

HU e-ISSN 1787-2413

Miskolc Mathematical Notes

A Publication of the University of Miskolc

VOLUME 14 (2013), NUMBER 3



MISKOLC UNIVERSITY PRESS



CONTENTS

<i>M. U. Akhmet and A. Kashkynbayev</i> Nonautonomous transcritical and pitchfork bifurcations in impulsive systems	737
<i>H. Ansari-Toroghy, F. Farshadifar, and Sh.Habibi</i> The cozero-divisor graph relative to finitely generated modules	749
<i>Abdelouaheb Ardjouni, Ahcene Djoudi</i> Existence and positivity of solutions for a totally nonlinear neutral periodic differential equation	757
<i>Muhammad Arshad, Abdullah Shoaib, Mujahid Abbas, and Akbar Azam</i> Fixed points of a pair of Kannan type mappings on a closed ball in ordered partial metric spaces	769
<i>Imre Ferenc Barna and László Mátyás</i> Analytic solutions for the one-dimensional compressible Euler equation with heat conduction and with different kind of equations of state . .	785
<i>Huseyin Bereketoglu and Gizem S. Oztepe</i> Convergence of the solution of an impulsive differential equation with piecewise constant arguments	801
<i>Bayram Çekim</i> New kinds of matrix polynomials	817
<i>Ivan Chajda, Miroslav Kolařík, and Helmut Länger</i> Algebras assigned to ternary relations	827
<i>T. Dehghan-Zadeh, Hongbo Hua, A. R. Ashrafi, and N. Habibi</i> Remarks on a conjecture about Randić index and graph radius	845
<i>Yan Dong</i> Gradient estimates in Morrey spaces of weak solutions to quasilinear parabolic systems of Hörmander's vector fields	851
<i>Zuzana Farkasová and Danica Jakubíková-Studenovská</i> On radical classes of monounary algebras	871



CONTENTS

S. Fegyverneki

A simple robust estimation for parameters of Cauchy distribution 887

Zrinka Franušić

Diophantine quadruples in the ring of integers of the pure cubic field $Q(\sqrt[3]{2})$ 893

Birol Gunduz and Sezgin Akbulut

Strong convergence of an explicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces 905

Birol Gunduz and Sezgin Akbulut

Strong and Δ -convergence theorems in hyperbolic spaces 915

Eszter K. Horváth

Islands: from coding theory to enumerative combinatorics and to lattice theory – overview and open problems 927

Shuliang Huang and Öznur Gölbaşı

On Lie ideals and generalized derivations of $\ast$ -prime rings 941

Nurettin Irmak

Balancing with balancing powers 951

Refik Keskin, Olcay Karaath, and Zafer Şiar

Positive integer solutions of the Diophantine equations $x^2 - 5F_nxy - 5(-1)^n y^2 = \pm 5^r$ 959

H. Khatibzadeh

Global behavior of solutions to a second order delay differential equation 973

Xhevat Z. Krasniqi

On absolute generalized Hausdorff summability of orthogonal series . . . 981

V. Lampret

An acceleration of convergence to some generalized-Euler-constant function 991



CONTENTS

Changwen Li, Xuemei Zhang, and Xiaolan Yi On τ -supplemented subgroups of finite groups	997
V. V. Marynets and K. V. Marynets On Goursat-Darboux boundary-value problem for systems of non-linear differential equations of hyperbolic type	1009
Ali Olgun On bivariate Meyer-König and Zeller operators	1021
Recep Sahin and Sebahattin İkikardes Squares of congruence subgroups of the Extended Modular Group	1031
Jiangtao Shi and Cui Zhang Finite groups in which some particular subgroups are TI-subgroups	1037
Mevlüt Tunç On some integral inequalities via h -convexity	1041
Burcu Nişancı Türkmen Modules that have a Rad-supplement in every cofinite extension	1059
Marjan Uddin and Sirajul Haq On the numerical solution of generalized nonlinear Schrodinger equation using RBFs	1067
Aynur Yalçın On generalizations of two curious divisibility properties	1085
B. Yılmaz, G. Başcanbaz-Tunca, and A. Aral Remark on weighted approximation properties of generalized Picard oper- ators	1091
Huilin Zhu, Gökhan Soydan, and Wei Qin A note on two Diophantine equations $x^2 \pm 2^a p^b = y^4$	1105



NONAUTONOMOUS TRANSCRITICAL AND PITCHFORK BIFURCATIONS IN IMPULSIVE SYSTEMS

M. U. AKHMET AND A. KASHKYNBAYEV

Received 3 September, 2013

Abstract. For the first time analogues of nonautonomous transcritical and pitchfork bifurcations are investigated for impulsive systems.

2010 Mathematics Subject Classification: 34A37; 34C23; 34D05; 37B55; 37G35; 34D45; 37C70; 37C75

Keywords: nonautonomous bifurcation theory, impulsive differential equations, attractive solution, repulsive solution, Pitchfork bifurcation, transcritical bifurcation

1. INTRODUCTION

Investigation of states of time dependent dynamical systems leads to nonautonomous problems in the form of the equations of perturbed motions. If the model depends on parameters, it is the main object of nonautonomous bifurcation theory to describe the qualitative changes when these parameters are varied. Extending nonautonomous bifurcation theory to impulsive systems is a contemporary problem. In continuous dynamical systems, this theory is studied in [10, 11, 13, 14, 16, 17].

Many evolutionary processes in the real world are characterized by sudden changes at certain times. These changes are called impulsive phenomena [4, 5, 12, 18, 19], which are widespread in modeling of mechanics, electronics, biology, neural networks, medicine, and social sciences [1, 2, 8, 15, 18–21]. An impulsive differential equation is one of the basic instruments to understand better the role of discontinuity in the real world problems. There are qualitative studies on asymptotic properties of the quasilinear impulsive differential equations and ordinary differential equations [3–6, 9, 12, 19]. The main novelty of this paper is the study of the analogues of nonautonomous pitchfork and transcritical bifurcations in scalar impulsive systems, in which the properties of the system depend on the entire time, both of past and future.

The second author was supported in part by the TUBITAK, the Scientific and Technological Research Council of Turkey, BIDEB 2215 Program.

This paper is organized as follows. In Section 2 we give basic definitions and definitions of all-time, past and future attractors. In Section 3 results on linearized attractivity and repulsivity are presented. Section 4 and Section 5 is devoted to impulsive analogues of nonautonomous transcritical and pitchfork bifurcations, respectively.

2. PRELIMINARIES

We denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{Z}$ the set of integers and write $\mathbb{R}_k^- := [-\infty, k)$ and $\mathbb{R}_k^+ := [k, \infty)$ for given $k \in \mathbb{R}$. In this section we introduce the concepts of attractive and repulsive solutions, which are used to analyze asymptotic behavior of impulsive non-autonomous systems. This paper is concerned with systems of the type

$$\begin{aligned} \dot{x} &= f(t, x), \\ \Delta x|_{t=\theta_i} &= J_i(x), \end{aligned} \quad (2.1)$$

where $\Delta x|_{t=\theta_i} := x(\theta_i+) - x(\theta_i)$, $x(\theta_i+) = \lim_{t \rightarrow \theta_i^+} x(t)$. The system (2.1) is defined on the set $\Omega = I \times \mathbb{A} \times G$ where $G \subseteq \mathbb{R}^n$, I is the interval of the form $I = \mathbb{R}$, $I = \mathbb{R}_k^-$ or $I = \mathbb{R}_k^+$, respectively. θ is a nonempty sequence with the set of indexes $\mathbb{A}$ such that $|\theta_i| \rightarrow \infty$ as $|i| \rightarrow \infty$. Let $\phi(t, t_0, x_0)$ be solution of (2.1) which is continuable and unique on I .

Denote by $PC(\mathbb{R}, \theta)$ the space of piecewise left continuous functions with discontinuity of the first kind at points in θ . The Euclidean space $\mathbb{R}^n$ is equipped with the norm $\|\cdot\|$, and write $B_\epsilon(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| < \epsilon\}$ for arbitrary ϵ -neighborhood of some point $x_0 \in \mathbb{R}^n$. We use Hausdorff semi-distance between nonempty set A and B as $d(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b)$. For arbitrary nonempty set $X \subset \mathbb{R}^n$ define $\phi(t, t_0, X) := \bigcup_{x_0 \in X} \phi(t, t_0, x_0)$.

A set $M \subset I \times \mathbb{R}^n$ is called nonautonomous set if for all $t \in I$, the t -fibers $M(t) := \{x \in \mathbb{R}^n : (t, x) \in M\}$ are nonempty. M is said to be compact if all t -fibers are compact. M is said to be invariant if $\phi(t, t_0, M(t_0)) = M(t)$ for all $t, t_0 \in I$.

Asymptotic properties of continuous dynamics and dynamics with discontinuous are the same. In what follows we use definitions of attractivity and repulsivity without any changes from [17].

Definition 1. Let $I = \mathbb{R}$, A and R be compact and invariant nonautonomous sets and $\psi : \mathbb{R} \rightarrow \mathbb{R}^n$ be a solution of the system (2.1).

- A is called all-time attractor if there exists an $\eta > 0$ with

$$\lim_{t \rightarrow \infty} \sup_{t_0 \in \mathbb{R}} d(\phi(t + t_0, t_0, B_\eta(A(t_0))), A(t + t_0)) = 0.$$

The supremum of all positive η with this property is called all-time attraction radius of A and denoted by $\mathcal{A}_A^\pm$.

- ψ is called all-time attractive if graph ψ is an all-time attractor.

- R is called all-time repeller if there exists an $\eta > 0$ with

$$\lim_{t \rightarrow \infty} \sup_{t_0 \in \mathbb{R}} d(\phi(t_0 - t, t_0, B_\eta(R(t_0))), R(t_0 - t)) = 0.$$

The supremum of all positive η with this property is called all-time repulsion radius of R and denoted by $\mathcal{R}_R^\pm$.

- ψ is called all-time repulsive if graph ψ is an all-time repeller.

In what follows we use past attractivity and repulsivity in the asymptotic analysis of the systems that depend on the past. Note that a past attractor is a local form of a pullback attractor [7].

Definition 2. Let $I = \mathbb{R}_k^-$, A and R be compact and invariant nonautonomous sets and $\psi : \mathbb{R} \rightarrow \mathbb{R}^n$ be a solution of the system (2.1).

- A is called past attractor if there exists an $\eta > 0$ with

$$\lim_{t \rightarrow \infty} d(\phi(t_0, t_0 - t, B_\eta(A(t_0 - t))), A(t_0)) = 0 \text{ for all } t_0 \in I.$$

The supremum of all positive η with this property is called past attraction radius of A and is denoted by $\mathcal{A}_A^-$.

- ψ is called past attractive if the graph ψ is a past attractor.
- R is called past repeller if there exists an $\eta > 0$ with

$$\lim_{t \rightarrow \infty} d(\phi(t_0 - t, t_0, B_\eta(R(t_0))), R(t_0 - t)) = 0 \text{ for all } t_0 \in I.$$

The supremum of all positive η such that there exists a $\widehat{k} \in I$ with

$$\lim_{t \rightarrow \infty} d(\phi(t_0 - t, t_0, B_\eta(R(t_0))), R(t_0 - t)) = 0 \text{ for all } t_0 \leq \widehat{k}$$

is called past repulsion radius of R and denoted by $\mathcal{R}_R^-$.

- ψ is called past repulsive if graph ψ is an past repeller.

Definition 3. Let $I = \mathbb{R}_k^+$, A and R be compact and invariant nonautonomous sets and $\psi : \mathbb{R} \rightarrow \mathbb{R}^n$ be a solution of the system (2.1).

- A is called future attractor if there exists an $\eta > 0$ with

$$\lim_{t \rightarrow \infty} d(\phi(t + t_0, t_0, B_\eta(A(t_0))), A(t + t_0)) = 0 \text{ for all } t_0 \in I.$$

The supremum of all positive η such that there exists a $\widehat{k} \in I$ with

$$\lim_{t \rightarrow \infty} d(\phi(t + t_0, t_0, B_\eta(A(t_0))), A(t + t_0)) = 0 \text{ for all } t_0 \geq \widehat{k}$$

is called future attraction radius of A and is denoted by $\mathcal{A}_A^+$.

- ψ is called future attractive if graph ψ is a future attractor.

- R is called future repeller if there exists an $\eta > 0$ with

$$\lim_{t \rightarrow \infty} d(\phi(t_0, t_0 + t, B_\eta(R(t + t_0))), R(t_0)) = 0 \text{ for all } t_0 \in I.$$

The supremum of all positive η with this property is called future repulsion radius of R and is denoted by $\mathcal{R}_R^+$.

- ψ is called future repulsive if graph ψ is a future repeller.

From the definitions given above it follows that every future attractive solution is uniformly asymptotically stable. Moreover, every all-time attractor (repeller, respectively) is both a past attractor (repeller, respectively) and a future attractor (repeller, respectively).

3. LINEARIZED ATTRACTIVITY AND REPULSIVITY

In this section we study linearized systems with definitions provided in the previous section which play great role in the stability analysis of solutions of nonlinear impulsive systems with fixed moments of impulses. We consider the system with interval I of the form $\mathbb{R}$, $\mathbb{R}_k^-$ or $\mathbb{R}_k^+$, respectively, and let

$$\begin{aligned} \dot{x} &= A(t)x + F(t, x), \\ \Delta x|_{t=\theta_i} &= B_i x + I_i(x), \end{aligned} \quad (3.1)$$

where $A \in PC(I, \theta)$, matrices B_i satisfy $\det(B_i + I) \neq 0$, $F : I \times G \rightarrow \mathbb{R}^n$ and $I : \mathbb{A} \times G \rightarrow \mathbb{R}^n$. An infinite sequence θ_i satisfies $|\theta_i| \rightarrow \infty$ as $|i| \rightarrow \infty$. It is assumed that there exist positive constants $\underline{\theta}$ and $\bar{\theta}$ such that $\underline{\theta} \leq \theta_{i+1} - \theta_i \leq \bar{\theta}$. Denote $\phi(t, t_0, x_0)$ the solution of (3.1) and let $\Phi(t, s)$ be the fundamental matrix of the linearized system

$$\begin{aligned} \dot{x} &= A(t)x, \\ \Delta x|_{t=\theta_i} &= B_i x. \end{aligned} \quad (3.2)$$

A version of the next theorem for future time has been proved in [19]. We will extend this result for all-time attractiveness.

Theorem 1. *If there exist $\beta < 0$, $K \geq 1$ and $\delta > 0$ such that*

$$\|\Phi(t, s)\| \leq K e^{\beta(t-s)} \text{ for all } t \geq s,$$

and the functions $F(t, x)$ and $I_i(x)$ satisfy the inequalities

$$\|F(t, x)\| \leq l\|x\|, \quad \|I_i(x)\| \leq l\|x\| \quad (3.3)$$

for all $t \in I, i \in \mathbb{A}$ and $\|x\| < h, h > 0$. Then,

$$\|\phi(t, t_0, x_0)\| \leq \delta e^{(\beta + Kl + \frac{1}{\underline{\theta}} \ln(1 + Kl))(t - t_0)} \text{ for all } t, t_0 \in I \text{ with } t \geq t_0,$$

i.e., for sufficiently small values of l , the trivial solution of (3.1) is all-time (past, future, respectively) attractive.

Proof. An equivalent integral equation of the system (3.1) can be written as [4, 19]:

$$\begin{aligned} \phi(t, t_0, x_0) = \Phi(t, t_0)x_0 + \int_{t_0}^t \Phi(t, s)F(s, \phi(s, t_0, x_0))ds \\ + \sum_{t_0 \leq \theta_i < t} \Phi(t, \theta_i)I_i(\phi(\theta_i, t_0, x_0)) \end{aligned}$$

for all $t \geq t_0$. By using inequalities in (3.3) we get

$$\begin{aligned} \|\phi(t, t_0, x_0)\| \leq Ke^{\beta(t-t_0)}\|x_0\| + \int_{t_0}^t Ke^{\beta(t-s)}l\|\phi(s, t_0, x_0)\|ds \\ + \sum_{t_0 \leq \theta_i < t} Ke^{\beta(t-\theta_i)}l\|\phi(\theta_i, t_0, x_0)\| \end{aligned}$$

for all $t \geq t_0$ is fulfilled. The last expression can be rewritten as

$$\begin{aligned} e^{-\beta t}\|\phi(t, t_0, x_0)\| \leq Ke^{-\beta t_0}\|x_0\| + \int_{t_0}^t Kle^{-\beta s}\|\phi(s, t_0, x_0)\|ds \\ + \sum_{t_0 \leq \theta_i < t} Kle^{-\beta \theta_i}\|\phi(\theta_i, t_0, x_0)\| \end{aligned}$$

for all $t \geq t_0$. Hence, by Gronwall-Bellman lemma for piecewise continuous functions ([4, 19]) it follows that

$$\|\phi(t, t_0, x_0)\| \leq Ke^{(\beta+Kl)(t-t_0)}(1+Kl)^{i[t_0, t]}\|x_0\| \text{ for all } t \geq t_0. \quad (3.4)$$

By means of the inequality $\theta_{i+1} - \theta_i \geq \underline{\theta}$ one can see that

$$\|\phi(t, t_0, x_0)\| \leq Ke^{(\beta+Kl+\frac{1}{\underline{\theta}}\ln(1+Kl))(t-t_0)}\|x_0\| \text{ for all } t \geq t_0.$$

If l is small enough that

$$\beta + Kl + \frac{1}{\underline{\theta}}\ln(1+Kl) < 0 \text{ for } \beta < 0,$$

then the required result follows by choosing $\delta = Kh$. $\square$

Theorem 2. If there exist $\beta > 0$, $K \geq 1$ and $\delta > 0$ such that

$$\|\Phi(t, s)\| \leq Ke^{\beta(t-s)} \text{ for all } t \leq s,$$

and the functions $F(t, x)$ and $I_i(x)$ satisfy the inequalities

$$\|F(t, x)\| \leq l\|x\|, \quad \|I_i(x)\| \leq l\|x\| \quad (3.5)$$

for all $t \in I, i \in \mathbb{A}$ and $\|x\| < h, h > 0$. Then,

$$\|\phi(t, t_0, x_0)\| \leq \delta e^{(\beta-Kl+\frac{1}{\underline{\theta}}\ln(1-Kl))(t-t_0)} \text{ for all } t, t_0 \in I \text{ with } t \leq t_0,$$

i.e., for sufficiently small values of l , the trivial solution of (3.1) is all-time (past, future, respectively) repulsive.

Proof. An equivalent integral equation of the system (3.1) can be written as [4, 19]:

$$\begin{aligned} \phi(t, t_0, x_0) = & \Phi(t, t_0)x_0 + \int_{t_0}^t \Phi(t, s)F(s, \phi(s, t_0, x_0))ds \\ & - \sum_{t \leq \theta_i < t_0} \Phi(t, \theta_i)I_i(\phi(\theta_i, t_0, x_0)) \end{aligned}$$

for all $t \leq t_0$. By using inequalities in (3.3) we get

$$\begin{aligned} \|\phi(t, t_0, x_0)\| \leq & Ke^{\beta(t-t_0)}\|x_0\| + \int_{t_0}^t Ke^{\beta(t-s)}l\|\phi(s, t_0, x_0)\|ds \\ & + \sum_{t \leq \theta_i < t_0} Ke^{\beta(t-\theta_i)}l\|\phi(\theta_i, t_0, x_0)\| \end{aligned}$$

for all $t \leq t_0$ is fulfilled. The last expression can be rewritten as

$$\begin{aligned} e^{-\beta t}\|\phi(t, t_0, x_0)\| \leq & Ke^{-\beta t_0}\|x_0\| + \int_{t_0}^t Kle^{-\beta s}\|\phi(s, t_0, x_0)\|ds \\ & + \sum_{t \leq \theta_i < t_0} Kle^{-\beta \theta_i}\|\phi(\theta_i, t_0, x_0)\| \end{aligned}$$

for all $t \leq t_0$. Gronwall-Bellman lemma for piecewise continuous functions ([4, 19]) can be applied since l can be chosen that $Kl < 1$. Thus,

$$\|\phi(t, t_0, x_0)\| \leq Ke^{(\beta-Kl)(t-t_0)}(1-Kl)^{-i[t, t_0]}\|x_0\| \quad \text{for all } t \leq t_0. \quad (3.6)$$

By means of the inequality $\theta_{i+1} - \theta_i \geq \underline{\theta}$ one can see that

$$\|\phi(t, t_0, x_0)\| \leq Ke^{(\beta-Kl+\frac{1}{\underline{\theta}}\ln(1-Kl))(t-t_0)}\|x_0\| \quad \text{for all } t \leq t_0.$$

If l is small enough that

$$\beta - Kl + \frac{1}{\underline{\theta}}\ln(1-Kl) > 0 \quad \text{for } \beta > 0,$$

then the required results follow if we choose $\delta = Kh$ since $\|x_0\| < h$. $\square$

4. THE TRANSCRITICAL BIFURCATION

In this section we study impulsive analogue of the nonautonomous transcritical bifurcation. Let $x_- < 0 < x_+$ and $\alpha_- < \alpha_+$ be real numbers and I be interval of the

form $\mathbb{R}$, $\mathbb{R}_k^-$ or $\mathbb{R}_k^+$, respectively. Consider the system

$$\begin{aligned}\dot{x} &= a(t, \alpha)x + b(t, \alpha)x^2 + r(t, x, \alpha), \\ \Delta x|_{t=\theta_i} &= c_i(\alpha)x + d_i(\alpha)x^2 + e_i(x, \alpha),\end{aligned}\quad (4.1)$$

with piecewise continuous functions $a : I \times (\alpha_-, \alpha_+) \rightarrow \mathbb{R}$, $b : I \times (\alpha_-, \alpha_+) \rightarrow \mathbb{R}$ and $r : I \times (x_-, x_+) \times (\alpha_-, \alpha_+) \rightarrow \mathbb{R}$ satisfying $r(t, 0, \alpha) = 0$. $c : \mathbb{A} \times (\alpha_-, \alpha_+) \rightarrow \mathbb{R}$, $d : \mathbb{A} \times (\alpha_-, \alpha_+) \rightarrow \mathbb{R}$ and $e : \mathbb{A} \times (x_-, x_+) \times (\alpha_-, \alpha_+) \rightarrow \mathbb{R}$ with $c_i(\alpha) \neq -1$ and $e_i(0, \alpha) = 0$. An infinite sequence θ_i satisfies $|\theta_i| \rightarrow \infty$ as $|i| \rightarrow \infty$. It is assumed that there exist positive constants $\underline{\theta}$ and $\bar{\theta}$ such that $\underline{\theta} \leq \theta_{i+1} - \theta_i \leq \bar{\theta}$. Let $\Phi_\alpha(t, s)$ be the fundamental matrix of the linear system

$$\begin{aligned}\dot{x} &= a(t, \alpha)x, \\ \Delta x|_{t=\theta_i} &= c_i(\alpha)x.\end{aligned}\quad (4.2)$$

Assume that there exists $\alpha_0 \in (\alpha_-, \alpha_+)$ such that are two functions $\beta_1, \beta_2 : (\alpha_-, \alpha_+) \rightarrow \mathbb{R}$ which are either both monotone increasing or both monotone decreasing and $K \geq 1$ satisfying $\lim_{\alpha \rightarrow \alpha_0} \beta_1(\alpha) = \lim_{\alpha \rightarrow \alpha_0} \beta_2(\alpha) = 0$ and

$$\Phi_\alpha(t, s) \leq K e^{\beta_1(\alpha)(t-s)} \quad \text{for all } \alpha \in (\alpha_-, \alpha_+) \text{ and } t, s \in I \text{ with } t \geq s,$$

$$\Phi_\alpha(t, s) \leq K e^{\beta_2(\alpha)(t-s)} \quad \text{for all } \alpha \in (\alpha_-, \alpha_+) \text{ and } t, s \in I \text{ with } t \leq s.$$

The quadratic terms either fulfill

$$\begin{aligned}0 &< \liminf_{\alpha \rightarrow \alpha_0} \inf_{t \in I} b(t, \alpha) \leq \limsup_{\alpha \rightarrow \alpha_0} \sup_{t \in I} b(t, \alpha) < \infty, \\ 0 &< \liminf_{\alpha \rightarrow \alpha_0} \inf_{i \in \mathbb{A}} d_i(\alpha) \leq \limsup_{\alpha \rightarrow \alpha_0} \sup_{i \in \mathbb{A}} d_i(\alpha) < \infty\end{aligned}\quad (4.3)$$

or

$$\begin{aligned}-\infty &< \liminf_{\alpha \rightarrow \alpha_0} \inf_{t \in I} b(t, \alpha) \leq \limsup_{\alpha \rightarrow \alpha_0} \sup_{t \in I} b(t, \alpha) < 0, \\ -\infty &< \liminf_{\alpha \rightarrow \alpha_0} \inf_{i \in \mathbb{A}} d_i(\alpha) \leq \limsup_{\alpha \rightarrow \alpha_0} \sup_{i \in \mathbb{A}} d_i(\alpha) < 0,\end{aligned}\quad (4.4)$$

and the remainders satisfy

$$\begin{aligned}\lim_{x \rightarrow 0} \sup_{\alpha \in (\alpha_0 - |x|, \alpha_0 + |x|)} \sup_{t \in I} \frac{|r(t, x, \alpha)|}{|x|^2} &= 0, \\ \lim_{x \rightarrow 0} \sup_{\alpha \in (\alpha_0 - |x|, \alpha_0 + |x|)} \sup_{i \in \mathbb{A}} \frac{|e_i(x, \alpha)|}{|x|^2} &= 0\end{aligned}\quad (4.5)$$

and there exists sufficiently small $l > 0$ such that

$$|r(t, x, \alpha)| < l|x|, \quad |e_i(x, \alpha)| < l|x|, \quad (4.6)$$

for all $\alpha \in (\alpha_-, \alpha_+)$, $t \in I$, $i \in \mathbb{A}$ and $x \in (x_-, x_+)$.

Theorem 3. Assume that above conditions hold for the system (4.1). Then there exist $\widehat{\alpha}_- < 0 < \widehat{\alpha}_+$ such that

- If the functions β_1 and β_2 are monotone increasing, the trivial solution is all-time (past, future, respectively) attractive for $\alpha \in (\widehat{\alpha}_-, \alpha_0)$ and all-time repulsive for $\alpha \in (\alpha_0, \widehat{\alpha}_+)$. The system (4.1) admits an all-time (past, future, respectively) bifurcation, since the corresponding radii of all-time (past, future, respectively) attraction and repulsion satisfy

$$\lim_{\alpha \nearrow \alpha_0} \mathcal{A}_0^\alpha = 0 \text{ and } \lim_{\alpha \searrow \alpha_0} \mathcal{R}_0^\alpha = 0. \quad (4.7)$$

- If the functions β_1 and β_2 are monotone decreasing, the trivial solution is all-time (past, future, respectively) repulsive for $\alpha \in (\widehat{\alpha}_-, \alpha_0)$ and all-time attractive for $\alpha \in (\alpha_0, \widehat{\alpha}_+)$. The system (4.1) admits an all-time (past, future, respectively) bifurcation, since the corresponding radii of all-time (past, future, respectively) attraction and repulsion satisfy

$$\lim_{\alpha \nearrow \alpha_0} \mathcal{R}_0^\alpha = 0 \text{ and } \lim_{\alpha \searrow \alpha_0} \mathcal{A}_0^\alpha = 0. \quad (4.8)$$

Proof. We prove the first part since the second part can be proven in a similar manner. Let ϕ_α be the general solution of the system (4.1). We may consider the case (4.3) since the functions β_1 and β_2 are monotone increasing. Choose $\widehat{\alpha}_- < \alpha_0 < \widehat{\alpha}_+$ such that

$$\begin{aligned} 0 < \inf_{\alpha \in (\widehat{\alpha}_-, \widehat{\alpha}_+), t \in I} b(t, \alpha) &\leq \sup_{\alpha \in (\widehat{\alpha}_-, \widehat{\alpha}_+), t \in I} b(t, \alpha) < \infty, \\ 0 < \inf_{\alpha \in (\widehat{\alpha}_-, \widehat{\alpha}_+), i \in \mathbb{A}} d_i(\alpha) &\leq \sup_{\alpha \in (\widehat{\alpha}_-, \widehat{\alpha}_+), i \in \mathbb{A}} d_i(\alpha) < \infty. \end{aligned} \quad (4.9)$$

By means of (4.6) and (4.9) one can see that Theorem 1 and Theorem 2 can be applied. Thus, we get attractivity and repulsivity of the trivial solution as it was required to show in the theorem. To prove relations (4.7) and (4.8) we assume to the contrary that

$$\gamma := \limsup_{\alpha \nearrow \alpha_0} \mathcal{A}_0^\alpha > 0.$$

By means of (4.5) and (4.9) one can show that there exist $\widetilde{\alpha} \in (\widehat{\alpha}_-, \alpha_0)$, $x_0 \in (0, \gamma)$ and $L \in (0, \frac{x_0}{4K})$ such that

$$b(t, \alpha)x^2 + r(t, x, \alpha) > L \text{ and } d_i(\alpha)x^2 + e_i(x, \alpha) > L \quad (4.10)$$

for all $t \in I$, $i \in \mathbb{A}$, $\alpha \in (\widetilde{\alpha}, \alpha_0)$ and $x_0 \in [\frac{x_0}{2K^2}, x_0]$. Next, fix $\widehat{\alpha} \in (\widetilde{\alpha}, \alpha_0)$ such that $\mathcal{A}_0^{\widehat{\alpha}} > x_0$ and $\beta_2(\widehat{\alpha}) \geq \beta := -\frac{2KL}{x_0} > -\frac{1}{2}$ so that $\phi_{\widehat{\alpha}}(t, t_0, x_0)$ is attracted to the origin. Denote $\psi(t) = \phi_{\widehat{\alpha}}(t, t_0, x_0)$. Then, there exists $t_2 \in I$, $t_2 > t_0$, such that $\psi(t_2) \leq \frac{x_0}{2K^2}$.

Choose minimal t_2 with this property, i.e., $\psi(t_2) > \frac{x_0}{2K^2}$ for all $t \in [t_0, t_2)$. Moreover, choose $t_1 \in [t_0, t_2)$ such that

$$\psi(t_1) = \frac{x_0}{2K} \text{ and } \psi(t) \in \left(\frac{x_0}{2K^2}, x_0 \right] \text{ for all } t \in [t_1, t_2).$$

We write integral equation of the system (4.1) at t_2 for fixed $\hat{\alpha}$ which start at point t_1 .

$$\begin{aligned} \psi(t_2) &= \Phi_{\hat{\alpha}}(t_2, t_1)\psi(t_1) + \int_{t_1}^{t_2} \Phi_{\hat{\alpha}}(t_2, s) (b(s, \hat{\alpha})(\psi(s))^2 + r(s, \psi(s), \hat{\alpha})) ds \\ &\quad + \sum_{t_1 \leq \theta_i < t_2} \Phi_{\hat{\alpha}}(t_2, \theta_i) (d_i(\hat{\alpha})(\psi(\theta_i))^2 + e_i(\psi(\theta_i), \hat{\alpha})) \\ &> \frac{x_0}{2K^2} e^{\beta(t_2-t_1)} + \frac{L}{K} \int_{t_1}^{t_2} e^{\beta(t_2-s)} ds \\ &= e^{\beta(t_2-t_1)} \left(\frac{x_0}{2K^2} + \frac{L}{K\beta} \right) - \frac{L}{K\beta} = \frac{x_0}{2K^2} \end{aligned}$$

which is a contradiction and proves that $\lim_{\alpha \nearrow \alpha_0} \mathcal{A}_0^\alpha = 0$. Analogously, one can show that $\lim_{\alpha \searrow \alpha_0} \mathcal{R}_0^\alpha = 0$. $\square$

5. THE PITCHFORK BIFURCATION

In this section we study impulsive analogue of the nonautonomous pitchfork bifurcation. Let $x_- < 0 < x_+$ and $\alpha_- < \alpha_+$ be real numbers and I be interval of the form $\mathbb{R}$, $\mathbb{R}_k^-$ or $\mathbb{R}_k^+$, respectively. Consider the system

$$\begin{aligned} \dot{x} &= a(t, \alpha)x + b(t, \alpha)x^3 + r(t, x, \alpha), \\ \Delta x|_{t=\theta_i} &= c_i(\alpha)x + d_i(\alpha)x^3 + e_i(x, \alpha), \end{aligned} \quad (5.1)$$

with piecewise continuous functions $a : I \times (\alpha_-, \alpha_+) \rightarrow \mathbb{R}$, $b : I \times (\alpha_-, \alpha_+) \rightarrow \mathbb{R}$ and $r : I \times (x_-, x_+) \times (\alpha_-, \alpha_+) \rightarrow \mathbb{R}$ satisfying $r(t, 0, \alpha) = 0$. $c : \mathbb{A} \times (\alpha_-, \alpha_+) \rightarrow \mathbb{R}$, $d : \mathbb{A} \times (\alpha_-, \alpha_+) \rightarrow \mathbb{R}$ and $e : \mathbb{A} \times (x_-, x_+) \times (\alpha_-, \alpha_+) \rightarrow \mathbb{R}$ with $c_i(\alpha) \neq -1$ and $e_i(0, \alpha) = 0$. An infinite sequence θ_i satisfies $|\theta_i| \rightarrow \infty$ as $|i| \rightarrow \infty$. It is assumed that there exist positive constants $\underline{\theta}$ and $\bar{\theta}$ such that $\underline{\theta} \leq \theta_{i+1} - \theta_i \leq \bar{\theta}$. Let $\Phi_\alpha(t, s)$ be the fundamental matrix of the linear system

$$\begin{aligned} \dot{x} &= a(t, \alpha)x, \\ \Delta x|_{t=\theta_i} &= c_i(\alpha)x. \end{aligned} \quad (5.2)$$

Assume that there exists $\alpha_0 \in (\alpha_-, \alpha_+)$ such that are two functions $\beta_1, \beta_2 : (\alpha_-, \alpha_+) \rightarrow \mathbb{R}$ which are either both monotone increasing or both monotone decreasing and $K \geq 1$ satisfying $\lim_{\alpha \rightarrow \alpha_0} \beta_1(\alpha) = \lim_{\alpha \rightarrow \alpha_0} \beta_2(\alpha) = 0$ and

$$\begin{aligned} \Phi_\alpha(t, s) &\leq K e^{\beta_1(\alpha)(t-s)} \text{ for all } \alpha \in (\alpha_-, \alpha_+) \text{ and } t, s \in I \text{ with } t \geq s, \\ \Phi_\alpha(t, s) &\leq K e^{\beta_2(\alpha)(t-s)} \text{ for all } \alpha \in (\alpha_-, \alpha_+) \text{ and } t, s \in I \text{ with } t \leq s. \end{aligned}$$

The quadratic terms either fulfills

$$\begin{aligned} 0 < \liminf_{\alpha \rightarrow \alpha_0} \inf_{t \in I} b(t, \alpha) &\leq \limsup_{\alpha \rightarrow \alpha_0} \sup_{t \in I} b(t, \alpha) < \infty, \\ 0 < \liminf_{\alpha \rightarrow \alpha_0} \inf_{i \in \mathbb{A}} d_i(\alpha) &\leq \limsup_{\alpha \rightarrow \alpha_0} \sup_{i \in \mathbb{A}} d_i(\alpha) < \infty \end{aligned} \quad (5.3)$$

or

$$\begin{aligned} -\infty < \liminf_{\alpha \rightarrow \alpha_0} \inf_{t \in I} b(t, \alpha) &\leq \limsup_{\alpha \rightarrow \alpha_0} \sup_{t \in I} b(t, \alpha) < 0, \\ -\infty < \liminf_{\alpha \rightarrow \alpha_0} \inf_{i \in \mathbb{A}} d_i(\alpha) &\leq \limsup_{\alpha \rightarrow \alpha_0} \sup_{i \in \mathbb{A}} d_i(\alpha) < 0, \end{aligned} \quad (5.4)$$

and the remainders satisfy

$$\begin{aligned} \lim_{x \rightarrow 0} \sup_{\alpha \in (\alpha_0 - x^2, \alpha_0 + x^2)} \sup_{t \in I} \frac{|r(t, x, \alpha)|}{|x|^3} &= 0, \\ \lim_{x \rightarrow 0} \sup_{\alpha \in (\alpha_0 - x^2, \alpha_0 + x^2)} \sup_{i \in \mathbb{A}} \frac{|e_i(x, \alpha)|}{|x|^3} &= 0 \end{aligned} \quad (5.5)$$

and there exists sufficiently small $l > 0$ such that

$$|r(t, x, \alpha)| < l|x|, \quad |e_i(x, \alpha)| < l|x|, \quad (5.6)$$

for all $\alpha \in (\alpha_-, \alpha_+)$, $t \in I$, $i \in \mathbb{A}$ and $x \in (x_-, x_+)$.

Theorem 4. Assume that above conditions hold for the system (5.1). Then there exist $\hat{\alpha}_- < 0 < \hat{\alpha}_+$ such that

- If the functions β_1 and β_2 are monotone increasing, the trivial solution is all-time (past, future, respectively) attractive for $\alpha \in (\hat{\alpha}_-, \alpha_0)$ and all-time repulsive for $\alpha \in (\alpha_0, \hat{\alpha}_+)$. The system (4.1) admits an all-time (past, future, respectively) bifurcation, since the corresponding radii of all-time (past, future, respectively) attraction and repulsion satisfy

$$\lim_{\alpha \nearrow \alpha_0} \mathcal{A}_0^\alpha = 0 \quad \text{and} \quad \lim_{\alpha \searrow \alpha_0} \mathcal{R}_0^\alpha = 0.$$

- If the functions β_1 and β_2 are monotone decreasing, the trivial solution is all-time (past, future, respectively) repulsive for $\alpha \in (\hat{\alpha}_-, \alpha_0)$ and all-time attractive for $\alpha \in (\alpha_0, \hat{\alpha}_+)$. The system (5.1) admits an all-time (past, future, respectively) bifurcation, since the corresponding radii of all-time (past, future, respectively) attraction and repulsion satisfy

$$\lim_{\alpha \nearrow \alpha_0} \mathcal{R}_0^\alpha = 0 \quad \text{and} \quad \lim_{\alpha \searrow \alpha_0} \mathcal{A}_0^\alpha = 0.$$

The proof of the theorem is similar to that of Theorem 3.

REFERENCES

- [1] M. Akhmet, *Nonlinear hybrid continuous/discrete-time models*. Atlantis Press, 2011.
- [2] M. U. Akhmet, "Perturbations and Hopf bifurcation of the planar discontinuous dynamical system," *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 60, no. 1, pp. 163–178, 2005.
- [3] M. U. Akhmet and M. A. Tleubergenova, "Asymptotic equivalence of the quasilinear impulsive differential equation and the linear ordinary differential equation," *Miskolc Math. Notes*, vol. 8, no. 2, pp. 117–121, 2007.
- [4] M. Akhmet, *Principles of discontinuous dynamical systems*. Berlin: Springer, 2010.
- [5] D. D. Bainov and P. S. Simeonov, *Impulsive differential equations. Asymptotic properties of the solutions*, ser. Series on Advances in Mathematics for Applied Sciences. Singapore: World Scientific, 1995, vol. 28.
- [6] M. A. Bilozerova and V. M. Evtukhov, "Asymptotic representations of solutions of the differential equation $y^{(n)} = \alpha_0 p(t) \prod_{i=0}^{n-1} \varphi_i(y^{(i)})$," *Math. Notes, Miskolc*, vol. 13, no. 2, pp. 249–270, 2012.
- [7] D. N. Cheban, P. E. Kloeden, and B. Schmalfuß, "Pullback attractors in dissipative nonautonomous differential equations under discretization," *J. Dyn. Differ. Equations*, vol. 13, no. 1, pp. 185–213, 2001.
- [8] J. Guckenheimer and R. F. Williams, "Structural stability of Lorenz attractors," *Publ. Math., Inst. Hautes Étud. Sci.*, vol. 50, pp. 59–72, 1979.
- [9] T. Hajba and M. Kovács, "Solution of the convex programming problem via second order differential equation system," *Math. Notes, Miskolc*, vol. 13, no. 1, pp. 23–37, 2012.
- [10] P. E. Kloeden, "Pitchfork and transcritical bifurcations in systems with homogeneous nonlinearities and an almost periodic time coefficient," *Commun. Pure Appl. Anal.*, vol. 3, no. 2, pp. 161–173, 2004.
- [11] P. E. Kloeden and M. Rasmussen, *Nonautonomous dynamical systems*, ser. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society (AMS), 2011, vol. 176.
- [12] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of impulsive differential equations*, ser. Series in Modern Applied Mathematics. Singapore: World Scientific, 1989, vol. 6.
- [13] J. A. Langa, J. C. Robinson, and A. Suárez, "Bifurcations in non-autonomous scalar equations," *J. Differ. Equations*, vol. 221, no. 1, pp. 1–35, 2006.
- [14] J. A. Langa, J. C. Robinson, and A. Suárez, "Stability, instability, and bifurcation phenomena in non-autonomous differential equations," *Nonlinearity*, vol. 15, no. 3, pp. 887–903, 2002.
- [15] P. Li, J. Cao, and Z. Wang, "Robust impulsive synchronization of coupled delayed neural networks with uncertainties," *Physica A*, vol. 373, pp. 261–272, 2007.
- [16] M. Rasmussen, *Attractivity and bifurcation for nonautonomous dynamical systems*, ser. Lecture Notes in Mathematics. Berlin: Springer, 2007, vol. 1907.
- [17] M. Rasmussen, "Nonautonomous bifurcation patterns for one-dimensional differential equations," *J. Differ. Equations*, vol. 234, no. 1, pp. 267–288, 2007.
- [18] M. Ronto and A. M. Samoilenko, *Numerical-analytic methods in the theory of boundary-value problems*. Singapore: World Scientific, 2000.
- [19] A. M. Samoilenko and N. A. Perestyuk, *Impulsive differential equations. Transl. from the Russian by Yury Chapovsky*, ser. World Scientific Series on Nonlinear Science. Series A. Singapore: World Scientific, 1995, vol. 14.
- [20] G. Zhang, Z. Liu, and Z. Ma, "Synchronization of complex dynamical networks via impulsive control," *Chaos*, vol. 17, no. 4, pp. 043 126, 9, 2007.
- [21] J. Zhou, L. Xiang, and Z. Liu, "Synchronization in complex delayed dynamical networks with impulsive effects," *Physica A*, vol. 384, pp. 684–692, 2007.

*Authors' addresses***M. U. Akhmet**

Middle East Technical University, Department of Mathematics, 06531 Ankara, Turkey

E-mail address: marat@metu.edu.tr

A. Kashkynbayev

Middle East Technical University, Department of Mathematics, 06531 Ankara, Turkey

E-mail address: ardaky@gmail.com



THE COZERO-DIVISOR GRAPH RELATIVE TO FINITELY GENERATED MODULES

H. ANSARI-TOROGHY, F. FARSHADIFAR, AND SH.HABIBI

Abstract. Let R be a commutative ring and let M be a finitely generated R -module. Let's denote the cozero-divisor graph of R by $\dot{\Gamma}(R)$. In this paper, we introduce a certain subgraph $\dot{\Gamma}_R(M)$ of $\dot{\Gamma}(R)$, called cozero-divisor graph relative to M , and obtain some related results.

2010 *Mathematics Subject Classification:* 05C75; 13A99; 05C99

Keywords: cozero-divisor, complete graph, finitely generated

1. INTRODUCTION

Throughout this paper, R will denote a commutative ring with identity. We denote the set of maximal ideals of R by $\text{Max}(R)$.

A *graph* G is defined as the pair $(V(G), E(G))$, where $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G . For two distinct vertices a and b of $V(G)$, the notation $a - b$ means that a and b are adjacent. A graph G is said to be *complete* if $a - b$ for all distinct $a, b \in V(G)$, and G is said to be *empty* if $E(G) = \emptyset$. Note that by this definition a graph may be empty even if $V(G) \neq \emptyset$. If $|V(G)| \geq 2$, a *path* from a to b is a series of adjacent vertices $a - v_1 - v_2 - \dots - v_n - b$. The *length of a path* is the number of edges it contains. A *cycle* is a path that begins and ends at the same vertex in which no edge is repeated, and all vertices other than the starting and ending vertex are distinct. If a graph G has a cycle, the *girth* of G (notated $g(G)$) is defined as the length of the shortest cycle of G ; otherwise, $g(G) = \infty$. A graph G is *connected* if for every pair of distinct vertices $a, b \in V(G)$, there exists a path from a to b . If there is a path from a to b with $a, b \in V(G)$, then the *distance from a to b* is the length of the shortest path from a to b and is denoted $d(a, b)$. If there is not a path between a and b , $d(a, b) = \infty$. The *diameter* of G is $\text{diam}(G) = \sup\{d(a, b) | a, b \in V(G)\}$.

The idea of a zero-divisor graph of a commutative ring was introduced by I. Beck in 1988 [8]. He assumes that all elements of the ring are vertices of the graph and was mainly interested in colorings and then this investigation of coloring of a commutative ring was continued by Anderson and Naseer in [4]. Anderson and Livingston [7], studied the zero-divisor graph whose vertices are the nonzero zero-divisors.

Let $Z(R)$ be the set of zero-divisors of R . The *zero-divisor graph* of R denoted by $\Gamma(R)$, is a graph with vertices $Z^*(R) = Z(R) \setminus \{0\}$ and for distinct $x, y \in Z^*(R)$ the vertices x and y are adjacent if and only if $xy = 0$. This graph turns out to exhibit properties of the set of the zero-divisors of a commutative ring. The zero-divisor graph helps us to study the algebraic properties of rings using graph theoretical tools. We can translate some algebraic properties of a ring to graph theory language and then the geometric properties of graphs help us explore some interesting results in algebraic structures of rings. The zero-divisor graph of a commutative ring has also been studied by several other authors (e.g., [5, 6, 10]).

In [2], Afkhami and Khashyarmansh introduced the *cozero-divisor graph* $\dot{\Gamma}(R)$ of R , in which the vertices are precisely the nonzero, non-unit elements of R , denoted $W^*(R)$, and two vertices x and y are adjacent if and only if $x \notin yR$ and $y \notin xR$.

Now let M be a finitely generated R -module. The purpose of this paper is to introduce a certain subgraph $\dot{\Gamma}_R(M)$ of $\dot{\Gamma}(R)$, called the *cozero-divisor graph relative to M* and obtain some results similar to those of [2] and [3]. This graph, with a different point of view, can be regarded as a reduction of $\dot{\Gamma}(R)$, namely, we have $\dot{\Gamma}_R(R) = \dot{\Gamma}(R)$.

2. AUXILIARY RESULTS

Let M be an R -module. The *support* of M is denoted by $Supp(M)$ and it is defined by

$$Supp(M) = \{P \in Spec(R) | Ann_R(N) \subseteq P \text{ for some cyclic submodule } N \text{ of } M\}.$$

In the rest of this paper $Max(Supp(M))$ (i.e., the set of all maximal elements in $Supp(M)$) is denoted by $Max(M)$.

The *Jacobson radical* of M is denoted by $J(M)$ and it is the intersection of all elements in $Max(M)$. Also, the union of all elements in $Max(M)$ is denoted by $N_R(M)$ [12].

M is said to be a *local module* if $|Max(M)| = 1$ [12].

The subset $W_R(M)$ of R is defined by $\{r \in R | rM \neq M\}$ [12] and set $W_R^*(M) = W_R(M) \setminus \{0\}$.

$Z_R(M) = \{r \in R | \text{the } R\text{-module endomorphism on } M \text{ defined by multiplication by } r \text{ is not injective}\}.$

Remark 1 (See [12]). Let M be an R -module. Then $W_R(M) \subseteq N_R(M)$ and we have equality if M is a finitely generated R -module.

Remark 2. $Max(M) \subseteq Max(R)$.

Proof. This follows immediately from the proof of [12, 1.4]. □

3. MAIN RESULTS

In the rest of this paper M is a finitely generated R -module.

Definition 1. We define the *cozero-divisor graph relative to M* , denoted by $\dot{\Gamma}_R(M)$ as a graph with vertices $W_R^*(M) = W_R(M) \setminus \{0\}$ and two distinct vertices r and s are adjacent if and only if $r \notin (sM :_R M)$ and $s \notin (rM :_R M)$.

Definition 2. We define the *strongly cozero-divisor graph relative to M* , denoted by $\tilde{\Gamma}_R(M)$ as a graph with vertices $W_R^*(M) = W_R(M) \setminus \{0\}$ and two distinct vertices r and s are adjacent if and only if $r \notin \sqrt{(sM :_R M)}$ and $s \notin \sqrt{(rM :_R M)}$.

The following example shows that $\dot{\Gamma}(R)$, $\dot{\Gamma}_R(M)$, and $\tilde{\Gamma}_R(M)$ are different.

Example 1. Set $R = \mathbb{Z}$ (here $\mathbb{Z}$ denotes the ring of integers) and $M = \mathbb{Z}_{12}$. Then $W_R^*(R) = \mathbb{Z} \setminus \{-1, 1, 0\}$ and $W_R^*(M) = \mathbb{Z} \setminus (\{m : (m, 12) = 1\} \cup \{0\})$, where $(m, 12)$ denotes the greatest common divisor of m and 12. The elements 8 and 12 are adjacent in $\dot{\Gamma}(R)$ but they are not adjacent in $\dot{\Gamma}_R(M)$. Also, 6 and 8 are adjacent in $\dot{\Gamma}_R(M)$ but they are not adjacent in $\tilde{\Gamma}_R(M)$. Moreover, 6 and 10 are adjacent in $\tilde{\Gamma}_R(R)$ but they are not adjacent in $\tilde{\Gamma}_R(M)$.

An R -module L is said to be a *multiplication module* if for every submodule N of L there exists an ideal I of R such that $N = IL$.

- Theorem 1.**
- (a) $\dot{\Gamma}_R(M)$ is a subgraph of $\dot{\Gamma}(R)$.
 - (b) $\tilde{\Gamma}_R(R)$ is a subgraph of $\dot{\Gamma}(R)$.
 - (c) If M is a faithful R -module, then $W_R^*(M) = W^*(R)$.
 - (d) If M is a faithful R -module, then $\tilde{\Gamma}_R(M) = \tilde{\Gamma}_R(R)$.
 - (e) If M is a faithful multiplication R -module, then $\dot{\Gamma}_R(M) = \dot{\Gamma}(R)$.

Proof. Parts (a) and (b) are clear.

(c) By part (a), $W_R^*(M) \subseteq W^*(R)$. Now let $r \in W^*(R)$ and $r \notin W_R^*(M)$. Then $rM = 0$. Thus by Nakayama's Lemma, $1 + rt \in \text{Ann}_R(M) = 0$. Hence $Rr = 0$, which is a contradiction.

(d) By part (c), $W_R^*(M) = W^*(R)$. Now let r and s be two distinct adjacent vertices of $\tilde{\Gamma}_R(R)$ and let $r \in \sqrt{(sM :_R M)}$. Then $r^n M \subseteq sM$ for some $n \in \mathbb{N}$. Thus by [11, Theorem 75], there exist $t \in R$ and $k \in \mathbb{N}$ such that $(r^{kn} + st)M = 0$. Since M is faithful, $r^{kn} + st = 0$ and so $r \in \sqrt{sR}$. This contradiction shows that $E(\tilde{\Gamma}_R(R)) \subseteq E(\dot{\Gamma}_R(M))$. The reverse inclusion is clear.

(e) By part (c), $W_R^*(M) = W^*(R)$. Now let r and s be two distinct adjacent vertices of $\dot{\Gamma}(R)$ and let $r \in (sM :_R M)$. Then $rM \subseteq sM$. Thus by [1], $Rr \subseteq sR$, which is a contradiction. Hence $E(\dot{\Gamma}(R)) \subseteq E(\dot{\Gamma}_R(M))$. The reverse inclusion is clear. $\square$

Remark 3. By using part (e) of Theorem 1, if $M = R$, then $\dot{\Gamma}_R(R) = \dot{\Gamma}(R)$.

We use the following lemma frequently.

Lemma 1. *Let M be an R -module and $P \in \text{Max}(M)$. Then $P = (PM :_R M)$.*

Proof. Assume $(PM :_R M) = R$ so that $PM = M$. Since M is finitely generated, there exists $x \in P$ such that $(1+x)M = 0$. Thus $1+x \in \text{Ann}_R(M)$ but by [12], $P \supseteq \text{Ann}_R(M)$. It follows that $1 \in P$, a contradiction. Now the results follows from $P \subseteq (PM :_R M)$ and Remark 2. $\square$

Proposition 1.

- (a) *The graph $\dot{\Gamma}_R(M)$ is not complete if and only if there exists an element $s \in W_R^*(M)$ such that $|(sM :_R M)| > 2$.*
- (b) *$\dot{\Gamma}_R(M)$ is complete if and only if $(sM :_R M) = \{0, s\}$ for all elements s in $W_R^*(M)$.*
- (c) *If R is an integral domain, then $\dot{\Gamma}_R(M)$ is not complete.*

Proof. Straightforward $\square$

Theorem 2. *$\dot{\Gamma}_R(M)$ is complete if and only if $\tilde{\Gamma}_R(M)$ is complete.*

Proof. The sufficiency is clear. Conversely, we assume that $\dot{\Gamma}_R(M)$ is complete and r, s be arbitrary distinct elements in $W_R^*(M)$ and $r \in \sqrt{(sM :_R M)}$. Then $r^n M \subseteq sM$ for some $n \in \mathbb{N}$. Since $\dot{\Gamma}_R(M)$ is complete, r^n and s are adjacent. But this is a contradiction by the above arguments. $\square$

We use the notation $\dot{\Gamma}_R(M) \setminus J(M)$ to denote a subgraph of $\dot{\Gamma}_R(M)$ with vertices $W_R^*(M) \setminus J(M)$.

Theorem 3. (a) *The graph $\dot{\Gamma}_R(M) \setminus J(M)$ is connected.*

(b) *If M is a non-local module, then $\text{diam}(\dot{\Gamma}_R(M) \setminus J(M)) \leq 2$.*

Proof. (a) If M is a local module, then $W_R^*(M) \setminus J(M)$ is a empty set, which is connected. So we assume that $|\text{Max}(M)| > 1$. Let r and s be arbitrary distinct elements in $W_R^*(M) \setminus J(M)$. Suppose that r is not adjacent to s . We may assume that $r \in (sM :_R M)$. Since $r \notin J(M)$, there exists $P \in \text{Max}(M)$ such that $r \notin P$. Thus $P \not\subseteq J(M) \cup (sM :_R M)$, otherwise, $P \subseteq J(M)$ or $P \subseteq (sM :_R M)$. In first case, $J(M) = P$ so that $|\text{Max}(M)| = 1$. In second case, $P = (sM :_R M)$ by Lemma 1. In either case we have a contradiction. Choose t in $P \setminus (J(M) \cup (sM :_R M))$. Now by using Lemma 1, we see that $r - t - s$ is the required path.

(b) This follows from the proof of part (a). $\square$

Corollary 1. *Let M be a non-local R -module with $J(M) = 0$. Then $\dot{\Gamma}_R(M)$ is connected and $\text{diam}(\dot{\Gamma}_R(M)) \leq 2$.*

Theorem 4. *Let M be a non-local module such that for every element $r \in J(M)$, there exist $P \in \text{Max}(M)$ and $s \in P \setminus J(M)$ with $r \notin (sM :_R M)$. Then $\dot{\Gamma}_R(M)$ is connected and $\text{diam}(\dot{\Gamma}_R(M)) \leq 3$.*

Proof. Suppose that $r, s \in W_R^*(M)$ and r is not adjacent to s . We may assume that $r \in (sM :_R M)$. Then, we have the following cases:

Case 1. Suppose that $s \in J(M)$. We claim that $r \in J(M)$. Otherwise there exists $P \in \text{Max}(M)$ such that $r \notin P$. Then $rM \subseteq sM \subseteq PM$. Thus by Lemma 1, $r \in (PM :_R M) = P$, a contradiction. Thus by hypothesis, there exists $t \in P \setminus J(M)$ for some $P \in \text{Max}(M)$ with $r \notin (tM :_R M)$. Also $t \notin (rM :_R M)$; otherwise, we have $tM \subseteq rM \subseteq sM$. Thus $t \in (sM :_R M) \subseteq (PM :_R M) = P$ for each $P \in J(M)$ so that $t \in J(M)$, a contradiction. Thus r is adjacent to t . By similar arguments, we see that t is adjacent to s . Hence $r - t - s$ is the required path.

Case 2. Suppose that $r, s \notin J(M)$. Then $r \notin P$, for some $P \in \text{Max}(M)$. If $P = (sM :_R M)$, then since $r \in (sM :_R M)$, we have a contradiction. Choose p in $P \setminus (sM :_R M)$. By similar arguments as in part (a), we see that $r - p - s$ is the desired path.

Case 3. Assume that $s \notin J(M)$ and $r \in J(M)$. By our assumption, there exists $q \in P \setminus J(M)$, for some $P \in \text{Max}(M)$ such that $r \notin (qM :_R M)$. We claim that $q \notin (rM :_R M)$. Otherwise, $qM \subseteq rM \subseteq PM$ for every $P \in \text{Max}(M)$. Thus by Lemma 1, $q \in (PM :_R M) = P$ for every $P \in \text{Max}(M)$, a contradiction. Hence r is adjacent to q . Further, $s \notin (qM :_R M)$. If $q \notin (sM :_R M)$, then we get the path $r - q - s$. Otherwise, we can apply case 2 for the elements q and s to get a path $q - u - s$ for some $u \in W_R^*(M)$. Hence we have $r - q - u - s$. $\square$

Theorem 5. Let M be a non-local module. Then $g(\dot{\Gamma}_R(M) \setminus J(M)) \leq 5$ or $g(\dot{\Gamma}_R(M) \setminus J(M)) = \infty$.

Proof. Use the technique of [2, 2.8] and apply Theorem 3. $\square$

Theorem 6. Let $|\text{Max}(M)| \geq 3$. Then $g(\dot{\Gamma}_R(M)) = 3$.

Proof. Clearly, $g(\dot{\Gamma}_R(M)) \geq 3$. Let P_1, P_2 , and P_3 be distinct elements of $\text{Max}(M)$. By Remark 2, $\text{Max}(M) \subseteq \text{Max}(R)$. Choose $a_i \in P_i \setminus \bigcup_{j=1}^3 P_j$, $1 \leq i \leq 3$ and $j \neq i$. Then by using 1, we see that $a_1 - a_2 - a_3 - a_1$ is a cycle. Therefore $g(\dot{\Gamma}_R(M)) = 3$. $\square$

For a graph G , let $\chi(G)$ denote the *chromatic number of the graph* G , i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. A *clique* of a graph is its complete subgraph and the number of vertices in the largest clique of G , denoted by $\text{clique}(G)$, is called the clique number of G .

Theorem 7. (a) Let R not be a field. Then if $\text{Max}(M)$ has an infinite number of maximal ideals, then $\text{clique}(\dot{\Gamma}_R(M))$ is also infinite; otherwise $\text{clique}(\dot{\Gamma}_R(M)) \geq |\text{Max}(M)|$.

(b) If $\chi(\dot{\Gamma}_R(M)) < \infty$, then $|\text{Max}(M)| < \infty$.

Proof. Use the technique of [2, 2.14]. $\square$

A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends.

Theorem 8. Assume that $|Max(M)| \geq 5$. Then $\dot{\Gamma}_R(M)$ is not planar.

Proof. Assume that $|Max(M)| \geq 5$. Choose $a_i \in m_i \setminus \cup_{j=1}^5 m_j$, where $m_i \in Max(M)$, $1 \leq i \leq 5$, and $j \neq i$. Then $a_i \notin (a_j M :_R M)$. Otherwise, $a_i \in (a_j M :_R M) \subseteq (m_j M :_R M) = m_j$ by Lemma 1. Similarly, $a_j \notin (a_i M :_R M)$. Hence m_1, m_2, m_3, m_4, m_5 forms a complete subgraph of $\dot{\Gamma}_R(M)$ which is isomorphic to K_5 . Thus by [9, p.153], $\dot{\Gamma}_R(M)$ is not planar. $\square$

For any vertex x of a connected graph G , the *eccentricity* of x , denoted by $e(x)$, is the maximum of the distances from x to the other vertices of G , and the minimum value of the eccentricity is the *radius* of G , which is denoted by $r(G)$.

Theorem 9. Let M be a non-local module with $J(M) = 0$. Then $r(\dot{\Gamma}_R(M)) = 2$ if and only if for each $t \in W_R^*(M)$, there exists $s \in W_R^*(M)$ such that t is not adjacent to s .

Proof. The proof is similar to that of [2, 3.14]. $\square$

Theorem 10. Let R be a Noetherian ring. If $\dot{\Gamma}_R(M)$ is totally disconnected, then M is a local module with maximal ideal of the form $(xM :_R M)$ for some $x \in W_R^*(M)$.

Proof. It is easy to see that M is a local module. Set $Max(M) = m$. Assume to contrary that m is not the form of $(rM :_R M)$ for every $r \in W_R^*(M)$. Set $A = \{(rM :_R M), r \in W_R^*(M)\}$. Then A has a maximal member, say $(\acute{r}M :_R M)$ for some $\acute{r} \in W_R^*(M)$. Choose $s \in m \setminus (\acute{r}M :_R M)$. We claim that $\acute{r} \notin (sM :_R M)$. Otherwise, we have $(\acute{r}M :_R M) \subseteq (sM :_R M)$, so $(\acute{r}M :_R M) = (sM :_R M)$ by maximality. Hence $s \in (\acute{r}M :_R M)$ so that $\acute{r}$ is adjacent to s , a contradiction. $\square$

Theorem 11. Assume that M is a non-local module. Then the following conditions are equivalent.

- (a) $\dot{\Gamma}_R(M) \setminus J(M)$ is complete bipartite.
- (b) $\dot{\Gamma}_R(M) \setminus J(M)$ is bipartite.
- (c) $\dot{\Gamma}_R(M) \setminus J(M)$ contains no triangles.

Proof. Use the technique of [3, 2.13]. $\square$

Proposition 2. If the graph $\dot{\Gamma}_R(M) \setminus J(M)$ is n -partite for some positive integer n , then $|Max(M)| \leq n$.

Proof. Assume to the contrary that $|Max(M)| > n$. Since $\dot{\Gamma}_R(M) \setminus J(M)$ is an n -partite graph, there are maximal ideals P_1 and P_2 of $Max_R(M)$ with $(rM :_R M) \subseteq P_1 \setminus P_2$ and $(sM :_R M) \subseteq P_2 \setminus P_1$, where r, s belong to the same part. But this implies that r is adjacent to s which is a contradiction. $\square$

Theorem 12. *Let M be an R -module with $\text{Max}(M) = \{m_1, m_2\}$. Then $\dot{\Gamma}_R(M) \setminus J(M)$ is a complete bipartite graph with parts $m_i \setminus J(M)$, $i = 1, 2$, if and only if every pair of ideals $(rM :_R M)$, $(sM :_R M)$ contained in $(m_1 \setminus J(M))$ or $(m_2 \setminus J(M))$, where $r, s \in R$, are totally ordered.*

Proof. Suppose that $\dot{\Gamma}_R(M) \setminus J(M)$ is a complete bipartite graph with parts $m_i \setminus J(M)$, $i = 1, 2$. Further assume to the contrary that there exist ideals $(rM :_R M)$, $(sM :_R M) \subseteq m_1 \setminus J(M)$ such that $(rM :_R M) \not\subseteq (sM :_R M)$ and $(sM :_R M) \not\subseteq (rM :_R M)$. We claim that r is adjacent to s in $m_1 \setminus J(M)$. Otherwise, without loss of generality, we assume that $r \in (sM :_R M)$. Then $r, s \in m_1 \setminus J(M)$ and we have $rM \subseteq (sM :_R M)M$. Thus $(rM :_R M) \subseteq ((sM :_R M)M :_R M) = (sM :_R M)$, a contradiction. Hence r is adjacent to s in $m_1 \setminus J(M)$, which is again a contradiction by hypothesis. Conversely, assume that $i \in \{1, 2\}$ and $(rM :_R M), (sM :_R M) \subseteq m_i \setminus J(M)$. We may assume that $(rM :_R M) \subseteq (sM :_R M)$. Then clearly, $r, s \in m_i \setminus J(M)$ and r is not adjacent. Now if $r \in m_1 \setminus m_2$ and $s \in m_2 \setminus m_1$, then by using 1, we see that r is adjacent to s . Therefore $\dot{\Gamma}_R(M) \setminus J(M)$ is a complete bipartite graph with parts $m_i \setminus J(M)$, $i = 1, 2$. $\square$

Theorem 13. *Let M be a faithful R -module and $Z_R(M) \neq W_R(M)$. Then $\dot{\Gamma}_R(M)$ is finite if and only if R is finite.*

Proof. Clearly if R is finite, then $\dot{\Gamma}_R(M)$ is finite. So we assume that $\dot{\Gamma}_R(M)$ is finite and show that R is finite. Suppose that R is infinite and look for a contradiction. By Remark 1, we have $Z_R(M) \subset W_R(M) = N_R(M)$. Choose $x \in W_R(M) \setminus Z_R(M)$. Since Rx is a finite R -module and $R \setminus W_R(M)$ is an infinite set, there exist distinct elements $r_1, r_2 \in R \setminus W_R(M)$ such that $r_1x = r_2x$. Therefore $(r_1 - r_2)x = 0$. Then we have $x((r_1 - r_2)M) = 0$. Since x is a nonzero-divisor on M , we have $(r_1 - r_2)M = 0$ so that $r_1 - r_2 \in \text{Ann}_R(M)$. Thus $r_1 = r_2$, a contradiction. $\square$

Corollary 2. *Let R be a domain and let $Z_R(M) = \{0\}$. If $\dot{\Gamma}_R(M)$ is a finite graph, then R is a field.*

Proof. If $W_R(M) \neq \{0\}$, then by Theorem 13, R is finite so that R is a field. Otherwise, if $W_R(M) = \{0\}$, then we have $W_R(M) = \cup_{p \in \text{Max}(M)} P = \{0\}$ by Remark 1. This implies that the zero ideal of R is a maximal ideal and hence R is a field. $\square$

Remark 4. One can see, by using the same technique, that the results about $\dot{\Gamma}_R(M)$ in this section is also true for $\tilde{\Gamma}_R(M)$.

REFERENCES

- [1] Z. Abd El-Bast and P. F. Smith, "Multiplication modules," *Commun. Algebra*, vol. 16, no. 4, pp. 755–779, 1988.
- [2] M. Afkhami and K. Khashyarmansh, "The cozero-divisor graph of a commutative ring," *Southeast Asian Bull. Math.*, vol. 35, no. 5, pp. 753–762, 2011.

- [3] M. Afkhami and K. Khashyarmansh, "On the cozero-divisor graphs of commutative rings and their complements," *Bull. Malays. Math. Sci. Soc. (2)*, vol. 35, no. 4, pp. 935–944, 2012.
- [4] D. D. Anderson and M. Naseer, "Beck's coloring of a commutative ring," *J. Algebra*, vol. 159, no. 2, pp. 500–514, 1993.
- [5] D. F. Anderson, A. Frazier, A. Lauve, and P. S. Livingston, "The zero-divisor graph of a commutative ring. II," in *Ideal theoretic methods in commutative algebra. Proceedings of the conference in honor of Professor James A. Huckaba's retirement, University of Missouri, Columbia, MO, USA*, ser. Lect. Notes Pure Appl. Math., D. D. Anderson, Ed. New York: Marcel Dekker, 2001, vol. 220, pp. 61–72.
- [6] D. F. Anderson, R. Levy, and J. Shapiro, "Zero-divisor graphs, von Neumann regular rings, and Boolean algebras," *J. Pure Appl. Algebra*, vol. 180, no. 3, pp. 221–241, 2003.
- [7] D. F. Anderson and P. S. Livingston, "The zero-divisor graph of a commutative ring," *J. Algebra*, vol. 217, no. 2, pp. 434–447, 1999.
- [8] I. Beck, "Coloring of commutative rings," *J. Algebra*, vol. 116, no. 1, pp. 208–226, 1988.
- [9] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*. New York: American Elsevier Publishing Co., 1976.
- [10] G. A. Cannon, K. M. Neuerburg, and S. P. Redmond, "Zero-divisor graphs of nearrings and semigroups," in *Nearrings and nearfields. Proceedings of the conference on nearrings and nearfields, Hamburg, Germany, July 27–August 3, 2003.*, H. Kiechle, Ed. Dordrecht: Springer, 2005, pp. 189–200.
- [11] I. Kaplansky, *Commutative rings*. Chicago: University of Chicago Press, 1978.
- [12] S. Yassemi, "Maximal elements of support and cosupport," [http://www.ictp.trieste.it/\\$\sim\\$pub_off](http://www.ictp.trieste.it/$\sim$pub_off).

Authors' addresses

H. Ansari-Toroghy

Department of pure Mathematics, Faculty of mathematical Sciences, University of Guilan, P. O. Box 41335-19141, Rasht, Iran.

E-mail address: ansari@guilan.ac.ir

F. Farshadifar

University of Farhangian, Tehran, Iran.

E-mail address: f.farshadifar@gmail.com

Sh.Habibi

Department of pure Mathematics, Faculty of mathematical Sciences, University of Guilan, P. O. Box 41335-19141, Rasht, Iran.

E-mail address: sh.habibi@guilan.ac.ir



EXISTENCE AND POSITIVITY OF SOLUTIONS FOR A TOTALLY NONLINEAR NEUTRAL PERIODIC DIFFERENTIAL EQUATION

ABDELOUAHEB ARDJOUNI, AHCENE DJOUDI

Received 25 March, 2013

Abstract. In this paper, we use a modification of Krasnoselskii's fixed point theorem introduced by Burton (see [6] Theorem 3) to establish new results on the existence and positivity of solutions for the totally nonlinear neutral periodic differential equation of the form

$$\begin{aligned}\frac{d}{dt}x(t) &= -a(t)x^3(t) + \frac{d}{dt}Q(t, x(g(t))) + G(t, x^3(t), x^3(g(t))), \\ x(t+T) &= x(t).\end{aligned}$$

We invert this equation to construct a sum of a completely continuous map and a large contraction which is suitable for the application of a modification of Krasnoselskii's theorem.

2010 *Mathematics Subject Classification:* 34A37; 34A12; 39A05

Keywords: fixed point theory, large contraction, totally nonlinear neutral differential equation, periodic

1. INTRODUCTION

In the past decades, very much attention has been given to neutral differential equations (see [1–16], [18] and the references therein). The main reason for this interest is that delay differential equations play an important role in applications. For instance, in biological applications, delay equations give better description of fluctuations in population than the ordinary ones. As known in Hale [13], Hale and Lunel [14] neutral delay differential equations appear as models of electrical networks which contain lossless transmission lines. Such networks arise, for example, in high speed computers where lossless transmission lines are used to interconnect switching circuits.

Existence, uniqueness, stability and positivity of solutions of functional differential equations are of great interest in mathematics and its applications to the modeling of various practical problems (see [1–16], [18]). Positivity is one of the most common and most important characteristics of mathematical models. In problem of economics, the positivity is quite important for processes that model interest rate dynamics on financial market, because the interest must be positive. Also, in fluid flow problems, densities, pressures, and concentrations are always positive.

The primary motivation for the present work are the papers by Y. M. Dib et al. [10] and Yankson [18]. In [10], the authors used Krasnoselskii's fixed point theorem to establish the existence of periodic solutions for the nonlinear neutral differential equation

$$\frac{d}{dt}x(t) = -a(t)x(t) + \frac{d}{dt}Q(t, x(t-g(t))) + G(t, x(t), x(t-g(t))), \quad (1.1)$$

where the nonlinear terms Q and G and the function a are continuous in all arguments. Also, the authors used the contraction mapping principle to show the uniqueness of periodic solutions and stability of the zero solution of (1.1).

In [18], the author used a modification of Krasnoselskii's fixed point theorem to establish the existence and positivity of solutions for the totally nonlinear neutral periodic differential equation

$$\begin{aligned} x'(t) &= -a(t)x^3(t) + c(t)x'(g(t))g'(t) + q(t, x^3(g(t))), \\ x(t+T) &= x(t), \end{aligned} \quad (1.2)$$

where $T > 0$ be fixed, the nonlinear term q is an L^1 -Carathéodory function and the function $a \in L^1[0, T]$ is bounded. Also, the author used the contraction mapping principle to show the uniqueness of periodic solutions of (1.2).

In this paper, we consider the totally nonlinear neutral periodic differential equation

$$\begin{aligned} \frac{d}{dt}x(t) &= -a(t)x^3(t) + \frac{d}{dt}Q(t, x(g(t))) + G(t, x^3(t), x^3(g(t))), \\ x(t+T) &= x(t), \end{aligned} \quad (1.3)$$

where $T > 0$ be fixed, the nonlinear terms Q and G are an L^1 -Carathéodory functions and the function $a \in L^1[0, T]$ is bounded. Our purpose here is to use a modification of Krasnoselskii's fixed point theorem due Burton (see [6] Theorem 3) to show the existence and positivity of periodic solutions for equation (1.3). Clearly, the present problem is totally nonlinear so that the variation of parameters can not be applied directly. Then, we resort to the idea of adding and subtracting a linear term. As noted by Burton in [6], the added term destroys a contraction already present in part of the equation but it replaces it with the so called a large contraction mapping which is suitable for fixed point theory. During the process we have to transform (1.3) into an integral equation written as a sum of two mappings; one is a large contraction and the other is completely continuous. After that, we use a variant of Krasnoselskii fixed point theorem, to show the existence and positivity of a periodic solution.

Note that in our consideration the neutral term $\frac{d}{dt}Q(t, x(g(t)))$ of (1.3) produces nonlinearity in the derivative term $\frac{d}{dt}x(g(t))$. The neutral term $\frac{d}{dt}x(g(t))$ of (1.2) in [18] enters linearly. As a consequence, our analysis is different from that in [18].

In Section 2, we present the inversion of totally nonlinear neutral periodic differential equation (1.3) and Krasnoselskii-Burton's fixed point theorem. For details

on Krasnoselskii-Burton theorem we refer the reader to [6, 7]. We present our main results on existence and positivity in Section 3.

2. PRELIMINARIES

The following definition is essential in our analysis.

Definition 1. A function $F : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is an L^1 -Carathéodory function if it satisfies the following conditions

- (c1) For each $z \in \mathbb{R}^n$, the mapping $t \rightarrow F(t, z)$ is Lebesgue measurable.
- (c2) For almost all $t \in [0, T]$, the mapping $z \rightarrow F(t, z)$ is continuous on $\mathbb{R}^n$.
- (c3) For each $r > 0$, there exists $f_r \in L^1([0, T], \mathbb{R})$ such that for almost all $t \in [0, T]$ and for all z such that $|z| < r$, we have $|F(t, z)| \leq f_r(t)$.

Define the set $P_T = \{\varphi \in C(\mathbb{R}, \mathbb{R}) : \varphi(t + T) = \varphi(t)\}$ and the norm $\|\varphi\| = \sup_{0 \leq t \leq T} |\varphi(t)|$. Then $(P_T, \|\cdot\|)$ is a Banach space. We will assume that the following conditions hold.

(h1) $a \in L^1_{Loc}(\mathbb{R}, \mathbb{R})$ is positive and bounded, satisfies $a(t + T) = a(t)$ for all t and

$$1 - e^{-\int_{t-T}^t a(s)ds} \equiv \frac{1}{\eta} \neq 0.$$

(h2) $g \in P_T$.

(h3) Q and G are an L^1 -Carathéodory functions, and $Q(t + T, x) = Q(t, x)$, $G(t + T, x, y) = G(t, x, y)$ for all t .

Lemma 1. Suppose that conditions (h1), (h2) and (h3) hold. Then $x \in P_T$ is a solution of equation (1.3) if and only if

$$\begin{aligned} x(t) = & \eta \int_{t-T}^t a(u) [x(u) - x^3(u)] e^{-\int_u^t a(s)ds} du + Q(t, x(g(t))) \\ & + \eta \int_{t-T}^t [G(u, x^3(u), x^3(g(u))) - a(u) Q(u, x(g(u)))] e^{-\int_u^t a(s)ds} du. \end{aligned} \quad (2.1)$$

Proof. Let $x \in P_T$ be a solution of (1.3). First we write this equation as

$$\begin{aligned} & \frac{d}{dt} \{x(t) - Q(t, x(g(t)))\} \\ & = -a(t) \{x(t) - Q(t, x(g(t)))\} + a(t) \{x(t) - x^3(t)\} \\ & \quad + G(t, x^3(t), x^3(g(t))) - a(t) Q(t, x(g(t))). \end{aligned}$$

Multiply both sides of the above equation by $e^{\int_0^t a(s)ds}$ and then integrate from $t - T$ to t to obtain

$$\int_{t-T}^t \left[(x(u) - Q(u, x(g(u)))) e^{\int_0^u a(s)ds} \right]' du$$

$$\begin{aligned}
&= \int_{t-T}^t a(u) [x(u) - x^3(u)] e^{\int_0^u a(s) ds} du \\
&+ \int_{t-T}^t [G(u, x^3(u), x^3(g(u))) - a(u) Q(u, x(g(u)))] e^{\int_0^u a(s) ds} du.
\end{aligned}$$

As a consequence, we arrive

$$\begin{aligned}
&(x(t) - Q(t, x(g(t)))) e^{\int_0^t a(s) ds} \\
&- (x(t-T) - Q(t-T, x(g(t-T)))) e^{\int_0^{t-T} a(s) ds} \\
&= \int_{t-T}^t a(u) [x(u) - x^3(u)] e^{\int_0^u a(s) ds} du \\
&+ \int_{t-T}^t [G(u, x^3(u), x^3(g(u))) - a(u) Q(u, x(g(u)))] e^{\int_0^u a(s) ds} du.
\end{aligned}$$

Dividing both sides of the above equation by $e^{\int_0^t a(s) ds}$ and the fact that $x(t) = x(t-T)$, $g(t) = g(t-T)$ and $Q(t, x) = Q(t-T, x)$, we obtain

$$\begin{aligned}
&x(t) - Q(t, x(g(t))) \\
&= \eta \int_{t-T}^t a(u) [x(u) - x^3(u)] e^{-\int_u^t a(s) ds} du \\
&+ \eta \int_{t-T}^t [G(u, x^3(u), x^3(g(u))) - a(u) Q(u, x(g(u)))] e^{-\int_u^t a(s) ds} du.
\end{aligned}$$

Since each step is reversible, the converse follows easily. This completes the proof. $\square$

Krasnoselskii (see [5, 17]) combined the contraction mapping theorem and Schauder's theorem and formulated the following hybrid and attractive result.

Theorem 1 (Krasnoselskii). *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A and B map $\mathbb{M}$ into S such that*

- (i) $x, y \in \mathbb{M}$, implies $Ax + By \in \mathbb{M}$,
- (ii) A is completely continuous, and
- (iii) B is a contraction with constant $\alpha < 1$.

Then there exists $z \in \mathbb{M}$ with $z = Az + Bz$.

This is a captivating result and has a number of interesting applications. In recent years much attention has paid to this theorem. Burton [5] observed that Krasnoselskii result can be more interesting in applications with certain changes and formulated in Theorem 3 below (see [4] for the proof).

Definition 2 (Large Contraction). Let (M, d) be a metric space and $B : \mathbb{M} \rightarrow \mathbb{M}$. B is said to be a large contraction if $\varphi, \psi \in \mathbb{M}$, with $\varphi \neq \psi$ then $d(B\varphi, B\psi) <$

$d(\varphi, \psi)$ and if for all $\varepsilon > 0$ there exists $\delta < 1$ such that

$$[\varphi, \psi \in \mathbb{M}, d(\varphi, \psi) \geq \varepsilon] \Rightarrow d(B\varphi, B\psi) \leq \delta d(\varphi, \psi).$$

Theorem 2. *Let $(\mathbb{M}, d)$ be a complete metric space and B be a large contraction. Suppose there is an $x \in \mathbb{M}$ and $L > 0$, such that $d(x, B^n x) \leq L$ for all $n \geq 1$. Then B has a unique fixed point in $\mathbb{M}$.*

Theorem 3 (Krasonskii-Burton). *Let $\mathbb{M}$ be a closed bounded convex non-empty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A, B map $\mathbb{M}$ into $\mathbb{M}$ and that*

- (i) $\forall x, y \in \mathbb{M} \Rightarrow Ax + By \in \mathbb{M}$,
- (ii) A is completely continuous, and
- (iii) B is a large contraction.

Then there is a $z \in \mathbb{M}$ with $z = Az + Bz$.

3. EXISTENCE RESULTS

We present our existence results in this section. To this end, we first define the operator H by

$$\begin{aligned} (H\varphi)(t) = & \eta \int_{t-T}^t a(u) [\varphi(u) - \varphi^3(u)] e^{-\int_u^t a(s) ds} du + Q(t, \varphi(g(t))) \\ & + \eta \int_{t-T}^t [G(u, \varphi^3(u), \varphi^3(g(u))) - a(u) Q(u, \varphi(g(u)))] e^{-\int_u^t a(s) ds} du, \end{aligned} \quad (3.1)$$

From Lemma 1 we see that fixed points of H are solutions of (1.3) and vice versa. In order to employ theorem 3 we need to express the operator H as the sum of two operators, one of which is completely continuous and the other of which is a large contraction. Let $(H\varphi)(t) = (A\varphi)(t) + (B\varphi)(t)$ where

$$\begin{aligned} (A\varphi)(t) = & Q(t, \varphi(g(t))) \\ & + \eta \int_{t-T}^t [G(u, \varphi^3(u), \varphi^3(g(u))) - a(u) Q(u, \varphi(g(u)))] e^{-\int_u^t a(s) ds} du, \end{aligned} \quad (3.2)$$

and

$$(B\varphi)(t) = \eta \int_{t-T}^t a(u) [\varphi(u) - \varphi^3(u)] e^{-\int_u^t a(s) ds} du. \quad (3.3)$$

We need the following condition on the nonlinear term Q .

(h4) The function $Q(t, x)$ is continuous in t and there exist bounded positive periodic functions $q_1, q_2 \in L^1[0, T]$, with period T , such that

$$|Q(t, x)| \leq q_1(t) |x| + q_2(t),$$

for all $x \in \mathbb{R}$.

Our first lemma in this section shows that $A : P_T \rightarrow P_T$ is completely continuous.

Lemma 2. *Suppose that conditions (h1) – (h4) hold. Then $A : P_T \rightarrow P_T$ is completely continuous.*

Proof. From (3.2) and conditions (h1) – (h3), it follows trivially that $e^{-\int_{u+T}^{t+T} a(s)ds} = e^{-\int_u^t a(s)ds}$. Consequently, we have that

$$(A\varphi)(t+T) = (A\varphi)(t).$$

That is, if $\varphi \in P_T$ then $A\varphi$ is periodic with period T .

To see that A is continuous. Let $\{\varphi_i\} \subset P_T$ be such that $\varphi_i \rightarrow \varphi$ as $i \rightarrow \infty$. By the Dominated Convergence Theorem,

$$\begin{aligned} & \lim_{i \rightarrow \infty} |(A\varphi_i)(t) - (A\varphi)(t)| \\ & \leq \lim_{i \rightarrow \infty} |Q(t, \varphi_i(g(t))) - Q(t, \varphi(g(t)))| \\ & \lim_{i \rightarrow \infty} \eta \int_{t-T}^t \{ |G(u, \varphi_i^3(u), \varphi_i^3(g(u))) - G(u, \varphi^3(u), \varphi^3(g(u)))| \\ & \quad + a(u) |Q(u, \varphi_i(g(u))) - Q(u, \varphi(g(u)))| \} e^{-\int_u^t a(s)ds} du \\ & = \lim_{i \rightarrow \infty} |Q(t, \varphi_i(g(t))) - Q(t, \varphi(g(t)))| \\ & \quad + \eta \int_{t-T}^t \left\{ \lim_{i \rightarrow \infty} |G(u, \varphi_i^3(u), \varphi_i^3(g(u))) - G(u, \varphi^3(u), \varphi^3(g(u)))| \right. \\ & \quad \left. + a(u) \lim_{i \rightarrow \infty} |Q(u, \varphi_i(g(u))) - Q(u, \varphi(g(u)))| \right\} e^{-\int_u^t a(s)ds} du \\ & = 0. \end{aligned}$$

Hence $A : P_T \rightarrow P_T$ is continuous.

Finally, we show that A is completely continuous. Let $\mathfrak{C} \subset P_T$ be a closed bounded subset and let C be such that $\|\varphi\| \leq C$ for all $\varphi \in \mathfrak{C}$. then

$$\begin{aligned} |(A\varphi)(t)| & \leq q_1^* C + q_2^* \\ & + \eta \int_{t-T}^t [|G(u, \varphi^3(u), \varphi^3(g(u)))| + a(u) |Q(u, \varphi(g(u)))|] e^{-\int_u^t a(s)ds} du \\ & \leq q_1^* C + q_2^* + \eta N \left\{ \int_{t-T}^t g_C(u) du + \int_{t-T}^t a(u) q_C(u) du \right\} \\ & \leq q_1^* C + q_2^* + \eta N \left\{ \int_{t-T}^t g_C(u) du + \alpha \int_{t-T}^t q_C(u) du \right\} \equiv K, \end{aligned}$$

where $q_1^* = \max_{u \in [0, T]} q_1(t)$, $q_2^* = \max_{u \in [0, T]} q_2(t)$, $N = \max_{u \in [t-T, t]} e^{-\int_u^t a(s)ds}$ and $\alpha = \max_{u \in [t-T, t]} a(u)$. And so, the family of functions $A\varphi$ is uniformly bounded.

Again, let $\varphi \in \mathfrak{C}$. Without loss of generality, we can pick $t_1 < t_2$ such that $t_2 - t_1 < T$. Then

$$|(A\varphi)(t_2) - (A\varphi)(t_1)|$$

$$\begin{aligned}
&= |Q(t_2, \varphi(t_2)) \\
&+ \eta \int_{t_2-T}^{t_2} [G(u, \varphi^3(u), \varphi^3(g(u))) - a(u) Q(u, \varphi(g(u)))] e^{-\int_u^{t_2} a(s) ds} du \\
&\quad - Q(t_1, \varphi(t_1)) \\
&- \eta \int_{t_1-T}^{t_1} [G(u, \varphi^3(u), \varphi^3(g(u))) - a(u) Q(u, \varphi(g(u)))] e^{-\int_u^{t_1} a(s) ds} du \Big|.
\end{aligned}$$

We can rewrite the left hand side as the sum of three integrals. We obtain the following

$$\begin{aligned}
&|(A\varphi)(t_2) - (A\varphi)(t_1)| \\
&\leq |Q(t_2, \varphi(t_2)) - Q(t_1, \varphi(t_1))| \\
&+ \eta \int_{t_1}^{t_2} [|G(u, \varphi^3(u), \varphi^3(g(u)))| + a(u) |Q(u, \varphi(g(u)))|] e^{-\int_u^{t_2} a(s) ds} du \\
&+ \eta \int_{t_2-T}^{t_1} [|G(u, \varphi^3(u), \varphi^3(g(u)))| + a(u) |Q(u, \varphi(g(u)))|] \\
&\times \left| e^{-\int_u^{t_2} a(s) ds} - e^{-\int_u^{t_1} a(s) ds} \right| du \\
&+ \eta \int_{t_1-T}^{t_2-T} [|G(u, \varphi^3(u), \varphi^3(g(u)))| + a(u) |Q(u, \varphi(g(u)))|] e^{-\int_u^{t_1} a(s) ds} du \\
&\leq |Q(t_2, \varphi(t_2)) - Q(t_1, \varphi(t_1))| + 2\eta N \left\{ \int_{t_1}^{t_2} g_C(u) + \alpha q_C(u) du \right\} \\
&+ \eta \int_{t_2-T}^{t_1} [g_C(u) + \alpha q_C(u)] \left| e^{-\int_u^{t_2} a(s) ds} - e^{-\int_u^{t_1} a(s) ds} \right| du.
\end{aligned}$$

Now $|Q(t_2, \varphi(t_2)) - Q(t_1, \varphi(t_1))| \rightarrow 0$ and $\int_{t_1}^{t_2} g_C(u) + \alpha q_C(u) du \rightarrow 0$ as $(t_2 - t_1) \rightarrow 0$. Also, since

$$\begin{aligned}
&\int_{t_2-T}^{t_1} [g_C(u) + \alpha q_C(u)] \left| e^{-\int_u^{t_2} a(s) ds} - e^{-\int_u^{t_1} a(s) ds} \right| du \\
&\leq \int_0^T [g_C(u) + \alpha q_C(u)] \left| e^{-\int_u^{t_2} a(s) ds} - e^{-\int_u^{t_1} a(s) ds} \right| du,
\end{aligned}$$

and $\left| e^{-\int_u^{t_2} a(s) ds} - e^{-\int_u^{t_1} a(s) ds} \right| \rightarrow 0$ as $(t_2 - t_1) \rightarrow 0$, then by the Dominated Convergence Theorem,

$$\int_{t_2-T}^{t_1} [g_C(u) + \alpha q_C(u)] \left| e^{-\int_u^{t_2} a(s) ds} - e^{-\int_u^{t_1} a(s) ds} \right| du \rightarrow 0,$$

as $(t_2 - t_1) \rightarrow 0$. Thus $|A\varphi(t_2) - A\varphi(t_1)| \rightarrow 0$ as $(t_2 - t_1) \rightarrow 0$ independently of $\varphi \in \mathfrak{C}$. As such, the family of functions $A\varphi$ is equicontinuous on $\mathfrak{C}$. By the Arzelà-Ascoli Theorem, A is completely continuous and the proof is complete. $\square$

Proposition 1. *Let $\|\cdot\|$ be the supremum norm,*

$$\mathbb{M} = \{\varphi \in S : \|\varphi\| \leq \sqrt{3}/3\},$$

and define $(\mathfrak{B}\varphi)(t) := \varphi(t) - \varphi^3(t)$. Then $\mathfrak{B}$ is a large contraction of the set $\mathbb{M}$.

Proof. For each $t \in \mathbb{R}$ we have, for φ, ψ real functions,

$$\begin{aligned} |(\mathfrak{B}\varphi)(t) - (\mathfrak{B}\psi)(t)| &= |\varphi(t) - \varphi^3(t) - \psi(t) + \psi^3(t)| \\ &= |\varphi(t) - \psi(t)| |1 - (\varphi^2(t) + \varphi(t)\psi(t) + \psi^2(t))|. \end{aligned}$$

Then for

$$|\varphi(t) - \psi(t)|^2 = \varphi^2(t) - 2\varphi(t)\psi(t) + \psi^2(t) \leq 2(\varphi^2(t) + \psi^2(t)),$$

and for $\varphi^2(t) + \psi^2(t) < 1$, we have

$$\begin{aligned} |(\mathfrak{B}\varphi)(t) - (\mathfrak{B}\psi)(t)| &= |\varphi(t) - \psi(t)| \left[1 - (\varphi^2(t) + \psi^2(t)) + |\varphi(t)\psi(t)| \right] \\ &\leq |\varphi(t) - \psi(t)| \left[1 - (\varphi^2(t) + \psi^2(t)) + \frac{\varphi^2(t) + \psi^2(t)}{2} \right] \\ &\leq |\varphi(t) - \psi(t)| \left[1 - \frac{\varphi^2(t) + \psi^2(t)}{2} \right]. \end{aligned}$$

Thus, $\mathfrak{B}$ is pointwise a large contraction. But application $\mathfrak{B}$ is still a large contraction for the supremum norm. For, let $\varepsilon \in (0, 1)$ be given and let $\varphi, \psi \in \mathbb{M}$ with $\|\varphi - \psi\| \geq \varepsilon$.

(a) Suppose that for some t we have $\varepsilon/2 \leq |\varphi(t) - \psi(t)|$. Then

$$(\varepsilon/2)^2 \leq |\varphi(t) - \psi(t)|^2 \leq 2(\varphi^2(t) + \psi^2(t)),$$

that is

$$\varphi^2(t) + \psi^2(t) \geq \varepsilon^2/8.$$

For all such t we have

$$|(\mathfrak{B}\varphi)(t) - (\mathfrak{B}\psi)(t)| \leq |\varphi(t) - \psi(t)| \left[1 - \frac{\varepsilon^2}{16} \right] \leq \|\varphi - \psi\| \left[1 - \frac{\varepsilon^2}{16} \right].$$

(b) Suppose that for some t we have

$$|\varphi(t) - \psi(t)| \leq \varepsilon/2.$$

Then

$$|(\mathfrak{B}\varphi)(t) - (\mathfrak{B}\psi)(t)| \leq |\varphi(t) - \psi(t)| \leq (1/2)\|\varphi - \psi\|.$$

Consequently, we obtain

$$\|\mathfrak{B}\varphi - \mathfrak{B}\psi\| \leq \max \left[1/2, 1 - \frac{\varepsilon^2}{16} \right] \|\varphi - \psi\|$$

$$\leq \left(1 - \frac{\varepsilon^2}{16}\right) \|\varphi - \psi\|.$$

□

For the rest of the paper we define

$$\mathbb{M} = \{\varphi \in S : \|\varphi\| \leq L\},$$

where $L = \sqrt{3}/3$.

We need the following condition on the nonlinear term G .

(h5) There exist positive periodic functions $g_1, g_2, g_3 \in L^1[0, T]$, with period T , such that

$$|G(t, x, y)| \leq g_1(t)|x| + g_2(t)|y| + g_3(t),$$

for all $x, y \in \mathbb{R}$.

Lemma 3. Suppose that conditions (h4) and (h5) hold. Also suppose there exist constants $\beta, \delta > 0$ and $J \geq 3$ such that

$$[g_1(t) + g_2(t)]L^3 + g_3(t) \leq \beta L a(t), \quad (3.4)$$

$$q_1(t)L + q_2(t) \leq \delta L, \quad (3.5)$$

and

$$J(2\delta + \beta) \leq 1. \quad (3.6)$$

For A defined by (3.3), if $\varphi \in \mathbb{M}$, then $|(A\varphi)(t)| \leq L/J < L$ for all t .

Proof. Let $\varphi \in \mathbb{M}$. Then $\|\varphi\| \leq L$. Thus for A defined by (3.2) we have that

$$\begin{aligned} |(A\varphi)(t)| &\leq |Q(t, \varphi(g(t)))| + \eta \int_{t-T}^t |G(u, \varphi^3(u), \varphi^3(g(u)))| e^{-\int_u^t a(s)ds} du \\ &\quad + \eta \int_{t-T}^t a(u) |Q(u, \varphi(g(u)))| e^{-\int_u^t a(s)ds} du. \end{aligned}$$

It follows from conditions (h4), (h5), (3.4)–(3.6) that

$$\begin{aligned} |(A\varphi)(t)| &\leq q_1(t)L + q_2(t) \\ &\quad + \eta \int_{t-T}^t \{[g_1(u) + g_2(u)]L^3 + g_3(u)\} e^{-\int_u^t a(s)ds} du \\ &\quad + \eta \int_{t-T}^t a(u) \{q_1(u)L + q_2(u)\} e^{-\int_u^t a(s)ds} du \\ &\leq \delta L + \beta L \eta \int_{t-T}^t a(u) e^{-\int_u^t a(s)ds} du + \delta L \eta \int_{t-T}^t a(u) e^{-\int_u^t a(s)ds} du \\ &\leq (2\delta + \beta)L \leq \frac{L}{J} < L. \end{aligned}$$

Therefore A maps $\mathbb{M}$ into itself. □

Lemma 4. Suppose (h1) – (h5) and (3.4)–(3.6) hold. For A, B defined in (3.2) and (3.3), if $\varphi, \psi \in \mathbb{M}$ are arbitrary, then

$$A\varphi + B\psi : \mathbb{M} \rightarrow \mathbb{M}.$$

Moreover, B is a large contraction on $\mathbb{M}$ with a unique fixed point in $\mathbb{M}$.

Proof. Let $\varphi, \psi \in M$ be arbitrary. Note first that $|\psi(t)| \leq \sqrt{3}/3$ implies

$$|\psi(t) - \psi^3(t)| \leq (2\sqrt{3})/9.$$

Using the definition of B , and the result of Lemma 4, we obtain

$$\begin{aligned} |(A\varphi)(t) + (B\psi)(t)| &\leq |(A\varphi)(t)| + |(B\psi)(t)| \\ &\leq \frac{\sqrt{3}}{3J} + \frac{2\sqrt{3}}{9} \leq L. \end{aligned}$$

Thus, $A\varphi + B\psi \in \mathbb{M}$.

Left to show that B is a large contraction with a unique fixed point in $\mathbb{M}$. Proposition 1 shows that $\psi - \psi^3$ is a large contraction in the supremum norm. For any ε , from the proof of that proposition, we have found a $\lambda < 1$, such that

$$\begin{aligned} |(B\psi)(t) - (B\varphi)(t)| &\leq \eta \int_{t-T}^t a(u) \lambda \|\psi - \varphi\| e^{-\int_u^t a(s) ds} du \\ &\leq \lambda \|\psi - \varphi\|. \end{aligned}$$

Further, since $0 \in \mathbb{M}$ the above inequality shows, when $\varphi = 0$, we see that $B : \mathbb{M} \rightarrow \mathbb{M}$. This completes the proof. $\square$

Theorem 4. Let $(S, \|\cdot\|)$ be the Banach space of continuous T -periodic real functions and $\mathbb{M} = \{\varphi \in S : \|\varphi\| \leq L\}$, where $L = \sqrt{3}/3$. Suppose (h1) – (h5) and (3.4)–(3.6) hold. Then equation (1.3) possesses a T -periodic solution φ in the subset $\mathbb{M}$.

Proof. By Lemma 1, φ is a solution of (1.3) if

$$\varphi = A\varphi + B\varphi,$$

where A and B are given by (3.2), (3.3) respectively. By Lemma 2, $A : \mathbb{M} \rightarrow \mathbb{M}$ is completely continuous. By Lemma 4, $A\varphi + B\psi \in \mathbb{M}$ whenever $\varphi, \psi \in \mathbb{M}$. Moreover, $B : \mathbb{M} \rightarrow \mathbb{M}$ is a large contraction. Clearly, all the hypotheses of Theorem 3 of Krasnoselskii-Burton are satisfied. Thus, there exists a fixed point $\varphi \in \mathbb{M}$ such that $\varphi = A\varphi + B\varphi$. Hence (1.3) has a T -periodic solution in $\mathbb{M}$. $\square$

For our last result, we give sufficient conditions under which there exists positive solutions of equation (1.3). We begin by defining some new quantities. Let

$$m = \min_{u \in [t-T, t]} e^{-\int_u^t a(s) ds}, \quad M = \max_{u \in [t-T, t]} e^{-\int_u^t a(s) ds}.$$

Given constants $0 < \mathfrak{L} < \mathfrak{K} \leq \sqrt{3}/3$, define the set

$$\mathbb{M}_1 = \{\varphi \in P_T : \mathfrak{L} \leq \varphi(t) \leq \mathfrak{K}, t \in [0, T]\}.$$

Assume that the following conditions hold.

(h6) There exist constants $0 < q^* < L^*$ such that $q^* \mathfrak{L} \leq Q(t, \rho) \leq L^* \mathfrak{K}$ for all $\rho \in \mathbb{M}_1$ and $t \in [0, T]$.

(h7) There exist constants $0 < \mathfrak{L} < \mathfrak{K}$ such that

$$\frac{(1 - q^*) \mathfrak{L}}{\eta m T} \leq a(u) [\sigma - \sigma^3] + G(u, \sigma^3, \rho^3) - a(u) Q(u, \rho) \leq \frac{(1 - L^*) \mathfrak{K}}{\eta M T},$$

for all $\sigma, \rho \in \mathbb{M}_1$ and $u \in [t - T, t]$.

Theorem 5. Suppose that conditions (h1) – (h4), (h6) and (h7) hold. Then there exists a positive solution of (1.3).

Proof. As in the proof of Theorem 4, we just need to show that condition (iii) of Theorem 3 is satisfied. Let $\varphi, \psi \in \mathbb{M}_1$. Then

$$\begin{aligned} & (A\varphi)(t) + (B\psi)(t) \\ &= Q(t, \psi(g(t))) + \eta \int_{t-T}^t \{a(u) [\varphi(u) - \varphi^3(u)] \\ &+ G(u, \varphi^3(u), \varphi^3(g(u))) - a(u) Q(u, \varphi(g(u)))\} e^{-\int_u^t a(s) ds} du \\ &\geq q^* \mathfrak{L} + \eta m T \frac{(1 - q^*) \mathfrak{L}}{\eta m T} = \mathfrak{L}. \end{aligned}$$

Likewise

$$(A\varphi)(t) + (B\psi)(t) \leq L^* \mathfrak{K} + \eta M T \frac{(1 - L^*) \mathfrak{K}}{\eta M T} = \mathfrak{K}.$$

By Theorem 3, the operator H has a fixed point in $\mathbb{M}_1$. This fixed point is a positive solution of (1.3) and the proof is complete. $\square$

REFERENCES

- [1] A. Ardjouni and A. Djoudi, “Existence of periodic solutions for a second order nonlinear neutral differential equation with functional delay,” *Electronic Journal of Qualitative Theory of Differential Equation*, vol. 2012, no. 31, pp. 1–9, 2012.
- [2] A. Ardjouni and A. Djoudi, “Existence of periodic solutions for totally nonlinear neutral differential equations with variable delay,” *Sarajevo J. Math.*, vol. 8, no. 1, pp. 107–117, 2012.
- [3] A. Ardjouni and A. Djoudi, “Existence of positive periodic solutions for a nonlinear neutral differential equation with variable delay,” *Appl. Math. E-Notes*, vol. 12, pp. 94–101, 2012.
- [4] T. A. Burton, “Integral equations, implicit functions, and fixed points,” *Proc. Am. Math. Soc.*, vol. 124, no. 8, pp. 2383–2390, 1996.
- [5] T. A. Burton, “A fixed-point theorem of Krasnoselskii,” *Appl. Math. Lett.*, vol. 11, no. 1, pp. 85–88, 1998.
- [6] T. A. Burton, “Liapunov functionals, fixed points, and stability by Krasnoselskii’s theorem,” *Nonlinear Stud.*, vol. 9, no. 2, pp. 181–190, 2002.

- [7] T. A. Burton, *Stability by fixed point theory for functional differential equations*. Mineola: Dover Publications, 2006.
- [8] F. Chen, "Positive periodic solutions of neutral Lotka-Volterra system with feedback control," *Appl. Math. Comput.*, vol. 162, no. 3, pp. 1279–1302, 2005.
- [9] F. Chen and J. Shi, "Periodicity in a nonlinear predator-prey system with state dependent delays," *Acta Math. Appl. Sin., Engl. Ser.*, vol. 21, no. 1, pp. 49–60, 2005.
- [10] Y. M. Dib, M. R. Maroun, and Y. N. Raffoul, "Periodicity and stability in neutral nonlinear differential equations with functional delay," *Electron. J. Differ. Equ.*, vol. 2005, p. 11, 2005.
- [11] M. Fan, K. Wang, P. J. Y. Wong, and R. P. Agarwal, "Periodicity and stability in periodic n -species Lotka-Volterra competition system with feedback controls and deviating arguments," *Acta Math. Sin., Engl. Ser.*, vol. 19, no. 4, pp. 801–822, 2003.
- [12] D. Hafsia and D. AHCÉNE, "Existence of periodic solutions for neutral nonlinear differential equations with variable delay," *Electron. J. Differ. Equ.*, vol. 2010, p. 8, 2010.
- [13] J. Hale, *Theory of functional differential equations*. 2nd ed., ser. Applied Mathematical Sciences. New York - Heidelberg - Berlin: Springer-Verlag, 1977, vol. 3.
- [14] J. K. Hale and S. M. Verduyn Lunel, *Introduction to functional differential equations*, ser. Applied Mathematical Sciences. New York: Springer-Verlag, 1993, vol. 99.
- [15] E. R. Kaufmann, "A nonlinear neutral periodic differential equation," *Electron. J. Differ. Equ.*, vol. 2010, p. 8, 2010.
- [16] Y. Raffoul, "Positive periodic solutions in neutral nonlinear differential equations," *Electron. J. Qual. Theory Differ. Equ.*, vol. 2007, p. 10, 2007.
- [17] D. R. Smart, *Fixed point theorems*, ser. Cambridge Tracts in Mathematics. London: Cambridge University Press, 1974, vol. 66.
- [18] E. Yankson, "Existence and positivity of solutions for a nonlinear periodic differential equation," *Arch. Math., Brno*, vol. 48, no. 4, pp. 261–270, 2012.

Author's address

Abdelouaheb Ardjouni, AHCENE Djoudi

Abdelouaheb Ardjouni, AHCENE Djoudi, Department of Mathematics, Faculty of Sciences, University of Annaba, P.O. Box 12 Annaba, Algeria

E-mail address: abd_ardjouni@yahoo.fr, adjoudi@yahoo.com



FIXED POINTS OF A PAIR OF KANNAN TYPE MAPPINGS ON A CLOSED BALL IN ORDERED PARTIAL METRIC SPACES

MUHAMMAD ARSHAD, ABDULLAH SHOAIB, MUJAHID ABBAS,
AND AKBAR AZAM

Received 21 March, 2013

Abstract. Unique common fixed point results for mappings satisfying locally contractive conditions on a closed ball in a complete ordered partial metric space have been established. The notion of dominated mappings of Economics, Finance, Trade and Industry is also been applied to approximate the unique solution to non linear functional equations. Our results improve some well-known, primary and conventional results.

2010 Mathematics Subject Classification: 46S40; 47H10; 54H25

Keywords: unique common fixed point, Kannan mapping, closed ball, dominated mapping, partial metric spaces

1. INTRODUCTION AND PRELIMINARIES

Let $T : X \longrightarrow X$ be a mapping. A point $x \in X$ is called a fixed point of T if $x = Tx$. Let x_0 be an arbitrarily chosen point in X . Define a sequence $\{x_n\}$ in X by a simple iterative method given by

$$x_{n+1} = Tx_n, \text{ where } n \in \{0, 1, 2, \dots\}.$$

Such a sequence is called a Picard iterative sequence and its convergence plays a very important role in proving an existence of a fixed point of a mapping T . A self mapping T on a metric space X is said to be a Banach contraction mapping if

$$d(Tx, Ty) \leq kd(x, y)$$

holds for all $x, y \in X$ where $0 \leq k < 1$. In 1969, Kannan obtained unique fixed point of a mapping $T : X \longrightarrow X$ satisfying:

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)] \quad (1.1)$$

for all $x, y \in X$ where $0 \leq k < \frac{1}{2}$ and X is a complete metric space. The mapping satisfying contractive condition 1.1 is called a Kannan mapping. It is observed that contractive condition 1.1 does not imply continuity of a mapping T on its domain, which differentiates its nature from Banach contraction mapping. It is important to

note that Kannan fixed point result has laid down the foundation of modern fixed points theory for contractive type mappings.

Fixed points results of mappings satisfying certain contractive conditions on the entire domain has been at the centre of rigorous research activity, for example (see [6–8, 11, 12]) and it has a wide range of applications in different areas such as non-linear and adaptive control systems, parameterize estimation problems, fractal image decoding, computing magnetostatic fields in a nonlinear medium, and convergence of recurrent networks, (see [15, 18, 20, 23]).

From the application point of view the situation is not yet completely satisfactory because it frequently happens that a mapping T is a contraction not on the entire space X but merely on a subset Y of X . However, if Y is closed and a Picard iterative sequence $\{x_n\}$ in X converges to some x in X then by imposing a subtle restriction on the choice of x_0 , one may force the Picard iterative sequence to stay eventually in Y . In this case, closedness of Y coupled with some suitable contractive condition establish the existence of a fixed point of T . Recently, Arshad et. al. [5] proved a significant result concerning the existence of common fixed point of mappings satisfying a contractive condition on a closed ball in a complete dislocated metric space. Other results on closed ball can be seen in [9, 10, 17, 22].

On the other hand, the notion of a partial metric space was introduced by Matthews in [19]. In partial metric spaces, the distance of a point from itself may not be zero. After the definition of partial metric space, Matthews proved the partial metric version of Banach fixed point theorem. Then, Oltra et al. [21] and Altun et al. [3] gave some generalizations of the result of Matthews (see also [4], [2], [13] and references mentioned therein). The dominated mapping, which satisfies the condition $fx \preceq x$, occurs very naturally in several practical problems. For example if x denotes the total quantity of food produced over a certain period of time and $f(x)$ gives the quantity of food consumed over the same period in a certain town, then we must have $fx \preceq x$. In this paper, we will exploit this concept for Kannan mappings [16] to generalize, extend and improve some classical fixed point results for two, three and four mappings in the framework of complete ordered partial metric space X . Our results not only extend some primary results to ordered partial metric spaces but also restrict the contractive conditions on a closed ball only. The concept of dominated mapping has been applied to approximate the unique solution of non linear functional equations. We have used weaker contractive condition and weaker restrictions to obtain unique fixed point. Our results do not exist even yet in metric spaces. Consistent with [1, 2, 6, 14] and [19], the following definitions and results will be needed in the sequel.

Definition 1 ([19]). Let $p : X \times X \rightarrow R^+$ be a function, where X is a nonempty set. Then p is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold.

(P₁) $p(x, x) = p(y, y) = p(x, y)$ if and only if $x = y$,

- (P₂) $p(x, x) \leq p(x, y)$,
 (P₃) $p(x, y) = p(y, x)$,
 (P₄) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) is then called a partial metric space.

Each partial metric p on X induces a T_0 topology p on X which has as a base the family of open balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Also $\overline{B(x_0, \varepsilon)} = \{y \in X : p(x, y) \leq p(x, x) + \varepsilon\}$ is a closed ball in (X, p) .

It is clear that if $p(x, y) = 0$, then from P_1 and P_2 , $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic example of a partial metric space is the pair (R^+, p) , where $p(x, y) = \max\{x, y\}$ for all $x, y \in R^+$.

Example 1 ([19]). If $X = \{[a, b] : a, b \in R, a \leq b\}$ then $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$ defines a partial metric p on X .

Definition 2 ([19]). Let (X, p) be a partial metric space, then we have the following.

A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ if and only if

$$\lim_{n \rightarrow \infty} p(x_n, x) = p(x, x).$$

A sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$ exists and is finite.

(X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X , there exist some $z \in X$ such that,

$$\lim_{n, m \rightarrow \infty} p(x_n, x_m) = \lim_{n \rightarrow \infty} p(x_n, z) = p(z, z).$$

If (X, p) is a partial metric space, then $p_s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, $x, y \in X$, is a metric on X .

Lemma 1 ([19]). Let (X, p) be a partial metric space.

$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p_s) .

(X, p) is complete if and only if the metric space (X, p_s) is complete. Furthermore, $\lim_{n \rightarrow \infty} p_s(x_n, z) = 0$ if and only if

$$p(z, z) = \lim_{n \rightarrow \infty} p(x_n, z) = \lim_{n, m \rightarrow \infty} p(x_n, x_m).$$

Definition 3 ([2]). Let X be a nonempty set. Then $(X, \preceq, p)$ is called an ordered partial metric space if:

- (i) p is a partial metric on X and (ii) $\preceq$ is a partial order on X .

Definition 4. Let $(X, \preceq)$ be a partial ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. We define $\nabla = \{(x, y) \in X \times X \mid x \text{ and } y \text{ are comparable}\}$.

Definition 5 ([1]). Let $(X, \leq)$ be a partially ordered set. A self mapping f on X is called dominated if $fx \leq x$ for each x in X .

Example 2 ([1]). Let $X = [0, 1]$ be endowed with the usual ordering and $f : X \rightarrow X$ be defined by $fx = x^n$ for some $n \in \mathbb{N}$. Since $fx = x^n \leq x$ for all $x \in X$, therefore f is a dominated map.

Definition 6. Let X be a non empty set and $T, f : X \rightarrow X$. A point $y \in X$ is called point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$. The mappings T, f are said to be weakly compatible if they commute at their coincidence point (i. e. $Tfx = fTx$ whenever $Tx = fx$). We require the following lemmas:

Lemma 2 ([14]). Let X be a non empty set and $f : X \rightarrow X$ a function. Then there exists a subset $E \subset X$ such that $fE = fX$ and $f : E \rightarrow X$ is one to one.

Lemma 3 ([6]). Let X be a non empty set and the mappings $S, T, f : X \rightarrow X$ have a unique point of coincidence v in X . If (S, f) and (T, f) are weakly compatible, then S, T, f have a unique common fixed point.

2. FIXED POINTS OF KANNAN MAPPINGS

In the following we present common fixed point theorems for the Kannan mappings on a closed ball in an ordered partial metric space.

Theorem 1. Let $(X, \leq, p)$ be a complete ordered partial metric space, $x_0, x, y \in X$, $r > 0$ and $S, T : X \rightarrow X$ be two dominated mappings. Suppose that there exists $t \in [0, \frac{1}{2})$ such that following conditions hold:

$$p(Sx, Ty) \leq t[p(x, Sx) + p(y, Ty)], \text{ for all } (x, y) \text{ in } (\overline{B(x_0, r)} \times \overline{B(x_0, r)}) \cap \nabla, \quad (2.1)$$

and

$$p(x_0, Sx_0) \leq (1 - \lambda)[r + p(x_0, x_0)], \quad (2.2)$$

where $\lambda = \frac{t}{1-t}$. Then there exists a point x^* such that $p(x^*, x^*) = 0$. Also if, for a nonincreasing sequence $\{x_n\}$ in $\overline{B(x_0, r)}$, $\{x_n\} \rightarrow u$ implies that $u \leq x_n$, then $x^* = Sx^* = Tx^*$. Moreover, x^* is unique, if for any two points x, y in $\overline{B(x_0, r)}$ there exists a point $z_0 \in \overline{B(x_0, r)}$ such that $z_0 \leq x$ and $z_0 \leq y$ and

$$p(x_0, Sx_0) + p(z, Tz) \leq p(x_0, z) + p(Sx_0, Tz) \quad (2.3)$$

for all $z \in \overline{B(x_0, r)}$ such that $z \leq Sx_0$.

Proof. Choose a point x_1 in X such that $x_1 = Sx_0$. As $Sx_0 \leq x_0$ so $(x_1, x_0) \in \nabla$ and let $x_2 = Tx_1$. Now $Tx_1 \leq x_1$ gives $(x_2, x_1) \in \nabla$. Continuing this process and having chosen x_n in X such that

$$x_{2i+1} = Sx_{2i} \text{ and } x_{2i+2} = Tx_{2i+1}, \text{ where } i = 0, 1, 2, \dots$$

we obtain $(x_{n+1}, x_n) \in \nabla$ for all $n \in N$. We will prove that $x_n \in \overline{B(x_0, r)}$ for all $n \in N$ by mathematical induction. By inequality 2.2

$$\begin{aligned} p(x_0, Sx_0) &\leq (1-\lambda)[r + p(x_0, x_0)] \\ &\leq r + p(x_0, x_0). \end{aligned}$$

It follows that $x_1 \in \overline{B(x_0, r)}$. Let $x_2, \dots, x_j \in \overline{B(x_0, r)}$ for some $j \in N$. If $j = 2i + 1$, then $(x_{2i+1}, x_{2i}) \in (\overline{B(x_0, r)} \times \overline{B(x_0, r)}) \cap \nabla$, where $i = 0, 1, 2, \dots, \frac{j-1}{2}$ so using inequality 2.1, we obtain

$$\begin{aligned} p(x_{2i+1}, x_{2i+2}) &= p(Sx_{2i}, Tx_{2i+1}) \\ &\leq t[p(x_{2i}, Sx_{2i}) + p(x_{2i+1}, Tx_{2i+1})] \end{aligned}$$

which implies that

$$\begin{aligned} p(x_{2i+1}, x_{2i+2}) &\leq \lambda p(x_{2i}, x_{2i+1}) \\ &\leq \lambda^2 p(x_{2i-1}, x_{2i}) \leq \dots \leq \lambda^{2i+1} p(x_0, x_1). \end{aligned} \quad (2.4)$$

If $j = 2i + 2$, then as $x_1, x_2, \dots, x_j \in \overline{B(x_0, r)}$ and $(x_{2i+2}, x_{2i+1}) \in (\overline{B(x_0, r)} \times \overline{B(x_0, r)}) \cap \nabla$, ($i = 0, 1, 2, \dots, \frac{j-2}{2}$). We obtain,

$$p(x_{2i+2}, x_{2i+3}) \leq \lambda^{2i+2} p(x_0, x_1). \quad (2.5)$$

Thus from inequality 2.4 and 2.5, we have

$$p(x_j, x_{j+1}) \leq \lambda^j p(x_0, x_1) \text{ for some } j \in N. \quad (2.6)$$

Now

$$\begin{aligned} p(x_0, x_{j+1}) &\leq p(x_0, x_1) + \dots + p(x_j, x_{j+1}) - [p(x_1, x_1) + \dots + p(x_j, x_j)] \\ &\leq p(x_0, x_1) + \dots + \lambda^j p(x_0, x_1) \quad (\text{by 2.6}) \\ &= p(x_0, x_1)[1 + \dots + \lambda^{j-1} + \lambda^j] \\ &\leq (1-\lambda)[r + p(x_0, x_0)] \frac{(1-\lambda^{j+1})}{1-\lambda} \\ &\leq r + p(x_0, x_0) \end{aligned}$$

gives $x_{j+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$ for all $n \in N$. Also $(x_{n+1}, x_n) \in (\overline{B(x_0, r)} \times \overline{B(x_0, r)}) \cap \nabla$ for all $n \in N$. It implies that

$$p(x_n, x_{n+1}) \leq \lambda^n p(x_0, x_1), \text{ for all } n \in N. \quad (2.7)$$

Also,

$$\begin{aligned} p(x_n, x_n) &\leq p(x_n, x_{n+1}) \\ &\leq \lambda^n p(x_0, x_1) \longrightarrow 0 \text{ as } n \rightarrow \infty \end{aligned} \quad (2.8)$$

Moreover,

$$\begin{aligned} p_s(x_{n+1}, x_n) &= 2p(x_n, x_{n+1}) - p(x_n, x_n) - p(x_{n+1}, x_{n+1}) \\ &\leq 2p(x_n, x_{n+1}) \\ &\leq 2\lambda^n p(x_0, x_1). \quad (\text{by 2.7}) \end{aligned} \quad (2.9)$$

So we have,

$$\begin{aligned} p_s(x_{n+k}, x_n) &\leq p_s(x_{n+k}, x_{n+k-1}) + \dots + p_s(x_{n+1}, x_n) \\ &\leq 2\lambda^{n+k-1} p(x_0, x_1) + \dots + 2\lambda^n p(x_0, x_1) \quad \text{by 2.9} \\ &= 2\lambda^n p(x_0, x_1) [\lambda^{k-1} + \lambda^{k-2} + \dots + 1] \\ &\leq 2\lambda^n p(x_0, x_1) \frac{(1 - \lambda^k)}{1 - \lambda} \longrightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence the sequence $\{x_n\}$ is a Cauchy sequence in $(\overline{B(x_0, r)}, p_s)$. By Lemma 1, $\{x_n\}$ is a Cauchy sequence in $(\overline{B(x_0, r)}, p)$. Therefore there exists a point $x^* \in \overline{B(x_0, r)}$ with $\lim_{n \rightarrow \infty} x_n = x^*$. Also $\lim_{n \rightarrow \infty} p_s(x_n, x^*) = 0$. Now by Lemma 1 and inequality 2.8 we have

$$p(x^*, x^*) = \lim_{n \rightarrow \infty} p(x_n, x^*) = \lim_{n \rightarrow \infty} p(x_n, x_n) = 0. \quad (2.10)$$

Moreover by the given assumption $x_n \rightarrow x^*$ as $n \rightarrow \infty$ implies that $(x^*, x_n) \in \nabla$. Also, $(x^*, x_{2n+1}) \in (\overline{B(x_0, r)} \times \overline{B(x_0, r)}) \cap \nabla$. Then,

$$\begin{aligned} p(x^*, Sx^*) &\leq p(x^*, x_{2n+2}) + p(x_{2n+2}, Sx^*) - p(x_{2n+2}, x_{2n+2}) \\ &\leq p(x^*, x_{2n+2}) + t[p(x_{2n+1}, Tx_{2n+1}) + p(x^*, Sx^*)], \\ (1-t)p(x^*, Sx^*) &\leq p(x^*, x_{2n+2}) + tp(x_{2n+1}, Tx_{2n+1}), \\ p(x^*, Sx^*) &\leq \frac{1}{1-t} p(x^*, x_{2n+2}) + \lambda p(x_{2n+1}, x_{2n+2}). \end{aligned}$$

On taking limit $n \rightarrow \infty$ and by using inequality 2.7 and 2.10 we obtain,

$$p(x^*, Sx^*) \leq 0,$$

and $x^* = Sx^*$. Similarly, by using

$$p(x^*, Tx^*) \leq p(x^*, x_{2n+1}) + p(x_{2n+1}, Tx^*) - p(x_{2n+1}, x_{2n+1}),$$

we can show that $x^* = Tx^*$. Hence S and T have a common fixed point in $\overline{B(x_0, r)}$. Let y be another point in $\overline{B(x_0, r)}$ such that $y = Sy = Ty$. If $x^* \leq y$, then

$$\begin{aligned} p(x^*, y) &= p(Sx^*, Ty) \\ &\leq t[p(x^*, Sx^*) + p(y, Ty)] \\ &= t[p(x^*, x^*) + p(y, y)] \\ &\leq tp(y, y) \quad (\text{by 2.10}) \end{aligned}$$

$$< p(y, y).$$

Using the fact that $\underline{p(y, y)} \leq p(x^*, y)$, we have $x^* = y$. Now if $x^* \neq y$, then there exists a point $z_0 \in \overline{B(x_0, r)}$ such that $(z_0, x^*) \in \nabla$ and $(z_0, y) \in \nabla$. Choose a point z_1 in X such that $z_1 = Tz_0$. As $Tz_0 \preceq z_0$ so $(z_1, z_0) \in \nabla$. Let $z_2 = Sz_1 \preceq z_1$ gives $(z_2, z_1) \in \nabla$. Continuing this process and having chosen z_n in X such that

$$z_{2i+1} = Tz_{2i}, z_{2i+2} = Sz_{2i+1} \text{ and } z_{2i+1} = Tz_{2i} \preceq z_{2i} \text{ where } i = 0, 1, 2, \dots$$

It follows that $z_{n+1} \preceq z_n \preceq \dots \preceq z_0 \preceq x^* \preceq x_n \dots \preceq x_0$. We will prove that $z_n \in \overline{B(x_0, r)}$ for all $n \in N$ by using mathematical induction. For $n = 1$. Now $(x_0, z_0) \in \overline{B(x_0, r)} \times \overline{B(x_0, r)} \cap \nabla$

$$\begin{aligned} p(Sx_0, Tz_0) &\leq t[p(x_0, x_1) + p(z_0, Tz_0)] \\ &\leq t[p(x_0, z_0) + p(x_1, Tz_0)] \quad (\text{by 2.3}) \\ &\leq \lambda p(x_0, z_0) \text{ and} \end{aligned} \tag{2.11}$$

$$\begin{aligned} p(x_0, z_1) &\leq p(x_0, x_1) + p(x_1, z_1) - p(x_1, x_1) \\ &\leq (1 - \lambda)[r + p(x_0, x_0)] + \lambda p(x_0, z_0) \quad (\text{by 2.2 and 2.11}) \\ &\leq (1 - \lambda)r + (1 - \lambda)p(x_0, x_0) + \lambda[r + p(x_0, x_0)] \quad (\text{as } z_0 \in \overline{B(x_0, r)}) \\ &\leq r + p(x_0, x_0) \end{aligned}$$

implies that $z_1 \in \overline{B(x_0, r)}$. Let $z_2, z_3, \dots, z_j \in \overline{B(x_0, r)}$ for some $j \in N$. Following similar arguments as we have used to prove inequality 2.6, we have

$$p(z_j, z_{j+1}) \leq \lambda^j p(z_0, z_1) \text{ for some } j \in N. \tag{2.12}$$

Note that if j is odd then, we have

$$\begin{aligned} p(x_{j+1}, z_{j+1}) &= p(Tx_j, Sz_j) \leq t[p(x_j, Tx_j) + p(z_j, Sz_j)] \\ &\leq t[\lambda^j p(x_0, x_1) + \lambda^j p(z_0, z_1)] \quad (\text{by 2.6 and 2.12}) \\ &\leq t\lambda^j [p(x_0, z_0) + p(x_1, z_1)] \quad (\text{by 2.3}) \\ &\leq t\lambda^j [p(x_0, z_0) + \lambda p(x_0, z_0)] \\ &= \lambda^{j+1} p(x_0, z_0). \end{aligned} \tag{2.13}$$

Similarly, if j is even then, we obtain

$$p(x_{j+1}, z_{j+1}) \leq \lambda^{j+1} p(x_0, z_0). \tag{2.14}$$

Now,

$$\begin{aligned} p(x_0, z_{j+1}) &\leq p(x_0, x_1) + p(x_1, x_2) + \dots + p(x_{j+1}, z_{j+1}) \\ &\leq p(x_0, x_1) + \lambda p(x_0, x_1) + \dots + \lambda^{j+1} p(x_0, z_0) \quad (\text{by 2.13 and 2.14}) \\ &\leq p(x_0, x_1)[1 + \lambda + \dots + \lambda^j] + \lambda^{j+1}[r + p(x_0, x_0)] \end{aligned}$$

$$\begin{aligned} &\leq (1-\lambda)[r + p(x_0, x_0)] \frac{(1-\lambda^{j+1})}{1-\lambda} + \lambda^{j+1}r + \lambda^{j+1}p(x_0, x_0) \\ &= r + p(x_0, x_0) \end{aligned}$$

gives $z_{j+1} \in \overline{B(x_0, r)}$. Hence $z_n \in \overline{B(x_0, r)}$ for all $n \in N$. Now inequality 2.12 can be written as

$$p(z_n, z_{n+1}) \leq \lambda^n p(z_0, z_1) \text{ for all } n \in N. \quad (2.15)$$

As $(z_0, x^*), (z_0, y) \in (\overline{B(x_0, r)} \times \overline{B(x_0, r)}) \cap \nabla$, so it follows that (z_n, x^*) and (z_n, y) are in $(\overline{B(x_0, r)} \times \overline{B(x_0, r)}) \cap \nabla$ for all $n \in N$. Then, for $i \in N$,

$$\begin{aligned} p(x^*, y) &= p(Tx^*, Ty) \\ &\leq p(Tx^*, Sz_{2i-1}) + p(Sz_{2i-1}, Ty) - p(Sz_{2i-1}, Sz_{2i-1}) \\ &\leq tp(x^*, x^*) + 2tp(z_{2i-1}, z_{2i}) + tp(y, y) \\ &\leq 2t\lambda^{2i-1}d_l(z_0, z_1) + tp(y, y). \quad (\text{by 2.10 and 2.15}) \end{aligned}$$

On taking limit as $i \rightarrow \infty$, we obtain,

$$p(x^*, y) \leq tp(y, y) < p(y, y),$$

a contradiction, so $x^* = y$. Hence x^* is a unique common fixed point of T and S in $\overline{B(x_0, r)}$. $\square$

Example 3. Let $X = R^+ \cup \{0\}$ be endowed with order $x \preceq y$ if $p(x, x) \leq p(y, y)$ and let $p : X \times X \rightarrow R^+ \cup \{0\}$ be the complete ordered partial metric on X defined by $p(x, y) = \max\{x, y\}$.

$$Sx = \begin{cases} \frac{x}{17} & \text{if } x \in [0, 1] \\ x - \frac{1}{5} & \text{if } x \in (1, \infty) \end{cases}$$

and

$$Tx = \begin{cases} \frac{3x}{17} & \text{if } x \in [0, 1] \\ x - \frac{1}{6} & \text{if } x \in (1, \infty). \end{cases}$$

Clearly, S and T are dominated mappings. Take, $t = \frac{3}{10} \in [0, \frac{1}{2})$, $x_0 = \frac{1}{2}$, $r = \frac{1}{2}$, we have $p(x_0, x_0) = \max\{\frac{1}{2}, \frac{1}{2}\} = \frac{1}{2}$, $\lambda = \frac{t}{1-t} = \frac{3}{7}$ and $\overline{B(x_0, r)} = [0, 1]$. Also

$$\begin{aligned} (1-\lambda)[r + p(x_0, x_0)] &= (1-\frac{3}{7})[\frac{1}{2} + \frac{1}{2}] = \frac{4}{7} \\ p(x_0, Sx_0) &= p(\frac{1}{2}, S\frac{1}{2}) = p(\frac{1}{2}, \frac{1}{17}) = \max\{\frac{1}{2}, \frac{1}{17}\} = \frac{1}{2} < \frac{4}{7} \\ \text{Also if } x, y &\in (1, \infty), \quad p(Sx, Ty) = \max\{x - \frac{1}{5}, y - \frac{1}{6}\} \end{aligned}$$

$$\begin{aligned}
&\geq \frac{3}{10}[x+y] \\
&= \frac{3}{10}[\max\{x, x - \frac{1}{5}\} + \max\{y, y - \frac{1}{6}\}] \\
p(Sx, Ty) &\geq t[p(x, Sx) + p(y, Ty)].
\end{aligned}$$

So the contractive condition does not hold on X . Now if $x, y \in \overline{B(x_0, r)}$, then

$$\begin{aligned}
p(Sx, Ty) &= \max\{\frac{x}{17}, \frac{3y}{17}\} = \frac{1}{17} \max\{x, 3y\} \\
&\leq \frac{3}{10}[x+y] \\
&= \frac{3}{10}[\max\{x, \frac{x}{17}\} + \max\{y, \frac{3y}{17}\}] \\
&= t[p(x, Sx) + p(y, Ty)].
\end{aligned}$$

Also, for all $z \in \overline{B(x_0, r)}$ such that $z \preceq Sx_0$, then,

$$p(x_0, Sx_0) + p(z, Tz) \leq p(x_0, z) + p(Sx_0, Tz)$$

Therefore, all the conditions of Theorem 1 are satisfied. Moreover, 0 is the unique common fixed point of S and T .

Corollary 1. Let $(X, \preceq, p)$ be a complete ordered partial metric space, $x_0, x, y \in X$, $r > 0$ and $S : X \rightarrow X$ be a dominated mapping. Suppose that there exists $t \in [0, \frac{1}{2})$ such that following conditions hold:

$$p(Sx, Sy) \leq t[p(x, Sx) + p(y, Sy)], \text{ for all } (x, y) \text{ in } (\overline{B(x_0, r)} \times \overline{B(x_0, r)}) \cap \nabla,$$

and

$$p(x_0, Sx_0) \leq (1 - \lambda)[r + p(x_0, x_0)],$$

where $\lambda = \frac{t}{1-t}$. Then there exists a point x^* such that $p(x^*, x^*) = 0$. Also if, for a nonincreasing sequence $\{x_n\}$ in $\overline{B(x_0, r)}$, $\{x_n\} \rightarrow u$ implies that $u \preceq x_n$, then $x^* = Sx^*$. Moreover, x^* is unique, if for any two points x, y in $\overline{B(x_0, r)}$ there exists a point $z_0 \in \overline{B(x_0, r)}$ such that $z_0 \preceq x$ and $z_0 \preceq y$ and

$$p(x_0, Sx_0) + p(z, Sz) \leq p(x_0, z) + p(Sx_0, Sz)$$

for all $z \in \overline{B(x_0, r)}$ such that $z \preceq Sx_0$.

Proof. In Theorem 1 take $T = S$ to get unique point $x^* \in \overline{B(x_0, r)}$ such that $x^* = Sx^*$. $\square$

In Theorem 1, the conditions 2.3 and 2.2 are imposed to restrict the condition 2.1 only for x, y in $\overline{B(x_0, r)}$ and Example 3 explains the utility of these restrictions. However, the following result relax the conditions 2.2 and 2.3 but impose the condition 2.1 for all comparable elements in the whole space X .

Theorem 2. Let $(X, \leq, p)$ be a complete ordered partial metric space and $S, T : X \rightarrow X$ be two dominated mappings. Suppose that there exists $t \in [0, \frac{1}{2})$ such that following condition holds for $x, y \in X$,

$$p(Sx, Ty) \leq t[p(x, Sx) + p(y, Ty)], \text{ for all } (x, y) \text{ in } \nabla.$$

Then there exists a point x^* such that $p(x^*, x^*) = 0$. Also if, for a nonincreasing sequence $\{x_n\}$ in X , $\{x_n\} \rightarrow u$ implies that $u \leq x_n$, then $x^* = Sx^* = Tx^*$. Moreover, x^* is unique, if for any two points x, y in X there exists a point $z_0 \in X$ such that $z_0 \leq x$ and $z_0 \leq y$.

In Theorem 1, the condition “for a nonincreasing sequence $\{x_n\} \rightarrow u$ implies that $u \leq x_n$ ”, the existence of z_0 and the condition 2.3 are imposed to restrict the condition 2.1 only for comparable elements. However, the following result relax these restrictions but impose the condition 2.1 for all elements in $\overline{B}(x_0, r)$.

Theorem 3. Let (X, p) be a complete partial metric space, $x_0, x, y \in X$, $r > 0$ and $S, T : X \rightarrow X$ be two mappings. Suppose that there exists $t \in [0, \frac{1}{2})$ such that following conditions hold

$$p(Sx, Ty) \leq t[p(x, Sx) + p(y, Ty)], \text{ for all } x, y \text{ in } \overline{B}(x_0, r).$$

$$p(x_0, Sx_0) \leq (1 - \lambda)[r + p(x_0, x_0)].$$

where $\lambda = \frac{t}{1-t}$. Then there exists a unique point x^* in $\overline{B}(x_0, r)$ such that $x^* = Sx^* = Tx^*$. Also $p(x^*, x^*) = 0$. Further S and T have no fixed point other than x^* .

Proof. By following the similar steps of Theorem 1, we can obtain $x^* = Sx^* = Tx^*$. Let $y = Ty$. Then y is the fixed point of T and it may not be the fixed point of S . Then,

$$\begin{aligned} p(x^*, y) &= p(Sx^*, Ty) \leq t[p(x^*, Sx^*) + p(y, Ty)] \\ &= t[p(x^*, x^*) + p(y, y)] \leq tp(y, y) \quad (\text{by 2.10}) \end{aligned}$$

This shows that $x^* = y$. Thus T has no fixed point other than x^* . Similarly S has no fixed point other than x^* . $\square$

Now we apply our Theorem 1 to obtain unique common fixed point of three mappings on a closed ball in complete partial ordered metric space. The technique and style of our proof is quite smart and innovative.

Theorem 4. Let $(X, \leq, p)$ be a ordered partial metric space, $x_0, x, y \in X$, $r > 0$ and S, T self mapping and f be a dominated mapping on X such that $SX \cup TX \subset fX$, $\overline{B}(fx_0, r) \subseteq fX$ and $(Tx, fx), (Sx, fx) \in \nabla$. Assume that the following conditions hold:

$$p(Sx, Ty) \leq k[p(fx, Sx) + p(fy, Ty)] \quad (2.16)$$

for all $(fx, fy) \in (\overline{B(fx_0, r)} \times \overline{B(fx_0, r)}) \cap \nabla$; where $0 \leq k < 1/2$,

$$p(fx_0, Sx_0) + p(fy, Ty) \leq p(fx_0, fy) + p(Sx_0, Ty) \quad (2.17)$$

for all $fy \in \overline{B(fx_0, r)}$ such that $fy \preceq Sx_0$,

$$p(fx_0, Tx_0) \leq (1 - \lambda)[r + p(fx_0, fx_0)] \quad (2.18)$$

where $\lambda = \frac{k}{1-k}$. If, for a nonincreasing sequence $\{x_n\}$ in $\overline{B(fx_0, r)}$, $\{x_n\} \rightarrow u$ implies that $u \preceq x_n$ and for any two points z and x in $\overline{B(fx_0, r)}$ there exists a point $y \in \overline{B(fx_0, r)}$ such that $y \preceq z$ and $y \preceq x$. If fX is complete subspace of X and (S, f) and (T, f) are weakly compatible, then S, T and f have a unique common fixed point fz in $\overline{B(fx_0, r)}$. Also $p(fz, fz) = 0$.

Proof. By Lemma 2, there exists $E \subset X$ such that $fE = fX$ and $f : E \rightarrow X$ is one-to-one. Now since $SX \cup TX \subset fX$, we define two mappings $g, h : fE \rightarrow fE$ by $g(fx) = Sx$ and $h(fx) = Tx$ respectively. Since f is one-to-one on E , then g, h are well-defined. As $(Sx, fx) \in \nabla$ implies that $(g(fx), fx) \in \nabla$ and $(Tx, fx) \in \nabla$ implies that $(h(fx), fx) \in \nabla$ therefore g and h are dominated maps. Now $fx_0 \in \overline{B(fx_0, r)} \subseteq fX$, then $fx_0 \in fX$. Let $y_0 = fx_0$, choose a point y_1 in fX such that $y_1 = h(y_0)$. As $h(y_0) \preceq y_0$, so $(y_1, y_0) \in \nabla$ and let $y_2 = g(y_1)$. Now $g(y_1) \preceq y_1$ gives $(y_2, y_1) \in \nabla$. Continuing this process and having chosen y_n in fX such that

$$y_{2i+1} = h(y_{2i}) \text{ and } y_{2i+2} = g(y_{2i+1}), \text{ where } i = 0, 1, 2, \dots,$$

then $y_{2i+1} = h(y_{2i}) \preceq y_{2i}$ implies $(y_{2i+1}, y_{2i}) \in \nabla$. Following similar arguments of Theorem 2, $y_n \in \overline{B(fx_0, r)}$. Also by inequality 2.17, we obtain for all $(fy, g(fx_0)) \in (\overline{B(fx_0, r)} \times \overline{B(fx_0, r)}) \cap \nabla$,

$$p(fx_0, g(fx_0)) + p(fy, h(fy)) \leq p(fx_0, fy) + p(g(fx_0), h(fy))$$

and by inequality 2.18, we have,

$$p(fx_0, h(fx_0)) \leq (1 - \lambda)[r + p(fx_0, fx_0)].$$

By using inequality 2.16, $(fx, fy) \in (\overline{B(fx_0, r)} \times \overline{B(fx_0, r)}) \cap \nabla$, we have,

$$p(g(fx), h(fy)) \leq k[p(fx, g(fx)) + p(fy, h(fy))]$$

As fX is a complete space, all conditions of Theorem 2 are satisfied, we deduce that there exists a unique common fixed point $fz \in \overline{B(fx_0, r)}$ of g and h . Also $p(fz, fz) = 0$. Now $fz = g(fz) = h(fz)$ or $fz = Sz = Tz = fz$. Thus fz is the point of coincidence of S, T and f . Let $v \in \overline{B(fx_0, r)}$ be another point of coincidence of f, S and T then there exist $u \in \overline{B(fx_0, r)}$ such that is $v = fu = Su = Tu$, which implies that $fu = g(fu) = h(fu)$. A contradiction as $fz \in \overline{B(fx_0, r)}$ is a unique common fixed point of g and h . Hence $v = fz$. Thus S, T and f have a unique point of coincidence $fz \in \overline{B(fx_0, r)}$. Now since (S, f) and (T, f) are weakly compatible, by Lemma 3 fz is a unique common fixed point of S, T and f . $\square$

Now we can apply our Theorem 2 to obtain unique common fixed point result of three mappings in complete partial ordered metric space.

Theorem 5. Let $(X, \preceq, p)$ be a ordered partial metric space, $x, y \in X$ and S, T self mapping and f be a dominated mapping on X such that $SX \cup TX \subset fX$ and $(Tx, fx), (Sx, fx) \in \nabla$. Assume that the following conditions hold:

$$p(Sx, Ty) \leq k[p(fx, Sx) + p(fy, Ty)]$$

for all $(fx, fy) \in \nabla$; where $0 \leq k < 1/2$. If, for a nonincreasing sequence $\{x_n\}$ in fX , $\{x_n\} \rightarrow u$ implies that $u \preceq x_n$ and for any two points z and x in fX there exists a point $y \in fX$ such that $y \preceq z$ and $y \preceq x$. If fX is complete subspace of X and (S, f) and (T, f) are weakly compatible, then S, T and f have a unique common fixed point fz in fX . Also $p(fz, fz) = 0$.

Now we can apply our Theorem 3 to obtain unique common fixed point of three mappings on closed ball in complete partial metric space.

Theorem 6. Let (X, p) be a partial metric space, $x_0, x, y \in X$, $r > 0$ and S, T and f be the self mappings on X such that $SX \cup TX \subset fX$, $\overline{B(fx_0, r)} \subseteq fX$. Assume that the following conditions hold:

$$p(Sx, Ty) \leq k[p(fx, Sx) + p(fy, Ty)]$$

for all $fx, fy \in \overline{B(fx_0, r)}$; where $0 \leq k < 1/2$,

$$p(fx_0, Tx_0) \leq (1 - \lambda)[r + p(fx_0, fx_0)]$$

where $\lambda = \frac{k}{1-k}$. If fX is complete subspace of X and (S, f) and (T, f) are weakly compatible, then S, T and f have a unique common fixed point fz in $\overline{B(fx_0, r)}$. Also $p(fz, fz) = 0$.

In the following theorem we use Theorem 3 to establish the existence of a unique common fixed point of four mappings on closed ball in complete partial metric space.

Theorem 7. Let (X, p) be a partial metric space, $x_0, x, y \in X$, $r > 0$ and S, T, g and f be self mappings on X such that $SX, TX \subset fX = gX$ and $\overline{B(fx_0, r)} \subseteq fX$. Assume that the following condition holds:

$$p(Sx, Ty) \leq k[p(fx, Sx) + p(gy, Ty)] \quad (2.19)$$

for all $fx, fy \in \overline{B(fx_0, r)}$, where $0 \leq k < 1/2$, and

$$p(fx_0, Sx_0) \leq (1 - \lambda)[r + p(fx_0, fx_0)] \quad (2.20)$$

where $\lambda = \frac{k}{1-k}$. If fX is complete subspace of X and (S, f) and (T, g) are weakly compatible, then S, T, f and g have a unique common fixed point fz in $\overline{B(fx_0, r)}$. Also $p(fz, fz) = 0$.

Proof. By Lemma 2, there exists $E_1, E_2 \subset X$ such that $fE_1 = fX = gX = gE_2$, $f : E_1 \rightarrow X$, $g : E_2 \rightarrow X$ are one to one. Now define the mappings $A, B : fE_1 \rightarrow fE_1$ by $A(fx) = Sx$ and $B(gx) = Tx$ respectively. Since f, g are one to one on E_1 , and E_2 respectively, then the mappings A, B are well-defined. As $fx_0 \in \overline{B(fx_0, r)} \subseteq fX$, then $fx_0 \in fX$. Let $y_0 = fx_0$, choose a point y_1 in fX such that $y_1 = A(y_0)$ and let $y_2 = B(y_1)$. Continuing this process and having chosen y_n in fX such that

$$y_{2i+1} = A(y_{2i}) \text{ and } y_{2i+2} = B(y_{2i+1}), \text{ where } i = 0, 1, 2, \dots,$$

Following similar arguments of Theorem 1, $y_n \in \overline{B(fx_0, r)}$. Also by inequality 2.20, we have,

$$p(fx_0, A(fx_0)) \leq (1 - \lambda)[r + p(fx_0, fx_0)]$$

By using inequality 2.19, for $fx, gy \in \overline{B(fx_0, r)}$, we have

$$p(A(fx), B(gy)) \leq k[p(fx, A(fx)) + p(gy, B(gy))]$$

As fX is a complete space, all conditions of Theorem 3 are satisfied, we deduce that there exists a unique common fixed point $fz \in \overline{B(fx_0, r)}$ of A and B . Further A and B have no fixed point other than fz . Also $p(fz, fz) = 0$. Now $fz = A(fz) = B(fz)$ or $fz = Sz = fz$. Thus fz is a point of coincidence of f and S . Let $w \in \overline{B(fx_0, r)}$ be another point of coincidence of S and f then there exist $u \in \overline{B(fx_0, r)}$ such that $w = fu = Su$, which implies that $fu = A(fu)$. A contradiction as $fz \in \overline{B(fx_0, r)}$ is a unique fixed point of A . Hence $w = fz$. Thus S and f have a unique point of coincidence $fz \in \overline{B(fx_0, r)}$. Since (S, f) are weakly compatible, by Lemma 3 fz is a unique common fixed point of S and f . As $fX = gX$ then there exist $v \in X$ such that $fz = gv$. Now as $A(fz) = B(fz) = fz \Rightarrow A(gv) = B(gv) = gv \Rightarrow Tv = gv$, thus gv is the point of coincidence of T and g . Now if $Tx = gx \Rightarrow B(gx) = gx$. A contradiction. This implies that $gv = gx$. As (T, g) are weakly compatible, we obtain gv , a unique common fixed point for T and g . But $gv = fz$. Thus S, T, g and f have a unique common fixed point $fz \in \overline{B(fx_0, r)}$. $\square$

Theorem 8. Let (X, p) be a partial metric space, $x, y \in X$ and S, T, g and f be self mappings on X such that $SX, TX \subset fX = gX$. Assume that the following condition holds:

$$p(Sx, Ty) \leq k[p(fx, Sx) + p(gy, Ty)]$$

for all $fx, fy \in fX$, where $0 \leq k < 1/2$. If fX is complete subspace of X and (S, f) and (T, g) are weakly compatible, then S, T, f and g have a unique common fixed point fz in fX . Also $p(fz, fz) = 0$.

We can obtain the unique point of coincidence results from Theorem 4 to Theorem 8. Unique point of coincidence result for Theorem 4 is given below.

Theorem 9. Let $(X, \preceq, p)$ be a ordered partial metric space, $x_0, x, y \in X$, $r > 0$ and S, T self mapping and f be a dominated mapping on X such that $SX \cup$

$TX \subset fX$, $\overline{B(fx_0, r)} \subseteq fX$ and $(Tx, fx), (Sx, fx) \in \nabla$. Assume that the following conditions holds:

$$p(Sx, Ty) \leq k[p(fx, Sx) + p(fy, Ty)]$$

for all $(fx, fy) \in (\overline{B(fx_0, r)} \times \overline{B(fx_0, r)}) \cap \nabla$; where $0 \leq k < 1/2$,

$$p(fx_0, Sx_0) + p(fy, Ty) \leq p(fx_0, fy) + p(Sx_0, Ty)$$

for all $fy \in \overline{B(fx_0, r)}$ such that $fy \preceq Sx_0$,

$$p(fx_0, Tx_0) \leq (1 - \lambda)[r + p(fx_0, fx_0)]$$

where $\lambda = \frac{k}{1-k}$. If, for a nonincreasing sequence $\{x_n\}$ in $\overline{B(fx_0, r)}$, $\{x_n\} \rightarrow u$ implies that $u \preceq x_n$ and for any two points z and x in $\overline{B(fx_0, r)}$ there exists a point $y \in \overline{B(fx_0, r)}$ such that $y \preceq z$ and $y \preceq x$. If fX is complete subspace of X , then S, T and f have a unique point of coincidence $fx \in \overline{B(fx_0, r)}$. Also $p(fx, fx) = 0$.

Unique point of coincidence result for Theorem 7 is given below.

Theorem 10. Let (X, p) be a partial metric space, $x_0, x, y \in X$, $r > 0$ and S, T, f and g be self mappings on X such that $SX, TX \subset fX = gX$ and $\overline{B(fx_0, r)} \subseteq fX$. Assume that the following condition holds:

$$p(Sx, Ty) \leq k[p(fx, Sx) + p(gy, Ty)]$$

for all $fx, fy \in \overline{B(fx_0, r)}$; where $0 \leq k < 1/2$, and

$$p(fx_0, Tx_0) \leq (1 - \lambda)[r + p(fx_0, fx_0)]$$

where $\lambda = \frac{k}{1-k}$. If fX is complete subspace of X , then S, T, f and g have a unique point of coincidence fx in $\overline{B(fx_0, r)}$. Also $p(fx, fx) = 0$.

Remark 1. We can obtain the metric version of all theorems, which are still not present in the literature.

3. CONCLUSION

Azam et. al. [9] very recently exploited the idea of fixed points and proved a significant result concerning the existence of fixed points for fuzzy mappings on closed ball in a complete metric space. We continue their investigations and in this paper, some unique common fixed point theorems for dominated mappings under Kannan contractive conditions on a closed ball in a complete ordered partial metric space $(X, \preceq, p)$ have been discussed. Our analysis is based on the simple observation that fixed point results can be deduced from fixed point theory of mappings on closed balls. Practically speaking there are many situations in which the mappings are not contractive on the whole space but instead they are contractive on its subsets. We feel that this aspect of finding the fixed points via closed balls were overlooked and our paper will bring a lot of interest in to this area. Furthermore, we have applied

the concept of dominated mappings in the process of investigating the existence of unique fixed point of contractive mappings on closed balls in the settings of ordered partial metric spaces.

REFERENCES

- [1] M. Abbas and S. Z. Németh, "Finding solutions of implicit complementarity problems by isotonicity of the metric projection," *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 75, no. 4, pp. 2349–2361, 2012.
- [2] I. Altun and A. Erduran, "Fixed point theorems for monotone mappings on partial metric spaces," *Fixed Point Theory Appl.*, vol. 2011, p. 10, 2011.
- [3] I. Altun and H. Simsek, "Some fixed point theorems on dualistic partial metric spaces," *J. Adv. Math. Stud.*, vol. 1, no. 1-2, pp. 1–8, 2008.
- [4] I. Altun, F. Sola, and H. Simsek, "Generalized contractions on partial metric spaces," *Topology Appl.*, vol. 157, no. 18, pp. 2778–2785, 2010.
- [5] M. Arshad, A. Shoaib, and I. Beg, "Fixed point of a pair of contractive dominated mappings on a closed ball in an ordered complete dislocated metric space," *Fixed Point Theory and Appl.*, vol. 2013, no. 115, p. 15, 2013.
- [6] M. Arshad, A. Azam, and P. Vetro, "Some common fixed point results in cone metric spaces," *Fixed Point Theory Appl.*, vol. 2009, p. 11, 2009.
- [7] H. Aydi, "Some fixed point results in ordered partial metric spaces," *J. Nonlinear Sci. Appl.*, vol. 4, pp. 210–217, 2011.
- [8] A. Azam and M. Arshad, "Common fixed points of generalized contractive maps in cone metric spaces," *Bull. Iran. Math. Soc.*, vol. 35, no. 2, pp. 255–264, 2009.
- [9] A. Azam, S. Hussain, and M. Arshad, "Common fixed points of chatterjea type fuzzy mappings on closed balls," *Neural Computing & Applications*.
- [10] A. Azam, M. Waseem, and M. Rashid, "Fixed point theorems for fuzzy contractive mappings in quasi-pseudo-metric spaces," *Fixed Point Theory Appl.*, vol. 2013, no. 27, 2013.
- [11] A. Azam and M. Arshad, "Kannan fixed point theorem on generalized metric spaces," *J. Nonlinear Sci. Appl.*, vol. 1, no. 1, pp. 45–48, 2008.
- [12] T. G. Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 65, no. 7, pp. 1379–1393, 2006.
- [13] M. Bukatin, R. Kopperman, S. Matthews, and H. Pajoohesh, "Partial metric spaces," *Am. Math. Mon.*, vol. 116, no. 8, pp. 708–718, 2009.
- [14] R. H. Haghi, S. Rezapour, and N. Shahzad, "Some fixed point generalizations are not real generalizations," *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 74, no. 5, pp. 1799–1803, 2011.
- [15] Y.-M. He and H.-J. Wang, "Fractal image decoding based on extended fixed point theorem," *Machine Learning and Cybernetics, International Conference*, pp. 4160–4163, 2006.
- [16] R. Kannan, "Some results on fixed points," *Bull. Calcutta Math. Soc.*, vol. 60, pp. 71–76, 1968.
- [17] M. A. Kutbi, J. Ahmad, N. Hussain, and M. Arshad, "Common fixed point results for mappings with rational expressions," *Abstract and Applied Analysis*, vol. 2013, Article ID 549518, p. 11, 2013.
- [18] K. Leibovic, "The principle of contraction mapping in nonlinear and adaptive control systems," *Automatic control, IEEE Transaction*, vol. 9, pp. 393–398, 1964.
- [19] S. G. Matthews, "Partial metric topology," in *Papers on general topology and applications. Papers from the 8th summer conference at Queens College, New York, NY, USA, June 18–20, 1992*. New York: The New York Academy of Sciences, 1994, pp. 183–197.

- [20] G. A. Medrano-Cerda, "A fixed point formulation to parameter estimation problems," in *Decision and control*, 1987, pp. 1468–1476.
- [21] S. Oltra and O. Valero, "Banach's fixed point theorem for partial metric spaces," *Rend. Ist. Mat. Univ. Trieste*, vol. 36, no. 1-2, pp. 17–26, 2004.
- [22] A. Shoib, M. Arshad, and J. Ahmad, "Fixed point results of locally cotractive mappings in ordered quasi-partial metric spaces," *The Scientific World Journal*, vol. 2013, Article ID 194897, p. 14, 2013.
- [23] J. E. Steck, "Convergence of recurrent networks as contraction mappings," *Neural Networks*, vol. 3, pp. 7–11, 1992.

Authors' addresses

Muhammad Arshad, Abdullah Shoaib

Department of Mathematics, International Islamic University, H-10, Islamabad - 44000, Pakistan.

E-mail address: marshad.zia@yahoo.com, abdullahshoaib15@yahoo.com

Mujahid Abbas

Centre for Advanced Studies in Mathematics and Department of Mathematics, Lahore University of Management Sciences, 54792-Lahore, Pakistan.

E-mail address: mujahid@lums.edu.pk

Akbar Azam

Department of Mathematics COMSATS Institute of Information Technology, Chack Shahzad, Islamabad - 44000, Pakistan.

E-mail address: akbar_azam@yahoo.com



ANALYTIC SOLUTIONS FOR THE ONE-DIMENSIONAL COMPRESSIBLE EULER EQUATION WITH HEAT CONDUCTION AND WITH DIFFERENT KIND OF EQUATIONS OF STATE

IMRE FERENC BARNA AND LÁSZLÓ MÁTYÁS

Received September 26, 2013

Abstract. We present analytic self-similar or traveling wave solutions for a one-dimensional coupled system of continuity, compressible Euler and heat conduction equations. Different kind of equations of state are investigated. In certain forms of the equation of state one can arrive to a picture regarding the long time behavior of density and pressure. The impact of these quantities on the evolution of temperature is also discussed.

2010 *Mathematics Subject Classification:* 35L65

Keywords: Euler equation, self-similar Ansatz, traveling waves

1. INTRODUCTION

Studying hydrodynamical systems with heat conduction is an important task, both from the application and from the theoretical side as well. There are numerous monographs and textbooks available which investigate three dimensional real fluid mechanics with heat transfer from the engineering point of view [24]. Usually, all the technical questions like the applied numerics, grids are analyzed in depth in these works.

In the following we will investigate the most simple one dimensional system of continuity, compressible and non-viscous Euler and heat conduction equations. We try to find analytic results. Firstly self-similar physically important diffusive solutions are investigated. If such solutions are not found, secondly traveling wave solutions are obtained and discussed.

There are basically two distinct very powerful set of trial functions available to investigate the global properties of the solutions of various single non-linear partial differential equations [4, 11, 23, 27]. One is the so-called self-similar solution which describes the intermediate asymptotics of a problem: they hold when the precise initial conditions are no longer important, but before the system has reached its final steady state. They are much simpler than the full solutions, so they are easier to understand and to study in different regions of parameter space. A final reason for

studying them is that they are solutions of a system of ordinary differential equations and hence do not suffer the extra inherent numerical problems of the full partial differential equations. In some cases self-similar solutions helps to understand diffusion-like properties or the existence of compact supports of the solution.

The other method is the traveling wave Ansatz which gives us a deeper insight into the wave properties of the system like the propagation speed which can be even time dependent.

In our former work the paradox of heat conduction was investigated with the self-similar Ansatz [6] and gave a new equation which has a finite propagation speed. Recently, a study about the generalized Cattaneo law is under publication where both the heat conduction coefficient and relaxation time have power law temperature dependence [7]. The results can be continuous or shock wave-like with compact support. With a straightforward generalization of the self-similar Ansatz even a partial differential equation(PDE) system of the 3 dimensional Navier-Stokes equations was successfully investigated [5]. Gilding and Kersner [11] studied large number of nonlinear diffusion-convection problems with the help of traveling waves.

As a beginning, without completeness we mention some analytic studies about hydrodynamical systems which are available in the literature.

Beyond traveling waves and self-similar Ansätze there are various examination techniques available in the literature. Manwai [25] studied the N -dimensional ($N \geq 1$) radial Navier-Stokes(NS) and Euler equation with different kind of viscosity and pressure dependences and presented analytical blow up solutions. His works are still 1+1 dimensional (one spatial and one time dimension) investigations. Another well established and popular investigation method is based on Lie algebras for which numerous studies are available. Some of them are treating even the three dimensional case, for more see [14]. Unfortunately, no explicit solutions are shown and analyzed there. Fushchich *et al.* [10] construct a complete set of $\tilde{G}(1, 3)$ -inequivalent Ansätze of codimension 1 for the NS system and they present 19 different analytical solutions for one or two space dimensions. The last solution is very close to our former result in [5], but they are not identical. Further two and three dimensional studies of the Navier-Stokes equation based on group analytical method were presented by Grassi [13]. They also presented solutions which look almost the same as ours [5], but they consider only two space dimensions.

Recently, Hu *et al.* [16] presents a study where symmetry reductions and exact solutions of the (2+1)-dimensional NS are investigated. Aristov and Polyanin [2] use various methods like Crocco transformation, generalized separation of variables or the method of functional separation of variables for the NS and present large number of new classes of exact solutions. Sedov in his classical work [23] (Page 120) presented analytic solutions for the three dimensional spherical NS equation where all three velocity components and the pressure have polar angle dependence (θ) only.

Even this kind of restricted symmetry led to a non-linear coupled ordinary differential equation system which has a very rich mathematical structure. Some similarity reduction solutions of the two dimensional incompressible NS equation was presented by Xia-Ju [17]. Additional solutions are available for the 2+1 dimensional NS also via symmetry reduction techniques by [9]. A full three dimensional Lie group analysis is available for the three dimensional Euler equation of gas dynamics, with polytropic EOS, unfortunately without any heat conduction mechanism [20]. Of course one may find numerical methods for solving equations of fluid dynamics [28]. The topic is quite important, there are interesting applications of the equations of fluids in granular matter [12].

To our knowledge there are very few analytic (self-similar or traveling wave) solutions known for any non-linear partial differential equation systems (PDEs) till today. Our experience shows that no such comparative studies are available for fluid mechanics with heat conduction in the literature, which we present on the next pages.

In the next section we outline our starting model and in the later subsections we give various solutions for three different polytropic equation of states (EOS), plus for the virial EOS. If our results meet some other theories (like porous media equation) then the corresponding literature is mentioned. The paper ends with a short summary.

2. THE BASIC MODEL

We start with the following one-dimensional compressible and non-viscous fluid with ordinary heat conduction which means the following coupled PDE system of continuity, Euler and heat conduction

$$\begin{aligned}\rho(x,t)_t + [\rho(x,t)v(x,t)]_x &= 0, \\ v(x,t)_t + v(x,t)v(x,t)_x &= -\frac{1}{\rho(x,t)}p(x,t)_x, \\ T(x,t)_t + v(x,t)T(x,t)_x &= \lambda T(x,t)_{xx},\end{aligned}\tag{2.1}$$

where ρ, v, T, P are the density, velocity, temperature and pressure field or distribution, respectively. Subscript means partial derivation with respect of time and the x to the Cartesian coordinate. We skip viscous velocity terms in the Euler and in the energy equation as well. As a starting point we include the polytropic EOS $p(x,t) = a\rho(x,t)^n$ where n is a real number and a is a material constant and λ is the heat conduction coefficient. After the investigation of this basic equation system we will introduce modifications for the equation of state.

In the following we are looking for the solution in the form of the self-similar Ansatz which is well-known from [4, 23, 27]

$$V(x,t) = t^{-\alpha} f\left(\frac{x}{t^\beta}\right) := t^{-\alpha} f(\eta)\tag{2.2}$$

where $V(x, t)$ can be an arbitrary variable of a PDE and t means time and x means spatial dependence. The similarity exponents α and β are of primary physical importance since α represents the rate of decay of the magnitude $V(x, t)$, while β is the rate of spread (or contraction if $\beta < 0$) of the space distribution as time goes on. The validity of this self-similar Ansatz also means that the solution has no characteristic time scale. The most powerful result of this Ansatz is the fundamental or Gaussian solution of the Fourier heat conduction equation (or for Fick's diffusion equation) with $\alpha = \beta = 1/2$. The function $f(\eta)$ is called the shape function.

For our system we use the following notations and shape functions

$$T(x, t) = t^{-\alpha} f\left(\frac{x}{t^\beta}\right), \quad v(x, t) = t^{-\delta} g\left(\frac{x}{t^\beta}\right), \quad \rho(x, t) = t^{-\gamma} h\left(\frac{x}{t^\beta}\right) \quad (2.3)$$

where the new variable is $\eta = x/t^\beta$. Calculating all the first time and spatial derivatives of (2.3) and putting them back to (2.1) after some algebra we get the following non-linear ordinary differential equation (ODE) system

$$\begin{aligned} -\gamma h - \frac{1}{2}\eta h' + gh' + hg' &= 0, \\ -\frac{1}{2}g - \frac{1}{2}\eta g' + gg' &= -a\left(\frac{1+\gamma}{\gamma}\right)h^{\left(\frac{1+\gamma}{\gamma}-2\right)}h', \\ -\alpha f - \frac{1}{2}\eta f' + gf' &= \lambda f'', \end{aligned} \quad (2.4)$$

where prime means derivation with respect to η . Among the initially free parameters α, β, γ, n we got the following constraints: α, γ are still arbitrary but $\beta = \delta = 1/2$ and $n = (1 + \gamma)/\gamma$. The material constants c, λ are still independent. Note, that the first and second equations of (2.1) are independent from the third one.

2.1. Aspects coming from the conservation law of density

The first equation is a conservation equation (without source term) and if $\gamma = 1/2$ it can be integrated

$$\left(\frac{\eta h}{2}\right)' = (gh)' \quad (2.5)$$

giving $g = \eta/2$ and $g' = 1/2$ when $h \neq 0$ and the integration constant was chosen zero. Plugging this result back into the second equation of (2.1).

$$-\frac{\eta}{4} = -3ahh' \quad (2.6)$$

where $n = (1 + \gamma)/\gamma = 3$. Which means that only the $p = a\rho^3$ ploytropic EOS can give self-similar solutions in this flow system. The solution of (2.6) can be obtained

via a simple integration giving the following expression

$$h = \sqrt{\left(\frac{4\eta^2 + 2c_1}{3a}\right)} \quad (2.7)$$

where c_1 is the integration constant (we will set it to 0) and a is still the material constant of the polytropic EOS. At last, let's investigate the third heat conduction equation. With the knowledge of the velocity its solution becomes quite simple

$$-\alpha f = \lambda f''. \quad (2.8)$$

The solution is well known

$$f = c_2 \cos\left(\frac{\alpha\eta}{\lambda}\right) + c_3 \sin\left(\frac{\alpha\eta}{\lambda}\right). \quad (2.9)$$

At last, we summarize all our results

$$g = \frac{\eta}{2}, \quad h = \sqrt{\left(\frac{4\eta^2 + 2c_1}{3a}\right)}, \quad f = c_2 \cos\left(\frac{\alpha\eta}{\lambda}\right) + c_3 \sin\left(\frac{\alpha\eta}{\lambda}\right) \quad (2.10)$$

where $n = 3$, $\gamma = \beta = \delta = 1/2$ and α is arbitrary. Now the velocity, density and the temperature field read

$$\begin{aligned} v(x, t) &= t^{-\delta} g(\eta) = \frac{x}{2t}, \\ \rho(x, t) &= t^{-\gamma} h(\eta) = t^{-\frac{1}{2}} \sqrt{\frac{4\left(\frac{x}{\sqrt{t}}\right)^2 + 2c_1}{3c}}, \\ T(x, t) &= t^{-\alpha} f(\eta) = t^{-\alpha} \left[c_2 \cos\left(\frac{\alpha x}{\lambda \sqrt{t}}\right) + c_3 \sin\left(\frac{\alpha x}{\lambda \sqrt{t}}\right) \right] \end{aligned} \quad (2.11)$$

and the applied polytropic EOS

$$p = a\rho^n = a(t^{-\gamma} h(\eta))^3 = a \left[t^{-\frac{1}{2}} \sqrt{\frac{4\left(\frac{x}{\sqrt{t}}\right)^2 + 2c_1}{3c}} \right]^3. \quad (2.12)$$

All the solutions have physical meaning for bounded regions and for sufficiently large times. We will see later, that according to the continuity equation, the velocity field is always a simple function of time and space coordinates, however the temperature distribution have a more complex form. In this case it is a quickly oscillating and decaying function or a slowly oscillating and slowly decaying function.

2.2. Aspects of the solution regarding velocity

We have the possibility to see that in certain cases, when the first equation is integrable, solutions of the system can be found. This was accessible when the pressure is proportional to the third power of the density. Now we try to find solutions when the pressure is proportional to the second power of the density

$$p = \frac{b}{2}\rho^2. \quad (2.13)$$

The role of b is to fix the proper physical dimension. In this case the second equation of (2.1) can be written

$$v_t + vv_x = -b\rho_x \quad (2.14)$$

If we consider that the system is not too far from the steady state, correspondingly the change in time of the gradient of the density is negligible, we get

$$v = -b\rho_x \cdot t + \hat{c}, \quad (2.15)$$

where $\hat{c}$ can be related to initial velocity and initial gradients at time t_0 if these terms do not cancel each other. We suppose that (2.15) holds for a longer time scale than the gradient would undergo of considerable changes. We are interested in a density profile in such a quasi-stationary state. The procedure can be also regarded as taking a first initial condition of the velocity field and iterate it successively [19]. Inserting (2.15) into the mass conservation formula we get

$$\rho_t - tb(\rho_x)^2 + \hat{c}\rho_x - tb\rho\rho_{xx} = 0. \quad (2.16)$$

If $\hat{c} = 0$ and dividing the equation with t we obtain the following equation

$$\frac{\rho_t}{t} - b(\rho_x)^2 - b\rho\rho_{xx} = 0. \quad (2.17)$$

Dividing the last equation by b and multiplying by 2, we have

$$\rho_{t_1} - 2(\rho_x)^2 - 2\rho(\rho_{xx}) = 0 \quad (2.18)$$

where $t_1 = bt^2/4$. This equation in certain aspects may resemble the Kardar-Parisi-Zhang(KPZ) equation [18] which describes the dynamics of surface growth. The original KPZ equation (where the third term doesn't contain the ρ variable) was investigated with the self-similar Ansatz by the recent authors [8] and found physically reasonable solutions which can be expressed with the error function.

The last equation can be brought to a more compact form

$$\rho_{t_1} = (\rho^2)_{xx}, \quad (2.19)$$

There is an even more important relationship with the nonlinear heat conduction or porous media equation which has the form

$$\rho_{t_1} = (\rho^m)_{xx}, \quad m > 1. \quad (2.20)$$

In [26] Zeldovich and Kompaneets have found the fundamental solution ρ_1 of this equation which we write in the following form:

$$\rho_1^{m-1} = t_1^{-\alpha(m-1)} \left(A^2 - B^2 x^2 t_1^{-2\beta} \right)_+ \quad (2.21)$$

where A is constant and the subscript $+$ at the bracket indicates that only physically relevant - positive - solutions are taken into account. Regarding the other constants we have

$$\alpha = \beta = \frac{1}{m+1}, \quad B^2 = \frac{m-1}{2m(m+1)}. \quad (2.22)$$

One can see that this solution has bounded support in x for any $t_1 > 0$ which is a hyperbolic property. Using comparison principle for such equations one can show this finite speed property for any initial condition having compact support. However, the fronts are not straight lines: $x(t_1) = \pm \frac{A}{B} t_1^\beta$, $\beta < 1$ so the speed of propagation $\dot{x}(t_1)$ goes to zero if t_1 goes to infinity. One can also see that ρ_1 is of source-type: $\rho_1(x, 0) = K_1 \delta(x)$.

In our case $m = 2$, consequently $\alpha = \beta = 1/3$, and the solution for the density distribution reads

$$\rho(x, t_1) = t_1^{-\frac{1}{3}} \left(A^2 - B^2 x^2 t_1^{-2/3} \right) = \left(\frac{b}{4} \right)^{-\frac{1}{3}} t^{-\frac{2}{3}} \left(A^2 - B^2 x^2 \left(\frac{b}{4} \right)^{-2/3} t^{-4/3} \right). \quad (2.23)$$

This solution gives us a density profile for a bounded spatial region after sufficiently long time. The velocity field is obtained via 2.15

$$v(x, t) = \frac{2x}{3t}. \quad (2.24)$$

For the temperature distribution the

$$T_t + \frac{2x}{3t} T_x = \lambda T_{xx} \quad (2.25)$$

PDE have to be solved. Considering the $T(x, t) = t^{-\gamma} f\left(\frac{x}{t^\omega}\right)$ the following ODE is derived

$$\lambda f'' - \frac{\eta f'}{6} + \gamma f = 0 \quad (2.26)$$

with arbitrary γ and for $\omega = 1/2$. The solutions are the Kummer M and Kummer U functions of the form of

$$f = c_1 M\left(\frac{1}{2} - 3\gamma, \frac{3}{2}, \frac{1}{12\lambda} \eta^2\right) \eta + c_2 U\left(\frac{1}{2} - 3\gamma, \frac{3}{2}, \frac{1}{12\lambda} \eta^2\right) \eta. \quad (2.27)$$

The mathematical properties of the Kummer function can be found in [1]. For completeness we give the temperature distribution as well,

$$T(x, t) = t^{-\gamma} \left[c_1 M\left(\frac{1}{2} - 3\gamma, \frac{3}{2}, \frac{x^2}{12\lambda t}\right) \frac{x}{\sqrt{t}} + c_2 U\left(\frac{1}{2} - 3\gamma, \frac{3}{2}, \frac{x^2}{12\lambda t}\right) \frac{x}{\sqrt{t}} \right]. \quad (2.28)$$

In contrast to the first case, we applied a simplification in the Euler equation (2.14) therefore we can consider solutions which are outside the scope of the self-similar class. Therefore, we may try to find other physically important solutions, like traveling-waves with the Ansatz of $\rho = h(\zeta) = h(x - ct)$ where c is usually the wave propagation velocity. (We still use h as shape function). However, if we consider (2.17), it can be rewritten in the following form

$$\rho_{t_2} - b(\rho_x)^2 - b\rho\rho_{xx} = 0 \quad (2.29)$$

where $t_2 = t^2/2$. We take the

$$\rho = h(\zeta) = h(x - ct_2) = h(x - at^2/2) \quad (2.30)$$

Ansatz where instead of the constant propagation speed c one may rather consider a kind of constant acceleration, and the constant c is renoted with a . We do this for the purpose of finding analytical solutions, and to get results closer to a physical interpretation.

Inserting this formula into (2.29) we obtain the following ODE:

$$bhh'' + h'(bh' + a) = 0. \quad (2.31)$$

The solution is the following

$$h(\zeta) = \frac{bc_1}{a} \left[\text{LambertW} \left(\frac{e^{\frac{-b^2c_1 + \zeta a^2 + c_2 a}{b^2c_1}}}{bc_1} \right) + 1 \right] \quad (2.32)$$

where c_1, c_2 are integration constants and the Lambert-W function, (which also called the omega function), is the inverse function of

$$f(W(x)) = W(x)e^{W(x)} \quad (2.33)$$

and can be evaluated with the following series expansion

$$W(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^{n-2}}{(n-1)!} x^n. \quad (2.34)$$

Additional properties of this function can be found in [22]. Banwell and Jayakumar [3] showed that a W-function describes the relation between voltage, current and resistance in a diode, and Packel and Yuen [21] applied the W-function to a ballistic projectile in the presence of air resistance. Other applications have been discovered in statistical mechanics, quantum chemistry, combinatorics, enzyme kinetics, the physiology of vision, the engineering of thin films, hydrology, and the analysis of algorithms [15]. Note, that in contrast to other special functions (e.g. Whitakker, Bessel, Kummer functions) the LambertW function is not so widely used or applied.

The final form of the density field is

$$\rho(x - at^2/2) = \frac{bc_1}{a} \left[\text{Lambert}W \left(\frac{e^{\frac{-b^2c_1 + (x-at^2)a^2 + c_2a}{b^2c_1}}}{bc_1} \right) + 1 \right]. \quad (2.35)$$

Figure 1 presents the density field of (2.35) for the $c_1 = c_2 = a = b = 1$ parameters. Note, that the solution is a steep wave front.

Inserting the space derivative of the density into (2.15) and multiplying with the time the velocity field reads the following

$$v(x - at^2/2) = - \frac{\text{Lambert}W \left(\frac{e^{\frac{-b^2c_1 + (x-at^2)a^2 + c_2a}{b^2c_1}}}{bc_1} \right) t}{\text{Lambert}W \left(\frac{e^{\frac{-b^2c_1 + (x-at^2)a^2 + c_2a}{b^2c_1}}}{bc_1} \right) + 1}. \quad (2.36)$$

Figure 2 presents the velocity field for the $c_1 = c_2 = a = b = 1$ parameters. Unfortunately, there is no traveling wave solution for the temperature distribution $T(x, t) = f(\xi) = f(x - at^2)$ because the velocity has an explicit time dependence which is transferred to the third equation of (2.1) giving the following expression $0 = -\lambda f'' - 2atf' + v(\xi)tf'$ which still depends on time and cannot be simplified to an ODE of $f(\xi)$.

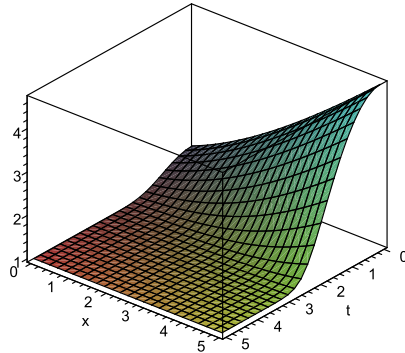


FIGURE 1. The density distribution of Eq. 2.35 for the $a = b = c_1 = c_2 = 1$ parameters.

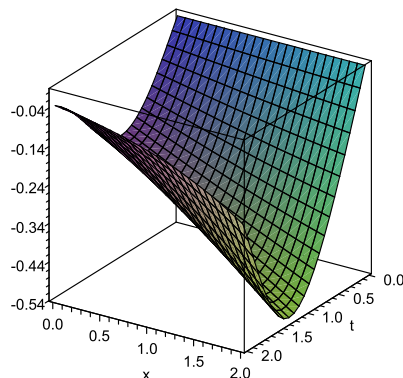


FIGURE 2. The velocity field of Eq. 2.36 for the $a = b = c_1 = c_2 = 1$ parameters.

2.3. The linear EOS case

Let's assume now the linear case of EOS where

$$p = A\rho \quad (2.37)$$

where A is till a constant just to fix the proper physical dimension. From the second case of Eq. (2.1) the velocity field can be obtained for small vs

$$v = -\frac{A\rho_x}{\rho}t. \quad (2.38)$$

From the continuity relation the following PDE is available for the density field

$$\frac{\rho_t}{t} - A\rho_{xx} = 0. \quad (2.39)$$

Note, that the former variable transformation $t_2 = t^2/2$ is still valid giving us the regular Fourier heat conduction or Fick's diffusion equation of

$$\rho_{t_2} - A\rho_{xx} = 0. \quad (2.40)$$

The solution is the well known Gaussian function of

$$\rho(x, t) = \frac{1}{\sqrt{At_2}} \exp\left(\frac{-x^2}{2At_2}\right) = \frac{1}{t} \sqrt{\frac{2}{A}} \exp\left(\frac{-x^2}{At^2}\right). \quad (2.41)$$

According to (2.38) the velocity field is $v(x, t) = 2x/t$. For the temperature distribution the following PDE have to be solved

$$T_t + 2\frac{x}{t}T_x = \lambda T_{xx}. \quad (2.42)$$

The solution is very similar to the former case of (2.25). Using the same notation the solution reads

$$f = c_1 M \left(\frac{1}{2} - \frac{1}{3}\gamma, \frac{3}{2}, \frac{3}{4\lambda}\eta^2 \right) \eta + c_2 U \left(\frac{1}{2} - \frac{1}{3}\gamma, \frac{3}{2}, \frac{3}{4\lambda}\eta^2 \right) \eta. \quad (2.43)$$

For completeness the temperature field is

$$T(x, t) = t^{-\gamma} \left[c_1 M \left(\frac{1}{2} - \frac{1}{3}\gamma, \frac{3}{2}, \frac{3x^2}{4\lambda t} \right) \frac{x}{\sqrt{t}} + c_2 U \left(\frac{1}{2} - \frac{1}{3}\gamma, \frac{3}{2}, \frac{3x^2}{4\lambda t} \right) \frac{x}{\sqrt{t}} \right]. \quad (2.44)$$

Figure 3 presents the Kummer M part of the whole solution with $c_1 = \gamma = \lambda = 1$, and $c_2 = 0$ parameters. The Kummer U part looks very similar. It is straightforward that the PDE of the density field (2.39) can be investigated with the traveling wave Ansatz (2.30), too. The resulting is the next linear ODE

$$Ah'' - ah' = 0. \quad (2.45)$$

The solution is obvious $h = c_1 + c_2 e^{\frac{ax}{A}}$ with the density field of $\rho(x, t) = c_1 + c_2 e^{\frac{a(x-at^2/2)}{A}}$. It turns out from (2.38) that $v = -at$. After trivial calculations from the heat conduction equation we get $T(x, t) = c_1 + c_2(x - at^2/2)$.

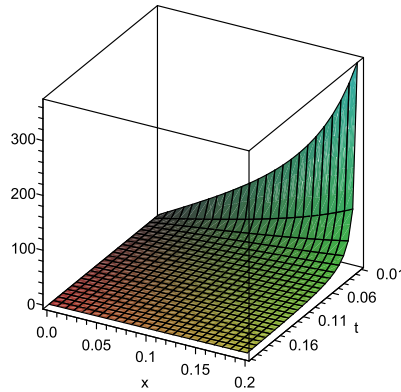


FIGURE 3. The temperature field of Eq. 2.44 only the Kummer M function is presented with the $\gamma = \lambda = c_1 = 1$ parameters.

2.4. Investigation of the virial EOS

We still consider the original system of (2.1), and change the EOS in the second equation. Outside the ideal gas and the polytropic equation of state the most important material law is the Van der Waals (VdW) formula for gases where $p(\rho, T) =$

$\frac{T\rho}{a-\rho} - b\rho^2$ where two of the constants (a, b) can be calculated from the critical point. One of them a is proportional to the universal gas constant times the molar mass of the fluid. Plugging this expression back to the Euler equation and calculating the derivatives we get constraints to the exponents of the Ansatz α, β, γ . For VdW EOS these constraints are contradictory, so no ODEs can be obtained, no self-similar solution can be evaluated.

We may go further and try the virial expansion which can be written in various forms, one possibility is the following

$$p = AT\rho(1 + B\rho + C\rho^2) \quad (2.46)$$

where A, B, C are constants and can be fixed from experiments. We try the simplest $B = C = 0$ case. Apply the Ansatz of Eq. (2.3). The universality relations are therefore $\alpha = 1$, $\beta = \gamma = \delta = 1/2$, which dictate the next ODE system

$$\begin{aligned} -\frac{h}{2} - \frac{1}{2}\eta h' + gh' + hg' &= 0, \\ -\frac{1}{2}g - \frac{1}{2}\eta g' + gg' &= -Af' - \frac{Afh'}{h}, \\ -f - \frac{1}{2}\eta f' + gf' &= \lambda f''. \end{aligned} \quad (2.47)$$

Note, that the first equation is a total derivative and can be integrated immediately which fixes $g = \eta/2$ as for the first investigated system in the Eq. (2.5). So, $v(x, t) = \frac{x}{2t}$. (We fixed the integration constant to zero.) Inserting this into the second and third equation of (2.47) we get

$$\begin{aligned} -\frac{\eta}{4} &= -Af' - \frac{Afh'}{h}, \\ -f &= \lambda f''. \end{aligned} \quad (2.48)$$

The solution for f reads:

$$f(\eta) = c_1 \sin\left(\frac{\eta}{\sqrt{\lambda}}\right) + c_2 \cos\left(\frac{\eta}{\sqrt{\lambda}}\right) \quad (2.49)$$

which means

$$T(x, t) = \frac{1}{t} \left[c_1 \sin\left(\frac{x}{\sqrt{\lambda t}}\right) + c_2 \cos\left(\frac{x}{\sqrt{\lambda t}}\right) \right] \quad (2.50)$$

Finally, the formal solution for $h(\eta)$ for the density shape function can be derived

$$h(\eta) = c_3 \int \frac{-Ac_1 \cos\left(\frac{z}{\sqrt{\lambda}}\right) + Ac_2 \sin\left(\frac{z}{\sqrt{\lambda}}\right) + \frac{z\sqrt{\lambda}}{4}}{\sqrt{\lambda}A \left[c_1 \sin\left(\frac{z}{\sqrt{\lambda}}\right) + c_2 \cos\left(\frac{z}{\sqrt{\lambda}}\right) \right]} dz. \quad (2.51)$$

For given parameters like $c_1 = 0, c_2 = c_3 = A = \lambda = 1$ the integral can be evaluated in a closed form of

$$h(\eta) = \frac{c_3(1 + ie^{i\eta})^{-\frac{\eta}{4}}(1 - ie^{i\eta})^{\frac{\eta}{4}}e^{-\frac{i}{4}[-4\eta - \text{dilog}(1 + ie^{i\eta}) + \text{dilog}(1 - ie^{i\eta})]}}{e^{2i\eta} + 1} \quad (2.52)$$

where i is the complex unit and $\text{dilog}(x)$ is the dilogarithm (or Spence's) function defined via the following integral

$$\text{dilog}(x) = - \int_0^x \frac{\ln|1-s|}{s} ds. \quad (2.53)$$

A detailed mathematical description of the properties of the dilogarithm function can be found in [1]. For completeness we give the final formula of the density

$$\rho(x, t) = t^{-\frac{1}{2}} \times \left[\frac{(1 + ie^{ix/\sqrt{t}})^{-\frac{x/\sqrt{t}}{4}}(1 - ie^{ix/\sqrt{t}})^{\frac{x/\sqrt{t}}{4}}e^{-\frac{i}{4}[-4x/\sqrt{t} - \text{dilog}(1 + ie^{ix/\sqrt{t}}) + \text{dilog}(1 - ie^{ix/\sqrt{t}})]}}{e^{2ix/\sqrt{t}} + 1} \right]$$

Note, that the velocity and the temperature distributions in the first and the last model are very similar to each other.

3. SUMMARY AND OUTLOOK

We investigated the basic one-dimension coupled PDE equations describing fluid flow with heat conduction and presented self-similar solutions for fluid density, flow velocity and temperature. In certain cases we have tried to find solutions for the system of equation presented in (2.1). In the case when $p \sim \rho^3$ the major achievement is that an exact solution is available and the long time decay of density and pressure is a power law of the time variable. The other situation when $p \sim \rho^2$ is more complicated from the point of view of exact solutions, consequently we tried to arrive to results iteratively. In this case solutions related to waves has been also obtained with the help of a special function. The following case when $p \sim \rho$ is again a nontrivial situation in the frame of eq. (2.1). At this point one can arrive to solutions iteratively, which in certain aspects may resemble the diffusivity in dilute systems. Moreover the long time behavior a kind of power law decay for density and pressure can be also observed.

Further possible study may be the investigation of the system (2.1), with other possible equations of state, which may be different than the ones considered here. The extension of Eq. (2.1) for real fluids (e.g. including a viscous term, temperature or density dependent viscosity etc.) and searching for possible analytic or eventually exact solutions is also a possible future problem to be investigated. Regarding computational methods we hope that our study can help to develop benchmark tests for various numerical codes or models.

4. ACKNOWLEDGEMENT

We thank for Prof. Robert Kersner for useful discussions.

REFERENCES

- [1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, 2nd ed. Dover, New York, John Wiley & Sons, 1968, vol. III.
- [2] S. N. Aristov and A. D. Polyanin, "New classes of exact solutions and some transformations of the Navier-Stokes equations," *Russ. J. Math. Phys.*, vol. 17, no. 1, pp. 1–18, 2010.
- [3] T. C. Banwell and A. Jayakumar, "Exact analytical solution for current flow through diode with series resistance," *Elect. Lett.*, vol. 36, no. 1, pp. 29–33, 2000.
- [4] G. I. Barenblatt, *Similarity, self-similarity, and intermediate asymptotics*. Transl. from Russian by Norman Stein, M. Van Dyke, Ed. New York, London: Consultants Bureau, 1979.
- [5] I. F. Barna, "Self-similar solutions of three-dimensional Navier-Stokes equation," *Commun. Theor. Phys.*, vol. 56, no. 4, pp. 745–750, 2011.
- [6] I. F. Barna and R. Kersner, "Heat conduction: a telegraph-type model with self-similar behavior of solutions," *J. Phys. A, Math. Theor.*, vol. 43, no. 37, p. 6, 2010.
- [7] I. F. Barna and R. Kersner, "Heat conduction: hyperbolic self-similar shock-waves in solids," *accepted in Shock Waves*, Preprint Server <http://arxiv.org/abs/1204.4386>.
- [8] I. F. Barna and L. Mátyás, "Self-similar solution of the kardar-parisi-zhang interface dynamic equation," Preprint Server <http://arxiv.org/abs/1112.2870?context=nlin.AO>.
- [9] K. Fakhar, T. Hayat, Y. Cheng, and N. Amin, "A note on similarity solutions of Navier-Stokes equations," *Commun. Theor. Phys.*, vol. 53, no. 3, pp. 575–578, 2010.
- [10] V. I. Fushchich, V. M. Shtelen, and S. L. Slavutskij, "Reduction and exact solutions of the Navier-Stokes equations," *J. Phys. A, Math. Gen.*, vol. 24, no. 5, pp. 971–984, 1991.
- [11] B. H. Gilding and R. Kersner, *Travelling waves in nonlinear diffusion-convection reaction*, ser. Progress in Nonlinear Differential Equations and their Applications. Basel: Birkhäuser, 2004, vol. 60.
- [12] G. Gradenigo, A. Sarraceno, D. Villamaina, and A. Puglisi, "Fluctuating hydrodynamics and correlation lengths in a driven granular fluid," *J. Stat. Mech.*, vol. P08017, 2011.
- [13] V. Grassi, R. A. Leo, G. Soliani, and P. Tempesta, "Temperature behavior of vortices of a 3D thermoconducting viscous fluid," *Physica*, vol. 286, pp. 79–84, 2000.
- [14] V. N. Grebenev, M. Oberlack, and A. N. Grishkov, "Lie algebra methods for the applications to the statistical theory of turbulence," *J. Nonlinear Math. Phys.*, vol. 15, no. 2, pp. 227–251, 2008.
- [15] B. Hayes, "Why W?" *Amer. Sci.*, vol. 93, pp. 104–109, 2005.
- [16] X. R. Hu, Z. Z. Dong, and F. Huang, "Reductions and exact solutions of the (2+1)-dimensional Navier-Stokes equations," *Z. Naturforschung A.*, vol. 65, pp. 504–509, 2010.
- [17] X. Jiao, "Some similarity reduction solutions to two-dimensional incompressible Navier-Stokes equation," *Commun. Theor. Phys.*, vol. 52, no. 3, pp. 389–394, 2009.
- [18] M. Kardar, G. Parisi, and Y. Zhang, "Dynamical scaling of growing interfaces," *Phys. Rev. Lett.*, vol. 56, pp. 889–892, 1986.
- [19] G. Micula and P. Pavel, *Differential and integral equations through practical problems and exercises*, ser. Kluwer Texts in the Mathematical Sciences. Dordrecht: Kluwer Academic Publishers, 1992, vol. 7.
- [20] M. Nadjafikhah, "Group analysis of three dimensional Euler equations of gas dynamics," Preprint Server <http://arxiv.org/abs/0908.3598>.
- [21] E. Packel and D. Yuen, "Projectile motion with resistance and the Lambert W function," *College Math. J.*, vol. 35, pp. 337–341, 2004.

- [22] G. Pólya and G. Szegő, *Aufgaben und Lehrsätze aus der Analysis. Bd. I: Reihen, Integralrechnung, Funktionentheorie. Bd. II: Funktionentheorie, Nullstellen, Polynome, Determinanten, Zahlentheorie.* S. Berlin: Julius Springer. (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. Bd. XIX, XX., 1925.
- [23] L. Sedov, *Similarity and Dimensional Methods in Mechanics.* CRC Press, 1993.
- [24] J. C. Tannehill, D. A. Anderson, and R. H. Pletcher, *Computational Fluid Mechanics and Heat Transfer.* Washington: Taylor and Francis, 1987.
- [25] M. Yuen, “Analytical solutions to the Navier-Stokes equations,” *J. Math. Phys.*, vol. 49, no. 11, pp. 113 102, 10, 2008.
- [26] Y. B. Zel’dovich and A. S. Kompaneets, “Collection dedicated to the 70th birthday of A. F. Joffe,” *Izdat. Akad. Nauk SSSR*, p. 61, 1950.
- [27] Y. B. Zel’dovich and Y. P. Raizer, *Physics of Shock Waves and High Temperature Hydrodynamic Phenomena.* New York: Academic Press, 1966.
- [28] O. C. Zienkiewicz, R. L. Taylor, and P. Nithiarasu, *The finite element method for fluid dynamics*, 6th ed. Amsterdam: Elsevier Butterworth-Heinemann, 2005.

Authors' addresses

Imre Ferenc Barna

Wigner Research Center of the Hungarian Academy of Sciences, H-1525 Budapest, P.P. Box 49, Hungary

Current address: Wigner Research Center of the Hungarian Academy of Sciences, H-1121 Budapest, Konkoly-Thege út 29-33, Hungary

E-mail address: barna.imre@wigner.mta.hu

László Mátyás

Sapientia University, Department Bioengineering, Libertății sq. 1, 530104 Miercurea Ciuc, Romania

Current address: Sapientia University, Department Bioengineering, Libertății sq. 1, 530104 Miercurea Ciuc, Romania

E-mail address: matyaslaszlo@sapientia.siculorum.ro



CONVERGENCE OF THE SOLUTION OF AN IMPULSIVE DIFFERENTIAL EQUATION WITH PIECEWISE CONSTANT ARGUMENTS

HUSEYIN BEREKETOGLU AND GIZEM S. OZTEPE

Received 18 September, 2012

Abstract. We show the existence of the unique solution of impulsive differential equation

$$\begin{cases} x'(t) = a(t)(x(t) - x(\lfloor t-1 \rfloor)) + f(t), & t \neq n \in \mathbb{Z}^+ = \{1, 2, \dots\}, t \geq 0, \\ \Delta x(t) = c_t x(t) + d_t, & t = n \in \mathbb{Z}^+, \end{cases}$$

with the initial conditions

$$x(-1) = x_{-1}, \quad x(0) = x_0,$$

where $\lfloor \cdot \rfloor$ denotes the floor integer function. Moreover, we obtain sufficient conditions for the asymptotic constancy of this equation and we compute, as $t \rightarrow \infty$, the limits of the solutions of the impulsive equation with $c_n = 0$ in terms of the initial conditions, a special solution of the corresponding adjoint equation and a solution of the corresponding difference equation.

2010 Mathematics Subject Classification: 34K06; 34K45

Keywords: asymptotic constancy, differential equation with piecewise constant argument, impulsive differential equation

1. INTRODUCTION

The theory of differential equations with piecewise constant arguments (*DEP-CA*) of the type

$$x'(t) = f(t, x(t), x(h(t)))$$

was initiated in [14, 40] where $h(t) = \lfloor t \rfloor$, $\lfloor t - n \rfloor$, $\lfloor t + n \rfloor$, etc. and $\lfloor \cdot \rfloor$ denotes the floor integer function. These types of equations have been intensively investigated for twenty five years. Systems described by *DEPCA* exist in a large area such as biomedicine, chemistry, physics and mechanical engineering. Busenberg and Cooke [13] first established a mathematical model with a piecewise constant argument for analyzing vertically transmitted diseases. Examples in practice include machinery driven by servo units, charged particles moving in a piecewise constantly varying electric field and elastic systems impelled by a Geneva wheel.

DEPCA are also closely related to difference and differential equations. So, they describe hybrid dynamical systems and combine the properties of both differential

and difference equations. The oscillation, periodicity and some asymptotic properties of various differential equations with piecewise constant arguments were methodically demonstrated in [1–5, 26–28, 35, 36, 38, 42]. Also, Wiener's book [43] is a distinguished source in this area.

Impulsive differential equations are a basic tool to study the dynamics of processes that are subjected to abrupt changes in their states. Theory of impulsive differential equations has been motivated by a number of applied problems such as control theory [24, 25], population dynamics [37], chemotherapeutic treatment in medicine [31] and some physics problems [32]. A significant development has been made in the mathematical theory of impulsive differential equations in the last two decades; see the monographs [7, 39].

But, there are only a few papers on impulsive differential equations with piecewise constant arguments (*IDEPCA*) [9, 29, 33, 44]. In [33], Li and Shen considered the problem

$$\begin{aligned} y'(t) &= f(t, y[t-k]), \quad t \neq n, \quad t \in J, \\ \Delta y(n^+) &= I_n(y(n)), \quad n = 1, 2, \dots, p, \quad y(0) = y(T). \end{aligned}$$

Using the method of upper and lower solutions, they proved that it has at least one solution. In [44], Wiener and Lakshmikantham established the existence and uniqueness of solutions of the initial value problem

$$x'(t) = f(x(t), x(g(t))), \quad x(0) = x_0,$$

and they also studied the cases of oscillation and stability, where f is a continuous function and $g: [0, \infty) \rightarrow [0, \infty)$, $g(t) \leq t$, is a step function. In [9] and [29], some qualitative aspects of advanced and delay *IDEPCA* are investigated. In [34], the authors investigated the impulsive stabilization of certain delay differential equations with piecewise constant argument by using Lyapunov function and analysis methods. They showed that some nonimpulsive systems can be stabilized by imposition of impulsive controls.

Lately, the problem of the asymptotic constancy of solutions was studied for some functional differential equations [6, 8, 11, 12, 15–23, 41] and as well the same problem has been considered for some impulsive delay differential equations [10, 30]. So, due to the practical reasons and the papers mentioned above one can be motivated to deal with the problem of asymptotic constancy of solutions of an impulsive differential equation with piecewise constant arguments.

In this paper, we consider the first order nonhomogeneous linear impulsive differential equation with piecewise constant argument

$$x'(t) = a(t)(x(t) - x(\lfloor t-1 \rfloor)) + f(t), \quad t \neq n \in \mathbb{Z}^+, \quad t \geq 0, \quad (1.1)$$

$$\Delta x(t) = c_t x(t) + d_t, \quad t = n \in \mathbb{Z}^+, \quad (1.2)$$

with the initial conditions

$$x(-1) = x_{-1}, \quad x(0) = x_0, \quad (1.3)$$

where $a(t)$ and $f(t)$ are continuous real valued functions on $[0, \infty)$, $c_n \in \mathbb{R} \setminus \{1\}$, $d_n \in \mathbb{R}$, $n \in \mathbb{Z}^+$, $\mathbb{Z}^+ = \{1, 2, \dots\}$, $x_0, x_{-1} \in \mathbb{R}$, $\Delta x(n) = x(n^+) - x(n^-)$, $x(n^+) = \lim_{t \rightarrow n^+} x(t)$, $x(n^-) = \lim_{t \rightarrow n^-} x(t)$ and $\lfloor \cdot \rfloor$ denotes the floor integer function.

The main purpose of this work is to obtain sufficient conditions for asymptotic constancy of the solution $x(t)$ of (1.1) – (1.3) and also, as $t \rightarrow \infty$, to compute the limit of the solution of the impulsive differential equation with piecewise constant argument

$$x'(t) = a(t)(x(t) - x(\lfloor t - 1 \rfloor)) + f(t), \quad t \neq n \in \mathbb{Z}^+, \quad t \geq 0, \quad (1.4)$$

$$\Delta x(t) = d_t, \quad t = n \in \mathbb{Z}^+, \quad (1.5)$$

in terms of initial conditions, the solution of an integral equation and the solution of a corresponding difference equation. To the best of author's knowledge, this problem has not been studied yet.

2. EXISTENCE OF SOLUTIONS

Definition 1. A function $x(t)$ defined on $[0, \infty)$ is said to be a solution of (1.1) – (1.3) if it satisfies the following conditions:

- (d₁) $x : [0, \infty) \rightarrow \mathbb{R}$ is continuous with the possible exception of the points $t \in \mathbb{Z}^+$,
- (d₂) $x(t)$ is right continuous and has left-hand limits at the points $t \in \mathbb{Z}^+$,
- (d₃) $x'(t)$ exists for every $t \in [0, \infty)$ with the possible exception of the points $t \in \mathbb{Z}^+$ where one-sided derivatives exist,
- (d₄) $x(t)$ satisfies (1.1) for any $t \in (0, \infty)$ with the possible exception of the points $t \in \mathbb{Z}^+$,
- (d₅) $x(t)$ satisfies (1.2) for every $t = n \in \mathbb{Z}^+$,
- (d₆) $x(-1) = x_{-1}$, $x(0) = x_0$.

Theorem 1. The initial value problem (1.1) – (1.3) has a unique solution $x(t)$ on $[0, \infty)$

$$\begin{aligned} x(t) = & \exp\left(\int_{\lfloor t \rfloor}^t a(u) du\right) z_{\lfloor t \rfloor} + \left(1 - \exp\left(\int_{\lfloor t \rfloor}^t a(u) du\right)\right) z_{\lfloor t \rfloor - 1} \\ & + \int_{\lfloor t \rfloor}^t \exp\left(\int_s^t a(u) du\right) f(s) ds \end{aligned} \quad (2.1)$$

where $\lfloor \cdot \rfloor$ denotes the floor integer function, $z_{\lfloor t \rfloor} = x(\lfloor t \rfloor)$ for $t \in [0, \infty)$ and it is a solution of the corresponding difference equation

$$\begin{aligned} z_{\lfloor t \rfloor + 1} = & \left((1 - c_{\lfloor t \rfloor + 1})^{-1} \exp \left(\int_{\lfloor t \rfloor}^{\lfloor t \rfloor + 1} a(u) du \right) \right) z_{\lfloor t \rfloor} \\ & + (1 - c_{\lfloor t \rfloor + 1})^{-1} \left(1 - \exp \left(\int_{\lfloor t \rfloor}^{\lfloor t \rfloor + 1} a(u) du \right) \right) z_{\lfloor t \rfloor - 1} \\ & + (1 - c_{\lfloor t \rfloor + 1})^{-1} \left(\int_{\lfloor t \rfloor}^{\lfloor t \rfloor + 1} \exp \left(\int_s^{\lfloor t \rfloor + 1} a(u) du \right) f(s) ds + d_{\lfloor t \rfloor + 1} \right). \end{aligned} \quad (2.2)$$

Proof. Let $x(t)$ be a solution of (1.1) on $n \leq t < n + 1$. Then, Eq.(1.1) reduces to linear ordinary differential equation

$$x'(t) = a(t)(x(t) - x(n-1)) + f(t).$$

When we solve this equation, we have

$$\begin{aligned} x(t) = & \exp \left(\int_n^t a(u) du \right) x(n) + \left(1 - \exp \left(\int_n^t a(u) du \right) \right) x(n-1) \\ & + \int_n^t \exp \left(\int_s^t a(u) du \right) f(s) ds, \quad n \leq t < n + 1. \end{aligned} \quad (2.3)$$

If we denote the solution $x(t)$ defined by (2.3) as $x_n(t)$, then $x_{n+1}(t)$ implies the solution of Eq.(1.1) on the interval $n + 1 \leq t < n + 2$:

$$\begin{aligned} x_{n+1}(t) = & \exp \left(\int_{n+1}^t a(u) du \right) x(n+1) + \left(1 - \exp \left(\int_{n+1}^t a(u) du \right) \right) x(n) \\ & + \int_{n+1}^t \exp \left(\int_s^t a(u) du \right) f(s) ds, \quad n + 1 \leq t < n + 2. \end{aligned} \quad (2.4)$$

Using impulse conditions (1.2) for $t = n + 1$, we have

$$\Delta x(n+1) = c_{n+1}x(n+1) + d_{n+1},$$

that is,

$$x(n+1)^+ - x(n+1)^- = c_{n+1}x(n+1) + d_{n+1}.$$

Since $x(t)$ is right continuous, the previous equality reduces to

$$x_n((n+1)^-) = (1 - c_{n+1})x_{n+1}(n+1) - d_{n+1}.$$

Considering this equality together with (2.3) and (2.4), we obtain the second order nonhomogeneous difference equation

$$\begin{aligned} x(n+1) &= (1 - c_{n+1})^{-1} \exp\left(\int_n^{n+1} a(u) du\right) x(n) \\ &+ (1 - c_{n+1})^{-1} \left(1 - \exp\left(\int_n^{n+1} a(u) du\right)\right) x(n-1) \\ &+ (1 - c_{n+1})^{-1} \left(\int_n^{n+1} \exp\left(\int_s^{n+1} a(u) du\right) f(s) ds + d_{n+1}\right). \end{aligned} \quad (2.5)$$

Let us define $x(\lfloor t \rfloor) = z_{\lfloor t \rfloor}$ for $t \in [0, \infty)$. So, $x(n) = z_n$ for $n = 0, 1, 2, \dots$, and Eq.(2.5) can be rewritten as

$$\begin{aligned} z_{n+1} &= (1 - c_{n+1})^{-1} \exp\left(\int_n^{n+1} a(u) du\right) z_n \\ &+ (1 - c_{n+1})^{-1} \left(1 - \exp\left(\int_n^{n+1} a(u) du\right)\right) z_{n-1} \\ &+ (1 - c_{n+1})^{-1} \left(\int_n^{n+1} \exp\left(\int_s^{n+1} a(u) du\right) f(s) ds + d_{n+1}\right) \end{aligned} \quad (2.6)$$

which implies Eq.(2.2) for $t \in [0, \infty)$.

Taking into account the initial conditions $x(-1) = x_{-1} = z_{-1}$ and $x(0) = x_0 = z_0$, the solution of difference equation (2.6) is obtained uniquely. So, the unique solution of (1.1) – (1.3) can be written as (2.1) on $[0, \infty)$. $\square$

In this while, we note the following statements:

a) Along this paper we will assume that all solutions of the difference equation (2.6) are bounded, that is, for any solution z_n of (2.6) there is a real positive constant L such that

$$|z_n| \leq L, \quad n \in \mathbb{Z}^+. \quad (2.7)$$

b) A straightforward verification shows that the solution of the initial value problem (1.1) – (1.3) satisfies the following integral equation

$$x(t) = \begin{cases} x_{-1}, & t = -1, \\ x_0 + \int_0^t a(s)x(s)ds - \int_0^t a(s)x(\lfloor s-1 \rfloor)ds + \int_0^t f(s)ds + \sum_{i=1}^{\lfloor t \rfloor} c_i z_i + \sum_{i=1}^{\lfloor t \rfloor} d_i, & t \geq 0, \end{cases} \quad (2.8)$$

where $z_i = x(i)$, $i = 1, 2, \dots, \lfloor t \rfloor$.

3. MAIN RESULTS

This section contains the statements of our main results.

Theorem 2. Let $a(t)$ and $f(t)$ be continuous functions on the interval $[0, \infty)$, $c : \mathbb{Z}^+ \rightarrow \setminus \{1\}$, $d : \mathbb{Z}^+ \rightarrow$ and $n \in \mathbb{Z}^+$.

If

$$(i) \int_0^\infty |a(s)|ds \leq K_1 < \infty,$$

$$(ii) \int_0^\infty |f(s)|ds \leq K_2 < \infty,$$

$$(iii) \prod_{i=1}^\infty (1 + |c_i|) \leq L_1 < \infty,$$

$$(iv) \sum_{i=1}^\infty |d_i| \leq L_2 < \infty,$$

then the solution $x(t)$ of (1.1) – (1.3) tends to a constant as $t \rightarrow \infty$, where K_j , L_j , $j = 1, 2$, are real positive constants.

Theorem 3. Suppose that all assumptions of Theorem 2, except (iii), are satisfied. Let $x(t)$ be the solution of (1.4) – (1.5) and $\lim_{t \rightarrow \infty} x(t) = l(x_0, x_{-1})$.

If

$$\int_t^{\lfloor t+1 \rfloor} |a(s)|ds \leq \rho < 1, \quad (3.1)$$

then

$$\begin{aligned} l(x_0, x_{-1}) = & \left(1 + \int_0^1 y(s)a(s)ds\right)x_0 + \left(\int_0^1 y(s)a(s)ds\right)x_{-1} \\ & + \int_0^\infty y(s)f(s)ds + \sum_{i=1}^\infty d_i + \sum_{i=1}^\infty (z_i - z_{i-1}) \int_i^{i+1} y(s)a(s)ds \end{aligned} \quad (3.2)$$

where y is a solution of the integral equation

$$y(t) = 1 + \int_t^{\lfloor t+1 \rfloor} y(s) a(s) ds, \quad t \geq 0, \quad (3.3)$$

and z_i is a solution of the difference equation (2.6) for $n = i$.

4. PROOFS

In this section the proofs of the main results are given.

4.1. Proof of Theorem 2.

For the proof of Theorem 2 we consider the following well known lemma [39]:

Lemma 1. Let a nonnegative piecewise continuous function $u(t)$ satisfy for $t \geq t_0$ the inequality

$$u(t) \leq \alpha + \int_{t_0}^t v(s) u(s) ds + \sum_{t_0 \leq \tau_i < t} \beta_i u(\tau_i)$$

where $\alpha \geq 0$, $\beta_i \geq 0$, $v(s) > 0$, τ_i are the first kind discontinuity points of the function $u(t)$. Then the following estimate holds for the function $u(t)$,

$$u(t) \leq \alpha \prod_{t_0 \leq \tau_i < t} (1 + \beta_i) \exp \left(\int_{t_0}^t v(s) ds \right).$$

Now we can give the proof of Theorem 2:

Proof of Theorem 2. Let $x(t)$ be the solution of (1.1) – (1.3). Then, from (2.8),

$$\begin{aligned} |x(t)| &\leq |x_0| + \int_0^t |a(s)| |x(s)| ds + \int_0^t |a(s)| |x(\lfloor s-1 \rfloor)| ds \\ &\quad + \int_0^t |f(s)| ds + \sum_{i=1}^{\lfloor t \rfloor} |c_i| |z_i| + \sum_{i=1}^{\lfloor t \rfloor} |d_i|. \end{aligned}$$

Considering $x(\lfloor s-1 \rfloor) = z_{\lfloor s-1 \rfloor}$, $s \in [0, \infty)$, and the boundedness of z_n , $n \in \mathbb{Z}^+$, we have

$$|x(t)| \leq |x_0| + \int_0^t |a(s)| |x(s)| ds + L \int_0^\infty |a(s)| ds$$

$$+ \int_0^{\infty} |f(s)| ds + \sum_{i=1}^{\lfloor t \rfloor} |c_i| |z_i| + \sum_{i=1}^{\infty} |d_i|.$$

By using (i), (ii), (iv) and (2.7), we obtain

$$|x(t)| \leq \alpha + \int_0^t |a(s)| |x(s)| ds + \sum_{i=1}^{\lfloor t \rfloor} |c_i| |z_i|$$

where $\alpha = |x_0| + LK_1 + K_2 + L_2$. Applying Lemma 1,

$$\begin{aligned} |x(t)| &\leq \alpha \prod_{i=0}^{\lfloor t \rfloor} (1 + |c_i|) \exp \left(\int_0^t |a(s)| ds \right) \\ &\leq \alpha \prod_{i=0}^{\infty} (1 + |c_i|) \exp \left(\int_0^{\infty} |a(s)| ds \right). \end{aligned}$$

Hence, by (i) and (iii)

$$|x(t)| \leq \alpha L_1 e^{K_1} = M, \quad t \geq 0, \quad (4.1)$$

where M is a positive real constant.

On the other hand,

$$\begin{aligned} |x(t) - x(s)| &\leq \int_s^t |a(u)| |x(u)| du + \int_s^t |a(u)| |x(\lfloor u - 1 \rfloor)| du \\ &\quad + \int_s^t |f(u)| du + \sum_{i=\lfloor s \rfloor + 1}^{\lfloor t \rfloor} |c_i| |z_i| + \sum_{i=\lfloor s \rfloor + 1}^{\lfloor t \rfloor} |d_i| \end{aligned} \quad (4.2)$$

for $0 \leq s < t < \infty$.

From (2.7) and (4.1), we obtain

$$|x(t) - x(s)| \leq (M + L) \int_s^{\infty} |a(u)| du + \int_s^{\infty} |f(u)| du + L \sum_{i=\lfloor s \rfloor + 1}^{\infty} |c_i| + \sum_{i=\lfloor s \rfloor + 1}^{\infty} |d_i|.$$

In this while, we note that

$$\sum_{i=1}^{\infty} |c_i| < \infty \quad (4.3)$$

since (iii) is valid. So, by using (4.3) and the conditions (i), (ii) and (iv), it is easy to verify that

$$\lim_{s \rightarrow \infty} |x(t) - x(s)| = 0.$$

Thus, by the Cauchy convergence criterion, $\lim_{t \rightarrow \infty} x(t) \in \mathbb{R}$. $\square$

4.2. Proof of Theorem 3.

For the proof of Theorem 3, it is necessary to prove the following theorem and lemmas.

Theorem 4. Suppose $a(t)$ is continuous and (3.1) is satisfied. Then, there is a unique bounded function $y \in PRC([0, \infty), \mathbb{R})$ such that (3.3) holds.

Proof. Denote the set of piecewise right continuous functions by $PRC([0, \infty), \mathbb{R})$, that is, $\varphi \in PRC$ means that $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is continuous for $t \in [0, \infty)$, $t \neq n$, $n = 1, 2, \dots$, and is continuous from the right for $t = n$, $n = 0, 1, 2, \dots$.

Now, let us take the space

$$B = \left\{ y \in PRC([0, \infty), \mathbb{R}) : |y|_B \leq \lambda, \lambda \geq \frac{1}{1-\rho} \right\}.$$

B is a Banach space with the norm

$$|y|_B = \sup_{t \geq 0} |y(t)|, \quad y \in B.$$

For $y \in B$ and $t \geq 0$, define

$$Ty(t) = 1 + \int_t^{\lfloor t+1 \rfloor} y(s)a(s)ds.$$

It can be easily shown that for every integer point n , $n = 0, 1, 2, \dots$,

$$Ty(n^+) = \lim_{t \rightarrow n^+} Ty(t) = Ty(n)$$

and

$$Ty(n^-) = \lim_{t \rightarrow n^-} Ty(t) = 1.$$

As well as, for $t_* \in (n, n+1)$, $n = 0, 1, 2, \dots$,

$$Ty(t_*^+) = Ty(t_*^-) = Ty(t_*).$$

So, $Ty \in PRC([0, \infty), \mathbb{R})$.

Also, from (3.1), it follows that

$$|Ty|_B \leq 1 + \rho|y|_B \leq \lambda.$$

Hence T maps B into itself. On the other hand, for $y, z \in B$

$$|Ty - Tz|_B \leq \rho|y - z|_B.$$

Since $\rho < 1$, $T : B \rightarrow B$ is a contraction. Therefore, the unique solution $y \in B$ of $Ty = y$ is the unique piecewise right continuous and bounded solution of (3.3). $\square$

Lemma 2. *Let $a(t)$ be continuous and (3.1) is true. Then, the solution y of the integral equation (3.3) satisfies the following integral equation*

$$\begin{cases} y'(t) = -y(t)a(t), & t \neq n, \\ \Delta y(n) = \int_n^{n+1} y(s)a(s)ds, & n \in \mathbb{Z}^+, \end{cases} \quad (4.4)$$

where $\Delta y(n) = y(n^+) - y(n^-)$, $y(n^+) = \lim_{t \rightarrow n^+} y(t)$ and $y(n^-) = \lim_{t \rightarrow n^-} y(t)$.

Proof. Taking the derivative of (3.3) for $t \in (n, n+1)$, $n \in \mathbb{Z}^+$, we obtain

$$y'(t) = -y(t)a(t).$$

On the other hand,

$$\begin{aligned} \Delta y(n) &= y(n^+) - y(n^-) \\ &= 1 + \int_n^{n+1} y(s)a(s)ds - 1 \\ &= \int_n^{n+1} y(s)a(s)ds. \end{aligned}$$

So, the proof of Lemma 2 is complete. $\square$

Now, let us denote the function

$$C(t) = y(t)x(t) - \int_t^{\lfloor t+1 \rfloor} y(s)a(s)x(\lfloor s-1 \rfloor)ds, \quad t \geq 0, \quad (4.5)$$

where y is the solution of the integral equation (3.3) and x is the solution of (1.4) – (1.5).

Lemma 3. *Suppose that $a(t)$ and $f(t)$ are continuous functions on $[0, \infty)$ and (3.1) is hold. Then,*

$$C(t) = C(0) + \int_0^t y(s)f(s)ds + \sum_{i=1}^{\lfloor t \rfloor} d_i + \sum_{i=1}^{\lfloor t \rfloor} (z_i - z_{i-1}) \int_i^{i+1} y(s)a(s)ds \quad (4.6)$$

where y is a solution of (3.3) and z_i is a solution of the difference equation (2.6) for $n = i$.

Proof. To obtain (4.6), it is enough to show that $C(t)$, defined by (4.6), satisfies

$$\begin{cases} C'(t) = y(t)f(t), & t \neq n, \quad t \geq 0, \\ \Delta C(n) = d_n + (z_n - z_{n-1}) \int_n^{n+1} y(s)a(s)ds, & n \in \mathbb{Z}^+. \end{cases} \quad (4.7)$$

For $t \in (n, n+1)$, (4.5) can be written as

$$C(t) = y(t)x(t) - x(n-1) \int_t^{n+1} y(s)a(s)ds. \quad (4.8)$$

By taking derivative of both sides of (4.8), we have

$$C'(t) = y'(t)x(t) + y(t)x'(t) + x(n-1)y(t)a(t).$$

Considering (1.4) and (4.4), we get

$$\begin{aligned} C'(t) &= -y(t)a(t)x(t) + y(t)\{a(t)x(t) - a(t)x(n-1) + f(t)\} + x(n-1)y(t)a(t) \\ &= y(t)f(t). \end{aligned}$$

Also, from (4.8),

$$\begin{aligned} \Delta C(n) &= C(n^+) - C(n^-) \\ &= y(n)x(n) - x(n-1) \int_n^{n+1} y(s)a(s)ds - y(n^-)x(n^-). \end{aligned} \quad (4.9)$$

Substituting

$$y(n^-) = y(n) - \int_n^{n+1} y(s)a(s)ds$$

and

$$x(n^-) = x(n) - d_n$$

into (4.9), we obtain

$$\Delta C(n) = d_n + (z_n - z_{n-1}) \int_n^{n+1} y(s)a(s)ds$$

where $z_n = x(n)$, $n \in \mathbb{Z}^+$. So, we obtain (4.7). Integrating both sides of (4.7), we get (4.6). $\square$

Now, we can prove Theorem 3:

Proof of Theorem 3. Let $x(t)$ be the solution of (1.4) – (1.5). For the proof, it is sufficient to show that

$$\lim_{t \rightarrow \infty} x(t) = C(0) + \int_0^\infty y(s)f(s)ds + \sum_{i=1}^\infty d_i + \sum_{i=1}^\infty (z_i - z_{i-1}) \int_i^{i+1} y(s)a(s)ds \quad (4.10)$$

where C is defined by (4.5) and z_i is a solution of the difference equation (2.6) for $n = i$. From (4.6), we have for $t \geq 0$,

$$\begin{aligned}
 & x(t) - C(0) - \int_0^\infty y(s) f(s) ds - \sum_{i=1}^\infty d_i - \sum_{i=1}^\infty (z_i - z_{i-1}) \int_i^{i+1} y(s) a(s) ds \\
 &= x(t) - \left(C(0) + \int_0^t y(s) f(s) ds + \sum_{i=1}^{\lfloor t \rfloor} d_i + \sum_{i=1}^{\lfloor t \rfloor} (z_i - z_{i-1}) \int_i^{i+1} y(s) a(s) ds \right) \\
 & - \int_t^\infty y(s) f(s) ds - \sum_{i=\lfloor t \rfloor+1}^\infty d_i - \sum_{i=\lfloor t \rfloor+1}^\infty (z_i - z_{i-1}) \int_i^{i+1} y(s) a(s) ds \\
 &= x(t) - C(t) - \int_t^\infty y(s) f(s) ds - \sum_{i=\lfloor t \rfloor+1}^\infty d_i - \sum_{i=\lfloor t \rfloor+1}^\infty (z_i - z_{i-1}) \int_i^{i+1} y(s) a(s) ds.
 \end{aligned}$$

Using (4.5), it follows for $t \geq 0$,

$$\begin{aligned}
 & x(t) - C(0) - \int_0^\infty y(s) f(s) ds - \sum_{i=1}^\infty d_i - \sum_{i=1}^\infty (z_i - z_{i-1}) \int_i^{i+1} y(s) a(s) ds \\
 &= x(t) - y(t)x(t) + \int_t^{\lfloor t+1 \rfloor} y(s) a(s) x(\lfloor s-1 \rfloor) ds - \int_t^\infty y(s) f(s) ds \\
 & - \sum_{i=\lfloor t \rfloor+1}^\infty d_i - \sum_{i=\lfloor t \rfloor+1}^\infty (z_i - z_{i-1}) \int_i^{i+1} y(s) a(s) ds. \tag{4.11}
 \end{aligned}$$

On the other hand, multiplying (3.3) by $x(t)$ we obtain

$$x(t) = y(t)x(t) - \int_t^{\lfloor t+1 \rfloor} y(s) a(s) x(t) ds$$

for $t \geq 0$. Substituting the last expression into (4.11), we find

$$\begin{aligned}
 & x(t) - C(0) - \int_0^\infty y(s) f(s) ds - \sum_{i=1}^\infty d_i - \sum_{i=1}^\infty (z_i - z_{i-1}) \int_i^{i+1} y(s) a(s) ds \\
 &= \int_t^{\lfloor t+1 \rfloor} y(s) a(s) (x(\lfloor s-1 \rfloor) - x(t)) ds - \int_t^\infty y(s) f(s) ds
 \end{aligned}$$

$$- \sum_{i=\lfloor t \rfloor + 1}^{\infty} d_i - \sum_{i=\lfloor t \rfloor + 1}^{\infty} (z_i - z_{i-1}) \int_i^{i+1} y(s) a(s) ds. \quad (4.12)$$

From (4.12), together with (2.7), (4.1) and the boundedness of $y(t)$ on $[0, \infty)$, we get

$$\begin{aligned} & \left| x(t) - C(0) - \int_0^t y(s) f(s) ds - \sum_{i=1}^{\infty} d_i - \sum_{i=1}^{\infty} (z_i - z_{i-1}) \int_i^{i+1} y(s) a(s) ds \right| \\ & \leq |y|_B (L + M) \int_t^{\lfloor t+1 \rfloor} |a(s)| ds + |y|_B \int_t^{\infty} f(s) ds \\ & \quad + \sum_{i=\lfloor t \rfloor + 1}^{\infty} |d_i| + 2L |y|_B \sum_{i=\lfloor t \rfloor + 1}^{\infty} \int_i^{i+1} |a(s)| ds \end{aligned}$$

where $|y|_B = \sup_{t \geq 0} |y(t)|$. Thus, it follows that (4.10) is correct. Taking into account (4.5), it is easily verified that the limit relation (4.10) is reduced to (3.2). So, the proof is completed. $\square$

ACKNOWLEDGEMENT

We are very grateful to referee for their affirmative contributions and valuable comments.

REFERENCES

- [1] A. R. Aftabizadeh and J. Wiener, "Oscillatory properties of first order linear functional differential equations," *Appl. Anal.*, vol. 20, pp. 165–187, 1985.
- [2] A. R. Aftabizadeh and J. Wiener, "Oscillatory and periodic solutions for systems of two first order linear differential equations with piecewise linear differential equations with piecewise constant argument," *Appl. Anal.*, vol. 26, no. 4, pp. 327–333, 1988.
- [3] A. R. Aftabizadeh, J. Wiener, and J.-M. Xu, "Oscillatory and periodic solutions of delay differential equations with piecewise constant argument," *Proc. Am. Math. Soc.*, vol. 99, pp. 673–679, 1987.
- [4] M. U. Akhmet, "Integral manifolds of differential equations with piecewise constant argument of generalized type," *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 66, no. 2, pp. 367–383, 2007.
- [5] M. U. Akhmet, "On the reduction principle for differential equations with piecewise constant argument of generalized type," *J. Math. Anal. Appl.*, vol. 336, no. 1, pp. 646–663, 2007.
- [6] O. Arino and M. Pituk, "More on linear differential systems with small delays," *J. Differ. Equations*, vol. 170, no. 2, pp. 381–407, 2001.

- [7] D. D. Bainov and P. S. Simeonov, *Impulsive differential equations. Asymptotic properties of the solutions*, ser. Advances in Mathematics for Applied Sciences. Singapore: World Scientific, 1995, vol. 28.
- [8] J. Baštinec, J. Diblík, and Z. Šmarda, "Convergence tests for one scalar differential equation with vanishing delay," *Arch. Math., Brno*, vol. 36, pp. 405–414, 2000.
- [9] H. Bereketoglu, G. Seyhan, and A. Ogun, "Advanced impulsive differential equations with piecewise constant arguments," *Math. Model. Anal.*, vol. 15, no. 2, pp. 175–187, 2010.
- [10] H. Bereketoglu and F. Karakoç, "Asymptotic constancy for impulsive delay differential equations," *Dyn. Syst. Appl.*, vol. 17, no. 1, pp. 71–83, 2008.
- [11] H. Bereketoglu and M. Pituk, "Asymptotic constancy for nonhomogeneous linear differential equations with unbounded delays," *Discrete Contin. Dyn. Syst.*, vol. 2003, pp. 100–107, 2003.
- [12] L. Berežanský, J. Diblík, M. Růžicková, and Z. Šutá, "Asymptotic convergence of the solutions of a discrete equation with two delays in the critical case," *Abstr. Appl. Anal.*, vol. 2011, p. 15, 2011.
- [13] S. Busenberg and K. L. Cooke, "Models of vertically transmitted diseases with sequential-continuous dynamics," in *Nonlinear phenomena in mathematical sciences, Proc. int. Conf., Arlington/Tex. 1980*, 1982, pp. 179–187.
- [14] K. L. Cooke and J. Wiener, "Retarded differential equations with piecewise constant delays," *J. Math. Anal. Appl.*, vol. 99, pp. 265–297, 1984.
- [15] J. Diblík, "A criterion of asymptotic convergence for a class of nonlinear differential equations with delay," *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 47, no. 6, pp. 4095–4106, 2001.
- [16] J. Diblík, M. Růžicková, Z. Šmarda, and Z. Šutá, "Asymptotic convergence of the solutions of a dynamic equation on discrete time scales," *Abstr. Appl. Anal.*, vol. 2012, p. 20, 2012.
- [17] J. Diblík, M. Růžicková, and Z. Šutá, "Asymptotic convergence of the solutions of a discrete equation with several delays," *Appl. Math. Comput.*, vol. 218, no. 9, pp. 5391–5401, 2012.
- [18] J. Diblík, M. Růžicková, and Z. Šutá, "Asymptotic convergence of the solutions of a discrete system with delays," *Appl. Math. Comput.*, vol. 219, no. 8, pp. 4036–4044, 2012.
- [19] J. Diblík, "Behaviour of solutions to linear differential equations with delay," *Arch. Math., Brno*, vol. 34, no. 1, pp. 31–47, 1998.
- [20] J. Diblík, "A criterion for convergence of solutions to homogeneous delay linear differential equations," *Ann. Pol. Math.*, vol. 72, no. 2, pp. 115–130, 1999.
- [21] J. Diblík, "Asymptotic convergence criteria of solutions of delayed functional differential equations," *J. Math. Anal. Appl.*, vol. 274, no. 1, pp. 349–373, 2002.
- [22] J. Diblík and M. Růžicková, "Convergence of the solutions of the equation $\dot{y}(t) = \beta(t)[y(t - \delta) - y(t - \tau)]$ in the critical case," *J. Math. Anal. Appl.*, vol. 331, no. 2, pp. 1361–1370, 2007.
- [23] J. Diblík, M. Růžicková, and Z. Šutá, "Asymptotical convergence of the solutions of a linear differential equation with delays," *Adv. Difference Equ.*, vol. 2010, p. 12, 2010.
- [24] R. K. George, A. K. Nandakumaran, and A. Arapostathis, "A note on controllability of impulsive systems," *J. Math. Anal. Appl.*, vol. 241, no. 2, pp. 276–283, 2000.
- [25] Z.-H. Guan, G. Chen, and T. Ueta, "On impulsive control of a periodically forced chaotic pendulum system," *IEEE Trans. Autom. Control*, vol. 45, no. 9, pp. 1724–1727, 2000.
- [26] I. Györi and G. Ladas, "Linearized oscillations for equations with piecewise constant arguments," *Differ. Integral Equ.*, vol. 2, no. 2, pp. 123–131, 1989.
- [27] I. Györi, "On approximation of the solutions of delay differential equations by using piecewise constant arguments," *Int. J. Math. Math. Sci.*, vol. 14, no. 1, pp. 111–126, 1991.
- [28] Y. K. Huang, "Oscillations and asymptotic stability of solutions of first order neutral differential equations with piecewise constant argument," *J. Math. Anal. Appl.*, vol. 149, no. 1, pp. 70–85, 1990.

- [29] F. Karakoc, H. Bereketoglu, and G. Seyhan, "Oscillatory and periodic solutions of impulsive differential equations with piecewise constant argument," *Acta Appl. Math.*, vol. 110, no. 1, pp. 499–510, 2010.
- [30] F. Karakoç and H. Bereketoglu, "Some results for linear impulsive delay differential equations," *Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal.*, vol. 16, no. 3, pp. 313–326, 2009.
- [31] A. Lakmeche and O. Arino, "Bifurcation of non trivial periodic solutions of impulsive differential equations arising chemotherapeutic treatment," *Dyn. Contin. Discrete Impulsive Syst.*, vol. 7, no. 2, pp. 265–287, 2000.
- [32] S. Lenci and G. Rega, "Periodic solutions and bifurcations in an impact-inverted pendulum under impulsive excitation," *Chaos Solitons Fractals*, vol. 11, no. 15, pp. 2453–2472, 2000.
- [33] J. Li and J. Shen, "Periodic boundary value problems of impulsive differential equations with piecewise constant argument," *J. Nat. Sci. Hunan Norm. Univ.*, vol. 25, no. 3, pp. 5–9, 2002.
- [34] X. Li, "Impulsive stabilization of certain delay differential equations with piecewise constant argument," *Int. J. Difference Equ.*, vol. 3, no. 2, pp. 267–276, 2008.
- [35] H. Liang and G.-Q. Wang, "Existence and uniqueness of periodic solutions for a delay differential equation with piecewise constant arguments," *Port. Math. (N.S.)*, vol. 66, no. 1, pp. 1–12, 2009.
- [36] Y. Muroya, "New contractivity condition in a population model with piecewise constant arguments," *J. Math. Anal. Appl.*, vol. 346, no. 1, pp. 65–81, 2008.
- [37] S. Nenov, "Impulsive controllability and optimization problems in population dynamics," *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 36, no. 7, pp. 881–890, 1999.
- [38] M. Pinto, "Asymptotic equivalence of nonlinear and quasi linear differential equations with piecewise constant arguments," *Math. Comput. Modelling*, vol. 49, no. 9–10, pp. 1750–1758, 2009.
- [39] A. M. Samoilenko and N. A. Perestyuk, *Impulsive differential equations*, ser. World Scientific Series on Nonlinear Science. Series A. Singapore: World Scientific, 1995, vol. 14.
- [40] S. M. Shah and J. Wiener, "Advanced differential equations with piecewise constant argument deviations," *Int. J. Math. Math. Sci.*, vol. 6, pp. 671–703, 1983.
- [41] J. Čermák, "Asymptotic behaviour of solutions to some differential equations with an unbounded delay," *Electron. J. Qual. Theory Differ. Equ.*, vol. 2000, p. 8, 2000.
- [42] J. Wiener, "Differential equations with piecewise constant delays," in *Trends in theory and practice of nonlinear differential equations, Proc. int. Conf., Arlington/Tex. 1982*, ser. Lect. Notes Pure Appl. Math., 1984, vol. 90, pp. 547–552.
- [43] J. Wiener, *Generalized solutions of functional differential equations*. Singapore: World Scientific, 1993.
- [44] J. Wiener and V. Lakshmikantham, "Differential equations with piecewise constant argument and impulsive equations," *Nonlinear Stud.*, vol. 7, no. 1, pp. 60–69, 2000.

Authors' addresses

Huseyin Bereketoglu

Ankara University, Faculty of Sciences, Department of Mathematics, TR-06100 Besevler, Ankara, Turkey

E-mail address: bereket@science.ankara.edu.tr

Gizem S. Oztepe

Ankara University, Faculty of Sciences, Department of Mathematics, TR-06100 Besevler, Ankara, Turkey

E-mail address: gseyhan@ankara.edu.tr (Corresponding Author)



NEW KINDS OF MATRIX POLYNOMIALS

BAYRAM ÇEKİM

Received 4 February, 2013

Abstract. In this study, we introduce new matrix polynomials and derive their properties such as explicit representations, recurrence relations and generating matrix functions.

2010 Mathematics Subject Classification: 33C45; 15A60

Keywords: matrix polynomials, recurrence relation, generating matrix functions

1. INTRODUCTION

Recently, mathematicians have interested in some properties of the orthogonal matrix polynomials. For example, Jódar and et. al have introduced and studied Laguerre matrix polynomials [8], Gegenbauer matrix polynomials [9], Hermite matrix polynomials [7], Chebyshev matrix polynomials [4] and Jacobi matrix polynomials [3]. Then the Konhauser matrix polynomials [11], the multivariable Humbert matrix polynomials [1] and the Bessel matrix polynomials [10] have been studied. Furthermore, one can find several papers concerning the orthogonal matrix polynomials (see [2, 12]). In this paper, we define new matrix polynomials and derive some of their properties.

Throughout this paper, for a matrix A in $\mathbb{C}^{r \times r}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A . Furthermore the identity matrix and the zero matrix of $\mathbb{C}^{r \times r}$ will be denoted by I and $\mathbf{0}$, respectively. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z , which are defined in an open set Ω of the complex plane and A is a matrix in $\mathbb{C}^{r \times r}$ with $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus in [6], it follows that:

$$f(A)g(A) = g(A)f(A). \quad (1.1)$$

Let A be a matrix in $\mathbb{C}^{r \times r}$ satisfying $(-k) \notin \sigma(A)$ for $k \in \mathbb{Z}^+$ and λ be a complex number whose real part is positive. Then the Laguerre matrix polynomials $L_n^{(A, \lambda)}(x)$ are defined by [8]:

$$L_n^{(A, \lambda)}(x) = \sum_{k=0}^n \frac{(-1)^k}{k! (n-k)!} (A+I)_n [(A+I)_k]^{-1} (\lambda x)^k, \quad n \in \mathbb{N}. \quad (1.2)$$

Also Laguerre matrix polynomials have the following generating matrix function:

$$\sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x)t^n = (1-t)^{-A-I} e^{-\frac{\lambda xt}{1-t}}; \quad x \in \mathbb{C}, t \in \mathbb{C}, |t| < 1. \quad (1.3)$$

2. DEFINITIONS AND PROPERTIES OF NEW MATRIX POLYNOMIALS

In this section, we define two new matrix polynomials $\left\{f_{n,m}^{(A,\lambda)}(x)\right\}_{n=0}^{\infty}$ and $\left\{g_{n,m}^{(A,\lambda)}(x)\right\}_{n=0}^{\infty}$ via following generating matrix functions

$$F^{(A,\lambda)}(x,t) = (1-t^m)^{-A} e^{-\frac{xt\lambda}{1-t^m}} = \sum_{n=0}^{\infty} f_{n,m}^{(A,\lambda)}(x)t^n; \quad |t| < 1 \quad (2.1)$$

$$G^{(A,\lambda)}(x,t) = (1+t^m)^{-A} e^{-\frac{xt\lambda}{1+t^m}} = \sum_{n=0}^{\infty} g_{n,m}^{(A,\lambda)}(x)t^n; \quad |t| < 1 \quad (2.2)$$

where A is a matrix in $\mathbb{C}^{r \times r}$, m is a positive natural number and λ is an arbitrary number, respectively. Here, (2.1) and (2.2) are matrix versions of polynomials given in [5].

Using (2.1) and (2.2), we obtain the following explicit representations for new matrix polynomials

$$\begin{aligned} f_{n,m}^{(A,\lambda)}(x) &= \sum_{i=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^{n-mi} (A + (n-mi)I)_i}{(n-mi)! i!} (\lambda x)^{n-mi}, \\ g_{n,m}^{(A,\lambda)}(x) &= \sum_{i=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^{n-(m-1)i} (A + (n-mi)I)_i}{(n-mi)! i!} (\lambda x)^{n-mi} \end{aligned} \quad (2.3)$$

where $[\cdot]$ denotes the integer part. Taking $m = 1$ in (2.3), (2.3) reduces Laguerre matrix polynomials as follows:

$$\begin{aligned} f_{n,1}^{(A,\lambda)}(x) &= \sum_{i=0}^n \frac{(-1)^{n-i} (A + (n-i)I)_i}{(n-i)! i!} (\lambda x)^{n-i} \\ &= \sum_{i=0}^n \frac{(-1)^i (A)_n (A)_i^{-1}}{(n-i)! i!} (\lambda x)^i = L_n^{(A-I,\lambda)}(x) \end{aligned}$$

where

$$\alpha \notin \mathbb{Z}^- \cup \{0\} \text{ for } \forall \alpha \in \sigma(A) \text{ and } \lambda \text{ be a complex parameter with } Re(\lambda) > 0 \quad (2.4)$$

(see [8]).

Differentiating (2.1) with respect to x , the $f_{n,m}^{(A,\lambda)}(x)$ matrix polynomials satisfy recurrence relations

$$\frac{\partial f_{n,m}^{(A,\lambda)}(x)}{\partial x} = -\lambda f_{n-1,m}^{(A+I,\lambda)}(x) \quad (2.5)$$

and a generalization of the above equation

$$\frac{\partial^k f_{n,m}^{(A,\lambda)}(x)}{\partial x^k} = (-\lambda)^k f_{n-k,m}^{(A+kI,\lambda)}(x) ; \quad n \geq k.$$

For the special case $m = 1$ in (2.5), it follows that

$$\frac{\partial L_n^{(A-I,\lambda)}(x)}{\partial x} = -\lambda L_{n-1}^{(A,\lambda)}(x)$$

where A and λ satisfy (2.4). Similarly, for $g_{n,m}^{(A,\lambda)}(x)$, we have

$$\begin{aligned} \frac{\partial g_{n,m}^{(A,\lambda)}(x)}{\partial x} &= -\lambda g_{n-1,m}^{(A+I,\lambda)}(x) \\ \frac{\partial^k g_{n,m}^{(A,\lambda)}(x)}{\partial x^k} &= (-\lambda)^k g_{n-k,m}^{(A+kI,\lambda)}(x) ; \quad n \geq k. \end{aligned}$$

On the other hand, differentiating (2.1) with respect to t , the $f_{n,m}^{(A,\lambda)}(x)$ matrix polynomials satisfy recurrence relation

$$\begin{aligned} n f_{n,m}^{(A,\lambda)}(x) - (n-m) f_{n-m,m}^{(A,\lambda)}(x) &= m A \left[f_{n-m,m}^{(A+I,\lambda)}(x) - f_{n-2m,m}^{(A+I,\lambda)}(x) \right] \\ &\quad - x \lambda \left[f_{n-1,m}^{(A+I,\lambda)}(x) + (m-1) f_{n-m-1,m}^{(A+I,\lambda)}(x) \right] \end{aligned} \quad (2.6)$$

where $n \geq 2m$. For the special case $m = 1$ in (2.6), it holds that

$$n L_n^{(A-I,\lambda)}(x) - (n-1) L_{n-1}^{(A-I,\lambda)}(x) = A \left[L_{n-1}^{(A,\lambda)}(x) - L_{n-2}^{(A,\lambda)}(x) \right] - x \lambda L_{n-1}^{(A,\lambda)}(x)$$

where $n \geq 2$, A and λ satisfy (2.4). For $f_{n,m}^{(A,\lambda)}(x)$, other recurrence relation via differentiating (2.1) with respect to t is

$$\begin{aligned} n f_{n,m}^{(A,\lambda)}(x) &= [m(A-2I) + 2nI] f_{n-m,m}^{(A,\lambda)}(x) - [m(A-2I) + nI] f_{n-2m,m}^{(A,\lambda)}(x) \\ &\quad - x \lambda \left[f_{n-1,m}^{(A,\lambda)}(x) + (m-1) f_{n-m-1,m}^{(A,\lambda)}(x) \right] \end{aligned} \quad (2.7)$$

where $n \geq 2m$. For $m = 1$, (2.7) reduces the recurrence relation satisfied by the Laguerre matrix polynomials as follows:

$$n L_n^{(A,\lambda)}(x) = [(A-I) + 2nI - x\lambda I] L_{n-1}^{(A,\lambda)}(x) - [(A-I) + nI] L_{n-2}^{(A,\lambda)}(x), \quad n \geq 2$$

where A and λ satisfy (2.4). On the one hand, for $g_{n,m}^{(A,\lambda)}(x)$, we derive the following relations:

$$n g_{n,m}^{(A,\lambda)}(x) = -m A g_{n-m,m}^{(A+I,\lambda)}(x) - x \lambda g_{n-1,m}^{(A+2I,\lambda)}(x) + (m-1) x \lambda g_{n-m-1,m}^{(A+2I,\lambda)}(x)$$

where $n \geq m + 1$ and

$$\begin{aligned} & [2(n-m)I + mA]g_{n-m,m}^{(A,\lambda)}(x) + [(n-2m)I + mA]g_{n-2m,m}^{(A,\lambda)}(x) \\ &= -x\lambda g_{n-1,m}^{(A,\lambda)}(x) + x\lambda(m-1)g_{n-m-1,m}^{(A,\lambda)}(x) - ng_{n,m}^{(A,\lambda)}(x) \end{aligned}$$

where $n \geq 2m$. Now, we shall prove interesting relations for $f_{n,m}^{(A,\lambda)}(x)$ and $g_{n,m}^{(A,\lambda)}(x)$. Starting from (2.1), we have

$$F^{(A,\lambda)}(x,t)F^{(A,\lambda)}(y,t) = (1-t^m)^{-2A}e^{-\frac{(x+y)t\lambda}{1-t^m}} = \sum_{n=0}^{\infty} f_{n,m}^{(2A,\lambda)}(x+y)t^n. \quad (2.8)$$

Using (2.1) in the left-hand side of (2.8), it follows that

$$\sum_{i=0}^n f_{n-i,m}^{(A,\lambda)}(x)f_{i,m}^{(A,\lambda)}(y) = f_{n,m}^{(2A,\lambda)}(x+y). \quad (2.9)$$

For $m = 1$, (2.9) reduces

$$\sum_{i=0}^n L_{n-i}^{(A-I,\lambda)}(x)L_i^{(A-I,\lambda)}(y) = L_n^{(2A-I,\lambda)}(x+y)$$

where A and λ satisfy (2.4) and $\mu \notin \mathbb{Z}^- \cup \{0\}$ for $\forall \mu \in \sigma(2A)$. In the same way, $g_{n,m}^{(A,\lambda)}(x)$ matrix polynomials satisfy

$$\sum_{i=0}^n g_{n-i,m}^{(A,\lambda)}(x)g_{i,m}^{(A,\lambda)}(y) = g_{n,m}^{(2A,\lambda)}(x+y).$$

Similarly, we have

$$\begin{aligned} \sum_{i=0}^n g_{n-i,m}^{(A,\lambda)}(x)g_{i,m}^{(B,\lambda)}(x) &= g_{n,m}^{(A+B,\lambda)}(2x), \\ \sum_{i=0}^n f_{n-i,m}^{(A,\lambda)}(x)f_{i,m}^{(B,\lambda)}(x) &= f_{n,m}^{(A+B,\lambda)}(2x) \end{aligned} \quad (2.10)$$

where $AB = BA$. For $m = 1$ in (2.10), we get

$$\sum_{i=0}^n L_{n-i}^{(A-I,\lambda)}(x)L_i^{(B-I,\lambda)}(x) = L_n^{(A+B-I,\lambda)}(2x)$$

where A and λ satisfy (2.4), $\nu \notin \mathbb{Z}^- \cup \{0\}$ for $\forall \nu \in \sigma(B)$ and $\eta \notin \mathbb{Z}^- \cup \{0\}$ for $\forall \eta \in \sigma(A+B)$. On the other hand, (2.8) can be written as

$$(1-t^m)^{-2A}e^{-\frac{xt\lambda}{1-t^m}} = e^{\frac{yt\lambda}{1-t^m}} \sum_{i=0}^{\infty} f_{i,m}^{(2A,\lambda)}(x+y)t^i$$

and then

$$\begin{aligned} \sum_{n=0}^{\infty} f_{n,m}^{(2A,\lambda)}(x)t^n &= \left[\sum_{n=0}^{\infty} \frac{\left(\frac{yt\lambda}{1-t^m}\right)^n}{n!} \right] \left[\sum_{i=0}^{\infty} f_{i,m}^{(2A,\lambda)}(x+y)t^i \right] \\ &= \left[\sum_{n=0}^{\infty} \frac{(yt\lambda)^n}{n!} \right] \left[\sum_{j=0}^{\infty} \frac{(n)_j}{j!} t^{mj} \right] \left[\sum_{i=0}^{\infty} f_{i,m}^{(2A,\lambda)}(x+y)t^i \right] \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{\lfloor \frac{n}{m} \rfloor} \sum_{i=0}^{n-mj} \frac{\lambda^{n-mj-i} y^{n-mj-i}}{(n-mj-i)!} \frac{(n-mj-i)_j}{j!} f_{i,m}^{(2A,\lambda)}(x+y)t^n. \end{aligned}$$

Comparing the coefficients of t^n , we have

$$f_{n,m}^{(2A,\lambda)}(x) = \sum_{j=0}^{\lfloor \frac{n}{m} \rfloor} \sum_{i=0}^{n-mj} \frac{\lambda^{n-mj-i} y^{n-mj-i}}{(n-mj-i)!} \frac{(n-mj-i)_j}{j!} f_{i,m}^{(2A,\lambda)}(x+y). \quad (2.11)$$

If $m = 1$ in (2.11), we obtain

$$L_n^{(2A-I,\lambda)}(x) = \sum_{j=0}^n \sum_{i=0}^{n-j} \frac{\lambda^{n-j-i} y^{n-j-i}}{(n-j-i)!} \frac{(n-j-i)_j}{j!} L_i^{(2A-I,\lambda)}(x+y)$$

where $\mu \notin \mathbb{Z}^- \cup \{0\}$ for $\forall \mu \in \sigma(2A)$. Similarly, for $g_{n,m}^{(A,\lambda)}(x)$, we get

$$g_{n,m}^{(2A,\lambda)}(x) = \sum_{j=0}^{\lfloor \frac{n}{m} \rfloor} \sum_{i=0}^{n-mj} \frac{(-1)^j \lambda^{n-mj-i} y^{n-mj-i}}{(n-mj-i)!} \frac{(n-mj-i)_j}{j!} g_{i,m}^{(2A,\lambda)}(x+y).$$

Now, let's give relations for the derivatives of new matrix polynomials. From (2.1), it follows that

$$\frac{\partial^s}{\partial x^s} F^{(A,\lambda)}(x,t) = (-1)^s t^s \lambda^s (1-t^m)^{-A-sI} e^{-\frac{xt\lambda}{1-t^m}}.$$

By the above equation and (2.1), we can write

$$\begin{aligned} \frac{\partial^s}{\partial x^s} F^{(A,\lambda)}(x,t) \frac{\partial^s}{\partial y^s} F^{(A,\lambda)}(y,t) &= \sum_{n=0}^{\infty} \lambda^{2s} f_{n,m}^{(2A+2sI,\lambda)}(x+y)t^{n+2s}, \\ \frac{\partial^s}{\partial x^s} \left[\sum_{n=0}^{\infty} f_{n,m}^{(A,\lambda)}(x)t^n \right] \frac{\partial^s}{\partial y^s} \left[\sum_{i=0}^{\infty} f_{i,m}^{(A,\lambda)}(y)t^i \right] &= \sum_{n=0}^{\infty} \lambda^{2s} f_{n,m}^{(2A+2sI,\lambda)}(x+y)t^{n+2s}, \\ \sum_{n=0}^{\infty} \sum_{i=0}^n D_x^s f_{n-i,m}^{(A,\lambda)}(x) D_y^s f_{i,m}^{(A,\lambda)}(y)t^n &= \sum_{n=2s}^{\infty} \lambda^{2s} f_{n-2s,m}^{(2A+2sI,\lambda)}(x+y)t^n \end{aligned}$$

where $D_x^s = \frac{d^s}{dx^s}$ and $D_y^s = \frac{d^s}{dy^s}$. Comparing the coefficients of t^n , we have

$$\sum_{i=0}^n D_x^s f_{n-i,m}^{(A,\lambda)}(x) D_y^s f_{i,m}^{(A,\lambda)}(y) = \lambda^{2s} f_{n-2s,m}^{(2A+2sI,\lambda)}(x+y) \quad ; \quad n \geq 2s. \quad (2.12)$$

If $m = 1$ in (2.12), (2.12) reduces

$$\sum_{i=0}^n D_x^s L_{n-i}^{(A-I,\lambda)}(x) D_y^s L_i^{(A-I,\lambda)}(y) = \lambda^{2s} L_{n-2s}^{(2A+(2s-1)I,\lambda)}(x+y) \quad ; \quad n \geq 2s$$

where A and λ satisfy (2.4) and $\mu \notin \mathbb{Z}^- \cup \{0\}$ for $\forall \mu \in \sigma(2A)$. In the same way, we get

$$\sum_{i=0}^n D_x^s g_{n-i,m}^{(A,\lambda)}(x) D_y^s g_{i,m}^{(A,\lambda)}(y) = \lambda^{2s} g_{n-2s,m}^{(2A+2sI,\lambda)}(x+y) \quad ; \quad n \geq 2s. \quad (2.13)$$

On the one hand, using (2.1) and (2.2), we can write

$$\frac{\partial^k}{\partial x^k} F^{(A,\lambda)}(x,t) \frac{\partial^k}{\partial x^k} G^{(A,\lambda)}(x,t) = \sum_{n=0}^{\infty} \lambda^{2k} f_{n,2m}^{(A+kI,\lambda)}(2x) t^{n+2k}.$$

Thus, it holds that

$$\sum_{i=0}^n D_x^k f_{n-i,m}^{(A,\lambda)}(x) D_x^k g_{i,m}^{(A,\lambda)}(x) = \lambda^{2k} f_{n-2k,2m}^{(A+kI,\lambda)}(2x) \quad ; \quad n \geq 2k. \quad (2.14)$$

The generalizations of (2.10) are as follows:

$$\begin{aligned} \sum_{i_1+i_2+\dots+i_k=n} f_{i_1,m}^{(A_1,\lambda)}(x_1) \dots f_{i_k,m}^{(A_k,\lambda)}(x_k) &= f_{n,m}^{(A_1+\dots+A_k,\lambda)}(x_1+\dots+x_k), \\ \sum_{i_1+i_2+\dots+i_k=n} g_{i_1,m}^{(A_1,\lambda)}(x_1) \dots g_{i_k,m}^{(A_k,\lambda)}(x_k) &= g_{n,m}^{(A_1+\dots+A_k,\lambda)}(x_1+\dots+x_k). \end{aligned}$$

where the matrices $A_1, \dots, A_k$ are assumed to be commutative. For $k = 0$ in (2.14), we have

$$\sum_{i=0}^n f_{n-i,m}^{(A,\lambda)}(x) g_{i,m}^{(A,\lambda)}(x) = f_{n,m}^{(A,\lambda)}(2x). \quad (2.15)$$

The generalization of (2.15) is

$$\begin{aligned} &\sum_{s=0}^n \left\{ \sum_{i_1+i_2+\dots+i_k=n-s} f_{i_1,m}^{(A_1,\lambda)}(x_1) \dots f_{i_k,m}^{(A_k,\lambda)}(x_k) \right. \\ &\quad \times \left. \sum_{j_1+j_2+\dots+j_k=n} g_{j_1,m}^{(A_1,\lambda)}(x_1) \dots g_{j_k,m}^{(A_k,\lambda)}(x_k) \right\} \end{aligned}$$

$$= \sum_{i_1+i_2+\dots+i_k=n} f_{i_1,2m}^{(A_1,\lambda)}(2x_1) \dots f_{i_k,2m}^{(A_k,\lambda)}(2x_k)$$

where the matrices $A_1, \dots, A_k$ are assumed to be commutative.

3. MULTILINEAR AND MULTILATERAL GENERATING MATRIX FUNCTIONS FOR $f_{n,m}^{(A,\lambda)}(x)$ AND $g_{n,m}^{(A,\lambda)}(x)$

In this section, we derive several families of bilinear and bilateral generating matrix functions for the new matrix polynomials generated by (2.1) and (2.2). We first state our result.

Theorem 1. *Corresponding to a non-vanishing function $\Omega_\mu(y_1, \dots, y_s)$ of s complex variables $y_1, \dots, y_s$ ($s \in \mathbb{N}$) and of complex order μ , let*

$$\Lambda_{\mu,v}(y_1, \dots, y_s; z) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+vk}(y_1, \dots, y_s) z^k; \quad (a_k \neq 0, \mu, v \in \mathbb{C}) \quad (3.1)$$

and

$$\Theta_{n,p,\mu,v}(x; y_1, \dots, y_s; \zeta) := \sum_{k=0}^{[n/p]} a_k f_{n-pk,m}^{(A,\lambda)}(x) \Omega_{\mu+vk}(y_1, \dots, y_s) \zeta^k \quad (3.2)$$

where $n, p \in \mathbb{N}$ and (as usual) $[\alpha]$ represents the greatest integer in $\alpha \in \mathbb{R}$. Then we have

$$\sum_{n=0}^{\infty} \Theta_{n,p,\mu,v} \left(x; y_1, \dots, y_s; \frac{\eta}{t^p} \right) t^n = (1-t^m)^{-A} e^{-\frac{xt\lambda}{1-t^m}} \Lambda_{\mu,v}(y_1, \dots, y_s; \eta) \quad (3.3)$$

provided that each member of (3.3) exists for $|t| < 1$ and $\operatorname{Re}(\lambda) > 0$.

Proof. For convenience, let S denote the first member of the assertion (3.3) of Theorem 1. Then, upon substituting for the polynomials $\Theta_{n,p,\mu,v}(x; y_1, \dots, y_s; \frac{\eta}{t^p})$ from the definition (3.2) into the left-hand side of (3.3), we obtain

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} a_k f_{n-pk,m}^{(A,\lambda)}(x) \Omega_{\mu+vk}(y_1, \dots, y_s) \eta^k t^{n-pk}. \quad (3.4)$$

Upon inverting the order of summation in (3.4), if we replace n by $n + pk$, we can write

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k f_{n,m}^{(A,\lambda)}(x) \Omega_{\mu+vk}(y_1, \dots, y_s) \eta^k t^n \\ &= \left(\sum_{n=0}^{\infty} f_{n,m}^{(A,\lambda)}(x) t^n \right) \left(\sum_{k=0}^{\infty} a_k \Omega_{\mu+vk}(y_1, \dots, y_s) \eta^k \right) \end{aligned}$$

$$= \left\{ (1-t^m)^{-A} e^{-\frac{xt\lambda}{1-t^m}} \right\} \Lambda_{\mu,v}(y_1, \dots, y_s; \eta),$$

which completes the proof of Theorem 1. $\square$

Corollary 1. *Corresponding to a non-vanishing function $\Omega_\mu(y_1, \dots, y_s)$ of s complex variables $y_1, \dots, y_s$ ($s \in \mathbb{N}$) and of complex order μ , let*

$$\Lambda_{\mu,v}(y_1, \dots, y_s; z) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+vk}(y_1, \dots, y_s) z^k \quad ; (a_k \neq 0, \mu, v \in \mathbb{C}) \quad (3.5)$$

and

$$\Theta_{n,p,\mu,v}(x; y_1, \dots, y_s; \zeta) := \sum_{k=0}^{[n/p]} a_k g_{n-pk,m}^{(A,\lambda)}(x) \Omega_{\mu+vk}(y_1, \dots, y_s) \zeta^k \quad (3.6)$$

where $n, p \in \mathbb{N}$ and $[\alpha]$ represents the greatest integer in $\alpha \in \mathbb{R}$. Then we have

$$\sum_{n=0}^{\infty} \Theta_{n,p,\mu,v} \left(x; y_1, \dots, y_s; \frac{\eta}{t^p} \right) t^n = (1+t^m)^{-A} e^{-\frac{xt\lambda}{1+t^m}} \Lambda_{\mu,v}(y_1, \dots, y_s; \eta) \quad (3.7)$$

provided that each member of (3.7) exists for $|t| < 1$ and $\text{Re}(\lambda) > 0$.

By expressing the multivariable function $\Omega_{\mu+vk}(y_1, \dots, y_s)$ ($k \in \mathbb{N}_0, s \in \mathbb{N}$) in terms of simpler function of one and more variables, we can give further applications of Theorem 1. For example, if we set $s = 1$ and $\Omega_{\mu+vk}(y) = g_{\mu+vk,m}^{(B,\gamma)}(y)$ in Theorem 1, where $g_{k,m}^{(B,\gamma)}(y)$ is defined by (2.2), then we obtain the following result which provides a class of bilateral generating functions for the $f_{n,m}^{(A,\lambda)}(x)$ and $g_{n,m}^{(A,\lambda)}(x)$.

Corollary 2. *If $\Lambda_{\mu,v}(y; z) := \sum_{k=0}^{\infty} a_k g_{\mu+vk,m}^{(B,\gamma)}(y) z^k$ where $(a_k \neq 0, \mu, v \in \mathbb{N}_0)$;*

and

$$\Theta_{n,p,\mu,v}(x; y; \zeta) := \sum_{k=0}^{[n/p]} a_k f_{n-pk,m}^{(A,\lambda)}(x) g_{\mu+vk,m}^{(B,\gamma)}(y) \zeta^k$$

where $n, p \in \mathbb{N}$, then we have

$$\sum_{n=0}^{\infty} \Theta_{n,p,\mu,v} \left(x; y; \frac{\eta}{t^p} \right) t^n = (1-t^m)^{-A} e^{-\frac{xt\lambda}{1-t^m}} \Lambda_{\mu,v}(y; \eta) \quad (3.8)$$

provided that each member of (3.8) exists.

Remark 1. Using the generating relation (2.2) for $g_{k,m}^{(B,\gamma)}(y)$ and taking $a_k = 1$, $\mu = 0$, $\nu = 1$, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} f_{n-pk,m}^{(A,\lambda)}(x) g_{k,m}^{(B,\gamma)}(y) \eta^k t^{n-pk} \\ &= (1-t^m)^{-A} e^{-\frac{xt\lambda}{1-t^m}} (1+\eta^m)^{-B} e^{-\frac{y\eta\gamma}{1+\eta^m}} \end{aligned}$$

where $|\eta| < 1$.

Choosing $s = 1$ and $\Omega_{\mu+\nu k}(y) = f_{\mu+\nu k,m}^{(B,\gamma)}(y)$, $(\mu, \nu \in \mathbb{N}_0)$, in Theorem 1, we obtain the following class of bilinear generating function for the $f_{n,m}^{(A,\lambda)}(x)$.

Corollary 3. If $\Lambda_{\mu,\nu}(y; z) := \sum_{k=0}^{\infty} a_k f_{\mu+\nu k,m}^{(B,\gamma)}(y) z^k$ where $(a_k \neq 0, \mu, \nu \in \mathbb{N}_0)$; and

$$\Theta_{n,p,\mu,\nu}(x; y; \zeta) := \sum_{k=0}^{[n/p]} a_k f_{n-pk,m}^{(A,\lambda)}(x) f_{\mu+\nu k,m}^{(B,\gamma)}(y) \zeta^k$$

where $n, p \in \mathbb{N}$, then we have

$$\sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu} \left(x; y; \frac{\eta}{t^p} \right) t^n = (1-t^m)^{-A} e^{-\frac{xt\lambda}{1-t^m}} \Lambda_{\mu,\nu}(y; \eta) \quad (3.9)$$

provided that each member of (3.9) exists.

Remark 2. Using Corollary 3 and taking $a_k = 1$, $\mu = 0$, $\nu = 1$, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} f_{n-pk,m}^{(A,\lambda)}(x) f_{k,m}^{(B,\gamma)}(y) \eta^k t^{n-pk} \\ &= (1-t^m)^{-A} e^{-\frac{xt\lambda}{1-t^m}} (1-\eta^m)^{-B} e^{-\frac{y\eta\gamma}{1-\eta^m}} \end{aligned}$$

where $|\eta| < 1$.

Now, we obtain the following class of bilinear generating function for the $g_{n,m}^{(A,\lambda)}(x)$.

Corollary 4. If

$$\Lambda_{\mu,\nu}^{n,p}(x; y; z) := \sum_{k=0}^{[n/p]} a_k g_{n-pk,m}^{(A+B,\lambda)}(2x) g_{\mu+\nu k,m}^{(C,\gamma)}(y) z^k,$$

where $a_k \neq 0$, $n, p \in \mathbb{N}$, $\mu, \nu \in \mathbb{N}_0$, $AB = BA$, then we have

$$\sum_{k=0}^n \sum_{l=0}^{[k/p]} a_l g_{n-k,m}^{(A,\lambda)}(x) g_{k-pl,m}^{(B,\lambda)}(x) g_{\mu+\nu l,m}^{(C,\gamma)}(y) z^l = \Lambda_{\mu,\nu}^{n,p}(x; y; z) \quad (3.10)$$

provided that each member of (3.10) exists.

Furthermore, for every suitable choice of the coefficients a_k ($k \in \mathbb{N}_0$), if the multivariable function $\Omega_{\mu+\psi k}(y_1, \dots, y_s)$, ($s \in \mathbb{N}$), is expressed as an appropriate product of several simpler functions, the assertions of Theorem 1 and Corollary 1 can be applied in order to derive various families of multilinear and multilateral generating functions for the $f_{n,m}^{(A,\lambda)}(x)$ and $g_{n,m}^{(A,\lambda)}(x)$.

REFERENCES

- [1] R. Aktaş, B. Çekim, and R. Şahin, "The matrix version for the multivariable Humbert polynomials," *Math. Notes, Miskolc*, vol. 13, no. 2, pp. 197–208, 2012.
- [2] B. Çekim, A. Altın, and R. Aktaş, "Some relations satisfied by orthogonal matrix polynomials," *Hacet. J. Math. Stat.*, vol. 40, no. 2, pp. 241–253, 2011.
- [3] E. Defez, L. Jódar, and A. Law, "Jacobi matrix differential equation, polynomial solutions, and their properties," *Comput. Math. Appl.*, vol. 48, no. 5-6, pp. 789–803, 2004.
- [4] E. Defez and L. Jódar, "Chebyshev matrix polynomials and second order matrix differential equations," *Util. Math.*, vol. 61, pp. 107–123, 2002.
- [5] G. B. Djordjević, "On the generalized Laguerre polynomials," *Fibonacci Q.*, vol. 39, no. 5, pp. 403–408, 2001.
- [6] N. Dunford and J. Schwartz, *Linear Operators*. New York: Interscience, 1957, vol. I.
- [7] L. Jódar and R. Company, "Hermite matrix polynomials and second order matrix differential equations," *Approximation Theory Appl.*, vol. 12, no. 2, pp. 20–30, 1996.
- [8] L. Jódar, R. Company, and E. Navarro, "Laguerre matrix polynomials and systems of second-order differential equations," *Appl. Numer. Math.*, vol. 15, no. 1, pp. 53–63, 1994.
- [9] L. Jódar, R. Company, and E. Ponsoda, "Orthogonal matrix polynomials and systems of second order differential equations," *Differ. Equ. Dyn. Syst.*, vol. 3, no. 3, pp. 269–288, 1995.
- [10] Z. M. G. Kishka, A. Shehata, and M. Abuldahab, "The generalized Bessel matrix polynomials," *J. Math. Comput. Sci.*, vol. 2, no. 2, pp. 305–316, 2012.
- [11] S. Varma, B. Çekim, and F. Taşdelen Yeşildal, "On Konhauser matrix polynomials," *Ars Comb.*, vol. 100, pp. 193–204, 2011.
- [12] S. Varma and F. Taşdelen, "Biorthogonal matrix polynomials related to Jacobi matrix polynomials," *Comput. Math. Appl.*, vol. 62, no. 10, pp. 3663–3668, 2011.

Author's address

Bayram Çekim

Gazi University, Faculty of Science, Department of Mathematics, Teknik Okullar TR-06500, Ankara, Turkey

E-mail address: bayramcekim@gazi.edu.tr



ALGEBRAS ASSIGNED TO TERNARY RELATIONS

IVAN CHAJDA, MIROSLAV KOLAŘÍK, AND HELMUT LÄNGER

Received 19 March, 2012

Abstract. We show that to every centred ternary relation T on a set A there can be assigned (in a non-unique way) a ternary operation t on A such that the identities satisfied by $(A; t)$ reflect relational properties of T . We classify ternary operations assigned to centred ternary relations and we show how the concepts of relational subsystems and homomorphisms are connected with subalgebras and homomorphisms of the assigned algebra $(A; t)$. We show that for ternary relations having a non-void median can be derived so-called median-like algebras $(A; t)$ which become median algebras if the median $M_T(a, b, c)$ is a singleton for all $a, b, c \in A$. Finally, we introduce certain algebras assigned to cyclically ordered sets.

2010 *Mathematics Subject Classification:* 08A02; 08A05

Keywords: ternary relation, betweenness, cyclic order, assigned operation, centre, median

In [2] and [3], the first and the third author showed that to certain relational systems $\mathcal{A} = (A; R)$, where $A \neq \emptyset$ and R is a binary relation on A , there can be assigned a certain groupoid $\mathcal{G}(A) = (A; \circ)$ which captures the properties of R . Namely, we have $x \circ y = y$ if and only if $(x, y) \in R$. In these papers we worked with so-called directed relational systems, i. e. for all $x, y \in A$ we have

$$U_R(x, y) := \{z \in A \mid (x, z), (y, z) \in R\} \neq \emptyset.$$

We are inspired by the idea of assigning a groupoid (called directoid) to a directed poset. This idea has its origin in the paper [6] by J. Ježek and R. Quackenbush. Then some structural properties of the assigned groupoid $\mathcal{G}(A)$ can be used for introducing certain structural properties of $\mathcal{A} = (A; R)$; in particular, we introduced congruences, quotient relational systems and homomorphisms which are in accordance with the corresponding concepts in $\mathcal{G}(A)$.

Hence, there arises the natural question if a similar way can be used for ternary relational systems and algebras with one ternary relation. In a particular case, such a

Support of the research of the first and third author by ÖAD, Cooperation between Austria and Czech Republic in Science and Technology, grant No. CZ 01/2011, of the first author by the Project CZ.1.07/2.3.00/20.0051 Algebraic Methods in Quantum Logics and of the second author by the Project CZ.1.07/2.3.00/20.0060 International Center for Information and Uncertainty is gratefully acknowledged.

correspondence exists. It is for the ternary relation “betweenness” and the so-called median algebras, see e.g. [1, 5] or [11].

However, there exist also other useful ternary relations for which a similar construction is not already derived, in particular the so-called cyclic orders, see e.g. [4, 7, 8] and [9].

Moreover, more general ternary relations were already investigated in [10] and [11] and hence our problem can be extended to a more general case than betweenness. However, to get a construction of a ternary operation, a certain restriction on the ternary relation is necessary.

In the following let A denote a fixed arbitrary non-empty set.

1. TERNARY OPERATIONS ASSIGNED TO TERNARY RELATIONS

We introduce the following concepts:

Definition 1. Let T be a ternary relation on A and $a, b \in A$. The set

$$Z_T(a, b) := \{x \in A \mid (a, x, b) \in T\}$$

is called the **centre of (a, b) with respect to T** . The ternary relation T on A is called **centred** if $Z_T(a, b) \neq \emptyset$ for all elements $a, b \in A$.

Definition 2. Let T be a ternary relation on A and $a, b, c \in A$. The set

$$M_T(a, b, c) := Z_T(a, b) \cap Z_T(b, c) \cap Z_T(c, a)$$

will be called the **median of (a, b, c) with respect to T** .

The concept of a median was originally introduced in lattices and structures derived from lattices. In particular, two sorts of medians are usually considered: $m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ and $M(x, y, z) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$.

Now we show that to every centred ternary relation there can be assigned ternary operations.

Definition 3. Let T be a centred ternary relation on A and t a ternary operation on A satisfying

$$t(a, b, c) \begin{cases} = b & \text{if } (a, b, c) \in T \\ \in Z_T(a, c) & \text{otherwise.} \end{cases}$$

Such an operation t is called **assigned to T** .

Remark 1. By definition, if T is a centred ternary relation on A and t assigned to T then $(a, t(a, b, c), c) \in T$ for all $a, b, c \in A$.

Lemma 1. Let T be a centred ternary relation on A and t an assigned operation. Let $a, b, c \in A$. Then $(a, b, c) \in T$ if and only if $t(a, b, c) = b$.

Proof. By Definition 3, if $(a, b, c) \in T$ then $t(a, b, c) = b$. Conversely, assume $(a, b, c) \notin T$. Then $t(a, b, c) \in Z_T(a, c)$. Now $t(a, b, c) = b$ would imply $(a, b, c) = (a, t(a, b, c), c) \in T$ contradicting $(a, b, c) \notin T$. Hence $t(a, b, c) \neq b$. $\square$

To illuminate the role of the median, let us consider the following example:

Example 1. Let $\mathcal{L} = (L; \vee, \wedge)$ be a lattice. Define a ternary operation T on L as follows:

$$(a, b, c) \in T \quad \text{if and only if} \quad a \wedge c \leq b \leq a \vee c.$$

Put $m(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ and $M(x, y, z) := (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$. If $p \in M_T(a, b, c)$ then $p \in Z_T(a, b)$, $p \in Z_T(b, c)$ and $p \in Z_T(c, a)$, i. e. $a \wedge b \leq p \leq a \vee b$, $b \wedge c \leq p \leq b \vee c$ and $c \wedge a \leq p \leq c \vee a$ whence $m(a, b, c) \leq p \leq M(a, b, c)$. This yields

$$M_T(a, b, c) = [m(a, b, c), M(a, b, c)],$$

the interval in $\mathcal{L}$. It is well-known that $m(x, y, z) = M(x, y, z)$ if and only if $\mathcal{L}$ is distributive. Hence, $\mathcal{L}$ is distributive if and only if $|M_T(a, b, c)| = 1$ for all $a, b, c \in L$.

The previous example was used in [5] for the definition of a median algebra. If $\mathcal{L}$ is a distributive lattice then the algebra $(L; m)$ is called the median algebra derived from $\mathcal{L}$. Of course, there exist median algebras which are not derived from a lattice, see [1] for details, but in every median algebra there can be introduced a ternary relation “between” by putting

$$(a, b, c) \in T_m \quad \text{if and only if} \quad m(a, b, c) = b.$$

In what follows, we show how this construction can be generalized and we get a characterization of some important properties of ternary relations by means of identities of their assigned operations.

Theorem 1. *A ternary operation t on A is assigned to some centred ternary relation T on A if and only if it satisfies the identity*

$$t(x, t(x, y, z), z) = t(x, y, z). \quad (1.1)$$

Proof. Let $a, b, c \in A$.

Assume that T is a ternary relation on A and t an assigned operation. If $(a, b, c) \in T$ then $t(a, b, c) = b$ and hence $t(a, t(a, b, c), c) = t(a, b, c)$. If $(a, b, c) \notin T$ then $t(a, b, c) \in Z_T(a, c)$ and hence $(a, t(a, b, c), c) \in T$ which yields $t(a, t(a, b, c), c) = t(a, b, c)$. Thus t satisfies identity (1.1).

Conversely, assume $t : A^3 \rightarrow A$ satisfies (1.1) and define $T := \{(x, y, z) \in A^3 \mid t(x, y, z) = y\}$. If $(a, b, c) \in T$ then $t(a, b, c) = b$ and, if $(a, b, c) \notin T$ then $(a, t(a, b, c), c) \in T$ whence $t(a, b, c) \in Z_T(a, c)$, i. e. t is assigned to T . $\square$

We can consider a number of properties of ternary relations which are used in [1–11] for “betweenness” and for “cyclic orders”.

Definition 4. Let T be a ternary relation on A . We call T

- **reflexive** if $|\{a, b, c\}| \leq 2$ implies $(a, b, c) \in T$;
- **symmetric** if $(a, b, c) \in T$ implies $(c, b, a) \in T$;

- **antisymmetric** if $(a, b, a) \in T$ implies $a = b$;
- **cyclic** if $(a, b, c) \in T$ implies $(b, c, a) \in T$;
- **R -transitive** if $(a, b, c), (b, d, e) \in T$ implies $(a, d, e) \in T$;
- **t_1 -transitive** if $(a, b, c), (a, d, b) \in T$ implies $(d, b, c) \in T$;
- **t_2 -transitive** if $(a, b, c), (a, d, b) \in T$ implies $(a, d, c) \in T$;
- **R -symmetric** if $(a, b, c) \in T$ implies $(b, a, c) \in T$;
- **R -antisymmetric** if $(a, b, c), (b, a, c) \in T$ implies $a = b$;
- **non-sharp** if $(a, a, b) \in T$ for all $a, b \in A$;
- **cyclically transitive** if $(a, b, c), (a, c, d) \in T$ implies $(a, b, d) \in T$.

Theorem 2. *Let T be a centred ternary relation on A and t an assigned operation. Then (i) – (xi) hold:*

(i) *T is reflexive if and only if t satisfies the identities*

$$t(x, x, y) = t(y, x, x) = t(y, x, y) = x.$$

(ii) *T is symmetric if and only if t satisfies the identity*

$$t(z, t(x, y, z), x) = t(x, y, z).$$

(iii) *T is antisymmetric if and only if t satisfies the identity*

$$t(x, y, x) = x.$$

(iv) *T is cyclic if and only if t satisfies the identity*

$$t(t(x, y, z), z, x) = z.$$

(v) *T is R -transitive if and only if t satisfies the identity*

$$t(x, t(t(x, y, z), u, v), v) = t(t(x, y, z), u, v).$$

(vi) *T is t_1 -transitive if and only if t satisfies the identity*

$$t(t(x, u, t(x, y, z)), t(x, y, z), z) = t(x, y, z).$$

(vii) *T is t_2 -transitive if and only if t satisfies the identity*

$$t(x, t(x, u, t(x, y, z)), z) = t(x, u, t(x, y, z)).$$

(viii) *T is R -symmetric if and only if t satisfies the identity*

$$t(t(x, y, z), x, z) = x.$$

(ix) *If t satisfies the identity*

$$t(t(x, y, z), x, z) = t(x, y, z)$$

then T is R -antisymmetric.

(x) *T is non-sharp if and only if t satisfies the identity*

$$t(x, x, y) = x.$$

(xi) *T is cyclically transitive if and only if t satisfies the identity*

$$t(x, t(x, y, t(x, z, u)), u) = t(x, y, t(x, z, u)).$$

Proof. Let $a, b, c, d, e \in A$.

(i) is clear.

(ii) t satisfies $t(z, t(x, y, z), x) = t(x, y, z)$ if and only if $(z, t(x, y, z), x) \in T$ for all $x, y, z \in A$.

“ $\Rightarrow$ ”: $(a, t(a, b, c), c) \in T$ and hence $(c, t(a, b, c), a) \in T$.

“ $\Leftarrow$ ”: If $(a, b, c) \in T$ then $(c, b, a) = (c, t(a, b, c), a) \in T$.

(iii) “ $\Rightarrow$ ”: $(a, t(a, b, a), a) \in T$ and hence $t(a, b, a) = a$.

“ $\Leftarrow$ ”: If $(a, b, a) \in T$ then $a = t(a, b, a) = b$.

(iv) t satisfies $t(t(x, y, z), z, x) = z$ if and only if $(t(x, y, z), z, x) \in T$ for all $x, y, z \in A$.

“ $\Rightarrow$ ”: $(a, t(a, b, c), c) \in T$ and hence $(t(a, b, c), c, a) \in T$.

“ $\Leftarrow$ ”: If $(a, b, c) \in T$ then $(b, c, a) = (t(a, b, c), c, a) \in T$.

(v) t satisfies $t(x, t(t(x, y, z), u, v), v) = t(t(x, y, z), u, v)$ if and only if $(x, t(t(x, y, z), u, v), v) \in T$ for all $x, y, z, u, v \in A$.

“ $\Rightarrow$ ”: $(a, t(a, b, c), c), (t(a, b, c), t(t(a, b, c), d, e), e) \in T$ and hence $(a, t(t(a, b, c), d, e), e) \in T$.

“ $\Leftarrow$ ”: If $(a, b, c), (b, d, e) \in T$ then $(a, d, e) = (a, t(t(a, b, c), d, e), e) \in T$.

(vi) t satisfies $t(t(x, u, t(x, y, z)), t(x, y, z), z) = t(x, y, z)$ if and only if $(t(x, u, t(x, y, z)), t(x, y, z), z) \in T$ for all $x, y, z, u \in A$.

“ $\Rightarrow$ ”: $(a, t(a, b, c), c), (a, t(a, d, t(a, b, c)), t(a, b, c)) \in T$ and hence $(t(a, d, t(a, b, c)), t(a, b, c), c) \in T$.

“ $\Leftarrow$ ”: If $(a, b, c), (a, d, b) \in T$ then $(d, b, c) = (t(a, d, t(a, b, c)), t(a, b, c), c) \in T$.

(vii) t satisfies $t(x, t(x, u, t(x, y, z)), z) = t(x, u, t(x, y, z))$ if and only if $(x, t(x, u, t(x, y, z)), z) \in T$ for all $x, y, z, u \in A$.

“ $\Rightarrow$ ”: $(a, t(a, b, c), c), (a, t(a, d, t(a, b, c)), t(a, b, c)) \in T$ and hence $(a, t(a, d, t(a, b, c)), c) \in T$.

“ $\Leftarrow$ ”: If $(a, b, c), (a, d, b) \in T$ then $(a, d, c) = (a, t(a, d, t(a, b, c)), c) \in T$.

(viii) t satisfies $t(t(x, y, z), x, z) = x$ if and only if $(t(x, y, z), x, z) \in T$ for all $x, y, z \in A$.

“ $\Rightarrow$ ”: $(a, t(a, b, c), c) \in T$ and hence $(t(a, b, c), a, c) \in T$.

“ $\Leftarrow$ ”: If $(a, b, c) \in T$ then $(b, a, c) = (t(a, b, c), a, c) \in T$.

(ix) If $(a, b, c), (b, a, c) \in T$ then $a = t(b, a, c) = t(t(a, b, c), a, c) = t(a, b, c) = b$.

(x) This is clear.

(xi) t satisfies $t(x, t(x, y, t(x, z, u)), u) = t(x, y, t(x, z, u))$ if and only if $(x, t(x, y, t(x, z, u)), u) \in T$ for all $x, y, z, u \in A$.

“ $\Rightarrow$ ”: $(a, t(a, b, t(a, c, d)), t(a, c, d)), (a, t(a, c, d), d) \in T$ and hence $(a, t(a, b, t(a, c, d)), d) \in T$.

“ $\Leftarrow$ ”: If $(a, b, c), (a, c, d) \in T$ then $t(a, b, d) = t(a, t(a, b, t(a, c, d)), d) = t(a, b, t(a, c, d)) = b$.

□

Lemma 2. *Let T be a ternary relation on A . Then*

$$|T| = \sum_{(a,b) \in A^2} |Z_T(a,b)|.$$

Proof.

$$T = \dot{\bigcup}_{(a,b) \in A^2} (\{a\} \times Z_T(a,b) \times \{b\}).$$

□

Corollary 1. *Let A be finite, $|A| = n$. If T is a centred ternary relation on A then $|T| \geq n^2$. Moreover, if T is centred then $|T| = n^2$ if and only if $|Z_T(x,y)| = 1$ for each $x, y \in A$.*

2. CONGRUENCES, HOMOMORPHISMS AND SUBSYSTEMS OF TERNARY RELATIONAL SYSTEMS

By a **ternary relational system** is meant a couple $\mathcal{T} = (A; T)$ where T is a ternary relation on A . $\mathcal{T}$ is called **centred** if T is centred. As shown in the previous section, to every centred ternary relational system $\mathcal{T} = (A; T)$ there can be assigned an algebra $\mathcal{A}(T) = (A; t)$ with one ternary operation $t : A^3 \rightarrow A$ such that t is assigned to T . Now, we can introduce an inverse construction. It means that to every algebra $\mathcal{A} = (A; t)$ of type (3) there can be assigned a ternary relational system $\mathcal{T}(\mathcal{A}) = (A; T_t)$ where T_t is defined by

$$T_t := \{(x, y, z) \in A^3 \mid t(x, y, z) = y\}. \quad (2.1)$$

Of course, an assigned ternary relational system $\mathcal{T}(\mathcal{A}) = (A; T_t)$ need not be centred. However, if $\mathcal{T} = (A; T)$ is a centred ternary relational system and $\mathcal{A}(T) = (A; t)$ an assigned algebra then T_t is centred despite the fact that t is not determined uniquely. In fact, we have $(a, b, c) \in T_t$ if and only if $t(a, b, c) = b$ if and only if $(a, b, c) \in T$. Hence, we have proved the following

Lemma 3. *Let $\mathcal{T} = (A; T)$ be a centred ternary relational system, $\mathcal{A}(T) = (A; t)$ an assigned algebra and $\mathcal{T}(\mathcal{A}(T)) = (A; T_t)$ the ternary relational system assigned to $\mathcal{A}(T)$. Then $\mathcal{T}(\mathcal{A}(T)) = \mathcal{T}$.*

The best known correspondence between centred ternary relational systems and corresponding algebras of type (3) is the case of “betweenness”-relations and median algebras which was initiated by J. R. Isbell [5] and essentially developed by H.-J. Bandelt and J. Hedlíková [1]. However, there are also some essential differences between relational systems and the corresponding algebras. For binary relational systems it was described by the first and the third author in [2]. In what follows, we are going to handle it for the ternary case.

If $\mathcal{T} = (A; T)$ is a ternary relational system and E an equivalence relation on A then the **quotient relational system** $\mathcal{T}/E$ is defined as the relational system $(A/E, T/E)$ where $T/E := \{([x]_E, [y]_E, [z]_E) \mid (x, y, z) \in T\}$. It is evident that E need not be a congruence on the assigned algebra $\mathcal{A}(T) = (A; t)$ and hence congruences on $\mathcal{T} = (A; T)$, respectively on $\mathcal{A}(T)$ are different concepts.

Similarly, by a **subsystem** of $\mathcal{T} = (A; T)$ is meant a couple of the form $(B, T|B)$ with a non-empty subset B of A and $T|B := T \cap B^3$. One can easily see that this need not be a subalgebra of $\mathcal{A}(T) = (A; t)$.

Finally, by a **homomorphism** of a ternary relational system $\mathcal{T} = (A; T)$ into a ternary relational system $\mathcal{S} = (B; S)$ is meant a mapping $h : A \rightarrow B$ satisfying

$$(a, b, c) \in T \implies (h(a), h(b), h(c)) \in S.$$

A homomorphism h is called **strong** if for each triple $(p, q, r) \in S$ there exists $(a, b, c) \in T$ such that $(h(a), h(b), h(c)) = (p, q, r)$.

Now, we define the following concept.

Definition 5. A **t -homomorphism** from a centred ternary relational system $\mathcal{T} = (A; T)$ to a ternary relational system $\mathcal{S} = (B; S)$ is a homomorphism from $\mathcal{T}$ to $\mathcal{S}$ such that there exists an algebra $(A; t)$ assigned to $\mathcal{T}$ such that $a, b, c, a', b', c' \in A$ and $(h(a), h(b), h(c)) = (h(a'), h(b'), h(c'))$ together imply $h(t(a, b, c)) = h(t(a', b', c'))$.

Theorem 3. Let $\mathcal{T} = (A; T)$ and $\mathcal{S} = (B; S)$ be centred ternary relational systems and $\mathcal{A}(T) = (A; t)$ and $\mathcal{B}(S) = (B; s)$ assigned algebras. Then every homomorphism from $\mathcal{A}(T)$ to $\mathcal{B}(S)$ is a t -homomorphism from $\mathcal{T}$ to $\mathcal{S}$.

Proof. Let $a, b, c, a', b', c' \in A$. If $(a, b, c) \in T$ then $t(a, b, c) = b$ and hence $s(h(a), h(b), h(c)) = h(t(a, b, c)) = h(b)$ showing $(h(a), h(b), h(c)) \in S$. Thus h is a homomorphism from $\mathcal{T}$ to $\mathcal{S}$.

Moreover, if $(h(a), h(b), h(c)) = (h(a'), h(b'), h(c'))$ then

$$h(t(a, b, c)) = s(h(a), h(b), h(c)) = s(h(a'), h(b'), h(c')) = h(t(a', b', c')).$$

Hence h is a t -homomorphism from $\mathcal{T}$ to $\mathcal{S}$. $\square$

The theorem just proved says that every homomorphism of assigned algebras is a t -homomorphism of the original relational systems. Now we can show under which conditions the converse assertion becomes true.

Theorem 4. Let $\mathcal{T} = (A; T)$ and $\mathcal{S} = (B; S)$ be centred ternary relational systems. Then for every strong t -homomorphism h from $\mathcal{T}$ to $\mathcal{S}$ with assigned algebra $\mathcal{A}(T) = (A; t)$ there exists an algebra $\mathcal{B}(S) = (B; s)$ assigned to $\mathcal{S}$ such that h is a homomorphism from $\mathcal{A}(T)$ to $\mathcal{B}(S)$.

Proof. Let h be a strong t -homomorphism from $\mathcal{T}$ to $\mathcal{S}$. By definition there exists an algebra $\mathcal{A}(T) = (A; t)$ assigned to $\mathcal{T}$ such that for all $a, b, c, a', b', c' \in A$ with

$(h(a), h(b), h(c)) = (h(a'), h(b'), h(c'))$ it holds $h(t(a, b, c)) = h(t(a', b', c'))$. Define a ternary operation s on B as follows:

$$s(h(x), h(y), h(z)) := h(t(x, y, z))$$

for all $x, y, z \in A$. Since h is strong and a t -homomorphism, s is correctly defined. For $a, b, c \in A$, if $(h(a), h(b), h(c)) \in S$ then there exists $(d, e, f) \in T$ such that $(h(d), h(e), h(f)) = (h(a), h(b), h(c))$. Now

$$s(h(a), h(b), h(c)) = h(t(a, b, c)) = h(t(d, e, f)) = h(e) = h(b).$$

If $(h(a), h(b), h(c)) \notin S$ then $(a, b, c) \notin T$ since h is a homomorphism from $\mathcal{T}$ to $\mathcal{S}$ and hence $t(a, b, c) \in Z_T(a, c)$, i. e. $(a, t(a, b, c), c) \in T$. Thus $(h(a), h(t(a, b, c)), h(c)) \in S$, i. e.

$$(h(a), s(h(a), h(b), h(c)), h(c)) \in S$$

whence $s(h(a), h(b), h(c)) \in Z_S(h(a), h(c))$. This shows that $\mathcal{B}(S)$ is an algebra assigned to $\mathcal{B}$. It is easy to see that h is a homomorphism from $\mathcal{A}(T)$ to $\mathcal{B}(S)$. $\square$

We are going to get connections between t -homomorphisms of relational systems and congruences on the assigned algebras.

Theorem 5. *Let $\mathcal{T} = (A; T)$, $\mathcal{S} = (B; S)$ be centred ternary relational systems. Then the following hold:*

- (i) *If h is a strong t -homomorphism from $\mathcal{T}$ to $\mathcal{S}$ then there exists an algebra $\mathcal{A}(T) = (A; t)$ assigned to $\mathcal{T}$ such that $\ker h \in \text{Con} \mathcal{A}(T)$.*
- (ii) *If $\mathcal{A}(T) = (A; t)$ is an algebra assigned to $\mathcal{T}$ and $\theta \in \text{Con} \mathcal{A}(T)$ then the canonical mapping $h : A \rightarrow A/\theta$ is a strong t -homomorphism from $\mathcal{T}$ onto $\mathcal{T}/\theta$.*

Proof. (i) Let h be a strong t -homomorphism from $\mathcal{T}$ to $\mathcal{S}$. By definition and Theorem 4, there exist assigned algebras $\mathcal{A}(T) = (A; t)$, respectively $\mathcal{B}(S) = (B; s)$ such that h is a homomorphism of $\mathcal{A}(T)$ to $\mathcal{B}(S)$ and hence $\ker h \in \text{Con} \mathcal{A}(T)$.

(ii) Let $\mathcal{A}(T) = (A; t)$ be an algebra assigned to $\mathcal{T}$, $\theta \in \text{Con} \mathcal{A}(T)$ and $h : A \rightarrow A/\theta$ denote the canonical mapping. By definition of T/θ , if $(a, b, c) \in T$ then $(h(a), h(b), h(c)) \in T/\theta$ and hence h is a homomorphism from $\mathcal{T}$ to $\mathcal{T}/\theta$. If, moreover, $a, b, c, a', b', c' \in A$ and $(h(a), h(b), h(c)) = (h(a'), h(b'), h(c'))$ then

$$h(t(a, b, c)) = t(h(a), h(b), h(c)) = t(h(a'), h(b'), h(c')) = h(t(a', b', c')).$$

Therefore h is a t -homomorphism from $\mathcal{T}$ onto $\mathcal{T}/\theta$. Obviously, h is strong. $\square$

Definition 6. Let $\mathcal{T} = (A; T)$ be a centred ternary relational system. An equivalence relation θ on A is called a **t -congruence** on $\mathcal{T}$ if there exists an algebra $\mathcal{A}(T) = (A; t)$ assigned to $\mathcal{T}$ such that $\theta \in \text{Con} \mathcal{A}(T)$. A subset B of A is called a **t -subsystem** of $\mathcal{T}$ if there exists an algebra $\mathcal{A}(T) = (A; t)$ assigned to $\mathcal{T}$ such that $(B; t)$ is a subalgebra of $\mathcal{A}(T)$.

Example 2. Consider $A = \{a, b, c, d\}$ and the ternary relation T on A defined as follows: $T := A \times \{d\} \times A$. Then $d \in Z_T(x, y)$ for each $x, y \in A$ and hence T is centred and its median is non-empty, in fact $M_T(x, y, z) = \{d\}$ for all $x, y, z \in A$. For $B = \{a, b, c\}$, $\mathcal{B} = (B; T|B)$ is a subsystem of $\mathcal{A} = (A; T)$ but it is not a t -subsystem. Namely, for every $x, y, z \in A$ t can be defined in the unique way as follows: $t(x, y, z) := d$. Hence, $(\{a, b, c\}; t)$ is not a subalgebra of $(A; t)$. On the contrary, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ are t -subsystems of $\mathcal{A}$.

Remark 2. Let $\mathcal{A} = (A; t)$, $\mathcal{B} = (B; s)$ be algebras of type (3) and $h : A \rightarrow B$ a homomorphism from $\mathcal{A}$ to $\mathcal{B}$. Put $\mathcal{T}(A) := (A; T_t)$ and $\mathcal{S}(B) := (B; S_s)$ where T_t , S_s are defined by (2.1). Then h need not be a t -homomorphism of $\mathcal{T}(A)$ to $\mathcal{S}(B)$, see the following example.

Example 3. Let $A = \{-1, 0, 1\}$, $B = \{1, 0\}$ and $t(x, y, z) = x \cdot y$, $s(x, y, z) = x \cdot y$, where “ $\cdot$ ” is the multiplication of integers. Let $h : A \rightarrow B$ be defined by $h(x) = |x|$. Then h is clearly a homomorphism from $\mathcal{A} = (A; t)$ to $\mathcal{B} = (B; s)$ and

$$T_t = (A \times \{0\} \times A) \cup (\{1\} \times A^2).$$

There exists exactly one algebra $(A; t^*)$ assigned to $\mathcal{T}(A)$, namely where

$$t^*(x, y, z) := \begin{cases} y & \text{if } y = 0 \text{ or } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now $h(-1) = h(1)$ but $h(t^*(-1, -1, 1)) = h(0) = 0 \neq 1 = h(1) = h(t^*(1, 1, 1))$. Thus h is not a t -homomorphism.

We can prove the following:

Theorem 6. *If $\mathcal{A} = (A; t)$ and $\mathcal{B} = (B; s)$ are algebras of type (3), $\mathcal{A}$ satisfies the identity*

$$t(x, t(x, y, z), z) = t(x, y, z)$$

and $\mathcal{T}(A) = (A; T_t)$ and $\mathcal{S}(B) = (B; S_s)$ denote the relational systems corresponding to $\mathcal{A}$ and $\mathcal{B}$, respectively, as defined by (2.1) then every homomorphism h from $\mathcal{A}$ to $\mathcal{B}$ is a t -homomorphism from $\mathcal{T}(A)$ to $\mathcal{S}(B)$.

Proof. Let $a, b, c, d, e, f \in A$. If $(a, b, c) \in \mathcal{T}(A)$ then $t(a, b, c) = b$ and hence

$$s(h(a), h(b), h(c)) = h(t(a, b, c)) = h(b)$$

whence $(h(a), h(b), h(c)) \in \mathcal{S}(B)$. This shows that h is a homomorphism from $\mathcal{T}(A)$ to $\mathcal{S}(B)$. Obviously, t is assigned to T_t . Finally, $(h(a), h(b), h(c)) = (h(d), h(e), h(f))$ implies

$$h(t(a, b, c)) = s(h(a), h(b), h(c)) = s(h(d), h(e), h(f)) = h(t(d, e, f))$$

which shows that h is a t -homomorphism from $\mathcal{T}(A)$ to $\mathcal{S}(B)$. $\square$

3. DERIVED BINARY SYSTEMS

Let T be a ternary relation on A and p an arbitrary, but fixed element of A . Then

$$R_T := \{(x, y) \in A^2 \mid (x, y, p) \in T\}$$

is called the binary relation **p -derived from T** . Moreover, put $x \circ y := t(x, y, p)$ for all $x, y \in A$ if T is centred and t is an assigned operation.

If T is reflexive then R_T is reflexive, too. If, moreover, T is centred then Theorem 2 implies $x \circ x = x$, the idempotency of the operation $\circ$ which is in accordance with (i) of Theorem 8 in [3].

Similarly, if T is R -symmetric then R_T is symmetric. If, moreover, T is centred then Theorem 2 implies $(x \circ y) \circ x = x$ which is identity (ii) of Theorem 8 in [3] characterizing symmetric binary relations (for directed relational systems).

If T is R -antisymmetric then R_T is antisymmetric. If, moreover, T is centred then Theorem 2 yields that $(x \circ y) \circ x = x \circ y$ which, if satisfied for all $p \in A$, is a sufficient condition for the antisymmetry of R_T . This condition is also a sufficient condition for the antisymmetry of binary relations (see (v) of Theorem 8 in [3]).

If T is R -transitive then R_T is transitive. If, moreover, T is centred then Theorem 2 implies $x \circ ((x \circ y) \circ u) = (x \circ y) \circ u$ which is just identity (iii) of Theorem 8 in [3] characterizing transitivity of binary relations.

Let us recall from [3] that a binary relation R on A is **(upward) directed** if

$$U_R(a, b) := \{x \in A \mid (a, x), (b, x) \in R\} \neq \emptyset \text{ for all } a, b \in A.$$

Although reflexivity, R -symmetry, R -antisymmetry and R -transitivity of a ternary relation T on A yields the corresponding property of R_T , we are not able to show that if T is centred then R_T is directed. However, our characterization of the corresponding properties for binary relations by means of the induced binary operations in [3] are possible for directed relations only.

Example 4. Put $A := \{x, y, z\}$ and

$$T := \{(x, z, y)\} \cup \{(a, y, b) \mid (a, b) \in A^2 \setminus \{(x, y)\}\}.$$

Then T is centred because $Z_T(x, y) = \{z\}$ and $Z_T(a, b) = \{y\}$ for $(a, b) \in A^2 \setminus \{(x, y)\}$. Put $p := y$ and consider the p -derived binary relation R_T on A . Then

$$x \circ (x \circ y) = t(x, t(x, y, y), y) = t(x, z, y) = z = t(x, y, y) = x \circ y,$$

but

$$y \circ (x \circ y) = t(y, t(x, y, y), y) = t(y, z, y) = y \neq z = t(x, y, y) = x \circ y.$$

Thus $y \circ (x \circ y) \neq (x \circ y)$. According to (ii) of Theorem 6 in [3], R_T is not directed.

Remark 3. Theorem 6 in [3] says that for a groupoid $(G; \circ)$ the following are equivalent:

- (i) There exists a directed relational system $(G; R)$ with a reflexive relation R such that $(G; \circ)$ corresponds to $(G; R)$.
- (ii) $(G; \circ)$ satisfies the identities $x \circ x = x$ and $x \circ (x \circ y) = y \circ (x \circ y) = x \circ y$.

We are going to show a sufficient condition for R_T to be directed.

Theorem 7. *Let T be a reflexive ternary relation on A such that $Z_T(a, c) \cap Z_T(b, c) \neq \emptyset$ for all $a, b, c \in A$. Let $p \in A$ and R_T denote the binary relation p -derived from T . Then R_T is directed.*

Proof. Due to the assumption, T is centred and hence we can consider a ternary operation t on A assigned to T such that $t(a, b, c) \in Z_T(a, c) \cap Z_T(b, c)$ if $(a, b, c) \in A^3 \setminus T$. Since T is reflexive, we have $x \circ x = t(x, x, p) = x$.

First assume $(x, y) \in R_T$. Then $(x, y, p) \in T$. Thus $t(x, y, p) = y$ and hence

$$x \circ (x \circ y) = t(x, t(x, y, p), p) = t(x, y, p) = x \circ y.$$

Since T is reflexive, we obtain

$$y \circ (x \circ y) = t(y, t(x, y, p), p) = t(y, y, p) = y = t(x, y, p) = x \circ y.$$

Now suppose $(x, y) \notin R_T$. Then

$$x \circ (x \circ y) = t(x, t(x, y, p), p) = t(x, y, p) = x \circ y.$$

Since $t(x, y, p) \in Z_T(x, p) \cap Z_T(y, p)$ we have also $t(x, y, p) \in Z_T(y, p)$ and hence

$$y \circ (x \circ y) = t(y, t(x, y, p), p) = t(x, y, p) = x \circ y.$$

We have shown that $\circ$ satisfies (ii) of Theorem 6 in [3]. Thus R_T is directed. $\square$

The converse assertion is also true. For a binary relation R on A and a fixed element $p \in A$ we define

$$T_p(R) := \{(x, y, p) \mid (x, y) \in R\} \cup \{(x, x, y) \mid x, y \in A\}. \quad (3.1)$$

Then we can prove

Proposition 1. *Let R be a reflexive binary relation on A , $p \in A$ and $T_p(R)$ defined by (3.1). Then $T_p(R)$ is a centred ternary relation on A and its p -derived binary relation is just R .*

Proof. It is evident that $T_p(R)$ is a ternary relation on A , its p -derived binary relation is just R and $Z_{T_p(R)}(x, y) \supseteq \{y\} \neq \emptyset$ for all $x, y \in A$, i. e. $T_p(R)$ is centred. $\square$

In what follows, we focus on the relation between ternary relations preserving a given function and properties of assigned operations.

Definition 7. Let T be a ternary relation and f an m -ary operation on A . We say that f **preserves** T if

$$(a_1, b_1, c_1), \dots, (a_m, b_m, c_m) \in T \text{ implies } (f(a_1, \dots, a_m), f(b_1, \dots, b_m), f(c_1, \dots, c_m)) \in T.$$

It is worth noticing that the set of all operations on A preserving a given relation T forms a so-called clone. This topic is intensively investigated in contemporary algebra.

Definition 8. Let f be an m -ary and g an n -ary operation on A . We say that f and g **commute with each other** if

$$f(g(x_{11}, \dots, x_{1n}), \dots, g(x_{m1}, \dots, x_{mn})) = g(f(x_{11}, \dots, x_{m1}), \dots, f(x_{1n}, \dots, x_{mn}))$$

for all $x_{11}, \dots, x_{1n}, \dots, x_{m1}, \dots, x_{mn} \in A$.

We remark that also the set of all operations on A commuting with a given operation f forms a clone. Our next task is to compare both of these concepts.

Lemma 4. *If T is a centred ternary relation, f an m -ary operation on A commuting with a ternary operation t assigned to T then f preserves T .*

Proof. Let t be a ternary operation assigned to T . Assume $(a_1, b_1, c_1), \dots, (a_m, b_m, c_m) \in T$. Let f commute with t . Then $t(a_i, b_i, c_i) = b_i$ for $i = 1, \dots, m$ and hence

$$\begin{aligned} & t(f(a_1, \dots, a_m), f(b_1, \dots, b_m), f(c_1, \dots, c_m)) \\ &= f(t(a_1, b_1, c_1), \dots, t(a_m, b_m, c_m)) \\ &= f(b_1, \dots, b_m) \end{aligned}$$

showing $(f(a_1, \dots, a_m), f(b_1, \dots, b_m), f(c_1, \dots, c_m)) \in T$. □

Clearly the sufficient condition used in the previous Lemma is not necessary. Such a condition is as follows.

Theorem 8. *If T is a centred ternary relation, f an m -ary operation on A and t a ternary operation assigned to T then f preserves T if and only if it satisfies the following identity:*

$$\begin{aligned} & t(f(x_1, \dots, x_m), f(t(x_1, y_1, z_1), \dots, t(x_m, y_m, z_m)), f(z_1, \dots, z_m)) \\ &= f(t(x_1, y_1, z_1), \dots, t(x_m, y_m, z_m)). \end{aligned} \tag{3.2}$$

Proof. Assume that f preserves T . Since t is assigned to T we have $(x_i, t(x_i, y_i, z_i), z_i) \in T$ for all $i = 1, \dots, m$. Hence

$$(f(x_1, \dots, x_m), f(t(x_1, y_1, z_1), \dots, t(x_m, y_m, z_m)), f(z_1, \dots, z_m)) \in T.$$

Thus (3.2) holds.

Conversely, assume that f satisfies (3.2) and $(a_1, b_1, c_1), \dots, (a_m, b_m, c_m) \in T$. Then

$$t(a_i, b_i, c_i) = b_i$$

for $i = 1, \dots, m$, and hence

$$\begin{aligned} & t(f(a_1, \dots, a_m), f(b_1, \dots, b_m), f(c_1, \dots, c_m)) \\ &= t(f(a_1, \dots, a_m), f(t(a_1, b_1, c_1), \dots, t(a_m, b_m, c_m)), f(c_1, \dots, c_m)) \\ &= f(t(a_1, b_1, c_1), \dots, t(a_m, b_m, c_m)) \\ &= f(b_1, \dots, b_m) \end{aligned}$$

proving $(f(a_1, \dots, a_m), f(b_1, \dots, b_m), f(c_1, \dots, c_m)) \in T$. Hence, f preserves T . $\square$

4. MEDIAN-LIKE ALGEBRAS

The concept of a median algebra was introduced by J. R. Isbell (see [5]) as follows: An algebra $\mathcal{A} = (A; t)$ of type (3) is called a **median algebra** if it satisfies the following identities:

- (M1) $t(x, x, y) = x$;
- (M2) $t(x, y, z) = t(y, x, z) = t(y, z, x)$;
- (M3) $t(t(x, y, z), v, w) = t(x, t(y, v, w), t(z, v, w))$.

It is well-known (see e.g. [1], [5]) that the ternary relation T_t on A assigned to t via (2.1) is centred and, moreover, $|M_{T_t}(a, b, c)| = 1$ for all $a, b, c \in A$. In fact, $t(a, b, c) \in M_{T_t}(a, b, c)$. In particular, having a distributive lattice $\mathcal{L} = (L; \vee, \wedge)$ then $m(x, y, z) = M(x, y, z)$ and putting $t(x, y, z) := m(x, y, z)$, one obtains a median algebra. Conversely, every median algebra can be embedded into a distributive lattice. Moreover, the assigned ternary relation T_t is the so-called “betweenness”, see [10] and [11].

In what follows, we focus on the case when $M_T(a, b, c) \neq \emptyset$ for all $a, b, c \in A$ and $t(a, b, c) \in M_T(a, b, c)$ also in case $|M_T(a, b, c)| \geq 1$.

Definition 9. A **median-like algebra** is an algebra $(A; t)$ of type (3) where t satisfies (M1) and (M2) and where there exists a centred ternary relation T on A such that $t(x, y, z) \in M_T(x, y, z)$ for all $x, y, z \in A$.

Theorem 9. An algebra $\mathcal{A} = (A; t)$ of type (3) is median-like if t satisfies (M1), (M2) and

$$t(x, t(x, y, z), y) = t(y, t(x, y, z), z) = t(z, t(x, y, z), x) = t(x, y, z). \quad (4.1)$$

Proof. If $T := \{(x, y, z) \in A^3 \mid t(x, y, z) = y\}$ then $t(x, y, z) \in M_T(x, y, z)$ for all $x, y, z \in A$. $\square$

Lemma 5. Every median algebra is a median-like algebra.

Proof. As shown in [5], identities (M1), (M2), (M3) are equivalent to the identity

$$t(x, t(x, z, w), t(y, z, w)) = t(x, z, w).$$

Putting $w = y$ and using (M1) and (M2), we derive

$$t(x, t(x, z, y), y) = t(x, t(x, z, y), t(y, z, y)) = t(t(x, x, y), z, y) = t(x, z, y)$$

whence (4.1) follows since according to (M2) we have $t(u, v, w) = t(x, y, z)$ for any permutation (u, v, w) of (x, y, z) . $\square$

The following examples show that a median-like algebra need not be a median algebra.

Example 5. Put $A := \{1, 2, 3, 4, 5\}$, let t denote the ternary operation on A defined by $t(x, x, y) = t(x, y, x) = t(y, x, x) := x$ for all $x, y \in A$ and $t(x, y, z) := \min(x, y, z)$ for all $x, y, z \in A$ with $x \neq y \neq z \neq x$ and put $T := \{(x, x, y) \mid x, y \in A\} \cup \{(y, x, x) \mid x, y \in A\} \cup \{(x, y, z) \in A^3 \mid y < x < z\} \cup \{(x, y, z) \in A^3 \mid y < z < x\}$. Then t satisfies (M1) and (M2) and $t(x, y, z) \in M_T(x, y, z)$ for all $x, y, z \in A$. This shows that $(A; t)$ is median-like. However, this algebra is not a median algebra since

$$t(t(1, 3, 4), 2, 5) = t(1, 2, 5) = 1 \neq 2 = t(1, 2, 2) = t(1, t(3, 2, 5), t(4, 2, 5))$$

and hence (M3) is not satisfied.

Example 6. Consider the lattice M_3 given in FIGURE 1 below.

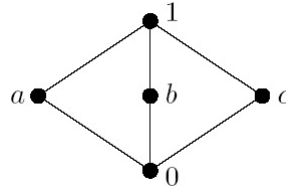


FIGURE 1.

Then M_3 is not distributive, $m(a, b, c) = 0$ and $M(a, b, c) = 1$. Define $(x, y, z) \in T$ if and only if $y \in [x \wedge z, x \vee z]$. Let t be an assigned operation defined as follows

$$t(x, y, z) := m(x, y, z).$$

Then $t(x, y, z) \in M_T(x, y, z)$ for all triples of elements x, y, z and hence $(M_3; t)$ is a median-like algebra. However, it is not a median algebra because identity (M3) is violated:

$$t(t(a, b, c), a, 1) = t(0, a, 1) = a \neq 1 = t(a, 1, 1) = t(a, t(b, a, 1), t(c, a, 1)).$$

The previous example motivated us to state a general construction for lattices which need not be neither distributive nor modular.

Theorem 10. Let $\mathcal{L} = (L; \vee, \wedge)$ be a lattice. Define $t_1(x, y, z) := m(x, y, z)$, $t_2(x, y, z) := M(x, y, z)$. Then $\mathcal{A}_1 := (L; t_1)$ and $\mathcal{A}_2 := (L; t_2)$ are median-like algebras. Moreover, the following conditions are equivalent

- (a) $\mathcal{A}_1 = \mathcal{A}_2$;
- (b) $\mathcal{A}_1$ is a median algebra;
- (c) $\mathcal{L}$ is distributive.

Proof. Since both $m(x, y, z)$ and $M(x, y, z)$ satisfy (M1) and (M2) and $m(x, y, z), M(x, y, z) \in [m(x, y, z), M(x, y, z)] = M_T(x, y, z)$ for $(x, y, z) \in L^3$ and $T := \{(x, y, z) \in L^3 \mid x \wedge z \leq y \leq x \vee z\}$, $\mathcal{A}_1, \mathcal{A}_2$ are median-like algebras. It is well-known that $m(x, y, z) = M(x, y, z)$ if and only if $\mathcal{L}$ is distributive which proves (a) $\Leftrightarrow$ (c). The implication (c) $\Rightarrow$ (b) is well-known (see e.g. [1], [5]). Finally, we prove (b) $\Rightarrow$ (c). Assume that (b) holds but (c) does not. Then $\mathcal{L}$ contains either $\mathcal{M}_3 = (\{0, a, b, c, 1\}; \vee, \wedge)$ or $\mathcal{N}_5 = (\{0, a, b, c, 1\}; \vee, \wedge)$ (with $a < c$) as a sublattice. In the first case we have

$$t(t(a, b, c), a, 1) = t(0, a, 1) = a \neq 1 = t(a, 1, 1) = t(a, t(b, a, 1), t(c, a, 1))$$

whereas in the second case

$$t(t(c, b, a), a, 1) = t(a, a, 1) = a \neq c = t(c, 1, a) = t(c, t(b, a, 1), t(a, a, 1))$$

which shows that (M3) does not hold. This is a contradiction to (b). Hence (c) holds. $\square$

Comparing our definition with Theorem 2, we conclude:

Corollary 2. An algebra $(A; t)$ of type (3) is median-like if t satisfies (M2) and if it is assigned to a centred antisymmetric or non-sharp ternary relation on A .

Let us mention that median-like algebras form a variety because they are defined by identities. Moreover, this variety is congruence distributive, i. e. $\text{Con}\mathcal{A}$ is distributive for every median-like algebra $\mathcal{A}$, because the operation t is a majority term, i. e. it satisfies by (M1) and (M2)

$$t(x, x, y) = t(x, y, x) = t(y, x, x) = x.$$

Theorem 11. let $\mathcal{L} = (L; \vee, \wedge)$ be a lattice and t a ternary operation on L satisfying (M1) and (M2) and $t(x, y, z) \in [m(x, y, z), M(x, y, z)]$ for all $x, y, z \in A$. Then $\mathcal{A} := (L; t)$ is a median-like algebra.

Proof. Put $T := \{(x, y, z) \in L^3 \mid x \wedge z \leq y \leq x \vee z\}$. Then $M_T(a, b, c) = [m(a, b, c), M(a, b, c)]$ for all $a, b, c \in L$. Hence $t(a, b, c) \in M_T(a, b, c)$ for all $a, b, c \in L$ showing that $\mathcal{A}$ is a median-like algebra. $\square$

5. CYCLIC ALGEBRAS

Apart from the “betweenness” relation, another ternary relation plays an important role in mathematics. It is the so-called **cyclic order**, see e.g. [4], [9] and references there.

Definition 10. A ternary relation T on A is called **asymmetric** if

$$(a, b, c) \in T \text{ for } a \neq b \neq c \text{ implies } (c, b, a) \notin T. \quad (5.1)$$

A ternary relation C on A is called a **cyclic order** if it is cyclic, asymmetric, cyclically transitive and satisfies $(a, a, a) \in C$ for each $a \in A$.

Remark 4. Let C be a cyclic order on a set A . Then $(a, b, a) \notin C$ for all $a, b \in A$ with $a \neq b$. Namely, if $(a, b, a) \in C$ then, by (5.1), $(a, b, a) \notin C$, a contradiction. Since C is cyclic, we have also $(a, a, b), (b, a, a) \notin C$.

Applying (5.1), we derive immediately

Lemma 6. A centred ternary relation T on A is asymmetric if and only if any assigned ternary operation t satisfies the implication:

$$(t(x, y, z) = y \text{ and } x \neq y \neq z) \implies t(z, y, x) \neq y. \quad (5.2)$$

Similarly as for “betweenness” relations, we can derive an algebra of type (3) for a centred cyclic order by means of its assigned operation.

Definition 11. A **cyclic algebra** is an algebra assigned to a cyclic relation.

Cyclic algebras can be characterized by certain identities and the implication (5.2) as follows.

Theorem 12. An algebra $\mathcal{A} = (A; t)$ of type (3) is a cyclic algebra if and only if it satisfies (5.2) and

$$\begin{aligned} t(x, t(x, y, z), z) &= t(x, y, z), \\ t(t(x, y, z), z, x) &= z, \\ t(x, t(x, y, t(x, z, u)), u) &= t(x, y, t(x, z, u)), \\ t(x, x, x) &= x. \end{aligned}$$

Proof. Assume that $\mathcal{A} = (A; t)$ satisfies the above identities and (5.2). By Theorem 1 and the first identity, t is an assigned operation of a certain centred ternary relation C on A . By Theorem 2 and the second and third identity, C is cyclic and cyclically transitive. The fourth identity gets $(x, x, x) \in C$ for each $x \in A$. Finally, Lemma 6 yields that C is asymmetric and hence a cyclic order on A . Of course, t is an assigned operation of C and hence $\mathcal{A} = (A; t)$ is a cyclic algebra.

The converse follows directly by Definition 11. □

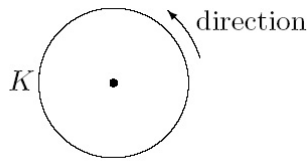


FIGURE 2.

Example 7. Let K be a circle in a plane with a given direction, see FIGURE 2. Define a ternary relation C on K as follows:

$$(a, a, a) \in C \text{ for each } a \in K \text{ and} \\ (a, b, c) \in C \text{ if } a \rightarrow b \text{ and } b \rightarrow c \text{ for } a \neq b \neq c.$$

It is an easy exercise to check that C is a cyclic order on K . If $a, b \in K$ then either $a = b$ and hence $Z_C(a, a) = \{a\}$ or $a \neq b$ thus $Z_C(a, b)$ equals the arc of K between a and b , i. e. it contains a continuum of points. Hence C is centred. For any assigned operation t , the algebra $\mathcal{A}(C) = (K; t)$ is a cyclic algebra.

REFERENCES

- [1] H.-J. Bandelt and J. Hedliková, “Median algebras,” *Discrete Math.*, vol. 45, pp. 1–30, 1983.
- [2] I. Chajda and H. Länger, “Quotients and homomorphisms of relational systems,” *Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math.*, vol. 49, no. 2, pp. 37–47, 2010.
- [3] I. Chajda and H. Länger, “Groupoids assigned to relational systems,” *Math. Bohem.*, vol. 138, no. 1, pp. 15–23, 2013.
- [4] I. Chajda and V. Novák, “On extensions of cyclic orders,” *Čas. Pěstování Mat.*, vol. 110, pp. 116–121, 1985.
- [5] J. R. Isbell, “Median algebra,” *Trans. Am. Math. Soc.*, vol. 260, pp. 319–362, 1980.
- [6] J. Ježek and R. Quackenbush, “Directoids: Algebraic models of up-directed sets,” *Algebra Univers.*, vol. 27, no. 1, pp. 49–69, 1990.
- [7] N. Megiddo, “Partial and complete cyclic orders,” *Bull. Am. Math. Soc.*, vol. 82, pp. 274–276, 1976.
- [8] G. Müller, “Lineare und zyklische ordnung,” *Prax. Math.*, vol. 16, pp. 261–269, 1974.
- [9] V. Novak, “Cyclically ordered sets,” *Czech. Math. J.*, vol. 32, pp. 460–473, 1982.
- [10] E. Pitcher and M. F. Smiley, “Transitivities of betweenness,” *Trans. Am. Math. Soc.*, vol. 52, pp. 95–114, 1942.
- [11] M. Sholander, “Medians and betweenness,” *Proc. Am. Math. Soc.*, vol. 5, pp. 801–807, 1954.

*Authors’ addresses***Ivan Chajda**

Palacký University Olomouc, Faculty of Science, Department of Algebra and Geometry, Třída 17. listopadu 12, 77146 Olomouc, Czech Republic
E-mail address: ivan.chajda@upol.cz

Miroslav Kolařík

Palacký University Olomouc, Faculty of Science, Department of Computer Science, Třída 17. listopadu 12, 77146 Olomouc, Czech Republic

E-mail address: miroslav.kolarik@upol.cz

Helmut Länger

Vienna University of Technology, Institute of Discrete Mathematics and Geometry, Wiedner Hauptstraße 8-10, 1040 Vienna, Austria

E-mail address: h.laenger@tuwien.ac.at



REMARKS ON A CONJECTURE ABOUT RANDIĆ INDEX AND GRAPH RADIUS

T. DEGHAN-ZADEH, HONGBO HUA, A. R. ASHRAFI, AND N. HABIBI

Received 5 December, 2012

Abstract. Let G be a nontrivial connected graph. The radius $r(G)$ of G is the minimum eccentricity among eccentricities of all vertices in G . The Randić index of G is defined as $R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u)d_G(v)}}$, and the Harmonic index is defined as $H(G) = \sum_{uv \in E(G)} \frac{2}{d_G(u) + d_G(v)}$, where $d_G(x)$ is the degree of the vertex x in G . In 1988, Fajtlowicz conjectured that for any connected graph G , $R(G) \geq r(G) - 1$. This conjecture remains still open so far. More recently, Deng et al. proved that this conjecture is true for connected graphs with cyclomatic number no more than 4 by means of Harmonic index. In this short paper, we use a class of composite graphs to construct infinite classes of connected graphs, with cyclomatic number greater than 4, for which the above conjecture holds. In particular, for any given positive odd number $k \geq 7$, we construct a connected graph with cyclomatic number k such that the above conjecture holds for this graph.

2010 *Mathematics Subject Classification*: 05C07; 05C12; 05C76

Keywords: Randić index, radius, graffiti conjecture, cyclomatic number, harmonic index

1. INTRODUCTION

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For a graph G , we let $d_G(v)$ be the degree of a vertex v in G and $d_G(u, v)$ be the distance between two vertices u and v in G . Other notation and terminology not defined here will conform to those in [3].

Topological indices are numerical parameters of a graph which characterize the topological structure of the graph. Topological indices are usually graph invariants associated with a graph. As one of the most well-known and successful topological indices, *Randić index*, defined as $R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u)d_G(v)}}$, was introduced by

Randić in [15]. Randić index gained much popularity during the past decades. Since its appearance, tremendous attention has been focused on the upper and lower bounds of this index. Bollobás and Erdős [2] proved that the Randić index of a graph of order

Research was supported in part by Natural Science Foundation of the Higher Education Institutions of Jiangsu Province (No. 12KJB110001) and NNSF of China (No. 11201227).

n without isolated vertices is bounded from below by $\sqrt{n-1}$; they left as an open problem to determine the minimum value of the Randić index for a graph G with given minimum degree $\delta(G)$. Delorme et al. [5] gave a partial solution to this open problem for the case of $\delta(G) = 2$. Furthermore, they completely solved this open problem for the case when G is a triangle-free graph with given minimum degree $\delta(G)$. Balister et al. [1] determined the maximal Randić index of a tree with a given number of vertices and leaves. About reviews of mathematical properties of the Randić index, the interested readers are referred to [12, 13]. Fajtlowicz [10] and Caporossi and Hansen [4] conjectured that the Randić index can be bounded from below in terms of the graph radius. In [8], Deng et al. gave a partial solution to this conjecture by studying the relationship between the Harmonic index and graph radius.

The *Harmonic index* of a connected graph G is defined by Fajtlowicz [9] as $H(G) = \sum_{uv \in E(G)} \frac{2}{d_G(u) + d_G(v)}$. Favaron et al. [11] considered the relationship between the Harmonic index and graph eigenvalues. Zhong [17] determined the minimum and maximum values of the Harmonic index for simple connected graphs and trees, and characterized the corresponding extremal graphs. Deng et al. [6] considered the relationship between the Harmonic index $H(G)$ and the chromatic number $\chi(G)$ and proved that $\chi(G) \leq 2H(G)$. This result strengthens a conjecture about the Randić index and the chromatic number which is based on the system AutoGraphiX. Deng et al. [7] gave a best possible lower bound for the Harmonic index of a graph and a triangle-free graph with minimum degree no less than two and characterized the corresponding extremal graphs, respectively.

In [10], Fajtlowicz proposed a conjecture concerning the relationship between the Randić index and graph radius, which reads as follows.

Conjecture 1 ([10]). *For any connected G , $R(G) \geq r(G) - 1$.*

Caporossi and Hansen [4] partially proved this conjecture by showing that $R(T) \geq r(T) + \sqrt{2} - \frac{3}{2}$ for any tree T . Liu and Gutman [14], and You and Liu [16] proved that the conjecture is true for unicyclic, bicyclic and tricyclic graphs.

In [8], Deng et al. proved the following result.

Theorem 1 ([8]). *Let G be a graph with cyclomatic number $k \geq 1$. Then $H(G) \geq r(G) - \frac{31}{105}(k-1)$. In particular, $H(G) > r(G) - 1$ for a graph with cyclomatic number no more than 4.*

Remark 1. For any connected graph G , since

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u)d_G(v)}} \geq \sum_{uv \in E(G)} \frac{2}{d_G(u) + d_G(v)} = H(G)$$

and by Theorem 1, $H(G) > r(G) - 1$ for a connected graph G with cyclomatic number no more than 4, Conjecture 1 holds for those connected graphs G with cyclomatic number no more than 4.

In this short paper, we use a class of composite graphs to construct infinite classes of connected graphs, with cyclomatic number greater than 4, for which Conjecture 1 holds. In particular, for any given positive odd number $k \geq 7$, we can construct a connected graph with cyclomatic number being k such that the above conjecture holds for this graph. Consequently, our results extend those of Deng et al. in [6].

2. MAIN RESULTS

We first introduce a class of composite graphs, with which we are able to construct our desired graphs for which Conjecture 1 holds. The *double graph* G^* of a given graph G is constructed by making two copies of (including the initial edge set of each) and adding edges u_1v_2 and u_2v_1 for every edge uv of G .

For each vertex u in G , we call the corresponding vertices u_1 and u_2 , in G^* , the *clone vertices* of u . If an n -vertex connected graph G has a vertex of degree $n - 1$, then G is said to be *well-connected*.

Concerning the Harmonic index and radius of double graphs, we have the following result.

Theorem 2. *Let G be a nontrivial connected graph of order n , and let G^* be its double graph. We have*

- (a) *If G is not well-connected, then $r(G^*) = r(G)$, and if G is well-connected, then $r(G^*) = r(G) + 1$;*
- (b) *$H(G^*) = 2H(G)$.*

Proof. It can be easily seen that if G is connected, then G^* is also connected. For the sake of convenience, we label all vertices of G as $v_1, \dots, v_n$. Suppose that x_i and y_i are the corresponding clone vertices, in G^* , of v_i for each $i = 1, \dots, n$. Given a vertex v_i in G . According to the definition of double graph, for any vertex v_j , different from v_i , in G , we have

$$d_{G^*}(x_i, x_j) = d_{G^*}(x_i, y_j) = d_{G^*}(y_i, x_j) = d_{G^*}(y_i, y_j) = d_G(v_i, v_j). \quad (2.1)$$

Moreover, we have $d_{G^*}(x_i) = d_{G^*}(y_i) = 2d_G(v_i)$ for $i = 1, \dots, n$. Furthermore, we have

$$d_{G^*}(x_i, y_i) = 2 \quad (2.2)$$

for $i = 1, \dots, n$, since there exists at least one vertex, say x_k (or y_k), such that both x_i and y_i are adjacent to x_k (or y_k).

If G is not well-connected, we have $ec_G(x_i) \geq 2$ for $i = 1, \dots, n$, that is, $r(G) \geq 2$. Combining this fact and Eqs (2.1) and (2.2), we have $ec_{G^*}(x_i) = ec_G(v_i) = ec_{G^*}(y_i)$ for any $i = 1, \dots, n$, i.e., $r(G) = r(G^*)$. If G is well-connected, then G has a vertex, say v_i , of degree $n - 1$. Thus, we have $ec_G(v_i) = 1$ and $r(G) = 1$. By

Eq. (2.2), we have $r(G^*) = ec_{G^*}(x_i) = d_{G^*}(x_i, y_i) = 2 = r(G) + 1$. This proves (a).

Now, we prove (b). Let $h_G(uv) = \frac{2}{d_G(u) + d_G(v)}$. Then $H(G) = \sum_{uv \in E(G)} h_G(uv)$.

For $1 \leq i, j \leq n$ and $i \neq j$,

$$h_{G^*}(x_i x_j) = \frac{2}{d_{G^*}(x_i) + d_{G^*}(x_j)} = \frac{1}{d_G(v_i) + d_G(v_j)} = \frac{1}{2} h_G(v_i v_j). \quad (2.3)$$

By symmetry, we have

$$h_{G^*}(x_i x_j) = h_{G^*}(x_i y_j) = h_{G^*}(y_i x_j) = h_{G^*}(y_i y_j) \quad (2.4)$$

for each $i, j = 1, \dots, n$ and $i \neq j$. By means of Eq.s (2.3) and (2.4),

$$\begin{aligned} H(G^*) &= 4 \sum_{x_i x_j \in E(G^*)} h_{G^*}(x_i x_j) \\ &= 4 \sum_{v_i v_j \in E(G)} \frac{1}{2} h_G(v_i v_j) \\ &= 2H(G), \end{aligned}$$

which proves (b). $\square$

We use Theorem 2 to prove the following consequence.

Theorem 3. *There exist infinite class of connected graphs, with cyclomatic number greater than 4, for which Conjecture 1 holds. In particular, for any given positive odd number $k \geq 7$, there exists a connected graph with cyclomatic number being k such that Conjecture 1 holds for this graph.*

Proof. We first take a connected graph G of order $n \geq 4$ and size m such that it is not well-connected and its cyclomatic number is less than or equal to 4. By Theorem 1, $H(G) > r(G) - 1$.

Now, we construct the double graph G^* of G . Since G is not well-connected, then by Theorem 2, $H(G^*) = 2H(G) > 2(r(G) - 1) = 2r(G^*) - 2 > r(G^*) - 1$, as $r(G^*) \geq 2$. Thus, $R(G^*) \geq H(G^*) > r(G^*) - 1$. Moreover, we can find that the cyclomatic number of G^* is $|E(G^*)| - |V(G^*)| + 1 = 4m - 2n + 1 \geq 4(n - 1) - 2n + 1 = 2n - 3 \geq 5$ for $n \geq 4$.

For a graph G , we now define its k -th iterated double graph G^{k*} as

$$G^{1*} = G^* \text{ and } G^{k*} = (G^{(k-1)*})^* \text{ for } k \geq 1$$

and $G^{0*} = G$ for consistence.

Note that G^* is also not well-connected. Repeatedly using Theorem 2, we can prove that

$$R(G^{k*}) \geq H(G^{k*}) = 2^k H(G) > 2^k (r(G) - 1) = 2^k (r(G^{k*}) - 1) > r(G^{k*}) - 1.$$

Similarly, the cyclomatic number of G^{k*} is $|E(G^{k*})| - |V(G^{k*})| + 1 = 4^k m - 2^k n + 1 \geq 4^k(n-1) - 2^k n + 1 = 2^k(2^k - 1)n - 4^k + 1 = 2^k[2^k(n-1) - n] + 1 \geq 2n - 3 \geq 5$ for $k \geq 1$ and $n \geq 4$.

Now, for a given positive odd number $k = 2k' + 1 \geq 7$, we construct a connected graph with cyclomatic number being k . In fact, we first take a graph G with k' vertices and k' edges. Let G^* be the double graph of G . Then the cyclomatic number of G^* is equal to $|E(G^*)| - |V(G^*)| + 1 = 2k' + 1 = k$. By the previous proof, we know that $R(G^*) \geq H(G^*) > r(G^*) - 1$. This completes the proof. $\square$

REFERENCES

- [1] P. Balister, B. Bollobás, and S. Gerke, "The generalized Randić index of trees," *J. Graph Theory*, vol. 56, no. 4, pp. 270–286, 2007.
- [2] B. Bollobás and P. Erdős, "Graphs of extremal weights," *Ars Comb.*, vol. 50, pp. 225–233, 1998.
- [3] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*. New York: American Elsevier Publishing Co., 1976.
- [4] G. Caporossi and P. Hansen, "Variable neighborhood search for extremal graphs. I: The AutoGraphiX system," *Discrete Math.*, vol. 212, no. 1–2, pp. 29–44, 2000.
- [5] C. Delorme, O. Favaron, and D. Rautenbach, "On the Randić index," *Discrete Math.*, vol. 257, no. 1, pp. 29–38, 2002.
- [6] H. Deng, S. Balachandran, S. K. Ayyaswamy, and Y. B. Venkatakrishnan, "On the harmonic index and the chromatic number of a graph," preprint.
- [7] H. Deng, Z. Tang, and R. Wu, "A lower bound for the harmonic index of a graph with minimum degree at least two," preprint.
- [8] H. Deng, Z. Tang, and J. Zhang, "On a conjecture of Randić index and graph radius," available online at: <http://arxiv.org/pdf/1210.2543.pdf>.
- [9] S. Fajtlowicz, "On conjectures of Graffiti. II," in *Combinatorics, graph theory, and computing, Proc. 18th Southeast. Conf., Boca Raton/Fl. 1987, Congr. Numerantium*, 1987, vol. 60, pp. 189–197.
- [10] S. Fajtlowicz, "On conjectures of Graffiti," *Discrete Math.*, vol. 72, no. 1–3, pp. 113–118, 1988.
- [11] O. Favaron, M. Mahio, and J. F. Saclé, "Some eigenvalue properties in graphs (Conjectures of Graffiti-II)," *Discrete Math.*, vol. 111, pp. 197–220, 1993.
- [12] I. Gutman and B. Furtula, Eds., *Recent Results in the Theory of Randić Index*, ser. Mathematical Chemistry Monograph No. 6. Kragujevac, 2008.
- [13] X. Li and I. Gutman, *Mathematical Aspects of Randić-type Molecular Structure Descriptors*, ser. Mathematical Chemistry Monographs No. 1. Kragujevac, 2006.
- [14] B. Liu and I. Gutman, "On a conjecture on Randić indices," *MATCH Commun. Math. Comput. Chem.*, vol. 62, no. 1, pp. 143–154, 2009.
- [15] M. Randić, "Characterization of molecular branching," *J. Am. Chem. Soc.*, vol. 97, pp. 6609–6615, 1975.
- [16] Z. You and B. Liu, "On a conjecture of the Randić index," *Discrete Appl. Math.*, vol. 157, no. 8, pp. 1766–1772, 2009.
- [17] L. Zhong, "The harmonic index for graphs," *Appl. Math. Lett.*, vol. 25, no. 3, pp. 561–566, 2012.

*Authors' addresses***T. Dehghan-Zadeh**

Department of Pure Mathematics, Faculty of Mathematical Science,, University of Kashan, Kashan
87317-51167, I. R. Iran

Hongbo Hua

Huaiyin Institute of Technology, Faculty of Mathematics and Physics, 223003 Huai'an City, PR
China

E-mail address: hongbo.hua@gmail.com

A. R. Ashrafi

Department of Pure Mathematics, Faculty of Mathematical Science,, University of Kashan, Kashan
87317-51167, I. R. Iran

E-mail address: ashrafi@kashanu.ac.ir

N. Habibi

Department of Mathematics, Faculty of Science,, University of Zanjan, Zanjan, I. R. Iran



GRADIENT ESTIMATES IN MORREY SPACES OF WEAK SOLUTIONS TO QUASILINEAR PARABOLIC SYSTEMS OF HÖRMANDER'S VECTOR FIELDS

YAN DONG

Received 16 April, 2013

Abstract. This paper is concerned with higher integrability for gradients of weak solutions to quasilinear parabolic systems of Hörmander's vector fields. We establish L^p estimates for gradients of weak solutions by deriving a parabolic Caccioppoli inequality and using the reverse Hölder inequality in parabolic cylinders, and then obtain L^p estimates for gradients of weak solutions to homogeneous parabolic system. At last higher integrability of gradients in Morrey space with $p \geq 2$ is proved.

2010 *Mathematics Subject Classification:* 35K55; 35B65

Keywords: quasilinear parabolic system, Hörmander's vector fields, higher integrability, Caccioppoli inequality, Morrey space, natural condition

1. INTRODUCTION

Let $X = (X_1, \dots, X_q)$ be a family of real smooth vector fields in a neighborhood $\tilde{\Omega}$ of some bounded domain $\Omega \subset \mathbb{R}^n$ ($q \leq n$) with the form

$$X_j = \sum_{k=1}^n b_{jk}(x) \frac{\partial}{\partial x_k}, \quad (b_{jk}(x) \in C^\infty(\Omega), j = 1, 2, \dots, q)$$

and satisfying Hörmander's condition, namely, the Lie algebra generated by the family $X_1, \dots, X_q$ at any point of $\tilde{\Omega}$ spans $\mathbb{R}^n$.

Regularity for weak solutions to parabolic equations and systems in Euclidean spaces (i.e., $q = n$, $X_i = \frac{\partial}{\partial x_i}$, $i = 1, 2, \dots, n$) had fairly complete results. When coefficients are discontinuous and belong to some VMO spaces, McBride in [15] derived the generalized Morrey estimates for gradients of weak solutions to linear parabolic systems. Giaquinta and Struwe in [12] treated partial regularity for weak solutions to quasilinear parabolic systems under the natural condition and got a sharp Hölder

This work was supported by the National Natural Science Foundation of China (Grant Nos. 11271299, 11001221), Natural Science Foundation Research Project of Shaanxi Province (2012JM1014) and NPU Foundation for Fundamental Research(NPU-FFR-JC20110271).

exponent by establishing a Caccioppoli inequality and the reverse Hölder inequality. For some other studies, we quote [2, 3, 11, 17] and references therein.

Divergence equations or systems constituted by Hörmander's vector fields have received extensive attention. Di Fazio and Fanciullo in [4] studied gradient estimates in Morrey spaces when $p = 2$ for weak solutions to linear degenerate elliptic system with coefficients in some VMO spaces. Dong and Niu in [6] generalized the result and derived gradient estimates in Morrey spaces when $p > 2$ for weak solutions to linear elliptic systems by establishing energy estimate and using the reverse Hölder inequality on the homogeneous spaces, and got Hölder regularity. Xu and Zuily in [20] attained interior regularity for weak solutions to quasilinear elliptic systems with natural condition. Partial Hölder regularity for weak solutions to quasilinear elliptic systems under quadratic natural growth condition and controllable growth condition was derived by Gao and Niu in [8]. Related results can also be found in [18, 19].

The purpose of this paper is to investigate quasilinear parabolic systems of Hörmander's vector fields

$$u_t^i + X_\alpha^* \left(a_i^{\alpha\beta}(z, u) X_\beta u^i \right) = g_i(z, u, Xu) + X_\alpha^* f_i^\alpha(z), \quad (1.1)$$

where $\alpha, \beta = 1, 2, \dots, q, i = 1, 2, \dots, N, X_j^* = -X_j + c_j (c_j = -\sum_{k=1}^n \frac{\partial b_{jk}}{\partial x_k} \in C^\infty(\Omega))$

is the transposed vector field of $X_j, z = (x, t) \in Q_T \subset \mathbb{R}^{n+1}, Q_T = \Omega \times (0, T), g_i(z, u, Xu)$ satisfies natural growth condition. We will establish higher integrability of gradients in the Morrey spaces $L_X^{p, \lambda}(p \geq 2)$ for weak solutions to (1.1).

Definition 1 (BMO and VMO spaces). For any $f \in L^1(Q_T)$, we set

$$\eta_R(f) = \sup_{z_0 \in Q_T, 0 \leq \rho \leq R} \left(\frac{1}{|Q_T \cap Q_\rho(z_0)|} \iint_{Q_T \cap Q_\rho(z_0)} |f(z) - f_{Q_T \cap Q_\rho(z_0)}(z)| dz \right),$$

where $f_{Q_T \cap Q_\rho(z_0)} = \frac{1}{|Q_T \cap Q_\rho(z_0)|} \iint_{Q_T \cap Q_\rho(z_0)} f(z) dz$, and the meaning of $Q_\rho(z_0)$ sees the next section. If $\sup_{R>0} \eta_R(f) < +\infty$, we say $f \in \text{BMO}(Q_T)$ (Bounded Mean Oscillation). Moreover, if $\eta_R(f) \rightarrow 0$ as $R \rightarrow 0$, then we say $f \in \text{VMO}(Q_T)$ (Vanishing Mean Oscillation).

Now we introduce the relevant assumptions.

(H1) (ellipticity condition) Let coefficients $a_i^{\alpha\beta}(z, u)$ satisfy the ellipticity condition, namely, there exists a constant $\delta > 0$ such that for any $\xi \in \mathbb{R}^{(q+1)N}$,

$$a_i^{\alpha\beta}(z, u) \xi_\alpha^i \xi_\beta^i \geq \delta |\xi|^2, \quad (1.2)$$

where $z = (x, t) \in Q_T, Q_T = \Omega \times (0, T), \Omega \subset \mathbb{R}^n$.

(H2) ($VMO \cap L^\infty$) Assume $a_i^{\alpha\beta}(z, u_0) \in VMO$ uniformly on $z \in Q_T$ for some fixed $u_0 \in \mathbb{R}^N$, that is,

$$\lim_{R \rightarrow 0} \eta_R(a_i^{\alpha\beta}(\cdot, u_0)) = 0,$$

and there exists a constant $L > 0$, such that for any $z \in Q_T$ and $u \in \mathbb{R}^N$, one has $|a_i^{\alpha\beta}(z, u)| \leq L$.

(H3) (Continuity) There exist a positive constant c and a continuous concave function $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\omega(0) = 0$, $0 \leq \omega \leq 1$, such that for any $z \in Q_T$, $u, v \in \mathbb{R}^N$,

$$|a_i^{\alpha\beta}(z, u) - a_i^{\alpha\beta}(z, v)| \leq c\omega(|u - v|^2).$$

(H4) (Natural growth condition) Let $u \in W_2^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$ (the definition of $W_2^{1,1}(Q_T, \mathbb{R}^N)$ sees Section 2), $M = \sup_{z \in Q_T} |u(z)|$ and $g_i(z, u, Xu)$ satisfy

$$|g_i(z, u, Xu)| \leq \mu(M) (|Xu|^2 + g^i(z)),$$

where $\mu(M) > 0$, $g^i(z) \in L_X^{m,\lambda}(Q_T)$, $f_i^\alpha(z) \in L_X^{m,\lambda}(Q_T)$, $i = 1, 2, \dots, N$, $m > p$ (for definition of p see Theorem 3.2), $0 < \lambda < Q + 2$ (Q is the homogeneous dimension relative to Ω), and $2\mu(M)M < \delta$.

Definition 2. If $u \in W_2^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$ satisfies

$$\iint_{Q_T} [u_t^i \varphi^i + a_i^{\alpha\beta}(z, u) X_\alpha \varphi^i X_\beta u^i] dz = \iint_{Q_T} [g_i(z, u, Xu) \varphi^i + f_i^\alpha(z) X_\alpha \varphi^i] dz,$$

for any $\varphi \in C_0^\infty(Q_T, \mathbb{R}^N)$, then u is said a weak solution to (1.1).

Now the main result of the paper is stated as follows.

Theorem 1. Suppose that assumptions (H1)-(H4) hold. If $u \in W_2^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$ is a weak solution of (1.1) in Q_T , then there exists a constant $\varepsilon_0 > 0$ such that for any $p \in [2, 2 + \frac{2Q}{Q+2}\varepsilon_0)$, $Q' \subset \subset Q'' \subset \subset Q_T$, we have $Xu \in L_{X,loc}^{p,\lambda}(Q_T)$ and

$$\|Xu\|_{L_X^{p,\lambda}(Q')} \leq c \left(\|Xu\|_{L^p(Q'')} + \sum_i \|g^i\|_{L_X^{p,\lambda}(Q_T)} + \|f\|_{L_X^{p,\lambda}(Q_T)} \right). \quad (1.3)$$

Remark 1. These conclusions are new even in quasilinear parabolic systems under Euclidean spaces. The proofs depend on some a priori estimates, e.g., parabolic Caccioppoli inequalities. We note that in the study of regularity to parabolic equations in Euclidean spaces, a normal tool is a parabolic Poincaré inequality. But for (1.1), a corresponding parabolic Poincaré inequality is still not proved, we introduce the average $\bar{u}(t)$ of $u(x, t)$ as done in [5] and [12], and apply (2.2) below to deduce a

parabolic Caccioppoli inequality. In order to get higher integrability, we employ a reverse Hölder inequality in parabolic cylinders.

The plan of the paper is organized as follows. In Section 2, we describe the notions of Carnot-Carathéodory distance, ball, doubling property, parabolic cylinder, and some function spaces related to vector fields, such as Morrey space, Sobolev space, and then recall the known iterative lemma, Sobolev-Poincaré inequality in Ω , reverse Hölder inequality relate to parabolic cylinder. In Section 3, we establish a parabolic Caccioppoli inequality with the average $\bar{u}(t)$ of $u(x, t)$ and get the L^p estimates of gradients for weak solutions (Theorem 3) by using the reverse Hölder inequality. In Section 4, higher integrability of gradients for weak solutions to homogeneous parabolic system are derived. In Section 5, the proof of Theorem 1 is given with the help of previous results.

2. PRELIMINARIES

Let us denote the commutator of vector fields $X_1, \dots, X_q$ by

$$X_\beta = [X_{\beta_d}, [X_{\beta_{d-1}}, \dots [X_{\beta_2}, X_{\beta_1}] \dots]],$$

where $\beta = (\beta_1, \dots, \beta_d)$, $1 \leq \beta_i \leq q$, $i = 1, \dots, d$, $|\beta| = d$, and recall that d is the length of X_β .

Definition 3. If $\{X_\beta(x_0)\}_{|\beta| \leq s}$ spans $\mathbb{R}^n$ at every $x_0 \in \Omega \subset \mathbb{R}^n$, then we say that the system $X = (X_1, \dots, X_q)$ satisfies Hörmander's condition of step s .

Following [20], we can assume that Hörmander's vector fields $X_1, \dots, X_q$ are free up to the order s .

Definition 4 (Carnot-Carathéodory distance). Let Ω be a bounded set in $\mathbb{R}^n$. An absolutely continuous curve $\gamma : [0, T] \rightarrow \Omega$ is called a sub-unit curve with respect to the system $X = (X_1, \dots, X_q)$, if $\gamma'(t)$ exists for a.e. $t \in [0, T]$ and satisfies

$$\langle \gamma'(t), \xi \rangle^2 \leq \sum_{j=1}^q \langle X_j(\gamma(t)), \xi \rangle^2, \quad \text{for any } \xi \in \mathbb{R}^n.$$

We denote the length of this curve by $l_S(\gamma) = T$. Given any $x, y \in \Omega$, let $\Phi(x, y)$ be the collection of all sub-unit curves connecting x and y , and define the Carnot-Carathéodory distance induced by X as follows

$$d_X(x, y) = \inf\{l_S(\gamma) : \gamma \in \Phi(x, y)\}.$$

With this distance, we denote a metric ball of radius R centered at x by

$$B_R(x) = B(x, R) = \{y \in \Omega : d(x, y) < R\}.$$

If one does not need to consider the center of the ball, then we also write B_R instead of $B(x, R)$.

It is well known that the doubling property (see [16]) for metric balls holds true, i.e., there exist positive constants c_D and R_D , such that for any $x_0 \in \Omega$, $0 < 2R < R_D$, $B(x_0, 2R) \subset \Omega$,

$$|B(x_0, 2R)| \leq c_D |B(x_0, R)|.$$

Furthermore, it follows that for any $R \leq R_D$ and $\tau \in (0, 1)$,

$$|B_{\tau R}| \geq c_D^{-1} \tau^Q |B_R|.$$

The number $Q = \log_2 c_D$ is called a locally homogeneous dimension relative to Ω . Clearly, $Q \geq n$.

As in [20] we assume that for any small positive R there exist two positive constants c_1 and c_2 , such that

$$c_1 R^Q \leq |B_R| \leq c_2 R^Q. \quad (2.1)$$

Throughout this paper, we denote $z_0 = (x_0, t_0) \in Q_T \subset \mathbb{R}^{n+1}$. A parabolic cylinder with vertex at z_0 is defined by

$$Q_R(z_0) = B_R(x_0) \times \left(t_0 - \frac{R^2}{2}, t_0 + \frac{R^2}{2} \right]$$

In what follows, let us denote $I_R(t_0) = (t_0 - \frac{R^2}{2}, t_0 + \frac{R^2}{2}]$, and the parabolic boundary of Q_R by

$$\partial_p Q_R(z_0) = \left(\partial B_R(x_0) \times \left(t_0 - \frac{R^2}{2}, t_0 + \frac{R^2}{2} \right] \right) \cup B_R(x_0) \times \left\{ t_0 - \frac{R^2}{2} \right\}.$$

We denote by $|B(x, R)|$ the Lebesgue measure of $B(x, R)$ in the n -dimensional space, and the Lebesgue measure of $Q_R(z_0)$ in the $(n+1)$ -dimensional space by $|Q_R(z_0)|$. To simplify the notations, in the sequel, $Q_R(z_0)$, $B_R(x_0)$, $I_R(t_0)$,

$\sqrt{\sum_{i=1}^q |X_i u|^2}$ and (x, t) are written as Q_R , B_R , I_R , $|Xu|$ and z , respectively.

Definition 5 (Morrey space $L_X^{p,\lambda}(Q_T)$). For $1 \leq p < +\infty$, $\lambda > 0$, we say that $f \in L^p(Q_T)$ belongs to the Morrey space $L_X^{p,\lambda}(Q_T)$ if

$$\|f\|_{L_X^{p,\lambda}} = \sup_{z_0 \in Q_T, 0 \leq \rho \leq d_0} \left(\frac{\rho^\lambda}{|Q_T \cap Q_\rho(z_0)|} \iint_{Q_T \cap Q_\rho(z_0)} |f|^p dz \right)^{\frac{1}{p}} < \infty,$$

where d_0 is the diameter of Q_T .

It is easy to prove as in [15] that the spaces $L_X^{p,\lambda}(Q_T)$ are Banach spaces.

Definition 6 (parabolic Sobolev space $W_p^{m,k}(Q_T)$). Let m and k be non-negative integers, $1 \leq p < +\infty$. The set

$$W_p^{m,k}(Q_T) = \{u; X_\alpha u, \partial_t^r u \in L^p(Q_T), 0 \leq |\alpha| \leq m, 0 \leq r \leq k\}$$

is called a parabolic Sobolev space related to Hörmander's vector fields, with the norm

$$\|u\|_{W_p^{m,k}(Q_T)} = \sum_{|\alpha| \leq m} \|X_\alpha u\|_{L^p(Q_T)} + \sum_{r \leq k} \|X_t^r u\|_{L^p(Q_T)}.$$

The space $W_{p,0}^{m,k}(Q_T)$ is the closure of $C_0^\infty(\bar{Q}_T)$ in $W_p^{m,k}(Q_T)$, where $C_0^\infty(\bar{Q}_T)$ is the set of all infinitely differentiable functions in $\bar{Q}_T$ vanishing on the parabolic boundary of Q_T .

Lemma 1 (see [13]). *Let $H(\rho)$ be a non-negative increasing function. Suppose that for any $0 < \rho < R \leq R_0 = \text{dist}(x_0, \partial\Omega)$,*

$$H(\rho) \leq A \left[\left(\frac{\rho}{R} \right)^a + \varepsilon \right] H(R) + BR^b,$$

where A, a and b are positive constants with $a > b$. Then there exist $\varepsilon_1 = \varepsilon_1(A, a, b) > 0$ and $c = c(A, a, b) > 0$, such that if $\varepsilon < \varepsilon_1$, then one has

$$H(\rho) \leq c \left[\left(\frac{\rho}{R} \right)^b H(R) + B\rho^b \right].$$

Lemma 2 (Sobolev-Poincaré inequality, see [9, 14]). *For any open set $\Omega', \bar{\Omega}' \subset \subset \Omega$, there exist positive constants R_0 and c , such that for any $0 < R \leq R_0$, $B_R \subset \Omega$, and $u \in C^\infty(\bar{B}_R)$, it holds*

$$\left(\frac{1}{|B_R|} \int_{B_R} |u - u_R|^{p'} dx \right)^{\frac{1}{p'}} \leq cR \left(\frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{1}{p}},$$

where $1 < p < Q$, $1 \leq p' \leq \frac{pQ}{Q-p}$, $u_R = \frac{1}{|B_R|} \int_{B_R} u(x) dx$, R_0 and c depend on Ω and Ω' .

In particular, if $p = p'$, then

$$\int_{B_R} |u - u_R|^p dx \leq cR^p \int_{B_R} |Xu|^p dx. \quad (2.2)$$

If $p = 2$, $p' = \frac{2Q}{Q-2}$, then

$$\left(\frac{1}{|B_R|} \int_{B_R} |u - u_R|^{\frac{2Q}{Q-2}} dx \right)^{\frac{Q-2}{2Q}} \leq cR \left(\frac{1}{|B_R|} \int_{B_R} |Xu|^2 dx \right)^{\frac{1}{2}}. \quad (2.3)$$

For $u \in C_0^\infty(\bar{B}_R)$, it follows

$$\left(\frac{1}{|B_R|} \int_{B_R} |u|^{p'} dx \right)^{\frac{1}{p'}} \leq cR \left(\frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{1}{p}}. \quad (2.4)$$

We need to define a parabolic distance d_p corresponding to d_X . For $(x, t), (y, s) \in Q_T$, set

$$d_p((x, t), (y, s)) = \sqrt{d_X(x, y)^2 + |t - s|}.$$

Denote a ball under the distance d_p :

$$B_p((x_0, t_0), R) = \{(x, t) \in Q_T : d_p((x_0, t_0), (x, t)) < R\}.$$

Since $B_p((x_0, t_0), R)$ is a homogeneous space ([1]) and it holds

$$Q_R(z) \subset B_p(z, 2R) \subset Q_{2R}(z),$$

we immediately follow the following result, which is the generalization on Q_T of results in [10].

Lemma 3 (reverse Hölder inequality). *Let $\hat{g}, \hat{f}$ be nonnegative on Q_T and satisfy*

$$\hat{g} \in L^{\hat{q}}(Q_T) \text{ and } \hat{f} \in L^r(Q_T), \quad 1 < \hat{q} < r.$$

Assume that there exist constants $b > 0$ and $\theta \in [0, 1)$ such that for any $Q_{2R} \subset Q_T$ the following inequality holds

$$\begin{aligned} \frac{1}{|Q_R|} \iint_{Q_R} \hat{g}^{\hat{q}} dz &\leq b \left[\left(\frac{1}{|Q_{4R/3}|} \iint_{Q_{4R/3}} \hat{g} dz \right)^{\hat{q}} + \frac{1}{|Q_{4R/3}|} \iint_{Q_{4R/3}} \hat{f}^{\hat{q}} dz \right] \\ &\quad + \theta \frac{1}{|Q_{4R/3}|} \iint_{Q_{4R/3}} \hat{g}^{\hat{q}} dz, \end{aligned}$$

then there exist positive constants ε_0 and $\theta_0 = \theta_0(\hat{q}, Q_T)$, such that if $\theta < \theta_0$, then $\hat{g} \in L_{loc}^{\hat{p}}(Q_T)$ for any $\hat{p} \in [\hat{q}, \hat{q} + \varepsilon_0)$, and

$$\left(\frac{1}{|Q_R|} \iint_{Q_R} \hat{g}^{\hat{p}} dz \right)^{\frac{1}{\hat{p}}} \leq c \left[\left(\frac{1}{|Q_{2R}|} \iint_{Q_{2R}} \hat{g}^{\hat{q}} dz \right)^{\frac{1}{\hat{q}}} + \left(\frac{1}{|Q_{2R}|} \iint_{Q_{2R}} \hat{f}^{\hat{p}} dz \right)^{\frac{1}{\hat{p}}} \right],$$

where c and ε_0 depend on $b, \hat{q}, \theta$ and Q .

Lemma 4 (Jensen inequality). *For any bounded open subset $U \subset \mathbb{R}^n$, suppose that $f : U \rightarrow \mathbb{R}$ is an integrable function and $\omega : \mathbb{R} \rightarrow \mathbb{R}$ is a concave function. Then*

$$\frac{1}{|U|} \iint_U \omega(f) dz \leq \omega \left(\frac{1}{|U|} \iint_U f dz \right).$$

3. PARABOLIC CACCIOPPOLI INEQUALITY AND L^p ESTIMATES

In this section, we first establish a parabolic Caccioppoli inequality (Theorem 2) for the weak solution u to (1.1) with the help of the average $\bar{u}(t)$ of u and then get a L^p estimates for the gradient of the weak solution (Theorem 3) by using Lemma 3.

Let two cut-off functions $\xi(x)$ and $\eta(t)$ satisfy that for any $0 < \rho < R$, $B_\rho \subset B_R \subset \Omega$,

$$\xi(x) \in C_0^\infty(B_R), \quad 0 \leq \xi \leq 1, \quad |X\xi| \leq \frac{c}{R-\rho}, \quad \text{and } \xi = 1 \text{ in } B_\rho,$$

$$\eta(t) = \begin{cases} \frac{2t-2(t_0-\frac{R^2}{2})}{R^2-\rho^2}, & t \in (t_0-\frac{R^2}{2}, t_0-\frac{\rho^2}{2}), \\ 1, & t \in [t_0-\frac{\rho^2}{2}, t_0+\frac{R^2}{2}]. \end{cases}$$

and set $\frac{1}{|B_R|} \int_{B_R} \xi^2 dx = N_1$. Similarly to [5, 12], we denote the average on B_R of $u(x, t)$ by

$$\bar{u}(t) = \left(\int_{B_R} \xi^2 dx \right)^{-1} \int_{B_R} u \xi^2 dx = \frac{1}{N_1 |B_R|} \int_{B_R} u \xi^2 dx.$$

Theorem 2 (parabolic Caccioppoli inequality). *Let $u \in W_2^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$ be a weak solution of (1.1). Then for any $0 < \rho < R$, $Q_R \subset Q_T$, one has*

$$\begin{aligned} & \sup_{I_\rho} \int_{B_\rho} |u - \bar{u}(t)|^2 dx + \iint_{Q_\rho} |Xu|^2 dz \\ & \leq \frac{c}{(R-\rho)^2} \iint_{Q_R} |u - \bar{u}(t)|^2 dz + c \iint_{Q_R} \left(\sum_i |g^i|^2 + |f|^2 \right) dz, \end{aligned} \quad (3.1)$$

where the positive constant c depends on Q, δ, L, M and $\mu(M)$, $I_\rho = [t_0 - \frac{\rho^2}{2}, t_0 + \frac{\rho^2}{2}]$.

Proof. Noting $B_\rho \subset B_R \subset \Omega$, multiplying both sides of (1.1) by $(u - \bar{u}(t))\xi^2(x)\eta(t)$, and integrating on $Q'_R = B_R(x_0) \times (t_0 - \frac{R^2}{2}, s]$ ($s \leq t_0 + \frac{R^2}{2}$), we get

$$\begin{aligned} & \iint_{Q'_R} \left[u_t^i + X_\alpha^* (a_i^{\alpha\beta} X_\beta u^i) \right] (u^i - \bar{u}^i(t)) \xi^2 \eta dz \\ & = \iint_{Q'_R} [g_i + X_\alpha^* f_i^\alpha] (u^i - \bar{u}^i(t)) \xi^2 \eta dz. \end{aligned} \quad (3.2)$$

By properties of $\xi(x)$ and $\eta(t)$, and the definition of $\bar{u}(t)$, we have

$$\int_{B_R} (u - \bar{u}(t)) \xi^2 \eta dx = \int_{B_R} u \xi^2 \eta dx - \bar{u}(t) \int_{B_R} \xi^2 \eta dx = 0,$$

and

$$\iint_{Q_R} \bar{u}(t)_t (u - \bar{u}(t)) \xi^2 \eta dz = \int_{I_R} \bar{u}(t)_t dt \int_{B_R} (u - \bar{u}(t)) \xi^2 \eta = 0.$$

It yields

$$\begin{aligned} \iint_{Q_R} u_t (u - \bar{u}(t)) \xi^2 \eta dz &= \iint_{Q_R} (u_t - \bar{u}(t)_t) (u - \bar{u}(t)) \xi^2 \eta dz \\ &= \iint_{Q_R} \left[\left(\frac{1}{2} |u - \bar{u}(t)|^2 \eta \right)_t \xi^2 - \frac{1}{2} |u - \bar{u}(t)|^2 \xi^2 \eta_t \right] dz. \end{aligned}$$

Since the left and right hand sides of (3.2) have the form

$$\begin{aligned} &\iint_{Q'_R} \left[u_t^i + X_\alpha^* (a_i^{\alpha\beta} X_\beta u^i) \right] (u^i - \bar{u}^i(t)) \xi^2 \eta dz \\ &= \iint_{Q'_R} \left[\left(\frac{1}{2} |u^i - \bar{u}^i(t)|^2 \eta \right)_t \xi^2 - \frac{1}{2} |u^i - \bar{u}^i(t)|^2 \xi^2 \eta_t + a_i^{\alpha\beta} \xi^2 \eta X_\alpha u^i X_\beta u^i \right. \\ &\quad \left. + 2a_i^{\alpha\beta} (u^i - \bar{u}^i(t)) \xi \eta X_\alpha \xi X_\beta u^i \right] dz \end{aligned}$$

and

$$\begin{aligned} &\iint_{Q'_R} [g_i + X_\alpha^* f_i^\alpha] (u^i - \bar{u}^i(t)) \xi^2 \eta dz \\ &= \iint_{Q'_R} \left[g_i (u^i - \bar{u}^i(t)) \xi^2 \eta + f_i^\alpha \xi^2 \eta X_\alpha u^i + 2\xi \eta (u^i - \bar{u}^i(t)) f_i^\alpha X_\alpha \xi \right] dz, \end{aligned}$$

respectively, it follows that (3.2) becomes

$$\begin{aligned} &\iint_{Q'_R} \left[\left(\frac{1}{2} |u^i - \bar{u}^i(t)|^2 \eta \right)_t \xi^2 + a_i^{\alpha\beta} \xi^2 \eta X_\alpha u^i X_\beta u^i \right] dz \\ &= \iint_{Q'_R} \left[\frac{1}{2} |u^i - \bar{u}^i(t)|^2 \xi^2 \eta_t - 2a_i^{\alpha\beta} (u^i - \bar{u}^i(t)) \xi \eta X_\alpha \xi X_\beta u^i \right] dz \\ &\quad + \iint_{Q'_R} \left[g_i (u^i - \bar{u}^i(t)) \xi^2 \eta + f_i^\alpha \xi^2 \eta X_\alpha u^i \right] dz \\ &\quad + \iint_{Q'_R} \left[2\xi \eta (u^i - \bar{u}^i(t)) f_i^\alpha X_\alpha \xi \right] dz. \end{aligned} \tag{3.3}$$

By (H1), (H4), $|u| \leq M$ and the Young inequality, it implies from (3.3) that

$$\begin{aligned}
& \iint_{Q'_R} \left(\frac{1}{2} |u - \bar{u}(t)|^2 \eta \right)_t \xi^2 dz + \delta \iint_{Q'_R} |Xu|^2 \xi^2 \eta dz \\
& \leq \iint_{Q'_R} \frac{1}{2} |u - \bar{u}(t)|^2 \xi^2 \eta_t dz + \varepsilon \iint_{Q'_R} |Xu|^2 \xi^2 \eta dz \\
& + c_\varepsilon \iint_{Q'_R} |u - \bar{u}(t)|^2 |X\xi|^2 \eta dz + 2\mu(M)M \iint_{Q'_R} |Xu|^2 \xi^2 \eta dz \\
& + \mu(M)c_\varepsilon \iint_{Q'_R} \sum_i |g^i|^2 \xi^2 \eta dz + \mu(M)\varepsilon \iint_{Q'_R} |u - \bar{u}(t)|^2 \xi^2 \eta dz \\
& + 2c_\varepsilon \iint_{Q'_R} |f|^2 \xi^2 \eta dz + \varepsilon \iint_{Q'_R} |Xu|^2 \xi^2 \eta dz.
\end{aligned} \tag{3.4}$$

Using

$$\iint_{Q'_R} \left(\frac{1}{2} |u - \bar{u}(t)|^2 \eta \right)_t \xi^2 dz = \int_{B_R} \frac{1}{2} |u - \bar{u}(s)|^2 \xi^2 \eta dx,$$

it follows by (3.4) that

$$\begin{aligned}
& \int_{B_R} \frac{1}{2} |u - \bar{u}(s)|^2 \xi^2 \eta dx + (\delta - 2(\varepsilon + \mu(M)M)) \iint_{Q'_R} |Xu|^2 \xi^2 \eta dz \\
& \leq \iint_{Q'_R} \frac{1}{2} |u - \bar{u}(t)|^2 \xi^2 \eta_t dz + c_\varepsilon \iint_{Q'_R} |u - \bar{u}(t)|^2 |X\xi|^2 \eta dz \\
& + 2c_\varepsilon \iint_{Q'_R} |f|^2 \xi^2 \eta dz + \mu(M)\varepsilon \iint_{Q'_R} |u - \bar{u}(t)|^2 \xi^2 \eta dz \\
& + \mu(M)c_\varepsilon \iint_{Q'_R} \sum_i |g^i|^2 \xi^2 \eta dz.
\end{aligned} \tag{3.5}$$

Because $2\mu(M)M < \delta$, we can choose a small positive ε such that $2(\varepsilon + \mu(M)M) < \delta$, and by (3.5) we obtain (3.1). The proof is completed. $\square$

Theorem 3. Let $u \in W_2^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$ be a weak solution of (1.1) in Q_T . Then there exists a positive constants ε_0 such that $Xu \in L_{loc}^p(Q_T)$, $p \in \left[2, 2 + \frac{2Q}{Q+2}\varepsilon_0\right)$, and for any $Q_R \subset Q_{2R} \subset Q_T$, it holds

$$\left(\frac{1}{|Q_R|} \iint_{Q_R} |Xu|^p dz \right)^{\frac{1}{p}} \leq$$

$$\leq c \left[\left(\frac{1}{|Q_{2R}|} \iint_{Q_{2R}} |Xu|^2 dz \right)^{\frac{1}{2}} + \left(\frac{1}{|Q_{2R}|} \iint_{Q_{2R}} \left(\sum_i |g^i|^2 + |f|^2 \right)^{\frac{p}{2}} dz \right)^{\frac{1}{p}} \right] \quad (3.6)$$

where the positive constant c depends on Q, δ, L, M and $\mu(M)$.

Proof. Denote $2^* = \frac{2Q}{Q-2}$, $\tilde{q} = \frac{2Q}{Q+2}$. Clearly,

$$\begin{aligned} & \iint_{Q_{4R/5}} |u - \bar{u}(t)|^2 dz \\ & \leq \sup_{I_{4R/5}} \left(\int_{B_{4R/5}} |u - \bar{u}(t)|^2 dx \right)^{\frac{1}{2}} \cdot \left(\int_{I_{4R/5}} \left(\int_{B_{4R/5}} |u - \bar{u}(t)|^2 dx \right)^{\frac{1}{2}} dt \right) \\ & \equiv A \cdot B. \end{aligned} \quad (3.7)$$

In virtue of (2.2) and (3.1),

$$\begin{aligned} A & \leq \frac{c}{R} \left(\iint_{Q_R} |u - \bar{u}(t)|^2 dz \right)^{\frac{1}{2}} + c \left(\iint_{Q_R} \left(\sum_i |g^i|^2 + |f|^2 \right) dz \right)^{\frac{1}{2}} \\ & \leq c \left(\iint_{Q_R} |Xu|^2 dz \right)^{\frac{1}{2}} + c \left(\iint_{Q_R} \left(\sum_i |g^i|^2 + |f|^2 \right) dz \right)^{\frac{1}{2}}. \end{aligned} \quad (3.8)$$

By (2.1), (2.2), (2.3) and the Hölder inequality, we have

$$\begin{aligned} B & \leq \int_{I_R} \left(\int_{B_R} |u - \bar{u}(t)|^{\tilde{q}} dx \right)^{\frac{1}{2\tilde{q}}} \left(\int_{B_R} |u - \bar{u}(t)|^{2^*} dx \right)^{\frac{1}{2 \cdot 2^*}} dt \\ & \leq c R^{\frac{1}{2}} \int_{I_R} \left(\int_{B_R} |Xu|^{\tilde{q}} dx \right)^{\frac{1}{2\tilde{q}}} \left(\int_{B_R} |Xu|^2 dx \right)^{\frac{1}{4}} dt \\ & \leq c R^{\frac{3}{2} - \frac{1}{Q}} \left(\iint_{Q_R} |Xu|^{\tilde{q}} dz \right)^{\frac{1}{2\tilde{q}}} \left(\iint_{Q_R} |Xu|^2 dz \right)^{\frac{1}{4}}. \end{aligned} \quad (3.9)$$

Inserting (3.8) and (3.9) into (3.7) follows

$$\begin{aligned} & \iint_{Q_{4R/5}} |u - \bar{u}(t)|^2 dz \leq c R^{\frac{3}{2} - \frac{1}{Q}} \left(\iint_{Q_R} |Xu|^2 dz \right)^{\frac{3}{4}} \cdot \left(\iint_{Q_R} |Xu|^{\tilde{q}} dz \right)^{\frac{1}{2\tilde{q}}} \\ & + c R^{\frac{3}{2} - \frac{1}{Q}} \left(\iint_{Q_R} |Xu|^{\tilde{q}} dz \right)^{\frac{1}{2\tilde{q}}} \left(\iint_{Q_R} |Xu|^2 dz \right)^{\frac{1}{4}} \cdot \left(\iint_{Q_R} \left(\sum_i |g^i|^2 + |f|^2 \right) dz \right)^{\frac{1}{2}} \\ & \equiv I + II. \end{aligned} \quad (3.10)$$

To estimate I and II , it uses the Young inequality to get

$$I \leq \varepsilon R^2 \iint_{Q_R} |Xu|^2 dz + c_\varepsilon R^{-\frac{4}{Q}} \left(\iint_{Q_R} |Xu|^{\tilde{q}} dz \right)^{\frac{2}{\tilde{q}}}, \quad (3.11)$$

$$\begin{aligned} II &\leq \varepsilon R^2 \iint_{Q_R} |Xu|^2 dz + \varepsilon c_\varepsilon R^{-\frac{4}{Q}} \left(\iint_{Q_R} |Xu|^{\tilde{q}} dz \right)^{\frac{2}{\tilde{q}}} \\ &\quad + c_\varepsilon R^2 \iint_{Q_R} \left(\sum_i |g^i|^2 + |f|^2 \right) dz. \end{aligned} \quad (3.12)$$

Putting (3.11) and (3.12) into (3.10),

$$\begin{aligned} \iint_{Q_{4R/5}} |u - \bar{u}(t)|^2 dz &\leq \varepsilon R^2 \iint_{Q_R} |Xu|^2 dz + c_\varepsilon R^{-\frac{4}{Q}} \left(\iint_{Q_R} |Xu|^{\tilde{q}} dz \right)^{\frac{2}{\tilde{q}}} \\ &\quad + c_\varepsilon R^2 \iint_{Q_R} \left(\sum_i |g^i|^2 + |f|^2 \right) dz. \end{aligned} \quad (3.13)$$

Let us return (3.1) with (3.13), then

$$\begin{aligned} \frac{1}{|Q_{3R/4}|} \iint_{Q_{3R/4}} |Xu|^2 dz &\leq \frac{\varepsilon}{|Q_R|} \iint_{Q_R} |Xu|^2 dz + c_\varepsilon \left(\frac{1}{|Q_R|} \iint_{Q_R} |Xu|^{\tilde{q}} dz \right)^{\frac{2}{\tilde{q}}} \\ &\quad + \frac{c_\varepsilon}{|Q_R|} \iint_{Q_R} \left(\sum_i |g^i|^2 + |f|^2 \right) dz. \end{aligned}$$

Denoting $\hat{g} = |Xu|^{\hat{q}}$ ($\hat{q} = \frac{2}{\tilde{q}} = \frac{Q+2}{Q} > 1$), $\hat{f} = \left(\sum_i |g^i|^2 + |f|^2 \right)^{\frac{Q}{Q+2}}$, $\theta = \varepsilon$, the above inequality is

$$\begin{aligned} \frac{1}{|Q_{3R/4}|} \iint_{Q_{3R/4}} \hat{g}^{\hat{q}} dz &\leq c \left[\left(\frac{1}{|Q_R|} \iint_{Q_R} \hat{g} dz \right)^{\hat{q}} + \frac{1}{|Q_R|} \iint_{Q_R} \hat{f}^{\hat{q}} dz \right] \\ &\quad + \frac{\theta}{|Q_R|} \iint_{Q_R} \hat{g}^{\hat{q}} dz. \end{aligned} \quad (3.14)$$

By Lemma 3, we know that there exist positive constant c and ε_0 , such that for $\hat{p} \in [\hat{q}, \hat{q} + \varepsilon_0)$,

$$\left(\frac{1}{|Q_R|} \iint_{Q_R} |Xu|^{\hat{p}\tilde{q}} dz \right)^{\frac{1}{\hat{p}}} \leq$$

$$\leq c \left[\left(\frac{1}{|Q_{2R}|} \iint_{Q_{2R}} |Xu|^2 dz \right)^{\frac{Q}{Q+2}} + \left(\frac{1}{|Q_{2R}|} \iint_{Q_{2R}} \left(\sum_i |g^i|^2 + |f|^2 \right)^{\frac{\hat{p}\tilde{q}}{2}} dz \right)^{\frac{1}{\hat{p}}} \right].$$

Setting $p = \hat{p}\tilde{q} \in \left[2, 2 + \frac{2Q}{Q+2}\varepsilon_0 \right)$, the proof is finished. $\square$

4. HOMOGENEOUS PARABOLIC SYSTEM

In this section, we study the following homogeneous parabolic system

$$u_t^i + X_\alpha^* (a_i^{\alpha\beta}(z, u) X_\beta u^i) = 0. \quad (4.1)$$

For convenience, we divide (4.1) into two parts. Concretely, let $v \in W_2^{1,1} \cap L^\infty$ be a weak solution of the following constant coefficients system:

$$\begin{cases} v_t^i + X_\alpha^* \left(\left(a_i^{\alpha\beta}(\cdot, \bar{u}(t)) \right)_R X_\beta v^i \right) = 0, & \text{in } Q_R, \\ v - u \in W_{2,0}^{1,1} \cap L^\infty, & \text{on } \partial_p Q_R, \end{cases} \quad (4.2)$$

then $w = u - v$ satisfies:

$$\begin{cases} w_t^i + X_\alpha^* \left(\left(a_i^{\alpha\beta}(\cdot, \bar{u}(t)) \right)_R X_\beta w^i \right) \\ = X_\alpha^* \left(\left(\left(a_i^{\alpha\beta}(\cdot, \bar{u}(t)) \right)_R - a_i^{\alpha\beta}(z, u) \right) X_\beta u^i \right), & \text{in } Q_R, \\ w \in W_{2,0}^{1,1} \cap L^\infty, & \text{on } \partial_p Q_R, \end{cases} \quad (4.3)$$

where $\left(a_i^{\alpha\beta}(\cdot, \bar{u}(t)) \right)_R = \frac{1}{|Q_R|} \iint_{Q_R} a_i^{\alpha\beta}(z, \bar{u}(t)) dz$.

Lemma 5 ([7]). *Let $v \in W_2^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$ be a weak solution of (4.2). Then for any $0 < \rho < R$, $Q_R \subset\subset Q_T$, one has*

$$\iint_{Q_\rho} |Xv|^2 dz \leq c \left(\frac{\rho}{R} \right)^{Q+2} \iint_{Q_R} |Xv|^2 dz.$$

Theorem 4. *Let $u \in W_2^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$ be a weak solution of (4.1). Then for any $0 < \rho < R$, $Q_R \subset\subset Q_T$, there exists $\varepsilon_0 > 0$, such that for any $p \in \left[2, 2 + \frac{2Q}{Q+2}\varepsilon_0 \right)$, $\frac{p-2}{p}(Q+2) < \mu < Q+2$, we have*

$$\iint_{Q_\rho} |Xu|^p dz \leq c \left(\frac{\rho}{R} \right)^{\frac{2(Q+2)-p(Q+2-\mu)}{2}} \iint_{Q_R} |Xu|^p dz.$$

Proof. When $\frac{1}{2}R \leq \rho < R$, the result is clearly true. Now it is enough to treat the case $\rho < \frac{1}{2}R$.

Multiplying both sides of (4.3) by w^i and integrating on Q_R , we obtain

$$\iint_{Q_R} \left(w_t^i w^i + \left(a_i^{\alpha\beta}(\cdot, \bar{u}(t)) \right)_R X_\beta w^i X_\alpha w^i \right) dz.$$

$$= \iint_{Q_R} \left(\left(a_i^{\alpha\beta}(\cdot, \bar{u}(t)) \right)_R - a_i^{\alpha\beta}(z, u) \right) X_\beta u^i X_\alpha w^i dz. \quad (4.4)$$

Noting $\iint_{Q_R} w_i^i w^i dz = \int_{B_R} dx \int_{t_0-R^2/2}^{t_0+R^2/2} w^i dw^i \geq 0$ and (H1), we get

$$\begin{aligned} \delta \iint_{Q_R} |Xw|^2 dz &\leq \iint_{Q_R} \left| \left(a_i^{\alpha\beta}(\cdot, \bar{u}(t)) \right)_R - a_i^{\alpha\beta}(z, \bar{u}(t)) \right| |Xu| |Xw| dz \\ &\quad + \iint_{Q_R} \left| a_i^{\alpha\beta}(z, \bar{u}(t)) - a_i^{\alpha\beta}(z, u) \right| |Xu| |Xw| dz \\ &= I + II. \end{aligned} \quad (4.5)$$

Because $a_i^{\alpha\beta}(z, u)$ satisfies (H2), it follows

$$\begin{aligned} &\iint_{Q_R} \left| a_i^{\alpha\beta}(z, \bar{u}(t)) - \left(a_i^{\alpha\beta}(\cdot, \bar{u}(t)) \right)_R \right|^{\frac{2p}{p-2}} dz \\ &\leq c \iint_{Q_R} \left| a_i^{\alpha\beta}(z, \bar{u}(t)) - \left(a_i^{\alpha\beta}(\cdot, \bar{u}(t)) \right)_R \right| dz \leq c |Q_R| \eta_R \left(a_i^{\alpha\beta} \right). \end{aligned}$$

By the Young inequality and the Hölder inequality, we have

$$\begin{aligned} I &\leq c_\varepsilon \left(\iint_{Q_R} \left| a_i^{\alpha\beta}(z, \bar{u}(t)) - \left(a_i^{\alpha\beta}(\cdot, \bar{u}(t)) \right)_R \right|^{\frac{2p}{p-2}} dz \right)^{\frac{p-2}{p}} \left(\iint_{Q_R} |Xu|^p dz \right)^{\frac{2}{p}} \\ &\quad + \varepsilon \iint_{Q_R} |Xw|^2 dz \\ &\leq c_\varepsilon \left(|Q_R| \eta_R \left(a_i^{\alpha\beta} \right) \right)^{\frac{p-2}{p}} \left(\iint_{Q_R} |Xu|^p dz \right)^{\frac{2}{p}} + \varepsilon \iint_{Q_R} |Xw|^2 dz. \end{aligned} \quad (4.6)$$

By the Young inequality, (H3), Lemma 4 and the Hölder inequality, it shows

$$\begin{aligned} II &\leq c_\varepsilon \iint_{Q_R} \left| a_i^{\alpha\beta}(z, \bar{u}(t)) - a_i^{\alpha\beta}(z, u) \right|^2 |Xu|^2 dz + \varepsilon \iint_{Q_R} |Xw|^2 dz \\ &\leq c_\varepsilon \iint_{Q_R} \omega^2 \left(|u - \bar{u}(t)|^2 \right) |Xu|^2 dz + \varepsilon \iint_{Q_R} |Xw|^2 dz \\ &\leq c_\varepsilon \left(\iint_{Q_R} \left(\omega \left(|u - \bar{u}(t)|^2 \right) \right)^{\frac{2p}{p-2}} dz \right)^{\frac{p-2}{p}} \left(\iint_{Q_R} |Xu|^p dz \right)^{\frac{2}{p}} \\ &\quad + \varepsilon \iint_{Q_R} |Xw|^2 dz \end{aligned}$$

$$\begin{aligned}
&\leq c_\varepsilon \left(|Q_R| \omega \left(\frac{1}{|Q_R|} \iint_{Q_R} |u - \bar{u}(t)|^2 dz \right) \right)^{\frac{p-2}{p}} \left(\iint_{Q_R} |Xu|^p dz \right)^{\frac{2}{p}} \\
&+ \varepsilon \iint_{Q_R} |Xw|^2 dz.
\end{aligned} \tag{4.7}$$

Inserting (4.6) and (4.7) into (4.5), we have

$$\begin{aligned}
&\delta \iint_{Q_R} |Xw|^2 dz \\
&\leq c_\varepsilon |Q_R| \left[\left(\eta_R(a_i^{\alpha\beta}) \right)^{\frac{p-2}{p}} + \left(\omega \left(\frac{1}{|Q_R|} \iint_{Q_R} |u - \bar{u}(t)|^2 dz \right) \right)^{\frac{p-2}{p}} \right] \\
&\cdot \left(\frac{1}{|Q_R|} \iint_{Q_R} |Xu|^p dz \right)^{\frac{2}{p}} + 2\varepsilon \iint_{Q_R} |Xw|^2 dz.
\end{aligned}$$

Let $G(z_0, R) = \left(\eta_R(a_i^{\alpha\beta}) \right)^{\frac{p-2}{p}} + \left(\omega \left(\frac{1}{|Q_R|} \iint_{Q_R} |u - \bar{u}(t)|^2 dz \right) \right)^{\frac{p-2}{p}}$ and choose ε small enough such that $\delta - 2\varepsilon > 0$, then the above can be rewritten as

$$\iint_{Q_R} |Xw|^2 dz \leq c G(z_0, R) |Q_R| \left(\frac{1}{|Q_R|} \iint_{Q_R} |Xu|^p dz \right)^{\frac{2}{p}}.$$

By the above, Lemma 5 and the Hölder inequality, we know that for any $\rho < \frac{1}{2}R$,

$$\begin{aligned}
&\iint_{Q_{2\rho}} |Xu|^2 dz \leq c \left(\frac{\rho}{R} \right)^{Q+2} \iint_{Q_R} |Xv|^2 dz + c \iint_{Q_R} |Xw|^2 dz \\
&\leq c \left(\frac{\rho}{R} \right)^{Q+2} \iint_{Q_R} |Xu|^2 dz + c \iint_{Q_R} |Xw|^2 dz \\
&\leq c |Q_R| \left[\left(\frac{\rho}{R} \right)^{Q+2} + G(z_0, R) \right] \left(\frac{1}{|Q_R|} \iint_{Q_R} |Xu|^p dz \right)^{\frac{2}{p}}.
\end{aligned} \tag{4.8}$$

Due to Theorem 3 with $g_i = f = 0$ and (4.8), it holds

$$\begin{aligned}
&\iint_{Q_\rho} |Xu|^p dz \leq c |Q_\rho| \left(\frac{1}{|Q_{2\rho}|} \iint_{Q_{2\rho}} |Xu|^2 dz \right)^{\frac{p}{2}} \\
&\leq c \left(\left(\frac{\rho}{R} \right)^{Q+2} + G(z_0, R) \right)^{\frac{p}{2}} \left(\frac{|Q_R|}{|Q_\rho|} \right)^{\frac{p-2}{2}} \iint_{Q_R} |Xu|^p dz.
\end{aligned}$$

Taking $H(\rho) = \left(|Q_\rho|^{\frac{p-2}{2}} \iint_{Q_\rho} |Xu|^p dz \right)^{\frac{2}{p}}$, $H(R) = \left(|Q_R|^{\frac{p-2}{2}} \iint_{Q_R} |Xu|^p dz \right)^{\frac{2}{p}}$, $a = Q + 2$, $B = 0$. We have by Lemma 1 with $b = \mu \left(\frac{p-2}{p} (Q + 2) \right) < \mu < Q + 2$ that

$$\left(|Q_\rho|^{\frac{p-2}{2}} \iint_{Q_\rho} |Xu|^p dz \right)^{\frac{2}{p}} \leq c \left(\frac{\rho}{R} \right)^\mu \left(|Q_R|^{\frac{p-2}{2}} \iint_{Q_R} |Xu|^p dz \right)^{\frac{2}{p}}.$$

Inserting $\frac{|Q_R|}{|Q_\rho|} \leq c \left(\frac{\rho}{R} \right)^{-Q-2}$ into the above, we get the result. $\square$

5. PROOFS OF THE MAIN RESULTS

In this section, we will prove Theorem 1. Let $v \in W_2^{1,1} \cap L^\infty$ be a weak solution of the following system:

$$\begin{cases} v_t^i + X_\alpha^* \left(a_i^{\alpha\beta}(\cdot, \bar{u}(t)) X_\beta v^i \right) = 0, & \text{in } Q_R, \\ v - u \in W_{2,0}^{1,1} \cap L^\infty, & \text{on } \partial_p Q_R, \end{cases} \quad (5.1)$$

then $w = u - v$ satisfies:

$$\begin{cases} w_t^i + X_\alpha^* \left(a_i^{\alpha\beta}(\cdot, \bar{u}(t)) X_\beta w^i \right) = g_i + X_\alpha^* f_i^\alpha, & \text{in } Q_R, \\ w \in W_{2,0}^{1,1} \cap L^\infty, & \text{on } \partial_p Q_R. \end{cases} \quad (5.2)$$

Lemma 6. *Let $w \in W_{2,0}^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$ be a weak solution of (5.2) in Q_T . Then there exists a positive constant ε_0 , such that $Xw \in L_{loc}^p(Q_T)$, $p \in \left[2, 2 + \frac{2Q}{Q+2} \varepsilon_0 \right)$, and it holds*

$$\iint_{Q_R} |Xw|^p dz \leq c \iint_{Q_{2R}} \left(\sum_i |g^i|^p + |f|^p \right) dz + c \iint_{Q_{2R}} |Xu|^p dz,$$

for any $Q_{2R} \subset\subset Q_T$.

Proof. Since $w \in W_{2,0}^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$, there exists a constant $M_1 > 0$, such that $|w| \leq M_1$.

Multiplying both sides of (1.1) by w^i and integrating on Q_{2R} leads to

$$\iint_{Q_{2R}} \left[w_t^i w^i + a_i^{\alpha\beta} X_\beta w^i X_\alpha w^i \right] dz = \iint_{Q_{2R}} \left[g_i w^i + f_i^\alpha X_\alpha w^i \right] dz.$$

Since

$$\iint_{Q_{2R}} w_t w dz = \int_{B_{2R}} dx \int_{I_{2R}} w \frac{\partial w}{\partial t} dt = \int_{B_{2R}} \frac{1}{2} w^2 dx \geq 0,$$

we have by (H1), (H4), the Young inequality and (2.4) that

$$\begin{aligned} \delta \iint_{Q_{2R}} |Xw|^2 dz &\leq \iint_{Q_{2R}} w_t w dz + \delta \iint_{Q_{2R}} |Xw|^2 dz \\ &\leq \iint_{Q_{2R}} \mu(M) \left(|Xu|^2 + \sum_i |g^i|^2 \right) |w| dz + c_\varepsilon \iint_{Q_{2R}} |f|^2 dz + \varepsilon \iint_{Q_{2R}} |Xw|^2 dz \\ &\leq \mu(M) M_1 \iint_{Q_{2R}} |Xu|^2 dz + c \iint_{Q_{2R}} \left(\sum_i |g^i|^2 + |f|^2 \right) dz + 2\varepsilon \iint_{Q_{2R}} |Xw|^2 dz, \end{aligned}$$

where ε is small enough such that $\delta - 2\varepsilon > 0$. It gets

$$\iint_{Q_{2R}} |Xw|^2 dz \leq c \iint_{Q_{2R}} |Xu|^2 dz + c \iint_{Q_{2R}} \left(\sum_i |g^i|^2 + |f|^2 \right) dz.$$

Using the above estimate into (3.6) and the Hölder inequality, it follows

$$\begin{aligned} &\frac{1}{|Q_R|} \iint_{Q_R} |Xw|^p dz \\ &\leq c \left[\frac{1}{|Q_{2R}|} \iint_{Q_{2R}} |Xu|^2 dz + \frac{1}{|Q_{2R}|} \iint_{Q_{2R}} \left(\sum_i |g^i|^2 + |f|^2 \right) dz \right]^{\frac{p}{2}} \\ &\quad + \frac{c}{|Q_{2R}|} \iint_{Q_{2R}} \left(\sum_i |g^i|^p + |f|^p \right) dz \\ &\leq \frac{c}{|Q_{2R}|} \iint_{Q_{2R}} |Xu|^p dz + \frac{c}{|Q_{2R}|} \iint_{Q_{2R}} \left(\sum_i |g^i|^p + |f|^p \right) dz. \end{aligned}$$

Hence the conclusion is proved. $\square$

Theorem 5. Suppose that assumptions (H1)-(H4) hold. Let $u \in W_2^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$ be a weak solution of (1.1). Then there exists a positive constant ε_0 , such that for any $p \in \left[2, 2 + \frac{2Q}{Q+2}\varepsilon_0\right)$, we have

$$Xu \in L_{X,loc}^{p,\lambda}(Q_T, \mathbb{R}^{qn}).$$

Moreover, there exists a constant $R_0 \leq d_0$ such that for any ρ and R , $0 < \rho < R \leq R_0$, $Q_{2R} \subset\subset Q_T$, one has

$$\|Xu\|_{L_X^{p,\lambda}(Q_\rho)}^p \leq c \left[\|Xu\|_{L^p(Q_R)}^p + \sum_i \|g^i\|_{L_X^{p,\lambda}}^p + \|f\|_{L_X^{p,\lambda}}^p \right].$$

Proof. By Lemma 6 and Theorem 4, it shows

$$\begin{aligned}
 & \iint_{Q_\rho} |Xu|^p dz \leq c \iint_{Q_\rho} |Xv|^p dz + c \iint_{Q_\rho} |Xw|^p dz \\
 & \leq c \left(\frac{\rho}{R} \right)^{\frac{2(Q+2)-p(Q+2-\mu)}{2}} \iint_{Q_R} |Xu|^p dz + c \iint_{Q_{2R}} \left(\sum_i |g^i|^p + |f|^p \right) dz \\
 & + c \iint_{Q_{2R}} |Xu|^p dz \\
 & \leq \left[c \left(\frac{\rho}{R} \right)^{\frac{2(Q+2)-p(Q+2-\mu)}{2}} + c \right] \iint_{Q_{2R}} |Xu|^p dz \\
 & + c R^{Q+2-\lambda} \left(\sum_i \|g^i\|_{L_X^{p,\lambda}}^p + \|f\|_{L_X^{p,\lambda}}^p \right).
 \end{aligned}$$

Denote $H(\rho) = \iint_{Q_\rho} |Xu|^p dz$, $H(R) = \iint_{Q_R} |Xu|^p dz$, $a = \frac{2(Q+2)-p(Q+2-\mu)}{2}$, $b = Q+2-\lambda$, $B = c \left(\sum_i \|g^i\|_{L_X^{p,\lambda}}^p + \|f\|_{L_X^{p,\lambda}}^p \right)$, $0 < \lambda < Q+2$. Then there exists μ , $Q+2-\frac{2\lambda}{p} < \mu < Q+2$, such that $a > b$. By Lemma 1, we have

$$\begin{aligned}
 & \iint_{Q_\rho} |Xu|^p dz \\
 & \leq c \left[\left(\frac{\rho}{R} \right)^{Q+2-\lambda} \iint_{Q_{2R}} |Xu|^p dz + \rho^{Q+2-\lambda} \left(\sum_i \|g^i\|_{L_X^{p,\lambda}}^p + \|f\|_{L_X^{p,\lambda}}^p \right) \right].
 \end{aligned}$$

This proof is completed. $\square$

Proof. (of Theorem 1) By Theorem 5 and the cutoff function technique, it is easy to see that Theorem 1 is true. $\square$

ACKNOWLEDGEMENT

The author is sincerely grateful to PengCheng Niu for giving some useful discussions.

REFERENCES

- [1] M. Bramanti and L. Brandolini, "Schauder estimates for parabolic nondivergence operators of Hörmander type," *J. Differ. Equations*, vol. 234, no. 1, pp. 177–245, 2007.
- [2] S. Campanato, "Equazioni paraboliche del secondo ordine e spazi $\mathfrak{L}^{2,\theta}(\Omega, \delta)$," *Ann. Mat. Pura Appl. (4)*, vol. 73, pp. 55–102, 1966.
- [3] S. Campanato, " L^p regularity for weak solutions of parabolic systems," *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.*, vol. 7, pp. 65–85, 1980.

- [4] G. Di Fazio and M. S. Fanciullo, “Gradient estimates for elliptic systems in Carnot-Carathéodory spaces,” *Commentat. Math. Univ. Carol.*, vol. 43, no. 4, pp. 605–618, 2002.
- [5] H. Dong and D. Kim, “Global regularity of weak solutions to quasilinear elliptic and parabolic equations with controlled growth,” *Commun. Partial Differ. Equations*, vol. 36, no. 10-12, pp. 1750–1777, 2011.
- [6] Y. Dong and P. Niu, “Estimates in Morrey spaces and Hölder continuity for weak solutions to degenerate elliptic systems,” *Manuscr. Math.*, vol. 138, no. 3-4, pp. 419–437, 2012.
- [7] Y. Dong, M. C. Zhu, and P. C. Niu, “Estimates in generalized Morrey spaces for weak solutions to divergence degenerate parabolic systems,” *arXiv: 1109.1929v1 Math. AP*, 2011.
- [8] D. Gao, P. Niu, and J. Wang, “Partial regularity for degenerate subelliptic systems associated with Hörmander’s vector fields,” *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 73, no. 10, pp. 3209–3223, 2010.
- [9] N. Garofalo and D.-M. Nhieu, “Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces,” *Commun. Pure Appl. Math.*, vol. 49, no. 10, pp. 1081–1144, 1996.
- [10] U. Gianazza, “Regularity for nonlinear equations involving square Hörmander operators,” *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 23, no. 1, pp. 49–73, 1994.
- [11] M. Giaquinta and E. Giusti, “Partial regularity for the solutions to nonlinear parabolic systems,” *Ann. Mat. Pura Appl. (4)*, vol. 97, pp. 253–266, 1973.
- [12] M. Giaquinta and M. Struwe, “On the partial regularity of weak solutions on nonlinear parabolic systems,” *Math. Z.*, vol. 179, pp. 437–451, 1982.
- [13] Q. Han and F. Lin, *Elliptic partial differential equations*, ser. Courant Lecture Notes in Mathematics. New York; Providence: NY: Courant Institute of Mathematical Sciences; RI: American Mathematical Society (AMS), 1997, vol. 1.
- [14] G. Lu, “The sharp Poincaré inequality for free vector fields: An endpoint result,” *Rev. Mat. Iberoam.*, vol. 10, no. 2, pp. 453–466, 1994.
- [15] M. McBride, “Estimates in the generalized Morrey spaces for linear parabolic systems,” *arXiv: 0811.3360 Math. AP*, 2010.
- [16] A. Nagel, E. M. Stein, and S. Wainger, “Balls and metrics defined by vector fields. I: Basic properties,” *Acta Math.*, vol. 155, pp. 103–147, 1985.
- [17] M. Struwe, “On the Hölder continuity of bounded weak solutions of quasilinear parabolic systems,” *Manuscr. Math.*, vol. 35, pp. 125–145, 1981.
- [18] S. Tang and P. Niu, “Morrey estimates for parabolic nondivergence operators of Hörmander type,” *Rend. Semin. Mat. Univ. Padova*, vol. 123, pp. 91–129, 2010.
- [19] J. Wang, P. Niu, and X. Cui, “ l^p estimates for weak solutions to nonlinear sub-elliptic systems related to Hörmander’s vector fields,” *Kyushu J. Math.*, vol. 63, pp. 301–314, 2009.
- [20] C.-J. Xu and C. Zuily, “Higher interior regularity for quasilinear subelliptic systems,” *Calc. Var. Partial Differ. Equ.*, vol. 5, no. 4, pp. 323–343, 1997.

Author’s address

Yan Dong

Northwestern Polytechnical University, Department of Applied Mathematics; Key Laboratory of Space Applied Physics and Chemistry, Ministry of Education, Xi’an, Shaanxi, 710129, China

E-mail address: dongyan@mail.nwpu.edu.cn



ON RADICAL CLASSES OF MONOUNARY ALGEBRAS

ZUZANA FARKASOVÁ AND DANICA JAKUBÍKOVÁ-STUDENOVSKÁ

Received 10 October, 2012

Abstract. Our work treats monounary algebras and their radical classes with respect to their homogeneity. Homogeneous radical classes and homomorphism-homogeneous radical classes of monounary algebras are characterized. Further, torsion classes of monounary algebras are described.

2010 Mathematics Subject Classification: 08A60; 08C99

Keywords: monounary algebra, homogeneous algebra, homomorphism-homogeneous algebra, homogeneous radical class, homomorphism-homogeneous radical class

1. INTRODUCTION

A major theme in universal algebra is the study of classes of algebras of the same type that are closed under one or more constructions [2]. For algebraic structures of a given type it might be useful to investigate various classes (e.g., varieties, torsion classes, radical classes, convexities) in order to classify the structures. By the classification we mean the division of algebraic structures into classes, according to their common features.

The general study of varieties of algebras with finitary operations was developed by G. Birkhoff in the 1930's. B.H. Neumann initiated the investigation of varieties of groups [20]. In the 70's J. Martinez began the study of varieties of ℓ -groups [16] and later he studied also torsion classes of ℓ -groups [17]. Further, torsion theory for acts (semigroup action on sets) was developed in [15]. Radical classes, as a generalization of torsion classes, of lattice ordered groups were introduced by J. Jakubík [8]. These classes were explored for different types of operators. Some other results for radical classes of ℓ -groups are found in [5, 25], for radical classes of generalized Boolean algebras in [9] and for radical classes of MV-algebras are available in [10].

The notion of homogeneity has been applied in different fields of mathematics in connection with deep results. As examples, it suffices to mention the investigation of

Supported by VEGA Grant 2/0194/10 and VVGS-PF-2012-50, respectively.

continuous geometries [26] or the study of homogeneous Boolean algebras [22]. Cameron and Nešetřil [3] introduced the notion of homomorphism-homogeneous structures; this notion is closely related to the classical notion of homogeneity. The importance of the notion of homomorphism-homogeneity for algebras is considerable, see e.g., [6, 13], or for relational structures [18, 19, 23].

Some properties of algebraic structures of a finite type can be described by means of analogous properties of certain assigned monounary algebras. So monounary algebras play a significant role in the study of algebraic structures (cf. monographs [4, 7, 12]). The advantage of monounary algebras is their relatively simple visualization, since they can be represented by oriented planar graphs. Monounary algebras are often considered as special types of automata [1, 21]. Also, they were studied in connection with other algebraic structures. For example, monounary algebras enriched with certain relations have been studied in [14, 24].

Several classes of monounary algebras (e.g., varieties, retract varieties, convexities) were described in [7]. The aim of this work is a description of radical classes of monounary algebras, where the radical classes are considered with respect to two types of homogeneity. Namely, we characterize homogeneous radical classes and homomorphism-homogeneous radical classes. In the end we deal also with homogeneous torsion classes and homomorphism-homogeneous torsion classes of monounary algebras.

2. BASIC NOTATIONS AND DEFINITIONS

Let $\mathbb{Z}$ be the set of all integers, $\mathbb{N}$ be the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Next, let us denote by *Card* the system of all cardinal numbers.

For the basic notions and definitions concerning monounary algebras we refer to [7]. Also, we will apply notions and definitions from [13]; let us recall some of them.

Definition 1. A *monounary algebra* is a pair (A, f) where A is a non-empty set and $f: A \rightarrow A$ is a unary operation on A .

Notice that we will consider the empty set $\emptyset$ as a monounary algebra and instead of (A, f) we will often write A as an abbreviation.

Let (A, f) be a monounary algebra. Then for $x \in A$, put $f^0(x) = x$ and $f^{-1}(x) = \{y \in A: f(y) = x\}$. If for some $n \in \mathbb{N}$, $f^{n-1}(x)$ is defined, then we set $f^n(x) = f(f^{n-1}(x))$.

An algebra (A, f) is called *connected*, if for each $x, y \in A$ there exist $m, n \in \mathbb{N}_0$ such that $f^m(x) = f^n(y)$.

A set $B \subseteq A$ is called a *connected component* of (A, f) , if B is a maximal connected subalgebra of A .

Let $x \in A$ be such that there exists $m \in \mathbb{N}$ with $f^m(x) = x$. Then x is called a *cyclic element* of A . If such $m \in \mathbb{N}$ does not exist then we say that x is *acyclic*. Let $n \in \mathbb{N}$ be the smallest integer such that $f^n(x) = x$. Then the set of elements

$C = \{x, f(x), \dots, f^{n-1}(x)\}$ is called a cycle of (A, f) and n is called the length of C ; so C is an n -element cycle of A .

We denote by $\mathbb{Z} = (\mathbb{Z}, f)$ a monounary algebra such that $f(x) = x + 1$ for all $x \in \mathbb{Z}$. Next, for $n \in \mathbb{N}$, let $\mathbb{Z}_n = (\mathbb{Z}_n, f)$ be a monounary algebra such that $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ and $f(i) = i + 1 \pmod{n}$ for each $i \in \mathbb{Z}_n$.

For $\alpha \in \text{Card} - \{0\}$ let $V_\alpha = (V_\alpha, f)$ be a connected monounary algebra possessing no cycle and such that $|f^{-1}(v)| = \alpha$ for every element $v \in V_\alpha$. We remark that such algebra exists for each $\alpha \in \text{Card} - \{0\}$ and it is unique up to isomorphism.

Let $A_i, i \in I$ be a system of mutually disjoint algebras. Denote by $\sum_{i \in I} A_i$ a disjoint union of algebras $A_i, i \in I$. If there exists an algebra A such that $A_i \cong A, i \in I$ and $|I| = k$, then instead of $\sum_{i \in I} A_i$ we write $k \cdot A$. Note that for $k = 0$ ($I = \emptyset$), $k \cdot A = \emptyset$ ($\sum_{i \in I} A_i = \emptyset$).

Let $A = (A, f)$ be a monounary algebra. For a connected component $S \subseteq A$ let $\text{cn}(S)$, the *cycle number* of S denote the length of the cycle in S . If S does not have any cycle, we set $\text{cn}(S) = \infty$.

We say that $a \in A$ is a *source* in A if $f^{-1}(a) = \emptyset$.

A *branch* in a monounary algebra A is a finite or infinite sequence $a_1, a_2, a_3, \dots$ such that

- $a_i = f(a_{i+1})$ for all $i \geq 1$,
- a_i is acyclic for all $i \geq 1$, and
- if the sequence is finite, that if it has the form $a_1, a_2, a_3, \dots, a_n$, then a_n is a source in A .

For an acyclic element $a \in A$, the *height* of a ($\text{ht}(a)$, for short) means the least number $k \geq 1$ such that $f^k(a)$ is a cyclic element. If no such k exists, we set $\text{ht}(a) = \infty$.

A connected monounary algebra A is said to be *regular* if

- A possesses a cycle, and
- either every branch in A is infinite, or every branch in A is finite and $\text{ht}(a) = \text{ht}(b)$ for all sources a, b in A .

Note that a connected algebra consisting of solely cyclic elements is regular.

If A is a regular algebra, then the *height* of A , denoted by $\text{ht}(A)$, is defined as follows:

- $\text{ht}(A) = 0$ if there are no acyclic elements in A (i.e., A is a cycle);
- $\text{ht}(A) = \infty$ if every branch starting at a cyclic element of A is infinite;
- otherwise, $\text{ht}(A)$ denotes the common height of all the sources in A .

Definition 2. A monounary algebra (A, f) is said to be *homogeneous* if for each $x, y \in A$ there is an automorphism φ of A such that $\varphi(x) = y$.

Definition 3. We say that a monounary algebra $A = (A, f)$ is *homomorphism-homogeneous* if every homomorphism $\varphi: B \rightarrow A$ from a finitely generated subalgebra B of A into A extends to an endomorphism of A .

We will consider the empty set $\emptyset$ as a homogeneous and homomorphism-homogeneous monounary algebra, too.

3. HOMOGENEOUS RADICAL CLASSES

In this section we characterize classes of monounary algebras which are radical with respect to homogeneity. Homogeneous monounary algebras have been described in the paper [11].

Definition 4. A non-empty class $\mathcal{K}$ of monounary algebras is called a *homogeneous radical class* (hr-class, for short) if it satisfies the following conditions:

- (0) $\mathcal{K}$ is closed with respect to isomorphisms,
- (1) whenever $A \in \mathcal{K}$ and B is a homogeneous subalgebra of A , then $B \in \mathcal{K}$,
- (2) for each monounary algebra D there exists its greatest homogeneous subalgebra belonging to $\mathcal{K}$.

Note that the condition (2) means that for each monounary algebra D , the system of all homogeneous subalgebras of D belonging to $\mathcal{K}$ has a greatest element. The following theorem is proved in [11].

Theorem 1 ([11]). *Let A be a monounary algebra. Then A is homogeneous if and only if either*

- (1) $A \cong \beta \cdot \mathbb{Z}_n$ for $\beta \in \text{Card}$, $n \in \mathbb{N}$,

or

- (2) $A \cong \beta \cdot \mathbb{V}_\alpha$ for $\alpha, \beta \in \text{Card}$, $\alpha > 0$.

Notice that the monounary algebra $\beta \cdot \mathbb{Z}_n$ is isomorphic to a monounary algebra consisting of β cycles of length n .

We show some necessary conditions for a class $\mathcal{K}$ to be homogeneous radical. First, it is proved that an hr-class does not contain a monounary algebra possessing a subalgebra isomorphic to $\mathbb{Z}$.

Lemma 1. *Let $\mathcal{K}$ be an hr-class of monounary algebras and A be a monounary algebra such that some its subalgebra is isomorphic to $\mathbb{Z}$. Then the algebra A does not belong to the class $\mathcal{K}$.*

Proof. By way of contradiction, suppose that $A \in \mathcal{K}$. Denote by B a subalgebra of A such that $B \cong \mathbb{Z}$. According to Theorem 1, B is a homogeneous subalgebra of A . Thus by (1) of Definition 4, $B \in \mathcal{K}$.

Let us consider a connected monounary algebra D without cyclic elements, having the property that $|f^{-1}(x)| = 2$ for only one element $x \in D$ and $|f^{-1}(y)| = 1$ for

each $y \in D - \{x\}$. Observe that D has two maximal homogeneous subalgebras D_i ($i = 1, 2$) belonging to the class $\mathcal{K}$ and $D_i \cong B$ ($i = 1, 2$). But D_i ($i = 1, 2$) is not the largest homogeneous subalgebra of D such that $D_i \in \mathcal{K}$. This contradicts the condition (2) of Definition 4 and completes the proof. $\square$

Lemma 2. *Let $\mathcal{K}$ be a class of monounary algebras satisfying (0) and (1) of Definition 4. Further, let there exist $A, B \in \mathcal{K}$ and $m, n \in \mathbb{N}$, $m \neq n$ such that the algebra A contains a subalgebra isomorphic to $\mathbb{Z}_m$ and the algebra B contains a subalgebra isomorphic to $\mathbb{Z}_n$. Then the class $\mathcal{K}$ does not satisfy (2) of Definition 4.*

Proof. By the assumption, $\mathbb{Z}_m$ and $\mathbb{Z}_n$ belong to the class $\mathcal{K}$. Let us take a monounary algebra D such that $D = \mathbb{Z}_m + \mathbb{Z}_n$. The algebras $\mathbb{Z}_m$ and $\mathbb{Z}_n$ are homogeneous subalgebras of D and $\mathbb{Z}_m, \mathbb{Z}_n \in \mathcal{K}$. These subalgebras of D are maximal but none of them is the largest homogeneous subalgebra of D belonging to $\mathcal{K}$. Thus (2) of Definition 4 fails to hold. $\square$

Corollary 1. *If a class $\mathcal{K}$ of monounary algebras satisfies (0) and (1) of Definition 4 and contains an algebra A such that A possesses cycles of different lengths, then the class $\mathcal{K}$ does not satisfy (2) of Definition 4.*

Proof. The assertion follows from Lemma 2 if we put $B = A$. $\square$

Lemma 3. *Let $\mathcal{K}$ be an hr-class of monounary algebras. If $\mathbb{Z}_n \in \mathcal{K}$, then for each cardinal α , $\alpha \cdot \mathbb{Z}_n \in \mathcal{K}$, too.*

Proof. Assume that $\mathbb{Z}_n \in \mathcal{K}$ and suppose, to the contrary, that $\alpha \cdot \mathbb{Z}_n \notin \mathcal{K}$ for some $\alpha \in \text{Card}$. Obviously, $\alpha \notin \{0, 1\}$. By (2) of Definition 4, $\alpha \cdot \mathbb{Z}_n$ contains a largest homogeneous subalgebra X such that $X \in \mathcal{K}$. Since $\alpha \cdot \mathbb{Z}_n \notin \mathcal{K}$, X is a proper subalgebra of $\alpha \cdot \mathbb{Z}_n$. Then there exists a homogeneous subalgebra Y of $\alpha \cdot \mathbb{Z}_n$ such that $Y \cong \mathbb{Z}_n \in \mathcal{K}$ and $Y \not\subseteq X$. The subalgebra Y of $\alpha \cdot \mathbb{Z}_n$ is incomparable with X because $Y \not\subseteq X$. Hence X cannot be the largest homogeneous subalgebra of $\alpha \cdot \mathbb{Z}_n$ such that $X \in \mathcal{K}$, which is a contradiction. $\square$

Theorem 2. *Let $\mathcal{K}$ be a class of monounary algebras closed with respect to isomorphisms and let $\emptyset \in \mathcal{K}$. Then $\mathcal{K}$ is an hr-class if and only if either*

- (a) *if $A \in \mathcal{K}$ then A possesses neither cycle nor subalgebra isomorphic to $\mathbb{Z}$, or*
- (b) *there exists $n \in \mathbb{N}$ such that*
 - (i) *$\alpha \cdot \mathbb{Z}_n \in \mathcal{K}$ for each $\alpha \in \text{Card}$, and*
 - (ii) *if $A \in \mathcal{K}$ then A possesses neither m -element cycle for $m \neq n$ nor subalgebra isomorphic to $\mathbb{Z}$.*

Proof. Suppose that $\mathcal{K}$ is an hr-class. From Lemma 1 it follows that $\mathcal{K}$ does not contain an algebra possessing a subalgebra isomorphic to $\mathbb{Z}$. Next, we distinguish two cases:

Case 1: Let for each $A \in \mathcal{K}$, A possess no cycle. According to Lemma 1, A does not contain a subalgebra which is isomorphic to $\mathbb{Z}$. Hence, the condition (a) is satisfied.

Case 2: Suppose that there exists an algebra $A_0 \in \mathcal{K}$ such that A_0 possesses a subalgebra isomorphic to $\mathbb{Z}_n$ for some $n \in \mathbb{N}$. By Corollary 1, A_0 does not contain any m -element cycle for $m \neq n$. Now take $A \in \mathcal{K}$. According to Lemma 2, A possesses no subalgebra isomorphic to $\mathbb{Z}_m$ for $m \neq n$. Thus, condition (ii) of case (b) is valid. Observe that $\mathbb{Z}_n \in \mathcal{K}$ because $A_0 \in \mathcal{K}$ and $\mathcal{K}$ satisfies (1) of Definition 4. So, condition (i) of case (b) follows from Lemma 3.

Next, we prove the converse implication of the theorem. The implication is trivial in the case (a). Now, assume that case (b) holds. By assumption, (0) of Definition 4 holds. Let $A \in \mathcal{K}$ and B be a homogeneous subalgebra of A . According to Theorem 1, $B \cong \beta \cdot \mathbb{Z}_n$ for some $\beta \in \text{Card}$. Then by condition (i) of case (b), $B \in \mathcal{K}$. Hence, $\mathcal{K}$ satisfies (1) of Definition 4. Let us consider an arbitrary monounary algebra D . If D has no subalgebra isomorphic to $\mathbb{Z}_n$ then $\emptyset$ is the largest homogeneous subalgebra of D such that $\emptyset \in \mathcal{K}$. Otherwise, denote by E the sum of all subalgebras of D isomorphic to $\mathbb{Z}_n$. Then E is the largest homogeneous subalgebra of D , $E \cong \alpha \cdot \mathbb{Z}_n$ for some nonzero cardinal α , thus $E \in \mathcal{K}$. This implies that $\mathcal{K}$ satisfies (2) of Definition 4. Therefore, $\mathcal{K}$ is an hr-class which concludes the proof. $\square$

For an hr-class $\mathcal{K}$ denote by $\mathcal{H}(\mathcal{K})$ the subclass of $\mathcal{K}$ such that $\mathcal{H}(\mathcal{K})$ consists of all homogeneous algebras of $\mathcal{K}$. Then $\mathcal{H}(\mathcal{K})$ is called a *homogeneous part* of $\mathcal{K}$. By Theorem 2, $\mathcal{H}(\mathcal{K})$ is an hr-class.

4. HOMOMORPHISM-HOMOGENEOUS RADICAL CLASSES

The present section is devoted to the characterization of classes of monounary algebras which are radical with respect to homomorphism-homogeneity of monounary algebras.

The following theorem is proved in the paper [13].

Theorem 3 ([13]). *Let A be a non-empty monounary algebra. Then A is homomorphism-homogeneous if and only if A satisfies one of the following conditions:*

- (1) *every branch in A is infinite,*
- (2) *every connected component in A is regular, and for any two connected components $S_1, S_2 \subseteq A$, if $\text{cn}(S_1) \mid \text{cn}(S_2)$ then $\text{ht}(S_1) \geq \text{ht}(S_2)$ or $\text{ht}(S_1) = 0$.*

Definition 5. A non-empty class $\mathcal{K}$ of monounary algebras will be called a *homomorphism-homogeneous radical class* (for short, Hhr-class) if it satisfies the following conditions:

- (0) $\mathcal{K}$ is closed with respect to isomorphisms,
- (1) whenever $A \in \mathcal{K}$ and B is a homomorphism-homogeneous subalgebra of A , then $B \in \mathcal{K}$,
- (2) for each monounary algebra D there exists its largest homomorphism-homogeneous subalgebra belonging to $\mathcal{K}$.

From the above definition immediately follows

Corollary 2. *Let $\mathcal{K}$ be a class of monounary algebras, closed with respect to isomorphisms and let $\emptyset \in \mathcal{K}$. Further, assume that each $A \in \mathcal{K}$ possesses neither cycle nor subalgebra isomorphic to $\mathbb{Z}$. Then $\mathcal{K}$ is an Hhr-class.*

We start by showing some necessary conditions for a class $\mathcal{K}$ to be homomorphism-homogeneous radical.

Lemma 4. *Let $\mathcal{K}$ be a class of monounary algebras satisfying (0) and (1) of Definition 5. Next, let A be a regular monounary algebra such that A has a unique branch and $\text{ht}(A) = 2$. If $A \in \mathcal{K}$ then $\mathcal{K}$ does not satisfy (2) of Definition 5.*

Proof. Assume that $A \in \mathcal{K}$ and suppose, on the contrary, that $\mathcal{K}$ satisfies (2) of Definition 5. Denote by B a regular subalgebra of A such that $\text{ht}(B) = 1$; and let C be the cycle of A . Note that B and C are homomorphism-homogeneous subalgebras of A . Thus by (1) of Definition 5, $B, C \in \mathcal{K}$. Let us consider a monounary algebra $D = D_1 + D_2$ such that $D_1 \cong A$ and $D_2 \cong B$. We show that D possesses no largest homomorphism-homogeneous subalgebra belonging to $\mathcal{K}$. This will lead to a contradiction. At first we make the following observations:

For $E = E_1 + E_2$ where $E_1 \cong E_2 \cong B$, we get that $E_1, E_2 \in \mathcal{K}$ according to (0) of Definition 5. Suppose that $E \notin \mathcal{K}$. Then E possesses a largest homomorphism-homogeneous subalgebra belonging to $\mathcal{K}$, denote it by $E_{\mathcal{K}}$. Since $E_{\mathcal{K}}$ is a proper subalgebra of E there exists an element $x \in E - E_{\mathcal{K}}$. Observe that there is a unique $i \in \{1, 2\}$ with $x \in E_i \not\subseteq E_{\mathcal{K}}$. On the other hand, $E_{\mathcal{K}}$ is incomparable with E_i . Thus $E_{\mathcal{K}}$ cannot be the largest homomorphism-homogeneous subalgebra of E belonging to $\mathcal{K}$, which is a contradiction. Hence, $E \in \mathcal{K}$.

Let $F = F_1 + F_2$ where $F_1 \cong A$ and $F_2 \cong C$. Then $F_1, F_2 \in \mathcal{K}$ according to (0) of Definition 5. Analogously as above we can show that $F \in \mathcal{K}$.

Note that D is not a homomorphism-homogeneous algebra and that D possesses two homomorphism-homogeneous subalgebras X, Y such that $X \cong E$ and $Y \cong F$. So, D contains two maximal homomorphism-homogeneous subalgebras belonging to $\mathcal{K}$, namely X and Y , but none of them is the largest. This contradicts (2) of Definition 5 and completes the proof. $\square$

Corollary 3. *Let $\mathcal{K}$ be a class of monounary algebras satisfying (0) and (1) of Definition 5 and A be a monounary algebra possessing a regular subalgebra of height greater than 1. If $A \in \mathcal{K}$ then $\mathcal{K}$ fails to be an Hhr-class.*

Proof. Assume that $A \in \mathcal{K}$ and suppose, to the contrary, that $\mathcal{K}$ is an Hhr-class. There exists a regular subalgebra B of A such that B has a unique branch and $\text{ht}(B) = 2$. According to Theorem 3, B is a homomorphism-homogeneous algebra. Then from the assumption, $B \in \mathcal{K}$. In view of Lemma 4, a class $\mathcal{K}$ does not satisfy (2) of Definition 5, a contradiction. $\square$

Lemma 5. *Let $\mathcal{K}$ be a class of monounary algebras satisfying (0) and (1) of Definition 5 and let A be a regular monounary algebra of nonzero height. If $\mathbb{Z}, A \in \mathcal{K}$ then $\mathcal{K}$ does not satisfy (2) of Definition 5.*

Proof. By way of contradiction, suppose that $\mathbb{Z}, A \in \mathcal{K}$ and $\mathcal{K}$ satisfies (2) of Definition 5. Corollary 3 implies that $\text{ht}(A) < 2$, hence $\text{ht}(A) = 1$. Consider an algebra $D = \mathbb{Z} + A$ which is not a homomorphism-homogeneous algebra in view of Theorem 3. According to (2) of Definition 5, D possesses a largest homomorphism-homogeneous subalgebra belonging to $\mathcal{K}$, denote it by $D_{\mathcal{K}}$. Observe that $D_{\mathcal{K}}$ is a proper subalgebra of D , therefore there exists an element $x \in D - D_{\mathcal{K}}$ and either $x \in \mathbb{Z}$ or $x \in A$. By Theorem 3, $\mathbb{Z}$ and A are homomorphism-homogeneous subalgebras of D . Assume that $x \in \mathbb{Z}$ (or $x \in A$). Since $\mathbb{Z} \not\subseteq D_{\mathcal{K}}$ (or $A \not\subseteq D_{\mathcal{K}}$), $D_{\mathcal{K}}$ cannot be the largest homomorphism-homogeneous subalgebra of D such that $D_{\mathcal{K}} \in \mathcal{K}$. Hence, D does not have a largest homomorphism-homogeneous subalgebra belonging to $\mathcal{K}$. This is a contradiction and completes the proof. $\square$

The following assertion follows from Lemmas 4 and 5.

Corollary 4. *Let $\mathcal{K}$ be an Hhr-class containing an algebra possessing a regular monounary subalgebra of nonzero height. Then $\mathcal{K}$ does not contain an algebra with a subalgebra isomorphic to $\mathbb{Z}$ and does not contain an algebra possessing a regular subalgebra of height greater than one.*

Next, if $\mathcal{K}$ is an Hhr-class such that $\mathbb{Z} \in \mathcal{K}$ then $\mathcal{K}$ fails to contain an algebra which possesses a regular subalgebra of nonzero height.

Notation 1. Let for $n \in \mathbb{N}$, $\mathcal{Z}_n^1$ be the class of all regular monounary algebras of height 1 and cycle number n . Next, let $\mathcal{Z}_n = \{A : A \cong \mathbb{Z}_n\}$.

Clearly, if $\mathcal{K}$ is an Hhr-class such that $\mathcal{K} \cap \mathcal{Z}_n \neq \emptyset$ then $\mathcal{Z}_n \subseteq \mathcal{K}$. We show that whenever $\mathcal{K}$ is an Hhr-class such that $\mathcal{K} \cap \mathcal{Z}_n^1 \neq \emptyset$ then $\mathcal{Z}_n^1 \subseteq \mathcal{K}$.

Definition 6. Let $A \in \mathcal{Z}_n^1$ for some $n \in \mathbb{N}$. Then the *width* of a cyclic element a in A denoted by $w(a)$ is defined as follows:

$$w(a) = |f^{-1}(a)| - 1.$$

Lemma 6. *Let $\mathcal{K}$ be an Hhr-class such that $\mathcal{Z}_n^1 \cap \mathcal{K} \neq \emptyset$ for some $n \in \mathbb{N}$. Then for each $\beta \in \text{Card} - \{0\}$ there exists an algebra $Y \in \mathcal{Z}_n^1 \cap \mathcal{K}$ such that $w(y) \geq \beta$ for each cyclic element $y \in Y$.*

Proof. By way of contradiction, suppose that there exists $\beta \in \text{Card} - \{0\}$ such that for every $Y \in \mathcal{Z}_n^1 \cap \mathcal{K}$, $w(y) < \beta$ for at least one cyclic element $y \in Y$.

Let us consider an algebra $D \in \mathcal{Z}_n^1$ such that $w(d) = \beta$ for each cyclic element $d \in D$. According to the assumption, $D \notin \mathcal{K} \cap \mathcal{Z}_n^1$, i.e., $D \notin \mathcal{K}$. By (2) of Definition 5, D possesses a largest homomorphism-homogeneous subalgebra belonging to $\mathcal{K}$; denote it by B . Then $B \neq \emptyset$ and $B \not\cong \mathbb{Z}_n$ because $\mathcal{K} \cap \mathcal{Z}_n^1 \neq \emptyset$. Also, $B \neq D$ because $D \notin \mathcal{K}$.

Let d_i , $i = 0, 1, \dots, n-1$, be the cyclic elements of D such that $d_i = f(d_{i-1})$ ($i-1$ is counted mod n). For d_i ($i = 0, 1, \dots, n-1$) denote by D_i , the set of all acyclic elements $x \in D$ such that $f(x) = d_i$, i.e., $D_i = f^{-1}(d_i) - \{d_{i-1}\}$.

Now we show that if for some $i = 0, 1, \dots, n-1$ there exists $x \in D_i$ such that $x \notin B$ then $D_i \cap B = \emptyset$.

If $|D_i| = 1$ then the assumption is obvious.

Let $|D_i| \geq 2$. Assume that $x \notin B$, $x \in D_i$ for some $i \in \{0, 1, \dots, n-1\}$ and suppose, to the contrary, that there exists an element y such that $y \in B \cap D_i$. Then $C = (B - \{y\}) \cup \{x\}$ is a subalgebra of D and $C \cong B$. By (0) of Definition 5, $C \in \mathcal{K}$. Hence, B cannot be the largest homomorphism-homogeneous subalgebra of D belonging to $\mathcal{K}$, a contradiction.

Because $B \neq D$ and $B \not\cong \mathbb{Z}_n$ there exists at least one $i \in \{0, 1, \dots, n-1\}$ such that $D_i \cap B = \emptyset$. On the other hand, for some $j \neq i$ ($j \in \{0, 1, \dots, n-1\}$), $\emptyset \neq D_j \subseteq B$. Otherwise, the assumption that $D_i \cap B = \emptyset$ for each $i = 0, 1, \dots, n-1$, would imply that $B \cong \mathbb{Z}_n$, which is a contradiction.

Next we show that D possesses a subalgebra C such that $C \cong B$ and $C \not\subseteq B$, ($C \neq B$). Then this will be a contradiction with (2) of Definition 5, and will complete the proof.

From the definition of D it follows that there exists an automorphism φ_0 of D such that $\varphi_0(j) = i$. Since B is a subalgebra of D , the mapping $\varphi = \varphi_0 \upharpoonright B$ is an embedding of B into D such that $\varphi(j) = i$. Put $C = \varphi(B)$. Then $\varphi(D_j) = D_i$. Let $c \in D_i$. Then $c \in \varphi(B) = C$, but $c \notin \emptyset = B \cap D_i$. This implies that $C \not\subseteq B$. By condition (0) of Definition 5 we have $C \in \mathcal{K}$, which completes the proof. $\square$

Lemma 7. Let $\mathcal{K}$ be a class of monounary algebras such that $\mathbb{Z}_n^1 \cap \mathcal{K} \neq \emptyset$ for some $n \in \mathbb{N}$. If $\mathcal{K}$ is an Hhr-class then $\mathbb{Z}_n^1 \subseteq \mathcal{K}$.

Proof. Assume that $\mathcal{K}$ is an Hhr-class and suppose, to the contrary, that $\mathbb{Z}_n^1 \not\subseteq \mathcal{K}$. Then there exists an algebra $A \in \mathbb{Z}_n^1$ such that $A \notin \mathcal{K}$. Let $\beta = \max\{w(a) : a \in A, a \text{ is cyclic}\}$. Denote by B , an algebra such that $B \in \mathbb{Z}_n^1$ and $w(b) = \beta$ for each cyclic element $b \in B$. Then according to Lemma 6, $B \in \mathcal{K}$. Observe, that B contains a homomorphism-homogeneous subalgebra X such that $X \cong A$. Because $\mathcal{K}$ is an Hhr-class, we get that $A \in \mathcal{K}$, which is a contradiction. $\square$

Definition 7. For a class $\mathcal{K}$ of monounary algebras denote

$$P(\mathcal{K}) = \{i \in \mathbb{N} : \mathbb{Z}_i^1 \cap \mathcal{K} \neq \emptyset\}$$

and $Q(\mathcal{K}) = \{i \in \mathbb{N} : \mathbb{Z}_i \cap \mathcal{K} \neq \emptyset\}$.

Note that if $\mathcal{K}$ is an Hhr-class of monounary algebras then obviously, $P(\mathcal{K}) \subseteq Q(\mathcal{K})$, and if $P(\mathcal{K}) \neq \emptyset$ then according to Lemma 7 we have:

$$P(\mathcal{K}) = \{i \in \mathbb{N} : \mathbb{Z}_i^1 \cap \mathcal{K} \neq \emptyset\} = \{i \in \mathbb{N} : \mathbb{Z}_i^1 \subseteq \mathcal{K}\}.$$

Further, $Q(\mathcal{K}) = \{i \in \mathbb{N} : \mathbb{Z}_i \cap \mathcal{K} \neq \emptyset\} = \{i \in \mathbb{N} : \mathbb{Z}_i \subseteq \mathcal{K}\}$.

Lemma 8. *Let $\mathcal{K}$ be an Hhr-class of monounary algebras such that $Q(\mathcal{K}) = Q \neq \emptyset$. Then $\sum_{i \in Q} \alpha_i \cdot \mathbb{Z}_i \in \mathcal{K}$ for each $\alpha_i \in \text{Card}$.*

Proof. By way of contradiction, suppose that there exists $\alpha_i \in \text{Card}$, $i \in Q$, such that $D = \sum_{i \in Q} \alpha_i \cdot \mathbb{Z}_i \notin \mathcal{K}$. Observe that D has to contain at least two connected components, since otherwise the assumption $D = \mathbb{Z}_i$ for some $i \in Q$ would imply that $\mathbb{Z}_i \notin \mathcal{K}$, a contradiction. By (2) of Definition 5, D possesses a largest homomorphism-homogeneous subalgebra belonging to $\mathcal{K}$, denote it by $D_{\mathcal{K}}$. Since $D \notin \mathcal{K}$, we have $D_{\mathcal{K}} \neq D$. Then D possesses a subalgebra $C \cong \mathbb{Z}_j$ for some $j \in Q$ such that $C \not\subseteq D_{\mathcal{K}}$. From the assumption that $j \in Q = \{i \in \mathbb{N} : \mathbb{Z}_i \subseteq \mathcal{K}\}$ it follows that $C \in \mathcal{K}$ and note that C is a homomorphism-homogeneous subalgebra of D . Then $D_{\mathcal{K}}$ is incomparable with C . Thus $D_{\mathcal{K}}$ cannot be the largest homomorphism-homogeneous subalgebra of D and this is a contradiction. $\square$

Lemma 9. *Let $\mathcal{K}$ be an Hhr-class with $Q(\mathcal{K}) \neq \emptyset$. Then $\mathcal{K}$ contains each A such that if B is a connected component of A then either $B \in \mathbb{Z}_i^1$ for some $i \in P(\mathcal{K})$ or $B \in \mathbb{Z}_j$ for some $j \in Q(\mathcal{K})$.*

Proof. If $P(\mathcal{K}) = \emptyset$ then the theorem follows from Lemma 8. Thus suppose that $P(\mathcal{K}) \neq \emptyset$ and let A be a monounary algebra such that if B is a connected component of A then either $B \in \mathbb{Z}_i^1$ for some $i \in P(\mathcal{K})$ or $B \in \mathbb{Z}_j$ for some $j \in Q(\mathcal{K})$. If A consists of cycles then the theorem follows from Lemma 8. So, suppose that there exists at least one acyclic element of A . By way of contradiction, suppose that $A \notin \mathcal{K}$. By (2) of Definition 5, A possesses the largest homomorphism-homogeneous subalgebra $A_{\mathcal{K}}$ such that $A_{\mathcal{K}} \in \mathcal{K}$. Denote by B the subalgebra of A such that B consists of all cyclic elements of A . According to Lemma 8 and (0) of Definition 5, $B \in \mathcal{K}$. Furthermore, B is a homomorphism-homogeneous subalgebra of A and then $B \subseteq A_{\mathcal{K}}$. Because $B \subseteq A_{\mathcal{K}} \neq A$, there exists an acyclic element $x \in A - A_{\mathcal{K}}$. Denote by X that connected component of A which contains the acyclic element x . Note that $X \in \mathbb{Z}_i^1$ for some $i \in P(\mathcal{K})$ and X is a homomorphism-homogeneous subalgebra of A . According to Lemma 7, $\mathbb{Z}_i^1 \subseteq \mathcal{K}$, so, $X \in \mathcal{K}$. Since $X \not\subseteq A_{\mathcal{K}}$, $A_{\mathcal{K}}$ is incomparable with X . Thus $A_{\mathcal{K}}$ cannot be the largest homomorphism-homogeneous subalgebra of A such that $A_{\mathcal{K}} \in \mathcal{K}$, a contradiction. $\square$

Lemma 10. *Let $\mathcal{K}$ be an Hhr-class such that $\mathbb{Z} \in \mathcal{K}$. Then $\mathcal{K}$ contains each A such that if B is a connected component of A then either $B \in \mathbb{Z}_i$ for some $i \in Q(\mathcal{K})$ or every branch in B is infinite and B possesses no cycle.*

Proof. Let us consider a monounary algebra A such that if B is a connected component of A then either $B \in \mathbb{Z}_i$ for some $i \in Q(\mathcal{K})$ or every branch in B is infinite and B possesses no cycle. Assume that $\mathcal{K}$ is an Hhr-class and suppose, to the contrary, that $A \notin \mathcal{K}$. By (2) of Definition 5, A possesses the largest homomorphism-homogeneous subalgebra $A_{\mathcal{K}}$ belonging to $\mathcal{K}$. Obviously, $\emptyset \neq A_{\mathcal{K}} \neq A$. Consider

the following two cases according to the definition of $Q(\mathcal{K})$.

Case 1: Let $Q(\mathcal{K}) = \emptyset$. Because $A_{\mathcal{K}} \neq A$, A possesses a subalgebra X such that $X \cong \mathbb{Z}$ and $X \not\subseteq A_{\mathcal{K}}$. The subalgebra X of A is homomorphism-homogeneous and by (0) of Definition 5, $X \in \mathcal{K}$. Since $X \not\subseteq A_{\mathcal{K}}$, $A_{\mathcal{K}}$ is incomparable with X . Thus $A_{\mathcal{K}}$ cannot be the largest homomorphism-homogeneous subalgebra of A such that $A_{\mathcal{K}} \in \mathcal{K}$, a contradiction.

Case 2: Let $Q(\mathcal{K}) \neq \emptyset$. If A possesses a subalgebra X such that $X \cong \mathbb{Z}$ and $X \not\subseteq A_{\mathcal{K}}$, then as in the previous case we get a contradiction. So, suppose that $A_{\mathcal{K}}$ possesses all acyclic elements of A . Then $A - A_{\mathcal{K}}$ contains an i -element cycle for some $i \in Q(\mathcal{K})$; denote it by Y . Because $\mathbb{Z}_i \subseteq \mathcal{K}$ for $i \in Q(\mathcal{K})$, $Y \in \mathcal{K}$. Note that Y is a homomorphism-homogeneous subalgebra of A . But then $A_{\mathcal{K}}$ is incomparable with Y , which is a contradiction and completes the proof. $\square$

The following two theorems present a characterization of Hhr-classes; in the first one we suppose that $\mathbb{Z}$ does not belong to the corresponding class $\mathcal{K}$ and that $Q(\mathcal{K}) \neq \emptyset$, while in the second one, $\mathbb{Z} \in \mathcal{K}$.

Theorem 4. *Let $\mathcal{K}$ be a class of monounary algebras such that $\mathbb{Z} \notin \mathcal{K}$ and $P(\mathcal{K}) \subseteq Q(\mathcal{K}) \neq \emptyset$. Next, let $\mathcal{K}$ satisfy (0) of Definition 5. Then $\mathcal{K}$ is an Hhr-class if and only if the following conditions hold:*

- (a) $\emptyset \in \mathcal{K}$.
- (b) *For every $A \in \mathcal{K}$, if B is a connected component of A then $B \in \mathbb{Z}_i^1$ for some $i \in P(\mathcal{K})$ or $B \in \mathbb{Z}_j$ for some $j \in Q(\mathcal{K})$ or B possesses neither cycle nor subalgebra isomorphic to $\mathbb{Z}$.*
- (c) *The class $\mathcal{K}$ includes all algebras A such that if B is a connected component of A then either $B \in \mathbb{Z}_i^1$ for some $i \in P(\mathcal{K})$ or $B \in \mathbb{Z}_j$ for some $j \in Q(\mathcal{K})$.*

Proof. " $\Rightarrow$ " Suppose that $\mathcal{K}$ is an Hhr-class. Then the condition (a) is obviously valid. From the assumption that $\mathbb{Z} \notin \mathcal{K}$ it follows that the class $\mathcal{K}$ does not contain an algebra having a subalgebra which is isomorphic to $\mathbb{Z}$.

By Corollary 3, the class $\mathcal{K}$ does not contain a monounary algebra possessing a regular subalgebra of height greater than one. Obviously, if some $i \notin P(\mathcal{K})$ then the class $\mathcal{K}$ does not contain a monounary algebra possessing a subalgebra from $\mathbb{Z}_i^1$. Also, if $j \notin Q(\mathcal{K})$ then $\mathcal{K}$ does not contain an algebra possessing a j -element cycle. Hence, the condition (b) is valid. The condition (3) results from Lemma 9.

" $\Leftarrow$ " Suppose that the conditions (a), (b) and (c) are true. Clearly, (0) of Definition 5 holds. Let $A \in \mathcal{K}$, and let B be an arbitrary homomorphism-homogeneous subalgebra of it. If $B = \emptyset$ then by the condition (a), $B \in \mathcal{K}$. Now suppose that $B \neq \emptyset$. Then by Theorem 3, if S is a connected component of B then either $S \in \mathbb{Z}_i^1$ for some $i \in Q(\mathcal{K})$ or $S \in \mathbb{Z}_j^1$ for some $j \in P(\mathcal{K})$. Thus $B \in \mathcal{K}$ in view of the condition (c). Hence (1) of Definition 5 holds. Next, we show that $\mathcal{K}$ satisfies (2) of Definition 5. Let D be an arbitrary monounary algebra. At first suppose that D contains neither subalgebra belonging to $\mathbb{Z}_i^1$ for each $i \in P(\mathcal{K})$ nor subalgebra belonging

to $\mathbb{Z}_j$ for each $j \in Q(\mathcal{K})$. Then the largest subalgebra of D belonging to $\mathcal{K}$ is $\emptyset$. Now suppose that D possesses the largest non-empty subalgebra E such that if X is a connected component of E then either $X \in \mathbb{Z}_i^1$ for some $i \in P(\mathcal{K})$ or $X \in \mathbb{Z}_j$ for some $j \in Q(\mathcal{K})$. By Theorem 3, E is a homomorphism-homogeneous algebra and by the condition (c), $E \in \mathcal{K}$. Then E is the largest homomorphism-homogeneous subalgebra of D belonging to $\mathcal{K}$. Thus (2) of Definition 5 holds for the class $\mathcal{K}$ and $\mathcal{K}$ is an Hhr-class. $\square$

Theorem 5. *Let $\mathcal{K}$ be a class of monounary algebras such that $\mathbb{Z} \in \mathcal{K}$ and let $\mathcal{K}$ satisfy (0) of Definition 5. Then $\mathcal{K}$ is an Hhr-class if and only if*

- (a) $\emptyset \in \mathcal{K}$.
- (b) *For all $A \in \mathcal{K}$, if B is a connected component of A then B possesses a subalgebra isomorphic to $\mathbb{Z}$ or $B \in \mathbb{Z}_i$ for some $i \in Q(\mathcal{K})$ or B possesses neither cycle nor subalgebra isomorphic to $\mathbb{Z}$.*
- (c) *The class $\mathcal{K}$ includes all monounary algebras A such that if B is a connected component of A then either $B \in \mathbb{Z}_i$ for some $i \in Q(\mathcal{K})$ or every branch in B is infinite and B possesses no cyclic element.*

Proof. " $\Rightarrow$ " Suppose that $\mathcal{K}$ is an Hhr-class. Then the condition (a) is obviously valid.

From Lemma 5 it follows, that $\mathcal{K}$ does not contain an algebra possessing a regular subalgebra of nonzero height. Thus $P(\mathcal{K}) = \emptyset$. Also, $\mathcal{K}$ does not contain an algebra with an i -element cycle for $i \notin Q(\mathcal{K})$. Hence the condition (b) holds.

According to Lemma 10, the condition (c) is valid.

" $\Leftarrow$ " Assume that the conditions (a), (b) and (c) are true. Obviously, (0) of Definition 5 holds. Let $A \in \mathcal{K}$ and B be its any homomorphism-homogeneous subalgebra. If $B = \emptyset$ then by the condition (a), $B \in \mathcal{K}$. So, suppose that $B \neq \emptyset$. Then according to Theorem 3, if S is a connected component of B then either $S \in \mathbb{Z}_i$ for some $i \in Q(\mathcal{K})$ or every branch in S is infinite and S possesses no cycle. Hence, $B \in \mathcal{K}$ in view of the condition (c). Thus (1) of Definition 5 holds.

Next, we show that $\mathcal{K}$ satisfies (2) of Definition 5. Let D be an arbitrary monounary algebra. Assume that D possesses neither i -element cycle for $i \in Q(\mathcal{K})$ nor subalgebra isomorphic to $\mathbb{Z}$. Then $\emptyset$ is the largest homomorphism-homogeneous subalgebra of D such that $\emptyset \in \mathcal{K}$. Now suppose that D possesses the largest non-empty subalgebra E such that if X is a connected component of E then either every branch in X is infinite and X possesses no cycle or $X \in \mathbb{Z}_i$ for some $i \in Q(\mathcal{K})$. By Theorem 3, E is a homomorphism-homogeneous algebra and by the condition (c), $E \in \mathcal{K}$. Then E is the largest homomorphism-homogeneous subalgebra of D belonging to $\mathcal{K}$. Thus (2) of Definition 5 holds and $\mathcal{K}$ is an Hhr-class. $\square$

Let $\mathcal{K}$ be an Hhr-class. Denote by $\mathcal{H}^*(\mathcal{K})$ the subclass of $\mathcal{K}$ such that $\mathcal{H}^*(\mathcal{K})$ consists of all homomorphism-homogeneous algebras of $\mathcal{K}$. Then $\mathcal{H}^*(\mathcal{K})$ is called a *homomorphism-homogeneous part* of $\mathcal{K}$. Note that $\mathcal{H}^*(\mathcal{K})$ is an Hhr-class.

Corollary 5. *Let $\mathcal{K}$ be an Hhr-class such that $\mathbb{Z} \notin \mathcal{K}$ and $Q(\mathcal{K}) \neq \emptyset$. Then $\mathcal{H}^*(\mathcal{K})$ is the class of all monounary algebras A such that if B is a connected component of A then either $B \in \mathbb{Z}_i^1$ for some $i \in P(\mathcal{K})$ or $B \in \mathbb{Z}_j$ for some $j \in Q(\mathcal{K})$.*

Proof. The assertion follows from Theorems 3 and 4. $\square$

Corollary 6. *Let $\mathcal{K}$ be an Hhr-class such that $\mathbb{Z} \in \mathcal{K}$. Then $\mathcal{H}^*(\mathcal{K})$ is the class of all monounary algebras A such that if B is a connected component of A then either $B \in \mathbb{Z}_i$ for some $i \in Q(\mathcal{K})$ or every branch in B is infinite and B possesses no cyclic element.*

Proof. The assertion follows from Theorems 3 and 5. $\square$

We will finish with a comparison of hr-classes and Hhr-classes as follows. All homogeneous algebras are homomorphism-homogeneous, too. But, in general an hr-class fails to be an Hhr-class. Assume that $\mathcal{K}$ is a class of monounary algebras which is closed with respect to isomorphisms, contains $\emptyset$ and that for all $A \in \mathcal{K}$, A possesses neither cycles nor subalgebra isomorphic to $\mathbb{Z}$. Then obviously, $\mathcal{K}$ is an hr-class and Hhr-class, too. The following assertions are consequence of Theorems 2 and 4.

Proposition 1. *Let $\mathcal{K}$ be an hr-class such that $\mathbb{Z}_n \in \mathcal{K}$ for some $n \in \mathbb{N}$. Then an hr-class $\mathcal{K}$ is an Hhr-class if and only if $\mathcal{K}$ satisfies the following conditions:*

- (1) $\mathcal{K}$ does not contain a monounary algebra possessing a regular subalgebra of height greater than one
- (2) if $\mathcal{K}$ contains an algebra A having a subalgebra $B \in \mathbb{Z}_n^1$ then $\mathcal{K}$ contains all algebras C such that whenever X is a connected component of C then $X \in \mathbb{Z}_n \cup \mathbb{Z}_n^1$.

Proposition 2. *Let $\mathcal{K}$ be an Hhr-class. Then $\mathcal{K}$ is an hr-class if and only if $\mathcal{K}$ does not contain a monounary algebra possessing a subalgebra isomorphic to $\mathbb{Z}$ and $|Q(\mathcal{K})| < 2$.*

5. TORSION CLASSES OF MONOUNARY ALGEBRAS CONCERNING HOMOGENEITY

In this section we will briefly characterize classes of monounary algebras which are torsion classes with respect to the homogeneity or homomorphism-homogeneity of monounary algebras.

Definition 8. A non-empty class $\mathcal{K}$ of monounary algebras will be called a *homogeneous torsion class* (for short, ht-class) if it satisfies the following conditions:

- (0) $\mathcal{K}$ is closed with respect to homomorphisms,
- (1) whenever $A \in \mathcal{K}$ and B is an homogeneous subalgebra of A , then $B \in \mathcal{K}$,
- (2) for each monounary algebra D there exists its largest homogeneous subalgebra belonging to $\mathcal{K}$.

For a class $\mathcal{K}$ of monounary algebras we will use the notions $P(\mathcal{K})$ and $Q(\mathcal{K})$ from the previous section. Note that from Theorem 2 it follows that if $\mathcal{K}$ is an hr-class then $|Q(\mathcal{K})| < 2$.

Theorem 6. *Let $\mathcal{K}$ be a class of monounary algebras closed with respect to homomorphisms. Then $\mathcal{K}$ is an ht-class if and only if $\mathcal{K}$ is an hr-class such that either $Q(\mathcal{K}) = \emptyset$ or $Q(\mathcal{K}) = \{1\}$.*

Proof. Suppose that $\mathcal{K}$ is an ht-class. Then obviously, $\mathcal{K}$ is an hr-class. The theorem trivially holds in the case when $|Q(\mathcal{K})| = 0$ or $Q(\mathcal{K}) = \{1\}$. Next, suppose that $|Q(\mathcal{K})| = 1$ and $1 \notin Q(\mathcal{K})$. The assumption that $n \in Q(\mathcal{K})$ for some $n \in \mathbb{N} - \{1\}$, would imply that $\mathbb{Z}_1 \in \mathcal{K}$ because $\mathbb{Z}_1$ is a homomorphic image of $\mathbb{Z}_n$. But then $\{1, n\} \subseteq Q(\mathcal{K})$, which is a contradiction. Conversely, it is clear that $\mathcal{K}$ satisfies each condition of Definition 8, i.e., $\mathcal{K}$ is an ht-class. $\square$

Definition 9. A non-empty class $\mathcal{K}$ of monounary algebras will be called a *homomorphism-homogeneous torsion class* (for short, Hht-class) if it satisfies the following conditions:

- (0) $\mathcal{K}$ is closed with respect to homomorphisms,
- (1) whenever $A \in \mathcal{K}$ and B is a homomorphism-homogeneous subalgebra of A , then $B \in \mathcal{K}$,
- (2) for each monounary algebra D there exists its largest homomorphism-homogeneous subalgebra belonging to $\mathcal{K}$.

It is easy to see the following assertion. Let $\mathcal{K}$ be a class of monounary algebras such that $\mathcal{K}$ is closed with respect to homomorphisms and $\emptyset \in \mathcal{K}$. Next, assume that each $A \in \mathcal{K}$ possess neither cycle nor subalgebra isomorphic to $\mathbb{Z}$. Then, by definition, $\mathcal{K}$ is an Hht-class.

Lemma 11. *Let $\mathcal{K}$ be an Hht-class. Then $\mathcal{K}$ fails to contain a monounary algebra possessing a subalgebra which is isomorphic to $\mathbb{Z}$.*

Proof. By way of contradiction suppose that $\mathcal{K}$ contains an algebra with a subalgebra isomorphic to $\mathbb{Z}$. Since $\mathcal{K}$ is an Hht-class, $\mathbb{Z} \in \mathcal{K}$. Observe that a regular monounary algebra which possesses only one infinite branch is a homomorphic image of $\mathbb{Z}$. By (0) of Definition 9, this regular algebra belongs to $\mathcal{K}$. According to Lemma 5, $\mathcal{K}$ fails to be an Hhr-class which is a contradiction and completes the proof. $\square$

Notation 2. Let for $i \in \mathbb{N}$ be $D_i = \{j \in \mathbb{N} : j \mid i\}$. Then for a class $\mathcal{K}$ of monounary algebras denote $Q^*(\mathcal{K}) = \bigcup_{i \in Q(\mathcal{K})} D_i$ and $P^*(\mathcal{K}) = \bigcup_{i \in P(\mathcal{K})} D_i$.

According to Lemma 11, $\mathbb{Z}$ does not belong to an Hht-class. Therefore the proof of the following characterization theorem is analogous to the proof of Theorem 4. It suffices to replace $Q(\mathcal{K})$ by $Q^*(\mathcal{K})$ and $P(\mathcal{K})$ by $P^*(\mathcal{K})$.

Theorem 7. *Let $\mathcal{K}$ be a class of monounary algebras such that $P(\mathcal{K}) \subseteq Q(\mathcal{K}) \neq \emptyset$. Next, let $\mathcal{K}$ satisfy (0) of Definition 9. Then $\mathcal{K}$ is an Hht-class if and only if the following conditions hold:*

- (a) $\emptyset \in \mathcal{K}$.
- (b) *For every $A \in \mathcal{K}$, if B is a connected component of A then $B \in \mathbb{Z}_i^1$ for some $i \in P^*(\mathcal{K})$ or $B \in \mathbb{Z}_j$ for some $j \in Q^*(\mathcal{K})$ or B possesses neither cycle nor subalgebra isomorphic to $\mathbb{Z}$.*
- (c) *The class $\mathcal{K}$ includes all algebras A such that if B is a connected component of A then either $B \in \mathbb{Z}_i^1$ for some $i \in P^*(\mathcal{K})$ or $B \in \mathbb{Z}_j$ for some $j \in Q^*(\mathcal{K})$.*

REFERENCES

- [1] W. Bartol, *Dynamic programs of computations*. Warszawa: PAN, 1974.
- [2] S. Burris and H. P. Sankappanavar, *A course in universal algebra*, ser. Graduate Texts in Mathematics. New York - Heidelberg Berlin: Springer-Verlag, 1981, vol. 78.
- [3] P. J. Cameron and J. Nešetřil, "Homomorphism-homogeneous relational structures," *Comb. Probab. Comput.*, vol. 15, no. 1-2, pp. 91–103, 2006.
- [4] J. Chvalina, *Functional graphs, quasiordered sets and commutative hypergroups*. Brno: Masaryk University, 1995.
- [5] P. Conrad, "K-radical classes of lattice ordered groups," *Algebra, Proc. Conf., Carbondale 1980, Lect. Notes Math.*, vol. 848, pp. 186–207, 1981.
- [6] I. Dolinka and D. Mašulović, "Remarks on homomorphism-homogeneous lattices and semilattices," *Monatsh. Math.*, vol. 164, no. 1, pp. 23–37, 2011.
- [7] D. Jakubíková-Studenovská and J. Pócs, *Monounary algebras*. Košice: Pavol Jozef Šafárik University, 2009.
- [8] J. Jakubík, "Radical mappings and radical classes of lattice ordered groups," *Symp. math. 21, Gruppi e Anelli ordin., Convegno 1975*, pp. 451–477, 1977.
- [9] J. Jakubík, "Radical classes of generalized Boolean algebras," *Czech. Math. J.*, vol. 48, no. 2, pp. 253–268, 1998.
- [10] J. Jakubík, "Radical classes of MV-algebras," *Czech. Math. J.*, vol. 49, no. 1, pp. 191–211, 1999.
- [11] D. Jakubíková-Studenovská, "Homogeneous monounary algebras," *Czech. Math. J.*, vol. 52, no. 2, pp. 309–317, 2002.
- [12] B. Jónsson, *Topics in universal algebra*, ser. Lecture Notes in Mathematics. Berlin-Heidelberg-New York: Springer-Verlag, 1972, vol. 250.
- [13] E. Jungábel and D. M. Mašulović, "Homomorphism-homogeneous monounary algebras," *Mathematica Slovaca*, to appear.
- [14] P. Körtesi, S. Radeleczki, and S. Szilágyi, "Congruences and isotone maps on partially ordered sets," *Math. Pannonica*, vol. 16, no. 1, pp. 39–55, 2005.
- [15] W. Lex and R. Wiegandt, "Torsion theory for acts," *Stud. Sci. Math. Hung.*, vol. 16, pp. 263–280, 1981.
- [16] J. Martinez, "Varieties of lattice-ordered groups," *Math. Z.*, vol. 137, pp. 265–284, 1974.
- [17] J. Martinez, "Torsion theory for lattice-ordered groups," *Czech. Math. J.*, vol. 25, pp. 284–299, 1975.
- [18] D. Mašulović, R. Nenadov, and N. Škorić, "On finite reflexive homomorphism-homogeneous binary relational systems," *Discrete Math.*, vol. 311, no. 21, pp. 2543–2555, 2011.

- [19] D. Mašulović and M. Pech, “Oligomorphic transformation monoids and homomorphism-homogeneous structures,” *Fundam. Math.*, vol. 212, no. 1, pp. 17–34, 2011.
- [20] H. Neumann, *Varieties of groups. (Ergebnisse der Mathematik und ihrer Grenzgebiete, Bd. 37.)*. Berlin-Heidelberg-New York: Springer-Verlag, 1967.
- [21] M. Novotný, “Mono-unary algebras in the work of Czechoslovak mathematicians,” *Arch. Math., Brno*, vol. 26, no. 2-3, pp. 155–164, 1990.
- [22] R. S. Pierce, “Some questions about complete Boolean algebras,” *Proc. Sympos. Pure Math.*, vol. 2, pp. 129–140, 1961.
- [23] M. Rusinov and P. Schweitzer, “Homomorphism-homogeneous graphs,” *J. Graph Theory*, vol. 65, no. 3, pp. 253–262, 2010.
- [24] J. Szígeti, “On well-orderd mono-unary algebras,” *Order*, vol. 7, no. 1, pp. 77–81, 1990.
- [25] D.-R. Ton, “Radical classes of ℓ -groups,” *Internat. J. Math. and Math. Sci.*, vol. 17, no. 1, pp. 129–142, 1994.
- [26] J. von Neumann, “Lectures on continuous geometries,” *Princeton University Press, Princeton*, pp. 1936–1937, 1967.

Authors' addresses

Zuzana Farkasová

P.J.Šafárik University, Institute of Mathematics, Jesenná 5, 04154 Košice, Slovakia

E-mail address: zuzana.farkasova@student.upjs.sk

Danica Jakubíková-Studenovská

P.J.Šafárik University, Institute of Mathematics, Jesenná 5, 04154 Košice, Slovakia

E-mail address: danica.studenovska@upjs.sk



A SIMPLE ROBUST ESTIMATION FOR PARAMETERS OF CAUCHY DISTRIBUTION

S. FEGYVERNEKI

Received 17 June, 2013

Abstract. We define robust estimators for the parameters of the Cauchy distribution based on the probability integral method. The proposed estimators have bounded influence functions and high breakdown points. These estimators are simple, robust and consistent, but asymptotically less efficient than the maximum likelihood estimators which are not robust. A simulation study for finite sample size shows that the efficiency of these robust estimators is rather similar to the maximum likelihood ones.

2010 Mathematics Subject Classification: 62E20; 62G35; 61E37

Keywords: M-estimates, robustness, Cauchy distributions

1. INTRODUCTION

Let $\{\xi_i, 1 \leq i \leq n\}$ be an i.i.d. sequence of random variables from the Cauchy distribution

$$G_0\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{x-\mu}{\sigma}\right), \quad (1.1)$$

where $\sigma > 0$ (see [8],[9],[10]).

The estimation of the parameters is a well discussed problem. Mostly, the location parameter μ and the scale parameter σ are estimated by the maximum likelihood (ML) estimators $\hat{\mu}$ and $\hat{\sigma}$, respectively. However, the ML estimators are conditional robust and the breakdown points are low. There exists other estimators, which are rather efficient or even unbiased, as linear estimators of order statistics (median, median absolute deviation).

In [2], [7], [9] Csernyák, Nagy and Steiner made an analysis how the I-divergence can be used to provide simple, efficient and robust estimator for parameters.

The aim of this short note is to make an analysis the probability integral transform estimators for the parameters μ and σ of the Cauchy distribution G_0 . We know that if the random variable ξ has strictly monotone increasing probability distribution

This research was (partially) carried out in the framework of the Center of Excellence of Mechatronics and Logistics at the University of Miskolc.

function F then the random variable $F(\xi)$ has uniform distribution on the interval $[0,1]$. Therefore

$$\begin{aligned} \sum_{i=1}^n \left(G_0 \left(\frac{\xi_i - \mu}{\sigma} \right) - \frac{1}{2} \right) &= 0, \\ \sum_{i=1}^n \left(\left(G_0 \left(\frac{\xi_i - \mu}{\sigma} \right) - \frac{1}{2} \right)^2 - \frac{1}{12} \right) &= 0. \end{aligned} \quad (1.2)$$

When the solutions T_n and s_n of this system of equations exist, T_n and s_n are called the probability integral transformation (PIT) estimators of the location and the scale parameter, respectively. These estimators are so-called M -estimators which are well-known in the context of robust estimation (see [3], [4], [5], [6]).

2. MAIN RESULTS

Theorem 1. Assume that G_0 is differentiable, strictly monotone increasing, then T_n and s_n are well defined, that is, (1.2) has a unique solution with $s_n > 0$.

The main steps of the proof: We follow [6]. The Jacobian of the map

$$(t, s) \longrightarrow \left(\int \psi \left(\frac{x-t}{s} \right) F(dx), \int \chi \left(\frac{x-t}{s} \right) F(dx) \right)$$

is

$$-\frac{1}{s} \begin{pmatrix} \int \psi'(y) dF & \int y \psi'(y) dF \\ \int \chi'(y) dF & \int y \chi'(y) dF \end{pmatrix}$$

with $y = \frac{x-t}{s}$. F is indifferently either the true or the empirical distribution.

We define a new probability measure F^* by

$$F^*(dy) = \frac{\psi'(y)}{E_F(\psi'(y))} F(dx);$$

then The Jacobian can be written as

$$-\frac{1}{s} E_F(\psi'(y)) \begin{pmatrix} 1 & E_{F^*}(y) \\ E_{F^*} \left(\frac{\chi'}{\psi'} \right) & E_{F^*} \left(y \frac{\chi'}{\psi'} \right) \end{pmatrix}$$

Its determinant

$$\left[\frac{E_F(\psi'(y))}{s} \right]^2 \text{cov}_{F^*} \left(y, \frac{\chi'}{\psi'} \right)$$

is strictly positive. The existence of a solution now follows from the observation that for each fixed s , the first component of the map has a unique zero at some $t = t(s)$ that depends continuously on s . We now conclude from the intermediate value theorem for continuous functions that the solution exists uniquely. The joint asymptotic distribution of (T_n, s_n) can be derived from a general result of Boos and Serfling [1].

Theorem 2. *The joint distribution of (T_n, s_n) converges to a normal one:*

$$\sqrt{n}((T_n, s_n) - (\mu, \sigma)) \xrightarrow{d} N(0, \Sigma)$$

where the covariance matrix Σ is given by

$$\Sigma = \sigma^2 \begin{pmatrix} \frac{\pi^2}{3} & 0 \\ 0 & \frac{\pi^4}{45} \end{pmatrix},$$

that is,

$$\Sigma \approx \sigma^2 \begin{pmatrix} 3.289868134 & 0 \\ 0 & 2.164646468 \end{pmatrix}.$$

The main steps of the proof: We use the general result of Boos and Serfling [1] on the central limit theorem for (T_n, s_n) for proving this theorem. T_n and s_n are the solutions of

$$\sum_{i=1}^n \psi\left(\frac{\xi_i - \mu}{\sigma}\right) = 0, \quad \sum_{i=1}^n \chi\left(\frac{\xi_i - \mu}{\sigma}\right) = 0,$$

where

$$\psi(x) = G_0(x) - 0.5$$

and

$$\chi(x) = \psi^2(x) - \frac{1}{12}.$$

The assumptions are true for the Lévy-Smirnov distribution [8], since ψ is monotone increasing and bounded and χ is bounded and negative at 0.

Thus it remains to derive the asymptotic covariance matrix $\Sigma = C^{-1}S[C^{-1}]^T$. Here

$$C = \begin{pmatrix} E\left(\frac{\partial}{\partial \mu} \psi\left(\frac{\xi - \mu}{\sigma}\right)\right) & E\left(\frac{\partial}{\partial \sigma} \psi\left(\frac{\xi - \mu}{\sigma}\right)\right) \\ E\left(\frac{\partial}{\partial \mu} \chi\left(\frac{\xi - \mu}{\sigma}\right)\right) & E\left(\frac{\partial}{\partial \sigma} \chi\left(\frac{\xi - \mu}{\sigma}\right)\right) \end{pmatrix}$$

and with random variable η , it has probability distribution G_0 , where $\mu = 0$ and $\sigma = 1$,

$$S = \begin{pmatrix} E(\psi^2(\eta)) & E(\psi(\eta)\chi(\eta)) \\ E(\psi(\eta)\chi(\eta)) & E(\chi^2(\eta)) \end{pmatrix}.$$

These terms can be evaluated by partial integration. The approximations made by program package Maple 15

$$C = -\frac{1}{\sigma} \begin{pmatrix} \frac{1}{2\pi} & 0 \\ 0 & \frac{1}{2\pi^2} \end{pmatrix}$$

and

$$S = \begin{pmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{180} \end{pmatrix}.$$

Comparing the asymptotic variances of the PIT estimators T_n and s_n with the ML estimators $\hat{\mu}$ and $\hat{\sigma}$, respectively, we derive the asymptotic relative efficiencies (ARE). It is known that the ML estimators are asymptotically efficient, having also an asymptotic normal distribution, with covariance matrix denoted by Σ .

For the robustness, we easily derive the breakdown point $\varepsilon^*(T_n)$ of the location estimator. We find that

$$\varepsilon^*(T_n) = 0.5.$$

This is the maximum for PIT estimator.

Because the function χ is symmetric we can use directly the result of Huber [6] on the breakdown point $\varepsilon^*(s_n)$.

Theorem 3. *If we choose the distribution G_0 with $\mu = 0$ and $\sigma = 1$ for the functions ψ and χ then the breakdown point $\varepsilon^*(s_n) = \frac{1}{3}$. This is the maximum for PIT estimators.*

We see that these properties are not similar to ML estimators because the function ψ is redescending.

3. NUMERICAL ALGORITHM

We propose an algorithm for estimating the location and the scale simultaneously. Let the general system be

$$\begin{aligned} \sum_{i=1}^n \psi\left(\frac{\xi_i - T}{s}\right) &= 0, \\ \sum_{i=1}^n \chi\left(\frac{\xi_i - T}{s}\right) &= 0, \end{aligned}$$

where the functions ψ and χ are given by (1.2).

Step 1: Preestimation of location and scale by $\max\{\xi_i\}$ and median absolute deviation (MAD), i.e.,

$$T_n^{(0)} = \max\{\xi_i\} \quad \text{and} \quad s_n^{(0)} = MAD\{\xi_i\}$$

Step 2: Estimation of location by

$$T_n^{(m+1)} = T_n^{(m)} + \frac{s_n^{(m)} \sum_{i=1}^n \psi\left(\frac{\xi_i - T_n^{(m)}}{s_n^{(m)}}\right)}{n}.$$

Step 3: Estimation of scale by

$$[s_n^{(m+1)}]^2 = \frac{12}{(n-1)} \sum_{i=1}^n \psi^2 \left(\frac{\xi_i - T_n^{(m+1)}}{s_n^{(m)}} \right) [s_n^{(m)}]^2.$$

Step 4: Stop or goto step 2.

Since the function $G_0(x)$ is an absolute continuous distribution function, the convergence of this iterative method follows from by Huber's result [6] Section 7.8, if $G_0(x)$ is strictly monotone increasing.

4. SIMULATION

The asymptotic results show that the PIT estimators are less efficient than the ML ones. But the efficiency for finite samples might be quite different. Therefore, we simulated finite samples of size n from the Cauchy distribution with $\mu = 0$ and $\sigma = 1$. From these samples the different estimators were calculated. From $n = 500$ and 1000 simulations the mean and the standard deviation were derived for each estimator, which were used for the comparison of their finite sample behaviour. We calculated the correlations of the paired estimators and the mean square error (MSE), too. These show that in pairwise comparison the behaviour of PIT and ML estimators are rather similar in finite. This is also indicated by their correlations.

REFERENCES

- [1] D. D. Boos and R. J. Serfling, "A note on differentials and the CLT and LIL for statistical functions, with application to M-estimates," *Ann. Stat.*, vol. 8, pp. 618–624, 1980.
- [2] L. Csernyák, "On the most frequent value and cohesion of probability functions," *Acta Geodæt., Geophys. et Mont. Acad. Sci. Hung.*, vol. 8, pp. 397–401, 1973.
- [3] S. Fegyverneki, "A special joint estimation of location and scale with applications," *Publ. Univ. Miskolc, Ser. D, Nat. Sci., Math.*, vol. 39, pp. 21–27, 1999.
- [4] S. Fegyverneki, "Robust estimators and probability integral transformations," *Math. Comput. Modelling*, vol. 38, no. 7-9, pp. 803–814, 2003.
- [5] F. R. Hampel, E. M. Ronchetti, P. J. Rousseeuw, and W. A. Stahel, *Robust statistics. The approach based on influence functions*, ser. Wiley Series in Probability and Mathematical Statistics. Probability and Mathematical Statistics. New York: John Wiley & Sons, 1986.
- [6] P. J. Huber, *Robust statistics.*, ser. Wiley Series in Probability and Mathematical Statistics. New York: John Wiley & Sons, 1981.
- [7] F. Nagy, "Parameter estimation of the Cauchy distribution in information theory approach," *Journal of Universal Computer Science*, vol. 12, pp. 1332–1344, 2006.
- [8] M. M. Rao and R. J. Swift, *Probability theory with applications*, 2nd ed., ser. Mathematics and its Applications (Springer). New York: Springer, 2006, vol. 582.
- [9] F. Steiner, "Most frequent value procedures (a short monograph)," *Geophysical Transactions*, vol. 34, pp. 139–260, 1988.
- [10] F. Steiner, *A geostatistika alapjai*. Budapest: Tankönyvkiadó, 1990.

Author's address

S. Fegyverneki

University of Miskolc, Department of Applied Mathematics, 3515 Miskolc-Egyetemáros, Hungary

E-mail address: `matfs@uni-miskolc.hu`



DIOPHANTINE QUADRUPLES IN THE RING OF INTEGERS OF THE PURE CUBIC FIELD $\mathbb{Q}(\sqrt[3]{2})$

ZRINKA FRANUŠIĆ

Received 16 April, 2013

Abstract. We show that in the ring of integers of the pure cubic field $\mathbb{Q}(\sqrt[3]{2})$ there exists a $D(w)$ -quadruple if and only if w can be represented as a difference of two squares of integers in $\mathbb{Q}(\sqrt[3]{2})$.

2010 *Mathematics Subject Classification:* 11D09; 11D79; 11R16

Keywords: Diophantine quadruples, pure cubic field

1. INTRODUCTION

Let R be a commutative ring with a unit 1 and let $w \in R$. A set of m distinct non-zero elements $\{w_1, \dots, w_m\} \subset R$, such that $w_i \cdot w_j + w$ is a perfect square in R for all $1 \leq i < j \leq m$, is called a *Diophantine m -tuple with the property $D(w)$* or a *$D(w)$ - m -tuple* in R . If $w = 1$ then these sets are often called *Diophantine m -tuples*. They are named after the 3rd-century Greek mathematician Diophantus of Alexandria who first studied these sets and constructed the set $\{1, 33, 68, 105\}$ with the property $D(256)$. Fermat found the set $\{1, 3, 8, 120\}$ - the first $D(1)$ -quadruple in $\mathbb{Z}$. Baker and Davenport in [3] showed that Fermat's quadruple cannot be extended to a $D(1)$ -quintuple. A folklore conjecture says that there is no $D(1)$ -quintuples in $\mathbb{Z}$. Dujella proved that there are only finitely many $D(1)$ -quintuples (see [8]).

The problem of existence of $D(w)$ -quadruples was mainly considered in rings of integers of number fields. It all started with the fact that there does not exist a $D(n)$ -quadruple in the ring of integers $\mathbb{Z}$ if $n \equiv 2 \pmod{4}$. This simple statement was observed independently by several authors (see [4, 13, 15]). On the other hand, Dujella [5] showed that if $n \not\equiv 2 \pmod{4}$ and $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$ then there exists a $D(n)$ -quadruple. It is interesting that the condition $n \not\equiv 2 \pmod{4}$ is equivalent to the condition that n is representable as a difference of the squares of two integers. An analogous result for Gaussian integers was also found by Dujella [7]. He proved that there does not exist a $D(a + bi)$ -quadruple in $\mathbb{Z}[i]$ if b is odd or

The author was supported by the Ministry of Science, Education and Sports, Republic of Croatia, Grant No. 037-0372781-2821.

$a \equiv b \equiv 2 \pmod{4}$, i.e. if $a + bi$ is not representable as a difference of the squares of two elements in $\mathbb{Z}[i]$, and in contrary if $a + bi$ is not of such form and $a + bi \notin \{2, -2, 1 + 2i, -1 - 2i, 4i, -4i\}$ then a $D(a + bi)$ -quadruple exists. Therefore, it is natural to state the following conjecture: *There exists a $D(w)$ -quadruple if and only if w can be represented as a difference of two squares, up to finitely many exceptions.*

So far, the conjecture was shown to be true in rings of integers of some real quadratic fields. More precisely, the author proved that there exist infinitely many $D(w)$ -quadruples if and only if w can be represented as a difference of two squares, in the ring of integers of $\mathbb{Q}(\sqrt{d})$ for $d = 2$ and for all positive integers d such that one of Pellian equations $x^2 - dy^2 = \pm 2$ is solvable or such that $x^2 - dy^2 = 4$ is solvable in odd numbers (see [10–12]). The assumption of solvability of these Pellian equations allows to derive an effective characterization of integers that are representable as a difference of two squares. Similar result for complex quadratic fields are harder to obtain. Several authors contributed to the characterization of elements z of $\mathbb{Z}[\sqrt{-2}]$ for which a Diophantine quadruple with the property $D(z)$ exists (see [1, 9, 16]). One important difference between real and complex quadratic fields is that in the real case there exist infinitely many units and the methods for the construction of Diophantine quadruples usually use elements with a small norm.

In this paper we prove the following conjecture for the ring of integers of the pure cubic field $\mathbb{Q}(\sqrt[3]{2})$:

Theorem 1. *If w can be represented as a difference of two squares of integers in $\mathbb{Q}(\sqrt[3]{2})$ then there exists infinitely many Diophantine quadruples with the property $D(w)$ in the ring of integers of $\mathbb{Q}(\sqrt[3]{2})$. If w is not a difference of two squares of integers then a $D(w)$ -quadruple does not exist in the ring of integers of $\mathbb{Q}(\sqrt[3]{2})$.*

The proof of the existence of $D(w)$ -quadruples is based on the description of a difference of two squares of integers in $\mathbb{Q}(\sqrt[3]{2})$ and on applying polynomial formulas for Diophantine quadruples. The first step in proving the non-existence of certain $D(w)$ -quadruples was made by Jukić Matić [14]. In the rings of integers of the cubic fields of the form $\mathbb{Q}(\sqrt[3]{d})$, where d is even, she described some elements w that cannot be written as a difference of two squares of integers and showed that $D(w)$ -quadruples do not exist. These results are complemented in Section 5 and lead to a proof of the second statement of Theorem 1. In Section 4, the first statement of Theorem 1 is proved by effective constructions of our objects - Diophantine quadruples. This was accomplished thanks to a nice characterization of differences of two squares in the ring of integers of $\mathbb{Q}(\sqrt[3]{2})$ (Section 3).

We continue by exposing some well known facts relevant to the proof of Theorem 1 in the following section.

2. PRELIMINARIES

Let us start with recalling some basic facts related to the structure of the ring of integers $\mathcal{O}_K$ of the pure cubic field $K = \mathbb{Q}(\sqrt[3]{2})$. Since $\sqrt[3]{2}$ is a root of the irreducible polynomial $x^3 - 2$ we have that $[K : \mathbb{Q}] = 3$. Also, $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$ is an integral basis for K , so $\mathcal{O}_K = \mathbb{Z}[1, \sqrt[3]{2}, \sqrt[3]{4}]$ (according to Theorem 7.3.2. in [2]).

As we said before, the methods for constructing $D(w)$ -quadruples use elements with a small norm. It is a well-known fact that $\mathbb{Q}(\sqrt[3]{2})$ possesses an unique fundamental unit $\eta > 1$ and it is $\eta = 1 + \sqrt[3]{2} + \sqrt[3]{4}$ (Theorem 13.6.2. in [2]). All fundamental units in $\mathbb{Q}(\sqrt[3]{2})$ are $\pm\eta$ and $\pm\eta^{-1} = \pm(1 - \sqrt[3]{2})$. Also, the important roles have all (algebraic) units of $\mathcal{O}_K$, i.e. $\pm\eta^n$, $n \in \mathbb{Z}$.

Following lemmas are valid in any commutative ring R with an unit.

Lemma 1 (Theorem 1, [6]). *Let $\mu, \kappa \in R$. The set*

$$\{\mu, \mu(3\kappa + 1)^2 + 2\kappa, \mu(3\kappa + 2)^2 + 2\kappa + 2, 9\mu(2\kappa + 1)^2 + 8\kappa + 4\} \quad (2.1)$$

has the property $D(2\mu(2\kappa + 1) + 1)$.

We emphasize that the set (2.1) is considered as a $D(2\mu(2\kappa + 1) + 1)$ -quadruple if it contains no equal elements or zero elements.

Lemma 2. *If $\{w_1, w_2, w_3, w_4\}$ is a $D(w)$ -quadruple in R , then $\{w_1u, w_2u, w_3u, w_4u\}$ is a $D(wu^2)$ -quadruple in R .*

3. ON DIFFERENCE OF TWO SQUARES

Here, we describe the set of all integers that are representable as a difference of squares of two integers in $\mathbb{Q}(\sqrt[3]{2})$.

Theorem 2. *Let $\alpha, \beta, \gamma \in \mathbb{Z}$. Then $w = \alpha + \beta\sqrt[3]{2} + \gamma\sqrt[3]{4}$ can be represented as a difference of squares of two integers in $\mathbb{Q}(\sqrt[3]{2})$ if and only if*

$$\alpha \equiv 0, 1, 3 \pmod{4}, \beta \equiv 0 \pmod{2} \text{ and } (\alpha, \beta, \gamma) \not\equiv (0, 0, 2) \pmod{4}.$$

Proof. Suppose that

$$\alpha + \beta\sqrt[3]{2} + \gamma\sqrt[3]{4} = (a_1 + b_1\sqrt[3]{2} + c_1\sqrt[3]{4})^2 - (a_2 + b_2\sqrt[3]{2} + c_2\sqrt[3]{4})^2,$$

for some $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{Z}$, then

$$\begin{aligned} \alpha &= a_1^2 - a_2^2 + 4(b_1c_1 - b_2c_2), \\ \beta &= 2(c_1^2 - c_2^2 + b_1a_1 - b_2a_2), \\ \gamma &= b_1^2 - b_2^2 + 2(a_1c_1 - a_2c_2). \end{aligned}$$

It is easy to see that $\alpha \equiv 0, 1, 3 \pmod{4}$ and $\beta \equiv 0 \pmod{2}$. Also, if we assume that $\alpha \equiv \beta \equiv 0 \pmod{4}$ and γ is even, then $a_1 \equiv a_2 \pmod{2}$, $b_1 \equiv b_2 \pmod{2}$, $c_1 \equiv c_2 \pmod{2}$. Hence, $\gamma = b_1^2 - b_2^2 + 2(a_1c_1 - a_2c_2) \equiv 0 \pmod{4}$, i.e. $\gamma \not\equiv 2 \pmod{4}$.

The converse of the statement is proved by a series of equalities:

$$\begin{aligned}
 &1 + 2\alpha + 2\beta \sqrt[3]{2} + (1 + 2\gamma) \sqrt[3]{4} = \\
 &(2\gamma - \alpha + 1 + (\alpha - \beta + 1) \sqrt[3]{2} + (\beta - \gamma) \sqrt[3]{4})^2 - (2\gamma - \alpha + (\alpha - \beta) \sqrt[3]{2} + (\beta - 1 - \gamma) \sqrt[3]{4})^2, \\
 &1 + 2\alpha + 2\beta \sqrt[3]{2} + 2\gamma \sqrt[3]{4} = (\alpha + 1 + \beta \sqrt[3]{2} + \gamma \sqrt[3]{4})^2 - (\alpha + \beta \sqrt[3]{2} + \gamma \sqrt[3]{4})^2, \\
 &4\alpha + 4\beta \sqrt[3]{2} + 4\gamma \sqrt[3]{4} = (\alpha + 1 + \beta \sqrt[3]{2} + \gamma \sqrt[3]{4})^2 - (\alpha - 1 + \beta \sqrt[3]{2} + \gamma \sqrt[3]{4})^2, \\
 &4\alpha + (4\beta + 2) \sqrt[3]{2} + 2\gamma \sqrt[3]{4} = (\alpha + \beta \sqrt[3]{2} + (\gamma + 1) \sqrt[3]{4})^2 - (\alpha + \beta \sqrt[3]{2} + \gamma \sqrt[3]{4})^2, \\
 &4\alpha + 2\beta \sqrt[3]{2} + (2\gamma + 1) \sqrt[3]{4} = (\alpha + (\beta + 1) \sqrt[3]{2} + \gamma \sqrt[3]{4})^2 - (\alpha + \beta \sqrt[3]{2} + \gamma \sqrt[3]{4})^2.
 \end{aligned}$$

□

4. THE EXISTENCE OF $D(w)$ -QUADRUPLES

In this section we construct a $D(w)$ -quadruple for all $w \in \mathcal{O}_K$ of the form given in Theorem 2. The following lemma serves as a prototype or template for constructing $D(w)$ -quadruples for the majority of cases.

Lemma 3. *Let $a, b, c \in \mathbb{Z}$ and $(a, b, c) \neq (0, -1, 0), (-1, 0, 0)$. The set*

$$\begin{aligned}
 D = \{ &1 - \sqrt[3]{2}, \\
 &-21 - 32a - 9a^2 - 40b - 36ab - 18b^2 - 40c - 36ac - 36bc - 36c^2 + \\
 &(-15 - 20a - 9a^2 - 32b - 18ab - 18b^2 - 40c - 36ac - 36bc - 18c^2) \sqrt[3]{2} + \\
 &(-11 - 20a - 9a^2 - 20b - 18ab - 9b^2 - 32c - 18ac - 36bc - 18c^2) \sqrt[3]{4}, \\
 &-16 - 26a - 9a^2 - 40b - 36ab - 18b^2 - 40c - 36ac - 36bc - 36c^2 + \\
 &(-12 - 20a - 9a^2 - 26b - 18ab - 18b^2 - 40c - 36ac - 36bc - 18c^2) \sqrt[3]{2} + \\
 &(-11 - 20a - 9a^2 - 20b - 18ab - 9b^2 - 26c - 18ac - 36bc - 18c^2) \sqrt[3]{4}, \\
 &-75 - 116a - 36a^2 - 160b - 144ab - 72b^2 - 160c - 144ac - 144bc - 144c^2 + \\
 &(-53 - 80a - 36a^2 - 116b - 72ab - 72b^2 - 160c - 144ac - 144bc - 72c^2) \sqrt[3]{2} + \\
 &(-44 - 80a - 36a^2 - 80b - 72ab - 36b^2 - 116c - 72ac - 144bc - 72c^2) \sqrt[3]{4} \}
 \end{aligned}$$

is a $D(3 + 4a + (2 + 4b) \sqrt[3]{2} + 4c \sqrt[3]{4})$ -quadruple in $\mathbb{Z}[1, \sqrt[3]{2}, \sqrt[3]{4}]$. The set

$$\begin{aligned}
 &\{-19 + 5 \sqrt[3]{2} + 8 \sqrt[3]{4}, -1749109 - 1388265 \sqrt[3]{2} - 1101867 \sqrt[3]{4}, \\
 &-1749060 - 1388224 \sqrt[3]{2} - 1101831 \sqrt[3]{4}, -6996319 - 5552983 \sqrt[3]{2} - 4407404 \sqrt[3]{4}\}
 \end{aligned}$$

is a $D(3 - 2 \sqrt[3]{2})$ -quadruple and

$$\begin{aligned}
 &\{-19 + 5 \sqrt[3]{2} + 8 \sqrt[3]{4}, -17524872 - 13909496 \sqrt[3]{2} - 11039975 \sqrt[3]{4}, \\
 &-17524709 - 13909365 \sqrt[3]{2} - 11039867 \sqrt[3]{4}, -70099143 - 55637727 \sqrt[3]{2} - 44159692 \sqrt[3]{4}\}
 \end{aligned}$$

is a $D(-1 + 2 \sqrt[3]{2})$ -quadruple.

Proof. Let $\mu = 1 - \sqrt[3]{2}$ and $\kappa = \alpha + \beta \sqrt[3]{2} + \gamma \sqrt[3]{4} \in \mathcal{O}_K$. Then

$$w = 2\mu(2\kappa + 1) + 1 = 4\alpha - 8\gamma + 3 + (-4\alpha + 4\beta - 2) \sqrt[3]{2} + (-4\beta + 4\gamma) \sqrt[3]{4}$$

is of the form $3 + 4a + (2 + 4b)\sqrt[3]{2} + 4c\sqrt[3]{4}$. Now, let us make sure that for given $a, b, c \in \mathbb{Z}$ there exist $\alpha, \beta, \gamma \in \mathbb{Z}$ such that $w = 3 + 4a + (2 + 4b)\sqrt[3]{2} + 4c\sqrt[3]{4}$. Indeed, the linear system

$$4\alpha - 8\gamma + 3 = 3 + 4a, \quad -4\alpha + 4\beta - 2 = 2 + 4b, \quad -4\beta + 4\gamma = 4c,$$

i.e. the system

$$\alpha - 2\gamma = a, \quad -\alpha + \beta = 1 + b, \quad -\beta + \gamma = c$$

has the integer solution

$$\alpha = -a - 2b - 2c - 2, \quad \beta = -a - b - 2c - 1, \quad \gamma = -a - b - c - 1.$$

By putting this solution into (2.1) we obtain the set D . This is a $D(w)$ -quadruple if it consists of non-zero distinct elements. Hence, we have to check whether there are equal elements or elements equal to zero in D and that requires solving 9 systems of three (nonlinear) equations. The results of the investigation are two integer solutions $(a, b, c) = (0, -1, 0), (-1, 0, 0)$ and corresponding sets $\{1 - \sqrt[3]{2}, 1 - \sqrt[3]{2}, 6 - 4\sqrt[3]{2}, 13 - 9\sqrt[3]{2}\}$ and $\{1 - \sqrt[3]{2}, 2 - 4\sqrt[3]{2}, 1 - \sqrt[3]{2}, 5 - 9\sqrt[3]{2}\}$ which do not represent $D(3 - 2\sqrt[3]{2})$ and $D(-1 + 2\sqrt[3]{2})$ -quadruple respectively (because they have two equal elements). This situation can be resolved by putting $w = 3 - 2\sqrt[3]{2} = u^8(2\mu(2\kappa + 1) + 1)$, where $u = \mu = 1 - \sqrt[3]{2}$. Indeed, this holds for $\kappa = -7390 - 5865\sqrt[3]{2} - 4655\sqrt[3]{4}$ and according to Lemmas 1 and 2 we obtain the set $\{-19 + 5\sqrt[3]{2} + 8\sqrt[3]{4}, -1749109 - 1388265\sqrt[3]{2} - 1101867\sqrt[3]{4}, \dots\}$. Similarly, we construct $D(-1 + 2\sqrt[3]{2})$ -quadruple since $-1 + 2\sqrt[3]{2} = u^8(2\mu(2\kappa + 1) + 1)$ for $u = \mu = 1 - \sqrt[3]{2}$ and $\kappa = -23391 - 18565\sqrt[3]{2} - 14735\sqrt[3]{4}$. $\square$

In what follows, the majority of cases are treated similarly by assuming that $w = u^2(2\mu(2\kappa + 1) + 1)$ for some $u, \mu \in \mathbb{Z}[1, \sqrt[3]{2}, \sqrt[3]{4}]$ and $\kappa \in \mathbb{Q}(\sqrt[3]{2})$. By multiplying the elements of the set (2.1) by u we obtain a $D(w)$ -quadruple up to finitely many exceptions. These exceptions are resolved separately as in Lemma 3. The results for the existence of $D(w)$ -quadruples for all w of the form $2a + 1 + 2b\sqrt[3]{2} + c\sqrt[3]{4}$, $a, b, c \in \mathbb{Z}$ are exposed in several tables in Section 6. To be more simple, we occasionally represent an integer $w = a + b\sqrt[3]{2} + c\sqrt[3]{4}$ as a triple $w = (a, b, c)$.

A Diophantine quadruple for each exception can be found as in Lemma 3. Precisely, each exception can be represented in a form $v^8u^2(2\mu(2\kappa + 1) + 1)$ or $v^{16}u^2(2\mu(2\kappa + 1) + 1)$, where μ and κ are given in tables for a corresponding exception and $v = 1 - \sqrt[3]{2}$. (Note that $w \cdot v^8 \equiv w \pmod{4}$ and $w \cdot v^{16} \equiv w \pmod{8}$).

Since, we have shown the existence of $D(w)$ -quadruples for $w = 2a + 1 + 2b\sqrt[3]{2} + c\sqrt[3]{4}$, by multiplying w with u^2 where $u = \sqrt[3]{2}$, $u = \sqrt[3]{4}$ and $u = 2$ we prove that there exist $D(4a + 2b\sqrt[3]{2} + (1 + 2c)\sqrt[3]{4})$, $D(4a + (2 + 4b)\sqrt[3]{2} + 4c\sqrt[3]{4})$ and $D(8a + 4 + 8b\sqrt[3]{2} + 4c\sqrt[3]{4})$ -quadruples, respectively. The remaining cases are given in the tables in Section 7.

The exceptions are not a problem and we treat them the same way as before. Hence, we have proved that if w is a difference of two squares of integers then there exists a $D(w)$ -quadruple and even more, there exist infinitely many $D(w)$ -quadruples. Indeed, if v is an algebraic unit so is v^{-1} and another $D(w)$ -quadruple is obtained by multiplying the elements of $D(wv^2)$ -quadruple by v^{-1} . So, the existence of infinitely many algebraic units in $\mathbb{Q}(\sqrt[3]{2})$ imply that there are infinitely many $D(w)$ -quadruples. One may ask whether distinct quadruples by multiplying by units appear. The answer is positive due to the fact that the first element of a Diophantine quadruple created as described above is of the form μu , where μ and u are taken from a finite set of values. For instance, see Lemma 3.

5. THE NONEXISTENCE OF $D(w)$ -QUADRUPLES

Lemma 4 (Lemma 3.1, Theorem 3.2, [14]). *If $w \in \mathcal{O}_K$ is of one of the following forms*

$$a + (2b + 1)\sqrt[3]{2} + c\sqrt[3]{4}, \quad 4a + 2 + b\sqrt[3]{2} + c\sqrt[3]{4},$$

then there exist no $D(w)$ -quadruples in $\mathcal{O}_K$.

Lemma 5. *If $w \in \mathcal{O}_K$ is of the form $4a + 4b\sqrt[3]{2} + (4c + 2)\sqrt[3]{4}$ then there exist no $D(w)$ -quadruples in $\mathcal{O}_K$.*

Proof. Let $a, b, c \in \mathbb{Z}$ and $w = 4a + 4b\sqrt[3]{2} + (4c + 2)\sqrt[3]{4}$. Assume that $\{w_1, w_2, w_3, w_4\}$ is a $D(w)$ -quadruple in $\mathcal{O}_K$. If $w_i = a_i + b_i\sqrt[3]{2} + c_i\sqrt[3]{4}$, i.e. $w_i = (a_i, b_i, c_i)$, $1 \leq i \leq 4$, then

$$w_i \cdot w_j \bmod 4 \in S = \{(0, 0, 2), (0, 0, 3), (0, 2, 2), (0, 2, 3), \\ (1, 0, 1), (1, 0, 2), (1, 2, 0), (1, 2, 3)\},$$

for $1 \leq i < j \leq 4$. We obtain that there do not exist four integers w_1, w_2, w_3, w_4 in $\mathcal{O}_K$ such that the above condition is fulfilled by testing all possibilities directly. To be more specific, we characterize congruence types modulo 4 of Diophantine pairs and triples. Let us recall, that the $D(w)$ -pair $\{w_1, w_2\}$ has a congruence type $[(\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2)]$ if $a_i \equiv \alpha_i \pmod{4}$, $b_i \equiv \beta_i \pmod{4}$, $c_i \equiv \gamma_i \pmod{4}$, $i = 1, 2$. Without loss of generality, we assume that $\{w_1, w_2\}$ is ordered in an ascending order according to a congruence type i.e. $\alpha_1\beta_1\gamma_1 \leq \alpha_2\beta_2\gamma_2$, where $\alpha\beta\gamma = \alpha \cdot 4^2 + \beta \cdot 4 + \gamma$ (- a number in the quaternary numeral system with digits α, β and γ). For instance, if $w_1 \bmod 4 = (0, 0, 1)$ and if $\{w_1, w_2\}$ is a $D(w)$ -pair, then

$$w_2 \bmod 4 \in S_1 = \{(2, 0, 0), (2, 0, 1), (2, 0, 2), (2, 0, 3), (2, 2, 0), (2, 2, 1), (2, 2, 2), \\ (2, 2, 3), (3, 0, 0), (3, 0, 1), (3, 0, 2), (3, 0, 3), (3, 2, 0), (3, 2, 1), (3, 2, 2), (3, 2, 3)\}.$$

Now, if $w_2 \bmod 4 = (2, 0, 0)$ and if $\{w_2, w_3\}$ is a $D(w)$ -pair, then

$$w_3 \bmod 4 \in S_2 = \{(2, 0, 1), (2, 0, 3), (2, 1, 1), (2, 1, 3), (2, 2, 1),$$

$$(2, 2, 3), (2, 3, 1), (2, 3, 3)\}.$$

So, if $\{w_1, w_2, w_3\}$ is a $D(w)$ -triple and if $\{w_1, w_2\}$ has a congruence type $[(0, 0, 1), (2, 0, 0)]$ then

$$w_3 \bmod 4 \in S_1 \cap S_2 = \{(2, 0, 1), (2, 0, 3), (2, 2, 1), (2, 2, 3)\}.$$

Finally, if $w_3 \bmod 4 = (2, 0, 1)$ and if $\{w_3, w_4\}$ is a $D(w)$ -pair, then

$$w_4 \bmod 4 \in S_3 = \{(2, 0, 2), (2, 2, 0), (2, 2, 2), (3, 1, 0), (3, 1, 2), (3, 3, 0), (3, 3, 2)\}.$$

Under assumptions that $\{w_1, w_2, w_3, w_4\}$ is a $D(w)$ -quadruple and $\{w_1, w_2, w_3\}$ has a congruence type $[(0, 0, 1), (2, 0, 0), (2, 0, 1)]$, it holds that

$$w_4 \in S_1 \cap S_2 \cap S_3 = \emptyset.$$

□

According to Theorem 2 and Lemmas 4, 5, we conclude the if $w \in \mathcal{O}_K$ is not representable a difference of two squares of integers, then a $D(w)$ -quadruple does not exist. That proves the second statement of the Theorem 1.

6. LIST OF TABLES FOR $D(2a + 1 + 2b\sqrt[3]{2} + c\sqrt[3]{4})$ -QUADRUPLES $D(4a + 3, 2b, c)$ -quadruples

w	μ	u	$k = (\alpha, \beta, \gamma)$	exceptions
$(3 + 4a, 4b, 4c)$	1	1	(a, b, c)	$(3, 0, 0), (-1, 0, 0)$
$(3 + 4a, 4b, 1 + 4c)$	$(1 - \sqrt[3]{2})^3$	$1 - \sqrt[3]{2}$	$(-318 - 281a - 354b - 446c, -252 - 223a - 281b - 354c, -200 - 177a - 223b - 281c)$	$(-37, 8, 17), (39, -12, -15)$
$(3 + 4a, 4b, 2 + 4c)$	$(1 - \sqrt[3]{2})^2$	1	$(6 + 5a + 6b + 8c, 5 + 4a + 5b + 6c, 4 + 3a + 4b + 5c)$	$(3, -4, 2), (-1, 4, -2)$
$(3 + 4a, 4b, 3 + 4c)$	$1 - \sqrt[3]{2}$	$1 - \sqrt[3]{2}$	$(-37 - 19a - 24b - 30c, -29 - 15a - 19b - 24c, -23 - 12a - 15b - 19c)$	$(-1, -8, 7), (3, 4, -5)$
$(3 + 4a, 2 + 4b, 4c)$	$1 - \sqrt[3]{2}$	1	$(-2 - a - 2b - 2c, -1 - a - b - 2c, -1 - a - b - c)$	$(3, -2, 0), (-1, 2, 0)$
$(3 + 4a, 2 + 4b, 1 + 4c)$	$(1 - \sqrt[3]{2})^2$	$1 - \sqrt[3]{2}$	$(128 + 73a + 92b + 116c, 102 + 58a + 73b + 92c, 81 + 46a + 58b + 73c)$	$(-13, -6, 13), (15, 2, -11)$
$(3 + 4a, 2 + 4b, 2 + 4c)$	$(1 - \sqrt[3]{2})^3$	1	$(-37 - 19a - 24b - 30c, -29 - 15a - 19b - 24c, -23 - 12a - 15b - 19c)$	$(-1, -6, 6), (3, 6, -6)$
$(3 + 4a, 2 + 4b, 3 + 4c)$	1	$1 - \sqrt[3]{2}$	$(12 + 5a + 6b + 8c, 10 + 4a + 5b + 6c, 8 + 3a + 4b + 5c)$	$(3, -6, 3), (-1, 2, -1)$

 $D(4a + 1, 4b, 4c)$ -quadruples

w	μ	u	$k = (\alpha, \beta, \gamma)$	exceptions
$(5 + 8a, 8b, 4c)$	2	1	$(a, b, \frac{5}{2})$	$(5, 0, 0), (-3, 0, 0)$
$(5 + 8a, 4 + 8b, 4c)$	2	$(1 - \sqrt[3]{2})^4$	$(20080 + 16001a + 20160b + 6350c, 15938 + 12700a + 16001b + 5040c, 25300 + 20160a + 25400b + \frac{16001}{2}c)$	$(5, 500, -400), (-3, -300, 240)$
$(1 + 8a, 4 + 8b, 4c)$	$2\sqrt[3]{2}$	1	$(\frac{b}{2}, \frac{5}{2}, a)$	$(1, 4, 0), (1, -4, 0)$
$(1 + 8a, 8b, 4c)$	$2\sqrt[3]{2}$	$(1 - \sqrt[3]{2})^4$	$(1587 + 12700a + \frac{16001}{2}b + 5040c, 2520 + 20160a + 12700b + \frac{16001}{2}c, 2000 + 16001a + 10080b + 6350c)$	$(-639, 104, 320), (641, 96, -480)$

 $D(4a + 1, 4b, 4c + 1)$ -quadruples

w	μ	u	$k = (\alpha, \beta, \gamma)$	exceptions
$(5 + 8a, 4b, 1 + 4c)$	$\sqrt[3]{2}$	$1 - \sqrt[3]{2}$	$(6 + 8a + 5b + 6c, 5 + 6a + 4b + 5c, 4 + 5a + 3b + 4c)$	$(5, 0, -3), (-3, -4, 5)$
$(1 + 8a, 4b, 1 + 4c)$	$\sqrt[3]{2}$	$(1 - \sqrt[3]{2})^5$	$(309581 + 375968a + 236845b + 298406c, 245715 + 298406a + 187984b + 236845c, 195024 + 236845a + 149203b + 187984c)$	$(-595, 980, -403), (1637, -1104, -155)$

 $D(4a + 1, 4b, 4c + 2)$ -quadruples

w	μ	u	$\kappa = (\alpha, \beta, \gamma)$	exceptions
$(1 + 8a, 4b, 2 + 4c)$	$\sqrt[3]{4}$	1	$(\frac{5}{2}a, b)$	$(1, 0, 2), (1, 0, -2)$
$(5 + 8a, 4b, 2 + 4c)$	$2 - \sqrt[3]{4}$	1	$(1 + 2a + b + \frac{5}{2}, 1 + a + b + \frac{5}{2}, 2 + 2a + b + c)$	$(5, 0, -2), (-3, 0, 2)$

$D(4a + 1, 4b, 4c + 3)$ -quadruples

w	μ	u	$\kappa = (\alpha, \beta, \gamma)$	exceptions
$(1 + 8a, 4b, 3 + 4c)$	$2 + \sqrt[3]{2} + \sqrt[3]{4}$	$1 - \sqrt[3]{2}$	$(1 + 2a + b + 2c, 1 + 2a + b + c, 1 + a + b + c)$	$(1, -4, 3), (1, 0, -1)$
$(5 + 8a, 4 + 8b, 3 + 8c)$	4	$1 + \sqrt[3]{2} + \sqrt[3]{4}$	$(-1 + \frac{a}{2} + b - 2c, -a + \frac{b}{2} + c, \frac{a}{2} - b + \frac{c}{2})$	$(5, 4, 3), (45, 36, 27)$ $(-35, -28, -21)$
$(5 + 8a, 8b, 3 + 8c)$	4	$(1 + \sqrt[3]{2} + \sqrt[3]{4})^5$	$(278 + \frac{521}{2}a - 279b - 62c, -248 - 31a + \frac{521}{2}b - 279c, 21 - \frac{279}{2}a - 31b + \frac{521}{2}c)$	$(5, 205949008, -1044421)$
$(5 + 8a, 4 + 8b, 7 + 8c)$	$2\sqrt[3]{4}$	$1 + \sqrt[3]{2} + \sqrt[3]{4}$	$(a - b + \frac{c}{2}, -2 + a + b - 2c, 1 - 2a + \frac{b}{2} + c)$	$(37, 28, 23), (-27, -20, -17)$
$(5 + 8a, 8b, 7 + 8c)$	$2\sqrt[3]{4}$	$(1 + \sqrt[3]{2} + \sqrt[3]{4})^5$	$(281 - 279a - 31b + \frac{521}{2}c, 217 + 521a - 279b - 62c, -527 - 62a + \frac{521}{2} - 279c)$	$(125880325, 0, 1096583)$

$D(4a + 1, 4b + 2, 4c)$ -quadruples

w	μ	u	$\kappa = (\alpha, \beta, \gamma)$	exceptions
$(1 + 8a, 2 + 4b, 4c)$	$\sqrt[3]{2}$	1	(b, c, a)	$(1, 2, 0), (1, -2, 0)$
$(5 + 8a, 2 + 4b, 4c)$	$2 - \sqrt[3]{4}$	$(1 + \sqrt[3]{2} + \sqrt[3]{4})^2$	$-3 - 6a + 3b + \frac{c}{2}, -1 + a - 3b + \frac{3}{2}c, 4 + 6a + b - 3c$	$(133, 106, 84), (13, 10, 8)$

$D(4a + 1, 4b + 2, 4c + 1)$ -quadruples

w	μ	u	$\kappa = (\alpha, \beta, \gamma)$	exceptions
$(5 + 8a, 6 + 8b, 1 + 4c)$	2	$1 - \sqrt[3]{2}$	$(8 + 5a + 6b + 2c, 7 + 4a + 5b + \frac{3}{2}c, 11 + 6a + 8b + \frac{5}{2}c)$	$(5, -10, 5), (-3, 6, -3)$
$(5 + 8a, 2 + 8b, 1 + 4c)$	2	$(1 - \sqrt[3]{2})^5$	$(269625 + 236845a + 298406b + 93992c, 214002 + 187984a + 236845b + \frac{149203}{2}c, 339707 + 298406a + 375968b + \frac{236845}{2}c)$	$(2605, -310, -1395), (-1563, 186, 837)$
$(1 + 8a, 6 + 8b, 1 + 4c)$	$2\sqrt[3]{2}$	$(1 + \sqrt[3]{2} + \sqrt[3]{4})^3$	$(-23 + 24a - \frac{35}{2}b + \frac{3}{2}c, 28 + 6a + 24b - \frac{35}{2}c, 6 - 35a + 3b + 12c)$	$(6529, 5182, 4113), (-4367, -3466, -2751)$
$(1 + 8a, 2 + 8b, 1 + 4c)$	$2\sqrt[3]{2}$	$1 - \sqrt[3]{2}$	$(2 + 4a + \frac{5}{2}b + \frac{3}{2}c, 4 + 6a + 4b + \frac{5}{2}c, 3 + 5a + 3b + 2c)$	$(9, 2, -7), (-7, -6, 9)$

$D(4a + 1, 4b + 2, 4c + 2)$ -quadruples

w	μ	u	$\kappa = (\alpha, \beta, \gamma)$	exceptions
$(1 + 8a, 2 + 4b, 2 + 4c)$	$\sqrt[3]{4} - \sqrt[3]{2}$	1	$(1 + 2a + b + 2c, 1 + 2a + b + c, 1 + a + b + c)$	$(1, 2, -2), (1, -2, 2)$
$(5 + 8a, 2 + 4b, 2 + 4c)$	$2 + \sqrt[3]{2} + \sqrt[3]{4}$	1	$(2a - b, b - c, -a + c)$	$(5, 2, 2), (-3, -2, -2)$

$D(4a + 1, 4b + 2, 4c + 3)$ -quadruples

$(a, b, c) \pmod{8}$	μ	u	$\kappa = (\alpha, \beta, \gamma)$	exceptions
$(5 + 8a, 2 + 4b, 3 + 4c)$	$2 - \sqrt[3]{4}$	$1 - \sqrt[3]{2}$	$36 + 24a + 15b + \frac{19}{2}c, 29 + 19a + 12b + \frac{15}{2}c, 46 + 30a + 19b + 12c$	$(13, -14, 3), (-11, 10, -1)$
$(1 + 8a, 2 + 4b, 3 + 4c)$	$\sqrt[3]{2}$	$1 + \sqrt[3]{2} + \sqrt[3]{4}$	$(1 - 4a + b + 2c, 2a - 2b + c, -1 + a + b - 2c)$	$(17, 14, 11), (-7, -6, -5)$

7. LIST OF TABLES FOR SOME $D(4a + 2b\sqrt[3]{2} + 2c\sqrt[3]{4})$ -QUADRUPLES

$D(8a, 4b, 4c)$ -quadruples

w	μ	u	$\kappa = (\alpha, \beta, \gamma)$	exceptions
$(8a, 8b, 8c)$	$1/2$	2	$(-1 + a, b, c)$	$(0, 0, 0)$ $(10, (8, 0, 0))$
$(8a, 4 + 8b, 8c)$	$(1 - \sqrt[3]{2})/2$	2	$(-1 - a - 2b - 2c, -a - b - 2c, -a - b - c)$	$(8, -4, 0), (0, 4, 0)$
$(8a, 8b, 4 + 8c)$	$(1 - \sqrt[3]{2})^2/2$	2	$(1 + 5a + 6b + 8c, 1 + 4a + 5b + 6c, 1 + 3a + 4b + 5c)$	$(8, -8, 4), (0, 8, -4)$
$(8a, 4 + 8b, 4 + 8c)$	$(1 - \sqrt[3]{2})^3/2$	2	$(-18 - 19a - 24b - 30c, -14 - 15a - 19b - 24c, -11 - 12a - 15b - 19c)$	$(0, -12, 12), (8, 12, -12)$

$D(4a, 4b + 2, 4c + 2)$ -quadruples

w	μ	u	$\kappa = (\alpha, \beta, \gamma)$	exceptions
$(8a, 2 + 8b, 2 + 4c)$	$1/\sqrt[3]{4}$	$\sqrt[3]{4}$	(c, a, b)	$(0, 2, 2), (0, 2, -2)$
$(4 + 8a, 2 + 8b, 2 + 4c)$	$(1 - \sqrt[3]{2})/\sqrt[3]{4}$	$\sqrt[3]{4}$	$(-2 - 2a - 2b - c, -1 - a - 2b - c, -1 - a - b - c)$	$(-4, 2, 2)$ $(4, 2, -2)$
$(8a, 6 + 8b, 2 + 4c)$	$(1 - \sqrt[3]{2})^2/\sqrt[3]{4}$	$\sqrt[3]{4}$	$6 + 6a + 8b + 5c, 5 + 5a + 6b + 4c, 4 + 4a + 5b + 3c)$	$(-8, 6, 2), (8, -2, -2)$
$(4 + 8a, 6 + 8b, 2 + 4c)$	$(1 - \sqrt[3]{2})^3/\sqrt[3]{4}$	$\sqrt[3]{4}$	$(-37 - 24a - 30b - 19c, -29 - 19a - 24b - 15c, -23 - 15a - 19b - 12c)$	$(-12, 14, -2), (12, -10, 2)$

$D(8a + 4, 8b + 4, 4c)$ -quadruples

w	μ	u	$\kappa = (\alpha, \beta, \gamma)$	exceptions
$(4 + 16a, 4 + 8b, 8c)$	$1/\sqrt[3]{4}$	2	(b, c, a)	$(4, 4, 0), (4, -4, 0)$
$(4 + 16a, 4 + 8b, 4 + 8c)$	$(1 - \sqrt[3]{2})/\sqrt[3]{4}$	2	$(-2 - 2a - b - 2c, -1 - 2a - b - c, -1 - a - b - c)$	$(4, 4, -4)$ $(4, -4, 4)$
$(12 + 16a, 4 + 8b, 8c)$	$(1 - \sqrt[3]{2})^2/\sqrt[3]{4}$	2	$(6 + 8a + 5b + 6c, 5 + 6a + 4b + 5c, 4 + 5a + 3b + 4c)$	$(12, 4, -8)$ $(-4, -4, 8)$
$(12 + 16a, 4 + 8b, 4 + 8c)$	$(1 - \sqrt[3]{2})^3/\sqrt[3]{4}$	2	$-37 - 30a - 19b - 24c, -29 - 24a - 15b - 19c, -23 - 19a - 12b - 15c)$	$(28, -4, -12)$ $(-20, 4, 12)$

REFERENCES

- [1] F. S. Abu Muriefah and A. Al-Rashed, "Some Diophantine quadruples in the ring $\mathbb{Z}[\sqrt{-2}]$," *Math. Commun.*, vol. 9, no. 1, pp. 1–8, 2004.
- [2] Ş. Alaca and K. S. Williams, *Introductory algebraic number theory*. Cambridge: Cambridge University Press, 2004.
- [3] A. Baker and H. Davenport, "The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$," *Q. J. Math., Oxf. II. Ser.*, vol. 20, pp. 129–137, 1969.
- [4] E. Brown, "Sets in which $xy + k$ is always a square," *Math. Comput.*, vol. 45, pp. 613–620, 1985.
- [5] A. Dujella, "Generalization of a problem of Diophantus," *Acta Arith.*, vol. 65, no. 1, pp. 15–27, 1993.
- [6] A. Dujella, "Some polynomial formulas for diophantine quadruples," *Grazer Math. Ber.*, vol. 328, pp. 25–30, 1996.
- [7] A. Dujella, "The problem of Diophantus and Davenport for Gaussian integers," *Glas. Mat., III. Ser.*, vol. 32, no. 1, pp. 1–10, 1997.
- [8] A. Dujella, "There are only finitely many Diophantine quintuples," *J. Reine Angew. Math.*, vol. 566, pp. 183–214, 2004.
- [9] A. Dujella and I. Soldo, "Diophantine quadruples in $\mathbb{Z}[\sqrt{-2}]$," *An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat.*, vol. 18, no. 1, pp. 81–98, 2010.
- [10] Z. Franušić, "Diophantine quadruples in the ring $\mathbb{Z}[\sqrt{s}]$," *Math. Commun.*, vol. 9, no. 2, pp. 141–148, 2004.
- [11] Z. Franušić, "Diophantine quadruples in $\mathbb{Z}[\sqrt{4k+3}]$," *Ramanujan J.*, vol. 17, no. 1, pp. 77–88, 2008.
- [12] Z. Franušić, "A diophantine problem in $\mathbb{Z}\left[(1 + \sqrt{d})/2\right]$," *Stud. Sci. Math. Hung.*, vol. 46, no. 1, pp. 103–112, 2009.
- [13] H. Gupta and K. Singh, "On k-triad sequences," *Int. J. Math. Math. Sci.*, vol. 8, pp. 799–804, 1985.
- [14] L. J. Matic, "Non-existence of certain Diophantine quadruples in rings of integers of pure cubic fields," *Proc. Japan Acad., Ser. A*, vol. 88, no. 10, pp. 163–167, 2012.
- [15] S. P. Mohanty and A. M. S. Ramasamy, "On $P_{r,k}$ sequences," *Fibonacci Q.*, vol. 23, pp. 36–44, 1985.
- [16] I. Soldo, "On the existence of Diophantine quadruples in $\mathbb{Z}[\sqrt{-2}]$," *Miskolc Math. Notes*, vol. 14, no. 1, pp. 265–277, 2013.

Author's address

Zrinka Franušić

University of Zagreb, Department of Mathematics, Bijenička 30, 10000 Zagreb, Croatia

E-mail address: fran@math.hr



STRONG CONVERGENCE OF AN EXPLICIT ITERATION PROCESS FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN CONVEX METRIC SPACES

BIROL GUNDUZ AND SEZGIN AKBULUT

Received 30 November, 2012

Abstract. In this paper, we study strong convergence of an iteration process for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces. We prove some strong convergence theorems for this iterative algorithm. Our results improve and extend the corresponding results of Kettapun et al. [2].

2010 *Mathematics Subject Classification:* 47H05; 47H09; 47H10

Keywords: iteration process, convex metric spaces, asymptotically quasi-nonexpansive mappings, common fixed point, strong convergence.

1. INTRODUCTION AND BASIC DEFINITIONS

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers and $I = \{1, 2, \dots, k\}$, the set of first k natural numbers. Denote by $F(T)$ the set of fixed points of T and by $F := \bigcap_{i=1}^k F(T_i)$ the set of common fixed points of a finite family $\{T_i : i \in I\}$.

Takahashi [10] introduced initially a notion of convex metric space and studied the fixed point theory for nonexpansive mappings in such a setting. Later on, many authors discussed the existence of the fixed point and the convergence of the iterative processes for various mappings in convex metric spaces.

We recall some definitions in a metric space (X, d) .

Definition 1. Let (X, d) be a metric space and $T : X \rightarrow X$ be a mapping. The map T is said to be:

- (1) Nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$.
- (2) Quasi-nonexpansive if $d(Tx, p) \leq d(x, p)$ for all $x \in X$ and $p \in F(T)$.
- (3) Asymptotically nonexpansive [1] if there exists $k_n \in [0, \infty)$ for all $n \in \mathbb{N}$ with $\lim_{n \rightarrow \infty} k_n = 0$ such that $d(T^n x, T^n y) \leq (1 + k_n) d(x, y) \forall x, y \in X$.
- (4) Asymptotically quasi-nonexpansive if there exists $k_n \in [0, \infty)$ for all $n \in \mathbb{N}$ with $\lim_{n \rightarrow \infty} k_n = 0$ such that $d(T^n x, p) \leq (1 + k_n) d(x, p) \forall x \in X, \forall p \in F(T)$.

From above definitions, the following diagram of implications holds and none of arrows is reversible (see, for example, [1, 2, 6, 12]).

$$\begin{array}{ccc} \text{Nonexpansive} & \implies & \text{Quasi-nonexpansive} \\ \Downarrow & & \Downarrow \\ \text{Asymptotically nonexpansive} & \implies & \text{Asymptotically} \\ & & \text{quasi-nonexpansive} \end{array}$$

Definition 2 ([10]). A convex structure in a metric space (X, d) is a mapping $W : X \times X \times [0, 1] \rightarrow X$ satisfying, for all $x, y, u \in X$ and all $\lambda \in [0, 1]$,

$$d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y).$$

A metric space together with a convex structure is called a convex metric space.

A nonempty subset C of X is said to be convex if $W(x, y; \lambda) \in C$ for all $(x, y; \lambda) \in C \times C \times [0, 1]$.

Definition 3 ([3]). Let (X, d) be a metric space and q be a fixed element of X . A q -starshaped structure in X is a mapping $W : X \times X \times [0, 1] \rightarrow X$ satisfying, for all $x, y \in X$ and all $\lambda \in [0, 1]$,

$$d(q, W(x, y; \lambda)) \leq \lambda d(q, x) + (1 - \lambda) d(q, y).$$

A metric space together with a q -starshaped structure is called a q -starshaped metric space.

Clearly, a convex metric space is a q -starshaped metric space but the converse is not true in general.

The convex metric space is a more general space and each normed linear space is a special example of a convex metric space. But there are many examples of convex metric spaces which are not embedded in any normed linear space.

Example 1 ([7]). Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$. For each $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$ and $\lambda \in [0, 1]$, we define a mapping $W : X \times X \times [0, 1] \rightarrow X$ by

$$W(x, y; \lambda) = \left(\lambda x_1 + (1 - \lambda) y_1, \frac{\lambda x_1 x_2 + (1 - \lambda) y_1 y_2}{\lambda x_1 + (1 - \lambda) y_1} \right)$$

and define a metric $d : X \times X \rightarrow [0, \infty)$ by

$$d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|.$$

Then (X, d, W) is a convex metric space, but it is not a normed space.

In 2008, Khan et al. [4] introduced an iterative process for a finite family of mappings as follows:

Let C be a convex subset of a Banach space X and let $\{T_i : i \in I\}$ be a family of self-mappings of C . Suppose that $\alpha_{in} \in [0, 1]$, for all $n = 1, 2, \dots$ and $i = 1, 2, \dots, k$.

For $x_1 \in C$, let $\{x_n\}$ be the sequence generated by the following algorithm:

$$x_{n+1} = (1 - \alpha_{kn}) x_n + \alpha_{kn} T_k^n y_{(k-1)n},$$

$$\begin{aligned}
y_{(k-1)n} &= (1 - \alpha_{(k-1)n})x_n + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n}, \\
y_{(k-2)n} &= (1 - \alpha_{(k-2)n})x_n + \alpha_{(k-2)n}T_{k-2}^n y_{(k-3)n}, \\
&\vdots \\
y_{2n} &= (1 - \alpha_{2n})x_n + \alpha_{2n}T_2^n y_{1n} \\
y_{1n} &= (1 - \alpha_{1n})x_n + \alpha_{1n}T_1^n y_{0n},
\end{aligned} \tag{1.1}$$

where $y_{0n} = x_n$ for all n . The iterative process (1.1) is the generalized form of the modified Mann (one-step) iterative process by Schu [8], the modified Ishikawa (two-step) iterative process by Tan and Xu [11], and the three-step iterative process by Xu and Noor [13].

In [3], Khan and Ahmed transformed this process into convex metric spaces.

Recently, Kettapun et al. [2] introduced a new iteration process for a finite family of mappings as follows:

$$\begin{aligned}
x_1 \in C, \quad x_{n+1} &= (1 - \alpha_{kn})y_{(k-1)n} + \alpha_{kn}T_k^n y_{(k-1)n}, \\
y_{(k-1)n} &= (1 - \alpha_{(k-1)n})y_{(k-2)n} + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n}, \\
y_{(k-2)n} &= (1 - \alpha_{(k-2)n})y_{(k-3)n} + \alpha_{(k-2)n}T_{k-2}^n y_{(k-3)n}, \\
&\vdots \\
y_{2n} &= (1 - \alpha_{2n})y_{1n} + \alpha_{2n}T_2^n y_{1n}, \\
y_{1n} &= (1 - \alpha_{1n})y_{0n} + \alpha_{1n}T_1^n y_{0n},
\end{aligned} \tag{1.2}$$

where $y_{0n} = x_n$ for all n .

Equivalence of the iteration schemes (1.1) and (1.2) is a problem worth investigating, assuming that the same point $x_1 \in C$ is initiated for various mappings.

Now, we transform iteration process (1.2) to the case of a family of asymptotically quasi-nonexpansive mappings in convex metric spaces as follows:

Definition 4. Let (X, d) be a convex metric space with convex structure W and $T_i : X \rightarrow X$ be a finite family of asymptotically quasi-nonexpansive mappings. Suppose that $\alpha_{in} \in [0, 1]$, for all $n = 1, 2, \dots$ and $i = 1, 2, \dots, k$. For any given $x_1 \in X$, we define iteration process $\{x_n\}$ as follows.

$$\begin{aligned}
x_{n+1} &= W(T_k^n y_{(k-1)n}, y_{(k-1)n}, \alpha_{kn}), \\
y_{(k-1)n} &= W(T_{k-1}^n y_{(k-2)n}, y_{(k-2)n}, \alpha_{(k-1)n}), \\
y_{(k-2)n} &= W(T_{k-2}^n y_{(k-3)n}, y_{(k-3)n}, \alpha_{(k-2)n}), \\
&\vdots \\
y_{2n} &= W(T_2^n y_{1n}, y_{1n}, \alpha_{2n}), \\
y_{1n} &= W(T_1^n y_{0n}, y_{0n}, \alpha_{1n}),
\end{aligned} \tag{1.3}$$

where $y_{0n} = x_n$ for all n .

Lemma 1 ([5]). *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative sequences satisfying*

$$\sum_{n=0}^{\infty} b_n < \infty, \sum_{n=0}^{\infty} c_n < \infty, a_{n+1} = (1 + b_n)a_n + c_n, n \geq 0.$$

Then

- i) $\lim_{n \rightarrow \infty} a_n$ exists,*
- ii) if $\liminf_{n \rightarrow \infty} a_n = 0$ then $\lim_{n \rightarrow \infty} a_n = 0$.*

Remark 1. It is easy to verify that Lemma 1 (ii) holds under the hypothesis $\limsup_{n \rightarrow \infty} a_n = 0$ as well. Therefore, the condition (ii) in Lemma 1 can be reformulated as follows:

- ii)' if either $\liminf_{n \rightarrow \infty} a_n = 0$ or $\limsup_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.*

The following proposition was proved by Khan and Ahmed [3] for one family of asymptotically quasi-nonexpansive mappings.

Proposition 1 ([3]). *Let X be a convex metric space and $T_i : X \rightarrow X$ be a finite family of asymptotically quasi-nonexpansive mappings with $F := \bigcap_{i=1}^k F(T_i)$. Then, there exist a point $p \in F$ and a sequence $\{r_n\} \subset [0, \infty)$ with $\lim_{n \rightarrow \infty} r_n = 0$ such that*

$$d(T_i^n x, p) \leq (1 + r_n)d(x, p),$$

for all $x \in K$, for each $i \in I$.

2. MAIN RESULTS

We begin with a technical result.

Lemma 2. *Let (X, d, W) be a convex metric space with convex structure W and $T_i : X \rightarrow X$ be a finite family of asymptotically quasi-nonexpansive mappings with $F \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty} r_n < \infty$ and $\{x_n\}$ is as in (1.3). Then*

- (i) $d(x_{n+1}, p) \leq (1 + r_n)^k d(x_n, p)$, for all $p \in F$ and for each $n \in N$,*
- (ii) there exists a constant $M > 0$ such that, for all $n, m \in N$ and for every $p \in F$, $d(x_{n+1}, p) \leq M d(x_n, p)$.*

Proof. (i) For all $p \in F$, we have from Proposition 1 that

$$\begin{aligned} d(y_{1n}, p) &= d(W(T_1^n y_{0n}, y_{0n}, \alpha_{1n}), p) \leq \alpha_{1n} d(T_1^n y_{0n}, p) + (1 - \alpha_{1n}) d(y_{0n}, p) \\ &\leq \alpha_{1n} (1 + r_n) d(x_n, p) + (1 - \alpha_{1n}) d(x_n, p) \\ &\leq (1 + \alpha_{1n} r_n) d(x_n, p) \\ &\leq (1 + r_n) d(x_n, p). \end{aligned}$$

Assume that $d(y_{jn}, p) \leq (1 + r_n)^j d(x_n, p)$ holds for some $1 \leq j \leq k - 2$. Then

$$\begin{aligned} d(y_{(j+1)n}, p) &= d\left(W\left(T_{j+1}^n y_{jn}, y_{jn}, \alpha_{(j+1)n}\right), p\right) \\ &\leq \alpha_{(j+1)n} d\left(T_{j+1}^n y_{jn}, p\right) + (1 - \alpha_{(j+1)n}) d(y_{jn}, p) \\ &\leq \alpha_{(j+1)n} (1 + r_n) d(y_{jn}, p) + (1 - \alpha_{(j+1)n}) d(y_{jn}, p) \\ &\leq (1 + r_n) d(y_{jn}, p) \\ &\leq (1 + r_n) (1 + r_n)^j d(x_n, p) \\ &= (1 + r_n)^{j+1} d(x_n, p). \end{aligned}$$

Therefore, by mathematical induction, we obtain

$$d(y_{in}, p) \leq (1 + r_n)^i d(x_n, p), \text{ for } i = 1, 2, \dots, k - 1. \quad (2.1)$$

Now, we have from (2.1) that

$$\begin{aligned} d(x_{n+1}, p) &= d\left(W\left(T_k^n y_{(k-1)n}, y_{(k-1)n}, \alpha_{kn}\right), p\right) \\ &\leq \alpha_{kn} d\left(T_k^n y_{(k-1)n}, p\right) + (1 - \alpha_{kn}) d(y_{(k-1)n}, p) \\ &\leq \alpha_{kn} (1 + r_n) d(y_{(k-1)n}, p) + (1 - \alpha_{kn}) d(y_{(k-1)n}, p) \\ &\leq (1 + r_n) d(y_{(k-1)n}, p) \\ &\leq (1 + r_n) (1 + r_n)^{k-1} d(x_n, p) \\ &= (1 + r_n)^k d(x_n, p). \end{aligned}$$

(ii) If $t \geq 0$, then $1 + t \leq e^t$ and so, $(1 + t)^k \leq e^{kt}$, $k = 1, 2, \dots$. Thus, from part (i), we get

$$\begin{aligned} d(x_{n+m}, p) &\leq (1 + r_{n+m-1})^k d(x_{n+m-1}, p) \\ &\leq \exp\{kr_{n+m-1}\} d(x_{n+m-1}, p) \leq \dots \leq \exp\left\{k \sum_{i=1}^{n+m-1} r_i\right\} d(x_n, p) \\ &\leq \exp\left\{k \sum_{i=1}^{\infty} r_i\right\} d(x_n, p). \end{aligned}$$

Setting $M = \exp\left\{k \sum_{i=1}^{\infty} r_i\right\} < \infty$, completes the proof. $\square$

Lemma 3. Let (X, d, W) be a convex metric space with convex structure W and $T_i : X \rightarrow X$ be a finite family of asymptotically quasi-nonexpansive mappings with $F \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty} r_n < \infty$ and $\{x_n\}$ is as in (1.3). If $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ where $d(x, F) = \inf\{d(x, p) : p \in F\}$, then $\{x_n\}$ is a Cauchy sequence.

Proof. We use Lemma 2 (ii) to prove that $\{x_n\}$ is a Cauchy sequence. From $\lim_{n \rightarrow \infty} d(x_n, F) = 0$, for each $\varepsilon > 0$ there exists $n_1 \in N$ such that

$$d(x_n, F) < \frac{\varepsilon}{M+1} \quad \forall n \geq n_1.$$

Thus, there exists $q \in F$ such that

$$d(x_n, q) < \frac{\varepsilon}{M+1} \quad \forall n \geq n_1. \quad (2.2)$$

Using Lemma 2 (ii) and (2.2), we obtain

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, q) + d(x_n, q) \leq M d(x_n, q) + d(x_n, q) \\ &= (M+1) d(x_n, q) \\ &< (M+1) \left(\frac{\varepsilon}{M+1} \right) = \varepsilon \end{aligned}$$

for all $n, m \geq n_1$. Therefore $\{x_n\}$ is a Cauchy sequence. $\square$

We now state and prove the main theorem of this section.

Theorem 1. *Let (X, d, W) be a convex metric space with convex structure W and $T_i : X \rightarrow X$ be a finite family of asymptotically quasi-nonexpansive mappings with $F \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty} r_n < \infty$ and $\{x_n\}$ is as in (1.3). Then if $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ where $d(x, F) = \inf\{d(x, p) : p \in F\}$, then $\{x_n\}$ is a Cauchy sequence.*

(i) $\liminf_{n \rightarrow \infty} d(x_n, F) = \limsup_{n \rightarrow \infty} d(x_n, F) = 0$ if converges to a unique point in F .

(ii) $\{x_n\}$ converges to a unique point in F if X is complete and either $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$ or $\limsup_{n \rightarrow \infty} d(x_n, F) = 0$.

Proof. (i) Let $p \in F$. Since $\{x_n\}$ converges to p , $\lim_{n \rightarrow \infty} d(x_n, p) = 0$. So, for a given $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$d(x_n, p) < \varepsilon \quad \forall n \geq n_0.$$

Taking infimum over $p \in F$, we have

$$d(x_n, F) < \varepsilon \quad \forall n \geq n_0.$$

This means $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ so that

$$\lim_{n \rightarrow \infty} \inf d(x_n, F) = \lim_{n \rightarrow \infty} \sup d(x_n, F) = 0.$$

(ii) Suppose that X is complete and $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$ or $\limsup_{n \rightarrow \infty} d(x_n, F) = 0$. Then, we have from Lemma 1 (ii) and Remark 1 that $\lim_{n \rightarrow \infty} d(x_n, F) = 0$. From the completeness of X and Theorem 3, we get that $\lim_{n \rightarrow \infty} x_n$ exists and equals $q \in X$, say. Moreover, since the set F of fixed points of

asymptotically quasi-nonexpansive mappings is closed $q \in F$ from $\lim_{n \rightarrow \infty} d(x_n, F) = 0$. That is, $q \in F$. Hence $\{x_n\}$ converges to a unique point in F . $\square$

Remark 2. Since a Banach space and each of its convex subsets are convex metric spaces, so Theorem 1 reduces to Theorem 3.2 in [2]. Also, Corollary 3.3 of Kettapun, Kananthai and Suantai [2] is special case of Theorem 1.

3. APPLICATIONS

Now, we give a couple of applications of Theorem 1.

Theorem 2. Let (X, d, W) be a complete convex metric space with convex structure W and $T_i : X \rightarrow X$ be a finite family of asymptotically quasi-nonexpansive mappings with $F \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty} r_n < \infty$ and $\{x_n\}$ is as in (1.3). Assume that the following two conditions hold.

$$i) \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0, \quad (3.1)$$

ii) the sequence $\{y_n\}$ in X satisfying $\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = 0$ implies

$$\liminf_{n \rightarrow \infty} d(y_n, F) = 0 \text{ or } \limsup_{n \rightarrow \infty} d(y_n, F) = 0. \quad (3.2)$$

Then $\{x_n\}$ converges to a unique point in F .

Proof. From (3.1) and (3.2), we have that

$$\liminf_{n \rightarrow \infty} d(x_n, F) = 0 \text{ or } \limsup_{n \rightarrow \infty} d(x_n, F) = 0.$$

Therefore, we obtain from Theorem 1 (ii) that the sequence $\{x_n\}$ converges to a unique point in F . $\square$

Theorem 3. Let (X, d, W) be a complete convex metric space with convex structure W and $T_i : X \rightarrow X$ be a finite family of asymptotically quasi-nonexpansive mappings satisfying $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$ with $F \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty} r_n < \infty$ and $\{x_n\}$ is as in (1.3). If one of the following is true, then the sequence $\{x_n\}$ converges to a unique point in F .

i) If there exists a nondecreasing function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$, $g(t) > 0$ for all $t \in (0, \infty)$ such that $d(x_n, T_i x_n) \geq g(d(x_n, F))$ for all $n \geq 1$ (see Senter and Dotson [9])

ii) There exists a function $f : [0, \infty) \rightarrow [0, \infty)$ which is right continuous at 0, $f(0) = 0$ and $f(d(x_n, T_i x_n)) \geq d(x_n, F)$ for all $n \geq 1$.

Proof. First suppose that (i) holds. Then

$$\lim_{n \rightarrow \infty} g(d(x_n, F)) \leq \lim_{n \rightarrow \infty} d(x_n, T_i x_n).$$

So, $\lim_{n \rightarrow \infty} g(d(x_n, F)) = 0$; and properties of g imply $\lim_{n \rightarrow \infty} d(x_n, F) = 0$.

Now all the conditions of Theorem 1 are satisfied, therefore by its conclusion $\{x_n\}$ converges to a point of F .

Next, assume (ii). Then

$$\lim_{n \rightarrow \infty} d(x_n, F) \leq \lim_{n \rightarrow \infty} f(d(x_n, T_i x_n)) = f\left(\lim_{n \rightarrow \infty} d(x_n, T_i x_n)\right) = f(0) = 0.$$

Again in this case, $\lim_{n \rightarrow \infty} d(x_n, F) = 0$. Thus $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$ or $\limsup_{n \rightarrow \infty} d(x_n, F) = 0$. By Theorem 1, $\{x_n\}$ converges to a point of F . $\square$

Remark 3. In the Banach space setting, Theorem 3 reduces to the improvement of Theorem 4.1 in [2].

Remark 4. We can prove all the results obtained so far in the context of a q -starshaped metric space with suitable changes. We leave the details to the reader.

REFERENCES

- [1] K. Goebel and W. A. Kirk, "A fixed point theorem for asymptotically nonexpansive mappings," *Proc. Am. Math. Soc.*, vol. 35, pp. 171–174, 1972.
- [2] A. Kettapun, A. Kananthai, and S. Suantai, "A new approximation method for common fixed points of a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces," *Comput. Math. Appl.*, vol. 60, no. 5, pp. 1430–1439, 2010.
- [3] A. R. Khan and M. A. Ahmed, "Convergence of a general iterative scheme for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces and applications," *Comput. Math. Appl.*, vol. 59, no. 8, pp. 2990–2995, 2010.
- [4] A. R. Khan, A.-A. Domlo, and H. Fukhar-Ud-Din, "Common fixed points Noor iteration for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces," *J. Math. Anal. Appl.*, vol. 341, no. 1, pp. 1–11, 2008.
- [5] Q. Liu, "Iterative sequences for asymptotically quasi-nonexpansive mappings with error member," *J. Math. Anal. Appl.*, vol. 259, no. 1, pp. 18–24, 2001.
- [6] W. V. Petryshyn and T. E. jun. Williamson, "Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings," *J. Math. Anal. Appl.*, vol. 43, pp. 459–497, 1973.
- [7] G. S. Saluja and H. K. Nashine, "Convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces," *Opusc. Math.*, vol. 30, no. 3, pp. 331–340, 2010.
- [8] J. Schu, "Weak and strong convergence to fixed points of asymptotically nonexpansive mappings," *Bull. Aust. Math. Soc.*, vol. 43, no. 1, pp. 153–159, 1991.
- [9] H. F. Senter and W. G. jun. Dotson, "Approximating fixed points of nonexpansive mappings," *Proc. Am. Math. Soc.*, vol. 44, pp. 375–380, 1974.
- [10] W. Takahashi, "A convexity in metric space and nonexpansive mappings. I," *Kōdai Math. Semin. Rep.*, vol. 22, pp. 142–149, 1970.
- [11] K.-K. Tan and H.-K. Xu, "Fixed point iteration processes for asymptotically nonexpansive mappings," *Proc. Am. Math. Soc.*, vol. 122, no. 3, pp. 733–739, 1994.
- [12] Y.-X. Tian, "Convergence of an Ishikawa type iterative scheme for asymptotically quasi-nonexpansive mappings," *Comput. Math. Appl.*, vol. 49, no. 11–12, pp. 1905–1912, 2005.
- [13] B. Xu and M. Aslam Noor, "Fixed-point iterations for asymptotically nonexpansive mappings in Banach spaces," *J. Math. Anal. Appl.*, vol. 267, no. 2, pp. 444–453, 2002.

*Authors' addresses***Birol Gunduz**

E-mail address: birolgndz@gmail.com

Department of Mathematics, Faculty of Science and Art, Erzincan University, Erzincan, 24000, Turkey.

Sezgin Akbulut

E-mail address: sezginakbulut@atauni.edu.tr

Department of Mathematics, Faculty of Science, Ataturk University, Erzurum, 25240, Turkey.



STRONG AND Δ -CONVERGENCE THEOREMS IN HYPERBOLIC SPACES

BIROL GUNDUZ AND SEZGIN AKBULUT

Received 6 June, 2013

Abstract. In this paper, we prove some strong and Δ -convergence theorems for two finite families of nonexpansive maps on a hyperbolic space. Our iterative scheme is independent and simpler than the Ishikawa type iteration process. Our results refine and generalize several recent and comparable results in uniformly convex Banach spaces as well as $CAT(0)$ spaces.

2010 Mathematics Subject Classification: 47H10; 49M05

Keywords: Kohlenbach hyperbolic space, Δ -convergence, strong convergence, nonexpansive map, common fixed point

1. INTRODUCTION

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers and $I = \{1, 2, \dots, N\}$, the set of first N natural numbers. Let (X, d) be a metric space and K be a nonempty subset of X . A selfmap T on K is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$. Denote by $F(T)$ the set of fixed points of T and by $F = \bigcap_{i=1}^N (F(T_i) \cap F(S_i))$ the set of common fixed points of two finite families of mappings $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$.

Given x_0 in K (a subset of Banach space), Mann and Ishikawa iteration schemes for a mapping $T : K \rightarrow K$ read as follows:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad (1.1)$$

and

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \end{aligned} \quad (1.2)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$.

The concept of Δ -convergence in general metric spaces was coined by Lim [18]. Kirk and Panyanak [14] specialized this concept to $CAT(0)$ spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting. Dhompongsa and Panyanak [5] continued to work in this direction and studied the Δ -convergence of Picard, Mann and Ishikawa iterates in $CAT(0)$ spaces

(Theorems 3.1, 3.2 and 3.3 respectively in [5]). While acknowledging their contribution, we note that their schemes involve one mapping. The case of two mappings in iteration schemes has also remained under study since Das and Debata [4] gave and studied a two mappings scheme on the pattern of the Ishikawa scheme:

$$\begin{aligned}x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T x_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n S x_n, \quad n \geq 1.\end{aligned}\tag{1.3}$$

This scheme reduces to Ishikawa's scheme when $S = T$, and to Mann's Iteration scheme when $S = I$. In 2011, Khan and Abbas [11] obtained some results for two mappings using the Ishikawa-type iteration scheme (1.3) in $CAT(0)$ spaces. Note that the two mappings case, that is, approximating the common fixed points, has its own importance as it has a direct link with the minimization problem, see for example Takahashi [22].

On the other hand, let us state without error terms the iteration scheme studied by Yao and Chen [23] for common fixed points of two mappings:

$$x_{n+1} = \alpha_n x_n + \beta_n T x_n + \gamma_n S x_n, \tag{1.4}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are in $(0, 1)$ and $\alpha_n + \beta_n + \gamma_n = 1$.

We, in this paper, translate the iteration scheme (1.4) for two finite families of nonexpansive mappings in a subset K of a hyperbolic space X as under:

$$x_{n+1} = W\left(T_n x_n, W\left(x_n, S_n x_n, \frac{\beta_n}{1 - \alpha_n}\right), \alpha_n\right) \tag{1.5}$$

where $T_n = T_n \pmod{N}$ and $S_n = S_n \pmod{N}$, $0 < a \leq \alpha_n, \beta_n \leq b < 1$ and satisfy $\alpha_n + \beta_n < 1$.

Remark 1. It is worth mentioning that the iteration scheme (1.5) coincide with the iteration scheme (1.4) of Yao and Chen [23] when $W(x, y, \alpha) = \alpha x + (1 - \alpha)y$, $S_i = S$ and $T_i = T$ for all $i \in I$, and X is a Banach space. Moreover, it provides iteration scheme of Abbas and Khan [1] in a $CAT(0)$ space if $W(x, y, \alpha) = \alpha x \oplus (1 - \alpha)y$ and $S_i = S$, $T_i = T$ for all $i \in I$.

Remark 2. Note that this scheme reduces to Mann's iteration scheme when either $S_i = T$ or $S_i = I$ or $T_i = I$ for all $i \in I$, and is independent and simpler than both the Ishikawa scheme and (1.3).

The aim of this article is to analyze Δ -convergence as well as strong convergence through algorithm (1.5) for two finite families of nonexpansive maps in the more general setup of hyperbolic spaces. Our results can be viewed as refinement and generalization of several well-known results in $CAT(0)$ spaces and uniformly convex Banach spaces.

2. HYPERBOLIC SPACE

We work in the setting of hyperbolic spaces as introduced by Kohlenbach [15]. It is worth noting that they are different from Gromov hyperbolic spaces [3] or from other notions of hyperbolic space that can be found in the literature (see for example [7, 13, 19]).

A hyperbolic space [15] is a triple (X, d, W) where (X, d) is a metric space and $W : X^2 \times [0, 1] \rightarrow X$ is such that

- W1. $d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha) d(u, y)$
- W2. $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$
- W3. $W(x, y, \alpha) = W(y, x, (1 - \alpha))$
- W4. $d(W(x, z, \alpha), W(y, w, \alpha)) \leq \alpha d(x, y) + (1 - \alpha) d(z, w)$

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$.

The convexity mapping W was first considered by Takahashi in [21], where a triple (X, d, W) satisfying (W1) is called a convex metric space (see, more details [8]). If (X, d, W) satisfies (W1)-(W3), then we obtain the notion of space of hyperbolic type in the sense of Goebel and Kirk [7]. (W4) was already considered by Itoh [9] under the name "condition III" and it is used by Reich and Shafrir [19] and Kirk [13] to define their notions of hyperbolic space. Hyperbolic spaces are a natural generalization of both uniformly convex normed spaces and CAT(0) spaces.

A hyperbolic space (X, d, W) is said to be uniformly convex [20] if for all $u, x, y \in X$, $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that

$$\left. \begin{array}{l} d(x, u) \leq r \\ d(y, u) \leq r \\ d(x, y) \geq \varepsilon r \end{array} \right\} \Rightarrow d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq (1 - \delta)r.$$

A map $\eta : (0, \infty) \times (0, 2] \rightarrow (0, 1]$ which provides such a $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$, is called modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε).

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X . For $x \in X$, define a continuous functional $r(\cdot, \{x_n\}) : X \rightarrow [0, \infty)$ by $r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n)$. The asymptotic radius $\rho = r(\{x_n\})$ of $\{x_n\}$ is given by $\rho = \inf\{r(x, \{x_n\}) : x \in X\}$. The asymptotic center of a bounded sequence $\{x_n\}$ with respect to a subset K of X is defined as follows:

$$A_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \leq r(y, \{x_n\}) \text{ for any } y \in K\}.$$

The set of all asymptotic centers of $\{x_n\}$ is denoted by $A(\{x_n\})$.

It has been shown in [17] that bounded sequences have unique asymptotic center with respect to closed convex subsets in a complete and uniformly convex hyperbolic space with monotone modulus of uniform convexity.

A sequence $\{x_n\}$ in X is said to Δ -converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$ [14]. In this case, we write $\Delta\text{-}\lim_n x_n = x$.

Recall that Δ -convergence coincides with weak convergence in Banach spaces with Opial's property [16].

The proofs of the following two lemmas can be found in Khan et al. [10].

Lemma 1 ([10]). *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η . Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n \rightarrow \infty} d(x_n, x) \leq r$, $\limsup_{n \rightarrow \infty} d(y_n, x) \leq r$ and $\lim_{n \rightarrow \infty} d(W(x_n, y_n, \alpha_n), x) = r$ for some $r \geq 0$, then $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$.*

Lemma 2 ([10]). *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in K such that $\lim_{m \rightarrow \infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m \rightarrow \infty} y_m = y$.*

3. STRONG AND Δ -CONVERGENCE THEOREMS IN HYPERBOLIC SPACES

Lemma 3. *Let K be a closed and convex subset of a hyperbolic space X and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on K such that $F \neq \emptyset$. Then for the sequence $\{x_n\}$ in (1.5), we have $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for each $p \in F$.*

Proof. Let $p \in F$. It follows from (1.5) that

$$\begin{aligned} d(x_{n+1}, p) &= d\left(W\left(T_n x_n, W\left(x_n, S_n x_n, \frac{\beta_n}{1 - \alpha_n}\right), \alpha_n\right), p\right) \\ &\leq \alpha_n d(T_n x_n, p) + (1 - \alpha_n) d\left(W\left(x_n, S_n x_n, \frac{\beta_n}{1 - \alpha_n}\right), p\right) \\ &\leq \alpha_n d(x_n, p) + \beta_n d(x_n, p) + (1 - \alpha_n - \beta_n) d(S_n x_n, p) \\ &\leq \alpha_n d(x_n, p) + \beta_n d(x_n, p) + (1 - \alpha_n - \beta_n) d(x_n, p). \end{aligned}$$

That is

$$d(x_{n+1}, p) \leq d(x_n, p).$$

This gives that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for each $p \in F$. $\square$

We need the following lemma in order to prove some strong and Δ -convergence theorems.

Lemma 4. *Let K be a closed and convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $\{T_i : i \in I\}$ and*

$\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on K such that $F \neq \emptyset$. Then for the sequence $\{x_n\}$ in (1.5), we have

$$\lim_{n \rightarrow \infty} d(x_n, T_l x_n) = \lim_{n \rightarrow \infty} d(x_n, S_l x_n) = 0 \text{ for each } l \in I.$$

Proof. From Lemma 3, we know that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for each $p \in F$. We suppose that $\lim_{n \rightarrow \infty} d(x_n, p) = c$, that is,

$$\lim_{n \rightarrow \infty} d(x_{n+1}, p) = \lim_{n \rightarrow \infty} d\left(W\left(T_n x_n, W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) = c \quad (3.1)$$

Since T_n is nonexpansive, we have $d(T_n x_n, p) \leq d(x_n, p)$ for each $p \in F$. Taking $\limsup$ on both sides, we obtain

$$\limsup_{n \rightarrow \infty} d(T_n x_n, p) \leq c \quad (3.2)$$

Observe that

$$\begin{aligned} d\left(W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) &\leq \frac{\beta_n}{1-\alpha_n} d(x_n, p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(S_n x_n, p) \\ &\leq \frac{\beta_n}{1-\alpha_n} d(x_n, p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(x_n, p) \\ &= d(x_n, p) \end{aligned}$$

This gives

$$\limsup_{n \rightarrow \infty} d\left(W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \leq c. \quad (3.3)$$

By using (3.1)–(3.3), and Lemma 1, we get

$$\lim_{n \rightarrow \infty} d\left(T_n x_n, W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right)\right) = 0. \quad (3.4)$$

Now

$$\begin{aligned} d(x_{n+1}, T_n x_n) &= d\left(W\left(T_n x_n, W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), T_n x_n\right) \\ &\leq \alpha_n d(T_n x_n, T_n x_n) + (1-\alpha_n) d\left(T_n x_n, W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right)\right) \end{aligned}$$

gives by (3.4) that

$$\lim_{n \rightarrow \infty} d(x_{n+1}, T_n x_n) = 0 \quad (3.5)$$

Moreover,

$$\begin{aligned} d(x_{n+1}, p) &= d\left(W\left(T_n x_n, W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) \\ &\leq \alpha_n d(T_n x_n, p) + (1-\alpha_n) d\left(W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n d(x_n, p) + d\left(W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\
&\quad - \alpha_n d\left(W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\
&\leq \alpha_n d(x_n, p) + d\left(W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\
&\quad - \alpha_n \left[\frac{\beta_n}{1-\alpha_n} d(x_n, p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(S_n x_n, p) \right] \\
&\leq \alpha_n d(x_n, p) + d\left(W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\
&\quad - \alpha_n \left[\frac{\beta_n}{1-\alpha_n} d(x_n, p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(x_n, p) \right] \\
&= d\left(W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right)
\end{aligned}$$

yields that

$$c \leq \liminf_{n \rightarrow \infty} d\left(W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right). \quad (3.6)$$

Combining (3.3) and (3.6), we have

$$\lim_{n \rightarrow \infty} d\left(W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) = c \quad (3.7)$$

Again by Lemma 1 (with $\limsup_{n \rightarrow \infty} d(S_n x_n, p) \leq c$), we get

$$\lim_{n \rightarrow \infty} d(x_n, S_n x_n) = 0. \quad (3.8)$$

Furthermore,

$$d\left(W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), x_n\right) \leq \frac{\beta_n}{1-\alpha_n} d(x_n, x_n) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(x_n, S_n x_n)$$

gives by (3.8),

$$\lim_{n \rightarrow \infty} d\left(W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), x_n\right) = 0. \quad (3.9)$$

Consider

$$\begin{aligned}
d(x_{n+1}, x_n) &= d\left(W\left(T_n x_n, W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), x_n\right) \\
&\leq \alpha_n d(T_n x_n, x_n) + (1-\alpha_n) d\left(W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), x_n\right) \\
&\leq \alpha_n \{d(x_{n+1}, x_n) + d(T_n x_n, x_{n+1})\} \\
&\quad + (1-\alpha_n) d\left(W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), x_n\right).
\end{aligned}$$

From the above inequality, the following important inequality is obtained:

$$d(x_{n+1}, x_n) \leq \frac{b}{1-b} d(T_n x_n, x_{n+1}) + d\left(W\left(x_n, S_n x_n, \frac{\beta_n}{1-\alpha_n}\right), x_n\right).$$

Taking limsup on both sides in the above inequality and then using (3.5) and (3.9), we have

$$\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = 0$$

which implies that

$$\lim_{n \rightarrow \infty} d(x_{n+l}, x_n) = 0 \text{ for } l < N. \quad (3.10)$$

Since $d(x_n, T_n x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, T_n x_n)$, therefore it follows from (3.5) and (3.10) that

$$\lim_{n \rightarrow \infty} d(x_n, T_n x_n) = 0. \quad (3.11)$$

For each $l \in I$, we have

$$\begin{aligned} d(x_n, T_{n+l} x_n) &\leq d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l} x_{n+l}) + d(T_{n+l} x_{n+l}, T_{n+l} x_n) \\ &\leq d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l} x_{n+l}) + d(x_{n+l}, x_n) \\ &\leq 2d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l} x_{n+l}). \end{aligned}$$

Thus, from (3.10) and (3.11), we have

$$\lim_{n \rightarrow \infty} d(x_n, T_{n+l} x_n) = 0 \text{ for each } l \in I. \quad (3.12)$$

Since the sequence $\{d(x_n, T_l x_n)\}$ is a subsequence of $\bigcup_{i=1}^N \{d(x_n, T_{n+i} x_n)\}$ and $\lim_{n \rightarrow \infty} d(x_n, T_{n+l} x_n) = 0$ for each $l \in I$, therefore

$$\lim_{n \rightarrow \infty} d(x_n, T_l x_n) = 0 \text{ for each } l \in I. \quad (3.13)$$

For each $l \in I$, similar to the proof of (3.12), we have $\lim_{n \rightarrow \infty} d(x_n, S_{n+l} x_n) = 0$ for each $l \in I$. Consequently, we obtain

$$\lim_{n \rightarrow \infty} d(x_n, S_l x_n) = 0 \text{ for each } l \in I. \quad (3.14)$$

The proof is completed. $\square$

Now we give Δ -convergence theorems for two finite families of nonexpansive mappings.

Theorem 1. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on K such that $F \neq \emptyset$. Then the sequence $\{x_n\}$ defined in (1.5), Δ -converges to a common fixed point of $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$.*

Proof. It follows from Lemma 3 that $\{x_n\}$ is bounded. Therefore $\{x_n\}$ has a unique asymptotic center, that is, $A(\{x_n\}) = \{x\}$. Assume that $\{u_n\}$ is any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Then by Lemma 4, we have $\lim_{n \rightarrow \infty} d(u_n, T_l u_n) = \lim_{n \rightarrow \infty} d(u_n, S_l u_n) = 0$ for each $l = 1, 2, \dots, N$. Now we prove that u is the common fixed point of $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$.

Define a sequence $\{v_m\}$ in K by $v_m = T_m u$, where $T_m = T_{m \pmod N}$.

Clearly,

$$\begin{aligned} d(v_m, u_n) &\leq d(T_m u, T_m u_n) + d(T_m u_n, T_{m-1} u_n) + \dots + d(T u_n, u_n) \\ &\leq d(u, u_n) + \sum_{i=1}^{m-1} d(u_n, T_i u_n). \end{aligned}$$

Thus, we have

$$r(v_m, \{u_n\}) = \limsup_{n \rightarrow \infty} d(v_m, u_n) \leq \limsup_{n \rightarrow \infty} d(u, u_n) = r(u, \{u_n\}).$$

This implies that $|r(v_m, \{u_n\}) - r(u, \{u_n\})| \rightarrow 0$ as $m \rightarrow \infty$. By Lemma 2, we obtain $T_{m \pmod N} u = u$, which implies that u is the common fixed point of $\{T_i : i \in I\}$. Similarly, we can show that u is the common fixed point of $\{S_i : i \in I\}$. Therefore $u \in F$. Moreover, $\lim_{n \rightarrow \infty} d(x_n, u)$ exists by Lemma 3.

Assume $x \neq u$. By the uniqueness of asymptotic centers,

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(u_n, u) &< \limsup_{n \rightarrow \infty} d(u_n, x) \\ &\leq \limsup_{n \rightarrow \infty} d(x_n, x) \\ &< \limsup_{n \rightarrow \infty} d(x_n, u) \\ &= \limsup_{n \rightarrow \infty} d(u_n, u) \end{aligned}$$

a contradiction. Thus $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ Δ -converges to a common fixed point of $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$. $\square$

Khan and Fukhar-ud-din [12], introduced the so-called condition (A') and gave a slightly improved version of it in [6] as follows:

Two mappings $T, S : K \rightarrow K$ with $F \neq \emptyset$ are said to satisfy the Condition (A') [12] if there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$\text{either } d(x, Tx) \geq f(d(x, F)) \text{ or } d(x, Sx) \geq f(d(x, F))$$

for all $x \in K$, where $d(x, F) = \inf\{d(x, p) : p \in F\}$.

We can modify this definition for two finite families of mappings as follows. Let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive mappings of K with nonempty fixed points set F . These families are said to satisfy Condition (B)

on K if there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$\text{either } \max_{i \in I} d(x, T_i x) \geq f(d(x, F)) \text{ or } \max_{i \in I} d(x, S_i x) \geq f(d(x, F))$$

for all $x \in K$. The Condition (B) reduces to the Condition (A') when $T_1 = T_2 = \dots = T_N = T$ and $S_1 = S_2 = \dots = S_N = S$ for all $i \in I$.

A sequence $\{x_n\}$ in a metric space X is said to be Fejér monotone with respect to K (a subset of X) if $d(x_{n+1}, p) \leq d(x_n, p)$ for all $p \in K$ and for all $n \geq 1$.

For further development, we need the following technical result.

Lemma 5 ([2]). *Let K be a nonempty closed subset of a complete metric space (X, d) and $\{x_n\}$ be Fejér monotone with respect to K . Then $\{x_n\}$ converges to some $p \in K$ if and only if $\lim_{n \rightarrow \infty} d(x_n, K) = 0$.*

Lemma 6. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on K such that $F \neq \emptyset$. Then the sequence $\{x_n\}$ defined in (1.5) converges strongly to $p \in F$ if and only if $\lim_{n \rightarrow \infty} d(x_n, F) = 0$.*

Proof. It follows from Lemma 3 that $\{x_n\}$ is Fejér monotone with respect to F and $\lim_{n \rightarrow \infty} d(x_n, F)$ exists. Now applying the above Lemma, we obtain the result. $\square$

We use the Condition (B) to study strong convergence of $\{x_n\}$ defined in (1.5). It is worth noting that, in the case of two finite families of nonexpansive self mappings $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$, the Condition (B) is weaker than the compactness of K .

Theorem 2. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on K such that $F \neq \emptyset$. Suppose that $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ satisfy Condition (B). Then the sequence $\{x_n\}$ defined in (1.5) converges strongly to $p \in F$.*

Proof. By Lemma 3, $\lim_{n \rightarrow \infty} d(x_n, F)$ exists for all $p \in F$. Also, by Lemma 4, $\lim_{n \rightarrow \infty} d(x_n, T_l x_n) = \lim_{n \rightarrow \infty} d(x_n, S_l x_n) = 0$ for each $l \in I$. It follows from Condition (B) that $\lim_{n \rightarrow \infty} f(d(x_n, F)) = 0$. Since f is nondecreasing with $f(0) = 0$, we have $\lim_{n \rightarrow \infty} d(x_n, F) = 0$. Now, Theorem 6 implies that $\{x_n\}$ converges strongly to a point p in F . $\square$

Although the following are corollaries to our Theorems 1-2, yet they are new in the literature.

Theorem 3. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let T*

and S be two nonexpansive selfmaps on K such that $F := F(T) \cap F(S) \neq \emptyset$. Let the sequence $\{x_n\}$ defined by

$$x_{n+1} = W\left(Tx_n, W\left(x_n, Sx_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right) \quad (3.15)$$

Then $\{x_n\}$ Δ -converges to a common fixed point of $\{T_i : i \in I\}$.

Proof. Choose $S_i = S$ and $T_i = T$ for all $i \in I$ in the theorem 1. $\square$

Theorem 4. Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let T and S be two nonexpansive selfmaps satisfying the Condition (A') on K such that $F := F(T) \cap F(S) \neq \emptyset$. Then the sequence $\{x_n\}$ defined in (3.15) converges strongly to $p \in F$.

Remark 3. (1) Theorem 1 and Theorem 2 extends the corresponding results of Abbas and Khan [1] in two ways: (i) from two nonexpansive mappings to two finite families of nonexpansive mappings (ii) from $CAT(0)$ space to general setup of uniformly convex hyperbolic spaces.

(2) Theorem 1 sets analogue of [11, Theorem 1], for two finite families of nonexpansive maps on unbounded domain in a uniformly convex hyperbolic space X .

(3) Theorems 1–2 contain the corresponding theorems proved for the Mann iteration scheme when $S_i = Id$ or $T_i = Id$ (Id is the identity mapping) for all $i \in I$.

(4) In view of simplicity of the iterative process (1.5) as compared with (1.2) and (1.3), our results improve and generalize the results of Dhompongsa and Panyanak [5] and Khan and Abbas [11].

REFERENCES

- [1] M. Abbas and S. H. Khan, "Some δ -convergence theorems in $cat(0)$ spaces," *Hacet. J. Math. Stat.*, vol. 40, no. 4, pp. 563–569, 2011.
- [2] H. H. Bauschke and P. L. Combettes, *Convex analysis and monotone operator theory in Hilbert spaces*, ser. CMS Books in Mathematics. Berlin: Springer, 2011.
- [3] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, ser. Grundlehren der Mathematischen Wissenschaften. Berlin: Springer, 1999, vol. 319.
- [4] G. Das and J. P. Debata, "Fixed points of quasicontractive mappings," *Indian J. Pure Appl. Math.*, vol. 17, pp. 1263–1269, 1986.
- [5] S. Dhompongsa and B. Panyanak, "On Δ -convergence theorems in $CAT(0)$ spaces," *Comput. Math. Appl.*, vol. 56, no. 10, pp. 2572–2579, 2008.
- [6] H. Fukhar-Ud-Din and S. H. Khan, "Convergence of iterates with errors of asymptotically quasicontractive mappings and applications," *J. Math. Anal. Appl.*, vol. 328, no. 2, pp. 821–829, 2007.
- [7] K. Goebel and W. A. Kirk, "Iteration processes for nonexpansive mappings," *Contemp. Math.*, vol. 21, pp. 115–123, 1983.
- [8] B. Gunduz and S. Akbulut, "Strong convergence of an explicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces," *Miskolc Math. Notes*, Accepted.

- [9] S. Itoh, "Some fixed point theorems in metric spaces," *Fundam. Math.*, vol. 102, pp. 109–117, 1979.
- [10] A. R. Khan, H. Fukhar-ud-din, and M. A. A. Khan, "An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces," *Fixed Point Theory Appl.*, vol. 2012, p. 12, 2012.
- [11] S. H. Khan and M. Abbas, "Strong and Δ -convergence of some iterative schemes in CAT(0) spaces," *Comp. Math. Appl.*, vol. 61, pp. 109–116, 2011.
- [12] S. H. Khan and H. Fukhar-ud-din, "Weak and strong convergence of a scheme with errors for two nonexpansive mappings," *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 61, no. 8, pp. 1295–1301, 2005.
- [13] W. A. Kirk, "Krasnoselskii's iteration process in hyperbolic space," *Numer. Funct. Anal. Optim.*, vol. 4, pp. 371–381, 1982.
- [14] W. A. Kirk and B. Panyanak, "A concept of convergence in geodesic spaces," *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 68, no. 12, pp. 3689–3696, 2008.
- [15] U. Kohlenbach, "Some logical metatheorems with applications in functional analysis," *Trans. Am. Math. Soc.*, vol. 357, no. 1, pp. 89–128, 2005.
- [16] T. Kuczumow, "An almost convergence and its applications," *Ann. Univ. Mariae Curie-Skłodowska, Sect. A*, vol. 32, pp. 79–88, 1978.
- [17] L. Leuştean, "Nonexpansive iterations in uniformly convex w -hyperbolic spaces," in *Nonlinear analysis and optimization I. Nonlinear analysis. A conference in celebration of Alex Ioffe's 70th and Simeon Reich's 60th birthdays, Haifa, Israel, June 18–24, 2008*, ser. Contemporary Mathematics. Providence, RI; Ramat-Gan: American Mathematical Society (AMS); Bar-Ilan University, 2010, vol. 513, pp. 193–210.
- [18] T.-C. Lim, "Remarks on some fixed point theorems," *Proc. Am. Math. Soc.*, vol. 60, pp. 179–182, 1977.
- [19] S. Reich and I. Shafrir, "Nonexpansive iterations in hyperbolic spaces," *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 15, no. 6, pp. 537–558, 1990.
- [20] T. Shimizu and W. Takahashi, "Fixed points of multivalued mappings in certain convex metric spaces," *Topol. Methods Nonlinear Anal.*, vol. 8, no. 1, pp. 197–203, 1996.
- [21] W. Takahashi, "A convexity in metric spaces and nonexpansive mappings," *Kodai Math Sem Rep.*, vol. 22, pp. 142–149, 1970.
- [22] W. Takahashi, "Iterative methods for approximation of fixed points and their applications," *J. Oper. Res. Soc. Japan*, vol. 43, no. 1, pp. 87–108, 2000.
- [23] Y. Yao and R. Chen, "Weak and strong convergence of a modified Mann iteration for asymptotically nonexpansive mappings," *Nonlinear Funct. Anal. Appl.*, vol. 12, no. 2, pp. 307–315, 2007.

Authors' addresses

Birol Gunduz

Current address: Department of Mathematics, Faculty of Science and Art, Erzincan University, Erzincan, 24000, Turkey.

E-mail address: birolgndz@gmail.com

Sezgin Akbulut

Current address: Department of Mathematics, Faculty of Science, Ataturk University, Erzurum, 25240, Turkey.

E-mail address: sezginakbulut@atauni.edu.tr



ISLANDS: FROM CODING THEORY TO ENUMERATIVE COMBINATORICS AND TO LATTICE THEORY – OVERVIEW AND OPEN PROBLEMS

ESZTER K. HORVÁTH

Received 11 June, 2013

Abstract. As a tool to characterize instantaneous codes, Foldes and Singhi determined the maximum number of certain subsets of the set of the first n natural numbers in 2006. This motivated Czédli in 2009 to determine the maximum number of the analogous subsets of a rectangular grid. He called these subsets *islands*. The present paper summarizes recent results on this topic.

2010 *Mathematics Subject Classification:* 05D99; 06B99; 06A06

Keywords: island, lattice, tree, height function

1. INTRODUCTION

The topic we are going to survey goes back to two papers. Firstly, to characterize instantaneous codes by equalities, Foldes and Singhi [9] considered certain families of subsets of $\{1, 2, \dots, n\}$. They proved that the minimum number of equations needed equals the maximum size of these families, see [9, Theorem 4].

Secondly, soon afterwards, Czédli [4] studied analogous families of certain subsets of the rectangular grid $\{1, \dots, m\} \times \{1, 2, \dots, n\}$. He coined the name (2-dimensional) *rectangular islands* for these subsets. In this terminology, the subsets of Foldes and Singhi are 1-dimensional islands. By the main result of [4], the maximum number of rectangular islands on a m -by- n grid is

$$f(m, n) = \left\lfloor \frac{mn + m + n - 1}{2} \right\rfloor. \quad (1.1)$$

To be more precise, $f(m, n)$ is the maximum size of a family of rectangular islands on the m -by- n grid.

This research was supported by the NFSR of Hungary (OTKA), grant number K83219, and supported by the European Union and co-funded by the European Social Fund under the project “Telemedicine-focused research activities on the field of Mathematics, Informatics and Medical sciences” of project number “TÁMOP-4.2.2.A-11/1/KONV-2012-0073”.

1.1. *Remark (added in July 30th, 2013).*

After the present paper was submitted, new lattice theoretical approach of the island topic came to existence, see Czédli [2]. This approach is related to [5] and to [14].

The aim of the present paper is to overview the results and directions appeared in several branches of mathematics after the above-mentioned two papers.

2. THE DEFINITION OF ISLANDS

First, we fix a finite grid, consisting of e.g. squares or equilateral triangles. Its cells constitute the so-called *board*. The cells have *neighbours*. In each particular cases, we have to define exactly, which cells are considered neighbours.

We put a real number into each cell, these are the so-called *heights*. Actually we have a real valued *height function* on the set of cells. We fix a shape (consisting of cells), e.g. rectangle or triangle. We call this rectangle/triangle an *island* if its cells have greater heights than the heights in the neighboring cells.

We call a rectangle/triangle on a given board an *island*, if for the cell t , if we denote its height by a_t , then for each cell $\hat{t}$ neighbouring with a cell of the rectangle/triangle T , the inequality $a_{\hat{t}} < \min\{a_t : t \in T\}$ holds. For detailed didactical introduction for the maximum number of rectangular islands see e.g. [10]. The collection of all islands for a given height function is called island system. Also, we will use the following terminology: system of rectangular islands, system of triangular islands, etc.

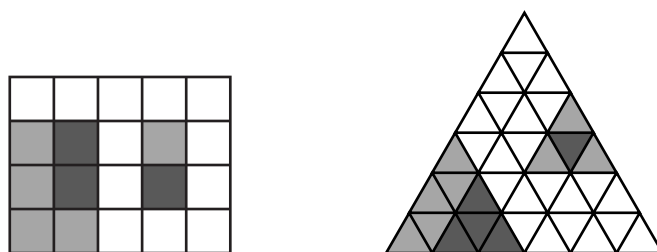


FIGURE 1. Rectangular and triangular islands

3. PROVING RESULTS ON THE MAXIMUM NUMBER OF ISLANDS

In this section, we present three proofs for formula (1.1) of Gábor Czédli [4] for rectangular islands. The presented methods are applicable for further cases, too.

3.1. Lower bound

We prove by induction on the number of cells that

$$f(m, n) \geq \left\lfloor \frac{mn + m + n - 1}{2} \right\rfloor.$$

If $m = 1$, $n = 1$ or $m = n = 2$, then it is easy to check that the statement is true. Let $m, n > 2$. First, we put maximally many islands into the rectangles of sizes $(m-2) \times n$, and $1 \times n$. Between these rectangles, we put one row of cells with heights smaller then the minimum of the heights in the two rectangles, furthermore we put even smaller heights outside of our $m \times n$ rectangle. Then we apply the induction hypothesis, i.e. the inequation for smaller rectangles:

$$\begin{aligned} f(m, n) &\geq f(m-2, n) + f(1, n) + 1 \geq \\ &\geq \left\lfloor \frac{(m-2)n + (m-2) + n - 1}{2} \right\rfloor + \left\lfloor \frac{n + 1 + n - 1}{2} \right\rfloor + 1 = \\ &= \left\lfloor \frac{(m-2)n + (m-2) + n - 1 + 2n}{2} \right\rfloor + 1 = \left\lfloor \frac{mn + m + n - 1}{2} \right\rfloor. \end{aligned}$$

3.2. Upper bound

3.2.1. Method A: lattice method

The original method was based on lattice theory, using the result of [3] that any two weak bases of a finite distributive lattice have the same number of elements. This method produced the result first. This proof can be read in [4], and provided several research directions in lattice theory, e.g. [5, 6, 14].

A little bit later, two other methods appeared in [1]. In this paper we present only these proofs (method B and method C, as follows).

3.2.2. Method B: induction

If $m = n = 1$, then the statement is obviously true. Let $m > 1$ or $n > 1$. The induction hypothesis: if $u < m$ or $v < n$, then for the rectangle R of size $u \times v$, $f(R) = f(u, v) \leq \frac{1}{2}(u+1)(v+1) - 1$.

Let $\mathcal{I}^*$ denote such system of rectangular islands that contains maximally many rectangular islands. Denote by $\max \mathcal{I}^*$ the set of all maximal rectangular islands for $\mathcal{I}^*$, i.e. those set of rectangular islands that have only one bigger rectangular island. For rectangle of size $u \times v$ the number of grid points is $\|R\| = (u+1)(v+1)$. Now

$$\begin{aligned} f(m, n) &= 1 + \sum_{R \in \max \mathcal{I}^*} f(R) \leq 1 + \sum_{R \in \max \mathcal{I}^*} \left(\frac{1}{2} \|R\| - 1 \right) = \\ &= 1 - |\max \mathcal{I}^*| + \frac{1}{2} \sum_{R \in \max \mathcal{I}^*} \|R\| \leq 1 - |\max \mathcal{I}^*| + \frac{1}{2} (m+1)(n+1). \end{aligned}$$

so we obtained $f(m, n) \leq 1 - |\max \mathcal{I}^*| + \frac{1}{2} (m+1)(n+1)$.

If we have at least two maximal rectangular islands, then the proof is ready.

If we have only one maximal rectangular island, then one of the following inequalities are true:

$$f(m, n) \leq 1 - |\max J^*| + \frac{1}{2}m(n+1) = 1 - 1 + \frac{1}{2}m(n+1) \leq \frac{1}{2}(m+1)(n+1) - 1.$$

$$f(m, n) \leq 1 - |\max J^*| + \frac{1}{2}(m+1)n = 1 - 1 + \frac{1}{2}(m+1)n \leq \frac{1}{2}(m+1)(n+1) - 1.$$

If we have no maximal rectangular island, then we have only one rectangular island, so the proof is also ready.

3.2.3. Method C: tree-graph method

Our rectangular islands constitute tree graph by inclusion. In Figure 2, the islands are represented by grey rectangles, labelled at their bottom right corner. The heights of cells are indicated at their top left corners. The island R_i and, if exists, R'_i occur at water level i .

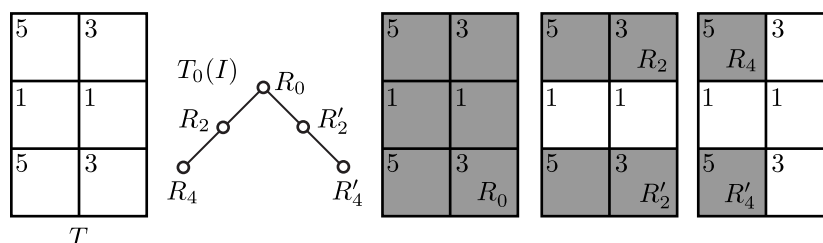


FIGURE 2. Islands and the corresponding tree

First, we need a Lemma.

Lemma 1. *Let T be a rooted tree such that any non-leaf node has at least 2 sons. Let ℓ be the number of leaves in T . Then $|V| \leq 2\ell - 1$.*

Proof. Let us direct the tree-graph from its root to the direction of its leaves. Then for the in-degrees and out-degrees the equation $\sum D^+ = \sum D^-$ holds. Now $\sum D^+ = |V| - 1$ because each node has father except for the root. Moreover $\sum D^- \geq 2(|V| - \ell)$ because all non-leaf node has at least two sons. So $\sum D^+ = |V| - 1 = \sum D^- \geq 2(|V| - \ell)$, i.e. we obtained $2\ell - 1 \geq |V|$. □

As we mentioned, our rectangular islands constitute tree-graphs by inclusion. For having at least binary tree-graph in all cases (condition of Lemma 1), we introduce the so-called "dummy island" in case the island shrinks when the water level increases. Figure 3 is derived from Figure 2. The dummy islands are depicted as thin, dark-grey rectangles.

We denote by s the number of the minimal rectangular islands, by d the number of dummy islands. By Figures 2 and 3, each minimal rectangular island covers at least four grid-points, each dummy island covers at least two grid-points.

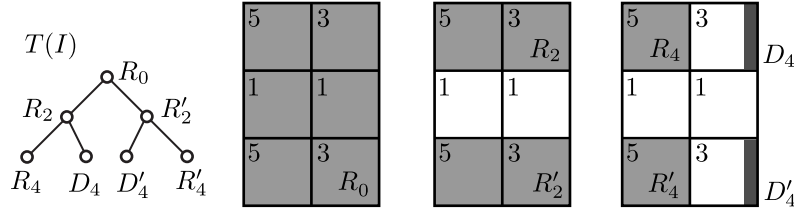


FIGURE 3. Islands and the corresponding tree with dummy islands D_4 and D'_4

This way we have covered not more than all the grid-points of the square grid, i. e.: $4s + 2d \leq (n + 1)(m + 1)$.

The number of leaves of is $\ell = s + d$. Hence by Lemma the number of islands is $|V| - d \leq (2\ell - 1) - d = 2s + d - 1 \leq \frac{1}{2}(n + 1)(m + 1) - 1$.

4. BRICK ISLANDS, HIGHER DIMENSIONS

In [4] a question was raised about the analogous problem in higher dimensions. Pluhár in [25] gave upper and lower bound for the maximum number of brick islands.

5. THE MAXIMUM NUMBER OF TRIANGULAR ISLANDS

Denote by $\text{tr}(n)$ the maximum number of triangular islands in the equilateral triangle of sidelength n . The following lower and upper bounds are proved in [13]:

$$\frac{n^2 + 3n}{5} \leq \text{tr}(n) \leq \frac{3n^2 + 9n + 2}{14}.$$

Furthermore, $\text{tr}(n) = \frac{3n^2 + 9n + 2}{14}$ for infinitely many n . It is proved in [17] that

$$\lim_{n \rightarrow \infty} \frac{\text{tr}(n)}{n^2} = \frac{3}{14}.$$

In [17] triangular islands were investigated not only on triangular, but also on trapezoid and parallelogram board, with triangular grid.

5.1. Problem

On triangular grid, not only triangular islands could be considered, but also other shapes, e.g. diamonds, parallelograms, trapezoids, hexagons.

6. THE MAXIMUM NUMBER OF SQUARE ISLANDS

It is proved in [12] that for the square grid of size $(m - 1) \times (n - 1)$ the maximum number of square islands $\text{sq}(m, n)$:

$$\frac{1}{3}(mn - 2m - 2n) \leq \text{sq}(m, n) \leq \frac{1}{3}(mn - 1).$$

This upper bound is sharp, whenever $m = n = 2^k$ for some $k \in \mathbb{N}$. Analogous upper and lower bounds are valid for the higher dimensional cases.

However, similarly to the triangular case, the lower and upper bounds are close, but exact formula for this case is also has not been proved.

7. EXACT RESULTS

The proofs of the following statements can be found in [1].

7.1. Peninsulas (semi islands)

Here we consider rectangular islands on the square grid that reaches at least one side of the rectangular board. The maximum number of such islands is:

$$p(m, n) = f(m, n) = \lfloor (mn + m + n - 1)/2 \rfloor.$$

7.2. Cylindric board, rectangular islands

We put square grid onto the surface of a cylinder. Denote by $h_1(m, n)$ the maximum number of rectangular islands: If $n \geq 2$, then $h_1(m, n) = \left\lfloor \frac{(m+1)n}{2} \right\rfloor$.

7.3. Cylindric board, cylindric and rectangular islands

It is possible to create cylindric islands on cylindric board. Denote by $h_2(m, n)$ the maximum number of rectangular or cylindric islands on the surface of a cylinder of height m . Of course, in this case, the maximum number of cylindric islands is more than in the previous case: If $n \geq 2$, then $h_2(m, n) = \left\lfloor \frac{(m+1)n}{2} \right\rfloor + \left\lfloor \frac{(m-1)}{2} \right\rfloor$.

7.4. Torus board, rectangular islands

We fold a torus from a rectangle of size $m \times n$. Denote by $t(m, n)$ the maximum number of rectangular islands on this board. If $m, n \geq 2$, then $t(m, n) = \left\lfloor \frac{mn}{2} \right\rfloor$.

7.5. Islands in Boolean algebras

Here the board consists of all vertices of a hypercube, i.e. the elements of a Boolean algebra $B = \{0, 1\}^n$. We consider two cells neighbouring if their Hamming distance is 1. Our islands are Boolean algebras. We denote the maximum number of islands in the Boolean algebra $B = \{0, 1\}^n$ by $b(n)$. Then, $b(n) = 1 + 2^{n-1}$.

8. THE MINIMUM SIZES OF MAXIMUM SYSTEMS OF ISLANDS

In general, a system of islands is maximal, if it cannot be extended to a larger system of islands.

Systems of rectangular islands on a $m \times n$ rectangle R constitute a partially ordered set I_R with respect to set inclusion. Let

$$g_r(m, n) = \min \{H : H \in \max(I_R)\}.$$

In [18] Lengvárszky proved that $g_r(m, n) = m + n - 1$.

Also, he treated the triangular case in [17] and if I_T is the partially ordered set of triangular islands on the triangular board and if

$$g_t(m, n) = \min \{H : H \in \max(I_T)\},$$

then he obtained that $g_t y(n) = n$. In [17], the minimum number of maximal systems of triangular islands are also estimated on diamond, trapezoid and parallelogram, giving possibility for further research.

The minimum cardinality of maximal systems of square islands on a n^2 board is considered in [19]. Let

$$g_s(n) = \min \{H : H \in \max(I_S)\}.$$

In [19] Lengvárszky proved that $g_s(n) = n$.

In [7] Eccles investigated the higher dimensional generalizations, and he proved formula for the minimal size of cuboid islands $g_d(m_1, \dots, m_d)$ in a cuboid of size $m_1 \times \dots \times m_d$, namely: $g_d(m_1, \dots, m_d) = \sum_{i=1}^d m_i - (d - 1)$. Moreover, he proved that in a cube of size m^d the minimal size of cubic islands is given by $g'_d(m, \dots, m) = m$.

9. CD-INDEPENDENCE AND CDW-INDEPENDENCE

It was observed that many subsets in island problems are CD-independent.

Let $\mathbb{P} = (P, \leq)$ be a partially ordered set and $a, b \in P$. The elements a and b are called *disjoint* and we write $a \perp b$ if

- either $\mathbb{P}$ has least element $0 \in P$ and $\inf\{a, b\} = 0$,
- or $\mathbb{P}$ is without 0 and the elements a and b have no common lowerbound.

Notice, that $a \perp b$ implies $x \perp y$ for all $x, y \in P$ with $x \leq a$ and $y \leq b$.

A nonempty set $X \subseteq P$ is called *CD-independent* if for any $x, y \in X$, $x \leq y$ or $y \leq x$, or $x \perp y$ holds. Maximal CD-independent sets (with respect to $\subseteq$) are called *CD-bases* in $\mathbb{P}$. In some papers, e.g. in [20, 21, 24] CD-independent sets are called *laminar* systems.

In [5] the authors showed that the CD-bases in a finite distributive lattice have the same number of elements, and conversely, if all finite lattices in a lattice variety have this property, then the variety must coincide with the variety of distributive lattices. From the proof of the main result of [5] it is clear that if we consider the maximum number of islands of arbitrary connected shape on the rectangular board, then it is bounded from above by the cardinality of a maximal chain of the powerset of the set of the cells, i.e. by $m \times n$.

In paper [14] it is shown that the CD bases of any poset $\mathbb{P}$ can be characterized as maximal chains in a related poset $\mathcal{D}(P)$. A nonempty set D of nonzero elements of P is called a *disjoint set* in $\mathbb{P}$, if $x \perp y$ holds for all $x, y \in D$, $x \neq y$; if the poset $\mathbb{P}$ contains 0-element, then $\{0\}$ is considered to be a disjoint set, too. Let $\mathcal{D}(P)$ denote

the set of all disjoint sets of $\mathbb{P}$. It is proved in [14] that if $\mathbb{P}$ is a complete lattice, then $\mathcal{D}(P)$ is also a lattice having a weak distributive property. If $\mathbb{P} = (P, \wedge)$ is a semilattice with 0, then for any $a, b \in P$ the relation $a \perp b$ means that $a \wedge b = 0$. Hence, a set $\{a_i \mid i \in I\}$ of nonzero elements is a disjoint system if and only if $a_i \wedge a_j = 0$, for all $i, j \in I, i \neq j$. A pair $a, b \in P$ with least upper bound $a \vee b$ in $\mathbb{P}$ is called a *distributive pair*, if $(c \wedge a) \vee (c \wedge b)$ exists in $\mathbb{P}$ for any $c \in P$, and $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$. We say that $(P, \wedge)$ is *dp-distributive* (*distributive with respect to disjoint pairs*), if any $a, b \in P$ with $a \wedge b = 0$ is a distributive pair. Two known lattice classes are pointed out where the CD-bases in finite lattices have the mentioned property: The first class is the graded, dp-distributive lattices, and the second class is obtained by generalizing the properties of the so-called interval lattices (having their origin in graph theory).

A family $\mathcal{H} \subseteq P$ is *weakly independent* if

$$H \leq \bigvee_{i \in I} H_i \implies \exists i \in I : H \leq H_i$$

holds for all $H \in \mathcal{H}, H_i \in \mathcal{H} (i \in I)$. If $\mathcal{H}$ is both CD-independent and weakly independent, then we say that $\mathcal{H}$ is *CDW-independent*.

It is proved in [6] that any two CDW-bases of a finite distributive lattice have the same number of elements. Moreover, if a lattice variety contains a nondistributive lattice, then there exists a finite lattice in this variety that has two CDW-bases with different number of elements.

9.1. Problem

It is conjectured that the formula of Czédli for the maximum number of rectangular islands (and other cases) can be proved by using the main result of [5] or [6], but by our knowledge, up to now nobody has elaborated the proof. We encourage the reader to do it, as different proofs might lead to different research directions.

10. HEIGHT FUNCTION WITH FINITE RANGE

If we put only finitely many heights into the square cells of a rectangle, we obtain a much more complicated problem for rectangular islands, which is solved only for one dimension. Namely, assume that there are n cells in a single row and each cell has to be of height at least 0 and at most h , and let $I(n, h)$ denote the maximum number of islands in this case. It was proved in [11] that

$$I(n, h) = n - \left\lfloor \frac{n}{2^h} \right\rfloor$$

if we have n cells in one row. Notice that the whole board is not necessarily an island, only if we do not use the height 0. Now from this formula it is easy to see that if the height is at least 1 and at most h , then the maximum number of islands is:

$$I'(n, h) = n + 1 - \left\lfloor \frac{n}{2^{h-1}} \right\rfloor$$

as it appears in [22].

In [22] the cases $2 \times n$ and $3 \times n$ are solved if the height is at least 1 and at most h . They obtained

$$I''(n, h) = \left\lfloor \frac{3n+1}{2} \right\rfloor + 1 + \left\lceil \frac{n}{2^{h-2}} \right\rceil$$

for the $2 \times n$ case and

$$I'''(n, h) = 2n + 2 + \left\lceil \frac{n}{2^{h-2}} \right\rceil$$

for the $3 \times n$ case.

10.1. Problem

The larger boards (m -by- n board with $m \geq 4$) were found problematic by [22].

11. ISLANDS AND CUTS OF LATTICE VALUED FUNCTIONS

A rectangular board equipped with a height function can be considered a fuzzy relation, see [16]. Fuzzy relations in many cases model real life better than crisp ones. Here we interpret the results of [16] with using the classical „height function” terminology only.

A *height function* h is a mapping from $\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ to $\mathbb{N}$, $h : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \mathbb{N}$. Actually, a *fuzzy relation* in [16] is a mapping from $\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ to $[0, 1]$, $h : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow [0, 1]$, but the results of [16] are valid if we consider co-domain $\mathbb{N}$.

The co-domain of the height function is the lattice $(\mathbb{N}, \leq)$, where $\mathbb{N}$ is the set of natural numbers under the usual ordering $\leq$ and suprema and infima are max and min, respectively.

For every $p \in \mathbb{N}$, the *cut of the height function*, p -cut of h is an ordinary relation h_p on $\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ defined by

$$(x, y) \in h_p \text{ if and only if } h(x, y) \geq p.$$

We say that two rectangles $\{\alpha, \dots, \beta\} \times \{\gamma, \dots, \delta\}$ and $\{\alpha_1, \dots, \beta_1\} \times \{\gamma_1, \dots, \delta_1\}$ are *distant* if they are disjoint and for every two cells, namely (a, b) from the first rectangle and (c, d) from the second, we have $(a - c)^2 + (b - d)^2 \geq 4$.

The height function h is called *rectangular* if for every $p \in \mathbb{N}$, every nonempty p -cut of h is a union of distant rectangles.

We denote by $\mathcal{I}_{rect}(h)$ the system of islands defined by the height function h .

In [16] it is proved that for every height function

$$h : \{1, 2, \dots, n\} \times \{1, 2, \dots, m\} \rightarrow \mathbb{N},$$

there is a rectangular height function

$$h^* : \{1, 2, \dots, n\} \times \{1, 2, \dots, m\} \rightarrow \mathbb{N},$$

such that $\mathcal{I}_{rect}(h) = \mathcal{I}_{rect}(h^*)$.

Moreover, in [16] there is a representation theorem, namely:

let $h : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \mathbb{N}$ be a rectangular height function. Then there is a lattice L and an L -valued fuzzy relation Φ , such that the cuts of Φ are precisely all islands of h .

It is proved in [15] that for every rectangular height function

$$h^* : \{1, 2, \dots, n\} \times \{1, 2, \dots, m\} \rightarrow \mathbb{N},$$

there is a rectangular height function

$$h^{**} : \{1, 2, \dots, n\} \times \{1, 2, \dots, m\} \rightarrow \mathbb{N},$$

such that $\mathcal{I}_{rect}(h^*) = \mathcal{I}_{rect}(h^{**})$ and in h^{**} every island appears exactly in one cut.

If a rectangular height function h^{**} has the property that each island appears exactly in one cut, then we call it *standard rectangular height function*.

We denote by $\Lambda_{max}(m, n)$ the maximum number of different nonempty p -cuts of a standard rectangular height function on the rectangular table of size $m \times n$. It is obtained in [15] that $\Lambda_{max}(m, n) = m + n - 1$. If $m, n \geq 3$ and the height function has maximally many rectangular islands, then the number of different nonempty cuts are strictly less than $\Lambda_{max}(m, n)$.

It is an interesting fact proved in [15] that if $m \geq 3$ and $n \geq 3$ and a height function $h : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \mathbb{N}$ has maximally many rectangular islands, then it has exactly two maximal islands.

We denote by $\Lambda_h^{cz}(m, n)$ the number of different nonempty cuts of a standard rectangular height function h in the case h has maximally many islands, i.e., when the number of islands is given by (1.1).

Let $h : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \mathbb{N}$ be a standard rectangular height function having maximally many islands $f(m, n)$. Then,

$$\Lambda_h^{cz}(m, n) \geq \lceil \log_2(m+1) \rceil + \lceil \log_2(n+1) \rceil - 1,$$

this lower bound is sharp, moreover if $m, n \geq 2$, then $\Lambda_h^{cz}(m, n) \leq \lfloor \frac{m+n+3}{2} \rfloor$, in addition for $m, n \geq 3$ this bound is also sharp, all proved also in [15]. This result might lead us closer to the solution to the two-dimensional problem of height function with finite range, mentioned in the former section.

12. CONTINUOUS CASE

Up to now in this paper, only discrete boards were considered. In [21], Lengvárszky and Pach investigated continuous board; real-valued height function f is defined on a closed rectangle $R \subset \mathbb{R}^n$. A rectangle $S \subset R$ is a f -island, if there exists an open set $G \subset R$ containing S such that $f(x) < \inf_S f$ for every $x \in G \setminus S$. The set of all f -islands for a fixed f are called a *system of rectangular islands*. In [21] it is proved that the size of a maximal system of rectangular islands is either countable or continuum, both exist.

In paper [24] Pach et al. investigated the maximum number of continuous islands of arbitrary form. From an obtained more general condition they derived that the cardinality of maximal laminar i.e. CD-independent system of closed discs in $\mathbb{R}^n$ is either countable or continuum. They also proved that all island systems are laminar, but not every laminar system is a system of islands for some height function. This paper looks for necessary conditions for a laminar system in order to be a system of islands.

The paper [20] investigates island systems with continuous height functions. It shows that these system of islands are strongly laminar, i.e. laminar with the property that every two distinct sets in it have disjoint boundaries, however not vice versa. It is shown also in this paper that in the discrete case, i.e. for rectangular islands with integer coordinates for a maximal rectangular system of islands with continuous height function $|\mathcal{H}|$ on an $m \times n$ grid we have $\left\lceil \frac{\min(m,n)}{4} \right\rceil \leq |\mathcal{H}| \leq \left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil$. In the continuous case this paper gives sufficient conditions for maximal strongly laminar systems to have cardinality countable or continuum.

13. GENERALIZATION

It is possible to generalize the notion of islands in such a way that formal concepts and prime implicants of Boolean functions are covered, the details can be read in [8].

An *island domain* is a pair $(\mathcal{C}, \mathcal{K})$, where $\mathcal{C} \subseteq \mathcal{K} \subseteq \mathcal{P}(U)$ for some nonempty finite set U such that $U \in \mathcal{C}$. By a *height function* we mean a map $h: U \rightarrow \mathbb{R}$.

We denote the cover relation of the poset $(\mathcal{K}, \subseteq)$ by $\prec$, and we write $K_1 \preceq K_2$ if $K_1 \prec K_2$ or $K_1 = K_2$.

We say that S is a *pre-island* with respect to the triple $(\mathcal{C}, \mathcal{K}, h)$, if every $K \in \mathcal{K}$ with $S \prec K$ satisfies

$$\min h(K) < \min h(S).$$

We say that S is an *island* with respect to the triple $(\mathcal{C}, \mathcal{K}, h)$, if every $K \in \mathcal{K}$ with $S \prec K$ satisfies

$$h(u) < \min h(S) \text{ for all } u \in K \setminus S.$$

The paper gives necessary condition for a system of pre-islands, called admissibility. This condition is not a sufficient one, however the paper proves that maximal pre-island systems and maximal admissible families are the same. An island domain

$(\mathcal{C}, \mathcal{K})$ is a *connective island domain* if

$$\forall A, B \in \mathcal{C} : (A \cap B \neq \emptyset \text{ and } B \not\subseteq A) \implies \exists K \in \mathcal{K} : A \subset K \subseteq A \cup B.$$

A result of [8] is that the following three conditions are equivalent for any island domain $(\mathcal{C}, \mathcal{K})$:

- (i) $(\mathcal{C}, \mathcal{K})$ is a connective island domain.
- (ii) Every system of pre-islands corresponding to $(\mathcal{C}, \mathcal{K})$ is CD-independent.
- (iii) Every system of pre-islands corresponding to $(\mathcal{C}, \mathcal{K})$ is CDW-independent.

In connective island domains the systems of pre-islands are exactly the admissible systems. A binary relation $\delta \subseteq \mathcal{C} \times \mathcal{C}$ is defined that expresses the fact that a set $B \in \mathcal{C}$ is in some sense close to a set $A \in \mathcal{C}$:

$$A\delta B \Leftrightarrow \exists K \in \mathcal{K} : A \subseteq K \text{ and } K \cap B \neq \emptyset.$$

We say that $A, B \in \mathcal{C}$ are *distant* if neither $A\delta B$ nor $B\delta A$ holds. The island domain $(\mathcal{C}, \mathcal{K})$ is called a *proximity domain*, if it is a connective island domain and the relation δ is symmetric for nonempty sets, that is

$$\forall A, B \in \mathcal{C} \setminus \{\emptyset\} : A\delta B \Leftrightarrow B\delta A.$$

It is proved in this paper that in proximity domains systems of islands are exactly the distant families.

14. DIDACTICAL ASPECTS

It is worth to introduce this topic to young students because there are several elementary problems that they can treat themselves. Some of these approaches can be found in [10], [11], [22] or [23].

ACKNOWLEDGEMENT

The author expresses her gratitude to all her coauthors on this topic.

REFERENCES

- [1] J. Barát, P. Hajnal, and E. K. Horváth, “Elementary proof techniques for the maximum number of islands,” *Eur. J. Comb.*, vol. 32, no. 2, pp. 276–281, 2011.
- [2] G. Czédli, “CD-independent subsets in meet-distributive lattices,” <http://arxiv.org/abs/1307.0900>.
- [3] G. Czédli, A. P. Huhn, and E. T. Schmidt, “Weakly independent subsets in lattices,” *Algebra Univers.*, vol. 20, pp. 194–196, 1985.
- [4] G. Czédli, “The number of rectangular islands by means of distributive lattices,” *Eur. J. Comb.*, vol. 30, no. 1, pp. 208–215, 2009.
- [5] G. Czédli, M. Hartmann, and E. T. Schmidt, “CD-independent subsets in distributive lattices,” *Publ. Math.*, vol. 74, no. 1-2, pp. 127–134, 2009.
- [6] G. Czédli and E. T. Schmidt, “CDW-independent subsets in distributive lattices,” *Acta Sci. Math.*, vol. 75, no. 1-2, pp. 49–53, 2009.
- [7] T. Eccles, “The minimum sizes of maximal systems of brick islands,” *Acta Sci. Math. (Szeged)*, vol. 78, pp. 375–387, 2012.

- [8] S. Foldes, E. K. Horváth, S. Radeleczki, and T. Waldhauser, "A general framework for island system," available at <http://arxiv.org/abs/1210.1741>.
- [9] S. Foldes and N. M. Singhi, "On instantaneous codes," *J. Comb. Inf. Syst. Sci.*, vol. 31, no. 1-4, pp. 307–316, 2006.
- [10] E. K. Horváth, "Survey on islands - a possible research topic, even for students," in *Interesting Mathematical Problems in Sciences and Everyday Life*. Szeged: University of Szeged, 2011, available at <http://www.model.u-szeged.hu/etc/edoc/imp/EHorvath/EHorvath.pdf>, vol. 011, p. 2.
- [11] E. K. Horváth, A. Máder, and A. Tepavčević, "One-dimensional Czedli-type islands," *College Math. J.*, vol. 42, pp. 374–378, 2011.
- [12] E. K. Horváth, G. Horváth, Z. Németh, and C. Szabó, "The number of square islands on a rectangular sea," *Acta Sci. Math.*, vol. 76, no. 1-2, pp. 35–48, 2010.
- [13] E. K. Horváth, Z. Németh, and G. Pluhár, "The number of triangular islands on a triangular grid," *Period. Math. Hung.*, vol. 58, no. 1, pp. 25–34, 2009.
- [14] E. K. Horváth and S. Radeleczki, "Notes on CD-independent subsets," *Acta Sci. Math.*, vol. 78, no. 1-2, pp. 3–24, 2012.
- [15] E. K. Horváth, B. Šešelja, and A. Tepavčević, "Cardinality of height function's range in case of maximally many rectangular islands – computed by cuts," *Cent. Eur. J. Math.*, vol. 11, no. 2, pp. 296–307, 2013.
- [16] E. K. Horváth, B. Šešelja, and A. Tepavčević, "Cut approach to islands in rectangular fuzzy relations," *Fuzzy Sets Syst.*, vol. 161, no. 24, pp. 3114–3126, 2010.
- [17] Z. Lengvárszky, "Notes on systems of triangular islands," *Acta Sci. Math.*, vol. 75, no. 3-4, pp. 369–376, 2009.
- [18] Z. Lengvárszky, "The minimum cardinality of maximal systems of rectangular islands," *Eur. J. Comb.*, vol. 30, no. 1, pp. 216–219, 2009.
- [19] Z. Lengvárszky, "The size of maximal systems of square islands," *Eur. J. Comb.*, vol. 30, no. 4, pp. 889–892, 2009.
- [20] Z. Lengvárszky, "System of islands with continuous height function," *Journal of Australian Mathematical Society*, accepted.
- [21] Z. Lengvárszky and P. P. Pach, "A note on systems of rectangular islands: the continuous case," *Acta Sci. Math.*, vol. 77, no. 1-2, pp. 27–34, 2011.
- [22] A. Máder and G. Makay, "The maximum number of rectangular islands," *The Teaching of Mathematics*, vol. 1461, no. 3, pp. 31–44, 2011.
- [23] A. Máder and R. Vajda, "Elementary approaches to the teaching of the combinatorial problem of rectangular islands," *Int. J. Comput. Math. Learn.*, vol. 15, pp. 267–281, 2010.
- [24] P. P. Pach, G. Pluhár, A. Pongrácz, and C. Szabó, "The possible number of islands on the sea," *J. Math. Anal. Appl.*, vol. 375, no. 1, pp. 8–13, 2011.
- [25] G. Pluhár, "The number of brick islands by means of distributive lattices," *Acta Sci. Math.*, vol. 75, no. 1-2, pp. 3–11, 2009.

Author's address

Eszter K. Horváth

University of Szeged, Department of Algebra and Number Theory, Aradi vértanúk tere 1., 6720 Szeged, Hungary

E-mail address: horeszt@math.u-szeged.hu



ON LIE IDEALS AND GENERALIZED DERIVATIONS OF *-PRIME RINGS

SHULIANG HUANG AND ÖZNUR GÖLBAŞI

Received 4 February, 2013

Abstract. Let $(R, *)$ be a 2-torsion free $*$ -prime ring with involution $*$ and center $Z(R)$, U a nonzero square closed $*$ -Lie ideal of R . An additive mapping $F : R \rightarrow R$ is called a generalized derivation if there exists a derivation $d : R \rightarrow R$ such that $F(xy) = F(x)y + xd(y)$. In the present paper, we prove that $U \subseteq Z(R)$ if any one of following conditions holds: 1) $[F(u), u] = 0$, 2) $[d(u), F(v)] = 0$, 3) $d(u)oF(v) = 0$, 4) $[d(u), F(v)] = \pm[u, v]$, 5) $d(u)oF(v) = \pm uov$, 6) $d(u)F(v) \pm uv \in Z(R)$, for all $u, v \in U$. Furthermore, an example is given to demonstrate that the $*$ -primeness hypothesis is not superfluous.

2010 *Mathematics Subject Classification:* 16W25; 16W10; 16N60; 16U80

Keywords: $*$ -prime rings, Lie ideals, derivations, generalized derivations

1. INTRODUCTION

Let R will be an associative ring with center Z . For any $x, y \in R$ the symbol $[x, y]$ represents commutator $xy - yx$. Recall that a ring R is prime if $xRy = 0$ implies $x = 0$ or $y = 0$. An additive mapping $*$: $R \rightarrow R$ is called an involution if $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or $*$ -ring. A ring with an involution is said to $*$ -prime if $xRy = xRy^* = 0$ or $xRy = x^*Ry = 0$ implies that $x = 0$ or $y = 0$. Every prime ring with an involution is $*$ -prime but the converse need not hold general. An example due to Oukhtite [7] justifies the above statement. That is, let R be a prime ring, $S = R \times R^o$ where R^o is the opposite ring of R . Define involution $*$ on S as $*(x, y) = (y, x)$. S is $*$ -prime, but not prime. This example shows that every prime ring can be injected in a $*$ -prime ring and from this point of view $*$ -prime rings constitute a more general class of prime rings. In all that follows the symbol $S_{a*}(R)$, first introduced by Oukhtite, will denote the set of symmetric and skew symmetric elements of R , i.e. $S_{a*}(R) = \{x \in R \mid x^* = \pm x\}$.

An additive subgroup U of R is said to be a Lie ideal of R if $[U, R] \subseteq U$. A Lie ideal is said to be a $*$ -Lie ideal if $U^* = U$. If U is a Lie (resp. $*$ -Lie) ideal of R , then U is called a square closed Lie (resp. $*$ -Lie) ideal of R if $x^2 \in U$ for all $x \in U$. An

additive mapping $d : R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \rightarrow R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation. Recently, in [4], Bresar defined the following notation. An additive mapping $F : R \rightarrow R$ is called a generalized derivation if there exists a derivation $d : R \rightarrow R$ such that

$$F(xy) = F(x)y + xd(y), \text{ for all } x, y \in R.$$

Basic examples are derivations and generalized inner derivations (i.e., maps of type $x \rightarrow ax + xb$ for some $a, b \in R$). One may observe that the concept of generalized derivations includes the concept of derivations and of the left multipliers (i.e., $F(xy) = F(x)y$ for all $x, y \in R$). Several authors studied commutativity in prime and semiprime rings admitting derivations and generalized derivations which satisfy appropriate algebraic conditions on suitable subsets of the rings. Generalized derivations have been primarily studied on operator algebras. Therefore any investigation from algebraic point of view might be interesting. Recently, some well-known results concerning prime rings have been proved for $*$ -prime ring by Oukhtite et al. (see, [5–9], where further references can be found). In [1] the authors explored the commutativity of the ring R satisfying one of the following conditions: (i) $d(x)oF(y) = 0$, (ii) $[d(x), F(y)] = 0$, (iii) $d(x)oF(y) = xoy$, (iv) $d(x)oF(y) = -xoy$, (v) $d(x)F(y) - xy \in Z$, (vi) $d(x)F(y) + xy \in Z$, (vii) $[d(x), F(y)] = [x, y]$ and (viii) $[d(x), F(y)] = -[x, y]$, for all x, y in some appropriate subset of the ring R . Motivated by this fact, we extend these theorems to a nonzero square closed $*$ -Lie ideal U and a generalized derivation of $*$ -prime rings.

2. PRELIMINARIES

Throughout the paper, we make some extensive use of the basic commutator identities:

$$\begin{aligned} [x, yz] &= y[x, z] + [x, y]z \\ [xy, z] &= [x, z]y + x[y, z] \\ xo(yz) &= (xoy)z - y[x, z] = y(xoz) + [x, y]z \\ (xy)oz &= x(yoz) - [x, z]y = (xoz)y + x[y, z]. \end{aligned}$$

Moreover, we shall require the following lemmas.

Lemma 1 (Lemma 4 in [8]). *Let R be a $*$ -prime ring with characteristic not two, U be a nonzero $*$ -Lie ideal of R and $a, b \in R$. If $aUb = a^*Ub = 0$, then $a = 0$ or $b = 0$ or $U \subseteq Z$.*

Lemma 2 (Lemma 2.7 in [2]). *Let R be a 2-torsion free $*$ -prime ring and U be a $*$ -Lie ideal of R . If $a \in R$ such that $[a, U] \subseteq Z$, then either $U \subseteq Z$ or $a \in Z$.*

The following Lemma is immediate consequence of Lemma 2.

Lemma 3. *Let R be a $*$ -prime ring with characteristic not two, U be a nonzero $*$ -Lie ideal of R . If $[U, U] \subseteq Z$, then $U \subseteq Z$.*

Lemma 4 (Lemma 2.4 in [6]). *Let R be a 2-torsion free $\ast$ -prime ring and U a nonzero $\ast$ -Lie ideal of R . If R admits nonzero derivation d which commutes with $\ast$ such that $d(U) \subset Z$, then $U \subseteq Z$.*

Lemma 5 (Theorem 1.1 in [6]). *Let R be a 2-torsion free $\ast$ -prime ring, U a nonzero $\ast$ -Lie ideal of R . If R admits nonzero derivation d which commutes with $\ast$ such that $d^2(U) = 0$, then $U \subseteq Z$.*

3. RESULTS

The following theorem gives a generalization of Posner's well known result [10, Theorem 1].

Theorem 1. *Let R be a 2-torsion free $\ast$ -prime ring, U a nonzero square closed $\ast$ -Lie ideal of R . If R admits derivations d_1, d_2 such that $\ast d_2 = d_2 \ast$ and $d_1 d_2(U) = 0$, then either $d_1 = 0$ or $d_2 = 0$ or $U \subseteq Z$.*

Proof. Notice that $uv + vu = (u + v)^2 - u^2 - v^2$, for all $u, v \in U$. Since $u^2 \in U$ for all $u \in U$, $uv + vu \in U$. Also $uv - vu \in U$, for all $u, v \in U$. Hence, we get $2uv \in U$, for all $u, v \in U$.

Now, we have $d_1 d_2(u) = 0$, for all $u \in U$. Replacing u by $2uv$ and using R is a 2-torsion free, we have $d_1(d_2(u)v + ud_2(v)) = 0$, and hence

$$d_2(u)d_1(v) + d_1(u)d_2(v) = 0, \text{ for all } u, v \in U.$$

Again replace v by $vd_2(w)$, $w \in [U, U]$ to get

$$d_1(u)vd_2^2(w) = 0, \text{ for all } u, v \in U, w \in [U, U].$$

Since U is a nonzero square closed $\ast$ -Lie ideal of R , we have $[U, U]$ is a nonzero square closed $\ast$ -Lie ideal of R , too. Hence using $\ast d_2 = d_2 \ast$ the last equation, we arrive at

$$d_1(u)Ud_2^2(w) = d_1(u)U \ast (d_2^2(w)) = 0, \text{ for all } u \in U, w \in [U, U] \cap S_{a_\ast}(R)$$

By Lemma 1, we get either $d_1(u) = 0$, for all $u \in U$ or $d_2^2(w) = 0$ for each $w \in [U, U] \cap S_{a_\ast}(R)$.

If $d_1(u) = 0$, for all $u \in U$, then we have $d_1 = 0$ or $U \subseteq Z$ by Lemma 4. Assume that $d_2^2(w) = 0$ for each $w \in [U, U] \cap S_{a_\ast}(R)$. Let $w \in [U, U]$ as $w + w^\ast, w - w^\ast \in [U, U] \cap S_{a_\ast}(R)$ and $d_2^2(w) = 0$. Hence we have $d_2^2([U, U]) = 0$, then we get $d_2 = 0$ or $[U, U] \subseteq Z$ by Lemma 5, and so $U \subseteq Z$ by Lemma 3. This completes the proof. $\square$

Theorem 2. *Let R be a 2-torsion free $\ast$ -prime ring, U a nonzero square closed $\ast$ -Lie ideal of R . If R admits a nonzero generalized derivation F associated with nonzero derivation d which commutes with $\ast$ such that $[F(u), u] = 0$, for all $u \in U$ then $U \subseteq Z$.*

Proof. Suppose that

$$[F(u), u] = 0, \text{ for all } u \in U. \quad (3.1)$$

Linearizing (3.1) and using this, we obtain

$$[F(u), v] + [F(v), u] = 0, \text{ for all } u, v \in U. \quad (3.2)$$

Replacing v by $2vu$ in (3.2) and using R is a 2-torsion free, we get

$$[F(u), vu] + [F(v)u + vd(u), u] = 0.$$

That is

$$\begin{aligned} & [F(u), v]u + v[F(u), u] + [F(v), u]u + \\ & F(v)[u, u] + v[d(u), u] + [v, u]d(u) \\ & = 0, \text{ for all } u, v \in U. \end{aligned}$$

Now combining (3.1) and (3.2), using hypothesis in the last equation, we find that

$$v[d(u), u] + [v, u]d(u) = 0, \text{ for all } u, v \in U. \quad (3.3)$$

Again replace v by $2vw$ in (3.3) and use (3.3), to get

$$[v, u]wd(u) = 0, \text{ for all } u, v, w \in U.$$

and so

$$[v, u]Ud(u) = 0, \text{ for all } u, v \in U. \quad (3.4)$$

Since U is a nonzero $*$ -Lie ideal of R yields that

$$[v, u]^*Ud(u) = 0, \text{ for all } v \in U, u \in U \cap S_{a_*}(R).$$

By Lemma 1, we get either $[v, u] = 0$, for all $v \in U$ or $d(u) = 0$ for each $u \in U \cap S_{a_*}(R)$. Let $u \in U$, as $u + u^*, u - u^* \in U \cap S_{a_*}(R)$ and $[v, u \pm u^*] = 0$, for all $v \in U$ or $d(u \pm u^*) = 0$. Hence we have $[v, u] = 0$ or $d(u) = 0$, for all $u, v \in U$. We obtain that U is union of two additive subgroups of U such that

$$K = \{u \in U \mid d(u) = 0\}$$

and

$$L = \{u \in U \mid [v, u] = 0, \text{ for all } v \in U\}.$$

Moreover, U is the set-theoretic union of K and L . But a group can not be the set-theoretic union of two proper subgroups. Hence $K = U$ or $L = U$. In the former case, we get $U \subseteq Z$ by Lemma 4. In the latter case, $[U, U] = (0)$. That is $U \subseteq Z$ by Lemma 3. This completes the proof. $\square$

Corollary 1. *Let R be a 2-torsion free $*$ -prime ring, U a nonzero square closed $*$ -Lie ideal of R . If R admits a nonzero derivation d which commutes with $*$ such that $[d(u), u] = 0$, for all $u \in U$ then $U \subseteq Z$.*

Theorem 3. *Let R be a 2-torsion free $\ast$ -prime ring and U a nonzero square closed $\ast$ -Lie ideal of R . If R admits a nonzero generalized derivation F associated with nonzero derivation d which commutes with $\ast$ such that $[d(u), F(v)] = 0$, for all $u, v \in U$, then $U \subseteq Z$.*

Proof. We have

$$[d(u), F(v)] = 0, \text{ for all } u, v \in U. \quad (3.5)$$

Replacing u by $2uw$, $w \in U$ in (3.5) and applying (3.5), we get

$$2(d(u)[w, F(v)] + [u, F(v)]d(w)) = 0, \text{ for all } u, v, w \in U.$$

Using the fact that R is a 2-torsion free ring, we have

$$d(u)[w, F(v)] + [u, F(v)]d(w) = 0, \text{ for all } u, v, w \in U. \quad (3.6)$$

Substituting $d(t)$, $t \in [U, U]$ for w in (3.6), we get

$$d(u)[d(t), F(v)] + [u, F(v)]d^2(t) = 0, \text{ for all } u, v \in U, t \in [U, U].$$

Using the hypothesis, the above relation yields that

$$[u, F(v)]d^2(t) = 0, \text{ for all } u, v \in U, t \in [U, U]. \quad (3.7)$$

Now, taking $2vu$ instead of u in (3.7) and using this equation, $\text{char} R \neq 2$, there by obtain

$$[v, F(v)]Ud^2(t) = 0, \text{ for all } v \in U, t \in [U, U].$$

Since $[U, U]$ is a nonzero square closed $\ast$ -Lie ideal of R and $\ast d = d\ast$, we find that

$$[v, F(v)]Ud^2(t) = [v, F(v)]U \ast (d^2(t)) = 0, \text{ for all } v \in U, t \in [U, U] \cap S_{a\ast}(R).$$

By Lemma 1, we get either $[v, F(v)] = 0$, for all $v \in U$, or $d^2(t) = 0$ for each $t \in [U, U] \cap S_{a\ast}(R)$. Let $t \in [U, U]$, as $t + t^\ast, t - t^\ast \in [U, U] \cap S_{a\ast}(R)$ and $d^2(t) = 0$. Hence we get $d^2(t) = 0$, for all $t \in [U, U]$. Thus we obtain that

$$[v, F(v)] = 0 \text{ or } d^2(t) = 0, \text{ for all } v \in U, t \in [U, U].$$

In the former case, $U \subseteq Z$ by Theorem 2. In the latter case, $[U, U] \subseteq Z$ by Lemma 5, and so $U \subseteq Z$ by Lemma 3. This completes the proof. $\square$

Proceeding on the same lines with some necessary variations in the proof of Theorem 3, we can prove the following theorem:

Theorem 4. *Let R be a 2-torsion free $\ast$ -prime ring and U a nonzero square closed $\ast$ -Lie ideal of R . If R admits nonzero derivations d which commutes with $\ast$ and g such that $[d(u), g(v)] = 0$, for all $u, v \in U$, then $U \subseteq Z$.*

Theorem 5. *Let R be a 2-torsion free $\ast$ -prime ring and U a nonzero square closed $\ast$ -Lie ideal of R . If R admits a nonzero generalized derivation F associated with nonzero derivation d which commutes with $\ast$ such that $d(u) \circ F(v) = 0$, for all $u, v \in U$, then $U \subseteq Z$.*

Proof. We have

$$d(u)oF(v) = 0, \text{ for all } u, v \in U. \quad (3.8)$$

Replacing u by u^2 in (3.8), we get

$$(d(u)oF(v))u + d(u)[u, F(v)] + u(d(u)oF(v)) - [u, F(v)]d(u) = 0, \text{ for all } u, v \in U$$

and so,

$$d(u)[u, F(v)] - [u, F(v)]d(u) = 0, \text{ for all } u, v \in U. \quad (3.9)$$

We can write $[d(u), [u, F(v)]] = 0$, for all $u, v \in U$ from (3.9). This yields that $[d(u), I_{F(v)}(u)] = 0$, where $I_{F(v)} : R \rightarrow R$, $I_{F(v)}(x) = [x, F(v)]$ is an inner derivation of R . By Theorem 4, we get either $I_{F(v)} = 0$ or $U \subseteq Z$. In the first case, $F(U) \subset Z$, and so $[u, F(u)] = 0$, for all $u \in U$. This implies that $U \subseteq Z$ by Theorem 2. This completes the proof. $\square$

Theorem 6. *Let R be a 2-torsion free $\ast$ -prime ring and U a nonzero square closed $\ast$ -Lie ideal of R . If R admits nonzero derivations d which commutes with $\ast$ and g such that $d(u)og(v) = 0$, for all $u, v \in U$, then $U \subseteq Z$.*

Proof. Using the same arguments as in the proof of Theorem 5, we get the required result. $\square$

Theorem 7. *Let R be a 2-torsion free $\ast$ -prime ring and U a nonzero square closed $\ast$ -Lie ideal of R . If R admits a nonzero generalized derivation F associated with nonzero derivation d which commutes with $\ast$ such that $[d(u), F(v)] = [u, v]$, for all $u, v \in U$, then $U \subseteq Z$.*

Proof. Replacing v by $2vw$, $w \in U$ and using $\text{char}R \neq 2$, we have

$$\begin{aligned} [d(u), F(v)]w + F(v)[d(u), w] + v[d(u), d(w)] + [d(u), v]d(w) - [u, v]w - v[u, v] \\ = 0. \end{aligned} \quad (3.10)$$

Applying hypothesis, we get

$$F(v)[d(u), w] + v[d(u), d(w)] + [d(u), v]d(w) - v[u, w] = 0, \text{ for all } u, v, w \in U. \quad (3.11)$$

Writing w for $d(u)$, $u \in [U, U]$, we obtain that

$$v[d(u), d^2(u)] + [d(u), v]d^2(u) - v[u, d(u)] = 0. \quad (3.12)$$

Taking $2vw$ for v and using $\text{char}R \neq 2$, (3.12), we see that

$$[d(u), v]Ud^2(u) = 0, \text{ for all } v \in U, u \in [U, U]. \quad (3.13)$$

Since $[U, U]$ is a nonzero square closed $\ast$ -Lie ideal of R and $\ast d = d\ast$, we can get $[d(u), v]U \ast (d^2(u)) = 0$, for all $v \in U, u \in [U, U] \cap S_{a\ast}(R)$. Hence we arrive at

$$[d(u), v]Ud^2(u) = [d(u), v]U \ast (d^2(u)) = 0, \text{ for all } v \in U, u \in [U, U] \cap S_{a\ast}(R). \quad (3.14)$$

By Lemma 1, we get either $[d(u), v] = 0$ or $d^2(u) = 0$, for each $u \in [U, U] \cap S_{a_\ast}(R)$. Let $u \in [U, U]$, as $u + u^\ast, u - u^\ast \in [U, U] \cap S_{a_\ast}(R)$ and $[d(u \pm u^\ast), v] = 0$ or $d^2(u \pm u^\ast) = 0$. Hence we have $[d(u), v] = 0$ or $d^2(u) = 0$, for all $u \in [U, U]$. We set

$$K = \{u \in [U, U] \mid [d(u), v] = 0, \text{ for all } v \in U\}$$

and

$$L = \{u \in [U, U] \mid d^2(u) = 0\}.$$

Clearly each of K and L is additive subgroup of $[U, U]$. Moreover, $[U, U]$ is the set-theoretic union of K and L . But a group can not be the set-theoretic union of two proper subgroups, hence $K = [U, U]$ or $L = [U, U]$. In the former case, we have $[d(u), u] = 0$ for all $u \in [U, U]$ and so using Corollary 1, $[U, U] \subseteq Z$. In the second case, we have $d^2([U, U]) = 0$. That is $[U, U] \subseteq Z$ by Lemma 5. Hence using Lemma 3, we get $U \subseteq Z$. This completes the proof. $\square$

Proceeding on the same lines with some necessary variations in the proof of Theorem 7, we can prove the following theorem which partially generalizes the result of [3, Theorem 1].

Theorem 8. *Let R be a 2-torsion free $\ast$ -prime ring and U a nonzero square closed $\ast$ -Lie ideal of R . If R admits nonzero derivations d which commutes with $\ast$ and g such that $[d(u), g(v)] = [u, v]$, for all $u, v \in U$, then $U \subseteq Z$.*

Theorem 9. *Let R be a 2-torsion free $\ast$ -prime ring and U a nonzero square closed $\ast$ -Lie ideal of R . If R admits a nonzero generalized derivation F associated with nonzero derivation d which commutes with $\ast$ such that $[d(u), F(v)] = -[u, v]$, for all $u, v \in U$, then $U \subseteq Z$.*

Proof. It can be proved using the techniques of Theorem 7. $\square$

Theorem 10. *Let R be a 2-torsion free $\ast$ -prime ring and U a nonzero square closed $\ast$ -Lie ideal of R . If R admits a nonzero generalized derivation F associated with nonzero derivation d which commutes with $\ast$ such that $d(u) \circ F(v) = u \circ v$, for all $u, v \in U$, then $U \subseteq Z$.*

Proof. By the hypothesis, we have

$$d(u) \circ F(v) - u \circ v = 0, \text{ for all } u, v \in U. \quad (3.15)$$

Replacing v by $2vw$, $w \in U$ in (3.15), we get

$$\begin{aligned} 2((d(u) \circ F(v))w - F(v)[d(u), w] + (d(u) \circ v)d(w) - v[d(u), d(w)] \\ - (u \circ v)w + v[u, w]) = 0. \end{aligned}$$

Using (3.15) and $\text{char} R \neq 2$, we obtain

$$-F(v)[d(u), w] + (d(u) \circ v)d(w) - v[d(u), d(w)] + v[u, w] = 0, \text{ for all } u, v, w \in U. \quad (3.16)$$

Substituting $d(u), u \in [U, U]$ for w in (3.16), we have

$$(d(u)ov)d^2(u) - v[d(u), d^2(u)] + v[u, d(u)] = 0, \text{ for all } u \in [U, U], v, w \in U. \quad (3.17)$$

Replacing v by $2wv, w \in U$ in (3.17), we get

$$2(w(d(u)ov)d^2(u) + [d(u), w]vd^2(u) - wv[d(u), d^2(u)] + wv[u, d(u)] = 0.$$

In the view of (3.17), the above expression yields that $2[d(u), w]vd^2(u) = 0$, for all $u \in [U, U], v, w \in U$. Since $\text{char} R \neq 2$, we arrive at

$$[d(u), w]Ud^2(u) = 0, \text{ for all } u \in [U, U], w \in U.$$

The proof is completed by using the same arguments in the equation (3.13) of Theorem 7. $\square$

Corollary 2. *Let R be a 2-torsion free $\ast$ -prime ring and U a nonzero square closed $\ast$ -Lie ideal of R . If R admits nonzero derivations d which commutes with $\ast$ and g such that $d(u)og(v) = uov$, for all $u, v \in U$, then $U \subseteq Z$.*

Proof. Using the same arguments as in the proof of Theorem 10, we get

$$[d(u), w]Ugd(u) = 0, \text{ for all } u \in [U, U], w \in U.$$

Using $\ast d = d\ast$, we arrive at

$$\ast([d(u), w])Ugd(u) = 0, \text{ for all } u \in [U, U], w \in U.$$

Arguing in the similar manner as we have done in the proof of Theorem 10, we arrive at $[d(u), u] = 0$ or $gd(u) = 0$, for all $u \in [U, U]$. In the former case, we have $U \subseteq Z$ by Corollary 1. In the second case, using Theorem 1, we find $U \subseteq Z$. $\square$

Theorem 11. *Let R be a 2-torsion free $\ast$ -prime ring and U a nonzero square closed $\ast$ -Lie ideal of R . If R admits a nonzero generalized derivation F associated with nonzero derivation d which commutes with $\ast$ such that $d(u)oF(v) = -uov$, for all $u, v \in U$, then $U \subseteq Z$.*

Proof. Similarly Theorem 10. $\square$

Theorem 12. *Let R be a 2-torsion free $\ast$ -prime ring and U a nonzero square closed $\ast$ -Lie ideal of R . If R admits a nonzero generalized derivation F associated with nonzero derivation d which commutes with $\ast$ such that $d(u)F(v) - uv \in Z$, for all $u, v \in U$, then $U \subseteq Z$.*

Proof. By the hypothesis, we have

$$d(u)F(v) - uv \in Z, \text{ for all } u, v \in U. \quad (3.18)$$

Replacing v by $2vw, w \in U$ in (3.18), we get

$$2((d(u)F(v) - uv)w + d(u)vd(w)) \in Z, \text{ for all } u, v, w \in U.$$

Since $\text{char} R \neq 2$, we have

$$(d(u)F(v) - uv)w + d(u)vd(w) \in Z, \text{ for all } u, v, w \in U. \quad (3.19)$$

Commuting this term with w and using the hypothesis, we obtain

$$[d(u)vd(w), w] = 0, \text{ for all } u, v, w \in U. \quad (3.20)$$

Substituting $2d(u)v, u \in [U, U]$ for v in (3.20) and using this equation, we get

$$[d(u), w]d(u)Ud(w) = 0, \text{ for all } w \in U, u \in [U, U].$$

Since U is a nonzero square closed $\ast$ -Lie ideal of R and using $\ast d = d\ast$ in the last equation, we arrive at

$$\begin{aligned} [d(u), w]d(u)Ud(w) &= [d(u), w]d(u)U \ast (d(w)) = 0, \\ &\text{for all } u \in [U, U], w \in U \cap S_{a\ast}(R). \end{aligned}$$

Arguing in the similar manner as we have done in the proof of Theorem 10 after (3.18) yields that either $[d(u), w]d(u) = 0$ or $d(w) = 0$, for each $w \in U, u \in [U, U]$.

Now, we assume that $[d(u), w]d(u) = 0$, for all $w \in U, u \in [U, U]$. Replacing w by $2wv, v \in U$ in this equation and using this, we have

$$[d(u), w]vd(u) = 0, \text{ for all } u \in [U, U], w, v \in U.$$

Again using $[U, U]$ is a nonzero square closed $\ast$ -Lie ideal of R and $\ast d = d\ast$ in the last equation, we get

$$[d(u), w]vd(u) = [d(u), w]v \ast (d(u)) = 0, \text{ for all } u \in U, w, v \in [U, U].$$

An application of similar techniques as used in the proof of Theorem 10, we obtain that $U \subseteq Z$. $\square$

Corollary 3. *Let R be a 2-torsion free $\ast$ -prime ring and U a nonzero square closed $\ast$ -Lie ideal of R . If R admits nonzero derivation d which commutes with $\ast$ such that $d(u)d(v) - uv \in Z$, for all $u, v \in U$, then $U \subseteq Z$.*

Theorem 13. *Let R be a 2-torsion free $\ast$ -prime ring and U a nonzero square closed $\ast$ -Lie ideal of R . If R admits a nonzero generalized derivation F associated with nonzero derivation d which commutes with $\ast$ such that $d(u)F(v) + uv \in Z$, for all $u, v \in U$, then $U \subseteq Z$.*

Proof. Similarly. $\square$

Corollary 4. *Let R be a 2-torsion free $\ast$ -prime ring and U a nonzero square closed $\ast$ -Lie ideal of R . If R admits nonzero derivation d which commutes with $\ast$ such that $d(u)d(v) + uv \in Z$, for all $u, v \in U$, then $U \subseteq Z$.*

The following example demonstrates that the above results are not true in the case of arbitrary rings.

Example 14. Let I be the ring of integers. Set $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in I \right\}$

and $U = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in I \right\}$. We define the following maps: $* \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$, $F \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 2b \\ 0 & 0 \end{pmatrix}$, $d \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. Then it is easy to see that U is a nonzero square closed $*$ -Lie ideal of R and F is a generalized derivation associated with a nonzero derivation d commuting with $*$. Moreover, it is straightforward to check that F satisfies the properties: 1) $[F(u), u] = 0$, 2) $[d(u), F(v)] = 0$, 3) $d(u) \circ F(v) = 0$, 4) $[d(u), F(v)] = \pm[u, v]$, 5) $d(u) \circ F(v) = \pm u \circ v$, 6) $d(u)F(v) \pm uv \in Z$, for all $u, v \in U$. However, $U \not\subseteq Z(R)$.

REFERENCES

- [1] M. Ashraf, A. Ali, and R. Rani, "On generalized derivations of prime rings," *Southeast Asian Bull. Math.*, vol. 29, no. 4, pp. 669–675, 2005.
- [2] M. Ashraf and A. Khan, "Commutativity of $*$ -prime rings with generalized derivations," *Rend. Semin. Mat. Univ. Padova*, vol. 125, pp. 71–79, 2011.
- [3] H. E. Bell and M. N. Daif, "On commutativity and strong commutativity-preserving maps," *Can. Math. Bull.*, vol. 37, no. 4, pp. 443–447, 1994.
- [4] M. Brešar, "On the distance of the composition of two derivations to the generalized derivations," *Glasg. Math. J.*, vol. 33, no. 1, pp. 89–93, 1991.
- [5] L. Oukhtite and S. Salhi, "On commutativity of σ -prime rings," *Glas. Mat., III. Ser.*, vol. 41, no. 1, pp. 57–64, 2006.
- [6] L. Oukhtite and S. Salhi, "Lie ideals and derivations of σ -prime rings," *Int. J. Algebra*, vol. 1, no. 1–4, pp. 25–30, 2007.
- [7] L. Oukhtite and S. Salhi, "On generalized derivations of σ -prime ring," *Afr. Diaspora J. Math.*, vol. 5, no. 1, pp. 21–25, 2007.
- [8] L. Oukhtite and S. Salhi, "Centralizing automorphisms and Jordan left derivations on σ -prime rings," *Adv. Algebra*, vol. 1, no. 1, pp. 19–26, 2008.
- [9] L. Oukhtite, S. Salhi, and L. Taoufiq, "Commutativity conditions on derivations and Lie ideals in σ -prime rings," *Beitr. Algebra Geom.*, vol. 51, no. 1, pp. 275–282, 2010.
- [10] E. C. Posner, "Derivations in prime rings," *Proc. Am. Math. Soc.*, vol. 8, pp. 1093–1100, 1958.

Authors' addresses

Shuliang Huang

Chuzhou University, Department of Mathematics, Chuzhou Anhui, 239012, P. R. China,
E-mail address: shulianghuang@sina.com

Öznur Gölbaşı

Cumhuriyet University, Faculty of Science, Department of Mathematics, 58140, Sivas - TURKEY
E-mail address: ogolbasi@cumhuriyet.edu.tr



BALANCING WITH BALANCING POWERS

NURETTIN IRMAK

Received 21 March, 2013

Abstract. In this paper, the Diophantine equation $B_1^k + B_2^k + \cdots + B_{n-1}^k = B_{n+1}^l + B_{n+2}^l + \cdots + B_{n+r}^l$ for the positive integer unknowns $n \geq 2$, k, l and r is studied in certain cases, where B_n denotes the n^{th} term of the balancing sequence.

2010 *Mathematics Subject Classification:* 11B39

Keywords: balancing numbers, powers

1. INTRODUCTION

The first definition of balancing numbers is essentially due to Finkelstein [5], though he named them numerical centers. A positive integer $n \geq 2$ is called balancing number if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \quad (1.1)$$

holds for some positive integer r which is called balancer corresponding to the balancing number n . The m^{th} term of the sequence of balancing numbers is denoted by B_m . It is known that the balancing numbers satisfy the recurrence

$$B_n = 6B_{n-1} - B_{n-2} \quad (1.2)$$

where initial conditions are defined by $B_0 = 0$ and $B_1 = 1$. The first few terms of the balancing sequence are

$$0, 1, 6, 35, 204, 1189, 6930, 40391, \dots$$

The Binet formula of the balancing numbers is given by

$$B_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n}{4\sqrt{2}},$$

where α and β are the roots of the characteristic polynomial $x^2 - 6x + 1$ of the recursive sequence (1.2).

Behera et al. [4] showed that the diophantine equation

$$F_1^k + F_2^k + \cdots + F_{n-1}^k = F_{n+1}^l + F_{n+2}^l + \cdots + F_{n+r}^l \quad (1.3)$$

has no solution in the positive integers n, r, k, l with $n \geq 2$ in the case

$$k \leq l \quad \text{and} \quad (k, l) = (2, 1), (3, 1), (3, 2). \quad (1.4)$$

Here F_n denotes the n^{th} Fibonacci number. They also conjectured in [4] that only the quadruple $(n, r, k, l) = (4, 3, 8, 2)$ satisfies (1.3). Their conjecture was proved by Alvarado et al. [2].

In this paper, we focus on the diophantine equation

$$B_1^k + B_2^k + \cdots + B_{n-1}^k = B_{n+1}^l + B_{n+2}^l + \cdots + B_{n+r}^l. \quad (1.5)$$

In the specific cases (1.4) we found no solutions. Based on Theorems 6-9 and an extended computer search, we conjectured the following.

Conjecture 1. *There is no quadruple (n, r, k, l) of positive integers ($n \geq 2$) which satisfies (1.5).*

2. LEMMAS

The proofs of the theorems use several statements collected in this section.

Lemma 1. *For any positive integer m*

- (a): $B_{m+1}B_{m-1} = (B_m + 1)(B_m - 1)$,
- (b): $B_{2m-1} = B_m^2 - B_{m-1}^2$,
- (c): $B_1 + B_3 + \cdots + B_{2m-1} = B_m^2$,
- (d): $B_1^2 + B_2^2 + \cdots + B_{m-1}^2 < B_{m+1} + B_{m+2} + \cdots + B_{2m-3}$ for $m \geq 5$,
- (e): $B_{2m+5}^2 + B_{2m+6}^2 + \cdots + B_{3m+4}^2 < B_1^3 + B_2^3 + \cdots + B_{2m+3}^3$ for $m \geq 2$,
- (f): $4B_m < B_{m+1} - B_m < 5B_m$

hold.

Proof. The first three identities appear in [3] and [6].

The fourth property (d) can be justified by induction on m . Clearly, $B_1^2 + B_2^2 + B_3^2 + B_4^2 < B_6 + B_7$. Now suppose that,

$$B_1^2 + B_2^2 + \cdots + B_{m-1}^2 < B_{m+1} + B_{m+2} + \cdots + B_{2m-3}.$$

Evaluating

$$B_1^2 + B_2^2 + \cdots + B_{m-1}^2 + B_m^2 < B_{m+1} + B_{m+2} + \cdots + B_{2m-3} + B_m^2$$

we see that $B_m^2 < B_{2m-1} + B_{2m-2} - B_{m+1}$ fulfills, and then

$$\begin{aligned} B_1^2 + B_2^2 + \cdots + B_m^2 &< B_{m+1} + B_{m+2} + \cdots + B_{2m-3} + B_{2m-1} + B_{2m-2} - B_{m+1} \\ &= B_{m+2} + B_{m+3} + \cdots + B_{2m-1} \end{aligned}$$

follow. Indeed, $B_m^2 - B_{2m-1} = B_{m-1}^2 < B_{2m-2} - B_{m-1}$.

The identity (e) can be seen by the same way, we leave it to the readers, as well as the proof of the statement (f). $\square$

Lemma 2. *Any positive integer n satisfies*

- (a): $\sum_{k=1}^n B_k = (B_{n+1} - B_n - 1) / 4,$
 (b): $\sum_{k=1}^n B_k^2 = (B_{2n+1} - (2n + 1)) / 32,$
 (c): $\sum_{k=1}^n B_k^3 = (B_{3n+3} - B_{3n} - 147(B_{n+1} - B_n) + 112) / 6272.$

Proof. Since the technique of the proofs of the cases (a), (b) and (c) coincide, we omit the first statement and start with the second one.

(b) By the Binet formula of the balancing numbers,

$$\begin{aligned} \sum_{k=1}^n B_k^2 &= \frac{1}{(\alpha - \beta)^2} \sum_{k=0}^{n-1} \alpha^{2k+2} + \beta^{2k+2} - 2 \\ &= \frac{1}{32} \left(\alpha^2 \frac{1 - \alpha^{2n}}{1 - \alpha^2} + \beta^2 \frac{1 - \beta^{2n}}{1 - \beta^2} - 2n \right) \\ &= \frac{1}{32} \left(\frac{\alpha^{2n+2} + \beta^{2n+2} - (\alpha^{2n} + \beta^{2n})}{32} - (2n + 1) \right) \\ &= (B_{n+1}^2 - B_n^2 - (2n + 1)) / 32 \\ &= (B_{2n+1} - (2n + 1)) / 32. \end{aligned}$$

(c) For the proof of the third identity, we use the Binet formula of balancing numbers again as follows.

$$\begin{aligned} \sum_{k=1}^n B_k^3 &= \frac{1}{(\alpha - \beta)^3} \sum_{k=0}^{n-1} \alpha^{3(k+1)} - \beta^{3(k+1)} - 3(\alpha^{k+1} - \beta^{k+1}) \\ &= \frac{1}{(\alpha - \beta)^3} \left\{ \alpha^3 \frac{1 - \alpha^{3n}}{1 - \alpha^3} - \beta^3 \frac{1 - \beta^{3n}}{1 - \beta^3} - 3 \left(\alpha \frac{1 - \alpha^n}{1 - \alpha} - \beta \frac{1 - \beta^n}{1 - \beta} \right) \right\} \\ &= \frac{1}{32} \left\{ \frac{B_{3(n+1)} - B_{3n} - 35}{196} - 3 \frac{B_{n+1} - B_n - 1}{4} \right\} \\ &= \frac{B_{3n+3} - B_{3n} - 147(B_{n+1} - B_n) + 112}{6272}. \end{aligned}$$

and the proof is completed. $\square$

Lemma 3. For any integer $u \geq 3$, the inequalities

$$\alpha^{u-0.99} < B_u < \alpha^{u-0.98}$$

hold.

Proof. See Lemma 4 in [1]. $\square$

Lemma 4. Suppose that $a > 0$ and $b \geq 0$ are real numbers and u_0 is a positive real number. Then $a\alpha^u + b \leq \alpha^{u+\kappa}$ holds for any $u \geq u_0$ where $\kappa = \log_\alpha \left(a + \frac{b}{\alpha^{u_0}} \right)$.

Proof. For the proof, see Lemma 2.4 in [4]. $\square$

3. THE RESULTS

Here we present four theorems and their proofs which confirm Conjecture 1.

Theorem 1. *The diophantine equation $B_1^k + B_2^k + \cdots + B_{n-1}^k = B_{n+1}^l + B_{n+2}^l + \cdots + B_{n+r}^l$ has no solution for any positive integers r and $n \geq 2$ if $k \leq l$.*

Proof. Applying Lemma 2(a) for $k \leq l$, we get

$$\begin{aligned} B_1^k + B_2^k + \cdots + B_{n-1}^k &\leq (B_1 + B_2 + \cdots + B_{n-1})^k = \left(\frac{B_n - B_{n-1} - 1}{4} \right)^k \\ &\leq \left(\frac{B_n - B_{n-1} - 1}{4} \right)^l < B_n^l, \end{aligned}$$

which completes the proof of Theorem 1. $\square$

Theorem 2. *The diophantine equation $B_1^2 + B_2^2 + \cdots + B_{n-1}^2 = B_{n+1} + B_{n+2} + \cdots + B_{n+r}$ has no solution in positive integers r and $n \geq 2$.*

Proof. By the virtue of Lemma 2(a) and (b), the equation

$$B_1^2 + B_2^2 + \cdots + B_{n-1}^2 = B_{n+1} + B_{n+2} + \cdots + B_{n+r} \quad (3.1)$$

is equivalent to

$$B_{2n-1} + 8(B_{n+1} - B_n) - (2n - 1) = 8(B_{n+r+1} - B_{n+r}).$$

For brevity, we use the following labels:

$$LS := B_{2n-1} + 8(B_{n+1} - B_n) - (2n - 1)$$

$$RS := 8(B_{n+r+1} - B_{n+r}).$$

Using Lemma 1(f),

$$B_{2n-1} < LS < B_{2n-1} + 40B_n$$

hold. The by Lemma 3,

$$\alpha^{2n-1.99} < LS < \alpha^{2n-1.98} + 40\alpha^{n-0.98} \leq \alpha^{2n-1.98} + \alpha^{n+1.12}.$$

According to the Lemma 4, if $a = b = 1$ we obtain $\kappa < 0.02$. Thus

$$\alpha^{2n-1.99} < LS < \alpha^{n+1.12} (\alpha^{n-3.1} + 1) < \alpha^{n+1.12} \alpha^{n-3.1} \alpha^{0.02}.$$

Consequently,

$$\alpha^{2n-1.99} < LS < \alpha^{2n-1.96}. \quad (3.2)$$

On the other hand, Lemma 1(f) implies

$$32B_{n+r} < RS < 40B_{n+r},$$

and then

$$\alpha^{n+r-0.99} \alpha^{1.95} = \alpha^{n+r+0.96} < RS < \alpha^{n+r+1.12} = \alpha^{n+r-0.98} \alpha^{2.1} \quad (3.3)$$

follow. Thus, from (3.2) and (3.3) we get

$$\max\{2n - 1.99, n + r + 0.96\} < \min\{2n - 1.96, n + r + 1.12\}$$

which yields the inequalities

$$2n - 1.99 < n + r + 1.12$$

and

$$n + r + 0.96 < 2n - 1.96.$$

Hence, the positive integer n and r satisfy

$$n - 3.11 < r < n - 2.92,$$

that is $r = n - 3$. If we plug $r = n - 3$ in (3.1), we obtain

$$B_1^2 + B_2^2 + \cdots + B_{n-1}^2 = B_{n+1} + B_{n+2} + \cdots + B_{2n-3} \quad (3.4)$$

which contradicts to Lemma 1(d). $\square$

Theorem 3. *The diophantine equation $B_1^3 + B_2^3 + \cdots + B_{n-1}^3 = B_{n+1} + B_{n+2} + \cdots + B_{n+r}$ has no solution for any positive integers r and $n \geq 2$.*

Proof. Obviously, $n = 2$ the case provides no solution. Therefore we may assume $n \geq 3$. By Lemma 2(a) and (c), the equation in the theorem is equivalent to

$$B_{3n} - B_{3(n-1)} + 1568B_{n+1} - 1715B_n + 147B_{n-1} + 112 = 1568(B_{n+r+1} - B_{n+r}). \quad (3.5)$$

Using the labels again, put

$$\begin{aligned} LS &:= B_{3n} - B_{3(n-1)} + 1568B_{n+1} - 1715B_n + 147B_{n-1} + 112, \\ RS &:= 1568(B_{n+r+1} - B_{n+r}). \end{aligned}$$

By Lemma 3, it is obvious that

$$196B_{3(n-1)} < LS < 197B_{3(n-1)}.$$

Further we have

$$\alpha^{3n-3.99}\alpha^{2.98} = \alpha^{3n-1.01} < LS < \alpha^{3n-0.98} = \alpha^{3n-3.98}\alpha^{3.00} \quad (3.6)$$

On the other hand, applying Lemma 1(f),

$$\alpha^{n+r+3.96} = 1568 \cdot 4 \cdot B_{n+r} < RS < 1568 \cdot 5 B_{n+r} = \alpha^{n+r+4.11} \quad (3.7)$$

hold. From (3.6) and (3.7) we get

$$\max\{3n - 1.01, n + r + 3.96\} < \min\{3n - 0.98, n + r + 4.11\},$$

and we deduce

$$2n - 5.12 < r < 2n - 4.94.$$

Since n and r are integers in this inequality, only $r = 2n - 5$ is possible (recall that $n \geq 3$). But it contradicts to

$$B_{3n} - B_{3(n-1)} + 1568B_{n+1} - 1715B_n + 147B_{n-1} + 112 < 1568(B_{3n-4} - B_{3n-5}).$$

Therefore, there is no solution which satisfies (3.5). $\square$

Theorem 4. *The diophantine equation $B_1^3 + B_2^3 + \cdots + B_{n-1}^3 = B_{n+1}^2 + B_{n+2}^2 + \cdots + B_{n+r}^2$ has no solution for any positive integers r and $n \geq 2$.*

Proof. For $2 \leq n \leq 4$, the statement is obvious. Assume now $n \geq 5$. The application of Lemma 2(b) and (c) converts the equation

$$B_1^3 + B_2^3 + \cdots + B_{n-1}^3 = B_{n+1}^2 + B_{n+2}^2 + \cdots + B_{n+r}^2 \quad (3.8)$$

to

$$\frac{B_{3n} - B_{3(n-1)} - 147(B_n - B_{n-1}) + 112}{32 \cdot 196} = \frac{1}{32} (B_{2(n+r)+1} - B_{2n+1} - 2r),$$

which is equivalent to

$$B_{3n} - B_{3(n-1)} - 147(B_n - B_{n-1}) + 196B_{2n+1} + 112 = 196(B_{2(n+r)+1} - 2r).$$

Put

$$\begin{aligned} LS &:= B_{3n} - B_{3(n-1)} - 147(B_n - B_{n-1}) + 196B_{2n+1} + 112, \\ RS &:= 196(B_{2(n+r)+1} - 2r). \end{aligned}$$

Now use Lemma 3 and Lemma 1(f) to obtain

$$\alpha^{2n+2r+3} < RS < \alpha^{2n+2r+3.02}. \quad (3.9)$$

Similarly,

$$\begin{aligned} 196B_{3(n-1)} &< LS < (197 + 196)B_{3(n-1)}, \\ \alpha^{3n-3-0.99}\alpha^{2.98} &= \alpha^{3n-1.01} < LS < \alpha^{3n-0.59} = \alpha^{3n-3-0.98}\alpha^{3.39} \end{aligned} \quad (3.10)$$

hold. From (3.9) and (3.10)

$$\max\{2n + 2r + 3, 3n - 1.01\} < \min\{2n + 2r + 3.02, 3n - 0.59\}$$

follows, which yields

$$n - 4.03 < 2r < n - 3.59.$$

Clearly, only $2r = n - 4$ is possible. Inserting $n = 2r + 4$ to (3.8), we obtain

$$B_1^3 + B_2^3 + \cdots + B_{2r+3}^3 = B_{2r+5}^2 + B_{2r+6}^2 + \cdots + B_{3r+4}^2,$$

and we arrived at a contradiction with Lemma 1(e). $\square$

ACKNOWLEDGEMENT

The authors would like to thank László Szalay for his helpful suggestions.

REFERENCES

- [1] M. Alp, N. Irmak, and L. Szalay, “Balancing diophantine triples,” *Acta Univ. Sapientiae, Math.*, vol. 4, no. 1, pp. 11–19, 2012.
- [2] S. D. Alvarado, A. Dujella, and F. Luca, “On a conjecture regarding balancing with powers of Fibonacci numbers,” *Integers*, vol. 12, no. 6, pp. 1127–1158, a2, 2012.
- [3] A. Behera and G. K. Panda, “On the square roots of triangular numbers,” *Fibonacci Q.*, vol. 37, no. 2, pp. 98–105, 1999.
- [4] A. Behera, K. Liptai, G. K. Panda, and L. Szalay, “Balancing with Fibonacci powers,” *Fibonacci Q.*, vol. 49, no. 1, pp. 28–33, 2011.
- [5] R. Finkelstein, “The house problem,” *Am. Math. Mon.*, vol. 72, pp. 1082–1088, 1965.
- [6] G. K. Panda, “Some fascinating properties of balancing numbers,” *Congr. Numerantium*, vol. 194, pp. 185–189, 2009.

Author’s address

Nurettin Irmak

Niğde University, Art and Science Faculty, Mathematics Department, 51110 Niğde, TURKEY

E-mail address: nirmak@nigde.edu.tr



POSITIVE INTEGER SOLUTIONS OF THE DIOPHANTINE EQUATIONS $x^2 - 5F_nxy - 5(-1)^ny^2 = \pm 5^r$

REFİK KESKİN, OLCAY KARAATLI, AND ZAFER ŞİAR

Received 11 July, 2012

Abstract. In this study, we consider the Diophantine equations given in the title and determine when these equations have positive integer solutions. Moreover, we find all positive integer solutions of them in terms of Fibonacci and Lucas numbers.

2010 Mathematics Subject Classification: 11D09; 11B37; 11B39

Keywords: Diophantine equations, Fibonacci numbers, Lucas numbers

1. INTRODUCTION

The problem of determining all integer solutions to Diophantine equations has gained a considerable amount of interest among the mathematicians and there is a very broad literature on this subject. In particular, several papers [3–6, 10, 11, 16, 21, 22, 26] deal with the Diophantine equation

$$ax^2 + bxy + cy^2 = d,$$

where a, b, c , and d are fixed integers. The name "Diophantine" comes from Diophantus, an Alexandrian mathematician of the third century A. D., who proposed many Diophantine problems; but such equations have a very long history, extending back to ancient Egypt, Babylonia, and Greece. The main problem when studying a given Diophantine equation is that whether a solution exists, and, in the case it exists, how many solutions there are. Moreover, another important problem closely related to the previous one is that whether there is a general form for the solutions. Further details on Diophantine equations can be found in [7, 8, 17–19, 23, 24].

In [12], the authors considered the Diophantine equations

$$x^2 - L_nxy + (-1)^ny^2 = \pm 5^r$$

under the assumptions that $n > 0$, $r > 1$. They determined when the Diophantine equations $x^2 - L_nxy + (-1)^ny^2 = \pm 5^r$ have positive integer solutions. Moreover, they found all positive integer solutions of these equations in terms of Fibonacci and Lucas numbers. Based on the above equations, in this study, we consider the

following Diophantine equations:

$$x^2 - 5F_n xy - 5(-1)^n y^2 = 5^r$$

and

$$x^2 - 5F_n xy - 5(-1)^n y^2 = -5^r$$

with $n > 1$, $r \geq 0$. The purpose of this study is to determine when the above equations have positive integer solutions and then find all positive integer solutions of them by using some properties of Fibonacci and Lucas sequences. As a reminder, for $n \geq 2$, the well-known Fibonacci and Lucas sequences are defined by $F_n = F_{n-1} + F_{n-2}$ and $L_n = L_{n-1} + L_{n-2}$, where $F_0 = 0$, $F_1 = 1$, and $L_0 = 2$, $L_1 = 1$, respectively. Also Fibonacci and Lucas numbers for negative subscripts are defined as $F_{-n} = (-1)^{n+1} F_n$ and $L_{-n} = (-1)^n L_n$ for $n \geq 1$. For more information about Fibonacci and Lucas sequences we refer the reader to [9, 15, 20, 25].

2. PRELIMINARIES

In this section, we give some theorems and identities to be used in the proofs of the main theorems. The following theorem is given in [6].

Theorem 1. *Let $k \geq 0$ be an integer. Then all nonnegative integer solutions of the equation $u^2 - 5v^2 = 4 \cdot 5^k$ are given by*

$$(u, v) = \begin{cases} (5^{(k+1)/2} F_{2m+1}, 5^{(k-1)/2} L_{2m+1}), & k \text{ is an odd integer} \\ (5^{k/2} L_{2m}, 5^{k/2} F_{2m}), & k \text{ is an even integer} \end{cases}$$

and all nonnegative integer solutions of the equation $u^2 - 5v^2 = -4 \cdot 5^k$ are given by

$$(u, v) = \begin{cases} (5^{(k+1)/2} F_{2m}, 5^{(k-1)/2} L_{2m}), & k \text{ is an odd integer} \\ (5^{k/2} L_{2m+1}, 5^{k/2} F_{2m+1}), & k \text{ is an even integer} \end{cases}$$

with $m \geq 0$.

Now we give the following two theorems from [14].

Theorem 2. *Let $m > 3$ be an integer and $F_n = F_m x^2$ for some integer x . Then $n = m$.*

Theorem 3. *Let $m \geq 2$ be an integer and $L_n = L_m x^2$ for some integer x . Then $n = m$.*

Now we recall some divisibility properties of Fibonacci and Lucas numbers. These properties are given in several sources such as [1, 15, 25]. Also one can find the proofs of the following theorems in [1, 13].

Theorem 4. *Let $m, n \in \mathbb{Z}$ and $n \geq 3$. Then $F_n | F_m$ if and only if $n | m$.*

Theorem 5. *Let $m, n \in \mathbb{Z}$ and $n \geq 2$. Then $L_n | F_m$ if and only if $n | m$ and m/n is an even integer.*

Theorem 6. Let $m, n \in \mathbb{Z}$ and $n \geq 2$. Then $L_n | L_m$ if and only if $n | m$ and m/n is an odd integer.

The following theorem is proved by Cohn in [2].

Theorem 7. Assume that $n \geq 1$. If $F_n = x^2$, then $n = 1, 2, 12$. If $F_n = 2x^2$, then $n = 3, 6$. If $L_n = x^2$, then $n = 1, 3$, and if $L_n = 2x^2$, then $n = 6$.

Now we give some well known identities involving Fibonacci and Lucas numbers. The identities are as follows:

$$L_m F_n - F_m L_n = 2(-1)^m F_{n-m}, \quad (2.1)$$

$$L_m L_n - 5F_m F_n = 2(-1)^m L_{n-m}, \quad (2.2)$$

$$L_m L_n + 5F_m F_n = 2L_{n+m}, \quad (2.3)$$

$$L_m F_n + F_m L_n = 2F_{n+m}, \quad (2.4)$$

$$L_{3n} = L_n(L_n^2 - 3(-1)^n), \quad (2.5)$$

$$F_{2n} = F_n L_n, \quad (2.6)$$

$$2 | F_n \iff 2 | L_n \iff 3 | n, \quad (2.7)$$

$$(F_n, L_n) = 1 \text{ or } (F_n, L_n) = 2, \quad (2.8)$$

$$L_n^2 - 5F_n^2 = 4(-1)^n, \quad (2.9)$$

$$F_{n+m}^2 - L_n F_{n+m} F_m + (-1)^n F_m^2 = (-1)^m F_n^2, \quad (2.10)$$

$$L_{n+m}^2 - 5F_n L_{n+m} F_m - 5(-1)^n F_m^2 = (-1)^m L_n^2, \quad (2.11)$$

and

$$5 \nmid L_n \quad (2.12)$$

for all $m, n \in \mathbb{Z}$.

The proof of the following theorem is given in [14].

Theorem 8. There is no integer x such that $L_n = 2L_m x^2$ for $m > 1$.

Before considering the main theorems of the paper, several theorems, which will be useful during the proofs of the main theorems, are needed.

Theorem 9. Let $m > 1$ be an integer and $F_n = 2L_m x^2$. Then $m = 3$, $x^2 = 1$, and $n = 6$ or $m = 6$, $x^2 = 4$, and $n = 12$.

Proof. Assume that $m > 1$ and $F_n = 2L_m x^2$. Then $L_m | F_n$ and therefore $n = 2mk$ for some natural number k by Theorem 5. Thus we get

$$F_n = F_{2mk} = F_{mk} L_{mk} = 2L_m x^2$$

by (2.6).

Firstly, assume that k is an even integer. Then we can write $(F_{mk}/L_m)L_{mk} = 2x^2$. It can be seen easily that if $(F_{mk}/L_m, L_{mk}) = d$, then $d = 1$ or $d = 2$ by (2.8). Assume that $d = 1$. Then

$$\frac{F_{mk}}{L_m} = u^2, L_{mk} = 2v^2 \quad (2.13)$$

or

$$\frac{F_{mk}}{L_m} = 2u^2, L_{mk} = v^2 \quad (2.14)$$

for some integers u and v with $\gcd(u, v) = 1$. Assume that (2.13) is satisfied. Then $mk = 6$ by Theorem 7. Since $m > 1$ and k is an even integer, we get $m = 3$ and $k = 2$. But this is impossible, since $u^2 = F_{mk}/L_m = F_6/L_3 = 2$. Assume that (2.14) is satisfied. Then $mk = 1$ or 3 by Theorem 7. But this contradicts the fact that k is an even integer. Assume that $d = 2$. Then

$$\frac{F_{mk}}{L_m} = 2u^2, L_{mk} = (2v)^2 \quad (2.15)$$

or

$$\frac{F_{mk}}{L_m} = (2u)^2, L_{mk} = 2v^2 \quad (2.16)$$

for some integers u and v with $\gcd(u, v) = 1$. A similar argument shows that (2.15) and (2.16) are impossible.

Secondly, assume that k is an odd integer. Then we can write $F_{mk}(L_{mk}/L_m) = 2x^2$. It can be easily seen that if $d = (F_{mk}, L_{mk}/L_m)$, then $d = 1$ or $d = 2$ by (2.8). Assume that $d = 1$. Then

$$F_{mk} = u^2, \frac{L_{mk}}{L_m} = 2v^2 \quad (2.17)$$

or

$$F_{mk} = 2u^2, \frac{L_{mk}}{L_m} = v^2 \quad (2.18)$$

for some integers u and v with $\gcd(u, v) = 1$. It can be seen that the identity (2.17) is impossible by Theorem 8. Assume that (2.18) is satisfied. Then $mk = m$, i.e., $k = 1$ by Theorem 3. Therefore $F_m = 2u^2$ and this shows that $m = 3$ or 6 by Theorem 7. If $m = 3$ or 6 , then it can be seen that $x^2 = 1$ and $n = 6$ or $x^2 = 4$ and $n = 12$, respectively. Assume that $d = 2$. Then

$$F_{mk} = 2u^2, \frac{L_{mk}}{L_m} = (2v)^2 \quad (2.19)$$

or

$$F_{mk} = (2u)^2, \frac{L_{mk}}{L_m} = 2v^2 \quad (2.20)$$

for some integers u and v with $\gcd(u, v) = 1$. The identity (2.19) is impossible by Theorem 3. The identity (2.20) is impossible by Theorem 8. This completes the proof. $\square$

Theorem 10. Let $m \geq 2$ and $F_n = L_m x^2$. Then $m = 2$, $x^2 = 1$, and $n = 4$ or $m = 3$, $x^2 = 36$, and $n = 12$ or $m = 12$, $x^2 = 144$, and $n = 24$.

Proof. Assume that $m \geq 2$ and $F_n = L_m x^2$. Then $L_m | F_n$ and therefore $n = 2mk$ for some natural number k by Theorem 5. Thus we get

$$F_n = F_{2mk} = F_{mk} L_{mk} = L_m x^2$$

by (2.6).

Firstly, assume that k is an odd integer. Then $F_{mk}(L_{mk}/L_m) = x^2$. It can be easily shown that if $d = (F_{mk}, L_{mk}/L_m)$, then $d = 1$ or $d = 2$ by (2.8). Assume that $d = 1$. Then we get $F_{mk} = u^2$, $L_{mk}/L_m = v^2$ for some integers u and v with $\gcd(u, v) = 1$. Since $m \geq 2$, we get $mk = 2, 12$ by Theorem 7. If $mk = 2$, then $m = 2$, $k = 1$ and therefore we get $n = 4$ and $x^2 = 1$. If $mk = 12$, then $m = 12$, $k = 1$ or $m = 4$, $k = 3$. If $m = 12$ and $k = 1$, then $n = 24$ and $x^2 = 144$. If $m = 4$ and $k = 3$, then $v^2 = L_{mk}/L_m = L_{12}/L_4 = 46$, which is impossible. Now assume that $d = 2$. Then it is easily seen that

$$F_{mk} = 2u^2, \quad \frac{L_{mk}}{L_m} = 2v^2 \quad (2.21)$$

for some integers u and v with $\gcd(u, v) = 1$. Since $L_{mk} = 2L_m v^2$ has no integer solution v for $m \geq 2$ by Theorem 8, the identity (2.21) is impossible.

Secondly, assume that k is an even integer. Then $(F_{mk}/L_m)L_{mk} = x^2$. It can be easily seen that if $d = (F_{mk}/L_m, L_{mk})$, then $d = 1$ or $d = 2$ by (2.8). Assume that $d = 1$. Then $F_{mk}/L_m = u^2$ and $L_{mk} = v^2$ for some positive integers u and v . Since $m \geq 2$, we get $mk = 3$ by Theorem 7. But this contradicts the fact that k is an even integer. Now assume that $d = 2$. Then it is easily seen that

$$\frac{F_{mk}}{L_m} = 2u^2, \quad L_{mk} = 2v^2 \quad (2.22)$$

for some integers u and v with $\gcd(u, v) = 1$. By Theorem 7, it follows that $mk = 6$ and therefore $k = 2$, $m = 3$. If $k = 2$ and $m = 3$, then we get $n = 12$ and $x^2 = 36$. This completes the proof. $\square$

Theorem 11. Let $r > 3$ and F_r be a prime number. If $F_n = F_r L_m x^2$ with $m \geq 2$, then $m = 2$, $n = 12$, $r = 4$, and $x^2 = 16$ or $m = r$, $n = 2m$, and $x^2 = 1$.

Proof. Assume that $F_n = F_r L_m x^2$. Then $L_m | F_n$ and therefore $n = 2mk$ for some natural number k by Theorem 5. Thus we get

$$F_n = F_{2mk} = F_{mk} L_{mk} = F_r L_m x^2$$

by (2.6).

Firstly, assume that k is an odd integer. Then $F_{mk}(L_{mk}/L_m) = F_r x^2$. It can be seen easily that if $(F_{mk}, L_{mk}/L_m) = d$, then $d = 1$ or $d = 2$ by (2.8). Assume that

$d = 1$. Since F_r is a prime number, we get

$$F_{mk} = u^2, \frac{L_{mk}}{L_m} = F_r v^2 \quad (2.23)$$

or

$$F_{mk} = F_r u^2, \frac{L_{mk}}{L_m} = v^2 \quad (2.24)$$

for some integers u and v . Assume that (2.23) is satisfied. Then $mk = 2$ or $mk = 12$ by Theorem 7. If $mk = 2$, then $m = 2$, $k = 1$, and therefore $F_r v^2 = L_{mk}/L_m = 1$, which is impossible. If $mk = 12$, then $m = 12$, $k = 1$ or $m = 4$, $k = 3$. A similar argument shows that $m = 12$, $k = 1$ and $m = 4$, $k = 3$ are impossible. Assume that (2.24) is satisfied. Then by Theorem 2 and Theorem 3, we get $mk = m = r$. This shows that $n = 2m$, and $x^2 = 1$ by Theorem 2. Assume that $d = 2$. Then, since F_r is odd prime number, it is seen that

$$F_{mk} = 2u^2, \frac{L_{mk}}{L_m} = 2F_r v^2 \quad (2.25)$$

or

$$F_{mk} = 2F_r u^2, \frac{L_{mk}}{L_m} = 2v^2 \quad (2.26)$$

for some integers u and v . Assume that (2.25) is satisfied. Then $mk = 3$ or $mk = 6$ by Theorem 7. If $mk = 3$, then $m = 3$, $k = 1$. If $m = 3$ and $k = 1$, then $F_r v^2 = L_{mk}/2L_m = 1/2$, which is impossible. If $mk = 6$, then $m = 6$, $k = 1$ or $m = 2$, $k = 3$. If $m = 6$, $k = 1$, then $F_r v^2 = L_{mk}/2L_m = 1/2$, which is impossible. If $k = 3$ and $m = 2$, then it can be seen that $n = 12$, $r = 4$, $x^2 = 16$. The identity (2.26) is impossible by Theorem 8.

Secondly, assume that k is an even integer. Then $(F_{mk}/L_m)L_{mk} = F_r x^2$. It can be easily seen that if $d = (F_{mk}/L_m, L_{mk})$, then $d = 1$ or $d = 2$ by (2.8). Assume that $d = 1$. Then

$$\frac{F_{mk}}{L_m} = u^2, L_{mk} = F_r v^2 \quad (2.27)$$

or

$$\frac{F_{mk}}{L_m} = F_r u^2, L_{mk} = v^2 \quad (2.28)$$

for some integers u and v . Assume that (2.27) is satisfied. Then by Theorem 10, it follows that $m = 2$, $k = 2$ or $m = 3$, $k = 4$ or $m = 12$, $k = 2$. But there is no solution of the equation $L_{mk} = F_r v^2$ in all situations. Assume that (2.28) is satisfied. Then $mk = 1$ or 3 by Theorem 7. But this is impossible since k is an even integer. Now assume that $d = 2$. Then we get

$$\frac{F_{mk}}{L_m} = 2u^2, L_{mk} = 2F_r v^2 \quad (2.29)$$

or

$$\frac{F_{mk}}{L_m} = 2F_r u^2, L_{mk} = 2v^2 \quad (2.30)$$

for some integers u and v . Assume that (2.29) is satisfied. Then $m = 3$, $k = 2$ or $m = 6$, $k = 2$ by Theorem 9. But there is no solution of the equation $L_{mk} = 2F_r v^2$ when $m = 3$, $k = 2$ or $m = 6$, $k = 2$. Assume that (2.30) is satisfied. Then $mk = 6$ by Theorem 7. It follows that $m = 3$, $k = 2$ and therefore $F_r u^2 = F_{mk}/2L_m = F_6/2L_3 = 1$. But this is impossible since F_r is a prime number. This completes the proof. $\square$

3. MAIN THEOREMS

In this section, we determine when the Diophantine equations

$$x^2 - 5F_n xy - 5(-1)^n y^2 = 5^r \quad (3.1)$$

and

$$x^2 - 5F_n xy - 5(-1)^n y^2 = -5^r \quad (3.2)$$

have positive integer solutions under the assumptions that $n \geq 1$, $r \geq 0$. After then, we find all positive integer solutions of the related Diophantine equations. Firstly, let us consider the above equations for $n = 1$. Then (3.1) and (3.2) turns into the equations

$$x^2 - 5xy + 5y^2 = 5^r$$

and

$$x^2 - 5xy + 5y^2 = -5^r,$$

respectively. Multiplying both sides of the above two equations by 4 and completing the square reduce them to the equations $u^2 - 5v^2 = 4 \cdot 5^r$ and $u^2 - 5v^2 = -4 \cdot 5^r$, respectively. So by means of Theorem 1, we can easily find all positive integer solutions of the equations $x^2 - 5xy + 5y^2 = 5^r$ and $x^2 - 5xy + 5y^2 = -5^r$. Now we will give all positive integer solutions of the equations $x^2 - 5xy + 5y^2 = 5^r$ and $x^2 - 5xy + 5y^2 = -5^r$ for the sake of completeness.

Theorem 12. *Let $r \geq 0$ be an integer. If r is an even integer, then all positive integer solutions of the equation $x^2 - 5xy + 5y^2 = 5^r$ are given by $(x, y) = (5^{r/2} L_{2m+1}, 5^{r/2} F_{2m})$ or $(x, y) = (5^{r/2} L_{2m-1}, 5^{r/2} F_{2m})$ with $m \geq 1$, and all positive integer solutions of the equation $x^2 - 5xy + 5y^2 = -5^r$ are given by $(x, y) = (5^{r/2} L_{2m+2}, 5^{r/2} F_{2m+1})$ or $(x, y) = (5^{r/2} L_{2m}, 5^{r/2} F_{2m+1})$ with $m \geq 0$. If r is an odd integer, then all positive integer solutions of the equation $x^2 - 5xy + 5y^2 = 5^r$ are given by $(x, y) = (5^{(r+1)/2} F_{2m+2}, 5^{(r-1)/2} L_{2m+1})$ with $m \geq 0$ or $(x, y) = (5^{(r+1)/2} F_{2m}, 5^{(r-1)/2} L_{2m+1})$ with $m \geq 1$, and all positive integer solutions of the equation $x^2 - 5xy + 5y^2 = -5^r$ are given by $(x, y) = (5^{(r+1)/2} F_{2m+1}, 5^{(r-1)/2} L_{2m})$ or $(x, y) = (5^{(r+1)/2} F_{2m-1}, 5^{(r-1)/2} L_{2m})$ with $m \geq 0$.*

From now on, we will assume that $n > 1$ and $k \geq 0$.

Theorem 13. *If n is an even integer, then all positive integer solutions of the equation $x^2 - 5F_nxy - 5(-1)^n y^2 = 5^{2k}$ are given by*

$$(x, y) = (5^k L_{2m+n}/L_n, 5^k F_{2m}/L_n)$$

with $m \geq 1$ and $n|m$, and if n is an odd integer, then all positive integer solutions of the equation $x^2 - 5F_nxy - 5(-1)^n y^2 = 5^{2k}$ are given by $(x, y) = (5^k L_{2m+n}/L_n, 5^k F_{2m}/L_n)$ or $(x, y) = (5^k L_{2m-n}/L_n, 5^k F_{2m}/L_n)$ with $m \geq 1$ and $n|m$.

Proof. Assume that $x^2 - 5F_nxy - 5(-1)^n y^2 = 5^{2k}$ for some positive integers x and y . Multiplying both sides of the equation by 4 and then completing the square give

$$(2x - 5F_ny)^2 - (25F_n^2 + 20(-1)^n) y^2 = 4 \cdot 5^{2k}. \quad (3.3)$$

It is easily seen from (2.9) that $25F_n^2 + 20(-1)^n = 5L_n^2$. Thus the equation (3.3) becomes $(2x - 5F_ny)^2 - 5(L_ny)^2 = 4 \cdot 5^{2k}$. By Theorem 1, we obtain $|2x - 5F_ny| = 5^k L_{2m}$ and $L_ny = 5^k F_{2m}$. Since $5 \nmid L_n$ and $L_ny = 5^k F_{2m}$, it follows that $L_n | F_{2m}$. Thus by Theorem 5, $n|2m$ and $2m/n$ is an even integer. This shows that $n|m$. Then from the equality $L_ny = 5^k F_{2m}$, we obtain $y = 5^k F_{2m}/L_n$. Assume that $2x - 5F_ny = 5^k L_{2m}$. Then substituting the value of y into $2x - 5F_ny = 5^k L_{2m}$, we obtain

$$x = \frac{5^k (L_{2m}L_n + 5F_nF_{2m})}{2L_n}.$$

By using the identity (2.3), we get $x = 5^k L_{2m+n}/L_n$. Now assume that $2x - 5F_ny = -5^k L_{2m}$. Then substituting the value of y into $2x - 5F_ny = -5^k L_{2m}$ and using the identity (2.2), we obtain $x = -5^k L_{n-2m}/L_n$. If n is even, then x is negative and we omit it. So n is odd and therefore $x = 5^k L_{2m-n}/L_n$. Thus when n is even, all positive integer solutions of the equation $x^2 - 5F_nxy - 5(-1)^n y^2 = 5^{2k}$ are given by $(x, y) = (5^k L_{2m+n}/L_n, 5^k F_{2m}/L_n)$ with $m \geq 1$, and when n is odd, all positive integer solutions of the equation $x^2 - 5F_nxy - 5(-1)^n y^2 = 5^{2k}$ are given by $(x, y) = (5^k L_{2m+n}/L_n, 5^k F_{2m}/L_n)$ or $(x, y) = (5^k L_{2m-n}/L_n, 5^k F_{2m}/L_n)$ with $m \geq 1$. Conversely, if n is an even integer and $(x, y) = (5^k L_{2m+n}/L_n, 5^k F_{2m}/L_n)$ with $m \geq 1$ and $n|m$, and if n is an odd integer and $(x, y) = (5^k L_{2m+n}/L_n, 5^k F_{2m}/L_n)$ or $(x, y) = (5^k L_{2m-n}/L_n, 5^k F_{2m}/L_n)$ with $m \geq 1$ and $n|m$, then by (2.11), it follows that $x^2 - 5F_nxy - 5(-1)^n y^2 = 5^{2k}$. $\square$

Theorem 14. *The equation $x^2 - 5F_nxy - 5(-1)^n y^2 = -5^{2k}$ has no positive integer solutions x and y .*

Proof. Assume that $x^2 - 5F_nxy - 5(-1)^n y^2 = -5^{2k}$ for some positive integers x and y . Multiplying both sides of the equation by 4 and then completing the square give

$$(2x - 5F_ny)^2 - (25F_n^2 + 20(-1)^n) y^2 = -4 \cdot 5^{2k}. \quad (3.4)$$

If we use the fact that $25F_n^2 + 20(-1)^n = 5L_n^2$ from (2.9), the equation (3.4) becomes $(2x - 5F_n y)^2 - 5(L_n y)^2 = -4 \cdot 5^{2k}$. By Theorem 1, we obtain $|2x - 5F_n y| = 5^k L_{2m+1}$ and $L_n y = 5^k F_{2m+1}$. Since $5 \nmid L_n$ and $L_n y = 5^k F_{2m+1}$, it follows that $L_n \mid F_{2m+1}$. Thus by Theorem 5, $n \mid 2m+1$ and $(2m+1)/n$ is an even integer, which is impossible. This completes the proof. $\square$

Theorem 15. *If n is an even integer, then the equation $x^2 - 5F_n xy - 5(-1)^n y^2 = 5^{2k+1}$ has no positive integer solutions x and y . If n is an odd integer, then all positive integer solutions of the equation $x^2 - 5F_n xy - 5(-1)^n y^2 = 5^{2k+1}$ are given by $(x, y) = (5^{k+1} F_{2m+1+n}/L_n, 5^k L_{2m+1}/L_n)$ with $m \geq 1$ and $n \mid 2m+1$ or $(x, y) = (5^{k+1} F_{2m+1-n}/L_n, 5^k L_{2m+1}/L_n)$ with $m > 1$ and $n \mid 2m+1$.*

Proof. Assume that $x^2 - 5F_n xy - 5(-1)^n y^2 = 5^{2k+1}$ for some positive integers x and y . Multiplying both sides of the equation by 4, we obtain $4x^2 - 20F_n xy - 20(-1)^n y^2 = 4 \cdot 5^{2k+1}$. Next completing the square and using the fact that $25F_n^2 + 20(-1)^n = 5L_n^2$ give

$$(2x - 5F_n y)^2 - 5(L_n y)^2 = 4 \cdot 5^{2k+1}.$$

By Theorem 1, we obtain $|2x - 5F_n y| = 5^{k+1} F_{2m+1}$ and $L_n y = 5^k L_{2m+1}$. Since $5 \nmid L_n$ and $L_n y = 5^k L_{2m+1}$, it follows that $L_n \mid L_{2m+1}$, which implies that $n \mid 2m+1$ and $(2m+1)/n$ is an odd integer by Theorem 6. This shows that n must be an odd integer. Then from the equality $L_n y = 5^k L_{2m+1}$, we obtain $y = 5^k L_{2m+1}/L_n$. Substituting the value of y into the equality $|2x - 5F_n y| = 5^{k+1} F_{2m+1}$ and using (2.1) and (2.4) give $x = 5^{k+1} F_{2m+1+n}/L_n$ or $x = 5^{k+1} F_{2m+1-n}/L_n$. So $(x, y) = (5^{k+1} F_{2m+1+n}/L_n, 5^k L_{2m+1}/L_n)$ with $m \geq 1$ and $n \mid 2m+1$ or $(x, y) = (5^{k+1} F_{2m+1-n}/L_n, 5^k L_{2m+1}/L_n)$ with $m > 1$ and $n \mid 2m+1$. Conversely, if n is an odd integer and

$$(x, y) = (5^{k+1} F_{2m+1+n}/L_n, 5^k L_{2m+1}/L_n)$$

with $m \geq 1$ and $n \mid 2m+1$ or $(x, y) = (5^{k+1} F_{2m+1-n}/L_n, 5^k L_{2m+1}/L_n)$ with $m > 1$ and $n \mid 2m+1$, then by (2.9) and (2.10), it follows that $x^2 - 5F_n xy - 5(-1)^n y^2 = 5^{2k+1}$. This completes the proof. $\square$

Theorem 16. *If n is an odd integer, then the equation $x^2 - 5F_n xy - 5(-1)^n y^2 = -5^{2k+1}$ has no positive integer solutions x and y . If n is an even integer, then all positive integer solutions of the equation $x^2 - 5F_n xy - 5(-1)^n y^2 = -5^{2k+1}$ are given by $(x, y) = (5^{k+1} F_{nt+n}/L_n, 5^k L_{nt}/L_n)$, where t is an odd integer.*

Proof. Assume that $x^2 - 5F_n xy - 5(-1)^n y^2 = -5^{2k+1}$ for some positive integers x and y . Multiplying both sides of the equation by 4 and then completing the square give

$$(2x - 5F_n y)^2 - 5(L_n y)^2 = -4 \cdot 5^{2k+1}.$$

By Theorem 1, we obtain $|2x - 5F_n y| = 5^{k+1} F_{2m}$ and $L_n y = 5^k L_{2m}$. Since $5 \nmid L_n$ and $L_n y = 5^k L_{2m}$, it follows that $L_n | L_{2m}$, which implies that $n | 2m$ and $2m/n$ is an odd integer by Theorem 6. Then $2m = nt$ for some odd integer t . This shows that n must be even. Then from the equality $L_n y = 5^k L_{2m}$, we obtain $y = 5^k L_{nt} / L_n$. Substituting the value of y into the equality $|2x - 5F_n y| = 5^{k+1} F_{2m}$ and using (2.1) and (2.4) give $x = 5^{k+1} F_{nt+n} / L_n$ or $x = 5^{k+1} F_{n-nt} / L_n$. Since $n | nt$, it follows that $n \leq nt$ and so the solution $x = 5^{k+1} F_{n-nt} / L_n$ is zero or negative. Therefore we omit it. Thus

$$(x, y) = (5^{k+1} F_{nt+n} / L_n, 5^k L_{nt} / L_n),$$

where t is an odd integer. Conversely, if n is an even integer and $(x, y) = (5^{k+1} F_{nt+n} / L_n, 5^k L_{nt} / L_n)$, where t is an odd integer, then by (2.9) and (2.10), it follows that $x^2 - 5F_n xy^2 - 5(-1)^n y^4 = -5^{2k+1}$. $\square$

Theorem 17. *If k is an even integer, then the equation $x^2 - 5F_n xy^2 - 5(-1)^n y^4 = 5^{2k}$ has positive integer solutions x and y only when $n = 2, 3, 12$. Moreover, the only positive integer solution of the equation $x^2 - 5F_2 xy^2 - 5y^4 = 5^{2k}$ is given by $(x, y) = (6.5^k, 5^{k/2})$, all positive integer solutions of the equation $x^2 - 5F_3 xy^2 + 5y^4 = 5^{2k}$ are given by $(x, y) = (19.5^k, 6.5^{k/2})$ or $(x, y) = (341.5^k, 6.5^{k/2})$, and the only positive integer solution of the equation $x^2 - 5F_{12} xy^2 - 5y^4 = 5^{2k}$ is given by $(x, y) = (103681.5^k, 12.5^{k/2})$. If k is an odd integer, then the equation $x^2 - 5F_n xy^2 - 5(-1)^n y^4 = 5^{2k}$ has positive integer solutions x and y only when $n = 5$, in which case all positive integer solutions are given by $(x, y) = (5^k, 5^{(k+1)/2})$ or $(x, y) = (124.5^k, 5^{(k+1)/2})$.*

Proof. Assume that $x^2 - 5F_n xy^2 - 5(-1)^n y^4 = 5^{2k}$ for some positive integers x and y . By Theorem 13, if n is even, then it follows that

$$(x, y^2) = (5^k L_{2m+n} / L_n, 5^k F_{2m} / L_n) \quad (3.5)$$

with $m \geq 1$ and $n | m$, and if n is odd, then we clearly have that

$$(x, y^2) = (5^k L_{2m+n} / L_n, 5^k F_{2m} / L_n) \quad (3.6)$$

or

$$(x, y^2) = (5^k L_{2m-n} / L_n, 5^k F_{2m} / L_n) \quad (3.7)$$

with $m \geq 1$ and $n | m$. Thus we obtain $y^2 = 5^k F_{2m} / L_n$. Now we divide the proof into two cases.

Case 1: Assume that k is an even integer. Then it follows that $F_{2m} = L_n u^2$ for some integer u . By Theorem 10, it is seen that $n = 2, m = 2$ or $n = 3, m = 6$ or $n = 12, m = 12$. Substituting these values of n and m into (3.5), (3.6), and (3.7), we obtain $(x, y) = (6.5^k, 5^{k/2})$ or $(x, y) = (19.5^k, 6.5^{k/2})$ or $(x, y) = (341.5^k, 6.5^{k/2})$ or $(x, y) = (103681.5^k, 12.5^{k/2})$, respectively.

Case 2: Assume that k is an odd integer. Then it follows that $L_n = 5F_{2m} v^2 = F_5 F_{2m} v^2$ for some integer v . By Theorem 11, it is seen that $n = 5$ and $m = 5$.

Substituting these values of n and m into (3.6), we obtain $(x, y) = (5^k, 5^{(k+1)/2})$ or $(x, y) = (124 \cdot 5^k, 5^{(k+1)/2})$. This completes the proof. $\square$

Theorem 18. Assume that n is an odd integer. If k is an odd integer, then the equation $x^2 - 5F_nxy^2 - 5(-1)^ny^4 = 5^{2k+1}$ has no positive integer solutions x and y . If k is an even integer, then there is only one positive integer solution of the equation $x^2 - 5F_nxy^2 - 5(-1)^ny^4 = 5^{2k+1}$ given by $(x, y) = (5^{k+1}F_n, 5^{k/2})$.

Proof. Assume that $x^2 - 5F_nxy^2 - 5(-1)^ny^4 = 5^{2k+1}$ for some positive integers x and y . By Theorem 15, if n is an odd integer, then it follows that $(x, y^2) = (5^{k+1}F_{2m+1+n}/L_n, 5^kL_{2m+1}/L_n)$ with $m \geq 1$ and $n|2m+1$ or $(x, y^2) = (5^{k+1}F_{2m+1-n}/L_n, 5^kL_{2m+1}/L_n)$ with $m > 1$ and $n|2m+1$. Thus we obtain $y^2 = 5^kL_{2m+1}/L_n$. Now we divide the proof into two cases.

Case 1: Assume that k is an even integer. Then from the equality $y^2 = 5^kL_{2m+1}/L_n$, it clearly follows that $L_{2m+1} = L_nu^2$ for some integer u . By Theorem 3, it is seen that $n = 2m+1$ and therefore $(x, y) = (5^{k+1}F_n, 5^{k/2})$.

Case 2: Assume that k is an odd integer. Since $y^2 = 5^kL_{2m+1}/L_n$, we clearly have that $L_{2m+1} = 5L_nv^2$ for some integer v . But this is impossible since $5 \nmid L_n$. This completes the proof. $\square$

Theorem 19. Assume that n is an even integer. If k is an odd integer, then the equation $x^2 - 5F_nxy^2 - 5(-1)^ny^4 = -5^{2k+1}$ has no positive integer solutions x and y . If k is an even integer, then there is only one positive integer solution of the equation $x^2 - 5F_nxy^2 - 5(-1)^ny^4 = -5^{2k+1}$ given by $(x, y) = (5^{k+1}F_n, 5^{k/2})$.

Proof. Assume that $x^2 - 5F_nxy^2 - 5(-1)^ny^4 = -5^{2k+1}$ for some positive integers x and y . By Theorem 16, if n is an even integer, then it follows that $(x, y^2) = (5^{k+1}F_{nt+n}/L_n, 5^kL_{nt}/L_n)$, where t is an odd integer. Thus $y^2 = 5^kL_{nt}/L_n$. Now we divide the proof into two cases.

Case 1: Assume that k is an even integer. Then it follows that $L_{nt} = L_nu^2$ for some integer u . By Theorem 3, it is seen that $nt = n$ and therefore $t = 1$. Thus we obtain $(x, y) = (5^{k+1}F_n, 5^{k/2})$.

Case 2: Assume that k is an odd integer. Then it follows that $L_{nt} = 5L_nv^2$ for some integer v . But this is impossible since $5 \nmid L_n$. This completes the proof. $\square$

Theorem 20. The equation $x^4 - 5F_nx^2y - 5(-1)^ny^2 = 5^{2k}$ has positive integer solutions x and y only when k is even and n is odd, in which case there is only one positive integer solution given by $(x, y) = (5^{k/2}, 5^kF_n)$.

Proof. Assume that $x^4 - 5F_nx^2y - 5(-1)^ny^2 = 5^{2k}$ for some positive integers x and y . By Theorem 13, if n is an even integer, then it follows that $(x^2, y) = (5^kL_{2m+n}/L_n, 5^kF_{2m}/L_n)$ with $m \geq 1$ and $n|m$, and if n is an odd integer, then it follows that $(x^2, y) = (5^kL_{2m+n}/L_n, 5^kF_{2m}/L_n)$ or $(x^2, y) = (5^kL_{2m-n}/L_n, 5^kF_{2m}/L_n)$ with $m \geq 1$ and $n|m$. From now on, we divide the proof into two cases.

Case 1: Assume that n is an even integer. Then it follows that $x^2 = 5^k L_{2m+n}/L_n$. Here if k is an even integer, then we get $L_{2m+n} = L_n u^2$ for some integer u . By Theorem 3, it is seen that $n = n + 2m$ and thus $m = 0$, which contradicts the fact that $m \geq 2$. If k is an odd integer, then a simple computation shows that $L_{2m+n} = 5L_n v^2$ for some integer v , which is impossible since $5 \nmid L_n$.

Case 2: Assume that n is an odd integer. Here we are not interested in the solution $x^2 = 5^k L_{2m+n}/L_n$. Because it is clear from Case 1 that, we have no solutions. Thus it follows that $x^2 = 5^k L_{2m-n}/L_n$. Here if k is an even integer, then we get $L_{2m-n} = L_n u^2$ for some integer u . By Theorem 3, it is seen that $2m - n = n$ and therefore $m = n$. Thus the only one positive integer solution is given by $(x, y) = (5^{k/2}, 5^k F_n)$. If k is an odd integer, then a simple computation shows that $L_{2m-n} = 5L_n v^2$ for some integer v , which is impossible since $5 \nmid L_n$. This completes the proof. $\square$

Theorem 21. *Assume that n is odd. If k is an even integer, then the equation $x^4 - 5F_n x^2 y - 5(-1)^n y^2 = 5^{2k+1}$ has positive integer solutions x and y only when $n = 5$, in which case all positive integer solutions are given by $(x, y) = (5^{(k+2)/2}, 5^k)$ or $(x, y) = (5^{(k+2)/2}, 124 \cdot 5^k)$. If k is an odd integer, then the equation $x^4 - 5F_n x^2 y - 5(-1)^n y^2 = 5^{2k+1}$ has positive integer solutions x and y only when $n = 3$, in which case all positive integer solutions are given by $(x, y) = (6 \cdot 5^{(k+1)/2}, 19 \cdot 5^k)$ or $(x, y) = (6 \cdot 5^{(k+1)/2}, 341 \cdot 5^k)$.*

Proof. Assume that $x^4 - 5F_n x^2 y - 5(-1)^n y^2 = 5^{2k+1}$ for some positive integers x and y . By Theorem 15, if n is an odd integer, then it follows that $(x^2, y) = (5^{k+1} F_{2m+1+n}/L_n, 5^k L_{2m+1}/L_n)$ with $m \geq 1$ and $n \mid 2m + 1$ or $(x^2, y) = (5^{k+1} F_{2m+1-n}/L_n, 5^k L_{2m+1}/L_n)$ with $m > 1$ and $n \mid 2m + 1$. Assume that $x^2 = 5^{k+1} F_{2m+1+n}/L_n$. If k is even, then we get $F_{2m+1+n} = 5L_n u^2$ for some integer u . Since n is odd, it follows that $n = 5$, $m = 2$ by Theorem 11. Thus we obtain $(x, y) = (5^{(k+2)/2}, 5^k)$. If k is odd, then we get $F_{2m+1+n} = L_n v^2$ for some integer v . Since n is odd, it is seen that $n = 3$, $m = 4$ by Theorem 10. Thus we obtain $(x, y) = (6 \cdot 5^{(k+1)/2}, 19 \cdot 5^k)$. Now assume that $x^2 = 5^{k+1} F_{2m+1-n}/L_n$. If k is even, then we get $F_{2m+1-n} = 5L_n u^2 = F_5 L_n u^2$ for some integer u . Since n is odd, this shows that $n = 5$, $m = 7$ by Theorem 11. Thus we obtain $(x, y) = (5^{(k+2)/2}, 124 \cdot 5^k)$. If k is odd, then it follows that $F_{2m+1-n} = L_n v^2$ for some integer v . Since n is odd, it is seen that $n = 3$, $m = 7$ by Theorem 10. Thus we obtain $(x, y) = (6 \cdot 5^{(k+1)/2}, 341 \cdot 5^k)$. This completes the proof. $\square$

Since the proof of the following theorem is similar to that of the above theorem, we omit it.

Theorem 22. *Assume that n is even. If k is an even integer, then the equation $x^4 - 5F_n x^2 y - 5(-1)^n y^2 = -5^{2k+1}$ has no positive integer solutions x and y . If k is an odd integer, then the equation $x^4 - 5F_n x^2 y - 5(-1)^n y^2 = -5^{2k+1}$ has positive*

integer solutions x and y only when $n = 2, 12$, in which cases there is only one solution given by $(x, y) = (5^{(k+1)/2}, 5^k)$ and $(x, y) = (12 \cdot 5^{(k+1)/2}, 5^k)$, respectively.

REFERENCES

- [1] L. Carlitz, "A note on Fibonacci numbers," *Fibonacci Q.*, vol. 2, no. 1, pp. 15–28, 1964.
- [2] J. Cohn, "Square fibonacci numbers, etc." *Fibonacci Q.*, vol. 2, pp. 109–113, 1964.
- [3] J. H. E. Cohn, "Twelve diophantine equations," *Arch. Math.*, vol. 65, no. 2, pp. 130–133, 1995.
- [4] J. H. E. Cohn, "The diophantine equation $x^4 - Dy^2 = 1$. II." *Acta Arith.*, vol. 78, no. 4, pp. 401–403, 1997.
- [5] T. W. Cusick, "The diophantine equation $x^4 - kx^2y^2 + y^4 = 1$," *Arch. Math.*, vol. 59, no. 4, pp. 345–347, 1992.
- [6] B. Demirtürk and R. Keskin, "Integer solutions of some Diophantine equations via Fibonacci and Lucas numbers," *J. Integer Seq.*, vol. 12, no. 8, p. 8, 2009.
- [7] M. J. jun. Jacobson and H. C. Williams, *Solving the Pell equation*, ser. CMS Books in Mathematics. New York: Springer, 2009.
- [8] H. W. jun. Lenstra, "Solving the Pell equation," *Notices Am. Math. Soc.*, vol. 49, no. 2, pp. 182–192, 2002.
- [9] D. Kalman and R. Mena, "The Fibonacci numbers – exposed," *Math. Mag.*, vol. 76, no. 3, pp. 167–181, 2003.
- [10] O. Karaatli and R. Keskin, "On some diophantine equations related to square triangular and balancing numbers," *J. Algebra Number Theory, Adv. Appl.*, vol. 4, no. 2, pp. 71–89, 2010.
- [11] R. Keskin, "Solutions of some quadratic Diophantine equations," *Comput. Math. Appl.*, vol. 60, no. 8, pp. 2225–2230, 2010.
- [12] R. Keskin and Z. Yosma, "Positive integer solutions of the Diophantine equations $x^2 - L_nxy + (-1)^n y^2 = \pm 5^r$," *submitted*.
- [13] R. Keskin and B. D. Bitim, "Fibonacci and Lucas congruences and their applications," *Acta Math. Sin., Engl. Ser.*, vol. 27, no. 4, pp. 725–736, 2011.
- [14] R. Keskin and Z. Yosma, "On Fibonacci and Lucas numbers of the form cx^2 ," *J. Integer Seq.*, vol. 14, no. 9, pp. article 11.9.3, 12, 2011.
- [15] T. Koshy, *Fibonacci and Lucas numbers with applications*, ser. Pure and Applied Mathematics. A Wiley-Interscience Series of Texts, Monographs, and Tracts. New York: Wiley, 2001.
- [16] K. Matthews, "The Diophantine equation $ax^2 + bxy + cy^2 = N$, $D = b^2 - 4ac > 0$," *J. Théor. Nombres Bordx.*, vol. 14, no. 1, pp. 257–270, 2002.
- [17] R. A. Mollin, "Quadratic Diophantine equations $x^2 - Dy^2 = c^n$," *Ir. Math. Soc. Bull.*, vol. 58, pp. 55–68, 2006.
- [18] L. J. Mordell, *Diophantine equations*, ser. Pure and Applied Mathematics. London-New York: Academic Press, 1969, vol. 30.
- [19] T. Nagell, *Introduction to number theory*. Stockholm-New York: Almqvist & Wiksell; John Wiley & Sons, Inc., 1951.
- [20] P. Ribenboim, *My numbers, my friends. Popular lectures on number theory*. New York: Springer, 2000.
- [21] J. P. Robertson, "Matthews' method for solving $ax^2 + bxy + cy^2 = N$," *manuscript*, <http://www.jpr2718.org/mtthbin.pdf>.
- [22] T. N. Shorey and C. L. Stewart, "Pure powers in recurrence sequences and some related diophantine equations," *J. Number Theory*, vol. 27, pp. 324–352, 1987.
- [23] T. N. Shorey and R. Tijdeman, *Exponential Diophantine equations*, ser. Cambridge Tracts in Mathematics. Cambridge: Cambridge University Press, 1986, vol. 87.

- [24] P. Stevenhagen, “A density conjecture for the negative Pell equation,” in *Computational algebra and number theory. Based on a meeting on computational algebra and number theory, held at Sydney University, Sydney, Australia, November of 1992*, ser. Math. Appl., Dordr. Dordrecht: Kluwer Academic Publishers, 1995, vol. 325, pp. 187–200.
- [25] S. Vajda, *Fibonacci & Lucas numbers, and the golden section. Theory and applications*, ser. Ellis Horwood Books in Mathematics and its Applications. Chichester-New York: Ellis Horwood Ltd.; Halsted Press, 1989.
- [26] G. Walsh, “A note on a theorem of Ljunggren and the diophantine equations $x^2 - kxy^2 + y^4 = 1, 4$,” *Arch. Math.*, vol. 73, no. 2, pp. 119–125, 1999.

Authors’ addresses

Refik Keskin

Sakarya University Faculty of Arts and Science TR54187 Sakarya Turkey

E-mail address: rkeskin@sakarya.edu.tr

Olcay Karaatlı

Sakarya University, Faculty of Arts and Science, TR54187, Sakarya, Turkey

E-mail address: okaraatli@sakarya.edu.tr

Zafer Şiar

Bilecik Şeyh Edebali University, Faculty of Arts and Science, TR11030, Bilecik, Turkey

E-mail address: zafer.siar@bilecik.edu.tr



GLOBAL BEHAVIOR OF SOLUTIONS TO A SECOND ORDER DELAY DIFFERENTIAL EQUATION

H. KHATIBZADEH

Received 4 February, 2013

Abstract. In this paper, we study the asymptotic behavior of eventually positive (negative) and oscillatory solutions to the following nonhomogeneous second order delay differential equation

$$\begin{cases} u''(t) = p(t)u(t - \tau) + f(t), \\ u(t) = \phi(t), \quad t \in [0, \tau), \end{cases}$$

where p and f are suitable real functions.

2010 Mathematics Subject Classification: 39A23; 47H05

Keywords: delay differential equation, asymptotic behavior, asymptotic stability, bounded solution

1. INTRODUCTION

Qualitative theory of delay differential, difference and functional equations is the subject of many recent publications. Specially, asymptotic behavior and oscillation of delay differential and difference equations of first and second order has been studied by many authors. We refer the reader to the interesting book by Gopalsamy [1] and to the recent articles [2–5]. In this paper, we consider the following second order delay differential equation

$$\begin{cases} u''(t) = p(t)u(t - \tau) + f(t), & t \geq \tau \\ u(t) = \phi(t), & t \in [0, \tau) \end{cases} \quad (1.1)$$

where $\tau > 0$ is a constant delay, and $p : [\tau, +\infty) \rightarrow \mathbb{R}^+$, $f : [\tau, +\infty) \rightarrow \mathbb{R}$, and $\phi : [0, \tau) \rightarrow \mathbb{R}$. In the special case $f(t) \equiv 0$ and $p(t) \equiv p > 0$, the equation (1.1) has the characteristic equation $\lambda^2 - pe^{-\lambda\tau} = 0$. This equation has negative roots for suitable constants p and τ . For example, if $\tau = 10$ and $p = \frac{1}{1000}$, then the equation $g(\lambda) = 1000\lambda^2 - e^{-10\lambda}$ has two negative roots $-\frac{1}{10} < \lambda_1 < 0$ and $-1 < \lambda_2 < -\frac{1}{10}$. Because $g(0) = -1 < 0$, $g(\frac{-1}{10}) = 10 - e > 0$ and $g(-1) = 1000 - e^{10} < 0$. Therefore $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are two bounded solutions of (1.1) and convergent to 0 as $t \rightarrow +\infty$. In this paper, our motivation is the study of the asymptotic behavior of solutions to (1.1)

in the case when f and p are nonconstant under the following suitable assumptions:

$$\int_{\tau}^{+\infty} tp(t)dt = +\infty, \quad (1.2)$$

$$\int_{\tau}^{+\infty} t|f(t)|dt < +\infty, \quad (1.3)$$

$$\limsup_{t \rightarrow +\infty} \int_{t-\tau}^t ds \int_{\tau}^s p(r)dr < 1, \quad (1.4)$$

$$\lim_{t \rightarrow +\infty} \frac{f(t)}{p(t)} = 0, \text{ where } p(t) > 0, \text{ for sufficiently large } t > 0. \quad (1.5)$$

Throughout the paper we assume that p , f , and ϕ are continuous. By a solution of equation (1.1), we mean a continuous function $u : [0, +\infty) \rightarrow \mathbb{R}$ which is twice continuously differentiable on $[\tau, +\infty)$ and satisfies the equation (1.1) for all $t \geq \tau$.

2. MAIN RESULTS

First, we prove two lemmas.

Lemma 1. *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be continuously differentiable and bounded from above, then $\liminf_{t \rightarrow +\infty} g'(t) \leq 0$.*

Proof. Suppose to the contrary, $\liminf_{t \rightarrow +\infty} g'(t) > \lambda > 0$. Then there exists $t_0 > 0$ such that for each $t \geq t_0$, $g'(t) > \lambda$. Integrating from t_0 to T , we have $g(T) - g(t_0) \geq \lambda(T - t_0)$. We get a contradiction by letting $T \rightarrow +\infty$. $\square$

Lemma 2. *Suppose that $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ is bounded from above and twice continuously differentiable such that*

$$g''(t) \geq -h(t), \quad \forall t \geq 0, \quad (2.1)$$

where $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous and $\int_0^{+\infty} th(t)dt < +\infty$. Then there exists $\lim_{t \rightarrow +\infty} g(t)$ and $\limsup_{t \rightarrow +\infty} g'(t) \leq 0$.

Proof. Integrating (2.1) from $t = S$ to $t = T$, where $S < T$, we get

$$g'(S) \leq g'(T) + \int_S^T h(t)dt.$$

Taking $\liminf$ as $T \rightarrow +\infty$, by Lemma 1, we get

$$g'(S) \leq \int_S^{+\infty} h(t)dt. \quad (2.2)$$

Taking $\limsup$ as $S \rightarrow +\infty$, we derive that: $\limsup_{t \rightarrow +\infty} g'(t) \leq 0$. Now, integrating (2.2) from $S = T_1$ to $S = T_2$, where $T_1 < T_2$, and applying Fubini's theorem, we obtain

$$\begin{aligned} g(T_2) &\leq g(T_1) + \int_{T_1}^{T_2} dS \int_S^{+\infty} h(t) dt \leq g(T_1) + \int_{T_1}^{+\infty} dS \int_S^{+\infty} h(t) dt \\ &= g(T_1) + \int_{T_1}^{+\infty} dt \int_{T_1}^t h(t) dS = g(T_1) + \int_{T_1}^{+\infty} (t - T_1) h(t) dt \\ &\leq g(T_1) + \int_{T_1}^{+\infty} t h(t) dt. \end{aligned}$$

Now, taking $\limsup$ as $T_2 \rightarrow +\infty$ and $\liminf$ as $T_1 \rightarrow +\infty$, the proof is complete. $\square$

Theorem 1. Suppose that u is a solution to (1.1).

- (1) If (1.2) and (1.3) are satisfied, then every eventually positive or eventually negative bounded solution to (1.1) converges to 0 as $t \rightarrow +\infty$.
- (2) If (1.4) and (1.5) are satisfied, then every oscillatory solution to (1.1) converges to 0 as $t \rightarrow +\infty$.

Proof. (1) Assume that $u(t)$ is eventually positive and bounded solution of (1.1). The same proof works for eventually negative solution of (1.1). Then for large t , $u''(t) \geq f(t) = f^+(t) - f^-(t) \geq -f^-(t)$, where $f^+(t) = \max\{f(t), 0\}$ and $f^-(t) = \max\{-f(t), 0\}$. By Lemma 2, there exists $\lim_{t \rightarrow +\infty} u(t) = l$ and $\limsup_{t \rightarrow +\infty} u'(t) \leq 0$. Suppose that $l > 0$, then there exists $t_0 > \tau$ such that for each $t > t_0$, $u(t - \tau) > \frac{l}{2}$. Integrating (1.1) from $t > t_0$ to T , we get

$$u'(T) - u'(t) \geq \frac{l}{2} \int_t^T p(s) ds - \int_t^T f^-(s) ds.$$

Taking $\limsup$ as $T \rightarrow +\infty$, we obtain

$$-u'(t) \geq \frac{l}{2} \int_t^{+\infty} p(s) ds - \int_t^{+\infty} f^-(s) ds, \quad \text{for } t > t_0.$$

If $\int_{\tau}^{\infty} p(s) ds = +\infty$, it is a contradiction; because $\int_{\tau}^{+\infty} f^-(t) dt < +\infty$. Otherwise, integrating from $t = t_0$ to $t = T$, we get

$$-u(T) + u(t_0) \geq \frac{l}{2} \int_{t_0}^T dt \int_t^{+\infty} p(s) ds - \int_{t_0}^T dt \int_t^{+\infty} f^-(s) ds.$$

Letting $T \rightarrow +\infty$, and by (1.2), (1.3) and Fubini's theorem, we get

$$-l + u(t_0) \geq \frac{l}{2} \int_{t_0}^{+\infty} dt \int_t^{+\infty} p(s) ds - \int_{t_0}^{+\infty} dt \int_t^{+\infty} f^-(s) ds$$

$$= \frac{l}{2} \int_{t_0}^{+\infty} (t - t_0) p(t) dt - \int_{t_0}^{+\infty} (t - t_0) f^-(t) dt = +\infty.$$

This is a contradiction. Then $l = 0$.

(2) Suppose that $u(t)$ oscillates. Then, there exists a sequence t_n of extreme points of $u(t)$ such that $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$. Let $\mu > 0$ such that

$\limsup_{t \rightarrow +\infty} \int_{t-\tau}^t ds \int_{\tau}^s p(r) dr < \mu < 1$ and let $0 < \delta < \frac{1-\mu}{1+\mu}$. The sequence $\{t_n\}$ of extreme points of u , has a subsequence $s_j = t_{n_j}$ such that:

1) $u(s_{2j}) \leq 0, u(s_{2j+1}) \geq 0$, for all $j \geq 1$.

2) s_{2j} is a minimum point of u in the interval $[s_{2j-1}, s_{2j+1}]$ and s_{2j+1} is a maximum point of u in the interval $[s_{2j}, s_{2j+2}]$, for all $j \geq 1$.

It is enough to prove, $u(s_j) \rightarrow 0$ as $j \rightarrow +\infty$. By (1.1), we have

$$u''(s_{2j}) = p(s_{2j})u(s_{2j} - \tau) + f(s_{2j}) \Rightarrow u(s_{2j} - \tau) \geq \frac{-f(s_{2j})}{p(s_{2j})} \quad (2.3)$$

$$\begin{aligned} u''(s_{2j+1}) &= p(s_{2j+1})u(s_{2j+1} - \tau) + f(s_{2j+1}) \\ &\Rightarrow u(s_{2j+1} - \tau) \leq \frac{-f(s_{2j+1})}{p(s_{2j+1})}. \end{aligned} \quad (2.4)$$

Give a subsequence s_{j_i} of s_j such that

$$\begin{aligned} s_{j_1} > \tau, s_{j_{i+1}} - s_{j_i} > \tau, \quad \frac{|f(t)|}{p(t)} < \delta^i, \quad \text{for each } t \geq s_{j_i}, \\ \text{and } \int_{s_n - \tau}^{s_n} ds \int_{\tau}^s p(r) dr < \mu < 1, \quad \text{for each } n \geq j_1. \end{aligned} \quad (2.5)$$

Integrating (1.1) on $[s_{j_i}, s]$, we get

$$u'(s) = u'(s) - u'(s_{j_i}) = \int_{s_{j_i}}^s p(r)u(r - \tau) dr + \int_{s_{j_i}}^s f(r) dr. \quad (2.6)$$

For large m such that $s_{2m} - \tau > s_{j_i}$, integrating (2.6) on $[s_{2m} - \tau, s_{2m}]$, and by (2.3), we obtain

$$\begin{aligned} u(s_{2m}) + \frac{f(s_{2m})}{p(s_{2m})} &\geq u(s_{2m}) - u(s_{2m} - \tau) = \\ &\int_{s_{2m} - \tau}^{s_{2m}} ds \int_{s_{j_i}}^s p(r)u(r - \tau) dr + \int_{s_{2m} - \tau}^{s_{2m}} ds \int_{s_{j_i}}^s f(r) dr. \end{aligned} \quad (2.7)$$

Integrating (2.6) on $[s_{2m+1} - \tau, s_{2m+1}]$, and by (2.4), we get

$$u(s_{2m+1}) + \frac{f(s_{2m+1})}{p(s_{2m+1})} \leq u(s_{2m+1}) - u(s_{2m+1} - \tau) =$$

$$\int_{s_{2m+1}-\tau}^{s_{2m+1}} ds \int_{s_{j_i}}^s p(r)u(r-\tau)dr + \int_{s_{2m+1}-\tau}^{s_{2m+1}} ds \int_{s_{j_i}}^s f(r)dr. \quad (2.8)$$

(2.7) and (2.8) imply that for sufficiently large n such that $s_n - \tau \geq s_{j_i}$

$$|u(s_n)| \leq \int_{s_n-\tau}^{s_n} ds \int_{s_{j_i}}^s p(r)|u(r-\tau)|dr + \int_{s_n-\tau}^{s_n} ds \int_{s_{j_i}}^s |f(r)|dr + \frac{|f(s_n)|}{p(s_n)}. \quad (2.9)$$

By (2.5) for each $n \geq j_i$, we have

$$\int_{s_n-\tau}^{s_n} ds \int_{s_{j_i}}^s p(r)dr < \mu$$

and

$$\int_{s_n-\tau}^{s_n} ds \int_{s_{j_i}}^s |f(r)|dr + \frac{|f(s_n)|}{p(s_n)} < \mu\delta^i + \delta^i.$$

Consider $\mu + \delta^i + \mu\delta^i < \mu + \delta + \mu\delta < 1$. From (2.9), we deduce that

$$|u(s_{j_2})| \leq M\mu + \delta + \mu\delta \leq (M+1)(\mu + \delta + \mu\delta),$$

where $M = \max_{0 \leq n \leq j_2} \{|u(s_n)|\}$. Suppose that for $j_2 \leq n \leq m$, we have

$$|u(s_n)| \leq (M+1)(\mu + \delta + \mu\delta),$$

then

$$\begin{aligned} & |u(s_{m+1})| \\ & \leq \int_{s_{m+1}-\tau}^{s_{m+1}} ds \int_{s_{j_1}}^s p(r)|u(r-\tau)|dr + \int_{s_{m+1}-\tau}^{s_{m+1}} ds \int_{s_{j_1}}^s |f(r)|dr + \frac{|f(s_{m+1})|}{p(s_{m+1})} \\ & \leq \max\{(M+1), |u(s_{m+1})|\}\mu + \delta + \mu\delta. \end{aligned}$$

If $|u(s_{m+1})| > M+1$, then

$$|u(s_{m+1})| \leq |u(s_{m+1})|\mu + \delta + \mu\delta.$$

Therefore

$$|u(s_{m+1})| \leq \frac{\delta + \mu\delta}{1 - \mu}.$$

This follows that $1 > \frac{\delta + \mu\delta}{1 - \mu} > M+1$. This is a contradiction. Therefore $|u(s_{m+1})| \leq M+1$. This implies that

$$|u(s_{m+1})| \leq (M+1)\mu + \delta + \mu\delta \leq (M+1)(\mu + \delta + \mu\delta).$$

Therefore, for all $n \geq j_2$, we get

$$|u(s_n)| \leq (M+1)(\mu + \delta + \mu\delta).$$

Now, by induction suppose that

$$|u(s_n)| \leq (\mu + \delta + \mu\delta)^k (M+1), \quad \text{for } n \geq j_{2k}. \quad (2.10)$$

By (2.9) for $n \geq j_{2k+2}$, we get

$$|u(s_n)| \leq \int_{s_n-\tau}^{s_n} ds \int_{s_{j_{2k+1}}}^s p(r)|u(r-\tau)|dr + \int_{s_n-\tau}^{s_n} ds \int_{s_{j_{2k+1}}}^s |f(r)|dr + \frac{|f(s_n)|}{p(s_n)}. \quad (2.11)$$

If $r \geq s_{j_{2k+1}}$, then $r - \tau \geq s_{j_{2k+1}} - \tau \geq s_{j_{2k}}$. By the hypothesis of induction and choosing the sequence s_n , we get

$$|u(r-\tau)| \leq (M+1)(\mu + \delta + \mu\delta)^k.$$

Now (2.11) implies that

$$\begin{aligned} |u(s_n)| &\leq (M+1)(\mu + \delta + \mu\delta)^k \mu + \delta^{2k+1} + \mu\delta^{2k+1} \\ &\leq (M+1)(\mu + \delta + \mu\delta)^k \mu + \delta^{k+1} + \mu\delta^{k+1} \leq (M+1)(\mu + \delta + \mu\delta)^{k+1}, \end{aligned} \quad (2.12)$$

for $n \geq j_{2k+2}$. This prove (2.10). The theorem is proved by letting $k \rightarrow +\infty$ in (2.10). $\square$

Corollary 1. *If the conditions (1.2), (1.3), (1.4) and (1.5) are satisfied, then every bounded solution to (1.1) converges to 0 as $t \rightarrow +\infty$.*

Now, we give some examples.

Example 1. Give $\tau = 1$, $\phi(t) = \frac{1}{t+1}$, $p(t) = \begin{cases} \frac{1}{t^2-1}, & t > 1 \\ 1, & t = 1, \end{cases}$

and $f(t) = \frac{t^2-4t-1}{t(t-1)(t+1)^3}$. The assumptions of Theorem 2.3 are satisfied, then every bounded and eventually positive (eventually negative) solution of (1.1) converges to 0 as $t \rightarrow +\infty$. As well as every its oscillatory solution converges to 0. For example, $u(t) = \frac{1}{t+1}$ is a bounded solution of (1.1) with the above conditions which converges to 0.

Example 2. $u(t) \equiv 1$ is a bounded and positive solution of

$$\begin{cases} u''(t) = p(t)u(t-\tau) - p(t), & t \geq \tau \\ u(t) = 1, & 0 \leq t \leq \tau \end{cases} \quad (2.13)$$

where $p : [\tau, +\infty) \rightarrow \mathbb{R}^+$. But neither [(1.2) and (1.3)] nor (1.5) are satisfied and $u(t)$ is not convergent to 0.

Example 3. $u(t) = \sin t$ is an oscillatory solution of

$$\begin{cases} u''(t) = u(t-\pi), & t \geq \pi \\ u(t) = \sin t, & 0 \leq t \leq \pi. \end{cases} \quad (2.14)$$

But (1.4) are not satisfied and $u(t)$ is not convergent to 0.

ACKNOWLEDGEMENT

The author would like to thank the referee for his (her) careful reading of the paper and for the valuable comments on the article's corrections.

REFERENCES

- [1] K. Gopalsamy, *Stability and oscillations in delay differential equations of population dynamics*, ser. Mathematics and its Applications. Dordrecht: Kluwer Academic Publishers, 1992, vol. 74.
- [2] J. R. Graef and C. Qian, "Global attractivity in differential equations with variable delays," *J. Aust. Math. Soc., Ser. B*, vol. 41, no. 4, pp. 568–579, 2000.
- [3] J. R. Graef, C. Qian, and B. Zhang, "Asymptotic behavior of solutions of differential equations with variable delays," *Proc. Lond. Math. Soc. (3)*, vol. 81, no. 1, pp. 72–92, 2000.
- [4] Y. Liu and W. Ge, "Asymptotic behavior of certain delay differential equations with forcing term," *J. Math. Anal. Appl.*, vol. 280, no. 2, pp. 350–363, 2003.
- [5] C. Qian and Y. Sun, "Global attractivity of solutions of nonlinear delay differential equations with a forcing term," *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 66, no. 3, pp. 689–703, 2007.

Author's address

H. Khatibzadeh

University of Zanjan, Department of Mathematics, Zanjan, Iran

E-mail address: hkhatibzadeh@znu.ac.ir



ON ABSOLUTE GENERALIZED HAUSDORFF SUMMABILITY OF ORTHOGONAL SERIES

XHEVAT Z. KRASNIQI

Received 21 February, 2012

Abstract. In this paper we prove two theorems on absolute generalized Hausdorff summability of orthogonal series. These theorems give some sufficient conditions in terms of coefficients of an orthogonal series under which it is absolute generalized Hausdorff summable. In addition, it is verified that several known results are corollaries of the new results.

2010 *Mathematics Subject Classification:* 42C15; 42C20; 40F05

Keywords: orthogonal series, absolute summability, Hausdorff matrices

1. INTRODUCTION

Hausdorff matrices have played an important role in summability theory and are linked with the moment problem for a finite interval. The matrices of such standard methods of summability as the Cesàro, the Hölder, the Euler and the weighted mean methods are all Hausdorff matrices.

On one hand, such matrices are used to define different types of absolute summability such as the absolute Nörlund summability, the absolute generalized Nörlund summability, the absolute Riesz summability, the absolute generalized Cesàro summability, and the absolute Euler summability. On the other hand, by many authors such notions are employed to study an interesting topic in theory of orthogonal series the so-called their absolute summability. For example, one can see the work of Tandori [19], Leindler [8–11], Okuyama and Tsuchikura [15], Okuyama [13, 14], Szalay [17], Billard [1], Grepačevskaya [4], Spevakov and Kudrajatsev [16]. In 2002 Okuyama [14] using generalized Nörlund means has proved two theorems which give sufficient conditions in terms of the coefficients of an orthogonal series under which it is absolute generalized Nörlund summable almost everywhere.

A generalization of a definition of absolute summability given by Borwien [2] is presented in the paper of Borwein et al in [3], called the absolute generalized Hausdorff summability. After some results proved by present author ([5–7]), which generalize some of the results of the above mentioned papers, we continue here to find

any sufficient condition under which an orthogonal series will be absolute generalized Hausdorff summable, and this is the main purpose of this paper.

2. NOTATIONS, NOTIONS AND KNOWN RESULTS

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with its partial sums $\{s_n\}$. Then, let p denotes the sequence $\{p_n\}$ of real numbers. For two given sequences p and q , the convolution $(p * q)_n$ is defined by

$$(p * q)_n = \sum_{m=0}^n p_m q_{n-m} = \sum_{m=0}^n p_{n-m} q_m.$$

When $(p * q)_n \neq 0$ for all n , the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$ obtained by putting

$$t_n^{p,q} = \frac{1}{(p * q)_n} \sum_{m=0}^n p_{n-m} q_m s_m.$$

The infinite series $\sum_{n=0}^{\infty} a_n$ is absolutely summable (N, p, q) if the series

$$\sum_{n=1}^{\infty} |t_n^{p,q} - t_{n-1}^{p,q}|$$

converges, and is written in brief

$$\sum_{n=0}^{\infty} a_n \in |N, p, q|.$$

The method $|N, p, q|$ of summability was introduced by Tanaka [18].

Let $\{\varphi_n(x)\}$ be an orthonormal system defined in the interval (a, b) . We assume that $f(x)$ belongs to $L^2(a, b)$ and

$$f(x) \sim \sum_{n=0}^{\infty} c_n \varphi_n(x), \quad (2.1)$$

where $c_n = \int_a^b f(x) \varphi_n(x) dx$, $(n = 0, 1, 2, \dots)$.

In [14] is written

$$R_n := (p * q)_n, \quad R_n^j := \sum_{m=j}^n p_{n-m} q_m$$

where

$$R_n^{n+1} = 0, \quad R_n^0 = R_n,$$

and the following two theorems are proved:

Theorem 1 ([14]). *If the series*

$$\sum_{n=1}^{\infty} \left\{ \sum_{j=1}^n \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |c_j|^2 \right\}^{\frac{1}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is summable $|N, p, q|$ almost everywhere.

Theorem 2 ([14]). *Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series $\sum_{n=1}^{\infty} |c_n|^2 \Omega(n) w(n)$ converges, then the orthogonal series $\sum_{n=0}^{\infty} c_n \varphi_n(x) \in |N, p, q|$ almost everywhere, where $w(n)$ is defined by $w(j) := j^{-1} \sum_{n=j}^{\infty} n^2 \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2$.*

The following definition is introduced by Borwein et al in [3]. Let $M := (a_{nk})$ ($n, k = 0, 1, \dots$) be a matrix. For the series $\sum_{n=0}^{\infty} a_n$ and its partial sums s_n let

$$\sigma_n = M(s_n) = \sum_{k=0}^{\infty} a_{nk} s_k.$$

Let

$$U_n = 1 - u_0 + \sum_{k=0}^n u_k \quad \text{where} \quad u_k > 0 \quad \text{for} \quad k = 0, 1, \dots,$$

and assume that γ is a real number and $\beta > 0$.

We say that the series $\sum_{n=0}^{\infty} a_n$ is absolute summable $|M, u_n, \gamma|_{\beta}$ or $\sum_{n=0}^{\infty} a_n \in |M, u_n, \gamma|_{\beta}$ if

$$\sum_{n=1}^{\infty} U_n^{\gamma\beta + \beta - 1} u_n^{1-\beta} |\sigma_n - \sigma_{n-1}|^{\beta} < \infty. \quad (2.2)$$

The main purpose of the present paper is to generalize Theorems 1 and 2 for $|M, u_n, \gamma|_{\beta}$ summability of the orthogonal series (2.1), where $1 \leq \beta \leq 2$. For the presentation of the main results we need to introduce some further notations and requirements. Firstly, we suppose that $M := (a_{nk})$ is a normal matrix, i.e. a lower triangular matrix of non-zero diagonal entries, and secondly, for it we associate another two lower matrices $\bar{M} := (\bar{a}_{nk})$ and $\hat{M} := (\hat{a}_{nk})$ as follows:

$$\bar{a}_{nk} := \sum_{i=k}^n a_{ni}, \quad n, i = 0, 1, 2, \dots$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \hat{a}_{nk} = \bar{a}_{nk} - \bar{a}_{n-1,k}, (n = 1, 2, \dots).$$

Throughout this paper, the letter K denotes a positive constant having different values in different contexts and depending only on k .

The following lemma due to Beppo Levi (see, for example [12]) is often used in the theory of functions. It will need us to prove main results.

Lemma 1. *If $f_n(t) \in L(E)$ are non-negative functions and*

$$\sum_{n=1}^{\infty} \int_E f_n(t) dt < \infty, \quad (2.3)$$

then the series

$$\sum_{n=1}^{\infty} f_n(t) dt$$

converges almost everywhere on E to a function $f(t) \in L(E)$. Moreover, the series (2.3) is also convergent to f in the norm of $L(E)$.

3. NEW RESULTS

We establish the following statement.

Theorem 3. *If the series*

$$\sum_{n=1}^{\infty} \left[\left(U_n^{\gamma+1-\frac{1}{\beta}} u_n^{\frac{1}{\beta}-1} \right)^2 \sum_{j=0}^n |\hat{a}_{n,j}|^2 |c_j|^2 \right]^{\frac{\beta}{2}}$$

converges for $1 \leq \beta \leq 2$, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is $|M, u_n, \gamma|_{\beta}$ -summable almost everywhere.

Proof. Let $1 < \beta < 2$. For the transform $\sigma_n(x) = M(s_n)(x)$ we have

$$\begin{aligned} \sigma_n(x) &= \sum_{k=0}^n a_{nk} s_k(x) = \sum_{k=0}^n a_{nk} \sum_{j=0}^k c_j \varphi_j(x) \\ &= \sum_{j=0}^n c_j \varphi_j(x) \sum_{k=j}^n a_{nk} = \sum_{j=0}^n \bar{a}_{nj} c_j \varphi_j(x) \end{aligned}$$

where $\sum_{j=0}^k c_j \varphi_j(x)$ is the partial sum of order k of the series (2.1).

Hence

$$\begin{aligned}
 \sigma_n(x) - \sigma_{n-1}(x) &= \sum_{j=0}^n \bar{a}_{nj} c_j \varphi_j(x) - \sum_{j=0}^{n-1} \bar{a}_{n-1,j} c_j \varphi_j(x) \\
 &= \bar{a}_{nn} c_n \varphi_n(x) + \sum_{j=0}^{n-1} (\bar{a}_{n,j} - \bar{a}_{n-1,j}) c_j \varphi_j(x) \\
 &= \hat{a}_{nn} c_n \varphi_n(x) + \sum_{j=0}^{n-1} \hat{a}_{n,j} c_j \varphi_j(x) = \sum_{j=0}^n \hat{a}_{n,j} c_j \varphi_j(x).
 \end{aligned}$$

Using the Hölder's inequality and orthogonality to the latter equality, with r and $s = \frac{2}{\beta}$ such that $r + s = rs$, we obtain

$$\begin{aligned}
 \int_a^b |\sigma_n(x) - \sigma_{n-1}(x)|^\beta dx &\leq (b-a)^{1-\frac{\beta}{2}} \left(\int_a^b |\sigma_n(x) - \sigma_{n-1}(x)|^2 dx \right)^{\frac{\beta}{2}} \\
 &= (b-a)^{1-\frac{\beta}{2}} \left(\int_a^b \left| \sum_{j=0}^n \hat{a}_{n,j} c_j \varphi_j(x) \right|^2 dx \right)^{\frac{\beta}{2}} \\
 &= (b-a)^{1-\frac{\beta}{2}} \left[\sum_{j=0}^n |\hat{a}_{n,j}|^2 |c_j|^2 \right]^{\frac{\beta}{2}}.
 \end{aligned}$$

Thus, the series

$$\begin{aligned}
 &\sum_{n=1}^{\infty} U_n^{\gamma\beta+\beta-1} u_n^{1-\beta} \int_a^b |\sigma_n(x) - \sigma_{n-1}(x)|^\beta dx \leq \\
 &\leq (b-a)^{1-\frac{\beta}{2}} \sum_{n=1}^{\infty} \left[\left(U_n^{\gamma+1-\frac{1}{\beta}} u_n^{\frac{1}{\beta}-1} \right)^2 \sum_{j=0}^n |\hat{a}_{n,j}|^2 |c_j|^2 \right]^{\frac{\beta}{2}} \quad (3.1)
 \end{aligned}$$

converges by the assumption since the last one does. Also we note that the functions $|\sigma_n(x) - \sigma_{n-1}(x)|$ are non-negative and integrable, therefore according to the Lemma 1 the series

$$\sum_{n=1}^{\infty} U_n^{\gamma\beta+\beta-1} u_n^{1-\beta} |\sigma_n(x) - \sigma_{n-1}(x)|^\beta$$

converges almost everywhere.

For $\beta = 1$ we use the Schwartz's inequality, until for $\beta = 2$ we use just the orthogonality. With this the proof is completed. $\square$

Now we shall prove a general theorem regarding to $|M, u_n, \gamma|_\beta$ -summability of an orthogonal series which involves a positive sequence with certain additional conditions.

For this reason first we denote

$$\phi^{(\beta, \gamma)}(j) := \frac{1}{j^{\frac{2}{\beta}-1}} \sum_{n=j}^{\infty} \left(n^{\frac{2}{\beta}-1} U_n^{\gamma+1-\frac{1}{\beta}} u_n^{\frac{1}{\beta}-1} |\hat{a}_{n,j}| \right)^2 \quad (3.2)$$

then the following theorem holds true.

Theorem 4. Let $1 \leq \beta \leq 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series

$$\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$$

converges.

If the series

$$\sum_{j=1}^{\infty} |c_j|^2 \Omega^{\frac{2}{\beta}-1}(j) \phi^{(\beta, \gamma)}(j)$$

converges, then

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \in |M, u_n, \gamma|_\beta$$

almost everywhere, where $\phi^{(\beta, \gamma)}(n)$ is defined by (3.2).

Proof. Applying the Hölder's inequality to the inequality (3.1) we get that

$$\begin{aligned} & \sum_{n=1}^{\infty} U_n^{\gamma\beta+\beta-1} u_n^{1-\beta} \int_a^b |\sigma_n(x) - \sigma_{n-1}(x)|^\beta dx \leq \\ & \leq K \sum_{n=1}^{\infty} \left[\left(U_n^{\gamma+1-\frac{1}{\beta}} u_n^{\frac{1}{\beta}-1} \right)^2 \sum_{j=0}^n |\hat{a}_{n,j}|^2 |c_j|^2 \right]^{\frac{\beta}{2}} \\ & = K \sum_{n=1}^{\infty} \frac{1}{(n\Omega(n))^{\frac{2-\beta}{2}}} \left[(n\Omega(n))^{\frac{2}{\beta}-1} \left(U_n^{\gamma+1-\frac{1}{\beta}} u_n^{\frac{1}{\beta}-1} \right)^2 \sum_{j=0}^n |\hat{a}_{n,j}|^2 |c_j|^2 \right]^{\frac{\beta}{2}} \\ & \leq K \left(\sum_{n=1}^{\infty} \frac{1}{(n\Omega(n))} \right)^{\frac{2-\beta}{2}} \left[\sum_{n=1}^{\infty} (n\Omega(n))^{\frac{2}{\beta}-1} \left(U_n^{\gamma+1-\frac{1}{\beta}} u_n^{\frac{1}{\beta}-1} \right)^2 \sum_{j=0}^n |\hat{a}_{n,j}|^2 |c_j|^2 \right]^{\frac{\beta}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq K \left\{ \sum_{j=1}^{\infty} |c_j|^2 \sum_{n=j}^{\infty} (n\Omega(n))^{\frac{2}{\beta}-1} \left(U_n^{\gamma+1-\frac{1}{\beta}} u_n^{\frac{1}{\beta}-1} \right)^2 |\hat{a}_{n,j}|^2 \right\}^{\frac{\beta}{2}} \\
&\leq K \left\{ \sum_{j=1}^{\infty} |c_j|^2 \left(\frac{\Omega(j)}{j} \right)^{\frac{2}{\beta}-1} \sum_{n=j}^{\infty} n^{2(\frac{2}{\beta}-1)} \left(U_n^{\gamma+1-\frac{1}{\beta}} u_n^{\frac{1}{\beta}-1} \right)^2 |\hat{a}_{n,j}|^2 \right\}^{\frac{\beta}{2}} \\
&= K \left\{ \sum_{j=1}^{\infty} |a_j|^2 \Omega^{\frac{2}{\beta}-1}(j) \phi^{(\beta,\gamma)}(j) \right\}^{\frac{\beta}{2}},
\end{aligned}$$

which is finite by assumption, and this completes the proof. $\square$

If $u_n = 1$ for $n = 0, 1, 2, \dots$, then (2.2) is equivalent to

$$\sum_{n=1}^{\infty} n^{\gamma\beta+\beta-1} |\sigma_n - \sigma_{n-1}|^{\beta} < \infty$$

which is the defining inequality in the definition of absolute summability given by Borwein [2]. Therefore from Theorems 3–4 we immediately obtain:

Corollary 1 ([6]). *If the series*

$$\sum_{n=1}^{\infty} \left[n^{2(\gamma+1-\frac{1}{\beta})} \sum_{j=0}^n |\hat{a}_{n,j}|^2 |c_j|^2 \right]^{\frac{\beta}{2}}$$

converges for $1 \leq \beta \leq 2$, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is $|M, \gamma|_{\beta}$ -summable almost everywhere.

Corollary 2 ([6]). *Let $1 \leq \beta \leq 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series*

$$\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$$

converges.

If the series

$$\sum_{j=1}^{\infty} |c_j|^2 \Omega^{\frac{2}{\beta}-1}(j) \theta^{(\beta,\gamma)}(j)$$

converges, then

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \in |M, \gamma|_{\beta}$$

almost everywhere, where $\phi^{(\beta, \gamma)}(n)$ is defined by

$$\theta^{(\beta, \gamma)}(j) := \frac{1}{j^{\frac{2}{\beta}-1}} \sum_{n=j}^{\infty} \left(n^{\gamma+\frac{1}{\beta}} |\hat{a}_{n,j}| \right)^2.$$

We note that Theorems 1-2 are corollaries of the main results as well. It is enough to put in Theorems 3-4: $u_n = 1$ for $n = 0, 1, 2, \dots$, $\gamma = 0$, $\beta = 1$, and $a_{nk} = \frac{p_{n-k}q_k}{R_n}$.

Namely, for such particular conditions we have

$$\begin{aligned} \hat{a}_{n,k} &= \bar{a}_{n,k} - \bar{a}_{n-1,k} = \sum_{j=k}^n a_{nj} - \sum_{j=k}^{n-1} a_{n-1,j} \\ &= \frac{1}{R_n} \sum_{j=k}^n p_{n-j} q_j - \frac{1}{R_{n-1}} \sum_{j=k}^{n-1} p_{n-1-j} q_j = \frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \Rightarrow \\ &\Rightarrow |\hat{a}_{n,k}|^2 = \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2, \end{aligned}$$

and similarly

$$\phi^{(1,0)}(j) := \frac{1}{j} \sum_{n=j}^{\infty} \left(n \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right) \right)^2,$$

which verify Theorems 1-2.

Remark 1. It should also noted that if we take the values of u_n , γ , β as above and:

- (i) $a_{nk} = \frac{p_{n-k}}{P_n}$
- (ii) $a_{nk} = \frac{q_k}{Q_n}$

one can obtain the results of [15], see also corollaries proved in [14].

REFERENCES

- [1] P. Billard, "Sur la sommabilité absolue des séries de fonctions orthogonales," *Bull. Sci. Math., II. Sér.*, vol. 85, pp. 29–33, 1961.
- [2] D. Borwein, "On strong ad absolute summability," *Proc. Glasg. Math. Assoc.*, vol. 4, pp. 122–139, 1960.
- [3] D. Borwein, F. P. Cass, and J. E. Sayre, "On absolute generalized Hausdorff summability," *Arch. Math.*, vol. 46, pp. 419–427, 1986.
- [4] L. V. Grepachevskaya, "Absolute summability of orthogonal series," *Mat. Sb. (N.S.)*, vol. 65, no. 107, pp. 370–389, 1964.
- [5] X. Z. Krasniqi, "On absolute weighted mean summability of orthogonal series," *Selçuk J. Appl. Math.*, vol. 12, no. 2, pp. 63–70, 2011.

- [6] X. Z. Krasniqi, "On $|A, \delta|_k$ -summability of orthogonal series," *Math. Bohem.*, vol. 137, no. 1, pp. 17–25, 2012.
- [7] X. Z. Krasniqi, "On absolute almost generalized Nörlund summability of orthogonal series," *Kyungpook Math. J.*, to appear.
- [8] L. Leindler, "On the absolute Riesz summability of orthogonal series," *Acta Sci. Math.*, vol. 46, pp. 203–209, 1983.
- [9] L. Leindler, "On the newly generalized absolute Riesz summability of orthogonal series," *Anal. Math.*, vol. 21, no. 4, pp. 285–297, 1995.
- [10] L. Leindler and K. Tandori, "On absolute summability of orthogonal series," *Acta Sci. Math.*, vol. 50, pp. 99–104, 1986.
- [11] L. Leindler, "Über die absolute Summierbarkeit der Orthogonalreihen," *Acta Sci. Math.*, vol. 22, pp. 243–268, 1961.
- [12] I. P. Natanson, *Theory of functions of a real variable (2 vols)*. New York: Frederick Ungar, 1955, 1961.
- [13] Y. Okuyama, "On the absolute Nörlund summability of orthogonal series," *Proc. Japan Acad., Ser. A*, vol. 54, pp. 113–118, 1978.
- [14] Y. Okuyama, "On absolute generalized Nörlund summability of orthogonal series," *Tamkang J. Math.*, vol. 33, no. 2, pp. 161–165, 2002.
- [15] Y. Okuyama and T. Tsuchikura, "On the absolute Riesz summability of orthogonal series," *Anal. Math.*, vol. 7, pp. 199–208, 1981.
- [16] V. N. Spevakov and A. B. Kudrjavcev, "Absolute summability of orthogonal series by Euler's method," *Math. Notes*, vol. 21, pp. 29–32, 1977.
- [17] I. Szalay, "On generalized absolute Cesàro summability of orthogonal series," *Acta Sci. Math.*, vol. 32, pp. 51–57, 1971.
- [18] M. Tanaka, "On generalized Nörlund methods of summability," *Bull. Aust. Math. Soc.*, vol. 19, pp. 381–402, 1978.
- [19] K. Tandori, "Über die orthogonalen Funktionen. IX." *Acta Sci. Math.*, vol. 21, pp. 292–299, 1960.

Author's address

Xhevat Z. Krasniqi

University of Prishtina, Department of Mathematics and Computer Sciences, Avenue Mother Teresa 5, 10000 Prishtinë, Republic of Kosova

E-mail address: xhevat.krasniqi@uni-pr.edu



AN ACCELERATION OF CONVERGENCE TO SOME GENERALIZED-EULER-CONSTANT FUNCTION

V. LAMPRET

Received 28 June, 2012

Abstract. For the generalized-Euler-constant function $\gamma(a)$,

$$\gamma(a) := \lim_{n \rightarrow \infty} \left(\sum_{k=0}^{n-1} \frac{1}{a+k} - \ln \frac{a+n-1}{a} \right) \quad (a > 0),$$

and for any positive integer $q \geq 2$, using the Bernoulli numbers B_{2m} , the sequences $n \mapsto \mathfrak{A}_n(a, q)$, $n \mapsto \mathfrak{B}_n(a, q)$ and $n \mapsto \mathfrak{C}_n(a, q)$, having the properties

$$\lim_{n \rightarrow \infty} n^{2q-2} [\gamma(a) - \mathfrak{A}_n(a, q)] = \frac{B_{2q-2}}{2q-2},$$

$$\lim_{n \rightarrow \infty} n^{2q-2} [\gamma(a) - \mathfrak{B}_n(a, q)] = -\left(1 - 2^{3-2q}\right) \frac{B_{2q-2}}{2q-2}$$

and

$$\lim_{n \rightarrow \infty} n^{2q-1} [\gamma(a) - \mathfrak{C}_n(a, q)] = \frac{1}{2} B_{2q-2},$$

are determined.

2010 *Mathematics Subject Classification:* 11B83; 40A05; 11B68; 33E20

Keywords: acceleration of convergence, generalized Euler's constant, generalized γ -sequence

1. INTRODUCTION

The gamma-sequence

$$y_n(a) = \sum_{k=0}^{n-1} \frac{1}{a+k} - \ln \frac{a+n-1}{a} \quad (n \in \mathbb{N}), \quad (1.1)$$

considered in [2, 3] is convergent for $a > 0$ and defines the generalized-Euler-constant function $\gamma(a)$,

$$\gamma(a) := \lim_{n \rightarrow \infty} y_n(a) \quad (1.2)$$

The name generalized-Euler-constant function has its origin in the identity $\gamma(1) = C$, where C is the Euler-Mascheroni constant. Several results on the rate of convergence of the sequence (1.1) have been established in the literature.

Recently, A. Sîntămărian [4] accelerated the convergence (1.2) using the Stolz-Cesaro limit theorem. In this reference the sequences

$$\alpha_{n,2}(a) := \sum_{k=0}^{n-1} \frac{1}{a+k} - \frac{1}{2(a+n-1)} + \frac{1}{12(a+n-1)^2} - \ln \left(\frac{a+n-1}{a} + \frac{1}{120a(a+n-1)^3} \right)$$

and

$$\beta_{n,2}(a) := \alpha_{n,2}(a) + \frac{1}{252(a+n-1)^6}$$

were considered and in Theorem 2 the equalities

$$\lim_{n \rightarrow \infty} n^6 [\gamma(a) - \alpha_{n,2}(a)] = \frac{1}{252} \quad (1.3)$$

and

$$\lim_{n \rightarrow \infty} n^8 [\beta_{n,2}(a) - \gamma(a)] = \frac{121}{28800} \quad (1.4)$$

were derived. Similarly, in Theorem 3, were considered some sequences $\alpha_{n,3}(a)$, $\beta_{n,3}(a)$ and $\delta_{n,3}(a)$ such that the following limits hold:

$$\lim_{n \rightarrow \infty} n^8 [\alpha_{n,3}(a) - \gamma(a)] = \frac{1}{240}, \quad (1.5)$$

$$\lim_{n \rightarrow \infty} n^{10} [\gamma(a) - \beta_{n,3}(a)] = \frac{1}{132} \quad (1.6)$$

and

$$\lim_{n \rightarrow \infty} n^{12} [\delta_{n,3}(a) - \gamma(a)] = \frac{174197}{8255520}. \quad (1.7)$$

In [4] the equalities above were demonstrated using rather tedious calculations.

The goal of this article is to complement/improve the results and the method of derivation as presented in [4]. In our paper we present an approach of incessant acceleration of the convergence (1.1) to any degree. We will present three classes of sequences converging to $\gamma(a)$ much faster than the original sequence $y_n(a)$ does.

2. PRELIMINARIES

Referring to (1.1), (1.2) and [1, Theorems 1–3], we have the following equalities¹

$$\gamma(a) = S_n(a, q) + R_n(a, q) \quad (n, q \in \mathbb{N}) \quad (2.1)$$

$$= \sigma_n(a, q) + \rho_n(a, q) \quad (n, q \in \mathbb{N}) \quad (2.2)$$

$$= S_n^*(a, q) + R_n^*(a, q) \quad (n, q \in \mathbb{N}) \quad (2.3)$$

with²

$$S_n(a, q) = \sum_{k=0}^{n-1} \frac{1}{a+k} - \ln \frac{a+n}{a} + \frac{1}{2(a+n)} + \sum_{j=1}^{q-1} \frac{B_{2j}}{2j(a+n)^{2j}}, \quad (2.4)$$

$$\sigma_n(a, q) = \sum_{k=0}^{n-1} \frac{1}{a+k} + \ln \left(\frac{a}{a+n-\frac{1}{2}} \right) - \sum_{i=1}^{q-1} \left(\frac{1-2^{1-2i}}{2i} \cdot \frac{B_{2i}}{(a+n-\frac{1}{2})^{2i}} \right) \quad (2.5)$$

and

$$\begin{aligned} S_n^*(a, q) &= \sum_{k=0}^{n-1} \frac{1}{a+k} - \ln \frac{a+n}{a} \\ &\quad - 1 - \ln \left(1 - \frac{1}{a+n+1} \right)^{a+n+1} + \frac{1}{2(a+n)} - \frac{1}{2} \ln \left(1 + \frac{1}{a+n} \right) \\ &\quad - \sum_{j=1}^{q-1} \frac{B_{2j}}{(2j)(2j-1)} \left[\frac{1}{(a+n)^{2j-1}} - \frac{1}{(a+n+1)^{2j-1}} - \frac{2j-1}{(a+n)^{2j}} \right]. \end{aligned} \quad (2.6)$$

The remainders are estimated as

$$|R_n(a, q)| < \frac{|B_{2q}|}{q(a+n)^{2q}}, \quad (2.7)$$

$$|\rho_n(a, q)| < \frac{|B_{2q}|}{q(a+n-\frac{1}{2})^{2q}} \quad (2.8)$$

¹The sequence $\sigma_n(a, q)$ in the expression (2.5) is given in the corrected form appearing in the proof of [1, Theorem 2], where in the first sum the start “ $k = 1$ ” should be replaced by “ $k = 0$ ” and where the summands in the third sum of $\sigma_n(a, q)$ are written incorrectly.

²By definition $\sum_{k=1}^m x_k = 0$ for $m < 1$.

and

$$|R_n^*(a, q)| < \frac{|B_{2q}|}{(a+n)^{2q+1}}. \quad (2.9)$$

Here, the symbol B_k means the k -th Bernoulli number,

$$\frac{te^{xt}}{e^t - 1} \equiv \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} \quad (x \in \mathbb{R}, |t| < 2\pi),$$

$B_k \equiv B_k(0)$, $B_k(x)$ is k -th Bernoulli polynomial.

3. AN ACCELERATION OF CONVERGENCE

Referring to (2.4)–(2.6) we make the following definition.

Definition 1. For any $a > 0$ and any integer $q \geq 2$ we consider the following sequences:

$$n \mapsto \mathfrak{A}_n(a, q) := S_n(a, q-1), \quad (3.1)$$

$$n \mapsto \mathfrak{B}_n(a, q) := \sigma_n(a, q-1) \quad (3.2)$$

and

$$n \mapsto \mathfrak{C}_n(a, q) := S_n^*(a, q-1). \quad (3.3)$$

Now, we are in the position to formulate the following result.

Theorem 1. For any positive a and any integer $q \geq 2$ we have the following limits:

$$\lim_{n \rightarrow \infty} n^{2q-2} [\gamma(a) - \mathfrak{A}_n(a, q)] = \frac{B_{2q-2}}{2q-2} =: L_{\mathfrak{A}}(q), \quad (3.4)$$

$$\lim_{n \rightarrow \infty} n^{2q-2} [\gamma(a) - \mathfrak{B}_n(a, q)] = -(1 - 2^{3-2q}) \frac{B_{2q-2}}{2q-2} =: L_{\mathfrak{B}}(q) \quad (3.5)$$

and

$$\lim_{n \rightarrow \infty} n^{2q-1} [\gamma(a) - \mathfrak{C}_n(a, q)] = \frac{1}{2} B_{2q-2} =: L_{\mathfrak{C}}(q). \quad (3.6)$$

Note that the limits are independent of a .

Proof. According to (2.1), (2.4) and (3.1), we have

$$\gamma(a) = \mathfrak{A}_n(a, q) + \frac{B_{2q-2}}{(2q-2)(a+n)^{2q-2}} + R_n(a, q),$$

for $n \geq 1$, $a > 0$ and $q \geq 2$. Consequently, using (2.7), the equality (3.4) follows. Similarly, referring to (2.2), (2.5) and (3.2), we get

$$\gamma(a) = \mathfrak{B}_n(a, q) - \frac{1 - 2^{3-2q}}{2q-2} \cdot \frac{B_{2q-2}}{(a+n-\frac{1}{2})^{2q-2}} + \rho_n(a, q),$$

for $n \geq 1$, $a > 0$ and $q \geq 2$. Thus, considering (2.8), we confirm (3.5). Finally, referring to (2.3), (2.6) and (3.3) we obtain

$$\begin{aligned} \gamma(a) &= \mathfrak{C}_n(a, q) + R_n^*(a, q) \\ &+ \frac{B_{2q-2}}{(2q-2)(2q-3)} \left[\frac{1}{(a+n+1)^{2q-3}} - \frac{1}{(a+n)^{2q-3}} + \frac{2q-3}{(a+n)^{2q-2}} \right], \end{aligned} \quad (3.7)$$

for $n \geq 1$, $a > 0$ and $q \geq 2$. Denoting $a+n=b$, $2q-3=m$ and using Taylor's formula of order 1 around b for the function $f(x) \equiv x^{-m}$, $(b+1)^{-m} = b^{-m} - mb^{-(m+1)} + \frac{1}{2}m(m+1)(b+\vartheta)^{-(m+2)}$, we obtain the equality

$$\frac{1}{(a+n+1)^{2q-3}} = \frac{1}{(a+n)^{2q-3}} - \frac{2q-3}{(a+n)^{2q-2}} + \frac{(2q-3)(2q-2)}{2(a+n+\vartheta)^{2q-1}}, \quad (3.8)$$

for some $\vartheta = \vartheta_n(a, q) \in (0, 1)$. From (3.7) and (3.8) we get the expression

$$\gamma(a) - \mathfrak{C}_n(a, q) = \frac{B_{2q-2}}{(2q-2)(2q-3)} \cdot \frac{(2q-3)(2q-2)}{2(a+n+\vartheta)^{2q-1}} + R_n^*(a, q),$$

which, recalling (2.9), demonstrates the relation (3.6). $\square$

Example 1. Referring to (3.4)–(3.6) and using [5] we obtain the following tables:

q	2	3	4	5	6	7
$L_{\mathfrak{A}}(q)$	$\frac{1}{12}$	$-\frac{1}{120}$	$\frac{1}{252}$	$-\frac{1}{240}$	$\frac{1}{132}$	$-\frac{691}{32760}$

TABLE 1. The type $\mathfrak{A}$ -limits; Theorem 1, Eq. (3.4).

q	2	3	4	5	6	7
$L_{\mathfrak{B}}(q)$	$-\frac{1}{24}$	$\frac{7}{960}$	$-\frac{31}{8064}$	$\frac{127}{30720}$	$-\frac{511}{67584}$	$\frac{1414477}{67092480}$

TABLE 2. The type $\mathfrak{B}$ -limits; Theorem 1, Eq. (3.5).

q	2	3	4	5	6	7
$L_{\mathfrak{C}}(q)$	$\frac{1}{12}$	$-\frac{1}{60}$	$\frac{1}{84}$	$-\frac{1}{60}$	$-\frac{5}{132}$	$-\frac{691}{5460}$

TABLE 3. The type $\mathfrak{C}$ -limits; Theorem 1, Eq. (3.6).

ACKNOWLEDGEMENT

The author express sincere thanks to the referee whose comments resulted in a significant improvement of the paper.

REFERENCES

- [1] V. Lampret, “A double inequality for a generalized-Euler-constant function,” *J. Math. Anal. Appl.*, vol. 381, no. 1, pp. 155–165, 2011.
- [2] A. Sîntămărian, “Some inequalities regarding a generalization of Euler’s constant,” *JIPAM, J. Inequal. Pure Appl. Math.*, vol. 9, no. 2, p. 7, 2008.
- [3] A. Sîntămărian, “A generalization of Euler’s constant,” *Numer. Algorithms*, vol. 46, no. 2, pp. 141–151, 2007.
- [4] A. Sîntămărian, “New sequences that converge to a generalization of Euler’s constant,” *Integers*, vol. 11, no. 2, pp. 127–138, A11, 2011.
- [5] S. Wolfram, *Mathematica, version 7.0*. Wolfram Research, Inc., 1988–2009.

Author’s address

V. Lampret

University of Ljubljana, Faculty of Civil and Geodetic Engineering, Jamova 2, 1000 Ljubljana, Slovenia

E-mail address: vito.lampret@fgg.uni-lj.si



ON τ -SUPPLEMENTED SUBGROUPS OF FINITE GROUPS

CHANGWEN LI, XUEMEI ZHANG, AND XIAOLAN YI

Received 30 January, 2013

Abstract. Let H be a subgroup of a finite group G . We say that: (1) H is τ -quasinormal in G if H permutes with every Sylow subgroup Q of G such that $(|H|, |Q|) = 1$ and $(|H|, |Q^G|) \neq 1$; (2) H is τ -supplemented in G if G has a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{\tau G}$, where $H_{\tau G}$ is the subgroup generated by all those subgroups of H which are τ -quasinormal in G . We investigate the influence of τ -supplemented subgroups on the structure of finite groups. Some recent known results are generalized and unified.

2010 *Mathematics Subject Classification:* 20D10; 20D20

Keywords: τ -supplemented, p -supersolvable supplement, p -nilpotent, saturated formation

1. INTRODUCTION

This paper deals with finite groups. We use the standard terminology as in [10]. G denotes always a group, $|G|$ is the order of G and the set of distinct primes dividing $|G|$ will be denoted by $\pi(G)$. A group G is called p -supersolvable if it is p -solvable and all its G -chief p -factors are cyclic. A group G is called p -nilpotent if it is p -solvable and all its G -chief p -factors are central in G . Obviously, a p -nilpotent group is also a p -supersolvable group and G is supersolvable (or nilpotent) if and only if G is p -supersolvable (or p -nilpotent) for any $p \in \pi(G)$. If $G = HK$ and K is p -supersolvable (or supersolvable, p -nilpotent), then we call that H has a p -supersolvable (or supersolvable, p -nilpotent) supplement K in G .

A subgroup H of a group G is said to be S -quasinormal (or S -permutable) in G if H permutes with all Sylow subgroups of G , i.e., $HS = SH$ for any Sylow subgroup S of G . This concept was first introduced by Kegel in [12]. Later, many authors generalized S -quasinormal concept; see, for example, [5, 13, 14, 34]. A subgroup H is said to be s -semipermutable in G if H permutes with every Sylow p -subgroup of G such that $(p, |H|) = 1$. More recently, Lukanenko and Skiba [19] introduced the concept of τ -quasinormal subgroup as follows: A subgroup H of G is said to be τ -quasinormal in G if H permutes with every Sylow subgroup Q of G such that $(|H|, |Q|) = 1$ and $(|H|, |Q^G|) \neq 1$. It is clear that s -semipermutability implies τ -quasinormality by definition; however, the converse is not true, as seen in [20, Example 1].

On the other hand, some subgroups with supplemented properties were introduced recently. In particular, Ballester-Bolinches, Wang and Guo [7] called that a subgroup H of a group G is c -supplemented in G if there is a subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the normal core of H in G . In 2007, Skiba [25] again gave the concept of S -supplemented subgroup as follows: A subgroup H of G is called S -supplemented in G if there exists a subgroup K such that $G = HK$ and $H \cap K \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are S -quasinormal in G . If we take $G = \langle a, b \mid a^{16} = b^4 = 1, ba = a^3b \rangle$, then $\langle b^2 \rangle$ is an S -supplemented subgroup of G . However $\langle b^2 \rangle$ is not c -supplemented in G . Hence, S -supplemented subgroups generalize c -supplemented subgroups.

There is no obvious general relationship between τ -quasinormal subgroups and S -supplemented subgroups. Hence it is meaningful to unify and generalize above series subgroups. On the basis of these definitions, we now introduce the following new concept:

Definition 1. A subgroup H of a group G is said to be τ -supplemented in G if G has a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{\tau G}$, where $H_{\tau G}$ is the subgroup generated by all those subgroups of H which are τ -quasinormal in G .

The next two examples show that the class of all τ -supplemented subgroups is wider than the class of all τ -quasinormal subgroups and the class of all S -supplemented subgroups.

Example 1. Let $G = S_4$ be the symmetric group of degree 4 and $H = \langle (14) \rangle$. Obviously, H is τ -supplemented in G . However, H is not τ -quasinormal in G .

Example 2. Let $G = \langle a, b, c \mid a^5 = b^4 = c^5 = 1, b^{-1}ab = a^2, [a, c] = [b, c] = 1 \rangle$ and $H = \langle b^2 \rangle$. It is easy to see that H is τ -supplemented in G , but not S -supplemented in G .

Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a formation provided that (i) if $G \in \mathcal{F}$ and $H \trianglelefteq G$, then $G/H \in \mathcal{F}$, and (ii) if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for any normal subgroups M, N of G . A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper, $\mathcal{U}$ and $\mathcal{N}$ will denote the class of all supersolvable groups and the class of all nilpotent groups, respectively. It is well known that both $\mathcal{U}$ and $\mathcal{N}$ are saturated formations. A chief factor H/K of a group G is called $\mathcal{F}$ -central provided $[H/K](G/C_G(H/K)) \in \mathcal{F}$. The product of all normal subgroups of G whose G -chief factors are $\mathcal{F}$ -central in G is called the $\mathcal{F}$ -hypercentre of G and denoted by $Z_{\mathcal{F}}(G)$. Noticing that for any $\mathcal{N}$ -central chief factor H/K of G we have $C_G(H/K) = G$. Hence the $\mathcal{N}$ -hypercentre of G coincides with the hypercentre $Z_{\infty}(G)$ of G . Another fact is that any chief factor of G under $Z_{\mathcal{U}}(G)$ is of prime order. The major aim of the present paper is to find cyclicity conditions for G -chief factors of normal subgroups of a group G by some

τ -supplemented subgroups. As their applications, we not only extend some known results in [1, 2, 4, 7, 9, 15, 16, 22, 28, 29, 32], but also give more simple proofs.

2. PRELIMINARIES

Lemma 1 ([19, Lemma 2.3]). *Let G be a group and $E \leq K \leq G$.*

- (1) *If E is τ -quasinormal in G , then E is τ -quasinormal in K .*
- (2) *Suppose that E is normal in G and $\pi(K/E) = \pi(K)$. If K is τ -quasinormal in G , then K/E is τ -quasinormal in G/E .*
- (3) *Suppose that E is normal in G . Then HE/E is τ -quasinormal in G/E for every τ -quasinormal subgroup E in G satisfying $(|H|, |E|) = 1$.*
- (4) *If E is τ -quasinormal in G and $E \leq O_p(G)$ for some prime p , then E is S -quasinormal in G .*

Lemma 2. *Let H be a τ -supplemented subgroup of a group G .*

- (1) *If $H \leq L \leq G$, then H is τ -supplemented in L .*
- (2) *If $E \trianglelefteq G$, $E \leq H \leq G$ and H is a p -group for some prime p , then H/E is τ -supplemented in G/E .*
- (3) *If H is a π -subgroup and E is a normal π' -subgroup of G , then HE/E is τ -supplemented in G/E .*

Proof. By the hypothesis, there is a subgroup K of G such that $G = HK$ and $H \cap K \leq H_{\tau G}$.

(1) $L = L \cap HK = H(L \cap K)$ and $H \cap (L \cap K) = H \cap K \leq H_{\tau G} \leq H_{\tau K}$. Hence H is τ -supplemented in L .

(2) We have $G/E = HK/E = H/E \cdot EK/E$ and $(H/E) \cap (KE/E) = (H \cap KE)/E = (H \cap K)E/E \leq H_{\tau G}E/E = H_{\tau G}/E \leq (H/E)_{\tau(G/E)}$ by Lemma 1. Hence H/E is τ -supplemented in G/E .

(3) Since $(|G : K|, |E|) = 1$ and $E \trianglelefteq G$, we have $E \leq K$. It is easy to see that $G/E = HE/E \cdot KE/E = HE/E \cdot K/E$ and $(HE/E) \cap (K/E) = (HE \cap K)/E = (H \cap K)E/E \leq H_{\tau G}E/E \leq (HE/E)_{\tau(G/E)}$ by Lemma 1. Hence HE/E is τ -supplemented in G/E . $\square$

Lemma 3 ([33, p.38, Theorem 7.19]). *Let H be a normal subgroup of G . Then $H \leq Z_{\mathcal{U}}(G)$ if and only if $H/\Phi(H) \leq Z_{\mathcal{U}}(G/\Phi(H))$.*

Lemma 4 ([26, Lemma A]). *Let E be a normal subgroup of a group G . Suppose that for every non-cyclic Sylow subgroup P of E , either all maximal subgroups of P or all cyclic subgroups of P of prime order and order 4 are S -supplemented in G . Then each G -chief factor below E is cyclic.*

Lemma 5 ([8, p.362, Proposition 3.11]). *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two saturated formations such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $Z_{\mathcal{F}_1}(G) \leq Z_{\mathcal{F}_2}(G)$.*

The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G . $F^*(G)$ is an important subgroup of G and it is a natural

generalization of $F(G)$. The definition and important properties can be found in [11, Chapter X].

Lemma 6 ([11, X, 13]). *Let G be a group. Then:*

- (1) *If $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*
- (2) *$C_G(F^*(G)) \leq F(G)$.*

Lemma 7 ([27, Theorem C]). *Let E be a normal subgroup of a group G . If $F^*(E) \leq Z_{\mathcal{U}}(G)$, then $E \leq Z_{\mathcal{U}}(G)$.*

Lemma 8 ([10, p.434, Satz 5.4 and p.281, Satz 5.2]). *If G is a group which is not p -nilpotent but all of its proper subgroups are p -nilpotent, then it is a minimal non-nilpotent group (that is, G is not nilpotent but all of its proper subgroups are nilpotent). Then*

- (1) *G has a normal Sylow p -subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q -subgroup for some prime $q \neq p$.*
- (2) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*
- (3) *If P is non-abelian and $p > 2$, then the exponent of P is p ; If P is non-abelian and $p = 2$, then the exponent of P is 4.*
- (4) *If P is abelian, then the exponent of P is p .*

Lemma 9 ([28, Lemma 2.8]). *Let M be a maximal subgroup of G , P a normal p -subgroup of G such that $G = PM$, where p is a prime. Then $P \cap M$ is a normal subgroup of G .*

Lemma 10 ([33, p.220, Theorem 6.3]). *Let P be a normal p -subgroup of G such that $|G/C_G(P)|$ is a power of p . Then $P \leq Z_{\infty}(G)$.*

3. MAIN RESULTS

Theorem 1. *Suppose that p is the smallest prime dividing the order of a group G and G has a normal subgroup E such that G/E is p -nilpotent. If every cyclic subgroup H of E with prime order p or order 4 (if $p = 2$) having no p -supersoluble supplement in G is τ -supplemented in G , then G is p -nilpotent.*

Proof. Suppose that the theorem is false, and let G be a counterexample of minimal order.

(1) The hypotheses are inherited by all proper subgroups of G and G is a group which is not nilpotent but whose proper subgroups are all nilpotent.

In fact, $\forall K < G$, since G/E is p -nilpotent, $K/(K \cap E) \cong KE/E$ is also p -nilpotent. Let H be any cyclic subgroup of $K \cap E$ with prime order p or order 4 (if $p = 2$). Obviously, H is a cyclic subgroup of E with prime order p or order 4. If H has a p -supersoluble supplement T in G , then H has a p -supersoluble supplement $T \cap K$ in K . If H is τ -supplemented in G , then H is τ -supplemented in K by Lemma

2(1). Thus K satisfies the hypotheses of the theorem. By the choice of G , K is p -nilpotent. Then G is a group which is not p -nilpotent but whose proper subgroups are all p -nilpotent. By Lemma 8, $G = PQ$ and $P \trianglelefteq G$.

(2) $G/(P \cap E)$ is p -nilpotent.

Since $G/P \cong Q$ is p -nilpotent, G/E is p -nilpotent and $G/(P \cap E) \lesssim G/P \times G/E$, we have $G/(P \cap E)$ is p -nilpotent.

(3) $P \leq E$.

If $P \not\leq E$, then $P \cap E < P$. So $Q(P \cap E) < QP = G$. Thus $Q(P \cap E)$ is nilpotent by Step (1) and so $Q(P \cap E) = Q \times (P \cap E)$. Since $G/(P \cap E) = P/(P \cap E) \cdot Q(P \cap E)/(P \cap E)$, it follows that $Q(P \cap E)/(P \cap E) \trianglelefteq G/(P \cap E)$ by Step (2). Now $Q \text{ char } Q(P \cap E) \trianglelefteq G$ implies that $G = P \times Q$, a contradiction.

(4) For every cyclic subgroup L of P with order p or 4, if there is a subgroup T of G such that $G = LT$, then $T = G$.

Let L be a cyclic subgroup of prime order p or of order 4 in P and assume that there exists a subgroup T of G such that $G = LT$. Obviously, $P = P \cap G = P \cap LT = L(P \cap T)$. Since $P/\Phi(P)$ is abelian, we have $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. By Step (1), $P \cap T \leq \Phi(P)$ or $P = (P \cap T)\Phi(P) = P \cap T$. If the former holds, then $L = P$ and so G is p -nilpotent by [24, P.280, Theorem 10.1.9], a contradiction. Hence $P = P \cap T$ and $T = G$.

(5) Every cyclic subgroup L of P with prime order p or of order 4 is S -quasinormal in G .

If L has a p -supersoluble supplement T in G , then $G = T$ is p -supersoluble by Step (4) and so G is p -nilpotent since p is the smallest prime dividing $|G|$, a contradiction. Thus we may assume all cyclic subgroups of P with order p or 4 are τ -supplemented in G . In view of Step (4), all cyclic subgroups of P with order p or 4 are τ -quasinormal in G . By Lemma 1(4), all cyclic subgroups of P with order p or 4 are S -quasinormal in G .

(6) Final contradiction.

For every $x \in P$, we have $| \langle x \rangle | = p$ or 4 by Step (1), and so $\langle x \rangle$ is S -quasinormal in G by Step (5). By [24, P.280, Theorem 10.1.9], we have $\langle x \rangle Q$ is a proper subgroup of G , and so $\langle x \rangle Q = \langle x \rangle \times Q$ by Step (1). Then we conclude that $G = P \times Q$, a contradiction. $\square$

Theorem 2. Suppose that P is a normal p -subgroup of a group G . If every cyclic subgroup of P with order p or 4 (if $p = 2$) having no p -supersoluble supplement in G is τ -supplemented in G , then $P \leq Z_{\mathcal{U}}(G)$.

Proof. We distinguish two cases:

Case I. $p = 2$.

Pick an arbitrary Sylow q -subgroup G_q of G , where $q \neq 2$. Consider the subgroup $W = G_q P$. Let L be a cyclic subgroup of P with order 2 or 4. If L has a 2-supersoluble supplement T in G , then L has a 2-supersoluble supplement $T \cap W$ in W . If L is τ -supplemented in G , then L is also τ -supplemented in W by Lemma

2(1). Applying Theorem 1, we have W is 2-nilpotent. Hence $W = P \times G_q$. This implies that $O^p(G) \leq C_G(P)$. In view of Lemma 10, $P \leq Z_\infty(G)$. Consequently, $P \leq Z_{\mathcal{U}}(G)$ from Lemma 5.

Case II. $p > 2$.

First suppose that some cyclic subgroup L of P with order p has a p -supersoluble supplement T in G . So $G = LT = PT$. If $T = G$, then $P \leq Z_{\mathcal{U}}(G)$ and we are done. Hence we may assume that $T < G$, which shows that $L \cap T = 1$ and T is a maximal subgroup of G . Clearly, $P = P \cap LT = L(P \cap T)$ and $P \cap T$ is a maximal subgroup of P . By Lemma 9, $P \cap T$ is normal in G . Since every cyclic subgroup of $P \cap T$ with order p having no p -supersoluble supplement in G is τ -supplemented in G , $P \cap T \leq Z_{\mathcal{U}}(G)$ by induction. Noticing that $P/(P \cap T)$ is a normal subgroup of $G/(P \cap T)$ with order p , we have $P/(P \cap T) \leq Z_{\mathcal{U}}(G/P \cap T)$. It follows that $P \leq Z_{\mathcal{U}}(G)$. Now we may assume that all cyclic subgroups of P with order p are τ -supplemented in G . From Lemma 1(4) we have that all cyclic subgroups of P with order p are S -supplemented in G since $P \leq O_p(G)$. In view of Lemma 4, we have also $P \leq Z_{\mathcal{U}}(G)$. $\square$

Theorem 3. *Let E be a normal subgroup of a group G . Suppose that for each $p \in \pi(E)$ and non-cyclic Sylow p -subgroup P of E , all cyclic subgroups of P with order p or 4 (if $p = 2$) having no p -supersoluble supplement in G are S -supplemented in G . Then $E \leq Z_{\mathcal{U}}(G)$.*

Proof. Let q be the smallest prime dividing $|E|$ and L a cyclic subgroup of the Sylow q -subgroup Q of E with order q or 4 (if $q = 2$). If L has a q -supersoluble supplement T in G , then L has a q -supersoluble supplement $T \cap E$ in E . If L is τ -supplemented in G , then L is also τ -supplemented in E by Lemma 2(1). In view of Theorem 1, E is q -nilpotent. Let $E_{q'}$ be the normal q' -complement of E . If $E = Q$, then $E \leq Z_{\mathcal{U}}(G)$ by Theorem 2. Hence we may assume that $E_{q'} \neq 1$. Since $E_{q'} \text{ char } E \trianglelefteq G$, $E_{q'} \trianglelefteq G$. By the hypothesis of the theorem, for each $p \in \pi(E_{q'})$ and non-cyclic Sylow p -subgroup P of $E_{q'}$, all cyclic subgroups of P with order p having no p -supersoluble supplement in G are S -supplemented in G . By induction, $E_{q'} \leq Z_{\mathcal{U}}(G)$. By Lemma 2(3), it is easy to see that all cyclic subgroups of $QE_{q'}/E_{q'}$ with order q or 4 (if $q = 2$) having no q -supersoluble supplement in $G/E_{q'}$ are S -supplemented in $G/E_{q'}$. By induction, we have also $E/E_{q'} \leq Z_{\mathcal{U}}(G/E_{q'})$. It follows that $E \leq Z_{\mathcal{U}}(G)$. $\square$

Corollary 1. *Let E be a normal subgroup of a group G . Suppose that every cyclic subgroup of each non-cyclic Sylow subgroup of E with prime order or order 4 having no supersoluble supplement in G are S -supplemented in G . Then $E \leq Z_{\mathcal{U}}(G)$.*

Theorem 4. *Let P be a normal p -subgroup of a group G , where p is a prime dividing the order of G . Suppose that every maximal subgroup of P having no p -supersoluble supplement in G is τ -supplemented in G . Then $P \leq Z_{\mathcal{U}}(G)$.*

Proof. We distinguish two cases:

Case I. $\Phi(P) \neq 1$.

Obviously, $P/\Phi(P)$ is a normal p -subgroup of $G/\Phi(P)$. Let $P_1/\Phi(P)$ be a maximal subgroup of $P/\Phi(P)$. Then P_1 is a maximal subgroup of P . If P_1 has a p -supersoluble supplement T in G , then $P_1/\Phi(P)$ has a p -supersoluble supplement $T\Phi(P)/\Phi(P)$ in $G/\Phi(P)$. If P_1 is τ -supplemented in G , then $P_1/\Phi(P)$ is τ -supplemented in $G/\Phi(P)$ by Lemma 2. Therefore, $G/\Phi(P)$ satisfies the hypothesis of the theorem. By induction, $P/\Phi(P) \leq Z_{\mathcal{U}}(G/\Phi(P))$. In view of Lemma 3, we have $P \leq Z_{\mathcal{U}}(G)$.

Case II. $\Phi(P) = 1$.

This shows that P is abelian. First suppose that some maximal subgroup V of P has a p -supersoluble supplement T in G . Then $G = VT = PT$ and $P \cap T \neq 1$. Since $P \cap T \trianglelefteq T$, we may assume that T has a minimal normal subgroup N contained in $P \cap T$. It is clear that $|N| = p$. From $G = PT$, we have N is also normal in G . With the similar argument in Case I, the hypothesis of the theorem holds for $(G/N, P/N)$. By induction, we have $P/N \leq Z_{\mathcal{U}}(G/N)$. It follows that $P \leq Z_{\mathcal{U}}(G)$. Now we may assume that every maximal subgroup of P is τ -supplemented in G . In view of Lemma 1, every maximal subgroup of P is S -supplemented in G since $P \leq O_p(G)$. Then we have also $P \leq Z_{\mathcal{U}}(G)$ by Lemma 4. $\square$

Corollary 2. *Let P be a normal p -subgroup of G . Suppose that every maximal subgroup of P having no supersoluble supplement in G is τ -supplemented in G . Then $P \leq Z_{\mathcal{U}}(G)$.*

In connection with Theorem 3 and Lemma 4 the following natural question arises:

Question. *Let E be a normal subgroup of a group G . Suppose that for every non-cyclic Sylow subgroup P of E all maximal subgroups of P having no supersoluble supplement in G are τ -supplemented in G . Is then $E \leq Z_{\mathcal{U}}(G)$?*

4. SOME APPLICATIONS

Theorem 5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup E such that $G/E \in \mathcal{F}$. If every maximal subgroup of each non-cyclic Sylow subgroup of $F(E)$ having no supersoluble supplement in G is τ -supplemented in G , then $G \in \mathcal{F}$.*

Proof. By Theorem 4, $F(E) \leq Z_{\mathcal{U}}(G)$. Since $\mathcal{U} \subseteq \mathcal{F}$, we have that $F(E) \leq Z_{\mathcal{F}}(G)$ by Lemma 5. In view of the solvability of E and Lemma 6, $F^*(E) = F(E) \leq Z_{\mathcal{F}}(G)$. By Lemma 7, $E \leq Z_{\mathcal{F}}(G)$. Since $G/Z_{\mathcal{F}}(G) \cong (G/E)/(Z_{\mathcal{F}}(G)/E) \in \mathcal{F}$, we have $G \in \mathcal{F}$. $\square$

Theorem 6. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If there is a normal subgroup E of a group G such that $G/E \in \mathcal{F}$ and every cyclic subgroup of E with*

prime order or order 4 having no supersoluble supplement in G is τ -supplemented in G , then $G \in \mathcal{F}$.

Proof. Since $E \leq Z_{\mathcal{U}}(G)$ by Theorem 3 and $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$ by Lemma 5, we have $E \leq Z_{\mathcal{F}}(G)$. Hence $G/Z_{\mathcal{F}}(G) \cong (G/E)/(Z_{\mathcal{F}}(G)/E) \in \mathcal{F}$. It follows that $G \in \mathcal{F}$. $\square$

Theorem 7. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. If there is a normal subgroup E such that $G/E \in \mathcal{F}$ and every cyclic subgroup of $F^*(E)$ with prime order or order 4 having no supersoluble supplement in G is τ -supplemented in G , then $G \in \mathcal{F}$.

Proof. By Theorem 3, $F^*(E) \leq Z_{\mathcal{U}}(G)$ and since $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$ by Lemma 5, we have $F^*(E) \leq Z_{\mathcal{F}}(G)$. In view of Lemma 7, $E \leq Z_{\mathcal{F}}(G)$. Hence $G \in \mathcal{F}$ since $G/E \in \mathcal{F}$. $\square$

Theorem 8. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup E such that $G/E \in \mathcal{F}$. If every cyclic subgroup of $F(E)$ with prime order or order 4 having no supersoluble supplement in G is τ -supplemented in G , then $G \in \mathcal{F}$.

Proof. Since E is solvable, $F^*(E) = F(E)$ by Lemma 6. Applying Theorem 7, we have $G \in \mathcal{F}$. $\square$

Corollary 3 ([22, Theorem 3.1]). Assume that G is solvable and every maximal subgroup of the Sylow subgroups of $F(G)$ is normal in G . Then G is supersoluble.

Corollary 4 ([4, Corollary 4.4]). Suppose that G is a solvable group with a normal subgroup E such that G/E is supersoluble. If all maximal subgroups of any Sylow subgroup of $F(E)$ are S -quasinormal in G , then G is supersoluble.

A subgroup H of a group G is c -normal in G if there is a normal subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the normal core of H in G .

Corollary 5 ([15, Theorem 2]). Let G be a group and E a solvable normal subgroup of G such that G/E is supersoluble. If every maximal subgroup of each Sylow subgroup of $F(E)$ is c -normal in G , then G is supersoluble.

Corollary 6 ([34, Theorem 2]). Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. Suppose that G is a group with a solvable normal subgroup E such that $G/E \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(E)$ are s -semipermutable in G , then $G \in \mathcal{F}$.

Corollary 7 ([16, Theorem 1.2]). Suppose that G is a solvable group with a normal subgroup E such that G/E is supersoluble. If every maximal subgroup of each Sylow subgroup of $F(E)$ is complement in G , then G is supersoluble.

Corollary 8 ([30, Theorem 1]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup E such that $G/E \in \mathcal{F}$. If every maximal subgroup of each Sylow subgroup of $F(E)$ is c -normal in G , then $G \in \mathcal{F}$.*

Corollary 9 ([1, Theorem 1.4]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a solvable group with a normal subgroup E such that $G/E \in \mathcal{F}$. If every maximal subgroup of each Sylow subgroup of $F(E)$ is S -quasinormal in G , then $G \in \mathcal{F}$.*

Corollary 10 ([28, Theorem 4.5]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup E such that $G/E \in \mathcal{F}$. If every maximal subgroup of each Sylow subgroup of $F(E)$ is c -supplemented in G , then $G \in \mathcal{F}$.*

Corollary 11 ([9, Theorem 1.6]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup E such that $G/E \in \mathcal{F}$. If every maximal subgroup of each Sylow subgroup of $F(E)$ is complemented in G , then $G \in \mathcal{F}$.*

A subgroup H is called Q -supplemented in a group G , if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is contained in $H_Q G$, where $H_Q G$ is the maximal quasinormal subgroup of G contained in H .

Corollary 12 ([21, Theorem 3.6]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup E such that $G/E \in \mathcal{F}$. If every maximal subgroup of each Sylow subgroup of $F(E)$ is Q -supplemented in G , then $G \in \mathcal{F}$.*

Corollary 13 ([34, Theorem 3]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If there is a normal subgroup E of G such that $G/E \in \mathcal{F}$ and every cyclic subgroup of E with prime order or order 4 is s -semipermutable in G , then $G \in \mathcal{F}$.*

Corollary 14 ([29, Theorem 4.2]). *If every cyclic subgroup of a group G with prime order or order 4 is c -normal in G , then G is supersolvable.*

Corollary 15 ([35, Theorem 3.1]). *Let G be a group and E a normal subgroup of a group G such that G/E is supersolvable. If every minimal subgroup of E is c -supplemented in G and if every cyclic subgroup of E with order 4 is c -normal in G , then G is supersolvable.*

Corollary 16 ([7, Theorem 4.1]). *If every cyclic subgroup of $G^{\mathcal{U}}$ with prime order or order 4 is c -supplemented in G , then G is supersolvable.*

Corollary 17 ([23, Theorem 3.9]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Then $G \in \mathcal{F}$ if and only if there is a normal subgroup E of G such that $G/E \in \mathcal{F}$ and the subgroups prime order or order 4 of E with are c -normal in G .*

Corollary 18 ([2, Theorem 1]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If there is a normal subgroup E of G such that $G/E \in \mathcal{F}$ and every cyclic subgroup of E with prime order or order 4 is S -quasinormal in G , then $G \in \mathcal{F}$.*

Corollary 19 ([6, Theorem 3.4]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If every cyclic subgroup of $G^{\mathcal{F}}$ with prime order or order 4 is c -normal in G , then $G \in \mathcal{F}$.*

Corollary 20 ([31, Theorem 3.2]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. If there is a normal subgroup E such that $G/E \in \mathcal{F}$ and the subgroups of $F^*(E)$ with prime order or order 4 are c -normal in G , then $G \in \mathcal{F}$.*

Corollary 21 ([18, Theorem 3.3]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. If there is a normal subgroup E such that $G/E \in \mathcal{F}$ and the subgroups of $F^*(E)$ with prime order or order 4 are S -quasinormal in G , then $G \in \mathcal{F}$.*

Corollary 22 ([32, Theorem 1.2]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. If there is a normal subgroup E such that $G/E \in \mathcal{F}$ and the subgroups of $F^*(E)$ with prime order or order 4 are c -supplemented in G , then $G \in \mathcal{F}$.*

Corollary 23 ([3, Corollary 1]). *Suppose that G is a group with a normal solvable subgroup E such that G/E is supersolvable. If every subgroup of $F(E)$ of prime order or order 4 is S -quasinormal in G , then G is supersolvable.*

Corollary 24 ([15, Theorem 3]). *Let G be a group and E a solvable normal subgroup of G such that G/E is supersolvable. If all minimal subgroups and all cyclic subgroups of $F(E)$ with order 4 are c -normal in G , then G is supersolvable.*

Corollary 25 ([28, Theorem 4.1]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup E such that $G/E \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups of $F(E)$ with order 4 is c -supplemented in G , then $G \in \mathcal{F}$.*

Corollary 26 ([30, Theorem 2]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup E such that $G/E \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups of $F(E)$ with order 4 is c -normal in G , then $G \in \mathcal{F}$.*

Corollary 27 ([17, Theorem 3]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. A group $G \in \mathcal{F}$ if and only if there is a solvable normal subgroup E of G such that $G/E \in \mathcal{F}$ and the subgroups of $F(E)$ with prime order or order 4 is c -normal in G .*

Corollary 28 ([3, Theorem]). *A group $G \in \mathcal{F}$ if and only if there is a solvable normal subgroup E of G such that $G/E \in \mathcal{F}$ and the subgroups of $F(E)$ with prime order or order 4 is S -quasinormal in G .*

Corollary 29 ([34, Theorem 4]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup E such that $G/E \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups of $F(E)$ with order 4 is s -semipermutable in G , then $G \in \mathcal{F}$.*

Acknowledgement. The authors would like to thank the referee who read the manuscript carefully and contributed a lot of valuable suggestions and useful comments. This research was partly supported by the Natural Science Foundation of China (grant no.11101369) and the Priority Academic Program Development of Jiangsu Higher Education Institutions.

REFERENCES

- [1] M. Asaad, "On maximal subgroups of Sylow subgroups of finite groups," *Commun. Algebra*, vol. 26, no. 11, pp. 3647–3652, 1998.
- [2] M. Asaad, A. Ballester-Bolinches, and M. C. Pedraza Aguilera, "A note on minimal subgroups of finite groups," *Commun. Algebra*, vol. 24, no. 8, pp. 2771–2776, 1996.
- [3] M. Asaad and P. Csörgö, "The influence of minimal subgroups on the structure of finite groups," *Arch. Math.*, vol. 72, no. 6, pp. 401–404, 1999.
- [4] M. Asaad, M. Ramadan, and A. Shaalan, "Influence of π -quasinormality on maximal subgroups of Sylow subgroups of Fitting subgroup of a finite group," *Arch. Math.*, vol. 56, no. 6, pp. 521–527, 1991.
- [5] A. Ballester-Bolinches and M. C. Pedraza-Aguilera, "Sufficient conditions for supersolubility of finite groups," *J. Pure Appl. Algebra*, vol. 127, no. 2, pp. 113–118, 1998.
- [6] A. Ballester-Bolinches and Y. Wang, "Finite groups with some c -normal minimal subgroups," *J. Pure Appl. Algebra*, vol. 153, no. 2, pp. 121–127, 2000.
- [7] A. Ballester-Bolinches, Y. Wang, and X. Guo, " c -supplemented subgroups of finite groups," *Glasg. Math. J.*, vol. 42, no. 3, pp. 383–389, 2000.
- [8] K. Doerk and T. Hawkes, *Finite soluble groups*, ser. De Gruyter Expositions in Mathematics. Berlin: W. de Gruyter, 1992, vol. 4.
- [9] X. Guo and K. P. Shum, "Complementarity of subgroups and the structure of finite groups," *Algebra Colloq.*, vol. 13, no. 1, pp. 9–16, 2006.
- [10] B. Huppert, *Endliche Gruppen. I.*, ser. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. Berlin-Heidelberg-New York: Springer-Verlag, 1967, vol. 134.
- [11] B. Huppert and N. Blackburn, *Finite groups. III*, ser. Grundlehren der Mathematischen Wissenschaften. Berlin-Heidelberg-New York: Springer-Verlag, 1982, vol. 243.
- [12] O. H. Kegel, "Sylow-Gruppen und Subnormalteiler endlicher Gruppen," *Math. Z.*, vol. 78, pp. 205–221, 1962.
- [13] C. Li, "Finite groups with some primary subgroups ss -quasinormally embedded," *Indian J. Pure Appl. Math.*, vol. 42, no. 5, pp. 291–306, 2012.
- [14] C. Li, "On weakly ss -permutable subgroups of finite groups," *Math. Notes, Miskolc*, vol. 12, no. 2, pp. 201–208, 2011.
- [15] D. Li and X. Guo, "The influence of c -normality of subgroups on the structure of finite groups. II," *Commun. Algebra*, vol. 26, no. 6, pp. 1913–1922, 1998.
- [16] D. Li and X. Guo, "On complemented subgroups of finite groups," *Chin. Ann. Math., Ser. B*, vol. 22, no. 2, pp. 249–254, 2001.
- [17] Y. Li, "Some notes on the minimal subgroups of Fitting subgroups of finite groups," *J. Pure Appl. Algebra*, vol. 171, no. 2-3, pp. 289–294, 2002.

- [18] Y. Li and Y. Wang, "The influence of minimal subgroups on the structure of a finite group," *Proc. Am. Math. Soc.*, vol. 131, no. 2, pp. 337–341, 2003.
- [19] V. O. Lukyanenko and A. N. Skiba, "On weakly τ -quasinormal subgroups of finite groups," *Acta Math. Hung.*, vol. 125, no. 3, pp. 237–248, 2009.
- [20] V. O. Lukyanenko and A. N. Skiba, "Finite groups in which τ -quasinormality is a transitive relation," *Rend. Semin. Mat. Univ. Padova*, vol. 124, pp. 231–246, 2010.
- [21] L. Miao, "Finite groups with some maximal subgroups of Sylow subgroups Q -supplemented," *Commun. Algebra*, vol. 35, no. 1, pp. 103–113, 2007.
- [22] M. Ramadan, "Influence of normality on maximal subgroups of Sylow subgroups of a finite group," *Acta Math. Hung.*, vol. 59, no. 1-2, pp. 107–110, 1992.
- [23] M. Ramadan, M. E. Mohamed, and A. A. Heliel, "On c -normality of certain subgroups of prime power order of finite groups," *Arch. Math.*, vol. 85, no. 3, pp. 203–210, 2005.
- [24] D. J. S. Robinson, *A course in theory of group*. Berlin: Springer-Verlag, 1982.
- [25] A. N. Skiba, "On weakly s -permutable subgroups of finite groups," *J. Algebra*, vol. 315, no. 1, pp. 192–209, 2007.
- [26] A. N. Skiba, "On two questions of L. A. Shemetkov concerning hypercyclically embedded subgroups of finite groups," *J. Group Theory*, vol. 13, no. 6, pp. 841–850, 2010.
- [27] A. N. Skiba, "A characterization of the hypercyclically embedded subgroups of finite groups," *J. Pure Appl. Algebra*, vol. 215, no. 3, pp. 257–261, 2011.
- [28] Y. Wang, H. Wei, and Y. Li, "A generalization of Kramer's theorem and its application," *Bull. Austral. Math. Soc.*, vol. 65, no. 3, pp. 467–475, 2002.
- [29] Y. Wang, " C -normality of groups and its properties," *J. Algebra*, vol. 180, no. 3, pp. 954–965, art. no. 0103, 1996.
- [30] H. Wei, "On c -normal maximal and minimal subgroups of Sylow subgroups of finite groups," *Commun. Algebra*, vol. 29, no. 5, pp. 2193–2200, 2001.
- [31] H. Wei, Y. Wang, and Y. Li, "On c -normal maximal and minimal subgroups of Sylow subgroups of finite groups. II," *Commun. Algebra*, vol. 31, no. 10, pp. 4807–4816, 2003.
- [32] H. Wei, Y. Wang, and Y. Li, "On c -supplemented maximal and minimal subgroups of Sylow subgroups of finite groups," *Proc. Am. Math. Soc.*, vol. 132, no. 8, pp. 2197–2204, 2004.
- [33] M. Weinstein, Ed., *Between nilpotent and solvable*. Passaic, New Jersey (USA): Polygonal Publishing House, 1982.
- [34] Q. Zhang and L. Wang, "The influence of s -semipermutable subgroups on the structure of finite groups," *Acta Math. Sin.*, vol. 48, no. 1, pp. 81–88, 2005.
- [35] X. Zhong and S. Li, "On c -supplemented minimal subgroups of finite groups," *Southeast Asian Bull. Math.*, vol. 28, no. 6, pp. 1141–1148, 2004.

Authors' addresses

Changwen Li

School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, 221116, China

E-mail address: lcw2000@126.com

Xuemei Zhang

Department of Basic Sciences, Yancheng Institute of Technology, Yancheng, 224051, China

E-mail address: xmzhang807@sohu.com

Xiaolan Yi

School of Science, Zhejiang Sci-Tech University, Hanzhou, 310018, China

E-mail address: yixiaolan2005@126.com



ON GOURSAT-DARBOUX BOUNDARY-VALUE PROBLEM FOR SYSTEMS OF NON-LINEAR DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

V. V. MARYNETS AND K. V. MARYNETS

Received 3 September, 2013

Abstract. We construct the modification of the two-sided method with improved characteristics of convergence for approximate integration of boundary-value problem for systems of second-order non-linear differential equations of the hyperbolic type on the plane when the bound of the domain of the independent variables consists of the pair of characteristics.

2010 Mathematics Subject Classification: 34B10; 34B15

Keywords: partial differential equations, Goursat-Darboux boundary-value problem, two-sided method, approximations

1. INTRODUCTION

In the theory of partial differential equations of the hyperbolic type the basic classical problems are already deeply studied. The domain of the independent variable is bounded by two pairs of parallel lines or by two pairs of characteristics, the pair of characteristics crossing "free" curve, etc [1, 2, 4].

Boundary-value problems for equations of hyperbolic type in the case of more complicated structure of the bound of the domain are studied less frequently.

In this work we generalize results obtained before in [6] and construct the modification of the two-sided method for approximate integration of the boundary-value problem for systems of second-order non-linear differential equations of hyperbolic type on the plane, when the bound of the domain consists of the pair of characteristics of the given system.

Note that constructive modifications of the numerical-analytic algorithm of investigation of the boundary-value problems for non-linear systems of ordinary differential equations subjected to different types of boundary restrictions have been studied in detail in [3, 7, 8].

Let us observe that the domain

$$D = \bigcup_s D_s \subset \mathbb{R}^2, s = 1, 2, 3,$$

where

$$D_1 = \{(x, y) | x \in (x_0, x_1], y \in (y_0, y_1]\},$$

$$D_2 = \{(x, y) | x \in [x_0, x_1], y \in (y_1, g_1(x))\},$$

$$D_3 = \{(x, y) | x \in (x_1, x_2], y \in (g_2(x), y_1]\},$$

$x_0 < x_1 < x_2$, $y_0 < y_1 < y_2$, $y = g_r(x)$ ($x = k_r(y)$), $x \in [x_{r-1}, x_r]$, $r = 1, 2$ are "free" curves [4], and $g_1(x_{r-1}) = y_r$, $g_2(x_r) = y_{r-1}$, $g'_r(x) > 0$.

The problem is to find the solution of the system of differential equations

$$L_2 U(x, y) = f(x, y, U(x, y)) := f[U(x, y)], \quad (1.1)$$

subjected to boundary restrictions of the form:

$$\begin{aligned} U(x_0, y) &= \Psi(y), U(x, y_0) = \Phi(x), (x, y) \in \overline{D}_1, \\ \Psi(y) &\in C^1[y_0, y_1], \Phi(x) \in C^1[x_0, x_1], \Psi(y_0) = \Phi(x_0), \end{aligned} \quad (1.2)$$

$$\begin{aligned} U(x, g_r(x)) &= \Omega_r(x), x \in [x_{r-1}, x_r], \Omega_r(x) \in C^1[x_{r-1}, x_r], r = 1, 2, \\ \Omega_1(x_0) &= \Psi(y_1), \Omega_2(x_1) = \Phi(x_1), \end{aligned} \quad (1.3)$$

in the space of vector-functions $C^*(\overline{D}) := C^{(1,1)}(D) \cap C(\overline{D})$, where

$$L_2 U(x, y) := U_{xy}(x, y) + A_1(x, y)U_x(x, y) + A_2(x, y)U_y(x, y),$$

$U(x, y) := (u_i(x, y))$, $f[U(x, y)] := (f_i[U(x, y)])$, $i = \overline{1, n}$, $\Omega_r(x) := (\omega_{i,r}(x))$, $\Psi(y) := (\psi_i(y))$, $\Phi(x) := (\phi_i(x))$ are some given vector-functions, $A_r(x, y) := (\delta_{i,j} a_{i,j}^{(r)})$, $r = 1, 2$, $j = \overline{1, n}$ are given matrixes and $\delta_{i,j}$ is the Kronecker symbol.

Function $U(x, y) = U_s(x, y)$, $(x, y) \in \overline{D}_s$, $s = 1, 2, 3$ is the solution of the boundary-value problem (1.1)-(1.3), where $U_1(x, y)$, $(x, y) \in \overline{D}_1$ is the solution of the Goursat problem (1.1), (1.2), $U_s(x, y)$, $s = 2, 3$, $(x, y) \in \overline{D}_s$, are the solutions of the Darboux problems (1.1), (1.3), and the following conditions

$$U_2(x, y_1) = U_1(x, y_1), U_3(x_1, y) = U_1(x_1, y), U_s(x, y) = (u_{s,i}(x, y))$$

take place.

Suppose that

$$A_1(x, y) \in C(D) \cap C^{(1,0)}(D_1 \cup D_3),$$

$$A_2(x, y) \in C(D) \cap C^{(0,1)}(D_1 \cup D_2),$$

$$f[U(x, y)] \in C(\overline{B}), f : \overline{B} \rightarrow \mathbb{R}^n, \overline{B} \subset \mathbb{R}^{n+2}.$$

It is easy to show that the boundary-value problem (1.1)-(1.3) can be written in the equivalent integral form

$$\begin{aligned} U_s(x, y) &= \Gamma_s(x, y) + \epsilon_s T_{1,s} F[U_1(\xi, \eta)] + T_s F[U_s(\xi, \eta)], \\ (x, y) &\in \overline{D}_s, s = 1, 2, 3, \end{aligned} \quad (1.4)$$

where

$$\epsilon_s = \begin{cases} 0, s = 1, \\ 1, s = 2, 3, \end{cases} \quad F[U(x, y)] := \begin{cases} F^*[U(x, y)], (x, y) \in D_1 \cup D_2, \\ F^{**}[U(x, y)], (x, y) \in D_3, \end{cases}$$

$$F^*[U(x, y)] := f[U(x, y)] + [A_{2y}(x, y) + A_1(x, y)A_2(x, y)]U(x, y),$$

$$F^{**}[U(x, y)] := F^*[U(x, y)] + [A_{1x}(x, y) - A_{2y}(x, y)]U(x, y),$$

$$T_1 F[U_1(\xi, \eta)] := \int_{x_0}^x \int_{y_0}^y K(x, y; \xi, \eta) F[U_1(\xi, \eta)] d\eta d\xi, (x, y) \in \overline{D}_1,$$

$$T_2 F[U_2(\xi, \eta)] := \int_{k_1(y)}^x \int_{y_1}^y K(x, y; \xi, \eta) F[U_2(\xi, \eta)] d\eta d\xi, (x, y) \in \overline{D}_2,$$

$$T_3 F[U_3(\xi, \eta)] := \int_{g_2(x)}^y \int_{x_1}^x K^{-1}(\xi, \eta; x, y) F[U_3(\xi, \eta)] d\xi d\eta, (x, y) \in \overline{D}_3,$$

$K(x, y; \xi, \eta) = (\delta_{i,j} k_{i,j}(x, y; \xi, \eta))$, $K^{-1}(x, y; \xi, \eta) = (\delta_{i,j} k_{i,j}^{-1}(x, y; \xi, \eta))$ are some matrixes,

$$k_{i,i}(x, y; \xi, \eta) := \exp \left(\int_x^\xi a_{i,i}^{(2)}(\tau, y) d\tau + \int_y^\eta a_{i,i}^{(1)}(\xi, \tau) d\tau \right),$$

$\Gamma_s(x, y) = (\gamma_{s,i}(x, y))$, $s = 1, 2, 3$ – are vector-functions,

$$\begin{aligned} \gamma_{1,i}(x, y) &:= \psi_i(y) \exp \left(\int_x^{x_0} a_{i,i}^{(2)}(\xi, y) d\xi \right) + \\ &+ \int_{x_0}^x k_{i,i}(x, y; \xi, \eta) [\phi_i'(\xi) + a_{i,i}^{(2)}(\xi, y_0) \phi_i(\xi)] d\xi, (x, y) \in \overline{D}_1, \end{aligned}$$

$$\begin{aligned} \gamma_{2,i}(x, y) &:= \omega_{i,1}(k_1(y)) \exp \left(\int_x^{k_1(y)} a_{i,i}^{(2)}(\xi, y) d\xi \right) + \\ &+ \int_{k_1(y)}^x k_{i,i}(x, y; \xi, y_0) [\phi_i'(\xi) + a_{i,i}^{(2)}(\xi, y_0) \phi_i(\xi)] d\xi, (x, y) \in \overline{D}_2, \end{aligned}$$

$$\begin{aligned} \gamma_{3,i}(x, y) &:= \omega_{i,2}(x) \exp \left(\int_y^{g_2(x)} a_{i,i}^{(1)}(x, \eta) d\eta \right) + \\ &+ \int_{g_2(x)}^y k_{i,i}^{-1}(x_0, \eta; x, y) [\psi_i'(\eta) + a_{i,i}^{(1)}(x_0, \eta) \psi_i(\eta)] d\eta, (x, y) \in \overline{D}_3, \end{aligned}$$

$$T_{1,2} F[U_1(\xi, \eta)] := \int_{k_1(y)}^x \int_{y_1}^y K(x, y; \xi, \eta) F[U_1(\xi, \eta)] d\eta d\xi, (x, y) \in \overline{D}_2,$$

$$T_{1,3} F[U_1(\xi, \eta)] := \int_{g_2(x)}^y \int_{x_0}^{x_1} K^{-1}(x, y; \xi, \eta) F[U_1(\xi, \eta)] d\xi d\eta, (x, y) \in \overline{D}_3.$$

Remark 1. If $A_{1x}(x, y) = A_{2y}(x, y)$, $(x, y) \in D$ then $F^*[U(x, y)] \equiv F^{**}[U(x, y)]$ and $K(x, y; \xi, \eta) \equiv K^{-1}(\xi, \eta; x, y)$.

According to the problem setting $U_{1x}(x, y_1) = U_{2x}(x, y_1)$ and $U_{1y}(x_1, y) = U_{3y}(x_1, y)$ when $x \in [x_0, x_1]$, $y \in [y_0, y_1]$ and

$$\begin{aligned} u_{1,i_y}(x, y_1) - u_{2,i_y}(x, y_1) &= \rho_{1,i} \exp \left(\int_x^{x_0} a_{i,i}^{(2)}(\xi, y_1) d\xi \right), x \in [x_0, x_1], \\ u_{1,i_x}(x_1, y) - u_{3,i_x}(x_1, y) &= \rho_{2,i} \exp \left(\int_y^{y_0} a_1(x_1, \eta) d\eta \right), y \in [y_0, y_1], \end{aligned} \quad (1.5)$$

where

$$\begin{aligned} \rho_{1,i} &:= \psi'_i(y_1) - k'_1(y_1) \left\{ \omega'_{i,1}(x_0) + a_{i,i}^{(2)}(x_0, y_1) \omega_{i,1}(x_0) - \right. \\ &\quad - \left[\phi'_i(x_0) + a_{i,i}^{(2)}(x_0, y_0) \phi_i(x_0) \right] \exp \left(\int_{y_1}^{y_0} a_{i,i}^{(1)}(x_0, \eta) d\eta \right) - \\ &\quad - \int_{y_0}^{y_1} [f_i(x_0, \eta, \psi_1(\eta), \dots, \psi_n(\eta)) + (a_{i,i\eta}^{(2)}(x_0, \eta) + a_{i,i}^{(1)}(x_0, \eta) a_{i,i}^{(2)}(x_0, \eta) \psi_i(\eta))] \\ &\quad \times \exp \left(\int_{y_1}^{y_0} a_{i,i}^{(1)}(x_0, \eta) d\eta \right) \Big\}, \\ \rho_{2,i} &:= \phi'_i(x_1) - \omega'_{i,2}(x_1) - g'_2(x_1) \left\{ a_{i,i}^{(1)}(x_1, y_0) \omega_{i,2}(x_1) - \right. \\ &\quad - \left[\psi'_i(x_0) + a_{i,i}^{(1)}(x_0, y_0) \psi_i(y_0) \right] \exp \left(\int_{x_1}^{x_0} a_{i,i}^{(2)}(\xi, y_0) d\xi \right) - \\ &\quad - \int_{x_0}^{x_1} [f_i(\xi, y_0, \phi_1(\xi), \dots, \phi_n(\xi)) + (a_{i,i\xi}^{(1)}(\xi, y_0) + a_{i,i}^{(2)}(\xi, y_0) \phi_i(\xi))] \\ &\quad \times \exp \left(\int_{x_1}^{\xi} a_{i,i}^{(2)}(\tau, y_0) d\tau \right) d\xi \Big\}. \end{aligned}$$

The following lemma holds.

Lemma 1. Let $f[U(x, y)] \in C(\overline{B})$, $A_r(x, y) \in C(D)$, $r = 1, 2$, $A_1(x, y) \in C^{(1,0)}(D_1 \cup D_3)$, $A_2(x, y) \in C^{(0,1)}(D_1 \cup D_2)$ and the boundary-value problem (1.1)-(1.3) has the solution.

The solutions of the boundary-value problem (1.1)-(1.3) are regular (it means that $U(x, y) \in C^*(\overline{D})$) if and only if the equality

$$\rho_{r,i} = 0,$$

is true, for all $r = 1, 2$ and $i = \overline{1, n}$.

In the other case equalities (1.5) hold and the solution of the problem (1.1)-(1.3) is irregular.

Definition 1. We say that the vector-function $F[U(x, y)] \in C_1(\overline{B})$ if it satisfies the following conditions [5]:

- (1) $F[U(x, y)] \in C(\overline{B})$,
- (2) there exists vector-function

$$H(x, y, U(x, y); V(x, y)) := H[U(x, y); V(x, y)]$$

in the space of vector-functions $C(\overline{B}_1)$, $\overline{B}_1 \subset \mathbb{R}^{2(n+1)}$, $Pr_{xOy} \overline{B}_1 = \overline{D}$, such that

- a) $H[U(x, y); V(x, y)] \equiv F[U(x, y)]$,
- b) for any pair of vector-functions $U(x, y), V(x, y) \in \overline{B}_1$ from the space $C(\overline{D})$ that satisfy condition

$$U(x, y) \geq V(x, y), (x, y) \in \overline{D},$$

in the domain $\overline{B}_1$, the inequality

$$H[U(x, y); V(x, y)] - H[V(x, y); U(x, y)] \geq 0 \quad (1.6)$$

is true,

- (3) vector-function $H[U(x, y); V(x, y)]$ satisfies the Lipschitz condition in the domain $\overline{B}_1$; it means that for any vector-functions $U_r(x, y), V_r(x, y) \in \overline{B}_1$, $r = 1, 2$ from the space $C(\overline{D})$ the following condition

$$|H[U_1(x, y); U_2(x, y)] - H[V_1(x, y); V_2(x, y)]| \leq L(|W_1(x, y)| + |W_2(x, y)|),$$

holds, where $W_r(x, y) := U_r(x, y) - V_r(x, y)$, $r = 1, 2$, $L = (\delta_{i,j}, l_{i,j})$ is the Lipschitz matrix, $l_{i,j} \geq 0$, $i, j = \overline{1, n}$.

If vector-function $f[U(x, y)] \in C(\overline{B})$ and has bounded partial derivatives of the first order on all arguments, starting from the third one, then $F[U(x, y)]$ always belongs to the space $C_1(\overline{B})$. The opposite statement isn't true.

Let us set sufficient conditions of existence and uniqueness of regular or irregular solution of the problem (1.1)-(1.3), when $(x, y) \in \overline{D}$.

Suppose that the vector-functions $Z_{s,p}(x, y) := (z_{s,i,p}(x, y))$ and $V_{s,p}(x, y) := (v_{s,i,p}(x, y)) \in C(\overline{D})$ belong to the domain $\overline{B}_1$, $s = 1, 2, 3$, $p \in \mathbb{N}$.

Let us put

$$\begin{aligned} W_{s,p}(x, y) &:= Z_{s,p}(x, y) - V_{s,p}(x, y), (x, y) \in \overline{D}_s, s = 1, 2, 3, \\ f_s^p(x, y) &:= H[Z_{s,p}(x, y); V_{s,p}(x, y)], \\ f_{s,p}(x, y) &:= H[V_{s,p}(x, y); Z_{s,p}(x, y)], \\ \alpha_{s,p}^*(x, y) &:= Z_{s,p}(x, y) - \Gamma_s(x, y) - \epsilon_s T_{1,s} f_1^p(\xi, \eta) - T_s f_s^p(\xi, \eta), \\ \beta_{s,p}^*(x, y) &:= V_{s,p}(x, y) - \Gamma_s(x, y) - \epsilon_s T_{1,s} f_{1,p}(\xi, \eta) - T_s f_{s,p}(\xi, \eta), \\ Z_{s,p}^*(x, y) &:= Z_{s,p}(x, y) - C_{s,p}(x, y) W_{s,p}(x, y), \\ V_{s,p}^*(x, y) &:= V_{s,p}(x, y) + Q_{s,p}(x, y) W_{s,p}(x, y), \\ (x, y) &\in \overline{D}_s, s = 1, 2, 3, p \in \mathbb{N}, \\ F_s^p(x, y) &:= H[Z_{s,p}^*(x, y); V_{s,p}^*(x, y)], \\ F_{s,p}(x, y) &:= H[V_{s,p}^*(x, y); Z_{s,p+1}(x, y)], \end{aligned} \quad (1.7)$$

$C_{s,p}(x, y) := (\delta_{i,j} c_{s,i,p}(x, y))$, $Q_{s,p}(x, y) := (\delta_{i,j} q_{s,i,p}(x, y))$ are arbitrary matrices with non-negative elements $c_{s,i,p}(x, y), q_{s,i,p}(x, y) \in C(\overline{D}_s)$, such that

$$c_{s,i,p}(x, y) \leq 0, 5, q_{s,i,p}(x, y) \leq 0, 5, (x, y) \in \overline{D}_s, s = 1, 2, 3, \quad (1.8)$$

for all $p \in \mathbb{N}$ and $i = \overline{1, n}$.

Let us build the sequences of vector-functions $\{Z_{s,p}(x, y)\}$ and $\{V_{s,p}(x, y)\}$ according to the formulas 1.9:

$$\begin{aligned} Z_{s,p+1}(x, y) &= \Gamma_s(x, y) + \epsilon_s T_{1,s} F_1^p(\xi, \eta) + T_s F_s^p(\xi, \eta), \\ V_{s,p+1}(x, y) &= \Gamma_s(x, y) + \epsilon_s T_{1,s} F_{1,p}(\xi, \eta) + T_s F_s^p(\xi, \eta), (x, y) \in \overline{D}_s, \end{aligned} \quad (1.9)$$

where as the zero approximation $Z_{s,0}(x, y), V_{s,0}(x, y) \in \overline{B}_1$ we take arbitrary vector-functions from $C(\overline{D}_s)$ satisfying conditions (1.2), (1.3) and inequalities

$$\begin{aligned} W_{s,0}(x, y) &\geq 0, \alpha_{s,0}^*(x, y) \geq 0, \\ \beta_{s,0}^*(x, y) &\leq 0, (x, y) \in \overline{D}_s, s = 1, 2, 3. \end{aligned} \quad (1.10)$$

Afterwards vector-functions $Z_{s,0}(x, y), V_{s,0}(x, y) \in C(\overline{D}_s)$ that satisfy conditions (1.2), (1.3), inequalities (1.10) and belong to the domain $\overline{B}_1$ will be called comparison functions of the problem (1.1)-(1.3).

From (1.9) follows that for $(x, y) \in \overline{D}_s, s = 1, 2, 3$ the equalities

$$\begin{aligned} Z_{s,p}(x, y) - Z_{s,p+1}(x, y) &= \alpha_{s,p}(x, y) := Z_{s,p}(x, y) - \Gamma_s(x, y) - \\ &\quad - \epsilon_s T_{1,s} F_1^p(\xi, \eta) - T_s F_s^p(\xi, \eta), \\ V_{s,p}(x, y) - V_{s,p+1}(x, y) &= \beta_{s,p}(x, y) := V_{s,p}(x, y) - \Gamma_s(x, y) - \\ &\quad - \epsilon_s T_{1,s} F_{1,p}(\xi, \eta) - T_s F_s^p(\xi, \eta), \end{aligned} \quad (1.11)$$

are true, where $\alpha_{s,p}(x, y) := (\alpha_{s,i,p}(x, y)), \beta_{s,p}(x, y) := (\beta_{s,i,p}(x, y)), i = \overline{1, n}$ are vector-functions and

$$W_{s,p+1}(x, y) = \epsilon_s T_{1,s} (F_1^p(\xi, \eta) - F_{1,p}(\xi, \eta)) + T_s (F_s^p(\xi, \eta) - F_{s,p}(\xi, \eta)), \quad (1.12)$$

$$\begin{aligned} \alpha_{s,p}(x, y) &= \epsilon_s T_{1,s} (F_1^p(\xi, \eta) - F_1^{p+1}(\xi, \eta)) + \\ &\quad + T_s (F_s^p(\xi, \eta) - F_s^{p+1}(\xi, \eta)), \\ \beta_{s,p}(x, y) &:= \epsilon_s T_{1,s} (F_{1,p}(\xi, \eta) - F_{1,p+1}(\xi, \eta)) + \\ &\quad + T_s (F_{s,p}(\xi, \eta) - F_{s,p+1}(\xi, \eta)). \end{aligned} \quad (1.13)$$

As (1.8), (1.10) hold we get that

$$V_{s,0}(x, y) \leq V_{s,0}^*(x, y) \leq Z_{s,0}^*(x, y) \leq Z_{s,0}(x, y), (x, y) \in \overline{D}_s, s = 1, 2, 3.$$

It means that

$$V_{s,0}^*(x, y), Z_{s,0}^*(x, y) \in \overline{B}_1$$

if only $V_{s,0}(x, y), Z_{s,0}(x, y) \in \overline{B}_1$.

Note that if

$$\alpha_{s,0}^*(x, y) \geq 0$$

then

$$\alpha_{s,0}(x, y) \geq 0.$$

But from (1.11) for $p = 0$ we get that

$$Z_{s,0}(x, y) - Z_{s,1}(x, y) \geq 0,$$

$$Z_{s,1}(x, y) - V_{s,0}(x, y) = -\beta_{s,0}^*(x, y) + \epsilon_s T_{1,s}(F_1^0(\xi, \eta) - f_{1,0}(\xi, \eta)) + T_s(F_s^0(\xi, \eta) - f_{s,0}(\xi, \eta)) \geq 0,$$

$Z_{s,1}(x, y) \in \overline{B}_1$ and

$$\beta_{s,0}^*(x, y) - \beta_{s,0}(x, y) \geq 0.$$

It means that

$$\beta_{s,0}(x, y) = V_{s,0}(x, y) - V_{s,1}(x, y) \leq 0.$$

As $Z_{s,0}(x, y) \geq V_{s,1}(x, y)$ then $V_{s,1}(x, y) \in \overline{B}_1$.

We choose matrixes $C_{s,0}(x, y)$ and $Q_{s,0}(x, y)$ in such way that for $(x, y) \in \overline{D}_s$, $s = 1, 2, 3$, the following conditions

$$\begin{aligned} Z_{s,0}(x, y) - Z_{s,1}(x, y) - C_{s,0}(x, y)W_{s,0}(x, y) &\geq 0, \\ V_{s,0}(x, y) - V_{s,1}(x, y) + Q_{s,0}(x, y)W_{s,0}(x, y) &\leq 0 \end{aligned} \quad (1.14)$$

hold.

It is easy to see that the inequality

$$\begin{aligned} V_{s,0}(x, y) - Z_{s,1}(x, y) + Q_{s,0}(x, y)W_{s,0}(x, y) &\leq 0, \\ (x, y) \in \overline{D}_s, s = 1, 2, 3, \end{aligned} \quad (1.15)$$

holds if and only if

$$W_{s,1}(x, y) \geq 0, (x, y) \in \overline{D}_s.$$

Indeed, if (1.15) holds, from (1.12) for $p = 0$, taking into account (1.6), we get that

$$W_{s,1}(x, y) \geq 0.$$

And if the last estimation is true then from the second inequality (1.14) we establish that condition (1.15) is true.

So

$$\begin{aligned} V_{s,0}(x, y) &\leq V_{s,1}(x, y) \leq Z_{s,1}(x, y) \leq Z_{s,0}(x, y), \\ (x, y) &\in \overline{D}_s, s = 1, 2, 3. \end{aligned}$$

Then from (1.13) for $p = 0$ we get

$$\alpha_{s,1}(x, y) \geq 0, \beta_{s,1}(x, y) \leq 0, (x, y) \in \overline{D}_s, s = 1, 2, 3.$$

Taking vector-functions $Z_{s,1}(x, y)$, $V_{s,1}(x, y) \in \overline{B}_1$ as the given ones and repeating all the above reasoning by the method of mathematical induction we get that if we choose matrices $C_{s,p}(x, y)$ and $Q_{s,p}(x, y)$ on every step of iterations (1.9) in such way that for $(x, y) \in \overline{D}_s$, $s = 1, 2, 3$ the conditions

$$\begin{aligned} Z_{s,p}(x, y) - Z_{s,p+1}(x, y) - C_{s,p}(x, y)W_{s,p}(x, y) &\geq 0, \\ V_{s,p}(x, y) - V_{s,p+1}(x, y) + Q_{s,p}(x, y)W_{s,p}(x, y) &\leq 0, p \in \mathbb{N}, \end{aligned} \quad (1.16)$$

hold, then in the domain $\overline{B}_1$ the inequalities

$$\begin{aligned} V_{s,p}(x, y) &\leq V_{s,p+1}(x, y) \leq Z_{s,p+1}(x, y) \leq Z_{s,p}(x, y), \\ \alpha_{s,p}(x, y) &\geq \alpha_{s,p}^*(x, y) \geq 0, \beta_{s,p}(x, y) \leq \beta_{s,p}^*(x, y) \leq 0, \\ (x, y) &\in \overline{D}_s, s = 1, 2, 3, \text{ for all } p \in \mathbb{N} \end{aligned} \quad (1.17)$$

are true.

Lemma 2. *If $F[U(x, y)] \in C_1(\overline{B})$ and there exist comparison vector-functions $Z_{s,0}(x, y)$, $V_{s,0}(x, y)$, $(x, y) \in \overline{D}_s$, $s = 1, 2, 3$ of the boundary-value problem (1.1)-(1.3) in the domain $\overline{B}_1$, then the set of matrixes $C_{s,p}(x, y)$, $Q_{s,p}(x, y) \in C(\overline{D}_s)$, that satisfy conditions (1.16), (1.8), in non-empty.*

Indeed, let us put

$$\begin{aligned} c_{s,i,p}(x, y) &= \begin{cases} \alpha_{s,i,p}^*(x, y) \rho_{s,i,p}^{-1}(x, y), W_{s,i,p}(x, y) \neq 0, \\ 0, W_{s,i,p}(x, y) = 0, \end{cases} \\ q_{s,i,p}(x, y) &= \begin{cases} -\beta_{s,i,p}^*(x, y) \rho_{s,i,p}^{-1}(x, y), W_{s,i,p}(x, y) \neq 0, \\ 0, W_{s,i,p}(x, y) = 0, (x, y) \in \overline{D}_s, s = 1, 2, 3, \end{cases} \\ \rho_{s,i,p}(x, y) &:= \alpha_{s,i,p}^*(x, y) - \beta_{s,i,p}^*(x, y) + W_{s,i,p}(x, y), p \in \mathbb{N}. \end{aligned}$$

It is obvious that the chosen non-negative functions $c_{s,i,p}(x, y)$, $q_{s,i,p}(x, y)$ satisfy conditions (1.8), and according to (1.11)

$$\begin{aligned} Z_{s,p}(x, y) - Z_{s,p+1}(x, y) - C_{s,p}(x, y) W_{s,p}(x, y) &\geq \\ &\geq (E - P_{s,p}(x, y)) \alpha_{s,p}(x, y) \geq 0, \\ V_{s,p}(x, y) - V_{s,p+1}(x, y) + Q_{s,p}(x, y) W_{s,p}(x, y) &\leq \\ &\leq (E - P_{s,p}(x, y)) \beta_{s,p}(x, y) \leq 0, \\ (x, y) &\in \overline{D}_s, s = 1, 2, 3, p \in \mathbb{N}, \end{aligned}$$

$$P_{s,p}(x, y) := \left(\delta_{i,j}(x, y) W_{s,i,p}(x, y) \rho_{s,i,p}^{-1}(x, y) \right), i, j = \overline{1, n} \text{ is a matrix.}$$

The following theorem is true.

Theorem 1. *Let the vector-function $F[U(x, y)] \in C_1(\overline{B})$, $A_1(x, y) \in C(D) \cap C^{(1,0)}(D_1 \cup D_3)$, $A_2(x, y) \in C(D) \cap C^{(0,1)}(D_1 \cup D_2)$ and there exist the comparison functions $Z_{s,0}(x, y)$, $V_{s,0}(x, y) \in \overline{D}_s$, $s = 1, 2, 3$ of the problem (1.1)-(1.3) in the domain $\overline{B}_1$.*

Then for the vector-functions $Z_{s,p}(x, y)$, $V_{s,p}(x, y)$ built according to (1.9), where non-negative matrixes $C_{s,p}(x, y)$, $Q_{s,p}(x, y) \in C(\overline{D}_s)$, $s = 1, 2, 3$ satisfy conditions (1.8), (1.10) in the domain $\overline{B}_1$, the inequalities (1.17) are true.

Let us show that these sequences of vector-functions $\{Z_{s,p}(x, y)\}$ and $\{V_{s,p}(x, y)\}$ converge uniformly in the domain $\overline{D}_s$, $s = 1, 2, 3$ to the unique solution of the system of integral equations (1.4).

As inequalities (1.17) in $\overline{B}_1$ hold it is sufficient to show that

$$\lim_{p \rightarrow \infty} W_{s,p}(x, y) = 0, \\ (x, y) \in \overline{D}_s, s = 1, 2, 3.$$

Let

$$\max_{s,i} \sup_{\overline{D}_s} |W_{s,i,0}(x, y)| = d, \\ \|L\| = l, \\ \max_{s,i,p} \sup_{\overline{D}_s} (1 - c_{s,i,p}(x, y) - q_{s,i,p}(x, y)) \leq q, \\ \max_{s,i} \sup_{\overline{D}_1 \cup \overline{D}_s} \{k_{i,i}(x, y; \xi, \eta), k_{i,i}^{-1}(\xi, \eta; x, y)\} \leq 0, 5K, s = 2, 3, \\ \max_{\overline{D}} \{1, \sup(y - y_0 + x - x_0)\} = \gamma.$$

Then from (1.12) by method of mathematical induction we get that the following estimations

$$\max_{s,i} \sup_{\overline{D}_s} W_{s,i,p}(x, y) := \|W_{s,p}(x, y)\| \leq \frac{1}{p!} (l q k \gamma n (y - y_0 + x - x_0))^p d, \quad (1.18)$$

so

$$\lim_{p \rightarrow \infty} Z_{s,p}(x, y) = \lim_{p \rightarrow \infty} V_{s,p}(x, y) = U_s(x, y)(x, y) \in \overline{D}_s, s = 1, 2, 3.$$

Passing to the limit in (1.9) for $p \rightarrow \infty$ we get that the limit functions $U_s(x, y)$ are the solutions of the systems of integral equations (1.4) for $(x, y) \in \overline{D}_s$, $s = 1, 2, 3$.

Theorem 2. *Let conditions of Theorem 1 hold.*

Then the sequences of vector-functions $\{Z_{s,p}(x, y)\}$, $\{V_{s,p}(x, y)\}$ build according to (1.9), where as a zero-approximation we choose the comparison vector-functions of the problem (1.1)-(1.3) and matrixes $C_{s,p}(x, y)$ and $Q_{s,p}(x, y)$, $s = 1, 2, 3$ satisfy conditions (1.8), (1.16):

- (1) *uniformly converge to the unique solution of the system of integral equations (1.4), for $(x, y) \in \overline{D}_s$, $s = 1, 2, 3$,*
- (2) *estimates (1.18) take place,*
- (3) *in the domain $\overline{B}_1$ the inequalities*

$$V_{s,p}(x, y) \leq V_{s,p+1}(x, y) \leq U_s(s, y) \leq Z_{s,p+1}(x, y) \leq Z_{s,p}(x, y), \quad (1.19) \\ (x, y) \in \overline{D}_s, s = 1, 2, 3, p \in \mathbb{N},$$

hold, where $U_s(x, y)$ is the unique solution of the system of integral equations (1.4), for $(x, y) \in \overline{D}_s$,

- (4) *the convergence of the iteration method (1.9), (1.8), (1.16) isn't slower than the convergence of the Pikar two-sided method (for $C_{s,p}(x, y) = Q_{s,p}(x, y) \equiv 0$).*

Proof. The uniqueness of the solution of the system of integral equations (1.4) for $(x, y) \in \overline{D}_s, s = 1, 2, 3$ can be proved from the opposite assumption.

Suppose that for any number p in some point $(x, y) \in \overline{D}_s$ $Z_{s,p}(x, y) < U_s(x, y)$. Then according to (1.17) for all $m \in \mathbb{N}$

$$U_s(x, y) > Z_{s,p}(x, y) \geq Z_{s,p+m}(x, y)$$

in the point $(x, y) \in \overline{D}_s$ and in this point the sequence of vector-functions $\{Z_{s,p+m}(x, y)\}$ for $m \rightarrow \infty$ does not converge to the solution $U_s(x, y)$. We got the contradiction. Analogically we prove that the inequality

$$V_{s,p}(x, y) \leq U_s(x, y), (x, y) \in \overline{D}_s, s = 1, 2, 3$$

holds.

Let $Z_{s,p}(x, y), V_{s,p}(x, y) \in \overline{B}_1$ are the comparison vector-functions of the problem (1.1)-(1.3). Let us put $\overline{Z}_{s,p+1}(x, y), \overline{V}_{s,p+1}(x, y)$ the following two-sided approximations to the solution of the systems of integral equations (1.4) build in order of the Picar method. Then for $(x, y) \in \overline{D}_s, s = 1, 2, 3$ according to (1.6), (1.8) we get

$$\begin{aligned} \overline{Z}_{s,p+1}(x, y) - Z_{s,p+1}(x, y) &= \epsilon_s T_{1,s}(f_1^p(\xi, \eta) - F_1^p(\xi, \eta)) + \\ &+ T_s(f_s^p(\xi, \eta) - F_s^p(\xi, \eta)) \geq 0, \overline{V}_{s,p+1}(x, y) - V_{s,p+1}(x, y) = \\ &= \epsilon_s T_{1,s}(f_{1,p}(\xi, \eta) - F_{1,p}(\xi, \eta)) + T_s(f_{s,p}(\xi, \eta) - F_{s,p}(\xi, \eta)) \leq 0. \end{aligned}$$

It means that the convergence of the method is not slower than the convergence of the Picar's two-sided method. $\square$

Let us remark that according to the forms of matrices $C_{s,p}(x, y)$ and $Q_{s,p}(x, y)$ in (1.9), (1.8), (1.10) we get different modifications of the two-sided method.

Corollary 1. *Let conditions of Theorem 2 hold.*

Then there exists the unique solution of the boundary-value problem (1.1)-(1.3) in the domain $\overline{D}$. Besides in the case, when $\rho_{r,i} = 0$ for all $r = 1, 2$ and $i = \overline{1, n}$ it will be regular, and in the other case, irregular.

Corollary 2. *Let $\Psi(y) = \Phi(x) = 0, (x, y) \in \overline{D}_1, \Omega_r(x) = 0, x \in [x_{r-1}, x_r], r = 1, 2$ and $F[U(x, y)] \in C_1(\overline{B})$, besides $F[U(x, y)] \equiv H[U(x, y); 0]$.*

Then if $F[0] \leq (\geq) 0$ in $\overline{B}_1$ the solution of (1.1)-(1.3) for $(x, y) \in \overline{D}$ satisfies the inequality

$$U(x, y) \leq (\geq) 0.$$

Together with (1.1) let us observe the following system

$$L_2 Z(x, y) = f_1(x, y, Z(x, y)) := f_1[Z(x, y)], f_1 : \overline{B} \rightarrow \mathbb{R}^n, \overline{B} \in \mathbb{R}^{n+2}. \quad (1.20)$$

We suppose that the right hand-sides of the systems (1.1), (1.20) satisfy the following conditions:

$$(1) f[U(x, y)] \in C_1(\overline{B}),$$

- (2) vector-function $f_1[Z(x, y)] \in C(\overline{B})$ and has bounded first order derivatives on all its arguments in $\overline{B}$, starting from the third one

$$\frac{\partial f_i[Z(x, y)]}{\partial z_j} := b_{i,j}(x, y) < \infty,$$

and $\forall (x, y, Z(x, y)) \in \overline{B}$ the inequalities

$$\begin{aligned} b_{i,j}(x, y) + \delta_{i,j}[a_{i,j_y}^{(2)}(x, y) + a_{i,j}^{(1)}(x, y)a_{i,j}^{(2)}(x, y)] &\geq 0, (x, y) \in D_1 \cup D_2, \\ b_{i,j}(x, y) + \delta_{i,j}[a_{i,j_x}^{(1)}(x, y) + a_{i,j}^{(1)}(x, y)a_{i,j}^{(2)}(x, y)] &\geq 0, (x, y) \in D_1 \cup D_3, \end{aligned} \quad (1.21)$$

hold,

- (3) for any vector-function $V(x, y) \in \overline{B}$ from $C^*(\overline{D})$

$$f_1[V(x, y)] \geq (\leq) f[V(x, y)]. \quad (1.22)$$

Theorem 3. Let $A_1 \in C(D) \cap C^{(1,0)}(D_1 \cup D_3)$, $A_2 \in C(D) \cap C^{(0,1)}(D_1 \cup D_2)$ and the right sides of the systems (1.1), (1.20) $f[U(x, y)]$ and $f_1[Z(x, y)]$ satisfy conditions (1)-(3) above and there exist comparison functions of the problems (1.1)-(1.3), (1.20), (1.2), (1.3) in the domain $\overline{B}_1$.

Then the solutions of these problems satisfy inequalities

$$U(x, y) \leq (\geq) Z(x, y), (x, y) \in \overline{D}. \quad (1.23)$$

Proof. According to Theorem 2 and Corollary 1 there exist unique solutions of the problems (1.1)-(1.3) and (1.20), (1.2), (1.3) (regular or irregular). So putting $W(x, y) := Z(x, y) - U(x, y)$ and using Lagrange theorem about complete increases we'll get

$$L_2 W(x, y) = A_3(x, y)W(x, y) + A_4(x, y), \quad (1.24)$$

where $A_3(x, y) = \tilde{b}_{i,j}(x, y)$, $i, j = \overline{1, n}$ matrix, $\tilde{b}_{i,j}(x, y)$ are derivatives of $b_{i,j}(x, y)$ for some fixed values $Z(x, y) \in \overline{B}$ and according to (1.22)

$$A_4(x, y) := f_1[U(x, y)] - f[U(x, y)] \geq (\leq) 0.$$

Vector-function $W(x, y)$ satisfies homogenous conditions (1.2), (1.3) and

$$\begin{aligned} F^*[W(x, y)] &:= [A_3(x, y) + A_{2y}(x, y) + A_1(x, y)A_2(x, y)]W(x, y) + A_4(x, y), \\ F^{**}[W(x, y)] &:= [A_3(x, y) + A_{1x}(x, y) + A_1(x, y)A_2(x, y)]W(x, y) + A_4(x, y) \\ &\equiv F^*[W(x, y)] + [A_{1x}(x, y) - A_{2y}(x, y)]W(x, y), \end{aligned}$$

i.e. according to (1.21)

$$F[W(x, y)] \equiv H[W(x, y); 0]$$

and

$$F[0] \geq (\leq) 0.$$

Using Corollary 2 we get that

$$W(x, y) \geq (\leq) 0, (x, y) \in \overline{D}.$$

It means that inequalities (1.23) hold. □

REFERENCES

- [1] L. Bers, F. John, and M. Schechter, Eds., *Partial differential equations. Proceeding of the Summer Seminar, Boulder, Colorado, 1957*. New York-London-Sydney: Interscience Publishers, a Division of John Wiley and Sons, 1964.
- [2] R. Courant, *Partial differential equations*, New York-London, 1962.
- [3] K. Marynets, “On the parametrization of nonlinear boundary value problems with nonlinear boundary conditions,” *Math. Notes, Miskolc*, vol. 12, no. 2, pp. 209–223, 2011.
- [4] V. V. Marynets, *Analytical methods in the theory of partial differential equations of hyperbolic type*. Hoverla, Uzhhorod, 2006.
- [5] V. V. Marynets’ and A. V. Dobryden’, “On a characteristic initial value problem,” *Nonlinear Oscil.*, vol. 4, no. 4, pp. 487–499, 2001.
- [6] V. V. Marynets and O. Y. Pytovka, “About one boundary-value problem for differential-functional equations of the hyperbolic type,” *Scientific Transactions of the Uzhhorod University, Series: Mathematics and Informatics*, vol. 20, pp. 79–89, 2010.
- [7] A. Rontó and M. Rontó, “Periodic successive approximations and interval halving,” *Math. Notes, Miskolc*, vol. 13, no. 2, pp. 459–482, 2012.
- [8] M. Rontó and K. Marynets, “On the parametrization of boundary-value problems with three-point non-linear restrictions,” *Math. Notes, Miskolc*, vol. 13, no. 1, pp. 91–106, 2012.

Authors’ addresses

V. V. Marynets

Uzhhorod National University, Ukraine

E-mail address: vasyi-marynets@rambler.ru

K. V. Marynets

Uzhhorod National University, Ukraine

E-mail address: katya_marynets@ukr.net



ON BIVARIATE MEYER-KÖNIG AND ZELLER OPERATORS

ALI OLGUN

Received 3 May, 2012

Abstract. This work relates to bivariate Meyer-König and Zeller operators, M_n , $n \in \mathbb{N}$ which are not a tensor product setting. We show the monotonicity of the sequence of operators for n under convexity, moreover we study the property of monotonicity in the sense of Li [9]. Finally, we provide an r th order generalization $M_n^{[r]}$ of M_n and also study approximation of $M_n^{[r]}$.

2010 *Mathematics Subject Classification:* 41A25; 41A36

Keywords: multivariate Meyer-König and Zeller operator, convexity, monotonicity, modulus of continuity

1. INTRODUCTION

The Cheney and Sharma modification of the well-known univariate Meyer-König and Zeller operators (MKZ) are defined as

$$M_n^*(f, x) = \sum_{k=0}^{\infty} m_{n,k}(x) f\left(\frac{k}{n+k}\right), \quad n \in \mathbb{N} \quad (1.1)$$

for $f \in C[0, 1]$ and $x \in [0, 1]$, $n \in \mathbb{N}$, where $m_{n,k}(x) = (1-x)^{n+1} \binom{n+k}{k} x^k$ [3].

The monotonic convergence of $\{M_n^*(f; x)\}_{n=1}^{\infty}$ under convexity was investigated in [2] by Cheney and Sharma by means of analytical approach, also studied in [8] by Khan, by means of probabilistic approach. Monotonic convergence of several approximation operators are studied deeply in [6] by Khan, Della-Vecchia and Fassih with probabilistic point of view. Some works related to MKZ operators can be viewed in [1–8, 13] and [10, 11].

We shall use the following standard notation. Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$, $\mathbf{k} = (k_1, k_2) \in \mathbb{N}_0^2$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} x_2^{k_2}$, $\mathbf{k}! = k_1! k_2!$, $|\mathbf{k}| = k_1 + k_2$, $|\mathbf{x}| =$

$x_1 + x_2$, $\mathbf{e}_i$ denotes the unit vector in $\mathbb{R}^2$. Furthermore $\binom{n}{\mathbf{k}} = \frac{n!}{\mathbf{k}!(n-|\mathbf{k}|)!}$, and

$$\sum_{\mathbf{k}=0}^{\infty} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty}.$$

Let $S_2 \subset \mathbb{R}^2$ be the open simplex defined by

$$S_2 = \{\mathbf{x} \in \mathbb{R}^2; \mathbf{x}_i \geq 0, i = 1, 2, |\mathbf{x}| < 1\}.$$

For a function $f \in C(S_2)$, bivariate MKZ operators are defined as

$$M_n(f, \mathbf{x}) = (1 - |\mathbf{x}|)^{n+1} \sum_{\mathbf{k}=0}^{\infty} f\left(\frac{\mathbf{k}}{n + |\mathbf{k}|}\right) \binom{n + |\mathbf{k}|}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}, \quad n \in \mathbb{N}, \quad (1.2)$$

where $C(S_2)$ denotes the space of continuous real valued functions defined on S_2 . It is obvious that M_n , $n \in \mathbb{N}$, are not a tensor product extension of the univariate MKZ operators M_n^* given by (1.2).

Now we give the following definitions which we shall use.

Definition 1. A continuous function f is said to be convex in $D \subset \mathbb{R}^m$, if

$$f\left(\sum_{i=1}^r \alpha_i \mathbf{x}_i\right) \geq \sum_{i=1}^r \alpha_i f(\mathbf{x}_i)$$

for every $x_1, x_2, \dots, x_r \in D$ and for every non-negative numbers $\alpha_1, \alpha_2, \dots, \alpha_r$ such that $\alpha_1 + \alpha_2 + \dots + \alpha_r = 1$.

Definition 2. A continuous function f from $D \subseteq \mathbb{R}^2$ into $\mathbb{R}$ is said to be Lipschitz continuous of order μ , $\mu \in (0, 1]$, if there exists a constant $A > 0$ such that for every $(x_1, x_2), (y_1, y_2) \in D$, f satisfies

$$|f(x_1, x_2) - f(y_1, y_2)| \leq A \sum_{i=1}^2 |x_i - y_i|^{\mu},$$

the set of Lipschitz continuous functions is denoted by $Lip_A(\mu, D)$.

In this work, we firstly show the monotonicity of the sequence of bivariate MKZ operators defined by (1.2) under convexity. Secondly we give a kind of monotonicity similar to the property given by Li in [9]. Namely, we show that if $f(\mathbf{x})$ is a non-negative function and $x_i^{-1} f(\mathbf{x})$ ($i = 1, 2$) is non-increasing for x_i on $(0, 1)$, then for each $n \geq 1$, $x_i^{-1} M_n(f; \mathbf{x})$ is also non-increasing for x_i on $(0, 1)$. Moreover we build an r -th order generalization $M_n^{[r]}$ of M_n analogues to Kirov and Popova's construction in [8] and investigate its approximation property.

2. MONOTONICITY FOR THE SEQUENCE OF BIVARIATE MEYER-KÖNIG AND ZELLER OPERATORS

In this section, we study the monotonic convergence of the sequence of bivariate MKZ operator under convexity. Note that monotonic convergence of univariate MKZ operator when f is convex, was first obtained by Cheney and Sharma in [3]. We note here that monotonic convergence of the multivariate Baskakov operator is studied in [2].

Theorem 1. *If f is convex, then $M_n(f; \mathbf{x})$ is strictly monotonically non-decreasing in n , unless f is the linear function (in which case $M_n(f; \mathbf{x}) = M_{n+1}(f; \mathbf{x})$ for all $n \in \mathbb{N}$).*

Proof. We have

$$\begin{aligned} M_n(f; \mathbf{x}) - M_{n+1}(f; \mathbf{x}) &= (1 - |\mathbf{x}|)^{n+1} \times \\ &\times \left\{ \sum_{k_2=1}^{\infty} \left[f\left(0, \frac{k_2}{n+k_2}\right) \binom{n+k_2}{k_2} - f\left(0, \frac{k_2}{n+1+k_2}\right) \binom{n+1+k_2}{k_2} \right] \mathbf{x}^{\mathbf{k}} \right. \\ &+ \sum_{k_1=1}^{\infty} \left[f\left(\frac{k_1}{n+k_1}, 0\right) \binom{n+k_1}{k_1} - f\left(\frac{k_1}{n+1+k_1}, 0\right) \binom{n+1+k_1}{k_1} \right] \mathbf{x}^{\mathbf{k}} \\ &+ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} f\left(\frac{\mathbf{k}}{n+1+|\mathbf{k}|}\right) \binom{n+1+|\mathbf{k}|}{\mathbf{k}} x_1^{k_1+1} x_2^{k_2} \\ &\left. + \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} f\left(\frac{\mathbf{k}}{n+1+|\mathbf{k}|}\right) \binom{n+1+|\mathbf{k}|}{\mathbf{k}} x_1^{k_1} x_2^{k_2+1} \right\}. \end{aligned}$$

Therefore the last equation reduces to the following:

$$\begin{aligned} M_n(f; \mathbf{x}) - M_{n+1}(f; \mathbf{x}) &= (1 - |\mathbf{x}|)^{n+1} \times \\ &\times \left\{ \sum_{k_2=1}^{\infty} \left[f\left(0, \frac{k_2}{n+k_2}\right) \binom{n+k_2}{k_2} - f\left(0, \frac{k_2}{n+1+k_2}\right) \binom{n+1+k_2}{k_2} \right] \right. \\ &+ f\left(0, \frac{k_2-1}{n+k_2}\right) \binom{n+k_2}{k_2-1} x_2^{k_2} + \sum_{k_1=1}^{\infty} \left[f\left(\frac{k_1}{n+k_1}, 0\right) \binom{n+k_1}{k_1} \right. \\ &- f\left(\frac{k_1}{n+1+k_1}, 0\right) \binom{n+1+k_1}{k_1} + f\left(\frac{k_1-1}{n+k_1}, 0\right) \binom{n+k_1}{k_1-1} \left. \right] x_1^{k_1} \\ &\left. + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \left[f\left(\frac{k_1-1}{n+|\mathbf{k}|}, \frac{k_2}{n+|\mathbf{k}|}\right) \binom{n+|\mathbf{k}|}{\mathbf{k}-\mathbf{e}_1} \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + f\left(\frac{k_1}{n+|\mathbf{k}|}, \frac{k_2-1}{n+|\mathbf{k}|}\right) \binom{n+|\mathbf{k}|}{\mathbf{k}-\mathbf{e}_2} \\
& + f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|}\right) \binom{n+|\mathbf{k}|}{\mathbf{k}} - f\left(\frac{\mathbf{k}}{n+1+|\mathbf{k}|}\right) \binom{n+1+|\mathbf{k}|}{\mathbf{k}} \Big] \mathbf{x}^{\mathbf{k}} \Big\}.
\end{aligned}$$

Let I_1 , I_2 and I_3 denote the followings

$$\begin{aligned}
I_1 &= f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|}\right) \binom{n+|\mathbf{k}|}{\mathbf{k}} - f\left(\frac{\mathbf{k}}{n+1+|\mathbf{k}|}\right) \binom{n+1+|\mathbf{k}|}{\mathbf{k}} \\
&+ f\left(\frac{\mathbf{k}-\mathbf{e}_1}{n+|\mathbf{k}|}\right) \binom{n+|\mathbf{k}|}{\mathbf{k}-\mathbf{e}_1} + f\left(\frac{\mathbf{k}-\mathbf{e}_2}{n+|\mathbf{k}|}\right) \binom{n+|\mathbf{k}|}{\mathbf{k}-\mathbf{e}_2}, \\
I_2 &= f\left(0, \frac{k_2}{n+k_2}\right) \binom{n+k_2}{k_2} - f\left(0, \frac{k_2}{n+1+k_2}\right) \binom{n+1+k_2}{k_2} \\
&+ f\left(0, \frac{k_2-1}{n+k_2}\right) \binom{n+k_2}{k_2-1}, \\
I_3 &= f\left(\frac{k_1}{n+k_1}, 0\right) \binom{n+k_1}{k_1} - f\left(\frac{k_1}{n+1+k_1}, 0\right) \binom{n+1+k_1}{k_1} \\
&+ f\left(\frac{k_1-1}{n+k_1}, 0\right) \binom{n+k_1}{k_1-1}.
\end{aligned}$$

For I_1 , we take

$$\alpha_1 = \frac{\binom{n+|\mathbf{k}|}{\mathbf{k}}}{\binom{n+1+|\mathbf{k}|}{\mathbf{k}}}, \quad \alpha_2 = \frac{\binom{n+|\mathbf{k}|}{\mathbf{k}-\mathbf{e}_1}}{\binom{n+1+|\mathbf{k}|}{\mathbf{k}}}, \quad \alpha_3 = \frac{\binom{n+|\mathbf{k}|}{\mathbf{k}-\mathbf{e}_2}}{\binom{n+1+|\mathbf{k}|}{\mathbf{k}}}.$$

Clearly α_1 , α_2 and α_3 are non-negative numbers, and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. On the other hand if we set

$$\mathbf{x}_1 = \frac{\mathbf{k}}{n+|\mathbf{k}|}, \quad \mathbf{x}_2 = \frac{\mathbf{k}-\mathbf{e}_1}{n+|\mathbf{k}|}, \quad \mathbf{x}_3 = \frac{\mathbf{k}-\mathbf{e}_2}{n+|\mathbf{k}|},$$

then it follows that

$$\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \alpha_3 \mathbf{x}_3 = \frac{\mathbf{k}}{n+1+|\mathbf{k}|}.$$

Hence, from the convexity of f we obtain that $I_1 \leq 0$. For I_2 , we set

$$\alpha_1 = \frac{\binom{n+k_2}{k_2}}{\binom{n+1+k_2}{k_2}}, \quad \alpha_2 = \frac{\binom{n+k_2}{k_2-1}}{\binom{n+1+k_2}{k_2}}$$

and

$$\mathbf{y}_1 = \left(0, \frac{k_2}{n+k_2}\right), \quad \mathbf{y}_2 = \left(0, \frac{k_2-1}{n+k_2}\right),$$

then, clearly, $\alpha_1, \alpha_2 \geq 0$, and $\alpha_1 + \alpha_2 = 1$. Moreover we have

$$\alpha_1 \mathbf{y}_1 + \alpha_2 \mathbf{y}_2 = \left(0, \frac{k_2}{n+k_2}\right).$$

Thus, from the convexity of f we obtain that $I_2 \leq 0$. Similarly we deduce that $I_3 \leq 0$. So we have proved $M_n(f; \mathbf{x}) \leq M_{n+1}(f; \mathbf{x})$ for all $n \in \mathbb{N}$. Using the similar arguments given in [2], the last part of the proof is given as follows: The equality of $M_n(f; \mathbf{x})$ and $M_{n+1}(f; \mathbf{x})$ can be satisfied only if $I_1 = 0$ for all $k_1, k_2 \in \mathbb{N}$ and $I_2 = I_3 = 0$ for all $k_1, k_2 \in \mathbb{N}_0$. But, if f is convex, then $I_1 = 0$ implies that its graph is a plane in triangle. $\square$

In the following, we show that the bivariate MKZ operators can retain a certain monotony, that is

Theorem 2. *Let f be defined on S_2 . If $f(\mathbf{x})$ is a non-negative function such that $\frac{f(\mathbf{x})}{x_i}$ ($i = 1, 2$) is non-increasing for x_i on $(0, 1)$, then for each $n \geq 1$, $\frac{M_n(f; \mathbf{x})}{x_i}$, is also non-increasing for x_i on $(0, 1)$.*

Proof. Straightforward computation gives that for $n \geq 1$ we have, for example with respect to x_1 ,

$$\begin{aligned} & \frac{\partial}{\partial x_1} \left(\frac{M_n(f; \mathbf{x})}{x_1} \right) = \\ &= \frac{\partial}{\partial x_1} \left\{ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(1-x_1-x_2)^{n+1}}{x_1} f\left(\frac{k_1}{n+k_1+k_2}, \frac{k_2}{n+k_1+k_2}\right) \binom{n+k_1+k_2}{k_1+k_2} x_1^{k_1} x_2^{k_2} \right\} \\ &= \frac{\partial}{\partial x_1} \left\{ \sum_{k_2=0}^{\infty} \frac{(1-x_1-x_2)^{n+1}}{x_1} f\left(0, \frac{k_2}{n+k_2}\right) \binom{n+k_2}{k_2} x_2^{k_2} + \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} \right. \\ & \quad \times (1-x_1-x_2)^{n+1} f\left(\frac{k_1}{n+k_1+k_2}, \frac{k_2}{n+k_1+k_2}\right) \binom{n+k_1+k_2}{k_1+k_2} x_1^{k_1-1} x_2^{k_2} \left. \right\} \\ &= \sum_{k_2=0}^{\infty} \frac{\partial}{\partial x_1} \left(\frac{(1-x_1-x_2)^{n+1}}{x_1} \right) f\left(0, \frac{k_2}{n+k_2}\right) \binom{n+k_2}{k_2} x_2^{k_2} + \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} \end{aligned}$$

$$\begin{aligned}
& \times f\left(\frac{k_1}{n+k_1+k_2}, \frac{k_2}{n+k_1+k_2}\right) \binom{n+k_1+k_2}{k_1+k_2} \frac{\partial}{\partial x_1} \left[(1-x_1-x_2)^{n+1} x_1^{k_1-1} x_2^{k_2} \right] \\
& = \sum_{k_2=0}^{\infty} \left(\frac{-(n+1)(1-x_1-x_2)^n x_1 - (1-x_1-x_2)^{n+1}}{x_1^2} \right) f\left(0, \frac{k_2}{n+k_2}\right) \binom{n+k_2}{k_2} x_2^{k_2} \\
& + \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} f\left(\frac{k_1}{n+k_1+k_2}, \frac{k_2}{n+k_1+k_2}\right) \binom{n+k_1+k_2}{k_1+k_2} x_2^{k_2} \\
& \times \left[-(n+1)(1-x_1-x_2)^n x_1^{k_1-1} + (1-x_1-x_2)^{n+1} (k_1-1) x_1^{k_1-2} \right] \\
& = \sum_{k_2=0}^{\infty} \frac{(1-x_1-x_2)^{n+1}}{x_1^2} \left(\frac{-(n+1)x_1}{1-x_1-x_2} - 1 \right) f\left(0, \frac{k_2}{n+k_2}\right) \binom{n+k_2}{k_2} x_2^{k_2} \\
& + \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} f\left(\frac{k_1}{n+k_1+k_2}, \frac{k_2}{n+k_1+k_2}\right) \binom{n+k_1+k_2}{k_1+k_2} (1-x_1-x_2)^n x_2^{k_2} \\
& \times \left[-n x_1^{k_1-1} - x_1^{k_1-1} + (k_1-1) x_1^{k_1-2} - (k_1-1) x_1^{k_1-1} - (k_1-1) x_1^{k_1-2} x_2 \right] \\
& \times x_1^{k_1-2} (1-x_2) (1-x_1-x_2)^n x_2^{k_2} \\
& = \sum_{k_2=0}^{\infty} \frac{(1-x_1-x_2)^{n+1}}{x_1^2} \left(-\frac{(1+nx_1-x_2)}{1-x_1-x_2} \right) f\left(0, \frac{k_2}{n+k_2}\right) \binom{n+k_2}{k_2} x_2^{k_2} \\
& + \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} f\left(\frac{k_1}{n+k_1+k_2}, \frac{k_2}{n+k_1+k_2}\right) \binom{n+k_1+k_2}{k_1+k_2} (1-x_1-x_2)^n x_2^{k_2} \\
& \times \left[-(n+k_1) x_1^{k_1-1} + (k_1-1) (1-x_2) x_1^{k_1-2} \right] \\
& = \sum_{k_2=0}^{\infty} \frac{(1-x_1-x_2)^n}{x_1^2} (- (1+nx_1-x_2)) f\left(0, \frac{k_2}{n+k_2}\right) \binom{n+k_2}{k_2} x_2^{k_2} \\
& + \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} f\left(\frac{k_1}{n+k_1+k_2}, \frac{k_2}{n+k_1+k_2}\right) \binom{n+k_1+k_2}{k_1+k_2} x_1^{k_1-1} (-n-k_1) x_2^{k_2} \\
& \times (1-x_1-x_2)^n + \sum_{k_1=2}^{\infty} \sum_{k_2=0}^{\infty} f\left(\frac{k_1}{n+k_1+k_2}, \frac{k_2}{n+k_1+k_2}\right) \binom{n+k_1+k_2}{k_1+k_2} \\
& = \sum_{k_2=0}^{\infty} \frac{-(1+nx_1-x_2)}{x_1^2} f\left(0, \frac{k_2}{n+k_2}\right) \binom{n+k_2}{k_2} (1-x_1-x_2)^n x_2^{k_2} \\
& - \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} f\left(\frac{k_1}{n+k_1+k_2}, \frac{k_2}{n+k_1+k_2}\right) \binom{n+k_1+k_2}{k_1+k_2} x_1^{k_1-1} (n+k_1) x_2^{k_2}
\end{aligned}$$

$$\begin{aligned}
& \times (1-x_1-x_2)^n + \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} f\left(\frac{k_1+1}{n+k_1+k_2}, \frac{k_2}{n+k_1+k_2}\right) \binom{n+k_1+1+k_2}{k_1+1+k_2} \\
& \times (1-x_2) x_1^{k_1-1} k_1 (1-x_1-x_2)^n x_2^{k_2} \\
& \leq - \sum_{k_2=0}^{\infty} \frac{(1+nx_1-x_2)}{x_1^2} f\left(0, \frac{k_2}{n+k_2}\right) \binom{n+k_2}{k_2} (1-x_1-x_2)^n x_2^{k_2} \\
& + \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} f\left(\frac{k_1+1}{n+k_1+k_2+1}, \frac{k_2}{n+k_1+k_2+1}\right) \binom{n+k_1+k_2}{k_1+k_2} \frac{n+k_1+k_2+1}{k_1+k_2+1} \frac{k_1}{x_1} \\
& \times (1-x_2) (1-x_1-x_2)^n x_1^{k_1} x_2^{k_2} - \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} f\left(\frac{k_1}{n+k_1+k_2}, \frac{k_2}{n+k_1+k_2}\right) \binom{n+k_1+k_2}{k_1+k_2} \\
& \times \frac{(n+k_1)}{k_1} \frac{k_1}{x_1} (1-x_1-x_2)^n x_1^{k_1} x_2^{k_2}
\end{aligned}$$

Since for $k_1, k_2 \geq 0$

$$\frac{n+k_1+k_2+1}{k_1+k_2+1} < \frac{n+k_1+k_2+1}{k_1+1}, \quad -\frac{n+k_1+k_2}{k_1} < -\frac{n+k_1}{k_1}$$

and

$$1+nx_1-x_2 \geq 0, \quad 1-x_2 \leq 1, \quad \text{for } x_1, x_2 \in S_2, \quad n \in \mathbb{N},$$

so we have for $n \geq 1$

$$\begin{aligned}
& \frac{\partial}{\partial x_1} \left(\frac{M_n(f; \mathbf{x})}{x_1} \right) \leq \\
& \leq - \sum_{k_2=0}^{\infty} \frac{(1+nx_1-x_2)}{x_1^2} f\left(0, \frac{k_2}{n+k_2}\right) \binom{n+k_2}{k_2} (1-x_1-x_2)^n x_2^{k_2} \\
& + \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} \left\{ f\left(\frac{k_1+1}{n+k_1+k_2+1}, \frac{k_2}{n+k_1+k_2+1}\right) \frac{n+k_1+k_2+1}{k_1+k_2+1} \right. \\
& \left. - f\left(\frac{k_1}{n+k_1+k_2}, \frac{k_2}{n+k_1+k_2}\right) \frac{(n+k_1+k_2)}{k_1} \right\} \\
& \times \binom{n+k_1+k_2}{k_1+k_2} \frac{k_1}{x_1} (1-x_1-x_2)^n x_1^{k_1} x_2^{k_2},
\end{aligned}$$

is non-positive, since f is a non-negative function, such that $\frac{f(\mathbf{x})}{x_i}$ ($i = 1, 2$) is non-increasing for x_i on $(0, 1)$. Similar calculations can be obtained for x_2 . $\square$

3. A GENERALIZATION

This section provides an r -th order generalization of the bivariate MKZ operators in the sense of Kirov and Popova's construction [8].

Let $C^r(S_2)$, $r \in \mathbb{N}_0$, denote the space of all functions f defined on S_2 and having all continuous partial derivatives up to order r . By $M_n^{[r]}$, we denote the following generalization of M_n . For $\mathbf{x}, \mathbf{t} \in S_2$,

$$M_n^{[r]}(f; \mathbf{x}) = M_n \left(P_{r, \frac{\mathbf{k}}{n+|\mathbf{k}|}}(\Delta \mathbf{x}, f) \right), \quad (3.1)$$

where

$$\begin{aligned} \Delta \mathbf{x} &:= (\Delta x_1, \Delta x_2) = \mathbf{x} - \frac{\mathbf{k}}{n+|\mathbf{k}|} = \left(x_1 - \frac{k_1}{n+|\mathbf{k}|}, x_2 - \frac{k_2}{n+|\mathbf{k}|} \right), \\ \nabla &= \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right), \\ (\Delta \mathbf{x} \cdot \nabla)^r &:= \sum_{i+j=r} \binom{r}{j} (\Delta x_1)^i (\Delta x_2)^j \frac{\partial^r}{\partial x_1^i \partial x_2^j}, \end{aligned} \quad (3.2)$$

$\binom{r}{j}$ are binomial coefficients, and

$$\begin{aligned} P_{r, \frac{\mathbf{k}}{n+|\mathbf{k}|}}(\Delta \mathbf{x}, f) &= f \left(\frac{\mathbf{k}}{n+|\mathbf{k}|} \right) + (\Delta \mathbf{x} \cdot \nabla) f \left(\frac{\mathbf{k}}{n+|\mathbf{k}|} \right) \\ &\quad + \frac{(\Delta \mathbf{x} \cdot \nabla)^2}{2!} f \left(\frac{\mathbf{k}}{n+|\mathbf{k}|} \right) + \dots + \frac{(\Delta \mathbf{x} \cdot \nabla)^r}{r!} f \left(\frac{\mathbf{k}}{n+|\mathbf{k}|} \right), \end{aligned} \quad (3.3)$$

the Taylor polynomial for f at $\frac{\mathbf{k}}{n+|\mathbf{k}|} \in S_2$.

Now we state the following pointwise estimate for $M_n^{[r]}$

Theorem 3. If $f \in C^r(S_2)$ and $\frac{\partial^r f}{\partial x_1^i \partial x_2^j} \in Lip_A(\gamma)$, $i+j=r$, then we have

$$\left| M_n^{[r]}(f; \mathbf{x}) - f(\mathbf{x}) \right| \leq \frac{2A}{(r-1)!} \frac{\gamma}{\gamma+r} B(\gamma, r) M_n(g; \mathbf{x}) \quad (3.4)$$

for $x \in S_2$, where $g(\mathbf{s}) = |\mathbf{x} - \mathbf{s}|^{r+\gamma}$, $B(\gamma, r)$ is the familiar beta function, $r, n \in \mathbb{N}_0$, $0 < \gamma \leq 1$ and $A > 0$.

Proof. From (3.1) and (3.3) we have

$$f(\mathbf{x}) - M_n^{[r]}(f; \mathbf{x}) = \sum_{\mathbf{k}=0}^{\infty} R_{r, \frac{\mathbf{k}}{n+|\mathbf{k}|}}(\Delta \mathbf{x}, f) \binom{n+|\mathbf{k}|}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1-|\mathbf{x}|)^{n+1}, \quad (3.5)$$

where

$$R_{r, \frac{\mathbf{k}}{n+|\mathbf{k}|}}(\Delta \mathbf{x}, f) := f(\mathbf{x}) - \sum_{h=0}^r \frac{(\Delta \mathbf{x} \cdot \nabla)^h}{h!} f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|}\right) \binom{n+|\mathbf{k}|}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1-|\mathbf{x}|)^{n+1}. \quad (3.6)$$

(3.6) can be given by

$$R_{r, \frac{\mathbf{k}}{n+|\mathbf{k}|}}(\Delta \mathbf{x}, f) = \frac{1}{(r-1)!} \int_0^1 (\Delta \mathbf{x} \cdot \nabla)^r \times \left[f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|} + t \Delta \mathbf{x}\right) - f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|}\right) \right] (1-t)^{r-1} dt. \quad (3.7)$$

Using (3.2), (3.7) results in

$$R_{r, \frac{\mathbf{k}}{n+|\mathbf{k}|}}(\Delta \mathbf{x}, f) = \frac{1}{(r-1)!} \int_0^1 \sum_{i+j=r} \binom{r}{j} (\Delta x_1)^i (\Delta x_2)^j \frac{\partial^r}{\partial x_1^i \partial x_2^j} \times \left[f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|} + t \Delta \mathbf{x}\right) - f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|}\right) \right] (1-t)^{r-1} dt. \quad (3.8)$$

Substituting (3.8) into (3.5) and taking into account that $\frac{\partial^r f}{\partial x_1^i \partial x_2^j} \in Lip_A(\gamma)$ we arrive at the following.

$$\begin{aligned} \left| f(\mathbf{x}) - M_n^{[r]}(f; \mathbf{x}) \right| &\leq \frac{A}{(r-1)!} \sum_{\mathbf{k}=0}^{\infty} (|\Delta x_1| + |\Delta x_2|)^r (|\Delta x_1|^\gamma + |\Delta x_2|^\gamma) \\ &\quad \times \binom{n+|\mathbf{k}|}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1-|\mathbf{x}|)^{n+1} \int_0^1 t^\gamma (1-t)^{r-1} dt \\ &\leq \frac{2A}{(r-1)!} \sum_{\mathbf{k}=0}^{\infty} (|\Delta x_1| + |\Delta x_2|)^{r+\gamma} \binom{n+|\mathbf{k}|}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1-|\mathbf{x}|)^{n+1} B(\gamma+1, r) \\ &\leq \frac{2A}{(r-1)!} \frac{\gamma}{\gamma+r} B(\gamma, r) \sum_{\mathbf{k}=0}^{\infty} \left| \mathbf{x} - \frac{\mathbf{k}}{n+|\mathbf{k}|} \right|^{r+\gamma} \binom{n+|\mathbf{k}|}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1-|\mathbf{x}|)^{n+1}, \end{aligned}$$

which proves (3.4). $\square$

For the uniform convergence of $M_n^{[r]}(f)$, let us consider the above mentioned function $g(\mathbf{s}) = |\mathbf{x} - \mathbf{s}|^{r+\gamma}$. Obviously $g(\mathbf{x}) = 0$ and g is continuous on S_2 . From multivariate extension of the Bohman-Korovkin theorem (see [12]) we have

$$\|M_n(g)\|_{C(S_2)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore by means of Theorem 3 we arrive at the following result

$$\left\| M_n^{[r]}(f) - f \right\|_{C(S_2)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

REFERENCES

- [1] A. Altın, O. Dođru, and F. Taşdelen, “The generalization of Meyer-König and Zeller operators by generating functions,” *J. Math. Anal. Appl.*, vol. 312, no. 1, pp. 181–194, 2005.
- [2] F. Cao, C. Ding, and Z. Xu, “On multivariate Baskakov operator,” *J. Math. Anal. Appl.*, vol. 307, no. 1, pp. 274–291, 2005.
- [3] E. W. Cheney and A. Sharma, “Bernstein power series,” *Can. J. Math.*, vol. 16, pp. 241–252, 1964.
- [4] O. Dođru and O. Duman, “Statistical approximation of Meyer-König and Zeller operators based on q -integers,” *Publ. Math.*, vol. 68, no. 1-2, pp. 199–214, 2006.
- [5] O. Dođru and V. Gupta, “Korovkin-type approximation properties of bivariate q -Meyer-König and Zeller operators,” *Calcolo*, vol. 43, no. 1, pp. 51–63, 2006.
- [6] M. Khan, B. Della Vecchia, and A. Fassih, “On the monotonicity of positive linear operators,” *J. Approx. Theory*, vol. 92, no. 1, pp. 22–37, art. no. at963 113, 1998.
- [7] R. A. Khan, “Some probabilistic methods in the theory of approximation operators,” *Acta Math. Acad. Sci. Hung.*, vol. 35, pp. 193–203, 1980.
- [8] G. Kirov and L. Popova, “A generalization of the linear positive operators,” *Math. Balk., New Ser.*, vol. 7, no. 2, pp. 149–162, 1993.
- [9] Z. Li, “Bernstein polynomials and modulus of continuity,” *J. Approx. Theory*, vol. 102, no. 1, pp. 171–174, art. no. jath.1999.3374, 2000.
- [10] A.-J. López-Moreno and F.-J. Muñoz-Delgado, “Asymptotic expansion of multivariate conservative linear operators,” *J. Comput. Appl. Math.*, vol. 150, no. 2, pp. 219–251, 2003.
- [11] T. Trif, “An elementary proof of the preservation of Lipschitz constants by the Meyer-König and Zeller operators,” *JIPAM, J. Inequal. Pure Appl. Math.*, vol. 4, no. 5, p. 3, 2003.
- [12] V. I. Volkov, “On the convergence of sequences of linear positive operators in the space of continuous functions of two variables,” *Dokl. Akad. Nauk SSSR*, vol. 115, pp. 17–19, 1957.
- [13] H. Wang, “Properties of convergence for the q -Meyer-König and Zeller operators,” *J. Math. Anal. Appl.*, vol. 335, no. 2, pp. 1360–1373, 2007.

Author’s address

Ali Olgun

Kırıkkale University, Faculty of Art and Science, Department of Mathematics, 71450, Yahşihan, Kırıkkale, Turkey

E-mail address: aliolgun71@gmail.com



SQUARES OF CONGRUENCE SUBGROUPS OF THE EXTENDED MODULAR GROUP

RECEP SAHIN AND SEBAHATTIN IKIKARDES

Received 6 June, 2013

Abstract. In this paper, we generalize some results related to the congruence subgroups of modular group Γ , given in [7] and [6] by Kiming, Schütt, and Verrill, to the extended modular group Π .

2010 *Mathematics Subject Classification:* 11F06

Keywords: modular group, extended modular group, principal congruence subgroup

1. INTRODUCTION

The modular group $\Gamma = PSL(2, \mathbb{Z})$ is the discrete subgroup of $PSL(2, \mathbb{R})$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad S(z) = -\frac{1}{z+1}.$$

Then modular group Γ has a presentation

$$\Gamma = \langle T, S \mid T^2 = S^3 = I \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_3.$$

The extended modular group $\Pi = PGL(2, \mathbb{Z})$ has been defined by adding the reflection $R(z) = 1/\bar{z}$ to the generators of the modular group Γ . The extended modular group Π has a presentation, see [5],

$$\Pi = \langle T, S, R \mid T^2 = S^3 = R^2 = (RT)^2 = (RS)^2 = I \rangle \cong D_2 *_{\mathbb{Z}_2} D_3.$$

Here T , S and R have matrix representations

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

respectively (in this work, we identify each matrix A in $GL(2, \mathbb{Z})$ with $-A$, so that they represent the same element of $PGL(2, \mathbb{Z})$). Thus the modular group $\Gamma = PSL(2, \mathbb{Z})$ is a subgroup of index 2 in the extended modular group Π .

Let us define Π^m as the subgroup generated by the m^{th} powers of all elements of Π , for some positive integer m . The subgroup Π^m is called the m^{th} power subgroup of Π . As fully invariant subgroups, they are normal in Π .

Then, power subgroups of the extended modular group Π were examined by Sahin, Ikikardes and Koruoglu in [10]. The authors showed that

$$\begin{aligned} |\Pi : \Pi^2| &= 4, \quad \Pi^2 = \Gamma^2, \\ \Pi^2 &= \langle S, TST \mid (S)^3 = (TST)^3 = I \rangle \cong \mathbb{Z}_3 * \mathbb{Z}_3, \end{aligned}$$

Also, from [5], we have the following. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represent a general element of Π . For each integer $N \geq 1$, we define

$$\begin{aligned} \Pi(N) &= \{A \in \Pi \mid a \equiv d \equiv \pm 1 \text{ and } b \equiv c \equiv 0 \pmod{N}\}, \\ \Gamma(N) &= \Pi(N) \cap \Gamma. \end{aligned}$$

These are normal subgroups of finite index in Π , and they are called as the *principal congruence subgroups*. If $N > 2$ then $\Pi(N) = \Gamma(N)$ and if $N = 2$ then $\Pi(2) \geq \Gamma(2) \geq \Pi(4) = \Gamma(4)$. A subgroup K of Π contains some $\Pi(N)$ if and only if it contains some $\Gamma(N)$. Such a subgroup K is called a *congruence subgroup*, and the *level* of K is the least n such that $\Pi(n) \leq K$. Any other subgroup of finite index in Π is called a *non-congruence subgroup*.

The most important of the congruence subgroups of Π are

$$\Pi_0(N) = \{A \in \Pi \mid c \equiv 0 \pmod{N}\}$$

and

$$\Pi^1(N) = \{A \in \Pi \mid a \equiv d \equiv \pm 1 \text{ and } c \equiv 0 \pmod{N}\}.$$

From [9], it is known that

$$\Pi_0(N) = \Gamma_0(N) \cup TR.\Gamma_0(N) \quad \text{and} \quad \Pi^1(N) = \Gamma^1(N) \cup TR.\Gamma^1(N).$$

Also, it is clear that $\Pi^1(N) \triangleleft \Pi_0(N)$ and for $N > 2$, $|\Pi_0(N) : \Pi^1(N)| = \varphi(N)/2$ where φ is the Euler Phi function (for the index $\Gamma^1(N)$ in $\Gamma_0(N)$, see [4]).

On the other hand, in [7] and [6], Kiming Schütt, and Verrill studied lifts of projective congruence subgroups. Now, we recall the following information from [7]. For a subgroup Λ of $SL(2, \mathbb{Z})$ denote by $\overline{\Lambda}$ the image of Λ in $PSL(2, \mathbb{Z})$. A lift of $\overline{\Lambda}$ is a subgroup of $SL(2, \mathbb{Z})$ that projects to $\overline{\Lambda}$ in $PSL(2, \mathbb{Z})$. A lift is called a congruence lift if it is a congruence subgroup.

In [7] and [6], the authors gave some consequences of their main results for the groups generated by squares of elements in congruence subgroups. These results are

a) $\Gamma(N)^2$ is a congruence if and only if $N \leq 2$.

b) All lifts of $\Gamma_0(N) \leq PSL(2, \mathbb{Z})$ are congruence subgroups of $SL(2, \mathbb{Z})$ if and only if either $N \in \{3, 4, 8\}$ or if $4 \nmid N$ and all odd prime divisors of N are congruent to 1 modulo 4.

c) All lifts of $\Gamma^1(N) \leq PSL(2, \mathbb{Z})$ are congruence subgroups of $SL(2, \mathbb{Z})$ if and only if $N \leq 4$.

The congruence and principal congruence subgroups (especially, $\Pi(2)$, $\Gamma(2)$, $\Gamma_0(N)$ and $\Gamma^1(N)$) of Γ and Π have been studied from various aspects in the literature, for example, number theory, modular forms, modular curves, Belyi's theory, graph theory, (please see [1], [2], [3] and [8]).

In this paper, we generalize the above results related with congruence subgroups of Γ , given in [7] and [6], to the extended modular group Π .

2. SQUARES OF CONGRUENCE SUBGROUPS OF Π

From [5], if $N > 2$ then $\Pi(N) = \Gamma(N)$ and so $\Pi(N)^2 = \Gamma(N)^2$. Thus, if $N > 2$ then $\Pi^2(N)$ is not a congruence. Also, from [10] and [5], $\Pi^2(1) = \Pi'$ and $\Pi(6) \leq \Pi^2(1)$ and so $\Pi^2(1)$ is a congruence subgroup. Therefore we need the following theorem.

Theorem 1. $\Pi(2)^2 = \Pi(4)$.

Proof. We know that the group structure of $\Pi(2)$ is

$$\begin{aligned} \Pi(2) = \langle TR, RSTS, RS^2TS^2 \mid (TR)^2 = (RSTS)^2 = (RS^2TS^2)^2 = I \rangle \\ \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2. \end{aligned}$$

Let $a = TR$, $b = RSTS$, $c = RS^2TS^2$. Then the quotient group $\Pi(2)/\Pi(2)^2$ is the group obtained by adding the relation $X^2 = I$ for all $X \in \Pi(2)$ to the relations of $\Pi(2)$. Thus we have

$$\Pi(2)/\Pi(2)^2 \cong \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^2 = (bc)^2 = \dots = I \rangle.$$

As $a^2 = b^2 = c^2 = I$, we obtain

$$\Pi(2)/\Pi(2)^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Therefore, we obtain $|\Pi(2) : \Pi(2)^2| = 8$.

Thus we use the Reidemeister-Schreier process to find the presentation of the subgroup $\Pi(2)^2$. Now we choose $\Sigma = \{I, a, b, c, ab, ac, bc, abc\}$ as a Schreier transversal for $\Pi(2)^2$. According to the Reidemeister-Schreier method, we can form all possible products :

$$\begin{array}{lll} I.a.(a)^{-1} = I, & I.b.(b)^{-1} = I, & I.c.(c)^{-1} = I, \\ a.a.(I)^{-1} = I, & a.b.(ab)^{-1} = I, & a.c.(ac)^{-1} = I, \\ b.a.(ab)^{-1} = baba, & b.b.(I)^{-1} = I, & b.c.(bc)^{-1} = I, \\ c.a.(ac)^{-1} = caca, & c.b.(bc)^{-1} = cbcb, & c.c.(I)^{-1} = I, \\ ab.a.(b)^{-1} = abab, & ab.b.(a)^{-1} = I, & ab.c.(abc)^{-1} = I, \\ ac.a.(c)^{-1} = acac, & ac.b.(abc)^{-1} = acbcb, & ac.c.(a)^{-1} = I, \\ bc.a.(abc)^{-1} = bcacba, & bc.b.(c)^{-1} = bcbc, & bc.c.(b)^{-1} = I, \\ abc.a.(bc)^{-1} = abcacb, & abc.b.(ac)^{-1} = abcbca, & abc.c.(ab)^{-1} = I, \end{array}$$

as $a^{-1} = a$, $b^{-1} = b$, and $c^{-1} = c$. Also, since $(baba)^{-1} = abab$, $(caca)^{-1} = acac$, $(cbcb)^{-1} = bcbc$, $(bcacba)^{-1} = abcacb$ and $(acbcba)^{-1} = abcbca$, the generators of $\Pi(2)^2$ are $abab = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$, $acac = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$, $bcbc = \begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}$, $abcacb = \begin{pmatrix} -7 & -12 \\ -4 & -7 \end{pmatrix}$ and $abcbca = \begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix}$.

From [7, Lemma 32], $\Pi(2)^2 = \Gamma(4)$. As $\Gamma(4) = \Pi(4)$, we obtain $\Pi(2)^2 = \Pi(4)$. $\square$

Using the above results, we have the following.

Proposition 1. $\Pi(N)^2$ is a congruence if and only if $N \leq 2$.

Now we present some results related with the congruence subgroups $\Pi_0(N)$ and $\Pi^1(N)$ of Π . To do this, we suppose that

$$A = \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix} \pmod{N}$$

is an element of $\Gamma_0(N)$. Then

$$TR.A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix} = \begin{pmatrix} -x & * \\ 0 & x^{-1} \end{pmatrix} \pmod{N}$$

is an element of $\Pi_0(N)$. Therefore

$$(TRA)^2 = \begin{pmatrix} -x & * \\ 0 & x^{-1} \end{pmatrix} \begin{pmatrix} -x & * \\ 0 & x^{-1} \end{pmatrix} = \begin{pmatrix} x^2 & * \\ 0 & x^{-2} \end{pmatrix} \pmod{N}$$

is an element of $\Gamma_0(N)^2$. Thus, we get $\Pi_0(N)^2 = \Gamma_0(N)^2$.

Similarly to the case $\Pi_0(N)$, if

$$B = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N}$$

is an element of $\Gamma^1(N)$, then

$$TR.B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & * \\ 0 & 1 \end{pmatrix} \pmod{N}$$

is an element of $\Pi^1(N)$. Therefore

$$(TRB)^2 = \begin{pmatrix} -1 & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N}$$

is an element of $\Gamma^1(N)^2$ and so we obtain $\Pi^1(N)^2 = \Gamma^1(N)^2$.

On the other hand, if $\Pi_0(N)$ and $\Pi^1(N)$ are not congruence, then $\Pi_0(N)^2$ and $\Pi^1(N)^2$ are not congruence, since any lift of $\Pi_0(N)$ (or $\Pi^1(N)$) necessarily contains $\Pi_0(N)^2$ (or $\Pi^1(N)^2$), from [7, Lemma 5]. Consequently, we have the following.

- Corollary 1.** *a) $\Pi_0(N)^2$ is not congruence if and only if either $N \notin \{3, 4, 8\}$ or if $4 \mid N$ and all odd prime divisors of N are congruent to 3 modulo 4.*
b) $\Pi^1(N)^2$ is not congruence if and only if $N > 4$.

REFERENCES

- [1] M. C. N. Cheng and A. Dabholkar, “Borcherds-Kac-Moody symmetry of $\mathcal{N} = 4$ dyons,” *Commun. Number Theory Phys.*, vol. 3, no. 1, pp. 59–110, 2009.
- [2] W. M. Goldman and W. D. Neumann, “Homological action of the modular group on some cubic moduli spaces,” *Math. Res. Lett.*, vol. 12, no. 4, pp. 575–591, 2005.
- [3] W. J. Harvey, “Teichmüller spaces, triangle groups and Grothendieck dessins,” in *Handbook of Teichmüller theory. Volume I*, ser. IRMA Lectures in Mathematics and Theoretical Physics, A. Papadopoulos, Ed. Zürich: European Mathematical Society (EMS), 2007, vol. 11, pp. 249–292.
- [4] I. Ivriissimtzis and D. Singerman, “Regular maps and principal congruence subgroups of Hecke groups,” *Eur. J. Comb.*, vol. 26, no. 3-4, pp. 437–456, 2005.
- [5] G. A. Jones and J. S. Thornton, “Automorphisms and congruence subgroups of the extended modular group,” *J. Lond. Math. Soc., II. Ser.*, vol. 34, pp. 26–40, 1986.
- [6] I. Kiming, “Lifts of projective congruence groups ii,” *Proc. Amer. Math. Soc.*, to appear.
- [7] I. Kiming, M. Schütt, and H. A. Verrill, “Lifts of projective congruence groups,” *J. Lond. Math. Soc., II. Ser.*, vol. 83, no. 1, pp. 96–120, 2011.
- [8] B. Köck and D. Singerman, “Real Belyi theory,” *Q. J. Math.*, vol. 58, no. 4, pp. 463–478, 2007.
- [9] R. S. Kulkarni, “An arithmetic-geometric method in the study of the subgroups of the modular group,” *Am. J. Math.*, vol. 113, no. 6, pp. 1053–1133, 1991.
- [10] R. Şahin, S. İkikardeş, and O. Koruoğlu, “On the power subgroups of the extended modular group $\overline{\Gamma}$,” *Turk. J. Math.*, vol. 28, no. 2, pp. 143–151, 2004.

Authors’ addresses

Recep Sahin

Balikesir University, Department of Mathematics, Cagis Kampusu, 10145 Balikesir, Turkey
E-mail address: rsahin@balikesir.edu.tr

Sebahattin İkikardes

Balikesir University, Department of Mathematics, Cagis Kampusu, 10145 Balikesir, Turkey
E-mail address: skardes@balikesir.edu.tr



FINITE GROUPS IN WHICH SOME PARTICULAR SUBGROUPS ARE TI-SUBGROUPS

JIANGTAO SHI AND CUI ZHANG

Received 6 June, 2013

Abstract. We prove that G is a group in which all noncyclic subgroups are TI-subgroups if and only if all noncyclic subgroups of G are normal in G . Moreover, we classify groups in which all subgroups of even order are TI-subgroups.

2010 Mathematics Subject Classification: 20D10

Keywords: noncyclic subgroup, subgroup of even order, TI-subgroup, normal

1. INTRODUCTION

All groups in this paper are considered to be finite. Let G be a group and M a subgroup of G . It is known that M is called a TI-subgroup of G if $M \cap M^g = 1$ or M for any $g \in G$.

In [5], Walls studied groups in which all subgroups are TI-subgroups. In [1], Guo, Li and Flavell classified groups in which all abelian subgroups are TI-subgroups. As a generalization of [1], we showed in [3] that G is a group in which all nonabelian subgroups are TI-subgroups if and only if all nonabelian subgroups of G are normal in G , and we showed in [4] that G is a group in which all nonnilpotent subgroups are TI-subgroups if and only if all nonnilpotent subgroups of G are normal in G .

In this paper, as a further generalization of above references, we first classify groups in which all noncyclic subgroups are TI-subgroups. We call such groups NCTI-groups. For NCTI-groups, we have the following result, the proof of which is given in Section 2.

Theorem 1. *A group G is an NCTI-group if and only if all noncyclic subgroups of G are normal in G .*

Remark 1. It is easy to show that any NCTI-group is solvable but it might not be supersolvable. For example, the alternating group A_4 is an NCTI-group but A_4 is nonsupersolvable.

The first author was supported by NSFC, Grant No. 11201401 and 11361075.

We say that a group G has nontrivial even order if $|G|$ is even but $|G| \neq 2$. In the following, we will classify groups in which all subgroups of even order are TI-subgroups. We call such groups EOTI-groups. For EOTI-groups, we have the following result, the proof of which is given in Section 3.

Theorem 2. *A group G is a EOTI-group if and only if one of the following statements holds:*

- (1) *all subgroups of G of nontrivial even order are normal in G .*
- (2) *$G = Z_p \rtimes \langle g \rangle$ is a Frobenius group with kernel Z_p and complement $\langle g \rangle$, where p is an odd prime and $o(g) = 2n$ for some $n > 1$.*

Remark 2. We can easily get that any EOTI-group is solvable. But the alternating group A_4 shows that a EOTI-group might not be supersolvable.

Arguing as in proof of Theorem 2, we can obtain the following three results, which are generalizations of Theorem 1, [3] and [4] respectively. Here we omit their proofs.

Theorem 3. *A group G is a group in which all noncyclic subgroups of even order are TI-subgroups if and only if all noncyclic subgroups of G of even order are normal in G .*

Theorem 4. *A group G is a group in which all nonabelian subgroups of even order are TI-subgroups if and only if all nonabelian subgroups of G of even order are normal in G .*

Theorem 5. *A group G is a group in which all nonnilpotent subgroups of even order are TI-subgroups if and only if all nonnilpotent subgroups of G of even order are normal in G .*

2. PROOF OF THEOREM 1

Proof. The sufficiency part is evident, we only need to prove the necessity part.

Let G be an NCTI-group. Assume that G has at least one nonnormal noncyclic subgroup. Choose M as a nonnormal noncyclic subgroup of G of largest order. Then, for any noncyclic subgroup K of G , we have that K is normal in G if $M < K$.

We claim that $M = N_G(M)$. Otherwise, assume that $M < N_G(M)$. Let L be a subgroup with $M < L \leq N_G(M)$ such that M has prime index in L . By the choice of M , we have that L is normal in G . Since M is not normal in G , $N_G(M) < G$. Take $h \in G \setminus N_G(M)$. We have $M^h < L^h = L$. Since M is a normal maximal subgroup of L and $M^h \neq M$, we have $L = MM^h$. Note that M is a TI-subgroup of G , which implies that $M \cap M^h = 1$. Then $|L| = |MM^h| = |M||M^h|$. It follows that $|M| = |M^h| = \frac{|L|}{|M|} = |L : M|$ is a prime, this contradicts that M is noncyclic. Thus $M = N_G(M)$.

Then, by the hypothesis, $M \cap M^a = 1$ for every $a \in G \setminus N_G(M) = G \setminus M$. We have that G is a Frobenius group with M being a Frobenius complement. Since G

is an NCTI-group, it follows that all nonabelian subgroups of G are TI-groups. By [3], we have that all nonabelian subgroups of G are normal in G . Then M is abelian. It follows that every Sylow subgroup of M must be cyclic by [2, Theorem 10.5.6]. This implies that M is cyclic, a contradiction. Thus G has no nonnormal noncyclic subgroup.

□

3. PROOF OF THEOREM 2

Proof. (1) We first prove the necessity part. Let G be a EOTI-group. Assume that G has at least one nonnormal subgroup of nontrivial even order. Let Q be any nonnormal subgroup of G of nontrivial even order. Choose L as a nonnormal subgroup of G of nontrivial even order of largest order such that $Q \leq L$.

We claim that $L = N_G(L)$. Assume that $L < N_G(L)$. Let K be a subgroup with $L < K \leq N_G(L)$ such that L has prime index in K . By the choice of L , we have that K is normal in G . Let y be an element of $G \setminus N_G(L)$. Then $L \cap L^y = 1$ by the hypothesis. It follows that $K = LL^y$ since L is a normal maximal subgroup of K . Then $|L| = |L^y| = |K : L|$ is a prime, a contradiction. Thus $L = N_G(L)$.

It follows that G is a Frobenius group with complement L . Let N be the Frobenius kernel of G . We have $G = N \rtimes L$.

We claim that L is a maximal subgroup of G . If L is not a maximal subgroup of G . Let M be a maximal subgroup of G such that $L < M$. It is obvious that M also has nontrivial even order. By the choice of L , we have that M is normal in G . Since G is a Frobenius group, it follows that either $M \leq N$ or $N < M$. If $M \leq N$, then $L < N$, a contradiction. If $N < M$, then $G = N \rtimes L \leq M$, again a contradiction. Thus L is a maximal subgroup of G .

It follows that N is a minimal normal subgroup of G . Since N is nilpotent by [2, Theorem 10.5.6], we can assume that $N = Z_p^m$, where $m \geq 1$ is a positive integer. Let d be an element of L of order 2. Since G is a Frobenius group and N is abelian, we have $e^d = e^{-1}$ for every $1 \neq e \in N$. Thus $\langle e \rangle \rtimes \langle d \rangle$ is a subgroup of G .

We claim that $\langle e \rangle \rtimes \langle d \rangle$ is normal in G . Note that $|\langle e \rangle \rtimes \langle d \rangle| = 2p$, where p is an odd prime. If $\langle e \rangle \rtimes \langle d \rangle$ is not normal in G . Arguing as the subgroup Q , there exists a nonnormal subgroup T of G of nontrivial even order of largest order such that $\langle e \rangle \rtimes \langle d \rangle \leq T$. And we have that T is also a Frobenius complement of G . But $N \cap T \geq \langle e \rangle \neq 1$, a contradiction. Thus $\langle e \rangle \rtimes \langle d \rangle$ is normal in G .

Since $\langle e \rangle \rtimes \langle d \rangle \not\leq N = Z_p^m$, we have $Z_p^m < \langle e \rangle \rtimes \langle d \rangle$. It follows that $m = 1$. Thus $N = Z_p$ is cyclic. By the N/C-theorem, since $C_G(N) = N$ it follows that L is isomorphic to a subgroup of $\text{Aut}(Z_p) \cong Z_{p-1}$. Then L is a cyclic group of order $2n$ for some $n > 1$. Assume that $L = \langle g \rangle$. We have $G = Z_p \rtimes \langle g \rangle$, where p is an odd prime and $o(g) = 2n$ for some $n > 1$.

(2) Now we prove the sufficiency part.

(i) If all subgroups of G of nontrivial even order are normal in G , then G is obviously a EOTI-group.

(ii) Assume that $G = Z_p \rtimes \langle g \rangle$ is a Frobenius group with kernel Z_p and complement $\langle g \rangle$, where p is an odd prime and $o(g) = 2n$ for some $n > 1$. Let S be any subgroup of G of even order. It is obvious that $S \cap Z_p = 1$ or Z_p .

If $S \cap Z_p = 1$, we have $S \leq \langle g \rangle^f$ for some $f \in G$. Obviously S is not normal in G . Then $N_G(S) = \langle g \rangle^f$. We have $S \cap S^x \leq \langle g \rangle^f \cap (\langle g \rangle^f)^x = 1$ for every $x \in G \setminus \langle g \rangle^f = G \setminus N_G(S)$.

If $S \cap Z_p = Z_p$, then $Z_p < S$. We have $S = S \cap G = S \cap (Z_p \rtimes \langle g \rangle) = Z_p \rtimes (S \cap \langle g \rangle)$. Then S is normal in G .

It follows that S is always a TI-subgroup of G whenever $S \cap Z_p = 1$ or Z_p . Then G is also a EOTI-group. □

ACKNOWLEDGEMENT

The authors are grateful to the referee who gives valuable comments and suggestions.

REFERENCES

- [1] X. Guo, S. Li, and P. Flavell, "Finite groups whose Abelian subgroups are TI-subgroups," *J. Algebra*, vol. 307, no. 2, pp. 565–569, 2007.
- [2] D. J. Robinson, *A course in the theory of groups*, 2nd ed., ser. Graduate Texts in Mathematics. New York: Springer-Verlag, 1995, vol. 80.
- [3] J. Shi and C. Zhang, "Finite groups in which all nonabelian subgroups are TI-subgroups," *J. Algebra Appl.*, vol. 13, 1350074, 2014.
- [4] J. Shi and C. Zhang, "A note on TI-subgroups of a finite group," *Algebra Colloq.*, to appear.
- [5] G. Walls, "Trivial intersection groups," *Arch. Math.*, vol. 32, pp. 1–4, 1979.

Authors' addresses

Jiangtao Shi

Yantai University, School of Mathematics and Information Science, 264005 Yantai, China

E-mail address: shijt@math.pku.edu.cn

Cui Zhang

Technical University of Denmark, Department of Applied Mathematics and Computer Science, DK-2800 Lyngby, Denmark

E-mail address: cuizhang2008@gmail.com



ON SOME INTEGRAL INEQUALITIES VIA h -CONVEXITY

MEVLÜT TUNÇ

Received 21 February, 2012

Abstract. In this paper, we establish some new inequalities for class of $SX(h, I)$ convex functions which are super-multiplicative or super-additive and nonnegative. And we also give some applications for special means.

2010 *Mathematics Subject Classification:* 26D15; 26A51

Keywords: Hadamard's inequality, h -convex, super-multiplicative, super-additive, similarly ordered

1. INTRODUCTION

In the sequel of the paper, I and J are intervals in $\mathbb{R}$, where I is a convex set, $(0, 1) \subseteq J$ and functions h and f are real non-negative functions defined on J and I , respectively.

The following definition is well known in the mathematical literature : [20] A function $f : I \rightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, where I is a convex set, is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad (1.1)$$

holds for all $x, y \in I$ and $t \in [0, 1]$. Geometrically, this means that if P, Q and R are three distinct points on the graph of f with Q between P and R , then Q is on or below the chord PR .

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2} \quad (1.2)$$

is known in the literature as Hadamard's inequality for convex function. Keep in mind that some of the classical inequalities for means can come from (1.2) for convenient particular selections of the function f . If f is concave, this double inequality holds in the opposite way.

The inequalities (1.2) which have been frequently used in a variety of settings, has come to a significant groundwork in mathematical analysis and optimization.

Many reports have provided new proofs, extensions and considering its refinements, generalizations, numerous interpolations and applications, for example, in the theory of special means and information theory. For some results on generalizations, extensions and applications of the Hermite-Hadamard inequalities, see [14, 15, 17], [2–4, 7, 10, 11, 21, 23, 24, 26], [5, 6, 16, 18, 19, 22].

Definition 1 ([15]). We say that $f : I \rightarrow \mathbb{R}$ is Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}. \quad (1.3)$$

Definition 2 ([14]). We say that $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a P -function or that f belongs to the class $P(I)$ if f is nonnegative and for all $x, y \in I$ and $t \in [0, 1]$, we have

$$f(tx + (1-t)y) \leq f(x) + f(y). \quad (1.4)$$

Definition 3 ([17]). Let $s \in (0, 1]$. A function $f : (0, \infty) \rightarrow [0, \infty]$ is said to be s -convex in the second sense if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y), \quad (1.5)$$

for all $x, y \in (0, b]$ and $t \in [0, 1]$. This class of s -convex functions is usually denoted by K_s^2 .

In 1978, Breckner introduced s -convex functions as a generalization of convex functions in [8]. Also, in that work Breckner proved the important fact that the set valued map is s -convex only if the associated support function is s -convex function in [9]. A number of properties and connections with s -convex in the first sense and its generalizations are discussed in the papers [12, 13, 17]. Of course, s -convexity means just convexity when $s = 1$.

The concept of h -convexity was introduced by Varošanec [26] and was generalized by Hány [16].

Definition 4 ([26]). Let $h : J \rightarrow \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. We say that $f : I \rightarrow \mathbb{R}$ is an h -convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y). \quad (1.6)$$

If inequality (1.6) is reversed, then f is said to be h -concave, i.e. $f \in SV(h, I)$. Obviously, if $h(t) = t$, then all nonnegative convex functions belong to $SX(h, I)$ and all nonnegative concave functions belong to $SV(h, I)$; if $h(t) = \frac{1}{t}$, then $SX(h, I) = Q(I)$; if $h(t) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(t) = t^s$, where $s \in (0, 1)$, then $SX(h, I) \supseteq K_s^2$.

Definition 5 ([16]). Let X be a real or complex topological vector space, $D \subset X$ be a nonempty open convex set and let $h : [0, 1] \rightarrow \mathbb{R}$ be a given function. We say that $f : D \rightarrow \mathbb{R}$ is an h -convex function if, for all $x, y \in D$ and $t \in [0, 1]$, we have

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y).$$

Definition 6 ([22, 26]). A function $h : J \rightarrow \mathbb{R}$ is said to be a super-multiplicative function if its values are positive and satisfy

$$h(xy) \geq h(x)h(y) \quad (1.7)$$

for all $x, y \in J$.

If inequality (1.7) is reversed, then h is said to be a sub-multiplicative function. If equality held in (1.7), then h is said to be a multiplicative function.

Definition 7 ([1]). A function $h : J \rightarrow \mathbb{R}$ is said to be a super-additive function if

$$h(x+y) \geq h(x) + h(y) \quad (1.8)$$

for all $x, y \in J$.

Definition 8 ([25]). Two functions $h : X \rightarrow \mathbb{R}$ and $g : X \rightarrow \mathbb{R}$ are said to be similarly ordered, shortly f s.o. g , if

$$(f(x) - f(y))(g(x) - g(y)) \geq 0 \quad (1.9)$$

for every $x, y \in X$.

Remark 1 ([26]). Let h be a non-negative function such that

$$h(\alpha) \geq \alpha \quad (1.10)$$

for all $\alpha \in (0, 1)$. For example, the function $h_k(x) = x^k$ where $k \leq 1$ and $x > 0$ has that property. If f is a non-negative convex function on I , then for $x, y \in I$, $\alpha \in (0, 1)$ we have

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) \leq h(\alpha)f(x) + h(1-\alpha)f(y). \quad (1.11)$$

So, $f \in SX(h, I)$. Similarly, if the function h has the property: $h(\alpha) \leq \alpha$ for all $\alpha \in (0, 1)$, then any non-negative concave function f belongs to the class $SV(h, I)$.

Proposition 1 ([26]). Let f and g be a similarly ordered functions on I , i.e.

$$(f(x) - f(y))(g(x) - g(y)) \geq 0, \quad (1.12)$$

for all $x, y \in I$. If $f \in SX(h_1, I)$, $g \in SX(h_2, I)$ and $h(\alpha) + h(1-\alpha) \leq c$ for all $\alpha \in (0, 1)$, where $h(t) = \max\{h_1(t), h_2(t)\}$ and c is a fixed positive number, then the product fg belongs to $SX(ch, I)$. If f and g are oppositely ordered, $f \in SV(h_1, I)$, $g \in SV(h_2, I)$ and $h(\alpha) + h(1-\alpha) \geq c$ for all $\alpha \in (0, 1)$, where $h(t) = \min\{h_1(t), h_2(t)\}$ and $c > 0$, then the product fg belongs to $SV(ch, I)$.

Up until now, there are many reports on the two convex function, two s -convex functions, two m -convex functions or on the product of the s -convex function with an ordinary convex function. And in this study, in addition to its predecessors, the new inequalities on the product of classes of h -convex function will be obtained by using the elementary analysis and the applications in the special means for the obtained inequalities will be provided. In this paper we will imply $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

2. MAIN RESULTS

The following inequalities is well known in the literature; For $\mu \leq \lambda$ and $\kappa \leq \varepsilon$ and $\mu, \lambda, \kappa, \varepsilon \in \mathbb{R}$,

$$\mu\varepsilon + \lambda\kappa \leq \mu\kappa + \lambda\varepsilon \quad (2.1)$$

$$\mu\kappa \leq \lambda\varepsilon \quad (2.2)$$

The inequality (2.1) is more useful than the inequality (2.2). In our proofs, the inequality (2.1) will be used.

Theorem 1. Let $f, g \in SX(h, I)$, h is super-multiplicative and f, g be similarly ordered functions on I for all $x, y \in I \subseteq \mathbb{R}$, and $(f, g)(x) \geq x$ and $h(t) \geq t$. Then for all $t \in [0, 1]$, we have;

$$\begin{aligned} & \frac{2a+3b}{6}(f(a)+g(a)) + \frac{3a+2b}{6}(f(b)+g(b)) \\ & \leq \frac{1}{b-a} \int_a^b (fg)(x) dx + (fg)(a) \int_0^1 [h(t(1-t)) + h(t^2)] dt \\ & + (fg)(b) \int_0^1 [h(t(1-t)) + h((1-t)^2)] dt. \end{aligned} \quad (2.3)$$

Proof. Since f, g are h -convex functions on I , we have

$$\begin{aligned} f(\alpha x + \beta y) & \leq h(\alpha)f(x) + h(\beta)f(y) \\ g(\alpha x + \beta y) & \leq h(\alpha)g(x) + h(\beta)g(y) \end{aligned}$$

for all $\alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$. Using the elementary inequality $\mu \leq \lambda$ and $\kappa \leq \varepsilon$ then $\mu\varepsilon + \lambda\kappa \leq \mu\kappa + \lambda\varepsilon$ for $\mu, \lambda, \kappa, \varepsilon \in \mathbb{R}$, so we get

$$\begin{aligned} & f(\alpha x + \beta y)[h(\alpha)g(x) + h(\beta)g(y)] \\ & + g(\alpha x + \beta y)[h(\alpha)f(x) + h(\beta)f(y)] \\ & \leq f(\alpha x + \beta y)g(\alpha x + \beta y) \\ & + [h(\alpha)f(x) + h(\beta)f(y)][h(\alpha)g(x) + h(\beta)g(y)] \end{aligned} \quad (2.4)$$

Using the other properties of f, g and h in Theorem 1, we get

$$f(\alpha x + \beta y)[h(\alpha)g(x) + h(\beta)g(y)]$$

$$\begin{aligned}
& + g(\alpha x + \beta y) [h(\alpha) f(x) + h(\beta) f(y)] \\
& = h(\alpha) g(x) f(\alpha x + \beta y) + h(\beta) g(y) f(\alpha x + \beta y) \\
& + h(\alpha) f(x) g(\alpha x + \beta y) + h(\beta) f(y) g(\alpha x + \beta y).
\end{aligned}$$

Since f , g and h are nonnegative functions, we write

$$\begin{aligned}
& h(\alpha) [g(x) f(\alpha x + \beta y) + f(x) g(\alpha x + \beta y)] \\
& + h(\beta) [g(y) f(\alpha x + \beta y) + f(y) g(\alpha x + \beta y)] \\
& \geq \alpha [(\alpha x + \beta y) g(x) + (\alpha x + \beta y) f(x)] \\
& + \beta [(\alpha x + \beta y) g(y) + (\alpha x + \beta y) f(y)] \\
& = \alpha [(\alpha x + \beta y) (g(x) + f(x))] + \beta [(\alpha x + \beta y) (g(y) + f(y))]. \quad (2.5)
\end{aligned}$$

For the right hand side of (2.4), again using the other properties of f , g and h in Theorem 1, we can write

$$\begin{aligned}
& f(\alpha x + \beta y) g(\alpha x + \beta y) \\
& + [h(\alpha) f(x) + h(\beta) f(y)] [h(\alpha) g(x) + h(\beta) g(y)] \\
& = f(\alpha x + \beta y) g(\alpha x + \beta y) + h^2(\alpha) f(x) g(x) + h^2(\beta) f(y) g(y) \\
& + h(\alpha) h(\beta) f(x) g(y) + h(\alpha) h(\beta) f(y) g(x) \\
& \leq f(\alpha x + \beta y) g(\alpha x + \beta y) + h^2(\alpha) f(x) g(x) + h^2(\beta) f(y) g(y) \\
& + h(\alpha \beta) f(x) g(y) + h(\alpha \beta) f(y) g(x) \\
& = f(\alpha x + \beta y) g(\alpha x + \beta y) + h^2(\alpha) f(x) g(x) + h^2(\beta) f(y) g(y) \\
& + h(\alpha \beta) (f(x) g(y) + f(y) g(x)) \\
& \leq f(\alpha x + \beta y) g(\alpha x + \beta y) + h^2(\alpha) f(x) g(x) + h^2(\beta) f(y) g(y) \\
& + h(\alpha \beta) (f(x) g(x) + f(y) g(y)) \\
& = f(\alpha x + \beta y) g(\alpha x + \beta y) + [h^2(\alpha) + h(\alpha \beta)] f(x) g(x) \\
& + [h^2(\beta) + h(\alpha \beta)] f(y) g(y) \quad (2.6)
\end{aligned}$$

Now by combining expression (2.5) and (2.6), we obtain

$$\begin{aligned}
& \alpha [(\alpha x + \beta y) (g(x) + f(x))] + \beta [(\alpha x + \beta y) (g(y) + f(y))] \\
& \leq f(\alpha x + \beta y) g(\alpha x + \beta y) + [h^2(\alpha) + h(\alpha \beta)] f(x) g(x) \\
& + [h^2(\beta) + h(\alpha \beta)] f(y) g(y). \quad (2.7)
\end{aligned}$$

If we choose $x = a$, $y = b$ and $\beta = 1 - \alpha$ in (2.7), we have

$$\begin{aligned}
& (\alpha^2 a + (1 - \alpha) b) (f(a) + g(a)) + (\alpha a + (1 - \alpha)^2 b) (f(b) + g(b)) \\
& \leq f(\alpha a + (1 - \alpha) b) g(\alpha a + (1 - \alpha) b) + [h^2(\alpha) + h(\alpha(1 - \alpha))] f(a) g(a) \\
& + [h^2(1 - \alpha) + h(\alpha(1 - \alpha))] f(b) g(b).
\end{aligned}$$

By multiplicatively of h , we deduce

$$\begin{aligned} & (\alpha^2 a + (1-\alpha)b)(f(a) + g(a)) + (\alpha a + (1-\alpha)^2 b)(f(b) + g(b)) \\ & \leq f(\alpha a + (1-\alpha)b)g(\alpha a + (1-\alpha)b) + [h(\alpha^2) + h(\alpha - \alpha^2)]f(a)g(a) \\ & + [h((1-\alpha)^2) + h(\alpha - \alpha^2)]f(b)g(b). \end{aligned}$$

By integrating the result with respect to α over $[0, 1]$, we obtain

$$\begin{aligned} & \left(a \int_0^1 \alpha^2 d\alpha + b \int_0^1 (1-\alpha) d\alpha \right) (f(a) + g(a)) \\ & + \left(a \int_0^1 \alpha d\alpha + b \int_0^1 (1-\alpha)^2 d\alpha \right) (f(b) + g(b)) \\ & = \frac{2a+3b}{6} (f(a) + g(a)) + \frac{3a+2b}{6} (f(b) + g(b)) \\ & \leq \frac{1}{b-a} \int_a^b fg(x) dx + f(a)g(a) \int_0^1 [h(\alpha^2) + h(\alpha - \alpha^2)] d\alpha \\ & + f(b)g(b) \int_0^1 [h((1-\alpha)^2) + h(\alpha - \alpha^2)] d\alpha \end{aligned}$$

which completes the proof. $\square$

Corollary 1. *If in (2.3) we take $h(t) = 1$, then we get an integral inequality for P -functions with launching of necessary mathematical operations,*

$$\begin{aligned} & \frac{2a+3b}{6} (f(a) + g(a)) + \frac{3a+2b}{6} (f(b) + g(b)) \\ & \leq \frac{1}{b-a} \int_a^b (fg)(x) dx + 2M(a, b). \end{aligned}$$

Corollary 2. *If in (2.3) we take $h(t) = t$, then we get an integral inequality for ordinary convex functions with launching of necessary mathematical operations,*

$$\begin{aligned} & \frac{2a+3b}{6} (f(a) + g(a)) + \frac{3a+2b}{6} (f(b) + g(b)) \\ & \leq \frac{1}{b-a} \int_a^b (fg)(x) dx + (fg)(a) \int_0^1 [(t-t^2) + (t^2)] dt \\ & + (fg)(b) \int_0^1 [(t-t^2) + ((1-t)^2)] dt \\ & = \frac{1}{b-a} \int_a^b (fg)(x) dx + \frac{M(a, b)}{2}. \end{aligned}$$

Corollary 3. If in (2.3) we take $h(t) = t^s$, then we obtain an integral inequality for s -convex functions in the second sense with use of the Beta function of Euler type

$$\begin{aligned} & \frac{2a+3b}{6} (f(a) + g(a)) + \frac{3a+2b}{6} (f(b) + g(b)) \\ & \leq \frac{1}{b-a} \int_a^b (fg)(x) dx + (fg)(a) \int_0^1 [t^s (1-t)^s + t^{2s}] dt \\ & \quad + (fg)(b) \int_0^1 [t^s (1-t)^s + (1-t)^{2s}] dt \\ & = \frac{1}{b-a} \int_a^b (fg)(x) dx + (fg)(a) [\beta(s+1, s+1) + \beta(2s+1, 1)] \\ & \quad + (fg)(b) [\beta(s+1, s+1) + \beta(1, 2s+1)] \\ & = \frac{1}{b-a} \int_a^b (fg)(x) dx + M(a, b) [\beta(s+1, s+1) + \beta(2s+1, 1)]. \end{aligned}$$

Theorem 2. Let $f, g \in SX(h, I)$, h is super-additive and nonnegative such that $h(\alpha) \geq \alpha$ and f, g be similarly ordered functions on I for all $x, y \in I \subseteq \mathbb{R}$. Then for all $\alpha \in (0, 1)$ and $\alpha + \beta = 1$ we have following inequality;

$$\begin{aligned} & \frac{M(a, b)}{6} + \frac{N(a, b)}{3} \\ & \leq h(1) \left[f(a) g(a) \int_0^1 h(\alpha) d\alpha + f(b) g(b) \int_0^1 h(1-\alpha) d\alpha \right]. \quad (2.8) \end{aligned}$$

Proof. Since f, g are h -convex functions on I , and using right hand side of (1.11), we have

$$\begin{aligned} \alpha f(x) + \beta f(y) & \leq h(\alpha) f(x) + h(\beta) f(y) \\ \alpha g(x) + \beta g(y) & \leq h(\alpha) g(x) + h(\beta) g(y) \end{aligned}$$

for all $\alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$. Using the elementary inequality $\mu \leq \lambda$ and $\kappa \leq \varepsilon$ then $\mu\varepsilon + \lambda\kappa \leq \mu\kappa + \lambda\varepsilon$ for $\mu, \lambda, \kappa, \varepsilon \in \mathbb{R}$, so we get

$$\begin{aligned} & [\alpha f(x) + \beta f(y)] [h(\alpha) g(x) + h(\beta) g(y)] \\ & + [\alpha g(x) + \beta g(y)] [h(\alpha) f(x) + h(\beta) f(y)] \\ & \leq [\alpha f(x) + \beta f(y)] [\alpha g(x) + \beta g(y)] \\ & + [h(\alpha) f(x) + h(\beta) f(y)] [h(\alpha) g(x) + h(\beta) g(y)]. \end{aligned} \quad (2.9)$$

Using the other properties of h in Theorem 2 on the left hand side of (2.9), we get

$$\begin{aligned} & [\alpha f(x) + \beta f(y)] [h(\alpha) g(x) + h(\beta) g(y)] \\ & + [\alpha g(x) + \beta g(y)] [h(\alpha) f(x) + h(\beta) f(y)] \\ & = \alpha h(\alpha) f(x) g(x) + \alpha h(\beta) f(x) g(y) \end{aligned}$$

$$\begin{aligned}
& + \beta h(\alpha) f(y) g(x) + \beta h(\beta) f(y) g(y) \\
& + \alpha h(\alpha) f(x) g(x) + \alpha h(\beta) f(y) g(x) \\
& + \beta h(\alpha) f(x) g(y) + \beta h(\beta) f(y) g(y) \\
& = 2\alpha h(\alpha) f(x) g(x) + \alpha h(\beta) [f(x) g(y) + f(y) g(x)] \\
& + \beta h(\alpha) [f(x) g(y) + f(y) g(x)] \\
& + 2\beta h(\beta) f(y) g(y),
\end{aligned}$$

if h be a non-negative function that $h(\alpha) \geq \alpha$

$$\begin{aligned}
& 2\alpha h(\alpha) f(x) g(x) + 2\beta h(\beta) f(y) g(y) \\
& + [\alpha h(\beta) + \beta h(\alpha)] [f(x) g(y) + f(y) g(x)] \\
& \geq 2\alpha^2 f(x) g(x) + 2\alpha\beta [f(x) g(y) + f(y) g(x)] + 2\beta^2 f(y) g(y). \quad (2.10)
\end{aligned}$$

However, using the other properties of f , g and h in Theorem 2 on the right hand side of (2.9), we get

$$\begin{aligned}
& [\alpha f(x) + \beta f(y)] [\alpha g(x) + \beta g(y)] \\
& + [h(\alpha) f(x) + h(\beta) f(y)] [h(\alpha) g(x) + h(\beta) g(y)] \\
& = \alpha^2 f(x) g(x) + \alpha\beta f(x) g(y) \\
& + \alpha\beta f(y) g(x) + \beta^2 f(y) g(y) \\
& + h^2(\alpha) f(x) g(x) + h(\alpha) h(\beta) f(x) g(y) \\
& + h(\alpha) h(\beta) f(y) g(x) + h^2(\beta) f(y) g(y) \\
& = \alpha^2 f(x) g(x) + \beta^2 f(y) g(y) \\
& + \alpha\beta [f(x) g(y) + f(y) g(x)] \\
& + h^2(\alpha) f(x) g(x) + h^2(\beta) f(y) g(y) \\
& + h(\alpha) h(\beta) [f(x) g(y) + f(y) g(x)] \\
& \leq \alpha^2 f(x) g(x) + \beta^2 f(y) g(y) + \alpha\beta [f(x) g(x) + f(y) g(y)] \\
& + h^2(\alpha) f(x) g(x) + h^2(\beta) f(y) g(y) \\
& + h(\alpha) h(\beta) [f(x) g(x) + f(y) g(y)] \\
& = [\alpha^2 + \alpha\beta + h^2(\alpha) + h(\alpha) h(\beta)] f(x) g(x) \\
& + [\alpha\beta + \beta^2 + h(\alpha) h(\beta) + h^2(\beta)] f(y) g(y) \\
& \leq [\alpha^2 + \alpha\beta + h(\alpha) h(\alpha + \beta)] f(x) g(x) \\
& + [\alpha\beta + \beta^2 + h(\beta) h(\alpha + \beta)] f(y) g(y) \\
& = [\alpha^2 + \alpha\beta + h(\alpha) h(1)] f(x) g(x) \\
& + [\alpha\beta + \beta^2 + h(\beta) h(1)] f(y) g(y). \quad (2.11)
\end{aligned}$$

Then by expression (2.10) and (2.11);

$$\begin{aligned} & \alpha^2 f(x)g(x) + 2\alpha\beta[f(x)g(y) + f(y)g(x)] + \beta^2 f(y)g(y) \\ & \leq [\alpha\beta + h(\alpha)h(1)]f(x)g(x) + [\alpha\beta + h(\beta)h(1)]f(y)g(y) \end{aligned} \quad (2.12)$$

by taking $x = a$, $y = b$ and $\beta = 1 - \alpha$ in (2.12), we have

$$\begin{aligned} & \alpha^2 f(a)g(a) + 2\alpha(1-\alpha)[f(a)g(b) + f(b)g(a)] + (1-\alpha)^2 f(b)g(b) \\ & \leq [\alpha(1-\alpha) + h(\alpha)h(1)]f(a)g(a) + [\alpha(1-\alpha) + h(1-\alpha)h(1)]f(b)g(b) \end{aligned}$$

By integrating the result with respect to α over $[0, 1]$, we get

$$\begin{aligned} & f(a)g(a) \int_0^1 \alpha^2 d\alpha + 2[f(a)g(b) + f(b)g(a)] \int_0^1 \alpha(1-\alpha) d\alpha \\ & + f(b)g(b) \int_0^1 (1-\alpha)^2 d\alpha \\ & = \frac{f(a)g(a) + f(b)g(b)}{3} + \frac{f(a)g(b) + f(b)g(a)}{3} \\ & \leq f(a)g(a) \left[\int_0^1 \alpha(1-\alpha) d\alpha + h(1) \int_0^1 h(\alpha) d\alpha \right] \\ & + f(b)g(b) \left[\int_0^1 \alpha(1-\alpha) d\alpha + h(1) \int_0^1 h(1-\alpha) dt \right] \\ & = \frac{f(a)g(a) + f(b)g(b)}{6} \\ & + h(1) \left[f(a)g(a) \int_0^1 h(\alpha) d\alpha + f(b)g(b) \int_0^1 h(1-\alpha) d\alpha \right] \end{aligned}$$

which completes the proof of (2.8). $\square$

Corollary 4. In Theorem 2, if we choose $h(t) = t$, then inequality of (2.8) brings inequality (1.12) down.

Corollary 5. In Theorem 2, if we choose $h(t) = t^s$, then we get an integral inequality for s -convex functions in the second sense;

$$\frac{M(a,b)}{6} + \frac{N(a,b)}{3} \leq \frac{M(a,b)}{s+1}. \quad (2.13)$$

And, in (2.13), if we choose $s = 1$, then the inequality of (2.13) above brings inequality (1.12) down.

Theorem 3. Let $f, g \in SX(h, I)$, h is super-additive and nonnegative such that $h(\alpha) \geq \alpha$ and f, g be similarly ordered functions on I for all $x, y \in I \subseteq \mathbb{R}$. Then for all $\alpha \in (0, 1)$ and $\alpha + \beta = 1$, we have

$$\frac{M(a,b)}{3} + \frac{N(a,b)}{6} \quad (2.14)$$

$$\leq h(1) \left[f(a)g(a) \int_0^1 h(\alpha) d\alpha + f(b)g(b) \int_0^1 h(1-\alpha) d\alpha \right].$$

Proof. As in the proof of the inequality (2.8), since f, g are h -convex functions on I , and using right hand side of (1.12), we have

$$\begin{aligned} \alpha f(x) + \beta f(y) &\leq h(\alpha) f(x) + h(\beta) f(y) \\ \alpha g(x) + \beta g(y) &\leq h(\alpha) g(x) + h(\beta) g(y) \end{aligned}$$

for all $\alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$. Using the elementary inequality $\mu \leq \lambda$ and $\kappa \leq \varepsilon$ then $\mu\varepsilon + \lambda\kappa \leq \mu\kappa + \lambda\varepsilon$ for $\mu, \lambda, \kappa, \varepsilon \in \mathbb{R}$, so we obtain

$$\begin{aligned} &[\alpha f(x) + \beta f(y)][h(\alpha)g(x) + h(\beta)g(y)] \\ &+ [\alpha g(x) + \beta g(y)][h(\alpha)f(x) + h(\beta)f(y)] \\ &\leq [\alpha f(x) + \beta f(y)][\alpha g(x) + \beta g(y)] \\ &+ [h(\alpha)f(x) + h(\beta)f(y)][h(\alpha)g(x) + h(\beta)g(y)]. \end{aligned} \quad (2.15)$$

Using the other properties of h in Theorem 3 on the left hand side of (2.15), we get

$$\begin{aligned} &[\alpha f(x) + \beta f(y)][h(\alpha)g(x) + h(\beta)g(y)] \\ &+ [\alpha g(x) + \beta g(y)][h(\alpha)f(x) + h(\beta)f(y)] \\ &\geq 2\alpha^2 f(x)g(x) + 2\alpha\beta[f(x)g(y) + f(y)g(x)] + 2\beta^2 f(y)g(y) \end{aligned} \quad (2.16)$$

However, using the other properties of f, g and h in Theorem 3 on the right hand side of (2.15), we obtain

$$\begin{aligned} &[\alpha f(x) + \beta f(y)][\alpha g(x) + \beta g(y)] \\ &+ [h(\alpha)f(x) + h(\beta)f(y)][h(\alpha)g(x) + h(\beta)g(y)] \\ &= \alpha^2 f(x)g(x) + \beta^2 f(y)g(y) + \alpha\beta[f(x)g(y) + f(y)g(x)] \\ &+ h^2(\alpha)f(x)g(x) + h^2(\beta)f(y)g(y) + h(\alpha)h(\beta)[f(x)g(y) + f(y)g(x)] \\ &\leq \alpha^2 f(x)g(x) + \beta^2 f(y)g(y) + \alpha\beta[f(x)g(x) + f(y)g(y)] \\ &+ h^2(\alpha)f(x)g(x) + h^2(\beta)f(y)g(y) + h(\alpha)h(\beta)[f(x)g(x) + f(y)g(y)] \\ &\leq h^2(\alpha)f(x)g(x) + h^2(\beta)f(y)g(y) + h(\alpha)h(\beta)[f(x)g(x) + f(y)g(y)] \\ &+ h^2(\alpha)f(x)g(x) + h^2(\beta)f(y)g(y) + h(\alpha)h(\beta)[f(x)g(x) + f(y)g(y)] \\ &= 2(h^2(\alpha) + h(\alpha)h(\beta))f(x)g(x) + 2(h^2(\beta) + h(\alpha)h(\beta))f(y)g(y) \\ &\leq 2h(\alpha)h(\alpha + \beta)f(x)g(x) + 2h(\beta)h(\alpha + \beta)f(y)g(y) \\ &= 2h(\alpha + \beta)[h(\alpha)f(x)g(x) + h(\beta)f(y)g(y)] \\ &= 2h(1)[h(\alpha)f(x)g(x) + h(\beta)f(y)g(y)] \end{aligned} \quad (2.17)$$

Then by expression (2.16) and (2.17);

$$\alpha^2 f(x)g(x) + \alpha\beta[f(x)g(y) + f(y)g(x)] + \beta^2 f(y)g(y)$$

$$\leq h(1)[h(\alpha)f(x)g(x) + h(\beta)f(y)g(y)] \quad (2.18)$$

by taking $x = a$, $y = b$ and $\beta = 1 - \alpha$ in (2.18), we have

$$\begin{aligned} & t^2 f(a)g(a) + t(1-t)[f(a)g(b) + f(b)g(a)] + (1-t)^2 f(b)g(b) \\ & \leq h(1)[h(\alpha)f(a)g(a) + h(1-\alpha)f(b)g(b)]. \end{aligned}$$

By integrating the result with respect to α over $[0, 1]$, we obtain

$$\begin{aligned} & f(a)g(a) \int_0^1 \alpha^2 d\alpha + 2[f(a)g(b) + f(b)g(a)] \int_0^1 \alpha(1-\alpha) d\alpha \\ & + f(b)g(b) \int_0^1 (1-\alpha)^2 d\alpha \\ & = \frac{f(a)g(a) + f(b)g(b)}{3} + \frac{f(a)g(b) + f(b)g(a)}{6} \\ & \leq h(1)[h(\alpha)f(a)g(a) + h(1-\alpha)f(b)g(b)] \\ & = h(1) \left[f(a)g(a) \int_0^1 h(\alpha) d\alpha + f(b)g(b) \int_0^1 h(1-\alpha) d\alpha \right] \end{aligned}$$

which completes the proof of (2.14). $\square$

Theorem 4. Let $f, g \in SX(h, I)$, h is super-additive. Then for all $\alpha \in (0, 1)$, we have following inequality;

$$\begin{aligned} & \left[f\left(\frac{a+b}{2}\right) \frac{g(a)+g(b)}{2} + g\left(\frac{a+b}{2}\right) \frac{f(a)+f(b)}{2} \right] \\ & \times \left[\int_0^1 h(\alpha) d\alpha + \int_0^1 h(1-\alpha) d\alpha \right] \\ & \leq (fg)\left(\frac{a+b}{2}\right) + h^2(1)(M(a, b) + N(a, b)). \end{aligned} \quad (2.19)$$

Proof. Since $f, g \in SX(h, I)$, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{\alpha a + (1-\alpha)b}{2} + \frac{(1-\alpha)a + \alpha b}{2}\right) \\ &\leq \frac{f(\alpha a + (1-\alpha)b) + f((1-\alpha)a + \alpha b)}{2} \\ &\leq \frac{h(\alpha)f(a) + h(1-\alpha)f(b) + h(1-\alpha)f(a) + h(\alpha)f(b)}{2} \\ &= \frac{[h(\alpha) + h(1-\alpha)][f(a) + f(b)]}{2} \\ g\left(\frac{a+b}{2}\right) &\leq \frac{[h(\alpha) + h(1-\alpha)][g(a) + g(b)]}{2} \end{aligned}$$

for all $\alpha \in (0, 1)$. Using the elementary inequality $\mu \leq \lambda$ and $\kappa \leq \varepsilon$ then $\mu\varepsilon + \lambda\kappa \leq \mu\kappa + \lambda\varepsilon$ for $\mu, \lambda, \kappa, \varepsilon \in \mathbb{R}$, so we get

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \frac{[h(\alpha) + h(1-\alpha)][g(a) + g(b)]}{2} \\ & + g\left(\frac{a+b}{2}\right) \frac{[h(\alpha) + h(1-\alpha)][f(a) + f(b)]}{2} \\ & = \left[f\left(\frac{a+b}{2}\right) \frac{g(a) + g(b)}{2} + g\left(\frac{a+b}{2}\right) \frac{f(a) + f(b)}{2} \right] [h(\alpha) + h(1-\alpha)] \\ & \leq f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) + \frac{[h(\alpha) + h(1-\alpha)]^2 [f(a) + f(b)][g(a) + g(b)]}{4} \\ & \leq f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) + \frac{[h(1)]^2 [f(a) + f(b)][g(a) + g(b)]}{4}, \end{aligned}$$

by integrating the result with respect to α over $[0, 1]$, we obtain (2.19). $\square$

Theorem 5. Let $f, g \in SX(h, I)$, and h be super-multiplicative. Then for all $\alpha \in (0, 1)$, we have following inequality;

$$\begin{aligned} & \frac{2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)}{M(a, b)} \\ & \leq \left[\int_0^1 h(\alpha^2) d\alpha + 2 \int_0^1 h(\alpha - \alpha^2) d\alpha + \int_0^1 h((1-\alpha)^2) d\alpha \right]. \end{aligned} \quad (2.20)$$

Proof. Since $f, g \in SX(h, I)$, we write

$$\begin{aligned} f\left(\frac{a+b}{2}\right) & \leq [h(\alpha) + h(1-\alpha)] \frac{f(a) + f(b)}{2} \\ g\left(\frac{a+b}{2}\right) & \leq [h(\alpha) + h(1-\alpha)] \frac{g(a) + g(b)}{2} \end{aligned}$$

for all $\alpha \in (0, 1)$. Since h is super-multiplicative function and f and g are similarly ordered functions, we get

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ & \leq \frac{[h(\alpha) + h(1-\alpha)]^2}{4} (f(a) + f(b))(g(a) + g(b)) \\ & = \frac{[h^2(\alpha) + 2h(\alpha)h(1-\alpha) + h^2(1-\alpha)]^2}{4} \\ & \quad \times (f(a)g(a) + f(b)g(b) + f(a)g(b) + f(b)g(a)) \end{aligned}$$

$$\leq \frac{\left[h(\alpha^2) + 2h(\alpha(1-\alpha)) + h((1-\alpha)^2) \right]^2 (f(a)g(a) + f(b)g(b))}{2}$$

by integrating the result with respect to α over $[0, 1]$, we obtain (2.20). $\square$

Corollary 6. *If in (2.20) we take $h(\alpha) = 1$, then we obtain an integral inequality for P -functions with launching of necessary mathematical operations,*

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq 2M(a, b)$$

Corollary 7. *If in (2.20) we take $h(\alpha) = \alpha$, then we obtain an integral inequality for ordinary convex functions with launching of necessary mathematical operations,*

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{M(a, b)}{2}.$$

Corollary 8. *If in (2.20) we take $h(\alpha) = \alpha^s$, then we obtain an integral inequality for s -convex functions in the second sense with the use of the Beta function of Euler type*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ & \leq \frac{M(a, b)}{2} \left[\int_0^1 \alpha^{2s} d\alpha + 2 \int_0^1 \alpha^s (1-\alpha)^s d\alpha + \int_0^1 (1-\alpha)^{2s} d\alpha \right] \\ & = \frac{M(a, b)}{2} \{ \beta(2s+1, 1) + 2\beta(2s+1, 2s+1) + \beta(1, 2s+1) \} \\ & = M(a, b) \{ \beta(2s+1, 1) + \beta(2s+1, 2s+1) \}. \end{aligned}$$

Theorem 6. *Let $I = [a, b] \subseteq \mathbb{R}$, $f, g \in SX(h, I)$, h is super-multiplicative and f, g are symmetric about $\frac{a+b}{2}$, then for all $x \in I$ and $\alpha \in [0, 1]$, we have the following inequalities;*

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x)g(x)dx \\ & \leq \frac{M(a, b) + N(a, b)}{4} \\ & \times \left[\int_0^1 h(\alpha^2) d\alpha + 2 \int_0^1 h(\alpha(1-\alpha)) d\alpha + \int_0^1 h((1-\alpha)^2) d\alpha \right] \end{aligned} \quad (2.21)$$

Proof. Since $f, g \in SX(h, I)$, we can write

$$\begin{aligned} f(\alpha a + (1-\alpha)b) + f((1-\alpha)a + \alpha b) & \leq [h(\alpha) + h(1-\alpha)][f(a) + f(b)] \\ g(\alpha a + (1-\alpha)b) + g((1-\alpha)a + \alpha b) & \leq [h(\alpha) + h(1-\alpha)][g(a) + g(b)] \end{aligned}$$

for all $\alpha \in [0, 1]$. Since h is a super-multiplicative function, and f and g are symmetric about $\frac{a+b}{2}$, we get

$$\begin{aligned} & [f(\alpha a + (1-\alpha)b) + f((1-\alpha)a + \alpha b)][g(\alpha a + (1-\alpha)b) + g((1-\alpha)a + \alpha b)] \\ &= 4f(\alpha a + (1-\alpha)b)g(\alpha a + (1-\alpha)b) \\ &\leq [h(\alpha) + h(1-\alpha)]^2[f(a) + f(b)][g(a) + g(b)] \\ &= [h^2(\alpha) + 2h(\alpha)h(1-\alpha) + h^2(1-\alpha)][M(a, b) + N(a, b)] \\ &\leq [h(\alpha^2) + 2h(\alpha(1-\alpha)) + h((1-\alpha)^2)][M(a, b) + N(a, b)]. \end{aligned}$$

By integrating the result with respect to α over $[0, 1]$, and taking into account the change of variable $x = \alpha a + (1-\alpha)b$, we obtain

$$\begin{aligned} & \frac{4}{b-a} \int_a^b f(x)g(x)dx \\ &\leq [M(a, b) + N(a, b)] \left[\int_0^1 h(\alpha^2) d\alpha + 2 \int_0^1 h(\alpha(1-\alpha)) d\alpha + \int_0^1 h((1-\alpha)^2) d\alpha \right] \end{aligned}$$

which completes the proof. $\square$

Corollary 9. *If in (2.21) we take $h(\alpha) = 1$, then we obtain an integral inequality for P -functions with launching of necessary mathematical operations,*

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq M(a, b) + N(a, b).$$

Corollary 10. *If in (2.21) we take $h(\alpha) = \alpha$, then we obtain an integral inequality for ordinary convex functions with the use of the necessary mathematical operations,*

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{M(a, b) + N(a, b)}{4}.$$

Corollary 11. *If in (2.21) we take $h(\alpha) = \alpha^s$, then we obtain an integral inequality for s -convex functions in the second sense the with use of the Beta function of Euler type*

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{M(a, b) + N(a, b)}{2} \{\beta(2s+1, 1) + \beta(2s+1, 2s+1)\}.$$

3. APPLICATIONS TO SOME SPECIAL MEANS

We now consider the applications of our Theorems to the following special means

The arithmetic mean: $A = A(a, b) := \frac{a+b}{2}$, $a, b \geq 0$,

The geometric mean: $G = G(a, b) := \sqrt{ab}$, $a, b \geq 0$,

The quadratic mean: $K = K(a, b) := \sqrt{\frac{a^2+b^2}{2}}$, $a, b \geq 0$,

The p -logarithmic mean: $L_p = L_p(a, b) := \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{1/p} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}$,
 $p \in \mathbb{R} \setminus \{-1, 0\}$; $a, b > 0$.

The following inequality is well known in the literature:

$$H \leq G \leq L \leq I \leq A \leq K$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_1 = A$, $L_0 = I$ and $L_{-1} = L$.

The following propositions hold:

Proposition 2. Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{Z}$, $|n| \geq 1$. Then we have:

$$\begin{aligned} & \frac{4}{3} A(a^{n+1}, b^{n+1}) + 2G^2(a, b) A(a^{n-1}, b^{n-1}) \\ & \leq L_{2n}^{2n}(a, b) + A(a^{2n}, b^{2n}). \end{aligned} \quad (3.1)$$

Proof. If we apply Theorem 1 for $f(x) = g(x) = x^n$, $h(\alpha) = \alpha$ where $x \in \mathbb{R}$, $n \in \mathbb{Z}$, $|n| \geq 1$, we get the proof (3.1). $\square$

Proposition 3. Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{Z}$, $|n| \geq 1$. Then we have:

$$G^2(a^n, b^n) \leq 2A(a^{2n}, b^{2n}). \quad (3.2)$$

Proof. The proof is immediate from Theorem 2 applied for $f(x) = g(x) = x^n$, $h(\alpha) = \alpha$ where $x \in \mathbb{R}$, $n \in \mathbb{Z}$, $|n| \geq 1$. $\square$

Proposition 4. Let $a, b \in \mathbb{R}$, $0 < a < b$. Then we have:

$$\frac{1}{G^2(a, b)} \leq K(a, b). \quad (3.3)$$

Proof. The assertion follows from Theorem 2 applied to $f(x) = g(x) = \frac{1}{x}$, $x \in [a, b]$ and $h(\alpha) = \alpha$. $\square$

Proposition 5. Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{Z}$, $|n| \geq 1$. Then we have:

$$G^2(a^n, b^n) \leq A(a^{2n}, b^{2n}) \quad (3.4)$$

Proof. The assertion follows from Theorem 3 applied to $f(x) = g(x) = \frac{1}{x}$, $x \in [a, b]$ and $h(\alpha) = \alpha$. $\square$

Proposition 6. Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{Z}$, $|n| \geq 1$. Then we have:

$$L_{2n}^{2n}(a, b) \leq \frac{A(a^{2n}, b^{2n}) + G^2(a^n, b^n)}{2}. \quad (3.5)$$

Proof. If we apply Theorem 6 for $f(x) = g(x) = x^n$, $h(\alpha) = \alpha$ where $x \in \mathbb{R}$, $n \in \mathbb{Z}$, $|n| \geq 1$, we get the proof (3.5). $\square$

Proposition 7. Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{Z}$, $|n| \geq 1$. Then we have:

$$G(a, b) \leq A(a, b) \quad (3.6)$$

Proof. If we apply Theorem 6 for $f(x) = g(x) = \frac{1}{x}$, $x \in [a, b]$ and $h(\alpha) = \alpha$, we get the proof (3.6). $\square$

REFERENCES

- [1] H. Alzer, "A superadditive property of Hadamard's gamma function," *Abh. Math. Semin. Univ. Hamb.*, vol. 79, no. 1, pp. 11–23, 2009.
- [2] M. Bessenyei, "Hermite-Hadamard-type inequalities for generalized convex functions," *JIPAM, J. Inequal. Pure Appl. Math.*, vol. 9, no. 3, p. 51, 2008.
- [3] M. Bessenyei, "The Hermite-Hadamard inequality on simplices," *Am. Math. Mon.*, vol. 115, no. 4, pp. 339–345, 2008.
- [4] M. Bessenyei, "The Hermite-Hadamard inequality in Beckenbach's setting," *J. Math. Anal. Appl.*, vol. 364, no. 2, pp. 366–383, 2010.
- [5] M. Bessenyei and Z. Páles, "Higher-order generalizations of Hadamard's inequality," *Publ. Math.*, vol. 61, no. 3-4, pp. 623–643, 2002.
- [6] M. Bessenyei and Z. Páles, "Characterizations of convexity via Hadamard's inequality," *Math. Inequal. Appl.*, vol. 9, no. 1, pp. 53–62, 2006.
- [7] M. Bombardelli and S. Varošanec, "Properties of h -convex functions related to the Hermite-Hadamard-Fejér inequalities," *Comput. Math. Appl.*, vol. 58, no. 9, pp. 1869–1877, 2009.
- [8] W. W. Breckner, "Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen," *Publ. Inst. Math., Nouv. Sér.*, vol. 23, pp. 13–20, 1978.
- [9] W. W. Breckner, "Continuity of generalized convex and generalized concave set-valued functions," *Rev. Anal. Numér. Théor. Approx.*, vol. 22, no. 1, pp. 39–51, 1993.
- [10] P. Burai and A. Hazy, "On Orlicz-convex functions," in *Proceedings of the 12th symposium of mathematics and its applications, "Politehnica" University of Timișoara, Timișoara, Romania, November 5–7, 2009*. Timișoara: Editura Politehnica, 2010, pp. 73–78.
- [11] P. Burai and A. Hazy, "On approximately h -convex functions," *J. Convex Anal.*, vol. 18, no. 2, pp. 447–454, 2011.
- [12] P. Burai, A. Hazy, and T. Juhász, "Bernstein-Doetsch type results for s -convex functions," *Publ. Math.*, vol. 75, no. 1-2, pp. 23–31, 2009.
- [13] P. Burai, A. Hazy, and T. Juhász, "On approximately Breckner s -convex functions," *Control Cybernet.*, vol. 40, no. 1, pp. 91–99, 2011.
- [14] S. S. Dragomir, J. E. Pečarić, and L. E. Persson, "Some inequalities of Hadamard type," *Soochow J. Math.*, vol. 21, no. 3, pp. 335–341, 1995.
- [15] E. K. Godunova and V. I. Levin, "Neravenstva dlja funkicii širokogo klassa, sodержascego vypuklye, monotonnye i nekotorye drugie vidy funkii," *Vycislitel. Mat. i Fiz. Mezvuzov. Sb. Nauc. Trudov, MGPI, Moskva*, pp. 138–142, 1985.
- [16] A. Hazy, "Bernstein-Doetsch type results for h -convex functions," *Math. Inequal. Appl.*, vol. 14, no. 3, pp. 499–508, 2011.
- [17] H. Hudzik and L. Maligranda, "Some remarks on s -convex functions," *Aequationes Math.*, vol. 48, no. 1, pp. 100–111, 1994.
- [18] J. Makó and Z. Páles, "Implications between approximate convexity properties and approximate Hermite-Hadamard inequalities," *Cent. Eur. J. Math.*, vol. 10, no. 3, pp. 1017–1041, 2012.
- [19] J. Makó and Z. Páles, "Approximate Hermite-Hadamard type inequalities for approximately convex functions," *Math. Inequal. Appl.*, vol. 16, no. 2, pp. 507–526, 2013.

- [20] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and new inequalities in analysis*, ser. Mathematics and Its Applications. East European Series. Dordrecht: Kluwer Academic Publishers, 1993, vol. 61.
- [21] M. E. Özdemir, M. Gürbüz, and A. O. Akdemir, “Inequalities for h -convex functions via further properties,” *RGMIA Research Report Collection*, vol. 14, article 22, 2011.
- [22] B. G. Pachpatte, *Mathematical inequalities*, ser. North-Holland Mathematical Library. Amsterdam: Elsevier, 2005, vol. 67.
- [23] M. Z. Sarikaya, A. Saglam, and H. Yildirim, “On some Hadamard-type inequalities for h -convex functions,” *J. Math. Inequal.*, vol. 2, no. 3, pp. 335–341, 2008.
- [24] M. Z. Sarikaya, E. Set, and M. E. Ozdemir, “On some new inequalities of Hadamard type involving h -convex functions,” *Acta Math. Univ. Comen., New Ser.*, vol. 79, no. 2, pp. 265–272, 2010.
- [25] H. J. Skala, “On the characterization of certain similarly ordered super-additive functionals,” *Proc. Am. Math. Soc.*, vol. 126, no. 5, pp. 1349–1353, 1998.
- [26] S. Varošanec, “On h -convexity,” *J. Math. Anal. Appl.*, vol. 326, no. 1, pp. 303–311, 2007.

Author's address

Mevlüt Tunç

Kilis 7 Aralık University, Faculty of Science and Arts, Department of Mathematics, Kilis, 79000, Turkey.

E-mail address: mevluttunc@kilis.edu.tr



MODULES THAT HAVE A RAD-SUPPLEMENT IN EVERY COFINITE EXTENSION

BURCU NIŞANCI TÜRKMEN

Received 30 January, 2013

Abstract. In this paper, we study modules with the properties (CRE) and (CREE), which are adapted Zöschinger's modules with the properties (E) and (EE). It is shown that: (1) a module M has the property (CREE) if and only if every submodule of M has the property (CRE); (2) a ring R is *Rad*-supplemented if and only if every left R -module has the property (CRE); (3) over a commutative Von Neumann regular ring a module M has the property (CRE) if and only if M is cofinitely injective.

2010 Mathematics Subject Classification: 16D10; 16N80

Keywords: *Rad*-supplement, cofinite extension, Von Neumann regular ring, semiperfect ring

1. INTRODUCTION

In this note, R is an associative ring with identity and all modules are unital left R -modules. Let N be an R -module. The notation $M \subseteq N$ means that M is a submodule of N . Here, N is called *an extension* of M . A module M is called *injective* if M is a direct summand of every extension N , i.e. $N = M + K$ and $N \cap K = 0$ for some submodule $K \subseteq N$. If every R -module is injective, then R is semisimple. The converse is true.

A submodule S of a module M is said to be *small* if $S + L \neq M$ for every proper submodule L of M . For every module M , the trivial submodule 0 is a small submodule of M . As a generalization of direct summands, we say that a submodule V of a module M is *supplement* of a submodule U of M if $M = U + V$ and $U \cap V \ll V$ as in [11]. Zöschinger generalized injective modules to modules with the property (E). He said that a module M has the property (E) if M has a supplement in every extension. He also said that a module M has the property (EE) if M has ample supplements in every extension, i.e. for $M \subseteq N$, if $N = M + K$, K contains a supplement of M in N ([13]). Every left R -module has the property (E) if and only if R is left perfect [4].

In [1], a submodule M of a module N is called *cofinite* if the factor module $\frac{N}{M}$ is finitely generated. It is easy to see that every submodule of a finitely generated module is cofinite. Adapting Zöschinger's module with the properties (E) and (EE),

Çalışıcı and Türkmen say that a module M has the property (CE) ((CEE)) if M has a supplement (ample supplements) in every cofinite extension. Here N is called *cofinite extension* of M if M is a cofinite submodule of N . They give a characterization of semiperfect rings via these modules.

For a module M , $Rad(M)$ will indicate the Jacobson radical of M . A submodule V of a module M is called *Rad-supplement* of a submodule U in M if $M = U + V$ and $U \cap V \subseteq Rad(V)$ [4, Theorem 10.14]. Since the Jacobson radical of a module is the sum of all small submodules, every supplement is *Rad-supplement*. A submodule U of M has ample *Rad-supplements* in M if every submodule V of M with $M = U + V$ contains a *Rad-supplement* in M . Now we have the following implications on submodules:

$$direct\ summand \implies supplement \implies Rad-supplement$$

In this paper, we investigate the properties (CRE) and (CREE) for modules as a generalization of the properties (CE) and (CEE), respectively. We show that a module has the property (CREE) if and only if every submodule has the property (CRE). We prove that every direct summand of a module M with the property (CRE) has this property. Using the property (CRE), we give a characterization of *Rad-supplemented* rings, which are a generalization of semiperfect rings.

2. MODULES WITH THE PROPERTIES (CRE) AND (CREE)

In this section, we define the concept of modules with the properties (CRE) and (CREE). We give various properties of these modules. We prove that every left R -module has the property (CRE) if and only if the ring R is *Rad-supplemented*.

Definition 1. Let M be an R -module. We say that M has *the property (CRE)* if M has a *Rad-supplement* in every cofinite extension, and M has *the property (CREE)* if M has ample *Rad-supplements* in every cofinite extension.

It is easy to see that every module with property (CE) has property (CRE). It is natural to pose the question whether there exists similar result for the properties (CEE) and (CREE). Before the end of this section, we shall give an example of a module which has the property (CRE) but not (CE).

Proposition 1. *Every radical module has the property (CRE).*

Proof. Let M be a module and N be any cofinite extension of M . Since $Rad(M) = M$, then $N = M + N$ and $M \cap N = M = Rad(M) \subseteq Rad(N)$. So N is a *Rad-supplement* of M in N . $\square$

By $P(M)$, we denote the sum of all radical submodules of the module M , that is, $P(M) = \sum \{U \subseteq M \mid Rad(U) = U\}$. A module M is *reduced* if $P(M) = 0$.

Since $P(M)$ is a radical submodule of M , we have the next result which is a direct consequence of Proposition 1.

Corollary 1. *For a module M , $P(M)$ has the property (CRE).*

Now we show that the property (CRE) is preserved by direct summands in the following proposition:

Proposition 2. *Every direct summand of a module with the property (CRE) has the property (CRE).*

Proof. Let N be a direct summand of M . Then there exists a submodule K of M such that $M = N \oplus K$. Let L be a cofinite extension of N , T be the external direct sum $L \oplus K$ and $\gamma : M \rightarrow T$ be the canonical embedding. Then $M \cong \gamma(M)$ has the property (CRE). We have

$$\frac{L}{N} \cong \frac{L \oplus K}{\gamma(M)}$$

is finitely generated. Since $\gamma(M)$ has the property (CRE), then there exists a submodule U of T such that $T = \gamma(M) + U$ and $\gamma(M) \cap U \subseteq \text{Rad}(U)$. Consider the projection $\pi : T \rightarrow L$. By this way, we have $N + \pi(U) = L$. Also $\ker(\pi) \subseteq \gamma(M)$, $\pi(\gamma(M) \cap U) \subseteq \pi(\gamma(M)) \cap \pi(U) = N \cap \pi(U) \subseteq \text{Rad}(\pi(U))$. Therefore $\pi(U)$ is a Rad -supplement of N in L . $\square$

In [3], Çalışıcı and Türkmen defined cofinitely injective modules, that is, a module M is called *cofinitely injective* if M is a direct summand of every cofinite extension.

Recall that a ring R is called *Von Neumann regular* if every element $a \in R$ can be written in the form axa , for some $x \in R$.

Proposition 3. *Let R be a commutative Von Neumann regular ring and M be an R -module. Then the following statements are equivalent:*

- (1) M has the property (CRE).
- (2) M is cofinitely injective.
- (3) M has the property (CE).

Proof. (1) $\implies$ (2) Suppose that M is a module with the property (CRE). Let N be any cofinite extension of M . So, there exists a Rad -supplement V of M in N . Since R is a commutative Von Neumann regular ring, $\text{Rad}(V) = 0$ by [5, 3.73 and 3.75]. So $N = M \oplus V$.

(2) $\implies$ (3) and (3) $\implies$ (1) are clear. $\square$

Zöschinger proved in [13] that a module has the property (EE) if and only if every submodule has the property (E). We give an analogous characterization of our modules.

Theorem 1. *A module M has the property (CREE) if and only if every submodule of M has the property (CRE).*

Proof. ($\implies$) Let M be a module with the property (CREE) and T be any submodule of M . For a cofinite extension N of T , let $F = \frac{M \oplus N}{H}$, where the submodule H is the set of all elements $(a, -a)$ of F with $a \in T$ and let $\alpha : M \longrightarrow F$ via $\alpha(m) = (m, 0) + H$, $\beta : N \longrightarrow F$ via $\beta(n) = (0, n) + H$ for all $m \in M, n \in N$. It is clear that α and β are monomorphisms. Thus we have the following pushout:

$$\begin{array}{ccc} T & \xrightarrow{\mu_1} & N \\ \downarrow \mu_2 & & \downarrow \beta \\ M & \xrightarrow{\alpha} & F \end{array}$$

where μ_1 and μ_2 are inclusion mappings. It is easy to prove that $F = \text{Im}(\alpha) + \text{Im}(\beta)$. Now we define $\gamma : F \longrightarrow \frac{N}{T}$ by $\gamma((m, n) + H) = n + T$ for all $(m, n) + H \in F$. Then γ is an epimorphism. Note that $\ker(\gamma) = \text{Im}(\alpha)$ and so $\frac{N}{T} \cong \frac{F}{\text{Im}(\alpha)}$ is finitely generated. Since α is a monomorphism, we have $M \cong \text{Im}(\alpha)$. By the assumption, $\text{Im}(\alpha)$ has the property (CREE). Then it follows immediately that $\text{Im}(\alpha)$ has a *Rad*-supplement V in F with $V \subseteq \text{Im}(\alpha)$, i.e., $F = \text{Im}(\alpha) + V$ and $\text{Im}(\alpha) \cap V \subseteq \text{Rad}(V)$. Then $N = T + \beta^{-1}(V)$ and $T \cap \beta^{-1}(V) \subseteq \text{Rad}(\beta^{-1}(V))$. Hence $\beta^{-1}(V)$ is a *Rad*-supplement of T in N .

($\impliedby$) Suppose that every submodule of M has the property (CRE). For a cofinite extension N of M , let $N = M + K$ for some submodule K of N . Then $\frac{N}{M} \cong \frac{K}{M \cap K}$ is finitely generated and so $M \cap K$ is a cofinite submodule of K . By the hypothesis, there exists a submodule V of K such that $K = (M \cap K) + V$ and $M \cap V \subseteq \text{Rad}(V)$. Note that $N = M + V$. It follows that V is a *Rad*-supplement of M in N . So M has the property (CREE). $\square$

Recall from [2] that a module M is (cofinitely) *Rad*-supplemented if every (cofinite) submodule has a *Rad*-supplement in M . It is clear that every finitely generated cofinitely *Rad*-supplemented module is *Rad*-supplemented.

Corollary 2. *Every submodule of a module with the property (CREE) has the property (CRE). Moreover, a module with the property (CREE) is cofinitely Rad-supplemented.*

Proposition 4. *Let M be a module and A be a submodule of B such that $\frac{B}{A}$ is noetherian. If A and $\frac{B}{A}$ have the property (CRE), then so is B .*

Proof. Let N be any cofinite extension of B . By the hypothesis, there is a *Rad*-supplement $\frac{V}{A}$ of $\frac{B}{A}$ in $\frac{N}{A}$. Note that $\frac{N}{B} \cong \frac{\frac{V}{A}}{\frac{B \cap V}{A}}$. Since $\frac{B}{A}$ is noetherian, then $\frac{V}{A}$ is finitely generated. Therefore V is a cofinite extension of A . Since A has the property (CRE), then A has a *Rad*-supplement in V , say K . We claim that K is a *Rad*-supplement of B in N . We have epimorphisms $f : K \longrightarrow \frac{V}{A}$ and $g : \frac{V}{A} \longrightarrow \frac{N}{B}$ such that $\ker(f) = K \cap A \subseteq \text{Rad}(K)$ and $\ker(g) = \frac{V}{A} \cap \frac{B}{A} \subseteq \text{Rad}(\frac{V}{A})$. Then

$g \circ f : K \longrightarrow \frac{N}{B}$ is an epimorphism such that $K \cap B = \text{Ker}(g \circ f) \subseteq \text{Rad}(K)$ by [12, Lemma 1.1]. Finally, $N = K + B$. $\square$

Proposition 5. *Let M be a module and K be a radical submodule of M . If $\frac{M}{K}$ has the property (CRE), then M has the property (CRE).*

Proof. Let U be any cofinite extension of M . Then $\frac{U}{M}$ is finitely generated. So $\frac{\frac{U}{K}}{\frac{M}{K}} \cong \frac{U}{M}$ is finitely generated. Thus $\frac{U}{K}$ is a cofinite extension of $\frac{M}{K}$. By the hypothesis, there exists a submodule $\frac{V}{K}$ of $\frac{U}{K}$ such that $\frac{U}{K} = \frac{M}{K} + \frac{V}{K}$ and $\frac{M}{K} \cap \frac{V}{K} \subseteq \text{Rad}(\frac{V}{K})$. Then we have $U = M + V$. Since K is radical, $M \cap V \subseteq \text{Rad}(V)$. So V is a *Rad*-supplement of M in U . Therefore M has the property (CRE). $\square$

Now we prove that the class of modules with the property (CRE) is closed under extensions if we impose certain conditions.

Theorem 2. *Let*

$$0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} L \longrightarrow 0$$

be a short exact sequence. Suppose that K is radical. If K and L have the property (CRE), so does M . If the sequence splits, the converse holds.

Proof. Without loss of generality; we assume that $K \subseteq M$. Since $\frac{M}{K} \cong L$ has the property (CRE) and K is radical, then we have M has the property (CRE) by Proposition 5. On the other hand, suppose that the sequence splits. Then $M \cong K \oplus L$. Let M be module with the property (CRE). So, K and L have the property (CRE) by Proposition 2. $\square$

Corollary 3. *Let M_i ($i \in I$) be any finite collection of radical modules and $M = M_1 \oplus M_2 \oplus \dots \oplus M_n$. Then M has the property (CRE) iff M_i has the property (CRE) for each $i = 1, 2, \dots, n$.*

Proof. The necessity follows from Proposition 2. Conversely, to prove that M has the property (CRE) it is sufficient by induction on n to prove that this is the case when $n = 2$. Thus suppose $M = M_1 \oplus M_2$. By using the following short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0.$$

Since M_1 is radical and M_1, M_2 have properties (CRE), then M has the property (CRE) by Theorem 2. $\square$

Let M be a module. A module N is called *maximal essential extension* of M if N is an essential extension of M , and whenever K is a proper extension of N , then K is not an essential extension of M ([8]).

Let M be a module. Then there exists a module N satisfying the following equivalent conditions:

- (1) N is an essential injective extension of M .
- (2) N is a maximal essential extension of M .

A module N satisfying above equivalent conditions is called *injective envelope* of M , denoted by $E(M)$ ([8, Theorem 2.21]).

Proposition 6. *Let M be a module. Then the following statements are equivalent.*

- (1) M has a *Rad-supplement* in every cofinite essential injective extension.
- (2) M has a *Rad-supplement* in every cofinite maximal essential extension.

Proof. (1) $\implies$ (2) is clear.

(2) $\implies$ (1) Let N be any cofinite essential extension of M , T be any cofinite maximal essential extension of M , $f : M \rightarrow N$, $g : M \rightarrow T$ be inclusion mappings and $I_T : T \rightarrow T$ be an identity mapping. Then we have the following commutative diagram with h necessarily monic:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow g & & \downarrow h \\ T & \xrightarrow{I_T} & T \end{array}$$

By the hypothesis, M has a *Rad-supplement* in T , say K , that is, $M + K = T$ and $M \cap K \subseteq \text{Rad}(K)$. Since $M \subseteq h(N)$, we obtain that $h(N) = h(N) \cap T = h(N) \cap (M + K) = M + (h(N) \cap K)$. Now, taking any $n \in N$, we have $h(N) = m + h(n_1) = h(m + n_1)$ where $m \in M$ and $h(n_1) \in h(N) \cap K$. So, $n = m + n_1 \in M + h^{-1}(K)$ since h is monic, and thus $M + h^{-1}(K) = N$. Moreover, $M \cap h^{-1}(K) = h^{-1}(M \cap K) \subseteq \text{Rad}(h^{-1}(K))$. Since $h^{-1}(M) = M$ as h is monic. Hence $h^{-1}(K)$ is a *Rad-supplement* of M in N . $\square$

Proposition 7. *Let M be a module with the property (CRE). If M is noetherian, then M has the property (CE).*

Proof. Let N be any cofinite extension of M . Since M has the property (CRE), then there exists a submodule K of N such that $N = M + K$ and $M \cap K \subseteq \text{Rad}(K)$. Since M is noetherian, then $M \cap K$ is finitely generated. So it is easy to see that $M \cap K \ll K$. So M has the property (CE). $\square$

A ring R is called *semiperfect* if $\frac{R}{\text{Rad}(R)}$ is semisimple and idempotents can be lifted modulo $\text{Rad}(R)$. It is well known that a ring R is semiperfect if and only if every left (or right) ideal has a supplement in ${}_R R$ (or R_R). A ring R is called *Rad-supplemented* if every left (or right) ideal has a *Rad-supplement* in ${}_R R$ (or R_R) (see [2]). Clearly, every semiperfect ring is *Rad-supplemented*, but a *Rad-supplemented* ring need not be semiperfect in general (see [2]).

It is proved in [3, Proposition 2.11] that a ring R is semiperfect if and only if every left R -module has the property (CE). Now, we give the characterization of the rings over which every module has the property (CRE).

Theorem 3. *The following statements are equivalent for a ring R :*

- (1) R is *Rad-supplemented*.
- (2) ${}_R R$ has ample *Rad-supplements* in every cofinite extension.
- (3) ${}_R R$ has ample *Rad-supplements* in every finitely generated extension.
- (4) Every left R -module has the property (CRE).
- (5) Every left R -module has the property (CREE).

Proof. (1) $\implies$ (4) Let M be an R -module and N be a cofinite extension of M . Then there exists a finitely generated submodule K of N such that $N = M + K$. Since R is a *Rad-supplemented* ring, then $M \cap K$ has a *Rad-supplement* in K , say T , i.e. $K = (M \cap K) + T$ and $(M \cap K) \cap T \subseteq \text{Rad}(T)$ by [10, Theorem 3.7]. It follows that $N = M + T$ and $M \cap T \subseteq \text{Rad}(T)$. This completes the proof.

(4) $\implies$ (1) Suppose that every left R -module has the property (CRE). Then every left ideal of R has a *Rad-supplement* in R as a left R -module. Hence ${}_R R$ is *Rad-supplemented*.

(2) $\implies$ (1) If ${}_R R$ has the property (CREE), it is *Rad-supplemented* by Corollary 2.

(5) $\implies$ (4) and (5) $\implies$ (3) $\implies$ (2) are clear. $\square$

A module M over a ring R is called *uniserial* if the lattice of submodules of M is a chain and M is called *serial* if M is a direct sum of uniserial modules. A ring R is called *left (right) uniserial* if the module ${}_R R$ (R_R) is uniserial. R is called a *uniserial ring* if R is both right and left uniserial. A uniserial domain R is called *nearly simple* if $\text{Rad}(R)$ is the unique nonzero two sided ideal of R and $\text{Rad}^2(R) \neq 0$ [7].

The following example shows that a module with the property (CRE) need not be (CE).

Example 1. (See[2]) Let $G = \{f : \mathbb{Q} \longrightarrow \mathbb{Q} \mid f(t) = at + b \text{ for } a, b \in \mathbb{Q} \text{ and } a > 0\}$ be the group of affine linear functions on the field of rational numbers $\mathbb{Q}$. Choose any irrational number $\epsilon \in \mathbb{R}$ and set $P = \{f \in G \mid \epsilon \leq f(\epsilon)\}$ and $P^+ = \{f \in G \mid \epsilon < f(\epsilon)\}$. Note that P , resp. P^+ , defines a left order on G . Take an arbitrary field F and consider the semigroup group ring $F[P]$ in which the right ideal $M = \sum_{g \in P^+} gF[P]$ is maximal. The set $F[P] \setminus M$ is a left and right ore set and the corresponding localization R is a nearly simple uniserial domain. Taking any nonzero element $r \in R$, $S = \text{End}(\frac{R}{rR})$ is a *Rad-supplemented* ring which is not semiperfect. Since S is a *Rad-supplemented* ring, then ${}_S S$ has the property (CRE) by Theorem 3. It is clear that ${}_S S$ has not the property (CE) by [3, Proposition 2.11]

The ring R is called *left hereditary* if and only if all left ideals of R are projective. It is well known that for a left hereditary ring R , every submodule of a free R -module is isomorphic to a direct sum of left ideals of R and hence is projective.

Corollary 4. *For a left hereditary ring R , the following statements are equivalent.*

- (1) R is semiperfect.
- (2) Every left R -module has the property (CRE).

Proof. (1) $\implies$ (2) Let M be any R -module and N be a cofinite extension of M . Then there exists a submodule K of N such that $N = M + K$. Since K is finitely generated, then so is $\frac{N}{M} \cong \frac{K}{M \cap K}$. Therefore $\frac{N}{M}$ has a projective cover because R is semiperfect. By [6, Lemma 4.40], there exists a submodule L of N such that $N = M + L$ and $M \cap L \ll L$. So $M \cap L \subseteq \text{Rad}(L)$. Thus M has the property (CRE).

(2) $\implies$ (1) By Theorem 3, R is Rad -supplemented. Applying [9, Corollary 2.22], it is semiperfect. $\square$

Acknowledgement. We would like to thank the referee for the valuable suggestions and comments.

REFERENCES

- [1] R. Alizade, G. Bilhan, and P. F. Smith, "Modules whose maximal submodules have supplements," *Commun. Algebra*, vol. 29, no. 6, pp. 2389–2405, 2001.
- [2] E. Büyükaşık and C. Lomp, "On a recent generalization of semiperfect rings," *Bull. Aust. Math. Soc.*, vol. 78, no. 2, pp. 317–325, 2008.
- [3] H. Çalışıcı and E. Türkmen, "Modules that have a supplement in every cofinite extension," *Georgian Math. J.*, vol. 19, no. 2, pp. 209–216, 2012.
- [4] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting modules. Supplements and projectivity in module theory*, ser. Frontiers in Mathematics. Basel: Birkhäuser, 2006.
- [5] T. Y. Lam, *Lectures on modules and rings*, ser. Graduate Texts in Mathematics. New York: Springer, 1999, vol. 189.
- [6] S. H. Mohamed and B. J. Müller, *Continuous and discrete modules*, ser. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 1990, vol. 147.
- [7] G. Puninski, "Some model theory over a nearly simple uniserial domain and decompositions of serial modules," *J. Pure Appl. Algebra*, vol. 163, no. 3, pp. 319–337, 2001.
- [8] D. W. Sharpe and P. Vamos, *Injective modules*, ser. Cambridge Tracts in Mathematics and mathematical Physics. Cambridge: At the University Press, 1972, vol. 62.
- [9] E. Türkmen and A. Pancar, "Some properties of Rad -supplemented modules," *International Journal of the Physical Sciences*, vol. 6, no. 35, pp. 7904–7909, 2011.
- [10] E. Türkmen and A. Pancar, "On cofinitely Rad -supplemented modules," *Int. J. Pure Appl. Math.*, vol. 53, no. 2, pp. 153–162, 2009.
- [11] R. Wisbauer, *Foundations of module and ring theory. A handbook for study and research. Revised and updated Engl. ed.*, ser. Algebra, Logic and Applications. Philadelphia: Gordon and Breach Science Publishers, 1991, vol. 3.
- [12] W. Xue, "Characterizations of semiperfect and perfect rings," *Publ. Mat., Barc.*, vol. 40, no. 1, pp. 115–125, 1996.
- [13] H. Zöschinger, "Moduln, die in jeder Erweiterung ein Komplement haben," *Math. Scand.*, vol. 35, pp. 267–287, 1975.

Author's address

Burcu Nişancı Türkmen

Amasya University, Faculty of Art and Science, Department of Mathematics, 05100, Amasya, Turkey

E-mail address: burcunisanacie@hotmail.com



ON THE NUMERICAL SOLUTION OF GENERALIZED NONLINEAR SCHRÖDINGER EQUATION USING RBFS

MARJAN UDDIN AND SIRAJUL HAQ

Received 21 February, 2012

Abstract. A meshfree technique based on radial basis functions (RBFs) is applied for the numerical solution of generalized nonlinear Schrödinger equation. The spatial derivatives are approximated by using the derivative of interpolation and a low order forward scheme is used to approximate the temporal derivative. Three test problems concerning the motion of solitary wave, interaction of two solitary waves and solution that blows up in finite time respectively are investigated, which examined the accuracy of the method in terms of the L_2 , L_∞ error norms, and the conservative quantities, I_1 , I_2 and I_3 . The results obtained from the method are compared with the exact solution and the earlier work in the literature.

2010 Mathematics Subject Classification: 65M06; 65M20

Keywords: generalized nonlinear Schrödinger equation, meshfree collocation method, radial basis functions

1. INTRODUCTION

The generalized nonlinear Schrödinger (GNS) equation given by

$$i \frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + q_1 |w|^2 w + q_2 |w|^4 w + i q_3 \frac{\partial(|w|^2)}{\partial x} w + i q_4 |w|^2 \frac{\partial w}{\partial x} = 0, \quad -\infty < x < \infty, \quad (1.1)$$

where $i = \sqrt{-1}$, w is a complex valued function of the spatial coordinate x and time t , the parameters q_1 , q_2 , q_3 and q_4 are real constants. The general solution of Eq. (1.1) does not belong to the class of integrable equations. However some cases of Eq. (1.1) are completely integrable, and has application in fluid mechanics, nonlinear optics and plasma physics (see [21, 22] and references therein). For the coefficients $q_1 \neq 0$, $q_2 \neq 0$ and $q_3 = q_4 = 0$ Eq. (1.1) reduces to nonlinear Schrödinger (NS) Equation. For certain values of the coefficients and some initial condition the solution of GNS equation experience finite time blow up [21]. Under the assumption that the solution $|w| \rightarrow 0$ as $|x| \rightarrow \infty$ GNS equation possess infinite number of conservation laws. However, rapidly decaying solution to Eq. (1.1) have three conservative laws

[21, 22] given by

$$I_1 = \int_{-\infty}^{\infty} |w|^2 dx, \quad (1.2)$$

$$I_2 = \int_{-\infty}^{\infty} \left[|w_x|^2 - \frac{1}{2}(2q_3 + q_4)|w|^2 \operatorname{Im}(ww_x^*) \right. \\ \left. - \frac{1}{2}q_1|w|^4 + \frac{1}{6}[q_3(2q_3 + q_4) - 2q_2]|w|^6 \right] dx, \quad (1.3)$$

$$I_3 = \int_{-\infty}^{\infty} [2\operatorname{Im}(ww_x^*) - q_3|w|^4] dx. \quad (1.4)$$

Although the GNS equation has been studied by many researchers (see [1, 12] and references therein), only a few numerical methods have been applied to solve the GNS equation. D. Pathria and J. LI. Morris [22] used the pseudo-spectral methods to obtain the solution of GNS equation, and discussed some of its properties, M. P. Robinson [23] used orthogonal spline collocation method for numerical solution of GNS equation. In some numerical studies of GNS equation the contribution from the nonlinear derivatives have been ignored [3, 4, 24]. More recently G. M. Muslu and H. A. Erbay [19] have been used Higher-order split-step method for the solution of GNS equation.

In the last decade, the theory of radial basis functions (RBFs) has enjoyed a great success as scattered data interpolating technique. A radial basis function, $\phi(x - x_j) = \phi(\|x - x_j\|)$, is a continuous spline which depends upon the separation distances of a subset of data centers, $X \subset \mathbb{R}^n$, $\{x_j \in X, j = 1, 2, \dots, N\}$. Due to spherical symmetry about the centers x_j , the RBFs are called radial. The distances, $\|x - x_j\|$, are usually taken to be the Euclidean metric. Hardy [11] was the first to introduced a general scattered data interpolation method, called radial basis functions method for the approximation of two-dimensional geographical surfaces. In 1982 Franke [8] in a review paper made the comparison among all the interpolation methods for scattered data sets available at that time, and the radial basis functions outperformed all the other methods regarding efficiency, stability and ease of implementations. Franke found that Hardy's multiquadrics (MQ) were ranked the best in accuracy, followed by thin plate splines (TPS). Despite MQ's excellent performance, it contains a shape parameter c , and the accuracy of MQ is greatly affected by the choice of shape parameter c whose optimal value is still unknown. Franke[7] used the formula $c^2 = (1.25)^2 d^2$ where d is the mean distance from each data point to its nearest neighbor. Hickernell and Hon [13] and Golberg et al. [9] had successfully used the technique of cross-validation to obtain an optimal value of the shape parameter. In 1990 radial basis functions scheme was introduced by Kansa [14] to solve partial differential equations. The existence, uniqueness and convergence of this method was discussed by Micchelli [18], Madych [17], Frank and Schaback [6]. It was studied by Micchelli in

1986 that for distinct interpolation points system obtained in multiquadric (MQ) method is always solvable. Very recently RBFs method have been used for the solutions of nonlinear PDEs (see [10,25,26]). Nicolas ali libre et.al [16] have used an adaptive scheme for nearly singular PDEs, Emdadi et. al [5] proposed a stable PDE solution method for large multiquadric shape parameters whereas a modified meshless control volume method was proposed by P. Orsini, H. Power [20]. G. Kosec and B. Sarler [15] have used local RBF collocation method for Darcy flow. S.N. Atluri et.al [2] have developed a meshless finite volume method through MLPG mixed approach. The RBFs scheme is truly a meshfree method which does not require the generation of a mesh, and since the MQ is infinitely differentiable, we can approximate the higher order spatial derivative directly by computing the derivative of the basis functions. However the RBFs method faces a serious ill-conditioning problem when we use RBFs as a global interpolant and increases the number of collocation points. There are different ways to overcome this problem e.g to use domain decomposition technique.

In this work, we apply meshfree collocation method based on three radial basis functions, MQ $\sqrt{r^2 + c^2}$ where c is a shape parameter, Spline basis (r^5) and TPS ($r^4 \log(r)$) for the numerical solution of generalized nonlinear Schrödinger (GNS) equation. GNS equation has an exact traveling solitary wave solution [19]

$$w(x, t) = \left[\frac{4}{4 + 3 \sinh^2(x - 2t - 15)} \right]^{1/2} \exp[i\phi(x, t)], \quad (1.5)$$

$$\phi(x, t) = 2 \tanh^{-1} \left[\frac{1}{2} \tanh(x - 2t - 15) \right] + x - 15. \quad (1.6)$$

This solitary wave is centered at $x_0 = 15$ and moving to the right with the velocity 2.

To avoid complex computation, we transform the GNS equation (1.1) into a nonlinear coupled equations, by decomposing w into its real and imaginary parts,

$$w(x, t) = u(x, t) + i v(x, t), \quad i^2 = -1, \quad (1.7)$$

where $u(x, t)$ and $v(x, t)$ are real functions. As a result we obtained the following coupled pair of equations

$$\frac{\partial u}{\partial t} + \frac{\partial^2 v}{\partial x^2} + q_1(u^2 + v^2)v + q_2(u^2 + v^2)^2v \quad (1.8)$$

$$+ q_3 \left(\frac{\partial(u^2 + v^2)}{\partial x} \right) u + q_4(u^2 + v^2) \frac{\partial u}{\partial x} = 0,$$

$$- \frac{\partial v}{\partial t} + \frac{\partial^2 u}{\partial x^2} + q_1(u^2 + v^2)u + q_2(u^2 + v^2)^2u \quad (1.9)$$

$$- q_3 \left(\frac{\partial(u^2 + v^2)}{\partial x} \right) v - q_4(u^2 + v^2) \frac{\partial v}{\partial x} = 0.$$

The structure of the present paper is organized as follows. In Section 2 we discuss the meshfree method. In Section 3 we deal the stability analysis. Section 4 is devoted to the numerical tests of the method on the problems related to the GNS equation. In Section 5, the results are concluded.

2. ANALYSIS OF THE METHOD

In this section, we consider a general time dependent boundary value problem

$$\frac{\partial u}{\partial t} + \mathcal{L}u = f(x, t), \quad x \in \Omega, \quad \mathcal{B}u = g(x), \quad x \in \partial\Omega, \quad (2.1)$$

where $\mathcal{L}$ and $\mathcal{B}$ are derivative and boundary operators respectively. Ω and $\partial\Omega$ represent interior and boundary of the domain respectively. Eq. (2.1) has to be supplemented by initial condition of the form $u(x, 0) = u_0(x)$. We use θ -weighted scheme for temporal derivative in the following form

$$\frac{u^{(n+1)} - u^{(n)}}{\delta t} + \theta \mathcal{L}u^{(n+1)} + (1 - \theta) \mathcal{L}u^{(n)} = f(x, t^{(n+1)}). \quad (2.2)$$

In the above equation δt is the time step, $u^{(n)}$ (n is non-negative integer) is the solution at time $t^{(n)} = n \delta t$, $0 \leq \theta \leq 1$. Let $\{x_i\}_{i=1}^{N_d}$ and $\{x_i\}_{i=N_d+1}^N$ be respectively interior and boundary points among the collocation points $\{x_i\}_{i=1}^N$ in the domain. The solution of Eq. (2.1) can be approximated by

$$u^{(n)}(x_i) = \sum_{j=1}^N \psi(r_{ij}) \lambda_j^{(n)}. \quad (2.3)$$

In the above equation $\psi(r_{ij})$ are radial basis functions with Euclidean norm $r_{ij} = \|P_i - P_j\|$ between the points P_i and P_j , $\{\lambda_j\}_{j=1}^N$ are constants to be determined. From Eqs. (2.2) and (2.3), we can write

$$\sum_{j=1}^N \left(\frac{\psi(r_{ij}) \lambda_j^{(n+1)} - \psi(r_{ij}) \lambda_j^{(n)}}{\delta t} + \theta \mathcal{L} \psi(r_{ij}) \lambda_j^{(n+1)} + (1 - \theta) \mathcal{L} \psi(r_{ij}) \lambda_j^{(n)} \right) = f(x, t^{(n+1)}), \quad i = 1, 2, \dots, N_d, \quad (2.4)$$

$$\sum_{j=1}^N \mathcal{B} \psi(r_{ij}) \lambda_j^{(n+1)} = g(x_i, t^{(n+1)}), \quad i = N_d + 1, \dots, N, \quad (2.5)$$

where $r_{ij} = \|P_i - P_j\|$. Eqs (2.4)-(2.5) are N equations in N unknown $\{\lambda_j\}_{j=1}^N$, which can be solved by using Gauss elimination method. We use $\theta = 0.5$ in our work.

2.1. GNS equation

The decomposed form of GNS equation can be written as

$$u_t + (2q_3 + q_4)u^2u_x = -v_{xx} - q_1(u^2 + v^2)v - q_2(u^2 + v^2)^2v - 2q_3uvv_x - q_4v^2u_x, \quad (2.6)$$

$$v_t + (2q_3 + q_4)v^2v_x = u_{xx} + q_1(u^2 + v^2)u + q_2(u^2 + v^2)^2u - 2q_3uvu_x - q_4u^2v_x, \quad a \leq x \leq b, \quad (2.7)$$

where q_1, q_2, q_3 and q_4 are real constants.

The boundary conditions are

$$\begin{aligned} u(a, t) &= f_1(a, t), u(b, t) = f_2(b, t), \\ v(a, t) &= g_1(a, t), v(b, t) = g_2(b, t), t > 0, \end{aligned} \quad (2.8)$$

and initial conditions

$$u(x, 0) = f(x, 0), \quad v(x, 0) = g(x, 0) \quad a \leq x \leq b. \quad (2.9)$$

From Eqs. (2.6), we can write

$$\begin{aligned} & \left(\frac{u^{(n+1)} - u^{(n)}}{\delta t} \right) + (3q_3 + q_4)\theta(u^2u_x)^{(n+1)} + (3q_3 + q_4)(1 - \theta)(u^2u_x)^{(n)} \\ & = (-v_{xx} - q_1(u^2 + v^2)v - q_2(u^2 + v^2)^2v - 2q_3uvv_x - q_4v^2u_x)^{(n)}, \end{aligned} \quad (2.10)$$

$$\begin{aligned} & \left(\frac{v^{(n+1)} - v^{(n)}}{\delta t} \right) + ((3q_3 + q_4)\theta(v^2v_x)^{(n+1)} + (3q_3 + q_4)(1 - \theta)(v^2v_x)^{(n)}) \\ & = (u_{xx} + q_1(u^2 + v^2)u + q_2(u^2 + v^2)^2u - 2q_3uvu_x - q_4u^2v_x)^{(n)}, \end{aligned} \quad (2.11)$$

where $t^{(n+1)} = t^{(n)} + \delta t$. Rearranging Eqs. (2.10)-(2.11), and substituting the linearised values of the nonlinear terms $(u^2u_x)^{(n+1)}$ and $(v^2v_x)^{(n+1)}$ given as

$$(u^2u_x)^{(n+1)} = (u^2)^{(n)}u_x^{(n+1)} + 2(u^{(n)}u_x^{(n)})u^{(n+1)} - 2(u^2)^{(n)}u_x^{(n)}, \quad (2.12)$$

$$(v^2v_x)^{(n+1)} = (v^2)^{(n)}v_x^{(n+1)} + 2(v^{(n)}v_x^{(n)})v^{(n+1)} - 2(v^2)^{(n)}v_x^{(n)}, \quad (2.13)$$

we get the following equations

$$\begin{aligned} & u^{(n+1)} + (3q_3 + q_4)\delta t \theta [(u^2)^{(n)}u_x^{(n+1)} + 2(u^{(n)}u_x^{(n)})u^{(n+1)}] \\ & = u^{(n)} + 2(3q_3 + q_4)\delta t \theta (u^2u_x)^{(n)} - (3q_3 + q_4)\delta t (1 - \theta)(u^2u_x)^{(n)} \\ & + \delta t [-v_{xx}^{(n)} - q_1((u^2)^{(n)} + (v^2)^{(n)})v^{(n)} - q_2((u^2)^{(n)} \\ & + (v^2)^{(n)})^2v^{(n)} - 2q_3(uv v_x)^{(n)} - q_4(v^2u_x)^{(n)}], \end{aligned} \quad (2.14)$$

$$v^{(n+1)} + (3q_3 + q_4)\delta t \theta [(v^2)^{(n)}v_x^{(n+1)} + 2(v^{(n)}v_x^{(n)})v^{(n+1)}] \quad (2.15)$$

$$\begin{aligned}
&= v^{(n)} + 2(3q_3 + q_4)\delta t \theta (v^2 v_x)^{(n)} - (3q_3 + q_4)\delta t (1 - \theta)(v^2 v_x)^{(n)} \\
&+ \delta t [(u_{xx})^{(n)} + q_1((u^2)^{(n)} + (v^2)^{(n)})u^{(n)} + q_2((u^2)^{(n)} \\
&+ (v^2)^{(n)})^2 u^{(n)} - 2q_3(uv u_x)^{(n)} - q_4(u^2 v_x)^{(n)}].
\end{aligned}$$

The RBFs approximations for the solutions u and v of Eqs. (2.6) are given by

$$u^{(n)}(x_i) = \sum_{j=1}^N \lambda_{1j}^{(n)} \psi(r_{ij}), \quad v^{(n)}(x_i) = \sum_{j=1}^N \lambda_{2j}^{(n)} \psi(r_{ij}), \quad i = 1, 2, \dots, N. \quad (2.16)$$

Using Eqs. (2.16), the system of Eqs. (2.14)-(2.15), along with the boundary conditions (2.8) can be written in matrix form as

$$\begin{aligned}
&(\mathbf{A} + (3q_3 + q_4)\delta t \theta ((\mathbf{u}^2)^{(n)} * \mathbf{D}_1 + 2\mathbf{u}^{(n)} \mathbf{u}_x^{(n)} * \mathbf{A})) \mathbf{\tilde{\lambda}}_1^{(n+1)} \\
&= [\mathbf{A} + 2(3q_3 + q_4)\delta t \theta (\mathbf{u}^2)^{(n)} * \mathbf{D}_1 - (3q_3 + q_4)\delta t (1 - \theta)(\mathbf{u}^2)^{(n)} * \mathbf{D}_1] \mathbf{\tilde{\lambda}}_1^{(n)} \\
&+ \delta t [-(\mathbf{v}_{xx})^{(n)} - q_1((\mathbf{u}^2)^{(n)} + (\mathbf{v}^2)^{(n)})\mathbf{v}^{(n)} - q_2((\mathbf{u}^2)^{(n)} \\
&+ (\mathbf{v}^2)^{(n)})^2 \mathbf{v}^{(n)} - 2q_3(\mathbf{u}\mathbf{v}\mathbf{v}_x)^{(n)} - q_4(\mathbf{v}^2 \mathbf{u}_x)^{(n)}] + \mathbf{f}^{(n+1)},
\end{aligned} \quad (2.17)$$

$$\begin{aligned}
&(\mathbf{A} + (3q_3 + q_4)\delta t \theta ((\mathbf{v}^2)^{(n)} * \mathbf{D}_1 + 2\mathbf{v}^{(n)} \mathbf{v}_x^{(n)} * \mathbf{A})) \mathbf{\tilde{\lambda}}_2^{(n+1)} \\
&= (\mathbf{A} + 2(3q_3 + q_4)\delta t \theta (\mathbf{v}^2)^{(n)} * \mathbf{D}_1 - (3q_3 + q_4)\delta t (1 - \theta)(\mathbf{v}^2)^{(n)} * \mathbf{D}_1) \mathbf{\tilde{\lambda}}_2^{(n)} \\
&+ \delta t ((\mathbf{u}_{xx})^{(n)} + q_1((\mathbf{u}^2)^{(n)} + (\mathbf{v}^2)^{(n)})\mathbf{u}^{(n)} + q_2((\mathbf{u}^2)^{(n)} \\
&+ (\mathbf{v}^2)^{(n)})^2 \mathbf{u}^{(n)} - 2q_3(\mathbf{u}\mathbf{v}\mathbf{u}_x)^{(n)} - q_4(\mathbf{u}^2 \mathbf{v}_x)^{(n)}) + \mathbf{g}^{(n+1)},
\end{aligned} \quad (2.18)$$

where ‘*’ stands for component by component multiplication and

$$\mathbf{A} = [\psi(r_{ij})]_{i,j=1}^N, \quad \mathbf{D}_1 = [\psi'(r_{ij})]_{i,j=1}^N.$$

In more compact form we can write Eqs. (2.17)-(2.18) as

$$\begin{aligned}
\lambda_1^{(n+1)} &= \mathbf{M}_1^{-1} \mathbf{N}_1 \lambda_1^{(n)} + \mathbf{M}_1^{-1} \mathbf{F}^{(n+1)}, \\
\lambda_2^{(n+1)} &= \mathbf{M}_2^{-1} \mathbf{N}_2 \lambda_2^{(n)} + \mathbf{M}_2^{-1} \mathbf{G}^{(n+1)},
\end{aligned} \quad (2.19)$$

where $\lambda_1^{(0)}$ and $\lambda_2^{(0)}$ are the values at the initial time step.

$$\begin{aligned}
\mathbf{M}_1 &= (\mathbf{A} + (3q_3 + q_4)\delta t \theta ((\mathbf{u}^2)^{(n)} * \mathbf{D}_1 + 2\mathbf{u}^{(n)} \mathbf{u}_x^{(n)} * \mathbf{A})), \\
\mathbf{M}_2 &= (\mathbf{A} + (3q_3 + q_4)\delta t \theta ((\mathbf{v}^2)^{(n)} * \mathbf{D}_1 + 2\mathbf{v}^{(n)} \mathbf{v}_x^{(n)} * \mathbf{A})), \\
\mathbf{N}_1 &= (\mathbf{A} + 2(3q_3 + q_4)\delta t \theta (\mathbf{u}^2)^{(n)} * \mathbf{D}_1 - (3q_3 + q_4)\delta t (1 - \theta)(\mathbf{u}^2)^{(n)} * \mathbf{D}_1), \\
\mathbf{N}_2 &= (\mathbf{A} + 2(3q_3 + q_4)\delta t \theta (\mathbf{v}^2)^{(n)} * \mathbf{D}_1 - (3q_3 + q_4)\delta t (1 - \theta)(\mathbf{v}^2)^{(n)} * \mathbf{D}_1), \\
\mathbf{f}^{(n+1)} &= (f_1^{(n+1)}, 0, 0, \dots, f_2^{(n+1)})^T, \quad \mathbf{g}^{(n+1)} = (g_1^{(n+1)}, 0, 0, \dots, g_2^{(n+1)})^T,
\end{aligned} \quad (2.20)$$

$$\begin{aligned}
\mathbf{F}^{(n+1)} &= \mathbf{f}^{(n+1)} + \delta t (-\mathbf{v}_{xx})^{(n)} - q_1 \left((\mathbf{u}^2)^{(n)} + (\mathbf{v}^2)^{(n)} \right) \mathbf{v}^{(n)} \\
&\quad - q_2 \left((\mathbf{u}^2)^{(n)} + (\mathbf{v}^2)^{(n)} \right)^2 \mathbf{v}^{(n)} \delta t \left[-2q_3 (\mathbf{u} \mathbf{v} \mathbf{v}_x)^{(n)} - q_4 (\mathbf{v}^2 \mathbf{u}_x)^{(n)} \right], \\
\mathbf{G}^{(n+1)} &= \mathbf{g}^{(n+1)} + \delta t ((\mathbf{u}_{xx})^{(n)} + q_1 \left((\mathbf{u}^2)^{(n)} + (\mathbf{v}^2)^{(n)} \right) \mathbf{u}^{(n)} \\
&\quad + q_2 \left((\mathbf{u}^2)^{(n)} + (\mathbf{v}^2)^{(n)} \right)^2 \mathbf{u}^{(n)} + \delta t \left[-2q_3 (\mathbf{u} \mathbf{v} \mathbf{u}_x)^{(n)} - q_4 (\mathbf{u}^2 \mathbf{v}_x)^{(n)} \right].
\end{aligned}$$

Eq. (2.16) can be written in matrix form as

$$\mathbf{u}^{(n)} = \mathbf{A} \lambda_1^{(n)}, \quad \mathbf{v}^{(n)} = \mathbf{A} \lambda_2^{(n)}. \quad (2.21)$$

Using Eq.(2.19) in Eq. (2.21), we get

$$\begin{aligned}
\mathbf{u}^{(n+1)} &= \mathbf{A} \mathbf{M}_1^{-1} \mathbf{N}_1 \mathbf{A}^{-1} \mathbf{u}^{(n)} + \mathbf{A} \mathbf{M}_1^{-1} \mathbf{F}^{(n+1)}, \\
\mathbf{v}^{(n+1)} &= \mathbf{A} \mathbf{M}_2^{-1} \mathbf{N}_2 \mathbf{A}^{-1} \mathbf{v}^{(n)} + \mathbf{A} \mathbf{M}_2^{-1} \mathbf{G}^{(n+1)}.
\end{aligned} \quad (2.22)$$

From here we can find the solution at any time level n .

3. STABILITY OF THE SCHEME

In this section, we discuss the stability analysis of the scheme (2.22), by using spectral norm of the amplification matrix. We assume that $\mathbf{u}, \mathbf{v}$ be the numerical and $\mathbf{u}^*, \mathbf{v}^*$ be the exact solutions of Eqs. (2.6). The error vectors $\varepsilon_i^{(n)}$ ($i = 1, 2$) are defined by $\varepsilon_1^{(n)} = \mathbf{u}^{(n)} - \mathbf{u}^{*(n)}$, $\varepsilon_2^{(n)} = \mathbf{v}^{(n)} - \mathbf{v}^{*(n)}$. Putting values from Eq. (2.22) in these equations, we arrive at the following equations

$$\begin{aligned}
\varepsilon_1^{(n+1)} &= \mathbf{u}^{(n+1)} - \mathbf{u}^{*(n+1)} = \mathbf{A} \mathbf{M}_1^{-1} \mathbf{N}_1 \mathbf{A}^{-1} \varepsilon_1^{(n)} = \mathbf{E}_1 \varepsilon_1^{(n)}, \\
\varepsilon_2^{(n+1)} &= \mathbf{v}^{(n+1)} - \mathbf{v}^{*(n+1)} = \mathbf{A} \mathbf{M}_2^{-1} \mathbf{N}_2 \mathbf{A}^{-1} \varepsilon_2^{(n)} = \mathbf{E}_2 \varepsilon_2^{(n)},
\end{aligned} \quad (3.1)$$

where $\mathbf{E}_1 = \mathbf{A} \mathbf{M}_1^{-1} \mathbf{N}_1 \mathbf{A}^{-1}$ and $\mathbf{E}_2 = \mathbf{A} \mathbf{M}_2^{-1} \mathbf{N}_2 \mathbf{A}^{-1}$ are the amplification matrices. For the scheme to remain stable, $\varepsilon_i^{(n)}$ must approach to zero, ($i = 1, 2$) as $n \rightarrow \infty$ i.e $\rho(\mathbf{E}_1) \leq 1$, $\rho(\mathbf{E}_2) \leq 1$, which is a condition of stability, where $\rho(\mathbf{E}_1)$ and $\rho(\mathbf{E}_2)$ represent spectral radii of the matrices $\mathbf{E}_1$ and $\mathbf{E}_2$ respectively.

TABLE 1. Spectral radii $\rho(\mathbf{E}_1)$ and $\rho(\mathbf{E}_2)$ for various values of MQ shape parameter c , at time $t = 0.2$, when time step size $\delta t = 0.0001$, spatial step size $\delta x = 0.4$, number of nodes $N=201$, spatial interval $[-5, 45]$, $q_1 = \frac{1}{2}$, $q_2 = \frac{-7}{4}$, $q_3 = -1$, $q_4 = -2$, corresponding to Problem 1.

c	$\rho(\mathbf{E}_1)$	$\rho(\mathbf{E}_2)$
0.10	1.00025	1.00038
0.20	1.00035	1.00043
0.30	1.00038	1.00032
0.40	1.00000	1.00032
0.50	1.00039	1.00034
0.80	1.00040	1.00035
0.90	1.00040	1.00036
1.50	1.00038	1.00038
2.00	1.00052	1.00164
2.50	1.00646	1.08955
3.00	3.34593	4.09048
3.30	46874.5	16416.02

TABLE 2. Conserved quantities I_1 , I_2 and I_3 for various values of MQ shape parameter c , at time $t = 0.2$, when time step size $\delta t = 0.0001$, spatial step size $\delta x = 0.4$, number of nodes $N=201$, spatial interval $[-5, 45]$, $q_1 = \frac{1}{2}$, $q_2 = \frac{-7}{4}$, $q_3 = -1$, $q_4 = -2$, corresponding to problem 1.

c	I_1	I_2	I_3	L_2
0.10	2.189675	2.435363	-5.569890	6.60587E-002
0.20	2.194867	2.570123	-5.732372	9.24601E-003
0.30	2.196505	2.673238	-5.823594	1.99296E-003
0.40	2.197113	2.722172	-5.862737	4.17283E-004
0.50	2.197341	2.742960	-5.878324	6.90097E-005
0.80	2.197476	2.756321	-5.887649	9.76190E-005
0.90	2.197482	2.756920	-5.888028	1.24626E-004
1.50	2.197488	2.757416	-5.888330	1.72375E-004
2.00	2.197489	2.757430	-5.888339	1.77317E-004
2.50	2.197489	2.757432	-5.888341	1.87258E-004
3.00	2.197487	2.757434	-5.888338	2.46310E-004
3.30	2.197487	2.757440	-5.888344	7.55716E-004
3.40	2.197467	2.757423	-5.888275	1.61953E-003
3.50	2.197461	2.757449	-5.888238	3.22212E-003
3.60	2.197450	2.757574	-5.888219	7.13120E-003
3.70	2.197212	2.759948	-5.887675	1.73985E-002
3.80	2.197143	2.855818	-5.886441	7.13152E-002
3.90	7.985062	9815.303	-28.67049	2.02353E+001

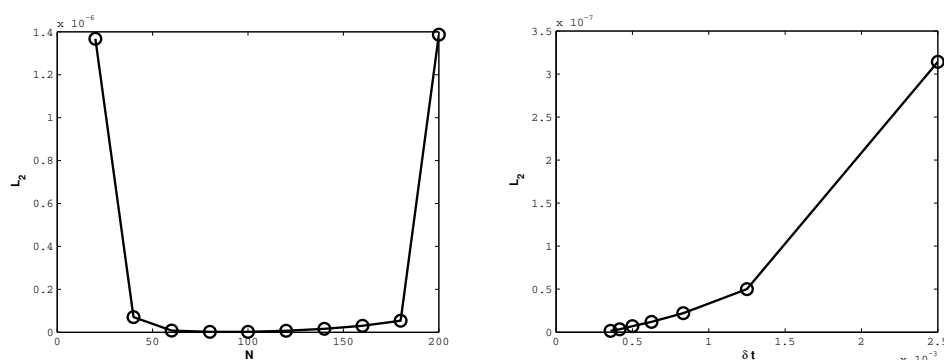


FIGURE 1. Plots, show L_2 error norms, versus N (Number of nodes), and temporal step size δt , at time $t = 0.2$, time step size $\delta t = 0.0001$, spatial step size $\delta x = 0.4$, number of nodes $N=201$, spatial interval $[-5,45]$, $q_1 = \frac{1}{2}$, $q_2 = \frac{-7}{4}$, $q_3 = -1$, $q_4 = -2$, corresponding to Problem 1.

4. NUMERICAL EXAMPLES

In this section, we apply the proposed method for the numerical solution of GNS equation. The accuracy of the meshfree method is tested in terms of the three invariants of GNS equation given in Eq. (1.2) and the error norms L_2 , L_∞ defined as

$$L_2 = \|w^* - w\|_2 = \left[\delta x \sum_{j=1}^N (w^* - w)^2 \right]^{1/2}, \quad (4.1)$$

$$L_\infty = \|w^* - w\|_\infty = \max_j |w^* - w|.$$

The tested problems are given below.

Problem 1: Single soliton

We consider GNS equation (1.1) and choose the initial condition from the exact solution (1.5)

$$w(x, 0) = \left[\frac{4}{4 + 3 \sinh^2(x - 15)} \right]^{1/2} \exp[i\phi(x, 0)]. \quad (4.2)$$

$$\phi(x, 0) = 2 \tanh^{-1} \left[\frac{1}{2} \tanh(x - 15) \right] + x - 15$$

In this problem the solution of GNS Eq. (1.1) represents a solitary wave initially centered at $x_0 = 15$. The exact values of the three invariants are $I_1 = 2 \log 3$, $I_2 = -3/2 + 3.875 \log 3$ and $I_3 = 4 - 9 \log 3$, respectively.

We solved the problem on the spatial interval, $-20 \leq x \leq 60$ and the time interval $[0, 3]$, using three radial basis function, MQ, r^5 and TPS. The L_2 , L_∞ error norms and the three invariants of motion I_1 , I_2 and I_3 are presented in Table 3. We observe that better results are obtained, when MQ is used as compared to spline basis r^5 and TPS, and all the three invariants are very well preserved.

In Fig. 2(A), The relation between L_2 error norm and the number of collocation points N , for fixed value of the time step $\delta t = 0.0001$, is presented. We observed that the solution converges for the choice of the nodes between, $N = 40$ to $N = 180$. In Fig. 2(B), L_2 error norm for various values of time step δt , and for fixed value of $N = 201$ is shown.

In Fig. 3(A)-3(B), motion of the solitary wave is shown. The soliton, which is initially centered at $x_0 = 15$, moves from left to right, with constant speed and amplitude. Hence the motion of solitary wave is well resolved by the present method. For the purpose of comparison, we presented the three invariants, I_1 , I_2 , I_3 and L_2 error norm, in Table 4. In this case, we chose constants $q_1 = \frac{1}{2}$, $q_2 = \frac{-7}{4}$, $q_3 = -1$, $q_4 = -2$, spatial interval $-5 \leq x \leq 45$, and time interval $[0, 3]$. Table 4, show that the present method preserved all the three invariants, when all the three radial basis functions are used, and are agreed with results obtained by orthogonal spline collocation method [23], however the present method conserved the second invariant I_2 more accurately than orthogonal spline collocation method [23]. In case of L_2 error norm, MQ radial basis function shown better accuracy than orthogonal spline collocation method [23].

TABLE 3. Error norms and Invariants for single soliton $|w|$, when time step size $\delta t = 0.0001$, spatial step size $\delta x = 0.4$, number of nodes $N = 201$, spatial interval $[-20, 60]$, $q_1 = \frac{1}{2}$, $q_2 = \frac{-7}{4}$, $q_3 = -1$, $q_4 = -2$, and MQ shape parameters $c_1 = c_2 = 2$, corresponding to the Problem 1.

t	MQ		r^5		TPS	
	L_∞	L_2	L_∞	L_2	L_∞	L_2
0.0001	1.55e-04	5.58e-08	1.55E-04	6.03E-07	1.55E-04	2.93E-06
0.2	2.45e-04	1.80e-04	7.60E-03	1.59E-04	2.10E-02	4.08E-04
0.4	2.34e-04	3.70e-04	1.18E-02	3.61E-04	3.46E-02	2.44E-03
0.6	4.24e-04	8.09e-04	1.37E-02	2.17E-03	4.15E-02	1.10E-02
0.8	3.50E-04	1.74E-03	1.54E-02	8.02E-03	4.84E-02	3.04E-02
1.0	4.62E-04	2.54E-03	1.68E-02	1.60E-02	5.60E-02	5.91E-02
t	MQ		r^5		TPS	
	I_1	I_2	I_3	I_1	I_2	I_3
0.0001	2.1972	2.7571	-5.8875	2.1972	2.7528	-5.8852
0.6	2.1980	2.7580	-5.8900	2.1979	2.7562	-5.8883
1.0	2.1986	2.7587	-5.8917	2.1984	2.7570	-5.8900
1.6	2.1994	2.7596	-5.8942	2.1992	2.7580	-5.8925
2.0	2.1999	2.7602	-5.8959	2.1997	2.7585	-5.8942
2.6	2.2007	2.7611	-5.8984	2.2005	2.7593	-5.8966
3.0	2.2012	2.7617	-5.9001	2.2010	2.7598	-5.8982

TABLE 4. Invariants and L_2 error norm for Single soliton, time step size $\delta t = 0.0001$, spatial step size $\delta x = 0.4$, number of nodes $N = 126$, spatial interval $[-5, 45]$, $q_1 = \frac{1}{2}$, $q_2 = \frac{-7}{4}$, $q_3 = -1$, $q_4 = -2$, and MQ shape parameters $c_1 = c_2 = 2$, corresponding to Problem 1.

t	I_1	I_2	I_3	L_2
MQ				
0	2.197225	2.757123	-5.887504	5.837E-011
1	2.198555	2.758655	-5.891696	1.837E-003
2	2.199890	2.760191	-5.895901	3.842E-003
3	2.201225	2.761726	-5.900107	5.691E-003
TPS				
0	2.197225	2.742761	-5.879475	1.797E-005
1	2.197724	2.761787	-5.888847	5.841E-002
2	2.200530	2.797950	-5.880088	3.963E-001
3	2.206579	2.854238	-5.848961	6.603E-001
r^5				
0	2.197225	2.752753	-5.885190	3.032E-008
1	2.198397	2.757003	-5.890038	1.557E-002
2	2.199696	2.758516	-5.894150	6.436E-002
3	2.201017	2.759837	-5.898085	9.878E-002
OSC [23]				
0	2.190546	2.861766	-5.873885	1.451E-002
1	2.192070	2.917542	-5.881301	1.837E-002
2	2.192500	2.970853	-5.882713	2.320E-002
3	2.193853	3.242514	-5.881409	3.390E-002

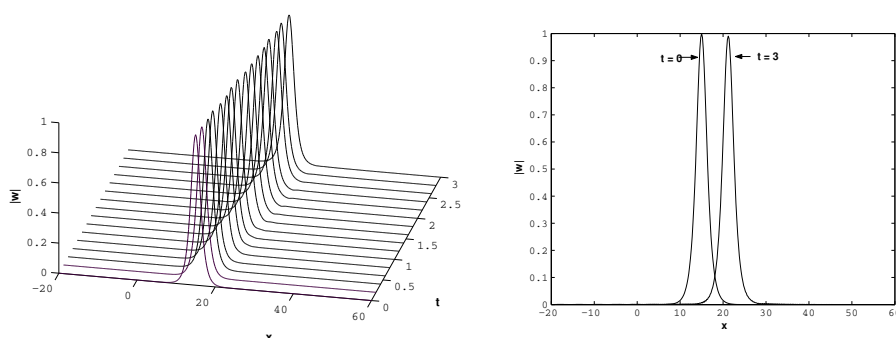


FIGURE 2. Motion of solitary wave $|w|$, for time step size $\delta t = 0.0001$, spatial step size $\delta x = 0.4$, number of nodes $N = 201$, spatial interval $[-20, 60]$, $q_1 = \frac{1}{2}$, $q_2 = \frac{-7}{4}$, $q_3 = -1$, $q_4 = -2$, and MQ shape parameters $c_1 = c_2 = 2$, corresponding to problem 1.

Problem 2: Blow-up

Here we present a test problem for the GNS equation. In [21] it has been shown that the exact solution of GNS will blow up in finite time for some specific values of the coefficients $q_1 = -2$, $q_2 = 20$, $q_3 = q_4 = 0$ and the Gaussian initial condition $w(x, 0) = e^{-x^2}$. For this problem the exact values of the three invariants are $I_1 = \sqrt{\pi}/2 \approx 1.253$, $I_2 = \sqrt{\pi}(9\sqrt{2} + 9 - 20\sqrt{6})/18 \approx -2.684$, and $I_3 = 0$. We applied two radial basis functions for the solution of the problem on the interval $[-7.5, 7.5]$ for time up to $t = 0.075$. The numerical values of the three invariants I_1 , I_2 , I_3 and the maximum modulus of the approximate solution $|w(x, 0)|$, for each time t , are shown in Table 5 and Fig. 4. The same problem has also been investigated numerically by G.M. Muslu [19] using Higher-order split-step Fourier method, and M. P. Robinson [23] using an orthogonal spline collocation method. The results obtained by these methods are given in Table 5. From the comparison, we observed that the results obtained by these different methods, about the predicted time of blow up are fully consistent, which show accuracy and correctness of these methods. By the present method the blowup is quite evident near the time $t=0.08$, and quite consistent with the results obtained in [23] and [19]. This blowup is accompanied by a radical change in the three invariants I_1 , I_2 and I_3 , and the maximum modulus of the approximate solution $|w(x, 0)|$, however I_2 changes more rapidly as compared to I_1 and I_3 .

TABLE 5. Variation of invariants and $|w(0, t)|$ in case of blow-up, time step size $\delta t = 0.0001$, spatial step size $\delta x = 0.1$, number of nodes $N = 151$, spatial interval $[-7.5, 7.5]$, $q_1 = -2$, $q_2 = 20$, $q_3 = 0$, $q_4 = 0$, and MQ shape parameters $c_1 = c_2 = 0.1$, corresponding to Problem 2.

t	I_1	I_2	I_3	$ w(0, t) $
MQ				
0.0	1.253314	-2.684666	0.000000	1.000000
0.01	1.253453	-2.687043	0.000000	1.007541
0.06	1.255209	-3.210414	0.000000	1.527028
0.07	1.267249	-34.599645	-0.000000	2.316339
0.75	1.613581	-500.855311	0.000000	3.115970
r^5				
0.0	1.253314	-2.684467	0.000000	1.000000
0.01	1.253454	-2.685862	0.000000	1.007411
0.06	1.255197	-2.892629	0.000000	1.526226
0.07	1.267809	-26.934744	0.000008	2.325083
0.075	1.582609	-412.744272	0.000001	3.046048
OSC [23]				
0.00	1.253314	-2.684467	0.000000	1.000000
0.01	1.253154	-2.682892	0.303(-13)	1.007286
0.06	1.253098	-2.681302	-0.864(-09)	1.525606
0.07	1.252911	-2.566558	0.220(-09)	2.368234
0.08	1.244698	34.08543	-0.675(-03)	3.856267
SFS [19]				
0.00	1.253314	-2.684467	-9.993(-17)	1.000000
0.01	1.253314	-2.684467	-6.651(-13)	1.007348
0.06	1.253314	-2.684467	-4.673(-12)	1.526243
0.07	1.253314	-2.684448	-4.395(-12)	2.376429
0.08	1.253335	-2.829258	-6.085(-10)	3.430374

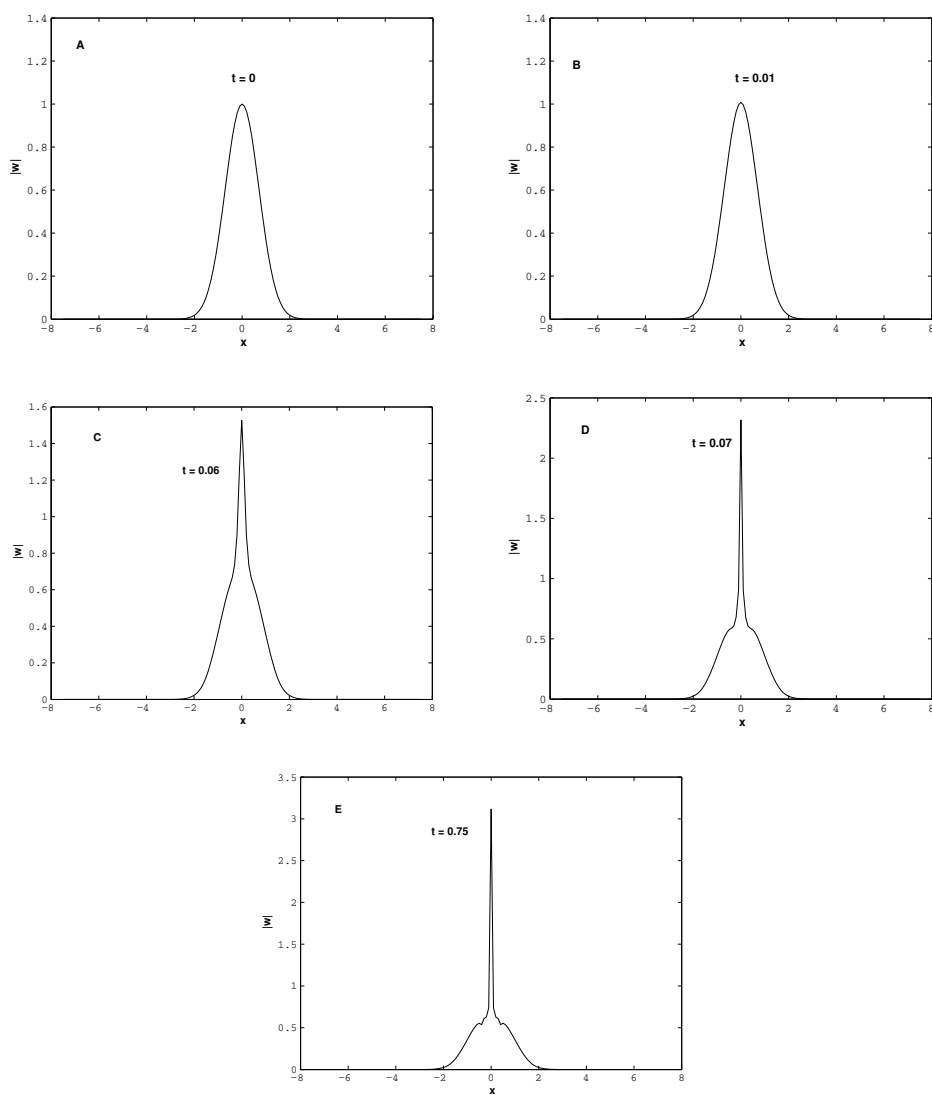


FIGURE 3. Graph of $|w(0, t)|$, at time $t = 0.01, 0.06, 0.07, 0.75$ in case of blow-up, time step size $\delta t = 0.0001$, spatial step size $\delta x = 0.1$, number of nodes $N = 151$, spatial interval $[-7.5, 7.5]$, $q_1 = -2$, $q_2 = 20$, $q_3 = 0$, $q_4 = 0$, and MQ shape parameters $c_1 = c_2 = 0.1$, corresponding to Problem 2.

Problem 3: Two soliton interaction

Here we consider the interaction of two solitary waves, with the initial condition of the form

$$w(x, 0) = 1/\sqrt{2} \operatorname{sech} \left[\frac{1}{2}(x - x_1) \right] \exp i \left[\frac{1}{4}(x - x_1) + \tanh \left(\frac{1}{2}(x - x_1) \right) \right] \\ + 1/2\sqrt{2} \operatorname{sech} \left[-\frac{1}{4}(x - x_2) \right] \exp i \left[-\frac{1}{2}(x - x_2) + \frac{1}{2} \tanh \left(\frac{1}{4}(x - x_2) \right) \right]. \quad (4.3)$$

This equation corresponds to sum of two solitary waves, the one initially located at $x_1 = 15$, while the other initially located at $x_2 = 35$ and moving in opposite directions. The problem is solved on the interval $-60 \leq x \leq 110$, for up to time $t = 30$ by the meshfree method. For $q_1 = 1, q_2 = 1, q_3 = -2$ and $q_4 = 0$, the interaction profile is shown in Fig. 6. We note that the two waves moves toward each other, collides with each other, and then moves away from each other as the time increases. We see that interaction, is consistent with [19]. The numerical values of the three invariants for this interaction of two waves up to time $t = 30$ are presented in Table 5, by using two radial basis functions MQ and r^5 . We can see that all the three invariants are preserved, and this behavior of the conserved quantity is a handy check on the accuracy of the present method.

5. CONCLUDING REMARKS

In this paper, a meshfree interpolation method using different types of RBFs is applied for the numerical solution of GNS equation. The method have shown excellent agreement with exact solution and the earlier work [19, 23]. As a whole the present method produces better results with ease of implementation. The technique used in this paper provides an efficient alternative for the solution of nonlinear partial differential equations.

TABLE 6. Invariants for two soliton, $\delta t = 0.0001, \delta x = 1, N = 171, [-60, 110], q_1 = 1, q_2 = 1, q_3 = -2, q_4 = 0$, and MQ shape parameters $c_1 = c_2 = 2$, corresponding to Problem 3.

t	MQ			r^5		
	I_1	I_2	I_3	I_1	I_2	I_3
0.0001	3.00168	0.44296	-0.75434	3.00168	0.44098	-0.75159
5	3.00180	0.44299	-0.75439	2.99974	0.44584	-0.75614
10	3.00188	0.44303	-0.75438	3.00073	0.44727	-0.75390
15	3.00197	0.44308	-0.75432	2.99976	0.44765	-0.75673
20	3.00107	0.44314	-0.75423	2.99979	0.44769	-0.75648
25	3.00214	0.44319	-0.75427	2.99989	0.44768	-0.75589
30	3.00224	0.44326	-0.75441	3.00008	0.44768	-0.75501

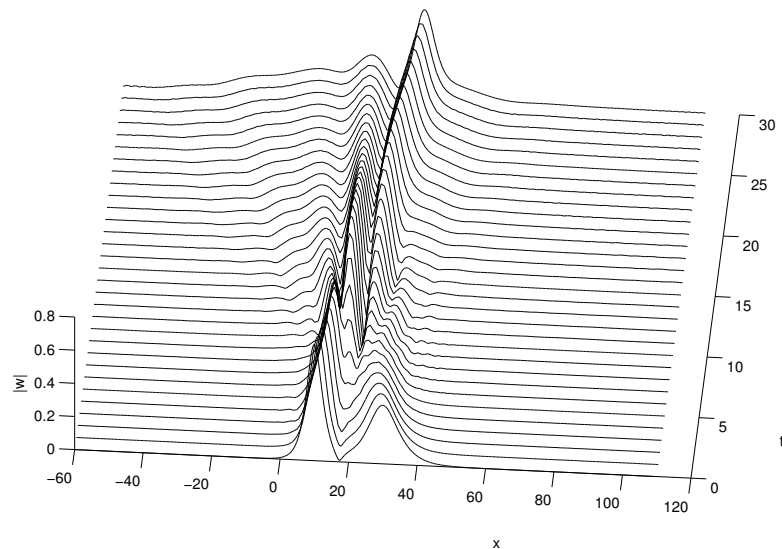


FIGURE 4. Interaction profile of two solitary waves, time step size $\delta t = 0.001$, spatial step size $\delta x = 1$, number of node $N = 171$, spatial interval $[-60, 110]$, $q_1 = 1$, $q_2 = 1$, $q_3 = -2$, $q_4 = 0$, and MQ shape parameters $c_1 = c_2 = 2$, corresponding to problem 3.

ACKNOWLEDGEMENT

The authors are thankful to the reviewers for their constructive comments.

REFERENCES

- [1] G. Adomian and R. Meyers, "Generalized nonlinear Schrödinger equation with time-dependent dissipation," *Appl. Math. Lett.*, vol. 8, no. 6, pp. 7–8, 1995.
- [2] S. N. Atluri, Z. D. Han, and A. M. Rajendran, "A new implementation of the meshless finite volume method, through the MLPG "mixed" approach," *CMES, Comput. Model. Eng. Sci.*, vol. 6, no. 6, pp. 491–513, 2004.
- [3] Q. Chang, E. Jia, and W. Sun, "Difference schemes for solving the generalized nonlinear Schrödinger equation," *J. Comput. Phys.*, vol. 148, no. 2, pp. 397–415, art. no. jcph.1998.6120, 1999.
- [4] A. Cloot, B. M. Herbst, and J. A. C. Weideman, "A numerical study of the nonlinear Schrödinger equation involving quintic terms," *J. Comput. Phys.*, vol. 86, no. 1, pp. 127–146, 1990.
- [5] A. Emdadi, E. J. Kansa, N. A. Libre, M. Rahimian, and M. Shekarchi, "Stable PDE solution methods for large multiquadric shape parameters," *CMES, Comput. Model. Eng. Sci.*, vol. 25, no. 1, pp. 23–41, 2008.
- [6] C. Franke and R. Schaback, "Convergence order estimates of meshless collocation methods using radial basis functions," *Adv. Comput. Math.*, vol. 8, no. 4, pp. 381–399, 1998.

- [7] R. Franke, "A critical comparison of some methods for interpolation of scattered data," Naval Postgraduate School, Tech. Rep. NPS-53-79-003, 1975.
- [8] R. Franke, "Scattered data interpolation: Tests of some methods," *Math. Comput.*, vol. 38, pp. 181–200, 1982.
- [9] M. A. Golberg, C. S. Chen, and S. R. Karur, "Improved multiquadric approximation for partial differential equations," *Engng. Anal. Bound. Elem.*, vol. 18, pp. 9–17, 1996.
- [10] S. Haq, S. I. Islam, and A. Ali, "A numerical meshfree technique for the solution of new equation," *CMES: Comput. Model. Engin. Sci.*, vol. 38, pp. 1–23, 2008.
- [11] R. L. Hardy, "Multiquadric equations of topography and other irregular surfaces," *Geo. Physcs. Res.*, vol. 176, pp. 1905–1915, 1971.
- [12] Q. He, "Domain decomposition method for nonlinear generalized Schrödinger-type systems: Semi-discrete problem," *Appl. Math. Comput.*, vol. 77, no. 1, pp. 33–52, 1996.
- [13] F. J. Hickernell and Y. C. Hon, "Radial basis function approximation of the surface wind field from scattered data," *International Journal of applied science and Computations*, vol. 4, pp. 221–247, 1998.
- [14] E. J. Kansa, "Multiquadrics - a scattered data approximation scheme with applications to computational fluid-dynamics. I: Surface approximations and partial derivative estimates," *Comput. Math. Appl.*, vol. 19, no. 8-9, pp. 127–145, 1990.
- [15] G. Kosec and B. Sarler, "Local RBF collocation method for darcy flow," *CMES: Comput. Model. Eng. Sci.*, vol. 25, pp. 197–207, 2008.
- [16] N. A. Libre, A. Emdadi, E. J. Kansa, M. Shekarchi, and M. Rahimian, "A fast adaptive wavelet scheme in rbf collocation for nearly singular potential pdes," *CMES: Comput. Model. Engin. Sci.*, vol. 38, pp. 263–284, 2008.
- [17] W. R. Madych and S. A. Nelson, "Multivariate interpolation and conditionally positive definite functions. II," *Math. Comput.*, vol. 54, no. 189, pp. 211–230, 1990.
- [18] C. A. Micchelli, "Interpolation of scattered data: distance matrices and conditionally positive definite functions," *Constr. Approx.*, vol. 2, pp. 11–22, 1986.
- [19] G. M. Muslu and H. A. Erbay, "Higher-order split-step Fourier schemes for the generalized nonlinear Schrödinger equation," *Math. Comput. Simul.*, vol. 67, no. 6, pp. 581–595, 2005.
- [20] P. Orsini, H. Power, and H. Morvan, "Improving volume element methods by meshless radial basis function techniques," *CMES, Comput. Model. Eng. Sci.*, vol. 23, no. 3, pp. 187–208, 2008.
- [21] D. Pathria, J. Li, and Morris, "Exact solutions for a generalized nonlinear Schrodinger equation," *Phys. Scripta*, vol. 39, pp. 673–679, 1989.
- [22] D. Pathria, J. Li, and Morris, "Pseudo-spectral solution of nonlinear Schrodinger equation," *J. Comput. Phys.*, vol. 87, pp. 108–125, 1990.
- [23] M. P. Robinson, "The solution of nonlinear Schrödinger equations using orthogonal spline collocation," *Comput. Math. Appl.*, vol. 33, no. 7, pp. 39–57, 1997.
- [24] Q. Sheng, A. Q. M. Khaliq, and E. A. Al-Said, "Solving the generalized nonlinear Schrödinger equation via quartic spline approximation," *J. Comput. Phys.*, vol. 166, no. 2, pp. 400–417, 2001.
- [25] M. Uddin, S. Haq, and Siraj-ul-Islam, "A mesh-free numerical method for solution of the family of Kuramoto-Sivashinsky equations," *Appl. Math. Comput.*, vol. 212, no. 2, pp. 458–469, 2009.
- [26] M. Uddin, S. Haq, and Siraj-Ul-Islam, "Numerical solution of complex modified Korteweg-de Vries equation by mesh-free collocation method," *Comput. Math. Appl.*, vol. 58, no. 3, pp. 566–578, 2009.

*Authors' addresses***Marjan Uddin**

University of Engineering and Technology Peshawar, Department of Basics Sciences, 25000, Peshawar, Pakistan

E-mail address: marjankhan1@hotmail.com (corresponding author)

Sirajul Haq

GIK Institute of Technology Topi, Faculty of Engineering Sciences, Topi, Pakistan

E-mail address: siraj_jcs@yahoo.co.in



ON GENERALIZATIONS OF TWO CURIOUS DIVISIBILITY PROPERTIES

AYNUR YALÇINER

Received 18 September, 2012

Abstract. In this paper, we extend two curious divisibility properties for the general second order linear recurrence $\{U_n(p, q)\}$. We also give new recursive identities for the general second linear recurrences $\{U_n(p, q)\}$ and $\{V_n(p, q)\}$. These results generalize the results given by E. Kılıç, "A matrix approach for generalizing two curious divisibility properties", Miskolc Math. Notes, vol. 13., No. 2, pp. 389-396, 2012.

2010 *Mathematics Subject Classification:* 11B37; 15B36

Keywords: binary linear recursion, divisibility property, matrix method

1. INTRODUCTION

Let p and q be nonzero integers such that $p^2 + 4q \neq 0$. For $n > 1$, the generalized Fibonacci sequence $\{U_n(p, q)\}$ and the generalized Lucas sequence $\{V_n(p, q)\}$ are defined by

$$U_n(p, q) = pU_{n-1}(p, q) + qU_{n-2}(p, q)$$

and

$$V_n(p, q) = pV_{n-1}(p, q) + qV_{n-2}(p, q),$$

where $U_0(p, q) = 0$, $U_1(p, q) = 1$ and $V_0(p, q) = 2$, $V_1(p, q) = p$, respectively.

Let α and β be the roots of the equation $x^2 - px - q = 0$. Then the Binet formulas of the sequences $\{U_n(p, q)\}$ and $\{V_n(p, q)\}$ are given by

$$U_n(p, q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n(p, q) = \alpha^n + \beta^n$$

If $p = q = 1$, then $U_n(1, 1) = F_n$ (n th Fibonacci number) and $V_n(1, 1) = L_n$ (n th Lucas number).

It is a well known fact that

$$\gcd(F_n, F_m) = F_{\gcd(n, m)}.$$

It is also known that F_{kn} is a multiple of F_n , for all integers k and n . In [9], the author showed that, for $n > 2$, the Fibonacci number F_m is a multiple of F_n^2 if and only if

m is multiply of nF_n (for more details see [4]). Also, in [1], the author obtained the following divisibility properties:

- i) $F_{kn-1} - F_{n-1}^k$ is divisible by F_n^2 ,
- ii) $F_{kn-2} - (-1)^{k+1} F_{n-2}^k$ is divisible by F_n^2 ,

where $n, k \geq 1$. Recently Kılıç [7] generalized these results for a general second order linear recursion $\{U_n(p, 1)\}$ as follows:

$$U_r^{k-1}(p, 1)U_{kn-r}(p, 1) - (-1)^{(r-1)(k+1)}U_{n-r}^k(p, 1) \text{ is divisible by } U_n^2(p, 1).$$

Furthermore, the author found new recursive identities for the general second order linear recurrences $\{U_n(p, 1)\}$ and $\{V_n(p, 1)\}$.

In this paper, for the case $q \neq 1$, we show that

$$U_r^{k-1}(p, q)U_{kn-r}(p, q) - (-1)^{(r-1)(k+1)}q^{r(k-1)}U_{n-r}^k(p, q) \text{ is divisible by } U_n^2(p, q).$$

To do that we use matrix methods. Matrix methods are useful tools for derivating some properties of linear recurrences (see [3, 5, 6, 8, 10]). We consider the quotient

$$\frac{U_r^{k-1}U_{kn-r} - (-1)^{(r-1)(k+1)}q^{r(k-1)}U_{n-r}^k}{U_n^2},$$

where $n, k \geq 1$. We define a generating matrix for this quotient for fixed n and increasing values of k . Then we give an explicit statement for the quotient. Also, by considering this explicit statement, we find new recursive identities for the general second order linear recurrences $\{U_n(p, q)\}$ and $\{V_n(p, q)\}$. Thus we obtain a generalization of the results given in [7].

Throughout this study, for simplicity, we will denote $U_n(p, q)$ by U_n and $V_n(p, q)$ by V_n .

2. MAIN RESULTS

Before we give our main results, we need some auxiliary results and definitions.

Denote the quotient $\left(U_r^{k-1}U_{kn-r} - (-1)^{(r-1)(k+1)}q^{r(k-1)}U_{n-r}^k\right)/U_n^2$ by $s(n, k)$. Define two matrices $H(n)$ and $G(n, k)$ of order 3 as follows:

$$H(n) = \begin{bmatrix} A_{n-1} & B_n & -(-q)^{n+r}U_r^2U_{n-r} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$G(n, k) = \begin{bmatrix} s(n, k+2) & t(n, k+2) & -(-q)^{n+r}U_r^2U_{n-r}s(n, k+1) \\ s(n, k+1) & t(n, k+1) & -(-q)^{n+r}U_r^2U_{n-r}s(n, k) \\ s(n, k) & t(n, k) & -(-q)^{n+r}U_r^2U_{n-r}s(n, k-1) \end{bmatrix},$$

where

$$A_{n-1} = U_r V_n - (-q)^r U_{n-r},$$

$$B_n = (-q)^r U_r V_n U_{n-r} - (-q)^n U_r^2$$

and

$$t(n, k) = \left((-q)^r U_r^{k-1} U_{kn} U_{n-r}^2 + (-1)^{r-1} q^{2n-r} U_r^{k+1} U_{n(k-2)} \right. \\ \left. + (-1)^{(r+1)(k-1)} q^{r(k-1)} U_r U_{2n} U_{n-r}^k \right) / U_n^3$$

Lemma 1. For $n \geq 1$, the eigenvalues of $H(n)$ are $U_r \alpha^n$, $U_r \beta^n$ and $-(-q)^r U_{n-r}$.

Proof. The characteristic polynomial of $H(n)$ is

$$x^3 - A_{n-1}x^2 - B_n x + (-q)^{n+r} U_r^2 U_{n-r} = 0$$

and it is factorized as

$$(x - U_r \alpha^n)(x - U_r \beta^n)(x + (-q)^r U_{n-r}) = 0,$$

as required. $\square$

Thus the first main result of this paper is the following.

Theorem 1. For $n > 1$,

$$H(n)^k = G(n, k).$$

Proof. In the proof, we will use induction on k . Since $G(n, 1) = H(n)$, the result is true when $k = 1$. Now assume that $H(n)^{k-1} = G(n, k-1)$. Then, by the definitions of $s(n, k)$ and $t(n, k)$, we have

$$A_{n-1}s(n, k+1) + t(n, k+1) = s(n, k+2)$$

and

$$B_n s(n, k+1) - (-q)^{n+r} U_r^2 U_{n-r} s(n, k) = t(n, k+2).$$

This completes the proof. $\square$

As a consequence of this theorem, we can see that the matrix $H(n)$ generate the $s(n, k)$. Since the elements of $H(n)$ are integers, the quotient $s(n, k)$ are integers.

Also from Theorem 1 in [2], we have the following result for the combinatorial representation of $s(n, k)$.

Corollary 1.

$$s(n, k) = \sum_{(l_1, l_2, l_3)} \binom{l_1 + l_2 + l_3}{l_1, l_2, l_3} (-1)^{(n+r-1)l_3} A_{n-1}^{l_1} B_n^{l_2} U_r^{2l_3} U_{n-r}^{l_3},$$

where the summation is over nonnegative integers satisfying $l_1 + 2l_2 + 3l_3 = k - 2$.

As another main result, we have the following theorem.

Theorem 2. For $n, k \geq 1$

$$\begin{aligned} (G(n, k))_{3,1} &= s(n, k) = \\ &= \frac{(-q)^{n-r} U_r^k U_{n(k-1)} + (-1)^k (-q)^{r(k-1)} U_n U_{n-r}^k + U_r^{k-1} U_{kn} U_{n-r}}{U_n^3}. \end{aligned}$$

Proof. Since the eigenvalues of $H(n)$ are distinct, $H(n)$ is diagonalizable as

$$V^{-1} H(n) V = D,$$

where

$$V = \begin{bmatrix} U_r^2 \alpha^{2n} & U_r^2 \beta^{2n} & q^{2r} U_{n-r}^2 \\ U_r \alpha^n & U_r \beta^n & -(-q)^r U_{n-r} \\ 1 & 1 & 1 \end{bmatrix},$$

and $D = \text{diag}(U_r \alpha^n, U_r \beta^n, -(-q)^r U_{n-r})$. Therefore, we obtain $V^{-1} H(n)^k V = D^k$. By Theorem 1, we write $V^{-1} G(n, k) V = D^k$. Then we have the following linear equation system:

$$\begin{aligned} g_{i1} U_r^2 \alpha^{2n} + g_{i2} U_r \alpha^n + g_{i3} &= U^{k+(3-i)} \alpha^{kn+(3-i)n} \\ g_{i1} U_r^2 \beta^{2n} + g_{i2} U_r \beta^n + g_{i3} &= U^{k+(3-i)} \beta^{kn+(3-i)n} \\ g_{i1} q^{2r} U_{n-r}^2 - g_{i2} (-q)^r U_{n-r} + g_{i3} &= (-1)^{(r-1)(k+3-i)} q^{kr+(3-i)r} U_{n-r}^{k+(3-i)} \end{aligned}$$

Using the identities

$$U_{n-r} U_{n+r} - U_n^2 = -(-q)^{n-r} U_r^2$$

and

$$q^r U_{n-r} + (-1)^r U_r V_n = (-1)^r U_{n+r},$$

the solution of the above linear equation system gives the claimed result. $\square$

By considering definition of $s(n, k)$, we have the following consequence of Theorem 2.

Corollary 2. Let n, k and r arbitrary integers. Then

$$U_{n-r} U_{kn} = U_n U_{kn-r} - (-q)^{n-r} U_r U_{n(k-1)}.$$

The next result presents a similar expression by considering generalized Lucas sequence $\{V_n\}$.

Theorem 3. For all integers n, k, r ,

$$U_{n-r} V_{kn} = U_n V_{kn-r} - (-q)^{n-r} U_r V_{n(k-1)}.$$

Proof. Using Binet formulas of the sequence $\{U_n\}$ and $\{V_n\}$, we have

$$\begin{aligned} U_n V_{kn-r} - (-q)^{n-r} U_r V_{n(k-1)} &= \\ &= (\alpha^{kn+n-r} - \beta^{kn+n-r} + \alpha^n \beta^{kn-r} - \alpha^{kn-r} \beta^n - (-q)^{n-r} \alpha^{kn-n+r} \\ &\quad + (-q)^{n-r} \beta^{kn-n+r} - (-q)^{n-r} \alpha^r \beta^{kn-n} + (-q)^{n-r} \alpha^{kn-n} \beta^r) / (\alpha - \beta) \end{aligned}$$

$$\begin{aligned}
&= \left(\alpha^{kn+n-r} - \beta^{kn+n-r} - (-q)^{n-r} \alpha^{kn-n+r} + (-q)^{n-r} \beta^{kn-n+r} \right) / (\alpha - \beta) \\
&= (\alpha^{n-r} - \beta^{n-r})(\alpha^{kn} + \beta^{kn}) / (\alpha - \beta) \\
&= U_{n-r} V_{kn}.
\end{aligned}$$

The proof is complete. $\square$

ACKNOWLEDGEMENT

The author would like to thank the referee for constructive comments, which improved the presentation of the paper. This work is supported by the Scientific Research Projects Office (BAP) of Selcuk University.

REFERENCES

- [1] M. Cavachi, "Some properties of the terms of the Fibonacci sequence," *Gaz. Mat., Bucur.*, vol. 85, pp. 290–293, 1980.
- [2] W. Y. Chen and J. D. Louck, "The combinatorial power of the companion matrix," *Linear Algebra Appl.*, vol. 232, pp. 261–278, 1996.
- [3] M. C. Er, "Sums of Fibonacci numbers by matrix methods," *Fibonacci Q.*, vol. 22, pp. 204–207, 1984.
- [4] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete mathematics: a foundation for computer science. 2nd ed.*, 2nd ed. Amsterdam: Addison-Wesley Publishing Group, 1994.
- [5] E. Kilic, "The generalized order- k Fibonacci-Pell sequence by matrix methods," *J. Comput. Appl. Math.*, vol. 209, no. 2, pp. 133–145, 2007.
- [6] E. Kiliç, "The generalized Fibonomial matrix," *Eur. J. Comb.*, vol. 31, no. 1, pp. 193–209, 2010.
- [7] E. Kiliç, "A matrix approach for generalizing two curious divisibility properties," *Math. Notes, Miskolc*, vol. 13, no. 2, pp. 389–396, 2012.
- [8] E. Kiliç and P. Stănică, "A matrix approach for general higher order linear recurrences," *Bull. Malays. Math. Sci. Soc. (2)*, vol. 34, no. 1, pp. 51–67, 2011.
- [9] Y. V. Matiyasevich, "Enumerable sets are diophantine," *Sov. Math., Dokl.*, vol. 11, pp. 354–358, 1970.
- [10] R. A. Rosenbaum, "An application of matrices to linear recursion relations," *Am. Math. Mon.*, vol. 66, pp. 792–793, 1959.

Author's address

Aynur Yalçın

Selcuk University, Science Faculty, Department of Mathematics, 42075, Campus, Konya, Turkey

E-mail address: ayalciner@selcuk.edu.tr



REMARK ON WEIGHTED APPROXIMATION PROPERTIES OF GENERALIZED PICARD OPERATORS

B. YILMAZ, G. BAŞCANBAZ-TUNCA, AND A. ARAL

Received 18 September, 2012

Abstract. In this article, we still hold the study of the generalized Picard operators $P_{\lambda,\beta}(f;q)$ depending on nonisotropic β -distance given in [3]. By continuing to deal with the nonisotropic weighted $L_{p,\beta}(\mathbb{R}^n)$ space defined in [17], we introduce a new weighted $L_{p,\beta}$ modulus of continuity depending on the nonisotropic distance to obtain the weighted rate of convergence. We show that weighted convergence rate of $P_{\lambda,\beta}(f;q)$ to f can be made better not only depending on the chosen q but also the choice of β . We get that $P_{\lambda,\beta}(f;q)$ satisfy the global smoothness preservation property via weighted $L_{p,\beta}$ modulus of continuity. Also we give direct approximation property of the generalized Picard operators $P_{\lambda,\beta}(f;q)$ with respect to nonisotropic weighted norm.

2010 *Mathematics Subject Classification:* 42B20; 45P05

Keywords: generalized Picard singular integral nonisotropic distance, Lebesgue point

1. INTRODUCTION

The q -analysis concept is comprehensive in approximation theory, especially in the study of various sequences of linear positive operators and singular integral operators, such as Picard and Gauss-Weierstrass singular integral operators (see [8, 10, 11]). We previously studied Picard singular operators in multivariate setting defined as in [17]. For a general framework related to the classical Picard operators [2, 4, 5, 13] and [14] may be referred. In [7] and [6], multivariate Picard operators and Gauss-Weierstrass operators with kernels including nonisotropic distance were introduced and pointwise convergence result were given. Latterly, in [8], Aral introduced a generalization of the Picard singular integral operators (see [4]) by using the q -analogue of the Euler Gamma integral and entitled that operators as q -Picard singular integral operators. Also, the author showed that these generalized operators have a more flexible rate of convergence than the classical Picard singular integral operators. In [3], Anastasiou and Aral introduced the multivariate variant of the q -Picard singular integral operators $P_{\lambda,\beta}(f)$ depending on nonisotropic norm (see Definition 3). They introduced a suitable modulus of continuity depending on nonisotropic distance with supremum norm to measure the rate of convergence. Also,

they proved the global smoothness preservation property of these operators. Also generalized potential-type integral operators with nonisotropic kernel was studied in [9].

In [17] we defined a weighted $L_{p,\beta}(\mathbb{R}^n)$ space (see Definition 2) and the concept of nonisotropic β -Lebesgue point. Then we obtained a pointwise convergence result for the family of $P_{\lambda,\beta}(f;q)$ to f for $f \in L_{p,\beta}(\mathbb{R}^n)$. We also studied the rate of this pointwise convergence. Convergence in the norm of this space was also studied.

In this work, we introduce a weighted $L_{p,\beta}$ modulus of continuity (see 2.1) depending on nonisotropic distance in order to measure the rate of convergence in the norm of $L_{p,\beta}$. Also, we show that the operators $P_{\lambda,\beta}(f;q)$ satisfy the global smoothness preservation property via new weighted $L_{p,\beta}$ modulus of continuity. Furthermore, we show that these operators retain direct approximation result in weighted $L_{p,\beta}(\mathbb{R}^n)$ space with respect to the nonisotropic weighted norm.

Now, we give the concept of the nonisotropic β -distance. Let $n \in \mathbb{N}$ and $\beta_1, \beta_2, \dots, \beta_n$ be positive numbers with $|\beta| = \beta_1 + \beta_2 + \dots + \beta_n$ and

$$\|\mathbf{x}\|_\beta = \left(|x_1|^{\frac{1}{\beta_1}} + \dots + |x_n|^{\frac{1}{\beta_n}} \right)^{\frac{|\beta|}{n}}, \quad \mathbf{x} \in \mathbb{R}^n.$$

The expression $\|\mathbf{x}\|_\beta$ is called the nonisotropic β -distance between $\mathbf{x}$ and $\mathbf{0}$. Note that this distance has the following properties of homogeneity for positive t :

$$\left(\left| t^{\beta_1} x_1 \right|^{\frac{1}{\beta_1}} + \dots + \left| t^{\beta_n} x_n \right|^{\frac{1}{\beta_n}} \right)^{\frac{|\beta|}{n}} = t^{\frac{|\beta|}{n}} \|\mathbf{x}\|_\beta.$$

Also, nonisotropic β -distance has following properties.

- (1) $\|\mathbf{x}\|_\beta = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$,
- (2) $\left\| t^{\beta} \mathbf{x} \right\|_\beta = t^{\frac{|\beta|}{n}} \|\mathbf{x}\|_\beta$,
- (3) $\|\mathbf{x} + \mathbf{y}\|_\beta \leq M_\beta (\|\mathbf{x}\|_\beta + \|\mathbf{y}\|_\beta)$,

where $\beta_{\min} = \min\{\beta_1, \beta_2, \dots, \beta_n\}$ and $M_\beta = 2^{\left(1 + \frac{1}{\beta_{\min}}\right) \frac{|\beta|}{n}}$, (see [16])

It can be seen that nonisotropic β -distance becomes the ordinary Euclidean distance $|\mathbf{x}|$ for $\beta_i = \frac{1}{2}$, $i = 1, 2, \dots, n$. Also, this distance does not satisfy the triangle inequality. Throughout the paper, we need the following notations of q -calculus.

The q -extension of exponential function e^x is

$$E_q(x) := \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q; q)_n} x^n = (-x; q)_{\infty}, \quad (1.1)$$

with $(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$ and $(-x; q)_{\infty} = \prod_{k=0}^{\infty} (1 + xq^k)$.

For $q > 0$, q -number is

$$[\lambda]_q = \begin{cases} \frac{1-q^\lambda}{1-q}, & q \neq 1 \\ \lambda, & q = 1 \end{cases}$$

for all nonnegative λ . If λ is an integer, i.e. $\lambda = n$ for some n , we write $[n]_q$ and call it q -integer. Also, we define a q -factorial as

$$[n]_q! = \begin{cases} [n]_q [n-1]_q \cdots [1]_q, & n = 1, 2, \dots \\ 1 & n = 0. \end{cases}$$

For integers $0 \leq k \leq n$, the q -binomial coefficients are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

For details see [15].

Another needed formula is q -extension of Euler integral representation for the gamma function given in [12] and [1] for $0 < q < 1$

$$c_q(x) \Gamma_q(x) = \frac{1-q}{\ln q^{-1}} q^{\frac{x(x-1)}{2}} \int_0^\infty \frac{t^{x-1}}{E_q((1-q)t)} dt, \quad \operatorname{Re} x > 0 \quad (1.2)$$

where $\Gamma_q(x)$ is the q -gamma function defined by

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x}, \quad 0 < q < 1$$

and $c_q(x)$ satisfies the following conditions:

- (1) $c_q(x+1) = c_q(x)$,
- (2) $c_q(n) = 1, n = 0, 1, 2, \dots$,
- (3) $\lim_{q \rightarrow 1^-} c_q(x) = 1$.

When $x = n+1$ with n is a nonnegative integer, we obtain

$$\Gamma_q(n+1) = [n]_q!. \quad (1.3)$$

In [3], Anastasiou and Aral introduced the q -Picard singular integral operators depending on nonisotropic distance taking the following lemma and (1.2) into consideration.

Lemma 1. For all $\lambda > 0, n \in \mathbb{N}$ and $\beta_i \in (0, \infty)$ ($i = 1, 2, \dots, n$) with $|\beta| = \beta_1 + \beta_2 + \dots + \beta_n$ we have

$$\frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t} = 1,$$

where

$$\mathcal{P}_\lambda(\beta, \mathbf{t}) = 1/E_q \left(\frac{(1-q) \|\mathbf{t}\|_\beta}{[\lambda]_q^{\frac{|\beta|}{n}}} \right), \quad (1.4)$$

and

$$c(n, \beta, q)^{-1} = \frac{n}{2|\beta|} \omega_{\beta, n-1} \Gamma_q(n) \frac{\ln q^{-1}}{(1-q) q^{\frac{n(n-1)}{2}}}. \quad (1.5)$$

Definition 1. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function. For $0 < q < 1$, $\lambda > 0$, $n \in \mathbb{N}$ and $\beta_i \in (0, \infty)$ ($i = 1, 2, \dots, n$) with $|\beta| = \beta_1 + \beta_2 + \dots + \beta_n$, the q -Picard singular integral of f depending on β -distance of f is

$$\begin{aligned} P_{\lambda, \beta}(f; q, \mathbf{x}) &\equiv P_{\lambda, \beta}(f; \mathbf{x}) \\ &:= \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} f(\mathbf{x} + \mathbf{t}) \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t}, \end{aligned} \quad (1.6)$$

where $\mathcal{P}_\lambda(\beta, \mathbf{t})$ and $c(n, \beta, q)$ are defined as in (1.4) and (1.5), respectively.

Note that, if we take $\beta_i = \frac{1}{2}$, $i = 1, 2, \dots, n$, it reduces to $P_{\lambda, \frac{1}{2}}(f; q, \mathbf{x})$ operators introduced in [8]. If we take $q \rightarrow 1$, then $P_{\lambda, \frac{1}{2}}(f; 1, \mathbf{x})$ operators are classical multivariate Picard singular integral (see [3]).

From [17], we have the following definition and subsequent result.

Definition 2. For fixed $1 \leq p < \infty$, we denote by $L_{p, \beta}(\mathbb{R}^n)$ the non-isotropic weighted space of all real valued functions f defined on $\mathbb{R}^n$ for which $\frac{f(\mathbf{x})}{1 + \|\mathbf{x}\|_\beta}$ is a function such that p -th power is Lebesgue integrable. The norm in $L_{p, \beta}(\mathbb{R}^n)$ is defined by the formula

$$\|f\|_{p, \beta} = \left(\int_{\mathbb{R}^n} \left| \frac{f(\mathbf{x})}{1 + \|\mathbf{x}\|_\beta} \right|^p d\mathbf{x} \right)^{\frac{1}{p}}.$$

Lemma 2. $P_{\lambda, \beta}(f)$ is a linear positive operator from the space $L_{p, \beta}(\mathbb{R}^n)$ into $L_{p, \beta}(\mathbb{R}^n)$. That is

$$\|P_{\lambda, \beta}(f)\|_{p, \beta} \leq K(n, \beta, q) \|f\|_{p, \beta},$$

where

$$\begin{aligned} K(n, \beta, q) &= \max\{1, M_\beta\} \left(1 + \frac{n}{2|\beta|} c(n, \beta, q) \omega_{\beta, n-1} c_q \left(n - \frac{n}{2|\beta|} + 2 \right) \right. \\ &\quad \times \Gamma_q \left(n - \frac{n}{2|\beta|} + 2 \right) \frac{\ln \frac{1}{q}}{1-q} q^{\frac{(n - \frac{n}{2|\beta|} + 2)(n - \frac{n}{2|\beta|} + 1)}{2}} \Big). \end{aligned}$$

2. WEIGHTED $L_{p,\beta}$ MODULUS OF CONTINUITY

We now introduce a weighted $L_{p,\beta}$ modulus of continuity depending on nonisotropic distance in order to measure the rate of convergence by $P_{\lambda,\beta}(f)$ in $L_{p,\beta}$ norm.

Definition 3. Let $f \in L_{p,\beta}(\mathbb{R}^n)$, $n \in \mathbb{N}$ and $\beta_i \in (0, \infty)$, $i = 1, \dots, n$, with $|\beta| = \beta_1 + \beta_2 + \dots + \beta_n$. For every $\delta > 0$, the nonisotropic weighted $L_{p,\beta}$ modulus of continuity of f is defined as

$$\omega_{p,\beta}(f; \delta) = \sup_{\|\mathbf{h}\|_\beta \leq \delta} \|f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})\|_{p,\beta}. \quad (2.1)$$

Analogously, the modulus of continuity defined above has the following property

Lemma 3. Let $f \in L_{p,\beta}(\mathbb{R}^n)$, $n \in \mathbb{N}$ and $\beta_i \in (0, \infty)$, $i = 1, \dots, n$, with $|\beta| = \beta_1 + \beta_2 + \dots + \beta_n$. For every $\delta > 0$ and $c > 0$, then

$$\omega_{p,\beta}\left(f; c^{\frac{|\beta|}{n}} \delta\right) \leq (1+c) \omega_{p,\beta}(f; \delta). \quad (2.2)$$

Proof. For $k \in \mathbb{Z}^+$, using similar arguments to that of in [3]. Using generalized Minkowsky inequality we have the following

$$\begin{aligned} \omega_{p,\beta}\left(f; k^{\frac{|\beta|}{n}} \delta\right) &= \sup_{\|\mathbf{h}\|_\beta \leq \delta} \left\| f\left(\mathbf{x} + k^\beta \mathbf{h}\right) - f(\mathbf{x}) \right\|_{p,\beta} = \\ &= \sup_{\|\mathbf{h}\|_\beta \leq \delta} \left(\int_{\mathbb{R}^n} \left| \frac{\sum_{s=1}^k \left[f\left(\mathbf{x} + s^\beta \mathbf{h}\right) - f\left(\mathbf{x} + (s-1)^\beta \mathbf{h}\right) \right]}{1 + \|\mathbf{x}\|_\beta} \right|^p d\mathbf{x} \right)^{\frac{1}{p}} = \\ &= \sup_{\|\mathbf{h}\|_\beta \leq \delta} \left\| \sum_{s=1}^k \left[f\left(\mathbf{x} + s^\beta \mathbf{h}\right) - f\left(\mathbf{x} + (s-1)^\beta \mathbf{h}\right) \right] \right\|_{p,\beta} \\ &\leq \sup_{\|\mathbf{h}\|_\beta \leq \delta} \sum_{s=1}^k \left\| \left[f\left(\mathbf{x} + s^\beta \mathbf{h}\right) - f\left(\mathbf{x} + (s-1)^\beta \mathbf{h}\right) \right] \right\|_{p,\beta} \\ &\leq \sum_{s=1}^k \sup_{\|s^\beta \mathbf{h} - (s-1)^\beta \mathbf{h}\|_\beta \leq \delta} \left(\int_{\mathbb{R}^n} \left| \frac{\left[f\left(\mathbf{x} + s^\beta \mathbf{h}\right) - f\left(\mathbf{x} + (s-1)^\beta \mathbf{h}\right) \right]}{1 + \|\mathbf{x}\|_\beta} \right|^p d\mathbf{x} \right)^{\frac{1}{p}} \\ &\leq k \omega_{p,\beta}(f; \delta), \end{aligned}$$

where $\|s^\beta \mathbf{h} - (s-1)^\beta \mathbf{h}\|_\beta \leq \|\mathbf{h}\|_\beta$, by $s^{\beta_i} - (s-1)^{\beta_i} \leq 1$ for $i = 1, 2, \dots, n$. Since $\omega_{p,\beta}(f; \delta)$ is a nondecreasing function of δ , then (2.2) is attained easily. $\square$

Now, we give an estimate by means of nonisotropic weighted $L_{p,\beta}$ modulus of continuity of f .

Theorem 1. *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$, $f \in L_{p,\beta}(\mathbb{R}^n)$ and $\omega_{p,\beta}(f; \delta) < \infty$. Then for any $\delta > 0$ we have*

$$\|P_{\lambda,\beta}(f) - f\|_{p,\beta} \leq \omega_{p,\beta}\left(f; [\lambda]_q^{\frac{|\beta|}{n}}\right) M(n, q, \beta)$$

where

$$M(n, q, \beta) = 1 + \frac{q^{\frac{n(n-1)}{2}} \Gamma_q\left(n + \frac{n}{|\beta|}\right) c_q\left(n + \frac{n}{|\beta|}\right)}{\Gamma_q(n) q^{\frac{(n + \frac{n}{|\beta|})(n + \frac{n}{|\beta|} - 1)}{2}}}$$

and $\omega_{p,\beta}(f; \cdot)$ is the nonisotropic weighted $L_{p,\beta}$ modulus of continuity of f given by (2.1).

Proof. From (1.4) and (1.6), by the same reasoning as in Theorem 1 of [3], we have

$$\begin{aligned} \|P_{\lambda,\beta}(f; \mathbf{x}) - f(\mathbf{x})\|_{p,\beta} &\leq \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left| \frac{f(\mathbf{x} + \mathbf{t}) - f(\mathbf{x})}{1 + \|\mathbf{x}\|_\beta} \right|^p d\mathbf{x} \right)^{\frac{1}{p}} \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t} \\ &\leq \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} \omega_{p,\beta}(f; \|\mathbf{t}\|_\beta) \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t}. \end{aligned} \quad (2.3)$$

Using (2.2) with $c = \frac{\|\mathbf{t}\|_\beta^{\frac{n}{|\beta|}}}{[\lambda]_q}$ for $\mathbf{t} \in \mathbb{R}^n$, we get that

$$\omega_{p,\beta}(f; \|\mathbf{t}\|_\beta) = \omega_{p,\beta}\left(f; \left(\frac{\|\mathbf{t}\|_\beta^{\frac{n}{|\beta|}}}{[\lambda]_q}\right)^{\frac{|\beta|}{n}} [\lambda]_q^{\frac{|\beta|}{n}}\right) \leq \left(1 + \frac{\|\mathbf{t}\|_\beta^{\frac{n}{|\beta|}}}{[\lambda]_q}\right) \omega_{p,\beta}\left(f; [\lambda]_q^{\frac{|\beta|}{n}}\right).$$

Therefore (2.3) results in

$$\begin{aligned} \|P_{\lambda,\beta}(f; \mathbf{x}) - f(\mathbf{x})\|_{p,\beta} &\leq \\ &\leq \omega_{p,\beta}\left(f; [\lambda]_q^{\frac{|\beta|}{n}}\right) \times \left\{ 1 + \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|+1}} \int_{\mathbb{R}^n} \|\mathbf{t}\|_\beta^{\frac{n}{|\beta|}} \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t} \right\}. \end{aligned}$$

Applying change of variable $\mathbf{t} = [\lambda]_q^\beta \mathbf{y}$, $d\mathbf{t} = [\lambda]_q^{|\beta|} d\mathbf{y}$, such that

$[\lambda]_q^\beta \mathbf{y} = ([\lambda]_q^\beta y_1, \dots, [\lambda]_q^\beta y_n)$ and using the generalized β -spherical coordinates and

finally taking (1.2) and (1.5) into consideration to the result, last inequality reduces to

$$\begin{aligned}
 & \|P_{\lambda, \beta}(f; \mathbf{x}) - f(\mathbf{x})\|_{p, \beta} \leq \\
 & \leq \omega_{p, \beta}\left(f; [\lambda]_q^{\frac{|\beta|}{n}}\right) \times \left\{1 + c(n, \beta, q) \int_{\mathbb{R}^n} \frac{\|\mathbf{y}\|_{\beta}^{\frac{n}{|\beta|}}}{E_q((1-q)\|\mathbf{y}\|_{\beta})} d\mathbf{y}\right\} \\
 & = \omega_{p, \beta}\left(f; [\lambda]_q^{\frac{|\beta|}{n}}\right) \times \left\{1 + c(n, \beta, q) \int_0^{\infty} \int_{S^{n-1}} \frac{u^{2|\beta|-1} u^2 \Omega_{\beta}(\theta) d\theta du}{E_q((1-q)u^{\frac{2|\beta|}{n}})}\right\} \\
 & = \omega_{p, \beta}\left(f; [\lambda]_q^{\frac{|\beta|}{n}}\right) \times \left\{1 + \frac{c(n, \beta, q)}{[\lambda]_q [\lambda]_q^{|\beta|}} \omega_{\beta, n-1} \int_0^{\infty} \frac{u^{n+\frac{n}{|\beta|}-1} du}{E_q((1-q)u)}\right\} \\
 & = \omega_{p, \beta}\left(f; [\lambda]_q^{\frac{|\beta|}{n}}\right) M(n, q, \beta),
 \end{aligned}$$

where

$$M(n, q, \beta) = 1 + \frac{q^{\frac{n(n-1)}{2}} \Gamma_q\left(n + \frac{n}{|\beta|}\right) c_q\left(n + \frac{n}{|\beta|}\right)}{\Gamma_q(n) q^{\frac{(n+\frac{n}{|\beta|})(n+\frac{n}{|\beta|}-1)}{2}}}.$$

□

Now, using weighted L_p -modulus of continuity defined by (2.1), we show that the q -Picard integral operators depending on the nonisotropic β -distance given by (1.6) satisfy the global smoothness preservation property.

Theorem 2. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$, and $\omega_{p, \beta}(f; \delta) < \infty$ for any $\delta > 0$ and $\beta_i \in (0, \infty)$, ($i = 1, \dots, n$) with $|\beta| = \beta_1 + \beta_2 + \dots + \beta_n$, such that $P_{\lambda, \beta}(f; \mathbf{x}) \in \mathbb{R}$ for $0 < q < 1$. Then we have

$$\omega_{p, \beta}(P_{\lambda, \beta}(f; \cdot); \delta) \leq L(n, q, \lambda, \beta) \omega_{p, \beta}(f; \delta), \quad (2.4)$$

where

$$L(n, q, \lambda, \beta) = \max\{1, M_{\beta}\} \times \left(1 + \frac{n}{2|\beta|} c(n, \beta, q) [\lambda]_q^{\frac{|\beta|}{n}} \omega_{\beta, n-1} \Gamma_q(n+1) \frac{\ln q^{-1}}{1-q} q^{-\frac{n(n+1)}{2}}\right)$$

in which M_{β} is given by property (3) of nonisotropic distance.

Proof. From (1.6) we can write

$$\mathcal{P}_{\lambda, \beta}(f; \mathbf{x} + \mathbf{h}) - \mathcal{P}_{\lambda, \beta}(f; \mathbf{x}) = \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} (f(\mathbf{x} + \mathbf{t} + \mathbf{h}) - f(\mathbf{x} + \mathbf{t})) \mathcal{P}_{\lambda}(\beta, \mathbf{t}) d\mathbf{t}.$$

Taking $L_{p,\beta}(\mathbb{R}^n)$ norm and using generalized Minkowsky inequality, we obtain that

$$\begin{aligned}
 & \left(\int_{\mathbb{R}^n} \left| \frac{\mathcal{P}_{\lambda,\beta}(f; \mathbf{x} + \mathbf{h}) - \mathcal{P}_{\lambda,\beta}(f; \mathbf{x})}{1 + \|\mathbf{x}\|_\beta} \right|^p d\mathbf{x} \right)^{\frac{1}{p}} = \\
 & = \left(\int_{\mathbb{R}^n} \left| \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} \frac{(f(\mathbf{x} + \mathbf{t} + \mathbf{h}) - f(\mathbf{x} + \mathbf{t})) \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t}}{1 + \|\mathbf{x}\|_\beta} \right|^p d\mathbf{x} \right)^{\frac{1}{p}} \\
 & = \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \left(\int_{\mathbb{R}^n} \left| \frac{1}{1 + \|\mathbf{x}\|_\beta} \int_{\mathbb{R}^n} (f(\mathbf{x} + \mathbf{t} + \mathbf{h}) - f(\mathbf{x} + \mathbf{t})) \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t} \right|^p d\mathbf{x} \right)^{\frac{1}{p}} \\
 & \leq \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left| \frac{(f(\mathbf{x} + \mathbf{t} + \mathbf{h}) - f(\mathbf{x} + \mathbf{t}))}{1 + \|\mathbf{x}\|_\beta} \right|^p d\mathbf{x} \right)^{\frac{1}{p}} \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t} \\
 & = \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left| \frac{f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})}{1 + \|\mathbf{x} - \mathbf{t}\|_\beta} \right|^p dx \right)^{\frac{1}{p}} \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t}
 \end{aligned}$$

From the property (3), we have

$$\begin{aligned}
 \frac{1}{1 + \|\mathbf{x}\|_\beta} \frac{1 + \|\mathbf{x}\|_\beta}{1 + \|\mathbf{x} - \mathbf{t}\|_\beta} &= \frac{1}{1 + \|\mathbf{x}\|_\beta} \frac{1 + \|\mathbf{x} - \mathbf{t} + \mathbf{t}\|_\beta}{1 + \|\mathbf{x} - \mathbf{t}\|_\beta} \\
 &\leq \frac{1}{1 + \|\mathbf{x}\|_\beta} \frac{1 + M_\beta (\|\mathbf{x} - \mathbf{t}\|_\beta + \|\mathbf{t}\|_\beta)}{1 + \|\mathbf{x} - \mathbf{t}\|_\beta} \\
 &= \frac{\max\{1, M_\beta\} (1 + \|\mathbf{x} - \mathbf{t}\|_\beta) (1 + \|\mathbf{t}\|_\beta)}{1 + \|\mathbf{x}\|_\beta (1 + \|\mathbf{x} - \mathbf{t}\|_\beta)}
 \end{aligned}$$

Using the above inequality we have

$$\begin{aligned}
 & \left(\int_{\mathbb{R}^n} \left| \frac{\mathcal{P}_{\lambda,\beta}(f; \mathbf{x} + \mathbf{h}) - \mathcal{P}_{\lambda,\beta}(f; \mathbf{x})}{1 + \|\mathbf{x}\|_\beta} \right|^p d\mathbf{x} \right)^{\frac{1}{p}} \leq \\
 & \leq \max\{1, M_\beta\} \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left| \frac{(f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}))}{1 + \|\mathbf{x}\|_\beta} \right|^p d\mathbf{x} \right)^{\frac{1}{p}} (1 + \|\mathbf{t}\|_\beta) \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t}
 \end{aligned}$$

$$\begin{aligned} &\leq \max \{1, M_\beta\} \|f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})\|_{p, \beta} \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} (1 + \|\mathbf{t}\|_\beta) \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t} \\ &= \max \{1, M_\beta\} \|f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})\|_{p, \beta} \left(1 + \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} \|\mathbf{t}\|_\beta \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t} \right). \end{aligned}$$

Using Lemma 1 and making change of variable $\mathbf{t} = [\lambda]_q^\beta \mathbf{x}$ we have

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} \left| \frac{\mathcal{P}_{\lambda, \beta}(f; \mathbf{x} + \mathbf{h}) - \mathcal{P}_{\lambda, \beta}(f; \mathbf{x})}{1 + \|\mathbf{x}\|_\beta} \right|^p d\mathbf{x} \right)^{\frac{1}{p}} \leq \\ &\leq \max \{1, M_\beta\} \|f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})\|_{p, \beta} \times \left(1 + c(n, \beta, q) [\lambda]_q^{\frac{|\beta|}{n}} \int_{\mathbb{R}^n} \frac{\|\mathbf{x}\|_\beta}{E_q((1-q)\|\mathbf{x}\|_\beta)} d\mathbf{x} \right). \end{aligned}$$

We use generalized β -spherical coordinates ([16]) and consider the transformation

$$\begin{aligned} x_1 &= (u \cos \theta_1)^{2\beta_1} \\ x_2 &= (u \sin \theta_1 \cos \theta_2)^{2\beta_2} \\ &\vdots \\ x_{n-1} &= (u \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \cos \theta_{n-1})^{2\beta_{n-1}} \\ x_n &= (u \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-1})^{2\beta_n}, \end{aligned}$$

where $0 \leq \theta_1, \theta_2, \dots, \theta_{n-2} \leq \pi$, $0 \leq \theta_{n-1} \leq 2\pi$, $u \geq 0$. Denoting the Jacobian of this transformation by $J_\beta(u, \theta_1, \dots, \theta_{n-1})$ we obtain

$$J_\beta(u, \theta_1, \dots, \theta_{n-1}) = u^{2|\beta|-1} \Omega_\beta(\theta),$$

where $\Omega_\beta(\theta) = 2^n \beta_1 \dots \beta_n \prod_{j=1}^{n-1} (\cos \theta_j)^{2\beta_j-1} (\sin \theta_j)^{\sum_{k=j}^{j+1} 2\beta_k-1}$. We can easily see that the integral

$$\omega_{\beta, n-1} = \int_{S^{n-1}} \Omega_\beta(\theta) d\theta$$

is finite, where S^{n-1} is the unit sphere in $\mathbb{R}^n$.

Thus we have

$$\left(\int_{\mathbb{R}^n} \left| \frac{\mathcal{P}_{\lambda, \beta}(f; \mathbf{x} + \mathbf{h}) - \mathcal{P}_{\lambda, \beta}(f; \mathbf{x})}{1 + \|\mathbf{x}\|_\beta} \right|^p d\mathbf{x} \right)^{\frac{1}{p}} \leq$$

$$\begin{aligned}
&\leq \max \{1, M_\beta\} \|(f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}))\|_{p,\beta} \\
&\times \left(1 + c(n, \beta, q) [\lambda]_q^{\frac{|\beta|}{n}} \int_0^\infty \int_{S^{n-1}} \frac{u^{2|\beta| + \frac{2|\beta|}{n} - 1} \Omega_\beta(\theta) d\theta du}{E_q\left((1-q)u^{\frac{2|\beta|}{n}}\right)} \right) \\
&\leq \max \{1, M_\beta\} \|(f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}))\|_{p,\beta} \\
&\times \left(1 + \frac{n}{2|\beta|} c(n, \beta, q) [\lambda]_q^{\frac{|\beta|}{n}} \omega_{\beta, n-1} \int_0^\infty \frac{u^n du}{E_q((1-q)u)} \right) \\
&\leq \max \{1, M_\beta\} \|(f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}))\|_{p,\beta} \\
&\times \left(1 + \frac{n}{2|\beta|} c(n, \beta, q) [\lambda]_q^{\frac{|\beta|}{n}} \omega_{\beta, n-1} \Gamma_q(n+1) \frac{\ln q^{-1}}{1-q} q^{-\frac{n(n+1)}{2}} \right)
\end{aligned}$$

Finally we find that

$$\|\mathcal{P}_{\lambda,\beta}(f; \mathbf{x} + \mathbf{h}) - \mathcal{P}_{\lambda,\beta}(f; \mathbf{x})\|_{p,\beta} \leq \|(f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}))\|_{p,\beta} L(n, q, \lambda, \beta)$$

proving (2.4). $\square$

Now we give direct approximation property of $P_{\lambda,\beta}(f; q)$ with respect to nonisotropic weighted norm.

Theorem 3. Let $f \in L_{p,\beta}(\mathbb{R}^n)$, $n \in \mathbb{N}$ and $\beta_i \in (0, \infty)$, $i = 1, \dots, n$, with $|\beta| = \beta_1 + \beta_2 + \dots + \beta_n$. We have

$$\lim_{\lambda \rightarrow 0} \|\mathcal{P}_{\lambda,\beta}(f; \mathbf{x}) - f(\mathbf{x})\|_{p,\beta} = 0.$$

Proof. Taking Lemma 1 and generalized Minkowsky inequality into consideration, we get

$$\begin{aligned}
\|P_{\lambda,\beta}(f; \mathbf{x}) - f(\mathbf{x})\|_{p,\beta} &\leq \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left| \frac{f(\mathbf{x} + \mathbf{t}) - f(\mathbf{x})}{1 + \|\mathbf{x}\|_\beta} \right|^p d\mathbf{x} \right)^{\frac{1}{p}} \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t} \\
&= \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} \|f(\mathbf{x} + \mathbf{t}) - f(\mathbf{x})\|_{p,\beta} \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t}.
\end{aligned}$$

For a $\delta > 0$ we can write

$$\begin{aligned}
\|P_{\lambda,\beta}(f; \mathbf{x}) - f(\mathbf{x})\|_{p,\beta} &\leq \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\|\mathbf{t}\|_\beta \leq \delta} \|f(\mathbf{x} + \mathbf{t}) - f(\mathbf{x})\|_{p,\beta} \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t} \\
&\quad + \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\|\mathbf{t}\|_\beta > \delta} \|f(\mathbf{x} + \mathbf{t}) - f(\mathbf{x})\|_{p,\beta} \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t}
\end{aligned}$$

$$= L_1(\lambda) + L_2(\lambda).$$

From Lemma 1 and definition of weighted $L_{p,\beta}$ modulus of continuity, we have

$$\begin{aligned} L_1(\lambda) &\leq \omega_{p,\beta}(f; \delta) \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\|\mathbf{t}\|_\beta \leq \delta} \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t} \\ &= \omega_{p,\beta}(f; \delta) \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\mathbb{R}^n} \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t} = 1, \end{aligned}$$

which implies that

$$\lim_{\lambda \rightarrow 0} L_1(\lambda) = 0. \quad (2.5)$$

Since

$$\begin{aligned} \|f(\mathbf{x} + \mathbf{t}) - f(\mathbf{x})\|_{p,\beta} &= \int_{\mathbb{R}^n} \left| \frac{f(\mathbf{x} + \mathbf{t}) - f(\mathbf{x})}{1 + \|\mathbf{x}\|_\beta} \right|^p d\mathbf{x} \\ &\leq \int_{\mathbb{R}^n} \left| \frac{f(\mathbf{x} + \mathbf{t})}{1 + \|\mathbf{x}\|_\beta} \right|^p d\mathbf{x} + \int_{\mathbb{R}^n} \left| \frac{f(\mathbf{x})}{1 + \|\mathbf{x}\|_\beta} \right|^p d\mathbf{x} \\ &\leq \max\{1, M_\beta\} (1 + \|\mathbf{t}\|_\beta) \|f\|_{p,\beta} + \|f\|_{p,\beta}, \end{aligned}$$

we can write

$$\begin{aligned} L_2(\lambda) &\leq (1 + \max\{1, M_\beta\}) \|f\|_{p,\beta} \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\|\mathbf{t}\|_\beta > \delta} \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t} \\ &\quad + \|f\|_{p,\beta} \frac{c(n, \beta, q)}{[\lambda]_q^{|\beta|}} \int_{\|\mathbf{t}\|_\beta > \delta} \|\mathbf{t}\|_\beta \mathcal{P}_\lambda(\beta, \mathbf{t}) d\mathbf{t}. \end{aligned}$$

Since the first and second term of above inequality tends to zero as $\lambda \rightarrow 0$ (see [17, p.508]), we get

$$\lim_{\lambda \rightarrow 0} L_2(\lambda) = 0. \quad (2.6)$$

From (2.5) and (2.6) we have desired result. $\square$

Remark 1. Using similar arguments in [3, p.825] we can easily see that $\omega_{p,\beta}(f; \delta)$ tends to zero as $\delta \rightarrow 0$. As a consequence of Theorem 1, we can say that resulting approximation process turns out to have a weighted rate of convergence which is

$[\lambda]_q^{\frac{|\beta|}{n}}$ at least as good as that classical Picard singular integral operators. Also this weighted convergence rate of $P_{\lambda,\beta}(f;q)$ to f can be made better not only depending on the chosen q but also the choice of β . This convergence is shown directly in Theorem 3.

REFERENCES

- [1] R. Álvarez Nodarse, M. K. Atakishiyeva, and N. M. Atakishiyev, "On q -extension of the Hermit Polynomials $H_n(x)$ with the continuous orthogonality property on $\mathbb{R}$," *Bol. Soc. Mat. Mexicana*, vol. 8, no. 3, pp. 127–139, 2002.
- [2] G. A. Anastassiou, "Global smoothness preservation by singular integrals," *Proyecciones*, vol. 14, no. 2, pp. 83–88, 1995.
- [3] G. A. Anastassiou and A. Aral, "Generalized Picard singular integrals," *Comput. Math. Appl.*, vol. 57, no. 5, pp. 821–830, 2009.
- [4] G. A. Anastassiou and S. G. Gal, *Approximation theory. Moduli of continuity and global smoothness preservation*. Boston: Birkhäuser, 2000.
- [5] G. A. Anastassiou and S. G. Gal, "Convergence of generalized singular integrals to the unit, univariate case," *Math. Inequal. Appl.*, vol. 3, no. 4, pp. 511–518, 2000.
- [6] A. Aral, "On convergence of singular integrals with non-isotropic kernels," *Commun. Fac. Sci. Univ. Ank., Sér. A1, Math. Stat.*, vol. 50, no. 1-2, pp. 83–93, 2001.
- [7] A. Aral, "On a generalized λ -Gauss Weierstrass singular integral," *Fasc. Math.*, vol. 35, pp. 23–33, 2005.
- [8] A. Aral, "On the generalized Picard and Gauss-Weierstrass singular integrals," *J. Comput. Anal. Appl.*, vol. 8, no. 3, pp. 249–261, 2006.
- [9] A. Aral, "Fine limits of generalized potential-type integral operators with nonisotropic kernel," *Miskolc Math. Notes*, vol. 8, no. 2, pp. 123–134, 2007.
- [10] A. Aral, "Pointwise approximation by the generalization of Picard and Gauss-Weierstrass singular integrals," *J. Concr. Appl. Math.*, vol. 6, no. 4, pp. 327–339, 2008.
- [11] A. Aral and S. G. Gal, " q -generalizations of the Picard and Gauss-Weierstrass singular integrals," *Taiwanese J. Math.*, vol. 12, no. 9, pp. 2501–2515, 2008.
- [12] N. M. Atakishiyev and M. K. Atakishiyeva, "A q -analogue of the Euler gamma integral," *Theor. Math. Phys.*, vol. 129, no. 1, pp. 1325–1334 (2001); translation from *teor. mat. fiz.* 129, no. 1, 20–30, 2001.
- [13] S. G. Gal, "Degree of approximation of continuous function by some singular integrals," *Rev. Anal. Numér. Théor. Approx.*, vol. XXVII, no. 1, pp. 251–261, 1998.
- [14] S. G. Gal, "Remark on degree of approximation of continuous function by some singular integrals," *Math. Nachr.*, vol. 164, pp. 197–199, 1998.
- [15] G. Gasper and M. Rahman, *Basic hypergeometric series.*, ser. Encyclopedia of Mathematics and Its Applications. Cambridge: Cambridge University Press, 1990, vol. 34.
- [16] V. P. Il'in, O. V. Besov, and S. M. Nikolsky, *The Integral Representation of Functions and Embedding Theorems*. Nauka, 1975.
- [17] B. Yilmaz, A. Aral, and G. Başcanbaz-Tunca, "Weighted approximation properties of generalized Picard operators," *J. Comput. Anal. Appl.*, vol. 13, no. 3, pp. 499–513, 2011.

*Authors' addresses***B. Yılmaz**

Ankara University, Faculty of Science, Department of Mathematics, 06100, Tandogan, Ankara, Turkey

E-mail address: basaryilmaz77@yahoo.com

G. Başcanbaz-Tunca

Ankara University, Faculty of Science, Department of Mathematics, 06100, Tandogan, Ankara, Turkey

E-mail address: tunca@science.ankara.edu.tr

A. Aral

Kırıkkale University, Faculty of Science and Arts, Department of Mathematics, Yahşihan, Kırıkkale, Turkey

E-mail address: aliaral73@yahoo.com



A NOTE ON TWO DIOPHANTINE EQUATIONS $x^2 \pm 2^a p^b = y^4$

HUILIN ZHU, GÖKHAN SOYDAN, AND WEI QIN

Received 21 February, 2012

Abstract. In this paper we gave some results of the two Diophantine equations $x^2 \pm 2^a p^b = y^4$, $x, y \in \mathbb{N}$, $\gcd(x, y) = 1$, $a, b \in \mathbb{Z}$, $a \geq 0$, $b \geq 0$, where p is an odd prime.

2010 *Mathematics Subject Classification:* 11D61; 11D41

Keywords: exponential Diophantine equation

1. INTRODUCTION

Let D denote an odd positive integer without any squared factor > 1 . In 1964, W.Ljunggren [15] showed that the Diophantine equation $x^2 + 4D = y^q$ has no solutions in rational integers if $q \not\equiv 3 \pmod{8}$ where the class number of $\mathbb{Q}(\sqrt{-D})$ is indivisible by the odd prime and also he showed that this equation has only a finite number of solutions in rational integers x and y and primes q for given D .

Let $\mathbb{Z}, \mathbb{N}$ be the sets of integers and positive integers respectively. Let p be a fixed odd prime. Recently many authors are interested in the equation

$$x^2 + \delta 2^a p^b = y^n, \quad x, y, a, b \in \mathbb{Z}, x, y \geq 1, \gcd(x, y) = 1, a, b \geq 0, n \geq 3, \quad (1.1)$$

where p is an odd prime and $\delta = 1$. In 2002, F.Luca [18] found all positive integer solutions (x, y, a, b, n) of $x^2 + 2^a 3^b = y^n$ with $n \geq 3$ and coprime x and y . In 2008, F.Luca and A.Togbé [19] solved $x^2 + 2^a 5^b = y^n$, $\gcd(x, y) = 1$. In 2009, F.Luca and A.Togbé [20] found all positive integer solutions of $x^2 + 2^a 13^b = y^n$, $\gcd(x, y) = 1$. I.N.Cangül, M.Demirci, F.Luca, Á.Pintér and G.Soydan [5] solved $x^2 + 2^a 11^b = y^n$, $\gcd(x, y) = 1$. Recently, in [23], G.Soydan, M.Ulas and H.Zhu found all positive integer solutions of $x^2 + 2^a 19^b = y^n$, $\gcd(x, y) = 1$. Obviously, the authors above researched the special cases of (1.1) and $p \in \{3, 5, 11, 13, 19\}$. In [4] A.Bérczes and I.Pink gave all the solutions of the Diophantine equation (1.1), when $\delta = 1$, $a = 0$ and b is even, where p is any prime in the interval $[2, 100]$ and $\gcd(x, y) = 1$. In [25] all the positive integer solutions (x, y, n) of the Diophantine equation $x^2 + a^2 = 2y^n$ with $a \in \{3, 4, \dots, 501\}$ were found under the conditions that $n \geq 3$ and that

The first author was supported in part by Fundamental Research Funds for the Central Universities, Grant No. 2011121039.

$\gcd(x, y) = 1$.

For general odd prime p and general n , (1.1) is difficult to solve completely, but for special n , maybe we can get some information. For example, when $n = 4$, (1.1) is rewritten as

$$x^2 + 2^a p^b = y^4, \quad x \geq 1, y > 1, \gcd(x, y) = 1, a \geq 0, b \geq 0. \quad (1.2)$$

where $\delta = 1$. Obviously, $y^4 - x^2$ can be factorized as $(y^2 - x)(y^2 + x)$. Our research is based on $(y^2 - x)(y^2 + x) = 2^a p^b$ and $\gcd(x, y) = 1$. Similarly, our research on another equation

$$x^2 - 2^a p^b = y^4, \quad x \geq 1, y \geq 1, \gcd(x, y) = 1, a \geq 0, b \geq 0 \quad (1.3)$$

is based on $(x - y^2)(x + y^2) = 2^a p^b$ and $\gcd(x, y) = 1$. When $b = 0$, (1.1) is solved by J.H.E. Cohn [8, 9], S.A.Arif and F.S.Abu Muriefah [1, 2] and M.Le [14]. When $a = 1$ and $b \geq 3$, we find (1.2) is concerned with the famous equation

$$x^2 - 2 = y^n, \quad x > 1, y \geq 1, n \geq 3. \quad (1.4)$$

This is still unsolved and is one of the most exciting questions on “classical Diophantine equations”. We do have good bounds for n , something like $n < 1237$ or so, see the Appendix written by S.Siksek in the GTM book of [7]. And it has been solved for “half” the primes (namely those $n \equiv 1 \pmod{3}$), by I.Chen [6]. Most people believe (1.4) has no solution, but this has not been proved up to day.

Now we introduce some notations and symbols. For any positive integer k , let

$$u_k = \frac{1}{2}(\rho^k + \bar{\rho}^k), \quad v_k = \frac{1}{2\sqrt{2}}(\rho^k - \bar{\rho}^k), 2 \nmid k, \quad (1.5)$$

$$U_k = \frac{1}{2}(\rho'^k + \bar{\rho}'^k), \quad V_k = \frac{1}{2\sqrt{2}}(\rho'^k - \bar{\rho}'^k), \quad (1.6)$$

where

$$\rho = 1 + \sqrt{2}, \bar{\rho} = 1 - \sqrt{2}, \rho' = 3 + 2\sqrt{2}, \bar{\rho}' = 3 - 2\sqrt{2}. \quad (1.7)$$

By basic properties of Pell equations [26], $(u, v) = (u_k, v_k) (k = 1, 3, 5, \dots)$, and $(U, V) = (U_k, V_k) (k = 1, 2, 3, \dots)$ are all solutions of equations

$$u^2 - 2v^2 = -1, u, v \in \mathbb{N}, \quad (1.8)$$

and

$$U^2 - 2V^2 = 1, U, V \in \mathbb{N}, \quad (1.9)$$

respectively.

2. STATEMENT OF THE RESULTS

Theorem 1. *If (1.4) has no solution, then all solutions of the equation*

$$x^2 + \delta 2^a p^b = y^4, \quad x \geq 1, y > 1, \gcd(x, y) = 1, a \geq 0, b \geq 0, \delta \in \{1, -1\}$$

are given as follows:

a	b	p	x	y	δ	restrictions on f, k, r, s, t
0	1	$2f^2 - 1$	$f^2 - 1$	f	1	$f \in \mathbb{N}, f \geq 2$
		$2f^2 + 1$	$f^2 + 1$	f	-1	$f \in \mathbb{N}$
0	2	u_k	$\frac{u_k^2 - 1}{2}$	v_k	1	$k \in \mathbb{N}, 2 \nmid k$
		U_k	$V_k^2 + 1$	V_k	-1	$k \in \mathbb{N}$
0	3	3	122	11	-1	-
0	5	23	6083	78	1	-
3	0	p	3	1	-1	-
3	1	$\frac{f^2 + 1}{2}$	$f^2 + 2$	f	-1	$f \in \mathbb{N}, f \geq 3$
		V_k	$u_k^2 + 2$	u_k	-1	$k \in \mathbb{N}, 2 \nmid k$
3	2	3	29	5	-1	-
3	3	13	5713	239	-1	-
4	3	5	129	11	-1	-
5	0	p	7	3	1	-
7	2	3	287	17	1	-
7	4	3	113	7	-1	-
9	3	17	4785	71	1	-
11	3	7	855	13	-1	-
$s + 2$	1	$2^s - f^2$	$2^{s+1} - f^2$	f	-1	$f, s \in \mathbb{N}, 2 \nmid f$
		$f^2 + 2^s$	$f^2 + 2^{s+1}$	f	-1	$f, s \in \mathbb{N}, 2 \nmid f$
$s + 4$	1	$2^s - 1$	$2^{2s+2} - 2^{s+2} - 1$	$2^{s+1} - 1$	1	$s \in \mathbb{N}, s \geq 2$
		$2^s - 1$	$ 2^{2s} - 2^{s+3} + 2^{s+1} + 1 $	$2^s + 1$	1	$s \in \mathbb{N}, s \geq 2$
$t + 2$	1	$ f^2 - 2^t $	$ f^2 - 2^{t+1} $	f	1	$f, t \in \mathbb{N}, 2 \nmid f$
$2^r + 4$	1	$2^{2^r} + 1$	$2^{2^{r+1}+2} + 2^{2^r+2} - 1$	$2^{2^r+1} + 1$	1	$r \in \mathbb{N}$
		$2^{2^r} + 1$	$2^{2^{r+1}} + 2^{2^r+3} - 2^{2^r+1} + 1$	$2^{2^r} - 1$	-1	$r \in \mathbb{N}$

3. PROOF OF THE THEOREM

We consider the equation

$$x^2 + \delta 2^a p^b = y^4, x \geq 1, y > 1, a \geq 0, b \geq 0, \gcd(x, y) = 1, \delta \in \{1, -1\} \quad (3.1)$$

where p is an odd prime. For the case $\delta = 1$ and $b = 0$, from [1, 2, 8, 9] and [14], we get $(x, y, a) = (7, 3, 5)$. For the case $\delta = -1$ and $b = 0$, we get $x^2 - 2^a = y^4$. From W.Ivorra [13] and S.Siksek [22], we know when $a \geq 2$, this equation has the solution $(x, y, a) = (3, 1, 3)$. When $a = 1$, from $\gcd(x, y) = 1$ we have $2 \nmid xy$ and $x^2 \equiv y^4 \equiv 1 \pmod{8}$, so $a \geq 3$, it is a contradiction. When $a = 0$, from P. Mihăilescu [21], this equation has no solutions.

When $a = 0$ and $b > 0$, from (3.1) we have $\delta p^b = (y^2 - x)(y^2 + x)$. From $\gcd(x, y) = 1$, we have $\gcd(\delta(y^2 - x), y^2 + x) = 1$. Otherwise $p|(y^2 - x)$ and $p|(y^2 + x)$, $p|2y^2$ and $p|2x$, $p|y$ and $p|x$. It is a contradiction with $\gcd(x, y) = 1$.

So we have

$$y^2 - x = \delta, \quad y^2 + x = p^b$$

and it leads to

$$2y^2 = p^b + 1. \quad (3.2)$$

and

$$p^b = 2y^2 + 1. \quad (3.3)$$

where $\delta = 1$ and $\delta = -1$, respectively. From Theorem 1.1 of M.A.Bennet [3], we know that when $b \geq 4$, (3.2) has no solutions. When $b = 3$, we get $(4y)^2 = (2p)^3 + 8$ and from J.Gebel [11] and J.London [17], we know that $p = 23, y = 78, x = 6083$. When $b = 1$, we obtain $p = 2f^2 - 1, x = f^2 - 1$, where $f \in \mathbb{N}, f \geq 2$. When $b = 2$, we obtain $p^2 - 2y^2 = -1$ and $p = u_k, y = v_k, x = \frac{u_k^2 - 1}{2}$ where $k \in \mathbb{N}, 2 \nmid k$.

From J.H.E.Cohn [10] and E.Herrmann [12], we know that when $b \geq 3$, (3.3) has the only solution $(p, b, y) = (3, 5, 11)$. When $b = 1$, we obtain $p = 2f^2 + 1, x = f^2 + 1$, where $f \in \mathbb{N}$. When $b = 2$, we obtain $p^2 - 2y^2 = 1$ and $p = U_k, y = V_k, x = V_k^2 + 1$, where $k \in \mathbb{N}$.

When $a > 0$ and $b > 0$, from (3.1) we have the factorization

$$\delta 2^a p^b = (y^2 - x)(y^2 + x), \quad 2 \nmid xy. \quad (3.4)$$

Because $\gcd(x, y) = 1$, we have $\gcd(\delta \frac{y^2 - x}{2}, \frac{y^2 + x}{2}) = 1$ and $\frac{y^2 - x}{2} \cdot \frac{y^2 + x}{2} = \delta 2^{a-2} p^b$. Therefore, we get

$$\frac{y^2 - x}{2} = \delta, \quad \frac{y^2 + x}{2} = 2^{a-2} p^b, \quad (3.5)$$

$$\frac{y^2 - x}{2} = \delta 2^{a-2}, \quad \frac{y^2 + x}{2} = p^b \quad (3.6)$$

and

$$\frac{y^2 - x}{2} = \delta p^b, \quad \frac{y^2 + x}{2} = 2^{a-2}. \quad (3.7)$$

In the following, we discuss the three cases:

Case 1: From (3.5), we get

$$y^2 - \delta = 2^{a-2} p^b, \quad (3.8)$$

For the case $\delta = 1$, we obtain

$$\frac{y-1}{2} \cdot \frac{y+1}{2} = 2^{a-4} p^b. \quad (3.9)$$

So from (3.9) we have

$$\frac{y-1}{2} = 1, \quad \frac{y+1}{2} = 2^{a-4} p^b \quad (3.10)$$

or

$$\frac{y-1}{2} = 2^{a-4}, \quad \frac{y+1}{2} = p^b \quad (3.11)$$

or

$$\frac{y-1}{2} = p^b, \quad \frac{y+1}{2} = 2^{a-4}. \quad (3.12)$$

(3.10) leads to $y = 3, a = 5, b = 0$. (3.11) leads to

$$p^b - 2^{a-4} = 1. \quad (3.13)$$

When $b \geq 2$, we have $a - 4 \geq 2$ and from P.Mihăilescu [21] we have $(p, a, b) = (3, 7, 2)$, the responding $(x, y) = (287, 17)$. When $b = 1$, $p = 2^{a-4} + 1$ is a Fermat prime. We have $p = 2^{2^r} + 1, a = 2^r + 4, y = 2^{2^r+1} + 1, x = 2^{2^r+1+2} + 2^{2^r+2} - 1$, where $r \in \mathbb{Z}$. (3.12) leads to

$$2^{a-4} - p^b = 1. \quad (3.14)$$

When $b \geq 2$, we have $a - 4 \geq 3$ and from P.Mihăilescu [21] we know (3.14) has no solutions. When $b = 1$, $p = 2^{a-4} - 1$ is a Mersenne prime. We have $p = 2^s - 1, a = s + 4, y = 2^{s+1} - 1, x = 2^{2s+2} - 2^{s+2} - 1$, where $s \in \mathbb{N}, s \geq 2$.

For the case $\delta = -1$, from (3.8) we get

$$y^2 + 1 = 2^{a-2} p^b. \quad (3.15)$$

Because $2 \nmid y$, we have $y^2 + 1 \equiv 2 \pmod{8}$. So $a = 3$ and

$$y^2 + 1 = 2p^b. \quad (3.16)$$

By using [16, 24], we know (3.16) with $b \geq 3$ has the solution $(y, p, b) = (239, 13, 4)$, the responding $x = 57123$. When $b = 1$, $p = \frac{f^2 + 1}{2}, x = f^2 + 2$, where $f \in \mathbb{N}, f \geq 3$. When $b = 2$, we have $y^2 - 2p^2 = -1$ and $y = u_k, p = v_k, x = u_k^2 + 2$, where $k \in \mathbb{N}, 2 \nmid k$.

Case 2: For the case $\delta = 1$, from (3.6) we get

$$y^2 - 2^{a-2} = p^b. \quad (3.17)$$

When $b = 1$, from (3.17) we have $p = |f^2 - 2^t|$, where $f, t \in \mathbb{N}, 2 \nmid f$. When $2 \mid b$, from (3.17) we have $\left(\frac{y - p^{\frac{b}{2}}}{2}\right)\left(\frac{y + p^{\frac{b}{2}}}{2}\right) = 2^{a-4}$ and $p^{\frac{b}{2}} = 2^{a-4} - 1$. If $b = 2$, then $p = 2^{a-4} - 1$, which is a Mersenne prime. We have $p = 2^s - 1, a = s + 4, y = 2^s + 1, x = |2^{2s} - 2^{s+3} + 2^{s+1} + 1|$, where $s \in \mathbb{N}, s \geq 2$. If $2 \nmid b$ and $b > 2$, then $a - 4 > 3$. From [21], we know it has no solution. So $2 \nmid b$ and $b \geq 3$. From W.Ivorra [13] and S.Siksek [22] we know (3.17) with $a - 2 \geq 2$ has the only solution $(y, p, a, b) = (71, 17, 9, 3)$, the responding $x = 4785$. There remains the following unsolved equation

$$y^2 - 2 = p^b, \quad 2 \nmid b, \quad b \geq 3. \quad (3.18)$$

For the case $\delta = -1$, from (3.6) we get

$$y^2 + 2^{a-2} = p^b. \quad (3.19)$$

When $b = 1$, from (3.19) we have $p = f^2 + 2^s, a = s + 2, x = f^2 + 2^{s+1}$, where $f, s \in \mathbb{N}, 2 \nmid f$. When $b = 2$, from (3.19) we have $\left(\frac{p-y}{2}\right)\left(\frac{p+y}{2}\right) = 2^{a-4}$ and $p = 2^{a-4} + 1$ is a Fermat prime. So we have $p = 2^{2^r} + 1, a = 2^r + 4, y = 2^{2^r} - 1, x = 2^{2^r+1} + 2^{2^r+3} - 2^{2^r+1} + 1$, where $r \in \mathbb{Z}$. If $b > 2$, from [1, 2, 8, 9] and [14], we know that (3.19) has the solutions $(y, p, a, b) = (5, 3, 3, 3), (7, 3, 7, 4), (11, 5, 4, 3)$, the responding $x = 29, 113, 129$.

Case 3: For the case $\delta = 1$ from (3.7), we also get (3.17) and discuss it similarly with Case 2.

For the case $\delta = -1$ from (3.7) we get

$$y^2 + p^b = 2^{a-2}. \quad (3.20)$$

When $b = 1$, we have $p = 2^s - f^2, a = s + 2, x = 2^{s+1} - f^2$, where $f, s \in \mathbb{N}, 2 \nmid f$. When $b \geq 2$, from Theorem 8.4 of M.A. Bennet [3], we see (3.20) has the solution $(y, p, a, b) = (13, 7, 11, 3)$, the responding $x = 855$. We complete the proof of the theorem.

ACKNOWLEDGEMENT

The authors would like to thank Professor Michael A. Bennett, Professor Yann Bugeaud, Professor Imin Chen and Professor Samir Siksek for their help.

REFERENCES

- [1] S. A. Arif and F. S. Abu Muriefah, "On the Diophantine equation $x^2 + 2^k = y^n$. II," *Arab J. Math. Sci.*, vol. 7, no. 2, pp. 67–71, 2001.
- [2] S. A. Arif and F. S. Abu Muriefah, "On the diophantine equation $x^2 + 2^k = y^n$," *Int. J. Math. Math. Sci.*, vol. 20, no. 2, pp. 299–304, 1997.
- [3] M. A. Bennett and C. M. Skinner, "Ternary Diophantine equations via Galois representations and modular forms," *Can. J. Math.*, vol. 56, no. 1, pp. 23–54, 2004.
- [4] A. Bérczes and I. Pink, "On the Diophantine equation $x^2 + p^{2k} = y^n$," *Arch. Math.*, vol. 91, no. 6, pp. 505–517, 2008.
- [5] I. N. Cangüla, M. Demirci, F. L. A. Pintér, and G. Soydan, "On the Diophantine equation $x^2 + 2^a \cdot 11^b = y^n$," *Fibonacci Q.*, vol. 48, no. 1, pp. 39–46, 2010.
- [6] I. Chen, "On the equations $a^2 - 2b^6 = c^p$ and $a^2 - 2 = c^p$," *LMS Journal of Comp. and Math.*, vol. 15, no. 1, pp. 158–171, 2012.
- [7] H. Cohen, *Number theory. Volume II: Analytic and modern tools*, ser. Graduate Texts in Mathematics. New York: Springer, 2007, vol. 240.
- [8] J. H. E. Cohn, "The diophantine equation $x^2 + 2^k = y^n$," *Arch. Math.*, vol. 59, no. 4, pp. 341–344, 1992.
- [9] J. H. E. Cohn, "The Diophantine equation $x^2 + 2^k = y^n$. II," *Int. J. Math. Math. Sci.*, vol. 22, no. 3, pp. 459–462, 1999.

- [10] J. H. E. Cohn, “The Diophantine equation $x^n = Dy^2 + 1$,” *Acta Arith.*, vol. 106, no. 1, pp. 73–83, 2003.
- [11] J. Gebel, A. Pethő, and H. G. Zimmer, “On Mordell’s equation,” *Compos. Math.*, vol. 110, no. 3, pp. 335–367, 1998.
- [12] E. Herrmann, I. Járási, and A. Pethő, “Note on J. H. E. Cohn’s paper “The Diophantine equation $x^n = Dy^2 + 1$,”,” *Acta Arith.*, vol. 113, no. 1, pp. 69–76, 2004.
- [13] W. Ivorra, “On the equations $x^p + 2^\beta y^p = z^2$ and $x^p + 2^\beta y^p = 2z^2$. (Sur les équations $x^p + 2^\beta y^p = z^2$ et $x^p + 2^\beta y^p = 2z^2$),” *Acta Arith.*, vol. 108, no. 4, pp. 327–338, 2003.
- [14] M. Le, “On Cohn’s conjecture concerning the Diophantine equation $x^2 + 2^m = y^n$,” *Arch. Math.*, vol. 78, no. 1, pp. 26–35, 2002.
- [15] W. Ljunggren, “On the Diophantine equation $Cx^2 + D = y^n$,” *Pac. J. Math.*, vol. 14, pp. 585–596, 1964.
- [16] W. Ljunggren, “Zur Theorie der Gleichung $x^2 + 1 = Dy^4$,” *Avh. Norske Vid. Akad. Oslo*, no. 5, pp. 1–27, 1942.
- [17] H. London and R. Finkelstein, *On Mordell’s equation $y^2 - k = x^3$* . Bowling Green, Ohio: Bowling Green State University, 1973.
- [18] F. Luca, “On the equation $x^2 + 2^a \cdot 3^b = y^n$,” *Int. J. Math. Math. Sci.*, vol. 29, no. 4, pp. 239–244, 2002.
- [19] F. Luca and A. Togbé, “On the Diophantine equation $x^2 + 2^a \cdot 5^b = y^n$,” *Int. J. Number Theory*, vol. 4, no. 6, pp. 973–979, 2008.
- [20] F. Luca and A. Togbé, “On the Diophantine equation $x^2 + 2^a 13^b = y^n$,” *Colloq. Math.*, vol. 116, no. 1, pp. 139–146, 2009.
- [21] P. Mihăilescu, “Primary cyclotomic units and a proof of Catalan’s conjecture,” *J. Reine Angew. Math.*, vol. 572, pp. 167–195, 2004.
- [22] S. Siksek, “On the Diophantine equation $x^2 = y^p + 2^k z^p$,” *J. Théor. Nombres Bordx.*, vol. 15, no. 3, pp. 839–846, 2003.
- [23] G. Soydan, M. Ulas, and H. Zhu, “On the Diophantine equation $x^2 + 2^a \cdot 19^b = y^n$,” *Indian J. Pure Appl. Math.*, vol. 43, no. 3, pp. 251–261, 2012.
- [24] C. Störmer, “Complete solution of the equation $m \arctan \frac{1}{x} + n \arctan \frac{1}{y} = k \frac{\pi}{4}$ in integers. (Solution complète en nombres entiers de l’équation $m \arctan \frac{1}{x} + n \arctan \frac{1}{y} = k \frac{\pi}{4}$),” *Bull. Soc. Math. Fr.*, vol. 27, pp. 160–170, 1899.
- [25] S. Tengely, “On the Diophantine equation $x^2 + a^2 = 2y^p$,” *Indag. Math., New Ser.*, vol. 15, no. 2, pp. 291–304, 2004.
- [26] D. T. Walker, “On the Diophantine equation $mX^2 - nY^2 = \pm 1$,” *Am. Math. Mon.*, vol. 74, pp. 504–513, 1967.

Authors’ addresses

Huilin Zhu

School of Mathematical Sciences, Xiamen University, 361005, Xiamen, P.R.China

E-mail address: hlzhu@xmu.edu.cn

Gökhan Soydan

İşıklar Air Force High School, 16039, Bursa, TURKEY

E-mail address: gsoydan@uludag.edu.tr

Wei Qin

Department of Mathematics, University of Illinois, Urbana, Champaign, 61801, Illinois, USA

E-mail address: qinweistudymaths@gmail.com

EDITOR-IN-CHIEF

Miklós Rontó, Institute of Mathematics,
University of Miskolc; H-3515 Miskolc–
Egyetemváros, Hungary;
matronto@uni-miskolc.hu

EDITORIAL BOARD

N. K. Agbeko, Institute of Mathematics,
University of Miskolc; H-3515 Miskolc–
Egyetemváros, Hungary;
matagbek@uni-miskolc.hu

Mátyás Domokos, A. Rényi Institute of Mathematics,
Hungarian Academy of Sciences;
P. O. Box 127, H-1053 Budapest, Hungary;
domokos@renyi.hu

Michal Fečkan, Comenius University, Department of Mathematical Analysis;
Mlynská dolina, 842 15 Bratislava, Slovakia;
Michal.Feckan@fmph.uniba.sk

Sándor Fegyverneki, Institute of Mathematics,
University of Miskolc; H-3515 Miskolc–
Egyetemváros, Hungary;
matfs@uni-miskolc.hu

Zoltán Füredi, A. Rényi Institute of Mathematics,
Hungarian Academy of Sciences; H-1364,
Budapest, Hungary, P. O. Box 127;
furedi.zoltan@renyi.mta.hu

Aurél Galántai, Óbuda University, John von Neumann Faculty of Informatics;
H-1034 Budapest, Bécsi u. 96/b, Hungary;
galantai.aurel@nik.uni-obuda.hu

Chaitan P. Gupta, Department of Mathematics,
University of Nevada; Reno, NV 89557, USA;
chaitang@hotmail.com

István Győri, Department of Mathematics and Computing,
University of Veszprém;
P. O. Box 158, H-8201 Veszprém, Hungary;
gyori@almos.vein.hu

László Hatvani, Bolyai Institute, University of Szeged;
1 Aradi vértanúk tere, H-6720 Szeged, Hungary;
hatvani@math.u-szeged.hu

Attila Házy, Institute of Mathematics,
University of Miskolc; H-3515 Miskolc–
Egyetemváros, Hungary;
matha@uni-miskolc.hu

Nino Partsvania, Andrea Razmadze Mathematical Institute;
2 University St., Tbilisi 0186 GEORGIA;
ninopa@rmi.ge

Nikolai Perestyuk, National Shevchenko University of Kiev,
Faculty of Mechanics and Mathematics; 64 Vladimirska St.,
01006 Kiev, Ukraine; pmo@univ.kiev.ua

András Rontó, Institute of Mathematics,
Academy of Sciences of Czech Republic;
Žižkova 22, 616 62 Brno, Czech Republic;
ronto@math.cas.cz

Nikolai Rozov, Moscow State University,
Faculty of Mechanics and Mathematics; The Main Building,
Vorobiovy Gory, 119899 Moscow, Russia;
rozov@rozov.mccme.ru

Anatoly Samoilenko, Institute of Mathematics,
National Academy of Sciences of Ukraine; 3 Tereshchenkivska St.,
01601 Kiev, Ukraine; sam@imath.kiev.ua

Jenő Szigeti, Institute of Mathematics, University of Miskolc;
H-3515 Miskolc–Egyetemváros, Hungary;
matszj@gold.uni-miskolc.hu

Jiří Šremr, Institute of Mathematics, Academy of Sciences of Czech Republic;
Žižkova 22, 616 62 Brno, Czech Republic;
sremr@ipm.cz

Sergei Trofimchuk, Instituto de Matemática y Física,
Universidad de Talca; 2 Norte 685, Talca, Chile;
trofimch@inst-mat.utalca.cl

Miskolc Mathematical Notes

Abstracted in *Zentralblatt für Mathematik*, *Mathematical Reviews*, and *Реферативный журнал „Математика”*

Responsible for publication: Rector of the University of Miskolc
Published by the Miskolc University Press under the leadership of Erzsébet Burmeister
Responsible for duplication: Works manager Erzsébet Pásztor
Number of copies printed: 100
Put to the Press on Dec 20, 2013
Number of permission: TU 2013-400-ME

HU e-ISSN 1787-2413