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GENERALIZED DERIVATIONS ON IDEALS OF PRIME RINGS

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Abstract. Let R be a prime ring. By a generalized derivation we mean an additive mapping $g : R \rightarrow R$ such that $g(xy) = g(x)y + xd(y)$ for all $x, y \in R$ where d is a derivation of R . In the present paper our main goal is to generalize some results concerning derivations of prime rings to generalized derivations of prime rings.

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Keywords: prime ring, derivation, generalized polynomial, generalized derivation

1. INTRODUCTION

Throughout this paper R always denotes an associative prime ring with center $Z(R)$, extended centroid C , Martindale quotients ring Q and Utumi quotients ring U . For any $x, y \in R$, the commutator of x and y denoted by $[x, y]$ is defined to be $xy - yx$. Recall that a ring R is prime if $xRy = 0$ implies $x = 0$ or $y = 0$. An additive mapping $\alpha : R \rightarrow R$ is called a derivation if $\alpha(xy) = \alpha(x)y + x\alpha(y)$ holds for all $x, y \in R$. The commutativity of prime rings with derivations was initiated by Posner [16]. Over the last two decades, a lot of work has been done on this subject (see [4, 7, 11, 16] where further references can be found). Following Brešar [4], $d : R \rightarrow R$ is called a *generalized derivation* if there exists a derivation α of R such that

$$d(xy) = d(x)y + x\alpha(y) \quad \text{for all } x, y \in R.$$

Hence the concept of generalized derivations covers both the concepts of a derivation and of a left multiplier that is, an additive mapping $f : R \rightarrow R$ satisfying $f(xy) = f(x)y$ for all $x, y \in R$. Basic examples are derivations and generalized inner derivations given by maps of type $f : R \ni x \mapsto ax + xb \in R$ for some $a, b \in R$.

In [9], Hvala initiated generalized derivations from the algebraic viewpoint. In [13], T.K. Lee extended the definition of generalized derivations as follows:

By a generalized derivation we mean an additive mapping $g : I \rightarrow U$ such that $g(xy) = g(x)y + xd(y)$ for all $x, y \in I$, where I is a dense right ideal of R and d is a derivation from I into U .

Moreover Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of U and thus all generalized derivations of R will be implicitly assumed to be defined on the whole U and obtained the following results:

Theorem 1 ([13], Theorem 3). *Every generalized derivation g on a dense right ideal of R can be uniquely extended to U and assumes the form $g(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U .*

In this paper we extend some well-known results concerning derivations of prime rings to generalized derivations of prime ring.

We note that if R has the property that $Rx = 0$ implies $x = 0$ and $h : R \rightarrow R$ is any function, $d : R \rightarrow R$ is any additive mapping such that $d(xy) = d(x)y + xh(y)$ for all $x, y \in R$, then d is uniquely determined by h and moreover h must be a derivation (see [4], Remark 1).

In all that follows, unless stated otherwise, R will be a prime ring. The related object we need to mention is the two-sided Quotient ring Q of a ring R , the two-sided Utumi quotient U of a ring R (sometimes, as in [3], U is called the maximal ring of quotients). The definitions, the axiomatic formulations and the properties of these quotient rings U and Q can be found in [2] and [3].

We make a frequent use of the theory of generalized polynomial identities and of the theory of differential identities (see [3, 5, 10, 12, 15]). In particular we need to recall that when R is a prime ring and I a nonzero two-sided ideal of R , then I , R , Q and U satisfy the same polynomial identities [5] and also the same differential identities [12].

We will also make frequent use of the following result due to Kharchenko [10] (see also [12]):

Let R be a prime ring, d a nonzero derivation of R and I a nonzero two-sided ideal of R . Let $f(x_1, \dots, x_n, d(x_1), \dots, d(x_n))$ be a differential identity on I , that is the relation

$$f(r_1, \dots, r_n, d(r_1), \dots, d(r_n)) = 0$$

holds for all $r_1, \dots, r_n \in I$.

One of the following holds:

1) Either d is an inner derivation in Q , the Martindale quotient ring of R , in the sense that there exists $q \in Q$ such that $d(x) = [q, x]$, for all $x \in R$, and I satisfies the generalized polynomial identity

$$f(r_1, \dots, r_n, [q, r_1], \dots, [q, r_n]);$$

2) or I satisfies the generalized polynomial identity

$$f(x_1, \dots, x_n, y_1, \dots, y_n).$$

In [14], T.K. Lee and W.K. Shiue proved a version of Kharchenko's theorem for generalized derivations and presented some results concerning certain identities with

generalized derivations. More detail about generalized derivations can be in [9, 13] and [14].

We recall some related known result in literature: We say that an additive map F acts as a homomorphism on a nonempty subset $T \subseteq R$, if $F(xy) = F(x)F(y)$ for all $x, y \in T$; F acts as an anti-homomorphism on T , if $F(xy) = F(y)F(x)$ for all $x, y \in T$; finally F acts as a Jordan homomorphism on T if $F(x^2) = F(x)^2$ for all $x \in T$. Obviously any additive mapping, which is a homomorphism or an anti-homomorphism, is a Jordan homomorphism. On the other hand, in [8] Herstein proved that in case R is a prime ring of characteristic different from 2, any Jordan homomorphism on R is either a homomorphism or an anti-homomorphism of R . In [17], Rehman proved:

Theorem 2 ([17], Theorem 1.2). *Let R be a prime ring of characteristic different from 2 and F a nonzero generalized derivation of R , with associated derivation d . If F acts as homomorphism or anti-homomorphism on a two-sided ideal of R , then R is commutative unless $d = 0$.*

Recently in [6], De Filippis extended the Rehman's result as follows:

Theorem 3 ([6], Theorem 2). *Let R be a prime ring, L a noncentral Lie ideal of R and F a nonzero generalized derivation of R . If F acts as a Jordan homomorphism on L , then either $F(x) = x$ for all $x \in R$, or $\text{char}(R) = 2$, R satisfies the standard identity $s_4(x_1, x_2, x_3, x_4)$, L is commutative and $u^2 \in Z(R)$, for all $u \in L$.*

By motivating above results, in the present paper our aim is to obtain a generalization of Rehman's one in [17], moreover this study is a partial generalization of the result in [6] (in case $I = L$ is a two-sided ideal of R).

Throughout the paper, we denote by I_{id} the identity map of a ring R (i.e., the map $I_{id} : R \rightarrow R$ defined by $I_{id}(x) = x$ for all $x \in R$).

2. RESULTS

In the following, we assume that R is a prime ring and that $Z(R)$ is the center of R without stated otherwise.

For the proof of our main results we need the following lemma.

Lemma 1. *Let R be a noncommutative prime ring with a generalized derivation d associated with a derivation α of R . Suppose that $0 \neq c$ is an element of R such that $cd(x) \in Z(R)$ for all $x \in R$. Then there exists $q \in U$ such that $d(x) = qx$ and $cq = 0$.*

Proof. By Theorem 1 we can write d as the form $d(x) = qx + \alpha(x)$, where $q \in U$. By the hypothesis we have $c(qx + \alpha(x)) \in Z(R)$ for all $x \in R$. Since R and U satisfy the same differential identity [12] we get

$$c(qx + \alpha(x)) \in C \quad \text{for all } x \in U. \quad (2.1)$$

Suppose first that $\alpha \neq 0$. By the result of modulo Kharchenko's Theorem [10] we can divide the proof into two cases.

Assume first that α is an inner derivation of U induced by an element $b \in U$, that is $[b, x]$, for all $x \in U$. In this case $d(x) = qx + [b, x]$. By the hypothesis we have $c(qx + [b, x]) \in C$ for all $x \in U$. Hence above relation implies that

$$[r, c(qx + [b, x])] = 0 \quad \text{for all } r, x \in U \quad (2.2)$$

and in particular $cq \in C$. Replacing x by b we get $cq[r, b] = 0$ for all $r \in U$. By the primeness of R we obtain that either $cq = 0$ or $b \in C$. Since $\alpha \neq 0$ we are forced to consider the first case. Let $cq = 0$. By (2.2) we get $[r, c[b, x]] = 0$ for all $r, x \in U$. Substituting xb for x in the last relation we have

$$c[b, x][r, b] = 0 \quad \text{for all } r, x \in U.$$

By the primeness of U and by the supposing on α the above relation implies that $c = 0$, a contradiction.

Assume now that α is not an inner derivation of U . By Kharchenko's Theorem in [10, 12], we get $c(qx + y) \in C$ for all $x, y \in U$. In particular we obtain that $cqx \in C$ for all $x \in U$. Since R is noncommutative prime ring and $cq \in C$ we arrive at $cq = 0$. By the last relation we get $cy \in C$ implying that $c = 0$, a contradiction.

Thanks to two contradictions we are forced to assume that $\alpha = 0$. So we get $d(x) = qx$ and using (2.1) we also obtain that $cq = 0$, as asserted. $\square$

Now we are ready to prove our main results. The following theorem may be considered as a generalization of [1], Theorem 3.4.

Theorem 4. *Let R be a prime ring with center $Z(R)$ and I be a nonzero ideal of R . If R admits a nonzero generalized derivation d of R , with associated derivation α such that $d(xy) - d(x)d(y) \in Z(R)$ or $d(xy) + d(x)d(y) \in Z(R)$ for all $x, y \in I$, then either R is commutative or $d = I_{id}$ or $d = -I_{id}$.*

Proof. As we have remarked above we may take a generalized derivation d as the form $d(x) = ax + \alpha(x)$ for all $x \in U$ where $a \in U$ and it is known that R and I satisfy the same differential identity [12]. So we may assume that R admits a generalized derivation such that $d(xy) - d(x)d(y) \in Z(R)$ or $d(xy) + d(x)d(y) \in Z(R)$ for all $x, y \in R$. For each $y \in R$ we consider two subsets $K_y = \{x \in R : d(xy) - d(x)d(y) \in Z(R)\}$ and $M_y = \{x \in R : d(xy) + d(x)d(y) \in Z(R)\}$. Then K_y and M_y are two additive subgroups of $(R, +)$ such that $(R, +) = K_y \cup M_y$; and since a group cannot be the union of two proper subgroups, we have that either $R = K_y$ or $R = M_y$ for all $y \in R$. Repeating the same above argument we obtain that either $R = \{y \in R : R = K_y\}$ or $R = \{y \in R : R = M_y\}$. Note that the second case can be reduced to the first case. Indeed, since $f = -d$ is also a generalized derivation of R associated with a derivation $\beta = -\alpha$ the latter case just means that $f(xy) - f(x)f(y) \in Z(R)$ for all $x, y \in R$. Thus we only need to handle the case that

$$d(xy) - d(x)d(y) \in Z(R) \quad \text{for all } x, y \in R.$$

If R is commutative we are done. So we may suppose that R is not commutative. For some $a \in U$ write $d(x) = ax + \alpha(x)$ in the last relation. Since R and U satisfy the same differential identity [12] we have

$$d(xy) - d(x)d(y) \in C \quad \text{for all } x, y \in U. \quad (2.3)$$

Take 1 instead of x in (2.3). Hence we get $(1-a)d(y) \in C$ for all $y \in U$.

First suppose that $a \neq 1$. In view of Lemma 1 there exists $q \in U$ such that $d(y) = qy$ for all $y \in U$ and $(1-a)q = 0$. By (2.3) we have $qxy - qxqy \in C$ and so $qx(1-q)y \in C$ for all $x, y \in U$. Since R is a noncommutative prime ring the last relation gives us that $q = 0$ or $q = 1$. The first case implies that $d = 0$, a contradiction. Moreover it is easily seen that $a = q$. Thus the second case gives a contradiction.

Now suppose that $a = 1$. By (2.3) we have

$$\alpha(x)\alpha(y) \in C \quad \text{for all } x, y \in U. \quad (2.4)$$

Applying Lemma 1 to (2.4), we obtain $\alpha(x)\alpha(y) = 0$ for all $x, y \in U$. Replacing x by xz in the last relation we get $\alpha(x)z\alpha(y) = 0$ for all $x, y, z \in U$. By the primeness of U we arrive at $\alpha = 0$. By the last relation and the assumption $a = 1$ we arrive at $d = I_id$, as asserted. $\square$

Theorem 5. *Let R be a prime ring with center $Z(R)$ and I be a nonzero ideal of R . If R admits a nonzero generalized derivation d of R , with associated derivation α such that $d(xy) - d(y)d(x) \in Z(R)$ or $d(xy) + d(y)d(x) \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Proof. In a similar manner as the proof of Theorem 4 we obtain that either $d(xy) - d(y)d(x) \in Z(R)$ for all $x, y \in R$ or $d(xy) + d(y)d(x) \in Z(R)$ for all $x, y \in R$. As stated before, since the second case can be reduced to the first case by using the observation in the proof of Theorem 4, we consider only the case

$$d(xy) - d(y)d(x) \in Z(R) \quad \text{for all } x, y \in R.$$

If R is commutative we are done. So we may suppose that R is not commutative. By Theorem 1, for some $a \in U$ write $d(x) = ax + \alpha(x)$ for all $x \in R$ and since R and U satisfy the same differential identity [12] we have

$$d(xy) - d(y)d(x) \in C \quad \text{for all } x, y \in U. \quad (2.5)$$

Substituting 1 for y in (2.5) we get $(1-a)d(x) \in C$ for all $x \in U$.

If $a \neq 1$, there exists $q \in U$ such that $d(x) = qx$ and $(1-a)q = 0$ by Lemma 1. Using this fact in (2.5) we have

$$qxy - qyqx \in C \quad \text{for all } x, y \in U.$$

Replacing x by xy we get $(qxy - qyqx)y \in C$ for all $x, y \in U$. Since $qxy - qyqx \in C$ and $(qxy - qyqx)y \in C$ for all $x, y \in U$, we see that for every $y \in U$, $qxy - qyqx = 0$ for all $x \in U$ or $y \in C$. Recall that R is noncommutative. So $qxy - qyqx = 0$ for all $x, y \in U$. Setting $x = 1$ in the last relation, we get $qU(1-q) = 0$. So the

last relation implies that $q = 0$ or $q = 1$. If $q = 0$, then $d = 0$, a contradiction to our hypothesis. If $q = 1$, then $xy - yx = 0$ for all $x, y \in U$ and hence R is commutative, a contradiction to our assumption.

Now let $a = 1$. Then by the hypothesis we have $xy + \alpha(x)y + x\alpha(y) - yx - y\alpha(x) - \alpha(y)x - \alpha(y)\alpha(x) \in C$ for all $x, y \in U$ yielding that

$$[x, y] + [\alpha(x), y] + [x, \alpha(y)] - \alpha(y)\alpha(x) \in C \quad \text{for all } x, y \in U. \quad (2.6)$$

If $\alpha = 0$, then (2.6) implies that $[x, y] \in C$ for all $x, y \in U$ which gives us that R is commutative, a contradiction. So we can assume that $\alpha \neq 0$. By Kharchenko's Theorem [10], if α is an inner derivation induced by an element $b \in U \setminus C$ such that $\alpha(x) = [b, x]$ for all $x \in U$ then replacing y by b in (2.6) we get $[x, b] + [\alpha(x), b] \in C$ for all $x \in U$. Taking xb instead of x we have $([x, b] + [\alpha(x), b])b \in C$ for all $x \in U$. Since $b \notin C$ we obtain $0 = [x, b] + [\alpha(x), b] = \alpha(x) + \alpha^2(x)$. Replacing x by $\alpha(x)$ in (2.6) and using the last relation we have $\alpha(x)\alpha(y) \in C$. Replacing y by yb in the last relation and using $b \notin C$ we get $\alpha(x)\alpha(y) = 0$ for all $x, y \in U$ yielding that $\alpha = 0$, a contradiction. If α is not inner, then by Kharchenko's Theorem in [10, 12], we get

$$[x, y] + [z, y] + [x, w] - wz \in C \quad \text{for all } x, y, z, w \in U.$$

In particular we obtain $[x, y] \in C$ for all $x, y \in U$ yielding that R is commutative, a contradiction. $\square$

Example 1. Let R_1 be any commutative and R_2 any noncommutative ring. Define the ring R as $R = R_1 \oplus R_2 = \{(a, b) : a \in R_1 \text{ and } b \in R_2\}$. It is clear that R is a noncommutative ring. Let δ be any derivation of R_1 . Define an additive map $\alpha : R \rightarrow R$ as $\alpha((a, b)) = (\delta(a), 0)$, where $(a, b) \in R$. One can be easily shown that α is a derivation on R . Then the map $d : R \rightarrow R$ defined as $d((a, b)) = (a + \delta(a), b)$ is a generalized derivation on R associated with the derivation α . It is easy to verify that d satisfies $d(xy) - d(x)d(y) \in Z(R)$ for all $x, y \in R$, but neither R is commutative, nor $d = 0$ nor $d = I_{id}$.

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A FIXED POINT THEOREM BY ALTERING DISTANCE TECHNIQUE IN COMPLETE METRIC SPACES

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Abstract. In this paper, we present a fixed point theorem for a new type of contractive mappings. Our main result extends and unifies some well-known results in the literature.

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Keywords: fixed point, Banach contraction principle, contractive type mapping, altering distance approach

1. INTRODUCTION

Banach contraction principle play an important role in several branches of mathematics. For instances, it has been used to study the existence of solutions of nonlinear Volterra and Fredholm integral equations, nonlinear integro-differential equations in Banach spaces and to prove the convergence of algorithms in computational mathematics. Because of its importance in mathematics, Banach contraction principle has been extended in many different directions, see [1–15] and references therein.

In [10] Khan et al. proved the following fixed point theorem.

Theorem 1 ([10]). *Let (X, d) be a complete metric space, and let $T : X \rightarrow X$ be a mapping satisfying*

$$\psi(d(Tx, Ty)) \leq k\psi(d(x, y)),$$

for each $x, y \in X$ and for some $k \in [0, 1)$ where $\psi : [0, \infty) \rightarrow [0, \infty)$ is a continuous and non-decreasing function such that $\psi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point $\bar{x} \in X$ such that $\lim_{n \rightarrow \infty} T^n x = \bar{x}$, for each $x \in X$.

Another generalization of the contraction principle was suggested by Alber and Guerre-Delabriere [1] in Hilbert spaces. Rhoades [15] has shown that the result that Alber and Guerre-Delabriere proved in [1] is also valid in complete metric spaces. We state the result of Rhoades in the following.

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Theorem 2 ([15]). *Let (X, d) be a complete metric space. Let $T : X \rightarrow X$ be a mapping which satisfies*

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)),$$

for each $x, y \in X$, where $\phi : [0, \infty) \rightarrow [0, \infty)$ is a continuous and non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point $\bar{x} \in X$ such that $\lim_{n \rightarrow \infty} T^n x = \bar{x}$, for each $x \in X$.

In fact, Alber and Guerre-Delabriere assumed an additional condition on T which is $\lim_{t \rightarrow \infty} \phi(t) = \infty$. Notice that Rhoades [15] obtained the result presented in Theorem 2 without using this particular assumption.

In 2008 Dutta et al. [6] proved the following generalization of Banach contraction mapping principle which includes the generalizations presented in Theorems 1 and 2.

Theorem 3 ([6]). *Let (X, d) be a complete metric space, and let $T : X \rightarrow X$ be a mapping satisfying*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)),$$

for each $x, y \in X$, where $\psi, \phi : [0, \infty) \rightarrow [0, \infty)$ are continuous and non-decreasing function such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point $\bar{x} \in X$ such that $\lim_{n \rightarrow \infty} T^n x = \bar{x}$, for each $x \in X$.

The purpose of this paper is to present a new generalization of the Banach contraction principle in terms of altering distances. Our theorem unifies and extends some well-known results in the literature including the above theorems.

2. MAIN RESULTS

We are ready to state our main result.

Theorem 4. *Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be a mapping satisfying*

$$u(d(Tx, Ty)) \leq v(d(x, y)), \text{ for each } x, y \in X,$$

where $u, v : [0, \infty) \rightarrow [0, \infty)$, $v(t) < u(t)$ for each $t > 0$ and $u(0) = v(0) = 0$. Assume either one of the following assumptions holds:

- (a) *u and v are continuous and if $\{u(x_n)\}$ be a non-increasing sequence then $\{x_n\}$ is bounded;*
- (b) *u is increasing, continuous, u^{-1} is continuous and v is continuous from the right;*
- (c) *u is increasing, continuous, v is continuous at 0, $\liminf_{t \rightarrow \infty} (t - u^{-1}(v(t))) > 0$ and $\limsup_{s \rightarrow r} v(s) < v(r)$ for each $r > 0$.*

Then T has a unique fixed point $\bar{x} \in X$ such that $\lim_{n \rightarrow \infty} T^n x = \bar{x}$, for each $x \in X$.

Proof. For any $x_0 \in X$, we construct the sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for $n \in \mathbb{N}$. From our assumption on T , we have

$$u(d(x_n, x_{n+1})) \leq v(d(x_{n-1}, x_n)), \text{ for each } n \in \mathbb{N}. \quad (2.1)$$

Since

$$v(d(x_{n-1}, x_n)) \leq u(d(x_{n-1}, x_n)), \text{ for each } n \in \mathbb{N} \quad (2.2)$$

then from (2.1) and (2.2), we get

$$u(d(x_n, x_{n+1})) \leq v(d(x_{n-1}, x_n)) \leq u(d(x_{n-1}, x_n)), \text{ for each } n \in \mathbb{N}. \quad (2.3)$$

From (2.3), we deduce that $\{u(d(x_n, x_{n+1}))\}$ is a non-negative non-increasing sequence and consequently there exists $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} u(d(x_n, x_{n+1})) = r. \quad (2.4)$$

We shall show that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (2.5)$$

Assume first that (a) holds. Then $\{d(x_n, x_{n+1})\}$ is a bounded subsequence. On the contrary, assume that

$$\lim_{k \rightarrow \infty} d(x_{n_k}, x_{n_k+1}) = s > 0, \quad (2.6)$$

for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Since u and v are continuous then from (2.3), (2.4) and (2.6), we have

$$u(s) = \lim_{k \rightarrow \infty} u(d(x_{n_k}, x_{n_k+1})) = \lim_{k \rightarrow \infty} v(d(x_{n_k}, x_{n_k+1})) = v(s),$$

and so $s = 0$, a contradiction. Thus (2.5) holds.

If (b) holds, since u is increasing and $\{u(d(x_n, x_{n+1}))\}$ is non-increasing then we get that the sequence $\{d(x_n, x_{n+1})\}$ is non-increasing and consequently there exists $s_0 \geq 0$ such that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = s_0$. Since u is continuous from (2.4) we deduce that

$$0 = \lim_{n \rightarrow \infty} u(d(x_n, x_{n+1})) = u(s_0),$$

and so $s_0 = 0$. Thus (2.5) holds in any case.

We next prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon, \text{ for each } k \in \mathbb{N}. \quad (2.7)$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (2.7). Then

$$d(x_{m(k)}, x_{n(k)-1}) < \epsilon. \quad (2.8)$$

Then, we have

$$\epsilon \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) < \epsilon + d(x_{n(k)-1}, x_{n(k)}).$$

Letting $k \rightarrow \infty$ and using (2.5),

$$\lim_{k \rightarrow \infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \quad (2.9)$$

Again,

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}),$$

and

$$d(x_{n(k)-1}, x_{m(k)-1}) \leq d(x_{n(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}).$$

Letting $k \rightarrow \infty$ in the above two inequalities and using (2.5) and (2.9), we get

$$\lim_{k \rightarrow \infty} d(x_{n(k)-1}, x_{m(k)-1}) = \epsilon. \quad (2.10)$$

Now, assume (a) holds. Let $0 < \alpha < 1$. Since u is continuous then from (2.1) and (2.9) there exists K such that

$$\alpha u(\epsilon) \leq u(d(x_{m(k)}, x_{n(k)})) \leq v(d(x_{m(k)-1}, x_{n(k)-1})),$$

for each $k \geq K$. Letting $k \rightarrow \infty$ and using (2.9) and (2.10), we obtain $\alpha u(\epsilon) \leq v(\epsilon)$. Since $0 < \alpha < 1$ is arbitrary, we get

$$u(\epsilon) \leq v(\epsilon),$$

which is a contradiction if $\epsilon > 0$ and so $\{x_n\}$ is a Cauchy sequence.

Now, assume that (b) holds. Since u is increasing then from our assumption on T , we get

$$d(Tx, Ty) \leq u^{-1}(v(d(x, y))) \leq d(x, y), \text{ for each } x, y \in X. \quad (2.11)$$

From (2.11) and for each $k \in \mathbb{N}$, we have

$$\begin{aligned} d(x_{m(k)}, x_{n(k)}) &\leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)}) \\ &\leq 2d(x_k, x_{k-1}) + u^{-1}(v(d(x_{m(k)}, x_{n(k)}))). \end{aligned} \quad (2.12)$$

Since v is continuous from the right and u^{-1} is continuous, then from (2.5), (2.7) and (2.9), we get (note that $v(\epsilon) < u(\epsilon)$)

$$\epsilon \leq u^{-1}(v(\epsilon)) < \epsilon,$$

a contradiction. Then $\{x_n\}$ is a Cauchy sequence.

Now suppose that (c) holds. To prove that $\{x_n\}$ is a Cauchy sequence let us, on the contrary, assume that $\limsup_{m,n \rightarrow \infty} d(x_m, x_n) > 0$. By the triangle inequality and using (2.11),

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{n+1}) + d(x_{n+1}, x_n) \\ &\leq d(x_m, x_{m+1}) + u^{-1}(v(d(x_m, x_n))) + d(x_{n+1}, x_n), \end{aligned}$$

for each $m, n \in \mathbb{N}$. Then for each $m, n \in \mathbb{N}$, we have

$$0 \leq d(x_m, x_n) - u^{-1}(v(d(x_m, x_n))) \leq d(x_m, x_{m+1}) + d(x_{n+1}, x_n). \quad (2.13)$$

From (2.5), we get

$$\lim_{m,n \rightarrow \infty} d(x_m, x_n) - u^{-1}(v(d(x_m, x_n))) = 0.$$

From (c) we deduce that $\{d(x_m, x_n)\}$ has a bounded subsequence. Let us denote $\lim_{k \rightarrow \infty} d(x_{m_k}, x_{n_k}) = r > 0$. Then from (2.13) and using the continuity of u , we get that

$$\lim_{k \rightarrow \infty} u(d(x_{m_k}, x_{n_k})) = u(r) \geq v(r),$$

a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence.

Since X is a complete metric space then $\{x_n\}$ is convergent. Let $\lim_{n \rightarrow \infty} x_n = \bar{x}$. From our assumption on T , we have

$$u(d(x_n, T\bar{x})) = u(d(Tx_{n-1}, T\bar{x})) \leq v(d(x_{n-1}, \bar{x})).$$

Letting $n \rightarrow \infty$ and using continuity of u and continuity of v at 0,

$$u(d(\bar{x}, T\bar{x})) \leq v(0) = 0,$$

which implies $u(d(\bar{x}, T\bar{x})) = 0$, that is, $T\bar{x} = \bar{x}$.

To prove the uniqueness of the fixed point, let us suppose that $\bar{x}$ and $\bar{y}$ are two fixed points of T . Then, we have

$$u(d(\bar{x}, \bar{y})) = u(d(T\bar{x}, T\bar{y})) \leq v(d(\bar{x}, \bar{y})),$$

and so $d(\bar{x}, \bar{y}) = 0$. Thus $\bar{x} = \bar{y}$. $\square$

Remark 1. Let $u(t) = \psi(t)$ and $v(t) = \psi(t) - \phi(t)$ for each $t \in [0, \infty)$ where $\psi, \phi : [0, \infty) \rightarrow [0, \infty)$ are continuous and increasing function such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. Then from Theorem 4 we get immediately the above mentioned result of Dutta and Choudhury.

Now we get the following fixed point theorem of integral type due to Branciari [2].

Corollary 1. *Let (X, d) be a complete metric space and let $k \in [0, 1)$, $T : X \rightarrow X$ be a mapping such that for each $x, y \in X$,*

$$\int_0^{d(fx, fy)} \varphi(t) dt \leq k \int_0^{d(x, y)} \varphi(t) dt,$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a Lebesgue-integrable mapping which is summable, non-negative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t) dt > 0$. Then T has a unique fixed point $\bar{x} \in X$ such that $\lim_{n \rightarrow \infty} T^n x = \bar{x}$, for each $x \in X$.

Proof. Let $u(t) = \int_0^t \varphi(x) dx$ and $v(t) = ku(t)$ for each $t \in [0, \infty)$ and apply Theorem 4. $\square$

The following result (due to Browder [3]) is another consequence of our main result.

Corollary 2. *Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be a mapping such that for each $x, y \in X$,*

$$d(Tx, Ty) \leq \psi(d(x, y)),$$

where $\psi : [0, \infty) \rightarrow [0, \infty)$ is non-decreasing and continuous from the right. Then T has a unique fixed point $\bar{x} \in X$ such that $\lim_{n \rightarrow \infty} T^n x = \bar{x}$, for each $x \in X$.

Corollary 3 ([7]). *Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be a mapping such that for each $x, y \in X$,*

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y),$$

where $\alpha : [0, \infty) \rightarrow [0, 1)$ satisfying

$$\alpha(t_n) \rightarrow 1 \Rightarrow t_n \rightarrow 0.$$

Then T has a unique fixed point $\bar{x} \in X$ such that $\lim_{n \rightarrow \infty} T^n x = \bar{x}$, for each $x \in X$.

Proof. Let $u(t) = t$ and $v(t) = \alpha(t)t$ and apply Theorem 4 (note that u and v satisfy (c)). $\square$

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ON COMMON FIXED POINT THEOREMS FOR (ψ, φ)-GENERALIZED f -WEAKLY CONTRACTIVE MAPPINGS

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Abstract. In this paper, we present some common fixed point theorems for (ψ, φ) -generalized f -weakly contractive mappings in metric and ordered metric spaces. Our results extend, generalize and improve some well-known results in the literature. Also, we give an example to illustrate our results.

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1. INTRODUCTION AND PRELIMINARIES

The first important result on fixed points for contractive type mapping was the much celebrated Banach's contraction principle by Banach [2] in 1922. After this, Kannan [9, 10] proved the following result:

Theorem 1. *Let (X, d) be a complete metric space. If $T : X \rightarrow X$ satisfies*

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)],$$

where $0 < k < \frac{1}{2}$ and $x, y \in X$, then T has a unique fixed point.

A similar type of contractive condition has been studied by Chatterjee [5] and he proved the following result:

Theorem 2. *Let (X, d) be a complete metric space. If $T : X \rightarrow X$ satisfies a C -contraction given as follows:*

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)],$$

where $0 < k < \frac{1}{2}$ and $x, y \in X$, then T has a unique fixed point.

Alber and Guerre-Delabriere [1] introduced the definition of weak Φ -contraction.

Definition 1. A self mapping T on a metric space X is called weak Φ -contraction if there exists a function $\Phi : [0, +\infty) \rightarrow [0, +\infty)$ such that for each $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \Phi(d(x, y)).$$

The notion of Φ -contraction and weak Φ -contraction has been studied by many authors, see [3, 12, 15, 17, 19]. In recent years, many results related to fixed point theorems in partially ordered metric spaces are given, for more details see [8, 12–16].

Choudhury in [6] introduced a generalization of C -contraction given by the following definition.

Definition 2 ([6]). Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said to be weakly C -contractive (or a weak C -contraction) if for all $x, y \in X$,

$$d(Tx, Ty) \leq \frac{1}{2}(d(x, Ty) + d(y, Tx)) - \varphi(d(x, Ty), d(y, Tx)),$$

where $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only $x = y = 0$.

In [6] the author proves that if X is complete then every weak C -contraction has a unique fixed point. Recently, Harjani et al, [8] presented this last result in the context of ordered metric spaces.

Chandok [4] introduced the following definition : A map $T : X \rightarrow X$ is generalized f -weakly contractive if for each $x, y \in X$,

$$d(Tx, Ty) \leq \frac{1}{2}(d(fx, Ty) + d(fy, Tx)) - \varphi(d(fx, Ty), d(fy, Tx)),$$

where $\varphi : [0, +\infty) \rightarrow [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only $x = y = 0$.

If $f = I_X$, the identity mapping, then generalized f -weakly contractive mapping is weakly C -contractive.

Khan et al. [11] introduced the concept of altering distance function as follows:

Definition 3 (altering distance function, [11]). The function $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

- (1) ψ is continuous and non-decreasing.
- (2) $\psi(t) = 0$ if and only if $t = 0$.

Following the above definitions, we introduce the following:

Definition 4. A map $T : X \rightarrow X$ is called (ψ, φ) -generalized f -weakly contractive if for each $x, y \in X$,

$$\psi(d(Tx, Ty)) \leq \psi\left(\frac{1}{2}[d(fx, Ty) + d(fy, Tx)]\right) - \varphi(d(fx, Ty), d(fy, Tx)),$$

where

- (1) $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is an altering distance function.
- (2) $\varphi : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function with $\varphi(t, s) = 0$ if and only if $t = s = 0$.

If $\psi(t) = t$, then (ψ, φ) -generalized f -weakly contractive mapping is generalized f -weakly contractive.

The aim of this paper is to study some common fixed point theorems for (ψ, φ) -generalized f -weakly contractive in metric and ordered metric spaces.

2. MAIN RESULTS

First, we state the following known definition:

Definition 5. Let X a non-empty set. A point $x \in X$ is a coincidence point (common fixed point) of $f : X \rightarrow X$ and $T : X \rightarrow X$ if $fx = Tx$ ($x = fx = Tx$). The pair $\{f, T\}$ is called commuting if $Tfx = fTx$ for all $x \in X$.

We start with a common fixed point theorem for (ψ, φ) -generalized f -weakly contractive mappings in complete metric spaces.

Theorem 3. Let (X, d) be a metric space. Let $f, T : X \rightarrow X$ satisfy $T(X) \subset f(X)$, $(f(X), d)$ is complete and

$$\psi(d(Tx, Ty)) \leq \psi\left(\frac{1}{2}[d(fx, Ty) + d(fy, Tx)]\right) - \varphi(d(fx, Ty), d(fy, Tx)), \quad (2.1)$$

for all $x, y \in X$, where

- (1) $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is an altering distance function,
- (2) $\varphi : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function with $\varphi(t, s) = 0$ if and only if $t = s = 0$, then T and f have a coincidence point in X . Further, if T and f commute at their coincidence points, then T and f have a common fixed point.

Proof. Let $x_0 \in X$. Since $T(X) \subset f(X)$, we can choose $x_1 \in X$, so that $fx_1 = Tx_0$. Since $Tx_1 \in f(X)$, there exists $x_2 \in X$ such that $fx_2 = Tx_1$. By induction, we construct a sequence $\{x_n\}$ in X such that $fx_{n+1} = Tx_n$, for every $n \in \mathbb{N}$. By inequality (2.1), we have

$$\begin{aligned} \psi(d(Tx_{n+1}, Tx_n)) &\leq \psi\left(\frac{1}{2}[d(fx_{n+1}, Tx_n) + d(fx_n, Tx_{n+1})]\right) \\ &\quad - \varphi(d(fx_{n+1}, Tx_n), d(fx_n, Tx_{n+1})) \\ &= \psi\left(\frac{1}{2}d(Tx_{n-1}, Tx_{n+1})\right) - \varphi(0, d(Tx_{n-1}, Tx_{n+1})) \\ &\leq \psi\left(\frac{1}{2}d(Tx_{n-1}, Tx_{n+1})\right) \\ &\leq \psi\left(\frac{1}{2}[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})]\right). \end{aligned} \quad (2.2)$$

Since ψ is a non-decreasing function, we get that

$$d(Tx_n, Tx_{n+1}) \leq d(Tx_{n-1}, Tx_n) \quad \text{for any } n \in \mathbb{N}^*. \quad (2.3)$$

Thus, $\{d(Tx_n, Tx_{n+1})\}$ is a monotone non-increasing sequence of non-negative real numbers and hence is convergent. Hence there is $r \geq 0$ such that

$$\lim_{n \rightarrow +\infty} d(Tx_n, Tx_{n+1}) = r.$$

Using a triangular inequality, we have

$$d(Tx_{n+1}, Tx_n) \leq \frac{1}{2}d(Tx_{n-1}, Tx_{n+1}) \leq \frac{1}{2}[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})].$$

Letting $n \rightarrow +\infty$, we get

$$r \leq \frac{1}{2} \lim_{n \rightarrow +\infty} d(Tx_{n-1}, Tx_{n+1}) \leq \frac{1}{2}r + \frac{1}{2}r,$$

that is $\lim_{n \rightarrow +\infty} d(Tx_{n-1}, Tx_{n+1}) = 2r$. Using the continuity of ψ and φ , and inequality (2.2), we have, letting $n \rightarrow +\infty$

$$\psi(r) \leq \psi(r) - \varphi(0, 2r),$$

and consequently, $\varphi(0, 2r) \leq 0$. Thus, by a property of φ , $r = 0$, so

$$\lim_{n \rightarrow +\infty} d(Tx_{n+1}, Tx_n) = 0. \quad (2.4)$$

Now, we show that $\{Tx_n\}$ is a Cauchy sequence. If otherwise, then there exists $\varepsilon > 0$ for which we can find subsequences $\{Tx_{m(k)}\}$ and $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ with $n(k) > m(k) > k$ such that for every k ,

$$d(Tx_{m(k)}, Tx_{n(k)}) \geq \varepsilon, \quad d(Tx_{m(k)}, Tx_{n(k)-1}) < \varepsilon. \quad (2.5)$$

By triangular inequality, we have from (2.5)

$$\begin{aligned} \varepsilon &\leq d(Tx_{m(k)}, Tx_{n(k)}) \\ &\leq d(Tx_{m(k)}, Tx_{n(k)-1}) + d(Tx_{n(k)-1}, Tx_{n(k)}) \\ &< \varepsilon + d(Tx_{n(k)-1}, Tx_{n(k)}). \end{aligned}$$

Using (2.4), we get

$$\lim_{k \rightarrow +\infty} d(Tx_{m(k)}, Tx_{n(k)}) = \lim_{k \rightarrow +\infty} d(Tx_{m(k)}, Tx_{n(k)-1}) = \varepsilon. \quad (2.6)$$

On the other hand,

$$\begin{aligned} d(Tx_{m(k)}, Tx_{n(k)-1}) &\leq d(Tx_{m(k)}, Tx_{m(k)-1}) + d(Tx_{m(k)-1}, Tx_{n(k)}) \\ &\quad + d(Tx_{n(k)}, Tx_{n(k)-1}), \end{aligned}$$

and

$$d(Tx_{m(k)-1}, Tx_{n(k)}) \leq d(Tx_{m(k)-1}, Tx_{m(k)}) + d(Tx_{m(k)}, Tx_{n(k)}).$$

Letting $k \rightarrow +\infty$ in the two above inequalities, we have thanks to (2.4) and (2.6),

$$\lim_{k \rightarrow +\infty} d(Tx_{m(k)-1}, Tx_{n(k)}) = \varepsilon. \quad (2.7)$$

From (2.1), we have

$$\begin{aligned}
\psi(\varepsilon) &\leq \psi(d(Tx_{m(k)}, Tx_{n(k)})) \\
&\leq \psi\left(\frac{1}{2}[d(fx_{m(k)}, Tx_{n(k)}) + d(fx_{n(k)}, Tx_{m(k)})]\right) \\
&\quad - \varphi(d(fx_{m(k)}, Tx_{n(k)}), d(fx_{n(k)}, Tx_{m(k)})) \\
&= \psi\left(\frac{1}{2}[d(Tx_{m(k)-1}, Tx_{n(k)}) + d(Tx_{n(k)-1}, Tx_{m(k)})]\right) \\
&\quad - \varphi(d(Tx_{m(k)-1}, Tx_{n(k)}), d(Tx_{n(k)-1}, Tx_{m(k)})).
\end{aligned}$$

Taking $k \rightarrow +\infty$, using the continuity of ψ and φ , we obtain from (2.6), (2.7)

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon, \varepsilon),$$

hence $\varphi(\varepsilon, \varepsilon) = 0$, so $\varepsilon = 0$, it is a contradiction. Thus $\{Tx_n\}$ is a Cauchy sequence. Since $fx_n = Tx_{n-1}$, hence $\{fx_n\}$ is a Cauchy sequence in $(f(X), d)$, which is complete. Thus there is $z \in X$ such that

$$\lim_{n \rightarrow +\infty} fx_n = fz. \quad (2.8)$$

Moreover, (2.4) reads

$$\lim_{n \rightarrow +\infty} d(fx_n, fx_{n+1}) = 0. \quad (2.9)$$

By (2.1), we have

$$\begin{aligned}
\psi(d(Tz, fx_{n+1})) &= \psi(d(Tz, Tx_n)) \\
&\leq \psi\left(\frac{1}{2}[d(fz, Tx_n) + d(fx_n, Tz)]\right) - \varphi(d(fz, Tx_n), d(fx_n, Tz)) \\
&= \psi\left(\frac{1}{2}[d(fz, fx_{n+1}) + d(fx_n, Tz)]\right) - \varphi(d(fz, fx_{n+1}), d(fx_n, Tz)),
\end{aligned}$$

and letting $n \rightarrow +\infty$, using the continuity of ψ and φ and by (2.8), (2.9), we find

$$\psi(d(Tz, fz)) \leq \psi\left(\frac{1}{2}d(Tz, fz)\right) - \varphi(0, d(fz, Tz)) \leq \psi\left(\frac{1}{2}d(Tz, fz)\right).$$

Consequently, $d(Tz, fz) \leq \frac{1}{2}d(Tz, fz)$, that is, $d(Tz, fz) = 0$, i.e. $Tz = fz$, hence z is a coincidence point of T and f . Now suppose that T and f commute at z . Let $w = Tz = fz$. Then $Tw = T(fz) = f(Tz) = fw$. By inequality (2.1)

$$\begin{aligned}
\psi(d(Tz, Tw)) &\leq \psi\left(\frac{1}{2}[d(fz, Tw) + d(fw, Tz)]\right) - \varphi(d(fz, Tw), d(fw, Tz)) \\
&= \psi\left(\frac{1}{2}[d(Tz, Tw) + d(Tw, Tz)]\right) - \varphi(d(Tz, Tw), d(Tw, Tz)) \\
&= \psi\left(\frac{1}{2}[d(Tz, Tw) + d(Tw, Tz)]\right) - \varphi(d(Tz, Tw), d(Tw, Tz)) \\
&= \psi(d(Tz, Tw)) - \varphi(d(Tz, Tw), d(Tw, Tz)).
\end{aligned}$$

This implies that $d(Tz, Tw) = 0$, by the property of φ . Therefore, $Tw = fw = w$. This completes the proof of Theorem 3. $\square$

Example 1. Let $X = [0, +\infty)$. Let d be defined by $d(x, y) = |x - y|$ for all $x, y \in X$. We set $fx = \frac{x}{2}$ and $Tx = \frac{x}{4}$ for all $x \in X$. It is clear that $T(X) \subset f(X)$ and $(f(X), d)$ is a complete metric space. Define $\psi : [0, +\infty) \rightarrow [0, +\infty)$ and $\varphi : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ by

$$\psi(t) = \frac{t}{2} \quad \text{and} \quad \varphi(t, s) = \frac{1}{16}(t + s).$$

It is obvious that ψ and φ satisfy the hypotheses of Theorem 3. We need to show that the inequality (2.1) holds for any $x, y \in X$. First, the left-hand side of (2.1) is

$$\psi(d(Tx, Ty)) = \frac{1}{8}|x - y|. \quad (2.10)$$

While, the right-hand side of (2.1) is

$$\begin{aligned} & \psi\left(\frac{1}{2}(d(fx, Ty) + d(fy, Tx))\right) - \varphi(d(fx, Ty), d(fy, Tx)) \\ &= \frac{1}{4}\left[\left|\frac{x}{2} - \frac{y}{4}\right| + \left|\frac{y}{2} - \frac{x}{4}\right|\right] - \frac{1}{16}\left[\left|\frac{x}{2} - \frac{y}{4}\right| + \left|\frac{y}{2} - \frac{x}{4}\right|\right] \\ &= \frac{3}{16}\left[\left|\frac{x}{2} - \frac{y}{4}\right| + \left|\frac{y}{2} - \frac{x}{4}\right|\right]. \end{aligned} \quad (2.11)$$

By symmetry of (2.10) and (2.11), and without loss of generality, we suppose that $x \geq y$. In particular, (2.10) reads

$$\psi(d(Tx, Ty)) = \frac{1}{8}(x - y).$$

We distinguish two cases:

- If $2y \geq x$. Here, we have from (2.11)

$$\begin{aligned} & \psi\left(\frac{1}{2}(d(fx, Ty) + d(fy, Tx))\right) - \varphi(d(fx, Ty), d(fy, Tx)) \\ &= \frac{3}{16}\left[\left|\frac{x}{2} - \frac{y}{4}\right| + \left|\frac{y}{2} - \frac{x}{4}\right|\right] = \frac{3}{16}\left[\left(\frac{x}{2} - \frac{y}{4}\right) + \left(\frac{y}{2} - \frac{x}{4}\right)\right] \\ &= \frac{3}{64}(x + y) \geq \frac{1}{8}(x - y) = \psi(d(Tx, Ty)). \end{aligned} \quad (2.12)$$

- If $2y < x$. Here, we have from (2.11)

$$\begin{aligned} & \psi\left(\frac{1}{2}(d(fx, Ty) + d(fy, Tx))\right) - \varphi(d(fx, Ty), d(fy, Tx)) \\ &= \frac{3}{16}\left[\left|\frac{x}{2} - \frac{y}{4}\right| + \left|\frac{y}{2} - \frac{x}{4}\right|\right] = \frac{3}{16}\left[\left(\frac{x}{2} - \frac{y}{4}\right) + \left(-\frac{y}{2} + \frac{x}{4}\right)\right] \\ &= \frac{9}{64}(x - y) \geq \frac{1}{8}(x - y) = \psi(d(Tx, Ty)). \end{aligned} \quad (2.13)$$

By (2.12) and (2.13), the inequality (2.1) is satisfied. Then by Theorem 3, T and f have a common fixed point, which is $z = 0$.

Corollary 1. *Let (X, d) be a complete metric space. If $T : X \rightarrow X$ satisfies*

$$\psi(d(Tx, Ty)) \leq \psi\left(\frac{1}{2}[d(x, Ty) + d(y, Tx)]\right) - \varphi(d(x, Ty), d(y, Tx)), \quad (2.14)$$

for all $x, y \in X$, where

- (1) $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is an altering distance function,
- (2) $\varphi : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function with $\varphi(t, s) = 0$ if and only if $t = s = 0$, then T has a unique fixed point.

Proof. It follows by taking $f = I_X$ in Theorem 3. The uniqueness of the fixed point follows by the following: suppose u and v are fixed points of T . By (2.14), we have

$$\begin{aligned} \psi(d(u, v)) &= \psi(d(Tu, Tv)) \\ &\leq \psi\left(\frac{1}{2}[d(u, Tv) + d(v, Tu)]\right) - \varphi(d(u, Tv), d(v, Tu)) \\ &= \psi\left(\frac{1}{2}[d(u, v) + d(v, u)]\right) - \varphi(d(u, v), d(v, u)) \\ &= \psi(d(u, v)) - \varphi(d(u, v), d(v, u)), \end{aligned}$$

which implies that $\varphi(d(u, v), d(v, u)) = 0$, and by a property of φ , we get $u = v$. $\square$

Corollary 2. *Let (X, d) be a metric space. If $T, f : X \rightarrow X$ are such that $T(X) \subset f(X)$, $(f(X), d)$ is complete and*

$$d(Tx, Ty) \leq \frac{1}{2}[d(fx, Ty) + d(fy, Tx)] - \varphi(d(fx, Ty), d(fy, Tx)), \quad (2.15)$$

for all $x, y \in X$, where $\varphi : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function with $\varphi(t, s) = 0$ if and only if $t = s = 0$, then T and f have a coincidence point in X . Further, if T and f commute at their coincidence points, then T and f have a common fixed point.

Proof. The proof follows by taking $\psi(t) = t$ in Theorem 3. $\square$

Corollary 3. *Let (X, d) be a complete metric space. If $T : X \rightarrow X$ satisfies for all $x, y \in X$*

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \varphi(d(x, Ty), d(y, Tx)), \quad (2.16)$$

where $\varphi : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function with $\varphi(t, s) = 0$ if and only if $t = s = 0$, then T has a unique fixed point.

Proof. It follows by taking $f = Id_X$ in Corollary 2. The uniqueness of the fixed point follows from Corollary 1. $\square$

- Remark 1.*
- Corollary 1 corresponds to Corollary 2.1 of Shatanawi [18].
 - Corollary 2 corresponds to Theorem 1 of Chandok [4].
 - Corollary 3 corresponds to Theorem 2.1 of Choudhury [6].

Now, we extend Theorem 3 and we prove a common fixed point theorem for f -non-decreasing generalized nonlinear contraction mappings in the context of ordered metric spaces.

Definition 6 ([7]). Suppose $(X, \leq)$ is a partially ordered set and $T, f : X \rightarrow X$. T is said to be monotone f -nondecreasing if for all $x, y \in X$,

$$fx \leq fy \quad \text{implies} \quad Tx \leq Ty. \quad (2.17)$$

If $f = I_X$ in Definition 6, then T is monotone non-decreasing.

Theorem 4. Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let f and T are self-mappings of X such that $T(X) \subset f(X)$, $f(X)$ is closed and T is f -non-decreasing mapping. Suppose that f and T satisfy for all $x, y \in X$, for which $f(x) \leq f(y)$

$$\psi(d(Tx, Ty)) \leq \psi\left(\frac{1}{2}[d(fx, Ty) + d(fy, Tx)]\right) - \varphi(d(fx, Ty), d(fy, Tx)) \quad (2.18)$$

where

- (1) $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is an altering distance function,
- (2) $\varphi : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function with $\varphi(x, y) = 0$ if and only if $x = y = 0$.

Also, suppose that if $\{f(x_n)\} \subset X$ is a non-decreasing sequence with $f(x_n) \rightarrow f(z)$ in $f(X)$, then $f(x_n) \leq f(z)$ and $f(z) \leq f(f(z))$ for every n .

If there exists $x_0 \in X$ with $fx_0 \leq Tx_0$, then T and f have a coincidence point. Further, if T and f commute at their coincidence points, then T and f have a common fixed point.

Proof. Let $x_0 \in X$ such that $fx_0 \leq Tx_0$. Since $T(X) \subset f(X)$, we can choose $x_1 \in X$, so that $fx_1 = Tx_0$. Since $Tx_1 \in f(X)$, there exists $x_2 \in X$ such that $fx_2 = Tx_1$. By induction, we construct a sequence $\{x_n\}$ in X such that

$$fx_{n+1} = Tx_n.$$

Since $fx_0 \leq Tx_0$, $Tx_0 = fx_1$, so $fx_0 \leq fx_1$. T is f -non-decreasing mapping, we get $Tx_0 \leq Tx_1$. Similarly $fx_1 \leq fx_2$, $Tx_1 \leq Tx_2$, hence $fx_2 \leq fx_3$. Continuing, we obtain

$$fx_0 \leq fx_1 \leq fx_2 \leq \dots \leq fx_n \leq fx_{n+1} \leq \dots$$

If for some n , $Tx_{n+1} = Tx_n$, then $Tx_{n+1} = fx_{n+1}$, i.e. T and f have a coincidence point x_{n+1} , and so we have the result. For the rest, assume that $d(Tx_n, Tx_{n+1}) > 0$

for all $n \in \mathbb{N}$. By (2.18), we have

$$\begin{aligned} \psi(d(Tx_n, Tx_{n+1})) &\leq \psi\left(\frac{1}{2}[d(fx_{n+1}, Tx_n) + d(fx_n, Tx_{n+1})]\right) \\ &\quad - \varphi(d(fx_{n+1}, Tx_n), d(fx_n, Tx_{n+1})) \\ &= \psi\left(\frac{1}{2}d(Tx_{n-1}, Tx_{n+1})\right) - \varphi(0, d(Tx_{n-1}, Tx_{n+1})) \\ &\leq \psi\left(\frac{1}{2}d(Tx_{n-1}, Tx_{n+1})\right) \\ &\leq \psi\left(\frac{1}{2}d(Tx_{n-1}, Tx_n) + \frac{1}{2}d(Tx_n, Tx_{n+1})\right). \end{aligned}$$

It follows that, for any $n \in \mathbb{N}^*$

$$d(Tx_n, Tx_{n+1}) \leq d(Tx_{n-1}, Tx_n).$$

Thus $\{d(Tx_n, Tx_{n+1})\}$ is a monotone non-increasing sequence, hence it is convergent. Now, proceeding as in Theorem 3, we can prove that

$$\lim_{n \rightarrow +\infty} d(Tx_n, Tx_{n+1}) = 0. \quad (2.19)$$

Moreover, $\{Tx_n\}$ is a Cauchy sequence. Since $Tx_n = fx_{n+1}$ and $f(X)$ is closed, so there exists $z \in X$ such that

$$\lim_{n \rightarrow +\infty} fx_n = fz. \quad (2.20)$$

Having in mind $\{fx_n\}$ is a non-decreasing sequence, so by (2.20) we have for every $n \in \mathbb{N}$

$$fx_n \leq fz, \quad f(z) \leq f(fz). \quad (2.21)$$

Having $fx_n \leq fz$, so from inequality (2.18), we have

$$\begin{aligned} \psi(d(fx_{n+1}, Tz)) &= \psi(d(Tx_n, Tz)) \\ &\leq \psi\left(\frac{1}{2}[d(fz, Tx_n) + d(fx_n, Tz)]\right) - \varphi(d(fz, Tx_n), d(fx_n, Tz)) \\ &= \psi\left(\frac{1}{2}[d(fz, fx_{n+1}) + d(fx_n, Tz)]\right) - \varphi(d(fz, fx_{n+1}), d(fx_n, Tz)). \end{aligned}$$

Taking $n \rightarrow +\infty$, using the continuity of ψ and φ , we get from (2.19), (2.20)

$$\psi(d(Tz, fz)) \leq \psi\left(\frac{1}{2}d(fz, fz)\right) - \varphi(0, d(fz, Tz)),$$

that is, $d(Tz, fz) = 0$, hence $Tz = fz$, so z is a coincidence point of T and f .

Now suppose that T and f commute at z . Let $w = Tz = fz$. Then $Tw = T(fz) = f(Tz) = fw$. From (2.21), we have $fz \leq f(fz) = fw$, so the inequality (2.18) gives us

$$\psi(d(Tz, Tw)) \leq \psi\left(\frac{1}{2}[d(fz, Tw) + d(fw, Tz)]\right) - \varphi(d(fz, Tw), d(fw, Tz))$$

$$\begin{aligned}
&= \psi\left(\frac{1}{2}[d(Tz, Tw) + d(Tw, Tz)]\right) - \varphi(d(Tz, Tw), d(Tw, Tz)) \\
&= \psi\left(\frac{1}{2}[d(Tz, Tw) + d(Tw, Tz)]\right) - \varphi(d(Tz, Tw), d(Tw, Tz)) \\
&= \psi(d(Tz, Tw)) - \varphi(d(Tz, Tw), d(Tw, Tz)).
\end{aligned}$$

This implies that $d(Tz, Tw) = 0$, by the property of φ . Therefore, $Tw = fw = w$. This completes the proof of Theorem 4. $\square$

Corollary 4. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let f and T are self-mappings of X such that $T(X) \subset f(X)$, $f(X)$ is closed and T is f -non-decreasing mapping. Assume that f and T satisfy for all $x, y \in X$, for which $f(x) \leq f(y)$*

$$d(Tx, Ty) \leq \frac{1}{2}[d(fx, Ty) + d(fy, Tx)] - \varphi(d(fx, Ty), d(fy, Tx)), \quad (2.22)$$

where $\varphi : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function with $\varphi(x, y) = 0$ if and only if $x = y = 0$.

Also, suppose that if $\{f(x_n)\} \subset X$ is a non-decreasing sequence with $f(x_n) \rightarrow f(z)$ in $f(X)$, then $f(x_n) \leq f(z)$ and $f(z) \leq f(f(z))$ for every n .

If there exists $x_0 \in X$ with $fx_0 \leq Tx_0$, then T and f have a coincidence point. Further, if T and f commute at their coincidence points, then T and f have a common fixed point.

Proof. It follows by taking $\psi(t) = t$ in Theorem 4. $\square$

Corollary 5. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T : X \rightarrow X$ be a monotone non-decreasing mapping. Suppose that T satisfies for all $x, y \in X$, for which $x \leq y$,*

$$\psi(d(Tx, Ty)) \leq \psi\left(\frac{1}{2}[d(x, Ty) + d(y, Tx)]\right) - \varphi(d(x, Ty), d(y, Tx)), \quad (2.23)$$

where

- (1) $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is an altering distance function,
- (2) $\varphi : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function with $\varphi(x, y) = 0$ if and only if $x = y = 0$.

Also suppose either

- (i) $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \rightarrow z$, then $x_n \leq z$ for every n , or
- (ii) T is continuous.

If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Proof. If (i) holds, then taking $f = I_X$ in Theorem 4, we get the result.
If (ii) holds, then proceeding as in Theorem 4 with $f = I_X$, we can prove that $\{Tx_n\}$ is a Cauchy sequence and

$$z = \lim_{n \rightarrow +\infty} x_{n+1} = \lim Tx_n = T(\lim_{n \rightarrow +\infty} x_n) = Tz.$$

Hence the proof is completed. $\square$

Corollary 6. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T : X \rightarrow X$ be a monotone non-decreasing mapping. Suppose that T satisfies for all $x, y \in X$, for which $x \leq y$,*

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \varphi(d(x, Ty), d(y, Tx)), \quad (2.24)$$

where $\varphi : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function with $\varphi(x, y) = 0$ if and only if $x = y = 0$.

Also, suppose either

- (i) *If $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \rightarrow z$, then $x_n \leq z$ for every n , or*
- (ii) *T is continuous.*

If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Proof. It follows by taking $\psi(t) = t$ in Corollary 5. $\square$

Remark 2. Corollary 6 corresponds to Theorem 2.1 and Theorem 2.2 of Harjani et al. [8].

Corollary 7. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T : X \rightarrow X$ be a monotone non-decreasing mapping. Suppose that T satisfies for all $x, y \in X$, for which $x \leq y$,*

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)], \quad (2.25)$$

where $0 < k < \frac{1}{2}$.

Also, suppose either

- (i) *If $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \rightarrow z$, then $x_n \leq z$ for every n , or*
- (ii) *T is continuous.*

If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Proof. It follows by taking $\varphi(t) = (\frac{1}{2} - k)t$ in Corollary 6. $\square$

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GENERALIZED A -STATISTICAL CONVERGENCE AND A KOROVKIN TYPE APPROXIMATION THEOREM FOR DOUBLE SEQUENCES

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Abstract. In this paper, we use the notion of ideal to obtain two more general concepts and call them as I - A -statistical convergence and ideal A -summability. We also use the the concept of I - A -statistical convergence to prove a Korovkin type approximation theorem for double sequences of positive linear operators and present an interesting example to show that our Korovkin theorem is stronger than those proved earlier.

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Keywords: ideal convergence, statistical convergence, A -summability, Korovkin theorem, double sequence

1. INTRODUCTION

In [12], Gadjiev and Orhan has given a more general form of the classical Korovkin theorem (see [14]) by means of statistical convergence. Later, using different summability methods, some Korovkin type theorems have been obtained by many authors in several ways. For instance, see [4], [6–9], [10]. In recent years, the studies on double sequences has a rapid growth. Some concepts related to the single sequences have been extended to double sequences. In this work, we first generalize the concepts of A -statistical convergence and statistical A -summability for double sequences via ideals, and we call them as $\mathcal{I}$ - A -statistical convergence and ideal A -summability. The another main purpose of this paper is to obtain a Korovkin type approximation theorem through the concept of $\mathcal{I}$ - A -statistical convergence. We refer the readers to [16] and [3] for more details on statistical convergence and ideal convergence of double sequences.

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2. $S_A^2(\mathcal{I})$ -CONVERGENCE AND $\mathcal{I}^2(A)$ -SUMMABILITY

Let $A = (a_{jk}^{mn})$, $m, n, j, k \in \mathbb{N}$, be a four dimensional matrix and $x = (x_{jk})$ be a double sequence. Then the double (transformed) sequence, $Ax := (y_{mn})$, is denoted by

$$y_{mn} := \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk}^{mn} x_{jk} \quad (2.1)$$

where it is assumed that the summation exists as a well-known Pringsheim convergence (or P-limit) for each $(m, n) \in \mathbb{N} \times \mathbb{N}$. Now, let $A = (a_{jk}^{mn})$ be a nonnegative RH -regular matrix (see, [13],[17]), $x = (x_{jk})$ be a double sequence and (y_{mn}) be defined as in (2.1). If (y_{mn}) is statistically convergent to L , then x is said to be statistically A -summable to L (see [1, 5]). Note that the concept of statistical A -summability for single sequences is introduced by Edely and Mursaleen in [11]. In this section, we first generalize the concepts of A -statistical convergence and statistical A -summability to $\mathcal{I}$ - A -statistical convergence and ideal A -summability for double sequences, when $A = (a_{jk}^{mn})$ is a nonnegative RH -regular matrix and $\mathcal{I}$ is an admissible ideal in $\mathbb{N} \times \mathbb{N}$. We further establish the relation between these more general forms of summability methods.

Definition 1. A double sequence $x = (x_{jk})$ is said to be $\mathcal{I}$ - A -statistically convergent (or briefly $S_A^2(\mathcal{I})$ -convergent) to a number L if for each $\varepsilon > 0$ and $\delta > 0$

$$\left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \sum_{(j,k): |x_{jk} - L| \geq \varepsilon} a_{jk}^{mn} \geq \delta \right\} \in \mathcal{I}.$$

In this case we write $S_A^2(\mathcal{I}) - \lim x = L$.

Note that $\mathcal{I}$ - A -statistically convergence for single sequences has been recently introduced by Savaş et al. (see [18]).

Definition 2. A double sequence $x = (x_{jk})$ is said to be ideal A -summable (or briefly $\mathcal{I}^2(A)$ -summable) to a number L if for each $\delta > 0$

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : |y_{mn} - L| \geq \delta\} \in \mathcal{I}.$$

In this case we write $\mathcal{I}^2(A) - \lim x = L$.

Remark 1. (i) Let $\mathcal{I}_0 := \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N}) (j, k \geq m(A) \Rightarrow (j, k) \notin A)\}$. If we take $\mathcal{I} = \mathcal{I}_0$ in Definition 1, then $S_A^2(\mathcal{I})$ -convergence is reduced to A -statistical convergence. Also for $A = C(1, 1)$, $S_A^2(\mathcal{I})$ -convergence becomes $S_2(\mathcal{I})$ -convergence introduced by Belen and Yildirim in [2].

(ii) Let $\mathcal{I}_{\delta_2} = \{K \subset \mathbb{N} \times \mathbb{N} : \delta_2(K) = 0\}$, where $\delta_2(K)$ is the double A -density of K .

If we take $\mathcal{I} = \mathcal{I}_{\delta_2}$ and $A = C(1, 1)$, which is the double Cesàro matrix, in Definition 2, then $\mathcal{I}^2(A)$ -summability coincides with statistical $C(1, 1)$ -summability introduced by Móricz in [15]. But the choice of $\mathcal{I} = \mathcal{I}_{\delta_2}$ in Definition 2 gives us the concept of statistical A -summability (see [1] and [5]).

Recall that a nontrivial ideal of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}$ for each $i \in \mathbb{N}$.

Theorem 1. *Let $A = (a_{jk}^{mn})$ be a nonnegative RH-regular matrix and $\mathcal{I}$ be a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$. If a double sequence is bounded and $S_A^2(\mathcal{I})$ -convergent to L , then it is $\mathcal{I}^2(A)$ -summable to L but not conversely.*

Proof. Let $x = (x_{jk})$ be bounded and $S_A^2(\mathcal{I})$ -convergent to L . Write $M := \sup_{j,k} |x_{jk} - L|$ and $K(\varepsilon) := \{(j, k), j \leq m, k \leq n : |x_{jk} - L| \geq \varepsilon\}$ for any $\varepsilon > 0$. Then, we have

$$\begin{aligned}
 |y_{mn} - L| &= \\
 &= \left| \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{mn} (x_{jk} - L) + L \left(\sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{mn} - 1 \right) \right| \\
 &\leq \left| \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{mn} (x_{jk} - L) \right| + |L| \left| \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{mn} - 1 \right| \\
 &\leq \left| \sum_{(j,k) \in K(\varepsilon)} a_{jk}^{mn} (x_{jk} - L) \right| + \left| \sum_{(j,k) \notin K(\varepsilon)} a_{jk}^{mn} (x_{jk} - L) \right| + |L| \left| \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{mn} - 1 \right| \\
 &\leq M \sum_{(j,k) \in K(\varepsilon)} a_{jk}^{mn} + \varepsilon \sum_{(j,k) \notin K(\varepsilon)} a_{jk}^{mn} + |L| \left| \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{mn} - 1 \right| \\
 &\leq B \left\{ \sum_{(j,k) \in K(\varepsilon)} a_{jk}^{mn} + \left| \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{mn} - 1 \right| \right\} + \varepsilon \|A\|
 \end{aligned}$$

where $\|A\| = \sup_{m,n} \sum_j \sum_k |a_{jk}^{mn}| < \infty$ and $B = \max(M, |L|)$. Now choose a $\delta > 0$ such that $\delta - \varepsilon \|A\| > 0$. Hence from the last inequality we obtain

$$\begin{aligned}
 &\{(m, n) \in \mathbb{N} \times \mathbb{N} : |y_{mn} - L| \geq \delta\} \\
 &\subset \left\{ n \in \mathbb{N} : \sum_{(j,k) \in K(\varepsilon)} a_{jk}^{mn} \geq \frac{\delta - \varepsilon \|A\|}{2B} \right\} \cup \left\{ n \in \mathbb{N} : \left| \sum_{j,k=1,1}^{\infty,\infty} a_{jk}^{mn} - 1 \right| \geq \frac{\delta - \varepsilon \|A\|}{2B} \right\} \\
 &=: K_1 \cup K_2.
 \end{aligned}$$

By using definition of $S_A^2(\mathcal{I})$ -convergence we get $K_1 \in \mathcal{I}$. Also since $\mathcal{I}$ is strongly admissible and A is RH -regular, we obtain that $\mathcal{I} - \lim_{m,n} \left| \sum_{j,k} a_{jk}^{mn} - 1 \right| = 0$, so $K_2 \in \mathcal{I}$. Hence, by definition of an ideal we conclude that $\mathcal{I}^2(A) - \lim x = L$. To show that the converse is not true in general, let $\mathcal{I}$ be a strongly admissible ideal, $A = C(1, 1)$ and $x = (x_{jk})$ be defined as

$$x_{jk} = (-1)^j, \text{ for all } k.$$

Since

$$P - \lim_{m,n} \frac{1}{mn} \sum_{j=1}^m \sum_{k=1}^n x_{jk} = 0,$$

x is $C(1, 1)$ -summable to zero (hence $\mathcal{I}^2(C(1, 1))$ -summable to zero), but obviously x is not $C(1, 1)^{\mathcal{I}}$ -statistical convergent. This completes the proof of theorem. $\square$

3. $S_A^2(\mathcal{I})$ -APPROXIMATION

In this section we prove a Korovkin type approximation theorem via the concept of $S_A^2(\mathcal{I})$ -convergence for a double sequence of positive linear operators defined on $C(K)$, where $C(K)$ is the space of all continuous real valued functions on any compact subset of the real two dimensional space. Note that $C(K)$ is a Banach space with the norm $\|\cdot\|_{C(K)}$ defined by

$$\|f\|_{C(K)} := \sup_{(x,y) \in K} |f(x, y)|, (f \in C(K)).$$

Let L be a linear operator from $C(K)$ into $C(K)$. Then as usual, we say that L is positive linear operator provided that $f \geq 0$ implies $Lf \geq 0$. Also, we denote the value of Lf at a point (x, y) by $L(f; x, y)$. Before proceeding further, we quote here the classical and statistical forms of Korovkin-type theorems introduced in [20] and [6], respectively.

Theorem 2. *Let $\{L_{jk}\}$ be a double sequence of positive linear operators acting from $C(K)$ into itself. Then for all $f \in C(K)$,*

$$P - \lim_{j,k} \|L_{jk}(f) - f\|_{C(K)} = 0$$

if and only if

$$P - \lim_{j,k} \|L_{jk}(f_i) - f_i\|_{C(K)} = 0 \quad \text{for } i = 0, 1, 2, 3,$$

where $f_0(x, y) = 1$, $f_1(x, y) = x$, $f_2(x, y) = y$, $f_3(x, y) = x^2 + y^2$.

Theorem 3. Let $A = (a_{jk}^{mn})$ be a nonnegative RH-regular matrix. Let $\{L_{jk}\}$ be a double sequence of positive linear operators acting from $C(K)$ into itself. Then for all $f \in C(K)$,

$$st_A^2 - \lim_{j,k} \|L_{jk}(f) - f\|_{C(K)} = 0$$

if and only if

$$st_A^2 - \lim_{j,k} \|L_{jk}(f_i) - f_i\|_{C(K)} = 0 \quad \text{for } i = 0, 1, 2, 3,$$

where $f_0(x, y) = 1$, $f_1(x, y) = x$, $f_2(x, y) = y$, $f_3(x, y) = x^2 + y^2$.

Now we give the main result of this section.

Theorem 4. Let $\{L_{jk}\}$ be a double sequence of positive linear operators acting from $C(K)$ into itself. Then for all $f \in C(K)$,

$$S_A^2(\mathcal{J}) - \lim_{j,k} \|L_{jk}(f) - f\|_{C(K)} = 0 \quad (3.1)$$

if and only if

$$S_A^2(\mathcal{J}) - \lim_{j,k} \|L_{jk}(f_i) - f_i\|_{C(K)} = 0 \quad \text{for } i = 0, 1, 2, 3, \quad (3.2)$$

where $f_0(x, y) = 1$, $f_1(x, y) = x$, $f_2(x, y) = y$, $f_3(x, y) = x^2 + y^2$.

Proof. Condition (3.2) follows immediately from condition (3.1) since each $f_i \in C(K)$, ($i = 0, 1, 2, 3$). Let us prove the converse. By the continuity of f on compact set K , we can write $|f(x, y)| \leq M$, where $M = \|f\|_{C(K)}$. Also since $f \in C(K)$, for every $\varepsilon > 0$, there is a number $\delta > 0$ such that $|f(u, v) - f(x, y)| < \varepsilon$ for all $(u, v) \in K$ satisfying $|u - x| < \delta$ and $|v - y| < \delta$. Hence we get

$$|f(u, v) - f(x, y)| < \varepsilon + \frac{2M}{\delta^2} \left\{ (u - x)^2 + (v - y)^2 \right\}. \quad (3.3)$$

Since $\{L_{jk}\}$ is linear and positive, we obtain from (3.3) that

$$\begin{aligned} & |L_{jk}(f; x, y) - f(x, y)| \\ & \leq L_{jk}(|f(u, v) - f(x, y)|; x, y) + |f(x, y)| |L_{jk}(f_0; x, y) - f_0(x, y)| \\ & \leq L_{jk}\left(\varepsilon + \frac{2M}{\delta^2} \left\{ (u - x)^2 + (v - y)^2 \right\}; x, y\right) \\ & \quad + |f(x, y)| |L_{jk}(f_0; x, y) - f_0(x, y)| \\ & \leq \varepsilon + (\varepsilon + M) |L_{jk}(f_0; x, y) - f_0| + \frac{2M}{\delta^2} |L_{jk}(f_3; x, y) - f_3(x, y)| \\ & \quad + \frac{4M}{\delta^2} |x| |L_{jk}(f_1; x, y) - f_1(x, y)| + \frac{4M}{\delta^2} |y| |L_{jk}(f_2; x, y) - f_2(x, y)| \\ & \quad + \frac{2M}{\delta^2} (x^2 + y^2) |L_{jk}(f_0; x, y) - f_0(x, y)| \end{aligned}$$

$$\begin{aligned}
&\leq \varepsilon + \left(\varepsilon + M + \frac{2M}{\delta^2} (C^2 + D^2) \right) |L_{jk}(f_0; x, y) - f_0(x, y)| \\
&\quad + \frac{2M}{\delta^2} |L_{jk}(f_3; x, y) - f_3(x, y)| + \frac{4MC}{\delta^2} |L_{jk}(f_1; x, y) - f_1(x, y)| \\
&\quad + \frac{4MD}{\delta^2} |L_{jk}(f_2; x, y) - f_2(x, y)|
\end{aligned}$$

where $C := \max |x|$, $D := \max |y|$. Taking supremum over $(x, y) \in K$ we get

$$\|L_{jk}(f) - f\|_{C(K)} \leq B \sum_{i=0}^3 \|L_{jk}(f_i(u, v); x, y) - f_i(x, y)\|_{C(K)}$$

where

$$B := \max \left\{ \varepsilon + M + \frac{2M}{\delta^2} (C^2 + D^2), \frac{2M}{\delta^2}, \frac{4MC}{\delta^2}, \frac{4MD}{\delta^2} \right\}.$$

For any $\sigma > 0$ define

$$\begin{aligned}
D &= \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|L_{jk}(f; x, y) - f(x, y)\|_{C(K)} \geq \sigma\} \\
D_i &= \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|L_{jk}(f_i; x, y) - f_i(x, y)\|_{C(K)} \geq \frac{\sigma}{4B}\}, \quad i = 0, 1, 2, 3.
\end{aligned}$$

Then $D \subset \bigcup_{i=0}^3 D_i$ and hence

$$\left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \sum_{(j,k) \in D} a_{jk}^{mn} \geq \eta \right\} \subset \bigcup_{i=0}^3 \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \sum_{(j,k) \in D_i} a_{jk}^{mn} \geq \eta \right\}$$

for any $\eta > 0$. From this inclusion and (3.2), we immediately get (3.1). This completes the proof of theorem. $\square$

We remark that if we take $\mathcal{I} = \mathcal{I}_0$ in Theorem 4, we obtain Theorem 3.

Now, we will show that Theorem 4 is stronger than its classical and statistical forms.

Example 1. Let $\mathcal{I}$ be a strongly admissible ideal and $A \in \mathcal{I}$ be infinite set. Further let $\lambda = (\lambda_m)$ and $\mu = (\mu_n)$ be two non-decreasing sequences of positive numbers tending to ∞ such that

$$\lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 1 \quad \text{and} \quad \mu_{n+1} \leq \mu_n + 1, \mu_1 = 1.$$

For the intervals $J_m := [m - \lambda_m + 1, m]$ and $I_n := [n - \mu_n + 1, n]$, consider the four dimensional matrix $A = (a_{jk}^{mn})$ and the double sequence $x = (x_{jk})$ given by

$$a_{jk}^{mn} = \begin{cases} \frac{1}{\lambda_m \mu_n} & \text{if } j \in J_m \text{ and } k \in I_n \\ 0 & \text{otherwise.} \end{cases}$$

and

$$x_{jk} = \begin{cases} jk & ; \text{if } m - [\sqrt{\lambda_m}] + 1 \leq j \leq m, n - [\sqrt{\mu_n}] + 1 \leq k \leq n, (m, n) \notin A \\ jk & ; \text{if } m - \lambda_m + 1 \leq j \leq m, n - \mu_n + 1 \leq k \leq n, (m, n) \in A \\ 0 & ; \text{otherwise} \end{cases} \quad (3.4)$$

respectively. In this case $S_A^2(\mathcal{I})$ -convergence coincides with $S_{\lambda, \mu}(\mathcal{I})$ -convergence introduced in [2]. Also x is neither P -convergent nor A -statistically convergent but $S_A^2(\mathcal{I}) - \lim x = 0$ (see again [2]). Now consider the following Bernstein operators (see [19]) given by

$$B_{jk}(f; x, y) = \sum_{p=0}^j \sum_{q=0}^k f\left(\frac{p}{j}, \frac{q}{k}\right) \binom{j}{p} \binom{k}{q} x^p (1-x)^{j-p} y^q (1-y)^{k-q},$$

where $0 \leq x, y \leq 1$ and $f \in C(I^2)$; $K := I^2 = [0, 1] \times [0, 1]$. Let $L_{jk} : C(I^2) \rightarrow C(I^2)$ be defined by

$$L_{jk}(f; x, y) = (1 + x_{jk}) B_{jk}(f; x, y) \quad (3.5)$$

where (x_{jk}) is defined as in (3.4). Then observe that

$$\begin{aligned} L_{jk}(f_0; x, y) &= (1 + x_{jk}) f_0(x, y) \\ L_{jk}(f_1; x, y) &= (1 + x_{jk}) f_1(x, y) \\ L_{jk}(f_2; x, y) &= (1 + x_{jk}) f_2(x, y) \\ L_{jk}(f_3; x, y) &= (1 + x_{jk}) \left(f_3(x, y) + \frac{x - x^2}{j} + \frac{y - y^2}{k} \right). \end{aligned}$$

Since $\mathcal{I}$ is strongly admissible ideal (in this case P -convergence implies $S_A^2(\mathcal{I})$ -convergence), we obtain

$$S_A^2(\mathcal{I}) - \lim_{j,k} \|L_{jk}(f_i) - f_i\|_{C(K)} = 0$$

for $i = 0, 1, 2, 3$. Thus (L_{jk}) satisfies the condition (3.2) of Theorem 4. Hence we have

$$S_A^2(\mathcal{I}) - \lim_{j,k} \|L_{jk}(f) - f\|_{C(K)} = 0,$$

for all $f \in C(K)$. But, since (x_{jk}) is not convergent in Pringsheims's sense and A -statistical sense, we see that Theorem 2 and Theorem 3 do not work for our operators defined by (3.5). Hence it is shown that our version is more general to deal with the situation when the operators L_{jk} do not satisfy the conditions of Theorem 2 and Theorem 3.

Finally by using trigonometric test functions, we give the statement of Theorem 4 for a double sequence of positive linear operators defined on $C^*(\mathbb{R}^2)$, the space of

all 2π -periodic and real valued continuous functions on $\mathbb{R}^2$. This space is equipped with the supremum norm

$$\|f\|_{C^*(\mathbb{R}^2)} := \sup_{(x,y) \in \mathbb{R}^2} |f(x,y)|, \quad (f \in C^*(\mathbb{R}^2)).$$

Then using the similar technique as in the proof of Theorem 4, one can also get the following result.

Theorem 5. *Let $\{L_{jk}\}$ be a double sequence of positive linear operator acting from $C^*(\mathbb{R}^2)$ into itself. Then for all $f \in C^*(\mathbb{R}^2)$*

$$S_A^2(\mathcal{I}) - \lim_{j,k} \|L_{jk}(f) - f\|_{C^*(\mathbb{R}^2)} = 0$$

if and only if

$$S_A^2(\mathcal{I}) - \lim_{j,k} \|L_{jk}(f_i) - f_i\|_{C^*(\mathbb{R}^2)} = 0,$$

where $f_0(x, y) = 1$, $f_1(x, y) = \sin x$, $f_2(x, y) = \sin y$, $f_3(x, y) = \cos x$, $f_4(x, y) = \cos y$.

If the ideal $\mathcal{I}$ in Theorem 5 is replaced by $\mathcal{I}_0$, we immediately get the following result proved in [4].

Corollary 1. *Let $\{L_{jk}\}$ be a double sequence of positive linear operator acting from $C^*(\mathbb{R}^2)$ into itself. Then for all $f \in C^*(\mathbb{R}^2)$*

$$st_A^2 - \lim_{j,k} \|L_{jk}(f) - f\|_{C^*(\mathbb{R}^2)} = 0$$

if and only if

$$st_A^2 - \lim_{j,k} \|L_{jk}(f_i) - f_i\|_{C^*(\mathbb{R}^2)} = 0,$$

where $f_0(x, y) = 1$, $f_1(x, y) = \sin x$, $f_2(x, y) = \sin y$, $f_3(x, y) = \cos x$, $f_4(x, y) = \cos y$.

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HYERS-ULAM STABILITY AND APPLICATIONS IN GAUGE SPACES

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Abstract. Using the weakly Picard operator technique, we will present some Ulam- Hyers stability results for operatorial equations and some applications in gauge spaces.

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1. INTRODUCTION

In 1959, G. Marinescu [10] extended the Banach Contraction Principle to locally convex spaces, while I. Colojoară [4] and N. Gheorghiu [7] to gauge spaces and R. J. Knill [9] to uniform spaces. In 1971, Cain and Nashed [3] extended the notion of contraction to Hausdorff locally convex linear spaces. They showed that on sequentially complete subset, the Banach Contraction Principle is still valid. V.G. Angelov [1] introduced the notion of generalized φ -contractive single-valued map in gauge spaces in 1987, meanwhile the concept for multivalued operators was given in 1998 (see V.G. Angelov [2]). In 2000, M. Frigon [6] introduced the notion of generalized contraction in gauge spaces and proved that every generalized contraction on a complete gauge space (sequentially complete gauge space) has a unique fixed point.

Definition 1. Let X be any set. A map $p : X \times X \rightarrow \mathbb{R}_+$ is called a pseudometric (or, a gauge) in X whenever

- (1) $p(x, y) \geq 0$, for all $x, y \in X$;
- (2) If $x = y$, then $p(x, y) = 0$;
- (3) $p(x, y) = p(y, x)$, for all $x, y \in X$;
- (4) $p(x, z) \leq p(x, y) + p(y, z)$, for every triple of point.

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Definition 2. A family $\mathcal{P} = \{p_\alpha\}_{\alpha \in A}$ of pseudometrics on X (or a gauge structure on X), where A is a directed set, is said to be separating if for each pair of points $x, y \in X$, with $x \neq y$, there is a $p_\alpha \in \mathcal{P}$ such that $p_\alpha(x, y) \neq 0$.

A pair $(X, \mathcal{P})$ of a nonempty set X and a separating gauge structure $\mathcal{P}$ on X is called a gauge space.

It is well known (see Dugundji [5], pages 198-204) that any family $\mathcal{P}$ of pseudometrics on a set X induces on X a uniform structure $\mathcal{U}$ and conversely, any uniform structure $\mathcal{U}$ on X is induced by a family of pseudometrics on X . In addition, we have that $\mathcal{U}$ is separating (or Hausdorff) if and only if $\mathcal{P}$ is separating. Thus we may identify the gauge spaces and the Hausdorff uniform spaces.

A sequence $(x_n)_{n \in \mathbb{N}}$ of elements in X is said to be Cauchy if for every $\varepsilon > 0$ and $\alpha \in A$, there is an N with $p_\alpha(x_n, x_{n+p}) \leq \varepsilon$ for all $n \geq N$ and $p \in \mathbb{N}$.

The sequence $(x_n)_{n \in \mathbb{N}}$ is called convergent if there exists an $x_0 \in X$ such that for every $\varepsilon > 0$ and $\alpha \in A$, there is an N with $p_\alpha(x_0, x_n) \leq \varepsilon$ for all $n \geq N$.

Definition 3. A gauge space is called sequentially complete if any Cauchy sequence is convergent.

A subset of X is said to be sequentially closed if it contains the limit of any convergent sequence of its elements.

For further details see J. Dugundji [5], A. Granas, J. Dugundji [8].

Let X be a nonempty set and $f : X \rightarrow X$ be an operator. Then $x \in X$ is called fixed point for f if and only if $x = f(x)$. The set $Fix(f) := \{x \in X \mid x = f(x)\}$ is called the fixed point set of f .

Definition 4. Let $(X, \mathcal{P})$ be a gauge space and let $f : (X, \mathcal{P}) \rightarrow (X, \mathcal{P})$ be a single-valued operator. By definition, f is weakly Picard (briefly WPO) operator if the sequence of successive approximations $f^n(x)$ converges for all $x \in X$ and the limit (which may depend on X) is a fixed point of f .

If f is WPO, then we consider the operator $f^\infty : (X, (\mathcal{P})) \rightarrow (X, (\mathcal{P}))$ defined by $f^\infty(x) = \lim_{n \rightarrow \infty} f^n(x)$.

Definition 5. Let $(X, \mathcal{P})$ be a gauge space and let $f : (X, \mathcal{P}) \rightarrow (X, \mathcal{P})$ be a WPO and $\psi = \{\psi_\alpha\}_{\alpha \in A}$ be a family of mappings such that $\psi_\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ increasing, continuous in 0 and $\psi_\alpha(0) = 0$. By definition the operator f is ψ_α -WPO if

$$p_\alpha(x, f^\infty(x)) \leq \psi_\alpha(p_\alpha(x, f(x))), \text{ for all } x \in X, \alpha \in A.$$

If there exists $c = \{c_\alpha\}_{\alpha \in A} \in (0, \infty)^A$ such that $\psi_\alpha(t) := c_\alpha \cdot t$, for each $t \in \mathbb{R}_+$ and $\alpha \in A$ then the operator f is c_α -WPO.

For the theory of weakly Picard operators, see [11] for the single-valued case.

The purpose of this paper is to present some results concerning the Hyers-Ulam stability of some operatorial inclusions (such as the fixed point inclusion, the coincidence point equation or inclusion, etc.) in gauge spaces, using the weakly Picard operator technique.

2. HYERS-ULAM STABILITY FOR FIXED POINT EQUATIONS

We will present first the concept of Hyers-Ulam stability in the setting of gauge spaces.

Definition 6. Let $(X, \mathcal{P})$ be a gauge space and let $f : (X, \mathcal{P}) \rightarrow (X, \mathcal{P})$ be a single-valued operator. The fixed point equation

$$x = f(x), \quad x \in X \quad (2.1)$$

is called generalized Hyers-Ulam stable if and only if there exists $\psi = \{\psi_\alpha\}_{\alpha \in A}$ a family of mappings, $\psi_\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ increasing, continuous in 0 and $\psi_\alpha(0) = 0$ such that for each $\varepsilon = \{\varepsilon_\alpha\}_{\alpha \in A} \in (0, \infty)^A$ and for each solution y^* of the inequation

$$p_\alpha(y, f(y)) \leq \varepsilon_\alpha, \quad \alpha \in A, \quad (2.2)$$

there exists a solution x^* of the fixed point equation (2.1) such that

$$p_\alpha(y^*, x^*) \leq \psi_\alpha(\varepsilon_\alpha), \quad \text{for all } \alpha \in A.$$

If there exists $c = \{c_\alpha\}_{\alpha \in A} \in (0, \infty)^A$ such that $\psi_\alpha(t) := c_\alpha \cdot t$, for each $t \in \mathbb{R}_+$ and $\alpha \in A$ then the fixed point equation (2.1) is said to be Hyers-Ulam stable.

We refer to [12] for the particular case of Hyers-Ulam stability in metric spaces. Our first abstract result is as follows.

Theorem 1. Let $(X, \mathcal{P})$ be a gauge space and let $f : (X, \mathcal{P}) \rightarrow (X, \mathcal{P})$ be a ψ_α -WPO. Then, the fixed point equation (2.1) is generalized Hyers-Ulam stable.

Proof. Let $\varepsilon = \varepsilon_\alpha \in (0, \infty)^A$ and let $y^* \in f^\infty(x, y)$ be an ε -solution of (2.2), i.e., $p_\alpha(y^*, f(y^*)) \leq \varepsilon_\alpha$, for all $\alpha \in A$. Since f is a ψ_α -WPO, for each $x \in X$ and $\alpha \in A$ we have

$$p_\alpha(x, f^\infty(x)) \leq \psi_\alpha(p_\alpha(x, f(x))).$$

Then choosing $x^* = f^\infty(y^*)$ we have

$$p_\alpha(y^*, x^*) = p_\alpha(y^*, f^\infty(y^*)) \leq \psi_\alpha(p_\alpha(y^*, f(y^*))) \leq \psi_\alpha(\varepsilon_\alpha).$$

Thus the fixed point equation (2.1) is generalized Hyers-Ulam stable. $\square$

In 1974, Tarafdar [13] expressed the notion of contraction in Hausdorff uniform spaces, using the observation that a uniformity on X determines a family of gauges $\{p_\alpha\}$. A Hyers-Ulam stability result for the case of Tarafdar contraction in gauge spaces is as follows.

Theorem 2. Let $(X, \mathcal{P})$ be a gauge space and let $f : (X, \mathcal{P}) \rightarrow (X, \mathcal{P})$ be an a_α -contraction, i.e. for every $\alpha \in A$ there exists $a = \{a_\alpha\}_{\alpha \in A} \in (0, 1)^A$ such that

$$p_\alpha(f(x), f(y)) \leq a_\alpha \cdot p_\alpha(x, y), \text{ for all } x, y \in X.$$

Then $F_f = \{x^*\}$ and the fixed point equation (2.1) is Hyers-Ulam stable.

Proof. From Tarafdar [13] we get that f has a unique fixed point $x^* \in X$ and, for each $x \in X$, we have that $f^n(x) \rightarrow x^*$. Thus, f is a Picard operator. Moreover, it is a c_α -WPO, with $c_\alpha := \frac{1}{1-a_\alpha}$. Applying Theorem 1 we obtain the conclusion. $\square$

An extension of the previous result concerns the case of graphic-contractions.

Theorem 3. Let $(X, \mathcal{P})$ be a sequentially complete gauge space. Let $f : (X, \mathcal{P}) \rightarrow (X, \mathcal{P})$ be an operator. If f is a graphic a_α -contraction, i.e., for every $\alpha \in A$ there exists $a = \{a_\alpha\}_{\alpha \in A} \in (0, 1)^A$ such that

$$p_\alpha(f^2(x), f(x)) \leq a_\alpha \cdot p_\alpha(x, f(x)), \text{ for all } x \in X$$

and f has closed graph, then $F_f \neq \emptyset$ and the equation (2.1) is Hyers-Ulam stable.

Proof. Let $x_0 \in X$ and $x_n \in f(x_{n-1}) = f^n(x_0)$, $n = 1, 2, \dots$. If m and n are positive integers, $m < n$, then for each $\alpha \in A$ we have:

$$\begin{aligned} p_\alpha(x_m, x_n) &= p_\alpha(f^m(x_0), f^n(x_0)) \\ &\leq p_\alpha(f^m(x_0), f^{m+1}(x_0)) + p_\alpha(f^{m+1}(x_0), f^{m+2}(x_0)) + \dots \\ &\quad + p_\alpha(f^{n-1}(x_0), f^n(x_0)) \\ &\leq a_\alpha p_\alpha(f^{m-1}(x_0), f^m(x_0)) + a_\alpha p_\alpha(f^m(x_0), f^{m+1}(x_0)) + \dots \\ &\quad + a_\alpha p_\alpha(f^{n-2}(x_0), f^{n-1}(x_0)) \\ &\leq a_\alpha^m p_\alpha(x_0, f(x_0)) + a_\alpha^{m+1} p_\alpha(x_0, f(x_0)) + \dots + a_\alpha^{n-1} p_\alpha(x_0, f(x_0)) \\ &= p_\alpha(x_0, f(x_0)) a_\alpha^m (1 + a_\alpha + \dots + a_\alpha^{n-m-1}) \\ &\leq p_\alpha(x_0, f(x_0)) a_\alpha^m \frac{1 - a_\alpha^{n-m}}{1 - a_\alpha}. \end{aligned}$$

Hence the sequence (x_n) is Cauchy, therefore (x_n) converges to a point $x^* \in X$. From the continuity of f we get that x^* is a fixed point for f . So, we have

$$p_\alpha(x_m, x_n) \leq p_\alpha(x_0, f(x_0)) a_\alpha^m \frac{1 - a_\alpha^{n-m}}{1 - a_\alpha}.$$

If we choose in the above inequality $m = 0$ and let $n \rightarrow \infty$ we obtain:

$$p_\alpha(x_0, x^*) \leq p_\alpha(x_0, f(x_0)) \frac{1}{1 - a_\alpha}, \text{ for all } \alpha \in A.$$

Thus f is a c_α -WPO with $c_\alpha := \frac{1}{1-a_\alpha}$. Therefore the second conclusion follows from Theorem 1. $\square$

3. APPLICATIONS

We will apply some of the above results to nonlinear integral equations on the real axis.

$$x(t) = \int_0^t K(t, s, x(s))ds + g(t), \quad t \in \mathbb{R}_+. \quad (3.1)$$

We give the notion of Hyers-Ulam stability for the integral equation.

Definition 7. The integral equation (3.1) is called Hyers-Ulam stable if and only if there exists $c = \{c_\alpha\}_{\alpha \in A} \in (0, \infty)^A$ such that for each $\varepsilon = \{\varepsilon_\alpha\}_{\alpha \in A} \in (0, \infty)^A$ and for any ε -solution y^* of (1) (i.e., any $y^* \in C([0, \infty], \mathbb{R}^n)$ which satisfies the inequality

$$|y^*(t) - \int_0^t K(t, s, x(s))ds - g(t)| \leq \varepsilon_\alpha, \quad \text{for each } t \geq 0 \quad (3.2)$$

there exists a solution x^* of the equation (3.1) such that

$$|y^*(t) - x^*(t)| \leq c_\alpha \cdot \varepsilon_\alpha, \quad \text{for each } t \geq 0.$$

Theorem 4. Consider equation (3.1). Suppose that:

- i) $K : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ are continuous;
- ii) there exists $k > 0$ such that

$$|K(t, s, u) - K(t, s, v)| \leq k|u - v|, \quad \text{for each } t, s \in \mathbb{R}_+, u, v \in \mathbb{R}^n;$$

Then the integral equation (3.1) has a unique solution x^* in $C([0, +\infty), \mathbb{R}^n)$ and equation (3.1) is Hyers-Ulam stable.

Proof. Let $X := C([0, +\infty), \mathbb{R}^n)$ and the family of pseudo-norms

$$\|x\|_n := \max_{t \in [0, n]} |x(t)|e^{-\tau t}, \quad \text{where } \tau > 0.$$

Define now $d_n(x, y) := \|x - y\|_n$ for $x, y \in X$.

Then $\mathcal{P} := (d_n)_{n \in \mathbb{N}^*}$ is family of gauges on X . Then $(X, \mathcal{P})$ is a complete gauge space.

Define $A : C([0, +\infty), \mathbb{R}^n) \rightarrow C([0, +\infty), \mathbb{R}^n)$, by the formula

$$Ax(t) := \int_0^t K(t, s, x(s))ds + g(t), \quad t \in \mathbb{R}_+.$$

For each $x, y \in X$ and for $t \in [0, n]$, we have successively:

$$\begin{aligned} |Ax(t) - Ay(t)| &\leq \int_0^t |K(t, s, x(s)) - K(t, s, y(s))|ds \leq \int_0^t k|x(s) - y(s)|ds \\ &= k \int_0^t |x(s) - y(s)|e^{-\tau s}e^{\tau s}ds \leq k \int_0^t e^{\tau s}(|x(s) - y(s)|e^{-\tau s})ds \\ &\leq kd_n(x, y) \int_0^t e^{\tau s}ds \leq \frac{k}{\tau}d_n(x, y)e^{\tau t}. \end{aligned}$$

Hence, for $\tau > k$ and denoting $L := \frac{k}{\tau} < 1$ we obtain

$$d_n(Ax, Ay) \leq L d_n(x, y), \text{ for each } x, y \in X.$$

The conclusion follows now from Theorem 2. $\square$

Consider now the following equation

$$x(t) = \int_{-t}^t K(t, s, x(s)) ds + g(t), \quad t \in \mathbb{R}. \quad (3.3)$$

Theorem 5. *Consider the equation (3.3). Suppose that:*

- i) $K : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g : \mathbb{R} \rightarrow \mathbb{R}^n$ are continuous;
- ii) there exists $k > 0$ such that

$$|K(t, s, u) - K(t, s, v)| \leq k|u - v|, \text{ for each } t, s \in \mathbb{R}, u, v \in \mathbb{R}^n;$$

Then the integral equation (3.3) has a unique solution x^* in $C(\mathbb{R}, \mathbb{R}^n)$ and equation (3.3) is Hyers-Ulam stable.

Proof. We consider the gauge space $X := (C(\mathbb{R}, \mathbb{R}^n), \mathcal{P} := (d_n)_{n \in \mathbb{N}})$ where

$$d_n(x, y) = \max_{-n \leq t \leq n} (|x(t) - y(t)| \cdot e^{-\tau|t|}), \quad \tau > 0,$$

and the operator $B : X \rightarrow X$ defined by

$$Bx(t) = \int_{-t}^t K(t, s, x(s)) ds + g(t).$$

From condition (ii), for $x, y \in X$, we have

$$\begin{aligned} |Bx(t) - By(t)| &\leq \int_{-t}^t k|x(s) - y(s)|e^{-\tau|s|}e^{\tau|s|}ds \leq \\ &k \int_{-t}^t e^{\tau|s|}(|x(s) - y(s)|e^{-\tau|s|})ds \leq k d_n(x, y) \left| \int_{-t}^t e^{\tau|s|}ds \right| \leq \\ &k d_n(x, y) \int_{-|t|}^{|t|} e^{\tau|s|}ds \leq \frac{2k}{\tau} d_n(x, y) e^{\tau|t|}, \quad t \in [-n; n]. \end{aligned}$$

Thus, for any $\tau \geq 2k$, if we denote $L := \frac{2k}{\tau} < 1$, we obtain

$$d_n(B(x), B(y)) \leq L d_n(x, y), \text{ for all } x, y \in E, \text{ and for } n \in \mathbb{N}.$$

The conclusion follows again by Theorem 2. $\square$

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SOME GENERALIZATIONS ON THE UNIVALENCE OF AN INTEGRAL OPERATOR AND QUASICONFORMAL EXTENSIONS

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Abstract. By using the method of Loewner chains, we establish some sufficient conditions for the analyticity and univalence of functions defined by an integral operator. Also, we refine the result to a quasiconformal extension criterion with the help of Beckers's method.

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1. INTRODUCTION

Let $\mathcal{A}$ the class of functions f which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$. We denote by $\mathcal{U}_r$ the open disk $\{z \in \mathbb{C} : |z| < r\}$, where $0 < r \leq 1$, by $\mathcal{U} = \mathcal{U}_1$ the open unit disk of the complex plane and by I the interval $[0, \infty)$.

Let k be constant in $[0, 1)$. Then a homeomorphism f of $G \subset \mathbb{C}$ is said to be k -quasiconformal, if $\partial_z f$ and $\partial_{\bar{z}} f$ in the distributional sense are locally integrable on G and fulfill the inequality $|\partial_{\bar{z}} f| \leq k |\partial_z f|$ almost everywhere in G . If we do not need to specify k , we will simply call f quasiconformal.

Three of the most important and known univalence criteria for analytic functions defined in the open unit disk were obtained by Nehari [14], Ozaki-Nunokawa [17] and Becker [3]. Some extensions of these three criteria were given by [15, 16, 21–25]. Furthermore a lot of univalence criteria have been obtained by different authors (see also [7–9]).

In the present investigation, we will obtain a number of new criteria for the functions defined by the integral operator $\mathcal{F}_\beta(z)$. Also, we obtain a refinement to a quasiconformal extension criterion of the main result.

2. PRELIMINARIES

Before proving our main theorem we present a brief summary of the method of Loewner chains and quasiconformal extension criterion.

A function $\mathcal{L}(z, t) : \mathcal{U} \times [0, \infty) \rightarrow \mathbb{C}$ is said to be *subordination chain* (or *Loewner chain*) if:

- (i) $\mathcal{L}(z, t)$ is analytic and univalent in $\mathcal{U}$ for all $t \geq 0$.
- (ii) $\mathcal{L}(z, t) \prec \mathcal{L}(z, s)$ for all $0 \leq t \leq s < \infty$, where the symbol " $\prec$ " stands for subordination.

To prove our results, we will need the following theorem due to Ch. Pommerenke [20].

Theorem 1. *Let $\mathcal{L}(z, t) = a_1(t)z + a_2(t)z^2 + \dots$, $a_1(t) \neq 0$ be analytic in $\mathcal{U}_r$ for all $t \in I$, locally absolutely continuous in I , and locally uniform with respect to $\mathcal{U}_r$. For almost all $t \in I$, suppose that*

$$z \frac{\partial \mathcal{L}(z, t)}{\partial z} = p(z, t) \frac{\partial \mathcal{L}(z, t)}{\partial t}, \quad \forall z \in \mathcal{U}_r \quad (2.1)$$

where $p(z, t)$ is analytic in $\mathcal{U}$ and satisfies the condition $\Re p(z, t) > 0$ for all $z \in \mathcal{U}$, $t \in I$. If $|a_1(t)| \rightarrow \infty$ for $t \rightarrow \infty$ and $\{\mathcal{L}(z, t)/a_1(t)\}$ forms a normal family in $\mathcal{U}_r$, then for each $t \in I$, the function $\mathcal{L}(z, t)$ has an analytic and univalent extension to the whole disk $\mathcal{U}$.

The method of constructing quasiconformal extension criteria is based on the following result of Becker (see [3], [4] and also [5]).

Theorem 2. *Suppose that $\mathcal{L}(z, t)$ is a Loewner chain for which the function $p(z, t)$ given in (2.1) satisfies the condition*

$$\begin{aligned} p(z, t) \in \mathcal{U}(k) &:= \left\{ w \in \mathbb{C} : \left| \frac{w-1}{w+1} \right| \leq k \right\} \\ &= \left\{ w \in \mathbb{C} : \left| w - \frac{1+k^2}{1-k^2} \right| \leq \frac{2k}{1-k^2} \right\}, \quad (0 \leq k < 1) \end{aligned}$$

for all $z \in \mathcal{U}$ and $t \geq 0$. Then $\mathcal{L}(z, t)$ admits a continuous extension to $\overline{\mathcal{U}}$ for each $t \geq 0$ and the function $F(z, \bar{z})$ defined by

$$F(z, \bar{z}) = \begin{cases} \mathcal{L}(z, 0), & \text{if } |z| < 1 \\ \mathcal{L}(\frac{z}{|z|}, \log |z|), & \text{if } |z| \geq 1 \end{cases}$$

is a k -quasiconformal extension of $\mathcal{L}(z, 0)$ to $\mathbb{C}$.

Examples of quasiconformal extension criteria can be found in [1], [2], [6], [13], [19] and more recently in [10–12].

3. MAIN RESULTS

In this section, using Theorem 1, we obtain certain sufficient conditions for the univalence of an integral operator.

Theorem 3. Let m be a positive real number and let α, β be complex numbers such that $\Re\alpha < 1/2$, $\Re\beta > 0$ and $f \in \mathcal{A}$. Let g and h be two analytic functions in $\mathcal{U}$, $g(z) = 1 + b_1z + \dots$, $h(z) = c_0 + c_1z + \dots$. If the following inequalities

$$\left| \frac{f'(z)}{g(z) - \alpha} - \frac{m-1}{2} \right| < \frac{m+1}{2}, \quad (3.1)$$

and

$$\begin{aligned} & \left| \left(\frac{f'(z)}{g(z) - \alpha} - 1 \right) |z|^{\beta(m+1)} \right. \\ & \quad \left. + (1 - |z|^{\beta(m+1)}) \left[2z^\beta \frac{f'(z)h(z)}{g(z) - \alpha} + \frac{1}{\beta} \frac{zg'(z)}{g(z) - \alpha} \right] \right. \\ & \quad \left. + \frac{z^{\beta+1} (1 - |z|^{\beta(m+1)})^2}{|z|^{\beta(m+1)}} \left[\frac{z^{\beta-1} f'(z)h^2(z)}{g(z) - \alpha} + \frac{1}{\beta} \left(\frac{g'(z)h(z)}{g(z) - \alpha} - h'(z) \right) \right] - \frac{m-1}{2} \right| \\ & \leq \frac{m+1}{2} \end{aligned} \quad (3.2)$$

are true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by

$$\mathcal{F}_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u) du \right]^{1/\beta} \quad (3.3)$$

is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Proof. We shall prove that there exists a real number r , $r \in (0, 1]$ such that the function $\mathcal{L} : \mathcal{U}_r \times I \rightarrow \mathbb{C}$, defined formally by

$$\mathcal{L}(z, t) = \left[\beta \int_0^{e^{-t}z} u^{\beta-1} f'(u) du + \frac{(e^{\beta mt} - e^{-\beta t}) z^\beta (g(e^{-t}z) - \alpha)}{1 + (e^{\beta mt} - e^{-\beta t}) z^\beta h(e^{-t}z)} \right]^{1/\beta} \quad (3.4)$$

is analytic in $\mathcal{U}_r$ for all $t \in I$.

Because $f \in \mathcal{A}$ we have

$$f(z) = z + a_2 z^2 + \dots + a_n z^n + \dots, \quad \forall z \in \mathcal{U}.$$

Let us denote by

$$\varphi_1(z, t) = \beta \int_0^{e^{-t}z} u^{\beta-1} f'(u) du. \quad (3.5)$$

We obtain $\varphi_1(z, t) = (e^{-t}z)^\beta + \frac{2\beta a_2}{\beta+1} (e^{-t}z)^{\beta+1} + \dots$ and we observe that

$$\varphi_1(z, t) = z^\beta \varphi_2(z, t) \quad (3.6)$$

where

$$\varphi_2(z, t) = e^{-\beta t} + \sum_{n=2}^{\infty} \frac{n\beta}{n+\beta-1} a_n e^{-(n+\beta-1)t} z^{n-1}. \quad (3.7)$$

The function φ_2 is analytic in $\mathcal{U}$ for all $t \in I$, since

$$\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n\beta}{n+\beta-1} a_n e^{-(n+\beta-1)t} \right|} = e^{-t} \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|a_n|}.$$

It is clear that if $z \in \mathcal{U}$, then $e^{-t}z \in \mathcal{U}$ for all $t \in I$ and because $f'(0) = 1$, there exists a disk $\mathcal{U}_{r_1}$, $0 < r_1 \leq 1$ in which $f'(e^{-t}z) \neq 0$ for all $t \geq 0$.

From the analyticity of f it follows that the function φ_3 is also analytic in $\mathcal{U}_{r_1}$, where

$$\varphi_3(z, t) = 1 + \left(e^{\beta m t} - e^{-\beta t} \right) z^{\beta} h(e^{-t}z). \quad (3.8)$$

We have $\varphi_3(0, t) = 1$ and then there exists a disk $\mathcal{U}_{r_2}$, $0 < r_2 \leq r_1$ in which $\varphi_3(z, t) \neq 0$ for all $t \geq 0$.

Then the function

$$\varphi_4(z, t) = \varphi_2(z, t) + \left(e^{\beta m t} - e^{-\beta t} \right) \frac{(g(e^{-t}z) - \alpha)}{\varphi_3(z, t)} \quad (3.9)$$

is also analytic in $\mathcal{U}_{r_2}$ and $\varphi_4(0, t) = (1 - \alpha)e^{\beta m t} + \alpha e^{-\beta t}$. From $\Re \alpha < 1/2$, $\Re \beta > 0$ we deduce that $\varphi_4(0, t) \neq 0$ for all $t \in I$. Therefore, there exists a disk $\mathcal{U}_r$, $0 < r \leq r_2$ in which $\varphi_4(0, t) \neq 0$ for all $t \in I$ and we can choose an analytic branch of $[\varphi_4(z, t)]^{1/\beta}$, denoted by $\varphi_5(z, t)$. We choose the uniform branch which is equal to $a_1(t) = \left[(1 - \alpha)e^{\beta m t} + \alpha e^{-\beta t} \right]^{1/\beta}$ at the origin, and for $a_1(t)$ we get $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$. Moreover, we have $a_1(t) \neq 0$ for all $t \geq 0$.

From (3.4)-(3.9) it follows that the relation (3.4) can be written as

$$\mathcal{L}(z, t) = z\varphi_5(z, t) \quad (3.10)$$

and hence we obtain that the function $\mathcal{L}(z, t)$ is analytic in $\mathcal{U}_r$,

$$\mathcal{L}(z, t) = a_1(t)z + \dots, \quad \forall z \in \mathcal{U}_r, \quad \forall t \in I.$$

$\mathcal{L}(z, t)$ is an analytic function in $\mathcal{U}_r$ for all $t \in I$ and then it follows that there is a number r_3 , $0 < r_3 < r$ and a positive constant $K = K(r_3)$ such that

$$\left| \frac{\mathcal{L}(z, t)}{a_1(t)} \right| < K, \quad \forall z \in \mathcal{U}_{r_3}, \quad t \geq 0.$$

Then, by Montel's theorem, it follows that $\left\{ \frac{\mathcal{L}(z, t)}{a_1(t)} \right\}_{t \geq 0}$ is a normal family in $\mathcal{U}_{r_3}$.

From (3.10) we have

$$\frac{\partial \mathcal{L}(z, t)}{\partial t} = z \frac{\partial \varphi_5(z, t)}{\partial t}. \quad (3.11)$$

It is clear that $\frac{\partial \varphi_5(z,t)}{\partial t}$ is an analytic function in $\mathcal{U}_{r_3}$ and then $\frac{\partial \mathcal{L}(z,t)}{\partial t}$ is also an analytic function in $\mathcal{U}_{r_3}$. Then, for all fixed numbers $T > 0$ and r_4 , $0 < r_4 < r_3$, there exists a constant $K_1 > 0$ (which depends on T and r_4) such that

$$\left| \frac{\partial \mathcal{L}(z,t)}{\partial t} \right| < K_1, \quad \forall z \in \mathcal{U}_{r_4} \text{ and } t \in [0, T].$$

Therefore, the function $\mathcal{L}(z,t)$ is locally absolutely continuous in $[0, \infty)$ and is locally uniform with respect to $\mathcal{U}_{r_4}$.

Since $\frac{\partial \mathcal{L}(z,t)}{\partial t}$ is analytic in $\mathcal{U}_{r_4}$, from (3.11) it follows that there is a number r_0 , $0 < r_0 < r_4$, such that $\frac{1}{z} \frac{\partial \mathcal{L}(z,t)}{\partial t} \neq 0$, $\forall z \in \mathcal{U}_{r_0}$, so the function

$$p(z,t) = z \frac{\partial \mathcal{L}(z,t)}{\partial z} \bigg/ \frac{\partial \mathcal{L}(z,t)}{\partial t}$$

is analytic in $\mathcal{U}_{r_0}$ for all $t \geq 0$.

In order to prove that the function $p(z,t)$ has an analytic extension with positive real part in $\mathcal{U}$ for all $t \geq 0$, it is sufficient to prove that the function $w(z,t)$ defined in $\mathcal{U}_{r_0}$ by

$$w(z,t) = \frac{p(z,t) - 1}{p(z,t) + 1}$$

can be extended analytically in $\mathcal{U}$, $|w(z,t)| < 1$ for all $z \in \mathcal{U}$ and $t \geq 0$.

After some calculations we obtain:

$$w(z,t) = \frac{2}{m+1} \mathcal{G}(z,t) - \frac{m-1}{m+1}, \quad (3.12)$$

where

$$\begin{aligned} \mathcal{G}(z,t) = & e^{-\beta(m+1)t} \left(\frac{f'(e^{-t}z)}{g(e^{-t}z) - \alpha} - 1 \right) \\ & + \left(1 - e^{-\beta(m+1)t} \right) \left[2e^{-\beta t} z^\beta \frac{f'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z) - \alpha} + \frac{e^{-t}z}{\beta} \frac{g'(e^{-t}z)}{g(e^{-t}z) - \alpha} \right] \\ & + \frac{e^{-\beta t} z^\beta \left(1 - e^{-\beta(m+1)t} \right)^2}{e^{-\beta(m+1)t}} \\ & \times \left[e^{-\beta t} z^\beta \frac{f'(e^{-t}z)h^2(e^{-t}z)}{g(e^{-t}z) - \alpha} \right. \\ & \left. + \frac{e^{-t}z}{\beta} \left(\frac{h(e^{-t}z)g'(e^{-t}z)}{g(e^{-t}z) - \alpha} - h'(e^{-t}z) \right) \right]. \end{aligned} \quad (3.13)$$

for $z \in \mathcal{U}$ and $t \geq 0$.

The inequality $|w(z, t)| < 1$ for all $z \in \mathcal{U}$ and $t \geq 0$, where $w(z, t)$ defined by (3.12), is equivalent to

$$\left| \mathcal{G}(z, t) - \frac{m-1}{2} \right| < \frac{m+1}{2}, \quad \forall z \in \mathcal{U} \text{ and } t \geq 0. \quad (3.14)$$

Define

$$\mathcal{H}(z, t) = \mathcal{G}(z, t) - \frac{m-1}{2}, \quad \forall z \in \mathcal{U} \text{ and } t \geq 0. \quad (3.15)$$

In view of (3.1) and (3.2), from (3.13) and (3.15) we have

$$|\mathcal{H}(z, 0)| = \left| \left(\frac{f'(z)}{g(z) - \alpha} - 1 \right) - \frac{m-1}{2} \right| < \frac{m+1}{2}. \quad (3.16)$$

Let $t > 0$, $z \in \mathcal{U} - \{0\}$. In this case the function $\mathcal{H}(z, t)$ is analytic in $\overline{\mathcal{U}}$ because $|e^{-t}z| \leq e^{-t} < 1$, for all $z \in \overline{\mathcal{U}}$. Using the maximum principle for $z \in \mathcal{U}$ and $t > 0$ we have

$$|\mathcal{H}(z, t)| < \max_{|\xi|=1} |\mathcal{H}(\xi, t)| = |\mathcal{H}(e^{i\theta}, t)|,$$

where $\theta = \theta(t)$ is a real number.

Let $u = e^{-t}e^{i\theta}$. We have $|u| = e^{-t}$ and $e^{-\beta(m+1)t} = (e^{-t})^{\beta(m+1)} = |u|^{\beta(m+1)}$. From (3.13), we have

$$\begin{aligned} |\mathcal{G}(e^{i\theta}, t)| &= \left| |u|^{\beta(m+1)} \left(\frac{f'(u)}{g(u) - \alpha} - 1 \right) \right. \\ &\quad + (1 - |u|^{\beta(m+1)}) \left[\frac{2u^\beta f'(u)h(u)}{g(u) - \alpha} + \frac{u}{\beta} \frac{g'(u)}{g(u) - \alpha} \right] \\ &\quad + \frac{u^\beta (1 - |u|^{\beta(m+1)})^2}{|u|^{\beta(m+1)}} \\ &\quad \left. \times \left[\frac{u^\beta f'(u)h^2(u)}{g(u) - \alpha} + \frac{u}{\beta} \left(\frac{h(u)g'(u)}{g(u) - \alpha} - h'(u) \right) \right] - \frac{m-1}{2} \right|. \end{aligned}$$

Since $u \in \mathcal{U}$, the inequality (3.2) implies that

$$|\mathcal{H}(e^{i\theta}, t)| \leq \frac{m+1}{2}, \quad (3.17)$$

and from (3.16) and (3.17) it follows that the inequality (3.14)

$$|\mathcal{H}(z, t)| = \left| \mathcal{G}(z, t) - \frac{m-1}{2} \right| < \frac{m+1}{2}$$

is satisfied for all $z \in \mathcal{U}$ and $t \in I$. Therefore $|w(z, t)| < 1$, for all $z \in \mathcal{U}$ and $t \geq 0$.

Since all the conditions of Theorem 1 are satisfied, we obtain that the function $\mathcal{L}(z, t)$ has an analytic and univalent extension to the whole unit disk $\mathcal{U}$, for all $t \in I$.

For $t = 0$ we have $\mathcal{L}(z, 0) = \mathcal{F}_\beta(z)$, for $z \in \mathcal{U}$ and therefore, the function $\mathcal{F}_\beta(z)$ is analytic and univalent in $\mathcal{U}$. $\square$

For $g = f'$ in Theorem 3, we obtain another univalence criterion as follows.

Corollary 1. *Let m be a positive real number and let α, β be complex numbers such that $\Re \alpha < 1/2$, $\Re \beta > 0$ and $f \in \mathcal{A}$. Let h be an analytic functions in $\mathcal{U}$, $h(z) = c_0 + c_1 z + \dots$. If the following inequalities*

$$\left| \frac{f'(z)}{f'(z) - \alpha} - \frac{m+1}{2} \right| < \frac{m+1}{2}, \quad (3.18)$$

and

$$\begin{aligned} & \left| \left(\frac{f'(z)}{f'(z) - \alpha} - 1 \right) |z|^{\beta(m+1)} \right. \\ & \quad \left. + (1 - |z|^{\beta(m+1)}) \left[2z^\beta \frac{f'(z)h(z)}{f'(z) - \alpha} + \frac{1}{\beta} \frac{zf''(z)}{f'(z) - \alpha} \right] \right. \\ & \quad \left. + \frac{z^{\beta+1} (1 - |z|^{\beta(m+1)})^2}{|z|^{\beta(m+1)}} \left[\frac{z^{\beta-1} f'(z)h^2(z)}{f'(z) - \alpha} + \frac{1}{\beta} \left(\frac{f''(z)h(z)}{f'(z) - \alpha} - h'(z) \right) \right] \right| \end{aligned} \quad (3.19)$$

$$\begin{aligned} & \left| -\frac{m-1}{2} \right| \\ & \leq \frac{m+1}{2} \end{aligned} \quad (3.20)$$

are true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.3) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

If we choose $h = f''$ in Corollary 1, we have another univalence criterion as follows.

Corollary 2. *Let m be a positive real number and let α, β be complex numbers such that $\Re \alpha < 1/2$, $\Re \beta > 0$ and $f \in \mathcal{A}$. Let h be an analytic functions in $\mathcal{U}$, $h(z) = c_0 + c_1 z + \dots$. If the following inequalities*

$$\left| \frac{f'(z)}{f'(z) - \alpha} - \frac{m+1}{2} \right| < \frac{m+1}{2}, \quad (3.21)$$

and

$$\begin{aligned} & \left| \left(\frac{f'(z)}{f'(z) - \alpha} - 1 \right) |z|^{\beta(m+1)} \right. \\ & \quad \left. + (1 - |z|^{\beta(m+1)}) \left[2z^\beta \frac{f'(z)h(z)}{f'(z) - \alpha} + \frac{1}{\beta} \frac{zf''(z)}{f'(z) - \alpha} \right] \right| \end{aligned}$$

$$+ \frac{z^{\beta+1} (1 - |z|^{\beta(m+1)})^2}{|z|^{\beta(m+1)}} \left[\frac{z^{\beta-1} f'(z) h^2(z)}{f'(z) - \alpha} + \frac{1}{\beta} \left(\frac{f''(z) h(z)}{f'(z) - \alpha} - h'(z) \right) \right] \quad (3.22)$$

$$\begin{aligned} & \left| -\frac{m-1}{2} \right| \\ & \leq \frac{m+1}{2} \end{aligned} \quad (3.23)$$

are true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.3) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Corollary 3. Let m be a positive real number and let α, β be complex numbers such that $\Re \alpha < 1/2$, $\Re \beta > 0$ and $f \in \mathcal{A}$. If the following inequalities

$$\left| \frac{f'(z)}{f'(z) - \alpha} - \frac{m+1}{2} \right| < \frac{m+1}{2}, \quad (3.24)$$

and

$$\begin{aligned} & \left| \left(\frac{f'(z)}{f'(z) - \alpha} - 1 \right) |z|^{\beta(m+1)} + (1 - |z|^{\beta(m+1)}) \left[\frac{1}{\beta} \frac{z f''(z)}{f'(z) - \alpha} \right] - \frac{m-1}{2} \right| \\ & \leq \frac{m+1}{2} \end{aligned} \quad (3.25)$$

are true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.3) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Proof. It results from Corollary 1 with $g = f'$ and $h = 0$. $\square$

If we consider $g(z) = f'$, $h(z) = -\frac{1}{2} \frac{f''}{f'}$, $\alpha = 0$, $\beta = 1$ in Theorem 3, we obtain another univalence criterion as follows.

Corollary 4. Let m be a positive real number and $f \in \mathcal{A}$. If the following inequality

$$\left| \frac{z^2 (1 - |z|^{m+1})^2}{|z|^{m+1}} \left(\frac{1}{2} \{f; z\} \right) - \frac{m-1}{2} \right| \leq \frac{m+1}{2} \quad (3.26)$$

where

$$\{f; z\} = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2$$

is true for all $z \in \mathcal{U}$, then the function $f(z)$ is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Setting $\alpha = 0$ in Corollary 3 we have another univalence criterion as follows.

Corollary 5. Let m be a positive real number and let β be complex number such that $\Re \beta > 0$ and $f \in \mathcal{A}$. If the following inequality

$$\left| \frac{(1 - |z|^{\beta(m+1)})}{\beta} \left(\frac{zf''(z)}{f'(z)} \right) - \frac{m-1}{2} \right| \leq \frac{m+1}{2} \quad (3.27)$$

is true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.3) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Corollary 6. Let m be a positive real number and let β be complex number with $\Re \beta > 0$ and $f \in \mathcal{A}$. If the following inequality

$$\left| \frac{(1 - |z|^{(m+1)\Re \beta})}{\Re \beta} \left(\frac{zf''(z)}{f'(z)} \right) \right| \leq 1 \quad (3.28)$$

is true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.3) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Proof. It can be proved (see [18]) that for $z \in \mathcal{U} \setminus \{0\}$, $\Re \beta > 0$ and $m \in \mathbb{R}_+$

$$\left| \frac{1 - |z|^{(m+1)\beta}}{\beta} \right| \leq \frac{1 - |z|^{(m+1)\Re \beta}}{\Re \beta}.$$

For $m \geq 1$, we have

$$\begin{aligned} & \left| \frac{1 - |z|^{(m+1)\beta}}{\beta} \left(\frac{zf''(z)}{f'(z)} \right) - \frac{m-1}{2} \right| \\ & \leq \left| \frac{1 - |z|^{(m+1)\beta}}{\beta} \left(\frac{zf''(z)}{f'(z)} \right) \right| + \frac{m-1}{2} \\ & \leq \frac{1 - |z|^{(m+1)\Re \beta}}{\Re \beta} \left| \frac{zf''(z)}{f'(z)} \right| + \frac{m-1}{2} \\ & \leq 1 + \frac{m-1}{2} = \frac{m+1}{2}. \end{aligned}$$

Since inequalities (3.1) and (3.2) are satisfied, making use of Theorem 3, we can conclude that the function $\mathcal{F}_\beta$ is analytic and univalent in $\mathcal{U}$. $\square$

Putting $g(z) = \left(\frac{f(z)}{z} \right)^2$, $h(z) = 0$, $\alpha = 0$, in Theorem 3, we get the univalence criterion as follows.

Corollary 7. Let m be a positive real number and let β be complex number such that $\Re \beta > 0$ and $f \in \mathcal{A}$. If the following inequalities

$$\left| \frac{z^2 f'(z)}{f^2(z)} - \frac{m+1}{2} \right| < \frac{m+1}{2}, \quad (3.29)$$

and

$$\left| \left(\frac{z^2 f'(z)}{f^2(z)} - 1 \right) |z|^{\beta(m+1)} + \frac{2(1 - |z|^{\beta(m+1)})}{\beta} \left(\frac{z f'(z)}{f(z)} - 1 \right) - \frac{m-1}{2} \right| \leq \frac{m+1}{2} \quad (3.30)$$

are true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ defined by (3.3) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Corollary 8. Let m be a positive real number and $f \in \mathcal{A}$. If the following inequality

$$\left| z(1 - |z|^{m+1}) \left(2f''(z) + \frac{f''(z)}{f'(z)} \right) + \frac{z^2(1 - |z|^{m+1})^2}{|z|^{m+1}} \left(\frac{(f''(z))^2}{f'(z)} + (f''(z))^2 - f'''(z) \right) - \frac{m-1}{2} \right| \leq \frac{m+1}{2} \quad (3.31)$$

is true for all $z \in \mathcal{U}$, then the function $f(z)$ is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Proof. It results from Corollary 2 with $\alpha = 0$, $\beta = 1$. □

Remark 1. (1) Putting $g(z) = f'(z)$, $h(z) = 0$, $\alpha = 0$, $\beta = m = 1$ in Theorem 3, we have Becker's criterion [3].

(2) If we consider $g(z) = f'(z)$, $h(z) = -\frac{1}{2} \frac{f''(z)}{f'(z)}$, $\alpha = 0$, $\beta = m = 1$ in Theorem 3, we obtain the univalence criterion due to Nehari [14].

(3) Setting $g(z) = \left(\frac{f(z)}{z} \right)^2$, $h(z) = \frac{1}{z} - \frac{f(z)}{z^2}$, $\alpha = 0$, $\beta = m = 1$ in Theorem 3, we get the univalence criterion due to Ozaki-Nunokawa [17].

(4) For $g(z) = f'(z)$, $h(z) = \frac{1}{z} - \frac{f(z)}{f'(z)}$, $\alpha = 0$, $\beta = m = 1$ in Theorem 3, we arrive at Goluzin's criterion for univalence [9].

(5) For $m = 1$ in Corollary 6, we obtain the univalence criterion due to Pascu [18].

(6) If we consider $g(z) = f'(z)$, $h(z) = 0$, $\beta = 1$ in Theorem 3, we have results of Raducanu et al. [23].

(7) Putting $\alpha = 0$, $\beta = m = 1$ in Theorem 3, we get the univalence criterion due to Ovesea-Tudor and Owa [16].

Example 1. Let the function

$$f(z) = \frac{z}{1 - \frac{z^2}{2}}. \quad (3.32)$$

Then f is univalent in $\mathcal{U}$ and the function

$$\mathcal{F}_2(z) = \left(2 \int_0^z u f'(u) du \right)^{\frac{1}{2}} \quad (3.33)$$

is analytic and univalent in $\mathcal{U}$.

Infact, from equality (3.29) for $m = 1$, we have

$$\frac{z^2 f'(z)}{f^2(z)} - 1 = \frac{z^2}{2}. \quad (3.34)$$

It is clear that the condition (3.29) of the Corollary 7 is satisfied for $m = 1$, and then the function f is univalent in $\mathcal{U}$.

Taking into account (3.34), the condition (3.30) of Corollary 7 becomes for $\beta = 2$, $m = 1$,

$$\begin{aligned} \left| \frac{z^2}{2} |z|^4 + (1 - |z|^4) \frac{2z^2}{2 - z^2} \right| &\leq \frac{|z|^6}{2} + 2(1 - |z|^4) |z|^2 \\ &= \frac{1}{2} (4|z|^2 - 3|z|^6) < 1 \end{aligned}$$

because the greatest value of the function $g(x) = 4x^2 - 3x^6$, for $x \in [0, 1]$ is taken for $x = \sqrt{\frac{2}{3}}$ and $g(\sqrt{\frac{2}{3}}) = \frac{24}{27}$. Therefore the function $\mathcal{F}_2(z)$ defined by (3.33) is analytic and univalent in $\mathcal{U}$.

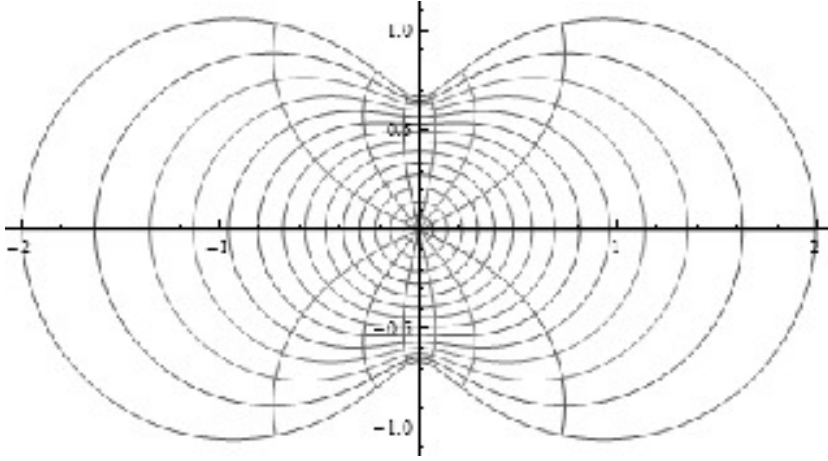


FIGURE 1. $f(z) = \frac{z}{1 - \frac{z^2}{2}}$

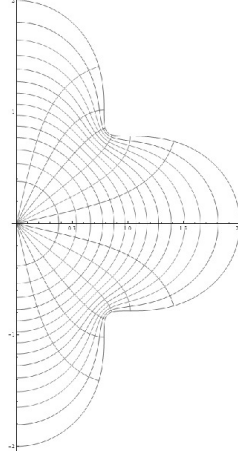


FIGURE 2. $\mathcal{F}_2(z) = \left(4 \int_0^z \frac{2+u^2}{(2-u^2)^2} du\right)^{\frac{1}{2}}$

4. QUASICONFORMAL EXTENSION CRITERION

In this section we will generalize the univalence condition given in Theorem 3 to a quasiconformal extension criterion.

Theorem 4. *Let m be a positive real number and let α, β be complex numbers such that $\Re \alpha < 1/2$, $\Re \beta > 0$, $f \in \mathcal{A}$ and $k \in [0, 1)$. Let g and h be two analytic functions in $\mathcal{U}$, $g(z) = 1 + b_1 z + \dots$, $h(z) = c_0 + c_1 z + \dots$. If the following inequalities*

$$\left| \frac{f'(z)}{g(z) - \alpha} - \frac{m+1}{2} \right| < k \frac{m+1}{2}, \quad (4.1)$$

and

$$\begin{aligned} & \left| \left(\frac{f'(z)}{g(z) - \alpha} - 1 \right) |z|^{\beta(m+1)} \right. \\ & \quad \left. + (1 - |z|^{\beta(m+1)}) \left[2z^\beta \frac{f'(z)h(z)}{g(z) - \alpha} + \frac{1}{\beta} \frac{zg'(z)}{g(z) - \alpha} \right] \right. \\ & \quad \left. + \frac{z^{\beta+1} (1 - |z|^{\beta(m+1)})^2}{|z|^{\beta(m+1)}} \left[\frac{z^{\beta-1} f'(z)h^2(z)}{g(z) - \alpha} + \frac{1}{\beta} \left(\frac{g'(z)h(z)}{g(z) - \alpha} - h'(z) \right) \right] - \frac{m-1}{2} \right| \\ & \leq k \frac{m+1}{2} \end{aligned} \quad (4.2)$$

is true for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta(z)$ given by (3.3) has a k -quasiconformal extension to $\mathbb{C}$.

Proof. Set

$$\mathcal{L}(z, t) = \left[\beta \int_0^{e^{-t}z} u^{\beta-1} f'(u) du + \frac{(e^{\beta m t} - e^{-\beta t}) z^{\beta} (g(e^{-t}z) - \alpha)}{1 + (e^{\beta m t} - e^{-\beta t}) z^{\beta} h(e^{-t}z)} \right]^{1/\beta} \quad (4.3)$$

In the proof of Theorem 3 has been shown that the function $\mathcal{L}(z, t)$ given by (4.3) is a subordination chain in $\mathcal{U}$. Then we have

$$\begin{aligned} \left| \frac{p(z, t) - 1}{p(z, t) + 1} \right| &= \left| \frac{2}{m+1} \left\{ e^{-\beta(m+1)t} \left(\frac{f'(e^{-t}z)}{g(e^{-t}z) - \alpha} - 1 \right) \right. \right. \\ &+ (1 - e^{-\beta(m+1)t}) \left[2e^{-\beta t} z^{\beta} \frac{f'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z) - \alpha} + \frac{e^{-t}z}{\beta} \frac{g'(e^{-t}z)}{g(e^{-t}z) - \alpha} \right] \\ &\quad + \frac{e^{-\beta t} z^{\beta} (1 - e^{-\beta(m+1)t})^2}{e^{-\beta(m+1)t}} \\ &\times \left[e^{-\beta t} z^{\beta} \frac{f'(e^{-t}z)h^2(e^{-t}z)}{g(e^{-t}z) - \alpha} + \frac{e^{-t}z}{\beta} \left(\frac{h(e^{-t}z)g'(e^{-t}z)}{g(e^{-t}z) - \alpha} - h'(e^{-t}z) \right) \right] \Big\} \\ &\quad \left. - \frac{m-1}{m+1} \right| \\ &\leq k. \end{aligned} \quad (4.4)$$

The right hand of (4.4) always less than or equal to k from (4.2) and therefore $\mathcal{F}_{\beta}$ can be extended to k quasiconformal mapping to $\mathbb{C}$ by Theorem 1 and Theorem 2. $\square$

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FORCED FERMI-PASTA-ULAM LATTICE MAPS

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Abstract. We prove existence and uniqueness results for periodic and quasiperiodic travelling waves of Fermi-Pasta-Ulam (FPU) type lattice maps. We also study quasiperiodic solutions of difference equations on Banach algebras as a natural generalizations of the FPU stationary lattice equations.

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1. INTRODUCTION

Dynamical systems on infinite lattices play an important role in applications to natural sciences. There are several well-known types of lattice equations such as sine-Gordon, Frenkel-Kontorova model, Klein-Gordon, Toda, discrete Schrödinger, Ablowitz-Ladik nonlinear Schrödinger, Fermi-Pasta-Ulam and discrete Nagumo equations. We refer the reader to the huge literature [1–4, 7, 8, 10, 12, 15, 16, 18, 21, 22]. Some of these lattice models are derived by spatial discretization of corresponding partial differential equations. In any case, they are defined as an infinite number of coupled nonlinear oscillators where the theory of ordinary differential equations in infinite dimensions can be applied [9, 17]. Of course, the dynamics of such systems is extremely rich. There are three important types of solutions: stationary ones, spatially localized ones, the so called breathers and the travelling waves.

This paper is devoted to the study of discrete FPU type lattices. The FPU model consists of a chain of particles connected by nonlinear springs

$$\ddot{u}_n = \phi(u_{n+1} - u_n) + \phi(u_{n-1} - u_n), \quad n \in \mathbb{Z} \quad (1.1)$$

for a smooth and odd function ϕ . Most of results are connected with Hamiltonian case presented by (1.1). But motivated by [13, 14], we consider in [11] a 1D (possible)

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damped FPU lattice forced by a travelling wave field:

$$\ddot{u}_n = \alpha(u_{n+1} + u_{n-1} - 2u_n) + \beta(u_{n+1} - u_n)^3 + \beta(u_{n-1} - u_n)^3 - \gamma \dot{u}_n + f \cos(\omega t + pn), \quad (1.2)$$

where $\alpha > 0$, $\beta > 0$, $\gamma \geq 0$, $\omega > 0$, $p \neq 0$, $f \neq 0$ are parameters. Analytical and numerical methods are used. This paper is a continuation of [11] in the sense that we also discretize (1.2) in time. Moreover, we consider general analytical nonlinearities (see (2.1) and (2.3)). First in Section 2, we study forced periodic and quasiperiodic travelling waves of such FPU lattice maps. We also mention a few words on their stationary solutions. Then in Section 3, we extend our method to nonlocal interactions. In Section 4, we end this paper with the investigation of general difference equations on Banach algebras, which are generalizations of stationary lattice equations. Proofs of our results are based on a majorant method [19]. Two illustrative examples are given as well.

2. 1D FORCED FPU LATTICE MAPS WITH LOCAL INTERACTIONS

In this section, we consider a discrete version of the following 1-dimensional damped FPU lattice forced by a travelling wave field (see [13, 14]):

$$\ddot{u}_n = \alpha(u_{n+1} - 2u_n + u_{n-1}) + \varphi_1(u_{n+1} - u_n) + \varphi_2(u_{n-1} - u_n) - \gamma \dot{u}_n + f \cos(\omega t + pn), \quad (2.1)$$

where $\alpha > 0$, $\gamma \geq 0$, $\omega > 0$, $p \neq 0$, $f \neq 0$ are parameters and φ_1, φ_2 are odd analytic functions with radius of convergence ρ_1, ρ_2 , respectively, such that

$$D\varphi_{1,2}(0) = 0. \quad (2.2)$$

Equation (2.1) of the form (1.2) is studied in [11]. We substitute the differentiation by the symmetric difference, i.e.

$$\dot{u}_n(t) \rightarrow u_n(t + 1/2) - u_n(t - 1/2), \quad \ddot{u}_n(t) \rightarrow u_n(t + 1) - 2u_n(t) + u_n(t - 1).$$

So we study the travelling waves of the system

$$\begin{aligned} u_n(t + 1) - 2u_n(t) + u_n(t - 1) = & \alpha(u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)) \\ & + \varphi_1(u_{n+1}(t) - u_n(t)) + \varphi_2(u_{n-1}(t) - u_n(t)) \\ & - \gamma(u_n(t + 1/2) - u_n(t - 1/2)) + f \cos(\omega t + pn). \end{aligned} \quad (2.3)$$

Putting $u_n(t) = U(\omega t + pn)$ for $U(z + \pi) = -U(z)$ in (2.3), we get

$$\begin{aligned} U(z + \omega) - 2U(z) + U(z - \omega) = & \alpha(U(z + p) - 2U(z) + U(z - p)) \\ & + \varphi_1(U(z + p) - U(z)) + \varphi_2(U(z - p) - U(z)) \\ & - \gamma(U(z + \omega/2) - U(z - \omega/2)) + f \cos z \end{aligned} \quad (2.4)$$

with $z = \omega t + pn$. We take Banach spaces

$$W := \left\{ U \in C(\mathbb{R}, \mathbb{R}) \mid U(z) = \sum_{k \in \mathbb{Z}} d_k e^{kiz}, \sum_{k \in \mathbb{Z}} |d_k| < \infty \right\},$$

$$X := \left\{ U \in C(\mathbb{R}, \mathbb{R}) \mid U(z) = \sum_{k \in \mathbb{Z}} c_k e^{(2k+1)iz}, \sum_{k \in \mathbb{Z}} |c_k| < \infty \right\}$$

with the norms

$$\|U\| := \sum_{k \in \mathbb{Z}} |d_k|, \quad \|U\| := \sum_{k \in \mathbb{Z}} |c_k|,$$

respectively. Obviously, $X \subset W$.

The following lemma is clear.

Lemma 1. *If $U_1, U_2 \in W$ then $U_1 U_2 \in W$ and $\|U_1 U_2\| \leq \|U_1\| \|U_2\|$. For each $k \in \mathbb{N}$, if $U_1, U_2, \dots, U_{2k+1} \in X$ then $U_1 U_2 \dots U_{2k+1} \in X$.*

By setting

$$\begin{aligned} \mathcal{K}U(z) &:= U(z + \omega) - 2U(z) + U(z - \omega) - \alpha(U(z + p) - 2U(z) + U(z - p)) \\ &\quad + \gamma(U(z + \omega/2) - U(z - \omega/2)), \end{aligned}$$

$$\mathcal{F}(U, f)(z) := \varphi_1(U(z + p) - U(z)) + \varphi_2(U(z - p) - U(z)) + f \cos z,$$

equation (2.4) has the form

$$\mathcal{K}U = \mathcal{F}(U, f).$$

Denote $\rho_0 := \min\{\rho_1, \rho_2\}$ and $B(\rho_0) := \{U \in X \mid \|U\| < \rho_0\}$.

We have the next result.

Lemma 2. *Function $\mathcal{F} : B(\rho_0/2) \times \mathbb{R} \rightarrow X$ fulfils*

$$\|\mathcal{F}(U, f)\| \leq \sum_{k=3}^{\infty} \frac{|D^k \varphi_1(0)| + |D^k \varphi_2(0)|}{k!} 2^k \|U\|^k + |f|, \quad (2.5)$$

$$\begin{aligned} &\|\mathcal{F}(U_1, f) - \mathcal{F}(U_2, f)\| \\ &\leq \|U_1 - U_2\| \sum_{k=3}^{\infty} \frac{|D^k \varphi_1(0)| + |D^k \varphi_2(0)|}{k!} 2^k \sum_{i=0}^{k-1} \|U_1\|^i \|U_2\|^{k-i-1}, \end{aligned} \quad (2.6)$$

$$\|\mathcal{F}(U, f_1) - \mathcal{F}(U, f_2)\| \leq |f_1 - f_2| \quad (2.7)$$

for any $U, U_1, U_2 \in B(\rho_0/2) \subset X$ and $f, f_1, f_2 \in \mathbb{R}$.

Proof. Since $\cos z = \frac{e^{iz} + e^{-iz}}{2}$, we get $\|\cos z\| = 1$ and (2.7) easily follows. Next we derive

$$U(z \pm p) - U(z) = \sum_{k \in \mathbb{Z}} c_k \left(e^{\pm(2k+1)ip} - 1 \right) e^{(2k+1)iz}$$

for any $U \in X$ with $U(z) = \sum_{k \in \mathbb{Z}} c_k e^{(2k+1)\iota z}$. So we obtain

$$\|U(z \pm p) - U(z)\| = \sum_{k \in \mathbb{Z}} |c_k| \left| e^{\pm(2k+1)\iota p} - 1 \right| \leq 2\|U\| \quad (2.8)$$

for any $U \in X$. For $U \in B(\rho_0)$ we estimate the Taylor series

$$\|\varphi_i(U)\| \leq \sum_{k=3}^{\infty} \frac{|D^k \varphi_i(0)|}{k!} \|U\|^k < \infty$$

for $i = 1, 2$. Note that since φ_1, φ_2 are odd, their even Taylor coefficients are zero. Then combining these estimates with (2.8) and Lemma 1, we arrive at (2.5).

Next arguing like for (2.8), we obtain

$$\|U_1(z \pm p) - U_1(z) - (U_2(z \pm p) - U_2(z))\| \leq 2\|U_1 - U_2\| \quad (2.9)$$

for any $U_1, U_2 \in X$. Then we apply Lemma 1, estimates (2.8), (2.9) and the formula

$$a^k - b^k = (a - b)(a^{k-1} + a^{k-2}b + \dots + b^{k-1}) \quad (2.10)$$

for any $a, b \in \mathbb{C}$, odd $k \in \mathbb{N}$ to obtain property (2.6). The proof is finished. $\square$

Next if $U \in X$ with $U(z) = \sum_{k \in \mathbb{Z}} c_k e^{(2k+1)\iota z}$, then

$$\mathcal{K}U(z) = \sum_{k \in \mathbb{Z}} \left(4\alpha \sin^2 \frac{2k+1}{2} p - 4 \sin^2 \frac{2k+1}{2} \omega + 2\iota \gamma \sin \frac{2k+1}{2} \omega \right) c_k e^{(2k+1)\iota z} \quad (2.11)$$

and so $\mathcal{K} \in L(X)$ with

$$\|\mathcal{K}\|_{L(X)} \leq 4\alpha + 4 + 2\gamma.$$

If

$$\Theta := \inf_{k \in \mathbb{Z}} \sqrt{\left(4\alpha \sin^2 \frac{2k+1}{2} p - 4 \sin^2 \frac{2k+1}{2} \omega \right)^2 + 4\gamma^2 \sin^2 \frac{2k+1}{2} \omega} > 0 \quad (2.12)$$

for a constant Θ depending on $\alpha, p, \omega, \gamma$, then $\mathcal{K}^{-1} \in L(X)$ is such that

$$\|\mathcal{K}^{-1}\|_{L(X)} \leq \frac{1}{\Theta}. \quad (2.13)$$

Now we can prove the following existence result on (2.4) when all parameters except f are fixed.

Theorem 1. Assume (2.12) along with

$$|f| < |f_l| \quad (2.14)$$

for f_l satisfying

$$A(r) := \Theta r - \sum_{k=3}^{\infty} \frac{|D^k \varphi_1(0)| + |D^k \varphi_2(0)|}{k!} 2^k r^k = |f_l|, \quad (2.15)$$

$$DA(r) = 0 \quad (2.16)$$

for some $r \in (0, \rho_0/2)$. Then (2.4) has a unique solution $U(f) \in \overline{B(\rho_f)}$ in a closed ball where $\rho_f < \rho_0/2$ is a smallest positive root of $A(r) = |f|$. Moreover, $U(f)$ can be approximated by an iteration process. Finally, it holds

$$\begin{aligned} \|U(f_1) - U(f_2)\| &\leq \frac{|f_1 - f_2|}{\Theta} \\ &\times \left(1 - \frac{1}{\Theta} \sum_{k=3}^{\infty} \frac{|\mathcal{D}^k \varphi_1(0)| + |\mathcal{D}^k \varphi_2(0)|}{(k-1)!} 2^k \rho_{\max\{|f_1|, |f_2|\}}^{k-1} \right)^{-1} \end{aligned} \quad (2.17)$$

for any $f_1, f_2 \in \mathbb{R}$ satisfying (2.14).

Proof. We rewrite (2.4) as a parametrized fixed point problem

$$U = \mathcal{R}(U, f) := \mathcal{K}^{-1} \mathcal{F}(U, f)$$

in $B(\rho_0/2) \subset X$. We already know that $\mathcal{R} : B(\rho_0/2) \times \mathbb{R} \rightarrow X$ is continuous and by (2.5), (2.13) such that

$$\|\mathcal{R}(U, f)\| \leq \frac{1}{\Theta} \left(\sum_{k=3}^{\infty} \frac{|\mathcal{D}^k \varphi_1(0)| + |\mathcal{D}^k \varphi_2(0)|}{k!} 2^k \|U\|^k + |f| \right).$$

Next, if there is $0 < \rho_f < \rho_0/2$ such that

$$A(\rho_f) = |f|, \quad (2.18)$$

then $\mathcal{R}(\cdot, f)$ maps $\overline{B(\rho_f)}$ into itself. So it remains to study (2.18). In order to find the largest f_l for which (2.18) has a solution $\rho_{f_l} > 0$, we need to solve $A(r) = |f_l|$ together with (2.16) for $r \in (0, \rho_0/2)$. This implies (2.14). Note that

$$\pm \mathcal{D}A(r) = \pm \Theta \mp \sum_{k=3}^{\infty} \frac{|\mathcal{D}^k \varphi_1(0)| + |\mathcal{D}^k \varphi_2(0)|}{(k-1)!} 2^k r^{k-1} > \mathcal{D}A(\rho_{f_l}) = 0$$

for $r \in (0, \rho_0/2)$, $\pm r < \pm \rho_{f_l}$, and $\lim_{r \rightarrow (\rho_0/2)^-} \mathcal{D}A(r) = -\infty$ (see Sections 7.21, 7.22 and 7.31 of [20]). Hence ρ_{f_l} is uniquely determined by (2.16). Moreover, continuity of $A(r)$ with $A(0) = 0$, $A(\rho_{f_l}) = |f_l|$ yield that $0 < \rho_f < \rho_{f_l}$ whenever $0 < |f| < |f_l|$, i.e. $\mathcal{D}A(\rho_f) > 0$. So assuming (2.12), (2.14) and by (2.2), we know that (2.18) has a positive solution $\rho_f < \rho_0/2$. We take the smallest one. So $\mathcal{R}(\cdot, f)$ maps $\overline{B(\rho_f)}$ into itself and, moreover, by (2.6), (2.13)

$$\|\mathcal{R}(U_1, f) - \mathcal{R}(U_2, f)\| \leq \frac{\|U_1 - U_2\|}{\Theta} \sum_{k=3}^{\infty} \frac{|\mathcal{D}^k \varphi_1(0)| + |\mathcal{D}^k \varphi_2(0)|}{(k-1)!} 2^k \rho_f^{k-1}$$

for any $U_1, U_2 \in \overline{B(\rho_f)}$. Hence, $\mathcal{R}(\cdot, f)$ is a contraction on $\overline{B(\rho_f)}$ with a contraction constant

$$\frac{1}{\Theta} \sum_{k=3}^{\infty} \frac{|\mathcal{D}^k \varphi_1(0)| + |\mathcal{D}^k \varphi_2(0)|}{(k-1)!} 2^k \rho_f^{k-1} = \frac{\Theta - \mathcal{D}A(\rho_f)}{\Theta} < \frac{\Theta - \mathcal{D}A(\rho_{f_l})}{\Theta} = 1.$$

The proof of the existence and uniqueness is finished by the Banach fixed point theorem [5]. Next, let $f_1, f_2 \in \mathbb{R}$ satisfy (2.14), then $U(f_i) \in \overline{B(\rho_{f_i})} \subset \overline{B(\rho_{f_3})}$ for $i = 1, 2$ and $f_3 := \max\{|f_1|, |f_2|\}$. Note f_3 satisfies (2.14). By (2.6), (2.7) and (2.13), we derive

$$\begin{aligned} & \|U(f_1) - U(f_2)\| = \|\mathcal{R}(U(f_1), f_1) - \mathcal{R}(U(f_2), f_2)\| \\ & \leq \|\mathcal{R}(U(f_1), f_1) - \mathcal{R}(U(f_2), f_1)\| + \|\mathcal{R}(U(f_2), f_1) - \mathcal{R}(U(f_2), f_2)\| \\ & \leq \frac{\|U(f_1) - U(f_2)\|}{\Theta} \sum_{k=3}^{\infty} \frac{|D^k \varphi_1(0)| + |D^k \varphi_2(0)|}{(k-1)!} 2^k \rho_{f_3}^{k-1} + \frac{|f_1 - f_2|}{\Theta} \end{aligned}$$

which implies (2.17). $\square$

Remark 1.

1. If $\gamma > 0$ and

$$\omega \in Q := \left\{ \frac{2l_1 + 1}{l_2} \pi \mid l_1, l_2 \in \mathbb{N} \right\},$$

then (2.12) holds and for $\omega = \frac{2l_1 + 1}{l_2} \pi$

$$\Theta \geq 2\gamma \inf_{k \in \mathbb{Z}} \left| \sin \left(\frac{2k+1}{2} \frac{2l_1 + 1}{l_2} \pi \right) \right| \geq 2\gamma \left| \sin \frac{\pi}{2l_2} \right| \geq \frac{2\gamma}{l_2}$$

since $\sin x \geq 2x/\pi$ for $x \in [0, \pi/2]$. So we can replace Θ with $2\gamma/l_2$ in the above considerations.

2. Clearly, if we change $p \leftrightarrow -p$, Θ remains the same and Theorem 1 holds as well. Thus we can assume $p > 0$. Next, each element $\frac{2l_1 + 1}{l_2} \pi$ of Q can be chosen such that $2l_1 + 1, l_2$ are relatively prime (their only common divisor is 1). In the next steps we always consider this form.

If $p = \frac{2k_1 + 1}{k_2} \pi \in Q$, then we know (cf. [6]) that there are integers a, b such that $a(2k_1 + 1) = 1 + 2bk_2$. Obviously, a is odd. So there exists $k \in \mathbb{Z}$ such that $2k + 1 = a(1 + 2k_2)$. Consequently,

$$(2k + 1)(2k_1 + 1) = 2bk_2(1 + 2k_2) + 2k_2 + 1 \equiv 1 \pmod{2k_2},$$

i.e. for such k it holds

$$\sin^2 \frac{2k+1}{2} p = \sin^2 \frac{\pi}{2k_2}.$$

In addition, on setting $l_i := i + k + 2ki$ for $i = 1, \dots, k_2 - 1$ we get

$$(2l_i + 1)(2k_1 + 1) = (2i + 1)(2k + 1)(2k_1 + 1) \equiv 2i + 1 \pmod{2k_2}$$

for each $i = 1, \dots, k_2 - 1$. Therefore,

$$\left\{ \sin^2 \frac{2k+1}{2} p \mid k \in \mathbb{Z} \right\} = \left\{ \sin^2 \frac{k\pi}{2k_2} \mid k = 1, 3, \dots, 2k_2 - 1 \right\}$$

for fixed $p = \frac{2k_1 + 1}{k_2} \pi \in Q$. Same discussion for $\omega \in Q$ yields:

If $p = \frac{2k_1+1}{k_2}\pi, \omega = \frac{2l_1+1}{l_2}\pi \in Q$ are such that

$$\left\{ \alpha \sin^2 \frac{k\pi}{2k_2} \mid k = 1, 3, \dots, 2k_2 - 1 \right\} \cap \left\{ \sin^2 \frac{l\pi}{2l_2} \mid l = 1, 3, \dots, 2l_2 - 1 \right\} = \emptyset,$$

then

$$\Theta \geq 2 \min_{\substack{k=1,3,\dots,2k_2-1 \\ l=1,3,\dots,2l_2-1}} \left| \alpha \sin^2 \frac{k\pi}{2k_2} - \sin^2 \frac{l\pi}{2l_2} \right| =: \Theta_1 > 0.$$

Thus for such p and ω we can replace Θ with Θ_1 in the above considerations.

3. One can investigate a discretization of equation (2.1) with more than one forcing which are of different periods, i.e.

$$\begin{aligned} \ddot{u}_n &= \alpha(u_{n+1} - 2u_n + u_{n-1}) + \varphi_1(u_{n+1} - u_n) + \varphi_2(u_{n-1} - u_n) \\ &\quad - \gamma \dot{u}_n + f_1 \cos(\omega_1 t + p_1 n) + f_2 \cos(\omega_2 t + p_2 n) \end{aligned}$$

by assuming $U(z_1 + \pi, z_2 + \pi) = -U(z_1, z_2)$ with $u_n(t) = U(\omega_1 t + p_1 n, \omega_2 t + p_2 n)$, and introducing Banach spaces

$$\begin{aligned} W &:= \left\{ U \in C(\mathbb{R}^2, \mathbb{R}) \mid U(z) = \sum_{k,l \in \mathbb{Z}} d_{kl} e^{i(kz_1 + lz_2)}, \sum_{k,l \in \mathbb{Z}} |d_{kl}| < \infty \right\}, \\ X &:= \left\{ U \in C(\mathbb{R}^2, \mathbb{R}) \mid U(z) = \sum_{k+l \in 2\mathbb{Z}+1} c_{kl} e^{i(kz_1 + lz_2)}, \sum_{k,l \in 2\mathbb{Z}+1} |c_{kl}| < \infty \right\} \end{aligned}$$

with the norms

$$\|U\| := \sum_{k,l \in \mathbb{Z}} |d_{kl}|, \quad \|U\| := \sum_{k,l \in 2\mathbb{Z}+1} |c_{kl}|,$$

respectively.

The other possibility is to consider forcing by a composed travelling wave field, i.e.

$$\begin{aligned} \ddot{u}_n &= \alpha(u_{n+1} - 2u_n + u_{n-1}) + \varphi_1(u_{n+1} - u_n) + \varphi_2(u_{n-1} - u_n) \\ &\quad - \gamma \dot{u}_n + f \cos(\omega_1 t + p_1 n) \cos(\omega_2 t + p_2 n) \end{aligned}$$

with $\omega_1, \omega_2 \in \mathbb{R}, \frac{\omega_1}{\omega_2} \notin \mathbb{Q}$, by assuming $U(z_1 + \pi, z_2) = -U(z_1, z_2) = U(z_1, z_2 + \pi)$ where $u_n(t) = U(\omega_1 t + p_1 n, \omega_2 t + p_2 n)$. In this case, one takes Banach spaces

$$\begin{aligned} W &:= \left\{ U \in C(\mathbb{R}^2, \mathbb{R}) \mid U(z) = \sum_{k,l \in \mathbb{Z}} d_{kl} e^{i(kz_1 + lz_2)}, \sum_{k,l \in \mathbb{Z}} |d_{kl}| < \infty \right\}, \\ X &:= \left\{ U \in C(\mathbb{R}^2, \mathbb{R}) \mid U(z) = \sum_{k,l \in \mathbb{Z}} c_{kl} e^{i(2k+1)z_1 + i(2l+1)z_2}, \sum_{k,l \in \mathbb{Z}} |c_{kl}| < \infty \right\} \end{aligned}$$

with the norms

$$\|U\| := \sum_{k,l \in \mathbb{Z}} |d_{kl}|, \quad \|U\| := \sum_{k,l \in \mathbb{Z}} |c_{kl}|,$$

respectively.

4. If u_n in (2.1) is independent of t , i.e. $u_n(t) = u_n$ for all $t \in \mathbb{R}$, equation (2.3) becomes a nonlinear difference equation

$$0 = \alpha(u_{n+1} - 2u_n + u_{n-1}) + \varphi_1(u_{n+1} - u_n) + \varphi_2(u_{n-1} - u_n) + f \cos pn \quad (2.19)$$

and by adding another forcing with different period, one can study the existence of quasiperiodic solution $\{u_n\}_{n \in \mathbb{Z}}$ of the infinite system of equations

$$0 = \alpha(u_{n+1} - 2u_n + u_{n-1}) + \varphi_1(u_{n+1} - u_n) + \varphi_2(u_{n-1} - u_n) + f_1 \cos p_1 n + f_2 \cos p_2 n, \quad n \in \mathbb{Z} \quad (2.20)$$

with $\frac{p_1}{p_2} \notin \mathbb{Q}$.

Example 1. Consider equation

$$\begin{aligned} U(z + \omega) - 2U(z) + U(z - \omega) &= \alpha(U(z + p) + U(z - p) - 2U(z)) \\ &+ \beta(U(z + p) - U(z))^3 + \beta(U(z - p) - U(z))^3 \\ &- \gamma(U(z + \omega/2) - U(z - \omega/2)) + f \cos z \end{aligned} \quad (2.21)$$

with parameters $\alpha > 0$, $\beta > 0$, $\gamma \geq 0$, $\omega > 0$, $p \neq 0$, $f \neq 0$.

Corollary 1. Assuming (2.12) and

$$|f| < \sqrt{\frac{\Theta^3}{108\beta}}, \quad (2.22)$$

equation (2.21) has a unique solution $U(f) \in \overline{B(\rho_f)} \subset X$ where

$$\rho_f = \sqrt{\frac{\Theta}{12\beta}} \sin \left(\frac{1}{3} \arcsin \left(|f| \sqrt{\frac{108\beta}{\Theta^3}} \right) \right). \quad (2.23)$$

Moreover, $U(f)$ can be approximated by an iteration process. Finally, it holds

$$\|U(f_1) - U(f_2)\| \leq \frac{|f_1 - f_2|}{\Theta \left(1 - 4 \sin^2 \left(\frac{1}{3} \arcsin \left(\max\{|f_1|, |f_2|\} \sqrt{\frac{108\beta}{\Theta^3}} \right) \right) \right)}$$

for any $f_1, f_2 \in \mathbb{R}$ satisfying (2.22).

Proof. In this case $\rho_0 = \infty$ and the corollary follows directly from Theorem 1. The smallest positive root ρ_f of equation

$$A(\rho_f) = \Theta \rho_f - 16\beta \rho_f^3 = |f|$$

is known to have a form of (2.23) [6]. □

3. 1D FORCED FPU LATTICE MAPS WITH NONLOCAL INTERACTIONS

In this section, we consider a discretization of 1-dimensional damped FPU lattice forced by a travelling wave field with nonlocal interactions:

$$\ddot{u}_n = \sum_{j \in \mathbb{Z}} \alpha_j (u_{n+j} - u_n) + \sum_{j \in \mathbb{Z}} \varphi_j (u_{n+j} - u_n) - \gamma \dot{u}_n + f \cos(\omega t + pn), \quad (3.1)$$

where $\alpha_j = \alpha_{-j} > 0$, $\alpha_0 = 0$ and

$$\sum_{j \in \mathbb{N}} \alpha_j < \infty,$$

$\gamma \geq 0$, $\omega > 0$, $p \neq 0$, $f \neq 0$ are parameters and φ_j is odd analytic function with radius of convergence ρ_j for $j \in \mathbb{Z} \setminus \{0\}$, $\varphi_0 \equiv 0$ such that

$$\rho_0 := \inf_{j \in \mathbb{Z} \setminus \{0\}} \rho_j > 0, \quad D\varphi_j(0) = 0, \quad j \in \mathbb{Z} \setminus \{0\}$$

and

$$\sum_{j \in \mathbb{Z} \setminus \{0\}} \sum_{k=3}^{\infty} \frac{|D^k \varphi_j(0)|}{k!} r^k < \infty$$

for all $r \in (0, \rho_0/2)$.

As in the previous section, we apply symmetric difference and study the equation

$$\begin{aligned} u_n(t+1) - 2u_n(t) + u_n(t-1) &= \sum_{j \in \mathbb{N}} \alpha_j (u_{n+j}(t) - 2u_n(t) + u_{n-j}(t)) \\ &+ \sum_{j \in \mathbb{N}} \varphi_j (u_{n+j}(t) - u_n(t)) + \varphi_{-j} (u_{n-j}(t) - u_n(t)) \\ &- \gamma (u_n(t+1/2) - u_n(t-1/2)) + f \cos(\omega t + pn). \end{aligned} \quad (3.2)$$

Putting $u_n(t) = U(\omega t + pn)$, $U(z + \pi) = -U(z)$ for $U \in B(\rho_0/2)$ in (3.2), we get

$$\begin{aligned} U(z + \omega) - 2U(z) + U(z - \omega) &= \sum_{j \in \mathbb{N}} \alpha_j (U(z + pj) - 2U(z) + U(z - pj)) \\ &+ \sum_{j \in \mathbb{N}} \varphi_j (U(z + pj) - U(z)) + \varphi_{-j} (U(z - pj) - U(z)) \\ &- \gamma (U(z + \omega/2) - U(z - \omega/2)) + f \cos z. \end{aligned} \quad (3.3)$$

with $z = \omega t + pn$. By setting

$$\begin{aligned} \mathcal{J}U(z) &:= U(z + \omega) - 2U(z) + U(z - \omega) \\ &\quad - \sum_{j \in \mathbb{N}} \alpha_j (U(z + pj) - 2U(z) + U(z - pj)) + \gamma(U(z + \omega/2) - U(z - \omega/2)), \\ \mathcal{G}(U, f)(z) &:= \sum_{j \in \mathbb{N}} \varphi_j(U(z + pj) - U(z)) + \varphi_{-j}(U(z - pj) - U(z)) + f \cos z, \end{aligned}$$

equation (3.3) has the form

$$\mathcal{J}U = \mathcal{G}(U, f).$$

Following the proof of Lemma 2, we have the following result.

Lemma 3. *Function $\mathcal{G} : B(\rho_0/2) \times \mathbb{R} \rightarrow X$ satisfies*

$$\begin{aligned} \|\mathcal{G}(U, f)\| &\leq \sum_{k=3}^{\infty} \frac{\sum_{j \in \mathbb{N}} |\mathbf{D}^k \varphi_j(0)| + |\mathbf{D}^k \varphi_{-j}(0)|}{k!} 2^k \|U\|^k + |f|, \\ &\quad \|\mathcal{G}(U_1, f) - \mathcal{G}(U_2, f)\| \\ &\leq \|U_1 - U_2\| \sum_{k=3}^{\infty} \frac{\sum_{j \in \mathbb{N}} |\mathbf{D}^k \varphi_j(0)| + |\mathbf{D}^k \varphi_{-j}(0)|}{k!} 2^k \sum_{i=0}^{k-1} \|U_1\|^i \|U_2\|^{k-i-1}, \\ &\quad \|\mathcal{G}(U, f_1) - \mathcal{G}(U, f_2)\| \leq |f_1 - f_2| \end{aligned}$$

for any $U, U_1, U_2 \in B(\rho_0/2)$ and $f, f_1, f_2 \in \mathbb{R}$.

Next if $U \in X$ with $U(z) = \sum_{k \in \mathbb{Z}} c_k e^{(2k+1)\iota z}$, then

$$\begin{aligned} \mathcal{J}U(z) &= \sum_{k \in \mathbb{Z}} \left(4 \sum_{j \in \mathbb{N}} \alpha_j \sin^2 \frac{2k+1}{2} pj - 4 \sin^2 \frac{2k+1}{2} \omega + 2\gamma \sin \frac{2k+1}{2} \omega \right) \\ &\quad \times c_k e^{(2k+1)\iota z}. \end{aligned}$$

Hence $\mathcal{J} \in L(X)$ with

$$\|\mathcal{J}\|_{L(X)} \leq 4 \sum_{j \in \mathbb{N}} \alpha_j + 4 + 2\gamma.$$

Moreover, assuming

$$\Gamma := \inf_{k \in \mathbb{Z}} \sqrt{\left(4 \sum_{j \in \mathbb{N}} \alpha_j \sin^2 \frac{2k+1}{2} pj - 4 \sin^2 \frac{2k+1}{2} \omega \right)^2 + 4\gamma^2 \sin^2 \frac{2k+1}{2} \omega} > 0, \quad (3.4)$$

$\mathcal{J}^{-1} : X \rightarrow X$ is such that

$$\|\mathcal{J}^{-1}\|_{L(X)} \leq \frac{1}{\Gamma}.$$

Summarising, we arrive at the following result.

Theorem 2. *Suppose (3.4). By replacing*

$$|D^k \varphi_1(0)| \leftrightarrow \sum_{j \in \mathbb{N}} |D^k \varphi_j(0)|, \quad |D^k \varphi_2(0)| \leftrightarrow \sum_{j \in \mathbb{N}} |D^k \varphi_{-j}(0)|, \quad \Theta \leftrightarrow \Gamma$$

for each $k = 1, 2, \dots$, the statements of Theorem 1 are valid for (3.3).

Remark 1.1. remains true with Γ instead of Θ .

4. QUASIPERIODIC SOLUTIONS OF DIFFERENCE EQUATIONS

Motivated by stationary FPU lattice equations (2.19) and (2.20), in this section, we study the existence of quasiperiodic solutions of a general nonlinear difference equation with a quasiperiodic perturbation. We take coefficients from Banach complex algebra V and introduce Banach space

$$W := \left\{ c = \{c_{kp}\}_{k,p \in \mathbb{Z}} \mid c_{kp} \in V, \forall k, p \in \mathbb{Z}, \sum_{k,p \in \mathbb{Z}} \|c_{kp}\| < \infty \right\}$$

with the norm

$$\|c\| := \sum_{k,p \in \mathbb{Z}} \|c_{kp}\|.$$

We shall seek the solutions in

$$S_{\omega_1 \omega_2} := \left\{ U = \{U_n\}_{n \in \mathbb{Z}} \mid U_n = \sum_{k,p \in \mathbb{Z}} c_{kp} e^{(\omega_1 k + \omega_2 p)in}, c \in W, \forall n \in \mathbb{Z} \right\}.$$

For each $U \in S_{\omega_1 \omega_2}$ there may be many different $c \in W$ such that

$$U_n = \sum_{k,p \in \mathbb{Z}} c_{kp} e^{(\omega_1 k + \omega_2 p)in}.$$

Therefore, we denote $U = U(c) = \{U_n\}_{n \in \mathbb{Z}} = \{U_n(c)\}_{n \in \mathbb{Z}}$ to emphasize which coefficients are considered. Moreover, we write $\|U(c)\| = \|c\|$ ($= \|U_n(c)\|$). On the other side, if $\|U\| = a$, then there exist coefficients $c \in W$ such that $U = U(c)$ and $\|c\| = a$. For the set $S_{\omega_1 \omega_2}$ we have the next statement which can be easily verified.

Lemma 4. *Let $X(x), Y(y) \in S_{\omega_1 \omega_2}$. Then the following holds true:*

- (1) $\|X(x)\| < \infty$,
- (2) *for all $\alpha, \beta \in \mathbb{C}$: $Z = \alpha X(x) + \beta Y(y) \in S_{\omega_1 \omega_2}$ and there exist coefficients $z \in W$ such that $Z = Z(z)$ and $z_{kp} = \alpha x_{kp} + \beta y_{kp}$ for each $k, p \in \mathbb{Z}$,*
- (3) $Z = \{Z_n\}_{n \in \mathbb{Z}} = \{X_n Y_n\}_{n \in \mathbb{Z}} \in S_{\omega_1 \omega_2}$ and there exists $z \in W$ such that $Z = Z(z)$ and $\|z\| \leq \|x\| \|y\|$,
- (4) *if $j \in \mathbb{Z}$ is arbitrary and fixed, then $Z = \{Z_n\}_{n \in \mathbb{Z}} = \{X_{n+j}\}_{n \in \mathbb{Z}} \in S_{\omega_1 \omega_2}$.*

Let $\omega_1, \omega_2 \in \mathbb{R}$ be such that $\frac{\omega_1}{\omega_2} \notin \mathbb{Q}$ and $j \in \mathbb{N}$ be fixed. Consider the following nonlinear difference equation

$$\mathcal{L}(\sigma)X = Y + F(X), \quad (\mathcal{L}(\sigma)X_n = Y_n + F_n(X_n), \forall n \in \mathbb{Z}) \quad (4.1)$$

where $X = \{X_n\}_{n \in \mathbb{Z}}, Y = \{Y_n\}_{n \in \mathbb{Z}} \in S_{\omega_1 \omega_2}$, $\mathcal{L}(\sigma) : S_{\omega_1 \omega_2} \rightarrow S_{\omega_1 \omega_2}$ is a linear difference operator given by

$$\mathcal{L}(\sigma) = \alpha_0 \sigma^j + \alpha_1 \sigma^{j-1} + \dots + \alpha_{j-1} \sigma + \alpha_j$$

with shift operator σ ($\sigma^m(X_n) = X_{n+m}$) and constants $\alpha_0, \dots, \alpha_j \in \mathbb{C}$, i.e.

$$\mathcal{L}(\sigma)X_n = \alpha_0 X_{n+j} + \alpha_1 X_{n+j-1} + \dots + \alpha_{j-1} X_{n+1} + \alpha_j X_n,$$

and $F : S_{\omega_1 \omega_2} \rightarrow S_{\omega_1 \omega_2}$ is a given nonlinear function such that

$$F_n(z) = \sum_{j=0}^{\infty} (a_j + b_j e^{i\omega_1 n} + c_j e^{i\omega_2 n}) z^j$$

with coefficients $a_j, b_j, c_j \in \mathbb{C}$ for each $j = 0, 1, 2, \dots$, which is analytic with radius of convergence ρ_n for each $n \in \mathbb{Z}$ such that $\inf_{n \in \mathbb{Z}} \rho_n > 0$. Let the radius of convergence $\tilde{\rho}$ of function

$$v(r) := \sum_{j=0}^{\infty} (|a_j| + |b_j| + |c_j|) r^j$$

be positive. Obviously, $\tilde{\rho} \leq \inf_{n \in \mathbb{Z}} \rho_n$. Denoting $B(\rho) := \{X \in S_{\omega_1 \omega_2} \mid \|X\| < \rho\}$ and $\mathcal{F}(X, Y) := Y + F(X)$ or equivalently $\mathcal{F}_n(X_n, Y_n) := Y_n + F_n(X_n)$ for each $n \in \mathbb{Z}$, equation (4.1) has the form

$$\mathcal{L}(\sigma)X = \mathcal{F}(X, Y), \quad (\mathcal{L}(\sigma)X_n = \mathcal{F}_n(X_n, Y_n), \forall n \in \mathbb{Z}) \quad (4.2)$$

and we have the following result.

Lemma 5. *Function $\mathcal{F} : B(\tilde{\rho}) \times B(\tilde{\rho}) \rightarrow S_{\omega_1 \omega_2}$ fulfils*

$$\|\mathcal{F}_n(X_n, Y_n)\| \leq \|Y_n\| + \sum_{j=0}^{\infty} (|a_j| + |b_j| + |c_j|) \|X_n\|^j, \quad (4.3)$$

$$\begin{aligned} & \|\mathcal{F}_n(X_n, Y_n) - \mathcal{F}_n(Z_n, Y_n)\| \\ & \leq \|X_n - Z_n\| \sum_{j=1}^{\infty} (|a_j| + |b_j| + |c_j|) \sum_{l=0}^{j-1} \|X_n\|^l \|Z_n\|^{j-l-1}, \end{aligned} \quad (4.4)$$

$$\|\mathcal{F}_n(X_n, Y_n) - \mathcal{F}_n(X_n, Z_n)\| \leq \|Y_n - Z_n\| \quad (4.5)$$

for each $n \in \mathbb{Z}$ and $X, Y, Z \in B(\tilde{\rho}) \subset S_{\omega_1 \omega_2}$.

Proof. The first and the last statements are trivial. The second one uses formula (2.10). $\square$

Next, if $X(x) \in S_{\omega_1\omega_2}$, i.e. $X_n = \sum_{k,p \in \mathbb{Z}} x_{kp} e^{(\omega_1 k + \omega_2 p)in}$, then

$$\begin{aligned} \mathcal{L}(\sigma)X_n &= \sum_{k,p \in \mathbb{Z}} x_{kp} \left(\alpha_0 e^{(\omega_1 k + \omega_2 p)in} + \dots + \alpha_{j-1} e^{(\omega_1 k + \omega_2 p)in} + \alpha_j \right) e^{(\omega_1 k + \omega_2 p)in} \\ &= \sum_{k,p \in \mathbb{Z}} \mathcal{L} \left(e^{(\omega_1 k + \omega_2 p)in} \right) x_{kp} e^{(\omega_1 k + \omega_2 p)in} \end{aligned}$$

and so

$$\|\mathcal{L}(\sigma)\|_{L(S_{\omega_1\omega_2})} \leq \sum_{l=0}^j |\alpha_l|.$$

If

$$\Theta := \inf_{\alpha \in [0, 2\pi]} |\mathcal{L}(e^{i\alpha})| > 0, \quad (4.6)$$

then $\mathcal{L}(\sigma)^{-1} \in L(S_{\omega_1\omega_2})$ is such that

$$\|\mathcal{L}(\sigma)^{-1}\|_{L(S_{\omega_1\omega_2})} \leq \frac{1}{\Theta}. \quad (4.7)$$

Now we state the sufficient condition for the existence of a quasiperiodic solution of equation (4.1) with given Y and function F . Note that it does not imply the uniqueness of the solution due to non-uniqueness of the coefficients y .

Theorem 3. *Let $Y = Y(y) \in S_{\omega_1\omega_2}$ be fixed and condition (4.6) be fulfilled. If there exists $0 < \tilde{r} < \tilde{\rho}$ such that*

$$\frac{\|Y\| + v(\tilde{r})}{\Theta} \leq \tilde{r}, \quad \frac{Dv(\tilde{r})}{\Theta} < 1, \quad (4.8)$$

then there exists at least one solution $X \in \overline{B(\tilde{r})} \subset S_{\omega_1\omega_2}$ of equation (4.1), i.e. there is $x \in W$ satisfying $\|x\| \leq \tilde{r}$ and $X = X(x)$. This solution can be approximated by an iteration process.

Proof. We formulate equation (4.2) as a parametrized fixed point problem

$$X = \mathcal{R}(X, Y) := \mathcal{L}(\sigma)^{-1} \mathcal{F}(X, Y)$$

in $B(\tilde{\rho}) \subset S_{\omega_1\omega_2}$. By (4.7) and (4.3) we know that

$$\|\mathcal{R}(X, Y)\| \leq \frac{\|Y\| + v(r)}{\Theta}$$

if $\|X\| \leq \tilde{r}$. Hence, if the first condition in (4.8) is satisfied, function $\mathcal{R}(\cdot, Y)$ maps $\overline{B(\tilde{r})}$ into itself. The second condition in (4.8) establishes the contractivity of $\mathcal{R}(\cdot, Y)$, since by (4.4)

$$\|\mathcal{R}(X, Y) - \mathcal{R}(Z, Y)\| \leq \frac{\|X - Z\|}{\Theta} \sum_{j=1}^{\infty} (|a_j| + |b_j| + |c_j|) \sum_{l=0}^{j-1} \|X\|^l \|Z\|^{j-l-1}$$

$$\leq \frac{Dv(r)}{\Theta} \|X - Z\|$$

for all $X, Z \in \overline{B(\tilde{r})}$. The rest follows from Banach fixed point theorem. $\square$

Example 2. Consider equation

$$X_{n+2} + aX_{n+1} + X_n = d_1 \cos \sqrt{2}n + d_2 \sin 3n + bX_n^3 \quad (4.9)$$

for each $n \in \mathbb{Z}$ with parameters $a \in \mathbb{R}$, $|a| > 2$, $b, d_1, d_2 \in \mathbb{C}$.

In this case, we have $\omega_1 = \sqrt{2}$, $\omega_2 = 3$, $\mathcal{L}(\sigma) = \sigma^2 + a\sigma + \sigma^0$,

$$W := l_1 = \left\{ c = \{c_{kp}\}_{k,p \in \mathbb{Z}} \mid c_{kp} \in \mathbb{C}, \forall k, p \in \mathbb{Z}, \sum_{k,p \in \mathbb{Z}} |c_{kp}| < \infty \right\},$$

$$Y_n = d_1 \cos \sqrt{2}n + d_2 \sin 3n = \sum_{k,p \in \mathbb{Z}} y_{kp} e^{(\sqrt{2}k+3p)\iota n}$$

with $y_{\pm 1,0} = d_1/2$, $y_{0,1} = -y_{0,-1} = d_2/2$ and $y_{kp} = 0$ for each $(k, p) \in \mathbb{Z}^2 \setminus \{(\pm 1, 0), (0, \pm 1)\}$, and $F_n(X_n) = bX_n^3$. Hence $v(r) = |b|r^3$ for any $r \geq 0$, $\|Y\| = |d_1| + |d_2|$ and we seek the solution X in space

$$S_{\omega_1 \omega_2} := \left\{ U = \{U_n\}_{n \in \mathbb{Z}} \mid U_n = \sum_{k,p \in \mathbb{Z}} c_{kp} e^{(\sqrt{2}k+3p)\iota n}, c \in W, \forall n \in \mathbb{Z} \right\}.$$

Corollary 2. *If*

$$|d_1| + |d_2| < \sqrt{\frac{4(|a|-2)^3}{27|b|}}, \quad (4.10)$$

then equation (4.9) has at least one quasiperiodic solution $X \in \overline{B(\tilde{r})} \subset S_{\omega_1 \omega_2}$ in the closed ball with radius

$$\tilde{r} = \sqrt{\frac{4(|a|-2)}{3|b|}} \sin \left(\frac{1}{3} \arcsin \left((|d_1| + |d_2|) \sqrt{\frac{27|b|}{4(|a|-2)^3}} \right) \right). \quad (4.11)$$

Proof. For $\mathcal{L}$ we have

$$\begin{aligned} |\mathcal{L}(e^{\iota \alpha})| &= |e^{2\iota \alpha} + a e^{\iota \alpha} + 1| \\ &= |(a + 2 \cos \alpha) \cos \alpha + \iota(a + 2 \cos \alpha) \sin \alpha| = |a + 2 \cos \alpha| \end{aligned}$$

and so $\Theta = |a| - 2 > 0$. Next, $v(r) = |b|r^3$ and conditions (4.8) are equivalent to

$$\frac{|d_1| + |d_2| + |b|\tilde{r}^3}{|a| - 2} \leq \tilde{r}, \quad \frac{3|b|\tilde{r}^2}{|a| - 2} < 1.$$

Moreover, the smaller $\tilde{r}$ satisfying these conditions, the more precise X . Thus from the first one it follows that $\tilde{r}$ has to be a positive real root of function $A(r) :=$

$|d_1| + |d_2| - r(|a| - 2) + |b|r^3$. The second one implies, it is the smallest one and nondegenerate. Assumption (4.10) is a necessary condition for $A(r)$ to have two distinct positive real roots. Finally, from [6] we know that $\tilde{r}$ is given by (4.11). $\square$

Remark 2.

1. The theory derived in this section can be extended to arbitrary number $q > 2$ of angle speeds $\omega_1, \dots, \omega_q$ assuming $\frac{\omega_i}{\omega_j} \notin \mathbb{Q}$ for each $i, j = 1, \dots, q, i \neq j$.
2. Example 2 is a very special application of the theory of Section 4. More generally, in (4.9) one can take $d_1, d_2 \in L(\mathbb{C}^N, \mathbb{C}^N)$, i.e. $N \times N$ complex-valued matrices. Then coefficient space V is a Banach complex algebra of $N \times N$ matrices.

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EXISTENCE RESULTS FOR NONLINEAR QUADRATIC FUNCTIONAL INTEGRAL EQUATIONS OF FRACTIONAL ORDERS

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Abstract. We present an existence theorem for at least one continuous solution for a nonlinear quadratic functional integral equation of fractional (arbitrary) order. Also, we prove the existence of maximal and minimal solutions for that equations.

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1. INTRODUCTION

It is well known that integral equations have many useful applications in describing numerous events and problems of real world. Moreover the theory of integral equations is rapidly developing using the tools of functional analysis, topology and fixed point theory. In particular, quadratic integral equations have many useful applications in the real world. For example, quadratic integral equations are often applicable in the theory of radiative transfer, the kinetic theory of gases, the theory of neutron transport, queuing theory and traffic theory. Many authors studied the existence of solutions for several classes of nonlinear quadratic integral equations (see e.g. [1–11] and [8–17, 19]). However, in most of the above literature, the main results are realized with the help of the technique associated with the measure of noncompactness. Instead of using the technique of measure of noncompactness we use Schauder fixed point theorem.

Let $\mathbb{R}$ be the set of real numbers whereas $I = [0, 1]$, $L_1 = L_1[0, 1]$ be the space of Lebesgue integrable functions on I .

Here, we prove the existence of at least one continuous solution of the quadratic functional integral equation of fractional order

$$x(t) = a(t) + g(t, x(\psi(t))) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\phi(s))) ds, \quad t \in I, \quad \alpha > 0 \quad (1.1)$$

and the existence of a continuous solution of the nonlinear functional differential equation of fractional-order

$${}_R D^\alpha x(t) = f(t, x(\phi(t))), \quad t \in I \quad \text{and} \quad x(0) = 0, \quad \alpha \in (0, 1) \quad (1.2)$$

(where ${}_R D^\alpha$ is the Riemann-Liouville fractional order derivative) will be given as an application. Also the results concerning the existence of continuous solution of the initial value problem

$$\frac{dx(t)}{dt} = f(t, x(\phi(t))), \quad x(0) = x_0, \quad (1.3)$$

will be given as another application. Finally, the existence of maximal and minimal solutions of (1.1) will be proved.

For $\psi(t) = \phi(t) = t$, J. Banas and B. Rzepka(see [8]) proved the existence of a nondecreasing continuous solution of (1.1) by using the technique of measure of noncompactness. The existence of continuous solutions for some quadratic integral equations was proved by using Schauder-Tychonoff fixed point theorem [25].

2. PRELIMINARIES AND DEFINITIONS

The existence results will be based on the following fixed-point theorems and definitions.

Theorem 1 (Schauder Fixed Point Theorem in [9]). *Let Q be a nonempty, convex, compact subset of a Banach space X , and $T : Q \rightarrow Q$ be a continuous map. Then T has at least one fixed point in Q .*

We shall collect the definitions of the fractional-order integral and differential operators. Let β be a positive real number

Definition 1. The fractional-order integral of order β of the function f is defined on $[a, b]$ by (see [20], [23], [22] and [26])

$$I_a^\beta f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s) ds, \quad (2.1)$$

and when $a = 0$, we have $I^\beta f(t) = I_0^\beta f(t)$.

Definition 2. The Riemann-Liouville fractional-order derivative of order $\beta \in (0, 1)$ of the function f is given by (see [20], [23], [22] and [26])

$${}_R D^\beta f(t) = \frac{d}{dt} I^{1-\beta} f(t).$$

For the properties of fractional calculus see [20], [23], [22] and [26] for example.

3. EXISTENCE OF CONTINUOUS SOLUTIONS

Now, equation (1.1) will be investigated under the assumptions:

- (i) $a : I \rightarrow \mathbb{R}$ is continuous and bound with $k_1 = \sup_{t \in I} |a(t)|$.
- (ii) $g : I \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and bounded with $k_2 = \sup_{(t,x) \in I \times \mathbb{R}} |g(t,x)|$.
- (iii) There exist two constants l_i , $i = 1, 2$ respectively satisfying
$$|g(t,x) - g(s,y)| \leq l_1 |t-s| + l_2 |x-y|$$
for all $t, s \in I$ and $x, y \in \mathbb{R}$.
- (iv) $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory condition (i.e. measurable in t for all $x : I \rightarrow \mathbb{R}$ and continuous in x for all $t \in I$).
- (v) There exist a function $m \in L_1$ and a constant b such that
$$|f(t,x)| \leq m(t) + b |x| \quad (\forall (t,x) \in I \times \mathbb{R})$$
and $k_3 = \sup_{t \in I} I^\beta m(t)$ for any $\beta \leq \alpha$.
- (vi) $\psi, \phi : I \rightarrow I$ are continuous.
- (vii) The inequality $k_2 b < \Gamma(1+\alpha)$ is satisfied.

Theorem 2. *Let the assumptions (i)-(vii) be satisfied. Then the quadratic functional integral equation (1.1) has at least one solution in the space $x \in C(I)$.*

Proof. Let $C = C(I)$ be the Banach space of all real functions defined and continuous on the interval I .

Fix a number $r > 0$ and consider the ball S_r in the space $C(I)$ defined as

$$S_r = \{x \in C(I) : |x(t)| \leq r \text{ for } t \in I\}.$$

Let T be the operator defined on S_r by the formula

$$(Tx)(t) = a(t) + g(t, x(\psi(t))) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\phi(s))) ds, \quad x \in S_r, t \in I.$$

Then, in view of our assumptions, for $x \in S_r$ and $t \in I$ we get

$$|Tx(t)| \leq |a(t)| + |g(t, x(\psi(t)))| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(\phi(s)))| ds \quad (3.1)$$

$$\leq k_1 + k_2 I^{\alpha-\beta} I^\beta m(t) + k_2 b \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(\phi(s))| ds \quad (3.2)$$

$$\leq k_1 + k_2 k_3 \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} ds + k_2 b r \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \quad (3.3)$$

$$\leq k_1 + \frac{k_2 k_3}{\Gamma(\alpha - \beta + 1)} + \frac{k_2 b r}{\Gamma(1+\alpha)}. \quad (3.4)$$

Hence, in view of assumption (vii) we have that T transforms the ball S_r into itself for

$$r = (k_1 + \frac{k_2 k_3}{\Gamma(\alpha - \beta + 1)})(1 - \frac{k_2 b}{\Gamma(1 + \alpha)})^{-1}.$$

Now, for t_1 and $t_2 \in I$ (without loss of generality assume that $t_1 < t_2$), we have

$$\begin{aligned} (Tx)(t_2) - (Tx)(t_1) &= a(t_2) - a(t_1) \\ &+ g(t_2, x(\psi(t_2))) I^\alpha f(t_2, x(\phi(t_2))) - g(t_1, x(\psi(t_1))) I^\alpha f(t_1, x(\phi(t_1))) \\ &+ g(t_1, x(\psi(t_1))) I^\alpha f(t_2, x(\phi(t_2))) - g(t_1, x(\psi(t_1))) I^\alpha f(t_2, x(\phi(t_2))) \\ &\leq a(t_2) - a(t_1) + [g(t_2, x(\psi(t_2))) - g(t_1, x(\psi(t_1)))] I^\alpha f(t_2, x(\phi(t_2))) \\ &\quad + g(t_1, x(\psi(t_1))) [I^\alpha f(t_2, x(\phi(t_2))) - I^\alpha f(t_1, x(\phi(t_1)))], \end{aligned}$$

but

$$\begin{aligned} I^\alpha f(t_2, x(\phi(t_2))) - I^\alpha f(t_1, x(\phi(t_1))) &= \int_0^{t_1} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\phi(s))) ds \\ &+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\phi(s))) ds - \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\phi(s))) ds \\ &\leq \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\phi(s))) ds + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\phi(s))) ds \\ &\quad - \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\phi(s))) ds = \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\phi(s))) ds. \end{aligned}$$

Then

$$\begin{aligned} |I^\alpha f(t_2, x(\phi(t_2))) - I^\alpha f(t_1, x(\phi(t_1)))| &\leq I_{t_1}^\alpha |f(t_2, x(\phi(t_2)))| \\ &\leq I_{t_1}^\alpha m(t_2) + b I_{t_1}^\alpha |x(\phi(t_2))| \leq I_{t_1}^{\alpha-\beta} I_{t_1}^\beta m(t_2) + b I_{t_1}^\alpha |x(\phi(t_2))| \\ &\leq k_3 \frac{(t_2 - t_1)^{\alpha-\beta}}{\Gamma(\alpha - \beta + 1)} + b r \frac{(t_2 - t_1)^\alpha}{\Gamma(\alpha + 1)}. \end{aligned}$$

Then we get

$$\begin{aligned} |(Tx)(t_2) - (Tx)(t_1)| & \tag{3.5} \\ &\leq |a(t_2) - a(t_1)| + [l_1 |t_2 - t_1| + l_2 |x(\psi(t_2)) - x(\psi(t_1))|] \\ &\quad I^\alpha |f(t_2, x(\phi(t_2)))| \\ &\quad + |g(t_1, x(\psi(t_1)))| (k_3 \frac{(t_2 - t_1)^{\alpha-\beta}}{\Gamma(\alpha - \beta + 1)} + b r \frac{(t_2 - t_1)^\alpha}{\Gamma(\alpha + 1)}) \end{aligned}$$

i.e.,

$$\begin{aligned} |(Tx)(t_2) - (Tx)(t_1)| & \\ &\leq |a(t_2) - a(t_1)| + [l_1 |t_2 - t_1| + l_2 |x(t_2) - x(t_1)|] \\ &\quad I^\alpha (m(t_2) + b |x(\phi(t_2))|) \end{aligned} \tag{3.6}$$

$$\begin{aligned}
& + k_2 k_3 \frac{(t_2 - t_1)^{\alpha - \beta}}{\Gamma(\alpha - \beta + 1)} + k_2 b r \frac{(t_2 - t_1)^\alpha}{\Gamma(\alpha + 1)} \\
& \leq |a(t_2) - a(t_1)| + \frac{k_3}{\Gamma(\alpha - \beta + 1)} [l_1 |t_2 - t_1| + l_2 |x(t_2) - x(t_1)|] \\
& + \frac{b r}{\Gamma(\alpha + 1)} [l_1 |t_2 - t_1| + l_2 |x(t_2) - x(t_1)|] + \frac{k_2 k_3}{\Gamma(\alpha - \beta + 1)} (t_2 - t_1)^{\alpha - \beta} \\
& + \frac{k_2 b r}{\Gamma(\alpha + 1)} (t_2 - t_1)^\alpha \rightarrow 0 \quad \text{as } t_2 \rightarrow t_1.
\end{aligned}$$

This means that the functions of TS_r are equi-continuous on I . Then by the Arzela-Ascoli Theorem [9] the closure of TS_r is compact.

It is clear that the set S_r is nonempty, bounded, closed and convex.

Assumptions (ii) and (iv) imply that $T : S_r \rightarrow C(I)$ is a continuous operator in x .

Since all conditions of the Schauder fixed-point theorem hold, then T has a fixed point in S_r . $\square$

4. SPECIAL CASES

Corollary 1. *Let the assumptions of Theorem 2 be satisfied (with $\psi(t) = \phi(t) = t$), then the fractional-order quadratic integral equation*

$$x(t) = a(t) + g(t, x(t)) \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} f(s, x(s)) ds$$

has at least one solution $x \in C$.

Corollary 2. *Let the assumptions of Theorem 2 be satisfied (with $g(t, x) = 1$), then the fractional-order integral equation*

$$x(t) = a(t) + \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} f(s, x(\phi(s))) ds$$

has at least one solution $x \in C$.

Letting $b = 0$ and $\phi(t) = t$ in Corollary 2, we obtain the same result as was proved in [18].

Now letting $\alpha, \beta \rightarrow 1$, we obtain

Corollary 3. *Let the assumptions of Theorem 2 be satisfied (with $g(t, x) = 1$, $a(t) = x_0$ and letting $\alpha, \beta \rightarrow 1$), then the integral equation*

$$x(t) = x_0 + \int_0^t f(s, x(\phi(s))) ds.$$

has at least one solution $x \in C$ which is equivalent to the initial value problem (1.3).

Letting $b = 0$ and $\phi(t) = t$ in Corollary 3 we obtain the Carathéodory Theorem (proved in [9]).

5. FRACTIONAL ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

For the initial value problem of the nonlinear fractional-order differential equation (1.2) we have the following theorem.

Theorem 3. *Let the assumptions of Theorem 2 be satisfied (with $a(t) = 0$ and $g(t, x(t)) = 1$), then the Cauchy type problem (1.2) has at least one solution $x \in C$.*

Proof. Integrating (1.2) we obtain the integral equation

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\phi(s))) ds, \quad t \in I \quad (5.1)$$

which by Theorem 2 has the desired solution.

Operating with ${}_R D^\alpha$ on (5.1) we obtain the initial value problem (1.2). So the equivalence between the initial value problem (1.2) and the integral equation (5.1) is proved and then the results follow from Theorem 2. $\square$

6. MAXIMAL AND MINIMAL SOLUTIONS

Definition 3 (see [21]). Let $q(t)$ be a solution of (1.1) Then $q(t)$ is said to be a maximal solution of (1.1) if every solution of (1.1) on I satisfies the inequality $x(t) \leq q(t)$, $t \in I$. A minimal solution $s(t)$ can be defined in a similar way by reversing the above inequality i.e. $x(t) \geq s(t)$, $t \in I$.

We need the following lemma to prove the existence of maximal and minimal solutions of (1.1).

Lemma 1. *Let $g(t, x)$, $f(t, x)$ satisfy the assumptions in Theorem 2 and let $x(t)$, $y(t)$ be continuous functions on I satisfying*

$$\begin{aligned} x(t) &\leq a(t) + g(t, x(\psi(t))) I^\alpha f(t, x(\phi(t))) \\ y(t) &\geq a(t) + g(t, y(\psi(t))) I^\alpha f(t, y(\phi(t))) \end{aligned}$$

where one of them is strict.

Suppose $f(t, x)$ is nondecreasing function in x . Then

$$x(t) < y(t). \quad (6.1)$$

Proof. Let the conclusion (6.1) be false; then there exists t_1 such that

$$x(t_1) = y(t_1), \quad t_1 > 0$$

and

$$x(t) < y(t), \quad 0 < t < t_1.$$

From the monotonicity of the function f in x , we get

$$\begin{aligned} x(t_1) &\leq a(t_1) + g(t_1, x(\psi(t_1))) I^\alpha f(t_1, x(\phi(t_1))) \\ &= a(t_1) + g(t_1, x(\psi(t_1))) \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\phi(s))) ds \\ &< a(t_1) + g(t_1, y(\psi(t_1))) \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(\phi(s))) ds \\ &< y(t_1). \end{aligned}$$

This contradicts the fact that $x(t_1) = y(t_1)$; then

$$x(t) < y(t).$$

□

Theorem 4. *Let the assumptions of Theorem 2 be satisfied. Furthermore, if $f(t, x)$ is nondecreasing functions in x , then there exist maximal and minimal solutions of (1.1).*

Proof. Firstly, we shall prove the existence of maximal solution of (1.1). Let $\epsilon > 0$ be given. Now consider the fractional-order quadratic functional integral equation

$$x_\epsilon(t) = a(t) + g_\epsilon(t, x_\epsilon(\psi(t))) I^\alpha f_\epsilon(t, x_\epsilon(\phi(t))), \quad (6.2)$$

where

$$f_\epsilon(t, x_\epsilon(\phi(t))) = f(t, x_\epsilon(\phi(t))) + \epsilon$$

and

$$g_\epsilon(t, x_\epsilon(\psi(t))) = g(t, x_\epsilon(\psi(t))) + \epsilon.$$

Clearly the functions $f_\epsilon(t, x_\epsilon)$ and $g_\epsilon(t, x_\epsilon)$ satisfy assumptions (ii), (iv) and

$$|g_\epsilon(t, x_\epsilon)| \leq M + \epsilon = M'.$$

$$|f_\epsilon(t, x_\epsilon)| \leq m(t) + \epsilon + b|x| = m'(t) + b|x|.$$

Therefore, equation (6.2) has a continuous solution $x_\epsilon(t)$ according to Theorem 2.

Let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$. Then

$$x_{\epsilon_1}(t) = a(t) + g_{\epsilon_1}(t, x_{\epsilon_1}(\psi(t))) I^\alpha f_{\epsilon_1}(t, x_{\epsilon_1}(\phi(t))),$$

$$\begin{aligned} x_{\epsilon_1}(t) &= a(t) + (g(t, x_{\epsilon_1}(\psi(t))) + \epsilon_1) I^\alpha (f(t, x_{\epsilon_1}(\phi(t))) + \epsilon_1), \\ &> a(t) + (g(t, x_{\epsilon_1}(\psi(t))) + \epsilon_2) I^\alpha (f(t, x_{\epsilon_1}(\phi(t))) + \epsilon_2), \end{aligned} \quad (6.3)$$

$$x_{\epsilon_2}(t) = a(t) + (g(t, x_{\epsilon_2}(\psi(t))) + \epsilon_2) I^\alpha (f(t, x_{\epsilon_2}(\phi(t))) + \epsilon_2). \quad (6.4)$$

Applying Lemma 1, then (6.3) and (6.4) imply

$$x_{\epsilon_2}(t) < x_{\epsilon_1}(t) \quad \text{for } t \in I.$$

As shown before in the proof of Theorem 2, the family of functions $x_\epsilon(t)$ defined by (6.2) is uniformly bounded and of equi-continuous functions. Hence by the Arzela-Ascoli Theorem, there exists a decreasing sequence ϵ_n such that $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$, and $\lim_{n \rightarrow \infty} x_{\epsilon_n}(t)$ exists uniformly in I . We denote this limit by $q(t)$. From the continuity of the functions f_{ϵ_n} and g_{ϵ_n} in the second argument, we get

$$q(t) = \lim_{n \rightarrow \infty} x_{\epsilon_n}(t) = a(t) + g(t, q(\psi(t))) I^\alpha f(t, q(\phi(t)))$$

which proves that $q(t)$ is a solution of (1.1).

Finally, we shall show that $q(t)$ is maximal solution of (1.1). To do this, let $x(t)$ be any solution of (1.1). Then

$$\begin{aligned} x_\epsilon(t) &= a(t) + g_\epsilon(t, x_\epsilon(\psi(t))) I^\alpha f_\epsilon(t, x_\epsilon(\phi(t))) \\ &> a(t) + g(t, x_\epsilon(\psi(t))) I^\alpha f(t, x_\epsilon(\phi(t))). \end{aligned}$$

and

$$x(t) = a(t) + g(t, x(\psi(t))) I^\alpha f(t, x(\phi(t))).$$

Applying Lemma 1, we get

$$x_\epsilon(t) > x(t) \quad \text{for } t \in I.$$

from the uniqueness of the maximal solution (see [21], [24]), it is clear that $x_\epsilon(t)$ tends to $q(t)$ uniformly in $t \in I$ as $\epsilon \rightarrow 0$.

In a similar way we can prove that there exists a minimal solution of (1.1). $\square$

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USING PERTURBATION METHODS AND LAPLACE–PADÉ APPROXIMATION TO SOLVE NONLINEAR PROBLEMS

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Abstract. In this paper, the perturbation method and Padé transformation are used to provide an approximate solution of elliptic integrals of the second kind and of complete integrals of the first kind. Besides, we used the obtained results to calculate an analytic expression for the period of a simple pendulum. The method has an acceptable accuracy for high values of the initial amplitude, compared to the relative error $< 1.7\%$ for initial angles $\theta \leq 70^\circ$.

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1. INTRODUCTION

Among others, the perturbation method is one of the pioneering techniques of approaching various kinds of nonlinear problems. This procedure was introduced by S. D. Poisson and extended by J. H. Poincaré. Although the method appeared in the early 19th century, the application of a perturbation procedure to solve nonlinear differential equations was used only a bit later. The most significant efforts were focused on celestial mechanics, fluid mechanics, and aerodynamics [7, 19].

In general, it is assumed that the differential equation can be expressed as the sum of two parts: a linear part and a nonlinear one. The nonlinear part is considered as a small perturbation through a small parameter (the perturbation parameter). The assumption that the nonlinear part is small compared to the linear is considered as a disadvantage of the method. There are other modern alternatives to find approximate solutions of the differential equations that describe some nonlinear problems such as those based on variational approaches [3, 17], Tanh method [9], Exp-function [24], Adomian's decomposition method [1, 4], parameter expansion [25], homotopy perturbation method [6, 8, 10, 11, 13–16, 18, 22, 23], homotopy perturbation transform Padé method (HPTPM) [12], homotopy perturbation transform method (HPTM) [20], among others.

This paper applies the PM method to find an approximation of the value of second kind elliptic integrals; this is done by solving a first order nonlinear differential equation, which depends on one perturbation parameter. We will see that our approximation has good accuracy, even for large values of the perturbation parameter. Finally, we will couple the PM and Padé methods to obtain an approximation of the oscillation period for a nonlinear pendulum.

This paper is organized as follows. In Section 2, a short introduction about the basic elements of elliptic integrals is provided. In Section 3, we introduce the basic idea of the PM method. Section 4 will provide a brief introduction to the Padé approximation. In Section 5, we provide an application of the PM method. Section 6 presents an approximate solution of the complete elliptic integral of the first kind by using Laplace–Padé approximation. In Section 7, we obtain the period for a pendulum in an approximate way. In Section 8, we summarize our findings. Finally, a brief conclusion is given in Section 9.

2. ELLIPTIC INTEGRALS AND THE PERIOD FOR A SIMPLE PENDULUM

Elliptic integrals are integrals without closed solution, they have been used extensively for evaluating integrals. For instance, integrals of the form

$$I = \int_0^t R\left(x, \sqrt{ax^4 + bx^3 + cx^2 + dx + e}\right) dx,$$

where R is a rational function of x and the radical; a, b, c, d , and e are constants, may be expressed in terms of elliptic integrals [2]. Furthermore, elliptic integrals are important because they are useful to solve physical problems, like the case of the magnetic vector potential for a circular current loop [2] and the orbital equation for a particle subject to certain central potentials [7].

Elliptic integrals are divided in three cases:

- (1) Incomplete elliptic integral of the first kind. Which is defined as

$$K(\epsilon, \phi) = \int_0^\phi \frac{d\theta}{\sqrt{1 - \epsilon^2 \sin^2 \theta}}, \quad 0 \leq \epsilon < 1, \quad (2.1)$$

where ϕ is the amplitude of $K(\epsilon, \phi)$, and ϵ is their modulus. For $\phi = \frac{\pi}{2}$, we have the complete elliptic integral of the first kind,

$$K(\epsilon) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \epsilon^2 \sin^2 \theta}}, \quad 0 \leq \epsilon < 1. \quad (2.2)$$

- (2) Incomplete elliptic integral of the second kind. It is defined as

$$E(\epsilon, \phi) = \int_0^\phi \sqrt{1 - \epsilon^2 \sin^2 \theta} d\theta, \quad 0 \leq \epsilon < 1. \quad (2.3)$$

For $\phi = \frac{\pi}{2}$, we have the complete elliptic integral of the second kind

$$E(\epsilon) = \int_0^{\pi/2} \sqrt{1 - \epsilon^2 \sin^2 \theta} d\theta, \quad 0 \leq \epsilon < 1. \quad (2.4)$$

(3) Incomplete elliptic integral of the third kind. It is defined as

$$\Pi(\epsilon, n, \phi) = \int_0^\phi \frac{d\theta}{(1 + n \sin^2 \theta) \sqrt{1 - \epsilon^2 \sin^2 \theta}}, \quad 0 \leq \epsilon < 1, \quad (2.5)$$

where n is a constant $n \neq 0$, that is, if $n = 0$ (2.5) would be identical to (2.1).

Again, for the case $\phi = \frac{\pi}{2}$, the complete elliptic integral of the third kind is obtained

$$\Pi(\epsilon, n) = \int_0^{\pi/2} \frac{d\theta}{(1 + n \sin^2 \theta) \sqrt{1 - \epsilon^2 \sin^2 \theta}}, \quad 0 \leq \epsilon < 1. \quad (2.6)$$

3. BASIC IDEA OF THE PERTURBATION METHOD

Let the differential equation of one-dimensional nonlinear system be in the form

$$L(x) + \epsilon N(x) = 0, \quad (3.1)$$

where we assume that x is a function of one variable $x = x(t)$, $L(x)$ is a linear operator which, in general, contains derivatives in terms of t . $N(x)$ is a nonlinear operator, and ϵ is a small parameter.

By considering the nonlinear term in (3.1) as a perturbation, we assume that solution (3.1) can be written as a power series in the small parameter ϵ

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \epsilon^3 x_3(t) + \dots \quad (3.2)$$

By substituting (3.2) into (3.1) and equating terms having identical powers of ϵ , we obtain a number of differential equations that can be integrated, recursively, to find the values for the functions $x_0(t), x_1(t), x_2(t), \dots$

4. PADÉ APPROXIMANT

A rational approximation to $f(x)$ on $[a, b]$ is the quotient of two polynomials $P_N(x)$ and $Q_M(x)$ of degrees N and M , respectively. We use the notation $R_{N,M}(x)$ to denote this quotient. The $R_{N,M}(x)$ Padé approximations of a function $f(x)$ are given by [5, 12, 21]

$$R_{N,M} = \frac{P_N(x)}{Q_M(x)} \quad \text{for } a \leq x \leq b. \quad (4.1)$$

The method of Padé requires that $f(x)$ and its derivative be continuous at $x = 0$. The polynomials used in (4.1) are

$$P_N(x) = p_0 + p_1 x + p_2 x^2 + \dots + p_N(x), \quad (4.2)$$

$$Q_M(x) = q_0 + q_1x + q_2x^2 + \cdots + q_M(x). \quad (4.3)$$

The polynomials in (4.2) and (4.3) are constructed so that $f(x)$ and $R_{N,M}(x)$ agree at $x = 0$ and their derivatives up to degree $N + M$ agree at $x = 0$. In the case $Q_0(x) = 1$, the approximation is just the Maclaurin expansion for $f(x)$. For a fixed value of $N + M$ the error is the smallest when $P_N(x)$ and $Q_M(x)$ have the same degree or when $P_N(x)$ has degree one higher than $Q_M(x)$.

Note that the constant coefficient of Q_M is $q_0 - 1$. This is permissible, because it can be noted that 0 and $R_{N,M}(x)$ do not change when both $P_N(x)$ and $Q_M(x)$ are divided by the same constant. Hence, the rational function $R_{N,M}(x)$ has $N + M + 1$ unknown coefficients. Assume that $f(x)$ is analytic and it has the Maclaurin expansion

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k + \cdots. \quad (4.4)$$

From the difference $f(x)Q_M(x) - P_N(x) = Z(x)$, we obtain

$$\left[\sum_{i=0}^{\infty} a_i x^i \right] \left[\sum_{i=0}^M q_i x^i \right] - \left[\sum_{i=0}^N p_i x^i \right] = \left[\sum_{i=N+M+1}^{\infty} c_i x^i \right]. \quad (4.5)$$

The lower index $j = N + M + 1$ in the summation on the right side of (4.5) is chosen because the first $N + M$ derivatives of $f(X)$ and $R_{N,M}(x)$ should converge at $x = 0$.

When the left side of (4.5) is multiplied and coefficients with powers of x^i are set equal to zero for $k = 0, 1, 2, \dots, N + M$, the results are a system of $N + M + 1$ linear equations

$$\begin{aligned} a_0 - p_0 &= 0, \\ q_1 a_0 + a_1 - p_1 &= 0, \\ q_2 a_0 + q_1 a_1 + a_2 - p_2 &= 0, \\ q_3 a_0 + q_2 a_1 + q_1 a_2 + a_3 - p_3 &= 0, \\ q_M a_{N-M} + q_{M-1} a_{N-M-1} + a_N - p_N &= 0, \end{aligned} \quad (4.6)$$

and

$$\begin{aligned} q_M a_{N-M+1} + q_{M-1} a_{N-M+2} + \cdots + q_1 a_N + a_{N+2} &= 0, \\ q_M a_{N-M+2} + q_{M-1} a_{N-M+3} + \cdots + q_1 a_{N+1} + a_{N+3} &= 0, \\ &\vdots \\ q_M a_N + q_{M-1} a_{N+1} + \cdots + q_1 a_{N+M+1} + a_{N+M} &= 0. \end{aligned} \quad (4.7)$$

Notice that, for each equation, the sum of the subscripts on the factors of each product is the same. This sum increases consecutively from 0 to $N + M$. The M equations in (4.7) involve only the unknowns $q_1, q_2, \dots, q_M$ and they must be solved first. Then the equations in (4.6) are used successively to find $p_1, p_2, \dots, p_N$ [12, 21].

5. APPLICATION OF PM METHOD TO SOLVE THE COMPLETE ELLIPTIC INTEGRALS OF THE SECOND KIND AND ELLIPTIC INTEGRAL OF THE FIRST KIND

This section deals with the solution of (2.2), (2.3), and (2.4) by following a procedure used in [22], where (2.3) is represented as follows

$$y(\phi) = \int_0^\phi \sqrt{1 - \epsilon^2 \sin^2 \theta} d\theta, \quad 0 \leq \phi \leq \pi/2. \quad (5.1)$$

This integral can be reformulated as a nonlinear differential equation

$$y'^2(\phi) - 1 + \epsilon^2 \sin^2 \phi = 0. \quad (5.2)$$

By evaluating the integral (5.1) at $\phi = 0$, the result will be zero. Therefore, the initial condition for the differential equation (5.2) should be $y(0) = 0$.

It is possible to find a handy solution for (5.2) by applying the PM method. To apply the method, first, we identify terms

$$L(\phi) = y'^2(\phi) - 1, \quad (5.3)$$

$$N(\phi) = \sin^2 \phi, \quad (\text{see (4.1)}), \quad (5.4)$$

and ϵ^2 with the PM parameter. Notice that, for this case, N is truly linear, showing the flexibility of the method.

By assuming a solution for (5.2), in the form

$$y(\phi) = v_0(\phi) + \eta v_1(\phi) + \eta^2 v_2(\phi) + \dots \quad (\eta = \epsilon^2) \quad (\text{see (3.2)}), \quad (5.5)$$

and equating terms with equal powers of η , it can be solved for $v_0(\phi), v_1(\phi), v_2(\phi), \dots$, and so on. As a result, a very good approximation is obtained considering terms up to third order.

$$\eta^0) \quad v_0' - 1 = 0, \quad (5.6)$$

$$\eta^1) \quad 2v_0'v_1' + \sin^2 \theta = 0, \quad (5.7)$$

$$\eta^2) \quad v_1'^2 + 2v_0'v_2' = 0, \quad (5.8)$$

$$\eta^3) \quad 2v_0'v_3' + 2v_1'v_2' = 0, \quad (5.9)$$

$\vdots$

In order to fulfil the initial condition from (5.2) ($y(0) = 0$), it follows that $v_0(0) = 0, v_1(0) = 0, v_2(0) = 0, v_3(0) = 0$, and so on. Thus, the results are

$$v_0(\phi) = \phi, \quad (5.10)$$

$$v_1(\phi) = -\frac{1}{4}\phi + \frac{1}{8}\sin 2\phi, \quad (5.11)$$

$$v_2(\phi) = -\frac{3}{64}\phi + \frac{1}{32}\sin 2\phi - \frac{1}{256}\sin 4\phi, \quad (5.12)$$

$$v_3(\phi) = -\frac{5}{96} \left[\frac{3}{8}\phi - \frac{1}{4}\sin 2\phi + \frac{1}{32}\sin 4\phi \right] + \frac{1}{96}\cos \phi \sin^5 \phi, \quad (5.13)$$

$\vdots$

By substituting (5.10), (5.11), (5.12), and (5.13) into (5.5); and considering that $E(\epsilon, \phi) = y(\phi)$, it follows that an approximate solution to the incomplete second kind elliptic integral would be

$$\begin{aligned} E(\eta, \phi) = & \phi + \eta \left[\frac{1}{8}\sin 2\phi - \frac{1}{4}\phi \right] \\ & + \eta^2 \left[-\frac{3}{64}\phi + \frac{1}{32}\sin 2\phi - \frac{1}{256}\sin 4\phi \right] \\ & + \eta^3 \left[-\frac{15}{768}\phi + \frac{5}{384}\sin 2\phi - \frac{5}{3072}\sin 4\phi + \frac{1}{96}\cos \phi \sin^5 \phi \right] \\ & + \dots \end{aligned} \quad (5.14)$$

For convenience, (5.14) is rewritten in the form

$$\begin{aligned} E(\eta, \phi) = & \phi \left[1 - \frac{1}{4}\eta - \frac{3}{64}\eta^2 - \frac{15}{768}\eta^3 + \dots \right] \\ & + \sin 2\phi \left[\frac{1}{8}\eta + \frac{1}{32}\eta^2 + \left(\frac{5}{384} + \frac{\sin^4 \phi}{192} \right) \eta^3 + \dots \right] \\ & - \sin 4\phi \left[\frac{1}{256}\eta^2 + \frac{5}{3072}\eta^3 + \dots \right] \\ & + \dots \end{aligned} \quad (5.15)$$

Evaluating (5.15) for $\phi = \pi/2$, it becomes the complete elliptic integral of the second kind (approximation (2.4))

$$E(\epsilon) = \frac{\pi}{2} \left[1 - \frac{1}{4}\epsilon^2 - \frac{3}{64}\epsilon^4 - \frac{15}{768}\epsilon^6 + \dots \right], \quad (5.16)$$

In order to find an approximate solution for integral (2.2), we use the following equation which relates the complete elliptic integrals [2]

$$K(\epsilon) = E(\epsilon) - \epsilon \frac{dE(\epsilon)}{d\epsilon}. \quad (5.17)$$

When substituting (5.16) into (5.17), it is possible to obtain the following expression for $K(\epsilon)$

$$K(\epsilon) = \frac{\pi}{2} \left[1 + \frac{1}{4}\epsilon^2 + \frac{9}{64}\epsilon^4 + \frac{15}{153}\epsilon^6 + \dots \right]. \quad (5.18)$$

6. AN APPROXIMATE SOLUTION BY USING LAPLACE–PADÉ TRANSFORMATION AND PM METHOD

In this section we will describe the use of Laplace transform and Padé transformation to deal with the truncated series (5.18) obtained by the PM method in order to improve the approximation and the proposed valid range for the same series [21].

First, Laplace transformation is applied to (5.18) and then $1/\epsilon$ is written instead of s in the obtained equation. Then, Padé approximant $[2/2]$ is applied and $1/s$ is written instead of ϵ . Finally, by using the inverse Laplace transformation, we obtain the modified approximate solution

$$K(\epsilon) = \frac{\pi}{2} \cosh\left(\frac{\sqrt{2}}{2}\epsilon\right). \quad (6.1)$$

7. CASE STUDY

One relevant application to be considered here, for the complete elliptic integral of the first kind (2.2), is the problem of calculating the period of a nonlinear pendulum. The simple pendulum's oscillatory motion is among the most investigated motions in physics; many nonlinear phenomena in several fields of science and technology are governed by pendulum-like differential equations.

For small amplitude oscillations, the approximate period of a simple pendulum is given by

$$T_0 \approx 2\pi \sqrt{\frac{l}{g}},$$

where l is the length and g is the acceleration of gravity.

The exact value of the period of oscillation is

$$T = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \epsilon^2 \sin^2 \theta}}; \quad \epsilon = \sin\left(\frac{\theta_0}{2}\right), \quad (7.1)$$

where θ_0 is the initial amplitude. It is clear that we can express (7.1) in terms of the complete integral of the first kind (2.2)

$$T = 4\sqrt{\frac{l}{g}} K(\epsilon), \quad (7.2)$$

our procedure allows obtaining two different approximations to the oscillation period for a pendulum. The first is obtained by substituting (5.18) into (7.2)

$$T = 2\pi \sqrt{\frac{l}{g}} \left[1 + \frac{1}{4}\epsilon^2 + \frac{9}{64}\epsilon^4 + \frac{15}{153}\epsilon^6 + \dots \right]. \quad (7.3)$$

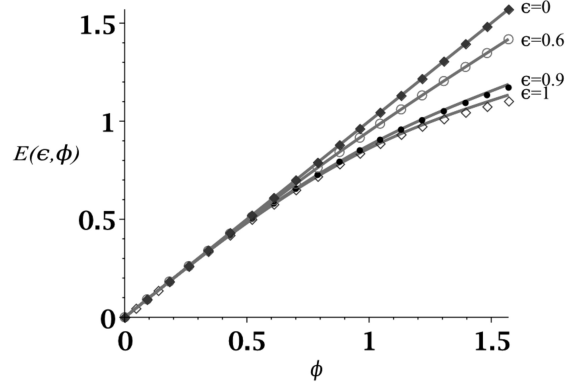


FIGURE 1. Exact numerical solution (solid line); approximate solutions (5.15) for $\epsilon = [0, 0.6, 0.9, 1]$ represented by: solid diamond, circle, solid circle, diamond, respectively.

The second is the result of substituting (6.1) into (7.2), which lead us to

$$T = 2\pi \sqrt{\frac{l}{g}} \cosh\left(\frac{\sqrt{2}}{2}\epsilon\right). \quad (7.4)$$

For comparison purposes, period (7.2), (7.3) and (7.4) will be normalized to

$$\hat{T} = \frac{1}{4} \sqrt{\frac{g}{l}} T. \quad (7.5)$$

In the next section, we will analyse and compare $\hat{T}$ for (7.2), (7.3) and (7.4).

8. DISCUSSION

Nonlinear phenomena appear in broad scientific fields like applied mathematics, physics, and engineering. Scientists in those disciplines constantly face the task of finding solutions of nonlinear ordinary differential equations. The fact that the PM method depends on a parameter which is assumed to be small, suggests that the method has limited range. In this work, the PM method has been applied to the problem of evaluating, approximately, complete or incomplete elliptic integrals of the second kind by using the nonlinear differential equation (5.2). From the obtained results, an approximation was achieved for the complete elliptic integral of the first kind and, as a consequence, for the period of a nonlinear pendulum. We will see that even though the PM does not provide a satisfactory approximation in the latter case, when it is coupled with Padé methods, we may obtain a better approximation.

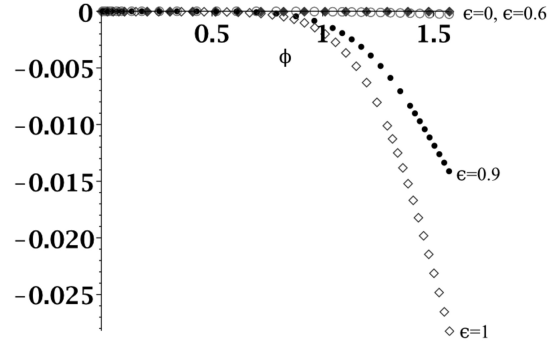


FIGURE 2. Relative error from approximated solutions given in Figure 1.

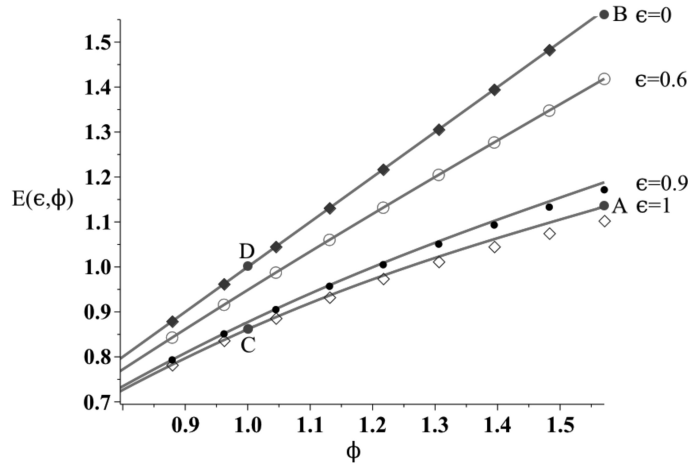


FIGURE 3. Zoom from figure 1.

8.1. Elliptic integrals of the second kind

For the case of elliptic integrals of the second kind, we obtained highly accurate approximations for both, complete and incomplete, integrals. From (5.16), for instance, in the case of the complete integral, the limit

$$\lim_{\epsilon \rightarrow 1} E(\epsilon) = 1.073,$$

means that the error is about 7%.

θ	Numerical	PM-Pad�	RE%	PM	RE%
0	1.570796	1.570796	0	1.570796	0
4	1.571275	1.571275	1.93e-05	1.572716	-0.0917
8	1.572712	1.572708	0.000308	1.578537	-0.37
10	1.573792	1.573780	0.000754	1.582968	-0.583
20	1.582843	1.582653	0.012	1.622204	-2.49
30	1.598142	1.597176	0.0605	1.698233	-6.26
40	1.620026	1.616958	0.189	1.832495	-13.1
50	1.648995	1.641458	0.457	2.065975	-25.3
60	1.685750	1.669998	0.934	2.470172	-46.5
70	1.731245	1.701771	1.7	3.161204	-82.6
80	1.786769	1.735863	2.85	4.317044	-142
90	1.854075	1.771271	4.47	6.197893	-234
100	1.935581	1.806932	6.65	9.169686	-374
110	2.034715	1.841752	9.48	13.730732	-575

TABLE 1. Numerical comparison for $\hat{T}$.

The PM method provides, in general, better approximations for small perturbation parameters when it includes the most number of terms from (3.2). Even so, it is common that the obtained approximations are appropriate for a small interval of the domain of the approximate solution. Fig. 3 represents, graphically, the reason why this method is efficient for the case of the elliptic integrals represented by (5.2). It can be seen that the range of values for $E(\epsilon)$ is $(1, \pi/2]$, that is, $1 < E(\epsilon) \leq 1.57$. The extreme values of that range are given by points A and B in Fig. 3 (where $\lim_{\epsilon \rightarrow 1} E(\epsilon) = 1$ and $E(0) = 1.57$). Notice that the range for the incomplete elliptical integral is even lower. For instance, points C and D represent the “extreme” values for $E(\epsilon, 1)$: $\lim_{\epsilon \rightarrow 1} E(\epsilon, 1) = 0.8414$ and $E(0, 1) = 1$.

Under these conditions it seems reasonable that just a few terms are required from (5.5) to obtain a good approximation of (5.2); because, since the beginning, the first approximation $v_0(\phi)$ (see (5.10)) is “close” to its final value $E(\epsilon, \phi)$. For instance, in the case of complete integrals, at the limit $\epsilon \rightarrow 1$, the first approximation provides the value $v_0(\pi/2) = 1.57$, not far of the true value $\lim_{\epsilon \rightarrow 1} E(\epsilon) = 1$. Moreover, for $E(\epsilon, 1)$, the $\lim_{\epsilon \rightarrow 1} E(\epsilon, 1) = 0.8414$ and $E(0, 1) = 1$ are even closer. Our results have an accuracy similar to other approximations reported in the literature [2].

8.2. Complete elliptic integral of the first kind and the nonlinear pendulum

Unlike what happens with the elliptic integrals depicted by differential equation (5.2), the complete elliptic integral of the first kind is not appropriately represented by (5.18). From Table 1 it can be noticed that normalized version of (7.3) is a proper

approximation for $\hat{T}$ only for angles within the 30° range, possessing a relative percentage error lower than six percent. From the same table can be seen that for angles of 80° the relative error is about -142%, and for angles of 110° the relative error is about -575%. Therefore, the approximation given by (7.3) is, in general, totally inadequate. Instead of adding terms to (5.18), Section 6 explained a technique that coupled the Padé transformation and the PM method giving as result an approximate value for $K(\epsilon)$ by (6.1) (see (7.4)).

[12] used a coupling of HPM, Laplace transformation, He's polynomials, and Padé technology in order to obtain series solution of nonlinear exponential boundary layer equation. Unlike the aforementioned method, this work proposes the use of PM and Laplace-Padé methods to deal with the truncated power series.

A noticeable difference can be seen in Table 1 for the percentage of the relative error at 20° , it is just 0.012%; for 80° we obtain an error that is about 2.85%. Thus, even for 110° , the obtained relative percentage error is 9.48%. Therefore, the combination of both techniques is not only novel but accurate. Finally note the simplicity of (6.1) compared to the infinite series in (5.18).

9. CONCLUSIONS

This work shows that some nonlinear problems can be adequately approximated by using the PM method, even for large values of the perturbation parameter; as it was done for the elliptic integrals described by (5.2). In fact the first approximation (5.10) is close to the true value for $E(\eta, \phi)$ (5.15), for every $\eta \in [0, 1]$ and $\phi \in [0, \pi/2]$ (see Fig. 1 and Fig. 3). This contributes to the success of the method in this case and it could be useful for similar cases, so our method can replace other sophisticated and difficult methods. Section 6 showed that even though the method is not adequate, as it can be seen for integral (2.2), it is possible to use a novel technique that coupled the PM method and the Padé transformation to improve the approximation (as it was proved for (6.1)). This result is much more accurate than (5.18), as it can be seen in Table 1. Besides, it is handy and useful for engineering applications.

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A NOTE ON GRÖBNER BASES AND GRAPH COLORINGS

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Abstract. In this paper, we correct a minor misstatement in [4], where J.A. De Loera demonstrates an explicit universal Gröbner basis of the radical ideal of a variety related to chromatic numbers. We show that this result does not hold when the base field is finite, and we correct it for this case.

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1. INTRODUCTION

The theory of Gröbner bases is a key computational tool for studying polynomial ideals. This theory has been introduced and developed by Buchberger in 1965 (see his PhD thesis [2]) and has been applied to the problem of graph coloring in [1]. A graph with n vertices may be represented by a polynomial in n variables. This polynomial lies in a particular ideal if and only if the graph is not k -colorable. Thus, the problem of k -coloring a graph is equivalent to an ideal membership problem. The concept of Gröbner bases may be applied to solve this problem. It has been shown in [4] that the Gröbner basis of the ideal corresponding to this problem is universal, i.e. it is a Gröbner basis for any monomial ordering.

Let $k, n \geq 2$ be two positive integers and K be an arbitrary field. Let $V(n, k)$ denote the set of vectors which have at most $k - 1$ distinct coordinates. Let also $J(n, k)$ be the vanishing ideal of $V(n, k)$. De Loera in [4] has proved the following theorem:

Theorem 1. *The set of polynomials*

$$\rho(n, k) = \left\{ \prod_{1 \leq r < s \leq k} (x_{i_r} - x_{i_s}) \mid 1 \leq i_1 < \dots < i_k \leq n \right\}$$

is a universal Gröbner basis for $J(n, k)$.

To prove this result, De Loera in his paper, on page 3, states that “... but no non-zero univariate polynomial belongs to $J(n, 2)$.” However, this claim (and thus this

theorem) holds only if K is infinite. In the following example, we show that Theorem 1 fails when K is a finite field.

Example 1. Let $K = \mathbb{F}_2 = \{0, 1\}$ be a field with two elements. Then $V(2, 2) = \{(0, 0), (1, 1)\}$, and therefore $x_1^2 - x_1$ and $x_2^2 - x_2$ belong to the ideal $J(2, 2) \subset \mathbb{F}_2[x_1, x_2]$. From the above notations, we have $\rho(2, 2) = \{x_1 - x_2\}$ which is not a universal Gröbner basis for $J(2, 2)$.

Extending Theorem 1 to the finite fields we prove the following theorem:

Theorem 2. *Let $q = p^e$ where p is a prime and e is a positive integer. Let also $K = \mathbb{F}_q$ be a finite field with q elements. Then, the set of polynomials*

$$\tau(n, k) = \left\{ \prod_{1 \leq r < s \leq k} (x_{i_r} - x_{i_s}) \mid 1 \leq i_1 < \dots < i_k \leq n \right\} \cup \{x_i^q - x_i \mid 1 \leq i \leq n\}$$

is a universal Gröbner basis for $J(n, k)$.

It is worth commenting that for the applications of Theorem 1 in [4], De Loera has used this theorem for infinite fields. Now, we give the structure of the paper. In Section 2 we prove Theorem 2. Section 3 is devoted to a correction of Example 3.4 in [4] on enumerating distinct colorings.

2. THE PROOF OF THEOREM 2

In this section, we prove Theorem 2, using the proof structure of Theorem 1 in [4]. We briefly state some necessary definitions.

Let $q = p^e$ where p is a prime and e is a positive integer. Let $K = \mathbb{F}_q$ be a finite field with q elements, $R = K[x_1, \dots, x_n]$ be a polynomial ring and $I = \langle f_1, \dots, f_t \rangle$ be the ideal of R generated by polynomials $f_1, \dots, f_t$. Let $f \in R$ and $<$ be a monomial ordering on R . The *leading monomial* of f is the greatest monomial (with respect to $<$) which appears in f , and we denote it by $\text{LM}(f)$. The *leading coefficient* of f , written $\text{LC}(f)$, is the coefficient of $\text{LM}(f)$ in f . The *leading term* of f is $\text{LT}(f) = \text{LC}(f)\text{LM}(f)$. The *leading term ideal* of I is defined as

$$\text{LT}(I) = \langle \text{LT}(f) \mid f \in I \rangle.$$

For a finite set $G \subset R$, we denote by $\text{LT}(G)$ the monomial ideal $\langle \text{LT}(g) \mid g \in G \rangle$. A finite subset of polynomials $G \subset I$ is called a *Gröbner basis* for I w.r.t. $<$ if $\text{LT}(I) = \text{LT}(G)$, see [3] for more details. A *universal Gröbner basis* for I is a finite subset of I which is a Gröbner basis w.r.t. any monomial ordering.

Proof of Theorem 2. The following lemma gives a set of conditions for a universal Gröbner basis ([4], Lemma 2.1).

Lemma 1. *Let $I \subset R$ be an ideal and let $G = \{g_1, \dots, g_t\} \subset I$ be a set of polynomials such that each g_i is a product of linear factors in $x_1, \dots, x_n$. Further, assume that for any $g \in G$ and for any permutation σ on the set $\{1, \dots, n\}$, we have $g(\sigma(x_1), \dots, \sigma(x_n)) \in G$. If G is a Gröbner basis for I w.r.t. a particular monomial ordering, then it is a universal Gröbner basis for I .*

In order to apply Lemma 1 to $\tau(n, k)$, we must prove the following three claims:

- Any $g \in \tau(n, k)$ factors into linear factors in R
- $g(\sigma(x_1), \dots, \sigma(x_n)) \in \tau$ for each $g \in \tau(n, k)$ and any permutation σ on the set $\{1, \dots, n\}$
- $\tau(n, k)$ is a Gröbner basis for $J(n, k)$ w.r.t. a particular monomial ordering.

For the first item, it is enough to prove that $x_i^k - x_i$ for any i factors into linear factors. This is deduced from the following lemma (see [6], Lemma 2.4).

Lemma 2. *With the above notations, the polynomial $x^q - x$ factors (into linear factors) in $K[x]$ as*

$$x^q - x = \prod_{a \in K} (x - a).$$

The second item follows from the structure of $\tau(n, k)$, and the fact that the elements of $\rho(n, k)$ are in bijection with the k element subsets of $\{x_1, \dots, x_n\}$. Now, we deal with the third item. We prove that $\tau(n, k)$ is a Gröbner basis for $J(n, k)$ w.r.t. the lexicographical ordering $<$ with $x_n < \dots < x_1$. Since $\tau(n, k) \subset J(n, k)$, it is enough to prove that the leading term of any polynomial in $J(n, k)$ is divisible by the leading term of a member of $\tau(n, k)$. For this, we proceed by a double induction on k and n like in the proof of Theorem 1 in [4]. Let $k = 2$ and n arbitrary. We know that

$$\tau(n, 2) = \{x_i - x_j \mid 1 \leq i < j \leq n\} \cup \{x_i^q - x_i \mid 1 \leq i \leq n\}$$

and $\text{LT}(\tau(n, 2)) = \{x_1, \dots, x_{n-1}, x_n^q\}$. Let $f \in J(n, 2)$ be a nonzero polynomial. If $\text{LT}(f)$ is divisible by any of the first $n - 1$ variables then $\text{LT}(f) \in \text{LT}(\tau(n, 2))$. Otherwise, f is a nonzero univariate polynomial in x_n (we consider it in $K[x_n]$). From the definition of $V(n, 2)$ we can conclude that $f(a) = 0$ for any $a \in K$. This implies that $x_n^q - x_n$ divides f , and therefore x_n^q divides $\text{LT}(f)$. By induction on k the result is true for $J(n, r)$ for where $k > r \geq 2$ and n arbitrary. We proceed by induction on n . We show that $J(k, k)$ is generated by the set $\{\prod_{1 \leq i \leq j \leq k} (x_i - x_j), x_1^q - x_1, \dots, x_k^q - x_k\}$. By Buchberger criterion and Buchberger first criterion (see [3], pages 85 and 104) we can prove easily that the set $B = \{x_1^q - x_1, \dots, x_k^q - x_k\}$ is a Gröbner basis for the ideal that it generates. Let f be an element of $J(k, k)$ and $\bar{f}$ be the remainder of the division of f by B . It is worth noting that since B is a Gröbner basis, this remainder is unique (see [3], Proposition 1 page 82). Since $\bar{f} \in J(k, k)$, regardless of whether $\bar{f}$ is zero or non-zero $\bar{f}(a_1, \dots, a_k) = 0$ for any $(a_1, \dots, a_k) \in V(k, k)$. In $V(k, k)$, every k -dimensional point has at most $k - 1$ distinct entries. Thus, if any two entries (such as x_i and x_j) are equal, even if the other $k - 2$ entries are distinct,

the point is still contained in $V(k, k)$. Therefore, since $\bar{f}$ vanishes on every point in $V(k, k)$ then we have $(x_i - x_j) \mid \bar{f}$ for $1 \leq i < j \leq k$, and $\bar{f} \in \tau(k, k)$. This settles the case $n = k$.

Now by induction hypothesis the result is true for $J(r, k)$ with $n > r \geq k$. We have to prove that $\text{LT}(\bar{f}) \in \text{LT}(\tau(n, k))$ for any $f \in J(n, k)$. Let $B = \{x_1^q - x_1, \dots, x_n^q - x_n\}$, which is a Gröbner basis for the ideal that it generates. Let $\bar{f}$ be the remainder of the division of f by B . We have $\bar{f} \in J(n, k)$. If $\bar{f} \neq 0$, we construct an auxiliary polynomial. Let $S \subseteq \{1, \dots, n-1\}$. We denote by $\bar{f}_S$ the polynomial obtained from $\bar{f}$ by substituting x_n for each variable x_i for $i \in S$. Thus, for a non-empty set S the polynomial $\bar{f}_S \in J(r, k)$ with $r = n - |S|$ where $|S|$ denotes the size of S . Let

$$g = \sum_{S \subseteq \{1, \dots, n-1\}} (-1)^{|S|} \bar{f}_S.$$

We claim that $\text{LT}(g) \in \text{LT}(\tau(n, k))$. Note that from the definition of $\bar{f}$ we can replace $\tau(n, k)$ by $\rho(n, k)$ in this claim. The rest of the proof is exactly the same as the latter part of the proof of Theorem 1 in [4]. However, for the sake of completeness, we provide it here. If we substitute x_n for any x_i with $1 \leq i \leq n-1$ then we get the zero polynomial (note that $\deg(\bar{f}) \leq q$). Thus $(x_1 - x_n) \cdots (x_{n-1} - x_n) \mid g$, and therefore we can write it as $g = (x_1 - x_n) \cdots (x_{n-1} - x_n)h$ for some polynomial $h \in R$. Since $g \in J(n, k)$ if we expand h as a polynomial in x_n , its coefficients L_i belongs to $J(n-1, k-1)$. By the induction hypothesis $\text{LT}(L_i) \in \text{LT}(\rho(n-1, k-1))$. We observe that $\text{LT}(h) = \text{LT}(L_j)x_n^j$ for some j and thus $\text{LT}(g) = x_1x_2 \cdots x_{n-1} \text{LT}(L_j)x_n^j$. Since $\text{LT}(L_j)$ is divisible by some element of $\text{LT}(\rho(n-1, k-1))$, then $x_1x_2 \cdots x_{n-1} \text{LT}(L_j)$ is divisible by some monomial in $\text{LT}(\rho(n, k))$ as desired.

If $\text{LT}(g) = \text{LT}(\bar{f})$ we are done. Otherwise, $\text{LT}(g) < \text{LT}(\bar{f})$ (since we use lexicographical ordering). But, in the definition of g the set S may be empty. In this case $\bar{f}_S = \bar{f}$ and we can write g as

$$g = \bar{f} + \sum_{S \neq \emptyset \text{ and } S \subseteq \{1, \dots, n-1\}} (-1)^{|S|} \bar{f}_S.$$

This follows that $\text{LT}(\bar{f}) = \text{LT}(\bar{f}_S)$ for a non-empty set $S \subseteq \{1, \dots, n-1\}$. We observe that $\bar{f}_S \in J(r, k)$ for $n > r \geq k$ and then $\text{LT}(\bar{f}) = \text{LT}(\bar{f}_S) \in \text{LT}(\rho(r, k))$ by the induction hypothesis. Finally, for $n > r \geq k$, we have $\rho(r, k) \subset \rho(n, k)$, and this ends the proof of the theorem. $\square$

3. ENUMERATING DISTINCT COLORINGS

In this section, we correct an error in Example 3.4 in [4] to compute the number of distinct 3-colorings of the two-by-four grid graph.

In [4], De Loera has applied Theorem 1 to the general question of enumerating distinct colorings of a graph (see Lemma 3). For this, we need some definitions. Let

us denote by $\pi(G, k)$ the number of distinct k -colorings of a graph G . Let also P_G be the polynomial associated with the labeling of a graph G , i.e. if $V = \{x_1, \dots, x_n\}$ is the set of vertices and $E(G)$ is the set of edges of G then

$$P_G = \prod_{i < j \text{ and } x_i x_j \in E(G)} (x_i - x_j).$$

Now we recall the definition of the degree of an ideal. Let $R = K[x_1, \dots, x_n]$ be a polynomial ring where K is an infinite field. Let X be a graded R -module and δ be a positive integer. We denote by X_δ the set of elements of X of degree δ . Let $I \subset R$ be a homogeneous ideal. The *Hilbert series* of I is the power series $HS_I(t) = \sum_{s=0}^{\infty} HF_I(s)t^s$ where $HF_I(s)$ (the Hilbert function of I) is the dimension of $(R/I)_s$ as a K -vector space.

Proposition 1. *We have $HS_I(t) = N(t)/(1-t)^d$ where $N(t)$ is a polynomial which is not multiple of $1-t$, and d is the dimension of I .*

For the proof of this proposition see [5], Theorem 7, Chapter 11. Now, using this proposition we could define the degree of an ideal.

Definition 1. The degree of the ideal I , noted by $\deg(I)$, is $N(1)$ where N is the numerator of HS_I .

We recall that the ideal $I : P_G^\infty$ is defined as

$$I : P_G^\infty = \{f \in R \mid f^m P_G \in I \text{ for some } m > 0\}.$$

Using these notations, we have the following result (see [4], Proposition 3.3).

Lemma 3. $\pi(G, k-1) = \deg(J(n, k) : P_G^\infty)$.

Example 2. In this example, we compute the number of distinct 3-colorings of the two-by-four grid graph H , and we correct an error of Example 3.4 in [4] to compute it. This graph has eight vertices $x_1, \dots, x_8$ and ten edges $x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7, x_7x_8, x_1x_8, x_2x_7, x_3x_6$. We have to compute the degree of the ideal $J(8, 4) : P_H^\infty$. In order to speed up the computation, De Loera has proposed to use the factorization of P_H to compute the generators of the saturation ideals $J(8, 4) : (x_i - x_j)^\infty$ for each of the edges of H . He has claimed that if one computes these ten ideals, then their intersection is precisely equal to $J(8, 4) : P_H^\infty$ (we denote this intersection by I). But, this equality does not hold¹. Using MAPLE11, we can compute I and its Hilbert series where the latter is equal to

$$(t^{13} + 6t^{12} + 22t^{11} + 55t^{10} + 106t^9 + 159t^8 + 190t^7 + 175t^6 + 126t^5 + 70t^4 + 35t^3 + 15t^2 + 5t + 1)/(1-t)^3.$$

¹After the submission of the paper, an anonymous referee pointed out that Example 3.4 in [4] remains true if we replace "intersection" by "sum". He/She provided also a Macaulay2 code to verify this statement, see <http://amirhashemi.iut.ac.ir/software.html>.

Therefore $\deg(I) = 966$ which is not equal to $\pi(H, 3)$, because we will see further that the number of distinct 3-colorings of H is 26. Let us see a simple example illustrating the difference between the above ideals. Let C_4 be the 4-cycle graph with the vertices $y_1, \dots, y_4$ and the edges $y_1y_2, y_2y_3, y_3y_4, y_4y_1$. We would like to compute $\pi(C_4, 2)$. We observe that $J(4, 3) : P_{C_4}^\infty$ is equal to $\langle y_1 - y_3, y_2 - y_4 \rangle$, i.e. $\pi(C_4, 2) = 1$. On the other hand,

$$\begin{aligned} \bigcap_{y_i y_j \in E(C_4)} J(4, 3) : (y_i - y_j)^\infty &= \langle y_1 - y_4, y_3 - y_4 \rangle \cap \langle y_1 - y_2, y_3 - y_4 \rangle \\ &\quad \cap \langle y_1 - y_3, y_2 - y_3 \rangle \cap \langle y_1 - y_4, y_2 - y_3 \rangle \\ &\quad \cap \langle y_1 - y_4, y_2 - y_4 \rangle \cap \langle y_2 - y_4, y_3 - y_4 \rangle \\ &\quad \cap \langle y_1 - y_3, y_2 - y_4 \rangle \end{aligned}$$

which is not equal to $J(4, 3) : P_{C_4}^\infty$. The Hilbert series of this intersection is equal to $(t^3 + 3t^2 + 2t + 1)/(1 - t)^2$, and therefore its degree is 7.

Now, we compute $\pi(H, 3)$. Computing $J(8, 4) : P_H^\infty$ is not feasible in less than 12 hours (timings in this paper were conducted on a personal computer with 3.2GHz, 2×Intel(R)-Xeon(TM) Quad core, 24 GB RAM and 64 bits under the Linux operating system). In order to speed up the computation, we use the following simple result (see [3], Theorem 11, page 196).

Lemma 4. *Let $L \subset R$ be a radical ideal and $f \in R$. Let $L \cap \langle f \rangle = \langle g_1, \dots, g_\ell \rangle$. Then $\{g_1/f, \dots, g_\ell/f\}$ is a generating set for the ideal $L : f^\infty$.*

Proof. It is enough to prove that any polynomial $g \in L : f^\infty$ belongs to $\langle g_1/f, \dots, g_\ell/f \rangle$. We know that $gf^m \in L$ for some integer m . This follows that $(gf)^m \in L$, and therefore $gf \in L \cap \langle f \rangle$. Thus, $g \in \langle g_1/f, \dots, g_\ell/f \rangle$. $\square$

We can compute $J(8, 4) \cap \langle P_H \rangle$ and then a generating set for $J(8, 4) : P_H^\infty$ in 2152.549 seconds. The Hilbert series of this ideal is equal to $(8t^3 + 12t^2 + 5t + 1)/(1 - t)^3$, and therefore its degree is equal to 26.

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ON THE POSITIVE LINEAR SET VALUED MAPS

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Abstract. In this paper positive linear set valued maps defined on the cone are studied. The representation theorem for positive linear set valued maps is given and Lipschitz continuity of these maps is proved. The estimations of upper and lower norms of the positive linear set valued maps are obtained.

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1. INTRODUCTION

It is known that the set valued analogues of continuous linear operators are set valued maps the graph of which are cones or vector subspaces. These kinds of set valued maps are called processes or linear processes respectively (see, e.g. [10, 11]). They provide a good framework for investigation of general homogenous systems which arise in economics, mechanics and functional analysis (see, e.g. [2, 3, 7, 12]). Differentials of the set valued maps provide examples of closed convex processes (see, e.g. [2, 4]). Various properties of the convex processes are investigated in [1–3, 5, 6, 8–12]. Regularity and continuity of the convex processes are discussed in [2, 3, 5, 6, 8, 9, 12]. In [1] controllability property of the differential inclusion is studied where the right hand side of the given differential inclusion is a convex process. In this paper positive linear set valued maps are defined which satisfy additivity and positive homogeneity conditions. The paper is organized as follows:

In Section 2 the representation theorem for a positive linear set valued map is given (Theorem 1). In Section 3 the Lipschitz continuity of the positive linear set valued maps is proved (Theorem 2). In Section 4 the evaluations of upper and lower norms of the positive linear set valued maps are obtained (Theorem 3).

Let $\mathbb{R}^n$ be the n -dimensional Euclidean space, $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, and $a_i \in X$, $i = 1, 2, \dots, n$, are nonzero linearly independent vectors. Denote

$$\mathbb{R}_+^n = \{\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n : \xi_i \geq 0, i = 1, 2, \dots, n\},$$

$$K = \left\{ x = \sum_{i=1}^n \xi_i a_i : (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}_+^n \right\}, \quad (1.1)$$

$$E^n = \left\{ x = \sum_{i=1}^n \xi_i a_i : (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n \right\}, \quad (1.2)$$

$$\alpha = \left(\sum_{i=1}^n \|a_i\|_X^2 \right)^{\frac{1}{2}}. \quad (1.3)$$

It is obvious that $K \subset X$ is a cone and every $x \in K$ has a unique representation in the form $x = \sum_{i=1}^n \xi_i a_i$, where $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}_+^n$. Similarly, every $x \in E^n$ also

has a unique representation in the form $x = \sum_{i=1}^n \xi_i a_i$, where $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$.

For given $x \in E^n$ we will denote $\mathbf{x}_n = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$ and $\|\mathbf{x}_n\| = \left(\sum_{i=1}^n \xi_i^2 \right)^{\frac{1}{2}}$.

For each $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$ we define a function $f(\cdot) : \mathbb{R}^n \rightarrow [0, \infty)$, setting

$$f(\xi) = f(\xi_1, \xi_2, \dots, \xi_n) = \left\| \sum_{i=1}^n \xi_i a_i \right\|_X. \quad (1.4)$$

Let

$$S^n = \{ \xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n : \|\xi\| = 1 \},$$

$$S_+^n = \{ \xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}_+^n : \|\xi\| = 1 \}.$$

It is clear that $S_+^n \subset \mathbb{R}^n$, $S^n \subset \mathbb{R}^n$ are compact sets, $S_+^n \subset S^n$, $f(\cdot) : \mathbb{R}^n \rightarrow [0, \infty)$ is a continuous function and

$$f(\xi) = \left\| \sum_{i=1}^n \xi_i a_i \right\|_X > 0 \text{ for every } \xi = (\xi_1, \xi_2, \dots, \xi_n) \in S^n.$$

Hence there exist $\gamma > 0$ and $\beta > 0$ such that

$$\min \{ f(\xi) : \xi \in S^n \} = \gamma, \quad (1.5)$$

$$\min \{ f(\xi) : \xi \in S_+^n \} = \beta. \quad (1.6)$$

It is obvious that $\beta \geq \gamma > 0$.

Proposition 1. *The inequality*

$$\gamma \|\mathbf{x}_n\| \leq \|x\|_X \leq \alpha \|\mathbf{x}_n\| \quad (1.7)$$

is satisfied for every $x \in E^n$, and the inequality

$$\beta \|\mathbf{x}_n\| \leq \|x\|_X \leq \alpha \|\mathbf{x}_n\| \quad (1.8)$$

holds for every $x \in K$, where $\alpha > 0$, $\gamma > 0$ and $\beta > 0$ are defined by (1.3), (1.5) and (1.6) respectively.

Proof. Let $x \in E^n$. Then $x = \sum_{i=1}^n \xi_i a_i$, where $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$, and hence $\mathbf{x}_n = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$. Thus, using Cauchy-Schwarz inequality, we obtain from (1.3) that

$$\|x\|_X = \left\| \sum_{i=1}^n \xi_i a_i \right\|_X \leq \sum_{i=1}^n |\xi_i| \|a_i\|_X \leq \left(\sum_{i=1}^n \xi_i^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^n \|a_i\|_X^2 \right)^{\frac{1}{2}} = \alpha \|\mathbf{x}_n\|. \quad (1.9)$$

Now, let again $x = \sum_{i=1}^n \xi_i a_i \in E^n$ be an arbitrarily chosen vector where $x \neq 0$. Then $\mathbf{x}_n = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$, $\mathbf{x}_n \neq 0$ and

$$\|x\|_X = \left\| \sum_{i=1}^n \xi_i a_i \right\|_X = \|\mathbf{x}_n\| \cdot \left\| \sum_{i=1}^n \frac{\xi_i}{\|\mathbf{x}_n\|} \cdot a_i \right\|_X. \quad (1.10)$$

Since $\left(\frac{\xi_1}{\|\mathbf{x}_n\|}, \frac{\xi_2}{\|\mathbf{x}_n\|}, \dots, \frac{\xi_n}{\|\mathbf{x}_n\|} \right) \in S^n$, we have from (1.4) and (1.5) that

$$\left\| \sum_{i=1}^n \frac{\xi_i}{\|\mathbf{x}_n\|} \cdot a_i \right\|_X \geq \gamma. \quad (1.11)$$

(1.10) and (1.11) imply

$$\|x\|_X \geq \gamma \|\mathbf{x}_n\|. \quad (1.12)$$

If $x = 0$, then inequality (1.12) is satisfied trivially. From (1.9) and (1.12) we obtain the proof of the inequality (1.7).

Since $K \subset E^n$, (1.7) gives

$$\|x\|_X \leq \alpha \|\mathbf{x}_n\| \quad (1.13)$$

for every $x \in K$, where $\alpha > 0$ is defined by (1.3).

Let, now, $x = \sum_{i=1}^n \xi_i a_i \in K$ be an arbitrarily chosen vector such that $x \neq 0$. Then $\mathbf{x}_n = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}_+^n$, $\mathbf{x}_n \neq 0$ and

$$\|x\|_X = \left\| \sum_{i=1}^n \xi_i a_i \right\|_X = \|\mathbf{x}_n\| \cdot \left\| \sum_{i=1}^n \frac{\xi_i}{\|\mathbf{x}_n\|} \cdot a_i \right\|_X. \quad (1.14)$$

Since $\left(\frac{\xi_1}{\|\mathbf{x}_n\|}, \frac{\xi_2}{\|\mathbf{x}_n\|}, \dots, \frac{\xi_n}{\|\mathbf{x}_n\|}\right) \in S_+^n$, we have from (1.4) and (1.6) that

$$\left\| \sum_{i=1}^n \frac{\xi_i}{\|\mathbf{x}_n\|} \cdot a_i \right\|_X \geq \beta. \quad (1.15)$$

(1.14) and (1.15) imply

$$\|x\|_X \geq \beta \|\mathbf{x}_n\|. \quad (1.16)$$

The inequality (1.16) is also verified, if $x = 0$. (1.13) and (1.16) yield the proof of the inequality (1.8). $\square$

Proposition 2. *The cone $K \subset X$ defined by relation (1.1) is a pointed cone, i.e. K is closed, convex, and $K \cap (-K) = \{0_X\}$.*

Proof. Convexity of the cone K is obvious. Let us prove closedness. Let $x^{(m)} = \sum_{i=1}^n \xi_i^{(m)} a_i \in K$ for every $m = 1, 2, \dots$ and $x^{(m)} \rightarrow x^*$ as $m \rightarrow \infty$. By virtue of Proposition 1 we have

$$\|\mathbf{x}_n^{(m)}\| \leq \frac{1}{\beta} \|x^{(m)}\|_X \quad (1.17)$$

for every $m = 1, 2, \dots$ where $\mathbf{x}_n^{(m)} = (\xi_1^{(m)}, \xi_2^{(m)}, \dots, \xi_n^{(m)})$.

Since $x^{(m)} \rightarrow x^*$ as $m \rightarrow \infty$, then it follows from (1.17) that there exists an $r > 0$ such that

$$\|\mathbf{x}_n^{(m)}\| \leq r$$

for every $m = 1, 2, \dots$, and hence

$$\sum_{i=1}^n \left(\xi_i^{(m)}\right)^2 \leq r^2 \quad (1.18)$$

for every $m = 1, 2, \dots$. (1.18) implies that the sequences $\{\xi_i^{(m)}\}_{m=1}^\infty$ are bounded for every $i = 1, 2, \dots, n$. Hence there exists a subsequence $\{\xi_i^{(m_j)}\}_{j=1}^\infty$ such that $\xi_i^{(m_j)} \rightarrow \xi_i^*$ as $j \rightarrow \infty$ for every $i = 1, 2, \dots, n$. It is obvious that $\xi_i^* \geq 0$ for every $i = 1, 2, \dots, n$.

Denote $y_* = \sum_{i=1}^n \xi_i^* a_i$. Then $y_* \in K$ and $x^{(m_j)} \rightarrow y_*$ as $j \rightarrow \infty$, where $x^{(m_j)} = \sum_{i=1}^n \xi_i^{(m_j)} a_i$. Since $x^{(m)} \rightarrow x^*$ as $m \rightarrow \infty$, we get $x_* = y_* \in K$.

Let us prove the validity of the relation

$$K \cap (-K) = \{0_X\}.$$

Let us assume contrary, i.e. let there exists $x_* \in K$ such that $x_* \neq 0$ and $-x_* \in K$. Then there exist unique $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}_+^n$ and $\eta = (\eta_1, \eta_2, \dots, \eta_n) \in \mathbb{R}_+^n$ such that

$$x_* = \sum_{i=1}^n \xi_i a_i, \quad -x_* = \sum_{i=1}^n \eta_i a_i. \quad (1.19)$$

(1.19) yields

$$\sum_{i=1}^n (\xi_i + \eta_i) a_i = 0. \quad (1.20)$$

Since the vectors $a_1, a_2, \dots, a_n$ are linearly independent, $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}_+^n$ and $\eta = (\eta_1, \eta_2, \dots, \eta_n) \in \mathbb{R}_+^n$, (1.20) implies that $\xi_i = 0, \eta_i = 0$ for every $i = 1, 2, \dots, n$, and hence $x_* = 0$. This contradicts with $x_* \neq 0$. $\square$

The family of all non-empty, compact, and convex subsets of Y is denoted by $P_{kc}(Y)$. The Hausdorff distance between the sets $U \subset Y$ and $V \subset Y$ is denoted by $h_Y(U, V)$ and defined as

$$h_Y(U, V) = \max \left\{ \sup_{u \in U} d_Y(u, V), \sup_{v \in V} d_Y(v, U) \right\},$$

where $d_Y(u, V) = \inf \{\|u - v\|_Y : v \in V\}$ (see, e.g., [2]). It is known that $(P_{kc}(Y), h_Y(\cdot, \cdot))$ is a metric space.

Now let us formulate some properties of convex sets. For $U \in P_{kc}(Y)$, $V \in P_{kc}(Y)$, and $\lambda \in \mathbb{R}^1$ we denote

$$U + V = \{u + v : u \in U, v \in V\}, \quad \lambda U = \{\lambda u : u \in U\}.$$

It is obvious that if $U \in P_{kc}(Y)$, $V \in P_{kc}(Y)$, $\lambda \in \mathbb{R}^1$, then $\lambda(U + V) = \lambda U + \lambda V$. If $U \in P_{kc}(Y)$, $\lambda \in \mathbb{R}_+^1$, $\mu \in \mathbb{R}_+^1$, then $(\lambda + \mu)U = \lambda U + \mu U$.

For a given $U \in P_{kc}(Y)$ we denote

$$\|U\| = \max \{\|u\|_Y : u \in U\}.$$

It is not a difficult matter to verify that $\|U\| = 0$ iff $U = \{0_Y\}$, $\|U + V\| \leq \|U\| + \|V\|$ for every $U \in P_{kc}(Y)$ and $V \in P_{kc}(Y)$, $\|\lambda U\| = |\lambda| \|U\|$ for every $\lambda \in \mathbb{R}^1$ and $U \in P_{kc}(Y)$. Let

$$(P_{kc}(Y))^n = \underbrace{P_{kc}(Y) \times P_{kc}(Y) \times \dots \times P_{kc}(Y)}_{n \text{ times}}.$$

For $\lambda \in \mathbb{R}_+^1$, $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}_+^n$, $\mathcal{U} = (U_1, U_2, \dots, U_n) \in (P_{kc}(Y))^n$, $\mathcal{V} = (V_1, V_2, \dots, V_n) \in (P_{kc}(Y))^n$ we denote

$$\mathcal{U} + \mathcal{V} = (U_1 + V_1, U_2 + V_2, \dots, U_n + V_n),$$

$$\begin{aligned}\lambda \mathcal{U} &= (\lambda U_1, \lambda U_2, \dots, \lambda U_n), \\ \|\mathcal{U}\|_C &= \left(\sum_{i=1}^n \|U_i\|^2 \right)^{\frac{1}{2}}, \\ \langle \xi, \mathcal{U} \rangle &= \sum_{i=1}^n \xi_i U_i.\end{aligned}$$

Proposition 3. Let $\mathcal{U} = (U_1, U_2, \dots, U_n) \in (P_{kc}(Y))^n$, $\mathcal{V} = (V_1, V_2, \dots, V_n) \in (P_{kc}(Y))^n$, $\lambda \in \mathbb{R}_+^1$, $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}_+^n$. Then

$$\|\mathcal{U} + \mathcal{V}\|_C \leq \|\mathcal{U}\|_C + \|\mathcal{V}\|_C, \quad (1.21)$$

$$\|\lambda \mathcal{U}\|_C = \lambda \cdot \|\mathcal{U}\|_C, \quad (1.22)$$

$$\|\langle \xi, \mathcal{U} \rangle\| \leq \|\xi\| \cdot \|\mathcal{U}\|_C. \quad (1.23)$$

Proof. The proofs of (1.21) and (1.22) are obvious. To prove inequality (1.23) it is enough to use Cauchy-Schwarz inequality. Thus,

$$\begin{aligned}\|\langle \xi, \mathcal{U} \rangle\| &= \left\| \sum_{i=1}^n \xi_i U_i \right\| \leq \sum_{i=1}^n \xi_i \|U_i\| \\ &\leq \left(\sum_{i=1}^n (\xi_i)^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^n \|U_i\|^2 \right)^{\frac{1}{2}} = \|\xi\| \cdot \|\mathcal{U}\|_C.\end{aligned}$$

□

2. POSITIVE LINEAR SET VALUED MAPS

Definition 1. Let X and Y be normed spaces, $K \subset X$ be defined by (1.1), $F(\cdot) : K \rightarrow P_{kc}(Y)$ be a given set valued map. If

$$F(\lambda x_1 + \mu x_2) = \lambda F(x_1) + \mu F(x_2)$$

for every $x_1 \in K$, $x_2 \in K$, $\lambda \in \mathbb{R}_+^1$ and $\mu \in \mathbb{R}_+^1$, then $F(\cdot)$ is called a positive linear set valued map.

Let us give the representation theorem for positive linear set valued maps.

Theorem 1. Let $K \subset X$ be defined by (1.1), $F(\cdot) : K \rightarrow P_{kc}(Y)$ be a given positive linear set valued map. Then there exists a unique $\mathcal{F} = (F_1, F_2, \dots, F_n) \in (P_{kc}(Y))^n$ such that

$$F(x) = \langle \mathbf{x}_n, \mathcal{F} \rangle \quad (2.1)$$

and

$$\|F(x)\| \leq \|\mathbf{x}_n\| \cdot \|\mathcal{F}\|_C \leq \frac{1}{\beta} \|\mathcal{F}\|_C \cdot \|x\|_X \quad (2.2)$$

for every $x \in K$, where $\beta > 0$ is defined by (1.6).

Proof. Denote $F_i = F(a_i)$ ($i = 1, 2, \dots, n$) and $\mathcal{F} = (F_1, F_2, \dots, F_n)$. Since $F_i \in P_{kc}(Y)$, then $\mathcal{F} \in (P_{kc}(Y))^n$.

Let $x \in K$. Then there exists a unique $(\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}_+^n$ such that $x = \sum_{i=1}^n \xi_i a_i$, and hence $\mathbf{x}_n = (\xi_1, \xi_2, \dots, \xi_n)$. Since the set valued map $F(\cdot)$ is positive linear, we obtain

$$F(x) = F\left(\sum_{i=1}^n \xi_i a_i\right) = \sum_{i=1}^n F(\xi_i a_i) = \sum_{i=1}^n \xi_i F(a_i) = \sum_{i=1}^n \xi_i F_i = \langle \mathbf{x}_n, \mathcal{F} \rangle.$$

The validity of inequality (2.2) follows from Proposition 1 and Proposition 3.

Now let us prove the uniqueness of representation (2.1). Let there exists $\mathcal{E} = (E_1, E_2, \dots, E_n) \in (P_{kc}(Y))^n$ such that for every $x \in K$

$$F(x) = \langle \mathbf{x}_n, \mathcal{E} \rangle. \quad (2.3)$$

From (2.1) and (2.3) it follows that

$$\langle \mathbf{x}_n, \mathcal{F} \rangle = \langle \mathbf{x}_n, \mathcal{E} \rangle$$

for every $x \in K$ and consequently

$$\sum_{i=1}^n \xi_i F_i = \sum_{i=1}^n \xi_i E_i \quad (2.4)$$

for every $(\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}_+^n$.

Taking $\xi_{i_*} = 1$, $\xi_i = 0$ if $i \neq i_*$, we get from (2.4) that $F_{i_*} = E_{i_*}$, and hence $F_i = E_i$ for every $i = 1, 2, \dots, n$. Thus $\mathcal{E} = \mathcal{F}$, and representation (2.1) is unique. $\square$

Theorem 1 implies the validity of the following corollary.

Corollary 1. Let $K \subset X$ be defined by (1.1), $F(\cdot) : K \rightarrow P_{kc}(\mathbb{R}^1)$ be a given positive linear set valued map. Then there exists a unique $\mathcal{F} = ([\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots, [\alpha_n, \beta_n]) \in (P_{kc}(\mathbb{R}^1))^n$ such that

$$F(x) = \sum_{i=1}^n \xi_i \cdot [\alpha_i, \beta_i] = \left[\sum_{i=1}^n \xi_i \cdot \alpha_i, \sum_{i=1}^n \xi_i \cdot \beta_i \right]$$

for every $x \in K$, where $(\xi_1, \xi_2, \dots, \xi_n) = \mathbf{x}_n$ and $(\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}_+^n$.

3. LIPSCHITZ CONTINUITY OF THE POSITIVE LINEAR SET VALUED MAPS

The following proposition characterizes Lipschitz continuity of the positive linear set valued maps.

Theorem 2. Let $K \subset X$ be defined by (1.1), $F(\cdot) : K \rightarrow P_{kc}(Y)$ be a given positive linear set valued map. Then the set valued map $F(\cdot)$ is Lipschitz continuous with Lipschitz constant $\frac{1}{\gamma} \|\mathcal{F}\|_C$ that is

$$h_Y \left(F \left(x^{(1)} \right), F \left(x^{(2)} \right) \right) \leq \frac{1}{\gamma} \|\mathcal{F}\|_C \cdot \left\| x^{(1)} - x^{(2)} \right\|_X$$

for every $x^{(1)} \in K$, $x^{(2)} \in K$, where $\gamma > 0$ is defined by (1.5), $\mathcal{F} = (F_1, F_2, \dots, F_n) \in (P_{kc}(Y))^n$, $F_i \in P_{kc}(Y)$ ($i = 1, 2, \dots, n$) are defined in Theorem 1.

Proof. According to Theorem 1, there exists a unique $\mathcal{F} = (F_1, F_2, \dots, F_n) \in (P_{kc}(Y))^n$ such that for every $x \in K$

$$F(x) = \langle \mathbf{x}_n, \mathcal{F} \rangle. \quad (3.1)$$

Let $x^{(1)} \in K$, $x^{(2)} \in K$, $\mathbf{x}_n^{(1)} = (\xi_1^{(1)}, \xi_2^{(1)}, \dots, \xi_n^{(1)}) \in \mathbb{R}_+^n$, $\mathbf{x}_n^{(2)} = (\xi_1^{(2)}, \xi_2^{(2)}, \dots, \xi_n^{(2)}) \in \mathbb{R}_+^n$. Then we get from (3.1) that

$$F(x^{(1)}) = \langle \mathbf{x}_n^{(1)}, \mathcal{F} \rangle = \sum_{i=1}^n \xi_i^{(1)} F_i, \quad (3.2)$$

$$F(x^{(2)}) = \langle \mathbf{x}_n^{(2)}, \mathcal{F} \rangle = \sum_{i=1}^n \xi_i^{(2)} F_i. \quad (3.3)$$

Let $u_* \in F(x^{(1)})$ be an arbitrarily chosen vector. Then by virtue of (3.2) there exist $f_1 \in F_1$, $f_2 \in F_2$, $\dots$, $f_n \in F_n$ such that

$$u_* = \sum_{i=1}^n \xi_i^{(1)} f_i. \quad (3.4)$$

Now let

$$v_* = \sum_{i=1}^n \xi_i^{(2)} f_i. \quad (3.5)$$

Then according to (3.3) we have $v_* \in F(x^{(2)})$. From (3.4) and (3.5) it follows that

$$\begin{aligned} \|u_* - v_*\|_Y &= \left\| \sum_{i=1}^n (\xi_i^{(1)} f_i - \xi_i^{(2)} f_i) \right\|_Y \leq \sum_{i=1}^n \left| \xi_i^{(1)} - \xi_i^{(2)} \right| \|f_i\|_Y \\ &\leq \sum_{i=1}^n \left| \xi_i^{(1)} - \xi_i^{(2)} \right| \|F_i\|. \end{aligned} \quad (3.6)$$

(3.6) and Cauchy-Schwarz inequality yield

$$\|u_* - v_*\|_Y \leq \left(\sum_{i=1}^n (\xi_i^{(1)} - \xi_i^{(2)})^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^n \|F_i\|^2 \right)^{\frac{1}{2}} = \|\mathcal{F}\|_C \cdot \|\mathbf{x}_n^{(1)} - \mathbf{x}_n^{(2)}\|. \quad (3.7)$$

Since $x^{(1)} - x^{(2)} \in E^n$, where E^n is defined by (1.2), Proposition 1 implies that

$$\|\mathbf{x}_n^{(1)} - \mathbf{x}_n^{(2)}\| \leq \frac{1}{\gamma} \|x^{(1)} - x^{(2)}\|_X, \quad (3.8)$$

where $\gamma > 0$ is defined by (1.5). (3.7) and (3.8) yield that

$$\|u_* - v_*\|_Y \leq \frac{1}{\gamma} \|\mathcal{F}\|_C \cdot \|x^{(1)} - x^{(2)}\|_X. \quad (3.9)$$

It follows from (3.9) that

$$d_Y(u_*, F(x^{(2)})) \leq \frac{1}{\gamma} \|\mathcal{F}\|_C \cdot \|x^{(1)} - x^{(2)}\|_X. \quad (3.10)$$

Since $u_* \in F(x^{(1)})$ is an arbitrarily chosen vector, then we get from (3.10) that

$$\sup \left\{ d_Y(u, F(x^{(2)})) : u \in F(x^{(1)}) \right\} \leq \frac{1}{\gamma} \|\mathcal{F}\|_C \cdot \|x^{(1)} - x^{(2)}\|_X. \quad (3.11)$$

Analogously it is possible to prove that

$$\sup \left\{ d_Y(v, F(x^{(1)})) : v \in F(x^{(2)}) \right\} \leq \frac{1}{\gamma} \|\mathcal{F}\|_C \cdot \|x^{(1)} - x^{(2)}\|_X. \quad (3.12)$$

(3.11) and (3.12) imply the proof of the proposition. $\square$

Proposition 4. Let $K \subset X$ be defined by (1.1), $F(\cdot) : K \rightarrow P_{kc}(Y)$ be a given positive linear set valued map. Then

$$gr F(\cdot) = \{(x, y) \in K \times Y : y \in F(x)\}$$

is a closed convex cone.

Proof. The proof that $gr F(\cdot)$ is a convex cone is obvious. Closedness of $gr F(\cdot)$ follows from Theorem 2, asserting that the set valued map $F(\cdot) : K \rightarrow P_{kc}(Y)$ is continuous. $\square$

By virtue of [2], [11], and Proposition 4 we conclude that the positive linear set valued map $F(\cdot) : K \rightarrow P_{kc}(Y)$ is a closed convex processes.

4. THE NORM OF THE POSITIVE LINEAR SET VALUED MAPS

According to [2] and [6] the upper and lower norms of the positive linear set valued map $F(\cdot) : K \rightarrow P_{kc}(Y)$, where $K \subset X$ is given by relation (1.1), are respectively defined as

$$\|F(\cdot)\|_u = \sup_{x \in K \setminus \{0_X\}} \frac{\|F(x)\|}{\|x\|_X} = \sup_{x \in K \setminus \{0_X\}} \frac{\sup\{\|f\|_Y : f \in F(x)\}}{\|x\|_X},$$

$$\|F(\cdot)\|_l = \sup_{x \in K \setminus \{0_X\}} \frac{d_Y(0_Y, F(x))}{\|x\|_X} = \sup_{x \in K \setminus \{0_X\}} \frac{\inf\{\|f\|_Y : f \in F(x)\}}{\|x\|_X}.$$

It is obvious that $\|F(\cdot)\|_l \leq \|F(\cdot)\|_u$. The following theorem gives estimations of upper and lower norms of the positive linear set valued map $F(\cdot) : K \rightarrow P_{kc}(Y)$.

Theorem 3. *Let $K \subset X$ be defined by (1.1), $F(\cdot) : K \rightarrow P_{kc}(Y)$ be a given positive linear set valued map. Then*

$$\max \left\{ \frac{m_i}{\|a_i\|_X} : i = 1, 2, \dots, n \right\} \leq \|F(\cdot)\|_l \leq \|F(\cdot)\|_u \leq \frac{1}{\beta} \|\mathcal{F}\|_C,$$

where $\beta > 0$ is defined by (1.6), $\mathcal{F} = (F_1, F_2, \dots, F_n) \in (P_{kc}(Y))^n$, $F_i \in P_{kc}(Y)$ ($i = 1, 2, \dots, n$) are defined in Theorem 1, $m_i = d_Y(0_Y, F_i)$, $a_i \in X$, $i = 1, 2, \dots, n$, are linearly independent vectors given in (1.1).

Proof. Applying Theorem 1 we have

$$\begin{aligned} \|F(\cdot)\|_u &= \sup_{x \in K \setminus \{0_X\}} \frac{\|F(x)\|}{\|x\|_X} \\ &\leq \sup_{x \in K \setminus \{0_X\}} \frac{1}{\beta} \|\mathcal{F}\|_C \|x\|_X \cdot \frac{1}{\|x\|_X} = \frac{1}{\beta} \|\mathcal{F}\|_C. \end{aligned} \quad (4.1)$$

Let us choose an $x \in K \setminus \{0_X\}$. Then there exists a unique $(\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}_+^n$ such

that $x = \sum_{i=1}^n \xi_i a_i$, and hence $\mathbf{x}_n = (\xi_1, \xi_2, \dots, \xi_n)$ and $\|\mathbf{x}_n\| = \left(\sum_{i=1}^n \xi_i^2 \right)^{\frac{1}{2}}$.

Using Theorem 1 we have

$$\begin{aligned} d_Y(0_Y, F(x)) &= \min\{\|f\|_Y : f \in F(x)\} = \min\{\|f\|_Y : f \in \langle \mathbf{x}_n, \mathcal{F} \rangle\} \\ &= \min \left\{ \left\| \sum_{i=1}^n \xi_i f_i \right\|_Y : f_i \in F_i \right\} \end{aligned} \quad (4.2)$$

where $\mathcal{F} = (F_1, F_2, \dots, F_n) \in (P_{kc}(Y))^n$, $F_i \in P_{kc}(Y)$ ($i = 1, 2, \dots, n$) are defined in Theorem 1. Since $x = \sum_{i=1}^n \xi_i a_i$, we obtain from (4.2) that

$$\begin{aligned} \frac{d_Y(0_Y, F(x))}{\|x\|_X} &= \frac{\min \left\{ \left\| \sum_{i=1}^n \xi_i f_i \right\|_Y : f_i \in F_i \right\}}{\left\| \sum_{i=1}^n \xi_i a_i \right\|_X} \\ &= \frac{\min \left\{ \left\| \sum_{i=1}^n \frac{\xi_i}{\|\mathbf{x}_n\|} f_i \right\|_Y : f_i \in F_i \right\}}{\left\| \sum_{i=1}^n \frac{\xi_i}{\|\mathbf{x}_n\|} a_i \right\|_X}. \end{aligned} \quad (4.3)$$

Since $x \in K \setminus \{0_X\}$ is arbitrarily chosen, $\left(\frac{\xi_1}{\|\mathbf{x}_n\|}, \frac{\xi_2}{\|\mathbf{x}_n\|}, \dots, \frac{\xi_n}{\|\mathbf{x}_n\|} \right) \in S_+^n$, (4.3) implies that

$$\begin{aligned} \|F(\cdot)\|_I &= \sup_{x \in K \setminus \{0_X\}} \frac{d_Y(0_Y, F(x))}{\|x\|_X} \\ &= \sup_{x \in K \setminus \{0_X\}} \frac{\min \left\{ \left\| \sum_{i=1}^n \frac{\xi_i}{\|\mathbf{x}_n\|} f_i \right\|_Y : f_i \in F_i \right\}}{\left\| \sum_{i=1}^n \frac{\xi_i}{\|\mathbf{x}_n\|} a_i \right\|_X} \\ &= \sup_{(\theta_1, \theta_2, \dots, \theta_n) \in S_+^n} \frac{\min \left\{ \left\| \sum_{i=1}^n \theta_i f_i \right\|_Y : f_i \in F_i \right\}}{\left\| \sum_{i=1}^n \theta_i a_i \right\|_X}. \end{aligned} \quad (4.4)$$

Let $(\theta_1^*, \theta_2^*, \dots, \theta_n^*) = (0, \dots, 0, 1, 0, \dots, 0) \in S_+^n$ where 1 is i -th coordinate. Then it follows from (4.4) that

$$\|F(\cdot)\|_I = \sup_{(\theta_1, \theta_2, \dots, \theta_n) \in S_+^n} \frac{\min \left\{ \left\| \sum_{i=1}^n \theta_i f_i \right\|_Y : f_i \in F_i \right\}}{\left\| \sum_{i=1}^n \theta_i a_i \right\|_X}$$

$$\begin{aligned}
& \min \left\{ \left\| \sum_{i=1}^n \theta_i^* f_i \right\|_Y : f_i \in F_i \right\} \\
& \geq \frac{\left\| \sum_{i=1}^n \theta_i^* a_i \right\|_X}{\left\| a_i \right\|_X} \\
& = \frac{\min \{ \|f_i\|_Y : f_i \in F_i \}}{\|a_i\|_X} = \frac{m_i}{\|a_i\|_X}.
\end{aligned} \tag{4.5}$$

Since the inequality (4.5) holds for every $i = 1, 2, \dots, n$, then we have from (4.5) that

$$\|F(\cdot)\|_I \geq \max \left\{ \frac{m_i}{\|a_i\|_X} : i = 1, 2, \dots, n \right\}. \tag{4.6}$$

The inequalities (4.1) and (4.6) complete the proof. $\square$

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TRIBONACCI NUMBERS WITH INDICES IN ARITHMETIC PROGRESSION AND THEIR SUMS

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Abstract. In this paper, we give a recurrence relation for the Tribonacci numbers with indices in arithmetics progression, $\{T_{rn+s}\}$ for $0 \leq s < n$. We find sums of $\{T_{rn}\}$ for arbitrary integer r via matrix methods.

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1. INTRODUCTION

For $n \geq 3$, the Tribonacci numbers are defined by

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad (1.1)$$

with initial conditions $T_0 = 0, T_1 = T_2 = 1$.

The Binet formula of the Tribonacci numbers is given by,

$$\begin{aligned} T_n &= c_1 \alpha^{n+1} + c_2 \beta^{n+1} + c_3 \gamma^{n+1} \\ &= c_4 \alpha^n + c_5 \beta^n + c_6 \gamma^n \end{aligned}$$

where $c_1 = \frac{1}{(\alpha-\beta)(\alpha-\gamma)}$, $c_2 = \frac{1}{(\beta-\alpha)(\beta-\gamma)}$, $c_3 = \frac{1}{(\gamma-\alpha)(\gamma-\beta)}$, $c_4 = \frac{1}{-\alpha^2+4\alpha-1}$, $c_5 = \frac{1}{-\beta^2+4\beta-1}$, $c_6 = \frac{1}{-\gamma^2+4\gamma-1}$ and α, β, γ are the roots of the equation $x^3 - x^2 - x - 1 = 0$. The Tribonacci sequence is a special generalization of Fibonacci sequence. Many authors studied the Tribonacci sequence and its various properties. For example, in [11], the author found a formula for the Tribonacci numbers by using analytic methods. In [13], the author studied relationships between third order sequences with different initial conditions. And in [10], the author investigated the Tribonacci numbers' De moivre-type identities. In [5], the author found some formulas and identities and generating matrices for the sequences $\{T_n\}, \{T_{4n}\}$ and their sums. In [9], one can find the well-known identities for Tribonacci sequences. For example,

the generating matrix of Tribonacci sequence is given by,

$$Q^n = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{bmatrix}.$$

For more details, we refer to [1, 2, 11, 12].

The matrix method is very useful method in order to obtain some identities for special sequences. For example, using matrix methods, the authors obtained some identities for various special sequences (see [3, 4, 6–8]).

In this study, we will consider certain subsequences of the Tribonacci sequence of form $\{T_{rn+s}\}$ for arbitrary integers r, s with $0 \leq s < r$. After, we will find some formulas for sums of the sequence $\{T_{rn}\}$ by using matrix methods.

Now we are giving a lemma and a theorem for the subsequence of the Tribonacci sequences to determine the recurrence relation.

Lemma 1. *Let α, β and γ are roots of $x^3 - x^2 - x - 1 = 0$. Then, for any positive integer r , $(\alpha^r + \beta^r + \gamma^r)$, $(\alpha\beta)^r + (\alpha\gamma)^r + (\beta\gamma)^r$ and $(\alpha\beta\gamma)^r$ are always integers.*

Proof. We will use the induction method. Obviously, $\alpha + \beta + \gamma = 1$, $\alpha\beta + \beta\gamma + \alpha\gamma = -1$ and $\alpha\beta\gamma = 1$. Now, suppose that $(\alpha^n + \beta^n + \gamma^n)$, $(\alpha\beta)^n + (\alpha\gamma)^n + (\beta\gamma)^n$ and $(\alpha\beta\gamma)^n$ are the integers for $0 < n \leq r$. Then,

$$\begin{aligned} (\alpha^r + \beta^r + \gamma^r)(\alpha + \beta + \gamma) &= \alpha^{r+1} + \beta^{r+1} + \gamma^{r+1} + \alpha^{r-1}(\alpha\beta + \alpha\gamma) \\ &\quad + \beta^{r-1}(\alpha\beta + \beta\gamma) + \gamma^{r-1}(\alpha\gamma + \beta\gamma) \end{aligned}$$

and since $\alpha\beta + \beta\gamma + \alpha\gamma = -1$,

$$\begin{aligned} (\alpha^r + \beta^r + \gamma^r)(\alpha + \beta + \gamma) &= \alpha^{r+1} + \beta^{r+1} + \gamma^{r+1} + \alpha^{r-1}(-1 - \beta\gamma) \\ &\quad + \beta^{r-1}(-1 - \alpha\gamma) + \gamma^{r-1}(-1 - \alpha\beta) \\ &= \alpha^{r+1} + \beta^{r+1} + \gamma^{r+1} - (\alpha^{r-1} + \beta^{r-1} + \gamma^{r-1}) \\ &\quad - (\alpha\beta\gamma)(\alpha^{r-2} + \beta^{r-2} + \gamma^{r-2}). \end{aligned}$$

Finally, we obtain

$$\alpha^{r+1} + \beta^{r+1} + \gamma^{r+1} = \alpha^r + \beta^r + \gamma^r + \alpha^{r-1} + \beta^{r-1} + \gamma^{r-1} + \alpha^{r-2} + \beta^{r-2} + \gamma^{r-2}.$$

Since $(\alpha^r + \beta^r + \gamma^r)$, $(\alpha^{r-1} + \beta^{r-1} + \gamma^{r-1})$ and $(\alpha^{r-2} + \beta^{r-2} + \gamma^{r-2})$ are the integers, $\alpha^{r+1} + \beta^{r+1} + \gamma^{r+1}$ is the integer. Similarly, it can be shown that the terms of $(\alpha\beta)^r + (\alpha\gamma)^r + (\beta\gamma)^r$ and $(\alpha\beta\gamma)^r$ are the integers. $\square$

Theorem 1. *For $n > 2$ and the integers r, s such that $0 \leq s < r$,*

$$\begin{aligned} T_{rn+s} &= (\alpha^r + \beta^r + \gamma^r) T_{r(n-1)+s} - ((\alpha\beta)^r + (\alpha\gamma)^r + (\beta\gamma)^r) T_{r(n-2)+s} \\ &\quad + (\alpha\beta\gamma)^r T_{r(n-3)+s}, \end{aligned} \quad (1.2)$$

where α, β and γ are the roots of characteristic polynomial of the Tribonacci sequence $x^3 - x^2 - x - 1$.

Proof. In order to prove the claim, we will use the Binet formula of the Tribonacci sequence. If we evaluate the right hand side of (1.2), then

$$\begin{aligned} & (\alpha^r + \beta^r + \gamma^r) T_{r(n-1)+s} \\ & - ((\alpha\beta)^r + (\alpha\gamma)^r + (\beta\gamma)^r) T_{r(n-2)+s} + (\alpha\beta\gamma)^r T_{r(n-3)+s} \\ & = (\alpha^r + \beta^r + \gamma^r) (c_1 \alpha^{rn+s} + c_2 \beta^{rn+s} + c_3 \gamma^{rn+s}) \\ & - ((\alpha\beta)^r + (\alpha\gamma)^r + (\beta\gamma)^r) (c_1 \alpha^{r(n-1)+s} + c_2 \beta^{r(n-1)+s} + c_3 \gamma^{r(n-1)+s}) \\ & + (\alpha\beta\gamma)^r (c_1 \alpha^{r(n-2)+s} + c_2 \beta^{r(n-2)+s} + c_3 \gamma^{r(n-2)+s}), \end{aligned}$$

where $c_1 = \frac{1}{(\alpha-\beta)(\alpha-\gamma)}$, $c_2 = \frac{1}{(\beta-\alpha)(\beta-\gamma)}$ and $c_3 = \frac{1}{(\gamma-\alpha)(\gamma-\beta)}$. If we simplify the above equation, then we obtain

$$\begin{aligned} & (\alpha^r + \beta^r + \gamma^r) T_{r(n-1)+s} \\ & - ((\alpha\beta)^r + (\alpha\gamma)^r + (\beta\gamma)^r) T_{r(n-2)+s} + (\alpha\beta\gamma)^r T_{r(n-3)+s} \\ & = c_1 \alpha^{r(n+1)} + c_2 \beta^{r(n+1)} + c_3 \gamma^{r(n+1)} \\ & = T_{rn+s}. \end{aligned}$$

Thus the proof is complete. $\square$

When $r = 1$ in the above theorem, we obtain the well-known Tribonacci recurrence.

2. SUMS OF TERMS OF THE SUBSEQUENCE $\{T_{rn}\}$

In this section, we will compute sums of the Tribonacci numbers $\{T_{rn}\}$ and their sums by matrix methods.

$$S_{n,r} = \sum_{k=0}^n T_{rk} \quad (2.1)$$

where r is an integer, and define matrices F_r and $G_{n,r}$ as shown,

$$F_r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & C_r^{(1)} & -C_r^{(2)} & C_r^{(3)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$G_{n,r} = \begin{bmatrix} T_r + T_r^* & 0 & 0 & 0 \\ S_{n,r} & T_{r(n+1)} & -C_r^{(2)} T_{rn} + C_r^{(3)} T_{r(n-1)} & C_r^{(3)} T_{rn} \\ S_{n-1,r} & T_{rn} & -C_r^{(2)} T_{r(n-1)} + C_r^{(3)} T_{r(n-2)} & C_r^{(3)} T_{r(n-1)} \\ S_{n-2,r} & T_{r(n-1)} & -C_r^{(2)} T_{r(n-2)} + C_r^{(3)} T_{r(n-3)} & C_r^{(3)} T_{r(n-2)} \end{bmatrix},$$

where $C_r^{(1)} = \alpha^r + \beta^r + \gamma^r$, $C_r^{(2)} = (\alpha\beta)^r + (\beta\gamma)^r + (\alpha\gamma)^r$ and $C_r^{(3)} = (\alpha\beta\gamma)^r$. For $n \geq 3$, the sequence $\{T_r^*\}$ is defined as follows

$$T_n^* = -T_{n-1}^* - T_{n-2}^* + T_{n-3}, \quad (2.2)$$

where initial conditions $T_0^* = T_1^* = 0, T_2^* = 1$.

Theorem 2. For $n \geq 3$,

$$T_r F_r^n + T_r^* F_r^{n-1} = G_{n,r}.$$

Proof. The proof follows from the induction method. $\square$

After some computations, the eigenvalues of matrix F_r are $\alpha^r, \beta^r, \gamma^r$ and 1. Define two matrices L_r and W_r as follows

$$L_r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha^k & 0 & 0 \\ 0 & 0 & \beta^k & 0 \\ 0 & 0 & 0 & \gamma^k \end{bmatrix} \text{ and } W_r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{-1}{C_r^{(1)} - C_r^{(2)}} & \alpha^{2k} & \beta^{2k} & \gamma^{2k} \\ \frac{-1}{C_r^{(1)} - C_r^{(2)}} & \alpha^k & \beta^k & \gamma^k \\ \frac{-1}{C_r^{(1)} - C_r^{(2)}} & 1 & 1 & 1 \end{bmatrix}.$$

Theorem 3. If $n > 0$, then

$$S_{n,r} = \frac{1}{C_r^{(1)} - C_r^{(2)}} \left\{ T_{r(n+1)} - (C_r^{(2)} - 1) T_{rn} + C_r^{(3)} T_{r(n-1)} - (T_r + T_r^*) \right\}.$$

Proof. Since α, β are γ different zero, then $\det L_k \neq 0$. One can check that

$$T_r F_r^n W_r = T_r W_r L_r^n$$

and

$$T_r^* F_r^{n-1} W_r = T_r^* W_r L_r^{n-1}.$$

If we sum both equations side by side, we obtain that

$$(T_r F_r^n + T_r^* F_r^{n-1}) W_r = W_r (T_r L_r^n + T_r^* L_r^{n-1}).$$

By Theorem 2, we deduce

$$G_{n,r} W_r = W_r (T_r L_r^n + T_r^* L_r^{n-1})$$

Equating the (2,1)– elements of each sides of the above equation completes the proof. $\square$

In the above theorem, we give a formula for sum of the terms of the sequence $\{T_{rn}\}$ for arbitrary r and for the generating matrix of the sums. Now we define two

new matrices K_r and $R_{n,r}$ in order to give a Binet like-formula for the sum of the terms of the sequence $\{T_{rn}\}$,

$$K_r = \begin{bmatrix} C_r^{(1)} + 1 & -C_r^{(1)} - C_r^{(2)} & C_r^{(2)} + 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$R_{n,r} = \begin{bmatrix} S_{n+1,r} & X_{n,r} & Y_{n,r} & -S_{n,r} \\ S_{n,r} & X_{n-1,r} & Y_{n-1,r} & -S_{n-1,r} \\ S_{n-1,r} & X_{n-2,r} & Y_{n-2,r} & -S_{n-2,r} \\ S_{n-2,r} & X_{n-3,r} & Y_{n-3,r} & -S_{n-3,r} \end{bmatrix}$$

where

$$X_{n,r} = -\left(C_r^{(1)} + C_r^{(2)}\right) S_{n,r} + \left(C_r^{(2)} + 1\right) S_{n-1,r} - S_{n-2,r},$$

$$Y_{n,r} = \left(C_r^{(2)} + 1\right) S_{n,r} + S_{n-1,r},$$

and $S_{n,r}$ was defined by (2.1).

Theorem 4. For $n \geq 3$,

$$T_r K_r^n + T_r^* K_r^{n-1} = R_{n,r} \quad (2.3)$$

where $\{T_r\}$ is the Tribonacci sequence and the sequence $\{T_r^*\}$ is defined as in (2.2).

Proof. The proof follows from the induction method. $\square$

Now we define two matrices Δ_1 and D_r as follow

$$\Delta_1 = \begin{bmatrix} \alpha^{3r} & \beta^{3r} & \gamma^{3r} & 1 \\ \alpha^{2r} & \beta^{2r} & \gamma^{2r} & 1 \\ \alpha^r & \beta^r & \gamma^r & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } D_r = \begin{bmatrix} \alpha^r & 0 & 0 & 0 \\ 0 & \beta^r & 0 & 0 \\ 0 & 0 & \gamma^r & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Theorem 5. For $n > 4$,

$$S_{n,r} = T_r \left(\frac{\alpha^{r(n+2)}}{(\alpha^r - 1)(\alpha^r - \beta^r)(\alpha^r - \gamma^r)} + \frac{\beta^{r(n+2)}}{(\beta^r - 1)(\beta^r - \alpha^r)(\beta^r - \gamma^r)} \right. \\ \left. + \frac{\gamma^{r(n+2)}}{(\gamma^r - 1)(\gamma^r - \alpha^r)(\gamma^r - \beta^r)} \right) + \\ T_r^* \left(\frac{\alpha^{r(n+1)}}{(\alpha^r - 1)(\alpha^r - \beta^r)(\alpha^r - \gamma^r)} + \frac{\beta^{r(n+1)}}{(\beta^r - 1)(\beta^r - \alpha^r)(\beta^r - \gamma^r)} \right. \\ \left. + \frac{\gamma^{r(n+1)}}{(\gamma^r - 1)(\gamma^r - \alpha^r)(\gamma^r - \beta^r)} \right).$$

Proof. The characteristic equation of the matrix K_r is

$$x^4 - (C_r^{(1)} + 1)x^3 + (C_r^{(1)} + C_r^{(2)})x^2 - (C_r^{(1)} + 1)x + 1 = 0.$$

Computing roots of the equation, we obtain α^r , β^r , γ^r and 1. So the matrix K_r is diagonalizable. It can be shown that

$$T_r K_r^n \Delta_1 = T_r \Delta_1 D_r^n$$

and

$$T_r^* K_r^{n-1} \Delta_1 = T_r^* \Delta_1 D_r^{n-1}.$$

If we sum the both equalities, we obtain

$$(T_r K_r^n + T_r^* K_r^{n-1}) \Delta_1 = \Delta_1 (T_r D_r^n + T_r^* D_r^{n-1}).$$

Using (2.3), we deduce that

$$R_{n,r} \Delta_1 = \Delta_1 (T_r D_r^n + T_r^* D_r^{n-1}).$$

So, we have the following equation system,

$$\begin{aligned} \alpha^{3r} x(r)_{i,1} + \alpha^{2r} x(r)_{i,2} + \alpha^r x(r)_{i,3} + x(r)_{i,4} &= T_r \alpha^{r(n-i)+4r} + T_r^* \alpha^{r(n-i-1)+4r} \\ \beta^{3r} x(r)_{i,1} + \beta^{2r} x(r)_{i,2} + \beta^r x(r)_{i,3} + x(r)_{i,4} &= T_r \beta^{r(n-i)+4r} + T_r^* \beta^{r(n-i-1)+4r} \\ \gamma^{3r} x(r)_{i,1} + \gamma^{2r} x(r)_{i,2} + \gamma^r x(r)_{i,3} + x(r)_{i,4} &= T_r \gamma^{r(n-i)+4r} + T_r^* \gamma^{r(n-i-1)+4r} \\ x(r)_{i,1} + x(r)_{i,2} + x(r)_{i,3} + x(r)_{i,4} &= T_r + T_r^* \end{aligned}$$

where $R_{n,r} = [x(r)_{i,j}]$. In order to obtain $x(r)_{2,1}$, we use Cramer's method

$$x(r)_{2,1} = \frac{\det \begin{bmatrix} T_r \alpha^{r(n+2)} + T_r^* \alpha^{r(n+1)} & \alpha^{2r} & \alpha^r & 1 \\ T_r \beta^{r(n+2)} + T_r^* \beta^{r(n+1)} & \beta^{2r} & \beta^r & 1 \\ T_r \gamma^{r(n+2)} + T_r^* \gamma^{r(n+1)} & \gamma^{2r} & \gamma^r & 1 \\ T_r + T_r^* & 1 & 1 & 1 \end{bmatrix}}{\det \begin{bmatrix} \alpha^{3r} & \beta^{3r} & \gamma^{3r} & 1 \\ \alpha^{2r} & \beta^{2r} & \gamma^{2r} & 1 \\ \alpha^r & \beta^r & \gamma^r & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}}.$$

If we expand the first column of the matrix,

$$\det \begin{bmatrix} T_r \alpha^{r(n+2)} + T_r^* \alpha^{r(n+1)} & \alpha^{2r} & \alpha^r & 1 \\ T_r \beta^{r(n+2)} + T_r^* \beta^{r(n+1)} & \beta^{2r} & \beta^r & 1 \\ T_r \gamma^{r(n+2)} + T_r^* \gamma^{r(n+1)} & \gamma^{2r} & \gamma^r & 1 \\ T_r + T_r^* & 1 & 1 & 1 \end{bmatrix}$$

$$= \det \begin{bmatrix} T_r \alpha^{r(n+2)} & \alpha^{2r} & \alpha^r & 1 \\ T_r \beta^{r(n+2)} & \beta^{2r} & \beta^r & 1 \\ T_r \gamma^{r(n+2)} & \gamma^{2r} & \gamma^r & 1 \\ T_r & 1 & 1 & 1 \end{bmatrix} + \det \begin{bmatrix} T_r^* \alpha^{r(n+1)} & \alpha^{2r} & \alpha^r & 1 \\ T_r^* \beta^{r(n+1)} & \beta^{2r} & \beta^r & 1 \\ T_r^* \gamma^{r(n+1)} & \gamma^{2r} & \gamma^r & 1 \\ T_r^* & 1 & 1 & 1 \end{bmatrix},$$

and after some simplifications, we obtain $S_{n,r}$ as

$$\begin{aligned} x(r)_{2,1} &= S_{n,r} \\ &= T_r \left(\frac{\alpha^{r(n+2)}}{(\alpha^r - 1)(\alpha^r - \beta^r)(\alpha^r - \gamma^r)} + \frac{\beta^{r(n+2)}}{(\beta^r - 1)(\beta^r - \alpha^r)(\beta^r - \gamma^r)} \right. \\ &\quad \left. + \frac{\gamma^{r(n+2)}}{(\gamma^r - 1)(\gamma^r - \alpha^r)(\gamma^r - \beta^r)} \right) + \\ &\quad T_r^* \left(\frac{\alpha^{r(n+1)}}{(\alpha^r - 1)(\alpha^r - \beta^r)(\alpha^r - \gamma^r)} + \frac{\beta^{r(n+1)}}{(\beta^r - 1)(\beta^r - \alpha^r)(\beta^r - \gamma^r)} \right. \\ &\quad \left. + \frac{\gamma^{r(n+1)}}{(\gamma^r - 1)(\gamma^r - \alpha^r)(\gamma^r - \beta^r)} \right). \end{aligned}$$

□

2.1. Determinantal representations

In this section, we give some relationships between the sequence $\{T_{rn}\}$, the sum of terms of the sequence and the permanents of certain matrices. We define $n \times n$ matrices $H_{n,r}$, $Z_{n,r}$ as follows

$$H_{n,r} = [u(r)_{i,j}] = \begin{cases} C_r^{(1)} & \text{if } i = j, \\ -C_r^{(2)} & \text{if } j = i + 1, \\ C_r^{(3)} & \text{if } j = i + 2, \\ 1 & \text{if } i = j + 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$Z_{n,r} = \begin{bmatrix} 1 & 1 & & \cdots & 1 \\ 1 & & & & \\ 0 & & H_{n-1} & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}$$

We present some relationships with the following theorem.

Theorem 6. For $n > 1$

- (i) $T_r \text{per} H_{n,r} + T_r^* \text{per} H_{n-1,r} = T_{r(n+1)}$
- (ii) $T_r \text{per} Z_{n,r} + T_r^* \text{per} Z_{n-1,r} = \sum_{j=1}^n T_{rj}$

Proof. (i) If we expand the permanent of matrix $H_{n,r}$ according to the first column, then we obtain

$$\text{per}H_{n+1,r} = C_r^{(1)} \text{per}H_{n,r} - C_r^{(2)} \text{per}H_{n-1,r} + C_r^{(3)} \text{per}H_{n-1,r}.$$

Since $\text{per}H_{n,r}$ and $\{T_{rn}\}$ have the same recurrence relation and initial conditions, the proof is complete.

(ii) It can be proven similarly to the first identity. $\square$

3. COMBINATORIAL REPRESENTATIONS

In this part, we give some combinatorial representations of the terms of the sum of subsequence of the Tribonacci sequence and the entries of the n^{th} power of the companion matrix

$$A_k(k_1, k_2, \dots, k_k) = \begin{bmatrix} k_1 & k_2 & \cdots & k_k \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

In [2], we can see the following results;

Theorem 7. The (i, j) entry $a_{i,j}^{(n)}(k_1, k_2, \dots, k_k)$ in matrix $A_k^{(n)}(k_1, k_2, \dots, k_k)$ is given by following formula:

$$\begin{aligned} & a_{i,j}^{(n)}(k_1, k_2, \dots, k_k) \\ &= \sum_{(t_1, t_2, \dots, t_k)} \frac{t_j + t_{j+1} + \cdots + t_k}{t_1 + t_2 + \cdots + t_k} \times \binom{t_1 + t_2 + \cdots + t_k}{t_1, t_2, \dots, t_k} k_1^{t_1} \cdots k_k^{t_k} \end{aligned} \quad (3.1)$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + kt_k = n - i + j$, and the coefficient in (3.1) is defined to be 1 if $n = i - j$.

Corollary 1. Let $S_{n,r}$ be the sum of Tribonacci numbers where the subscripts form an arithmetic sequence. Then

$$\begin{aligned} S_{n,r} = \sum_{(r_1, r_2, r_3, r_4)} \binom{r_1 + r_2 + r_3 + r_4}{r_1, r_2, r_3, r_4} (C_r^{(1)} + 1)^{r_1} (-C_r^{(1)} + C_r^{(2)})^{r_2} \\ \times (-C_r^{(2)} + 1)^{r_3} (-1)^{r_4} \end{aligned}$$

where the summation is over nonnegative integers satisfying $r_1 + 2r_2 + 3r_3 + 4r_4 = n - 1$.

Corollary 2. Let T_n be the n^{th} Tribonacci number. Then

$$T_{rn} = \sum_{(t_1, t_2, t_3)} \binom{t_1 + t_2 + t_3}{t_1, t_2, t_3} (C_r^{(1)})^{t_1} (-C_r^{(2)})^{t_2} (C_r^{(3)})^{t_3}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + 3t_3 = n - 1$.

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FIXED POINT THEOREM ON PARTIAL METRIC SPACES INVOLVING RATIONAL EXPRESSIONS

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Abstract. We establish a fixed point theorem involving a rational expression in a complete partial metric space. Our result generalizes a well-known result in (usual) metric spaces. Also, we introduce an example to illustrate the usability of our result.

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1. INTRODUCTION AND PRELIMINARIES

The notion of partial metric space was introduced by Matthews [10, 11] in 1992. In fact, a partial metric space is a generalization of usual metric spaces in which $d(x, x)$ are no longer necessarily zero. After this remarkable contribution, many authors focused on partial metric spaces and its topological properties (see e.g. [1–9, 12, 13]).

The definition of partial metric space is given by Matthews (see e.g. [10, 11]) as follows:

Definition 1. Let X be a nonempty set and let $p : X \times X \rightarrow [0, \infty)$ satisfy

$$(PM1) \quad x = y \Leftrightarrow p(x, x) = p(y, y) = p(x, y)$$

$$(PM2) \quad p(x, x) \leq p(x, y)$$

$$(PM3) \quad p(x, y) = p(y, x)$$

$$(PM4) \quad p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$$

for all x, y and $z \in X$. Then the pair (X, p) is called a partial metric space (in short PMS) and p is called a partial metric on X .

Notice that for a partial metric p on X , the function $d_p : X \times X \rightarrow \mathbb{R}^+$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \quad (1.1)$$

is a (usual) metric on X . Observe that each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p -balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$,

where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Similarly, closed p -ball is defined as $B_p[x, \varepsilon] = \{y \in X : p(x, y) \leq p(x, x) + \varepsilon\}$

Some basic concepts on partial metric spaces are defined as follows:

Definition 2 (See e.g. [10, 11]).

- (i) A sequence $\{x_n\}$ in a PMS (X, p) converges to $x \in X$ if and only if $p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n)$.
- (ii) A sequence $\{x_n\}$ in a PMS (X, p) is called Cauchy if and only if $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$ exists (and finite).
- (iii) A PMS (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p , to a point $x \in X$ such that $p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m)$.
- (iv) A mapping $f : X \rightarrow X$ is said to be continuous at $x_0 \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon)$.

The following lemmas play important role in the proofs of our main result.

Lemma 1 (See e.g. [10, 11]). (A) A sequence $\{x_n\}$ is Cauchy in a PMS (X, p) if and only if $\{x_n\}$ is Cauchy in a metric space (X, d_p) ,
 (B) A PMS (X, p) is complete if and only if a metric space (X, d_p) is complete.
 Moreover,

$$\lim_{n \rightarrow \infty} d_p(x, x_n) = 0 \Leftrightarrow p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n) = \lim_{n, m \rightarrow \infty} p(x_n, x_m) \quad (1.2)$$

Lemma 2 (See e.g. [4]). Let (X, p) be a PMS. Then

- (A) If $p(x, y) = 0$ then $x = y$,
- (B) If $x \neq y$, then $p(x, y) > 0$.

Lemma 3 (See e.g. [4]). Let $x_n \rightarrow z$ as $n \rightarrow \infty$ in a PMS (X, p) where $p(z, z) = 0$. Then $\lim_{n \rightarrow \infty} p(x_n, y) = p(z, y)$ for every $y \in X$.

2. THE MAIN RESULT

In this section we establish our main theorem which gives conditions for existence and uniqueness of a fixed point for a certain type operators on partial metric spaces.

Theorem 1. Let (X, p) be a complete partial metric space and $T : X \rightarrow X$ be a mapping satisfying

$$\psi(p(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)), \forall x, y \in X, \quad (2.1)$$

where $M(x, y)$ is given by

$$M(x, y) = \max \left\{ p(y, Ty) \frac{1 + p(x, Tx)}{1 + p(x, y)}, p(x, y) \right\}$$

and $\psi : [0, \infty) \rightarrow [0, \infty)$ is a continuous and monotone non-decreasing function with $\psi(t) = 0$ if and only if $t = 0$ and $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Proof. Let x_0 be an arbitrary point in X . We construct the sequence $\{x_n\}$ in X as follows:

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, 3, \dots$$

If there exists n such that $x_n = x_{n+1}$ then x_n is a fixed point of T . Now, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Letting $x = x_{n-1}$ and $y = x_n$ in (2.1), we have

$$\psi(p(x_n, x_{n+1})) = \psi(p(Tx_{n-1}, Tx_n)) \leq \psi(M(x_{n-1}, x_n)) - \varphi(M(x_{n-1}, x_n))$$

where

$$M(x_{n-1}, x_n) = \max \left\{ p(x_n, Tx_n) \frac{1 + p(x_{n-1}, Tx_{n-1})}{1 + p(x_{n-1}, x_n)}, p(x_{n-1}, x_n) \right\}.$$

Hence, we obtain

$$\begin{aligned} \psi(p(x_n, x_{n+1})) &\leq \psi(\max\{p(x_n, x_{n+1}), p(x_{n-1}, x_n)\}) \\ &\quad - \varphi(\max\{p(x_n, x_{n+1}), p(x_{n-1}, x_n)\}) \end{aligned} \quad (2.2)$$

If $p(x_n, x_{n+1}) > p(x_{n-1}, x_n)$, then from (2.2), we have

$$\psi(p(x_n, x_{n+1})) \leq \psi(p(x_n, x_{n+1})) - \varphi(p(x_n, x_{n+1})) < \psi(p(x_n, x_{n+1}))$$

which is a contradiction since $p(x_n, x_{n+1}) > 0$ by Lemma 2. So, we have $p(x_n, x_{n+1}) \leq p(x_{n-1}, x_n)$, that is, $\{p(x_n, x_{n+1})\}$ is a non-increasing sequence of positive real numbers. Thus, there exists $L \geq 0$ such that

$$\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = L \quad (2.3)$$

Suppose that $L > 0$. Taking the upper limit in (2.2) as $n \rightarrow \infty$ and using (2.3) and the properties of ψ , φ , we have

$$\psi(L) \leq \psi(L) - \liminf_{n \rightarrow \infty} \varphi(p(x_{n-1}, x_n)) \leq \psi(L) - \varphi(L) < \psi(L)$$

which is a contradiction. Therefore

$$\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0 \quad (2.4)$$

Due to (1.1), we have $d_p(x_n, x_{n+1}) \leq 2p(x_n, x_{n+1})$

$$\lim_{n \rightarrow \infty} d_p(x_n, x_{n+1}) = 0 \quad (2.5)$$

Now, we prove that

$$\lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0.$$

Suppose the contrary, that is,

$$\lim_{n, m \rightarrow \infty} p(x_n, x_m) \neq 0.$$

Then there exists $\epsilon > 0$ for which we can find two subsequences $\{x_{m(k)}\}, \{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which

$$n(k) > m(k) > k, \quad p(x_{n(k)}, x_{m(k)}) \geq \epsilon. \quad (2.6)$$

This means that

$$p(x_{n(k)-1}, x_{m(k)}) < \varepsilon. \quad (2.7)$$

From (2.6) and (2.7), we have

$$\begin{aligned} \varepsilon &\leq p(x_{n(k)}, x_{m(k)}) \leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) - p(x_{n(k)-1}, x_{n(k)-1}) \\ &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) \\ &< \varepsilon + p(x_{n(k)}, x_{n(k)-1}) \end{aligned}$$

Taking $k \rightarrow \infty$ and using (2.4), we get

$$\lim_{k \rightarrow \infty} p(x_{n(k)}, x_{m(k)}) = \varepsilon \quad (2.8)$$

By the triangle inequality, we have

$$\begin{aligned} p(x_{n(k)}, x_{m(k)}) &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) - p(x_{n(k)-1}, x_{n(k)-1}) \\ &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) \\ &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) \\ &\quad - p(x_{m(k)-1}, x_{m(k)-1}) \\ &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) \end{aligned}$$

and

$$\begin{aligned} p(x_{n(k)-1}, x_{m(k)-1}) &\leq p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)-1}) - p(x_{n(k)}, x_{n(k)}) \\ &\leq p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)-1}) \\ &\leq p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)}) + p(x_{m(k)}, x_{m(k)-1}) \\ &\quad - p(x_{m(k)}, x_{m(k)}) \\ &\leq p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)}) + p(x_{m(k)}, x_{m(k)-1}) \end{aligned}$$

Taking $k \rightarrow \infty$ in the above two inequalities and using (2.4), (2.8), we get

$$\lim_{k \rightarrow \infty} p(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon \quad (2.9)$$

Now from (2.1), we have

$$\begin{aligned} \psi(p(x_{m(k)}, x_{n(k)})) &= \psi(p(Tx_{m(k)-1}, Tx_{n(k)-1})) \\ &\leq \psi(M(x_{m(k)-1}, x_{n(k)-1})) - \varphi(p(x_{m(k)-1}, x_{n(k)-1})) \end{aligned} \quad (2.10)$$

where

$$\begin{aligned} M(x_{m(k)-1}, x_{n(k)-1}) &= \\ &= \max \left\{ p(x_{n(k)-1}, x_{n(k)}) \frac{1 + p(x_{m(k)-1}, x_{m(k)})}{1 + p(x_{m(k)-1}, x_{n(k)-1})}, p(x_{m(k)-1}, x_{n(k)-1}) \right\} \end{aligned}$$

By (2.4), (2.8) and (2.9), we have

$$\lim_{k \rightarrow \infty} M(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \quad (2.11)$$

Now, passing to the upper limit when $k \rightarrow \infty$ in (2.10) and using (2.8), (2.11) and the properties of ψ , φ , we have

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \lim_{k \rightarrow \infty} \inf \varphi(M(x_{m(k)-1}, x_{n(k)-1})) \leq \psi(\varepsilon) - \varphi(\varepsilon) < \psi(\varepsilon)$$

which is a contradiction. So, we have

$$\lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0.$$

Since $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$ exists and finite, we conclude that (x_n) is a Cauchy sequence in (X, p) .

Due to (1.1), we have $d_p(x_n, x_m) \leq 2p(x_n, x_m)$. Therefore

$$\lim_{n, m \rightarrow \infty} d_p(x_n, x_m) = 0. \quad (2.12)$$

Thus, by Lemma 1, $\{x_n\}$ is a Cauchy sequence in both (X, d_p) and (X, p) .

Since (X, p) is a complete partial metric space, then there exists $x \in X$ such that $\lim_{n \rightarrow \infty} p(x_n, x) = p(x, x)$. Since $\lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0$, then again by Lemma 1, we have $p(x, x) = 0$. Let us now prove that x is a fixed point of T . Suppose that $Tx \neq x$. From (2.1) and Lemma 3, we have

$$\begin{aligned} \psi(p(x_n, Tx)) &= \psi(p(Tx_{n-1}, Tx)) \\ &\leq \psi\left(\max\left\{p(x, Tx) \frac{1 + p(x_{n-1}, Tx_{n-1})}{1 + p(x_{n-1}, x)}, p(x_{n-1}, x)\right\}\right) \\ &\quad - \varphi\left(\max\left\{p(x, Tx) \frac{1 + p(x_{n-1}, Tx_{n-1})}{1 + p(x_{n-1}, x)}, p(x_{n-1}, x)\right\}\right) \end{aligned} \quad (2.13)$$

Letting $n \rightarrow \infty$ in the above inequality and regarding the property of ϕ, ψ , we obtain

$$\psi(p(x, Tx)) \leq \psi(p(x, Tx)) - \varphi(p(x, Tx)) < \psi(p(x, Tx))$$

which is a contradiction. Thus $Tx = x$, that is, x is fixed point of T . Finally, we shall prove the uniqueness. Suppose that y is another fixed point of T such that $x \neq y$. From (2.1),

$$\begin{aligned} \psi(p(x, y)) &= \psi(p(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)) \\ &=> \psi(p(x, y)) - \varphi(p(x, y)) < \psi(p(x, y)) \end{aligned}$$

which is a contradiction since $p(x, y) > 0$. Hence $x = y$.

The proof is completed. $\square$

Example 1. Let $X = [0, 1]$ and $p(x, y) = \max\{x, y\}$ then (X, p) is a PMS. Suppose $T : X \rightarrow X$ such that $Tx = \frac{x^2}{1+x}$ for all $x \in X$ and $\phi(t) : [0, \infty) \rightarrow [0, \infty)$ such that $\psi(t) = \frac{t}{1+t}$ and $\phi(t) = t$. Without loss of generality assume $x \geq y$. Then, we have

$$p(Tx, Ty) = \max \left\{ \frac{x^2}{1+x}, \frac{y^2}{1+y} \right\} = \frac{x^2}{1+x}$$

On the other hand,

$$\max \left\{ p(y, Ty) \frac{1 + p(x, Tx)}{1 + p(x, y)}, p(x, y) \right\} = \max \left\{ y \frac{1+x}{1+x}, x \right\} = x$$

Combining the observations above, we get

$$p(Tx, Ty) = \max \left\{ \frac{x^2}{1+x}, \frac{y^2}{1+y} \right\} = \frac{x^2}{1+x} \leq x - \frac{x}{1+x} = \frac{x^2}{1+x}.$$

Thus, it satisfies all conditions of Theorem 1. Hence, T has a unique fixed point, indeed $x = 0$ is the required point.

In Theorem 1, taking $\psi(t) = t$ for all $t \in [0, \infty)$ and $\varphi(t) = (1-k)t$ for all $t \in [0, \infty)$ with $k \in (0, 1)$, we get the following result.

Corollary 1. *Let (X, d) be a complete partial metric space and $T : X \rightarrow X$ be a mapping satisfying*

$$p(Tx, Ty) \leq k \max \left\{ p(y, Ty) \frac{1 + p(x, Tx)}{1 + p(x, y)}, p(x, y) \right\} \quad (2.14)$$

where $k \in (0, 1)$. Then T has a unique fixed point.

Example 2. Let $X = [0, 1]$. Define $T : X \times X \rightarrow X$ by $Tx = \frac{x}{2}$. Also, define $p : X \times X \rightarrow \mathbf{R}^+$ by $p(x, y) = \max\{x, y\}$, then (X, p) is a complete partial metric space and

$$p(Tx, Ty) \leq \frac{1}{2} \max \left\{ p(y, Ty) \frac{1 + p(x, Tx)}{1 + p(x, y)}, p(x, y) \right\}.$$

Thus by Corollary 1, T has a unique fixed point. Here 0 is the unique fixed point of T .

Example 3. Let $X = [0, \infty)$. Define $T : X \times X \rightarrow X$ by $Tx = 2x$. Also, define $p : X \times X \rightarrow \mathbf{R}^+$ by $p(x, y) = \max\{x, y\}$, then (X, p) is a complete partial metric space. It is clear that Matthew's Theorem (analog of Banach Contraction Mapping principle) does not work. Indeed, without loss of generality, we may assume that $x \leq y$.

Then

$$p(Tx, Ty) = 2y > ky = kp(x, y)$$

for any $k \in [0, 1)$.

However, for $k = \frac{1}{3}$, we have

$$2y = p(Tx, Ty) \leq \frac{1}{3}2y \frac{1+2x}{1+x} = k \max \left\{ p(y, Ty) \frac{1+p(x, Tx)}{1+p(x, y)}, p(x, y) \right\}.$$

Thus by Corollary 1, T has a unique fixed point. Here 0 is the unique fixed point of T .

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TWO NOVEL AGGREGATION-BASED ALGEBRAIC MULTIGRID METHODS

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Abstract. In the last two decades, substantial effort has been devoted to solve large systems of linear equations with algebraic multigrid (AMG) method. Usually, these systems arise from discretizing partial differential equations (PDE) which we encounter in engineering problems. The main principle of this methodology focuses on the elimination of the so-called algebraic smooth error after the smoother has been applied. Smoothed aggregation style multigrid is a particular class of AMG method whose coarsening process differs from the classic AMG. It is also a very popular and effective iterative solver and preconditioner for many problems. In this paper, we present two kinds of novel methods which both focus on the modification of the aggregation algorithm, and both lead a better performance while apply to several problems, such as Helmholtz equation.

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Keywords: algebraic multigrid, smoothed aggregation, preconditioner

1. INTRODUCTION

In this study we consider multigrid methods for solving large and sparse $n \times n$ linear systems

$$Au = b$$

arising from the discretization of partial differential equations.

Multigrid methods are widely popular as linear solvers because they can achieve linear algorithmic scalability for many practical problems. A very important ingredient of multigrids is the choice of both the relaxation and the correction scheme. The relaxation scheme utilizes a simple iterative solver such as the Jacobi or (symmetric) Gauss-seidel method as the smoother, and is aimed at eliminating the high-frequency components of the error. Afterwards, the low-frequency components of the error are damped via the coarse-grid correction, which consists of solving approximately the residual equation on a coarser grid.

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In order to construct the coarse-grid correction, we need to define several multigrid components, that are the hierarchy of grids, the smoothing and transfer (prolongation/restriction) operators, and a solver for the coarsest grid. Both theory and several reported experiments indicate that the rate of convergence of multigrid methods may be independent on the number of unknowns, and the complexity increases only linearly with the problem size.

In AMG methods, the grids and the operators are selected using only information from the entries of the coefficient matrix, and not using physical meshes like in Geometric multigrids (GMG). It has been shown that AMG often inherits the excellent algorithmic scalability of GMG, without requiring any geometric information. For this reason, the development of efficient and reliable AMG methods has received considerable research attention in the past decade.

Variants of AMG methods based on smoothed aggregation (referred to as SA methods) have been introduced in [17, 18]. The main algorithmic difference between the two approaches consists in the distinction between *coarse* and *fine* variables. In classic AMG techniques, the fine-level variables are splitted into two disjoint subsets, the so-called C/F splitting where C represent the variables on the coarser level. Vice versa, in the SA method the coarsening process is defined by generating aggregates rather than by constructing the C/F splitting. All unknowns in the same aggregate are strongly dependent on each other in order to guarantee the accuracy of the interpolation.

Recent trend of development of SA methods moved in two directions: 1. Use the near-null space information to accelerate the standard algorithm, such as in [3, 6]. 2. Construct more accurate interpolation operators, such as in [6, 12]. In this paper, we follow the second direction and we investigate a different strategy to construct efficient interpolation operators that can also preserve low iteration cost and storage requirements.

The paper is organized as follows. In Section 2, we briefly recall SA methods and we introduce the standard notation that is used throughout the remainder of the paper. In Section 3, we focus on the description of two modified aggregation algorithms. Finally, we demonstrate the performance of our new method in Section 4.

2. PRELIMINARIES

We briefly recall the standard SA method and its principles following [3, 17]. Let us assume that the coefficient matrix A of the linear system is of order $n = n_1$ and it arises from the discretization of a PDE. A hierarchy of coarse-level matrices is defined as

$$A_{l+1} = (S_l P_{l+1}^l)^T A_l S_l P_{l+1}^l, A_1 = A,$$

for $l = 1, \dots, L-1$. A simple and proper choice for the prolongation smoother is Richardson's method with a particular step size:

$$S_l = I - \frac{4}{3\lambda_l} A_l,$$

where λ_l is an upper bound on the spectral radius of the matrix on level l , that is $\rho(A_l) \leq \lambda_l$. Notice that this smoother is different from the relaxation smoother mentioned in Section 1, like the Jacobi smoother.

Suppose that we are given a smoothing procedure for the projected system $A_l x_l = b_l$, at each level $l \in \{1, \dots, L\}$, of the form

$$x_l \leftarrow (I - R_l A_l) x_l + R_l b_l.$$

Here, R_l is some approximation of the inverse of A_l for $l = 1, \dots, L-1$, which we consider again to be the Richardson iteration ($R_l = s_l I$, where $s_l \approx \frac{1}{\rho(A_l)}$). Assume for simplicity to use a direct solver at the coarsest level: $R_L = A_L^{-1}$. We assume that

$$\lambda_{\min}(I - R_l A_l) \geq 0 \quad \text{and} \quad \lambda_{\min}(R_l) \geq \frac{1}{C_R^2 \rho(A_l)}$$

for constant $C_R > 0$ independent of the level l . This assumption enables us to make use of the existing convergence estimates for Richardson's iteration.

The SA method can be formally viewed as a standard variational multigrid method with prolongator of the form $S_l P_{l+1}^l$. One SA iteration for solving $A_1 x_1 = b_1$ is represented by $x_1 \leftarrow AMG(x_1, b_1)$. We set $AMG = AMG_1$, where $AMG_l(\cdot, \cdot)$, for $l = 1, \dots, L-p = p1$, is defined recursively as follows.

Since the prolongation smoother S_l has been formally introduced before, now we

Algorithm 1 AMG_l

- (1) Presmoothing: Apply v presmoothings to $A_l x_l = b_l$ of the form $x_l \leftarrow (I - R_l A_l) x_l + R_l b_l$
 - (2) Coarse-grid correction:
 - (a) Set $b_{l+1} = (S_l P_{l+1}^l)^T (b_l - A_l x_l)$.
 - (b) If $l+1 = L$, solve $A_{l+1} x_{l+1} = b_{l+1}$ by a direct method, Otherwise set $x_{l+1} = 0$ and perform γ iterations of $x_{l+1} \leftarrow AMG_{l+1}(x_{l+1}, b_{l+1})$
 - (c) Correct the solution on level l : $x_l \leftarrow x_l + (S_l P_{l+1}^l) x_{l+1}$.
 - (3) Postsmoothing: apply v postsmoothings to $A_l x_l = b_l$ of the form $x_l \leftarrow (I - R_l A_l) x_l + R_l b_l$.
- AMG_L returns, simply, $x_L = A_L^{-1} b_L$.
-

focus on the construction of the tentative prolongator operators P_{l+1}^l . For illustration,

here we present the simplest prolongator for the 1D Laplace equation discretized on a mesh consisting of $n_1 = 3^{L-1}n_L$ nodes:

$$P_{l+1}^l = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & \ddots & \\ & & & & & \ddots & \\ & & & & & & 1 \\ & & & & & & 1 \\ & & & & & & 1 \end{pmatrix}.$$

For later reference, we outline the standard SA setup in Algorithm 2 below.

The prolongator P_{l+1}^l and the $n_{l+1} \times r$ matrix B_{l+1} are such that

Algorithm 2

Given A_1 , and L , do the following for $l = 1, \dots, L-1$:

- (1) Construct $\{C_i^l\}_{i=1}^{N_l}$ based on A_l . ($\{C_i^l\}_{i=1}^{N_l}$, a disjoint covering for all nodes.)
 - (2) Construct P_{l+1}^l based on $\{C_i^l\}_{i=1}^{N_l}$.
 - (3) Construct the prolongation smoother: S_l .
 - (4) Construct the coarse matrix: $A_{l+1} = (S_l P_{l+1}^l)^T A_l S_l P_{l+1}^l$.
-

$$P_{l+1}^l B_{l+1} = B_l,$$

where B_l has been constructed during the setup of P_l^{l-1} or it has been given if $l = 1$. A typical example for B_l is $B_1 = (1, \dots, 1)^T$ or $B_1 = \text{Kernel}(A)$ if A is the discrete representation of a continuous differential operator with an appropriate null space.

The prolongator P_{l+1}^l is constructed from a given system of aggregates $\{C_i^l\}_{i=1}^{N_l}$ which form a disjoint covering of the set $\{1, \dots, n_l\}$. Next, we present an algorithm introduced in [19] for generating aggregates based on information on the structure of the matrix A_l .

For a given parameter θ , the strongly coupled neighborhood of the node i is defined as

$$N_i^l(\theta) = \{j : |a_{ij}| \geq \theta \sqrt{a_{ii}a_{jj}}\} \cup \{i\}.$$

This algorithm generates as output a disjoint covering $\{C_i^l\}_{i=1}^{N_l}$ of the set $\{1, \dots, n_l\}$.

Algorithm 3

Let A_l be a matrix of order n_l and $\theta \in [0, 1)$. Generate a disjoint covering $\{C_i^l\}_{i=1}^{N_l}$ of the set $\{1, \dots, n_l\}$ as follows.

Aggregate (U).

Initialization: set $U = \{1, \dots, n_l\}$, $i = 1$ and $j = 0$.

Step 1: disjoint strongly-coupled neighborhoods are selected as the initial approximation of the covering:

```
{
  for( $i \in U$ )
  {
    if( $N_i^l(\theta) \subset U$ )
      { $j = j + 1$ ;  $C_j^l = N_i^l(\theta)$ ;  $U = U - C_j^l$ ;}
  }
```

Step 2: each remaining $i \in U$ is added to one of the selected sets to which it is strongly connected, if possible:

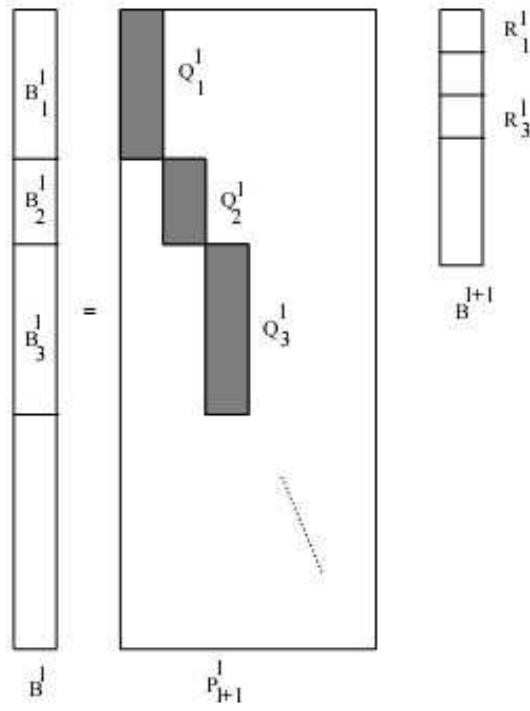
```
  for( $k \leq j$ )  $\overline{C_k^l} = C_k^l$ ;
  for( $i \in U$ )
  {
    for( $k \leq j$ )
    {
      if( $N_i^l(\theta) \cap \overline{C_k^l} \neq \emptyset$ )
        { $C_k^l = C_k^l \cup \{i\}$ ;  $U = U - \{i\}$ ;}
    }
  }
```

Step 3: combine the remaining $i \in U$ into aggregates that consist of subsets of strongly coupled neighborhoods:

```
  for( $i \in U$ ) until  $U = \emptyset$ 
  {
     $j = j + 1$ ;  $C_j^l = N_i^l(\theta) \cap U$ ;  $U = U - C_j^l$ ;
  }
```

The columns of P_{l+1}^l associated with the aggregate C_i^l are formed by restriction of the rows of B_l onto the aggregate C_i^l . Each aggregate gives rise to r degrees of freedom on the coarse-grid.

The detailed algorithm follows here. For ease of presentation, we assume that the fine level unknowns are numbered lexicographically within each aggregate. A different order can be used by renumbering them. The algorithm and figure described

FIGURE 1. The structure of the tentative prolongator P_{l+1}^l .

above are based on the assumption that the fine level unknowns are numbered consecutively within each aggregate. The typical structure of a tentative prolongator is described below, where we denote by $\times$ one nonzero element in the matrix.

$$\begin{pmatrix} \times & & & & & & \\ \times & & & & & & \\ & \times & & & & & \\ & \times & & & & & \\ \times & & & & & & \\ & & \times & & & & \\ & \times & & & & & \\ & & \times & & & & \\ & & \times & & & & \\ & & \times & & & & \\ & & & \times & & & \\ & & \times & & & & \\ & & & & \times & & \\ & & & & & \times & \end{pmatrix}.$$

Algorithm 4

For the given system of aggregates $\{C_i^l\}_{i=1}^{N_l}$ and the $n_l \times r$ matrix satisfying $P_l^1 B_l = B_1$, we create a prolongator, a matrix B_{l+1} and the unknowns on level $l+1$ as follows:

- (1) Let d_i denote the number of unknowns associated with aggregate C_i^l . Split the $n_l \times r$ matrix B_l into blocks B_i^l of size $d_i \times r$, $i = 1, \dots, N_l$, each corresponding to the unknowns on an aggregate C_i^l .
- (2) A reduced QR decomposition is applied to $B_i^l = Q_i^l R_i^l$, where Q_i^l is an $d_i \times r$ orthogonal matrix, and R_i^l is an $r \times r$ upper triangular matrix.
- (3) Create the tentative prolongator $P_{l+1}^l = \text{diag}(Q_1^l, \dots, Q_{N_l}^l)$, and set

$$B_{l+1} = \begin{pmatrix} R_1^l \\ R_2^l \\ \dots \\ R_{N_l}^l \end{pmatrix}.$$

- (4) For each aggregate C_i^l , the coarsening gives rise to r unknowns on the coarse level (the i -th block column of P_{l+1}^l).

In Algorithm 4, the QR decomposition has been used to generate the prolongator P_{l+1}^l . A basic prolongator P_{l+1}^l may be computed to be piecewise constant, which leads to the simplest definition:

$$(P_{l+1}^l)_{ij} = \begin{cases} 1, & i \in C_j^l, \\ 0, & \text{otherwise.} \end{cases}$$

However, this choice is not efficient in general and the computation is still time consuming. A different way to generate the prolongator P_{l+1}^l has been considered in [6]. It uses the singular value decomposition for minimizing the interpolation error. In Algorithm 4, we need to change the step 2 as:

The reduced singular value decomposition is applied to $B_i^l = Q_i^l T_i^l M_i^l$, $R_i^l = T_i^l M_i^l$, where Q_i^l is an $d_i \times r$ matrix, and R_i^l is an $r \times r$ matrix.

For ease of discussion, in this paper we simply denote this algorithm as the SVSA method. See [6] for more details.

3. TWO NOVEL AGGREGATION SCHEMES

We briefly describe a general measure of strength of connection between two unknowns. In [2], the authors propose a general definition of strength of connection and an algorithm, which is not restricted to M -matrices.

We define the column vector $G^{(i)}$ first, with entries $(G^{(i)})_j = (A^{-1})_{ij}$. Next, we define the strength measure as follows:

$$S_{ij} = \frac{\|G^{(i)} - (G^{(i)})_j I^{(j)}\|_A}{\|G^{(i)}\|_A},$$

where $I^{(j)}$ is the j th canonical unit vector.

It is evident that this measure is not a practical option to implement, as $G^{(i)}$ is the column vector of A^{-1} . A natural choice is to apply the relaxation scheme of the AMG solver on the Gauss-Jordan system $AG = I$ columnwise, that is $AG^{(i)} = I^{(i)}$ for each node i .

Now, we can give the full algorithmic computation of the matrix S_{ij} as follows:
for each node i , $1 \leq i \leq n$:

- (1) relax μ times on $AG^{(i)} = I^{(i)}$, starting from a zero initial guess,
- (2) for each j such that the approximation to $(G^{(i)})_j$ is non-zero, compute

$$S_{ij} = \frac{\|\hat{G}^{(i)} - (\hat{G}^{(i)})_j I^{(j)}\|_A}{\|\hat{G}^{(i)}\|_A}. \quad (3.1)$$

Once the measure of strength of connection has been computed, the set of nodes that are strongly dependent on node i is defined as

$$S_i = \{j : S_{ij} - 1 \geq \theta \max_{k \neq i} \{S_{ik} - 1\}\} \quad (3.2)$$

In [2], the authors define the strong connections based on the entries of A^{-1} instead of those of A ; then they present the theoretical analysis, and finally the practical algorithmic implementation. The key point in [2] is to replace the traditional definition of strong dependence in classic AMG with the Formula 3.1, which produces efficient AMG V-cycles for many problems.

Inspired by this analysis, we aim at applying the Formula 3.2 to our SA method. As we mentioned before, clearly the connection between two nodes in the same aggregate should be as strong as possible to make the interpolation more accurate. The aggregation Algorithm 3 is based on the definition of strongly coupled neighborhood:

$$N_i^l(\theta) = \{j : |a_{ij}| \geq \theta \sqrt{a_{ii}a_{jj}}\} \cup \{i\}.$$

Here, we utilize the Formula 3.2 to the Algorithm 3 and we have a modified aggregation scheme, which results into Algorithm 5:

Generally, as output of this algorithm, a disjoint covering is constructed by Algorithm 5. Intuitively, the nodes in the same aggregate inherit the strong connectivity from Formula 3.2, which does favor better performance of the AMG method.

It is not difficult to establish an AMG scheme with the components defined above, once an appropriate smoother and cycling strategy are selected. Then we have a novel

Algorithm 5

Input: $U = \{1, \dots, n_l\}$, strong coupling threshold θ

Output: a disjoint covering $\{C_i^l\}_{i=1}^{N_l}$ of the set $\{1, \dots, n_l\}$

Definition: for all $i \in U$, $N_i^l(\theta) = \{j : S_{ij} - 1 \geq \theta \max_{k \neq i} \{S_{ik} - 1\}\}$, where S_{ij} is defined by Formula 3.1

Algorithm: the loop process is the same as Algorithm 3, the difference is the definition of S_{ij} .

SA method whose prolongator differs from the one of the traditional SA method because of the different aggregation scheme. The numerical results in the next section will show better performance of our novel methods.

Similarly to what we did in Algorithm 5, here we also propose another novel aggregation scheme. Let us go back to the standard aggregation Algorithm 3 and highlight the step 1. The strongly coupled neighborhood of each node has been generated and disjoint strongly coupled neighborhoods are selected as initial approximation of the covering. We noticed that the set U is changing over the recursion. If we calculate the strongly coupled neighborhood only in the remaining set U , to be more specific, we define:

$$M_i^l(\theta) = \{j : |a_{ij}| \geq \theta \sqrt{a_{ii}a_{jj}}\} \cup \{i\}$$

$$N_i^l(\theta) = M_i^l(\theta) \cap U.$$

Another variant of standard aggregation scheme results. In step 1, the assumption $N_i^l(\theta) \subset U$ is not necessary anymore for the definition of $N_i^l(\theta)$. Algorithm 6 follows below:

Algorithm 6

Input: $U = \{1, \dots, n_l\}$, strong coupling threshold θ

Output: a disjoint covering $\{C_i^l\}_{i=1}^{N_l}$ of the set $\{1, \dots, n_l\}$

Definition: for all $i \in U$, $M_i^l(\theta) = \{j : |a_{ij}| \geq \theta \sqrt{a_{ii}a_{jj}}\} \cup \{i\}$, $N_i^l(\theta) = M_i^l(\theta) \cap U$

Step 1: {
 $j = 0; i = 1;$
 for($i \in U$)
 {
 $j = j + 1, C_j^l = N_i^l(\theta), U = U - C_j^l$
 }
 }

After step 1, the strongly coupled neighborhood of each node is either $\emptyset$ or it is the covering set. The latter case means that the algorithm stops. From Algorithm 6,

we clearly see that all the nodes in the same aggregate are mutually coupled, contrast to Algorithm 3 and 5. Additionally, we achieve some gaining in time, leading potentially to overall better performances.

4. NUMERICAL TESTS

We illustrate the performance of our two methods on different matrix problems, also in comparison against the standard SA method in the first subsection and the classic AMG method in the second subsection.

In our experiments, we used an adaptive scheme which consists of applying a few iterations of relaxation to the homogeneous problem, $Ae = 0$, to expose the slowly converging components of the error. The residuals produced by the iterations are then used to update the approximate solution. We combined the SA method with this adaptive scheme to yield an enhanced SA method. Notice that this approach is different from the pure αSA method described in [3], which has an expensive setup cost and high efficiency, that cannot be achieved in our problems. For the sake of simplicity and cost effectiveness, we applied several steps of a weighted Jacobi smoother to generate some near-null space vectors that were used as initial approximations.

Next, we introduce some parameters for the construction of the SA scheme. We used V(1,1) with weighted Jacobi smoother, and we ran five post-smoothing steps with a weighed Jacobi smoother. Here we have to point out that, according to the basic theory of MG methods, it is a fact that no value of ω effectively eliminates the smooth, or low-frequency, error components. However, it is possible to reduce them significantly. We denote by R_ω the error propagation matrix of the weighted Jacobi method. The eigenvalues of the iteration matrix are given by

$$\lambda_k(R_\omega) = 1 - 2\omega \sin^2\left(\frac{k\pi}{2n}\right), \quad 1 \leq k \leq n-1$$

so that

$$\lambda_{n/2}(R_\omega) = -\lambda_n(R_\omega).$$

The solution of this equation gives the value $\omega = \frac{2}{3}$, which is our choice here. Another parameter is the strength threshold θ which is set to 0.25 in our experiments. In Algorithm 4, we used the singular value decomposition instead of the QR decomposition to achieve better performance. The traditional SA method with singular value decomposition was denoted as SVSA; the variant of SVSA based on Algorithm 5 is denoted in the forthcoming experiments as SVSA1, and that based on Algorithm 6 as SVSA2. All our numerical tests are performed using the MATLAB environment version 7.8.0 (R2009a) run on the Windows 7 operating system.

For each experiment shown in the following tables, we report the following performance measures:

- (1) the time cost for the construction of the AMG preconditioner (column "P-T") and for solving the linear system (column "I-T");

- (2) the memory burden (column "M-cost"), calculated as the sum of the number of nonzero entries in the projected matrix A at each level divided by the number of nonzeros in the coefficient matrix A ;
- (3) the number of iterations (column "Its") required by the Krylov subspace method to reduce the initial residual by six orders of magnitude. The symbol "-1" indicates a failure in the solving phase due to two possible reasons, either the preconditioner is ill-conditioned and the iterative solution was not started, or the number of iterations was larger than 2000.
- (4) The reduction factor (column "r-ratio"), which gives the ratio of the sum of the number of unknowns at all levels of the initial matrices to the number of unknowns in the original system.

4.1. Helmholtz equation

In this section we report on experiments for solving sparse linear systems resulting from the five point finite difference approximation of the 2D Helmholtz equation

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - k^2 u = f \quad \text{in } \Omega = (0, 1) \times (0, 1)$$

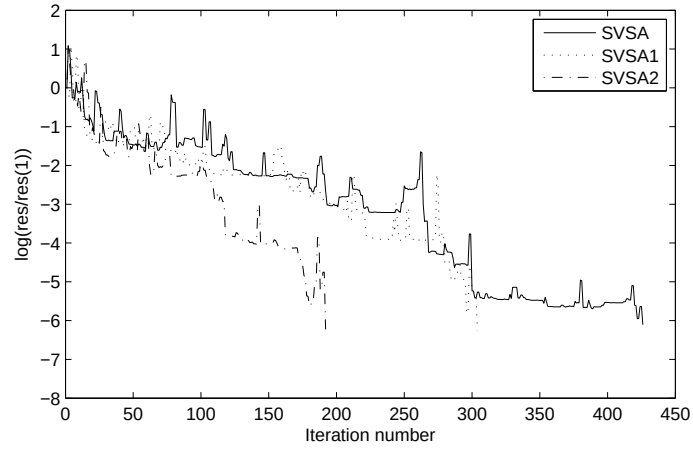
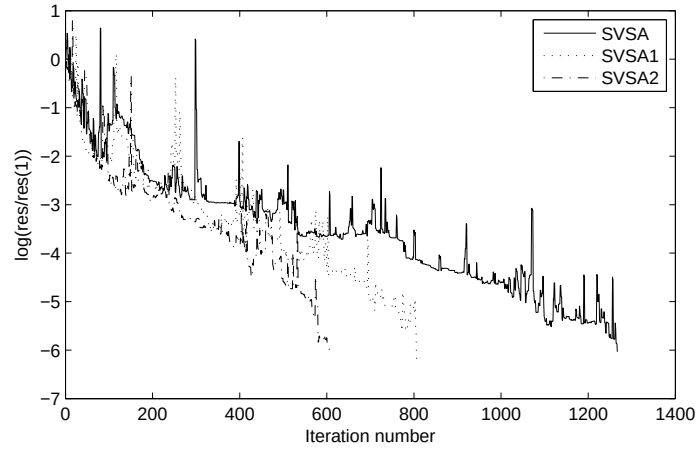
with the right function $f = \sin(\pi x) \sin(\pi y) \sin(\sqrt{2}\pi x) \sin(\sqrt{3}\pi y)$ and $u(x, y) = 0$ everywhere on $\partial\Omega$.

For Helmholtz equation, empirically, we keep $\frac{k}{n} = 0.625$ constant for the iterative solution. While $k = 20, 30$, the grid number is $n = 32, 48$. Here we use a multigrid-preconditioned Krylov subspace method. The Bi-CGSTAB method is the Krylov subspace method of choice in our experiments for the Helmholtz equation, especially for the problem without damping.

TABLE 1. Experiments with SA-preconditioned Bi-CGSTAB method for Helmholtz equation.

	Method	Its	M-cost	r-ratio	P-T	I-T
k=20	SVSA	209.5	1.2845	1.1719	127.1	11.2
	SVSA1	141.5	1.0929	1.0605	292.1	7.8
	SVSA2	88.5	1.6859	1.5000	127.1	6.9
k=30	SVSA	620.5	1.2931	1.1719	699.0	47.5
	SVSA1	383.5	1.0791	1.0512	2032.8	25.0
	SVSA2	293	1.6907	1.5000	715.5	67.5

Here, for comparison we also try to solve the system using another Krylov subspace method, which is GMRES with restart equal to 20.

FIGURE 2. Table 1, $k=20$.FIGURE 3. Table 1, $k=30$.

4.2. Experiments with Harwell-Boeing matrices.

Here, in Table 3, we present the comparison of classic AMG and our new method on five matrices. These matrices arise from different applications: SHERMAN1 and SAYLR3 come from computational fluid dynamics problems, PLBUCKLE and BCSSTK08 from structural problems and CAGE8 from directed weighted graphs.

The column "cond" shows the condition number of the matrix, which is a measure of the sensitivity of linear solvers to numerical errors. We can see from the Table 3

TABLE 2. SA-preconditioned GMRES method for Helmholtz equation.

	Method	Its	M-cost	r-ratio	P-T	I-T
k=20	SVSA	2800	1.2845	1.1719	134.0	82.3
	SVSA1	1179	1.0929	1.0605	317.9	33.3
	SVSA2	856	1.6859	1.5000	135.7	33.9
k=30	SVSA	15738	1.2931	1.1719	698.0	646.2
	SVSA1	3519	1.0791	1.0512	1959.3	117.8
	SVSA2	2062	1.6907	1.5000	669.8	240.4

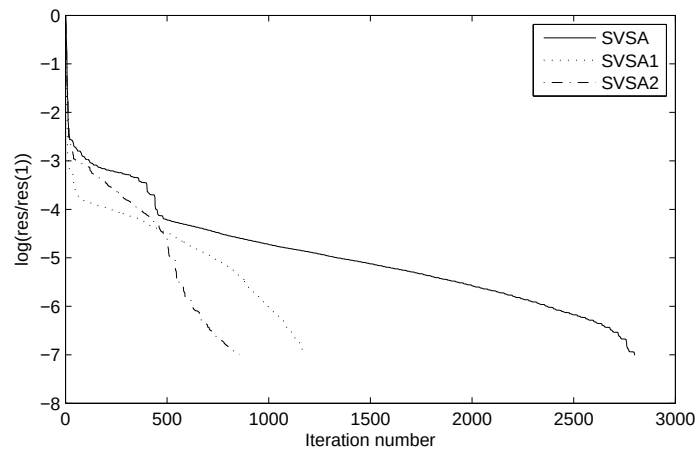


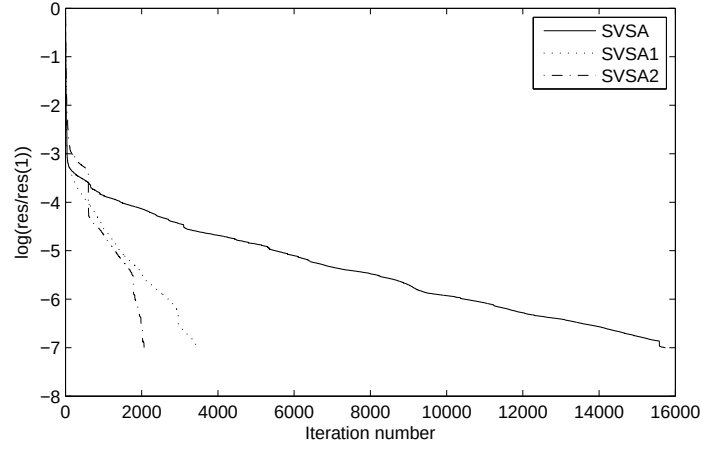
FIGURE 4. Table 2, k=20.

that some linear systems are ill-conditioned and thus difficult to solve. In the third column, called "Method", "C-AMG" denotes the classic AMG.

The results highlight the robustness of the SVSA2 preconditioner, which delivered faster and more stable convergence. There are three cases where C-AMG diverge and SVSA2 converges within a few iteration steps and requires less time cost. Even on the two problems where C-AMG converges, our method is more efficient with respect to memory cost, number of iterations and solution time.

5. CONCLUSION

All the numerical experiments reported in this paper illustrate an interesting behavior of the three proposed aggregation schemes. In most of our runs, the SVSA2 method decreases the number of iterations of the standard SA method by a factor of two, or sometimes larger. Also the solution time is moderately reduced for some

FIGURE 5. Table 2, $k=30$.

Matrix	cond	Method	P-T	I-T	Its	r-ratio	M-cost
SHERMAN1	2.2575×10^4	C-AMG	68.809	0.55672	-1	-1	-1
		SVSA2	40.885	0.86451	6.5	1.684	1.6672
PLBUCKLE	4.128×10^6	C-AMG	73.739	9.5594	60.5	1.3378	1.4889
		SVSA2	56.905	3.3188	51	1.3261	1.2119
BCSSTK08	4.7262×10^7	C-AMG	62.33	0.75017	-1	-1	-1
		SVSA2	54.329	0.63049	4	1.7486	1.8298
CAGE8	5.8238×10	C-AMG	59.569	0.12071	1	1.2512	1.9009
		SVSA2	43.33	0.048572	1	1.2148	1.3515
SAYLR3	INF	C-AMG	73.839	0.55805	-1	-1	-1
		SVSA2	42.71	0.65484	6.5	1.684	1.6672

TABLE 3. Performance comparison of the two methods C-AMG and SVSA2.

problems. The only flaw are the larger r-ratio and M-cost, which means extra storage cost. However, it is relatively small in comparison to the order of matrix A .

A significant disadvantage of SVSA1 is its computational complexity in computing the aggregates, being inherited from [2]. In Algorithm 5, the calculation of strong neighborhood requires the number of relaxation to be small. Now, searching for an optimal number of relaxations steps in Formula 3.1 is still a problem under investigation.

The results suggest that the two novel methods proposed in this paper can improve the performance of the SA method to some extent for various problems. The whole

SA scheme includes many ingredients, such as the adaptive scheme and the choice of the smoother. Different choices lead to different performance for each problem, but the key point is the interplay between interpolation and adaptive method, which is a subject of ongoing research.

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SHARP FUNCTION ESTIMATES AND BOUNDEDNESS FOR TOEPLITZ TYPE OPERATOR RELATED TO GENERAL SINGULAR INTEGRAL OPERATOR

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Abstract. In this paper, we establish the sharp maximal function estimates for the Toeplitz type operator related to the singular integral operator with general kernel. As an application, we obtain the boundedness of the operator on Lebesgue, Morrey and Triebel-Lizorkin spaces.

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1. INTRODUCTION

As the development of singular integral operators (see [7, 19, 20]), their commutators have been well studied. In [4, 17, 18] the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chanillo (see [2]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [3, 8, 14] the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) spaces are obtained. In [1], some singular integral operators with general kernel are introduced, and the boundedness for the operators and their commutators generated by BMO and Lipschitz functions are obtained (see [1, 10]). In [9, 11], some Toeplitz type operators related to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by BMO and Lipschitz functions are obtained. In this paper, we will study the Toeplitz type operators generated by the singular integral operators with general kernel and the Lipschitz and BMO functions.

2. PRELIMINARIES

First, let us introduce some notations. Throughout this paper, Q will denote a cube of $\mathbb{R}^n$ with sides parallel to the axes. For any locally integrable function f , the sharp

maximal function of f is defined by

$$M^\#(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well-known that (see [7, 19])

$$M^\#(f)(x) \approx \sup_{Q \ni x} \inf_{c \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that f belongs to $BMO(R^n)$ if $M^\#(f)$ belongs to $L^\infty(R^n)$ and define $\|f\|_{BMO} = \|M^\#(f)\|_{L^\infty}$. It has been known that (see [19])

$$\|f - f_{2^k Q}\|_{BMO} \leq Ck \|f\|_{BMO}.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For $\eta > 0$, let $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < n$ and $1 \leq r < \infty$, set

$$M_{\eta,r}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-r\eta/n}} \int_Q |f(y)|^r dy \right)^{1/r}.$$

The A_p weight is defined by (see [7]), for $1 < p < \infty$,

$$A_p = \left\{ w \in L^1_{loc}(R^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty \right\}$$

and

$$A_1 = \{w \in L^p_{loc}(R^n) : M(w)(x) \leq Cw(x), a.e.\}.$$

For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}(R^n)$ be the homogeneous Triebel-Lizorkin space (see [14]).

For $\beta > 0$, the Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$\|f\|_{Lip_\beta} = \sup_{\substack{x,y \in R^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty.$$

Definition 1. Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \leq D\varphi(t) \text{ for } t \geq 0.$$

Let f be a locally integrable function on R^n . Set, for $1 \leq p < \infty$,

$$\|f\|_{L^{p,\varphi}} = \sup_{x \in R^n, d > 0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} |f(y)|^p dy \right)^{1/p},$$

where $Q(x, d) = \{y \in R^n : |x - y| < d\}$. The generalized Morrey space is defined by

$$L^{p,\varphi}(R^n) = \{f \in L^1_{loc}(R^n) : \|f\|_{L^{p,\varphi}} < \infty\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n) = L^{p,\delta}(R^n)$, which is the classical Morrey space (see [15, 16]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n) = L^p(R^n)$, which is the Lebesgue space (see [13]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [5, 6, 12, 13]).

In this paper, we will study some singular integral operators as following (see [1]).

Definition 2. Let $T : S \rightarrow S'$ be a linear operator such that T is bounded on $L^2(R^n)$ and there exists a locally integrable function $K(x, y)$ on $R^n \times R^n \setminus \{(x, y) \in R^n \times R^n : x = y\}$ such that

$$T(f)(x) = \int_{R^n} K(x, y) f(y) dy$$

for every bounded and compactly supported function f , where K satisfies: there is a sequence of positive constant numbers $\{C_j\}$ such that for any $j \geq 1$,

$$\int_{2^j|y-z| < |x-y|} (|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)|) dx \leq C,$$

and

$$\left(\int_{2^j|z-y| \leq |x-y| < 2^{j+1}|z-y|} (|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)|)^q dy \right)^{1/q} \leq C_j (2^j |z - y|)^{-n/q'},$$

where $1 < q' < 2$ and $1/q + 1/q' = 1$.

Moreover, let b be a locally integrable function on R^n . The Toeplitz type operator related to T is defined by

$$T_b = \sum_{k=1}^m T^{k,1} M_b T^{k,2},$$

where $T^{k,1}$ are T or $\pm I$ (the identity operator), $T^{k,2}$ are the bounded linear operators on $L^p(R^n)$ for $1 < p < \infty$, $k = 1, \dots, m$, $M_b(f) = bf$.

Note that the classical Calderón-Zygmund singular integral operator satisfies Definition 2 with $C_j = 2^{-j\delta}$ (see [7, 19]). And note that the commutator $[b, T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz type operator T_b . The Toeplitz type operator T_b are the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [17, 18]). The main purpose of this paper is to

prove the sharp maximal inequalities for the Toeplitz type operator T_b . As the application, we obtain the the L^p -norm inequality, Morrey and Triebel-Lizorkin spaces boundedness for the Toeplitz type operators T_b .

3. THEOREMS

We shall prove the following theorems.

Theorem 1. *Let T be the singular integral operator as in Definition 2, the sequence $\{C_j\} \in l^1$, $0 < \beta < 1$, $q' \leq s < \infty$ and $b \in Lip_\beta(R^n)$. If $g \in L^u(R^n)$ ($1 < u < \infty$) and $T_1(g) = 0$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$M^\#(T_b(f))(\tilde{x}) \leq C \|b\|_{Lip_\beta} \sum_{k=1}^m M_{\beta,s}(T^{k,2}(f))(\tilde{x}).$$

Theorem 2. *Let T be the singular integral operator as in Definition 2, the sequence $\{2^{j\beta} C_j\} \in l^1$, $0 < \beta < 1$, $q' \leq s < \infty$ and $b \in Lip_\beta(R^n)$. If $g \in L^u(R^n)$ ($1 < u < \infty$) and $T_1(g) = 0$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$\sup_{Q \ni \tilde{x}} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T_b(f)(x) - C_0| dx \leq C \|b\|_{Lip_\beta} \sum_{k=1}^m M_s(T^{k,2}(f))(\tilde{x}).$$

Theorem 3. *Let T be the singular integral operator as in Definition 2, the sequence $\{j C_j\} \in l^1$, $q' \leq s < \infty$ and $b \in BMO(R^n)$. If $g \in L^u(R^n)$ ($1 < u < \infty$) and $T_1(g) = 0$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$M^\#(T_b(f))(\tilde{x}) \leq C \|b\|_{BMO} \sum_{k=1}^m M_s(T^{k,2}(f))(\tilde{x}).$$

Theorem 4. *Let T be the singular integral operator as in Definition 2, the sequence $\{C_j\} \in l^1$, $0 < \beta < \min(1, n/q')$, $q' < p < n/\beta$, $1/r = 1/p - \beta/n$ and $b \in Lip_\beta(R^n)$. If $g \in L^u(R^n)$ ($1 < u < \infty$) and $T_1(g) = 0$, then T_b is bounded from $L^p(R^n)$ to $L^r(R^n)$.*

Theorem 5. *Let T be the singular integral operator as in Definition 2, the sequence $\{C_j\} \in l^1$, $0 < \beta < \min(1, n/q')$, $q' < p < n/\beta$, $1/r = 1/p - \beta/n$, $0 < D < 2^n$ and $b \in Lip_\beta(R^n)$. If $g \in L^u(R^n)$ ($1 < u < \infty$) and $T_1(g) = 0$, then T_b is bounded from $L^{p,\varphi}(R^n)$ to $L^{r,\varphi}(R^n)$.*

Theorem 6. *Let T be the singular integral operator as in Definition 2, the sequence $\{2^{j\beta} C_j\} \in l^1$, $0 < \beta < \min(1, n/q')$, $q' < p < n/\beta$ and $b \in Lip_\beta(R^n)$. If $g \in L^u(R^n)$ ($1 < u < \infty$) and $T_1(g) = 0$, then T_b is bounded from $L^p(R^n)$ to $\dot{F}_p^{\beta,\infty}(R^n)$.*

Theorem 7. Let T be the singular integral operator as in Definition 2, the sequence $\{jC_j\} \in l^1$, $q' < p < n/\beta$ and $b \in BMO(R^n)$. If $g \in L^u(R^n)$ ($1 < u < \infty$) and $T_1(g) = 0$, then T_b is bounded on $L^p(R^n)$.

Theorem 8. Let T be the singular integral operator as in Definition 2, the sequence $\{jC_j\} \in l^1$, $q' < p < n/\beta$, $0 < D < 2^n$ and $b \in BMO(R^n)$. If $g \in L^u(R^n)$ ($1 < u < \infty$) and $T_1(g) = 0$, then T_b is bounded on $L^{p,\varphi}(R^n)$.

4. PROOFS OF THE THEOREMS

To prove the theorems, we need the following lemmas.

Lemma 1. (see [1]) Let T be the singular integral operator as in Definition 2. Then T is bounded on $L^p(R^n)$ for $1 < p < \infty$.

Lemma 2. (see [14]) For $0 < \beta < 1$ and $1 < p < \infty$, we have

$$\begin{aligned} \|f\|_{\dot{F}_p^{\beta,\infty}} &\approx \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| dx \right\|_{L^p} \\ &\approx \left\| \sup_{Q \ni \cdot} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c| dx \right\|_{L^p}. \end{aligned}$$

Lemma 3. (see [7]) Let $0 < p < \infty$ and $w \in \cup_{1 \leq r < \infty} A_r$. Then, for any smooth function f for which the left-hand side is finite,

$$\int_{R^n} M(f)(x)^p w(x) dx \leq C \int_{R^n} M^\#(f)(x)^p w(x) dx.$$

Lemma 4. (see [2]). Suppose that $0 < \eta < n$, $1 \leq s < p < n/\eta$ and $1/r = 1/p - \eta/n$. Then

$$\|M_{\eta,s}(f)\|_{L^r} \leq C \|f\|_{L^p}.$$

Lemma 5. Let $1 < p < \infty$, $0 < D < 2^n$. Then, for any smooth function f for which the left-hand side is finite,

$$\|M(f)\|_{L^{p,\varphi}} \leq C \|M^\#(f)\|_{L^{p,\varphi}}.$$

Proof. For any cube $Q = Q(x_0, d)$ in R^n , we know $M(\chi_Q) \in A_1$ for any cube $Q = Q(x, d)$ by [7]. Noticing that $M(\chi_Q) \leq 1$ and $M(\chi_Q)(x) \leq \frac{d^n}{(|x-x_0|-d)^n}$ if $x \in Q^c$, by Lemma 3, we have, for $f \in L^{p,\varphi}(R^n)$,

$$\begin{aligned} \int_Q M(f)(x)^p dx &= \int_{R^n} M(f)(x)^p \chi_Q(x) dx \\ &\leq \int_{R^n} M(f)(x)^p M(\chi_Q)(x) dx \leq C \int_{R^n} M^\#(f)(x)^p M(\chi_Q)(x) dx \end{aligned}$$

$$\begin{aligned}
&= C \left(\int_Q M^\#(f)(x)^p M(\chi_Q)(x) dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} M^\#(f)(x)^p M(\chi_Q)(x) dx \right) \\
&\leq C \left(\int_Q M^\#(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} M^\#(f)(x)^p \frac{|Q|}{|2^{k+1}Q|} dx \right) \\
&\leq C \left(\int_Q M^\#(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q} M^\#(f)(x)^p 2^{-kn} dy \right) \\
&\leq C \|M^\#(f)\|_{L^{p,\varphi}}^p \sum_{k=0}^{\infty} 2^{-kn} \varphi(2^{k+1}d) \\
&\leq C \|M^\#(f)\|_{L^{p,\varphi}}^p \sum_{k=0}^{\infty} (2^{-n}D)^k \varphi(d) \\
&\leq C \|M^\#(f)\|_{L^{p,\varphi}}^p \varphi(d),
\end{aligned}$$

thus

$$\left(\frac{1}{\varphi(d)} \int_Q M(f)(x)^p dx \right)^{1/p} \leq C \left(\frac{1}{\varphi(d)} \int_Q M^\#(f)(x)^p dx \right)^{1/p}$$

and

$$\|M(f)\|_{L^{p,\varphi}} \leq C \|M^\#(f)\|_{L^{p,\varphi}}.$$

This finishes the proof. $\square$

Lemma 6. Let $0 < D < 2^n$, $1 \leq s < p < n/\eta$ and $1/r = 1/p - \eta/n$. Then

$$\|M_{\eta,s}(f)\|_{L^{r,\varphi}} \leq C \|f\|_{L^{p,\varphi}}.$$

Lemma 7. Let T be the bounded linear operators on $L^q(R^n)$ for any $1 < q < \infty$. Then, for $1 < p < \infty$, $0 < D < 2^n$,

$$\|T(f)\|_{L^{p,\varphi}} \leq C \|f\|_{L^{p,\varphi}}.$$

The proofs of the last two Lemmas are similar to that of Lemma 5 by Lemma 4, we omit the details.

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0 , the following inequality holds:

$$\frac{1}{|Q|} \int_Q |T_b(f)(x) - C_0| dx \leq C \|b\|_{Lip_\beta} \sum_{k=1}^m M_{\beta,s}(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1, \dots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$T_b(f)(x) = T_{b-b_Q}(f)(x) = T_{(b-b_Q)\chi_{2Q}}(f)(x) + T_{(b-b_Q)\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x).$$

Then

$$\begin{aligned} \frac{1}{|Q|} \int_Q |T_b(f)(x) - f_2(x_0)| dx &\leq \frac{1}{|Q|} \int_Q |f_1(x)| dx \\ &\quad + \frac{1}{|Q|} \int_Q |f_2(x) - f_2(x_0)| dx = I_1 + I_2. \end{aligned}$$

For I_1 , by Hölder's inequality, we obtain

$$\begin{aligned} &\frac{1}{|Q|} \int_Q |T^{k,1} M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)| dx \\ &\leq \left(\frac{1}{|Q|} \int_{R^n} |T^{k,1} M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|^s dx \right)^{1/s} \\ &\leq C |Q|^{-1/s} \left(\int_{R^n} |M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|^s dx \right)^{1/s} \\ &\leq C |Q|^{-1/s} \left(\int_{2Q} (|b(x) - b_Q| |T^{k,2}(f)(x)|)^s dx \right)^{1/s} \\ &\leq C |Q|^{-1/s} \|b\|_{Lip_\beta} |2Q|^{\beta/n} |Q|^{1/s-\beta/n} \left(\frac{1}{|Q|^{1-s\beta/n}} \int_Q |T^{k,2}(f)(x)|^s dx \right)^{1/s} \\ &\leq C \|b\|_{Lip_\beta} M_{\beta,s}(T^{k,2}(f))(\tilde{x}), \end{aligned}$$

thus

$$\begin{aligned} I_1 &\leq \sum_{k=1}^m \frac{1}{|Q|} \int_Q |T^{k,1} M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)| dx \\ &\leq C \|b\|_{Lip_\beta} \sum_{k=1}^m M_{\beta,s}(T^{k,2}(f))(\tilde{x}). \end{aligned}$$

For I_2 , by the boundedness of T and recalling that $s > q'$, we get, for $x \in Q$,

$$\begin{aligned} &|T^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x) - T^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x_0)| \\ &\leq \int_{(2Q)^c} |b(y) - b_{2Q}| |K(x, y) - K(x_0, y)| |T^{k,2}(f)(y)| dy \\ &= \sum_{j=1}^{\infty} \int_{2^j d \leq |y-x_0| < 2^{j+1} d} |b(y) - b_{2Q}| |K(x, y) - K(x_0, y)| |T^{k,2}(f)(y)| dy \\ &\leq C \|b\|_{Lip_\beta} \sum_{j=1}^{\infty} |2^{j+1} Q|^{\beta/n} \left(\int_{2^j d \leq |y-x_0| < 2^{j+1} d} |K(x, y) - K(x_0, y)|^q dy \right)^{1/q} \end{aligned}$$

$$\begin{aligned}
& \times \left(\int_{2^{j+1}Q} |T^{k,2}(f)(y)|^{q'} dy \right)^{1/q'} \\
& \leq C \|b\|_{Lip_\beta} \sum_{j=1}^{\infty} |2^{j+1}Q|^{\beta/n} C_j (2^j d)^{-n/q'} |2^{j+1}Q|^{1/q' - \beta/n} \\
& \quad \times \left(\frac{1}{|2^{j+1}Q|^{1-s\beta/n}} \int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s dy \right)^{1/s} \\
& \leq C \|b\|_{Lip_\beta} M_{\beta,s}(T^{k,2}(f))(\tilde{x}) \sum_{j=1}^{\infty} C_j \\
& \leq C \|b\|_{Lip_\beta} M_{\beta,s}(T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_2 & \leq \frac{1}{|Q|} \int_Q \sum_{k=1}^m |T^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x) \\
& \quad - T^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x_0)| dx \\
& \leq C \|b\|_{Lip_\beta} \sum_{k=1}^m M_{\beta,s}(T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

These complete the proof of Theorem 1. $\square$

Proof of Theorem 2. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0 , the following inequality holds:

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |T_b(f)(x) - C_0| dx \leq C \|b\|_{Lip_\beta} \sum_{k=1}^m M_s(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1, \dots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. For $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$, write

$$T_b(f)(x) = T_{b-b_Q}(f)(x) = T_{(b-b_Q)\chi_{2Q}}(f)(x) + T_{(b-b_Q)\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x)$$

and

$$\begin{aligned}
\frac{1}{|Q|^{1+\beta/n}} \int_Q |T_b(f)(x) - f_2(x_0)| dx & \leq \frac{1}{|Q|^{1+\beta/n}} \int_Q |f_1(x)| dx \\
& + \frac{1}{|Q|^{1+\beta/n}} \int_Q |f_2(x) - f_2(x_0)| dx = I_3 + I_4.
\end{aligned}$$

By using the same argument as in the proof of Theorem 1, we get

$$I_3 \leq \sum_{k=1}^m \frac{C}{|Q|^{\beta/n}} \|b\|_{Lip_\beta} |2Q|^{\beta/n} |Q|^{-1/s} \left(\int_{2Q} |T^{k,2}(f)(x)|^s dx \right)^{1/s}$$

$$\begin{aligned}
&\leq C \|b\|_{Lip_\beta} \sum_{k=1}^m \left(\frac{1}{|2Q|} \int_{2Q} |T^{k,2}(f)(x)|^s dx \right)^{1/s} \\
&\leq C \|b\|_{Lip_\beta} \sum_{k=1}^m M_s(T^{k,2}(f))(\tilde{x}), \\
I_4 &\leq \sum_{k=1}^m \frac{1}{|Q|^{1+\beta/n}} \int_Q \sum_{j=1}^\infty \int_{2^j d \leq |y-x_0| < 2^{j+1}d} |b(y) - b_{2Q}| \\
&\quad \times |K(x, y) - K(x_0, y)| |T^{k,2}(f)(y)| dy dx \\
&\leq \sum_{k=1}^m \frac{C}{|Q|^{1+\beta/n}} \int_Q \sum_{j=1}^\infty \|b\|_{Lip_\beta} |2^{j+1}Q|^{\beta/n} \left(\int_{2^{j+1}Q} |T^{k,2}(f)(y)|^{q'} dy \right)^{1/q'} \\
&\quad \times \left(\int_{2^j d \leq |y-x_0| < 2^{j+1}d} |K(x, y) - K(x_0, y)|^q dy \right)^{1/q} dx \\
&\leq C \|b\|_{Lip_\beta} \sum_{k=1}^m |Q|^{-\beta/n} \sum_{j=1}^\infty |2^{j+1}Q|^{\beta/n} C_j (2^j d)^{-n/q'} |2^{j+1}Q|^{1/q'} \\
&\quad \times \left(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s dy \right)^{1/s} \\
&\leq C \|b\|_{Lip_\beta} \sum_{k=1}^m M_s(T^{k,2}(f))(\tilde{x}) \sum_{j=1}^\infty 2^{j\beta} C_j \\
&\leq C \|b\|_{Lip_\beta} \sum_{k=1}^m M_s(T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

This completes the proof of Theorem 2. $\square$

Proof of Theorem 3. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0 , the following inequality holds:

$$\frac{1}{|Q|} \int_Q |T_b(f)(x) - C_0| dx \leq C \|b\|_{BMO} \sum_{k=1}^m M_s(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T^{k,1}$ are T ($k = 1, \dots, m$). Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. For $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$, similar to the proof of Theorem 1, we have

$$\begin{aligned}
T_b(f)(x) &= T_{b-b_Q}(f)(x) = T_{(b-b_Q)\chi_{2Q}}(f)(x) \\
&\quad + T_{(b-b_Q)\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x)
\end{aligned}$$

and

$$\begin{aligned} & \frac{1}{|Q|} \int_Q |T_b(f)(x) - f_2(x_0)| dx \\ & \leq \frac{1}{|Q|} \int_Q |f_1(x)| dx + \frac{1}{|Q|} \int_Q |f_2(x) - f_2(x_0)| dx = I_5 + I_6. \end{aligned}$$

For I_5 , choose $1 < r < s$, by Hölder's inequality and the boundedness of T , we obtain

$$\begin{aligned} & \frac{1}{|Q|} \int_Q |T^{k,1} M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)| dx \\ & \leq \left(\frac{1}{|Q|} \int_{R^n} |T^{k,1} M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|^r dx \right)^{1/r} \\ & \leq C |Q|^{-1/r} \left(\int_{R^n} |M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|^r dx \right)^{1/r} \\ & \leq C |Q|^{-1/r} \left(\int_{2Q} |T^{k,2}(f)(x)|^s dx \right)^{1/s} \left(\int_{2Q} |b(x) - b_Q|^{sr/(s-r)} dx \right)^{(s-r)/sr} \\ & \leq C \|b\|_{BMO} \left(\frac{1}{|Q|} \int_{2Q} |T^{k,2}(f)(x)|^s dx \right)^{1/s} \\ & \leq C \|b\|_{BMO} M_s(T^{k,2}(f))(\tilde{x}), \end{aligned}$$

thus

$$\begin{aligned} I_5 & \leq \sum_{k=1}^m \frac{1}{|Q|} \int_Q |T^{k,1} M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)| dx \\ & \leq C \|b\|_{BMO} \sum_{k=1}^m M_s(T^{k,2}(f))(\tilde{x}). \end{aligned}$$

For I_6 , recalling that $s > q'$, taking $1 < p < \infty$, $1 < r < s$ with $1/p + 1/q + 1/r = 1$, by the boundedness of T , we get, for $x \in Q$,

$$\begin{aligned} & |T^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x) - T^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x_0)| \\ & \leq \sum_{j=1}^{\infty} \int_{2^j d \leq |y-x_0| < 2^{j+1} d} |K(x, y) - K(x_0, y)| |b(y) - b_{2Q}| |T^{k,2}(f)(y)| dy \\ & \leq \sum_{j=1}^{\infty} \left(\int_{2^j d \leq |y-x_0| < 2^{j+1} d} |K(x, y) - K(x_0, y)|^q dy \right)^{1/q} \\ & \quad \times \left(\int_{2^{j+1} Q} |b(y) - b_Q|^p dy \right)^{1/p} \left(\int_{2^{j+1} Q} |T^{k,2}(f)(y)|^r dy \right)^{1/r} \end{aligned}$$

$$\begin{aligned}
&\leq C \|b\|_{BMO} \sum_{j=1}^{\infty} C_j (2^j d)^{-n/q'} j (2^j d)^{n/p} (2^j d)^{n/s} \\
&\quad \cdot \left(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s dy \right)^{1/s} \\
&\leq C \|b\|_{BMO} M_s(T^{k,2}(f))(\tilde{x}) \sum_{j=1}^{\infty} j C_j \\
&\leq C \|b\|_{BMO} M_s(T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_6 &\leq \frac{1}{|Q|} \int_Q \sum_{k=1}^m |T^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x) - T^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x_0)| dx \\
&\leq C \|b\|_{BMO} \sum_{k=1}^m M_s(T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

This completes the proof of Theorem 3. $\square$

Proof of Theorem 4. Choose $q' < s < p$ in Theorem 1, we have, by Lemma 1, 3 and 4,

$$\begin{aligned}
\|T_b(f)\|_{L^r} &\leq \|M(T_b(f))\|_{L^r} \leq C \|M^\#(T_b(f))\|_{L^r} \\
&\leq C \|b\|_{Lip_\beta} \sum_{k=1}^m \|M_{\beta,s}(T^{k,2}(f))\|_{L^r} \leq C \|b\|_{Lip_\beta} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^p} \\
&\leq C \|b\|_{Lip_\beta} \|f\|_{L^p}.
\end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 5. Choose $q' < s < p$ in Theorem 1, we have, by Lemma 5, 6 and 7,

$$\begin{aligned}
\|T_b(f)\|_{L^{r,\varphi}} &\leq \|M(T_b(f))\|_{L^{r,\varphi}} \leq C \|M^\#(T_b(f))\|_{L^{r,\varphi}} \\
&\leq C \|b\|_{Lip_\beta} \sum_{k=1}^m \|M_{\beta,s}(T^{k,2}(f))\|_{L^{r,\varphi}} \leq C \|b\|_{Lip_\beta} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^{p,\varphi}} \\
&\leq C \|b\|_{Lip_\beta} \|f\|_{L^{p,\varphi}}.
\end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 6. Choose $q' < s < p$ in Theorem 2, we have, by Lemma 1, 2 and 3,

$$\begin{aligned} \|T_b(f)\|_{\dot{F}_p^{\beta,\infty}} &\leq C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T_b(f)(x) - C_0| dx \right\|_{L^p} \\ &\leq C \|b\|_{Lip_\beta} \sum_{k=1}^m \|M_s(T^{k,2}(f))\|_{L^p} \leq C \|b\|_{Lip_\beta} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^p} \\ &\leq C \|b\|_{Lip_\beta} \|f\|_{L^p}. \end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 7. Choose $q' < s < p$ in Theorem 3, we have, by Lemma 1, 3 and 4,

$$\begin{aligned} \|T_b(f)\|_{L^p} &\leq \|M(T_b(f))\|_{L^p} \leq C \|M^\#(T_b(f))\|_{L^p} \\ &\leq C \|b\|_{BMO} \sum_{k=1}^m \|M_s(T^{k,2}(f))\|_{L^p} \leq C \|b\|_{BMO} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^p} \\ &\leq C \|b\|_{BMO} \|f\|_{L^p}. \end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 8. Choose $q' < s < p$ in Theorem 3, we have, by Lemma 5, 6 and 7,

$$\begin{aligned} \|T_b(f)\|_{L^{p,\varphi}} &\leq \|M(T_b(f))\|_{L^{p,\varphi}} \leq C \|M^\#(T_b(f))\|_{L^{p,\varphi}} \\ &\leq C \|b\|_{BMO} \sum_{k=1}^m \|M_s(T^{k,2}(f))\|_{L^{p,\varphi}} \leq C \|b\|_{BMO} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^{p,\varphi}} \\ &\leq C \|b\|_{BMO} \|f\|_{L^{p,\varphi}}. \end{aligned}$$

This completes the proof. $\square$

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ALMOST SURE CONVERGENCE OF WEIGHTED SUMS

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Abstract. Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed random variables and $\{a_{i,n}, 1 \leq i \leq n\}$ be a triangular array of constants. In this short paper, we establish a general almost sure convergence theorem for the weighted sum $S_n = \sum_{i=1}^n a_{i,n} X_i$. Our results improves the works of Sung [14]. Furthermore, almost sure convergence theorems of S_n for negatively associated random variables, martingale difference sequence and ρ -mixing sequence are obtained.

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1. INTRODUCTION

Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed random variables and $\{a_{i,n}, 1 \leq i \leq n\}$ be a triangular array of constants. Define a weighted sum by

$$S_n = \sum_{i=1}^n a_{i,n} X_i.$$

Note that many useful linear statistics are in this form, e.g., least squares estimators, nonparametric regression function estimators, jackknife estimates and so on. Studies on strong laws of S_n are of great interest not only in probability theory but also in statistics.

1.1. Independent random variables

There is a sizable literature on the almost sure (a.s.) convergence of weighted sums S_n when $\{X, X_n, n \geq 1\}$ are assumed to be independent random variables (see for example Choi and Sung [3], Chow [4], Chow and Lai [5], Stout [13], Sung [14, 15], Teicher [16], and Thrum [17]).

Firstly, let us recall the following result of Teicher.

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Theorem 1 (Teicher [16], P.341). *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^p < \infty$, for some $1 \leq p \leq 2$. If*

$$\max_{1 \leq i \leq n} |a_{i,n}| = O\left(1/(n^{1/p} \log n)\right), \quad (1.1)$$

then $S_n \rightarrow 0$ a.s.

Choi and Sung [3], Sung [15] and Sung [14] ($p = 1$, $1 < p < 2$ and $p = 2$, respectively) improved the above work of Teicher.

Theorem 2 ([3], Theorem 5). *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$. If*

$$\max_{1 \leq i \leq n} |a_{i,n}| = O(1/n), \quad (1.2)$$

then $S_n \rightarrow 0$ a.s.

Theorem 3 ([15], Theorem 1). *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^p < \infty$ for some $1 \leq p < 2$. If*

$$\max_{1 \leq i \leq n} |a_{i,n}| = O\left(\frac{1}{n^{1/p} (\log n)^{1-1/p}}\right), \quad (1.3)$$

then $S_n \rightarrow 0$ a.s.

Theorem 4 ([14], Corollary 4). *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^2 < \infty$. If*

$$\max_{1 \leq i \leq n} |a_{i,n}| = o\left(\frac{1}{\sqrt{n \log n}}\right), \quad (1.4)$$

then $S_n \rightarrow 0$ a.s.

By comparing Theorem 3 and Theorem 4, it is natural to ask

Problem 1. *whether the condition (1.4) can be replaced by*

$$\max_{1 \leq i \leq n} |a_{i,n}| = O\left(\frac{1}{\sqrt{n \log n}}\right).$$

1.2. Negatively associated random variables

Definition 1. A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA), if for every pair of disjoint subset A and B of $\{1, 2, \dots, n\}$ and any real nondecreasing coordinatewise functions f_1 on $\mathbb{R}^A$ and f_2 on $\mathbb{R}^B$

$$\text{Cov}(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0$$

whenever f_1 and f_2 are such that covariance exists. An infinity family of random variables is negatively associated if every finite subfamily is negatively associated.

The notion of negative association was first introduced by Alam and Saxena [1]. Joag-Dev and Proschan [9] showed that many well known multivariate distributions possess the NA property. Some examples include: (a) multinomial, (b) convolution of unlike multinomial, (c) multivariate hypergeometric, (d) Dirichlet, (e) Dirichlet compound multinomial, (f) negatively correlated normal distribution, (g) permutation distribution, (h) random sampling without replacement, and (i) joint distribution of ranks. Because of its wide applications in multivariate statistical analysis and system reliability, the notion of negative association has received considerable attention recently.

In particular, we refer Jing and Liang [8] for the strong limit theorems for weighted sums of negatively associated random variables as follows.

Theorem 5 ([8], Theorem 2.1). *Suppose that $\mathbb{E}|X|^p < \infty$ for some $p > 0$ and $\mathbb{E}X_1 = 0$ if $p > 1$. Assume that $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ satisfies*

$$\max_{1 \leq i \leq n} |a_{i,n}| = O(n^{-1/p}).$$

- (1) if $p > 2$ and $\sum_{i=1}^n a_{i,n}^2 = o((\log n)^{-1})$, then $S_n \rightarrow 0$ a.s.
- (2) if $p > 2$ and $\sum_{i=1}^n a_{i,n}^2 = O((\log n)^{-1})$, then $S_n = O(1)$ a.s.
- (3) if $0 < p \leq 2$ and $\sum_{i=1}^n |a_{i,n}|^p = O(n^{-\delta})$ for some $\delta > 0$, then $S_n \rightarrow 0$ a.s.

Based on the above works, we want to ask

Problem 2. *whether the similar strong limits for the weight sums of negative associated random variables hold only in the condition to bound $\max_{1 \leq i \leq n} |a_{i,n}|$.*

So, in this paper, the main purpose is to establish a very general almost sure convergence for weighted sums, which only depends on the bound of the weights $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ and on the almost sure convergence of partial sums. From the general result, Problem 1 and Problem 2 are answered, and some further strong limits theorem are obtained which improve or supplement some relevant results.

2. A GENERAL THEOREM

The main result in the paper is the following very general theorem, which includes many of the results about almost sure convergence, but the proof is very elementary.

Theorem 6. *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed random variables. Suppose that $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ satisfies*

$$\max_{1 \leq i \leq n} |a_{i,n}| \leq b_n^{-1}.$$

where $0 < b_n \uparrow +\infty$. If $\{Z_n\}$ satisfies

$$Z_n := b_n^{-1}(X_1 + \cdots + X_n) \rightarrow 0 \quad \text{a.s.}, \quad (2.1)$$

then we have

$$S_n = \sum_{i=1}^n a_{i,n} X_i \rightarrow 0 \quad a.s.$$

Proof. Without loss of generality, we can suppose that $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ satisfies $a_{1,n} \geq a_{2,n} \geq \dots \geq a_{n,n}$. In fact, if the order does not be satisfied, then we can rearrange the sequence $\{a_{i,n} X_i, 1 \leq i \leq n, n \geq 1\}$.

For $1 \leq i \leq n$, let $\tilde{a}_{i,n} = a_{i,n} b_n$ (which implies $|\tilde{a}_{i,n}| \leq 1$) and

$$c_{i,n} = \begin{cases} \frac{b_i}{b_n} (\tilde{a}_{i,n} - \tilde{a}_{i+1,n}), & \text{if } 1 \leq i \leq n-1 \\ \tilde{a}_{n,n}, & \text{if } i = n. \end{cases}$$

Then we have

$$\begin{cases} c_{i,n} \geq 0, & \text{if } 1 \leq i \leq n-1; \\ \sum_{i=1}^n |c_{i,n}| \leq 3; \\ \lim_{n \rightarrow \infty} |c_{i,n}| = 0, & \text{for every fixed } i, \end{cases} \quad (2.2)$$

and

$$S_n = \sum_{i=1}^n a_{i,n} X_i = \frac{1}{b_n} \sum_{i=1}^n \tilde{a}_{i,n} X_i = \sum_{i=1}^n c_{i,n} Z_i. \quad (2.3)$$

From (2.2), we can claim that for every sequences of real numbers r_n with $r_n \rightarrow 0$, the following holds

$$\sum_{i=1}^n c_{i,n} r_i \rightarrow 0. \quad (2.4)$$

Let us give the proof of (2.4). For any $\varepsilon > 0$, there exists a positive constant N_ε , such that $|r_n| < \varepsilon$, when $n > N_\varepsilon$. So, we have

$$\left| \sum_{i=1}^n c_{i,n} r_i \right| \leq \left| \sum_{i=1}^N c_{i,n} r_i \right| + \left| \sum_{i=N+1}^n c_{i,n} r_i \right| < \left| \sum_{i=1}^N c_{i,n} r_i \right| + 3\varepsilon. \quad (2.5)$$

Furthermore, from the condition that $\lim_{n \rightarrow \infty} |c_{i,n}| = 0$ for every i , we have

$$\left| \sum_{i=1}^N c_{i,n} r_i \right| \rightarrow 0 \quad (2.6)$$

which together with (2.5) implies (2.4). Hence by the condition (2.1) and the relation (2.3), we have $S_n \rightarrow 0$ a.s. $\square$

3. SOME APPLICATIONS

3.1. Independent random variables

In this subsection, we obtain the almost sure convergence theorems for the weighted sums, which improve Theorem 3 and Theorem 4.

Corollary 1. *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^2 < \infty$. Suppose that $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ satisfies*

$$\max_{1 \leq i \leq n} |a_{i,n}| \leq b_n^{-1} = o\left((n \log \log n)^{-1/2}\right). \quad (3.1)$$

Then we have

$$S_n = \sum_{i=1}^n a_{i,n} X_i \rightarrow 0 \quad a.s.$$

Proof. By the Hartman-Wintner law of the iterated logarithm for the i.i.d sequence and the condition (3.1), we have

$$b_n^{-1}(X_1 + X_2 + \cdots + X_n) \rightarrow 0, \quad a.s.$$

which yields Corollary 1 from Theorem 6. $\square$

Remark 1. By taking $b_n = \sqrt{n \log n}$, it is easy to see that Corollary 1 improves Sung's result (see Theorem 4).

Corollary 2. *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^p < \infty$ for some $0 < p < 2$. Suppose that $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ satisfies*

$$\max_{1 \leq i \leq n} |a_{i,n}| \leq b_n^{-1} = O\left(n^{-1/p}\right). \quad (3.2)$$

Then we have

$$S_n = \sum_{i=1}^n a_{i,n} X_i \rightarrow 0 \quad a.s.$$

Proof. By the Marcinkiewicz-Zygmund strong law of large numbers, the proof of the corollary can be completed. $\square$

Remark 2. Obviously, Corollary 2 improves the result in Theorem 3.

Remark 3. Recently, Bai and Cheng [2] and Cuzick [6] proved the Marcinkiewicz-Zygmund strong laws of large numbers for the weighted sums

$$n^{-1/p} \sum_{i=1}^n a_{i,n} X_i \rightarrow 0, \quad a.s.$$

when $\{X, X_n, n \geq 1\}$ is a sequence of i.i.d. random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^\beta < \infty$ and $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ is an array of constants satisfying

$$A_\alpha = \limsup_{n \rightarrow \infty} A_{\alpha,n} < \infty, \quad A_{\alpha,n}^\alpha = n^{-1} \sum_{i=1}^n |a_{i,n}|^\alpha,$$

where $1 \leq p < 2$, $0 < \alpha, \beta < \infty$ and $1/p = 1/\alpha + 1/\beta$. Bai and Cheng [2] proved it for $1 < p < 2$ by the Bernstein inequality and Cuzick [6] for $p = 1$.

Note that the order of the moment β is higher than the rate of convergence p , so Corollary 2 improves the above results.

3.2. Negatively associated sequence

Corollary 3. *Let $\{X, X_n, n \geq 1\}$ be a sequence of strictly stationary negatively associated random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^2 < \infty$. Suppose that $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ satisfies*

$$\max_{1 \leq i \leq n} |a_{i,n}| \leq b_n^{-1} = o\left((n \log \log n)^{-1/2}\right). \quad (3.3)$$

Then we have

$$S_n = \sum_{i=1}^n a_{i,n} X_i \rightarrow 0 \quad a.s.$$

Proof. In [12], the authors obtained the following result: for any $0 < \varepsilon < 1/30$,

$$\limsup_{n \rightarrow \infty} \frac{|X_1 + \cdots + X_n|}{(2\mathbb{E}X^2 n \log \log n)^{1/2}} \leq 1 + 8\varepsilon, \quad a.s.$$

Hence from the condition (3.3), we have

$$b_n^{-1}(X_1 + X_2 + \cdots + X_n) \rightarrow 0, \quad a.s.$$

Based on Theorem 6, we complete the proof of this corollary. $\square$

Remark 4. By comparing Theorem 5 with Corollary 3, one can find that Corollary 3 complement the works of Jing and Liang for $p = 2$.

3.3. Martingale difference sequence

Corollary 4. *Let $\{X_n, n \geq 0\}$ be a stationary stochastic sequence such that $\mathbb{E}(X_n | \mathcal{F}_{n-1}) = 0$ a.s., all $n \geq 1$, and $\mathbb{E}X_0^2 = 0$. Here $\mathcal{F}_n$ is the σ -field generated by $Z_i, i \leq n$, where $\{\cdots, Z_{-1}, Z_0, Z_1, \cdots\}$ is a stationary stochastic sequence such that $X_k = \phi(Z_k, Z_{k-1}, \cdots)$, $k \geq 0$, for some measurable ϕ . We suppose also that*

$$n^{-1} \sum_{i=1}^n \mathbb{E}(X_i^2 | \mathcal{F}_{i-1}) \rightarrow 1 \quad a.s.$$

Suppose that $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ satisfies

$$\max_{1 \leq i \leq n} |a_{i,n}| \leq b_n^{-1} = o\left((n \log \log n)^{-1/2}\right). \quad (3.4)$$

Then we have

$$S_n = \sum_{i=1}^n a_{i,n} X_i \rightarrow 0 \quad a.s.$$

Proof. Under the assumptions of Corollary 4, we know (see [7, Theorem 1])

$$\limsup_{n \rightarrow \infty} \frac{|X_1 + \cdots + X_n|}{(2n \log \log n)^{1/2}} = 1 \quad a.s.$$

So the desired result can be easily obtained. $\square$

Remark 5. If we take $Z_i = X_i$ and assume that $\{X_i\}$ is a stationary ergodic martingale difference sequence, then the condition $n^{-1} \sum_{i=1}^n \mathbb{E}(X_i^2 | \mathcal{F}_{i-1}) \rightarrow 1$ a.s. can follow as a consequence of the ergodic theorem.

Corollary 5. Let $\{X_n, \mathcal{F}_n, n \geq 1\}$ denote a sequence of identically distributed martingale differences. Suppose that

- (i) $\mathbb{E}|X_1|^p < \infty$ for some $0 < p < 2$ and $p \neq 1$;
- (ii) $\mathbb{P}(|X_n| \geq x | X_1, \dots, X_{n-1}) \leq \mathbb{P}(|X_1| \geq x)$ a.s. for $p = 1$.

Let $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of constants satisfying

$$\max_{1 \leq i \leq n} |a_{i,n}| \leq b_n^{-1} = O\left(n^{-1/p}\right). \quad (3.5)$$

then if (i) or (ii) holds we have

$$S_n = \sum_{i=1}^n a_{i,n} X_i \rightarrow 0 \quad a.s.$$

Proof. Using the well-know Marcinkiewicz's strong law of large numbers of martingale differences (see [10, P. 53]), the proof is done. $\square$

3.4. ρ -mixing sequence

A sequence of random variables $\{X_n, n \geq 1\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called ρ -mixing if the maximal correlation coefficient

$$\rho(n) = \sup_{k \geq 1, X \in L^2(\mathcal{F}_1^k), Y \in L^2(\mathcal{F}_{k+n}^\infty)} |Cov(X, Y)| / \|X\|_2 \|Y\|_2 \rightarrow 0$$

as $n \rightarrow \infty$, where $\mathcal{F}_n^m$ is the σ -field generated by the random variables $X_n, X_{n+1}, \dots, X_m$. Here $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$.

Corollary 6. Let $\{X, X_n, n \geq 1\}$ denote a ρ -mixing sequence of identically distributed random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^p < \infty$ for some $1 \leq p < 2$. Suppose that

$$\sum_{n=1}^{\infty} \rho(2^n) < \infty$$

and $\{a_{i,n}, 1 \leq i \leq n, n \geq 1\}$ satisfies

$$\max_{1 \leq i \leq n} |a_{i,n}| \leq b_n^{-1} = O(n^{-1/p}). \quad (3.6)$$

Then we have

$$S_n = \sum_{i=1}^n a_{i,n} X_i \rightarrow 0 \quad a.s.$$

Proof. Under the conditions of the corollary, the proof can be completed by using the Marcinkiewicz-Zygmund law of large numbers for ρ -mixing sequences (see [11, Corollary 3.1]). $\square$

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EXISTENCE AND HYERS-ULAM STABILITY RESULTS FOR A COINCIDENCE PROBLEM WITH APPLICATIONS

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Abstract. The purpose of the work is to present some Hyers-Ulam stability results for the coincidence point problem associated to a single-valued operator problem. As an application, a Hyers-Ulam stability theorem for a initial value problem associated to a differential equation is given.

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1. INTRODUCTION

Let (X, d) be a metric space and $f : X \rightarrow X$ an operator. We denote by

$$Fix(f) := \{x \in X \mid f(x) = x\},$$

the fixed point set of the operator f . By definition, f is called a weakly Picard operator if the sequence $(f^n(x))_{n \in \mathbb{N}}$, of successive approximations converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f . For example, self Caristi type operators and self graphic contractions on complete metric spaces are examples of weakly Picard operators (see [3], [4]).

If f is weakly Picard operator, then we define the operator $f^\infty : X \rightarrow X$ defined by $f^\infty(x) := \lim_{n \rightarrow \infty} f^n(x)$. It is clear that $f^\infty(X) = Fix(f)$. Moreover, f^∞ is a set retraction of X to $Fix(f)$.

If f is weakly Picard operator and $Fix(f) = \{x^*\}$, then by definition f is a Picard operator. In this case f^∞ is the constant operator, $f^\infty(x) = x^*$, for all $x \in X$. Self Banach contractions, Kannan contractions and Ćirić-Reich-Rus contractions on complete metric spaces are nice examples of Picard operators (see [3], [4]).

The following concepts are important in our consideration, see [4].

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Definition 1. Let $f : X \rightarrow X$ be a weakly Picard operator and $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ an increasing function which is continuous in 0 and $\psi(0) = 0$. By definition the operator f is ψ -weakly Picard operator if

$$d(x, f^\infty(x)) \leq \psi(d(x, f(x))), \text{ for all } x \in X.$$

In particular, if $\psi(t) = c \cdot t$ with $c > 0$ then we say that f is c -weakly Picard operator.

For some examples of weakly Picard operators and ψ -weakly Picard operators see [1].

Example 1. Let (X, d) be a complete metric space and $f : X \rightarrow X$ an operator with closed graphic. We suppose that f is a graphic α -contraction, i.e.,

$$d(f^2(x), f(x)) \leq \alpha d(x, f(x)), \text{ for all } x \in X.$$

Then f is a c -weakly Picard operator, with $c = \frac{1}{1-\alpha}$.

Definition 2. Let (X, d) be a metric space. A function $f : X \rightarrow X$ is a φ -contraction if φ is increasing and $(\varphi^n(t)) \rightarrow 0, n \rightarrow \infty$ for all $t \geq 0$ and

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \text{ for all } x, y \in X.$$

Theorem 1 ([3]). Let (X, d) be a complete metric space and $f : X \rightarrow X$ a φ -contraction. Then f is a Picard operator.

Definition 3. Let (X, d) be a metric space and $f : X \rightarrow X$ be an operator. By definition, the fixed point equation

$$x = f(x) \tag{1.1}$$

is said to be generalized Hyers-Ulam stable if there exists a function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that is increasing, continuous in 0 with $\psi(0) = 0$, such that for each $\varepsilon > 0$ and each solution y^* of the inequation

$$d(y, f(y)) \leq \varepsilon \tag{1.2}$$

there exists a solution x^* of the equation (1.1) such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

If $\psi(t) = ct$, for each $t \in \mathbb{R}_+$ (for some $c > 0$), then the equation (1.1) is said to be Hyers-Ulam stable.

Theorem 2 (see [4]). Let (X, d) be a metric space. If $f : X \rightarrow X$ is a ψ -weakly Picard operator, then the fixed point equation (1.1) is generalized Hyers-Ulam stable. In particular, if f is c -weakly Picard operator, then the equation (1.1) is Hyers-Ulam stable.

Proof. Let $\varepsilon > 0$ and y^* a solution of (1.2). Since f is ψ -weakly Picard operator, we have that

$$d(x, f^\infty(x)) \leq \psi(d(x, f(x))), \text{ for all } x \in X.$$

If we take $x := y^*$ and $x^* := f^\infty(y)$, we have that $d(y^*, x^*) \leq \psi(\varepsilon)$. So, the fixed point equation (1.1) is generalized Hyers-Ulam stable. $\square$

Let (X, d) and (Y, ρ) be two metric spaces and $f, g : X \rightarrow Y$ two operators. Let us consider the following coincidence point problem

$$f(x) = g(x) \quad (1.3)$$

Definition 4 ([4]). Let (X, d) and (Y, ρ) be two metric spaces and $f, g : X \rightarrow Y$ be two operators. The coincidence problem (1.3) is called generalized Hyers-Ulam stable if and only if there exists $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that is increasing, continuous in 0 and $\psi(0) = 0$ such that for every $\varepsilon > 0$ and for each solution u^* of the inequality

$$\rho(f(u), g(u)) \leq \varepsilon \quad (1.4)$$

there exists a solution x^* of (1.3) such that

$$d(u^*, x^*) \leq \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$ then the coincidence point (1.3) is said to be Hyers-Ulam stable.

2. HYERS-ULAM STABILITY FOR COINCIDENCE EQUATIONS

Our main abstract result is an existence and Hyers-Ulam stability result for the coincidence point problem.

Theorem 3. Let $A \neq \emptyset$ be an arbitrary set and let (M, d) be a metric space. Let $S, T : A \rightarrow M$ such that $S(A) \subset T(A)$ and $(T(A), d)$ is a complete subspace of M . Suppose that there exists a function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with φ increasing and $(\varphi^n(t)) \rightarrow 0$, $n \rightarrow \infty$, for all $t \in \mathbb{R}_+$ such that

$$d(Sx, Sy) \leq \varphi(d(Tx, Ty)), \text{ for all } x, y \in A.$$

Then:

- a) $C(S, T) \neq \emptyset$;
- b) If additionally, we suppose that the function $\beta(t) := t - \varphi(t)$ is increasing and bijective and there exists $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ increasing, continuous in 0 and $\psi(0) = 0$ such that

$$d(y, S(T^{-1}(y))) \leq \psi(d(Ty, Sy)), \text{ for all } y \in T(A), \quad (2.1)$$

then the coincidence point problem (1.3) is $(\beta^{-1} \circ \psi)$ -generalized Hyers-Ulam stable.

Proof. a) The proof is organized in several steps. Let $f := S \circ T^{-1}$. We prove:

i) f is a singlevalued operator on $T(A)$;

Let $y_1, y_2 \in f(x)$. We get $y_1 \in S(T^{-1}(x))$ and $y_2 \in S(T^{-1}(x))$. So exists $u_1, u_2 \in T^{-1}(x)$ such that $y_1 = S(u_1)$ and $y_2 = S(u_2)$. Because $u_1, u_2 \in T^{-1}(x)$ we have $T(u_1) = x$ and $T(u_2) = x$. Then we have:

$$d(y_1, y_2) = d(Su_1, Su_2) \leq \varphi(d(Tu_1, Tu_2)) = \varphi(0).$$

Taking into account that φ is increasing and $(\varphi^n(t)) \rightarrow 0, n \rightarrow \infty$, for all $t \in \mathbb{R}_+$ we deduce that $\varphi(0) = 0$.

We get that $d(y_1, y_2) = 0$. So $y_1 = y_2$ and thus $f(x)$ is a singleton.

ii) $f : T(A) \rightarrow T(A)$;

Let $x \in T(A)$. Then exists $a \in A$ such that $x = T(a)$. So we have $a \in T^{-1}(x) \implies S(a) \subseteq S(T^{-1}(x)) \implies S(a) \subseteq f(x)$. Since f is a singlevalued operator we get $S(a) = f(x) \implies f(x) = S(a) \subseteq S(A) \subseteq T(A)$.

iii) $f : T(A) \rightarrow T(A)$ is a φ -contraction;

Let $x_1, x_2 \in T(A)$ and $u_1, u_2 \in A$ such that $u_1 \in T^{-1}(x_1)$ and $u_2 \in T^{-1}(x_2)$. Then we have:

$$\begin{aligned} d(f(x_1), f(x_2)) &= d(S(T^{-1}(x_1)), S(T^{-1}(x_2))) = d(Su_1, Su_2) \leq \\ &\leq \varphi(d(Tu_1, Tu_2)) = \varphi(d(x_1, x_2)). \end{aligned}$$

So f is self φ -contraction on the complete metric space $(T(A), d)$.

iv) We can apply now Theorem 1 for f and we deduce that f is a Picard operator. So we get that there exists a unique $y^* \in T(A)$ such that

$$y^* = f(y^*) = S(T^{-1}(y^*)).$$

Let $x^* = T^{-1}(y^*)$. Then $y^* = T(x^*)$ and so we get $y^* = S(x^*)$. Hence we conclude

$$S(x^*) = T(x^*) = y^*.$$

b) We prove that the coincidence point problem is generalized Hyers-Ulam stable.

Let $\varepsilon > 0$ and $v^* \in X$ be such that $d(T(v^*), S(v^*)) \leq \varepsilon$.

If we take into account of (2.1), we have

$$d(v^*, f(v^*)) = d(v^*, S(T^{-1}(v^*))) \leq \psi(d(S(v^*), T(v^*))) \leq \psi(\varepsilon).$$

So we get

$$\begin{aligned} d(v^*, y^*) &= d(v^*, f(y^*)) \\ &\leq d(f(v^*), v^*) + d(f(v^*), f(y^*)) \leq \psi(\varepsilon) + \varphi(d(v^*, y^*)). \end{aligned}$$

Then

$$\beta((d(v^*, y^*))) \leq \psi(\varepsilon)$$

Hence we get that

$$d(v^*, y^*) \leq (\beta^{-1} \circ \psi)(\varepsilon).$$

Thus, the coincidence point problem (1.3) is $(\beta^{-1} \circ \psi)$ - generalized Hyers-Ulam stable. $\square$

Remark 1. Our theorem is an extension of Goebel's Theorem (see [2]), which can be obtained from our result by taking $\varphi(t) = kt$ for each $t \in \mathbb{R}_+$ (for some $k \in [0, 1)$).

We will present now an application of Theorem 3.

Theorem 4. Consider the differential equation

$$x' = f(t, x) \quad (2.2)$$

with the initial condition

$$x(0) = \xi. \quad (2.3)$$

Suppose that the function f is defined in the half-plane $t \geq 0$, $-\infty < x < +\infty$ and satisfies following conditions:

- i) $f(t, x)$ is a continuous function of x for almost all $t \geq 0$;
- ii) $f(t, x)$ is a measurable function of t for all $x \in \mathbb{R}$;
- iii) Lipschitz inequality, i.e.

$$|f(t, x) - f(t, y)| \leq L(t)|x - y|,$$

where L is locally integrable function on the interval $(0, \infty)$;

- iv) $\int_0^t f(\tau, 0) d\tau = O(e^{\int_0^t L(\tau) d\tau})$;
- v) $f(t, \gamma u) \geq \gamma f(t, u)$ for all $\gamma \geq 1, t > 0, u \in \mathbb{R}$

Then, the differential equation (2.2) has, for every $\xi \in \mathbb{R}$, a unique solution and the equation (2.2) is Hyers-Ulam stable.

Proof. Let us consider the set

$$A = \{x \in C[0, \infty) : x(t) = O(e^{\int_0^t L(\tau) d\tau})\}.$$

We define the operators $S, T : A \rightarrow B$ by

$$(Sx)(t) = \left\{ \int_0^t f(\tau, x(\tau)) d\tau + \xi \right\} e^{-p \int_0^t L(\tau) d\tau},$$

$$(Tx)(t) = x(t) e^{-p \int_0^t L(\tau) d\tau},$$

where B is a Banach space of bounded continuous functions on $[0, \infty)$ with the norm $\|x\| = \sup_{(0, \infty)} |x(t)|$ and $p > 1$. By simple calculation we have

$$|(Sx)(t) - (Sy)(t)| \leq \frac{1}{p} \|Tx - Ty\|$$

and further $\|Sx - Sy\| \leq \frac{1}{p} \|Tx - Ty\|$. But $S(A) \subset T(A)$ and $T(A)$ is a complete subspace of B . By Theorem 3 there exists $\bar{x} \in A$ such that $S(\bar{x}) = T(\bar{x})$. From this we have

$$\bar{x}(t) = \int_0^t f(\tau, \bar{x}(\tau)) d\tau + \xi.$$

Since T is a single-valued operator, $\bar{x}$ is unique. Then the differential equation (2.2) has for every $\xi \in \mathbb{R}$ a unique solution with the initial condition $\bar{x}(0) = \xi$.

Next we prove that the equation (2.2) is Hyers-Ulam stable.

We have $(T^{-1}y)(t) = y(t) \cdot e^{p \int_0^t L(\tau) d\tau}$. We prove that $d(y, S(T^{-1}(y))) \leq \alpha d(Ty, Sy)$, for all $y \in T(A)$. We obtain that

$$\begin{aligned} S(T^{-1}(y))(t) &= \left\{ \int_0^t f(\tau, (T^{-1}y)(\tau)) d\tau + \xi \right\} \cdot e^{-p \int_0^t L(\tau) d\tau} = \\ &= e^{-p \int_0^t L(\tau) d\tau} \left\{ \int_0^t f(\tau, y(\tau)) e^{p \int_0^t L(\tau) d\tau} d\tau + \xi \right\}. \end{aligned}$$

By calculations we get

$$\begin{aligned} &|y(t) - S(T^{-1}(y))(t)| \\ &= \left| y(t) - e^{-p \int_0^t L(\tau) d\tau} \left\{ \int_0^t f(\tau, y(\tau)) e^{p \int_0^t L(\tau) d\tau} d\tau + \xi \right\} \right| \leq \\ &\leq \left| y(t) - e^{-p \int_0^t L(\tau) d\tau} \left\{ e^{p \int_0^t L(\tau) d\tau} \int_0^t f(\tau, y(\tau)) d\tau + \xi \right\} \right| = \\ &= \left| y(t) - \int_0^t f(\tau, y(\tau)) d\tau - e^{-p \int_0^t L(\tau) d\tau} \xi \right| = \\ &= \left| -e^{p \int_0^t L(\tau) d\tau} \left\{ -y(t) e^{-p \int_0^t L(\tau) d\tau} + \right. \right. \\ &\quad \left. \left. e^{-p \int_0^t L(\tau) d\tau} \int_0^t f(\tau, y(\tau)) d\tau + e^{-2p \int_0^t L(\tau) d\tau} \xi \right\} \right| \leq \\ &\leq e^{p \int_0^t L(\tau) d\tau} \left| -y(t) e^{-p \int_0^t L(\tau) d\tau} + \right. \\ &\quad \left. e^{-p \int_0^t L(\tau) d\tau} \int_0^t f(\tau, y(\tau)) d\tau + e^{-p \int_0^t L(\tau) d\tau} \xi \right| = \\ &= e^{p \int_0^t L(\tau) d\tau} |(Sy)(t) - (Ty)(t)|. \end{aligned}$$

Since, all the condition of Theorem 3 hold, then the equation (2.2) is Hyers-Ulam stable. $\square$

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NOTE ON THE q -STANCU-SCHURER OPERATOR

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Abstract. In this paper, we introduce a generalization of the Stancu- Schurer operators based on q -integers and get a Bohman-Korovkin type approximation theorem of these operators. We also compute the rate of convergence by using the first modulus of smoothness and give some numerical result of operators based on Matlab algorithms.

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1. PRELIMINARIES

After the construction of q -Bernstein operators by Lupas [13] in 1987 and then by Phillips [12] many authors studied new classes of q -generalized operators. To show the extend of this research direction, we mention in the following some achievements in this field. In [1] O. Agratini introduced a new class of q -Bernstein-type operators which fix certain polynomials and studied the limit of iterates of Lupas q -analogue of the Bernstein operators. In [3] Aral and Dođru obtained the uniform approximation of q -Bleimann- Butzer-Hahn (BBH) operators and in [5] O. Dođru and V. Gupta studied the monotonicity properties and the Voronovskaja type asymptotic estimate of these operators. See also the recent paper [2].

T. Trif [11] investigated Meyer-Kőnig and Zeller (MKZ) operators based on q -integers. Some approximation properties of q -MKZ operators were investigated by W. Heping in [10]. O. Dođru and O. Duman introduced also a new generalization of Meyer-Kőnig and Zeller operators and studied some statistical approximation properties in [4]. O. Dođru and Gupta [6] constructed a q -type generalization of Meyer-Kőnig and Zeller operators in bivariate case.

A new q -generalization of Meyer-Kőnig and Zeller type operators was constructed by Dođru and Muraru for improve the rate of convergence, see [7].

Gupta and Heping introduced a q -analogue of Bernstein-Durrmeyer operators in [9] and in 2009 Gupta and Finta [8] studied some local and global approximation properties for q -Durrmeyer operators. In [10] Gupta and Radu constructed a q -analogue of

Baskakov-Kantorovich operators and investigated their weighted statistical approximation properties.

For our study we remind first some elements of q-Calculus. For any fixed real number $q > 0$, the q-integer $[k]_q$, for $k \in N$ is defined by

$$[k]_q = \begin{cases} (1 - q^k)/(1 - q), & q \neq 1 \\ k, & q = 1 \end{cases}$$

The q-factorial integer and q-binomial coefficients are :

$$[k]_q! = \begin{cases} [k][k-1]\dots[1], & k = 1, 2, \dots \\ 1, & k = 0 \end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \quad (n \geq k \geq 0)$$

The q-analogue of $(x - a)^n$ is the polynomial

$$(x - a)_q^n = \begin{cases} 1, & \text{if } n = 0 \\ (x - a)(x - qa)\dots(x - q^{n-1}a), & \text{if } n \geq 1 \end{cases}$$

As usual, we note with $C([a, b])$, the space of all real valued continuous functions defined on $[a, b]$. The space is endowed with usual norm $\|\cdot\|$ given by $\|f\| = \sup_{x \in [a, b]} |f(x)|$.

Let $p \in N$ be fixed. In 1962 Schurer introduced and studied the Bernstein-Schurer operators $\tilde{B}_{m,p} : C([0, p+1]) \rightarrow C([0, 1])$ defined for any $m \in N$ and any function $f \in C([0, p+1])$ as follows

$$\tilde{B}_{m,p}(f; x) = \sum_{k=0}^{m+p} \binom{m+p}{k} x^k (1-x)^{m+p-k} f\left(\frac{k}{m}\right)$$

where $\tilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k}$ are the fundamental Schurer's polynomials.

Let p be a non-negative integer and let α, β be a real parameters satisfying the conditions $0 \leq \alpha \leq \beta$.

The Schurer-Stancu operator $\tilde{S}_{m,p}^{(\alpha,\beta)} : C([0, 1+p]) \rightarrow C([0, 1])$ is defined for any $m \in N^*$, any $f \in C([0, 1+p])$ and any $x \in [0, 1]$ by

$$\left(\tilde{S}_{m,p}^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right).$$

Lemma 1. For any $x \in [0, 1+p]$ the following relations

- (1) $\left(\tilde{S}_{m,p}^{(\alpha,\beta)} e_0\right)(x) = 1$
- (2) $\left(\tilde{S}_{m,p}^{(\alpha,\beta)} e_1\right)(x) = \frac{m+p}{m+\beta} x + \frac{\alpha}{m+\beta}$

$$(3) \left(\tilde{S}_{m,p}^{(\alpha,\beta)} e_2 \right)(x) = \frac{1}{(m+\beta)^2} \{ (m+p)^2 x^2 + (m+p)x(1-x) + 2\alpha(m+p)x + \alpha^2 \}$$

hold where $e_j(x) = x^j$ ($j \in N$) denotes like usually test monomials.

Our aim is to introduce a q -analogue of the above operators. We investigate the approximation properties of this class and we estimate the rate of convergence by using modulus of continuity .

2. APPROXIMATION PROPERTIES OF q -SCHURER-STANCU OPERATORS

We will consider in the following that $q \in (0, 1)$.

For any $m \in N$, p a fixed non negative number and α, β a real parameters satisfying the conditions $0 \leq \alpha \leq \beta$, we construct the class of generalized q -Schurer-Stancu operators $(S_{m,p,q}^{(\alpha,\beta)} f)(x) : C[0, 1+p] \rightarrow C[0, 1]$ as follows:

$$(S_{m,p,q}^{(\alpha,\beta)} f)(x) = \sum_{k=0}^{m+p} \left[\begin{matrix} m+p \\ k \end{matrix} \right]_q x^k \prod_{s=0}^{m+p-k-1} (1 - q^s x) f\left(\frac{[k+\alpha]_q}{[m+\beta]_q}\right)$$

for any $x \in [0, 1]$.

Lemma 2. For the polynomials defined above the following properties hold

$$\begin{aligned} (1) \quad & (S_{m,p,q}^{(\alpha,\beta)} e_0)(x) = 1 \\ (2) \quad & (S_{m,p,q}^{(\alpha,\beta)} e_1)(x) = \frac{[\alpha]_q}{[m+\beta]_q} + \frac{q^\alpha [m+p]_q}{[m+\beta]_q} x \\ (3) \quad & (S_{m,p,q}^{(\alpha,\beta)} e_2)(x) = \\ & = \frac{[\alpha]_q^2}{[m+\beta]_q^2} + \frac{q^{2\alpha} [m+p]_q^2}{[m+\beta]_q^2} x^2 + \frac{2q^\alpha [\alpha]_q [m+p]_q x}{[m+\beta]_q^2} + \frac{q^{2\alpha} [m+p]_q}{[m+\beta]_q^2} x(1-x). \end{aligned}$$

Proof. (1) We use the known identity

$$\sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_q x^k (1-x)^{n-k}_q = 1$$

which can be proved by induction with respect to n . The left hand side represents $(B_{n,q} e_0)(x)$ where $B_{n,q}$ is the q -analogue of Bernstein operator introduced by Phillips.

If we choose $n = m + p$, taking into account that

$$(1-x)^{m+p-k}_q = \prod_{s=0}^{m+p-k-1} (1 - q^s x)$$

we get

$$\sum_{k=0}^{m+p} \left[\begin{matrix} m+p \\ k \end{matrix} \right]_q x^k \prod_{s=0}^{m+p-k-1} (1 - q^s x) = 1$$

Consequently we obtain

$$(S_{m,p,q}^{(\alpha,\beta)} e_0)(x) = \sum_{k=0}^{m+p} \left[\begin{matrix} m+p \\ k \end{matrix} \right] x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) = 1.$$

$$\begin{aligned} (2) \quad (S_{m,p,q}^{(\alpha,\beta)} e_1)(x) &= \\ &= \sum_{k=0}^{m+p} \left[\begin{matrix} m+p \\ k \end{matrix} \right]_q x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) \frac{[k+\alpha]_q}{[m+\beta]_q} \\ &= \frac{1}{[m+\beta]_q} \sum_{k=0}^{m+p} \frac{[m+p]_q! [k+\alpha]_q}{[k]_q! [m+p-k]_q!} x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) \\ &= \frac{1}{[m+\beta]_q} \sum_{k=0}^{m+p} \frac{[m+p]_q! ([\alpha]_q + q^\alpha [k]_q)}{[k]_q! [m+p-k]_q!} x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) \\ &= \frac{[\alpha]_q}{[m+\beta]_q} + \frac{q^\alpha [m+p]_q}{[m+\beta]_q} x \sum_{k=1}^{m+p} \frac{[m+p-1]_q!}{[k-1]_q! [m+p-k]_q!} x^{k-1} \prod_{s=0}^{m+p-k-1} (1-q^s x) \\ &= \frac{[\alpha]_q}{[m+\beta]_q} + \frac{q^\alpha [m+p]_q}{[m+\beta]_q} x. \end{aligned}$$

We choose q a sequence that $q_m^\alpha \rightarrow 1$, for $m \rightarrow \infty$ and we have

$$(S_{m,p,q}^{(\alpha,\beta)} e_1)(x) \rightarrow x.$$

$$\begin{aligned} (3) \quad (S_{m,p,q}^{(\alpha,\beta)} e_2)(x) &= \\ &= \sum_{k=0}^{m+p} \left[\begin{matrix} m+p \\ k \end{matrix} \right]_q x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) \frac{[k+\alpha]_q^2}{[m+\beta]_q^2} \\ &= \frac{1}{[m+\beta]_q^2} \sum_{k=0}^{m+p} \frac{[m+p]_q! [k+\alpha]_q [k+\alpha]_q}{[k]_q! [m+p-k]_q!} x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) \\ &= \frac{1}{[m+\beta]_q^2} \sum_{k=0}^{m+p} \frac{[m+p]_q! ([\alpha]_q + q^\alpha [k]_q) [k+\alpha]_q}{[k]_q! [m+p-k]_q!} x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) \\ &= \frac{1}{[m+\beta]_q^2} \sum_{k=0}^{m+p} \frac{[m+p]_q! [\alpha]_q [k+\alpha]_q}{[m+p-k]_q! [k]_q!} x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) \\ &\quad + \frac{q^\alpha}{[m+\beta]_q^2} \sum_{k=0}^{m+p} \frac{[k]_q ([\alpha]_q + q^\alpha [k]_q) [m+p]_q!}{[m+p-k]_q! [k]_q!} x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) \end{aligned}$$

$$\begin{aligned}
&= \frac{[\alpha]}{[m+\beta]_q^2} \sum_{k=0}^{m+p} \frac{[m+p]_q! ([\alpha]_q + q^\alpha [k]_q)}{[m+p-k]_q! [k]_q!} x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) \\
&\quad + \frac{q^\alpha [\alpha]_q}{[m+\beta]_q^2} \sum_{k=0}^{m+p} \frac{[k]_q [m+p]_q!}{[m+p-k]_q! [k]_q!} x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) \\
&\quad + \frac{q^\alpha [\alpha]_q}{[m+\beta]_q^2} \sum_{k=0}^{m+p} \frac{[k]_q^2 [m+p]_q!}{[m+p-k]_q! [k]_q!} x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) \\
&= \frac{[\alpha]_q}{[m+\beta]_q^2} ([\alpha]_q + q^\alpha [m+p]_q x) + \frac{q^\alpha [\alpha]_q [m+p]_q x}{[m+\beta]_q^2} \\
&\quad + \frac{q^{2\alpha}}{[m+\beta]_q^2} \sum_{k=0}^{m+p} \frac{(q[k-1]_q + 1)[k]_q [m+p]_q!}{[m+p-k]_q! [k]_q!} x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) \\
&= \frac{[\alpha]_q}{[m+\beta]_q^2} ([\alpha]_q + q^\alpha [m+p]_q x) \\
&\quad + \frac{q^\alpha [\alpha]_q [m+p]_q x}{[m+\beta]_q^2} + \frac{q^{2\alpha}}{[m+\beta]_q^2} \sum_{k=0}^{m+p} \frac{(q[k-1]_q [k]_q [m+p]_q!)}{[m+p-k]_q! [k]_q!} x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) \\
&\quad + \frac{q^{2\alpha}}{[m+\beta]_q^2} \sum_{k=0}^{m+p} \frac{[k]_q [m+p]_q!}{[m+p-k]_q! [k]_q!} x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) \\
&= \frac{[\alpha]}{[m+\beta]_q^2} ([\alpha]_q + q^\alpha [m+p]_q x) + \frac{q^\alpha [\alpha]_q [m+p]_q x}{[m+\beta]_q^2} \\
&\quad + \frac{x^2 [m+p]_q [m+p-1]_q q^{2\alpha+1}}{[m+\beta]_q^2} + \frac{q^{2\alpha} [m+p]_q}{[m+\beta]_q^2} x \\
&= \frac{[\alpha]^2}{[m+\beta]_q^2} + \frac{q^{2\alpha} [m+p]_q^2}{[m+\beta]_q^2} x^2 + \frac{2q^\alpha [\alpha]_q [m+p]_q x}{[m+\beta]_q^2} + \frac{q^{2\alpha} [m+p]_q}{[m+\beta]_q^2} x(1-x).
\end{aligned}$$

We used that $[m+p-1]_q = [m+p]_q - 1$. □

3. ESTIMATION OF CONVERGENCE

We will estimate the rate of convergence in terms of modulus of continuity. Let $f \in C([0, b])$. The modulus of continuity of f denoted by $\omega_f(\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by the relation

$$\omega_f(\delta) = \sup_{|y-x| \leq \delta} |f(y) - f(x)|, \quad x, y \in [0, b].$$

It is well-known that for $f \in C([0, b])$

$$\lim_{\delta \rightarrow 0} \omega_f(\delta) = 0,$$

and for any $\delta > 0$

$$|f(y) - f(x)| \leq \omega_f(\delta) \left(\frac{|y - x|}{\delta} + 1 \right). \quad (3.1)$$

The next result is based on Popoviciu technique for the considered operators in the terms of first order modulus of continuity.

Theorem 1. *If $f \in C([0, 1 + p])$ then*

$$\left| (S_{m,p,q}^{(\alpha,\beta)} f)(x) - f(x) \right| \leq 2\omega_f(\delta_m) \quad (3.2)$$

takes place, when

$$\delta_m = \frac{1}{\sqrt{[m]_q}} \left(\frac{p}{1 - q^m} + \frac{1}{2} \sqrt{\frac{1 - q^{m+p}}{1 - q^m}} \right), \quad q \in (0, 1).$$

Proof. Since $\frac{[m+p]_q!}{[m+p-k]_q![k]_q!} > 0$, $x^k > 0$, $\prod_{s=0}^{m+p-k-1} (1 - q^s x) > 0$ we have

$$\begin{aligned} & \left| (S_{m,p,q}^{(\alpha,\beta)} f)(x) - f(x) \right| \leq \\ & \leq \sum_{k=0}^{m+p} \left| f\left(\frac{[k+\alpha]_q}{[m+p]_q}\right) - f(x) \right| \frac{[m+p]_q!}{[m+p-k]_q![k]_q!} x^k \prod_{s=0}^{m+p-k-1} (1 - q^s x) \end{aligned}$$

Using (3.1) we have

$$\begin{aligned} & \left| (S_{m,p,q}^{(\alpha,\beta)} f)(x) - f(x) \right| \leq \\ & \leq \omega_f(\delta) \left\{ \frac{1}{\delta} \sum_{k=0}^{m+p} \left| \frac{[k+\alpha]_q}{[m+p]_q} - x \right| \frac{[m+p]_q!}{[m+p-k]_q![k]_q!} x^k \prod_{s=0}^{m+p-k-1} (1 - q^s x) \right. \\ & \quad \left. + \sum_{k=0}^{m+p} \frac{[m+p]_q!}{[m+p-k]_q![k]_q!} x^k \prod_{s=0}^{m+p-k-1} (1 - q^s x) \right\} \\ & = \omega_f(\delta) \left\{ \frac{1}{\delta} \sum_{k=0}^{m+p} \left| \frac{[k+\alpha]_q}{[m+p]_q} - x \right| \frac{[m+p]_q!}{[m+p-k]_q![k]_q!} x^k \right. \\ & \quad \left. \cdot \prod_{s=0}^{m+p-k-1} (1 - q^s x) + (S_{m,p,q}^{(\alpha,\beta)} e_0)(x) \right\} \end{aligned}$$

Using Cauchy-Schwartz inequality we have

$$\begin{aligned} & \left| (S_{m,p,q}^{(\alpha,\beta)} f)(x) - f(x) \right| \leq \\ & \leq \omega_f(\delta) \left\{ \frac{1}{\delta} \left(\sum_{k=0}^{m+p} \left(\frac{[k+\alpha]_q}{[m+p]_q} - x \right)^2 \frac{[m+p]_q!}{[m+p-k]_q! [k]_q!} x^k \prod_{s=0}^{m+p-k} (1-q^s x) \right)^{1/2} + 1 \right\} \\ & \leq \omega_f(\delta) \left\{ \frac{1}{\delta} \left((S_{m,p,q}^{(\alpha,\beta)} e_2)(x) - 2x(S_{m,p,q}^{(\alpha,\beta)} e_1)(x) + x^2(S_{m,p,q}^{(\alpha,\beta)} e_0)(x) \right)^{1/2} + 1 \right\} \end{aligned}$$

Finally, taking into account that $0 < x < 1$

$$\begin{aligned} & \left| (S_{m,p,q}^{(\alpha,\beta)} f)(x) - f(x) \right| \leq \\ & \leq \omega_f(\delta) \left\{ \frac{1}{\delta} \left(\frac{[\alpha]^2}{[m+\beta]^2} + \frac{q^{2\alpha}[m+p]^2}{[m+\beta]^2} x^2 + \frac{2q^\alpha[\alpha][m+p]x}{[m+\beta]^2} + \frac{q^{2\alpha}[m+p]}{[m+\beta]^2} x(1-x) \right. \right. \\ & \quad \left. \left. - 2x \left(\frac{[\alpha]}{[m+\beta]} + xq^\alpha \frac{[m+p]}{[m+\beta]} \right) + x^2 \right)^{1/2} + 1 \right\} \\ & = \omega_f(\delta) \left\{ \frac{1}{\delta} \left(x^2 \left(q^\alpha \frac{[m+p]_q}{[m+\beta]_q} - 1 \right)^2 + q^{2\alpha} x(1-x) \frac{[m+p]_q}{[m+\beta]_q^2} \right. \right. \\ & \quad \left. \left. + 2x \frac{[\alpha]}{[m+\beta]} \left(q^\alpha \frac{[m+p]}{[m+\beta]} - 1 \right) + \frac{[\alpha]^2}{[m+\beta]^2} \right)^{1/2} + 1 \right\} \end{aligned} \quad (3.3)$$

Using the inequality $(a^2 + b^2)^{1/2} \leq |a| + |b|$ and that $x \in [0, 1]$, $x(1-x) \leq 1/4$ and $q^\alpha \leq 1$, the relation (3.3) becomes

$$\begin{aligned} & \left| (S_{m,p,q}^{(\alpha,\beta)} f)(x) - f(x) \right| \leq \\ & \leq \omega_f(\delta) \left\{ \frac{1}{\delta} \left(\left(q^\alpha \frac{[m+p]_q}{[m+\beta]_q} - 1 \right)^2 + \frac{1}{4} q^{2\alpha} \frac{[m+p]_q}{[m+\beta]_q^2} \right. \right. \\ & \quad \left. \left. + 2 \frac{[\alpha]_q}{[m+\beta]_q} \left(q^\alpha \frac{[m+p]_q}{[m+\beta]_q} - 1 \right) + \frac{[\alpha]_q^2}{[m+\beta]_q^2} \right)^{1/2} + 1 \right\} \\ & \leq \omega_f(\delta) \left\{ \frac{1}{\delta} \left(\left(q^\alpha \frac{[m+p]_q}{[m+\beta]_q} - 1 + \frac{[\alpha]_q}{[m+\beta]_q} \right)^2 + \frac{1}{4} q^{2\alpha} \frac{[m+p]_q}{[m+\beta]_q^2} \right)^{1/2} + 1 \right\} \\ & \leq \omega_f(\delta) \left\{ \frac{1}{\delta} \left(\left| q^\alpha \frac{[m+p]_q}{[m+\beta]_q} - 1 + \frac{[\alpha]}{[m+\beta]_q} \right| + \frac{1}{2} q^\alpha \frac{\sqrt{[m+p]_q}}{[m+\beta]_q} \right) + 1 \right\} \\ & \leq \omega_f(\delta) \left\{ \frac{1}{\delta} \left(\frac{q^m}{1-q^{m+\beta}} \left| q^\beta - q^{\alpha+p} \right| + \frac{1}{2} q^\alpha \frac{\sqrt{1-q} \sqrt{1-q^{m+p}}}{1-q^{m+\beta}} \right) + 1 \right\} \end{aligned}$$

$$\leq \omega_f(\delta) \left\{ \frac{1}{\delta} \left(\frac{q^m}{1-q^m} + \frac{1}{2} \frac{\sqrt{1-q} \sqrt{1-q^{m+p}}}{1-q^m} \right) + 1 \right\} \quad (3.4)$$

From $\frac{q^m(1-q^p)}{1-q^m} \leq \frac{p}{\sqrt{[m]_q}}$ it follows that

$$\frac{q^m}{1-q^m} \leq \frac{p}{(1-q^p)\sqrt{[m]_q}} \leq \frac{p}{(1-q^m)\sqrt{[m]_q}} \quad (p < m).$$

The relation (3.4) becomes

$$\begin{aligned} & \left| (S_{m,p,q}^{(\alpha,\beta)} f)(x) - f(x) \right| \leq \\ & \leq \omega_f(\delta) \left\{ \frac{1}{\delta} \left(\frac{p}{(1-q^m)\sqrt{[m]_q}} + \frac{1}{2} \frac{1}{\sqrt{[m]_q}} \sqrt{\frac{1-q^{m+p}}{1-q^m}} \right) + 1 \right\} \\ & = \omega_f(\delta) \left\{ \frac{1}{\delta} \left(\frac{1}{\sqrt{[m]}} \left(\frac{p}{1-q^m} + \frac{1}{2} \sqrt{\frac{1-q^{m+p}}{1-q^m}} \right) \right) + 1 \right\} \end{aligned}$$

Let $q = q_m$ satisfy $0 < q_m < 1$, $\lim_{m \rightarrow \infty} q_m = 1$ and $\lim_{m \rightarrow \infty} q_m^m = c < 1$.

Choosing $\delta = \delta_m = \left(\frac{1}{\sqrt{[m]}} \left(\frac{p}{1-q^m} + \frac{1}{2} \sqrt{\frac{1-q^{m+p}}{1-q^m}} \right) \right)$ as $\delta_m \rightarrow 0$ when $m \rightarrow \infty$ we obtain the desired result. $\square$

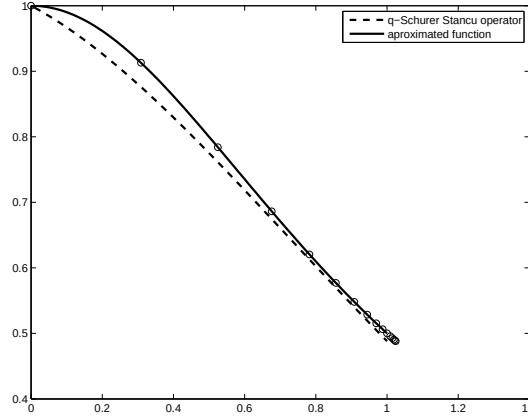


FIGURE 1. The approximated function $f(x) = 1/(1+x^2)$, $x \in (0, 1)$ with the q-Schurer-Stancu operator, $q=0.7$ $\alpha = 1/40$, $\beta = 1/6$, $m = 10$

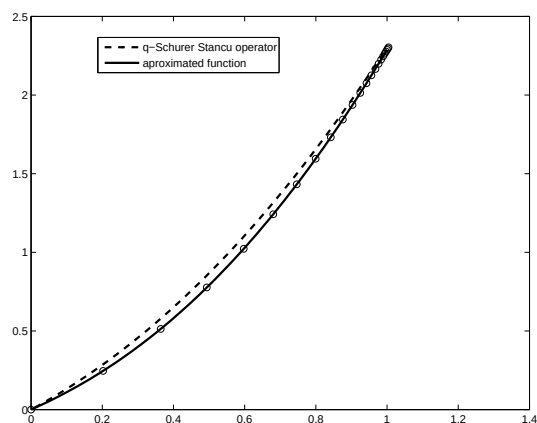


FIGURE 2. The approximated function $f(x) = \sin(x)e^x$ with the q -Schurer-Stancu operator, $x \in (0, 1)$, $q=0.8$, $\alpha = 1/40$, $\beta = 1/6$, $m = 20$

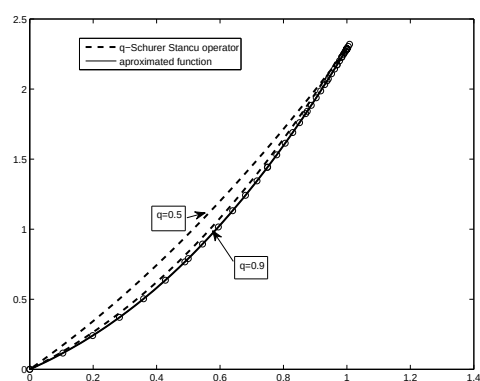


FIGURE 3.

In the following we give some numerical results regarding the approximation properties of q -Schurer-Stancu operators using Matlab algorithms for construction of operators.

In Fig.3 we notice that for the same function, when $0 < q < 1$ is increasing, the curve of operator is closer, as follows from the convergence properties of operators given in Section 2. We notice also that for $0 < \alpha, \beta < 1$ the operator is interpolator on ends.

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A NEW APPLICATION OF ALMOST INCREASING SEQUENCES

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Abstract. In the present paper we have proved a more general theorem dealing with $|A, p_n|_k$ summability by using almost increasing sequence. This theorem also includes several known results.

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Keywords: summability factors, absolute matrix summability, almost increasing sequence, infinite series

1. INTRODUCTION

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Let $\sum a_n$ be a given infinite series with partial sums (s_n) , and let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^n a_{nv}s_v, \quad n = 0, 1, \dots \quad (1.1)$$

The series $\sum a_n$ is said to be summable $|A|_k, k \geq 1$, if (see [9])

$$\sum_{n=1}^{\infty} n^{k-1} |\bar{\Delta} A_n(s)|^k < \infty, \quad (1.2)$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^n p_v \rightarrow \infty \quad \text{as } n \rightarrow \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \quad (1.3)$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v \quad (1.4)$$

defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n) , generated by the sequence of coefficients (p_n) (see [6]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k, k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |\Delta \sigma_{n-1}|^k < \infty, \quad (1.5)$$

and it is said to be summable $|A, p_n|_k, k \geq 1$, if (see [8])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |\bar{\Delta} A_n(s)|^k < \infty, \quad (1.6)$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

In the special case when $p_n = 1$ for all n , $|A, p_n|_k$ summability is the same as $|A|_k$ summability. Also if we take $a_{nv} = \frac{p_v}{P_n}$, then $|A, p_n|_k$ summability is the same as $|\bar{N}, p_n|_k$ summability.

In [5], Bor has proved the following theorem for $|\bar{N}, p_n|_k$ summability factors of infinite series.

Theorem 1. *Let (X_n) be an almost increasing sequence and let there be sequences (β_n) and (λ_n) such that*

$$|\Delta \lambda_n| \leq \beta_n, \quad (1.7)$$

$$\beta_n \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (1.8)$$

$$\sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty, \quad (1.9)$$

$$|\lambda_n| X_n = O(1). \quad (1.10)$$

and

$$\sum_{v=1}^n \frac{|t_v|^k}{v} = O(X_n) \quad \text{as } n \rightarrow \infty, \quad (1.11)$$

where (t_n) is the n -th $(C, 1)$ mean of the sequence (na_n) . Suppose further, the sequence (p_n) is such that

$$P_n = O(np_n), \quad (1.12)$$

$$P_n \Delta p_n = O(p_n p_{n+1}). \quad (1.13)$$

Then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is summable $|\bar{N}, p_n|_k, k \geq 1$.

Remark 1. It should be noted that, from the hypotheses of Theorem 1, (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$ (see [3]).

2. THE MAIN RESULT

The aim of this paper is to generalize Theorem 1 for absolute matrix summability. Before stating the main theorem we must first introduce some further notations. Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^n a_{ni}, \quad n, v = 0, 1, \dots \quad (2.1)$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \dots \quad (2.2)$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^n a_{nv}s_v = \sum_{v=0}^n \bar{a}_{nv}a_v \quad (2.3)$$

and

$$\bar{\Delta}A_n(s) = \sum_{v=0}^n \hat{a}_{nv}a_v. \quad (2.4)$$

Now, we shall prove the following theorem.

Theorem 2. Let (X_n) be an almost increasing sequence. The conditions (1.7)–(1.13) of Theorem 1 and

$$\sum_{v=1}^n \frac{p_v}{P_v} = O(X_n) \quad \text{as } n \rightarrow \infty, \quad (2.5)$$

are satisfied. If $A = (a_{nv})$ is a positive normal matrix such that

$$\bar{a}_{no} = 1, \quad n = 0, 1, \dots, \quad (2.6)$$

$$a_{n-1,v} \geq a_{nv}, \quad \text{for } n \geq v + 1, \quad (2.7)$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \quad (2.8)$$

$$|\hat{a}_{n,v+1}| = O(v |\Delta_v(\hat{a}_{nv})|), \quad (2.9)$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is summable $|A, p_n|_k, k \geq 1$.

It should be noted that if we take $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem 1. We need the following lemmas for the proof of our theorem.

Lemma 1. ([7]) If (X_n) an almost increasing sequence, then under the conditions (1.8)–(1.9) we have that

$$nX_n\beta_n = O(1), \quad (2.10)$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \quad (2.11)$$

Lemma 2 ([4]). If the conditions (1.12) and (1.13) are satisfied, then $\Delta(P_n/p_n n^2) = O(1/n^2)$.

Proof of Theorem 2. Let (T_n) denotes A-transform of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{n p_n}$. Then we have by (2.3) and (2.4)

$$\bar{\Delta}T_n = \sum_{v=1}^n \hat{a}_{nv} \frac{a_v P_v \lambda_v}{v p_v}.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned} \bar{\Delta}T_n &= \sum_{v=1}^n \hat{a}_{nv} \frac{v a_v P_v \lambda_v}{v^2 p_v} \\ &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} P_v \lambda_v}{v^2 p_v} \right) \sum_{r=1}^v r a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n^2 p_n} \sum_{r=1}^n r a_r \\ &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} P_v \lambda_v}{v^2 p_v} \right) (v+1) t_v + \frac{a_{nn} P_n \lambda_n}{n^2 p_n} (n+1) t_n \\ &= \frac{a_{nn} P_n \lambda_n}{n^2 p_n} (n+1) t_n + \sum_{v=1}^{n-1} \Delta_v (\hat{a}_{nv}) \frac{(v+1)}{v^2} \frac{P_v \lambda_v}{p_v} t_v \\ &\quad + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} P_v}{p_v} \Delta \lambda_v t_v \frac{(v+1)}{v^2} + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \Delta \left(\frac{P_v}{v^2 p_v} \right) t_v (v+1) \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}. \end{aligned}$$

Since

$$|T_n(1) + T_n(2) + T_n(3) + T_n(4)|^k \leq 4^k (|T_n(1)|^k + |T_n(2)|^k + |T_n(3)|^k + |T_n(4)|^k)$$

to complete the proof of the theorem it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4. \quad (2.12)$$

Firstly, by using Abel's transformation, we have that

$$\begin{aligned}
 \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,1}|^k &= O(1) \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{k-1} a_{nn}^k \left(\frac{P_n}{p_n} \right)^k |\lambda_n|^k \frac{|t_n|^k}{n^k} \\
 &= O(1) \sum_{n=1}^m |\lambda_n|^{k-1} |\lambda_n| \frac{|t_n|^k}{n} \\
 &= O(1) \sum_{n=1}^m |\lambda_n| \frac{|t_n|^k}{n} \\
 &= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^n \frac{|t_v|^k}{v} + O(1) |\lambda_m| \sum_{n=1}^m \frac{|t_n|^k}{n} \\
 &= O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| X_n + O(1) |\lambda_m| X_m \\
 &= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) |\lambda_m| X_m = O(1) \quad \text{as } m \rightarrow \infty,
 \end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 1.

Now, using the fact that $P_v = O(vp_v)$ by (1.12), we have that

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,2}|^k = O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_v| |t_v| \right)^k$$

Now, applying Hölder's inequality with indices k and k' , where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\begin{aligned}
 \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,2}|^k &= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_v|^k |t_v|^k \right) \\
 &\quad \times \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right)^{k-1} \\
 &= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_v|^k |t_v|^k \right) \\
 &= O(1) \sum_{v=1}^m |\lambda_v|^k |t_v|^k \sum_{n=v+1}^{m+1} |\Delta_v(\hat{a}_{nv})| \\
 &= O(1) \sum_{v=1}^m |\lambda_v| |t_v|^k a_{vv} = O(1) \sum_{v=1}^m |\lambda_v| |t_v|^k \frac{p_v}{P_v}
 \end{aligned}$$

$$\begin{aligned}
&= O(1) \sum_{v=1}^{m-1} \Delta |\lambda_v| \sum_{r=1}^v \frac{p_r}{P_r} |t_r|^k + O(1) |\lambda_m| \sum_{v=1}^m \frac{p_v}{P_v} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta \lambda_v| X_v + O(1) |\lambda_m| X_m \\
&= O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) |\lambda_m| X_m \\
&= O(1) \quad \text{as } m \rightarrow \infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 1.

Now, using Hölder's inequality we have that

$$\begin{aligned}
&\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,3}|^k \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\Delta \lambda_v| |t_v| \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| \beta_v |t_v|^k \right) \times \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| \beta_v \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| \beta_v |t_v|^k \right) \\
&= O(1) \sum_{v=1}^m \beta_v |t_v|^k \sum_{n=v+1}^{m+1} |\hat{a}_{n,v+1}| = O(1) \sum_{v=1}^m v \beta_v \frac{1}{v} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v \beta_v) \sum_{r=1}^v \frac{1}{r} |t_r|^k + O(1) m \beta_m \sum_{v=1}^m \frac{1}{v} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta(v \beta_v)| X_v + O(1) m \beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v |\Delta \beta_v| X_v + O(1) \sum_{v=1}^{m-1} \beta_{v+1} X_{v+1} + O(1) m \beta_m X_m \\
&= O(1) \quad \text{as } m \rightarrow \infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 1.

Finally, since $\Delta(\frac{P_v}{v^2 p_v}) = O(\frac{1}{v^2})$, as in $T_{n,1}$ we have that

$$\begin{aligned}
& \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |T_n(4)|^k \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\lambda_{v+1}| \frac{|t_v|}{v} \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\lambda_{v+1}|^k \frac{|t_v|^k}{v} \right) \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| \frac{1}{v} \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\lambda_{v+1}|^k \frac{|t_v|^k}{v} \right) \left(\sum_{v=1}^{n-1} |\Delta_v \hat{a}_{nv}| \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v \hat{a}_{nv}| |\lambda_{v+1}|^k \frac{|t_v|^k}{v} \right) \\
&= O(1) \sum_{v=1}^m \frac{|\lambda_{v+1}|}{v} |t_v|^k \sum_{n=v+1}^{m+1} |\hat{a}_{n,v+1}| \\
&= O(1) \sum_{v=1}^m \frac{|\lambda_{v+1}|}{v} |t_v|^k = O(1), \quad \text{as } m \rightarrow \infty
\end{aligned}$$

Therefore we get

$$\sum_{n=1}^m \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,r}|^k = O(1) \quad \text{as } m \rightarrow \infty, \quad \text{for } r = 1, 2, 3, 4.$$

This completes the proof of Theorem 2. $\square$

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A-STATISTICAL CONVERGENCE OF MITTAG-LEFFLER OPERATORS

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Abstract. In this paper we introduce the Mittag-Leffler operators, which includes the modified Szász-Mirakjan operators. We obtain the transformation properties and compute the rate of convergence by using modulus of continuity. Furthermore we give the A -statistical approximation theorem for these operators.

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1. INTRODUCTION

The function defined by [11]

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (z \in \mathbb{C}; \Re(\alpha) > 0)$$

is known as the Mittag-Leffler function. The two-index Mittag-Leffler function is defined by [14]

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (z, \beta \in \mathbb{C}; \Re(\alpha) > 0).$$

Note that $E_{\alpha, 1}(z) = E_{\alpha}(z)$ and

$$E_{1,1}(z) = e^z, \quad E_{1,2}(z) = \frac{e^z - 1}{z}, \quad E_{1,m+1}(z) = \frac{e^z - \sum_{k=0}^{m-1} \frac{z^k}{k!}}{z^m}.$$

Moreover, for $|z| < 2\pi$, we have

$$\frac{1}{E_{1,2}(z)} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad \frac{1}{E_{1,m+1}(z)} = \sum_{n=0}^{\infty} B_n^{(m)} \frac{z^n}{n!}$$

where the coefficients (B_n) are the familiar Bernoulli numbers and $(B_n^{(m)})$ are the generalized Bernoulli numbers (see [2]).

Let (b_n) be a sequence of positive real numbers and let $\beta > 0$ be fixed. For all $n \in \mathbb{N}$, we introduce the Mittag-Leffler operators by

$$L_n^{(\beta)}(f; x) = \frac{1}{E_{1, \beta}\left(\frac{nx}{b_n}\right)} \sum_{k=0}^{\infty} f\left(\frac{k}{n}b_n\right) \frac{(nx)^k}{b_n^k \Gamma(k + \beta)}, \quad (1.1)$$

where $f \in E := \left\{ f \in C[0, +\infty) : \lim_{x \rightarrow +\infty} \frac{f(x)}{1+x^2} \text{ is finite} \right\}$ and $C[0, +\infty)$ denotes the space of continuous functions defined on $[0, +\infty)$. Recall that the Banach lattice E is endowed with the norm

$$\|f\|_* := \sup_{x \in [0, +\infty)} \frac{|f(x)|}{1+x^2}.$$

It is obvious that the operators $L_n^{(\beta)}(f; x)$ defined in (1.1) are linear and positive.

Note that for $\beta = 1$, we have

$$L_n^{(1)}(f; x) = e^{-nx/b_n} \sum_{k=0}^{\infty} f\left(\frac{k}{n}b_n\right) \frac{(nx)^k}{b_n^k k!} = S_n(f; x)$$

where the operators S_n are the modified Szász-Mirakjan operators considered in [1].

By direct computations one can state the following lemma;

Lemma 1. Let $\psi_x^2(t) = (t-x)^2$. Then, for each $x \geq 0$ and $n \in \mathbb{N}$, we have

$$(a) \quad L_n^{(\beta)}(1; x) = 1,$$

$$(b) \quad \left| L_n^{(\beta)}(t; x) - x \right| \leq \frac{|1-\beta|b_n}{n},$$

$$(c)$$

$$\begin{aligned} \left| L_n^{(\beta)}(t^2; x) - x^2 \right| &\leq \frac{(2|1-\beta|+1)b_n}{n}x \\ &\quad + \frac{(2(1-\beta)^2 + |1-\beta| + |1-\beta||\beta-2|)b_n^2}{n^2} \end{aligned}$$

$$(d)$$

$$\begin{aligned} L_n^{(\beta)}(\psi_x^2; x) &\leq \frac{(4|1-\beta|+1)b_n}{n}x \\ &\quad + \frac{(2(1-\beta)^2 + |1-\beta| + |1-\beta||\beta-2|)b_n^2}{n^2}. \end{aligned}$$

Proof. Since

$$\sum_{k=0}^{\infty} \frac{(nx)^k}{b_n^k \Gamma(k+\beta)} = E_{1,\beta} \left(\frac{nx}{b_n} \right),$$

then $L_n^{(\beta)}(1; x) = 1$. Using the fact that $\Gamma(k+\beta) = (k+\beta-1)\Gamma(k+\beta-1)$, we get

$$\begin{aligned} L_n^{(\beta)}(t; x) &= \frac{1}{E_{1,\beta} \left(\frac{nx}{b_n} \right)} \sum_{k=1}^{\infty} \frac{k b_n}{n} \frac{(nx)^k}{b_n^k \Gamma(k+\beta)} \\ &= \frac{1}{E_{1,\beta} \left(\frac{nx}{b_n} \right)} \sum_{k=1}^{\infty} \frac{[(k+\beta-1)+1-\beta] b_n}{n} \frac{(nx)^k}{b_n^k (k+\beta-1) \Gamma(k+\beta-1)} \\ &= x + \frac{1}{E_{1,\beta} \left(\frac{nx}{b_n} \right)} \sum_{k=1}^{\infty} \frac{1-\beta}{n} \frac{b_n (nx)^k}{b_n^k \Gamma(k+\beta)}. \end{aligned} \quad (1.2)$$

Hence

$$\left| L_n^{(\beta)}(t; x) - x \right| = \frac{|1-\beta| b_n}{n} \frac{1}{E_{1,\beta} \left(\frac{nx}{b_n} \right)} \sum_{k=1}^{\infty} \frac{(nx)^k}{b_n^k \Gamma(k+\beta)} \leq \frac{|1-\beta| b_n}{n}.$$

Similarly, by $k(k-1) = (k+\beta-1)(k+\beta-2) + 2(1-\beta)k + (1-\beta)(\beta-2)$ and $\Gamma(k+\beta) = (k+\beta-1)(k+\beta-2)\Gamma(k+\beta-2)$, we get

$$\begin{aligned} L_n^{(\beta)}(t^2; x) &= \frac{1}{E_{1,\beta} \left(\frac{nx}{b_n} \right)} \sum_{k=1}^{\infty} \left(\frac{k}{n} b_n \right)^2 \frac{(nx)^k}{b_n^k \Gamma(k+\beta)} \\ &= \frac{1}{E_{1,\beta} \left(\frac{nx}{b_n} \right)} \sum_{k=1}^{\infty} \frac{(k(k-1)+k) b_n^2}{n^2} \frac{(nx)^k}{b_n^k \Gamma(k+\beta)} \\ &= \frac{1}{E_{1,\beta} \left(\frac{nx}{b_n} \right)} \sum_{k=2}^{\infty} \frac{k(k-1) b_n^2}{n^2} \frac{(nx)^k}{b_n^k \Gamma(k+\beta)} + \frac{b_n L_n^{(\beta)}(t; x)}{n} \\ &= \frac{1}{E_{1,\beta} \left(\frac{nx}{b_n} \right)} \sum_{k=2}^{\infty} \frac{(k+\beta-1)(k+\beta-2) b_n^2}{n^2} \\ &\quad \times \frac{(nx)^k}{b_n^k (k+\beta-1)(k+\beta-2) \Gamma(k+\beta-2)} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}\right)} \sum_{k=2}^{\infty} \frac{2(1-\beta)kb_n^2}{n^2} \frac{(nx)^k}{b_n^k \Gamma(k+\beta)} \\
& + \frac{(1-\beta)(\beta-2)b_n^2}{n^2 E_{1,\beta}\left(\frac{nx}{b_n}\right)} \sum_{k=2}^{\infty} \frac{(nx)^k}{b_n^k \Gamma(k+\beta)} + \frac{b_n L_n^{(\beta)}(t;x)}{n}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\left| L_n^{(\beta)}(t^2;x) - x^2 \right| & \leq \frac{2|1-\beta|b_n}{n E_{1,\beta}\left(\frac{nx}{b_n}\right)} \sum_{k=2}^{\infty} \frac{kb_n}{n} \frac{(nx)^k}{b_n^k \Gamma(k+\beta)} \\
& + \frac{|1-\beta||\beta-2|b_n^2}{n^2 E_{1,\beta}\left(\frac{nx}{b_n}\right)} \sum_{k=2}^{\infty} \frac{(nx)^k}{b_n^k \Gamma(k+\beta)} + \frac{b_n \left| L_n^{(\beta)}(t;x) \right|}{n} \\
& \leq \frac{(2|1-\beta|+1)b_n}{n} \left| L_n^{(\beta)}(t;x) \right| + \frac{|1-\beta||\beta-2|b_n^2}{n^2}.
\end{aligned}$$

Using (1.2), we obtain

$$\begin{aligned}
\left| L_n^{(\beta)}(t^2;x) - x^2 \right| & \leq \frac{(2|1-\beta|+1)b_n}{n} x \\
& + \frac{\left(2(1-\beta)^2 + |1-\beta| + |1-\beta||\beta-2| \right) b_n^2}{n^2}.
\end{aligned}$$

Finally,

$$\begin{aligned}
& L_n^{(\beta)}(\psi_x^2;x) \\
& \leq \left| L_n^{(\beta)}(t^2;x) - x^2 \right| + 2x \left| L_n^{(\beta)}(t;x) - x \right| + x^2 \left| L_n^{(\beta)}(1;x) - 1 \right| \\
& \leq \frac{(4|1-\beta|+1)b_n}{n} x + \frac{\left(2(1-\beta)^2 + |1-\beta| + |1-\beta||\beta-2| \right) b_n^2}{n^2}
\end{aligned}$$

which completes the proof. $\square$

We organize the paper as follows: In Section 2, we give the transformation properties of the operators $L_n^{(\beta)}$ and compute the rate of convergence by using the modulus of continuity. In Section 3, we prove an A -statistical Korovkin type approximation theorem.

2. TRANSFORMATION PROPERTIES AND RATE OF CONVERGENCE

We start with the following lemma, which proves that $L_n^{(\beta)}$ maps E into itself.

Lemma 2. Let $\left(\frac{b_n}{n}\right)$ be a bounded sequence of positive numbers and $\beta > 0$ be fixed. Then there exists a constant $M(\beta)$ such that, for $w(x) = (1 + x^2)^{-1}$, we have

$$w(x)L_n^{(\beta)}\left(\frac{1}{w}; x\right) \leq M(\beta)$$

holds for all $x \in [0, \infty)$ and $n \in \mathbb{N}$. Furthermore, for all $f \in E$, we have

$$\left\|L_n^{(\beta)}(f)\right\|_* \leq M(\beta) \|f\|_*.$$

Proof. Using Lemma 1, we can write that

$$\begin{aligned} w(x)L_n^{(\beta)}\left(\frac{1}{w}; x\right) &= \frac{1}{1+x^2} \left[L_n^{(\beta)}(1; x) + L_n^{(\beta)}(t^2; x) \right] \\ &\leq \frac{1}{1+x^2} \left[1 + x^2 + \frac{(2|1-\beta|+1)b_n}{n}x \right. \\ &\quad \left. + \frac{(2(1-\beta)^2 + |1-\beta| + |1-\beta||\beta-2|)b_n^2}{n^2} \right] \\ &\leq M(\beta). \end{aligned}$$

On the other hand

$$w(x) \left| L_n^{(\beta)}(f; x) \right| = w(x) \left| L_n^{(\beta)}\left(w \frac{f}{w}; x\right) \right| \leq \|f\|_* w(x) L_n^{(\beta)}\left(\frac{1}{w}; x\right) \leq M(\beta) \|f\|_*.$$

Taking supremum over $x \in [0, \infty)$ in the above inequality, gives the result. $\square$

Now, recall that the usual modulus of continuity of f on the closed interval $[0, B]$ is defined by

$$\omega_B(f, \delta) = \sup_{\substack{|t-x| \leq \delta \\ x, t \in [0, B]}} |f(t) - f(x)|.$$

It is well known that, for a function $f \in E$, we have $\lim_{\delta \rightarrow \infty} \omega_B(f, \delta) = 0$.

The next theorem gives the rate of convergence of the operators $L_n^{(\beta)}(f; x)$ to $f(x)$, for all $f \in E$.

Theorem 1. Let $\beta > 0$ be fixed, $\left(\frac{b_n}{n}\right)$ be a bounded sequence of positive numbers, $f \in E$ and $\omega_{B+1}(f, \delta)$ ($B > 0$) be its modulus of continuity on the finite interval $[0, B+1] \subset [0, \infty)$. Then

$$\left\|L_n^{(\beta)}(f; x) - f(x)\right\|_{C[0, B]} \leq M_f(\beta, B) \delta_n(\beta, B) + 2\omega_{B+1}(f, \delta_n^{1/2}(\beta, B))$$

where $\delta_n(\beta, B) = N_f(\beta, B) \frac{b_n}{n} \left[1 + \frac{b_n}{n} \right]$,

$$N_f(\beta, B) = \max \left\{ (4|1-\beta|+1)B, \left(2(1-\beta)^2 + |1-\beta| + |1-\beta||\beta-2| \right) \right\}$$

and $M_f(\beta, B)$ is an absolute constant depending on f, β and B .

Proof. Let $\beta > 0$ be fixed. For $x \in [0, B]$ and $t \leq B+1$, we have the inequality

$$|f(t) - f(x)| \leq \omega_{B+1}(f, |t-x|) \leq \left(1 + \frac{|t-x|}{\delta} \right) \omega_{B+1}(f, \delta) \quad (2.1)$$

where $\delta > 0$. On the other hand, for $x \in [0, B]$ and $t > B+1$, using the fact that $t-x > 1$, we have

$$|f(t) - f(x)| \leq A_f(1+x^2+t^2) \leq A_f(2+3x^2+2(t-x)^2) \leq 6A_f(1+B^2)(t-x)^2 \quad (2.2)$$

By (2.1) and (2.2), we get for all $x \in [0, B]$ and $t \geq 0$ that

$$|f(t) - f(x)| \leq 6A_f(1+B^2)(t-x)^2 + \left(1 + \frac{|t-x|}{\delta} \right) \omega_{B+1}(f, \delta).$$

Therefore

$$\begin{aligned} & \left| L_n^{(\beta)}(f; x) - f(x) \right| \\ & \leq 6A_f(1+B^2) L_n^{(\beta)}((t-x)^2; x) + \left(1 + \frac{L_n^{(\beta)}(|t-x|; x)}{\delta} \right) \omega_{B+1}(f, \delta). \end{aligned}$$

Applying Cauchy-Schwarz inequality and Lemma 1, we get

$$\begin{aligned} & \left| L_n^{(\beta)}(f; x) - f(x) \right| \\ & \leq 6A_f(1+B^2) L_n^{(\beta)}(\psi_x^2; x) + \left(1 + \frac{[L_n^{(\beta)}(\psi_x^2; x)]^{1/2}}{\delta} \right) \omega_{B+1}(f, \delta) \\ & \leq 6A_f(1+B^2) \\ & \quad \times \left[(4|1-\beta|+1)B \frac{b_n}{n} + \left(2(1-\beta)^2 + |1-\beta| + |1-\beta||\beta-2| \right) \frac{b_n^2}{n^2} \right] \\ & \quad + \left(1 + \frac{\left[(4|1-\beta|+1)B \frac{b_n}{n} + \left(2(1-\beta)^2 + |1-\beta| + |1-\beta||\beta-2| \right) \frac{b_n^2}{n^2} \right]^{1/2}}{\delta} \right) \\ & \quad \times \omega_{B+1}(f, \delta) \leq M_f(\beta, B) \delta_n(\beta, B) + 2\omega_{B+1}(f, (\delta_n(\beta, B))^{1/2}), \end{aligned}$$

where

$$N_f(\beta, B) = \max \left\{ (4|1-\beta|+1)B, \left(2(1-\beta)^2 + |1-\beta| + |1-\beta||\beta-2| \right) \right\},$$

$$M_f(\beta, B) = 6A_f(1+B^2) \text{ and } \delta_n(\beta, B) = N_f(\beta, B) \frac{b_n}{n} \left[1 + \frac{b_n}{n} \right].$$

Whence the result follows. $\square$

3. A-STATISTICAL CONVERGENCE

Recently, A -statistical convergence of linear positive operators have been an active research area (see [3–5, 12]). We start to this section by recalling concepts of A -statistical convergence.

Let $A = (a_{jk})$ be a non-negative regular summability matrix.

Definition 1. The A -density of a subset K of $\mathbb{N}$ is given by

$$\delta_A(K) = \lim_j \sum_{k \in K} a_{j,k}, \quad (3.1)$$

provided that limit exists (see [7]).

Definition 2. A sequence $x = (x_n)$ is said to be A -statistically convergent to l and denoted by $st_A\text{-}\lim x = l$ if for every $\varepsilon > 0$, $\delta_A \{n \in \mathbb{N} : |x_n - l| \geq \varepsilon\} = 0$ (see [6, 13]).

Taking $A = C_1$, the Cesaro matrix of order one in (3.1), A -statistical convergence reduces to statistical convergence [8, 10]. Taking $A = I$, the identity matrix then A -statistical convergence reduces to ordinary convergence. Kolk [9] proved that in the case of $\lim_j \max_n |a_{j,n}| = 0$, A -statistical convergence is stronger than ordinary convergence.

Now let $A = (a_{jn})$ be a non-negative regular summability matrix. Assume that $(b_n)_{n \in \mathbb{N}}$ is a sequence in $[0, \infty)$ satisfying

$$st_A\text{-}\lim_n \frac{b_n}{n} = 0. \quad (3.2)$$

Then we have

$$st_A\text{-}\lim_n \left(\frac{b_n}{n} \right)^2 = 0. \quad (3.3)$$

Such a sequence $(b_n)_{n \in \mathbb{N}}$ satisfying (3.2), can be constructed as follows: Take $A = C_1$, and define

$$b_n := \begin{cases} n, & \text{if } n = m^2 \ (m \in \mathbb{N}) \\ n^{1/3}, & \text{otherwise.} \end{cases} \quad (3.4)$$

Then clearly $st_{C_1}\text{-}\lim \frac{b_n}{n} = st\text{-}\lim \frac{b_n}{n} = 0$.

Theorem 2. Let $A = (a_{jk})$ be a non-negative regular summability matrix and $\beta > 0$ be fixed. If

$$st_A\text{-}\lim_n \frac{b_n}{n} = 0$$

then

$$st_A\text{-}\lim_n \left\| L_n^{(\beta)}(f; x) - f(x) \right\|_{C[0, B]} = 0$$

holds for every $f \in E$.

Proof. Given $r > 0$ choose $\varepsilon > 0$ such that $\varepsilon < r$. For fixed $\beta > 0$, define the following sets:

$$\begin{aligned} U &:= \{n : \delta_n(\beta, B) \geq r\}, \\ U_1 &:= \left\{n : \frac{b_n}{n} \geq \frac{r - \varepsilon}{2N_f(\beta, B)}\right\}, \\ U_2 &:= \left\{n : \left(\frac{b_n}{n}\right)^2 \geq \frac{r - \varepsilon}{2N_f(\beta, B)}\right\}, \end{aligned}$$

where $N_f(\beta, B)$ and $\delta_n(\beta, B)$ be the same as in Theorem 1. Then it is clear that $U \subseteq U_1 \cup U_2$, which gives

$$\sum_{k \in U} a_{jk} \leq \sum_{k \in U_1} a_{jk} + \sum_{k \in U_2} a_{jk}. \quad (3.5)$$

Letting $j \rightarrow \infty$ in (3.5) and using (3.2) and (3.3), we have $\lim_j \sum_{k \in U} a_{jk} = 0$. This proves that $st_A\text{-}\lim_n \delta_n(\beta, B) = 0$ which also implies

$$st_A\text{-}\lim_n \omega_{B+1}(f, \delta_n^{1/2}(\beta, B)) = 0.$$

Using Theorem 1 we get the result. $\square$

Remark that choosing the sequence $(b_n)_{n \in \mathbb{N}}$ as in (3.4), the statistical approximation results in Theorem 2 works, however its classical case does not work since $\left(\frac{b_n}{n}\right)_{n \in \mathbb{N}}$ is not convergent in the ordinary sense.

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THE VORONOVSKAJA TYPE THEOREM FOR A GENERAL CLASS OF SZÁSZ-MIRAKJAN OPERATORS

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Abstract. The paper is devoted to defining a new general class of linear and positive operators depending on a certain function φ . This new class of linear and positive operators generalizes the Szász-Mirakjan operators. For this new class of operators we establish a convergence theorem and the evaluation of the rate of convergence, in terms of the modulus of continuity.

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1. INTRODUCTION

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In this section we recall some results from [12], which we shall use in the present paper.

Let I, J be real intervals and $I \cap J \neq \emptyset$. For any $n, k \in \mathbb{N}_0$, $n \neq 0$ consider the functions $\varphi_{n,k} : J \rightarrow \mathbb{R}$, with the property that $\varphi_{n,k}(x) \geq 0$, for any $x \in J$ and the linear positive functionals $A_{n,k} : E(I) \rightarrow \mathbb{R}$.

For any $n \in \mathbb{N}$ define the operator $L_n : E(I) \rightarrow F(J)$, by

$$(L_n f)(x) = \sum_{k=0}^{\infty} \varphi_{n,k}(x) A_{n,k}(f), \quad (1.1)$$

where $E(I)$ is a linear space of real-valued functions defined on I , for which the operators (1.1) are convergent and $F(J)$ is a subset of the set of real-valued functions defined on J .

Remark 1. [12] The operators $(L_n)_{n \in \mathbb{N}}$ are linear and positive on $E(I \cap J)$.

For $n, i \in \mathbb{N}_0$, $n \neq 0$, let the function ψ_x be defined by $\psi_x(t) = t - x$, $(x, t) \in [0, +\infty[\times [0, +\infty[$ and define $T_{n,i}^*$ by

$$(T_{n,i}^* L_n)(x) = n^i (L_n \psi_x^i)(x) = n^i \sum_{k=0}^{\infty} \varphi_{n,k}(x) A_{n,k}(\psi_x^i), \quad x \in I \cap J. \quad (1.2)$$

In what follows $s \in \mathbb{N}_0$ is even and we suppose that the following two conditions hold:

- there exists the smallest $\alpha_s, \alpha_{s+2} \in [0, +\infty[$, so that

$$\lim_{n \rightarrow \infty} \frac{(T_{n,j}^* L_n)(x)}{n^{\alpha_j}} = B_j(x) \in \mathbb{R}, \quad (1.3)$$

for any $x \in I \cap J$ and $j \in \{s, s+2\}$,

$$\alpha_{s+2} < \alpha_s + 2 \quad (1.4)$$

- $I \cap J$ is an interval.

Theorem 1 ([12]). *Let $f \in E(I)$ be a function. If $x \in I \cap J$ and f is s times differentiable in a neighborhood of x , $f^{(s)}$ is continuous in x , then*

$$\lim_{n \rightarrow \infty} n^{s-\alpha_s} \left((L_n f)(x) - \sum_{i=0}^s \frac{f^{(i)}(x)}{n^i i!} (T_{n,i}^* L_n)(x) \right) = 0. \quad (1.5)$$

Assume that f is s times differentiable on I and there exists an interval $K \subset I \cap J$ such that, there exists $n(s) \in \mathbb{N}$ and the constants $k_j \in \mathbb{R}$ depending on K , so that for $n \geq n(s)$ and $x \in K$, the following

$$\frac{(T_{n,j}^* L_n)(x)}{n^{\alpha_j}} \leq k_j \quad (1.6)$$

holds, for $j \in \{s, s+2\}$.

Then, the convergence expressed by (1.5) is uniform on K and

$$\begin{aligned} n^{s-\alpha_s} \left| (L_n f)(x) - \sum_{i=0}^s \frac{f^{(i)}(x)}{n^i i!} (T_{n,i}^* L_n)(x) \right| \\ \leq \frac{1}{s!} (k_s + k_{s+2}) \omega_1 \left(f^{(s)}; \frac{1}{\sqrt{n^{2+\alpha_s-\alpha_{s+2}}}} \right), \end{aligned} \quad (1.7)$$

for any $x \in K$, $n \geq n(s)$, where $\omega_1(f; \delta)$ denotes the modulus of continuity [1], (the first order modulus of smoothness) of the function f .

In [11], C. Mortici defined the sequence of operators

$$\varphi S_n : C^2([0, +\infty]) \rightarrow C^\infty([0, +\infty]),$$

given by

$$(\varphi S_n f)(x) = \frac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k f\left(\frac{k}{n}\right), \quad (1.8)$$

for any $x \in [0, +\infty[$ and $n \in \mathbb{N}$, where $\varphi : \mathbb{R} \rightarrow]0, +\infty[$ is an analytic function. These are called the φ -Szász-Mirakjan operators, because in the case when $\varphi(y) = e^y$, they reduce to the classical Mirakjan-Favard-Szász operators [5, 10] and [15].

Remark 2. Similar generalization of this type are the operators defined and studied by Jakimovski and Leviatan [7] or the operators defined by Baskakov in 1957 (see, e.g., the book [2], subsection 5.3.11, p. 344, where they are attributed to Mastroianni). Another recently similar generalization of this type are the operators defined and studied by authors in [3, 4, 8, 9] and [13].

Remark 3. The classical Mirakjan-Favard-Szász operators $S_n : C_2([0, +\infty]) \rightarrow C([0, +\infty])$ are defined by

$$(S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad (1.9)$$

where

$$C_2([0, +\infty]) = \left\{ f \in C([0, +\infty]) : \lim_{x \rightarrow \infty} \frac{f(x)}{1+x^2} \text{ exists and is finite} \right\}.$$

In the following, we shall use the classical definition of Mirakjan-Favard-Szász operators, i.e., $f \in C_2([0, +\infty])$.

The purpose of this paper is to introduce a new linear positive operators of Szász-Mirakjan type, in order to use it in the theory of uniform approximation of functions, which generalize some new, respectively older results, cited at the adequate moment. We shall prove uniform convergence, Voronovskaja type formulas and the order of approximation, for these new linear positive operators.

2. AUXILIARY RESULTS

We consider an analytic function $\varphi : [0, +\infty[\rightarrow]0, +\infty[$ and it follows

$$\varphi(x) = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} x^k. \quad (2.1)$$

By differentiation of the relation (2.1), we get

$$\varphi^{(1)}(x) = \sum_{k=1}^{\infty} \frac{\varphi^{(k)}(0)}{(k-1)!} x^{k-1} \quad (2.2)$$

and

$$\varphi^{(2)}(x) = \sum_{k=2}^{\infty} \frac{\varphi^{(k)}(0)}{(k-2)!} x^{k-2}. \quad (2.3)$$

We consider the function $\alpha : [0, +\infty[\rightarrow \mathbb{R}$ defined by

$$\alpha(x) = \frac{\varphi^{(1)}(x)}{\varphi(x)} - 1, \quad (2.4)$$

for any $x \in [0, +\infty[$.

Remark 4. From (2.4) it follows that α is also an analytic function.

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence, so that for any $n \in \mathbb{N}$, $a_n > 0$ and we assume that the function φ has the properties

$$\lim_{x \rightarrow \infty} \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} = 1 \quad (2.5)$$

and

$$\lim_{x \rightarrow \infty} \left(\frac{a_n}{n} \right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} = 1, \quad (2.6)$$

where $x \in [0, +\infty[$.

Lemma 1. *The following identity*

$$\varphi(x) = c \cdot e^{x + \int_0^x \alpha(t) dt} \quad (2.7)$$

holds, for any $x \in [0, +\infty[$, where $c > 0$.

Proof. From (2.4), we get

$$\varphi^{(1)}(x) - (1 + \alpha(x))\varphi(x) = 0,$$

for any $x \in [0, +\infty[$. This relation is equivalent to

$$\left(\varphi(x) \cdot e^{-x - \int_0^x \alpha(t) dt} \right)^{(1)} = 0,$$

for any $x \in [0, +\infty[$. Because $\varphi : [0, +\infty[\rightarrow]0, +\infty[$ exists $c > 0$, so that

$$\varphi(x) \cdot e^{-x - \int_0^x \alpha(t) dt} = c,$$

for any $x \in [0, +\infty[$.

The relation (2.7) follows immediately from the above equality. $\square$

3. THE φ -SZÁSZ-MIRAKJAN TYPE OPERATORS

We define the sequence of operators $\varphi S_n : C_2([0, +\infty]) \rightarrow C([0, +\infty])$, given by

$$(\varphi S_n f)(x) = \frac{1}{\varphi(a_n x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k f\left(\frac{k}{n}\right), \quad (3.1)$$

for any $x \in [0, +\infty[$ and $n \in \mathbb{N}$. The operators (3.1) are called the φ -Szász-Mirakjan type operators, because in the case when $a_n = n$, for any $n \in \mathbb{N}$, the φ -Szász-Mirakjan operators [11] are obtained. In the case when $a_n = n$, for any $n \in \mathbb{N}$ and $\varphi(x) = e^x$, for any $x \in [0, +\infty[$, the operators (3.1) become the classical Mirakjan-Favard-Szász operators.

Lemma 2. Let $e_j(x) = x^j$, $j \in \{0, 1, 2\}$ be the test functions. The φ -Szász-Mirakjan type operators satisfy

- i) $(\varphi S_n e_0)(x) = 1$,
 - ii) $(\varphi S_n e_1)(x) = \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x$,
 - iii) $(\varphi S_n e_2)(x) = \left(\frac{a_n}{n}\right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} x^2 + \frac{1}{n} \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x$,
- for any $x \in [0, +\infty[$ and $n \in \mathbb{N}$.

Proof. We get

$$(\varphi S_n e_0)(x) = \frac{1}{\varphi(a_n x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k = 1,$$

taking (2.1) into account.

Next, we get

$$\begin{aligned} (\varphi S_n e_1)(x) &= \frac{1}{\varphi(a_n x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k \frac{k}{n} \\ &= \frac{a_n}{n} \frac{x}{\varphi(a_n x)} \sum_{k=1}^{\infty} \frac{\varphi^{(k)}(0)}{(k-1)!} (a_n x)^{k-1} = \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x, \end{aligned}$$

taking (2.2) into account. In a similar way, one obtains

$$\begin{aligned} (\varphi S_n e_2)(x) &= \frac{1}{\varphi(a_n x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k \left(\frac{k}{n}\right)^2 \\ &= \frac{1}{n^2} \frac{1}{\varphi(a_n x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k (k(k-1) + k) \\ &= \left(\frac{a_n}{n}\right)^2 \frac{x^2}{\varphi(a_n x)} \sum_{k=2}^{\infty} \frac{\varphi^{(k)}(0)}{(k-2)!} (a_n x)^{k-2} + \frac{a_n}{n^2} \frac{x}{\varphi(a_n x)} \sum_{k=1}^{\infty} \frac{\varphi^{(k)}(0)}{(k-1)!} (a_n x)^{k-1} \\ &= \left(\frac{a_n}{n}\right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} x^2 + \frac{1}{n} \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x, \end{aligned}$$

taking (2.2) and (2.3) into account. □

Lemma 3. Let $e_j(x) = x^j$, $j \in \{3, 4\}$ be the test functions. The φ -Szász-Mirakjan type operators satisfy

$$(\varphi S_n e_3)(x) = \left(\frac{a_n}{n}\right)^3 \frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)} x^3 + \frac{3}{n} \left(\frac{a_n}{n}\right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} x^2 + \frac{1}{n^2} \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x \quad (3.2)$$

and

$$(\varphi S_n e_4)(x) = \left(\frac{a_n}{n}\right)^4 \frac{\varphi^{(4)}(a_n x)}{\varphi(a_n x)} x^4 + \frac{6}{n} \left(\frac{a_n}{n}\right)^3 \frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)} x^3 + \frac{7}{n^2} \left(\frac{a_n}{n}\right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} x^2 + \frac{1}{n^3} \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x, \quad (3.3)$$

for any $x \in [0, +\infty[$ and $n \in \mathbb{N}$.

Proof. Because the function φ is analytic by differentiation, it follows

$$\varphi^{(3)}(x) = \sum_{k=3}^{\infty} \frac{\varphi^{(k)}(0)}{(k-3)!} x^{k-3}, \quad \varphi^{(4)}(x) = \sum_{k=4}^{\infty} \frac{\varphi^{(k)}(0)}{(k-4)!} x^{k-4}.$$

Using the above relations and (2.1), (2.2), then from (3.1), it follows

$$\begin{aligned} (\varphi S_n e_3)(x) &= \frac{1}{\varphi(a_n x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k \left(\frac{k}{n}\right)^3 \\ &= \frac{1}{n^3} \frac{1}{\varphi(a_n x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k \cdot u_1(k) \\ &= \left(\frac{a_n}{n}\right)^3 \frac{x^3}{\varphi(a_n x)} \sum_{k=3}^{\infty} \frac{\varphi^{(k)}(0)}{(k-3)!} (a_n x)^{k-3} \\ &\quad + \frac{3}{n} \left(\frac{a_n}{n}\right)^2 \frac{x^2}{\varphi(a_n x)} \sum_{k=2}^{\infty} \frac{\varphi^{(k)}(0)}{(k-2)!} (a_n x)^{k-2} + \frac{1}{n^2} \frac{a_n}{n} \frac{x}{\varphi(a_n x)} \sum_{k=1}^{\infty} \frac{\varphi^{(k)}(0)}{(k-1)!} (a_n x)^{k-1} \\ &= \left(\frac{a_n}{n}\right)^3 \frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)} x^3 + \frac{3}{n} \left(\frac{a_n}{n}\right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} x^2 + \frac{1}{n^2} \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x, \end{aligned}$$

where $u_1(k) = k(k-1)(k-2) + 3k(k-1) + k$.

$$\begin{aligned} (\varphi S_n e_4)(x) &= \frac{1}{\varphi(a_n x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k \left(\frac{k}{n}\right)^4 \\ &= \frac{1}{n^4} \frac{1}{\varphi(a_n x)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k \cdot u_2(k) \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{a_n}{n}\right)^4 \frac{x^4}{\varphi(a_n x)} \sum_{k=4}^{\infty} \frac{\varphi^{(k)}(0)}{(k-4)!} (a_n x)^{k-4} + \frac{6}{n} \left(\frac{a_n}{n}\right)^3 \frac{x^3}{\varphi(a_n x)} \sum_{k=3}^{\infty} \frac{\varphi^{(k)}(0)}{(k-3)!} (a_n x)^{k-3} \\
&+ \frac{7}{n^2} \left(\frac{a_n}{n}\right)^2 \frac{x^2}{\varphi(a_n x)} \sum_{k=2}^{\infty} \frac{\varphi^{(k)}(0)}{(k-2)!} (a_n x)^{k-2} + \frac{1}{n^3} \frac{a_n}{n} \frac{x}{\varphi(a_n x)} \sum_{k=1}^{\infty} \frac{\varphi^{(k)}(0)}{(k-1)!} (a_n x)^{k-1} \\
&= \left(\frac{a_n}{n}\right)^4 \frac{\varphi^{(4)}(a_n x)}{\varphi(a_n x)} x^4 + \frac{6}{n} \left(\frac{a_n}{n}\right)^3 \frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)} x^3 + \frac{7}{n^2} \left(\frac{a_n}{n}\right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} x^2 \\
&+ \frac{1}{n^3} \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x,
\end{aligned}$$

where $u_2(k) = k(k-1)(k-2)(k-3) + 6k(k-1)(k-2) + 7k(k-1) + k$. $\square$

Lemma 4. For any $x \in [0, +\infty[$ and $n \in \mathbb{N}$, the following

$$(T_{n,0}^* \varphi S_n)(x) = 1, \quad (3.4)$$

$$(T_{n,1}^* \varphi S_n)(x) = n \left(\frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} - 1 \right) x, \quad (3.5)$$

$$\begin{aligned}
(T_{n,2}^* \varphi S_n)(x) &= n^2 \left(\left(\left(\frac{a_n}{n} \right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} - 2 \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} + 1 \right) x^2 \right. \\
&\quad \left. + \frac{1}{n} \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x \right), \quad (3.6)
\end{aligned}$$

$$\begin{aligned}
(T_{n,4}^* \varphi S_n)(x) &= n^4 \left(\left(\left(\frac{a_n}{n} \right)^4 \frac{\varphi^{(4)}(a_n x)}{\varphi(a_n x)} - 4 \left(\frac{a_n}{n} \right)^3 \frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)} \right. \right. \\
&\quad \left. \left. + 6 \left(\frac{a_n}{n} \right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} - 4 \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} + 1 \right) x^4 \right. \\
&\quad \left. + \frac{6}{n} \left(\left(\frac{a_n}{n} \right)^3 \frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)} - 2 \left(\frac{a_n}{n} \right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} + \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} \right) x^3 \right. \\
&\quad \left. + \frac{1}{n^2} \left(7 \left(\frac{a_n}{n} \right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} - 4 \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} \right) x^2 + \frac{1}{n^3} \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x \right)
\end{aligned} \quad (3.7)$$

hold.

Proof. Taking into account relations (1.2), (3.1), Lemma 2 and Lemma 3, we get

$$(T_{n,0}^* \varphi S_n)(x) = (\varphi S_n e_0)(x) = 1,$$

$$(T_{n,1}^* \varphi S_n)(x) = n(\varphi S_n \psi_x)(x) = n((\varphi S_n e_1)(x) - x(\varphi S_n e_0)(x))$$

$$= n \left(\frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} - 1 \right) x,$$

$$\begin{aligned} (T_{n,2}^* \varphi S_n)(x) &= n^2 (\varphi S_n \psi_x^2)(x) = n^2 ((\varphi S_n e_2)(x) - 2x(\varphi S_n e_1)(x) + x^2(\varphi S_n e_0)(x)) \\ &= n^2 \left(\left(\left(\frac{a_n}{n} \right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} - 2 \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} + 1 \right) x^2 + \frac{1}{n} \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x \right), \end{aligned}$$

$$\begin{aligned} (T_{n,4}^* \varphi S_n)(x) &= n^4 (\varphi S_n \psi_x^4)(x) = n^4 ((\varphi S_n e_4)(x) - 4x(\varphi S_n e_3)(x) \\ &\quad + 6x^2(\varphi S_n e_2)(x) - 4x^3(\varphi S_n e_1)(x) + x^4(\varphi S_n e_0)(x)) \\ &= n^4 \left(\left(\left(\frac{a_n}{n} \right)^4 \frac{\varphi^{(4)}(a_n x)}{\varphi(a_n x)} - 4 \left(\frac{a_n}{n} \right)^3 \frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)} \right. \right. \\ &\quad \left. \left. + 6 \left(\frac{a_n}{n} \right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} - 4 \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} + 1 \right) x^4 \right. \\ &\quad \left. + \frac{6}{n} \left(\left(\frac{a_n}{n} \right)^3 \frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)} - 2 \left(\frac{a_n}{n} \right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} + \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} \right) x^3 \right. \\ &\quad \left. + \frac{1}{n^2} \left(7 \left(\frac{a_n}{n} \right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} - 4 \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} \right) x^2 + \frac{1}{n^3} \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} x \right). \end{aligned}$$

□

Remark 5. Taking (2.5) and (2.6) into account, for $x \in [0, +\infty[$ we have

$$\lim_{n \rightarrow \infty} \left(\left(\frac{a_n}{n} \right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} - 2 \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} + 1 \right) = 0.$$

In the following, we assume that there exist γ, δ , so that $0 < \gamma \leq 1, \delta \leq 2, \gamma < \delta$, and the function φ verifies the conditions

$$\lim_{n \rightarrow \infty} n^\gamma \left(\left(\frac{a_n}{n} \right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} - 2 \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} + 1 \right) = \beta_2(x), \quad (3.8)$$

$$\lim_{n \rightarrow \infty} n^{\delta-1} \left(\left(\frac{a_n}{n} \right)^3 \frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)} - 2 \left(\frac{a_n}{n} \right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} + \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} \right) = \beta_3(x) \quad (3.9)$$

and

$$\lim_{n \rightarrow \infty} n^\delta \left(\left(\frac{a_n}{n} \right)^4 \frac{\varphi^{(4)}(a_n x)}{\varphi(a_n x)} - 4 \left(\frac{a_n}{n} \right)^3 \frac{\varphi^{(3)}(a_n x)}{\varphi(a_n x)} + 6 \left(\frac{a_n}{n} \right)^2 \frac{\varphi^{(2)}(a_n x)}{\varphi(a_n x)} \right) \quad (3.10)$$

$$\left. -4 \frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} + 1 \right) = \beta_4(x),$$

for any $x \in [0, +\infty[$, where $\beta_2, \beta_3, \beta_4$ are functions, $\beta_2, \beta_3, \beta_4 : [0, +\infty[\rightarrow \mathbb{R}$.

Lemma 5. *For any $x \in [0, +\infty[$, the following identities*

$$\lim_{n \rightarrow \infty} (T_{n,0}^* \varphi S_n)(x) = 1, \quad (3.11)$$

$$\lim_{n \rightarrow \infty} \frac{(T_{n,2}^* \varphi S_n)(x)}{n^{2-\gamma}} = \beta_2(x) \cdot x^2 + \epsilon(\gamma) \cdot x, \quad (3.12)$$

$$\lim_{n \rightarrow \infty} \frac{(T_{n,4}^* \varphi S_n)(x)}{n^{4-\delta}} = \beta_4(x) \cdot x^4 + 6\beta_3(x) \cdot x^3 + 3\eta(\delta) \cdot x^2 \quad (3.13)$$

hold and there exists $n_0 \in \mathbb{N}$, so that

$$(T_{n,0}^* \varphi S_n)(x) = 1 = k_0, \quad (3.14)$$

$$\frac{(T_{n,2}^* \varphi S_n)(x)}{n^{2-\gamma}} \leq m_2(K) \cdot b^2 + b + 1 = k_2, \quad (3.15)$$

$$\frac{(T_{n,4}^* \varphi S_n)(x)}{n^{4-\delta}} \leq m_4(K) \cdot b^4 + 6m_3(K) \cdot b^3 + 3b^2 + 1 = k_4, \quad (3.16)$$

for any $x \in K = [0, b]$, $b > 0$, any $n \in \mathbb{N}$, $n \geq n_0$ and $m_2(K) := \sup_{x \in K} |\beta_2(x)|$,

$m_3(K) := \sup_{x \in K} |\beta_3(x)|$, $m_4(K) := \sup_{x \in K} |\beta_4(x)|$, where $\epsilon(\gamma) = \begin{cases} 1, & \gamma = 1 \\ 0, & 0 < \gamma < 1 \end{cases}$

and $\eta(\delta) = \begin{cases} 1, & \delta = 2 \\ 0, & \delta < 2. \end{cases}$

Proof. The identities (3.11)-(3.13) follow from Lemma 4, while (3.14)-(3.16) follow from (3.11)-(3.13) by taking the definition of the limit into account. $\square$

Let us assume that $I = J = [0, +\infty[$, $E(I) = C_2([0, +\infty[)$, $F(J) = C([0, +\infty[)$, the functions $\varphi_{n,k} : [0, +\infty[\rightarrow \mathbb{R}$ let be defined by $\varphi_{n,k}(x) = \frac{1}{\varphi(a_n x)} \frac{\varphi^{(k)}(0)}{k!} (a_n x)^k$, for any $x \in [0, +\infty[$, any $n, k \in \mathbb{N}_0$, $n \neq 0$ and the functionals $A_{n,k} : C_2([0, +\infty[) \rightarrow \mathbb{R}$ let be defined by $A_{n,k}(f) = f\left(\frac{k}{n}\right)$, for any $n, k \in \mathbb{N}_0$, $n \neq 0$. In this way we get the φ -Szász-Mirakjan type operators.

Theorem 2. *Let the function $f \in E(I)$ be given. If $x \in [0, +\infty[$, f is s times differentiable in a neighborhood of x , then:*

$$\lim_{n \rightarrow \infty} (\varphi S_n f)(x) = f(x), \quad (3.17)$$

if $s = 0$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} n^\gamma \left((\varphi S_n f)(x) - f(x) - \left(\frac{a_n}{n} \frac{\varphi^{(1)}(a_n x)}{\varphi(a_n x)} - 1 \right) x f^{(1)}(x) \right) \\ = \frac{1}{2} (\beta_2(x) \cdot x^2 + \epsilon(\gamma) \cdot x) f^{(2)}(x), \end{aligned} \quad (3.18)$$

if $s = 2$.

If f is s times differentiable on $[0, +\infty[$, then the convergence from (3.17) and (3.18) is uniform on any compact interval $K = [0, b] \subset [0, +\infty[$.

Moreover, we get

$$|(\varphi S_n f)(x) - f(x)| \leq (1 + k_2) \omega_1 \left(f; \frac{1}{\sqrt{n^\gamma}} \right), \quad (3.19)$$

for any $f \in C_2([0, +\infty[)$, any $x \in K$ and $n \in \mathbb{N}$, $n \geq n_0$.

Proof. It follows from Theorem 1, with $\alpha_0 = 0$, $\alpha_2 = 2 - \gamma$ and $\alpha_4 = 4 - \delta$, Lemma 4 and Lemma 5. $\square$

Remark 6. If $a_n = n$, for any $n \in \mathbb{N}$ and $\varphi(x) = e^x$, for any $x \in [0, +\infty[$ we get the well-known results for the classical Szász-Mirakjan operators. In this case $\gamma = 1$, $\delta = 2$, $\beta_2(x) = \beta_3(x) = \beta_4(x) = 0$, for any $x \in [0, +\infty[$ and $k_2 = b$, $k_4 = 3b^2 + b$, (see [12]).

4. OTHER FORM FOR THE SZÁSZ-MIRAKJAN OPERATORS

In this section, we assume that $c = 1$ in the relation (2.7). In [6] or [14], we can find some known results about differentiation of composed functions.

Theorem 3 ([14]). *Let I, J be two intervals of real numbers and let $f : I \rightarrow \mathbb{R}$, $g : J \rightarrow \mathbb{R}$ be two real-valued functions, such that $f(I) \subset J$. If f is k times differentiable on I and g is k times differentiable on J , $k \in \mathbb{N}$, then the function $g \circ f : I \rightarrow \mathbb{R}$ is also k times differentiable on I and the following identity*

$$\begin{aligned} (g \circ f)^{(k)}(x) = \sum_{i=1}^k \left(g^{(i)} \circ f \right)(x) \sum_{\substack{a_1 + a_2 + \dots + a_k = i \\ a_1 + 2a_2 + \dots + ka_k = k}} \frac{k!}{a_1! a_2! \dots a_k!} \\ \cdot \left(\frac{f^{(1)}(x)}{1!} \right)^{a_1} \left(\frac{f^{(2)}(x)}{2!} \right)^{a_2} \dots \left(\frac{f^{(k)}(x)}{k!} \right)^{a_k} \end{aligned} \quad (4.1)$$

holds, for any $x \in I$.

Theorem 4. If $k \in \mathbb{N}$, then the following identity

$$\begin{aligned} \varphi^{(k)}(0) = \sum_{i=1}^k \sum_{\substack{a_1+a_2+\dots+a_k=i \\ a_1+2a_2+\dots+ka_k=k}} \frac{k!}{a_1!a_2!\dots a_k!} \left(\frac{1+\alpha(0)}{1!} \right)^{a_1} \\ \cdot \left(\frac{\alpha^{(1)}(0)}{2!} \right)^{a_2} \cdot \dots \cdot \left(\frac{\alpha^{(k-1)}(0)}{k!} \right)^{a_k} \end{aligned} \quad (4.2)$$

holds.

Proof. One applies Theorem 3, with $g(x) = e^x$, for $x \in \mathbb{R}$ and $f(x) = x + \int_0^x \alpha(t) dt$, for $x \in [0, +\infty[$. $\square$

Remark 7. The relation (3.1) represents the form of φ -Szász-Mirakjan type operators, by using the function φ . By replacing the form of $\varphi^{(k)}(0)$, from the relation (4.2) in (3.1), we get the form of the φ -Szász-Mirakjan type operators, where appears the function α .

Now we can give some applications.

Example 1. If $a_n = n$, $n \in \mathbb{N}$ and $\alpha(x) = 0$, for any $x \in [0, +\infty[$, from (4.2) follows that $\varphi^{(k)}(0) = 1$, for any $k \in \mathbb{N}$. In this case, we get the classical Szász-Mirakjan operators.

Example 2. Suppose that $\alpha(x) = \frac{1}{(x+1)^2}$, for any $x \in [0, +\infty[$. Then $\alpha^{(p)}(x) = \frac{(-1)^p(p+1)!}{(x+1)^{p+2}}$, $\varphi(x) = e^{x+1-\frac{1}{x+1}}$, for any $x \in [0, +\infty[$, any $p \in \mathbb{N}_0$ and we get

$$\varphi^{(k)}(0) = \sum_{i=1}^k \sum_{\substack{a_1+a_2+\dots+a_k=i \\ a_1+2a_2+\dots+ka_k=k}} \frac{k!}{a_1!a_2!\dots a_k!} 2^{a_1} (-1)^{1a_2+2a_3+\dots+(k-1)a_k}.$$

Taking into account $a_1 + 2a_2 + \dots + ka_k = k$ and $a_1 + a_2 + \dots + a_k = i$, we get

$$\varphi^{(k)}(0) = \sum_{i=1}^k \sum_{\substack{a_1+a_2+\dots+a_k=i \\ a_1+2a_2+\dots+ka_k=k}} \frac{k!}{a_1!a_2!\dots a_k!} (-1)^{k-i} 2^{a_1},$$

from where, it follows

$$\varphi^{(k)}(0) = (-1)^k \sum_{i=1}^k \sum_{\substack{a_1+a_2+\dots+a_k=i \\ a_1+2a_2+\dots+ka_k=k}} (-1)^i \frac{k!}{a_1!a_2!\dots a_k!} 2^{a_1}. \quad (4.3)$$

If we consider $a_n = n + \frac{1}{n}$, for any $n \in \mathbb{N}$, the operators defined by (3.1) become

$$(\varphi S_n f)(x) = e^{\frac{1}{(n+\frac{1}{n})x+1} - (n+\frac{1}{n})x-1} \sum_{k=0}^{\infty} \frac{(-1)^k \sum_{\substack{i=1 \\ a_1+a_2+\dots+a_k=i \\ a_1+2a_2+\dots+ka_k=k}}^k (-1)^i \frac{k!}{a_1!a_2!\dots a_k!} 2^{a_1}}{k!} \cdot \left(\left(n + \frac{1}{n} \right) x \right)^k f\left(\frac{k}{n} \right),$$

for any $f \in C_2([0, +\infty[)$, $x \in [0, +\infty[$ and $n \in \mathbb{N}$.

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SOME COMMON FIXED POINT RESULTS FOR WEAKLY COMPATIBLE MAPPINGS IN CONE METRIC TYPE SPACE

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Abstract. In this paper we consider cone metric type spaces which are introduced as a generalization of symmetric and metric spaces by Khamsi and Hussain in 2010. Then we prove several common fixed point for weakly compatible mappings in cone metric type spaces. All results are proved in the settings of a solid cone, without the assumption of continuity of the mappings.

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1. INTRODUCTION

In 1922, Banach proved the famous contraction mapping principle [4]. Afterward, other authors considered various definitions of contractive mappings and proved several fixed and common fixed point theorems [6, 13, 19, 25]. In 1976, Jungck [18] proved a common fixed point theorem for two commuting mappings. This theorem has many applications but it requires the continuity of one of the two mappings. In 1996, Jungck [16] defined a pair of self-mappings to be weakly compatible if they commute at their coincidence points. In the sequel, Jungck and Rhoades [17] proved some fixed and common fixed point theorems for noncommuting and compatible mappings in metric spaces.

In 2007, Huang and Zhang [14] introduced cone metric spaces and proved some fixed point theorems. Several fixed and common fixed point results in cone metric spaces are proved in [1, 5, 9–12, 21, 22, 24, 26, 27].

In 1931, Wilson [28] introduced symmetric spaces, as metric-like spaces lacking the triangle inequality. Recently, Khamsi and Hussain [20, 21] defined a new type of spaces which they called cone metric type spaces. Afterward, other authors proved fixed point theorems in metric type space and cone metric type spaces [7, 15, 23]. The purpose of this paper is to generalize and unify the common fixed point theorems for weakly compatible mappings of Abbas and Jungck [1], Abbas and Rhoades [2], Arshad et al. [3], Huang and Zhang [14], on cone metric type spaces.

2. PRELIMINARIES

We recall some definitions and results that we will use in the sequel. Throughout this article we denote by $\mathbb{R}$ the set of all real numbers and by $\mathbb{N}$ the set of positive integers.

Definition 1 (See [28]). Let X be a nonempty set. Suppose that the mapping $D : X \times X \rightarrow [0, +\infty)$ satisfies

$$(S1) \quad D(x, y) = 0 \iff x = y;$$

$$(S2) \quad D(x, y) = D(y, x),$$

for all $x, y \in X$. Then D is called a symmetric on X and (X, D) is called a symmetric space.

Definition 2 (See [8, 14]). Let E be a real Banach space and P a subset of E . Then P is called a cone if and only if

(a) P is closed, non-empty and $P \neq \{0\}$;

(b) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$;

(c) if $x \in P$ and $-x \in P$, then $x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by

$$x \leq y \iff y - x \in P.$$

We shall write $x < y$ if $x \leq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in \text{int}P$ (where $\text{int}P$ is the interior of P). If $\text{int}P \neq \emptyset$, the cone P is called solid. The cone P is called normal if there is a number $k > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \implies \|x\| \leq k\|y\|.$$

The least positive number satisfying the above condition is called the normal constant of P . In the sequel we always suppose that E is a real Banach space, P is a solid cone in E , and $\leq$ is a partial ordering with respect to P .

Example 1. (See [24])

(i) Let $E = C_{\mathbb{R}}[0, 1]$ with the supremum norm and $P = \{f \in E : f \geq 0\}$. Then, P is a normal cone with normal constant $k = 1$.

(ii) Let $E = C_{\mathbb{R}}^1[0, 1]$ with the norm $\|f\| = \|f\|_{\infty} + \|f'\|_{\infty}$ and consider the cone $P = \{f \in E : f \geq 0\}$. Then P is a non-normal cone.

Definition 3 (See [14]). Let X be a nonempty set. Suppose that the mapping $d : X \times X \rightarrow E$ satisfies

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then, d is called a cone metric on X and (X, d) is called a cone metric space.

Example 2. (See [14]) Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E \mid x, y \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}$ and $d : X \times X \rightarrow E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 4 (See [7, 20, 21]). Let X be a nonempty set, $K \geq 1$ be a real number and E a real Banach space with cone P . Suppose that the mapping $d : X \times X \rightarrow E$ satisfies

(cd1) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(cd2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(cd3) $d(x, z) \leq K(d(x, y) + d(y, z))$ for all $x, y, z \in X$.

Then (X, d, K) is called a cone metric type space. Obviously, for $K = 1$, a cone metric type space is a cone metric space.

Example 3. (See [7]) Let $B = \{e_i \mid i = 1, \dots, n\}$ be an orthonormal basis of $\mathbb{R}^n$ with inner product $(\cdot, \cdot)$ and $p > 0$. Define

$$X_p = \{[x] \mid x : [0, 1] \rightarrow \mathbb{R}^n, \int_0^1 |(x(t), e_j)|^p dt \in \mathbb{R}, \quad j = 1, 2, \dots, n\},$$

where $[x]$ represents the class of equivalence of x with respect to relation of functions equal almost everywhere. Let $E = \mathbb{R}^n$ and

$$P_B = \{y \in \mathbb{R}^n \mid (y, e_i) \geq 0, \quad i = 1, 2, \dots, n\}$$

be a solid cone. Define $d : X_p \times X_p \rightarrow P_B \subset \mathbb{R}^n$ by

$$d(f, g) = \sum_{i=1}^n e_i \int_0^1 |((f - g)(t), e_i)|^p dt, \quad f, g \in X_p.$$

Then (X_p, d, K) is a cone metric type space with $K = 2^{p-1}$.

We define convergence in cone metric type spaces as in the cone metric spaces.

Definition 5 (See [20]). Let (X, d, K) be a cone metric type space, $\{x_n\}$ a sequence in X and $x \in X$.

(i) $\{x_n\}$ converges to x if for every $c \in E$ with $0 \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$, and we write $\lim_{n \rightarrow +\infty} d(x_n, x) = 0$

(ii) $\{x_n\}$ is called a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$, and we write $\lim_{n, m \rightarrow +\infty} d(x_n, x_m) = 0$.

Lemma 1 (See [7]). Let (X, d, K) be a cone metric type space over-ordered real Banach space E . Then the following properties are often used, particularly in dealing with cone metric type spaces in which the cone need not be normal.

(P₁) If $u \leq v$ and $v \ll w$, then $u \ll w$.

(P₂) If $0 \leq u \ll c$ for each $c \in \text{int}P$, then $u = 0$.

(P₃) If $u \leq \lambda u$ where $u \in P$ and $0 \leq \lambda < 1$, then $u = 0$.

(P₄) Let $x_n \rightarrow 0$ in E and $0 \ll c$. Then there exists a positive integer n_0 such that $x_n \ll c$ for each $n > n_0$.

Definition 6 (See [17]). Let f and g be two self-mappings defined on a set X . If $fw = gw = z$ for some $z \in X$, then w is called a coincidence point of f and g , and z is called a point of coincidence of f and g . The mappings f and g are said to be weakly compatible if they commute at every coincidence point, that is, if $fgw = gfw$ for all coincidence points w .

Lemma 2 (See [1]). Let f and g be weakly compatible self-mappings on a set X . If f and g have a unique point of coincidence $z = fw = gw$, then z is the unique common fixed point of f and g .

3. MAIN RESULTS

The following theorem, that extends and improves Theorem 2 of [3] and Corollary 2.10 of [2] in a cone metric type space, is our main result.

Theorem 1. Let (X, d, K) be a cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose that f, g and T are three self-mappings on X , satisfying $f(X) \cup g(X) \subset T(X)$, and

$$d(fx, gy) \leq p(x, y)d(Tx, Ty) + q(x, y)d(fx, Tx) + r(x, y)d(gy, Ty) + 2t(x, y)\left[\frac{d(fx, Ty) + d(gy, Tx)}{2}\right], \quad (3.1)$$

for all $x, y \in X$, where $p, q, r, t : X \times X \rightarrow [0, \frac{1}{K})$ are real functions such that

$$\sup_{x, y \in X} \{Kp(x, y) + (K + 1)\max\{q(x, y), r(x, y)\} + (K^2 + K)t(x, y)\} \leq \lambda < 1. \quad (3.2)$$

If one of $f(X)$, $g(X)$ or $T(X)$ is a complete subspace of X , then $\{f, T\}$ and $\{g, T\}$ have a unique point of coincidence in X . Moreover if $\{f, T\}$ and $\{g, T\}$ are weakly compatible, then f, g and T have a unique common fixed point.

Proof. Suppose x_0 is an arbitrary point of X . Since $f(X) \subset T(X)$, there exists $x_1 \in X$ such that $fx_0 = Tx_1 = y_1$. Since $g(X) \subset T(X)$, there exists $x_2 \in X$ such that $gx_1 = Tx_2 = y_2$. If we continue in this manner, then

$$\begin{array}{lll} \exists x_{2n+1} \in X & \text{such that} & y_{2n+1} = fx_{2n} = Tx_{2n+1} \\ \exists x_{2n+2} \in X & \text{such that} & y_{2n+2} = gx_{2n+1} = Tx_{2n+2}, \end{array}$$

for $n = 0, 1, \dots$.

In the sequel we denote $p(x, y), q(x, y), r(x, y), t(x, y)$ respectively with p, q, r, t . Now, we have

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) &= d(fx_{2n}, gx_{2n+1}) \\ &\leq pd(Tx_{2n}, Tx_{2n+1}) + qd(fx_{2n}, Tx_{2n}) \end{aligned}$$

$$\begin{aligned}
& + rd(gx_{2n+1}, Tx_{2n+1}) \\
& + t[d(fx_{2n}, Tx_{2n+1}) + d(gx_{2n+1}, Tx_{2n})] \\
& = pd(y_{2n}, y_{2n+1}) + qd(y_{2n+1}, y_{2n}) + rd(y_{2n+2}, y_{2n+1}) \\
& + t[d(y_{2n+1}, y_{2n+1}) + d(y_{2n+2}, y_{2n})] \\
& \leq pd(y_{2n}, y_{2n+1}) + qd(y_{2n+1}, y_{2n}) + rd(y_{2n+2}, y_{2n+1}) \\
& + tK[d(y_{2n+2}, y_{2n+1}) + d(y_{2n+1}, y_{2n})] \\
& = (p + q + tK)d(y_{2n}, y_{2n+1}) + (r + tK)d(y_{2n+1}, y_{2n+2}),
\end{aligned}$$

which implies that

$$d(y_{2n+1}, y_{2n+2}) \leq \frac{p + q + tK}{1 - r - tK} d(y_{2n}, y_{2n+1}).$$

Similarly,

$$d(y_{2n+3}, y_{2n+2}) \leq \frac{p + r + tK}{1 - q - tK} d(y_{2n+2}, y_{2n+1}).$$

Since $\lambda < 1$, from

$$Kp(x, y) + Kq(x, y) + \lambda r(x, y) + K^2t(x, y) + \lambda Kt(x, y) \leq \lambda$$

and

$$Kp(x, y) + \lambda q(x, y) + Kr(x, y) + K^2t(x, y) + \lambda Kt(x, y) \leq \lambda,$$

that holds by relation (3.2), we have

$$K \frac{p(x, y) + q(x, y) + Kt(x, y)}{1 - r(x, y) - Kt(x, y)} \leq \lambda$$

and

$$K \frac{p(x, y) + r(x, y) + Kt(x, y)}{1 - q(x, y) - Kt(x, y)} \leq \lambda$$

holds for all $x, y \in X$. Therefore

$$d(y_n, y_{n+1}) \leq \frac{\lambda}{K} d(y_{n-1}, y_n) \quad \text{for all } n \in \mathbb{N}$$

and hence

$$d(y_n, y_{n+1}) \leq \frac{\lambda}{K} d(y_{n-1}, y_n) \leq \cdots \leq \left(\frac{\lambda}{K}\right)^n d(y_0, y_1).$$

Now, for $m > n$ we have

$$\begin{aligned}
d(y_n, y_m) & \leq Kd(y_n, y_{n+1}) + K^2d(y_{n+1}, y_{n+2}) + \cdots + K^{m-n}d(y_{m-1}, y_m) \\
& \leq \frac{1}{K^{n-1}} (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) d(y_0, y_1) \\
& \leq \frac{1}{K^{n-1}} \frac{\lambda^n}{1 - \lambda} d(y_0, y_1) \rightarrow 0 \quad \text{in } E \text{ as } n \rightarrow +\infty.
\end{aligned}$$

Now, by (P_1) and (P_4) , it follows that for every $c \in \text{int}P$ there exists a positive integer n_0 such that $d(y_n, y_m) \ll c$ for every $m > n > n_0$, so $\{y_n\}$ is a Cauchy sequence. Suppose that $T(X)$ is a complete subspace of X , then $\{y_n\}$ is convergent in $T(X)$ and there exists $v \in X$ such that $\lim_{n \rightarrow +\infty} Tx_{2n} = \lim_{n \rightarrow +\infty} y_{2n} = v$. Since T is a self-mapping, there exists $u \in X$ such that $Tu = v$. Now, we prove that $fu = v$. By (3.1), we obtain

$$\begin{aligned} d(fu, v) &\leq K[d(fu, gx_{2n+1}) + d(gx_{2n+1}, v)] \\ &\leq K[pd(Tu, Tx_{2n+1}) + qd(fu, Tu) + rd(gx_{2n+1}, Tx_{2n+1}) \\ &\quad + t[d(fu, Tx_{2n+1}) + d(gx_{2n+1}, Tu)]] + Kd(gx_{2n+1}, v) \\ &= K[pd(v, y_{2n+1}) + qd(fu, v) + rd(y_{2n+2}, y_{2n+1}) \\ &\quad + t[d(fu, y_{2n+1}) + d(y_{2n+2}, v)]] + Kd(y_{2n+2}, v) \\ &\leq pKd(v, y_{2n+1}) + qKd(fu, v) + rKd(y_{2n+2}, y_{2n+1}) \\ &\quad + tK^2[d(fu, v) + d(v, y_{2n+1})] + tKd(y_{2n+2}, v) + Kd(y_{2n+2}, v) \\ &= (pK + tK^2)d(v, y_{2n+1}) + rKd(y_{2n+2}, y_{2n+1}) + \\ &\quad K(t+1)d(y_{2n+2}, v) + (qK + tK^2)d(fu, v), \end{aligned}$$

which implies

$$\begin{aligned} (1 - qK - tK^2)d(fu, v) &\leq (pK + tK^2)d(v, y_{2n+1}) + rKd(y_{2n+2}, y_{2n+1}) \\ &\quad + K(t+1)d(y_{2n+2}, v). \end{aligned}$$

Now, using (3.2), we have that

$$(1 - \lambda)d(fu, v) \leq \lambda d(v, y_{2n+1}) + \lambda d(y_{2n+2}, y_{2n+1}) + (\lambda + K)d(y_{2n+2}, v)$$

holds for all $n \in \mathbb{N}$. Since $\{y_n\}$ converges to v and $d(y_{2n+2}, y_{2n+1}) \rightarrow 0$ in E as $n \rightarrow +\infty$, for every $c \in \text{int}P$ there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$, we have

$$\begin{aligned} d(v, y_{2n+1}) &\ll \frac{(1 - \lambda)c}{3\lambda}, \\ d(y_{2n+2}, y_{2n+1}) &\ll \frac{(1 - \lambda)c}{3\lambda}, \\ d(y_{2n+2}, v) &\ll \frac{(1 - \lambda)c}{3(\lambda + K)}. \end{aligned}$$

It follows that $d(fu, v) \ll c$ for every $c \in \text{int}P$, and by (P_2) we have $d(fu, v) = 0$, that is, $fu = v$. So, we have $fu = Tu = v$, that is, v is a point of coincidence of the mappings f and T , and u is a coincidence point of the mapping f and T .

Similarly, by (3.1) and (3.2) we conclude $d(v, gu) \ll c$ for every $c \in \text{int}P$, and we have $d(gu, v) = 0$ by (P_2) , that is, $gu = v$. So, we have $gu = Tu = v$, that is, v is a point of coincidence of the mappings g and T , and u is a coincidence point of the mappings g and T . Hence $fu = gu = Tu = v$.

Now we shall show that v is the unique point of coincidence of the pairs $\{f, T\}$ and $\{g, T\}$. Let v' be also a point of coincidence of these three mappings, then $fu' = gu' = Tu' = v'$ for $u' \in X$. From (3.1), we have

$$\begin{aligned} d(v, v') &= d(fu, gu') \\ &\leq pd(Tu, Tu') + qd(fu, Tu) + rd(gu', Tu') \\ &\quad + t[d(fu, Tu') + d(gu', Tu)] \\ &= pd(v, v') + qd(v, v) + rd(v', v') + t[d(v, v') + d(v', v)] \\ &= (p + 2t)d(v, v') \\ &\leq \lambda d(v, v'), \end{aligned}$$

and (by (P_3)) it follows that $v = v'$. If the pairs $\{f, T\}$ and $\{g, T\}$ are weakly compatible, then v is the unique common fixed point of f , g and T by Lemma (2). $\square$

The following result is a consequence of Theorem 1.

Theorem 2. Let (X, d, K) be a cone metric type space and P a solid cone. Suppose that f , g and T are three self-mappings on X , satisfying $f(X) \cup g(X) \subset T(X)$, and

$$d(fx, gy) \leq \alpha M_{x,y}(f, g, T), \quad (3.3)$$

where $\alpha \in (0, \min\{\frac{1}{K+1}, \frac{2}{K(K+1)}\})$ for $K \geq 1$ and

$$M_{x,y}(f, g, T) \in \left\{ d(Tx, Ty), d(fx, Tx), d(gy, Ty), \frac{d(fx, Ty) + d(gy, Tx)}{2} \right\}, \quad (3.4)$$

for all $x, y \in X$. If one of $f(X)$, $g(X)$ or $T(X)$ is a complete subspace of X , then $\{f, T\}$ and $\{g, T\}$ have a unique point of coincidence in X . Moreover if $\{f, T\}$ and $\{g, T\}$ are weakly compatible, then f , g and T have a unique common fixed point.

Proof. First, we consider the following subsets of $X \times X$:

$$A = \{(x, y) \in X \times X : d(fx, gy) \leq d(Tx, Ty)\};$$

$$B = \{(x, y) \in X \times X \setminus A : d(fx, gy) \leq d(fx, Tx)\};$$

$$C = \{(x, y) \in X \times X \setminus (A \cup B) : d(fx, gy) \leq d(gy, Ty)\},$$

and we define the functions $p, q, r, t : X \times X \rightarrow [0, 1/K]$ as following

$$\begin{aligned} p(x, y) &= \begin{cases} \alpha, & \text{if } (x, y) \in A \\ 0, & \text{otherwise} \end{cases}, & q(x, y) &= \begin{cases} \alpha, & \text{if } (x, y) \in B \\ 0, & \text{otherwise} \end{cases}, \\ r(x, y) &= \begin{cases} \alpha, & \text{if } (x, y) \in C \\ 0, & \text{otherwise} \end{cases}, & t(x, y) &= \begin{cases} \alpha/2, & \text{if } (x, y) \in (A \cup B \cup C)^c \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Now, the mappings f , g and T satisfy the conditions (3.1) and (3.2) with respect to the previous functions if $\alpha < \min\{\frac{1}{K+1}, \frac{2}{K(K+1)}\}$, and so Theorem 2 follows by Theorem 1. $\square$

Remark 1. Proceeding as in the proof of Theorem 1, we can prove that Theorem 2 holds for $\alpha < \frac{2}{K(K+1)}$ also when $1 \leq K \leq 2$. So Theorem 2 extends Corollary 2.5 of [2] in the setting of a cone metric type space.

The following results is obtained from Theorem 1.

Corollary 1. *Let (X, d, K) be a cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose that the mappings f , g and T are three self-maps on X , satisfying*

$f(X) \cup g(X) \subset T(X)$, and

$$d(fx, gy) \leq pd(Tx, Ty) + qd(fx, Tx) + rd(gy, Ty) + t[d(fx, Ty) + d(gy, Tx)], \quad (3.5)$$

for all $x, y \in X$, where $p, q, r, t \in [0, \frac{1}{K})$ and

$$Kp + (K+1)\max\{q, r\} + (K^2 + K)t < 1. \quad (3.6)$$

If one of $f(X)$, $g(X)$ or $T(X)$ is a complete subspace of X , then $\{f, T\}$ and $\{g, T\}$ have a unique point of coincidence in X . Moreover if $\{f, T\}$ and $\{g, T\}$ are weakly compatible, then f , g and T have a unique common fixed point.

Corollary 2. *Let (X, d, K) be a cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose that f and T are two self-mappings on X , satisfying $f(X) \subset T(X)$ and*

$$d(fx, fy) \leq pd(Tx, Ty) + qd(fx, Tx) + rd(fy, Ty) + t[d(fx, Ty) + d(fy, Tx)], \quad (3.7)$$

for all $x, y \in X$, where $p, q, r, t \in [0, \frac{1}{K})$ and

$$Kp + (K+1)\max\{q, r\} + (K^2 + K)t < 1. \quad (3.8)$$

If one of $f(X)$ or $T(X)$ is a complete subspace of X , then $\{f, T\}$ has a unique point of coincidence in X . Moreover if $\{f, T\}$ is weakly compatible, then f and T have a unique common fixed point.

Proof. In (3.5), set $f = g$. It follows from Corollary 1 that $\{f, T\}$ have a unique common fixed point v . $\square$

Corollary 3. *Let (X, d, K) be a cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose that f and T are two self-mappings on X , satisfying $f(X) \subset T(X)$ and*

$$\begin{aligned} d(fx, fy) &\leq a_1d(Tx, Ty) + a_2d(fx, Tx) + a_3d(fy, Ty) \\ &\quad + a_4d(fx, Ty) + a_5d(fy, Tx), \end{aligned} \quad (3.9)$$

for all $x, y \in X$, where $a_i \geq 0$ for $i = 1, 2, \dots, 5$ and

$$Ka_1 + (K+1)\max\{a_2, a_3\} + (K^2 + K)a_4 < 1,$$

$$Ka_1 + (K + 1)\max\{a_2, a_3\} + (K^2 + K)a_5 < 1. \quad (3.10)$$

If one of $f(X)$ or $T(X)$ is a complete subspace of X , then $\{f, T\}$ has a unique point of coincidence in X . Moreover if $\{f, T\}$ is weakly compatible, then f and T have a unique common fixed point.

Proof. See [7]. □

Remark 2. Corollaries 2.11 and 2.12 in [2] can be generalized into cone metric type space by Corollary 2. Also, some results in [7] can be obtained by our corollaries.

The following corollary is obtained from Theorem 2.

Corollary 4. Let (X, d, K) be a cone metric type space and P a solid cone. Suppose that f and T are two self-mappings on X , satisfying $f(X) \subset T(X)$, and

$$d(fx, fy) \leq \alpha M_{x,y}(f, T), \quad (3.11)$$

where $\alpha \in (0, \min\{\frac{1}{K+1}, \frac{2}{K(K+1)}\})$ for $K \geq 1$ and

$$M_{x,y}(f, T) \in \left\{ d(Tx, Ty), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2} \right\}, \quad (3.12)$$

for all $x, y \in X$. If one of $f(X)$ or $T(X)$ is a complete subspace of X , then $\{f, T\}$ have a unique point of coincidence in X . Moreover if $\{f, T\}$ is weakly compatible, then f and T have a unique common fixed point.

Proof. In Theorem 2, set $f = g$. □

Example 4. Let $X = E = \mathbb{R}$ and $P = [0, +\infty)$. Suppose that $d(x, y) = |x - y|^2$ for all $x, y \in X$. Then (X, d, K) is a cone metric type space with $K = 2$ by Minkowski inequality. Let $f, T : X \rightarrow X$ be two mappings defined as follows

$$fx = \frac{1}{\sqrt{4}}(2x + 3) \quad \text{and} \quad Tx = 2x + 3,$$

where $x \in X$. Since $f(X) = T(X) = X$, we have $f(X) \subset T(X)$. Also,

$$d(fx, fy) = \left| \frac{1}{\sqrt{4}}(2x + 3) - \frac{1}{\sqrt{4}}(2y + 3) \right|^2 = \frac{1}{4}d(Tx, Ty)$$

and so (3.11) holds with $\alpha = \frac{1}{4}$. According to Corollary (4), $\{f, T\}$ have a unique point of coincidence in X . Indeed $v = 0$ is the unique point of coincidence of $\{f, T\}$ and $u = \frac{-3}{2}$ is a coincidence point of the mappings f and T .

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A COMMON FIXED POINT THEOREM FOR MULTIVALUED MONOTONE MAPPINGS IN ORDERED PARTIAL METRIC SPACES

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Abstract. In this paper, we use the order relation on partial metric spaces defined by [1] to prove some new fixed point theorems for multivalued monotone mappings in ordered partial metric spaces.

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1. INTRODUCTION

In the last years, the extension of the theory of fixed point to generalized structures as cone metrics, partial metric spaces and quasi-metric spaces has received a lot of attention. One of the most interesting is partial metric space. Partial metric space is a generalized metric space in which each object does not necessarily have to have a zero distance from itself [8]. A motivation behind introducing the concept of a partial metric was to obtain appropriate mathematical models in the theory of computation and, in particular, give a modified version of the Banach contraction principle, more suitable in this context [8]. Subsequently, Valero [11], Oltra and Valero [9] and Altun et al [4] gave some generalizations of the result of Matthews. Romaguera [10] proved the Caristi type fixed point theorem on this space. Lately, I. Altun and M. Imdad [1] have introduced a partial ordering on uniform spaces utilizing E -distance function and have used the same to prove a fixed point theorem for single-valued non-decreasing mappings on ordered uniform spaces. In this paper, we use the partial ordering on partial metric spaces which is defined by [1], so we prove some fixed point theorems of multivalued monotone mappings which are given for ordered metric spaces in [12] on ordered uniform spaces.

First, we recall some definitions and results needed in the sequel. The reader interested in fixed point theory in partial metric spaces is referred to the work of [2, 6, 8–11] and references therein.

In what follows $\mathbb{N}$ will denote the set of all natural numbers and $\mathbb{R}^+$ the set of all positive real numbers.

A partial metric on a nonempty set X is a mapping $p : X \times X \rightarrow \mathbb{R}^+$ such that for all $x, y, z \in X$:

- (p₁) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,
- (p₂) $p(x, x) \leq p(x, y)$,
- (p₃) $p(x, y) = p(y, x)$,
- (p₄) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X . It is clear that, if $p(x, y) = 0$, then from (p₁) and (p₂) $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. Other examples of the partial metric spaces which are interesting from a computational point of view may be found in [7] and [8].

Let (X, d) and (X, p) be a metric space and partial metric space, respectively.

Lemma 1. Mappings $\rho_i : X \times X \rightarrow \mathbb{R}^+$ ($i \in \{1, 2, 3\}$) defined by

$$\begin{aligned}\rho_1(x, y) &= d(x, y) + p(x, y) \\ \rho_2(x, y) &= d(x, y) + \max\{\omega(x), \omega(y)\} \\ \rho_3(x, y) &= d(x, y) + a\end{aligned}$$

define partial metrics on X , where $\omega : X \rightarrow \mathbb{R}^+$ is an arbitrary function and $a \geq 0$.

In addition in above Lemma the partial metric ρ_3 is a special case of ρ_2 , and ρ_2 is a special case of ρ_1 .

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p - balls

$$\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\},$$

where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 1. A subset A of a partial metric space (X, p) is called open if for every $x \in A$ there exists $r > 0$ such that $B_p(x, r) \subset A$.

Let τ_p be the set of all open subsets X , then τ_p is a topology on X (induced by the partial metric p).

Definition 2. A sequence $\{x_n\}$ in a partial metric space (X, p) converges to x if and only if $p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x)$. That is, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$p(x, x_n) < p(x, x) + \varepsilon \quad \forall n \geq n_0.$$

Definition 3. A sequence $\{x_n\}$ in a partial metric (X, p) is called a Cauchy sequence if there exists $\lim_{n,m \rightarrow \infty} p(x_n, x_m)$ which is finite.

A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p , to a point $x \in X$ such that $p(x, x) = \lim_{n,m \rightarrow \infty} p(x_n, x_m)$.

Suppose that $\{x_n\}$ is a sequence in partial metric space (X, p) , then we define $L(x_n) = \{x | x_n \rightarrow x\}$.

The following example shows that every convergent sequence $\{x_n\}$ in a partial metric space X may not be Cauchy. In particular, it shows that the limit of a convergent sequence is not unique.

Example 1. Let $X = [0, \infty)$ and $p(x, y) = \max\{x, y\}$. Let

$$x_n = \begin{cases} 0, & n = 2k, \\ 1, & n = 2k + 1. \end{cases}$$

Then clearly it is convergent sequence and for every $x \geq 1$ we have $\lim_{n \rightarrow \infty} p(x_n, x) = p(x, x)$, therefore $L(x_n) = [1, \infty)$. But $\lim_{n,m \rightarrow \infty} p(x_n, x_m)$ does not exist.

The following Lemma shows that under certain conditions the limit is unique.

Lemma 2. Let $\{x_n\}$ be a convergent sequence in partial metric space X such that $x_n \rightarrow x$ and $x_n \rightarrow y$. If

$$\lim_{n \rightarrow \infty} p(x_n, x_n) = p(x, x) = p(y, y),$$

then $x = y$.

Proof. As

$$p(x, y) \leq p(x, x_n) + p(x_n, y) - p(x_n, x_n),$$

therefore

$$p(x_n, x_n) \leq p(x, x_n) + p(x_n, y) - p(x, y).$$

By given assumptions, we have $\lim_{n \rightarrow \infty} p(x_n, x) = p(x, x)$, $\lim_{n \rightarrow \infty} p(x_n, y) = p(y, y)$, and $\lim_{n \rightarrow \infty} p(x_n, x_n) = p(x, x)$. Therefore

$$p(x, x) \leq p(x, x) + p(y, y) - p(x, y)$$

which shows that $p(y, y) \leq p(x, y) \leq p(y, y)$. Also, the inequality

$$p(x, x) \leq p(x, y) \leq p(x, x)$$

follow directly from the first part of the proof by replacing x and y , and using property (p3). Thus $p(x, x) = p(x, y) = p(y, y)$, therefore $x = y$. $\square$

Lemma 3. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in partial metric space X such that*

$$\lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n \rightarrow \infty} p(x_n, x_n) = p(x, x),$$

and

$$\lim_{n \rightarrow \infty} p(y_n, y) = \lim_{n \rightarrow \infty} p(y_n, y_n) = p(y, y),$$

then $\lim_{n \rightarrow \infty} p(x_n, y_n) = p(x, y)$. In particular, $\lim_{n \rightarrow \infty} p(x_n, z) = p(x, z)$ for every $z \in X$.

Proof. As $\{x_n\}$ and $\{y_n\}$ converge to a $x \in X$ and $y \in X$ respectively, therefore for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$p(x, x_n) < p(x, x) + \frac{\epsilon}{2}, \quad p(y, y_n) < p(y, y) + \frac{\epsilon}{2}, \quad p(x, x_n) < p(x_n, x_n) + \frac{\epsilon}{2}$$

and

$$p(y, y_n) < p(y_n, y_n) + \frac{\epsilon}{2}$$

for $n \geq n_0$. Now

$$\begin{aligned} p(x_n, y_n) &\leq p(x_n, x) + p(x, y_n) - p(x, x) \\ &\leq p(x_n, x) + p(x, y) + p(y, y_n) - p(y, y) - p(x, x) \\ &< p(x, y) + \frac{\epsilon}{2} + \frac{\epsilon}{2} = p(x, y) + \epsilon, \end{aligned}$$

and so we have

$$p(x_n, y_n) - p(x, y) < \epsilon.$$

Also,

$$\begin{aligned} p(x, y) &\leq p(x, x_n) + p(x_n, y) - p(x_n, x_n) \\ &\leq p(x, x_n) + p(x_n, y_n) + p(y_n, y) - p(y_n, y_n) - p(x_n, x_n) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} + p(x_n, y_n) = p(x_n, y_n) + \epsilon \end{aligned}$$

implies that

$$p(x, y) - p(x_n, y_n) < \epsilon.$$

Hence for all $n \geq n_0$, we have $|p(x_n, y_n) - p(x, y)| < \epsilon$. Hence the result follows. $\square$

Lemma 4. *If p is a partial metric on X , then mappings $p^s, p^m : X \times X \rightarrow \mathbb{R}^+$ given by*

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

and

$$p^m(x, y) = \max \{ p(x, y) - p(x, x), p(x, y) - p(y, y) \}$$

define equivalent metrics on X .

Proof. It is easy to see that p^s and p^m are metrics on X . Obviously,

$$p^m(x, y) \leq p^s(x, y)$$

for every $x, y \in X$. As for every nonnegative real numbers a and b , we have $a + b \leq 2 \max\{a, b\}$, therefore

$$\begin{aligned} p^s(x, y) &= 2p(x, y) - p(x, x) - p(y, y) \\ &\leq 2 \max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\} = 2p^m(x, y). \end{aligned}$$

Hence

$$\frac{1}{2}p^s(x, y) \leq p^m(x, y) \leq p^s(x, y).$$

So p^s and p^m are equivalent. $\square$

Lemma 5 ([8],[9]). *Let (X, p) be a partial metric space.*

(a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s) .

(b) A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Furthermore, $\lim_{n \rightarrow \infty} p^s(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m).$$

Lemma 6. *If $\{x_n\}$ is a convergent sequence in (X, p^s) , then it is a convergent sequence in the partial metric space (X, p) .*

Proof. As, $\lim_{n \rightarrow \infty} p^s(x_n, x) = 0$, and $p(x_n, x_n) \leq p(x_n, x)$. Therefore

$$0 \leq p(x_n, x) - p(x, x) \leq p^s(x_n, x)$$

implies that

$$0 \leq \limsup_{n \rightarrow \infty} p(x_n, x) - p(x, x) \leq \lim_{n \rightarrow \infty} p^s(x_n, x)$$

and consequently, $\lim_{n \rightarrow \infty} p(x_n, x) = p(x, x)$. $\square$

2. MAIN RESULT

We begin this section giving the concept of weakly increasing mappings (see [5]).

Definition 4. Let $(X, \preceq)$ be a partially ordered set. Two mappings $S, T : X \rightarrow X$ are said to be weakly increasing if $Sx \preceq TSx$ and $Tx \preceq STx$ for all $x \in X$.

Note that, two weakly increasing mappings need not be nondecreasing. There exist some examples to illustrate this fact in [3].

Definition 5. Let $(X, \preceq)$ be a partially ordered set. If there exists a partial metric p in X such that $x \preceq y$ implies that $p(x, x) \leq p(y, y)$ for $x, y \in X$ then we said that partial metric p have p -property.

Example 2. Let $X = \mathbb{R}^+$ and $\leq$ be a partially ordered on X . Define $p(x, y) = \max\{x, y\}$. It is easy to see that partial metric p have p -property.

Example 3. Let $X = \mathbb{R}^+$ and $\leq$ be a partially ordered on X . Define $p(x, y) = \max\{(\frac{1}{2})^x, (\frac{1}{2})^y\}$. It is easy to see that partial metric p have not p -property.

Lemma 7. Let (X, p) be a partial metric space and $\varphi : X \rightarrow \mathbb{R}$. Define the relation $\leq$ on X as follows:

$$x \leq y \iff x = y \text{ or } p(x, y) - p(x, x) \leq \varphi(x) - \varphi(y).$$

Then $\leq$ is a (partial) order on X induced by φ .

Proof. i) Since $x = x$ hence $x \leq x$.

ii) Let $x \leq y$ then $p(x, y) - p(x, x) \leq \varphi(x) - \varphi(y)$. Also, if $y \leq x$ then $p(x, y) - p(y, y) \leq \varphi(y) - \varphi(x)$. Therefore,

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \leq 0,$$

thus $x = y$.

iii) Let $x \leq y$ then $p(x, y) - p(x, x) \leq \varphi(x) - \varphi(y)$. Also, if $y \leq z$ then $p(y, z) - p(y, y) \leq \varphi(y) - \varphi(z)$. Therefore,

$$p(x, z) \leq p(x, y) + p(y, z) - p(y, y) \leq \varphi(x) - \varphi(z) - p(x, x),$$

thus $x \leq z$. □

Example 4. Let $X = \{a, b, c\}$. Define $p(a, a) = 1, p(b, b) = 2, p(c, c) = 4, p(a, b) = p(b, a) = 2, p(a, c) = p(c, a) = 4, p(b, c) = p(c, b) = 4$. Define $\varphi(a) = 5, \varphi(b) = 3, \varphi(c) = 1$. It is easy to see that $a \leq b \leq c$. That is $\leq$ is a (partial) order on X induced by φ .

Theorem 1. Let (X, p) be a complete partial metric space such that partial metric p in X have p -property and $\varphi : X \rightarrow \mathbb{R}$ be a function which is bounded from below and $\leq$ the order introduced by φ . Let $T : X \rightarrow 2^X$ be a multivalued mapping, $[x, \infty) = \{y \in X : x \leq y\}$ and $M = \{x \in X \mid T(x) \cap [x, \infty) \neq \emptyset\}$. Suppose that

- (i) T is upper semicontinuous, that is, $x_n \in X$ and $y_n \in T(x_n)$ with $x_n \rightarrow x_0$ and $y_n \rightarrow y_0$, implies $y_0 \in T(x_0)$,
- (ii) $M \neq \emptyset$,
- (iii) for each $x \in M$, $T(x) \cap M \cap [x, \infty) \neq \emptyset$.

Then T has a fixed point x^* and there exists a sequence $\{x_n\}$ with

$$x_{n-1} \leq x_n \in T(x_n), \quad n = 1, 2, 3, \dots$$

such that $x_n \rightarrow x^*$. Moreover if φ is lower semicontinuous, then $x_n \leq x^*$ for all n .

Proof. By the condition (ii), take $x_0 \in M$. From (iii), there exist $x_1 \in T(x_0) \cap M$ and $x_0 \leq x_1$. Again from (iii), there exist $x_2 \in T(x_1) \cap M$. Thus $x_1 \leq x_2$. Continuing this procedure we get a sequence $\{x_n\}$ satisfying

$$x_{n-1} \leq x_n \in T(x_{n-1}), \quad n = 1, 2, 3, \dots$$

So by the definition of $\leq$, we have

$$0 \leq p(x_{n-1}, x_n) - p(x_{n-1}, x_{n-1}) \leq \varphi(x_{n-1}) - \varphi(x_n),$$

that is $\dots \varphi(x_2) \leq \varphi(x_1) \leq \varphi(x_0)$, that is, the sequence $\{\varphi(x_n)\}$ is a nonincreasing sequence in $\mathbb{R}$. Since φ is bounded from below, $\{\varphi(x_n)\}$ is convergent and hence it is Cauchy, that is, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$ we have $|\varphi(x_m) - \varphi(x_n)| < \varepsilon$. Since $x_n \leq x_m$, we have $x_n = x_m$ or $p(x_n, x_m) - p(x_n, x_n) \leq \varphi(x_n) - \varphi(x_m)$. Therefore,

$$\begin{aligned} p(x_n, x_m) - p(x_n, x_n) &\leq \varphi(x_n) - \varphi(x_m) \\ &= |\varphi(x_n) - \varphi(x_m)| \\ &< \varepsilon. \end{aligned}$$

On the other hand, since partial metric p has p -property we have

$$p(x_n, x_m) - p(x_m, x_m) \leq p(x_n, x_m) - p(x_n, x_n) < \varepsilon.$$

Therefore

$$\begin{aligned} p^s(x_n, x_m) &= 2p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m) \\ &< 2\varepsilon. \end{aligned}$$

That is $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s) . Since (X, p) is complete then from Lemma 5, the sequence $\{x_n\}$ converges in the metric space (X, p^s) , say $\lim_{n \rightarrow \infty} p^s(x_n, x^*) = 0$. Again from Lemma 5, we have

$$p(x^*, x^*) = \lim_{n \rightarrow \infty} p(x_n, x^*) = \lim_{n, m \rightarrow \infty} p(x_n, x_m).$$

Since T is upper semicontinuous, $x^* \in T(x^*)$.

Moreover, when φ is lower semicontinuous, by Lemma 3 and above equality for each n we have:

$$\begin{aligned} p(x_n, x^*) - p(x_n, x_n) &= \lim_{m \rightarrow \infty} p(x_n, x_m) - p(x_n, x_n) \\ &\leq \lim_{m \rightarrow \infty} \sup(\varphi(x_n) - \varphi(x_m)) \\ &= \varphi(x_n) - \lim_{m \rightarrow \infty} \inf \varphi(x_m) \\ &\leq \varphi(x_n) - \varphi(x^*). \end{aligned}$$

So $x_n \leq x^*$, for all n . □

Similarly, we can prove the following.

Theorem 2. Let (X, p) be a complete partial metric space such that partial metric p in X have p -property and $\varphi : X \rightarrow \mathbb{R}$ be a function which is bounded from below and $\leq$ the order introduced by φ . Let $T : X \rightarrow 2^X$ be a multivalued mapping, $(-\infty, x] = \{y \in X : y \leq x\}$ and $M = \{x \in X \mid T(x) \cap (-\infty, x] \neq \emptyset\}$. Suppose that

- (i) T is upper semicontinuous, that is, $x_n \in X$ and $y_n \in T(x_n)$ with $x_n \longrightarrow x_0$ and $y_n \longrightarrow y_0$, implies $y_0 \in T(x_0)$,
- (ii) $M \neq \emptyset$,
- (iii) for each $x \in M$, $T(x) \cap M \cap (-\infty, x] \neq \emptyset$.

Then T has a fixed point x^* and there exists a sequence $\{x_n\}$ with

$$x_{n-1} \geq x_n \in T(x_{n-1}), \quad n = 1, 2, 3, \dots$$

such that $x_n \longrightarrow x^*$. Moreover if φ is upper semicontinuous, then $x^* \leq x_n$ for all n .

Theorem 3. Let (X, p) be a complete partial metric space such that partial metric p in X have p -property and $\varphi : X \longrightarrow \mathbb{R}$ be a function which is bounded below and $\leq$ the order introduced by φ . Let $S, T : X \longrightarrow X$ are two weakly increasing mappings, then

- (i) there exists a sequence $\{x_n\}$ with

$$x_n \leq x_{n+1}, \quad n = 1, 2, 3, \dots \quad \text{and} \quad \lim_{n \rightarrow \infty} Sx_{2n} = x^* \quad , \quad \lim_{n \rightarrow \infty} Tx_{2n+1} = x^*,$$

- (ii) if S, T are continuous in (X, p^s) then x^* is common fixed point of S, T .

Proof. Let x_0 be an arbitrary point of X . Set $Sx_0 = x_1$ and $Tx_1 = x_2$. We can define a sequence in X as follows:

$$Sx_{2n} = x_{2n+1} \quad \text{and} \quad Tx_{2n+1} = x_{2n+2} \quad \text{for} \quad n \in \{0, 1, \dots\}.$$

Note that, since S and T are weakly increasing, we have

$$x_1 = Sx_0 \leq TSx_0 = Tx_1 = x_2 \leq STx_1 = Sx_2 = x_3$$

and continuing this process we have

$$x_1 \leq x_2 \leq \dots \leq x_n \leq x_{n+1} \leq \dots$$

So by the definition of $\leq$, we have

$$0 \leq p(x_n, x_{n+1}) - p(x_n, x_n) \leq \varphi(x_n) - \varphi(x_{n+1}),$$

that is $\dots \varphi(x_{n+1}) \leq \varphi(x_n)$, that is, the sequence $\{\varphi(x_n)\}$ is a nonincreasing sequence in $\mathbb{R}$. Since φ is bounded from below, $\{\varphi(x_n)\}$ is convergent and hence it is Cauchy, that is, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$ we have $|\varphi(x_m) - \varphi(x_n)| < \varepsilon$. Since $x_n \leq x_m$, we have $x_n = x_m$ or $p(x_n, x_m) - p(x_n, x_n) \leq \varphi(x_n) - \varphi(x_m)$. Therefore,

$$\begin{aligned} p(x_n, x_m) - p(x_n, x_n) &\leq \varphi(x_n) - \varphi(x_m) \\ &= |\varphi(x_n) - \varphi(x_m)| \\ &< \varepsilon. \end{aligned}$$

On the other hand, since partial metric p has p -property we have

$$p(x_n, x_m) - p(x_m, x_m) \leq p(x_n, x_m) - p(x_n, x_n) < \varepsilon.$$

Therefore

$$\begin{aligned} p^s(x_n, x_m) &= 2p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m) \\ &< 2\varepsilon. \end{aligned}$$

That is $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s) . Since (X, p) is complete then from Lemma 5, the sequence $\{x_n\}$ converges in the metric space (X, p^s) , say $\lim_{n \rightarrow \infty} p^s(x_n, x^*) = 0$. Again from Lemma 5, we have

$$p(x^*, x^*) = \lim_{n \rightarrow \infty} p(x_n, x^*) = \lim_{n, m \rightarrow \infty} p(x_n, x_m).$$

(ii)

$$Sx^* = \lim_{n \rightarrow \infty} Sx_{2n} = \lim_{n \rightarrow \infty} x_{2n+1} = x^*,$$

and

$$Tx^* = \lim_{n \rightarrow \infty} Tx_{2n+1} = \lim_{n \rightarrow \infty} x_{2n+2} = x^*.$$

□

Example 5. Now, consider the $X = [0, 1]$. Define $\varphi : X \rightarrow \mathbb{R}$ by $\varphi(x) = 1 - x$ and define $S, T : X \rightarrow X$ by $Sx = \frac{x+1}{2}$ and $Tx = \frac{2x+1}{3}$. If $p(x, y) = \max\{x, y\}$ then it is easy to see that $\leq$ is a partial order relation on X induced by φ , $Tx \leq STx$ and $Sx \leq TSx$, that is S, T are two weakly increasing mappings.

Hence all of the conditions of Theorem 3 are hold. That is there exists $x^* = 1 \in X$ such that $Tx^* = Sx^* = x^* = 1$.

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GLOBAL OPTIMAL SOLUTIONS FOR NONCYCLIC MAPPINGS IN G -METRIC SPACES

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Abstract. In this paper, the existence of solutions of some minimization problems for noncyclic mappings in G -metric spaces is studied. Our results can be considered as an extension of Abkar and Gabeleh's result [*Global Optimal Solutions of Noncyclic Mappings in Metric Spaces*, J. Optim. Theory. Appl. **153** (2011), 298–305] to the case of G -metric spaces.

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1. INTRODUCTION

In 2011, Abkar et al. [2] studied the existence of solutions of some specific minimization problems for noncyclic mappings in metric spaces. In 2006, Mustafa et al. [11] introduced the G -metric spaces as a generalization of the notion of metric spaces. Fixed point results and other results in G -metric spaces have been proved by a number of authors, see, e.g., [1, 3–5, 12, 14, 15]. In this paper we investigate some minimization problems for noncyclic mappings in G -metric spaces. This work extends results of Abkar et al. [2] to the case of G -metric spaces.

2. PRELIMINARIES

Throughout this paper, N is the set of all natural numbers and R is the set of all real numbers. Generalizations of the notion of a metric space have been proposed by Gabler [8, 9] and by Dhage [6, 7]. Mustafa et al. [11] introduced a more appropriate notion of a generalized metric space as following.

Definition 1. Let X be a nonempty set, and $G : X \times X \times X \rightarrow R^+$ be a function satisfying the following conditions:

- (1) $G(x, y, z) = 0$ if $x = y = z$,
- (2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

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- (3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
- (4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$,
- (5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$,

The function G is called a generalized metric, or, a G -metric on X , and the pair (X, G) is called a G -metric space.

Example 1. ([11, Example 6.3]) Let (X, d) be a metric space and define the functions G_s and G_m with

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z), \quad \forall x, y, z \in X$$

$$G_m(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}, \quad \forall x, y, z \in X$$

Then (X, G_s) and (X, G_m) are G -metric space.

Now, we recall some of the basic concepts for G -metric spaces from ([11]).

Definition 2. Let (X, G) be a G -metric space, and $\{x_n\}$ be a sequence of points of X , we say that $\{x_n\}$ is G -convergent to x and write $x_n \xrightarrow{G} x$ if $\lim_{n,m \rightarrow \infty} G(x, x_n, x_m) = 0$, that is, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq n_0$.

Proposition 1. Let (X, G) be a G -metric space, then the following are equivalent.

- (1) $\{x_n\}$ is G -convergent to x .
- (2) $\lim_{n \rightarrow \infty} G(x, x_n, x_n) = 0$.
- (3) $\lim_{n \rightarrow \infty} G(x, x, x_n) = 0$.

Definition 3. Let (X, G) be a G -metric space, a $\{x_n\}$ is called G -Cauchy for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq n_0$ that is $\lim_{n,m,l \rightarrow \infty} G(x_n, x_m, x_l) = 0$

Proposition 2. Let (X, G) be a G -metric space, then the following are equivalent.

- (1) $\{x_n\}$ is G -Cauchy.
- (2) For any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq n_0$

Definition 4. Let (X_1, G_1) and (X_2, G_2) be G -metric spaces. A function $f : (X_1, G_1) \rightarrow (X_2, G_2)$ is G -continuous at a point $a \in X$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X_1$, $G_1(a, x, y) < \delta$ implies $G_2(f(a), f(x), f(y)) < \epsilon$. A function f is G -continuous on X if and only if it is G -continuous at all $a \in X$.

Proposition 3. Let (X_1, G_1) and (X_2, G_2) be G -metric spaces. A function $f : (X_1, G_1) \rightarrow (X_2, G_2)$ is G -continuous at a point $x \in X$ if and only if whenever $\{x_n\}$ is G -convergent to x , $\{f(x_n)\}$ is G -convergent to $f(x)$.

Definition 5. A G -metric space (X, G) is said to be G -complete if every G -Cauchy sequence in (X, G) is G -convergent in (X, G) .

Definition 6. Let (X, G) be a G -metric space. A G -Ball with center x_0 and radius r is

$$B_G(x_0, r) = \{x \in X : G(x_0, y, y) < r\}.$$

Definition 7. Let (X, G) be a G -metric space and $\epsilon > 0$ be given, then a set $A \subset X$ is called ϵ -net of (X, G) if given any x there is at last one point $a \in A$ such that $x \in B_G(a, \epsilon)$. If the A is finite then A is called a finite ϵ -net of (X, G) . Note that if A is an ϵ -net then $X = \bigcup_{a \in A} B_G(a, \epsilon)$.

Definition 8. A G -metric space (X, G) is called G -totally bounded if for every $\epsilon > 0$ there exists a finite ϵ -net.

Definition 9. A G -metric space (X, G) is called G -compact space if it is G -complete and G -totally bounded.

Proposition 4. Let (X, G) be a G -metric space, then the following are equivalent.

- (1) (X, G) is a G -compact space.
- (2) (X, G) is G -sequentially compact, that is, if the sequence $\{x_n\} \subset X$ is such that $\sup\{G(x_n, x_m, x_l) : n, m, l \in \mathbb{N}\} < \infty$, then $\{x_n\}$ has a G -convergent subsequence.

Theorem 1 ([12], Theorem 2.1). Let (X, G) be a G -metric space and $T : X \rightarrow X$ be a mapping which satisfies the following condition, for all $x, y, z \in X$,

$$G(T(x), T(y), T(z)) \leq k \max\{G(x, y, z), G(x, T(x), T(x)), G(y, T(y), T(y)), G(z, T(z), T(z)), G(x, T(y), T(y)), G(y, T(z), T(z)), G(z, T(x), T(x))\}, \quad (2.1)$$

where $k \in [0, 1/2)$. Then T has a unique fixed point (say u) and T is G -continuous at u .

Definition 10. Let A, B, C be subsets of a G -metric space (X, G) . A mapping $T : A \cup B \cup C \rightarrow A \cup B \cup C$ is called relatively G -nonexpansive if

$$G(T(x), T(y), T(z)) \leq G(x, y, z), \quad \forall (x, y, z) \in A \times B \times C.$$

Definition 11. Let (X, G) be a G -metric space and $A, B, C \subset X$, then

$$\text{dist}(A, B, C) = \inf\{G(a, b, c) : a \in A, b \in B, c \in C\}.$$

Example 2. Let R be equipped with the usual metric, and $A = [-1, 0]$ and $B = N_o$ and $C = N_e$ where N_o and N_e are the set of odd natural numbers and even natural numbers, respectively. Let $G_m(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\}$, then $\text{dist}(A, B, C) = 2$.

Definition 12. Let (X, G) be a G -metric space and $A, B, C \subset X$, $T : A \cup B \cup C \rightarrow A \cup B \cup C$ is said noncyclic mapping, if

$$T(A) \subset A, \quad T(B) \subset B, \quad T(C) \subset C.$$

We consider the following minimization problem: Find

$$\begin{aligned} \min_{a \in A} \{G(a, T(a), T(a))\}, \quad \min_{b \in B} \{G(b, T(b), T(b))\}, \\ \min_{b \in B} \{G(c, T(c), T(c))\}, \quad \min_{(a,b,c) \in A \times B \times C} \{G(a, b, c)\} \end{aligned} \quad (2.2)$$

We say that $(x^*, y^*, z^*) \in A \times B \times C$ is a solution of above problem, if

$$Tx^* = x^*, \quad Ty^* = y^*, \quad Tz^* = z^*,$$

and

$$G(x^*, y^*, z^*) = \text{dist}(A, B, C).$$

Definition 13. Let (X, G) be a G -metric space and $A, B, C \subset X$, we set

$$A_0 = \{a \in A : G(a, b, c) = \text{dist}(A, B, C), \text{ for some } b \in B, c \in C\}$$

$$B_0 = \{b \in B : G(a, b, c) = \text{dist}(A, B, C), \text{ for some } a \in A, c \in C\}$$

$$C_0 = \{c \in C : G(a, b, c) = \text{dist}(A, B, C), \text{ for some } a \in A, b \in B\}$$

Definition 14. Let (X, G) be a G -metric space and A, B, C be nonempty subsets of X , with $A_0 \neq \emptyset$. We say that A, B, C have P -property iff

$$\begin{cases} G(x_1, y_1, z_1) = \text{dist}(A, B, C) \\ G(x_2, y_2, z_2) = \text{dist}(A, B, C) \\ G(x_3, y_3, z_3) = \text{dist}(A, B, C) \end{cases}$$

then

$$G(x_1, x_2, x_3) = G(y_1, y_2, y_3) = G(z_1, z_2, z_3),$$

where $x_1, x_2, x_3 \in A_0$ and $y_1, y_2, y_3 \in B_0$ and $z_1, z_2, z_3 \in C_0$.

The above definition were found in the case of metric space in ([13]).

Example 3. Let A, B, C be nonempty subsets of a G -metric space (X, G) such that $A_0 \neq \emptyset$ and $\text{dist}(A, B, C) = 0$, then A, B, C have P -property.

Definition 15. Let (X, G) be a G -metric space and $T : X \rightarrow X$ be a mapping. T is called expansive if for all $x, y, z \in X$,

$$G(T(x), T(y), T(z)) \geq G(x, y, z).$$

Definition 16. Let (X, G) be a G -metric space and $T : X \rightarrow X$ be a mapping. T is said to be asymptotically regular iff $\lim_{n \rightarrow \infty} G(T^n x, T^{n+1} x, T^{n+1} x) = 0$, for all $x \in X$.

3. MAIN RESULTS

We start this section with the following theorem.

Theorem 2. *Let A, B, C be nonempty and closed subsets of a G -complete space (X, G) such that $A_0 \neq \emptyset$ and A, B, C satisfies the P -property. Let $T : A \cup B \cup C \rightarrow A \cup B \cup C$ be a noncyclic mapping. Suppose that*

- (1) $T|_A$ be a mapping which satisfies in (2.1).
- (2) T is relatively G -nonexpansive.

Then the minimization problem (2.2) has a solution.

Proof. If $x \in A_0$, then there exist $y \in B$ and $z \in C$ such that $G(x, y, z) = \text{dist}(A, B, C)$. Since T is relatively G -nonexpansive then

$$G(T(x), T(y), T(z)) \leq G(x, y, z) = \text{dist}(A, B, C)$$

Hence $Tx \in A_0$.

Let $x_0 \in A_0$ by Theorem 1 if $x_n = T^n(x_0)$ then $x_n \xrightarrow{G} x^*$ where x^* is unique fixed point of T in A . Since $x_0 \in A_0$ there exist $y_0 \in B$ and $z_0 \in C$ such that $G(x_0, y_0, z_0) = \text{dist}(A, B, C)$. Since $x_1 = Tx_0 \in A_0$, there exist $y_1 \in B$ and $z_1 \in C$ such that $G(x_1, y_1, z_1) = \text{dist}(A, B, C)$. Using this process, we have a sequence $\{y_n\}$ in B and $\{z_n\}$ in C such that

$$G(x_n, y_n, z_n) = \text{dist}(A, B, C) \quad \forall n \in N \cup \{0\}.$$

Since A, B, C have the P -property, we have for all $m, n, l \in N \cup \{0\}$

$$G(x_n, x_m, x_l) = G(y_n, y_m, y_l) = G(z_n, z_m, z_l).$$

This implies that $\{y_n\}$ and $\{z_n\}$ are G -Cauchy sequences, and there exist $y^* \in B$ and $z^* \in C$ such that $y_n \xrightarrow{G} y^*$ and $z_n \xrightarrow{G} z^*$. Thus

$$G(x^*, y^*, z^*) = \lim_{n \rightarrow \infty} G(x_n, y_n, z_n) = \text{dist}(A, B, C)$$

Since

$$G(T(x^*), T(y^*), T(z^*)) \leq G(x^*, y^*, z^*) = \text{dist}(A, B, C)$$

Therefore by the P -property, we have

$$G(x^*, T(x^*), T(x^*)) = G(y^*, T(y^*), T(y^*)) = G(z^*, T(z^*), T(z^*))$$

Thus $(x^*, y^*, z^*) \in A \cup B \cup C$ is a solution of the minimization problem (2.2). $\square$

Example 4. Let R be equipped with the usual metric, and $G_m(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\}$. Let $A = [-2, 0]$ and $B = \{1\}$ and $C = [2, 3]$. It is obvious that

$A_0 = \{0\}, B_0 = \{1\}, C_0 = \{2\}$. Define $T : A \cup B \cup C \rightarrow A \cup B \cup C$ with

$$T(x) = \begin{cases} \frac{x}{4} & x \in A \\ 1 & x \in B \\ \frac{x+2}{2} & x \in C \end{cases}$$

It is easy to check that all the conditions of Theorem 2 hold. Therefore, the minimization problem (2.2) has a solution $(x^*, y^*, z^*) = (0, 1, 2)$.

Theorem 3. Let A, B, C be nonempty subsets of a G -complete space (X, G) such that A is G -compact and B and C are G -closed. Let $A_0 \neq \emptyset$ and A, B, C satisfy the P -property. Let $T : A \cup B \cup C \rightarrow A \cup B \cup C$ be a noncyclic mapping. Then the minimization problem (2.2) has a solution provided that the following conditions are satisfied:

- (1) T is relatively G -nonexpansive.
- (2) $T|_A$ is a G -expansive.
- (3) $T|_B$ and $T|_C$ be mappings which satisfy in (2.1).

Proof. If $x \in A_0$, and $x_{n+1} = Tx_n, (n \in N \cup \{0\})$. By argument similar in the proof of Theorem 2 we obtain that $T(A_0) \subset A_0$ and there exist y_n in B and z_n in C such that

$$G(x_n, y_n, z_n) = \text{dist}(A, B, C) \quad \forall n \in N \cup \{0\}.$$

Since A is G -compact, by Proposition 4 there exist a subsequence $\{x_{n_k}\}$ of the $\{x_n\}$ such that $x_{n_k} \xrightarrow{G} x^* \in A$. Since A, B, C satisfy the P -property,

$$G(x_{n_k}, x_{n_s}, x_{n_l}) = G(y_{n_k}, y_{n_s}, y_{n_l}) = G(z_{n_k}, z_{n_s}, z_{n_l}), \quad (k, s, l \in N).$$

This implies that $\{y_n\}$ and $\{z_n\}$ are G -Cauchy sequences and there exist $y^* \in B$ and $z^* \in C$ such that $y_{n_k} \xrightarrow{G} y^*$ and $z_{n_k} \xrightarrow{G} z^*$. Thus

$$G(x^*, y^*, z^*) = \lim_{n \rightarrow \infty} G(x_{n_k}, y_{n_k}, z_{n_k}) = \text{dist}(A, B, C)$$

Now we prove that $x^*, y^*, z^* \in F(T)$. Since T is relatively G -nonexpansive,

$$G(T^2(x^*), T^2(y^*), T^2(z^*)) = G(T(x^*), T(y^*), T(z^*)) = \text{dist}(A, B, C).$$

Since A, B, C satisfy the P -property, we have

$$G(x^*, T(x^*), T(x^*)) = G(y^*, T(y^*), T(y^*)) = G(z^*, T(z^*), T(z^*)),$$

and

$$\begin{aligned} G(T(x^*), T^2(x^*), T^2(x^*)) &= G(T(y^*), T^2(y^*), T^2(y^*)) \\ &= G(T(z^*), T^2(z^*), T^2(z^*)). \end{aligned}$$

Now let $Ty^* \neq T^2y^*$, since $T|_B$ satisfies in (2.1),

$$G(T(y^*), T(T(y^*)), T(T(y^*))) \leq kG(y^*, T(y^*), T(y^*))$$

Thus since $T|_A$ is a G -expansive, we have

$$\begin{aligned} G(T(y^*), T^2(y^*), T^2(y^*)) &= G(T(y^*), T(T(y^*)), T(T(y^*))) \\ &\leq kG(y^*, T(y^*), T(y^*)) \\ &= kG(x^*, T(x^*), T(x^*)) \\ &\leq kG(T(x^*), T^2(x^*), T^2(x^*)) \\ &= kG(T(y^*), T^2(y^*), T^2(y^*)), \end{aligned}$$

which is a contraction. Therefore $Ty^* = T^2y^*$. A similar argument implies that $Tz^* = T^2z^*$. Thus $x^* = T(x^*)$ and $y^* = T(y^*)$ and $z^* = T(z^*)$. $\square$

Example 5. Let $X = R^3$ and

$$G((x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)) = \max\{G_m(x_1, x_2, x_3), G_m(y_1, y_2, y_3), G_m(z_1, z_2, z_3)\},$$

where $G_m(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\}$. Let $A = \{(x, 0, 0) : -1 \leq x \leq 0\}$ and $B = \{(0, y, 0) : 0 \leq y \leq 1\}$ and $C = \{(0, 0, z) : -1 \leq z \leq 1\}$. It is obvious that $A_0 = B_0 = C_0 = \{(0, 0, 0)\}$ and $\text{dist}(A, B, C) = 0$, therefore A, B, C have the P -property. Define $T : A \cup B \cup C \rightarrow A \cup B \cup C$ with

$$T(x, 0, 0) = (-x, 0, 0), \quad T(0, y, 0) = (0, \frac{y}{4}, 0) \quad \text{and} \quad T(0, 0, z) = (0, 0, \frac{z}{4}).$$

It is easy to check that all the conditions of Theorem 3 hold. Therefore the minimization problem (2.2) has a solution $x^* = y^* = z^* = (0, 0, 0)$.

Theorem 4. Let A, B, C be nonempty subsets of a G -complete space (X, G) such that A is G -compact and B and C are G -closed. Let $A_0 \neq \emptyset$ and A, B, C satisfy the P -property. Let $T : A \cup B \cup C \rightarrow A \cup B \cup C$ be a noncyclic mapping. Then the minimization problem (2.2) has a solution provided that the following conditions are satisfied:

- (1) T is relatively G -nonexpansive.
- (2) $T|_A$ is G -continuous and asymptotically regular.

Proof. Let $\{x_n\}_G, \{y_n\}, \{z_n\}, \{x_{n_k}\}, \{y_{n_k}\}, \{z_{n_k}\}, x^*, y^*$ and z^* be as in Theorem 3. We have $x_{n_k} \xrightarrow{G} x^* \in A$, $y_{n_k} \rightarrow y^* \in B$, $z_{n_k} \xrightarrow{G} z^* \in C$ and $G(x^*, y^*, z^*) = \text{dist}(A, B, C)$. From Proposition 3, since $T|_A$ is G -continuous, we have

$$x_{n_k+1} = T(x_{n_k}) \xrightarrow{G} T(x^*).$$

Also by the asymptotic regularity of $T|_A$, we obtain

$$\begin{aligned} G(x^*, T(x^*), T(x^*)) &= \lim_{k \rightarrow \infty} G(x_{n_k}, T(x_{n_k}), T(x_{n_k})) \\ &= \lim_{k \rightarrow \infty} G(T^{n_k}(x_0), T^{n_k+1}(x_0), T^{n_k+1}(x_0)) \\ &= 0. \end{aligned}$$

This implies that $T(x^*) = x^*$. Since T is relatively G –nonexpansive, we have

$$G(T(x^*), T(y^*), T(z^*)) \leq G(x^*, y^*, z^*) = \text{dist}(A, B, C)$$

Therefore by the P –property, we have

$$G(x^*, T(x^*), T(x^*)) = G(y^*, T(y^*), T(y^*)) = G(z^*, T(z^*), T(z^*))$$

Hence $T(y^*) = y^*$ and $T(z^*) = z^*$. □

QUESTION: In 2011, Karapinar [10] obtain some common fixed point results in partial metric spaces. Can one study the minimization problem (2.2) for two mappings in partial metric spaces?

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ON THE EXISTENCE OF DIOPHANTINE QUADRUPLES IN $\mathbb{Z}[\sqrt{-2}]$

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Abstract. By the work of Abu Muriefah, Al-Rashed, Dujella and the author, the problem of the existence of $D(z)$ -quadruples in the ring $\mathbb{Z}[\sqrt{-2}]$ has been solved, except for the cases

$$\begin{aligned} z &= 24a + 2 + (12b + 6)\sqrt{-2}, z = 24a + 5 + (12b + 6)\sqrt{-2}, \\ z &= 48a + 44 + (24b + 12)\sqrt{-2}. \end{aligned}$$

In this paper, we present some new formulas for $D(z)$ -quadruples in these remaining cases, involving some congruence conditions modulo 11 on integers a and b . We show the existence of $D(z)$ -quadruple for significant proportion of the remaining three cases.

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1. INTRODUCTION

Let z be an element of a commutative ring R . A *Diophantine quadruple* with the property $D(z)$, or a $D(z)$ -*quadruple*, is a set D of four non-zero elements of R with the property that the product of any two distinct elements of this set increased by z is a square of some element in R . Any set D satisfying this condition is called a *set with the property $D(z)$* .

Greek mathematician Diophantus of Alexandria found four positive rational numbers $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ with the property $D(1)$ (see [4]). The first $D(1)$ -quadruple in integers, the set $\{1, 3, 8, 120\}$, was found by Fermat (see [4, 15]). Baker and Davenport proved that Fermat's set cannot be extended to a $D(1)$ -quintuple in integers (see [2]). In 2004, Dujella proved that in $\mathbb{Z}$ there does not exist a $D(1)$ -sextuple and there are only finitely many $D(1)$ -quintuples (see [8]).

Many authors considered the problem of the existence of Diophantine quadruples in integers. Brown (see [3]) proved that if $n \in \mathbb{Z}$, $n \equiv 2 \pmod{4}$, then there does not exist a $D(n)$ -quadruple. On the other hand, Dujella (see [5]), proved that if $n \not\equiv 2 \pmod{4}$ and if $n \notin S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists at least one $D(n)$ -quadruple, and he conjectured that there does not exist a $D(n)$ -quadruple for $n \in S$.

Dujella also considered the existence of $D(a + b\sqrt{-1})$ -quadruples in the ring of Gaussian integers. In [6], it was shown that if b is odd or $a \equiv b \equiv 2 \pmod{4}$, then there does not exist a $D(a + b\sqrt{-1})$ -quadruple, and if $z = a + b\sqrt{-1}$ is not of that form and $z \notin \{\pm 2, \pm 1 \pm 2i, \pm 4i\}$, then there exist at least two distinct Diophantine quadruples with the property $D(z)$. Franušić also gave some results on Diophantine quadruples in Gaussian integers (see [13]), and in [11, 12, 14] completely solved the analogous problem in some real quadratic fields.

In addition to results of Dujella and Franušić in Gaussian integers, the case of the ring $\mathbb{Z}[\sqrt{-2}]$ of integers in the quadratic field $\mathbb{Q}(\sqrt{-2})$ is the only case of a complex quadratic field studied until now. It started by the work of Abu Muriefah and Al-Rashed in [1]. Dujella and Soldo extended the results of Abu Muriefah and Al-Rashed and obtained several new polynomial formulas for Diophantine quadruples with the property $D(a + b\sqrt{-2})$, for integers a and b satisfying certain congruence conditions. Their main result is

Theorem 1 ([10], Theorem 1.1). *Let $z \in \mathbb{Z}[\sqrt{-2}]$. If z is of the form $z = a + (2b + 1)\sqrt{-2}$ or $z = 4a + (4b + 2)\sqrt{-2}$, $a, b \in \mathbb{Z}$, then there does not exist a $D(z)$ -quadruple. If z is not of that form, then there exists at least one $D(z)$ -quadruple, except maybe if z has one of the following forms:*

$$\begin{aligned} z &= 24a + 2 + (12b + 6)\sqrt{-2}, z = 24a + 5 + (12b + 6)\sqrt{-2}, \\ z &= 48a + 44 + (24b + 12)\sqrt{-2}, \end{aligned}$$

or if $z \in \{-1, 1 \pm 2\sqrt{-2}\}$.

In this paper we considered the existence of $D(z)$ -quadruple for $z = 24a + 2 + (12b + 6)\sqrt{-2}$, $z = 24a + 5 + (12b + 6)\sqrt{-2}$, $z = 48a + 44 + (24b + 12)\sqrt{-2}$, these are cases not covered by the research described in [10]. By [9, Theorem 1], each of these elements can be represented as a difference of two squares of elements in $\mathbb{Z}[\sqrt{-2}]$. Thus, in an analogy to what is known in $\mathbb{Z}$ (see [5]) and certain quadratic fields (see [6, 11, 12, 14]), it seems reasonable to expect that for such z 's there exists at least one $D(z)$ -quadruple, with perhaps finitely many exceptions.

The methods for the construction of Diophantine quadruples usually use elements with small norm. We proved that for such z 's, quadruples cannot contain some elements with very small norm, as in the formulas in Sections 3 and 4 of [10].

Moreover, for finding formulas for Diophantine quadruples, we used the method described in details in [10, Section 2]. Using the fact that numbers 11 and 22 factorizes in $\mathbb{Z}[\sqrt{-2}]$ as $11 = (3 + \sqrt{-2})(3 - \sqrt{-2})$ and $22 = (2 + 3\sqrt{-2})(2 - 3\sqrt{-2})$, we found some formulas for quadruples containing the elements $2 \pm 3\sqrt{-2}$ with norm 22 and $3 \pm \sqrt{-2}$ with norm 11. By studying them carefully, we obtained our main results, covered by Propositions 3, 4, and 5, presented in Section 3. Namely, we showed the existence of $D(z)$ -quadruples for significant proportion of the remaining

three cases. These new formulas then necessarily involve some congruence conditions modulo 11 on a and b , which makes those cases harder to handle.

We also specify very simple result which will be used later on.

Lemma 1. *Let $\{z_1, z_2, z_3, z_4\} \subset \mathbb{Z}[\sqrt{-2}]$ be a set with the property $D(z)$ and $w \in \mathbb{Z}[\sqrt{-2}]$. Then*

- (i) *the set $\{\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4\}$ has the property $D(\bar{z})$,*
- (ii) *the set $\{z_1w, z_2w, z_3w, z_4w\}$ has the property $D(zw^2)$.*

Note that by multiplying elements of a $D(24a + 2 + (12b + 6)\sqrt{-2})$ -quadruple by $\sqrt{-2}$, by Lemma 1(ii) we obtain (with obvious substitution) a $D(48a + 44 + (24b + 12)\sqrt{-2})$ -quadruple. Thus, it is sufficient to consider $D(z)$ -quadruples for $z = 24a + 2 + (12b + 6)\sqrt{-2}$ and $z = 24a + 5 + (12b + 6)\sqrt{-2}$.

2. ELEMENTS OF SMALL NORM

Concerning to three unsolved cases from Theorem 1, it can be shown that such quadruples cannot contain some elements with small norm.

Proposition 1. *Let $z = 24a + 5 + (12b + 6)\sqrt{-2}$, $a, b \in \mathbb{Z}$ and $s \in \{\pm 1, \pm 3, \pm 4, \pm 1 \pm \sqrt{-2}, \pm 2 \pm \sqrt{-2}, \pm 1 \pm 2\sqrt{-2}\}$. Then there does not exist a $D(z)$ -quadruple of the form $\{s, t, u, v\}$ in $\mathbb{Z}[\sqrt{-2}]$.*

Proof. Let $z = 24a + 5 + (12b + 6)\sqrt{-2}$, $a, b \in \mathbb{Z}$.

(1) Suppose that $s = 1$ and $\{1, t, u, v\}$ is $D(z)$ -quadruple in $\mathbb{Z}[\sqrt{-2}]$. Then there exists an element $u_1 \in \mathbb{Z}[\sqrt{-2}]$ such that $t + z = u_1^2$. We will look this equation modulo 4. Since

$$u_1^2 \equiv 0, 1, 2, 3 + 2\sqrt{-2} \pmod{4},$$

$$z \equiv 1 + 2\sqrt{-2} \pmod{4},$$

from $t = u_1^2 - z$, we obtain $t \equiv t' \pmod{4}$, where

$$t' \in S = \{2, 2\sqrt{-2}, 1 + 2\sqrt{-2}, 3 + 2\sqrt{-2}\}.$$

Similarly, $u \equiv u' \pmod{4}$, $v \equiv v' \pmod{4}$, where $u', v' \in S$.

On the other hand, there exist $u_2, u_3, u_4 \in \mathbb{Z}[\sqrt{-2}]$, such that $tu + z = u_2^2$, $tv + z = u_3^2$, $uv + z = u_4^2$. In a same way, we obtain $tu \equiv (tu)' \pmod{4}$, $tv \equiv (tv)' \pmod{4}$, $uv \equiv (uv)' \pmod{4}$, where $(tu)', (ts)', (us)' \in S$.

This means that there exist three (equal or distinct) elements of the set S , such that the product of any two of them is some element of S . It is easy to check that this is not possible.

From Lema 1(ii) for $w = -1$, it follows that there does not exist a $D(z)$ -quadruple of the form $\{-1, t, u, v\}$.

(2) If we suppose that $s = 4$ and $\{4, t, u, v\}$ is $D(z)$ -quadruple in $\mathbb{Z}[\sqrt{-2}]$, then there exists an element $w_1 \in \mathbb{Z}[\sqrt{-2}]$ such that $4t + z = w_1^2$. Since, $w_1^2 - z \equiv t_1 \pmod{4}$, where $t_1 \in S$, from $4t = w_1^2 - z$ we obtain a contradiction. The nonexistence of a $D(z)$ -quadruple of the form $\{-4, t, u, v\}$ immediately follows from Lemma 1(ii).

(3) Let $s = 1 \pm \sqrt{-2}$ and $\{s, t, u, v\}$ be a $D(z)$ -quadruple in $\mathbb{Z}[\sqrt{-2}]$. Then there exists an element $w_2 \in \mathbb{Z}[\sqrt{-2}]$ such that $st + z = w_2^2$. We have

$$\begin{aligned} w^2 &\equiv 0, 1 \pmod{s}, \\ z &\equiv 2 \pmod{s}. \end{aligned}$$

Then from $st = w_2^2 - z$ we get a contradiction.

Since $1 + \sqrt{-2}$ or $1 - \sqrt{-2}$ is a divisor of any element of the set

$$T = \{\pm 3, -1 \pm \sqrt{-2}, \pm 2 \pm \sqrt{-2}, \pm 1 \pm 2\sqrt{-2}\},$$

we conclude that there does not exist $D(z)$ -quadruple of the form $\{s, t, u, v\}$, $s \in T$. $\square$

Proposition 2. Let $z = 24a + 2 + (12b + 6)\sqrt{-2}$, $a, b \in \mathbb{Z}$ and $s \in \{\pm 1, \pm 3, \pm 4, \pm 1 \pm \sqrt{-2}, \pm 2 \pm \sqrt{-2}, \pm 1 \pm 2\sqrt{-2}\}$. Then there does not exist a $D(z)$ -quadruple of the form $\{s, t, u, v\}$ in $\mathbb{Z}[\sqrt{-2}]$.

Proof. Let $z = 24a + 2 + (12b + 6)\sqrt{-2}$, $a, b \in \mathbb{Z}$.

(1) Suppose that $s = 1$ and $\{1, t, u, v\}$ is $D(z)$ -quadruple in $\mathbb{Z}[\sqrt{-2}]$. Then there exists an element $u_1 \in \mathbb{Z}[\sqrt{-2}]$ such that $t + z = u_1^2$. We will look this equation modulo 8. We have

$$u_1^2 \equiv 0, 1, 4, 6, 1 + 4\sqrt{-2}, 2 + 4\sqrt{-2}, 7 + 2\sqrt{-2}, 7 + 6\sqrt{-2} \pmod{8}.$$

From $12b + 6 \equiv 2, 6 \pmod{8}$, we conclude:

(i) If $z \equiv 2 + 2\sqrt{-2} \pmod{8}$, then from $t = u_1^2 - z$ we get $t \equiv t' \pmod{8}$, where

$$t' \in S_1$$

$$= \{5, 2\sqrt{-2}, 7 + 2\sqrt{-2}, 5 + 4\sqrt{-2}, 2 + 6\sqrt{-2}, 4 + 6\sqrt{-2}, 6 + 6\sqrt{-2}, 7 + 6\sqrt{-2}\}.$$

Similarly, $u \equiv u' \pmod{8}$, $v \equiv v' \pmod{8}$, $tu \equiv (tu)' \pmod{8}$, $tv \equiv (tv)' \pmod{8}$, $uv \equiv (uv)' \pmod{8}$, where $u', v', (tu)', (tv)', (uv)' \in S_1$.

(ii) If $z \equiv 2 + 6\sqrt{-2} \pmod{8}$, in the same way we get $t', u', v', (tu)', (tv)', (uv)' \in S_2$, where

$$S_2 = \{5, 6\sqrt{-2}, 2 + 2\sqrt{-2}, 4 + 2\sqrt{-2}, 6 + 2\sqrt{-2}, 7 + 2\sqrt{-2}, 5 + 4\sqrt{-2}, 7 + 6\sqrt{-2}\}.$$

Now, in both cases, it is easy to check that there does not exist three (equal or distinct) elements of the sets S_1 and S_2 , such that the product of any two of them is some element of those sets. Using the Lemma 1(ii), we conclude that there does not exist a $D(z)$ -quadruple of the form $\{-1, t, u, v\}$.

(2) If we suppose that $s = 4$ and $\{4, t, u, v\}$ is $D(z)$ -quadruple in $\mathbb{Z}[\sqrt{-2}]$, then there exists an element $w_1 \in \mathbb{Z}[\sqrt{-2}]$ such that $4t + z = w_1^2$. Since,

$$\begin{aligned} w_1^2 &\equiv 0, 1, 2, 3 + 2\sqrt{-2} \pmod{4}, \\ z &\equiv 2 + 2\sqrt{-2} \pmod{4}, \end{aligned}$$

we conclude that $w_1^2 - z \equiv 1, 2\sqrt{-2}, 2 + 2\sqrt{-2}, 3 + 2\sqrt{-2} \pmod{4}$. Now, from $4t = w_1^2 - z$, we get a contradiction. Now, Lema 1(ii) implies that there does not exist a $D(z)$ -quadruple of the form $\{-4, t, u, v\}$.

(3) Let $s = 1 \pm \sqrt{-2}$ and $\{s, t, u, v\}$ be a $D(z)$ -quadruple in $\mathbb{Z}[\sqrt{-2}]$. Since $z \equiv 2 \pmod{s}$, the proof is the same as in case (3) in Proposition 1. $\square$

On the other hand, by following the approach from [7], we were able to prove that there exist infinitely many $D(z)$ -quadruples for z of studied forms, containing elements ± 2 and $\pm 2\sqrt{-2}$.

We consider the existence of $D(z)$ -quadruples of the form $\{u, v, u + v + 2r, 4u + v + 4r\}$, $u, v, r \in \mathbb{Z}[\sqrt{-2}]$. The method is based on factorization $z = st, s, t \in \mathbb{Z}[\sqrt{-2}]$. For arbitrary u , let v be an element such that $uv + z = r^2$. The set $\{u, v, u + v + 2r\}$ has the property $D(z)$. Indeed,

$$\begin{aligned} u(u + v + 2r) + z &= (u + r)^2, \\ v(u + v + 2r) + z &= (v + r)^2. \end{aligned}$$

If we apply this construction to the Diophantine pair $\{u, u + v + 2r\}$ we obtain the set $\{u, u + v + 2r, 4u + v + 4r\}$ also with the property $D(z)$. Therefore, the set

$$\{u, v, u + v + 2r, 4u + v + 4r\}$$

has the property $D(z)$, if and only if

$$v(4u + v + 4r) + z = y^2, \quad y \in \mathbb{Z}[\sqrt{-2}].$$

That is equivalent to

$$(v + 2r - y)(v + 2r + y) = 3st.$$

If we set

$$\begin{aligned} v + 2r - y &= s, \\ v + 2r + y &= 3t, \end{aligned} \tag{2.1}$$

we obtain

$$2v + 4r = s + 3t. \tag{2.2}$$

In the first case we set $u = 2$. Using $2v + z = r^2$, from (2.2) it follows that

$$(r + 2)^2 = (s + 3)(t + 1) + 1.$$

Now, we can choose $r + 2 = (t + 1)m \pm 1$ or $r + 2 = (s + 3)m \pm 1$, for $m \in \mathbb{Z}[\sqrt{-2}]$.

Let $r + 2 = (s + 3)m - 1$. Then

$$t + 1 = (s + 3)m^2 - 2m,$$

$$z = st = s(s+3)m^2 - (2m+1)s.$$

If we choose $m = 1 - \sqrt{-2}$ and $s = 6p + 4 + (6q + 3)\sqrt{-2}$, $p, q \in \mathbb{Z}$, we obtain the set

$$\begin{aligned} &\{2, -18p + 12q - 15 + (-6p - 18q - 13)\sqrt{-2}, \\ &-6p + 36q + 7 + (-18p - 6q - 21)\sqrt{-2}, \\ &6p + 60q + 33 + (-30p + 6q - 29)\sqrt{-2}\}, \end{aligned} \quad (2.3)$$

with the property

$$\begin{aligned} &D(60p - 36p^2 + 312q + 288pq + 72q^2 + 98 \\ &+ (-156p - 72p^2 + 60q - 72pq + 144q^2 - 54)\sqrt{-2}). \end{aligned}$$

Let $r + 2 = (t + 1)m + 1$. For $m = 2$ and $t = 24p + 5 + (12q + 6)\sqrt{-2}$, $p, q \in \mathbb{Z}$, we get the set

$$\begin{aligned} &\{2, -12p - 2 + (-6q - 3)\sqrt{-2}, \\ &84p + 22 + (42q + 21)\sqrt{-2}, \\ &180p + 50 + (90q + 45)\sqrt{-2}\}, \end{aligned} \quad (2.4)$$

with the property

$$\begin{aligned} &D(1080p + 2304p^2 - 1152q - 1152q^2 - 163 \\ &+ (1152p + 540q + 2304pq + 270)\sqrt{-2}). \end{aligned}$$

Now, let $u = 2\sqrt{-2}$. From (2.1) and $2\sqrt{-2}v + z = r^2$, we obtain

$$(r + 2\sqrt{-2})^2 = (s + 3\sqrt{-2})(t + \sqrt{-2}) - 2.$$

If we set $r + 2\sqrt{-2} = (s + 3\sqrt{-2})m + \sqrt{-2}$, for $m = 2 - \sqrt{-2}$ and $s = 12p + 6 + (12q + 1)\sqrt{-2}$, $p, q \in \mathbb{Z}$, we get the set

$$\begin{aligned} &\{2\sqrt{-2}, -6p + 96q + 35 + (-48p - 6q - 21)\sqrt{-2}, \\ &42p + 144q + 75 + (-72p + 42q - 17)\sqrt{-2}, \\ &90p + 192q + 115 + (-96p + 90q - 9)\sqrt{-2}\}, \end{aligned} \quad (2.5)$$

with the property

$$\begin{aligned} &D(816p + 288p^2 + 840q + 2304pq - 576q^2 + 314 \\ &+ (-420p - 576p^2 + 816q + 576pq + 1152q^2 - 30)\sqrt{-2}). \end{aligned}$$

Similarly, if $r + 2\sqrt{-2} = (s + 3\sqrt{-2})m - \sqrt{-2}$, for $m = 1 - \sqrt{-2}$ and $s = 6p + 3 + (6q + 1)\sqrt{-2}$, $p, q \in \mathbb{Z}$, we obtain the set

$$\begin{aligned} &\{2\sqrt{-2}, -18p + 12q - 7 + (-6p - 18q - 15)\sqrt{-2}, \\ &-6p + 36q + 15 + (-18p - 6q - 17)\sqrt{-2}, \\ &6p + 60q + 37 + (-30p + 6q - 15)\sqrt{-2}\}, \end{aligned} \quad (2.6)$$

with the property

$$D(60p - 36p^2 + 240q + 288pq + 72q^2 + 53 \\ + (-120p - 72p^2 + 60q - 72pq + 144q^2 - 30)\sqrt{-2}).$$

By Lemma 1(ii) we obtain $D(z)$ -quadruples with elements $-2, -2\sqrt{-2}$.

Remark 1. While, for example, the property

$$D(1080p + 2304p^2 - 1152q - 1152q^2 - 163 \\ + (1152p + 540q + 2304pq + 270)\sqrt{-2})$$

of the set (2.4) can be expressed as $D(24a + 5 + (12b + 6)\sqrt{-2})$, where

$$\begin{aligned} a &= 45p + 96p^2 - 48q - 48q^2 - 7, \\ b &= 96p + 45q + 192pq + 22, \end{aligned} \quad (2.7)$$

for the most of integer pairs (a, b) there does not exist integers p and q such that a and b can be represented by (2.7). Similar situation appears while considering properties of all the other sets. This means that obtained results do not cover any of the remaining unsolved cases of Theorem 1 (regardless of the fact that for arbitrary p and q such sets generate infinitely many quadruples with the certain property).

3. $D(z)$ -QUADRUPLES WITH AN ELEMENT OF NORM 11

In this section, according to three unsolved cases

$$\begin{aligned} z &= 24a + 2 + (12b + 6)\sqrt{-2}, z = 24a + 5 + (12b + 6)\sqrt{-2}, \\ z &= 48a + 44 + (24b + 12)\sqrt{-2}, \end{aligned}$$

of Theorem 1, we consider the existence of a $D(z)$ -quadruple of the form $\{u, v, u + v + 2r, u + 4v + 4r\}$, $u, v, r \in \mathbb{Z}[\sqrt{-2}]$. We obtain some new results by considering a and b modulo 11, using the method in detail described in [10, Section 2]. Anyway, here we will specify that method again, for the convenience of the reader.

Let $\{u, v\}$ be an arbitrary pair with the property $D(z)$, for $z \in \mathbb{Z}[\sqrt{-2}]$. It means that

$$uv + z = r^2,$$

for $r \in \mathbb{Z}[\sqrt{-2}]$. Similarly as in Section 2, the only condition such that the set

$$\{u, v, u + v + 2r, u + 4v + 4r\}$$

has the property $D(z)$ is equal to

$$u(u + 4v + 4r) + z = y^2, \quad (3.1)$$

i.e.

$$3z = (u + 2r - y)(u + 2r + y).$$

This means that there is an element $e \in \mathbb{Z}[\sqrt{-2}]$ such that

$$u + 2r - y = e,$$

$$u + 2r + y = \frac{3z}{e},$$

which gives

$$2u + 4r = \frac{3z\bar{e}}{N(e)} + e, \quad (3.2)$$

where $\bar{e}$ is the conjugate of an element e and $N(e) = e \cdot \bar{e}$ is the norm of e . Suppose that z and e are given. Then, if we look at equation (3.2) modulo 4, we get a condition for the form of u . We choose a u of small norm satisfying this condition. Now, it is easy to find the form of r . It remains to satisfy the condition that $v = (r^2 - z)/u \in \mathbb{Z}[\sqrt{-2}]$, which is equivalent to the divisibility condition

$$N(u) \mid (r^2 - z) \cdot \bar{u}.$$

Proposition 3. *If z is of the form $z = 24a + 5 + (12b + 6)\sqrt{-2}$, then there exists at least one Diophantine quadruple with the property $D(z)$, for any $a, b \in \mathbb{Z}$, except maybe for $a \equiv a' \pmod{11}$, $b \equiv b' \pmod{11}$, where*

$$(a', b') \in \{(0, 3), (0, 4), (0, 6), (0, 7), (1, 1), (1, 5), (1, 9), (2, 1), (2, 2), (2, 8), (2, 9), (3, 5), (4, 3), (4, 5), (4, 7), (5, 0), (5, 2), (5, 8), (5, 10), (6, 0), (6, 5), (6, 10), (7, 2), (7, 5), (7, 8), (8, 4), (8, 5), (8, 6), (9, 1), (9, 3), (9, 7), (9, 9), (10, 0), (10, 4), (10, 6), (10, 10)\}.$$

Proof. Let $z = 24a + 5 + (12b + 6)\sqrt{-2}$, $a, b \in \mathbb{Z}$, and $e = 3$. We choose $u = 2 + 3\sqrt{-2}$ and $N(u) = 22$. Now, from (3.2), it follows that

$$r = 6a + 1 + 3b\sqrt{-2}.$$

From the condition $uv + z = r^2$, it follows that

$$\begin{aligned} v = & \frac{1}{11} (36a^2 - 12a + 108ab - 18b^2 - 18b - 22) \\ & + \frac{1}{11} (-54a^2 + 18a + 36ab + 27b^2 - 6b) \sqrt{-2}. \end{aligned}$$

We get the set

$$\begin{aligned}
& \{2 + 3\sqrt{-2}, \frac{1}{11}(36a^2 - 12a + 108ab - 18b^2 - 18b - 22) \\
& + \frac{1}{11}(-54a^2 + 18a + 36ab + 27b^2 - 6b)\sqrt{-2}, \\
& \frac{1}{11}(36a^2 + 120a + 108ab - 18b^2 - 18b + 22) \\
& + \frac{1}{11}(-54a^2 + 18a + 36ab + 27b^2 + 60b + 33)\sqrt{-2}, \\
& \frac{1}{11}(144a^2 + 216a + 432ab - 72b^2 - 72b - 22) \\
& + \frac{1}{11}(-216a^2 + 72a + 144ab + 108b^2 + 108b + 33)\sqrt{-2}\},
\end{aligned} \tag{3.3}$$

with the property $D(z)$ in $\mathbb{Q}(\sqrt{-2})$. It remains to check for which pairs (a, b) modulo 11, set (3.3) has the elements in the ring $\mathbb{Z}[\sqrt{-2}]$. Equivalently, we have to find all pairs (a, b) which satisfy the condition

$$N(u)|(r^2 - z) \cdot \bar{u},$$

i.e.

$$\begin{aligned}
72a^2 - 24a + 216ab - 36b^2 - 36b - 44 &\equiv 0 \pmod{11}, \\
-108a^2 + 36a + 72ab + 54b^2 - 12b &\equiv 0 \pmod{11}.
\end{aligned}$$

It is easy to check that the above condition is satisfied for all $(a, b) = (11k + a', 11l + b')$, $k, l \in \mathbb{Z}$, where

$$\begin{aligned}
(a', b') \in \{ & (0, 0), (0, 10), (1, 2), (1, 3), (2, 5), (2, 6), (3, 8), (3, 9), (4, 0), (4, 1), \\
& (5, 3), (5, 4), (6, 6), (6, 7), (7, 9), (7, 10), (8, 1), (8, 2), (9, 4), (9, 5), \\
& (10, 7), (10, 8) \}.
\end{aligned}$$

Let us determine the pairs (a, b) for which set (3.3) has at least two equal elements or some elements equal to zero. If $(a, b) = (11k, 11l)$, the above cases appear if $(k, l) = (0, 0)$. But for that pair the set $\{1 - 3\sqrt{-2}, -3 + \sqrt{-2}, 6 + 2\sqrt{-2}, 5 + 9\sqrt{-2}\}$ has the property $D(5 + 6\sqrt{-2})$. If $(a, b) = (11k, 11l + 10)$, the above cases appear if $(k, l) = (0, -1)$. But in this case, the set $\{1 + 3\sqrt{-2}, -3 - \sqrt{-2}, 6 - 2\sqrt{-2}, 5 - 9\sqrt{-2}\}$ has the property $D(5 - 6\sqrt{-2})$.

Similarly, if $u = 2 - 3\sqrt{-2}$, we obtain another set with the property $D(z)$ in $\mathbb{Q}(\sqrt{-2})$. The set has the elements in the ring $\mathbb{Z}[\sqrt{-2}]$ for new pairs $(a, b) = (11k + a', 11l + b')$, $k, l \in \mathbb{Z}$, where

$$\begin{aligned}
(a', b') \in \{ & (1, 7), (1, 8), (2, 4), (3, 1), (3, 2), (4, 9), (4, 10), (5, 6), (5, 7), (6, 3), \\
& (6, 4), (7, 0), (7, 1), (8, 8), (8, 9), (9, 6), (10, 2), (10, 3) \},
\end{aligned}$$

and contains four distinct elements.

Let $e = 1 + \sqrt{-2}$ and $u = 3 + \sqrt{-2}$ with $N(u) = 11$. In the similar way, we also get a set with the property $D(z)$ in $\mathbb{Q}(\sqrt{-2})$. If $(a, b) = (11k + a', 11l + b')$, $k, l \in \mathbb{Z}$, and

$$(a', b') \in \{(0, 9), (1, 6), (2, 3), (3, 0), (4, 8), (5, 5), (6, 2), (8, 7), (10, 1)\},$$

then the set contains four distinct elements in the ring $\mathbb{Z}[\sqrt{-2}]$. If we put $u = 3 - \sqrt{-2}$, a new set has all distinct elements in the ring $\mathbb{Z}[\sqrt{-2}]$ for new pairs $(a, b) = (11k + a', 11l + b')$, $k, l \in \mathbb{Z}$ and

$$(a', b') \in \{(0, 5), (0, 8), (1, 0), (2, 0), (3, 3), (3, 6), (4, 6), (5, 1), (5, 9), (6, 1), (7, 4), (7, 7), (8, 10), (9, 2), (9, 10), (10, 5)\}.$$

Now, we take $e = 1 - \sqrt{-2}$ and $u = 3 + \sqrt{-2}$. We obtain another set with the property $D(z)$ in $\mathbb{Q}(\sqrt{-2})$. It is easy to check that for all $(a, b) = (11k + a', 11l + b')$, $k, l \in \mathbb{Z}$, where

$$(a', b') \in \{(0, 2), (1, 10), (2, 7), (2, 10), (3, 4), (3, 7), (4, 4), (6, 9), (7, 3), (7, 6), (8, 0), (8, 3), (9, 0), (9, 8)\},$$

the set contains four distinct elements in the ring $\mathbb{Z}[\sqrt{-2}]$. Finally, if we choose $u = 3 - \sqrt{-2}$, we get the new set and new pairs $(a, b) = (11k + a', 11l + b')$,

$$(a', b') \in \{(0, 1), (1, 4), (3, 10), (4, 2), (6, 8), (10, 9)\},$$

with four distinct elements in the ring $\mathbb{Z}[\sqrt{-2}]$. □

Proposition 4. *If z is of the form $z = 24a + 2 + (12b + 6)\sqrt{-2}$, then there exists at least one Diophantine quadruple with the property $D(z)$, for any $a, b \in \mathbb{Z}$, except maybe for $a \equiv a' \pmod{11}$, $b \equiv b' \pmod{11}$, where*

$$(a', b') \in \{(0, 3), (0, 5), (0, 7), (1, 0), (1, 2), (1, 8), (1, 10), (2, 0), (2, 5), (2, 10), (3, 2), (3, 5), (3, 8), (4, 4), (4, 5), (4, 6), (5, 1), (5, 3), (5, 7), (5, 9), (6, 0), (6, 4), (6, 6), (6, 10), (7, 3), (7, 4), (7, 6), (7, 7), (8, 1), (8, 5), (8, 9), (9, 1), (9, 2), (9, 8), (9, 9), (10, 5)\}.$$

Proof. Let $z = 24a + 2 + (12b + 6)\sqrt{-2}$, $a, b \in \mathbb{Z}$, and $e = 2 + \sqrt{-2}$. We choose $u = 3 - \sqrt{-2}$ and $N(u) = 11$. We get the set

$$\begin{aligned} & \{3 - \sqrt{-2}, \frac{1}{11}(126a^2 + 180ab - 63b^2 - 66b - 23) \\ & + \frac{1}{11}(-90a^2 + 66a + 126ab + 45b^2 - 15)\sqrt{-2}, \\ & \frac{1}{11}(126a^2 + 132a + 180ab - 63b^2 - 32) \\ & + \frac{1}{11}(-90a^2 + 126ab + 45b^2 - 66b + 18)\sqrt{-2}, \\ & \frac{1}{11}(504a^2 + 264a + 720ab - 252b^2 - 132b - 15) \\ & + \frac{1}{11}(-360a^2 + 132a + 504ab + 180b^2 + 132b + 17)\sqrt{-2}\}, \end{aligned} \quad (3.4)$$

with the property $D(z)$ in $\mathbb{Q}(\sqrt{-2})$. It is easy to check that the set (3.4) has all distinct elements in the ring $\mathbb{Z}[\sqrt{-2}]$ for all $(a, b) = (11k + a', 11l + b')$, $k, l \in \mathbb{Z}$, where

$$\begin{aligned} (a', b') \in \{ & (0, 2), (0, 9), (1, 1), (1, 5), (2, 4), (2, 8), (3, 0), (3, 7), (4, 3), (4, 10), \\ & (5, 2), (5, 6), (6, 5), (6, 9), (7, 1), (7, 8), (8, 0), (8, 4), (9, 3), (9, 7), \\ & (10, 6), (10, 10)\}. \end{aligned}$$

Similarly, if $u = 3 + \sqrt{-2}$, we obtain another set with the property $D(z)$ in $\mathbb{Q}(\sqrt{-2})$. That set has the elements in the ring $\mathbb{Z}[\sqrt{-2}]$ for new pairs $(a, b) = (11k + a', 11l + b')$, $k, l \in \mathbb{Z}$, where

$$\begin{aligned} (a', b') \in \{ & (0, 4), (1, 6), (2, 3), (2, 9), (3, 6), (4, 8), (5, 0), (5, 5), (6, 2), \\ & (6, 8), (7, 5), (7, 10), (8, 2), (8, 7), (9, 4), (9, 10), (10, 1), (10, 7)\}, \end{aligned}$$

and contains four distinct elements.

Let $e = 2 - \sqrt{-2}$ and $u = 3 + \sqrt{-2}$. In a same way, we also get a set with the property $D(z)$ in $\mathbb{Q}(\sqrt{-2})$. If $(a, b) = (11k + a', 11l + b')$, $k, l \in \mathbb{Z}$, and

$$\begin{aligned} (a', b') \in \{ & (0, 1), (0, 8), (1, 9), (2, 2), (2, 6), (3, 3), (3, 10), (4, 0), (4, 7), \\ & (5, 4), (5, 8), (6, 1), (7, 2), (7, 9), (8, 6), (8, 10), (10, 0), (10, 4)\}, \end{aligned}$$

then the set contains four distinct elements in the ring $\mathbb{Z}[\sqrt{-2}]$. If we put $u = 3 - \sqrt{-2}$, a new set has all distinct elements in the ring $\mathbb{Z}[\sqrt{-2}]$ for new pairs $(a, b) = (11k + a', 11l + b')$, $k, l \in \mathbb{Z}$ and

$$\begin{aligned} (a', b') \in \{ & (0, 6), (1, 4), (2, 1), (2, 7), (3, 4), (4, 2), (5, 10), \\ & (7, 0), (8, 3), (8, 8), (9, 0), (9, 6), (10, 3), (10, 9)\}. \end{aligned}$$

Now, we take $e = \sqrt{-2}$ and $u = 3 - \sqrt{-2}$. We obtain another set with the property $D(z)$ in $\mathbb{Q}(\sqrt{-2})$. It is easy to check that for all $(a, b) = (11k + a', 11l + b')$, $k, l \in \mathbb{Z}$, where

$$(a', b') \in \{(0, 0), (1, 3), (3, 9), (4, 1), (6, 7), (9, 5), (10, 8)\},$$

the set contains elements of the ring $\mathbb{Z}[\sqrt{-2}]$. If $(a, b) = (11k, 11l)$, then there appear two equal elements for $(k, l) = (0, 0)$. But for that pair the set $\{1 - 3\sqrt{-2}, 2\sqrt{-2}, 9 + \sqrt{-2}, 17 + 9\sqrt{-2}\}$ has the property $D(2 + 6\sqrt{-2})$. If we choose $u = 3 + \sqrt{-2}$, we get the new set and new pairs $(a, b) = (11k + a', 11l + b')$,

$$(a', b') \in \{(0, 10), (1, 7), (3, 1), (4, 9), (6, 3), (10, 2)\},$$

with all elements in the ring $\mathbb{Z}[\sqrt{-2}]$. Moreover, if $(a, b) = (11k, 11l + 10)$, then the set contains two equal elements for $(k, l) = (0, -1)$. But the set $\{1 + 3\sqrt{-2}, -2\sqrt{-2}, 9 - \sqrt{-2}, 17 - 9\sqrt{-2}\}$ has the property $D(2 - 6\sqrt{-2})$. □

As we mentioned, Proposition 4 immediately implies

Proposition 5. *If z is of the form $z = 48a + 44 + (24b + 12)\sqrt{-2}$, then there exists at least one Diophantine quadruple with the property $D(z)$, for any $a, b \in \mathbb{Z}$, except maybe for $a \equiv a' \pmod{11}$, $b \equiv b' \pmod{11}$, where*

$$\begin{aligned} (a', b') \in \{ & (0, 5), (1, 1), (1, 2), (1, 8), (1, 9), (2, 1), (2, 5), (2, 9), (3, 3), (3, 4), \\ & (3, 6), (3, 7), (4, 0), (4, 4), (4, 6), (4, 10), (5, 1), (5, 3), (5, 7), (5, 9), \\ & (6, 4), (6, 5), (6, 6), (7, 2), (7, 5), (7, 8), (8, 0), (8, 5), (8, 10), (9, 0), \\ & (9, 2), (9, 8), (9, 10), (10, 3), (10, 5), (10, 7)\}. \end{aligned}$$

Remark 2. Concerning the possible exceptions from Propositions 3, 4, and 5, for certain numbers z (e.g. $z = 5 + 42\sqrt{-2}$, $5 + 78\sqrt{-2}$, $29 + 114\sqrt{-2}$, $197 + 78\sqrt{-2}$ of the exceptional form in Proposition 3) we were not able to find any $D(z)$ -quadruple.

On the other hand, for all exceptional pairs (a', b') from Proposition 3, we were able to find at least one z of the form $z = 24a + 5 + (12b + 6)\sqrt{-2}$, $a \equiv a' \pmod{11}$, $b \equiv b' \pmod{11}$ for which there exist a $D(z)$ -quadruple (in some cases we can find infinitely many such z 's by using formulas from Remark 1). Some of the observed quadruples contain an element of norm 19. A systematic study of such quadruples would involve additional congruence conditions modulo 19 on a and b , so the results become too complicated to state explicitly and do not lead to the complete solution of the problem. Similar results and problems appear when considering exceptional cases from Propositions 4, and 5.

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SIMPLEXES AND THEIR APPLICATIONS - A SHORT SURVEY

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Abstract. We investigate different types of simplexes (linear algebraic, affine, geometric), solve some extremal problems and state general conjectures concerning these notions in discrete geometry and matroid theory, finally present some applications in chemistry and physics.

Connections to homogeneous system of linear equations are also mentioned.

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1. DEFINITIONS

Since there are many different notions of **simplexes** in use in mathematics, first we have to present the definitions we need.

Definition 1. (Algebraic simplexes) In the n -dimensional linear (Euclidean) space $\mathbb{R}^n$ any set of vectors $\mathcal{S} \subset \mathbb{R}^n$ is called a **(linear) algebraic simplex** iff $\mathcal{S}$ is minimal linear dependent, i.e. $\mathcal{S}$ is linearly dependent but each proper subset $\mathcal{T} \subsetneq \mathcal{S}$ is independent.

Remark 1. The set $\mathcal{S} = \{0\}$ fulfills the previous definition, so it may be called (the only) *1-element simplex*. Though this extremal case is not interesting in applications, but we need it in Remark 2.

The corresponding notion in *matroids* is called a **circuit**. Clearly linear algebraic simplexes are just matroidal circuits in special matroids.

Definition 2. (Geometric simplexes) Any set of $n + 1$ points in $\mathbb{R}^n$ in general position is called a **geometric simplex**.

Describing *affine simplexes* runs by induction on the number of points. The definition might be easier if we first assume that one of the points is located at the origin, the other cases come after by translations.

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Definition 3. (Affine simplexes) Any finite set $\mathcal{S} = \{\underline{s}_1, \underline{s}_2, \dots, \underline{s}_k\} \subset \mathbb{R}^n$ is an **affine simplex** if and only if $k \geq 3$ and

o) in the case $k = 3$ the points $\{\underline{s}_1, \underline{s}_2, \underline{s}_3\}$ must be colinear,

i) in the case $\underline{s}_1 = \underline{0}$ and $k \geq 4$ we require the *vectors*

$$\mathcal{S}^- := \{\underline{s}_2, \underline{s}_3, \dots, \underline{s}_k\} \quad (1.1)$$

to be linearly dependent, and **no** proper subset of (the original) $\mathcal{S}$ is an *affine* simplex,

ii) in the general case $k \geq 4$ the set of *vectors*

$$\mathcal{S}^- := \{\underline{s}_2 - \underline{s}_1, \underline{s}_3 - \underline{s}_1, \dots, \underline{s}_k - \underline{s}_1\} \quad (1.2)$$

is required to be linearly dependent, but $\mathcal{S}$ is not allowed to contain any *affine* simplex as a proper subset.

Cases $n = 3$ and $n = 4$ can be formulated in easier way.

Example 1. (i) A set of points $\mathcal{S} \subset \mathbb{R}^2$ is an **affine**

- ▷ 3 -element **simplex** iff $\mathcal{S}$ is three colinear points,
- ▷ 4 -element simplex iff $\mathcal{S}$ is any four points but none three of them are colinear,
- ▷ there are no other affine simplexes in $\mathbb{R}^2$.

(ii) A set of points $\mathcal{S} \subset \mathbb{R}^3$ is an **affine**

- ▷ 3 -element **simplex** iff $\mathcal{S}$ is three colinear points,
- ▷ 4 -element simplex iff $\mathcal{S}$ is any four coplanar points but none three of them are colinear,
- ▷ 5 -element simplex iff $\mathcal{S}$ is any five points but none four are coplanar (and thus none three of them are colinear),
- ▷ there are no other affine simplexes in $\mathbb{R}^3$.

The connection between linear algebraic and affine simplexes will be discussed in Definition 6 and Proposition 2. From affine spaces we can move towards hypergraphs (set systems):

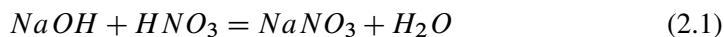
Definition 4. Let $(\mathcal{V}, \mathcal{E})$ be any hypergraph. Any edge (set of vertices) $E \in \mathcal{E}$ is called a **hypergraph simplex** if it is *inclusion-minimal*, that is E does not contain any other edge $F \in \mathcal{E}$:

$$F \not\subseteq E \quad \text{for } F \in \mathcal{E} . \quad (1.3)$$

2. APPLICATIONS

One of the practical applications of **linear algebraic simplexes** is in stoichiometry when studying *minimal reactions* and *mechanisms*.

Consider for example the chemical reaction



which corresponds to the vector-equation

$$[1, 0, 1, 1]^T + [1, 1, 0, 3]^T = [0, 1, 1, 3]^T + [2, 0, 0, 1]^T \quad (2.2)$$

or

$$[1, 0, 1, 1]^T + [1, 1, 0, 3]^T - [0, 1, 1, 3]^T - [2, 0, 0, 1]^T = [0, 0, 0, 0]^T \quad (2.3)$$

where the first coordinates correspond to (the number) of H in each molecule, the second ones to N , the third ones to Na and the fourth ones to O . In other words we could say that in $\mathbb{R}^4$ we use the (linear algebraic) "basis" $B = \{H, N, Na, O\}$ instead of the usual one $\{\underline{e}_1, \underline{e}_2, \underline{e}_3, \underline{e}_4\}$.

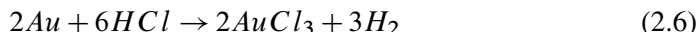
In general: let the chemical species ("molecules") $A_1, A_2, \dots, A_n$ consist of chemical elements $E_1, E_2, \dots, E_m$ as

$$A_j = \sum_{i=1}^m a_{i,j} E_i \quad \text{where } a_{i,j} \in \mathbb{N} \text{ for } j = 1, 2, \dots, n. \quad (2.4)$$

Write $\underline{A}_j$ for the vector $[a_{1,j}, a_{2,j}, \dots, a_{m,j}]^T$. Using the *law of mass-balance* we know that there (might) exists a chemical reaction between the species $\{A_j : j \in S\}$ for any $S \subseteq \{1, 2, \dots, n\}$ if and only if the homogeneous linear equation

$$\sum_{j \in S} x_j \underline{A}_j = \underline{0} \quad (2.5)$$

has nontrivial solution for some $x_j \in \mathbb{R}$, $j \in S$, that is if the vector set $\{\underline{A}_j : j \in S\}$ is *linearly dependent*. (Each x_j can be assumed an integer by (2.4).) The vectors $\underline{A}_j$ with *positive* coefficients $x_j > 0$ in (2.5) denote the species the reaction starts with (on the left-hand-side of (2.1)), while the vectors $\underline{A}_j$ with *negative* coefficients $x_j < 0$ in (2.5) denote the species the reaction ends with (on the right-hand-side). Of course the reactions obtained in the above way are only *possibilities*, e.g. the reaction



does not occur under normal conditions.

Considering a reaction like (2.1) or (2.6) one may ask whether the number of different kinds of molecules can be reduced, or, in other words, could react the remaining molecules if one of them is missing? Are the remaining vectors linearly dependent if one of (or more) them is/are missing? In the language of mathematics we can say:

Definition 5. The reaction in (2.5) is called a **minimal reaction** if for no $T \subsetneq S$ might there be any reaction among the species $\{A_j : j \in T\}$; that is if the vector set $\{\underline{A}_j : j \in T\}$ is *linearly independent* for any $T \subsetneq S$. $\square$

The above (chemical) notion explains the need of Definition 1 of linear algebraic simplexes. Indeed, the set of vectors, which correspond to molecules involved in a minimal reaction, form a (linear algebraic) simplex!

In our third example the species $A_1 = C$, $A_2 = O$, $A_3 = CO$ and $A_4 = CO_2$ determine the vectors $\underline{A}_1 = [1, 0]$, $\underline{A}_2 = [0, 1]$, $\underline{A}_3 = [1, 1]$ and $\underline{A}_4 = [1, 2]$. The vectorset $H = \{\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4\}$ contains the simplexes

$$\{\underline{A}_1, \underline{A}_2, \underline{A}_3\}, \{\underline{A}_1, \underline{A}_2, \underline{A}_4\}, \{\underline{A}_1, \underline{A}_3, \underline{A}_4\} \text{ and } \{\underline{A}_2, \underline{A}_3, \underline{A}_4\}.$$

After solving the corresponding equations (2.5), we have the following (complete) list of minimal reactions, corresponding to the above simplexes:

$$C + O = CO, C + 2O = CO_2, O + CO = CO_2 \text{ and } C + CO_2 = 2CO. \quad (2.7)$$

In general, instead of considering (2.5) for some subsets S , we are invited to investigate the "full" homogeneous system of linear equations

$$\sum_{j=1}^n x_j \underline{A}_j = \underline{0}, \quad \text{equivalently} \quad \mathbf{A} \cdot \underline{X} = \underline{0} \quad (2.8)$$

where

$$\mathbf{A} := [\underline{A}_1, \underline{A}_2, \dots, \underline{A}_n] \in \mathbb{R}^{m \times n} \quad \text{and} \quad \underline{X} = [x_1, \dots, x_n] \in \mathbb{R}^n. \quad (2.9)$$

The solution vectors $\underline{X}$ of (2.8) are *any* (possible) reactions among the species $\{A_1, A_2, \dots, A_n\}$.

Denote now the *set of (all) solutions* of (2.8) by $\mathbf{M}_\mathbf{A}$:

$$\mathbf{M}_\mathbf{A} := \{\underline{X} \in \mathbb{R}^n \mid \mathbf{A} \cdot \underline{X} = \underline{0}\}. \quad (2.10)$$

In practice, a chemical **mechanism** is a sequence, i.e. a *linear combination* of reactions (which correspond to solution-vectors $\underline{X} \in \mathbf{M}_\mathbf{A}$), so for any mechanism $\mathcal{M}$

$$\mathcal{M} = \sum_{\ell=1}^k m_\ell \underline{X}_\ell \quad (2.11)$$

for some vectors $\underline{X}_1, \dots, \underline{X}_\ell \in \mathbf{M}_\mathbf{A}$ and for some $k \in \mathbb{N}$. Since any mechanism satisfies the law of mass-balance, we must have

$$\mathbf{A} \cdot \mathcal{M} = \underline{0} \quad (2.12)$$

which clearly follows from the definition (2.10) of $\mathbf{M}_\mathbf{A}$ as

$$\mathbf{A} \cdot \mathcal{M} = \sum_{\ell=1}^k m_\ell \mathbf{A} \cdot \underline{X}_\ell = \underline{0} \quad (2.13)$$

In this way we get the second step of the **hierarchy**: the vectors $\underline{X}_\ell$ of the equality (2.12) are, in fact, all the solutions of the previous equality (2.8). The final connection among the coefficients $\underline{A}_j$ of (2.8) and *all* of the solutions $M := [m_1, \dots, m_\ell]$ of (2.12) has been raised and planned to investigate in [13].

The structure of minimal solutions of homogeneous systems of linear equations with connections to simplexes are discussed in [22].

Linear algebraic simplexes can also be used for finding *dimensionless groups* in dimensional analysis: we investigate the structure of composite measure units in physics (see e.g. [6, 11, 16] and [14]).

Let there be given m elementary quantities (mass, length, time, etc.) denoted by $E_1, \dots, E_m$ and n quantities $A_1, \dots, A_n$ ($n \in \mathbb{N}$ is any number) as

$$A_j = \prod_{i=1}^m E_i^{a_{i,j}}$$

where $a_{i,j} \in \mathbb{Z}$ for $j = 1, \dots, n$ and $i = 1, \dots, m$. Clearly we can assign the quantities A_j to the vectors

$$\mathbf{A}_j := [a_{1,j}, \dots, a_{m,j}]^T \in \mathbb{R}^m$$

for $j = 1, \dots, n$, which vectors are in the m -dimensional linear space $\mathbb{R}^m$. Now, a (possible) **dimensionless group** (real number) of the quantities $\{A_j : j \in S\}$ does exist for any $S \subseteq \{1, \dots, n\}$ iff the equality

$$\prod_{j \in S} A_j^{x_j} = 1 \quad (2.14)$$

holds, or equivalently (considering the exponents) the homogeneous system of linear equations

$$\sum_{j \in S} x_j \cdot A_j = \mathbf{0}$$

has nontrivial solution for $x_j \in \mathbb{R}$ ($j \in S$). That is, we again reached to the system of linear homogen equations and to the problem of finding simplexes!

For example, consider the flow of a fluid through a heated tube and the heat transfer between the pipe wall and the fluid. Consider now the following quantities, which we also write in vectors form in the base $\{m, \ell, t, T\}$:

$A_1 = d$ (ℓ)	tube diameter	$\mathbf{A}_1 = [0, 1, 0, 0]$	
$A_2 = v$ (s/t)	linear velocity	$\mathbf{A}_2 = [0, 1, -1, 0]$	
$A_3 = \rho$ (m/ℓ^3)	fluid density	$\mathbf{A}_3 = [1, -3, 0, 0]$	
$A_4 = \nu$ ($m/\ell t$)	viscosity	$\mathbf{A}_4 = [1, -1, -1, 0]$	(2.15)
$A_5 = \kappa$ ($A/t^2 T$)	heat capacity	$\mathbf{A}_5 = [0, 2, -2, -1]$	
$A_6 = \lambda$ ($m/t^3 T$)	heat transfer coeff.	$\mathbf{A}_6 = [1, 0, -3, -1]$	
$A_7 = \mu$ ($m\ell/t^3 T$)	thermal conductivity	$\mathbf{A}_7 = [1, 1, -3, -1]$	

For example, one minimal dimensionless group X_1 of the seven ones is the following:

$$X_1 = [0, 0, 0, 1, 1, 0, -1]^T$$

which corresponds to the equality

$$\nu \cdot \kappa = \mu \cdot c$$

for some constant $c \in \mathbb{R}$.

3. BASIC PROPERTIES

For investigating the common properties of the different Definitions 1 through 4, we denote the ground set with $\mathcal{R}$ in each case.

The following fundamental property is valid for *all of these* definitions.

Proposition 1. (i) For any set $X \subset \mathcal{R}$ either X is a simplex itself, or X strictly contains a simplex $Y \subsetneq X$ or X is strictly contained in a simplex $X \subsetneq Z$. Exactly one of the previous three cases yields for all subset $X \subset \mathcal{R}$.

(ii) (follows from (i)) The simplexes in $\mathcal{R}$ form a Sperner system, i.e. for each two simplexes $S_1, S_2 \subset \mathcal{R}$ ($S_1 \neq S_2$) we have

$$S_1 \not\subseteq S_2 \quad \text{and} \quad S_2 \not\subseteq S_1. \quad (3.1)$$

The main problem of our investigations in the last decade was the following:

Problem 1. For given natural numbers $m, n \in \mathbb{N}$ what is the **minimal** and **maximal** possible value of the number of simplexes $\mathcal{S} \subset \mathcal{H}$ if $\mathcal{H}$ can be any set $\mathcal{H} \subset \mathbb{R}^n$ satisfying that $\mathcal{H}$ spans $\mathbb{R}^n$ and $|\mathcal{H}| = m$, **and** what are the extremal sets $\mathcal{H}$ (having least/maximal number of simplexes $\mathcal{S} \subset \mathcal{H}$) ?

This problem can be raised for any kind of simplexes.

For technical simplifications we introduce

Notation 1. For any finite set $\mathcal{H} \subset \mathbb{R}^n$ we denote by $\mathbf{simp}_\ell(\mathcal{H})$ the number of linear algebraic and by $\mathbf{simp}_a(\mathcal{H})$ the number of affine simplexes $\mathcal{S} \subset \mathcal{H}$ contained in $\mathcal{H}$. For $2 \leq k \leq n+1$ we denote by $\mathbf{simp}_k(\mathcal{H})$ the number of k -element (linear algebraic) simplexes in $\mathcal{H}$.

The following assumption (which is originally *not* supposed in Problem 1), plays a crucial role in later investigations:

Condition 1. $\mathcal{H}$ contains no parallel vectors, especially $\underline{0} \notin \mathcal{H}$.

Remark 2. This Condition is equivalent to " $\mathcal{H}$ does not contain 1- and 2-element linear algebraic simplexes" (see Remark 1).

Recall, that *parallel vectors* in $\mathcal{H}$ represent isomers or different doses of molecules, etc.

In the shade of the above Condition we can replace each vector by its scalar multiplier and reduce the dimension as follows.

Definition 6. For any $\underline{h} \in \mathbb{R}^n$ we let

$$\Lambda \underline{h} := \{\lambda \cdot \underline{h} : \lambda \in \mathbb{R}, \lambda \neq 0\} \quad (3.2)$$

and for $\mathcal{H} \subset \mathbb{R}^n$

$$\Lambda \mathcal{H} := \{\Lambda \underline{h} : \underline{h} \in \mathcal{H}\}. \quad (3.3)$$

and

$$\mathcal{H}^{\mathcal{P}} := \Lambda \mathcal{H} \cap \mathcal{P} \quad (3.4)$$

for an $n - 1$ -dimensional affine hyperplane $\mathcal{P} \subset \mathbb{R}^n$ which is *not* parallel to any $\underline{h} \in \mathcal{H}$.

(For finite or at most countable set $\mathcal{H}$ such a $\mathcal{P}$ always exists.)

Proposition 2. *There is a bijective correspondence between $\mathcal{H}^{\mathcal{P}} \subset \mathcal{P}$ (i.e. $\mathcal{H}^{\mathcal{P}} \subset \mathbb{R}^{n-1}$) and $\mathcal{H} \subset \mathbb{R}^n$, and similarly between $\text{simp}_a(\mathcal{H}^{\mathcal{P}})$ and $\text{simp}_\ell(\mathcal{H})$, so*

$$|\mathcal{H}^{\mathcal{P}}| = |\mathcal{H}| \quad \text{and} \quad \text{simp}_a(\mathcal{H}^{\mathcal{P}}) = \text{simp}_\ell(\mathcal{H}). \quad (3.5)$$

(Actually we map $\mathcal{H}$ into the projective space $P(\mathbb{R}^n)$. It is not too hard to see, that the sets $\mathcal{S}^{\mathcal{P}} \subset \mathcal{P}$ form *affine simplexes* (in the sense of Definition 3) for each linear algebraic simplex $\mathcal{S} \subseteq \mathcal{H}$. The cases $n = 3$ and $n = 4$ in Definition 1 are clear 2 and 3 -dimensional concepts.

The other notions of simplexes (in matroids and in hypergraphs) are pure mathematical generalizations only at this moment.

4. ON THE NUMBER OF SIMPLEXES

First we present some results concerning Problem 1.

In [17] and [19] a polynomial algorithm for listing all simplexes, contained in a given arbitrary finite set $\mathcal{H} \subset \mathbb{R}^n$, and related questions were discussed. Though only linear algebraic simplexes are mentioned in those papers, the algorithm works for *any kind of* set system, satisfying (i) of Proposition 1, too. The ground idea of our algorithm was to modify (in- or decrease) the actually tested set (whether or not it is a simplex) to search for all simplexes in $\mathcal{H}$. Another approaches to and applications of the algorithmic problems can be found eg. in [1, 2, 4, 10].

We have the following theoretical results. First, in Theorems 1 through 4 we deal with linear algebraic simplexes, we list the results on other kinds of simplexes after.

Theorem 1 ([7]). *For any $\mathcal{H} \subseteq \mathbb{R}^n$ of fixed size, such that $\mathcal{H}$ spans $\mathbb{R}^n$, $\text{simp}(\mathcal{H})$ is maximal if and only if any n vectors of $\mathcal{H}$ are linearly independent.*

Remark 3. Such sets $\mathcal{H}$ of any size can be easily constructed using the property of Vandermonde determinants: $\mathcal{H} = \{h_1, \dots, h_m\}$ where $h_i = [1, i, i^2, \dots, i^{n-1}]^T$ ($i \leq m = |\mathcal{H}|$).

Corollary 1. *For any $\mathcal{H} \subseteq \mathbb{R}^n$, $|\mathcal{H}| = m$ and $\mathcal{H}$ spans $\mathbb{R}^n$ we have*

$$\text{simp}(\mathcal{H}) \leq \binom{m}{n+1} \quad (4.1)$$

and equality holds if and only if any n vectors of $\mathcal{H}$ are linearly independent.

Though for each simplex $\mathcal{S} \subseteq \mathcal{H}$ in $\mathbb{R}^n$ we have $|\mathcal{S}| \leq n + 1$ and by Proposition 1.(ii) the set of simplexes forms a Sperner system, Theorem 1 and Corollary 1 are *not* direct consequences of Sperner's well known theorem, since we have to ensure that *each* $(n + 1)$ -element subset of $\mathcal{H}$ form a simplex. Moreover, our specific (short) proof reveals the inner structure of simplexes and the reason for the uniqueness of the maximal construction.

The other direction is much harder.

Theorem 2 ([7]). *For any $\mathcal{H} \subseteq \mathbb{R}^n$ of fixed size such that $\mathcal{H}$ spans $\mathbb{R}^n$, $\text{simp}(\mathcal{H})$ is minimal if and only if $\mathcal{H}$ consists of n collections of parallel vectors of sizes differing by at most one from each other.*

The proofs of Theorems 1 and 2 run by modifying the vectors in $\mathcal{H}$ to increase $\text{simp}(\mathcal{H})$ as long as possible. Again, this construction works for other kinds of simplexes, too, e.g. for matroids in Theorems 5 through 11.

Corollary 2. *For any $\mathcal{H} \subseteq \mathbb{R}^n$, if $\mathcal{H}$ spans $\mathbb{R}^n$ and $|\mathcal{H}| = m$ where $m = a \cdot n + b$, $0 < b < n$ we have*

$$b \cdot \binom{a+1}{2} + (n-b) \cdot \binom{a}{2} \leq \text{simp}(\mathcal{H}). \quad (4.2)$$

In the special case when m is divisible by n we can write

$$n \cdot \binom{\frac{m}{n}}{2} \leq \text{simp}(\mathcal{H}). \quad (4.3)$$

Let us emphasize that $\mathcal{H}$ above **may** contain parallel vectors, moreover the minimal construction is built up from as many parallel vectors as possible.

In the forthcoming research we investigated the case when $\mathcal{H}$ **may not** contain parallel vectors.

Our first result is in dimension $n = 3$.

Theorem 3 ([8]). *For any $\mathcal{H} \subseteq \mathbb{R}^3$ of fixed size not equal to 3, 4 or 7, such that $\mathcal{H}$ spans $\mathbb{R}^3$ and no parallel vectors are allowed in $\mathcal{H}$, $\text{simp}(\mathcal{H})$ is minimal if and only if $\mathcal{H}$ is contained in two intersecting planes (their intersection vector belongs to $\mathcal{H}$), one of which is of size 3. In other words: precisely when $\mathcal{H}$ contains 3 linearly independent vectors $\{u_1, u_2, u_3\}$, another vector v colinear with u_1 and u_2 and the rest $\mathcal{H} \setminus \{u_1, u_2, u_3, v\}$ colinear with u_2 and u_3 .*

Corollary 3. *For any $\mathcal{H} \subseteq \mathbb{R}^3$ such that $\mathcal{H}$ spans $\mathbb{R}^3$, no parallel vectors are allowed in $\mathcal{H}$ and $|\mathcal{H}| = m \geq 4$ we have*

$$\binom{m-2}{3} + 1 + \binom{m-3}{2} \leq \text{simp}(\mathcal{H})$$

and equality holds just in the case described in the previous Theorem 3.

In the proof we used the technique described in Definition 6 for reducing the dimension from $\mathcal{H} \subseteq \mathbb{R}^3$ to $\mathcal{H}^{\mathcal{P}} \subset \mathbb{R}^2$, in the meanwhile we transformed linear algebraic simplexes into affine ones (see Definition 1). The minimal configurations for $|\mathcal{H}| = 3, 4, 7$ are almost completely described in [8], P.Sellers drew our attention to the missing one in a personal communication.

For the general problem for minimizing $\text{simp}(\mathcal{H})$ when $\mathcal{H}$ spans $\mathbb{R}^n$ and no parallel vectors allowed in $\mathcal{H} \subset \mathbb{R}^n$ we have the below conjecture (thanks to student **Jianzhong Meng** for his help with some calculations with Maple).

Conjecture 1 ([8]). *In the case of $\mathcal{H} \subset \mathbb{R}^n$, $\mathcal{H}$ spans $\mathbb{R}^n$ and no parallel vectors are allowed in $\mathcal{H}$, the minimum for $\text{simp}(\mathcal{H})$ is attained precisely for the following configurations:*

- (i) *if n is even, then $\mathcal{H}$ contains n linearly independent vectors $\{u_i : i = 1, \dots, n\}$ and the remaining divided as evenly as possible between the planes $\{[u_i, u_{i+1}] : i = 1, 3, \dots, n-1\}$,*
- (ii) *if n is odd, then $\mathcal{H}$ again contains n linearly independent vectors $\{u_i : i = 1, \dots, n\}$, one special vector in the plane $[u_{n-1}, u_n]$ and finally the remaining vectors divided as evenly as possible between the planes $\{[u_i, u_{i+1}] : i = 1, 3, \dots, n-2\}$ with lower indices having precedence.*

Our latest result is for the dimension $n = 4$, we again used the technique described in Definition 6 for reducing the dimension from $\mathcal{H} \subseteq \mathbb{R}^4$ to $\mathcal{H}^{\mathcal{P}} \subset \mathbb{R}^3$, dealing with affine simplexes in $\mathbb{R}^3$:

Theorem 4 ([15]). *Assuming $\mathcal{H} \subset \mathbb{R}^4$, no parallel vectors allowed in $\mathcal{H}$ but $\mathcal{H}$ spans $\mathbb{R}^4$ (i.e. $\mathcal{H}^{\mathcal{P}} \subset \mathbb{R}^3$ is not coplanar) and $|\mathcal{H}| = m \geq 24$ we have*

$$\binom{\lfloor m/2 \rfloor}{3} + \binom{\lceil m/2 \rceil}{3} \leq \text{simp}(\mathcal{H}) \quad (4.4)$$

and equality holds just in the case if $\mathcal{H}^{\mathcal{P}}$ is contained in two skew (detour) lines and these lines contain $\lfloor m/2 \rfloor$ and $\lceil m/2 \rceil$ many points of $\mathcal{H}^{\mathcal{P}}$.

The general problem for $\mathcal{H} \subset \mathbb{R}^n$ assuming that $\mathcal{H}$ spans $\mathbb{R}^n$ and no parallel vectors are in $\mathcal{H}$, still remains open.

For **matroids** we have the following results and open problem:

Theorem 5 ([3]). *If $m > n + 1$, then only the uniform matroid $U_{m,n}$ contains the maximum number of circuits, $\binom{n+1}{m}$. If $m = n + 1$, all matroids of size m and of rank n contain exactly 1 circuit.*

Theorem 6 ([3]). *Only the uniform matroid $U_{m,n}$ contains the maximum number of bases, namely $\binom{m}{n}$.*

Theorem 7 ([3]). *For each m and n , there is a unique matroid M_o of size m and of rank n containing the minimum number of bases, namely 1, when we allow loops in the matroid.*

Theorem 8 ([3]). Any matroid M of size m and of rank n contains at least $n - m$ circuits. A matroid contains exactly $m - n$ circuits if and only if the circuits of the matroid are pairwise disjoint.

Remark 4. The matroid M_o mentioned above also contains exactly $m - n$ pairwise disjoint circuits, i.e. loops.

Theorem 9 ([3]). Suppose that there are no large circuits and no loops in the matroid M , and let $\{a_1, a_2, \dots, a_n\}$ be any fixed base. If ϑ_i denotes the number of elements in M parallel to a_i (including a_i itself) for $i = 1, 2, \dots, n$, then M contains the minimum number of circuits iff $|\vartheta_i - \vartheta_j| \leq 1$ for $i \neq j$.

Corollary 4 ([3]). The minimum number of circuits in a matroid of size m and of rank n , where $m = an + b$ ($0 \leq b < n$), is

$$b \cdot \binom{a+1}{2} + (n-b) \cdot \binom{a}{2}$$

and in particular, if m is a multiple of n :

$$n \cdot \binom{\frac{m}{n}}{2}.$$

Theorem 10 ([3]). *a)* For $m < 2n$, a matroid of size m and rank n contains the minimum number of circuits iff all its circuits are disjoint.

b) For $m \geq 2n$, a matroid contains the minimum number of circuits iff it contains only 2-element circuits (i.e. parallel elements), and the sizes of the equivalence classes of parallel elements differ by at most 1.

Remark 5. (i) There are many matroids satisfying a) while the matroids described in b) are, in fact, isomorphic.

(ii) The last part of the above proof describes uniquely the structures of matroids containing the minimum number of circuits when $m \geq 2n$.

Theorem 11 ([3]). A matroid M of size m and rank n contains the minimum number of bases iff it has a base $\{a_1, a_2, \dots, a_n\}$ such that all other elements in M are parallel to a_1 .

Corollary 5 ([3]). The minimum number of bases is $m - n + 1$, and the minimum configuration is unique.

Problem 2 ([3], still open). Characterize the matroids with the minimum number of circuits and bases, when neither parallel elements nor loops are allowed.

Our results concerning **hypergraphs** are planned to be published in [21].

Geometric simplexes and their chemical applications are discussed e.g. in [5, 9, 18].

The strong connection between minimal reactions and minimal solutions of the homogeneous system of linear equations (2.5) and (2.8) was studied first in [12] and extended in [22].

In [20] the related notion of *valuation operators* is discussed in detail.

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REGULARIZATION FOR A NONLINEAR BACKWARD PARABOLIC PROBLEM WITH CONTINUOUS SPECTRUM OPERATOR

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Abstract. We study the backward parabolic problem for a nonlinear parabolic equation of the form $u_t + Au(t) = f(t, u(t))$, $u(T) = \varphi$, where A is a positive self-adjoint unbounded operator and f is a Lipschitz function. The problem is ill-posed, in the sense that if the solution does exist, it will not depend continuously on the data. To regularize the problem, we use the quasi-boundary method and the quasi-reversibility method to establish a modified problem. We present approximated solutions that depend on a small parameter $\epsilon > 0$ and give error estimates for our regularization.

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1. INTRODUCTION

Let $K = \Re$ and let H be a Hilbert space on K with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Let $A : D(A) \rightarrow H$ be a self-adjoint operator defined on a subspace $D(A)$ of the vector space H , such that $-A$ generates a contraction semi-group on H . Consider the backward parabolic equation of finding a function $u : [0, T] \rightarrow H$, such that

$$u_t + Au(t) = f(t, u(t)), \quad 0 < t < T, \quad (1.1)$$

$$u(T) = \varphi, \quad (1.2)$$

where $\varphi \in H$ is a prescribed final value and $f : \mathbb{R} \times H \rightarrow H$ is a given Lipschitz function. We can rewrite the above problem as the following integral equation (see, e.g., [1], chapter 4)

$$u(t) = S(T-t)^{-1}\varphi - \int_t^T S(s-t)^{-1}f(s, u(s))ds, \quad (1.3)$$

where $S(t), t \geq 0$ is the semigroup (generated by $-A$) which is defined precisely later. This problem is well known to be severely ill-posed; i.e., solutions do not always exist, and when they do exist, they do not depend continuously on the given data. In addition, the ill-posed problem is very sensitive to the measurement errors (see, e.g., [2]). The final data is usually the result of discretely experimental measurements and thus patched into L^2 -functions which resulted an error. With the natural error, the normal computation is intractable, that require some special regularization methods. Backward parabolic equations are very important in various practical situations and there have been many articles devoted to this problem. The linear homogeneous case $f = 0$ of this problem has been considered by many authors, using many different approaches. In the case when A has discrete spectrum, backward problems have been studied in many recent papers, such as [5–7, 9, 10, 12]. For some of works on the continuous spectrum of A , we refer the reader to N.Boussetila and F. Rebbani [3, 4], Denche and S. Djeddar [7], N.H. Tuan and D.D. Trong [13, 14]. As far as we know, we did not find any results concerned with nonlinear backward Cauchy problems with spectrum continuous operator. In this paper, we are going to modify the quasi-boundary value (QBV) and the quasi-reversibility methods to solve (1.1)–(1.2). In fact, we shall establish approximated solutions on the interval $[0, T + m]$ (instead of $[0, T]$) with $m > 0$. Moreover, we shall prove that this idea the same stability magnitude order as the one in the case of QBV method (we can see the same idea in [8]).

The paper is organized into sections as follows: In Section 3, we shall regularize the homogeneous problem. In Section 4, we describe our regularization method for the nonlinear case.

2. AUXILIARY RESULTS

In this section, we present the notation and the functional setting which will be use in this paper to prepare some material which will be used in our analysis.

Let a be a positive number. We denote by $\{E_\lambda, \lambda \geq a\}$ the spectral resolution of the identity associated to A .

We denote by $S(t) = e^{-tA} = \int_a^\infty e^{-t\lambda} dE_\lambda \in \mathcal{L}(H), t \geq 0$, the C_0 -semigroup generated by $-A$. Some basic properties of $S(t)$ are listed in the following theorem.

Theorem 1 (See [11], Ch.2, Theorem 6.13, p.74). *For the family of operators $S(t), t \geq 0$ the following properties are valid:*

- (1) $\|S(t)\| \leq 1$, for all $t \geq 0$;
- (2) the function $t \mapsto S(t), t > 0$, is analytic;
- (3) for every real $r \geq 0$ and $t > 0$, the operator $S(t) \in \mathcal{L}(H, \mathcal{D}(A^r))$;
- (4) for every $x \in \mathcal{D}(A^r), r \geq 0$ we have $S(t)A^r x = A^r S(t)x$.

Definition 1. Let $A : D(A) \subset H \rightarrow H$ be a self-adjoint operator on the Hilbert space H and let $g : \mathbb{R} \rightarrow K$ be a piecewise continuous function. We set

$$D(g(A)) = \{u \in H : \int_a^{+\infty} |g(\lambda)|^2 d\|E_\lambda u\|^2 < \infty\}$$

and define the linear operator $g(A) : D(A) \subset H \rightarrow H$ by the formula

$$g(A)u = \int_a^{+\infty} g(\lambda) dE_\lambda u,$$

for all $u \in D(g(A))$.

3. REGULARIZATION OF THE HOMOGENEOUS PROBLEM

In this section, we shall consider the homogeneous problem

$$u_t + Au(t) = 0, \quad 0 < t < T, \quad (3.1)$$

$$u(T) = \varphi, \quad (3.2)$$

It is useful to state that the admissible set refers to the final value φ . The following lemma gives an answer to this question.

Lemma 1. *Problem (3.1)–(3.2) has a solution if and only if*

$$\int_a^\infty e^{2\lambda T} d\|E_\lambda \varphi\|^2 < \infty$$

and its unique solution is represented by

$$u(t) = e^{(T-t)A} \varphi. \quad (3.3)$$

If the problem (1.1) admits a solution u then this solution can be represented by

$$u(t) = e^{(T-t)A} \varphi = \int_a^\infty e^{\lambda(T-t)} dE_\lambda \varphi. \quad (3.4)$$

Since $t < T$, we know from (3.4) that the terms $e^{-(t-T)\lambda}$ is the source of the instability. So, to regularize problem (3.4), we should replace it by the better terms. Let φ and φ_ϵ denote the exact and measured data at $t = T$, respectively, which satisfy

$$\|\varphi - \varphi_\epsilon\| \leq \epsilon,$$

where ϵ is a noise level. In this paper, we perturbed the final condition $u(T) = \varphi$ to form an approximate nonlocal problem depending on a small parameter. We introduced the regularized problem with boundary condition containing a derivative of the same order than the equation as the following equation

$$\begin{aligned} v_t^\epsilon + Av^\epsilon &= 0, \quad 0 < t < T + m \\ \epsilon v_t^\epsilon(0) + v^\epsilon(T + m) &= \varphi_\epsilon \end{aligned}$$

where $m > 0$ is a fixed number. From [7], the approximated solution v_ϵ corresponding to the final value φ_ϵ is given as

$$v^\epsilon(t) = \int_a^\infty \frac{e^{-\lambda t}}{\epsilon\lambda + e^{-\lambda(T+m)}} dE_\lambda \varphi_\epsilon, \quad t \in [0, T+m]. \quad (3.5)$$

To get an error estimate for $\|v^\epsilon(t+m) - u(t)\|$, we will use the function

$$u^\epsilon(t) = \int_a^\infty \frac{e^{-\lambda t}}{\epsilon\lambda + e^{-\lambda(T+m)}} dE_\lambda \varphi, \quad t \in [0, T+m]. \quad (3.6)$$

Theorem 2. Assume that u has the eigenfunction expansion $u(t) = \int_0^\infty dE_\lambda u(t)$.

a) Assume that there exist a positive constant C_1 such that $\|Au(0)\| \leq C_1$. Then for every $t \in [0, T]$,

$$\|v^\epsilon(t+m) - u(t)\| \leq \epsilon^{\frac{t+m}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})} \right)^{\frac{T-t}{T+m}} + \frac{C_1}{\ln(\frac{T+m}{\epsilon})}. \quad (3.7)$$

b) Assume that there exist positive constants m and C_2 such that

$$\int_a^\infty \lambda^2 e^{2(t+m)\lambda} d\|E_\lambda u(0)\|^2 \leq C_2^2. \quad (3.8)$$

Then for every $t \in [0, T]$,

$$\|v^\epsilon(t+m) - u(t)\| \leq (C_2 + 1) \epsilon^{\frac{t+m}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})} \right)^{\frac{T-t}{T+m}}. \quad (3.9)$$

Proof. First, we prove the following useful lemma.

Lemma 2. Let $s, t, \epsilon, m, \xi, \lambda$ be real numbers such that $0 \leq t \leq s \leq T$, $\lambda \in (a, \infty)$ and $0 < \epsilon < eT$. Then the following estimate holds for all $\lambda \in (a, \infty)$

$$\frac{e^{-(t+m)\lambda}}{\epsilon\lambda + e^{-(T+m)\lambda}} \leq \epsilon^{\frac{t-T}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})} \right)^{\frac{T-t}{T+m}}. \quad (3.10)$$

Proof. For $\lambda > 0$, we define the function

$$h(\lambda) = \frac{1}{\epsilon\lambda + e^{-\lambda(T+m)}}.$$

Then

$$h(\lambda) \leq h\left(\frac{\ln(\frac{T+m}{\epsilon})}{T+m}\right) = \frac{T+m}{\epsilon(1 + \ln(\frac{T+m}{\epsilon}))} \quad \epsilon \in (0, eT).$$

Hence

$$\frac{1}{\epsilon\lambda + e^{-(T+m)\lambda}} \leq \frac{1}{\epsilon \ln(\frac{T+m}{\epsilon})}. \quad (3.11)$$

Using (3.11), we obtain

$$\begin{aligned}
 \frac{e^{-(t+m)\lambda}}{\epsilon\lambda + e^{-(T+m)\lambda}} &= \frac{e^{-(t+m)\lambda}}{(\epsilon\lambda + e^{-(T+m)\lambda})^{\frac{t+m}{T+m}} (\epsilon + e^{-(T+m)\lambda})^{\frac{T-t}{T+m}}} \\
 &\leq \frac{1}{(\epsilon\lambda + e^{-(T+m)\lambda})^{\frac{T-t}{T+m}}} \\
 &\leq \left(\frac{T+m}{\epsilon \ln(\frac{T+m}{\epsilon})} \right)^{\frac{T-t}{T+m}} \\
 &= \epsilon^{\frac{t-T}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})} \right)^{\frac{T-t}{T+m}}.
 \end{aligned}$$

□

Now, we prove Theorem 2. The proof is divided into three steps.

Step 1. Estimate $\|v^\epsilon(t+m) - u^\epsilon(t+m)\|$. For every $\lambda \in (0, \infty)$,

$$v^\epsilon(t+m) - u^\epsilon(t+m) = \int_a^\infty \frac{e^{-\lambda(t+m)}}{\epsilon\lambda + e^{-\lambda(T+m)}} dE_\lambda(\varphi_\epsilon - \varphi)$$

we have

$$\begin{aligned}
 \|v^\epsilon(t+m) - u^\epsilon(t+m)\|^2 &= \int_a^\infty \left(\frac{e^{-\lambda(t+m)}}{\epsilon\lambda + e^{-\lambda(T+m)}} \right)^2 d\|E_\lambda(\varphi_\epsilon - \varphi)\|^2 \\
 &\leq \epsilon^{\frac{2t-2T}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})} \right)^{\frac{2T-2t}{T+m}} \int_a^\infty d\|E_\lambda(\varphi_\epsilon - \varphi)\|^2 \\
 &\leq \epsilon^{\frac{2t-2T}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})} \right)^{\frac{2T-2t}{T+m}} \|\varphi_\epsilon - \varphi\|^2 \\
 &\leq \epsilon^{\frac{2t-2T}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})} \right)^{\frac{2T-2t}{T+m}} \epsilon^2 \\
 &\leq \epsilon^{\frac{2t+2m}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})} \right)^{\frac{2T-2t}{T+m}}.
 \end{aligned}$$

Thus

$$\|v^\epsilon(t+m) - u^\epsilon(t+m)\| \leq \epsilon^{\frac{t+m}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})} \right)^{\frac{T-t}{T+m}}.$$

Step 2. We estimate $\|u^\epsilon(t+m) - u(t)\|$ if $\|Au(0)\| \leq C_1$.

$$u^\epsilon(t+m) - u(t) = \int_a^\infty \left(\frac{e^{-\lambda(t+m)}}{\epsilon\lambda + e^{-\lambda(T+m)}} - e^{(T-t)\lambda} \right) dE_\lambda \varphi$$

$$= \int_a^\infty \frac{\epsilon \lambda e^{(T-t)\lambda}}{\epsilon \lambda + e^{-(T+m)\lambda}} dE_\lambda \varphi$$

Since (3.4), we have $u(0) = \int_a^\infty e^{\lambda T} dE_\lambda \varphi$. Thus $dE_\lambda \varphi = e^{-\lambda T} dE_\lambda u(0)$. This implies that

$$u^\epsilon(t+m) - u(t) = \int_a^\infty \frac{\epsilon \lambda e^{-t\lambda}}{\epsilon \lambda + e^{-(T+m)\lambda}} dE_\lambda u(0).$$

Applying the inequality (3.11), we obtain

$$\begin{aligned} \|u^\epsilon(t+m) - u(t)\|^2 &= \int_a^\infty \left(\frac{\epsilon \lambda e^{-t\lambda}}{\epsilon \lambda + e^{-(T+m)\lambda}} \right)^2 d\|E_\lambda u(0)\|^2 \\ &\leq \left(\frac{\epsilon}{\epsilon \ln(\frac{T+m}{\epsilon})} \right)^2 \int_a^\infty \lambda^2 d\|E_\lambda u(0)\|^2 \\ &\leq \left(\frac{1}{\ln(\frac{T+m}{\epsilon})} \right)^2 \|Au(0)\|^2. \end{aligned}$$

Therefore

$$\|u^\epsilon(t+m) - u(t)\| \leq \frac{1}{\ln(\frac{T+m}{\epsilon})} \|Au(0)\|.$$

It implies that

$$\begin{aligned} \|v^\epsilon(t+m) - u(t)\| &\leq \|v^\epsilon(t+m) - u^\epsilon(t+m)\| + \|u^\epsilon(t+m) - u(t)\| \\ &\leq \epsilon^{\frac{t+m}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})} \right)^{\frac{T-t}{T+m}} + \frac{1}{\ln(\frac{T+m}{\epsilon})} \|Au(0)\|. \end{aligned}$$

Step 3. Estimate $\|u^\epsilon(t+m) - u(t)\|$ if $\int_a^\infty \lambda^2 e^{2(t+m)\lambda} d\|E_\lambda u(0)\|^2 \leq C_2^2$.

Since $u^\epsilon(t+m) - u(t) = \int_a^\infty \frac{\epsilon \lambda}{\epsilon \lambda + e^{-(T+m)\lambda}} dE_\lambda u(0)$, we have

$$\begin{aligned} u^\epsilon(t+m) - u(t) &= \int_a^\infty \frac{\epsilon \lambda}{\epsilon \lambda + e^{-(T+m)\lambda}} dE_\lambda u(0) \\ &= \int_a^\infty \frac{\epsilon \lambda e^{-(t+m)\lambda}}{\epsilon \lambda + e^{-(T+m)\lambda}} e^{(t+m)\lambda} dE_\lambda u(0). \end{aligned}$$

Then

$$\begin{aligned} \|u^\epsilon(t+m) - u(t)\|^2 &= \int_a^\infty \left(\frac{\epsilon e^{-(t+m)\lambda}}{\epsilon \lambda + e^{-(T+m)\lambda}} \right)^2 \left(\lambda e^{(t+m)\lambda} \right)^2 d\|E_\lambda u(0)\|^2 \\ &\leq \epsilon^2 \epsilon^{\frac{2t-2T}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})} \right)^{\frac{2T-2t}{T+m}} \int_a^\infty \lambda^2 e^{2(t+m)\lambda} d\|E_\lambda u(0)\|^2 \\ &= \epsilon^{\frac{2t+2m}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})} \right)^{\frac{2T-2t}{T+m}} \int_a^\infty \lambda^2 e^{2(t+m)\lambda} d\|E_\lambda u(0)\|^2. \end{aligned}$$

Thus

$$\|u^\epsilon(t+m) - u(t)\| \leq \epsilon^{\frac{t+m}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})} \right)^{\frac{T-t}{T+m}} \sqrt{\int_a^\infty \lambda^2 e^{2(t+m)\lambda} d\|E_\lambda u(0)\|^2}.$$

Applying the triangle inequality, we obtain

$$\begin{aligned} \|v^\epsilon(t+m) - u(t)\| &\leq \|v^\epsilon(t+m) - u^\epsilon(t+m)\| + \|u^\epsilon(t+m) - u(t)\| \\ &\leq \epsilon^{\frac{t+m}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})} \right)^{\frac{T-t}{T+m}} \\ &\quad + \epsilon^{\frac{t+m}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})} \right)^{\frac{T-t}{T+m}} \sqrt{\int_a^\infty \lambda^2 e^{2(t+m)\lambda} d\|E_\lambda u(0)\|^2} \\ &\leq (C_2 + 1) \epsilon^{\frac{t+m}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})} \right)^{\frac{T-t}{T+m}}. \end{aligned}$$

□

4. REGULARIZATION OF THE NONLINEAR PROBLEM

In this section, we shall approximate the problem (1.1)–(1.2) by the following problem

$$\frac{d}{dt} u^\epsilon(t) + A_\epsilon u^\epsilon(t) = B(\epsilon, t) f(t, u^\epsilon(t)), \quad t \in (0, T), \quad (4.1)$$

$$u^\epsilon(T) = \varphi, \quad (4.2)$$

where $A_\epsilon, B(\epsilon, t)$ are defined in (4.3) and (4.4). For every $v \in H$ having the expansion $v = \int_a^{+\infty} dE_\lambda v$, we define

$$\begin{aligned} S(t)v &= \int_a^{+\infty} e^{-t\lambda} dE_\lambda v. \\ A_\epsilon(v) &= -\frac{1}{T+m} \int_a^{+\infty} \ln(\epsilon + e^{-(T+m)\lambda}) dE_\lambda v. \end{aligned} \quad (4.3)$$

$$B(\epsilon, t)(v) = \int_a^{+\infty} (1 + \epsilon e^{(T+m)\lambda})^{\frac{t-T}{T+m}} dE_\lambda v, \quad t \in [0, T]. \quad (4.4)$$

$$(\epsilon I + S(T+m))^{\frac{t-T}{T+m}} v = \int_a^{+\infty} \frac{dE_\lambda v}{(\epsilon + e^{-(T+m)\lambda})^{\frac{T-t}{T+m}}} \quad (4.5)$$

Notice that if $f = 0$ then the problem (4.1)–(4.2) has been studied in [4]. The main theorem is as follows

Theorem 3. *Let $\varphi \in H$. Then the problem (4.1)–(4.2) has a unique solution $u^\epsilon \in H$. Let $m > 0$ be a positive number. Let $u \in C([0, T]; H)$ be a solution of*

(1.1)–(1.3). Assume that u has the eigenfunction expansion $u(t) = \int_a^{+\infty} dE_\lambda u(t)$ satisfying $\int_a^{+\infty} e^{2(T+m)\lambda} d\|E_\lambda u(t)\|^2 < \infty$ for every $t \in (0, T]$. Let φ_ϵ be a measured data such that $\|\varphi_\epsilon - \varphi\| \leq \epsilon$ where $\epsilon \in (0, \min\{T, 1 - e^{-Ta}\})$. Using φ_ϵ we can construct a function $U^\epsilon : [0, T] \rightarrow H$ such that

$$\|U^\epsilon(t) - u(t)\| \leq (1 + B)e^{k(T-t)} \epsilon^{\frac{t+m}{T+m}}, \quad \forall t \in (0, T], \quad (4.6)$$

where $\epsilon \in (0, \min\{T, 1 - e^{-Ta}\})$, U^ϵ is a solution of (4.1) with $U^\epsilon(T) = \varphi_\epsilon$ and

$$B = \sup_{t \in [0, T]} \sqrt{\int_a^{+\infty} e^{2(T+m)\lambda} d\|E_\lambda u(t)\|^2}$$

Remark 1. The estimate (4.6) for $t = 0$ is $\epsilon^{\frac{m}{T+m}}$ which depends on m . To improve this, we estimate $u(0)$ by another direction. In fact, for $\epsilon > 0$, we can choose $t_\epsilon \in (0, \epsilon)$ such that

$$\|u(0) - U^\epsilon(t_\epsilon)\| \leq \left[(1 + B)e^{kT} + T\|f(0)\| \right] \frac{T+1}{T} \sqrt{\frac{T}{\ln(\frac{1}{\epsilon})}}.$$

Proof. First, we introduce some useful lemmas for the proof of main results in this paper.

Lemma 3. Let $\epsilon > 0$ and $0 < t < s < T$. Let A_ϵ be defined in (4.3) where $\epsilon \in (0, 1 - e^{-Ta})$. Let $B(\epsilon, t)$ be defined as in (4.4). Then the following inequalities hold:

- a) $\|(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}\| \leq \epsilon^{\frac{t-T}{T+m}}$
- b) $\|S(T-s)(\epsilon I + S(T+m))^{\frac{t}{T}-1}\| \leq \epsilon^{\frac{t-s}{T+m}}$.
- c) $\|A_\epsilon\| \leq \frac{1}{T} \ln(\frac{1}{\epsilon})$.
- d) $\|B(\epsilon, t)\| \leq 1$.

Proof. a) Let $v \in H$ and let $v = \int_a^{+\infty} dE_\lambda v$ be the eigenfunction expansion of v . We have

$$(\epsilon I + S(T+m))^{\frac{t-T}{T+m}} v = \int_a^{+\infty} \frac{dE_\lambda v}{(\epsilon + e^{-(T+m)\lambda})^{\frac{T-t}{T+m}}}.$$

Then

$$\begin{aligned} \|(\epsilon I + S(T+m))^{\frac{t-T}{T+m}} v\|^2 &= \int_a^{+\infty} \frac{d\|E_\lambda v\|^2}{(\epsilon + e^{-(T+m)\lambda})^{\frac{2T-2t}{T+m}}} \\ &\leq \int_a^{+\infty} \frac{d\|E_\lambda v\|^2}{\epsilon^{\frac{2T-2t}{T+m}}} = \epsilon^{\frac{2t-2T}{T+m}} \|v\|^2. \end{aligned}$$

Therefore, we obtain

$$\|(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}\| \leq \epsilon^{\frac{t-T}{T+m}}.$$

b) First, let $v \in H$, we get

$$\begin{aligned}
 & \|S(T-s)(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}(v)\|^2 \\
 &= \int_a^{+\infty} e^{2(s-T)\lambda} (\epsilon + e^{-(T+m)\lambda})^{\frac{2t-2T}{T+m}} d\|E_\lambda v\|^2 \\
 &= \int_a^{+\infty} (\epsilon e^{(T+m)\lambda} + 1)^{\frac{2s-2T}{T+m}} (\epsilon + e^{-(T+m)\lambda})^{\frac{2t-2s}{T+m}} d\|E_\lambda v\|^2 \\
 &\leq \int_a^{+\infty} (\epsilon + e^{-(T+m)\lambda})^{\frac{2t-2s}{T+m}} d\|E_\lambda v\|^2 \\
 &\leq \int_a^{+\infty} \epsilon^{\frac{2t-2s}{T+m}} d\|E_\lambda v\|^2 \\
 &= \epsilon^{\frac{2t-2s}{T+m}} \|v\|^2.
 \end{aligned}$$

Then, the following inequality is obtained

$$\|S(T-s)(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}\| \leq \epsilon^{\frac{t-s}{T+m}}.$$

c) We have

$$\|A_\epsilon(v)\|^2 = \frac{1}{(T+m)^2} \int_a^{+\infty} \ln^2\left(\frac{1}{\epsilon + e^{-(T+m)\lambda}}\right) d\|E_\lambda v\|^2.$$

Since $\epsilon \in (0, 1 - e^{-Ta})$, we obtain $\epsilon + e^{-(T+m)\lambda} < 1$, $\forall \lambda \geq a$. But

$$0 < \ln\left(\frac{1}{\epsilon + e^{-(T+m)\lambda}}\right) < \ln\left(\frac{1}{\epsilon}\right).$$

It follows that

$$\|A_\epsilon(v)\|^2 \leq \frac{1}{(T+m)^2} \ln^2\left(\frac{1}{\epsilon}\right) \int_a^{+\infty} d\|E_\lambda v\|^2 \leq \frac{1}{T^2} \ln^2\left(\frac{1}{\epsilon}\right) \|v\|^2.$$

d) Taking $v \in H$, we have

$$\|B(\epsilon, t)(v)\|^2 = \int_a^{+\infty} (1 + \epsilon e^{(T+m)\lambda})^{\frac{2t-2T}{T+m}} d\|E_\lambda v\|^2 \leq \int_a^{+\infty} d\|E_\lambda v\|^2 = \|v\|^2,$$

which finishes the proof. $\square$

Lemma 4. Let $\varphi \in H$ and let $f : \mathbb{R} \times H \rightarrow H$ be a continuous operator satisfying $\|f(t, w) - f(t, v)\| \leq k\|w - v\|$ for a $k > 0$ independent of $w, v \in H, t \in \mathbb{R}$. Then problem (4.1)–(4.2) has a unique solution $u^\epsilon \in C([0, T]; H)$ for any $0 < \epsilon < 1 - e^{-Ta}$.

Proof. The proof of Lemma 4 is divided into three steps.

Step 1.

For $w \in C([0, T]; H)$, we define

$$F(w)(t) = (\epsilon I + S(T + m))^{\frac{t-T}{T+m}} \left[\varphi - \int_t^T S(T-s) f(s, w(s)) ds \right]. \quad (4.7)$$

For every $w, v \in C([0, T]; H)$, we shall prove that

$$\|F^n(w)(t) - F^n(v)(t)\| \leq \left(\frac{k(T-t)}{\epsilon} \right)^n \frac{C^n}{n!} \|w - v\|, \quad (4.8)$$

where $C = \max\{T, 1\}$ and $\|\cdot\|$ is the sup norm in $C([0, T]; H)$. Using the induction method we can verify the latter inequality. We have

$$F(v)(t) = (\epsilon I + S(T + m))^{\frac{t-T}{T+m}} \left[\varphi - \int_t^T S(T-s) f(s, v(s)) ds \right]. \quad (4.9)$$

Using (4.7), (4.9), Lemma 3 and the Lipschitz property of f , we have

$$\begin{aligned} \|F(w)(t) - F(v)(t)\| &= \left\| \int_t^T S(T-s) (\epsilon I + S(T + m))^{\frac{t-T}{T+m}} (f(s, w(s)) - f(s, v(s))) ds \right\| \\ &\leq \frac{k}{\epsilon} \int_t^T \|w(s) - v(s)\| ds \leq C \frac{k}{\epsilon} (T-t) \|w - v\|. \end{aligned}$$

Thus, (4.8) holds for $n = 1$. Suppose that (4.8) holds for $n = j$. We prove that it also holds for $n = j + 1$. In fact, we have

$$\begin{aligned} &\|F^{j+1}(w)(t) - F^{j+1}(v)(t)\| \\ &= \left\| \int_t^T S(T-s) (\epsilon I + S(T + m))^{\frac{t-T}{T+m}} (f(F^j w)(s) - f(F^j v)(s)) ds \right\| \\ &\leq \frac{1}{\epsilon} (T-t) k \int_t^T \|F^j(w)(s) - F^j(v)(s)\|^2 ds \\ &\leq \left(\frac{k}{\epsilon} \right)^{(j+1)} \frac{(T-t)^{j+1}}{(j+1)!} C^{j+1} \|w - v\|. \end{aligned}$$

Therefore, by the induction principle, we have (4.8) for all $w, v \in C([0, T]; H)$. We consider $F : C([0, T]; H) \rightarrow C([0, T]; H)$. Since $\lim_{n \rightarrow \infty} \left(\frac{kT}{\epsilon} \right)^n \frac{C^n}{n!} = 0$, there exists

a positive integer n_0 such that F^{n_0} is a contraction. It follows that the equation $F^{n_0}(w) = w$ has a unique solution $u^\epsilon \in C([0, T]; H)$.

We claim that $F(u^\epsilon) = u^\epsilon$. In fact, one has $F(F^{n_0}(u^\epsilon)) = F(u^\epsilon)$. Hence $F^{n_0}(F(u^\epsilon)) = F(u^\epsilon)$. By the uniqueness of the fixed point of F^{n_0} , one has $F(u^\epsilon) = u^\epsilon$, i.e., the equation $F(w) = w$ has a unique solution $u^\epsilon \in C([0, T]; H)$.

Step 2.

Suppose u^ϵ is the unique solution of the integral equation (4.7), then u^ϵ is also a solution of the (4.1)–(4.2).

In fact, we have

$$u^\epsilon(t) = (\epsilon I + S(T + m))^{\frac{t-T}{T+m}} \left[\varphi - \int_t^T S(T-s)f(s, u^\epsilon(s))ds \right], \quad t \in [0, T].$$

Taking the derivative of $u^\epsilon(t)$, and by direct computation, we get

$$\frac{d}{dt}u^\epsilon(t) = A_\epsilon u^\epsilon(t) + B(\epsilon, t)f(t, u^\epsilon(t)).$$

Now, we are clear to see that

$$u^\epsilon(T) = \varphi.$$

Hence, u^ϵ is a solution of problem (4.1)–(4.2).

Step 3. The problem (4.1)–(4.2) has at most one solution in $C([0, T]; H)$.

Let u and v be two solutions of problem (4.1)–(4.2) such that $u, v \in C([0, T]; H)$. First, we denote $g : \mathbb{R} \times H \rightarrow H$ by

$$g(t, u(t)) = B(\epsilon, t)f(t, u(t)).$$

Next, because the property of function f , we have for any $w, v \in H$

$$\|g(t, w(t)) - g(t, v(t))\| \leq \|B(\epsilon, t)\| \|f(t, w) - f(t, v)\| \leq k \|w - v\|.$$

Let b be a positive constant, we put

$$w(t) = e^{-b(t-T)}(u(t) - v(t)) \quad b > 0.$$

By calculating directly, we can get

$$w_t + A_\epsilon w(t) - bw(t) = e^{b(t-T)} \left(g(t, e^{-b(t-T)}u(t)) - g(t, e^{-b(t-T)}v(t)) \right).$$

It follows that

$$\begin{aligned} & \langle w_t(t) + A_\epsilon w(t) - bw(t), w(t) \rangle \\ &= \langle e^{b(t-T)} \left(g(t, e^{-b(t-T)}u(t)) - g(t, e^{-b(t-T)}v(t)) \right), w(t) \rangle. \end{aligned}$$

Using the Lipschitz property of f , we have

$$| \langle e^{b(t-T)} \left(g(t, e^{-b(t-T)}u(t)) - g(t, e^{-b(t-T)}v(t)) \right), w(t) \rangle | \leq k \|w(t)\|^2.$$

So we obtain

$$\langle e^{b(t-T)} (g(t, e^{-b(t-T)}u(t)) - g(t, e^{-b(t-T)}v(t))), w(t) \rangle \geq -k \|w(t)\|^2.$$

Using Lemma 3 c), we have

$$|\langle A_\epsilon w(t), w(t) \rangle| \leq \frac{1}{T} \ln\left(\frac{1}{\epsilon}\right) \|w(t)\|^2,$$

which gives

$$\langle A_\epsilon w(t), w(t) \rangle \geq -\frac{1}{T} \ln\left(\frac{1}{\epsilon}\right) \|w(t)\|^2.$$

This implies that

$$\frac{1}{2} \frac{d}{dt} \|w(t)\|^2 \geq b \|w(t)\|^2 - k \|w(t)\|^2 - \frac{1}{T} \ln\left(\frac{1}{\epsilon}\right) \|w(t)\|^2.$$

Let any $t_1 \in [0, T]$. Taking the integral with respect to t from t_1 to T , we get

$$\|w(T)\|^2 - \|w(t_1)\|^2 \geq 2 \int_{t_1}^T (b - k - \frac{1}{T} \ln\left(\frac{1}{\epsilon}\right)) \|w(t)\|^2 dt.$$

Choosing $b = k + \frac{1}{T} \ln\left(\frac{1}{\epsilon}\right)$ and noting that $w(T) = 0$, we get $w(t_1) = 0$. Hence, $w(t) = 0$ or $u(t) = v(t)$, $\forall t \in [0, T]$. This completes the proof of step 3. $\square$

Lemma 5. *The (unique) solution of problem (4.1)–(4.2) depends continuously (in $C([0, T]; H)$) on φ .*

Proof. Let u and v be two solutions of problem (4.1)–(4.2) corresponding to the final values φ and ω respectively. We have

$$\begin{aligned} u(t) - v(t) &= (\epsilon I + S(T + m))^{\frac{t-T}{T+m}} (\varphi - \omega) \\ &\quad - \int_t^T S(T-s) (\epsilon I + S(T + m))^{\frac{t-T}{T+m}} (f(u(s)) - f(v(s))) ds. \end{aligned}$$

Applying Lemma 3 and the Lipchitz property of f we get

$$\begin{aligned} \|u(t) - v(t)\| &\leq \|(\epsilon I + S(T + m))^{\frac{t-T}{T+m}} (\varphi - \omega)\| \\ &\quad + \left\| \int_t^T S(T-s) (\epsilon I + S(T + m))^{\frac{t-T}{T+m}} (f(u(s)) - f(v(s))) ds \right\| \\ &\leq \epsilon^{\frac{t-T}{T+m}} \|\varphi - \omega\| + k \int_t^T \epsilon^{\frac{t-s}{T+m}} \|u(s) - v(s)\| ds. \end{aligned}$$

Therefore

$$\epsilon^{\frac{-t}{T+m}} \|u(t) - v(t)\| \leq \epsilon^{\frac{-T}{T+m}} \|\varphi - \omega\| + k \int_t^T \epsilon^{\frac{-s}{T+m}} \|u(s) - v(s)\| ds.$$

Applying Gronwall's inequality, we obtain

$$\|u(t) - v(t)\| \leq \epsilon^{\frac{t-T}{T+m}} e^{k(T-t)} \|\varphi - \omega\|.$$

So, the solution of the problem (4.1)–(4.2) is depended continuously on φ . $\square$

Now, we turn to

Proof of Theorem 3. In view of (1.3), we have

$$u(t) = S(t-T)\varphi - \int_t^T S(t-s)f(u(s))ds.$$

It follows that

$$\begin{aligned} & S(T-t)(\epsilon I + S(T+m))^{(t-T)/T+m} u(t) \\ &= (\epsilon I + S(T+m))^{\frac{t-T}{T+m}} \varphi - \int_t^T S(T-s)(\epsilon I + S(T+m))^{\frac{t-T}{T+m}} f(u(s))ds. \end{aligned}$$

Applying Lemma 3 and the inequality $1 - (1+x)^{-\alpha} \leq x\alpha$, ($x, \alpha > 0$) we get

$$\begin{aligned} \|u(t) - u^\epsilon(t)\| &\leq \int_t^T \|S(T-s)(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}\| \|f(u(s)) - f(u^\epsilon(s))\| ds \\ &\quad + \|(I - S(T-t)(\epsilon I + S(T+m))^{\frac{t-T}{T+m}})u(t)\| \\ &\leq k \int_t^T \epsilon^{\frac{t-s}{T+m}} \|u(s) - u^\epsilon(s)\| ds \\ &\quad + \sqrt{\int_a^{+\infty} \left(1 - (1 + \epsilon e^{(T+m)\lambda})^{\frac{t-T}{T+m}}\right)^2 d\|E_\lambda u(t)\|^2} \\ &\leq k \epsilon^{\frac{t+m}{T+m}} \int_t^T \epsilon^{\frac{s+m}{T+m}} \|u(s) - u^\epsilon(s)\| ds \\ &\quad + \epsilon^{\frac{T-t}{T+m}} \sqrt{\int_a^{+\infty} e^{2(T+m)\lambda} d\|E_\lambda u(t)\|^2}. \end{aligned}$$

Noting that $0 < \epsilon < \epsilon^{\frac{t+m}{T+m}}$ ($0 < \epsilon < 1$), we obtain

$$\epsilon^{\frac{-t-m}{T+m}} \|u(t) - u^\epsilon(t)\| \leq B + k \int_t^T \epsilon^{\frac{-s-m}{T+m}} \|u(s) - u^\epsilon(s)\| ds.$$

Using Gronwall's inequality we obtain

$$\epsilon^{\frac{-t-m}{T+m}} \|u(t) - u^\epsilon(t)\| \leq B e^{k(T-t)}.$$

Therefore

$$\|u(t) - u^\epsilon(t)\| \leq B e^{k(T-t)} \epsilon^{\frac{t+m}{T+m}}.$$

It follows from Lemma 5 that

$$\|U^\epsilon(t) - u^\epsilon(t)\| \leq \epsilon^{\frac{t-T}{T+m}} e^{k(T-t)} \|\varphi_\epsilon - \varphi\| \leq e^{k(T-t)} \epsilon^{\frac{t+m}{T+m}}.$$

Therefore

$$\begin{aligned} \|U^\epsilon(t) - u(t)\| &\leq \|U^\epsilon(t) - u^\epsilon(t)\| + \|u^\epsilon(t) - u(t)\| \\ &\leq (1 + B) e^{k(T-t)} \epsilon^{\frac{t+m}{T+m}}, \end{aligned}$$

for every $t \in (0, T)$.

This completes the proof of Theorem 3. □

Proof of Remark 1. For $t \in (0, T)$, considering the function $h(t) = \frac{\ln t}{t} - \frac{\ln \epsilon}{T}$, we have $h(\epsilon) > 0$, $\lim_{t \rightarrow 0} h(t) = -\infty$, $h'(t) > 0$, ($0 < t < \epsilon$). It follows that the equation $h(t) = 0$ has a unique solution t_ϵ in $(0, \epsilon)$. Since $\frac{\ln t_\epsilon}{t_\epsilon} = \frac{\ln \epsilon}{T}$, the inequality $\ln t > -\frac{1}{t}$ gives $t_\epsilon < \sqrt{\frac{T}{\ln \frac{1}{\epsilon}}}$.

We have $u(t_\epsilon) - u(0) = \int_0^{t_\epsilon} u'(t) dt$. Hence $\|u(0) - u(t_\epsilon)\| \leq t_\epsilon \sup_{t \in [0, T]} \|u'(t)\|$. On the other hand, one has

$$\begin{aligned} \|u'(t)\| &\leq \|Au(t)\| + \|f(u(t))\| \leq \sqrt{\int_a^{+\infty} \lambda^2 d\|E_\lambda u(t)\|^2 + k\|u(t)\| + \|f(0)\|} \\ &\leq \frac{1}{T} \sqrt{\int_a^{+\infty} e^{2T\lambda} d\|E_\lambda u(t)\|^2 + k\|u(t)\| + \|f(0)\|} \\ &\leq \left(\frac{1}{T} + k\right) B + \|f(0)\|. \end{aligned}$$

It follows that $\|u(0) - u(t_\epsilon)\| \leq \left[\left(\frac{1}{T} + k\right) B + \|f(0)\|\right] t_\epsilon$. From (4.3), (4.4) and the definition of t_ϵ we get

$$\|u(0) - U^\epsilon(t_\epsilon)\| \leq \|u(0) - u(t_\epsilon)\| + \|u(t_\epsilon) - U^\epsilon(t_\epsilon)\|$$

$$\begin{aligned}
&\leq t_\epsilon \sup_{t \in [0, T]} \|u'(t)\| + (1+B)e^{k(T-t_\epsilon)} \epsilon^{\frac{t_\epsilon+m}{T+m}} \\
&\leq \left[\left(\frac{1}{T} + k \right) B + \|f(0)\| \right] t_\epsilon + (1+B)e^{kT} \epsilon^{\frac{t_\epsilon}{T}}.
\end{aligned}$$

Since $\epsilon^{\frac{t_\epsilon}{T}} = t_\epsilon$ and $(\frac{1}{T} + k)B \leq \frac{(1+B)e^{kT}}{T}$ we have

$$\begin{aligned}
\|u(0) - U^\epsilon(t_\epsilon)\| &\leq \left[(1+B)e^{kT} + T\|f(0)\| \right] \frac{T+1}{T} t_\epsilon \\
&\leq \left[(1+B)e^{kT} + T\|f(0)\| \right] \frac{T+1}{T} \sqrt{\frac{T}{\ln(\frac{1}{\epsilon})}}.
\end{aligned}$$

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INTERVAL OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH NONLINEAR DAMPING

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Abstract. Using generalized Riccati transformations, we derive new interval oscillation criteria for a class of general type second-order nonlinear differential equations with nonlinear damping. Examples are also given to illustrate the results.

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1. INTRODUCTION

In 1836 Sturm introduced the concept of oscillation when he studied the problem of the heat transmission, oscillation theory has been an important area of research in the qualitative theory of ordinary differential equations. Usually a qualitative approach is concerned with the behavior of solutions of a given differential equation and does not seek explicit solutions. Since then, oscillation behavior of solutions to different classes of linear and nonlinear ordinary differential equations have attracted the attention of many researchers.

In this paper we are concerned with the oscillatory behavior of the nonlinear second order differential equation with damping term

$$(r(t)k_1(x, x'))' + p(t)k_2(x, x')x' + q(t)f(x) = 0, \quad t \geq t_0, \quad (1.1)$$

where $t_0 \geq 0$ is a fixed real number, $p, q \in C([t_0, \infty), R)$, $r \in C^1([t_0, \infty), (0, \infty))$, $f \in C(R, R)$, $k_1 \in C^1(R^2, R)$ and $k_2 \in C(R^2, R)$. As usual, a function $x : [t_0, t_1) \rightarrow (-\infty, \infty)$, $t_1 > t_0$, is called a solution of Eq. (1.1) if $x(t)$ satisfies Eq. (1.1) for all $t \in [t_0, t_1)$. In what follows, we always assume that solutions of this equation are continuable, that is, they exist for all $t \geq t_0$. A nonconstant continuable solution $x(t)$ of Eq.(1.1) is called proper if $\sup\{|x(t)| : t \geq t_0\} > 0$. A proper solution $x(t)$ is called oscillatory if it does not have the largest zero, otherwise it is called nonoscillatory. The equation (1.1) is called oscillatory if all its proper solutions are oscillatory.

The oscillation problem for Eq. (1.1) and its various particular cases has been the subject of intensive studies in recent years (see, for example ([1–42]) and the references cited therein).

In 2005, Tiryaki and Zafer [30] established several new interval oscillation criteria for Eq. (1.1) by making use of a generalized Riccati substitutions

$$w(t) = \rho(t) \left[\frac{r(t)k_1(x(t), x'(t))}{x(t)} + g(t) \right] \quad (1.2)$$

and

$$w(t) = \rho(t) \left[\frac{r(t)k_1(x(t), x'(t))}{f(x(t))} + g(t) \right] \quad (1.3)$$

where $\rho \in C^1([t_0, \infty), (0, \infty))$ and $g \in C^1([t_0, \infty), R)$. The results in [30] extend, improve, and generalize some known oscillation criteria in the literature. As pointed out earlier Tiryaki and Zafer [29], equation (1.1) deserves attention simply because it includes (as special case) quite a large class of equations in the literature, and therefore more research is necessary to enlarge the families of the functions k_1 and k_2 . Motivated by the works of Tiryaki and Zafer ([29], [30]), we establish some sufficient conditions which guarantee the oscillatory of all the solutions of Eq. (1.1). We also want to emphasize that the results in this work neither include, nor are included by the results in [30] and are interesting in their own right. At the end, examples are given to illustrate the theoretical analysis of this work.

Let $D = \{(t, s) : t \geq s \geq t_0\}$. A function $H \in C(D, R)$ is said to belong to a function class F if

- (i) $H(t, t) = 0$ for $t \geq t_0$ and $H(t, s) > 0$ for $t > s \geq t_0$;
- (ii) H has continuous partial derivatives $\partial H / \partial t$ and $\partial H / \partial s$ satisfying

$$\frac{\partial H}{\partial t}(t, s) = h_1(t, s)\sqrt{H(t, s)} \quad \text{and} \quad \frac{\partial H}{\partial s}(t, s) = -h_2(t, s)\sqrt{H(t, s)},$$

where $h_1, h_2 \in L_{loc}(D, R)$.

2. OSCILLATION RESULTS FOR $f(x)$ WITHOUT MONOTONOCITY

In this section we consider the oscillation of Eq. (1.1) when the function $f(x)$ is not monotone. In this case we shall assume that the following conditions are satisfied:

- (A1) $\frac{f(x)}{x} \geq K$ for some constant $K > 0$ and all $x \in R \setminus \{0\}$;
- (A2) $k_1^2(u, v) \leq \alpha_1 v k_1(u, v)$ for some constant $\alpha_1 > 0$ and all $(u, v) \in R^2$;
- (A3) $uv k_2(u, v) \geq \alpha_2 u k_1(u, v)$ for some constant $\alpha_2 > 0$ and all $(u, v) \in R^2$.

Theorem 1. *Let (A1)-(A3) hold. Suppose that there exists an interval $(a, b) \subset [t_0, \infty)$ such that $p(t) \geq 0$ and $q(t) \geq 0$ for all $t \in (a, b)$, and that there exist $c \in (a, b)$,*

$H \in F$ and $\rho \in C^1([t_0, \infty), (0, \infty))$ such that

$$\begin{aligned} & \frac{1}{H(c, a)} \int_a^c H(s, a) K \rho(s) q(s) ds + \frac{1}{H(b, c)} \int_c^b H(b, s) K \rho(s) q(s) ds \\ & > \frac{\alpha_1}{4H(c, a)} \int_a^c \rho(s) r(s) \left[h_1(s, a) + Q_1(s) \sqrt{H(s, a)} \right]^2 ds \\ & + \frac{\alpha_1}{4H(b, c)} \int_c^b \rho(s) r(s) \left[h_2(b, s) - Q_1(s) \sqrt{H(b, s)} \right]^2 ds, \end{aligned} \quad (2.1)$$

where

$$Q_1(s) = \left(\frac{\rho'(s)}{\rho(s)} - \frac{\alpha_2 p(s)}{r(s)} \right). \quad (2.2)$$

Then every solution of Eq. (1.1) has a zero in (a, b) .

Proof. For the sake of contradiction, assume that there exists a solution $x(t)$ of Eq. (1.1) such that $x(t) \neq 0$ for all $t \in (a, b)$. Introduce a new function w by

$$w(t) = \rho(t) \frac{r(t) k_1(x(t), x'(t))}{x(t)}, \quad t \in (a, b). \quad (2.3)$$

Note that w is well-defined because $x(t) \neq 0$ for all $t \in (a, b)$. Differentiating (2.3), we obtain

$$\begin{aligned} w'(t) &= \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) p(t) \frac{k_2(x(t), x'(t)) x'(t)}{x(t)} - \rho(t) q(t) \frac{f(x(t))}{x(t)} \\ &\quad - \rho(t) \frac{r(t) k_1(x(t), x'(t)) x'(t)}{x^2(t)}. \end{aligned} \quad (2.4)$$

Using (A1)-(A3) in (2.4), one has

$$w'(t) \leq -K \rho(t) q(t) + \left(\frac{\rho'(t)}{\rho(t)} - \frac{\alpha_2 p(t)}{r(t)} \right) w(t) - \frac{1}{\alpha_1 \rho(t) r(t)} w^2(t). \quad (2.5)$$

Multiplying (2.5) by $H(t, s)$, integrating it with respect to s from c to t for $t \in [c, b)$, and using (i) and (ii), we easily obtain

$$\begin{aligned} & \int_c^t H(t, s) K \rho(s) q(s) ds \\ & \leq - \int_c^t H(t, s) w'(s) ds + \int_c^t H(t, s) Q_1(s) w(s) ds - \int_c^t \frac{H(t, s) w^2(s)}{\alpha_1 \rho(s) r(s)} ds \\ & = H(t, c) w(c) \end{aligned}$$

$$\begin{aligned}
& - \int_c^t \left\{ H(t, s) \frac{w^2(s)}{\alpha_1 \rho(s) r(s)} + \left[h_2(t, s) \sqrt{H(t, s)} - Q_1(s) H(t, s) \right] w(s) \right\} ds \\
& = H(t, c) w(c) \\
& - \int_c^t \left[\sqrt{\frac{H(t, s)}{\alpha_1 \rho(s) r(s)}} w(s) + \frac{1}{2} \sqrt{\alpha_1 \rho(s) r(s)} \left(h_2(t, s) - Q_1(s) \sqrt{H(t, s)} \right) \right]^2 ds \\
& + \frac{\alpha_1}{4} \int_c^t \rho(s) r(s) \left[h_2(t, s) - Q_1(s) \sqrt{H(t, s)} \right]^2 ds \\
& \leq H(t, c) w(c) + \frac{\alpha_1}{4} \int_c^t \rho(s) r(s) \left[h_2(t, s) - Q_1(s) \sqrt{H(t, s)} \right]^2 ds. \tag{2.6}
\end{aligned}$$

Similarly, multiplying (2.5) by $H(s, t)$ and integrating from t to c for $t \in (a, c]$, one obtains

$$\begin{aligned}
& \int_t^c H(s, t) K \rho(s) q(s) ds \\
& \leq -H(c, t) w(c) + \frac{\alpha_1}{4} \int_t^c \rho(s) r(s) \left[h_1(s, t) + Q_1(s) \sqrt{H(s, t)} \right]^2 ds. \tag{2.7}
\end{aligned}$$

Letting $t \rightarrow a^+$ in (2.7) and $t \rightarrow b^-$ in (2.6), dividing (2.6) and (2.7) by $H(b, c)$ and $H(c, a)$, respectively, and then adding them, we have that

$$\begin{aligned}
& \frac{1}{H(c, a)} \int_a^c H(s, a) K \rho(s) q(s) ds + \frac{1}{H(b, c)} \int_c^b H(b, s) K \rho(s) q(s) ds \\
& \leq \frac{\alpha_1}{4H(c, a)} \int_a^c \rho(s) r(s) \left[h_1(s, a) + Q_1(s) \sqrt{H(s, a)} \right]^2 ds \\
& + \frac{\alpha_1}{4H(b, c)} \int_c^b \rho(s) r(s) \left[h_2(b, s) - Q_1(s) \sqrt{H(b, s)} \right]^2 ds,
\end{aligned}$$

which contradicts the assumption (2.1). The proof is complete. $\square$

Thus by Theorem 1, we have the following oscillation result:

Theorem 2. *Let (A1)-(A3) hold. Suppose that there exists an interval $(a, b) \subset [t_0, \infty)$ such that $p(t) \geq 0$ and $q(t) \geq 0$ for all $t \in (a, b)$. If, for every $T \geq t_0$, there*

exist $H \in F$, $\rho \in C^1([t_0, \infty), (0, \infty))$, and $a, b, c \in \mathbb{R}$ such that $T \leq a < c < b$ and (2.1) holds, then every solution of Eq. (1.1) is oscillatory.

Theorem 3. Let (A1)-(A3) hold, and $p(t) \geq 0$ and $q(t) \geq 0$ for all $t \in [t_0, \infty)$. If

$$\limsup_{t \rightarrow \infty} \int_l^t \left\{ H(s, l) K \rho(s) q(s) - \frac{\alpha_1 \rho(s) r(s)}{4} \left[h_1(s, l) + Q_1(s) \sqrt{H(s, l)} \right]^2 \right\} ds > 0 \quad (2.8)$$

and

$$\limsup_{t \rightarrow \infty} \int_l^t \left\{ H(t, s) K \rho(s) q(s) - \frac{\alpha_1 \rho(s) r(s)}{4} \left[h_2(t, s) - Q_1(s) \sqrt{H(t, s)} \right]^2 \right\} ds > 0, \quad (2.9)$$

for some $H \in F$, $\rho \in C^1([t_0, \infty), (0, \infty))$ and for each $l \geq t_0$, where $Q_1(s)$ is as in (2.2), then every solution of Eq. (1.1) is oscillatory.

Proof. For any $T \geq t_0$, let $a = T$. In (2.8), we choose $l = a$. Then there exists $c > a$ such that

$$\int_a^c \left\{ H(s, a) K \rho(s) q(s) - \frac{\alpha_1 \rho(s) r(s)}{4} \left[h_1(s, a) + Q_1(s) \sqrt{H(s, a)} \right]^2 \right\} ds > 0. \quad (2.10)$$

In (2.9), we choose $l = c$. Then there exists $b > c$ such that

$$\int_c^b \left\{ H(b, s) K \rho(s) q(s) - \frac{\alpha_1 \rho(s) r(s)}{4} \left[h_2(b, s) - Q_1(s) \sqrt{H(b, s)} \right]^2 \right\} ds > 0. \quad (2.11)$$

Combining (2.10) and (2.11), we obtain (2.1). The required conclusion thus follows from Theorem 2. The proof is complete. $\square$

Corollary 1. If the inequality in (A3) is actually an equality, we can eliminate the sign condition on $p(t)$ in the above theorems.

Remark 1. If (A3) hold with equality ($\alpha_2 = 1$), then Theorem 1, 2 and 3 reduce to Theorem 3.3, 3.4 and Corollary 3.2, respectively, obtained by Tiryaki and Zafer [30] with $g(t) = 0$ in the substitution (1.2).

Remark 2. If we use the substitution (1.2) instead of (2.3) and the condition (A3) is replaced with

$$uvk_2(u, v) \geq \alpha_2 k_1^2(u, v), \alpha_2 > 0, \quad (C1)$$

then Theorem 1, 2 and 3 becomes Theorem 3.1, Corollary 3.1 and Theorem 3.2 of Tiryaki and Zafer [30], respectively.

Remark 3. We note that the case $p(t) \leq 0$ can be handled in a similar manner by reversing the inequality sign in (A3).

Remark 4. If $k_1(u, v) = v$, $p(t) \equiv 0$, $\rho(t) = 1$ and $f(x) = x$, we also recover Corollary 2.2, Theorem 2.1 and Corollary 2.4 of Kong [12], from Theorem 1, 2 and 3, respectively.

It is easy to see that when $H(t, s) = H(t - s) \in F$, $h_1(t - s) \equiv h_2(t - s)$. From now on, we denote them by $h(t - s)$. Furthermore, the subclass of F containing such $H(t - s)$ will be denoted by F_0 . Applying Theorem 2 to F_0 we have the following result:

Theorem 4. Suppose that (A1)-(A3) hold, and $p(t) \geq 0$ and $q(t) \geq 0$ for all $t \in (a, b)$. If for each $T \geq t_0$, there exist $\rho \in C^1([t_0, \infty), (0, \infty))$, $H \in F_0$, and real numbers a, c such that $T \leq a < c$ and

$$\begin{aligned} & \int_a^c H(s-a) K[\rho(s)q(s) + \rho(2c-s)q(2c-s)] ds \\ & > \frac{\alpha_1}{4} \int_a^c [\rho(s)r(s) + \rho(2c-s)r(2c-s)] h^2(s-a) ds \\ & + \frac{\alpha_1}{2} \int_a^c [\rho(s)r(s)Q_1(s) - \rho(2c-s)r(2c-s)Q_2(s)] h(s-a) \sqrt{H(s-a)} ds \\ & + \frac{\alpha_1}{4} \int_a^c [\rho(s)r(s)Q_1^2(s) + \rho(2c-s)r(2c-s)Q_2^2(s)] H(s-a) ds, \end{aligned} \quad (2.12)$$

where $Q_2(s) = \left(\frac{\rho'(2c-s)}{\rho(2c-s)} - \frac{\alpha_2 p(2c-s)}{r(2c-s)} \right)$, then every solution of Eq. (1.1) is oscillatory.

Proof. Let $c = \frac{a+b}{2}$. Then

$$H(b-c) = H(c-a) = H\left(\frac{b-a}{2}\right),$$

and for any $w \in L[a, b]$, one has

$$\int_c^b w(s) ds = \int_a^c w(2c-s) ds.$$

Therefore,

$$\int_c^b H(b-s) K \rho(s) q(s) ds = \int_a^c H(s-a) K \rho(2c-s) q(2c-s) ds$$

and

$$\int_c^b \rho(s) r(s) h^2(b-s) ds = \int_a^c \rho(2c-s) r(2c-s) h^2(s-a) ds$$

Thus, (2.12) yields (2.1), and Eq. (1.1) is oscillatory by Theorem 2. $\square$

For different choices of $H(t, s)$, we will obtain different sufficient conditions for oscillatory behavior of solutions of Eq. (1.1). Let $H(t, s) = (t-s)^\lambda$, $t \geq s \geq t_0$, where $\lambda > 1$ is a constant. Then, we obtain the following useful oscillation criterion.

Corollary 2. Assume that (A1)-(A3) hold, and $p(t) \geq 0$ and $q(t) \geq 0$ for all $t \in [t_0, \infty)$. Then every solution of Eq. (1.1) is oscillatory provided that for each $l \geq t_0$ and for some $\lambda > 1$, there exists a function $\rho \in C^1([t_0, \infty), (0, \infty))$ such that the following two inequalities hold:

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_l^t (s-l)^\lambda \left\{ K \rho(s) q(s) - \frac{\alpha_1 \rho(s) r(s)}{4} \left[\frac{\lambda}{(s-l)} + Q_1(s) \right]^2 \right\} ds > 0 \quad (2.13)$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_l^t (t-s)^\lambda \left\{ K \rho(s) q(s) - \frac{\alpha_1 \rho(s) r(s)}{4} \left[\frac{\lambda}{(t-s)} - Q_1(s) \right]^2 \right\} ds > 0. \quad (2.14)$$

Define

$$R(t) = \int_l^t \frac{1}{r(s)} ds, \quad t \geq l \geq t_0,$$

and set

$$H(t, s) = [R(t) - R(s)]^\lambda, \quad t \geq t_0$$

where $\lambda > 1$ is a constant.

If we choose $\rho(t) = 1$, by Theorem 3, we have the following oscillatory criterion.

Theorem 5. Let (A1)-(A3) hold, $p(t) \geq 0$ and $q(t) \geq 0$ for all $t \in [t_0, \infty)$, and $\lim_{t \rightarrow \infty} R(t) = \infty$. Then every solution of Eq. (1.1) is oscillatory provided that for

each $l \geq t_0$ and for some $\lambda > 1$, the following two inequalities hold:

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{ \left(Kq(s) - \frac{\alpha_1 \alpha_2^2 p^2(s)}{4r(s)} \right) [R(s) - R(l)]^\lambda \right. \\ \left. + \frac{\alpha_1 \alpha_2 \lambda p(s)}{2r(s)} [R(s) - R(l)]^{\lambda-1} \right\} ds > \frac{\alpha_1 \lambda^2}{4(\lambda-1)} \end{aligned} \quad (2.15)$$

and

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{ \left(Kq(s) - \frac{\alpha_1 \alpha_2^2 p^2(s)}{4r(s)} \right) [R(t) - R(s)]^\lambda \right. \\ \left. - \frac{\alpha_1 \alpha_2 \lambda p(s)}{2r(s)} [R(t) - R(s)]^{\lambda-1} \right\} ds > \frac{\alpha_1 \lambda^2}{4(\lambda-1)} \end{aligned} \quad (2.16)$$

Proof. Since $H(t, s) = [R(t) - R(s)]^\lambda$, it is easy to see that

$$h_1(t, s) = \lambda [R(t) - R(s)]^{(\lambda-2)/2} \frac{1}{r(t)},$$

and

$$h_2(t, s) = \lambda [R(t) - R(s)]^{(\lambda-2)/2} \frac{1}{r(s)}.$$

Noting that

$$\begin{aligned} \int_l^t r(s) h_1^2(s, l) ds &= \int_l^t r(s) \lambda^2 [R(s) - R(l)]^{\lambda-2} \frac{1}{r^2(s)} ds \\ &= \frac{\lambda^2}{\lambda-1} [R(t) - R(l)]^{\lambda-1} \end{aligned}$$

and

$$\begin{aligned} \int_l^t r(s) h_2^2(t, s) ds &= \int_l^t r(s) \lambda^2 [R(t) - R(s)]^{\lambda-2} \frac{1}{r^2(s)} ds \\ &= \frac{\lambda^2}{\lambda-1} [R(t) - R(l)]^{\lambda-1}, \end{aligned}$$

In view of $\lim_{t \rightarrow \infty} R(t) = \infty$, it follows that

$$\lim_{t \rightarrow \infty} \frac{\alpha_1}{4R^{\lambda-1}(t)} \int_l^t r(s) h_1^2(s, l) ds = \frac{\alpha_1 \lambda^2}{4(\lambda-1)} \quad (2.17)$$

and

$$\lim_{t \rightarrow \infty} \frac{\alpha_1}{4R^{\lambda-1}(t)} \int_l^t r(s) h_2^2(t, s) ds = \frac{\alpha_1 \lambda^2}{4(\lambda-1)}. \quad (2.18)$$

From (2.15) and (2.17) we have that

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{ H(s, l) Kq(s) - \frac{\alpha_1 r(s)}{4} \left[h_1(s, l) - \frac{\alpha_2 p(s)}{r(s)} \sqrt{H(s, l)} \right]^2 \right\} ds \\ &= \limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{ [R(s) - R(l)]^\lambda Kq(s) + \frac{\alpha_1 \alpha_2 \lambda p(s)}{2r(s)} [R(s) - R(l)]^{\lambda-1} \right. \\ & \quad \left. - \frac{\alpha_1 \alpha_2^2 p^2(s)}{4r(s)} [R(s) - R(l)]^\lambda \right\} ds - \lim_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \frac{\alpha_1}{4} r(s) h_1^2(s, l) ds \\ &= \limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{ \left(Kq(s) - \frac{\alpha_1 \alpha_2^2 p^2(s)}{4r(s)} \right) [R(s) - R(l)]^\lambda \right. \\ & \quad \left. + \frac{\alpha_1 \alpha_2 \lambda p(s)}{2r(s)} [R(s) - R(l)]^{\lambda-1} \right\} ds - \frac{\alpha_1 \lambda^2}{4(\lambda-1)} > 0, \end{aligned}$$

i.e., (2.8) holds. Similarly, (2.16) implies that (2.9) holds. By Theorem 3, Eq. (1.1) is oscillatory. The proof is complete. $\square$

Remark 5. If $k_1(u, v) = v$, $p(t) \equiv 0$, $\rho(t) = 1$ and $f(x) = x$, we recover Theorem 2.2 and Theorem 2.3 of Kong [12], from Theorem 4 and 5, respectively.

3. OSCILLATION RESULTS FOR $f(x)$ WITH MONOTONOCITY

We shall assume that the following conditions are satisfied:

- (A4) $f(x)$ is differentiable, $xf(x) \neq 0$ and $f'(x) \geq L$ for some constant $L > 0$ and all $x \in R \setminus \{0\}$;
- (A5) $vf(u)k_2(u, v) \geq \alpha_3 f(u)k_1(u, v)$ for some $\alpha_3 > 0$ and for all $(u, v) \in R^2$.

Theorem 6. Let (A2), (A4) and (A5) hold. Suppose that there exists an interval $(a, b) \subset [t_0, \infty)$ such that $p(t) \geq 0$ for all $t \in (a, b)$, and that there exist $c \in (a, b)$, $H \in F$ and $\rho \in C^1([t_0, \infty), (0, \infty))$ such that

$$\frac{1}{H(c, a)} \int_a^c H(s, a) \rho(s) q(s) ds + \frac{1}{H(b, c)} \int_c^b H(b, s) \rho(s) q(s) ds$$

$$\begin{aligned}
&> \frac{\alpha_1}{4LH(c,a)} \int_a^c \rho(s)r(s) \left[h_1(s,a) + Q_3(s)\sqrt{H(s,a)} \right]^2 ds \\
&+ \frac{\alpha_1}{4LH(b,c)} \int_c^b \rho(s)r(s) \left[h_2(b,s) - Q_3(s)\sqrt{H(b,s)} \right]^2 ds, \quad (3.1)
\end{aligned}$$

where

$$Q_3(s) = \left(\frac{\rho'(s)}{\rho(s)} - \frac{\alpha_3 p(s)}{r(s)} \right). \quad (3.2)$$

Then every solution of (1.1) has a zero in (a,b) .

Proof. For the sake of contradiction, assume that there exists a solution $x(t)$ of (1.1) such that $x(t) \neq 0$ for all $t \in (a,b)$. Introduce a new function w by

$$w(t) = \rho(t) \frac{r(t)k_1(x(t), x'(t))}{f(x(t))}, \quad t \in (a,b). \quad (3.3)$$

In view of Eq. (1.1), it follows from (3.3) that

$$\begin{aligned}
w'(t) &= \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) p(t) \frac{k_2(x(t), x'(t))x'(t)}{f(x(t))} - \rho(t) q(t) \\
&\quad - \rho(t) \frac{r(t)k_1(x(t), x'(t))f'(x(t))x'(t)}{f^2(x(t))}. \quad (3.4)
\end{aligned}$$

Using (A2), (A4) and (A5) in (3.4), we easily obtain

$$w'(t) \leq -\rho(t)q(t) + \left(\frac{\rho'(t)}{\rho(t)} - \frac{\alpha_3 p(t)}{r(t)} \right) w(t) - \frac{L}{\alpha_1 \rho(t)r(t)} w^2(t). \quad (3.5)$$

The rest of the proof is similar to that of Theorem 1. $\square$

Theorem 7. Suppose that (A2), (A4) and (A5) hold, and $p(t) \geq 0$ for all $t \in (a,b)$. If, for every $T \geq t_0$, there exist $H \in F$, $\rho \in C^1([t_0, \infty), (0, \infty))$, and $a, b, c \in \mathbb{R}$ such that $T \leq a < c < b$ and (3.1) holds, then every solution of Eq. (1.1) is oscillatory.

Theorem 8. Let (A2), (A4) and (A5) hold, and $p(t) \geq 0$ for all $t \in [t_0, \infty)$. If

$$\limsup_{t \rightarrow \infty} \int_l^t \left\{ H(s,l) \rho(s)q(s) - \frac{\alpha_1 \rho(s)r(s)}{4L} \left[h_1(s,l) + Q_3(s)\sqrt{H(s,l)} \right]^2 \right\} ds > 0$$

and

$$\limsup_{t \rightarrow \infty} \int_l^t \left\{ H(t,s) \rho(s)q(s) - \frac{\alpha_1 \rho(s)r(s)}{4L} \left[h_2(t,s) - Q_3(s)\sqrt{H(t,s)} \right]^2 \right\} ds > 0,$$

for some $H \in F$, $\rho \in C^1([t_0, \infty), (0, \infty))$ and for each $l \geq t_0$, then every solution of Eq. (1.1) is oscillatory.

Corollary 3. *If the inequality in (A5) is actually an equality, we can eliminate the sign condition on $p(t)$ in the above theorems.*

Remark 6. If (A5) hold with equality ($\alpha_3 = 1$), then Theorem 6, 7 and 8 reduce to Theorem 2.3, 2.4 and Corollary 2.2, respectively, obtained by Tiryaki and Zafer [30] with $g(t) = 0$ in the substitution (1.3).

Remark 7. If we use the substitution (1.3) instead of (3.3) and the condition (A5) is replaced with

$$vf(u)k_2(u, v) \geq \alpha_3 k_1^2(u, v), \alpha_3 > 0, \quad (C2)$$

then Theorem 6, 7 and 8 becomes Theorem 2.1, Corollary 2.1 and Theorem 2.2 of Tiryaki and Zafer [30], respectively.

Remark 8. We note that the case $p(t) \leq 0$ can be handled in a similar manner by reversing the inequality sign in (A5).

Theorem 9. *Suppose that (A2), (A4) and (A5) hold, and $p(t) \geq 0$ for all $t \in (a, b)$. If for each $T \geq t_0$, there exist $\rho \in C^1([t_0, \infty), (0, \infty))$, $H \in F_0$, and real numbers a, c such that $T \leq a < c$ and*

$$\begin{aligned} & \int_a^c H(s-a) [\rho(s)q(s) + \rho(2c-s)q(2c-s)] ds \\ & > \frac{\alpha_1}{4L} \int_a^c [\rho(s)r(s) + \rho(2c-s)r(2c-s)] h^2(s-a) ds \\ & + \frac{\alpha_1}{2L} \int_a^c [\rho(s)r(s)Q_3(s) - \rho(2c-s)r(2c-s)Q_4(s)] h(s-a) \sqrt{H(s-a)} ds \\ & + \frac{\alpha_1}{4L} \int_a^c [\rho(s)r(s)Q_3^2(s) + \rho(2c-s)r(2c-s)Q_4^2(s)] H(s-a) ds, \end{aligned}$$

where $Q_4(s) = \left(\frac{\rho'(2c-s)}{\rho(2c-s)} - \frac{\alpha_3 p(2c-s)}{r(2c-s)} \right)$, then every solution of Eq. (1.1) is oscillatory.

Corollary 4. *Assume that (A2), (A4) and (A5) hold, and $p(t) \geq 0$ for all $t \in [t_0, \infty)$. Then every solution of Eq. (1.1) is oscillatory provided that for each $l \geq t_0$ and for some $\lambda > 1$, there exists a function $\rho \in C^1([t_0, \infty), (0, \infty))$ such that the following two inequalities hold:*

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_l^t (s-l)^\lambda \left\{ \rho(s)q(s) - \frac{\alpha_1 \rho(s)r(s)}{4L} \left[\frac{\lambda}{(s-l)} + Q_3(s) \right]^2 \right\} ds > 0$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_l^t (t-s)^{\lambda} \left\{ \rho(s)q(s) - \frac{\alpha_1 \rho(s)r(s)}{4L} \left[\frac{\lambda}{(t-s)} - Q_3(s) \right]^2 \right\} ds > 0.$$

Remark 9. If $k_1(u, v) = v$, $p(t) \equiv 0$, $\rho(t) = 1$ and $f(x) = x$, we also recover Corollary 2.2, Theorem 2.1 and Corollary 2.4, Theorem 2.2, Theorem 2.3 of Kong [12], from Theorem 6, 7, 8 and 9, Corollary 3, respectively.

Example 1. Consider the equation

$$\left((1 + \cos^2 t) \frac{x'}{1 + x^2} \right)' + \frac{1}{1 + x^2} x' + q(t)f(x) = 0 \quad (3.6)$$

where

$$k_1(u, v) = \frac{v}{1 + u^2}, \quad k_2(u, v) = \frac{1}{1 + u^2}, \quad f(x) = x(1 + x^4), \quad (\alpha_1 = \alpha_2 = \alpha_3 = 1),$$

$$q(t) = \begin{cases} 7(t - 3n), & 3n \leq t \leq 3n + 1 \\ 7(-t + 3n + 2), & 3n + 1 < t \leq 3n + 2 \\ g(t), & 3n + 2 < t \leq 3n + 3, \end{cases}$$

$g(t) = -n(3n + 3 - t)(t - 3n - 2)$ and $n \in N_0 = \{0, 1, \dots\}$. For any $T \geq 0$, there exists $n \in N_0$ such that $3n \geq T$. Let $a = 3n$, $c = 3n + 1$, $b = 2c - a$, $\rho(t) = 1$ and $H(t - s) = (t - s)^2$. Since $f'(x) = 1 + 5x^4 \geq 1 = L$ and $h(t - s) = 2$, we get

$$\begin{aligned} & \int_a^c H(s - a)[q(s) + q(2c - s)] ds \\ &= \int_{3n}^{3n+1} (s - 3n)^2 [7(s - 3n) + 7(3n + 2 - s)] ds \\ &= 14 \int_{3n}^{3n+1} (s - 3n)^2 ds = 14/3 \end{aligned}$$

and

$$\begin{aligned} & \frac{\alpha_1}{4L} \int_a^c [r(s) + r(2c - s)] h^2(s - a) ds \\ &+ \frac{\alpha_1}{2L} \int_a^c [-p(s) + p(2c - s)] h(s - a) \sqrt{H(s - a)} ds \end{aligned}$$

$$\begin{aligned}
& + \frac{\alpha_1}{4L} \int_a^c \left[\frac{p^2(s)}{r(s)} + \frac{p^2(2c-s)}{r(2c-s)} \right] H(s-a) ds \\
& = \int_{3n}^{3n+1} [2 + \cos^2 s + \cos^2 (6n+2-s)] ds \\
& + \frac{1}{4} \int_{3n}^{3n+1} \left[\frac{1}{1 + \cos^2 s} + \frac{1}{1 + \cos^2 (6n+2-s)} \right] (s-3n)^2 ds \\
& \leq \int_{3n}^{3n+1} 4 ds + \frac{1}{2} \int_{3n}^{3n+1} (s-3n)^2 ds = 4 + \frac{1}{6} = 25/6 < 14/3.
\end{aligned}$$

Hence, (3.6) is oscillatory by Theorem 9. The condition (C2) in Remark 7 is not satisfied. Therefore, the results of [30] can not be used to Eq. (3.6).

If we take $f(x) = x(3 + 2\cos x)$ in Eq. (3.6), we find that every solution of (3.6) is oscillatory by Theorem 4. Since the condition (C1) in Remark 2 is not satisfied, the results of [30] can not be used to Eq. (3.6) with $f(x) = x(3 + 2\cos x)$.

Example 2. Consider the equation

$$\left((2 + \sin t) \frac{x'}{1 + x^2} \right)' - \frac{4 + 2 \sin t}{1 + x^2} x' + (2 + \sin t) x(7 + x^4) = 0 \quad (3.7)$$

where

$$\begin{aligned}
k_1(u, v) &= \frac{v}{1 + u^2}, \quad k_2(u, v) = \frac{1}{1 + u^2}, \quad (\alpha_1 = \alpha_2 = 1), \\
f(x) &= x(7 + x^4), \quad (K = 7).
\end{aligned}$$

Let $H(t, s) = (t - s)^2$, $\rho(t) = 1$. Since $p(t) = -4 - 2 \sin t < 0$ and $\alpha_1 \alpha_2 p(t) + r(t) < 0$, we see that Eq. (3.7) is oscillatory by Theorem 3 and Remark 3.

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COUPLED FIXED POINT THEOREMS AND APPLICATIONS TO PERIODIC BOUNDARY VALUE PROBLEMS

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Abstract. In this paper we present some existence, uniqueness and Hyers-Ulam stability results for the coupled fixed point of a pair of contractive type operators on complete metric spaces. The approach is based on a Perov type fixed point theorem for contractions. Some applications to integral equations and to boundary value problems are also given.

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1. INTRODUCTION

The classical Banach contraction principle is a very useful tool in nonlinear analysis with many applications to operatorial equations, fractal theory, optimization theory and other topics. Banach contraction principle was extended for singlevalued contraction on spaces endowed with vector-valued metrics by Perov in [7], while the case of multivalued contractions is treated in A. Petruşel [8].

In the study of the existence of fixed points for an operator, it is useful to consider a more general concept, namely coupled fixed points. The concept of coupled fixed point for continuous and discontinuous operators was introduced in 1987 by D. Guo and V. Lakshmikantham (see [5]) in connection with coupled quasisolutions of an initial value problem for ordinary differential equations.

Let X be a nonempty set. A mapping $d : X \times X \rightarrow \mathbb{R}^m$ is called a vector-valued metric on X if the following properties are satisfied:

- (a) $d(x, y) \geq 0$ for all $x, y \in X$; if $d(x, y) = 0$, then $x = y$;
- (b) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$.

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A set endowed with a vector-valued metric d is called generalized metric space. The notions of convergent sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

We denote by $M_{mm}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements and by I the identity $m \times m$ matrix. If $x, y \in \mathbb{R}^m$, $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$, then, by definition:

$$x \leq y \text{ if and only if } x_i \leq y_i \text{ for } i \in \{1, 2, \dots, m\}.$$

Notice that we will make an identification between row and column vectors in $\mathbb{R}^m$.

For the proof of the main results we need the following theorems. A classical result in matrix analysis is the following theorem (see [1], [10], [13]).

Theorem 1. *Let $A \in M_{mm}(\mathbb{R}_+)$. The following assertions are equivalents:*

- (i) A is convergent towards zero;
- (ii) $A^n \rightarrow 0$ as $n \rightarrow \infty$;
- (iii) The eigenvalues of A are in the open unit disc, i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(A - \lambda I) = 0$;
- (iv) The matrix $(I - A)$ is nonsingular and

$$(I - A)^{-1} = I + A + \dots + A^n + \dots; \quad (1.1)$$

- (v) The matrix $(I - A)$ is nonsingular and $(I - A)^{-1}$ has nonnegative elements;

- (vi) $A^n q \rightarrow 0$ and $qA^n \rightarrow 0$ as $n \rightarrow \infty$, for each $q \in \mathbb{R}^m$.

We recall now Perov's fixed point theorem (see [7]).

Theorem 2 (Perov). *Let (X, d) be a complete generalized metric space and the operator $f : X \rightarrow X$ with the property that there exists a matrix $A \in M_{mm}(\mathbb{R})$ such that $d(f(x), f(y)) \leq Ad(x, y)$ for all $x, y \in X$.*

If A is a matrix convergent towards zero, then:

- (i) $\text{Fix}(f) = \{x^*\}$;
- (ii) the sequence of successive approximations $(x_n)_{n \in \mathbb{N}}$, $x_n = f^n(x_0)$ is convergent and has the limit x^* , for all $x_0 \in X$;
- (iii) one has the following estimation

$$d(x_n, x^*) \leq A^n (I - A)^{-1} d(x_0, x_1); \quad (1.2)$$

- (iv) if $g : X \rightarrow X$ is an operator such that there exist $y^* \in \text{Fix}(g)$ and $\eta \in (\mathbb{R}_+^m)^*$ with $d(f(x), g(x)) \leq \eta$, for each $x \in X$, then

$$d(x^*, y^*) \leq (I - A)^{-1} \eta;$$

- (v) if $g : X \rightarrow X$ is an operator and there exists $\eta \in (\mathbb{R}_+^m)^*$ such that

$d(f(x), g(x)) \leq \eta$, for all $x \in X$, then for the sequence $y_n := g^n(x_0)$ we have the following estimation

$$d(y_n, x^*) \leq (I - A)^{-1} \eta + A^n (I - A)^{-1} d(x_0, x_1). \quad (1.3)$$

For related results to Perov's fixed point theorem and for some generalizations and applications of it we refer to [3], [4], [9].

Let (X, d) be a metric space. We will focus our attention to the following system of operatorial equations:

$$\begin{cases} u = A(u, v) \\ v = A(v, u) \end{cases} \quad (1.4)$$

where $A : X \times X \rightarrow X$ is a given operator.

By definition, a solution $(u, v) \in X \times X$ of the above system is called a coupled fixed point for A (see also [5], [6]).

In this paper we present some coupled fixed points results for contractive type single-valued operators on spaces endowed with vector-valued metrics and, as an application, we discuss the existence, uniqueness and Hyers-Ulam stability of the solution of a periodic boundary value problem related to a system of differential equations. The approach is based on Perov-type fixed point theorem for contractions in metric spaces endowed with vector-valued metrics.

2. EXISTENCE, UNIQUENESS AND STABILITY RESULTS FOR COUPLED FIXED POINTS

Definition 1. Let (X, d) be a generalized metric space and $f : X \rightarrow X$ be an operator. Then, the fixed point equation

$$x = f(x) \quad (2.1)$$

is said to be generalized Hyers-Ulam stable if there exists an increasing function $\psi : \mathbb{R}_+^m \rightarrow \mathbb{R}_+^m$, continuous in 0 with $\psi(0) = 0$, such that, for any $\varepsilon := (\varepsilon_1, \dots, \varepsilon_m)$ with $\varepsilon_i > 0$ for $i \in \{1, \dots, m\}$ and any solution $y^* \in X$ of the inequation

$$d(y, f(y)) \leq \varepsilon \quad (2.2)$$

there exists a solution x^* of (2.1) such that

$$d(x^*, y^*) \leq \psi(\varepsilon). \quad (2.3)$$

In particular, if $\psi(t) = C \cdot t$, $t \in \mathbb{R}_+^m$ (where $C \in M_{mm}(\mathbb{R}_+)$), then the fixed point equation (2.1) is called Hyers-Ulam stable.

We remind a direct consequence of Perov's fixed point theorem.

Theorem 3. Let (X, d) be a generalized metric space and let $f : X \rightarrow X$ be an operator with the property that there exists a matrix $A \in M_{mm}(\mathbb{R})$ such that A converges to zero and

$$d(f(x), f(y)) \leq Ad(x, y), \text{ for all } x, y \in X.$$

Then the fixed point equation

$$x = f(x), x \in X$$

is Hyers-Ulam stable.

Definition 2. Let (X, d) be a metric space and let $T_1, T_2 : X \times X \rightarrow X$ be two operators. Then the operatorial equations system

$$\begin{cases} x = T_1(x, y) \\ y = T_2(x, y) \end{cases} \quad (2.4)$$

is said to be Hyers-Ulam stable if there exist $c_1, c_2, c_3, c_4 > 0$ such that for each $\varepsilon_1, \varepsilon_2 > 0$ and each solution-pair $(u^*, v^*) \in X \times X$ of the inequations:

$$\begin{aligned} d(u^*, T_1(u^*, v^*)) &\leq \varepsilon_1 \\ d(v^*, T_2(u^*, v^*)) &\leq \varepsilon_2 \end{aligned} \quad (2.5)$$

there exists a solution $(x^*, y^*) \in X \times X$ of (2.4) such that

$$\begin{aligned} d(u^*, x^*) &\leq c_1 \varepsilon_1 + c_2 \varepsilon_2 \\ d(v^*, y^*) &\leq c_3 \varepsilon_1 + c_4 \varepsilon_2 \end{aligned} \quad (2.6)$$

For related results regarding Hyers-Ulam stability of the operatorial equations see [11].

We recall the following existence, uniqueness, data dependence and Hyers-Ulam stability theorem for the coupled fixed point of a pair of singlevalued operators (see [12]).

Theorem 4. Let (X, d) be a complete metric space, $T_1, T_2 : X \times X \rightarrow X$ such that

$$\begin{aligned} d(T_1(x, y), T_1(u, v)) &\leq k_1 d(x, u) + k_2 d(y, v) \\ d(T_2(x, y), T_2(u, v)) &\leq k_3 d(x, u) + k_4 d(y, v) \end{aligned} \quad (2.7)$$

for all $(x, y), (u, v) \in X \times X$. We suppose that $A = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$ converges to zero. Then

(i) there exists a unique element $(x^*, y^*) \in X \times X$ such that

$$\begin{cases} x^* = T_1(x^*, y^*) \\ y^* = T_2(x^*, y^*) \end{cases} \quad (2.8)$$

(ii) the sequence $(T_1^n(x, y), T_2^n(x, y))_{n \in \mathbb{N}}$ converges to (x^*, y^*) as $n \rightarrow \infty$, where

$$\begin{aligned} T_1^{n+1}(x, y) &:= T_1^n(T_1(x, y), T_2(x, y)) \\ T_2^{n+1}(x, y) &:= T_2^n(T_1(x, y), T_2(x, y)) \end{aligned} \quad (2.9)$$

for all $n \in \mathbb{N}^*$.

(iii) we have the following estimation

$$\begin{pmatrix} d(T_1^n(x_0, y_0), x^*) \\ d(T_2^n(x_0, y_0), y^*) \end{pmatrix} \leq A^n (I - A)^{-1} \begin{pmatrix} d(x_0, T_1(x_0, y_0)) \\ d(y_0, T_2(x_0, y_0)) \end{pmatrix} \quad (2.10)$$

(iv) let $F_1, F_2 : X \times X \rightarrow X$ be two operators such that, there exist $\eta_1, \eta_2 > 0$ with

$$\begin{aligned} d(T_1(x, y), F_1(x, y)) &\leq \eta_1 \\ d(T_2(x, y), F_2(x, y)) &\leq \eta_2 \end{aligned} \quad (2.11)$$

for all $(x, y) \in X \times X$. If $(a^*, b^*) \in X \times X$ is such that

$$\begin{cases} a^* = F_1(a^*, b^*) \\ b^* = F_2(a^*, b^*) \end{cases} \quad (2.12)$$

then

$$\begin{pmatrix} d(a^*, x^*) \\ d(b^*, y^*) \end{pmatrix} \leq (I - A)^{-1} \eta \quad (2.13)$$

where $\eta := \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$.

(v) let $F_1, F_2 : X \times X \rightarrow X$ be two operators such that, there exist $\eta_1, \eta_2 > 0$ with

$$\begin{aligned} d(T_1(x, y), F_1(x, y)) &\leq \eta_1 \\ d(T_2(x, y), F_2(x, y)) &\leq \eta_2 \end{aligned} \quad (2.14)$$

for all $(x, y) \in X \times X$ and considering the sequence $(F_1^n(x, y), F_2^n(x, y))_{n \in \mathbb{N}}$, where

$$\begin{aligned} F_1^{n+1}(x, y) &:= F_1^n(F_1(x, y), F_2(x, y)) \\ F_2^{n+1}(x, y) &:= F_2^n(F_1(x, y), F_2(x, y)) \end{aligned} \quad (2.15)$$

for all $n \in \mathbb{N}^*$ and $\eta := \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$ then

$$\begin{pmatrix} d(F_1^n(x_0, y_0), x^*) \\ d(F_2^n(x_0, y_0), y^*) \end{pmatrix} \leq (I - A)^{-1} \eta + A^n (I - A)^{-1} \begin{pmatrix} d(x_0, T_1(x_0, y_0)) \\ d(y_0, T_2(x_0, y_0)) \end{pmatrix} \quad (2.16)$$

(vi) the operatorial equations system

$$\begin{cases} x = T_1(x, y) \\ y = T_2(x, y) \end{cases} \quad (2.17)$$

is Hyers-Ulam stable.

3. AN APPLICATION TO PERIODIC BOUNDARY VALUE PROBLEM

In this section we study the existence, uniqueness and Hyers-Ulam stability of a solution to a periodic boundary value problem as an application of the coupled fixed point Theorem 4 presented in Section 2. The approach is based on the application presented in [2].

We consider the periodic boundary value problem

$$\begin{cases} u' = f(t, u) + g(t, v) \\ v' = f(t, v) + g(t, u) \\ u(0) = u(T) \\ v(0) = v(T) \end{cases} \quad (3.1)$$

assuming that f, g are continuous functions and satisfy the following conditions:

Assumption 1. There exist $\lambda_1 > 0$, $\lambda_2 > 0$ and $\mu_1 > 0$, $\mu_2 > 0$, such that for all $u, v \in \mathbb{R}$, $v \leq u$,

$$0 \leq (f(t, u) + \lambda_1 u) - (f(t, v) + \lambda_1 v) \leq \mu_1(u - v) \quad (3.2)$$

$$-\mu_2(u - v) \leq (g(t, u) - \lambda_2 u) - (g(t, v) - \lambda_2 v) \leq 0, \quad (3.3)$$

where $S := \begin{pmatrix} \frac{\mu_1}{\lambda_1 + \lambda_2} & \frac{\mu_2}{\lambda_1 + \lambda_2} \\ \frac{\mu_2}{\lambda_1 + \lambda_2} & \frac{\mu_1}{\lambda_1 + \lambda_2} \end{pmatrix}$ is a matrix convergent to zero.

We study the existence of a solution of the following periodic system:

$$u' + \lambda_1 u - \lambda_2 v = f(t, u) + g(t, v) + \lambda_1 u - \lambda_2 v, \quad (3.4)$$

$$v' + \lambda_1 v - \lambda_2 u = f(t, v) + g(t, u) + \lambda_1 v - \lambda_2 u, \quad (3.5)$$

together with the periodicity conditions,

$$u(0) = u(T) \text{ and } v(0) = v(T). \quad (3.6)$$

This problem is equivalent to the integral equations:

$$u(t) = \int_0^T G_1(t, s)[f(s, u) + g(s, v) + \lambda_1 u - \lambda_2 v] \\ + G_2(t, s)[f(s, v) + g(s, u) + \lambda_1 v - \lambda_2 u] ds$$

$$v(t) = \int_0^T G_1(t, s)[f(s, v) + g(s, u) + \lambda_1 v - \lambda_2 u] \\ + G_2(t, s)[f(s, u) + g(s, v) + \lambda_1 u - \lambda_2 v] ds$$

where

$$G_1(t, s) = \begin{cases} \frac{1}{2} \left[\frac{e^{\sigma_1(t-s)}}{1-e^{\sigma_1 T}} + \frac{e^{\sigma_2(t-s)}}{1-e^{\sigma_2 T}} \right] & 0 \leq s < t \leq T \\ \frac{1}{2} \left[\frac{e^{\sigma_1(t+T-s)}}{1-e^{\sigma_1 T}} + \frac{e^{\sigma_2(t+T-s)}}{1-e^{\sigma_2 T}} \right] & 0 \leq t < s \leq T \end{cases}$$

$$G_2(t, s) = \begin{cases} \frac{1}{2} \left[\frac{e^{\sigma_2(t-s)}}{1-e^{\sigma_2 T}} - \frac{e^{\sigma_1(t-s)}}{1-e^{\sigma_1 T}} \right] & 0 \leq s < t \leq T \\ \frac{1}{2} \left[\frac{e^{\sigma_2(t+T-s)}}{1-e^{\sigma_2 T}} - \frac{e^{\sigma_1(t+T-s)}}{1-e^{\sigma_1 T}} \right] & 0 \leq t < s \leq T \end{cases}$$

Here, $\sigma_1 = -(\lambda_1 + \lambda_2)$ and $\sigma_2 = (\lambda_2 - \lambda_1)$.

We need to guarantee that $G_1(t, s) \geq 0$, $0 \leq t, s \leq T$, and $G_2(t, s) \leq 0$, $0 \leq t, s \leq T$, by choosing λ_1, λ_2 suitably. The following lemma addresses this issue.

Lemma 1 (Lakshmikantham, [2]). *If*

$$\ln \left(\frac{2e-1}{e} \right) \leq (\lambda_2 - \lambda_1)T \quad (3.7)$$

$$(\lambda_1 + \lambda_2)T \leq 1 \quad (3.8)$$

then $G_1(t, s) \geq 0$ for $0 \leq t, s \leq T$, and $G_2(t, s) \leq 0$ for $0 \leq t, s \leq T$.

Let $X = C(I, \mathbb{R})$ be the metric space of all continuous functions $u : I \rightarrow \mathbb{R}$, endowed with the metric $d(u, v) = \sup_{t \in I} |u(t) - v(t)|$, for $u, v \in X$.

For $x, y, u, v \in X$, we also denote $\tilde{d}((x, y), (u, v)) := \begin{pmatrix} d(x, u) \\ d(y, v) \end{pmatrix}$

Let us define $A : X \times X \rightarrow X$ for $t \in I$, by

$$A(u, v)(t) = \int_0^T G_1(t, s) [f(s, u) + g(s, v) + \lambda_1 u - \lambda_2 v] \\ + G_2(t, s) [f(s, v) + g(s, u) + \lambda_1 v - \lambda_2 u] ds$$

Note that if $(u, v) \in X \times X$ is a coupled fixed point of A , then we have

$$u(t) = A(u, v)(t) \text{ and } v(t) = A(v, u)(t), \text{ for all } t \in I.$$

Thus, (u, v) is a solution of (3.4)-(3.6).

For the proof of our main result we need the following notion.

Definition 3. The system

$$\begin{cases} u(t) = \int_0^T G_1(t,s)[f(s,u) + g(s,v) + \lambda_1 u - \lambda_2 v] \\ \quad + G_2(t,s)[f(s,v) + g(s,u) + \lambda_1 v - \lambda_2 u] ds \\ v(t) = \int_0^T G_1(t,s)[f(s,v) + g(s,u) + \lambda_1 v - \lambda_2 u] \\ \quad + G_2(t,s)[f(s,u) + g(s,v) + \lambda_1 u - \lambda_2 v] ds \end{cases} \quad (3.9)$$

is said to be Hyers-Ulam stable if there exist $c_1, c_2 > 0$ such that for each $\varepsilon_1, \varepsilon_2 > 0$ and each solution (x^*, y^*) of the following inequation system

$$\begin{cases} |x^*(t) - \int_0^T G_1(t,s)[f(s,x^*) + g(s,y^*) + \lambda_1 x^* - \lambda_2 y^*] \\ \quad + G_2(t,s)[f(s,y^*) + g(s,x^*) + \lambda_1 y^* - \lambda_2 x^*] ds| \leq \varepsilon_1 \\ |y^*(t) - \int_0^T G_1(t,s)[f(s,y^*) + g(s,x^*) + \lambda_1 y^* - \lambda_2 x^*] \\ \quad + G_2(t,s)[f(s,x^*) + g(s,y^*) + \lambda_1 x^* - \lambda_2 y^*] ds| \leq \varepsilon_2 \end{cases} \quad (3.10)$$

there exists a solution (u^*, v^*) of (3.9) such that

$$\begin{aligned} |u^*(t) - x^*(t)| &\leq c_1 \varepsilon_1 + c_2 \varepsilon_2 \\ |v^*(t) - y^*(t)| &\leq c_3 \varepsilon_1 + c_4 \varepsilon_2 \end{aligned}$$

Our main result is the following existence, uniqueness and Hyers-Ulam stability of a solution to a periodic boundary value problem.

Theorem 5. Consider the problem (3.1) with $f, g \in C(I \times \mathbb{R}, \mathbb{R})$ and suppose that the Assumption 1 is satisfied. If (3.7) and (3.8) are fulfilled, then:

(i) there exists a unique solution (u^*, v^*) of the periodic boundary value problem (3.1).

(ii) let $f_1, g_1 \in C(I \times \mathbb{R}, \mathbb{R})$ such that, there exist $\eta_1, \eta_2 > 0$ with

$$\begin{cases} |f(t,u) - f_1(t,u)| \leq \eta_1 \\ |g(t,u) - g_1(t,u)| \leq \eta_2 \end{cases}$$

for all $(t,u) \in I \times \mathbb{R}$. Let $(a^*, b^*) \in X \times X$ be a solution of the problem (3.1) with f replaced by f_1 and g replaced by g_1 . Then

$$\widetilde{d}((u^*, v^*), (a^*, b^*)) = \begin{pmatrix} d(a^*, u^*) \\ d(b^*, v^*) \end{pmatrix} \leq (I - S)^{-1} \eta$$

$$\text{where } \eta := \begin{pmatrix} (\eta_1 + \eta_2) \frac{1}{\lambda_2 - \lambda_1} \\ (\eta_1 + \eta_2) \frac{1}{\lambda_2 - \lambda_1} \end{pmatrix}.$$

(iii) the system (3.9) is Hyers-Ulam stable.

Proof. (i)

$$\begin{aligned}
d(A(x, y), A(u, v)) &= \sup_{t \in I} |A(x, y)(t) - A(u, v)(t)| \\
&= \sup_{t \in I} \int_0^T G_1(t, s) [f(s, x) + g(s, y) + \lambda_1 x - \lambda_2 y] \\
&\quad + G_2(t, s) [f(s, y) + g(s, x) + \lambda_1 y - \lambda_2 x] ds \\
&\quad - \int_0^T G_1(t, s) [f(s, u) + g(s, v) + \lambda_1 u - \lambda_2 v] \\
&\quad + G_2(t, s) [f(s, v) + g(s, u) + \lambda_1 v - \lambda_2 u] ds \\
&= \sup_{t \in I} \int_0^T G_1(t, s) \\
&\quad [[f(s, x) + g(s, y) + \lambda_1 x - \lambda_2 y] - [f(s, u) + g(s, v) + \lambda_1 u - \lambda_2 v]] \\
&\quad + G_2(t, s) [[f(s, y) + g(s, x) + \lambda_1 y - \lambda_2 x] - [f(s, v) + g(s, u) + \lambda_1 v - \lambda_2 u]] ds \\
&= \sup_{t \in I} \int_0^T G_1(t, s) \\
&\quad [[f(s, x) + g(s, y) + \lambda_1 x - \lambda_2 y] - [f(s, u) + g(s, v) + \lambda_1 u - \lambda_2 v]] \\
&\quad - G_2(t, s) [[f(s, v) + g(s, u) + \lambda_1 v - \lambda_2 u] - [f(s, y) + g(s, x) + \lambda_1 y - \lambda_2 x]] ds \\
&\leq \sup_{t \in I} \int_0^T G_1(t, s) [\mu_1(x - u) + \mu_2(v - y)] - G_2(t, s) [\mu_1(v - y) + \mu_2(x - u)] ds \\
&= [\mu_1 d(x, u) + \mu_2 d(y, v)] \cdot \sup_{t \in I} \left| \int_0^T [G_1(t, s) - G_2(t, s)] ds \right| \\
&= [\mu_1 d(x, u) + \mu_2 d(y, v)] \cdot \sup_{t \in I} \left| \int_0^t \frac{e^{\sigma_1(t-s)}}{1 - e^{\sigma_1 T}} ds + \int_t^T \frac{e^{\sigma_1(t+T-s)}}{1 - e^{\sigma_1 T}} ds \right| \\
&= \frac{\mu_1}{\lambda_1 + \lambda_2} d(x, u) + \frac{\mu_2}{\lambda_1 + \lambda_2} d(y, v).
\end{aligned}$$

In a similar way we get that

$$d(A(y, x), A(v, u)) \leq \frac{\mu_1}{\lambda_1 + \lambda_2} d(y, v) + \frac{\mu_2}{\lambda_1 + \lambda_2} d(x, u)$$

If we denote $k_1 := \frac{\mu_1}{\lambda_1 + \lambda_2}$ and $k_2 := \frac{\mu_2}{\lambda_1 + \lambda_2}$, we get that

$$\begin{aligned}
d(A(x, y), A(u, v)) &\leq k_1 d(x, u) + k_2 d(y, v) \quad \text{and} \\
d(A(y, x), A(v, u)) &\leq k_2 d(x, u) + k_1 d(y, v)
\end{aligned}$$

Then we have

$$\begin{pmatrix} d(A(x, y), A(u, v)) \\ d(A(y, x), A(v, u)) \end{pmatrix} \leq \begin{pmatrix} k_1 d(x, u) + k_2 d(y, v) \\ k_2 d(x, u) + k_1 d(y, v) \end{pmatrix} \\ = \begin{pmatrix} k_1 & k_2 \\ k_2 & k_1 \end{pmatrix} \begin{pmatrix} d(x, u) \\ d(y, v) \end{pmatrix} = S \cdot \tilde{d}((x, y), (u, v))$$

where S is a matrix convergent to zero.

Since the hypothesis of Theorem 4 is satisfied we get that the periodic boundary value problem (3.4) and (3.6) has a unique solution on I .

(ii) We define $T : X \times X \rightarrow X$, for $t \in I$, by

$$\begin{aligned} T(u, v)(t) = & \int_0^T G_1(t, s) [f_1(s, u) + g_1(s, v) + \lambda_1 u - \lambda_2 v] \\ & + G_2(t, s) [f_1(s, v) + g_1(s, u) + \lambda_1 v - \lambda_2 u] ds \end{aligned}$$

Then, after some calculation, we get:

$$\begin{aligned} & |A(u, v)(t) - T(u, v)(t)| \\ &= \left| \int_0^T G_1(t, s) [f(s, u) + g(s, v) + \lambda_1 u - \lambda_2 v] \right. \\ &\quad \left. + G_2(t, s) [f(s, v) + g(s, u) + \lambda_1 v - \lambda_2 u] ds \right. \\ &\quad \left. - \int_0^T G_1(t, s) [f_1(s, u) + g_1(s, v) + \lambda_1 u - \lambda_2 v] \right. \\ &\quad \left. + G_2(t, s) [f_1(s, v) + g_1(s, u) + \lambda_1 v - \lambda_2 u] ds \right| \\ &\leq \int_0^T |G_1(t, s)| \\ &\quad |[f(s, u) + g(s, v) + \lambda_1 u - \lambda_2 v] - [f_1(s, u) + g_1(s, v) + \lambda_1 u - \lambda_2 v]] \\ &\quad + G_2(t, s) [[f(s, v) + g(s, u) + \lambda_1 v - \lambda_2 u] \\ &\quad - [f_1(s, v) + g_1(s, u) + \lambda_1 v - \lambda_2 u]]| ds \\ &\leq (\eta_1 + \eta_2) \left(\int_0^T |G_1(t, s)| ds + \int_0^T |G_2(t, s)| ds \right). \end{aligned}$$

Taking the supremum after $t \in I$ we obtain that

$$d(A(u, v), T(u, v)) \leq (\eta_1 + \eta_2) \frac{1}{\lambda_2 - \lambda_1}.$$

In a similar way, we get that

$$d(A(v, u), T(v, u)) \leq (\eta_1 + \eta_2) \frac{1}{\lambda_2 - \lambda_1}.$$

Now the conclusion follows by Theorem 4 (iv).

(iii) By the first part of our proof and by Theorem 4 (vi) we get the conclusion. $\square$

Remark 1. From the proof of Theorem 5 and Theorem 4 (ii), (iii) we also get the following conclusions:

(a) the sequence $(A^n(u, v), A^n(v, u))_{n \in \mathbb{N}}$ converges in $X \times X$ to (u^*, v^*) as $n \rightarrow \infty$, where

$$(A^{n+1}(u, v), A^{n+1}(v, u)) := A^n(A(u, v), A(v, u))$$

for all $n \in \mathbb{N}^*$.

(b) we have the following estimation:

$$\begin{pmatrix} d(A^n(u_0, v_0), u^*) \\ d(A^n(v_0, u_0), v^*) \end{pmatrix} \leq S^n(I - S)^{-1} \begin{pmatrix} d(u_0, A(u_0, v_0)) \\ d(v_0, A(v_0, u_0)) \end{pmatrix}$$

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THE (P, Q) GENERALIZED ANTI-REFLEXIVE EXTREMAL RANK SOLUTIONS TO A SYSTEM OF MATRIX EQUATIONS

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Abstract. Let $n \times n$ complex matrices P and Q be nontrivial generalized reflection matrices, i.e., $P^* = P = P^{-1} \neq I_n$, $Q^* = Q = Q^{-1} \neq I_n$. A complex matrix A with order n is said to be a (P, Q) generalized anti-reflexive matrix, if $PAQ = -A$. We in this paper mainly investigate the (P, Q) generalized anti-reflexive maximal and minimal rank solutions to the system of matrix equation $AX = B$. We present necessary and sufficient conditions for the existence of the maximal and minimal rank solutions, with (P, Q) generalized anti-reflexive, of the system. Expressions of such solutions to this system are also given when the solvability conditions are satisfied. In addition, in correspondence with the minimal rank solution set to the system, the explicit expression of the nearest matrix to a given matrix in the Frobenius norm has been provided.

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1. INTRODUCTION

Throughout this paper, let $C_r^{n \times m}$ be the set of all $n \times m$ complex matrices with rank r , $UC^{n \times n}$ be the set of all $n \times n$ unitary matrices. Denote by I_n the identity matrix with order n . Let $J_n = (e_n e_{n-1} \dots e_1)$, where e_i is the i^{th} column of I_n . For a matrix A , A^* , A^+ , $\|A\|$ and $r(A)$ represent its conjugate transpose, Moore-Penrose inverse, Frobenius norm and rank, respectively.

Definition 1. Let $P, Q \in C^{n \times n}$ be nontrivial generalized reflection matrices, i.e., $P^* = P = P^{-1} \neq I_n$, $Q^* = Q = Q^{-1} \neq I_n$, then matrix $A \in C^{n \times n}$ is said to be the (P, Q) generalized reflexive (or anti-reflexive) matrix, if $PAQ = A$ (or $PAQ = -A$).

Obviously, if let $P = Q = J_n$ in Definition 1, then matrix A is the well-known centrosymmetric (or anti-centrosymmetric) matrix, which plays an important role in many areas (see, e.g., [6, 13, 16–19]), and has been widely and extensively studied (see, e.g., [1, 25, 28]). Moreover, let $P = Q$, then matrix A is called generalized centrosymmetric (or anti-centrosymmetric) matrix [12, 22]. As being the extensions

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of the centrosymmetric matrices and generalized centrosymmetric matrices, the generalized reflection matrices and generalized reflexive matrices have many special properties and practical applications, and have also been frequently investigated, see for instance, [3, 7, 8, 26, 27].

In matrix theory and applications, many problems are closely related to the ranks of some matrix expressions with variable entries, so it is necessary to explicitly characterize the possible ranks of the matrix expressions concerned. The study on the possible ranks of matrix equations can be traced back to the late 1970s (see, e.g. [2, 9–11, 23]). Recently, the extremal ranks, i.e. maximal and minimal ranks, of some matrix expressions have found many applications in control theory [4, 5], statistics, and economics (see, e.g. [14, 15, 21]).

In this paper, we consider the (P, Q) generalized anti-reflexive extremal rank solutions of the matrix equation

$$AX = B, \quad (1.1)$$

where A and B are given matrices in $C^{n \times m}$. In 1987, Uhlig [23] gave the maximal and minimal ranks of solutions to system (1.1). By applying the matrix rank method, recently, Tian [20] obtained the minimal rank of solutions to the matrix equation $A = BX + YC$. Xiao et al. [24] in 2009 considered the symmetric minimal rank solution to system (1.1). The (P, Q) generalized reflexive and anti-reflexive matrices with respect to the generalized reflection matrix dual (P, Q) are two classes of important matrices and have engineering and scientific applications. The (P, Q) generalized anti-reflexive maximal and minimal rank solutions of the matrix equation (1.1), however, has not been considered yet. In this paper, we will discuss this problem.

We also consider the matrix nearness problem

$$\min_{X \in S_m} \|X - \tilde{X}\|_F, \quad (1.2)$$

where $\tilde{X}$ is a given matrix in $C^{n \times m}$ and S_m is the minimal rank solution set of Eq. (1.1).

The matrix nearness problem (1.2) is the so-called optimal approximation problem, which has important application in practice, and has been discussed far and wide (see, e.g., [7, 8, 12, 25, 27] and the references therein).

We organize this paper as follows. In Section 2, we first establish a representation for the generalized reflection matrix dual (P, Q) . Then we give necessary and sufficient conditions for the existence of (P, Q) generalized anti-reflexive solution to (1.1). We also give the expressions of such solutions when the solvability conditions are satisfied. In Section 3 we establish formulas of maximal and minimal ranks of (P, Q) generalized anti-reflexive solutions to (1.1), and present the (P, Q) generalized anti-reflexive extremal rank solutions to (1.1). In Section 4 we present the expression of the optimal approximation solution to the set of the minimal rank solution.

2. (P, Q) GENERALIZED ANTI-REFLEXIVE SOLUTION TO (1.1)

In this section we first introduce some structure properties of the generalized reflection matrix dual (P, Q) and establish the representations of (P, Q) generalized anti-reflexive matrix. Then we give the necessary and sufficient conditions for the existence of and the expressions for the (P, Q) generalized anti-reflexive solution of Eq. (1.1).

Lemma 1 ([12]). *Given generalized reflection matrices $P, Q \in C^{n \times n}$. Let*

$$P_1 = \frac{I_n + P}{2} \in C_r^{n \times n}, \quad Q_1 = \frac{I_n + Q}{2} \in C_s^{n \times n}, \quad P_2 = \frac{I_n - P}{2}, \quad Q_2 = \frac{I_n - Q}{2}.$$

Then $r(P_2) = n - r$, $r(Q_2) = n - s$, and there exist column orthogonal matrices $U_{11} \in C^{n \times r}$, $U_{22} \in C^{n \times (n-r)}$, $V_{11} \in C^{n \times s}$, $V_{22} \in C^{n \times (n-s)}$, such that

$$P_1 = U_{11}U_{11}^*, \quad P_2 = U_{22}U_{22}^*, \quad Q_1 = V_{11}V_{11}^*, \quad Q_2 = V_{22}V_{22}^*. \quad (2.1)$$

Remark 1. Denote $U = [U_{11}, U_{22}]$, $V = [V_{11}, V_{22}]$, it follows from Lemma 1 that $U, V \in UC^{n \times n}$, and

$$P = U \begin{bmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{bmatrix} U^*, \quad Q = V \begin{bmatrix} I_s & 0 \\ 0 & -I_{n-s} \end{bmatrix} V^*, \quad (2.2)$$

where the symbols "0" stand for null matrices with associated orders (in the sequel, we always mark them like this).

Lemma 2 ([8]). *Let $A \in C^{n \times n}$ and generalized reflection matrices P, Q with the forms of (2.2), then A is the (P, Q) generalized anti-reflexive matrix if and only if*

$$A = U \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix} V^*, \quad (2.3)$$

where $M \in C^{r \times (n-s)}$, $N \in C^{(n-r) \times s}$ are arbitrary.

Given matrices $A_1 \in C^{m \times n}$, $B_1 \in C^{m \times p}$, by making generalized singular value decomposition to $[A_1, B_1]$, we have

$$A_1 = M_1 \Sigma_{A_1} U_1, \quad B_1 = M_1 \Sigma_{B_1} V_1 \quad (2.4)$$

where M_1 is a $m \times m$ nonsingular matrix, $U_1 \in UC^{n \times n}$, $V_1 \in UC^{p \times p}$,

$$\Sigma_{A_1} = \begin{bmatrix} I & 0 & 0 \\ 0 & S_{A_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_1 - s_1 \\ s_1 \\ k_1 - r_1 \\ m - k_1 \end{matrix}, \quad \Sigma_{B_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{B_1} & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_1 - s_1 \\ s_1 \\ k_1 - r_1 \\ m - k_1 \end{matrix},$$

$k_1 = r[A_1, B_1]$, $s_1 = r(A_1) + r(B_1) - r[A_1, B_1]$, $S_{A_1} = \text{diag}(\alpha_1, \dots, \alpha_{s_1})$, $S_{B_1} = \text{diag}(\beta_1, \dots, \beta_{s_1})$, $r_1 = r(A_1)$, $0 < \alpha_{s_1} \leq \dots \leq \alpha_1 < 1$, $0 < \beta_1 \leq \dots \leq \beta_{s_1} < 1$, $\alpha_i^2 + \beta_i^2 = 1$, $i = 1, \dots, s_1$.

Lemma 3. Given matrices $A_1 \in C^{m \times n}$, $B_1 \in C^{m \times p}$, the generalized singular value decomposition of the matrix pair $[A_1, B_1]$ is given by (2.4), then matrix equation $A_1 X = B_1$ is consistent, if and only if

$$r[A_1, B_1] = r(A_1), \quad (2.5)$$

and the expression of its general solution is

$$X = U_1^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_1}^{-1} S_{B_1} \\ Y_{31} & Y_{32} \end{bmatrix} V_1, \quad (2.6)$$

where $Y_{31} \in C^{(n-r_1) \times (p-s_1)}$, $Y_{32} \in C^{(n-r_1) \times s_1}$ are arbitrary.

Proof. With (2.4)(2.4) we have

$$r(B_1 - A_1 X) = r(M_1 \Sigma_{B_1} V_1 - M_1 \Sigma_{A_1} U_1 X) = r(\Sigma_{B_1} - \Sigma_{A_1} U_1 X V_1^*).$$

Let $Y = U_1 X V_1^*$ and Partition Y with $Y = (Y_{ij})_{3 \times 3}$, then

$$\Sigma_{B_1} - \Sigma_{A_1} Y = \begin{bmatrix} -Y_{11} & -Y_{12} & -Y_{13} \\ -S_{A_1} Y_{21} & S_{B_1} - S_{A_1} Y_{22} & -S_{A_1} Y_{23} \\ 0 & 0 & I_{B_1} \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_1 - s_1 \\ s_1 \\ k_1 - r_1 \\ m - k_1 \end{matrix}. \quad (2.7)$$

Noting that Y_{ij} ($i = 1, 2, j = 1, 2, 3$) are arbitrary, then

$$\min r(B_1 - A_1 X) = \min r(\Sigma_{B_1} - \Sigma_{A_1} Y) = k_1 - r_1 = r(A_1, B_1) - r(A_1).$$

$A_1 X = B_1$ is solvable in $C^{n \times p}$ if and only if $\min r(B_1 - A_1 X) = 0$. Then matrix equation $A_1 X = B_1$ is consistent, if and only if (2.5) holds. In this case, from (2) and $Y = U_1 X V_1^*$, its general solution can be expressed as (2.6). The proof is completed. $\square$

Assume the given generalized reflection matrices P, Q with the forms of (2.2). Let

$$AU = [A_2, A_3], \quad BV = [B_2, B_3], \quad (2.8)$$

where $A_2 \in C^{m \times r}$, $A_3 \in C^{m \times (n-r)}$, $B_2 \in C^{m \times s}$, $B_3 \in C^{m \times (n-s)}$, and the generalized singular value decomposition of the matrix pair $[A_2, B_3]$, $[A_3, B_2]$ are, respectively,

$$A_2 = M_2 \Sigma_{A_2} U_2, \quad B_3 = M_2 \Sigma_{B_3} V_2, \quad (2.9)$$

$$A_3 = M_3 \Sigma_{A_3} U_3, \quad B_2 = M_3 \Sigma_{B_2} V_3, \quad (2.10)$$

where $U_2 \in UC^{r \times r}$, $V_2 \in UC^{(n-s) \times (n-s)}$, $U_3 \in UC^{(n-r) \times (n-r)}$, $V_3 \in UC^{s \times s}$, nonsingular matrices $M_2, M_3 \in C^{m \times m}$, $k_2 = r[A_2, B_3]$, $r_2 = r(A_2)$, $s_2 = r(A_2) + r(B_3) - r[A_2, B_3]$, and $k_3 = r[A_3, B_2]$, $r_3 = r(A_3)$, $s_3 = r(A_3) + r(B_2) - r[A_3, B_2]$,

$$\Sigma_{A_2} = \begin{bmatrix} I & 0 & 0 \\ 0 & S_{A_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_2 - s_2 \\ s_2 \\ k_2 - r_2 \\ m - k_2 \end{matrix}, \quad \Sigma_{B_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{B_3} & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_2 - s_2 \\ s_2 \\ k_2 - r_2 \\ m - k_2 \end{matrix},$$

$$\Sigma_{A_3} = \begin{bmatrix} I & 0 & 0 \\ 0 & S_{A_3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_3 - s_3 \\ s_3 \\ k_3 - r_3 \\ m - k_3 \end{matrix}, \quad \Sigma_{B_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{B_2} & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_3 - s_3 \\ s_3 \\ k_3 - r_3 \\ m - k_3 \end{matrix},$$

Then we can establish the existence theorems as follows.

Theorem 1. Let $A, B \in C^{m \times n}$ and generalized reflection matrices P, Q of size n be known. Suppose generalized reflection matrices P, Q with the forms of (2.2), AU, BU have the partition forms of (2.8), and the generalized singular value decompositions of the matrix pair $[A_2, B_3]$ and $[A_3, B_2]$ are given by (2.9) and (2.10). Then the equation (1.1) has a (P, Q) generalized anti-reflexive solution X if and only if

$$r[A_2, B_3] = r(A_2), \quad r[A_3, B_2] = r(A_3), \quad (2.11)$$

and its general solution can be expressed as

$$X = U \begin{bmatrix} 0 & U_2^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ Z_{31} & \bar{Z}_{32} \end{bmatrix} V_2 \\ U_3^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ W_{31} & \bar{W}_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} V^*, \quad (2.12)$$

where $Z_{31} \in C^{(r-r_2) \times (n-s-s_2)}$, $Z_{32} \in C^{(r-r_2) \times s_2}$, $W_{31} \in C^{(n-r-r_3) \times (s-s_3)}$, $W_{32} \in C^{(n-r-r_3) \times s_3}$ are arbitrary.

Proof. Suppose the matrix equation (1.1) has a solution X that is (P, Q) generalized anti-reflexive, then it follows from Lemma 2 that there exist $M \in C^{r \times (n-s)}$, $N \in C^{(n-r) \times s}$ satisfying

$$X = U \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix} V^* \quad \text{and} \quad AX = B \quad (2.13)$$

By (2.8), that is

$$[A_2 \ A_3] \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix} = [B_2 \ B_3], \quad (2.14)$$

i.e.

$$A_2 M = B_3, \quad A_3 N = B_2. \quad (2.15)$$

Therefore by Lemma 3, (2.11) holds, and

$$M = U_2^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ Z_{31} & \bar{Z}_{32} \end{bmatrix} V_2, \quad N = U_3^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ W_{31} & \bar{W}_{32} \end{bmatrix} V_3, \quad (2.16)$$

where $Z_{31} \in C^{(r-r_2) \times (n-s-s_2)}$, $Z_{32} \in C^{(r-r_2) \times s_2}$, $W_{31} \in C^{(n-r-r_3) \times (s-s_3)}$, $W_{32} \in C^{(n-r-r_3) \times s_3}$ are arbitrary. Substituting (2.16) into (2.13) yields that the (P, Q) generalized anti-reflexive solution X of the matrix equation (1.1) can be represented by (2.12). The proof is completed. $\square$

3. (P, Q) GENERALIZED ANTI-REFLEXIVE EXTREMAL RANK SOLUTIONS TO (1.1)

In this section, we first derive the formulas of the maximal and minimal ranks of (P, Q) generalized anti-reflexive solutions of (1.1), then present the expressions of (P, Q) generalized anti-reflexive maximal and minimal rank solutions to (1.1).

Theorem 2. Suppose that the matrix equation (1.1) has a (P, Q) generalized anti-reflexive solution X and Ω is the set of all (P, Q) generalized anti-reflexive solutions of (1.1). Then the extreme ranks of X are as follows:

(1) The maximal rank of X is

$$\min\{n-s, r-r(A_2)+r(B_3)\} + \min\{s, n-r-r(A_3)+r(B_2)\}. \quad (3.1)$$

The general expression of X satisfying (3.1) is

$$X = U \begin{bmatrix} 0 & U_2^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} V_2 \\ U_3^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ W_{31} & W_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} V^*, \quad (3.2)$$

where $Z_{31} \in C^{(r-r_2) \times (n-s-s_2)}$, $W_{31} \in C^{(n-r-r_3) \times (s-s_3)}$ are chosen such that $r(Z_{31}) = \min(r-r_2, n-s-s_2)$, $r(W_{31}) = \min(n-r-r_3, s-s_3)$, $Z_{32} \in C^{(r-r_2) \times s_2}$, $W_{32} \in C^{(n-r-r_3) \times s_3}$ are arbitrary.

(2) The minimal rank of X is

$$\min_{X \in \Omega} r(X) = r(B_2) + r(B_3). \quad (3.3)$$

The general expression of X satisfying (3.3) is

$$X = U \begin{bmatrix} 0 & U_2^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ 0 & Z_{32} \end{bmatrix} V_2 \\ U_3^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ 0 & W_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} V^*, \quad (3.4)$$

where $Z_{32} \in C^{(r-r_2) \times s_2}$, $W_{32} \in C^{(n-r-r_3) \times s_3}$ are arbitrary.

Proof. (1) By (2.12),

$$\max_{X \in \Omega} r(X) = \max_{Z_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ Z_{31} & \hat{Z}_{32} \end{bmatrix} + \max_{W_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ W_{31} & \hat{W}_{32} \end{bmatrix}, \quad (3.5)$$

$$\max_{Z_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ Z_{31} & \hat{Z}_{32} \end{bmatrix} = s_2 + \min\{r - r_2, n - s - s_2\} \quad (3.6)$$

$$= \min\{n - s, r - r_2 + s_2\} = \min\{n - s, r - r(A_2) + r(B_3)\},$$

and

$$\max_{W_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ W_{31} & \hat{W}_{32} \end{bmatrix} = s_3 + \min\{n - r - r_3, s - s_3\} \quad (3.7)$$

$$= \min\{s, n - r - r_3 + s_3\} = \min\{s, n - r - r(A_3) + r(B_2)\}.$$

Taking (3.6) and (3.7) into (3.5) yields (3.1).

According to the general expression of the solution in Theorem 1, it is easy to verify the rest of part in (1).

(2) By (2.12),

$$\min_{X \in \Omega} r(X) = \min_{Z_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ Z_{31} & \hat{Z}_{32} \end{bmatrix} + \min_{W_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ W_{31} & \hat{W}_{32} \end{bmatrix}, \quad (3.8)$$

$$\min_{Z_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ Z_{31} & \hat{Z}_{32} \end{bmatrix} = s_2 = r(B_3) \quad (3.9)$$

and

$$\min_{W_{31}} r \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ W_{31} & \hat{W}_{32} \end{bmatrix} = s_3 = r(B_2). \quad (3.10)$$

Taking (3.9) and (3.10) into (3.8) yields (3.3).

According to the general expression of the solution in Theorem 1, it is easy to verify the rest of part in (2). The proof is completed. $\square$

4. THE EXPRESSION OF THE OPTIMAL APPROXIMATION SOLUTION TO THE SET OF THE MINIMAL RANK SOLUTION

Let $\Omega = \{X : AX = B, PXQ = -X\}$ be the solution set and $S_m = \{X \in \Omega : r(X) = \min\{r(X') : X' \in \Omega\}\}$ the set of minimal rank solutions. From (3.4), When the solution set S_m is nonempty, it is easy to verify that S_m is a closed convex set, therefore there exists a unique solution $\hat{X}$ to the matrix nearness Problem (1.2).

Theorem 3. Given matrix $\tilde{X}$, and the other given notations and conditions are the same as in Theorem 1. Let

$$U^* \tilde{X} V = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{21} & \tilde{X}_{22} \end{bmatrix}, \quad \tilde{X}_{12} \in C^{r \times (n-s)}, \quad \tilde{X}_{21} \in C^{(n-r) \times s}, \quad (4.1)$$

and we denote

$$U_2 \tilde{X}_{12} V_2^* = \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ \tilde{Z}_{21} & \tilde{Z}_{22} \\ \tilde{Z}_{31} & \tilde{Z}_{32} \end{bmatrix}, \quad U_3 \tilde{X}_{21} V_3^* = \begin{bmatrix} \tilde{W}_{11} & \tilde{W}_{12} \\ \tilde{W}_{21} & \tilde{W}_{22} \\ \tilde{W}_{31} & \tilde{W}_{32} \end{bmatrix}. \quad (4.2)$$

If S_m is nonempty, then Problem (1.2) has a unique $\hat{X}$ which can be represented as

$$\hat{X} = U \begin{bmatrix} 0 & U_2^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ 0 & \tilde{Z}_{32} \end{bmatrix} V_2 \\ U_3^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ 0 & \tilde{W}_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} V^*, \quad (4.3)$$

where $\tilde{Z}_{32}, \tilde{W}_{32}$ are the same as in (4.2).

Proof. When S_m is nonempty, it is easy to verify from (3.4) that S_m is a closed convex set. Since $C^{n \times n}$ is a uniformly convex Banach space under Frobenius norm, there exists a unique solution for Problem (1.2). By Theorem 2, for any $X \in S_m$, X can be expressed as

$$X = U \begin{bmatrix} 0 & U_2^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ 0 & \tilde{Z}_{32} \end{bmatrix} V_2 \\ U_3^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ 0 & \tilde{W}_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} V^*, \quad (4.4)$$

where $\tilde{Z}_{32} \in C^{(r-r_2) \times s_2}$, $\tilde{W}_{32} \in C^{(n-r-r_3) \times s_3}$ are arbitrary.

Using the invariance of the Frobenius norm under unitary transformations, we have

$$\begin{aligned} \|X - \tilde{X}\|^2 &= \left\| \begin{bmatrix} 0 & U_2^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ 0 & \tilde{Z}_{32} \end{bmatrix} V_2 \\ U_3^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ 0 & \tilde{W}_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} - U^* \tilde{X} V \right\|^2 \\ &= \|Z_{32} - \tilde{Z}_{32}\|^2 + \|W_{32} - \tilde{W}_{32}\|^2 + \|S_{A_2}^{-1} S_{B_3} - \tilde{Z}_{22}\|^2 \\ &\quad + \|S_{A_3}^{-1} S_{B_2} - \tilde{W}_{22}\|^2 + \|\tilde{X}_{11}\|^2 + \|\tilde{X}_{22}\|^2 + \|\tilde{Z}_{11}\|^2 + \|\tilde{Z}_{12}\|^2 \\ &\quad + \|\tilde{Z}_{21}\|^2 + \|\tilde{Z}_{31}\|^2 + \|\tilde{W}_{11}\|^2 + \|\tilde{W}_{12}\|^2 + \|\tilde{W}_{21}\|^2 + \|\tilde{W}_{31}\|^2. \end{aligned}$$

Therefore, $\|X - \tilde{X}\|$ reaches its minimum if and only if

$$Z_{32} = \tilde{Z}_{32}, \quad W_{32} = \tilde{W}_{32}. \quad (4.5)$$

Substituting (4.5) into (4.4) yields (4.3). The proof is completed. $\square$

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A NOTE ON DERIVATIONS IN MV-ALGEBRAS

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Abstract. The aims of this paper are introduce the notions of symmetric bi-derivation and generalized derivation in MV -algebras and investigate some of their properties.

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Keywords: (linearly ordered) MV -algebra, symmetric bi-derivation, trace of symmetric bi-derivation, (additive, isotone) symmetric bi-derivation, generalized derivation

1. INTRODUCTION

In [5], C. C. Chang invented the notion of MV -algebra in order to provide an algebraic proof of the completeness theorem of infinite valued Lukasiewicz propositional calculus. Recently, algebraic theory of MV -algebras is intensively studied, see [3, 6, 7, 9].

Let R be a ring. An additive mapping $d : R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. During the past few decades there has been on ongoing interest concerning the relation ship between the commutativity of a ring and the existance of certain specific types of derivations of R .

The concept of a symmetric bi-derivation has been introduced by Maksa [8]. Let R be a ring. A mapping $B : R \times R \rightarrow R$ is said to be symmetric if $B(x, y) = B(y, x)$ holds for all pairs $x, y \in R$. A mapping $f : R \rightarrow R$ defined by $f(x) = B(x, x)$, where $B : R \times R \rightarrow R$ is a symmetric mapping, is called the trace of B . A symmetric bi-additive (i. e. additive in both arguments) mapping $D : R \times R \rightarrow R$ is called a symmetric bi-derivation if $D(xy, z) = D(x, z)y + xD(y, z)$ is fullfilled for all $x, y, z \in R$. In recent years, many mathematicians studied the commutativity of prime and semi-prime rings admitting suitably-constrained symmetric bi-derivations. In [4], Y. Çeven applied the notion of symmetric bi-derivation in ring and near ring theory to lattices. In this paper, we introduce the notion of symmetric bi-derivation in MV -algebras and investigate some of its properties.

M. Bresar [2] defined the following notation. An additive mapping $f : R \rightarrow R$ is called a generalized derivation if there exists a derivation $d : R \rightarrow R$ such that

$f(xy) = f(x)y + xd(y)$ for all $x, y \in R$. One may observe that the concept of derivations, also of the left multipliers when $d = 0$.

In [1], N. O. Alshehri introduced the concept of derivation in MV -algebras and discussed some related properties. In this paper, we introduce the notions of symmetric bi-derivation and generalized derivation in MV -algebras and investigate some of their properties.

2. PRELIMINARIES

Definition 1 ([9]). An MV -algebra is a structure $(M, +, *, 0)$ where $+$ is a binary operation, $*$ is a unary operation and 0 is a constant such that the following axioms are satisfied for any $a, b \in M$,

(MV1) $(M, +, 0)$ is a commutative monoid,

(MV2) $(a^*)^* = a$,

(MV3) $0^* + a = 0^*$,

(MV4) $(a^* + b)^* + b = (b^* + a)^* + a$.

If we define the constant $1 = 0^*$ and the auxiliary $\odot, \vee, \wedge$ by $a \odot b = (a^* + b^*)^*$, $a \vee b = a + (b \odot a^*)$, $a \wedge b = a \odot (b \oplus a^*)$ then $(M, \odot, 1)$ is a commutative monoid and the structure $(M, \vee, \wedge, 0, 1)$ is a bounded distributive lattice. Also, we define the binary operation $\ominus$ by $x \ominus y = x \odot y^*$. A subset of X an MV -algebra M is called subalgebra of M if and only if X is closed under the MV -operations defined in M . In any MV -algebras one can define a partial order $\leq$ by putting $x \leq y$ if and only if $x \wedge y = x$ for each $x, y \in M$. If the order relation $\leq$, defined over M , is total then we say that M is linearly ordered. For an MV -algebra M , if we define $B(M) = \{x \in M : x + x = x\} = \{x \in M : x \odot x = x\}$. Then $(B(M), +, *, 0)$ is both largest subalgebra of M and a Boolean algebra.

An MV -algebra M has the following properties for all $x, y, z \in M$,

- (1) $x + 1 = 1$,
- (2) $x + x^* = 1$,
- (3) $x + x^* = 0$,
- (4) If $x + y = 0$, then $x = y = 0$,
- (5) If $x \odot y = 1$, then $x = y = 1$,
- (6) If $x \leq y$, then $x \vee z \leq y \vee z$ and $x \wedge z \leq y \wedge z$,
- (7) If $x \leq y$, then $x + z \leq y + z$ and $x \odot z \leq y \odot z$,
- (8) $x \leq y$ if and only if $y^* \leq x^*$,
- (9) $x + y = y$ if and only if $x \odot y = x$.

Theorem 1 ([5]). The following conditions are equivalent for all $x, y \in M$,

- (i) $x \leq y$,
- (ii) $y + x^* = 1$,
- (iii) $x \odot y^* = 0$.

Definition 2 ([5]). Let M be an MV -algebra and I be a nonempty subset of M . Then we say that I is an ideal if the following conditions are satisfied,

- (i) $0 \in I$,
- (ii) $x, y \in I$ imply $x \oplus y \in I$,
- (iii) $x \in I$ and $y \leq x$ imply $y \in I$.

Proposition 1 ([5]). Let M be a linearly ordered MV -algebra, then $x + y = x + z$ and $x + z \neq 1$ imply that $y = z$.

Definition 3 ([1]). Let M be an MV -algebra and $d : M \rightarrow M$ be a function. We called d a derivation of M , if it satisfies the following condition for all $x, y \in M$,

$$d(x \odot y) = (dx \odot y) + (x \odot dy)$$

Definition 4. Let M be an MV -algebra. A mapping $D : M \times M \rightarrow M$ is called symmetric if $D(x, y) = D(y, x)$ holds for all $x, y \in M$.

Definition 5. Let M be an MV -algebra. A mapping $d : M \rightarrow M$ defined by $d(x) = D(x, x)$ is called trace of D , where $D : M \times M \rightarrow M$ is a symmetric mapping.

We often abbreviate $d(x)$ to dx .

3. SYMMETRIC BI-DERIVATION OF MV-ALGEBRAS

Definition 6. Let M be an MV -algebra and $D : M \times M \rightarrow M$ be a symmetric mapping. We call D a symmetric bi-derivation on M , if it satisfies the following condition,

$$D(x \odot y, z) = (D(x, z) \odot y) + (x \odot D(y, z))$$

for all $x, y, z \in M$.

Obviously, a symmetric bi-derivation D on M satisfies the relation $D(x, y \odot z) = (D(x, y) \odot z) + (y \odot D(x, z))$ for all $x, y, z \in M$.

Example 1. Let $M = \{0, a, b, 1\}$. Consider the following tables:

+	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

*	0	a	b	1
0	1	b	a	0

Then $(M, +, *, 0)$ is an MV -algebra. Define a map $D : M \times M \rightarrow M$ by

$$D(x, y) = \begin{cases} b, & (x, y) = (b, b), (b, 1), (1, b) \\ 0, & \text{otherwise} \end{cases}$$

Then we can see that D is a symmetric bi-derivation of M .

Proposition 2. *Let M be an MV -algebra, D be a symmetric bi-derivation on M and d be a trace of D . Then, for all $x \in M$,*

- (i) $d0 = 0$,
- (ii) $dx \odot x^* = x \odot dx^* = 0$,
- (iii) $dx = dx + (x \odot D(x, 1))$,
- (iv) $dx \leq x$,
- (v) If I is an ideal of an MV -algebra, $d(I) \subseteq I$.

Proof. (i) For all $x \in M$,

$$\begin{aligned} D(x, 0) &= D(x, 0 \odot 0) = (D(x, 0) \odot 0) + (0 \odot D(x, 0)) \\ &= 0 + 0 = 0. \end{aligned}$$

Since d is the trace of D ,

$$\begin{aligned} d0 &= D(0, 0) = D(0 \odot 0, 0) = (D(0, 0) \odot 0) + (0 \odot D(0, 0)) \\ &= 0 + 0 = 0. \end{aligned}$$

(ii) For all $x \in M$,

$$\begin{aligned} 0 &= D(x, 0) = D(x, x \odot x^*) \\ &= (D(x, x) \odot x^*) + (x \odot D(x, x^*)) \end{aligned}$$

and so, $dx \odot x^* = 0$ and $x \odot D(x, x^*) = 0$.

Similarly, $x \odot dx^* = 0$ for all $x \in M$.

(iii) For all $x \in M$,

$$\begin{aligned} dx &= D(x, x) = D(x, x \odot 1) = (D(x, x) \odot 1) + (x \odot D(x, 1)) \\ &= dx + (x \odot D(x, 1)) \end{aligned}$$

(iv) For all $x \in M$,

$$\begin{aligned} 1 &= 0^* = (dx \odot x^*)^* = \left[\left((dx)^* + (x^*)^* \right)^* \right]^* \\ &= (dx)^* + x \end{aligned}$$

From Theorem 1 (ii), $dx \leq x$ for all $x \in M$.

(v) Let $y \in d(I)$, then $d(x) = y$ for some $x \in I$. From (iv), $d(x) \leq x$ and so $y \in I$, since I is an ideal of M . Hence $d(I) \subseteq I$. $\square$

Corollary 1. For all $x \in M$, since $x \odot D(x, x^*) = 0$, we get $D(x, x^*) \leq x^*$ and $x \leq (D(x, x^*))^*$.

For all $x, y \in M$, since

$$0 = D(x \odot x^*, y) = (D(x, y) \odot x^*) + (x \odot D(x^*, y))$$

we get, $D(x, y) \leq x$ and $D(x^*, y) \leq x^*$.

Similarly, $D(x, y) \leq y$ and $D(x, y^*) \leq y^*$ for all $x, y \in M$.

Proposition 3. Let M be an MV -algebra, D be a symmetric bi-derivation on M and d be a trace of D . If $x \leq y$ for $x, y \in M$, then the followings hold:

- (i) $d(x \odot y^*) = 0$,
- (ii) $dy^* \leq x^*$,
- (iii) $dx \odot dy^* = 0$.

Proof. (i) Let $x \leq y$, for $x, y \in M$. From (7), since $x \odot y^* \leq y \odot y^* = 0$, we get $x \odot y^* = 0$. Since $d0 = 0$, we have $d(x \odot y^*) = 0$.

(ii) Let $x \leq y$, for $x, y \in M$. Since $x \odot dy^* \leq y \odot dy^* \leq y \odot y^* = 0$, we get $x \odot dy^* = 0$ and so $dy^* \leq x^*$.

(iii) Since $x \leq y$, we get $dx \leq y$ and so $dx \odot dy^* \leq y \odot dy^* \leq y \odot y^* = 0$. Hence $dx \odot dy^* = 0$. $\square$

Proposition 4. Let M be an MV -algebra, D be a symmetric bi-derivation on M and d be a trace of D . The the followings hold:

- (i) $dx \odot dx^* = 0$,
- (ii) $dx^* = (dx)^*$ if and only if d is the identity on M .

Proof. (i) Since $dx \odot dy^* = 0$, replacing y by x , we get $dx \odot dx^* = 0$.

(ii) Since $x \odot dy^* = 0$ for $x, y \in M$, we get $x \odot dx^* = x \odot (dx)^* = 0$. Since $x \leq dx$ and $dx \leq x$, we have $x = dx$. Hence d is the identity on M .

If d is the identity on M , $dx^* = (dx)^*$ for all $x \in M$. $\square$

Definition 7. Let M is an MV -algebra, D be a symmetric bi-derivation on M . If $x \leq y$ implies $D(x, z) \leq D(y, z)$ for all $x, y, z \in M$, D is called an isotone.

If d is the trace of D and D is an isotone, $x \leq y$ implies $d(x) \leq d(y)$ for all $x, y \in M$.

Example 2. Let M be an MV -algebra as in Example 1. Define a map $D : M \times M \rightarrow M$ by

$$D(x, y) = \begin{cases} 0, & (x, y) \in \{(0, 0), (a, 0), (0, a), (b, 0), (0, b), (1, 0), (0, 1), (a, b), (b, a)\} \\ b, & (x, y) \in \{(b, b), (b, 1), (1, b)\} \\ a, & (x, y) \in \{(a, a), (1, a), (a, 1)\} \\ 1, & (x, y) \in \{(1, 1)\} \end{cases}$$

Then we can see that D is an isotone symmetric bi-derivation on M . Since $d0 = 0$, $d1 = 1$, $da = a$ and $db = b$, d is the identity on M and so $x \leq y$ implies $d(x) \leq$

$d(y)$ for all $x, y \in M$.

In Example 1, $b \leq 1$, $D(b, 1) = b$, $D(1, 1) = 0$, but $0 \leq b$. That is, D is not isotone.

Proposition 5. *Let M be an MV -algebra, D be a symmetric bi-derivation on M and d be a trace of D . If $dx^* = dx$ for all $x \in M$, then the followings hold:*

- (i) $d1 = 0$,
- (ii) $dx \odot dx = 0$,
- (iii) *If D is an isotone on M , then $d = 0$.*

Proof. (i) Replacing x by 0 in $dx^* = dx$, we get $d1 = 0$.
(ii) For all $x \in M$, $dx \odot dx = dx \odot dx^* = 0$.
(iii) Let D is an isotone on M . For $x \in M$, since $dx \leq d1 = 0$, we get $dx = 0$. Thus $d = 0$. $\square$

Definition 8. Let M be an MV -algebra and D be a symmetric mapping on M . If $D(x + y, z) = D(x, z) + D(y, z)$ for all $x, y, z \in M$, D is called bi-additive mapping.

Theorem 2. *Let M be an MV -algebra, D be a bi-additive symmetric bi-derivation on M and d be a trace of D . Then $d(B(M)) \subseteq B(M)$.*

Proof. Let $y \in d(B(M))$. Thus $y = d(x)$ for some $x \in B(M)$. Then

$$\begin{aligned} y + y &= dx + dx = D(x, x) + D(x, x) = D(x + x, x) \\ &= D(x, x) = y. \end{aligned}$$

Hence $y \in B(M)$. That is, $d(B(M)) \subseteq B(M)$. $\square$

Theorem 3. *Let M be a linearly ordered MV -algebra, D be a bi-additive symmetric bi-derivation on M and d be a trace of D . Then $d = 0$ or $d1 = 1$.*

Proof. Since $x + x^* = 1$ and $x + 1 = 1$ for all $x \in M$,

$$d1 = D(1, 1) = D(x + x^*, 1) = D(x, 1) + D(x^*, 1)$$

and

$$\begin{aligned} d1 &= D(1, 1) = D(x + 1, 1) \\ &= D(x, 1) + d1 \end{aligned}$$

If $d1 \neq 1$, Proposition 1, we get $D(x^*, 1) = d1$. Replacing x by 1, we get $d1 = 0$.

For all $x \in M$,

$$0 = d1 = D(x, 1) + d1 = D(x, 1)$$

and

$$D(x, 1) = D(x, x + 1) = dx = D(x, 1) = dx.$$

Thus $dx = 0$ for all $x \in M$. That is, $d = 0$. $\square$

Theorem 4. Let M be a linearly ordered MV -algebra, D_1 and D_2 bi-additive symmetric bi-derivations on M and d_1, d_2 be traces of D_1, D_2 , respectively. If $d_1 d_2 = 0$ where $(d_1 d_2)(x) = d_1(d_2 x)$ for all $x \in M$, then $d_1 = 0$ or $d_2 = 0$.

Proof. Let $d_1 d_2 = 0$ and $d_2 \neq 0$. Thus $d_2 1 = 1$. For all $x \in M$,

$$0 = (d_1 d_2)(x) = d_1(d_2 x) = d_1(d_2 x + (x \odot D_2(x, 1))).$$

Also, since $d_2 1 = 1$, we have

$$\begin{aligned} D_2(x, 1) &= D_2(x \odot 1, 1) = (D_2(x, 1) \odot 1) + (x \odot D_2(1, 1)) \\ &= D_2(x, 1) + x \end{aligned}$$

From (9), we get $x \odot D_2(x, 1) = x$.

Thus,

$$\begin{aligned} 0 &= d_1(d_2 x + x) = D_1(d_2 x + x, d_2 x + x) \\ &= D_1(d_2 x, d_2 x) + D_1(d_2 x, x) + D_1(x, d_2 x) + D_1(x, x) \\ &= D_1(d_2 x, x) + D_1(x, d_2 x) + d_1 x. \end{aligned}$$

From (4), we get $D_1(d_2 x, x) = 0$ or $d_1 x = 0$ for all $x \in M$.

Let $D_1(d_2 x, x) = 0$ for all $x \in M$. Replacing x by 1, we get $D_1(1, 1) = 0$, that is, $d_1 1 = 0$. For all $x \in M$,

$$0 = d_1 1 = D_1(x + 1, 1) = D_1(x, 1) + d_1 1$$

and so, $D_1(x, 1) = 0$. Therefore,

$$0 = D_1(x, 1) = D_1(x, x + 1) = d_1 x + d_1 1 = d_1 x.$$

Thus $d_1 = 0$. □

4. GENERALIZED DERIVATIONS ON MV-ALGEBRAS

Definition 9. Let M be an MV -algebra. A mapping $f : M \rightarrow M$ is called generalized derivation on M if there exists a derivation $d : M \rightarrow M$ such that

$$f(x \odot y) = (f(x) \odot y) + (x \odot d(y)),$$

for all $x, y \in M$.

Example 3. Let M be an MV -algebra in Example 1. Define a function $d : M \rightarrow M$ as the following,

$$d(x) = \begin{cases} 0, & x = 0, a, 1 \\ b, & x = b \end{cases}$$

Example 4. It is obvious that d is derivation on M . If we define f by

$$f(x) = \begin{cases} 0, & x = 0, a \\ b, & x = b, 1 \end{cases}$$

Then f is generalized derivation determined by d on M . Also, f is derivation on M .

Example 5. Let $M = \{0, a, b, c, d, 1\}$. Consider the following tables:

+	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	c	d	c	1	1
b	b	d	b	1	d	1
c	c	c	1	c	1	1
d	d	1	d	1	1	1
1	1	1	1	1	1	1

*	0	a	b	c	d	1
0	1	d	c	b	a	0

Then $(M, +, *, 0)$ is an MV -algebra. Define a function $d : M \rightarrow M$ as the following

$$d(x) = \begin{cases} 0, & x = 0, a, c \\ b, & x = b, d, 1 \end{cases}$$

It is obvious that d is derivation on M . If we define a function f by $f(x) = x$, for all $x \in M$.

Then f is generalized derivation determined by d on M . But, since

$$\begin{aligned} f(ac) &= f(a)c + af(c) \\ &= ac + ac = a + a = c \end{aligned}$$

and $f(ac) = f(c) = a$, f is not derivation on M .

Proposition 6. Let M be an MV -algebra, f be a generalized derivation determined by d on M . Then the followings hold for all $x \in M$,

- (i) $f(0) = 0$,
- (ii) $f(x) \odot x^* = 0$,
- (iii) $f(x) = f(x) + (x \odot d(1))$,
- (iv) $f(x) \leq x$,
- (v) If I is an ideal of an MV -algebra, then $f(I) \subseteq I$.

Proof. (i) $f(0) = f(0 \odot 0) = (f(0) \odot 0) + (0 \odot d(0)) = 0$.

(ii) For all $x \in M$,

$$0 = f(0) = f(x \odot x^*) = (f(x) \odot x^*) + (x \odot d(x^*))$$

and so, $f(x) \odot x^* = 0$.

(iii) For all $x \in M$,

$$\begin{aligned} f(x) &= f(x \odot 1) = (f(x) \odot 1) + (x \odot d(1)) \\ &= f(x) + (x \odot d(1)). \end{aligned}$$

(iv) For all $x \in M$,

$$1 = 0^* = (f(x) \odot x^*)^* = (f(x))^* + x$$

From Theorem 1 (ii), $f(x) \leq x$ for all $x \in M$.

(v) Let $y \in f(I)$, then $d(x) = y$ for some $x \in I$. From (iv), $f(x) \leq x$ and so $y \in I$, since I is an ideal of M . Hence $f(I) \subseteq I$. $\square$

Corollary 2. Let M be an MV-algebra, f be a generalized derivation determined by d on M . If $x \leq y$ for some $x, y \in M$, then the followings hold,

(i) $f(x \odot y^*) = 0$,

(ii) $f(x) \leq y$,

(iii) $f(x) \odot f(y^*) = 0$,

(iv) $f(x^*) = (f(x))^*$ if and only if f is the identity on M .

Definition 10. Let M is an MV-algebra, f be a generalized derivation determined by d on M . If $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in M$, f is called an isotone.

Example 6. In Example 5, since f is an identity function, f is isotone.

Proposition 7. Let M be an MV-algebra, f be a generalized derivation determined by d on M . If $f(x^*) = f(x)$ for all $x \in M$, then the followings hold,

(i) $f(1) = 0$,

(ii) $f(x) \odot f(x) = 0$,

(iii) If f is an isotone on M , then $f = 0$.

Proof. It is clear. $\square$

Definition 11. Let M be an MV-algebra and f be a generalized derivation determined by d on M . If $f(x + y) = f(x) + f(y)$ for all $x, y \in M$, f is called additive generalized derivation on M .

Example 7. In Example 4, f is additive generalized derivation on M .

Theorem 5. Let M be an MV-algebra and f be a nonzero additive derivation on M . Then $f(B(M)) \subseteq B(M)$.

Proof. Let $y \in f(B(M))$. Thus $y = f(x)$ for some $x \in B(M)$. Then

$$y + y = f(x) + f(x) = f(x + x) = f(x) = y$$

Hence $y \in B(M)$. That is, $f(B(M)) \subseteq B(M)$. $\square$

Theorem 6. Let f be an additive generalized derivation on a linearly ordered MV-algebra M . Then either $f = 0$ or $f(1) = 1$.

Proof. Let f be an additive generalized derivation on a linearly ordered MV -algebra M . Hence

$$f(1) = f(x + x^*) = f(x) + f(x^*)$$

and

$$f(1) = f(x + 1) = f(x) + f(1)$$

for all $x \in M$. If $f(1) \neq 1$, from Proposition 1, we get $f(1) = 0$. Therefore

$$0 = f(1) = f(1) + f(x) = f(x)$$

for all $x \in M$. That is, $f = 0$. □

Corollary 3. *Let M be a linearly ordered MV -algebra and f additive generalized derivation determined by d on M . If $f^2 = 0$ where $f^2(x) = f(f(x))$ for all $x \in M$, then $f = 0$.*

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A CLASS OF $2n$ TH-ORDER DIFFERENTIAL OPERATOR WITH EIGENPARAMETER-DEPENDENT BOUNDARY AND TRANSMISSION CONDITIONS

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Abstract. In this paper, we investigate a class of $2n$ th-order regular differential operator with eigenparameter-dependent boundary conditions and transmission conditions at an interior discontinuous point. By constructing a new linear operator A associated with the problem, we prove that the operator A is self-adjoint in a suitable Hilbert space H , and the eigenvalues of the problem coincide with those of A . In terms of basic solutions of differential equation, we show that the eigenvalues of this problem coincide with the zeros of the entire function $\det \Phi(1, \lambda)$, and obtain that the operator A has only point spectrum.

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1. INTRODUCTION

The Sturm-Liouville theory is one of the most actual and extensively developing fields of theoretical and applied mathematics. In recent years, highly important results in this field have been obtained for the case when the eigenparameter appears not only in the differential equation but also in the boundary conditions. The literature on such results is voluminous and we refer to [1, 2, 4, 5, 7, 8, 11, 13, 14]. While the general theory and methods of such second-order boundary value problems are highly developed, very little is known about the general character of the higher order problems.

While dealing with interior discontinuousness, some boundary value problems with transmission conditions arise in heat and mass transfer, vibrating string problems, diffraction problems and various physical transfer problems [3, 6, 9]. Also, some problems with transmission conditions which arise in mechanics [12]. In this

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paper we shall investigate a class of $2n$ th-order boundary value problems with eigenparameter boundary conditions and transmission conditions at an interior discontinuous point. The goal of this study is to extend some results of the standard S-L problems to the higher order discontinuous case.

We consider a $2n$ th-order differential operator with eigenparameter boundary conditions and transmission conditions, and define a new linear operator A associated with the problem in a suitable Hilbert space H , and prove the operator A is self-adjoint, the eigenvalues of the problem coincide with those of A . We construct basic solutions, and show that the eigenvalues of this problem coincide with the zeros of the entire function $\det \Phi(1, \lambda)$, and get that the operator A has only point spectrum.

This paper is organized as follows. In Section 1, we introduce the problem considered, a formulae of the operator A in the suitable Hilbert space H is constructed. In Section 2, we prove the operator A is self-adjoint. Basic solutions are achieved in Section 3, and show that the eigenvalues of this problem coincide with the zeros of the entire function $\det \Phi(1, \lambda)$. In Section 4, we prove that the operator A has only point spectrum.

In this paper, we investigate a discontinuous boundary value problem which consists of the differential equation

$$lu := -(p(x)u^{(n)})^{(n)} + q(x)u = \lambda u, x \in J, \quad (1.1)$$

where $J = [-1, 0) \cup (0, 1]$, $p(x) = p_1^2$ for $x \in [-1, 0)$, $p(x) = p_2^2$ for $x \in (0, 1]$, p_1, p_2 are nonzero real numbers; $q(x) \in L^1(J, R)$, $\lambda \in \mathbb{C}$ is the so-called eigenparameter; with boundary conditions

$$l_i u := a_i u^{(i-1)}(-1) + a_{2n+1-i} u^{(2n-i)}(-1) = 0, i = 1, 2, \dots, n, \quad (1.2)$$

the eigenparameter-dependent boundary conditions

$$\begin{aligned} l_{n+i} u := & \lambda(b'_i u^{(i-1)}(1) + b'_{2n+1-i} u^{(2n-i)}(1)) \\ & + b_i u^{(i-1)}(1) + b_{2n+1-i} u^{(2n-i)}(1) = 0, i = 1, 2, \dots, n, \end{aligned} \quad (1.3)$$

and transmission conditions at the interior discontinuous point

$$l_{2n+i} u := u^{(i-1)}(0+) - \sum_{j=1}^{2n} c_{ij} u^{(j-1)}(0-) = 0, i = 1, 2, \dots, 2n, \quad (1.4)$$

where $a_i, b_i, b'_i (i = 1, 2, \dots, 2n)$ are real numbers, $C = (c_{ij})$ is $2n \times 2n$ real matrix, we assume that

$$a_{n+i} \neq 0 (i = 1, 2, \dots, n), \quad \det C = \rho^n, \rho > 0, \quad C^T Q C = \rho Q, \quad (1.5)$$

$$\theta_i = \begin{vmatrix} b'_i & b'_{2n+1-i} \\ b_i & b_{2n+1-i} \end{vmatrix} > 0, i = 1, 2, \dots, n, \quad (1.6)$$

where Q is $2n \times 2n$ real matrix as follows

$$Q = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & \cdots & & \\ & 1 & & & \\ -1 & & & & \end{pmatrix}.$$

Let

$$C_u(x) = (u(x), u'(x), \dots, u^{(2n-1)}(x))^T. \quad (1.7)$$

In order to investigate the problem (1.1)-(1.4), we define a new inner product in $L^2(J)$ as

$$\langle f, g \rangle_1 = \frac{1}{p_1^2} \int_{-1}^0 f_1(x) \overline{g_1(x)} dx + \frac{1}{p_2^2 \rho} \int_0^1 f_2(x) \overline{g_2(x)} dx, \forall f, g \in L^2(J), \quad (1.8)$$

where $f_1(x) = f(x)|_{[-1,0]}$, $f_2(x) = f(x)|_{(0,1]}$. It is easy to verify that $H_1 = (L^2(J), \langle \cdot, \cdot \rangle_1)$ is a Hilbert space.

2. THE SELF-ADJOINTNESS OF OPERATOR A

In this section, we introduce the special inner product in the Hilbert space $H = H_1 \oplus \mathbb{C}^n$, where $H_1 = (L^2(J), \langle \cdot, \cdot \rangle_1)$, $\mathbb{C}$ denotes the set of complex numbers. A symmetric linear operator A is defined on this Hilbert space such that problem (1.1)-(1.4) can be considered as the eigenvalue problem of this operator. Here, we prove that the operator A is self-adjoint, not only symmetric.

We define an inner product in H by

$$\begin{aligned} \langle F, G \rangle &= \langle f, g \rangle_1 + \frac{1}{\rho} \sum_{i=1}^n (-1)^i \frac{1}{\theta_i} h_i \bar{k}_i, \\ \forall f, g \in H_1, h_i, k_i &\in \mathbb{C}, i = 1, 2, \dots, n, \end{aligned} \quad (2.1)$$

for

$$F = (f, h_1, h_2, \dots, h_n), G = (g, k_1, k_2, \dots, k_n) \in H.$$

In the Hilbert space H , we consider the operator A which is defined by

$$\begin{aligned} D(A) &= \{(f, h_1, h_2, \dots, h_n) \in H \mid f_1^{(i-1)} \in AC_{loc}((-1, 0)), \\ &\quad f_2^{(i-1)} \in AC_{loc}((0, 1)), i = 1, 2, \dots, 2n, lf \in H_1, \\ &\quad l_j f = 0, j = 1, 2, \dots, n, C_f(0+) = C \cdot C_f(0-), \\ &\quad h_i = b'_i f^{(i-1)}(1) + b'_{2n+1-i} f^{(2n-i)}(1), i = 1, 2, \dots, n\}, \\ AF &= (lf, -(b_1 f(1) + b_{2n} f^{(2n-1)}(1)), -(b_2 f'(1) + b_{2n-1} f^{(2n-2)}(1)), \\ &\quad \dots, -(b_n f^{(n-1)}(1) + b_{n+1} f^{(n)}(1))), \end{aligned} \quad (2.2)$$

$$F = (f, b'_1 f(1) + b'_{2n} f^{(2n-1)}(1), b'_2 f'(1) + b'_{2n-1} f^{(2n-2)}(1), \dots, b'_n f^{(n-1)}(1) + b'_{n+1} f^{(n)}(1)) \in D(A).$$

For convenience, $\forall F = (f, h_1, h_2, \dots, h_n) \in D(A)$, let

$$\begin{aligned} M_i(f) &= b_i f^{(i-1)}(1) + b_{2n+1-i} f^{(2n-i)}(1), i = 1, 2, \dots, n, \\ M'_i(f) &= b'_i f^{(i-1)}(1) + b'_{2n+1-i} f^{(2n-i)}(1), i = 1, 2, \dots, n, \end{aligned} \quad (2.3)$$

so by (1.1), (1.3) we have

$$\begin{aligned} AF &= (lf, -M_1(f), -M_2(f), \dots, -M_n(f)) \\ &= (\lambda f, \lambda M'_1(f), \lambda M'_2(f), \dots, \lambda M'_n(f)) = \lambda F. \end{aligned} \quad (2.4)$$

Now we can rewrite the considered problem (1.1)-(1.4) in the operator form $AF = \lambda F$.

From (1.6), (2.3) and (2.4), by direct calculation we obtain the following Lemma.

Lemma 1. *If the functions $f(x)$ and $g(x)$ are differential on the interval $[0, 1]$, then*

$$W(f, \bar{g}; 1) = \sum_{i=1}^n (-1)^i \frac{1}{\theta_i} M_i(f) M'_i(\bar{g}) - \sum_{i=1}^n (-1)^i \frac{1}{\theta_i} M'_i(f) M_i(\bar{g}),$$

where

$$\begin{aligned} W(f, g; x) &= f(x)g^{(2n-1)}(x) - f'(x)g^{(2n-2)}(x) + \dots + (-1)^{n-1} f^{(n-1)}(x)g^{(n)}(x) \\ &\quad + (-1)^n f^{(n)}(x)g^{(n-1)}(x) + \dots + f^{(2n-2)}(x)g'(x) - f^{(2n-1)}(x)g(x) \\ &= \sum_{i=1}^n (-1)^{i+1} (f^{(i-1)}(x)g^{(2n-i)}(x) - f^{(2n-i)}(x)g^{(i-1)}(x)) = C_f^T(x) Q C_g(x). \end{aligned}$$

Proof. By (1.6), (2.3) and (2.4), we have

$$\begin{aligned} &\sum_{i=1}^n (-1)^i \frac{1}{\theta_i} M_i(f) M'_i(\bar{g}) - \sum_{i=1}^n (-1)^i \frac{1}{\theta_i} M'_i(f) M_i(\bar{g}) \\ &= \sum_{i=1}^n (-1)^i \\ &\quad \frac{1}{\theta_i} ((b_i f^{(i-1)}(1) + b_{2n+1-i} f^{(2n-i)}(1))(b'_i \bar{g}^{(i-1)}(1) + b'_{2n+1-i} \bar{g}^{(2n-i)}(1)) \\ &\quad - (b'_i f^{(i-1)}(1) + b'_{2n+1-i} f^{(2n-i)}(1))(b_i \bar{g}^{(i-1)}(1) + b_{2n+1-i} \bar{g}^{(2n-i)}(1))) \\ &= \sum_{i=1}^n (-1)^i \end{aligned}$$

$$\begin{aligned} & \frac{1}{\theta_i} (b_i b'_{2n+1-i} - b'_i b_{2n+1-i}) (f^{(i-1)}(1) \bar{g}^{(2n-i)}(1) - f^{(2n-i)}(1) \bar{g}^{(i-1)}(1)) \\ &= \sum_{i=1}^n (-1)^{i+1} (f^{(i-1)}(1) \bar{g}^{(2n-i)}(1) - f^{(2n-i)}(1) \bar{g}^{(i-1)}(1)) = W(f, \bar{g}; 1). \end{aligned}$$

□

Lemma 2. *The eigenvalues of the problem (1.1)-(1.4) coincide with those of operator A , and its eigenfunctions are the first component of corresponding eigenfunctions of the operator A .*

Lemma 3. *The domain $D(A)$ of the operator A is dense in H .*

Proof. Suppose that there is an element $F = (f, h_1, h_2, \dots, h_n) \in H$, which is orthogonal to all $U = (u, M'_1(u), M'_2(u), \dots, M'_n(u)) \in D(A)$ in Hilbert space H , i.e.

$$\begin{aligned} \langle F, U \rangle &= \frac{1}{p_1^2} \int_{-1}^0 f_1(x) \overline{u_1(x)} dx + \frac{1}{p_2^2 \rho} \int_0^1 f_2(x) \overline{u_2(x)} dx + \\ & \quad \frac{1}{\rho} \sum_{i=1}^n (-1)^i \frac{1}{\theta_i} h_i M'_i(\bar{u}). \end{aligned} \quad (2.5)$$

Let $\widetilde{C_0^\infty}$ be the set of all functions defined on $[-1, 0) \cup (0, 1]$ such that

$$\phi(x) = \begin{cases} \varphi_1(x), & x \in [-1, 0), \\ \varphi_2(x), & x \in (0, 1], \end{cases}$$

for $\varphi_1(x) \in C_0^\infty[-1, 0)$, $\varphi_2(x) \in C_0^\infty(0, 1]$. As usual, from the well-known fact that $C_0^\infty(a, b)$ is dense in the Hilbert space $L^2(a, b)$ [14, p96], it follows that the set $\widetilde{C_0^\infty}$ is dense in the Hilbert space H_1 . Since $\widetilde{C_0^\infty} \oplus 0^n \subset D(A)$ ($0 \in \mathbb{C}$) and $U = (u(x), 0, \dots, 0) \in \widetilde{C_0^\infty} \oplus 0^n$ is orthogonal to F , i.e.

$$\langle F, U \rangle = \frac{1}{p_1^2} \int_{-1}^0 f_1(x) \overline{u_1(x)} dx + \frac{1}{p_2^2 \rho} \int_0^1 f_2(x) \overline{u_2(x)} dx = \langle f, u \rangle_1 = 0. \quad (2.6)$$

Therefore, (2.6) means that $f(x)$ is orthogonal to the subspace $\widetilde{C_0^\infty}$ which is dense everywhere in the Hilbert space H_1 , so $f(x)$ is null element of H_1 , putting $f(x) = 0$ in (2.5), we have

$$\frac{1}{\rho} \sum_{i=1}^n (-1)^i \frac{1}{\theta_i} h_i M'_i(\bar{u}) = 0,$$

for all $u \in H_1$, such that $U \in D(A)$. So for all $G_1 = (g(x), M'_1(g), 0, \dots, 0) \in D(A)$,

$$\langle F, G_1 \rangle = \langle f, g \rangle_1 + \frac{1}{\rho \theta_1} h_1 M'_1(\bar{g}) = 0,$$

since $M'_1(g)$ can be chosen arbitrarily, so $h_1 = 0$. Similarly $h_2 = h_3 = \cdots = h_n = 0$. Hence $F = (0, 0, \dots, 0)$ is the null element of the Hilbert space H . Thus, the orthogonal complement of $D(A)$ consists of only the null element, and therefore is dense in the Hilbert space H . $\square$

Theorem 1. *The operator A is self-adjoint in H .*

Proof. Let F and G be arbitrary elements of $D(A)$, by partial integrations we obtain

$$\begin{aligned} \langle AF, G \rangle &= \langle F, AG \rangle + W(f, \bar{g}; 0-) - W(f, \bar{g}; -1) + \frac{1}{\rho} W(f, \bar{g}; 1) \\ &\quad - \frac{1}{\rho} W(f, \bar{g}; 0+) + \frac{1}{\rho} \sum_{i=1}^n (-1)^i \frac{1}{\theta_i} M'_i(f) M_i(\bar{g}) \\ &\quad - \frac{1}{\rho} \sum_{i=1}^n (-1)^i \frac{1}{\theta_i} M_i(f) M'_i(\bar{g}). \end{aligned} \quad (2.7)$$

Since f and g satisfy the boundary conditions (1.2), and $a_{n+i} \neq 0 (i = 1, 2, \dots, n)$, it follows that

$$W(f, \bar{g}; -1) = 0. \quad (2.8)$$

From the transmission conditions (1.4) we get

$$\begin{aligned} W(f, \bar{g}; 0+) &= C_f^T(0+) Q C_{\bar{g}}(0+) = C_f^T(0-) C^T Q C C_{\bar{g}}(0-) \\ &= \rho C_f^T(0-) Q C_{\bar{g}}(0-) = \rho W(f, \bar{g}; 0-). \end{aligned} \quad (2.9)$$

Further, putting (2.8) and (2.9) into (2.7), and applying Lemma 1, we get the required equality

$$\langle AF, G \rangle = \langle F, AG \rangle \quad (F, G \in D(A)),$$

so A is symmetric.

It remains to show that if $\langle AF, W \rangle = \langle F, U \rangle$ for all $F = (f(x), M'_1(f), M'_2(f), \dots, M'_n(f)) \in D(A)$ then $W \in D(A)$ and $AW = U$. Where $W = (w(x), h_1, h_2, \dots, h_n)$, $U = (u(x), k_1, k_2, \dots, k_n)$, i.e.

- (1) $w_1^{(i-1)} \in AC_{loc}((-1, 0))$, $w_2^{(i-1)} \in AC_{loc}((0, 1))$, $i = 1, 2, \dots, 2n$, and $lw \in H_1$;
- (2) $h_i = M'_i(w) = b'_i w^{(i-1)}(1) + b'_{2n+1-i} w^{(2n-i)}(1)$, $i = 1, 2, \dots, n$;
- (3) $l_i w = a_i w^{(i-1)}(-1) + a_{2n+1-i} w^{(2n-i)}(-1) = 0$, $i = 1, 2, \dots, n$;
- (4) $C_w(0+) = C \cdot C_w(0-)$;
- (5) $u(x) = lw$;
- (6) $k_i = -M_i(w) = -(b_i w^{(i-1)}(1) + b_{2n+1-i} w^{(2n-i)}(1))$, $i = 1, 2, \dots, n$.

For an arbitrary $F \in \widetilde{\mathbb{C}}_0^\infty \oplus 0^n \subset D(A)$, from $\langle AF, W \rangle = \langle F, U \rangle$, we have

$$\frac{1}{p_1^2} \int_{-1}^0 (lf) \bar{w} dx + \frac{1}{p_2^2 \rho} \int_0^1 (lf) \bar{w} dx = \frac{1}{p_1^2} \int_{-1}^0 f \bar{u} dx + \frac{1}{p_2^2 \rho} \int_0^1 f \bar{u} dx,$$

that is $\langle lf, w \rangle_1 = \langle f, u \rangle_1$. According to normal Sturm-Liouville theory, (1) and (5) hold. By (5), the equation $\langle AF, W \rangle = \langle F, U \rangle, \forall F \in D(A)$ becomes

$$\begin{aligned} & \frac{1}{p_1^2} \int_{-1}^0 (lf) \bar{w} dx + \frac{1}{p_2^2 \rho} \int_0^1 (lf) \bar{w} dx - \frac{1}{\rho} \sum_{i=1}^n (-1)^i M_i(f) \bar{h}_i \\ &= \frac{1}{p_1^2} \int_{-1}^0 f(l\bar{w}) dx + \frac{1}{p_2^2 \rho} \int_0^1 f(l\bar{w}) dx + \frac{1}{\rho} \sum_{i=1}^n (-1)^i M'_i(f) \bar{k}_i, \end{aligned}$$

so

$$\langle lf, w \rangle_1 = \langle f, lw \rangle_1 + \frac{1}{\rho} \sum_{i=1}^n (-1)^i \frac{1}{\theta_i} M_i(f) \bar{h}_i + \frac{1}{\rho} \sum_{i=1}^n (-1)^i \frac{1}{\theta_i} M'_i(f) \bar{k}_i.$$

However

$$\begin{aligned} \langle lf, w \rangle_1 &= \frac{1}{p_1^2} \int_{-1}^0 (lf) \bar{w} dx + \frac{1}{p_2^2 \rho} \int_0^1 (lf) \bar{w} dx \\ &= \frac{1}{p_1^2} \int_{-1}^0 (-p_1^2 f^{(2n)} + q(x)f) \bar{w} dx + \frac{1}{p_2^2 \rho} \int_0^1 (-p_2^2 f^{(2n)} + q(x)f) \bar{w} dx \\ &= \frac{1}{p_1^2} \int_{-1}^0 f(l\bar{w}) dx + \frac{1}{p_2^2 \rho} \int_0^1 f(l\bar{w}) dx + W(f, \bar{w}; 0-) - W(f, \bar{w}; -1) \\ &\quad + \frac{1}{\rho} W(f, \bar{w}; 1) - \frac{1}{\rho} W(f, \bar{w}; 0+) \\ &= \langle f, lw \rangle_1 + W(f, \bar{w}; 0-) - W(f, \bar{w}; -1) + \frac{1}{\rho} W(f, \bar{w}; 1) - \frac{1}{\rho} W(f, \bar{w}; 0+). \end{aligned}$$

Hence,

$$\begin{aligned} \langle lf, w \rangle_1 &= \langle f, lw \rangle_1 + W(f, \bar{w}; 0-) - W(f, \bar{w}; -1) \\ &\quad + \frac{1}{\rho} W(f, \bar{w}; 1) - \frac{1}{\rho} W(f, \bar{w}; 0+), \end{aligned}$$

so

$$\begin{aligned} & \frac{1}{\rho} \sum_{i=1}^n (-1)^i \frac{1}{\theta_i} M_i(f) \bar{h}_i + \frac{1}{\rho} \sum_{i=1}^n (-1)^i \frac{1}{\theta_i} M'_i(f) \bar{k}_i \\ &= W(f, \bar{w}; 0-) - W(f, \bar{w}; -1) + \frac{1}{\rho} W(f, \bar{w}; 1) - \frac{1}{\rho} W(f, \bar{w}; 0+). \end{aligned} \tag{2.10}$$

By Naimark Patching Lemma [10], there is an $F \in D(A)$, such that

$$\begin{aligned} f^{(i-1)}(-1) &= f^{(i-1)}(0-) = f^{(i-1)}(0+) = 0, i = 1, 2, \dots, 2n, \\ f(1) &= -b'_{2n}, f^{(i)}(1) = 0, i = 1, 2, \dots, 2n-2, f^{(2n-1)}(1) = b'_1, \end{aligned}$$

For such an F , $M'_i(f) = 0, i = 1, 2, \dots, n, M_1(f) = \theta_1, M_i(f) = 0, i = 2, \dots, 2n$. Then from (2.10) we have $h_1 = M'_1(w)$. Similarly, we can prove $h_i = M'_i(w), i = 2, \dots, n$. So the equalities (2) hold.

Similarly, we can prove (6).

Next choose $F \in D(A)$, such that

$$\begin{aligned} f^{(i-1)}(0+) &= f^{(i-1)}(0-) = f^{(i-1)}(1) = 0, i = 1, 2, \dots, 2n, f(-1) = a_{2n}, \\ f^{(i-1)}(-1) &= 0, i = 2, \dots, 2n-1, f^{(2n-1)}(-1) = -a_1, \end{aligned}$$

thus $M_i(f) = M'_i(f) = 0, i = 1, 2, \dots, n$. Then from (2.10) we have

$$W(f, \bar{w}, -1) = a_1 \bar{w}(-1) + a_{2n} \bar{w}^{(2n-1)}(-1) = 0,$$

that is $l_1 \bar{w} = 0$, so $l_1 w = 0$. Similarly, we can prove $l_i w = 0, i = 2, \dots, n$. So the equalities (3) hold.

Next choose $F \in D(A)$, such that

$$\begin{aligned} f^{(i-1)}(-1) &= f^{(i-1)}(1) = 0, i = 1, 2, \dots, 2n, f^{(i-1)}(0+) = 0, i = 1, 2, \dots, \\ 2n-1, f^{(2n-1)}(0+) &= \rho, f^{(i-1)}(0-) = (-1)^i c_{1, 2n+1-i}, i = 1, 2, \dots, 2n, \end{aligned}$$

thus $M_i(f) = M'_i(f) = 0, i = 1, 2, \dots, n$. Then from (2.10) we have

$$W(f, \bar{w}; 0+) = \rho W(f, \bar{w}; 0-).$$

However, $C = (c_{ij})$ is $2n \times 2n$ real matrix, then $w(0+) = \sum_{i=1}^{2n} c_{1i} w^{(i-1)}(0-)$. Using the same method, we can prove

$$w^{(k-1)}(0+) = \sum_{i=1}^{2n} c_{ki} w^{(i-1)}(0-), k = 2, 3, \dots, 2n,$$

that is $C_w(0+) = C \cdot C_w(0-)$. So the equality (4) holds.

From the above synthesis proof, we can know A is a self-adjoint operator. $\square$

From the properties of self-adjoint operators, we have

Corollary 1. All the eigenvalues of the considered problem (1.1)-(1.4) are real.

Corollary 2. Let λ_1 and λ_2 be two different eigenvalues of the problem (1.1)-(1.4), then the corresponding eigenfunctions $f(x)$ and $g(x)$ are orthogonal in the sense of

$$\frac{1}{p_1^2} \int_{-1}^0 f \bar{g} dx + \frac{1}{p_2^2 \rho} \int_0^1 f \bar{g} dx + \frac{1}{\rho} \sum_{i=1}^n (-1)^i \frac{1}{\theta_i}$$

$$(b'_i f^{(i-1)}(1) + b'_{2n+1-i} f^{(2n-i)}(1))(b'_i \bar{g}^{(i-1)}(1) + b'_{2n+1-i} \bar{g}^{(2n-i)}(1)) = 0.$$

Consequently, the eigenfunctions of the problem (1.1)-(1.4) corresponding to the different eigenvalues are not orthogonal in the usual sense in the Hilbert space H .

3. CONSTRUCTION OF BASIC SOLUTIONS

In terms of existence and uniqueness theorem in ordinary differential equation theory, we shall define two group basic solutions $\phi_1(x, \lambda), \phi_2(x, \lambda), \dots, \phi_n(x, \lambda)$ and $\chi_1(x, \lambda), \chi_2(x, \lambda), \dots, \chi_n(x, \lambda)$ of the equation (1.1) on whole $J = [-1, 0) \cup (0, 1]$.

Let $\phi_{11}(x, \lambda), \phi_{12}(x, \lambda), \dots, \phi_{1n}(x, \lambda)$ and $\chi_{11}(x, \lambda), \chi_{12}(x, \lambda), \dots, \chi_{1n}(x, \lambda)$ be solutions of the equation (1.1) in the interval $[-1, 0)$ satisfying the following initial conditions

$$(C_{\phi_{11}}, C_{\phi_{12}}, \dots, C_{\phi_{1n}}, C_{\chi_{11}}, C_{\chi_{12}}, \dots, C_{\chi_{1n}})(-1, \lambda) = \begin{pmatrix} a_{2n} & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & a_{2n-1} & \dots & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{n+1} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & -a_n & 1 & \dots & 0 & 0 \\ 0 & -a_2 & \dots & 0 & 0 & \dots & 1 & 0 \\ -a_1 & 0 & \dots & 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \quad (3.1)$$

As same as above, we let $\phi_{21}(x, \lambda), \phi_{22}(x, \lambda), \dots, \phi_{2n}(x, \lambda)$ and $\chi_{21}(x, \lambda), \chi_{22}(x, \lambda), \dots, \chi_{2n}(x, \lambda)$ be solutions of the equation (1.1) satisfying the following initial conditions

$$(C_{\phi_{21}}, C_{\phi_{22}}, \dots, C_{\phi_{2n}}, C_{\chi_{21}}, C_{\chi_{22}}, \dots, C_{\chi_{2n}})(0, \lambda) = C \cdot (C_{\phi_{11}}, C_{\phi_{12}}, \dots, C_{\phi_{1n}}, C_{\chi_{11}}, C_{\chi_{12}}, \dots, C_{\chi_{1n}})(0, \lambda). \quad (3.2)$$

The Wronskian $w(\phi_{i1}, \dots, \phi_{in}, \chi_{i1}, \dots, \chi_{in})(x, \lambda) (i = 1, 2)$ are independent of the variable x , and are entire functions of parameter λ .

Let

$$w_i(\lambda) = w(\phi_{i1}, \dots, \phi_{in}, \chi_{i1}, \dots, \chi_{in})(x, \lambda) (i = 1, 2).$$

Since $w_i(\lambda)$ are independent of the variable x , by (3.1), (3.2), we have

$$\begin{aligned} w_1(\lambda) &= w(\phi_{11}, \phi_{12}, \dots, \phi_{1n}, \chi_{11}, \chi_{12}, \dots, \chi_{1n})(x, \lambda) \\ &= \det(C_{\phi_{11}}, C_{\phi_{12}}, \dots, C_{\phi_{1n}}, C_{\chi_{11}}, C_{\chi_{12}}, \dots, C_{\chi_{1n}})(x, \lambda) \\ &= \det(C_{\phi_{11}}, C_{\phi_{12}}, \dots, C_{\phi_{1n}}, C_{\chi_{11}}, C_{\chi_{12}}, \dots, C_{\chi_{1n}})(-1, \lambda) \\ &= a_{n+1}a_{n+2} \dots a_{2n} \neq 0, \end{aligned} \quad (3.3)$$

and

$$\begin{aligned}
 w_2(\lambda) &= w(\phi_{21}, \phi_{22}, \dots, \phi_{2n}, \chi_{21}, \chi_{22}, \dots, \chi_{2n})(x, \lambda) \\
 &= \det(C_{\phi_{21}}, C_{\phi_{22}}, \dots, C_{\phi_{2n}}, C_{\chi_{21}}, C_{\chi_{22}}, \dots, C_{\chi_{2n}})(x, \lambda) \\
 &= \det(C_{\phi_{21}}, C_{\phi_{22}}, \dots, C_{\phi_{2n}}, C_{\chi_{21}}, C_{\chi_{22}}, \dots, C_{\chi_{2n}})(0, \lambda) \\
 &= \det(C \cdot (C_{\phi_{11}}, C_{\phi_{12}}, \dots, C_{\phi_{1n}}, C_{\chi_{11}}, C_{\chi_{12}}, \dots, C_{\chi_{1n}})(0, \lambda)) \\
 &= (\det C)w_1(\lambda) = \rho^n w_1(\lambda) \neq 0.
 \end{aligned} \tag{3.4}$$

So the functions $\phi_{21}(x, \lambda), \phi_{22}(x, \lambda), \dots, \phi_{2n}(x, \lambda)$ and $\chi_{21}(x, \lambda), \chi_{22}(x, \lambda), \dots, \chi_{2n}(x, \lambda)$ are linearly independent in the interval $(0, 1]$.

Let

$$\begin{aligned}
 \phi_1(x, \lambda) &= \begin{cases} \phi_{11}(x, \lambda), & x \in [-1, 0), \\ \phi_{21}(x, \lambda), & x \in (0, 1]; \end{cases}, \phi_2(x, \lambda) = \begin{cases} \phi_{12}(x, \lambda), & x \in [-1, 0), \\ \phi_{22}(x, \lambda), & x \in (0, 1]; \end{cases}, \\
 \dots\dots, \phi_n(x, \lambda) &= \begin{cases} \phi_{1n}(x, \lambda), & x \in [-1, 0), \\ \phi_{2n}(x, \lambda), & x \in (0, 1]; \end{cases},
 \end{aligned}$$

and

$$\begin{aligned}
 \chi_1(x, \lambda) &= \begin{cases} \chi_{11}(x, \lambda), & x \in [-1, 0), \\ \chi_{21}(x, \lambda), & x \in (0, 1]; \end{cases}, \chi_2(x, \lambda) = \begin{cases} \chi_{12}(x, \lambda), & x \in [-1, 0), \\ \chi_{22}(x, \lambda), & x \in (0, 1]; \end{cases}, \\
 \dots\dots, \chi_n(x, \lambda) &= \begin{cases} \chi_{1n}(x, \lambda), & x \in [-1, 0); \\ \chi_{2n}(x, \lambda), & x \in (0, 1]. \end{cases}.
 \end{aligned}$$

Moreover, $\phi_1(x, \lambda), \phi_2(x, \lambda), \dots, \phi_n(x, \lambda)$ and $\chi_1(x, \lambda), \chi_2(x, \lambda), \dots, \chi_n(x, \lambda)$ satisfy the boundary conditions (1.2) and transmission conditions (1.4). They are independent of x , and are entire functions of parameter λ .

Lemma 4. *The following determinant is equal to $w_1(\lambda)$, or $-w_1(\lambda)$, i.e.*

$$\begin{vmatrix} l_1 \chi_{11} & l_1 \chi_{12} & \dots & l_1 \chi_{1n} \\ l_2 \chi_{11} & l_2 \chi_{12} & \dots & l_2 \chi_{1n} \\ \dots & \dots & \dots & \dots \\ l_n \chi_{11} & l_n \chi_{12} & \dots & l_n \chi_{1n} \end{vmatrix} = w_1(\lambda), \text{ or } = -w_1(\lambda),$$

if $n = 1, 4, 5, 8, 9, \dots$, then the determinant is equal to $w_1(\lambda)$;

if $n = 2, 3, 6, 7, \dots$, then it is equal to $-w_1(\lambda)$.

Where

$$\begin{aligned}
 l_1 \chi_{1i} &= a_1 \chi_{1i}(-1, \lambda) + a_{2n} \chi_{1i}^{(2n-1)}(-1, \lambda), i = 1, 2, \dots, n, \\
 l_2 \chi_{1i} &= a_2 \chi'_{1i}(-1, \lambda) + a_{2n-1} \chi_{1i}^{(2n-2)}(-1, \lambda), i = 1, 2, \dots, n, \\
 &\dots\dots\dots \\
 l_n \chi_{1i} &= a_n \chi_{1i}^{(n-1)}(-1, \lambda) + a_{n+1} \chi_{1i}^{(n)}(-1, \lambda), i = 1, 2, \dots, n.
 \end{aligned}$$

Proof. By (3.1), we have

$$\begin{vmatrix} l_1\chi_{11} & l_1\chi_{12} & \cdots & l_1\chi_{1n} \\ l_2\chi_{11} & l_2\chi_{12} & \cdots & l_2\chi_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ l_n\chi_{11} & l_n\chi_{12} & \cdots & l_n\chi_{1n} \end{vmatrix} = \begin{vmatrix} a_1\chi_{11} + a_{2n}\chi_{11}^{(2n-1)} & a_1\chi_{12} + a_{2n}\chi_{12}^{(2n-1)} & \cdots & a_1\chi_{1n} + a_{2n}\chi_{1n}^{(2n-1)} \\ a_2\chi'_{11} + a_{2n-1}\chi_{11}^{(2n-2)} & a_2\chi'_{12} + a_{2n-1}\chi_{12}^{(2n-2)} & \cdots & a_2\chi'_{1n} + a_{2n-1}\chi_{1n}^{(2n-2)} \\ \cdots & \cdots & \cdots & \cdots \\ a_n\chi_{11}^{(n-1)} + a_{n+1}\chi_{11}^{(n)} & a_n\chi_{12}^{(n-1)} + a_{n+1}\chi_{12}^{(n)} & \cdots & a_n\chi_{1n}^{(n-1)} + a_{n+1}\chi_{1n}^{(n)} \end{vmatrix} \\ = \begin{vmatrix} 0 & \cdots & 0 & a_{2n} \\ 0 & \cdots & a_{2n-1} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+1} & \cdots & 0 & 0 \end{vmatrix} = a_{n+1}a_{n+2}\cdots a_{2n}, \text{ or } = -a_{n+1}a_{n+2}\cdots a_{2n},$$

if $n = 1, 4, 5, 8, 9, \dots$, then it is equal to $a_{n+1}a_{n+2}\cdots a_{2n}$, i.e. $w_1(\lambda)$; if $n = 2, 3, 6, 7, \dots$, then it is equal to $-a_{n+1}a_{n+2}\cdots a_{2n}$, i.e. $-w_1(\lambda)$. We complete the proof. $\square$

Lemma 5. Let

$$u(x) = \begin{cases} u_1(x), & x \in [-1, 0), \\ u_2(x), & x \in (0, 1]; \end{cases}$$

be any solution of the equation $ly = \lambda y$, then it can be represented as

$$u(x) = \begin{cases} d_1\phi_{11} + \cdots + d_n\phi_{1n} + d_{n+1}\chi_{11} + \cdots + d_{2n}\chi_{1n}, & x \in [-1, 0), \\ d_{2n+1}\phi_{21} + \cdots + d_{3n}\phi_{2n} + d_{3n+1}\chi_{21} + \cdots + d_{4n}\chi_{2n}, & x \in (0, 1]; \end{cases},$$

where $d_i \in \mathbb{C} (i = 1, 2, \dots, 4n)$. If $u(x)$ satisfies the transmission conditions (1.4), then $d_1 = d_{2n+1}, d_2 = d_{2n+2}, \dots, d_{2n} = d_{4n}$.

Proof. Let $u(x)$ is represented in the form

$$u(x) = \begin{cases} d_1\phi_{11} + \cdots + d_n\phi_{1n} + d_{n+1}\chi_{11} + \cdots + d_{2n}\chi_{1n}, & x \in [-1, 0), \\ d_{2n+1}\phi_{21} + \cdots + d_{3n}\phi_{2n} + d_{3n+1}\chi_{21} + \cdots + d_{4n}\chi_{2n}, & x \in (0, 1]; \end{cases}.$$

Apply the transmission conditions (1.4) to this representation of $u(x)$. Since

$$u^{(i-1)}(0+) = \sum_{j=1}^{2n} c_{ij}u^{(j-1)}(0-), i = 1, 2, \dots, 2n,$$

i.e.

$$C_u(0+) = C \cdot C_u(0-),$$

$$\begin{aligned}
& \begin{pmatrix} d_{2n+1}\phi_{21} + \cdots + d_{3n}\phi_{2n} + d_{3n+1}\chi_{21} + \cdots + d_{4n}\chi_{2n} \\ d_{2n+1}\phi'_{21} + \cdots + d_{3n}\phi'_{2n} + d_{3n+1}\chi'_{21} + \cdots + d_{4n}\chi'_{2n} \\ \dots \\ d_{2n+1}\phi_{21}^{(2n-1)} + \cdots + d_{3n}\phi_{2n}^{(2n-1)} + d_{3n+1}\chi_{21}^{(2n-1)} + \cdots + d_{4n}\chi_{2n}^{(2n-1)} \end{pmatrix} (0, \lambda) \\
&= C \begin{pmatrix} d_1\phi_{11} + \cdots + d_n\phi_{1n} + d_{n+1}\chi_{11} + \cdots + d_{2n}\chi_{1n} \\ d_1\phi'_{11} + \cdots + d_n\phi'_{1n} + d_{n+1}\chi'_{11} + \cdots + d_{2n}\chi'_{1n} \\ \dots \\ d_1\phi_{11}^{(2n-1)} + \cdots + d_n\phi_{1n}^{(2n-1)} + d_{n+1}\chi_{11}^{(2n-1)} + \cdots + d_{2n}\chi_{1n}^{(2n-1)} \end{pmatrix} (0, \lambda),
\end{aligned}$$

we rewrite it in the following form

$$\begin{aligned}
& (C_{\phi_{21}}, C_{\phi_{22}}, \dots, C_{\phi_{2n}}, C_{\chi_{21}}, C_{\chi_{22}}, \dots, C_{\chi_{2n}})(0, \lambda) \begin{pmatrix} d_{2n+1} \\ d_{2n+2} \\ \dots \\ d_{4n} \end{pmatrix} \\
&= C \cdot (C_{\phi_{11}}, C_{\phi_{12}}, \dots, C_{\phi_{1n}}, C_{\chi_{11}}, C_{\chi_{12}}, \dots, C_{\chi_{1n}})(0, \lambda) \begin{pmatrix} d_1 \\ d_2 \\ \dots \\ d_{2n} \end{pmatrix},
\end{aligned}$$

taking into account the initial conditions (3.2) for $\phi_{21}(x, \lambda), \dots, \phi_{2n}(x, \lambda)$ and $\chi_{21}(x, \lambda), \dots, \chi_{2n}(x, \lambda)$, we have

$$\begin{aligned}
& C \cdot (C_{\phi_{11}}, C_{\phi_{12}}, \dots, C_{\phi_{1n}}, C_{\chi_{11}}, C_{\chi_{12}}, \dots, C_{\chi_{1n}})(0, \lambda) \begin{pmatrix} d_{2n+1} \\ d_{2n+2} \\ \dots \\ d_{4n} \end{pmatrix} \\
&= C \cdot (C_{\phi_{11}}, C_{\phi_{12}}, \dots, C_{\phi_{1n}}, C_{\chi_{11}}, C_{\chi_{12}}, \dots, C_{\chi_{1n}})(0, \lambda) \begin{pmatrix} d_1 \\ d_2 \\ \dots \\ d_{2n} \end{pmatrix}.
\end{aligned}$$

So

$$C \cdot (C_{\phi_{11}}, C_{\phi_{12}}, \dots, C_{\phi_{1n}}, C_{\chi_{11}}, C_{\chi_{12}}, \dots, C_{\chi_{1n}})(0, \lambda) \begin{pmatrix} d_{2n+1} - d_1 \\ d_{2n+2} - d_2 \\ \dots \\ d_{4n} - d_{2n} \end{pmatrix} = 0. \quad (3.5)$$

Since

$$\det(C(C_{\phi_{11}}, C_{\phi_{12}}, \dots, C_{\phi_{1n}}, C_{\chi_{11}}, C_{\chi_{12}}, \dots, C_{\chi_{1n}})(0, \lambda)) = \rho^n w_1(\lambda) \neq 0,$$

the above equation (3.5) has only zero solution, $d_1 = d_{2n+1}, d_2 = d_{2n+2}, \dots, d_{2n} = d_{4n}$. □

For convenience, we let

$$B_\lambda = \begin{pmatrix} \lambda b'_1 + b_1 & \cdots & 0 & 0 & \cdots & \lambda b'_{2n} + b_{2n} \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \lambda b'_n + b_n & \lambda b'_{n+1} + b_{n+1} & \cdots & 0 \end{pmatrix},$$

and

$$\det \Phi(1, \lambda) = \begin{vmatrix} l_{n+1}\phi_{21} & l_{n+1}\phi_{22} & \cdots & l_{n+1}\phi_{2n} \\ l_{n+2}\phi_{21} & l_{n+2}\phi_{22} & \cdots & l_{n+2}\phi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ l_{2n}\phi_{21} & l_{2n}\phi_{22} & \cdots & l_{2n}\phi_{2n} \end{vmatrix},$$

so

$$\det \Phi(1, \lambda) = \det(B_\lambda \cdot (C_{\phi_{21}}, C_{\phi_{22}}, \cdots, C_{\phi_{2n}})(1, \lambda)),$$

where

$$l_{n+1}\phi_{2i} = (\lambda b'_1 + b_1)\phi_{2i}(1, \lambda) + (\lambda b'_{2n} + b_{2n})\phi_{2i}^{(2n-1)}(1, \lambda), i = 1, 2, \cdots, n,$$

$$l_{n+2}\phi_{2i} = (\lambda b'_2 + b_2)\phi'_{2i}(1, \lambda) + (\lambda b'_{2n-1} + b_{2n-1})\phi_{2i}^{(2n-2)}(1, \lambda), i = 1, 2, \cdots, n,$$

.....

$$l_{2n}\phi_{2i} = (\lambda b'_n + b_n)\phi_{2i}^{(n-1)}(1, \lambda) + (\lambda b'_{n+1} + b_{n+1})\phi_{2i}^{(n)}(1, \lambda), i = 1, 2, \cdots, n.$$

Theorem 2. The eigenvalues of the problem (1.1)-(1.4) coincide with the zeros of the entire function $\det \Phi(1, \lambda)$.

Proof. Let λ_0 be an eigenvalue and $u_0(x)$ be any corresponding eigenfunction. Show that $\det \Phi(1, \lambda_0) = 0$. If not, $\det \Phi(1, \lambda_0) \neq 0$. From (3.3) and (3.4) we know that $w_1(\lambda_0) \neq 0$ and $w_2(\lambda_0) \neq 0$. Consequently, each group of functions $\phi_{11}(x, \lambda_0), \cdots, \phi_{1n}(x, \lambda_0), \chi_{11}(x, \lambda_0), \cdots, \chi_{1n}(x, \lambda_0)$ and $\phi_{21}(x, \lambda_0), \cdots, \phi_{2n}(x, \lambda_0), \chi_{21}(x, \lambda_0), \cdots, \chi_{2n}(x, \lambda_0)$ would be linearly independent on $[-1, 0)$ and $(0, 1]$, respectively. Therefore eigenfunction $u_0(x)$ might be represented in the form

$$u_0(x) = \begin{cases} (d_1\phi_{11} + \cdots + d_n\phi_{1n} + d_{n+1}\chi_{11} \\ + \cdots + d_{2n}\chi_{1n})(x, \lambda_0), & x \in [-1, 0); \\ (d_{2n+1}\phi_{21} + \cdots + d_{3n}\phi_{2n} + d_{3n+1}\chi_{11} \\ + \cdots + d_{4n}\chi_{2n})(x, \lambda_0), & x \in (0, 1]; \end{cases}$$

where at least one of the constants $d_1, d_2, \cdots, d_{4n}$ is not zero. By substituting this representation in the conditions (1.2)-(1.4), we obtain a system of linear, homogeneous equations for the variables of the constants $d_1, d_2, \cdots, d_{4n}$. The considered equations are

$$l_k(u_0(x)) = 0, \quad k = 1, 2, \cdots, 4n. \quad (3.6)$$

In fact

$$\begin{aligned}
 l_i u_0 &= \sum_{k=1}^n d_{n+k} (a_i \chi_{1k}^{(i-1)}(-1) + a_{2n+1-i} \chi_{1k}^{(2n-i)}(-1)) = 0, i = 1, 2, \dots, n, \\
 l_{n+i} u_0 &= \sum_{k=1}^n d_{2n+k} ((\lambda_0 b'_i + b_i) \phi_{2k}^{(i-1)}(1) + (\lambda_0 b'_{2n+1-i} + b_{2n+1-i}) \phi_{2k}^{(2n-i)}(1)) \\
 &\quad + \sum_{k=1}^n d_{3n+k} ((\lambda_0 b'_i + b_i) \chi_{2k}^{(i-1)}(1) + (\lambda_0 b'_{2n+1-i} + b_{2n+1-i}) \chi_{2k}^{(2n-i)}(1)) = 0, \\
 &\quad i = 1, 2, \dots, n,
 \end{aligned}$$

$$\begin{aligned}
 l_{2n+i} u_0 &= - \sum_{k=1}^n d_k \phi_{2k}^{(i-1)}(0) - \sum_{k=1}^n d_{n+k} \chi_{2k}^{(i-1)}(0) + \sum_{k=1}^n d_{2n+k} \phi_{2k}^{(i-1)}(0) \\
 &\quad + \sum_{k=1}^n d_{3n+k} \chi_{2k}^{(i-1)}(0), i = 1, 2, \dots, 2n.
 \end{aligned}$$

By Lemma 4, we obtain that the determinant of this system is equal to

$$(-1)^{n^2} w_1(\lambda_0) w_2(\lambda_0) \det \Phi(1, \lambda_0) \neq 0,$$

or is equal to

$$(-1)^{n^2+1} w_1(\lambda_0) w_2(\lambda_0) \det \Phi(1, \lambda_0) \neq 0,$$

where

$$\begin{aligned}
 l_{n+1} \chi_{2i} &= (\lambda b'_1 + b_1) \chi_{2i}(1, \lambda) + (\lambda b'_{2n} + b_{2n}) \chi_{2i}^{(2n-1)}(1, \lambda), i = 1, 2, \dots, n, \\
 l_{n+2} \chi_{2i} &= (\lambda b'_2 + b_2) \chi'_{2i}(1, \lambda) + (\lambda b'_{2n-1} + b_{2n-1}) \chi_{2i}^{(2n-2)}(1, \lambda), i = 1, 2, \dots, n, \\
 &\quad \dots\dots\dots \\
 l_{2n} \chi_{2i} &= (\lambda b'_n + b_n) \chi_{2i}^{(n-1)}(1, \lambda) + (\lambda b'_{n+1} + b_{n+1}) \chi_{2i}^{(n)}(1, \lambda), i = 1, 2, \dots, n, \\
 \phi_{2k}^{(i-1)} &= \phi_{2k}^{(i-1)}(0), \chi_{2k}^{(i-1)} = \chi_{2k}^{(i-1)}(0), k = 1, 2, \dots, n, i = 1, 2, \dots, 2n.
 \end{aligned}$$

Therefore, the system (3.6) has the only trivial solution $d_1 = d_2 = \dots = d_{4n} = 0$. Thus, we get the contradiction.

On the contrary, if $\det \Phi(1, \lambda_0) = 0$, the homogeneous equations for the variables of the constants $c_1, c_2, \dots, c_{2n}$,

$$\Phi(1, \lambda) \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_{2n} \end{pmatrix} = 0$$

has nonzero solution $(c'_1, c'_2, \dots, c'_{2n})^T$. By Lemma 5, let

$$y(x) = \begin{cases} (c'_1 \phi_{11} + \dots + c'_n \phi_{1n} + c'_{n+1} \chi_{11} + \dots + c'_{2n} \chi_{1n})(x, \lambda), & x \in (-1, 0), \\ (c'_1 \phi_{21} + \dots + c'_n \phi_{2n} + c'_{n+1} \chi_{21} + \dots + c'_{2n} \chi_{2n})(x, \lambda), & x \in (0, 1]; \end{cases}$$

then $y(x)$ is the nonzero solution of equation $l(y) = \lambda y$, which satisfy conditions (1.3)-(1.4). so λ is the eigenvalue of the problem (1.1)-(1.4). Which completes the proof. $\square$

Corollary 3. *The eigenvalues of the problem (1.1)-(1.4) are real and form a finite or infinite sequence without finite accumulation point.*

4. COMPLETENESS OF EIGENFUNCTIONS

In this section, in terms of new Hilbert space H and new operator A , we investigate the problem (1.1)-(1.4), and obtain the following conclusion.

Theorem 3. *The operator A has only point spectrum, i.e. $\sigma(A) = \sigma_p(A)$.*

Proof. Let $\gamma \in \sigma_p(A)$. We only show $\gamma \in \rho(A)$. Since A is self-adjoint, we only consider a real γ . We investigate the equation $(A - \gamma)Y = F$, where $F = (f, h_1, \dots, h_n) \in H, \gamma \in \mathbb{R}$.

Consider the following problem

$$\begin{cases} ly - \gamma y = f, x \in [-1, 0) \cup (0, 1], \\ l_i y = a_i y^{(i-1)}(-1) + a_{2n+1-i} y^{(2n-i)}(-1) = 0, i = 1, 2, \dots, n, \\ l_{2n+i} y = y^{(i-1)}(0+) - \sum_{j=1}^{2n} c_{ij} y^{(j-1)}(0-) = 0, i = 1, 2, \dots, 2n; \end{cases} \quad (4.1)$$

and equations

$$M_i(y) + \gamma M'_i(y) = -h_i, i = 1, 2, \dots, n. \quad (4.2)$$

Let $u(x)$ be the solution of the equation $lu - \gamma u = 0$, satisfying

$$\begin{aligned} u^{(i-1)}(-1) &= a_{2n+1-i}, u^{(2n-i)}(-1) = -a_i, i = 1, 2, \dots, n, \\ u^{(i-1)}(0+) &= \sum_{j=1}^{2n} c_{ij} u^{(j-1)}(0-), i = 1, 2, \dots, 2n. \end{aligned}$$

In fact

$$u(x) = \begin{cases} u_1(x), & x \in [-1, 0), \\ u_2(x), & x \in (0, 1]; \end{cases}$$

where $u_1(x)$ is the unique solution of the initial-value problem

$$\begin{cases} -p_1^2 u^{(2n)}(x) + q(x)u(x) = \gamma u(x), x \in [-1, 0), \\ u^{(i-1)}(-1) = a_{2n+1-i}, u^{(2n-i)}(-1) = -a_i, i = 1, 2, \dots, n; \end{cases}$$

and $u_2(x)$ is the unique solution of the initial-value problem

$$\begin{cases} -p_2^2 u^{(2n)}(x) + q(x)u(x) = \gamma u(x), x \in (0, 1], \\ u^{(i-1)}(0+) = \sum_{j=1}^{2n} c_{ij} u^{(j-1)}(0-), i = 1, 2, \dots, 2n; \end{cases}.$$

Let

$$w(x) = \begin{cases} w_1(x), & x \in [-1, 0), \\ w_2(x), & x \in (0, 1]; \end{cases}$$

be a special solution of $lw - \gamma w = f$ satisfying

$$\begin{cases} a_i w^{(i-1)}(-1) + a_{2n+1-i} w^{(2n-i)}(-1) = 0, \\ w^{(i-1)}(0+) = \sum_{j=1}^{2n} c_{ij} w^{(j-1)}(0-), i = 1, 2, \dots, 2n; \end{cases}.$$

By Lemma 5, Equation (4.1) has the general solution in the form

$$y(x) = \begin{cases} (d_1 \phi_{11} + \dots + d_n \phi_{1n} + d_{n+1} \chi_{11} \\ + \dots + d_{2n} \chi_{1n})(x, \gamma) + w_1, & x \in [-1, 0), \\ (d_1 \phi_{21} + \dots + d_n \phi_{2n} + d_{n+1} \chi_{21} \\ + \dots + d_{2n} \chi_{2n})(x, \gamma) + w_2, & x \in (0, 1]; \end{cases}, \quad (4.3)$$

where $d_i \in \mathbb{C} (i = 1, 2, \dots, 2n)$.

Since γ is not eigenvalue of the problem (1.1)-(1.4), at least one of the following expressions is not zero

$$\gamma(b'_i u^{(i-1)}(1) + b'_{2n+1-i} u^{(2n-i)}(1)) + b_i u^{(i-1)}(1) + b_{2n+1-i} u^{(2n-i)}(1), \quad (4.4)$$

$$i = 1, 2, \dots, n.$$

The second, third, $\dots$, n th components of the equation $(A - \gamma)Y = F$ involves the equation (4.2), i.e.

$$\begin{aligned} (b_i y^{(i-1)}(1) + b_{2n+1-i} y^{(2n-i)}(1)) + \gamma(b'_i y^{(i-1)}(1) \\ + b'_{2n+1-i} y^{(2n-i)}(1)) = -h_i, i = 1, 2, \dots, n. \end{aligned} \quad (4.5)$$

Substituting (4.3) into (4.5), we get

$$\begin{aligned} & (b_i \phi_{21}^{(i-1)}(1) + b_{2n+1-i} \phi_{21}^{(2n-i)}(1) + \gamma(b'_i \phi_{21}^{(i-1)}(1) + b'_{2n+1-i} \phi_{21}^{(2n-i)}(1)))d_1 \\ & + (b_i \phi_{22}^{(i-1)}(1) + b_{2n+1-i} \phi_{22}^{(2n-i)}(1) + \gamma(b'_i \phi_{22}^{(i-1)}(1) + b'_{2n+1-i} \phi_{22}^{(2n-i)}(1)))d_2 \\ & + \dots + (b_i \phi_{2n}^{(i-1)}(1) + b_{2n+1-i} \phi_{2n}^{(2n-i)}(1) + \gamma(b'_i \phi_{2n}^{(i-1)}(1) + b'_{2n+1-i} \phi_{2n}^{(2n-i)}(1)))d_n \\ & + (b_i \chi_{21}^{(i-1)}(1) + b_{2n+1-i} \chi_{21}^{(2n-i)}(1) + \gamma(b'_i \chi_{21}^{(i-1)}(1) + b'_{2n+1-i} \chi_{21}^{(2n-i)}(1)))d_{n+1} \\ & + (b_i \chi_{22}^{(i-1)}(1) + b_{2n+1-i} \chi_{22}^{(2n-i)}(1) + \gamma(b'_i \chi_{22}^{(i-1)}(1) + b'_{2n+1-i} \chi_{22}^{(2n-i)}(1)))d_{n+2} \end{aligned}$$

$$\begin{aligned}
& + \cdots + (b_i \chi_{2n}^{(i-1)}(1) + b_{2n+1-i} \chi_{2n}^{(2n-i)}(1) + \gamma(b'_i \chi_{21}^{(i-1)}(1) + b'_{2n+1-i} \chi_{2n}^{(2n-i)}(1))) d_{2n} \\
& = -h_i - (b_i w_2^{(i-1)}(1) + b_{2n+1-i} w_2^{(2n-i)}(1)) - \gamma(b'_i w_2(1) + b'_{2n+1-i} w_2^{(2n-i)}(1)), \\
& \qquad \qquad \qquad i = 1, 2, \dots, n.
\end{aligned}$$

We rewrite the above equations in following form, i.e.

$$\begin{aligned}
& B_\gamma(C_{\phi_{21}}(1, \gamma), C_{\phi_{22}}(1, \gamma), \dots, C_{\phi_{2n}}(1, \gamma)) \begin{pmatrix} d_1 \\ d_2 \\ \dots \\ d_{2n} \end{pmatrix} \\
& = \begin{pmatrix} -h_1 \\ -h_2 \\ \dots \\ -h_n \end{pmatrix} - B_\gamma \begin{pmatrix} w_2(1) \\ w'_2(1) \\ \dots \\ w_2^{(2n-1)}(1) \end{pmatrix}.
\end{aligned}$$

The coefficient determinant of the equations about $d_1, d_2, \dots, d_{2n}$ is $\det \Phi(1, \gamma)$. Since γ is not an eigenvalue, $\det \Phi(1, \gamma) \neq 0$, we know that $d_1, d_2, \dots, d_{2n}$ is uniquely solvable. So the general solution of the boundary value problem (4.1) is uniquely determined.

The above argument shows that $(A - \gamma I)^{-1}$ is defined on all of H . We obtain that $(A - \gamma I)^{-1}$ is bounded by Theorem 1 and by the closed Graph Theorem. Thus $\gamma \in \rho(A)$. Hence, $\sigma(A) = \sigma_p(A)$. $\square$

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SOME SHARP FUNCTION ESTIMATES FOR VECTOR-VALUED MULTILINEAR INTEGRAL OPERATOR

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Abstract. In this paper, we establish some sharp estimates for certain vector-valued multilinear integral operators. The operators include Littlewood-Paley operators, Marcinkiewicz operators and the Bochner-Riesz operator. As an application, we obtain the (L^p, L^q) -norm inequality for the vector-valued multilinear operators.

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1. INTRODUCTION

As a part of the development of singular integral operators, their commutators and multilinear operators have been well studied in [2–5, 14–18]. In [8], Hu and Yang obtained a variant sharp estimate for the multilinear singular integral operators. In [18], C. Pérez and R. Trujillo-Gonzalez obtained a sharp weighted estimate for the vector-valued singular integral operators and their commutators. The main purpose of this paper is to prove the sharp estimates for some multilinear operators related to certain integral operators. The integral operators include Littlewood-Paley operator, Marcinkiewicz operator and the Bochner-Riesz operator. As an applications, we obtain the (L^p, L^q) -norm inequalities for the multilinear operators.

2. NOTATIONS AND THEOREMS

Let m_j ($j = 1, \dots, l$) be positive integers with $m_1 + \dots + m_l = m$, $A_j : R^n \rightarrow \mathcal{C}$ ($j = 1, \dots, l$), $F_t : R^n \times R^n \times [0, +\infty) \rightarrow \mathcal{C}$ and $(x, y, t) \mapsto F_t(x, y)$ be some locally integrable functions. Set

$$F_t(f)(x) := \int_{R^n} F_t(x, y) f(y) dy$$

and

$$F_t^A(f)(x) := \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, y)}{|x - y|^m} F_t(x, y) f(y) dy$$

for every bounded and compactly supported function f , where

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha| \leq m_j} \frac{1}{\alpha!} D^\alpha A_j(y) (x - y)^\alpha.$$

In addition F_t satisfies the following condition: for fixed $\varepsilon > 0$ and $0 \leq \delta < n$,

$$\|F_t(x, y)\| \leq C |x - y|^{-n+\delta}$$

and

$$\|F_t(y, x) - F_t(z, x)\| \leq C |y - z|^\varepsilon |x - z|^{-n-\varepsilon+\delta}$$

if $2|y - z| \leq |x - z|$. For $1 < r < \infty$ and for the vector-valued multilinear operators $|T^A(f)|_r$ and $|T(f)|_r$ the bounded and compactly supported functions on $R^n \rightarrow H$ related to F_t^A are defined by

$$|T^A(f)(x)|_r := \left(\sum_{i=1}^{\infty} (T^A(f_i)(x))^r \right)^{1/r} \quad \text{and} \quad |T(f)(x)|_r := \left(\sum_{i=1}^{\infty} |T(f_i)(x)|^r \right)^{1/r},$$

where

$$T^A(f_i)(x) := \|F_t^A(f_i)(x)\|, \quad T(f_i)(x) := \|F_t(f_i)(x)\|.$$

Here $\|\cdot\|$ is the norm of the Banach space $H := \{\text{continuous and bounded } h : R^n \rightarrow \mathcal{C}\}$ with norm $\|h\|$ such that, for each fixed $x \in R^n$, $F_t(f)(x)$ and $F_t^A(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H . Set

$$|f(x)|_r := \left(\sum_{i=1}^{\infty} |f_i(x)|^r \right)^{1/r}.$$

Suppose that $|T|_r$ is bounded from $L^p(R^n)$ to $L^q(R^n)$ for any $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$.

If $m = 0$, T^A is just the vector-valued multilinear commutator of T and A (see [15]). If $m > 0$, T^A is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2–5, 14]). The main purpose of this paper is to prove a sharp inequality for the vector-valued multilinear integral operators T^A . As an application, we obtain a (L^p, L^q) -norm inequality for the vector-valued multilinear operators. In Section 4, we give some examples.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function $f : R^n \rightarrow \mathcal{C}$, define $f_Q = |Q|^{-1} \int_Q f(x) dx$ and

$$f^\#(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

and set $\|f\|_{BMO} := \|f^\#\|_{L^\infty}$. We write $f \in BMO(R^n)$ if $\|f\|_{BMO} < \infty$. It is well-known that (see [7])

$$f^\#(x) \approx \sup_{Q \ni x} \inf_{c \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

For $1 \leq p < \infty$ and $0 \leq \delta < n$, let

$$M_{\delta,p}(f)(x) := \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-p\delta/n}} \int_Q |f(y)|^p dy \right)^{1/p}.$$

We shall prove the following theorems.

Theorem 1. *If $1 < r < \infty$, $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ ($j = 1, \dots, l$), then there exists a constant $C > 0$ such that for any $f = \{f_i\} \in C_0^\infty(R^n)$, $1 < s < n/\delta$ and $\tilde{x} \in R^n$,*

$$|(T^A(f)|_r)^\#(\tilde{x})| \leq C \prod_{j=1}^l \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).$$

Theorem 2. *If $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ ($j = 1, \dots, l$), then $|T^A|_r : L^p(R^n) \rightarrow L^q(R^n)$ is bounded for $1 < r < \infty$, $1 < p < n/\delta$ and $1/p - 1/q = \delta/n$, that is*

$$\||T^A(f)|_r\|_{L^q} \leq C \prod_{j=1}^l \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \||f|_r\|_{L^p}.$$

3. PROOFS OF THE THEOREMS

To prove the theorems, we need the following lemmas.

Lemma 1 (see [4]). *Let $A : R^n \rightarrow \mathbb{C}$ be a locally integrable function and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \leq C |x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x, y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Lemma 2 (see [1, 6]). *Suppose that $1 < r < \infty$, $0 \leq \delta < n$, $1 \leq s < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then*

$$\|M_{\delta,s}(|f|_r)\|_{L^q} \leq C \||f|_r\|_{L^p}.$$

Proof of Theorem 1. It suffices to prove for $f = \{f_i\} \in C_0^\infty(R^n)$ and some constant C_0 , the following inequality:

$$\frac{1}{|Q|} \int_Q \|T^A(f)(x)\|_r - C_0 dx \leq C \prod_{j=1}^l \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).$$

Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!} (D^\alpha A_j)_{\tilde{Q}} x^\alpha$, then $R_{m_j}(A_j; x, y) = R_{m_j}(\tilde{A}_j; x, y)$ and $D^\alpha \tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We split $f = g + h = \{g_i\} + \{h_i\}$ for $g_i = f_i \chi_{\tilde{Q}}$ and $h_i = f_i \chi_{R^n \setminus \tilde{Q}}$. Write

$$\begin{aligned} F_t^A(f_i)(x) &= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x-y|^m} F_t(x, y) f_i(y) dy \\ &= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x-y|^m} F_t(x, y) h_i(y) dy \\ &\quad + \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m} F_t(x, y) g_i(y) dy \\ &\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \tilde{A}_1(y) F_t(x, y) g_i(y) dy \\ &\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{A}_2(y) F_t(x, y) g_i(y) dy \\ &\quad + \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{A}_1(y) D^{\alpha_2} \tilde{A}_2(y)}{|x-y|^m} F_t(x, y) g_i(y) dy, \end{aligned}$$

then, by Minkowski's inequality,

$$\begin{aligned} &\frac{1}{|Q|} \int_Q \left| \|T^A(f)(x)\|_r - \|T^{\tilde{A}}(h)(x_0)\|_r \right| dx \\ &\leq \frac{1}{|Q|} \int_Q \left| \|F_t^A(f)(x)\|_r - \|F_t^{\tilde{A}}(h)(x_0)\|_r \right| dx \\ &\leq \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^\infty \|F_t^A(f_i)(x) - F_t^{\tilde{A}}(h_i)(x_0)\|_r^r \right)^{1/r} dx \\ &\leq \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^\infty \left\| \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m} F_t(x, y) g_i(y) dy \right\|_r^r \right)^{1/r} dx \end{aligned}$$

$$\begin{aligned}
& + \frac{C}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \left\| \sum_{|\alpha_1|=m_1} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \tilde{A}_1(y) F_t(x, y) g_i(y) dy \right\|^r \right)^{1/r} dx \\
& + \frac{C}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \left\| \sum_{|\alpha_2|=m_2} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{A}_2(y) F_t(x, y) g_i(y) dy \right\|^r \right)^{1/r} dx \\
& + \frac{C}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \left\| \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{A}_1(y) D^{\alpha_2} \tilde{A}_2(y)}{|x-y|^m} F_t(x, y) g_i(y) dy \right\|^r \right)^{1/r} dx \\
& + \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \left\| F_t^{\tilde{A}}(h_i)(x) - F_t^{\tilde{A}}(h_i)(x_0) \right\|^r \right)^{1/r} dx \\
& := I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Now, let us estimate I_1 , I_2 , I_3 , I_4 and I_5 , respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, we get by Lemma 1,

$$R_m(\tilde{A}_j; x, y) \leq C |x-y|^m \sum_{|\alpha_j|=m} \|D^{\alpha_j} A_j\|_{BMO},$$

thus, by the (L^s, L^q) -boundedness of $|T|_r$ with $1 < s < n/\delta$ and $1/q = 1/s - \delta/n$, we obtain

$$\begin{aligned}
I_1 & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \frac{1}{|Q|} \int_Q |T(g)(x)|_r dx \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \left(\frac{1}{|Q|} \int_Q |T(g)(x)|_r^q dx \right)^{1/q} \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) |Q|^{-1/q} \left(\int_{\tilde{Q}} |f(x)|_r^s dx \right)^{1/s}
\end{aligned}$$

$$\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).$$

For I_2 , denoting $s = pq$ for $1 < p < n/\delta$, $q > 1$, $1/q + 1/q' = 1$ and $1/u = 1/p - \delta/n$, we get, by Hölder's inequality,

$$\begin{aligned} I_2 &\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|} \int_Q |T(D^{\alpha_1} \tilde{A}_1 g)(x)|_r dx \\ &\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} \tilde{A}_1 g)(x)|_r^u dx \right)^{1/u} \\ &\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1|=m_1} |Q|^{-1/u} \left(\int_{R^n} (|D^{\alpha_1} \tilde{A}_1(x)| |g(x)|_r)^p dx \right)^{1/p} \\ &\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{\tilde{Q}} |D^{\alpha_1} \tilde{A}_1(x)|^{pq'} dx \right)^{1/pq'} \\ &\quad \times \left(\frac{1}{|Q|^{1-s\delta/n}} \int_{\tilde{Q}} |f(x)|_r^{pq} dx \right)^{1/pq} \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^{\alpha} A_j\|_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}). \end{aligned}$$

For I_3 , similar to the proof of I_2 , we get

$$I_3 \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^{\alpha} A_j\|_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).$$

Similarly, for I_4 , denoting $s = pq_3$ for $1 < p < n/\delta$, $q_1, q_2, q_3 > 1$, $1/q_1 + 1/q_2 + 1/q_3 = 1$ and $1/u = 1/p - \delta/n$, we obtain

$$\begin{aligned} I_4 &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|} \int_Q |T(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 g)(x)|_r dx \\ &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 g)(x)|_r^u dx \right)^{1/u} \\ &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1/t} \left(\int_{R^n} (|D^{\alpha_1} \tilde{A}_1(x) D^{\alpha_2} \tilde{A}_2(x)| |g(x)|_r)^p dx \right)^{1/p} \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{\tilde{Q}} |D^{\alpha_1} \tilde{A}_1(x)|^{pq_1} dx \right)^{1/pq_1} \\
&\quad \left(\frac{1}{|Q|} \int_{\tilde{Q}} |D^{\alpha_2} \tilde{A}_2(x)|^{pq_2} dx \right)^{1/pq_2} \times \left(\frac{1}{|Q|^{1-s\delta/n}} \int_{\tilde{Q}} |f(x)|_r^{pq_3} dx \right)^{1/pq_3} \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^{\alpha} A_j\|_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).
\end{aligned}$$

For I_5 , we write

$$\begin{aligned}
&F_t^{\tilde{A}}(h_i)(x) - F_t^{\tilde{A}}(h_i)(x_0) \\
&= \int_{R^n} \left(\frac{F_t(x, y)}{|x-y|^m} - \frac{F_t(x_0, y)}{|x_0-y|^m} \right) \prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y) h_i(y) dy \\
&+ \int_{R^n} (R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; x_0, y)) \frac{R_{m_2}(\tilde{A}_2; x, y)}{|x_0-y|^m} F_t(x_0, y) h_i(y) dy \\
&+ \int_{R^n} (R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y)) \frac{R_{m_1}(\tilde{A}_1; x_0, y)}{|x_0-y|^m} F_t(x_0, y) h_i(y) dy \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left[\frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} F_t(x, y) \right. \\
&\quad \left. - \frac{R_{m_2}(\tilde{A}_2; x_0, y)(x_0-y)^{\alpha_1}}{|x_0-y|^m} F_t(x_0, y) \right] \times D^{\alpha_1} \tilde{A}_1(y) h_i(y) dy \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[\frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} F_t(x, y) \right. \\
&\quad \left. - \frac{R_{m_1}(\tilde{A}_1; x_0, y)(x_0-y)^{\alpha_2}}{|x_0-y|^m} F_t(x_0, y) \right] \times D^{\alpha_2} \tilde{A}_2(y) h_i(y) dy \\
&+ \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} F_t(x, y) - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} F_t(x_0, y) \right] \\
&\quad \times D^{\alpha_1} \tilde{A}_1(y) D^{\alpha_2} \tilde{A}_2(y) h_i(y) dy \\
&= I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)}.
\end{aligned}$$

By Lemma 1 and the following inequality (see [19])

$$|b_{Q_1} - b_{Q_2}| \leq C \log(|Q_2|/|Q_1|) \|b\|_{BMO} \text{ for } Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$\begin{aligned} |R_m(\tilde{A}; x, y)| &\leq C|x-y|^m \sum_{|\alpha|=m} (||D^\alpha A||_{BMO} + |(D^\alpha A)_{\tilde{Q}(x,y)} - (D^\alpha A)_{\tilde{Q}}|) \\ &\leq Ck|x-y|^m \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}. \end{aligned}$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the conditions on F_t ,

$$\begin{aligned} ||I_5^{(1)}|| &\leq C \int_{R^n} \left(\frac{|x-x_0|}{|x_0-y|^{m+n+1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon-\delta}} \right) \\ &\quad \times \prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y) |h_i(y)| dy \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) \\ &\quad \times \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} k^2 \left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta}} \right) |f_i(y)| dy \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) \\ &\quad \times \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-\varepsilon k}) \frac{1}{|2^k\tilde{Q}|^{1-\delta/n}} \int_{2^k\tilde{Q}} |f_i(y)| dy, \end{aligned}$$

thus, by Minkowski' inequality,

$$\begin{aligned} \left(\sum_{i=1}^{\infty} ||I_5^{(1)}||^r \right)^{1/r} &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) \\ &\quad \times \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-\varepsilon k}) \frac{1}{|2^k\tilde{Q}|^{1-\delta/n}} \int_{2^k\tilde{Q}} |f(y)|_r dy \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) M_{\delta,1}(|f|_r)(\tilde{x}). \end{aligned}$$

For $I_5^{(2)}$, by the formula (see [4]):

$$R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y) = \sum_{|\beta| < m} \frac{1}{\beta!} R_{m-|\beta|}(D^\beta \tilde{A}; x, x_0)(x - y)^\beta$$

and Lemma 1, we have

$$|R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y)| \leq C \sum_{|\beta| < m} \sum_{|\alpha|=m} |x - x_0|^{m-|\beta|} |x - y|^{|\beta|} \|D^\alpha A\|_{BMO},$$

thus

$$\begin{aligned} & \left(\sum_{i=1}^{\infty} \|I_5^{(2)}\|^r \right)^{1/r} \\ & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} k \frac{|x - x_0|}{|x_0 - y|^{n+1-\delta}} |f(y)|_r dy \\ & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,1}(|f|_r)(\tilde{x}). \end{aligned}$$

Similarly,

$$\left(\sum_{i=1}^{\infty} \|I_5^{(3)}\|^r \right)^{1/r} \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,1}(|f|_r)(\tilde{x}).$$

For $I_5^{(4)}$, we get

$$\begin{aligned} & \left(\sum_{i=1}^{\infty} \|I_5^{(4)}\|^r \right)^{1/r} \\ & \leq C \sum_{|\alpha_1|=m_1} \int_{R^n \setminus \tilde{Q}} \left\| \frac{(x - y)^{\alpha_1} F_t(x, y)}{|x - y|^m} - \frac{(x_0 - y)^{\alpha_1} F_t(x_0, y)}{|x_0 - y|^m} \right\| \\ & \quad \times |R_{m_2}(\tilde{A}_2; x, y)| |D^{\alpha_1} \tilde{A}_1(y)| |h(y)|_r dy \\ & \quad + C \sum_{|\alpha_1|=m_1} \int_{R^n \setminus \tilde{Q}} |R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y)| \\ & \quad \times \frac{|(x_0 - y)^{\alpha_1}| |F_t(x_0, y)|}{|x_0 - y|^m} |D^{\alpha_1} \tilde{A}_1(y)| |h(y)|_r dy \\ & \leq C \sum_{|\alpha|=m_2} \|D^\alpha A_2\|_{BMO} \sum_{|\alpha_1|=m_1} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-\varepsilon k}) \end{aligned}$$

$$\begin{aligned}
& \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_1} \tilde{A}_1(y)|^{s'} dy \right)^{1/s'} \left(\frac{1}{|2^k \tilde{Q}|^{1-s\delta/n}} \int_{2^k \tilde{Q}} |f(y)|_r^s dy \right)^{1/s} \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).
\end{aligned}$$

Similarly,

$$\left(\sum_{i=1}^{\infty} \|I_5^{(5)}\|^r \right)^{1/r} \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).$$

For $I_5^{(6)}$, taking $q_1, q_2 > 1$ such that $1/s + 1/q_1 + 1/q_2 = 1$, then

$$\begin{aligned}
& \left(\sum_{i=1}^{\infty} \|I_5^{(6)}\|^r \right)^{1/r} \\
& \leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{R^n \setminus \tilde{Q}} \left\| \frac{(x-y)^{\alpha_1+\alpha_2} F_t(x,y)}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1+\alpha_2} F_t(x_0,y)}{|x_0-y|^m} \right\| \\
& \quad \times |D^{\alpha_1} \tilde{A}_1(y)| |D^{\alpha_2} \tilde{A}_2(y)| |f(y)|_r dy \\
& \leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-\varepsilon k}) \left(\frac{1}{|2^k \tilde{Q}|^{1-s\delta/n}} \int_{2^k \tilde{Q}} |f(y)|_r^s dy \right)^{1/s} \\
& \quad \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_1} \tilde{A}_1(y)|^{q_1} dy \right)^{1/q_1} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_2} \tilde{A}_2(y)|^{q_2} dy \right)^{1/q_2} \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).
\end{aligned}$$

Thus

$$I_5 \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).$$

This completes the proof of Theorem 1. $\square$

Proof of Theorem 2. We choose $1 < s < p$ in Theorem 1 and by using Lemma 2, we get

$$\|T^A(f)\|_{L^q} \leq C \|(T^A(f))^\#\|_{L^q} \leq C \prod_{j=1}^l \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) \|M_{\delta,s}(|f|_r)\|_{L^q}$$

$$\leq C \prod_{j=1}^l \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) \|f\|_{L^p}.$$

This finishes the proof. $\square$

4. APPLICATIONS

Now we give some applications of the Theorems:

Application 1. Littlewood-Paley operators.

Fixed $0 \leq \delta < n$, $\varepsilon > 0$ and $\mu > (3n + 2 - 2\delta)/n$. Let ψ be a fixed function which satisfies the following properties:

- (1) $\int_{\mathbb{R}^n} \psi(x) dx = 0$,
- (2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1-\delta)}$,
- (3) $|\psi(x+y) - \psi(x)| \leq C|y|^\varepsilon(1 + |x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$.

We denote $\Gamma(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley multilinear operators are defined by

$$g_\psi^A(f)(x) := \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

$$S_\psi^A(f)(x) := \left[\int \int_{\Gamma(x)} |F_t^A(f)(x, y)|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2}$$

and

$$g_\mu^A(f)(x) := \left[\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} |F_t^A(f)(x, y)|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2},$$

where

$$F_t^A(f)(x) := \int_{\mathbb{R}^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, y)}{|x - y|^m} \psi_t(x - y) f(y) dy,$$

$$F_t^A(f)(x, y) := \int_{\mathbb{R}^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, z)}{|x - z|^m} f(z) \psi_t(y - z) dz$$

and $\psi_t(x) := t^{-n+\delta} \psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define

$$g_\psi(f)(x) := \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

$$S_\psi(f)(x) := \left(\int \int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}$$

and

$$g_\mu(f)(x) := \left(\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} |F_t(f)(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2},$$

which are the Littlewood-Paley operators (see[20]). Let H be the space

$$H := \left\{ h : \|h\| = \left(\int_0^\infty |h(t)|^2 dt/t \right)^{1/2} < \infty \right\}$$

or

$$H := \left\{ h : \|h\| = \left(\int \int_{R_+^{n+1}} |h(y,t)|^2 dy dt / t^{n+1} \right)^{1/2} < \infty \right\},$$

then, for each fixed $x \in R^n$, $F_t^A(f)(x)$ and $F_t^A(f)(x, y)$ may be viewed as the mapping from $[0, +\infty)$ to H , and it is clear that

$$g_\psi^A(f)(x) = \|F_t^A(f)(x)\|, \quad g_\psi(f)(x) = \|F_t(f)(x)\|,$$

$$S_\psi^A(f)(x) = \left\| \chi_{\Gamma(x)} F_t^A(f)(x, y) \right\|, \quad S_\psi(f)(x) = \left\| \chi_{\Gamma(x)} F_t(f)(y) \right\|$$

and

$$g_\mu^A(f)(x) = \left\| \left(\frac{t}{t + |x - y|} \right)^{n\mu/2} F_t^A(f)(x, y) \right\|,$$

$$g_\mu(f)(x) = \left\| \left(\frac{t}{t + |x - y|} \right)^{n\mu/2} F_t(f)(y) \right\|.$$

It is easily to see that g_ψ , S_ψ and g_μ satisfy the conditions of Theorems 1 and 2 (see [9–11]), thus Theorems 1 and 2 hold for g_ψ^A , S_ψ^A and g_μ^A .

Application 2. Marcinkiewicz operators.

Fixed $0 \leq \delta < n$, Fix $\lambda > \max(1, 2n/(n + 2 - 2\delta))$ and $0 < \gamma \leq 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$. The Marcinkiewicz multilinear operators are defined by

$$\mu_\Omega^A(f)(x) := \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

$$\mu_S^A(f)(x) := \left[\int \int_{\Gamma(x)} |F_t^A(f)(x, y)|^2 \frac{dy dt}{t^{n+3}} \right]^{1/2}$$

and

$$\mu_\lambda^A(f)(x) := \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\lambda} |F_t^A(f)(x, y)|^2 \frac{dy dt}{t^{n+3}} \right]^{1/2},$$

where

$$F_t^A(f)(x) := \int_{|x-y| \leq t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y) dy$$

and

$$F_t^A(f)(x, y) := \int_{|y-z| \leq t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; y, z)}{|y-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-\delta-1}} f(z) dz.$$

Set

$$F_t(f)(x) := \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y) dy.$$

We also define

$$\begin{aligned} \mu_\Omega(f)(x) &:= \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2}, \\ \mu_S(f)(x) &:= \left(\int \int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dy dt}{t^{n+3}} \right)^{1/2} \end{aligned}$$

and

$$\mu_\lambda(f)(x) := \left(\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} |F_t(f)(y)|^2 \frac{dy dt}{t^{n+3}} \right)^{1/2},$$

which are the Marcinkiewicz operators (see[21]). Let H be the space

$$H := \left\{ h : \|h\| = \left(\int_0^\infty |h(t)|^2 dt / t^3 \right)^{1/2} < \infty \right\}$$

or

$$H := \left\{ h : \|h\| = \left(\int \int_{\mathbb{R}_+^{n+1}} |h(y, t)|^2 dy dt / t^{n+3} \right)^{1/2} < \infty \right\}.$$

Then, it is clear that

$$\mu_\Omega^A(f)(x) = \|F_t^A(f)(x)\|, \quad \mu_\Omega(f)(x) = \|F_t(f)(x)\|,$$

$$\mu_S^A(f)(x) = \left\| \chi_{\Gamma(x)} F_t^A(f)(x, y) \right\|, \quad \mu_S(f)(x) = \left\| \chi_{\Gamma(x)} F_t(f)(y) \right\|$$

and

$$\begin{aligned} \mu_\lambda^A(f)(x) &= \left\| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} F_t^A(f)(x, y) \right\|, \\ \mu_\lambda(f)(x) &= \left\| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} F_t(f)(y) \right\|. \end{aligned}$$

It is easily to see that μ_Ω , μ_S and μ_λ satisfy the conditions of Theorems 1 and 2 (see [12, 21]), thus Theorems 1 and 2 hold for μ_Ω^A , μ_S^A and μ_λ^A .

Application 3. Bochner-Riesz operator .

Let $\delta > (n-1)/2$, $B_t^\delta(f)(\xi) = (1-t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n} B^\delta(z/t)$ for $t > 0$. Set

$$F_{\delta,t}^A(f)(x) := \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, y)}{|x-y|^m} B_t^\delta(x-y) f(y) dy,$$

The maximal Bochner-Riesz multilinear operator are defined by

$$B_{\delta,*}^A(f)(x) := \sup_{t>0} |B_{\delta,t}^A(f)(x)|.$$

We also define

$$B_{\delta,*}(f)(x) := \sup_{t>0} |B_t^\delta(f)(x)|,$$

which is the maximal Bochner-Riesz operator (see [13]). Let H be the space $H := \{h : \|h\| = \sup_{t>0} |h(t)| < \infty\}$, then

$$B_{\delta,*}^A(f)(x) = \|B_{\delta,t}^A(f)(x)\|, \quad B_{\delta,*}(f)(x) = \|B_t^\delta(f)(x)\|.$$

It is easily to see that $B_{\delta,*}^A$ satisfies the conditions of Theorems 1 and 2 (see [22]), thus Theorems 1 and 2 hold for $B_{\delta,*}^A$.

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