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CONTENTS

<i>Miroslav Kureš</i>	
AGMP 2012: Preface	397
<i>Josef Mikeš, Bogdan Balcerzak, and Miroslav Kureš</i>	
In memory of Leonid Evtushik and Jan Kubarski	401
<i>Paolo Giordano</i>	
Fermat reals: infinitesimals without Logic	407
<i>Ivan Kolář</i>	
On the functorial prolongations of fiber bundles	423
<i>Korbinian Münster and Ivo Sachs</i>	
On homotopy algebras and quantum string field theory	433
<i>Rachel Nicks and Gareth Parry</i>	
Geometrical issues in the continuum mechanics of defective solid crystals	445
<i>Maria Clara Nucci</i>	
Symmetries for thought	461
<i>Sergey E. Stepanov and Josef Mikeš</i>	
Betti and Tachibana numbers	475
<i>Ján Brajerčík and Milan Demko</i>	
Second order natural Lagrangians on coframe bundles	487
<i>Miroslav Doupovec and Włodzimierz M. Mikulski</i>	
On symmetrization of higher order jets	495
<i>Demeter Krupka, Zbyněk Urban, and Jana Volná</i>	
Variational projectors in fibred manifolds	503
<i>Jan Kurek and Włodzimierz M. Mikulski</i>	
The natural liftings of connections to tensor powers of the cotangent bundle	517
<i>Miroslav Kureš</i>	
Lagrangians on iterated tangent bundles and the Hamiltonization	525
<i>Erico Tanaka and Demeter Krupka</i>	
On the structure of Finsler and areal spaces	539

Jiří Tomáš	
Bundles of (p, A) -covelocities and (p, A) -jets	547
Olga Belova	
The third type bunch of connections induced by an analog of Norden's normalization for the Grassmann-like manifold of centered planes . .	557
Olena Chepurna and Irena Hinterleitner	
On the mobility degree of (pseudo-) Riemannian spaces with respect to concircular mappings	561
Hana Chudá and Mohsen Shiha	
Conformal holomorphically projective mappings satisfying a certain initial condition	569
Irena Hinterleitner and Josef Mikeš	
Geodesic mappings of (pseudo-) Riemannian manifolds preserve class of differentiability	575
Jaroslav Hrdina	
Geometry of almost Cliffordian manifolds: Nijenhuis tensor	583
Marek Jukl and Lenka Juklová	
On decomposition problems on manifolds with a special differential operator	591
Jury Shevchenko and Elena Skrydlova	
Reduction of the centroprojective connection of the projective group to the fundamental-group connection of a surface	601
Elena Stepanova and Marie Chodorová	
On the degree of geodesic mobility of Riemannian manifolds	609
Irina I. Tsyganok and Sergey E. Stepanov	
On complete totally umbilical and maximal space-like surfaces in pseudo- Riemannian manifolds	613
Alena Vanžurová	
On metrizability of a class of 2-manifolds with linear connection	621
Füsün Özen Zengin and Bahar Kirik	
Conformal mappings of nearly quasi-Einstein manifolds	629
Roman Cherniha	
Conditional symmetries for boundary value problems: new definition and its application for nonlinear problems with Neumann conditions . .	637
Markus Hunziker, Mark R. Sepanski, and Ronald J. Stanke	
Global Lie symmetries of a system of Schrödinger equations and the oscil- lator representation	647
Marek Z. Elżanowski and Serge Preston	
On continuously defective elastic crystals	659

<i>Radka Malíková</i>	
Galilei invariance of the Helmholtz morphism	671
<i>Petar Pavlov, Svetlana Lilkova-Markova, Borislav Nakov, and Janitza Kehajova</i>	
A geometric oriented approach in drawing the simulation model of small free angular vibrations of a body	679
<i>Samuli Pupponen and Jukka Tuomela</i>	
Configuration space analysis of rigid mechanisms	685
<i>Martin Swaczyna and Petr Volný</i>	
Geometric concept of isokinetic constraint for a system of particles	697
<i>Petr Vašík</i>	
Higher order connections on Lie groupoid: application to materials	705
<i>Jan Kubarski</i>	
Cyclic Čech–Hochschild bicomplex	713
<i>Alexander Zuevsky</i>	
Identities for modular forms generated by vertex algebras	721

AGMP 2012: PREFACE

MIROSLAV KUREŠ

It was in October 2007 that two representatives of the Brno school of differential geometry attended the *Algebra, Geometry and Mathematical Physics* (AGMP) conference, and I had the honour to be one of them. The venue of the then third year of the conference was Göteborg, Sweden where interesting topics such as deformation theory and quantization, Hopf algebras, quantum algebras, integrable systems, Lie theory and applications, and others appeared. Dubbed Baltic-Nordic Workshop, the conference also served as a meeting point with mathematicians and physicists from areas — somewhat illogically — distant to us. Cooperation was initiated and further annual conference at Tartu, Będlewo, Tjärnö, and Mulhouse were always attended by someone from Brno. This was also reflected in closer personal ties and, as an honour, we could then accept an offer to hold the eighth AGMP conference in Brno made to me and my colleagues, Petr Vašík and Jaroslav Hrdina, by the main organizer Prof. Eugen Paal from Technical University in Tallinn during our stay at Tallinn in 2011. Thus, the basis of an organising committee was formed.

We believe that trying to organize this year's conference with a touch of originality was a step in the right direction. The idea of our specific contribution to the conference was twofold. First to introduce to it Czech specialists in differential geometry, a field with a living tradition in the Czech Republic. Thus, thanks our obliging more noted colleagues, new names appeared at the conference such as Ivan Kolář, Josef Mikeš, Martin Markl, Demeter Krupka, Olga Rossi, and many other colleagues from abroad with long cooperation with Czech universities. Second, we wanted to enrich the conference thematically, too: we did this by increasing emphasis on jet spaces, Weil bundles, new geometric views at infinitesimals, quantum theory of strings, kinematic theory of defective crystals, and other topics.

The participants might want to recall — and other readers perhaps to learn about — the conference plenary lectures:

- ◇ Paolo Giordano: *Infinitesimals without logic*
- ◇ Natalia Iyudu: *Golod–Shafarevich algebras, NCCI and RCI*
- ◇ Ivan Kolář: *On the interrelations between jet spaces and Weil bundles*
- ◇ Hirokazu Nishimura: *Axiomatic differential geometry — towards model categories of differential geometry*
- ◇ Maria Clara Nucci: *Symmetries for thought*
- ◇ Gareth Parry: *Geometrical issues in the continuum mechanics of defective solid crystals*
- ◇ Ivo Sachs: *On homotopy algebras and quantum string field theory*
- ◇ Sergey Stepanov: *Tachibana numbers*
- ◇ Karl Strambach: *Geodesics and foundation of geometry*

Let us also list the specialised sections with parallel programmes:

- ◇ *Deformation theory and quantization*
- ◇ *Geometry of manifolds with connection*
- ◇ *Geometry in mechanics and thermodynamics*
- ◇ *Jordan algebras and applications*
- ◇ *Jet theory and Weil bundles*
- ◇ *Noncommutative algebra and geometry*
- ◇ *Operads, Feynman transform and string field theory*
- ◇ *Symmetry and integrability Variational geometry*

The conference was held from 12th to 14th September 2012 at Brno University of Technology. The organizing committee members included M. Kureš (Chair), J. Hrdina, E. Paal, P. Sehnalová, A. Stolin, P. Vašík, J. Zatočilová. Much support for the organizers came from Miroslav Doupovec, the Dean of the BUT Faculty of Mechanical Engineering, and Josef Šlapal, the Director of the Institute of Mathematics of the faculty. We would also like to express our gratitude to the city of Brno and the South Moravian Region for their help, too. Already during the conference, positive reactions could be observed to the somewhat non-traditional mix of the traditional topics of the AGMP series with those presented by the Czech participants: more discussions took place in the corridors than usual at events of this type with the organizers receiving positive responses to the programme structure. The total conference attendance was 111 participants from 28 countries. Knowing that it is only the time that will show whether the contacts initiated are going to last and result in valuable research cooperation, we still had an impression that the circumstances were favourable. Speaking about the accompanying events, a walk in downtown Brno should be mentioned even if spoiled by the rain. Another event of the social programme was a conference dinner at the *U královny Elišky* wine bar with a cimbalom band. We will be glad if the participants feel that the positive memories of the conference prevail hoping that any inconveniences will be forgiven and forgotten.

This extraordinary issue of *Miskolc Mathematical Notes* is a selection of the papers submitted based on careful reviews. The issue editorial board worked efficiently and it is my pleasant duty to thank all its members as listed in the imprint.

The order of papers in this issue is according to the thematic topic and is as follows.

- I. Main Contributions — Plenary Lectures pp. 407–486
- II. Jet Theory, Weil Bundles, Variational Geometry pp. 487–555
- III. Riemannian Geometry, Geometry of Manifolds
with Connection pp. 557–636
- IV. Symmetry and Integrability, Geometry of Mechanics
and Thermodynamics pp. 637–711
- V. Noncommutative Algebra and Geometry pp. 713–727

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IN MEMORY OF LEONID EVTUSHIK AND JAN KUBARSKI

JOSEF MIKEŠ, BOGDAN BALCERZAK, AND MIROSLAV KUREŠ

For the AGMP-8 Conference in Brno, we were trying to gather experts in differential geometry and colleagues from abroad having long-term cooperation with Czech universities. Among them were two professors, Leonid Evtushik and Jan Kubarski, both good friends of Czech mathematicians and the Czech Republic. They had even visited Brno before. We also planned for further cooperation with them, Leonid was to continue research cooperation with Josef Mikeš and Jan was to attend our Spring School of Invariant Theory in Písek (April 2013). Unfortunately, none of this had happened as both excellent mathematicians and genuine friends died unexpectedly.

The following lines in their memory should remind you of their great importance for mathematics — we think it notable that Brno was the venue of their last conference and, thus, their participation will stay forever in our memories.

LEONID E. EVTUSHIK, 1931–2013

We lost Leonid Evgenyevich Evtushik on 15th February 2013. We will miss this great scientist, honorary professor of the Lomonosov Moscow State University, and excellent man for a long time.

L. E. Evtushik was born on 25th May 1931 in Priluky in the centre of the Chernigov region in Ukraine. In 1949 he graduated there from the secondary school and passed entrance exam to the Faculty of Mechanics and Mathematics of the Lomonosov Moscow State University. Then, his scientific life was connected with this university:

In 1954, he received a diploma with distinction and began studying for a PhD degree at the Faculty of Mechanics and Mathematics. Mr. Evtushik also obtained the scientific degree of “Doctor of Science” being appointed professor at the Faculty of Mechanics and Mathematics of MSU. His scientific interests were formed by such excellent geometers as S.P. Finnikov, G.F. Laptev (his PhD supervisor), A.M. Vasilev, founder of the MSU oldest scientific seminar on classical and modern differential geometry; Prof. Evtushik actively participated in this seminar and gradually became their successor passing on their scientific tradition. He was interested especially in the theory of higher-order differential geometric structures, which describes invariant structures in mathematical analysis such as differential equations, operators, Lagrangians, Hamiltonians, etc. His scientific work was devoted to perfecting the

Cartan–Laptev method based on a theoretic categorical approach, which agrees well with the differential-algebraic apparatus of jets and external algebras.

In recent years, the main activities of L. E. Evtushik have concentrated on a large-scale general geometric problem – structuring and inclusive modern description of a generic Cartan–Laptev method and mission-critical applications to the invariant structures of analysis and geometry of arbitrarily high orders on smooth manifolds.

The main results of L. Evtushik are the following:

1. For multidimensional Lagrangians of the second order an invariant connection of the Cartan type was constructed in a special principal bundle with the complete system of structural equations and corresponding invariants,
2. A completely new detailed theory was developed of higher-order non-linear connections in an arbitrary principal and associated bundles; this theory has very interesting and promising specification for a natural category of principal bundles - frame bundles of higher orders on smooth manifolds,
3. On this basis, the most natural approach was developed to constructing the geometry of higher-order ordinary differential equations and its implementation for the third, fourth and fifth orders,
4. A method of constructing an invariant form and structural equations of pseudo-groups of transformations was given; it is more natural than the Cartan one,
5. In the monograph *Theory of differential geometric structures on smooth manifolds* (Problems of Geometry, 9, 199X), the current construction of the Cartan–Laptev method is given as a universal methodological platform for 1., 2., 3., 4.; this method is presented as an apparatus of structural differential forms and structural equations of differential geometric structures in their most general sense based on the theory of fiber bundles and calculus of invariant differential forms and jets of maps, as well as the theoretic categorical approach,
6. It was shown that a general theory of Lie differentiation in its invariant form is automatically built into the theory of connections and its structural equations; it represents a fruitful approach to the description of the invariant Hamiltonian systems for Lagrangians of higher order.

Besides L. E. Evtushik's individual research, we should also mention his scientific and organizational work of the whole scientific school, referred to as the Cartan–Laptev school, for the organization and functioning of which he was fully responsible. He also organized a large international conference at MSU dedicated to the 90th anniversary of L. F. Laptev, which was the beginning of regular Laptev Readings, held in various geometric centers of the country, with the participation of geometries from Italy, Yugoslavia, Hungary, the Czech Republic, Poland, Norway, and USA. The scientific achievements of L. E. Evtushik won great acclaim at the international congress on the occasion of the 200th anniversary of the birth of N. I. Lobachevsky, and in 2009, he was awarded the Anasilaus medal in Italy. His scientific activities were

supported by many grants and recalled at international mathematical congresses, geometric conferences (Zürich, Berlin, Vienna, Budapest, Brno, Olomouc, Prague, Belgrade, Warsaw) and numerous geometrical conferences in the country. He published over 100 publications on this topic and more than 20 PhD students defended their dissertations.

The teaching activities of Prof. Evtushik were no less diverse: a long-term chairman of the Methodical Commission of the All Union School of Mathematics at Moscow State University, a member of the Dissertation Council of the MSU and Moscow State Pedagogical University, part-time head of the department of computer science at the University of Tver and professor at the Penza Pedagogical University, a member of the Committee on Education and science of the State Duma of Russia, diverse teaching activities in geometry and mathematical analysis, supervision of PhD students. For his pedagogical gift, L. E. Evtushik was awarded a degree of Honorary Professor of Moscow State University.

In addition to research and teaching, Prof. Evtushik engaged in travel during large part of his life. The geography of his travels covers the entire Eurasian continent. From Spain and Portugal to China, from Norway to Sicily and Peloponnese. All the republics of the Soviet Union, from the Baltic and Ukraine to Azerbaijan and Kyrgyzstan. And of course the whole of Russia, from the mountains of Chechnya to the northern seas, from Kaliningrad (Königsberg) to Irkutsk and Vladivostok. And everywhere Leonid visited places connected with the history of mathematics and the lives of outstanding mathematicians. In Sicily, he found the tomb of Archimedes, in Norway, he travelled to places associated with Sophus Lie, in Germany, he visited the Klein memorial in Erlangen. But most of all he was attracted by places related to Elie Cartan. He visited twice the home of the outstanding mathematician in a small French town of Dolomieu (near Lyon). In Russia, he went to the homes of Chebyshev, Ostrogradsky and other mathematicians, who brought glory of the national science, and of course in Kazan' where N. I. Lobachevsky created his non-Euclidean world. Professor Evtushik travelled to the home of Michail Lomonosov, the founder of Moscow University, where he spent his entire scientific life.

We have lost a great scientist and teacher and a fine man.

JAN KUBARSKI, 1950–2013

Jan Kubarski was born on 26 December 1950 in Łódź, Poland. In 1969, after completing a Secondary Technical School of Mechanical Engineering No. 4 in Łódź (class in cinematic techniques), he entered the Faculty of Mathematics, Physics and Chemistry at the Łódź University, where he studied mathematics. After obtaining his degree, Jan Kubarski began his PhD studies at the Mathematical Institute of the Polish Academy of Sciences in Warsaw, where in 1977 he defended his PhD Thesis entitled *About Groupoids and Lie Algebroids* (in Polish) under the supervision of Professor Włodzimierz Waliszewski. At that time Lie groupoids and Lie algebroids

were hardly studied in Poland. It was Professor Ivan Kolář (Masaryk University, Brno) who introduced Jan Kubarski to the subject and who acted as an informal co-supervisor of the thesis. He received the habilitation degree in mathematics at the Łódź University in 1993 for his dissertation, *The Chern-Weil homomorphism of regular Lie algebroids*. On 21 December 2007, President of Poland Professor Lech Kaczyński conferred upon him the degree of professor of mathematical sciences.

In 1977, he started to work at the Institute of Mathematics of the Łódź University of Technology where, from 1997 until his death, he worked as an associate professor. In 2000–2002, he held the office of vice-dean for science of the Faculty of Technical Physics, Information Technology and Applied Mathematics, Łódź University of Technology.

He worked in differential geometry, with particular interest in characteristic classes in categories of Lie groupoids and Lie algebroids. The most valuable results obtained by Jan Kubarski are those that concern the primary and secondary characteristic classes, and also a paper on Bott’s vanishing theorem for Lie algebroids. His habilitation thesis was devoted to the construction and research of the Chern–Weil homomorphism for regular Lie algebroids. Some of his works are devoted to differential spaces and foliations with singularities in the sense of Stefan. His latest research mainly focused on the study of cyclic Čech–Hochschild bicomplexes.

Jan Kubarski published nearly 50 scientific research articles. He is the author of many reviews in Mathematical Reviews of the AMS and Zentralblatt MATH and co-author of Encyclopedia of Mathematics, famous in Poland (*Encyklopedia szkolna. Matematyka*, in Polish).

He actively participated in many international conferences delivering lectures, plenary lectures, seminars and training sessions. He was a co-organizer of nine conferences in the *Geometry and Topology of Manifolds* cycle and conferences such as: *Forum of Geometry* (Warsaw, 2000), *Lie Algebroids and Related Topics in Differential Geometry* (Warsaw, 2000), *C*-algebras and Elliptic Theory. III* (Będlewo, Poland, 2009), *Geometry of Manifolds and Mathematical Physics* (Kraków, 2011). He participated in editing several volumes of the monograph conference proceedings in such journals as *Universitatis Iagellonicae Acta Mathematica*, *Banach Center Publications*, *Central European Journal of Mathematics*, and *Travaux Mathématiques*. He maintained scientific contacts with research centres in Poland and in the world — in particular with Masaryk University in Brno (Ivan Kolář), Moscow State University (Alexander S. Mishchenko), University of Paul Sabatier in Toulouse (Jean Pradines), National Technical University of Ukraine, “Kiev Polytechnic Institute” (Roman Kadobianski) and Marche Polytechnic University in Ancona (Nicolae Teleman). He supervised three PhD students in mathematics.

Jan Kubarski was an enthusiast of computer technology and a great fan of photography, travel, history of Poland, and ancient history. In particular, he was interested

in the history of the beginnings of mathematics in the great civilizations of the Bronze Age and Iron Ages.

He died unexpectedly on 8th June 2013 in Łódź.

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FERMAT REALS: INFINITESIMALS WITHOUT LOGIC

PAOLO GIORDANO

Abstract. We review the theory of Fermat reals and Fermat extensions, a relatively new theory of nilpotent infinitesimals which does not need any background in Mathematical Logic. We focus on some differences from Nonstandard Analysis and Synthetic Differential Geometry using the viewpoint of intuitive interpretation and applicability in Physics. Finally, we state some open problems.

2000 *Mathematics Subject Classification:* Primary 14B10; Secondary 26E30, 37P20, 03H10

Keywords: infinitesimals methods, non-Archimedean mathematics, superreal rings, Fermat reals

1. IMAGINE A STUDENT. . .

Let me start with a short, fictitious, but hopefully meaningful story. Imagine a student taking first year courses in Physics and Calculus. In Calculus she would see something like

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x : \left| \frac{x}{c} \right| < \delta \implies \left| \frac{\frac{1}{\sqrt{1-\frac{x^2}{c^2}}} - 1 - \frac{x^2}{2c^2}}{x^2} c^2 \right| < \varepsilon.$$

However, in Physics she would frequently work with formulas similar to Einstein's formulas

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1 + \frac{v^2}{2c^2}, \quad \sqrt{1-h_{44}(x)} = 1 - \frac{1}{2}h_{44}(x) \quad (1.1)$$

$$f(x, t + \tau) = f(x, t) + \tau \cdot \frac{\partial f}{\partial t}(x, t), \quad (1.2)$$

that explicitly use infinitesimals like $v/c \ll 1$, $\tau \ll 1$ or $h_{44}(x) \ll 1$, such that, e. g., $h_{44}(x)^2 = 0$, see, e. g., [6] and [5]. Of course, our student asks herself whether the two lecturers use the same ring of scalars or not. She senses that there are inconsistencies, and she can hence try to search for a solution to this clash of methodologies.

P. Giordano has been supported by Grants M1247-N13 and P25116-N25 of the Austrian Science Fund FWF.

However, some students frequently adopt solutions that refer to the fact that different courses require different mentalities, and hence different answers according to different professors. So, let us assume that our student starts to search on the Internet (and also in the library...) for a method to address this *inconsistency between intuition and formal Mathematics*. After a few steps, she finds that probably the difficulties can be resolved studying Nonstandard Analysis (NSA, [1, 17, 23]) and Synthetic Differential Geometry (SDG, [2, 20–22]). Our student begins to study these wonderful and formally powerful theories, but since she has a great critical mind, she quickly finds that these theories are not as intuitively clear as expected:

NSA: Let $I \in {}^*\mathbb{R}$ be an infinite number, where ${}^*\mathbb{R} \supseteq \mathbb{R}$ is the totally ordered field of NSA. Therefore, we have $|I| > n$ for all $n \in \mathbb{N}$. In NSA it is very easy to prove that every finite number $r \in {}^*\mathbb{R}$, i.e. such that r is not infinite, is infinitely close to a standard real number ${}^\circ r \in \mathbb{R}$, called its *standard part*. The property of being infinitely close is denoted by $r \simeq {}^\circ r$ and indicates that $r - {}^\circ r$ is infinitesimal, i.e. $|r - {}^\circ r| < \frac{1}{n}$ for all $n \in \mathbb{N}_{\neq 0}$. It is also very easy to prove that $-1 \leq \sin(I) \leq 1$ is a finite hyperreal, so that we get $\sin(I) \simeq {}^\circ \sin(I)$. What is the intuitive meaning of this real number ${}^\circ \sin(I)$? It could be questioned whether this is an objection to NSA, or in fact to standard real analysis based as it is on infinitary hypotheses common in Zermelo-Fraenkel set theory. This is because the same question can be asked of a real number I which is so large as to be inexpressible by even a computer the size of the universe in the span of the total time allotted to our civilization. If one cannot, even in principle, express such a real number I , what could possibly be the intuitive meaning of the value of the sine function at I ? Does the usual geometrical interpretation with the unit circle of $\sin(I) = \sin(J)$, $J = I \bmod 2\pi$, work in this case?

NSA: Similarly to the extension $\mathbb{R} \mapsto {}^*\mathbb{R}$, every subset $X \subseteq \mathbb{R}$ can be extended to ${}^*X \subseteq {}^*\mathbb{R}$. This extension operation has wonderful formal properties, since it preserves all the logical operations. Let $e \in {}^*\mathbb{R}_{\neq 0}$ be a non zero infinitesimal, and $[x] \in {}^*\mathbb{N}$ be the integer part of $x \in {}^*\mathbb{R}$. Then, it is not hard to prove (see, e.g., [4, 24] and references therein) that $\mathcal{U} := \{X \subseteq \mathbb{N} \mid [\frac{1}{e}] \in {}^*X\}$ is an ultrafilter on $\mathbb{N}$, and using $\mathcal{U}$ it is possible to construct a non-measurable set without using the axiom of choice. Connes [3] expressed the opinion that the possibility of easily producing a non-measurable set by every non zero infinitesimal shows that every example of infinitesimal in NSA is not “completely knowable”. On the other hand, this possibility is used by Tao [25] to prove the existence of non-measurable sets. The meaning of Connes’ criticism to NSA is hence largely debated; for a detailed analysis see the recent paper [19].

NSA: The hyperreal field ${}^*\mathbb{R}$ can be defined as a quotient field $\mathbb{R}^{\mathbb{N}} / \sim$, where two sequences $(x_n)_{n \in \mathbb{N}} \sim (y_n)_{n \in \mathbb{N}}$ are intuitively said to be equivalent if $x_n =$

y_n on a large set of indexes $n \in \mathbb{N}$. To formalize this notion of “large subset of natural numbers”, one can use the notion of ultrafilter $\mathcal{U}$ by saying that A is large if $A \in \mathcal{U}$, where $A \subseteq \mathbb{N}$. Let

$$P(A) := \lim_{n \rightarrow +\infty} \frac{\text{card}(A_{\leq n})}{n+1}$$

be the density of A (in the case that this limit exists), see, e. g., [26]. The density $P(A)$ is usually interpreted as the finitely additive uniform probability of picking a number from A . It is not hard to prove that

$$\forall \varepsilon > 0 \exists S \in \mathcal{U} : P(S) < \varepsilon.$$

Using this set of indexes $S \subseteq \mathbb{N}$ and any hyperreal $x = [(x_n)_{n \in \mathbb{N}}]_{\sim} \in {}^*\mathbb{R}$, we can easily construct another sequence $(y_n)_{n \in \mathbb{N}}$ generating the same number, i. e. such that $x = [(y_n)_{n \in \mathbb{N}}]_{\sim}$, but such that $P[x_n = y_n] < 10^{-100}$. See [7, 10].

NSA: Even if we consider a hyperreal $x = [(x_n)_{n \in \mathbb{N}}]_{\sim} \in {}^*\mathbb{R}$ infinitely close to a standard real number $x \simeq r \in \mathbb{R}$, in general, it is not possible to prove that $\lim_{n \rightarrow +\infty} x_n = r$. Only if ${}^*\mathbb{R}$ is defined using a particular type of ultrafilter, whose existence follows, for example, by assuming the continuum hypothesis, we can prove that $x_n \rightarrow r$ along a subsequence $n \in S \in \mathcal{U}$. In any event, even with the better case we can define an infinitesimal hyperreal $x = [(x_n)_{n \in \mathbb{N}}]_{\sim}$ that goes to zero along $S \in \mathcal{U}$, but goes to $+\infty$ for $n \in \mathbb{N} \setminus S$ and such that $P(S) < 10^{-100}$ whereas $P(\mathbb{N} \setminus S) > 1 - 10^{-100}$. See [15].

SDG: In SDG we have nilpotent infinitesimals, i. e. numbers of a ring $h \in R$ such that $h^n = 0$ for some $n \in \mathbb{N}$. Therefore, we do not have a field, but this lacking is necessary for formulas such as (1.2). The order relation in the ring of scalars R is only a partial order and not a total order, so that one cannot deduce $x = y$ from $x \leq y$ and $y \leq x$. Moreover, in SDG for every infinitesimal h both $h \geq 0$ and $h \leq 0$ hold. Our student does not understand how to reconcile this order property with the tens of intuitive drawings of infinitesimal objects she must reproduce in the study of Physics, in particular since the length of every infinitesimal segment is positive and not negative.

SDG: SDG is so beautiful and powerful that it permits the development of several differential geometry topics in a cartesian closed framework, i. e., not only for ordinary smooth manifolds but also for infinite dimensional spaces such as the space of all smooth functions between two manifolds. On the other hand, it seems impossible to consider a physical theory dealing with a physical infinitesimal constant taken from the ring R of SDG (e. g., Planck’s constant to study the relationships between Quantum and Classical Mechanics). Indeed, in this theory it is only possible to prove that

$$\neg \neg \exists h \in R : h \simeq 0,$$

and this does not imply that infinitesimal numbers exist in the ring R , because SDG is incompatible with classical logic and admits models only in intuitionistic logic, in which the logic law $\neg\neg\mathcal{A} \Rightarrow \mathcal{A}$ does not hold. These models must be constructed using a non trivial amount of Topos theory which has been judged by the best researchers in SDG as “more complicated” as compared to other cartesian closed theories of manifolds. See [22].

Our imaginary student is even more confused. These are surely the most beautiful theories of infinitesimals currently available in mathematics. Surely they are the most formally powerful ones. However, shouldn't it be possible to develop another construction, perhaps less powerful, but simpler and *always intuitively clear*? Wouldn't it be possible to define a ring of numbers with infinitesimals within “standard mathematics,” without it being necessary to have a deep background in Mathematical Logic?

Since Einstein wrote (1.2) exactly using an equality sign, if we want to be faithful to his writings and intuition, we are forced to consider a ring with nilpotent infinitesimals (let $f(x, t) = t^2$ at $t = 0$ in (1.2) to deduce that, necessarily, $\tau^2 = 0$). Can we have a theory similar to SDG, but always intuitively clear, that is compatible with classical logic and deals with easier models? Can we start from an ordinary smooth manifold M and extend it with something like $\bullet M \supseteq M$, similar to what we can do in NSA, but with the addition of “nilpotent infinitely close new points”? Our final aim would be to develop Differential Geometry also based on infinite dimensional spaces like in SDG, so we can obtain a theory with new results and not only useful and more elegant reformulations of well-known classical theorems. If our student's passion is still alive, after all these questions, the best we can hope for is that she try her own solution. Indeed, this is a natural step, since two different languages ($\varepsilon - \delta$ and informal infinitesimals) which describe a sufficiently large part of nature must be strongly related to each other. Using only elementary analysis, after a couple of decades and with the usual small amount of inspiration and a large amount of perspiration, she was successful in creating a new useful theory.

Our student called the new ring of scalars $\bullet\mathbb{R}$ *the ring of Fermat reals*, because “Fermat would surely have liked it”. For more formal motivations concerning this name, see [11, 12].

2. THE RING OF FERMAT REALS

We start from the idea that a smooth ($\mathcal{C}^\infty$) function $f : \bullet\mathbb{R} \rightarrow \bullet\mathbb{R}$, where $\bullet\mathbb{R} \supseteq \mathbb{R}$ is our new ring we have still to define, is actually *equal* to its tangent straight line in the first order neighborhood, e. g., of the point $x = 0$. Formally, we wish to write

$$\forall h \in D : f(h) = f(0) + h \cdot f'(0) \quad (2.1)$$

where D is the subset of $\bullet\mathbb{R}$ which defines the above-mentioned neighborhood of $x = 0$. The equality (2.1) can be seen as a first-order Taylor's formula without remainder because, intuitively, we think that $h^2 = 0$ for any $h \in D$ (indeed the

property $h^2 = 0$ defines the first order neighborhood of $x = 0$ in $\bullet\mathbb{R}$. These almost trivial considerations lead us to understand many things: $\bullet\mathbb{R}$ must necessarily be a ring and not a field because in a field the equation $h^2 = 0$ implies $h = 0$; moreover we will surely have some limitation in the extension of some function from $\mathbb{R}$ to $\bullet\mathbb{R}$. For example, we will necessarily have limitations in extending the square root because, using this function with the usual properties, the equation $h^2 = 0$ implies $|h| = 0$. On the other hand, we are also led to ask whether (2.1) uniquely determines the derivative $f'(0)$. Indeed, even if it is true that we cannot simplify by h , we know that the polynomial coefficients of a Taylor's formula are unique in classical analysis. In fact, we will prove that

$$\exists! m \in \mathbb{R} \quad \forall h \in D : f(h) = f(0) + h \cdot m, \quad (2.2)$$

that is, the slope of the tangent is uniquely determined if it is an ordinary real number. We will call formulas like (2.2) *derivation formulas*.

If we try to construct a model for the formula (2.2), a natural idea is to think our new numbers in $\bullet\mathbb{R}$ as equivalence classes $[h]$ of usual functions $h : \mathbb{R} \rightarrow \mathbb{R}$. In this way, we may hope both to include the real field using classes generated by constant functions, and that the class generated by $h(t) = t$ could be a first order infinitesimal number.

Remark 1. Sometimes, but not always, we will use a notation like $h_t := h(t)$ for real functions of the real variable t . This makes it possible to decrease the number of parenthesis used in formulas and to leave the classical notation $f(x)$ for functions of the form $f : \bullet\mathbb{R} \rightarrow \bullet\mathbb{R}$.

To understand how to define this equivalence relation, we have to think of (2.1) in the following sense:

$$f(h_t) \sim f(0) + h_t \cdot f'(0), \quad (2.3)$$

where the idea is that we are going to define $\sim$. If we think h_t “sufficiently similar to t ”, we can define $\sim$ so that (2.3) is equivalent to

$$\lim_{t \rightarrow 0^+} \frac{f(h_t) - f(0) - h_t \cdot f'(0)}{t} = 0,$$

that is,

$$x \sim y : \iff \lim_{t \rightarrow 0^+} \frac{x_t - y_t}{t} = 0. \quad (2.4)$$

In this way, (2.3) is very near to the definition of differentiability for f at 0.

It is important to note that, because of de L'Hôpital's theorem, we have the isomorphism

$$\mathcal{C}^1(\mathbb{R}, \mathbb{R}) / \sim \simeq \mathbb{R}[x] / (x),$$

the left hand side is (isomorphic to) the usual tangent bundle of $\mathbb{R}$ and thus we obtain nothing new. It is not easy to understand what set of functions we have to choose for

x, y in (2.4) so as to obtain a non trivial structure. The first idea is to take continuous functions at $t = 0$, instead of more regular ones like $\mathcal{C}^1$ -functions. In this way, we see that, e. g., $h_k(t) = |t|^{1/k}$ becomes a k -th order nilpotent infinitesimal because $h^{k+1} \sim 0$. For almost all the results presented in this paper, continuous functions at $t = 0$ work well. However, only in proving the non-trivial property

$$(\forall x \in {}^\bullet\mathbb{R} : x \cdot f(x) = 0) \implies \forall x \in {}^\bullet\mathbb{R} : f(x) = 0 \quad (2.5)$$

we can see that it does not suffice to take continuous functions at $t = 0$. Another problem, necessarily connected with the basic idea (2.1), is that the use of nilpotent infinitesimals frequently leads to considering terms like $h_1^{i_1} \cdot \dots \cdot h_n^{i_n}$. For this type of products, the first problem is to know whether $h_1^{i_1} \cdot \dots \cdot h_n^{i_n} \neq 0$ and what is the order k of this new infinitesimals, that is, for what k we have $(h_1^{i_1} \cdot \dots \cdot h_n^{i_n})^k \neq 0$ but $(h_1^{i_1} \cdot \dots \cdot h_n^{i_n})^{k+1} = 0$. We will have a good frame if we are able to solve these problems starting from the order of each infinitesimal h_j and from the values of the powers $i_j \in \mathbb{N}$. On the other hand, almost all the examples of nilpotent infinitesimals are sums of terms of the form $h(t) = t^\alpha$, with $0 < \alpha < 1$. These functions also have very good properties in dealing with products of powers. It is for these reasons that we shall focus our attention on the following family of functions $x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ in the definition (2.4) of $\sim$.

Definition 2. We say that x is a *little-oh polynomial*, and we write $x \in \mathbb{R}_o[t]$ iff

- (1) $x : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$
- (2) We can write

$$x(t) = r + \sum_{i=1}^k \alpha_i \cdot t^{a_i} + o(t) \quad \text{as } t \rightarrow 0^+$$

for suitable

$$\begin{aligned} k &\in \mathbb{N} \\ r, \alpha_1, \dots, \alpha_k &\in \mathbb{R} \\ a_1, \dots, a_k &\in \mathbb{R}_{\geq 0}. \end{aligned}$$

Hence, a little-oh polynomial $x \in \mathbb{R}_o[t]$ is a polynomial function with real coefficients, in the real variable $t \geq 0$, with generic positive powers of t , and up to a little-oh function as $t \rightarrow 0^+$. Simple examples of little-oh polynomials are the following: $x(t) = 1 + t + t^{1/2} + t^{1/3} + o(t)$ and $x(t) = r + o(t)$.

Definition 3. Let $x, y \in \mathbb{R}_o[t]$. Then we say that $x \sim y$ or that $x = y$ in ${}^\bullet\mathbb{R}$ iff $x_t = y_t + o(t)$ as $t \rightarrow 0^+$. Because it is easy to prove that $\sim$ is an equivalence relation, we can define the quotient ring ${}^\bullet\mathbb{R} := \mathbb{R}_o[t] / \sim$, where in $\mathbb{R}_o[t]$ we consider the pointwise ring operations. We will use the notation $x = [x_t] \in {}^\bullet\mathbb{R}$ for the equivalence class generated by the little-oh polynomial $t \in \mathbb{R}_{\geq 0} \mapsto x_t \in \mathbb{R}$. Moreover:

- (1) We define the standard part map as ${}^\circ(-) : x \in {}^\bullet\mathbb{R} \mapsto {}^\circ x = x(0) \in \mathbb{R}$
- (2) $dt_a := [t^{\frac{1}{a}}] \in {}^\bullet\mathbb{R}$ for all $a \in \mathbb{R}_{\geq 0}$.
- (3) Let $x = [x_t], y = [y_t] \in {}^\bullet\mathbb{R}$. Then we say that $x < y$ iff $x \neq y$ and there exists $z \in \mathbb{R}_o[t]$ such that

$$\exists \delta \in \mathbb{R}_{>0} \forall t \in (0, \delta] : x_t \leq y_t + z_t$$

$$z_t = o(t) \quad \text{as } t \rightarrow 0^+.$$

- (4) Let $A \subseteq \mathbb{R}^n$ be an open subset, $f \in \mathcal{C}^\infty(A, \mathbb{R})$ a smooth function. Define ${}^\bullet A$ like in (2.8) (or, equivalently, as ${}^\bullet A = A_o[t]/\sim$, where $A_o[t]$ is the set of little-oh polynomials taking values in A). For $x \in {}^\bullet A$ define

$${}^\bullet f(x) := [f(x_t)] \in {}^\bullet\mathbb{R}.$$

Using these definitions, we have all we need to characterize rings like ${}^\bullet\mathbb{R}$ using the following axioms.

Axiom 1 (decomposition). *${}^\bullet\mathbb{R}$ is a commutative ring with unity.*

Every Fermat real $x \in {}^\bullet\mathbb{R}$ can be written, in a unique way, as

$$x = {}^\circ x + \sum_{i=1}^N \alpha_i \cdot dt_{a_i}, \quad (2.6)$$

where ${}^\circ x, \alpha_i, a_i \in \mathbb{R}$ are standard reals, $a_1 > a_2 > \dots > a_N \geq 1, \alpha_i \neq 0$. The term ${}^\circ x \in \mathbb{R}$ is called *standard part* of x , and $\alpha_i =: {}^\circ x_i$ its i -th standard part. Vice versa, any writing of the type (2.6), which is called the *decomposition* of x , gives a Fermat real, so that e.g. ${}^\bullet\mathbb{R} \supseteq \mathbb{R}$ and ${}^\circ r = r$ for all $r \in \mathbb{R}$.

Axiom 2 (base infinitesimals). *The terms dt_a verify the following properties*

$$\begin{aligned} dt_a \cdot dt_b &= dt_{\frac{ab}{a+b}} \\ (dt_a)^p &= dt_{\frac{a}{p}} \quad \forall p \in \mathbb{R}_{\geq 1} \\ dt_a &= 0 \quad \forall a \in \mathbb{R}_{<1}. \end{aligned} \quad (2.7)$$

Therefore, among Fermat reals we also have nilpotent infinitesimals, like $x = 3 dt_2$, since $x^3 = 27 dt_{\frac{2}{3}} = 0$. These are exactly the same type of infinitesimals used by Einstein in formulas like (1.1). We will simply use the symbol dt for dt_1 . Intuitively speaking, looking at (2.7), we can also intuitively say that the greater is a and the greater is the nilpotent infinitesimal dt_a . Our axiom on the total order relation $<$ will confirm this intuition. This relation should not be confused with the following notion.

Axiom 3 (order of infinitesimals). *The order $\omega(x) =: a_1$ (see (2.6)) can be interpreted as the leading term in the decomposition and hence it has the following*

expected properties

$$\omega(x + y) \leq \max[\omega(x), \omega(y)]$$

$$\frac{1}{\omega(x \cdot y)} = \frac{1}{\omega(x)} + \frac{1}{\omega(y)},$$

whenever x, y are infinitesimals such that $x + y \neq 0$, respectively $x \cdot y \neq 0$.

In the decomposition (2.6), the term $\alpha_i =: \omega_i(x)$ will be called the i -th order of x . Directly from (2.6) it is not hard to prove that if $k \in \mathbb{N}_{>1}$, then $x^k = 0$ if and only if $\omega(x) < k$. Nilpotent Fermat reals can be thought of as non zero numbers which are so small that a suitable power of them gives zero.

Axiom 4 (ideals of infinitesimals). For $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, the set

$$D_a := \{x \in {}^\bullet\mathbb{R} \mid {}^\circ x = 0, \omega(x) < a + 1\}$$

is an ideal. Moreover, for $k \in \mathbb{N}_{\geq 1}$, we have $D_k = \{x \in {}^\bullet\mathbb{R} \mid x^{k+1} = 0\}$.

We will simply use D for D_1 . The ideal D_k is, therefore, a perfect candidate for the k -th order infinitesimal neighborhood of zero, with no k -th order Taylor formula having a remainder, since $x^{k+1} = 0$. This is indeed the subject of the next

Axiom 5 (Taylor formulas). Set ${}^\bullet\mathbb{R}^d := {}^\bullet\mathbb{R} \times \dots^d \times {}^\bullet\mathbb{R}$, then every ordinary smooth function $f \in \mathcal{C}^\infty(A, \mathbb{R})$ defined on an open set $A \subseteq \mathbb{R}^d$ can be extended to the set

$${}^\bullet A := \{x \in {}^\bullet\mathbb{R}^d \mid {}^\circ x \in A\}, \quad (2.8)$$

$${}^\bullet f : {}^\bullet A \rightarrow {}^\bullet\mathbb{R},$$

obviously obtaining a true extension, i. e., ${}^\bullet f(x) = f(x)$ if $x \in A$. Moreover, the following Taylor formula

$$\forall h \in D_k^d : {}^\bullet f(x + h) = \sum_{\substack{j \in \mathbb{N}^d \\ |j| \leq k}} \frac{h^j}{j!} \cdot \frac{\partial^{|j|} f}{\partial x^j}(x) \quad (2.9)$$

holds, where $x \in A$ is a standard point, and $D_k^d = D_k \times \dots^d \times D_k$.

Therefore, smooth functions become exactly equal to polynomials of degree k in the infinitesimal k -th order neighborhood $x + D_k$. In particular, $f(x + h) = f(x) + h \cdot f'(x)$ for $h \in D$, i. e. every smooth function is equal to its tangent line in a first order infinitesimal neighborhood. Einstein's formulas (1.1) are particular cases of this infinitesimal Taylor formula.

Axiom 6 (cancellation laws). Let $h_1, \dots, h_n \in D_\infty$, $i_1, \dots, i_n \in \mathbb{N}$, $x \in {}^\bullet\mathbb{R}$, then we have

$$(1) \quad h_1^{i_1} \cdot \dots \cdot h_n^{i_n} = 0 \text{ if and only if } \sum_{k=1}^n \frac{i_k}{\omega(h_k)} > 1.$$

- (2) x is invertible if and only if ${}^\circ x \neq 0$.
- (3) If $x \cdot r = x \cdot s$ in ${}^\bullet\mathbb{R}$, where $r, s \in \mathbb{R}$ and $x \neq 0$, then $r = s$.

If you are scared of working in a ring instead of a field, these laws allow for efficient work with this type of infinitesimals. If x is invertible, and proceeding like in the case of formal power series, it is not hard to prove that

$$\frac{1}{x} = \frac{1}{{}^\circ x} \cdot \sum_{j=0}^{+\infty} (-1)^j \cdot \left(\sum_{i=1}^N \frac{\alpha_i}{{}^\circ x} \cdot dt_{a_i} \right),$$

where the series is really a finite sum due to nilpotency.

Axiom 7 (total order relation). *The ring of Fermat reals is totally ordered by the relation $<$. This relation verifies the following properties: let $x, y \in {}^\bullet\mathbb{R}$, if ${}^\circ x \neq {}^\circ y$, then*

$$x < y \iff {}^\circ x < {}^\circ y.$$

Vice versa, if ${}^\circ x = {}^\circ y$, then

- (1) If $\omega(x) > \omega(y)$, then $x > y$ if and only if ${}^\circ x_1 > 0$.
- (2) If $\omega(x) = \omega(y)$, then

$$\begin{aligned} {}^\circ x_1 > {}^\circ y_1 &\implies x > y \\ {}^\circ x_1 < {}^\circ y_1 &\implies x < y. \end{aligned}$$

The axiom gives an effective criterion to decide whether $x < y$ or not. Indeed:

- (1) first of all $x < y$ is equivalent to $0 < y - x$, so we can describe the algorithm for the case $0 < x$, $x \in {}^\bullet\mathbb{R} \setminus \mathbb{R}$ only (from the first part of the previous axiom it follows that $<$ extends the usual order relation on $\mathbb{R}$). If the standard part ${}^\circ x \neq 0$, then the order relation can be decided on the basis of this standard part only. e. g. $2 + dt_2 > 0$ and $1 + dt_2 < 3 + dt$.
- (2) Otherwise, if the standard part ${}^\circ x = 0$, we look at the order $\omega(x)$ and at the first standard part ${}^\circ x_1$, which is the coefficient of the biggest infinitesimals in the decompositions of x : because $\omega(x) > \omega(0) = 0$, we have $x > 0$ iff ${}^\circ x_1 > 0$. e. g. $3 dt_2 > 0$; $dt_2 > a dt$ for every $a \in \mathbb{R}$; $0 < dt < dt_2 < dt_3 < \dots < dt_k$ for every k .

For a proof that these axioms characterize the structure $({}^\bullet\mathbb{R}, +, \cdot, <, {}^\circ(-), dt_{(-)})$ up to isomorphisms of ordered rings, see [8]. For a proof that indeed in the ring ${}^\bullet\mathbb{R}$ these axioms hold, see [11, 12].

More advanced axioms are needed to deal with (quasi-standard) smooth functions that are more general than extension ${}^\bullet f$ of standard smooth functions f , e. g. like the very simple $g(x) = x + dt$; see [10, 12] for more details. Clearly, we can define the absolute value, powers and logarithms of invertible Fermat reals and generalize their usual properties, [12]. We can also define meaningful metrics on ${}^\bullet\mathbb{R}$ and roots of (nilpotent!) infinitesimals, and prove applications to fractional derivatives, [16].

3. GEOMETRICAL REPRESENTATION

In our introductory fictitious story, our student was in search of a new theory which, on the one hand, can resolve the inconsistency between the intuition of the Physics' course and the formal Mathematics of the Calculus course. On the other hand, she was searching for a theory which is always intuitively clear. We can say that our student is looking for a theory which can keep a good dialectic between provable formal properties and their intuitive meanings. In this direction, we can see the possibility to find a geometrical representation of Fermat reals.

The idea is that, to any Fermat real $x \in {}^\bullet\mathbb{R}$, we can associate the function

$$t \in \mathbb{R}_{\geq 0} \mapsto {}^\circ x + \sum_{i=1}^N {}^\circ x_i \cdot t^{1/\omega_i(x)} \in \mathbb{R} \quad (3.1)$$

where N is, of course, the number of addends in the decomposition of x . Therefore, a geometric representation of this function is also a geometric representation of the number x because different Fermat reals have different decompositions, see Axiom 1 (decomposition). equality in ${}^\bullet\mathbb{R}$ depends only on the germ generated by each little-oh polynomial (see Definition 3), we can represent each $x \in {}^\bullet\mathbb{R}$ using only the first small part of the function (3.1).

Definition 4. If $x \in {}^\bullet\mathbb{R}$ and $\delta \in \mathbb{R}_{>0}$, then

$$\text{graph}_\delta(x) := \left\{ \left({}^\circ x + \sum_{i=1}^N {}^\circ x_i \cdot t^{1/\omega_i(x)}, t \right) \mid 0 \leq t < \delta \right\}$$

where N is the number of addends in the decomposition of x .

Note that the value of the function is placed in the abscissa position so that the correct representation of $\text{graph}_\delta(x)$ is given by Figure 3.1.

This interchange of abscissa and ordinate in the $\text{graph}_\delta(x)$ makes it possible to represent this graph as a line tangent to the classical straight line $\mathbb{R}$ and hence to have a better graphical picture. Finally, note that if $x \in \mathbb{R}$ is a standard real, then $N = 0$ and the $\text{graph}_\delta(x)$ is a vertical line passing through ${}^\circ x = x$.

The following theorem makes it possible to represent the Fermat reals geometrically.

Theorem 5. If $\delta \in \mathbb{R}_{>0}$, then the function

$$x \in {}^\bullet\mathbb{R} \longmapsto \text{graph}_\delta(x) \subset \mathbb{R}^2$$

is injective. Moreover if $x, y \in {}^\bullet\mathbb{R}$, then we can find $\delta \in \mathbb{R}_{>0}$ (depending on x and y) such that $x < y$ if and only if

$$\forall p, q, t : (p, t) \in \text{graph}_\delta(x), (q, t) \in \text{graph}_\delta(y) \implies p < q. \quad (3.2)$$

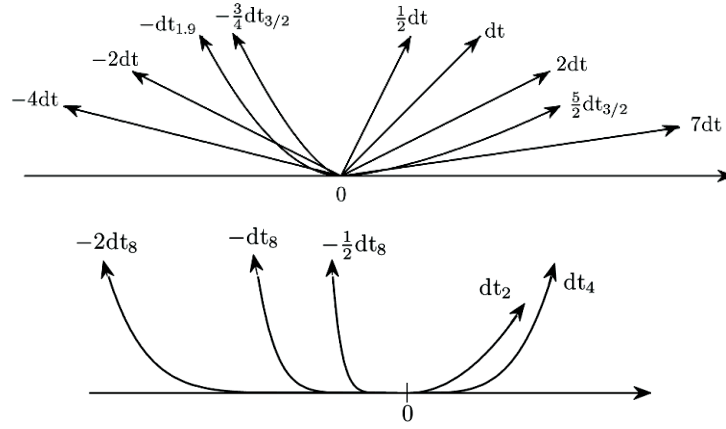
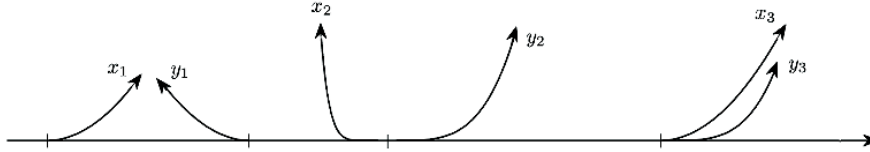


FIGURE 3.1. Some first and higher order infinitesimals

For a proof of this theorem, see [11]. See Figure 3.2 for the meaning of condition (3.2).

FIGURE 3.2. Different cases in which $x_i < y_i$

4. COMPUTER IMPLEMENTATION

The definition of the ring of Fermat reals is highly constructive. Therefore, using object oriented programming, it is not hard to write a computer code corresponding to ${}^\bullet\mathbb{R}$. We implemented a first version of this software using Matlab R2010b.

The constructor of a Fermat real is $x = \text{FermatReal}(s, w, r)$, where s is the $n + 1$ double vector of standard parts ($s(1)$ is the standard part ${}^\circ x$) and w is the double vector of orders ($w(1)$ is the order $\omega(x)$ if $x \in {}^\bullet\mathbb{R} \setminus \mathbb{R}$, otherwise $w = []$ is the empty vector). The last input r is a logical variable and assumes the value `true` if we want that the display of the number x is implemented using the Matlab `rats` function for both its standard parts and orders. In this way, the number will be displayed using continued fraction approximations and, therefore, in many cases, the calculations will be exact. These inputs are the basic methods of every Fermat real,

and can be accessed using the `subhref`, and `subasgn`, notations `x.stdParts`, `x.orders`, `x.rats`. The function `w=orders(x)` gives exactly the double vector `x.orders` if $x \in \bullet\mathbb{R} \setminus \mathbb{R}$ and 0 otherwise.

The function `dt(a)`, where a is a double, constructs the Fermat real dt_a . Because we have overloaded all the algebraic operations, like $x+y$, $x*y$, $x-y$, $-x$, $x==y$, $x\sim y$, $x<y$, $x\leq y$, $x\wedge y$, we can define a Fermat real, e.g., using an expression of the form $x=2+3*dt(2)-1/3*dt(1)$, which corresponds to

```
x=FermatReal([2 3 -1/3],[2 1],true).
```

We have also implemented the function `y=decomposition(x)`, which gives the decomposition of the Fermat real x , `abs(x)`, `log(x)`, `exp(x)`, `isreal(x)`, `isinfinesimal(x)`, `isinvertible(x)`.

The ratio x/y has been implemented for x and y infinitesimals and $y\sim 0$, or in case y is invertible. Finally, the function `y=ext(f,x)`, corresponds to $\bullet f(x)$ and has been implemented using the evaluation of the symbolic Taylor formula of the inline function f .

The functions x^p , `sqrt(x)` and `nthroot(x,n)` have been implemented both for x infinitesimal or invertible using the formulas derived in [16].

Using these tools, we can easily find, e.g., that

$$\frac{\sin(\sqrt{dt_3 + 2 dt_2})}{\cos(\sqrt[3]{-4 dt})} = dt_6 + dt_3 - \frac{2}{3} dt_2 + \frac{1096}{2787} dt_{\frac{6}{5}} + \frac{1234}{913} dt.$$

The Matlab source code is freely available under open-source license, and can be requested from the author of the present paper.

5. APPLICATIONS AND DEVELOPMENTS

The simplicity of the ring of Fermat reals permits an easy extension of this approach along different directions.

We can see many informal uses of infinitesimals in Physics with this new point of view. Frequently, these informal calculations can be faithfully repeated using $\bullet\mathbb{R}$, without any changes. This can be thought of as a partial proof that this theory provides a meaningful way to help us solve our modeling problems without forcing us to completely change our methodological approach. Formalizing these physical models is a good way to learn the properties of Fermat reals. On the other hand, working in a rigorous mathematical theory of infinitesimals allows us to gain a great formal power and to discover the physical meaning of several informal approximations. For example, in [11] it is proven that the wave equation is *equivalent* to a suitable condition of infinitesimal oscillations of a string. Other elementary examples have also been developed in [11]: the deduction of the heat equation, a study of the electric dipole, the Newtonian limit in relativity, the curvature of a smooth curve, the area of the circle and volumes of revolution, the stretching of a spring and others.

The theory of Fermat reals is strongly inspired by SDG and, indeed, it can be deeply developed to consider infinitesimal methods that are valid in both finite and infinite dimensional spaces. Every smooth manifold M can be extended using this type of infinitesimals analogously to what we have previously shown for the extension $\bullet\mathbb{R} \supseteq \mathbb{R}$, and obtaining a smooth embedding $\bullet M \supseteq M$. More generally, this extension is applicable to every diffeological space [18] obtaining a functor with very good preservation properties. The category of diffeological spaces is cartesian closed and embeds the category of smooth manifolds, so that these Fermat extensions can also be applied to infinite dimensional function spaces. In this framework, we can define tangent vectors of the space M as smooth maps of the form $t : D \rightarrow \bullet M$, i. e. as infinitesimal linear curves traced on $\bullet M$. We can also intrinsically define the sum of two tangent vectors as the diagonal of the infinitesimal parallelogram generated by these tangent vectors. We can prove that a vector field can be defined as an infinitesimal smooth transformation of the space $\bullet M$ into itself, i. e. as $V : D \rightarrow \bullet M^{\bullet M}$ such that $V(0) = 1_{\bullet M}$. We can define infinitesimal integral curves and prove their existence even in infinite dimensional spaces of functions such as $\mathcal{C}^\infty(\bullet M, \bullet N)$. These are only a few examples from a very long ongoing project. For more details, see [10, 13, 14].

6. OPEN PROBLEMS AND IDEAS

The following describes some open problems and ideas for the interested reader to undertake. Please feel free to contact the author to obtain more feedback, considering that some of these ideas are being investigated within the ongoing projects P25116-N25 and P25311-N25 of the Austrian Science Fund FWF.

6.1. *Infinities and nilpotent infinitesimals*

Almost every theory of actual infinitesimals is usually well coupled with a corresponding theory of infinite numbers. In the case of a field, this coupling is naturally provided by the reciprocal function. These infinitely large numbers can be fruitfully applied to the formulation of integrals as infinite sums, limits at infinite points, hyperfinite subdivisions of the real line, studies of singularities, and generalized functions, to cite only a few. Of course, in the case of nilpotent infinitesimals, the trivial possibility that infinities are reciprocals of infinitesimals does not hold because, in that case, $h^2 = 0$ would imply $h = 0$. Is there some meaningful way to bypass the impossibility to have infinities as reciprocals of nilpotent infinitesimals? What property of multiplication of numbers should be weakened to allow a coexistence between these infinities and nilpotent infinitesimals? Is the corresponding formalism sufficiently easy to use and intuitively meaningful? Can these infinities be used to define a suitable class of generalized functions? Hint: the reciprocal of $h = [t] \in \bullet\mathbb{R}$ is necessarily generated by $t \in \mathbb{R}_{>0} \mapsto t^{-1} \in \mathbb{R}$.

6.2. Weil functors and infinitesimals whose product is not zero

In the ring $\bullet\mathbb{R}$, the product of any two first order infinitesimals $h, k \in D$ is zero: $h \cdot k = 0$. As proved in [11, Theorem 24] this is a general consequence of the total order property, but having infinitesimals h, k such that $h^2 = 0 = k^2$ and $h \cdot k \neq 0$ would be useful for studying the double tangent bundle (see [20]). An idea to explore, inspired by rings like $\mathbb{R}[t, s]/\langle t^2 = 0, s^2 = 0 \rangle$, can be roughly stated as “two first order infinitesimals $(h_t)_t$ and $(k_s)_s$ have a non zero product $(h_t \cdot k_s)_{t,s}$ if they depend on two independent variables t and s ”. A possible formalization of this idea can be sketched in the following way. Instead of little-oh polynomials, let us consider maps of the form $x : \mathbb{R}_{\geq 0}^v \rightarrow \mathbb{R}$ (v depending on x) such that

$$x(t_1, \dots, t_v) = r + \sum_{j=1}^k \alpha_j \cdot t_1^{a_{1j}} \cdot \dots \cdot t_v^{a_{vj}} + o(t_1) + \dots + o(t_v), \quad (6.1)$$

Now the analogue of the equality in $\bullet\mathbb{R}$ is that $x \sim y$ if and only if x and y are both defined on the same domain $\mathbb{R}_{\geq 0}^v$ and $x(t_1, \dots, t_v) = y(t_1, \dots, t_v) + o(t_1) + \dots + o(t_v)$ as $t_k \rightarrow 0^+$ for all k . This idea seems positive for two reasons: first, if we define a new ring in this way, considering only the subring of all the maps $\mathbb{R}_o[t_i]$ which only depend on one variable t_i , we obtain a ring $\bullet\mathbb{R}[t_i]$ isomorphic to the present $\bullet\mathbb{R}$. Second, if we consider $h(t_1, t_2) := t_1$ and $k(t_1, t_2) := t_2$, then we have $h^2 \sim 0$ and $k^2 \sim 0$, but not $h \cdot k \sim 0$. Of course, from [11, Theorem 24] it follows that every subring $\bullet\mathbb{R}[t_i]$ is totally ordered, but the entire ring cannot be totally ordered. Assuming that this new class of little-oh polynomials works as a sufficiently good theory, is there a good representation of every Weil functor using this new ring? For a representation of a subclass of Weil functors using $\bullet\mathbb{R}$, see [9].

6.3. Perturbation theory

Several classical methods used in perturbation theory seem to simplify if we take the small perturbation parameter ε as a non zero nilpotent infinitesimal in the ring of Fermat reals, i. e. if $\varepsilon \in \bullet\mathbb{R}_{\neq 0}$, $\varepsilon^n = 0$. For example, we can more easily use the nilpotency property of ε instead of using big-oh asymptotic estimates; we can also take advantage of exact Taylor formulas in $\bullet\mathbb{R}$, i. e. with nilpotent infinitesimal increments and no remainder. More generally, we can use finite sums in the ring $\bullet\mathbb{R}$ instead of a (convergent or formal) power series, and we can also take advantage of the existence of arbitrary roots of nilpotent infinitesimals (see [16]). Do these properties permit an easier calculus of classical perturbation methods?

6.4. Example in (elementary) Physics

If you have an example, or even only an idea, of how to use the ring of Fermat in (elementary) Physics, please feel free to contact the author. We are planning to write a textbook with an introduction to this ring and its first properties and applications.

Even in the case of a new example in elementary Physics, your name will be clearly cited as the author of the example. For examples in this direction, see [11].

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ON THE FUNCTORIAL PROLONGATIONS OF FIBER BUNDLES

IVAN KOLÁŘ

Abstract. We study the prolongation of various geometric structures on a fibered manifold with respect to the Weil functor T^A or a fiber product preserving bundle functor F on the category of all fibered manifolds with m -dimensional bases and their morphisms with local diffeomorphisms as base maps. Special attention is paid to connections, vector bundles, principal bundles and weak principal bundles.

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There are two important classes of geometric functors that can be characterized in terms of Weil algebras. About 1987, it was clarified that the product preserving bundle functors on the category $\mathcal{M}f$ of all manifolds and all smooth maps coincide with the Weil functors T^A , see [2] for a survey. Further, in 1999 W. Mikulski and the author, [4], analogously described the fiber product preserving bundle functors F on the category $\mathcal{FM}_m$ of fibered manifolds with m -dimensional bases and their morphisms with local diffeomorphisms as base maps.

In the present paper, we discuss the application of both types of functors to various geometric structures on fibered manifolds and we underline the differences between both cases. In Section 1, we summarize the basic facts about T^A (called Case 1) and F (called Case 2). The prolongation of vector bundles in both cases is discussed in Section 2. In Section 3, we consider a general connection Γ on a fibered manifold $p: Y \rightarrow M$ and its prolongation $\mathcal{T}^A \Gamma$ to $T^A p: T^A Y \rightarrow T^A M$. Section 4 is devoted to the prolongation of Γ to $FY \rightarrow M$. Our presentation is systematically based on the concept of flow natural map ψ_Y^F of F over Y . In both cases, special attention is paid to linear connections.

The prolongation $\mathcal{T}^A \Gamma$ of a principal connection Γ on a principal bundle $P \rightarrow M$ is discussed in Section 5. We have already pointed out, [1], that the case of $FP \rightarrow M$ is much more complicated. In Section 6, we characterize some induced connections by the properties related with our original concept of weak principal bundle, [2].

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from the book [3].

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1. TWO TYPES OF BUNDLE FUNCTORS

We recall that a Weil algebra is a finite dimensional, commutative, associative and unital algebra of the form $A = \mathbb{R} \times N$, where N is the ideal of all nilpotent elements of A . Since N is finite dimensional, there exists an integer r such that $N^{r+1} = 0$. The smallest r with this property is called the order of A . On the other hand, the dimension wA of the vector space N/N^2 is the width of A . We say that a Weil algebra of width k and order r is a Weil (k, r) -algebra, [2].

The simplest example of a Weil (k, r) -algebra is

$$\mathbb{D}_k^r = \mathbb{R}[x_1, \dots, x_k] / \langle x_1, \dots, x_k \rangle^{r+1} = J_0^r(\mathbb{R}^k, \mathbb{R}).$$

For $k = r = 1$, $\mathbb{D}_1^1 = \mathbb{D}$ is the algebra of Study numbers. In [2], we deduced

Lemma 1. *Every Weil (k, r) -algebra is a factor algebra of $\mathbb{D}_k^r$. If $\varrho, \sigma: \mathbb{D}_k^r \rightarrow A$ are two algebra epimorphisms, then there exists an algebra isomorphism $\chi: \mathbb{D}_k^r \rightarrow \mathbb{D}_k^r$ such that $\varrho = \sigma \circ \chi$.*

We apply the covariant approach to Weil functors, [2, 3].

Definition 1. Two maps $\gamma, \delta: \mathbb{R}^k \rightarrow M$ determine the same A -velocity $j^A \gamma = j^A \delta$ if for every smooth function $\varphi: M \rightarrow \mathbb{R}$,

$$\varrho(j_0^r(\varphi \circ \gamma)) = \varrho(j_0^r(\varphi \circ \delta)). \quad (1.1)$$

By Lemma 1, this is independent of the choice of ϱ .

We say that

$$T^A M = \{j^A \gamma; \gamma: \mathbb{R}^k \rightarrow M\} \quad (1.2)$$

is the bundle of all A -velocities on M . For every smooth map $f: M \rightarrow N$, we define $T^A f: T^A M \rightarrow T^A N$ by

$$T^A f(j^A \gamma) = j^A (f \circ \gamma). \quad (1.3)$$

Clearly, $T^A \mathbb{R} = A$.

Case 1. We say that (1.2) and (1.3) represent the covariant approach to Weil bundles. The following result is a fundamental assertion, see [2] for a survey.

Theorem 1. *The product preserving bundle functors on $\mathcal{M}f$ are in bijection with T^A . The natural transformations $T^{A_1} \rightarrow T^{A_2}$ are in bijection with the algebra homomorphisms $\mu: A_1 \rightarrow A_2$.*

We write $\mu_M: T^{A_1} M \rightarrow T^{A_2} M$ for the value of $\mu: A_1 \rightarrow A_2$ on M . If A_i is a Weil (k_i, r_i) -algebra, $i = 1, 2$, then there exists a polynomial map $\bar{\mu}: \mathbb{R}^{k_2} \rightarrow \mathbb{R}^{k_1}$ such that

$$\mu_M(j^{A_1} \gamma) = j^{A_2}(\gamma \circ \bar{\mu}), \quad \gamma: \mathbb{R}^{k_1} \rightarrow M. \quad (1.4)$$

The iteration $T^{A_2} T^{A_1}$ corresponds to the tensor product of A_1 and A_2 . The algebra exchange homomorphism $\text{ex}: A_1 \otimes A_2 \rightarrow A_2 \otimes A_1$ defines a natural exchange transformation $T^{A_2} T^{A_1} \rightarrow T^{A_1} T^{A_2}$. We have $T = T^{\mathbb{D}}$.

The canonical exchange $\kappa_M^A: T^A TM \rightarrow T T^A M$ is called flow natural. Indeed, if Fl_t^X is the flow of a vector field $X: M \rightarrow TM$, then

$$\mathcal{T}^A X = \frac{\partial}{\partial t} \Big|_0 T^A(Fl_t^X): T^A M \rightarrow T T^A M$$

is the flow prolongation of X . On the other hand, $T^A X: T^A M \rightarrow T^A TM$ is the functorial prolongation of X . One deduces easily, [2],

$$\mathcal{T}^A X = \kappa_M^A \circ T^A X. \quad (1.5)$$

Case 2. Consider now a bundle functor F on $\mathcal{F}\mathcal{M}_m$ preserving fiber products. Examples are the r th jet prolongation $J^r Y$, $V^A Y = \cup_{x \in M} T^A(Y_x)$, $\cup_{x \in M} J_x^r(M, Y_x)$ and iterations.

We say F is of the base order r if for two $\mathcal{F}\mathcal{M}_m$ -morphisms $\varphi, \psi: Y \rightarrow Z$ with base maps $\underline{\varphi}, \underline{\psi}: M \rightarrow N$, $j_x^r \underline{\varphi} = j_x^r \underline{\psi}$ and $\varphi|_{Y_x} = \psi|_{Y_x}$ imply $F\varphi|_{F_x Y} = F\psi|_{F_x Y}$, $x \in \bar{M}$.

Write $\mathcal{M}f_m$ for the category of m -dimensional manifolds and their local diffeomorphisms. The construction of product fibered manifolds defines injection $\iota: \mathcal{M}f_m \times \mathcal{M}f \rightarrow \mathcal{F}\mathcal{M}_m$, $\iota(M, N) = (M \times N \rightarrow M)$, $\iota(f_1, f_2) = f_1 \times f_2$, $f_1: M \rightarrow M'$, $f_2: N \rightarrow N'$.

W. Mikulski and the author deduced, [4], that the bundle functors $\Phi = F \circ \iota$ on $\mathcal{M}f_m \times \mathcal{M}f$ are in bijection with the pairs (A, H) , where A is a Weil algebra and $H: G_m^r \rightarrow \text{Aut } A$ is a group homomorphism of the r th jet group in dimension m into the group of all algebra automorphisms of A . Since $H(g): A \rightarrow A$ is an algebra automorphism for every $g \in G_m^r$, we have the induced action $H_N(g) = H(g)_N: T^A N \rightarrow T^A N$ of G_m^r on $T^A N$. Then $\Phi(M, N)$ is the associated fiber bundle $P^r M[T^A N]$. For a local diffeomorphism $f_1: M \rightarrow M'$ and a smooth map $f_2: N \rightarrow N'$,

$$\Phi(f_1, f_2) = P^r f_1[T^A f_2]: \Phi(M, N) \rightarrow \Phi(M', N'), \quad (1.6)$$

where $P^r f_1: P^r M \rightarrow P^r M'$ is the induced local isomorphism of principal fiber bundles and $T^A f_2: T^A N \rightarrow T^A N'$ is a G_m^r -equivariant map, [2, 4].

Then the functor F is determined by adding an equivariant algebra homomorphism $t: \mathbb{D}_m^r \rightarrow A$, where $\text{Aut } \mathbb{D}_m^r = G_m^r$. We have

$$FY = \{\{u, Z\} \in P^r M[T^A Y], t_M u = T^A p(Z), u \in P^r M, Z \in T^A Y\}, \quad (1.7)$$

where $t_M: T_m^r M \rightarrow T^A M$ and $P^r M \subset T_m^r M$. For an $\mathcal{F}\mathcal{M}_m$ -morphism $f: Y \rightarrow Y'$ over $\underline{f}: M \rightarrow M'$, Ff is the restriction of $\Phi(\underline{f}, f)$ to FY . In the product case $Y = \bar{M} \times N$, we have

$$F(M \times N) = P^r M[T^A N]. \quad (1.8)$$

If we consider another fibered manifold $Y' \rightarrow M$ over M and $\underline{f} = \text{id}_M$, we have

$$Ff(\{u, Z\}) = \{u, T^A f(Z)\}. \quad (1.9)$$

Formally, t induces a natural map

$$\tilde{t}_Y: J^r Y \rightarrow FY, \quad \{u, Z\} \mapsto \{u, t_Y(Z)\}. \quad (1.10)$$

Geometrically, we interpret a section $s: M \rightarrow Y$ as a morphism $\tilde{s}: \tilde{M} \rightarrow Y$, where $\tilde{M} = (M \xrightarrow{\text{id}} M)$ is the “doubled” manifold. Then $F\tilde{s}$ is identified with $j^r s$ and $\tilde{t}_Y(j_x^r s) = (F\tilde{s})(x)$.

2. PROLONGATION OF VECTOR BUNDLES

Consider a vector bundle $p: E \rightarrow M$. The vector addition in E and the multiplication of vectors by real numbers are two maps

$$a: E \times_M E \rightarrow E, \quad m: \mathbb{R} \times E \rightarrow E. \quad (2.1)$$

Applying T^A , we obtain

$$T^A a: T^A E \times_{T^A M} T^A E \rightarrow T^A E, \quad T^A m: A \times T^A E \rightarrow T^A E. \quad (2.2)$$

If we use $\mathbb{R} \subset A$ and express all the basic properties of a vector bundle in terms of diagrams, we prove

Proposition 1. $T^A p: T^A E \rightarrow T^A M$ is also a vector bundle. □

However, our concept of A -velocity offers a more geometric proof, that is, one of jet-like character. We have

$$T^A E = \{j^A \gamma, \gamma: \mathbb{R}^k \rightarrow E\}.$$

For $c \in \mathbb{R}$, we define $cj^A \gamma(\tau) = j^A(c\gamma(\tau))$, $\tau \in \mathbb{R}^k$. If $p \circ \gamma_1 = p \circ \gamma_2$, we set

$$j^A(\gamma_1(\tau)) + j^A(\gamma_2(\tau)) = j^A(\gamma_1(\tau) + \gamma_2(\tau))$$

with addition in the individual fibers of E . Then we verify easily that $T^A E$ is a vector bundle.

We underline that the map $T^A m: A \times T^A E \rightarrow T^A E$ introduces an action of the Weil algebra A on $T^A E$, which is important in several applications, [2].

In the case of $F = (A, H, t)$, we can apply F directly to the first formula in (2.1). This yields

$$Fa: FE \times_M FE \rightarrow FE \quad (2.3)$$

with the fiber product over M , not over $T^A M$ as in (2.2). Further, F can be applied to fibered manifolds only. Hence, we have to rewrite the second formula from (2.1) as

$$m: (M \times \mathbb{R}) \times_M E \rightarrow E. \quad (2.4)$$

Using (1.8), we find

$$Fm: P^r M[A] \times_M FE \rightarrow FE \quad (2.5)$$

with the action of G_m^r on $A = \mathbb{R} \times N$ determined by H . But $\mathbb{R}$ is invariant with respect to every $H(g)$, so that $M \times \mathbb{R} \subset P^r M[A]$. Restricting (2.5) to this subspace,

we obtain the multiplication by real scalars on FE . A direct discussion of all the related diagrams imply

Proposition 2. $FE \rightarrow M$ is also a vector bundle. $\square$

Even here, it is more geometric to apply the jet like approach based on (1.7). Two $V_1, V_2 \in F_x E$ can be written as

$$V_1 = \{u, Z_1\}, \quad V_2 = \{u, Z_2\}, \quad Z_1, Z_2 \in T^A E \quad (2.6)$$

with the same $u \in P_x^r M$. Then $T^A p(Z_1) = t_M u = T^A p(Z_2)$ imply that Z_1 and Z_2 lie in the same fiber of $T^A E$ over $t_M u \in T^A M$. But $T^A E \rightarrow T^A M$ is a vector bundle and t is invariant. By (1.4), $V_1 + V_2 = \{u, Z_1 + Z_2\}$ is defined in an intrinsic way. This leads directly to another proof of our assertion.

3. INDUCED CONNECTIONS ON $T^A Y \rightarrow T^A M$

Consider a general connection Γ on Y as a lifting map $\Gamma: Y \times_M TM \rightarrow TY$. In [3], we constructed the induced general connection $\mathcal{T}^A \Gamma$ on $T^A p: T^A Y \rightarrow T^A M$ by a commutative diagram

$$\begin{array}{ccccc} T^A Y & \times_{T^A M} & T^A TM & \xrightarrow{T^A \Gamma} & T^A TY \\ \parallel & & \downarrow \kappa_M^A & & \downarrow \kappa_Y^A \\ T^A Y & \times_{T^A M} & TT^A M & \xrightarrow{\mathcal{T}^A \Gamma} & TT^A Y \end{array} \quad (3.1)$$

Proposition 3. Let Γ be a linear connection on the vector bundle $p: E \rightarrow M$. Then $\mathcal{T}^A \Gamma$ is also linear.

Proof. Linearity of Γ means

$$\Gamma(u_1, X) + \Gamma(u_2, X) = \Gamma(u_1 + u_2, X), \quad u_1, u_2 \in E_x, \quad X \in T_x M.$$

We have

$$\mathcal{T}^A \Gamma: T^A E \times_{T^A M} TT^A M \rightarrow TT^A E$$

and

$$\mathcal{T}^A \Gamma(j^A \gamma, j^A \xi) = j^A \Gamma(\gamma, \xi),$$

$$\gamma: \mathbb{R}^k \rightarrow E, \quad \xi: \mathbb{R}^k \rightarrow TM, \quad p \circ \gamma = \pi \circ \xi, \quad \pi: TM \rightarrow M.$$

Then

$$\begin{aligned} \mathcal{T}^A \Gamma(j^A \gamma_1 + j^A \gamma_2, j^A \xi) &= j^A \Gamma(\gamma_1(\tau) + \gamma_2(\tau), \xi(\tau)) = \\ &= j^A \Gamma(\gamma_1, \xi) + j^A \Gamma(\gamma_2, \xi) = \mathcal{T}^A \Gamma(j^A \gamma_1, j^A \xi) + \mathcal{T}^A \Gamma(j^A \gamma_2, j^A \xi). \end{aligned}$$

This completes the proof. $\square$

Example 1. We present the case $T^A = T$ in coordinates. Let x^i, y^p be some fiber coordinates on Y and

$$dy^p = F_i^p(x, y) dx^i \quad (3.2)$$

be the equations of a general connection Γ . If x_1^i, y_1^p are the additional coordinates on TY , then the additional equations of $\mathcal{T}\Gamma$ are, see [2],

$$dy_1^p = \left(\frac{\partial F_i^p}{\partial x^j} x_1^j + \frac{\partial F_i^p}{\partial y^q} y_1^q \right) dx^i + F_i^p dx_1^i. \quad (3.3)$$

In the linear case, we have

$$\begin{aligned} dy^p &= \Gamma_{qi}^p(x) y^q dx^i, \\ dy_1^p &= \left(\frac{\partial \Gamma_{qi}^p}{\partial x^j} y^q x_1^j + \Gamma_{qi}^p y_1^q \right) dx^i + \Gamma_{qi}^p y^q dx_1^i, \end{aligned} \quad (3.4)$$

that is linear in y^p and y_1^p .

4. INDUCED CONNECTIONS ON $FY \rightarrow M$

It was clarified in several concrete problems that if F is of base order r , we need an auxiliary linear splitting $\Lambda: TM \rightarrow J^r TM$ to construct an induced connection $\mathcal{F}(\Gamma, \Lambda): FY \times_M TM \rightarrow TFY$, [3]. Consider a vector field $\xi: M \rightarrow TM$ and its Γ -lift $\Gamma\xi: Y \rightarrow TY$. The flow prolongation $\mathcal{F}(\Gamma\xi): FY \rightarrow TFY$ depends on $j^r \xi$ only. This yields $\mathcal{F}\Gamma: FY \times_M J^r TM \rightarrow TFY$ linear in $J^r TM$. Then $\mathcal{F}(\Gamma, \Lambda) = \mathcal{F}\Gamma \circ (\text{id}_{FY}, \Lambda)$.

It is useful to describe this construction by using the flow natural map ψ_Y^F . In Section 1, we constructed $t_{TM}: J^r TM \rightarrow FTM$. Consider a projectable vector field $\eta: Y \rightarrow TY$ over $\xi: M \rightarrow TM$. If we interpret η as an $\mathcal{F}\mathcal{M}$ -morphism $\eta: Y \rightarrow TY$ over id_M , we can construct $F\eta: FY \rightarrow FTY$. By [2], there is a unique map

$$\psi_Y^F: FTY \times_{FTM} J^r TM \rightarrow TFY \quad (4.1)$$

satisfying

$$\mathcal{F}\eta = \psi_Y^F(F\eta, j^r \xi) \quad (4.2)$$

for every η . This map is linear in $J^r TM$. Hence

$$\mathcal{F}(\Gamma, \Lambda) = \psi_Y^F(F(\Gamma\xi), j^r \xi) \circ \Lambda = \mathcal{F}\Gamma \circ (\text{id}_{FY}, \Lambda). \quad (4.3)$$

In the case of a vector bundle E , ψ_E^F is linear also in FTE , [2]. This implies directly

Proposition 4. *If Γ is linear, then $\mathcal{F}(\Gamma, \Lambda)$ is linear for every Λ .*

5. THE PRINCIPAL BUNDLE $T^A P \rightarrow T^A M$

Consider a principal G -bundle $p: P \rightarrow M$ and write $u \cdot g, u \in P, g \in G$ for the action of G on P . We shall use the same symbol for the induced action of $T^A G$ on $T^A p: T^A P \rightarrow T^A M$,

$$j^A u(\tau) \cdot j^A g(\tau) = j^A(u(\tau) \cdot g(\tau)), \quad \tau \in \mathbb{R}^k. \quad (5.1)$$

This is a right action. If $j^A u_1(\tau)$ and $j^A u_2(\tau)$ satisfy $p \circ u_1 = p \circ u_2$, then the relation

$$u_2(\tau) = u_1(\tau) \cdot g(\tau)$$

implies that $T^A P \rightarrow T^A M$ is a principal $T^A G$ -bundle.

The induced action of G on TP is

$$\left(\frac{\partial}{\partial t}\Big|_0 v(t)\right) \cdot g = \frac{\partial}{\partial t}\Big|_0 (v(t) \cdot g), \quad v: \mathbb{R} \rightarrow P. \quad (5.2)$$

A principal connection $\Gamma: P \times_M TM \rightarrow TP$ is characterized by

$$\Gamma\left(u, \frac{\partial}{\partial t}\Big|_0 \gamma(t)\right) \cdot g = \Gamma\left(u \cdot g, \frac{\partial}{\partial t}\Big|_0 \gamma(t)\right), \quad \gamma: \mathbb{R} \rightarrow M. \quad (5.3)$$

The induced action of $T^A G$ on $TT^A P$ is

$$\left(j^A \frac{\partial}{\partial t}\Big|_0 u(\tau, t)\right) \cdot j^A g(\tau) = \frac{\partial}{\partial t}\Big|_0 j^A(u(\tau, t) \cdot g(\tau)), \quad (5.4)$$

$u: \mathbb{R}^k \times \mathbb{R} \rightarrow P, g: \mathbb{R}^k \rightarrow G$.

Proposition 5. *If Γ is a principal connection on $P \rightarrow M$, then $\mathcal{T}^A \Gamma$ is a principal connection on $T^A P \rightarrow T^A M$.*

Proof. For $u: \mathbb{R}^k \rightarrow P, g: \mathbb{R}^k \rightarrow G, \gamma: \mathbb{R}^k \times \mathbb{R} \rightarrow M$, we obtain by (5.3) and (5.4)

$$\begin{aligned} \mathcal{T}^A \Gamma\left(j^A(u(\tau)) \cdot j^A g(\tau), \frac{\partial}{\partial t}\Big|_0 j^A \gamma(\tau, t)\right) &= \\ &= \mathcal{T}^A \Gamma\left(j^A(u(\tau)), \frac{\partial}{\partial t}\Big|_0 j^A \gamma(\tau, t)\right) \cdot j^A g(\tau). \end{aligned}$$

This completes the proof of the proposition. $\square$

In the case of a bundle $P[S]$ associated to P with respect to a left action $G \times S \rightarrow S$, every principal connection Γ on P induces a general connection Γ_S on $P[S]$. If $\Gamma(u, X) = \frac{\partial}{\partial t}\Big|_0 \gamma(t), \gamma: \mathbb{R} \rightarrow P$, then

$$\Gamma_S(\{u, a\}, X) = \frac{\partial}{\partial t}\Big|_0 \{\gamma(t), a\}, \quad a \in S. \quad (5.5)$$

Consider the induced action $T^A G \times T^A S \rightarrow T^A S$. Analogously to Proposition 5, we deduce

Proposition 6. We have $\mathcal{T}^A(\Gamma_S) = (\mathcal{T}^A\Gamma)_{T^A S}$.

6. WEAK PRINCIPAL BUNDLES

It is well known that the r th jet prolongation $J^r P \rightarrow M$ of a principal G -bundle $P \rightarrow M$ is not a principal bundle, [2]. When investigating the functor $F = (A, H, t)$, we realized that some general phenomena are reflected in the concept of the weak principal bundle, [2]. Let K be a Lie group.

Definition 2. A fibered manifold $p: C \rightarrow M$ is called a group bundle of type K if each fiber is a Lie group and for every $x \in M$ there exists a neighbourhood U such that $p^{-1}(U) \approx U \times K$.

The group compositions form a base preserving morphism $v: C \times_M C \rightarrow C$.

Lemma 2. $Fv: FC \times_M FC \rightarrow FC$ is a group bundle of type $T^A K$.

Proof. By locality, we may assume $C = M \times K$. Write $\kappa: K \times K \rightarrow K$ for the group composition in K . Consider $c_1, c_2 \in F_x C$, $c_1 = \{u, Z_1\}$, $c_2 = \{u, Z_2\}$, $u \in P_x^r M$, $Z_1, Z_2 \in T^A K$. Using (1.9), we obtain

$$F\kappa(\{u, Z_1\}, \{u, Z_2\}) = \{u, T^A \kappa(Z_1, Z_2)\}.$$

The lemma is now proved. $\square$

Definition 3. A fibered manifold $Q \rightarrow M$ is called a weak principal bundle with structure group bundle $C \rightarrow M$ if we are given a base-preserving morphism $\varrho: Q \times_M C \rightarrow Q$ such that each group C_x acts simply transitively on the right on Q_x .

Clearly, the principal bundle is a weak principal bundle, the group bundle of which is the product $M \times K$.

Example 2. We have $F(M \times G) = P^r M[T^A G]$. This is a group bundle of type $T^A G$. Applying F to $\varrho: P \times_M (M \times G) \rightarrow P$, we obtain $F\varrho: FP \times_M P^r M[T^A G] \rightarrow FP$. This defines a weak principal bundle structure on FP for every principal bundle P .

Proposition 7. For a weak principal bundle $\varrho: Q \times_M C \rightarrow Q$, $F\varrho: FQ \times_M FC \rightarrow FQ$ is also a weak principal bundle.

Proof. By locality, the problem reduces to Example 2. $\square$

Definition 4. A connection $\Delta: C \times_M TM \rightarrow TC$ is called a group connection if $\Delta(c_i, X) = \frac{\partial}{\partial t}|_0 u_i(t)$, $i = 1, 2$, imply $\Delta(\kappa(c_1, c_2), X) = \frac{\partial}{\partial t}|_0 \kappa(u_1(t), u_2(t))$ for every $c_1, c_2 \in C_x$, $X \in T_x M$.

Definition 5. A connection $\Gamma: Q \times_M TM \rightarrow TQ$ is called weak principal with respect to a group connection $\Delta: C \times_M TM \rightarrow TC$ if $\Gamma(u, X) = \frac{\partial}{\partial t}|_0 u(t)$, $\Delta(c, X) = \frac{\partial}{\partial t}|_0 c(t)$ implies

$$\Gamma(\varrho(u(0), c(0)), X) = \frac{\partial}{\partial t}|_0 \varrho(u(t), c(t)), u: \mathbb{R} \rightarrow Q, c: \mathbb{R} \rightarrow C.$$

Clearly, if $C = M \times K$ is the product bundle, a weak principal connection Γ on Q with respect to the product connection on C is principal.

Using the ideas of Section 4, one deduces directly

Proposition 8. *If Γ is a weak principal connection on Q with respect to a group connection Δ on C , then $\mathcal{F}(\Gamma, \Lambda)$ is weak principal with respect to $\mathcal{F}(\Delta, \Lambda)$ for every Λ .*

Remark. In the theory of prolongation of geometric object fields, one modifies the weak principal bundle $J^r P$ into a principal bundle by constructing $W^r P = J^r P \times_M P^r M$, [3]. In [1], A. Cabras and the author have described a general procedure of this type.

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ON HOMOTOPY ALGEBRAS AND QUANTUM STRING FIELD THEORY

KORBINIAN MÜNSTER AND IVO SACHS

Abstract. In this paper, we revisit the existence, background independence and uniqueness of closed, open and open-closed bosonic- and topological string field theory, using the machinery of homotopy algebra. In the theory of classical open- and closed strings, the space of inequivalent open string field theories is isomorphic to the space of classical closed string backgrounds. We then discuss obstructions of these moduli spaces at the quantum level. For the quantum theory of closed strings, uniqueness on a given background follows from the decomposition theorem for loop homotopy algebras. We also address the question of background independence of closed string field theory.

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1. INTRODUCTION

The standard formulation of classical string theory consists of a set of rules to compute scattering amplitudes for a set of n (excited) strings typically propagating on a D -dimensional Minkowski space-time M_D . This prescription involves an integration over the moduli space of disks with n punctures for open strings (or spheres with n punctures for the closed strings). Comparing this with the approach taken for point particles the situation in string theory seems incomplete. Indeed, for point particles one starts with an action principle and then obtains the classical scattering amplitudes by solving the equations of motions deriving from this action. Since the various string excitations ought to be interpreted as particles one would hope to be able to apply the same procedure for the scattering of strings. The aim of string field theory is precisely to provide such an action principle so that the set of rules to compute scattering amplitudes for strings follow from this action. Since the string consists of a infinite linear superpositions of point particle excitations one would expect that such an action may be rather complicated. Yet the first construction of a consistent classical string field theory of interacting open strings [11] has a remarkably simple algebraic structure of a differential graded algebra (DGA) together with a non-degenerate odd symplectic form.

The geometric approach for the construction of string field theory [13, 14], starts with a decomposition of the relevant moduli space of Riemann surfaces into elementary vertices and graphs. The condition that the moduli space is covered exactly once implies that the geometric vertices satisfy a classical Batalin-Vilkovisky master equation. From this one then anticipates that any string field theory action should realize some homotopy algebra. The subject of this talk is to investigate to what extent this algebraic structure is useful, and to determine certain additional properties that should be satisfied by any consistent string field theory. In particular, it is of interest to know in what sense string field theory is unique. Another related issue stems from the fact that the construction of string field theory assumes that the string propagates in a certain string background whose geometry is that of Minkowski space. However, since string theory includes gravity, this background is dynamical. The question of background independence of this construction is thus relevant.

To set the stage, let us start with the well understood case of a single point particle propagating on a non-compact manifold M_D with a pseudo-Riemannian metric g . The world line of the particle is described by a curve $\phi : [a, b] \rightarrow M_D$ that extremizes the action

$$S[\phi, h] = \int_{[a, b]} \frac{1}{\sqrt{h_{tt}}} g(\dot{\phi}, \dot{\phi}) dt$$

where h_{tt} is a non-dynamical "metric" on the world line that can be set to 1 by a suitable reparametrization of t . Similarly, for an open string we have a map $\phi : \Sigma = [a, b] \times [c, d] \rightarrow M_D$ that extremizes the action

$$S[\phi, h] = \int_{\Sigma} \sqrt{h} h^{ij} g(\partial_i \phi, \partial_j \phi) \quad (1.1)$$

so that the area is minimal. If the Riemann curvature of M_D vanishes, then the action (1.1) is invariant under conformal mappings of the *world sheet* Σ . For $[a, b] = [-\infty, \infty]$, we can conformally map Σ to a disk with 2 punctures. Analogously, a world sheet describing $n - 1$ strings joining into one can be mapped into a disk with n punctures. In order to specify which particles (or string excitations) are involved in the scattering amplitude, we need to endow the puncture with additional structures. This is done by attaching conformal tensors $\{V_i[\phi]\}$ built out of the maps ϕ evaluated at the puncture and the coefficients of the Laurent polynomial of ϕ evaluated in local coordinates. The amplitude is then expressed in terms of the n -point correlator

$$\langle V_{i_1}(z_1), \dots, V_{i_n}(z_n) \rangle, \quad (1.2)$$

with respect to the (formal) Gaussian measure defined by $S[\phi]$. In fact, the correlator, (1.2) which is called a conformal field theory correlator in physics, is not quite what one needs. In order to get the string scattering amplitude we need to integrate over the moduli space of the punctured disk. Now, since the action $S[\phi, h]$ is invariant under diffeomorphisms on the world sheet Σ as well as under Weyl re-scalings of the world sheet metric h , we really want to integrate over the $(n - 3)$ -dimensional gauge-fixed

moduli space M_{n-3} (for a review, see, e. g., [12] and references therein). Treating the gauge-fixed action using the standard BRST formalism, we end up with an action $S[\phi, c, b]$ including odd world sheet tensor fields (BRST ghosts) together with an odd differential Q_o that generates the odd symmetry transformations of the gauge fixed action. Similarly, the insertions at the punctures of Σ contain added Laurent coefficients of the b and c ghosts. The string amplitude can be written schematically as in Figure 1, where the $n - 3$ meromorphic vector fields v_i are constant near the



FIGURE 1. Sketch of the CFT realization of the scattering amplitude of n open strings.

puncture P_i , and cannot be extended to the whole disk. These vector fields generate translations in the moduli space; they move the punctures. Concretely, this amplitude becomes

$$\int_{M_{n-3}} ds_1 \dots ds_{n-3} \langle b(v_1) \dots b(v_{n-3}) V_{i_1}[\phi, b, c](z_1) \dots V_{i_n}[\phi, b, c](z_n) \rangle, \quad (1.3)$$

where the correlator is evaluated with respect to the measure obtained from the world sheet action $S[\phi, c, b]$. What we have just described is what is usually referred to as the operator formalism of the world sheet conformal field theory (CFT), which dresses the geometric amplitudes (punctured disks) with the physical states (particles). The amplitudes (1.3) are well defined on the cohomology of Q_o .

The purpose of string field theory is two-fold. First to reproduce these amplitudes in terms of vertices and graphs built from them and second to generalize the amplitudes (1.3) on $\text{coh}(Q_o)$ to the module A_o of all conformal tensors with suitable regularity conditions. At the geometrical level, the simplest possible construction would be that of a single vertex of 3 joining strings which has no moduli, with all amplitudes recovered from graphs built from 3-vertices. This is indeed possible for the open bosonic string [11]. However, the decomposition of moduli space is not unique so that other realizations are possible where higher order vertices are needed to recover the amplitudes (1.3). In any case, the geometric vertices in any consistent decomposition form a BV algebra.

The world sheet CFT then defines a morphism of BV algebras between the set of geometric vertices $\{\mathcal{V}_n\}$, and the dressed "physical" vertices. It also provides us with an inner product on the graded module A_o generated by the conformal tensors $V_i[\phi, b, c]$ of the (ϕ, b, c) -CFT inserted at the origin in the local coordinate z around a puncture P on the disk. With the help of the latter, we can interpret the set of physical vertices as multilinear maps $m_i : A_o^{\otimes i} \rightarrow A_o$, $m_1 = Q_o$, with some further

symmetry properties implied by the cyclic symmetry of the vertices. We denote by $C(A_o)$ the space of such multilinear maps on A_o . It is then not hard to see that the BV-master equation implies that the maps m_i define an A_∞ -structure. One way to see this is to define a coderivation M of degree 1 on the tensor algebra $TA_o = \bigoplus_n A_o^{\otimes n}$ with components

$$(M)_{n,u} = \sum_{\substack{r+s+t=n \\ r+1+t=u}} \mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}$$

Imposing vanishing of the graded commutator $[M, M]$, we obtain a characterization of all differentials compatible with the A_∞ -structure.

The classical solutions of the string field theory action defined by the maps m_i together with $\langle \cdot, \cdot \rangle$ are given by the Maurer–Cartan elements, $M(e^{\psi_0}) = 0$.

There is an analogous story for classical closed strings obtained from the above by replacing the punctured disk by a punctured sphere with world sheet conformal field theory $S[\Phi, c, \bar{c}, b, \bar{b}]$ and dressed by conformal tensors $V_i[\Phi, b, \bar{b}, c, \bar{c}]$ where $b, \bar{b}, c$ and $\bar{c}$ depend holomorphically and anti-holomorphically on the world sheet coordinates z and $\bar{z}$, respectively. The CFT then provides a morphism between the set of geometric vertices and the (dressed) physical vertices of closed string field theory. The latter can again be interpreted as maps, l_i on the graded symmetric module $SA_c = \bigoplus_n A_c^{\wedge n}$. Accordingly, they realize an L_∞ algebra (A_c, L) , with $[L, L] = 0$.

Finally, we let open and closed strings interact with each other. The open closed vertices consist of disks with punctures on the boundary as well as on the disk. These vertices realize an L_∞ morphism F , between the closed and open sector taken separately,

$$(A_c, L) \xrightarrow{F} (\text{Coder}^{\text{cycl}}(TA_o), d_h, [\cdot, \cdot]). \quad (1.4)$$

This is the open-closed homotopy algebra of Kajiura and Stasheff [5].

Remark 1. Note that, while the geometric decomposition of the moduli spaces appearing in the construction of string field theory just reviewed is independent of the details of M_D , the operator formalism makes explicit use of the geometry of M_D as well as possible other background fields inserted at the punctures. In particular, the module A of conformal tensors typically depends on these data. This is where the background dependence enters in the construction of string field theory. This is in contrast to, e. g., General Relativity where the action does not depend on any background metric on M_D .

A natural question that arises in the above context is whether, for a given background (in the sense just described), the generalization of (1.3) as well as its closed string version is unique. For classical string field theory, the answer to this question is affirmative, as follows from the decomposition theorem [5] for homotopy algebras. This theorem establishes an isomorphism between a given homotopy algebra and the

direct sum of a linear contractible algebra and a minimal model. In the context of string field theory, the structure maps of the minimal model are given by (1.3).

In this talk, we discuss the following generalizations of the results reviewed above:

- ◇ classification of inequivalent deformations of classical open string field theory;
- ◇ background independence of closed string field theory;
- ◇ decomposition theorem for quantum closed string field theory;
- ◇ quantization of the open closed homotopy algebra.

2. RESULTS

Let us start with non-trivial deformations of open string field theory. That is we consider *continuous* deformations of the worldsheet CFT that do not preserve Q_o and (1.3) simultaneously. The usefulness of the homotopy formulation of SFT in this respect is that this problem can be formulated as a cohomology problem. Indeed, since any consistent open string field theory realizes an A_∞ algebra, i.e. defines a coderivation M of degree 1 on the tensor algebra TA_o with $[M, M] = 0$, any infinitesimal deformation $M + \delta M$ satisfies $d_H(\delta M) \equiv [M, \delta M] = 0$. For a given worldsheet CFT, one would therefore like to determine $\text{coh}(d_H)$. The outcome of this analysis is contained in

Theorem 1 ([8]). *Let $S[\phi, c, b]$ be the open string world sheet CFT on M_D , A_o the corresponding module of conformal tensors, Q_o the BRST differential, and (1.3) the corresponding string amplitudes on $\text{coh}(Q_o)$. Then the only non-trivial infinitesimal deformations of $S[\phi, c, b]$ preserving A_o are infinitesimal deformations of the closed string background in the relative cohomology of Q_c ,*

$$\text{coh}(d_H) \cong \text{coh}(b_0 - \bar{b}_0, Q_c).$$

Remark 2. A particular class of deformations that do not preserve Q_o and (1.3) are shifts in the open string background $\phi_0 \rightarrow \phi_0 + \epsilon \delta \phi$ with $M(e^{\phi_0 + \epsilon \delta \phi}) = O(\epsilon^2)$. Such transformations are, however, d_H -exact as are all field redefinitions of ϕ . From a physics perspective, an interesting fact implied by theorem 1 is that open string theory already contains the complete information of the particle content of *closed* string theory.

Proof. The proof of this assertion proceeds via a detailed analysis of the deformations of the CFT correlator (1.3). $\square$

Given the isomorphism between the cohomologies, one may wonder whether this isomorphism holds for finite deformations. On the closed string side, finite deformations correspond to classical solutions of the closed string field theory equation of motion, that is Maurer–Cartan elements $L(e^\Phi) = 0$, whereas finite deformations of open string field theory are Maurer–Cartan elements of $[\cdot, \cdot]$ on $\{M \in \text{Coder}^{\text{cycl}}(TA_o)\}$, that is, $[M, M] = 0$. A classic theorem of Kontsevich then guarantees isomorphism at the

$$\begin{array}{ccc}
S_{CFT} & \xrightarrow[\mathcal{V}_n \mapsto l_n]{\text{operator formalism}} & \{l_n\}_{n \in \mathbb{N}} \\
\downarrow S_{CFT} \rightarrow S_{CFT} + \int \Phi_0 & & \downarrow L \rightarrow e^{-\Phi_0} \circ L \circ e^{\Phi_0} \\
S'_{CFT} & \xrightarrow[\mathcal{V}_n \mapsto l'_n]{\text{operator formalism}} & \{l'_n\}_{n \in \mathbb{N}}
\end{array}$$

FIGURE 2. Background independence requires that the L_∞ maps $\{l'_n\}$ obtained upon conjugation by the MC-element e^{Φ_0} are equivalent to those obtained from the world sheet CFT in the background Φ_0 .

finite level, or more precisely that the moduli spaces of two L_∞ -algebras connected by a L_∞ -quasi-isomorphism are isomorphic. Thus, we have

Corollary 1. *Let $\mathcal{M}(A_c, L)$ and $\mathcal{M}(\text{Coder}^{\text{cycl}}(TA_o), [\cdot, \cdot])$ be the moduli space of Maurer–Cartan elements obtained by moding out L - and $[\cdot, \cdot]$ -gauge transformations respectively, then we have*

$$\mathcal{M}(A_c, L) \cong \mathcal{M}(\text{Coder}^{\text{cycl}}(TA_o), d_h, [\cdot, \cdot]).$$

We will return to the question whether this isomorphism survives quantization below, but first we would like to turn to background independence of closed string field theory. As mentioned above, for a given background the operator, formalisms realize a certain L_∞ algebra. Furthermore, for a given classical solution Φ_0 in this field theory, we then obtain a new homotopy algebra upon conjugation by this Maurer–Cartan element. Background independence then would imply that the structure maps of the minimal model obtained from this homotopy algebra are equivalent to the amplitudes (1.3) obtained with the measure of the world-sheet CFT $S[\Phi, c, \bar{c}, b, \bar{b}]$ in the new background (see figure 2).

We can answer this question by addressing the cohomology problem on $\{L \in \text{Coder}^{\text{sym}}(SA_c)\}$. The bracket $[\cdot, \cdot]$ on $\text{Coder}(SA)$ induces the Chevalley-Eilenberg differential $d_C = [L, \cdot]$ on the deformation complex. The analysis proceeds in close analogy with that for open string theory with the result,

Proposition 1. *Let $S[\Phi, c, \bar{c}, b, \bar{b}]$ be the closed string world sheet CFT on M_D , A_c the corresponding module of conformal tensors and Q_c the BRST differential. Then*

$$\text{coh}(d_c) = \emptyset.$$

An immediate consequence of this proposition is that the diagram in figure 2 commutes, which, in turn, implies independence under shifts in the background that preserve A_c .

Remark 3. We should note that generic shifts in the background Φ will not preserve the module A_c .

Let us now return to the decomposition theorem, which states that a homotopy algebra defined on a certain complex can be decomposed into the direct sum of a minimal and a linear contractible part. By definition, the linear contractible part is just a complex with vanishing cohomology, whereas the minimal part is a homotopy algebra of the same type as the initial one but without differential [6]. Furthermore, the initial and the decomposed algebras are isomorphic in the appropriate sense. Clearly, the minimal part can be extracted from the decomposed algebra by projection and, thus, the decomposition theorem implies the minimal model theorem.

The relevance of the minimal model theorem in physics is as follows: Suppose that the vertices of some field theory satisfy the axioms of some homotopy algebra. Then the minimal model describes the corresponding S -matrix amplitudes [3, 4]. Furthermore, the S -matrix amplitudes and the field theory vertices are quasi-isomorphic, which implies that their respective moduli spaces are isomorphic (this follows in general from the minimal model theorem).

Now we conclude that string field theory is unique up to isomorphisms on a fixed conformal background (CFT): In string field theory, the differential is generically given by the BRST charge Q . Furthermore the CFT determines the S -matrix amplitudes. Thus a conformal background determines the minimal and the linear contractible part, which implies uniqueness up to isomorphisms.

An explicit construction of the decomposition model is known for the classical algebras (A_∞ and L_∞) [3, 4]. In the following, we construct the decomposition model for quantum closed string field theory, formulated in the framework of IBL_∞ -algebras (see, e. g., [2, 10] for a definition).

Quantum closed string field theory has the algebraic structure of a loop homotopy Lie-algebra $(A, \mathfrak{L})$ [7], i.e.

$$\mathfrak{L} = \sum \hbar^g L^g + \hbar \Omega^{-1}, \quad \mathfrak{L}^2 = 0, \quad (2.1)$$

where $L^g = D(l^g) \in \text{Coder}^{\text{cycl}}(SA)$ and $\Omega^{-1} = D(\omega^{-1}) \in \text{Coder}^2(SA)$ is the lift of the inverse of the odd symplectic structure (D denotes the lift from multilinear maps to coderivations). We define $l_q := \sum_g \hbar^g l^g$. The differential on A is given by

$d = l_{cl} \circ i_1$. Furthermore, we abbreviate the collection of multilinear maps without the differential by $l_q^* := l_q - d$.

Definition 1. A pre-Hodge decomposition of A is a map $h : A \rightarrow A$ of degree minus one which is compatible with the symplectic structure and squares to zero.

For a given pre-Hodge decomposition of A , we define the map

$$P = 1 + dh + hd, \quad (2.2)$$

and

$$g := -\omega \circ d \quad \text{and} \quad g^{-1} := h \circ \omega^{-1} \in A^{\wedge 2}, \quad (2.3)$$

where the symplectic structure ω and its inverse ω^{-1} are considered as a map from A to A^* and A^* to A , respectively. We define *trees* constructed recursively from l_q^* and h via

$$\mathbb{T}_q = h \circ l_q^* \circ e^{1+\mathbb{T}_q} \quad \text{and} \quad \mathbb{T}_q \circ i_1 = 0. \quad (2.4)$$

Theorem 2 ([9]). *Let $(A, \mathfrak{L} = D(d + l_q^* + \hbar\omega^{-1}))$ be a loop homotopy Lie algebra. For a given pre-Hodge decomposition h , there is an associated loop homotopy Lie algebra*

$$\bar{\mathfrak{L}} = D\left(d + \overset{(P)}{\mathbb{T}}_q \circ e^{\hbar g^{-1}} + \hbar\bar{\omega}^{-1}\right), \quad (2.5)$$

where $\bar{\omega}^{-1} = P^{\wedge 2}(\omega^{-1})$ and $\overset{(P)}{\mathbb{T}}_q \circ e^{\hbar g^{-1}}$ represents the graphs with a single output labeled by P . Furthermore, there is an IBL_∞ -isomorphism from $(A, \bar{\mathfrak{L}})$ to $(A, \mathfrak{L})$. d is called the linear contractible part and $\overset{(P)}{\mathbb{T}}_q \circ E(\hbar g^{-1}) + \hbar\bar{\omega}^{-1}$ the minimal part.

Proof. The proof follows from explicit verification, using equation (2.1), (2.2) and (2.4). $\square$

Finally, we describe the quantum generalization of the classical open-closed homotopy algebra (OCHA) of Kajiura and Stasheff. As already alluded to in the introduction, the OCHA can be described by an L_∞ -morphism, N , mapping from the closed string algebra (A_c, L) to the deformation complex of the open string algebra $(\text{Coder}^{\text{cycl}}(TA_o), d_h, [\cdot, \cdot])$, i. e.,

$$e^N \circ L = D(d_h + [\cdot, \cdot]) \circ e^N,$$

or equivalently

$$N \circ L = d_h \circ N + \frac{1}{2}[N, N] \circ \Delta, \quad (2.6)$$

where N describes the open-closed vertices and the comultiplication $\Delta : TA \rightarrow TA \otimes TA$ is defined by

$$\Delta(a_1 \otimes \dots \otimes a_n) = \sum_{i=0}^n (a_1 \otimes \dots \otimes a_i) \otimes (a_{i+1} \otimes \dots \otimes a_n).$$

In a similar way, one can describe the QOCHA by an IBL_∞ -morphism from the loop homotopy Lie algebra $(A_c, \mathfrak{L})$ of closed strings to the involutive Lie bialgebra $(\mathcal{A}_o, d_h, [\cdot, \cdot], \delta)$, where $^* \mathcal{A}_o := \text{Hom}^{\text{cycl}}(TA_o, \mathbb{k})$. The operation

$$\delta : \mathcal{A}_o \rightarrow \mathcal{A}_o^{\wedge 2},$$

is defined by

$$(\delta f)(a_1, \dots, a_n)(b_1, \dots, b_m) := (-1)^f \sum_{i=1}^n \sum_{j=1}^m (-1)^\epsilon \cdot f(e_k, a_i, \dots, a_n, a_1, \dots, a_{i-1}, e^k, b_j, \dots, b_m, b_1, \dots, b_{j-1}), \quad (2.7)$$

where $(-1)^\epsilon$ denotes the Koszul sign, $\{e_i\}$ is a basis of A_o and $\{e^i\}$ is the corresponding dual basis satisfying $\omega_o(i e, e^j) = \delta^j_i$. This operation can be interpreted geometrically as the sewing of open strings on one boundary component. In [1, 2] it has been shown that $(\mathcal{A}_o, d_h, [\cdot, \cdot], \delta)$ defines an involutive Lie bialgebra, a special case of an IBL_∞ -algebra. In the language of IBL_∞ -algebras, this is equivalent to the statement that

$$\mathfrak{L}_o := D(d_h + [\cdot, \cdot] + \hbar \delta)$$

squares to zero.

Definition 2 ([10]). The quantum open-closed homotopy algebra is defined by an IBL_∞ -morphism from a loop homotopy Lie algebra $(A_c, \mathfrak{L}_c)$ to the involutive Lie bialgebra $(\mathcal{A}_o, \mathfrak{L}_o)$, i.e.

$$e^n \circ \mathfrak{L}_c = \mathfrak{L}_o \circ e^n \quad (2.8)$$

The maps n describe the open-closed vertices to all orders in $\hbar$.

Equation (2.8) can be recast so that the five distinct sewing operations in open-closed string field theory become apparent:

$$\begin{aligned} n \circ \mathfrak{L}_c + \frac{\hbar}{2} (n \circ D(e_i) \wedge n \circ D(e^i)) \circ \Delta \\ = \mathfrak{L}_o \circ n + \frac{1}{2} D([\cdot, \cdot]) \circ (n \wedge n) \circ \Delta - ((D([\cdot, \cdot]) \circ n) \wedge n) \circ \Delta. \end{aligned} \quad (2.9)$$

In equation (2.9), e_i and e^i denote a basis and corresponding dual basis of A_c w.r.t. the symplectic structure ω_c . Obviously we recover the OCHA of equation (2.6) in the limit $\hbar \rightarrow 0$.

Similarly to the classical case, the morphism e^n is a quasi-isomorphism which implies isomorphism of the corresponding moduli spaces, i.e.

$$\mathcal{M}(A_c, \mathfrak{L}_c) \cong \mathcal{M}(A_o, \mathfrak{L}_o).$$

*In the quantum case it is more convenient to work with $\text{Hom}^{\text{cycl}}(TA_o, \mathbb{k})$ rather than with $\text{Coder}^{\text{cycl}}(TA_o)$

Theorem 3 ([10]). *The moduli space of any loop homotopy Lie algebra is empty,*

$$\mathcal{M}(A_c, \mathfrak{L}_c) = \emptyset.$$

Proof. The proof follows from considering the order $\hbar$ term of the Maurer Cartan equation for a general ansatz. This equation, together with the non-degeneracy of the symplectic form implies triviality of the cohomology, which in turn implies that $\mathcal{M}(A_c, \mathfrak{L}_c) = \emptyset$. $\square$

Remark 4. The story is different for the topological string, where the symplectic structure ω degenerates on-shell. Under this condition, Theorem 3 does not hold anymore, which implies consistency of open topological string theory at the quantum level in contrast to bosonic string theory.

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GEOMETRICAL ISSUES IN THE CONTINUUM MECHANICS OF DEFECTIVE SOLID CRYSTALS

RACHEL NICKS AND GARETH PARRY

Abstract. We shall outline geometrical and algebraic ideas which appear to lie at the foundation of the theory of defective crystals that was introduced by Davini [5] in 1986. The focus of the paper will be on the connection between continuous and discrete models of such crystals, approached by consideration of the symmetries inherent in these models. To begin with, we review briefly the results of analysis of variational problems where relevant functionals have the symmetry of perfect (as opposed to defective) crystals, in order to motivate the subsequent study of symmetry in the case when defects are present. In the body of the paper we indicate how the theory of Lie groups, and their discrete subgroups, relates to this geometrical theory of defects, and discuss types of symmetry that occur.

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1. INTRODUCTION

In Davini's [5] model of defective crystals, the kinematical state of the crystal corresponds to a distribution of three vector fields in a region $\Omega \subseteq \mathbb{R}^3$. Thus the defective crystal state, Σ say, is represented by

$$\Sigma = \{\ell_a(\cdot), \Omega; \quad a = 1, 2, 3\}, \quad (1.1)$$

where $\ell_1(\cdot), \ell_2(\cdot), \ell_3(\cdot)$ are three 'lattice' vector fields defined and linearly independent at each $x \in \Omega$. One imagines that these three smooth fields also determine the local discrete structure at any point $x \in \Omega$. Thus, in the particular case where the crystal is 'perfect', the lattice vector fields are independent of the point x , $\ell_a(x) = e_a$ say, $x \in \Omega$, $a = 1, 2, 3$, where e_1, e_2, e_3 is a basis of $\mathbb{R}^3$, and it is common to associate the perfect crystal lattice

$$L \equiv \{x : x = n_1 e_1 + n_2 e_2 + n_3 e_3, \quad n_1, n_2, n_3 \in \mathbb{Z}\}, \quad (1.2)$$

(the translational symmetries of which are e_1, e_2, e_3) with these 'constant' fields. The manner in which states Σ determine corresponding discrete structures, in the general case, will be made explicit later.

The second author is the corresponding author.

So far as continuum mechanical theories based on kinematical models such as (1.1) are concerned, it is a central task to prescribe how quantities such as stress, energy, etc., depend on the geometrical fields – this is a specification of the *constitutive* behaviour of the material. We shall not be concerned here with detailed mechanical theories, but remark that there are two experimentally well documented types of kinematical change (i. e. change of crystal state) which are generally reckoned to play a significant role in any mechanical theory which purports to describe these materials. The first of these types of change of state is *elastic deformation*: two crystal states, Σ and $\Sigma' \equiv \{\ell'_a(\cdot), \Omega'\}; a = 1, 2, 3\}$ are said to be elastically related to one another if there exists a smooth invertible mapping $\mathbf{u} : \Omega \rightarrow \mathbf{u}(\Omega) = \Omega'$ such that

$$\ell'_a(\mathbf{u}(\mathbf{x})) = \nabla \mathbf{u}(\mathbf{x}) \ell_a(\mathbf{x}), \quad a = 1, 2, 3. \quad (1.3)$$

It is commonly held that a change of state which is inelastic (i. e., not an elastic deformation) is irreversible, in a thermodynamic sense. (We do not embrace that prejudice here). The second type is a particular inelastic change of state: in *slip*, or *rearrangement of the lattice vector fields*, thin layers of the crystal slide one over the other with the ‘internal structure’ (i. e. the distribution of fields in each layer) unchanged in the process. We shall not need to be mathematically precise regarding this type of change of state in this paper (see Chipot and Kinderlehrer [4], Fonseca and Parry [9] for such treatments).

We proceed as follows: in the next section we motivate the consideration of symmetry in defective crystals by reviewing results for a variational problem that corresponds to minimizing a functional which represents the energy of a perfect crystal (in that the corresponding energy density has perfect crystal symmetries). It turns out that the symmetry properties determine *qualitative* properties of the set of minimizers. Subsequently, we indicate how ‘elastic invariants’ may be constructed from the lattice vector fields and their derivatives – the invariants are measures of inelastic change (since they are invariant under elastic deformation). It turns out that certain rearrangements of the lattice vector fields also preserve the elastic invariants, so that the ‘slip’ type of change of state has an abstract status in this kinematical theory, subordinate to the elastic invariants.

The main part of the paper describes how one associates a discrete structure with the continuous lattice vector fields of (1.1), how a simple constitutive assumption regarding the energy density of a defective crystal leads to the theory of Lie groups and their discrete subgroups, how perfect crystal symmetries are generalized to the defective crystal case. It will turn out that the symmetries with which we are concerned correspond to changes in sets of generators of certain discrete Lie groups. We indicate, briefly, that particular symmetries (those which extend to automorphisms of the corresponding discrete group) extend to symmetries of the continuous model (the automorphisms of the ambient Lie group), following Gorbatsevich [10] in part. This last observation is important from the continuum mechanics point of view, for it shows that certain discrete symmetries are embedded in elastic deformation, so

that these *discrete* geometrical changes can themselves be regarded as elastic, even though elastic deformation is a continuum concept.

2. A VARIATIONAL PROBLEM FOR PERFECT CRYSTALS

In the context of elastic changes of state determined by a deformation gradient $F(\mathbf{x}) \equiv \nabla \mathbf{u}(\mathbf{x})$, cf. (1.3), Chipot and Kinderlehrer [4] considered the following variational problem: for a given non-singular 3×3 matrix A , find

$$E_\Omega(A) \equiv \inf_{\mathcal{A}(A)} \int_\Omega w(\nabla \mathbf{u}(\mathbf{x})) \, d\mathbf{x}, \quad (2.1)$$

where

$$\mathcal{A}(A) \equiv \{\mathbf{u} \in W^{1,\infty}(\Omega; \mathbb{R}^3); \det \nabla \mathbf{u} > 0 \text{ a.e. in } \Omega, \mathbf{u} = A\mathbf{x} \text{ on } \partial\Omega\}, \quad (2.2)$$

and where the prescribed *energy density* function w has the properties

$$w(F) \geq 0, \quad w(\mathbb{I}) = 0, \quad \lim_{\det F \rightarrow 0} w(F) = \infty, \quad w(F) = w(QFH), \quad (2.3)$$

whenever Q is proper orthogonal, $\det(F) > 0$, and H is an element of a particular group conjugate to $\text{GL}_3(\mathbb{Z})$. The requirement on the form of the matrix H relates to the assumption that the material under consideration is a perfect crystal — it derives from two facts: first that if $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is an integral basis of L , cf. (1.2), then $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ is also a basis of L if and only if $\mathbf{e}'_a = \gamma_{ab}\mathbf{e}_b$, $a, b = 1, 2, 3$, where $\gamma \equiv (\gamma_{ab}) \in \text{GL}_3(\mathbb{Z})$; second that the crystal structure corresponding to deformation gradient F is a perfect lattice with basis vectors $F\mathbf{e}_1, F\mathbf{e}_2, F\mathbf{e}_3$. They find that:

- (i) $E_\Omega(A) = |\Omega|\tilde{w}(A)$ whenever $\Omega \subset \mathbb{R}^3$ is a bounded domain with $\partial\Omega$ of measure zero;
- (ii) $\tilde{w}$ is the greatest convex function less than the function ϕ defined by

$$\phi(\det A) = \inf_{\det F = \det A} w(F); \quad (2.4)$$

- (iii) For all uniformly bounded (in L^∞) minimizing sequences $\{\nabla \mathbf{u}_n\}$ of (2.1), with corresponding Young measure $\{\mu_{\mathbf{x}}\}_{\mathbf{x} \in \Omega}$, the *average* limiting Cauchy stress, defined by

$$\overline{T}(\mathbf{x}) = \int_{M^{3 \times 3}} \frac{1}{\det F} \frac{\partial w}{\partial F}(F) F^T d\mu_{\mathbf{x}}, \quad (2.5)$$

where F^T denote the transpose of F , $M^{3 \times 3}$ is the set of all 3×3 matrices, is a pressure, i. e., $\overline{T}(\mathbf{x}) = \alpha(\mathbf{x})\mathbb{I}$, for some $\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}$, $\mathbb{I}$ the 3×3 identity.

In particular, according to (iii) the ‘perfect’ symmetry of the energy density, expressed through the requirement $w(F) = w(QFH)$, cf. (2.3), implies that the crystal cannot sustain any stress other than a pressure i. e., it cannot sustain shear stresses in equilibrium. This result is contradicted by experiment, so one is forced to concede that real crystals are not perfect, and one is forced to consider what symmetry properties should apply to energy density functions appropriate for defective crystals.

Note that Chipot and Kinderlehrer's results were extended by Fonseca and Parry [9] to the case where the class of competitor functions $\mathcal{A}(A)$ includes rearrangements of the crystal state as well as elastic deformations.

3. INVARIANT INTEGRALS

Since inelastic changes of state play an essential role in the mechanics of defective crystals, it is important to consider the following question: given two crystal states, how does one determine whether or not those states are elastically related to one another. To discuss this we construct elastic invariants, which are unchanged in elastic deformation — then if two states are elastically related, it is a necessary condition that these objects match in the two states. As an example, consider the scalar fields

$$S_{ab} \equiv \nabla \wedge \mathbf{d}_a \cdot \mathbf{d}_b / \det(\{\mathbf{d}_a\}), \quad a, b = 1, 2, 3, \quad (3.1)$$

where $\mathbf{d}_1(\cdot), \mathbf{d}_2(\cdot), \mathbf{d}_3(\cdot)$ are fields dual to the lattice vector fields, and

$$\det(\{\mathbf{d}_a\}) \equiv \mathbf{d}_1 \cdot \mathbf{d}_2 \wedge \mathbf{d}_3.$$

One calculates that if (1.3) holds, and if S'_{ab} is calculated from the fields $\ell'_a(\cdot)$, $a = 1, 2, 3$, then

$$S'_{ab}(\mathbf{u}(x)) = S_{ab}(x), \quad a, b = 1, 2, 3. \quad (3.2)$$

Therefore

$$\text{range}_{y \in \Omega'} S'_{ab}(y) = \text{range}_{x \in \Omega} S_{ab}(x), \quad a, b = 1, 2, 3, \quad (3.3)$$

and each of the nine quantities $\text{range}_{x \in \Omega} S_{ab}(x)$ is an elastic invariant. Likewise, for example, if ζ is a circuit in Ω , one calculates that $\oint_{\zeta} \mathbf{d}_a \cdot d\mathbf{x}$, $a = 1, 2, 3$, is an elastic invariant integral. In fact, there exists an infinite number of elastic invariant objects, there is a finite functional basis for these quantities, see Olver [16], Davini and Parry [6, 7]. So one obtains a finite number of conditions necessary that two crystal states be elastically related.

Consider the particular case where $\Sigma = \{\mathbf{e}_a, \Omega; a = 1, 2, 3\}$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is a basis of $\mathbb{R}^3$ with $\mathbf{e}_1 \cdot \mathbf{e}_2 \wedge \mathbf{e}_3 = 1$, so that the lattice vector fields are constant in Ω . Then all of the quantities S_{ab} , $a, b = 1, 2, 3$, are zero, and it turns out that $\Sigma' \equiv \{\ell'_a(\cdot), \Omega\}$ (we choose $\Omega' = \Omega$ for simplicity) is such that all the elastically invariant objects match those of Σ if and only if

$$\mathbf{d}'_a(x) = \nabla \tau_a(x), \quad x \in \Omega, \quad a = 1, 2, 3; \quad \mathbf{d}_1(x) \cdot \mathbf{d}_2(x) \wedge \mathbf{d}_3(x) = 1, \quad x \in \Omega, \quad (3.4)$$

for some potentials τ_1, τ_2, τ_3 such that $\nabla \tau_1, \nabla \tau_2, \nabla \tau_3$ are linearly independent at each $x \in \Omega$. The important point to note is that Σ and Σ' are *not* elastically related to each other, in this case. Indeed, Σ' is elastically related to $\tilde{\Sigma} \equiv \{\mathbf{e}_a, \tau(\Omega)\}$ where $\tau \equiv (\tau_a)$, but $\tilde{\Sigma}$ is not elastically related to Σ because, if that were so, there would exist $\mathbf{u} : \Omega \rightarrow \mathbf{u}(\Omega) = \tau(\Omega)$ such that $\nabla \mathbf{u} = \mathbb{I}$, from (1.3). Then we would have that $\mathbf{u} = \mathbf{x} + \mathbf{c}$, $\mathbf{c} \in \mathbb{R}^3$, $\mathbf{u}(\Omega) = \tau(\Omega) = \Omega + \mathbf{c}$, which is false for general choices of τ . One says that Σ and $\tilde{\Sigma}$ are related by *slip*, or *rearrangement of the lattice vector*

fields, since Σ and $\tilde{\Sigma}$ are defined on regions which have the same volume (because $\det \nabla \tau = 1$ in Ω), and the corresponding fields $\ell_a(\cdot)$, $\tilde{\ell}_a(\cdot)$ are restrictions of fields which have the same constant values in $\mathbb{R}^3$. (This is just a *particular example* of slip).

The slip mechanism is the archetypical inelastic change of state in phenomenological theories of defective crystals. The remarks above that indicate the slips are inelastic changes of state which preserve the elastic invariants – and this is so.

4. ENERGY DENSITY FOR DEFECTIVE CRYSTALS, SYMMETRY OF ENERGY DENSITY

We assume that point values of quantities associated with the lattice vector fields determine the values of a corresponding energy density function w , and we make what seems to be the simplest extrapolation of elasticity theory that accounts for the existence of continuous distributions of defects. We assume that

$$w = w(\{\ell_a\}, S). \quad (4.1)$$

In (4.1) $\{\ell_a\}$ denotes a set of vectors $\ell_1, \ell_2, \ell_3 \in \mathbb{R}^3$, assumed to consist of values of the fields $\ell_1(\cdot), \ell_2(\cdot), \ell_3(\cdot)$ at some point, the origin $\mathbf{0}$ say, and S lies in a particular class of 3×3 matrices. The field $S(\cdot) \equiv (S_{ab}(\cdot))$, defined through the duals of the lattice vector fields by (3.1), measures the non-commutativity of the fields $\{\ell_a(\cdot)\}$ to some extent, so it is nonzero in the case of a defective crystal, by definition. $S(\cdot)$ is called the *dislocation density tensor* (even though it represents nine scalar fields). As an argument of the energy density w , we are given just one value of the field $S(\cdot)$ and no information regarding any of the derivatives of $S(\cdot)$ – so we make the *major assumption* that the field $S(\cdot)$ is constant in Ω . The fact that $S(\cdot)$ is constant restricts the possible values that the constant $S = S(\mathbf{0})$, say, may take, but we do not make this explicit.

In a different language, the arguments of the energy density are determined by a ‘uniform crystal state,’ which is a crystal state where the lattice vector fields have constant dislocation density S , and $\ell_a(\mathbf{0}) = \ell_a$, $a = 1, 2, 3$.

In the particular case where $S = 0$, so the crystal is ‘perfect’, it is a traditional a priori assumption that the symmetries of the energy density $w(\{\ell_a\}, 0)$ correspond to symmetries of the perfect lattice L with integral basis vectors ℓ_1, ℓ_2, ℓ_3 . Thus the symmetries of the *continuum* energy density are derived from an associated *discrete* structure, L in this case. We shall consider, in the following sections, ways of associating a discrete structure, D , say, with given arguments $\{\ell_a\}, S$ of the energy density, in the general case where $S \neq 0$. Also, we shall take the point of view that if different arguments $(\{\ell'_a\}, S')$ lead to the same discrete structure D , then

$$w(\{\ell_a\}, S) = w(\{\ell'_a\}, S'), \quad (4.2)$$

and say that such equivalent arguments, $\{\ell_a\}, S$ and $\{\ell'_a\}, S'$, represent ‘symmetries’ of the energy density (this argument mimics what is done when $S = 0$). We shall also

be concerned with whether or not such equivalent arguments correspond to uniform crystal states which are elastically related to one another, for in the case when such arguments are so related there will be a precise sense in which particular discrete symmetries of (4.2) represent elastic changes of state.

5. DEFECTIVE ANALOGUES OF PERFECT LATTICES AND LINKS TO THE THEORY OF LIE GROUPS

We are interested in uniform defective crystal configurations where the lattice vector fields $\{\ell_a(\cdot)\}$ are defined in $\Omega \equiv \mathbb{R}^3$, and are such that the dislocation density tensor $S(\cdot)$ is constant in $\mathbb{R}^3$. The condition that S is constant is an integrability condition which guarantees that for given lattice vector fields, the partial differential system

$$\ell_a(\psi(x, y)) = \nabla_1 \psi(x, y) \ell_a(x), \quad a = 1, 2, 3, \quad (5.1)$$

where $\nabla_1 \psi(\cdot, \cdot)$ denotes the gradient of ψ with respect to its first argument, has a solution for the function ψ . Moreover, the function $\psi : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ can be taken to satisfy the properties required for it to be a Lie group composition function, i. e.

$$\begin{aligned} \psi(\mathbf{0}, x) &= \psi(x, \mathbf{0}) = x, \\ \psi(x, x^{-1}) &= \psi(x^{-1}, x) = \mathbf{0}, \\ \psi(\psi(x, y), z) &= \psi(x, \psi(y, z)), \end{aligned}$$

where $\mathbf{0}$ is the group identity element and x^{-1} is the unique inverse of the element x [17, 21]. Given an appropriate value of the dislocation density tensor S one can specify a corresponding Lie group G by constructing fields $\ell_a(\cdot)$, $a = 1, 2, 3$ such that the dual fields satisfy (3.1) and then solving (5.1) for the group composition function ψ .

Since the dislocation density tensor S is an elastic invariant, the composition function ψ obtained from (5.1) is just one amongst the infinite number of those which may be found by making different choices of the lattice vector fields, given a value of S .

Each Lie group $G = (\mathbb{R}^3, \psi)$ has a corresponding Lie algebra $\mathfrak{g}$ consisting of the vector space $\mathbb{R}^3$, here, with Lie bracket operation $[\cdot, \cdot] : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$[x, y] = C_{ijk} x_j y_k e_i, \quad x, y \in \mathbb{R}^3, \quad (5.2)$$

where $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{R}^3$, $x = x_j e_j$, $y = y_j e_j$, C_{ijk} are the structure constants given by

$$C_{ijk} = \frac{\partial^2 \psi_i}{\partial x_j \partial y_k}(\mathbf{0}, \mathbf{0}) - \frac{\partial^2 \psi_i}{\partial x_k \partial y_j}(\mathbf{0}, \mathbf{0}), \quad (5.3)$$

where $\psi(x, y) = \psi_i(x, y) e_i$. The structure constants with respect to the basis $\{e_1, e_2, e_3\}$ are related to the components of the dislocation density tensor by

$$C_{kij} \ell_{rj}(\mathbf{0}) \ell_{sk}(\mathbf{0}) = \epsilon_{prs} S_{kp} \ell_{ki}(\mathbf{0}), \quad (5.4)$$

where ϵ_{rij} is the permutation symbol and $\ell_r(\mathbf{0}) = \ell_{rj}(\mathbf{0})e_j$, see Elzanowski and Parry [8].

Given a value of S , for a particular choice of ψ (equivalently G), the corresponding geometrical structure D , which we take to be the defective crystal analogue of the perfect lattice, is constructed as follows. Let $\ell_a(\cdot)$, $a = 1, 2, 3$, satisfy (5.1) and let v_1, v_2, v_3 be given real numbers. Define the integral curve through x_0 of the field $v_a \ell_a(\cdot)$ to be the solution $\{x(t) : t \in \mathbb{R}\}$ of the ordinary differential equation $\dot{x}(t) = v_a \ell_a(x(t))$, $x(0) = x_0$. Note that $v := v_a \ell_a(\mathbf{0})$ determines the field $v_a \ell_a(\cdot)$ by (5.1). One defines the mapping $\exp(v) : G \rightarrow G$, and the group element $e^{(v)}$, by

$$\exp(v)(x_0) = x(1), \quad e^{(v)} = \exp(v)(\mathbf{0}) \quad (5.5)$$

It is standard result of Lie group theory that

$$\exp(v)(x) = \psi(e^{(v)}, x), \quad (5.6)$$

which states that the flow along the integral curves of the lattice vector fields corresponds to group multiplication by the group element $e^{(v)}$.

The set of points of the geometrical structure D can be built through an iteration process. One begins with the origin then adds the six points which are reached by flowing along each of the vector fields $\ell_a(\cdot)$, $a = 1, 2, 3$, forwards and backwards one 'unit', i.e. $e^{(\ell_a)}$ and $e^{-(\ell_a)}$, $a = 1, 2, 3$, where $\ell_a = \ell_a(\mathbf{0})$. One repeats this process starting at each of these six points, and continues in this way. By (5.6) the structure, D that one obtains in this way is the subgroup of G that is generated by the group elements $e^{(\ell_1)}, e^{(\ell_2)}, e^{(\ell_3)}$ (noting that $e^{-(\ell_a)}$ is the inverse of $e^{(\ell_a)}$ in the group G).

Remark 1. In the case of perfect crystals $S = 0$, and one can choose $\ell_a(x) \equiv \ell_a(\mathbf{0}) \equiv \ell_a$ as a solution of (3.1) where $\{\ell_1, \ell_2, \ell_3\}$ is a basis of $\mathbb{R}^3$. Then $\psi(x, y) = x + y$ is a solution of (5.1) which has the properties of a Lie group composition function. Flow along the lattice vector fields corresponds in this case to translation by ℓ_1, ℓ_2, ℓ_3 . Successive translations produce the perfect lattice L as in (1.2). Hence the geometrical structure D produced by the iteration process described above is a generalisation of the perfect lattice L .

Let G be a three-dimensional Lie group with group multiplication ψ and write $\psi(x, y) \equiv xy$ as an alternative notation. Let $(x, y) = x^{-1}y^{-1}x, y$ denote the commutator of two group elements $x, y \in G$ and denote by G' the commutator subgroup of G - that is, the group generated by all commutators of elements of G . Notice that $(e^{(\ell_a)}, e^{(\ell_b)}) = e^{-(\ell_a)}e^{-(\ell_b)}e^{(\ell_a)}e^{(\ell_b)}$ is the group element obtained by successive flows along the vector fields $\ell_b(\cdot), \ell_a(\cdot), -\ell_b(\cdot), -\ell_a(\cdot)$ so there is a clear correspondence between the commutator of group elements $e^{(\ell_a)}, e^{(\ell_b)}$ and the construction of the Burgers vector in mechanics.

5.1. Conditions for discreteness

In general, the geometric structures generated by the iterative process outlined above are not discrete sets of points. The structures D are discrete when D is a uniform discrete subgroup of the Lie group G (i.e. the left coset space G/D is compact). This is a generalisation of the requirement in the perfect crystal case that $\mathbb{R}^3/L$ (the unit cell of the lattice L with appropriate identification of boundary points) is compact. According to Auslander et al. [1] there are three classes of three-dimensional Lie group which contain uniform discrete subgroups. These are a certain class of nilpotent groups and two non-isomorphic classes of solvable Lie groups. Recall that a three-dimensional Lie group G is nilpotent if $(G', G) = \mathbf{0}$, i.e. elements of the commutator subgroup G' commute with elements of the group G . Also a three-dimensional Lie group G is solvable if $(G', G') = \mathbf{0}$, i.e. elements of the commutator subgroup commute with each other.

We next summarize conditions on the dislocation density tensor which guarantee that the structures D are discrete, and give the forms of D explicitly, in the cases D is a uniform discrete subgroup of a three-dimensional nilpotent or (one type of) solvable Lie group. We also discuss the symmetries of D .

6. THE NILPOTENT CASE

Suppose that G is a three-dimensional nilpotent Lie group which contains a uniform discrete subgroup D . Mal'cev [12] shows that D must be generated by three elements and that the Lie algebra $\mathfrak{g}$ corresponding to G must have rational structure constants with respect to some basis. Taking an equivalent approach, following Thurston [22], it can be shown, by assuming first that there is a non-zero minimum distance between the points of D (generated by the iterative process outlined in section 2) and second that the points $e^{(\ell_a)}$, $a = 1, 2, 3$, are 'close enough' to the origin (see Parry [17] for details) that the Lie group G must be nilpotent and that the dislocation density has the form

$$S_{ab} = \lambda v_a v_b \quad a, b = 1, 2, 3, \quad (6.1)$$

where $\lambda \in \mathbb{Q}$, and v_1, v_2, v_3 are relatively prime integers [3, 17].

Now recall that, given a value of the dislocation density tensor S of the form (6.1), there are infinitely many possible choices of corresponding nilpotent Lie groups G corresponding to different choices of ψ . We make a canonical choice as follows. Let ψ be the group composition function for a three dimensional nilpotent Lie group G and let $e^{(\cdot)} : \mathfrak{g} \rightarrow G$ be the exponential function as defined by (5.5). The Campbell-Baker-Hausdorff formula gives an explicit expression for c in the relation $e^{(c)} = e^{(a)}e^{(b)}$ for $a, b \in \mathfrak{g}$. For three-dimensional nilpotent Lie groups

$$c = a + b + \frac{1}{2} [a, b], \quad (6.2)$$

and $c(a, b)$ satisfies the required conditions for it to be a group composition function on $\mathbb{R}^3$. Thus for any G in the relevant isomorphism class of Lie groups one can

define a canonical Lie group J by taking the group operation in J to be given by $\psi'(a, b) = c$. With this choice of composition function the one parameter subgroups of J are straight lines through the origin and $e^{(x)} = x$ (Lie group and Lie algebra elements can be identified).

When the components of the dislocation density tensor have the form (6.1) the structure constants with respect to the basis e_1, e_2, e_3 where $e_a = \ell_a(\mathbf{0})$, are given by

$$C_{ijk} = \lambda v_i v_r \epsilon_{rjk}. \quad (6.3)$$

Then the composition function in J has the form (dropping the prime in ψ')

$$\psi(x, y) = x + y + \frac{1}{2} \lambda v(v \cdot x \wedge y) \quad x, y \in \mathbb{R}^3, \quad (6.4)$$

where $v = v_r e_r$.

Let D be the discrete subgroup of J which is generated by $e^{(\ell_1)}, e^{(\ell_2)}, e^{(\ell_3)}$. The translation group T of D is defined by

$$T = \{t \in J : \text{if } d \in D, d + t \in D\}. \quad (6.5)$$

Let $\lambda = p/q \in \mathbb{Q}$ where p and $q \in \mathbb{Z}$ have no common factors and define $v = v_1 v_2 v_3$. Also define the integer k as

$$k = \begin{cases} p & \text{if } v \text{ is even or if } (v \text{ is odd and } p \in 4\mathbb{Z}) \\ \frac{p}{2} & \text{if } v \text{ is odd and } p \in 2\mathbb{Z}, p \notin 4\mathbb{Z} \\ 2p & \text{if } v \text{ is odd and } p \text{ is odd.} \end{cases} \quad (6.6)$$

Cermelli and Parry [3] show that if k is even then $T = D$ and T consists of all integer linear combinations of $e_1, e_2, e_3, \lambda v/k$. Thus the points of D form a three-dimensional lattice, as an integral basis of $T = D$ may be found in terms of $e_1, e_2, e_3, \lambda v/k$. Also, if k is odd they show that T consists of all integer linear combinations of $2e_1, 2e_2, 2e_3, \lambda v/k$ and D/T has four elements which may be written as $T, \alpha T, \beta T, \alpha\beta T$ for some $\alpha, \beta \in D$. Thus the points of D form a 4-lattice in the sense of Pitteri and Zanzotto [20].

6.1. Canonical Coordinates and Symmetries of D

The uniform discrete subgroup $D \subset J$ is generated by three elements $e^{(\ell_1)}, e^{(\ell_2)}, e^{(\ell_3)}$. According to Mal'cev [12], in a three dimensional nilpotent Lie group with structure constants given by (6.3) there are generators of D and corresponding Lie algebra elements $c_1, c_2, c_3 \in \mathbb{R}^3$ such that for some integer k_c

$$[c_1, c_2] = k_c c_3, \quad [c_1, c_3] = [c_2, c_3] = \mathbf{0}.$$

Recalling that group and algebra elements may be identified in J , Mal'cev shows further that any $d \in D$ can be written uniquely in the form

$$d = c_1^\alpha c_2^\beta c_3^\gamma, \quad \alpha, \beta, \gamma \in \mathbb{Z}.$$

(This relation is a generalisation of the expression of a point x of a perfect lattice with basis ℓ_1, ℓ_2, ℓ_3 as $x = \alpha\ell_1 + \beta\ell_2 + \gamma\ell_3$, $\alpha, \beta, \gamma \in \mathbb{Z}$.) Also note that, for $x, y \in J$, $x^{-1} = -x$, etc., and one can show that the commutator and Lie bracket can also be identified. In particular,

$$(c_1, c_2) = c_3^{k_c}, \quad (c_1, c_3) = (c_2, c_3) = \mathbf{0},$$

so the commutator subgroup D' is generated by $c_3^{k_c}$. The commutator $(c_1, c_2) = c_3^{k_c}$ represents flow along the vector fields defined by the Lie algebra elements $c_2, c_1, -c_2, -c_1$ successively, because Lie group and algebra elements can be identified in this canonical representation. The corresponding Burgers vector is an integer multiple of the third vector c_3 and one may picture this lack of commutativity as a 'screw dislocation'.

Let the discrete subgroup $D \subset J$ be generated by c_1, c_2, c_3 where these three elements provide a canonical basis for D as described above. Parry and Sigrist [19] compute the necessary and sufficient conditions that elements $e_1, e_2, e_3 \in D$ also generate D . These changes of generators are the geometrical symmetries of the set of points D . The computations in [19] use the fact that the translation group consists of all integer linear combinations of c_1, c_2, c_3 in the case that k_c is even and consists of all integer linear combinations of $2c_1, 2c_2$ and c_3 in the case that k_c is odd.

Let D_e denote the group generated by $e_1, e_2, e_3 \in D$. In the case where the (canonical) generators of D satisfy $(c_1, c_2) = c_3^{k_c}$ with k_c even there exists a matrix γ of integers with determinant Γ , say, such that $e_i = \gamma_{ij}c_j$ and $\gamma^{-1} = \pm \frac{1}{\Gamma}(p_{lk})$ where $p_{lk} \in \mathbb{Z}$. We define $v_i = p_{3i}$. (There are similar results in the case k_c odd). Parry and Sigrist [19] prove the following result regarding the geometrical symmetries of D (we give the results only for the case k_c even).

Theorem 2. $D_e = D$ if and only if there exist integers l_0, m_0, Γ , and a matrix $A \in \text{GL}_3(\mathbb{Z})$ with third row of relatively prime integers v_1, v_2, v_3 with $v := v_1 v_2 v_3$ such that if k_c is even, then

$$\begin{pmatrix} 1 & 0 & -l_0 \\ 0 & 1 & -m_0 \\ 0 & 0 & \pm\Gamma \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = A \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \quad (6.7)$$

where if $k_c \in 4\mathbb{Z}$ then $\text{hcf}(k_c, \Gamma) = 1$ or if $k_c \in 2\mathbb{Z}$, $k_c \notin 4\mathbb{Z}$ then either (v is even and $\text{hcf}(k_c, \Gamma) = 1$) or (v is odd and $\text{hcf}(k_c, \Gamma) = 2$).

7. THE SOLVABLE CASES

According to Auslander et al [1] there are two non-isomorphic classes of three-dimensional solvable Lie groups which have uniform discrete subgroups. They call these two classes S_1 and S_2 . Group elements $x \in S_p$, $p = 1, 2$, are identified with points of $\mathbb{R}^3$ by expressing them as $x = x_i e_i$ with respect to some basis e_1, e_2, e_3 . Auslander et al. [1] choose to represent the group elements as 4×4 matrices (still

parameterised by (x_1, x_2, x_3) and these matrix representations form an isomorphic group $S_{p,m}$ where the matrix representation of $\mathbf{x} \in S_p$ is $r_m(\mathbf{x}) \in S_{p,m}$ which has the form

$$r_m(\mathbf{x}) \equiv \left(\begin{array}{c|cc} \phi(x_3) & 0 & x_1 \\ & 0 & x_2 \\ \hline 0 & 0 & 1 \\ & 0 & x_3 \\ 0 & 0 & 0 & 1 \end{array} \right), \quad \mathbf{x} \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3, \quad \phi(x_3) = \begin{pmatrix} a(x_3) & b(x_3) \\ c(x_3) & d(x_3) \end{pmatrix}.$$

Here, $\{\phi(x_3) : x_3 \in \mathbb{R}\}$ is a one parameter subgroup of the unimodular group $\text{SL}_2(\mathbb{R})$ and $\phi(1) \in \text{SL}_2(\mathbb{Z})$. Let us define

$$\phi(1) \equiv \theta = \begin{pmatrix} a(1) & b(1) \\ c(1) & d(1) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1. \quad (7.1)$$

If $\text{tr}(\theta) > 2$ then the eigenvalues of θ are real and distinct and the group of matrices of the form $r_m(\mathbf{x})$ is isomorphic to S_1 . We discuss S_1 only. (S_2 will be treated elsewhere). Group multiplication ψ in S_1 is defined as follows:

$$r_m(\mathbf{x})r_m(\mathbf{y}) = r_m(\psi(\mathbf{x}, \mathbf{y})), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^3 \quad (7.2)$$

Let us also define

$$\phi'(0) = \mathcal{A} = \begin{pmatrix} a'(0) & b'(0) \\ c'(0) & d'(0) \end{pmatrix}, \quad \text{where } ' \text{ denotes } \frac{d}{dx}. \quad (7.3)$$

It is shown in [14] that $\text{tr}(\mathcal{A}) = 0$ and $\det(\mathcal{A}) \neq 0$ and that the dislocation density tensor is given by the following symmetric matrix:

$$S = \begin{pmatrix} -b'(0) & a'(0) & 0 \\ a'(0) & c'(0) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (7.4)$$

The discrete subgroup D of S_1 is generated by e_1, e_2, e_3 . Let these vectors have matrix representations $r_m(e_1) := B$, $r_m(e_2) := C$ and $r_m(e_3) := A$. Note that A depends on the matrix $\theta = \phi(1)$. The matrix group D_m generated by A, B, C is isomorphic to D . The commutators of the generating matrix elements satisfy

$$(A, B) = B^{1-d} C^c, \quad (A, C) = B^b C^{1-a}, \quad (B, C) = 0, \quad (7.5)$$

and from this one can show that $D = (\mathbb{Z}^3, \psi)$. Thus the discrete structure which is the analogue of perfect lattice L is the cubic lattice $\mathbb{Z}^3$ with group multiplication ψ defined by (7.2).

7.1. Symmetries of D

The set of global symmetries of a discrete subgroup $D \subset S_1$ corresponds to the choices of three elements $g_1, g_2, g_3 \in D$ such that the group generated by these elements, denoted $G = gp(g_1, g_2, g_3)$ in [14,15] equals D . These are the changes of generators which preserves the integer lattice $\mathbb{Z}^3$ in S_1 . It turns out that the conditions on g_1, g_2, g_3 that are necessary and sufficient that $G = D$ are the same as those that

are necessary and sufficient that the commutator subgroups G' and D' are equal. Here, we merely note that the elements $\{g_1, g_2, g_3\}$ can be put into canonical form by a sequence of the following operations: permuting a pair of elements, inversion of a single element, multiplication of precisely one element by another. Then, explicit formulae given in [14, 15] provide conditions, on the canonical set of elements so obtained, which are necessary and sufficient in order that the set generate D .

8. ELASTIC SYMMETRIES OF DEFECTIVE CRYSTALS

In Sections 6 and 7 we have summarised results about the nature and geometrical symmetries of the discrete structures D which we take to be the defective crystal analogues of the perfect lattice whose symmetries are widely used in classical crystallography. These structures D are embedded in a continuous Lie group, just as a perfect lattice L is embedded in $\mathbb{R}^3$ with addition as the group multiplication. Recall that the geometrical symmetries of a perfect lattice L with basis vectors $\{\ell_1, \ell_2, \ell_3\}$ are the bijections of L which preserve addition. These are given by $\phi_L(\ell_a) = \gamma_{ab}\ell_b$, $a, b = 1, 2, 3$ where $\gamma = (\gamma_{ab}) \in \text{GL}_3(\mathbb{Z})$. Note that these bijections $\phi_L : L \rightarrow L$ extend uniquely to smooth bijections of $\mathbb{R}^3$ that preserve addition so that every geometrical symmetry of L represents a (restriction of an) elastic deformation.

In contrast, in the case of the uniform defective crystal whose discrete structures D are discrete subgroups of nilpotent or solvable Lie groups, the analogous results do not hold. There is a difference between the set of ‘symmetries’ (the changes of generators summarised in Sections 6 and 7 which preserve the elements of D) and the subset of these symmetries which preserve the group structure of D (these are the automorphisms of D). We consider now whether or not the automorphisms of D extend uniquely to automorphisms of the ambient Lie group. First, an elastic deformation provides an isomorphism of Lie groups, see [18]. Recall, too, that if $\phi : G \rightarrow H$ is a Lie group homomorphism, and $\mathfrak{g}, \mathfrak{h}$ are the corresponding Lie algebras, then $\nabla\phi(\mathbf{0})$ is a Lie algebra homomorphism and

$$\phi(e^{(\mathbf{v})}) = e^{(\nabla\phi(\mathbf{0})\mathbf{v})}, \quad \mathbf{v} \in \mathfrak{g}, \quad (8.1)$$

where the exponential on the left hand side is the exponential which maps $\mathfrak{g}$ to G and that on the right hand side maps $\mathfrak{h}$ to H .

Next, the nilpotent group G , and S_1 , are exponential groups in the sense that each group element is the exponential of some element in the corresponding algebra. So the automorphisms of G and S_1 are straightforwardly calculated.

The automorphisms of *discrete subgroups* of G and S_1 are calculated as follows. Given a discrete subgroup D which is generated by the three elements d_1, d_2, d_3 , $i, j = 1, 2, 3$, one may compute the subset of the geometrical symmetries which represent automorphisms of D using a result of Magnus, Karrass and Solitar [11]. Thus, a mapping $\phi_D : \{d_1, d_2, d_3\} \rightarrow D$ is an automorphism of D if and only if for

any word w in the generators such that $w(d_1, d_2, d_3) = \mathbf{0}$,

$$w(\phi_D(d_1), \phi_D(d_2), \phi_D(d_3)) = 0 \text{ and } w(\phi_D^{-1}(d_1), \phi_D^{-1}(d_2), \phi_D^{-1}(d_3)) = 0, \quad (8.2)$$

where ϕ_D^{-1} is the mapping that takes $\phi_D(d_1)$ to d_1 , $\phi_D(d_2)$ to d_2 and $\phi_D(d_3)$ to d_3 . Thus, we determine the conditions that must be satisfied by ϕ_D by taking the words w in (8.2) to be the relevant commutator relations.

8.1. Automorphisms of nilpotent discrete subgroups

Let $D \subset J$ be the discrete subgroup generated by the canonical set of elements c_1, c_2, c_3 with $(c_1, c_2) = c_3^k$ and $(c_1, c_3) = (c_2, c_3) = \mathbf{0}$. From (8.1) the automorphisms of D (and J) are linear mappings since $e^{(x)} = x$ in J . It is shown in [18] that the automorphisms of D have the form $\phi_D : D \rightarrow D$ where $\phi(c_i) = c_1^{\alpha_i} c_2^{\beta_i} c_3^{\gamma_i}$, $i = 1, 2, 3$ and

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \beta_1 & \beta_2 & 0 \\ \gamma_1 & \gamma_2 & \epsilon \end{pmatrix}$$

where $\epsilon = \pm 1$, $\alpha_1 \beta_2 - \alpha_2 \beta_1 = \epsilon$ and γ_1, γ_2 are arbitrary integers.

In this case, a theorem of Mal'cev [12] states that every automorphism of D extends uniquely to an automorphism of J . Thus every symmetry of D which corresponds to an automorphism of D is an elastic symmetry of D in that it is the restriction of an elastic deformation of J . Symmetries of D which do not satisfy (8.2) cannot be extended to automorphisms of J , so they may be called inelastic symmetries.

8.2. Automorphisms of solvable discrete subgroups

In the recent work [13] we showed that $\phi_D : D_m \rightarrow D_m$ is an automorphism if $\phi_D(A) = A^{\alpha_1} B^{\beta_1} C^{\lambda_1}$, $\phi_D(B) = A^{\alpha_2} B^{\beta_2} C^{\lambda_2}$, $\phi_D(C) = A^{\alpha_3} B^{\beta_3} C^{\lambda_3}$, where $\alpha_1 = \xi = \pm 1$, $\alpha_2 = \alpha_3 = 0$, β_1, γ_1 arbitrary and

$$\chi := \begin{pmatrix} \beta_2 & \beta_3 \\ \gamma_2 & \gamma_3 \end{pmatrix} \in \text{GL}_2(\mathbb{Z}) \quad \text{is such that} \quad \theta^\xi \chi = \chi \theta. \quad (8.3)$$

There is an algorithm for determining the matrices χ which satisfy (8.3) for a given matrix θ , described in [13], following the method used in [2]. Note that from (7.2) and (7.5), it is the dislocation density tensor that determines the automorphisms of D , via (8.3). $\text{Aut}(D)$ is a finitely generated infinite group.

In the case where D is a uniform discrete subgroup of S_1 a theorem of Gorbatsovich [10] implies that every automorphism of D extends uniquely to an automorphism of S_1 . The automorphisms of the discrete group D are therefore restrictions of elastic deformations.

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SYMMETRIES FOR THOUGHT

MARIA CLARA NUCCI

In memory of my Hungarian grandmother Clara.

Abstract. This paper covers several topics that involve symmetries of differential equations: from the connection between Lie symmetries and Jacobi last multiplier with a detour to the inverse problem of calculus of variations, to the importance of maximal group of Noether symmetries on the road from classical to quantum mechanics.

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1. INTRODUCTION

Symmetries are everywhere, yet symmetry is an ambiguous word and can be misleading. The symmetries that are the subject of this paper are the Lie symmetries admitted by a differential equation, those symmetries that generate a Lie algebra. Sophus Lie himself provided a link between his namesake symmetries and the solutions of a differential equation, or its conservation laws by means of the Jacobi last multiplier [29]. Lie symmetries are also related to the inverse problem of calculus of variations because of the properties of the Jacobi last multiplier. Since conservation laws can be obtained without even a Lagrangian, the importance of Noether symmetries seems to be diminished, yet we have recently linked them to the problem of quantization [11, 36–38].

In the next section, we will recall the main properties of the Jacobi last multiplier, its connection to Lie symmetries, and the inverse problem of calculus of variations, namely the possibility of finding one (or more) Lagrangians.

In section 3, the extraordinary journey taken by three Lie symmetries, i. e.,

$$\Gamma_1 = \partial_x - \partial_y, \quad \Gamma_2 = x\partial_x + y\partial_y, \quad \Gamma_3 = x^2\partial_x - y^2\partial_y, \quad (1.1)$$

a representation of $\mathfrak{sl}(2, \mathbb{R})$, is narrated. It begins with the hydrodynamic equations as formulated by Riemann and ends with the Schrödinger equation. We anticipate that during this journey, more Lie symmetries will be picked up and consequently

a representation of $\mathfrak{sl}(3, \mathbb{R})$ is obtained. Also the Schwarzian derivative [14] and Calogero's goldfish [4, 35] are encountered.

2. LIE SYMMETRIES, JACOBI LAST MULTIPLIER, LAGRANGIANS

The method of the Jacobi last multiplier* [15–17] and [18][†] provides a means to determine all the solutions of the partial differential equation

$$\mathcal{A}f = \sum_{i=1}^n a_i(x_1, \dots, x_n) \frac{\partial f}{\partial x_i} = 0 \quad (2.1)$$

or its equivalent associated Lagrange's system

$$\frac{dx_1}{a_1} = \frac{dx_2}{a_2} = \dots = \frac{dx_n}{a_n}. \quad (2.2)$$

In fact, if one knows the Jacobi last multiplier and $n - 2$ functionally independent solutions, then the last solution can be obtained by a quadrature. The Jacobi last multiplier M is given by[‡]

$$\frac{\partial(f, \omega_1, \omega_2, \dots, \omega_{n-1})}{\partial(x_1, x_2, \dots, x_n)} = M \mathcal{A}f, \quad (2.3)$$

where

$$\frac{\partial(f, \omega_1, \omega_2, \dots, \omega_{n-1})}{\partial(x_1, x_2, \dots, x_n)} = \det \begin{pmatrix} \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \\ \frac{\partial \omega_1}{\partial x_1} & & \frac{\partial \omega_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \omega_{n-1}}{\partial x_1} & \dots & \frac{\partial \omega_{n-1}}{\partial x_n} \end{pmatrix} = 0 \quad (2.4)$$

and $\omega_1, \dots, \omega_{n-1}$ are $n - 1$ solutions of (2.1) or, equivalently, first integrals of (2.2) independent of each other. This means that M is a function of the variables $(x_1, \dots, x_n)$ and depends on the chosen $n - 1$ solutions, in the sense that it varies as they vary. The essential properties of the Jacobi last multiplier were proven by Jacobi himself and they are:

*Many authors have dealt with the Jacobi last multiplier, and an up-to-date (2004) nearly complete list can be found in [34]. It ranges from the 1871 paper by Laguerre [27] and the seminal 1874 paper by Lie [29] to the 2003 review paper by Berrone e Giacomini [2]. A missed reference in [34] is Section 2.11 in the 2001 book by Goriely [10].

[†]An English translation is now available [19].

[‡] M is not zero, yet connects two quantities that are zero.

- (a) If one selects a different set of $n - 1$ independent solutions $\eta_1, \dots, \eta_{n-1}$ of equation (2.1), then the corresponding last multiplier N is linked to M by the relationship

$$N = M \frac{\partial(\eta_1, \dots, \eta_{n-1})}{\partial(\omega_1, \dots, \omega_{n-1})}.$$

- (b) Given a non-singular transformation of variables

$$\tau : (x_1, x_2, \dots, x_n) \longrightarrow (x'_1, x'_2, \dots, x'_n),$$

then the last multiplier M' of $\mathcal{A}'F = 0$ is given by:

$$M' = M \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(x'_1, x'_2, \dots, x'_n)},$$

where M obviously comes from the $n - 1$ solutions of $\mathcal{A}F = 0$ which correspond to those chosen for $\mathcal{A}'F = 0$ through the inverse transformation τ^{-1} .

- (c) One can prove that each multiplier M is a solution of the following linear partial differential equation:

$$\sum_{i=1}^n \frac{\partial(Ma_i)}{\partial x_i} = 0; \quad (2.5)$$

vice versa, every solution M of this equation is a Jacobi last multiplier.

- (d) If one knows two Jacobi last multipliers M_1 and M_2 of equation (2.1), then their ratio is a solution ω of (2.1), or, equivalently, a first integral of (2.2). Naturally, the ratio may be quite trivial, namely a constant. *Vice versa*, the product of a multiplier M_1 times any solution ω yields another last multiplier $M_2 = M_1\omega$.

Since the existence of a solution/first integral is consequent upon the existence of symmetry, an alternative formulation in terms of symmetries was provided by Lie [29, 30]. A clear treatment of the formulation in terms of solutions/first integrals and symmetries is given by Bianchi [3]. If we know $n - 1$ symmetries of (2.1)/(2.2), say

$$\Gamma_i = \sum_{j=1}^n \xi_{ij}(x_1, \dots, x_n) \partial_{x_j}, \quad i = 1, \dots, n - 1, \quad (2.6)$$

a Jacobi last multiplier is given by $M = \Delta^{-1}$, provided that $\Delta \neq 0$, where

$$\Delta = \det \begin{pmatrix} a_1 & \cdots & a_n \\ \xi_{1,1} & \cdots & \xi_{1,n} \\ \vdots & & \vdots \\ \xi_{n-1,1} & \cdots & \xi_{n-1,n} \end{pmatrix}. \quad (2.7)$$

There is an obvious corollary to the results of Jacobi mentioned above. In the case that there exists a constant multiplier, the determinant is a first integral. This result is potentially very useful in the search for first integrals of systems of ordinary differential equations. In particular, if each component of the vector field of the equation

of motion is missing the variable associated with that component, i. e., $\partial a_i / \partial x_i = 0$, the last multiplier is a constant, and any other Jacobi last multiplier is a first integral.

Another property of the Jacobi last multiplier is its (almost forgotten) relationship with the Lagrangian, $L = L(t, x, \dot{x})$, for any second-order equation

$$\ddot{x} = \phi(t, x, \dot{x}) \quad (2.8)$$

i. e. [18, Lecture 10], [52]

$$M = \frac{\partial^2 L}{\partial \dot{x}^2} \quad (2.9)$$

where $M = M(t, x, \dot{x})$ satisfies the equation

$$\frac{d}{dt}(\log M) + \frac{\partial \phi}{\partial \dot{x}} = 0. \quad (2.10)$$

Then, equation (2.8) becomes the Euler–Lagrangian equation:

$$-\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) + \frac{\partial L}{\partial x} = 0. \quad (2.11)$$

The proof is given by taking the derivative of (2.11) by $\dot{x}$ and showing that this yields (2.10). If one knows a Jacobi last multiplier, then L can be obtained by a double integration, namely,

$$L = \int \left(\int M \, d\dot{x} \right) d\dot{x} + \ell_1(t, x)\dot{x} + \ell_2(t, x), \quad (2.12)$$

where ℓ_1 and ℓ_2 are functions of t and x which have to satisfy a single partial differential equation related to (2.8) [40]. As it was shown in [40], ℓ_1, ℓ_2 are related to the gauge function $G = G(t, x)$. In fact, we may assume

$$\begin{aligned} \ell_1 &= \frac{\partial G}{\partial x}, \\ \ell_2 &= \frac{\partial G}{\partial t} + \ell_3(t, x), \end{aligned} \quad (2.13)$$

where ℓ_3 has to satisfy the partial differential equation mentioned and G is obviously arbitrary.

It was shown in [39] that Jacobi last multiplier yields the Lagrangian for any equation of even order*

$$u^{(2n)} = F(x, u, u', u'', \dots, u^{(2n-1)}), \quad (2.14)$$

since it can be derived from the following formula

$$M^{1/n} = \frac{\partial^2 L}{\partial (u^{(n)})^2}, \quad (2.15)$$

where M is the Jacobi last multiplier of equation (2.14) and L is its Lagrangian. This formula was given by Jacobi himself in [17] p. 364.

*We use a prime to indicate the derivative with respect to x .

We recall that Fels has proven [7] that the Lagrangian is unique in the case of fourth-order equation if it exists. In the case of equations of sixth and eighth order, the uniqueness was proven by Juráš [20].

In [47], Tonti provided a brief historical survey of the inverse problem of calculus of variations. He begins with the year 1887 when both Helmholtz [13] and Volterra [50] published their work. Unfortunately, no mention is made of Jacobi's work. That historical survey should have at least begun — if not with the year 1845 when Jacobi's paper [17] was published in Crelle's journal — with the year 1884 when Jacobi's Dynamics Lectures delivered at the University of Königsberg in the Winter Semester 1842–1843, were finally published [18] with a foreword by Weierstrass.

As pointed out by Tonti [47], many authors have dealt with the inverse problem of calculus of variations by either using a formal approach or an operatorial approach following on the steps of either Helmholtz or Volterra, for example, [1, 31, 43, 46, 48, 49] and many others.

We do not underestimate the research of these very distinguished authors. Yet, when feasible — a single equation, a non-dissipative system [41] — we prefer to follow Jacobi since his last multiplier has a direct link to conservation laws and symmetries that are the essential elements that, in our opinion, make the difference between a mathematical abstraction and a physical concreteness.

It is a matter for historians alike to find out why Darboux [5], Helmholtz [13], Koenigsberger himself* and many other successive authors, e. g., Douglas [6] and Havas† [12], never acknowledged the use of the Jacobi last multiplier in order to find Lagrangians of a second-order equation. Sonin did [45, p. 10], although very few authors [26] cite his 1886 Russian paper.

Volterra knew Jacobi's work, especially [18], which he mentions in 1887 in one of his earlier works [50, p. 280]. Therefore, Volterra knew Lecture 10 of [18] since he cites p. 78 of precisely this Lecture in [50]. Maybe he overlooked the following pages, especially p. 82, where Jacobi wrote his formula (2.9) that links the last multiplier to the Lagrangian of any second-order equation. Also, in his 1906 address at the Congress of Italian Naturalists [51], Volterra wrote: *una delle più celebri scoperte del matematico Jacobi, quella del principio dell'ultimo moltiplicatore*.‡

*In 1902–1903, Koenigsberger wrote Helmholtz's biography [24] — which in 1906 was (abridged) translated into English with a Preface by Lord Kelvin [23] — after he wrote his 1901 book on Mechanics [21]. Neither book cites the connection between Jacobi last multiplier and Lagrangians. In 1904, Koenigsberger wrote Jacobi's biography [22] where the Jacobi last multiplier is extensively described.

†Havas even cites the book by Whittaker [52] but only in connection with the formulation of Lagrangian equations.

‡“One of the most celebrated discoveries by the mathematician Jacobi, that of the principle of the last multiplier.” (tr. by MCN)

3. FROM RIEMANN TO SCHRÖDINGER

In [8], it was found that the following linear second-order hyperbolic partial differential equation

$$(x + y)^2 u_{xy} + B_1(x + y)(u_x + u_y) + B_2 u = 0, \quad (3.1)$$

that encases the hydrodynamic equations as formulated by Riemann [8, 44] admits the following Lie symmetries for general values of the parameters B_1, B_2 :

$$\begin{aligned} \Lambda_1 &= \partial_x - \partial_y, & \Lambda_2 &= x\partial_x + y\partial_y, & \Lambda_3 &= x^2\partial_x - y^2\partial_y - B_1(x - y)u\partial_u, \\ \Lambda_4 &= u\partial_u, & \Lambda_\infty &= f(x, y)\partial_u, \end{aligned} \quad (3.2)$$

where $f(x, y)$ is any solution to (3.1). We emphasize that Λ_4 and Λ_∞ are the symmetries that are always admitted by any linear homogeneous partial differential equation. It is obvious that taking the independent variables only in (3.2) allows the identification of $\Lambda_1, \Lambda_2, \Lambda_3$ with $\Gamma_1, \Gamma_2, \Gamma_3$, respectively. We now assume that $y = y(x)$ and search for an ordinary differential equation of second order, say $y'' = F(x, y, y')$, that admits the three Lie symmetries (1.1). We obtain* a family of equations characterized by a parameter A , i. e.:

$$y'' = \frac{y'}{x + y} \left(A\sqrt{y'} + 2y' - 2 \right). \quad (3.3)$$

It is known [28] that if we solve this equation for the parameter A , i. e.

$$A = \frac{y''(x + y) - 2y'^2 + 2y'}{y'\sqrt{y'}}, \quad (3.4)$$

and then derive once with respect to x , a third-order equation is obtained, i. e.

$$y''' = \frac{3y''^2}{2y'}, \quad (3.5)$$

that admits a six-dimensional Lie symmetry algebra [9] generated by

$$\partial_x, \quad x\partial_x, \quad x^2\partial_x, \quad \partial_y, \quad y\partial_y, \quad y^2\partial_y. \quad (3.6)$$

Equation (3.5) contains the Schwarzian derivative [14] and thus, it is connected to both a linear second-order equation and a Riccati equation. In particular, the transformation [14]

$$y = \int \frac{1}{r^2} dx + a_3 \quad (3.7)$$

yields $r'' = 0$, so that $r = a_1 x + a_2$, and the general solution to (3.3) is

$$y = -\frac{1}{a_1(a_1 x + a_2)} + a_3, \quad a_3 = \frac{2a_2 - A}{2a_1}, \quad (3.8)$$

where a_1 and a_2 are arbitrary parameters.

*We employ our own *ad hoc* interactive REDUCE programs [33] in order to find Lie symmetries.

The calculation of the Jacobi last multiplier requires that the differential equation (3.3) be written as a system of first-order equations (2.2), i. e.

$$\frac{dt}{1} = \frac{dy}{y'} = \frac{dy'}{\frac{y'}{x+y} (A\sqrt{y'} + 2y' - 2)}, \quad (3.9)$$

then we can determine three Jacobi last multipliers for equation (3.3) by using the determinant (2.7) and two of the symmetries (1.1), i. e.

$$\Delta_{12} = \det \begin{pmatrix} 1 & y' & \frac{y'}{x+y} (A\sqrt{y'} + 2y' - 2) \\ 1 & -1 & 0 \\ x & y & 0 \end{pmatrix}, \quad (3.10)$$

$$\Delta_{13} = \det \begin{pmatrix} 1 & y' & \frac{y'}{x+y} (A\sqrt{y'} + 2y' - 2) \\ 1 & -1 & 0 \\ x^2 & -y^2 & -2y'(x+y) \end{pmatrix}, \quad (3.11)$$

and

$$\Delta_{23} = \det \begin{pmatrix} 1 & y' & \frac{y'}{x+y} (A\sqrt{y'} + 2y' - 2) \\ x & y & 0 \\ x^2 & -y^2 & -2y'(x+y) \end{pmatrix}. \quad (3.12)$$

Therefore, the three Jacobi last multipliers are:

$$JLM_{1,2} = \frac{1}{\Delta_{12}} = \frac{1}{y'(A\sqrt{y'} + 2y' - 2)}, \quad (3.13)$$

$$JLM_{1,3} = \frac{1}{\Delta_{13}} = \frac{1}{y'(4y - Ay\sqrt{y'} + 4xy' + Ax\sqrt{y'})}, \quad (3.14)$$

$$JLM_{2,3} = \frac{1}{\Delta_{23}} = \frac{1}{y'(2x^2y' - 2y^2 - Axy\sqrt{y'})}. \quad (3.15)$$

From property (d) in Section 2, we have that the ratio of two Jacobi last multipliers is a first integral of equation (3.3) and, therefore, three first integrals can be obtained, i. e.

$$\begin{aligned} Int_1 &= \frac{JLM_{1,2}}{JLM_{1,3}} = \frac{y'(4y - Ay\sqrt{y'} + 4xy' + Ax\sqrt{y'})}{y'(A\sqrt{y'} + 2y' - 2)}, \\ Int_2 &= \frac{JLM_{1,2}}{JLM_{2,3}} = \frac{y'(2x^2y' - 2y^2 - Axy\sqrt{y'})}{y'(A\sqrt{y'} + 2y' - 2)}, \\ Int_3 &= \frac{JLM_{1,3}}{JLM_{2,3}} = \frac{y'(2x^2y' - 2y^2 - Axy\sqrt{y'})}{y'(4y - Ay\sqrt{y'} + 4xy' + Ax\sqrt{y'})}. \end{aligned}$$

Three Lagrangians $L_{1,2}$, $L_{1,3}$, and $L_{2,3}$ can be obtained from (2.12) and the Jacobi last multipliers $JLM_{1,2}$, $JLM_{1,3}$, $JLM_{2,3}$, respectively. Then, applying Noether's theorem [32] yields that each of them admits only one Lie symmetry among those in (1.1) as Noether symmetry, and they are different from each other. In particular, we obtain that $L_{1,2}$ admits Γ_1 , $L_{1,3}$ admits Γ_2 , and $L_{2,3}$ admits Γ_3 . Since the expressions of those three Lagrangians are very long, we do not write them down here. They are available in REDUCE format upon request.

If $A = 0$, then equation (3.3) becomes:

$$y'' = \frac{2y'}{x+y} (y' - 1) \quad (3.16)$$

and admits five further Lie symmetries, namely,

$$\begin{aligned} \Gamma_4 &= \frac{1}{x+y} (\partial_x + \partial_y), & \Gamma_5 &= \frac{1}{x+y} (y\partial_x - x\partial_y), \\ \Gamma_6 &= \frac{xy}{x+y} (\partial_x + \partial_y), & \Gamma_7 &= \frac{xy}{x+y} (x\partial_x - y\partial_y), \\ \Gamma_8 &= \frac{xy}{x+y} (x^2\partial_x + y^2\partial_y). \end{aligned} \quad (3.17)$$

This equation is related to Calogero's goldfish [4], namely it is equation (24) in [35] with $a_{12} = -1$ and $w_2 = -y$.

The eight Lie operators Γ_i , $i = 1, 8$ generate an eight-dimensional Lie symmetry algebra $\mathfrak{sl}(3, \mathbb{R})$ which implies that equation (3.16) is linearizable by means of a point transformation [30]. In order to find the linearizing transformation, we have to look for an abelian intransitive subalgebra of $\mathfrak{sl}(3, \mathbb{R})$ and, following Lie's classification of two-dimensional algebras in the real plane [30], we have to transform it into the canonical form

$$\partial_{\tilde{y}}, \quad \tilde{x}\partial_{\tilde{y}} \quad (3.18)$$

with $\tilde{y}$ and $\tilde{x}$ the new dependent and independent variables, respectively. We have found two* such subalgebras, although they are obviously related since there exists only one abelian intransitive subalgebra of $\mathfrak{sl}(3, \mathbb{R})$ [53]. One such subalgebra is that generated by Γ_4 and $\Gamma_{10} = 2\Gamma_5 - \Gamma_1$ while the other subalgebra is that generated by Γ_8 and $\Gamma_9 = -2\Gamma_7 + \Gamma_3$. Then, it is easy to derive that

$$\{\Gamma_4, \Gamma_{10}\} \implies \tilde{x} = y - x, \quad \tilde{y} = -xy, \quad (3.19)$$

while

$$\{\Gamma_8, \Gamma_9\} \implies \tilde{x} = \frac{x-y}{xy}, \quad \tilde{y} = -\frac{1}{xy}. \quad (3.20)$$

*Actually, there is also a third one, i.e., that generated by Γ_7 and $\Gamma_1 - \Gamma_5$. The further analysis of this case is analogous to the other two and we omit it here at the request of an anonymous Referee.

Equation (3.16) becomes

$$\frac{d^2 \tilde{y}}{d\tilde{x}^2} = 0 \quad (3.21)$$

by means of both transformations.

We can also determine many Jacobi last multipliers for equation (3.16) by using the determinant (2.7) with two of the Lie point symmetries Γ_j , $j = 1, \dots, 10$. For example:

$$\begin{aligned} JLM_{1,2} &= \frac{1}{2y'(y' - 1)}, \\ JLM_{1,3} &= \frac{1}{4y'(xy' + y)}, \\ JLM_{2,3} &= \frac{1}{2y'(x^2y' - y^2)}, \\ JLM_{1,8} &= \frac{(x + y)^2}{(y^2 - x^2y')(x^2y'^2 + y^2y' + 4xyy' + x^2y' + y^2)}, \\ JLM_{8,9} &= JLM_{8,3} = \frac{(x + y)^2}{(y^2 - x^2y')^3}, \\ JLM_{4,10} &= JLM_{4,5} = \frac{(x + y)^2}{(1 - y')^3}, \\ JLM_{1,9} &= JLM_{3,10} = \frac{1}{2} JLM_{4,8} = \frac{1}{2} \frac{(x + y)^2}{(y' - 1)(y^2 - x^2y')(xy' + y)}, \\ JLM_{4,6} &= -\frac{(x + y)^2}{(1 - y')^2(xy' + y)}, \\ JLM_{3,5} &= \frac{(x + y)^2}{(y' - 1)(y^2 - x^2y')^2}. \end{aligned}$$

We observe that

$$\Delta_{67} = \Delta_{68} = \Delta_{78} = 0.$$

Then, it is easy to show [42] that $\{\Gamma_6, \Gamma_7, \Gamma_8\}$ are a representation of the complete symmetry group of equation (3.16), i. e. the group of the symmetries which completely specifies the differential equation under consideration [25].

We can now derive many Lagrangians for equation (3.16) from (2.12). In particular, the Lagrangian

$$L_{4,10} = \frac{1}{2} \frac{(x + y)^2}{1 - y'} \quad (3.22)$$

corresponds to the Jacobi last multiplier $JLM_{4,10}$, and admits five Noether symmetries and corresponding first integrals, i. e.

$$\begin{aligned}
 \Gamma_4 &\Rightarrow \frac{xy' + y}{1 - y'} \\
 \Gamma_5 &\Rightarrow -\frac{(y' - 2)x^2y' - 2xyy' - (2y' - 1)y^2}{(2(y' - 1))^2} \\
 -3\Gamma_6 + 2\Gamma_2 &\Rightarrow \frac{(y^2 - x^2y')(y + xy')}{(3(y' - 1))^2} \\
 \Gamma_9 &\Rightarrow -\frac{(y^2 - x^2y')^2}{4(y' - 1)^2} \\
 \Gamma_1 &\Rightarrow \frac{y'(x + y)^2}{(y' - 1)^2}.
 \end{aligned} \tag{3.23}$$

Also the Lagrangian

$$L_{8,9} = \frac{1}{2} \frac{(x + y)^2}{x^4(y^2 - x^2y')} \tag{3.24}$$

admits five Noether symmetries and corresponding first integrals, i. e.

$$\begin{aligned}
 \Gamma_{10} &\Rightarrow \frac{(y' - 1)^2}{4(y^2 - x^2y')^2} \\
 -3\Gamma_6 + \Gamma_2 &\Rightarrow \frac{(y + xy')(y' - 1)}{3(y^2 - x^2y')^2} \\
 \Gamma_7 &\Rightarrow \frac{(y + xy')^2}{2(y^2 - x^2y')^2} \\
 \Gamma_8 &\Rightarrow \frac{y + xy'}{x^2y' - y^2} \\
 \Gamma_3 &\Rightarrow \frac{y'(x + y)^2}{(y^2 - x^2y')^2}.
 \end{aligned} \tag{3.25}$$

Since equation (3.16) is related to the free-particle equation (3.21), then we can derive the corresponding Schrödinger equation by means of the five Noether symmetries admitted by the Lagrangian $L_{4,10}$ [11, 36–38] in spite of the fact that the transformation (3.19) is not linear.

Indeed, we have determined a linear parabolic partial differential equation

$$\phi_{xx} + 2\phi_{xy} + \phi_{yy} + 2i\frac{-x(x + y)^2 + i}{x + y}\phi_x + 2i\frac{y(x + y)^2 + i}{x + y}\phi_y = 0 \tag{3.26}$$

that admits the following Lie symmetries

$$\begin{aligned} \Gamma_4, \quad \Gamma_5 + \mathrm{i}xy\phi\partial_\phi, \quad -3\Gamma_6 + 2\Gamma_2 + \frac{1}{4}(x - y + \mathrm{i}x^2y^2)\phi\partial_\phi, \quad \Gamma_9, \\ \Gamma_1 + \mathrm{i}xy\phi\partial_\phi, \quad \phi\partial_\phi, \quad \alpha(x, y)\partial_\phi, \end{aligned} \quad (3.27)$$

where $\alpha(x, y)$ is any solution to (3.26). It is obvious that the Noether symmetries (3.23) admitted by the Lagrangian $L_{4,10}$ of equation (3.16) correspond to the five Lie symmetries of equation (3.26). We can also transform the parabolic equation (3.26) into its canonical form with the help of MAPLE, i. e.

$$2\mathrm{i}\psi_t + \frac{1}{(2q - t)^2}\psi_{qq} + 2\frac{\mathrm{i}q(2q - t)^2 - 1}{(2q - t)^3}\psi_q = 0 \quad (3.28)$$

with $q = y, t = y - x$. A further trivial substitution, i. e.

$$\psi = \sqrt{2q - t} \exp\left(-\frac{\mathrm{i}q^2}{6}(4q - 3t)\right) \Psi,$$

eliminates the first derivative ψ_q in (3.28) and yields

$$2\mathrm{i}\Psi_t + \frac{1}{(2q - t)^2}\Psi_{qq} - \left(\frac{2\mathrm{i}}{2q - t} + \frac{3}{(2q - t)^4}\right)\Psi = 0. \quad (3.29)$$

Another linear parabolic partial differential equation can be derived from the Lagrangian $L_{8,9}$ and its admitted five Noether symmetries (3.25), i. e.

$$\begin{aligned} \frac{x^4}{y^4}\phi_{xx} + 2\frac{x^2}{y^2}\phi_{xy} + \phi_{yy} + \frac{2}{y^5(x + y)}(\mathrm{i}x^2 + 2\mathrm{i}xy + \mathrm{i}y^2 + x^4y + 2x^3y^2)\phi_x \\ + \frac{2}{xy^4(x + y)}(-\mathrm{i}x^2 - 2\mathrm{i}xy - \mathrm{i}y^2 + 2x^2y^3 + xy^4)\phi_y = 0 \end{aligned} \quad (3.30)$$

that admits the following Lie symmetries

$$\begin{aligned} \Gamma_{10} + \frac{\mathrm{i} + xy(y - x)}{4x^2y^2}\phi\partial_\phi, \quad -3\Gamma_6 + \Gamma_2, \quad \Gamma_7, \quad \Gamma_8, \quad \Gamma_3 - \frac{\mathrm{i}}{xy}\phi\partial_\phi, \\ \phi\partial_\phi, \quad \beta(x, y)\partial_\phi, \end{aligned} \quad (3.31)$$

where $\beta(x, y)$ is any solution to (3.30). We can also transform the parabolic equation (3.30) into its canonical form with the help of MAPLE, i. e.

$$2\mathrm{i}\psi_t + \frac{q^6}{(2 + tq)^2}\psi_{qq} + \frac{2q^2}{(2 + tq)^3}(q^3(3 + tq) + \mathrm{i}(2 + tq)^2)\psi_q = 0 \quad (3.32)$$

with $q = x, t = \frac{x-y}{xy}$. A further trivial substitution, i. e.

$$\psi = \frac{\sqrt{2 + qt}}{q^{3/2}} \exp\left(\frac{\mathrm{i}}{6q^3}(3tq + 4)\right) \Psi,$$

eliminates the first derivative ψ_q in (3.32) and yields

$$2i\psi_t + \frac{q^6}{(2+qt)^2}\psi_{qq} + q\left(\frac{2i}{2+qt} - \frac{3q^3}{(2+qt)^4}\right)\psi = 0. \quad (3.33)$$

The journey is over.

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BETTI AND TACHIBANA NUMBERS

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Abstract. We present a rough classification of differential forms on a Riemannian manifold, we consider definitions and properties of conformal Killing forms on a compact Riemannian manifold and define Tachibana numbers as an analog of the well known Betti numbers. We state the conditions that characterize these numbers. In the last section we show connections between the Betti and Tachibana numbers.

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1. INTRODUCTION

The the paper is based on our plenary lecture at the International Conference AGMP-8 (Brno, 12–14 September 2012). The paper is organized as follows. In the first section, we introduce some notations and give some basic definitions of the theory of differential forms on a Riemannian manifold (M, g) and natural operators on forms. In addition, we present a rough classification of differential forms on a Riemannian manifold. In the next two paragraphs of the paper we consider properties of harmonic and conformal Killing forms on an n -dimensional compact Riemannian manifold (M, g) and determine the Tachibana number $t_r(M)$ as an analog of the Betti number $b_r(M)$ of (M, g) for $1 \leq r \leq n-1$. Moreover, we state some conditions that characterize these numbers. In the last section we formulate results on relationships between Betti and Tachibana numbers.

2. ON A CLASSIFICATION OF DIFFERENTIAL FORMS ON RIEMANNIAN MANIFOLD

In this section we will show a rough classification of differential forms on a Riemannian manifold, which will be useful for the establishment of the desired results. Moreover, we want to fix some notations and basic concepts.

Let (M, g) be an n -dimensional compact and orientable Riemannian manifold with the Levi–Civita connection ∇ and let $\Omega^r(M)$ denote the vector space of smooth

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r -forms on M for $1 \leq r \leq n-1$. We define an inner product on $\Omega^r(M)$ by the formula: $\langle \omega, \omega' \rangle = \int_M g(\omega, \omega') d \text{ vol}$ for any $\omega, \omega' \in \Omega^r(M)$. The inner product structures on $\Omega^r(M)$ allow us to define the exterior co-differential operator $d^*: \Omega^r(M) \rightarrow \Omega^{r-1}(M)$ as a formal adjoint to the well known exterior differential operator $d: \Omega^r(M) \rightarrow \Omega^{r+1}(M)$ by the following formula $\langle d\omega, \theta \rangle = \langle \omega, d^*\theta \rangle$.

More than thirty years ago, Bourguignon [3] considered the space of Riemannian natural (with respect to isometric diffeomorphisms) first-order differential operators on $\Omega^r(M)$ with values in the space of homogeneous tensors on M . He proved the existence of a basis of this space which consists of three operators $\{D_1, D_2, D_3\}$ where $D_1 := d$ and $D_2 := d^*$. As for the third operator D_3 , Bourguignon said that D_3 does not have any geometric interpretation for $r > 1$. It was also pointed out that, in the case $r = 1$, the kernel of D_3 consists of infinitesimal conformal transformations of M .

By way of specification of Bourguignon's result, we showed [14, 15] that

$$\begin{aligned} D_1 &= \frac{1}{p+1} d, \\ D_2 &= \frac{1}{n-p+1} g \wedge d^*, \\ D_3 &= \nabla - \frac{1}{p+1} d - \frac{1}{n-p+1} g \wedge d^* \end{aligned}$$

and proved [14, 15] that the kernel of the third basis operator D_3 consists of conformal Killing r -forms. These forms were introduced by Tachibana [22] and Kashiwada [8] as a natural generalization of *conformal Killing vector fields*, which are also called infinitesimal conformal transformations [12]. The space of conformal Killing r -forms will be denoted by $\mathbf{T}^r(M, \mathbb{R})$.

In addition, we mention that the kernel of the first basis differential operator D_1 consists of closed r -forms and the kernel of the second basis differential operator D_2 consists of co-closed r -forms. The two vector spaces of closed and co-closed r -forms will be denoted by $\mathbf{D}^r(M, \mathbb{R})$ and $\mathbf{F}^r(M, \mathbb{R})$.

The condition $\omega \in \ker D_1 \cap \ker D_2$ characterizes the form ω as a *harmonic form* [11, p. 204], therefore the vector space $\mathbf{H}^r(M, \mathbb{R})$ of harmonic r -forms is defined as

$$\mathbf{H}^r(M, \mathbb{R}) = \mathbf{D}^r(M, \mathbb{R}) \cap \mathbf{F}^r(M, \mathbb{R}).$$

The condition $\omega \in \ker D_2 \cap \ker D_3$ characterizes the r -form as a *co-closed conformal Killing form*, which is also called a *Killing form* (see [4, p. 65–66] and [16]). Therefore, the vector space $\mathbf{K}^r(M, \mathbb{R})$ of co-closed conformal Killing r -forms is defined as

$$\mathbf{K}^r(M, \mathbb{R}) = \mathbf{T}^r(M, \mathbb{R}) \cap \mathbf{F}^r(M, \mathbb{R}).$$

Co-closed conformal Killing forms can be regarded as a generalization of Killing vector fields that define infinitesimal isometric transformations [12].

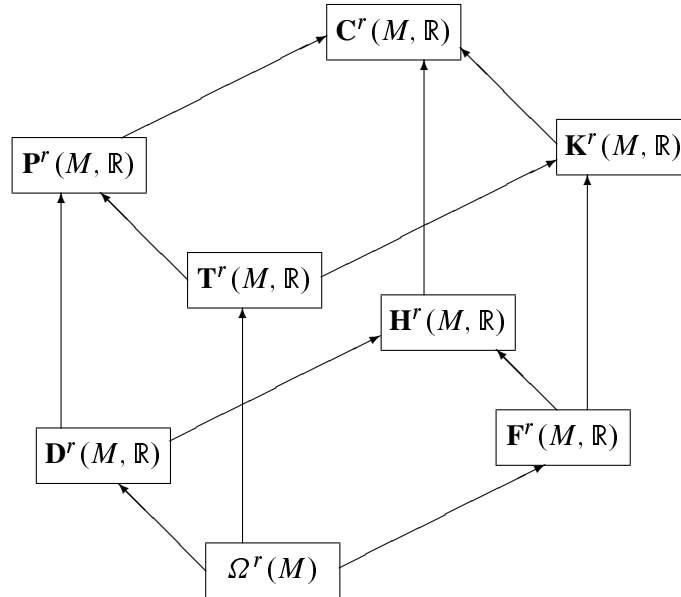
The condition $\omega \in \ker D_1 \cap \ker D_3$ characterizes ω as a *closed conformal Killing form*. Therefore, the space $\mathbf{F}^r(M, \mathbb{R})$ of closed conformal Killing r -forms is defined as (see [14])

$$\mathbf{F}^r(M, \mathbb{R}) = \mathbf{D}^r(M, \mathbb{R}) \cap \mathbf{T}^r(M, \mathbb{R}).$$

Closed conformal Killing forms (which are also called planar forms) are a generalization of closed conformal Killing vector fields, which are also called local gradient infinitesimal conformal transformations [12]. We denote by $\mathbf{C}^r(M, \mathbb{R})$ the subspace of $\Omega^r(M)$ which consists of parallel r -forms, which are also called covariant constant r -forms. It is clear that

$$\mathbf{C}^r(M, \mathbb{R}) = \mathbf{K}^r(M, \mathbb{R}) \cap \mathbf{P}^r(M, \mathbb{R}) \subset \mathbf{H}^r(M, \mathbb{R}).$$

Using definitions and propositions which we have formulated above, we obtain the 3D-diagram of inclusions of subspaces of differential r -forms on an n -dimensional ($1 \leq r \leq n-1$) Riemannian manifold (M, g) .



Here, for instance, the arrow $\mathbf{F}^r(M, \mathbb{R}) \rightarrow \mathbf{K}^r(M, \mathbb{R})$ means that the vector space $\mathbf{K}^r(M, \mathbb{R})$ is subspace of $\mathbf{F}^r(M, \mathbb{R})$.

3. HARMONIC FORMS AND BETTI NUMBERS

One of the motivations for studying Tachibana numbers of a compact Riemannian manifold (M, g) is their close relation to Betti numbers. In this section we will

present a brief review of the results of the theory of harmonic forms and Betti numbers.

The *Laplacian on forms* $\Delta: \Omega^r(M) \rightarrow \Omega^r(M)$, also called the *Hodge Laplacian*, is defined in terms of the natural Riemannian operators d and d^* as $\Delta = d d^* + d^* d$ (see [3, p. 52], [2, p. 377], [6, p. 316], [11, p. 204]). The operator Δ is a nonnegative elliptic second order linear differential operator and its kernel is finite dimensional on a compact Riemannian manifold (M, g) .

Two following equivalent conditions define $\omega \in \Omega^r(M)$ as a *harmonic form* $d\omega = 0$ and $d^*\omega = 0$ or $\Delta\omega = 0$ on a compact Riemannian manifold (M, g) (see [11, p. 204]).

If we denote by $\mathbf{H}^r(M, \mathbb{R})$ the vector space of harmonic r -forms on a compact Riemannian manifold (M, g) , then, by Hodge theory (see [7]), the r th *Betti number* $b_r(M)$ of (M, g) is defined by the equation $b_r(M) = \dim \mathbf{H}^r(M, \mathbb{R}) < \infty$.

The Hodge Laplacian commutes $*\Delta = \Delta*$ with the well known (see [3, p. 33], [11, p. 203]) Hodge star operator $*\Omega^r(M) \rightarrow \Omega^{n-r}(M)$, which implies the following isomorphism $*\mathbf{H}^r(M, \mathbb{R}) \rightarrow \mathbf{H}^{n-r}(M, \mathbb{R})$. From the isomorphism we obtain (see [11, p. 389]) the *Poincaré duality theorem* $b_r(M) = b_{n-r}(M)$.

4. CONFORMAL KILLING FORMS AND TACHIBANA NUMBERS

Before explaining the main results of our paper, we want to point out that they are extensions of our results in [14–20] for conformal Killing forms and Tachibana numbers. In this section we will present a brief review of these results.

Throughout in this section we let (M, g) be an n -dimensional compact and oriented Riemannian manifold. For any linear differential operator D , the inner product structures on $\Omega^r(M)$ allow us to define the formal adjoint operator D^* to D (see [3, p. 460]). In the case of the first order natural Riemannian (with respect to isometric diffeomorphisms) operator D , we can define a *strong Laplacian* $D^* \circ D$ (see [2, p. 377], [6, p. 316–317]). The first simple property of these operators comes from the fact that any strong Laplacian is a non-negative elliptic second order linear differential operator (see [6, p. 314–315]) with a finite dimensional kernel (see [3, pp. 461–463]). For the other properties of these operators, see [2, 6, 11, 24].

We showed in [17] that the formal adjoint operator D_3^* to D_3 determined by the formula $D_3^* = \nabla^* - \frac{1}{p+1} d^* - \frac{1}{n-p+1} d \circ \text{trace}$ and the Tachibana Laplacian $\square = D_3^* \circ D_3$ have the form

$$\square = D_3^* \circ D_3 = \frac{1}{r(r+1)} \left(\bar{\Delta} - \frac{1}{r+1} d^* \circ d - \frac{1}{n-r+1} d \circ d^* \right). \quad (4.1)$$

The symbol $\bar{\Delta}$ is called the *Bochner rough Laplacian* and $\bar{\Delta} := \nabla^* \circ \nabla$ where we denote the formal adjoint of ∇ by ∇^* (see [3, p. 52], [2, p. 377]).

We proved the following three propositions (see [17]):

$$\omega \in \mathbf{T}^r(M, \mathbb{R}) \Leftrightarrow \omega \in \ker(D_3^* D_3); \quad (4.2)$$

$$\omega \in \mathbf{K}^r(M, \mathbb{R}) \Leftrightarrow \omega \in \ker(D_3^* D_3) \cap \ker d^*; \quad (4.3)$$

$$\omega \in \mathbf{P}^r(M, \mathbb{R}) \Leftrightarrow \omega \in \ker(D_3^* D_3) \cap \ker d. \quad (4.4)$$

It is known that the kernel of the Tachibana Laplacian $\square = D_3^* \circ D_3$ has a finite dimension and, therefore, we concluded that (see [20])

$$\dim \mathbf{T}^r(M, \mathbb{R}) = \dim_{\mathbb{R}}(\ker D_3^* D_3) = t_r(M) < \infty;$$

$$\dim \mathbf{K}^r(M, \mathbb{R}) = k_r(M) < \infty;$$

$$\dim \mathbf{P}^r(M, \mathbb{R}) = p_r(M) < \infty$$

on a compact Riemannian manifold (M, g) . The numbers $t_r(M)$, $k_r(M)$ and $p_r(M)$ we have called the *Tachibana number*, the *Killing number* and the *planarity number* of a compact Riemannian manifold (M, g) , respectively (see [18]). These numbers satisfy the following duality properties $t_r(M) = t_{n-r}(M)$ and $p_r(M) = k_{n-r}(M)$ for all $r = 1, \dots, n-1$. These equalities are analogues of the Poincaré duality for Betti numbers and corollaries of the following isomorphisms (see [9, 14, 15])

$$*: \mathbf{T}^r(M, \mathbb{R}) \rightarrow \mathbf{T}^{n-r}(M, \mathbb{R}) \text{ and } *: \mathbf{P}^r(M, \mathbb{R}) \rightarrow \mathbf{K}^{n-r}(M, \mathbb{R}).$$

Moreover, Tachibana numbers $t_r(M)$ are conformal scalar invariants, the Killing number $k_r(M)$ and the planarity number $p_r(M)$ are projective scalar invariants of a Riemannian manifold (M, g) for all $r = 1, \dots, n-1$. In addition, we mention that the first proposition is a corollary of conformal invariance of conformal Killing r -forms (see [1]). The second proposition of our theorem is a corollary of projective invariance of closed and co-closed conformal Killing r -forms (see [16]).

Let (M, g) be an n -dimensional connected Riemannian manifold. Then the Tachibana number $t_r(M)$, the Killing number $k_r(M)$ and the planarity number $p_r(M)$ of (M, g) satisfy the following inequalities

$$0 \leq t_r(M) \leq \frac{(n+2)!}{(r+1)!(n-r+1)!}; \quad 0 \leq k_r(M) \leq \frac{(n+1)!}{(r+1)!(n-r)!}$$

$$0 \leq p_r(M) \leq \frac{(n+1)!}{r!(n-r+1)!};$$

for all $r = 1, \dots, n-1$. Moreover, any of two numbers $p_r(M)$ and $k_r(M)$ is maximal if and only if (M, g) is a Riemannian manifold with positive constant curvature (see [10, 14, 16]). In addition, the Tachibana number $t_r(M)$ is maximal if and only if (M, g) is conformal flat Riemannian manifold (see [8, 13]).

5. TACHIBANA AND BETTI NUMBERS

In this section we will show some relationships between Tachibana and Betti numbers.

First, we formulate the *vanishing theorems* of Betti and Tachibana numbers. Let (M, g) be an n -dimensional compact oriented Riemannian manifold and let $\mathfrak{R}: \Omega^2(M) \rightarrow \Omega^2(M)$ be the standard symmetric Riemannian curvature operator of (M, g) (see [11, p. 35–36]).

Assuming $\mathfrak{R} \geq 0$, we have $b_r(M) \leq \frac{n!}{r!(n-r)!} = b_r(T^n)$ where T_n is a flat Riemannian n -torus and $r = 1, \dots, n-1$ (see [11, p. 212]). Moreover, an n -dimensional compact Riemannian manifold with the positive curvature operator $\mathfrak{R}$ is a spherical space form (see [5]) and its Betti numbers $b_1(M), \dots, b_{n-1}(M)$ are zeros (see [11, p. 212]). In addition, we proved (see [20]) that, in this case, an arbitrary conformal Killing r -form ω is uniquely decomposed in the form $\omega' + \omega''$ where ω' is a Killing and ω'' is a closed conformal Killing r -forms on (M, g) for all $r = 1, \dots, n-1$ and hence $t_r(M) = k_r(M) + p_r(M)$.

On the other hand, if $\mathfrak{R}$ is non-positive, then $t_r(M) \leq \frac{n!}{r!(n-r)!} = t_r(T^n)$ and, if $\mathfrak{R} < 0$ somewhere, then $t_r(M) = 0$ for $r = 1, \dots, n-1$ (see [20]).

Second, we consider Betti and Tachibana numbers of a compact conformally flat Riemannian manifold and prove the following theorem as a corollary of above results.

Theorem 1. *Let (M, g) be an n -dimensional ($n \geq 3$) compact conformally flat Riemannian manifold with the positive or negative definite Ricci tensor Ric then $t_r(M) \cdot b_h(M) = 0$ for all $h, r = 1, \dots, n-1$.*

Proof. Let one the two following conditions be satisfied $\text{Ric} > 0$ or $\text{Ric} < 0$ at every point of (M, g) . If we suppose that (M, g) is conformally flat Riemannian manifold of dimension $n \geq 3$, then, in the first case, we have $\mathfrak{R} > 0$ and Betti numbers $b_1(M) = \dots = b_{n-1}(M) = 0$ because there are no non-zero harmonic r -forms for all $r = 1, \dots, n-1$ (see [4, pp. 78-79]). On the other hand, in the second case, we have $\mathfrak{R} < 0$ and Tachibana numbers $t_1(M) = \dots = t_{n-1}(M) = 0$ because there are no non-zero conformal Killing r -forms for all $r = 1, \dots, n-1$ (see [19]). Therefore, we conclude that $t_r(M) \cdot b_h(M) = 0$ for all $h, r = 1, \dots, n-1$, which finishes the proof of the theorem. $\square$

Next, we consider the case of an even-dimensional compact and oriented conformally flat Riemannian manifold. In this special case, the following theorem is true.

Theorem 2. *Let (M, g) be a $2r$ -dimensional compact and oriented conformally flat Riemannian manifold. Then, the following propositions are true.*

- (1) *If the scalar curvature $s \neq 0$ and the Tachibana number $t_r(M) \neq 0$, then the Betti number $b_r(M) = 0$. Moreover, if s is a positive constant, then*

$$t_r(M) = k_r(M) + p_r(M)$$

for $k_r(M) = \dim_{\mathbb{R}}(\ker D_3^* D_3 \cap \text{Im } d^*)$ and $p_r(M) = \dim_{\mathbb{R}}(\ker D_3^* D_3 \cap \text{Im } d)$.

- (2) *If the scalar curvature $s \neq 0$ and the Betti number $b_r(M) \neq 0$, then the Tachibana number $t_r(M) = 0$.*

- (3) If the scalar curvature $s = 0$, then the Betti number $b_r(M) \neq 0$ if and only if the Tachibana number $t_r(M) \neq 0$ and, moreover,

$$t_r(M) = b_r(M) = (2r)!(r!)^{-2}.$$

Proof. First of all, we recall, that the Hodge Laplacian operator Δ admits the well known Weitzenböck decomposition (see [3, p. 53])

$$\Delta\omega = \bar{\Delta}\omega + F_r(\omega), \quad (5.1)$$

where (see [4, pp. 60–61])

$$F_r(\omega)(X_1, \dots, X_r) = \sum_{\alpha=1}^r \text{Ric}(e_j, X_\alpha)\omega(X_1, \dots, X_{\alpha-1}, e_j, X_{\alpha+1}, \dots, X_r) \\ - \sum_{\substack{1, \dots, r \\ \alpha < \beta}} R(e_j, e_k, X_\alpha, X_\beta)\omega(X_1, \dots, X_{\alpha-1}, e_j, X_{\alpha+1}, \dots, X_{\beta-1}, e_k, X_{\beta+1}, \dots, X_r)$$

for an arbitrary $\omega \in \Omega^r(M)$; $X_1, \dots, X_r \in C^\infty TM$, for any orthonormal basis $\{e_1, \dots, e_n\}$, and the curvature tensor R and the Ricci tensor Ric of (M, g) . As a consequence of this fact, we obtain from (4.1) the following

$$\square \omega = \frac{1}{r(r+1)} \left(\Delta\omega - F_r(\omega) - \frac{1}{r+1} d^* d\omega - \frac{1}{n-r+1} d d^* \omega \right) \quad (5.2)$$

for any $\omega \in \Omega^r(M)$. In particular, we have

$$\Delta\omega = F_r(\omega) + \frac{1}{r+1} d^* d\omega + \frac{1}{n-r+1} d d^* \omega \quad (5.3)$$

for an arbitrary conformal Killing r -form ω .

Next, if we suppose that our Riemannian manifold (M, g) has the dimension $2r$ and can be reduced to an Euclidean space by suitable conformal transformation the metric tensor g , then we have the equality (see [4, p. 79])

$$F_r(\omega) = \frac{1}{2(2r-1)} s \cdot \omega, \quad (5.4)$$

After that, from (5.3) and (5.4) we can deduce the equation

$$\Delta\omega = \frac{r+1}{2r(2r-1)} s \cdot \omega, \quad (5.5)$$

where s is the scalar curvature of (M, g) . In particular, if $s \neq 0$ we deduce from (5.5) that $\Delta\omega = 0$ if and only if $\omega = 0$. Moreover, if we suppose that s is a positive constant, we can rewrite the equation (5.5) in the form

$$\omega = d^* d\omega' + d d^* \omega' \quad (5.6)$$

where $\omega' = \frac{2r(2r-1)}{(r+1)s} \omega$. The equality (5.6) is the well known Hodge – de Ram decomposition where there is no harmonic form. From (5.6) we conclude the following orthogonal decomposition

$$\mathbf{T}^r(M, \mathbb{R}) = \mathbf{P}^r(M, \mathbb{R}) \oplus \mathbf{K}^r(M, \mathbb{R}) \quad (5.7)$$

where

$$\mathbf{K}^r(M, \mathbb{R}) = \{\omega \in \Omega^r(M) \mid \omega \in \ker D_3^* D_3 \cap \operatorname{Im} d^*\} \quad (5.8)$$

and

$$\mathbf{P}^r(M, \mathbb{R}) = \{\omega \in \Omega^r(M) \mid \omega \in \ker D_3^* D_3 \cap \operatorname{Im} d\} \quad (5.9)$$

From (5.7) – (5.9), we obtain the following equalities $t_r(M) = k_r(M) + p_r(M)$, $k_r(M) = \dim_{\mathbb{R}}(\ker D_3^* D_3 \cap \operatorname{Im} d^*)$ and $p_r(M) = \dim_{\mathbb{R}}(\ker D_3^* D_3 \cap \operatorname{Im} d)$, which finish the proof of the first assertion of the theorem.

Next, from (4.1), (5.1) and (5.4) we infer

$$\frac{s}{2(2r-1)} \omega = \frac{r}{r+1} \Delta \omega - r(r+1) \square \omega \quad (5.10)$$

for any $\omega \in \Omega^r(M)$.

Then, for an arbitrary harmonic r -form ω we have

$$\frac{s}{2(2r-1)} \omega = -r(r+1) \square \omega. \quad (5.11)$$

In particular, if $s \neq 0$ we deduce from (5.5) that $\square \omega = 0$ if and only if $\omega = 0$.

Finally, we consider a $2r$ -dimensional compact conformally flat Riemannian manifold with zero scalar curvature. In this case, from (5.10) we obtain $\Delta \omega = (r+1)^2 \square \omega$. If we suppose that $b_r(M) \neq 0$, then there exists a nonzero harmonic r -form ω such that $\Delta \omega = (r+1)^2 \square \omega = 0$. Then, ω is conformal Killing and from (5.1) we obtain the equation $\nabla \omega = 0$. In this case we have $b_r(M) = t_r(M) = \frac{n!}{r!(n-r)!} = (2r)! \cdot (r!)^{-2}$, because a parallel form is completely determined by its value at the point. This finishes the proof of the theorem. $\square$

As a consequence of the previous theorem we obtain the following corollary.

Corollary 1. *Let (M, g) be a $2r$ -dimensional compact and oriented conformally flat Riemannian manifold with constant nonzero scalar curvature s . Then, any nonzero conformal Killing r -form is an eigenform of the Hodge Laplacian Δ corresponding to the eigen-value $\lambda = (r+1)(2r(2r-1))^{-1}s$ for $s > 0$ and any nonzero harmonic r -form is an eigenform of the Tachibana Laplacian $\square$ corresponding to the eigenvalue $\lambda = -((2r(r+1)(2r-1))^{-1}s$ for $s < 0$.*

Proof. It is well known that, if a nonzero r -form ω satisfies the equation $\Delta \omega = \lambda \omega$ for a constant $\lambda > 0$, it is called an eigenform of the Laplacian Δ corresponding to the eigen-value λ . Then, if we suppose that s is a positive constant, then, from (5.5), we conclude that the conformal Killing r -form ω is an eigenform of Δ corresponding to the eigenvalue $\lambda = (r+1)(2r(2r-1))^{-1}s$ (see also [21]).

Moreover, if we suppose that s is a negative constant, then, from (5.11), we conclude that the harmonic r -form ω is an eigenform of $\square$ corresponding to the eigenvalue $\lambda = -((2r(r+1)(2r-1))^{-1}s$. $\square$

Next, we consider conformal Killing and harmonic forms on a Riemannian manifold of constant sectional curvature, which is an example of a conformally flat Riemannian manifold.

Theorem 3. *Let (M, g) be an n -dimensional compact and oriented Riemannian manifold of constant nonzero sectional curvature C . Then, we have three propositions.*

- (1) *A nonzero r -form ω is harmonic if and only if ω is an eigenform of the Tachibana Laplacian $\square$ corresponding to the eigenvalue $\lambda = -(n-r)(r+1)^{-1}C$ for $C < 0$.*
- (2) *Any nonzero closed conformal Killing r -form ω is an eigenform of the Hodge Laplacian Δ corresponding to the eigenvalue $\lambda = r(n-r+1)C$ for $C > 0$. For $n < 2r$ converse is true.*
- (3) *Any nonzero Killing r -form ω is an eigenform of the Hodge Laplacian Δ corresponding to the eigenvalue $\lambda = (n-r)(r+1)C$ for $C > 0$. For $n > 2r$ converse is true.*

Proof. Let (M, g) be an n -dimensional compact and oriented Riemannian manifold with constant sectional curvature C . Then (see [9]), we have the identity

$$F_r(\omega) = r(n-r)C \cdot \omega \quad (5.12)$$

for an arbitrary $\omega \in \Omega^r(M)$. In this case we can rewrite (5.2) in the form

$$\square \omega = \frac{1}{r(r+1)} \left(\Delta \omega - r(n-r)C \cdot \omega - \frac{1}{r+1} d^* d \omega - \frac{1}{n-r+1} d d^* \omega \right). \quad (5.13)$$

From this, we see $d\omega = 0$ and $d^*\omega = 0$ hold if and only if

$$\square \omega = (n-r)(r+1)^{-1}C \cdot \omega$$

holds. On the other hand, we can rewrite (5.2) in two different forms

$$\square \omega = \frac{1}{r(r+1)} \left(\frac{n-r}{n-r+1} \Delta \omega - r(n-r)C \cdot \omega - \frac{n-2r}{(r+1)(n-r+1)} d^* d \omega \right) \quad (5.14)$$

and

$$\square \omega = \frac{1}{r(r+1)} \left(\frac{r}{r+1} \Delta \omega - r(n-r)C \cdot \omega + \frac{n-2r}{(r+1)(n-r+1)} d d^* \omega \right). \quad (5.15)$$

From (5.14) we can conclude that if $d\omega = 0$ and $\square \omega = 0$ hold, then $\Delta \omega = r(n-r+1)C \cdot \omega$ holds. At the same time, from (5.15) we conclude that if $d^*\omega = 0$ and $\square \omega = 0$ hold, then $\Delta \omega = (n-r)(r+1)C \cdot \omega$ holds. It is easy to see that converses are true only for cases when $n < 2r$ and $n > 2r$, respectively. $\square$

In the following theorem, we study compact Riemannian manifolds whose Ricci tensor is negative and positive semi-definite and we give the lower bound for the Betti number $b_r(M)$ and the Tachibana number $t_r(M)$, respectively. More precisely, we have

Theorem 4. *Let (M, g) be a compact n -dimensional Riemannian manifold satisfy one of the following conditions:*

- (1) *the Ricci tensor Ric of (M, g) is negative semi-definite and the first Tachibana number $t_1(M) = h \leq n$;*
- (2) *the Ricci tensor Ric of (M, g) is positive semi-definite and the Betti number $b_1(M) = h \leq n$. Then,*

$$\frac{h!}{r!(h-r)!} \leq b_r(M) = t_r(M) \leq \frac{n!}{r!(h-r)!}$$

for $1 \leq r < h = b_1(M) \leq n$. If $\text{Ric} \leq 0$ and $t_1(M) = n$ or $\text{Ric} \geq 0$ and $b_1(M) = n$, then (M, g) is a flat Riemannian n -torus.

Proof. Let (M, g) be a compact n -dimensional Riemannian manifold and $t_1(M) = h \neq 0$. Then there are h linearly independent nonzero conformal Killing 1-forms $\omega_1, \dots, \omega_h$. We denote by $X_1, \dots, X_h$ the dual conformal Killing vector fields, i. e. $\omega_a(Y) = g(Y, X_a)$ for $a = 1, \dots, h$. If we suppose that the Ricci tensor Ric is negative semi-definite, then $X_1, \dots, X_h$ are parallel (see [4, p. 53–55]), i. e. $\nabla X_1 = 0, \dots, \nabla X_h = 0$. In this case, the following 1-forms $\omega_1, \dots, \omega_h$ are parallel and, hence, are harmonic 1-forms. Using this result, we conclude that $\theta_{i_1 i_2 \dots i_r} = \omega_{i_1} \wedge \omega_{i_2} \wedge \dots \wedge \omega_{i_r}$ for any $1 \leq i_1 < \dots < i_r \leq h$ are parallel r -forms and, hence, conformal Killing r -forms. Since these $\theta_{i_1 i_2 \dots i_r}$ are $\frac{h!}{r!(h-r)!}$ linearly independent conformal Killing r -forms, we have $t_r(M) \geq \frac{h!}{r!(h-r)!}$ for $r < h$. Moreover, we have $b_r(M) \geq \frac{h!}{r!(h-r)!}$ for $r < h$ because an arbitrary parallel r -form ω is harmonic. On the other hand, let (M, g) be a compact n -dimensional Riemannian manifold with $b_1(M) = h \neq 0$. Then, there are h linearly independent nonzero harmonic 1-forms $\omega_1, \dots, \omega_h$. We denote by $X_1, \dots, X_h$ the dual harmonic vector fields, i. e. $\omega_a(Y) = g(Y, X_a)$ for $a = 1, \dots, h$. If we suppose that the Ricci tensor Ric is positive semi-definite, then $X_1, \dots, X_h$ are parallel (see [4, p. 53–55]), i. e. $\nabla X_1 = 0, \dots, \nabla X_h = 0$. From this we can conclude that $b_r(M) \geq \frac{h!}{r!(h-r)!}$ for $r < h$ and hence $t_r(M) \geq \frac{h!}{r!(h-r)!}$. Next, if $\text{Ric} \leq 0$ and $t_1(M) = n$ or $\text{Ric} \geq 0$ and $b_1(M) = n$, then (M, g) has a parallel frame. The curvature tensor of (M, g) vanishes in this frame, so (M, g) is a flat torus (see [11, p. 208]) and hence $b_r(M) = t_r(M) = \frac{h!}{r!(h-r)!}$. This concludes the proof of the theorem. $\square$

Finally, we prove the following theorem and formulate its corollary.

Theorem 5. *Let (M, g) be a compact n -dimensional Riemannian manifold with its zero first Betti number $b_1(M)$ and non-zero first planarity number $p_1(M)$. Then, Betti numbers $b_2(M), \dots, b_{n-1}(M)$ equal zeros, too.*

Proof. If $b_1(M) = 0$ and $p_1(M) \neq 0$, then there exists a non-zero exact conformal Killing 1-form $\omega = df$ such that $\nabla \text{grad} f = -n^{-1} \Delta f \cdot g$ for a smooth function f . In this case, due to Tashiro theorem (see [23]), (M, g) is conformally diffeomorphic to a Euclidean sphere $\mathbb{S}^n$ and hence $b_r(M) = b_r(\mathbb{S}^n) = 0$ for $r = 1, \dots, n-1$. This finishes the proof. $\square$

Corollary 2. *Let (M, g) be a compact n -dimensional Riemannian manifold such that its Betti number $b_{n-1}(M) = 0$ and Killing number $k_{n-1}(M) \neq 0$. Then, Betti numbers $b_1(M), \dots, b_{n-2}(M)$ equal zeros, too.*

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SECOND ORDER NATURAL LAGRANGIANS ON COFRAME BUNDLES

JÁN BRAJERČÍK AND MILAN DEMKO

Abstract. We study the structure of second order natural Lagrangians on the bundle of linear coframes F^*X over an n -dimensional manifold X . They are identified with the corresponding differential invariants which can be obtained by the factorization method. We give an explicit description of these differential invariants in terms of their bases. For construction of natural Lagrangians, the canonical odd n -form on F^*X is also introduced.

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1. INTRODUCTION AND PRELIMINARIES

The aim of this paper is to characterize all global, second order Lagrangians on coframe bundles F^*X , invariant with respect to diffeomorphisms of X . Such Lagrangians are called *natural*. Natural Lagrangians on coframe bundles play an important role in several theories, such as teleparallel description of gravity (see, e. g., [11]). A characteristic property of a natural Lagrangian is that it is a *differential invariant*. The domain of definition of second order natural Lagrangians on coframe bundles, the bundle J^2F^*X , can equivalently be understood as the type fibre of J^2F^*X .

As usual, we denote by $\mathbb{R}$ the field of real numbers. The r -th differential group L_n^r of $\mathbb{R}^n$ is the Lie group of invertible r -jets with source and target at the origin $0 \in \mathbb{R}^n$. The group multiplication in L_n^r is defined by the composition of jets. The *first (second) canonical coordinates* are denoted by $a_{j_1}^i, a_{j_1 j_2}^i, \dots, a_{j_1 j_2 \dots j_r}^i$ ($b_{j_1}^i, b_{j_1 j_2}^i, \dots, b_{j_1 j_2 \dots j_r}^i$), where $1 \leq i \leq n, 1 \leq j_1 \leq j_2 \leq \dots \leq j_r \leq n$. The canonical jet projection of L_n^r onto L_n^1 is a Lie group homomorphism. Denoting by K_n^r its *kernel*, we can represent the differential group L_n^r as a *semi-direct product* of L_n^1 and K_n^r , $L_n^r = L_n^1 \times_s K_n^r$. For generalities on spaces of jets and their mappings, differential groups, their actions, etc., we refer to [6, 10].

By a *left G -manifold* we mean a smooth manifold endowed with a left action of a Lie group G . Let P and Q be two left L_n^r -manifolds. A smooth mapping $F : P \rightarrow$

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Q is called a *differential invariant* if it is L_n^r -equivariant, i.e., $F(g \cdot p) = g \cdot F(p)$ for all $g \in L_n^r$ and $p \in P$.

If L_n^r acts on manifold Q from the left via its subgroup L_n^1 , and $F : P \rightarrow Q$ is a differential invariant, then the restriction of the group action of L_n^r to K_n^r gives the condition $F(g \cdot p) = F(p)$. This is the basic idea of the *orbit reduction method*, first time used by Krupka in [7], which utilizes the semi-direct product of L_n^r to compute the differential invariants.

Let us consider a left L_n^1 -manifold Q , and denote by $T_n^r Q$ the manifold of r -jets with source $0 \in \mathbb{R}^n$ and target in Q . According to the general theory of prolongations of left G -manifolds, $T_n^r Q$ has a (canonical) structure of a left L_n^{r+1} -manifold. To define this structure, denote by t_x the translation of $\mathbb{R}^n$ defined by $t_x(y) = y - x$. Consider elements $q \in T_n^r Q$, $q = J_0^r \gamma$, and $a \in L_n^{r+1}$, $a = J^{r+1} \alpha$. If we denote $\bar{\alpha}_x = t_x \circ \alpha \circ t_{-\alpha^{-1}(x)}$, setting $\bar{\alpha}(x) = J_0^1 \bar{\alpha}_x$ we get an element of the group L_n^1 . Then the formula

$$a \cdot q = J_0^r (\bar{\alpha} \cdot (\gamma \circ \alpha^{-1})) \quad (1.1)$$

defines a left action of the differential group L_n^{r+1} on $T_n^r Q$. The formula (1.1) is usually called the *prolongation formula* for the action of the group L_n^1 on Q . The left L_n^{r+1} -manifold $T_n^r Q$ is called the *r -jet prolongation* of the left L_n^1 -manifold Q .

Let X be an n -dimensional manifold. By an *r -frame* at a point $x \in X$ we mean an invertible r -jet with source $0 \in \mathbb{R}^n$ and target x . The set of r -frames together with its natural structure of a principal L_n^r -bundle with base X is denoted by $F^r X$, and is called the *bundle of r -frames* over X . For $r = 1$, we get the *bundle of linear frames*, and write $F^1 X = FX$. If Q is a left L_n^1 -manifold, then the bundle with type fibre Q , associated with FX is denoted by $F_Q X$. Formula (1.1) defines on $J^r F_Q X$ the structure of a fibre bundle with type fibre $T_n^r Q$, associated with the bundle $F^{r+1} X$ (see, e. g., [9]).

An *r -coframe* at $x \in X$ is an invertible r -jet with source x and target $0 \in \mathbb{R}^n$. If $r = 1$, we speak of *linear coframes*. We denote by $F^* X$ the set of all linear coframes at all $x \in X$. The right action of L_n^1 on $F^* X$ is given by

$$F^* X \times L_n^1 \ni (J_x^1 \xi, J_0^1 \alpha) \mapsto J_x^1 \xi \cdot J_0^1 \alpha = J_x^1 (\alpha^{-1} \circ \xi) \in F^* X. \quad (1.2)$$

With this action, $F^* X$ is a right principal L_n^1 -bundle.

$F^* X$ can also be considered as a fibre bundle with type fibre L_n^1 associated with L_n^1 -bundle FX , i.e., we have an identification $F^* X = F_{L_n^1} X$. Induced left action of the group L_n^1 on the type fibre L_n^1 , is given by

$$L_n^1 \times L_n^1 \ni (J_0^1 \alpha, J_0^1 \eta) \mapsto J_0^1 (\eta \circ \alpha^{-1}) \in L_n^1. \quad (1.3)$$

In the canonical coordinates (p_j^i) on the type fibre L_n^1 of $F^* X$, (1.3) is expressed as

$$p_j^i (J_0^1 (\eta \circ \alpha^{-1})) = p_s^i (J_0^1 \eta) a_j^s (J_0^1 \alpha^{-1}) = p_s^i (J_0^1 \eta) b_j^s (J_0^1 \alpha), \quad (1.4)$$

or simply by

$$\bar{p}_j^i = p_s^i b_j^s. \quad (1.5)$$

(1.5) is called the *coframe action* of L_n^1 on itself.

$J^r F^* X$ denotes the r -jet prolongation of $F^* X$. Since $F^* X = F_{L_n^1} X$, $J^r F^* X$ has the structure of a fibre bundle with type fibre $T_n^r L_n^1$, associated with $F^{r+1} X$. The action of L_n^{r+1} on $T_n^r L_n^1$ can be obtained by prolongation of the action (1.4) of L_n^1 on L_n^1 using the formula (1.1).

Let us consider a left action of the general linear group L_n^1 on the real line $\mathbb{R}$ defined by $L_n^1 \times \mathbb{R} \ni (a, t) \mapsto |\det a^{-1}| \cdot t \in \mathbb{R}$. The real line, endowed with this action, is an L_n^1 -manifold, denoted by $\tilde{\mathbb{R}}$. A differential invariant with values in $\tilde{\mathbb{R}}$ is called a *scalar invariant*. General result on the structure of natural Lagrangians says that, given L_n^1 -manifold Q , there is a one-to-one correspondence between natural Lagrangians on $J^r F_Q X$ and differential invariants $\mathcal{I} : T_n^r Q \rightarrow \tilde{\mathbb{R}}$ (see, e. g., [9]). This means that for finding second order natural Lagrangians of coframe bundles it is sufficient to describe all scalar invariants of $T_n^2 L_n^1$ associated with the coframe action (1.5).

2. BASIS OF THE SECOND ORDER INVARIANTS

We are interested in differential invariants $F : P \rightarrow Q$ with values in L_n^1 -manifold Q , which can be viewed as manifold with action of L_n^r via its subgroup L_n^1 . In such case, each differential invariant $F : P \rightarrow Q$ has the form $F = f \circ \pi$, where $\pi : P \rightarrow P/K_n^r$ is the canonical projection onto the orbit space, and $f : P/K_n^r \rightarrow Q$ is a uniquely determined L_n^1 -equivariant mapping (see [10]).

Indeed, in this scheme P/K_n^r is considered with the quotient topology, but it is not necessarily a smooth manifold. The quotient projection π is continuous but not necessarily smooth. If P/K_n^r has a smooth structure such that π is a submersion, we call π the *basis of differential invariants* on P . The general concepts on equivariant mappings, related with a normal subgroup of a Lie group, and corresponding assertions with the proofs can be found in [4, 7].

Our aim is to find a basis of differential invariants on $T_n^2 L_n^1$. We use the expression of the action of L_n^3 on $T_n^2 L_n^1$ (see [3]), obtained as prolongation of the coframe action of L_n^1 on L_n^1 (1.5),

$$\begin{aligned} \bar{p}_j^i &= p_s^i b_j^s, \\ \bar{p}_{j,k}^i &= p_{s,t}^i b_k^t b_j^s + p_s^i b_{jk}^s, \\ \bar{p}_{j,kl}^i &= p_{s,tu}^i b_l^u b_k^t b_j^s + p_{s,t}^i (b_{kl}^t b_j^s + b_k^t b_{jl}^s + b_l^t b_{jk}^s) + p_s^i b_{jkl}^s. \end{aligned} \quad (2.1)$$

Restricting to the kernel K_n^3 of the projection $\pi_n^{3,1} : L_n^3 \rightarrow L_n^1$ we obtain the group action of K_n^3 on $T_n^2 L_n^1$ induced by the coframe action (see [3])

$$\begin{aligned}\bar{p}_j^i &= p_j^i, \\ \bar{p}_{j,k}^i &= p_{j,k}^i + p_s^i b_{jk}^s, \\ \bar{p}_{j,kl}^i &= p_{j,kl}^i + p_{j,t}^i b_{kl}^t + p_{s,k}^i b_{jl}^s + p_{s,l}^i b_{jk}^s + p_s^i b_{jkl}^s.\end{aligned}\tag{2.2}$$

We denote by q_j^i the inverse matrix of the matrix p_j^i ; thus, $q_j^i : T_n^2 L_n^1 \rightarrow \mathbb{R}$ are functions such that $q_s^i p_j^s = \delta_j^i$, where δ_j^i denotes the Kronecker symbol. *Symmetrization* (*antisymmetrization*) in some indices $j, k, l, \dots$ is denoted by writing a bar (a tilde) over these indices, i.e., by writing $\bar{j}, \bar{k}, \bar{l}, \dots$ ($\tilde{j}, \tilde{k}, \tilde{l}, \dots$).

We also introduce the following functions on $T_n^2 L_n^1$:

$$\begin{aligned}I_{j,k}^i(p_b^a, p_{b,c}^a, p_{b,cd}^a) &= q_m^i p_{j,\bar{k}}^m, \\ I_{j,kl}^i(p_b^a, p_{b,c}^a, p_{b,cd}^a) &= q_m^i p_{j,\tilde{k}l}^m + q_m^i q_t^s (p_{\tilde{s},\tilde{j}}^m p_{\bar{k},\bar{l}}^t - p_{\tilde{s},\tilde{k}}^m p_{\bar{j},\bar{l}}^t).\end{aligned}\tag{2.3}$$

Lemma 1. K_n^3 -orbits in $T_n^2 L_n^1$ induced by the coframe action of L_n^1 on L_n^1 are defined by the equations

$$p_j^i = c_j^i, \quad I_{j,k}^i(p_b^a, p_{b,c}^a, p_{b,cd}^a) = c_{j,k}^i, \quad I_{j,kl}^i(p_b^a, p_{b,c}^a, p_{b,cd}^a) = c_{j,kl}^i,$$

where $c_j^i, c_{j,k}^i, c_{j,kl}^i \in \mathbb{R}$ are constants satisfying $\det c_j^i \neq 0$, $c_{j,k}^i + c_{k,j}^i = 0$, $c_{j,kl}^i + c_{k,jl}^i = 0$, and $c_{j,kl}^i - c_{j,lk}^i = 0$.

Proof. Since the coordinates b_{jk}^s are symmetric in j, k , from the action (2.2) we can write $b_{jk}^s = q_i^s (\bar{p}_{j,\bar{k}}^i - p_{j,\bar{k}}^i)$, and substituting this expression of b_{jk}^s in (2.2), we get

$$q_i^s (\bar{p}_{j,k}^i - p_{j,k}^i) = q_i^s (\bar{p}_{j,\bar{k}}^i - p_{j,\bar{k}}^i).$$

Now we compare the tensor on the left hand side with its symmetric part. Using $\bar{q}_i^s = q_i^s$, it gives us $\bar{q}_i^s \bar{p}_{j,\bar{k}}^i = q_i^s p_{j,\bar{k}}^i$. Thus, for the functions $I_{j,k}^s$, defined by (2.3), we have $I_{j,k}^s(\bar{p}_b^a, \bar{p}_{b,c}^a, \bar{p}_{b,cd}^a) = I_{j,k}^s(p_b^a, p_{b,c}^a, p_{b,cd}^a)$. If we denote $\Delta_{jkl}^s = q_i^s (\bar{p}_{j,kl}^i - p_{j,kl}^i - p_{j,t}^i b_{kl}^t - p_{t,k}^i b_{jl}^t - p_{t,l}^i b_{jk}^t)$, expressing b_{jk}^s from (2.2), we analogously obtain that $\Delta_{jkl}^s = \Delta_{j\bar{k}\bar{l}}^s$. Using the Young decomposition of the tensor Δ_{jkl}^s , this equation is equivalent to $I_{j,kl}^s(\bar{p}_b^a, \bar{p}_{b,c}^a, \bar{p}_{b,cd}^a) = I_{j,kl}^s(p_b^a, p_{b,c}^a, p_{b,cd}^a)$. Therefore the functions $I_{j,k}^s$ and $I_{j,kl}^s$ are invariant with respect to the action (2.2). Finally, it is easy to see that the functions $I_{j,k}^i, I_{j,kl}^i$ are antisymmetric in the indices j, k , and the functions $I_{j,kl}^i$ are symmetric in the indices k, l . $\square$

Corollary 1. *The mappings p_j^i , $I_{j,k}^i$, $I_{j,kl}^i$ represent a basis of second order invariants of coframes with values in left L_n^1 -manifold.*

3. SCALAR INVARIANTS OF COFRAMES

In order to obtain scalar invariants of L_n^3 on $T_n^2 L_n^1$ it is sufficient to consider L_n^1 -equivariant mappings defined on $T_n^2 L_n^1 / K_n^3$ (see [1]). Let us define the functions $\mathfrak{I}_{j,k}^i$, $\mathfrak{I}_{j,kl}^i$, on $T_n^2 L_n^1$, by

$$\begin{aligned}\mathfrak{I}_{j,k}^i &= q_j^s q_k^t p_{s,\tilde{t}}^i, \\ \mathfrak{I}_{j,kl}^i &= q_j^s q_k^t q_l^u (p_{s,\tilde{t}u}^i + q_m^v (p_{\tilde{v},\tilde{s}}^i p_{t,\tilde{u}}^m - p_{\tilde{v},\tilde{t}}^i p_{s,\tilde{u}}^m))\end{aligned}\quad (3.1)$$

Theorem. (a) *The functions $\mathfrak{I}_{j,k}^i$, $\mathfrak{I}_{j,kl}^i$ on $T_n^2 L_n^1$ are invariant with respect to the prolonged coframe action.*

(b) *Any function on $T_n^2 L_n^1$, invariant with respect to the prolonged coframe action, is a differentiable function of the functions $\mathfrak{I}_{j,k}^i$, and $\mathfrak{I}_{j,kl}^i$.*

Proof. (a) The group $L_n^1 \simeq L_n^3 / K_n^3$ acts in orbit space $T_n^2 L_n^1 / K_n^3$, with coordinates $I_{j,k}^i$ and $I_{j,kl}^i$, by

$$\bar{I}_{j,k}^i = a_r^i b_j^s b_k^t I_{s,t}^r, \quad \bar{I}_{j,kl}^i = a_r^i b_j^s b_k^t b_l^u I_{s,tu}^r. \quad (3.2)$$

Using relations $a_r^i = \bar{q}_l^i p_r^l$, and $b_j^s = q_m^s \bar{p}_j^m$, obtained from (1.5), in (3.2), we have

$$\bar{p}_i^a \bar{q}_b^j \bar{q}_c^k \bar{I}_{j,k}^i = p_r^a q_b^s q_c^t I_{s,t}^r, \quad \bar{p}_i^a \bar{q}_b^j \bar{q}_c^k \bar{q}_d^l \bar{I}_{j,kl}^i = p_r^a q_b^s q_c^t q_d^u I_{s,tu}^r,$$

which describes L_n^1 -invariant objects in $T_n^2 L_n^1 / K_n^3$. Applying (2.3), we get

$$p_r^i q_j^s q_k^t I_{s,t}^r = \mathfrak{I}_{j,k}^i, \quad p_r^i q_j^s q_k^t q_l^u I_{s,tu}^r = \mathfrak{I}_{j,kl}^i,$$

where $\mathfrak{I}_{j,k}^i$, $\mathfrak{I}_{j,kl}^i$ are given by (3.1).

(b) The statement follows from the invariance theory. $\square$

Lemma 2. *The function $\mathfrak{I}_0 : T_n^2 L_n^1 \ni q \mapsto \mathfrak{I}_0(q) = |\det q_j^i(q)| \in \tilde{\mathbb{R}}$ is a differential invariant.*

Proof. Obviously, the function $\mathfrak{I}_0$ is smooth, and for every $a \in L_n^3$, and every $q \in T_n^2 L_n^1$ we have $\mathfrak{I}_0(a \cdot q) = |\det a^{-1}| \cdot \mathfrak{I}_0(q)$. $\square$

Corollary 2. *Every differential invariant $\mathfrak{I} : T_n^2 L_n^1 \rightarrow \tilde{\mathbb{R}}$ is the product of some differentiable function of $\mathfrak{I}_{j,k}^i$, $\mathfrak{I}_{j,kl}^i$, and the function $\mathfrak{I}_0$.*

4. SECOND ORDER NATURAL LAGRANGIANS OF COFRAMES

Our aim in this Section is to characterize all Lagrangians on J^2F^*X , invariant with respect to all diffeomorphisms of X . First, we recall the main concepts to this purpose.

We present basic definitions in full generality (for odd base forms). If the underlying manifold X is orientable, odd base forms may be replaced by ordinary forms. The concept of volume form is needed for integration on not necessarily orientable manifold.

Any chart (U, φ) , $\varphi = (x^i)$, on X , induces the *fibre*d chart (V, ψ) , $\psi = (x^i, z_j^i)$, on F^*X . By setting $z_j^i w_k^j = \delta_k^i$ we define another coordinates w_k^j on F^*X . With a chart (V, ψ) we also associate the object

$$\tilde{\omega}_{(V, \psi)} = |\det z_j^i| \cdot \tilde{\varphi} \otimes dx^1 \wedge dx^2 \wedge \dots \wedge dx^n, \quad (4.1)$$

where $\tilde{\varphi}$ is a *field of odd scalars* on X , associated with (U, φ) (see [8]). It is easily seen that (4.1) represents a globally defined odd base form on F^*X ; we denote this form by $\tilde{\omega}$, and call it the *canonical odd n -form on F^*X* . This form has the following properties:

- (1) For each coframe field $\xi : W \rightarrow F^*X$, where W is an open set on X , the pullback $\xi^*\tilde{\omega}$ is an odd volume form on W .
- (2) The construction of $\tilde{\omega}$ does not depend on the orientability of the base manifold X . In the case of orientable and oriented manifolds X , $\tilde{\omega}$ is equivalent to an (ordinary) n -form on F^*X .
- (3) The form $\tilde{\omega}$ is *diff X -invariant*, i.e., if $F^*\alpha$ denotes the canonical lift of a diffeomorphism $\alpha : W \rightarrow X$ to F^*X , then $(F^*\alpha)^*\tilde{\omega} = \tilde{\omega}$ for all α .

It should be pointed out that odd n -forms $\xi^*\tilde{\omega}$ may be used as volume forms for integration on the base manifold X . In particular, these forms naturally appear as components of Lagrangians for variational problems for coframe fields.

The canonical jet projection $\tau^2 : J^2F^*X \rightarrow X$ is, for every $J_x^2\xi \in J^2F^*X$, defined by $\tau^2(J_x^2\xi) = x$. A second order Lagrangian for F^*X is any τ^2 -horizontal n -form λ defined on the second jet prolongation J^2F^*X of F^*X . In the chart (V^2, ψ^2) , $\psi^2 = (x^i, z_j^i, z_{j,k}^i, z_{j,kl}^i)$, on J^2F^*X , associated with (V, ψ) , a Lagrangian λ has an expression $\lambda = \mathcal{L} \cdot \tilde{\varphi} \otimes \omega_0$, where $\omega_0 = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$, $\tilde{\varphi}$ is a field of odd scalars, and $\mathcal{L} : V^2 \rightarrow \mathbb{R}$ is the component of λ with respect to (V, ψ) (the Lagrange function associated with (V, ψ)).

We say that a second order Lagrangian λ is *natural*, if for every diffeomorphism $\alpha : W \rightarrow X$, where W is an open set in X , the canonical lift $F^*\alpha$ of α to F^*X is an invariance transformation of λ , i.e.,

$$(J^2F^*\alpha)^*\lambda = \lambda$$

on the corresponding open set in J^2F^*X . An application of a general result, mentioned at the end of Section 1, to the structure considered in this paper gives us that there is a one-to-one correspondence between natural Lagrangians on J^2F^*X and differential invariants $\mathcal{I} : T_n^2 L_n^1 \rightarrow \mathbb{R}$.

We denote by $\mathcal{A}_{\text{diff}X}^*$ the algebra of diff X -invariant functions on J^2F^*X . In any chart (V, ψ) , $\psi = (x^i, z_j^i)$, on F^*X , define the functions $\mathcal{L}_{j,k}^i, \mathcal{L}_{j,kl}^i$, by

$$\begin{aligned}\mathcal{L}_{j,k}^i &= w_j^s w_k^t z_{\tilde{s},\tilde{t}}^i, \\ \mathcal{L}_{j,kl}^i &= w_j^s w_k^t w_l^u (z_{\tilde{s},\tilde{t},\tilde{u}}^i + w_m^v (z_{\tilde{v},\tilde{s}}^i z_{\tilde{t},\tilde{u}}^m - z_{\tilde{v},\tilde{t}}^i z_{\tilde{s},\tilde{u}}^m)),\end{aligned}\quad (4.2)$$

(compare with (3.1)). The functions $\mathcal{L}_{j,k}^i, \mathcal{L}_{j,kl}^i$, in coordinates expressed by (4.2), are globally well defined functions on J^2F^*X .

Lemma 3. (a) *The functions $\mathcal{L}_{j,k}^i, \mathcal{L}_{j,kl}^i$ belong to $\mathcal{A}_{\text{diff}X}^*$.*

(b) *Every function $\mathcal{L}$ from $\mathcal{A}_{\text{diff}X}^*$ can be locally written as a differentiable function of the functions $\mathcal{L}_{j,k}^i, \mathcal{L}_{j,kl}^i$.*

The following Theorem is an immediate consequence of the invariance theory.

Theorem. *Every natural Lagrangian λ on J^2F^*X is of the form*

$$\lambda = \mathcal{L}\tilde{\omega},$$

where $\mathcal{L} \in \mathcal{A}_{\text{diff}X}^*$ and $\tilde{\omega}$ is canonical odd n -form of F^*X .

Remark 1. Starting with the frame action of L_n^1 on L_n^1 , in the corresponding coordinates given by $\bar{p}_j^i = a_k^i p_j^k$ (compare with (1.5)), by the similar procedure it is possible to obtain *differential invariants of frames*. Second order differential invariants, including second order natural Lagrangians of frames, were described in [2]. Orbit reduction method gives us an alternative expression of natural Lagrangians of frames in comparison with [5].

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ON SYMMETRIZATION OF HIGHER ORDER JETS

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Abstract. We discuss geometric constructions transforming r -th order semiholonomic or non-holonomic jets into holonomic ones.

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Denote by $\mathcal{M}f$ the category of smooth manifolds and all smooth maps, by $\mathcal{M}f_m$ the subcategory of m -dimensional manifolds and their local diffeomorphisms and by $\mathcal{FM}_m$ the category of fibered manifolds with m -dimensional bases and fibered maps over local diffeomorphisms. Roughly speaking, the construction of higher order jets can be interpreted as a bundle functor on the category $\mathcal{FM}_m$ or on the product category $\mathcal{M}f_m \times \mathcal{M}f$.

First, the r -th *holonomic prolongation* $J^r Y$ of a fibered manifold $Y \rightarrow M$ is defined as the space of all r -jets of local sections of Y . Then J^r is a bundle functor on $\mathcal{FM}_m$ transforming a fibered manifold $Y \rightarrow M$ into its r -jet prolongation $J^r Y$ (with the projection onto Y) and any $\mathcal{FM}_m$ -morphism $\varphi : Y_1 \rightarrow Y_2$ covering $\varphi : M_1 \rightarrow M_2$ into $J^r \varphi : J^r Y_1 \rightarrow J^r Y_2$, $J^r \varphi(j_x^r \sigma) = j_{\varphi(x)}^r(\varphi \circ \sigma \circ \varphi^{-1})$.

Second, for every two manifolds M and N , the *bundle of holonomic jets* $J^r(M, N)$ is the space of all r -jets of M into N . Given a local diffeomorphism $f : M_1 \rightarrow M_2$ and a map $g : N_1 \rightarrow N_2$, we have the induced map $J^r(f, g) : J^r(M_1, N_1) \rightarrow J^r(M_2, N_2)$ defined by $J^r(f, g)(X) = (j_y^r g) \circ X \circ (j_x^r f)^{-1}$, where x is the source and y is the target of $X \in J^r(M_1, N_1)$. Then J^r is a bundle functor defined on the product category $\mathcal{M}f_m \times \mathcal{M}f$.

Constructions of semiholonomic jets $\bar{J}^r$ and nonholonomic jets $\tilde{J}^r$ can be also interpreted as bundle functors on $\mathcal{FM}_m$ or on $\mathcal{M}f_m \times \mathcal{M}f$, see Section 1. The aim of this paper is to study natural transformations

$$\bar{J}^r \rightarrow J^r \quad \text{and} \quad \tilde{J}^r \rightarrow J^r$$

which will be called in short *symmetrizations* of semiholonomic or nonholonomic jets.

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We recall that the theory of higher order jets was established by C. Ehresmann [3], see also [4, 10, 12]. Now the theory of jets is a very powerful tool in many areas of differential geometry, mainly in the theory of connections, bundle functors and natural differential operators, see e.g. [1, 2, 5, 6, 9]. It is also well known that the theory of jets forms the theoretical background in calculus of variations, field theories and many areas of mathematical physics. We point out that the idea of symmetrization plays an important role not only in the theory of jets, but also in current physical theories. Moreover, the most important role in differential geometry and in mathematical physics is played by classical holonomic jets. That is why it is useful to study symmetrizations of higher order jets.

1. PRELIMINARIES

In what follows, we denote by $\mathbb{N}$ the set of positive integers and by $\mathcal{FM}_{m,n}$ the category of fibered manifolds with m -dimensional bases, n -dimensional fibers and local fibered diffeomorphisms. All manifolds and maps are assumed to be infinitely differentiable.

We recall that the r -th *nonholonomic prolongation* $\tilde{J}^r Y$ of a fibered manifold $Y \rightarrow M$ is defined by iteration

$$\tilde{J}^1 Y = J^1 Y, \quad \tilde{J}^r Y = J^1(\tilde{J}^{r-1} Y \rightarrow M),$$

which yields natural identification $\tilde{J}^r(\tilde{J}^s Y) = \tilde{J}^{r+s} Y$. Clearly, we have the canonical inclusion $J^r Y \subset \tilde{J}^r Y$ given by $j_x^r s \mapsto j_x^1(u \mapsto j_u^{r-1} s)$ for every local section s of Y . The r -th *semiholonomic prolongation* $\bar{J}^r Y \rightarrow M$ is defined by the following induction. Write $\bar{J}^1 Y = J^1 Y$ and assume we have defined $\bar{J}^{r-1} Y \subset \tilde{J}^{r-1} Y$ such that the restriction of $\beta_{\tilde{J}^{r-2} Y} : \tilde{J}^{r-1} Y \rightarrow \tilde{J}^{r-2} Y$ maps $\bar{J}^{r-1} Y$ into $\bar{J}^{r-2} Y$, where $\beta_Y : J^1 Y \rightarrow Y$ means the projection. Then we define

$$\bar{J}^r Y = \{U \in J^1 \bar{J}^{r-1} Y; \beta_{\bar{J}^{r-1} Y}(U) = J^1 \beta_{\tilde{J}^{r-2} Y}(U) \in \bar{J}^{r-1} Y\}.$$

Obviously, J^r , $\bar{J}^r$ and $\tilde{J}^r$ are bundle functors on $\mathcal{FM}_m$ and for $r > 1$ we have $J^r Y \subset \bar{J}^r Y \subset \tilde{J}^r Y$. Denoting by (x^i, y^p) the canonical coordinates on Y , the induced coordinates on $J^1 Y$ are $y_i^p = \frac{\partial y^p}{\partial x^i}$. The canonical coordinates on $\tilde{J}^r Y$ can be introduced by the following induction. First, assume we have the coordinates $(x^i, y_{i_1 \dots i_{r-1}}^p)$ on $\tilde{J}^{r-1} Y$, where $i_1, \dots, i_{r-1} \in \{0, 1, \dots, m\}$. Then the induced coordinates on $\tilde{J}^r Y$ are

$$x^i, \quad y_{i_1 \dots i_{r-1} 0}^p = y_{i_1 \dots i_{r-1}}^p, \quad y_{i_1 \dots i_{r-1} i}^p = \frac{\partial}{\partial x^i} y_{i_1 \dots i_{r-1}}^p.$$

Next, the semiholonomic prolongation $\bar{J}^r Y$ can be characterized by the following condition: $y_{i_1 \dots i_r}^p = y_{j_1 \dots j_r}^p$ provided the sequences obtained from $(i_1, \dots, i_r)$ and

$(j_1, \dots, j_r)$ by deleting all zeros with preserving the order of nonzero indices coincide. So the local coordinates on $\bar{J}^r Y$ are $(x^i, y_{i_1 \dots i_s}^p)$, $s = 0, \dots, r$. Finally, the holonomic prolongation $J^r Y$ is characterized by full symmetry in all subscripts.

Every bundle functor F on $\mathcal{F}\mathcal{M}_m$ induces a bundle functor F_0 on the product category $\mathcal{M}f_m \times \mathcal{M}f$ by $F_0(M, N) = F(M \times N \rightarrow M)$. Using such a point of view, the bundles of r -th semiholonomic or nonholonomic jets on $\mathcal{M}f_m \times \mathcal{M}f$ are defined as the r -th semiholonomic or nonholonomic prolongation of the product fibered manifold $M \times N \rightarrow M$, respectively,

$$\bar{J}_0^r(M, N) = \bar{J}^r(M \times N \rightarrow M), \quad \tilde{J}_0^r(M, N) = \tilde{J}^r(M \times N \rightarrow M).$$

In what follows, we omit the subscript 0 at the jet functors on $\mathcal{M}f_m \times \mathcal{M}f$, i.e. we write $\bar{J}^r(M, N) = \bar{J}_0^r(M, N)$, $\tilde{J}^r(M, N) = \tilde{J}_0^r(M, N)$.

Clearly, the jet functors J^r , $\bar{J}^r$ and $\tilde{J}^r$ on $\mathcal{F}\mathcal{M}_m$ preserve fiber products and the corresponding induced jet functors on $\mathcal{M}f_m \times \mathcal{M}f$ preserve products in the second factor. By [7], every fiber product preserving bundle functor F on $\mathcal{F}\mathcal{M}_m$ of order r can be characterized by a triple (A, H, t) , where A is a Weil algebra of order r , $H : G_m^r \rightarrow \text{Aut} A$ is a group homomorphism of the r -jet group in dimension m into the group of algebra automorphisms of A and $t : \mathbb{D}_m^r \rightarrow A$ is an equivariant algebra homomorphism, $\mathbb{D}_m^r = J_0^r(\mathbb{R}^m, \mathbb{R})$. According to [7], bundle functors G on $\mathcal{M}f_m \times \mathcal{M}f$ of order r in the first factor and preserving products in the second factor are characterized by pairs (A, H) , where A and H are as above. Moreover, if $F = (A, H, t)$ is a functor on $\mathcal{F}\mathcal{M}_m$, then the induced functor F_0 on $\mathcal{M}f_m \times \mathcal{M}f$ is of the form $F_0 = (A, H)$.

We recall that an r -th order nonholonomic connection on a fibered manifold $Y \rightarrow M$ is a section $\Gamma : Y \rightarrow \tilde{J}^r Y$. Such a connection is called semiholonomic or holonomic, if it has values in $\bar{J}^r Y$ or $J^r Y$, respectively. For $r = 1$ we obtain the concept of a general connection $Y \rightarrow J^1 Y$. Further, the product of two connections $\Gamma_1 : Y \rightarrow \tilde{J}^r Y$ and $\Gamma_2 : Y \rightarrow \tilde{J}^s Y$ is a connection $\Gamma_1 * \Gamma_2 : Y \rightarrow \tilde{J}^{r+s} Y$ defined by $\Gamma_1 * \Gamma_2 := \tilde{J}^s \Gamma_1 \circ \Gamma_2$. Finally, Ehresmann prolongation of a connection $\Gamma : Y \rightarrow J^1 Y$ is the r -th order connection $\Gamma^{(r-1)} : Y \rightarrow \tilde{J}^r Y$ defined by $\Gamma^{(1)} := \Gamma * \Gamma$, $\Gamma^{(r-1)} := \Gamma^{(r-2)} * \Gamma$. It is well known that Ehresmann prolongation $\Gamma^{(r-1)}$ is semiholonomic.

2. SYMMETRIZATION OF JETS ON $\mathcal{F}\mathcal{M}_{m,n}$

By a symmetrization of semiholonomic jets $\bar{J}^r Y$ we understand an $\mathcal{F}\mathcal{M}_{m,n}$ -natural transformation $\bar{J}^r \rightarrow J^r$ being the identity map on $J^r Y$ for any $Y \in \text{Obj } \mathcal{F}\mathcal{M}_{m,n}$. Quite analogously, we can define the symmetrization of nonholonomic jets $\tilde{J}^r Y$ and also the symmetrization of jets $\bar{J}^r(M, N)$ and $\tilde{J}^r(M, N)$.

First, for $r = 2$, we have a well-known symmetrization of second order semiholonomic jets

$$C^{(2)} : \bar{J}^2 Y \rightarrow J^2 Y, \quad (x^i, y^p, y_i^p, y_{ij}^p) \mapsto (x^i, y^p, y_i^p, y_{ij}^p + y_{ji}^p).$$

In the general case, we proved in [1]

Theorem 1. *All $\mathcal{FM}_{m,n}$ -natural transformations $A : \bar{J}^r Y \rightarrow J^r Y$ are of the form*

- (1) *For $m = 1$ and arbitrary natural r , $A = \text{id} : \bar{J}^r Y = J^r Y \rightarrow J^r Y$.*
- (2) *For $r = 1$ and $m \geq 2$, $A = \text{id} : \bar{J}^1 Y = J^1 Y \rightarrow J^1 Y$.*
- (3) *For $r = 2$ and $m \geq 2$, $A = C^{(2)} : \bar{J}^2 Y \rightarrow J^2 Y$.*
- (4) *For $r \geq 3$ and $m \geq 2$ there is no $\mathcal{FM}_{m,n}$ -natural transformation A in question.*

Proposition 1. *For $r \geq 2$ and $m \geq 2$ there is no $\mathcal{FM}_{m,n}$ -natural transformation $\tilde{J}^r Y \rightarrow J^r Y$.*

By Theorem 1 and Proposition 1, to define symmetrizations of higher order semiholonomic and nonholonomic jets, it is unavoidable to use an additional geometric object. Using a projectable classical linear connection Σ on Y , we defined natural transformations

$$C^{(r)}(\Sigma) : \bar{J}^r Y \rightarrow J^r Y \quad \text{and} \quad s^{(r)}(\Sigma) : \tilde{J}^r Y \rightarrow J^r Y \quad \text{for any } r \quad (1)$$

depending on Σ , see [1]. We also proved that $C^{(2)}(\Sigma) = C^{(2)}$, which means that $C^{(r)}(\Sigma)$ generalizes $C^{(2)}$ for $r \geq 3$.

Moreover, the second author [11] introduced the symmetrization of semiholonomic jets $\bar{J}^r Y$ by means of a classical linear connection ∇ on the base manifold. He solved a more general problem on the existence of natural transformations of two fiber product preserving bundle functors on $\mathcal{FM}_m$. Given two fiber product preserving bundle functors $F = (A, H, t)$ and $F^1 = (A^1, H^1, t^1)$ of order r on $\mathcal{FM}_m$, he first defined a quasimorphism $\nu : (A, H, t) \rightarrow (A^1, H^1, t^1)$ as a $\text{GL}(m)$ -invariant homomorphism $\nu : A \rightarrow A^1$ such that $t^1 = \nu \circ t$. Then he proved that there is an $\mathcal{FM}_{m,n}$ -natural transformation $FY \rightarrow F^1 Y$ depending on a classical linear connection on the base of Y if and only if there is a quasi-morphism $(A, H, t) \rightarrow (A^1, H^1, t^1)$. In particular, the Weil algebra of J^r is $\mathbb{D}_m^r = J_0^r(\mathbb{R}^m, \mathbb{R})$ and the Weil algebra of $\bar{J}^r$ is $\bar{\mathbb{D}}_m^r = \bar{J}_0^r(\mathbb{R}^m, \mathbb{R})$, see also (3). By [11], the usual symmetrization $\bar{\mathbb{D}}_m^r \rightarrow \mathbb{D}_m^r$ is a quasimorphism of the corresponding triples. This yields that given a classical linear connection ∇ on M , there is an $\mathcal{FM}_{m,n}$ -natural transformation

$$S^{(r)}(\nabla) : \bar{J}^r Y \rightarrow J^r Y \quad \text{for any } r \quad (2)$$

depending on ∇ . In [11], it is also proved that for $r \geq 2$ it is not possible to symmetrize nonholonomic jets $\tilde{J}^r Y$ by means of a classical linear connection on the base of Y .

3. SYMMETRIZATION OF JETS ON $\mathcal{M}f_m \times \mathcal{M}f_n$

We start with several auxiliary assertions. From Proposition 3 in [1], we obtain directly

Proposition 2. *Let $m, n, r \in \mathbb{N}$, $m \geq 2$ and $r \geq 3$. Then there is no $\mathcal{M}f_m \times \mathcal{M}f_n$ -natural operator A transforming connections $\Gamma : M \times N \rightarrow J^1(M \times N)$ on $\text{pr}_1 : M \times N \rightarrow M$ into r -th order holonomic connections $A(\Gamma) : M \times N \rightarrow J^r(M \times N)$ extending Γ (i. e., such that $\pi_1^r \circ A(\Gamma) = \Gamma$, where $\pi_1^r : J^r(M \times N) \rightarrow J^1(M \times N)$ is the jet projection).*

Write

$$\bar{A}^r = \bar{J}_0^r(\mathbb{R}^m, \mathbb{R}) = \bigoplus_{k=0}^r \otimes^k \mathbb{R}^{m*}, \quad A^r = J_0^r(\mathbb{R}^m, \mathbb{R}) = \bigoplus_{k=0}^r S^k \mathbb{R}^{m*}. \quad (3)$$

Lemma 1. *Let $C_1, C_2 : \times^n \bar{A}^r \rightarrow A^q$ be $\text{GL}(m)$ -invariant maps such that $C_1|_{\times^n A^r} = C_2|_{\times^n A^r}$. Then $C_1 = C_2$*

Proof. For $n = 2$ and $q = r$ this is exactly Lemma 2 in [11]. The proof for any n and q is a direct modification of the proof of Lemma 2 from [11]. $\square$

Proposition 3. *Let $m, n, r, q \in \mathbb{N}$. Let $D_1, D_2 : \bar{J}^r \rightarrow J^q$ be $\mathcal{M}f_m \times \mathcal{M}f_n$ -natural transformations such that $D_1|_{J^r(M, N)} = D_2|_{J^r(M, N)}$. Then $D_1 = D_2$.*

Proof. Define $E_i : \times^n \bar{A}^r = \bar{J}_0^r(\mathbb{R}^m, \mathbb{R}^n) \rightarrow \times^n A^q = J_0^q(\mathbb{R}^m, \mathbb{R}^n)$ as the restrictions of $D_i : \bar{J}^r(\mathbb{R}^m, \mathbb{R}^n) \rightarrow J^q(\mathbb{R}^m, \mathbb{R}^n)$, $i = 1, 2$. Because of the $\mathcal{M}f_m \times \mathcal{M}f_n$ -invariance of D_1 and D_2 it remains to show that $\text{pr}_l \circ E_1 = \text{pr}_l \circ E_2$ for $l = 1, \dots, n$, where $\text{pr}_l : \times^n A^q \rightarrow A^q$ is the projection on the l th factor. But from the assumption of our proposition, we claim that $E_1, E_2 : \times^n \bar{A}^r \rightarrow \times^n A^q$ are $\text{GL}(m)$ -invariant maps such that $E_1|_{\times^n A^r} = E_2|_{\times^n A^r}$. So the proposition is an immediate consequence of Lemma 1. $\square$

Proposition 4. *Let $m, n, r \in \mathbb{N}$, $m \geq 2$ and $r \geq 3$. There is no $\mathcal{M}f_m \times \mathcal{M}f_n$ -natural transformation $A : \bar{J}^r \rightarrow J^r$ such that $A|_{J^r(M, N)} = \text{id}$ for any $(M, N) \in \text{Obj}(\mathcal{M}f_m \times \mathcal{M}f_n)$.*

Proof. Suppose that such A exists. We modify the respective part of the proof of Theorem 3 from [1]. Given a connection $\Gamma : M \times N \rightarrow J^1(M \times N)$ on $\text{pr}_1 : M \times N \rightarrow M$, we have an r -th order holonomic connection $A(\Gamma) := A \circ \Gamma^{(r-1)} : M \times N \rightarrow J^r(M \times N) = J^r(M, N)$, where $\Gamma^{(r-1)} : M \times N \rightarrow \bar{J}^r(M \times N) = \bar{J}^r(M, N)$ is the r -th Ehresmann prolongation of Γ . Next, we have two $\mathcal{M}f_m \times \mathcal{M}f_n$ -natural transformations $D_1 = \pi_1^r \circ A : \bar{J}^r \rightarrow J^1$ and $D_2 = \bar{\pi}_1^r : \bar{J}^r \rightarrow J^1$, where $\bar{\pi}_1^r$ and $\pi_1^r : J^r \rightarrow J^1$ are jet projections. By the assumption of A , we see $D_1|_{J^r(M, N)} = D_2|_{J^r(M, N)}$. By Proposition 3 we have $D_1 = D_2$. Then $\pi_1^r \circ A = \bar{\pi}_1^r$, so that $\pi_1^r \circ A(\Gamma) = \pi_1^r \circ A \circ \Gamma^{(r-1)} = \bar{\pi}_1^r \circ \Gamma^{(r-1)} = \Gamma$. This yields that connection $A(\Gamma)$ extends Γ . But this is impossible because of Proposition 2. $\square$

The book [6] proves

Proposition 5. *For $r \geq 2$ the only $\mathcal{M}f_m \times \mathcal{M}f_n$ -natural transformations $A : J^r \rightarrow J^r$ are the identity and the contraction. For $r = 1$ we have*

$$J^1(M, N) = \text{Hom}(TM, TN),$$

and all $\mathcal{M}f_m \times \mathcal{M}f_n$ -natural transformations $J^1 \rightarrow J^1$ form the one-parameter family of homotheties $X \mapsto cX$, $c \in \mathbb{R}$.

The main result of this section is

Theorem 2. (1) *For $r = 1$ and $m, n \in \mathbb{N}$, all $\mathcal{M}f_m \times \mathcal{M}f_n$ -natural transformations $\bar{J}^1 = J^1 \rightarrow J^1$ form the one-parameter family of homotheties.*
 (2) *For $m = 1$, $r \geq 2$ and $n \in \mathbb{N}$, the only $\mathcal{M}f_m \times \mathcal{M}f_n$ -natural transformations $\bar{J}^r = J^r \rightarrow J^r$ are the identity and the contraction.*
 (3) *For $m \geq 2$, $r = 2$ and $n \in \mathbb{N}$, the only $\mathcal{M}f_m \times \mathcal{M}f_n$ -natural transformation $\bar{J}^2 \rightarrow J^2$ are the classical symmetrization $\bar{J}^2 \rightarrow J^2$ of semiholonomic 2-jets and the trivial one $w \mapsto j_x^2([y])$, $w \in \bar{J}_x^2(M, N)_y$, $x \in M$, $y \in N$, where $[y] : M \rightarrow \{y\} \subset N$ denotes the constant map.*
 (4) *For $m \geq 2$, $r \geq 3$ and $n \in \mathbb{N}$, the only $\mathcal{M}f_m \times \mathcal{M}f_n$ -natural transformation $\bar{J}^r \rightarrow J^r$ is the trivial one $w \mapsto j_x^r([y])$, $w \in \bar{J}_x^r(M, N)_y$, $x \in M$, $y \in N$.*

Proof. The parts (1) and (2) are immediate consequences of Proposition 5. Part (3) follows from Propositions 3 and 5. Indeed, by these propositions we have that the number of such natural transformations is at least 2. Quite similarly, part (4) is the consequence of Propositions 3, 5 and 4. $\square$

Proposition 6. *For $m \geq 2$, $r \geq 2$ and $n \in \mathbb{N}$, the only $\mathcal{M}f_m \times \mathcal{M}f_n$ -natural transformation $\tilde{J}^r \rightarrow J^r$ is the contraction.*

Proof. For $r \geq 3$ the statement follows from Theorem 2 and the case $r = 2$ is examined in [8]. $\square$

Using the naturality with respect to $\mathcal{M}f_m \times \mathcal{M}f$, we obtain easily from Theorem 2 the following

Corollary 1. (1) *For $r = 1$ and $m \in \mathbb{N}$, all $\mathcal{M}f_m \times \mathcal{M}f$ -natural transformations $\bar{J}^1 = J^1 \rightarrow J^1$ form the one-parameter family of homotheties.*
 (2) *For $m = 1$ and $r \geq 2$, the only $\mathcal{M}f_m \times \mathcal{M}f$ -natural transformations $\bar{J}^r \rightarrow J^r$ are the identity and the contraction.*
 (3) *For $m \geq 2$ and $r = 2$, the only $\mathcal{M}f_m \times \mathcal{M}f$ -natural transformation $\bar{J}^2 \rightarrow J^2$ are the classical symmetrization $\bar{J}^2 \rightarrow J^2$ of semiholonomic 2-jets and the trivial one $w \mapsto j_x^2([y])$, $w \in \bar{J}_x^2(M, N)_y$, $x \in M$, $y \in N$, where $[y] : M \rightarrow \{y\} \subset N$ denotes the constant map.*
 (4) *For $m \geq 2$ and $r \geq 3$, the only $\mathcal{M}f_m \times \mathcal{M}f$ -natural transformation $\bar{J}^r \rightarrow J^r$ is the trivial one $w \mapsto j_x^r([y])$, $w \in \bar{J}_x^r(M, N)_y$, $x \in M$, $y \in N$.*

Corollary 2. *For $m \geq 2$ and $r \geq 2$, the only $\mathcal{M}f_m \times \mathcal{M}f$ -natural transformation $\tilde{J}^r \rightarrow J^r$ is the contraction.*

In the rest of this section, we show that it is possible to symmetrize higher order semiholonomic jets on $\mathcal{M}f_m \times \mathcal{M}f_n$ by means of a classical linear connection ∇ on M . Indeed, we define easily the symmetrization of semiholonomic jets

$$S_0^{(r)}(\nabla) : \bar{J}^r(M, N) \rightarrow J^r(M, N) \quad \text{for any } r$$

depending on a classical linear connection ∇ on M . Let $S^{(r)}(\nabla) : \bar{J}^r(M \times N) \rightarrow J^r(M \times N)$ be the symmetrization (2) for the product fibered manifold $M \times N \rightarrow M$. This is exactly the $\mathcal{M}f_m \times \mathcal{M}f_n$ -natural transformation $S_0^{(r)}(\nabla)$

$$\bar{J}^r(M, N) = \bar{J}^r(M \times N \rightarrow M) \xrightarrow{S^{(r)}(\nabla)} J^r(M \times N \rightarrow M) = J^r(M, N).$$

By [11], for $r \geq 2$ it is not possible to define symmetrization $\tilde{J}^r Y \rightarrow J^r Y$ of nonholonomic jets on $\mathcal{F}\mathcal{M}_m$ by means of a classical linear connection ∇ on M . On the other hand, we have the $\mathcal{M}f_m \times \mathcal{M}f_n$ -natural operator

$$s_0^{(r)} : \tilde{J}^r(M, N) \rightarrow J^r(M, N), \quad w \mapsto j_x^r([y]), \quad w \in \tilde{J}_x^r(M, N)_y, \quad x \in M, y \in N.$$

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VARIATIONAL PROJECTORS IN FIBRED MANIFOLDS

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Abstract. The aim of this paper is to give a survey of recent developments in global variational geometry, and in particular, to complete the results on the construction of classes (terms) in the variational sequences related to higher-order variational problems on fibred spaces. Explicit description of the first order variational sequences is given as an example.

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1. INTRODUCTION

This paper contains the essential parts of the lecture presented by the first author at the workshop Algebra, Geometry and Mathematical Physics, Brno University of Technology, September 12–14, 2012, and prepared in collaboration with Zbyněk Urban and Jana Volná. The aim is to give a survey of the basic concepts and theorems of global variational geometry, and in particular to complete the construction of the terms (classes) in the variational sequences related to higher-order variational problems on fibred spaces (Section 3).

Our main references for general higher-order variational theory on smooth manifolds (Section 2) are the papers Garcia [4], Goldschmidt and Sternberg [5], Krupka [7], and Trautman [15], where the geometric setting for the theory of variational functionals has been established. Later, essential contributions were obtained by Anderson and Duchamp [2], and Dedecker [3]; a broad, conceptually more complete treatment, including invariance of variational functionals, was presented by Olver [14]. For contemporary status of the theory, we refer to the survey chapters in the book Krupka and Saunders [11], where many other relevant references can be found.

For the notion of the *interior Euler–Lagrange operator*, as defined by a chart expression, we refer to Anderson [1], and Krbek and Musilová [6] (see also other sources quoted therein). In global variational theory, we relate this notion to different terms (classes) in the variational sequence; to this purpose, however, the existing theory should be completed by proving the coordinate independence of the interior Euler–Lagrange operator, and then also by a description of its *image*. In this paper, we apply a coordinate-free definition given by Krupka and Šeděnková-Volná [13], and

Volná and Urban [16], and extend the description of the interior Euler-Lagrange operators by the discussion of the image (Section 3). We also briefly discuss the classes entering the first order variational sequence in field theory. In order to avoid misunderstanding with the Euler-Lagrange mapping, we change terminology and call the (global version of) interior Euler-Lagrange operators the *variational projectors*.

From now on, we suppose we are given a fibred manifold Y with base X and projection π and set $n = \dim X$ and $n + m = \dim Y$. We denote by $J^r Y$ the r -jet prolongation of Y . An element of the set $J^r Y$ is usually denoted by $J_x^r \gamma$; the mapping $x \rightarrow J^r \gamma(x) = J_x^r \gamma$ is the r -jet prolongation of γ . The canonical source projection $J_x^r \gamma \rightarrow x$ of $J^r Y$ onto X (resp. the canonical jet projection $J_x^r \gamma \rightarrow J_x^s \gamma$ of $J^r Y$ onto $J^s Y$, $0 \leq s \leq r$) is denoted by π^r (resp. $\pi^{r,s}$). The r -jet prolongation $J^r \mathcal{E}$ of a π -projectable vector field $\mathcal{E}$, is a vector field on $J^r Y$ whose flow is the canonical jet prolongation of the flow of $\mathcal{E}$. For any open set $W \subset Y$, we set $W^r = (\pi^{r,0})^{-1}(W)$ and denote by $\Omega^r W$ the exterior algebra of differential forms on the set W^r .

If (V, ψ) , $\psi = (x^i, y^\sigma)$, is a fibred chart on Y , then the associated fibred chart on $J^r Y$ is denoted by (V^r, ψ^r) , $\psi^r = (x^i, y^\sigma, y_{i_1}^\sigma, y_{i_1 i_2}^\sigma, \dots, y_{i_1 i_2 \dots i_r}^\sigma)$. When we use a multi-index notation, we write $\psi^r = (x^i, y_I^\sigma)$, where $I = (i_1 i_2 \dots i_s)$, and the length $s = |I|$ of I satisfies $0 \leq |I| \leq r$. We also write $Ii = (i_1 i_2 \dots i_s i)$. Differential forms, locally generated by differential forms $\omega_I^\sigma = dy_I^\sigma - y_{Ii}^\sigma dx^i$, $0 \leq |I| \leq r-1$, and $d\omega_I^\sigma$, $|I| = r-1$, are called *contact forms*. The canonical decomposition theorem (cf. e.g. Krupka [9]) for a q -form ρ on $J^r Y$ says that $(\pi^{r+1,r})^* \rho = h\rho + p_1\rho + p_2\rho + \dots + p_q\rho$, where the component $p_k\rho$ contains exactly k exterior factors ω_I^σ .

Throughout, standard concepts of analysis of differential forms on smooth manifolds are used; $d\rho$ is the exterior derivative of a form ρ , and $i_\xi\rho$ (resp. $\partial_\xi\rho$) is the contraction (resp. Lie derivative) of ρ by a vector field ξ .

2. VARIATIONAL FUNCTIONALS

We introduce a morphism of exterior algebras related with a fibred manifold structure.

Lemma 1. *Let W be an open set in the fibred manifold Y . Then there exists a unique morphism $\Omega^r W \ni \rho \rightarrow h\rho \in \Omega^{r+1} W$ of exterior algebras, such that for any fibred chart (V, ψ) , $\psi = (x^i, y^\sigma)$, with $V \subset W$, and any differentiable function $f : W^r \rightarrow \mathbb{R}$,*

$$hf = f \circ \pi^{r+1,r}, \quad hdf = d_i f \cdot dx^i, \quad (2.1)$$

where

$$d_i f = \frac{\partial f}{\partial x^i} + \sum_{j_1 \leq j_2 \leq \dots \leq j_k} \frac{\partial f}{\partial y_{j_1 j_2 \dots j_k}^\sigma} y_{j_1 j_2 \dots j_k i}^\sigma. \quad (2.2)$$

The morphism h is called the *horizontalisation*. The symbol d_i in (2.2) is called the *i-th formal derivative operator* relative to the fibred chart (V, ψ) . It is important for practical calculations that $hdx^i = dx^i$ and $hdy_I^\sigma = y_{Ii}^\sigma dx^i$.

A *Lagrangian* for Y is a π^r -horizontal n -form λ on the r -jet prolongation $J^r Y$ of Y . The number r is the *order* of λ . The pair (Y, λ) is sometimes called the *variational structure*. In a fibred chart (V, ψ) , $\psi = (x^i, y^\sigma)$, λ has an expression $\lambda = \mathcal{L}\omega_0$, where $\omega_0 = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$ and $\mathcal{L} : V^r \rightarrow \mathbb{R}$ is the *component* of λ with respect to (V, ψ) (the *Lagrange function*).

Let Ω be a *piece* of X with boundary $\partial\Omega$, let $\Gamma_\Omega Y$ be the set of sections of Y , defined on Ω . A Lagrangian λ of order r defines the *variational functional*

$$\Gamma_\Omega Y \ni \gamma \rightarrow \lambda_\Omega(\gamma) = \int_\Omega J^r \gamma^* \lambda \in \mathbb{R}. \quad (2.3)$$

The main objective of the variational theory is to study the properties of this real-valued function, such as *extremals* and *invariance conditions*. To investigate λ_Ω , we apply *variational methods* based on variations of each individual section γ of Y , and induced variations of the value $\lambda_\Omega(\gamma)$. Given a π -projectable vector field Ξ on Y , and its π -projection ξ , we can construct 1-parameter family of sections $\gamma_t = \alpha_t^\Xi \gamma \alpha_{-t}^\xi$, using the flows α_t^Ξ and α_t^ξ of Ξ and ξ . Then, differentiating the function $t \rightarrow \lambda_\Omega(\gamma_t)$ at $t = 0$, we get another variational functional

$$\Gamma_\Omega Y \ni \gamma \rightarrow \left(\frac{d}{dt} \lambda_\Omega(\gamma_t) \right)_0 = \int_\Omega J^r \gamma^* \partial_{J^r \Xi} \lambda \in \mathbb{R}, \quad (2.4)$$

the *first variation* of (2.3), *induced by* Ξ . For further computations of the Lie derivative $\partial_{J^r \Xi} \lambda$ in this expression, we need the following result.

Theorem 1 (Lepage equivalents of a Lagrangian). *Let λ be a Lagrangian of order r for Y . There exists an integer s and an n -form ρ on $J^s Y$ such that:*

- (a) $J^s \gamma^* \rho = J^r \gamma^* \lambda$ for all sections γ of Y .
- (b) $J^s \gamma^* i_\Xi d\rho = 0$ for all sections γ of Y and all $\pi^{s,0}$ -vertical vector fields Ξ on $J^s Y$.

Conditions (a) and (b) can be equivalently expressed in terms of the horizontalisation h as $h\rho = \lambda$ and $hi_\Xi d\rho = 0$, respectively.

Any form ρ , satisfying conditions (a) and (b), is called a *Lepage equivalent* of the Lagrangian λ . Comparing condition (a) with formula (2.3), we see that a Lepage equivalent ρ defines the same variational functional as λ .

There are many interesting examples of Lepage equivalents (see references in Krupka, Krupková and Saunders [12]). We mention here the *principal Lepage equivalent* of a *second-order* Lagrangian $\lambda = \mathcal{L}\omega_0$,

$$\Theta = \mathcal{L}\omega_0 + \left(\left(\frac{\partial \mathcal{L}}{\partial y_i^\sigma} - d_j \frac{\partial \mathcal{L}}{\partial y_{ij}^\sigma} \right) \omega^\sigma + \frac{\partial \mathcal{L}}{\partial y_{ij}^\sigma} \omega_j^\sigma \right) \wedge \omega_i, \quad (2.5)$$

where $\omega^\sigma = dy^\sigma - y_l^\sigma dx^l$, $\omega_i^\sigma = dy_i^\sigma - y_{il}^\sigma dx^l$, and $\omega_i = i_{\partial/\partial x^i} \omega_0$ (Krupka [7]). The Lepage form Θ is important in applications. If λ is of the first order, then Θ reduces to the well-known *Poincaré-Cartan form* introduced by Garcia [4].

Theorem 2 (Infinitesimal first variation formula). *Let λ be a Lagrangian of order r for Y , and let ρ be any Lepage equivalent of λ , defined on $J^s Y$. Then, for any section γ and any π -projectable vector field Ξ on Y ,*

$$J^r \gamma^* \partial_{J^r \Xi} \lambda = J^s \gamma^* i_{J^s \Xi} d\rho + dJ^s \gamma^* i_{J^s \Xi} \rho. \quad (2.6)$$

Formula (2.6) is called the *infinitesimal first variation formula*. It corresponds with the integrand of the “classical” (integral) first variation formula, but does not have a classical analogue. Using the horizontalisation, (2.6) can be equivalently written as

$$\partial_{J^r \Xi} \lambda = h i_{J^s \Xi} d\rho + h d i_{J^s \Xi} \rho. \quad (2.7)$$

The *Euler-Lagrange form* of a Lagrangian λ of order r is an $(n+1)$ -form E_λ on $J^{2r} Y$, defined by

$$E_\lambda = E_\sigma(\mathcal{L}) \omega^\sigma \wedge \omega_0, \quad (2.8)$$

where $E_\sigma(\mathcal{L})$ are the *Euler-Lagrange expressions*,

$$E_\sigma(\mathcal{L}) = \sum_{l=0}^r (-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial \mathcal{L}}{\partial y_{p_1 p_2 \dots p_l}^\sigma}. \quad (2.9)$$

It can be verified that the first summand on the right-hand side of (2.6) represents the Euler–Lagrange form: by a direct computation $J^s \gamma^* i_{J^s \Xi} d\rho = J^{s+1} \gamma^* i_{J^{s+1} \Xi} E_\lambda$. The *boundary term* $dJ^s \gamma^* i_{J^s \Xi} \rho$ is an *exact* differential form; in the *integral* first variation formula, it transforms, via the Stokes’ theorem, to the boundary $\partial\Omega$.

Suppose that the Euler–Lagrange form E_λ is of order s . Then a section γ is called an *extremal*, if E_λ vanishes along the s -jet prolongation $J^s \gamma$; γ is an extremal if and only if its restriction to any coordinate neighborhood satisfies the *Euler-Lagrange equations* $E_\sigma(\mathcal{L}) = 0$.

The $\mathbb{R}$ -linear mapping $\lambda \rightarrow E_\lambda$, assigning to a Lagrangian its Euler–Lagrange form, is called the *Euler-Lagrange mapping*. The Lagrangians, belonging to its *kernel*, are called (*variationally*) *trivial*, or *null*.

Theorem 3 (Trivial Lagrangians). *For a Lagrangian λ of order r , the following two conditions are equivalent:*

- (a) $E_\lambda = 0$.
- (b) *To each point $y_0 \in Y$, there exist a fibred chart (V, ψ) at y_0 and an $(n-1)$ -form η defined on $V^{r-1} \subset J^{r-1} Y$ such that*

$$J^r \gamma^* \lambda = dJ^{r-1} \gamma^* \eta \quad (2.10)$$

for all sections γ , defined on $U = \pi(V) \subset X$.

Condition (b) is equivalent to the equation $\lambda = h d\eta$ on V^r .

By a *source form* on $J^s Y$, we mean a differential $(n + 1)$ -form ε expressible in any fibred chart (V, ψ) , $\psi = (x^i, y^\sigma)$, as

$$\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0. \quad (2.11)$$

Clearly, this definition is invariant under transformations of fibred coordinates. Every Euler–Lagrange form is a source form. A source form is said to be *variational* if it is expressible as $\varepsilon = E_\lambda$ for some Lagrangian λ . The problem as to find conditions under which a source form is variational is the *inverse problem* of the calculus of variations.

To study the *invariant* variational structure, let us first mention, for the record, some general differential-geometric notions that are needed, sometimes with slight modifications, in proofs. Let X be a manifold, W an open set in X and $\alpha : W \rightarrow X$ a diffeomorphism, let ρ be a p -form on X . We say that ρ is *invariant with respect to* α , if its pull-back $\alpha^* \rho$ coincides with ρ ,

$$\alpha^* \rho = \rho. \quad (2.12)$$

We also say that α is an *invariance transformation* of ρ . This definition immediately extends to vector fields. Let ξ be a vector field on X , α_t^ξ its flow and α_t^ξ its local 1-parameter group. ξ is said to be the *generator of invariance transformations* of ρ if for all (t, x) from the domain of the flow α_t^ξ ,

$$(\alpha_t^\xi)^* \rho(x) = \rho(x). \quad (2.13)$$

Lemma 2. *For every point (t, x) from the domain of the definition of the flow of the vector field ξ ,*

$$\frac{d}{dt} (\alpha_t^\xi)^* \rho(x) = ((\alpha_t^\xi)^* \partial_\xi \rho)(x). \quad (2.14)$$

Lemma 3. *The following two conditions are equivalent:*

- (a) ξ generates invariance transformations of ρ .
- (b) The Lie derivative of ρ by ξ vanishes,

$$\partial_\xi \rho = 0. \quad (2.15)$$

Let λ be a Lagrangian of order r for Y , $\alpha : W \rightarrow Y$ an automorphism of the fibred manifold Y , and $J^r \alpha : W^r \rightarrow J^r Y$ the r -jet prolongation of α . We say that α is an *invariance transformation* of λ , if $(J^r \alpha)^* \lambda = \lambda$. A *generator of invariance transformations* of λ is a π -projectable vector field Ξ on Y whose local one-parameter group consists of invariance transformations of λ . The definitions naturally extend to *r-jet prolongations* $J^r \Xi$ of Ξ .

In the following lemma, we use the multi-index notation. We introduce a class of vector fields on $J^r Y$, preserving the contact forms $\omega_I^\sigma = dy_I^\sigma - y_{Ii}^\sigma dx^i$, $0 \leq |I| \leq r - 1$. We say that a vector field Z on $J^r Y$ *preserves contact forms* if, for any contact

form ρ , the Lie derivative $\partial_Z \rho$ is again a contact form; we also say that Z is a *contact symmetry* (see Krupka, Krupkova, Prince and Sarlet [10]).

Lemma 4. *For any π -projectable vector field Ξ on Y , the r -jet prolongation $Z = J^r \Xi$ is a contact symmetry.*

Lemma 5. *Let λ be a Lagrangian of order r for Y .*

(a) *A π -projectable vector field Ξ on Y generates invariance transformations of λ if and only if*

$$\partial_{J^r \Xi} \lambda = 0. \quad (2.16)$$

(b) *Generators of invariance transformations of λ constitute a subalgebra of the algebra of the vector fields on $J^r Y$.*

In terminology used by Trautman [15] (see also Krupka [8]), equation (2.16) is the *Noether equation*. This equation represents a relation between the Lagrangian λ and the generator Ξ of invariance transformations. Given λ , we can use the Noether equation to determine the generators Ξ . On the other hand, given a Lie algebra of π -projectable vector fields Ξ , one can use the corresponding Noether equations to determine invariant Lagrangians λ .

Theorem 4. *Suppose that a Lagrangian λ is invariant with respect to a π -projectable vector field Ξ . Then for any Lepage equivalent ρ of order s of λ ,*

$$J^s \gamma^* i_{J^s \Xi} d\rho + dJ^s \gamma^* i_{J^s \Xi} \rho = 0 \quad (2.17)$$

for every section γ of Y .

By a *conserved current* for a section $\gamma \in \Gamma_\Omega(Y)$, we mean any $(n-1)$ -form η on $J^r Y$ such that

$$dJ^r \gamma^* \eta = 0. \quad (2.18)$$

We call formula (2.18) the *conservation law equation*, or a *conservation law* for the section γ .

The following assertion says that *extremals* of invariant Lagrangians satisfy, in addition to the Euler–Lagrange equations, also some other conditions in the form of conservation law equations.

Theorem 5 (First theorem of Emmy Noether). *Let λ be a Lagrangian, ρ a Lepage equivalent of λ defined on $J^s Y$ and let γ be an extremal. Then, for every generator Ξ of invariance transformations of λ ,*

$$dJ^s \gamma^* i_{J^s \Xi} \rho = 0. \quad (2.19)$$

One can use invariance of variational functionals in a different way than in Theorem 5. Namely, the infinitesimal first variation formula (Theorem 2) shows that the property of a Lagrangian to be invariant reduces the number of Euler–Lagrange equations.

Theorem 6. *Let λ be an invariant Lagrangian with respect to a vector field Ξ , and suppose that a section γ satisfies the conservation law*

$$dJ^s \gamma^* i_{J^s \Xi} \rho = 0 \quad (2.20)$$

for a Lepage equivalent ρ of λ of order s . Then, for any fibred chart (V, ψ) , the associated Euler–Lagrange expressions are linearly dependent along γ .

Let $\alpha : W \rightarrow Y$ be an automorphism of Y and let ε be a source form on $J^s Y$. We say that α is an *invariance transformation* of ε , if $J^s \alpha^* \varepsilon = \varepsilon$. A *generator of invariance transformations* of ε is a π -projectable vector field on Y whose local one-parameter group consists of invariance transformations of ε .

Lemma 6 (Noether–Bessel–Hagen equation). *Let ε be a source form of order s for Y .*

(a) *A π -projectable vector field Ξ on Y is the generator of invariance transformations of ε if and only if*

$$\partial_{J^r \Xi} \varepsilon = 0. \quad (2.21)$$

(b) *Generators of invariance transformations of ε constitute a subalgebra of the algebra of vector fields on $J^r Y$.*

To study invariant Euler–Lagrange forms, note the following identity.

Lemma 7. *Let λ be a Lagrangian of order r for Y , let α be an automorphism of Y , and let E_λ be the Euler–Lagrange form of λ . Then*

$$J^{2r} \alpha^* E_\lambda = E_{J^r \alpha^* \lambda}. \quad (2.22)$$

Lemma 8. (a) *Every invariance transformation of a Lagrangian λ is an invariance transformation of the Euler–Lagrange form E_λ .*

(b) *For every invariance transformation α of E_λ , the Lagrangian $\lambda - J^r \alpha^* \lambda$ is variationally trivial.*

We can generalize Noether’s theorem to invariance transformations of the Euler–Lagrange forms and conservation equations. However, since the proof is based on the theorem on the kernel of the Euler–Lagrange mapping $\lambda \rightarrow E_\lambda$, which is of local character (Theorem 3), the assertion we obtain is also of local character. Let Θ_λ denote the principal Lepage equivalent of λ (see Krupka, Krupková, Saunders [12]); for second-order Lagrangians, Θ_λ was defined in (2.5). Note, however, that for general orders Θ_λ is *not* necessarily defined globally.

Theorem 7. *Let λ be a Lagrangian of order r , γ an extremal, and Ξ the generator of invariance transformations of the Euler–Lagrange form E_λ . Then, for every point $y_0 \in Y$, there exists a fibred chart (V, ψ) at y_0 and an $(n - 1)$ -form η defined on V^{r-1} such that*

$$dJ^{2r} \gamma^* (i_{J^{2r} \Xi} \Theta_\lambda - \eta) = 0 \quad (2.23)$$

on $U = \pi(V)$.

Now, we study symmetries of solutions of the Euler–Lagrange equations. Let λ be a Lagrangian of order r for Y . Suppose that we have an extremal γ of λ ; thus, γ satisfies the Euler–Lagrange equations

$$E_\lambda \circ J^{2r} \gamma = 0. \quad (2.24)$$

An automorphism $\alpha : W \rightarrow Y$ of Y with projection α_0 is said to be a *symmetry* of γ , if the section $\alpha\gamma\alpha_0^{-1}$ is also an extremal, i.e.,

$$E_\lambda \circ J^{2r}(\alpha\gamma\alpha_0^{-1}) = 0. \quad (2.25)$$

We say that a π -projectable vector field Ξ *generates symmetries* of γ if its local one-parameter group consists of symmetries of γ .

To study symmetries, we need a lemma on pushforward vector fields. Let X be any manifold, ξ a vector field on X , $W \subset X$ an open set, and $\alpha : W \rightarrow X$ a diffeomorphism. The *pushforward vector field* of ξ by α is the vector field

$$\xi^{(\alpha)}(x) = T_{\alpha^{-1}(x)}\alpha \cdot \xi(\alpha^{-1}(x)). \quad (2.26)$$

Lemma 9. *Let X be a manifold, W an open set in X , ξ a vector field on X , $\alpha : W \rightarrow X$ a diffeomorphism, and ρ a p -form. Then*

$$i_\xi \alpha^* \rho = \alpha^* i_{\xi^{(\alpha)}} \rho. \quad (2.27)$$

The following theorem says that invariance transformations of the Euler–Lagrange form E_λ permute extremals of the variational structure (Y, λ) and give us examples of symmetries of extremals.

Theorem 8. *An invariance transformation of the Euler–Lagrange form E_λ is a symmetry of every extremal γ of λ .*

The following theorem describes the properties of individual extremals.

Theorem 9. *Let λ be a Lagrangian of order r , let s be the order of the Euler–Lagrange form E_λ , and let γ be an extremal. Then a π -projectable vector field Ξ generates symmetries of γ if and only if*

$$E_{\partial_{J^r \Xi} \lambda} \circ J^s \gamma = 0. \quad (2.28)$$

3. VARIATIONAL SEQUENCES AND VARIATIONAL PROJECTORS

We have already defined *contact forms* on the r -jet prolongation $J^r Y$. Evidently, every q -form, where $q \geq n + 1$, $n = \dim X$, would be contact. Now, we introduce, for $(n + k)$ -forms, $k \geq 1$, a modified concept of a strongly contact form. We say that an $(n + k)$ -form ρ on $J^r Y$ is *strongly contact* if, for every point $J_x^r \gamma \in J^r Y$, there exist a fibred chart (V, ψ) , $\psi = (x^i, y^\sigma)$, on Y , an integer $s \geq r$, and a contact $(n + k - 1)$ -form η on V^s with $p_{k-1}\eta = 0$ such that

$$p_k((\pi^{s,r})^* \rho - d\eta) = 0. \quad (3.1)$$

This condition is equivalent to saying that $(\pi^{s,r})^* \rho$ can be expressed in the form

$$(\pi^{s,r})^* \rho = \mu + d\eta, \quad (3.2)$$

where μ and η satisfy $p_k \mu = 0$ and $p_{k-1} \eta = 0$.

Let $q \geq 0, k \geq 1$. If $0 \leq q \leq n$, denote by Θ_q^r the sheaf of contact forms on $J^r Y$; if $q > n$, then let Θ_q^r be the sheaf of strongly contact forms on $J^r Y$. We get an exact subsequence

$$0 \rightarrow \Theta_1^r \rightarrow \Theta_2^r \rightarrow \Theta_3^r \rightarrow \dots \rightarrow \Theta_M^r \rightarrow 0 \quad (3.3)$$

of the de Rham sequence of sheaves of Abelian groups on $J^r Y$,

$$0 \rightarrow \mathbb{R} \rightarrow \Omega_0^r \rightarrow \Omega_1^r \rightarrow \Omega_2^r \rightarrow \dots \rightarrow \Omega_M^r \rightarrow \Omega_{M+1}^r \rightarrow \dots \rightarrow \Omega_N^r \rightarrow 0, \quad (3.4)$$

with the sequence mappings equal to the exterior derivative operator d . Then numbers N and M are given by

$$N = \dim J^r Y, \quad M = n \binom{n+r-1}{n} + 2n - 1. \quad (3.5)$$

We call (3.3) the *contact subsequence* of the de Rham sequence. The quotient sequence

$$0 \rightarrow \mathbb{R} \rightarrow \Omega_0^r \rightarrow \Omega_1^r / \Theta_1^r \rightarrow \dots \rightarrow \Omega_M^r / \Theta_M^r \rightarrow \Omega_{M+1}^r \rightarrow \dots \rightarrow \Omega_N^r \rightarrow 0 \quad (3.6)$$

is also exact; we call the quotient sequence the *variational sequence* (of order r). The class $[\rho]$ of a form $\rho \in \Omega_n^r$ ($\rho \in \Omega_{n+1}^r, \rho \in \Omega_{n+2}^r$) is called the *Lagrangian* (*Euler-Lagrange class*, *Helmholtz class*) of ρ . The quotient mappings $E : \Omega_k^r / \Theta_k^r \rightarrow \Omega_{k+1}^r / \Theta_{k+1}^r$ are defined by

$$E([\rho]) = [d\rho]. \quad (3.7)$$

The mapping $E : \Omega_n^r / \Theta_n^r \rightarrow \Omega_{n+1}^r / \Theta_{n+1}^r$, assigning to a Lagrangian $\lambda = [\rho]$ its Euler-Lagrange class $E(\lambda) = [d\rho]$, is called the *Euler-Lagrange mapping*; one can extend the definition and call each of the quotient mappings in (3.6) the *Euler-Lagrange morphisms*.

Theorem 10. *The variational sequence is an acyclic resolution of the constant sheaf $\mathbb{R}$.*

In the well-known sense, Theorem 10 allows us to identify the cohomology of the sequence of global sections of (3.6) with the De Rham cohomology of the underlying fibred manifold Y (see Krupka [9]). In particular, we get cohomology conditions for the existence of global variational principles for systems of differential equations on a manifold, expressed by means of a source form.

Now, we explain without proofs the basic results on variational projectors (Volná and Urban [16]).

Since our definition of the variational projectors is inductive, we first need the following lemma on the structure of 1-contact $(n+1)$ -forms.

Lemma 10. *Let ρ be a 1-contact $\pi^{r+1,r}$ -horizontal $(n+1)$ -form on $W^{r+1} \subset J^{r+1}Y$ expressed in a fibred chart (V, ψ) , $\psi = (x^i, y^\sigma)$, by*

$$\rho = \sum_{0 \leq |J| \leq r} A_\sigma^J \omega_J^\sigma \wedge \omega_0. \quad (3.8)$$

Then, there exist a unique 1-contact ω^σ -generated $(n+1)$ -form $I_1\rho$ on $W^{2r+1} \subset J^{2r+1}Y$, and a 1-contact n -form $J_1\rho$ and a 2-contact $(n+1)$ -form $K_1\rho$ defined over the chart neighborhood V such that

$$(\pi^{2r+1,r+1})^* \rho = I_1\rho - dJ_1\rho + K_1\rho, \quad (3.9)$$

where

$$\begin{aligned} I_1\rho &= B_\sigma \omega^\sigma \wedge \omega_0, \\ J_1\rho &= \sum_{0 \leq |J| \leq r-1} B_\sigma^{Ji} \omega_J^\sigma \wedge \omega_i, \quad K_1\rho = \sum_{0 \leq |J| \leq r-1} p dB_\sigma^{Ji} \wedge \omega_J^\sigma \wedge \omega_i, \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} B_\sigma &= \sum_{0 \leq s \leq r} (-1)^s d_{i_1} d_{i_2} \dots d_{i_s} A_\sigma^{i_1 i_2 \dots i_s}, \\ B_\sigma^{Ji} &= \sum_{1 \leq q \leq r-k} (-1)^{q+1} d_{i_{k+2}} d_{i_{k+3}} \dots d_{i_{k+q}} A_\sigma^{J i i_{k+2} i_{k+3} \dots i_{k+q}}, \end{aligned} \quad (3.11)$$

where $|J| = k = 0, 1, 2, \dots, r-1$.

The form $I_1\rho$, defined by Lemma 10 on $(n+1)$ -forms ρ in a unique way, can be extended to arbitrary $(n+k)$ -forms. We proceed as follows.

Let $k > 1$. Let W be an open subset of Y . Suppose we have a k -contact $(n+k)$ -form ρ defined on $W^{r+1} \subset J^{r+1}Y$. Let $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_k$ be arbitrary π -vertical vector fields on $W \subset Y$. Then, we define a k -contact $(n+k)$ -form $I_k\rho$ on $J^{2r+1}Y$ by

$$\begin{aligned} & i_{J^{2r+1}\mathcal{E}_k} \dots i_{J^{2r+1}\mathcal{E}_2} i_{J^{2r+1}\mathcal{E}_1} I_k\rho \\ &= \frac{1}{k} \left(i_{J^{2r+1}\mathcal{E}_k} i_{J^{2r+1}\mathcal{E}_{k-1}} \dots i_{J^{2r+1}\mathcal{E}_3} i_{J^{2r+1}\mathcal{E}_2} I_{k-1}(i_{J^{r+1}\mathcal{E}_1}\rho) \right. \\ & \quad - i_{J^{2r+1}\mathcal{E}_k} i_{J^{2r+1}\mathcal{E}_{k-1}} \dots i_{J^{2r+1}\mathcal{E}_3} i_{J^{2r+1}\mathcal{E}_1} I_{k-1}(i_{J^{r+1}\mathcal{E}_2}\rho) \\ & \quad - i_{J^{2r+1}\mathcal{E}_k} i_{J^{2r+1}\mathcal{E}_{k-1}} \dots i_{J^{2r+1}\mathcal{E}_4} i_{J^{2r+1}\mathcal{E}_1} i_{J^{2r+1}\mathcal{E}_2} I_{k-1}(i_{J^{r+1}\mathcal{E}_3}\rho) \\ & \quad - \dots \\ & \quad - i_{J^{2r+1}\mathcal{E}_k} i_{J^{2r+1}\mathcal{E}_1} i_{J^{2r+1}\mathcal{E}_{k-2}} \dots i_{J^{2r+1}\mathcal{E}_3} i_{J^{2r+1}\mathcal{E}_2} I_{k-1}(i_{J^{r+1}\mathcal{E}_{k-1}}\rho) \\ & \quad \left. - i_{J^{2r+1}\mathcal{E}_1} i_{J^{2r+1}\mathcal{E}_{k-1}} \dots i_{J^{2r+1}\mathcal{E}_3} i_{J^{2r+1}\mathcal{E}_2} I_{k-1}(i_{J^{r+1}\mathcal{E}_k}\rho) \right) \end{aligned}$$

and an $\mathbb{R}$ -linear mapping $\Omega_{n+k}^r W \ni \rho \rightarrow \mathcal{I}\rho \in \Omega_{n+k}^{2r+1} W$ by

$$\mathcal{I}\rho = I_k p_k \rho. \quad (3.12)$$

It should be pointed out that the form $\mathcal{J}\rho \in \Omega_{n+k}^{2r+1}W$ is defined over the same open set W as ρ . The following theorem has a fundamental meaning for applications of the theory of variational sequences.

Theorem 11. *Let $k \geq 1$ be an integer. Let $\pi : Y \rightarrow X$ be a fibred manifold over n -dimensional manifold X , and let W be an open set in Y .*

- (a) *For every form $\rho \in \Omega_{n+k}^r W$, $\mathcal{J}\rho$ belongs to the same class as $(\pi^{2r+1,r})^* \rho$.*
- (b) *The kernel of the mapping $\Omega_{n+k}^r W \ni \rho \rightarrow \mathcal{J}\rho \in \Omega_{n+k}^{2r+1}W$ coincides with the Abelian group $\Theta_{n+k}^r W$.*
- (c) *$\mathcal{J}$ satisfies, up to the canonical jet projection,*

$$\mathcal{J} \circ \mathcal{J} = \mathcal{J}. \quad (3.13)$$

Remark. Theorem 11, (b) states that the kernel $\text{Ker } \mathcal{J}$ of the morphism of Abelian groups $\Omega_{n+k}^r W \ni \rho \rightarrow \mathcal{J}\rho \in \Omega_{n+k}^{2r+1}W$ coincides with the subgroup $\Theta_{n+k}^r W \subset \Omega_{n+k}^r W$ of *contact forms*. Consequently, the image $\text{Im } \mathcal{J}$ is canonically isomorphic with the quotient group $\Omega_{n+k}^r W / \Theta_{n+k}^r W$. Under this isomorphism, the class $[\rho]$ of a form $\rho \in \Omega_{n+k}^r W$ can be canonically identified with a form $\mathcal{J}\rho \in \Omega_{n+k}^{2r+1}W$. In particular, the quotient group can be characterized as a subgroup of the group $\Omega_{n+k}^{2r+1}W$; this gives a way to compute the classes. One can find equations of the subgroups; however, in general, an explicit solution of these equations is still an open problem.

4. EXAMPLE: FIRST ORDER VARIATIONAL SEQUENCE IN FIELD THEORY

We give some remarks on the structure of classes in the variational sequence on J^1Y over an n -dimensional base. These expressions allow us to study the local inverse problem for systems of differential equations and are also related with the meaning of cohomological conditions arising from the variational sequence theory.

(a) Classes of n -forms (Lagrangians)

The class of a form $\rho \in \Omega_n^1 W$ is canonically identified with an n -form $h\rho \in \Omega_n^2 W$. Conversely, an n -form $\lambda \in \Omega_n^2 W$ is a class if and only if the equation $\lambda = h\rho$ has a solution ρ . One can show that λ is a class if and only if $\lambda = \mathcal{L}\omega_0$, where $\mathcal{L}$ is a polynomial

$$\begin{aligned} \mathcal{L} = & \mathcal{L}_0 + P_{\sigma_1}^{j_1 i_1} y_{j_1 i_1}^{\sigma_1} + P_{\sigma_1}^{j_1 i_1 j_2 i_2} y_{j_1 i_1}^{\sigma_1} y_{j_2 i_2}^{\sigma_2} + \\ & \dots + P_{\sigma_1}^{j_1 i_1 j_2 i_2} \dots P_{\sigma_n}^{j_n i_n} y_{j_1 i_1}^{\sigma_1} y_{j_2 i_2}^{\sigma_2} \dots y_{j_n i_n}^{\sigma_n} \end{aligned} \quad (4.1)$$

with the coefficients depending on x^i, y^v, y_j^v , symmetric in j_l, i_l , and such that $P_{\sigma_1}^{j_1 i_1 j_2 i_2} \dots P_{\sigma_k}^{j_k i_k} = 0$ whenever one cannot choose $i_1, i_2, \dots, i_k$ mutually different.

(b) *Classes of $(n + k)$ -forms (source forms)*

The class of an $(n + 1)$ -form $\rho \in \Omega_{n+1}^1 W$ is canonically identified with a form $\mathcal{I}\rho \in \Omega_{n+1}^3 W$, and conversely, an $(n + 1)$ -form $\varepsilon \in \Omega_{n+1}^3 W$ is a class if and only if the equation $\varepsilon = \mathcal{I}\rho$ has a solution ρ or, which is the same, ε should satisfy *integrability conditions* for this equation. Proceeding in the same way as in (a), one can describe these conditions explicitly. The resulting expression for ε is polynomial in y_{ji}^σ and y_{jik}^σ whose coefficients should satisfy a system of first order partial differential equations. The classes of $(n + k)$ -forms for general k have the same structure.

(c) *The Euler–Lagrange mapping*

The mapping E of the quotient group $\Omega_n^1 W / \Theta_n^1 W$ into $\Omega_{n+1}^1 W / \Theta_{n+1}^1 W$ is expressed by familiar formulas with some restrictions. Writing a class λ as above, we get the corresponding class as an $(n + 1)$ -form $E(\lambda) = E_\sigma(\mathcal{L})\omega^\sigma \wedge \omega_0$, where

$$E_\sigma(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial y^\sigma} - d_i \frac{\partial \mathcal{L}}{\partial y_i^\sigma} + d_i d_j \frac{\partial \mathcal{L}}{\partial y_{ij}^\sigma}. \quad (4.2)$$

In addition, however, one should substitute for $\mathcal{L}$ from (4.1).

(d) *The Helmholtz form*

Given a source form $\varepsilon \in \Omega_{n+1}^3 W$ such that $\varepsilon = \mathcal{I}\rho$ (i.e., belonging to the quotient group $\Omega_{n+1}^1 W / \Theta_{n+1}^1 W$), then the class $E(\varepsilon) = \mathcal{I}d\rho$ belongs to $\Omega_{n+2}^1 W / \Theta_{n+2}^1 W$, and it can equivalently be characterized as a form from the space of forms $\Omega_{n+2}^3 W$. Writing $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0$, we get a globally defined $(n + 2)$ -form

$$E(\varepsilon) = \left(H_{\sigma\nu}(\varepsilon)\omega^\nu + H_{\sigma\nu}^{p_1}(\varepsilon)\omega_{p_1}^\nu + H_{\sigma\nu}^{p_1 p_2}(\varepsilon)\omega_{p_1 p_2}^\nu + H_{\sigma\nu}^{p_1 p_2 p_3}(\varepsilon)\omega_{p_1 p_2 p_3}^\nu \right) \wedge \omega^\sigma \wedge \omega_0, \quad (4.3)$$

where the components are the *Helmholtz expression*

$$\begin{aligned} H_{\sigma\nu}(\varepsilon) &= \frac{\partial \varepsilon_\sigma}{\partial y^\nu} - \frac{\partial \varepsilon_\nu}{\partial y^\sigma} + d_{p_1} \frac{\partial \varepsilon_\nu}{\partial y_{p_1}^\sigma} - d_{p_1} d_{p_2} \frac{\partial \varepsilon_\nu}{\partial y_{p_1 p_2}^\sigma} + d_{p_1} d_{p_2} d_{p_3} \frac{\partial \varepsilon_\nu}{\partial y_{p_1 p_2 p_3}^\sigma}, \\ H_{\sigma\nu}^{p_1}(\varepsilon) &= \frac{\partial \varepsilon_\sigma}{\partial y_{p_1}^\nu} + \frac{\partial \varepsilon_\nu}{\partial y_{p_1}^\sigma} - 2d_{p_2} \frac{\partial \varepsilon_\nu}{\partial y_{p_1 p_2}^\sigma} + 3d_{p_2} d_{p_3} \frac{\partial \varepsilon_\nu}{\partial y_{p_1 p_2 p_3}^\sigma}, \\ H_{\sigma\nu}^{p_1 p_2}(\varepsilon) &= \frac{\partial \varepsilon_\sigma}{\partial y_{p_1 p_2}^\nu} - \frac{\partial \varepsilon_\nu}{\partial y_{p_1 p_2}^\sigma} + 3d_{p_3} \frac{\partial \varepsilon_\nu}{\partial y_{p_1 p_2 p_3}^\sigma}, \\ H_{\sigma\nu}^{p_1 p_2 p_3}(\varepsilon) &= \frac{\partial \varepsilon_\sigma}{\partial y_{p_1 p_2 p_3}^\nu} + \frac{\partial \varepsilon_\nu}{\partial y_{p_1 p_2 p_3}^\sigma}, \end{aligned}$$

with ε satisfying the integrability condition for the equation $\varepsilon = \mathcal{I}\rho$.

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THE NATURAL LIFTINGS OF CONNECTIONS TO TENSOR POWERS OF THE COTANGENT BUNDLE

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Abstract. We reduce the problem of finding all $\mathcal{M}f_m$ -natural operators $B : Q \rightsquigarrow Q(\otimes^k T^*)$ lifting classical linear connections ∇ on m -manifolds M into classical linear connections $B_M(\nabla)$ on the k th tensor power $\otimes^k T^*M$ of the cotangent bundle T^*M of M to the fairly known description of all $\mathcal{M}f_m$ -natural operators $D : Q \rightsquigarrow \otimes^p T^* \otimes \otimes^q T$ transforming classical linear connections ∇ on m -manifolds M into tensor fields $D_M(\nabla)$ of types (q, p) on M .

1. INTRODUCTION

All manifolds and maps are assumed to be smooth (of class C^∞). All manifolds are assumed to be Hausdorff, finite dimensional and without boundaries. The category of m -dimensional manifolds and their embeddings is denoted by $\mathcal{M}f_m$.

In [3], M. Kureš described completely all $\mathcal{M}f_m$ -natural operators $B : Q \rightsquigarrow QT^*$ lifting classical linear connections ∇ on m -manifolds M into classical linear connections $B_M(\nabla)$ on the cotangent bundle T^*M of M .

In the present note, we study a similar problem for $\mathcal{M}f_m$ -natural operators $B : Q \rightsquigarrow Q(\otimes^k T^*)$ transforming classical linear connections ∇ on m -manifolds M into classical linear connections $B_M(\nabla)$ on the k th tensor power $\otimes^k T^*M$ of T^*M .

An example of an $\mathcal{M}f_m$ -natural operator $Q \rightsquigarrow Q(\otimes^k T^*)$ is the following construction. Let ∇ be a classical linear connection on M . We extend ∇ to the linear connection $D = \otimes^k \nabla^*$ in the vector bundle $E = \otimes^k T^*M \rightarrow M$. Then, owing to [1], we have a classical linear connection (we will denote it by $\nabla^{(k)}$) on $E = \otimes^k T^*M$ corresponding to (D, ∇) .

The main result of the paper says that if $k \geq 3$, then the $\mathcal{M}f_m$ -natural operators $B : Q \rightsquigarrow Q(\otimes^k T^*)$ are in bijection with the systems $D = (D_j)_{j=1,\dots,5}$ of $\mathcal{M}f_m$ -natural operators $D_j : Q \rightsquigarrow \otimes^{p_j} T^* \otimes \otimes^{q_j} T$ transforming classical linear connections ∇ on m -manifolds M into tensor fields of types $(q_1, p_1) = (1, 2)$, $(q_2, p_2) = (0, k + 2)$, $(q_3, p_3) = (k, k + 2)$ and $(q_4, p_4) = (q_5, p_5) = (k, k + 1)$ on M . The case of $k = 2$ is also studied.

Such a characterization is sufficient because a complete description of all $\mathcal{M}f_m$ -natural operators $Q \rightsquigarrow \otimes^p T^* \otimes \otimes^q T$ is fairly known (see Remark 1).

A general concept of natural operators can be found in [2]. In this paper, we will use some partial definitions of natural operators only. In Section 2, we recall these partial definitions for the readers' convenience. In Sections 3 and 4, we made some preparations. The main theorem is presented in Section 5.

2. BASIC DEFINITIONS

We will use the following partial definitions of natural operators only. (The general concept of natural operators can be found in [2].)

Definition 1. An $\mathcal{M}f_m$ -natural operator $B : \underline{Q} \rightsquigarrow \underline{Q}(\otimes^k T^*)$ transforming classical linear connections ∇ on m -manifolds M into classical linear connections $B_M(\nabla)$ on $\otimes^k T^*M$ is an $\mathcal{M}f_m$ -invariant system $B = \{B_M\}_{M \in \text{obj}(\mathcal{M}f_m)}$ of regular operators (functions)

$$B_M : \underline{Q}(M) \rightarrow \underline{Q}(\otimes^k T^*M)$$

for any m -manifold M , where $\underline{Q}(M)$ is the set of all classical linear connections on M . The $\mathcal{M}f_m$ -invariance of B means that if $\nabla_1 \in \underline{Q}(M_1)$ and $\nabla_2 \in \underline{Q}(M_2)$ are φ -related by an embedding $\varphi : M_1 \rightarrow M_2$ between m -manifolds (i. e., φ is (∇_1, ∇_2) -affine), then $B_{M_1}(\nabla_1)$ and $B_{M_2}(\nabla_2)$ are $\otimes^k T^*\varphi$ -related. The regularity of B means that B_M transforms smoothly parametrized families of connections into smoothly parametrized ones.

For example, the family $B^{(k)} : \underline{Q} \rightsquigarrow \underline{Q}(\otimes^k T^*)$ of operators

$$B_M^{(k)} : \underline{Q}(M) \rightarrow \underline{Q}(\otimes^k T^*M), \quad B_M^{(k)}(\nabla) := \nabla^{(k)}, \quad \nabla \in \underline{Q}(M)$$

(see Section 1), for any m -manifolds M , is an $\mathcal{M}f_m$ -natural operator.

Definition 2. An $\mathcal{M}f_m$ -natural operator $C : \underline{Q} \rightsquigarrow (\otimes^k T^*, \otimes^p T^* \otimes \otimes^q T)$ transforming classical linear connections ∇ on m -manifolds M into base preserving fibred maps $C_M(\nabla) : \otimes^k T^*M \rightarrow \otimes^p T^*M \otimes \otimes^q TM$ is an $\mathcal{M}f_m$ -invariant system $C = \{C_M\}_{M \in \text{obj}(\mathcal{M}f_m)}$ of regular operators

$$C_M : \underline{Q}(M) \rightarrow C_M^\infty(\otimes^k T^*M, \otimes^p T^*M \otimes \otimes^q TM)$$

for any m -manifold M , where $\underline{Q}(M)$ is as in Definition 1 and

$$C_M^\infty(\otimes^k T^*M, \otimes^p T^*M \otimes \otimes^q TM)$$

is the set of all base preserving fibred maps $\otimes^k T^*M \rightarrow \otimes^p T^*M \otimes \otimes^q TM$. The $\mathcal{M}f_m$ -invariance of C means that if $\nabla_1 \in \underline{Q}(M_1)$ and $\nabla_2 \in \underline{Q}(M_2)$ are φ -related by an embedding $\varphi : M_1 \rightarrow M_2$ between m -manifolds, then fibred maps $C_{M_1}(\nabla_1) : \otimes^k T^*M_1 \rightarrow \otimes^p T^*M_1 \otimes \otimes^q TM_1$ and $C_{M_2}(\nabla_2) : \otimes^k T^*M_2 \rightarrow \otimes^p T^*M_2 \otimes \otimes^q TM_2$ are also φ -related (i. e., $C_{M_2}(\nabla_2) \circ \otimes^k T^*\varphi = (\otimes^p T^*\varphi \otimes \otimes^q T\varphi) \circ C_{M_1}(\nabla_1)$). The regularity means almost the same as in Definition 1.

For example, the curvature tensor $\mathcal{R}(\nabla)$ of a classical linear connection ∇ determines an $\mathcal{M}f_m$ -natural operator $C^{\mathcal{R}} : \underline{Q} \rightsquigarrow (\otimes^k T^*, \otimes^3 T^* \otimes T)$. More precisely, the family $C^{\mathcal{R}} : \underline{Q} \rightsquigarrow (\otimes^k T^*, \otimes^3 T^* \otimes TM)$ of operators $C_M^{\mathcal{R}} : \underline{Q}(M) \rightarrow C_M^\infty(\otimes^k T^* M, \otimes^3 T^* M \otimes TM)$ given by

$$C_M^{\mathcal{R}}(\nabla) := \mathcal{R}(\nabla) \circ \pi_M : \otimes^k T^* M \rightarrow \otimes^3 T^* M \otimes TM, \nabla \in \underline{Q}(M)$$

(where $\pi_M : \otimes^k T^* M \rightarrow M$ is the bundle projection), for any m -manifold M , is an $\mathcal{M}f_m$ -natural operator.

Definition 3. An $\mathcal{M}f_m$ -natural transformation $C : \otimes^k T^* \rightarrow \otimes^p T^* \otimes \otimes^q T$ is an $\mathcal{M}f_m$ -natural operator $C = \{C_M\} : \underline{Q} \rightsquigarrow (\otimes^k T^*, \otimes^p T^* \otimes \otimes^q T)$ such that $C_M : \underline{Q}(M) \rightarrow C_M^\infty(\otimes^k T^* M, \otimes^p T^* M \otimes \otimes^q TM)$ is a constant function for any $M \in \text{obj}(\mathcal{M}f_m)$.

For example, the family $C : \otimes^k T^* \rightarrow \otimes^{2k} T^*$ of fibred maps $C_M : \otimes^k T^* M \rightarrow \otimes^{2k} T^* M$ given by

$$C_M(v) := v \otimes v, v \in \otimes^k T_x^* M, x \in M,$$

for any m -manifold M , is an $\mathcal{M}f_m$ -natural transformation.

Definition 4. An $\mathcal{M}f_m$ -natural operator ($\mathcal{M}f_m$ -natural tensor) $D : \underline{Q} \rightsquigarrow \otimes^p T^* \otimes \otimes^q T$ transforming classical linear connections ∇ on m -manifolds M into tensor fields $D_M(\nabla)$ of type $\otimes^p T^* \otimes \otimes^q T$ on M is an $\mathcal{M}f_m$ -invariant system of regular operators

$$D_M : \underline{Q}(M) \rightarrow C_M^\infty(\otimes^p T^* M \otimes \otimes^q TM)$$

for any m -manifold M , where $C_M^\infty(\otimes^p T^* M \otimes \otimes^q TM)$ is the space of all tensor fields of type $\otimes^p T^* \otimes \otimes^q T$ on M . The $\mathcal{M}f_m$ -invariance of D means that if $\nabla_1 \in \underline{Q}(M_1)$ and $\nabla_2 \in \underline{Q}(M_2)$ are φ -related by an embedding $\varphi : M_1 \rightarrow M_2$, then $D_{M_1}(\nabla_1)$ and $D_{M_2}(\nabla_2)$ are also φ -related (i.e., $(\otimes^p T^* \varphi \otimes \otimes^q T \varphi) \circ D_{M_1}(\nabla_1) = D_{M_2}(\nabla_2) \circ \varphi$). The regularity means almost the same as in Definition 1.

For example, the family $\mathcal{R} : \underline{Q} \rightsquigarrow \otimes^3 T^* \otimes T$ of operators $\mathcal{R}_M : \underline{Q}(M) \rightarrow C_M^\infty(\otimes^3 T^* M \otimes TM)$ given by

$$\mathcal{R}_M(\nabla) = \text{the curvature tensor of } \nabla \in \underline{Q}(M),$$

for any m -manifold M , is an $\mathcal{M}f_m$ -natural operator.

Remark 1. A full description of all $\mathcal{M}f_m$ -natural operators $Q_\tau \rightsquigarrow \otimes^p T^* \otimes \otimes^q T$ transforming torsion free classical connections on m -manifolds into tensor fields of types (q, p) can be found in the Lemma in [2, Section 33.4]. The following is the description. Each covariant derivative of the curvature

$$\mathcal{R}(\nabla) \in C_M^\infty(\wedge^2 T^* M \otimes T^* M \otimes TM)$$

of a classical linear connection ∇ is an $\mathcal{M}f_m$ -natural tensor. Further, every tensor multiplication of two natural tensors and every contraction on one covariant and one contravariant entry of a natural tensor give a new natural tensor. Finally, we can tensor any natural tensor with a connection independent natural tensor, we can permute any number of entries in the tensor product and we can repeat these steps and take linear combinations. In this way, we can obtain any natural tensor of types (q, p) depending on a torsion free classical linear connection. Modifying the proof of the above fact, one can see that all natural tensors of a (not necessarily torsion free) classical linear connection ∇ can be obtained, provided we also include the torsion tensor $\mathcal{T}(\nabla)$ and their covariant derivatives to the above procedure.

3. A PROPOSITION

We have the following description of $\mathcal{M}f_m$ -natural operators $Q \rightsquigarrow Q(\otimes^k T^*)$ by means of systems of $\mathcal{M}f_m$ -natural operators $Q \rightsquigarrow (\otimes^k T^*, \otimes^p T^* \otimes \otimes^q T)$.

Proposition 1. *The $\mathcal{M}f_m$ -natural operators $B : Q \rightsquigarrow Q(\otimes^k T^*)$ are in bijection with the systems $C = (C_j)_{j=1,\dots,8}$ of $\mathcal{M}f_m$ -natural operators $C_1 : Q \rightsquigarrow (\otimes^k T^*, \otimes^2 T^* \otimes T)$, $C_2 : Q \rightsquigarrow (\otimes^k T^*, T^* \otimes \otimes^{k+1} T)$, $C_3 : Q \rightsquigarrow (\otimes^k T^*, T^* \otimes \otimes^{k+1} T)$, $C_4 : Q \rightsquigarrow (\otimes^k T^*, \otimes^{2k+1} T)$, $C_5 : Q \rightsquigarrow (\otimes^k T^*, \otimes^{k+2} T^*)$, $C_6 : Q \rightsquigarrow (\otimes^k T^*, \otimes^{k+1} T^* \otimes \otimes^k T)$, $C_7 : Q \rightsquigarrow (\otimes^k T^*, \otimes^{k+1} T^* \otimes \otimes^k T)$ and $C_8 : Q \rightsquigarrow (\otimes^k T^*, \otimes^k T^* \otimes \otimes^{2k} T)$.*

Proof. Let ∇ be a classical linear connection on an m -manifold M . Then, for any point $v \in \otimes^k T_x^* M$, $x \in M$, we have the standard linear isomorphisms

$$T_v(\otimes^k T^* M) \cong H_v^\nabla \oplus V_v(\otimes^k T^* M) \cong T_x M \oplus (\otimes^k T_x^* M)$$

canonically depending on ∇ , where $H_v^\nabla \subset T_v(\otimes^k T^* M)$ is the ∇ -horizontal subspace and $V_v(\otimes^k T^* M) \subset T_v(\otimes^k T^* M)$ is the vertical subspace. So, by an obvious linear algebra, we have the identification (linear isomorphism)

$$\begin{aligned} T_v^*(\otimes^k T^* M) \otimes T_v^*(\otimes^k T^* M) \otimes T_v(\otimes^k T^* M) \cong \\ (T_x^* M \otimes T_x^* M \otimes T_x M) \oplus (T_x^* M \otimes (\otimes^k T_x^* M)^* \otimes T_x M) \oplus \\ ((\otimes^k T_x^* M)^* \otimes T_x^* M \otimes T_x M) \oplus ((\otimes^k T_x^* M)^* \otimes (\otimes^k T_x^* M)^* \otimes T_x M) \oplus \\ (T_x^* M \otimes T_x^* M \otimes \otimes^k T_x^* M) \oplus (T_x^* M \otimes (\otimes^k T_x^* M)^* \otimes \otimes^k T_x^* M) \oplus \\ ((\otimes^k T_x^* M)^* \otimes T_x^* M \otimes \otimes^k T_x^* M) \oplus ((\otimes^k T_x^* M)^* \otimes (\otimes^k T_x^* M)^* \otimes \otimes^k T_x^* M) \end{aligned}$$

canonically depending on ∇ or (simply) the one

$$\begin{aligned} T_v^*(\otimes^k T^* M) \otimes T_v^*(\otimes^k T^* M) \otimes T_v(\otimes^k T^* M) \cong \\ (\otimes^2 T_x^* M \otimes T_x M) \oplus (T_x^* M \otimes \otimes^{k+1} T_x M) \oplus \\ (T_x^* M \otimes \otimes^{k+1} T_x M) \oplus (\otimes^{2k+1} T_x M) \oplus \\ (\otimes^{k+2} T_x^* M) \oplus (\otimes^{k+1} T_x^* M \otimes \otimes^k T_x M) \oplus \\ (\otimes^{k+1} T_x^* M \otimes \otimes^k T_x M) \oplus (\otimes^k T_x^* M \otimes \otimes^{2k} T_x M). \end{aligned}$$

Consequently, any $\mathcal{M}f_m$ -natural operator $B : \underline{Q} \rightsquigarrow \underline{Q}(\otimes^k T^* M)$ can be identified with the system $C = (C_i)_{i=1,\dots,8}$ of $\mathcal{M}f_m$ -natural operators $C_1 : \underline{Q} \rightsquigarrow (\otimes^k T^*, \otimes^2 T^* \otimes T)$, $C_2 : \underline{Q} \rightsquigarrow (\otimes^k T^*, T^* \otimes \otimes^{k+1} T)$, $C_3 : \underline{Q} \rightsquigarrow (\otimes^k T^*, T^* \otimes \otimes^{k+1} T)$, $C_4 : \underline{Q} \rightsquigarrow (\otimes^k T^*, \otimes^{2k+1} T)$, $C_5 : \underline{Q} \rightsquigarrow (\otimes^k T^*, \otimes^{k+2} T^*)$, $C_6 : \underline{Q} \rightsquigarrow (\otimes^k T^*, \otimes^{k+1} T^* \otimes \otimes^k T)$, $C_7 : \underline{Q} \rightsquigarrow (\otimes^k T^*, \otimes^{k+1} T^* \otimes \otimes^k T)$ and $C_8 : \underline{Q} \rightsquigarrow (\otimes^k T^*, \otimes^k T^* \otimes \otimes^{2k} T)$ defined by

$$((C_i)_M(\nabla)(v))_{i=1,\dots,8} = \delta_M(\nabla)(v)$$

modulo the last identification, $v \in \otimes^k T_x^* M$, $x \in M$, where $\delta_M(\nabla) : \otimes^k T^* M \rightarrow T^*(\otimes^k T^* M) \otimes T^*(\otimes^k T^* M) \otimes T(\otimes^k T^* M)$ is the tensor field of type $T^* \otimes T^* \otimes T$ on $\otimes^k T^* M$ with $B_M(\nabla) = \nabla^{(k)} + \delta_M(\nabla)$. Here, $\nabla^{(k)}$ are the classical linear connections on $\otimes^k T^* M$ corresponding to $(\otimes^k \nabla^*, \nabla)$ as in Section 1.

The proof of the proposition is complete. $\square$

4. SOME LEMMAS

To obtain a description of $\mathcal{M}f_m$ -natural operators $\underline{Q} \rightsquigarrow \underline{Q}(\otimes^k T^*)$ by means of systems of $\mathcal{M}f_m$ -natural operators $\underline{Q} \rightsquigarrow \otimes^p T^* \otimes \otimes^q T$, we prove some lemmas.

Lemma 1. *If $q > p$, then any $\mathcal{M}f_m$ -natural operator $C : \underline{Q} \rightsquigarrow (\otimes^k T^*, \otimes^p T^* \otimes \otimes^q T)$ is the zero one (i. e., $C_M(\nabla)(v) = 0$ for all $M \in \text{obj}(\mathcal{M}f_m)$, $\nabla \in \underline{Q}(M)$, $v \in \otimes^k T_x^* M$, $x \in M$).*

Proof. We have to show that $C_M(\nabla)(v) = 0 \in \otimes^p T_x^* M \otimes \otimes^q T_x M$ for all m -manifolds M , $\nabla \in \underline{Q}(M)$ and $v \in \otimes^k T_x^* M$, $x \in M$. Using the invariance of C with respect to the manifold charts, we may assume $M = \mathbb{R}^m$, $x = 0$.

By the non-linear Petree theorem (see [2]), one can assume that the Christoffel symbols ∇_{jk}^i of ∇ in the usual coordinates on $\mathbb{R}^m$ are polynomials (of arbitrary degree d) with coefficients $\nabla_{jk;\alpha}^i$, i. e.

$$\nabla = (\nabla_{jk;\alpha}^i)_{i,j,k=1,\dots,m, \alpha \in \mathbb{N}^m, |\alpha| \leq d}$$

or in short $\nabla = (\nabla_{jk;\alpha}^i)$. The regularity of C yields that $C_{\mathbb{R}^m}((\nabla_{jk;\alpha}^i))(v)$ is smooth in $((\nabla_{jk;\alpha}^i), v)$ for any (fixed) d .

Using the invariance of C with respect to the homotheties $\frac{1}{t}id_{\mathbb{R}^m}$ for $t > 0$, we get the homogeneity condition

$$C_{\mathbb{R}^m}((\nabla_{jk;\alpha}^i))(v) = t^{q-p} C_{\mathbb{R}^m}((t^{1+|\alpha|}\nabla_{jk;\alpha}^i))(t^k v).$$

Putting $t \rightarrow 0$, we get $C_{\mathbb{R}^m}((\nabla_{jk;\alpha}^i))(v) = 0$ as $q - p > 0$. $\square$

Lemma 2. *If $k \geq 2$, the $\mathcal{M}f_m$ -natural operators $C : Q \rightsquigarrow (\otimes^k T^*, \otimes^{q+1} T^* \otimes \otimes^q T)$ are in bijection with the $\mathcal{M}f_m$ -natural operators $D : Q \rightsquigarrow \otimes^{q+1} T^* \otimes \otimes^q T$.*

More precisely, if $k \geq 2$, any $\mathcal{M}f_m$ -natural operator $C : Q \rightsquigarrow (\otimes^k T^*, \otimes^{q+1} T^* \otimes \otimes^q T)$ defines an $\mathcal{M}f_m$ -natural operator $D^C : Q \rightsquigarrow \otimes^{q+1} T^* \otimes \otimes^q T$,

$$D_M^C(\nabla)(x) := C_M(\nabla)(0),$$

$0 \in \otimes^k T_x^* M$ (the zero element), $x \in M$, $M \in \text{obj}(\mathcal{M}f_m)$, $\nabla \in \underline{Q}(M)$. Conversely, any $\mathcal{M}f_m$ -natural operator $D : Q \rightsquigarrow \otimes^{q+1} T^* \otimes \otimes^q T$ defines an $\mathcal{M}f_m$ -natural operator $C^D : Q \rightsquigarrow (\otimes^k T^*, \otimes^{q+1} T^* \otimes \otimes^q T)$, $C_M^D(\nabla)(v) = D_M(\nabla)(x)$, $v \in \otimes^k T_x^* M$, $x \in M$, $M \in \text{obj}(\mathcal{M}f_m)$, $\nabla \in \underline{Q}(M)$. Moreover, the correspondences $C \rightarrow D^C$ and $D \rightarrow C^D$ are mutually inverse.

Proof. Let $C : Q \rightsquigarrow (\otimes^k T^*, \otimes^{q+1} T^* \otimes \otimes^q T)$ be an $\mathcal{M}f_m$ -natural operator. We use the notation from the proof of Lemma 1. The invariance of C with respect to the homotheties $\frac{1}{t}id_{\mathbb{R}^m}$ gives the homogeneity condition

$$t C_{\mathbb{R}^m}((\nabla_{jk;\alpha}^i))(v) = C_{\mathbb{R}^m}((t^{1+|\alpha|}\nabla_{jk;\alpha}^i))(t^k v).$$

Then, (if $k \geq 2$) $C_{\mathbb{R}^m}((\nabla_{jk;\alpha}^i))(v) = C_{\mathbb{R}^m}((\nabla_{jk;\alpha}^i))(0)$ because of the homogeneous function theorem. So, $C_M(\nabla)(v) = C_M(\nabla)(0)$ for all $M \in \text{obj}(\mathcal{M}f_m)$, $\nabla \in \underline{Q}(M)$, $v \in \otimes^k T_x^* M$, $x \in M$. Consequently, the correspondences $C \rightarrow D^C$ and $D \rightarrow C^D$ are mutually inverse. $\square$

Lemma 3. *If $k \geq 3$, the $\mathcal{M}f_m$ -natural operators $C : Q \rightsquigarrow (\otimes^k T^*, \otimes^{k+2} T^*)$ are in bijection with the pairs (E, F) of $\mathcal{M}f_m$ -natural operators $E : Q \rightsquigarrow \otimes^{k+2} T^*$ and $F : Q \rightsquigarrow \otimes^{k+2} T^* \otimes \otimes^k T$.*

More precisely, any $\mathcal{M}f_m$ -natural operator $C : Q \rightsquigarrow (\otimes^k T^*, \otimes^{k+2} T^*)$ defines an $\mathcal{M}f_m$ -natural operator $E^C : Q \rightsquigarrow \otimes^{k+2} T^*$, $E_M^C(\nabla)(x) := C_M(\nabla)(0)$, $M \in \text{obj}(\mathcal{M}f_m)$, $x \in M$, $\nabla \in \underline{Q}(M)$, $0 \in \otimes^k T_x^* M$ (the zero) and an $\mathcal{M}f_m$ -natural operator $F^C : Q \rightsquigarrow \otimes^{k+2} T^* \otimes \otimes^k T$,

$$\langle F_M^C(\nabla)(x), v \rangle := \left. \frac{d}{d\tau} \right|_{\tau=0} C_M(\nabla)(\tau v),$$

$M \in \text{obj}(\mathcal{M}f_m)$, $\nabla \in \underline{Q}(M)$, $v \in \otimes^k T_x^* M$, $x \in M$, $0 \in \otimes^k T_x^* M$ (the zero). Conversely, any pair (E, F) of an $\mathcal{M}f_m$ -natural operator $E : Q \rightsquigarrow \otimes^{k+2} T^*$ and

an $\mathcal{M}f_m$ -natural operator $F : Q \rightsquigarrow \otimes^{k+2}T^* \otimes \otimes^k T$ defines an $\mathcal{M}f_m$ -natural operator $C^{(E,F)} : Q \rightsquigarrow (\otimes^k T^*, \otimes^{k+2}T^*)$,

$$C_M^{(E,F)}(\nabla)(v) := E_M(\nabla)(x) + \langle F_M(\nabla)(x), v \rangle,$$

$M \in \text{obj}(\mathcal{M}f_m)$, $\nabla \in \underline{Q}(M)$, $v \in \otimes^k T_x^* M$, $x \in M$. Moreover, if $k \geq 3$, the correspondences $C \rightarrow (E^C, F^C)$ and $(E, F) \rightarrow C^{(E,F)}$ are mutually inverse.

If $k = 2$, the $\mathcal{M}f_m$ -natural operators $C : Q \rightsquigarrow (\otimes^2 T^*, \otimes^4 T^*)$ are in bijection with the triples (E, F, G) of $\mathcal{M}f_m$ -natural operators $E : Q \rightsquigarrow \otimes^4 T^*$ and $F : Q \rightsquigarrow \otimes^4 T^* \otimes \otimes^2 T$ and $\mathcal{M}f_m$ -natural transformations $G : \otimes^2 T^* \rightarrow \otimes^4 T^*$.

More precisely, any $\mathcal{M}f_m$ -natural operator $C : Q \rightsquigarrow (\otimes^2 T^*, \otimes^4 T^*)$ defines an $\mathcal{M}f_m$ -natural operator $E^C : Q \rightsquigarrow \otimes^4 T^*$ and an $\mathcal{M}f_m$ -natural operator $F^C : Q \rightsquigarrow \otimes^4 T^* \otimes \otimes^2 T$ (the same as above but for $k = 2$) and an $\mathcal{M}f_m$ -natural transformation $G^C : \otimes^2 T^* \rightarrow \otimes^4 T^*$,

$$G_M^C(v) := \frac{1}{2} \frac{d^2}{d\tau^2} \Big|_{\tau=0} C_M(\nabla)(\tau v),$$

$M \in \text{obj}(\mathcal{M}f_m)$, $v \in \otimes^2 T_x^* M$, $x \in M$, $\nabla \in \underline{Q}(M)$ (the definition of $G_M^C(v)$ is independent of the choice of $\nabla \in \underline{Q}(M)$). Conversely, any triple (E, F, G) of an $\mathcal{M}f_m$ -natural operator $E : Q \rightsquigarrow \otimes^4 T^*$ and an $\mathcal{M}f_m$ -natural operator $F : Q \rightsquigarrow \otimes^4 T^* \otimes \otimes^2 T$ and an $\mathcal{M}f_m$ -natural transformation $G : \otimes^2 T^* \rightarrow \otimes^4 T^*$ defines an $\mathcal{M}f_m$ -natural operator $C^{(E,F,G)} : Q \rightsquigarrow (\otimes^2 T^*, \otimes^4 T^*)$,

$$C_M^{(E,F,G)}(\nabla)(v) := E_M(\nabla)(x) + \langle F_M(\nabla)(x), v \rangle + G_M(v),$$

$M \in \text{obj}(\mathcal{M}f_m)$, $\nabla \in \underline{Q}(M)$, $v \in \otimes^2 T_x^* M$, $x \in M$. Moreover, if $k = 2$, the correspondences $C \rightarrow (E^C, F^C, G^C)$ and $(E, F, G) \rightarrow C^{(E,F,G)}$ are mutually inverse.

Proof. Let $C : Q \rightsquigarrow (\otimes^k T^*, \otimes^{k+2} T^*)$ be an $\mathcal{M}f_m$ -natural operator. We use the notations from the proof of Lemma 1. The invariance of C with respect to the homotheties gives the homogeneity condition

$$t^{k+2} C_{\mathbb{R}^m}((\nabla_{jk;\alpha}^i))(v) = C_{\mathbb{R}^m}((t^{1+|\alpha|} \nabla_{jk;\alpha}^i))(t^k v).$$

Then, by virtue of the homogeneous function theorem and the Taylor formula for $\tau \rightarrow C_{\mathbb{R}^m}((\nabla_{jk;\alpha}^i))(\tau v)$ in $\tau_0 = 0$ and evaluated at $\tau = 1$, we have

$$\begin{aligned} C_{\mathbb{R}^m}((\nabla_{jk;\alpha}^i))(v) &= C_{\mathbb{R}^m}((\nabla_{jk;\alpha}^i))(0) + \frac{d}{d\tau} \Big|_{\tau=0} C_{\mathbb{R}^m}((\nabla_{jk;\alpha}^i))(\tau v) \\ &\quad + \frac{1}{2} \frac{d^2}{d\tau^2} \Big|_{\tau=0} C_{\mathbb{R}^m}((\nabla_{jk;\alpha}^i))(\tau v) \end{aligned}$$

for any $(\nabla_{jk;\alpha}^i)$ and $v \in \otimes^k T_0^* \mathbb{R}^m$. Moreover, $\frac{d^2}{d\tau^2} \Big|_{\tau=0} C_{\mathbb{R}^m}((\nabla_{jk;\alpha}^i))(\tau v)$ is zero if $k \geq 3$ and it is independent of $(\nabla_{jk;\alpha}^i)$ if $k = 2$.

So, we have

$$C_M(\nabla)(v) = C_M(\nabla)(0) + \frac{d}{d\tau} \Big|_{\tau=0} C_M(\nabla)(\tau v) + \frac{1}{2} \frac{d^2}{d\tau^2} \Big|_{\tau=0} C_M(\nabla)(\tau v)$$

for any $M \in \text{obj}(\mathcal{M}f_m)$, $\nabla \in \underline{Q}(M)$, $v \in \otimes^k T_x^* M$, $x \in M$. Moreover, the expression $\frac{d^2}{d\tau^2} \Big|_{\tau=0} C_M(\nabla)(\tau v)$ is zero if $k \geq 3$ and it is independent of ∇ if $k = 2$. Therefore, the definition of $G_M^C(v)$ (for $k = 2$) is really independent of the choice of ∇ and the correspondences $C \rightarrow (E^C, F^C)$ and $(E, F) \rightarrow C^{(E, F)}$ for $k \geq 3$ are mutually inverse and the correspondences $C \rightarrow (E^C, F^C, G^C)$ and $(E, F, G) \rightarrow C^{(E, F, G)}$ for $k = 2$ are also mutually inverse. $\square$

5. THE MAIN RESULT

The main result of the present note is the following theorem, which is an immediate consequence of Proposition 1 and Lemmas 1, 2, and 3.

- Theorem 1.** (1) If $k \geq 3$, the $\mathcal{M}f_m$ -natural operators $B : Q \rightsquigarrow Q(\otimes^k T^*)$ are in bijection with the systems $D = (D_j)_{j=1, \dots, 5}$ of $\mathcal{M}f_m$ -natural operators $D_1 : Q \rightsquigarrow \otimes^2 T^* \otimes T$, $D_2 : Q \rightsquigarrow \otimes^{k+2} T^*$, $D_3 : Q \rightsquigarrow \otimes^{k+2} T^* \otimes \otimes^k T$, $D_4 : Q \rightsquigarrow \otimes^{k+1} T^* \otimes \otimes^k T$ and $D_5 : Q \rightsquigarrow \otimes^{k+1} T^* \otimes \otimes^k T$.
- (2) If $k = 2$, the $\mathcal{M}f_m$ -natural operators $B : Q \rightsquigarrow Q(\otimes^k T^*)$ are in bijection with the systems $D = (D_j)_{j=1, \dots, 6}$ of $\mathcal{M}f_m$ -natural operators $D_1 : Q \rightsquigarrow \otimes^2 T^* \otimes T$, $D_2 : Q \rightsquigarrow \otimes^4 T^*$, $D_3 : Q \rightsquigarrow \otimes^4 T^* \otimes \otimes^2 T$, $D_4 : Q \rightsquigarrow \otimes^3 T^* \otimes \otimes^2 T$, $D_5 : Q \rightsquigarrow \otimes^3 T^* \otimes \otimes^2 T$ and $\mathcal{M}f_m$ -natural transformations $D_6 : \otimes^2 T^* \rightarrow \otimes^4 T^*$.

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LAGRANGIANS ON ITERATED TANGENT BUNDLES AND THE HAMILTONIZATION

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Abstract. The structure of higher order iterated tangent bundles is studied. Canonical affinors and canonical vector fields on these bundles are geometrically described and differential operators are introduced. Furthermore, possibilities of a generalization of Lagrangian and Hamiltonian formalisms are investigated and a procedure of the Hamiltonization is demonstrated.

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INTRODUCTION

Differential geometry provides a good framework for studying Lagrangian and Hamiltonian formalisms of classical mechanics. The approach using semisprays in the place of differential equations brings forth results effectively. Sprays were introduced in the paper of Ambrose *et al.*, [2], for homogeneous second-order differential equations. Grifone's results, [6], make it possible to evaluate a unique Euler-Lagrange vector field solving fundamental dynamic equations and to construct a connection from a Lagrangian in such a way that the solutions of these equations and the paths of the connection are the same. There are two natural generalizations of the tangent bundle TM . The first one is represented by tangent bundles of higher order $T^r M$ and corresponds with the classical higher-order calculus of variations. Over the past years, results have been published of de León *et al.*, [3], dealing with semisprays and mutual relations on $T^r M$ and motivated by mechanics. The second generalization is represented by higher order iterated tangent bundles. In this paper, we start an investigation into the possibilities of an application of the above approach in this case. First of all, it is necessary to build up a good grounding in a calculus accompanied by a discussion of the structure of iterated tangent bundles. Algebraic properties of investigated higher-order objects are described by Weil algebras as well as by their simplicial structure. The starting points of this interesting view are summarized in the book by White, [19]. The extensive applicability of affinors and generalized Liouville's vector field in differential geometry and mechanics is

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also known. We construct canonical affinors and vector fields on $\overset{\circ}{T}^r M$ comparing them with the classification of natural objects on Weil bundles due to Kolář in [7, 10]. The techniques of finding natural objects are also studied in [9]. Next, we introduce differential operators on a bundle in question. Our definition is a generalization of so-called Tulczyjew's differential operator though our concept is related more significantly to a certain sequence of canonical vector fields. The correspondence between such sequences and ways of differentiating through so-called holonomizing projections is one of the main results of this paper. The new results dealing with the formalisms of mechanics are contained in Section 3. They represent a generalization of the Legendre–Ostrogradskii transformation for the case of iterated tangent bundles. By this, we have introduced Lagrangian and Hamiltonian formalism on iterated structures.

1. ITERATED TANGENT BUNDLES

1.1. Preliminaries and motivation

Let M, N be two smooth manifolds, $\dim M = m$, $\dim N = n$. As usually, we denote by $J_a^r(M, N)$ the set of all r -jets from M to N with the source $a \in M$. $J_a^r(M, N)$ is a smooth manifold. The manifold $J_0^1(\mathbb{R}, M)$ is just the *tangent bundle* TM on M . So, we consider $\pi^1: TM \rightarrow M$ the tangent bundle of a smooth m -dimensional manifold M . Further, the *second iterated tangent bundle* $TTM = T(TM)$ obtained by the additional application of the functor T possesses the following bundle structures: $\pi^2: TTM \rightarrow M$, $\pi_1^1 := \pi_{TM}^1: T(TM) \rightarrow (TM)$, $\pi^1 := T\pi_M^1: T(TM) \rightarrow T(M)$. Given some local coordinates x^i on M , let us denote x^i, y^i the induced coordinates on TM and x^i, y^i, X^i, Y^i the induced coordinates on TTM . Then $\pi^2: (x^i, y^i, X^i, Y^i) \mapsto (x^i)$, $\pi_1^1: (x^i, y^i, X^i, Y^i) \mapsto (x^i, y^i)$, $\pi^1: (x^i, y^i, X^i, Y^i) \mapsto (x^i, X^i)$.

For the mechanical interpretation, we recall Tulczyjew's example of the bundle TTM presented in the monograph of Abraham and Marsden, [1]. The equilibrium configuration of an elastic beam in a Euclidian space with no external forces is a straight line l . Small deflections induced by external forces and torques can be represented by points of a plane M perpendicular to the line l . The distance measured along l from an arbitrary reference point is denoted by s . We select a section of the beam corresponding to an interval $[s_1, s_2]$ and assume that the external forces and bending torques are applied to the ends of the section. The configuration manifold of the section of the beam is the product $TM \times TM$ with coordinates $(x^i, y^i, \bar{x}^i, \bar{y}^i)$, $i = 1, 2$. In the limit $s_2 \rightarrow s_1$, the configuration manifold is the bundle TTM with coordinates (x^i, y^i, X^i, Y^i) .

If we generalize, we obtain the (r th) iterated tangent bundle

$$\overset{\circ}{T}^r M = \underbrace{T \dots T}_{r \text{ times}} M = \underbrace{J_0^1(\mathbb{R}, J_0^1(\dots(J_0^1(\mathbb{R}, M)) \dots))}_{r \text{ times}}$$

by the multiple application of J_0^1 .

1.2. Iterated tangent bundles and nonholonomic tangent bundles

The functor $\tilde{J}^r$ is defined recurrently as $\tilde{J}^1 = J^1$ and

$$\tilde{J}^r(M, N) = J^1(M, \tilde{J}^{r-1}(M, N)).$$

We obtain nonholonomic jets in the sense of Ehresmann, [5]. If we take the subset of nonholonomic jets of $\tilde{J}^r(\mathbb{R}, M)$ with the source $0 \in \mathbb{R}$, we have a *nonholonomic tangent bundle of order r* $\tilde{T}^r M = \tilde{J}_0^r(\mathbb{R}, M)$.

There is a subtle difference between the nonholonomic tangent bundle and the iterated tangent bundle. We can see it through the following definition of the natural equivalence (see Kolář, [8]) $\epsilon_r: \tilde{T}^r \rightarrow \overset{\circ}{T}^r$. We define it by induction. For $r = 1$, ϵ_1 is the identity. Let t_u denote the translation on $\mathbb{R}^k$ transforming 0 into u . If $\sigma_{r-1}: \mathbb{R} \rightarrow \tilde{J}^{r-1}(\mathbb{R}, M)$ is a section, then

$$\tau_{r-1}(u) := \sigma_{r-1}(u) \circ j_0^{r-1}(t_u) \in \tilde{T}^{r-1} M.$$

Hence, $\epsilon_{r-1}(\tau_{r-1}(u)) \in \overset{\circ}{T}^{r-1} M$ and we define

$$\epsilon_r(j_0^1 \sigma_r) := j_0^1 \epsilon_{r-1}(\tau_{r-1}(u)).$$

Thus, local coordinates of $\overset{\circ}{T}^r M$ correspond to the local coordinates of $\tilde{T}^r M$. For more details about local expressions, see [15].

1.3. Weil bundles

We only remark that $\overset{\circ}{T}^r$ and $\tilde{T}$ are examples of Weil (i. e. product preserving) functors. We refer to [9] for a good introduction to the theory of Weil bundles.

1.4. Simplicial structure

For specifications of many general results dealing with Weil bundles, we often need a clear description of them in local coordinates. The use of simplices introduced in White's book, [19], solves this problem adequately just for the case of iterated tangent bundles. We remark that it is also possible to use a more general concept of so-called colored simplices for a wide class of Weil bundles, [13]. A $(r-1)$ -simplex is the collection of all non-empty subsets of the set of vertices $\{1, \dots, r\}$. An h -face γ_h of the simplex, $0 \leq h \leq r-1$, is a subset with $h+1$ elements. We shall designate γ_h as an ordered r -tuple with $a_p \in \{0, 1\}$, $p = 1, \dots, r$ and $h+1$ of the elements equals to unity. This corresponds to the subset containing each s such that $a_s = 1$. The zero tuple is excluded. We write the *composition* of two disjoint faces γ_h, γ_c

as $\gamma_h + \gamma_c$. An *orientation* for a given simplex is an *ordering* of the vertices. The *standard orientation* is given by the arrangement $\{1, \dots, r\}$. Hence we regard such simplices as *(standardly) oriented*. On the other hand, if we define the simplex as the collection of non-empty subsets of an set of r indistinguishable elements, it is no orientation for it and we call it *non-oriented*.

White's original procedure of associating simplices with iterated tangent bundles consists in constructing vector spaces of functions from simplices to $\mathbb{R}^m$, a definition of a left action of the r th differential group on the fiber product of the frame bundle and this space, etc. It is very exhaustive, but it is a rather long formal process and we do not go into details here. Roughly speaking, it is essential that the local coordinates correspond to faces of simplices in the following way. Given some local coordinates x^i on M and t on $\mathbb{R}$, the iterated differentiation of $x^i(t)$ determines the induced coordinates $x^i, y_{\gamma_h}^i, h = 0, \dots, r-1$ on $\overset{\circ}{T}^r M = \underbrace{T \dots T}_r M$. In the face γ_h , the elements represent the order of the differentiation. Thus, $a_s = 1$ means the differentiation $\frac{d}{dt}$ at the s th iteration.

1.5. Projections

For every $s, 0 \leq s \leq r$, we denote by $\pi^s: \overset{\circ}{T}^s M \rightarrow M$ the canonical projection to the base. Further, we denote (the notation was first used in [11] for $s = 2$):

$\pi_b^s := \pi_{\overset{\circ}{T}^b M}^s: \overset{\circ}{T}^s(\overset{\circ}{T}^b M) \rightarrow \overset{\circ}{T}^b M$ the projection with $\overset{\circ}{T}^b M$ as the base space; such projections are sometimes called *direct projections*, e. g. in [17]. Iterated tangent bundles together with direct projections represent a so-called *full tangential resolution* for a manifold M .

$\pi^s := \overset{\circ}{T}^a \pi^s: \overset{\circ}{T}^a(\overset{\circ}{T}^s M) \rightarrow \overset{\circ}{T}^a M$ the induced projection originating by the posterior application of the functor $\overset{\circ}{T}^a$; these regular projections are sometimes viewed only as “*projections*” (the inverted commas e. g. in [17]) or *lateral projections* (e. g. in [16]).

$\pi_b^s := \overset{\circ}{T}^a \pi_{\overset{\circ}{T}^b M}^s$ the general case containing both previous cases; if a or b equals zero, we do not write them.

Projections $\overset{\circ}{T}^p M \rightarrow \overset{\circ}{T}^q M$ are of a type π_b^s or they are a composition of projections of such types. The *composition* $\pi_{b_1}^{s_1} \circ \pi_{b_2}^{s_2}$ is defined only for $a_1 + b_1 + s_1 = a_2 + b_2$ and properties of it are derived in [14]. We note that there are also non-simplifiable compositions of projections of the type π_b^s .

1.6. Structure of subbundles

For $s = 1, \dots, r$ and $\omega = 1, \dots, s$, let us denote by $\overset{\omega}{T}^s M$ the subset of all $Z \in \overset{\circ}{T}^s M$ satisfying the ω -condition

$$\omega-1\pi_{s-\omega}^1(Z) = {}_{k-1}\pi_{s-k}^s(Z)$$

for all k , $\omega \leq k \leq s$. Of course, $\overset{\omega}{T^s}M$ has a bundle structure (moreover, it is a Weil bundle, because it preserves products) and there are canonical inclusions

$$\overset{1}{T^s}M \subset \dots \subset \overset{s}{T^s}M$$

and $\overset{s}{T^s}M = \overset{\circ}{T^s}M$. Indeed, if we take $f: N \rightarrow M$, then $\overset{\circ}{T^s}f: \overset{\circ}{T^s}N \rightarrow \overset{\circ}{T^s}M$ and $\overset{\circ}{T^s}f(\overset{\omega}{T^s}N) = \overset{\omega}{T^s}M$. In the jet formulation, we easily see, that $\overset{1}{T^r}M$ coincides with semi-holonomic tangent bundle of order r . Furthermore, we can even regard $\overset{1}{T^r}M$ as the holonomic tangent bundle of order r because, in the special case of one-dimensional velocities, the difference has only a formal character. For details, see e. g. [14] and [15]. We shall write $\bar{T}^rM$ instead $\overset{1}{T^r}M$.

1.7. Simplices and Weil algebras associated with subbundles

It is known that h -faces, $h = 1, \dots, r-1$, i. e. the coordinates, indices of them have the same number of units, identify in $\bar{T}^rM$. Analogously, the ω -condition implies the identification of coordinates with the same number of units in first $r - \omega + 1$ indices and with identical rests.

In other words, we have for $\overset{\omega}{T^r}M$, $1 < \omega < r$ two subsimplices: the non-oriented $(r - \omega)$ -simplex and the oriented $(\omega - 2)$ -simplex. Faces γ_h induce faces $\bar{\gamma}_v, \gamma_w$ in these subsimplices, $v + w + 1 = h$; we write $\gamma_h = (\bar{\gamma}_v, \gamma_w)$. As we show, we obtain directly the Weil algebra $\overset{\omega}{\mathbb{D}}^r$ of the Weil bundle $\overset{\omega}{T^r}M$. Evaluating the product of two elements and matching the coefficients, we obtain generating equations for the ideal $\mathfrak{i}$ as

$$\sum_{\bar{\gamma}_v} \sum_{\rho \in \mathbb{P}(\bar{\gamma}_v, 2)} t^{(\bar{\gamma}_{v_1}, \gamma_{w_1})} t^{(\bar{\gamma}_{v_2}, \gamma_{w_2})} = \sum_{\bar{\gamma}_v} \sum_{\rho \in \mathbb{P}(\bar{\gamma}_v, 2)} t^{(\bar{\gamma}_v, \gamma_w)}$$

for all decompositions of γ_w to $\gamma_{w_1}, \gamma_{w_2}$ (where $\mathbb{P}(\bar{\gamma}_v, 2)$ is the set of all decompositions of v -face to 2 faces $\bar{\gamma}_{v_1}, \bar{\gamma}_{v_2}$, $v_1 + v_2 + 1 = v$, i. e., we sum as to all such decompositions $\rho \in \mathbb{P}(\bar{\gamma}_v, 2)$) and

$$t^{(\bar{\gamma}_{v_1}, \gamma_{w_1})} t^{(\bar{\gamma}_{v_2}, \gamma_{w_2})} = 0$$

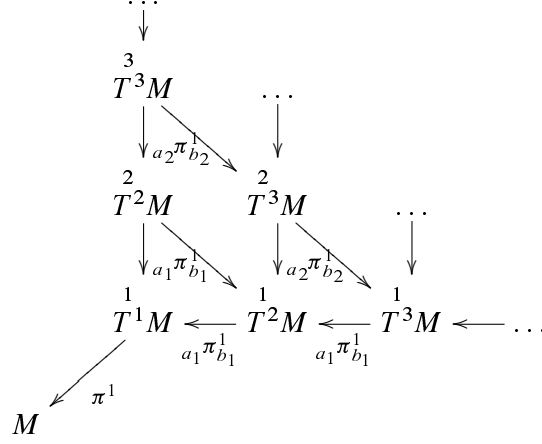
for non-disjoint faces $\gamma_{w_1}, \gamma_{w_2}$ or for $v_1 + v_2 > r - \omega - 1$. Then

$$\overset{\omega}{\mathbb{D}}^r = \mathbb{R}[t^{\gamma_0}, \dots, t^{\gamma_{r-1}}] / \mathfrak{i}.$$

1.8. Holonomizing projections

The sequence of functors $\overset{1}{\rightarrow} T^s \subset \dots \subset \overset{s}{\rightarrow} T^s$ is respected, be it any choice of projections

$a_{s-1}\pi_{b_{s-1}}^1: \overset{\circ}{T}^s M \rightarrow \overset{\circ}{T}^{s-1} M, \dots, a_1\pi_{b_1}^1: \overset{\circ}{T}^2 M \rightarrow \overset{\circ}{T}^1 M$ as we see in the following diagram:



We define the *holonomizing projection* $\mu: \overset{s}{T} M \rightarrow \overset{1}{T} M$ (with respect to the projections $a_{s-1}\pi_{b_{s-1}}^1: \overset{\circ}{T}^s M \rightarrow \overset{\circ}{T}^{s-1} M, \dots, a_1\pi_{b_1}^1: \overset{\circ}{T}^2 M \rightarrow \overset{\circ}{T}^1 M$) by

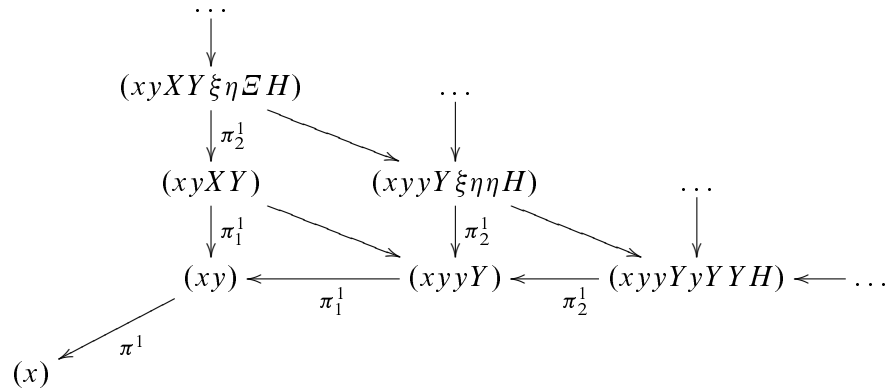
$$a_1\pi_{b_1}^1 \circ \dots \circ a_{s-1}\pi_{b_{s-1}}^1 \circ \mu = a_1\pi_{b_1}^1 \circ \dots \circ a_{s-1}\pi_{b_{s-1}}^1$$

and it is expressed by the composition of slanting arrows in the diagram.

Altogether, we have $s!$ holonomizing projections $\overset{s}{T} M \rightarrow \overset{1}{T} M$.

1.8.1. Example

For example, we take $s = 3$ and the direct projections π_2^1, π_1^1 . If we have local coordinates on $\overset{\circ}{T}^3 M$ as $x^i, y_{100}^i = y^i, y_{010}^i = X^i, y_{110}^i = Y^i, y_{001}^i = \xi^i, y_{101}^i = \eta^i, y_{011}^i = \Xi^i, y_{111}^i = H^i$, the coordinate version of the previous diagram is:



2. AFFINORS, VECTOR FIELDS AND DIFFERENTIAL OPERATORS

2.1. Canonical affinors

Let $\hat{\pi}^s$ denote a family of projections ${}_{a_1}\pi_{b_1}^1: \overset{\circ}{T}^r M \rightarrow \overset{\circ}{T}^{r-1} M, \dots, {}_{a_s}\pi_{b_s}^1: \overset{\circ}{T}^r M \rightarrow \overset{\circ}{T}^{r-1} M, 1 \leq s \leq r, a_1 + b_1 = \dots = a_s + b_s = r - 1$.

Lemma 1. *Every family $\hat{\pi}^s$ determines a unique projection $\overset{\circ}{T}^r M \rightarrow \overset{\circ}{T}^{r-s} M$.*

Proof. The proof is easy from the simplicial point of view. The projections $\overset{\circ}{T}^r M \rightarrow \overset{\circ}{T}^{r-1} M$ mean projections of $(r-1)$ -simplex onto $(r-2)$ -faces. If we have s projections $\overset{\circ}{T}^r M \rightarrow \overset{\circ}{T}^{r-1} M, s < r$, an $(r-s-1)$ -face or a projection $\overset{\circ}{T}^r M \rightarrow \overset{\circ}{T}^{r-s} M$ is determined. If $s = r$, we have the projection $\pi^r: \overset{\circ}{T}^r M \rightarrow M$. $\square$

For every such a family $\hat{\pi}^s$, we have the canonical monomorphism

$$\mathbf{i}: V^{\hat{\pi}^s} \overset{\circ}{T}^r M \longrightarrow T \overset{\circ}{T}^r M$$

and the canonical epimorphism

$$\mathbf{s}: T \overset{\circ}{T}^r M \longrightarrow T \overset{\circ}{T}^{r-s} M \times_{\overset{\circ}{T}^{r-s} M} \overset{\circ}{T}^r M$$

of vector bundles over $\overset{\circ}{T}^r M$, where $V^{\hat{\pi}^s}$ means the verticality with respect to all projections belonging to the family $\hat{\pi}^s$ and the fibration $\overset{\circ}{T}^r M \rightarrow \overset{\circ}{T}^{r-s} M$ is through the determined projection from the previous lemma. There is also the canonical isomorphism (in fact the vertical lift)

$$\mathbf{h}: T \overset{\circ}{T}^{r-s} M \times_{\overset{\circ}{T}^{r-s} M} \overset{\circ}{T}^r M \longrightarrow V^{\hat{\pi}^s} \overset{\circ}{T}^r M.$$

Since, by an *affinor* A on a manifold Y , we mean a $(1,1)$ -tensor field, which we can identify with a linear morphism $A: TY \rightarrow TY$ over id_M , the canonical affinor is defined as

$$A := \mathbf{i} \circ \mathbf{h} \circ \mathbf{s}.$$

For fixed r, s we obtain $\binom{r}{s}$ affinors in this way. Particularly, for $s = 1$ we obtain r affinors. We shall call them the *primary affinors*. We obtain all generators of natural affinors (except the identical affinor) in this way, where by a *natural affinor* on a natural bundle F over m -dimensional manifolds we mean a system of $(1,1)$ -tensor fields $A_M: TFM \rightarrow TFM$ for every m -dimensional manifold M satisfying $TFf \circ A_M = A_N \circ TFf$ for every local diffeomorphism $f: M \rightarrow N$. All natural affinors on iterated tangent bundles are described in [12], where it is proved:

Proposition 1. *All natural affinors on $\overset{\circ}{T}^r M$ constitute a 2^r -parameter family linearly generated by the identical affinor A_0 and by affinors A_{γ_c} , where γ_c represent*

all c -faces, $0 \leq c \leq r-1$, which work in this way:

$$\begin{aligned} dy_{\gamma_c}^i &= dx^i \\ dy_{\gamma_c + \gamma_h}^i &= dy_{\gamma_h}^i, \quad h \leq r - c - 2. \end{aligned}$$

Indeed, we have 2^r canonical affinors on $\overset{\circ}{T}^r M$ including the identical affinor. It is essential that the classification of all natural objects makes assurance double sure that our list of canonical affinors is complete. Evidently,

$$A_{\gamma_h} \circ A_{\gamma_c} = A_{\gamma_h + \gamma_c}$$

for disjoint faces γ_h, γ_c , elsewhere the composition vanishes. It follows that it is possible to generate all canonical affinors by the iterations of primary affinors.

2.2. Canonical vector fields

Now we construct canonical vector fields on $\overset{\circ}{T}^r M$. For a projection $\pi_b^1: \overset{\circ}{T}^r M \rightarrow \overset{\circ}{T}^{r-1} M$, let α denote the canonical section of the vector bundle $T\overset{\circ}{T}^{r-1} M \times_{\overset{\circ}{T}^{r-1} M} \overset{\circ}{T}^r M$. The canonical vector field is defined as

$$\zeta := \mathbf{i} \circ \mathbf{h} \circ \alpha.$$

We shall call such vector fields the *primary vector fields*. For the construction of subsequent canonical vector fields, we apply canonical affinors to primary vector fields.

We remark that a *natural vector field* on a natural bundle F over m -dimensional manifolds is a system of vector fields $\zeta_M: FM \rightarrow TFM$ for every m -dimensional manifold M satisfying $TFf \circ \zeta_M = \zeta_N \circ TFf$ for every local diffeomorphism $f: M \rightarrow N$. The natural vector fields on F can be interpreted as so-called absolute natural operators $C^\infty TM \rightarrow C^\infty TFM$ transforming vector fields on M into vector fields on FM . For more details about natural objects see [9]. Let us add that we have entailed the geometrical interpretation of generators of natural vector fields by our construction. Now, we take the sequence $\zeta_1, \dots, \zeta_r$ of canonical vector fields with these properties:

- (i) ζ_1 is a primary vector field;
- (ii) ζ_{s+1} is obtained from ζ_s by applying a primary affinor (for every $s = 1, \dots, r-1$).

Then we call this sequence the *normal sequence of canonical vector fields*. By some combinatorics, we see that we have constructed $r2^{r-1}$ canonical vector fields, which we can order to $r!$ normal sequences. We need the following assertion.

Proposition 2. *Every normal sequence of canonical vector fields $\zeta_1, \dots, \zeta_r$ on $\overset{\circ}{T}^r M$ determines a unique holonomizing projection $\mu: \overset{\circ}{T}^r M \rightarrow \bar{T}^r M$.*

Proof. The normal sequence of canonical vector fields $\zeta_1, \dots, \zeta_r$ is given by a primary vector field ζ_1 and by primary affinors $A_2, \dots, A_r$, where $\zeta_s = A_s(\zeta_{s-1})$ for $s = 2, \dots, r$. Every such affinor has a form A_{γ_0} . We know that, for non-disjoint faces γ_h, γ_c , the composition of affinors $A_{\gamma_h} \circ A_{\gamma_c}$ vanishes. It follows that affinors $A_2, \dots, A_r$ are different and are related to $r - 1$ different projections $\mathring{T}^r M \rightarrow \mathring{T}^{r-1} M$. Further, the vector field ζ_1 is also related to a projection $\mathring{T}^r M \rightarrow \mathring{T}^{r-1} M$. Of course, this projection is different from all those mentioned above: if not, the application of such an affinor gives the zero vector field. So, we have r different projections $\mathring{T}^r M \rightarrow \mathring{T}^{r-1} M$. Hence, we have families of projections $\hat{\pi}^s, s = 1, \dots, r$, which we construct gradually by adding projections related to $A_2, \dots, A_r$ to the projection related to ζ_1 . In Lemma 1, it was proved that every family $\hat{\pi}^s$ determines a unique projection $\mathring{T}^r M \rightarrow \mathring{T}^{r-s} M$. We obtain projections

$$\mathring{T}^r M \rightarrow \mathring{T}^{r-1} M, \mathring{T}^r M \rightarrow \mathring{T}^{r-2} M, \dots, \mathring{T}^r M \rightarrow \mathring{T}^1 M, \mathring{T}^r M \rightarrow M$$

in this way. They determine uniquely projections

$$\mathring{T}^r M \rightarrow \mathring{T}^{r-1} M, \mathring{T}^{r-1} M \rightarrow \mathring{T}^{r-2} M, \dots, \mathring{T}^2 M \rightarrow \mathring{T}^1 M, \mathring{T}^1 M \rightarrow M.$$

We fix the choice of projections $a_{r-1}\pi_{b_{r-1}}^1: \mathring{T}^r M \rightarrow \mathring{T}^{r-1} M, \dots, a_1\pi_{b_1}^1: \mathring{T}^2 M \rightarrow \mathring{T}^1 M$ (the projection $\mathring{T}^1 M \rightarrow M$ is unique trivially). So, the holonomizing projection $\mu: \mathring{T}^r M \rightarrow \bar{T}^r M$ is determined, cf. Section 1.8. $\square$

Remark 1. Conversely, every holonomizing projection determines a unique normal sequence of canonical vector fields.

2.3. Differential operators

We introduce the following differential operator d_T on $\mathring{T}^r M$ related to a given normal sequence of canonical vector fields $\zeta_1, \dots, \zeta_r$. Let μ be the holonomizing projection with respect to bundle projections $a_s\pi_{b_s}^1: \mathring{T}^{s+1} M \rightarrow \mathring{T}^s M, s = 1, \dots, r - 1$ derived from $\zeta_1, \dots, \zeta_r$ (Proposition 2). Let $\psi_r: \mathring{T}^r M \rightarrow T(\mathring{T}^{r-1} M)$, be the mapping given by $\mu(\tau_r) \mapsto j_0^1 \tau_{r-1}$. (See Sections 1.2 and 1.8 for the definitions of τ_r and μ .) ψ_r can be viewed as a generalization of the map ϕ_r using by Tulczyjew in [18]. Then, locally,

$$\psi_r(x^i, y_{\bar{\gamma}_h}^i) = y_{\bar{\gamma}_0}^i \frac{\partial}{\partial x^i} + y_{\bar{\gamma}_1}^i \sum_{\gamma_0} \frac{\partial}{\partial y_{\gamma_0}^i} + \dots + y_{\bar{\gamma}_r}^i \sum_{\gamma_{r-1}} \frac{\partial}{\partial y_{\gamma_{r-1}}^i},$$

where $\bar{\gamma}_0, \dots, \bar{\gamma}_{r-1}$ are the faces belonging to $\bar{T}^r M = \mu(\mathring{T}^r M)$ and $\bar{\gamma}_r$ is the face with nothing but units. We use this map to construct the differential operator d_T , which maps each function F on $\mathring{T}^r M$ into a function $d_T F$ on $\mathring{T}^{r+1} M$, defined by

$$d_T F(\tau_{r+1}) = dF(\tau_r)(\psi_r \circ \mu(\tau_{r+1})).$$

Since $d_T(FG) = (d_T F)G + F(d_T G)$, d_T extends to p -forms on $\overset{\circ}{T}^r M$ in such a way that $d_T d = d d_T$. Further, $d_T^s := \underbrace{d_T \circ d_T}_{r \text{ times}}$.

Remark 2. As we have $r!$ holonomizing projections related to $r!$ normal sequences of canonical vector fields, we have also $r!$ operators d_T , in other words $r!$ ways of differentiating. Nevertheless, a hard and fast rule of using them is respecting the described correspondence. On purpose, we desist from more complicated denotation with an indexing of d_T purposely.

3. LEGENDRE–OSTROGRADSKII TRANSFORMATION

After preceding preparation, we can now formulate the foundations of formalisms of mechanics on $\overset{\circ}{T}^r M$. Let us consider a Lagrangian $\mathcal{L}: \overset{\circ}{T}^r M \rightarrow \mathbb{R}$. For vector fields $\zeta_s: \overset{\circ}{T}^r M \rightarrow T\overset{\circ}{T}^r M$, $s = 1, \dots, r$ from a normal sequence of canonical vector fields, we have induced vector fields $\bar{\zeta}_s: \overset{\circ}{T}^r M \rightarrow T\overset{\circ}{T}^r M$ such that $\bar{\zeta}_s \circ \mu = T\mu \circ \zeta_s$. We define the *energy* as

$$E := \sum_{\zeta_1, \dots, \zeta_r} \sum_{s=1}^r \frac{(-1)^{s-1}}{(r-s)!s!} d_T^{s-1}(\bar{\zeta}_s \mathcal{L}) - \mathcal{L},$$

where $\sum_{\zeta_1, \dots, \zeta_r}$ denotes the sum over all normal sequences of vector fields.

Proposition 3. *The energy is a function on $V^{\hat{\pi}^{2r-2}} \dots V^{\hat{\pi}^r} \overset{\circ}{T}^r M$.*

Proof. $\mathcal{L}$ is a function on $\overset{\circ}{T}^r M$ and $\bar{\zeta}_s \mathcal{L}$ is also a function on $\overset{\circ}{T}^r M$ for an arbitrary $s = 1, \dots, r$. We transfer to the bundle $\overset{\circ}{T}^{r+1} M$ by the application of d_T . Since we have

$$d_T F = y_{\gamma_0}^i \frac{\partial F}{\partial x^i} + y_{\gamma_1}^i \sum_{\gamma_0} \frac{\partial F}{\partial y_{\gamma_0}^i} + \dots + y_{\gamma_{r+1}}^i \sum_{\gamma_r} \frac{\partial F}{\partial y_{\gamma_r}^i}$$

for a arbitrary function $F: \overset{\circ}{T}^r M \rightarrow \mathbb{R}$, we have $y_{\gamma_0}^i, \dots, y_{\gamma_r}^i$ as coordinates on $\overset{\circ}{T}^r M$ and, in addition, we have only new coordinates $y_{\gamma_{r+1}}^i$, which are on $\overset{\circ}{T}^{r+1} M$. So, we are on a vertical bundle, more precisely on the bundle, which is vertical with respect to all r projections $\overset{\circ}{T}^r M \rightarrow \overset{\circ}{T}^{r-1} M$. This verticality we have denoted $V^{\hat{\pi}^r}$. Analogously, we obtain the verticality $V^{\hat{\pi}^{r+1}}$ by the application of $d_T^2 = d_T \circ d_T$, etc. Finally, for d_T^{r-1} we are really on the bundle $V^{\hat{\pi}^{2r-2}} \dots V^{\hat{\pi}^r} \overset{\circ}{T}^r M$. $\square$

For every normal sequence of canonical vector fields we define the *partial momenta* $\wp_i^{\bar{\gamma}_h}$, $h = 0, \dots, r-1$ as

$$\wp_i^{\bar{\gamma}_h} := \sum_{c=0}^{r-h-1} \frac{(-1)^c}{(r-h-c-1)!(h+c+1)!} d_T^c \frac{\partial \mathcal{L}}{\partial y_{\gamma_{h+c}}^i}.$$

If we evaluate all partial momenta for all normal sequences of canonical vector fields, it is evident that we obtain partial momenta, indices of them are the same faces. We define the (total) momenta $p_i^{\gamma_h}$, $h = 0, \dots, r-1$ as the sum of the partial momenta with equal indices, i. e. $p_i^{\bar{\gamma}_h} := \sum \wp_i^{\bar{\gamma}_h}$.

Paper [19] studies a so-called *bundle of boundaries* $B^{r-1}M$. In the simplicial language, we obtain it by removing the maximal face from the associated $(r-1)$ -simplex. Geometrically,

$$B^{r-1}M := \tilde{T}^{r-2}M \rightarrow \times_r \overset{\circ}{T}^{r-1}M$$

where $\tilde{T}^{r-2}M \rightarrow \times_r$ denotes the fiber (Whitney) product over $\overset{\circ}{T}^{r-2}M$ with respect to r different projections $\overset{\circ}{T}^r M \rightarrow \overset{\circ}{T}^{r-1}M$. It is known that $\overset{\circ}{T}^r M \rightarrow B^{r-1}M$ is the affine bundle. We discover easily:

Proposition 4. *The partial momenta represent local coordinates on $T^*\bar{T}^{r-1}M$. The momenta represent local coordinates on $T^*B^{r-1}M$.*

Proof. We recall that the classical way of defining the cotangent bundle is the union of vector spaces over the reals, which are composed of differentials at a point of a given manifold. So, the first part of the assertion follows immediately from the definition of the partial momenta because the indices of which are the faces belonging to $\bar{T}^r M = \mu(\overset{\circ}{T}^r M)$. We see the second part equally clearly; it is sufficient to recall the geometrical definition of $B^{r-1}M$. $\square$

The natural generalization of the Legendre transformation to higher-order theories is called the *Legendre–Ostrogradskii transformation* and it is well-known for the bundle $T^r M$; it enables the Hamiltonization of mechanical problems (see, e. g., [4]). If we take the two preceding assertions into this consideration, we have immediately the main theoretical result:

Corollary 1. *If $\mathcal{L}$ is a regular Lagrangian, then the introduced energy and momenta enable the generalized Legendre–Ostrogradskii transformation*

$$\widetilde{\text{Leg}}: V^{\hat{\pi}^{2r-2}} \dots V^{\hat{\pi}^r} \overset{\circ}{T}^r M \rightarrow T^*B^{r-1}M,$$

which is a natural generalization of the Legendre–Ostrogradskii transformation $\text{Leg}: T^{2r-1}M \rightarrow T^*T^{r-1}M$.

For the completion, we state the following essential formula.

Proposition 5. *The local coordinate expression of the energy is*

$$E = \sum_{h=0}^{r-1} p_i^{\gamma_h} y_{\gamma_h}^i - \mathcal{L}.$$

Proof. The equalities

$$\begin{aligned}
E &= \sum_{\zeta_1, \dots, \zeta_r} \sum_{s=1}^r \frac{(-1)^{s-1}}{(r-s)!s!} d_T^{s-1}(\bar{\zeta}_s \mathcal{L}) - \mathcal{L} \\
&= \sum_{\zeta_1, \dots, \zeta_r} \sum_{s=1}^r \frac{(-1)^{s-1}}{(r-s)!s!} d_T^{s-1} \left(\sum_{h=0}^{r-s} y_{\bar{\gamma}_h}^i \frac{\partial \mathcal{L}}{\partial y_{\bar{\gamma}_{h+s-1}}^i} \right) - \mathcal{L} \\
&= \sum_{\zeta_1, \dots, \zeta_r} \sum_{h=0}^{r-1} \sum_{c=0}^{r-h-1} \frac{(-1)^c}{(r-c-1)!(c+1)!} d_T^c \left(y_{\bar{\gamma}_h}^i \frac{\partial \mathcal{L}}{\partial y_{\bar{\gamma}_{h+c}}^i} \right) - \mathcal{L} \\
&= \sum_{\zeta_1, \dots, \zeta_r} \sum_{h=0}^{r-1} y_{\bar{\gamma}_h}^i \sum_{c=0}^{r-h-1} \frac{(-1)^c}{(r-h-c-1)!(h+c+1)!} d_T^c \left(\frac{\partial \mathcal{L}}{\partial y_{\bar{\gamma}_{h+c}}^i} \right) - \mathcal{L} \\
&= \sum_{\zeta_1, \dots, \zeta_r} \sum_{h=0}^{r-1} y_{\bar{\gamma}_h}^i \delta_i^{\bar{\gamma}_h} - \mathcal{L} = \sum_{h=0}^{r-1} p_i^{\gamma_h} y_{\gamma_h}^i - \mathcal{L}
\end{aligned}$$

lead one immediately to the formula given above. $\square$

Example. We take $\mathring{T}^3 M$ and $m = 1$ with local coordinates $x, y_{100} = y, y_{010} = X, y_{110} = Y, y_{001} = \xi, y_{101} = \eta, y_{011} = \mathcal{E}, y_{111} = H$. Given is also the Lagrangian $\mathcal{L} = (2y + X)\eta + H^2$.

We have 12 canonical vector fields in 6 normal sequences:

$$\zeta_1 = y \frac{\partial}{\partial y} + Y \frac{\partial}{\partial Y} + \eta \frac{\partial}{\partial \eta} + H \frac{\partial}{\partial H}, \quad \zeta_2 = y \frac{\partial}{\partial \eta} + Y \frac{\partial}{\partial H}, \quad \zeta_3 = y \frac{\partial}{\partial H}, \quad (1)$$

$$\zeta_1 = y \frac{\partial}{\partial y} + Y \frac{\partial}{\partial Y} + \eta \frac{\partial}{\partial \eta} + H \frac{\partial}{\partial H}, \quad \zeta_2 = y \frac{\partial}{\partial Y} + \eta \frac{\partial}{\partial H}, \quad \zeta_3 = y \frac{\partial}{\partial H}, \quad (2)$$

$$\zeta_1 = X \frac{\partial}{\partial X} + Y \frac{\partial}{\partial Y} + \mathcal{E} \frac{\partial}{\partial \mathcal{E}} + H \frac{\partial}{\partial H}, \quad \zeta_2 = X \frac{\partial}{\partial \mathcal{E}} + Y \frac{\partial}{\partial H}, \quad \zeta_3 = X \frac{\partial}{\partial H}, \quad (3)$$

$$\zeta_1 = X \frac{\partial}{\partial X} + Y \frac{\partial}{\partial Y} + \mathcal{E} \frac{\partial}{\partial \mathcal{E}} + H \frac{\partial}{\partial H}, \quad \zeta_2 = X \frac{\partial}{\partial Y} + \mathcal{E} \frac{\partial}{\partial H}, \quad \zeta_3 = X \frac{\partial}{\partial H}, \quad (4)$$

$$\zeta_1 = \xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} + \mathcal{E} \frac{\partial}{\partial \mathcal{E}} + H \frac{\partial}{\partial H}, \quad \zeta_2 = \xi \frac{\partial}{\partial \mathcal{E}} + \eta \frac{\partial}{\partial H}, \quad \zeta_3 = \xi \frac{\partial}{\partial H}, \quad (5)$$

$$\zeta_1 = \xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} + \mathcal{E} \frac{\partial}{\partial \mathcal{E}} + H \frac{\partial}{\partial H}, \quad \zeta_2 = \xi \frac{\partial}{\partial \eta} + \mathcal{E} \frac{\partial}{\partial H}, \quad \zeta_3 = \xi \frac{\partial}{\partial H}. \quad (6)$$

Every sequence determines a holonomizing projection μ and it provides us with d_T and with $\bar{\xi}_s$, $s = 1, 2, 3$. We have

$$\bar{\xi}_1 = y \frac{\partial}{\partial y} + Y \frac{\partial}{\partial Y} + Y \frac{\partial}{\partial Y} + H \frac{\partial}{\partial H}, \quad \bar{\xi}_2 = y \frac{\partial}{\partial Y} + Y \frac{\partial}{\partial H}, \quad \bar{\xi}_3 = y \frac{\partial}{\partial H} \quad (1)$$

and we evaluate

$$\sum_{s=1}^3 \frac{(-1)^{s-1}}{(3-s)!s!} d_T^{s-1}(\bar{\xi}_s \mathcal{L}) = y \left(\eta + \frac{1}{3} \ddot{H} \right) + Y \left(-\frac{1}{3} \dot{H} \right) + H \left(\frac{1}{3} H \right).$$

$\dot{H}$, $\ddot{H}$ are new coordinates on $\overset{\circ}{T}^4 M$, $\overset{\circ}{T}^5 M$, respectively. The expressions in the brackets are the partial momenta. Analogously, we obtain

$$y \left(-\frac{1}{2} \eta + \frac{1}{3} \ddot{H} \right) + \eta \left(y + \frac{1}{2} X - \frac{1}{3} \dot{H} \right) + H \left(\frac{1}{3} H \right), \quad (2)$$

$$X \left(\frac{1}{2} \eta + \frac{1}{3} \ddot{H} \right) + Y \left(-\frac{1}{3} \dot{H} \right) + H \left(\frac{1}{3} H \right), \quad (3)$$

$$X \left(\frac{1}{2} \eta + \frac{1}{3} \ddot{H} \right) + \mathcal{E} \left(-\frac{1}{3} \dot{H} \right) + H \left(\frac{1}{3} H \right), \quad (4)$$

$$\xi \left(\frac{1}{3} \ddot{H} \right) + \mathcal{E} \left(-\frac{1}{3} \dot{H} \right) + H \left(\frac{1}{3} H \right), \quad (5)$$

$$\xi \left(-\frac{3}{2} \eta + \frac{1}{3} \ddot{H} \right) + \eta \left(y + \frac{1}{2} X - \frac{1}{3} \dot{H} \right) + H \left(\frac{1}{3} H \right). \quad (6)$$

We sum the corresponding partial momenta to obtain the energy as

$$\begin{aligned} E = & y \left(\frac{1}{2} \eta + \frac{2}{3} \ddot{H} \right) + X \left(\eta + \frac{2}{3} \ddot{H} \right) + \xi \left(-\frac{3}{2} \eta + \frac{2}{3} \ddot{H} \right) + Y \left(-\frac{2}{3} \dot{H} \right) \\ & + \eta \left(2y + X - \frac{2}{3} \dot{H} \right) + \mathcal{E} \left(-\frac{2}{3} \dot{H} \right) + H(2H) - (2y + X)\eta - H^2. \end{aligned}$$

The expressions in the brackets are the momenta $p_{100} = p$, $p_{010} = q$, $p_{001} = r$, $p_{110} = P$, $p_{101} = Q$, $p_{011} = R$, $p_{111} = \Pi$. We evaluate the Hamiltonian $\mathcal{H} = yp + Xq + \xi r + YP + \eta Q + \mathcal{E}R + \frac{1}{4}\Pi^2 - (2y + X)\eta$.

Finally, let us consider the standard holonomic case. If we choose a holonomizing projection μ , e. g. $\mu: (xyXY\xi\eta\mathcal{E}H) \mapsto (xyYyYyYH)$, we have the Lagrangian $\mu(\mathcal{L}) = 3yY + H^2$. We can evaluate energy $(2y\ddot{H} - 2Y\dot{H} + H^2)$ and Hamiltonian $(yp + YP + \frac{1}{4}\Pi^2 - 3yY)$ in the classical way and verify that they represent the holonomized E and $\mathcal{H}$, respectively.

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ON THE STRUCTURE OF FINSLER AND AREAL SPACES

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Abstract. We study underlying geometric structures for integral variational functionals, depending on submanifolds of a given manifold. Applications include (first order) variational functionals of Finsler and areal geometries with integrand the Hilbert 1-form, and admit immediate extensions to higher-order functionals.

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1. INTRODUCTION

This paper is a contribution to the theory of integral variational functionals, depending on submanifolds of a given manifold X . The theory is based on geometric notions such as the bundles of (skew-symmetric) multivectors, and Grassmann fibrations. Conceptually, it extends local parametric integrals of Finsler–Kawaguchi and areal geometries (see, e. g., Chern, Chen, Lam [1], Davies [3], Kawaguchi [4], and Tamassy [6]) to global functionals, depending on (global) submanifolds. In Section 2 we summarize integration theory of differential forms along submanifolds. Section 3 is devoted to vector bundles of k -vectors; we show how mappings of Euclidean spaces into manifolds (*parametrisations*) can be lifted to the bundles of k -vectors. In Section 4 we introduce, using the Plücker embedding, underlying spaces for parameter-invariant variational problems, the Grassmann fibrations. In Section 5 we show that any k -form on the Grassmann fibration defines an integral variational functional, depending on k -dimensional submanifolds. An example is the *Hilbert form*, a well-known first-order construction in Finsler geometry and its generalisations (Chern, Chen, Lam [1], Crampin, Saunders [2]).

It should be pointed out that the theory can be further generalised. To this end, one should consider higher-order Grassmann fibrations endowed with Lagrangians satisfying the relevant homogeneity conditions (Zermelo conditions, see, e. g., Saunders [5], and Urban and Krupka [8]).

2. INTEGRATION OVER SUBMANIFOLDS

Let X be an n -dimensional manifold, S a subset of X , $x_0 \in S$ a point. A chart $(U, \varphi), \varphi = (x^i)$, at x_0 is a *submanifold chart* for S , if there exists a non-negative integer $k \leq n$ such that $\varphi(U \cap S) = \{x \in U \mid x^{k+1}(x) = c_1, x^{k+2}(x) = c_2, \dots, x^n(x) = c_{n-k}\}$. If such a chart exists, we say that S is a *submanifold* of X at the point x_0 ; k is the *dimension* of S at x_0 . If such a submanifold chart exists at every point of X , we say S is a *submanifold* of X and call k the *dimension* of S .

Denote by $(t^1, t^2, \dots, t^n)$ the canonical coordinates on the Euclidean space $\mathbb{R}^n$, and $\mathbb{R}_{(-)}^n = \{t_0 \in \mathbb{R}^n \mid t^n(t_0) \leq 0\}$, $\partial\mathbb{R}_{(-)}^n = \{t_0 \in \mathbb{R}_{(-)}^n \mid t^n(t_0) = 0\}$. $\mathbb{R}_{(-)}^n$ is the *halfspace* of $\mathbb{R}^n$, $\partial\mathbb{R}_{(-)}^n$ is the *boundary* of $\mathbb{R}_{(-)}^n$. Let Ω be a non-void subset of X , and $x_0 \in \Omega$ a point. A chart (U, φ) at x_0 is said to be *adapted* to Ω , if the set $\varphi(U \cap \Omega)$ is an open set in $\mathbb{R}_{(-)}^n$. Ω is a *piece* of X , if it is compact and each point $x \in \Omega$ admits a chart, adapted to Ω .

Let η be a k -form on X . Our aim now will be to introduce an integral of η on a piece of a k -dimensional submanifold S (k -piece of a X). Express η in a submanifold chart $(U, \varphi), \varphi = (x^i)$, as $\eta = \eta_{i_1 i_2 \dots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}$. Then, restricting η to S we get from the equations $x^{k+1} = 0, x^{k+2} = 0, \dots, x^n = 0$

$$\eta = f dx^1 \wedge dx^2 \wedge \dots \wedge dx^k,$$

where we write $f = f(x^{i_1}, x^{i_2}, \dots, x^{i_k})$ for the component of η restricted to S . From now on we suppose that S is *orientable*, and is endowed with an orientation $\text{Or}_S X$; only submanifold charts on X belonging to $\text{Or}_S X$ are used. The integral of η on a compact set $\Omega \subset S$ is defined in a standard way. There exist a finite family $\{(U_1, \varphi_1), (U_2, \varphi_2), \dots, (U_N, \varphi_N)\}$ of submanifold charts on X , such that the family $\{U_1 \cap S, U_2 \cap S, \dots, U_N \cap S\}$ covers Ω . Let $\{\chi_1, \chi_2, \dots, \chi_N\}$ be a partition of unity, subordinate to this covering. Then,

$$\int_{\Omega} \eta = \sum_{j=1}^N \int_{\text{supp } \chi_j \cap \Omega} \chi_j \eta.$$

The following basic properties of the integral are needed in the calculus of variations.

Lemma 1 (transformation of integration domain). *Let X and Y be two smooth n -dimensional oriented manifolds, $\alpha : X \rightarrow Y$ an orientation-preserving diffeomorphism. Then*

$$\int_{\Omega} \eta = \int_{\alpha^{-1}(\Omega)} \alpha * \eta$$

for any compact set $\Omega \subset S$ and any continuous differential n -form on Y .

Lemma 2 (Leibniz rule). *Let X be an oriented n -dimensional manifold, η_t a family of n -forms on X , differentiable on a real parameter t , $\Omega \subset S$ a compact set.*

Then, the function $I \ni t \mapsto \int_{\Omega} \eta_t \in \mathbb{R}$ is differentiable, and

$$\frac{d}{dt} \int_{\Omega} \eta_t = \int_{\Omega} \frac{d\eta_t}{dt}.$$

Lemma 3 (Stokes formula). *Let X be an n -dimensional manifold, S a k -dimensional oriented submanifold of X , η a $(k-1)$ -form on X . Let Ω be a piece of S with boundary $\partial\Omega$, endowed with induced orientation. Then*

$$\int_{\partial\Omega} \eta = \int_{\Omega} d\eta.$$

3. BUNDLES OF k -VECTORS

Let X be an n -dimensional manifold, $\Lambda^k T_x X$ the k -exterior product of the tangent space $T_x X$, $x \in X$ a point. We put

$$\Lambda^k TX = \bigcup_{x \in X} \Lambda^k T_x X.$$

This set has a natural vector bundle structure over X , with type fibre $\Lambda^k \mathbb{R}^n$. We denote by τ^k the vector bundle projection of $\Lambda^k TX$.

Let X (resp. Y) be a smooth manifold of dimension n (resp. m), and let $f : X \rightarrow Y$ be a differentiable mapping. Choose a point $x \in X$ and a k -vector $\Xi \in \Lambda^k T_x X$. Then, choose a chart (U, φ) , $\varphi = (x^i)$, at x and a chart (V, ψ) , $\psi = (y^\sigma)$, at $f(x) \in Y$ such that $f(U) \subset V$. Expressing Ξ in components and setting

$$\begin{aligned} \Lambda^k T_x f \cdot \Xi &= \frac{1}{(k!)^2} \left(\frac{\partial y^{\sigma_1} f \varphi^{-1}}{\partial x^{i_1}} \right)_{\varphi(x)} \left(\frac{\partial y^{\sigma_2} f \varphi^{-1}}{\partial x^{i_2}} \right)_{\varphi(x)} \cdots \left(\frac{\partial y^{\sigma_k} f \varphi^{-1}}{\partial x^{i_k}} \right)_{\varphi(x)} \\ &\quad \cdot \Xi^{i_1 i_2 \dots i_k} \left(\frac{\partial}{\partial y^{\sigma_1}} \right)_{f(x)} \wedge \left(\frac{\partial}{\partial y^{\sigma_2}} \right)_{f(x)} \wedge \cdots \wedge \left(\frac{\partial}{\partial y^{\sigma_k}} \right)_{f(x)}, \end{aligned}$$

we get a k -vector $\Lambda^k T_x f \cdot \Xi \in \Lambda^k T_{f(x)} Y$, and a vector bundle homomorphism $\Lambda^k T f : \Lambda^k TX \rightarrow \Lambda^k TY$ over f (the lift of f).

It is easily seen that differentiable mappings of a Euclidean space into a manifold can be canonically lifted to the bundles of k -vectors. For this purpose we use the canonical k -vector field on $\mathbb{R}^n$

$$\mathbb{R}^n \ni t \rightarrow \theta(t) = \frac{1}{k!} \varepsilon^{i_1 i_2 \dots i_k} \left(\frac{\partial}{\partial t^{i_1}} \right)_t \wedge \left(\frac{\partial}{\partial t^{i_2}} \right)_t \wedge \cdots \wedge \left(\frac{\partial}{\partial t^{i_k}} \right)_t \in \Lambda^k T \mathbb{R}^n.$$

Identifying $\Lambda^k T \mathbb{R}^n$ with $\mathbb{R}^n \times \Lambda^k \mathbb{R}^n$, the canonical section becomes the mapping $t \rightarrow (t, \varepsilon^{i_1 i_2 \dots i_k})$.

Consider a differentiable mapping $f : U \rightarrow Y$, where U is an open subset of $\mathbb{R}^n$. For any point $t \in U$, $\Lambda^k T_t f \cdot \theta(t)$ is an element of the vector space $\Lambda^k T_{f(t)} Y$. We

get the *canonical lift* $\Lambda^k f$ of f to $\Lambda^k TY$, defined by

$$\Lambda^k f = \Lambda^k Tf \cdot \theta.$$

The canonical lift of the *parametrisation* $U \ni t \rightarrow (\psi^{-1} \circ \iota_{k,m})(t) \in V \cap S$ is expressed in a chart (V, ψ) , $\psi = (y^\sigma)$, as

$$\begin{aligned} & \Lambda^k(\psi^{-1} \circ \iota_{k,m})(t) \\ &= \left(\frac{\partial}{\partial y^1} \right)_{\psi^{-1} \circ \iota_{k,m}(t)} \wedge \left(\frac{\partial}{\partial y^2} \right)_{\psi^{-1} \circ \iota_{k,m}(t)} \wedge \dots \wedge \left(\frac{\partial}{\partial y^k} \right)_{\psi^{-1} \circ \iota_{k,m}(t)}. \end{aligned} \quad (3.1)$$

Formula (3.1) also defines the mapping $V \ni y \rightarrow (\Lambda^k \psi)(y) \in (\tau^k)^{-1}(V)$ by

$$\Lambda^k \psi = \Lambda^k(\psi^{-1} \circ \iota_{k,m}) \circ \text{pr}_{m,k} \psi, \quad (3.2)$$

the *canonical section along* S , associated with (V, ψ) . $\Lambda^k \psi$ is expressed by

$$\begin{aligned} (y^1, y^2, \dots, y^k, y^{k+1}, y^{k+2}, \dots, y^m) &\rightarrow \Lambda^k(\psi^{-1} \circ \iota_{k,m})(y^1, y^2, \dots, y^k) \\ &= ((y^1, y^2, \dots, y^k, 0, 0, \dots, 0), (1, 0, 0, \dots, 0)). \end{aligned}$$

Writing in the multi-index notation $((\tau^r)^{-1}(V), \Phi)$, $\Phi = (\dot{y}^I)$, and setting $I_0 = (1, 2, \dots, k)$, we get the image of this mapping as a subset of $(\tau^r)^{-1}(V)$, defined by the equations $y^{k+1} = 0, y^{k+2} = 0, \dots, y^m = 0, \dot{y}^{I_0} = 1, \dot{y}^I = 0, I \neq I_0$.

Lemma 4. *Let (V, ψ) , $\psi = (y^\sigma)$, and $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{y}^\sigma)$, be two charts on Y , adapted to S , such that $V \cap \bar{V} \neq \emptyset$.*

(1) *The canonical sections along S satisfy*

$$\Lambda^k \bar{\psi} = \det \left(\frac{\partial y^i}{\partial \bar{y}^j} \right)_{\bar{\psi}(y)} \Lambda^k \psi.$$

(2) *The differential forms dy^σ and $d\dot{y}^I$ satisfy $(\Lambda^k \psi)^* d\dot{y}^i = d\dot{y}^i$, $1 \leq i \leq k$, $(\Lambda^k \psi)^* d\dot{y}^v = 0$, $k+1 \leq v \leq m$, $(\Lambda^k \psi)^* d\dot{y}^I = 0$. In particular, on the set $V \cap \bar{V}$,*

$$\begin{aligned} & (\Lambda^k \bar{\psi})^* d\bar{y}^1 \wedge d\bar{y}^2 \wedge \dots \wedge d\bar{y}^k \\ &= \det \left(\frac{\partial \bar{y}^i}{\partial y^j} \right)_{\psi(y)} (\Lambda^k \psi)^* dy^1 \wedge dy^2 \wedge \dots \wedge dy^k. \end{aligned} \quad (3.3)$$

4. GRASSMANN FIBRATIONS

Consider the vector bundle $\Lambda^k TY$ and the subset $\Lambda_0^k TY \subset \Lambda^k TY$, consisted of *non-zero* k -vectors. We have an equivalence relation on $\Lambda_0^k TY$ “ $\mathcal{E}_1$ is equivalent with $\mathcal{E}_2$, if there exists a real number $\lambda > 0$ such that $\mathcal{E}_1 = \lambda \mathcal{E}_2$ ”. The quotient set has the structure of a fibration over Y , called the *Grassmann fibration* of degree k , and is denoted by $G^k Y$.

To describe the structure of the set $G^k Y$, we proceed similarly as in the case of classical projective spaces. If in a chart (V, ψ) , $\psi = (y^\sigma)$,

$$\mathcal{E}_i = \frac{1}{k!} \mathcal{E}_i^{\sigma_1 \sigma_2 \dots \sigma_k} \left(\frac{\partial}{\partial y^{\sigma_1}} \right)_y \wedge \left(\frac{\partial}{\partial y^{\sigma_2}} \right)_y \wedge \dots \wedge \left(\frac{\partial}{\partial y^{\sigma_k}} \right)_y, \quad i = 1, 2,$$

are two nonzero k -vectors, then $\mathcal{E}_1$ is equivalent with $\mathcal{E}_2$ if and only if in this chart, $\mathcal{E}_1^{\sigma_1 \sigma_2 \dots \sigma_k} = \lambda \mathcal{E}_2^{\sigma_1 \sigma_2 \dots \sigma_k}$ for some $\lambda > 0$ and all $\sigma_1, \sigma_2, \dots, \sigma_k$. We denote $V^{v_1 v_2 \dots v_k} = \{\mathcal{E} \in (\tau^k)^{-1}(V) \mid \mathcal{E}^{v_1 v_2 \dots v_k} > 0\}$. Then, a k -vector belonging to the set $V^{v_1 v_2 \dots v_k} \subset \Lambda_0^k T Y$ can be expressed by

$$\begin{aligned} \mathcal{E} &= \mathcal{E}^{v_1 v_2 \dots v_k} \left(\frac{\partial}{\partial y^{v_1}} \right)_y \wedge \left(\frac{\partial}{\partial y^{v_2}} \right)_y \wedge \dots \wedge \left(\frac{\partial}{\partial y^{v_k}} \right)_y \\ &+ \frac{1}{k!} \sum_{(\tau_1 \tau_2 \dots \tau_k) \neq (v_1 v_2 \dots v_k)} \mathcal{E}^{\tau_1 \tau_2 \dots \tau_k} \left(\frac{\partial}{\partial y^{\tau_1}} \right)_y \wedge \left(\frac{\partial}{\partial y^{\tau_2}} \right)_y \wedge \dots \wedge \left(\frac{\partial}{\partial y^{\tau_k}} \right)_y \end{aligned}$$

(no summation through $v_1, v_2, \dots, v_k$). Denoting by $\text{sgn } \mathcal{E}^{v_1 v_2 \dots v_k}$ the sign of the component $\mathcal{E}^{v_1 v_2 \dots v_k}$, we can write $\mathcal{E}^{v_1 v_2 \dots v_k} = \text{sgn } \mathcal{E}^{v_1 v_2 \dots v_k} \cdot |\mathcal{E}^{v_1 v_2 \dots v_k}|$ and

$$\begin{aligned} \mathcal{E} &= \text{sgn } \mathcal{E}^{v_1 v_2 \dots v_k} \cdot |\mathcal{E}^{v_1 v_2 \dots v_k}| \left(\frac{\partial}{\partial y^{v_1}} \right)_y \wedge \left(\frac{\partial}{\partial y^{v_2}} \right)_y \wedge \dots \wedge \left(\frac{\partial}{\partial y^{v_k}} \right)_y \\ &+ \frac{|\mathcal{E}^{v_1 v_2 \dots v_k}|}{k!} \sum \frac{\mathcal{E}^{\tau_1 \tau_2 \dots \tau_k}}{|\mathcal{E}^{v_1 v_2 \dots v_k}|} \left(\frac{\partial}{\partial y^{\tau_1}} \right)_y \wedge \left(\frac{\partial}{\partial y^{\tau_2}} \right)_y \wedge \dots \wedge \left(\frac{\partial}{\partial y^{\tau_k}} \right)_y, \end{aligned}$$

with the summation through $(\tau_1 \tau_2 \dots \tau_k) \neq (v_1 v_2 \dots v_k)$. But $\text{sgn } \mathcal{E}^{v_1 v_2 \dots v_k} = 1$, so we see the class of $\mathcal{E}$ can be represented as

$$\begin{aligned} [\mathcal{E}] &= \left(\frac{\partial}{\partial y^{v_1}} \right)_y \wedge \left(\frac{\partial}{\partial y^{v_2}} \right)_y \wedge \dots \wedge \left(\frac{\partial}{\partial y^{v_k}} \right)_y \\ &+ \frac{1}{k!} \sum \frac{\mathcal{E}^{\tau_1 \tau_2 \dots \tau_k}}{\mathcal{E}^{v_1 v_2 \dots v_k}} \left(\frac{\partial}{\partial y^{\tau_1}} \right)_y \wedge \left(\frac{\partial}{\partial y^{\tau_2}} \right)_y \wedge \dots \wedge \left(\frac{\partial}{\partial y^{\tau_k}} \right)_y. \end{aligned}$$

We set for any $\mathcal{E} \in V^{v_1 v_2 \dots v_k}$

$$\begin{aligned} w^\sigma(\mathcal{E}) &= y^\sigma(\mathcal{E}), \quad w^{v_1 v_2 \dots v_k}(\mathcal{E}) = \dot{y}^{v_1 v_2 \dots v_k}(\mathcal{E}), \\ w^{\sigma_1 \sigma_2 \dots \sigma_k}(\mathcal{E}) &= \frac{\dot{y}^{\sigma_1 \sigma_2 \dots \sigma_k}(\mathcal{E})}{\dot{y}^{v_1 v_2 \dots v_k}(\mathcal{E})}, \quad (\sigma_1 \sigma_2 \dots \sigma_k) \neq (v_1 v_2 \dots v_k). \end{aligned} \quad (4.1)$$

The pair $(V^{v_1 v_2 \dots v_k}, \Psi^{v_1 v_2 \dots v_k})$, $\Psi^{v_1 v_2 \dots v_k} = (w^\sigma, w^{v_1 v_2 \dots v_k}, w^{\sigma_1 \sigma_2 \dots \sigma_k})$, where the indices satisfy $(\sigma_1 \sigma_2 \dots \sigma_k) \neq (v_1 v_2 \dots v_k)$, is a chart on $\Lambda_0^k T Y$; we call this chart $(v_1 v_2 \dots v_k)$ -associated with (V, ψ) . The pair $(V^{v_1 v_2 \dots v_k}, W^{v_1 v_2 \dots v_k})$, $W^{v_1 v_2 \dots v_k} = (w^\sigma, w^{\sigma_1 \sigma_2 \dots \sigma_k})$, $(\sigma_1 \sigma_2 \dots \sigma_k) \neq (v_1 v_2 \dots v_k)$, is a fibred chart on

$G^k Y$. Writing formulas (4.1) in a different way, we have the transformation equations

$$w^\sigma = y^\sigma, \quad w^{v_1 v_2 \dots v_k} = \dot{y}^{v_1 v_2 \dots v_k}, \quad w^{\sigma_1 \sigma_2 \dots \sigma_k} = \frac{\dot{y}^{\sigma_1 \sigma_2 \dots \sigma_k}}{\dot{y}^{v_1 v_2 \dots v_k}}.$$

The projection $\kappa^k : \Lambda^k T Y \rightarrow G^k Y$ of $\Lambda^k T Y$ onto $G^k Y$ is the Cartesian projection $(w^\sigma, w^{v_1 v_2 \dots v_k}, w^{\sigma_1 \sigma_2 \dots \sigma_k}) \rightarrow (w^\sigma, w^{\sigma_1 \sigma_2 \dots \sigma_k})$. Combining $\Lambda^k(\psi^{-1} \iota_{k,m})$ and κ^k we get the canonical lift of $\psi^{-1} \iota_{k,m}$ to the Grassmann fibration,

$$G^k(\psi^{-1} \iota_{k,m}) = \kappa^k \circ \Lambda^k(\psi^{-1} \iota_{k,m}). \quad (4.2)$$

Lemma 5. *Let (V, ψ) , $\psi = (y^\sigma)$, and $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{y}^\sigma)$, be two rectangle charts, adapted to S at a point $y \in Y$. Suppose that (V, ψ) and $(\bar{V}, \bar{\psi})$ are consistently oriented. Then*

$$G_k(\bar{\psi}^{-1} \iota_{k,m}) = G^k(\psi^{-1} \iota_{k,m}). \quad (4.3)$$

We set

$$G^k S = \{[\mathcal{E}] \in G^k Y \mid [\mathcal{E}] = G^k(\psi^{-1} \iota_{k,m})(\text{pr}_{m,k} \psi(y)), y \in S\}. \quad (4.4)$$

To a given chart (V, ψ) , $\psi = (y^\sigma)$, we associate the induced chart $((\tau^k)^{-1}(V), \Phi)$, $\Phi = (y^\sigma, \dot{y}^{\sigma_1 \sigma_2 \dots \sigma_k})$, on $\Lambda^k T Y$; the associated charts on the Grassmann fibration $G^k Y$ are $(V_0^{v_1 v_2 \dots v_k}, W^{v_1 v_2 \dots v_k})$,

$$W^{v_1 v_2 \dots v_k} = (w^\sigma, w^{\sigma_1 \sigma_2 \dots \sigma_k}),$$

with $(\sigma_1 \sigma_2 \dots \sigma_k) \neq (v_1 v_2 \dots v_k)$. Then, it is easily seen that each of the charts $(V_0^{v_1 v_2 \dots v_k}, W^{v_1 v_2 \dots v_k})$ is adapted to the submanifold $G^k S$.

Theorem 1. *Suppose S is oriented. Then, the subset $G^k S$ of the Grassmann fibration $G^k Y$ is a k -dimensional oriented submanifold, diffeomorphic with S .*

Theorem 1 allows us to integrate over k -dimensional submanifolds of Y directly on the Grassmann fibration $G^k Y$.

5. VARIATIONAL FUNCTIONALS DEPENDING ON SUBMANIFOLDS

As before, we write $G^k S$ (resp., $G^k \Omega$) for the canonical lift of a k -dimensional submanifold $S \subset Y$ (resp., k -piece $\Omega \subset Y$) to the Grassmann fibration $G^k Y$. Denote by $\Gamma^k Y$ the set of all k -pieces Ω of the manifold Y .

Let η be a k -form on the Grassmann fibration $G^k Y$. The form η defines the *variational functional*

$$\Gamma^k Y \ni \Omega \rightarrow \eta_\Omega(S) = \int_{G^k \Omega} \eta \in \mathbb{R}. \quad (5.1)$$

We roughly describe in this paper this construction for $k = 1$, representing variational functionals of *Finsler geometry* in terms of differential forms (cf. Urban and

Krupka [7]). Consider the tangent bundle $\Lambda^1 TY = TY$, a chart (V, ψ) , $\psi = (y^\sigma)$, on Y , and the associated chart $(\tau^1)^{-1}(V)$, $\Psi = (y^\sigma, \dot{y}^\sigma)$, on TY . A function $F : TY \rightarrow \mathbb{R}$ satisfies the *homogeneity condition*, if it satisfies

$$F(\lambda\xi) = \lambda F(\xi)$$

for all tangent vectors ξ and every positive $\lambda \in \mathbb{R}$. The same can be stated in coordinates, requiring that

$$F(y^\nu, \lambda \dot{y}^\nu) = \lambda F(y^\nu, \dot{y}^\nu).$$

Theorem 2.

(1) For any function $F : TY \rightarrow \mathbb{R}$, the chart expressions

$$\eta = \frac{\partial F}{\partial \dot{y}^\nu} dy^\nu \quad (5.2)$$

define a global 1-form on TY .

(2) If F satisfies the homogeneity condition, then η is projectable on the Grassmann fibration $G^1 TY$.

(3) If F satisfies the homogeneity condition, then, for any curve $\zeta : I \rightarrow Y$

$$(\Lambda^1 \zeta) * \eta = (F \circ \Lambda^1 \zeta) dt. \quad (5.3)$$

The form η (5.2) is known as the *Hilbert form* (Chern, Chen and Lam [1], Crampin and Saunders [2]). Theorem 2 (2) characterizes its basic property when F is positive homogeneous: namely, in this case the Hilbert form is defined on the *Grassmann fibration* $G^1 TY$. One can also easily verify that η is a special case of the *Lepage-Cartan form*. This fact completely determines the behaviour of the variational functional (5.1) under variations of submanifolds, extremal submanifolds, and their invariance properties.

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BUNDLES OF (p, A) -COVELOCITIES AND (p, A) -JETS

JIRÍ TOMÁŠ

Abstract. Let M be an m -dimensional manifold, $P^r M$ the frame bundle, $A = \mathbb{D}_k^r/I = \mathbb{R} \oplus N_A$ a Weil algebra and $p : G_k^r \rightarrow \text{Aut } A$ a Lie group homomorphism. For a Lie subgroup $G_{p,k}^r \subseteq G_k^r$ and for $m \geq k$, we define the concept of a (p, A) -covelocality extending restrictions $T_x^A f|_{\text{Orb}(G_{p,k}^r, j^A \varphi_x)}$ of A -covelocities to $G_{p,k}^r$ -orbits with respect to the left action $\ell : G_m^r \times N_A \rightarrow N_A$ determined by p . Further, we introduce bundle functors J_p^A of (p, A) -jets defined on $\mathcal{M}f_m \times \mathcal{M}f$ and give their geometrical description.

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1. PRELIMINARIES

We give a contribution concerning the geometry of Weil functors and their morphisms. The starting points are the concepts of an r -jet, a jet space $J^r(M, N)$, and a bundle functor, all defined in [9]. As usual, we denote by M an m -dimensional manifold and by N a smooth manifold. All manifolds are considered to be C^∞ -manifolds. We denote by $\mathcal{M}f_m$ the category of m -dimensional manifolds with local diffeomorphisms, by $\mathcal{M}f$ the category of smooth manifolds and smooth maps and by $\mathcal{F}\mathcal{M}$ the category of fibered manifolds with smooth fibered maps. Analogously, we denote by $\mathcal{F}\mathcal{M}_m$ the category of fibered manifolds with m -dimensional bases together with fibered maps covering local diffeomorphisms. Recall that a bundle functor defined on the category $\mathcal{M}f_m$ is said to be a natural bundle [9]. Further, we recall the jet functor defined on the category $\mathcal{M}f_m \times \mathcal{M}f$ which assigns the space of r -jets $J^r(M, N)$ of smooth maps $M \rightarrow N$ to any pair $(M, N) \in \text{Obj}(\mathcal{M}f_m \times \mathcal{M}f)$ and the map $J^r(g, h) : J^r(M_1, N_1) \rightarrow J^r(M_2, N_2)$ defined by $j_x^r f \mapsto j_{f(x)}^r h \circ j_x^r f \circ (j_x^r g)^{-1}$ to any pair $(g, h) \in \text{Morph}(\mathcal{M}f_m \times \mathcal{M}f)$.

Among bundle functors, we recall the product preserving bundle functors defined on the category $\mathcal{M}f$. The classical result by Kainz, Michor and others [6, 7, 9] reads that such functors coincide with Weil functors. Denoting them by T^A , we involve the associated Weil algebras to their notation. The restriction of a Weil functor to $\mathcal{M}f_m$ is

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said to be a Weil bundle. There are many authors studying their geometry, e. g., Kolář, Mikulski, Shurygin, Bushueva and many others ([5, 8–10, 14, 15]). Further, there are authors applying and connecting Weil functors with problems from the theory of Lie groups, e. g., Alonso, Muriel, and Muñoz Rodríguez [1, 2, 12]. Further, there are papers applying Weil functors in the theory of jets, e. g., Kureš [11], who also studies their applications in the theoretical mechanics.

From the algebraic point of view, Weil functors are studied by Bertram (e. g. [3, 4]) who generalizes their definition substituting general fields and rings for reals in the definition of the associated Weil algebra. Weil functors are also applied in the research of geometrical categories (e. g., Nishimura [13]).

Let us resume briefly the elementary concepts of the Weil theory. Consider the algebra $\mathcal{E}(k)$ of germs of smooth functions defined on $\mathbb{R}^k$ at zero. A Weil algebra can be defined either as $A = \mathbb{R} \oplus N_A$ for the ideal N_A of nilpotent elements (the so called nilpotent ideal) or as a quotient $A = \mathcal{E}(k)/I$ by an ideal I of finite codimension. It can also be defined as a quotient $A = \mathbb{D}_k^r/J$ of the so called jet algebra $\mathbb{D}_k^r$ or, in other words, as the algebra of polynomials of k indeterminates of order at most r factorized by some of its ideal J . Finally, we define $\text{width}(A)$ as $\dim(N_A/N_A^2)$ and $\text{height}(A)$ as the minimal r for which $A = \mathbb{D}_k^r/J$.

We apply the covariant approach to the definition of Weil functors. This comes out from the I -factorization of germs as follows. Two germs $\text{germ}_0 g : \mathbb{R}_0^k \rightarrow M$ and $\text{germ}_0 h : \mathbb{R}_0^k \rightarrow M$ taking the same value x at $0 \in \mathbb{R}^k$ are said to be I -equivalent if and only if $\text{germ}_x \gamma \circ \text{germ}_0 g - \text{germ}_x \gamma \circ \text{germ}_0 h \in I$ for any function $\gamma : M \rightarrow \mathbb{R}$ defined near x . Classes of such equivalence relation denoted by $j^A g$ form the space $T^A M$. For a smooth map $\varphi : M \rightarrow N$, the map $T^A \varphi : T^A M \rightarrow T^A N$ is defined by $T^A \varphi(j^A g) := j^A(\varphi \circ g)$.

We recall that natural transformations $\tilde{t}_M : T^B M \rightarrow T^A M$ bijectively correspond to homomorphisms $t : B \rightarrow A$ and consequently to the so-called B -admissible A -velocities defined in [8] as follows. Let $A = \mathcal{E}(k)/I$ and $B = \mathcal{E}(p)/J$ be Weil algebras considered as quotients of germ algebras. For a smooth map $f : \mathbb{R}_0^p \rightarrow \mathbb{R}_0^k$ an A -velocity $j^A f$ is said to be B -admissible if and only if the condition of admissibility

$$\text{germ}_0 \varphi \in J \Rightarrow \text{germ}_0(\varphi \circ f) \in I \quad (1.1)$$

is satisfied for every smooth function $\varphi : \mathbb{R}^p \rightarrow \mathbb{R}$. Further, every B -admissible A -velocity $j^A f$ is assigned bijectively a natural transformation $\tilde{t}_M : T^B M \rightarrow T^A M$ defined by $\tilde{t}_M(j^B \varphi) = t_M^{j^A f}(j^B \varphi) = j^A(\varphi \circ f)$. It follows that automorphisms of A are determined by reparametrizations of indeterminates $\tau_1, \dots, \tau_k \in \mathbb{D}_k^r$ satisfying the admissibility condition (1.1).

Let $G_m^r = \text{inv } J_0^r(\mathbb{R}^m, \mathbb{R}^m)_0$ with the composition of jets be a general jet group. Recall the identification of G_m^r with the group $\text{Aut}(\mathbb{D}_m^r)$ of Weil algebra automorphisms assigning to every $j_0^r g \in G_m^r$ the automorphism of $\mathbb{D}_m^r$ defined by $j_0^r \eta \mapsto$

$j_0^r \eta \circ (j_0^r g)^{-1}$ for any $j_0^r \eta \in \mathbb{D}_m^r$. For a Weil algebra $A = \mathbb{D}_m^r/I$ and the projection homomorphism $p : \mathbb{D}_m^r \rightarrow A$, Alonso defined the subgroup $G^A \subseteq G_m^r \simeq \text{Aut}(\mathbb{D}_m^r)$ and its normal subgroup G_A ([1]) as follows

$$\begin{aligned} G_A &= \{j_0^r g \in G_m^r; p \circ j_0^r g = p\}, \\ G^A &= \{j_0^r g \in G_m^r; \text{Ker}(p \circ j_0^r g) = \text{Ker}(p)\}. \end{aligned} \quad (1.3)$$

He also proved the identification $G^A/G_A \simeq \text{Aut } A$ of Lie groups.

In [16], we have defined the spaces $T^{A*}M$ of A -covelocities on M by $T^{A*}M = \{T_x^A f : T_x^A M \rightarrow T_0^A \mathbb{R} \simeq N_A; x \in M\}$ and for local diffeomorphisms $g : M \rightarrow N$ the maps $T^{A*}g : T^{A*}M \rightarrow T^{A*}N$ by $T^{A*}g(T_x^A f) = T_x^A f \circ (T_x^A g)^{-1}$. For $A = \mathbb{D}_k^r$, we write $T_k^{r*}M$ and $T_k^{r*}g$ (see [9]). In [16], we have proved the following statement.

- (a) *Let $A = \mathbb{D}_k^r/I$ and $m = \dim M \geq k$. Then, the spaces $T^{A*}M$ with the maps $T^{A*}g$ form the natural bundle $P^r M[N_m^r, \ell]$ identified with $T_m^{r*}M$ where N_m^r is the nilpotent ideal of $\mathbb{D}_m^r$ and $\ell : G_m^r \times N_m^r$ is the left action defined by $\ell(j_0^r g, j_0^r \eta) = j_0^r \eta \circ (j_0^r g)^{-1}$*

For any $(M, N) \in \text{Obj}(\mathcal{M}f_m) \times \text{Obj}(\mathcal{M}f)$, define $J^A(M, N) = \{T_x^A f; f : M \rightarrow N\}$. For a local diffeomorphism $g : M_1 \rightarrow M_2$ and a smooth map $h : N_1 \rightarrow N_2$, define the map $J^A(g, h) : J^A(M_1, N_1) \rightarrow J^A(M_2, N_2)$ by

$$J^A(g, h)(T_x^A f) = T_{f(x)}^A h \circ T_x^A f \circ (T_x^A g)^{-1}.$$

Then, it holds

- (b) *For $m \geq k$ and $A = \mathbb{D}_k^r/I$ the space $J^A(M, N)$ is identified with $J^r(M, N)$.*

Proof. For any $M \in \text{Obj}(\mathcal{M}f_m)$, consider the bundle functor $G_M : \mathcal{M}f \rightarrow \mathcal{F}\mathcal{M}$ defined by $G_M N = J^r(M, N)$ on objects and by $G_M h = J^r(\text{id}_M, h)$ on morphisms.

The assertion (a) implies

$$J^r(M, N) = G_M N = P^r N[(T_k^{r*}M)^n, \ell_M] = P^r N[(T^{A*}M)^n, \ell_M],$$

where $\ell_M : G_m^r \times (T_k^{r*}M)^n \rightarrow (T_k^{r*}M)^n$ is defined by $\ell_M(j_0^r g, (T_k^r)_x f) = (T_k^r)_0 g \circ (T_k^r)_x f = T_0^A g \circ T_x^A f$. It follows $J^r(M, N) = J^A(M, N)$. For morphisms, we have $J^A(g, h)(T_x^A f) = T_{f(x)}^A (h \circ f \circ g^{-1}) = j_{g(x)}^r (h \circ f \circ g^{-1}) = J^r(g, h)(j_x^r f)$ if we denote by g^{-1} the inverse to g considered near x . $\square$

2. BUNDLES OF (p, A) -COVELOCITIES AND (p, A) -JETS

Let $\hat{p} : G_k^r \rightarrow G^A$ be a G_A -stabilizing Lie group homomorphism and $p : G_k^r \rightarrow \text{Aut}(A)$ its corestriction in the sense of (1.3). For $m \geq k$, we define and investigate the spaces $T_p^{A*}M$ of (p, A) -covelocities.

For the present, suppose $m = k$. Consider the left action ℓ of G^A with its effective left action $\bar{\ell}$ of $\text{Aut } A$ on $T_x^A M$

$$\ell(j_0^r g, j^A \varphi_x) = \bar{\ell}(\pi(j_0^r g), j^A \varphi_x) = j^A \varphi_x \circ (j_0^r g)^{-1} \quad (2.1)$$

if $\pi : G^A \rightarrow \text{Aut } A$ denotes the projection Lie group homomorphism. For a Lie subgroup $H \subseteq \text{Aut } A$, let us denote by $\text{Orb}(H, j^A \varphi_x)$ the H -orbit of $j^A \varphi_x$ with respect to $\bar{\ell}$ restricted to $H \times T_x^A M$. Since the elements of $\text{Aut } A$ determine natural transformations over $T^A M$ determined by (2.1), the values of $T_x^A f$ on the whole $\text{Orb}(\text{Aut } A, j^A \varphi_x)$ are determined by the value $T_x^A f(j^A \varphi_x)$. Indeed, we have

$$\begin{aligned} T_x^A f \circ \ell(j_0^r g, j^A \varphi_x) &= T_x^A f \circ \bar{\ell}(\pi(j_0^r g), j^A \varphi_x) = \\ &= \bar{\ell}(\pi(j_0^r g), T_x^A f(j^A \varphi_x)) = \ell(j_0^r g, T_x^A f(j^A \varphi_x)). \end{aligned} \quad (2.2)$$

We are searching for the greatest subgroup $H \subseteq \text{Aut } A$ satisfying the existence of a map $X : \text{reg } T_x^A M \rightarrow N_A$ prolongating for $j^A \varphi_x \in \text{reg } T_x^A M$ the restrictions $T_x^A f|_{\text{Orb}(H, j^A \varphi_x)}$ in the sense of the following formula

$$X(j^A \psi_x) = T_x^A \hat{f}(j^A \psi_x) = T_x^A f(j^A \varphi_x) \circ p((j_0^r \hat{\varphi}_x)^{-1} \circ j_0^r \hat{\psi}_x) \quad (2.3)$$

for suitable $T_x^A \hat{f} \in T_x^{A*} M$ and any $j_0^r \hat{\varphi}_x \in j^A \varphi_x$ and $j_0^r \hat{\psi}_x \in j^A \psi_x$. It is easy to see that

$$j^A \varphi_{x,1} = j^A \varphi_{x,2} \text{ implies } (j_0^r \varphi_{x,1})^{-1} \circ j_0^r \varphi_{x,2} \in G_A \quad (2.4)$$

and thus, the formula (2.3) is correct.

It follows from (2.2) and (2.3) that $H \subseteq p(G_k^r)$. Insisting on the condition $H = p(G_k^r)$, one deduces easily that (2.3) works only if an additional assumption concerning p is satisfied. It reads as follows

$$\pi \circ (\hat{p} \circ \hat{p})(j_0^r g) = p(j_0^r g) \text{ for any } \hat{p} \in p. \quad (2.5)$$

We remark that this case leads to the concept of a p -vertical A -covelocality, which was presented in [16].

Coming back to (p, A) -covelocities, we define the Lie subgroups

$$\begin{aligned} \bar{G}_k^{r,p} &= \{j_0^r g \in G_k^r, \pi \circ (\hat{p} \circ \hat{p})(j_0^r g) = p(j_0^r g) \text{ for any } \hat{p} \in p\}, \\ G_k^{r,p} &= \bar{G}_k^{r,p} \cap p(G_k^r) \end{aligned} \quad (2.6)$$

of G_k^r determined by $p : G_k^r \rightarrow \text{Aut } A$ under discussion. We put $H = G_k^{r,p}$.

On the other hand, a (p, A) -covelocality will be a map $X : \text{reg } T_x^A M \rightarrow N_A$ sharing the same value with some A -covelocality over a selected element from $\text{reg } T_x^A M$ and uniquely extended to the whole $\text{reg } T_x^A M$ by (2.3). We give a definition connecting both of those approaches.

Definition 1. Let $A = \mathbb{D}_k^r / I$, $p : G_k^r \rightarrow \text{Aut } A$ be a Lie group homomorphism and $k = m = \dim M$. A map $X : \text{reg } T_x^A M \rightarrow T_0^A \mathbb{R} = N_A$ is said to be a

(p, A) -covelocality at $x \in M$, i. e., an element of $(T_p^{A*})_x M$ if some of the following conditions is satisfied

- (i) for every $j^A \varphi_x \in \text{reg } T_x^A M$, there is a covelocity $T_x^A f \in T_x^{A*} M$ satisfying $X(j^A \varphi_x) = T_x^A f(j^A \varphi_x)$ and for any $j^A \psi_x \in \text{reg } T_x^A M$, the value $X(j^A \psi_x)$ is obtained from $X(j^A \varphi_x)$ by the following extension

$$X(j^A \psi_x) = X(j^A \varphi_x) \circ p((j_0^r \hat{\varphi}_x)^{-1} \circ j_0^r \hat{\psi}_x), \quad j_0^r \hat{\varphi}_x \in j^A \varphi_x \\ \text{and } j_0^r \hat{\psi}_x \in j^A \psi_x. \quad (2.7)$$

- (ii) for every $j^A \varphi_x \in \text{reg } T_x^A M$, there is $T_x^A f \in T_x^{A*} M$ such that the restrictions of X and $T_x^A f$ to $\text{Orb}(G_k^{r,p}, j^A \varphi_x)$ coincide and the extensions of X to other $G_k^{r,p}$ -orbits of $\text{reg } T_x^A M$ are obtained by (2.7).

Then, $T_p^{A*} M = \cup_{x \in M} (T_p^{A*})_x M$ is said to be the space of (p, A) -covelocities on M . For a local diffeomorphism $g : M \rightarrow N$, there is a map $T_p^{A*} g : T_p^{A*} M \rightarrow T_p^{A*} N$ defined as follows

$$(T_p^{A*} g)(X) = X \circ (T_x^A g)^{-1} \text{ for } X \in (T_p^{A*})_x M \quad (2.8)$$

We note that (2.4) implies the correctness of (2.7) in the sense of its independency on the choice of $j_0^r \hat{\varphi}_x \in j^A \varphi_x$ and $j_0^r \hat{\psi}_x \in j^A \psi_x$.

It is easy to see that for any $p : G_k^r \rightarrow \text{Aut } A$ satisfying the assumption (2.5), p -vertical A -covelocities coincide with more general objects of (p, A) -covelocities. In other cases, the difference corresponds to the difference between (2.1) and (2.7), namely in the transformations of values within $p(G_k^r)$ -orbits. One can consider the substitution of $p(j_0^r g) = p((j_0^r \varphi_x)^{-1} \circ j_0^r \psi_x)$ for $j_0^r g = (j_0^r \varphi_x)^{-1} \circ j_0^r \psi_x$ in (2.1) as an addition of a deformation within orbits expressed by p .

Proposition 1. For $m = k = \text{width}(A)$, the system of spaces $T_p^{A*} M$ with T_p^{A*} -maps forms the structure of a natural bundle $P^r M[N_A, \ell]$ with the standard fiber formed by N_A . The left action $\ell : G_m^r \times N_A \rightarrow N_A$ is defined as follows

$$\ell(j_0^r g, j^A \alpha) = p_A \circ r(j_0^r \alpha, (p(j_0^r g))^{-1}) = j^A \alpha \circ (p(j_0^r g))^{-1} \quad (2.9)$$

where r is obtained in the obvious way from the standard right action of G_m^r on $P_x^r M$ considered on $M_x = \mathbb{R}_0^m$. Further, $p_A : \mathbb{D}_m^r \rightarrow A$ denotes the projection homomorphism of Weil algebras.

Proof. The identification $H : T_p^{A*} M \rightarrow P^r M[N_A, \ell]$ is given by the assignment $X \mapsto \{j_0^r \alpha_x, X(j_0^r \alpha_x)\}$ and H^{-1} by the assignment $\{j_0^r \alpha_x, j^A \eta\} \mapsto (j_0^r \alpha_x \mapsto j^A \eta)$. We must verify that the map H is well defined. Indeed, we have

$$\{j_0^r \alpha_x \circ j_0^r g, X(j_0^r \alpha_x \circ j_0^r g)\} = \{j_0^r \alpha_x \circ j_0^r g, X(j_0^r \alpha_x) \circ p(j_0^r g)\} \\ = \{r(j_0^r \alpha_x, j_0^r g), \ell((j_0^r g)^{-1}, X(j_0^r \alpha_x))\}.$$

Clearly, $\{j_0^r \alpha_x, j^A \eta\}$ and $\{j_0^r \alpha_x \circ j_0^r g, j^A \eta \circ p(j_0^r g)\}$ determine the same (p, A) -covelocity by its definition.

Furthermore, morphisms of associated bundles $\{P^r g, \text{id}_{N_A}\}$ are identified with maps $T_p^{A*} g$ since every $\{j_0^r \alpha_x, T_x^A f(j_0^r \alpha_x)\}$ is assigned $\{j_0^r (g \circ \alpha_x), T_x^A f(j_0^r \alpha_x)\} = \{j_0^r (g \circ \alpha_x), T_x^A f \circ (T_x^A g)^{-1}(j_0^r (g \circ \alpha_x))\}$. This completes the proof. $\square$

There is an easy extension of the result from $m = k = \text{width } A$ to the cases of $m > k$. We can replace $A = \mathbb{D}_k^r / I$ by $\tilde{A} = \mathbb{D}_m^r / (I \vee \langle \tau_{k+1}, \dots, \tau_m \rangle)$ adding formally $m - k$ indeterminates. As for the subgroups G^A or $G_A \subseteq G_k^r$, replace them by $G^{\tilde{A}}$ and $G_{\tilde{A}} \subset G_m^r$ as follows. For any $j_0^r \tilde{g} \in G_m^r$, put

$$\begin{aligned} j_0^r \tilde{g} \in G^{\tilde{A}} \text{ or } j_0^r \tilde{g} \in G_{\tilde{A}} \text{ if and only if } \text{pr}_1 \circ j_0^r \tilde{g} \circ j_0^r i_1 \in G^A \\ \text{or } \text{pr}_1 \circ j_0^r \tilde{g} \circ j_0^r i_1 \in G_A \text{ and } \text{pr}_2 \circ j_0^r \tilde{g} \circ j_0^r i_1 = j_0^r 0^{m-k} \end{aligned} \quad (2.10)$$

for the zero function on $\mathbb{R}^k$, the inclusion $i_1 : \mathbb{R}^k \rightarrow \mathbb{R}^m$ defined by $(x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0)$ and the projection $\text{pr}_2 : \mathbb{R}^k \times \mathbb{R}^{m-k} \rightarrow \mathbb{R}^{m-k}$.

Let us replace $p : G_k^r \rightarrow \text{Aut}(A)$ by a Lie group homomorphism $\tilde{p} : G_m^r \rightarrow \text{Aut}(\tilde{A})$. Consider any free basis $\mathcal{B}_1$ of generators of $G_k^r \simeq G_k^r \times \{j_0^r \text{id}_{\mathbb{R}^{m-k}}\} \subseteq G_m^r$ and complete it to a free basis of generators $\mathcal{B}$ of G_m^r . Then, we define $\tilde{p} : G_m^r \rightarrow \text{Aut}(\tilde{A})$ by $j_0^r g \times \{j_0^r \text{id}_{\mathbb{R}^{m-k}}\} \mapsto p(j_0^r g)$ for $j_0^r g \in G_k^r$ and by $j_0^r \tilde{g} \mapsto j_0^r \text{id}_{\mathbb{R}^m}$ in case of $j_0^r \tilde{g} \in \mathcal{B} - \mathcal{B}_1$. Further, the left action from (2.9) is replaced by $\ell : G_m^r \times N_{\tilde{A}} \rightarrow N_{\tilde{A}}$, which is defined as follows

$$\ell(j_0^r g, j^{\tilde{A}} \alpha) = \ell(j_0^r g, j^{\tilde{A}} \alpha) \quad (2.11)$$

$$\begin{aligned} &= p_{\tilde{A}} \circ r(j_0^r \alpha, (\tilde{p}(j_0^r g))^{-1}) \\ &= j^{\tilde{A}} \alpha \circ (\tilde{p}(j_0^r g))^{-1}. \end{aligned} \quad (2.12)$$

So, we can modify the extension of (p, A) -covelocity X from $j^A \varphi_x \in \text{reg } T_x^A M$ to another value $j^{\tilde{A}} \psi_x \in \text{reg } T_x^{\tilde{A}} M$ from (2.7) as follows

$$X(j^{\tilde{A}} \psi_x) = X(j^A \varphi_x) \circ (\tilde{p}(j_0^r (\hat{\psi}_x^{-1} \circ \hat{\varphi}_x))^{-1} \quad (2.13)$$

where $j_0^r \hat{\varphi}_x$ and $j_0^r \hat{\psi}_x$ are formally considered as arbitrary elements of $j^{\tilde{A}} \varphi_x$ and $j^{\tilde{A}} \psi_x \in \text{reg } T_x^{\tilde{A}} M$. Since the definition (2.8) of T_p^{A*} -maps remains unchanged, we can reformulate Proposition 1 as follows

Proposition 2. *For $m \geq k$ and a Lie group homomorphism $p : G_k^r \rightarrow \text{Aut}(A)$, the system of spaces $T_p^{A*} M$ of (p, A) -covelocities with their T_p^{A*} -maps is identified with the system of spaces $T_{\tilde{p}}^{\tilde{A}*} M$ with their $T_{\tilde{p}}^{\tilde{A}*}$ -maps and consequently to a natural bundle $T_p^{A*} M = P^r M[N_A, \ell]$ with the standard fiber formed by $N_A \simeq N_A \vee \langle \tau_{k+1}, \dots, \tau_m \rangle$ and the left action ℓ defined by (2.12).*

Let us define the space $J_p^A(M, N)$ of (p, A) -jets.

Definition 2. Let $A = \mathbb{D}_k^r/I$ and $p : G_k^r \rightarrow \text{Aut}(A)$ be as above, $m \geq k = \text{width}(A)$. For $(M, N) \in \text{Obj}(\mathcal{M}f_m) \times \text{Obj}(\mathcal{M}f)$, define $(J_p^A)_x(M, N)_y = \{X : \text{reg } T_x^A M \rightarrow T_y^A N; \text{ for all } j^A \varphi_x \in \text{reg } T_x^A M, \text{ there is } T_x^A f \in J_x^A(M, N)_y \text{ satisfying } X(j^A \varphi_x) = T_x^A f(j^A \varphi_x) \text{ and } X(j^A \psi_x) = X(j^A \varphi_x) \circ \tilde{p}((j_0^r \hat{\varphi}_x)^{-1} \circ j_0^r \hat{\psi}_x) \text{ for any } j_0^r \hat{\varphi}_x \in j^A \varphi_x, j_0^r \hat{\psi}_x \in j^A \psi_x\}$. We define the space $J_p^A(M, N)$ of (p, A) -jets setting

$$J_p^A(M, N) := \bigcup_{x \in M, y \in N} (J_p^A)_x(M, N)_y.$$

For a local diffeomorphism $g : M_1 \rightarrow M_2$ and a smooth map $h : N_1 \rightarrow N_2$, define the map $J^A(g, h) : J_p^A(M_1, N_1) \rightarrow J_p^A(M_2, N_2)$ by

$$J^A(g, h)(X) = T_{f(x)}^A h \circ X \circ (T_x^A g)^{-1}$$

for all $X \in (J_p^A)_x(M, N)_y$.

In the very end, we give the geometrical description of the spaces $J^A(M, N)$. Denote by $\mathcal{NE}(A)$ the group of natural equivalences over T^A identified with the group $\text{Aut}(A)$.

Proposition 3 (Kolář and Mikulski [10]). *Every bundle functor F defined on the product category $\mathcal{M}f_m \times \mathcal{M}f$ of order r in the first factor and preserving products in the second factor is of the form $F(M, N) = P^r M[G^F N, H_N^F] = P^r M[T^A N, H_N^F]$ for a product preserving bundle functor $G^F = T^A$ defined on objects by $G^F N = F_0(\mathbb{R}^m, N) = T^A N$, on morphisms by $G^F f = F_0(\text{id}_{\mathbb{R}^m}, f) = T^A f$ and for a Lie group homomorphism $H^F : G_m^r \rightarrow \mathcal{NE}(A) \simeq \text{Aut}(A)$ defined by $H_N^F(j_0^r g)(a) = F_0(g, \text{id}_N)(a)$ for every $a \in G^F N \simeq T^A N$.*

Proposition 4. *For $m \geq k$, the spaces $J_p^A(M, N)$ with J_p^A -maps form a bundle functor on the product category $\mathcal{M}f_m \times \mathcal{M}f$ satisfying $J_p^A(M, \mathbb{R})_0 = T_p^{A*} M$ and $J_p^A(g, \text{id}_{\mathbb{R}})|_{\text{Obj}(\mathcal{M}f_m) \times \{\mathbb{R}_0\}} = T_p^{A*} g$. Moreover, J_p^A preserves products in the second factor and $J_p^A(M, N) \simeq P^r M[G^F N, H_N^F]$ for $G^F = T^A$ and $H^F \simeq \ell$ for ℓ defined by (2.9) or (2.12).*

Proof. We are searching for a bundle functor of the required kind on J_p^A extending T_p^{A*} . We prove

$$J_p^A(M, N) \simeq P^r N[(T_p^{A*} M)^n, \bar{\ell}] \rightarrow M \times N,$$

where $\bar{\ell} : G_n^r \times (T_p^{A*} M)^n \rightarrow (T_p^{A*} M)^n$ is defined in the following way. Let us identify $(\{j_0^r \alpha_x^1, j^A \theta^1\}, \dots, \{j_0^r \alpha_x^n, j^A \theta^n\}) \in (T_p^{A*} M)_x^n \simeq (P_x^r M[N_A, \ell])^n$ with $\{j_0^r \alpha_x, j^A \eta^1, \dots, j^A \eta^n\} \in P_x^r M \times_{G_m^r} N_A^n$ where $j_0^r \alpha_x \in P_x^r M$ is arbitrary and

$j^A \eta^1, \dots, j^A \eta^n \in N_A$ are obtained by $j^A \eta^i = \ell(j_0^r h_i^{-1}, j^A \theta^i)$ for $j_0^r h_i \in G_m^r$ satisfying $j_0^r \alpha_x = r(j_0^r \alpha_x^i, j_0^r h_i)$. In this identification, we put

$$\begin{aligned} \bar{\ell}(j_0^r g, \{j_0^r \alpha_x, j^A \eta^1, \dots, j^A \eta^n\}) = \\ = \{j_0^r \alpha_x, T_0^A g^1(j^A \eta^1, \dots, j^A \eta^n), \dots, T_0^A g^n(j^A \eta^1, \dots, j^A \eta^n)\}. \end{aligned} \quad (2.14)$$

It is clear that, for any $X \in J_x^A(M, N)_y$ and $j_0^r \gamma_y \in P_y^r N$, we have $(T_0^A \gamma_y)^{-1} X \in ((T_p^{A*})_x M)^n$. Thus, $X \simeq \{j_0^r \gamma_y, (T_0^A \gamma_y)^{-1}(X)\} \in P_y^r N \times_{G_n^r} ((T_p^{A*})_x M)^n$ with respect to the standard right action of G_n^r on $P^r N$ and the left action $\bar{\ell}$ of G_n^r on $(T_p^{A*} M)^n$ given by the composition of maps $T_0^A g : T_0^A \mathbb{R}^n \rightarrow T_0^A \mathbb{R}^n$ determined by $j_0^r g \in G_n^r$ with elements of $(T_p^{A*} M)^n$, which can be also expressed by (2.14).

One can observe easily the smoothness of $\bar{\ell}$ from (2.14). Further, there is the obvious projection $P^r N[(T_p^{A*} M)^n, \bar{\ell}] \rightarrow M \times N$ compatible with that of $J_p^A(M, N) \rightarrow M \times N$, which proves its smoothness.

The maps $J_p^A(g, h)$ coincide with the compositions of the pairs of smooth maps $J_p^A(\text{id}_M, h) \simeq \{P^r h, \text{id}_{(T_p^{A*} M)^n}\}$ and $J_p^A(g, \text{id}_N) \simeq \{\text{id}_{P^r N}, T_p^{A*} g\}$. The verification of the localization condition from the definition of a bundle functor (see [9]) is easy. The last assertion is a direct consequence of Proposition 3. $\square$

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THE THIRD TYPE BUNCH OF CONNECTIONS INDUCED BY AN ANALOG OF NORDEN'S NORMALIZATION FOR THE GRASSMANN-LIKE MANIFOLD OF CENTERED PLANES

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Abstract. The Grassmann-like manifold of centered m -planes (i.e. passing through a fixed point) and the principal bundle over it are considered in n -dimensional projective space. An analogy of Norden's construction of strong normalization induces bunches of connections of three types in the fibering associated to the Grassmann-like manifold which are related to each other. The third type bunch of group connections is constructed and a unique connection from this bunch is distinguished.

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Keywords: projective space, Grassmann-like manifold of centered planes, Norden's normalization, object of group connection, average connection, bunch of connections

In the n -dimensional projective space P_n , we consider the moving frame $\{A, A_I\}$ ($I, \dots = \overline{1, n}$) with the derivation formulas

$$\begin{aligned}dA &= \theta A + \omega^I A_I, \\dA_I &= \theta A_I + \omega_I^J A_J + \omega_I A,\end{aligned}$$

where the form θ plays the role of a proportionality factor and the structure forms $\omega^I, \omega_J^I, \omega_I$ of the projective group $GP(n)$, which acts effectively on P_n , satisfy the Cartan equations

$$\begin{aligned}D\omega^I &= \omega^J \wedge \omega_J^I, \\D\omega_J^I &= \omega_J^K \wedge \omega_K^I + \delta_J^I \omega_K \wedge \omega^K + \omega_J \wedge \omega^I, \\D\omega_I &= \omega_I^J \wedge \omega_J.\end{aligned}$$

We consider the Grassmann-like manifold $\text{Gr}^*(m, n)$ [3] of centered m -dimensional planes. This manifold is given by the equations

$$\omega^a = \Lambda_\alpha^a \omega^\alpha + \Lambda_\alpha^{ab} \omega_b^\alpha \quad (a, \dots = \overline{1, m}; \quad \alpha, \dots = \overline{m+1, n}),$$

where the components of the first order fundamental object $\Lambda = \{\Lambda_\alpha^a, \Lambda_\alpha^{ab}\}$ satisfy the differential equations

$$\begin{aligned}\Delta \Lambda_\alpha^a + \Lambda_\alpha^{ab} \omega_b + \omega_\alpha^a &= \Lambda_{\alpha\beta}^a \omega^\beta + \Lambda_{\alpha\beta}^{ab} \omega_b^\beta, \\ \Delta \Lambda_\alpha^{ab} &= \bar{\Lambda}_{\alpha\beta}^{ab} \omega^\beta + \Lambda_{\alpha\beta}^{abc} \omega_c^\beta.\end{aligned}$$

The principal fiber bundle $G^*(\text{Gr}^*(m, n))$ with the structure equations

$$\begin{aligned}D\omega^\alpha &= \omega^\beta \wedge \omega_\beta^\alpha + \Lambda_\beta^a \omega^\beta \wedge \omega_a^\alpha + \Lambda_\beta^{ab} \omega_b^\beta \wedge \omega_a^\alpha, \\ D\omega_a^\alpha &= \omega_a^b \wedge \omega_b^\alpha + \omega_a^\beta \wedge \omega_\beta^\alpha + \omega_a \wedge \omega^\alpha; \\ D\omega_b^a &= \omega_b^c \wedge \omega_c^a + (\delta_b^a \Lambda_\alpha^c \omega_c + \delta_b^a \omega_\alpha + \Lambda_\alpha^a \omega_b) \wedge \omega^\alpha \\ &\quad + (\delta_b^a \Lambda_\alpha^{ec} \omega_e - \delta_b^c \omega_\alpha^a + \Lambda_\alpha^{ac} \omega_b) \wedge \omega_c^\alpha, \\ D\omega_\beta^\alpha &= \omega_\beta^\gamma \wedge \omega_\gamma^\alpha + (\delta_\beta^\alpha \Lambda_\gamma^a \omega_a + \delta_\beta^\alpha \omega_\gamma + \delta_\gamma^\alpha \omega_\beta) \wedge \omega^\gamma + (\delta_\beta^\alpha \Lambda_\gamma^{ba} \omega_b + \delta_\gamma^\alpha \omega_\beta^a) \wedge \omega_a^\gamma, \\ D\omega_\alpha^a &= \omega_\alpha^b \wedge \omega_b^a + \omega_\alpha^\beta \wedge \omega_\beta^a + (\Lambda_\beta^a \omega_\alpha) \wedge \omega^\beta + (\Lambda_\beta^{ab} \omega_\alpha) \wedge \omega_b^\beta, \\ D\omega_a &= \omega_a^b \wedge \omega_b + \omega_a^\alpha \wedge \omega_\alpha, \\ D\omega_\alpha &= \omega_\alpha^a \wedge \omega_a + \omega_\alpha^\beta \wedge \omega_\beta\end{aligned}$$

is constructed over the manifold $\text{Gr}^*(m, n)$, the stationarity subgroup G^* of the centered plane L_m^* is the typical fiber. In the principal fiber bundle the connection is given in G. F. Laptev's [3] method by the field of the connection object Γ . This object contains the subobject

$$\Gamma_1 = \{\Gamma_{b\alpha}^a, L_{b\alpha}^{ac}, \Gamma_{\beta\gamma}^\alpha, L_{\beta\gamma}^{\alpha a}\}.$$

An analog of strong Norden's normalization for this manifold is carried out. It consists of the fields of the planes C_{n-m-1} and N_{m-1} :

$$L_m^* \cap C_{n-m-1} = \emptyset, \quad A \notin N_{m-1} \subset L_m^*.$$

The planes C_{n-m-1} and N_{m-1} will be set by the points $B_\alpha = A_\alpha + \lambda_\alpha^a A_a + \lambda_\alpha A$ and $B_a = A_a + \lambda_a A$, correspondingly.

The differential equations for the components of the clothing quasitensor $\lambda = \{\lambda_\alpha^a, \lambda_\alpha, \lambda_a\}$ have the form

$$\begin{aligned}\Delta \lambda_\alpha^a + \omega_\alpha^a &= \lambda_{\alpha\beta}^a \omega^\beta + \lambda_{\alpha\beta}^{ab} \omega_b^\beta, \\ \Delta \lambda_\alpha + \lambda_\alpha^a \omega_a + \omega_\alpha &= \lambda_{\alpha\beta} \omega^\beta + \lambda_{\alpha\beta}^a \omega_a^\beta, \\ \Delta \lambda_a + \omega_a &= \lambda_{a\alpha} \omega^\alpha + \lambda_{a\alpha}^b \omega_b^\alpha.\end{aligned}$$

This normalization induces the connections of three types [1, 3] $\overset{01}{\Gamma}, \overset{02}{\Gamma}, \overset{03}{\Gamma}$ in the fibering associated with the manifold $\text{Gr}^*(m, n)$. Moreover,

$$\overset{01}{\Gamma} = \frac{1}{2} \left(\overset{02}{\Gamma} + \overset{03}{\Gamma} \right).$$

Thus the first type connection is average [4] with respect to the connections of the remaining types.

In [2] it is proved that this clothing of the manifold $\text{Gr}^*(m, n)$ induces the first type bunch of connections in the fibering $G^*(\text{Gr}^*(m, n))$. From this bunch a unique first type connection is distinguished. In [1], because the covariant derivatives are tensors, we construct the second type bunch of connections. A unique second type connection is distinguished from this bunch.

These bunches of connections [1, 2] have identical parameters. These parameters are the components of the subobject Γ_1 . In consideration of the correlation between the group connections of three types, we can construct the third type bunch of connections with the object

$$\overset{3}{\Gamma} = 2\overset{1}{\Gamma} - \overset{2}{\Gamma}.$$

The dependent components of this bunch are

$$\begin{aligned} \overset{3}{\Gamma}_{\alpha\beta}^a &= \lambda_\gamma^a \Gamma_{\alpha\beta}^\gamma - \lambda_\alpha^b \Gamma_{b\beta}^a - \lambda_{\alpha\beta}^a + 2\lambda_\alpha \mu_\beta^a, \\ \overset{3}{L}_{\alpha\beta}^{ab} &= \lambda_\gamma^a L_{\alpha\beta}^{\gamma b} - \lambda_\alpha^c L_{c\beta}^{ab} - \lambda_{\alpha\beta}^{ab} + 2\lambda_\alpha^b \lambda_\beta^a - 2\lambda_\alpha \Lambda_\beta^{ab}, \\ \overset{3}{L}_{a\alpha} &= \lambda_b \Gamma_{a\alpha}^b + 2\lambda_a \lambda_b \Lambda_\alpha^b + 2\lambda_a \eta_\alpha - \lambda_{a\alpha}, \\ \overset{3}{\Pi}_{a\alpha}^b &= \lambda_c L_{a\alpha}^{cb} + 2\lambda_a \lambda_c \Lambda_\alpha^{cb} + 2\delta_a^b \eta_\alpha - \lambda_{a\alpha}^b, \\ \overset{3}{L}_{\alpha\beta} &= \lambda_\gamma \Gamma_{\alpha\beta}^\gamma - \lambda_\alpha^a \lambda_b \Gamma_{a\beta}^b + 2\lambda_\alpha \lambda_\beta - \lambda_{\alpha\beta} + \lambda_\alpha^a \lambda_{a\beta} - 2\lambda_\alpha^a \lambda_a \eta_\beta - 2\lambda_\alpha^a \Lambda_\beta^b \lambda_a \lambda_b, \\ \overset{3}{\Pi}_{\alpha\beta}^a &= \lambda_\gamma L_{\alpha\beta}^{\gamma a} - \lambda_\alpha^b \lambda_c L_{b\beta}^{ca} - 2\lambda_\alpha^b \lambda_b \lambda_c \Lambda_\beta^{ca} + 2\lambda_\alpha^a \lambda_b \lambda_\beta^b - \chi_{\alpha\beta}^a + \lambda_\alpha^b \lambda_{b\beta}^a, \end{aligned}$$

where

$$\begin{aligned} \mu_\alpha^a &= \lambda_\alpha^a - \Lambda_\alpha^a, \\ \eta_\alpha &= \lambda_\alpha - \lambda_\alpha^a \lambda_a. \end{aligned}$$

Taking into account the covered subobject $\overset{0}{\Gamma}_1$ [3]

$$\begin{aligned} \overset{0}{\Gamma}_{b\alpha}^a &= -\delta_b^a \lambda_\alpha + \mu_\alpha^a \lambda_b + \delta_b^a \mu_\alpha^c \lambda_c, & \overset{0}{L}_{b\alpha}^{ac} &= \delta_b^c \lambda_\alpha^a - (\delta_b^a \Lambda_\alpha^{ec} + \delta_b^e \Lambda_\alpha^{ac}) \lambda_e, \\ \overset{0}{\Gamma}_{\beta\gamma}^\alpha &= -\delta_\gamma^\alpha \lambda_\beta - \delta_\beta^\alpha \lambda_\gamma + \delta_\beta^\alpha \mu_\gamma^a \lambda_a, & \overset{0}{L}_{\beta\gamma}^{\alpha a} &= -\delta_\gamma^\alpha \lambda_\beta^a - \delta_\beta^\alpha \Lambda_\gamma^{ba} \lambda_b, \end{aligned}$$

we have a unique connection in the third type bunch of connections:

$$\begin{aligned}
\overset{03}{\Gamma}_{\alpha\beta}^a &= -\lambda_{\alpha\beta}^a - \mu_{\beta}^a \lambda_{\alpha}^b \lambda_b + \mu_{\beta}^a \lambda_{\alpha} - \Lambda_{\beta}^a \lambda_{\alpha}, \\
\overset{03}{L}_{\alpha\beta}^{ab} &= -\lambda_{\alpha\beta}^{ab} + \Lambda_{\beta}^{ab} \lambda_{\alpha}^c \lambda_c - 2\Lambda_{\beta}^{ab} \lambda_{\alpha}, \\
\overset{03}{L}_{a\alpha} &= -\lambda_{a\alpha} + \lambda_a \lambda_{\alpha}, \\
\overset{03}{\Pi}_{a\alpha}^b &= -\lambda_{a\alpha}^b - \delta_a^b \lambda_c \lambda_{\alpha}^c + 2\delta_a^b \lambda_{\alpha}, \\
\overset{03}{L}_{\alpha\beta} &= -\lambda_{\alpha\beta} + \lambda_{\alpha}^a \lambda_{a\beta} + \mu_{\beta}^a \lambda_a \lambda_{\alpha} - \lambda_{\alpha}^a \lambda_a \lambda_{\beta}, \\
\overset{03}{\Pi}_{\alpha\beta}^a &= -\chi_{\alpha\beta}^a - \Lambda_{\beta}^{ba} \lambda_b \lambda_{\alpha} + \lambda_{b\beta}^a \lambda_{\alpha}^b - \lambda_{\beta} \lambda_{\alpha}^a + \lambda_{\alpha}^a \lambda_{\beta}^b \lambda_b.
\end{aligned}$$

Theorem. *The analog of the strong Norden's normalization induces $(n-m)(m+1)(m^2 + (n-m)^2)$ -parameter third type bunch of group connections*

$$\overset{3}{\Gamma} = \left\{ \Gamma_1, \overset{3}{\Gamma}_{\alpha\beta}^a, \overset{3}{L}_{\alpha\beta}^{ab}, \overset{3}{L}_{a\alpha}, \overset{3}{\Pi}_{a\alpha}^b, \overset{3}{L}_{\alpha\beta}, \overset{3}{\Pi}_{\alpha\beta}^a \right\}$$

in the fibering $G^*(\text{Gr}^*(m, n))$. The unique third type connection

$$\overset{03}{\Gamma} = \left\{ \overset{0}{\Gamma}_{b\alpha}^a, \overset{0}{L}_{c\alpha}^{ab}, \overset{0}{\Gamma}_{\beta\gamma}^{\alpha}, \overset{0}{L}_{\beta\gamma}^{\alpha a}, \overset{03}{\Gamma}_{\alpha\beta}^a, \overset{03}{L}_{\alpha\beta}^{ab}, \overset{03}{L}_{a\alpha}, \overset{03}{\Pi}_{a\alpha}^b, \overset{03}{L}_{\alpha\beta}, \overset{03}{\Pi}_{\alpha\beta}^a \right\}$$

is distinguished from this bunch.

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ON THE MOBILITY DEGREE OF (PSEUDO-) RIEMANNIAN SPACES WITH RESPECT TO CONCIRCULAR MAPPINGS

OLENA CHEPURNA AND IRENA HINTERLEITNER

Abstract. In this paper we study the mobility degree of (pseudo-) Riemannian spaces with respect to concircular mappings. We assume that the smoothness class of differentiability is C^2 .

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Keywords: geodesic mappings, (pseudo-)Riemannian manifold, smoothness class

1. INTRODUCTION

Under a geodesic circle we understand a curve for which the first curvature is constant and the second curvature is zero. K. Yano [14] introduced a conformal mapping of (pseudo-) Riemannian spaces which preserves geodesic circles and is called *concircular*.

These mappings are studied in many papers. In the present paper, we show results connected with basic notations under the conditions of minimal differentiability of metrics and geometric objects which define concircular mappings and also concircular vector fields.

2. FUNDAMENTAL EQUATIONS OF CONCIRCULAR MAPPINGS

Let $V_n = (M, g)$ and $\bar{V}_n = (\bar{M}, \bar{g})$ be n -dimensional (pseudo-) Riemannian manifolds with the metric tensors g and $\bar{g}$, respectively, $n > 2$.

Definition 1. A *conformal mapping* is a diffeomorphism of V_n onto $\bar{V}_n$ such that for all points $x \in M$ ($\equiv \bar{M}$) the following relation is satisfied

$$\bar{g}(x) = e^{2\sigma(x)} g(x), \quad (2.1)$$

where σ is a function on M .

If σ is constant, then the mapping is *homothetic*, and, moreover, if $\sigma = 0$, then the mapping is *isometric*. See [1, 7, 9, 10, 12].

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As we have checked (see [14], [9, p. 117]), if a pseudo-Riemannian space admits concircular mappings, then the function of conformality $\vartheta \stackrel{\text{def}}{=} e^{-\sigma}$ satisfies

$$\nabla \nabla \vartheta = \rho \cdot g, \quad (2.2)$$

where ρ is a function and ∇ is the Levi-Civita connection with respect to the metric g . In a local coordinate neighbourhood (U, x) , $U \subset M$, it has the form $\nabla_i \vartheta_j = \rho g_{ij}$, where g_{ij} are components of g and $\vartheta_j = \nabla_j \vartheta$. A vector field ϑ_i is called *equidistant* (Sinyukov [12, p. 92], see [9, p. 82]).

The integrability conditions of the last set of equations read

$$\vartheta_\alpha R_{ijk}^\alpha = g_{ij} \nabla_k \rho - g_{ik} \nabla_j \rho, \quad (2.3)$$

where R_{ijk}^h are components of the Riemann tensor of V_n . Using contraction, we get:

$$\nabla_i \rho = -\frac{1}{n-1} \vartheta_\alpha R_i^\alpha, \quad (2.4)$$

where $R_i^h = g^{h\alpha} R_{\alpha i}$ and $R_{ij} = R_{i\alpha j}^\alpha$ are components of the Ricci tensor on V_n .

Remark 1. In many papers, the Ricci tensor was defined with the opposite sign, for example, [2–8, 12].

Contracting the integrability condition (2.3) with $g^{i\beta} \vartheta_\beta$, we obtain easily $\nabla_k \rho = B \vartheta_k$, where B is a function. Because ϑ_k is gradient-like: $\vartheta_k = \nabla_k \vartheta$, then it implies that $\rho = \rho(\vartheta)$ and $B = B(\vartheta)$.

After this, the condition (2.3) acquires the following form

$$\vartheta_\alpha R_{ijk}^\alpha = B (g_{ij} \vartheta_k - g_{ik} \vartheta_j). \quad (2.5)$$

As was shown earlier [13] (see [3, 4, 6, 9]), these equations are satisfied if

$$V_n, \bar{V}_n \in C^2 \text{ (i. e. } g_{ij}(x), \bar{g}_{ij}(x) \in C^2), \vartheta(x) \in C^3, \vartheta_i(x) \in C^2 \text{ and } \varrho(x) \in C^1.$$

3. FUNDAMENTAL EQUATIONS OF CONCIRCULAR MAPPINGS FOR MINIMAL DIFFERENTIABLE CONDITIONS

We can write formula (2.2) in the following form

$$\nabla_j \vartheta^i \equiv \frac{\partial \vartheta^i}{\partial x^j} + \Gamma_{\alpha j}^i \vartheta^\alpha = \varrho \cdot \delta_j^i, \quad (3.1)$$

where $\vartheta^i = g^{i\alpha} \vartheta_\alpha$, δ_j^i is the Kronecker symbol and Γ_{ij}^h are the Christoffel symbols. It is easily seen that formulas (3.1) and also (2.2) are true when

$$V_n, \bar{V}_n \in C^1 \text{ (i. e. } g_{ij}(x), \bar{g}_{ij}(x) \in C^1), \vartheta(x) \in C^2, \vartheta_i(x) \in C^1 \text{ and } \varrho(x) \in C^0.$$

The following lemma holds.

Lemma 1 (Hinterleitner and Mikeš [2]). *Let $\lambda^h \in C^1$ be a vector field and ρ a function. If*

$$\frac{\partial \lambda^h}{\partial x^i} - \rho \delta_i^h \in C^1,$$

then $\lambda^h \in C^2$ and $\rho \in C^1$.

If $\Gamma_{ij}^h \in C^1$ holds, which is equivalent to $V_n \in C^2$ (i. e., $g_{ij} \in C^2$), then from formula (3.1) follows $\frac{\partial \vartheta^i}{\partial x^j} - \varrho \cdot \delta_j^i \in C^1$, and from Lemma 1 we get:

$$\vartheta^i(x) \in C^2 (\equiv \vartheta_i(x) \in C^2 \equiv \vartheta(x) \in C^3) \text{ and } \varrho(x) \in C^1.$$

From this viewpoint, we specify and generalize the results involving concircular vector fields below. Evidently, in this case, the above formulas from (2.3) to (2.5) are satisfied.

The system of equations

$$\begin{aligned} \nabla_i \vartheta_j &= \rho \cdot g_{ij}, \\ \nabla_i \rho &= -\frac{1}{n-1} \vartheta_\alpha R_i^\alpha \end{aligned} \quad (3.2)$$

is closed. It is a system of linear differential equations with respect to the co-vector ϑ_i and function ϱ , of Cauchy type, in first order covariant derivatives with coefficients uniquely determined by the metric g of the (pseudo-) Riemannian space V_n . For any family of initial values $\vartheta_i(x_0) = \vartheta_i^\circ$ and $\rho(x_0) = \rho^\circ$ of the functions under consideration in the given point x_0 , it admits at most one solution. Consequently, the number of free parameters in the general solution of the system is at most $n + 1$. See [6, 13].

Definition 2. The upper bound for the number of substantial parameters in the general solution of the system of equations (2.2) is called *the mobility degree under concircular mappings* of the (pseudo-) Riemannian manifold V_n .

Since the system is linear, it admits at most $n + 1$ linearly independent solutions corresponding to constant coefficients. It is obvious that the mobility degree under concircular mappings of the space coincides with the cardinality of the system of independent (substantial) concircular vector fields of the space.

It is known that only spaces with constant curvature admit the maximal number of $n + 1$ linearly independent concircular vector fields. Hence, under concircular mappings, only the spaces of constant curvature have the maximal mobility degree. This holds locally.

It follows from the analysis of the system of equations (3.2) that if $V_n \in C^r$, $r \geq 2$, then $\vartheta_i \in C^r$ and $\rho \in C^{r-1}$. It follows that the function ϑ belongs to C^{r+1} . From this and the formula (2.1), we obtain the following theorem.

Theorem 1. *If the (pseudo-) Riemannian manifold V_n ($V_n \in C^r$, $r \geq 2$, $n > 2$) admits a concircular mapping onto $\bar{V}_n \in C^2$, then $\bar{V}_n$ belongs to C^r . Moreover, the function ϑ of conformality V_n and $\bar{V}_n$: $\bar{g} = \vartheta^{-2} \cdot g$ belongs to C^{r+1} .*

We suppose that the differentiability class r is equal to $2, 3, \dots, \infty, \omega$, where ∞ and ω denote infinitely differentiable and real analytic functions, respectively.

We can construct examples of such concircular mappings $V_n \rightarrow \bar{V}_n$ in the form of equidistant metrics, see [9, p. 79]:

$$\bar{g} = \frac{1}{\vartheta^2} \cdot g, \quad g = \pm(dx^1)^2 + \text{const} \cdot \sqrt{|\vartheta'|} \cdot d\tilde{s}^2,$$

where $d\tilde{s}^2(x^2, \dots, x^n)$ is a C^r metric of an $(n-1)$ -dimensional (pseudo-) Riemannian space $\bar{V}_{n-1}$ and $\vartheta(x^1)$ is a C^{r+1} function and $\vartheta > 0$, $\vartheta' \neq 0$.

4. A (PSEUDO-) RIEMANNIAN SPACE WHICH ADMITS AT LEAST TWO LINEARLY INDEPENDENT CONCIRCULAR VECTOR FIELDS

Below we prove the following properties of concircular fields.

Lemma 2. *The non-vanishing concircular vector field $\vartheta_i(x)$ can be equal to zero only on point sets of zero measure.*

Proof. Let us suppose that Lemma 2 is not true. Thus there exists a point $x_0 \in M$ in the neighborhood $U_{x_0} \subset M$ of which the concircular vector field $\vartheta_i(x)$ is vanishing. From (3.2) follows that $\rho(x) = 0$ on U_{x_0} . From that follows the initial conditions at the point x_0 : $\vartheta_i(x_0) = 0$ and $\rho(x_0) = 0$. The system of linear equations (3.2) with these initial conditions has only the trivial solution $\vartheta_i(x) = 0$ and $\rho(x) = 0$ on all of M . $\square$

By mathematical induction we have the following lemma.

Lemma 3. *The set of r ($r < n$) linear independent concircular vector fields*

$$\{\vartheta_i^1, \vartheta_i^2, \dots, \vartheta_i^r\} \quad (4.1)$$

on V_n can be linearly dependent only on point sets of zero measure.

Proof. Successively we are able to substitute $r = 1, 2, \dots, n-1$. Let (4.1) be linearly independent (excluding at point sets of zero measure) concircular vector fields on V_n which satisfy the equations

$$\vartheta_{i,j}^s = \rho g_{ij},$$

where ρ^s are functions on V_n .

Let these vectors be linearly independent at the point $x_0 \in M$, then these are linearly independent at a point x in a certain neighborhood U_{x_0} . Finally, let ϑ_i be a concircular vector field on M and

$$\vartheta_i(x) = \sum_{s=1}^r \alpha^s(x) \cdot \vartheta_i^s(x) \text{ for } x \in U_{x_0} \quad (4.2)$$

where $\overset{s}{\alpha}(x)$ are functions on U_{x_o} . Because $\overset{s}{\vartheta}_i(x) \in C^1$, the functions $\overset{s}{\alpha}(x)$ are differentiable. Covariantly differentiating (4.2) with respect to x^j we find

$$(\rho - \sum_{s=1}^r \overset{s}{\alpha} \cdot \overset{s}{\rho}) g_{ij} = \sum_{s=1}^r \nabla_j \overset{s}{\alpha} \cdot \overset{s}{\vartheta}_i.$$

This implies that $\rho = \sum_{s=1}^r \overset{s}{\alpha} \cdot \overset{s}{\rho}$ and $\nabla_j \overset{s}{\alpha} = 0$ (i. e., $\overset{s}{\alpha} = \text{const}$) on U_{x_o} .

For the initial conditions

$$\begin{aligned} \vartheta_i(x_o) &= \sum_{s=1}^r \overset{s}{\alpha} \cdot \overset{s}{\vartheta}_i(x_o), \\ \rho(x_o) &= \sum_{s=1}^r \overset{s}{\alpha} \cdot \overset{s}{\rho}(x_o), \end{aligned}$$

the equations (3.2) have only one solution: $\vartheta_i(x) = \sum_{s=1}^r \overset{s}{\alpha} \cdot \overset{s}{\vartheta}_i(x)$ on V_n . $\square$

We are going to prove the following

Theorem 2. *If a (pseudo-) Riemannian space $V_n \in C^2$ ($n > 2$) admits at least two linearly independent concircular vector fields $\vartheta_i(x) \in C^1$ with constant coefficients, then B is a constant, uniquely determined by the metric of the space V_n .*

Remark 2. In [6] and [4, p. 88] a similar theorem was published, but the proof was done only for $V_n \in C^3$, $\vartheta_i(x) \in C^3$ and $\varrho(x) \in C^2$, and, moreover, it has local validity. This also concerns the following Theorems 3, 4 and 5. On the basis of Lemmas 2 and 3 these Theorems are valid globally.

Proof. Assume in V_n exist at least two linearly independent concircular vector fields with constant coefficients ϑ_i and $\tilde{\vartheta}_i$, with correspondent functions B and $\tilde{B}$, respectively. Then the following is satisfied (see (3.1)):

$$\vartheta_\alpha R_{ijk}^\alpha = B(g_{ij} \vartheta_k - g_{ik} \vartheta_j), \quad (4.3)$$

$$\tilde{\vartheta}_\alpha R_{ijk}^\alpha = \tilde{B}(g_{ij} \tilde{\vartheta}_k - g_{ik} \tilde{\vartheta}_j). \quad (4.4)$$

Multiplying (4.3) by $\tilde{\vartheta}_\alpha g^{\alpha k}$ and contracting over k we get by (4.4)

$$(B - \tilde{B})(g_{ij} \vartheta_\alpha \tilde{\vartheta}^\alpha - \tilde{\vartheta}_i \vartheta_j) = 0.$$

Suppose $B \neq \tilde{B}$. Then $g_{ij} \vartheta_\alpha \tilde{\vartheta}^\alpha - \tilde{\vartheta}_i \vartheta_j = 0$. From the last formula we get $\vartheta_\alpha \tilde{\vartheta}^\alpha = 0$ and $\tilde{\vartheta}_i \vartheta_j = 0$, a contradiction, since the vector fields are non-zero.

Hence $B = \tilde{B}$ holds. That is, the function B is uniquely defined by the metric of the space V_n itself. Because ϑ_k and $\tilde{\vartheta}_k$ are gradient-like covector fields ($\vartheta_k = \nabla_k \vartheta$ and $\tilde{\vartheta}_k = \nabla_k \tilde{\vartheta}$) from the equality $B = \tilde{B}$ the fact $B(\vartheta) = \tilde{B}(\tilde{\vartheta})$ follows. Note that ϑ and $\tilde{\vartheta}$ are independent variables, then from this fact follows: B is constant. $\square$

Note that the above theorem is analogous to some results proven earlier under the additional assumptions $V_n, \bar{V}_n \in C^3$, [5, 6, 13].

Theorem 3. *There are no (pseudo-) Riemannian spaces $V_n \in C^2$, except spaces of constant curvature, which admit more than $(n - 2)$ linearly independent concircular vector fields $\vartheta_i(x) \in C^1$ corresponding to constant coefficients.*

Remark 3. In [4, p. 86], [3, 5], a similar theorem was published but the proof was done only for $V_n \in C^3$, $\vartheta_i(x) \in C^3$ and $\varrho_i(x) \in C^2$.

Proof. Let us suppose the opposite. Let V_n be a space which is not of constant curvature and yet admits more than $(n - 2)$ linearly independent concircular vector fields with constant coefficients. The conditions (2.5) read

$$\vartheta_\alpha Z_{ijk}^\alpha = 0, \quad (4.5)$$

where

$$Z_{ijk}^h \stackrel{\text{def}}{=} R_{ijk}^h - B(\delta_k^h g_{ij} - \delta_j^h g_{ik}).$$

We can write the tensor Z_{ijk}^h as

$$Z_{ijk}^h = \sum_{s=1}^m b_s^h \varOmega_{ijk}^s,$$

where b_s^h are some linearly independent vectors, and $\varOmega_{ijk}^s$ are linearly independent tensors. Since V_n is not of constant curvature, $m \geq 2$ holds.

From the conditions (4.5), we obtain

$$\vartheta_\alpha b_1^\alpha = 0, \quad \vartheta_\alpha b_2^\alpha = 0, \quad \dots, \quad \vartheta_\alpha b_m^\alpha = 0. \quad (4.6)$$

Since $m \geq 2$, among the equations of the system (4.6) there are at least two substantial equations. From the previous facts it follows that there exist less or equal to $n - 2$ linearly independent vector fields ϑ_i , a contradiction. This proves Theorem 3. $\square$

From Theorem 3 and results in [6], the following two theorems are obtained:

Theorem 4. *Let $V_n \in C^2$, $(n > 2)$, be (pseudo-) Riemannian spaces in which there are $(n - 2)$ linearly independent concircular vector fields $\vartheta_i(x) \in C^1$. Then the Riemannian tensor has the following expression*

$$R_{hijk} = B(g_{hk}g_{ij} - g_{hj}g_{ik}) + e(a_h b_i - a_i b_h)(a_j b_k - a_k b_j),$$

where a_i and b_i are non-colinear and pairwise orthogonal covectors, $e = \pm 1$, and $B = \text{const}$.

Theorem 5. *The (pseudo-) Riemannian space $V_n \in C^3$ ($n > 3$) admits $(n - 2)$ linearly independent concircular vector fields $\vartheta_i(x) \in C^1$ if and only if in V_n the relations [11]*

$$\begin{aligned} R_{hijk} &= B(g_{hk}g_{ij} - g_{hj}g_{ik}) + e(a_h b_i - a_i b_h)(a_j b_k - a_k b_j), \\ a_{i,j} &= \overset{1}{\xi_j} a_i + \overset{2}{\xi_j} b_i + c_i a_j; \\ b_{i,j} &= \overset{3}{\xi_j} a_i + \overset{4}{\xi_j} b_i + c_i b_j; \\ c_{i,j} &= \overset{5}{\xi_j} a_i + \overset{6}{\xi_j} b_i + c_i c_j - B g_{ij} \end{aligned}$$

are satisfied, where a_i and b_i are non-collinear and pairwise orthogonal covectors; $c_i, \overset{s}{\xi_j}$ ($s = 1, \dots, 6$) are some covectors; $e = \pm 1$, and $B = \text{const}$.

Remark. This theorem was proved locally for $V_n \in C^3$, $\vartheta_i \in C^3$, $Q \in C^2$, in [6]. The detailed local proof is contained in the dissertation [3, p. 94-95], [4, p. 88-92].

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CONFORMAL HOLOMORPHICALLY PROJECTIVE MAPPINGS SATISFYING A CERTAIN INITIAL CONDITION

HANA CHUDÁ AND MOHSEN SHIHA

Abstract. In this paper we study conformal holomorphically projective mappings between conformal e -Kähler manifolds $K_n=(M, g, F)$ and $\bar{K}_n=(\bar{M}, \bar{g}, \bar{F})$, i.e. diffeomorphisms $f: M \rightarrow \bar{M}$ satisfying $f = f_1 \circ f_2 \circ f_3$, where f_1, f_3 are conformal mappings and f_2 is a holomorphically projective mapping between e -Kähler manifolds (i.e. Kähler, pseudo-Kähler and hyperbolic Kähler manifolds).

Suppose that the initial condition $f^* \bar{g} = k \cdot g$ is satisfied at a point $x_0 \in M$ and that at this point the Weyl conformal tensor satisfies a certain inequality. We prove that the mapping f is then necessarily conformal.

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Keywords: conformal mappings, holomorphically projective mappings, conformal holomorphically projective mappings, conformal e -Kähler manifolds, initial conditions

1. INTRODUCTION

One may say that the pioneering work in conformal and projective geometry was done by H. Weyl [19] and T. Thomas [17]. The topic of the holomorphically projective (HP) mappings was introduced (for classical, elliptic) Kähler manifolds $\mathbb{K}_n^-$ by T. Otsuki and Y. Tashiro [13], for hyperbolic Kähler manifolds $\mathbb{K}_n^+$ by M. Prvanović [14], and for parabolic Kähler manifolds $\mathbb{K}_n^0$ by V. V. Vishnevskij [18]. See, e.g., [1, 7, 10, 11, 16, 20].

Let us mention that geodesic, conformally geodesic and holomorphically projective mappings were studied under a certain additional condition based on the proportionality of the metrics. It turns out that even under this condition, the mapping is a homothety. See, e.g., [2–6, 8, 12].

In this paper we consider the following question, whether properties of conformal geodesic mappings are applicable for the composition of the conformal and holomorphically projective mappings — *conformal holomorphically projective mappings* of conformal e -Kähler manifolds.

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An analysis of the HP mappings of e -Kähler manifolds in terms of differentiability is presented in paper by I. Hinterleitner [9]. If the contrary is not specified, consideration is given in the tensor form in the class of real sufficiently smooth functions, the dimension $n \geq 4$, and is not mentioned specially. All the spaces are assumed to be connected.

2. MAIN PROPERTIES OF KÄHLER AND CONFORMAL KÄHLER MANIFOLDS

We introduce in the following definition generalizations of (pseudo-) Kähler, conformal Kähler and Hermitian manifolds.

Definition 1. An n -dimensional (pseudo-) Riemannian manifold (M, g) is called an e -Hermite manifold $H_n = (M, g, F)$ if besides the metric tensor g , a tensor field $F (\neq \text{Id})$ of type $(1,1)$ is given on manifold M_n , such that $F^2 = e \text{Id}$, $e = \pm 1$, and $g(X, FX) = 0$ for all tangent vector X . Moreover, if $\nabla F = 0$ then H_n is e -Kähler manifold $K_n = (M, g, F)$.

We remark, that for $e = -1$ are manifolds H_n and K_n (pseudo-) Hermitian and (pseudo-) Kähler manifolds, respectively, F is (almost) complex structure. For $e = 1$ we get hyperbolic Hermitian and hyperbolic Kähler manifold, respectively, F is (almost) product structure. See [9, 12, 12, 16, 20].

We remind the fundamental knowledge of a conformal mapping, that to be found in many monographs, see [6, 7, 12, 16, 20].

Definition 2. A diffeomorphism f between pseudo-Riemannian manifolds $V_n = (M, g)$ and $\bar{V}_n = (\bar{M}, \bar{g})$ is called a *conformal mapping*, if f preserves angles between all (smooth) curves on V_n ($n \geq 2$).

Equivalently, a mapping $f: V_n \rightarrow \bar{V}_n$ is conformal if and only if $\bar{g} = \rho \cdot g$, where ρ is a nowhere zero function on M and we will again suppose, $\bar{M} = M$.

From the equation $\bar{g} = \rho \cdot g$ it follows that

$$(\bar{\nabla} - \nabla)_X X = 2 \sigma(X) \cdot X - g(X, X) \cdot \Sigma, \quad (2.1)$$

where ∇ and $\bar{\nabla}$ are the Levi-Civita connections on V_n and $\bar{V}_n$, respectively, $\sigma(X) = \frac{1}{2} \nabla_X \ln |\rho|$, $\sigma(X) = g(X, \Sigma)$ and X is an arbitrary tangent vector.

Let us recall a definition of a *conformal Weyl tensor* C on V_n ($n \geq 3$):

$$C_{ijk}^h = R_{ijk}^h + \delta_j^h L_{ik} - \delta_k^h L_{ij} + L_j^h g_{ik} + L_k^h g_{ij},$$

where $L_{ij} = -\frac{1}{n-2} (R_{ij} - \frac{R}{2(n-1)} g_{ij})$, $L_i^h = g^{h\alpha} L_{\alpha i}$, R_{ijk}^h are the components of the Riemannian tensor of (M, g) , $R_{ij} = R_{i\alpha j}^\alpha$ are the components of the Ricci tensor, $R = R_{\alpha\beta} g^{\alpha\beta}$ is the scalar curvature and g^{ij} are components of the inverse matrix of g_{ij} .

If there is a conformal mapping $V_n \rightarrow \bar{V}_n$ ($n > 2$), then the conformal Weyl tensor remains invariant (i. e. $\bar{C} = C$). The converse is not true.

For $n > 3$, a pseudo-Riemannian space is locally *conformally flat* if and only if, the conformal Weyl tensor vanishes ($C = 0$).

Definition 3. A conformal e -Kähler manifold is conformally equivalent to e -Kähler manifold.

Clearly, any conformal e -Kähler manifold K_n may be considered as an e -Hermite manifold and it may be characterised by an e -Hermite structure. This structure has the following properties (for $e = -1$, see [15]):

$$\nabla_Y F(X) = \varphi(X) \cdot Y - g(X, Y) \cdot \Phi + \varphi(FX) \cdot (FY) + g(FX, Y) \cdot F\Phi,$$

where $\varphi(X) = g(X, \Phi) = \nabla_X \mathcal{F}$, $\mathcal{F}$ is a function on M and X, Y are tangent vector fields.

3. HOLOMORPHICALLY PROJECTIVE MAPPINGS OF e -KÄHLER MANIFOLDS

Assume the e -Kähler manifolds $K_n = (M, g, F)$ and $\bar{K}_n = (\bar{M}, \bar{g}, \bar{F})$ with metrics g and $\bar{g}$, structures F and $\bar{F}$, the Levi-Civita connections ∇ and $\bar{\nabla}$, respectively. Likewise, as in [13]

Definition 4. A curve ℓ in K_n which is given by the equation $\ell = \ell(t)$, $\lambda = d\ell/dt$ ($\neq 0$), $t \in I$, where t is a parameter is called *analytically planar*, if under the parallel translation along the curve, the tangent vector λ belongs to the two-dimensional distribution $D = \text{Span}\{\lambda, F\lambda\}$ generated by λ and its conjugate $F\lambda$, that is, it satisfies $\nabla_t \lambda = a(t)\lambda + b(t)F\lambda$, where $a(t)$ and $b(t)$ are some functions of the parameter t .

Particularly, in the case $b(t) = 0$, an analytically planar curve is a geodesic.

Definition 5. A diffeomorphism $f: K_n \rightarrow \bar{K}_n$ is called a *holomorphically projective mapping* if f maps any analytically planar curve in K_n onto an analytically planar curve in $\bar{K}_n$.

Let there exist a HP mapping $f: K_n = (M, g, F) \rightarrow \bar{K}_n = (\bar{M}, \bar{g}, \bar{F})$. Since f is a diffeomorphism, we can suppose local coordinate charts on M or $\bar{M}$, respectively, such that locally $f: K_n \rightarrow \bar{K}_n$ maps points onto points with the same coordinates, and $\bar{M} = M$.

A manifold K_n admits a holomorphically projective mapping onto $\bar{K}_n$ if and only if the following equations [12]

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X) \cdot Y + \psi(Y) \cdot X + e\psi(FX) \cdot FY + e\psi(FY) \cdot FX \quad (3.1)$$

hold for any tangent fields X, Y and where ψ is a differential form. If $\psi \equiv 0$ than f is *affine* or *trivially holomorphically projective*. Beside these facts it was proved [12], that $\bar{F} = \pm F$; for this reason we can suppose that $\bar{F} = F$.

The *holomorphically projective tensor*, which is defined by the following form,

$$P_{ijk}^h = R_{ijk}^h + \frac{1}{n+2} \left(\delta_k^h R_{ij} - \delta_j^h R_{ik} - e F_k^h R_{i\alpha} F_j^\alpha + e F_j^h R_{i\alpha} F_k^\alpha + 2e F_i^h R_{j\alpha} F_k^\alpha \right)$$

is invariant with respect to holomorphically projective mappings, i. e. $\bar{P} = P$.

It is known, that an e -Kähler K_n is a manifold of the constant holomorphically projective curvature if and only if the holomorphically projective tensor vanishes ($P = 0$).

4. CONFORMAL HOLOMORPHICALLY PROJECTIVE MAPPINGS

After we have sketched some basic properties of holomorphically projective and conformal mappings, let us focus our attention to the already mentioned conformal holomorphically projective ones. In papers [8, 9] by I. Hinterleitner so called conformal projective mappings were studied. These mappings are closely related to our subject. Inspired by her observations, we will derive some further results on them.

Definition 6. A diffeomorphism $f: K_n \rightarrow \bar{K}_n$ is called a *conformal holomorphically projective mapping* if $f = f_1 \circ f_2 \circ f_3$, where

$$\begin{aligned} f_1: K_n = (M, g, F) &\rightarrow {}^1K_n = (M, {}^1g, F) \quad \text{is a conformal mapping,} \\ f_2: {}^1K_n = (M, {}^1g, F) &\rightarrow {}^2K_n = (M, {}^2g, F) \quad \text{is a HP mapping and} \\ f_3: {}^2K_n = (M, {}^2g, F) &\rightarrow \bar{K}_n = (M, \bar{g}, F) \quad \text{is a conformal mapping.} \end{aligned}$$

Evidently, K_n and $\bar{K}_n$ are conformal e -Kähler manifolds, and ${}^1K_n$ and ${}^2K_n$ are e -Kähler manifolds. We will assume, that structures F are on the same manifold M . We have the following theorem.

Theorem 1. A diffeomorphism $f: K_n = (M, g, F) \rightarrow \bar{K}_n = (M, \bar{g}, \bar{F})$ is a conformal holomorphically projective mapping if and only if for each vector field X the following condition holds

$$(\bar{\nabla} - \nabla)_X X = 2\psi(X) \cdot X + e 2\psi(FX) \cdot FX + g(X, X) \cdot \Sigma + \bar{g}(X, X) \cdot \Omega, \quad (4.1)$$

where ψ is a differential 1-form, Σ and Ω are vector fields and there exist the functions ϱ_1 , ϱ_2 and ϱ_3 on the manifold M such that for each field X ,

$$\nabla_X \varrho_1 = g(X, \Sigma), \quad \nabla_X \varrho_2 = \bar{g}(X, \Omega), \quad \nabla_X \varrho_3 = \psi(X).$$

Proof. The necessity of (4.1) and the existence of the functions ϱ_1 , ϱ_2 and ϱ_3 follow from the relations (2.1) and (3.1). The conditions are sufficient due to the following observation.

Suppose the conditions (4.1) are satisfied. Then one may construct metrics ${}^1g = \exp(-2\varrho_1) \cdot g$ and ${}^2g = \exp(2\varrho_2) \cdot \bar{g}$. Computing the difference between the Levi-Civita connections associated to ${}^1g$ and ${}^2g$, we get formula, thus according to (3.1), the spaces ${}^1K_n$ and ${}^2K_n$ are in HP correspondence. $\square$

It is evident that the relation of “being conformal holomorphically projective equivalent” is symmetric and reflexive. Unfortunately, the conformal holomorphically projective mappings do not form a group because of lack of transitivity — the relation is not an equivalence relation.

5. CONFORMAL HP MAPPINGS WITH INITIAL CONDITIONS

We generalized the following Theorem:

Theorem 2 (Chudá and Mikeš [4]). *Let f be a holomorphically projective mapping between e -Kähler manifolds (M, g, F) and $(M, \bar{g}, \bar{F})$, $x_0 \in M$ and $\bar{x}_0 = f(x_0)$. Suppose that the initial condition $\bar{g}(\bar{x}_0) = k \cdot g(x_0)$ is satisfied for a $k \in \mathbb{R}$. If the holomorphically projective tensor does not vanish at x_0 , then the mapping f provides a homothety between (M, g, F) and $(M, \bar{g}, \bar{F})$, i. e. $\bar{g} = k \cdot g$, $k = \text{const}$.*

We introduce: $Q_{ijk}^h = \delta_j^h g_{ik} - \delta_k^h g_{ij} - \frac{n-1}{3e} \cdot (F_j^h F_{ik} - F_k^h F_{ij} + 2F_i^h F_{jk})$, where $F_{ik} = g_{i\alpha} F_k^\alpha$, and we prove the following lemma:

Lemma 1. *If x_0 be a fixed point on e -Kähler manifold K_n and $P(x_0) = 0$, then at the point x_0 the formula $C_{hijk} = \frac{R}{n(n+2)} \cdot Q_{hijk}$ holds.*

Proof. Let at the point x_0 the holomorphically projective tensor vanishes, i. e. $P_{ijk}^h = 0$. After contraction of g^{ij} we persuade, that $R_{ij} = \frac{R}{n} \cdot g_{ij}$, and finally we get $C_{hijk} = \frac{R}{n(n+2)} \cdot Q_{hijk}$. $\square$

Theorem 3. *Let f be a conformal holomorphically projective mapping between two conformal e -Kähler manifolds (M, g, F) and $(M, \bar{g}, \bar{F})$. If the metrics are proportional at the point x_0 , i. e. $\bar{g}_{ij}(x_0) = \mu \cdot g_{ij}(x_0)$, $\mu \in \mathbb{R}$ and there exists no $\alpha \in \mathbb{R}$ so that $C_{hijk}(x_0) = \alpha \cdot Q_{hijk}(x_0)$, then f is conformal.*

Proof. Let K_n admit a conformal holomorphically projective mapping f onto $\bar{K}_n$ and at the point $x_0 \in M$, the inequality $C_{hijk} \neq \alpha \cdot Q_{hijk}$ holds. Because the mapping $f_1: K_n \rightarrow {}^1K_n$ is a conformal, then for metrics ${}^1g_{ij} = f \cdot g_{ij}$ and for conformal Weyl tensor ${}^1C_{ijk}^h = C_{ijk}^h$ hold.

Therefore the inequality $C_{hijk} \neq \alpha \cdot Q_{hijk}$ from K_n has on manifold ${}^1K_n$ form: ${}^1C \neq {}^1\alpha \cdot {}^1Q$. Based on the contraposition of Lemma 1 we known, that the holomorphically projective tensor ${}^1P \neq 0$.

Consequently are satisfied the conditions from Theorem 2, the mapping $f_2: {}^1K_n \rightarrow {}^2K_n$ is homothetic. Moreover at the point x_0 the following assertion holds ${}^1g_{ij}(x_0) = \varrho \cdot {}^2g_{ij}(x_0)$, i. e. mapping $f = f_1 \circ f_2 \circ f_3$ is conformal. $\square$

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GEODESIC MAPPINGS OF (PSEUDO-) RIEMANNIAN MANIFOLDS PRESERVE CLASS OF DIFFERENTIABILITY

IRENA HINTERLEITNER AND JOSEF MIKEŠ

Abstract. In this paper, we prove that geodesic mappings of (pseudo-) Riemannian manifolds preserve the class of differentiability (C^r , $r \geq 1$). Also, if the Einstein space V_n admits a non-trivial geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_n \in C^1$, then $\bar{V}_n$ is an Einstein space. If a four-dimensional Einstein space with non-constant curvature globally admits a geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_4 \in C^1$, then the mapping is affine and, moreover, if the scalar curvature is non-vanishing, then the mapping is homothetic, i. e. $\bar{g} = \text{const} \cdot g$.

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Keywords: geodesic mappings, (pseudo-) Riemannian manifold, smoothness class, Einstein manifold

1. INTRODUCTION

The paper is devoted to the geodesic mapping theory of (pseudo-) Riemannian manifolds with respect to differentiability of their metrics. Most of the results in this area are formulated for “sufficiently” smooth, or analytic, geometric objects, as usual in differential geometry. It can be observed in most of the monographs and researches dedicated to the study of the theory of geodesic mappings and transformations, see [1, 3, 5–11, 13–19, 23–36].

Let $V_n = (M, g)$ and $\bar{V}_n = (\bar{M}, \bar{g})$ be (pseudo-) Riemannian manifolds, where M and $\bar{M}$ are n -dimensional manifolds with dimension $n \geq 2$, g and $\bar{g}$ are metrics. All the manifolds are assumed to be connected.

Definition 1. A diffeomorphism $f: V_n \rightarrow \bar{V}_n$ is called a *geodesic mapping* of V_n onto $\bar{V}_n$ if f maps any geodesic in V_n onto a geodesic in $\bar{V}_n$.

Hinterleitner and Mikeš [11] have proved the following theorem:

Theorem 1. *If the (pseudo-) Riemannian manifold V_n ($V_n \in C^r$, $r \geq 2$, $n \geq 2$) admits a geodesic mapping onto $\bar{V}_n \in C^2$, then $\bar{V}_n$ belongs to C^r .*

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Here and later, $V_n = (M, g) \in C^r$ means that $g \in C^r$, i. e., in a coordinate neighborhood (U, x) for the components of the metric g , $g_{ij}(x) \in C^r$ holds. If $V_n \in C^r$, then $M \in C^{r+1}$. This means that the atlas on the manifold M has the differentiability class C^{r+1} , i. e., for non-disjoint charts (U, x) and (U', x') on $U \cap U'$, it is true that the transformation $x' = x'(x) \in C^{r+1}$.

We suppose that the differentiability class r is equal to $0, 1, 2, \dots, \infty, \omega$, where $0, \infty$ and ω denote continuous, infinitely differentiable, and real analytic functions, respectively.

In the paper, we prove more general results. The following theorem holds:

Theorem 2. *If the (pseudo-) Riemannian manifold V_n ($V_n \in C^r$, $r \geq 1$, $n \geq 2$) admits a geodesic mapping onto $\bar{V}_n \in C^1$, then $\bar{V}_n$ belongs to C^r .*

Briefly, this means that *the geodesic mapping preserves the class of smoothness of the metric*.

Remark 1. It's easy to prove that the Theorems 1 and 2 are valid also for $r = \infty$ and for $r = \omega$. This follows from the theory of solvability of differential equations. Of course, we can apply this theorem only locally, because differentiability is a local property.

Remark 2. To require $V_n, \bar{V}_n \in C^1$ is a minimal requirement for geodesic mappings.

T. Levi-Civita [13] found metrics (*Levi-Civita metrics*) which admit geodesic mappings, see [1, 5], [25, p. 173], [27, p. 325]. From these metrics, we can easily see examples of non-trivial geodesic mappings $V_n \rightarrow \bar{V}_n$, where

- $V_n, \bar{V}_n \in C^r$ and $\notin C^{r+1}$ for $r \in \mathbb{N}$;
- $V_n, \bar{V}_n \in C^\infty$ and $\notin C^\omega$;
- $V_n, \bar{V}_n \in C^\omega$.

2. GEODESIC MAPPINGS OF EINSTEIN MANIFOLDS

These results may be applied to geodesic mappings of Einstein manifolds V_n onto pseudo-Riemannian manifolds $\bar{V}_n \in C^1$.

Geodesic mappings of Einstein spaces have been studied by many authors starting by A. Z. Petrov (see [27]). Einstein spaces V_n are characterized by the condition $\text{Ric} = \text{const} \cdot g$.

An Einstein space V_3 is a space of constant curvature. It is known that Riemannian spaces of constant curvature form a closed class with respect to geodesic mappings (Beltrami theorem [5, 23, 25, 27, 29, 31]). In 1978 (see [15] and PhD. thesis [14], and see [16, 20, 22], [23, p. 125], [25, p. 188]), Mikeš proved that under the conditions $V_n, \bar{V}_n \in C^3$, the following theorem holds (locally):

Theorem 3. *If the Einstein space V_n admits a non-trivial geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_n$, then $\bar{V}_n$ is an Einstein space.*

Many properties of Einstein spaces appear when $V_n \in C^3$ and $n > 3$. Moreover, it is known (D. M. DeTurck and J. L. Kazdan [4], see [2, p. 145]), that Einstein space V_n belongs to C^ω , i. e., for all points of V_n a local coordinate system x exists, for which $g_{ij}(x) \in C^\omega$ (*analytic coordinate system*).

It implies global validity of Theorem 3 and, on the basis of Theorem 2, the following more general theorem holds:

Theorem 4. *If the Einstein space V_n admits a nontrivial geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_n \in C^1$, then $\bar{V}_n$ is an Einstein space.*

The present Theorem is true globally, because the function Ψ which determines the geodesic mapping is real analytic on an analytic coordinate system and so $\psi (= \nabla \Psi)$ is vanishing only on a point set of zero measure. This simplifies the proof given in [11].

Finally, based on the results (see [16, 20–22], [23, p. 128], [25, p. 194]) for geodesic mappings of four-dimensional Einstein manifolds, the following theorem holds:

Theorem 5. *If a four-dimensional Einstein space V_4 with non-constant curvature globally admits a geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_4 \in C^1$, then the mapping is affine and, moreover, if the scalar curvature is non-vanishing, then the mapping is homothetic, i. e. $\bar{g} = \text{const} \cdot g$.*

3. GEODESIC MAPPING THEORY FOR $V_n \rightarrow \bar{V}_n$ OF CLASS C^1

Let us briefly recall some main facts of geodesic mapping theory of (pseudo-) Riemannian manifolds which were found by T. Levi-Civita [13], L. P. Eisenhart [5, 6] and N. S. Sinyukov [31], see [1, 9–11, 14, 16, 18, 19, 23, 25–32, 34–36]. In these results, no details about the smoothness class of the metric were stressed. They were formulated “for sufficiently smooth” geometric objects.

Since a geodesic mapping $f: V_n \rightarrow \bar{V}_n$ is a diffeomorphism, we can suppose $\bar{M} = M$. A (pseudo-) Riemannian manifold $V_n = (M, g)$ admits a geodesic mapping onto $\bar{V}_n = (M, \bar{g})$ if and only if the *Levi-Civita equations*

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X \quad (3.1)$$

hold for any tangent fields X, Y and where ψ is a differential form on M . Here, ∇ and $\bar{\nabla}$ are Levi-Civita connections of g and $\bar{g}$, respectively. If $\psi \equiv 0$, then f is *affine* or *trivially geodesic*.

Let (U, x) be a chart from the atlas on M . Then, equation (3.1) on U has the following local form: $\bar{\Gamma}_{ij}^h = \Gamma_{ij}^h + \psi_i \delta_j^h + \psi_j \delta_i^h$, where Γ_{ij}^h and $\bar{\Gamma}_{ij}^h$ are the Christoffel symbols of V_n and $\bar{V}_n$, ψ_i are components of ψ and δ_i^h is the Kronecker delta. Equations (3.1) are equivalent to the following Levi-Civita equations

$$\nabla_k \bar{g}_{ij} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik} \quad (3.2)$$

where $\bar{g}_{ij}$ are components of $\bar{g}$.

It is known that

$$\psi_i = \partial_i \Psi, \quad \Psi = \frac{1}{2(n+1)} \ln \left| \frac{\det \bar{g}}{\det g} \right|, \quad \partial_i = \frac{\partial}{\partial x^i}.$$

N.S. Sinyukov proved that the Levi-Civita equations (3.1) and (3.2) are equivalent to ([31, p. 121], [16], [23, p. 108], [25, p. 167], [29, p. 63]):

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik}, \quad (3.3)$$

where

$$(a) \ a_{ij} = e^{2\Psi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}; \quad (b) \ \lambda_i = -e^{2\Psi} \bar{g}^{\alpha\beta} g_{\beta i} \psi_\alpha. \quad (3.4)$$

From (3.3) follows $\lambda_i = \partial_i (\frac{1}{2} a_{\alpha\beta} g^{\alpha\beta})$, $(g^{ij}) = (g_{ij})^{-1}$ and $(\bar{g}^{ij}) = (\bar{g}_{ij})^{-1}$.

On the other hand [29, p. 63]:

$$\bar{g}_{ij} = e^{2\Psi} \hat{g}_{ij}, \quad \Psi = \frac{1}{2} \ln \left| \frac{\det \hat{g}}{\det g} \right|, \quad (\hat{g}_{ij}) = (g^{i\alpha} g^{j\beta} a_{\alpha\beta})^{-1}. \quad (3.5)$$

We can rewrite equations (3.3) and (3.4) in the following equivalent form (see [18], [25, p. 150]):

$$\nabla_k a^{ij} = \lambda^i \delta_k^j + \lambda^j \delta_k^i, \quad (3.6)$$

where

$$(a) \ a^{ij} = e^{2\Psi} \bar{g}^{ij} \quad \text{and} \quad (b) \ \lambda^i = -\psi_\alpha a^{\alpha i}. \quad (3.7)$$

Evidently, it follows

$$\lambda^i = \frac{1}{2} g^{ik} \partial_k (a^{\alpha\beta} g_{\alpha\beta}). \quad (3.8)$$

The above formulas (3.1), (3.2), (3.3), (3.6), are the criterion for geodesic mappings $V_n \rightarrow \bar{V}_n$ globally as well as locally. These formulas are true only under the condition $V_n, \bar{V}_n \in C^1$.

4. GEODESIC MAPPING THEORY FOR $V_n \in C^2 \rightarrow \bar{V}_n \in C^1$

In this section, we prove the main Theorem 2 from above. It is easy to see that Theorem 2 follows from Theorem 1 and the following theorem.

Theorem 6. *If $V_n \in C^2$ admits a geodesic mapping onto $\bar{V}_n \in C^1$, then $\bar{V}_n \in C^2$.*

Proof. Below, we prove Theorem 6.

4.1. We will suppose that the (pseudo-) Riemannian manifold $V_n \in C^2$ admits the geodesic mapping onto the (pseudo-) Riemannian manifold $\bar{V}_n \in C^1$. Furthermore, we can assume that $\bar{M} = M$.

We study the coordinate neighborhood (U, x) of any point $p = (0, 0, \dots, 0)$ at M . Evidently, components $g_{ij}(x) \in C^2$ and $\bar{g}_{ij}(x) \in C^1$ on $U \subset M$. On (U, x) , formulas (3.1)–(3.8) hold. From that fact, it follows that the functions $g^{ij}(x) \in C^2$, $\bar{g}^{ij}(x) \in C^1$, $\Psi(x) \in C^1$, $\psi_i(x) \in C^0$, $a^{ij}(x) \in C^1$, $\lambda^i(x) \in C^0$, and $\Gamma_{ij}^h(x) \in C^1$, where $\Gamma_{ij}^h = \frac{1}{2} g^{hk} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$ are Christoffel symbols.

4.2. It is easy to see that in a neighborhood of the point p in $V_n \in C^r$ there exist a semigeodesic coordinate system (U, x) for which the metric $g \in C^r$ has the following form (see [5], [25, p. 64])

$$ds^2 = e(dx^1)^2 + g_{ab}(x^1, \dots, x^n)dx^a dx^b, \quad e = \pm 1, \quad a, b > 1. \quad (4.1)$$

Evidently, for $a > 1$:

$$g_{11} = g^{11} = e = \pm 1, \quad g_{1a} = g^{1a} = 0 \quad \text{and} \quad \Gamma_{11}^1 = \Gamma_{1a}^1 = \Gamma_{11}^a = 0. \quad (4.2)$$

We can construct such a coordinate system using a coordinate transformation of class C^{r+1} for a basis of non-isotropic hypersurfaces $\Sigma \in C^{r+1}$ in a neighborhood of $p \in \Sigma$. Moreover, we can assume at p that

$$g_{ij}(0) = e_i \delta_{ij}; \quad e_i = \pm 1. \quad (4.3)$$

4.3. We write equations (3.6) in the following form

$$\partial_k a^{ij} = \lambda^i \delta_k^j + \lambda^j \delta_k^i - a^{i\alpha} \Gamma_{\alpha k}^j - a^{j\alpha} \Gamma_{\alpha k}^i. \quad (4.4)$$

Because $a^{ij} \in C^1$ and $\Gamma_{\alpha k}^j \in C^1$ from equation (4.4), we have the existence of the derivative immediately

$$\partial_{kl} a^{ii}, \partial_{kk} a^{ii}, \partial_{ki} a^{ii} (\equiv \partial_{ik} a^{ii}), \partial_{kl} a^{ij}, \partial_{kk} a^{ij}, \partial_{ki} a^{ij} (\equiv \partial_{ik} a^{ij}),$$

for each set of different indices i, j, k, l . Derivatives do not depend on the order because they are continuous functions.

We compute formula (4.4) for $i = j = k$ and for $i \neq j = k$:

$$\partial_i a^{ii} = 2\lambda^i - 2a^{i\alpha} \Gamma_{\alpha i}^i \quad \text{and} \quad \partial_k a^{ik} = \lambda^i - a^{k\alpha} \Gamma_{\alpha k}^i - a^{i\alpha} \Gamma_{\alpha k}^k$$

where, for an index k , we do not carry out the Einstein summation and after eliminating λ^i , we obtain

$$\frac{1}{2} \partial_i a^{ii} - \partial_k a^{ik} = a^{k\alpha} \Gamma_{\alpha k}^i + a^{i\alpha} \Gamma_{\alpha k}^k - a^{i\alpha} \Gamma_{\alpha i}^i \quad (4.5)$$

Because there exists the partial derivative $\partial_{ik} a^{ii}$, formula (4.5) implies the existence of the partial derivatives $\partial_{kk} a^{ik}$.

4.4. In the semigeodesic coordinate system (4.1), we compute (4.4) for $i = j = k = 1$: $\lambda^1 = \frac{1}{2} \partial_1 a^{11}$, and from (3.8): $\lambda^1 = \frac{1}{2} \partial_1 (a^{11} + e a^{\alpha\beta} g_{\alpha\beta})$, we obtain $\partial_1 (a^{\alpha\beta} g_{\alpha\beta}) = 0$. Here and later $\alpha, \beta > 1$.

Further (4.4) for $i = j = 1$ and $k = 2$, we have the following expression $\partial_1 a^{12} + a^{1\gamma} \Gamma_{\gamma 1}^2 + a^{2\gamma} \Gamma_{\gamma 1}^1 = \lambda^2$. Using (3.8), we have

$$\partial_1 a^{12} = \frac{1}{2} g^{2\gamma} \cdot \partial_\gamma (a^{11} + a^{\alpha\beta} g_{\alpha\beta}) - a^{1\gamma} \Gamma_{\gamma 1}^2, \quad \gamma > 1,$$

and after integration, we obtain

$$\begin{aligned} a^{12} = & \frac{1}{2} \left(\int_0^{x^1} g^{2\gamma}(\tau^1, x^2, \dots, x^n) d\tau^1 \right) \cdot \partial_\gamma(a^{\alpha\beta} \cdot g_{\alpha\beta}) \\ & + \frac{1}{2} \int_0^{x^1} g^{2\gamma}(\tau^1, x^2, \dots, x^n) \cdot \partial_\gamma a^{11} d\tau^1 \\ & - \int_0^{x^1} a^{1\gamma} \Gamma_{\gamma 1}^2 d\tau^1 + A(x^2, \dots, x^n). \end{aligned} \quad (4.6)$$

As $a^{12}(0, x^2, \dots, x^n) \equiv A(x^2, \dots, x^n)$, the function $A \in C^1$.

After differentiating the formula (4.6) by x^2 and using the law of commutation of derivatives and integrals, see [12, p. 300], we can see that

$$\frac{\partial}{\partial x^2} \left\{ \left(\int_0^{x^1} g^{2\gamma}(\tau^1, x^2, \dots, x^n) d\tau^1 \right) \cdot \partial_\gamma(a^{\alpha\beta} \cdot g_{\alpha\beta}) \right\} \quad (4.7)$$

exists. From (4.5) for $i = 2$ and $k = c \neq 2$, we obtain $\partial_c a^{c2} = \frac{1}{2} \partial_2 a^{22} + a^{c\delta} \Gamma_{\delta c}^2 + a^{2\delta} \Gamma_{\delta c}^c - a^{2\delta} \Gamma_{\delta 2}^2$. Using this formula, we can rewrite the bracket (4.7) in the following form

$$\left\{ \left(\int_0^{x^1} g^{2\gamma}(\tau^1, x^2, \dots, x^n) d\tau^1 \right) \cdot g_{2\gamma} \cdot \partial_2 a^{22} + f \right\},$$

where f is the rest of this parenthesis, which is evidently differentiable by x^2 .

Since the parenthesis and also the coefficients by $\partial_2 a^{22}$ are differentiable with respect to x^2 , there exists $\partial_{22} a^{22}$ if

$$\left(\int_0^{x^1} g^{2\gamma}(\tau^1, x^2, \dots, x^n) d\tau^1 \right) \cdot g_{2\gamma} \neq 0.$$

Using (3.3), this inequality is true for all x in a neighborhood of the point p excluding the point for which $x^1 = 0$.

For these reasons, in this domain, there exists the derivative $\partial_{22} a^{22}$ as well as all second derivatives a^{ij} . This follows from the derivative of the formula (4.5).

So, $a^{ij} \in C^2$ and $\lambda^i \in C^1$, from the formula (3.7b), it follows $\psi_i \in C^1$ and it means that $\Psi \in C^2$. From (3.7a) follows $\bar{g}^{ij} \in C^2$ and also $\bar{g}_{ij} \in C^2$. This is a proof of Theorem 6. $\square$

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GEOMETRY OF ALMOST CLIFFORDIAN MANIFOLDS: NIJENHUIS TENSOR

JAROSLAV HRDINA

Abstract. We generalize some classical results on Nijenhuis tensor for an almost Cliffordian manifold based on arbitrary Clifford algebra and suggest its relations with the integrability of the corresponding G -structure. We prove the set of properties for Nijenhuis tensors with respect to arbitrary Clifford algebra.

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1. ALMOST CLIFFORDIAN MANIFOLDS

Let $\mathcal{O} = \mathcal{C}\ell(s, t)$ be a Clifford algebra. If M is an km -dimensional manifold, where $k = 2^{s+t}$ and $m \in \mathbb{N}$, then an almost Clifford manifold is given by a reduction of the structure group $\mathrm{GL}(km, \mathbb{R})$ of the principal frame bundle of M to

$$\mathrm{GL}(m, \mathcal{O}) = \{A \in \mathrm{GL}(km, \mathbb{R}) \mid AI = IA, I \in \mathcal{O}\},$$

where $\mathcal{O}$ is arbitrary Clifford algebra. In other words, an almost Clifford manifold is a smooth manifold equipped by the set of anti commuting and commuting affinors $I_i, i = 1, \dots, t, I_i^2 = -E$ and $J_j, j = 1, \dots, s, J_j^2 = E$ such that the free associative unitary algebra generated by $\langle I_i, J_j, E \rangle$ is isomorphically equivalent to $\mathcal{O}$. In particular, on the elements of this reduced bundle, one can define affinors in the form $F_1, \dots, F_k$ globally.

Definition 1. Let M be a smooth manifold such that $\dim(M) = m$. Let A be a smooth ℓ -dimensional ($\ell < m$) vector subbundle in $T^*M \otimes TM$ such that the identity affnor $E = \mathrm{id}_{TM}$ restricted to $T_x M$ belongs to $A_x M \subset T_x^* M \otimes T_x M$ at each point $x \in M$. We say that M is equipped with an ℓ -dimensional A -structure.

It is easy to see that an almost Clifford structure is not an A -structure because the affinors in the form $F_0, \dots, F_\ell \in A$ have to be defined only locally.

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Definition 2. The A -structure, where A is a Clifford algebra $\mathcal{O}$, is called an almost Cliffordian manifold.

In particular, the almost Clifford and almost Cliffordian structures are G -structures based on Clifford algebras. Two most important examples are an almost hypercomplex geometry and an almost quaternionic geometry, which are based on Clifford algebra $\mathcal{C}\ell(0, 2)$. Note that the geometric property of an almost hypercomplex structure reads that there is no nontrivial G -invariant subspace $\mathcal{D}$ in $\mathbb{V} \otimes \wedge^2 \mathbb{V}^*$ because the first prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$ vanishes. For an almost quaternionic structure, the situation is more complicated, because $\mathfrak{g}^{(1)} = \mathbb{V}^*$ and there is a class of these structures indexed by $\mathbb{V}$, see [1]. Note that for Cliffordian structures based on $\mathcal{C}\ell(0, 3)$ is $\mathfrak{g}^{(1)} = \mathbb{V}^*$ and there is a class of these structures indexed by $\mathbb{V}$ too, see [2].

2. NIJENHUIS TENSOR

The Nijenhuis tensor plays an important role in the theory of integrability. As a classical concept, Nijenhuis introduced $N_J \in \wedge^2 T^*M \otimes TM$ of an almost complex structure $J \in T^*M \otimes TM$. This tensor is an obstruction for an almost complex structure which distinguished it from the complex structure, i. e. their integrability. Recall that Nijenhuis tensor $N(P, Q) \in \wedge^2 T^*M \otimes TM$ for a pair of tensors $P, Q \in T^*M \otimes TM$ is given by the expression

$$\begin{aligned} N(P, Q)(X, Y) = & [PX, QY] - P[QX, Y] - Q[X, PY] + [QX, PY] \\ & - Q[PX, Y] - P[X, QY] + (PQ + QP)[X, Y]. \end{aligned}$$

An almost quaternionic manifold M is integrable if and only if the Nijenhuis tensors $N(I, I)$ and $N(J, J)$ vanish, where $I, J, IJ \in T^*M \otimes TM$ is a quaternionic structure. Let us finally note that in [3], the author proved a similar fact for Clifford algebra $\mathcal{C}\ell(0, 3)$.

If $P = Q$, then, by straightforward computing,

$$N(P, P)(X, Y) = 2([PX, PY] - P[PX, Y] - P[X, PY] + P^2[X, Y])$$

and if $P = E$, where E is an identity, then

$$N(E, Q)(X, Y) = 0.$$

Following [6, 7], we reformulate Nijenhuis tensor calculus with respect to arbitrary Clifford algebra. In those papers, a set of classical results for $\mathcal{C}\ell(0, 2)$ and $\mathcal{C}\ell(2, 0)$ was proved. In fact, most of the arguments work generally, but we have to formulate and prove them for any Clifford algebra. First of all, following papers [6, 7], we recall the operation $\bar{\wedge}$

$$(S \bar{\wedge} P)(X, Y) := S(PX, Y) + S(X, PY) \quad (1)$$

and

$$(P \bar{\wedge} S)(X, Y) := PS(X, Y) \quad (2)$$

where $S \in \otimes^2 \mathbb{V} \otimes \mathbb{V}^*$ and $N \in \mathbb{V} \otimes \mathbb{V}^*$. Now, one can easily check the following identities

$$N(L, QP) + N(Q, LP) = N(L, QP) \bar{\wedge} P + L \bar{\wedge} N(Q, P) + Q \bar{\wedge} N(L, P) \quad (3)$$

$$(S \bar{\wedge} Q) \bar{\wedge} P - (S \bar{\wedge} P) \bar{\wedge} Q = S \bar{\wedge} QP - S \bar{\wedge} PQ \quad (4)$$

$$(L \bar{\wedge} S) \bar{\wedge} P = L \bar{\wedge} (S \bar{\wedge} P) \quad (5)$$

Lemma 1. *Let $\mathcal{O} = \mathcal{C}\ell(s, t)$ be a Clifford algebra. If $F, G \in \mathcal{O}$ such that $F \neq G$, then the following identities hold:*

$$N(FG, F) = -\frac{1}{2}N(F, G) \bar{\wedge} F - \frac{1}{2}G \bar{\wedge} N(F, F) + \frac{1}{4}N(F, F) \bar{\wedge} G \quad (6)$$

$$0 = N(F, G) \bar{\wedge} F + 2F \bar{\wedge} N(F, G) + G \bar{\wedge} N(F, F) + \frac{1}{2}N(F, F) \bar{\wedge} G \quad (7)$$

$$N(G, H) = N(F, G) \bar{\wedge} G + G \bar{\wedge} N(F, G) + F \bar{\wedge} N(G, G). \quad (8)$$

Proof. Putting $L = Q = F$, $P = G$ in (3), we find

$$N(F, FG) + N(F, FG) = N(F, F) \bar{\wedge} G + F \bar{\wedge} N(F, G) + F \bar{\wedge} N(F, G),$$

that is,

$$N(FG, F) = \frac{1}{2}N(F, F) \bar{\wedge} G + F \bar{\wedge} N(F, G). \quad (9)$$

Putting $L = G$, $P = Q = F$ in (3), we find

$$N(G, F^2) + N(F, GF) = N(G, F) \bar{\wedge} F + G \bar{\wedge} N(F, F) + F \bar{\wedge} N(G, F),$$

that is,

$$N(F, GF) = N(G, F) \bar{\wedge} F + G \bar{\wedge} N(F, F) + F \bar{\wedge} N(G, F). \quad (10)$$

Adding (9) and (10) and dividing the sum by 2, we find

$$N(FG, F) = -\frac{1}{2}N(F, G) \bar{\wedge} F - \frac{1}{2}G \bar{\wedge} N(F, F) + \frac{1}{4}N(F, F) \bar{\wedge} G$$

and subtracting (10) from (9), we find

$$N(F, G) \bar{\wedge} F + 2F \bar{\wedge} N(F, G) + G \bar{\wedge} N(F, F) + \frac{1}{2}N(F, F) \bar{\wedge} G = 0 \quad (11)$$

Finally, putting $L = P = G$, $Q = F$ in (3), we find

$$N(F, G^2) + N(G, FG) = N(F, G) \bar{\wedge} G + F \bar{\wedge} N(G, G) + G \bar{\wedge} N(F, G),$$

that is,

$$N(G, H) = N(F, G) \bar{\wedge} G + G \bar{\wedge} N(F, G) + F \bar{\wedge} N(G, G).$$

□

Lemma 2. Let $\mathcal{O} = \mathcal{C}\ell(s, t)$ be a Clifford algebra. If $F, G \in \mathcal{O}$ such that $F \neq G$, then the following identity holds:

$$\begin{aligned} \epsilon_1 N(F, F) - \epsilon_2 N(G, G) + N(H, F) \bar{\wedge} G + G \bar{\wedge} N(H, F) \\ - N(G, H) \bar{\wedge} F + F \bar{\wedge} N(G, H) + 2H \bar{\wedge} N(F, G) = 0, \end{aligned} \quad (12)$$

where $H = FG$ and $\epsilon_i = 1$ for $K^2 = -1$ and $\epsilon_i = -1$ for $K^2 = 1$.

Proof. Putting $L = FG$, $Q = F$, $P = G$ in (3), we find

$$\begin{aligned} N(FG, FG) - N(F, FGG) = N(FG, F) \bar{\wedge} G + FG \bar{\wedge} N(F, G) \\ + F \bar{\wedge} N(FG, G), \end{aligned}$$

that is,

$$\begin{aligned} N(H, H) = \epsilon N(F, F) + N(FG, F) \bar{\wedge} G + FG \bar{\wedge} N(F, G) \\ + F \bar{\wedge} N(FG, G), \end{aligned} \quad (13)$$

where $\epsilon = 1$ for $G^2 = -1$ and $\epsilon = -1$ for $G^2 = 1$.

Putting $L = FG$, $Q = G$, $P = F$ in (3), we find

$$\begin{aligned} N(FG, GF) - N(G, FGF) = N(FG, G) \bar{\wedge} F + FG \bar{\wedge} N(G, F) \\ + G \bar{\wedge} N(FG, F), \end{aligned}$$

that is,

$$N(H, H) = \epsilon N(G, G) - N(G, H) \bar{\wedge} F - H \bar{\wedge} N(F, G) - G \bar{\wedge} N(H, F), \quad (14)$$

where $\epsilon = 1$ for $G^2 = -1$ and $\epsilon = -1$ for $G^2 = 1$. Thus, from (13) and (14), we find

$$\begin{aligned} N(H, H) = \frac{1}{2} \{ \epsilon N(F, F) + \epsilon N(G, G) + N(H, F) \bar{\wedge} G \\ + G \bar{\wedge} N(H, F) - N(G, H) \bar{\wedge} F + F \bar{\wedge} N(G, H) + 2H \bar{\wedge} N(F, G) \} = 0 \end{aligned}$$

and

$$\begin{aligned} \epsilon N(F, F) - \epsilon N(G, G) + N(H, F) \bar{\wedge} G \\ + G \bar{\wedge} N(H, F) - N(G, H) \bar{\wedge} F + F \bar{\wedge} N(G, H) + 2H \bar{\wedge} N(F, G) = 0, \end{aligned}$$

which completes the proof. □

Lemma 3. Let $\mathcal{O} = \mathcal{C}\ell(s, t)$ be a Clifford algebra. If $F \in \mathcal{O}$, then the following identity holds:

$$N(F, F) \bar{\wedge} F = -2F \bar{\wedge} N(F, F).$$

Proof. One can easily check that putting $L = P = Q = F$ gives $N(F, F^2) + N(F, F^2) = N(F, F) \bar{\wedge} F + F \bar{\wedge} N(F, F) + F \bar{\wedge} N(F, F)$ □

Theorem 1. *Let $\mathcal{O}$ be a Clifford algebra $\mathcal{C}\ell(s, t)$ and let $F, G \in \mathcal{O}$ such that $F \neq G$. If the Nijenhuis tensors $N(F, F)$ and $N(G, G)$ vanish, then $N(FG, FG)$ vanishes.*

Proof. Since $N(F, F) = 0$, we have from (9) $N(H, F) = F \bar{\wedge} N(F, G)$, and from (11),

$$N(F, G) \bar{\wedge} F = -2F \bar{\wedge} N(F, G). \quad (15)$$

Since $N(G, G) = 0$, we have from (9), where we changed F and G ,

$$N(F, G) \bar{\wedge} G = -2G \bar{\wedge} N(F, G) \quad (16)$$

and from (11), where we changed F and G ,

$$N(F, G) \bar{\wedge} G = -2G \bar{\wedge} N(F, G). \quad (17)$$

Now, if we substitute $N(F, F) = 0$, $N(G, G) = 0$, and (16) into (12), then the part containing ϵ_i vanishes and the proof is correct for any Clifford algebra, i. e., we find

$$\begin{aligned} & (F \bar{\wedge} N(F, G)) \bar{\wedge} G + G \bar{\wedge} (F \bar{\wedge} N(F, G)) \\ & - (G \bar{\wedge} N(F, G)) \bar{\wedge} F - F \bar{\wedge} (G \bar{\wedge} N(F, G)) + 2H \bar{\wedge} N(F, G) = 0, \end{aligned}$$

from which

$$(F \bar{\wedge} N(F, G)) \bar{\wedge} G - (G \bar{\wedge} N(F, G)) \bar{\wedge} F = 0, \quad (18)$$

since

$$G \bar{\wedge} (F \bar{\wedge} N(F, G)) = -F \bar{\wedge} (G \bar{\wedge} N(F, G)) = -H \bar{\wedge} N(F, G)$$

by virtue of $GF = -FG = -H$. Now, using (5), (15) and (17), we find, from (18)

$$\begin{aligned} & F \bar{\wedge} (N(F, G) \bar{\wedge} G) - G \bar{\wedge} (N(F, G) \bar{\wedge} F) = 0, \\ & -2F \bar{\wedge} (G \bar{\wedge} N(F, G)) + 2G \bar{\wedge} (F \bar{\wedge} N(F, G)) = 0, \\ & -4FG \bar{\wedge} N(F, G) = 0, \end{aligned}$$

that is,

$$H \bar{\wedge} N(F, G) = 0. \quad (19)$$

Since $H^2 = -1$, we have from (19) $N(F, G) = 0$. $\square$

Corollary 1. *Let $\mathcal{O}$ be a Clifford algebra $\mathcal{C}\ell(s, t)$. If the Nijenhuis tensors $N(I_i, I_i)$ vanish, where I_i are the algebra generators of $\mathcal{O}$, then*

$$N(F_i, F_j) = 0,$$

where F_i are vector space generators.

3. CLASSES OF SUBORDINATED CONNECTIONS

Recall the concept of A -planar curves on A -structures equipped with the linear connection ∇ . For any tangent vector $X \in T_x M$, we shall write $A_x(X)$ for the vector subspace

$$A_x(X) = \{F_i(X) | F_i \in A_x M\} \subset T_x M$$

and call it the A -hull of the vector X . Similarly, A -hull of a vector field is a subbundle in TM obtained pointwise. Let M be a smooth manifold equipped with an A -structure and a linear connection ∇ . A smooth curve $c : \mathbb{R} \rightarrow M$ is said to be A -planar if

$$\nabla_{\dot{c}} \dot{c} \in A(\dot{c}).$$

In [5], the authors proved a set of facts about the class of $\mathcal{D}$ -connections. The theorems below, about Cliffordian structures, are proved in paper [5] and some examples of this concept can be found in papers [2, 4]. The theorems about $\mathcal{D}$ -connections can be found in [1].

Following [4, 5], we have a set of results on Clifford and Cliffordian manifolds.

Corollary 2. *Let M be a smooth manifold equipped with a G -structure, where $G = \text{GL}(n, \mathcal{O})$, $\mathcal{O} = \mathcal{C}\ell(s, t)$, $s + t > 1$, i. e. an almost Clifford manifold. Then the G -structure is of type 1 and there exists a unique $\mathcal{D}$ -connection.*

One can see that an almost Cliffordian manifold M is given as a G -structure provided that there is a reduction of the structure group of the principal frame bundle of M to $G := \text{GL}(m, \mathcal{O}) \text{GL}(1, \mathcal{O}) = \text{GL}(m, \mathcal{O}) \times \text{GL}(1, \mathcal{O})$, the action of G on $T_x M$ looks like QXq , where $Q \in \text{GL}(m, \mathcal{O})$, $q \in \text{GL}(1, \mathcal{O})$, where the right action of $\text{GL}(1, \mathcal{O})$ is blockwise. In this case the tensor fields in the form $F_1, \dots, F_k$ can be defined only locally. It is easy to see that the Lie algebra $\mathfrak{gl}(m, \mathcal{O})$ of a Lie group $\text{GL}(m, \mathcal{O})$ is of the form

$$\mathfrak{gl}(m, \mathcal{O}) = \{A \in \mathfrak{gl}(km, \mathbb{R}) | AI_i = I_i A, AJ_j = J_j A\}$$

and the Lie algebra $\mathfrak{g}$ of a Lie group $\text{GL}(m, \mathcal{O}) \text{GL}(1, \mathcal{O})$ is of the form $\mathfrak{g} = \mathfrak{gl}(m, \mathcal{O}) \oplus \mathfrak{gl}(1, \mathcal{O})$.

Let us note that the cases of $\mathcal{C}\ell(s, t)$, where $s + t = 2$, were studied in [6, 7] and the case of $\mathcal{C}\ell(0, 3)$ was studied in a detailed way in [2].

Corollary 3. *Let M be an almost Cliffordian manifold based on Clifford algebra $\mathcal{O} = \mathcal{C}\ell(s, t)$, where $\dim(M) \geq 2(s + t)$, i. e., a smooth manifold equipped with a G -structure, where $G = \text{GL}(n, \mathcal{O}) \text{GL}(1, \mathcal{O})$ or equivalently an A -structure, where $A = \mathcal{O}$. Then the class of $\mathcal{D}$ -connections preserves A and shares the same A -planar curves, which are isomorphic to $(\mathbb{R}^{km})^*$.*

4. CONCLUSION

From the classical theory, the Nijenhuis tensor is a part of a torsion of any almost complex connection ∇ and is J -antilinear in each argument ,

$$N_J(X, Y) = T_{\nabla}(X, Y) + J T_{\nabla}(JX, Y) + J T_{\nabla}(X, JY) - T_{\nabla}(JX, JY)$$

and the connection ∇ called minimal such that $N_J = 4T_{\nabla}$, i.e. the structure is integrable if and only if N_J vanishes. Let M be an almost quaternionic manifold or an almost quaternionic manifold of the second kind (paraquaternionic). In [6] and [7], the authors proved that the structure is integrable if and only if the structure tensor $T_Q = N(I, I) + N(J, J) + N(K, K)$ vanishes. In this case, there is the class of $\mathcal{D}$ -connections without torsion.

A similar fact was proved for an almost Cliffordian manifold, where $\mathcal{C}\ell(0, 3)$, see [3]. In this article, the authors proved that the structure tensor which is locally generated by F_i is given by

$$T_Q = \sum_{i=1}^6 N(F_i, F_i) + \sum_{i=1}^6 N(F_i, F_i) \partial(\alpha_a \otimes F_a),$$

where ∂ denotes Spencer's operator of alternation. The following step is to find a description of the structure preserving connection based on the Nijenhuis tensor for any Cliffordian manifold.

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ON DECOMPOSITION PROBLEMS ON MANIFOLDS WITH A SPECIAL DIFFERENTIAL OPERATOR

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Abstract. This paper deals with the properties of a special differential operator with respect to the general decomposition of tensor fields on manifolds with affine connection. It is shown that properties of a special differential operator are transferred to the components of a given decomposition.

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1. INTRODUCTION

The trace decomposition of tensors on an n -dimensional Riemannian manifold M_n was described by Weyl [8]. This decomposition problem may be naturally generalized for certain cases (see for example [2, 4–7, 9]).

In this paper, we generalize some results from [3] concerning decompositions of tensors and tensor fields. Crasmareanu [1] has studied decompositions of tensor fields for the case of covariant derivative of such tensors which satisfy certain conditions. We consider a special differential operator and generalize those results.

2. GENERAL DECOMPOSITION OF TENSORS

It is well known that on every manifold M_n , a positive definite metric g determining the structure of a Riemannian manifold may be introduced. For practical reasons (e. g., in theoretical physics), the pseudo-Riemannian manifolds with an indefinite pseudometric g are considered, too. In what follows, we will use in both cases the terms of *Riemannian manifold* and *metric*.

Let M_n be a manifold with a metric g . We introduce the following notation: For every ordered system of indices $i_1, i_2, \dots, i_q$, $1 \leq i_\rho \leq n$, and every pair (i_ρ, i_σ) , $\rho < \sigma$, of them, we denote by

$$M_{i_1, \dots, i_{\rho-1} \overset{\rho}{\vee} i_{\rho+1} \dots i_{\sigma-1} \overset{\sigma}{\vee} i_{\sigma+1} \dots i_q}^{(\rho, \sigma)}$$

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the tensor of the type $(0, q - 2)$, where the ρ -th and the σ -th indices are omitted.

Then we have the following theorem:

Theorem 1. *Let $T_{i_1 \dots i_q}$ be a $(0, q)$ -tensor and*

$$(i_{k_1}, i_{l_1}), (i_{k_2}, i_{l_2}), \dots, (i_{k_s}, i_{l_s}), \quad s \leq \frac{1}{2} q(q - 1), \quad (2.1)$$

be pairs of indices satisfying $k_\sigma < l_\sigma$. Then there exists the decomposition of the tensor $T_{i_1 \dots i_q}$ of the form

$$T_{i_1 \dots i_q} = \tilde{T}_{i_1 \dots i_q} + \sum_{\sigma=1}^s g_{i_{k_\sigma} i_{l_\sigma}} \cdot M_{i_1, \dots, i_{k_\sigma-1} \quad i_{k_\sigma+1} \dots i_{l_\sigma-1} \quad i_{l_\sigma+1} \dots i_q}^{(k_\sigma, l_\sigma)}, \quad (2.2)$$

where the tensor $\tilde{T}$ satisfies

$$\tilde{T}_{i_1 \dots i_{k_\sigma} \dots i_{l_\sigma} \dots i_q} \cdot g^{i_{k_\sigma} i_{l_\sigma}} = 0, \quad (2.3)$$

for all indices (2.1). The tensor $\tilde{T}$ is uniquely determined and $M^{(k_\sigma, l_\sigma)}$ are certain tensors of the type $(0, q - 2)$.

Proof. Let M_n be a Riemannian manifold with metric g . Now, let us consider an a priori chosen point x_0 at M_n . Values of metric g as well as all the given tensors will be considered in x_0 .

Taking a coordinate system in which the matrix of g fulfills

$$g_{ij} = g_0 \cdot D, \quad \text{where } g_0 \in R, \quad \text{and } D = \text{diag}(\pm 1, \pm 1, \dots, \pm 1), \quad (2.4)$$

we obtain the inverse matrix g^{-1} in the form of

$$g^{ij} = g_0^{-1} \cdot D. \quad (2.5)$$

The existence of the coordinated system above is guaranteed only for the concrete point x_0 . A metric (2.4) exists globally in Euclidean and pseudo-Euclidean spaces.

Therefore, the relation (2.3) may be written as

$$\sum_{k_\sigma, l_\sigma=1}^n \tilde{T}_{i_1 \dots i_{k_\sigma} \dots i_{l_\sigma} \dots i_q} \cdot g_{i_{k_\sigma} i_{l_\sigma}} = 0. \quad (2.6)$$

Let us define the inner product $\circ$ on T_q^0 by

$$\overset{1}{T} \circ \overset{2}{T} \stackrel{\text{def}}{=} \sum_{i_1, \dots, i_q=1}^n \overset{1}{T}_{i_1 \dots i_q} \cdot \overset{2}{T}_{i_1 \dots i_q}. \quad (2.7)$$

Now, let us construct a linear subspace $\overset{*}{T} \subseteq T_p^0$ which is generated by all the tensors of the form

$$\sum_{\sigma=1}^s g_{i_{k_\sigma} i_{l_\sigma}} \cdot M_{i_1, \dots, i_{k_\sigma-1} \quad i_{k_\sigma+1} \dots i_{l_\sigma-1} \quad i_{l_\sigma+1} \dots i_q}^{(k_\sigma, l_\sigma)}. \quad (2.8)$$

Considering a linear subspace in T_p^0 which contains all the tensors fulfilling (2.3), we obtain $\tilde{T} \circ \tilde{T}^* = 0$ for any tensors $\tilde{T}$ of this subspace in the obvious way. Therefore, this subspace is contained in the orthogonal complement of the subspace $\tilde{T}^*$, which implies the unicity of the decomposition (2.2) of the tensor $T_{i_1 \dots i_p}$. $\square$

3. SPECIAL DECOMPOSITIONS OF TENSORS

Comparing the decomposition in the following theorem with decompositions in Theorem 1 (and Theorem 4) as well as in the fundamental theorem proved by Weyl (see [8]), we may remark that in the following one, not only tensor $\tilde{T}$ is uniquely determined but also all tensors $\overset{(*)}{M}$.

Theorem 2. Let $T_{i_1 i_2 \dots i_q}$ be a $(0, q)$ -tensor and

$$(i_1, i_2), (i_1, i_3), \dots, (i_1, i_q) \quad (3.1)$$

be some pairs of indices. Then for $n > q - 1$ there exists the unique decomposition of $T_{i_1 \dots i_q}$ of the form

$$T_{i_1 \dots i_q} = \tilde{T}_{i_1 \dots i_q} + \sum_{\sigma=2}^n g_{i_1 i_\sigma} \cdot \overset{(1, \sigma)}{M}_{i_2, \dots, i_{\sigma-1} i_{\sigma+1} \dots i_q}^\sigma, \quad (3.2)$$

where $\tilde{T}_{i_1 \dots i_q}$ satisfies

$$\tilde{T}_{i_1 \dots i_\sigma \dots i_q} \cdot g^{i_1 i_\sigma} = 0, \quad (3.3)$$

for any pairs of indices (3.1). The tensor $\tilde{T}$ and all $(0, q - 2)$ -tensors $\overset{(i_1 i_\sigma)}{M}$ are uniquely determined.

Proof. With respect to Theorem 1 we are only to prove the uniqueness of the tensors $\overset{(i_1 i_\sigma)}{M}$. Contracting (3.2) with $g^{i_1 i_s}$ for $s = 2, \dots, n$ and using (3.3) for the tensor $\tilde{T}$, we obtain the following system of linear equations

$$\begin{aligned} T_{i_1 \dots i_q} g^{i_1 i_s} &= 0 + n \cdot \overset{(1, \sigma)}{M}_{i_2, \dots, i_{\sigma-1} i_{\sigma+1} \dots i_q}^\sigma \\ &+ \sum_{\sigma=2, \sigma \neq s}^q \overset{(i_1 i_\sigma)}{M}_{i_2, \dots, i_{\sigma-1} i_{\sigma+1} \dots i_{s-1} i_{s+1} \dots i_q}^\sigma. \end{aligned} \quad (3.4)$$

It follows from the fixed point theorem that such system of equations with variables $\overset{(i_1 i_2)}{M}, \dots, \overset{(i_1 i_n)}{M}$ has for $n > q - 1$ exactly one solution. $\square$

Let us recall the raising indices of tensors. If given a tensor T of the type $(0, p+q)$, we may construct a tensor of the type (p, q) by the following:

$$T_{j_1 \dots j_q}^{i_1 \dots i_p} \stackrel{\text{def}}{=} g^{i_1 \alpha_1} g^{i_2 \alpha_2} \dots g^{i_p \alpha_p} T_{\alpha_1 \dots \alpha_p j_1 \dots j_q}. \quad (3.5)$$

Raising indices, we obtain from Theorem 2 the known unique decomposition of the tensor $T_{i_2 \dots i_p}^{i_1}$:

$$T_{i_2 \dots i_p}^{i_1} = \tilde{T}_{i_2 \dots i_p}^{i_1} + \delta_{i_2}^{i_1} M_{i_3 \dots i_p}^{12} + \delta_{i_3}^{i_1} M_{i_2 i_4 \dots i_p}^{13} + \dots + \delta_{i_p}^{i_1} M_{i_2 \dots i_{p-1}}^{1p}, \quad (3.6)$$

where the tensor $\tilde{T}$ is traceless, i. e.,

$$\tilde{T}_{\dots \alpha \dots}^{\alpha} = 0 \quad (3.7)$$

and the tensor $\tilde{T}$ as well as the $(0, p-1)$ -tensors $\overset{1*}{M}$ are uniquely determined.

As we have mentioned above, the decomposition (3.6) is presented by Weyl in [8], but the uniqueness of tensors $\overset{1*}{M}$ is not contained there.

In view of (3.5), Theorem 1 implies immediately

Theorem 3. Let $T_{i_1 \dots i_p}^{j_1 \dots j_q}$ be a (p, q) -tensor and

$$(i_{k_1}, j_{l_1}), (i_{k_2}, j_{l_2}), \dots, (i_{k_s}, j_{l_s}), \quad s \leq p, q, \quad (3.8)$$

be pairs of indices satisfying $1 \leq k_\sigma \leq p$, $1 \leq l_\sigma \leq q$.

Then there exists a decomposition of $T_{i_1 \dots i_p}^{j_1 \dots j_q}$ of the form

$$T_{j_1 \dots j_q}^{i_1 \dots i_p} = \tilde{T}_{j_1 \dots j_q}^{i_1 \dots i_p} + \sum_{\sigma=1}^s \delta_{j_{l_\sigma}}^{i_{k_\sigma}} \cdot \overset{(k_\sigma)}{M}_{(l_\sigma) j_1, \dots, j_{l_\sigma-1} j_{l_\sigma+1}, \dots, j_q}^{i_1, \dots, i_{k_\sigma-1} i_{k_\sigma+1}, \dots, i_p}, \quad (3.9)$$

where the tensor $\tilde{T}$ satisfies

$$\tilde{T}_{j_1 \dots j_{l_\sigma} \dots j_q}^{i_1 \dots i_{k_\sigma} \dots i_p} \cdot \delta_{i_{k_\sigma}}^{j_{l_\sigma}} = 0, \quad (3.10)$$

for any indices (3.8). The tensor $\tilde{T}$ is uniquely determined and $\overset{(k_\sigma)}{M}_{(l_\sigma)}$ are certain tensors of the type $(p-1, q-1)$.

The condition (3.10) means that the tensor $\tilde{T}$ is traceless over any pair of indices (3.8). This may be expressed by

$$\tilde{T}_{j_1 \dots j_{l_\sigma-1} j_{l_\sigma+1} \dots j_q}^{i_1 \dots i_{k_\sigma-1} i_{k_\sigma+1} \dots i_p} = 0. \quad (3.11)$$

Let us remark that Theorem 3 holds on every manifold since the metric tensor (which was used in the proof of Theorem 1) may be constructed in any case.

Immediately, we obtain the following Theorem, which is presented in [3, 6].

Theorem 4. Let $T_{i_1 \dots i_p}^{j_1 \dots j_p}$ be a tensor of the type (p, q) . If $n + 1 \geq p + q$, then there exists the unique decomposition of $T_{i_1 \dots i_p}^{j_1 \dots j_p}$ in the following form.

$$T_{i_1 \dots i_p} = \tilde{T}_{i_1 \dots i_p} + \sum_{t=1}^{\min\{p, q\}} \sum_{\oplus} \delta_{j_{\sigma_1}}^{i_{\rho_1}} \delta_{j_{\sigma_2}}^{i_{\rho_2}} \dots \delta_{j_{\sigma_t}}^{i_{\rho_t}} M_{\dots}^{\star}, \quad (3.12)$$

where

$$\begin{aligned} \oplus &= \left\{ \begin{array}{ll} \rho_1, \rho_2, \dots, \rho_t = 1, 2, \dots, p & (\rho_1 < \rho_2 < \dots < \rho_t) \\ \sigma_1, \sigma_2, \dots, \sigma_t = 1, 2, \dots, q & (\sigma_i \text{ are mutually different}) \end{array} \right\}, \\ \star &= \left\{ \begin{array}{cccc} \rho_1 & \rho_2 & \dots & \rho_t \\ \sigma_1 & \sigma_2 & \dots & \sigma_t \end{array} \right\} \end{aligned} \quad (3.13)$$

and tensors $\tilde{T}_{i_1 \dots i_p}$ and $M^{\star}$ are traceless.

4. SPECIAL DIFFERENTIAL OPERATOR L

Now, let us introduce the following concept on tensors of the type (p, q) .

Definition 1. A map $L : T_q^p \mapsto T_q^p$ is said to be a *special differential operator* if for any tensors $\overset{1}{T}, \overset{2}{T}$ the following properties are satisfied:

$$\begin{aligned} L(\overset{1}{T} \pm \overset{2}{T}) &= L\overset{1}{T} \pm L\overset{2}{T}, \\ L(\overset{1}{T} \cdot \overset{2}{T}) &= L\overset{1}{T} \cdot \overset{2}{T} + \overset{1}{T} \cdot L\overset{2}{T}, \\ Lg &= 0; \quad L\delta = 0. \end{aligned} \quad (4.1)$$

By δ we denote the Kronecker tensor.

Let us give some examples of the operator L :

- (a) ∇ — the covariant derivative, i. e., ∇ is an affine connection on M_n
- (b) $\circ R$ — “semisymmetric” operator; $T \circ R$ is in the coordinate description defined by

$$\nabla_l \nabla_m T_{i_1 \dots i_q}^{h_1 \dots h_p} - \nabla_m \nabla_l T_{i_1 \dots i_q}^{h_1 \dots h_p}.$$

Using the Bianci identity, we may write $T \circ R$ in the form

$$-T_{i_1 \dots i_q}^{\alpha \dots h_p} R_{\alpha lm}^{h_1} - \dots - T_{i_1 \dots i_q}^{h_1 \dots \alpha} R_{\alpha lm}^{h_p} + T_{\alpha \dots i_q}^{h_1 \dots h_p} R_{i_1 lm}^{\alpha} + \dots + T_{i_1 \dots \alpha}^{h_1 \dots h_p} R_{i_q lm}^{\alpha}, \quad (4.2)$$

where R_{ijk}^h are components of a Riemannian tensor R on M_n .

- (c) $\circ Z$ — “pseudosymmetric” operator; this operator is defined analogously as the previous one — in the formula (4.2), we replace the tensor R by the tensor Z defined as follows

$$Z_{ijk}^h = R_{ijk}^h - B(\delta_k^h g_{ij} - \delta_j^h g_{ik}),$$

where B is a function on M_n .

5. ON MANIFOLDS WITH $LT = 0$

It is well known that a symmetric manifold in the sense of Cartan is characterized by the condition

$$\nabla R = 0.$$

Projective symmetric and conformal symmetric manifolds are characterized by the conditions

$$\nabla W = 0 \text{ and } \nabla C = 0, \text{ respectively;}$$

by W and C we denote the Weyl tensors of projective and conformal curvature, respectively.

Semisymmetric and pseudosymmetric manifolds are characterized by the conditions

$$R \circ R = 0 \text{ and } R \circ Z = 0, \text{ respectively.}$$

Further, projective and conformal semisymmetric manifolds are characterized by

$$W \circ R = 0 \text{ and } C \circ R = 0, \text{ respectively.} \quad (5.1)$$

Replacing in (5.1) the Riemannian tensor by the Weyl tensor, we obtain the characterization of projective pseudosymmetric and conformal pseudosymmetric manifolds.

It follows from these remarks that it is useful to investigate manifolds with a general condition $LT = 0$.

Applying Definition 1 of the operator L and Theorem 1, we may prove the following lemmas.

Lemma 1. *Let T be a tensor of the type $(0, p)$ with the decomposition (2.2) for the selected pairs of indices (2.1). If $L\tilde{T} = 0$ and $L\tilde{M}^* = 0$, where $*$ = (k_σ, l_σ) , then $LT = 0$.*

Lemma 2. *Let T be a tensor of the type $(0, p)$ with the decomposition (2.2) for the selected pairs of indices (2.1). If $LT = 0$ then $L\tilde{T} = 0$.*

Consider a $C^\infty(M)$ -module of k -differential forms $\Omega^k(M)$ on a manifold M_n . Then for any k -differential form $\alpha \in \Omega^k(M)$, the special differential operator L may be determined by the relation $LT = \alpha T$.

Let a 1-differential form $\omega \in \Omega^1(M)$ with

$$\nabla_X T = \omega(X) \cdot T \quad (5.2)$$

be given (see [1]). Then we speak about *T-recurrency*. The following special cases are well-known:

- ◇ $\nabla R = \omega \cdot R$, the *Walker recurrency* (or simply *recurrency*),
- ◇ $\nabla W = \omega \cdot W$, the *projective recurrency*,
- ◇ $\nabla C = \omega \cdot C$, the *conformal recurrency*,
- ◇ $\nabla \text{Ric} = \omega \cdot \text{Ric}$, the *Ricci recurrency*.

Therefore, it will be useful to deal with the case (5.2).

Considering a 1-differential form ω with (5.2) and choosing indices according to the Theorem 2, we may prove the following theorem.

Theorem 5. *Let T be a $(0, p)$ -tensor decomposed in the sense of (3.2). Then*

$$LT = \omega T, \quad (5.3)$$

if and only if

$$L\tilde{T} = \omega\tilde{T} \quad \text{and} \quad L\overset{1\sigma}{M} = \omega\overset{1\sigma}{M} \quad (5.4)$$

where ω is a k -differential form in $\Omega^k(M)$.

Proof. Consider a decomposition (3.2) of T for pairs of indices (3.1). First, let us suppose

$$L\tilde{T} = \omega\tilde{T} \quad \text{and} \quad L\overset{1\sigma}{M} = \omega\overset{1\sigma}{M}. \quad (5.5)$$

Substituting (5.5) into (3.2), we obtain immediately $LT = \omega T$.

Now, let us suppose $LT = \omega T$. Applying L to the expression (3.2) of T , we have

$$\begin{aligned} LT_{i_1 \dots i_p} &= L\tilde{T}_{i_1 \dots i_p} + g_{i_1 i_2} \overset{12}{L\tilde{M}}_{i_3 \dots i_p} + g_{i_1 i_3} \overset{13}{L\tilde{M}}_{i_2 i_4 \dots i_p} + \dots \\ &\quad + g_{i_1 i_p} \overset{1p}{L\tilde{M}}_{i_2 \dots i_{p-1}} \end{aligned} \quad (5.6)$$

i. e.

$$\begin{aligned} \omega T &= L\tilde{T}_{i_1 \dots i_p} + g_{i_1 i_2} \overset{12}{\omega\tilde{M}}_{i_3 \dots i_p} + g_{i_1 i_3} \overset{13}{\omega\tilde{M}}_{i_2 i_4 \dots i_p} \\ &\quad + \dots + g_{i_1 i_p} \overset{1p}{\omega\tilde{M}}_{i_2 \dots i_{p-1}}. \end{aligned} \quad (5.7)$$

Applying the form ω to the expression of T in the form (3.2), we obtain

$$\begin{aligned} \omega T &= L\tilde{T}_{i_1 \dots i_p} + g_{i_1 i_2} \overset{12}{\omega\tilde{M}}_{i_3 \dots i_p} + g_{i_1 i_3} \overset{13}{\omega\tilde{M}}_{i_2 i_4 \dots i_p} \\ &\quad + \dots + g_{i_1 i_p} \overset{1p}{\omega\tilde{M}}_{i_2 \dots i_{p-1}}. \end{aligned} \quad (5.8)$$

Comparing (5.7) and (5.8), we have

$$\begin{aligned} (L\tilde{T} - \omega\tilde{T}) &+ (g_{i_1 i_2} L\overset{12}{M}_{i_3 \dots i_p} - g_{i_1 i_2} \omega\overset{12}{M}_{i_3 \dots i_p}) \\ &+ (g_{i_1 i_3} L\overset{13}{M}_{i_2 i_4 \dots i_p} - g_{i_1 i_3} \omega\overset{13}{M}_{i_2 i_4 \dots i_p}) + \\ &+ \dots + (g_{i_1 i_p} L\overset{1p}{M}_{i_2 \dots i_{p-1}} - g_{i_1 i_p} \omega\overset{1p}{M}_{i_2 \dots i_{p-1}}) = 0. \end{aligned} \quad (5.9)$$

Contracting (5.9) with $g^{i_1 i_\sigma}$ for $\sigma = 2, \dots, n$, we deduce a system of linear equations as follows

$$n \cdot (L\overset{1\sigma}{M} - \omega\overset{1\sigma}{M}) + \sum_{\tau=2, \tau \neq \sigma}^p (L\overset{1\tau}{M} - \omega\overset{1\tau}{M}) = 0.$$

For $n > p - 1$, such a system has the unique solution $L\overset{1\sigma}{M} - \omega\overset{1\sigma}{M} = 0$ for $\sigma = 2, \dots, n$, which implies that $L\tilde{T} - \omega\tilde{T} = 0$. $\square$

The following theorem is the consequence of Theorem 5.

Theorem 6. *Let T be a $(0, p)$ -tensor decomposed in the sense of (3.2). Then*

$$LT = 0$$

if and only if $L\tilde{T} = 0$ and $L\overset{1\sigma}{M} = 0$.

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REDUCTION OF THE CENTROPROJECTIVE CONNECTION OF THE PROJECTIVE GROUP TO THE FUNDAMENTAL-GROUP CONNECTION OF A SURFACE

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Abstract. The projective group effectively acting in the multi-measured projective space is represented by the principal bundle of the centroprojective frames, in which a symmetrical centroprojective connection is defined. A surface is considered in the projective space. The principal bundle above the surface with the typical fiber — a subgroup of the stationarity for the centered tangent plane to the surface in a fixed point is considered. Fundamental-group connection is given in this fibering. It consists of tangent affine and centroprojective, normal linear and affine-group connections. It is shown that the centroprojective connection in the projective group is reduced to the fundamental-group connection in the principal bundle associated with the surface in the projective space.

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1. CENTROPROJECTIVE CONNECTION IN THE PROJECTIVE GROUP

In n -dimensional projective space P_n , let us consider the moving frame $\{A, A_I\}$ ($I, \dots = \overline{1, n}$) with the derivation formulas

$$dA = \vartheta A + \theta^I A_I, \quad dA_I = \vartheta A_I + \theta_I^J A_J + \theta_I A, \quad (1.1)$$

where the form ϑ plays the role of a proportionality factor and the structure forms $\theta^I, \theta_J^I, \theta_I$ of the projective group $GP(n)$, which acts effectively in the space P_n , satisfy the Cartan equations (see, e. g., [3])

$$D\theta^I = \theta^J \wedge \theta_J^I, \quad (1.2_1)$$

$$D\theta_J^I = \theta_J^K \wedge \theta_K^I + \theta^K \wedge \theta_{JK}^I, \quad (1.2_2)$$

$$D\theta_I = \theta_I^J \wedge \theta_J; \quad (1.2_3)$$

$$\theta_{JK}^I = -\delta_J^I \theta_K - \delta_K^I \theta_J. \quad (1.3)$$

The equations (1.2) are the structure equations of the principal fiber bundle of the centroprojective frames $C_{n(n+1)}(P_n)$, whose base is the projective space P_n (a region circumscribed by a point A). The centroprojective (coaffine) group $C_{n(n+1)} =$

$GA^*(n) \subset GP(n)$ is the typical fiber. This group acts in any centroprojective space P_n^0 that is the space P_n with the fixed point A .

Proposition 1. *The fibering $C_{n(n+1)}(P_n)$ of the centroprojective frames has the principal quotient bundle $L_{n^2}(P_n)$ with the structure equations (1.2₁), (1.2₂) the same base P_n , and typical fiber $L_{n^2} = GL(n)$, where $GL(n)$ is the linear quotient group acting [2] ineffectively in the pencil of lines passing through the point A of the projective space P_n (this group acts in the projective quotient space $P_{n-1} = P_n/A$ (see, e. g., [1])).*

We give a centroprojective connection on the principal fiber bundle $C_{n(n+1)}(P_n)$ using Laptev–Lumiste method [4, 6, 8] by the forms

$$\hat{\theta}_J^I = \theta_J^I - \Pi_{JK}^I \theta^K, \quad \hat{\theta}_I = \theta_I - \Pi_{IJ} \theta^J, \quad (1.4)$$

where the components of the centroprojective connection object $\Pi = \{\Pi_{JK}^I, \Pi_{IJ}\}$ satisfy the differential equations

$$\Delta \Pi_{JK}^I + \theta_{JK}^I = \Pi_{JKL}^I \theta^L, \quad (1.5_1)$$

$$\Delta \Pi_{IJ} + \Pi_{IJ}^K \theta_K = \Pi_{IJK} \theta^K. \quad (1.5_2)$$

The tensor operator Δ acts by

$$\Delta \Pi_{JK}^I = d \Pi_{JK}^I + \Pi_{JK}^L \theta_L^I - \Pi_{LK}^I \theta_J^L - \Pi_{JL}^I \theta_K^L.$$

Proposition 2. *The centroprojective connection object Π is a quasitensor containing the quasitensor Π_{JK}^I which determines an affine (special linear) connection on the fibering $L_{n^2}(P_n)$ of linear frames.*

The forms of centroprojective connection (1.4) satisfy the structure equations

$$D \hat{\theta}_J^I = \hat{\theta}_J^K \wedge \hat{\theta}_K^I + K_{JKL}^I \theta^K \wedge \theta^L, \quad (1.6_1)$$

$$D \hat{\theta}_I = \hat{\theta}_I^J \wedge \hat{\theta}_J + K_{IJK} \theta^J \wedge \theta^K. \quad (1.6_2)$$

The components of the centroprojective curvature object $K = \{K_{JKL}^I, K_{IJK}\}$ are defined by

$$K_{JKL}^I = \Pi_{J[KL]}^I - \Pi_{J[K}^M \Pi_{ML]}^I, \quad K_{IJK} = \Pi_{I[JK]} - \Pi_{I[J}^L \Pi_{LK]}.$$

The square brackets are alternation in the extreme indices in these brackets. These components satisfy the differential comparisons modulo basis forms θ^I of the space P_n :

$$\Delta K_{JKL}^I \cong 0, \quad \Delta K_{IJK} + K_{IJK}^L \theta_L \cong 0.$$

Proposition 3. *The centroprojective curvature object K is a tensor containing the subtensor of the affine curvature K_{JKL}^I .*

We put the affine connection forms (1.4₁) into the structure equations (1.2₁) of the base P_n

$$D\theta^I = \theta^J \wedge \hat{\theta}_J^I + S_{JK}^I \theta^J \wedge \theta^K, \quad (1.7)$$

where $S_{JK}^I = \Pi_{[JK]}^I$ are the components of the affine torsion object of the centroprojective connection. The differential equations (1.5₁) with the symmetry of the forms θ_{JK}^I (1.3) in the lower indices imply the comparisons $\Delta S_{JK}^I \cong 0$.

Proposition 4. *The bundle $C_{n(n+1)}(P_n)$ of centroprojective frames with given centroprojective connection is the space $C_{n(n+1),n}$ of the centroprojective connection with closed structure equations (1.6), (1.7) which contain the affine torsion tensor S_{JK}^I and the centroprojective curvature object K . The space $C_{n(n+1),n}$ has the quotient space of an affine connection $L_{n^2,n}$ (1.7), (1.6₁) with the torsion S_{JK}^I and curvature K_{JKL}^I tensors.*

According to [5], we can introduce the torsion object (centroprojective torsion object) of the centroprojective connection $S = \{S_{JK}^I, S_{IJ}\}$, where $S_{IJ} = \Pi_{[IJ]}$. The differential equations (1.5₂) imply

$$\Delta S_{IJ} + S_{IJ}^K \theta_K \cong 0.$$

Proposition 5. *The centroprojective torsion object S is a tensor containing the affine torsion subtensor S_{JK}^I .*

Definition. The centroprojective connection is an affine symmetric connection (an affine torsion-free connection) if $S_{JK}^I = 0$. The centroprojective connection is a symmetric connection (a centroprojective torsion-free connection) if $S = 0$.

Conclusion 1. *The forms θ_{JK}^I (1.3) in the differential equations (1.5₁) for the components of the affine subconnection object Π_{JK}^I are symmetric forms. Therefore, the affine connection have to be symmetric connection ($\Pi_{[JK]}^I = 0$). There are the symmetric components Π_{JK}^I in the differential equations (1.5₂) for the components Π_{IJ} of the object Π hence Π_{IJ} are symmetric components ($\Pi_{[IJ]} = 0$). Therefore, we can put only symmetric centroprojective connection into the projective group $GP(n) = C_{n(n+1)}(P_n)$.*

2. FUNDAMENTAL-GROUP CONNECTION ASSOCIATED WITH A SURFACE

In the projective space P_n we shall consider m -dimensional surface S_m ($1 \leq m < n$) as the family of the centered tangent planes T_m . Let us partition the indices set as:

$$I = (i, a); \quad i, \dots = \overline{1, m}; \quad a, \dots = \overline{m+1, n}.$$

Let us put the tops A, A_i of the moving frame $\{A, A_I\}$ on the tangent plane T_m so that the top A coincides with the tangent point. According to (1.1), let us write the

equations of the surface S_m in the form

$$\theta^a = 0, \quad (2.1_1)$$

$$\theta_i^a = \Lambda_{ij}^a \theta^j. \quad (2.1_2)$$

Closing the equations (2.1₁) we obtain $\Lambda_{[ij]}^a = 0$. Prolonging (2.1₂) we have

$$\Delta \Lambda_{ij}^a \cong 0, \quad (2.2_1)$$

$$\Delta \Lambda_{ij}^a = \partial \Lambda_{ij}^a + \Lambda_{ij}^b \omega_b^a - \Lambda_{kj}^a \omega_i^k - \Lambda_{ik}^a \omega_j^k, \quad (2.2_2)$$

where $\partial = d|_{S_m}$, $\omega = \theta|_{S_m}$, the symbol $\cong$ is the comparison modulo basis forms θ^i of the surface S_m .

Eliminating the principal forms θ^i , θ^a , θ_i^a of the equations (2.1) for the surface S_m from the structure forms θ^I , θ_J^I , θ_I of the projective group $GP(n)$ we keep the secondary forms. They are called the fibre forms on the surface S_m . The basis forms θ^i and fiber forms ω_j^i , ω_i , ω_b^a , ω_a^i , ω_a satisfy the structure equations [1, 9]

$$D\theta^i = \theta^j \wedge \omega_j^i; \quad (2.3)$$

$$D\omega_j^i = \omega_j^k \wedge \omega_k^i + \theta^k \wedge \omega_{jk}^i, \quad (2.4_1)$$

$$\omega_{jk}^i = \Lambda_{jk}^a \omega_a^i - \delta_j^i \omega_k - \delta_k^i \omega_j; \quad (2.4_2)$$

$$D\omega_i = \omega_i^j \wedge \omega_j + \theta^i \wedge \omega_{ij}, \quad (2.5_1)$$

$$\omega_{ij} = \Lambda_{ij}^a \omega_a; \quad (2.5_2)$$

$$D\omega_b^a = \omega_b^c \wedge \omega_c^a + \theta^i \wedge \omega_{bi}^a, \quad \omega_{bi}^a = -\Lambda_{ij}^a \omega_b^j - \delta_b^a \omega_i; \quad (2.6)$$

$$D\omega_a^i = \omega_a^j \wedge \omega_j^i + \omega_a^b \wedge \omega_b^i + \theta^j \wedge \omega_{aj}^i, \quad (2.7_1)$$

$$\omega_{aj}^i = -\delta_j^i \omega_a; \quad (2.7_2)$$

$$D\omega_a = \omega_a^i \wedge \omega_i + \omega_a^b \wedge \omega_b. \quad (2.8)$$

We obtain the structure equations (2.3)–(2.8) of the principal bundle $G_r(S_m)$ associated with S_m . The surface S_m is a base of the principal bundle $G_r(S_m)$. The subgroup of stationarity $G_r \subset GP(n)$ of the centered tangent plane T_m is a typical fiber. We have

$$r = \dim G_r = n(n+1) - m(n-m).$$

Associated fibering $G_r(S_m)$ has 4 simple quotient principal bundles [10] over the base S_m with the structure equations:

- (1) (2.3), (2.4) — the tangent linear frame fibering $L_{m^2}(S_m)$ with the typical fiber $L_{m^2} = GL(m)$, where $GL(m)$ is linear quotient group acting ineffectively on the bunch of tangent lines passing through the center A (on the quotient space $T_{m-1} = T_m/A$);

- (2) (2.3)–(2.5) — the tangent centroprojective frame fibering $C_{m(m+1)}(S_m)$ with the typical fiber $C_{m(m+1)} = GA^*(m)$, where $GA^*(m)$ is centroprojective (coaffine) quotient group acting effectively on the centered tangent plane T_m ;
- (3) (2.3), (2.6) — the normal linear frame fibering $L_{(n-m)^2}(S_m)$ with the typical fiber $L_{(n-m)^2} = GL(n-m)$, where $GL(n-m)$ is linear quotient group acting ineffectively in the $(n-m-1)$ -dimensional projective space $P_{n-m-1} = P_n/T_m$ (see, e. g., [1]);
- (4) (2.3), (2.4), (2.6), (2.7) — the fibering $H_{m^2-mn+n^2}(S_m)$ with the typical fiber $H_{m^2-mn+n^2}$. This fiber is:
- (a) stationarity subgroup for the centered tangent plane to m -dimensional surface of n -dimensional affine space A_n ,
 - (b) stationarity subgroup for the tangent straight lines subbunch on the straight line bunch of the space P_n with the center A ,
 - (c) stationarity subgroup for the quotient plane $T_{m-1} = T_m/A$ in P_n/A ,
 - (d) an affine quotient group [11] of the projective subgroup G_r .

According to the Laptev–Lumiste method, a connection in the principal bundle $G_r(S_m)$ is defined by the forms

$$\begin{aligned}\tilde{\omega}_j^i &= \omega_j^i - \Gamma_{jk}^i \theta^k, & \tilde{\omega}_i &= \omega_i - \Gamma_{ij} \theta^j, \\ \tilde{\omega}_b^a &= \omega_b^a - \Gamma_{bi}^a \theta^i, & \tilde{\omega}_a^i &= \omega_a^i - \Gamma_{aj}^i \theta^j, & \tilde{\omega}_a &= \omega_a - \Gamma_{ai} \theta^i,\end{aligned}\quad (2.9)$$

and the components of the fundamental-group connection object

$$\Gamma = \{\Gamma_{jk}^i, \Gamma_{ij}, \Gamma_{bi}^a, \Gamma_{aj}^i, \Gamma_{ai}\}$$

satisfy the differential equations [9]

$$\Delta \Gamma_{jk}^i + \omega_{jk}^i = \Gamma_{jkl}^i \theta^l, \quad (2.10_1)$$

$$\Delta \Gamma_{ij} + \Gamma_{ij}^k \omega_k + \omega_{ij} = \Gamma_{ijk} \theta^k, \quad (2.10_2)$$

$$\Delta \Gamma_{bi}^a + \omega_{bi}^a = \Gamma_{bij}^a \theta^j, \quad (2.10_3)$$

$$\Delta \Gamma_{aj}^i + \Gamma_{aj}^b \omega_b^i - \Gamma_{kj}^i \omega_a^k + \omega_{aj}^i = \Gamma_{ajk}^i \theta^k, \quad (2.10_4)$$

$$\Delta \Gamma_{ai} + \Gamma_{ai}^j \omega_j + \Gamma_{ai}^b \omega_b - \Gamma_{ji} \omega_a^j = \Gamma_{aij} \theta^j. \quad (2.10_5)$$

The connection object Γ has 4 simple subobjects [10]: Γ_{jk}^i — the tangent affine connection object, $\{\Gamma_{jk}^i, \Gamma_{ij}\}$ — the tangent centroprojective connection object, Γ_{bi}^a — the normal linear connection object, $\{\Gamma_{jk}^i, \Gamma_{bi}^a, \Gamma_{aj}^i\}$ — affine-group connection object. These subobjects give the fundamental-group connections in the corresponding quotient fiberings of the associated fibering $G_r(S_m)$.

Conclusion 2. *The forms ω_{jk}^i (2.4₂), ω_{ij} (2.5₂) from the differential equations (2.10₁) and (2.10₂) for the components of the centroprojective connection object*

$\{\Gamma_{jk}^i, \Gamma_{ij}\}$ are symmetric forms. Therefore, on the surface S_m we can consider only connection centroprojective torsion-free: $\Gamma_{[jk]}^i = 0, \Gamma_{[ij]} = 0$.

The fundamental-group connection forms (2.9) satisfy the structure equations

$$\begin{aligned} D\tilde{\omega}_j^i &= \tilde{\omega}_j^k \wedge \tilde{\omega}_k^i + R_{jkl}^i \theta^k \wedge \theta^l, & D\tilde{\omega}_i &= \tilde{\omega}_i^j \wedge \tilde{\omega}_j + R_{ijk} \theta^j \wedge \theta^k, \\ D\tilde{\omega}_b^a &= \tilde{\omega}_b^c \wedge \tilde{\omega}_c^a + R_{bij}^a \theta^i \wedge \theta^j, & D\tilde{\omega}_a^i &= \tilde{\omega}_a^j \wedge \tilde{\omega}_j^i + \tilde{\omega}_a^b \wedge \tilde{\omega}_b^i + R_{ajk}^i \theta^j \wedge \theta^k, \\ D\tilde{\omega}_a &= \tilde{\omega}_a^i \wedge \tilde{\omega}_i + \tilde{\omega}_a^b \wedge \tilde{\omega}_b + R_{aij} \theta^i \wedge \theta^j, \end{aligned}$$

where the components of curvature object

$$R = \{R_{jkl}^i, R_{ijk}, R_{bij}^a, R_{ajk}^i, R_{aij}\}$$

of the fundamental-group connections Γ are expressed by the formulas

$$\begin{aligned} R_{jkl}^i &= \Gamma_{j[kl]}^i - \Gamma_{j[k}^m \Gamma_{m]l}^i, & R_{ijk} &= \Gamma_{i[jk]} - \Gamma_{i[j}^l \Gamma_{l]k}, & R_{bij}^a &= \Gamma_{b[ij]}^a - \Gamma_{b[i}^c \Gamma_{c]j}^a, \\ R_{ajk}^i &= \Gamma_{a[jk]}^i - \Gamma_{a[j}^l \Gamma_{l]k}^i - \Gamma_{a[j}^b \Gamma_{bk]}^i, & R_{aij} &= \Gamma_{a[ij]} - \Gamma_{a[i}^k \Gamma_{kj]} - \Gamma_{a[i}^b \Gamma_{bj]}. \end{aligned}$$

These components satisfy the differential comparisons [10]

$$\begin{aligned} \Delta R_{jkl}^i &\cong 0, & \Delta R_{ijk} + R_{ijk}^l \omega_l &\cong 0, & \Delta R_{bij}^a &\cong 0, \\ \Delta R_{ajk}^i - R_{ljk}^i \omega_a^l + R_{ajk}^b \omega_b^i &\cong 0, & \Delta R_{aij} + R_{aij}^k \omega_k + R_{aij}^b \omega_b - R_{kij} \omega_a^k &\cong 0. \end{aligned}$$

Theorem 1. *The curvature object R of the fundamental-group connection Γ is a tensor containing:*

- (1) the curvature tensor R_{jkl}^i of the tangent affine connection Γ_{jk}^i ,
- (2) the curvature tensor $\{R_{jkl}^i, R_{ijk}\}$ of the tangent centroprojective connection $\{\Gamma_{jk}^i, \Gamma_{ij}\}$,
- (3) the curvature tensor R_{bij}^a of the normal linear connection Γ_{bi}^a ,
- (4) the curvature tensor $\{R_{jkl}^i, R_{bij}^a, R_{ajk}^i\}$ of the affine-group connection $\{\Gamma_{jk}^i, \Gamma_{bi}^a, \Gamma_{aj}^i\}$.

3. REDUCTION OF THE CENTROPROJECTIVE CONNECTION

We define a symmetric centroprojective connection object $\Pi = \{\Pi_{JK}^I, \Pi_{IJ}\}$. By the partition of each index into two indices the object Π will consist from the following essential components

$$\Pi = \{\Pi_{jk}^i, \Pi_{ij}^a, \Pi_{aj}^i, \Pi_{bi}^a, \Pi_{ab}^i, \Pi_{bc}^a, \Pi_{ij}, \Pi_{ai}, \Pi_{ab}\}.$$

In the differential equations (1.5) for the components of the object Π we expand the action of the operator Δ and partition the indices. We take the equations (2.1) of

the surface S_m , apply the operator Δ to the subobjects and write result in the comparisons form for the components analogical to the components of the fundamental object Λ_{ij}^a and connection object Γ :

$$\Delta \bar{\Pi}_{ij}^a \cong 0, \quad (3.1_1)$$

$$\Delta \bar{\Pi}_{jk}^i + \bar{\Pi}_{jk}^a \omega_a^i - \delta_j^i \omega_k - \delta_k^i \omega_j \cong 0, \quad (3.1_2)$$

$$\Delta \bar{\Pi}_{bi}^a - \bar{\Pi}_{ji}^a \omega_b^j - \delta_b^a \omega_i \cong 0, \quad (3.1_3)$$

$$\Delta \bar{\Pi}_{aj}^i + \bar{\Pi}_{aj}^b \omega_b^i - \bar{\Pi}_{kj}^i \omega_a^k - \delta_j^i \omega_a \cong 0, \quad (3.1_4)$$

$$\Delta \bar{\Pi}_{ij} + \bar{\Pi}_{ij}^k \omega_k + \bar{\Pi}_{ij}^a \omega_a \cong 0, \quad (3.1_5)$$

$$\Delta \bar{\Pi}_{ai} - \bar{\Pi}_{ji} \omega_a^j + \bar{\Pi}_{ai}^j \omega_j + \bar{\Pi}_{ai}^b \omega_b \cong 0. \quad (3.1_6)$$

In (3.1) $\bar{\Pi} = \Pi|_{S_m}$. From the coincidence of the differential comparisons (2.2₁) and (3.1₁) we have

Lemma 1. *The subobject Π_{ij}^a (of the centroprojective connection object Π without the torsion) restricted to the surface S_m is identified with the fundamental object Λ_{ij}^a :*

$$\bar{\Pi}_{ij}^a = \Lambda_{ij}^a. \quad (3.2)$$

Comparing the differential equations (2.10) with the forms (2.4₂)–(2.7₂) and the comparisons (3.1₂)–(3.1₆) by means of Lemma 1 we obtain the equalities

$$\Gamma_{jk}^i = \bar{\Pi}_{jk}^i, \quad \Gamma_{ij} = \bar{\Pi}_{ij}, \quad \Gamma_{bi}^a = \bar{\Pi}_{bi}^a, \quad \Gamma_{aj}^i = \bar{\Pi}_{aj}^i, \quad \Gamma_{ai} = \bar{\Pi}_{ai}.$$

Theorem 2. *The symmetric centroprojective connection on the projective group GP(n) effectively acting in projective space P_n is reduced to the fundamental-group connection on the surface S_m of the space P_n .*

Remark 1. The components $\Pi_{ab}^i, \Pi_{bc}^a, \Pi_{ab}$ of the connection object Π are not used (compare [7, 12]) for the reduction of the object Π to the object Γ .

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ON THE DEGREE OF GEODESIC MOBILITY OF RIEMANNIAN MANIFOLDS

ELENA STEPANOVA AND MARIE CHODOROVÁ

Abstract. We will consider the definition of the degree r of geodesic mobility of a Riemannian manifold and will prove that the well-known results on the geodesic mobility are essentially local. In the last section of our paper, we will prove that the geodesic mobility of an n -dimensional closed, simply-connected Riemannian manifold with positive constant sectional curvature satisfies the identity $r = (n + 1)(n + 2)/2, n \geq 2$.

2000 *Mathematics Subject Classification:* 53C20; 53C21; 53C22

Keywords: Riemannian manifold, geodesic mapping, degree of geodesic mobility

1. INTRODUCTION

The solving of the problem of geodesic mapping of Riemannian manifolds was started by Levi-Civita (see [5]) who studied the trajectory equivalence of dynamical systems. The definition of the degree of geodesic mobility of a Riemannian manifold was introduced by Sinyukov (see [12, p. 137], see [2, 8], [11, p. 168]) as a numerical characteristic of the cardinality of the geodesic class. The method for finding the degree of geodesic mobility is based on an analysis of the integrability conditions of fundamental equations of geodesic mappings (see [12, pp. 137–147]). Interesting results of global geodesic mappings for special Riemannian manifolds are formulated in papers [1, 6–11].

In this paper, we will prove that the well-known results on the geodesic mobility are essentially local and that the geodesic mobility r of an n -dimensional closed, simply-connected Riemannian manifold with positive constant sectional curvature satisfies the identity

$$r = (n + 1)(n + 2)/2.$$

2. THE DEGREE OF GEODESIC MOBILITY

Suppose that (M, g) is an n -dimensional Riemannian manifold and ∇ is the Levi-Civita connection corresponding to the metric tensor $g = (g_{ij}), n \geq 2$.

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A mapping of the Riemannian manifold (M, g) to a Riemannian manifold (M', g') is called *geodesic* if it takes every geodesic of (M, g) to a geodesic of (M', g') .

It is well known (see, for example [12, p. 137], [8], [11, p. 168]) that the Riemannian manifold (M, g) admits a geodesic mapping if and only if there exists a covariant regular symmetric tensor field $a = (a_{ij})$ on this manifold such that

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik} \quad (2.1)$$

where $\lambda_i = \frac{1}{2} \nabla_i (\text{trace}_g a)$. If $\lambda_i = 0$, then the geodesic mapping is called *trivial*.

The equations (2.1) have been completed by Sinyukov to a system of differential equations of Cauchy type in covariant derivatives (see [12, p. 134]; [8]; [11, p. 168]). In particular, Sinyukov has proved that if the system of differential equations (2.1) is a totally-integrable system of differential equations in a neighborhood of an arbitrary point of Riemannian manifold (M, g) , then (M, g) is a manifold of constant curvature (see [12, p. 138]).

The *degree of geodesic mobility* r of an n -dimensional Riemannian manifold (M, g) is the dimension of linear space of solutions of system of differential equations (2.1). The notion of geodesic mobility of a Riemannian manifold was introduced by Sinyukov (see [12, p. 137], [8]). From Sinyukov's theorem, we conclude that the n -dimensional Riemannian manifolds of constant curvature and only these manifolds have the maximal degree $r = (n + 1)(n + 2)/2$ for $n \geq 2$.

On the other hand, an arbitrary Riemannian manifold has mobility no smaller than 1, because system (2.1) has at least one solution of the form $a = \alpha g$ for an arbitrary real constant α on this manifold. Therefore, we can conclude that $1 \leq r \leq (n + 1)(n + 2)/2$ for any n -dimensional Riemannian manifold (M, g) .

The result formulated in Sinyukov's theorem and other results on the geodesic mobility (see [11, pp. 147-149]) are essentially local. For example, we consider an n -dimensional hyperbolic space, which is a Riemannian manifold (M, g) of constant negative curvature. In this case, we have $r = (n + 1)(n + 2)/2$. If we factorize hyperbolic space by a suitable discrete group of motions (see [13]), we obtain a compact manifold of constant negative curvature.

It is well known (see [7]) that on a compact Riemannian manifold with non positive constant curvature, any tensor field $a = (a_{ij})$ satisfying the equations (2.1) has the trivial form $a = \text{const} \cdot g$. Therefore, this manifold does not admit a "globally defined" non-trivial geodesic mapping, i. e. $r = 1$. Finally, we can conclude that the number $r = (n + 1)(n + 2)/2$ is a *local* numerical characteristic of the cardinality of the geodesic class of a hyperbolic space. On the other hand, if we consider the number r as a *global* numerical characteristic of the cardinality of the geodesic class of a compact hyperbolic space, then $r = 1$.

3. THE MAIN THEOREM

In this section, we shall prove the main proposition of our paper.

Theorem. Let (M, g) be an n -dimensional closed and simply-connected Riemannian manifold with positive constant sectional curvature and r be the degree of geodesic mobility as a global numerical characteristic of the cardinality of the geodesic class of (M, g) . Then

$$r = (n + 1)(n + 2)/2.$$

Proof. Let $\mathbb{R}^{n+1}$ be an $(n+1)$ -dimensional Euclidean space endowed with the inner product $\langle \cdot, \cdot \rangle$. If $\{x^\alpha\}$ for $\alpha, \beta, \gamma = 1, \dots, n, n+1$ are Cartesian orthogonal coordinates in $\mathbb{R}^{n+1}$ then

$$\mathbb{S}^n : (x^1)^2 + \dots + (x^n)^2 + (x^{n+1})^2 = 1$$

is a unit sphere in $\mathbb{R}^{n+1}$. If the sphere $\mathbb{S}^n$ is represented locally by $x^\alpha = f^\alpha(u^k)$ in terms of its local coordinates $\{u^k\}$, then putting $f_k^\alpha = \partial_k x^\alpha$ for $\partial_k = \partial/\partial u^k$, we can define the second fundamental form of the sphere $\mathbb{S}^n$ by the Gauss equations (see [14])

$$f_{ij}^\alpha = \nabla_i f_j^\alpha = \partial_i f_j^\alpha - f_k^\alpha \Gamma_{ji}^k + \Gamma_{\beta\gamma}^\alpha f_i^\beta f_j^\gamma$$

where ∇_i is the van der Waerden–Bortolotti covariant differentiation along $\mathbb{S}^n$, Γ_{ji}^k are the Christoffel symbols of the metric tensor $g_{ij} = f_i^\alpha f_j^\beta g_{\alpha\beta}$ on $\mathbb{S}^n$, $\Gamma_{\beta\gamma}^\alpha$ are the Christoffel symbols of the metric tensor $g_{\alpha\beta}$ on a $(n+1)$ -dimensional Euclidean space. In a Cartesian orthogonal system, $g_{\alpha\beta} = \delta_{\alpha\beta}$ and $\Gamma_{\beta\gamma}^\alpha = 0$, where $\delta_{\alpha\beta}$ is Kronecker's delta.

It is well known that $\mathbb{S}^n$ is a totally umbilical submanifold and hence $f_{ij}^\alpha = g_{ij} N^\alpha$, where $N^\alpha = -x^\alpha$ are the components of the unit normal vector $N \in C^\infty(T\mathbb{S}^n)^\perp$.

If $A = (A_{\alpha\beta})$ is a constant symmetric tensor, then we can define a tensor field $a = (a_{ij})$ on $\mathbb{S}^n$ by the following identities $a_{ij} = A_{\alpha\beta} f_i^\alpha f_j^\beta$. Applying ∇_k to these identities, we obtain

$$\begin{aligned} \nabla_k a_{ij} &= f_k^\gamma (\nabla_\gamma A_{\alpha\beta}) f_i^\alpha f_j^\beta + A_{\alpha\beta} (f_{ki}^\alpha f_j^\beta + f_i^\alpha f_{kj}^\beta) \\ &= A_{\alpha\beta} (N^\alpha f_j^\beta g_{ik} + N^\beta f_i^\alpha g_{jk}). \end{aligned}$$

If we suppose that $\lambda_j = A_{\alpha\beta} N^\alpha f_j^\beta$, then we can rewrite the last equations in the following form

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik},$$

which shows that $\mathbb{S}^n$ admits a globally defined geodesic mapping because $a = (a_{ij})$ is a tensor field globally defined on $\mathbb{S}^n$.

The number of independent components of the constant symmetric tensor $A = (A_{\alpha\beta})$ is equal to $(n+1)(n+2)/2$. Therefore, we can determine a sharp upper bound on the degree of geodesic mobility r as a *global* numerical characteristic of the cardinality of the geodesic class for a unit sphere $\mathbb{S}^n$. In addition, we recall that Hopf has shown (see [3, 4]) that a closed, simply-connected Riemannian manifold

with constant sectional curvature 1 is necessarily isometric to the unit sphere S^n , equipped with its standard metric. This finishes the proof. $\square$

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ON COMPLETE TOTALLY UMBILICAL AND MAXIMAL SPACE-LIKE SURFACES IN PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. Space-like surfaces with special second fundamental forms have been an important tool in the study of pseudo-Riemannian manifolds. This paper focuses on adapting some theorems of Riemannian geometry in the large to the study of geometry of space-like surfaces of pseudo-Riemannian manifolds, specifically, various comparison theorems that use sectional and Ricci curvatures. The purpose of this paper is to establish a global geometry of complete space-like totally umbilical and maximal surfaces in pseudo-Riemannian manifolds.

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Keywords: pseudo-Riemannian manifold, Lorentzian manifold, de Sitter space, space-like totally umbilical and maximal hypersurfaces

1. DEFINITIONS AND NOTATIONS

First, we set up our terminology and notations (see [2, 4]). Let (M, g) be an n -dimensional connected pseudo-Riemannian manifold (also called a semi-Riemannian manifold) of class C^∞ whose metric g has the signature (p, q) for $p + q = n$ and M' an n' -dimensional differentiable manifold of class C^∞ imbedded in (M, g) with an imbedding map $f : M' \rightarrow M$. We call the image $f(M')$ a surface (also called a submanifold) in (M, g) and identify it with the manifold M' .

The differential df of the imbedding map $f : M' \rightarrow M$ will denote by f_* , so that a vector field X' in TM' corresponds to a vector field f_*X' in TM .

We denote by $g' = g(f_*, f_*)$ the metric tensor $g' = f^*g$ induced in M' from g by f , where f^* is the mapping conjugate of f_* . Then, the surface M' is called *space-like* if the metric tensor g' is positive definite and hence $n' \leq q$. In this case, (M', g') is a Riemannian manifold and for each of its points x , there exists a normal subspace of a tangent space $T_x M$ such that the following orthogonal decomposition $T_x M' = (T_x M')^\perp \oplus T_x M'$ is true.

We denote by ∇' the covariant differential operator corresponding to the Riemannian metric g' . Then, the *second fundamental form* Q of the surface (M', g') is defined by the formula

$$Q(X', Y') := (\nabla'_{X'} f_*)Y' = \nabla_{f_*X'} f_*Y' - f_*(\nabla'_{X'} Y')$$

for any $X', Y' \in C^\infty TM'$. The Gauss curvature equation for (M', g') has the form

$$R'(X', Y', V', W') = R(f_*X', f_*Y', f_*V', f_*W') + g(Q(X', W'), Q(Y', V')) - g(Q(Y', W'), Q(X', V')) \quad (1.1)$$

for all $X', Y', V', W' \in C^\infty TM'$, where R' is the curvature tensor of (M', g') . In particular, if (M, g) is a pseudo-Riemannian manifold of constant curvature C , then (1.1) can be rewritten in the form

$$R'(X', Y', V', W') = C \cdot (g'(X', W')g'(Y', V') - g'(Y', W')g'(X', V')) + g(Q(X', W'), Q(Y', V')) - g(Q(Y', W'), Q(X', V')). \quad (1.2)$$

A surface (M', g') is said to be totally umbilical in (M, g) if its second fundamental form satisfies $Q = g' \cdot H$ where H is the *mean curvature vector* defined by the formula $n \cdot H = \text{trace}_{g'} Q$. In this case, (1.1) can be rewritten as the following equations

$$R'(X', Y', V', W') = R(f_*X', f_*Y', f_*V', f_*W') + g(H, H) \cdot (g'(X', W')g'(Y', V') - g'(Y', W')g'(X', V')) \quad (1.3)$$

for all $X', Y', V', W' \in C^\infty TM'$. In addition, if (M, g) is a manifold of constant curvature C , then from (1.2) and (1.3), we obtain

$$R'(X', Y', V', W') = (C + g(H, H)) \cdot (g'(X', W')g'(Y', V') - g'(Y', W')g'(X', V')) \quad (1.4)$$

and hence, (M', g') is a manifold of constant curvature $C' = C + g(H, H)$.

On the other hand, from (1.3), we obtain

$$\sec'(x, \pi) = \sec(x, \pi) + g(H_x, H_x) \quad (1.5)$$

where

$$\begin{aligned} \sec'(x, \pi) &= \sec'(X'_x \wedge Y'_x) \\ &:= - \frac{R'(X'_x, Y'_x, X'_x, Y'_x)}{g'(X'_x, X'_x)g'(Y'_x, Y'_x) - g'(X'_x, Y'_x)^2} \end{aligned}$$

and

$$\begin{aligned} \sec(x, \pi) &= \sec(X'_x \wedge Y'_x) \\ &:= - \frac{R(f_*X'_x, f_*Y'_x, f_*X'_x, f_*Y'_x)}{g(f_*X'_x, f_*X'_x)g(f_*Y'_x, f_*Y'_x) - g(f_*X'_x, f_*Y'_x)^2} \end{aligned}$$

are space-like sectional curvatures of (M', g') and (M, g) at $x \in M'$ with respect to the space-like nondegenerated plane $\pi = \text{span}\{X'_x, Y'_x\} \subset T_x M'$.

In particular, let (M', g') be an n -dimensional space-like totally umbilical hypersurface (also called a hypersubmanifold) in a *Lorentzian manifold* (M, g) . It means that (M, g) is a $(n + 1)$ -dimensional pseudo-Riemannian manifold (M, g) with metric g of Lorentzian signature $(-, +, \dots, +)$. In this case, we can rewrite the equality (1.5) in the following form

$$\sec'(x, \pi) = \sec(x, \pi) - |H^2| \quad (1.6)$$

where $|H^2| = |g(H, H)|$ is a *mean curvature* of (M', g') .

Next, a surface (M', g') is said to be *maximal* (in contrast to the Riemannian case where it is called the *minimal* surface) if its mean curvature vector $H = 0$. In particular, if (M', g') is an n -dimensional space-like maximal hypersurface in a Lorentzian manifold (M, g) , the following equation can be obtained from (1.1)

$$\text{Ric}'(Y', V') = \text{Ric}(f_* Y', f_* V') + R(\mathcal{N}, f_* Y', f_* V', \mathcal{N}) + g'(\mathcal{A} Y', \mathcal{A} V') \quad (1.7)$$

where Ric' and Ric are Ricci tensors of (M', g') and (M, g) , respectively. Moreover, $\mathcal{N}$ is the (globally defined) unitary time-like normal vector field on M' and $\mathcal{A}$ is the shape operator of (M', g') in (M, g) with respect to $\mathcal{N}$ such that $Q(Y', V') = -g'(\mathcal{A} Y', V')\mathcal{N}$. In particular, from (1.7), we obtain

$$\text{Ric}'(V'_x) = \text{Ric}(f_* V'_x) - \sec(x, \pi) + \|\mathcal{A} V'\|^2 \cdot \|V'_x\|^{-2} \quad (1.8)$$

where $\text{Ric}'(V'_x) = \text{Ric}'(V'_x, V'_x) \cdot \|V_x\|^{-2}$ and $\text{Ric}(f_* V'_x) = \text{Ric}'(f_* V'_x, f_* V'_x) \cdot \|V_x\|^{-2}$ are Ricci curvatures at $x \in M'$ with respect to the nonzero vector $V'_x \in T_x M'$,

$$\|\mathcal{A} V'\|^2 = g'(\mathcal{A} V'_x, \mathcal{A} V'_x)$$

and

$$\|V_x\|^2 = g'(V'_x, V'_x) = g(f_* V'_x, f_* V'_x).$$

Moreover, here, $\sec(x, \pi) := \sec(\mathcal{N}_x \wedge f_* V'_x)$ is a *time-like sectional curvature* of the *time-like plane* $\pi = \text{span}\{\mathcal{N}_x, f_* V'_x\}$ and $V'_x \in T_x M'$ is a nonzero vector. Next, along with (1.8) from (1.7), we obtain the equation

$$s = s' + 2 \text{Ric}(\mathcal{N}, \mathcal{N}) + \text{trace}(\mathcal{A}^2) \quad (1.9)$$

where s' and s are scalar curvatures of (M', g') and (M, g) , respectively.

Finally, (M', g') is said to be *totally geodesic* if its second fundamental form Q vanishes identically. Obviously, such a surface is totally umbilical and maximal at the same time. In this paper, we exclude such surfaces from consideration.

2. TOTALLY UMBILICAL SURFACES IN PSEUDO-RIEMANNIAN MANIFOLDS

Let (M', g') be a n -dimensional complete space-like totally umbilical surface in a pseudo-Riemannian manifold (M, g) . In this case, we have

Theorem 2.1. *Let (M', g') be a complete space-like totally umbilical surface in a pseudo-Riemannian manifold (M, g) .*

- (1) *If the mean curvature vector H of (M', g') and the sectional curvature $\sec$ of (M, g) satisfy the inequality*

$$\sec(\pi) \leq -g(H, H)$$

for all plane sections π of TM' , then (M', g') is diffeomorphic to $\mathbb{R}^n$.

- (2) *If there exists a constant $\delta > 0$ such that the mean curvature vector H of (M', g') and the sectional curvature $\sec$ of (M, g) satisfy the inequality*

$$\sec(\pi) \geq -g(H, H) + \delta$$

for all plane sections π of TM' , then (M', g') is compact, its diameter satisfies the inequality $\text{diam } M' \leq \pi/\sqrt{\delta}$ and the first fundamental group is finite.

Proof. Firstly, we recall that Mayer's theorem states (see [5, pp. 212–213]) that a complete Riemannian manifold (M', g') is compact if there exists a positive constant δ such that $\sec'(x, \pi) \geq \delta > 0$ for an arbitrary plane $\pi = \text{span}\{X'_x, Y'_x\}$ at each point $x \in M'$. Moreover, its diameter satisfies the inequality $\text{diam } M' \leq \pi/\sqrt{\delta}$ and the first fundamental group is finite.

Secondly, by (1.3), the inequality $\sec'(x, \pi) \geq \delta > 0$ can be rewritten in the form

$$\sec(x, \pi) \geq -g(H_x, H_x) + \delta.$$

Thirdly, if a complete Riemannian manifold (M', g') has nonpositive sectional curvature at all plane sections, then (M', g') is diffeomorphic to $\mathbb{R}^n$ (see [5, p. 201]). After these remarks, the theorem statements become obvious. $\square$

Next, from Theorem 2.1 and (1.4), we conclude that if (M, g) is a manifold of constant curvature C such that $C + g(H, H) > 0$, then its complete space-like totally umbilical surface (M', g') is compact. In addition, we recall that any compact, simply-connected Riemannian manifold (M', g') with constant sectional curvature $C' > 0$ is necessarily isometric to a Euclidian sphere $\mathbb{S}^n$ of radius $(C')^{-1/2}$, equipped with its standard metric (see [5, p. 217–218]). Thus, using Theorem 2.1 and the result above, we can formulate the following corollary.

Corollary 2.1. *Let (M', g') be a complete space-like totally umbilical surface in a pseudo-Riemannian manifold (M, g) of constant curvature C . If the mean curvature vector H of (M', g') satisfies the inequality*

$$g(H, H) + C > 0,$$

then (M', g') is compact and its first fundamental group is finite. Further, suppose that (M', g') is simply-connected, then (M', g') is isometric to a Euclidian sphere $\mathbb{S}^n$ of radius $(g(H, H) + C)^{-1/2}$.

Now, we suppose that (M', g') is an n -dimensional space-like totally umbilical hypersurface in a Lorentzian manifold (M, g) . In this case, from the inequality $\sec(x, \pi) \leq |H_x|^2$, we obtain $\sec'(x, \pi) \leq 0$ for an arbitrary plane $\pi \subset T_x M'$ at each point $x \in M'$. It follows that (M', g') is diffeomorphic to $\mathbb{R}^n$ (see [5, p. 201]).

On the other hand, if the inequality $\sec(x, \pi) > |H_x|^2 + \delta$ holds for some positive constant δ , then from (1.5), we obtain $\sec'(x, \pi) \geq \delta > 0$. Therefore, as a consequence of Mayer's theorem (see [5, pp. 212–213]), we have the following result.

Corollary 2.2. *Let (M', g') be a complete space-like totally umbilical hypersurface in a Lorentzian manifold (M, g) .*

- (1) *If the mean curvature $|H^2|$ of (M', g') and the sectional curvature $\sec$ of (M, g) satisfy the inequality*

$$\sec(\pi) \leq |H^2|$$

for all plane sections π of TM' , then (M', g') is diffeomorphic to $\mathbb{R}^n$.

- (2) *If there exists a positive constant δ such that the mean curvature $|H^2|$ of (M', g') and the sectional curvature $\sec$ of (M, g) satisfy the inequality*

$$\sec(\pi) \geq |H^2| + \delta$$

for all plane sections π of TM' , then (M', g') is compact, its diameter satisfies the inequality

$$\text{diam } M' \leq \frac{\pi}{\sqrt{\delta}}$$

and the first fundamental group is finite.

We define a de Sitter space $\mathbb{S}_1^{n+1}(C)$ as a Lorentzian manifold (M, g) with positive constant sectional curvature C and recall that any space-like compact hypersurface (M', g') is a de Sitter space $\mathbb{S}_1^{n+1}(C)$ diffeomorphic to a Euclidian sphere $\mathbb{S}^n$. In particular, compact totally umbilical space-like hypersurfaces in $\mathbb{S}_1^{n+1}(C)$ are round n -spheres (see [1, 8]). Now, we can formulate the following corollary.

Corollary 2.3. *Let (M', g') be a complete space-like totally umbilical hypersurface in $\mathbb{S}_1^{n+1}(C)$. If the mean curvature $|H^2|$ of (M', g') satisfies the inequality*

$$|H^2| < C,$$

then (M', g') is a round n -sphere. Furthermore, if (M', g') is simply-connected, then (M', g') is isometric to a Euclidian sphere $\mathbb{S}^n$ of radius $(C - |H^2|)^{-1/2}$.

Montiel (see [8]) proved that every compact space-like surface in $\mathbb{S}_1^{n+1}(C)$ of constant mean curvature is totally umbilical. Based on these results and Corollary 2.4, we can formulate the following obvious corollary.

Corollary 2.4. *Let (M', g') be a compact simply-connected space-like hypersurface in a de Sitter space $\mathbb{S}_1^{n+1}(C)$. If the mean curvature $|H^2|$ of (M', g') is constant and*

$$|H^2| < C,$$

then (M', g') is isometric to a Euclidian sphere $\mathbb{S}^n$ of radius $(C - |H^2|)^{-1/2}$.

3. MAXIMAL SPACE-LIKE HYPERSURFACES IN LORENTZIAN MANIFOLDS

Let (M', g') be an n -dimensional complete space-like maximal hypersurface in a Lorentzian manifold (M, g) .

Theorem 3.1. *Let (M', g') be a space-like complete non-totally geodesic, maximal hypersurface in a Lorentzian manifold (M, g) . If there exists a positive constant δ such that the space-like Ricci curvature and time-like sectional curvature of (M, g) satisfy the inequality*

$$\text{Ric}(f_* V'_x) \geq \sec(\mathcal{N}_x \wedge f_* V'_x) + \delta$$

for any vector $V'_x \in T_x M'$ at each point $x \in M'$, then (M', g') is compact, its diameter satisfies the inequality

$$\text{diam } M' \leq \frac{\pi}{\sqrt{\delta}}$$

and its first fundamental group is finite.

Proof. Firstly, from (1.8), we obtain

$$\begin{aligned} \text{Ric}'(V'_x) &= \text{Ric}(f_* V'_x) - \sec(\mathcal{N}_x \wedge f_* V'_x) + \|\mathcal{A}V'\|^2 \cdot \|V'_x\|^{-2} \\ &\geq \text{Ric}(V'_x) - \sec(\mathcal{N}_x \wedge f_* V'_x) \end{aligned}$$

for the nonzero vector $V'_x \in T_x M'$. Secondly, if we suppose that

$$\text{Ric}(f_* V'_x) - \sec(\mathcal{N}_x \wedge f_* V'_x) \geq \delta > 0$$

for some positive constant δ and an arbitrary nonzero vector $V'_x \in T_x M'$ at each point $x \in M'$, then $\text{Ric}'(V'_x) \geq \delta > 0$. This means that (M', g') is compact, its diameter satisfies the inequality

$$\text{diam } M' \leq \frac{\pi}{\sqrt{\delta}}$$

and the first fundamental group is finite (see [5, p. 216]). In particular, if we suppose for (M, g) that its all time-like sectional curvatures $\sec(x, \pi) \leq -\delta < 0$ and space-like Ricci curvatures $\text{Ric}(V) \geq 0$ for all space-like nonzero vectors V , then (1.8) implies that

$$\text{Ric}'(V'_x) = \text{Ric}(f_* V'_x) - \sec(\mathcal{N}_x \wedge f_* V'_x) + \|\mathcal{A}V'\|^2 \cdot \|V'_x\|^{-2} \geq \delta > 0$$

and hence, (M', g') is compact. This completes the proof. $\square$

Next, from (1.9), we obtain

$$s' = s + 2 \operatorname{Ric}(\mathcal{N}, \mathcal{N}) + \sum_{i=1, \dots, n} (\lambda_i)^2,$$

where $\lambda_1, \dots, \lambda_n$ are eigenvalues of $\mathcal{A}$ which are called the *principal curvatures* of (M', g') . In addition, we recall that (M, g) is said to satisfy the *strong energy condition* or the *time-like convergence condition* (see [6, p. 95]) if $\operatorname{Ric}(V_x, V_x) \geq 0$ for every time-like vector $V_x \in T_x M$ at each point $x \in M$. Then, the following corollary holds.

Corollary 3.1. *Let (M', g') be a space-like and maximal hypersurface Lorentzian manifold (M, g) which satisfies the strong energy condition, then $s' \geq s$ for the scalar curvatures s' and s of (M', g') and (M, g) , respectively. The equality holds on a totally geodesic hypersurface.*

Cheng and Yau [3] and T. Ishihara [7] proved that a complete maximal space-like submanifold (M', g') of $\mathbb{S}_1^{n+1}(C)$ is totally geodesic. Then, the Ricci curvature Ric' of a space-like hypersurface (M', g') in $\mathbb{S}_1^{n+1}(C)$ has the form

$$\operatorname{Ric}'(X') = (n-1)C > 0$$

for any nonzero $X' \in TM'$. This inequality is a necessary condition for a complete Riemannian manifold (M', g') to be compact (see [5, p. 216]). Thus, using this fact and Mayer's theorem (see [5, pp. 212–213]), we can formulate the following corollary.

Corollary 3.2. *If (M', g') is a space-like complete maximal hypersurface in a de Sitter space-time $\mathbb{S}_1^{n+1}(C)$, then (M', g') is compact, totally geodesic and its first fundamental group is finite. Further suppose that (M', g') is simply-connected then (M', g') is isometric to a Euclidian sphere $\mathbb{S}^n$ of radius $C^{-1/2}$.*

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ON METRIZABILITY OF A CLASS OF 2-MANIFOLDS WITH LINEAR CONNECTION

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Abstract. In [17], we proved a theorem which shows how to find, under particular assumptions that guarantee metrizable (among others, recurrency of the curvature is necessary), all (at least local) pseudo-Riemannian metrics compatible with a given torsion-free linear connection without flat points in a domain of two-dimensional manifold. The result has the form of an implication only. In general, if there are flat points, or if curvature is not recurrent, we cannot give any good answer as it can also be demonstrated by examples. Note that in higher dimension, the problem of metrizable is not easy to solve, [6, 7]. Here, we try to apply this apparatus to the class of (torsion-free, locally homogeneous) connections with constant Christoffels in open domains of 2-manifolds (called *connections of Type A* in [1, 8]).

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1. INTRODUCTION

In [17], we developed a simple method which helps to decide whether a given linear connection on a 2-dimensional manifold arises as a Riemannian connection of some pseudo-Riemannian metric. Note that in an arbitrary dimension, the problem is difficult to solve. In the real analytic Riemannian case, probably the most effective method was offered by O. Kowalski in [6, 7], see also [14–16]. In the simplest case, for nowhere flat two-manifolds, we formulated the necessary and sufficient conditions for local metrizable in [17], and in favourable case, we described all compatible metrics in terms of the Ricci tensor. The purpose of the paper is to show a particular application, related to the class of locally homogeneous connections in 2-manifolds, [1, 8, 11]. First, recall some notation. Let (M_n, ∇) be a manifold with a linear connection and let R denote the corresponding curvature tensor, $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]}Z$ for X, Y, Z from $\mathcal{X}(M)$. The induced map $Z \mapsto R(X, Y)Z$, $R(X, Y): T_x M \rightarrow T_x M$ for X, Y, Z from $T_x M$ is linear and skew-symmetric. The Ricci tensor Ric in type $(0, 2)$ is a

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trace of the endomorphism: $\text{Ric}(Y, Z) = \text{Tr}\{X \mapsto R(X, Y)Z\}$, $X, Y, Z \in \mathfrak{X}(M)$, hence it carries less information than R in general. The formula

$$\text{Ric}(Z, Y) - \text{Ric}(Y, Z) = \text{Tr } R(Y, Z)$$

holds [10, p. 14].

1.1. The Ricci tensor of a pseudo-Riemannian manifold

On a pseudo-Riemannian manifold (M_n, g) with the metric tensor g , besides the curvature R in type (1, 3), we consider the type (0, 4) tensor introduced, up to a sign, as $\tilde{R}(X, Y, Z, W) = g(R(X, Y)Z, W)$, usually also called the curvature tensor; the relations

$$\tilde{R}(X, Y, Z, W) = \tilde{R}(Z, W, X, Y) = -\tilde{R}(Y, X, Z, W) = -\tilde{R}(X, Y, W, Z)$$

hold. In a coordinate system $(U, (x^i))$ based at a point $x \in M$, components R_{hijk} of $\tilde{R}$ and R_{ijk}^ℓ of R satisfy $R_{hijk} = R_{jkh i} = -R_{ihjk} = -R_{hikj}$, $R_{hijk} = g_{hs} R_{ijk}^s$, and $g^{\ell h} R_{hijk} = R_{ijk}^\ell$. On (M_n, g) , the Ricci tensor of type (1, 1) has components $R_j^i = g_s^i R_{sj}$, and the scalar curvature ϱ is its trace,

$$\varrho = \text{Tr Ric} = R_s^s = g^{ij} R_{ij}.$$

Recall that a Riemannian manifold (M_n, g) is an *Einstein space* if there is a real constant λ such that $\text{Ric}(X, Y) = \lambda g(X, Y)$. (M_n, g) is *isotropic in a point* $x \in M$ if the curvature is the same constant, $K(x)$, on every (two-plane) section, and *isotropic* if it is isotropic in every point, [3]. If x is an isotropic point of (M_n, g) , then the following formula holds in x in any local coordinates around x : $R_{hijk} = K(g_{hj}g_{ik} - g_{hi}g_{jk})$.

From now on, suppose $n = 2$. Let (x^1, x^2) denote local coordinates in a coordinate neighborhood U of a manifold (M_2, g) . In dimension two, the non-zero components of $\tilde{R}$ are equal R_{1212} up to a sign, and the sectional curvature

$$K(x) = \frac{R_{1212}}{g_{11}g_{22} - g_{12}g_{21}}$$

is sometimes called the *Gaussian curvature* ([12, p. 62,], [4, p. 137]). Any two-dimensional manifold is isotropic and the formula $R_{hijk} = K(g_{hj}g_{ik} - g_{hi}g_{jk})$ holds [4, p. 137]. Components of the Riemann curvature R of type (1, 3) are related to the Ricci tensor by $R_{hjk}^i = \delta_j^i R_{kh} - \delta_k^i R_{jh}$, [9, 12]. We easily check by direct evaluation (cf. [12, 17]):

Lemma 1. *The curvature tensor of a pseudo-Riemannian two-manifold (M_2, g) satisfies*

$$R_{hjk}^i = K(x)(\delta_k^i g_{hj} - \delta_j^i g_{hk})$$

where $K(x)$ is the Gaussian curvature and the Ricci tensor is a multiple of the metric tensor:

$$\text{Ric} = Kg = \frac{1}{2}\varrho g, \quad g = \frac{2}{\varrho} \text{Ric}$$

where $\varrho(x)$ is the scalar curvature.

Consequently, the Ricci tensor of the Levi–Civita connection of a 2-dimensional pseudo-Riemannian manifold (M_2, g) is always symmetric and is proportional to the metric tensor. As a corollary, the Ricci tensor is *non-degenerate* for a nowhere flat (M_2, g) .

Obviously, (M_2, g) is an Einstein space if and only if it has constant (scalar or sectional) curvature (cf. [2, p. 44]). If this is the case, then (due to $\nabla g = 0$) the Ricci tensor is covariantly constant, $\nabla \text{Ric} = 0$, and *vice versa*.

Lemma 2. ([17]) *The Ricci tensor of a nowhere flat 2-dimensional pseudo-Riemannian manifold (M_2, g) with non-zero sectional curvature is recurrent, i. e.*

$$\nabla \text{Ric} = \omega \otimes \text{Ric}, \quad (1.1)$$

and the corresponding (non-zero) 1-form is exact, $\omega = df$ for some function f .

Cf. [5, I, p. 280].

1.2. Two-manifolds with a linear connection

Let us now pay attention to a torsion-free linear connection ∇ on a 2-manifold M_2 . As is well known, in this case, the curvature tensor R of type $(1, 3)$ can be completely recovered from the Ricci tensor,

$$R(X, Y)Z = \text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y$$

for $X, Y, Z \in \mathfrak{X}(M)$. In components $R_{hjk}^i = \partial_j^i R_{hk} - \partial_k^i R_{jh}$; $R_{hij}^i = R_{jh}$ for $j \neq i$ and $R_{hjj}^i = 0$. We get immediately

Lemma 3. *In (M_2, ∇) , $R = 0$ if and only if $\text{Ric} = 0$.*

Lemma 4. *For (M_2, ∇) , the Ricci tensor Ric is recurrent if and only if R is recurrent.*

Proof. If Ric is recurrent, then there is a one-form $\omega = \omega_j dx^j$ such that in local coordinates,

$$\nabla_\ell R_{hjk}^i = \delta_j^i \nabla_\ell R_{kh} - \delta_k^i \nabla_\ell R_{jh} = \delta_j^i \omega_\ell R_{kh} - \delta_k^i \omega_\ell R_{jh} = \omega_\ell R_{hjk}^i$$

and, hence, $\nabla R = \omega \otimes R$. The converse is similar: if R is recurrent, then $\nabla_\ell R_{jk} = \omega_\ell R_{kij}^i = \omega_\ell R_{jk}$, and $\nabla \text{Ric} = \omega \otimes \text{Ric}$. $\square$

Since, in the case $R = 0$, the tensor Ric is in fact also *recurrent* ($\omega = 0$), *recurrency is a necessary condition* for metrization of two-manifolds.

2. METRIZABILITY OF 2-MANIFOLDS WITH LINEAR CONNECTION

As far as $R_{hjj}^i = 0$ and $R_{hij}^i = R_{jh}$ hold for $j \neq i$, the curvature tensor of a linear connection ∇ on M_2 is completely determined by its Ricci tensor; explicitly,

$$\begin{aligned} R_{11} &= -R_{112}^2 = R_{121}^2, \\ R_{21} &= -R_{121}^1 = R_{112}^1, \\ R_{12} &= -R_{212}^2 = R_{221}^2, \\ R_{22} &= -R_{221}^1 = R_{212}^1. \end{aligned}$$

It is now clear that there are two necessary conditions for local metrizability of a symmetric connection on a pseudo-Riemannian 2-manifold: the Ricci tensor must be symmetric and must also be recurrent for some closed 1-form (Lemma 2). Moreover, Ric may be degenerate only if $R = 0$ holds, and if this is the case, then $\text{Ric} = 0$. For the sake of global metrizability, the corresponding 1-form must also be exact. As is well known, a flat connection is always globally metrizable; we can even prescribe the signature.

When the Ricci tensor (or equivalently, the curvature) is non-zero in one fixed point $x_0 \in M$, then due to continuity, it is non-zero in some neighborhood of x_0 . Note that the subset of non-flat points is open. The following can be proved:

Theorem 1. *Let (M_2, ∇) be a 2-manifold with a torsion-free linear connection such that the Ricci tensor is symmetric, $R_{ij} = R_{ji}$, recurrent, i. e. (1.1) holds for some 1-form ω , and the regularity condition $|R_{ij}| \neq 0$ is satisfied. Then, at least locally, there is a metric compatible with the connection.*

Theorem 2. *Let (M_2, ∇) be a two-dimensional manifold with a complete torsion-free linear connection and with the curvature R non-zero everywhere. If the Ricci tensor of ∇ is regular, symmetric, and (1.1) holds where the 1-form is gradient, i. e., $\omega = df$ for some function f , then $g = e^{-f} \cdot \text{Ric}$ is a global metric tensor such that $\nabla g = 0$.*

Hence, a now-where flat complete torsion-free linear connection on M_2 is metrizable if and only if its Ricci tensor is symmetric, regular and recurrent with the corresponding 1-form being exact. If this is the case and $\nabla \text{Ric} = df \otimes \text{Ric}$ holds for some smooth function f , then there is a family of (mutually homothetic) global metrics corresponding to ∇ , $g_t = \exp(-f + t) \cdot \text{Ric}$, $t \in \mathbb{R}$.

The cases

- (i) $\text{Ric} = 0$ identically

and

- (ii) Ric is symmetric, recurrent and non-degenerate

are always favourable for local metrizability of a 2-manifold. Isolated flat points do not matter. But, as examples show, the Ricci tensor might vanish in some “greater” subset and then we can hardly give any reasonable prediction.

3. TORSION-FREE CONNECTIONS WITH CONSTANT CHRISTOFFELS IN DIMENSION 2

Consider $n = 2$. Let us apply the above methods, cf. [17], in the case of torsion-free connections with constant Christoffels on 2-manifolds (motivation for examining this class comes from classification of locally homogeneous linear connections, [1, 8, 11]). A torsion-free connection ∇ in a domain $U \subset M_2$ is given uniquely by a family of functions (components) $A(u, v), \dots, F(u, v)$ in u, v :

$$\nabla_{\partial u} \partial u = A \partial u + B \partial v, \quad \nabla_{\partial u} \partial v = C \partial u + D \partial v, \quad \nabla_{\partial v} \partial v = E \partial u + F \partial v,$$

where $\partial u = \frac{\partial}{\partial u}$, similarly for v , and (u, v) are local coordinates in U (i. e. $\Gamma_{11}^1 = A$, $\Gamma_{11}^2 = B$ etc.). In the case of constant Christoffels, the curvature and the Ricci tensor are constant,

$$\begin{aligned} R_{11} &= B(F - C) + D(A - D), \\ R_{12} &= R_{21} = CD - BE, \\ R_{22} &= E(A - D) + C(F - C), \end{aligned} \tag{3.1}$$

and constant components of the covariant derivative ∇Ric are

$$\begin{aligned} -R_{11;1} &= 2(AR_{11} + BR_{12}), \\ -R_{11;2} &= 2(CR_{11} + DR_{12}), \\ -R_{12;1} &= CR_{11} + (A + D)R_{12} + BR_{22}, \\ -R_{12;2} &= ER_{11} + (C + F)R_{12} + DR_{22}, \\ -R_{22;1} &= 2(CR_{12} + DR_{22}), \\ -R_{22;2} &= 2(ER_{12} + FR_{22}). \end{aligned}$$

Lemma 5. *A torsion-free linear connection on $M_2 = \mathbb{R}^2[u, v]$ with constant Christoffels is locally flat in the following cases:*

- (a) $B = D = 0$, $A, C, E, F \in \mathbb{R}$ are related by $AE - C^2 + CF = 0$;
- (b) $D \neq 0$, $C = BE/D$, $A, B, E, F \in \mathbb{R}$ are related by $D^2A - EB^2 + DBF - D^3 = 0$.

Proof. Suppose $\text{Ric} = 0$ holds. From technical reasons, let us distinguish the cases $D = 0$, $D \neq 0$ in our considerations (although D is not a preferable coefficient, we can start with something else, e.g. with C ; the main purpose is only to prepare the situation for application of Maple).

Let $D = 0$. Then, the condition $R_{12} = 0$ implies $BE = 0$. If $E = 0$ as well, then we get $B(F - C) = 0 = C(F - C)$ from (3.1) and further either $B = C = 0$ holds, we get the subcase

(a1) $B = C = D = E = 0$, $A, F \in \mathbb{R}$,

or $F = C$, we obtain the subcase

(a2) $D = E = 0$, $F = C$, $A, B, C \in \mathbb{R}$;

the condition from (a) obviously holds. If $D = B = 0$, then the equation $C(F - C) + AE = 0$ must hold, case (a).

Now let $D \neq 0$. We get $C = BE/D$ from $R_{12} = 0$. The other two components of the Ricci tensor vanish if and only if the following holds:

$$\begin{aligned} B(F - BE/D) + D(A - D) &= 0, \\ (BE/D)(F - BE/D) + E(A - D) &= 0. \end{aligned}$$

This system is equivalent to the unique third-order algebraic equation $D^2A - EB^2 + DBF - D^3 = 0$, hence (b) is checked. $\square$

Note that if we choose $D \neq 0$ and either A, B, E or B, E, F , we calculate the remaining coefficients uniquely. It means:

Corollary 1. *The class of torsion-free linear connections with constant Christoffels contains a three-parameter family of metrizable locally flat connections.*

Step by step, the following can be proved, [13].

Lemma 6. *If the Ricci tensor of a torsion-free linear connection is recurrent on $M_2 = \mathbb{R}^2[u, v]$, has constant Christoffels and satisfies $R_{12} = 0$ then either Ric vanishes or is degenerate.*

Lemma 7. *If the Ricci tensor of a torsion-free linear connection with constant Christoffels on $M_2 = \mathbb{R}^2[u, v]$ is symmetric, recurrent, non-vanishing and non-degenerate then the components must satisfy $R_{11} \cdot R_{22} \neq 0$.*

Hence, if a torsion-free connection with constant Christoffels on M_2 and non-vanishing Ricci tensor should be symmetric and recurrent, then it satisfies necessarily $R_{11} \cdot R_{12} \cdot R_{22} \neq 0$.

Lemma 8. *If (M_2, ∇) is a 2-manifold with a torsion-free linear connection which has constant Christoffels, the corresponding Ricci tensor is recurrent and all its components are non-vanishing on M_2 , then Ric is degenerate.*

Corollary 2. *In the above notation, exactly the following choices of constants give rise to a metrizable connection ∇ on M_2 :*

- (1) $B = C = D = E = 0$, $A, F \in \mathbb{R}$;
- (2) $D = E = 0$, $F = C$, $A, B, C \in \mathbb{R}$;
- (3) $B = D = 0$, $A, C, E, F \in \mathbb{R}$ satisfying the equality $AE - C^2 + CF = 0$;
- (4) $D \neq 0$, $C = BE/D$, $A, B, E, F \in \mathbb{R}$ satisfying the equality $D^2A - EB^2 + DBF - D^3 = 0$.

Theorem 3. *The only (locally) metrizable connections in the class of torsion-free linear connections with constant Christoffels defined on 2-manifolds are exactly locally flat connections.*

Note that Theorem 3 follows also from the results published in [1].

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CONFORMAL MAPPINGS OF NEARLY QUASI-EINSTEIN MANIFOLDS

FÜSUN ÖZEN ZENGİN AND BAHAR KIRIK

Abstract. In this paper, we consider a conformal mapping between two nearly quasi-Einstein manifolds V_n and $\bar{V}_n$. We find some properties of this transformation from V_n to $\bar{V}_n$ and some theorems are proved.

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Keywords: Nearly quasi-Einstein manifold, conformal mapping, conharmonic mapping, $\sigma(\text{Ric})$ -vector field

1. INTRODUCTION

A non-flat n -dimensional Riemannian or a semi-Riemannian manifold (M, g) , ($n > 2$) is said to be an Einstein manifold if the condition

$$S(X, Y) = \frac{r}{n}g(X, Y) \quad (1.1)$$

holds on M , where S and r denote the Ricci tensor and the scalar curvature of (M, g) , respectively. Einstein manifolds play an important role in Riemannian Geometry, as well as in general theory of relativity. For this reason, these manifolds have been studied by many authors.

A non-flat n -dimensional Riemannian manifold (M, g) , ($n > 2$) is defined to be a quasi-Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the following condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) \quad (1.2)$$

where $a, b \in \mathbb{R}$ and A is a non-zero 1-form such that $g(X, U) = A(X)$ for all vector fields X on M , [5]. Then A is called the associated 1-form and U is called the generator of the manifold. Also M. C. Chaki and R. K. Maity [2] studied the quasi-Einstein manifolds by considering a and b as scalars such that $b \neq 0$ and U as a unit vector field.

A non-flat n -dimensional Riemannian manifold (M, g) , ($n > 2$) is called a nearly quasi-Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and

satisfies the following condition

$$S(X, Y) = ag(X, Y) + bE(X, Y) \quad (1.3)$$

where a and b are non-zero scalars and E is a non-zero symmetric tensor of type $(0,2)$, [4]. Then E is called the associated tensor and a and b are called the associated scalars of M . An n -dimensional nearly quasi-Einstein manifold is denoted by $N(QE)_n$. An example of $N(QE)_4$ has been given in [4].

Putting $X = Y = e_i$ in (1.3), we get

$$r = na + b\tilde{E}. \quad (1.4)$$

Here r is the scalar curvature of $N(QE)_n$ and $\tilde{E} = E(e_i, e_i)$ where $\{e_i\}$, $i = 1, 2, \dots, n$ is an orthonormal basis of the tangent space at each point of the manifold.

In this paper, we investigate a conformal mapping between two nearly quasi-Einstein manifolds.

2. CONFORMAL MAPPINGS OF NEARLY QUASI-EINSTEIN MANIFOLDS

In this section, we suppose that V_n and $\bar{V}_n$, ($n > 2$) are two nearly quasi-Einstein manifolds with metrics g and $\bar{g}$, respectively.

Definition 1. A conformal mapping is a diffeomorphism of V_n onto $\bar{V}_n$ such that

$$\bar{g} = e^{2\sigma} g \quad (2.1)$$

where σ is a function on V_n . If σ is constant, then it is called a homothetic mapping. In local coordinates, (2.1) is written as

$$\bar{g}_{ij}(x) = e^{2\sigma(x)} g_{ij}(x), \quad \bar{g}^{ij} = e^{-2\sigma} g^{ij}. \quad (2.2)$$

Besides those equations, we have the Christoffel symbols, the components of the curvature tensor, the Ricci tensor, and the scalar curvature, respectively

$$\bar{\Gamma}_{ij}^h = \Gamma_{ij}^h + \delta_i^h \sigma_j + \delta_j^h \sigma_i - \sigma^h g_{ij}, \quad (2.3)$$

$$\begin{aligned} \bar{R}_{ijk}^h &= R_{ijk}^h + \delta_k^h \sigma_{ij} - \delta_j^h \sigma_{ik} + g^{ha} (\sigma_{ak} g_{ij} - \sigma_{aj} g_{ik}) \\ &\quad + \Delta_1 \sigma (\delta_k^h g_{ij} - \delta_j^h g_{ik}), \end{aligned} \quad (2.4)$$

$$\bar{S}_{ij} = S_{ij} + (n-2)\sigma_{ij} + (\Delta_2 \sigma + (n-2)\Delta_1 \sigma) g_{ij}, \quad (2.5)$$

$$\bar{r} = e^{-2\sigma} (r + 2(n-1)\Delta_2 \sigma + (n-1)(n-2)\Delta_1 \sigma), \quad (2.6)$$

where $S_{ij} = R_{ij\alpha}^\alpha$, $r = S_{\alpha\beta} g^{\alpha\beta}$, $\sigma_i = \frac{\partial \sigma}{\partial x^i} = \nabla_i \sigma$, $\sigma^h = \sigma_\alpha g^{\alpha h}$ and

$$\sigma_{ij} = \nabla_j \nabla_i \sigma - \nabla_i \sigma \nabla_j \sigma, \quad (2.7)$$

$\Delta_1 \sigma$ and $\Delta_2 \sigma$ are the first and the second Beltrami's symbols which are determined by

$$\Delta_1 \sigma = g^{\alpha\beta} \nabla_\alpha \sigma \nabla_\beta \sigma, \quad \Delta_2 \sigma = g^{\alpha\beta} \nabla_\beta \nabla_\alpha \sigma \quad (2.8)$$

where ∇ is the covariant derivative according to the Riemannian connection in V_n . We denote the objects of space conformally corresponding to V_n by a bar, i. e., $\bar{V}_n$. If $\bar{V}_n$ is a $N(QE)_n$, then we have, from (1.3), (2.2), and (2.5),

$$\bar{b}\bar{E}_{ij} = bE_{ij} + (n-2)\sigma_{ij} + (\Delta_2\sigma + (n-2)\Delta_1\sigma + a - \bar{a}e^{2\sigma})g_{ij}. \quad (2.9)$$

Definition 2. A vector field ξ in a Riemannian manifold M is called torse-forming if it satisfies the condition $\nabla_X\xi = \rho X + \phi(X)\xi$ where $X \in TM$, $\phi(X)$ is a linear form and ρ is a function, [12]. In the local transcription, this reads

$$\nabla_i\xi^h = \rho\delta_i^h + \xi^h\phi_i \quad (2.10)$$

where ξ^h and ϕ_i are the components of ξ and ϕ , and δ_i^h is the Kronecker symbol. A torse-forming vector field ξ is called recurrent if $\rho = 0$; concircular if the form ϕ_i is a gradient covector, i. e., there is a function $\psi(x)$ such that $\phi = d\psi(x)$; convergent, if it is concircular and $\rho = \text{const} \cdot \exp(\psi)$.

Therefore, recurrent vector fields are characterized by the following equation

$$\nabla_X\xi = \phi(X)\xi. \quad (2.11)$$

Also, from the Definition 2., for a concircular vector field ξ , we get

$$(\nabla_Y\xi)X = \rho g(X, Y) \quad (2.12)$$

for all $X, Y \in TM$. A Riemannian space with a concircular vector field is called equidistant, [10, 11].

Conformal mappings of Riemannian spaces (or semi-Riemannian spaces) have been studied by many authors, [1, 3, 6, 9]. In this section, we investigate the conformal mappings of nearly quasi-Einstein manifolds preserving the associated tensor E .

Theorem 1. *If V_n admits a conformal mapping preserving the associated tensor E and the associated scalar b , then V_n is an equidistant manifold.*

Proof. Suppose that V_n admits a conformal mapping preserving the associated tensor E and the associated scalar b . Using (2.9), we obtain

$$(n-2)\sigma_{ij} + (\beta + a - \bar{a}e^{2\sigma})g_{ij} = 0 \quad (2.13)$$

where $\beta = \Delta_2\sigma + (n-2)\Delta_1\sigma$. In this case, we get

$$\sigma_{ij} = \alpha g_{ij} \quad (2.14)$$

where $\alpha = \frac{1}{n-2}(\bar{a}e^{2\sigma} - a - \beta)$ is a function. Putting $\xi = -\exp(-\sigma)$ and using (2.7), (2.12) and (2.14), we get that V_n is an equidistant manifold. Hence, the proof is complete. $\square$

Theorem 2. *An equidistant manifold V_n admits a conformal mapping preserving the associated tensor E if the associated scalars $\bar{a}$ and $\bar{b}$ satisfy both of the conditions*

$$\begin{aligned}\bar{b} &= b, \\ \bar{a} &= e^{-2\sigma}(a + \gamma),\end{aligned}$$

where $\gamma = \left(\frac{n-1}{n}\right)[2\Delta_2\sigma + (n-2)\Delta_1\sigma]$.

Proof. Suppose that V_n is an equidistant manifold. Then, there exists a concircular vector field ξ satisfying the condition (2.12), that is, we have

$$\nabla_j \xi_i = \rho g_{ij} \quad (2.15)$$

where $\xi_i \equiv \nabla_i \xi$. Putting $\sigma = -\ln(-\xi(x))$ and using the condition (2.5), we obtain

$$\bar{S}_{ij} = S_{ij} + \gamma g_{ij} \quad (2.16)$$

where $\gamma = \left(\frac{n-1}{n}\right)[2\Delta_2\sigma + (n-2)\Delta_1\sigma]$. Considering (1.3) in (2.16) and using (2.2), we get

$$\bar{a}e^{2\sigma}g_{ij} + \bar{b}\bar{E}_{ij} = (a + \gamma)g_{ij} + bE_{ij}. \quad (2.17)$$

Taking $\bar{a} = e^{-2\sigma}(a + \gamma)$ and $\bar{b} = b$, (2.17) implies that $\bar{E}_{ij} = E_{ij}$. This completes the proof. $\square$

The conharmonic transformation is a conformal transformation preserving the harmonicity of a certain function. If the conformal mapping is also conharmonic, then we have, [8]

$$\nabla_i \sigma^i + \frac{1}{2}(n-2)\sigma^i \sigma_i = 0. \quad (2.18)$$

Theorem 3. *Let V_n be a conformal mapping with preservation of the associated tensor E and the associated scalar b . A necessary and sufficient condition for this conformal mapping to be conharmonic is that the associated scalar $\bar{a}$ be transformed by $\bar{a} = e^{-2\sigma}a$.*

Proof. Suppose that V_n admits a conformal mapping preserving the associated tensor E and the associated scalar b . Using (2.7), (2.8) and (2.9), we obtain

$$(n-2)\nabla_j \nabla_i \sigma - (n-2)\sigma_i \sigma_j + [\nabla_h \sigma^h + (n-2)\sigma^h \sigma_h + a - \bar{a}e^{2\sigma}]g_{ij} = 0. \quad (2.19)$$

Multiplying (2.19) by g^{ij} , we get

$$\nabla_h \sigma^h + \frac{1}{2}(n-2)\sigma^h \sigma_h + \frac{n}{2(n-1)}(a - \bar{a}e^{2\sigma}) = 0. \quad (2.20)$$

If this mapping is conharmonic, using (2.18) in (2.20), we obtain $\bar{a} = e^{-2\sigma}a$. The converse is also true. This completes the proof. $\square$

Definition 3. A $\varphi(\text{Ric})$ -vector field is a vector field on an n -dimensional Riemannian manifold (M, g) and Levi-Civita connection ∇ , which satisfies the condition

$$\nabla \varphi = \mu \text{Ric} \quad (2.21)$$

where μ is a constant and Ric is the Ricci tensor, [7]. When (M, g) is an Einstein space, the vector field φ is concircular. Moreover, when $\mu = 0$, the vector field φ is covariantly constant. In local coordinates, (2.21) can be written as

$$\nabla_j \varphi_i = \mu S_{ij} \quad (2.22)$$

where S_{ij} denote the components of the Ricci tensor and $\varphi_i = \varphi^\alpha g_{i\alpha}$.

Suppose that V_n admits a $\sigma(\text{Ric})$ -vector field. Then, we have

$$\nabla_j \sigma_i = \mu S_{ij} \quad (2.23)$$

where μ is a constant. Now, we can state the following theorem.

Theorem 4. *Let us consider the conformal mapping (2.1) of a nearly quasi-Einstein manifold V_n with constant associated scalars being also conharmonic with the $\sigma(\text{Ric})$ -vector field. A necessary and sufficient condition for the length of σ to be constant is that the trace of the associated tensor E of V_n be constant.*

Proof. We consider that the conformal mapping (2.1) of a nearly quasi-Einstein manifold V_n admitting a $\sigma(\text{Ric})$ -vector field is also conharmonic. In this case, comparing (2.18) and (2.23), we get

$$r = \frac{(2-n)}{2\mu} \sigma^i \sigma_i \quad (2.24)$$

where r is the scalar curvature of V_n . If V_n is of the constant associated scalars, from (1.4) and (2.24), we find

$$\tilde{E} = \frac{1}{b} \left(\frac{(2-n)}{2\mu} \sigma^i \sigma_i - na \right).$$

If the length of σ is constant, then $\sigma^i \sigma_i = c$, where c is a constant. Thus, we can see that $\tilde{E}$ is constant. The converse is also true. Hence, the proof is complete. $\square$

3. AN EXAMPLE OF A NEARLY QUASI-EINSTEIN MANIFOLD

In this section, we consider a Riemannian metric g on $\mathbb{R}^4$ by the formula

$$ds^2 = g_{ij} dx^i dx^j = (x^4)^{\frac{4}{3}} [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2 \quad (3.1)$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of $\mathbb{R}^4$. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor,

the Ricci tensor and the scalar curvature are

$$\begin{aligned} \Gamma_{14}^1 &= \Gamma_{24}^2 = \Gamma_{34}^3 = \frac{2}{3x^4}, \quad \Gamma_{11}^4 = \Gamma_{22}^4 = \Gamma_{33}^4 = -\frac{2}{3}(x^4)^{\frac{1}{3}}, \\ R_{1441} &= R_{2442} = R_{3443} = -\frac{2}{9(x^4)^{2/3}}, \\ R_{1221} &= R_{1331} = R_{2332} = \frac{4}{9}(x^4)^{2/3}, \\ S_{11} &= S_{22} = S_{33} = \frac{2}{3(x^4)^{2/3}}, \quad S_{44} = -\frac{2}{3(x^4)^2}, \quad r = \frac{4}{3(x^4)^2}. \end{aligned} \quad (3.2)$$

Therefore $\mathbb{R}^4$ with the considered metric is a Riemannian manifold (M_4, g) of non-vanishing scalar curvature. Let us now consider the associated scalars a , b , and the associated tensor E as follows:

$$a = -\frac{2}{3(x^4)^2}, \quad b = -\frac{1}{9x^4} \quad (3.3)$$

and

$$E_{ij}(x) = \begin{cases} -12(x^4)^{1/3} & \text{for } i = j = 1, 2, 3 \\ 0 & \text{for } i = j = 4 \text{ and } i \neq j \end{cases} \quad (3.4)$$

at any point $x \in M$. To verify the relation (1.3), it is sufficient to check the relations $S_{ii} = ag_{ii} + bE_{ii}$, $i = 1, 2, 3, 4$ since for the other cases, (1.3) holds trivially. From (3.2), (3.3), and (3.4), we obtain

$$R.H.S \text{ of } S_{11} = ag_{11} + bE_{11} = \frac{2}{3(x^4)^{2/3}} = S_{11}.$$

Similarly, S_{22} , S_{33} , and S_{44} are also satisfied. Hence, (M_4, g) endowed with the metric (3.1) is a $N(QE)_4$ with the conditions (3.3) and (3.4).

Let (M_4, g) endowed with the metric (3.1) be a conformal mapping with preservation of the associated tensor E and the associated scalar b . Also, we choose σ and $\bar{a}$ as follows:

$$\sigma = \ln(x^1 x^2 x^3), \quad \bar{a} = -\frac{2}{3(x^1 x^2 x^3 x^4)^2} \quad (3.5)$$

where $x^1, x^2, x^3 > 0$. Now, we show that these choices satisfy Theorem 3.

From (3.5), we get $\nabla_i \sigma = \frac{\partial \sigma}{\partial x^i} = \sigma_i = \frac{1}{x^i}$ for $i = 1, 2, 3$ and $\sigma_4 = 0$. Moreover, the only non-vanishing covariant derivatives of σ_i ($i = 1, 2, 3, 4$) are

$$\nabla_1 \sigma_4 = \nabla_4 \sigma_1 = -\frac{2}{3x^1 x^4}, \quad (3.6)$$

$$\nabla_2 \sigma_4 = \nabla_4 \sigma_2 = -\frac{2}{3x^2 x^4}, \quad (3.7)$$

$$\nabla_3 \sigma_4 = \nabla_4 \sigma_3 = -\frac{2}{3x^3 x^4}, \quad (3.8)$$

and

$$\nabla_1 \sigma_1 = -\frac{1}{(x^1)^2}, \quad \nabla_2 \sigma_2 = -\frac{1}{(x^2)^2}, \quad \nabla_3 \sigma_3 = -\frac{1}{(x^3)^2}. \quad (3.9)$$

Using (3.6)–(3.9), we find

$$g^{11} \nabla_1 \sigma_1 + g^{11} \sigma_1 \sigma_1 = 0 \quad (3.10)$$

and similarly the other cases hold. Therefore, the condition (2.18) is satisfied.

Moreover, from (3.3) and (3.5), we obtain

$$\bar{a} e^{2\sigma} = -\frac{2}{3(x^1 x^2 x^3 x^4)^2} \times e^{2 \ln(x^1 x^2 x^3)} = -\frac{2}{3(x^4)^2} = a. \quad (3.11)$$

From (3.10) and (3.11), we see that the equation (2.20) is satisfied. Hence, Theorem 3 holds for (M_4, g) endowed with the metric (3.1) and the conditions (3.3) and (3.5).

Now, we also show that (M_4, g) endowed with the metric (3.1) is not a quasi-Einstein manifold.

If possible, we have $S_{ij} = a g_{ij} + b A_i A_j$, where $i, j = 1, 2, 3, 4$. For $i = j$, we get from this relation

$$S_{ii} = a g_{ii} + b A_i A_i, \quad (3.12)$$

for all $i = 1, 2, 3, 4$. Since $S_{ii} \neq 0$ and $g_{ii} \neq 0$, we can choose $a \neq 0$, $b \neq 0$ and $A_i \neq 0$ for all $i = 1, 2, 3, 4$ such that (3.12) holds. However, for these values of a , b and A_i and for $i \neq j$, the equation $S_{ij} = a g_{ij} + b A_i A_j$ cannot be satisfied because for $i \neq j$, $S_{ij} = g_{ij} = 0$ but $A_i \neq 0$.

Therefore, (M_4, g) is not a quasi-Einstein manifold. Thus, a $N(QE)_n$ is not necessarily a quasi-Einstein manifold.

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CONDITIONAL SYMMETRIES FOR BOUNDARY VALUE PROBLEMS: NEW DEFINITION AND ITS APPLICATION FOR NONLINEAR PROBLEMS WITH NEUMANN CONDITIONS

ROMAN CHERNIHA

Abstract. A new definition of conditional invariance for boundary-value problems (BVPs) is proposed. Its relationship with the definitions given earlier for Lie's invariance, is shown and a wider applicability is demonstrated via a non-trivial example. The definition is applied to reduction of BVPs with the basic reaction-diffusion-convection equation, which is used in physically and biologically motivated problems. The constraints on the functions arising in the boundary conditions are established and the nonlinear BVPs in question are thus reduced to those based on linear second-order ODEs. Moreover, the exact solution of a nonlinear BVP with zero Neumann conditions is found.

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1. INTRODUCTION

Nowadays Lie symmetries are widely applied to partial differential equations (including multi-component systems of multidimensional PDEs), notably, to their reductions to ordinary differential equations (ODEs) and constructing exact solutions. There are a huge number of papers and many excellent books (see, e. g., [3, 4, 12, 18, 19] and papers cited therein) devoted to such applications. Over recent decades, other symmetry methods based on the classical Lie–Ovsiannikov method have been derived. The Bluman–Cole method of non-classical symmetry (other widely used terminology is Q -conditional symmetry) is the most noted among them and [3] summarizes the results obtained by means of this approach for scalar PDEs while in recent papers [6, 7] this method is applied to non-linear PDE systems. Further generalizations of the classical Lie–Ovsiannikov method were introduced by many authors (see, e. g., [11] for details).

However, a PDE (a system of PDEs) cannot model any real process without additional condition(s) for the unknown function(s). On the other hand, the relevant boundary-value problem (BVP) based on the given PDE (system of PDEs) can describe many real processes arising in nature and society. One may note that the symmetry-based methods were not widely used for solving BVPs. The obvious

reason follows from the following observation: the relevant boundary and initial conditions are usually not invariant under any transformations, i. e., they don't admit any symmetry of the governing PDE(s). Nevertheless, there are some classes of BVPs which can be solved by means of the Lie symmetry based algorithm. This algorithm uses the notion of Lie's invariance of BVP in question. Probably, the first rigorous definition of Lie's invariance for BVPs was formulated by G.W. Bluman in 1970s [2] (the definition and several examples are summarized in the book [3]). This definition was used (explicitly or implicitly) in several papers to derive exact solutions of some BVPs. It should be noted that Ibragimov's definition of BVP invariance [14] (see also his recent paper [15]), which was formulated independently, is equivalent to Bluman's. On the other hand, one notes that Bluman's definition is not suitable for all types of boundary conditions. Notably, the definition doesn't work in the case of boundary conditions involving points at infinity (e. g., $x \rightarrow \infty$) and conditions on the moving surfaces (e. g., the Stefan conditions). J. R. King was probably the first to note the problem with points at infinity and suggested the appropriate substitution to transform such points to regular those [16]. In [20] (several examples are summarized in [1]), the way was proposed of defining Lie's invariance on the moving boundaries in the case of some BVPs arising in hydrodynamics. In our recent papers [8–10], a new definition of Lie's invariance of BVP with a wide range of boundary conditions (including those on non-regular manifolds and moving surfaces) was formulated in two-dimensional and multidimensional cases, respectively. Moreover, an algorithm of the group classification for the given class of BVPs was created for the first time. The definition and algorithm were applied to a class of nonlinear two-dimensional and multidimensional BVPs of Stefan type with the aim to show their efficiency. However, there are many realistic BVPs that cannot be solved using any definition of Lie's invariance of BVP (a simple example will be presented in the next section), hence, definitions involving more general types of symmetries should be proposed.

In section 2, new definition of conditional invariance of BVPs with a wide range of boundary conditions is presented. The known earlier definitions of Lie's invariance follow as particular cases from this definition but not vice versa. In section 3, to demonstrate the efficiency of the definition, BVPs based on the widely used nonlinear reaction-diffusion-convection equations with the Neumann boundary conditions are examined. Reductions of the BVPs in question with correctly-specified boundary conditions to BVPs for ODEs are obtained and examples of exact solutions are presented. As a conclusion, we summarize the results obtained.

2. LIE'S AND CONDITIONAL INVARIANCE OF BVPs

First of all, to avoid possible complications we restrict ourselves to the case when the basic equation of BVP is a two-dimensional evolution PDE of k th-order ($k \geq 2$).

In this case, the relevant BVP may be formulated as follows:

$$u_t = F(x, u, u_x, \dots, u_x^{(k)}), \quad (t, x) \in \Omega \subset \mathbb{R}^2 \quad (2.1)$$

$$s_a(t, x) = 0: B_a(t, x, u, u_x, \dots, u_x^{(k_a)}) = 0, \quad a = 1, 2, \dots, p, \quad k_a < k \quad (2.2)$$

where F and B_a are smooth functions in the corresponding domains, Ω is a domain with smooth boundaries and $s_a(t, x)$ are smooth curves. Hereafter the subscripts t and x denote differentiation with respect to these variables, $u_x^{(j)} = \frac{\partial^j u}{\partial x^j}$, $j = 1, 2, \dots, k$. We assume that BVP (2.1)–(2.2) has a classical solution (in the usual sense).

Consider the infinitesimal generator

$$X = \xi^0(t, x) \frac{\partial}{\partial t} + \xi^1(t, x) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u} \quad (2.3)$$

Hereafter, ξ^0 , ξ^1 and η are known smooth functions. Let us assume that this operator defines a Lie symmetry acting on both (t, x, u) -space as well as on its projection to (t, x) -space. Let X_k be the k th prolongation of the generator X calculated by the well-known prolongation formulae (see, e. g. [18, 19]).

Definition 1. [4] The Lie symmetry X (2.3) is admitted by the boundary value problem (2.1)–(2.2) if and only if:

- (a) $X_k \left(F(x, u, u_x, \dots, u_x^{(k)}) - u_t \right) = 0$ when u satisfies (2.1);
- (b) $X(s_a(t, x)) = 0$ when $s_a(t, x) = 0$, $a = 1, \dots, p$;
- (c) $X_{k_a} \left(B_a(t, x, u, u_x, \dots, u_x^{(k_a)}) \right) = 0$ when $B_a|_{s_a(t, x)=0} = 0$, $a = 1, \dots, p$.

Because BVP (2.1)–(2.2) involves only the simplest boundary conditions, Definition 1 cannot be applied to BVP with boundary conditions defined on the non-regular manifolds and those on the unknown moving surfaces. In the recent papers [8, 9], it was shown how this definition can be extended to the classes of BVPs with more complicated boundary and initial conditions. Nevertheless, the relevant definitions coincide with one another if the given BVP is based on the two-dimensional PDE of the form (2.1), here the definition from paper [9] will be used below for further generalization.

Consider a BVP for the evolution equation (2.1) involving conditions (2.2) and some other types of the boundary and initial conditions, namely:

$$S(t, x) = 0: B^l \left(t, x, u, \dots, u_x^{(k_l)}, S, \frac{\partial S}{\partial t}, \frac{\partial S}{\partial x} \right) = 0, \quad l = 1, \dots, n_1, \quad (2.4)$$

and

$$\gamma_c(t, x) = \infty: \Gamma_c \left(t, x, u, u_x, \dots, u_x^{(k_c)} \right) = 0, \quad c = 1, 2, \dots, n_2. \quad (2.5)$$

Here $k_l < k, k_c < k, n_1$ and n_2 are the given numbers, $\gamma_c(t, x)$ are the known functions, while the function $S(t, x)$ defining free boundary surface must be found (in contrast to the known functions $s_a(t, x)$ in (2.2) !). We assume that all functions arising in (2.1)–(2.2) and (2.4)–(2.5) and the number of boundary and initial conditions are such that the classical solution exists for this BVP.

Let us assume that the operator

$$Q = \xi^0(t, x, u) \frac{\partial}{\partial t} + \xi^1(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u} \quad (2.6)$$

is a Q -conditional symmetry of PDE (2.2), i. e., the following well-known criterion is satisfied

$$\frac{Q}{k} F \equiv \frac{Q}{k} \left(u_t - F \left(t, x, u, u_x, \dots, u_x^{(k)} \right) \right) \Big|_M = 0, \quad (2.7)$$

where $\frac{Q}{k}$ is the k th prolongation of Q and the manifold $M = \{F = 0, Q(u) \equiv \xi^0(t, x)u_t + \xi^1(t, x)u_x - \eta(t, x, u) = 0\}$.

Definition 2. BVP (2.1)–(2.2) and (2.4)–(2.5) is Q -conditionally invariant under operator (2.6) if:

- (a) the criterion (2.7) is fulfilled;
- (b) $Q(s_a(t, x)) = 0$ when $s_a(t, x) = 0$, $B_a|_{s_a(t, x)=0} = 0$, $a = 1, \dots, p$;
- (c) $\frac{Q}{k_a} \left(B_a \left(t, x, u, u_x, \dots, u_x^{(k_a)} \right) \right) = 0$ when $B_a|_{s_a(t, x)=0} = 0$, $a = 1, \dots, p$;
- (d) $\frac{Q}{k_l} \left(B^l \left(t, x, u, \dots, u_x^{(k_l)} \right), S, \frac{\partial S}{\partial t}, \frac{\partial S}{\partial x} \right) = 0$ when $B^l|_{S(t, x)=0} = 0$, $l = 1, \dots, n_1$;
- (e) $Q_*(\gamma_c^*(\tau, y)) = 0$ when $\gamma_c^*(\tau, y) = 0$, $c = 1, 2, \dots, n_2$;
- (f) $\frac{Q}{k_c} \left(\Gamma_c^* \left(\tau, y, u, u_y, \dots, u_y^{(k_c)} \right) \right) = 0$ when $\Gamma_c^*|_{\gamma_c^*(\tau, y)=0} = 0$, $c = 1, \dots, n_2$,

where Q_* , Γ_c^* and $\gamma_c^*(\tau, y)$ are operators (2.6), the functions Γ_c and $\frac{1}{\gamma_c(t, x)}$, respectively, expressed via the new independent variables

$$\tau = \begin{cases} t, & \text{if } \frac{\partial \gamma_c(t, x)}{\partial x} \neq 0, \\ \frac{1}{\gamma_c(t, x)}, & \text{if } \frac{\partial \gamma_c(t, x)}{\partial x} = 0 \end{cases} \quad y = \begin{cases} x, & \text{if } \frac{\partial \gamma_c(t, x)}{\partial x} = 0, \\ \frac{1}{\gamma_c(t, x)}, & \text{if } \frac{\partial \gamma_c(t, x)}{\partial x} \neq 0. \end{cases} \quad (2.8)$$

Remark 1. In item (f) of Definition 2, we take into account that any physically motivated boundary condition (2.5) contains the expression $\gamma_c(t, x) = \infty$, which is equivalent either to $\gamma_c^0(t) = \infty$ or to $\gamma_c^1(x) = \infty$. It means that the form of the substitution does not depend on the choice of partial derivatives in (2.8) because the assumption $\frac{\partial \gamma_c(t, x)}{\partial x} \neq 0$ automatically leads to $\frac{\partial \gamma_c(t, x)}{\partial t} = 0$ and vice versa.

It should be noted that this definition coincides with Definition 1 if the operator (2.6) is simultaneously Lie's symmetry (2.3) and there are no boundary conditions of the form (2.4) and (2.5).

Nevertheless, Definition 2 seems to be a direct generalization of the definition from [9, p. 98], such generalization makes it possible to extend essentially the applicability of symmetries for BVPs. A simple example is presented below.

Example 1. Consider the reaction-diffusion-convection equation

$$u_t = u_{xx} + \lambda u u_x + u(1 - u), \quad (2.9)$$

where λ is an arbitrary non-zero constant. Equation (2.9) was introduced in [17] as a generalization of the well-known Fisher equation (i. e., (2.9) with $\lambda = 0$) by adding the simple convection term, therefore, being named the Murray equation. As follows from [5], this equation is invariant with respect to the trivial Lie algebra spanned by the operators $P_t = \partial/\partial t$ and $P_x = \partial/\partial x$. Let us consider the BVP on the semi-infinite space interval $(0; +\infty)$ for the Murray equation (2.9) with boundary conditions

$$x = 0 : u = \varphi_1(t), \quad x = \infty : u = \varphi_2(t), \quad (2.10)$$

where $\varphi_1(t), \varphi_2(t)$ are arbitrary smooth functions.

Let us apply items (b) and (c) from Definition 1 to a linear combination of the operators P_t and P_x , say $X = \lambda_0 P_t + \lambda_1 P_x$. One easily checks that item (b) immediately leads to the restriction $\lambda_1 = 0$, i. e., the BVP in question can be invariant only under the operator P_t generating the one-parameter time translation group. Now we apply item (c) to the first boundary condition from (2.10) and obtain the restriction $\varphi_1'(t) = 0$. One also notes by direct checking that the second boundary condition from (2.10) is invariant under time translation provided $\varphi_2'(t) = 0$. It means that BVP (2.9)–(2.10) is not invariant under any linear combination of the operators P_t and P_x , excluding a special case when both functions $\varphi_1(t)$ and $\varphi_2(t)$ are constants. Moreover, in this special case, the invariant solution of this problem must be stationary (the operator P_t generates such solutions only), i. e., the result is rather trivial.

It turns out that BVP (2.9)–(2.10) with arbitrary smooth functions $\varphi_1(t)$ and $\varphi_2(t)$ is Q -conditionally invariant according to Definition 2. In fact, equation (2.9) admits the Q -conditional symmetry [5]

$$Q = \partial_t - \left(\lambda u + \frac{1}{\lambda} \right) \partial_x + (u - u^2) \partial_u. \quad (2.11)$$

Because each conditional symmetry can be multiplied by an arbitrary smooth function, for instance $f(x)$, this operator can be rewritten as

$$Q = f(x) \partial_t - f(x) \left(\lambda u + \frac{1}{\lambda} \right) \partial_x + f(x) (u - u^2) \partial_u. \quad (2.12)$$

Now one sees that item (a) from Definition 2 is fulfilled by the correct choice of the operator Q , while the first boundary condition from (2.10) satisfies items (b) and (c) provided $f(0) = 0$. Note that there is no need to calculate any prolongation of the operator Q in this case because conditions (2.10) do not involve derivatives of u . Finally, we check whether items (e) and (f) are fulfilled because the second boundary

condition from (2.10) involves a non-regular manifold (item (d) is not applicable for BVP in question). Applying formulae (2.8) to operator (2.12), one obtains

$$Q_* = f(1/y)\partial_t + f(1/y)y^2\left(\lambda u + \frac{1}{\lambda}\right)\partial_y + f(1/y)(u - u^2)\partial_u. \quad (2.13)$$

Simultaneously, the relevant boundary condition takes the form

$$y = 0 : u = \varphi_2(t). \quad (2.14)$$

So, items (e) and (f) are fulfilled provided $f(1/y) \rightarrow 0$ if $y \rightarrow 0$, i.e. $f(x) \rightarrow 0$ if $x \rightarrow \infty$.

Thus, BVP (2.9)–(2.10) is Q -conditionally invariant under operator (2.12) if $f(0) = 0$ and $f(x) \rightarrow 0$ if $x \rightarrow \infty$ (in particular, one may set $f(x) = xe^{-x}$).

3. CONDITIONAL INVARIANCE AND EXACT SOLUTIONS OF SOME NON-LINEAR BVPs

Here we consider some BVPs with the basic reaction-diffusion-convection equation

$$u_t = [u^m u_x]_x + \lambda u^m u_x + (\lambda_1 u^{m+1} + \lambda_2)(u^{-m} - \lambda_3), \quad (3.1)$$

where λ_k , $k = 1, 2, 3$ and $m \neq -1, 0$ are arbitrary constants. This equation with correctly-specified coefficients generates several equations, which are used in physically and biologically motivated problems (see examples in [11]). Let us formulate a BVP with the governing equation in the domain $\Omega = \{(t, x) : t > 0, x \in (x_1, x_2), x_1 < x_2 \in \mathbb{R}\}$ using the initial profile

$$t = 0 : u = \varphi_0(x) \quad (3.2)$$

$$x = x_a : u_x = \varphi_a(t), a = 1, 2, \quad (3.3)$$

where $\varphi_0(x)$, $\varphi_a(t)$ are the known smooth functions. So, (3.1)–(3.3) is the nonlinear BVP, which is the known subject for investigation in pure and applied mathematics. In [11], it was proved that (3.1) admits the Q -conditional symmetry

$$Q = \partial_t + (\lambda_1 u + \lambda_2 u^{-m})\partial_u, \quad (3.4)$$

which is not equivalent to any Lie's symmetry provided $\lambda_2 \neq 0$. Because each conditional symmetry can be multiplied by an arbitrary smooth function, for instance $f(t)$, the operator

$$Q = f(t)\partial_t + f(t)(\lambda_1 u + \lambda_2 u^{-m})\partial_u \quad (3.5)$$

will be examined instead of (3.4).

Now we apply Definition 2 to BVP (3.1)–(3.3) in order to obtain the correctly-specified constraints when this problem is conditionally invariant under operator (3.5). Obviously, the first item is fulfilled by the correct choice of the operator. Item (b) is satisfied provided $f(0) = 0$ which leads to the restriction on the operator.

A non-trivial result is obtained by applying item (c). For the initial profile the condition is automatically fulfilled because $Q(u - \varphi_0(x)) \equiv 0$ if $f(0) = 0$. However,

the story is different with the boundary conditions (3.3). In fact, calculating the first prolongation of operator (3.5) according to the item (c) with $k_a = 1$ and acting on $u_x = \varphi_a(t)$, one obtains two first-order ODEs

$$\varphi_a(t) \left(\lambda_1 - m \lambda_2 (u(t, x_a))^{-m-1} \right) = \dot{\varphi}_a(t), \quad a = 1, 2. \quad (3.6)$$

to find the functions $\varphi_a(t)$. Because the BVP in question does not involve any conditions of the form (2.4)–(2.5), there is no need to examine items (d)–(f). Thus, we obtain

Proposition 1. *BVP (3.1)–(3.3) is Q -conditionally invariant under operator (3.5) if and only if $f(0) = 0$ and constraints (3.6) hold.*

One may note that constraints (3.6) are two additional boundary conditions because they involve the function $u(t, x)$. Hence, it may lead to the non-existence of a classical solution to the BVP in question. Fortunately, there are cases when constraints (3.6) do not produce any boundary conditions:

- (i) $\varphi_a(t) = 0, a = 1, 2$;
- (ii) $\lambda_2 = 0$.

In case (i), BVP with the zero Neumann conditions

$$x = x_a : u_x = 0, a = 1, 2, \quad (3.7)$$

are obtained. Such conditions (zero flux on boundaries) are widely used in physical and biological problems.

In case (ii), BVP takes the form

$$u_t = [u^m u_x]_x + \lambda u^m u_x + \lambda_1 u(1 - \lambda_3 u^m), \quad (3.8)$$

$$x = x_a : u_x = C_a \exp(\lambda_1 t), \quad a = 1, 2, \quad (3.9)$$

where $C_a, a = 1, 2$ are arbitrary constants, and the initial profile is (3.2).

Now we demonstrate how the result obtained above can be applied to reducing the BVP in question and finding exact solution in case (i). The following ansatz is constructed using the standard procedure, i. e., solving the linear equation $Q(u) = 0$ for operator (3.5):

$$V(t, x) \equiv u^{m+1} = \begin{cases} \lambda_2(m+1)t + \varphi(x), & \lambda_1 = 0, \\ \varphi(x)e^{\lambda_1(m+1)t} - \frac{\lambda_2}{\lambda_1}, & \lambda_1 \neq 0 \end{cases} \quad (3.10)$$

where $\varphi(x)$ is an unknown function. Substituting (3.10) with $\lambda_1 \neq 0$ into the govern equation (3.1), one arrives at the ordinary differential equation (ODE)

$$\varphi_{xx} + \lambda \varphi_x - (m+1)\lambda_1 \lambda_3 \varphi = 0. \quad (3.11)$$

Simultaneously, ansatz (3.10) reduces the boundary conditions (3.7) to the form

$$x = x_a : \varphi_x = 0, \quad a = 1, 2. \quad (3.12)$$

Finally, the reduction requires the initial profile to be correctly-specified as follows

$$t = 0 : u = \varphi_0(x) \equiv \left(\varphi(x) - \frac{\lambda_2}{\lambda_1} \right)^{\frac{1}{m+1}}. \quad (3.13)$$

Thus, the Sturm-Liouville problem (3.11)–(3.12) is obtained to find the function $\varphi(x)$. One easily proves that its non-trivial solution exists only for $\delta = \lambda^2 + 4\lambda_1(m+1)\lambda_3 < 0$ and a discrete set of values x_1 and x_2 . To avoid cumbersome calculations, we present the following particular solution of problem (3.11)–(3.12)

$$\varphi(x) = C_1 \exp\left(-\frac{\lambda}{2}x\right) \left(\cos \frac{\sqrt{-\delta}}{2}x + \frac{\lambda}{\sqrt{-\delta}} \sin \frac{\sqrt{-\delta}}{2}x \right), \quad (3.14)$$

which is valid on the space interval (x_1, x_2) with $x_a = \frac{2}{\sqrt{-\delta}}\pi n_a$, $n_a \in \mathbb{Z}$ and $n_1 < n_2$.

Thus, substituting (3.14) into the second ansatz from (3.10), one immediately arrives at the exact solution

$$u = \left(C_1 \exp\left(-\frac{\lambda}{2}x + \lambda_1(m+1)t\right) \left(\cos \frac{\sqrt{-\delta}}{2}x + \frac{\lambda}{\sqrt{-\delta}} \sin \frac{\sqrt{-\delta}}{2}x \right) - \frac{\lambda_2}{\lambda_1} \right)^{\frac{1}{m+1}}$$

of the nonlinear BVP (3.1) with $\lambda_1 \neq 0$, (3.7) and (3.13) in the domain $\Omega = \{(t, x) : t > 0, x \in (x_1, x_2), x_a = \frac{2}{\sqrt{-\delta}}\pi n_a, n_a \in \mathbb{Z}\}$.

Finally, it should be noted that BVP (3.8), (3.9) and (3.2) can be examined in a similar way.

4. CONCLUSIONS

In this paper, a new definition of conditional invariance for BVPs is proposed. Its relationship with the definitions, given earlier for Lie's invariance, is shown and the applicability of the Q -conditional invariance is demonstrated. In particular, the example of BVP, which is Q -conditionally invariant under the given operator and is not invariant under any Lie symmetry operator, is presented. The definition is applied to the reduction of BVPs with the basic reaction-diffusion-convection equation (3.1). The constraints on the functions arising in the boundary conditions and allowing the reduction to the linear BVP with the basic second-order ODE are established. Moreover, the exact solution of the nonlinear BVP with zero Neumann conditions is found.

In conclusion, it should be noted that in [13] an algorithm was proposed allowing one to construct so-called side infinitesimals of the form (2.3) for BVP involving two-dimensional evolution equations. We assume that those 'side infinitesimals' are connected with the operators of Q -conditional symmetry, however, this is rather difficult to check because the examples presented in [13] are very cumbersome.

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GLOBAL LIE SYMMETRIES OF A SYSTEM OF SCHRÖDINGER EQUATIONS AND THE OSCILLATOR REPRESENTATION

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Abstract. We realize the oscillator representation of the metaplectic group $\mathrm{Mp}(n)$ in the space of solutions to a system of Schrödinger type equations on $\mathbb{R}^n \times \mathrm{Sym}(n, \mathbb{R})$. Our realization has particularly simple intertwining maps to the realizations given by Kashiwara and Vergne.

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1. INTRODUCTION

The origin of Lie theory lies in the symmetry analysis of systems of partial differential equations. These ideas have proved to be one of the most fruitful tools in the study of representation theory. Generalizing work in [21, 22] and using techniques similar to those found in [14], this paper employs Lie symmetry analysis to study the system of partial differential equations

$$\begin{aligned} 4s\partial_{t_{ii}}f(x, t) + \partial_{x_i}^2f(x, t) &= 0, \quad 1 \leq i \leq n \\ 2s\partial_{t_{ij}}f(x, t) + \partial_{x_i}\partial_{x_j}f(x, t) &= 0, \quad 1 \leq i < j \leq n \end{aligned} \quad (*)$$

with $s \in i\mathbb{R}^\times$. Here, $x = (x_i)$ and $t = (t_{ij})$ are the canonical coordinates on $\mathbb{R}^n$ and the space of real symmetric matrices $\mathrm{Sym}(n, \mathbb{R})$, respectively. Lie's prolongation method calculates the infinitesimal symmetries of system (*) to be the Jacobi Lie algebra $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R}) \ltimes \mathfrak{h}_{2n+1}$, where $\mathfrak{sp}(n, \mathbb{R})$ is the symplectic Lie algebra on $\mathbb{R}^{2n}$ and $\mathfrak{h}_{2n+1}$ is the $(2n+1)$ -dimensional Heisenberg Lie algebra, plus an infinite dimensional Lie algebra reflecting the fact that system (*) is linear. In particular, the space of all complex-valued functions $f \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathrm{Sym}(n, \mathbb{R}))$ satisfying system (*) carries a representation of $\mathfrak{g}$. The $\mathfrak{g}$ -action on $\mathcal{C}^\infty(\mathbb{R}^n \times \mathrm{Sym}(n, \mathbb{R}))$ does not exponentiate to a global action of the Jacobi group $G^J = \mathrm{Sp}(n, \mathbb{R}) \ltimes H_{2n+1}$ or any cover group. However, as we will explain below, there are canonical $\mathfrak{g}$ -invariant subspaces $\mathcal{J}'_\pm \subseteq \mathcal{C}^\infty(\mathbb{R}^n \times \mathrm{Sym}(n, \mathbb{R}))$ such that the $\mathfrak{g}$ -action on $\mathcal{J}'_\pm$ does exponentiate to a global action of the group $G = \mathrm{Mp}(n) \ltimes H_{2n+1}$, where $\mathrm{Mp}(n)$ is the metaplectic group, i. e., the double cover of $\mathrm{Sp}(n, \mathbb{R})$. The end result of our analysis shows that

the space of solutions to system (*) in $\mathcal{J}'_{\pm}$ gives a realization of the oscillator representation (or its dual depending on the sign of σ) of $\mathrm{Mp}(n)$ with particularly simple intertwining maps to two realizations given by Kashiwara and Vergne in [16].

One of the most ubiquitous representations found in the literature is the oscillator representation ω , often called the metaplectic or Segal–Shale–Weil representation, a genuine representation of G . We mention only briefly some of the settings in which this representation appears and refer the reader to [4] for a thorough development.

From classical number theory, the invariance properties of Jacobi theta functions [8] are found by lifting such functions to G^J . This lift, in turn, utilizes the oscillator representation [5]. A complete treatment of theta functions appears in [15] and many more results demonstrating the interplay between ω and aspects of number theory can be found in [17, 18, 27].

The quantization procedure in theoretical physics associates classical geometric systems to quantum mechanical systems and is very well studied ([11, 24–26, 28], and [1]). For example, the oscillator representation arises in quantum mechanics when one quantizes a single particle structure [20]. The representation ω is constructed and then used to establish results about the inducibility of a field automorphism by a unitary operator in all quantizations [23]. Another application of ω appears in quantum optics. In [2], the tensor product of ω with discrete series representations of $\mathrm{SU}(1, 1)$ admits squeezed coherent states. The broader role that ω plays in physics can be found in [7, 12].

In the representation theory, the oscillator representation is used to construct other important representations. For instance, the representations of G^J ($n = 1$) with non-trivial central character are realized as products of representations of $\mathrm{Mp}(1)$ and the oscillator representation [5]. In the well-known article [16], the k -fold tensor product $\otimes_k \omega$ is decomposed into irreducible unitary representations. First conjectured by Kashiwara and Vergne and later proved by Enright and Parthasarathy [9], all irreducible unitary highest weight representations for which the Verma module $N(\lambda + \rho)$ is reducible (i. e., λ is a reduction point) are found in $\otimes_k \omega$ for some k . In a similar vein, it is shown in [13] that every genuine discrete series representation of $\mathrm{Mp}(n)$ appears in $(\otimes_k \omega) \otimes (\otimes_m \omega^*)$, for some k and m . Finally, if F is a finite field, irreducible representations of $\mathrm{GL}(2, F)$ can be constructed by using the Weil representation ([6]), the restriction of ω to $\mathrm{SL}(2, F)$. For F , a non-Archimedean local field, the same is true of many supercuspidal irreducible representations of $\mathrm{GL}(2, F)$.

Given the manifold applications of ω , it may be helpful to identify some canonical realizations. A standard realization of ω arises via the Stone – von Neumann theorem as an intertwining operator between equivalent irreducible unitary representations of H_{2n+1} on $L^2(\mathbb{R}^n)$ ([10] and, in more generality, [27]). A second realization is the Fock model, where ω is realized as an integral operator on a reproducing kernel space. Motivated by Lie’s prolongation method ([19]), we induce from a subgroup of G and use a system of Schrödinger type equations to find a subspace on which

the action is irreducible. In [3], a reproducing space of holomorphic functions on $\mathrm{Sp}(n, \mathbb{R})/U(n) \times U(n)$ is shown to satisfy analogous differential equations (if one replaces real with complex differentiation), but no unitary action on that space is provided.

We now give an overview of our results. Kashiwara and Vergne give an embedding of the tensor product of the oscillator representation into a subspace of sections of vector bundles over Siegel upper half-space, $\mathfrak{H}_n$, and also into a subspace of certain principal series representations. For instance, in the case of the even part of the oscillator representation, they construct the maps

$$\begin{array}{ccc} \mathcal{I}'_+ \subseteq \mathcal{C}^\infty(\mathrm{Sym}(n, \mathbb{R})) & \xleftarrow{\mathcal{F}_1} & \mathcal{S}_+(\mathbb{R}^n) \\ & \searrow \mathrm{BV} \quad \swarrow \mathcal{F}_0 & \\ & \mathcal{O}(\mathfrak{H}_n) & \end{array}$$

where $\mathcal{S}(\mathbb{R}^n)$ denotes the set of Schwartz functions on $\mathbb{R}^n$, $\mathcal{S}_+(\mathbb{R}^n)$ denotes the even Schwartz functions, $\mathcal{I}'_+$ denotes the image of $\mathcal{S}_+(\mathbb{R}^n)$ under the map $\mathcal{F}_1 = \mathrm{BV} \circ \mathcal{F}_0$ (with $\mathcal{C}^\infty(\mathrm{Sym}(n, \mathbb{R}))$ being the noncompact picture of a certain principal series representation of the metaplectic group $\mathrm{Mp}(n)$), and the maps are given by the equalities $(\mathrm{BV} \Psi)(t) = \lim_{Y \rightarrow 0+} \Psi(t + iY)$ and

$$\begin{aligned} (\mathcal{F}_0 \psi)(Z) &= \int_{\mathbb{R}^n} \psi(\xi) e^{\frac{i}{2} \xi Z \xi^T} d\xi, \\ (\mathcal{F}_1 \psi)(t) &= \int_{\mathbb{R}^n} \psi(\xi) e^{\frac{i}{2} \xi t \xi^T} d\xi, \end{aligned}$$

where $\mathbb{R}^n$ is identified with $M_{1 \times n}(\mathbb{R})$, $\psi \in \mathcal{S}_+(\mathbb{R}^n)$, $Z \in \mathfrak{H}_n$, $t \in \mathrm{Sym}(n, \mathbb{R})$, $\Psi \in \mathrm{Im}(\mathcal{F}_0) \subseteq \mathcal{O}(\mathfrak{H}_n)$, and $\lim_{Y \rightarrow 0+}$ denotes the limit as $Y \rightarrow 0$ with $Y \in \mathrm{Sym}(n, \mathbb{R})$ and $Y > 0$.

Turning to our realization in a special case, we have a commutative diagram

$$\begin{array}{ccc} & \mathcal{D}'_+ & \\ \mathcal{E} \swarrow & & \searrow \mathcal{E} \\ \mathcal{I}'_+ & \xleftarrow{\mathcal{H}} & \mathcal{S}_+(\mathbb{R}^n) \end{array}$$

where $\mathcal{D}'_+ \subseteq \mathcal{C}^\infty(\mathbb{R}^n \times \mathrm{Sym}(n, \mathbb{R}))$ is a subset of smooth functions (coming from an induced representation via the noncompact picture) f in the kernel of the operators $4\pi^2 i \partial_{t_{ij}} - \partial_{x_i} \partial_{x_j}$, $i \neq j$, and $8\pi^2 i \partial_{t_{ii}} - \partial_{x_i}^2$ with $f(\cdot, t) \in \mathcal{S}_+(\mathbb{R}^n)$ for each $t \in \mathrm{Sym}(n, \mathbb{R})$ and $\mathcal{I}'_+ \subseteq \mathcal{C}^\infty(\mathrm{Sym}(n, \mathbb{R}))$ consisting of the set of Fourier transforms of Schwartz functions pulled back as measures on $\{-y^T y \mid y \in \mathbb{R}^n\} \subseteq \mathrm{Sym}(n, \mathbb{R})$. The maps $\mathcal{E}$ and $\mathcal{H}$ are particularly simple:

$$(\mathcal{E} f)(x) = \hat{f}(x, 0) \text{ and } (\mathcal{H} f)(t) = f(0, t)$$

(with the Fourier transform given by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi x^T} dx$). There is an explicit integral formula for $\mathcal{E}^{-1}$ given by

$$(\mathcal{E}^{-1}\psi)(x, t) = \int_{\mathbb{R}^n} f(\xi) e^{\frac{i}{2}\xi t \xi^T} e^{2\pi i \xi x^T} d\xi,$$

which gives rise to a formula for $\mathcal{H} = \mathcal{F}_1$ as $(\mathcal{H}\psi)(t) = \int_{\mathbb{R}^n} f(\xi) e^{\frac{i}{2}\xi t \xi^T} d\xi$. An inverse for $\mathcal{G}$ can be given by viewing elements of $\mathcal{I}'_+$ as tempered distributions on $\text{Sym}(n, \mathbb{R})$, applying a Fourier transform and taking a limit using approximations to a δ -function.

The highest weight vector in $\mathcal{D}'_+$ is the function $f_+ \in \mathcal{C}^\infty(\mathbb{R}^n \times \text{Sym}(n, \mathbb{R}))$ defined as

$$f_+(x, t) = \det(I_n - it)^{-\frac{1}{2}} e^{-2\pi^2 x(I_n - it)^{-1} x^T}.$$

The corresponding vector in $\mathcal{I}'_+$ is $f_+(0, t) = \det(I_n - it)^{-\frac{1}{2}}$ and in $\mathcal{S}_+(\mathbb{R}^n)$ is

$$\hat{f}_+(\xi, 0) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}\|\xi\|^2}.$$

Note that the choice of, say, $s = 2\pi^2 i$ gives rise to the dual representation and Schrödinger-like partial differential operators with the lowest weight representations.

The above commutative diagram fits on top of the Kashiwara–Vergne picture to give the following combined commutative diagram.

$$\begin{array}{ccc} & \mathcal{D}'_+ & \\ \mathcal{G} \swarrow & & \searrow \mathcal{E} \\ \mathcal{I}'_+ & \xleftarrow{\mathcal{H}=\mathcal{F}_1} & \mathcal{S}_+(\mathbb{R}^n) \\ \nwarrow \text{BV} & & \swarrow \mathcal{F}_0 \\ & \mathcal{O}(\mathfrak{H}_n) & \end{array}$$

There is a similar picture for the odd part of the oscillator representation that fits in with the Kashiwara–Vergne realization in an analogous way. The main change is that $\mathcal{G}$ is replaced by the operator $(\mathcal{G}_n f)(t) = \nabla_{\mathbb{R}^n} f(0, t)$ and $\mathcal{H}$ by the operator

$$(\mathcal{H}_n f)(t) = \nabla \left(\int_{\mathbb{R}^n} f(\xi) e^{\frac{i}{2}\xi t \xi^T} e^{2\pi i \xi x^T} d\xi \right) \Big|_{x=0}.$$

2. A DOUBLE COVER OF THE JACOBI GROUP

With respect to the standard symplectic form $J_{n+1} = \begin{pmatrix} 0 & -I_{n+1} \\ I_{n+1} & 0 \end{pmatrix}$, the *Jacobi group* is

$$G^J = \text{Sp}(n+1, \mathbb{R}) \cap \left\{ \begin{pmatrix} * & * \\ 0_{1 \times (2n+1)} & 1 \end{pmatrix} \right\} \cong \text{Sp}(n, \mathbb{R}) \ltimes H_{2n+1},$$

where H_{2n+1} is the $2n + 1$ dimensional real Heisenberg Lie group. It has Lie algebra $\mathfrak{g} \cong \mathfrak{sp}(n, \mathbb{R}) \ltimes \mathfrak{h}_{2n+1}$, where $\mathfrak{h}_{2n+1}$ is the $2n + 1$ dimensional real Heisenberg Lie algebra. We write $\mathfrak{H}_n$ for the *Siegel upper half-space* consisting of the set of $Z = X + iY$ with $X, Y \in \text{Sym}(n, \mathbb{R})$ and $Y > 0$ (positive definite), which carries a transitive action by $\text{Sp}(n, \mathbb{R})$ by linear fractional transformations, $g \cdot Z = (AZ + B)(CZ + D)^{-1}$ where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n, \mathbb{R})$. Note that the stabilizer of iI_n in $\text{Sp}(n, \mathbb{R})$ is a maximal compact subgroup $K \cong U(n)$.

The main object of study is the double cover of G^J ,

$$G = \text{Mp}(n) \ltimes H_{2n+1}.$$

Here, the action of $\text{Mp}(n)$ on H_{2n+1} factors through its projection to $\text{Sp}(n, \mathbb{R})$ and we realize the metaplectic group $\text{Mp}(n)$ as the set of pairs (g, ε) with $g \in \text{Sp}(n, \mathbb{R})$ and smooth $\varepsilon : \mathfrak{H}_n \rightarrow \mathbb{C}$ satisfying $\varepsilon(Z)^2 = \det(CZ + D)$. The group law on $\text{Mp}(n)$ is given by $(g_1, \varepsilon_1) \cdot (g_2, \varepsilon_2) = (g_1 g_2, Z \mapsto \varepsilon_1(g_2 \cdot Z) \varepsilon_2(Z))$, and to be explicit, the group law on $\text{Mp}(n) \ltimes H_{2n+1}$ is given by

$$((g_1, \varepsilon_1), h_1) \cdot ((g_2, \varepsilon_2), h_2) = ((g_1, \varepsilon_1) \cdot (g_2, \varepsilon_2), g_2^{-1} h_1 g_2 h_2).$$

We also need to extend the definition of ε from $\mathfrak{H}_n$ to $\text{Sym}(n, \mathbb{R})$ almost everywhere. For this, let $\varepsilon : \text{Sym}(n, \mathbb{R}) \rightarrow \mathbb{C}$ be given by $\varepsilon(X) = \lim_{Y \rightarrow 0^+} \varepsilon(X + iY)$ (here $Y \rightarrow 0^+$ denotes $Y \rightarrow 0$ with $Y > 0$), which will be defined when $\det(CX + D) \neq 0$. In particular, it is easy to see that $\varepsilon(X)$ exists and is given by

$$\varepsilon(X) = i^l \sqrt{|\det(CX + D)|},$$

where $l = l(\varepsilon, X)$ is one of the two choices (determined precisely by ε and X) of $l \in \mathbb{Z}_4$ for which $(-1)^l = \text{sgn}(\det(CX + D))$. In a similar manner, we extend to an almost everywhere action of $\text{Sp}(n, \mathbb{R})$ on $\text{Sym}(n, \mathbb{R})$ given by $g \cdot X = (AX + B)(CX + D)^{-1}$ for $X \in \text{Sym}(n, \mathbb{R})$ when $\det(CX + D) \neq 0$.

3. THE INDUCED REPRESENTATIONS

Consider the subalgebra $\bar{\mathfrak{p}}$ of $\mathfrak{g}$ given as the semidirect product of the maximal parabolic subalgebra $\bar{\mathfrak{p}}_{\text{sp}}$ of $\mathfrak{sp}(n, \mathbb{R})$ consisting of lower $n \times n$ block-diagonal matrices and the maximal abelian subalgebra $\mathfrak{w}$ of $\mathfrak{h}_{2n+1}$ isomorphic to $\mathbb{R}^n \times \mathbb{R}$ that is stabilized by $\bar{\mathfrak{p}}_{\text{sp}}$. Write the Langlands decomposition for $\bar{\mathfrak{p}}_{\text{sp}}$ as $\bar{\mathfrak{p}}_{\text{sp}} = \mathfrak{m}\mathfrak{a}\bar{\mathfrak{n}}$. Write $\bar{P} = M\bar{A}\bar{N} \ltimes W$ for the corresponding groups. It turns out that $M \cong \{(A, c) \mid A \in GL(n, \mathbb{R}), \det A \in \{\pm 1\}, c^2 = \det A^{-1}\}$, so the component group M/M_0 is isomorphic to $\mathbb{Z}_4$, $A \cong \mathbb{R}^+$, $\bar{N} \cong \text{Sym}(n, \mathbb{R})$, and $W \cong \mathbb{R}^{n+1}$.

For $q \in \mathbb{Z}$ (determined only mod 4), $r \in \mathbb{C}$, and $s \in \mathbb{C}$, we define a character $\chi_{q,r,s} : \bar{P} \rightarrow \mathbb{C}$ by $\chi_{q,r,s}(A, c) = c^q$ on M , $\chi_{q,r,s}(a) = a^{rn}$ on A , and $\chi_{q,r,s}$ is

trivial on W and $\bar{N}$. We study the induced representation

$$\begin{aligned} I(q, r, s) &= \text{Ind}_{\bar{P}}^G \chi_{q,r,s} \\ &= \{ \mathcal{C}^\infty \phi : G \rightarrow \mathbb{C} \mid \phi(gp) = \chi_{q,r,s}(p)^{-1} \phi(g) \text{ for } g \in G, p \in \bar{P} \} \end{aligned}$$

with group action $(g \cdot \phi)(g') = \phi(g^{-1}g')$.

We will also have an opportunity to make use of two related induced representations of $\text{Mp}(n)$. To this end, define a character and an n -dimensional representation of MAN , $\chi_{q,r} : MAN \rightarrow \mathbb{C}$ and $\pi_{q,r} : MAN \rightarrow \text{GL}(n, \mathbb{C})$, by $\chi_{q,r} = \chi_{q,r,s}|_{MAN}$ and $\pi_{q,r}((A, c)a\bar{n}) = \chi_{q,r}((A, c)a\bar{n})A^{-1}$. The associated induced representations are $I(q, r) = \text{Ind}_{MAN}^{\text{Mp}(n)} \chi_{q,r}$ and $I_n(q, r) = \text{Ind}_{MAN}^{\text{Mp}(n)} \pi_{q,r}$.

4. NONCOMPACT PICTURES

The *noncompact realization* of these induced representations amounts to restricting the functions ϕ to the copy of $\mathbb{R}^n \times \text{Sym}(n, \mathbb{R})$ in G given by the exponential of the nilpotent radical of the opposite parabolic to $\bar{\mathfrak{p}}_{\text{sp}}$ in $\mathfrak{sp}(n, \mathbb{R})$ and the corresponding complementary invariant abelian subalgebra to $\mathfrak{w}$ in $\mathfrak{h}_{2n+1}$. We write

$$I'(q, r, s) \subsetneq \mathcal{C}^\infty(\mathbb{R}^n \times \text{Sym}(n, \mathbb{R}))$$

for the resulting representation. When necessary, we will coordinatize $\text{Sym}(n, \mathbb{R})$ as $\mathbb{R}^{n(n+1)/2}$ by writing $B = (t_{ij})$ with $t_{ij} = t_{ji}$.

It is straightforward to check that for $f \in I'(q, r, s)$, the action of $g = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \varepsilon \right)$ in $\text{Mp}(n)$ on f is given by

$$\begin{aligned} ((g, \varepsilon) \cdot f)(x, t) &= i^{lq} |\det(A - tC)|^r e^{-sx C(A-tC)^{-1} x^T} \\ &\quad \cdot f(x \left(-C^T t + A^T \right)^{-1}, (A - tC)^{-1}(tD - B)) \end{aligned}$$

when $\det(A - tC) \neq 0$ and where $l \in \mathbb{Z}_4$ satisfies

$$\varepsilon(g^{-1} \cdot t) = i^l |\det(A - tC)|^{-1/2}.$$

For $h = (x_0, y_0, z_0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, the action of $h \in H_{2n+1}$ on f is given by $(h \cdot f)(x, t) = e^{s(2xx_0^T + z_0 - x_0 t x_0^T - y_0 x_0^T)} \cdot f(x - y_0 - x_0 t, t)$.

In a similar fashion, we also have the noncompact realizations of $I(q, r)$ and $I_n(q, r)$ given by restriction to $N \cong \text{Sym}(n, \mathbb{R})$. We denote these realizations by $I'(q, r) \subseteq \mathcal{C}^\infty(\text{Sym}(n, \mathbb{R}))$ and $I'_n(q, r) \subseteq \mathcal{C}^\infty(\text{Sym}(n, \mathbb{R}), \mathbb{C}^n)$. There are $\text{Mp}(n)$ -intertwining maps

$$\mathcal{G} : I'(q, r, s) \rightarrow I'(q, r), \mathcal{G}_n : I'(q, r, s) \rightarrow I'_n(q, r - n^{-1})$$

given by mappings $f \rightarrow f(0, \cdot)$ and $f \rightarrow \nabla f(0, \cdot)$, respectively. Using the noncompact realization, a Lie algebra calculation shows the following.

Theorem 1. *For $r = -1/2$, the set of functions $f \in I'(q, r, s)$ satisfying the system of partial differential equations from system (*) is G -invariant.*

For the rest of the paper, we assume $r = -1/2$. Now let $s = i\sigma \neq 0$ be purely imaginary and write $\varepsilon_\sigma = \text{sgn}(\sigma)$. Define

$$\mathcal{D}' \subseteq I'(q, r, s) \subseteq \mathcal{C}^\infty(\mathbb{R}^n \times \text{Sym}(n, \mathbb{R}))$$

to be the space of functions $f \in I'(q, r, s)$ that satisfy the system of partial differential equations from system (*) with $f(\cdot, 0) \in \mathcal{S}(\mathbb{R}^n)$, a Schwartz function. Write $\mathcal{D}'_+$ and $\mathcal{D}'_-$ for the functions in $\mathcal{D}'$ that are even (odd) in x for each $t \in \text{Sym}(n, \mathbb{R})$. A nontrivial argument using Fourier analysis shows the following.

Theorem 2. *The space $\mathcal{D}'$ is G -invariant and, for $q = -\text{sgn } \sigma$, $\mathcal{D}'_\pm$ is nonzero.*

For example, when $\sigma > 0$ and $q \equiv -1$, $f_+ \in \mathcal{D}'_+$ with

$$f_+(x, t) = \varepsilon_t (iI_n)^{-1} e^{-\sigma x(I_n + it)^{-1} x^T},$$

where $\varepsilon_t(Z)$ is the analytic continuation of the function $Z \mapsto \sqrt{\det(I_n + tZ)}$ to $Z \in \mathfrak{H}_n$ for sufficiently small Z . Similarly, for $\alpha \in \mathbb{C}^n$, $f_{+, \alpha} \in \mathcal{D}'_-$ where

$$f_{+, \alpha}(x, t) = \varepsilon_t (iI_n)^{-1} (x(I_n + it)^{-1} \alpha^T) e^{-\sigma x(I_n + it)^{-1} x^T}.$$

When $\sigma < 0$ and $q \equiv 1$, f_- and $f_{-, \alpha}$ can be defined using complex conjugation.

5. RESTRICTION TO $t = 0$

In order to guarantee the existence of nontrivial solutions in $\mathcal{D}'$, we assume $q = -\text{sgn } \sigma$ for the rest of the paper. By Fourier analysis, the map from $\mathcal{D}'$ to $\mathcal{S}(\mathbb{R}^n)$ given by restriction to $t = 0$ is injective. Following this map with the Fourier transform gives the injective map: $\mathcal{E} : \mathcal{D}' \rightarrow \mathcal{S}(\mathbb{R}^n)$ defined by $(\mathcal{E}f)(x) = \hat{f}(x, 0)$. We also write $\mathcal{S} = \text{Im}(\mathcal{E})$ and $\mathcal{S}_+$ and $\mathcal{S}_-$ for the images of $\mathcal{D}'_+$ and $\mathcal{D}'_-$, respectively. We make $\mathcal{S}$ into a G -module by requiring $\mathcal{E}$ to be an intertwining isomorphism. Finally, consider the dilation operator defined by $(Tf)(x) = f(|\sigma|^{1/2} x / \pi \sqrt{2})$. Making T into an intertwining map, we get an equivalent action on $T(\mathcal{S}) \subseteq \mathcal{S}(\mathbb{R}^n)$. Using standard analysis arguments and the definitions, we get the following result.

Theorem 3. *If $s = i\sigma$ with $\sigma < 0$, $T(\mathcal{S})$ is a dense $\text{Mp}(n)$ -invariant subspace in the oscillator representation. If $\sigma > 0$, $T(\mathcal{S})$ is isomorphic to the dual to the oscillator representation.*

In either case, this action completes to a unitary representation on $L^2(\mathbb{R}^n)$ and decomposes as a direct sum of irreducible representations via the set of odd and even functions, $L^2(\mathbb{R}^n) = L^2(\mathbb{R}^n)_+ \oplus L^2(\mathbb{R}^n)_-$.

6. RESTRICTION TO $x = 0$

Recall that we have an $\text{Mp}(n)$ -intertwining map $\mathcal{G} : I'(q, r, s) \rightarrow I'(q, r)$ given by $(\mathcal{G}f)(t) = f(0, t)$ and an intertwining map $\mathcal{G}_n : I'(q, r, s) \rightarrow I'_n(q, r - \frac{1}{n})$ given by $(\mathcal{G}_n f)(t) = \nabla f(0, t)$. By restricting to $\mathcal{D}'$ and pre-composing with $\mathcal{E}^{-1}$,

we obtain $\text{Mp}(n)$ -maps $\mathcal{H} : \mathcal{S} \rightarrow I'(q, r)$ and $\mathcal{H}_n : \mathcal{S} \rightarrow I'_n(q, r - n^{-1})$. A Fourier analysis shows they are given by the formulas

$$(\mathcal{H}f)(t) = \int_{\mathbb{R}^n} f(\xi) e^{\frac{\pi^2}{s} \xi t \xi^T} d\xi,$$

$$(\mathcal{H}_n f)(t) = \nabla \left(\int_{\mathbb{R}^n} f(\xi) e^{\frac{\pi^2}{s} \xi t \xi^T} e^{2\pi i \xi x^T} d\xi \right) \Big|_{x=0}.$$

Clearly, $\mathcal{S}_- \subseteq \ker \mathcal{H}$ and $\mathcal{S}_+ \subseteq \ker \mathcal{H}_n$ (equivalently, $\mathcal{D}'_- \subseteq \ker \mathcal{G}$ and $\mathcal{D}'_+ \subseteq \ker \mathcal{G}_n$). Showing that these are the entire kernels involves inverting $\mathcal{H}|_{\mathcal{S}_+}$ and $\mathcal{H}_n|_{\mathcal{S}_-}$ (equivalently, $\mathcal{G}|_{\mathcal{D}'_+}$ and $\mathcal{G}_n|_{\mathcal{D}'_-}$). A straightforward Fourier analysis requires a bit more care due to the fact that the images usually do not have sufficient decay properties to be L^1 or L^2 functions (unless $n = 1$, see [22]). In fact, if we could view $f \in \mathcal{D}' \subseteq I'(q, r, s)$ as a tempered distribution $f(x, \cdot) \in \mathcal{S}'(\text{Sym}(n, \mathbb{R}) \cong \mathbb{R}^{n(n+1)/2})$ and writing $\mathcal{F}$ for the Fourier transform on $\mathcal{S}(\text{Sym}(n, \mathbb{R}))$ given by $(\mathcal{F}f)(\tau) = \int_{\text{Sym}(n, \mathbb{R})} f(t) e^{-2\pi i \text{tr}(t\tau)} dt$, it would be possible to show that $\mathcal{F}f$ is supported on $\{\tau \in \text{Sym}(n, \mathbb{R}) \mid \tau_{i,i} \tau_{j,j} = \tau_{ij}^2 \text{ all } i \neq j\}$. This is, of course, a rank of at most one condition on $\text{Sym}(n, \mathbb{R})$.

As a result, it will be useful to consider the cone defined by the function $\theta : \mathbb{R}^n \rightarrow \text{Sym}(n, \mathbb{R})$ given by $\theta(y) = \pi y^T y / 2\sigma$. For $f \in \mathcal{D}' \subseteq I'(q, r, s)$ and each $x \in \mathbb{R}^n$, it is possible to show that $f(x, \cdot)$ may be viewed as a tempered distribution on $\text{Sym}(n, \mathbb{R})$ given by

$$\langle f(x, \cdot), \phi \rangle = \int_{\text{Sym}(n, \mathbb{R})} f(x, t) \phi(t) dt$$

for each $\phi \in \mathcal{S}(\text{Sym}(n, \mathbb{R}))$. Its Fourier transform $\mathcal{F}f(x, \cdot) \in \mathcal{S}'(\text{Sym}(n, \mathbb{R}))$ is then shown to be $\langle \mathcal{F}f(x, \cdot), \phi \rangle = \int_{\mathbb{R}^n} \hat{f}(\xi, 0) (\phi \circ \theta)(\xi) e^{2\pi i \xi x^T} d\xi$ and is supported on $\text{Im } \theta$. Similarly, for each $1 \leq j \leq n$, $\partial_{x_j} f(x, \cdot)$ may be viewed as a tempered distribution on $\text{Sym}(n, \mathbb{R})$ given by

$$\langle \partial_{x_j} f(x, \cdot), \phi \rangle = \int_{\text{Sym}(n, \mathbb{R})} \partial_{x_j} f(x, t) \phi(t) dt$$

for an arbitrary $\phi \in \mathcal{S}(\text{Sym}(n, \mathbb{R}))$. Its Fourier transform is then shown to be $\mathcal{F}(\partial_{x_j} f)(x, \cdot) \in \mathcal{S}'(\text{Sym}(n, \mathbb{R}))$ given by

$$\langle \mathcal{F}(\partial_{x_j} f)(x, \cdot), \phi \rangle = 2\pi i \int_{\mathbb{R}^n} \xi_j \hat{f}(\xi, 0) (\phi \circ \theta)(\xi) e^{2\pi i \xi x^T} d\xi$$

and is also supported on $\text{Im } \theta$.

Again, using Fourier analysis, it is then possible to show that $\mathcal{H}|_{\mathcal{S}_+}$ is injective and $\mathcal{H}_n|_{\mathcal{S}_-}$ is injective (equivalently, $\mathcal{G}|_{\mathcal{D}'_+}$ is injective and $\mathcal{G}_n|_{\mathcal{D}'_-}$ is injective). If we write $\mathcal{I}'_{\pm}$ for the image of $\mathcal{D}'_{\pm}$ under $\mathcal{G}$ and $\mathcal{G}_n$, respectively (alternately, the image of $\mathcal{S}_{\pm}$ under $\mathcal{H}$ and $\mathcal{H}_n$, respectively), and embed $\mathcal{S} \hookrightarrow \mathcal{S}'(\text{Sym}(n, \mathbb{R}))$

via θ by mapping $\psi \rightarrow \langle \psi, \cdot \rangle$ where $\langle \psi, \phi \rangle = \int_{\mathbb{R}^n} \psi(\xi) (\phi \circ \theta)(\xi) d\xi$ and embed $\mathcal{S} \hookrightarrow \mathcal{S}'(\text{Sym}(n, \mathbb{R}), \mathbb{R}^n)$ via θ by mapping $\psi \rightarrow \langle \psi, \cdot \rangle$, where

$$\langle \psi, \phi \rangle = \left(\int_{\mathbb{R}^n} \xi_j \psi(\xi) (\phi \circ \theta)(\xi) d\xi \right)_{j=1, \dots, n}$$

for $\phi \in \mathcal{S}(\text{Sym}(n, \mathbb{R}))$, we can prove:

Theorem 4. *The spaces $\mathcal{J}'_{\pm} \subseteq I'(q, r)$ are given explicitly by*

$$\begin{aligned} \mathcal{J}'_+ &= \{ \mathcal{F}\psi \mid \psi \in \mathcal{S} \subseteq \mathcal{S}'(\text{Sym}(n, \mathbb{R})) \}, \\ \mathcal{J}'_- &= \{ \mathcal{F}\psi \mid \psi \in \mathcal{S} \subseteq \mathcal{S}'(\text{Sym}(n, \mathbb{R}), \mathbb{R}^n) \}. \end{aligned}$$

7. K -FINITE VECTORS

If $M \in M_n(\mathbb{C})$ and p is a complex valued polynomial on $\mathbb{R}^n$, define $\tilde{p}(x, M)$ by

$$\tilde{p}(x, M) = e^{|\sigma|xMx^T} p(\partial_x) \left(e^{-|\sigma|xMx^T} \right),$$

with $p(\partial_x)$ representing the constant coefficient differential operator obtained by replacing x_j by ∂_{x_j} . For p of the form x^α , $\tilde{p}$ defines a generalization of the Hermite polynomials. Making use of the previous theorems and knowledge of the oscillator representation, we get the following description of K -finite vectors.

Theorem 5. *Up to constant multiples, the highest ($\sigma < 0$) and lowest ($\sigma > 0$) K -finite vectors of $(\mathcal{D}'_+)_K$ are given by the functions f_- and f_+ , respectively.*

The highest and lowest K -type vectors of $(\mathcal{D}'_-)_K$ consist of the functions $f_{-,a}$ and $f_{+,a}$, respectively, for $a \in \mathbb{C}^n$.

In general, the K -finite vectors in $\mathcal{D}'$ consist of the functions $f_{-,p}$ and $f_{+,p}$, where

$$f_{-,p}(x, t) = \overline{\varepsilon_t(iI_n)}^{-1} \tilde{p}(x, (I_n - it)^{-1}) e^{\sigma x(I_n - it)^{-1} x^T}$$

and

$$f_{+,p}(x, t) = \varepsilon_t(iI_n)^{-1} \tilde{p}(x, (I_n + it)^{-1}) e^{-\sigma x(I_n + it)^{-1} x^T},$$

p being a complex valued polynomial on $\mathbb{R}^n$.

Corollary 1. *The spaces of the highest ($\sigma < 0$) and lowest ($\sigma > 0$) K -finite vectors of $(\mathcal{J}'_+)_K$ are spanned by the functions $f_{\text{sgn } \sigma}$ given by $f_-(0, t) = \overline{\varepsilon_t(iI_n)}^{-1}$ and $f_+(0, t) = \varepsilon_t(iI_n)^{-1}$, respectively. The highest ($\sigma < 0$) and lowest ($\sigma > 0$) K -type vectors of $(\mathcal{J}'_-)_K$ are given by the functions $f_{\text{sgn } \sigma, a}$, where*

$$f_{-,a}(t) = \overline{\varepsilon_t(iI_n)}^{-1} a(I_n - it)^{-1}$$

and

$$f_{+,a}(t) = \varepsilon_t (iI_n)^{-1} a (I_n + it)^{-1}$$

for $a \in \mathbb{R}^n$.

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ON CONTINUOUSLY DEFECTIVE ELASTIC CRYSTALS

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Abstract. We analyze mathematical underpinnings of Davini's theory of defective crystals [1] when the defectiveness of a kinematic state may be material point dependent. We show how the underlying space can be identified with a suitably chosen homogeneous space and how the uniformly defective structure is just a special case.

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INTRODUCTION

The kinematic theory of continuously defective elastic crystal bodies was originally proposed by Davini [1] and developed over the last two decades by him and Parry, and his collaborators (see, for example, [2, 3, 7]). The key assumption of this approach is that the state of a defective (continuous) crystal is defined by three linearly independent and smoothly varying over the body, $\mathbb{R}^3$ in our case, *lattice vector fields* which are to represent an averaged micro (atomic) structure. The defectiveness of such a state is described by the *dislocation density tensor*, an object which measures the first order interrelations of the given lattice vector fields.

The main focus of the theory has been so far on uniform defective crystals, that is, the states characterized by the property that the dislocation density tensor is constant throughout the body. The reason for this is that when the given vector fields are such that the corresponding dislocation density tensor is position independent, the underlying space (the body) can be equipped with a Lie group structure in such a way that the lattice vector fields are right invariant under its action on itself, [7]. The availability of this "additional" structure allows one to use the power of the theory of Lie groups and algebras to analyze in a systematic way the properties of such uniform defective crystal states, in particular, the question of symmetries, both local and global, discrete and continuous (see for example [8]). In contrast, when the lattice vector fields are such that the dislocation density tensor is material point dependent, the Lie group structure is no longer available.

In this short note, we look closer at the states with a non-uniform dislocation density tensor. We show that although the underlying space $\mathbb{R}^3$ cannot indeed be assigned a Lie group structure, it can be identified with the properly defined homogeneous

space the total space of which is a Lie group. The Lie algebra of this group is homomorphic to a Lie subalgebra of smooth vector fields on $\mathbb{R}^3$ generated uniquely by the three lattice vector fields. The homogeneous space structure of $\mathbb{R}^3$ collapses to a Lie group when the Lie algebra generated by the lattice vector fields is of dimension three and the corresponding dislocation density tensor is base point independent, thus proving that the uniformly defective state is just a special case of a general kinematic state of a defective crystal.

The paper is divided into four short sections. The first two sections are dedicated to the presentation of the foundations of Davini's theory [1] and the analysis of the uniformly defective states, [7]. In Section 3, we deal with the non-uniform case. In the last section, we show how the dislocation density tensor relates to the intrinsic characteristic of the homogeneous space associated with the lattice vector fields.

1. CONTINUOUS ELASTIC CRYSTALS

Let the kinematic state of a continuous solid crystal body be given by three linearly independent vector fields $l_i : \mathbb{R}^3 \rightarrow T\mathbb{R}^3$, $i = 1, 2, 3$, where $T\mathbb{R}^3$ denotes the tangent space to $\mathbb{R}^3$. In other words, the state of a continuous elastic crystal is defined by a global smooth section $\mathbf{l} : \mathbb{R}^3 \rightarrow L(\mathbb{R}^3)$ of the bundle of the linear frames of $\mathbb{R}^3$, [5], called a *continuous lattice* or simply a *lattice*. Invoking the Euclidean structure of $\mathbb{R}^3$, the lattice $\mathbf{l}(x)$ induces the dual frame $\mathbf{d} : \mathbb{R}^3 \rightarrow L(\mathbb{R}^3)$ such that $d_i(x) \cdot l_j(x) = \delta_{ij}$, $i, j = 1, 2, 3$, $x \in \mathbb{R}^3$, where δ_{ij} denotes the usual Kronecker's delta. In this context, the “defectiveness” of the lattice $\mathbf{l}(x)$ can be characterized by the *dislocation density tensor* (ddt) the components of which are defined by the equations

$$n(x)S_{ij}(x) = \nabla \wedge d_i(x) \cdot d_j(x), \quad i, j = 1, 2, 3, \quad x \in \mathbb{R}^3, \quad (1.1)$$

where $n(x) := d_1(x) \cdot d_2(x) \wedge d_3(x)$ is the lattice volume element. Looking closer at the properties of the dislocation density tensor, we note that if the defining frame field $\mathbf{l}(x)$ is holonomic (integrable), the corresponding dislocation density tensor vanishes everywhere. In fact, the opposite is also true. That is, if $S_{ij}(x)$ is identically zero, the lattice $\mathbf{l}(x)$ it represents is holonomic, [2]. In particular, the dislocation density tensor of the *ideal lattice* defined by a constant orthonormal frame $\mathbf{e}(x)$ vanishes identically.

Having two crystalline structures, say $\mathbf{l}(x)$ and $\tilde{\mathbf{l}}(x)$, we state that they are *elastically related* if there exists a diffeomorphism $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that

$$\tilde{l}_i(\phi(x)) = \phi_*(l_i(x)), \quad i = 1, 2, 3, \quad x \in \mathbb{R}^3 \quad (1.2)$$

where $\phi_* : T\mathbb{R}^3 \rightarrow T\mathbb{R}^3$ denotes the tangent map of ϕ . Thus, any diffeomorphism of $\mathbb{R}^3$, when applied to a continuous lattice via (1.2), induces an elastically related structure. It is not true, however, that any two (smooth) crystalline structures are elastically related unless additional conditions are met. Indeed, as shown in [1], one of the properties of an elastic deformation is that it preserves the defectiveness of the given continuous lattice. Namely, given a diffeomorphism $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, the lattice

$\mathbf{l}(x)$, and the elastically induced lattice $\tilde{\mathbf{l}}(\phi(x)) = \phi_*(\mathbf{l}(x))$,

$$\tilde{S}_{ij}(\phi(x)) = S_{ij}(x), \quad i, j = 1, 2, 3, \quad x \in \mathbb{R}^3 \quad (1.3)$$

where $\tilde{S}_{ij}(x)$ are the components of the dislocation density tensor of the new structure*.

The ideal lattice, in fact any holonomic frame field, is an example of a continuous elastic crystal with a uniform (point independent) dislocation density tensor. More generally, a continuous lattice is called *uniformly defective* if its dislocation density tensor $S_{ij}(x)$ is material point independent. It is easy to see from the equation (1.3) that if two uniformly defective lattices are elastically related, they have the same dislocation density tensor. The reader should, however, be cautioned that two uniformly defective lattices with the same dislocation density tensor are not guaranteed to be elastically related. Indeed, such lattices may be related by the *neutral deformation* [1], which preserves defectiveness of a lattice but is not elastic in the sense that it does not come from a diffeomorphism of the underlying space $\mathbb{R}^3$ in our case.

In the next section, we shall look closer at the uniformly defective continuous lattices summarizing how the fact that the dislocation density tensor is uniform allows one to introduce a nontrivial group structure on the underlying space $\mathbb{R}^3$.

2. UNIFORMLY DEFECTIVE STRUCTURES

Consider a uniformly defective continuous lattice $\mathbf{l}(x)$ with the dislocation density tensor S_{ij} and such that $l_i(\mathbf{0}) = \mathbf{e}_i$, $i = 1, 2, 3$, at the origin $\mathbf{0} \in \mathbb{R}^3$. Motivated by the fact that S_{ij} is constant and that any elastic deformation preserves it, we ask if given an arbitrary point $u \in \mathbb{R}^3$ there exists a diffeomorphism $\psi(\cdot, u) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $\psi(\mathbf{0}, u) = u$ and

$$l_i(\psi(x, u)) = \nabla_1 \psi(x, u) l_i(x), \quad i = 1, 2, 3, \quad x, u \in \mathbb{R}^3 \quad (2.1)$$

where $\nabla_1 \psi(\cdot, u)$ denotes the Jacobian of ψ with respect to the (first) variable. The assumption that the dislocation density tensor S_{ij} is constant turns out to be an integrability condition for the system of equations (2.1), [7], guaranteeing that there is a smooth and invertible (in each variable separately) solution $\psi : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which is associative and such that for every $x \in \mathbb{R}^3$ there exists $x^{-1} \in \mathbb{R}^3$ with the property that

$$\psi(x, x^{-1}) = \psi(x^{-1}, x) = \mathbf{0}, \quad (2.2)$$

where

$$\psi(\mathbf{0}, x) = \psi(x, \mathbf{0}) = x. \quad (2.3)$$

*A homogeneous deformation $\phi(x) = \mathbf{\Gamma}x$ is a simple example of an elastic deformation, where $\mathbf{\Gamma} \equiv \{\gamma_i^j\}$ is a non-singular 3×3 matrix. Indeed, as easily confirmed by the direct calculations, the deformed lattice $\hat{l}_i(x) = \gamma_i^j l_j(\mathbf{\Gamma}^{-1}x)$ has the dislocation density tensor $\hat{S}_{ij}(x) = S_{ij}(\mathbf{\Gamma}^{-1}x)$ for every $x \in \mathbb{R}^3$.

In other words, any uniformly defective lattice is elastically self-similar. Conversely, if, for some continuous lattice $\mathbf{l}(x)$, the system of equations (2.1) has an associative solution ψ satisfying (2.2) and (2.3), the lattice must be uniformly defective.

The associative solution ψ to the system of equations (2.1) can be viewed as a group multiplication on the underlying space $\mathbb{R}^3$ with the identity element at the origin $\mathbf{0} \in \mathbb{R}^3$. In fact, as the mapping ψ is smooth and invertible in each argument, it introduces a Lie group structure on $\mathbb{R}^3$ such that the vector fields $l_i(x)$, $i = 1, 2, 3$ are right invariant under its action. This implies that the algebra of vectors fields generated by $l_i(x)$, $i = 1, 2, 3$, with the standard Lie bracket $[l_i, l_j] := (l_i \cdot \nabla)l_j - (l_j \cdot \nabla)l_i$ is isomorphic to the Lie algebra of the left invariant vector fields on $\mathbb{R}^3$ viewed as a group with the multiplication $\psi : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Let C_{kij} denote the corresponding Lie algebra constants, that is, $[l_i, l_j] = C_{kij}l_k$, $i, j, k = 1, 2, 3$, where the summation convention over the repeated indices is enforced. It can be shown, [3], that the dislocation density tensor is such that

$$\epsilon_{jkl}S_{ij} = C_{ikl} \quad (2.4)$$

where ϵ_{jkl} is the classical alternating tensor.

To end this section, we present a simple example illustrating the concepts introduced so far.

Example 1. Let us consider a continuous lattice defined by the frame

$$l_1(x) = \mathbf{e}_1, \quad l_2(x) = \mathbf{e}_2, \quad l_3(x) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \mathbf{e}_3. \quad (2.5)$$

As the Lie brackets are $[l_1, l_2] = 0$, $[l_1, l_3] = l_1$ and $[l_2, l_3] = l_2$, the only non-zero Lie algebra constants are $C_{113} = C_{223} = 1$. Hence, the dislocation density tensor

$$S_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.6)$$

To find a suitable group operation $\psi : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$, we must solve the system of equations (2.1). It can easily be shown that

$$\psi(u, v) = u + (0, 0, v_3) + e^{u_3}(v_1, v_2, 0), \quad u, v \in \mathbb{R}^3 \quad (2.7)$$

is indeed a solution. Calculating its Jacobian with respect to its first variable, we obtain

$$\nabla_1 \psi(u, v) = \begin{pmatrix} 1 & 0 & v_1 e^{u_3} \\ 0 & 1 & v_2 e^{u_3} \\ 0 & 0 & 1 \end{pmatrix} \quad (2.8)$$

confirming, that the vector fields generating the given lattice are right invariant under the action of this group.

3. NON-UNIFORMLY DEFECTIVE STRUCTURES

We start our analysis of crystal structures which may not be uniformly defective by looking at a specific example of a continuum lattice with a variable dislocation density tensor.

Example 2. Consider the crystalline structure of an elastic body given by three linearly independent vector fields

$$l_1(x) = \mathbf{e}_1, \quad l_2(x) = \mathbf{e}_2 - x_2 \mathbf{e}_1, \quad l_3(x) = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \mathbf{e}_3, \quad x \in \mathbb{R}^3. \quad (3.1)$$

Using the definition of the dislocation density tensor (1.1), we see that it is position dependent as

$$S_{ij}(x) = \begin{pmatrix} -x_2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3. \quad (3.2)$$

Looking back at the identity (2.4), we conclude that either the relation between the Lie algebra constants and the dislocation density tensor is no longer valid and/or the given vector fields do not generate a 3-dim Lie algebra. Indeed, calculating the corresponding Lie brackets, one obtains that

$$[l_1(x), l_2(x)] = 0, \quad [l_1(x), l_3(x)] = l_1(x), \quad [l_2(x), l_3(x)] = \mathbf{e}_2 \quad (3.3)$$

where the vector field $\mathbf{e}_2 \equiv l_2(x) + x_2 l_1(x)$ cannot be represented as a linear combination of the given vector fields (3.1). Investigating this further, one discovers that as

$$[\mathbf{e}_2, l_1(x)] = 0, \quad [\mathbf{e}_2, l_2(x)] = -l_1(x), \quad [\mathbf{e}_2, l_3(x)] = \mathbf{e}_2, \quad (3.4)$$

the smallest Lie algebra of vector fields on $\mathbb{R}^3$ the given frame field belongs to is the four dimensional Lie algebra generated by

$$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2 - x_2 \mathbf{e}_1, x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \mathbf{e}_3. \quad (3.5)$$

Thus, it appears that no group structure can be given to the underlying space $\mathbb{R}^3$. However, one may ask if there is an algebraic or a differential structure which $\mathbb{R}^3$ can be equipped with, above and beyond its natural Euclidean structure. Moreover, one wonders how the dislocation density tensor relates to any intrinsic characteristics of such a structure[†]. We shall investigate these questions in the sequel.

Consider a continuous lattice defined by a frame field $l : \mathbb{R}^3 \rightarrow L(\mathbb{R}^3)$ such that $l_i(\mathbf{0}) = \mathbf{e}_i$, $i = 1, 2, 3$. We assume that the lattice fields $l_i(x)$, $i = 1, 2, 3$, generate a k -dimensional Lie subalgebra, say $\mathfrak{l}$, of the algebra $\mathcal{X}(\mathbb{R}^3)$ of all smooth vector fields on $\mathbb{R}^3$, where $3 \leq k < \infty$. Hence, the *lattice algebra* $\mathfrak{l}$ is spanned by the vector fields, say $l_1, l_2, \dots, l_k$, where for the consistency of our presentation, the vector fields are ordered so that $l_i = l_i(x)$, $i = 1, 2, 3$, unless stated otherwise. We

[†]Note that the relation (2.4) is still valid if the Lie algebra constants are replaced by the variable coefficients $\hat{C}_{kij}(x)$ such that $[l_i, l_j] = \hat{C}_{kij}(x)l_k$. See also Proposition 3.

postulate also that all generators of the subalgebra $\mathfrak{l}$ are complete vector fields on the manifold $\mathbb{R}^3$, which implies that the algebra $\mathfrak{l}$ consists entirely of complete vector fields, [4]. Moreover, there exists a Lie group acting on $\mathbb{R}^3$ the Lie algebra of which is isomorphic to $\mathfrak{l}$, [4, 6].

Theorem 1. *Consider a continuous lattice defined by three linearly independent smooth vector fields $l_i : \mathbb{R}^3 \rightarrow T\mathbb{R}^3$, $i = 1, 2, 3$. Let $\mathfrak{l} \subset \mathcal{X}(\mathbb{R}^3)$ be the smallest algebra of vector fields containing the given lattice vector fields. Assume that $\mathfrak{l}$ is finite-dimensional and complete. Then, there exists a simply connected Lie group G contained in $\text{Diff}(\mathbb{R}^3)$ as an abstract subgroup and such that the natural action $\Lambda : G \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of the group G on $\mathbb{R}^3$ is smooth and the algebra $\mathfrak{l}$ is homomorphic to the Lie algebra, say $\mathfrak{g}$, of the group G .*

Given the smooth action Λ , there exists a homomorphism $\lambda : G \rightarrow \text{Diff}(\mathbb{R}^3)$ from the group G into the group of all diffeomorphisms of $\mathbb{R}^3$ such that

$$\lambda(a)(x) = \Lambda(a, x), \quad a \in G, x \in \mathbb{R}^3. \quad (3.6)$$

If, in addition, the action Λ is effective, the homomorphism λ identifies the group G with a subgroup, say $\lambda(G) \subset \text{Diff}(\mathbb{R}^3)$. Given $x \in \mathbb{R}^3$, consider a smooth mapping $\Lambda(x) : G \rightarrow \mathbb{R}^3$ such that

$$\Lambda(x)(g) := \Lambda(g, x) \quad (3.7)$$

for any $g \in G$. The mapping $\Lambda(x)$ maps the group G onto the orbit $G(x)$ of the point x (under the action Λ). Indeed, $y \in G(x)$ if and only if there exists $a \in G$ such that $y = \Lambda(x)(a) \equiv \Lambda(a, x)$. Moreover, the mapping $\Lambda(x)$ is a morphism (but not necessarily an isomorphism) of the action of G on itself (by left translations) into the action of Λ on $\mathbb{R}^3$. Correspondingly, there exists a relation between the Lie algebra $\mathfrak{g}$ and the algebra of all smooth vector fields $\mathcal{X}(\mathbb{R}^3)$. To illustrate this fact, let us define the map $d\lambda : \mathfrak{g} \rightarrow \mathcal{X}(\mathbb{R}^3)$ by requiring that

$$d\lambda(\mathfrak{v})(x) := d_e \Lambda(x)(\mathfrak{v}) \quad (3.8)$$

for any $\mathfrak{v} \in \mathfrak{g}$ and any $x \in \mathbb{R}^3$, where $d_e \Lambda(x)$ denotes the tangent map of $\Lambda(x)$ at the identity $e \in G$. Note that, by (3.8), $d\lambda(\mathfrak{v})$ is a smooth vector field on $\mathbb{R}^3$ while $d_e \Lambda(x)(\mathfrak{g})$ is a subset of the tangent space $T_x \mathbb{R}^3$. In addition, [4],

Proposition 1. *The mapping $d\lambda : \mathfrak{g} \rightarrow \mathcal{X}(\mathbb{R}^3)$ is a homomorphism of Lie algebras. In fact, $d\lambda(\mathfrak{g}) = \mathfrak{l}$.*

Remark 1. The action Λ is a generalization of the action ψ of $\mathbb{R}^3$ on itself when the lattice is uniformly defective. Indeed, to illustrate this claim, let us revisit Example 1. There, we shall look at $\mathbb{R}^3$ both as a Lie group with the Lie algebra $\mathfrak{g} \equiv \mathbb{R}^3$ generated by the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and as the space (manifold) the group is acting on. The action Λ of the additive group $\mathbb{R}^3$ on the space $\mathbb{R}^3$ is given by

$$\Lambda(a, u) = (a_1 + u_1 e^{a_3}, a_2 + u_2 e^{a_3}, a_3 + u_3) \quad (3.9)$$

where for the clarity of the presentation, $a \in \mathbb{R}^3$ denotes a group element while $u \in \mathbb{R}^3$ is a point in the space acted upon. To identify the homomorphism $d\lambda$ defined by (3.8), we calculate that

$$d_0\Lambda(u) = d_a\Lambda(u)|_{a=0} = \begin{pmatrix} 1 & 0 & u_1e^{a_3} \\ 0 & 1 & u_2e^{a_3} \\ 0 & 0 & 1 \end{pmatrix} \Big|_{a=0} = \begin{pmatrix} 1 & 0 & u_1 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.10)$$

Hence, given an arbitrary element $\mathbf{v} = (v_1, v_2, v_3)$ of the Lie algebra $\mathfrak{g}$, we have

$$d\lambda(\mathbf{v})(u) = d_0\Lambda(u)(\mathbf{v}) = \begin{pmatrix} 1 & 0 & u_1 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3 \begin{pmatrix} u_1 \\ u_2 \\ 1 \end{pmatrix} \quad (3.11)$$

which is exactly the subalgebra $\mathfrak{l} \subset \mathcal{X}(\mathbb{R}^3)$ of example 1, that is, the corresponding lattice algebra; see (2.5).

Given a k -parameter Lie group G acting on $\mathbb{R}^3$, where the Lie algebra $\mathfrak{g}$ of G is homomorphic to the lattice algebra $\mathfrak{l}$, consider a point, say $x_0 \in \mathbb{R}^3$, and let G_{x_0} be the *isotropy group* of the action Λ at x_0 . That is, let

$$G_{x_0} := \{g \in G : \Lambda(g, x_0) = x_0\}, \quad (3.12)$$

where the projection $\Lambda(x_0) : G \rightarrow \mathbb{R}^3$ given by (3.7) is such that $\Lambda(x_0)(G_{x_0}) = x_0$. If the action Λ is transitive, the rank of the projection $\Lambda(x_0)$ is constant, [4], which, in turn, allows one to identify $\mathbb{R}^3$ with the quotient space G/G_{x_0} . Indeed, consider the mapping $\widehat{\Lambda(x_0)} : G/G_{x_0} \rightarrow \mathbb{R}^3$ such that

$$\widehat{\Lambda(x_0)}(hG_{x_0}) = \Lambda(x_0)(h) = \Lambda(h, x_0). \quad (3.13)$$

where hG_{x_0} denotes the left co-set of G_{x_0} under the left translation of the group G . It can easily be shown that $\widehat{\Lambda(x_0)}$ is a diffeomorphism and that the isotropy group at a point, say $y \in \mathbb{R}^3$ such that $y = \Lambda(h, x_0)$ for some $h \in G$, is the conjugate of G_{x_0} , i. e., if $g_0 \in G_{x_0}$, then

$$\Lambda(hg_0h^{-1}, y) = \Lambda(hg_0h^{-1}, \Lambda(h, x_0)) = \Lambda(hg_0, x_0) = \Lambda(h, x_0) = y. \quad (3.14)$$

Summarizing what we have just discussed, we state

Theorem 2. *Consider a continuous lattice defined by three linearly independent smooth vector fields $l_i : \mathbb{R}^3 \rightarrow T\mathbb{R}^3$, $i = 1, 2, 3$, where $\mathfrak{l} \subset \mathcal{X}(\mathbb{R}^3)$ is the corresponding lattice algebra. Then, if the induced action $\Lambda : G \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is transitive, the underlying space $\mathbb{R}^3$ can be identified with the homogeneous space G/G_0 where the subgroup $G_0 \subset G$ is the isotropy group of the action Λ at the origin of $\mathbb{R}^3$.*

Note that if the isotropy group G_0 is trivial and/or a normal subgroup of the group G , the homogeneous space G/G_0 is a group proving that the corresponding continuous lattice is uniformly defective. Note also that, as the isotropy groups at two different points of the underlying space are conjugate of each other, the identification

(diffeomorphism) $\widehat{\Lambda(x_0)} : G/G_{x_0} \rightarrow \mathbb{R}^3$ is a base point dependent. Indeed, consider $y \in \mathbb{R}^3$ and let $h \in G$ be such that $y = \Lambda(h, x_0)$. Then $\Lambda(y)(g) = \Lambda(g, y) = \Lambda(g, \Lambda(h, x_0)) = \Lambda(gh, x_0) = \Lambda(x_0)(gh)$ for any $g \in G$. In other words,

$$\Lambda(y) = \Lambda(\Lambda(h, x_0)) = \Lambda(x_0) \circ R_h : G \rightarrow \mathbb{R}^3 \quad (3.15)$$

where $R_h : G \rightarrow G$ denotes here the right translation by $h \in G$.

Example 3. We shall revisit now Example 2 where the continuous lattice $\mathbf{l}$ was given by the frame (3.1) with the corresponding four-dimensional lattice algebra $\mathbf{l}$ of vector fields on $\mathbb{R}^3$ generated by (3.5). Thus, using the fact that the group of affine transformations of $\mathbb{R}^3$ can be realized as a subgroup of the general linear group $GL(4, \mathbb{R})$, it is easy to show that the general element $\mathfrak{v}$ of the algebra $\mathfrak{g}$ of the subgroup $G \subset \text{Diff}(\mathbb{R}^3)$ acting on $\mathbb{R}^3$ can be represented as

$$\mathfrak{v} = \begin{pmatrix} q & -r & 0 & t \\ 0 & q & 0 & s+r \\ 0 & 0 & 0 & q \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad q, r, s, t \in \mathbb{R}, \quad (3.16)$$

where

$$\mathfrak{v} = t\mathbf{e}_1 + s\mathbf{e}_2 + r(\mathbf{e}_2 - x_2\mathbf{e}_1) + q(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \mathbf{e}_3), \quad x_1, x_2 \in \mathbb{R}. \quad (3.17)$$

One can also show that the corresponding, but re-parameterized, four parameter subgroup G of $\text{Diff}(\mathbb{R}^3)$ (or rather its $GL(4, \mathbb{R})$ representation) is given by the elements

$$g(q, r, \alpha, \beta) = \begin{pmatrix} e^q & -re^q & 0 & \alpha \\ 0 & e^q & 0 & \beta \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.18)$$

This means that the smooth action Λ of the group G on $\mathbb{R}^3$ takes the form

$$\Lambda(g(q, r, \alpha, \beta), (u_1, u_2, u_3)) = \begin{pmatrix} u_1 e^q - u_2 r e^q + \alpha \\ u_2 + \beta \\ u_3 + q \end{pmatrix} \quad (3.19)$$

where $(u_1, u_2, u_3) \in \mathbb{R}^3$. The corresponding homomorphism $d\lambda$ of the algebras $\mathfrak{g}$ and $\mathbf{l}$ can now be easily evaluated as

$$d\lambda(\cdot)(u) = d_e \Lambda(u) = \begin{pmatrix} u_1 & -u_2 & 1 & 0 \\ u_2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (3.20)$$

where the identity element e of the group G is attained when all the parameters vanish. Note that the isotropy group of the action Λ at the origin $\mathbf{0} \in \mathbb{R}^3$ is represented

by

$$\begin{pmatrix} 1 & -r & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.21)$$

4. DISLOCATION DENSITY TENSOR

When a continuous lattice is uniformly defective, its dislocation density tensor S_{ij} can be represented as a linear combination of the Lie algebra constants of the defining lattice frame $l_i(x)$, $i = 1, 2, 3$, see (2.4). On the other hand, when the lattice is non-uniform, that is, when the dislocation density tensor is material point dependent, the relation (2.4) is no longer valid. In this section, we shall show how to re-interpret this relation so as to make it a natural generalization of the uniformly defective case.

To this end, note first that the projection $\Lambda(0) : G \rightarrow \mathbb{R}^3$, which we shall denote by Λ_0 , commutes with the left action of G as

$$\Lambda_0(gh) = \Lambda(gh, 0) = \Lambda(g, \Lambda(h, 0)) = \Lambda(g, \Lambda_0(h)) \quad (4.1)$$

for any $g, h \in G$. Given the lattice subalgebra $\mathfrak{l} \subset \mathfrak{X}(\mathbb{R}^3)$, we can lift its generators $l_i(x)$, $i = 1, \dots, k$ to the Lie algebra $\mathfrak{g}$ of the group G using the homomorphism $d\lambda$, (3.8). That is, let $\mathfrak{l}_i \in \mathfrak{g}$ be such that $d\lambda(\mathfrak{l}_i) = l_i(x)$, $i = 1, \dots, k$. Hence, using the relation (3.15), we obtain that

$$d\lambda(\mathfrak{l}_i)(x) = d_e \Lambda(x)(\mathfrak{l}_i) = d_e(\Lambda_0 \circ R_h)(\mathfrak{l}_i) = d_h \Lambda_0 \circ d_e R_h(\mathfrak{l}_i) \quad (4.2)$$

where $x = \Lambda(h, 0)$ and $h \in G$. This proves, in fact, that given the generators $\mathfrak{l}_i$ of the Lie algebra $\mathfrak{g}$ and extending them by the right translation to the whole group G , one obtains a right-invariant frame field $\tilde{\mathfrak{l}}_i(h) := d_e R_h(\mathfrak{l}_i)$ such that it projects onto the generators of the lattice algebra $\mathfrak{l}$. Namely,

$$d\Lambda_0(\tilde{\mathfrak{l}}_i(h)) = l_i(\Lambda(h, 0)) \quad (4.3)$$

for any $i = 1, \dots, k$ and $h \in G$. In particular, the first three vector fields $\tilde{\mathfrak{l}}_i : G \rightarrow TG$, corresponding to the lattice frame $l_i : \mathbb{R}^3 \rightarrow T\mathbb{R}^3$, form a 3-dimensional, right invariant, distribution $\tilde{\mathfrak{D}} : G \rightarrow TG$ (of vector spaces) in TG . That is,

$$\tilde{\mathfrak{D}}(h) = d_e R_h(\tilde{\mathfrak{D}}(e)). \quad (4.4)$$

The projection of the distribution $\tilde{\mathfrak{D}}$ by the tangent map $d\Lambda_0$ is, by the definition of Λ_0 and the explicit construction of the frame field $\tilde{\mathfrak{l}}_i(h)$, $i = 1, 2, 3$, the tangent space $T\mathbb{R}^3$. This immediately implies

Proposition 2. *Let $\Lambda : G \times \mathbb{R}^3$ be a transitive action of a Lie group G on $\mathbb{R}^3$. Given a complete lattice frame $\mathbf{l} : \mathbb{R}^3 \rightarrow L(\mathbb{R}^3)$ such that the Lie algebra $\mathfrak{g} \equiv T_e G$ is isomorphic to the lattice algebra $\mathfrak{l} \subset \mathfrak{X}(\mathbb{R}^3)$, the tangent space TG can be*

decomposed into the direct sum of the right-invariant distribution $\tilde{\mathfrak{D}}$ and the tangent space to the fibers of the projection $G \rightarrow G/G_0$. That is, at any point $p \in G$

$$T_p G = \tilde{\mathfrak{D}}(p) \oplus T_p(pG_0). \quad (4.5)$$

Moreover, this direct sum is right invariant under the natural action of the isotropy group G_0 .

Indeed, consider an arbitrary $x \in \mathbb{R}^3$ and $h \in G$ such that $\Lambda(h, 0) = x$. Knowing that the mapping $\hat{\Lambda}_0$, see (3.13), identifying the underlying space $\mathbb{R}^3$ with the homogeneous space G/G_0 , is a diffeomorphism, one can easily see that the inverse image $\hat{\Lambda}_0^{-1}(x) = hG_0$ is a submanifold. Thus, the kernel

$$\ker d_{hG_0} \hat{\Lambda}_0 = \ker d_h \Lambda_0 = T_p(pG_0). \quad (4.6)$$

This and the fact that $d_h \Lambda_0(\mathfrak{D}(h)) = T_{\Lambda(h,0)} \mathbb{R}^3$ prove finally the decomposition (4.5)[‡].

In general, the vector space $\mathfrak{D} := \tilde{\mathfrak{D}}(e) \subset \mathfrak{g}$ is not a subalgebra[§], thus, the commutators $[l_i, l_j]$, $i, j = 1, 2, 3$, although in the algebra $\mathfrak{g}$, are not necessarily elements of the vector space $\mathfrak{D}$. Looking closer at the commutators of the vectors defining the vector space $\mathfrak{D}$, we know that

$$[l_i, l_j] = C_{mij} l_m, \quad i = 1, 2, 3, \quad m = 1, \dots, k \quad (4.7)$$

where C_{mij} are the Lie algebra constants of the algebra $\mathfrak{g}$ and not all the coefficients C_{mij} , $m = 4, \dots, k$, $i, j = 1, 2, 3$, vanish. The linear combination of the first three vectors is in $\mathfrak{D}$. The remaining vectors $l_{ij}^r := C_{mij} l_m$, $m \geq 4$, although not in $\mathfrak{D}$, are elements of the algebra $\mathfrak{g}$ and can therefore be presented as a sum of an element from $\mathfrak{D}$ and an element from $T_e(G_0)$ (see footnote 4). That is,

$$l_{ij}^r = C_{mij}^r l_m + v_{ij} \quad (4.8)$$

where the vertical vector $v_{ij} \in T_e(G_0)$ and where the constants C_{mij}^r , $i, j, m = 1, 2, 3$, are no longer the Lie algebra constants. In summary, for any pair of vectors $l_i, l_j \in \mathfrak{D} \subset \mathfrak{g}$, the corresponding commutator can be represented as

$$[l_i, l_j] = C_{mij} l_m + (C_{mij}^r l_m + v_{ij}), \quad i, j, m = 1, 2, 3. \quad (4.9)$$

Note, however, that as the first part of the decomposition is G right invariant, the second part is not. Therefore, extending the vectors l_i , $i = 1, 2, 3$, to the whole group G by the right translation, we obtain that the corresponding vector fields $\tilde{l}_i$ have commutators which can be represented as

$$[\tilde{l}_i(h), \tilde{l}_j(h)] = C_{mij} \tilde{l}_m(h) + (C_{mij}^r \tilde{l}_m(h) + v_{ij}(h)), \quad i, j, m = 1, 2, 3 \quad (4.10)$$

[‡]The distribution $\tilde{\mathfrak{D}} : G \rightarrow TG$ defines a (right) invariant connection on the bundle $G \rightarrow G/G_0 \cong \mathbb{R}^3$, see [5]. The curvature of this connection is an invariant of the non-uniformly defective crystal states, the issue which we will look closer at in the forthcoming work.

[§]However, the Lie algebra $\mathfrak{g} = \mathfrak{D} \oplus \mathfrak{g}_0$, where $\mathfrak{g}_0 = T_e(G_0)$, is the Lie algebra of the isotropy group G_0 .

where $h \in G$ and the coefficients C_{mij} are the Lie algebra constants. Indeed, although a bracket of two right invariant vector fields is right invariant and the right translations $d_e R_h$ are linear maps, the decomposition (4.5) of the tangent space TG into the distribution $\tilde{\mathfrak{D}}$ and the tangent space to the cosets of the isotropy group G_0 is not. That is, when the right translations of the remainder vector $\mathfrak{l}_{ij}^r$ get decomposed, the decomposition is, in general, point h dependent causing the coefficients $C_{mij}^r(h)$ to be different at different points of the group G . Note that as the decomposition (4.5) is G_0 (isotropy group) right invariant, the coefficients $C_{mij}^r(h)$ are constant along the fibers $T_p(pG_0)$, $p \in G$. Note also that should the coefficients $C_{mij}^r(h)$ be shown to be group element independent, the vector space $\mathfrak{D}$ would be a Lie subalgebra of the algebra $\mathfrak{g}$.

The above analysis leads one to a straightforward generalization of the relation (2.4).

Proposition 3. *Consider a continuous lattice $l_i(x)$, $i = 1, 2, 3$, not necessarily uniformly defective, and let $S_{ij}(x)$ be its dislocation density tensor. Then*

$$\epsilon_{jkl} S_{ij}(x) = \hat{C}_{ikl}(x), \quad i, j, k = 1, 2, 3, \quad x \in \mathbb{R}^3 \quad (4.11)$$

where

$$\hat{C}_{kij}(x) = C_{kij} + C_{kij}^r(h) \quad (4.12)$$

for $x = \Lambda(h, 0)$.

Indeed, the key point to consider is the fact that the Lie algebra homomorphism $d\lambda : \mathfrak{g} \rightarrow \mathcal{X}(\mathbb{R}^3)$ takes a Lie bracket into a Lie bracket of its image and that its kernel is the tangent space to cosets of the isotropy group G_0 . Namely,

$$d\lambda([l_i, l_j])(x) = d_h \Lambda_0([\tilde{l}_i, \tilde{l}_j])(x) = [l_i(x), l_j(x)] = \hat{C}_{mij}(x) l_m(x). \quad (4.13)$$

The rest follows the argument of the uniformly defective case [3]. To see how this works in the case of a specific non-uniformly defective lattice, the reader may revisit Example 2 where the only nonzero coefficients $\hat{C}_{kij}(x)$ are $\hat{C}_{113}(x) = \hat{C}_{223}(x) = 1$ and $\hat{C}_{123}(x) = x_2$.

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GALILEI INVARIANCE OF THE HELMHOLTZ MORPHISM

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Abstract. We study the Helmholtz morphism in terms of the variational sequences. We find conditions for the Helmholtz form to be invariant with respect to the Galilei group.

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1. INTRODUCTION

In this paper, we shall use the framework of the theory of variational sequences on fibred manifolds introduced by Krupka [2, 3]. The variational sequence is a quotient sequence of the de Rham sequence such that one of the morphisms is the *Euler–Lagrange morphism* $\mathcal{E}_1: \lambda \rightarrow E_\lambda$, assigning to a Lagrangian, i.e. one-form $\lambda = L dt$, its Euler–Lagrange form, i.e. two-form $E_\lambda = E_\sigma(L) dq^\sigma \wedge dt$, where $E_\sigma(L)$ are the Euler–Lagrange expressions

$$E_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\sigma}.$$

The next morphism $\mathcal{E}_2: E \rightarrow H_E$, called the *Helmholtz morphism*, assigns to a two-form $E = E_\sigma dq^\sigma \wedge dt$ a three-form H_E

$$\begin{aligned} H_E = & \frac{1}{2} \left(\frac{\partial E_\sigma}{\partial q^v} - \frac{\partial E_v}{\partial q^\sigma} - \frac{1}{2} \frac{d}{dt} \left(\frac{\partial E_\sigma}{\partial \dot{q}^v} - \frac{\partial E_v}{\partial \dot{q}^\sigma} \right) \right) \omega^v \wedge \omega^\sigma \wedge dt \\ & + \frac{1}{2} \left(\frac{\partial E_\sigma}{\partial \dot{q}^v} + \frac{\partial E_v}{\partial \dot{q}^\sigma} - \frac{d}{dt} \left(\frac{\partial E_\sigma}{\partial \ddot{q}^v} + \frac{\partial E_v}{\partial \ddot{q}^\sigma} \right) \right) \dot{\omega}^v \wedge \omega^\sigma \wedge dt \\ & + \frac{1}{2} \left(\frac{\partial E_\sigma}{\partial \ddot{q}^v} - \frac{\partial E_v}{\partial \ddot{q}^\sigma} \right) \ddot{\omega}^v \wedge \omega^\sigma \wedge dt. \end{aligned} \quad (1.1)$$

called *Helmholtz form*.

While the Euler–Lagrange morphism $\mathcal{E}_1$ is well-understood, much less is known about the Helmholtz morphism. In the present paper, we study the Galilei invariance of the Helmholtz morphism.

In Section 2, we recall the basic structures and notations and briefly introduce the variational sequence, according to [2, 3, 5], and we recall known properties of the

Helmholtz morphism [6, 7]. In Section 3, we recall the basic concepts of the theory of symmetries of differential forms, the Noether theorem, and the Galilei group of transformations. We present a theorem on the structure of Helmholtz forms invariant with respect to the Galilei group.

2. THE HELMHOLTZ MORPHISM

Let $\pi: Y \rightarrow X$ be a smooth fibred manifold, $\dim X = 1$, $\dim Y = m + 1$, and $\pi_r: J^r Y \rightarrow X$, $r \geq 1$, its jet prolongations. Denote by $\pi_{r,s}: J^r Y \rightarrow J^s Y$, $r > s \geq 0$, canonical jet projections. A mapping $\gamma: W \rightarrow Y$, where W is an open subset of X , is called a *section* of the manifold $\pi: Y \rightarrow X$ if $\gamma \circ \pi = \text{id}_W$.

Fibred coordinates on Y are denoted by (t, q^σ) , $1 \leq \sigma \leq m$, associated coordinates on $J^r Y$ are denoted by (t, q_i^σ) , $1 \leq \sigma \leq m$, $0 \leq i \leq r$. We usually use the notation $q_0^\sigma = q^\sigma$, $q_1^\sigma = \dot{q}^\sigma$, $q_2^\sigma = \ddot{q}^\sigma$, $q_3^\sigma = \ddot{\ddot{q}}^\sigma$.

A vector field ξ on $J^r Y$ is called π_r -vertical if $T\pi_r \cdot \xi = 0$, and π_r -projectable if there exists a vector field ξ_0 on X such that $T\pi_r \cdot \xi = \xi_0 \circ \pi_r$.

A differential q -form ($q > 1$) η on $J^r Y$ is called *contact* if $J^r \gamma^* \eta = 0$ for every section γ of π , *horizontal* or *0-contact* if $i_\xi \eta = 0$ for every vertical vector field ξ on $J^r Y$, and *k-contact*, $1 \leq k \leq q$, if for every vertical vector field ξ , $i_\xi \eta$ is $(k-1)$ -contact. If lifted to $J^{r+1} Y$, every q -form η on $J^r Y$ can be canonically decomposed into a sum of k -contact components, η_k , where $k = 0, 1, \dots, q$. We write $\eta_k = p_k \eta$, and $p_0 = h$, then,

$$\pi_{r+1,r}^* \eta = h\eta + p_1 \eta + \dots + p_q \eta. \quad (2.1)$$

A contact q -form is called *strongly contact* if $\pi_{r+1,r}^* \eta = p_q \eta$.

A general framework for our exposition is the *variational sequence* [2, 3].

Let $\Omega_{0,c}^r = \{0\}$, and let $\Omega_{p,c}^r$ be the sheaf of contact p -forms, if $p \leq n$, or the sheaf of strongly contact p -forms, if $p > n$, on $J^r Y$. Set

$$\Theta_p^r = \Omega_{p,c}^r + d\Omega_{p-1,c}^r,$$

where $d\Omega_{p-1,c}^r$ is the image sheaf of $\Omega_{p-1,c}^r$ by the exterior derivative d . We get an exact sequence of soft sheaves

$$0 \longrightarrow \Theta_1^r \longrightarrow \Theta_2^r \longrightarrow \Theta_3^r \longrightarrow \dots,$$

where the morphisms are the exterior derivative, i. e., a subsequence of the De Rham sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega_0^r \longrightarrow \Omega_1^r \longrightarrow \Omega_2^r \longrightarrow \Omega_3^r \longrightarrow \dots$$

The quotient sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega_0^r \longrightarrow \Omega_1^r / \Theta_1^r \longrightarrow \Omega_2^r / \Theta_2^r \longrightarrow \Omega_3^r / \Theta_3^r \longrightarrow \dots$$

is also exact. It is called the *variational sequence* of order r on π . The variational sequence is an acyclic resolution of the constant sheaf $\mathbb{R}$ over Y . We denote by

$$\mathcal{E}_p: \Omega_p^r / \Theta_p^r \rightarrow \Omega_{p+1}^r / \Theta_{p+1}^r$$

the quotient mapping. The class of a form $\rho \in \Omega_p^r$ is denoted by $[\rho]$. Hence, $\mathcal{E}_p([\rho]) = [d\rho]$.

The quotient mapping

$$\mathcal{E}_1 : \Omega_1^r / \Theta_1^r \rightarrow \Omega_2^r / \Theta_2^r$$

then identifies with the *Euler–Lagrange mapping*. The quotient mapping

$$\mathcal{E}_2 : \Omega_2^r / \Theta_2^r \rightarrow \Omega_3^r / \Theta_3^r$$

is called the *Helmholtz mapping*. The image of a class $[\rho] \in \Omega_2^r / \Theta_2^r$, i. e., the class $[d\rho] \in \Omega_3^r / \Theta_3^r$ is called *Helmholtz class*.

Due to the exactness of the variational sequence, if $[\alpha] \in \Omega_2^r / \Theta_2^r$ is such that

$$\mathcal{E}_2([\alpha]) = [d\alpha] = 0, \quad (2.2)$$

there exists $[\rho] \in \Omega_1^r / \Theta_1^r$ such that $[\alpha] = [d\rho] = \mathcal{E}_1([\rho])$, i. e. $[\alpha]$ is the image by the Euler–Lagrange mapping of a class $[\rho]$. In other words, the class $[\alpha]$ is locally variational – it comes from a class $[\rho]$ that has the meaning of a *local Lagrangian*. If, moreover, $H^2Y = \{0\}$, a *global* Lagrangian exists. Condition (2.2) for “local variationality” then provides Helmholtz conditions (of order r).

Classes in the variational sequence can be represented by differential forms [1, 4]. We shall use the representation by so-called *source forms*, $(q - 1)$ -contact q -forms belonging to the ideal generated by contact forms

$$\omega^\sigma = dq^\sigma - \dot{q}^\sigma dt, \quad \dot{\omega}^\sigma = d\dot{q}^\sigma - \ddot{q}^\sigma dt, \quad \dots, \quad \omega_{r-1}^\sigma = dq_{r-1}^\sigma - \dot{q}_r^\sigma dt \quad (2.3)$$

where $1 \leq i \leq m$.

Source forms for classes $[\rho] \in \Omega_1^r / \Theta_1^r$ are *horizontal forms* $\lambda = Ldt$, called *Lagrangians*. Source forms for classes $[\alpha] \in \Omega_2^r / \Theta_2^r$ are two-forms $E = E_i \omega^i \wedge dt$, called *dynamical forms* (corresponding to *differential equations*). Note that, in this representation, if $[\rho]$ is represented by λ , then $[d\rho] = \mathcal{E}_1([\rho])$ is represented by the dynamical form E_λ , the Euler–Lagrange form of λ . If $[\alpha] \in \Omega_2^r / \Theta_2^r$ is represented by a dynamical form E , $[d\alpha] = \mathcal{E}_2([\alpha])$ is represented by a source three-form H_E , the *Helmholtz-form* of E . As shown in [3], H_E is then given by (1.1).

Source forms representing elements in Ω_3^r / Θ_3^r are called *Helmholtz-like forms*. We shall be interested in Helmholtz-like forms of order 3 (in particular, they correspond to second order ordinary differential equations). In coordinates,

$$H = H_{\sigma\nu}^0 \omega^\nu \wedge \omega^\sigma \wedge dt + H_{\sigma\nu}^1 \dot{\omega}^\nu \wedge \omega^\sigma \wedge dt + H_{\sigma\nu}^2 \ddot{\omega}^\nu \wedge \omega^\sigma \wedge dt \quad (2.4)$$

where $H_{\sigma\nu}^0 = -H_{\nu\sigma}^0$, $H_{\sigma\nu}^1 = H_{\nu\sigma}^1$, $H_{\sigma\nu}^2 = -H_{\nu\sigma}^2$.

In [6, 7] was studied and solved the question when a three-form corresponds to a system of differential equations (such form is called *locally Helmholtz*). This problem is closely related to the question of existence of a closed counterpart of a three-form. This problem is solved for second-order Helmholtz-like forms in [6] and for third-order Helmholtz-like forms in [7].

3. GALILEI INVARIANCE OF THE HELMHOLTZ FORM

Let ξ be a projectable vector field on Y , η a differential form on $J^r Y$. ξ is called *point symmetry* of η , if

$$\partial_{J^r \xi} \eta = 0 \quad (3.1)$$

where ∂ denotes the Lie derivative. If η is a Helmholtz-like form of order three (2.4) the symmetry conditions (3.1) reads as follows:

$$\begin{aligned} 0 = & H_{\sigma\rho}^0 \frac{\partial \xi^\rho}{\partial q^v} + H_{\rho v}^0 \frac{\partial \xi^\rho}{\partial q^\sigma} + H_{\sigma v}^0 \frac{\partial \xi^0}{\partial t} + H_{\sigma\rho}^1 \frac{\partial}{\partial q^v} \left(\frac{d\xi^\rho}{dt} - \dot{q}^\rho \frac{d\xi^0}{dt} \right) \\ & + H_{\sigma\rho}^2 \frac{\partial}{\partial q^v} \left(\frac{d^2 \xi^\rho}{dt^2} - \frac{d^2 \xi^0}{dt^2} \dot{q}^\rho - 2 \frac{d\xi^0}{dt} \ddot{q}^\rho \right) \\ & + \frac{\partial H_{\sigma v}^0}{\partial t} \xi^0 + \frac{\partial H_{\sigma v}^0}{\partial q^\rho} \xi^\rho + \frac{\partial H_{\sigma v}^0}{\partial \dot{q}^\rho} \left(\frac{d\xi^\rho}{dt} - \dot{q}^\rho \frac{d\xi^0}{dt} \right) \\ & + \frac{\partial H_{\sigma v}^0}{\partial \ddot{q}^\rho} \left(\frac{d^2 \xi^\rho}{dt^2} - \frac{d^2 \xi^0}{dt^2} \dot{q}^\rho - 2 \frac{d\xi^0}{dt} \ddot{q}^\rho \right) \\ & + \frac{\partial H_{\sigma v}^0}{\partial \ddot{q}^\rho} \left(\frac{d^3 \xi^\rho}{dt^3} - \frac{d^3 \xi^0}{dt^3} \dot{q}^\rho - 3 \frac{d^2 \xi^0}{dt^2} \ddot{q}^\rho - 3 \frac{d\xi^0}{dt} \ddot{q}^\rho \right) \end{aligned} \quad (3.2)$$

$$\begin{aligned} 0 = & H_{\sigma\rho}^0 \frac{\partial \xi^\rho}{\partial \dot{q}^v} - H_{\rho\sigma}^0 \frac{\partial \xi^\rho}{\partial \dot{q}^v} \\ & + H_{\sigma v}^1 \frac{\partial \xi^0}{\partial t} + H_{\rho v}^1 \frac{\partial \xi^\rho}{\partial q^\sigma} + H_{\sigma\rho}^1 \frac{\partial}{\partial \dot{q}^v} \left(\frac{d\xi^\rho}{dt} - \dot{q}^\rho \frac{d\xi^0}{dt} \right) \\ & + H_{\sigma\rho}^2 \frac{\partial}{\partial \dot{q}^v} \left(\frac{d^2 \xi^\rho}{dt^2} - \frac{d^2 \xi^0}{dt^2} \dot{q}^\rho - 2 \frac{d\xi^0}{dt} \ddot{q}^\rho \right) \\ & + \frac{\partial H_{\sigma v}^1}{\partial t} \xi^0 + \frac{\partial H_{\sigma v}^1}{\partial q^\rho} \xi^\rho + \frac{\partial H_{\sigma v}^1}{\partial \dot{q}^\rho} \left(\frac{d\xi^\rho}{dt} - \dot{q}^\rho \frac{d\xi^0}{dt} \right) \\ & + \frac{\partial H_{\sigma v}^1}{\partial \ddot{q}^\rho} \left(\frac{d^2 \xi^\rho}{dt^2} - \frac{d^2 \xi^0}{dt^2} \dot{q}^\rho - 2 \frac{d\xi^0}{dt} \ddot{q}^\rho \right) \\ & + \frac{\partial H_{\sigma v}^1}{\partial \ddot{q}^\rho} \left(\frac{d^3 \xi^\rho}{dt^3} - \frac{d^3 \xi^0}{dt^3} \dot{q}^\rho - 3 \frac{d^2 \xi^0}{dt^2} \ddot{q}^\rho - 3 \frac{d\xi^0}{dt} \ddot{q}^\rho \right) \end{aligned} \quad (3.3)$$

$$\begin{aligned} 0 = & H_{\sigma\rho}^0 \frac{\partial \xi^\rho}{\partial \ddot{q}^v} - H_{\rho\sigma}^0 \frac{\partial \xi^\rho}{\partial \ddot{q}^v} + H_{\sigma\rho}^1 \frac{\partial}{\partial \ddot{q}^v} \left(\frac{d\xi^\rho}{dt} - \dot{q}^\rho \frac{d\xi^0}{dt} \right) \\ & + H_{\sigma v}^2 \frac{\partial \xi^0}{\partial t} + H_{\rho v}^2 \frac{\partial \xi^\rho}{\partial q^\sigma} + H_{\sigma\rho}^2 \frac{\partial}{\partial \ddot{q}^v} \left(\frac{d^2 \xi^\rho}{dt^2} - \frac{d^2 \xi^0}{dt^2} \dot{q}^\rho - 2 \frac{d\xi^0}{dt} \ddot{q}^\rho \right) \\ & + \frac{\partial H_{\sigma v}^2}{\partial t} \xi^0 + \frac{\partial H_{\sigma v}^2}{\partial q^\rho} \xi^\rho + \frac{\partial H_{\sigma v}^2}{\partial \dot{q}^\rho} \left(\frac{d\xi^\rho}{dt} - \dot{q}^\rho \frac{d\xi^0}{dt} \right) \end{aligned} \quad (3.4)$$

$$\begin{aligned}
& + \frac{\partial H_{\sigma v}^2}{\partial \ddot{q}^\rho} \left(\frac{d^2 \xi^\rho}{dt^2} - \frac{d^2 \xi^0}{dt^2} \dot{q}^\rho - 2 \frac{d \xi^0}{dt} \ddot{q}^\rho \right) \\
& + \frac{\partial H_{\sigma v}^2}{\partial \ddot{\ddot{q}}^\rho} \left(\frac{d^3 \xi^\rho}{dt^3} - \frac{d^3 \xi^0}{dt^3} \dot{q}^\rho - 3 \frac{d^2 \xi^0}{dt^2} \ddot{q}^\rho - 3 \frac{d \xi^0}{dt} \ddot{\ddot{q}}^\rho \right)
\end{aligned}$$

The Galilei group on $\mathbb{R}^4$ is the 10-parametric transformation group, generated by the vector fields

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial q^\sigma}, \quad -\varepsilon_{\sigma\nu\rho} q^\nu \frac{\partial}{\partial q^\rho}, \quad t \frac{\partial}{\partial q^\sigma}, \quad \sigma = 1, 2, 3 \quad (3.5)$$

for the time translation, the space translations, the space rotations, and the Galilei transformations, respectively.

Theorem 1. *A Helmholtz-like form H (2.4) is invariant under the Galilei group of transformations if and only if the component $H_{\sigma v}^0$ of H takes the form:*

$$H_{\sigma v}^0 = \varepsilon_{\sigma\nu\rho} f(\eta) \ddot{q}^\rho + \varepsilon_{\sigma\nu\rho} g(\bar{\eta}) \ddot{\ddot{q}}^\rho \quad (3.6)$$

where $\eta = \sum_{\kappa=1}^3 (\ddot{q}^\kappa)^2$, $\bar{\eta} = \sum_{\kappa=1}^3 (\ddot{\ddot{q}}^\kappa)^2$, and components $H_{\sigma v}^1$, $H_{\sigma v}^2$ of H vanish.

Proof. Substituting the generators of the Galilei group (3.5) into (3.2)–(3.4) we obtain

$$\begin{aligned}
\frac{\partial H_{\sigma v}^0}{\partial t} &= 0 & \frac{\partial H_{\sigma v}^0}{\partial q^\rho} &= 0 & \frac{\partial H_{\sigma v}^0}{\partial \dot{q}^\rho} &= 0 \\
\frac{\partial H_{\sigma v}^1}{\partial t} &= 0 & \frac{\partial H_{\sigma v}^1}{\partial q^\rho} &= 0 & \frac{\partial H_{\sigma v}^1}{\partial \dot{q}^\rho} &= 0 \\
\frac{\partial H_{\sigma v}^2}{\partial t} &= 0 & \frac{\partial H_{\sigma v}^2}{\partial q^\rho} &= 0 & \frac{\partial H_{\sigma v}^2}{\partial \dot{q}^\rho} &= 0
\end{aligned} \quad (3.7)$$

and

$$\begin{aligned}
\varepsilon_{\theta\rho\nu} H_{\sigma\rho}^0 + \varepsilon_{\theta\rho\sigma} H_{\rho\nu}^0 + \varepsilon_{\theta\rho\kappa} \frac{\partial H_{\sigma v}^0}{\partial \ddot{q}^\rho} \ddot{q}^\kappa + \varepsilon_{\theta\rho\kappa} \frac{\partial H_{\sigma v}^0}{\partial \ddot{\ddot{q}}^\rho} \ddot{\ddot{q}}^\kappa &= 0 \\
\varepsilon_{\theta\rho\sigma} H_{\rho\nu}^1 + \varepsilon_{\theta\rho\kappa} \frac{\partial H_{\sigma v}^1}{\partial \ddot{q}^\rho} \ddot{q}^\kappa + \varepsilon_{\theta\rho\kappa} \frac{\partial H_{\sigma v}^1}{\partial \ddot{\ddot{q}}^\rho} \ddot{\ddot{q}}^\kappa &= 0 \\
\varepsilon_{\theta\rho\sigma} H_{\rho\nu}^2 + \varepsilon_{\theta\rho\kappa} \frac{\partial H_{\sigma v}^2}{\partial \ddot{q}^\rho} \ddot{q}^\kappa + \varepsilon_{\theta\rho\kappa} \frac{\partial H_{\sigma v}^2}{\partial \ddot{\ddot{q}}^\rho} \ddot{\ddot{q}}^\kappa &= 0
\end{aligned} \quad (3.8)$$

From equalities (3.7), we get that $H_{\sigma v}^0$, $H_{\sigma v}^1$ and $H_{\sigma v}^2$ do not depend on $t, q^\rho, \dot{q}^\rho$. From (3.8), we get formula (3.6) and the vanishing of the components $H_{\sigma v}^1$, $H_{\sigma v}^2$. $\square$

Theorem 2. *Let H be a Helmholtz-like form (2.4) which is locally Helmholtz. H is invariant under the Galilei group if and only if the component $H_{\sigma v}^0$ of H takes the form*

$$H_{\sigma v}^0 = \varepsilon_{\sigma\nu\rho} c \ddot{q}^\rho \quad (3.9)$$

where c is constant and the components $H_{\sigma v}^1$, $H_{\sigma v}^2$ of H vanish. The corresponding dynamical form takes the form

$$E = -\frac{2}{3}\varepsilon_{\sigma\nu\rho} c q^\nu \ddot{q}^\rho \omega^\sigma \wedge dt \quad (3.10)$$

Proof. H is locally Helmholtz, i. e., it corresponds to a system of differential equations, if and only if its components satisfy the following identities (see [7]):

$$\begin{aligned} \frac{\partial H_{\sigma v}^2}{\partial \ddot{q}^\rho} &= 0 & \frac{\partial H_{\sigma v}^0}{\partial \ddot{q}^\rho} + \frac{1}{2} \frac{\partial H_{\sigma \rho}^2}{\partial \dot{q}^\nu} &= 0 \\ \left(\frac{\partial H_{\sigma v}^1}{\partial \ddot{q}^\rho} - \frac{\partial H_{\sigma v}^2}{\partial \ddot{q}^\rho} \right)_{[\nu\rho]} &= 0 & \left(\frac{\partial H_{\sigma v}^1}{\partial \ddot{q}^\rho} - \frac{d}{dt} \frac{\partial H_{\sigma v}^1}{\partial \ddot{q}^\rho} \right)_{(\nu\rho)} &= 0 \\ \left(\frac{\partial H_{\sigma v}^0}{\partial \ddot{q}^\rho} + \frac{1}{2} \frac{\partial H_{\sigma v}^2}{\partial q^\rho} + \frac{1}{2} \frac{\partial H_{\nu\rho}^2}{\partial q^\sigma} - \frac{1}{2} \frac{d}{dt} \left(3 \frac{\partial H_{\sigma v}^0}{\partial \ddot{q}^\rho} + \frac{\partial H_{\sigma v}^2}{\partial \dot{q}^\rho} \right) \right)_{[\nu\rho]} &= 0, \\ \left(\frac{\partial H_{\sigma v}^1}{\partial \dot{q}^\rho} - \frac{1}{2} \frac{d}{dt} \frac{\partial H_{\sigma v}^1}{\partial \ddot{q}^\rho} + \frac{1}{2} \frac{d^2}{dt^2} \frac{\partial H_{\sigma v}^1}{\partial \ddot{q}^\rho} \right)_{[\nu\rho]} &= 0, \\ \left(\frac{\partial H_{\sigma v}^0}{\partial \dot{q}^\rho} - \frac{1}{2} \left(\frac{\partial H_{\sigma \rho}^1}{\partial q^\nu} - \frac{\partial H_{\nu\rho}^1}{\partial q^\sigma} \right) - \frac{d}{dt} \left(\frac{\partial H_{\sigma v}^0}{\partial \ddot{q}^\rho} - \frac{1}{2} \frac{\partial H_{\sigma v}^2}{\partial q^\rho} \right) + \right. & \\ \left. + \frac{d^2}{dt^2} \frac{\partial H_{\sigma v}^0}{\partial \ddot{q}^\rho} \right)_{(\nu\rho)} &= 0, \\ \left(\frac{\partial H_{\sigma v}^0}{\partial q^\rho} - \frac{1}{3} \frac{d}{dt} \frac{\partial H_{\sigma v}^0}{\partial \dot{q}^\rho} + \frac{1}{3} \frac{d^2}{dt^2} \left(\frac{\partial H_{\sigma v}^0}{\partial \ddot{q}^\rho} + \frac{\partial H_{\sigma v}^2}{\partial q^\rho} \right) - \right. & \\ \left. - \frac{1}{3} \frac{d^3}{dt^3} \frac{\partial H_{\sigma v}^0}{\partial \ddot{q}^\rho} \right)_{[\nu\rho]} &= 0. \end{aligned} \quad (3.11)$$

The corresponding dynamical form is then in coordinates given by the formula (see [6]):

$$E = \left(\tilde{E}_\sigma - \frac{d\tilde{E}_\sigma^1}{dt} + \frac{d^2\tilde{E}_\sigma^2}{dt^2} \right) \omega^\sigma \wedge dt \quad (3.12)$$

where

$$\begin{aligned} \tilde{E}_\sigma &= 2q^\nu \int_0^1 H_{\nu\sigma}^0(t, u q^\rho, u \dot{q}^\rho, u \ddot{q}^\rho) u \, du \\ &\quad - \dot{q}^\nu \int_0^1 H_{\sigma\nu}^1(t, u q^\rho, u \dot{q}^\rho, u \ddot{q}^\rho) u \, du \\ &\quad - \ddot{q}^\nu \int_0^1 H_{\sigma\nu}^2(t, u q^\rho, u \dot{q}^\rho, u \ddot{q}^\rho) u \, du \end{aligned}$$

and

$$\begin{aligned}\tilde{E}_\sigma^1 &= q^\nu \int_0^1 H_{\nu\sigma}^1(t, u q^\rho, u \dot{q}^\rho, u \ddot{q}^\rho) u \, du, \\ \tilde{E}_\sigma^2 &= q^\nu \int_0^1 H_{\nu\sigma}^2(t, u q^\rho, u \dot{q}^\rho, u \ddot{q}^\rho) u \, du.\end{aligned}$$

From equalities (3.7) and (3.11), we get that $H_{\sigma\nu}^0$, $H_{\sigma\nu}^1$ and $H_{\sigma\nu}^2$ do not depend on $t, q^\rho, \dot{q}^\rho, \ddot{q}^\rho$. From (3.8), we get (3.9). Substituting (3.9) into the (3.12), we get (3.10). $\square$

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A GEOMETRIC ORIENTED APPROACH IN DRAWING THE SIMULATION MODEL OF SMALL FREE ANGULAR VIBRATIONS OF A BODY

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AND JANITZA KEHAJOVA

Abstract. The paper provides a geometry-oriented way of drawing the input block of the simulation model for studying the small free vertical angular vibrations of the body. The simulation model is created by one of the main tools of the software package MATLAB SIMULINK. The geometry of the input block of the simulation model looks like the geometry of the dynamic model, allowing easy and attractive introduction of various geometric, inertial, elastic and viscous characteristics. The model described in the paper is made with a licensed version of the software package MATLAB, purchased with funds from a research project on which the authors of the paper have worked over the last two years.

2000 Mathematics Subject Classification: 34B10, 34B15

Keywords: dynamic model, mathematical model, simulation model, small angular vibrations, elastic-viscous characteristics

1. INTRODUCTION

The input block for which geometry-oriented approach to creation is offered is part of a generalized simulation model for studying small free vertical angular vibrations of the body. The standard structure of a similar model to study the dynamic behavior of any physical object consists of two parts:

- ◇ A system of connected blocks to integrate the differential equation describing the movement of the object;
- ◇ A system of blocks for visualizing the results.

A standard simulation model of an analysis of the dynamics of a material object or a system with one degree of freedom is shown on Figure 1, but without the left input block, which will be discussed below.

The characteristics of the material objects or the system and the relationships limiting their movement are usually given by a separately compiled text file produced by the programming language of the software package MATLAB [1]. Prior to starting the simulation model, a input file for the mentioned characteristics should be started.

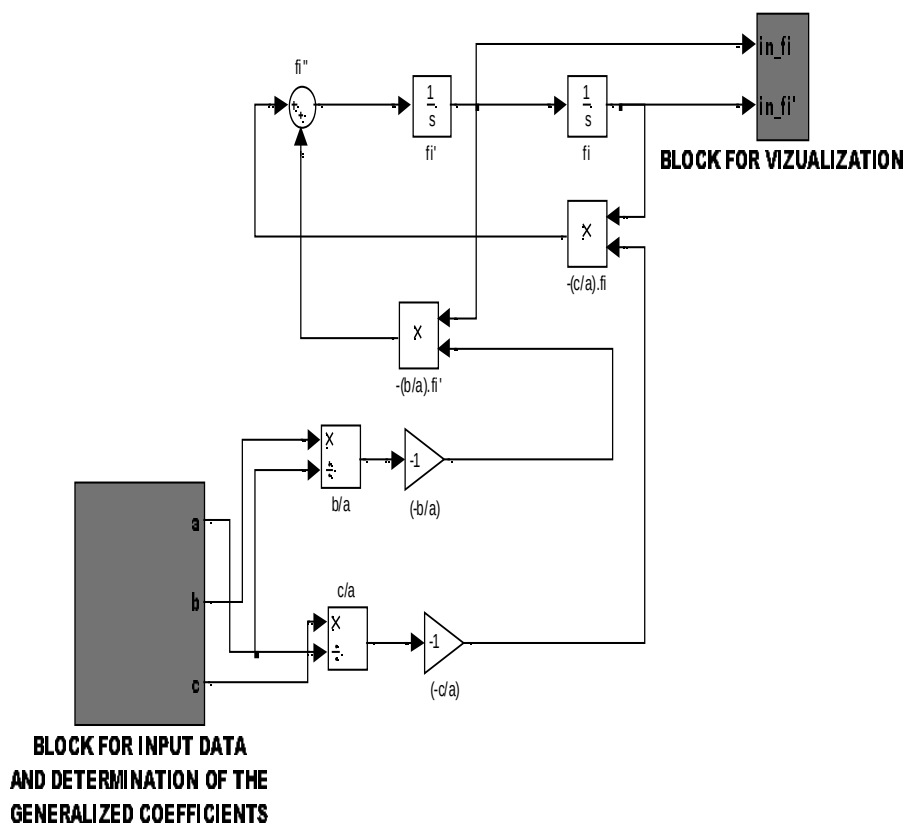


FIGURE 1. A simulation model for studying the dynamics of a vibrating system with 1 DOF.

This naturally leads to a number of inconveniences. Each exit from the system imposes a new startup of the text file or of a file, saving the previously entered input data. Many inconveniences appear when changing one or several inputs. It is not easy to check the value of the individual characteristics, especially if several files with similar names of the individual variables are open in the working window of the program. The text and the simulation file should always be transferred together. The absence of one of them would prevent the right solution of a dynamic problem.

All these inconveniences in the paper can be removed by establishing an additional block of input data. This approach is implemented when testing the small free angular oscillation of a body, but can also be applied to the forced oscillations. The architecture of the block, composed in the middle of the SIMULINK, resembles the dynamic model for the study of the small angular vibrations of the body [2]. It allows

an easy and convenient entry or modification of the physic-mechanical characteristics of the vibrating system.

2. PRESENTATION

The construction of a simulation model is the third stage in studying the dynamics of an oscillating system. Prior to this, an idealized dynamic model is created along with a corresponding mathematical model of the vibrations. The dynamic model of small angular vibrations of a body in the vertical plane is described in [2], and the deductively derived model of free ones is shown on Figure 2.

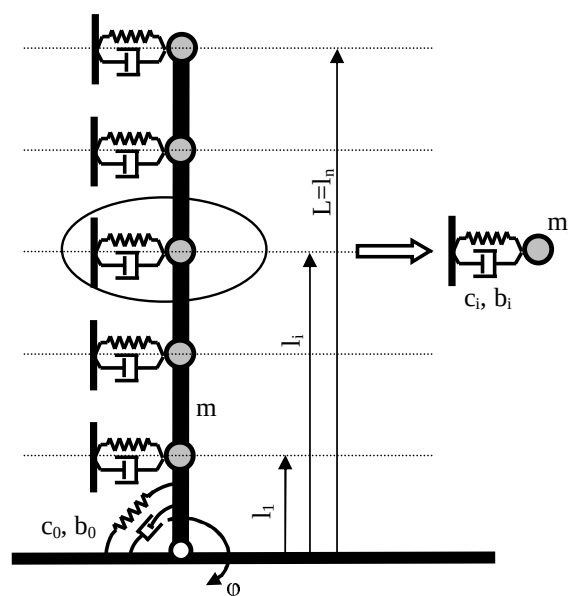


FIGURE 2. Dynamic model for free small angular vibrations of a body in the vertical plane.

The corresponding mathematical model, obtained by the Lagrange equations of second class, in a generalized coordinate (the angle), in which the vibrating body deviates from the vertical axis, is

$$a\ddot{\varphi} + b\dot{\varphi} + c\varphi = 0. \quad (2.1)$$

In the equation, a , b and c denote the generalized inertial, the dissipative, and the elastic coefficients respectively.

The general appearance of the simulation model for analyzing the small free angular vibrations of a body is that of Figure 1, plus the block of input data. Like any standard simulation model for analyzing the dynamics of systems with one degree of

freedom, it contains the basic blocks – block for integration of the differential equation and block for visualization of the results. The unit of integration shows that the generalized coefficients a , b and c must be known.

In the standard simulation programs, they can be found through the input program, because they are a function of the input data by the formulas

$$\begin{aligned} a &= \frac{1}{3}m\ell^2 + \sum m_i \ell_i^2, \\ b &= b_0 + \sum b_i \ell_i^2, \\ c &= c_0 + \sum c_i \ell_i^2 - g \left(m \frac{\ell}{2} + \sum m_i \ell_i \right). \end{aligned} \quad (2.2)$$

As mentioned in the paper, the compilation of an additional file is ignored and, instead of that, a block for input data and for calculating the coefficients by the formulas (2.2) is added. The idea is that the incoming block should be similar to the real dynamic model. This is why the blocks are in a form resembling the respective elements of the vibrating system. The main part of this input block of the simulation model of a system consisting of a vibrating body and two concentrated masses is shown on Figure 3.

The type of all blocks in the subsystem is constant because the values of the parameters they introduce are constant too. The name of each block contains the parameter, which is also related to equality sign. By clicking on the desired block, the corresponding window opens, and the respective characteristic can be entered or confirmed. The entered characteristics appear as values in the center of each block. The units in which each characteristic is entered, can be written in a specific location in the subsystem. By default, the base units in the system SI are used for length, mass, spring stiffness and a linear coefficient of resistance damper.

The blocks for the geometric characteristics are vertically elongated rectangles reminding of the measurement lines in the dynamic model. The block for mass of the vibrating rod is also vertically extended rectangle, so that it can match the rod in the dynamic model. The blocks for entering the concentrated masses are adjoined to the last block squares. The blocks for entering the characteristics of the elastic-viscous sets along the length of the body are complemented by blocks in the form of spring and damper.

Output block with a certain number is attached to each constant block type. The numbers of the outputs are used as inputs for the next subsystem — the subsystem for calculation of the generalized coefficients of equations (2.1) and (2.2).

3. CONCLUSION

The compilation of a separate module for data entry in the construction of a simulation model, for the study of the dynamics of an oscillating system, has the following advantages in comparison with the standard simulation model:

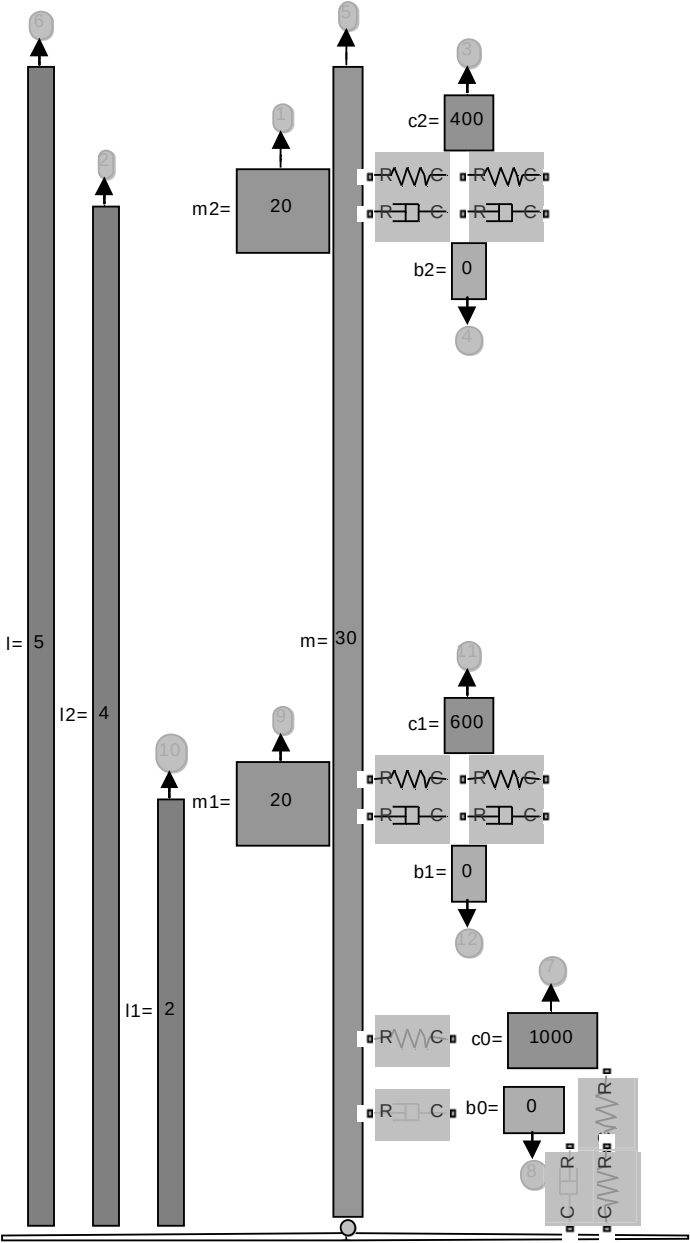


FIGURE 3. Part of the input block of the simulation model for analyzing the free small angular vibrations of a body in vertical plane.

- ◇ When entering data, the user sees the real dynamic model in the individual components in which he enters or changes certain parameters;
- ◇ It is easy to solve the problem of optimal dynamic synthesis.

The model described in the report is made with a licensed version of the software package MATLAB, purchased with funds from a research project in which the authors of the paper have worked in the last two years.

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CONFIGURATION SPACE ANALYSIS OF RIGID MECHANISMS

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Abstract. In mechanical engineering and robotics, the main problem is usually to solve the equations of motion of a given rigid mechanism. The rigid mechanisms are sometimes restricted or controlled to move along a constrained path. In engineering sciences, the most common constraints are called *ideal joint constraints*. The ideal joint constraints are generally *holonomic* constraints involving only the coordinates and orientation of a mechanism with respect to a fixed coordinate frame. An important aspect is that ideal joint constraints can be formulated as a set of polynomial equations. This means that the configuration spaces of typical rigid mechanisms can be treated as *algebraic varieties* and the components of a constraint function as generators of a polynomial ideal. This geometry algebra equivalence and the advances of computational commutative algebra and algebraic geometry gives us means to actually *compute* the properties of configuration spaces.

2000 Mathematics Subject Classification: 58D30, 00A69, 14-00, 14Q99

Keywords: mechanisms, robotics, holonomic constraints, algebraic geometry, computational algebraic geometry, Gröbner basis

1. INTRODUCTION

The equations of motion arising from Lagrangian mechanics for multibody systems are typically differential algebraic equations (DAE) where algebraic equations in our case determine the holonomic constraints. The equations of motion in Lagrangian mechanics in the holonomic case are generally of the form

$$\begin{aligned} f(t, u, \dot{u}, \ddot{u}, \lambda) &= 0, \\ g(u) &= 0, \end{aligned} \tag{1.1}$$

where $u : I \mapsto g^{-1}(0) \subset \mathbb{R}^k$ is the trajectory of the system, λ denotes the Lagrangian multipliers defining the holonomic constraint forces and $g^{-1}(0)$ is usually the analytic or algebraic variety defining the configuration space. In this paper we will introduce modern methods of computational algebraic geometry to study the configuration space as an algebraic variety. We will present relevant theorems and give three easy examples of planar mechanisms and their configuration space analysis. In computations, we have used the well established computer program SINGULAR [9]. In most of the computations the key is to compute the Gröbner bases of a given ideal in a particular monomial ordering.

2. PRELIMINARY DEFINITIONS

Let $\mathbb{A} = \mathbb{K}[x_1, \dots, x_k]$ be the ring of polynomials with coefficient field $\mathbb{K}$. We will always assume that $\mathbb{K}$ is algebraically closed. We can look at the vector components of the constraint map $g : \mathbb{R}^k \mapsto \mathbb{R}^n$ as functions or as generators of an ideal $\mathcal{I} = \langle g_1, \dots, g_n \rangle \subset \mathbb{A}$. For standard facts about the ideal theory we refer to [3, 4, 8, 10]. Since the polynomial rings that we consider are Noetherian, we know that *every* ideal in $\mathbb{A}$ is finitely generated. The geometric object corresponding to an ideal $\mathcal{I} \subset \mathbb{A}$ is its *variety* $V(\mathcal{I}) \subset \mathbb{K}^k$ which is the vanishing set of all polynomials in $\mathcal{I} \subset \mathbb{A}$. We will also frequently use the fact that any radical ideal can be written uniquely as a finite intersection of *prime ideals*. This prime decomposition implies directly the decomposition of the corresponding variety into its *irreducible* parts which is called the irreducible decomposition of a variety.

Let us then present the tools which allow us to do local analysis on varieties and distinguish between different types of singularities.

Definition 1. (Local ring) A local ring R is a ring which has exactly one maximal ideal.

Remark 1. In this paper we look at localizations of polynomial rings with respect to a point/maximal ideal. We consider equivalence classes of polynomials giving always the same value when evaluated at $V(\mathcal{I})$. The equivalence classes are given by

$$[f] = \{g \in \mathbb{A} \mid f - g \in I(V(\mathcal{I}))\}, \quad (2.1)$$

where $I(V(\mathcal{I}))$ denotes the *ideal of a variety* $V(\mathcal{I})$. As usual, the *coordinate ring* $\mathbb{K}(V(\mathcal{I}))$ is

$$\mathbb{K}(V(\mathcal{I})) = \{[f] \mid f \in \mathbb{A}\}. \quad (2.2)$$

Definition 2. (Localization of $V(\mathcal{I})$ at a point) Let $\mathfrak{m}_p$ be the maximal ideal $\mathfrak{m}_p = \langle x_1 - p_1, \dots, x_n - p_n \rangle \subset \mathbb{A}$ where $p = (p_1, \dots, p_n) \in V(\mathcal{I}) \subset \mathbb{K}^n$. Then we can write the localization of $\mathbb{A}$ at p as

$$\mathcal{O}_p = \{f/g \mid f \in \mathbb{A}, g \notin \mathfrak{m}_p\} \quad (2.3)$$

The unique maximal ideal in this case is

$$\mathfrak{m}_p = \{f/g \mid f \in \mathfrak{m}_p, g \notin \mathfrak{m}_p\} \quad (2.4)$$

The *localization* of $V(\mathcal{I})$ at p is the ideal

$$\mathcal{O}_{V,p} = \{f/g \mid f \in \mathbb{K}(V(\mathcal{I})), g \notin \mathfrak{m}_p\} \subset \mathcal{O}_p. \quad (2.5)$$

Notice that $\mathcal{I} \subset \mathfrak{m}_p$ and $\mathcal{O}_{V,p}$ is a local ring which is a subring of a $\mathcal{O}_p$ with maximal ideal $\mathfrak{m}_p \mathcal{O}_{V,p} = \mathfrak{M}_p$.

3. SINGULARITIES AND DIMENSION

In this section we will briefly present the relevant definitions and theorems in order to compute our examples. Remember that the embedding dimension $\text{edim}(\mathcal{O}_{V,p}) = \dim_K(M_p/M_p^2)$ of an algebraic variety is the minimal number of generators of M_p . Particularly important is that the Krull dimension of an ideal can be easily computed if the elements of the Gröbner basis of an ideal is known.

Definition 3 (Singular and regular points of a variety). Suppose that $\mathcal{I}$ is a radical ideal. The local ring $\mathcal{O}_{V,p}$ is a *regular local ring* if

$$\dim_K(\mathcal{O}_{V,p}) = \text{edim}(\mathcal{O}_{V,p}) = \dim(T_p V(\mathcal{I})),$$

where $T_p V(\mathcal{I})$ denotes the tangent space of $V(\mathcal{I})$ at p . If the point p is not regular, it is *singular*.

The last equation in Definition 3 gives us the actual means to compute the singular points [5, 7, 8, 11].

Theorem 1 (Jacobian criterion). Let $\mathcal{I} = \langle g_1, \dots, g_n \rangle \subset \mathbb{A}$ be a radical ideal and suppose that $V(\mathcal{I}) \subset \mathbb{K}^k$ is equidimensional* and $\dim(V(\mathcal{I})) = k - \ell$. Then the singular variety of $V(\mathcal{I})$ is

$$S(V(\mathcal{I})) = V(\mathcal{I} + I_\ell(dg)) = V(\mathcal{I}) \cap V(I_\ell(dg)) \subset \mathbb{K}^k.$$

Here $I_\ell(dg)$ denotes the ℓ th Fitting ideal of the Jacobian of the constraint map g generated by $\ell \times \ell$ minors of dg . From this, it follows that, if $p \in S(V(\mathcal{I}))$, then $\mathcal{O}_{V,p}$ is not a regular local ring. Moreover, if $1 \in \mathcal{I} + I_\ell(dg)$, then the variety $V(\mathcal{I})$ is naturally smooth since $V(1) = \emptyset$.

Let us then introduce another important object in our analysis - the *tangent cone* [3, 4]. We can rearrange any polynomial $f \in \mathbb{A}$ by total degree d as a linear combination $f = f_{p,0} + f_{p,1} + \dots + f_{p,d}$, where

$$f_{p,k} = \sum_{|\alpha|=k} c_\alpha (x - p)^\alpha.$$

Here, α is a multi index. Let us denote by $f_{p,\min}$ the nonzero homogeneous component of smallest degree.

Definition 4 (tangent cone). Suppose that $V(\mathcal{I}) \subset \mathbb{K}^k$ is an affine variety and let $p \in V(\mathcal{I})$. Let $\mathcal{I}_{p,\min}$ be the ideal generated by the minimal parts of polynomials in $\mathcal{I}$. The *tangent cone* of $V(\mathcal{I})$ at p , denoted by $C_p(V(\mathcal{I}))$, is

$$C_p(V(\mathcal{I})) = V(\mathcal{I}_{p,\min}), \quad (3.1)$$

The following theorem allows us also to distinguish between singular and regular points [3, 4].

*A variety is called *equidimensional* if all of its irreducible components have the same dimension.

Theorem 2. *The following conditions are equivalent:*

- (1) $p \in V$ is a regular point of V ;
- (2) $\dim(C_p(V)) = \dim(T_p V)$;
- (3) $C_p(V) = T_p V$.

Finally, let us present a theorem to distinguish certain types of singularities [12].

Theorem 3. *Suppose that $\mathcal{I} \subset \mathbb{K}[x_1, \dots, x_k]$ is a ideal where $\mathbb{K}$ is algebraically closed. Let $p \in V(\mathcal{I})$ be a singular point of $V(\mathcal{I})$ and $\mathcal{O}_p$ be the local ring at p . If the prime decomposition of the radical of $\mathcal{O}_{V,p}$ in the local ring is*

$$\sqrt{\mathcal{O}_{V,p}} = \mathcal{I}_1 \cap \dots \cap \mathcal{I}_r \subset \mathcal{O}_p,$$

then the corresponding irreducible varieties $V(\mathcal{I}_i)$ of prime ideals $\mathcal{I}_i$ represent varieties passing through the singular point p and the varieties intersect at this point. However, if the prime decomposition of $\mathcal{O}_{V,p}$ is $\sqrt{\mathcal{O}_{V,p}} = \mathcal{I}$, then $\mathcal{I}$ is an integral domain and the point p is a singularity of an irreducible variety $V(\mathcal{I})$.

4. EXAMPLES

Here we apply the previous theorems to three easy examples.

4.1. Simple slider-crank mechanisms

Let us consider a mechanism constructed from two and three bars attached to each other with a revolute joint and suppose that the last bar is constrained to move on x -axis.

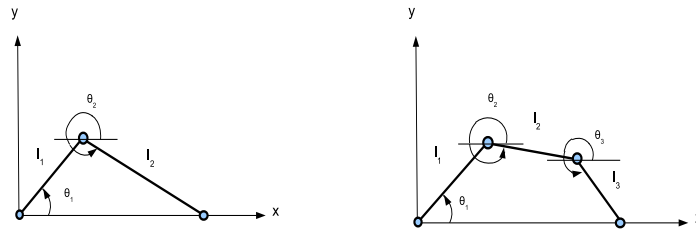


FIGURE 1. On the left the 2-bar slider-crank mechanism. On the right the 3-bar slider crank mechanism.

4.1.1. 2-bar slider-crank mechanism

The constraint mapping $g : \mathbb{R}^2 \mapsto \mathbb{R}$ for the 2-bar slider-crank mechanism is then

$$g(\theta_1, \theta_2) = l_1 \sin(\theta_1) + l \sin(\theta_2).$$

The configuration space is then the analytic variety $g^{-1}(0)$. With substitutions $c_i = \cos(\theta_i)$ and $s_i = \sin(\theta_i)$, the constraint equations take the form

$$p_1 = l_1 s_1 + l_2 s_2 = 0, \quad p_2 = c_1^2 + s_1^2 - 1 = 0, \quad p_3 = c_2^2 + s_2^2 - 1 = 0.$$

The configuration space is then $V(\langle p_1, p_2, p_3 \rangle) = V(\mathcal{I})$. Now it is easy to check that $\mathcal{I}$ is a radical ideal and, moreover, $\dim(V(\mathcal{I})) = 1$. Let us then deduce the necessary condition for singularities from Theorem 1 in (l_1, l_2) -space. Since $\dim(V(\mathcal{I})) = 1$, the singular variety of $V(\mathcal{I})$ is

$$S(V(\mathcal{I})) = V(\mathcal{I} + \mathfrak{l}_3(dp))$$

Decomposing the Fitting ideal $\mathfrak{l}_3(dp)$ gives

$$\sqrt{\mathfrak{l}_3(dp)} = \mathcal{I}_1 \cap \dots \cap \mathcal{I}_6,$$

Only the ideal $\mathcal{I}_5 = \langle c_1, c_2 \rangle$ corresponds to a physically feasible solution so we only need to look at the intersection of $V(\mathcal{I}_5)$ and $V(\mathcal{I})$. To find the necessary condition, we compute the Gröbner basis of $S = \mathcal{I} + \mathcal{I}_5$ in the ring $\mathbb{Q}[(c_1, s_1, c_2, s_2), (l_1, l_2)]$ with the above *elimination ordering*. The computation shows

$$E = S \cap \mathbb{Q}[l_1, l_2] = \langle (l_1 + l_2)(l_1 - l_2) \rangle,$$

Hence $l_1 = l_2$ is a necessary condition for singularities in (l_1, l_2) -space. Next we choose for example $l_1 = l_2 = 1$ and compute the actual singular points. This gives

$$\sqrt{S} = \langle s_2 + 1, c_2, s_1 + s_2, c_1 \rangle \cap \langle s_2 - 1, c_2, s_1 + s_2, c_1 \rangle.$$

Thus the singular points are

$$V(S) = V(\sqrt{S}) = \{(0, 1, 0, -1), (0, -1, 0, 1)\} = q_1 \cup q_2.$$

Let us then carry out a local analysis for variety $V(\mathcal{I})$ at the singular point $q_2 = (c_1, s_1, c_2, s_2) = (0, -1, 0, 1)$. Now we can compute the tangent cone and get

$$C_{q_2}(V(\mathcal{I})) = V(\langle c_1, c_2, (s_1 - 1)^2 - (s_2 + 1)^2 \rangle).$$

Near q_2 , the variety $V(\mathcal{I})$ looks like two lines $s_1 - 1 = \pm(s_2 + 1)$ intersecting in the plane $c_1 = c_2 = 0$. Next, we consider $\mathcal{I} = \langle p_1, p_2, p_3 \rangle$ in the local ring $\mathcal{O}_{q_2}$ using *local ordering* and compute the prime decomposition of $\mathcal{O}_{V, q_2}$. As expected, we have

$$\mathcal{O}_{V, q_2} = H_1 \cap H_2.$$

By theorem 3, at least two irreducible varieties/motion modes pass through q_2 . In fact, when we compute in global ordering after substitution $l_1 = l_2 = 1$, we simply have

$$\mathcal{J} = \mathcal{J}_1 \cap \mathcal{J}_2 = \langle c_2^2 + s_2^2 - 1, c_1 - c_2, s_1 + s_2 \rangle \cap \langle c_2^2 + s_2^2 - 1, c_1 + c_2, s_1 + s_2 \rangle,$$

$$V(\mathcal{J}) = V(\mathcal{J}_1) \cup V(\mathcal{J}_2) \text{ and } S(V(\mathcal{J})) = V(\mathcal{J}_1) \cap V(\mathcal{J}_2).$$

4.1.2. 3-bar slider-crank mechanism

Let us then do similar analysis for 3-bar slider crank mechanism. With substitutions $c_i = \cos(\theta_i)$ and $s_i = \sin(\theta_i)$, as before, the constraint equations take the form

$$p_1 = l_1 s_1 + l_2 s_2 + l_3 s_3 = 0, \quad p_{i+1} = c_i^2 + s_i^2 - 1 = 0, \quad 1 \leq i \leq 3.$$

Again, we check that $\mathcal{J}$ is radical and compute $\dim(V(\mathcal{J})) = 2$. The singular variety is then

$$S(V(\mathcal{J})) = V(\mathcal{J} + \mathfrak{l}_4(dp)).$$

Again, we first analyze the Fitting ideal $\mathfrak{l}_4(dp)$ and find

$$\sqrt{\mathfrak{l}_4(dp)} = \mathcal{J}_1 \cap \dots \cap \mathcal{J}_{11}.$$

Only one of the prime components $\mathcal{J}_9 = \langle c_1, c_2, c_3 \rangle$ is physically relevant so we compute the Gröbner basis of $S = \mathcal{J} + \mathcal{J}_9$ in the ring

$$\mathbb{Q}[(c_1, s_1, c_2, s_2, c_3, s_3), (l_1, l_2, l_3)]$$

with the above elimination ordering. Now we find

$$E = S \cap \mathbb{Q}[l_1, l_2, l_3] = \langle (l_1 + l_2 + l_3)(l_1 - l_2 - l_3)(l_1 + l_2 - l_3)(l_1 - l_2 + l_3) \rangle$$

$$= \langle h_1 h_2 h_3 h_4 \rangle$$

The necessary conditions for singularities are thus $h_2 = 0$, $h_3 = 0$ or $h_4 = 0$. Let us choose $l_1 = 2$ and $l_2 = l_3 = 1$ so that $h_2 = 0$ is fulfilled. This gives

$$\sqrt{S} = \langle c_1, s_1 - 1, c_2, s_2 + 1, c_3, s_3 + 1 \rangle \cap \langle c_1, s_1 - 1, c_2, s_2 + 1, c_3, s_3 + 1 \rangle.$$

The singular points are thus

$$V(S) = V(\sqrt{S}) = \{(0, 1, 0, -1, 0, -1), (0, -1, 0, 1, 0, 1)\} = q_1 \cup q_2.$$

Let us then investigate locally the variety at point q_2 . Computing the tangent cone gives

$$C_{q_2}(V(\mathcal{J})) = V(\langle c_1, c_2, c_3, 2(s_1 + 1)^2 - (s_2 - 1)^2 - (s_3 - 1)^2 \rangle).$$

Note that $2(s_1 + 1)^2 - (s_2 - 1)^2 - (s_3 - 1)^2 = 0$ actually gives a cone in the (s_1, s_2, s_3) space. Moreover, when we compute the prime decomposition of the radical of localization $\mathcal{O}_{V, q_2}$ of $V(\mathcal{I})$ at $\mathcal{O}_{q_2}$, we find that

$$\sqrt{\mathcal{O}_{V, q_2}} = \mathcal{O}_{V, q_2}$$

Thus the singularity is not an intersection of different motion modes/irreducible varieties. It is still possible to visualize the configuration spaces of 2-bar and 3-bar slider crank mechanisms. In the first case, the configuration space "breaks" naturally to two irreducible varieties. In the 3-bar case, such separation does not exist.

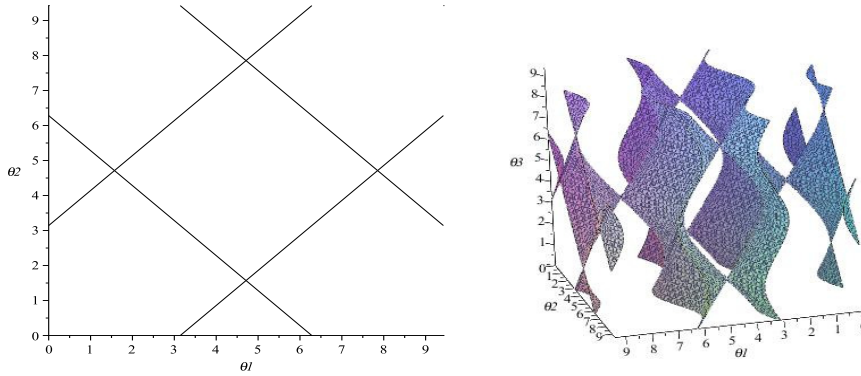


FIGURE 2. On the left, the configuration space of 2-bar slider-crank mechanism in the (θ_1, θ_2) -space. On the right, the configuration space of 3-bar slider crank mechanism in the $(\theta_1, \theta_2, \theta_3)$ -space.

Remark 2. The plots agree with our computational results. Also, in the 2-bar slider crank case, there are no regular solutions $c : I \mapsto g^{-1}(0)$ for equations of motion (1.1) through the singular point $q_2 = c(t_0)$ from one motion mode to another since, for such solutions, automatically, $c'(t_0) = (0, 0)$. However, in the case of the 3-bar slider crank, such a problem does not exist.

4.2. Closed four bar mechanism

Let us finally consider the closed four bar mechanism. To simplify the analysis, we assume that one bar with a length of unity is fixed on the x -axis. In this case, the constraint equations are given by

$$\begin{aligned} p_1 &= l_1 c_1 + l_2 c_2 + l_3 c_3 - 1 = 0, \\ p_2 &= l_1 s_1 + l_2 s_2 + l_3 s_3 = 0, \\ p_{2+i} &= c_i^2 + s_i^2 - 1 = 0, \quad 1 \leq i \leq 3. \end{aligned}$$

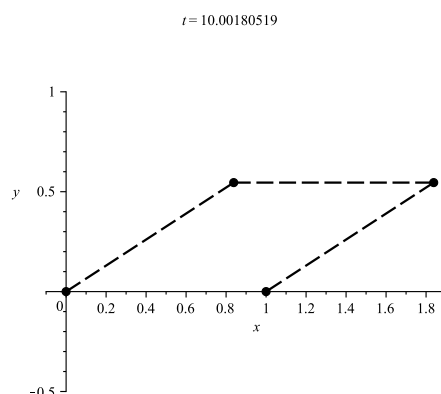


FIGURE 3. Simple closed four bar mechanism.

Setting $\mathcal{I} = \langle p_1, \dots, p_5 \rangle$, we compute $\dim(V(\mathcal{I})) = 1$. The singular variety is thus

$$S(V(\mathcal{I})) = V(\mathcal{I} + \mathfrak{l}_5(dp)).$$

The prime decomposition of $\sqrt{\mathfrak{l}_5(dp)}$ yields

$$\sqrt{\mathfrak{l}_5(dp)} = \mathfrak{I}_1 \cap \dots \cap \mathfrak{I}_{10}.$$

The only physically relevant component is given by

$$\mathfrak{I}_8 = \langle -s_2c_3 + c_2s_3, -s_1c_3 + c_1s_3, -s_1c_2 + c_1s_2 \rangle$$

Computing the Gröbner basis of $S = \mathcal{I} + \mathfrak{I}_8$ in the ring

$$\mathbb{Q}[(c_1, s_1, c_2, s_2, c_3, s_3), (l_1, l_2, l_3)]$$

we find

$$E = S \cap \mathbb{Q}[l_1, l_2, l_3] = \langle k_1 \cdot \dots \cdot k_9 \rangle.$$

The necessary condition for $V(S) \neq \emptyset$ in (l_1, l_2, l_3) -space is $k_i = 0$, where the nine polynomials k_i are of the form

$$k_i = l_1 \pm l_2 \pm l_3 \pm 1.$$

Let us choose $l_1 = l_2 = l_3 = 1$ so that $k_4 = l_1 - l_2 + l_3 - 1 = 0$. After substitutions we compute

$$\begin{aligned}\sqrt{S} &= \mathcal{J}_1 \cap \mathcal{J}_2 \cap \mathcal{J}_3 \\ \mathcal{J}_1 &= \langle c_1 + 1, s_1, c_2 - 1, s_2, c_3 - 1, s_3 \rangle \\ \mathcal{J}_2 &= \langle c_1 + 1, s_1, c_2 - 1, s_2, c_3 + 1, s_3 \rangle \\ \mathcal{J}_3 &= \langle c_1 + 1, s_1, c_2 + 1, s_2, c_3 - 1, s_3 \rangle.\end{aligned}$$

Also, after substitutions, the constraint ideal decomposes as $\sqrt{\mathcal{J}} = \mathcal{J} = \mathcal{J}_1 \cap \mathcal{J}_2 \cap \mathcal{J}_3$ and easy computation shows that the singularities q_i are the intersections

$$\begin{aligned}q_1 &= V(\mathcal{J}_1) = V(\mathcal{J}_1) \cap V(\mathcal{J}_3) = (-1, 0, 1, 0, 1, 0) \\ q_2 &= V(\mathcal{J}_2) = V(\mathcal{J}_2) \cap V(\mathcal{J}_3) = (1, 0, 1, 0, -1, 0) \\ q_3 &= V(\mathcal{J}_3) = V(\mathcal{J}_1) \cap V(\mathcal{J}_2) = (1, 0, -1, 0, 1, 0).\end{aligned}$$

Let us still do the local analysis for $V(\mathcal{J})$ for example at q_1 . The tangent cone $C_{q_1}(V(\mathcal{J}))$ is

$$C_{q_1}(V(\mathcal{J})) = V(\langle s_1, s_2, s_3, c_1 + c_2 + c_3 - 1, (c_2 - 1)(c_3 - 1) \rangle)$$

The singularity looks now again like the intersection of two lines in the hyper plane $s_1 = s_2 = s_3 = 0$. Then, the computation in the local ring shows that

$$\mathcal{O}_{V, q_1} = H_1 \cap H_2$$

This confirms that the two irreducible varieties $V(\mathcal{J}_1)$ and $V(\mathcal{J}_2)$ intersect at point q_1 as Theorem 3 suggests.

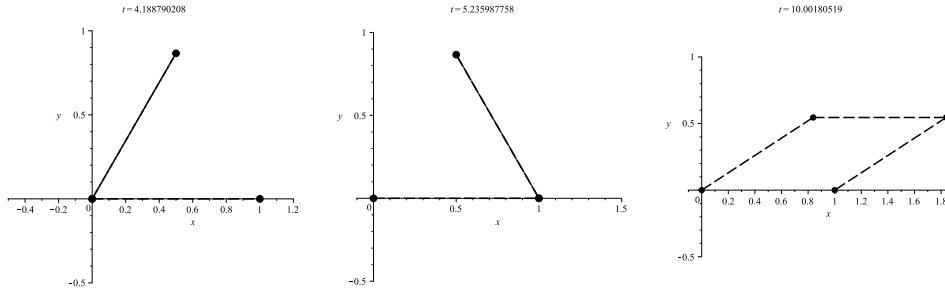


FIGURE 4. On the left the degenerate motion mode $V(\mathcal{J}_1)$. In the middle the degenerate motion mode $V(\mathcal{J}_2)$. On the right the "actual" four bar motion mode $V(\mathcal{J}_3)$. In both degenerate motion modes, two bars are fixed on the horizontal axis and two bars rotate freely together. In the "actual" motion mode, the mechanism rotates as one parallelogram.

Remark 3. Notice again that there are no regular solutions $c : I \mapsto g^{-1}(0)$ for (1.1) which would take the mechanism through q_1 from $V(\mathcal{J}_3)$ to $V(\mathcal{J}_1)$ and again through q_2 from $V(\mathcal{J}_3)$ to $V(\mathcal{J}_1)$. At singular points $c(t_i) = q_i$, $1 \leq i \leq 3$, automatically, $c'(t_i) = (0, 0, 0)$ since the one dimensional varieties intersect transversally.

5. CONCLUSIONS

In this paper, we successfully analyzed configuration spaces of three simple mechanisms with computational algebraic geometry. The methods can be applied to more complicated mechanisms as well [1, 2, 15]. When we are able to represent the configuration space as an algebraic variety, we can use the tools of computational algebraic geometry to make both global and local statements about the configuration space. The local analysis for analytic varieties has been previously investigated in [13, 14] and the Gröbner bases methods in [6]. The dimension or mobility in particular motion mode $V(\mathcal{J})$ can be regarded as the Krull dimension of the corresponding ideal $\mathcal{J}$. The Jacobian criterion together with elimination theory provides us with the means to determine whether and under which conditions, configuration space singularities exist. The tangent cone and the localization of motion mode $V(\mathcal{J})$ at a singular point gives us the means to investigate the nature of the singularity.

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GEOMETRIC CONCEPT OF ISOKINETIC CONSTRAINT FOR A SYSTEM OF PARTICLES

MARTIN SWACZYNA AND PETR VOLNÝ

Abstract. The paper deals with the geometric concept of mechanical systems of N particles. The systems are modelled on the Cartesian product $\mathbb{R} \times \mathbb{X}^N$ and its first jet prolongation $J^1(\mathbb{R} \times \mathbb{X}^N) = \mathbb{R} \times T\mathbb{X}^N$, where $\mathbb{X}$ is a 3-dimensional Riemannian manifold with a metric G . The kinetic energy T of the system of N -particles is interpreted by means of the weighted quadratic form $\mathcal{Q}_{\mathcal{G}}$ associated with the weighted metric tensor $\mathcal{G}$ which arises from the original metric tensor G and the system of N particles $m_1, \dots, m_N$. A requirement for the kinetic energy of the system of N particles to be constant is regarded as a nonholonomic, so-called isokinetic constraint and it is defined as a fibered submanifold $\mathbb{T}$ of the jet space $\mathbb{R} \times T\mathbb{X}^N$ endowed with a certain distribution $\mathcal{C}$ called canonical distribution, which has the meaning of generalized admissible displacements of the system of particles subject to the isokinetic constraint. Vector generators of the canonical distribution are found.

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Keywords: mechanical systems of particles, kinetic energy, metric tensor, nonholonomic constraints, isokinetic constraints, isokinetic canonical distribution

1. INTRODUCTION

Our motivation for the study of systems of particles whose movement is restricted by the a priori requirement that kinetic energy of the system remains constant comes from our previous investigations of a simple mechanical system of one particle moving with constant speed in the homogeneous gravitational field [7] and in the central gravitational field inspired by [3]. This theoretical problem of classical mechanics has the origin at the end of the 17th century (Jacob Bernoulli, Pierre Varignon). However, for a full understanding of all aspects of the problem, it was necessary to correctly interpret the requirement of constant instantaneous speed of a particle as a nonlinear nonholonomic constraint and to apply modern geometric approach based on the concept of nonholonomic constraints as a certain submanifold in the “velocity” space.

The condition preserving constant kinetic energy of systems of particles (isokinetic constraint) arose as a natural generalization of the condition of constant speed of one particle (isotachyonic constraint). For further study of these “isokinetically”

constrained systems, it is necessary to propose a suitable approach to isokinetic constraint. First, it was the need to precisely analyze the concept of kinetic energy of systems of particles, to find its geometric essence, and then to interpret the isokinetic constraint as the corresponding submanifold in the corresponding evolution space and, finally, to adopt general geometric theory of nonholonomic constraints [1, 2] and [4–6]. In the future, we will try to apply this general concept to a non-trivial example, so-called “isokinetic two body problem”, where speeds of two gravitationally interacting particles are not constant but the total kinetic energy of this system remains constant.

2. GEOMETRIC MODEL OF THE SYSTEM OF N PARTICLES

A typical configuration space for the system of one particle in classical mechanics is the Euclidean space $\mathbb{E}_3$ or, in general, some 3-dimensional smooth manifold $\mathbb{X}$. A suitable geometric structure for modelling motions of the particle is the fibered manifold $\pi : \mathbb{R} \times \mathbb{X} \rightarrow \mathbb{R}$ and its first jet prolongation $\pi_1 : J^1(\mathbb{R} \times \mathbb{X}) = \mathbb{R} \times T\mathbb{X} \rightarrow \mathbb{R}$, $T\mathbb{X}$ is the tangent space to $\mathbb{X}$. The spaces $\mathbb{R} \times \mathbb{X}$, $T\mathbb{X}$, $\mathbb{R} \times T\mathbb{X}$ will be called *space of events*, *phase space* and *evolution space* for the system of one particle, respectively. Let us denote by $(U_\kappa, \varphi_\kappa)$ some local chart on $\mathbb{X}$, $U_\kappa \subset \mathbb{X}$ is an open subset and $\varphi_\kappa = (q^1, q^2, q^3)$ are called generalized curvilinear local coordinates on $\mathbb{X}$. There naturally arise the following local coordinates: (t, q^s) on $\mathbb{R} \times \mathbb{X}$, $(q^s, \dot{q}^s)$ on $T\mathbb{X}$, and $(t, q^s, \dot{q}^s)$ on $\mathbb{R} \times T\mathbb{X}$, $1 \leq s \leq 3$; t is a global coordinate on $\mathbb{R}$ (time axis).

Let us consider a system of N particles. The set of all possible configurations of this system, configuration space $\mathbb{Y}$, will be treated in “separable” form of the Cartesian product $\mathbb{Y} = \mathbb{X} \times \mathbb{X} \times \dots \times \mathbb{X} = \mathbb{X}^N$, where each $\mathbb{X}$ represents the configuration space of a single particle, $\dim \mathbb{X} = 3$. We use the notation ${}^i\mathbb{X}$ for the configuration space of the i -th particle. There arise naturally the projections $\text{pr}_i : \mathbb{Y} \rightarrow {}^i\mathbb{X}$, $\text{pr}_i(Y) = {}^iX$, Y is instantaneous configuration of the system and ${}^iX$ is the corresponding position of the i -th particle. The configuration space $\mathbb{Y} = \mathbb{X}^N$ has the structure of the smooth manifold given by the atlas $At \mathbb{Y} = \{(U_{\kappa_1 \dots \kappa_N}, \varphi_{\kappa_1 \dots \kappa_N})\} = \{({}^1U_{\kappa_1} \times \dots \times {}^N U_{\kappa_N}, {}^1\varphi_{\kappa_1} \times \dots \times {}^N \varphi_{\kappa_N})\}$, where $({}^iU_{\kappa_i}, {}^i\varphi_{\kappa_i})$, ${}^i\varphi_{\kappa_i} = ({}^iq^1, {}^iq^2, {}^iq^3)$ is some coordinate system on ${}^i\mathbb{X}$ for each i , $1 \leq i \leq N$.

Thus, the instantaneous position Y of the system is described by the $3N$ -tuple $(q^1(Y), q^2(Y), \dots, q^{3N}(Y))$, such that every triple $(q^{3i-2}(Y), q^{3i-1}(Y), q^{3i}(Y)) = ({}^iq^1({}^iX), {}^iq^2({}^iX), {}^iq^3({}^iX))$ relates to the position ${}^iX$ of the i -th particle.

Let $I \subset \mathbb{R}$ be such a time interval during which there is no collision of particles. By a regular *motion* of the system of N particles (motion without collisions) in time interval I we mean the twice differentiable mapping $c : I \ni t \mapsto c(t) \in \mathbb{X}^N$. Hence, c is the curve on $\mathbb{X}^N$. Especially, $\text{pr}_i \circ c = {}^ic : I \rightarrow {}^i\mathbb{X}$ represents the regular motion of the i -th particle, it is a curve in ${}^i\mathbb{X}$.

The space $\mathbb{R} \times \mathbb{X}^N$ and $\mathbb{R} \times {}^i\mathbb{X}$ is called the *space of events* of the system of N particles and the *space of events* of the i -th particle, respectively. The spaces of events

together with the projections $\text{pr}_0 = \pi : \mathbb{R} \times \mathbb{X}^N \rightarrow \mathbb{R}$, ${}^i\text{pr}_0 = {}^i\pi : \mathbb{R} \times {}^i\mathbb{X} \rightarrow \mathbb{R}$ have the structure of the fibered manifolds over 1-dimensional base $\mathbb{R}$; it is a basic underlying structure for modeling motions of particles without collisions. Fibers $\pi^{-1}(t)$ and ${}^i\pi^{-1}(t)$ at any time t are just the configuration spaces $\mathbb{X}^N$ and ${}^i\mathbb{X}$.

Let $Y \in \mathbb{X}^N$, $Y = ({}^1X, \dots, {}^NX)$ be some configuration of the mechanical system of N particles. The tangent vector space $T_Y\mathbb{X}^N$ to the configuration space $\mathbb{X}^N$ at the configuration Y and the tangent vector spaces $T_{{}^iX}{}^i\mathbb{X}$ to the configuration spaces ${}^i\mathbb{X}$ of single particles at the positions ${}^iX$ have the meaning of the *space of velocities* of the system of N particles at the configuration Y and *spaces of velocities* of single particles at the corresponding positions ${}^iX$. Since $T_Y\mathbb{X}^N = T_{{}^1X}{}^1\mathbb{X} \times \dots \times T_{{}^NX}{}^N\mathbb{X}$, every velocity vector $\mathbf{v} \in T_Y\mathbb{X}^N$ consists of the velocity vectors ${}^i\mathbf{v}$ of the single particles, i.e. $\mathbf{v} = ({}^1\mathbf{v}, \dots, {}^N\mathbf{v})$, ${}^i\mathbf{v} \in T_{{}^iX}{}^i\mathbb{X}$. In some coordinate system $(U, \varphi) \in \text{At } \mathbb{Y}$, $U = U_{\kappa_1 \dots \kappa_N} \subset \mathbb{Y}$, $\varphi = \varphi_{\kappa_1 \dots \kappa_N} = ({}^iq^s)$, $1 \leq i \leq N$, $1 \leq s \leq 3$, $\mathbf{v} = {}^iv^s(\partial/\partial {}^iq^s)_Y = {}^i\dot{q}^s(\mathbf{v})(\partial/\partial {}^iq^s)_Y$, where ${}^iv^s = {}^i\dot{q}^s(\mathbf{v})$ are components of $\mathbf{v} = ({}^i\mathbf{v}) = ({}^1\mathbf{v}, \dots, {}^N\mathbf{v})$, and where every triple $({}^iv^1, {}^iv^2, {}^iv^3) = ({}^i\dot{q}^1, {}^i\dot{q}^2, {}^i\dot{q}^3)$ represents components of the velocity vector ${}^i\mathbf{v}$ of the i -th particle with respect to the basis $\{(\partial/\partial {}^iq^s)_{X_i}\}$ of the space $T_{{}^iX}{}^i\mathbb{X}$.

Put $T\mathbb{X}^N = \cup_{Y \in \mathbb{X}^N} T_Y\mathbb{X}^N$ and $TU = \cup_{Y \in U} T_Y\mathbb{X}^N$. Denote $\tau : T\mathbb{X}^N \rightarrow \mathbb{X}^N$ a surjective mapping defined as $\mathbf{v} \in T\mathbb{X}^N \mapsto \tau(\mathbf{v}) = Y$ such that $\mathbf{v} \in T_Y\mathbb{X}^N$. Quite analogously, for the i -th particle, $\tau_i : T^i\mathbb{X} \rightarrow {}^i\mathbb{X}$, $\tau_i({}^i\mathbf{v}) = {}^iX$, ${}^i\mathbf{v} \in T_{{}^iX}{}^i\mathbb{X}$. The set $T\mathbb{X}^N$ has the structure of a smooth manifold of dimension $6N$ such that, for every coordinate system (U, φ) , $\varphi = ({}^iq^s)$, on $\mathbb{X}^N$ is $(TU, T\varphi)$, $T\varphi = ({}^iq^s, {}^i\dot{q}^s)$, coordinate system on $T\mathbb{X}^N$. The manifold $T\mathbb{X}^N$ is called *tangent fibered space* of the configuration manifold $\mathbb{X}^N$ and plays the role of the *phase space*, space of all possible configurations and of all possible velocities of the mechanical system of N particles. Similarly, the manifolds $T^i\mathbb{X}$ have the meaning of phase spaces for single particles.

The first jet prolongation $J^1(\mathbb{R} \times \mathbb{X}^N)$ of the space of events $\mathbb{R} \times \mathbb{X}^N$ is the manifold $\mathbb{R} \times T\mathbb{X}^N$ called *evolution space* of the system of N particles. Associated coordinates on $\mathbb{R} \times T\mathbb{X}^N$ are denoted by $(t, {}^iq^s, {}^i\dot{q}^s)$. There naturally arise the projections $\pi_1 : \mathbb{R} \times T\mathbb{X}^N \rightarrow \mathbb{R}$, and $\pi_{1,0} : \mathbb{R} \times T\mathbb{X}^N \rightarrow \mathbb{R} \times \mathbb{X}^N$, $\pi_{1,0} = \text{id}_{\mathbb{R}} \times \tau$. The evolution space and the corresponding projections can be also distinguished for the single particles, then, upright index i is used.

3. GEOMETRIC APPROACH TO THE KINETIC ENERGY

Suppose that there exists on each factor $\mathbb{X}$ a Riemannian metric tensor G , a symmetric bilinear non-degenerate form $G_X : T_X\mathbb{X} \times T_X\mathbb{X} \rightarrow \mathbb{R}$, which is at each point $X \in \mathbb{X}$ positive definite, i.e., the inner product between tangent vectors. We use the notation ${}^iG$ for the specification of the metric tensor G on the factor ${}^i\mathbb{X}$. Coordinate representation of the metric tensor ${}^iG$ in the chart $(T^iU_{\kappa_i}, T^i\varphi_{\kappa_i})$ on $T^i\mathbb{X}$ associated with the chart $({}^iU_{\kappa_i}, {}^i\varphi_{\kappa_i})$, ${}^i\varphi_{\kappa_i} = ({}^iq^s)$ on $\mathbb{X}$ is ${}^iG_{X_i}({}^i\mathbf{u}, {}^i\mathbf{v}) =$

${}^i G_{kl} {}^i \dot{q}^k ({}^i \mathbf{u}) {}^i \dot{q}^l ({}^i \mathbf{v}) = {}^i G_{kl} {}^i \dot{q}^k {}^i \dot{q}^l$, ${}^i \mathbf{u}, {}^i \mathbf{v} \in T_{X_i} {}^i \mathbb{X}$. The associated quadratic form Q_{iG} will be denoted by ${}^i Q$. By Sylvester's law of inertia, there exists a basis of the tangent space $T_{iX} {}^i \mathbb{X}$ and a local chart $({}^i U_0, {}^i \varphi_0) \in At {}^i \mathbb{X}$, ${}^i X \in {}^i U_0$, ${}^i \varphi_0 = ({}^i w^s)$ so that the quadratic form ${}^i Q$ becomes diagonal

$${}^i Q = \delta_{kl} {}^i \dot{w}^k {}^i \dot{w}^l = ({}^i \dot{w}^1)^2 + ({}^i \dot{w}^2)^2 + ({}^i \dot{w}^3)^2. \quad (3.1)$$

On the configuration space $\mathbb{X}^N$, there naturally arises a metric tensor $\mathcal{G} = \sum_{i=1}^N {}^i G$ and the associated quadratic form $\mathcal{Q}_{\mathcal{G}} = \sum_{i=1}^N {}^i Q$, more precisely,

$$\mathcal{G}(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^N {}^i G({}^i \mathbf{u}, {}^i \mathbf{v}), \quad \mathcal{Q}_{\mathcal{G}}(\mathbf{v}, \mathbf{v}) = \sum_{i=1}^N {}^i Q({}^i \mathbf{v}, {}^i \mathbf{v}).$$

Another metric tensor and corresponding quadratic form on $\mathbb{X}^N$ directly associated with the system of N particles with masses $m_1, \dots, m_N$ are

$$\bar{\mathcal{G}} = \frac{\sum_{i=1}^N m_i {}^i G}{\sum_{i=1}^N m_i}, \quad \bar{\mathcal{Q}}_{\bar{\mathcal{G}}} = \frac{\sum_{i=1}^N m_i {}^i Q}{\sum_{i=1}^N m_i},$$

the so-called *weighted metric tensor* and *weighted quadratic form* associated with the system of N particles considered.

Let us denote by $M = \sum_{i=1}^N m_i$ the total mass of the system. The total kinetic energy of the system of N particles with masses $m_1, \dots, m_N$ can be interpreted by means of

$$T = \frac{1}{2} M \bar{\mathcal{Q}}_{\bar{\mathcal{G}}}(\mathbf{v}) = \frac{1}{2} M \bar{\mathcal{G}}(\mathbf{v}, \mathbf{v}) = \frac{1}{2} \sum_{i=1}^N m_i {}^i Q({}^i \mathbf{v}) = \frac{1}{2} \sum_{i=1}^N m_i {}^i G({}^i \mathbf{v}, {}^i \mathbf{v}), \quad (3.2)$$

or in some coordinates on $T\mathbb{X}^N$

$$T = \sum_{i=1}^N \frac{1}{2} m_i {}^i G_{kl} {}^i \dot{q}^k {}^i \dot{q}^l. \quad (3.3)$$

Quite analogously, one can introduce the total kinetic energy $T^{m_{j_1}, \dots, m_{j_K}}$ of any subsystem of K particles $m_{j_1}, \dots, m_{j_K}$ of the system $m_1, \dots, m_N$ by means of the corresponding weighted quadratic form associated with the subsystem considered.

4. GEOMETRIC CONCEPT OF ISOKINETIC CONSTRAINTS

4.1. Nonholonomic constraints

By constraints we mean conditions in the form of equations which restrict the possible configurations and velocities of mechanical systems of N particles. By nonholonomic constraints we mean a system of non-integrable differential equations defined on the evolution space $\mathbb{R} \times T\mathbb{X}^N$.

From the geometric point of view, p nonholonomic constraints generate certain submanifold, called *constraint submanifold* $\mathbb{H} \subset \mathbb{R} \times T\mathbb{X}^N$, $\dim \mathbb{H} = 6N + 1 - p$, which is fibered over $\mathbb{R} \times \mathbb{X}^N$, and also over $\mathbb{R}$, i.e., there arises the fibered manifold $\bar{\pi}_{1,0} : \mathbb{H} \rightarrow \mathbb{R} \times \mathbb{X}^N$, where $\bar{\pi}_{1,0}$ is the restriction of the projection $\pi_{1,0} : \mathbb{R} \times T\mathbb{X}^N \rightarrow \mathbb{R} \times \mathbb{X}^N$ onto $\mathbb{H}$, and the fibered manifold $\bar{\pi}_1 : \mathbb{H} \rightarrow \mathbb{R}$ where $\bar{\pi}_1 = \pi_1|_{\mathbb{H}}$. We denote by $\iota : \mathbb{H} \rightarrow \mathbb{R} \times T\mathbb{X}^N$ the canonical embedding. Especially, constraint submanifold of the form $\mathbb{H} = \mathbb{R} \times \mathbb{H}_0$, where $\mathbb{H}_0 \subset T\mathbb{X}^N$ represents *time independent* nonholonomic constraints.

Let $\mathbb{H}$ be the constraint submanifold $\mathbb{H} \subset \mathbb{R} \times T\mathbb{X}^N$. At each point $H \in \mathbb{H}$, there exists a local chart (V, χ) on $\mathbb{R} \times T\mathbb{X}^N$, $V = I \times TU$, $\chi = \text{id}_I \times T\varphi$, where $I \subset \mathbb{R}$ is a time interval and $(TU, T\varphi)$ is some coordinate system on $T\mathbb{X}^N$, $T\varphi = ({}^i q^s, {}^i \dot{q}^s)$, $1 \leq i \leq N$, $1 \leq s \leq 3$, such that on $\mathbb{H}_V = \mathbb{H} \cap V$ the submanifold $\mathbb{H}$ is locally expressed by the equations

$$f^a(t, {}^i q^s, {}^i \dot{q}^s) = 0, \quad 1 \leq a \leq p, \quad (4.1)$$

where $\text{rank}\left(\frac{\partial f^a}{\partial {}^i \dot{q}^s}\right) = p$.

The equations can be locally solved with respect to some selected p coordinates ${}^i \dot{q}^s$. Assume that $p < N$, then, without loss of generality, one can assort some p particles $\{m_{i_a}\}$, $i_a \in \{1, 2, \dots, N\}$, $1 \leq a \leq p$, and, for each particle m_{i_a} , one can choose some s_{i_a} component of its velocity ${}^{i_a} \mathbf{v}$, $s_{i_a} \in \{1, 2, 3\}$. Hence, we obtain equations of constraints in normal (explicit) form

$${}^{i_a} \dot{q}^{s_{i_a}} = g^a(t, {}^i q^s, {}^j \dot{q}^s, {}^{i_a} \bar{\dot{q}}^{\bar{s}_{i_a}}), \quad (4.2)$$

with $1 \leq j \leq N$, $j \neq i_a$, $1 \leq \bar{s}_{i_a} \leq 3$, $\bar{s}_{i_a} \neq s_{i_a}$, where ${}^{i_a} \bar{\dot{q}}^{s_{i_a}}$ are expressed as functions g^a of remaining coordinates.

4.2. Isokinetic constraint submanifold

Let us consider a situation such that the movement of the system of N particles is restricted by the a priori condition that the total kinetic energy T of the system remains constant and it equals to nonzero initial total kinetic energy T_0 of the system given by initial positions and initial velocities of single particles.

In some local chart (V, χ) on $\mathbb{R} \times T\mathbb{X}^N$, $V = I \times TU$, $I \subset \mathbb{R}$ a time interval, $\chi = \text{id}_I \times T\varphi$, where $(TU, T\varphi)$ is some coordinate system on $T\mathbb{X}^N$, $T\varphi = ({}^i q^s, {}^i \dot{q}^s)$, $1 \leq i \leq N$, $1 \leq s \leq 3$, the requirement for constant total kinetic energy of the system has the form

$$T^{m_1, \dots, m_N} = T = \sum_{i=1}^N \frac{1}{2} m_i {}^i G_{kl} ({}^i q^1, {}^i q^2, {}^i q^3) {}^i \dot{q}^k {}^i \dot{q}^l = T_0 = \text{const}. \quad (4.3)$$

This condition represents one nonholonomic constraint and generates a constraint submanifold $\mathbb{T} \subset \mathbb{R} \times T\mathbb{X}^N$ of the dimension $6N$, called *isokinetic constraint submanifold*. Since $\mathbb{T} = \mathbb{R} \times \mathbb{T}_0$, where $\mathbb{T}_0 \subset T\mathbb{X}^N$, it is time-independent nonholonomic constraint. Equation (4.3) is then local expression of the submanifold $\mathbb{T}_V = \mathbb{T} \cap V$ or $\mathbb{T}_{0U} = \mathbb{T}_0 \cap TU$. The rank condition (4.1) is evidently satisfied,

$$\text{rank} \left(\frac{\partial f}{\partial \dot{q}^l} \right) = \text{rank} (m_i{}^i G_{kl}{}^i \dot{q}^k) = 1, \quad (4.4)$$

since the metric tensors ${}^i G$ are positive and the total kinetic energy remains constant and nonzero, thus, the system cannot stop.

In order to obtain the normal form of the equation of isokinetic constraint submanifold $\mathbb{T}$ we will suppose that there exist some time intervals during which some particle is located just in such a coordinate chart on $\mathbb{X}$ in which the corresponding particular quadratic form Q has the diagonal form (3.1). Without loss of generality, let us assume that, for instance, in time interval I_J , J -th particle is located in coordinate chart $({}^J U_0, {}^J \varphi_0)$, ${}^J \varphi_0 = ({}^i w^s)$ on ${}^i \mathbb{X}$. Denote by (W, ψ) the chart on $\mathbb{R} \times T\mathbb{X}^N$, $W = I_J \times T\bar{U}$, $\psi = \text{id}_{I_J} \times T\bar{\varphi}$, where $(T\bar{U}, T\bar{\varphi})$ is such a coordinate system on $T\mathbb{X}^N$ that its J -th component is just the chart $({}^J U_0, {}^J \varphi_0)$. Hence, the segment $\mathbb{T}_W = \mathbb{T} \cap W$ of the isokinetic submanifold $\mathbb{T}$ is locally expressed by the equation

$$\sum_{i=1, i \neq J}^N \frac{1}{2} m_i{}^i G_{kl}{}^i \dot{q}^k{}^i \dot{q}^l + \frac{1}{2} m_J \left(({}^J \dot{w}^1)^2 + ({}^J \dot{w}^2)^2 + ({}^J \dot{w}^3)^2 \right) = T_0. \quad (4.5)$$

Finally, for instance, ${}^J \dot{w}^3$ can be expressed explicitly

$$\begin{aligned} {}^J \dot{w}^3 &= g_{\pm} \\ &= \pm \frac{1}{\sqrt{m_J}} \sqrt{2T_0 - \sum_{\substack{i=1 \\ i \neq J}}^N m_i{}^i G_{kl}{}^i \dot{q}^k{}^i \dot{q}^l - m_J ({}^J \dot{w}^1)^2 - m_J ({}^J \dot{w}^2)^2}, \end{aligned} \quad (4.6)$$

thus the local isokinetic constraint $\mathbb{T}_W$ splits into two components $\mathbb{T}_W^+$ (${}^J \dot{w}^3 > 0$) and $\mathbb{T}_W^-$ (${}^J \dot{w}^3 < 0$).

4.3. Isokinetic canonical distribution

A fundamental object in the geometry of nonholonomic constraints is the *canonical distribution* discovered in [4], naturally associated with the constraint submanifold $\mathbb{H}$. It is a vector distribution $\mathcal{C} \subset T\mathbb{H}$ of the dimension $6N - 2p + 1$ on the constraint submanifold $\mathbb{H}$ related to p nonholonomic constraints (4.1). This distribution has the meaning of generalized possible displacements of mechanical systems constrained to $\mathbb{H}$. Particularly, its π_1 -vertical subdistribution represents the space of

$\mathbb{H}$ -admissible virtual displacements and its $\pi_{1,0}$ -vertical subdistribution represents the space of $\mathbb{H}$ -admissible virtual velocities. The generators of the canonical distribution $\mathcal{C}$ for the case of general constraint submanifold are calculated in [4]. These formulas contain the equations of constraints in the normal form (4.2). The canonical distribution associated with the isokinetic constraint (4.3) is called *isokinetic canonical distribution* and denoted by $\mathcal{C}_{\mathbb{T}}$, evidently $\mathcal{C}_{\mathbb{T}} \subset T\mathbb{T}$.

If we take into account equation (4.6) of local isokinetic submanifold $\mathbb{T}_W$ or more concretely, its “positive” component $\mathbb{T}_W^+$ (${}^J\dot{w}^3 = g_+$), we obtain the following vector generators of the local isokinetic canonical distribution $\mathcal{C}_{\mathbb{T}_W^+}$

$$\begin{aligned}\xi_0 &= \frac{\partial}{\partial t} + \frac{2T_0}{m_J g_+} \frac{\partial}{\partial {}^J w^3}, \\ {}^i \xi_r &= \frac{\partial}{\partial {}^i q^r} - \frac{m_i}{m_J g_+} {}^i G_{rs} {}^i \dot{q}^s \frac{\partial}{\partial {}^J w^3}, \\ {}^J \xi_1 &= \frac{\partial}{\partial {}^J w^1} - \frac{{}^J \dot{w}^1}{g_+} \frac{\partial}{\partial {}^J w^3}, \\ {}^J \xi_2 &= \frac{\partial}{\partial {}^J w^2} - \frac{{}^J \dot{w}^2}{g_+} \frac{\partial}{\partial {}^J w^3}, \\ {}^i \tilde{\xi}_r &= \frac{\partial}{\partial {}^i \dot{q}^r}, \\ {}^J \tilde{\xi}_1 &= \frac{\partial}{\partial {}^J \dot{w}^1}, \\ {}^J \tilde{\xi}_2 &= \frac{\partial}{\partial {}^J \dot{w}^2},\end{aligned}$$

where $1 \leq i \leq N$, $i \neq J$, $1 \leq r, s \leq 3$. Note that the rank of every local isokinetic canonical distribution $\mathcal{C}_{\mathbb{T}_W}$ is equal to $6N - 1$. The meaning of generators is the following: ξ_0 represents isokinetically admissible time translation of the system, ${}^i \xi_r$ have the meaning of components of isokinetically admissible virtual displacements of single particles (except the J th particle), ${}^J \xi_1, {}^J \xi_2$ are the components of isokinetically admissible virtual displacements of the J th particle, ${}^i \tilde{\xi}_r$ play the role of the components of isokinetically admissible virtual velocities of single particles (except the J th particle) and, finally, ${}^J \tilde{\xi}_1, {}^J \tilde{\xi}_2$ represent the components of isokinetically admissible virtual velocities of the J th particle.

All the above considerations and calculations can also be applied in the case of the so-called *partial isokinetic constraint*, when the a priori condition keeps constant the total kinetic energy $T^{m_{j_1}, \dots, m_{j_K}} = T_0^{m_{j_1}, \dots, m_{j_K}}$ of any selected subsystem of K -particles $m_{j_1}, \dots, m_{j_K}$.

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HIGHER ORDER CONNECTIONS ON LIE GROUPOID: APPLICATION TO MATERIALS

PETR VAŠÍK

Abstract. We present a possibility of material representation by a higher order connection on a Lie groupoid instead of a representation by principal connections on a principal bundle. We also prove some interesting properties of higher order connections on a Lie groupoid induced by principal connections with respect to the material setting.

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1. INTRODUCTION

We present a method of deciding whether the constitutional equations of certain material are equivalent. We deal with a material whose representation is given by different isomorphic principal bundles with conjugate structure groups and by means of a Lie groupoid associated to these settings we compare the connections induced by constitutive equations on these principal bundles. We use the notion of semi-holonomy of a higher order connection on a Lie groupoid, which was studied in [5]. We also compare the results with those obtained in [3] for connections on general bundles.

2. LIE GROUPOIDS

We start with a definition of a general groupoid, see [2].

Definition 1. A groupoid is defined as a pair (Φ, B) of a total set Φ and a base set B endowed with two submersions

$$\alpha : \Phi \rightarrow B \text{ and } \beta : \Phi \rightarrow B$$

called the source and target maps, respectively, and a binary operation defined for those ordered pairs $(y, z) \in \Phi \times \Phi$ such that $\alpha(z) = \beta(y)$. This operation must satisfy the following properties:

(1) Associativity:

$$(xy)z = x(yz),$$

whenever the products are defined;

- (2) Existence of identities: for each $b \in B$ there exists an element $\text{id}_b \in \Phi$, called the identity at b , such that $z \text{id}_b = z$ whenever $\alpha(z) = b$, and $\text{id}_b z = z$ whenever $\beta(z) = b$;
- (3) Existence of inverse: for each $z \in \Phi$ there exists a (unique) inverse z^{-1} such that

$$zz^{-1} = \text{id}_{\beta(z)} \text{ and } z^{-1}z = \text{id}_{\alpha(z)}.$$

Definition 2. A groupoid (Φ, B) is said to be transitive if for each pair of points $a, b \in B$ there exists at least one element $z \in \Phi$ such that

$$\alpha(z) = a \text{ and } \beta(z) = b.$$

Definition 3. A groupoid (Φ, B) is a Lie groupoid if the total set Φ and the base set B are differentiable manifolds, the projections α, β are smooth and so are the operations of composition and of inverse.

Remark 1. Note that any transitive Lie groupoid induces naturally a set of isomorphic principal bundles with mutually conjugate structure groups, more precisely for every Lie groupoid Φ and every $x \in B$

$$\Phi_x := \{\theta \in \Phi \mid \alpha(\theta) = x\}$$

is a principal bundle, whose structure group G_x is the isotropy group of Φ over x . On the other hand, any principal bundle $\pi : P \rightarrow B$ with structure group G induces a Lie groupoid $PP^{-1} := (P \times P)/G$, where the equivalence class is given by $pq^{-1} := (p, q) \sim (pg, qg)$ for $p, g \in P$ and $g \in G$, see e. g. [2, 4] for further details. Let us just note that the projections in this case are then defined by

$$\alpha(pq^{-1}) = \pi(p) \text{ and } \beta(pq^{-1}) = \pi(q).$$

3. CONNECTIONS ON LIE GROUPOIDS

First, following the paper [5], let us identify the jet prolongation $J^r(B \times \Phi)$ of a fibered manifold $B \times \Phi \rightarrow B$ with the set $J^r(B, \Phi)$ of r -jets of mappings $B \rightarrow \Phi$. On the nonholonomic jet prolongation $\tilde{J}^r(B \times \Phi)$ for integers $r \geq q \geq 0$ we denote by π_q^r the target surjection $\pi_q^r : \tilde{J}^r(B, \Phi) \rightarrow \tilde{J}^q(B, \Phi)$ with π_r^r being the identity on $\tilde{J}^r(B, \Phi)$. Together with π_q^r we have also the surjections $J^k \pi_{q-k}^{r-k} : \tilde{J}^r(B, \Phi) \rightarrow \tilde{J}^q(B, \Phi)$. Then the following holds, [5].

Lemma 1. *The element $X \in \tilde{J}^r(B, \Phi)$ is semiholonomic if and only if*

$$(J^k \pi_{q-k}^{r-k})(X) = \pi_q^r(X) \text{ for any integers } 1 \leq k \leq q \leq r. \quad (1)$$

The space of semiholonomic r -jets will be denoted by $\tilde{J}^r(B, \Phi)$.

The notion of a higher order connection on a Lie groupoid was established by C. Ehresmann in [1]. Let $\sim : B \rightarrow \Phi$ denote an inclusion of the manifold units into the groupoid and let us consider the projections π_k^r as above. We use the notation of [5].

Definition 4. A nonholonomic, semiholonomic or holonomic connection of order $r \geq 1$ on Φ is a smooth map

$$\Gamma : B \rightarrow \tilde{J}^r(B, \Phi), \quad B \rightarrow \bar{J}^r(B, \Phi) \text{ or } B \rightarrow J^r(B, \Phi),$$

respectively, satisfying

$$\pi_0^r \Gamma = \sim, \quad (j^r \alpha) \Gamma(x) = j_x^r(u \rightarrow u), \quad (j^r \beta) \Gamma(x) = j_x^r(u \rightarrow x)$$

for all $u, x \in B$.

Remark 2. According to [6], let us consider the set

$$\begin{aligned} \tilde{Q}^r(\Phi) = \{X \in \tilde{J}^r(B, \Phi) \mid \pi_0^r X = \sim(x), \\ (j^r \alpha)X = j_x^r(u \rightarrow u), (j^r \beta)X = j_x^r(u \rightarrow x), (\alpha(X) = x)\}, \end{aligned}$$

where α is the source map. Then $\alpha : \tilde{Q}^r(\Phi) \rightarrow B$ is a fibered manifold and the r th order connections are the sections $B \rightarrow \tilde{Q}^r(\Phi)$.

It is well known that for $r = 1$ this corresponds to the standard notion of a connection on any of the principal bundles determined by Φ . Recall that in the language of jet prolongations, a principal connection on a principal bundle is defined as follows. Let us consider a principal bundle (P, p, B, G) , where $p : P \rightarrow B$ is a fibered manifold, G is a Lie group and by r we denote the principal right action $r : P \times G \rightarrow P$ and write $r^g = r(-, g) : P \rightarrow P$ for $g \in G$. We also denote by r the canonical right action $r : J^1 P \times G \rightarrow J^1 P$ given by $r^g(j_x^1 s) = j_x^1(r^g \circ s)$ for all $g \in G$ and $j_x^1 s \in J^1 P$. A principal connection Γ on a principal fiber bundle P with a principal action r is an r -equivariant section $\Gamma : P \rightarrow J^1 P$ of the first jet prolongation $J^1 P \rightarrow P$.

The above definition together with the constructions mentioned in Remark 1 proves the following claim.

Proposition 1. *A principal connection on a principal bundle $P \rightarrow B$ induces naturally a first order connection on the Lie groupoid PP^{-1} and any first order connection on a Lie groupoid Φ induces a principal connection on the principal bundle Φ_x for any $x \in B$.*

The following concept of a construction of higher order connection can be found in [5]. Let now $\xi = \xi^{(1)} : x \rightarrow j_x^1 \xi_x$ be a first order connection on Φ and define for each integer $r \geq 1$ the map

$$\xi^{(r)} : B \rightarrow \tilde{J}^r(B, \Phi), \quad x \rightarrow j_x^1(u \rightarrow j_u^{r-1}[\xi_x(u)]).$$

Then for any r th order connection Γ and first order connection ξ on Φ the map

$$\Gamma * \xi : B \rightarrow \tilde{J}^{r+1}(B, \Phi), \quad x \rightarrow j_x^1 \Gamma \cdot \xi^{(r+1)}(x)$$

is well defined connection on Φ of order $r + 1$. Note that $\Gamma' = \Gamma * \pi_1^r \Gamma$ is called the prolongation of Γ . Furthermore, given r first order connections $\xi_1, \xi_2, \dots, \xi_r$ on Φ ,

we can define recurrently the r th order connection on Φ as a composition $\xi_1 * \dots * \xi_r$. On the other hand, given an r th order connection Γ on Φ , we can define r first order connections

$$\xi_s = \xi_s(\Gamma) : B \rightarrow J^1(B, \Phi) : x \rightarrow (j_1 \pi_0^{(s-1)}) \pi_s^r \Gamma(x)$$

for $s = 1, \dots, r$.

Using this notation, let us mention the classification property of higher order connections on a Lie groupoid Φ , [5].

Theorem 1. *If Γ is a semiholonomic connection on Φ , then all $\xi_s(\Gamma)$ are equal, i.e. $\Gamma = \xi * \dots * \xi$. Moreover, a connection $\xi * \dots * \xi$ is holonomic if and only if ξ is curvature free.*

To recall some further properties of higher order connections on a Lie groupoid Φ we have to mention the following notions, [5]. Let us denote by $G = G(\Phi)$ the isotropy group bundle and by $L = L(\Phi)$ the isotropy Lie algebra bundle attached to Φ , i. e.

$$G_x = \{\theta \in \Phi \mid \alpha(\theta) = \beta(\theta) = x\} \text{ and } L_x = T_{\tilde{x}}(G_x),$$

where $\tilde{x} \in \Phi$ is an image of $x \in B$ under the inclusion $\sim$. Then the following holds, [5].

Theorem 2. *Every second order connection Γ on Φ is uniquely determined by two first order connections $\xi_1(\Gamma)$, $\xi_2(\Gamma)$ and a linear map*

$$A(\Gamma) : TB \otimes TB \rightarrow L(\Phi).$$

Now if Γ and $\bar{\Gamma}$ are two r th order connections on Φ , we can consider the composition $\bar{\Gamma} \cdot \Gamma^{-1} : x \rightarrow \bar{\Gamma}(x) \cdot \Gamma^{-1}(x)$. To generalize the linear map $A(\Gamma)$ from the previous theorem we put

$$A(\Gamma) = \Gamma \cdot [\xi_1(\Gamma) * \dots * \xi_r(\Gamma)]^{-1}.$$

Then the following holds, [5].

Theorem 3. *Let Γ be an r th order connection on Φ . Then Γ is uniquely determined by $\xi_1(\Gamma), \dots, \xi_r(\Gamma)$ and $A(\Gamma)$. Moreover, Γ is semiholonomic if and only if all $\xi_s(\Gamma)$ are equal, and $A(\Gamma)$ is semiholonomic.*

Finally, let us recall that two r th order connections $\Gamma, \bar{\Gamma}$ on Φ are said to be equivalent in the q -th order ($1 \leq q < r$) if

$$\pi_q^{r,C} \Gamma = \pi_q^{r,C} \bar{\Gamma}$$

for all decreasing sequences $C = \{r \geq c_1 > \dots > c_{r-q} \geq 1\}$, where

$$\pi_q^{r,c_i} \Gamma = (j^{c_i} \pi_{q-c_i}^{r-c_i}) \Gamma.$$

Especially they are equivalent in the $(r-1)$ -st order if $\bar{\Gamma} \cdot \Gamma^{-1}$ is a section in

$$L(\Phi) \otimes (\otimes^r T^*B).$$

4. APPLICATIONS TO MATERIALS

In material sciences, the basic setting is often given in the form of a principal bundle, for the case of so called Cosserat media see [2], endowed with a principal connection obtained from a constitutive equation. Let us consider the following case.

Proposition 2. *Let a material admit different settings in the form of r mutually isomorphic principal bundles (P_i, B, G_i) with the same base manifold B and with conjugate structure groups G_i . Let each of the settings be endowed with a material connection Γ_i . The set of settings $\{(P_i, B, G_i), \Gamma_i, i = 1, \dots, r\}$ is equivalent to a setting given by a Lie groupoid $\Phi = P_k P_k^{-1}$ for some $k \in \{1, \dots, r\}$ and an r th order connection Γ on Φ . Moreover, if Γ is semiholonomic, then the connections Γ_i are generated by equivalent constitutive equations.*

Proof. According to Remark 1, each principal bundle (P_i, B, G_i) induces Lie groupoid $P_i P_i^{-1}$. Obviously, as the principal bundles are isomorphic with conjugate structure groups, Lie groupoids $P_i P_i^{-1}$ and $P_j P_j^{-1}$ are isomorphic for any $i, j \in \{1, \dots, r\}$. Furthermore, each of the connections Γ_i corresponds to a connection on the Lie groupoid $P_i P_i^{-1}$ and thus, up to a Lie groupoid isomorphism, they correspond to r first order connection Γ_i on a Lie groupoid $P_k P_k^{-1}$ for certain $k \in \{1, \dots, r\}$. Then

$$\Gamma = \Gamma_1 * \dots * \Gamma_r.$$

According to Theorem 1, if Γ is semiholonomic, then all generating connections Γ_i are equal and thus they were generated by equivalent constitutive equations. $\square$

Let us consider the case when the connection Γ is not semiholonomic, i. e. the generating connections are not all equal. Then we can consider $r!$ connections of order r obtained from r first order connections by changing the order in the expression $\Gamma_1 * \dots * \Gamma_r$. Let us denote such connections by $\bar{\Gamma}^i, i \in I = \{1, \dots, r!\}$. Then the following holds.

Proposition 3. *If $\bar{\Gamma}^i$ and $\bar{\Gamma}^j$ are equivalent in $(r-1)$ -st order for some couple $(i, j) \in I \times I, i \neq j$, then there exist at least 2 couples of equal generating connections, i. e. there are 2 couples of equivalent corresponding constitutive equations.*

Proof. If $\bar{\Gamma}^i$ and $\bar{\Gamma}^j$ are equivalent in $(r-1)$ -st order, then

$$(j^k \pi_{r-k-1}^{r-k}) \bar{\Gamma}^i = (j^k \pi_{r-k-1}^{r-k}) \bar{\Gamma}^j$$

for $k = 0, 1, \dots, r-1$. Particularly,

$$\pi_{r-1}^r \bar{\Gamma}^i = \pi_{r-1}^r \bar{\Gamma}^j,$$

i. e. for the first order generating connections it holds that $\Gamma_l^i = \Gamma_l^j$ for $l = 1, \dots, r-1$ and $\Gamma_r^i \neq \Gamma_r^j$. But this means from the construction of $\bar{\Gamma}^i$ and $\bar{\Gamma}^j$

that there exist connections Γ_m^i, Γ_n^j different from Γ_r^i, Γ_r^j such that the relations

$$\Gamma_r^i = \Gamma_m^i \text{ and } \Gamma_r^j = \Gamma_n^j$$

hold. □

By analogous yet more combinatoric considerations we obtain

Proposition 4. *If k connections $\bar{\Gamma}^{i_1}, \dots, \bar{\Gamma}^{i_k}, \{i_1, \dots, i_k\} \subset I$, are equivalent in $(r - 1)$ th order, then there exist at least k couples of equal generating connections, i. e. there are k couples of equivalent corresponding constitutive equations.*

Let us denote by $\langle \Gamma_1 * \dots * \Gamma_k \rangle$ the set of all permutations of k first order connections $\Gamma_1, \dots, \Gamma_k$. Then by excluding one of the equal connections and by iterating the process described in Proposition 4 we obtain the following assertion, which reduces the number of material connections to just those corresponding to non-equivalent constitutive equations.

Corollary 1. *Let us suppose that there is no pair of connections equivalent in $(l_m - 1)$ -st order among the connections $\langle \Gamma_{l_1} * \dots * \Gamma_{l_m} \rangle$, $l_1 < l_2 < \dots < l_m$, $l_i \in \{1, \dots, r\}$, $i = 1, \dots, m$, $m < r$, respectively, and for any $n = m + 1, \dots, r$ there exists a pair of connections equivalent in $(n - 1)$ th order. Then the appropriate material representation is given by a Lie groupoid and a nonholonomic connection of order l_m .*

Finally, we show an analogue and a generalization of a result proved in [3]. Indeed, in [3] we handled second order connections on fibered manifolds, while Theorem 3 gives us the possibility to prove similar result for r th order connections on a Lie groupoid Φ . Note that for $r > 2$ there is no similar identification of connections on fibered manifolds. First, one can define the relation on the space of r th order nonholonomic connections on Φ , for our purpose we identify such space with $(r + 1)$ -tuples $(\Gamma_1, \dots, \Gamma_r, A(\Gamma))$ as in Theorem 3. We say that the elements $(\Gamma_1, \dots, \Gamma_r, A(\Gamma)), (\bar{\Gamma}_1, \dots, \bar{\Gamma}_r, A(\bar{\Gamma}))$ are equivalent if and only if $\Gamma_1 = \bar{\Gamma}_1, \dots, \Gamma_r = \bar{\Gamma}_r$. It is easy to see that this is an equivalence relation and we denote by $[\theta] = [(\Gamma_1, \dots, \Gamma_r, A(\Gamma))]$ a class of this equivalence. Finally the class $[\theta]$ consists of semiholonomic connections if and only if $\Gamma_1 = \dots = \Gamma_r$ for any $(\Gamma_1, \dots, \Gamma_r, A(\Gamma)) \in [\theta]$. Furthermore, $A(\Gamma)$ is semiholonomic.

Proposition 5. *Let Φ be a Lie groupoid and $[\theta]$ a class of r th order connections. Then the constitutive equations on a principal bundle Φ_x corresponding to first order connections $\Gamma_1, \dots, \Gamma_r$ on Φ are in the same projective class if and only if $[\theta]$ is semiholonomic.*

Proof. If the element $[(\Gamma_1, \dots, \Gamma_r, A(\Gamma))]$ belongs to $[\theta]$, then from Theorem 3 the semiholonomity is equivalent to the property $\Gamma_1 = \dots = \Gamma_r$. In particular, r constitutive equations determine r projectively equivalent connections of the first order. □

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CYCLIC ČECH–HOCHSCHILD BICOMPLEX

JAN KUBARSKI

Abstract. In classical differential topology and geometry, the double Čech-de Rham complex for a good covering $\mathfrak{U}$ of M corresponds to the single de Rham complex. When passing to noncommutative geometry on smooth manifolds (i.e. to noncommutative geometry of the algebra $C^\infty(M)$ of smooth functions on M), the single Connes complex is the counterpart of the single de Rham complex. We can ask: which bicomplex corresponds to the single Connes complex in noncommutative geometry of smooth manifolds? We give an answer to this question defining the Cyclic Čech–Hochschild bicomplex for a good covering of a smooth manifold. We pose fundamental problems on the total homology of this bicomplex.

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1. ČECH – DE RHAM BICOMPLEX FOR A GOOD COVERING

In differential geometry and topology of smooth manifolds, a fundamental role is played by the de Rham complex $(\Omega^*(M), d)$, $\dim M = n$,

$$\Omega^0(M) \xrightarrow{d} \dots \xrightarrow{d^{q-1}} \Omega^q(M) \xrightarrow{d^q} \Omega^{q+1}(M) \xrightarrow{d^{q+1}} \dots \xrightarrow{d} \Omega^n(M) \rightarrow 0$$

of differential forms and the de Rham differential operator $d = d_{dR}$. The cohomology of this complex is the de Rham cohomology of the manifold M ,

$$\mathbf{H}_{dR}^q(M) = \ker d^q / \operatorname{Im} d^{q-1}.$$

The Čech – de Rham bicomplex for a good covering $\mathfrak{U}$ of M corresponds to the single de Rham complex,

$$(C^*(\mathfrak{U}, \Omega^*), \delta, \tilde{d})$$

(over the first quadrant) where $C(\mathfrak{U}, \Omega) = \{C^p(\mathfrak{U}, \Omega^q)\}_{p \geq 0, q \geq 0}$, $\delta = \delta_{\check{C}}$ is the Čech differential and $\tilde{d} = (-1)^p d$ on $C^p(\mathfrak{U}, \Omega^*)$, $\delta\tilde{d} + \tilde{d}\delta = 0$,

$$\begin{array}{ccccccc}
 & & \uparrow & & & & \\
 \longrightarrow & C^{p-1}(\mathfrak{U}, \Omega^{q+2}) & \longrightarrow & & & & \\
 & \uparrow \tilde{d} & & & \uparrow & & \\
 \longrightarrow & C^{p-1}(\mathfrak{U}, \Omega^{q+1}) & \xrightarrow{\delta} & C^p(\mathfrak{U}, \Omega^{q+1}) & \longrightarrow & & \\
 & \uparrow \tilde{d} & & \uparrow \tilde{d} & & \uparrow & \\
 \longrightarrow & C^{p-1}(\mathfrak{U}, \Omega^q) & \xrightarrow{\delta} & C^p(\mathfrak{U}, \Omega^q) & \xrightarrow{\delta} & C^{p+1}(\mathfrak{U}, \Omega^q) & \longrightarrow \\
 & \uparrow & & \uparrow & & \uparrow \tilde{d} &
 \end{array}$$

The total differential $D = \delta + \tilde{d} : C^{(r)} \rightarrow C^{(r+1)}$ is homogeneous of degree +1 with respect to the total gradation

$$C^{(r)} = \bigoplus_{p+q=r} C^p(\mathfrak{U}, \Omega^q)$$

along the lines $p + q = \text{const.}$

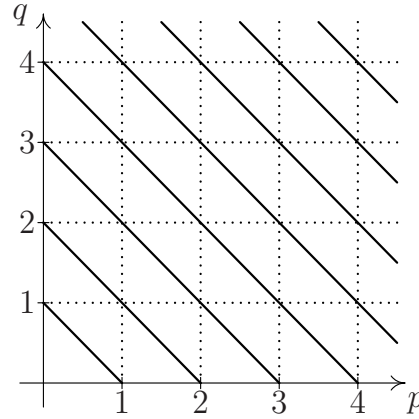


FIGURE 1. Total gradation along $p + q = \text{const.}$

In spite of the fact that rows and columns of the double Čech – de Rham bicomplex are exact, the total cohomology of this bicomplex is, on the one hand, isomorphic to the de Rham cohomology and on the other hand to the Čech cohomology:

$$\mathbf{H}_D^{(r)} = \mathbf{H}_{DR}^r(M) = \check{\mathbf{H}}^r(M).$$

Therefore, the characteristic classes of M (and more generally of any vector bundle or any G -principal fibre bundle) could be determined by cochain forms defined locally from the double Čech – de Rham bicomplex. See the results by Bott [1], Bott and Tu [2, pp. 304–305], and Sharygin [11].

The Bott, Bott and Tu, and Sharygin approach consists in producing local formulas in the Čech-de Rham bicomplex by using transition functions $g_{ij} : U_i \cap U_j \rightarrow G$ (forming a cocycle, i. e., $g_{ij}g_{jk} = g_{ik}$) with values in a suitable Lie group G . The transition functions g_{ij} are obtained from a system of local trivializations of a suitable principal fibre bundle.

All such formulas for characteristic classes obtained for chains defined for a good covering are called *local formulas for characteristic classes*.

And what is the corresponding object in noncommutative geometry? The counterpart of the de Rham complex is the *Connes complex* for an associative algebra A with a unit over the field $K = \mathbb{R}$ or $\mathbb{C}$, which lies at the foundations of noncommutative geometry [3–5].

2. SINGLE CONNES COMPLEX

Put crudely, the (abstract) *Connes complex* consists of cyclic Hochschild chains and bar differentials

$$(C_*(A) / \text{Im}(1 - T_*), [b_*]) \cong (C_*^\lambda(A) = \ker(1 - T_*), b'_*)$$

where $C_q(A) = \bigotimes^{q+1}(A)$, which determines the cyclic homology $\mathbf{H}_*^\lambda(A)$ [10]. For a review of abstract Hochschild and Cyclic Homology, see [8, 12].

Now, we turn to noncommutative geometry on smooth manifolds, i. e. to the algebra $A = C^\infty(M)$ of smooth functions on M .

The relation between the above-stated abstract Hochschild theory of an arbitrary algebra A and the “smooth theory” for manifolds uses the tensor power $\widehat{\bigotimes}^{q+1} C^\infty(M)$ in the category of topological linear spaces, following the introduction of the Fréchet topology in the algebra of smooth functions $C^\infty(M)$ (see, for example, [6]). We have the following identification of $\widehat{\bigotimes}^{q+1} C^\infty(M)$ with the space of smooth functions on the cartesian product M^{q+1} :

$$\begin{aligned} \widehat{\bigotimes}^{q+1} C^\infty(M) &= C_q(C^\infty(M)) := \{\phi^q : M^{q+1} \rightarrow \mathbb{R}\}, \\ f_0 \otimes \dots \otimes f_q(x_0, \dots, x_q) &= f_0(x_0) \cdot \dots \cdot f_q(x_q). \end{aligned}$$

The above-said allows us to apply all the cyclic homology differentials like b , b' , B etc. to Hochschild q -chains $C_q(C^\infty(M))$. The *bar differential*

$$b'_q : C_q(C^\infty(M)) \rightarrow C_{q-1}(C^\infty(M))$$

is defined by the formula

$$(b'_q(\phi^q))(x_0, \dots, x_{q-1}) = \sum_{i=0}^{q-1} (-1)^i \phi^q(x_0, \dots, x_i, x_i, \dots, x_{q-1}).$$

We introduce the *graded cyclic permutation*

$$\begin{aligned} T_q : C_q(C^\infty(M)) &\rightarrow C_q(C^\infty(M)), \\ T_q(\phi^q)(x_0, \dots, x_q) &= (-1)^q \phi^q(x_1, \dots, x_q, x_0). \end{aligned}$$

Definition 2.1. A q -chain ϕ^q is called *cyclic* if $T_q(\phi^q) = \phi^q$, i.e.

$$\phi^q \text{ is cyclic} \quad \equiv \quad \phi^q \in \ker(1 - T_q).$$

The space of cyclic q -chains is denoted by $C_q^\lambda(C^\infty(M))$.

The space of cyclic chains is stable under the bar differential:

$$b'(C_q^\lambda(C^\infty(M))) \subset C_{q-1}^\lambda(C^\infty(M)).$$

Definition 2.2. The complex $(C_*^\lambda(C^\infty(M)), b')$ is called the *cyclic complex* or *Connes complex* of the manifold M . Its k th homology group is denoted by $\mathbf{H}_k^\lambda(C^\infty(M))$:

$$\mathbf{H}_k^\lambda(C^\infty(M)) := \mathbf{H}_k(C_*^\lambda(C^\infty(M)), b').$$

The homology of this complex was given by Connes (for compact manifolds) and by Teleman (for paracompact manifolds). See also [9].

Theorem 2.3 (Connes–Teleman theorem). *For a paracompact manifold M , we have*

$$\mathbf{H}_0^\lambda(C^\infty(M)) = \Omega^0(M) = C^\infty(M),$$

and for $i \geq 1$,

$$\begin{aligned} \mathbf{H}_{2i}^\lambda(C^\infty(M)) \\ = \mathbf{H}^0(M) \oplus \mathbf{H}^2(M) \oplus \mathbf{H}^4(M) \oplus \dots \oplus \mathbf{H}^{2i-2}(M) \oplus \Omega^{2i}(M) / d[\Omega^{2i-1}(M)], \end{aligned}$$

$$\begin{aligned} \mathbf{H}_{2i+1}^\lambda(C^\infty(M)) \\ = \mathbf{H}^1(M) \oplus \mathbf{H}^3(M) \oplus \mathbf{H}^5(M) \oplus \dots \oplus \mathbf{H}^{2i-1}(M) \oplus \Omega^{2i+1}(M) / d[\Omega^{2i}(M)]. \end{aligned}$$

In particular, if $\dim M$ is even, $\dim M = 2m$, then

$$\begin{aligned} \mathbf{H}_{2m+2}^\lambda(C^\infty(M)) &= \mathbf{H}_{2m+4}^\lambda(C^\infty(M)) = \dots \\ &= \mathbf{H}^0(M) \oplus \mathbf{H}^2(M) \oplus \mathbf{H}^4(M) \oplus \dots \oplus \mathbf{H}^{2m}(M), \end{aligned}$$

while if $\dim M$ is odd, $\dim M = 2m + 1$, then

$$\begin{aligned} \mathbf{H}_{2m+2}^\lambda(C^\infty(M)) &= \mathbf{H}_{2m+4}^\lambda(C^\infty(M)) = \dots \\ &= \mathbf{H}^0(M) \oplus \mathbf{H}^2(M) \oplus \mathbf{H}^4(M) \oplus \dots \oplus \mathbf{H}^{2m}(M). \end{aligned}$$

Therefore, the whole algebra of characteristic classes (Chern character) is sitting in one homology space, $\mathbf{H}_{2m+2}^\lambda(C^\infty(M))$.

Conclusion 2.4. If $U \subset M$ is a contractible open subset ($U \cong \mathbb{R}^m$), then $\mathbf{H}^k(U) = 0$ for $k > 0$, $\mathbf{H}^0(U) = \mathbb{R}$, therefore

$$\begin{aligned}\mathbf{H}_{2i}^\lambda(C^\infty(U)) &= \mathbb{R} \oplus (\Omega^{2i}(U) / d[\Omega^{2i-1}(U)]) \\ &= \mathbb{R} \oplus (\Omega^{2i}(U) / \ker d^{2i}) = \mathbb{R} \oplus \operatorname{Im} d^{2i}, \quad i \geq 1, \\ \mathbf{H}_0^\lambda(C^\infty(U)) &= C^\infty(U), \\ \mathbf{H}_{2i-1}^\lambda(C^\infty(U)) &= (\Omega^{2i-1}(U) / \ker d_{2i-1}) = \operatorname{Im} d_{2i-1}.\end{aligned}$$

3. ČECH–HOCHSCHILD BICOMPLEX FOR A GOOD COVERING

In classical differential topology and geometry, the single de Rham complex corresponds to the double Čech – de Rham complex for a good covering $\mathfrak{U}$ of M . And what is the case in noncommutative geometry of smooth manifolds?

We give an answer to this question. A double complex counterpart in noncommutative geometry of smooth manifolds has been defined by N. Teleman.

During his visit to the Institute of Mathematics of the Lodz University of Technology in December 2009 at the invitation of the author, N. Teleman suggested the definition of the *Cyclic Čech–Hochschild bicomplex* for a good covering of a manifold, which corresponds to the single Connes complex, and posed the problem of computing its total homology in connection with the problem of computing characteristic classes.

Definition 3.1 (Čech–Hochschild bicomplex). By the *Čech–Hochschild bicomplex* for a good covering $\mathfrak{U}$ of M , we mean a bicomplex (over the first quadrant) of vector spaces

$$\left(\{\check{C}H_{p,q}^\lambda(\mathfrak{U}, \mathbb{R})\}_{p \geq 0, q \geq 0}, \tilde{d}, b' \right)$$

where

(a)

$$\begin{aligned}\check{C}H_{p,q}(\mathfrak{U}, \mathbb{R}) &= \bigoplus_{(i_0, \dots, i_p) \in I^{p+1}} C_{[i_0, \dots, i_p], q}(\mathfrak{U}, \mathbb{R}), \\ C_{[i_0, \dots, i_p], q}(\mathfrak{U}, \mathbb{R}) &= \left\{ \phi_{i_0 \dots i_p}^q : (U_{i_0 \dots i_p})^{q+1} \rightarrow \mathbb{R} \right\},\end{aligned}$$

(b) b' is the bar differential for $p \geq 0, q \geq 1$,

$$\begin{aligned}b'_{p,q} : \check{C}H_{p,q}(\mathfrak{U}, V) &\rightarrow \check{C}H_{p,q-1}(\mathfrak{U}, V), \\ (b'_{p,q}(\phi_p^q))_{i_0, \dots, i_p}(x_0, \dots, x_{q-1}) &= \sum_{i=0}^{q-1} (-1)^i \phi_{i_0 \dots i_p}^q(x_0, \dots, x_i, x_i, \dots, x_{q-1}).\end{aligned}$$

(c) $\tilde{\delta}$ is a sign modification of the Čech differential for $p, q \geq 0$,

$$\begin{aligned}\delta^{p,q} : \check{C}H_{p,q}(\mathfrak{U}, V) &\rightarrow \check{C}H_{p+1,q}(\mathfrak{U}, V), \\ \delta^{p,q}(\phi^q)_{i_0, \dots, i_{p+1}} &= \sum_{r=0}^{p+1} (-1)^r \phi^q_{i_0, \dots, \hat{i}_r, \dots, i_{p+1}}, \\ \tilde{\delta}^{p,q} &= (-1)^q \delta^{p,q}.\end{aligned}$$

It is a standard calculation that $\tilde{\delta}b' + b'\tilde{\delta} = 0$. We modify the sign of δ , not of b' , because in the standard CC -bicomplex in cyclic homology, we have “ $-b'$ ” everywhere and, technically, the CC -bicomplex is used in this theory.

We take the graded cyclic permutation

$$\begin{aligned}T_{p,q} : \check{C}H_{p,q}(\mathfrak{U}, \mathbb{R}) &\rightarrow \check{C}H_{p,q}(\mathfrak{U}, \mathbb{R}), \\ T_{p,q}(\phi^q)_{i_0, \dots, i_p}(x_0, \dots, x_q) &= (-1)^q \phi^q_{i_0 \dots i_p}(x_1, \dots, x_q, x_0),\end{aligned}$$

and the *cyclic functions*

$$\check{C}H_{p,q}^\lambda(\mathfrak{U}, \mathbb{R}) = \{\phi \in \check{C}H_{p,q}(\mathfrak{U}, \mathbb{R}) : T_{p,q}(\phi) = \phi\} = \ker(1 - T_q).$$

The spaces of cyclic chains are stable under the bar and Čech differentials,

$$b'[\check{C}H_{p,q}^\lambda(\mathfrak{U}, \mathbb{R})] \subset \check{C}H_{p,q-1}^\lambda(\mathfrak{U}, \mathbb{R}), \quad \delta^{p,q}[\check{C}H_{p,q}^\lambda(\mathfrak{U}, \mathbb{R})] \subset \check{C}H_{p+1,q}^\lambda(\mathfrak{U}, \mathbb{R}).$$

Thus, we introduce a subcomplex $(\check{C}H^\lambda(\mathfrak{U}, \mathbb{R}), \tilde{\delta}, b')$, $\check{C}H^\lambda = \ker(1 - T)$, of the Čech–Hochschild bicomplex.

Definition 3.2 (Cyclic Čech–Hochschild bicomplex). By the name *Cyclic Čech–Hochschild bicomplex*, we mean the bicomplex $(\check{C}H^\lambda(\mathfrak{U}, \mathbb{R}), \tilde{\delta}, b')$,

$$\begin{array}{ccccccc} & & & & \downarrow & & \\ & & & & \xrightarrow{\tilde{\delta}} & \check{C}H_{p+2,q+1}^\lambda(\mathfrak{U}, \mathbb{R}) & \longrightarrow \\ & & \downarrow b' & & \downarrow b' & & \\ & \xrightarrow{\tilde{\delta}} & \check{C}H_{p,q}^\lambda(\mathfrak{U}, \mathbb{R}) & \xrightarrow{\tilde{\delta}} & \check{C}H_{p+1,q}^\lambda(\mathfrak{U}, \mathbb{R}) & \longrightarrow \\ \downarrow b' & & \downarrow b' & & \downarrow b' & & \\ \rightarrow & \check{C}H_{p-1,q-1}^\lambda(\mathfrak{U}, \mathbb{R}) & \xrightarrow{\tilde{\delta}} & \check{C}H_{p,q-1}^\lambda(\mathfrak{U}, \mathbb{R}) & \xrightarrow{\tilde{\delta}} & \check{C}H_{p+1,q-1}^\lambda(\mathfrak{U}, \mathbb{R}) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \end{array}$$

The total differential

$$D = \tilde{\delta} + b' : C^{[r]} \rightarrow C^{[r+1]}$$

is homogeneous of degree $+1$ with respect to the total gradation

$$C^{[r]} = \prod_{p-q=r} \check{C}H_{p,q}^{\lambda}(\mathfrak{U}, \mathbb{R})$$

along the lines $p - q = \text{const}$

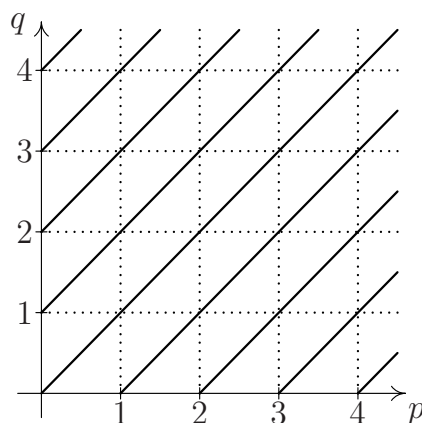


FIGURE 2. Total gradation along the lines $p - q = \text{const}$

(we observe that on each line $p - q = \text{const}$, we have infinitely many vector spaces and we must use the cartesian product of vector spaces, and not the direct sum; see [7, Th. 3.2]). Since $D^2 = 0$, we can define the total homology

$$\mathbf{H}^{[r]}(\check{C}H^{\lambda}(\mathfrak{U}, \mathbb{R})).$$

First main problem. Calculate the total homology of the Cyclic Čech–Hochschild bicomplex.

We observe that the homology of the columns of the Cyclic Čech–Hochschild bicomplex is the homology of the suitable Connes complex for contractible sets $U_{i_0 \dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}$, therefore, the above mentioned Connes–Teleman theorem on homology of the complex $C_{*}^{\lambda}(C^{\infty}(U_{i_0 \dots i_p}))$ is very useful.

It turns out that the total homology of the cyclic Čech–Hochschild bicomplex $(\check{C}H_{p,q}^{\lambda}, \tilde{\delta}, b')$ is very rich [7], so it is reasonable to look for representations of characteristic classes in the homology of this bicomplex.

Second main problem. Find a Cyclic Čech–Hochschild chain which represents characteristic classes.

This problem will be discussed elsewhere.

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IDENTITIES FOR MODULAR FORMS GENERATED BY VERTEX ALGEBRAS

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Abstract. We review our recent results in the vertex algebra approach in construction of identities for modular forms.

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1. RECENT RESULTS

The purpose of this review article is to describe our recent results as well as to announce results to appear towards understanding of the vertex operator origin of fundamental identities for modular forms. In particular, we review the results presented in the papers [6–11] accomplished in collaboration with M. P. Tuite (National University of Ireland, Galway).

Vertex algebras [1] give rise to non-trivial relations among the representation theory, analytic number theory, and algebraic geometry. In particular, they serve as an algebraic tool in generating modular forms as higher characters for vertex operator algebra modules. In the physics language, this corresponds to multiple-point correlation functions for appropriate conformal field theories defined on compact Riemann surfaces.

A complete description of all n -point functions for the Heisenberg bosonic and lattice VOAs in [4] and for fermionic vertex operator superalgebras in [5] has been given. Furthermore, employing the lattice bosonized version of these VOSAs leads to new Frobenius-Fay-type and Jacobi product identities for genus g half-integral theta series and Prime forms [6–8, 10, 11]. For further discussion of computation of vertex operator superalgebra n -point functions and their modular properties, see [12].

1.1. *The structure of twisted modules for vertex operator superalgebras (VOSA)*

First, we have to describe suitable vertex operator modules and corresponding intertwining operators. The algebra of intertwining operators $\mathcal{Y}_g(v, z)$ for the Heisenberg vertex operator algebra extended by its modules generated by a family g of continuous automorphisms is constructed explicitly. We introduced a $\mathbb{C}$ -extension of

the notion of a generalized vertex algebra [2] and proved that the intertwining operators constructed satisfy this definition. Explicit relations between shifted and twisted VOSA modules is also established [9].

1.2. *Geometry of reconstructing kernels on Riemann surfaces*

Reconstruction kernels (e. g. Cauchy kernel) play a very important role in computation of correlation functions for vertex operator modules. In particular, the Szegő kernel is suitable for the description of fermionic (half-integral graded) modules. Two alternative procedures of sewing (generalizing Yamada's construction) Riemann surfaces of arbitrary genres are developed. We prove the modular properties and find explicit formulas for the Szegő kernel on a Riemann surface of genus $g_1 + g_2$ (or $g + 1$) obtained by using the corresponding kernels on two original genres g_1 and g_2 Riemann surfaces [8].

1.3. *Computation of correlation functions in conformal field theory*

In a rigorous and constructive manner, we introduce the partition functions (graded dimensions) and correlations n -point functions (higher characters) for a twisted module M_f of a vertex operator (super) algebra V with the formal parameter associated to a complex parameter on a Riemann surface $\Sigma^{(g)}$ of genus $g \geq 1$ obtained by the surface surgery. A modification of the Zhu reduction procedure expressing a correlation $n + 1$ -point function via a finite sum of n -point functions with twisted higher Weierstrass functions as coefficients is introduced. In this algebraic approach, we can find closed formulas and explicitly establish modular properties of bosonic and fermionic characters on the torus, $q = e^{2\pi i\tau}$, $z_i \in \Sigma^{(g)}$, $v_i \in V$ [5, 6]:

$$Z_{M_f}^{(1)}(v_1, z_1, \dots, v_n, z_n; q) = \text{STr}_{M_f} \left(\mathcal{Y}_f(v_1, z_1) \dots \mathcal{Y}_f(v_n, z_n) q^{L_f(0) - C/24} \right),$$

and at genus two [7, 10]. Modularity properties of n -point functions with respect to appropriate group follows explicitly from the vertex operator derivation. The final expressions are given by the determinants of the matrices with elements being coefficients in the expansions of the regular parts of corresponding differentials (Bergman kernel A_a , $a = 1, 2$ for bosons, or Szegő kernel Q for fermions) [5, 7, 8, 10]. Deep combinatorial properties of these formulas are established by means of the Principal MacMahon Theorem [6]. The 1-point function for the Virasoro vector satisfies the genus two Ward identity [8, 9]. The partition functions on Riemann surfaces of arbitrary genres formed in the Schottky uniformization were calculated [6].

1.4. *Twisted modular forms*

The theory of twisted elliptic quasi-modular forms with two real torsion characteristics associated with two parameters of the character over a vertex operator superalgebra twisted module is developed. Twisted Eisenstein series and higher Weierstrass

functions are introduced and modularity with respect to the group $\mathrm{SL}(2, \mathbb{Z})$ is proved [5]. This theory can be generalized to the theory of twisted Jacobi forms.

2. IDENTITIES FOR GENUS g PARTITION FUNCTIONS

In [6, 11], we consider the definition and computation of the genus g partition function (in the multiple torus sewing and Schottky formations of a genus g Riemann surface) for bosonic and fermionic vertex models.

2.1. Genus g partition function in sewing two Riemann surfaces

We define [11] a genus g partition function by inductively sewing together lower genus 1-point functions $Z_M^{(g_i)}(u, z_i)$, $i = 1, 2$, $z_i \in \Sigma^{(g_i)}$, u belongs to the n th grading subspace of V , for a V -module M :

$$Z_M^{(g)}(\epsilon) = \sum_{n \geq 0} \epsilon^n \sum_{u \in V_{[n]}} Z_M^{(g_1)}(u, z_1) Z_M^{(g_2)}(\bar{u}, z_2),$$

where $g = g_1 + g_2$ and z_1, z_2 are insertion points, ϵ is the sewing parameter, and $\bar{u}, u$ are related by a non-degenerate bilinear invariant form on V .

For instance, with suitable local coordinates in the neighbourhood of these points, we find for the Heisenberg VOA M_2 , [11]

Theorem 1.

$$Z_{M_2}^{(g)}(\epsilon) = \frac{1}{\det(I - Q_{(g_1, g_2)})} Z_{M_2}^{(g_1)} Z_{M_2}^{(g_2)},$$

where $Z_{M_2}^{(g)}(\epsilon)$ is a non-vanishing holomorphic function on the sewing domain and is automorphic with respect to $\mathrm{SL}(2, \mathbb{Z}) \times \dots \times \mathrm{SL}(2, \mathbb{Z}) \subset \mathrm{Sp}(2g, \mathbb{Z})$ with automorphy factor $\det(C \Omega^{(g)} + D)^{-1}$, and a multiplier system, $Q_{(g_1, g_2)}$ is a moment matrix containing genus g sewing data, and $\Omega^{(g)}$ is a genus g period matrix.

Consider the canonical embedding of $\gamma_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \mathrm{Sp}(2g_1, \mathbb{Z})$,

$$\gamma_1 \hookrightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & I_{g_2} & 0 & 0 \\ C_1 & 0 & D_1 & 0 \\ 0 & 0 & 0 & I_{g_2} \end{pmatrix} \in \mathrm{Sp}(2(g_1 + g_2), \mathbb{Z}).$$

Then, we prove the following identities [11]

Theorem 2.

$$\det(I - \gamma_1 \cdot Q_{(g_1, g_2)}) = \det(I - Q_{(g_1, g_2)}) \frac{\det(C \Omega^{(g_1 + g_2)} + D)}{\det(C_1 \Omega^{(g_1)} + D_1)},$$

with $\gamma_1 \cdot Q_{(g_1, g_2)} = Q_{(g_1, g_2)}(\gamma_1 \cdot \Omega^{(g_1)}, \Omega^{(g_2)})$.

2.2. Genus g partition function for g sewn tori

One can find explicit formulas for various objects defined on the Riemann surface such as the bilinear form of the second kind, a basis of g holomorphic 1-forms and the genus g period matrix in terms of either the Canonical or Schottky parametrizations [6].

We define the genus g partition function for a VOSA V in the canonical sewing scheme in terms of genus zero $2g$ -point correlation functions as follows:

$$Z_V^{(g)}(\rho_i, A_{\pm i}) = \left\langle \mathbf{1}, \prod_{i=1}^g \sum_{n_i \geq 0} \rho_i^{n_i} \sum_{v_i \in V^{(n)}} Y(v_i, A_{-i}) Y(\bar{v}_i, A_i) \mathbf{1} \right\rangle,$$

where $\mathbf{1}$ is the vacuum state in V , $\bar{v}_i$ is dual to v_i with respect to a non-degenerate bilinear invariant form on V , and $\rho_i, i = 1, \dots, g$ $A_{\pm i}$ are the corresponding sewing parameters (see [6] for details).

In particular, the genus g partition function can be computed for the rank 2 Heisenberg VOA by means of the MacMahon Master Theorem [10]. We then find for the Heisenberg VOA M_2 in canonical sewing of g tori, [6]

Theorem 3.

$$Z_{M_2}^{(g)}(\rho_i, A_{\pm i}) = \frac{1}{\det(I - R^{\text{Can}})},$$

where R^{Can} is the canonical moment matrix. Furthermore, $\det(I - R^{\text{Can}})$ is a holomorphic non-vanishing function. In general, the genus g Heisenberg generating function is expressed in terms of a permanent of genus g bilinear forms of the second kind.

Alternatively, we may repeat this by using a definition of the genus g partition function in terms of Schottky parameters which involve sums or products over the Schottky group [6]. We then find

Theorem 4. *The genus g partition function is*

$$Z_{M_2}^{(g)}(q_i, a_{\pm i}) = \frac{1}{\det(I - R^{\text{Sch}})},$$

where R^{Sch} is the Schottky moment matrix, and $q_i, a_{\pm i}$ are the corresponding sewing parameters. Furthermore, $\det(I - R^{\text{Sch}})$ is a holomorphic non-vanishing function, and the genus g Heisenberg generating function is expressed in terms of a permanent of genus g bilinear forms of the second kind.

Then, it is natural to formulate the following

Conjecture. $\det(I - R^{\text{Can}}) = \det(I - R^{\text{Sch}})$.

2.3. The Montonen–Zograf product formula

The determinant $\det(I - R^{\text{Sch}})$ can also be re-expressed in terms of an infinite product formula originally calculated by Montonen. A similar product formula was subsequently found by Zograf. This has been recently related by McIntyre and Takhtajan to Mumford’s theorem concerning the absence of a global section on moduli space for the canonical line bundle. Expressing the corresponding moments in terms of an $\text{SL}(2, \mathbb{C})$ representation, we derive in [6] the identity

$$\det(I - R^{\text{Sch}}) = \prod_{m \geq 1} \prod_{\gamma^\alpha \in \Gamma} (1 - q_\alpha^m)^2,$$

where the inner product ranges over the primitive classes $\gamma^\alpha \neq 1$ of the Schottky group Γ , i. e. $\gamma^\alpha \neq \gamma^k$ for any $\gamma \in \Gamma$ for $k \neq 1$. Each such element has a multiplier q_α where $\gamma^\alpha = \text{diag}(q_\alpha, 1)$.

3. GENUS $g \geq 1$ GENERALIZATIONS OF CLASSICAL IDENTITIES

Bosonization formulas for n -point functions provide us with new genus g generalizations of the classical elliptic Frobenius–Fay Trisecant identities for modular forms [5] as well as of Jacobi product identity [7]. For instance, on the torus, we obtain

$$\det(\mathbf{M}_\mathbf{D}) = \frac{\vartheta\left(\begin{smallmatrix} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{smallmatrix}\right) \left(\sum_{i=1}^r m_i z_i - \sum_{j=1}^s n_j z'_j, \tau\right)}{\vartheta\left(\begin{smallmatrix} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{smallmatrix}\right) (0, \tau)} \cdot \frac{\prod_{1 \leq i < k \leq r} K^{(1)}(z_i - z_k, \tau)^{m_i m_k} \prod_{1 \leq j < l \leq s} K^{(1)}(z'_j - z'_l, \tau)^{n_j n_l}}{\prod_{1 \leq i \leq r, 1 \leq j \leq s} K^{(1)}(z_i - z'_j, \tau)^{m_i n_j}},$$

where the matrix $\mathbf{M}_\mathbf{D}$ contains information on vertex operator expansions in torus n -point functions, and this formula relates it to product of powers of prime forms $K^{(1)}$ and theta functions.

For the torus case, the boson–fermion correspondence results in the classical (torus case) Jacobi triple identity [3]. From the vertex operator superalgebra side, its essence lies in the Mason–Tuite formula [4] relating torus lattice n -point functions and products of modular forms. For genus $g \geq 2$, Prime form similar identities can be conjectured. Nevertheless, for $g \geq 2$, we could make use of the genus one identities to derive higher genus analogues of Jacobi product identities [7, 10]. In particular, at genus two, we get an explicit analogue of the classical (torus case, [3]) Jacobi triple

identity [7]:

$$\frac{\Theta^{(2)}\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)(\Omega)}{\vartheta^{(1)}\left(\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right)(\tau_1) \vartheta^{(1)}\left(\begin{smallmatrix} a_2 \\ b_2 \end{smallmatrix}\right)(\tau_2)} = \det\left(I - A_1^{(1)} A_2^{(1)}\right)^{1/2} \det(I - Q).$$

Here, $\vartheta^{(1)}\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)(\tau)$, and $\Theta^{(2)}\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)(\Omega)$ are genus one and genus two theta-functions correspondingly, and Ω is a period matrix. Similar formulas in self-sewing of a Riemann surface (in ρ -formalism) are given in [10]. Generalizations for this type of identities are available for genus g [11].

4. PERSPECTIVES

The results obtained stimulate further interest in application of algebraic technology of vertex algebras to number theory. We work now on procedures described above for super Riemann surfaces, on the identities for modular forms, θ -functions and primary forms (in particular, generalizations of Macdonald–Kac identities) at genres $g \geq 2$, the partition and n -point correlation functions, generation of twisted Jacobi forms, relations with co-operads, cluster algebras and structures generalizing vertex algebras [12], and applications of the theory of vertex algebras to Langlands program.

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The order of papers in this issue is according to the thematic topic and is as follows.

- I. Main Contributions — Plenary Lectures pp. 407–486
- II. Jet Theory, Weil Bundles, Variational Geometry pp. 487–555
- III. Riemannian Geometry, Geometry of Manifolds
with Connection pp. 557–636
- IV. Symmetry and Integrability, Geometry of Mechanics
and Thermodynamics pp. 637–711
- V. Noncommutative Algebra and Geometry pp. 713–727

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