseteq (N : M)^{n-1} N$.

(iv) Let $a, b \in (N : M)$. If $abx \notin (N : M)^{n-1} N$, then since $rsx \in N$, $(a+r)(b+s)x \in N$. we show that $(a+r)(b+s)x \notin (N : M)^{n-1} N$.

If $(a+r)(b+s)x \in (N : M)^{n-1} N$, then $rsx + rbx + asx + abx \in (N : M)^{n-1} N$, and so by parts (i), (iii), $rsx + rbx + asx \in (N : M)^{n-1} N$. Hence $abx \in (N : M)^{n-1} N$, which is impossible. Thus $(a+r)(b+s)x \notin (N : M)^{n-1} N$. Therefore $(a+r)(b+s)x \in N \setminus (N : M)^{n-1} N$ and so $(a+r)x \in N$ or $(b+s)x \in N$ or $(a+r)(b+s) \in (N : M)$, which implies $rx \in N$ or $sx \in N$ or $rs \in (N : M)$, and this is a contradiction. Then $abx \in (N : M)^{n-1} N$ and so $(N : M)^2 x \subseteq (N : M)^{n-1} N$.

(v) If for some $b \in (N : M)$ and $y \in N$, $rby \notin (N : M)^{n-1} N$, then $r(s+b)(x+y) \in N$. By parts (i), (ii), (iii), $rsx + rsy + rbx \in (N : M)^{n-1} N$ and since $rby \notin (N : M)^{n-1} N$, then $r(s+b)(x+y) \notin (N : M)^{n-1} N$. Hence $r(x+y) \in N$ or $(s+b)(x+y) \in N$ or $r(s+b) \in (N : M)$. Then $rx \in N$ or $sx \in N$ or $rs \in (N : M)$, which is a contradiction. Consequently $r(N : M)N \subseteq (N : M)^{n-1} N$ and similarly $s(N : M)N \subseteq (N : M)^{n-1} N$.

Now we prove the theorem. Let $a, b \in (N : M)$ and $y \in N$. If $aby \notin (N : M)^{n-1} N$, then obviously $(a+r)(b+s)(x+y) \in N$. If $(a+r)(b+s)(x+y) \in (N : M)^{n-1} N$, then by previous parts $aby = (a+r)(b+s)(x+y) - (abx + asx + asy + rbx +$

$rbx + rsx + rsy \in (N : M)^{n-1}N$, which is impossible. Thus $(a+r)(b+s)(x+y) \notin (N : M)^{n-1}N$ and so $(a+r)(b+s)(x+y) \in N \setminus (N : M)^{n-1}N$. Hence $(a+r)(x+y) \in N$ or $(b+s)(x+y) \in N$ or $(a+r)(b+s) \in (N : M)$. Therefore $rx \in N$ or $sx \in N$ or $rs \in (N : M)$, which is impossible. Consequently $(N : M)^2N \subseteq (N : M)^{n-1}N$. $\square$

Corollary 4. *Let $n > 3$ and M be a nonzero torsion-free Noetherian R -module. Then a submodule is 2-absorbing if and only if it is n -weakly prime.*

Proof. Let N be an n -weakly prime submodule. By Theorem 3, it is enough to prove that $(N : M)^{n-1}N \neq (N : M)^2N$. On the contrary suppose that $(N : M)^{n-1}N = (N : M)^2N$. Then by Nakayama's lemma there exists $a \in (N : M)^{n-3}$ such that $(a-1)(N : M)^2N = 0$. As M is torsion-free, we have $a = 1$, or $(N : M) = 0$ or $N = 0$.

If $a = 1$, then $N = M$, which is impossible. Evidently $N = 0$ is 2-absorbing. Now suppose $(N : M) = 0$. Assume $rsx \in N$, where $r, s \in R$ and $x \in M$. If $rsx \neq 0$, then $rsx \in N \setminus (N : M)^{n-1}N$, and since N is n -weakly prime, the proof is clear in this case.

In case $rsx = 0$, then $rs = 0 \in (N : M)$, or $x = 0 \in N$. $\square$

Proposition 2. *Let $x \in M$ and $a \in R$.*

- (i) *If $\text{ann}_M(a) \subseteq aM$, then the submodule aM is 2-absorbing if and only if it is n -weakly prime.*
- (ii) *If $\text{ann}_R(x) \subseteq (Rx : M)$, then the submodule Rx is 2-absorbing if and only if Rx is n -weakly prime.*

Proof. (i) Let M be an n -weakly prime submodule and $r, s \in R$ and $x \in M$ with $rsx \in aM$. If $rsx \notin (aM : M)^{n-1}aM$, then $rs \in (aM : M)$ or $rx \in aM$ or $sx \in aM$. Therefore assume $rsx \in (aM : M)^{n-1}aM$. Clearly $r(s+a)x = rsx + rax \in aM$. If $r(s+a)x \notin (aM : M)^{n-1}aM$, then $r(s+a) \in (aM : M)$ or $rx \in aM$ or $(s+a)x \in aM$. So as $a \in (aM : M)$, $rs \in (aM : M)$ or $rx \in aM$ or $sx \in aM$.

Now suppose that $r(s+a)x \in (aM : M)^{n-1}aM$. Then since $rsx \in (aM : M)^{n-1}aM$, for some $y \in (aM : M)^{n-1}M$, we have $arx = ay$ and so $a(rx-y) = 0$. Hence $rx-y \in \text{ann}_M(a) \subseteq aM$ and $y \in (aM : M)^{n-1}M = (aM : M)^{n-2}(aM : M)M \subseteq aM$. Thus $rx \in aM$.

(ii) Let Rx be an n -weakly prime submodule and $r, s \in R, y \in M$ with $rsy \in Rx$. Since Rx is n -weakly prime, we may assume $rsy \in (Rx : M)^{n-1}Rx$. Evidently $rs(x+y) \in Rx$. If $rs(x+y) \notin (Rx : M)^{n-1}Rx$, then $rs \in (Rx : M)$ or $r(x+y) \in Rx$ or $s(x+y) \in Rx$. Hence $rs \in (Rx : M)$ or $ry \in Rx$ or $sy \in Rx$.

Now let $rs(x+y) \in (Rx : M)^{n-1}Rx$. Then as $rsy \in (Rx : M)^{n-1}Rx$, $rsx \in (Rx : M)^{n-1}Rx$ and so $rsx = tx$, for some $t \in (Rx : M)^{n-1} \subseteq (Rx : M)$. Hence $rs-t \in \text{ann}(x) \subseteq (Rx : M)$ and thus $rs \in (Rx : M)$. $\square$

Example 3. Let R be a unique factorization domain, p an irreducible element of R , and $M = R \oplus R$.

- (a) The submodule $N = p^2M$ is 2-absorbing.
- (b) The submodule $N = p^3M$ is neither 2-absorbing, nor 2-weakly prime.

Proof. (a) Consider $ab(c, d) \in N$, where $a, b, c, d \in R$. Then a straightforward calculation shows that $a(c, d) \in N$ or $b(c, d) \in N$ or $p^2 \mid ab$.

(b) If N is 2-absorbing, then by Proposition 1(i), $(N : M)$ is 2-absorbing and evidently $p^3 \in (N : M)$, therefore $p^2 \in (N : M)$. Then $p^2(1, 0) \in N = p^3M$. Hence there exists $t \in R$ with $p^2 = p^3t$. Then $pt = 1$, which is impossible. Therefore N is not 2-absorbing and by Proposition 2(i), N is not 2-weakly prime. $\square$

Recall that the set of zero divisors of M , denoted by $Z(M)$ is defined by $Z(M) = \{r \in R \mid \exists 0 \neq x \in M, rx = 0\}$.

The following result studies the behavior of n -weakly prime submodules under localization. Its proof is not difficult and we leave it to the reader.

Proposition 3. *Let S be a multiplicatively closed subset of R .*

- (i) *If N is an n -weakly prime submodule of M with $S^{-1}N \neq S^{-1}M$, then $S^{-1}N$ is an n -weakly prime submodule of $S^{-1}M$.*
- (ii) *Let N be an n -weakly prime submodule of M with $Z(\frac{M}{N}) \cap S = \emptyset$. Then $S^{-1}N$ is an n -weakly prime submodule of $S^{-1}M$ and $(S^{-1}N)^c = N$. Moreover $S^{-1}(N : M) = (S^{-1}N : S^{-1}M)$.*

We can introduce the concept of n -weak prime as follows:

A proper submodule N of M will be called *n -weakly prime*, if for $r, s \in R$ and $x \in M$, $rsx \in N \setminus (N : M)^{n-1}N$ implies that $rx \in N$ or $sx \in N$.

Then similar to the proof of Theorem 3, Corollary 4 and Proposition 2 we can prove the following results:

- (1) Let N be a submodule of M with $(N : M)^2N \not\subseteq (N : M)^{n-1}N$. Then N is weakly prime if and only if it is n -weak prime.
- (2) Let $n > 3$ and M be a nonzero torsion-free Noetherian R -module. Then a submodule is weakly prime if and only if it is n -weak prime.
- (3) Let $a \in R$ with $ann_M(a) \subseteq aM$. Then aM is a weakly prime if and only if it is n -weak prime.

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Authors' addresses

S. Moradi

Department of Mathematics, College of Sciences, Shiraz University, Shiraz, 71457-44776, Iran

A. Azizi

Department of Mathematics, College of Sciences, Shiraz University, Shiraz, 71457-44776, Iran

E-mail address: aazizi@shirazu.ac.ir, a_azizi@yahoo.com



COMMENTS ON “NEW HYPERGEOMETRIC IDENTITIES ARISING FROM GAUSS’S SECOND SUMMATION THEOREM”

MEDHAT K. RAKHA, ARJUN K. RATHIE, PURNIMA CHOPRA,
AND RICHARD B. PARIS

Received 4 May, 2011

Abstract. In 1997, Exton [J. Comput. Appl. Math. **88** (1997) 269–274] obtained a general transformation involving hypergeometric functions by elementary manipulation of series. A number of hypergeometric identities that had not been previously recorded in the literature were then deduced by application of Gauss’ second summation theorem and other known hypergeometric summation theorems. However, some of the results stated by Exton contain errors. It is the purpose of this note to present the corrected forms of these hypergeometric identities.

2000 *Mathematics Subject Classification:* 33C20; 68Q40

Keywords: generalized hypergeometric series, hypergeometric identities

1. INTRODUCTION

The generalized hypergeometric function with p numeratorial and q denominatorial parameters is defined by the series [2, p. 41]

$${}_pF_q \left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} ; x \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_p)_n}{(b_1)_n (b_2)_n \dots (b_q)_n} \frac{x^n}{n!}, \quad (1.1)$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol (or ascending factorial). When $q = p$ this series converges for $|x| < \infty$, but when $q = p - 1$ convergence occurs when $|x| < 1$. However, when only one of the parameters a_j is a negative integer or zero, then the series (1.1) terminates and so always converges since it becomes a polynomial in x of degree $-a_j$.

In [1], Exton obtained a number of hypergeometric identities which had not been previously recorded in the literature. Exton based his investigation on the following general transformation which he obtained by elementary manipulation of series [1, Eq. (1.8)]

$$\sum_{n=0}^{\infty} \frac{(c_1)_n \dots (c_p)_n}{(d_1)_n \dots (d_q)_n} \frac{(\frac{1}{2}a)_n (-2x)^n}{n!} {}_{p+1}F_q \left[\begin{matrix} a+2n, c_1+n, \dots, c_p+n \\ d_1+n, \dots, d_q+n \end{matrix} ; x \right] \quad (1.2)$$

$$= {}_{2p+1}F_{2q} \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}c_1, \dots, \frac{1}{2}c_p, \frac{1}{2}(1+c_1), \dots, \frac{1}{2}(1+c_p) \\ \frac{1}{2}d_1, \dots, \frac{1}{2}d_q, \frac{1}{2}(1+d_1), \dots, \frac{1}{2}(1+d_q) \end{matrix} ; -4^{p-q}x^2 \right].$$

The domain of validity in the x -plane depends on the particular values of p , q and the various parameters. For example, if $p = q$ then (1.2) holds in general for $|x| < 1$. When one of the numeratorial parameters c_j equals a negative integer, the resulting expansion (1.2) involves *finite* sums and convergence at $x = \pm 1$ is assured. Then, upon letting $x = \pm 1$ in (1.2) followed by application of Gauss' second summation theorem and other well-known summation formulas listed in [2, Appendix III], Exton deduced several interesting hypergeometric identities.

2. EXTON'S RESULTS IN CORRECTED FORM

We have discovered that some of these results are incorrect. Following the same method used by Exton, notably the equation (1.2), we find that his corrected results are as stated below. In the following N denotes a positive integer.

Exton's result (2.5) should read

$$\begin{aligned} & \frac{(b-a)_N}{(b)_N} {}_3F_2 \left[\begin{matrix} -N, \frac{1}{2}a, 1+a-b \\ \frac{1}{2}(1+a-b-N), \frac{1}{2}(2+a-b-N) \end{matrix} ; \frac{1}{2} \right] \\ &= {}_3F_2 \left[\begin{matrix} -\frac{1}{2}N, \frac{1}{2}-\frac{1}{2}N, \frac{1}{2}a \\ \frac{1}{2}b, \frac{1}{2}+\frac{1}{2}b \end{matrix} ; -1 \right]; \end{aligned}$$

Exton's result (3.1) should read

$$\begin{aligned} & \frac{(1+a)_N}{(1+\frac{1}{2}a)_N} {}_2F_1 \left[\begin{matrix} -N, \frac{1}{2}a \\ \frac{1}{2}+\frac{1}{2}a \end{matrix} ; \frac{1}{2} \right] \\ &= {}_3F_2 \left[\begin{matrix} -\frac{1}{2}N, \frac{1}{2}-\frac{1}{2}N, \frac{1}{2}a \\ \frac{1}{2}(1+a+N), \frac{1}{2}(2+a+N) \end{matrix} ; -1 \right]; \end{aligned}$$

Exton's result (3.2) should read

$$\begin{aligned} & \frac{(1+a)_N(1+\frac{1}{2}a-b)_N}{(1+\frac{1}{2}a)_N(1+a-b)_N} {}_3F_2 \left[\begin{matrix} -N, \frac{1}{2}a, b \\ \frac{1}{2}+\frac{1}{2}a, b-\frac{1}{2}a-N \end{matrix} ; \frac{1}{2} \right] \\ &= {}_5F_4 \left[\begin{matrix} -\frac{1}{2}N, \frac{1}{2}-\frac{1}{2}N, \frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}+\frac{1}{2}b \\ \frac{1}{2}(1+a-b), \frac{1}{2}(2+a-b), \frac{1}{2}(1+a+N), \frac{1}{2}(2+a+N) \end{matrix} ; -1 \right]; \end{aligned}$$

Exton's result (3.3) should read

$$\begin{aligned} & \frac{(1+a)_N}{(\frac{1}{2}+\frac{1}{2}a)_N} {}_1F_0 \left[\begin{matrix} -N \\ - \end{matrix} ; \frac{1}{2} \right] = \frac{2^{-N}(1+a)_N}{(\frac{1}{2}+\frac{1}{2}a)_N} \\ &= {}_4F_3 \left[\begin{matrix} -\frac{1}{2}N, \frac{1}{2}-\frac{1}{2}N, \frac{1}{2}a, 1+\frac{1}{4}a \\ \frac{1}{4}a, \frac{1}{2}(1+a+N), \frac{1}{2}(2+a+N) \end{matrix} ; -1 \right]. \end{aligned}$$

We also note that Exton's results for the equations (2.9), (3.4) and (3.6) are correct, but his (3.7) is not a new result since the right-hand side is the same as that in (3.6) and so yields a simple identity between two ${}_2F_1(\frac{1}{2})$ functions. In addition, we cannot derive the results in (3.5), (3.8) and (3.9), nor could we verify them numerically. They should therefore be taken to be incorrect.

ACKNOWLEDGEMENT

The authors wish to acknowledge the helpful suggestions made by the referee.

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Authors' addresses

Medhat K. Rakha

Department of Mathematics, College of Science, Sultan Qaboos University, P.O. Box 36, Al-Khoud 123, Muscat, Oman

E-mail address: medhat@squ.edu.om

Arjun K. Rathie

Department of Mathematics, Vedant College of Engineering Technology, Tulsi, Bundi, Rajasthan State, India

E-mail address: akrathie@rediffmail.com

Purnima Chopra

Department of Mathematics, Marudhar Engineering College, Raisar, Bikaner, Rajasthan State, India

E-mail address: purnimachopra@rediffmail.com

Richard B. Paris

School of Computing, University of Abertay Dundee, Dundee DD1 1HG, UK

E-mail address: r.paris@abertay.ac.uk



ON THE PARAMETRIZATION OF BOUNDARY-VALUE PROBLEMS WITH THREE-POINT NON-LINEAR RESTRICTIONS

MIKLÓS RONTÓ AND K. MARYNETS

Received 19 December, 2011

Abstract. We obtain some results concerning the solutions of certain types of three-point non-linear boundary-value problems, subject to non-linear boundary conditions. We reduce the given problem to a parametrized one with linear two-point boundary restrictions containing some artificially introduced parameters. By the study of the transformed two-point problem we justify our method, which is based upon a special type of approximations constructed in analytic form.

2000 Mathematics Subject Classification: 34B15; 34B08

Keywords: three-point non-linear boundary-value problem, successive approximations, numerical-analytic method, determining system

1. INTRODUCTION

The non-linear three-point boundary-value problems were studied earlier mostly in the case of linear boundary restrictions. They were investigated in [2],[4], and [5] by using parametrization and the so-called numerical analytic method based upon successive approximations [7].

According to the basic idea of the above mentioned method, at first the original boundary-value problem is transformed to a two-point one which is replaced by the problem for the "perturbed" differential equation containing some new artificially introduced parameters, whose numerical values should be determined. The solutions of the parametrized problem are built in analytic form by successive iterations with all iterations depending upon the artificially introduced parameters.

Here we give a possible approach to transform a three-point boundary-value problem with non-linear boundary restrictions into a two-point one with linear boundary conditions using ideas from [1,3,5].

2. PROBLEM SETTING

We consider the three-point boundary-value problem subject to the nonlinear boundary conditions

$$\frac{dx(t)}{dt} = f(t, x(t)), t \in [0, T], x \in \mathbb{R}^n, \quad (2.1)$$

$$g(x(0), x(t_1), x(T)) = 0, t_1 \in (0, T). \quad (2.2)$$

Here, we suppose that the functions

$$f : [0, T] \times D \rightarrow \mathbb{R}^n$$

and

$$g : D \times D \times D \rightarrow \mathbb{R}^n, (n \geq 2)$$

are continuous, where $D \subset \mathbb{R}^n$ is a closed and bounded domain.

The problem is the determination of a continuously differentiable solution of the system of differential equations (2.1) satisfying non-linear three-point boundary restrictions (2.2).

3. CONSTRUCTION OF AN EQUIVALENT PROBLEM WITH LINEAR TWO-POINT BOUNDARY CONDITIONS

To pass to the linear two-point boundary conditions from (2.2), we replace the values of the components of the solution of (2.1), (2.2) at the points $t = 0$, $t = t_1$ and $t = T$ by the following parameters:

$$\begin{aligned} z &:= x(0) = \text{col}(x_1(0), x_2(0), \dots, x_n(0)) = \text{col}(z_1, z_2, \dots, z_n), \\ \eta &:= x(t_1) = \text{col}(x_1(t_1), x_2(t_1), \dots, x_n(t_1)) = \text{col}(\eta_1, \eta_2, \dots, \eta_n), \\ \lambda &:= x(T) = \text{col}(x_1(T), x_2(T), \dots, x_n(T)) = \text{col}(\lambda_1, \lambda_2, \dots, \lambda_n). \end{aligned} \quad (3.1)$$

Let us rewrite the boundary conditions (2.2) in the form:

$$Ax(0) + Cx(T) + g(x(0), x(t_1), x(T)) = Ax(0) + Cx(T), \quad (3.2)$$

where A is some given matrix and $C \equiv I_n$, where I_n is a unit $n \times n$ matrix.

Using parametrization (3.1), the non-linear three-point boundary restrictions (3.2) can be written as a two-point ones:

$$Ax(0) + x(T) = Az + \lambda - g(z, \eta, \lambda). \quad (3.3)$$

Let us put

$$d(z, \eta, \lambda) := Az + \lambda - g(z, \eta, \lambda). \quad (3.4)$$

Taking into account (3.4), the parametrized boundary conditions (3.3) can be rewritten in the form:

$$Ax(0) + x(T) = d(z, \eta, \lambda). \quad (3.5)$$

Let us consider the special case of (3.5) where we take the zero matrix instead of A :

$$\begin{aligned} x(0) &= z, \\ x(T) &= d(z, \eta, \lambda). \end{aligned} \quad (3.6)$$

So, instead of the original three-point boundary-value problem with non-linear boundary conditions (2.1), (2.2) we study an equivalent parametrized two-point one, containing linear separated boundary restrictions (2.1), (3.6).

Remark 1. The set of the solutions of the non-linear three-point boundary-value problem (2.1), (2.2) coincides with the set of the solutions of the two-point problem (2.1), (3.6) satisfying additional conditions (3.1).

4. CONSTRUCTION OF THE SUCCESSIVE APPROXIMATIONS

Let us introduce the vector

$$\delta_D(f) := \frac{1}{2} \left[\max_{(t,x) \in [0,T] \times D} f(t,x) - \min_{(t,x) \in [0,T] \times D} f(t,x) \right]. \quad (4.1)$$

The original boundary-value problem (2.1), (2.2) is such that the subset

$$D_\beta := \left\{ z \in D : B \left(z, \max_{t \in [0,T]} \left| z + \frac{t}{T} [d(z, \eta, \lambda) - z] \right| \right) \subset D, \forall \eta, \lambda \in D \right\}$$

is non-empty

$$D_\beta \neq \emptyset. \quad (4.2)$$

Assume that the function $f(t, x)$ satisfies Lipschitz condition of the form

$$|f(t, u) - f(t, v)| \leq K |u - v|, \quad (4.3)$$

for all $t \in [0, T]$, $\{u, v\} \subset D$ with some non-negative constant matrix $K = (k_{ij})_{i,j=1}^n$.

Moreover, we suppose that the spectral radius $r(K)$ of the matrix K satisfies the following inequality

$$r(K) < \frac{10}{3T}. \quad (4.4)$$

Let us associate to the parametrized boundary-value problem (2.1), (3.6) the sequence of functions:

$$\begin{aligned} x_m(t, z, \eta, \lambda) &:= z + \int_0^t f(s, x_{m-1}(s, z, \eta, \lambda)) ds - \\ &\quad - \frac{t}{T} \int_0^T f(s, x_{m-1}(s, z, \eta, \lambda)) ds + \frac{t}{T} [d(z, \eta, \lambda) - z], \end{aligned} \quad (4.5)$$

where $m = 1, 2, 3, \dots$,

$$x_0(t, z, \eta, \lambda) = z + \frac{t}{T} [d(z, \eta, \lambda) - z] \in D_\beta, \quad (4.6)$$

$\eta = \text{col}(\eta_1, \eta_2, \dots, \eta_n) \in D$, $\lambda = \text{col}(\lambda_1, \lambda_2, \dots, \lambda_n) \in D$,
 $x_m(t, z, \eta, \lambda) = \text{col}(x_{m,1}(t, z, \eta, \lambda), x_{m,2}(t, z, \eta, \lambda), \dots, x_{m,n}(t, z, \eta, \lambda))$ and z, η, λ
 are considered as parameters.

It is easy to check that the functions $x_m(t, z, \eta, \lambda)$ satisfy linear parametrized boundary conditions (3.6) for all $m \geq 1$, $z \in D_\beta$, $\eta \in D$, $\lambda \in D$.

The following statement establishes the convergence of the sequence (4.5).

Theorem 1. Assume that the function $f : [0, T] \times D \rightarrow \mathbb{R}^n$ on the right side of the system of differential equations (2.1) and the parametrized boundary restrictions (3.6) satisfy conditions (4.2)–(4.4).

Then for all fixed $z \in D_\beta$, $\eta \in D$, $\lambda \in D$:

- (1) The functions of the sequence (4.5) are continuously differentiable and satisfy the parametrized boundary conditions (3.6):

$$\begin{aligned} x_m(0, z, \eta, \lambda) &= z, \\ x_m(T, z, \eta, \lambda) &= d(z, \eta, \lambda), \end{aligned} \quad (4.7)$$

$m = 1, 2, 3, \dots$

- (2) The sequence of functions (4.5) for $t \in [0, T]$ converges uniformly as $m \rightarrow \infty$ to the limit function

$$x^*(t, z, \eta, \lambda) = \lim_{m \rightarrow \infty} x_m(t, z, \eta, \lambda). \quad (4.8)$$

- (3) The limit function $x^*(t, z, \eta, \lambda)$ satisfies the parametrized linear two-point boundary conditions:

$$\begin{aligned} x^*(0, z, \eta, \lambda) &= z, \\ x^*(T, z, \eta, \lambda) &= d(z, \eta, \lambda). \end{aligned}$$

- (4) The limit function (4.8) for all $t \in [0, T]$ is a unique continuously differentiable solution of the integral equation

$$x(t) = z + \int_0^t f(s, x(s)) ds - \frac{t}{T} \int_0^T f(s, x(s)) ds + \frac{t}{T} [d(z, \eta, \lambda) - z], \quad (4.9)$$

i. e. it is the solution of the Cauchy problem for the modified system of differential equations:

$$\frac{dx}{dt} = f(t, x) + \Delta(z, \eta, \lambda), \quad (4.10)$$

$$x(0) = z, \quad (4.11)$$

where

$$\Delta(z, \eta, \lambda) := \frac{1}{T} [d(z, \eta, \lambda) - z] - \frac{1}{T} \int_0^T f(s, x(s)) ds. \quad (4.12)$$

(5) The following error estimate holds:

$$|x^*(t, z, \eta, \lambda) - x_m(t, z, \eta, \lambda)| \leq \frac{20}{9}t \left(1 - \frac{t}{T}\right) Q^m (I_n - Q)^{-1} \delta_D(f), \quad (4.13)$$

where matrix

$$Q := \frac{3T}{10} K. \quad (4.14)$$

Proof. We will prove that the sequence of functions (4.5) is a Cauchy sequence in the Banach space $C([0, T], \mathbb{R}^n)$. First we show that $x_m(t, z, \eta, \lambda) \in D$, for all $(t, z, \eta, \lambda) \in [0, T] \times D_\beta \times D \times D$, $m \in \mathbb{N}$.

Indeed, using the estimates of Lemma 2.3 from [7] (see also Lemma 3 [6] and Lemma 2 [4]):

$$\left| \int_0^t \left[f(\tau) - \frac{1}{T} \int_0^T f(s) ds \right] d\tau \right| \leq \frac{1}{2} \alpha_1(t) \left[\max_{t \in [0, T]} f(t) - \min_{t \in [0, T]} f(t) \right], \quad (4.15)$$

where

$$\alpha_1(t) = 2t \left(1 - \frac{t}{T}\right), \quad |\alpha_1(t)| \leq \frac{T}{2}, \quad t \in [0, T], \quad (4.16)$$

relation (4.5) for $m = 0$ implies that:

$$\begin{aligned} |x_1(t, z, \eta, \lambda) - x_0(t, z, \eta, \lambda)| &\leq \\ &\leq \left| \int_0^t \left[f(t, z) - \frac{1}{T} \int_0^T f(s, z) ds \right] dt \right| \leq \\ &\leq \alpha_1(t) \delta_D(f) \leq \frac{T}{2} \delta_D(f). \end{aligned} \quad (4.17)$$

Therefore, by virtue of (4.17), we conclude that $x_1(t, z, \eta, \lambda) \in D$ whenever $(t, z, \eta, \lambda) \in [0, T] \times D_\beta \times D \times D$.

By induction we can easily establish that all functions (4.5) are also contained in the domain $D \forall m = 1, 2, 3, \dots, t \in [0, T], z \in D_\beta, \eta \in D, \lambda \in D$.

Now, consider the difference of functions:

$$\begin{aligned} x_{m+1}(t, z, \eta, \lambda) - x_m(t, z, \eta, \lambda) &= \\ &= \int_0^t [f(s, x_m(s, z, \eta, \lambda)) - f(s, x_{m-1}(s, z, \eta, \lambda))] ds - \\ &\quad - \frac{t}{T} \int_0^T [f(s, x_m(s, z, \eta, \lambda)) - f(s, x_{m-1}(s, z, \eta, \lambda))] ds, \end{aligned} \quad (4.18)$$

$m = 1, 2, 3, \dots$ and introduce the notation:

$$r_m(t, z, \eta, \lambda) := |x_m(t, z, \eta, \lambda) - x_{m-1}(t, z, \eta, \lambda)|, \quad m = 1, 2, 3, \dots$$

By virtue of the estimation (4.15) and of the Lipschitz condition (4.3), we have:

$$r_{m+1}(t, z, \eta, \lambda) \leq K \left[\left(1 - \frac{t}{T}\right) \int_0^t r_m(s, z, \eta, \lambda) ds + \frac{t}{T} \int_t^T r_m(s, z, \eta, \lambda) ds \right], \quad (4.19)$$

$\forall m = 0, 1, 2, \dots$

According to (4.17)

$$r_1(t, z, \eta, \lambda) = |x_1(t, z, \eta, \lambda) - x_0(t, z, \eta, \lambda)| \leq \alpha_1(t) \delta_D(f). \quad (4.20)$$

By virtue of the statement of Lemma 3 [4] of the form

$$\alpha_{m+1}(t) \leq \frac{10}{9} \left(\frac{3}{10} T \right)^m \alpha_1(t), m = 0, 1, 2, \dots \quad (4.21)$$

obtained for sequence of functions

$$\alpha_{m+1}(t) = \left(1 - \frac{t}{T}\right) \int_0^t \alpha_m(s) ds + \frac{t}{T} \int_t^T \alpha_m(s) ds, m = 0, 1, 2, \dots, \quad (4.22)$$

$$\alpha_0(t) = 1, \alpha_1(t) = 2t \left(1 - \frac{t}{T}\right),$$

from (4.19) for $m = 1$ follows:

$$r_2(t, z, \eta, \lambda) \leq K \delta_D(f) \left[\left(1 - \frac{t}{T}\right) \int_0^t \alpha_1(s) ds + \frac{t}{T} \int_t^T \alpha_1(s) ds \right] \leq K \alpha_2(t) \delta_D(f).$$

By induction using (4.22), we can easily obtain that

$$r_{m+1}(t, z, \eta, \lambda) \leq K^m \alpha_{m+1}(t) \delta_D(f), \quad (4.23)$$

$m = 0, 1, 2, \dots$, where $\alpha_{m+1}(t)$, $\alpha_m(t)$ are calculated according to (4.22), and $\delta_D(f)$ is given by (4.1).

By virtue of the estimate (4.21) from (4.23) we have

$$r_{m+1}(t, z, \eta, \lambda) \leq \frac{10}{9} \alpha(t) \left[Q^m \delta_D(f) + K Q^{m-1} |d(z, \eta, \lambda) - z| \right], \quad (4.24)$$

$\forall m = 1, 2, 3, \dots$, where the matrix Q is given by (4.14).

Therefore, in view of (4.24)

$$\begin{aligned} |x_{m+j}(t, z, \eta, \lambda) - x_m(t, z, \eta, \lambda)| &\leq |x_{m+j}(t, z, \eta, \lambda) - x_{m+j-1}(t, z, \eta, \lambda)| + \\ &\quad + |x_{m+j-1}(t, z, \eta, \lambda) - x_{m+j-2}(t, z, \eta, \lambda)| + \dots + \\ &\quad + |x_{m+1}(t, z, \eta, \lambda) - x_m(t, z, \eta, \lambda)| = \\ &= \sum_{i=1}^j r_{m+i}(t, z, \eta, \lambda) \leq \frac{10}{9} \alpha_1(t) \sum_{i=1}^j Q^{m+i} \delta_D(f) = \\ &= \frac{10}{9} \alpha_1(t) Q^m \sum_{i=0}^{j-1} Q^i \delta_D(f). \end{aligned} \quad (4.25)$$

Since, due to the condition (4.4), the maximum eigenvalue of the matrix Q of the form (4.14) does not exceed the unity, we have

$$\sum_{i=0}^{j-1} Q^i \leq (I_n - Q)^{-1}, \quad \lim_{m \rightarrow \infty} Q^m = O_n,$$

where O_n is a zero $n \times n$ matrix.

Therefore we conclude from (4.25) that, according to the Cauchy criterium, the sequence $\{x_m(t, z, \eta, \lambda)\}$ of the form (4.5) uniformly converges in the domain $(t, z, \eta, \lambda) \in [0, T] \times D_\beta \times D \times D$ to the limit function $x^*(t, z, \eta, \lambda)$. Since all functions $x_m(t, z, \eta, \lambda)$ of the sequence (4.5) satisfy the boundary conditions (3.6) for all values of the artificially introduced parameters, the limit function $x^*(t, z, \eta, \lambda)$ also satisfies these conditions. Passing to the limit as $m \rightarrow \infty$ in equality (4.5) we show that the limit function satisfies both the integral equation (4.9) and the Cauchy problem (4.10), (4.11), where $\Delta(z, \eta, \lambda)$ is given by (4.12). $\square$

Consider the Cauchy problem

$$\frac{dx}{dt} = f(t, x) + \mu, \quad t \in [0, T] \quad (4.26)$$

$$x(0) = z, \quad (4.27)$$

where $\mu = \text{col}(\mu_1, \dots, \mu_n)$ is a control parameter.

Theorem 2. *Under the conditions of Theorem 1, the solution $x = x(\cdot, z, \eta, \lambda, \mu)$ of the initial value problem (4.26), (4.27) satisfies the boundary conditions (3.6) if and only if $x = x(\cdot, z, \eta, \lambda, \mu)$ coincides with the limit function $x^*(\cdot, z, \eta, \lambda, \mu)$ of the sequence (4.5). Moreover*

$$\mu = \mu_{z, \eta, \lambda} = \frac{1}{T} [d(z, \eta, \lambda) - z] - \frac{1}{T} \int_0^T f(s, x^*(s, z, \eta, \lambda)) ds. \quad (4.28)$$

Proof. Sufficiency. Let us suppose that μ on the right side of the system of differential equations (4.26) is given by (4.28). By virtue of Theorem 1, the limit function (4.8) of the sequence (4.5) is the unique solution of the boundary-value problem (4.26), (3.6) for fixed values of parameters z, η and λ when $\mu = \mu_{z, \eta, \lambda}$. Furthermore the limit function $x^*(t, z, \eta, \lambda, \mu)$ satisfies initial conditions (4.27), i.e. it is a solution of the Cauchy problem (4.26), (4.27) when $\mu = \mu_{z, \eta, \lambda}$.

Necessity. Let us fix an arbitrary $\bar{\mu} \in \mathbb{R}^n$ and assume that the initial value problem (4.29), (4.27):

$$\frac{dx}{dt} = f(t, x) + \bar{\mu}, \quad t \in [0, T] \quad (4.29)$$

has a solution $\bar{x}(t)$, that satisfies the two-point linear boundary conditions (3.6). Then $\bar{x}(t)$ satisfies an integral equation:

$$\bar{x}(t) = z + \int_0^t f(s, \bar{x}(s)) ds + \bar{\mu}t \quad (4.30)$$

for all $t \in [0, T]$.

When $t = T$ from (4.30) we get:

$$T\bar{\mu} = \bar{x}(T) - z - \int_0^T f(s, \bar{x}(s))ds. \quad (4.31)$$

Under the assumption $\bar{x}(t)$ satisfies boundary restrictions (3.6):

$$\bar{x}(T) = d(z, \eta, \lambda), \quad (4.32)$$

and the initial condition

$$\bar{x}(0) = z.$$

Substituting (4.32) in (4.31) we get:

$$\bar{\mu} = \frac{1}{T}d(z, \eta, \lambda) - \frac{1}{T}z - \frac{1}{T} \int_0^T f(s, \bar{x}(s))ds. \quad (4.33)$$

On the other hand, it is proved that the limit function $x^*(\cdot, z, \lambda, \mu)$ is the solution of the initial value problem (4.26), (4.27) for $\mu = \mu_{z, \eta, \lambda}$ given by formula (4.28) and satisfies boundary conditions (3.6).

By analogy

$$x^*(t, z, \eta, \lambda, \mu) = z + \int_0^t f(s, x^*(s, z, \eta, \lambda, \mu))ds + \mu_{z, \eta, \lambda}t, \quad (4.34)$$

$$T\mu_{z, \eta, \lambda} = x^*(T, z, \eta, \lambda, \mu) - z - \int_0^T f(s, x^*(s, z, \eta, \lambda, \mu))ds, \quad (4.35)$$

$$x^*(T, z, \eta, \lambda, \mu) = d(z, \eta, \lambda), \quad (4.36)$$

$$x^*(0, z, \eta, \lambda, \mu) = z.$$

By virtue of (4.34)–(4.36) it is easy to get that

$$\mu_{z, \eta, \lambda} = \frac{1}{T}d(z, \eta, \lambda) - \frac{1}{T}z - \frac{1}{T} \int_0^T f(s, x^*(s, z, \eta, \lambda, \mu))ds. \quad (4.37)$$

Substituting (4.33) in (4.30) and (4.37) in (4.34), we get that for all $t \in [0, T]$

$$\bar{x}(t) = z + \int_0^t f(s, \bar{x}(s))ds - \frac{t}{T} \int_0^T f(s, \bar{x}(s))ds + \frac{t}{T}[d(z, \lambda) - z], \quad (4.38)$$

$$\begin{aligned} x^*(t, z, \eta, \lambda, \mu) &= z + \int_0^t f(s, x^*(s, z, \eta, \lambda, \mu))ds - \\ &\quad - \frac{t}{T} \int_0^T f(s, x^*(s, z, \eta, \lambda, \mu))ds + \frac{t}{T}[d(z, \eta, \lambda) - z]. \end{aligned} \quad (4.39)$$

Using Theorem 1 $\bar{x}(t) \in D$ and $x^*(t, z, \eta, \lambda, \mu) \in D$. Taking into account (4.38), (4.39) it follows that

$$\begin{aligned} x^*(t, z, \eta, \lambda, \mu) - \bar{x}(t) = & \int_0^t [f(s, x^*(s, z, \eta, \lambda, \mu)) - f(s, \bar{x}(s))] ds - \\ & - \frac{t}{T} \int_0^T [f(s, x^*(s, z, \eta, \lambda, \mu)) - f(s, \bar{x}(s))] ds. \end{aligned} \quad (4.40)$$

By the Lipschitz condition (4.3) from the relation (4.40), we get that the function

$$\omega(t) = |x^*(t, z, \eta, \lambda, \mu) - \bar{x}(t)|, t \in [0, T] \quad (4.41)$$

satisfies integral inequalities:

$$\begin{aligned} \omega(t) \leq K \left[\int_0^t \omega(s) ds + \frac{t}{T} \int_0^T \omega(s) ds \right] \leq \\ \leq K\alpha_1(t) \max_{s \in [0, T]} \omega(s), t \in [0, T], \end{aligned} \quad (4.42)$$

where $\alpha_1(t)$ is given by (4.16).

Using (4.42) recursively, we come to an inequality:

$$\omega(t) \leq K^m \alpha_m(t) \max_{s \in [0, T]} \omega(s), t \in [0, T], \quad (4.43)$$

where $m \in \mathbb{N}$ and functions $\alpha_m(t)$ are given by formula (4.22).

Taking into account (4.21), from (4.43) for each $m \in \mathbb{N}$ we get an estimation:

$$\omega(t) \leq K\alpha_1(t) \frac{10}{9} \left(\frac{3T}{10} K \right)^{m-1} \cdot \max_{s \in [0, T]} \omega(s), t \in [0, T]. \quad (4.44)$$

By passing to the limit $m \rightarrow \infty$ in the last inequality and by virtue of (4.4), we come to the conclusion that

$$\max_{s \in [0, T]} \omega(s) \leq Q^m \max_{s \in [0, T]} \omega(s) \rightarrow 0.$$

It means that the function $\bar{x}(t)$ coincides with $x^*(t, z, \eta, \lambda, \mu)$. Starting with (4.33) and (4.37), we come to the conclusion that $\bar{\mu} = \mu_{z, \eta, \lambda}$. $\square$

Let's find the relation of the limit function $x = x^*(t, z, \eta, \lambda)$ of the sequence (4.5) to the solution of the parametrized two-point linear boundary-value problem (2.1), (3.6) or to the equivalent three-point non-linear problem (2.1), (2.2).

Theorem 3. Assume that the conditions (4.2)–(4.4) are satisfied for the original boundary-value problem (2.1), (2.2).

Then the triplex $(x^*(\cdot, z^*, \eta^*, \lambda^*), \eta^*, \lambda^*)$ is the solution of the parametrized boundary-value problem (2.1), (3.6) if and only if $z^* = (z_1^*, z_2^*, \dots, z_n^*)$,

$\eta^* = (\eta_1^*, \eta_2^*, \dots, \eta_n^*)$, $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_n^*)$ satisfy a determining system of algebraic or transcendental equations

$$\Delta(z, \eta, \lambda) = \frac{1}{T} [d(z, \eta, \lambda) - z] - \frac{1}{T} \int_0^T f(s, x^*(s, z, \eta, \lambda)) ds = 0, \quad (4.45)$$

$$x^*(t_1, z, \eta, \lambda) = \eta, \quad (4.46)$$

$$x^*(T, z, \eta, \lambda) = \lambda. \quad (4.47)$$

Proof. It suffices to apply Theorem 2 and to notice that the differential equation in (4.10) coincides with (2.1) if and only if triplex (z^*, η^*, λ^*) satisfies an equation

$$\Delta(z^*, \eta^*, \lambda^*) = 0.$$

Taking into account (3.1) and the equivalence of (2.1), (2.2) and (2.1), (3.6), it is clear that $(x^*(\cdot, z^*, \eta^*, \lambda^*), \eta^*, \lambda^*)$ coincides with the solution of the parametrized boundary-value problem (2.1), (3.1), (3.6) if and only if $(x^*(\cdot, z^*, \eta^*, \lambda^*), \eta^*, \lambda^*)$ satisfies equations

$$x^*(t_1, z, \eta^*, \lambda^*) = \eta^*,$$

$$x^*(T, z, \eta^*, \lambda^*) = \lambda^*.$$

It means that the triplex $(x^*(\cdot, z^*, \eta^*, \lambda^*), \eta^*, \lambda^*)$ is the solution of the parametrized boundary-value problem (2.1), (3.6) if and only if (4.45), (4.46), (4.47) are hold. $\square$

The next statement proves that the system of determining equations (4.45)–(4.47) defines all possible solutions of the original three-point boundary-value problem (2.1), (2.2).

Lemma 1. *Let all conditions of Theorem 1 be satisfied. Furthermore there exist some vectors $z \in D_\beta$, $\eta \in D$ and $\lambda \in D$ that satisfy the system of determining equations (4.45)–(4.47).*

Then the non-linear three-point boundary-value problem (2.1), (2.2) has solution $x(\cdot)$ such that:

$$\begin{aligned} x(0) &= z, \\ x(t_1) &= \eta, \\ x(T) &= \lambda. \end{aligned} \quad (4.48)$$

Moreover this solution is given by formula

$$x(t) = x^*(t, z, \eta, \lambda), t = [0, T], \quad (4.49)$$

where $x^(t, z, \eta, \lambda)$ is the limit function of the sequence (4.5). And if the boundary-value problem (2.1), (2.2) has a solution $x(\cdot)$, then this solution is given by (4.49), and the system of determining equations (4.45)–(4.47) is satisfied when*

$$\begin{aligned} z &= x(0), \\ \eta &= x(t_1), \\ \lambda &= x(T). \end{aligned}$$

Proof. We will apply Theorems 2 and 3. If there exist such $z \in D_\beta$, $\eta \in D$ and $\lambda \in D$ that satisfy determining system (4.45), (4.46), (4.47), then according to Theorem 3 function (4.49) is a solution of the original boundary-value problem (2.1), (2.2). On the other hand, if $x(\cdot)$ is the solution of the original boundary-value problem (2.1), (2.2), then this function is the solution of the Cauchy problem (4.26), (4.27) for

$$\begin{aligned}\mu &= 0, \\ z &= x(0).\end{aligned}$$

As $x(\cdot)$ satisfies boundary restrictions (2.2) and equivalent conditions (3.5), by virtue of Theorem 2 an equality (4.49) is hold. Besides,

$$\begin{aligned}\mu &= \mu_{z,\eta,\lambda} = 0, \\ z &= x(0),\end{aligned}\tag{4.50}$$

where vectors η , λ are defined by (3.1).

But $\mu_{z,\eta,\lambda}$ is given by formula (4.28), that's why the first equation (4.45) of the determining system is satisfied, if

$$\begin{aligned}z &= x(0), \\ \eta &= x(t_1), \\ \lambda &= x(T) : \end{aligned}$$

$$\Delta(z, \eta, \lambda) = 0.\tag{4.51}$$

From (3.6) follows, that the next two equations (4.46), (4.47) of the determining system are hold, too. So we specified such triplexes $(z, \eta, \lambda) = (x(0), x(t_1), x(T))$, that satisfy system of determining equations (4.45)–(4.47), that proves Lemma. $\square$

Remark 2. The main difficulty of the realization of this method is to find the limit function $x^*(\cdot, z, \eta, \lambda)$. But in most cases this problem can be solved using the properties of the approximate solution $x_m(\cdot, z, \eta, \lambda)$ built in an analytic form.

For $m \geq 1$ let us define the function $\Delta_m : D_\beta \times D \times D \rightarrow \mathbb{R}^n$ by formula

$$\Delta_m(z, \eta, \lambda) := \frac{1}{T} [d(z, \lambda) - z] - \frac{1}{T} \int_0^T f(s, x_m(s, z, \eta, \lambda)) ds,\tag{4.52}$$

where z , η and λ are given by the relation (3.1). To investigate the solubility of the parametrized boundary-value problem (2.1), (3.6) we observe an approximate determining system of algebraic or transcendental equations of the form

$$\Delta_m(z, \eta, \lambda) = \frac{1}{T} [d(z, \eta, \lambda) - z] - \frac{1}{T} \int_0^T f(s, x_m(s, z, \eta, \lambda)) ds = 0,\tag{4.53}$$

$$x_m(t_1, z, \eta, \lambda) = \eta,\tag{4.54}$$

$$x_m(T, z, \eta, \lambda) = \lambda,\tag{4.55}$$

where $x_m(\cdot, z, \eta, \lambda)$ is a vector-function, that defines with the recursive relation (4.5). Increasing m , the systems (4.45)–(4.47) and (4.53), (4.54) are close enough to

provide the needed precision of the approximate solution of the original boundary–value problem (2.1), (2.2).

5. EXAMPLE

Consider the system

$$\begin{cases} \frac{dx_1}{dt} = 0.05x_2 - 0.005t^2 + 0.1 = f_1(t, x_1, x_2), \\ \frac{dx_2}{dt} = -x_2^2 + 0.5x_1 + 0.01t^4 + 0.15t = f_2(t, x_1, x_2), \end{cases} \quad (5.1)$$

where $t \in [0, \frac{1}{2}]$, with non–linear two–point boundary conditions

$$\begin{cases} g_1(x(0), x(\frac{1}{4}), x(1)) := x_1(\frac{1}{2}) + x_2^2(0) - x_1(\frac{1}{4}) - 0.025 = 0, \\ g_2(x(0), x(\frac{1}{4}), x(1)) := x_1(0) + x_2(\frac{1}{2}) - x_2(0) - 0.025 = 0. \end{cases} \quad (5.2)$$

It is easy to check that an exact solution of the problem (5.1), (5.2) are the functions

$$\begin{cases} x_1^* = 0.1t, \\ x_2^* = 0.1t^2. \end{cases} \quad (5.3)$$

Suppose that the boundary–value problem (5.1), (5.2) is considered in the domain

$$D = \{(x_1, x_2) : |x_1| \leq 0.42, |x_2| \leq 0.4\}. \quad (5.4)$$

Boundary conditions (5.2) can be rewritten in the form

$$Cx\left(\frac{1}{2}\right) + g\left(x(0), x\left(\frac{1}{4}\right)x\left(\frac{1}{2}\right)\right) = Cx\left(\frac{1}{2}\right), \quad (5.5)$$

where

$$C = I_2, \\ I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$g\left(x(0), x\left(\frac{1}{4}\right)x\left(\frac{1}{2}\right)\right) = \text{col}\left(g_1\left(x(0), x\left(\frac{1}{4}\right)x\left(\frac{1}{2}\right)\right), g_2\left(x(0), x\left(\frac{1}{4}\right)x\left(\frac{1}{2}\right)\right)\right).$$

Let us replace the values of the components of the solution of the boundary–value problem (5.1), (5.2) at the points $t = 0$, $t = \frac{1}{4}$ and $t = \frac{1}{2}$ by parameters z_1, z_2, η_1, η_2 and λ_1, λ_2 :

$$\begin{aligned} z &:= x(0) = \text{col}(x_1(0), x_2(0)) = \text{col}(z_1, z_2), \\ \eta &:= x\left(\frac{1}{4}\right) = \text{col}\left(x_1\left(\frac{1}{4}\right), x_2\left(\frac{1}{4}\right)\right) = \text{col}(\eta_1, \eta_2), \\ \lambda &:= x\left(\frac{1}{2}\right) = \text{col}\left(x_1\left(\frac{1}{2}\right), x_2\left(\frac{1}{2}\right)\right) = \text{col}(\lambda_1, \lambda_2). \end{aligned} \quad (5.6)$$

Using (5.6), the boundary restrictions (5.5) can be rewritten as

$$\begin{aligned} x(0) &= z, \\ x\left(\frac{1}{2}\right) &= \lambda - g(z, \eta, \lambda). \end{aligned} \quad (5.7)$$

Let us put

$$d(z, \eta, \lambda) := \lambda - g(z, \eta, \lambda). \quad (5.8)$$

where z, η and λ are given by (5.6).

Using (5.8), the parametrized boundary conditions (5.7) can be written in the form:

$$\begin{aligned} x(0) &= z, \\ x\left(\frac{1}{2}\right) &= d(z, \eta, \lambda). \end{aligned} \quad (5.9)$$

It is easy to check that the matrix K from the Lipschitz condition (4.3) is

$$K = \begin{pmatrix} 0 & 0.05 \\ 0.5 & 1 \end{pmatrix},$$

and

$$r(K) < 1.03 < \frac{10}{3T},$$

when $T = \frac{1}{2}$.

Vector $\delta_D(f)$ can be chosen as

$$\delta_D(f) \leq \begin{pmatrix} 0.03125 \\ 0.515 \end{pmatrix}.$$

Domain D_β is defined by inequalities:

$$\begin{aligned} |0.045 + z_2^2| &\leq 0.42, \\ |-z_1 + z_2 + 0.025| &\leq 0.4. \end{aligned}$$

One can verify that, for the parametrized boundary-value problem in this example, all of the necessary conditions are fulfilled. So, we can proceed with the application of the numerical-analytic scheme described above and thus construct the sequence of approximate solutions.

The components of the iteration sequence (4.5) for the boundary-value problem (5.1) under the linear parametrized two-point boundary conditions (5.9) have the form

$$\begin{aligned} x_{m,1}(t, z, \eta, \lambda) &:= z_1 + \int_0^t f_1(s, x_{m-1,1}(s, z, \eta, \lambda), x_{m-1,2}(s, z, \eta, \lambda)) ds - \\ &- 2t \int_0^{\frac{1}{2}} f_1(s, x_{m-1,1}(s, z, \eta, \lambda), x_{m-1,2}(s, z, \eta, \lambda)) ds + \\ &+ 2t(z_2^2 + \eta_1 + 0.025 - z_1), \end{aligned} \quad (5.10)$$

$$\begin{aligned} x_{m,2}(t, z, \eta, \lambda) &:= z_2 + \int_0^t f_2(s, x_{m-1,1}(s, z, \eta, \lambda), x_{m-1,2}(s, z, \eta, \lambda)) ds - \\ &- 2t \int_0^{\frac{1}{2}} f_2(s, x_{m-1,1}(s, z, \eta, \lambda), x_{m-1,2}(s, z, \eta, \lambda)) ds + \\ &+ 2t(0.025 - z_1), \end{aligned} \quad (5.11)$$

where $m = 1, 2, 3, \dots$,

$$x_{0,1}(t, z, \eta, \lambda) = z_1 + 2t(z_2^2 + \eta_1 + 0.025 - z_1), \quad (5.12)$$

$$x_{0,2}(t, z, \eta, \lambda) = z_2 + 2t(0.025 - z_1). \quad (5.13)$$

The system of approximate determining equations (4.53)–(4.55) depending on the number of iterations for the given example is

$$\begin{aligned} \Delta_{m,1}(z, \eta, \lambda) = -2 \int_0^{\frac{1}{2}} f_1(s, x_{m-1,1}(s, z, \eta, \lambda), x_{m-1,2}(s, z, \eta, \lambda)) ds + \\ + 2(z_2^2 + \eta_1 + 0.025 - z_1) = 0, \end{aligned} \quad (5.14)$$

$$\begin{aligned} \Delta_{m,2}(z, \eta, \lambda) = -2 \int_0^{\frac{1}{2}} f_2(s, x_{m-1,1}(s, z, \eta, \lambda), x_{m-1,2}(s, z, \eta, \lambda)) ds + \\ + 2(0.025 - z_1) = 0, \end{aligned} \quad (5.15)$$

$$x_{m,1}\left(\frac{1}{4}, z, \eta, \lambda\right) = \eta_1, x_{m,2}\left(\frac{1}{4}, z, \eta, \lambda\right) = \eta_2, \quad (5.16)$$

$$x_{m,1}\left(\frac{1}{2}, z, \eta, \lambda\right) = \lambda_1, x_{m,2}\left(\frac{1}{2}, z, \eta, \lambda\right) = \lambda_2. \quad (5.17)$$

Using (5.10)–(5.13), as a result of the first iteration ($m = 1$), by Maple 13 we get:

$$\begin{aligned} x_{11} = z_1 - 0.001666666667t^3 + 0.00125t^2 - 0.05t^2z_1 + \\ + 0.04979166666t - 1.975tz_1 + 2tz_2^2 + 2t\eta_1, \end{aligned}$$

$$\begin{aligned} x_{12} = z_2 + 0.002t^5 - 0.0008333333332t^3 + 0.06666666666t^3z_1 - \\ - 1.333333333t^3z_1^2 + 0.5t^2z_2^2 + 0.5t^2\eta_1 + 0.875t^2 - 0.5t^2z_1 - \\ - 0.05t^2z_2 + 2t^2z_2z_1 - 1.766666667tz_1 - 0.25tz_2^2 + \\ + 0.006333333333t + 0.3333333334tz_1^2 - 0.25t\eta_1 + \\ + 0.025z_2t - tz_2z_1, \end{aligned}$$

for all $t \in [0, \frac{1}{2}]$.

The system (5.14)–(5.17), after the first iteration, has the form

$$\begin{aligned} \Delta_{1,1}(z, \eta, \lambda) := -0.0500263021 - 1.975937500z_1 - \\ - 0.002083333334z_1^2 + 2.001041667\eta_1 + 0.004166666666z_2z_1 - \\ - 0.05010416666z_2 + 2.001041667z_2^2 = 0, \end{aligned} \quad (5.18)$$

$$\begin{aligned} \Delta_{1,2}(z, \eta, \lambda) := 0.1631250001z_2z_1^2 - 0.04177083332z_2\eta_1 - \\ - 0.1471874999z_2^2z_1 - 0.00208333338z_1^2\eta_1 - 0.000833333351z_1^3z_2 + \\ + 0. - 6250000002z_1^2z_2^2 + 0.0005208333338\eta_1^2 + 0.01989583334\eta_1z_1 + \\ + 0.001041666668\eta_1z_2^2 + 0.00416666667z_2^3z_1 - 0.04177083332z_2^3 + \\ + 0.002116402121z_1^4 - 0.04257275137z_1^3 + 0.0005208333338z_2^4 + \\ + 0.006258188451 + 0.004166666670\eta_1z_2z_1 + 0.3158912369z_1^2 - \\ - 0.9658448663z_2z_1 - 2.263765001z_1 + 0.01775263207z_2 - \\ - 0.2503388207\eta_1 + 0.7538330543z_2^2 = 0, \end{aligned} \quad (5.19)$$

$$0.503125z_1 + 0.01249999999 + 0.5z_2^2 + 0.5\eta_1 = \eta_1, \quad (5.20)$$

$$1.003125z_2 + 0.007041015625 - 0.4718750001z_1 + 0.06250000002z_1^2 - 0.03125z_2^2 - 0.03125\eta_1 - 0.125z_2z_1 = \eta_2. \quad (5.21)$$

The computation shows that the approximate solutions of the approximate determining system (5.18)–(5.21) are

$$\begin{aligned} z_1 &:= z_{11} = -1.732102940 \cdot 10^{-8}, \\ z_2 &:= z_{12} = -0.000005209304726, \\ \eta_1 &:= \eta_{11} = 0.02499998258, \\ \eta_2 &:= \eta_{12} = 0.006254548758, \\ \lambda_1 &:= \lambda_{11} = 0.0499999826, \\ \lambda_2 &:= \lambda_{12} = 0.02499480802. \end{aligned}$$

The first approximation for the first and second components of the solution is

$$\begin{aligned} x_{11} &= -1.73210294 \cdot 10^{-8} - 0.001666666667t^3 + \\ &\quad + 0.001250000866t^2 + 0.09979166608t, \\ x_{12} &= -0.000005209304726 + 0.002t^5 - 0.0008333344879t^3 + \\ &\quad + 0.1000002604t^2 + 0.00008323804922t. \end{aligned}$$

The graphs of first approximation and the exact solution of the original boundary-value problem are shown on Figure 1. The error of the first approximation is

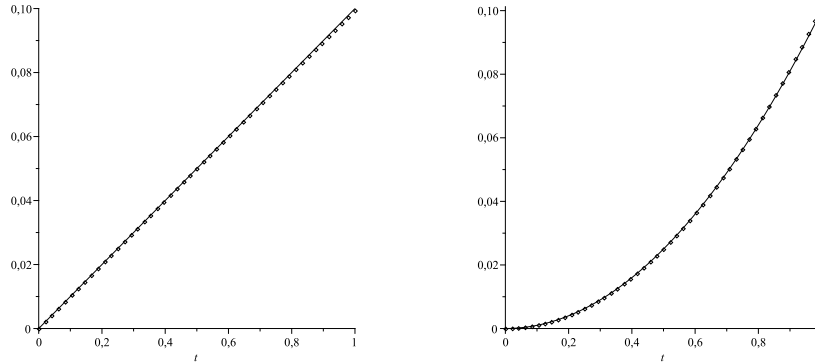


FIGURE 1. The first components of the exact solution (solid line) and its first approximation (drawn with dots)

$$\max_{t \in [0, \frac{1}{2}]} |x_1^*(t) - x_{11}(t)| \leq 1.0041 \cdot 10^{-5},$$

$$\max_{t \in [0, \frac{1}{2}]} |x_2^*(t) - x_{12}(t)| \leq 6,8 \cdot 10^{-6}.$$

The error of the third approximation is

$$\max_{t \in [0, \frac{1}{2}]} |x_1^*(t) - x_{31}(t)| \leq 1.51 \cdot 10^{-9},$$

$$\max_{t \in [0, \frac{1}{2}]} |x_2^*(t) - x_{32}(t)| \leq -1.264 \cdot 10^{-9}.$$

Continuing our iterations we will get more precise approximate solutions of the original boundary-value problem.

ACKNOWLEDGEMENT

This research was carried out as part of the TAMOP-4.2.1.B-10/2/KONV-2010-0001 project supported by the European Union and co-financed by the European Social Fund.

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Authors' addresses

Miklós Rontó

Department of Analysis, University of Miskolc, 3515, Miskolc Egyetemváros
E-mail address: matronto@uni-miskolc.hu

K. Marynets

Uzhhorod National University, Ukraine
E-mail address: katya_marinets@ukr.net



ON THE EDGE-TO-VERTEX GEODETIC NUMBER OF A GRAPH

A. P. SANTHAKUMARAN AND J. JOHN

Received 4 May, 2011

Abstract. Let $G = (V, E)$ be a connected graph with at least three vertices. For vertices u and v in G , the *distance* $d(u, v)$ is the length of a shortest $u - v$ path in G . A $u - v$ path of length $d(u, v)$ is called a $u - v$ *geodesic*. For subsets A and B of V , the distance $d(A, B)$, is defined as $d(A, B) = \min \{d(x, y) : x \in A, y \in B\}$. A $u - v$ path of length $d(A, B)$ is called an $A - B$ *geodesic* joining the sets $A, B \subseteq V$, where $u \in A$ and $v \in B$. A vertex x is said to *lie on* an $A - B$ geodesic if x is a vertex of an $A - B$ geodesic. A set $S \subseteq E$ is called an edge-to-vertex geodetic set if every vertex of G is either incident with an edge of S or lies on a geodesic joining a pair of edges of S . The edge-to-vertex geodetic number $g_{ev}(G)$ of G is the minimum cardinality of its edge-to-vertex geodetic sets and any edge-to-vertex geodetic set of cardinality $g_{ev}(G)$ is an edge-to-vertex geodetic basis of G . Any edge-to-vertex geodetic basis is also called a g_{ev} -set of G . It is shown that if G is a connected graph of size q and diameter d , then $g_{ev}(G) \leq q - d + 2$. It is proved that, for a tree T with $q \geq 2$, $g_{ev}(T) = q - d + 2$ if and only if T is a caterpillar. For positive integers r, d and $l \geq 2$ with $r \leq d \leq 2r$, there exists a connected graph G with $rad G = r$, $diam G = d$ and $g_{ev}(G) = l$. Also graphs G for which $g_{ev}(G) = q, q - 1$ or $q - 2$ are characterized.

2000 *Mathematics Subject Classification:* 05C12*Keywords:* distance, geodesic, edge-to-vertex geodetic basis, edge-to-vertex geodetic number

1. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q , respectively. We consider connected graphs with at least three vertices. For basic definitions and terminologies we refer to [1, 6]. For vertices u and v in a connected graph G , the *distance* $d(u, v)$ is the length of a shortest $u - v$ path in G . An $u - v$ path of length $d(u, v)$ is called an $u - v$ *geodesic*. A vertex v is said to *lie on* an $x - y$ geodesic P if v is a vertex of P including the vertices x and y . A vertex v is an *internal vertex* of an $x - y$ path P if v is a vertex of P and $v \neq x, y$. An edge e of G is an *internal edge* of an $x - y$ path P if e is an edge of P with both its ends internal vertices of P . An edge e is a *pendant edge* if one of its ends is of degree 1. For a vertex v of G , the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v . The minimum eccentricity among the vertices of G is the *radius*, $rad G$ and the

maximum eccentricity is its *diameter*, $\text{diam } G$ of G . A *double star* is a tree of diameter 3. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v . A vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete.

The *closed interval* $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G , while for $S \subseteq V$, $I[S] = \bigcup_{x, y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if

$I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set*. The geodetic number of a graph was introduced in [7] and further studied in [2],[3] and [4]. It was shown in [7] that determining the geodetic number of a graph is an NP-hard problem. The forcing geodetic number of graph was introduced and studied in [5]. The connected geodetic number of graph was studied in [11]. The upper connected geodetic number and forcing connected geodetic number of a graph were studied in [12].

The edge geodetic number of a graph was studied by in [9]. An *edge geodetic set* of a connected graph G with at least two vertices is a set $S \subseteq V$ such that every edge of G is contained in a geodesic joining some pair of vertices in S . The *edge geodetic number* $g_1(G)$ of G is the minimum order of its edge geodetic sets and any edge geodetic set of order $g_1(G)$ is an *edge geodetic basis* of G .

Consider the graph G given in Figure 1. The sets $S = \{v_3, v_5\}$ and $S_1 = \{v_1, v_2, v_4\}$ are minimum geodetic set and minimum edge geodetic set of G respectively so that $g(G) = 2$ and $g_1(G) = 3$. These concepts have many applications in location the-

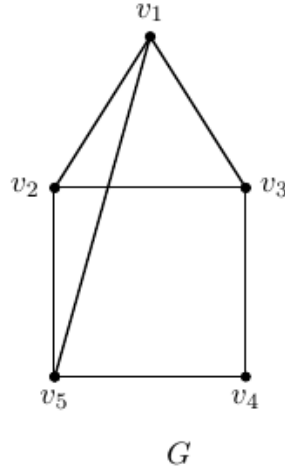


FIGURE 1.

ory and convexity theory. There are interesting applications of these concepts to the problem of designing the route for a shuttle and communication network design. We

further extend these concepts to the edge set of G and present several interesting results in [10].

Throughout the following G denotes a connected graph with at least three vertices.

For subsets A and B of V , the *distance* $d(A, B)$ is defined as $d(A, B) = \min \{d(x, y) : x \in A, y \in B\}$. A $u-v$ path of length $d(A, B)$ is called an $A-B$ *geodesic* joining the sets A, B , where $u \in A$ and $v \in B$. A vertex x is said to lie on an $A-B$ geodesic if x is a vertex of an $A-B$ geodesic. For $A = \{u, v\}$ and $B = \{z, w\}$ with uv and zw edges, we write an $A-B$ geodesic as $uv-zw$ geodesic and $d(A, B)$ as $d(uv, zw)$.

For the graph G given in Figure 2 with $A = \{v_4, v_5\}$ and $B = \{v_1, v_2, v_7\}$, the paths $P : v_5, v_6, v_7$ and $Q : v_4, v_3, v_2$ are the only two $A-B$ geodesics so that $d(A, B) = 2$. A set $S \subseteq E$ is called an *edge-to-vertex geodetic set* if every vertex of G is either

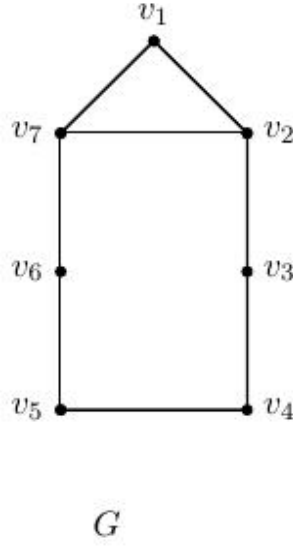


FIGURE 2.

incident with an edge of S or lies on a geodesic joining a pair of edges of S . The *edge-to-vertex geodetic number* $g_{ev}(G)$ of G is the minimum cardinality of its edge-to-vertex geodetic sets and any edge-to-vertex geodetic set of cardinality $g_{ev}(G)$ is an *edge-to-vertex geodetic basis* of G .

For the graph G given in Figure 3, the three $v_1v_6-v_3v_4$ geodesics are $P : v_1, v_2, v_3$; $Q : v_1, v_2, v_4$; and $R : v_6, v_5, v_4$ with each of length 2 so that $d(v_1v_6, v_3v_4) = 2$. Since the vertices v_2 and v_5 lie on the $v_1v_6-v_3v_4$ geodesics P and R respectively, $S = \{v_1v_6, v_3v_4\}$ is an edge-to-vertex geodetic basis of G so that $g_{ev}(G) = 2$. For the graph G given in Figure 2, $S_1 = \{v_1v_2, v_1v_7, v_4v_5\}$ and $S_2 = \{v_1v_2, v_4v_5, v_6v_7\}$ are two g_{ev} -sets of G . Thus there can be more than one g_{ev} -set of G .

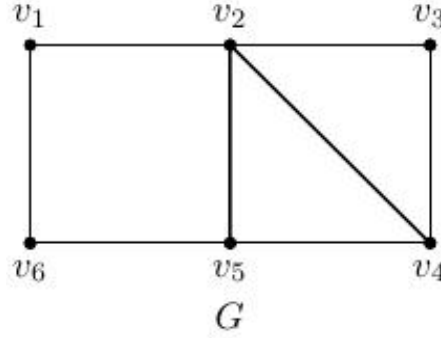


FIGURE 3.

For a connected graph G of size $q \geq 2$, it is clear that $2 \leq g_{ev}(G) \leq q$. Further, these bounds for $g_{ev}(G)$ are sharp. For the star $G = K_{1,q}$ ($q \geq 2$), it is clear that the set of all edges is the unique edge-to-vertex geodetic set so that $g_{ev}(G) = q$. The set of two end-edges of a path P of length at least 2 is its unique edge-to-vertex geodetic basis so that $g_{ev}(P) = 2$. Thus the star $K_{1,q}$ has the largest possible edge-to-vertex geodetic number q and the paths of length at least 2 have the smallest edge-to-vertex geodetic number 2.

An edge of a connected graph G is called an *extreme edge* of G if one of its ends is an extreme vertex of G . An edge e of a connected graph G is an *edge-to-vertex geodetic edge* in G if e belongs to every edge-to-vertex geodetic basis of G . If G has a unique edge-to-vertex geodetic basis S , then every edge in S is an edge-to-vertex geodetic edge of G .

For the graph G given in Figure 4, $S = \{ux, zv\}$ is the unique edge-to-vertex geodetic basis so that both the edges in S are edge-to-vertex geodetic edges of G . For the graph G given in Figure 5, $S_1 = \{v_1v_2, v_6v_7, v_7v_8\}$, $S_2 = \{v_1v_2, v_5v_6, v_7v_8\}$ and $S_3 = \{v_1v_2, v_5v_8, v_6v_7\}$ are the only g_{ev} -sets of G so that every g_{ev} -set contains the edge v_1v_2 . Hence the edge v_1v_2 is the unique edge-to-vertex geodetic edge of G . The following theorems from [10] are used in the sequel.

Theorem 1. *If v is an extreme vertex of a connected graph G , then every edge-to-vertex geodetic set contains at least one extreme edge that is incident with v .*

Theorem 2. *Every pendant edge of a connected graph G belongs to every edge-to-vertex geodetic set of G .*

Theorem 3. *For a non-trivial tree T with k end-vertices, $g_{ev}(T) = k$ and the set of all pendant edges of T is the unique edge-to-vertex geodetic basis of T .*

Theorem 4. *For the complete graph K_p ($p \geq 4$) with p even, $g_{ev}(K_p) = p/2$.*

Theorem 5. *For the cycle C_p ($p \geq 4$), $g_{ev}(C_p) = \begin{cases} 2 & \text{if } p \text{ is even} \\ 3 & \text{if } p \text{ is odd.} \end{cases}$*

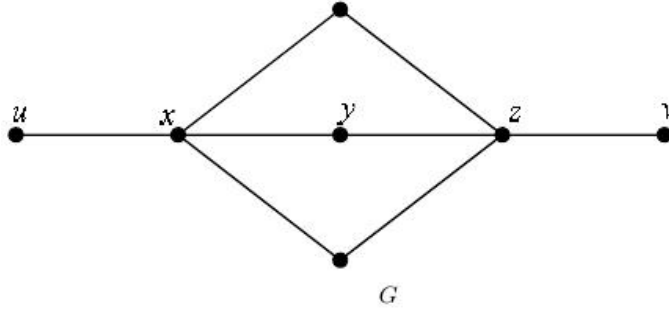


FIGURE 4.

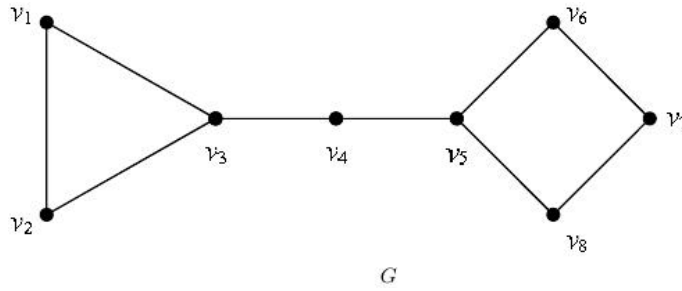


FIGURE 5.

2. THE EDGE-TO-VERTEX GEODETIC NUMBER AND DIAMETER OF A GRAPH

If G is a connected graph of size $q \geq 2$, then $2 \leq g_{ev}(G) \leq q$. An improved upper bound for the edge-to-vertex geodetic number of a graph can be given in terms of its size q and diameter d .

Theorem 6. *For a connected graph G with $q \geq 2$, $g_{ev}(G) \leq q - d + 2$, where d is the diameter of G .*

Proof. Let u and v be vertices of G for which $d(u, v) = d$, where d is the diameter of G and let $P : u = v_0, v_1, v_2, \dots, v_d = v$ be a $u - v$ path of length d . Let $e_i = v_{i-1}v_i$ ($1 \leq i \leq d$). Let $S = E(G) - \{v_1v_2, v_2v_3, \dots, v_{d-2}v_{d-1}\}$. Let x be a vertex of G . If $x = v_i$ ($1 \leq i \leq d-1$), then x lies on the $e_1 - e_d$ geodesic $P_1 : v_1, v_2, \dots, v_{d-1}$. If $x \neq v_i$ ($1 \leq i \leq d-1$), then x is incident with an edge of S . Therefore, S is an edge-to-vertex geodetic set of G . Consequently, $g_{ev}(G) \leq |S| = q - d + 2$. $\square$

Remark 1. The bound in Theorem 6 is sharp. For the star $G = K_{1,q}$ ($q \geq 2$), $d = 2$ and $g_{ev}(G) = q$, by Theorem 3 so that $g_{ev}(G) = q - d + 2$.

We give below a characterization theorem for trees.

A *caterpillar* is a tree for which the removal of all end-vertices leaves a path.

Theorem 7. *Let $q \geq 2$. For any tree T with diameter d , $g_{ev}(T) = q - d + 2$ if and only if T is a caterpillar.*

Proof. Let $P : v_0, v_1, \dots, v_{d-1}, v_d$ be a diametral path of length d . Let $e_i = v_{i-1}v_i$ ($1 \leq i \leq d$) be the edges of the diametral path P . Let k be the number of pendant edges of T and l be the number of internal edges of T other than e_i ($2 \leq i \leq d-1$). Then $d - 2 + l + k = q$. By Theorem 3, $g_{ev}(T) = k$ and so $g_{ev}(T) = q - d + 2 - l$. Hence $g_{ev}(T) = q - d + 2$ if and only if $l = 0$, if and only if all internal vertices of T lie on the diametral path P , if and only if T is a caterpillar. $\square$

The following theorem gives a realization result.

Theorem 8. *For each triple d, k, q of integers with $2 \leq k \leq q - d + 2$, $d \geq 4$ and $q - d + k + 1 > 0$, there exists a connected graph G of size q with $\text{diam } G = d$ and $g_{ev}(G) = k$.*

Proof. Let $2 \leq k = q - d + 2$. Let G be the graph obtained from the path P of length d by adding $q - d$ new vertices to P and joining them to a cut-vertex of P . Then G is a tree of size q and $\text{diam } G = d$. By Theorem 3, $g_{ev}(G) = q - d + 2 = k$. Now, let $2 \leq k < q - d + 2$.

Case 1. $q - d - k + 1$ is even. Let $(q - d - k + 1) \geq 2$. Let $n = \frac{q-d-k+1}{2}$. Then $n \geq 1$. Let $P_d : u_0, u_1, \dots, u_d$ be a path of length d . Add new vertices $v_1, v_2, \dots, v_{k-2}$ and $w_1, w_2, \dots, w_n$ and join each v_i ($1 \leq i \leq k-2$) with u_1 and also join each w_i ($1 \leq i \leq n$) with u_1 and u_3 in P_d . Now, join w_1 with u_2 and we obtain the graph G in Figure 6. Then G has size q and diameter d . By Theorem 2, all the pendant edges u_1v_i ($1 \leq i \leq k-2$), u_0u_1 and $u_{d-1}u_d$ lie in every edge-to-vertex geodetic set of G . Let $S = \{u_1v_1, u_1v_2, \dots, u_1v_{k-2}, u_0u_1, u_{d-1}u_d\}$ be the set of all pendant edges of G . Then it is clear that S is an edge-to-vertex geodetic set of G and so $g_{ev}(G) = k$.

Case 2. $q - d - k + 1$ is odd. Let $q - d - k + 1 \geq 5$. Let $m = \frac{q-d-k}{2}$. Then $m \geq 2$. Let $P_d : u_0, u_1, \dots, u_d$ be a path of length d . Add new vertices $v_1, v_2, \dots, v_{k-2}$ and $w_1, w_2, \dots, w_m$ and join each v_i ($1 \leq i \leq k-2$) with u_1 and also join each w_i ($1 \leq i \leq m$) with u_1 and u_3 in P_d . Now join w_1 and w_2 with u_2 and we obtain the graph G in Figure 7. Then G has size q and diameter d . Now, as in Case 1, $S = \{u_1v_1, u_1v_2, \dots, u_1v_{k-2}, u_0u_1, u_{d-1}u_d\}$ is an edge-to-vertex geodetic set of G so that $g_{ev}(G) = k$. Let $q - d - k + 1 = 1$. Let $P_d : u_0, u_1, \dots, u_d$ be a path of length d . Add new vertices $v_1, v_2, \dots, v_{k-2}$ and w_1 and join each v_i ($1 \leq i \leq k-2$) with u_1 and also join w_1 with u_1 and u_3 in P_d , there by obtaining the graph G in Figure 8. Then the graph is of size q and diameter d . Now, as in Case 1, $S = \{u_1v_1, u_1v_2, \dots, u_1v_{k-2}, u_0u_1, u_{d-1}u_d\}$ is an edge-to-vertex geodetic set of G so that $g_{ev}(G) = k$.

Now, let $q - d - k + 1 = 3$. Let $P_d : u_0, u_1, \dots, u_d$ be a path of length d . Add new vertices $v_1, v_2, v_3, \dots, v_{k-2}, w_1$ and w_2 and join each v_i ($1 \leq i \leq k-2$) with u_1 and

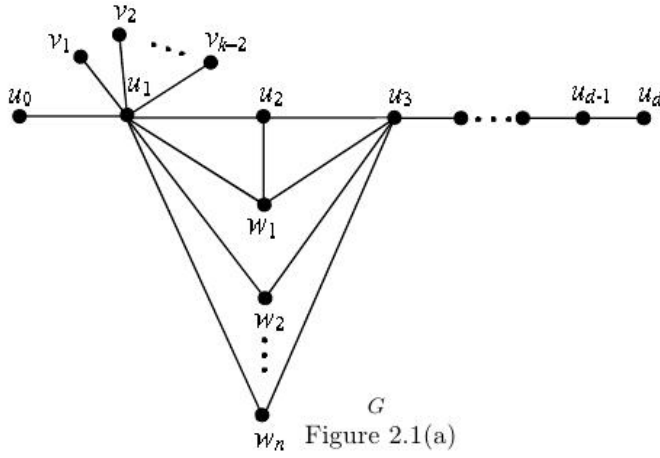


Figure 2.1(a)

FIGURE 6.

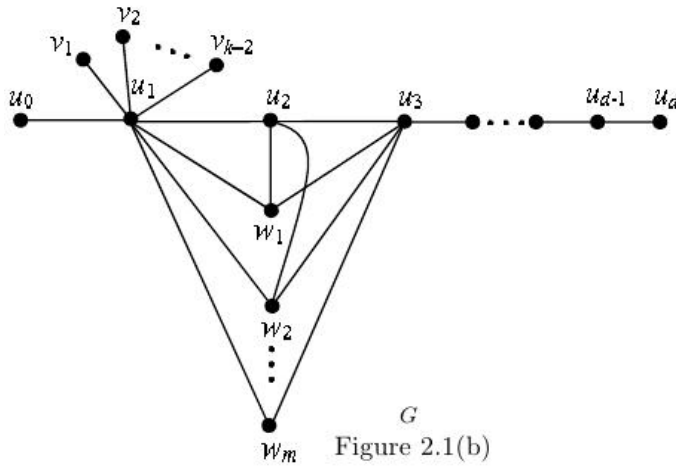


Figure 2.1(b)

FIGURE 7.

also join w_1 and w_2 with u_1 and u_3 and obtain the graph G in Figure 9. Then G has size q and diameter d . Now, as in Case 1, $S = \{u_1v_1, u_1v_2, \dots, u_1v_{k-2}, u_0u_1, u_{d-1}u_d\}$ is an edge-to-vertex geodetic set of G so that $g_{ev}(G) = k$. $\square$

For every connected graph, $rad\ G \leq diam\ G \leq 2\ rad\ G$. Ostrand [8] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter, respectively, of some connected graph. Now, Ostrand's theorem can be extended so that the edge-to-vertex geodetic number can also be prescribed.

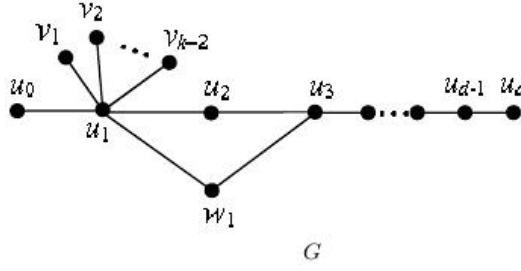


FIGURE 8.

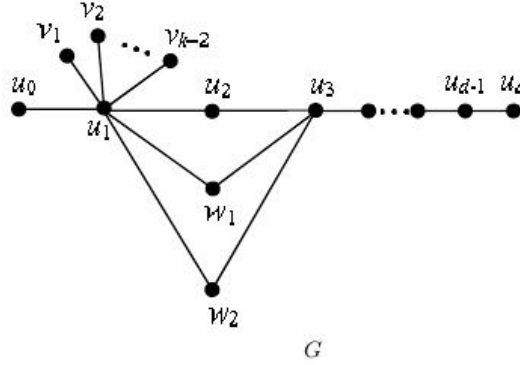


FIGURE 9.

Theorem 9. For positive integers r, d and $l \geq 2$ with $r \leq d \leq 2r$, there exists a connected graph G with $\text{rad } G = r$, $\text{diam } G = d$ and $g_{ev}(G) = l$.

Proof. When $r = 1$, we let $G = K_{2l}$ or $G = K_{1,l}$ according to whether $d = 1$ or $d = 2$ respectively. Then the result follows from Theorem 4 and Theorem 3 respectively. Let $r \geq 2$. If $r = d$ and $l = 2$, let $G = C_{2r}$. Then by Theorem 5, $g_{ev}(G) = 2 = l$. Let $l \geq 3$. Let $C_{2r} : u_1, u_2, \dots, u_{2r}, u_1$ be the cycle of order $2r$. Let G be the graph obtained by adding the new vertices $x_1, x_2, \dots, x_{l-1}$ and joining each x_i ($1 \leq i \leq l-1$) with u_1 and u_2 of C_{2r} . The graph G is shown in Figure 10. It is easily verified that the eccentricity of each vertex of G is r so that $\text{rad } G = \text{diam } G = r$. Let $S = \{u_1x_1, u_1x_2, \dots, u_1x_{l-2}, u_2x_{l-1}\}$. It is clear that S is not an edge-to-vertex geodetic set of G . However, $S \cup \{u_{r+1}u_{r+2}\}$ is an edge-to-vertex geodetic set of G . Since $x_1, x_2, \dots, x_{l-1}$ are the only extreme vertices of G , it follows from Theorem 1 that $g_{ev}(G) = l$.

Let $r < d$. If $l = 2$, then take G to be any path on at least three vertices. Let $l \geq 3$. Let $C_{2r} : v_1, v_2, \dots, v_{2r}, v_1$ be a cycle of order $2r$ and let $P_{d-r+1} : u_0, u_1, u_2, \dots, u_{d-r}$ be a path of order $d-r+1$. Let H be the graph obtained from C_{2r} and u_0 in P_{d-r+1} by identifying v_1 in C_{2r} and u_0 in P_{d-r+1} . Now, add $(l-3)$ new vertices

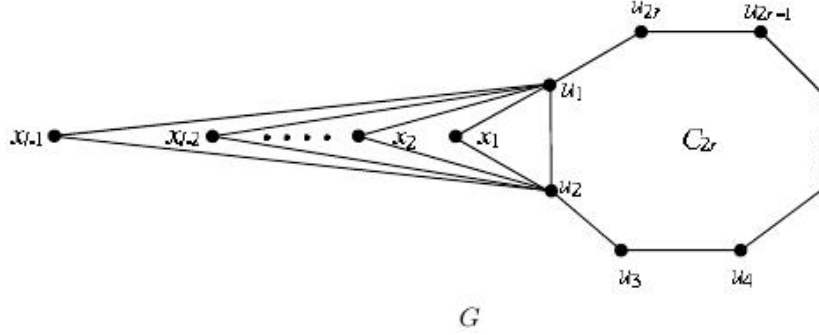


FIGURE 10.

$w_1, w_2, \dots, w_{l-3}$ to H and join each vertex w_i ($1 \leq i \leq l-3$) to the vertex u_{d-r-1} and obtain the graph G of Figure 11. Then $\text{rad } G = r$ and $\text{diam } G = d$. Let $S = \{u_{d-r-1}w_1, u_{d-r-1}w_2, \dots, u_{d-r-1}w_{l-3}, u_{d-r-1}u_{d-r}\}$ be the set of pendant edges of G . By Theorem 2, S is contained in every edge-to-vertex geodetic set of G . It is clear that S is not an edge-to-vertex geodetic set of G . It is also seen that $S \cup \{e\}$, where $e \in E(G) - S$ is not an edge-to-vertex geodetic set of G . However, the set $S_1 = S \cup \{v_r v_{r+1}, v_{r+1} v_{r+2}\}$ is an edge-to-vertex geodetic set of G so that $g_{ev}(G) = l - 2 + 2 = l$. $\square$

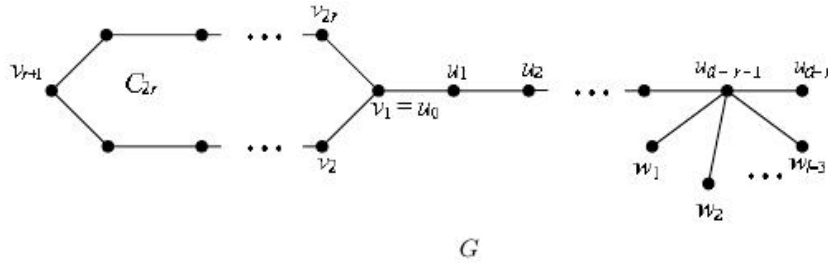


FIGURE 11.

3. GRAPHS G WITH $g_{ev}(G) = q, q-1$ AND $q-2$

In the following we characterize graphs G for which $g_{ev}(G) = q, q-1$ or $q-2$. Let G be a graph. A subset $M \subseteq E(G)$ is called a *matching* of G if no pair of edges in M are incident. The maximum size of such M is called the *matching number* of G and is denoted by $\alpha'(G)$. An *edge covering* of G is subset $K \subseteq E(G)$ such that each vertex of G is an end of some edge in K . The number of edges in a minimum edge covering of G , denoted by $\beta'(G)$, is the *edge covering number* of G . The well-known Gallai's theorem states that if $q \geq 1$, then $\alpha'(G) + \beta'(G) = p$. Since every

edge covering for G is an edge-to-vertex geodetic set, we have the following.

Lemma A. *For any graph G , $g_{ev}(G) \leq \beta'(G) = p - \alpha'(G)$.*

We will make use of this lemma in the sequel. The proofs of the next two theorems are straightforward.

Theorem 10. *If G is a connected graph such that it is not a star, then $g_{ev}(G) \leq q - 1$.*

Theorem 11. *For any connected graph G , $g_{ev}(G) = q$ if and only if G is a star.*

Theorem 12. *Let G be a connected graph which is not a tree. Then $g_{ev}(G) \leq q - 2$ ($q \geq 4$).*

Proof. Since $G \neq C_3$ and it has atleast one cycle, $\alpha'(G) \geq 2$. Thus, by Lemma A, $g_{ev}(G) \leq p - \alpha'(G) \leq q - \alpha'(G) \leq q - 2$. $\square$

Theorem 13. *For any connected graph G with $q \geq 3$, $g_{ev}(G) = q - 1$ if and only if G is either C_3 or a double star.*

Proof. If G is C_3 , then $g_{ev}(G) = 2 = q - 1$. If G is a double star, then by Theorem 3, $g_{ev}(G) = q - 1$. Conversely, let $g_{ev}(G) = q - 1$. If G is a tree, then from Lemma A it follows that $\alpha'(G) \leq 2$. If $\alpha'(G) = 1$, then G is a star, which is impossible due to Theorem 11. So $\alpha'(G) = 2$, which implies that G is a double star. If G is not a tree, then $g_{ev}(G) = q - 1 \geq p - 1$. Again by Lemma A, $\alpha'(G) = 1$, which is the case only when $G = C_3$. Thus the proof is complete. $\square$

Theorem 14. *Let G be a connected graph with $q \geq 4$, which is not a cycle and not a tree and let $C(G)$ be the length of a smallest cycle. Then $g_{ev}(G) \leq q - C(G) + 1$ if $C(G)$ is odd, and $g_{ev}(G) \leq q - C(G) + 2$ if $C(G)$ is even.*

Proof. Let $C(G)$ denote the length of a smallest cycle in G and let C be a cycle of length $C(G)$. We consider two cases.

Case 1. $C(G)$ is odd. First suppose that $C(G) = 3$. Let $C : v_1, v_2, v_3, v_1$ be a cycle of length 3. Since G is not a cycle, there exists a vertex v in G such that v is not on C and v is adjacent to v_1 , say. Let $S = E(G) - \{v_1v_2, v_1v_3\}$. Then every vertex of G lies on an edge of S and so S is an edge-to-vertex geodetic set of G set of G . Thus $g_{ev}(G) \leq q - 2 = q - C(G) + 1$.

Next suppose that $C(G) \geq 5$. Let $C : v_1, v_2, \dots, v_k, v_{k+1}, v_{k+2}, \dots, v_{2k+1}, v_1$ be a cycle of least length $C(G) = 2k + 1$. Since G is not a cycle, there exists a vertex v in G such that v is not on C and v is adjacent to v_1 , say. We claim that $d(vv_1, v_{k+1}v_{k+2}) = k$. Since $P : v_1, v_2, v_3, \dots, v_{k+1}$ is a path of length k on C , it follows that $d(vv_1, v_{k+1}v_{k+2}) \leq k$. If $d(vv_1, v_{k+1}v_{k+2}) \leq k - 1$, then at least one

of $d(v_1, v_i)$ and $d(v, v_i)$ for $i = k+1, k+2$ is less than or equal to $k-1$. First suppose that $d(v_1, v_{k+1}) \leq k-1$. Let Q be a $v_1 - v_{k+1}$ shortest path of length at most $k-1$ different from P . Hence there exists at least one vertex of Q that is not on P and since the length of Q is at most $k-1$, it follows that a cycle of length at most $2k-1$ is formed. This is a contradiction to $C(G) = 2k+1$. Thus $d(v_1, v_{k+1}) = k$. Similarly we can prove that $d(v_1, v_{k+2}) = k$.

Next, suppose that $d(v, v_{k+1}) \leq k-1$. Since $P' : v, v_1, v_2, v_3, \dots, v_{k+1}$ is a path of length $k+1$, it follows that $d(v, v_{k+1}) \leq k+1$. Then, as above, a cycle of length at most $2k$ is formed and this is a contradiction. Hence $d(v, v_{k+1}) = k$ or $k+1$. Similarly we can prove that $d(v, v_{k+2}) = k$ or $k+1$. Since $d(v_1, v_{k+1}) = d(v_1, v_{k+2}) = k$, it follows that $d(vv_1, v_{k+1}v_{k+2}) = k$.

Now, let $S = (E(G) - E(C)) \cup \{v_{k+1}v_{k+2}\}$. It is clear that the vertices $v_2, v_3, \dots, v_k, v_{k+3}, v_{k+4}, \dots, v_{2k+1}$ on the cycle C lie on the $vv_1 - v_{k+1}v_{k+2}$ geodesic on the cycle and all the other vertices of G are incident with an edge of S . Thus S is an edge-to-vertex geodetic set of G and so $g_{ev}(G) \leq q - C(G) + 1$.

Case 2. $C(G)$ is even. First suppose that $C(G) = 4$. Let $C : v_1, v_2, v_3, v_4, v_1$ be a cycle of length 4. Since G is not a cycle, there exists a vertex v in G such that v is not on C and v is adjacent to v_1 , say. Let $S = E(G) - \{v_1v_2, v_1v_4\}$. Then every vertex of G lies on an edge of S and so S is an edge-to-vertex geodetic set of G . Thus $g_{ev}(G) \leq q - 2 = q - C(G) + 2$.

Next suppose that $C(G) \geq 6$. Let $C : v_1, v_2, \dots, v_k, v_{k+1}, v_{k+2}, \dots, v_{2k}, v_1$ be a cycle of least length $C(G) = 2k$. Since G is not a cycle, there exists a vertex v in G such that v is not on C and v is adjacent to v_1 , say. We claim that $d(vv_1, v_kv_{k+1}) = d(vv_1, v_{k+1}v_{k+2}) = k-1$. Since $Q : v_1, v_2, v_3, \dots, v_k$ and $Q' : v_1, v_{2k}, v_{2k-1}, \dots, v_{k+3}, v_{k+2}$ are paths of length $k-1$ on C , it follows that $d(vv_1, v_kv_{k+1}) = d(vv_1, v_{k+1}v_{k+2}) \leq k-1$. If $d(vv_1, v_kv_{k+1}) \leq k-2$ or $d(vv_1, v_{k+1}v_{k+2}) \leq k-2$, then proceeding as in Case 1, a cycle of length at most $2k-3$ or $2k-2$ or $2k-1$ is formed as the case may be, contradicting that the least length of a cycle is $2k$. Thus $d(vv_1, v_kv_{k+1}) = d(vv_1, v_{k+1}v_{k+2}) = k-1$.

Now, if we let $S = (E(G) - E(C)) \cup \{v_kv_{k+1}, v_{k+1}v_{k+2}\}$, then the vertices $v_2, v_3, \dots, v_{k-1}$ lie on the $vv_1 - v_kv_{k+1}$ geodesic on C , the vertices $v_{k+3}, v_{k+4}, \dots, v_{2k}$ lie on the $vv_1 - v_{k+1}v_{k+2}$ geodesic on C and all the other vertices of G are incident with an edge of S . Thus S is an edge-to-vertex geodetic set of G and so $g_{ev}(G) \leq q - C(G) + 2$. $\square$

Theorem 15. *If G is a connected graph of size $q \geq 4$ and not a tree such that $g_{ev}(G) = q - 2$, then G is unicyclic.*

Proof. Let G have more than one cycle. Then $q \geq p + 1$ and so $p - 1 \leq q - 2 = g_{ev}(G) \leq p - \alpha'(G)$, by Lemma A. Hence $\alpha'(G) = 1$ and so G must be either a star or the cycle C_3 , a contradiction. $\square$

Denote by $\mathfrak{S}$ the two classes of graphs given in Figure 12.

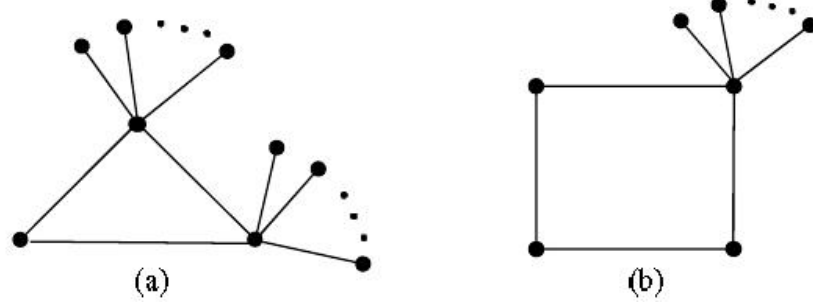


FIGURE 12.

Theorem 16. *For a connected graph G , $g_{ev}(G) = q - 2$ ($q \geq 4$) if and only if G is C_4 or C_5 or $K_{1,q-1} + e$ or caterpillar with $d = 4$ or the class of graphs given in family $\mathfrak{S}$ of Figure 12.*

Proof. For $G = C_4$ or C_5 , the result follows from Theorem 5. For a caterpillar of diameter 4, the result follows from Theorem 3. For $G = K_{1,q-1} + e$, it follows from Theorem 1 that the set of all end edges of G together with e forms an edge-to-vertex geodetic basis so that $g_{ev}(G) = q - 2$. Further, it is easily verified that $g_{ev}(G) = q - 2$ for the graphs given in family $\mathfrak{S}$ of Figure 12.

Now, let G be a connected graph such that $g_{ev}(G) = q - 2$. By Theorem 15, G is either a tree or unicyclic. If G is a tree, then from Lemma A it follows that $\alpha'(G) \leq 3$. By Theorems 12 and 13, $\alpha' > 2$. So $\alpha' = 3$, which implies that G is a Caterpillar of diameter 4. If G is unicyclic, by Lemma A, $\alpha'(G) \leq 2$. Let C_k be the unique cycle of G . We have $k \leq 5$ since otherwise $\alpha'(G) \geq \alpha'(C_k) \geq 3$. Therefore, we have the following three cases:

Case 1. $k = 5$. Then G cannot have any other vertices since otherwise $\alpha'(G) \geq 3$. Therefore $G = C_5$.

Case 2. $k = 4$. If $G = C_4$, we are done. So, let $G \neq C_4$. Because $\alpha'(G) \leq 2$, only one of the vertices of C_4 , say v , is of degree more than 2 and moreover all the neighbors of v are of degree 1. Thus G should be a graph like Figure 12(b).

Case 3. $k = 3$. Since $g_{ev}(C_3) = 2 = q - 1$, we have $G \neq C_3$. Let $V(C_3) = \{v_1, v_2, v_3\}$. We note that if $u \in V(G) - V(C_3)$, then $\deg u = 1$. Otherwise, there are $u_1, u_2 \in V(G) - V(C_3)$ such that u_1 is adjacent to both u_2 and v_1 , say. Then it is easily seen that $E(G) - \{u_1 v_1, v_1 v_2, v_1 v_3\}$ is an edge-to-vertex geodetic set, which implies $g_{ev}(G) \leq q - 3$. Further, at least one of the v_i s should be of degree 2. Otherwise, $E(G) - E(C_3)$ is an edge-to-vertex geodetic set, which is impossible. Thus G should be either $K_{1,q} + e$ or a graph like Figure 12(a). $\square$

ACKNOWLEDGEMENT

The authors express their gratitude to the referees for the useful suggestions.

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*Authors' addresses***A. P. Santhakumaran**

Department of Mathematics, St. Xavier's College (Autonomous), Palayamkottai - 627 002, India.

E-mail address: apskumar1953@yahoo.co.in

J. John

Department of Mathematics, A.C. College of Engineering and Technology, Karaikudi - 630 004, India.



THE INTERSECTION OF THE COMPATIBLE LINEAR EXTENSIONS OF A NATURAL PARTIAL ORDER

SZILVIA SZILÁGYI

Received 16 January, 2012

Abstract. For an acyclic function $f : A \longrightarrow A$ we define a natural f -compatible partial order $\hat{f}$ on A and determine the intersection of the f -compatible linear extensions of $\hat{f}$.

2000 Mathematics Subject Classification: 06A06; 06F99.

Keywords: compatible partial order, compatible linear extension

1. INTRODUCTION

One of the central problems in the theory of ordered algebras is to find necessary and sufficient conditions for the existence of a compatible linear extension of r in a partially ordered algebraic structure $(A, F, \leq_r)$. If $F = \emptyset$, then Szpilrajn proved that any partial order r ($\leq_r$) on a set A can always be extended to a linear order R and any partial order is the intersection of its linear extensions (see [5]). In the present paper we consider a partial order r on the set A and an order endomorphism $f : A \longrightarrow A$ with the natural compatibility property: $x \leq_r y$ implies $f(x) \leq_r f(y)$ for all $x, y \in A$. Clearly, the pair (A, f) is a unary algebra and the above f -compatibility condition allows us to view the triple $(A, f, \leq_r)$ as a partially ordered unary algebra. For $F = \{f\}$ ($f : A \rightarrow A$ is a unary operation) the above mentioned extension problem has been thoroughly investigated. Szigeti and Nagy proved that the partial order r of a unary algebra $(A, f, \leq_r)$ has an f -compatible linear extension if and only if the function $f : A \longrightarrow A$ is acyclic (see [3]). For an acyclic $(A, f, \leq_r)$ the intersection of the f -compatible linear extensions of r is determined in [2]. For an arbitrary $(A, f, \leq_r)$ the maximal f -compatible partial order extensions of r and the intersection are investigated in [1] and [4]. The aim of the present paper is to determine the intersection of the f -compatible linear extensions of $\hat{f}$, where the f -compatible partial order $\hat{f}$ on A can be defined in a natural way starting from an acyclic function $f : A \longrightarrow A$.

2. PRELIMINARIES

Let (A, f) be a (mono-)unary algebra. A partial order r on A is called *f-compatible* if the function (unary operation) $f : A \rightarrow A$ is an order endomorphism with respect to r . We shall make use of the following notation

$$\mathcal{L}(A, f, \leq_r) = \{R \mid r \subseteq R \subseteq A \times A \text{ is an } f\text{-compatible linear order on } A\}.$$

The intersection

$$r_f = \mathbf{cl}(A, f, \leq_r) = \bigcap_{R \in \mathcal{L}(A, f, \leq_r)} R$$

is called the *closure* of r with respect to f . Indeed, the above definition gives a closure operator (with the monotone, idempotent and extensive properties) on the set of the f -compatible partial orders of A .

Definition 1 (see [3]). Let $A \neq \emptyset$ be a set and $N \geq 0$ be an integer. A function $f : A \rightarrow A$ takes N steps on the element $x \in A$, if

$$x, f(x), f^2(x), \dots, f^N(x)$$

are different elements in A , and

$$f^{N+1}(x) = f^N(x)$$

(by convention $f^0(x) = x$). Take $N = \infty$ if

$$f^m(x) \neq f^n(x)$$

for all integers $0 \leq m < n$.

Definition 2 (see [3]). The function $f : A \rightarrow A$ is called *acyclic*, if for each element $x \in A$ there is an integer $0 \leq N = N(x) \leq \infty$ such that f takes $N = N(x)$ steps on x .

A partially ordered unary algebra $(A, f, \leq_r)$ is called *acyclic*, if $f : A \rightarrow A$ is acyclic. It is easy to see that any linearly ordered unary algebra is acyclic. If $(A, f, \leq_r)$ is acyclic, then r_f is an f -compatible partial order on A .

Lemma 1 (see [2]). Let $(A, f, \leq_r)$ be an acyclic partially ordered unary algebra. If $a, b \in A$ and $a \leq_r f(a)$ or $f(a) \leq_r a$, then $(a, b) \in r_f$ implies that $a = b$ or $f^m(a) \leq_r f^m(b)$ and $f^m(a) \neq f^m(b)$ for some integer $m \geq 0$.

3. THE ACYCLIC PARTIALLY ORDERED UNARY ALGEBRA $(A, f, \hat{f})$

Let

$$\langle x \rangle_f = \{x, f(x), f^2(x), \dots\}$$

denote the f -orbit of x and define the following reflexive and transitive relation $\hat{f} \subseteq A \times A$ as follows:

$$x \hat{f} y \Leftrightarrow \langle x \rangle_f \subseteq \langle y \rangle_f.$$

Clearly, $\langle x \rangle_f \subseteq \langle y \rangle_f$ if and only if $x \in \langle y \rangle_f$, i.e. we can find an integer $k \geq 0$ such that $x = f^k(y)$.

Proposition 1. *If (A, f) is a unary algebra, then the following are equivalent.*

- (1) *f is an acyclic function.*
- (2) *$\hat{f}$ is antisymmetric.*

Proof. (1) $\Rightarrow$ (2): Suppose that $x \hat{f} y$ and $y \hat{f} x$ for $x, y \in A$. Then $\langle x \rangle_f = \langle y \rangle_f$ imply that $x = f^m(y)$ and $y = f^n(x)$ for some $m \geq 0$ and $n \geq 0$. Thus

$$x = f^m(f^n(x)) = f^{m+n}(x),$$

whence

$$x = f(x) = \dots = f^n(x) = \dots = f^{n+m}(x)$$

follows from the acyclic property of f . Obviously,

$$x = f^n(x) = y.$$

(2) $\Rightarrow$ (1): Suppose that $f^m(x) = f^n(x)$ for some integers $0 \leq m < n$. Clearly, $\langle f^{m+1}(x) \rangle_f \subseteq \langle f^m(x) \rangle_f$ and $\langle f^m(x) \rangle_f \subseteq \langle f^{m+1}(x) \rangle_f$ is a consequence of

$$f^m(x) = f^{n-m-1}(f^{m+1}(x)).$$

Thus we have

$$f^m(x) \hat{f} f^{m+1}(x) \quad \text{and} \quad f^{m+1}(x) \hat{f} f^m(x).$$

Since $\hat{f}$ is antisymmetric, we get $f^m(x) = f^{m+1}(x)$ and

$$f^m(x) = f^{m+1}(x) = \dots = f^n(x).$$

□

Proposition 2. *Let (A, f) be an acyclic unary algebra. Then the partial order $\hat{f}$ is f -compatible.*

Proof. If the function f is acyclic, then $\hat{f}$ is a partial order by Proposition 1. If $x \hat{f} y$ for $x, y \in A$, then we can find an integer $m \geq 0$ such that

$$x = f^m(y),$$

whence $f(x) = f^m(f(y))$ and $f(x) \hat{f} f(y)$ follows. □

4. THE INTERSECTION

Proposition 3. *If (A, f) is an acyclic unary algebra and $\langle x \rangle_f \cap \langle y \rangle_f \neq \emptyset$ for $x, y \in A$, then there exists a unique $z \in A$ such that*

$$\langle x \rangle_f \cap \langle y \rangle_f = \langle z \rangle_f.$$

This element $z = x \triangle y$ is called the f -intersection of x and y .

Proof. Since $\langle x \rangle_f \cap \langle y \rangle_f \neq \emptyset$, we can define an integer n as follows:

$$n = \min\{k \geq 0 \mid f^k(x) \in \langle y \rangle_f\}.$$

We claim that $\langle x \rangle_f \cap \langle y \rangle_f = \langle z \rangle_f$ for $z = f^n(x)$. Obviously,

$$\langle z \rangle_f \subseteq \langle x \rangle_f \cap \langle y \rangle_f.$$

On the other hand if $u \in \langle x \rangle_f \cap \langle y \rangle_f$ then $u = f^k(x) \in \langle y \rangle_f$ for some $k \geq 0$. Thus $k \geq n$ and $u = f^{k-n}(f^n(x)) = f^{k-n}(z) \in \langle z \rangle_f$. The fact that there is only one $z \in A$ with

$$\langle x \rangle_f \cap \langle y \rangle_f = \langle z \rangle_f$$

is a consequence of Proposition 1. □

Definition 3. Let (A, f) be an acyclic unary algebra. If $\langle x \rangle_f \cap \langle y \rangle_f \neq \emptyset$ for $x, y \in A$, then define the distance of x and y as follows:

$$\delta(x, y) = |(\langle x \rangle_f \setminus \langle y \rangle_f) \cup (\langle y \rangle_f \setminus \langle x \rangle_f)| = |\langle x \rangle_f \setminus \langle y \rangle_f| + |\langle y \rangle_f \setminus \langle x \rangle_f|.$$

We note that

$$\delta(x, y) = |\langle x \rangle_f \setminus \langle x \Delta y \rangle_f| + |\langle y \rangle_f \setminus \langle x \Delta y \rangle_f|.$$

immediately follows from

$$\langle x \rangle_f \setminus \langle y \rangle_f = \langle x \rangle_f \setminus (\langle x \rangle_f \cap \langle y \rangle_f) = \langle x \rangle_f \setminus \langle x \Delta y \rangle_f$$

and

$$\langle y \rangle_f \setminus \langle x \rangle_f = \langle y \rangle_f \setminus \langle x \Delta y \rangle_f.$$

Proposition 4. If (A, f) is an acyclic unary algebra and $\langle x \rangle_f \cap \langle y \rangle_f \neq \emptyset$ for $x, y \in A$, then

$$|\langle x \rangle_f \setminus \langle y \rangle_f| = |\langle x \rangle_f \setminus \langle x \Delta y \rangle_f| = \min\{k \geq 0 \mid f^k(x) \in \langle y \rangle_f\} \leq N(x),$$

where the integer $N(x)$ is the number of f -steps on x .

Proof. If $n = \min\{k \geq 0 \mid f^k(x) \in \langle y \rangle_f\}$, then we have

$$x \Delta y = f^n(x)$$

(see the proof of Proposition 3). We distinguish two cases.

Case 1 If $N(x) = \infty$, then $n < N(x)$ and

$$\langle x \rangle_f \setminus \langle f^n(x) \rangle_f = \{f^k(x) \mid k \geq 0\} \setminus \{f^k(x) \mid k \geq n\}.$$

Thus

$$|\langle x \rangle_f \setminus \langle f^n(x) \rangle_f| = |\{x, f(x), \dots, f^{n-1}(x)\}| = n.$$

Case 2 If $N = N(x) < \infty$ and $N < n$, then

$$\langle x \rangle_f \cap \langle y \rangle_f = \{x, f(x), \dots, f^N(x)\} \cap \langle y \rangle_f = \emptyset,$$

a contradiction. Thus $n \leq N$ and

$$\langle x \rangle_f \setminus \langle f^n(x) \rangle_f = \{x, f(x), \dots, f^N(x)\} \setminus \{f^n(x), \dots, f^N(x)\},$$

i.e.

$$|\langle x \rangle_f \setminus \langle f^n(x) \rangle_f| = |\{x, f(x), \dots, f^{n-1}(x)\}| = n.$$

□

The next theorem gives a complete description of the intersection of the f -compatible linear extensions of the partial order $\hat{f}$.

Theorem 1. *If (A, f) is an acyclic unary algebra and $x, y \in A$, then:*

$$(x, y) \in (\hat{f})_f \Leftrightarrow x = y, \text{ or } \langle x \rangle_f \cap \langle y \rangle_f \neq \emptyset \text{ and } \delta(x, x \triangle y) < \delta(y, x \triangle y).$$

Proof. Since $\langle f(x) \rangle_f \subseteq \langle x \rangle_f$ (i.e. $f(x) \leq_{\hat{f}} x$) for all $x \in A$, the application of Lemma 1 gives that

$$(\hat{f})_f = \{(x, y) \in A \times A \mid (\exists m) 0 \leq m, f^m(x) \hat{f} f^m(y), f^m(x) \neq f^m(y)\} \cup \{(x, x) \mid x \in A\}.$$

If $(x, y) \in (\hat{f})_f$ and $x \neq y$, then $\langle f^m(x) \rangle_f \subseteq \langle f^m(y) \rangle_f$ and $f^m(x) \neq f^m(y)$ for some integer $m \geq 0$. Suppose that

$$k = \delta(x, x \triangle y) \geq \delta(y, x \triangle y) = l.$$

Since $f^m(x) \in \langle y \rangle_f$, we have $m \geq k$ (see Proposition 4). Now

$$f^m(x) = f^{m-k}(f^k(x)) = f^{m-k}(x \triangle y)$$

and

$$f^m(y) = f^{m-l}(f^l(y)) = f^{m-l}(x \triangle y)$$

imply that

$$f^m(y) = f^{k-l}(f^m(x)).$$

It follows that $\langle f^m(y) \rangle_f \subseteq \langle f^m(x) \rangle_f$. The antisymmetric property of $\hat{f}$ gives that $f^m(x) = f^m(y)$, a contradiction.

If $\langle x \rangle_f \cap \langle y \rangle_f \neq \emptyset$ and

$$m = \delta(x, x \triangle y) < \delta(y, x \triangle y) = l,$$

then

$$f^m(x) = x \triangle y = f^l(y) = f^{l-m}(f^m(y))$$

ensures that $\langle f^m(x) \rangle_f \subseteq \langle f^m(y) \rangle_f$. On the other hand, using Proposition 4,

$$m < l \leq N(y)$$

implies that

$$f^m(y) \neq f^l(y) = f^m(x).$$

Thus we have $(x, y) \in (\hat{f})_f$.

□

ACKNOWLEDGEMENT

This research was carried out as part of the TAMOP- 4.2.1.B-10/2/KONV-2010-0001 project with support from the European Union, co-financed by the European Social Fund.

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Author's address

Szilvia Szilágyi

University of Miskolc, Institute of Mathematics, Miskolc-Egyetemváros, 3515 Miskolc, Hungary

E-mail address: matszisz@uni-miskolc.hu



NONLOCAL CAUCHY PROBLEMS FOR FRACTIONAL EVOLUTION EQUATIONS INVOLVING VOLTERRA-FREDHOLM TYPE INTEGRAL OPERATORS

JINRONG WANG, WEI WEI, AND MICHAL FEČKAN

Received 19 December, 2011

Abstract. In this paper, nonlocal Cauchy problems for fractional evolution equations involving Volterra-Fredholm type integral operators are investigated. Some new existence theorems of mild solutions are presented by using fractional calculus, Hölder inequality, Beta function and fixed point theorems.

2000 Mathematics Subject Classification: 26A33; 47J35

Keywords: fractional evolution equations, nonlocal Cauchy problems, mild solutions, analytic compact semigroup, fixed point theorems

1. INTRODUCTION

The fractional differential equations has recently been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, economy and science. We can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. [8, 11, 12, 14, 22, 23]. There has been a significant development in fractional differential equations. For more details on fractional calculus theory, one can see the monographs of Diethelm [9], Kilbas et al. [17], Lakshmikantham et al. [19], Michalski [24], Miller and Ross [25], Podlubny [29] and Tarasov [30]. In the last years, the theory of fractional differential equations attracted the attention of many authors (see for instance [1–7, 10, 13, 15, 16, 21, 26, 27, 32, 34, 35] and references therein).

However, to our knowledge, nonlocal Cauchy problems for fractional evolution equations involving Volterra-Fredholm type integral operators has not been discussed extensively. Motivated by the above mentioned works (including our papers [6,

The first and second authors acknowledge the support by Key Projects of Science and Technology Research in the Chinese Ministry of Education (211169), Natural Science Foundation of Guizhou Province (2010, No.2142). The third author acknowledges the support by Grants VEGA-MS 1/0507/11, VEGA-SAV 2/0124/10 and APVV-0134-10.

20, 21, 26, 31–33, 35]), the main purpose of this paper is to consider the following nonlocal Cauchy problems for fractional evolution equations involving Volterra–Fredholm type integral operators such as

$$\begin{cases} {}^c D_t^q x(t) = -Ax(t) + t^n f(t, x(t), (Kx)(t), (Hx)(t)), \\ t \in J = [0, T], n \in \mathbb{Z}^+, q \in (0, 1), \\ x(0) = g(x) + x_0, \end{cases} \quad (1.1)$$

where the fractional derivative ${}^c D_t^q$ is understood here in the Caputo sense, $-A : D(A) \rightarrow X$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{S(t), t \geq 0\}$, the Volterra type integral operator K and Fredholm type integral operator H are defined by

$$(Kx)(t) = \int_0^t k(t, s, x(s))ds, \quad (Hx)(t) = \int_0^T h(t, s, x(s))ds.$$

The function $f : J \times X_\alpha \times X_\alpha \times X_\alpha \rightarrow X$ (or X_α , or X_μ) is continuous where $X_\alpha = D(A^\alpha)$, $0 \leq \mu \leq \alpha \leq 1$, is a Banach space with the norm $\|x\|_\alpha = \|A^\alpha x\|$ for $x \in X_\alpha$. f , k , h and g are specified latter. It is easy to see that term t^n appears before the nonlinear term f . We remark that this term t^n will help us to overcome the essential difficult caused by the singular term $(t-s)^{q-1}$ in the formula of the solutions due to the well known Beta function.

In the present paper, we discuss the existence and uniqueness of mild solutions for system (1.1). Our results cover the cases for the nonlinear term f taking values in the spaces such as X , X_α , X_μ , where $0 \leq \mu \leq \alpha \leq 1$, the nonlocal term g is linear completely continuous or satisfies the Lipschitz continuous condition. The main techniques used here are Hölder's inequality, Beta function via Banach contraction principle, Schauder's fixed point theorem for compact maps and Sadovskii's fixed point theorem for condensing maps.

The rest of this paper is organized as follows. In Section 2, we give some known preliminary results on the fraction powers of the generator of an analytic compact semigroup and introduce the mild solution of system (1.1). In Section 3, we study the existence of mild solutions for system (1.1) by using fractional calculus, Hölder inequality via Banach contraction principle, Schauder's fixed point theorem and Sadovskii's fixed point theorem. At last, an example is given to demonstrate the applicability of our result.

2. PRELIMINARIES

In this section, we introduce some facts about the fractional powers of the generator of a compact analytic semigroup, the Riemann-Liouville fractional integral operator that are used throughout this paper.

We denote by X a Banach space with norm $\|\cdot\|$ and $-A : D(A) \rightarrow X$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{S(t), t \geq 0\}$. This means that there exists $M > 1$ such that $\|S(t)\| \leq M$.

We assume without loss of generality that $0 \in \rho(A)$. This allows us to define the fractional power A^α for $0 \leq \alpha \leq 1$, as a closed linear operator on its domain $D(A^\alpha)$ with inverse $A^{-\alpha}$ (see [28]).

In the sequel, we will also use $\|l\|_{L^p(J, R^+)}$ to denote the $L^p(J, R^+)$ norm of l whenever $l \in L^p(J, R^+)$ for some p with $1 \leq p < \infty$. We will set $\alpha \in [0, 1]$ and denote by C_α , the Banach space $C(J, X_\alpha)$ endowed with supnorm given by $\|x\|_\infty = \sup_{t \in J} \|x\|_\alpha$, for $x \in C_\alpha$.

Let us recall the following known definitions. For more details, see [17].

Definition 1. The fractional integral of order γ with lower limit zero for a function l is defined as

$$I^\gamma l(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{l(s)}{(t-s)^{1-\gamma}} ds, \quad t > 0, \gamma > 0,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2. The Riemann-Liouville derivative of order γ with lower limit zero for a function $l : [0, \infty) \rightarrow R$ can be written as

$${}^L D_t^\gamma l(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{l(s)}{(t-s)^{\gamma+1-n}} ds, \quad t > 0, n-1 < \gamma < n.$$

Definition 3. The Caputo derivative of order γ for a function $l : [0, \infty) \rightarrow R$ can be written as

$${}^c D_t^\gamma l(t) = {}^L D_t^\gamma \left(l(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} l^{(k)}(0) \right), \quad t > 0, n-1 < \gamma < n.$$

Remark 1. (i) If $l \in C^1[0, \infty)$, then

$${}^c D_t^\gamma l(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{l'(s)}{(t-s)^\gamma} ds = I^{1-\gamma} l'(t), \quad t > 0, 0 < \gamma < 1.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If l is an abstract function with values in X , then integrals which appear in Definitions 1 and 2 are taken in Bochner's sense.

Motivated by Definition 3.1 of [35], we adopt the following concept of mild solution for our problem.

Definition 4. By the mild solution of system (1.1), we mean that the function $x : J \rightarrow X_\alpha$ which satisfies

$$x(t) = \mathcal{T}(t)[x_0 + g(x)] + \int_0^t (t-s)^{q-1} s^n \mathcal{S}(t-s) f(s, x(s), (Kx)(s), (Hx)(s)) ds$$

for any $t \in J$, where

$$\mathcal{T}(t) = \int_0^\infty \xi_q(\theta) S(t^q \theta) d\theta, \quad \mathcal{S}(t) = q \int_0^\infty \theta \xi_q(\theta) S(t^q \theta) d\theta,$$

and

$$\xi_q(\theta) = \frac{1}{q} \theta^{-1-\frac{1}{q}} \varpi_q(\theta^{-\frac{1}{q}}) \geq 0,$$

where

$$\varpi_q(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-qn-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \quad \theta \in (0, \infty).$$

ξ_q is a probability density function defined on $(0, \infty)$, that is

$$\xi_q(\theta) \geq 0, \quad \theta \in (0, \infty) \quad \text{and} \quad \int_0^\infty \xi_q(\theta) d\theta = 1.$$

Remark 2. It is not difficult to verify that for $v \in [0, 1]$,

$$\int_0^\infty \theta^v \xi_q(\theta) d\theta = \int_0^\infty \theta^{-qv} \varpi_q(\theta) d\theta = \frac{\Gamma(1+v)}{\Gamma(1+qv)}.$$

The following results are very useful and will be used throughout this paper.

Lemma 1 (Lemma 2.9, [32]). *The operators $\mathcal{T}$ and $\mathcal{S}$ have the following properties:*

- (1) *For fixed $t \geq 0$, $\mathcal{T}(t)$ and $\mathcal{S}(t)$ are linear and bounded operators, that is, for any $x \in X$,*

$$\|\mathcal{T}(t)x\| \leq M\|x\|, \quad \|\mathcal{S}(t)x\| \leq \frac{M}{\Gamma(q)}\|x\|.$$

- (2) *$\{\mathcal{T}(t), t \geq 0\}$ and $\{\mathcal{S}(t), t \geq 0\}$ are strongly continuous.*
- (3) *For every $t > 0$, $\mathcal{T}(t)$ and $\mathcal{S}(t)$ are also compact operators.*
- (4) *For any $x \in X$, $\beta \in [0, 1]$ and $\alpha \in [0, 1]$, we have*

$$A\mathcal{S}(t)x = A^{1-\beta}\mathcal{S}(t)A^\beta x, \quad t \in J,$$

$$\|A^\alpha \mathcal{S}(t)\| \leq \frac{M_\alpha q \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} t^{-\alpha q}, \quad 0 < t \leq T.$$

- (5) *For fixed $t \geq 0$ and any $x \in X_\alpha$, we have*

$$\|\mathcal{T}(t)x\|_\alpha \leq M\|x\|_\alpha, \quad \|\mathcal{S}(t)x\|_\alpha \leq \frac{M}{\Gamma(q)}\|x\|_\alpha.$$

- (6) *For a positive number μ with $0 \leq \mu \leq \alpha \leq 1$, fixed $t \geq 0$ and any $x \in X_\mu$, we have*

$$\|\mathcal{T}(t)x\|_\alpha \leq \|A^{\alpha-\mu}\| M\|x\|_\mu, \quad \|\mathcal{S}(t)x\|_\alpha \leq \|A^{\alpha-\mu}\| \frac{M}{\Gamma(q)}\|x\|_\mu.$$

- (7) $\mathcal{T}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$, $t > 0$ is uniformly continuous, that is for each fixed $t > 0$, and $\epsilon > 0$, there exists $h > 0$ such that

$$\|\mathcal{T}_\alpha(t + \epsilon) - \mathcal{T}_\alpha(t)\|_\alpha < \epsilon, \quad \text{for } t + \epsilon \geq 0 \quad \text{and} \quad |\epsilon| < h,$$

$$\|\mathcal{S}_\alpha(t + \epsilon) - \mathcal{S}_\alpha(t)\|_\alpha < \epsilon, \quad \text{for } t + \epsilon \geq 0 \quad \text{and} \quad |\epsilon| < h.$$

where

$$\mathcal{T}_\alpha(t) = \int_0^\infty \xi_q(\theta) S_\alpha(t^q \theta) d\theta, \quad \mathcal{S}_\alpha(t) = q \int_0^\infty \theta \xi_q(\theta) S_\alpha(t^q \theta) d\theta.$$

3. EXISTENCE OF MILD SOLUTIONS

In this section, we give theorems for the existence and uniqueness of the mild solutions of system (1.1).

We first make the following assumptions.

[Hf1]: $f : J \times X_\alpha \times X_\alpha \times X_\alpha \rightarrow X$ is continuous and there exist $m_1, m_2, m_3 > 0$ such that

$$\|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)\| \leq m_1 \|x_1 - y_1\|_\alpha + m_2 \|x_2 - y_2\|_\alpha + m_3 \|x_3 - y_3\|_\alpha$$

for all $x_i, y_i \in X_\alpha$, $i = 1, 2, 3$ and $t \in J$.

[Hk1]: Let $D_k = \{(t, s) \in R^2; 0 \leq s \leq t \leq T\}$. The function $k : D_k \times X_\alpha \rightarrow X_\alpha$ is continuous and there exists a $m_k(t, s) \in C(D_k, R^+)$ such that

$$\|k(t, s, x) - k(t, s, y)\|_\alpha \leq m_k(t, s) \|x - y\|_\alpha$$

for each $(t, s) \in D_k$ and $x, y \in X_\alpha$. We set

$$K^* = \max_{t \in J} \int_0^t m_k(t, s) ds.$$

[Hh1]: Let $D_h = \{(t, s) \in R^2; 0 \leq s, t \leq T\}$. The function $h : D_h \times X_\alpha \rightarrow X_\alpha$ is continuous and there exists a $m_h(t, s) \in C(D_h, R^+)$ such that

$$\|h(t, s, x) - h(t, s, y)\|_\alpha \leq m_h(t, s) \|x - y\|_\alpha$$

for each $(t, s) \in D_h$ and $x, y \in X_\alpha$. We set

$$H^* = \max_{t \in J} \int_0^T m_h(t, s) ds.$$

[Hg1]: $g : C_\alpha \rightarrow X_\alpha$ and there exists a constant $l_g > 0$ such that

$$\|g(x) - g(y)\|_\alpha \leq l_g \|x - y\|_\infty, \quad \text{for arbitrary } x, y \in C_\alpha.$$

[HΩ]: A constant $\Omega_{n, \alpha, q, T}$ defined by

$$\Omega_{n, \alpha, q, T} = M l_g + \frac{M_\alpha q \Gamma(2 - \alpha) B(q, n + 1)}{\Gamma(1 + q(1 - \alpha))} T^{n + (1 - \alpha)q} (m_1 + K^* m_2 + H^* m_3)$$

satisfies $\Omega_{n, \alpha, q, T} < 1$, where $B(\cdot, \cdot)$ denotes Beta function.

Now we are ready to give our first result which is based on the Banach contraction mapping principle.

Theorem 1. *Assume that [Hf1], [Hk1], [Hh1], [Hg1] and [HΩ] are satisfied. If $x_0 \in X_\alpha$ then system (1.1) has a unique mild solution $x \in C_\alpha$.*

Proof. Define the function $\Gamma : C_\alpha \rightarrow C_\alpha$ by

$$\begin{aligned} (\Gamma x)(t) &= \mathcal{T}(t)[x_0 + g(x)] \\ &\quad + \int_0^t (t-s)^{q-1} s^n \mathcal{S}(t-s) f(s, x(s), (Kx)(s), (Hx)(s)) ds, t \in J. \end{aligned} \quad (3.1)$$

Note that Γ is well defined on C_α . Now, take $t \in J$ and $x, y \in C_\alpha$. Then we have

$$\begin{aligned} \|(\Gamma x)(t) - (\Gamma y)(t)\|_\alpha &\leq \|\mathcal{T}(t)(g(x) - g(y))\|_\alpha \\ &\quad + \int_0^t (t-s)^{q-1} s^n \|\mathcal{S}(t-s)[f(s, x(s), (Kx)(s), (Hx)(s)) \\ &\quad - f(s, y(s), (Ky)(s), (Hy)(s))]\|_\alpha ds \\ &\leq M \|g(x) - g(y)\|_\alpha \\ &\quad + \int_0^t (t-s)^{q-1} s^n \|A^\alpha \mathcal{S}(t-s)\| \|f(s, x(s), (Kx)(s), (Hx)(s)) \\ &\quad - f(s, y(s), (Ky)(s), (Hy)(s))\| ds, \end{aligned}$$

which according to [Hf1], [Hk1], [Hh1], [Hg1], (4)-(5) of Lemma 1 and Hölder inequality, gives

$$\begin{aligned} \|(\Gamma x)(t) - (\Gamma y)(t)\|_\alpha &\leq M l_g \|x - y\|_\infty \\ &\quad + M_\alpha q t^{-\alpha q} \frac{\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} m_1 \int_0^t (t-s)^{q-1} s^n \|x(s) - y(s)\|_\alpha ds \\ &\quad + M_\alpha q t^{-\alpha q} \frac{\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} m_2 \int_0^t (t-s)^{q-1} s^n \|(Kx)(s) - (Ky)(s)\|_\alpha ds \\ &\quad + M_\alpha q t^{-\alpha q} \frac{\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} m_3 \int_0^t (t-s)^{q-1} s^n \|(Hx)(s) - (Hy)(s)\|_\alpha ds \\ &\leq M l_g \|x - y\|_\infty + M_\alpha q t^{-\alpha q} \frac{\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} (m_1 + K^* m_2 + H^* m_3) \\ &\quad \times \|x - y\|_\infty \int_0^t (t-s)^{q-1} s^n ds \\ &\leq \left\{ M l_g + \frac{M_\alpha q \Gamma(2-\alpha) B(q, n+1)}{\Gamma(1+q(1-\alpha))} t^{n+(1-\alpha)q} (m_1 + K^* m_2 + H^* m_3) \right\} \\ &\quad \times \|x - y\|_\infty, \end{aligned}$$

due to

$$\int_0^t (t-s)^{q-1} s^n ds = B(q, n+1) t^{n+q},$$

$\|(Kx)(s) - (Ky)(s)\|_\alpha \leq K^* \|x - y\|_\infty$ and $\|(Hx)(s) - (Hy)(s)\|_\alpha \leq H^* \|x - y\|_\infty$.
Therefore, we can deduce that

$$\begin{aligned} & \|\Gamma x - \Gamma y\|_\infty \\ & \leq \left\{ Ml_g + \frac{M_\alpha q \Gamma(2-\alpha) B(q, n+1)}{\Gamma(1+q(1-\alpha))} t^{n+(1-\alpha)q} (m_1 + K^* m_2 + H^* m_3) \right\} \\ & \quad \times \|x - y\|_\infty \leq \Omega_{n,\alpha,q,T} \|x - y\|_\infty. \end{aligned}$$

Hence, $[\text{H}\Omega]$ allows us to conclude in view of the contraction mapping principle, that Γ has a unique fixed point $x \in C_\alpha$, and

$$x(t) = \mathcal{T}(t)[x_0 + g(x)] + \int_0^t (t-s)^{q-1} s^n \mathcal{S}(t-s) f(s, x(s), (Kx)(s), (Hx)(s)) ds$$

which is the unique mild solution of system (1.1). $\square$

Our second result uses Schauder's fixed point theorem.

We assume the following conditions.

[Hf2]: $f : J \times X_\alpha \times X_\alpha \times X_\alpha \rightarrow X_\alpha$ is continuous and there exists a positive function $\rho \in L^p(J, R^+)$ for some $p \in (\frac{1}{q}, \infty)$ such that

$$\|f(t, x, y, z)\|_\alpha \leq \rho(t)$$

for all $x, y, z \in X_\alpha$ and $t \in J$.

[Hk2]: The function $k : D_k \times X_\alpha \rightarrow X_\alpha$ is continuous and there exists $L_1 > 0$ such that

$$\|k(t, s, x) - k(t, s, y)\|_\alpha \leq L_1 \|x - y\|_\alpha$$

for each $(t, s) \in D_k$ and $x, y \in X_\alpha$.

[Hh2]: The function $h : D_h \times X_\alpha \rightarrow X_\alpha$ is continuous and there exists $L_2 > 0$ such that

$$\|h(t, s, x) - h(t, s, y)\|_\alpha \leq L_2 \|x - y\|_\alpha$$

for each $(t, s) \in D_h$ and $x, y \in X_\alpha$.

[Hg2]: $g : C_\alpha \rightarrow X_\alpha$ is compact continuous and there exist $\beta_1 \geq 0, \beta_2 \geq 0$ such that

$$\|g(x)\|_\alpha \leq \beta_1 \|x\|_\infty + \beta_2.$$

Now we are ready to state and prove the following existence result.

Theorem 2. Assume that the conditions [Hf2], [Hk2], [Hh2], [Hg2] are satisfied. If $x_0 \in X_\alpha$ then system (1.1) has at least one mild solution on J provided that

$$M\beta_1 < 1.$$

Proof. Define the function $F : C_\alpha \rightarrow C_\alpha$ by

$$(Fx)(t) = \mathcal{T}(t)[x_0 + g(x)] + \int_0^t (t-s)^{q-1} s^n \mathcal{S}(t-s) f(s, x(s), (Kx)(s), (Hx)(s)) ds,$$

and for $n \in \mathbb{Z}^+$, we choose r such that

$$r \geq \frac{1}{1-M\beta_1} \left[M(\|x_0\|_\alpha + \beta_2) + \frac{M}{\Gamma(q)} B \left(\frac{pq-1}{p-1}, \frac{(n+1)p-1}{p-1} \right)^{\frac{p-1}{p}} T^{\frac{pq+n p-1}{p}} \|\rho\|_{L^p(J, R^+)} \right].$$

Let $B_r = \{x \in C_\alpha \mid \|x\|_\infty \leq r\}$. Then we proceed in three steps.

Step 1. We show that $FB_r \subset B_r$. Let $x \in B_r$. Then for $t \in J$, using (5) of Lemma 1 and Hölder inequality, we have

$$\begin{aligned} \|(Fx)(t)\|_\alpha &\leq \|\mathcal{T}(t)(x_0 + g(x))\|_\alpha \\ &\quad + \int_0^t (t-s)^{q-1} s^n \|\mathcal{S}(t-s) f(s, x(s), (Kx)(s), (Hx)(s))\|_\alpha ds \\ &\leq M(\|x_0\|_\alpha + \|g(x)\|_\alpha) \\ &\quad + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n \|f(s, x(s), (Kx)(s), (Hx)(s))\|_\alpha ds, \end{aligned}$$

which according to [Hf2], [Hg2] and $pq > 1$ ($\Leftrightarrow \frac{(q-1)p}{p-1} > -1$), gives

$$\begin{aligned} \|(Fx)(t)\|_\alpha &\leq M(\|x_0\|_\alpha + \beta_1 \|x\|_\infty + \beta_2) + \frac{M}{\Gamma(q)} \left(\int_0^t (t-s)^{q-1} s^n \rho(s) ds \right) \\ &\leq M(\|x_0\|_\alpha + \beta_1 \|x\|_\infty + \beta_2) + \frac{M}{\Gamma(q)} \left(\int_0^t (t-s)^{\frac{(q-1)p}{p-1}} s^{\frac{np}{p-1}} ds \right)^{\frac{p-1}{p}} \\ &\quad \times \left(\int_0^t \rho(s)^p ds \right)^{\frac{1}{p}} \\ &\leq M(\|x_0\|_\alpha + \beta_1 \|x\|_\infty + \beta_2) + \frac{M}{\Gamma(q)} B \left(\frac{pq-1}{p-1}, \frac{(n+1)p-1}{p-1} \right)^{\frac{p-1}{p}} \\ &\quad \times t^{\frac{pq+np-1}{p}} \|\rho\|_{L^p(J, R^+)} \leq r, \text{ for } t \in J. \end{aligned}$$

Hence, we deduce $\|Fx\|_\infty \leq r$.

Step 2. We prove that F is continuous. Let $\{x_m\}$ be a sequence of B_r such that $x_m \rightarrow x$ in B_r . It comes from the continuity of k , h and assumptions [Hk2], [Hh2]

that

$$\begin{aligned} \int_0^s k(s, \tau, x_m(\tau)) d\tau &\rightarrow \int_0^s k(s, \tau, x(\tau)) d\tau, \\ \int_0^T h(s, \tau, x_m(\tau)) d\tau &\rightarrow \int_0^T h(s, \tau, x(\tau)) d\tau \end{aligned}$$

uniformly in $s \in J$ on C_α . Then,

$$f(s, x_m(s), (Kx_m)(s), (Hx_m)(s)) \rightarrow f(s, x(s), (Kx)(s), (Hx)(s)) \quad (3.2)$$

as $m \rightarrow \infty$, because the function f is continuous on $J \times X_\alpha \times X_\alpha$.

Further, one has

$$g(x_m) \rightarrow g(x) \text{ as } m \rightarrow \infty \quad (3.3)$$

because g is continuous on C_α .

Now for $t \in J$, according to [Hf2], [Hg2], (5) of Lemma 1 and Hölder inequality, we have

$$\begin{aligned} &\|(Fx_m)(t) - (Fx)(t)\|_\alpha \leq \|\mathcal{T}(t)(g(x_m) - g(x))\|_\alpha \\ &\quad + \int_0^t (t-s)^{q-1} s^n \left\| \mathcal{S}(t-s)[f(s, x_m(s), (Kx_m)(s), (Hx_m)(s)) \right. \\ &\quad \quad \left. - f(s, x(s), (Kx)(s), (Hx)(s))] \right\|_\alpha ds \\ &\leq M \|g(x_m) - g(x)\|_\alpha \\ &\quad + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n \|f(s, x_m(s), (Kx_m)(s), (Hx_m)(s)) \\ &\quad \quad - f(s, x(s), (Kx)(s), (Hx)(s))\|_\alpha ds \\ &\leq M \|g(x_m) - g(x)\|_\alpha + \frac{M}{\Gamma(q)} \left(\int_0^t (t-s)^{\frac{(q-1)p}{p-1}} s^{\frac{np}{p-1}} ds \right)^{\frac{p-1}{p}} \\ &\quad \times \left(\int_0^t \|f(s, x_m(s), (Kx_m)(s), (Hx_m)(s)) \right. \\ &\quad \quad \left. - f(s, x(s), (Kx)(s), (Hx)(s))\|^p ds \right)^{\frac{1}{p}} \\ &\leq M \|g(x_m) - g(x)\|_\alpha + \frac{M}{\Gamma(q)} B \left(\frac{pq-1}{p-1}, \frac{(n+1)p-1}{p-1} \right)^{\frac{p-1}{p}} t^{\frac{pq+n-1}{p}} \\ &\quad \times \left(\int_0^t \|f(s, x_m(s), (Kx_m)(s), (Hx_m)(s)) \right. \\ &\quad \quad \left. - f(s, x(s), (Kx)(s), (Hx)(s))\|^p ds \right)^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned}
& -f(s, x(s), (Kx)(s), (Hx)(s))\|^p ds \Big)^{\frac{1}{p}} \\
& \leq M \|g(x_m) - g(x)\|_\alpha + \frac{M}{\Gamma(q)} B \left(\frac{pq-1}{p-1}, \frac{(n+1)p-1}{p-1} \right)^{\frac{p-1}{p}} T^{\frac{pq+n-1}{p}} \\
& \quad \times \left(\int_0^T \|f(s, x_m(s), (Kx_m)(s), (Hx_m)(s)) \right. \\
& \quad \left. - f(s, x(s), (Kx)(s), (Hx)(s))\|^p ds \right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore, using (3.2), (3.3), [Hf2] and the Lebesgue Dominated Convergence Theorem, it can easily be shown that

$$\lim_{m \rightarrow \infty} \|Fx_m - Fx\|_\infty = 0, \text{ as } m \rightarrow \infty.$$

That is, F is continuous.

Step 3. We show that F is compact. To this end, we use the famous Ascoli-Arzelà's theorem. We first prove that $\{(Fx)(t) \mid x \in B_r\}$ is relatively compact in X_α , for all $t \in J$. Obviously, $\{(Fx)(0) \mid x \in B_r\}$ is compact. Let $t \in (0, T]$. For each $h \in (0, t)$, arbitrary $\delta > 0$ and $x \in B_r$, we defined the operator F_h by

$$\begin{aligned}
(F_{h,\delta}x)(t) &= \mathcal{T}(t)[x_0 + g(x)] + S(h^q\delta) \int_0^{t-h} (t-s)^{q-1} s^n \\
& \quad \times \left(q \int_\delta^\infty \theta \xi_q(\theta) S((t-s)^q\theta - h^q\delta) d\theta \right) f(s, x(s), (Kx)(s), (Hx)(s)) ds \\
&= \mathcal{T}(t)[x_0 + g(x)] \\
& \quad + q \int_0^{t-h} \int_\delta^\infty \theta (t-s)^{q-1} s^n \xi_q(\theta) S((t-s)^q\theta) \\
& \quad \times f(s, x(s), (Kx)(s), (Hx)(s)) d\theta ds.
\end{aligned}$$

From above expression, we can see that the sets $\{(F_{h,\delta}x)(t) \mid x \in B_r\}$ are also relatively compact in X_α since the operator $S_\alpha(h^q\delta)$, $h^q\delta > 0$ are compact in X_α . Moreover, using [Hf2] and Hölder inequality, we have

$$\begin{aligned}
& \|(Fx)(t) - (F_{h,\delta}x)(t)\|_\alpha \\
& \leq q \left\| \int_0^t \int_0^\delta \theta (t-s)^{q-1} s^n \xi_q(\theta) S((t-s)^q\theta) \right. \\
& \quad \left. \times f(s, x(s), (Kx)(s), (Hx)(s)) d\theta ds \right\|_\alpha
\end{aligned}$$

$$\begin{aligned}
& + q \left\| \int_0^t \int_\delta^\infty \theta(t-s)^{q-1} s^n \xi_q(\theta) S((t-s)^q \theta) \right. \\
& \quad \times f(s, x(s), (Kx)(s), (Hx)(s)) d\theta ds \\
& - \int_0^{t-h} \int_\delta^\infty \theta(t-s)^{q-1} s^n \xi_q(\theta) S((t-s)^q \theta) \\
& \quad \times f(s, x(s), (Kx)(s), (Hx)(s)) d\theta ds \left. \right\|_\alpha \\
& \leq q \int_0^t \int_0^\delta \theta(t-s)^{q-1} s^n \xi_q(\theta) \|S((t-s)^q \theta) \\
& \quad \times f(s, x(s), (Kx)(s), (Hx)(s))\|_\alpha d\theta ds \\
& + q \int_{t-h}^t \int_\delta^\infty \theta(t-s)^{q-1} s^n \xi_q(\theta) \|S((t-s)^q \theta) \\
& \quad \times f(s, x(s), (Kx)(s), (Hx)(s))\|_\alpha d\theta ds \\
& \leq qM \int_0^t \int_0^\delta \theta(t-s)^{q-1} s^n \xi_q(\theta) \rho(s) d\theta ds \\
& + qM \int_{t-h}^t \int_\delta^\infty \theta(t-s)^{q-1} s^n \xi_q(\theta) \rho(s) d\theta ds \\
& \leq qM \left(\int_0^t (t-s)^{q-1} s^n \rho(s) ds \right) \int_0^\delta \theta \xi_q(\theta) d\theta \\
& + qM \left(\int_{t-h}^t (t-s)^{q-1} s^n \rho(s) ds \right) \int_0^\infty \theta \xi_q(\theta) d\theta.
\end{aligned}$$

It comes from

$$\int_0^\infty \theta \xi_q(\theta) d\theta = \frac{1}{\Gamma(1+q)},$$

and

$$\begin{aligned}
\int_0^t (t-s)^{q-1} s^n \rho(s) ds & \leq \left(\int_0^t (t-s)^{\frac{(q-1)p}{p-1}} s^{\frac{np}{p-1}} ds \right)^{\frac{p-1}{p}} \left(\int_0^t \rho(s)^p ds \right)^{\frac{1}{p}} \\
& \leq B \left(\frac{pq-1}{p-1}, \frac{(n+1)p-1}{p-1} \right)^{\frac{p-1}{p}} t^{\frac{pq+n p-1}{p}} \|\rho\|_{L^p(J, \mathbb{R}^+)},
\end{aligned}$$

and for a fixed $\chi \in \left(1, \frac{p-1}{(1-q)p}\right)$,

$$\begin{aligned} \int_{t-h}^t (t-s)^{q-1} s^n \rho(s) ds &\leq \left(\int_{t-h}^t (t-s)^{\frac{(q-1)p}{p-1}} s^{\frac{np}{p-1}} ds \right)^{\frac{p-1}{p}} \left(\int_{t-h}^t \rho(s)^p ds \right)^{\frac{1}{p}} \\ &\leq \left(\int_{t-h}^t ds \right)^{\frac{\chi(p-1)}{(\chi-1)p}} \left(\int_{t-h}^t (t-s)^{\frac{\chi(q-1)p}{p-1}} s^{\frac{\chi np}{p-1}} ds \right)^{\frac{p-1}{\chi p}} \|\rho\|_{L^p(J, R^+)} \\ &\leq h^{\frac{\chi(p-1)}{(\chi-1)p}} B \left(\frac{\chi(q-1)p}{p-1} + 1, \frac{\chi np}{p-1} + 1 \right)^{\frac{p-1}{\chi p}} t^{\frac{\chi p((q-1)p+n)}{p-1} + 1} \|\rho\|_{L^p(J, R^+)}, \end{aligned}$$

that

$$\begin{aligned} &\|(Fx)(t) - (F_{h,\delta}x)(t)\|_\alpha \\ &\leq qMB \left(\frac{pq-1}{p-1}, \frac{(n+1)p-1}{p-1} \right)^{\frac{p-1}{p}} t^{\frac{pq+np-1}{p}} \|\rho\|_{L^p(J, R^+)} \int_0^\delta \theta \xi_q(\theta) d\theta \\ &\quad + \frac{M}{\Gamma(q)} B \left(\frac{\chi(q-1)p}{p-1} + 1, \frac{\chi np}{p-1} + 1 \right)^{\frac{p-1}{\chi p}} t^{\frac{\chi p((q-1)p+n)}{p-1} + 1} \\ &\quad \times \|\rho\|_{L^p(J, R^+)} h^{\frac{\chi(p-1)}{(\chi-1)p}} \rightarrow 0 \end{aligned}$$

as $\delta \rightarrow 0^+$ and $h \rightarrow 0^+$. Therefore, $\{(Fx)(t) \mid x \in B_r\}$ is relatively compact in X_α for all $t \in (0, T]$ and since it is compact at $t = 0$ we have the relatively compactness in X_α for all $t \in J$.

Next, let us prove that $F(B_r)$ is equicontinuous. For $0 \leq t_2 < t_1 \leq T$, we have

$$\begin{aligned} \|(Fx)(t_1) - (Fx)(t_2)\|_\alpha &\leq \|(\mathcal{T}(t_1) - \mathcal{T}(t_2))[x_0 - g(x)]\|_\alpha \\ &\quad + \left\| \int_0^{t_2} (t_1-s)^{q-1} s^n [\mathcal{S}(t_1-s) - \mathcal{S}(t_2-s)] \right. \\ &\quad \quad \times f(s, x(s), (Kx)(s), (Hx)(s)) ds \Big\|_\alpha \\ &\quad + \left\| \int_0^{t_2} [(t_1-s)^{q-1} - t_2-s)^{q-1}] s^n \mathcal{S}(t_2-s) \right. \\ &\quad \quad \times f(s, x(s), (Kx)(s), (Hx)(s)) ds \Big\|_\alpha \\ &\quad + \left\| \int_{t_2}^{t_1} (t_1-s)^{q-1} s^n \mathcal{S}(t_1-s) f(s, x(s), (Kx)(s), (Hx)(s)) ds \right\|_\alpha. \end{aligned}$$

Denote

$$\begin{aligned}
I_1 &= \|(\mathcal{T}(t_1) - \mathcal{T}(t_2))[x_0 - g(x)]\|_\alpha, \\
I_2 &= \left\| \int_0^{t_2} (t_1 - s)^{q-1} s^n [\mathcal{S}(t_1 - s) - \mathcal{S}(t_2 - s)] \right. \\
&\quad \left. \times f(s, x(s), (Kx)(s), (Hx)(s)) ds \right\|_\alpha, \\
I_3 &= \left\| \int_0^{t_2} [(t_1 - s)^{q-1} - (t_2 - s)^{q-1}] s^n \mathcal{S}(t_2 - s) \right. \\
&\quad \left. \times f(s, x(s), (Kx)(s), (Hx)(s)) ds \right\|_\alpha, \\
I_4 &= \left\| \int_{t_2}^{t_1} (t_1 - s)^{q-1} s^n \mathcal{S}(t_1 - s) f(s, x(s), (Kx)(s), (Hx)(s)) ds \right\|_\alpha.
\end{aligned}$$

Now, we need to check that I_1, I_2, I_3, I_4 tend to 0 independently of $x \in B_r$ when $t_1 \rightarrow t_2$.

In fact, by the compactness of the set $g(B_r)$ in view of (2) of Lemma 1, one can deduce that $\lim_{t_1 \rightarrow t_2} I_1 = 0$ uniformly.

Next for $0 < h < t_2$, when $t_2 > 0$, we similarly derive

$$\begin{aligned}
I_2 &\leq \int_0^{t_2} (t_1 - s)^{q-1} s^n \|\mathcal{S}_\alpha(t_1 - s) - \mathcal{S}_\alpha(t_2 - s)\|_\alpha \\
&\quad \times \|f(s, x(s), (Kx)(s), (Hx)(s))\|_\alpha ds \\
&\leq \int_0^{t_2-h} (t_2 - s)^{q-1} s^n \rho(s) \|\mathcal{S}_\alpha(t_1 - s) - \mathcal{S}_\alpha(t_2 - s)\|_\alpha ds \\
&\quad + \int_{t_2-h}^{t_2} (t_2 - s)^{q-1} s^n \rho(s) \|\mathcal{S}_\alpha(t_1 - s) - \mathcal{S}_\alpha(t_2 - s)\|_\alpha ds \\
&\leq \max_{s \in [0, t_2-h]} \|\mathcal{S}_\alpha(t_1 - s) - \mathcal{S}_\alpha(t_2 - s)\|_\alpha \int_0^{t_2-h} (t_2 - s)^{q-1} s^n \rho(s) ds \\
&\quad + \frac{2M}{\Gamma(q)} \int_{t_2-h}^{t_2} (t_2 - s)^{q-1} s^n \rho(s) ds \\
&\leq \max_{s \in [0, t_2-h]} \|\mathcal{S}_\alpha(t_1 - s) - \mathcal{S}_\alpha(t_2 - s)\|_\alpha B \left(\frac{pq-1}{p-1}, \frac{(n+1)p-1}{p-1} \right)^{\frac{p-1}{p}} \\
&\quad \times t_2^{\frac{pq+nq-1}{p}} \|\rho\|_{L^p(J, R^+)} \\
&\quad + \frac{2M}{\Gamma(q)} h^{\frac{\chi(p-1)}{(\chi-1)p}} B \left(\frac{\chi(q-1)p}{p-1} + 1, \frac{\chi np}{p-1} + 1 \right)^{\frac{p-1}{\chi p}} t_2^{\frac{\chi p((q-1)p+n)}{p-1} + 1} \|\rho\|_{L^p(J, R^+)}
\end{aligned}$$

from which we deduce that $\lim_{(h,t_2) \rightarrow (0,t_1)} I_2 = 0$ uniformly, since by (7) of Lemma 1.

Using the inequality $(x - y)^a \leq x^a - y^a$ for any $x, y \geq 0$ and $a > 1$, analogically we derive

$$\begin{aligned}
I_3 &\leq \frac{M}{\Gamma(q)} \int_0^{t_2} |(t_1 - s)^{q-1} - (t_2 - s)^{q-1}| s^n \\
&\quad \times \|f(s, x(s), (Kx)(s), (Hx)(s))\|_\alpha ds \\
&\leq \frac{M}{\Gamma(q)} \int_0^{t_2} |(t_1 - s)^{q-1} - (t_2 - s)^{q-1}| s^n \rho(s) ds \\
&\leq \frac{M}{\Gamma(q)} \left(\int_0^{t_2} \rho(s)^p ds \right)^{\frac{1}{p}} \left(\int_0^{t_2} s^{\frac{np}{p-1}} |(t_1 - s)^{q-1} - (t_2 - s)^{q-1}|^{\frac{p}{p-1}} ds \right)^{\frac{p-1}{p}} \\
&\leq \frac{M}{\Gamma(q)} \left(\int_0^{t_2} s^{\frac{np}{p-1}} \left((t_2 - s)^{\frac{(q-1)p}{p-1}} - (t_1 - s)^{\frac{(q-1)p}{p-1}} \right) ds \right)^{\frac{p-1}{p}} \|\rho\|_{L^p(J, R^+)} \\
&\leq \frac{M}{\Gamma(q)} \left[B \left(\frac{pq-1}{p-1}, \frac{(n+1)p-1}{p-1} \right)^{\frac{p-1}{p}} \left(t_2^{\frac{pq+n-1}{p}} - t_1^{\frac{pq+n-1}{p}} \right)^{\frac{p-1}{p}} \right. \\
&\quad \left. + \left(\int_{t_2}^{t_1} s^{\frac{np}{p-1}} (t_1 - s)^{\frac{(q-1)p}{p-1}} ds \right)^{\frac{p-1}{p}} \right] \|\rho\|_{L^p(J, R^+)} \\
&\leq \frac{M}{\Gamma(q)} \|\rho\|_{L^p(J, R^+)} \left[B \left(\frac{pq-1}{p-1}, \frac{(n+1)p-1}{p-1} \right)^{\frac{p-1}{p}} \right. \\
&\quad \times \left(t_2^{\frac{pq+n-1}{p}} - t_1^{\frac{pq+n-1}{p}} \right)^{\frac{p-1}{p}} \\
&\quad \left. + (t_1 - t_2)^{\frac{\chi(p-1)}{(\chi-1)p}} B \left(\frac{\chi(q-1)p}{p-1} + 1, \frac{\chi np}{p-1} + 1 \right)^{\frac{p-1}{\chi p}} t_1^{\frac{\chi p((q-1)p+n)}{p-1} + 1} \right].
\end{aligned}$$

Thus, $\lim_{t_1 \rightarrow t_2} I_3 = 0$ uniformly.

Finally,

$$\begin{aligned}
I_4 &\leq \int_{t_2}^{t_1} (t_1 - s)^{q-1} s^n \|\mathcal{S}(t_1 - s) f(s, x(s), (Kx)(s), (Hx)(s))\|_\alpha ds \\
&\leq \frac{M}{\Gamma(q)} \int_{t_2}^{t_1} (t_1 - s)^{q-1} s^n \rho(s) ds \\
&\leq \frac{M}{\Gamma(q)} \left(\int_{t_2}^{t_1} s^{\frac{np}{p-1}} (t_1 - s)^{\frac{(q-1)p}{p-1}} ds \right)^{\frac{p-1}{p}} \|\rho\|_{L^p(J, R^+)}
\end{aligned}$$

$$\begin{aligned} &\leq \frac{M}{\Gamma(q)} (t_1 - t_2)^{\frac{\chi(p-1)}{(\chi-1)p}} B \left(\frac{\chi(q-1)p}{p-1} + 1, \frac{\chi np}{p-1} + 1 \right)^{\frac{p-1}{\chi p}} \\ &\quad \times t_1^{\frac{\chi p((q-1)p+n)}{p-1} + 1} \|\rho\|_{L^p(J, \mathbb{R}^+)}, \end{aligned}$$

from which we deduce that $\lim_{t_1 \rightarrow t_2} I_4 = 0$ uniformly.

In summary, we have proven that $F(B_r)$ is relatively compact, for $t \in J$, $\{F x \mid x \in B_r\}$ is a family of equicontinuous functions. Hence by the Arzela-Ascoli Theorem, F is compact. By Schauder fixed point theorem F has a fixed point $x \in B_r$. Consequently, system (1.1) has at least one mild solution on J . $\square$

Our next result is based on the following well-known fixed point theorem.

Lemma 2 ([18]). *Let Γ be a condensing operator on a Banach space X . If $\Gamma(\mathcal{B}) \subset \mathcal{B}$ for a convex, closed and bounded set $\mathcal{B}$ of X , then Γ has a fixed point in $\mathcal{B}$.*

Now, we assume the following conditions and apply the above fixed point theorem.

[Hf3]: (1) There exists μ with $0 \leq \mu \leq \alpha \leq 1$ such that $f : J \times X_\alpha \times X_\alpha \times X_\alpha \rightarrow X_\mu$ is continuous and there exist $L_f^{(1)}, L_f^{(2)}, L_f^{(3)} > 0$ such that

$$\begin{aligned} \|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)\|_\mu &\leq L_f^{(1)} \|x_1 - y_1\|_\alpha + L_f^{(2)} \|x_2 - y_2\|_\alpha \\ &\quad + L_f^{(3)} \|x_3 - y_3\|_\alpha \end{aligned}$$

for all $x_i, y_i \in X_\alpha, i = 1, 2, 3$ and $t \in J$.

(2) There exist two positive constants $c^{(1)}, d^{(1)}$ such that for each $(t, x, y, z) \in J \times X_\alpha \times X_\alpha \times X_\alpha$

$$\|f(t, x, y, z)\|_\mu \leq c^{(1)} (\|x\|_\alpha + \|y\|_\alpha + \|z\|_\alpha) + d^{(1)}.$$

[Hk3]: (1) The function $k : D_k \times X_\alpha \rightarrow X_\alpha$ is continuous and there exists a $L_k^{(1)} > 0$ such that for $(t, s) \in D_k$ and $x, y \in X_\alpha$,

$$\left\| \int_0^t [k(t, s, x) - k(t, s, y)] ds \right\|_\alpha \leq L_k^{(1)} \|x - y\|_\alpha.$$

(2) There exists a constant $L_k^{(2)} > 0$ such that for $(t, s) \in D_k$ and $x, y \in X_\alpha$,

$$\left\| \int_0^t k(t, s, x) ds \right\|_\alpha \leq L_k^{(2)} (1 + \|x\|_\alpha).$$

[Hh3]: (1) The function $h : D_h \times X_\alpha \rightarrow X_\alpha$ is continuous and there exists a $L_h^{(1)} > 0$ such that for $(t, s) \in D_h$ and $x, y \in X_\alpha$,

$$\left\| \int_0^T [h(t, s, x) - h(t, s, y)] ds \right\|_\alpha \leq L_h^{(1)} \|x - y\|_\alpha.$$

(2) There exists a constant $L_h^{(2)} > 0$ such that for $(t, s) \in D_h$ and $x, y \in X_\alpha$,

$$\left\| \int_0^T h(t, s, x) ds \right\|_\alpha \leq L_h^{(2)} (1 + \|x\|_\alpha).$$

[Hg3]: $g : C_\alpha \rightarrow X_\alpha$ is compact continuous and there exists a nondecreasing function $\phi : R^+ \rightarrow R^+$ such that for all $x \in C_\alpha$,

$$\|g(x)\|_\alpha \leq \phi(\|x\|_\infty), \text{ and } \lim_{l \rightarrow +\infty} \inf \frac{\phi(l)}{l} = \delta < \infty.$$

Now we are ready to state and prove the following existence result.

Theorem 3. Assume that the conditions [Hf3], [Hk3], [Hh3], [Hg3] are satisfied. If $x_0 \in X_\alpha$ then system (1.1) admits at least one mild solution on J provided that

$$M \left\{ \delta + \frac{\|A^{\alpha-\mu}\| \Gamma(n)}{\Gamma(n+q+1)} [c^{(1)}(1 + L_k^{(2)} + L_h^{(2)})] T^{n+q} \right\} < 1 \quad (3.4)$$

and

$$\frac{M \|A^{\alpha-\mu}\| \Gamma(n)}{\Gamma(n+q+1)} T^{n+q} (L_f^{(1)} + L_f^{(2)} L_k^{(1)} + L_f^{(3)} L_h^{(1)}) < 1. \quad (3.5)$$

Proof. Define the operator $\Gamma : C_\alpha \rightarrow C_\alpha$ given by (3.1). For each positive number l , let $B_l = \{x \in C_\alpha \mid \|x\|_\infty \leq l\}$ then, for each l , B_l is obviously a bounded closed convex set in C_α .

First, we claim that $\Gamma(B_l) \subset B_l$ for some $l > 0$. If it is not true, then for each $l > 0$, there would exist $x_l \in B_l$ and $t_l \in J$ such that $\|(\Gamma x_l)(t_l)\|_\alpha > l$. However, on the other hand, by [Hf3], [Hk3] and [Hh3]

$$\begin{aligned} l &\leq \|(\Gamma x_l)(t_l)\|_\alpha \leq \|\mathcal{T}(t)(x_0 + g(x_l))\|_\alpha \\ &+ \int_0^{t_l} (t_l - s)^{q-1} s^n \|\mathcal{S}(t_l - s) f(s, x_l(s), (Kx_l)(t), (Hx_l)(t))\|_\alpha ds \\ &\leq M(\|x_0\|_\alpha + \|g(x_l)\|_\alpha) \\ &+ \frac{M \|A^{\alpha-\mu}\|}{\Gamma(q)} [c^{(1)}(l + L_k^{(2)}(1 + l) + L_h^{(2)}(1 + l) + d^{(1)})] \int_0^{t_l} (t_l - s)^{q-1} s^n ds \\ &\leq M(\|x_0\|_\alpha + \phi(l)) \\ &+ \frac{\|A^{\alpha-\mu}\| M}{\Gamma(q)} [c^{(1)}(l + L_k^{(2)}(1 + l) + L_h^{(2)}(1 + l) + d^{(1)})] B(q, n+1) t_l^{n+q} \\ &\leq M(\|x_0\|_\alpha + \phi(l)) \\ &+ \frac{M \|A^{\alpha-\mu}\| \Gamma(n)}{\Gamma(n+q+1)} [c^{(1)}(l + L_k^{(2)}(1 + l) + L_h^{(2)}(1 + l) + d^{(1)})] T^{n+q}. \end{aligned}$$

Dividing both sides by l and taking the lower limit as $l \rightarrow +\infty$, we obtain

$$1 \leq M \left\{ \delta + \frac{\|A^{\alpha-\mu}\| \Gamma(n)}{\Gamma(n+q+1)} [c^{(1)}(1 + L_k^{(2)} + L_h^{(2)})] T^{n+q} \right\},$$

which contradicts the expression (3.4). Thus, for some positive number l , $\Gamma(B_l) \subset B_l$.

We decompose $\Gamma = \Gamma_1 + \Gamma_2$ as

$$(\Gamma_1 x)(t) = \mathcal{T}(t)[x_0 + g(x)],$$

$$(\Gamma_2 x)(t) = \int_0^t (t-s)^{q-1} s^n \mathcal{S}(t-s) f(s, x(s), (Kx)(s), (Hx)(s)) ds.$$

Second, we show that Γ_1 is compact continuous and Γ_2 is a contraction. By [Hg3], we can infer that Γ_1 is compact continuous on X_α . Next, we prove that Γ_2 is a contraction on B_l . In fact, for each $t \in J$, $x, y \in B_l$, by [Hg3] we have

$$\begin{aligned} & \|(\Gamma_2 x)(t) - (\Gamma_2 y)(t)\|_\alpha \\ & \leq \|A^{\alpha-\mu}\| \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n \|f(s, x(s), (Kx)(s), (Hx)(s)) \\ & \quad - f(s, y(s), (Ky)(s), (Hy)(s))\|_\mu ds \\ & \leq \|A^{\alpha-\mu}\| \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n [L_f^{(1)} \|x(s) - y(s)\|_\alpha \\ & \quad + L_f^{(2)} \|(Kx)(s) - (Ky)(s)\|_\alpha + L_f^{(3)} \|(Hx)(s) - (Hy)(s)\|_\alpha] ds \\ & \leq \|A^{\alpha-\mu}\| \frac{M}{\Gamma(q)} (L_f^{(1)} + L_f^{(2)} L_k^{(1)} + L_f^{(3)} L_h^{(1)}) \int_0^t (t-s)^{q-1} s^n ds \|x - y\|_\infty \\ & \leq \|A^{\alpha-\mu}\| \frac{M \Gamma(n)}{\Gamma(n+q+1)} T^{n+q} (L_f^{(1)} + L_f^{(2)} L_k^{(1)} + L_f^{(3)} L_h^{(1)}) \|x - y\|_\infty. \end{aligned}$$

Thus,

$$\begin{aligned} \|\Gamma_2 x - \Gamma_2 y\|_\infty & \leq \frac{M \|A^{\alpha-\mu}\| \Gamma(n)}{\Gamma(n+q+1)} T^{n+q} (L_f^{(1)} + L_f^{(2)} L_k^{(1)} + L_f^{(3)} L_h^{(1)}) \\ & \quad \times \|x - y\|_\infty, \end{aligned}$$

which implies that Γ_2 is a contraction by (3.5).

At last, we can conclude that $\Gamma = \Gamma_1 + \Gamma_2$ is a condensing map on B_l . By Lemma 2, system (1.1) admits at least one mild solution on J . $\square$

4. AN EXAMPLE

In this section, we present an example, which indicate how our theorems can be applied to concrete problems.

Consider the following problem:

$$\left\{ \begin{array}{l} {}^c D_t^q x(t, y) - \Delta x(t, y) \\ = \left(\frac{e^{-t}}{e^t + e^{-t}} + e^{-t} \right) \cos \left[x(t, y) + \int_0^t \sin(t+s)x(s, y)ds \right. \\ \left. + \int_0^T \cos(ts)x(s, y)ds \right], \\ y \in \Omega, t, s \in (0, T], q = \frac{19}{20}, \\ x(t, y)|_{y \in \partial\Omega} = 0, \quad t > 0, \\ x(0, y) = \int_\Omega \int_0^T h(t, y) \log(1 + |x(t, \xi)|^{\frac{1}{2}}) dt d\xi, \end{array} \right. \quad (4.1)$$

where Δ is the Laplace operator in $\mathbb{R}^3$, $\Omega \subset \mathbb{R}^3$ is a bounded domain, $\partial\Omega \in C^3$, and $h(t, y) \in C(J \times \bar{\Omega})$.

We apply Theorem 2 by taking $X = L^2(\Omega)$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, and $Ax = -\Delta x$ for $x \in D(A)$ and set $\alpha = 0$.

Define $x(t)(y) = x(t, y)$, $(Kx)(t)(y) = \int_0^t \sin(t+s)x(s, y)ds$, $(Hx)(t)(y) = \int_0^T \cos(ts)x(s, y)ds$, and

$$\begin{aligned} & f(t, x(t), (Kx)(t), (Hx)(t))(y) \\ &= \left(\frac{e^{-t}}{e^t + e^{-t}} + e^{-t} \right) \cos \left[x(t) + \int_0^t \sin(t+s)x(s)ds + \int_0^T \cos(ts)x(s)ds \right](y), \end{aligned}$$

$$g(x)(y) = \int_\Omega \int_0^T h(t, y) \log(1 + |x(t, \xi)|^{\frac{1}{2}}) dt d\xi, y \in \bar{\Omega}, x \in C(J, X).$$

Then, A generates a compact analytic semigroup in X with $M = 1$, and

$$\|f(t, x(t), (Kx)(t), (Hx)(t))\| \leq \rho(t) = \left(\frac{e^{-t}}{e^t + e^{-t}} + e^{-t} \right) (\text{mes}(\Omega))^{\frac{1}{2}}$$

with $\rho \in L^p(J, \mathbb{R}^+)$, $p = 10$. Moreover, g is compact (see [20]). Next, using $\log(1+a) \leq a$ for any $a \geq 0$, we derive

$$\int_\Omega \log(1 + |x(\xi)|^{\frac{1}{2}}) d\xi \leq \int_\Omega |x(\xi)|^{\frac{1}{2}} d\xi \leq (\text{mes}(\Omega))^{\frac{3}{4}} \|x\|^{\frac{1}{2}} \leq \frac{(\text{mes}(\Omega))^{\frac{3}{4}}}{2} (1 + \|x\|)$$

for any $x \in X$. Hence using $\|x(t, \cdot)\| \leq \|x\|_\infty$ for any $x \in C(J, X)$, we obtain

$$\|g(x)\| \leq \frac{T}{2} (\text{mes}(\Omega))^{\frac{7}{4}} \max_{t \in J, y \in \bar{\Omega}} |h(t, y)| (1 + \|x\|_\infty), x \in C(J, X).$$

Thus problem (4.1) can be rewritten as

$$\left\{ \begin{array}{l} {}^c D_t^q x(t) = -Ax(t) + t^n f(t, x(t), (Kx)(t), (Hx)(t)), \\ t \in J, n \in \mathbb{Z}^+, q \in (0, 1), \\ x(0) = g(x) + x_0. \end{array} \right.$$

Obviously, $q = \frac{19}{20} > \frac{1}{10} = \frac{1}{p}$. Furthermore, if T and $h(t, y)$ satisfy

$$T(\text{mes}(\Omega))^{\frac{7}{4}} \max_{t \in J, y \in \bar{\Omega}} |h(t, y)| < 2$$

then all the assumptions given in Theorem 2 are verified. Therefore, the problem (4.1) has at least one mild solution.

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*Authors' addresses***JinRong Wang**

Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, P.R. China

E-mail address: wjr9668@126.com

Wei Wei

Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, P.R. China

E-mail address: wwei@gzu.edu.cn

Michal Fečkan

Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynská dolina, 842 48 Bratislava, Slovakia and Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia

E-mail address: Michal.Feckan@fmph.uniba.sk



POSITIVE SOLUTIONS OF BVPS FOR INFINITE DIFFERENCE EQUATIONS WITH ONE-DIMENSIONAL p -LAPLACIAN

JIANYE XIA AND YUJI LIU

Received 4 May, 2011

Abstract. Sufficient conditions guaranteeing the existence of three positive solutions of the multi-point boundary value problem for the infinite difference equation

$$\begin{cases} \Delta[p(n)\phi(\Delta x(n))] + f(n, x(n), \Delta x(n)) = 0, & n \in N_0, \\ x(0) - \sum_{n=1}^{\infty} \alpha_n x(n) = 0, \\ \lim_{n \rightarrow +\infty} \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} - \sum_{n=1}^{\infty} \beta_n x(n) = 0, \end{cases}$$

are established using a fixed point theorem. It is the purpose of this paper to show that this approach of obtaining positive solutions of BVPs by using multi-fixed-point theorems can be extended to infinite difference equations containing the nonlinear operator $\Delta[p\phi(\Delta x)]$. The possible solutions of this BVP are not concave if $p(n) \neq \text{constant}$.

2000 *Mathematics Subject Classification:* 34B10; 34B15; 35B10

Keywords: one-dimension p -Laplacian infinite difference equation, multi-point boundary value problem, positive solution, fixed point theorem

1. INTRODUCTION

In recent years, there have been many papers interested in proving the existence of positive solutions of the boundary value problems (BVPs for short) of the finite difference equations since these BVPs have extensive applications, see the papers [1, 5–9, 11–15, 18, 19], [10] and the references therein.

Recently, the authors [2–4, 16, 17] studied the existence of solutions of the boundary value problems for infinite difference equations. In [17], the existence of multiple positive solutions of boundary value problems for the second-order discrete equations

$$\begin{cases} \Delta^2 x(n-1) - p\Delta x(n-1) - qx(n-1) + f(n, x(n)) = 0, & n \in N, \\ \alpha x(0) - \beta \Delta x(0) = 0, \\ \lim_{n \rightarrow +\infty} x(n) = 0, \end{cases} \quad (1.1)$$

Supported by Natural Science Foundation of Hunan province, P.R.China(No:06JJ5008) and Natural Science Foundation of Guangdong province(No:7004569).

was investigated by using the cone compression and expansion and fixed point theorems in Fréchet spaces, where $\alpha > 0, \beta > 0, p > 0, q > 0$ and f is a continuous function.

In paper [2], the existence of solutions of a class of the infinite time scale boundary value problems was considered. It is easy to see that the results of [2] can be applied to the following BVP for the infinite difference equation

$$\begin{cases} \Delta^2 x(n) + f(n, x(n)) = 0, & n \in N_0, \\ x(0) = 0, \\ x(n) \text{ is bounded.} \end{cases}$$

The methods are based upon a growth argument and the upper and lower solutions methods.

In [12] the existence of at least three positive solutions for the following BVP of the finite difference equations

$$\begin{cases} \Delta[\phi(\Delta x(n))] + f(n, x(n), \Delta x(n)) = 0, & n \in [0, N], \\ x(0) - \sum_{i=1}^m \alpha_i x(n_i) = 0, \\ x(N+2) - \sum_{i=1}^m \beta_i x(n_i) = 0 \end{cases} \quad (1.2)$$

was proved under some assumptions.

Motivated by above mentioned two papers, the purpose of this paper is to investigate the multi-point boundary value problem of the second order infinite p -Laplacian difference equation

$$\begin{cases} \Delta[p(n)\phi(\Delta x(n))] + f(n, x(n), \Delta x(n)) = 0, & n \in N_0, \\ x(0) - \sum_{n=1}^{\infty} \alpha_n x(n) = 0, \\ \lim_{n \rightarrow +\infty} \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} - \sum_{n=1}^{\infty} \beta_n x(n) = 0, \end{cases} \quad (1.3)$$

where N_0 denotes the set of all nonnegative integers, $p(n) > 0, \alpha_n \geq 0, \beta_n \geq 0$ for all $n \in N_0$ and satisfy

$$\begin{aligned} \sum_{n=1}^{\infty} \alpha_n &< 1, \\ \sum_{n \in N} \alpha_n \sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))} &< \infty \end{aligned}$$

and

$$\frac{\sum_{n=1}^{\infty} \beta_n}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1}\left(\frac{1}{p(j)}\right) + \sum_{n=1}^{\infty} \beta_n \sum_{j=0}^{n-1} \phi^{-1}\left(\frac{1}{p(j)}\right) < 1,$$

and

$$\sum_{s=0}^{\infty} \frac{1}{\phi^{-1}(p(s))} = \infty, \quad (1.4)$$

f is a Caratheodory function, i.e., f satisfies that for each $r > 0$ there exist a real number sequence $\{\phi_r(n)\}$ with $\sum_{n=0}^{\infty} \phi_r(n) < \infty$ such that

$$f\left(n, \left(1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}\right)x, \frac{1}{\phi^{-1}(p(n))}y\right) \leq \phi_r(n), \quad n \in N_0, |x| \leq r, |y| \leq r,$$

ϕ is defined by $\phi(x) = |x|^{p-2}x$ with $p > 1$, its inverse function is denoted by $\phi^{-1}(x) = |x|^{q-2}x$ with $1/p + 1/q = 1$, $\Delta x(n) = x(n+1) - x(n)$. We establish sufficient conditions for the existence of at least three positive solutions of BVP(3).

It is easy to see that the positive solutions of BVP(1) are bounded, the positive solutions of BVP(3) may be unbounded since 1.4 holds. The results in this article generalize the theorems in [12] to the infinite case and the fixed point theorem used is different from those used in [3, 12, 17]. The most interesting part in this article is the construction of the nonlinear operator and the cone, this method is not found in known papers.

The remainder of this paper is organized as follows: in Section 2, we first give some lemmas, then the main result (Theorem 1) and its proof are presented. An example is given in Section 3 to illustrate the main result.

2. MAIN RESULTS

In this section, we first present some background definitions in Banach spaces and state three important fixed point theorems and lemmas. Then the main result is given and proved.

Definition 1. Let X be a real Banach space. The nonempty convex closed subset P of X is called a cone in X if $ax \in P$ for all $x \in P$ and $a \geq 0$, $x \in X$ and $-x \in X$ imply $x = 0$.

Definition 2. A map $\psi : P \rightarrow [0, +\infty)$ is a nonnegative continuous concave or convex functional map provided ψ is nonnegative, continuous and satisfies $\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y)$, or $\psi(tx + (1-t)y) \leq t\psi(x) + (1-t)\psi(y)$, for all $x, y \in P$ and $t \in [0, 1]$.

Definition 3. An operator $T; X \rightarrow X$ is completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Definition 4. Let $a, b, c, d, h > 0$ be positive constants, α, ψ be two nonnegative continuous concave functionals on the cone P , γ, β, θ be three nonnegative continuous convex functionals on the cone P . Define the convex sets as follows:

$$\begin{aligned} P_c &= \{x \in P : \|x\| < c\}, \\ P(\gamma, \alpha; a, c) &= \{x \in P : \alpha(x) \geq a, \gamma(x) \leq c\}, \\ P(\gamma, \theta, \alpha; a, b, c) &= \{x \in P : \alpha(x) \geq a, \theta(x) \leq b, \gamma(x) \leq c\}, \\ Q(\gamma, \beta; d, c) &= \{x \in P : \beta(x) \leq d, \gamma(x) \leq c\}, \end{aligned}$$

$$Q(\gamma, \beta, \psi; h, d, c) = \{x \in P : \psi(x) \geq h, \beta(x) \leq d, \gamma(x) \leq c\}.$$

Lemma 1 ([7]). *Let X be a real Banach space, P be a cone in X , α, ψ be two nonnegative continuous concave functionals on the cone P , γ, β, θ be three nonnegative continuous convex functionals on the cone P . There exist constant $M > 0$ such that*

$$\alpha(x) \leq \beta(x), \|x\| \leq M\gamma(x) \text{ for all } x \in P.$$

Furthermore, Suppose that $h, d, a, b, c > 0$ are constants with $d < a$. Let $T : \overline{P_c} \rightarrow \overline{P_c}$ be a completely continuous operator. If

(C1) $\{y \in P(\gamma, \theta, \alpha; a, b, c) | \alpha(x) > a\} \neq \emptyset$ and

$$\alpha(Tx) > a \text{ for every } x \in P(\gamma, \theta, \alpha; a, b, c);$$

(C2) $\{y \in Q(\gamma, \theta, \psi; h, d, c) | \beta(x) < d\} \neq \emptyset$ and

$$\beta(Tx) < d \text{ for every } x \in Q(\gamma, \theta, \psi; h, d, c);$$

(C3) $\alpha(Ty) > a$ for $y \in P(\gamma, \alpha; a, c)$ with $\theta(Ty) > b$;

(C4) $\beta(Tx) < d$ for each $x \in Q(\gamma, \beta; d, c)$ with $\psi(Tx) < h$,
then T has at least three fixed points y_1, y_2 and y_3 such that

$$\beta(y_1) < d, \alpha(y_2) > a, \beta(y_3) > d, \alpha(y_3) < a.$$

Choose

$$X = \left\{ \begin{array}{l} x(n) \in R, n \in N_0 \\ \text{there exist the limits} \\ \{x(n)\} : \lim_{n \rightarrow \infty} \frac{x(n)}{\sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))} + 1}, \\ \lim_{n \rightarrow \infty} \phi^{-1}(p(n)) \Delta x(n) \end{array} \right\}.$$

Define the norm

$$\|x\| = \max \left\{ \sup_{n \in N_0} \frac{|x(n)|}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}, \sup_{n \in N_0} \phi^{-1}(p(n)) |\Delta x(n)| \right\}.$$

It is easy to see that X is a real Banach space.

Denote $[a, b] = \{a, a+1, \dots, b\}$ for $a \leq b$ and $[a, b] = \emptyset$ for $a > b$ with $a, b \in N_0$, N denotes the set of all positive integeres. Let $k_1, k_2 \in N$ with $k_1 < k_2$. Let

$$\mu = \frac{1}{\max\{1, \phi^{-1}(p(k_1-1))\} \left(1 + \sum_{n=0}^{k_2-1} \frac{1}{\phi^{-1}(p(n))}\right)}. \quad (2.1)$$

Choose

$$P = \left\{ x \in X : \begin{array}{l} x(n) \geq 0 \text{ for all } n \in N_0, \\ \min_{n \in [k_1, k_2]} \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \geq \mu \sup_{n \in N_0} \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \end{array} \right\},$$

It is easy to see that P is a nontrivial cone in X .

Let $h(n) \not\equiv 0 (n \in N_0)$ be a nonnegative sequence with $\sum_{n \in N_0} h(n)$ converging, consider the following BVP

$$\begin{cases} \Delta[p(n)\phi(\Delta y(n))] + h(n) = 0, & n \in N_0, \\ y(0) - \sum_{n=1}^{\infty} \alpha_n y(n) = 0, \\ \lim_{n \rightarrow \infty} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} - \sum_{n=1}^{\infty} \beta_n y(n) = 0, \end{cases} \quad (2.2)$$

Set

$$\delta = \frac{\phi\left(\frac{\sum_{n=1}^{\infty} \beta_n}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1}\left(\frac{1}{p(j)}\right) + \sum_{n=1}^{\infty} \beta_n \sum_{j=0}^{n-1} \phi^{-1}\left(\frac{1}{p(j)}\right)\right)}{1 - \phi\left(\frac{\sum_{n=1}^{\infty} \beta_n}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1}\left(\frac{1}{p(j)}\right) + \sum_{n=1}^{\infty} \beta_n \sum_{j=0}^{n-1} \phi^{-1}\left(\frac{1}{p(j)}\right)\right)}. \quad (2.3)$$

Lemma 2. If y is a solution of BVP(6), then $y(n) \geq 0$ and $\Delta y(n) \geq 0$ for all $n \in N_0$ and there exists a unique number $A_h \in [0, \delta \sum_{s=0}^{\infty} h(s)]$ such that

$$y(n) = \frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1}\left(\frac{A_h + \sum_{s=j}^{\infty} h(s)}{p(j)}\right) + \sum_{j=0}^{n-1} \phi^{-1}\left(\frac{A_h + \sum_{s=j}^{\infty} h(s)}{p(j)}\right). \quad (2.4)$$

Proof. Since $\Delta[p(n)\phi(\Delta y(n))] = -h(n) \leq 0$ for all $n \in N_0$ and $\sum_{n \in N_0} h(n)$ converges, we get that $p(n)\phi(\Delta y(n))$ is decreasing and there exists the limit $\lim_{n \rightarrow \infty} p(n)\phi(\Delta y(n))$. Then $\phi^{-1}(p(n))\Delta y(n)$ is decreasing and there exists the limit $\lim_{n \rightarrow \infty} \phi^{-1}(p(n))\Delta y(n)$.

Since there exists the limit $\lim_{n \rightarrow \infty} \phi^{-1}(p(n))\Delta y(n) = c$, we can prove

$$\lim_{n \rightarrow \infty} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} = c.$$

In fact, if $c = 0$, then for any $\epsilon > 0$ there exists $H > 0$ such that

$$\phi^{-1}(p(n))|\Delta y(n)| < \frac{\epsilon}{2}, n \geq H.$$

It follows that

$$|y(n)| \leq |y(H)| + \sum_{s=H}^{n-1} |\Delta y(s)| \leq |y(H)| + \frac{\epsilon}{2} \sum_{s=H}^{n-1} \frac{1}{\phi^{-1}(p(s))}, n \geq H.$$

Then

$$\frac{|y(n)|}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}$$

$$\begin{aligned}
&\leq \frac{|y(H)|}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} + \frac{1}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \frac{\epsilon}{2} \sum_{s=H}^{n-1} \frac{1}{\phi^{-1}(p(s))} \\
&< \frac{|y(H)|}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} + \frac{\epsilon}{2}, \quad n \geq H.
\end{aligned}$$

Since $\sum_{s=0}^{\infty} \frac{1}{\phi^{-1}(p(s))} = \infty$, we can choose $H' > H$ large enough so that

$$\frac{|y(n)|}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \leq \frac{|y(H)|}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} + \frac{\epsilon}{2} < \epsilon, \quad n \geq H',$$

which implies that

$$\lim_{n \rightarrow \infty} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} = 0.$$

If $c \neq 0$, then $\lim_{t \rightarrow \infty} (\phi^{-1}(p(n))\Delta y(n) - c) = 0$. It follows that

$$\lim_{t \rightarrow \infty} \phi^{-1}(p(n))\Delta \left[y(n) - c \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))} \right] = 0.$$

Then we get similarly that

$$\lim_{t \rightarrow \infty} \frac{y(n) - c \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} = 0.$$

Together with $\sum_{s=0}^{\infty} \frac{1}{\phi^{-1}(p(s))} = \infty$, it follows that

$$\lim_{n \rightarrow \infty} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} = c.$$

It follows from 2.2 that there exist $A, B \in R$ such that

$$p(n)\phi(\Delta y(n)) = A + \sum_{s=n}^{\infty} h(s), \quad (2.5)$$

and

$$y(n) = B + \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \left(A + \sum_{s=n}^{\infty} h(s) \right) \right), \quad n \in N_0.$$

By the boundary conditions in 2.2, we get

$$B = B \sum_{n=1}^{\infty} \alpha_n + \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \left(A + \sum_{s=j}^{\infty} h(s) \right) \right).$$

It follows that

$$B = \frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \left(A + \sum_{s=j}^{\infty} h(s) \right) \right).$$

So

$$\begin{aligned} y(n) &= \frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \left(A + \sum_{s=j}^{\infty} h(s) \right) \right) \\ &\quad + \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \left(A + \sum_{s=j}^{\infty} h(s) \right) \right). \end{aligned}$$

From 2.5, one sees that $\lim_{n \rightarrow \infty} p(n) \phi(\Delta y(n)) = A$.

Then $\lim_{n \rightarrow \infty} \phi^{-1}(p(n)) \Delta y(n) = \phi^{-1}(A)$. So

$$\lim_{n \rightarrow \infty} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} = \phi^{-1}(A).$$

Boundary conditions in 2.2 imply that

$$\begin{aligned} \phi^{-1}(A) &= \frac{\sum_{n=1}^{\infty} \beta_n}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{A + \sum_{s=j}^{\infty} h(s)}{p(j)} \right) \\ &\quad + \sum_{n=1}^{\infty} \beta_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{A + \sum_{s=j}^{\infty} h(s)}{p(j)} \right). \end{aligned} \quad (2.6)$$

Set

$$\begin{aligned} G(x) &= 1 - \frac{\sum_{n=1}^{\infty} \beta_n}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \left(1 + \frac{\sum_{s=j}^{\infty} h(s)}{x} \right) \right) \\ &\quad - \sum_{n=1}^{\infty} \beta_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \left(1 + \frac{\sum_{s=j}^{\infty} h(s)}{x} \right) \right). \end{aligned}$$

It is easy to see that $G(x)$ is continuous on $(-\infty, 0)$ and $(0, \infty)$, respectively, $G(x)$ is increasing on $(-\infty, 0)$ and $(0, \infty)$, respectively.

Since

$$\begin{aligned} &\lim_{x \rightarrow -\infty} G(x) \\ &= 1 - \frac{\sum_{n=1}^{\infty} \beta_n}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \right) - \sum_{n=1}^{\infty} \beta_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \right) > 0, \end{aligned}$$

and

$$\lim_{x \rightarrow 0^-} G(x) = +\infty,$$

we get $G(x)$ has no zero point on $(-\infty, 0)$. On the other hand, we have

$$\begin{aligned} & G\left(\delta \sum_{s=0}^{\infty} h(s)\right) \\ &= 1 - \frac{\sum_{n=1}^{\infty} \beta_n}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1}\left(\frac{1}{p(j)} \left(1 + \frac{\sum_{s=j}^{\infty} h(s)}{\delta \sum_{s=0}^{\infty} h(s)}\right)\right) \\ &\quad - \sum_{n=1}^{\infty} \beta_n \sum_{j=0}^{n-1} \phi^{-1}\left(\frac{1}{p(j)} \left(1 + \frac{\sum_{s=j}^{\infty} h(s)}{\delta \sum_{s=0}^{\infty} h(s)}\right)\right) \\ &\geq 1 - \frac{\sum_{n=1}^{\infty} \beta_n}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1}\left(\frac{1}{p(j)} \left(1 + \frac{1}{\delta}\right)\right) \\ &\quad - \sum_{n=1}^{\infty} \beta_n \sum_{j=0}^{n-1} \phi^{-1}\left(\frac{1}{p(j)} \left(1 + \frac{1}{\delta}\right)\right) \\ &= 1 - \phi^{-1}\left(1 + \frac{1}{\delta}\right) \left(\frac{\sum_{n=1}^{\infty} \beta_n}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1}\left(\frac{1}{p(j)}\right) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \beta_n \sum_{j=0}^{n-1} \phi^{-1}\left(\frac{1}{p(j)}\right)\right) \\ &= 0, \end{aligned}$$

and

$$\lim_{x \rightarrow 0^+} G(x) = -\infty,$$

Hence we find that there exists a unique $\xi \in (0, \delta \sum_{s=0}^{\infty} h(s)]$ such that $G(\xi) = 0$. It follows that there exists unique $A_h \in [0, \delta \sum_{s=0}^{\infty} h(s)]$ satisfying 2. Hence 2.4 holds for $A_h \in [0, \delta \sum_{s=0}^{\infty} h(s)]$. It is easy to see from 2.4 that $y(n) \geq 0$ and $\Delta y(n) \geq 0$ for all $n \in N_0$. The proof is complete. $\square$

Lemma 3. Choose integers $k_1, k_2 \in N_0$ with $1 < k_1 + 1 < k_2$. Let μ be defined by 2.1. Suppose y is a solution of BVP(6). Then

$$\min_{n \in [k_1, k_2]} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \geq \mu \sup_{n \in N_0} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}. \quad (2.7)$$

Proof. Since $\Delta[p(n)\phi(\Delta y(n))] = -h(n) \leq 0$ for all $n \in N_0$, we see that $p(n)\phi(\Delta y(n))$ is decreasing. Then $\phi^{-1}(p(n))\Delta y(n)$ is decreasing.

It follows from Lemma 1 that $y(n) \geq 0$ and $\Delta y(n) \geq 0$ for all $n \in N_0$ and there exists the limit

$$\sup_{n \in N_0} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}.$$

To complete the proof of 2.7, we consider two cases:

Case 1. there is $n_0 \in N_0$ such that

$$\sup_{n \in N_0} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} = \frac{y(n_0)}{1 + \sum_{s=0}^{n_0-1} \frac{1}{\phi^{-1}(p(s))}}.$$

For $n_1, n, n_2 \in N_0$ with $n_1 < n < n_2$, we have

$$\begin{aligned} & \left(\sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n_1-1} \frac{1}{\phi^{-1}(p(s))} \right) \\ & \frac{y(n_2) - y(n)}{\sum_{s=0}^{n_2-1} \frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} + y(n_1) - y(n) \\ & = \frac{\left(\sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n_1-1} \frac{1}{\phi^{-1}(p(s))} \right) (y(n_2) - y(n))}{\sum_{s=0}^{n_2-1} \frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \\ & + \frac{\left(\sum_{s=0}^{n_2-1} \frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))} \right) (y(n_1) - y(n))}{\sum_{s=0}^{n_2-1} \frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \\ & = \frac{\left(\sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n_1-1} \frac{1}{\phi^{-1}(p(s))} \right) \sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))} \phi^{-1}(p(s)) \Delta y(s)}{\sum_{s=0}^{n_2-1} \frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \\ & - \frac{\left(\sum_{s=0}^{n_2-1} \frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))} \right) \sum_{s=n_1}^{n-1} \frac{1}{\phi^{-1}(p(s))} \phi^{-1}(p(s)) \Delta y(s)}{\sum_{s=0}^{n_2-1} \frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}. \end{aligned}$$

Since $\phi^{-1}(p(n))\Delta y(n)$ is decreasing, we get $\phi^{-1}(p(s))\Delta y(s) \leq \phi^{-1}(p(k))\Delta y(k)$ for all $s \geq k$. So there there is λ such that

$$\phi^{-1}(p(s))\Delta y(s) \leq \phi^{-1}(p(k))\Delta y(k), \quad s \geq k.$$

Then we get

$$\frac{\left(\sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n_1-1} \frac{1}{\phi^{-1}(p(s))} \right) \sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))} \phi^{-1}(p(s)) \Delta y(s)}{\sum_{s=0}^{n_2-1} \frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}$$

$$\begin{aligned}
&= \frac{\sum_{s=n_1}^{n-1} \frac{1}{\phi^{-1}(p(s))} \sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))} \phi^{-1}(p(s)) \Delta y(s)}{\sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}} \\
&\leq \frac{\lambda \sum_{s=n_1}^{n-1} \frac{1}{\phi^{-1}(p(s))} \sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}}{\sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}} \\
&= \frac{\lambda \sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))} \sum_{s=n_1}^{n-1} \frac{1}{\phi^{-1}(p(s))}}{\sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}} \\
&\leq \frac{\sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))} \sum_{s=n_1}^{n-1} \frac{1}{\phi^{-1}(p(s))} \phi^{-1}(p(s)) \Delta y(s)}{\sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}}.
\end{aligned}$$

So

$$\left(\sum_{s=n_1}^{n-1} \frac{1}{\phi^{-1}(p(s))} \right) \frac{y(n_2) - y(n)}{\sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}} + y(n_1) - y(n) \leq 0.$$

It follows that

$$y(n) \geq \frac{\sum_{s=n}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}}{\sum_{s=n_1}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}} y(n_1) + \frac{\sum_{s=n_1}^{n-1} \frac{1}{\phi^{-1}(p(s))}}{\sum_{s=n_1}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}} y(n_2). \quad (2.8)$$

If $n_0 = k_1$, we get

$$\begin{aligned}
&\min_{n \in [k_1, k_2]} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \geq \frac{y(k_1)}{1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}} \\
&= \frac{y(n_0)}{1 + \sum_{s=0}^{n_0-1} \frac{1}{\phi^{-1}(p(s))}} \frac{1 + \sum_{s=0}^{k_1-1} \frac{1}{\phi^{-1}(p(s))}}{1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}} \\
&\geq \frac{1}{\phi^{-1}(p(k_1-1)) \left(1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))} \right)} \sup_{n \in N_0} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \\
&\geq \mu \sup_{n \in N_0} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}.
\end{aligned}$$

If $n_0 > k_1$, choose $n_1 = k_1 - 1$, $n = k_1$ and $n_2 = n_0$, by using 2.8 we have

$$\begin{aligned}
y(k_1) &= \frac{\sum_{s=k_1}^{n_0-1} \frac{1}{\phi^{-1}(p(s))}}{\sum_{s=k_1-1}^{n_0-1} \frac{1}{\phi^{-1}(p(s))}} y(k_1 - 1) + \frac{\sum_{s=k_1-1}^{k_1-1} \frac{1}{\phi^{-1}(p(s))}}{\sum_{s=k_1-1}^{n_0-1} \frac{1}{\phi^{-1}(p(s))}} y(n_0) \\
&\geq \frac{\sum_{s=k_1-1}^{k_1-1} \frac{1}{\phi^{-1}(p(s))}}{\sum_{s=k_1-1}^{n_0-1} \frac{1}{\phi^{-1}(p(s))}} y(n_0).
\end{aligned}$$

Then

$$\begin{aligned}
& \min_{n \in [k_1, k_2]} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \\
& \geq \frac{y(k_1)}{1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}} \\
& \geq \frac{1 + \sum_{s=0}^{n_0-1} \frac{1}{\phi^{-1}(p(s))}}{1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}} \frac{\sum_{s=k_1-1}^{k_1-1} \frac{1}{\phi^{-1}(p(s))}}{\sum_{s=k_1-1}^{n_0-1} \frac{1}{\phi^{-1}(p(s))}} \frac{y(n_0)}{1 + \sum_{s=0}^{n_0-1} \frac{1}{\phi^{-1}(p(s))}} \\
& \geq \frac{1}{\phi^{-1}(p(k_1-1)) \left(1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}\right)} \sup_{n \in N_0} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \\
& \geq \mu \sup_{n \in N_0} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}.
\end{aligned}$$

If $n_0 < k_1$, we have

$$\begin{aligned}
& \min_{n \in [k_1, k_2]} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \\
& \geq \frac{y(k_1)}{1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}} \\
& \geq \frac{1}{1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}} y(n_0) \\
& \geq \frac{1 + \sum_{s=0}^{n_0-1} \frac{1}{\phi^{-1}(p(s))}}{1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}} \frac{y(n_0)}{1 + \sum_{s=0}^{n_0-1} \frac{1}{\phi^{-1}(p(s))}} \\
& \geq \frac{1}{1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}} \sup_{n \in N_0} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \\
& \geq \mu \sup_{n \in N_0} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}.
\end{aligned}$$

Case 2. $\sup_{n \in N_0} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} = \lim_{n \rightarrow \infty} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}.$

Choose $n' > k_2$, similarly to Case 1 we can prove that

$$\min_{n \in [k_1, k_2]} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \geq \mu \frac{y(n')}{1 + n'}.$$

Let $n' \rightarrow \infty$, one sees

$$\min_{n \in [k_1, k_2]} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \geq \mu \sup_{n \in N_0} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}.$$

From Cases 1 and 2, we get 2.7. The proof is complete. $\square$

Define the functionals on $P : P \rightarrow R$ by

$$\gamma(x) = \sup_{n \in N_0} \phi^{-1}(p(n)) |\Delta x(n)|, \quad x \in P,$$

$$\beta(x) = \sup_{n \in N_0} \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}, \quad x \in P,$$

$$\theta(x) = \sup_{n \in N_0} \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}, \quad x \in P,$$

$$\alpha(x) = \min_{n \in [k_1, k_2]} \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}, \quad x \in P,$$

$$\psi(x) = \min_{n \in [k_1, k_2]} \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}, \quad x \in P.$$

Lemma 4. *If y is a solution of BVP(6), we have $\|y\| \leq M\gamma(y)$ for all $y \in P$, where*

$$M = \max \left\{ 1, \sup_{n \in N_0} \frac{\sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))} + \frac{\sum_{n \in N} \alpha_n \sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))}}{1 - \sum_{i=1}^{\infty} \alpha_i} \right\}. \quad (2.9)$$

Proof. Since y is the solution of BVP(6), we get

$$\begin{aligned} |y(n)| &= |y(n) - y(0) + y(0)| \\ &\leq \left| \sum_{i=0}^{n-1} \Delta y(i) \right| + \left| \frac{y(0) - \sum_{n \in N} \alpha_n y(0)}{1 - \sum_{n \in N} \alpha_n} \right| \\ &= \left| \sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))} \phi^{-1}(p(i)) \Delta y(i) \right| + \frac{\sum_{n \in N} \alpha_n |y(n) - y(0)|}{1 - \sum_{n \in N} \alpha_n} \\ &\leq \sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))} \sup_{n \in N_0} \phi^{-1}(p(n)) |\Delta y(n)| + \left| \frac{\sum_{n \in N} \alpha_n [y(n) - y(0)]}{1 - \sum_{i=1}^{\infty} \alpha_i} \right| \\ &\leq \left(\sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))} + \frac{\sum_{n \in N} \alpha_n \sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))}}{1 - \sum_{i=1}^{\infty} \alpha_i} \right) \sup_{n \in N_0} \phi^{-1}(p(n)) |\Delta y(n)|. \end{aligned}$$

It follows that

$$\begin{aligned}
& \frac{y(n)}{1 + \sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))}} \\
& \leq \frac{\sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))} + \frac{\sum_{n \in N} \alpha_n \sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))}}{1 - \sum_{i=1}^{\infty} \alpha_i}}{1 + \sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))}} \sup_{n \in N_0} \phi^{-1}(p(n)) |\Delta y(n)| \\
& \leq \sup_{n \in N_0} \frac{\sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))} + \frac{\sum_{n \in N} \alpha_n \sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))}}{1 - \sum_{i=1}^{\infty} \alpha_i}}{1 + \sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))}} \sup_{n \in N_0} \phi^{-1}(p(n)) |\Delta y(n)|.
\end{aligned}$$

we get that

$$\begin{aligned}
\|y\| &= \max \left\{ \sup_{n \in N_0} \frac{|y(n)|}{1 + \sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))}}, \sup_{n \in N_0} \phi^{-1}(p(n)) |\Delta y(n)| \right\} \\
&\leq \max \left\{ 1, \sup_{n \in N_0} \frac{\sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))} + \frac{\sum_{n \in N} \alpha_n \sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))}}{1 - \sum_{i=1}^{\infty} \alpha_i}}{1 + \sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))}} \right\} \\
&\quad \sup_{n \in N_0} \phi^{-1}(p(n)) |\Delta y(n)| \\
&= M\gamma(y).
\end{aligned}$$

Then $\|y\| \leq M\gamma(y)$ for all solutions y of BVP(6). The proof is complete. $\square$

For $x \in P$, define $(Tx)(n)$ by

$$\begin{aligned}
(Tx)(n) &= \frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{A_x + \sum_{s=j}^{\infty} f(s, x(s), \Delta x(s))}{p(j)} \right) \\
&\quad + \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{A_x + \sum_{s=j}^{\infty} f(s, x(s), \Delta x(s))}{p(j)} \right), \quad n \in N_0,
\end{aligned}$$

where A_x satisfies

$$\begin{aligned}
\phi^{-1}(A_x) &= \frac{\sum_{n=1}^{\infty} \beta_n}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{A_x + \sum_{s=j}^{\infty} f(s, x(s), \Delta x(s))}{p(j)} \right) \\
&\quad + \sum_{n=1}^{\infty} \beta_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{A_x + \sum_{s=j}^{\infty} f(s, x(s), \Delta x(s))}{p(j)} \right).
\end{aligned}$$

One sees easily that

$$\begin{cases} \Delta[\phi(\Delta(Tx)(n))] + f(n, x(n), \Delta x(n)) = 0, & n \in N_0, \\ (Tx)(0) - \sum_{i=1}^{\infty} \alpha_i (Tx)(i) = 0, \\ \lim_{n \rightarrow \infty} \frac{(Tx)(n)}{1 + \sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))}} - \sum_{i=1}^{\infty} \beta_i (Tx)(i) = 0. \end{cases} \quad (2.10)$$

Let δ be defined by 2.3. By Lemma 2, we have

$$A_x \in \left[0, \delta \sum_{n=0}^{\infty} f(n, x(n), \Delta x(n)) \right]. \quad (2.11)$$

Lemma 5. Let $V = \{x \in X : \|x\| < l\} (l > 0)$. If $\left\{ \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} : x \in V \right\}$ and $\{\phi^{-1}(p(n))\Delta x(n) : x \in V\}$ are both equiconvergent at infinity, where

$$V_1 =: \left\{ \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} : x \in V \right\} \cup \{\phi^{-1}(p(n))\Delta x(n) : x \in V\}$$

is called equiconvergent at infinity if and only if for all $\epsilon > 0$, there exists $N = N(\epsilon) > 0$ such that for all $x \in V$, it holds that

$$\left| \frac{x(n_1)}{1 + \sum_{s=0}^{n_1-1} \frac{1}{\phi^{-1}(p(s))}} - \frac{x(n_2)}{1 + \sum_{s=0}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}} \right| < \epsilon,$$

and

$$|\phi^{-1}(p(n_1))\Delta x(n_1) - \phi^{-1}(p(n_2))\Delta x(n_2)| < \epsilon \quad n_1, n_2 > N.$$

Then V is pre-compact on X .

Proof. The proof is similar to that of the proof of a Lemma in [14] and is omitted. $\square$

Lemma 6. The following facts hold.

- (i) $Tx \in P$ for each $x \in P$;
- (ii) x is a solution of BVP(3) if and only if x is a solution of the operator equation $x = Tx$;
- (iii) $T : P \rightarrow P$ is completely continuous;

Proof. (i) Note the definition of P . For $x \in P$, Lemma 2 and Lemma 3 imply that $Tx \in P$.

(ii) It is easy to see that x is a solution of BVP(3) if and only if x is a solution of the operator equation $x = Tx$.

(iii) It suffices to prove that T is continuous on P and T maps bounded subsets into pre-compact sets. We divide the proof into four steps:

Step 1. For each bounded subset $D \subset P$, prove that $\{A_x : x \in \overline{D}\}$ is bounded in R .

Denote

$$L_1 = \sup \left\{ \max_{n \in N_0} \frac{|x(n)|}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}, \sup_{n \in N_0} \phi^{-1}(p(n)) |\Delta x(n)| : x \in \overline{D} \right\}$$

and

$$f_{L_1}(j) = \max_{|x|, |y| \leq L_1} \left| f \left(j, \left(1 + \sum_{s=0}^{j-1} \frac{1}{\phi^{-1}(p(s))} \right) x, \frac{y}{\phi^{-1}(p(j))} \right) \right|.$$

Since f is a Caratheodory function, it follows from 2.11 that

$$0 \leq A_x \leq \delta \sum_{j=0}^{\infty} f_{L_1}(j) < \infty.$$

Hence $\{A_x : x \in \overline{D}\}$ is bounded in R .

Step 2. For each bounded subset $D \subset P$, and each $x_0 \in D$, prove that T is continuous at x_0 .

For $x_0 \in D$ and $x_n \in D$ with $x_n \rightarrow x_0 (n \rightarrow +\infty)$ in D .

Denote $u_n(k) = (Tx_n)(k)$, $u_0(k) = (Tx_0)(k)$ for all $k \in N_0$. We prove that T is continuous at x_0 , i.e., $u_n \rightarrow u_0 (n \rightarrow +\infty)$. Let A_{x_0} be defined by

$$\begin{aligned} & \phi^{-1}(A_{x_0}) \\ &= \frac{\sum_{n=1}^{\infty} \beta_n}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{A_{x_0} + \sum_{s=j}^{\infty} f(s, x_0(s), \Delta x_0(s))}{p(j)} \right) \\ & \quad + \sum_{n=1}^{\infty} \beta_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{A_{x_0} + \sum_{s=j}^{\infty} f(s, x_0(s), \Delta x_0(s))}{p(j)} \right). \end{aligned}$$

First, we prove that A_x are continuous in x , i.e.,

$$A_{x_n} \rightarrow A_{x_0}, \quad n \rightarrow +\infty.$$

It follows from Step 1 that A_{x_n} is bounded. Without loss of generality, suppose that $A_{x_n} \rightarrow \bar{A} \neq A_{x_0}$.

It is easy to see that

$$\begin{aligned} & \lim_{n \rightarrow +\infty} u_n(k) \\ &= \lim_{n \rightarrow +\infty} \left[\frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{A_{x_n} + \sum_{s=j}^{\infty} f(s, x_n(s), \Delta x_n(s))}{p(j)} \right) \right. \\ & \quad \left. + \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{A_{x_n} + \sum_{s=j}^{\infty} f(s, x_n(s), \Delta x_n(s))}{p(j)} \right) \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{\bar{A} + \lim_{n \rightarrow +\infty} \sum_{s=j}^{\infty} f(s, x_n(s), \Delta x_n(s))}{p(j)} \right) \\
&\quad + \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{\bar{A} + \lim_{n \rightarrow +\infty} \sum_{s=j}^{\infty} f(s, x_n(s), \Delta x_n(s))}{p(j)} \right) \\
&= \frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{\bar{A} + \sum_{s=j}^{\infty} f(s, x_0(s), \Delta x_0(s))}{p(j)} \right) \\
&\quad + \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{\bar{A} + \sum_{s=j}^{\infty} f(s, x_0(s), \Delta x_0(s))}{p(j)} \right) \\
&= \bar{u}(k).
\end{aligned}$$

One sees that $\bar{A} = \lim_{n \rightarrow \infty} \phi^{-1}(p(n)) \Delta \bar{u}(n)$ and $\bar{u}$ satisfies

$$\bar{u}(0) - \sum_{n=1}^{\infty} \alpha_n \bar{u}(n) = 0, \quad \lim_{n \rightarrow \infty} \frac{\bar{u}(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} - \sum_{i=1}^{\infty} \beta_i \bar{u}(i) = 0.$$

So

$$\begin{aligned}
\phi^{-1}(\bar{A}) &= \frac{\sum_{n=1}^{\infty} \beta_n}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{\bar{A} + \sum_{s=j}^{\infty} f(s, x_0(s), \Delta x_0(s))}{p(j)} \right) \\
&\quad + \sum_{n=1}^{\infty} \beta_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{\bar{A} + \sum_{s=j}^{\infty} f(s, x_0(s), \Delta x_0(s))}{p(j)} \right).
\end{aligned}$$

It follows from Lemma 2 that $\bar{A} = A_{x_0}$. Hence

$$A_{x_n} \rightarrow \bar{A} = A_{x_0}, \quad n \rightarrow +\infty.$$

This together with the continuous property of f implies that T is continuous at x_0 .

Step 3. For each bounded subset $\Omega \subset P$, prove that $T\Omega$ is bounded.

In fact, for each bounded subset $\Omega \subseteq D$, and $x \in \Omega$. Suppose

$$||x|| = \max \left\{ \sup_{n \in N_0} \frac{|x(n)|}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}, \max_{n \in [0, N+1]} \phi^{-1}(p(n)) |\Delta x(n)| \right\} \leq M_1$$

and Step 1 implies that there exist constants $M_2 > 0$ such that $|A_x| < M_2$ for all $x \in \Omega$. Then

$$\frac{|(Tx)(n)|}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}$$

$$\begin{aligned}
 &= \frac{1}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \\
 &\left(\frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{A_x + \sum_{s=j}^{\infty} f(s, x(s), \Delta x(s))}{p(j)} \right) \right. \\
 &\quad \left. + \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{A_x + \sum_{s=j}^{\infty} f(s, x(s), \Delta x(s))}{p(j)} \right) \right) \\
 &\leq \frac{1}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \\
 &\left(\frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{M_2 + \sum_{s=j}^{\infty} f(s, x(s), \Delta x(s))}{p(j)} \right) \right. \\
 &\quad \left. + \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{M_2 + \sum_{s=j}^{\infty} f(s, x(s), \Delta x(s))}{p(j)} \right) \right) \\
 &\leq \phi^{-1} \left(M_2 + \sum_{s=0}^{\infty} f_{M_1}(s) \right) \\
 &\frac{\frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \right) + \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \right)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \\
 &\leq \phi^{-1} \left(M_2 + \sum_{s=0}^{\infty} f_{M_1}(s) \right) \\
 &\sup_{n \in N_0} \frac{\frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \right) + \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \right)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \\
 &=: M_3,
 \end{aligned}$$

where $f_{M_1}(j) = \max_{|x| \leq M_1, |y| \leq M_1 \leq M_1} \left| f \left(j, \left(1 + \sum_{s=0}^{j-1} \frac{1}{\phi^{-1}(p(s))} \right) x, \frac{y}{\phi^{-1}(p(j))} \right) \right|$. Similarly, one has that

$$\phi^{-1}(p(n)) |\Delta(Tx)(n)| = \left| \phi^{-1} \left(A_x + \sum_{j=n}^{\infty} f(j, x(j), \Delta x(j)) \right) \right|$$

$$\leq \phi^{-1} \left(M_2 + \sum_{j=0}^{\infty} f_{M_1}(j) \right) =: M_4.$$

It follows that $T\Omega$ is bounded.

Step 4. For each bounded subset $\Omega \subset P$, prove that $T\Omega$ is pre-compact.

Similarly to Step 3, we see $|A_x| \leq M_2$, we get

$$A_x + \sum_{n=0}^{\infty} f(n, x(n), \Delta x(n)) \leq \phi(M_4).$$

Note $\phi^{-1}(x) = |x|^{q-2}x$. Then there $\xi \in \left[A_x, A_x + \sum_{j=n}^{\infty} f(j, x(j), \Delta x(j)) \right]$ such that

$$\begin{aligned} & |\phi^{-1}(p(n))\Delta(Tx)(n) - \phi^{-1}(A_x)| \\ &= \left| \phi^{-1} \left(A_x + \sum_{j=n}^{\infty} f(j, x(j), \Delta x(j)) \right) - \phi^{-1}(A_x) \right| \\ &= (q-1)\xi^{q-2} \sum_{j=n}^{\infty} f(j, x(j), \Delta x(j)) \\ &\leq (q-1)\phi(M_4)^{q-2} \sum_{j=n}^{\infty} f(j, x(j), \Delta x(j)) \\ &\leq (q-1)\phi(M_4)^{q-2} \sum_{j=n}^{\infty} f_{M_1}(j) \\ &\rightarrow 0 \text{ uniformly as } n \rightarrow \infty. \end{aligned}$$

For any $\epsilon > 0$, there exists $N_{1,\epsilon} > 0$ such that

$$|\phi^{-1}(p(n_1))\Delta(Tx)(n_1) - \phi^{-1}(p(n_2))\Delta(Tx)(n_2)| < \epsilon, \quad n > N_{1,\epsilon}. \quad (2.12)$$

Since $\lim_{n \rightarrow \infty} \phi^{-1}(p(n))\Delta(Tx)(n) = \phi^{-1}(A_x)$ uniformly, we know from the same methods used in the proof of Lemma 2 that

$$\lim_{n \rightarrow \infty} \frac{|(Tx)(n)|}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} = \phi^{-1}(A_x) \text{ uniformly.}$$

So there exists $N_{2,\epsilon} > 0$ such that

$$\left| \frac{(Tx)(n_1)}{1 + \sum_{s=0}^{n_1-1} \frac{1}{\phi^{-1}(p(s))}} - \frac{(Tx)(n_2)}{1 + \sum_{s=0}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}} \right| < \epsilon, \quad n > N_{2,\epsilon}. \quad (2.13)$$

Choose $N_\epsilon = \max \{N_{1,\epsilon}, N_{2,\epsilon}\}$. Then

$$|\phi^{-1}(p(n_1)\Delta(Tx)(n_1) - \phi^{-1}(p(n_2)\Delta(Tx)(n_2))| < \epsilon,$$

and

$$\left| \frac{(Tx)(n_1)}{1 + \sum_{s=0}^{n_1-1} \frac{1}{\phi^{-1}(p(s))}} - \frac{(Tx)(n_2)}{1 + \sum_{s=0}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}} \right| < \epsilon, \quad n > N_\epsilon.$$

One knows that $T\Omega$ is pre-compact. Lemma 5 with Steps 1, 2, 3 and 4 imply that T is completely continuous. $\square$

Theorem 1. Choose $k_1, k_2 \in N$ with $k_1 < k_2$. Let μ be defined by 2.1, δ by 2.3 and M by 2.9. Suppose that there exist positive constants e_1, e_2, c such that

$$c \geq \frac{e_2}{\mu} > e_2 > e_1 > 0.$$

Let

$$\begin{aligned} Q &= \phi\left(\frac{c}{M}\right) \frac{1}{1+\delta}; \\ W &= \phi\left(\frac{e_2\left(1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}\right)}{\sum_{j=0}^{k_1-1} \phi^{-1}\left(\frac{\sum_{s=k_1}^{k_2} \frac{1}{2s+1}}{p(j)}\right)}\right); \\ E &= \frac{1}{1+\delta} \phi\left(\frac{e_1}{\sup_{n \in N_0} \frac{\frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1}\left(\frac{1}{p(j)}\right) + \sum_{j=0}^{n-1} \phi^{-1}\left(\frac{1}{p(j)}\right)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}}\right). \end{aligned}$$

If $Q > W$ and

(A1) $f\left(n, \left(1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}\right)u, \frac{v}{\phi^{-1}(p(n))}\right) \leq \frac{Q}{2^{n+1}}$ for all $n \in N_0, u \in [0, c], v \in [0, c]$;

(A2) $f\left(n, \left(1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}\right)u, \frac{v}{\phi^{-1}(p(n))}\right) \geq \frac{W}{2^{n+1}}$ for all $n \in [k_1, k_2], u \in [e_2, \frac{e_2}{\mu}], v \in [0, c]$;

(A3) $f\left(n, \left(1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}\right)u, \frac{v}{\phi^{-1}(p(n))}\right) \leq \frac{E}{2^{n+1}}$ for all $n \in N_0, u \in [0, e_1], v \in [0, c]$;

then BVP(3) has at least three positive solutions x_1, x_2, x_3 such that

$$\sup_{n \in N_0} \frac{x_1(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} < e_1, \quad \min_{n \in [k_1, k_2]} \frac{x_2(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} > e_2, \quad (2.14)$$

and

$$\sup_{n \in N_0} \frac{x_3(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} > e_1, \quad \min_{n \in [k_1, k_2]} \frac{x_3(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} < e_2. \quad (2.15)$$

Proof. Let X , P and T be defined above. We complete the proof by using Lemma 1. By the definitions, it is easy to see that α, ψ are two nonnegative continuous concave functionals on the cone P , γ, β, θ are three nonnegative continuous convex functionals on the cone P .

One sees $\alpha(x) \leq \beta(x)$ for all $x \in P$. From Lemma 4, we have $\|x\| \leq M\gamma(x)$ for all $x \in P$.

Lemma 6 implies that $x = x(n)$ is a solution of BVP(3) if and only if x is a solution of the operator equation $x = Tx$ and $T : P \rightarrow P$ is completely continuous.

Corresponding to Lemma 1, choose

$$h = \mu e_1, \quad d = e_1, \quad a = e_2, \quad b = \frac{e_2}{\mu}, \quad c = c.$$

Now, we prove that all conditions of Lemma 1 hold. One sees that $0 < d < a$. The remainder is divided into five steps.

Step 1. Prove that $T : \overline{P_c} \rightarrow \overline{P_c}$;

For $x \in \overline{P_c}$, we have $\|x\| \leq c$. Then

$$\begin{aligned} 0 &\leq \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \leq c, \quad n \in N_0, \\ 0 &\leq \phi^{-1}(p(n))\Delta x(n) \leq c \text{ for } n \in N_0. \end{aligned}$$

So (A1) implies that

$$\begin{aligned} &f(n, x(n), \Delta x(n)) \\ &= f\left(n, \left(1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}\right) \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}, \frac{1}{\phi^{-1}(p(n))} \phi^{-1}(p(n))\Delta x(n)\right) \\ &\leq \frac{Q}{2^{n+1}}, \quad n \in N_0. \end{aligned}$$

It follows from Lemma 6 that $Tx \in P$. One sees from Lemma 1 that

$$0 \leq A_x \leq \delta \sum_{j=0}^{\infty} f(j, x(j), \Delta x(j)). \quad (2.16)$$

We have that

$$\phi^{-1}(p(n))|\Delta(Tx)(n)| = \left| \phi^{-1} \left(A_x + \sum_{j=n}^{\infty} f(j, x(j), \Delta x(j)) \right) \right|$$

$$\begin{aligned}
&\leq \phi^{-1} \left((1+\delta) \sum_{j=0}^{\infty} f(j, x(j), \Delta x(j)) \right) \\
&\leq \phi^{-1} \left((1+\delta) \sum_{j=0}^{\infty} \frac{Q}{2^{j+1}} \right) \\
&\leq \phi^{-1}(Q(1+\delta)) \leq c.
\end{aligned}$$

From Lemma 5, we have

$$\begin{aligned}
\frac{|(Tx)(n)|}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} &\leq M \sup_{n \in N_0} \phi^{-1}(p(n)) |\Delta(Tx)(n)| \\
&\leq M \phi^{-1}((1+\delta)Q) \leq c.
\end{aligned}$$

It follows that

$$\|Tx\| = \max \left\{ \max_{n \in N_0} \frac{|(Tx)(n)|}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}, \max_{n \in N_0} \phi^{-1}(p(n)) |\Delta(Tx)(n)| \right\} \leq c.$$

Then $T : \overline{P_c} \rightarrow \overline{P_c}$.

Step 2. Prove that

$$\begin{aligned}
&\{y \in P(\gamma, \theta, \alpha; a, b, c) | \alpha(x) > a\} \\
&= \left\{ y \in P \left(\gamma, \theta, \alpha; e_2, \frac{e_2}{\mu}, c \right) | \alpha(x) > e_2 \right\} \neq \emptyset
\end{aligned}$$

and $\alpha(Tx) > e_2$ for every $x \in P \left(\gamma, \theta, \alpha; e_2, \frac{e_2}{\mu}, c \right)$;

By the definition of μ , we can choose A, B such that

$$\begin{aligned}
A &\in \left[\left(1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))} \right) e_2, \left(1 + \sum_{s=0}^{k_1-1} \frac{1}{\phi^{-1}(p(s))} \right) \frac{e_2}{\mu} \right], \\
B &\leq \min \left\{ \frac{e_2}{\mu}, \frac{A}{\mu \left(1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))} \right)} \right\}
\end{aligned}$$

and

$$|A - B| < \frac{c}{\max\{\phi^{-1}(p(k_1-1)), \phi^{-1}(p(k_2))\}}.$$

Let

$$x(n) = \begin{cases} A, & n \in [k_1, k_2], \\ B, & n \notin [k_1, k_2]. \end{cases}$$

Then $x \in P$ and

$$\alpha(x) = \min_{n \in [k_1, k_2]} \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \geq \frac{A}{\left(1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}\right)} > e_2,$$

$$\begin{aligned} \theta(x) &= \sup_{n \in N_0} \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \\ &\leq \max \left\{ B, \frac{A}{\left(1 + \sum_{s=0}^{k_1-1} \frac{1}{\phi^{-1}(p(s))}\right)}, \frac{B}{\left(1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}\right)} \right\} \leq \frac{e_2}{\mu} = b, \end{aligned}$$

and

$$\gamma(x) = \sup_{n \in N_0} \phi^{-1}(p(n)) |\Delta x(n)| = \max\{\phi^{-1}(p(k_1-1)), \phi^{-1}(p(k_2))\} |A - B| < c.$$

It follows that $\{y \in P(\gamma, \theta, \alpha; a, b, c) | \alpha(x) > a\} \neq \emptyset$.

For $x \in P(\gamma, \theta, \alpha; a, b, c)$, one has that

$$\alpha(x) = \min_{n \in [k_1, k_2]} \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \geq e_2,$$

$$\theta(x) = \sup_{n \in N_0} \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \leq \frac{e_2}{\mu},$$

and

$$\gamma(x) = \sup_{n \in N_0} \phi^{-1}(p(n)) |\Delta x(n)| \leq c.$$

Then

$$e_2 \leq \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \leq \frac{e_2}{\mu}, \quad n \in [k_1, k_2], \quad 0 \leq \phi^{-1}(p(n)) \Delta x(n) \leq c.$$

Thus (A2) implies that

$$f(n, x(n), \Delta x(n)) \geq \frac{W}{2^{n+1}}, \quad n \in [k_1, k_2].$$

We get

$$\begin{aligned} \alpha(Tx) &= \min_{n \in [k_1, k_2]} \frac{(Tx)(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \\ &> \frac{(Tx)(k_1)}{1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}} \\ &= \frac{1}{1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}} \left[\frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \right] \end{aligned}$$

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{A_x + \sum_{s=j}^{\infty} f(s, x(s), \Delta x(s))}{p(j)} \right) \\
 & + \sum_{j=0}^{k_1-1} \phi^{-1} \left(\frac{A_x + \sum_{s=j}^{\infty} f(s, x(s), \Delta x(s))}{p(j)} \right) \Big] \\
 & \geq \frac{1}{1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}} \sum_{j=0}^{k_1-1} \phi^{-1} \left(\frac{\sum_{s=j}^{\infty} f(s, x(s), \Delta x(s))}{p(j)} \right) \\
 & \geq \frac{1}{1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}} \sum_{j=0}^{k_1-1} \phi^{-1} \left(\frac{\sum_{s=k_1}^{k_2} f(s, x(s), \Delta x(s))}{p(j)} \right) \\
 & \geq \frac{1}{1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}} \sum_{j=0}^{k_1-1} \phi^{-1} \left(\frac{\sum_{s=k_1}^{k_2} \frac{W}{2^{s+1}}}{p(j)} \right) \\
 & = e_2.
 \end{aligned}$$

This completes Step 2.

Step 3. Prove that

$$\{y \in Q(\gamma, \theta, \psi; h, d, c) | \beta(x) < d\} = \{y \in Q(\gamma, \theta, \psi; \mu e_1, e_1, c) | \beta(x) < e_1\} \neq \emptyset$$

and

$$\beta(Tx) < e_1 \text{ for every } x \in Q(\gamma, \theta, \psi; h, d, c) = Q(\gamma, \theta, \psi; \mu e_1, e_1, c);$$

Similarly to Step 2, we can see that $\{y \in Q(\gamma, \theta, \psi; h, d, c) | \beta(x) < d\} \neq \emptyset$.

For $x \in Q(\gamma, \theta, \psi; h, d, c)$, one has that

$$\psi(x) = \min_{n \in [k_1, k_2]} \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \geq \mu e_1$$

$$\theta(x) = \sup_{n \in N_0} \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \leq d = e_1,$$

and

$$\gamma(x) = \sup_{n \in N_0} \phi^{-1}(p(n)) |\Delta x(n)| \leq c.$$

Hence we get that

$$0 \leq \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \leq e_1, \quad n \in N_0; \quad 0 \leq \phi^{-1}(p(n)) \Delta x(n) \leq c, \quad n \in N_0.$$

Then (A3) implies that

$$f(n, x(n), \Delta x(n)) \leq \frac{E}{2^{n+1}}, \quad n \in N_0.$$

So 2.16 implies that

$$\begin{aligned}
\beta(Tx) &= \sup_{n \in N_0} \frac{(Tx)(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \\
&= \sup_{n \in N_0} \frac{1}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \left[\frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \right. \\
&\quad \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{A_x + \sum_{s=j}^{\infty} f(s, x(s), \Delta x(s))}{p(j)} \right) \\
&\quad \left. + \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{A_x + \sum_{s=j}^{\infty} f(s, x(s), \Delta x(s))}{p(j)} \right) \right] \\
&< \sup_{n \in N_0} \frac{\phi^{-1}((1+\delta) \sum_{s=0}^{\infty} f(s, x(s), \Delta x(s)))}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \left[\frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \right. \\
&\quad \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \right) + \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \right) \left. \right] \\
&\leq \sup_{n \in N_0} \frac{\phi^{-1}((1+\delta) \sum_{s=0}^{\infty} \frac{E}{2^{j+1}})}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \left[\frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \right) \right. \\
&\quad \left. + \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \right) \right] \\
&\leq \phi^{-1}((1+\delta)E) \\
&\quad \sup_{n \in N_0} \frac{\frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \right) + \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \right)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \\
&\leq e_1 = d.
\end{aligned}$$

This completes Step 3.

Step 4. Prove that $\alpha(Ty) > a$ for $y \in P(\gamma, \alpha; a, c)$ with $\theta(Ty) > b$;

For $x \in P(\gamma, \alpha; a, c) = P(\gamma, \alpha; e_2, c)$ with $\theta(Tx) = \beta(Tx) > b = \frac{e_2}{\mu}$, we have that

$$\begin{aligned}
\alpha(x) &= \min_{n \in [k_1, k_2]} \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \geq e_2, \\
\gamma(x) &= \sup_{n \in N_0} \phi^{-1}(p(n)) |\Delta x(n)| \leq c,
\end{aligned}$$

$$\sup_{n \in N_0} \frac{(Tx)(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} > \frac{e_2}{\mu}.$$

Then

$$\alpha(Tx) = \min_{n \in [k_1, k_2]} \frac{(Tx)(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \geq \mu \beta(Tx) > e_2 = a.$$

This completes Step 4.

Step 5. Prove that $\beta(Tx) < d$ for each $x \in Q(\gamma, \beta; d, c)$ with $\psi(Tx) < h$.

For $x \in Q(\gamma, \beta; d, c)$ with $\psi(Tx) < h$, we have

$$\begin{aligned} \gamma(x) &= \sup_{n \in N_0} \phi^{-1}(p(n)) |\Delta x(n)| \leq c, \\ \beta(x) &= \sup_{n \in N_0} \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \leq d = e_1, \\ \psi(Tx) &= \min_{n \in [k_1, k_2]} \frac{(Tx)(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} < h = \mu e_1. \end{aligned}$$

Then

$$\beta(Tx) = \sup_{n \in N_0} \frac{(Tx)(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} \leq \frac{1}{\mu} \min_{n \in [k_1, k_2]} \frac{(Tx)(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} < e_1 = d.$$

This completes the Step 5.

Then Lemma 1 implies that T has at least three fixed points y_1, y_2 and y_3 such that

$$\beta(y_1) < e_1, \alpha(y_2) > e_2, \beta(y_3) > e_1, \alpha(y_3) < e_2.$$

Hence BVP(3) has three positive solutions y_1, y_2 and y_3 satisfying 2.14 and 2.15. The proof is complete. $\square$

3. AN EXAMPLE

In this section, we present an example to illustrate Theorem 1.

Example 1. Consider the following BVP

$$\begin{cases} \Delta^2 x(n) = -f(n, x(n), \Delta x(n)), & n \in N_0, \\ x(0) = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} x(n), \\ \lim_{n \rightarrow \infty} \frac{x(n)}{1+n} = \sum_{n=1}^{\infty} \frac{1}{3^{n+1}} x(n), \end{cases} \quad (3.1)$$

where $f : N_0 \times [0, +\infty)^2 \rightarrow [0, +\infty)$ is a Caratheodory function.

Corresponding to BVP(3), $p(n) \equiv 1, \alpha_n = \frac{1}{2^{n+1}}, \beta_n = \frac{1}{3^{n+1}}$ and $\phi(x) = x$.

Choose the constant $k_1 = 10, k_2 = 10000, e_1 = 100, e_2 = 5400, c = 3688400 \times 2^{10009}$. It is easy to see that

$$\begin{aligned} \mu &= \frac{1}{\max\{\phi^{-1}(p(k_1 - 1)), 1\} \left(1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))}\right)} = \frac{1}{10001}, \\ M &= \max \left\{ 1, \sup_{n \in N_0} \frac{\sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))} + \frac{\sum_{n \in N} \alpha_n \sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))}}{1 - \sum_{i=1}^{\infty} \alpha_i} \right\} = \frac{5}{4}, \\ \delta &= \frac{\phi \left(\frac{\sum_{n=1}^{\infty} \beta_n}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \right) + \sum_{n=1}^{\infty} \beta_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \right) \right)}{1 - \phi \left(\frac{\sum_{n=1}^{\infty} \beta_n}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \right) + \sum_{n=1}^{\infty} \beta_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \right) \right)} \\ &= 1, \\ Q &= \phi \left(\frac{c}{M} \right) \frac{1}{1 + \delta} = 1498400 \times 2^{10009}; \\ W &= \phi \left(\frac{e_2 \left(1 + \sum_{s=0}^{k_2-1} \frac{1}{\phi^{-1}(p(s))} \right)}{\phi^{-1} \left(\sum_{s=k_1}^{k_2} \frac{1}{2^{s+1}} \right) \sum_{j=0}^{k_1-1} \phi^{-1} \left(\frac{1}{p(j)} \right)} \right) = \frac{10001 \times 540 \times 2^{10009}}{2^{10000} - 2^9}; \\ E &= \frac{1}{1 + \delta} \phi \left(\frac{e_1}{\sup_{n \in N_0} \frac{\frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \right) + \sum_{j=0}^{n-1} \phi^{-1} \left(\frac{1}{p(j)} \right)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}} \right) = 40. \end{aligned}$$

So $Q > W$ and $c > \frac{e_2}{\mu} > e_2 > e_1 > 0$. If

$$f_0(x) = \begin{cases} 20, & x \in [0, 100], \\ 20 + (x - 100) \frac{\frac{1498400 \times 2^{10009} + \frac{10001 \times 540 \times 2^{10009}}{2^{10000} - 2^9}}{2} - 20}{5400 - 100}, & x \in [100, 5400], \\ \frac{\frac{15300}{49} - \frac{5}{343}}{5404 - 1554} (x - 1554) + \frac{5}{343}, & x \in [5400, 5400 \times 10001], \\ \frac{1498400 \times 2^{10009} + \frac{10001 \times 540 \times 2^{10009}}{2^{10000} - 2^9}}{2}, & x \in [5400 \times 10001, +\infty), \end{cases}$$

and

$$f(n, x, y) = \frac{f_0 \left(\frac{x}{1+n} \right)}{2^{n+1}} + \frac{|y|}{343 \times 3688400}.$$

Then

$$f(n, (1+n)x, y) = \frac{f_0(x)}{2^{n+1}} + \frac{|y|}{343 \times 3688400}.$$

It is easy to check that if

- (A1) $f(n, (1+n)u, v) \leq \frac{Q}{2^{n+1}}$ for all $n \in N_0, u \in [0, c], v \in [0, c]$;
 (A2) $f(n, (1+n)u, v) \geq \frac{w}{2^{n+1}}$ for all $n \in [k_1, k_2], u \in [e_2, e_2\mu], v \in [0, c]$;
 (A3) $f(n, (1+n)u, v) \leq \frac{E}{2^{n+1}}$ for all $n \in N_0, u \in [0, e_1], v \in [0, c]$;
 then Theorem 1 implies that BVP(21) has at least three positive solutions such that

$$\sup_{n \in N_0} \frac{x_1(n)}{1+n} < 100, \quad \min_{n \in [10, 10000]} \frac{x_2(n)}{1+n} > 5400,$$

and

$$\sup_{n \in N_0} \frac{x_3(n)}{1+n} > 100, \quad \min_{n \in [10, 10000]} \frac{x_3(n)}{1+n} < 5400.$$

Remark 1. BVP(21) in Example 1 can not be solved by the theorems in [2–4, 16, 17].

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Authors' addresses

Jianye Xia

Department of Applied Mathematics, Guangdong University of Finance, Guangzhou 510000, P.R.China

E-mail address: liuyuji888@sohu.com

Yuji Liu

Department of Mathematics, Guangdong University of Business Studies Guangzhou 510000, P.R.China



CORRIGENDUM

TO

On Riemannian manifolds admitting W_2 -curvature tensor, Miskolc Math. Notes,
Vol. 12(2011), No. 2, pp. 255–264.

by

Füsun Özen Zengin

1. In Abstract: The last sentence was changed as "In a radiative perfect fluid, it is shown that the sectional curvature is constant"

2. In Theorem 5: The sentence of Theorem 5 must be changed as "For a $(W_2FS)_4$, the energy momentum-tensor satisfying the Einstein's equations with a cosmological constant is locally symmetric".

3. In the Proof of Theorem 5: (3.3) and (3.4) are true. (3.5)-(3.7) must be changed as the followings

$$\nabla_k T_{ij} = -\frac{1}{4k}(\nabla_k r)g_{ij} \quad (3.5)$$

Since r is constant in $(W_2FS)_4$,

$$\nabla_k r = 0 \quad (3.6)$$

then (3.5) reduces to

$$\nabla_k T_{ij} = 0$$

4. In Theorem 6: The sentence of theorem 6 must be changed as the following:
"For a $(W_2S)_4$ whose the Ricci tensor is quadratic Killing, a necessary and sufficient condition the energy-momentum tensor satisfying the Einstein's equations with a cosmological constant be a quadratic Killing tensor is that the scalar curvature of this space must be constant"

5. In the Proof of Theorem 6: The first sentence must be changed as "If T_{ij} and R_{ij} are quadratic Killing tensors, (3.5) reduces to"

6. After the Proof of Theorem 6, (before the equation (3.9)), the first sentence must be changed as

"Now, we consider that a radiative perfect fluid in $(W_2S)_4$."

7. In Theorem 7: In Theorem 7 and the proof of this theorem (in the first sentence of the proof) $(W_2FS)_4$

must be changed as $(W_2 S)_4$.

8. In Theorem 8:

The sentence of this theorem must be changed as

"For a radiative perfect fluid in $(W_2 F S)_4$, the sectional curvature is constant."

9. In the Proof of Theorem 8: It must be deleted (3.20), (3.21) and (3.22). The first sentence must be "For a radiative perfect fluid in $(W_2 F S)_4$, the scalar curvature of this space is constant." The numbers of equations (3.23)-(3.26) must be (3.21)-(3.23).

EDITOR-IN-CHIEF

Miklós Rontó, Institute of Mathematics,
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Moscow, Russia; rozov@rozov.mccme.ru

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Kiev, Ukraine; sam@imath.kiev.ua

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Hungary; stoyan@cs.elte.hu

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versity of Miskolc; H-3515 Miskolc–Egye-
temváros, Hungary;
matszj@gold.uni-miskolc.hu

Jiří Šremr, Institute of Mathematics, Acade-
my of Sciences of Czech Republic; Žižkova
22, 616 62 Brno, Czech Republic;
sremr@ipm.cz

Zsolt Tuza, A. Rényi Institute of Mathe-
matics, Hungarian Academy of Sciences;
P. O. Box 127, H-1053 Budapest, Hungary;
tuza@sztaki.hu

Miskolc Mathematical Notes

Abstracted in *Zentralblatt für Mathematik*, *Mathematical Reviews*, and *Реферативный журнал „Математика”*

Responsible for publication: Rector of the University of Miskolc
Published by the Miskolc University Press under the leadership of Burmeister Erzsébet
Responsible for duplication: Works manager Erzsébet Pásztor
Number of copies printed: 200
Put to the Press on June 12, 2012
Number of permission: TU 2012-238-ME

HU e-ISSN 1787-2413