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# REGULARITY AND ENTROPY SOLUTIONS OF SOME ELLIPTIC EQUATIONS

M. ALIZADEH AND M. ALIMOHAMMADY

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*Abstract.* Here, we would study regularity of solutions and existence at least one entropy solution for $L^1$-data and duality solution. This result would improve some results of Laplacian differential equations.

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*Keywords:* Fourier transformation, regularity, duality solutions, entropy solutions

## 1. Introduction

First, we summarize some result notions of Schwratz space and tempered distributions. The schwartz space $\mathcal{S}(\mathbb{R}^n)$ is a topological vector space of all $f : \mathbb{R}^n \to \mathbb{C}$ such that $f \in C^\infty(\mathbb{R}^n)$ and $x^\alpha \partial^\beta f(x)$ is bounded. For every pair of multi-induces $\alpha, \beta \in \mathbb{N}^n$, we set

$$|||f|||_{\alpha,\beta} := \sup_x |x^\alpha \partial^\beta f|$$

which induces a family of semi-norms on $\mathcal{S}(\mathbb{R}^n)$. A tempered distribution is a continuous linear functional $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ and $\mathcal{S}'(\mathbb{R}^n)$ is the space of all tempered distributions. $\mathcal{D}(\mathbb{R}^n)$ is the space $C_c^\infty(\mathbb{R}^n)$ endowed with the topology in which $f_n \to 0$ means that, there is a compact set $K$; such that $Supp f_n \subseteq K$ $(n = 1, 2, ...)$ and for each $\alpha \in \mathbb{N}^n$, $D^\alpha f_n \to 0$ uniformly.
The fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ is the function $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\widehat{f}(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{-ikx} dx.$$

It is well known that
1)$\widehat{\ }$: $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous one to one.
2) $\widehat{\partial^\alpha f}(k) = (ik)^\alpha \widehat{f}(k)$.
3)$\widehat{(-ix)^\beta f}(k) = \partial^\beta \widehat{f}(k)$.
For regularity of Laplacian, Ma and Thompson [8], Ma [7] proved regularity, where

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$f \in C[0,1]$. Moreover, Lee and Sim [6] proved it but for $f \in L^1(0,1)$. Recently, interior regularity have studied by many mathematician:

M. Cozzi [4] studied regularity theory of weak solutions for the second order linear elliptic differential equations $-div(A(.)\nabla u) = f$ in $\Omega$, where $\Omega$ is an open bounded subset of $\mathbb{R}^n$ and $A = [a_{ij}]$ is $n \times n$ matrix uniformly elliptic, $a_{ij} \in C^{0,1}_{loc}(\Omega)$ and $f \in L^2(\Omega)$. In fact, it is proved that for any $\Omega' \subset\subset \Omega$, $||u||_{H^2(\Omega)} \leq C(||u||_{L^2(\Omega)} + ||f||_{L^2(\Omega)})$. Moreover, it was shown the interior $H^{2s-\epsilon}$ regularity for weak solutions of some linear elliptic differential equations.

J. Siljander, J. M. Urbano [11] studied the Serrin-type interior regulatity result.

$$u \in L^{2+\epsilon}_{loc}(\Omega_T) \Rightarrow \text{regularity}$$

for a weak solution in the energy space $L^\infty_t L^2_x$ satisfying in appropriate vorticity estimates for

$$\partial_t u + (u.\nabla)u + \nabla p = 0 \text{ and } \operatorname{div} u = 0.$$

S. Gustafson and co authors [5] gave an interior regularity criteria for suitable weak solutions of the 3D Navier-Stokes equations. In fact they considered the regularity problem for a suitable weak solution $(u,p) : \Omega \times I \to \mathbb{R}^3 \times \mathbb{R}$ of three-dimensional incompressible Navier-Stokes equations (NS)

$$\begin{cases} u_t - \Delta u + (u.\nabla)u + \nabla p = f & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \times I \end{cases}$$

and proved $u \in L^\infty(Q_{z,r})$ for some $B_{x,r} \times (t - r^2, t) = Q_{z,r} \subseteq \Omega \times I$, $r > 0$.

These brand of problems have potential applications to the modeling of combustion, thermal explosions, nonlinear heat generation, gravitational equilibrium of polytropic stars, glaciology, non-Newtonian fluids, and the flow through porous media.

In this paper, we would study the regular property of

$$-\Delta_\alpha u + \lambda u = f, \tag{1.1}$$

in $\mathcal{D}'(\Omega)$, where we define $\Delta_\alpha := \alpha_1 \frac{\partial^2}{\partial x_1^2} + ... + \alpha_n \frac{\partial^2}{\partial x_n^2}$ and $\alpha = (\alpha_1, ..., \alpha_n)$, $(\alpha_i > 0, \forall i)$, our aim is to show $u \in C^\infty(\mathbb{R}^n)$.

## 2. Results

Here, we use of $\approx$ for equivalent norms.

**Lemma 1.** *Suppose that $m \in \mathbb{Z}$, $\lambda > 0$ and let $u, f \in \mathcal{S}'(\mathbb{R}^n)$ satisfy (1.1) as distributions. If $f \in W^{m,2}(\mathbb{R}^n)$, then $u \in W^{m+2,2}(\mathbb{R}^n)$ and there exists a constant $C$ such that $\|u\|_{W^{m+2,2}} \leq C\|f\|_{W^{m,2}}$.*

*Proof.* Taking the Fourier transform of (1.1), we have

$$\mathcal{F}(-\alpha_1 u_{x_1x_1} - ... - \alpha_n u_{x_nx_n} + \lambda u) = \mathcal{F}(f),$$

$$(\alpha_1\xi_1^2+\ldots+\alpha_n\xi_n^2+\lambda)\mathcal{F}(u)=\mathcal{F}(f).$$

From theorem 5.2.3 of [3] for any $m\in\mathbb{Z}$ and $a,b>0$ we note to the following equivalent norms:

$$W^{m,2}(\mathbb{R}^n):=\{u\in\mathcal{S}'(\mathbb{R}^n);\mathcal{F}^{-1}[(a+b|\xi|^2)^{\frac{m}{2}}\mathcal{F}(u)]\in L^2(\mathbb{R}^n)\}$$
$$\|u\|_{W^{m,2}}\approx\|\mathcal{F}^{-1}[(a+b|\xi|^2)^{\frac{m}{2}}\mathcal{F}(u)]\|_{L^2},\quad u\in W^{m,2}(\mathbb{R}^n).$$

By using Parseval theorem

$$\|\mathcal{F}^{-1}[(a+b|\xi|^2)^{\frac{m}{2}}\mathcal{F}(u)]\|_{L^2}=\|(a+b|\xi|^2)^{\frac{m}{2}}\mathcal{F}(u)\|_{L^2}.$$

Thus,

$$\|(a+b|\xi|^2)^{\frac{m}{2}}\mathcal{F}(u)\|_{L^2}\approx\|(\alpha_1\xi_1^2+\ldots+\alpha_n\xi_n^2+\lambda)^{\frac{m}{2}}\mathcal{F}(u)\|_{L^2}\tag{2.1}$$

and

$$(\alpha_1\xi_1^2+\ldots+\alpha_n\xi_n^2+\lambda)^{\frac{m+2}{2}}\mathcal{F}(u)=(\alpha_1\xi_1^2+\ldots+\alpha_n\xi_n^2+\lambda)^{\frac{m}{2}}\mathcal{F}(f)$$

so the result follows from (2.1).

□

We now consider the case of a general domain $\Omega$

**Theorem 1.** *Suppose that $\lambda\in\mathbb{R}$ and $u,f\in\mathcal{D}'(\Omega)$ satisfy the equation (1.1) in $\mathcal{D}'(\Omega)$.*
*(i) If $f\in W^{m,2}_{loc}(\Omega)$ and $u\in W^{n,2}_{loc}(\Omega)$ for some $m\geq 0$ and $n\in\mathbb{Z}$, then $u\in W^{m+2,2}_{loc}(\Omega)$ and for every $\Omega_2\subset\subset\Omega_1\subset\subset\Omega$, there exists a constant $C$ (depending only on $m$, $\Omega_2$ and $\Omega_1$) such that $\|u\|_{W^{m+2,2}(\Omega_2)}\leq C(\|f\|_{W^{m,2}(\Omega_1)}+\|u\|_{W^{n,2}(\Omega_1)})$.*
*(ii) If $f\in C^\infty(\Omega)$ and $u\in W^{n,2}_{loc}(\Omega)$ for some $n\in\mathbb{Z}$ then $u\in C^\infty(\Omega)$.*

*Proof.* We proceed in two steps.
**Step 1:** Consider $M''\subset\subset M'\subset\subset\Omega$ and $k\in\mathbb{Z}$. If $u\in W^{k,2}(M')$ and $f\in W^{k-1,2}(M')$ solve the equation (1.1) in $\mathcal{D}'(\Omega)$, thus, $u\in W^{k+1,2}(M'')$ and there exists $C$ such that $\|u\|_{W^{k+1,2}(M'')}\leq C(\|f\|_{W^{k-1,2}(M')}+\|u\|_{W^{k,2}(M')})$. To show this, consider $\rho\in C_c^\infty(\mathbb{R}^n)$ such that $\rho\equiv 1$ on $M''$ and supp $\rho\subset M'$ and define $v\in\mathcal{D}'(\mathbb{R}^n)$ by $v=\rho u$, i.e.

$$(v,\varphi)_{D'(\mathbb{R}^n),D(\mathbb{R}^n)}=(u,\rho\varphi)_{D'(M'),D(M')}.$$

Clearly $v\in W^{k,2}(\mathbb{R}^n)$ and $\|v\|_{W^{k,2}(\mathbb{R}^n)}\leq C\|u\|_{W^{k,2}(M')}$.
$v$ solves the equation

$$-\Delta_\alpha v+v=T_1+T_2+T_3\tag{2.2}$$

in $\mathcal{D}'(\mathbb{R}^n)$, where the distributions $T_1,T_2$, and $T_3$ are defined by

$$(T_1,\varphi)_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)}=(f+(1-\lambda)u,\rho\varphi)_{\mathcal{D}'(M'),\mathcal{D}(M')},$$
$$(T_2,\varphi)_{D'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)}=-(u,\Delta_\alpha\rho.\varphi)_{\mathcal{D}'(M'),\mathcal{D}(M')},$$

$$(T_3,\varphi)_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)} = -2(u,(\alpha.\nabla\rho)\varphi)_{\mathcal{D}'(M'),\mathcal{D}(M')},$$

for every $\varphi \in C_c^\infty(\mathbb{R}^n)$, since

$$\begin{aligned}(-\Delta_\alpha v + v,\varphi)_{D'(\mathbb{R}^n),D(\mathbb{R}^n)} &= (-\alpha_1 v_{x_1x_1} - \ldots - \alpha_n v_{x_nx_n} + v,\varphi)_{D'(\mathbb{R}^n),D(\mathbb{R}^n)}\\ = -(\alpha_1 &\left[(u,\rho_{x_1x_1}\varphi)_{D'(M'),D(M')} + 2(u,\rho_{x_1}\varphi)_{D'(M'),D(M')} + (u_{x_1x_1},\rho\varphi)_{D'(M'),D(M')}\right]\\ +\ldots + \alpha_n &\left[(u,\rho_{x_nx_n}\varphi)_{D'(M'),D(M')} + 2(u,\rho_{x_n}\varphi)_{D'(M'),D(M')} + (u_{x_nx_n},\rho\varphi)_{D'(M'),D(M')}\right])\\ +(u,\rho\varphi)_{D'(M'),D(M')} &= (-\alpha_1 u_{x_1x_1} - \ldots - \alpha_n u_{x_nx_n} + \lambda u,\rho\varphi)_{D'(M'),D(M')}\\ -(u,(\alpha_1\rho_{x_1x_1} + \ldots + \alpha_n\rho_{x_nx_n})\varphi)_{D'(M'),D(M')} &- 2(u,(\alpha_1\rho_{x_1} + \ldots + \alpha_n\rho_{x_n})\varphi)_{D'(M'),D(M')}.\end{aligned}$$

Thus $T_j \in W^{k-1,2}(\mathbb{R}^n)$ and

$$\|T_j\|_{W^{k-1,2}(\mathbb{R}^n)} \le C(\|f\|_{W^{k-1,2}(M')} + \|u\|_{W^{k,2}(M')}),$$

for $j = 1,2,3$. Applying (2.1) and lemma 1, we deduce $v \in W^{k+1,2}(\mathbb{R}^n)$ and $\|v\|_{W^{k+1,2}(\mathbb{R}^n)} \le C(\|f\|_{W^{k-1,2}(M')} + \|u\|_{W^{k,2}(M')})$.

**Step 2:** (Conclusion) Without loss of generality, we may assume $n = -\ell \le 0$. Let $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega$. Consider a family $(M_j)_{0\le j\le m+\ell+1}$ of open subsets of $\Omega$, such that

$$\Omega_2 = M_{m+\ell+1} \subset\subset \ldots \subset\subset M_0 \subset\subset \Omega_1$$

(one constructs easily such a family). It follows from Step 1 that $u \in W^{-\ell+1,2}(M_0)$ and can

$$\begin{aligned}\|u\|_{W^{-\ell+1,2}(M_0)} &\le C(\|f\|_{W^{-\ell-1,2}(\Omega_1)} + \|u\|_{W^{-\ell,2}(\Omega_1)})\\ &\le C(\|f\|_{W^{m,2}(\Omega_1)} + \|u\|_{W^{n,2}(\Omega_1)}).\end{aligned} \tag{2.3}$$

(2.3) and lemma 1 imply that $u \in W^{-\ell+2,2}(M_1)$ and

$$\begin{aligned}\|u\|_{W^{-\ell+2,2}(M_1)} &\le C(\|f\|_{W^{-\ell,2}(M_0)} + \|u\|_{W^{-\ell+1,2}(M_0)})\\ &\le C(\|f\|_{W^{m,2}(\Omega_1)} + \|u\|_{W^{n,2}(\Omega_1)}).\end{aligned} \tag{2.4}$$

Iterating the above argument, $u \in W^{m+2,2}(M_{m+\ell+1}) = W^{m+2,2}(\Omega_2)$ and that there exists $C$ in which

$$\|u\|_{W^{m+2,2}(\Omega_2)} \le C(\|f\|_{W^{m,2}(\Omega_1)} + \|u\|_{W^{n,2}(\Omega_1)}).$$

Hence, property (i) satisfies since $\Omega_1$ and $\Omega_2$ are arbitrary. Property (ii) follows from Property (i) and $C^\infty(\Omega) = \bigcap_{m\ge 0} W^{m,2}_{loc}(\Omega)$. □

Before paying to entropy solutions, we remember some notions:

Let $f, g$ and $q$ be functions in $L^\infty(\Omega)$, $u$ and $v$ be the solutions of

$$\begin{cases} -div(A(x)\nabla u) + q(x)u = f & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega\end{cases} \tag{2.5}$$

and

$$\begin{cases} -div(A^*(x)\nabla v) + q(x)v = g & \text{in } \Omega\\ v = 0 & \text{on } \partial\Omega\end{cases} \tag{2.6}$$

respectively, where $A^*$ is the transposed matrix of $A$ and

$$A(x)y.y \geq \alpha|y|^2, |A(x)| \leq \beta \tag{2.7}$$

for every $y \in \mathbb{R}^n$, $0 < \alpha \leq \beta$. It is well known that (2.5) has a weak unique solution (Theorem 1.6.1 [1]). Since both $u$ and $v$ belong to $H_0^1(\Omega)$, $u$ can be chosen as test function in the formulation of weak solution for $v$ and vice versa. One obtains

$$\int fv = \int A(x)\nabla u.\nabla v + \int q(x)u.v = \int A^*(x)\nabla v.\nabla u + \int q(x)v.u = \int ug$$

for every $f, g \in L^\infty(\Omega)$, where $u$ and $v$ solve the corresponding problems with data $f$ and $g$ respectively. $u, v \in L^\infty(\Omega)$ (Theorem 2.3 [9]), but we remark that the two integrals are well-defined also if $f \in L^1(\Omega)$ and $u \in L^1(\Omega)$ (always maintaining the assumption that $g$ and so $v$ is a bounded function). This fact inspired to Guido Stampacchia the following definition of solution for (2.5) if the datum is in $L^1(\Omega)$.

**Definition 1.** Suppose that $f \in L^1(\Omega)$. A function $u \in L^1(\Omega)$ is called a *duality solution* with datum $f$ if one has $\int ug = \int fv$, for every $g \in L^\infty(\Omega)$, where $v$ is the solution of

$$\begin{cases} -div(A^*(x)\nabla v) + q(x)v = g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

**Theorem 2** (Stampacchia, theorem 3.3 of [9])**.** *For $f \in L^1(\Omega)$ there exists a unique duality solution with datum $f$. Furthermore, $u \in L^q(\Omega)$ for every $q < \frac{N}{N-2}$.*

*Remark* 1. In special case if

$$A = \begin{bmatrix} \alpha_1 & 0 & 0 & . & . & . & 0 \\ 0 & \alpha_2 & 0 & . & . & . & 0 \\ . & & & & & & . \\ . & & & & & & . \\ . & & & & & & . \\ 0 & 0 & . & . & . & 0 & \alpha_n \end{bmatrix}$$

and $q(x) = \lambda$, problems (2.5) and (2.6) change to

$$\begin{cases} -\Delta_\alpha u + \lambda u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta_\alpha v + \lambda v = g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

Respectively.

**Definition 2.** For $k > 0$, set

$$T_k(s) := \max\{-k, \min\{s, k\}\}$$

and

$$\tau_0^{1,2} = \{u : \Omega \longrightarrow \mathbb{R} \, measurable : T_k(u) \in H_0^1(\Omega), \forall k > 0\},$$

It is well known that $\nabla T_k(u) = \nabla u \chi_{\{|u| \leq k\}}$ .

**Lemma 2.** *Suppose that $u \in \tau_0^{1,2}(\Omega)$, $q \in L^\infty(\Omega)$ and (2.7) valid. Then there is $\alpha > 0$ in which*

$$\alpha \int_\Omega |\nabla T_k(u)|^2 \leq \int_\Omega A(x)\nabla u.\nabla T_k(u) + \int_\Omega q(x)uT_k(u).$$

*Proof.* By (2.7)

$$\begin{aligned}\int_\Omega A(x)\nabla u.\nabla T_k(u) + \int_\Omega q(x)uT_k(u) &\geq \gamma \int_\Omega |\nabla u|^2 + \int_\Omega q(x)uT_k(u)\\ &\geq \gamma \int_\Omega |\nabla u|^2 - \int_\Omega |q(x)||u||T_k(u)|,\end{aligned}$$

since $|T_k(u)| \leq |u|$, $|q(x)| \leq b$ (almost every where) for a suitable $b > 0$ and from Poincaré inequality

$$\begin{aligned}\gamma \int_\Omega |\nabla u|^2 - \int_\Omega |q(x)||u||T_k(u)| &\geq \gamma \int_\Omega |\nabla u|^2 - \int_\Omega |q(x)||u|^2\\ &\geq \gamma \int_\Omega |\nabla u|^2 - bC_q \int_\Omega |\nabla u|^2 = \alpha \int_\Omega |\nabla T_k(u)|^2,\end{aligned}$$

for $\alpha := \gamma - bC_q$, where $C_q$ is multiplier in Poincaré inequality. □

**Definition 3.** Suposse that $f \in L^1(\Omega)$. A function $u \in \tau_0^{1,2}(\Omega)$ is called an enteropy solution of (2.5) if

$$\int_\Omega A(x)\nabla u.\nabla T_k(u-\varphi) + \int_\Omega q(x)uT_k(u-\varphi) \leq \int_\Omega fT_k(u-\varphi), \tag{2.8}$$

for every $k > 0$ and for every $\varphi$ in $H_0^1(\Omega) \cap L^\infty(\Omega)$.

**Theorem 3.** *Suppose that $f \in L^1(\Omega)$. Then there exists an entropy solution $u$ for (2.5).*

*Proof.* We do by approximation; Suppose that $f_n = T_n(f)$ and by the Lax-Miligram theorem, there exists a weak solution $u_n$ for

$$\begin{cases} -div(A(x)\nabla u_n) + q(x)u_n = f_n & \text{in } \Omega\\ u_n = 0 & \text{on } \partial\Omega.\end{cases}$$

Let $k > 0$. Taking $T_k(u_n)$ as test function and using of lemma 2,

$$\begin{aligned}\alpha \int_\Omega |\nabla T_k(u_n)|^2 &\leq \int_\Omega A(x)\nabla u_n.\nabla T_k(u_n) + \int_\Omega q(x)u_nT_k(u_n)\\ &= \int_\Omega f_nT_k(u_n) \leq k||f||_{L^1(\Omega)}.\end{aligned}$$

Therefore, $(T_k(u_n))_n$ is bounded in $H_0^1(\Omega)$ for a fixed $k$. This implies that there exists a function $v_k \in H_0^1(\Omega)$ such that, up to subsequences $T_k(u_n)$ converges to $v_k$

weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. From lemma 2, one can deduce that

$$\int_\Omega |\nabla(u_n - u_m)|^q \leq C_q \| f_n - f_m \|^q_{L^1(\Omega)}$$

and since $(f_n)_n$ is a Cauchy sequence in $L^1(\Omega)$, so $(u_n)_n$ is a Cauchy sequence in $W_0^{1,q}(\Omega)$ and then $u_n$ converges strongly to a suitable $u \in W_0^{1,q}(\Omega)$. For every $q < \frac{N}{N-1}$, $\nabla u_n$ converges to $\nabla u$ almost everywhere in $\Omega$. Thus, $T_k(u_n)$ converges strongly to $T_k(u)$ in $L^2(\Omega)$, and so $v_k = T_k(u)$. Therefore, by Fatou lemma,

$$\alpha \int_\Omega |\nabla T_k(u_n)|^2 \leq \liminf_{n\to+\infty} \alpha \int_\Omega |\nabla T_k(u_n)|^2 \leq k||f||_{L^1(\Omega)},$$

which implies that $u$ belongs to $\tau_0^{1,2}$. Fix $k > 0$, $\varphi$ in $H_0^1(\Omega) \cap L^\infty(\Omega)$, and $v := T_k(u_n - \varphi)$ as test function in the weak formulation of (2.5). Then

$$\int_\Omega A(x)\nabla u_n . \nabla T_k(u_n - \varphi) + \int_\Omega q(x) u_n T_k(u_n - \varphi) = \int_\Omega f_n T_k(u_n - \varphi).$$

For the right hand side we have $T_n \longrightarrow I$ as $n \longrightarrow \infty$ and $f_n = T_n(f) \longrightarrow f$. Thus, $f_n \longrightarrow f$ point wise in $L^1(\Omega)$ and $|f_n T_k(u_n - \varphi)| \leq 2k|f|$. Lebesgue theorem implies that

$$\lim_{n\to+\infty} \int_\Omega f_n T_k(u_n - \varphi) = \int_\Omega f T_k(u - \varphi),$$

while the left hand side can be rewritten as

$$\int_\Omega A(x)\nabla T_k(u_n - \varphi).\nabla T_k(u_n - \varphi) + \int_\Omega A(x)\nabla\varphi . T_k(u_n - \varphi)$$
$$+ \int_\Omega q(x) u_n T_k(u_n - \varphi).$$

The first term is non-negative, thus, the almost everywhere convergence of $\nabla u_n$ to $\nabla u$ follows by Fatou lemma,

$$\int_\Omega A(x)\nabla T_k(u - \varphi).\nabla T_k(u - \varphi) \leq \liminf_{n\to+\infty} (\int_\Omega A(x)\nabla T_k(u_n - \varphi).\nabla T_k(u_n - \varphi).$$

For the second, since $u_n$ converges to $u$ in $H_0^1(\Omega)$ so $u_n - \varphi$ to $u - \varphi$ in $H_0^1(\Omega)$, then $T_k(u_n - \varphi)$ to $T_k(u - \varphi)$ in $H_0^1(\Omega)$ ) and since $-\nabla(A(x)\nabla\varphi) \in H_0^1$

$$< -\nabla(A(x)\nabla\varphi), T_k(u_n - \varphi) > \longrightarrow < -\nabla(A(x)\nabla\varphi), T_k(u - \varphi) >$$

i.e.

$$\int_\Omega A(x)\nabla\varphi . T_k(u - \varphi) = \lim_{n\to+\infty} \int_\Omega A(x)\nabla\varphi . T_k(u_n - \varphi).$$

For third term since

$$|q(x) u_n T_k(u_n - \varphi)| \leq 2k \|q\|_\infty |u|$$

by Lebesgue dominated convergence theorem

$$\int_\Omega q(x)uT_k(u-\varphi) = \lim_{n\to+\infty}\int_\Omega q(x)u_nT_k(u_n-\varphi).$$

Then by cancelling equal terms:

$$\int_\Omega A(x)\nabla u.\nabla T_k(u-\varphi) + \int_\Omega q(x)uT_k(u-\varphi) \le \int_\Omega fT_k(u-\varphi),$$

so $u$ is an entropy solution of (2.5). □

**Theorem 4.** *Let $f \in L^1(\Omega)$ and $u$ be an entropy solution of (2.5) with datum $f$. Then $u$ belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$ and it is a distributional solution for (2.5).*

*Proof.* Taking $\varphi = 0$ in (2.8)

$$\alpha\int_\Omega |\nabla T_k(u)|^2 \le \int_\Omega A(x)\nabla u.\nabla T_k(u) + \int_\Omega q(x)uT_k(u) = \int_\Omega fT_k(u) \le k||f||_{L^1(\Omega)}.$$

Proof of Theorem 4.1 in [9] shows that $u \in W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$. We now fix $h > 0$ and choose $\varphi = T_h(u)$ as test function in (2.8). Then

$$\int_\Omega A(x)\nabla u.\nabla T_k(u-T_h(u)) + \int_\Omega q(x)uT_k(u-T_h(u)) \le \int_\Omega fT_k(u-T_h(u)).$$

Moreover,

$$T_k(u-T_h(u)) = \begin{cases} u-T_h(u) & -k \le u-T_h(u) \le k, \\ k & k \le u-T_h(u), \\ -k & u-T_h(u) \le -k, \end{cases}$$

where

$$u-T_h(u) = \begin{cases} 0 & -h \le u \le h, \\ u-h & h \le u, \\ u+h & u \le -h. \end{cases}$$

Therefore, if $|u| \le h$, then $T_k(u-T_h(u)) = 0$. Moreover, if $h-k \le |u| \le h+k$, then $T_k(u-T_h(u)) = u-T_h(u)$. Thus,

$$\int_{\{h-k\le|u|\le h+k\}} A(x)\nabla u.\nabla u + \int_{\{|u|\ge h\}} q(x)uT_k(u-T_h(u))$$
$$= \int_{\{|u|\ge h\}} fT_k(u-T_h(u)) \le k\int_{\{|u|\ge h\}} |f|.$$

Defining $A_h = \{|u| \ge h\}$, $m(A_h) \longrightarrow 0$ as $h \longrightarrow \infty$ (since $u \in W_0^{1,1}(\Omega)$, thus, in $L^1(\Omega)$). From $f \in L^1(\Omega)$,

$$\lim_{h\to+\infty}\int_{\{|u|\ge h\}} |f| = 0,$$

hence by recalling (2.7)

$$\lim_{h\to+\infty}\int_{\{h-k\le|u|\le h+k\}}|\nabla u|^2=0. \tag{2.9}$$

For $h>0$, $\eta$ in $C_0^1(\Omega)$ and $\varphi=T_h(u)-\eta$ as test function in the entropy formulation (2.8), where $k=\|\eta\|_{L^\infty(\Omega)}$ then

$$\int_\Omega A(x)\nabla u.\nabla T_k(u-T_h(u)+\eta)+\int_\Omega q(x)uT_k(u-T_h(u)+\eta)\le\int_\Omega fT_k(u-T_h(u)+\eta).$$

By Lebesgue dominated theorem and choice of $k$

$$\lim_{h\to+\infty}\int_\Omega fT_k(u-T_h(u)+\eta)=\int_\Omega fT_k(\eta)=\int_\Omega f\eta.$$

For the left hand side, using again the choice of $k$

$$\int_{\{|u|\le h\}}(A(x)\nabla u.\nabla T_k(u-T_h(u)+\eta)+\int_{\{|u|\le h\}}q(x)uT_k(u-T_h(u)+\eta))$$
$$+\int_{\{|u|\ge h\}}(A(x)\nabla u.\nabla T_k(u-T_h(u)+\eta)+\int_{\{|u|\ge h\}}q(x)uT_k(u-T_h(u)+\eta)).$$

Since $A$ is bounded, so $u\in W_0^{1,1}(\Omega)$ and $\eta\in C_0^1(\Omega)$. For $\{|u|\le h\}$ we have $T_k(u-T_h(u)+\eta)=T_k(\eta)$. Thus, by Lebesgue dominated theorem

$$\lim_{h\to+\infty}\int_{\{|u|\le h\}}A(x)\nabla u.\nabla T_k(\eta)=\lim_{h\to+\infty}\int_{\{|u|\le h\}}A(x)\nabla u.\nabla\eta=\int_\Omega A(x)\nabla u.\nabla\eta.$$

Similarly

$$\int_{\{|u|\le h\}}q(x)uT_k(u-T_h(u)+\eta)=\int_{\{|u|\le h\}}q(x)uT_k(\eta)=\int_{\{|u|\le h\}}q(x)u\eta.$$

Then

$$\lim_{h\to+\infty}\int_{\{|u|\le h\}}q(x)u\eta=\int_\Omega q(x)u\eta.$$

Since

$$\{|u-T_h(u)+\eta|\le k,|u|\ge h\}\subseteq\{h-2k\le|u|\le h+2k\}$$

by (2.7) and choice of $k$

$$\int\limits_{\{|u|\ge h\}}A(x)\nabla u.\nabla T_k(u-T_h(u)+\eta)\le|\int\limits_{\{|u|\ge h\}}A(x)\nabla u.\nabla T_k(u-T_h(u)+\eta)|$$
$$\le\int\limits_{\{|u|\ge h\}}A(x)|\nabla u|.|\nabla T_k(u-T_h(u)+\eta)|\le\beta\int\limits_{\{h-2k\le|u|\le h+2k\}}|\nabla u|(|\nabla u|+|\nabla\eta|).$$

Thus by (2.9) and Hölder Inequality

$$\beta \int_{\{h-2k\le|u|\le h+2k\}} |\nabla u|(|\nabla u|+|\nabla\eta|) \le \beta \int_{\{h-2k\le|u|\le h+2k\}} (|\nabla u|^2+|\nabla u|.|\nabla\eta|)$$
$$\le \beta[\int_{\{h-2k\le|u|\le h+2k\}} |\nabla u|^2]^{\frac{1}{2}}.[\int_{\{h-2k\le|u|\le h+2k\}} |\nabla\eta|^2]^{\frac{1}{2}} = 0.$$

Therefore,

$$\lim_{h\to+\infty} \int_{\{|u|\ge h\}} (A(x)\nabla u.\nabla T_k(u-T_h(u)+\eta) = 0$$

and

$$|q(x)uT_k(u-T_h(u)+\eta))| \le k\|q\|_\infty|u|$$

so $q(x)uT_k(u-T_h(u)+\eta)) \in L^1(\Omega)$. Thus, $\lim_{h\to+\infty}\int_{\{|u|\ge h\}} k\|q\|_\infty|u| = 0$ and

$$\int_{\{|u|\ge h\}} q(x)uT_k(u-T_h(u)+\eta)) = 0$$

Putting together the results,

$$\int_\Omega (A(x)\nabla u.\nabla\eta + q(x)u\eta) \le \int_\Omega f\eta,$$

for any $\eta \in C_0^1(\Omega)$. Exchanging $\eta$ with $-\eta$ we obtain the reverse inequality so that $u$ is a distributional solution of (2.5). □

Finally, we would show uniqueness of entropy solution.

**Theorem 5.** *Let $f \in L^1(\Omega)$. Then the entropy solution of (2.5) is unique.*

*Proof.* We proceed in three steps.
**Step 1** (An entropy solution is a duality solution): Consider $g$ is in $L^\infty(\Omega)$ and $v$ is a weak solution of

$$\begin{cases} -div(A^*(x)\nabla v)+q(x)v = g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

According to the Stampacchia's theorem [2], $v \in L^\infty(\Omega)$. We repeat the proof of Theorem 4. By choosing $\varphi = T_h(u)-v$ in the entropy formulation, for $h>0$ and $k = \|v\|_{L^\infty(\Omega)}$:

$$\int_\Omega A(x)\nabla u.\nabla T_k(u-T_h(u)+v) + \int_\Omega q(x)uT_k(u-T_h(u)+v) \le \int_\Omega fT_k(u-T_h(u)+v).$$

Similar to theorem 4, from Lebesgue dominated theorem and choose of $k$,

$$\lim_{h\to+\infty}\int_\Omega fT_k(u-T_h(u)+v) = \int_\Omega fv,$$

Moreover, the left hand side can be rewritten as

$$\int_{\{|u|\le h\}}(A(x)\nabla u.\nabla v+q(x)uv)+\int_{\{|u|\ge h\}}A(x)\nabla u.\nabla T_k(u-T_h(u)+v)$$
$$+\int_{\{|u|\ge h\}}q(x)uT_k(u-T_h(u)+v).$$

For the second and the third term, similar to the proof of Theorem 4 (using (2.9))

$$\lim_{h\to+\infty}\int_{\{|u|\ge h\}}A(x)\nabla u.\nabla T_k(u-T_h(u)+v)+\int_{\{|u|\ge h\}}q(x)uT_k(u-T_h(u)+v)=0,$$

and the first term can be rewritten as

$$\int_{\{|u|\le h\}}(A(x)\nabla u.\nabla v+q(x)uv)=\int_{\Omega}(A(x)\nabla T_h(u).\nabla v+q(x)T_h(u)v)$$
$$=\int_{\Omega}(A^*(x)\nabla v.\nabla T_h(u)+q(x)vT_h(u))=\int_{\Omega}gT_h(u),$$

since $T_h(u)\in H_0^1(\Omega)$ can be chosen as test function in the problem solved by $v$. Then, by Lebesgue dominated theorem,

$$\int_{\{|u|\le h\}}(A(x)\nabla u.\nabla v+q(x)uv)=\lim_{h\to+\infty}\int_{\{|u|\le h\}}(A(x)\nabla u.\nabla v+q(x)uv)=\int_{\Omega}gu.$$

Putting together the results, we obtain

$$\int_{\Omega}gu\le\int_{\Omega}fv.$$

Exchanging $g$ with $-g$ (and so $v$ with $-v$, by linearity), we obtain the reverse inequality, therefore, $u$ is a duality solution of (2.5).

**Step 2** (An entropy solution is a solution obtained by approximation (See [10])): Suppose that $f_n\in L^\infty(\Omega)$ be a sequence of functions that converges to $f\in L^1(\Omega)$, and suppose that $u_n$ be the solution of

$$\begin{cases}-div(A^*(x)\nabla u_n)+q(x)u_n=f_n & \text{in } \Omega\\ u_n=0 & \text{on } \partial\Omega\end{cases}$$

$u_n\in H_0^1(\Omega)\cap L^\infty(\Omega)$, so that $\varphi=u_n$ is an admissible choice in the entropy formulation for $u$. Thus,

$$\int_{\Omega}A(x)\nabla u.\nabla T_k(u-u_n)+\int_{\Omega}q(x)uT_k(u-u_n)\le\int_{\Omega}fT_k(u-u_n).$$

On the other hand, $T_k(u-u_n)$ belongs to $H_0^1(\Omega)$ and so it can be chosen as test function in the weak formulation for $u_n$. Then

$$\int_{\Omega}A(x)\nabla u_n.\nabla T_k(u-u_n)+\int_{\Omega}q(x)u_nT_k(u-u_n)=\int_{\Omega}f_nT_k(u-u_n).$$

Thus,

$$\int_\Omega A(x)\nabla(u-u_n).\nabla T_k(u-u_n)+\int_\Omega q(x)(u-u_n)T_k(u-u_n)\leq\int_\Omega(f-f_n)T_k(u-u_n).$$

By (2.7) and lemma 2

$$\alpha\int_\Omega|\nabla T_k(u-u_n)|^2\leq\int_\Omega A(x)\nabla(u-u_n).\nabla T_k(u-u_n)+\int_\Omega q(x)(u-u_n)T_k(u-u_n)$$
$$\leq k\|(f-f_n)\|_{L^1(\Omega)}.$$

approaching $n\longrightarrow\infty$ , $T_k(u-u_n)\longrightarrow 0$ in $H_0^1(\Omega)$ and this implies that $u_n$ converges to the entropy solution $u$. From solutions obtained by approximation are unique, hence, the entropy solution $u$ is unique.

**Step 3** (There exists at most an entropy solution): Here, we follow [2]. Suppose that $u$ and $v$ be two entropy solutions of (2.5), with the same datum $f$, and let $h>k>0$. Then $\varphi=T_h(v)$ is admissible in the entropy formulation for $u$ and $\varphi=T_h(u)$ is admissible in the entropy formulation for $v$. Thus,

$$\int_\Omega A(x)\nabla u.\nabla T_k(u-T_h(v))+\int_\Omega q(x)uT_k(u-T_h(v))\leq\int_\Omega fT_k(u-T_h(v)),$$

and

$$\int_\Omega A(x)\nabla v.\nabla T_k(v-T_h(u))+\int_\Omega q(x)vT_k(v-T_h(u))\leq\int_\Omega fT_k(v-T_h(u)).$$

Summing these two inequalities,

$$\int_\Omega A(x)\nabla u.\nabla T_k(u-T_h(v))+\int_\Omega q(x)uT_k(u-T_h(v))$$
$$+\int_\Omega A(x)\nabla v.\nabla T_k(v-T_h(u))+\int_\Omega q(x)vT_k(v-T_h(u)),$$

in the left hand side is less than or equal to

$$\int_\Omega f(T_k(u-T_h(v))+T_k(v-T_h(u)))$$

in the right hand side. From oddness of $T_k(s)$ and Lebesgue dominated theorem,

$$\lim_{h\to+\infty}\int_\Omega f(T_k(u-T_h(v))+T_k(v-T_h(u)))=0.$$

Hence,

$$\limsup_{h\to+\infty}\int_\Omega A(x)\nabla u.\nabla T_k(u-T_h(v))+\int_\Omega q(x)uT_k(u-T_h(v))$$
$$+\int_\Omega A(x)\nabla v.\nabla T_k(v-T_h(u))+\int_\Omega q(x)uT_k(v-T_h(u))\leq 0.$$

For the sake of simplicity we will suppose from now on that $u \geq 0$ and $v \geq 0$, since the proof turns out to be considerably simplified. We refer to [2] for the proof in the general case of changing sign solutions. We set

$$\Omega = \{u \leq h, v \leq h\} \cup \{u > h, v \leq h\} \cup \{v > h\} = E_0^h \cup F_1^h \cup F_2^h,$$

and

$$\Omega = \{v \leq h, u \leq h\} \cup \{v > h, u \leq h\} \cup \{u > h\} = E_0^h \cup F_3^h \cup F_4^h.$$

Then

$$\begin{aligned}\int_{E_0^h} A(x)\nabla u.\nabla T_k(u - T_h(v)) + \int_{E_0^h} q(x)uT_k(u - T_h(v))\\ = \int_{E_0^h} A(x)\nabla u.\nabla T_k(u - v) + \int_{E_0^h} q(x)uT_k(u - v).\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{E_0^h} A(x)\nabla v.\nabla T_k(v - T_h(u)) + \int_{E_0^h} q(x)vT_k(v - T_h(u))\\ = \int_{E_0^h} A(x)\nabla v.\nabla T_k(v - u) + \int_{E_0^h} q(x)vT_k(v - u),\end{aligned}$$

On $F_1^h$,

$$\int_{F_1^h} A(x)\nabla u.\nabla T_k(u - T_h(v)) = \int_{\{u>h, v\leq h, 0\leq u-v\leq k\}} A(x)\nabla u.\nabla(u - v).$$

On $\{u > h, v \leq h, 0 \leq u - v \leq k\}$ it valid $h < u \leq h + k$ and $h - k < v \leq h$, so

$$\left|\int_{F_1^h} A(x)\nabla u.\nabla T_k(u - T_h(v))\right| \leq \beta \int_{\{h<u\leq h+k, h-k<v\leq h\}} |\nabla u||\nabla v|.$$

By (2.9)

$$\lim_{h\to+\infty} \int_{\{h<u\leq h+k\}} |\nabla u|^2 = 0,$$

and

$$\lim_{h\to+\infty} \int_{\{h-k<v\leq h\}} |\nabla v|^2 = 0,$$

Hence, by Hölder inequality

$$\lim_{h\to+\infty} \left|\int_{F_1^h} A(x)\nabla u.\nabla T_k(u - T_h(v))\right| = 0;$$

and $|q(x)uT_k(u - T_h(v))| \leq k\|q\|_\infty |u|$, so $q(x)uT_k(u - T_h(v)) \in L^1(\Omega)$. Thus,

$$\lim_{h\to+\infty} \int_{F_1^h} k\|q\|_\infty |u| = 0,$$

hence,

$$\int_{F_1^h} q(x)uT_k(u-T_h(v)) = 0.$$

Repeating the same for $F_3^h$ we have

$$\lim_{h\to+\infty}\left|\int_{F_3^h} A(x)\nabla v.\nabla T_k(v-T_h(u))\right| = 0.$$

and

$$\int_{F_3^h} q(x)vT_k(v-T_h(u)) = 0$$

Moreover on $F_2^h$,

$$\begin{aligned}&\int_{F_2^h}(A(x)\nabla u.\nabla T_k(u-T_h(v)) + q(x)uT_k(u-T_h(v)))\\ &\quad= \int_{\{v>h,0\le u<h+k\}}(A(x)\nabla u.\nabla u) + q(x)uu) \ge 0,\end{aligned}$$

and similarly on $F_4^h$,

$$\begin{aligned}&\int_{F_4^h}(A(x)\nabla v.\nabla T_k(v-T_h(u)) + q(x)vT_k(v-T_h(u)))\\ &\quad= \int_{\{u>h,0\le v<h+k\}}(A(x)\nabla v.\nabla v) + q(x)vv) \ge 0,\end{aligned}$$

Putting the results together,

$$\limsup_{h\to+\infty}\int_{E_0^h}(A(x)\nabla(u-v).\nabla T_k(u-v) + q(x)(u-v)T_k(u-v)) \le 0,$$

which, by Fatou lemma, implies, from $E_0^h$ "fills" $\Omega$ as $h \longrightarrow +\infty$,

$$0 \le \int_{\Omega}(A(x)\nabla(u-v).\nabla T_k(u-v) + q(x)(u-v)T_k(u-v)) \le 0,$$

Using (2.7) and lemma 2 we have $\nabla T_k(u-v) \equiv 0$, thus $u = v$. □

*Remark* 2. In special case if

$$A = \begin{bmatrix} \alpha_1 & 0 & 0 & . & . & . & 0 \\ 0 & \alpha_2 & 0 & . & . & . & 0 \\ . & & & & & & . \\ . & & & & & & . \\ . & & & & & & . \\ 0 & 0 & . & . & . & 0 & \alpha_n \end{bmatrix}$$

and $q(x) = \lambda$, problem (2.5) can be rewritten

$$\begin{cases} -div(A(x)\nabla u)+q(x)u=f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases} \Longrightarrow \begin{cases} -\Delta_\alpha u+\lambda u=f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases}$$

Then above theorems are satisfied for this problem.

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# TRACES OF PERMUTING GENERALIZED $N$-DERIVATIONS OF RINGS

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*Abstract.* Let $n \geq 1$ be a fixed positive integer and $R$ be a ring. A permuting $n$-additive map $\Omega : R^n \to R$ is known to be permuting generalized $n$-derivation if there exists a permuting $n$-derivation $\Delta : R^n \to R$ such that $\Omega(x_1, x_2, \cdots, x_i x_i^{'}, \cdots, x_n) = \Omega(x_1, x_2, \cdots, x_i, \cdots, x_n)x_i^{'} + x_i \Delta(x_1, x_2, \cdots, x_i^{'}, \cdots, x_n)$ holds for all $x_i, x_i^{'} \in R$. A mapping $\delta : R \to R$ defined by $\delta(x) = \Delta(x, x, \cdots, x)$ for all $x \in R$ is said to be the trace of $\Delta$. The trace $\omega$ of $\Omega$ can be defined in the similar way. The main result of the present paper states that if $R$ is a $(n+1)!$-torsion free semi-prime ring which admits a permuting $n$-derivation $\Delta$ such that the trace $\delta$ of $\Delta$ satisfies $[[\delta(x), x], x] \in Z(R)$ for all $x \in R$, then $\delta$ is commuting on $R$. Besides other related results it is also shown that in a $n!$-torsion free prime ring if the trace $\omega$ of a permuting generalized $n$-derivation $\Omega$ is centralizing on $R$, then $\omega$ is commuting on $R$.

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## 1. Introduction

Throughout $R$ will denote an associative ring with center $Z(R)$. For any $x, y \in R$, $xy - yx$ denote the commutator $[x, y]$. A ring $R$ is said to be prime (resp. semi-prime) if $aRb = \{0\}$ implies either $a = 0$ or $b = 0$ (resp. $aRa = \{0\}$ implies $a = 0$). Let $m \geq 1$ be a fixed positive integer. A map $f : R \to R$ is said to be centralizing (resp. commuting) on $R$ if $[f(x), x] \in Z(R)$ (resp. $[f(x), x] = 0$) holds for all $x \in R$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Following [4], an additive mapping $F : R \longrightarrow R$ is said to be a generalized derivation on $R$ if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Suppose $n$ is a fixed positive integer and $R^n = R \times R \times \cdots \times R$. A map $\Delta : R^n \to R$ is said to be permuting if the relation $\Delta(x_1, x_2, \cdots, x_n) = \Delta(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)})$ holds for all $x_i \in R$ and for every permutation $\{\pi(1), \pi(2), \cdots, \pi(n)\}$. The concept of derivation and symmetric bi-derivation was generalized by Park [7] as follows: a permuting map $\Delta : R^n \to R$ is said to be a permuting $n$-derivation if $\Delta$ is $n$-additive (i.e.; additive

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in each coordinate) and $\Delta(x_1,x_2,\cdots,x_ix_i^{'},\cdots,x_n) = x_i\Delta(x_1,x_2,\cdots,x_i^{'},\cdots,x_n) + \Delta(x_1,x_2,\cdots,x_i,\cdots,x_n)x_i^{'}$ holds for all $x_i,x_i^{'} \in R$. A 1-derivation is a derivation and a 2-derivation is a symmetric bi-derivation while a 3-derivation is known as permuting tri-derivation.

A well known result due to Posner [8] states that a prime ring $R$ which admits a non-zero centralizing derivation is commutative. In fact, this result initiated the study of centralizing and commuting mappings in rings. Since then, several authors have done a great deal of work concerning commutativity of prime and semi-prime rings admitting different kinds of maps which are centralizing or commuting on some appropriate subsets of $R$ (see [5,6] and [7] for further references). Let $n \geq 2$ be a fixed integer and a map $\delta : R \to R$ defined by $\delta(x) = \Delta(x,x,\cdots,x)$ for all $x \in R$, where $\Delta : R^n \to R$ is a permuting map, be the trace of $\Delta$. Moreover, it can be easily seen that $\Delta(x_1,x_2,\cdots,-x_i,\cdots,x_n) = -\Delta(x_1,x_2,\cdots,x_i,\cdots,x_n)$ for all $x_i \in R, i = 1,2,\cdots,n$.

Motivated by the concept of generalized derivation in ring, we introduce the notion of permuting generalized $n$-derivation in ring. Let $n \geq 1$ be a fixed positive integer. A permuting $n$-additive map $\Omega : R^n \to R$ is known to be permuting generalized $n$-derivation if there exists a permuting $n$-derivation $\Delta : R^n \to R$ such that $\Omega(x_1,x_2,\cdots,x_ix_i^{'},\cdots,x_n) = \Omega(x_1,x_2,\cdots,x_i,\cdots,x_n)x_i^{'} + x_i\Delta(x_1,x_2,\cdots,x_i^{'},\cdots,x_n)$ holds for all $x_i,x_i^{'} \in R$. For an example of permuting generalized $n$-derivation, let $n \geq 1$ be a fixed positive integer and $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a,b,c \in \mathbb{C} \right\}$ where $\mathbb{C}$ is a complex field. Consider permuting $n$-derivation $\Delta$ as above and define $\Omega : R^n \to R$ such that

$$\Omega\left(\begin{pmatrix} 0 & a_1 & b_1 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{pmatrix},\cdots,\begin{pmatrix} 0 & a_n & b_n \\ 0 & 0 & c_n \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & a_1\cdots a_n \\ 0 & 0 & c_1\cdots c_n \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\Omega$ is a permuting generalized $n$-derivation on $R$ associated with a permuting $n$-derivation $\Delta$ on $R$.

Let $\omega : R \to R$ such that $\omega(x) = \Omega(x,x,\cdots,x)$. Then $\omega$ is known as the trace of $\Omega$. A permuting $n$-additive map $\Lambda : R^n \to R$ is said to be a permuting left $n$-multiplier (resp. permuting right $n$-multiplier) if $\Lambda(x_1,x_2,\cdots,x_ix_i^{'},\cdots,x_n) = \Lambda(x_1,x_2,\cdots,x_i,\cdots,x_n)x_i^{'}$ (resp. $\Lambda(x_1,x_2,\cdots,x_ix_i^{'},\cdots,x_n) = x_i\Lambda(x_1,x_2,\cdots,x_i^{'},\cdots,x_n)$) holds for all $x_i,x_i^{'} \in R$. If $\Lambda$ is both permuting left $n$-multiplier as well as right $n$-multiplier, then $\Lambda$ is called a permuting $n$-multiplier.

Motivated by the results due to Posner [8], Vukman obtained some results concerning the trace of symmetric bi-derivation in prime ring (see [9, 10]). Ashraf [1] proved similar results for semi-prime ring. In the year 2009, Park [7] introduced the concept of symmetric permuting $n$-derivation and obtained some results related to

the commuting traces of permuting $n$-derivations in rings. Further, the first author together with Jamal and Parveen [2, 3] obtained commutativity of rings admitting $n$-derivations whose traces satisfy certain polynomial conditions.

The main objective of this paper is to find the analogous results for permuting generalized $n$-derivation in the setting of prime and semi-prime rings. In fact, our theorems present a wide generalization of the results obtained in [1], Theorem 2.1, [7], Theorem 2.3, [7], Theorem 2.5, [9], Theorem 1, [9], Theorem 2, [10], Theorem 2 etc.

## 2. Results

We begin with the following known results which are frequently used in our discussion.

**Lemma 1** (Lemma 2.4 in [7]). *Let $n$ be a fixed positive integer and let $R$ be a $n!$-torsion free ring. Suppose that $y_1, y_2, \cdots, y_n \in R$ satisfy $\lambda y_1 + \lambda^2 y_2 + \cdots + \lambda^n y_n = 0$ ( $or \in Z(R)$) for $\lambda = 1, 2, \cdots, n$. Then $y_i = 0$ ($or$ $y_i \in Z(R)$) for all $i$.*

**Lemma 2** (Theorem 2.3 in [7]). *Let $n \geq 2$ be a fixed positive integer and $R$ be a non-commutative $n!$-torsion free prime ring. Suppose that there exists a permuting $n$-derivation $\Delta : R^n \to R$ such that the trace $\delta$ of $\Delta$ is commuting on $R$. Then we have $\Delta = 0$.*

**Lemma 3** (Theorem 2.6 in [7]). *Let $n \geq 2$ be a fixed positive integer and $R$ be a $n!$-torsion free prime ring. Suppose that there exists a non-zero permuting $n$-derivation $\Delta : R^n \to R$ such that the trace $\delta$ of $\Delta$ is centralizing on $R$ then $R$ is commutative.*

As stated in the beginning, there has been a great deal of work concerning centralizing and commuting mappings. The following result shows that if the trace $\delta$ of a permuting $n$-derivation $\Delta$ is centralizing on $R$ then it is commuting on $R$. In fact, we prove rather a more general result:

**Theorem 1.** *Let $n \geq 2$ be a fixed positive integer and $R$ be a $(n+1)!$-torsion free semi-prime ring admitting a permuting $n$-derivation $\Delta$ such that the trace $\delta$ of $\Delta$ satisfies $[[\delta(x), x], x] = 0$ for all $x \in R$. Then $\delta$ is commuting on $R$.*

*Proof.* From our hypothesis we have

$$[[\delta(x), x], x] = 0 \text{ for all } x \in R. \tag{2.1}$$

An easy computation shows that the traces $\delta$ of $\Delta$ satisfies the following relations

$$\delta(x+y) = \delta(x) + \delta(y) + \sum_{r=1}^{n-1} \binom{n}{r} h_r(x, y) \text{ for all } x, y \in R$$

where $h_r(x, y) = \Delta(\underbrace{x, x, \cdots, x}_{(n-r)-\text{times}}, \underbrace{y, y, \cdots, y}_{r-\text{times}})$.

Consider a positive integer $k, 1 \leq k \leq n+1$. Replacing $x$ by $x+ky$ in equation (2.1), we obtain

$$kQ_1(x,y)+k^2Q_2(x,y)+\ldots+k^{n+1}Q_{n+1}(x,y)=0 \text{ for all } x,y \in R,$$

where $Q_i(x,y)$ denotes the sum of the terms in which $y$ appears $i$ times. By (2.1) and Lemma 1, we have for all $x,y \in R$,

$$[[\delta(x),x],y]+[[\delta(x),y],x]+n[[\Delta(x,x,\ldots,y),x],x]=0. \tag{2.2}$$

Replacing $y$ by $xy$ in (2.2) we get

$$\begin{aligned}
0 &= [[\delta(x),x],xy]+[[\delta(x),xy],x]+n[[\Delta(x,x,\ldots,xy),x],x] \\
&= [[\delta(x),x],xy]+[[\delta(x),xy],x]+n[[x\Delta(x,x,\ldots,y),x],x] \\
&\quad +n[[\delta(x)y,x],x] \\
&= [[\delta(x),x],xy]+[x[\delta(x),y],x]+[[\delta(x),x]y,x] \\
&\quad +n[x[\Delta(x,x,\ldots,y),x],x]+n[\delta(x)[y,x],x]+n[[\delta(x),x]y,x] \\
&= [[\delta(x),x],x]y+x[[\delta(x),x],y]+x[[\delta(x),y],x]+[x,x][\delta(x),y] \\
&\quad +[\delta(x),x][y,x]+[[\delta(x),x],x]y+nx[[\Delta(x,x,\ldots,y),x],x] \\
&\quad +n\delta(x)[[y,x],x]+n[\delta(x),x][y,x]+n[\delta(x),x][y,x]+n[[\delta(x),x],x]y.
\end{aligned}$$

Using (2.1) and (2.2) we find that

$$(2n+1)[\delta(x),x][y,x]+n\delta(x)[[y,x],x]=0 \text{ for all } x,y \in R. \tag{2.3}$$

Similarly, replacing $y$ by $yx$ in (2.2), one can get

$$(2n+1)[y,x][\delta(x),x]+n[[y,x],x]\delta(x)=0 \text{ for all } x,y \in R. \tag{2.4}$$

Replacing $y$ by $yz$ in (2.3), we have

$$\begin{aligned}
0 &= (2n+1)[\delta(x),x][yz,x]+n\delta(x)[[yz,x],x] \\
&= (2n+1)\{[\delta(x),x][y,x]z+[\delta(x),x]y[z,x]\}+n\delta(x)y[[z,x],x] \\
&\quad +n\delta(x)[y,x][z,x]+n\delta(x)[y,x][z,x]+n\delta(x)[[y,x],x]z.
\end{aligned}$$

Using equation (2.3)

$$(2n+1)[\delta(x),x]y[z,x]+n\delta(x)y[[z,x],x]+2n\delta(x)[y,x][z,x]=0.$$

Replacing $y$ by $\delta(x)$ in the above relation we find that

$$(2n+1)[\delta(x),x]\delta(x)[z,x]+n\delta(x)^2[[z,x],x]+2n\delta(x)[\delta(x),x][z,x]=0. \tag{2.5}$$

From (2.3) we have $n\delta(x)^2[[y,x],x]=-(2n+1)\delta(x)[\delta(x),x][y,x]$. Now using this relation in (2.5) we get

$$\begin{aligned}
0 &= (2n+1)[\delta(x),x]\delta(x)[z,x]-(2n+1)\delta(x)[\delta(x),x][z,x] \\
&\quad +2n\delta(x)[\delta(x),x][z,x] \qquad (2.6) \\
&= \{(2n+1)[\delta(x),x]\delta(x)-\delta(x)[\delta(x),x]\}[z,x].
\end{aligned}$$

Similarly using (2.4) one can easily obtain

$$\{(2n+1)\delta(x)[\delta(x),x]-[\delta(x),x]\delta(x)\}[z,x]=0. \tag{2.7}$$

Adding (2.6) and (2.7) we arrive at

$$2n\{[\delta(x),x]\delta(x)+2n\delta(x)[\delta(x),x]\}[z,x]=0.$$

Since $2n$ divides $(n+1)!$, we find that $R$ is $2n$-torsion free and hence for all $x,z\in R$,

$$\{[\delta(x),x]\delta(x)+\delta(x)[\delta(x),x]\}[z,x]=0. \tag{2.8}$$

Using (2.8) in (2.6) we obtain $(2n+2)[\delta(x),x]\delta(x)[z,x]=0$ for all $x,z\in R$. Since $2(n+1)$ divides $(n+1)!$, we find that $R$ is $2(n+1)$-torsion free and hence for all $x,z\in R$,

$$[\delta(x),x]\delta(x)[z,x]=0 \text{ for all } x,z\in R.$$

Substituting $yz$ for $z$ we get $[\delta(x),x]\delta(x)y[z,x]=0$ for all $x,y,z\in R$. Replacing $z$ by $\delta(x)$ we obtain $0=[\delta(x),x]\delta(x)y[\delta(x),x]\delta(x)$. Semiprimeness of $R$ yields

$$[\delta(x),x]\delta(x)=0 \text{ for all } x\in R. \tag{2.9}$$

Similarly application of (2.7) and (2.8) yields that

$$\delta(x)[\delta(x),x]=0 \text{ for all } x\in R. \tag{2.10}$$

Replacing $x$ by $x+ky$ in equation (2.10) where $1\le k\le 2n$ and implementing Lemma 1

$$\delta(x)[\delta(x),y]+n\delta(x)[\Delta(x,x,\dots,y),x]+n\Delta(x,x,\dots,y)[\delta(x),x]=0. \tag{2.11}$$

Replacing $y$ by $yx$

$$\begin{aligned}0&=\delta(x)[\delta(x),yx]+n\delta(x)[y\delta(x)+\Delta(x,x,\dots,y)x,x]\\&\quad+n\{y\delta(x)+\Delta(x,x,\dots,y)x\}[\delta(x),x]\\&=\delta(x)y[\delta(x),x]+\delta(x)[\delta(x),y]x+n\delta(x)y[\delta(x),x]\\&\quad+n\delta(x)[y,x]\delta(x)+n\delta(x)[\Delta(x,x,\dots,y),x]x\\&\quad+ny\delta(x)[\delta(x),x]+n\Delta(x,x,\dots,y)x[\delta(x),x].\end{aligned}$$

From (2.11) we have

$$-n\Delta(x,x,\dots,y)[\delta(x),x]x=\delta(x)[\delta(x),y]x+n\delta(x)[\Delta(x,x,\dots,y),x]x. \tag{2.12}$$

Using (2.10) and (2.12) in the above relation, we get

$$\begin{aligned}0&=(n+1)\delta(x)y[\delta(x),x]+n\delta(x)[y,x]\delta(x)+n\Delta(x,x,\dots,y)x[\delta(x),x]\\&\quad-n\Delta(x,x,\dots,y)[\delta(x),x]x\\&=(n+1)\delta(x)y[\delta(x),x]+n\delta(x)[y,x]\delta(x)-n\Delta(x,x,\dots,y)[[\delta(x),x]x].\end{aligned}$$

This gives that

$$(n+1)\delta(x)y[\delta(x),x]+n\delta(x)[y,x]\delta(x)=0 \text{ for all } x,y\in R. \tag{2.13}$$

Substituting $xy$ for $y$ in (2.13)

$$(n+1)\delta(x)xy[\delta(x),x]+n\delta(x)x[y,x]\delta(x)=0 \text{ for all } x,y\in R. \tag{2.14}$$

Left multiply (2.13) by $x$, we obtain

$$(n+1)x\delta(x)y[\delta(x),x]+nx\delta(x)[y,x]\delta(x)=0 \text{ for all } x,y\in R. \tag{2.15}$$

Combining (2.14) and (2.15), we get

$$(n+1)[\delta(x),x]y[\delta(x),x]+n[\delta(x),x][y,x]\delta(x)=0 \text{ for all } x,y\in R. \tag{2.16}$$

Replacing $y$ by $yz$ in (2.4), we obtain

$$\begin{aligned}0&=(2n+1)[yz,x][\delta(x),x]+n[[yz,x],x]\delta(x)\\&=(2n+1)[yz,x][\delta(x),x]+n[y[z,x],x]\delta(x)+n[[y,x]z,x]\delta(x)\\&=(2n+1)y[z,x][\delta(x),x]+(2n+1)[y,x]z[\delta(x),x]+ny[[z,x],x]\delta(x)\\&\quad+n[y,x][z,x]\delta(x)+n[y,x][z,x]\delta(x)+n[[y,x],x]z\delta(x).\end{aligned}$$

Using (2.4) we get,

$$(2n+1)[y,x]z[\delta(x),x]+2n[y,x][z,x]\delta(x)+n[[y,x],x]z\delta(x)=0.$$

Replacing $y$ by $\delta(x)$ in the above relation we get

$$(2n+1)[\delta(x),x]z[\delta(x),x]+2n[\delta(x),x][z,x]\delta(x)=0 \text{ for all } x,z\in R. \tag{2.17}$$

Combining equations (2.16) and (2.17) we find that

$$\begin{aligned}0&=(2n+1)[\delta(x),x]z[\delta(x),x]-2(n+1)[\delta(x),x]z[\delta(x),x]\\&=[\delta(x),x]z[\delta(x),x] \text{ for all } x,z\in R.\end{aligned}$$

Since $R$ is semi-prime, we get $[\delta(x),x]=0$, for all $x\in R$. □

**Theorem 2.** *Let $n\geq 2$ be a fixed positive integer and $R$ be a $(n+1)!$-torsion free semi-prime ring admitting a permuting $n$-derivation $\Delta$ such that the trace $\delta$ of $\Delta$ satisfies $[[\delta(x),x],x]\in Z(R)$ for all $x\in R$. Then $\delta$ is commuting on $R$.*

*Proof.* Replace $x$ by $x+ky$ for $1\leq k\leq n+1$ in the given condition to find that

$$kQ_1(x,y)+k^2Q_2(x,y)+\ldots+k^{n+1}Q_{n+1}(x,y)\in Z(R) \text{ for all } x,y\in R,$$

where $Q_i(x,y)$ denotes the sum of the terms in which $y$ appears $i$ times. By Lemma 1, we have for all $x,y\in R$,

$$[[\delta(x),x],y]+[[\delta(x),y],x]+n[[\Delta(x,x,\ldots,y),x],x]\in Z(R). \tag{2.18}$$

Again replacing $y$ by $xy$ in the above expression, we get

$$\begin{aligned}&x([[\delta(x),x],y]+[[\delta(x),y],x]+n[[\Delta(x,x,\ldots,y),x],x])+(n+2)[[\delta(x),x],x]y\\&\qquad+(2n+1)[\delta(x),x][y,x]+n\delta(x)[[y,x],x]\in Z(R).\end{aligned}$$

Combining (2.18) with the latter relation, we find that

$$(3n+3)[[\delta(x),x],x][y,x]+(3n+1)[\delta(x),x][[y,x],x]+n\delta(x)[[[y,x],x],x]=0. \tag{2.19}$$

Further replace $y$ by $\delta(x)$ in (2.19) to get

$$(6n+4)[[\delta(x),x],x][\delta(x),x]=0.$$

On commuting with $x$, we find that

$$(6n+4)[[\delta(x),x],x]^2=0. \tag{2.20}$$

Next, on replacing $y$ by $[\delta(x),x]$ in (2.19) and using the given condition, we have

$$(3n+3)[[\delta(x),x],x]^2=0. \tag{2.21}$$

Now combine (2.20) and (2.21) to get $2[[\delta(x),x],x]^2=0$.
Since, $R$ is (n+1)!-torsion free and also the center of semi-prime ring is free from nilpotent element, we have $[[\delta(x),x],x]=0$. From Theorem 1, $\delta$ is commuting on $R$. □

Combining Theorem 2 with Lemma 3, we can prove the following:

**Corollary 1.** *Let $n\geq 2$ be a fixed positive integer and $R$ be a $(n+1)!$-torsion free semi-prime ring admitting a non-zero permuting $n$-derivation $\Delta$ such that the trace $\delta$ satisfies $[[\delta(x),x],x]\in Z(R)$ for all $x\in R$. Then $R$ is commutative.*

**Theorem 3.** *Let $n\geq 1$ be a fixed positive integer and $R$ be a non-commutative $n!$-torsion free prime ring admitting a permuting generalized $n$-derivation $\Omega$ with associated $n$-derivation $\Delta$ such that the trace $\omega$ of $\Omega$ is commuting on $R$. Then $\Omega$ is a left $n$-multiplier on $R$.*

*Proof.* Our hypothesis yields that

$$[\omega(x),x]=0 \text{ for all } x\in R. \tag{2.22}$$

It can be easily seen that

$$\omega(x+y)=\omega(x)+\omega(y)+\sum_{r=1}^{n-1}\binom{n}{r}p_r(x,y) \text{ for all } x,y\in R$$

where $p_r(x,y)=\Omega(\underbrace{x,x,\cdots,x}_{(n-r)-\text{times}},\underbrace{y,y,\cdots,y}_{r-\text{times}})$.
Substituting $x+\lambda y$, where $\lambda$ $(1\leq\lambda\leq n)$ is a positive integer, in place of $x$ in the above equation we obtain

$$\begin{aligned}0&=[\omega(x+\lambda y),x+\lambda y]\\&=[\omega(x)+\omega(\lambda y)+\sum_{r=1}^{n-1}\binom{n}{r}p_r(x,\lambda y),x+\lambda y].\end{aligned}$$

Using (2.22), we have

$$\begin{aligned}0 &= \lambda\{[\omega(x),y]+\binom{n}{1}[p_1(x,y),x]\}+\lambda^2\{\binom{n}{1}[p_1(x,y),y]\\ &+\binom{n}{2}[p_2(x,y),x]\}+\cdots+\lambda^n\{[\omega(y),x]\\ &+\binom{n}{n-1}[p_{n-1}(x,y),x]\} \quad \text{for all } x,y\in R.\end{aligned}$$

Implementing Lemma 1 we get

$$\begin{aligned}0 &= [\omega(x),y]+\binom{n}{1}[p_1(x,y),x]\\ &= [\omega(x),y]+n[\Omega(x,x,\cdots,y),x].\end{aligned}$$

Replacing $y$ by $yx$ we obtain

$$\begin{aligned}0 &= y[\omega(x),x]+[\omega(x),y]x+n[\Omega(x,x,\cdots,y)x+y\Delta(x,x,\cdots,x),x]\\ &= [\omega(x),y]x+ny[\delta(x),x]+n[y,x]\delta(x)+n[\Omega(x,x,\cdots,y),x]x\\ &= n[y,x]\delta(x)+ny[\delta(x),x].\end{aligned}$$

Again replacing $y$ by $zy$ for any $z\in R$ we have $[z,x]y\delta(x)=0$ for all $x,y,z\in R$. Since $R$ is prime we find that for any $x\notin Z(R),\delta(x)=0$. Now for any $y\in Z(R)$ and $x\notin Z(R)$, $x+\lambda y\notin Z(R)$. Hence,

$$\begin{aligned}0 &= \delta(x+\lambda y)\\ &= \lambda p_1(x,y)+\cdots+\lambda^{n-1}p_{n-1}(x,y)+\lambda^n\delta(y).\end{aligned}$$

Using Lemma 2 we obtain $\delta(y)=0$ for any $y\in Z(R)$. Thus, $\delta(x)=0$ for all $x\in R$. Lemma 1 yields that $\Delta=0$. This implies that $\Omega$ acts as a left $n$-multiplier. □

**Theorem 4.** *Let $n\geq 2$ be a fixed positive integer and $R$ be an $n!$-torsion free semi-prime ring admitting a permuting generalized $n$-derivation $\Omega$ with associated $n$-derivation $\Delta$ such that the trace $w$ of $\Omega$ is centralizing on $R$. Then $w$ is commuting on $R$.*

*Proof.* It is given that $[w(x),x]\in Z(R)$ for all $x\in R$. Using the similar arguments as used in Theorem 2, we obtain

$$[w(x),y]+n[\Omega(x,x,...,y),x]\in Z(R) \text{ for all } x,y\in R. \tag{2.23}$$

Replacing $y$ by $yx$, we obtain

$$y[w(x),x]+[w(x),y]x+n[\Omega(x,x,...,y),x]x+ny[\delta(x),x]+n[y,x]\delta(x)\in Z(R).$$

Now in view of (2.23), we find that

$$0 = [y,x][w(x),x] + n[y,x][\delta(x),x] + ny[[\delta(x),x],x] \\ + n[y,x][\delta(x),x] + n[[y,x],x]\delta(x) \text{ for all } x,y \in R. \tag{2.24}$$

Again replace $y$ by $w(x)y$ to get

$$\begin{aligned} 0 = {} & w(x)[y,x][w(x),x] + [w(x),x]y[w(x),x] + nw(x)[y,x][\delta(x),x] \\ & + n[w(x),x]y[\delta(x),x] + nw(x)y[[\delta(x),x],x] + nw(x)[y,x][\delta(x),x] \\ & + n[w(x),x]y[\delta(x),x] + n[w(x),x][y,x]\delta(x) + nw(x)[[y,x],x]\delta(x) \\ & + n[w(x),x][y,x]\delta(x) + n[[w(x),x],x]y\delta(x). \end{aligned}$$

Using (2.24) and the given condition, we find that

$$[w(x),x]y[w(x),x] + 2n[w(x),x]y[\delta(x),x] + 2n[w(x),x][y,x]\delta(x) = 0. \tag{2.25}$$

Further, replacing $y$ by $[w(x),x]^2$ in (2.25) and using the given condition, we have

$$[w(x),x]^4 + 2n[w(x),x]^3[\delta(x),x] = 0 \text{ for all } x \in R. \tag{2.26}$$

Again, replace $y$ by $yz$ in (2.25) and use (2.25), to get

$$2n[w(x),x][y,x]z\delta(x) = 0 \text{ for all } x,y,z \in R. \tag{2.27}$$

Next, we replace $y$ by $w(x)$ and $z$ by $[w(x),x]$ to find that $2n[w(x),x]^3\delta(x) = 0$ for all $x \in R$. On commuting the latter relation with $x$ and using the given condition, we have

$$2n[w(x),x]^3[\delta(x),x] = 0 \text{ for all } x \in R. \tag{2.28}$$

From (2.26) and (2.28), we find that $[w(x),x]^4 = 0$. Since the center of a semi-prime ring does not contain any nilpotent element, we get $[w(x),x] = 0$. □

**Corollary 2.** *Let $n \geq 2$ be a fixed positive integer and $R$ be a non-commutative $n!$-torsion free semi-prime ring admitting a permuting generalized $n$-derivation $\Omega$ with associated $n$-derivation $\Delta$ such that the trace $w$ of $\Omega$ is centralizing on $R$. Then $\Omega$ is a left $n$-multiplier on $R$.*

*Proof.* By Theorem 3 and Theorem 2, we get the required result. □

In conclusion, if we look at Theorem 2 closely, it is tempting to conjecture as follows:

**Conjecture 1.** *Let $R$ be a semi-prime ring with suitable torsion restrictions and $\Delta$ be a non-zero permuting $n$-derivation. Suppose that for some integer $m \geq 1$, $\delta_m(x) \in Z(R)$ for all $x \in R$ where $\delta_{k+1}(x) = [\delta_k(x),x]$ for $k > 1$ and $\delta_1(x) = \delta(x)$ stands for the trace of $\Delta$. Then $[\delta(x),x] = 0$ for all $x \in R$.*

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# LINK BETWEEN HOSOYA INDEX AND FIBONACCI NUMBERS

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*Abstract.* Let $G$ be a graph and $Z(G)$ be its Hosoya index. We show how the Hosoya index can be a good tool to establish some new identities involving Fibonacci numbers. This permits to extend Hillard and Windfeldt work.

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*Keywords:* Fibonacci numbers, matching, Hosoya index, paths

## 1. Introduction

We denote by $G = (V(G), E(G))$ a simple undirected graph, $V(G)$ is the set of its vertices and $E(G)$ is the set of its edges. The order of $G$ is $|V(G)|$ and the size of $G$ is $|E(G)|$. For a vertex $v$ of $G$, $N(v)$ is the set of vertices adjacent to $v$, $\deg(v) := |N(v)|$ is the degree of $v$; $Link(v)$ is the set of edges incident to $v$. An edge $\{u,v\}$ of $G$ is denoted $uv$. A path $P_n$, from a vertex $v_1$ to a vertex $v_n$, $n \geq 2$, is a sequence of vertices $v_1,\dots,v_n$ and edges $v_i v_{i+1}$, for $i = 1,\dots,n-1$; for simplicity we denote it by $v_1 \cdots v_n$. We extend the definition of $P_n$ to $n = 0$ and $n = 1$ by setting $P_0$ is empty and $P_1$ is a single vertex, we add the convention that $P_n P_0 = P_0 P_n = P_n$ for all $n \geq 1$.

The graph $G - v$ is obtained from $G$ by removing the vertex $v$ and all edges of $G$ which are incident to $v$. For an edge $e$ of $G$, we denote by $G - e$ the graph obtained from $G$ by removing $e$. The contraction of a graph $G$, associated to an edge $e$, is the graph $G/e$ obtained by removing $e$ and identifying the vertices $u$ and $v$ incident to $e$ and replacing them by a single vertex $v'$ where any edges incident to $u$ or $v$ are redirected to $v'$. We then say that we contract in $G$ the adjacent vertices $u$, $v$ into the vertex $v'$.

The well-known Fibonacci sequence $\{F_n\}$ is defined as $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. The Fibonacci numbers are connected to the element of Pascal's triangle using the following identity

$$F_{n+1} = \sum_{k=0}^{n} \left. \begin{matrix} n-k \\ k \end{matrix} \right).$$

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For some results and properties related to Fibonacci numbers, see for instance [1]. Many fields widely applies this sequence, particularly in physics and chemistry [10].

A matching $M$ of a graph $G$ is a subset of $E(G)$ such that no two edges in $M$ share a vertex in $G$. A matching of $G$ is also called an independent edge set of $G$. A $k$-matching of a graph $G$ is a matching of $G$ of cardinal $k$, it is then an independent edge set of $G$ of cardinal $k$. We denote by $m(G,k)$ the number of $k$-matchings of $G$ with the convention $m(G,0)=1$. Note that $m(G,1)=|E(G)|$ and when $k>n/2$, $m(G,k)=0$.

The Hosoya index of a graph $G$, denoted by $Z(G)$, is an index introduced by Hosoya [9] as follows.

$$Z(G)=\sum_{k=0}^{\lfloor n/2\rfloor} m(G,k),$$

where $n=|V(G)|$, $\lfloor n/2\rfloor$ stands for the integer part of $n/2$. This index has several applications in molecular chemistry such as boiling point, entropy or heat of vaporization. The literature includes many papers dealing the Hosoya index [2,3,6].

## 2. Preliminary results

Before proving our main results, we first list the following results. From the definition of the Hosoya index, it is not difficult to deduce the following Lemma.

**Lemma 1** ([7])**.** *Let $G$ be a graph, we have*
(1) *If $uv\in E(G)$, then $Z(G)=Z(G-uv)+Z(G-\{u,v\})$.*
(2) *If $v\in V(G)$, then $Z(G)=Z(G-v)+\sum_{w\in N_G(v)}Z(G-\{w,v\})$.*
(3) *If $G_1,G_2,\ldots,G_t$ are the components of $G$ then $Z(G)=\prod_{k=1}^{t}Z(G_k)$.*

Lemma 1 allows us to compute $Z(G)$ for any graph recursively.

The following theorem gives a relation between Hosoya index and Fibonacci number (see [5], [7]).

**Theorem 1.** *Let $P_n$ be a path on $n$ vertices, then $Z(P_n)=F_{n+1}$.*

## 3. Main results

In this section, inspired by [4], we give another proofs of well-known identities (see Lemmas 2 and 3). Our goal is to prove the formula of Lemma 3 via Theorem 1. For this we give a direct proof of Lemma 2. We also establish two new identities, given in Theorems 2 and 3.

The following identity shows the relation between independent edge subsets in $P_{r_1+r_2}$, $P_{r_1}$ and $P_{r_2}$ for $r_1$ and $r_2$ two non-negative integers.

**Lemma 2.** *Let $r_1$, $r_2$ be two non-negative integers, then*

$$Z(P_{r_1+r_2})=Z(P_{r_1})Z(P_{r_2})+Z(P_{r_1-1})Z(P_{r_2-1}).$$

*Proof.* Let $v_1 \cdots v_{r_1+r_2}$be a path $P_{r_1+r_2}$ partitioned into two paths $P_{r_1}$ represented by a sequence of vertices $v_1 \cdots v_{r_1}$ and $P_{r_2}$ represented by a sequence of vertices $v_{r_1+1} \cdots v_{r_1+r_2}$. The vertices $v_{r_1}$ and $v_{r_1+1}$ share the same edge in $P_{r_1+r_2}$. The Hosoya number $Z(P_{r_1+r_2})$ of the path $P_{r_1+r_2}$ represents the number of independent edge subsets between the vertices of $P_{r_1+r_2}$, this index can be written as $Z(P_{r_1+r_2}) = |M| + |M'|$, where:

- $M$ is the set of independent edge subsets of $P_{r_1+r_2}$ such that the edge $v_{r_1}v_{r_1+1}$ does not belong to any independent edge subset of $M$, that means all independent edge subsets of $M$ are in $P_{r_1}$ and $P_{r_2}$, so $|M| = Z(P_{r_1})Z(P_{r_2})$.
- $M'$ is the set of independent edge subsets of $P_{r_1+r_2}$ such that the edge $v_{r_1}v_{r_1+1}$ belongs to all independent edge subsets of $M'$, that means the others independent edges of every subset of $M'$ are in $P_{r_1-1}$ and $P_{r_2-1}$, so $|M| = Z(P_{r_1-1})Z(P_{r_2-1})$.

□

**Lemma 3.** *Let $k,n$ be two integers such that $1 \leq k \leq n$. Then*

$$F_{n+1} = F_k F_{n-k} + F_{k+1} F_{n-k+1}.$$

*Proof.* Consider a path $P_n = v_1 \cdots v_n$ on $n$ vertices. We set $r_1 := k$, $r_2 := n-k$, and use Theorem 1 and Lemma 2. □

We introduce a new identity of Fibonacci numbers which generalize identities of Fibonacci numbers given in [8].

For every integer $s \geq 2$, let $\Omega_s$ be the set of $\varepsilon := (\varepsilon_1, \varepsilon_2, ..., \varepsilon_s)$ with $\varepsilon_i \in \{-1,0,1\}$ $(1 \leq i \leq s)$ such that :

(1) The number of $\varepsilon_i = 0$ is even. Let $2h(\varepsilon)$ this number.
(2) If $2h(\varepsilon) = 0$, then $\varepsilon_i = 1$ for all $i$.
(3) If $2h(\varepsilon) \neq 0$, let $L_\varepsilon := \{s_1, s_2, \ldots, s_{2h(\varepsilon)} : s_1 < s_2 < \ldots < s_{2h(\varepsilon)} \; and \; \varepsilon_{s_i} = 0 \; for \; all \; i \in \{1,2,\ldots,2h(\varepsilon)\}\}$.
For all $l \in L_\varepsilon$, we have :
- $l$ is even $\Longrightarrow$ $\varepsilon_i = 1$ for all $s_l < i < s_{l+1}$;
- $l$ is odd $\Longrightarrow$ $\varepsilon_i = -1$ for all $s_l < i < s_{l+1}$;
- $\varepsilon_i = 1$ for all $i < s_1$ or $i > s_{2h(\varepsilon)}$.

For example,

$$\Omega_2 = \{(1,1),(0,0)\},$$
$$\Omega_3 = \{(1,1,1),(1,0,0),(0,0,1),(0,-1,0)\},$$
$$\Omega_4 = \{(1,1,1,1),(0,0,1,1),(1,0,0,1),(1,1,0,0),(0,0,0,0),(0,-1,0,1),$$
$$(1,0,-1,0),(0,-1,-1,0)\}.$$

- The lines of the following table represent the elements of $\Omega_5$,

$$\left|\begin{array}{ccccc||ccccc}
1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & -1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0
\end{array}\right|$$

- The lines of the following table represent the elements of $\Omega_6$,

$$\left|\begin{array}{cccccc||cccccc}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 \\
1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\
1 & 1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & -1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 \\
1 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right|$$

**Theorem 2.** *For any positive integers $r_i$ $(1 \leq i \leq s)$ and any integer $s \geq 2$, we have*

$$F_{r_1+r_2+\cdots+r_s+1} = \sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_s)\in\Omega_s} \prod_{i=1}^{s} F_{r_i+\varepsilon_i}, \tag{3.1}$$

*Proof.* Let $P_{r_1+r_2+\cdots+r_s}$ be a path with $r_1+r_2+\cdots+r_s$ vertices. We subdivide $P_{r_1+r_2+\cdots+r_s}$ in consecutive blocs of paths $P_{r_i}$ with $r_i$ vertices $(1 \leq i \leq s)$, see Figure 1.

In one hand side, by Theorem 1, we have $Z(P_{r_1+\cdots+r_s}) = F_{r_1+\cdots+r_s+1}$. In the other hand $Z(P_{r_1+r_2+\cdots+r_s})$ is the number of independent edge subsets in $P_{r_1+r_2+\cdots+r_s}$. So, $Z(P_{r_1+r_2+\cdots+r_s}) = \sum_{i=0}^{s-1} |M_k|$ where $M_k$ is the set of independent edge subsets $I$ in $P_{r_1+r_2+\cdots+r_s}$ such that it exists exactly $k$ edges between blocs of paths $P_{r_i}$ $(1 \leq i \leq s)$ which belong to $I$.

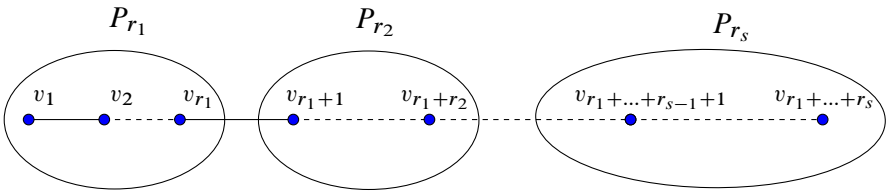


FIGURE 1. Path $P_{r_1+r_2+\ldots+r_s}$ divided in consecutive blocs of paths $P_{r_i}$ with $r_i$ vertices $(1 \leq i \leq s)$.

$M_0$ is the set of independent edge subsets $I$ in $P_{r_1+r_2+\cdots+r_s}$ such that doesn't contain any edge between blocs of paths $P_{r_i}$ $(1 \leq i \leq s)$, so all these independent edge subsets are in blocs $P_{r_i}$ $(1 \leq i \leq s)$. Hence, $|M_0| = \prod_{i=1}^{s} F_{r_i+1}$.

$M_1$ is the set of independent edge subsets $I$ in $P_{r_1+r_2+\cdots+r_s}$ such that it exists only one edge between blocs of paths $P_{r_i}$ $(1 \leq i \leq s)$ which belong to $I$. Let $H$ be a subset of $M_1$ containing the edge $v_{r_1+\cdots+r_k} v_{r_1+\cdots+r_k+1}$ $(1 \leq k \leq s-1)$ in all of its independent edge subsets. We contract the adjacent vertices $v_{r_1+\cdots+r_k}, v_{r_1+\cdots+r_k+1}$ in $P_{r_1+\cdots+r_s}$ into one vertex $v'$ and $P_{r_1+\cdots+r_s-1}$ is a new path after contraction composed of consecutive blocs of paths $P_{r_1}, P_{r_2}, \ldots, P_{r_{k-1}}, P_{r_k-1}, v', P_{r_{k+1}-1}, P_{r_{k+2}}, \ldots, P_{r_s}$. A path $P_{r_1+\cdots+r_s-1}$ does not contain any edge between blocs which belong to independent edge subsets of $H$, so $|H| = F_{r_1+1} \times F_{r_2+1} \times \cdots \times F_{r_{k-1}+1} \times F_{r_k} \times F_2 \times F_{r_{k+1}} \times F_{r_{k+2}+1} \times \cdots \times F_{r_s+1}$. Thus, $|M_1| = \sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_s)\in\Delta_1} \prod_{i=1}^{s} F_{r_i+\varepsilon_i}$ where $\Delta_1$ is the set of $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s)$ such that for $1 \leq i \leq s$, $\varepsilon_i \in \{0,1\}$ and $\varepsilon_1\varepsilon_2\cdots\varepsilon_s$ forms a sequence such that there is only one pair of zeros and this pair is of the form $(\varepsilon_l, \varepsilon_{l+1})$.

$M_2$ is the set of independent edge subsets $I$ in $P_{r_1+r_2+\cdots+r_s}$ such that it exists exactly two edges between blocs of paths $P_{r_i}$ $(1 \leq i \leq s)$ which belong to $I$. As for computing of $|M_1|$ and using the contraction method for the two edges between blocs of paths $P_{r_i}$ $(1 \leq i \leq s)$, we have $|M_2| = \sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_s)\in\Delta_2} \prod_{i=1}^{s} F_{r_i+\varepsilon_i}$ where $\Delta_2$ is the set of $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s) \in \Omega_s$ such that for $1 \leq i \leq s$, $\varepsilon_i \in \{-1, 0, 1\}$ and $\varepsilon_1\varepsilon_2\cdots\varepsilon_s$ forms a sequence such that there is only one pair of zeros $\{\varepsilon_i, \varepsilon_{i+2}\}$ and $\varepsilon_{i+1} = -1$, or only two pairs of zeros $\{\varepsilon_i, \varepsilon_{i+1}\}, \{\varepsilon_{i+1+k}, \varepsilon_{i+2+k}\}$ $(1 \leq k$ and $i+k+2 \leq s)$ and $\varepsilon_l = 1$ for all $l \in \{1, 2, \ldots, i-1, i+2, i+3, \ldots, i+k, i+k+3, \ldots, s\}$.

For $M_k$ $(3 \leq k \leq s-2)$, using the contraction method for $k$ edges between blocs of paths $P_{r_i}$ $(1 \leq i \leq s)$, we have $|M_k| = \sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_s)\in\Delta_k} \prod_{i=1}^{s} F_{r_i+\varepsilon_i}$ where $\Delta_k = \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s) \in \Omega_s : for\ all\ s_i \in L_\varepsilon (1 \leq i \leq 2h(\varepsilon)), \sum_{l=1}^{h(\varepsilon)} (s_{2l} - s_{2l-1}) = k\}$ which represents all sequences of $\Omega_s$ such that the sum of the difference of the position of each pair of 0 is equal to $k$.

We finish by $M_{s-1}$ which is the set of matchings $I$ in $P_{r_1+r_2+\cdots+r_s}$ such that it exists exactly $s-1$ edges between blocs of paths $P_{r_i}$ $(1 \leq i \leq s)$ which belong to

$I$. In this case, except the paths $P_{r_1}, P_{r_s}$ that lose one vertex after a contraction all others paths $P_{r_i}$ $(2 \leq i \leq s-1)$ lose two vertices after contraction method. Thus, $|M_{s-1}| = F_{r_1} F_{r_s} \prod_{i=2}^{s-1} F_{r_i - 1}$.

Note that $\{\Delta_k : 1 \leq k \leq s-1\}$ is a partition of $\Omega_s$. Hence, the identity (3.1) holds. □

The following corollaries are the main results given by [8].

**Corollary 1.** *For any non-negative integers $r$ and $t$, we have*

$$F_{r+t} = F_{r+1} F_t + F_r F_{t-1}. \tag{3.2}$$

*Proof.* From Theorem 2 with $s = 2$ and $\Omega_2 = \{(1,1), (0,0)\}$, we obtain the following identity $F_{r_1+r_2+1} = F_{r_1+1} F_{r_2+1} + F_{r_1} F_{r_2}$. We put $r = r_1$ and $t = r_2 + 1$ and we conclude. □

**Corollary 2.** *For any non-negative integers $u$, $v$ and $w$, we have*

$$F_{u+v+w} = F_{u+1} F_{v+1} F_{w+1} + F_u F_v F_w - F_{u-1} F_{v-1} F_{w-1}.$$

*Proof.* From Theorem 2 with $s = 3$ and $\Omega_3 = \{(1,1,1), (1,0,0), (0,0,1), (0,-1,0)\}$, we obtain the following identity $F_{r_1+r_2+r_3+1} = F_{r_1+1} F_{r_2+1} F_{r_3+1} + F_{r_1+1} F_{r_2} F_{r_3} + F_{r_1} F_{r_2} F_{r_3+1} + F_{r_1} F_{r_2-1} F_{r_3}$. We put $u = r_1$, $v = r_2$ and $w = r_3 + 1$ and using $F_t = F_{t+1} - F_{t-1}$, we have :

$$\begin{aligned}
F_{u+v+w} &= F_{u+1} F_{v+1} F_w + F_{u+1} F_v F_{w-1} + F_u F_v F_w + F_u F_{v-1} F_{w-1} \\
&= F_{u+1} F_{v+1} (F_{w+1} - F_{w-1}) + F_{u+1} F_v F_{w-1} + F_u F_v F_w \\
&+ (F_{u+1} - F_{u-1}) F_{v-1} F_{w-1} \\
&= F_{u+1} F_{v+1} F_{w+1} - F_{u+1} F_{v+1} F_{w-1} + F_{u+1} F_v F_{w-1} + F_u F_v F_w \\
&+ F_{u+1} F_{v-1} F_{w-1} - F_{u-1} F_{v-1} F_{w-1} \\
&= F_{u+1} F_{v+1} F_{w+1} + F_u F_v F_w - F_{u-1} F_{v-1} F_{w-1} + F_{u+1} F_{v-1} F_{w-1} \\
&+ F_{u+1} F_v F_{w-1} - F_{u+1} F_{v+1} F_{w-1} \\
&= F_{u+1} F_{v+1} F_{w+1} + F_u F_v F_w - F_{u-1} F_{v-1} F_{w-1} \\
&+ F_{u+1} (F_{v-1} + F_v - F_{v+1}) F_{w-1} \\
&= F_{u+1} F_{v+1} F_{w+1} + F_u F_v F_w - F_{u-1} F_{v-1} F_{w-1}.
\end{aligned}$$

□

**Corollary 3.** *For any non-negative integers $a$, $b$, $c$ and $d$, we have*

$$\begin{aligned}
&F_{a+b+c+d+1} \\
&= F_{a+1} F_{b+1} F_{c+1} F_{d+1} + F_a F_b F_c F_d + F_{a+1} F_b F_c F_{d+1} + F_{a+1} F_{b+1} \\
&+ F_c F_d + F_a F_{b-1} F_c F_{d+1} + F_a F_b F_{c+1} F_{d+1} \\
&+ F_{a+1} F_b F_{c-1} F_d + F_a F_{b-1} F_{c-1} F_d.
\end{aligned}$$

*Proof.* From Theorem 2, with $s = 4$ and $\Omega_4 = \{(1,1,1,1),(0,0,1,1),(1,0,0,1),(1,1,0,0),(0,0,0,0)\ ,(0,-1,0,1),(1,0,-1,0),(0,-1,-1,0)\}$, we obtain the identity. □

The following theorem is another identity of Fibonacci number which gives an equivalent of Theorem 2.

**Theorem 3.** *Let $s \geq 2$ be an integer. For any non-negative integer $r_i$ $(1 \leq i \leq s)$, we have*

$$F_{\sum_{i=1}^{s} r_i+1} = F_{\sum_{i=1}^{s-1} r_i+1} F_{r_s+1} + \sum_{i=0}^{s-2} \left[ \left( \prod_{j=1}^{i} F_{r_{s-j}-1} \right) F_{\sum_{j=1}^{s-i-2} r_j+1} F_{r_{s-i-1}} F_{r_s} \right].$$

*Proof.* As mentioned in Theorem 2, $F_{\sum_{i=1}^{s} r_i+1} = \sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_s)\in\Omega_s} \prod_{i=1}^{s} F_{r_i+\varepsilon_i}$. Then $F_{\sum_{i=1}^{s} r_i+1} = c_1 + c_0$ where $c_1$ corresponds to the case $\varepsilon_s = 1$ and $c_0$ to the case $\varepsilon_s = 0$. That means to count $F_{\sum_{i=1}^{s} r_i+1}$ we have two cases.

Case 1. $\varepsilon_s = 1$. Then for all s-uplet $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s)$ we obtain

$$c_1 = F_{r_s+1} \left( \sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{s-1})\in\Omega_{s-1}} \prod_{i=1}^{s-1} F_{r_i+\varepsilon_i} \right),$$

so for $\varepsilon_s = 1$ we have $c_1 = F_{\sum_{i=1}^{s-1} r_i+1} F_{r_s+1}$.

Case 2. $\varepsilon_s = 0$. Let $\varepsilon_{s-i-1} = 0$ with $i$ the smallest integer $k$, $0 \leq k \leq s-2$, such that $\varepsilon_{s-k-1} = 0$. So for $1 \leq j \leq i$ we have $\varepsilon_{s-j} = -1$. Hence,

$$\begin{aligned} c_0 &= \sum_{i=0}^{s-2} \left[ \left( \prod_{j=1}^{i} F_{r_{s-j-1}} \right) F_{r_{s-i-1}} F_{r_s} \sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{s-i-2})\in\Omega_{s-i-2}} \prod_{j=1}^{s-i-2} F_{r_j+\varepsilon_j} \right] \\ &= \sum_{i=0}^{s-2} \left[ \left( \prod_{j=1}^{i} F_{r_{s-j}-1} \right) F_{\sum_{j=1}^{s-i-2} r_j+1} F_{r_{s-i-1}} F_{r_s} \right]. \end{aligned}$$

□

As an immediate consequence of Theorem 3 we have :

**Corollary 4.** *For any non-negative integers $s$ and $r$, we have*

$$F_{sr+1} = F_{r+1} F_{(s-1)r+1} + F_r^2 \sum_{i=0}^{s-2} F_{r-1}^i F_{(s-i-2)r+1}.$$

*Proof.* Use Theorem 3 with $r_1 = r_2 = \cdots = r_s = r$. □

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# AN ANALOG OF NEIFELD'S CONNECTION INDUCED ON THE SPACE OF CENTRED PLANES

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*Abstract.* The paper concerns to investigations in the field of differential geometry. It is realized by a method of prolongations and scopes of G.F. Laptev which generalizes a moving frame method and Cartan's exterior forms method and bases on calculation of exterior differential forms. The space $\Pi$ of centred $m$–planes is considered in projective space $P_n$. Principal fiber bundle is arised above $\Pi$. An analog of Neifeld's connection is given in this principal fibering. It is proved that the analog of the Norden normalization of the space of centred planes induces Neifeld's connection. The torsion of Neifeld's connection analog is introduced. It is shown, that this object is a tensor.

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*Keywords:* projective space, space of centred planes, principal fiber bundle, Neifeld's connection

## 1. Introduction

The object of research of this paper is an analog of Neifeld's connection on the space of centred planes.

We consider the space of centred planes as space of all $m$–dimensional centred planes in $n$–dimensional projective space. The non-classical analytical method is applied in this paper. It has advantage at allocation of subgroups and factor groups of a projective group. The connections in the fiberings associated with Grassmann manifold and the space of centred planes were investigated in [1] by such method. We show that an analog of Neifeld's connection is induced on the space of centred planes. By methods of tensor analysis E.G. Neifeld considered two dual linear connections (compare [3]), associated with normalized Grassmann manifold.

We put projective space $P_n$ to the moving frame $\{A, A_I\}$ $(I, J, K, ... = \overline{1,n})$ with the derivation formulae

$$dA = \theta A + \omega^I A_I, \quad dA_I = \theta A_I + \omega_I^J A_J + \omega_I A. \tag{1.1}$$

The Pfaffian forms $\omega^I, \omega_J^I, \omega_I$ satisfy the Cartan structure equations of the projective group $GP(n)$ (see, eg., [5]):

$$D\omega^I = \omega^J \wedge \omega_J^I, \quad D\omega_I = \omega_I^J \wedge \omega_J,$$

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$$D\omega_J^I = \omega_J^K \wedge \omega_K^I + \delta_J^I \omega_K \wedge \omega^K + \omega_J \wedge \omega^I. \tag{1.2}$$

## 2. The space of centred planes

The space $\Pi$ [1] of all centred $m$–dimensional planes $L_m^*$ is considered in $P_n$. Let's produce a specialization of the moving frame $\{A, A_a, A_\alpha\}$ $(a, \cdots = \overline{1,m}; \quad \alpha, \cdots = \overline{m+1,n})$ putting the top $A$ in the centre of $m$-dimensional plane $L_m$ and putting the tops $A_a$ on the plane $L_m$.

From the derivation formulae (1.1) it follows that the equations $\omega^\alpha = 0$, $\omega^a = 0$, $\omega_a^\alpha = 0$ are stationarity conditions of the centred planes $L_m^*$, i.e. forms $\omega^\alpha$, $\omega^a$, $\omega_a^\alpha$ are principal forms for the space $\Pi$. The basic forms of the space $\Pi$ satisfy the equations following from the structure Cartan equations (1.2)

$$\begin{aligned} D\omega^\alpha &= \omega^a \wedge \Omega_a^\alpha + \omega^\beta \wedge \Omega_\beta^\alpha, \\ D\omega^a &= \omega^b \wedge \Omega_b^a + \omega^\alpha \wedge \Omega_\alpha^a, \\ D\omega_a^\alpha &= \omega_b^\beta \wedge \Omega_{\beta a}^{\alpha b} + \omega^\alpha \wedge \Omega_a, \end{aligned} \tag{2.1}$$

where

$$\Omega_\beta^\alpha = \omega_\beta^\alpha, \quad \Omega_b^a = \omega_b^a, \quad \Omega_\alpha^a = \omega_\alpha^a, \quad \Omega_a = -\omega_a, \quad \Omega_{\beta a}^{\alpha b} = \delta_a^b \Omega_\beta^\alpha - \delta_\beta^\alpha \Omega_a^b. \tag{2.2}$$

*Remark* 1. The forms $\Omega_a^\alpha = \omega_a^\alpha$ are the basic-fibre forms.

Differentiating the forms (2.2) we obtain

$$D\Omega_b^a = \Omega_b^c \wedge \Omega_c^a + \omega^\alpha \wedge \Omega_{b\alpha}^a + \omega^c \wedge \Omega_{bc}^a + \omega_b^\alpha \wedge \Omega_\alpha^a, \tag{2.3}$$

$$D\Omega_\beta^\alpha = \Omega_\beta^\gamma \wedge \Omega_\gamma^\alpha + \omega^\gamma \wedge \Omega_{\beta\gamma}^\alpha + \omega^a \wedge \Omega_{\beta a}^\alpha - \omega_a^\alpha \wedge \Omega_\beta^a, \tag{2.4}$$

$$D\Omega_\alpha^a = \Omega_{b\alpha}^{\beta a} \wedge \Omega_\beta^b + \omega^a \wedge \Omega_\alpha, \tag{2.5}$$

$$D\Omega_a = \Omega_a^b \wedge \Omega_b + \omega_a^\alpha \wedge \Omega_\alpha, \tag{2.6}$$

where

$$\Omega_{\beta\gamma}^\alpha = -\delta_\beta^\alpha \omega_\gamma - \delta_\gamma^\alpha \omega_\beta, \quad \Omega_{\beta a}^\alpha = -\delta_\beta^\alpha \omega_a,$$
$$\Omega_{bc}^a = -\delta_b^a \omega_c - \delta_c^a \omega_b, \quad \Omega_{b\alpha}^a = -\delta_b^a \omega_\alpha, \quad \Omega_\alpha = -\omega_\alpha.$$

The principal fiber bundle $\mathfrak{L}(\Pi)$ is constructed over the space $\Pi$ of centred planes and the Lie group $\mathfrak{L}$ is the typical fiber. This group acts in the tangent space [2] to the space $\Pi$.

**Theorem 1.** *The principal fiber bundle $\mathfrak{L}(\Pi)$ contains the following factorfiberings:*

(1) *factorfibering of linear frames belonging to the centred plane $L_m^*$ which typical fiber is linear factorgroup acting in a bunch of lines belonging to a plane $L_m^*$, with the structure equations (2.1) and (2.3);*
(2) *factorfibering of normal linear frames with the structure equations (2.1) and (2.4);*

(3) *factorfibering of coaffine frames belonging to the plane $L^*_m$ which typical fiber is coaffine factorgroup acting in the centred plane with the structure equations (2.1), (2.3) and (2.6);*

(4) *maximal affine factorfibering which typical fiber is factorgroup acting in a bunch of lines with centre A; the factorfibering has structure equations (2.1) and (2.3 – 2.5).*

## 3. An analog of Neifeld's connection

In the principal fiber bundle we set an analog of Neifeld's connection [3] by the Laptev — Lumiste way.

Entering new forms

$$\tilde{\Omega}^a_b = \Omega^a_b - \Gamma^a_{b\alpha}\omega^\alpha - \Gamma^a_{bc}\omega^c - \Gamma^{ac}_{b\alpha}\omega^\alpha_c, \quad \tilde{\Omega}^\alpha_\beta = \Omega^\alpha_\beta - \Gamma^\alpha_{\beta\gamma}\omega^\gamma - \Gamma^\alpha_{\beta a}\omega^a - \Gamma^{\alpha a}_{\beta\gamma}\omega^\gamma_a,$$

$$\tilde{\Omega}^a_\alpha = \Omega^a_\alpha - \Gamma^a_{\alpha\beta}\omega^\beta - \Gamma^a_{\alpha b}\omega^b - \Gamma^{ab}_{\alpha\beta}\omega^\beta_b, \quad \tilde{\Omega}_a = \Omega_a - \Gamma_{a\alpha}\omega^\alpha - \Gamma_{ab}\omega^b - L^b_{a\alpha}\omega^\alpha_b$$

and finding their exterior differentials we get that connection in the principal fiber bundle $\mathfrak{L}(\Pi)$ is set with the help of a field of connection object

$$\Gamma = \{\Gamma^a_{b\alpha}, \Gamma^a_{bc}, \Gamma^{ac}_{b\alpha}, \Gamma^\alpha_{\beta\gamma}, \Gamma^\alpha_{\beta a}, \Gamma^{\alpha a}_{\beta\gamma}, \Gamma^a_{\alpha\beta}, \Gamma^a_{\alpha b}, \Gamma^{ab}_{\alpha\beta}, \Gamma_{a\alpha}, \Gamma_{ab}, L^b_{a\alpha}\}$$

on the base $\Pi$. We found differential equations of the connection object components:

$$\begin{gathered}
\Delta\Gamma^a_{b\alpha} - \Gamma^a_{bc}\Omega^c_\alpha - \Gamma^{ac}_{b\alpha}\Omega_c + \delta^a_b\Omega_\alpha = \Gamma^a_{b\alpha}\,|_\beta\,\omega^\beta + \Gamma^a_{b\alpha}\,|_c\,\omega^c + \Gamma^a_{b\alpha}\,|^c_\beta\,\omega^\beta_c,\\
\Delta\Gamma^a_{bc} + \delta^a_b\Omega_c + \delta^a_c\Omega_b = \Gamma^a_{bc}\,|_\alpha\,\omega^\alpha + \Gamma^a_{bc}\,|_e\,\omega^e + \Gamma^a_{bc}\,|^e_\alpha\,\omega^\alpha_e,\\
\Delta\Gamma^{ac}_{b\alpha} + \delta^c_b\Omega^a_\alpha = \Gamma^{ac}_{b\alpha}\,|_\beta\,\omega^\beta + \Gamma^{ac}_{b\alpha}\,|_e\,\omega^e + \Gamma^{ac}_{b\alpha}\,|^e_\beta\,\omega^\beta_e,\\
\Delta\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta a}\Omega^a_\gamma - \Gamma^{\alpha a}_{\beta\gamma}\Omega_a + \Omega^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma}\,|_\mu\,\omega^\mu + \Gamma^\alpha_{\beta\gamma}\,|_a\,\omega^a + \Gamma^\alpha_{\beta\gamma}\,|^a_\mu\,\omega^\mu_a,\\
\Delta\Gamma^\alpha_{\beta a} + \delta^\alpha_\beta\Omega_a = \Gamma^\alpha_{\beta a}\,|_\gamma\,\omega^\gamma + \Gamma^\alpha_{\beta a}\,|_b\,\omega^b + \Gamma^\alpha_{\beta a}\,|^b_\gamma\,\omega^\gamma_b,\\
\Delta\Gamma^{\alpha a}_{\beta\gamma} - \delta^\alpha_\gamma\Omega^a_\beta = \Gamma^{\alpha a}_{\beta\gamma}\,|_\mu\,\omega^\mu + \Gamma^{\alpha a}_{\beta\gamma}\,|_b\,\omega^b + \Gamma^{\alpha a}_{\beta\gamma}\,|^b_\mu\,\omega^\mu_b, \qquad (3.1)\\
\Delta\Gamma^a_{\alpha\beta} - \Gamma^a_{\alpha b}\Omega^b_\beta - \Gamma^{ab}_{\alpha\beta}\Omega_b - \Gamma^a_{b\beta}\Omega^b_\alpha + \Gamma^\gamma_{\alpha\beta}\Omega^a_\gamma = \Gamma^a_{\alpha\beta}\,|_\gamma\,\omega^\gamma + \Gamma^a_{\alpha\beta}\,|_b\,\omega^b + \Gamma^a_{\alpha\beta}\,|^b_\gamma\,\omega^\gamma_b,\\
\Delta\Gamma^a_{\alpha b} - \Gamma^a_{cb}\Omega^c_\alpha + \Gamma^\beta_{\alpha b}\Omega^a_\beta + \delta^a_b\Omega_\alpha = \Gamma^a_{\alpha b}\,|_\beta\,\omega^\beta + \Gamma^a_{\alpha b}\,|_c\,\omega^c + \Gamma^a_{\alpha b}\,|^c_\beta\,\omega^\beta_c,\\
\Delta\Gamma^{ab}_{\alpha\beta} + \Gamma^{\gamma b}_{\alpha\beta}\Omega^a_\gamma - \Gamma^{ab}_{c\beta}\Omega^c_\alpha = \Gamma^{ab}_{\alpha\beta}\,|_\gamma\,\omega^\gamma + \Gamma^{ab}_{\alpha\beta}\,|_c\,\omega^c + \Gamma^{ab}_{\alpha\beta}\,|^c_\gamma\,\omega^\gamma_c,\\
\Delta\Gamma_{a\alpha} - \Gamma_{ab}\Omega^b_\alpha + (\Gamma^b_{a\alpha} - L^b_{a\alpha})\Omega_b = \Gamma_{a\alpha}\,|_\beta\,\omega^\beta + \Gamma_{a\alpha}\,|_b\,\omega^b + \Gamma_{a\alpha}\,|^b_\beta\,\omega^\beta_b,\\
\Delta\Gamma_{ab} + \Gamma^c_{ab}\Omega_c = \Gamma_{ab}\,|_\alpha\,\omega^\alpha + \Gamma_{ab}\,|_c\,\omega^c + \Gamma_{ab}\,|^c_\alpha\,\omega^\alpha_c,\\
\Delta L^b_{a\alpha} + \Gamma^{cb}_{a\alpha}\Omega_c + \delta^b_a\Omega_\alpha = L^b_{a\alpha}\,|_\beta\,\omega^\beta + L^b_{a\alpha}\,|_c\,\omega^c + L^b_{a\alpha}\,|^c_\beta\,\omega^\beta_c.
\end{gathered}$$

There are Pfaffian derivatives in the right side of the equations (3.1).

**Theorem 2.** *The object of group connection $\Gamma$ contains four simple geometric subobjects $\Gamma_1 = \{\Gamma^{ac}_{b\alpha}, \Gamma^{a}_{bc}, \Gamma^{a}_{b\alpha}\}$, $\Gamma_2 = \{\Gamma^{\alpha a}_{\beta\gamma}, \Gamma^{\alpha}_{\beta a}, \Gamma^{\alpha}_{\beta\gamma}\}$, $\Gamma_3 = \{\Gamma_1, \Gamma_{a\alpha}, \Gamma_{ab}, Ł^{b}_{a\alpha}\}$, $\Gamma_4 = \{\Gamma_1, \Gamma_2, \Gamma^{a}_{\alpha b}, \Gamma^{ab}_{\alpha\beta}, \Gamma^{a}_{\alpha\beta}\}$ giving the connection in the corresponding factorfiberings.*

We realize an analogue of the strong Norden normalization [4] of the manifold by fields of the following geometrical images: $(n-m-1)$-dimensional plane $P_{n-m-1}$, not having the common points with the plane $L^*_m$, and $(m-1)$-dimensional plane $P_{m-1}$, belonging to the plane $L^*_m$ and not passing through its centre $A$.

We define the planes $P_{n-m-1}$ and $P_{m-1}$ by the points, respectively

$$B_\alpha = A_\alpha + \lambda^a_\alpha A_a + \lambda_\alpha A, \quad B_a = A_a + \lambda_a A.$$

Demanding a relative invariancy of the clothing planes we obtain the differential equations for the components of clothing geometrical object

$$\Delta\lambda^a_\alpha + \Omega^a_\alpha \equiv 0, \quad \Delta\lambda_\alpha - \lambda^a_\alpha \Omega_a - \omega_\alpha \equiv 0, \quad \Delta\lambda_a - \Omega_a \equiv 0. \tag{3.2}$$

The analog of the strong Norden normalization [4] of the spaces $\Pi$ allows to cover the components of the connection object $\Gamma$

$$\begin{gathered}
\Gamma^{ac}_{b\alpha} = \delta^c_b \lambda^a_\alpha, \quad \Gamma^{a}_{bc} = -\delta^a_b \lambda_c - \delta^a_c \lambda_b, \quad \Gamma^{a}_{b\alpha} = -\delta^a_b \mu_\alpha + \lambda^a_\alpha \lambda_b, \\
\Gamma^{\alpha a}_{\beta\gamma} = -\delta^\alpha_\gamma \lambda^a_\beta, \quad \Gamma^{\alpha}_{\beta a} = -\delta^\alpha_\beta \lambda_a, \quad \Gamma^{\alpha}_{\beta\gamma} = -\delta^\alpha_\gamma \lambda_\beta - \delta^\alpha_\beta \mu_\gamma, \\
\Gamma^{ab}_{\alpha\beta} = -\lambda^b_\alpha \lambda^a_\beta, \quad \Gamma^{a}_{\alpha b} = -\delta^a_b \mu_\alpha, \quad \Gamma^{a}_{\alpha\beta} = -\lambda_b \lambda^b_\alpha \lambda^a_\beta, \\
L^{b}_{a\alpha} = -\delta^b_a \lambda_\alpha, \quad \Gamma_{ab} = \lambda_a \lambda_b, \quad \Gamma_{a\alpha} = -\lambda_a \lambda_b \lambda^b_\alpha,
\end{gathered} \tag{3.3}$$

where $\mu_\alpha = \lambda_\alpha - \lambda^a_\alpha \lambda_a$.

The functions (3.3) by virtue of the comparisons (3.2) satisfy the differential equations (3.1) for the components of the connection object $\Gamma$.

Thus, we have

**Theorem 3.** *The analog of the strong Norden normalization of the space $\Pi$ induces analog of Neifeld's connection in the associated fibering $\mathfrak{L}(\Pi)$.*

## 4. A torsion of Neifeld's connection analog on the space of centred planes

Putting into the structure equations (2.1) of the basic forms $\omega^\alpha$, $\omega^a$ and $\omega^\alpha_a$ of the space $\Pi$ the connection forms $\tilde\Omega^a_b$, $\tilde\Omega^\alpha_\beta$, $\tilde\Omega^a_\alpha$, $\tilde\Omega_a$ we obtain the equations:

$$D\omega^\alpha = \omega^\beta \wedge \tilde\Omega^\alpha_\beta + \underline{\omega^a \wedge \omega^\alpha_a} + S^\alpha_{\beta\gamma}\omega^\beta \wedge \omega^\gamma + S^\alpha_{\beta a}\omega^\beta \wedge \omega^a + S^{\alpha a}_{\beta\gamma}\omega^\beta \wedge \omega^\gamma_a, \tag{4.1}$$

$$\begin{gathered}
D\omega^\alpha_a = \omega^\beta_b \wedge (\delta^b_a \tilde\Omega^\alpha_\beta - \delta^\alpha_\beta \tilde\Omega^b_a) + \omega^\alpha \wedge \tilde\Omega_a + S^\alpha_{a\beta\gamma}\omega^\beta \wedge \omega^\gamma + S^\alpha_{a\beta b}\omega^\beta \wedge \omega^b + \\
S^{\alpha b}_{a\beta\gamma}\omega^\beta \wedge \omega^\gamma_b + S^{\alpha b}_{a\beta c}\omega^c \wedge \omega^\beta_b + S^{\alpha bc}_{a\beta\gamma}\omega^\beta_b \wedge \omega^\gamma_c, \\
D\omega^a = \omega^b \wedge \tilde\Omega^a_b + \omega^\alpha \wedge \tilde\Omega^a_\alpha + S^a_{\alpha\beta}\omega^\alpha \wedge \omega^\beta + S^a_{\alpha b}\omega^\alpha \wedge \omega^b + S^{ab}_{\alpha\beta}\omega^\alpha \wedge \omega^\beta_b +
\end{gathered}$$

$$S^a_{bc}\omega^b\wedge\omega^c + S^{ac}_{b\alpha}\omega^b\wedge\omega^\alpha_c,$$

where

$$S^\alpha_{\beta\gamma} = \Gamma^\alpha_{[\beta\gamma]},\quad S^\alpha_{\beta a} = \Gamma^\alpha_{\beta a},\quad S^{\alpha a}_{\beta\gamma} = \Gamma^{\alpha a}_{\beta\gamma},$$

$$S^\alpha_{a\beta\gamma} = \delta^\alpha_{[\beta}\Gamma_{a\gamma]},\quad S^\alpha_{a\beta b} = \delta^\alpha_\beta\Gamma_{ab},\quad S^{\alpha b}_{a\beta\gamma} = \delta^\alpha_\beta L^b_{a\gamma} + \delta^\alpha_\gamma\Gamma^b_{a\beta} - \delta^b_a\Gamma^\alpha_{\gamma\beta},$$

$$S^{\alpha b}_{a\beta c} = \delta^\alpha_\beta\Gamma^b_{ac} - \delta^b_a\Gamma^\alpha_{\beta c},\quad S^{\alpha bc}_{a\beta c} = \delta^{[b}_a\Gamma^{\alpha c}_{\lfloor\beta\gamma]} - \delta^\alpha_{\lfloor\beta}\Gamma^{[bc]}_{a\gamma},$$

$$S^a_{\alpha\beta} = \Gamma^a_{[\alpha\beta]},\quad S^a_{\alpha b} = \Gamma^a_{\alpha b} - \Gamma^a_{b\alpha},\quad S^{ab}_{\alpha\beta} = \Gamma^{ab}_{\alpha\beta},\quad S^a_{bc} = \Gamma^a_{[bc]},\quad S^{ac}_{b\alpha} = \Gamma^{ac}_{b\alpha}.$$

Here the square brackets mean an alternation with the last indices and pairs of indices.

We represent the underlined term in formula (4.1) as

$$\omega^a\wedge\omega^\alpha_a = \delta^\alpha_\beta\delta^a_b\omega^b\wedge\omega^\beta_a.$$

Then the equation (4.1) takes the form

$$D\omega^\alpha = \omega^\beta\wedge\tilde{\Omega}^\alpha_\beta + S^{\alpha a}_{\beta b}\omega^b\wedge\omega^\beta_a + S^\alpha_{\beta\gamma}\omega^\beta\wedge\omega^\gamma + S^\alpha_{\beta a}\omega^\beta\wedge\omega^a + S^{\alpha a}_{\beta\gamma}\omega^\beta\wedge\omega^\gamma_a,$$

where we use the new components $S^{\alpha a}_{\beta b} = \delta^\alpha_\beta\delta^a_b$ of the torsion $S$.

The components of the torsion object $S$ satisfy the differential comparisons modulo the basic forms $\omega^\alpha$, $\omega^a$, $\omega^\alpha_a$:

$$\Delta S^\alpha_{\beta\gamma} - S^\alpha_{[\beta a}\omega^a_{\gamma]} + S^{\alpha a}_{[\beta\gamma]}\omega_a \equiv 0,\quad \Delta S^\alpha_{\beta a} - S^{\alpha b}_{\beta a}\omega_b \equiv 0,\quad \Delta S^{\alpha a}_{\beta\gamma} - S^{\alpha a}_{\gamma b}\omega^b_\beta \equiv 0,$$

$$\Delta S^{\alpha a}_{\beta b} \equiv 0,\quad \Delta S^\alpha_{a\beta\gamma} - S^\alpha_{a[\beta b}\omega^b_{\gamma]} + S^{\alpha b}_{a[\beta\gamma]}\omega_b - S^\alpha_{\beta\gamma}\omega_a \equiv 0,\quad \Delta S^\alpha_{a\beta b} - S^{\alpha c}_{a\beta b}\omega_c - S^\alpha_{\beta b}\omega_a \equiv 0,$$

$$\Delta S^{\alpha b}_{a\beta\gamma} - S^{\alpha b}_{a\gamma c}\omega^c_\beta + 2S^{\alpha cb}_{a\beta\gamma}\omega_c - S^{\alpha b}_{\beta\gamma}\omega_a \equiv 0,\quad \Delta S^{\alpha b}_{a\beta c} - S^{\alpha b}_{\beta c}\omega_b \equiv 0,\quad \Delta S^{\alpha bc}_{a\beta\gamma} \equiv 0,$$

$$\Delta S^a_{\alpha\beta} - S^a_{[\alpha b}\omega^b_{\beta]} + S^{ab}_{[\alpha\beta]}\omega_b + S^\gamma_{\alpha\beta}\omega^a_\gamma \equiv 0,\quad \Delta S^a_{\alpha b} + 2S^a_{bc}\omega^c_\alpha + S^\beta_{\alpha b}\omega^a_\beta - S^{ac}_{b\alpha}\omega_c \equiv 0,$$

$$\Delta S^{ab}_{\alpha\beta} - S^{ab}_{c\beta}\omega^c_\alpha + S^{\gamma b}_{\alpha\beta}\omega^a_\gamma \equiv 0,\quad \Delta S^a_{bc} \equiv 0,\quad \Delta S^{ac}_{b\alpha} + S^{\beta c}_{\alpha b}\omega^a_\beta \equiv 0.$$

From these comparisons we obtain the following theorem.

**Theorem 4.** *The torsion object* $S = \{S^{\alpha a}_{\beta b} = \delta^\alpha_\beta\delta^a_b, S^\alpha_{\beta\gamma}, S^\alpha_{\beta a}, S^{\alpha a}_{\beta\gamma}, S^\alpha_{a\beta\gamma}, S^\alpha_{a\beta b}, S^{\alpha b}_{a\beta\gamma}, S^{\alpha b}_{a\beta c}, S^{\alpha bc}_{a\beta\gamma}, S^a_{\alpha\beta}, S^a_{\alpha b}, S^{ab}_{\alpha\beta}, S^a_{bc}, S^{ac}_{b\alpha}\}$ *of the connection* $\Gamma$ *is a tensor. It contains three elementary subtensors and four simple subtensors.*

*Remark* 2. Since the torsion subobject $S^{\alpha a}_{\beta b} = \delta^\alpha_\beta\delta^a_b$ is a nonzero tensor, the connection $\Gamma$ is always a connection with torsion.

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# EQUIVALENCE OF STRONGLY CONNECTED GRAPHS AND BLACK-AND-WHITE 2-SAT PROBLEMS

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*Abstract.* Our goal is to create a propositional logic formula to model a directed graph and use a SAT solver to analyse it. This model is similar to the well-know one of Aspvall et al., but they create a directed graph from a 2-SAT problem, we generate a 2-SAT problem from a directed graph. In their paper if the 2-SAT problem is unsatisfiable, then the generated directed graph is strongly connected, in our case, if the directed graph is strongly connected, then the generated 2-SAT problem is a black-and-white 2-SAT problem, which has two solutions: where each variable is true (the white assignment), and where each variable is false (the black one). If we see a directed graph as a communication model of a network, then we can ask in our model whether a node can send a message to another one through the network. More specifically we can ask whether all nodes can send messages to all other ones, i.e., the graph is strongly connected or not.

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*Keywords:* black-and-white SAT problem, SAT representation, communication graph, connectivity test, strongly connected graph

## 1. Introduction

Propositional satisfiability is the problem of determining, for a formula of the propositional logic, if there is an assignment of truth values to its variables for which that formula evaluates to true. By SAT we mean the problem of propositional satisfiability for formulas in conjunctive normal form (CNF). SAT is one of the most-researched NP-complete problems [14] in several fields of computer science, including theoretical computer science, artificial intelligence, hardware design, and formal verification [1]. Modern sequential SAT solvers are based on the Davis-Putnam-Logemann-Loveland (DPLL) [11] algorithm. This algorithm performs Boolean constraint propagation (BCP) and variable branching, i.e., at each node of the search tree it selects a decision variable, assigns a truth value to it, and steps back when a conflict occurs.

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By $k$-SAT we mean the problem of propositional satisfiability for formulas in CNF, where each clause consists of at most $k$ literals. While 2-SAT is solvable in linear time [2], 3-SAT is NP-complete [9]. These days one of the most promising branch of mathematics is the idea, that we try to unify different mathematical theories, like in case of Langlands program [10], which relates algebraic number theory to automorphic forms and representation theory. Another nice example is the modularity theorem [16] (formerly called the Taniyama–Shimura–Weil conjecture), which states that elliptic curves over the field of rational numbers are related to modular forms. Without the modularity theorem Andrew Wiles could not prove Fermat's Last Theorem [17].

In this paper we show a link between directed graphs and propositional logic formulas. We prove a theorem which allows to use an algorithm from the field of propositional logic to check a graph property. Namely, we transform a graph into a SAT problem to check whether the graph is strongly connected or not.

The most prominent graph representations are:

- Implication graph [2] is a is a skew-symmetric directed graph, where vertices are literals (boolean variables, and their negation), edges represents implication. Note, that the binary clause $x \vee y$ is represented by two implications in the implication graph: $\neg x \supset y$, and $\neg y \supset x$, and so the implication graph is skew-symmetric, i.e., it is isomorphic to its own transpose graph.
- AIG, And-Inverter Graph [7] is directed acyclic graph where vertices are logical conjunction with two input edges, a marked edge means logical negation, the boolean variables are the input, the formula itself is the output.
- BDD, Reduced Ordered Binary Decision Diagram [5], which is a rooted, directed, acyclic graph consisting of vertices, which are boolean variables and terminal vertices, called 0-terminal, which terminates pathes where the formula evaluates to false; and 1-terminal, which terminates pathes, where the formula evaluates to true. Each non-terminal vertex has two child vertices called low child, corresponding edge is called 0-edge; and high child, corresponding edge is called 1-edge; which are possible values of the parent vertex. One has to merge any isomorphic subgraph and eliminate any vertex whose two children are isomorphic.
- ZDD (called also ZBDD in the literature), Zero-Suppressed Binary Decision Diagram [12], is a kind of binary decision diagram, where instead of the rule "eliminate any vertex whose two children are isomorphic" we use the rule "eliminate those vertices whose 1-edge points directly to 0-terminal". If a SAT problem has only a few solutions then ZDD is a better representation than BDD.

As we can see a great effort has been done in the direction from formulas to graphs. In this paper we study the other way, the direction from graphs to formulas. In our model vertices are boolean variables, and edges are implications. This means that our

model is similar to an implication graph, but in case of implication graphs vertices are literals. The intuition behind our model comes from the field of wireless sensor networks (WSN), where it is a relevant problem whether each sensor can communicate with all other ones through the network. If the network is represented by a directed graph where vertices are the sensor, and an edge represents that a sensor can send data to an other one, then this problem boils down to check whether the graph is strongly connected.

We wanted to solve this problem by a SAT solver, so we had to convert the above directed graph into a SAT problem. Since in our model each edge represents a logical implication we can generate a 2-SAT problem where each clause contains exactly one positive and one negative literal.

We have found that the graph is strongly connected if this 2-SAT problem has exactly two solutions: the first is the one where each boolean variable is true (which is called the white assignment), the second is the one where each boolean variable is false (which is called the black assignment). We call such a SAT problem to be a ”black-and-white” SAT problem.

This means that if we add the negation of these two solutions (the black one and the white one) to the generated SAT problem, then it will be unsatisfiable and will be not 2-SAT any more.

In other words, a SAT problem is a black-and-white SAT problem if and only if it is satisfiable and has only two solutions, the white assignment and the black one. So it becomes unsatisfiable if we add the negation of these assignments, which are two full length clauses, the clause which contains only negative literals (which is called the black clause), and the clause which contains only negative literals (which is called the white clause).

In the field of directed graphs the problem to check strongly connectedness is a linear time problem [15]. The black-and-white 2-SAT problem is also a linear time problem as we are going to show that later in this paper. The question arises, while should we transform a graph into a SAT problem to check a property which can be checked in linear time in both fields? We think that our model is a new link between the two fields: We learned that black-and-white 2-SAT problems and strongly connected graphs are equivalent. The black-and-white SAT problem appears also as a special case of weakly nondecisive SAT problems, see Lemma 6. in [3] (in that paper we did not use the term ”black-and-white SAT problem”, this term is introduced in this paper). This suggests two things: the black-and-white SAT problem could be an interesting problem in general, and since there are weakly nondecisive 3-SAT problems, which are also black-and-white, there might be a 3-SAT representation of directed graphs.

## 2. Logical model of a directed graph

Let $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ be a directed graph, where $\mathcal{V}$ is the set of vertices, and $E$ is the set of edges. We say that $\mathcal{D}$ is a communication graph if and only if the elements of $\mathcal{V}$ are atomic formulas or each vertex is labeled by a different atomic formula. Note, that any directed graph is a communication graph because we can label each vertex by a different atomic formula. In this paper we assume that the elements of $\mathcal{V}$ are atomic formulas. In other words, if $x$ is an element of $\mathcal{V}$, then for example $\neg x$ must not be an element of $\mathcal{V}$. From a communication graph we create the following model: We represent vertices by boolean variables, and edges by logical implication. The conjunction of these formulas gives the logical model of the directed graph. For example, if the vertex $x_1$ has edges to vertices $x_2$ and $x_3$, then the logical model is:

$$(x_1 \supset x_2) \wedge (x_1 \supset x_3). \tag{2.1}$$

This formula can be easily transformed to a 2-SAT problem by rewriting implications by the rule $x \supset y = \neg x \vee y$. Note, that although each edge in $\mathcal{D}$ is interpreted as logical implication, it is not an implication graph, because it does not contain negative literals. In other words, our model is not the same used by Aspvall et al. in [2]. They create a graph from a 2-SAT problem, we generate a 2-SAT problem from a communication graph. In their directed graph each variable is represented by a positive and a negative literal. In our case we have only positive literals in the directed graph.

We give the definition of our model in a more formal way. The 2-SAT representation $\mathcal{M}$ of a communication graph $\mathcal{D}$ is defined as follows:

$$\mathcal{M} := \bigwedge_{i=1, j=1, i \neq j}^{n} \mathcal{P}(x_i, x_j) \tag{2.2}$$

$$\mathcal{P}(x, y) = \begin{cases} \neg x \vee y, & \text{if } x \text{ has an edge to } y \\ \text{True}, & \text{otherwise} \end{cases} \tag{2.3}$$

Note, that $\mathcal{M}$ is a 2-SAT problem and has a nice property, each clause in it has exactly one positive and one negative literal, therefore, $\mathcal{M}$ is satisfiable. It has at least two solutions: one where each variable is true which is called the white assignment, and one where each variable is false which is called the black assignment.

If $\mathcal{M}$ has only these two solutions, and no other one, then $\mathcal{D}$ has an interesting property, namely it strongly connected. To check this we need the white-or-black constraint, which is the disjunction of the white assignment and the black assignment, that states that all vertices can be reached from all other ones. We denote the white-or-black constraint by $\mathcal{C}$ in this paper, and we define it as follows:

$$\mathcal{C} := \bigwedge_{i=1, j=1, i \neq j}^{n} x_i \supset x_j = (x_1 \wedge x_2 \wedge \cdots \wedge x_n) \vee (\neg x_1 \wedge \neg x_2 \wedge \cdots \wedge \neg x_n) \tag{2.4}$$

Its negation is called the black-and-white constraint, which is the conjunction of the black clause and the white clause:

$$\neg\mathcal{C} = (\neg x_1 \vee \neg x_2 \vee \cdots \vee \neg x_n) \wedge (x_1 \vee x_2 \vee \cdots \vee x_n) \tag{2.5}$$

By the formula $\mathcal{C}$ we state that there is a path from any vertex to any other one. To show that $\mathcal{D}$ is strongly connected we have to show that the logical representation of $\mathcal{D}$, which is $\mathcal{M}$, implies $\mathcal{C}$. Now we can use the fact that SAT solving is the dual of theorem proving. So instead of proving that "$\mathcal{D}$ is strongly connected if $\mathcal{M}$ implies $\mathcal{C}$", we prove that "$\mathcal{D}$ is strongly connected if $\mathcal{M} \wedge \neg\mathcal{C}$ is unsatisfiable". This means that we can use a SAT solver to check a property of a graph.

Motivated by this we define the following notion: $F$ is a black-and-white SAT problem if and only if $F$ is satisfiable and has exactly two solutions, the white and the black assignments, which implies that $F \wedge \neg\mathcal{C}$ is unsatisfiable. $F$ is a black-and-white 2-SAT problem if and only if $F$ is a 2-SAT and a black-and-white SAT problem. Lemma 6 in [3] suggests an alternative, more general definition: $F$ is a black-and-white SAT problem if and only if $F$ is satisfiable and has exactly two solutions, $A$ and $B$ such that $A = \neg B$. But we do not use this alternative definition in this paper.

First, we need an auxiliary lemma, which states that: there is a path from vertex $x_i$ to $x_j$ if and only if $x_i \supset x_j$ is subsumed by the logical model of the graph.

**Lemma 1.** *Let $\mathcal{D}$ be a communication graph. Let $\mathcal{M}$ be the 2-SAT representation of $\mathcal{D}$. Then $\mathcal{M}$ implies the formula $x_i \supset x_j$ if and only if there is path from vertex $x_i$ to $x_j$ in graph $\mathcal{D}$.*

*Proof.* We know that $\mathcal{M}$ is a special SAT instance where each clause contains exactly one positive and one negative literal, hence, each resolvent of clauses from $\mathcal{M}$ is a binary clause with one positive and one negative literal. The main idea of the proof is that if we have two clauses $\neg v_1 \vee v_2$ and $\neg v_2 \vee v_3$, i.e., there is a path in $\mathcal{D}$ from $v_1$ to $v_3$, then by resolution we can generate $\neg v_1 \vee v_3$. □

Based on this lemma we can prove the main theorem of this paper which states that the notion of strongly connected graphs and the notion of black-and-white 2-SAT problems are equivalent. This means that the 2-SAT representation of a strongly connected graph is a black-and-white 2-SAT problem, and the other way around, the directed graph representation of a black-and-white 2-SAT problem is a strongly connected graph.

**Theorem 1.** *Let $\mathcal{D}$ be a communication graph. Let $\mathcal{M}$ be the 2-SAT representation of $\mathcal{D}$. Then $\mathcal{M}$ is a black-and-white 2-SAT problem if and only if the graph $\mathcal{D}$ is strongly connected.*

*Proof.* Let $\mathcal{D}$ be a communication graph. Let $\mathcal{M}$ be the 2-SAT representation of $\mathcal{D}$. We show that both directions hold.

($\Rightarrow$): The main idea of the proof if the following, if a formula is satisfied by all solutions of a SAT problem, then it is implied by this SAT problem. We know, that $\mathcal{M}$

has only two solutions, the white and the black assignments, both of them satisfy all $x_i \supset x_j$ shaped formulas, i.e., they are implied by $\mathcal{M}$. From this and from Lemma 1. we obtain that in $\mathcal{D}$ there is a path from any vertex to any other one, i.e., $\mathcal{D}$ is strongly connected.

($\Leftarrow$): The main idea of the proof is the following, since $\mathcal{D}$ is strongly connected we know that there is a path $v_1 \supset v_2, v_2 \supset v_3 \ldots, v_{z-1} \supset v_z, v_z \supset v_1$ which is cyclic and contains all vertices from $\mathcal{D}$, the corresponding clause set is $\{(\neg v_1 \vee v_2), (\neg v_2 \vee v_3), \ldots, (\neg v_{z-1} \vee v_z), (\neg v_z \vee v_1)\}$, which is a subset of $\mathcal{M}$, and which can be satisfied only by the white and the black assignments. From this and since each clause in $\mathcal{M}$ is a binary clause, we obtain, that $\mathcal{M}$ is a black-and-white 2-SAT problem. □

To check whether a directed graph is strongly connected or not, we need linear time [15]. We are going to show that its 2-SAT representation can be solved also in linear time.

**Theorem 2.** *Let $F$ be a black-and-white 2-SAT problem. Then we need linear time to show that $F \wedge \neg C$ is unsatisfiable.*

*Proof.* The main idea of the proof is that any SAT solver which uses variable branching and BCP, can show that $F \wedge \neg C$ is unsatisfiable by using 1 variable branching and 2 BCP steps as follows: Variable branching will result in a unit. Without lost of generality let us assume that it is a positive one. Then BCP will generate other positive units, because the binary clauses in $F$ contain exactly one positive and one negative literal. BCP will finally result in a conflict, because $\neg C$ contains the negation of the white assignment. Then the other branch will result in a negative unit. This enables a BCP step which will generate negative units, and which will terminate in a conflict because $\neg C$ contains also the negation of the black assignment. Since BCP is a linear time method [19] we need linear time to show that $F \wedge \neg C$ is unsatisfiable. □

Note that DPLL algorithm is a suitable choice to solve the above problem because it uses variable branching and BCP. The question arises, why should we transform a graph into a SAT problem to check a property which can be checked in linear time in both fields? We think that our model is a new link between the two fields which might help to visualize SAT problems. This is not a problem in case of a 2-SAT problem, but in case of 3-SAT it is a problem. This paper and Lemma 6. in [3] together suggest that there might be a 3-SAT representation of directed graphs which is a black-and-white 3-SAT problem. If one finds that representation, then this could help to visualize 3-SAT problems as a directed graph.

## 3. Our model and the Aspvall model

In this section we show that the constraint used by our model is more general than the one used by the model of Aspvall et al. Fist we show some examples. In Figure 1.

we see a directed graph with 5 vertices. This graph is also a communication graph, since there is no negation sign in the vertices. A directed graph is strongly connected if there is a path between all pairs of vertices. A strongly connected component of a directed graph is a maximal strongly connected subgraph. There are 2 strongly connected components ([1, 2, 3, 5], [4]) in the following graph.

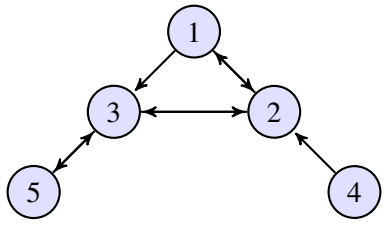


FIGURE 1. A directed graph with 5 vertices

The model of this graph is:

$$\begin{aligned}\mathcal{M} = (\neg x_1 \vee x_2) \wedge (\neg x_1 \vee x_3) \wedge (\neg x_2 \vee x_1) \wedge (\neg x_2 \vee x_3) \wedge \\ (\neg x_3 \vee x_2) \wedge (\neg x_3 \vee x_5) \wedge (\neg x_4 \vee x_2) \wedge (\neg x_5 \vee x_3)\end{aligned} \tag{3.1}$$

From $\mathcal{M}$ we can create $D(\mathcal{M})$ by following the construction defined in [2] by Aspvall et al. Construction steps are the following:

i. For each variable $x_i$, we add two vertices named $x_i$ and $\neg x_i$ to $D(\mathcal{F}_{\mathcal{M}})$.
ii. For each clause $(x_i \vee x_j)$ of $\mathcal{M}$, we add edges $\neg x_i \to x_j$ and $\neg x_j \to x_i$ to $D(\mathcal{M})$.

In Figure 2. we see the transformed $D(\mathcal{M})$ graph. Aspvall et al. [2] *(Theorem I.)* shows that $\mathcal{M}$ is satisfiable if and only if in $D(\mathcal{M})$ there is no vertex $x_i$ such that it is in the same strong component as its complement $\neg x_i$. We can be sure that this property holds for our model $\mathcal{M}$, because it is always satisfiable.

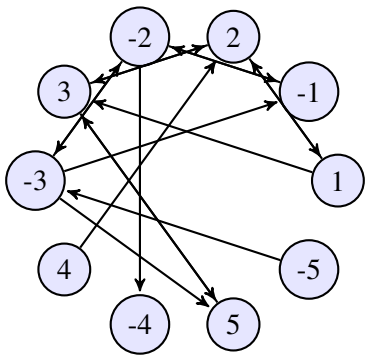


FIGURE 2. $D(\mathcal{F})$, where $\mathcal{F} = \mathcal{M}$

We define the so-called Aspvall kind constraints.

Let $\mathcal{C}_{ASP}(i,j) = (x_i \supset x_j) \wedge (x_i \supset x_j)$ , where $1 \leq i,j \leq n$ and $i \neq j$. From this we obtain after simplification that $\neg\mathcal{C}_{ASP}(i,j) = (x_i \vee x_j) \wedge (\neg x_j \vee \neg x_i)$. We can add these clauses to the model to check whether there is a directed path from $x_i$ vertex to $x_j$ and vice versa: $D(\mathcal{M} \wedge \neg\mathcal{C}_{ASP}(i,j))$.

We add two constraints to the example shown in Figure 1. The first one is $\mathcal{C}_{ASP}(1,4) = (x_1 \supset x_4) \wedge (x_4 \supset x_1)$. Let $\mathcal{F} = \mathcal{M} \wedge \neg\mathcal{C}_{ASP}(1,4)$. So we add these two clauses $(x_1 \vee x_4) \wedge (\neg x_1 \vee \neg x_4)$ and these 4 edges, see the red arrows in Figure 3. We see on Figure 3. that in graph $\mathcal{D}(\mathcal{F})$ there exists no vertex $x_i$ which is in the same strong component as its complement $\neg x_i$, hence, $\mathcal{F}$ is still satisfiable.

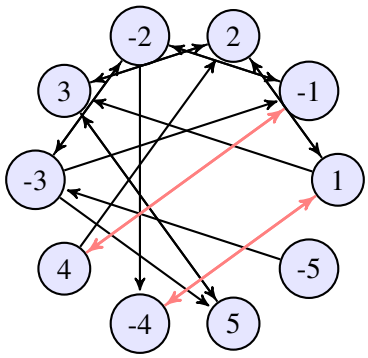


FIGURE 3. $D(\mathcal{F})$, where $\mathcal{F} = \mathcal{M} \wedge \neg\mathcal{C}_{ASP}(1,4)$

Now we add $\mathcal{C}_{ASP}(1,5) = (x_1 \supset x_5) \wedge (x_5 \supset x_1)$. Let $\mathcal{F}' = \mathcal{M} \wedge \neg\mathcal{C}_{ASP}(1,5)$. So we add these two clauses $(x_1 \vee x_5) \wedge (\neg x_1 \vee \neg x_5)$ and these 4 edges, see the red arrows in Figure 4. We can see on Figure 4. that in the graph $\mathcal{D}(\mathcal{F}')$ there exists a vertex $x_i$ which is in the same strong component as its complement $\neg x_i$, hence, $\mathcal{F}'$ is unsatisfiable.

In our approach $C$ is

$$\begin{aligned}(x_1 \supset x_2) \wedge (x_1 \supset x_3) \wedge (x_1 \supset x_4) \wedge \\ (x_2 \supset x_1) \wedge (x_2 \supset x_3) \wedge (x_2 \supset x_4) \wedge \\ (x_3 \supset x_1) \wedge (x_3 \supset x_2) \wedge (x_3 \supset x_4) \wedge \\ (x_4 \supset x_1) \wedge (x_4 \supset x_2) \wedge (x_4 \supset x_3)\end{aligned} \tag{3.2}$$

$$\begin{aligned}(\neg x_1 \vee x_2) \wedge (\neg x_1 \vee x_3) \wedge (\neg x_1 \vee x_4) \wedge (\neg x_1 \vee x_5) \wedge \\ (\neg x_2 \vee x_1) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_2 \vee x_4) \wedge (\neg x_2 \vee x_5) \wedge \\ (\neg x_3 \vee x_1) \wedge (\neg x_3 \vee x_2) \wedge (\neg x_3 \vee x_4) \wedge (\neg x_3 \vee x_5) \wedge \\ (\neg x_4 \vee x_1) \wedge (\neg x_4 \vee x_2) \wedge (\neg x_4 \vee x_3) \wedge (\neg x_4 \vee x_5) \wedge \\ (\neg x_5 \vee x_1) \wedge (\neg x_5 \vee x_2) \wedge (\neg x_5 \vee x_3) \wedge (\neg x_5 \vee x_4)\end{aligned} \tag{3.3}$$

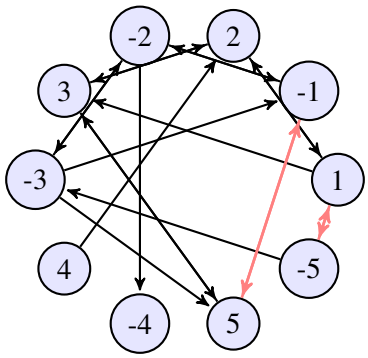


FIGURE 4. $D(\mathcal{F}')$, where $\mathcal{F}' = \mathcal{M} \wedge \neg\mathcal{C}_{ASP}(1,5)$

After simplification $\neg C$ is the following:

$$(x_1 \vee x_2 \vee x_3 \vee x_4 \vee x_5) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3 \vee \neg x_4 \vee \neg x_5) \tag{3.4}$$

For any $i$ and $j$ we have that $\neg\mathcal{C}_{ASP}(i,j)$ implies $\neg\mathcal{C}$, thus $\neg\mathcal{C}_{ASP}(i,j)$ subsumes $\neg\mathcal{C}$. This means that $\neg\mathcal{C}_{ASP}(i,j)$ defines a stronger constraint to the model, thus $\neg\mathcal{C}$ defines a more general constraint than $\neg\mathcal{C}_{ASP}(i,j)$ to check whether a graph is strongly connected or not.

## 4. AN APPLICATION

### 4.1. *Wireless sensor networks*

Ad hoc wireless sensor networks (WSN) are used widely (for example, in military to observe environment). They have the advantage that they consist of sensors with low energy consumption, which can be deployed easily in a cheap way on areas which are out-of-the-way. These sensors are the nodes of WSN. They are capable of processing some limited information and using wireless communication. A big effort has been to do research on how to deploy them in an optimal way to keep efficient energy consumption and communication. Although there are many WSN solutions, the deployment of a WSN is still an active research field [8].

One of the important property of an ad hoc wireless network is node density. The dense layout makes the following properties available: high fault tolerance, high-coverage characteristics, but also cause some problems. The interference is high near dense node areas, and there are a lot of collisions in the case of messaging, which requires complicated operations of a MAC protocol. Because of too many possible routes, routing needs lots of resources [4].

The aim of monitoring techniques is reducing the cost of the distributed algorithms interpreted on the network. The graph, which represents the network, has to be

thinned because of cost-reduction by techniques like disconnecting of nodes, removing links, changing scopes, etc., but the network-quality characteristics (like scalability, coverage, fault tolerance, etc.) must not fall below a required level. The overall aim is to create a scalable, fault-tolerant rare topology, where the degree of the nodes are low, the maximum load is low, energy consumption is low and the paths are short. The following techniques are used to create an optimal topology: reducing the scope of nodes, removing some nodes, introducing a dominating set of nodes, clustering, and adding some new nodes to gain all-all communication [13, 18].

### 4.2. *Connectivity test by SAT representation*

If we have a WSN, then we can redefine communication graph as follows, we say that $\mathcal{D}$ is a communication graph if and only if the elements of $\mathcal{V}$, the vertices, represent the sensor nodes of the WSN, and elements of $\mathcal{E}$, the edges, represent a one way communication between two nodes.

We intend to examine which sensors can communicate with which ones, thus we can create the communication graph and its 2-SAT representation. In this approach it can be checked fairly quickly in a given state of a randomly distributed sensor network (assuming all sensors are awaken) whether it can be ensured that each sensor can communicate with every other one [6].

For example let us model a randomly distributed heterogeneous sensor network which has 10 nodes. For representing the input clause set for a SAT solver, the

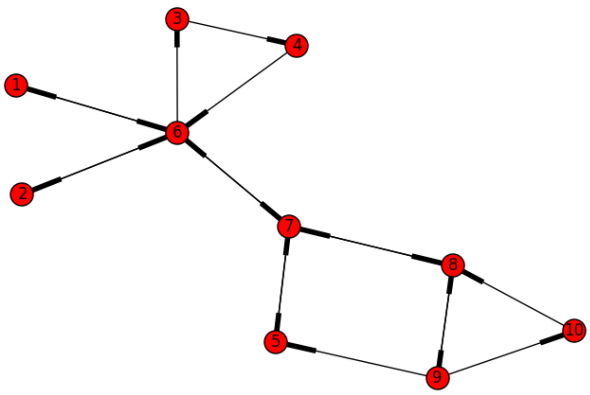


FIGURE 5. Heterogeneous sensor nodes with their communication graph

DIMACS CNF[1] format is commonly used, which references a Boolean variable by its (1-based) index. A negative literal is referenced by the negated reference to its variable. A clause is represented by a sequence of the references to its literals terminated by a ”0”.
DIMACS CNF format is the following: `p cnf 10 20`

```
c model
-1 6 0
-6 1 0
-2 6 0
-6 2 0
-6 3 0
-3 4 0
-4 6 0
-5 6 0
-6 7 0
-7 6 0
-5 7 0
-7 5 0
-7 8 0
-8 9 0
-9 8 0
-9 10 0
-10 8 0
-9 5 0
c constraint
1 2 3 4 5 6 7 8 9 10 0
-1 -2 -3 -4 -5 -6 -7 -8 -9 -10 0
```

Our main goal is to examine the produced DIMACS CNF file with `MiniSat 2.2.0`[2], which is a complete SAT solver, which returns unsatisfiable (UNSAT). Thus the model fulfills the requirements, namely communication is ensured between any two nodes. In this example the result is UNSAT, thus the represented communication graph is strongly connected.

This model can also give a more detailed analysis. For example, if we take the sensors out of the model one by one, and with the remaining we test the communication problem again between any two nodes. If the solver returns satisfiable (SAT) for the reduced model, then the currently removed node is extremely important to the sensor network, as its removal breaks the requirement of unhindered communication between any two sensors. The significance of this in graph theory is that the removal

[1] www.satlib.org/Benchmarks/SAT/satformat.ps
[2] http://minisat.se

of this node makes the graph not connected anymore. In our example the removal of nodes 3, 4, 5, 6, 7, 8 and 9 would return with SAT, thus the malfunction of those sensors means the communication in that sensor network is inadequate.

It is generally true that if the graph is strongly connected, then a DPLL-based SAT solver returns the following values.

- number of solutions : 0
- number of conflicts : 2
- number of decisions : 1
- number of unit propagations : $u = 2n$ , where $n$ the number of literals (number of vertices)

These results also show that the black-and-white 2-SAT problem with the black-and-white constraint can be solved in linear time since the number of decisions is 1, and the number of unit propagations is $2n$.

## 5. Conclusions and future work

In this paper we have introduced a SAT representation that can be used for modeling directed graphs. The only restriction is that the names of the vertices should be boolean variables. This model, which is a 2-SAT problem, makes it possible to check whether a graph is strongly connected by adding two clauses, the black and the white ones, to the model and asking a SAT solver whether this formula is unsatisfiable. The two clauses are a constraint which state that it is not true that there is a path from any vertex to any other one. This constraint is more general than the one used by Aspvall et al. We have shown that the representation of a strongly connected graph is a black-and-white 2-SAT problem. The black-and-white SAT problem appears also as a special case of weakly nondecisive SAT problems, see Lemma 6. in [3]. This suggests two things: the black-and-white SAT problem could be an interesting problem in general, and since there are weakly nondecisive 3-SAT problems, which are also black-and-white, there might be a 3-SAT representation of directed graphs. Our model can be applied also in a natural way to networks of sensors, such as wireless sensor networks. If we use a SAT solver to solve our model, then it gives back not only the solution but also lots of numbers. An interesting question is that how one can use these metrics, like *number of solutions, conflicts, decisions, unit propagations*, to say something about the topology of the graph.

It seems the the following generalized notion is also interesting: $F$ is a generalized black-and-white SAT problem if and only if $F$ is satisfiable and for any assignment $A$: if $A$ is a solution of $F$, then $F$ has another solution $B$ such that $A = \neg B$. For example, the empty clause set is a generalized black-and-white SAT problem. Another interesting example is the set of black and the white clauses. We are going to investigate this notion in a future study.

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# DERIVATIONS IN ŁUKASIEWICZ SEMIRINGS

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*Abstract.* An axiomatization of classical propositional logic is provided by means of Boolean algebras which are term equivalent to Boolean rings. This is important because rings form a classical part of algebra whose tools can be used for the investigations. The Łukasiewicz many-valued logic was axiomatized via so-called MV-algebras by C. C. Chang in 1950's. MV-algebras are successfully applied in the logic of quantum mechanics and hence they are considered as quantum structures. It is a natural question if also MV-algebras have their alter ego among classical structures. For this reason the so-called Łukasiewicz semirings were introduced by the first author and his collaborators in [3] – [4]. As shown, Łukasiewicz semirings are term equivalent to MV-algebras and we can use with advantage several developed tools for their study. In particular, we investigate derivations in semirings which were introduced formerly but here these semirings are enriched by an involution.

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*Keywords:* Łukasiewicz semiring, MV-algebra, derivation

## 1. Introduction

The concept of derivation in rings was introduced already in the 1960's. For semirings, it was defined in J. S. Golan's book ([9]). However, derivations were studied also in lattices (see e.g. [7] and [11]). Let us note that bounded distributive lattices are special semirings. Starting with 2010, several authors extended the study of derivations to certain algebras forming an algebraic semantic for non-classical logics including the logic of quantum mechanics. In this context let us mention the papers [1] and [10]. It was shown in [5] that every MV-algebra can be converted into a so-called Łukasiewicz semiring. The advantage of this approach is that we can apply also methods from the theory of semirings. This was done by the authors in [6].

The main difference between our approach to derivations in Łukasiewicz semirings and the approach by J. S. Golan in [9] is that in Łukasiewicz semirings we have a unary operation such that addition can be expressed by both multiplication and this

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unary operation. Moreover, Łukasiewicz semirings are additively idempotent and ordered by the induced semilattice ordering, see [1] – [4]. On the other hand, these semirings are not lattices and hence we cannot use results and methods from [7] or [11].

## 2. Łukasiewicz semirings

We start with the following definition:

**Definition 1.** A *Łukasiewicz semiring* is an algebra $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that

- $(R, +, 0)$ is a commutative monoid,
- $(R, \cdot, 1)$ is a commutative monoid,
- $(x + y) \cdot z \approx x \cdot z + y \cdot z$,
- $x \cdot 0 \approx 0$,
- $x + x \approx x$ and $x + 1 \approx 1$,
- $(x' \cdot y)' \cdot y \approx (y' \cdot x)' \cdot x$,
- $x \leq y$ implies $y' \leq x'$,
- $(x')' \approx x$.

An algebra $(R, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ satisfying the first four conditions is called a *commutative semiring*. So $(R, +)$ can be considered a join-semilattice and we denote by $\leq$ its corresponding partial order relation. We call it the *induced order* of $\mathbf{R}$. Put

$$x \wedge y := (x' + y')'$$

for all $x, y \in R$. Since $(R, +, 0, 1)$ is a bounded join-semilattice and $'$ is an antitone involution on $(R, \leq)$, $(R, +, \wedge, 0, 1)$ is a bounded lattice.

*Example* 1. The algebra $\mathbf{R} = ([0, 1], \vee, \odot, ', 0, 1)$ with

$$\begin{aligned} x \vee y &:= \max(x, y), \\ x \odot y &:= (x + y - 1) \vee 0, \\ x' &:= 1 - x \end{aligned}$$

for all $x, y \in [0, 1]$ is a Łukasiewicz semiring. Here $[0, 1]$ denotes the unit interval of the real numbers.

*Example* 2. For every positive integer $n$ the algebra $\mathbf{R}_n = (\{0, 1, 2, \ldots, n\}, \vee, \odot, ', 0, n)$ with

$$\begin{aligned} x \vee y &:= \max(x, y), \\ x \odot y &:= (x + y - n) \vee 0, \\ x' &:= n - x \end{aligned}$$

for all $x, y = 0, \ldots, n$ is a Łukasiewicz semiring. Note that in this case the greatest element is $n$.

*Example* 3. Every Boolean algebra $\mathbf{B} = (B, \vee, \wedge, ', 0, 1)$ is a Łukasiewicz semiring.

**Lemma 1.** *Let* $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ *be a Łukasiewicz semiring and* $a, b, c \in R$. *Then*

(i) $0' = 1$ *and* $1' = 0$,
(ii) $a \cdot a' = 0$,
(iii) $a \leq b$ *implies* $a \cdot c \leq b \cdot c$, *and hence* $a \cdot b \leq a \wedge b$, *especially,* $a \cdot a \leq a$,
(iv) $a \cdot (a + b)' = 0$,
(v) $a + b = ((a \cdot b')' \cdot b')'$,
(vi) $a \leq b$ *if and only if* $a \cdot b' = 0$.

*Proof.*

(i) This follows immediately from the fact that 0 is the smallest and 1 the greatest element of $(R, \leq)$ and that $'$ is an antitone involution of $(R, \leq)$.
(ii) According to (i) we have $a \cdot a' = (1 \cdot a')' \cdot a' = (a \cdot 0)' \cdot 0 = 0$.
(iii) $a \leq b$ implies $a + b = b$ whence $a \cdot c + b \cdot c = (a + b) \cdot c = b \cdot c$, i.e. $a \cdot c \leq b \cdot c$. Since $a \cdot b \leq a \cdot 1 = a$ and $a \cdot b \leq 1 \cdot b = b$, we have $a \cdot b \leq a \wedge b$.
(iv) According to (iii) and (ii) we have $a \cdot (a + b)' \leq (a + b) \cdot (a + b)' = 0$ and hence $a \cdot (a + b)' = 0$.
(v) According to (iv), (i) and (ii) we have

$$a + b = ((a \cdot (a + b)')' \cdot (a + b)')' = (((a + b) \cdot a')' \cdot a')' = ((b \cdot a')' \cdot a')' = ((a \cdot b')' \cdot b')'.$$

(vi) If $a \leq b$ then according to (iii) and (ii) we have $a \cdot b' \leq b \cdot b' = 0$ showing $a \cdot b' = 0$. If, conversely $a \cdot b' = 0$ then according to (v) and (i) we have $a \leq a + b = ((a \cdot b')' \cdot b')' = b$.

□

An element $a$ of $R$ is called *Boolean* if $a \cdot a = a$. This is equivalent to $a + a' = 1$ and to $a \wedge a' = 0$. Hence, $a$ is Boolean if and only if $a'$ has this property. The mentioned equivalence can be seen as follows: If $a \cdot a = a$ then $a + a' = ((a \cdot (a')')' \cdot (a')')' = ((a \cdot a)' \cdot a)' = (a' \cdot a)' = 0' = 1$. If, conversely, $a + a' = 1$ then $a \cdot a = a \cdot a + 0 = a \cdot a + a \cdot a' = a \cdot (a + a') = a \cdot 1 = a$. Let $\operatorname{Bool}\mathbf{R}$ denote the set of all Boolean elements of $R$. It is easy to see that $\operatorname{Bool}\mathbf{R}$ is a subuniverse of $(R, \cdot, ', 0, 1)$.

An *MV-algebra* is an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following identities:

$$(x \oplus y) \oplus z \approx x \oplus (y \oplus z),$$
$$x \oplus y \approx y \oplus x,$$
$$x \oplus 0 \approx x,$$
$$x \oplus 1 \approx 1,$$
$$\neg(\neg x) \approx x,$$
$$\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x.$$

Here and in the following $1 := \neg 0$. We define

$$\begin{aligned} x \otimes y &:= \neg(\neg x \oplus \neg y), \\ x \vee y &:= \neg(\neg x \oplus y) \oplus y, \\ x \wedge y &:= \neg(\neg x \vee \neg y) \end{aligned}$$

for all $x, y \in A$. Then $\otimes$ is distributive with respect to $\vee$. Moreover, for $x, y \in A$ we define $x \leq y$ if $\neg x \oplus y = 1$. Then $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice whose induced order coincides with $\leq$. Moreover $x \leq y$ implies $\neg y \leq \neg x$ $(x, y \in A)$.

It is known that Łukasiewicz semirings are in a natural one-to-one correspondence with MV-algebras (cf. e.g. [2], [3] and [8]). For the convenience of the reader we provide a short direct proof.

**Theorem 1.** *If* $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ *is a Łukasiewicz semiring and*

$$x \oplus y := (x' \cdot y')'$$

*for all* $x, y \in R$ *then* $\mathbb{M}(\mathbf{R}) := (R, \oplus, ', 0)$ *is an* MV*-algebra. If, conversely,* $\mathbf{A} = (A, \oplus, \neg, 0)$ *is an* MV*-algebra then* $\mathbb{R}(\mathbf{A}) := (A, \vee, \otimes, \neg, 0, 1)$ *is a Łukasiewicz semiring. This correspondence is one-to-one.*

*Proof.* First assume $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ to be a Łukasiewicz semiring and define $x \oplus y := (x' \cdot y')'$ for all $x, y \in R$. Then we have

$$\begin{aligned} (x \oplus y) \oplus z &\approx (((x' \cdot y')')' \cdot z')' \approx ((x' \cdot y') \cdot z')' \approx (x' \cdot (y' \cdot z'))' \approx \\ &\approx (x' \cdot ((y' \cdot z')')')' \approx x \oplus (y \oplus z), \\ x \oplus y &\approx (x' \cdot y')' \approx (y' \cdot x')' \approx y \oplus x, \\ x \oplus 0 &\approx (x' \cdot 0')' \approx (x' \cdot 1)' \approx (x')' \approx x, \\ x \oplus 1 &\approx (x' \cdot 1')' \approx (x' \cdot 0)' \approx 0' \approx 1, \\ (x')' &\approx x, \\ (x' \oplus y)' \oplus y &\approx (((((x')' \cdot y')')')' \cdot y')' \approx (((x')' \cdot y')' \cdot y')' \approx (((y')' \cdot x')' \cdot x')' \approx \\ &\approx (((((y')' \cdot x')')')' \cdot x')' \approx (y' \oplus x)' \oplus x. \end{aligned}$$

Hence $\mathbb{M}(\mathbf{R}) = (R, \oplus, ', 0)$ is an MV-algebra. Conversely, let $\mathbf{A} = (A, \oplus, \neg, 0)$ be an MV-algebra. Then $(A, \vee, 0)$ is a commutative monoid. Moreover,

$$\begin{aligned} (x \otimes y) \otimes z &\approx \neg(\neg(\neg(\neg x \oplus \neg y)) \oplus \neg z) \approx \neg((\neg x \oplus \neg y) \oplus \neg z) \approx \\ &\approx \neg(\neg x \oplus (\neg y \oplus \neg z)) \approx \neg(\neg x \oplus \neg(\neg(\neg y \oplus \neg z))) \approx x \otimes (y \otimes z), \\ x \otimes y &\approx \neg(\neg x \oplus \neg y) \approx \neg(\neg y \oplus \neg x) \approx y \otimes x, \\ x \otimes 1 &\approx \neg(\neg x \oplus \neg 1) \approx \neg(\neg x \oplus 0) \approx \neg(\neg x) \approx x. \end{aligned}$$

Hence, $(A, \otimes, 1)$ is a commutative monoid, too. It is clear that the partial order relations in $\mathbf{A}$ and $\mathbb{R}(\mathbf{A})$ coincide. Now for $a, b \in A$, $a \leq b$ implies $\neg b \leq \neg a$. Moreover,

we have

$$
\begin{aligned}
&(x \vee y) \otimes z \approx (x \otimes z) \vee (y \otimes z),\\
&\qquad x \otimes 0 \approx \neg(\neg x \oplus \neg 0) \approx \neg(\neg x \oplus 1) \approx \neg 1 \approx 0,\\
&\qquad x \vee x \approx x,\\
&\qquad x \vee 1 \approx 1,\\
&\neg(\neg x \otimes y) \otimes y \approx \neg(\neg(\neg(\neg(\neg(\neg x) \oplus \neg y))) \oplus \neg y) \approx \neg(\neg(\neg(\neg x) \oplus \neg y) \oplus \neg y) \approx\\
&\qquad\qquad \approx \neg(\neg(\neg(\neg y) \oplus \neg x) \oplus \neg x) \approx \neg(\neg(\neg(\neg(\neg(\neg y) \oplus \neg x))) \oplus \neg x) \approx\\
&\qquad\qquad \approx \neg(\neg y \otimes x) \otimes x.
\end{aligned}
$$

This shows that $\mathbb{R}(\mathbf{A}) = (A, \vee, \otimes, \neg, 0, 1)$ is a Łukasiewicz semiring. If $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ is a Łukasiewicz semiring, $\mathbb{M}(\mathbf{R}) := (R, \oplus, ', 0)$ and $\mathbb{R}(\mathbb{M}(\mathbf{R})) = (R, \vee, \otimes, ', 0, 0')$ then

$$
\begin{aligned}
&x \vee y \approx (x' \oplus y)' \oplus y \approx (((((x')' \cdot y')')')' \cdot y')' \approx ((x \cdot y')' \cdot y')' \approx x + y,\\
&x \otimes y \approx (x' \oplus y')' \approx (((x')' \cdot (y')')')' \approx x \cdot y,\\
&\quad 0' \approx 1,
\end{aligned}
$$

i.e. $\mathbb{R}(\mathbb{M}(\mathbf{R})) = \mathbf{R}$. If, finally, $\mathbf{A} = (A, \oplus, \neg, 0)$ is an MV-algebra, $\mathbb{R}(\mathbf{A}) := (A, \vee, \otimes, \neg, 0, 1)$ and $\mathbb{M}(\mathbb{R}(\mathbf{A})) = ((A, \oplus_1, \neg, 0)$ then

$$x \oplus_1 y \approx \neg(\neg x \otimes \neg y) \approx \neg(\neg(\neg(\neg x) \oplus \neg(\neg y))) \approx x \oplus y$$

and hence $\mathbb{M}(\mathbb{R}(\mathbf{A})) = \mathbf{A}$. □

Due to this correspondence, for every Łukasiewicz semiring $\mathbf{R} = (R, +, \cdot, ', 0, 1)$, $(R, +, \wedge, 0, 1)$ is a bounded distributive lattice.

We are going to show when an interval in a Łukasiewicz semiring can be converted into a bounded lattice with an antitone involution.

**Lemma 2.** *Let* $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ *be a Łukasiewicz semiring and* $b \in \operatorname{Bool}\mathbf{R}$ *and put*

$$
\begin{aligned}
x^* &:= x' \wedge b,\\
x \cap y &:= (x^* + y^*)^*
\end{aligned}
$$

*for all* $x, y \in [0, b]$. *Then* $([0, b], +, \cap, ^*, 0, b)$ *is a bounded lattice with an antitone involution.*

*Proof.* Since $b \in \operatorname{Bool}\mathbf{R}$, i.e. $b \wedge b' = 0$, we have

$$(x^*)^* = (x' \wedge b)' \wedge b = (x + b') \wedge b = (x \wedge b) + (b' \wedge b) = x + 0 = x$$

for all $x \in [0, b]$. Obviously, $^*$ is antitone. The rest of the proof is clear. □

**Lemma 3.** *Let* $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ *be a Łukasiewicz semiring and* $a \in \operatorname{Bool}\mathbf{R}$ *and put*

$$x^* := x' + a,$$
$$x \cap y := (x^* + y^*)^*$$

*for all* $x, y \in [a, 1]$. *Then* $([a, 1], +, \cap, ^*, a, 1)$ *is a bounded lattice with an antitone involution.*

*Proof.* Since $a \in \operatorname{Bool}\mathbf{R}$, i.e. $a + a' = 1$, we have

$$(x^*)^* = (x' + a)' + a = (x \wedge a') + a = (x + a) \wedge (a' + a) = x \wedge 1 = x$$

for all $x \in [a, 1]$. Obviously, $^*$ is antitone. The rest of the proof is clear. □

**Theorem 2.** *Let* $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ *be a Łukasiewicz semiring and* $a, b \in \operatorname{Bool}\mathbf{R}$, *assume* $a \leq b$ *and put*

$$x^* := (x' + a) \wedge b,$$
$$x \cap y := (x^* + y^*)^*$$

*for all* $x, y \in [a, b]$. *Then* $([a, b], +, \cap, ^*, a, b)$ *is a bounded lattice with an antitone involution.*

*Proof.* This follows from the fact that $(R, +, \wedge)$ is a distributive lattice. □

*Remark* 1. Theorem 2 remains valid if one defines $x^* := (x' \wedge b) + a$ for all $x \in [a, b]$. Note that $(x' \wedge b) + a \leq (x' + a) \wedge b$ for all $x \in R$.

## 3. Derivations

Although Łukasiewicz semirings are in a one-to-one correspondence with MV-algebras, we do not define derivations like in [1] or [10] since there the MV-operation $\oplus$ is used instead of $+$. Hence also our results differ essentially from those obtained in the mentioned papers.

Now we can adopt the definition of a derivation from [9].

**Definition 2.** Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring. A *derivation* on $\mathbf{R}$ is a mapping $d$ from $R$ to $R$ satisfying the following identities:

$$d(x + y) \approx dx + dy,$$
$$d(x \cdot y) \approx (dx) \cdot y + x \cdot (dy).$$

Let $\operatorname{Der}\mathbf{R}$ denote the set of all derivations on $\mathbf{R}$, for every $a \in R$ let $\bar{d}_a$ denote the mapping from $R$ to $R$ defined by $\bar{d}_a(x) := a \cdot x$ for all $x \in R$ and let id denote the identical mapping from $R$ to $R$. We call a *derivation* $d$ on $\mathbf{R}$ *principal* if it is of the form $\bar{d}_a$ for some $a \in R$. Hence $0 = \bar{d}_0$ and $\mathrm{id} = \bar{d}_1$ are principal derivations. For any mapping $d$ from $R$ to $R$ define $\operatorname{Fix} d := \{x \in R \mid dx = x\}$.

Here and in the following we write $dx$ instead of $d(x)$ if no confusion can arise.

**Lemma 4.** *Let* $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ *be a Łukasiewicz semiring,* $a, b \in R$ *and* $d, d_1, d_2 \in \operatorname{Der}\mathbf{R}$. *Then*

(i) $\bar{d}_a \in \operatorname{Der}\mathbf{R}$,
(ii) $d0 = 0$,
(iii) $da \leq a$,
(iv) $a \leq b$ *implies* $da \leq db$,
(v) $a \cdot (d1) \leq da$,
(vi) $d1 = 1$ *if and only if* $d = \mathrm{id}$,
(vii) $(d_1 a) \cdot (d_2 b) + (d_1 b) \cdot (d_2 a) \leq (d_1 \circ d_2)(a \cdot b)$ *where* $\circ$ *denotes the composition of mappings,*
(viii) $d_1 \circ d_2 \in \operatorname{Der}\mathbf{R}$ *if and only if for all* $x, y \in R$ *we have* $(d_1 x) \cdot (d_2 y) + (d_1 y) \cdot (d_2 x) \leq ((d_1 \circ d_2)x) \cdot y + x \cdot ((d_1 \circ d_2)y)$.

*Proof.*

(i) This is clear.
(ii) We have $d0 = d(0 \cdot 0) = (d0) \cdot 0 + 0 \cdot (d0) = 0$.
(iii) According to (ii) and Lemma 1 (ii) we have

$$(da) \cdot a' \leq d(a \cdot a') = d0 = 0$$

and hence $(da) \cdot a' = 0$ whence $da \leq a$ according to Lemma 1 (vi).
(iv) $a \leq b$ implies $da \leq da + db = d(a + b) = db$.
(v) We have $a \cdot (d1) \leq d(a \cdot 1) = da$.
(vi) If $d1 = 1$ then $a = a \cdot (d1) \leq d(a \cdot 1) = da \leq a$ according to (iii) showing $d = \mathrm{id}$. The converse is trivial.
(vii) We have

$$\begin{aligned}
&(d_1 a) \cdot (d_2 b) + (d_1 b) \cdot (d_2 a) \leq \\
&\leq (d_1 a) \cdot (d_2 b) + (d_1 b) \cdot (d_2 a) + ((d_1 \circ d_2)a) \cdot b + a \cdot ((d_1 \circ d_2)b) = \\
&= (d_1(d_2 a)) \cdot b + (d_2 a) \cdot (d_1 b) + (d_1 a) \cdot (d_2 b) + a \cdot (d_1(d_2 b)) = \\
&= d_1((d_2 a) \cdot b) + d_1(a \cdot (d_2 b)) = d_1((d_2 a) \cdot b + a \cdot (d_2 b)) = d_1(d_2(a \cdot b)) = \\
&= (d_1 \circ d_2)(a \cdot b).
\end{aligned}$$

(viii) According to the proof of (vii) we have

$$(d_1 \circ d_2)(a \cdot b) = (d_1 a) \cdot (d_2 b) + (d_1 b) \cdot (d_2 a) + ((d_1 \circ d_2)a) \cdot b + a \cdot ((d_1 \circ d_2)b).$$

□

**Lemma 5.** *Let* $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ *be a Łukasiewicz semiring and* $d \in \operatorname{Der}\mathbf{R}$. *Then the following are equivalent:*

(i) $d$ *is principal,*
(ii) $dx = x \cdot (d1)$ *for all* $x \in R$, *i.e.* $d = \bar{d}_{d1}$.

*Moreover, if $x \wedge (d1) \leq x \cdot (d1)$ for all $x \in R$ then $d$ is principal.*

*Proof.* The equivalence of (i) and (ii) is clear. If, finally, $x \wedge (d1) \leq x \cdot (d1)$ for all $x \in R$ then

$$dx \leq x \wedge (d1) \leq x \cdot (d1) \leq d(x \cdot 1) = dx$$

for all $x \in R$ and hence $d = \bar{d}_{d1}$. □

Lemma 5 shows that the Łukasiewicz semiring from Example 3 has only principal derivations since in this semiring multiplication coincides with lattice meet.

It was shown in [10], Lemma 5, that basic algebras have only principal derivations. We are going to show that this is not the case for Łukasiewicz semirings. In these semirings there exist non-principal derivations, see the following example.

*Example* 4. Consider the Łukasiewicz semiring $\mathbf{R} = (\{0,a,b,c,1\},+,\cdot,',0,1)$ with the following operations:

| $+$ | $0$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $c$ | $1$ |
| $a$ | $a$ | $a$ | $b$ | $c$ | $1$ |
| $b$ | $b$ | $b$ | $b$ | $c$ | $1$ |
| $c$ | $c$ | $c$ | $c$ | $c$ | $1$ |
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

| $\cdot$ | $0$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $0$ | $0$ | $0$ | $a$ |
| $b$ | $0$ | $0$ | $0$ | $a$ | $b$ |
| $c$ | $0$ | $0$ | $a$ | $b$ | $c$ |
| $1$ | $0$ | $a$ | $b$ | $c$ | $1$ |

| $x$ | $0$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|---|
| $x'$ | $1$ | $c$ | $b$ | $a$ | $1$ |

The principal derivations look as follows:

| $x$ | $0$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|---|
| $\bar{d}_0 x$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $\bar{d}_a x$ | $0$ | $0$ | $0$ | $0$ | $a$ |
| $\bar{d}_b x$ | $0$ | $0$ | $0$ | $a$ | $b$ |
| $\bar{d}_c x$ | $0$ | $0$ | $a$ | $b$ | $c$ |
| $\bar{d}_1 x$ | $0$ | $a$ | $b$ | $c$ | $1$ |

However, there exists a derivation $d$ of $\mathbf{R}$ which is not principal, namely the following:

| $x$ | $0$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|---|
| $dx$ | $0$ | $0$ | $0$ | $a$ | $a$ |

The Łukasiewicz semiring considered in Example 4 is isomorphic to the Łukasiewicz semiring $\mathbf{R}_4$ from Example 2. The following proposition shows that the situation described in Example 4 can be generalized to $\mathbf{R}_n$ for arbitrary $n > 1$.

**Proposition 1.** *For each $n > 1$ the Łukasiewicz semiring $\mathbf{R}_n$ has the non-principal derivation $d$ defined by*

$$d(x) := \begin{cases} 0 & \text{if } x < n-1 \\ 1 & \text{otherwise.} \end{cases}$$

*Proof.* Let $a,b \in \{0,\dots,n\}$. Then $d(a \vee b) = (da) \vee (db)$ since $(\{0,1,\dots,n\},\leq)$ is a chain. Moreover,

$$d(a \odot b) = \begin{cases} 0 & \text{if } a+b < 2n-1 \\ 1 & \text{otherwise,} \end{cases}$$

$$(da) \odot b = \begin{cases} 0 & \text{if } a < n-1 \text{ or } b < n \\ 1 & \text{if } a \geq n-1 \text{ and } b = n, \end{cases}$$

$$a \odot (db) = \begin{cases} 0 & \text{if } a < n \text{ or } b < n-1 \\ 1 & \text{if } a = n \text{ and } b \geq n-1. \end{cases}$$

Hence $d \in \operatorname{Der} \mathbf{R}_n$. Since

$$\bar{d}_a(b) = \begin{cases} 0 & \text{if } b \leq n-a \\ b-(n-a) & \text{otherwise,} \end{cases}$$

$d$ is not principal. □

For the unary operation $'$, we can prove the following theorem:

**Theorem 3.** *Let* $\mathbf{R} = (R,+,\cdot,',0,1)$ *be a Łukasiewicz semiring and* $d \in \operatorname{Der}\mathbf{R}$. *Then*

(i) $dx' = (dx)'$ *for all* $x \in R$ *if and only if* $d = \mathrm{id}$,
(ii) $(dx)\cdot(dx') \approx x\cdot(dx') \approx x'\cdot(dx) \approx 0$ *and hence* $dx' \leq (dx)'$ *for all* $x \in R$,
(iii) *if* $a \in \operatorname{Bool}\mathbf{R}$ *then* $da + da' = d1$,
(iv) $d((d1)') \leq (d1) \wedge (d1)'$,
(v) *if* $d1 \in \operatorname{Bool}\mathbf{R}$ *then* $d((d1)'\cdot x) = 0$ *for each* $x \in R$.

*Proof.*

(i) If $dx' = (dx)'$ for all $x \in R$ then for all $x \in R$ we have $x = (x')' \leq (dx')' = ((dx)')' = dx \leq x$ according to Lemma 4 (iii) and hence $d = \mathrm{id}$.
(ii) Due to Lemma 4 (iii) and Lemma 1 (iii) and (ii) we have $(dx)(dx') \leq x\cdot(dx') \leq x\cdot x' = 0$ and $x'\cdot(dx) \leq x'\cdot x = 0$ for all $x \in R$. By Lemma 1 (vi), $dx' \leq (dx)'$ for all $x \in R$.
(iii) If $a \in \operatorname{Bool}\mathbf{R}$ then $da + da' = d(a+a') = d1$.
(iv) Since $(d1)' \leq 1$, we have $d((d1)') \leq d1$ by Lemma 4 (iv). According to Lemma 4 (iii) we have $d((d1)') \leq (d1)'$. This shows $d((d1)') \leq (d1)\wedge(d1)'$.
(v) If $d1 \in \operatorname{Bool}\mathbf{R}$ then by (iv), Lemma 4 (iv) and Lemma 1 (ii) we have $d((d1)'\cdot x) = (d((d1)'))\cdot x + (d1)'\cdot(dx) = 0\cdot x + (d1)'\cdot(dx) = (dx)\cdot(d1)' \leq (d1)(d1)' = 0$ for all $x \in R$.

□

**Theorem 4.** *Let* $\mathbf{R} = (R,+,\cdot,',0,1)$ *be a Łukasiewicz semiring,* $a \in R$ *and* $d \in \operatorname{Der}\mathbf{R}$. *Then*

(i) $(\operatorname{Der}\mathbf{R},+)$ *is a join-semilattice with the least element* $\bar{d}_0$ *and the greatest element* id,

(ii) $\operatorname{Fix} d$ *is a subuniverse of* $(R,+,\cdot,0)$,
(iii) $(\operatorname{Fix} d)\cap d^{-1}(\{0\})=\{0\}$,
(iv) *if* $a\in\operatorname{Bool}\mathbf{R}$ *then* $\bar{d}_a\in\operatorname{End}(R,+,\cdot,0)$.

*Proof.*

(i) It is easy to see that $\operatorname{Der}\mathbf{R}$ is closed with respect to $+$. Hence $(\operatorname{Der}\mathbf{R},+)$ is a semilattice which is considered as a join-semilattice. We then have $d_1\le d_2$ if and only if $d_1x\le d_2x$ for all $x\in R$. Obviously, $\bar{d}_0,\mathrm{id}\in\operatorname{Der}\mathbf{R}$ and $\bar{d}_0$ is the smallest element of $(\operatorname{Der}\mathbf{R},\le)$. Because of Lemma 4 (iii), id is the greatest element of $(\operatorname{Der}\mathbf{R},\le)$.
(ii) If $a,b\in\operatorname{Fix} d$ then $d(a+b)=da+db=a+b$ and

$$d(a\cdot b)=(da)\cdot b+a\cdot(db)=a\cdot b+a\cdot b=a\cdot b.$$

Moreover, $d0=0$ according to Lemma 4 (ii).
(iii) $a\in(\operatorname{Fix} d)\cap d^{-1}(\{0\})$ implies $a=da=0$.
(iv) This can be proved by a straightforward computation.

□

For Łukasiewicz semirings we adopt the concept of an ideal as introduced for semirings in [9]. Hence, we define

**Definition 3.** Let $\mathbf{R}=(R,+,\cdot,',0,1)$ be a Łukasiewicz semiring. An *ideal* of $\mathbf{R}$ is a subset $I$ of $R$ satisfying

$$\begin{aligned}&0\in I,\\&\text{if } x,y\in I \text{ then } x+y\in I,\\&\text{if } x\in I \text{ and } y\in R \text{ then } x\cdot y\in I.\end{aligned}$$

Let $\operatorname{Id}\mathbf{R}$ denote the set of all ideals of $\mathbf{R}$.

It is well known and easy to prove that $(\operatorname{Id}\mathbf{R},\subseteq)$ is a complete lattice the with the smallest element $\{0\}$ and the greatest element $R$.

**Proposition 2.** *Let* $\mathbf{R}=(R,+,\cdot,',0,1)$ *be a Łukasiewicz semiring,* $d\in\operatorname{Der}\mathbf{R}$ *and* $I\in\operatorname{Id}\mathbf{R}$. *Then*

(i) $d^{-1}(\{0\})\in\operatorname{Id}\mathbf{R}$,
(ii) $d^{-1}(I):=\{x\in R\mid dx\in I\}$ *is a subuniverse of* $(R,+,\cdot,0)$.

*Proof.*

(i) We have $0\in d^{-1}(\{0\})$. Moreover, $d^{-1}(\{0\})$ is closed with respect to $+$. If $a\in d^{-1}(\{0\})$ and $b\in R$ then $d(a\cdot b)\le d(a\cdot 1)=da=0$ according to Lemma 1 (iii) and Lemma 4 (iv) and hence $d(a\cdot b)=0$ , i.e. $a\cdot b\in d^{-1}(\{0\})$.
(ii) The proof is straightforward.

□

Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a Łukasiewicz semiring and $\Theta \in \operatorname{Con}\mathbf{R}$. Denote by $[0]\Theta$ the class of 0 with respect to $\Theta$, the so-called *kernel* of $\Theta$. It is well-known that in any semiring, the kernel of any congruence is an ideal, but not every ideal is a kernel of some congruence, see e.g. [9]. Now let $d \in \operatorname{Der}\mathbf{R}$. We are interested in the question if $d^{-1}([0]\Theta) \in \operatorname{Id}\mathbf{R}$.

**Theorem 5.** *Let* $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ *be a Łukasiewicz semiring,* $\Theta \in \operatorname{Con}\mathbf{R}$ *and* $d \in \operatorname{Der}\mathbf{R}$. *Then*

(i) $d^{-1}([0]\Theta) := \{x \in R \mid dx \in [0]\Theta\} \in \operatorname{Id}\mathbf{R}$,
(ii) $d([0]\Theta) \subseteq [0]\Theta$ *and* $d([0]\Theta)$ *is a subuniverse of* $(R, +, 0)$.

*Proof.*

(i) If $I := d^{-1}([0]\Theta)$, $a, b \in I$ and $c \in R$ then $da, db \in [0]\Theta$ and hence

$$d(a+b) = da + db \in [0]\Theta,$$
$$d(a \cdot c) = d(a \cdot c) + 0 \in [d(a \cdot c) + da]\Theta = [da]\Theta = [0]\Theta$$

according to Lemma 1 (iii) and Lemma 4 (iv) showing $a + b, a \cdot c \in I$.

(ii) Clearly, $0 = d0 \in d([0]\Theta)$. If $a, b \in d([0]\Theta)$ then there exist $c, e \in [0]\Theta$ with $dc = a$ and $de = b$ and hence

$$a = dc = dc + 0 \in [dc + c]\Theta = [c]\Theta = [0]\Theta,$$
$$a + b = dc + de = d(c + e) \in d([0]\Theta)$$

according to Lemma 4 (iii).

□

**Theorem 6.** *Let* $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ *be a Łukasiewicz semiring,* $d \in \operatorname{Der}\mathbf{R}$, *assume* $d1 \in \operatorname{Bool}\mathbf{R}$, *put*

$$x^* := x' \wedge (d1),$$
$$x \cap y := (x^* + y^*)^*$$

*for all* $x, y \in d(R)$ *and assume* $x^* \in d(R)$ *for all* $x \in d(R)$. *Then* $(d(R), +, \cap, {}^*, 0, d1)$ *is a bounded lattice with an antitone involution.*

*Proof.* Since $d1 \in \operatorname{Bool}\mathbf{R}$, i.e. $(d1) \wedge (d1)' = 0$, we have

$$(x^*)^* = (x' \wedge (d1))' \wedge (d1) = (x + (d1)') \wedge (d1)$$
$$= (x \wedge (d1)) + ((d1)' \wedge (d1)) = x + 0 = x$$

for all $x \in d(R)$. Obviously, $^*$ is antitone. The rest of the proof is clear. □

**Theorem 7.** *Let* $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ *and* $\mathbf{S} = (S, +, \cdot, ', 0, 1)$ *be Łukasiewicz semirings. Then* $\operatorname{Der}(\mathbf{R} \times \mathbf{S}) = \operatorname{Der}\mathbf{R} \times \operatorname{Der}\mathbf{S}$.

*Proof.* Clearly, $\operatorname{Der}\mathbf{R}\times\operatorname{Der}\mathbf{S}\subseteq\operatorname{Der}(\mathbf{R}\times\mathbf{S})$. In order to prove the converse inclusion, let $d\in\operatorname{Der}(\mathbf{R}\times\mathbf{S})$. Then for each $x\in R$ we have $d(x,0)\leq(x,0)$ according to Lemma 4 (iii) and hence there exists some mapping $d_1$ from $R$ to $R$ satisfying $(d_1x,0)=d(x,0)$ for all $x\in R$. Analogously, there exists some mapping $d_2$ from $S$ to $S$ satisfying $(0,d_2y)=d(0,y)$ for all $y\in S$. Now let $a,b\in R$ and $c\in S$. Then

$$\begin{aligned}(d_1(a+b),0)&=d(a+b,0)=d((a,0)+(b,0))=d(a,0)+d(b,0)=\\&=(d_1a,0)+(d_1b,0)=(d_1a+d_1b,0),\end{aligned}$$

i.e. $d_1(a+b)=d_1a+d_1b$. Moreover,

$$\begin{aligned}(d_1(a\cdot b),0)&=d(a\cdot b,0)=d((a,0)\cdot(b,0))=(d(a,0))\cdot(b,0)+(a,0)\cdot(d(b,0))=\\&=(d_1a,0)\cdot(b,0)+(a,0)\cdot(d_1b,0)=((d_1a)\cdot b,0)+(a\cdot(d_1b),0)=\\&=((d_1a)\cdot b+a\cdot(d_1b),0),\end{aligned}$$

i.e. $d_1(a\cdot b)=(d_1a)\cdot b+a\cdot(d_1b)$. This shows $d_1\in\operatorname{Der}\mathbf{R}$. Analogously, $d_2\in\operatorname{Der}\mathbf{S}$. Finally,

$$d(a,c)=d((a,0)+(0,c))=d(a,0)+d(0,c)=(d_1a,0)+(0,d_2c)=(d_1a,d_2c)$$

proving $d\in\operatorname{Der}\mathbf{R}\times\operatorname{Der}\mathbf{S}$. Hence $\operatorname{Der}(\mathbf{R}\times\mathbf{S})\subseteq\operatorname{Der}\mathbf{R}\times\operatorname{Der}\mathbf{S}$ completing the proof of the theorem. □

**Proposition 3.** *Let* $\mathbf{R}=(R,+,\cdot,',0,1)$ *be a Łukasiewicz semiring and* $a\in\operatorname{Bool}\mathbf{R}$. *Then* $(\bar{d}_a(R),+,\cdot,0,a)$ *is a commutative semiring satisfying* $x+x\approx x$ *and* $x+a\approx a$.

*Proof.* Since $\bar{d}_a(x+y)=\bar{d}_ax+\bar{d}_ay$, $\bar{d}_a(x\cdot y)=a\cdot(x\cdot y)=(a\cdot x)\cdot(a\cdot y)=(\bar{d}_ax)\cdot(\bar{d}_ay)$ for all $x,y\in R$ and $\bar{d}_a0=a\cdot 0=0$ and $\bar{d}_a1=a\cdot 1=a$, $\bar{d}_a$ is a homomorphism from $(R,+,\cdot,0,1)$ onto $(\bar{d}_a(R),+,\cdot,0,a)$. □

## 4. $(f,g)$-DERIVATIONS

The concept of an $(f,g)$-derivation is mentioned in the monograph [9]. For Łukasiewicz semirings, it is specified as follows:

**Definition 4.** Let $\mathbf{R}=(R,+,\cdot,',0,1)$ be a Łukasiewicz semiring and $f,g\in\operatorname{End}\mathbf{R}$. An $(f,g)$-*derivation* on $\mathbf{R}$ is a mapping $d$ from $R$ to $R$ satisfying the following identities:

$$\begin{aligned}d(x+y)&\approx dx+dy,\\d(x\cdot y)&\approx(dx)\cdot f(y)+g(x)\cdot(dy).\end{aligned}$$

Let $\operatorname{Der}_{f,g}\mathbf{R}$ denote the set of all $(f,g)$-derivations on $\mathbf{R}$. Especially, $\operatorname{Der}_{\mathrm{id},\mathrm{id}}\mathbf{R}=\operatorname{Der}\mathbf{R}$.

The following example shows that there exist $(f,g)$-derivations on Łukasiewicz semirings which are not derivations.

*Example* 5. Let $M$ be a set with $|M| > 1$ and put $\mathbf{R} := (2^M, \cup, \cap, ', \varnothing, M)$. If $f \in \operatorname{End}\mathbf{R}$ then $f \in \operatorname{Der}_{f,f}\mathbf{R}$ since

$$f(A \cup B) = f(A) \cup f(B),$$
$$f(A \cap B) = f(A \cap B) \cup f(A \cap B) = (f(A) \cap f(B)) \cup (f(A) \cap f(B))$$

for all $A, B \in 2^M$. Now let $a \in M$ and $g$ denote the mapping from $2^M$ to $2^M$ defined by

$$g(A) := \begin{cases} M & \text{if } a \in A, \\ \varnothing & \text{otherwise} \end{cases}$$

($A \in 2^M$). Then $g \in \operatorname{End}\mathbf{R}$ (and hence $g \in \operatorname{Der}_{g,g}\mathbf{R}$) as can be seen from the following table:

| $A$ | $B$ | $g(A)$ | $g(B)$ | $g(A \cup B)$ | $g(A \cap B)$ | $g(A')$ |
|---|---|---|---|---|---|---|
| $\ni a$ | $\ni a$ | $M$ | $M$ | $M$ | $M$ | $\varnothing$ |
| $\ni a$ | $\not\ni a$ | $M$ | $\varnothing$ | $M$ | $\varnothing$ | $\varnothing$ |
| $\not\ni a$ | $\ni a$ | $\varnothing$ | $M$ | $M$ | $\varnothing$ | $M$ |
| $\not\ni a$ | $\not\ni a$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $M$ |

But $g \notin \operatorname{Der}\mathbf{R}$ since

$$g(\{a\} \cap \{x\}) = \varnothing \neq \{x\} = (g(\{a\}) \cap \{x\}) \cup (\{a\} \cap g(\{x\}))$$

for all $x \in M \setminus \{a\}$.

**Lemma 6.** *Let* $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ *be a Łukasiewicz semiring,* $f, g \in \operatorname{End}\mathbf{R}$, $a, b \in R$ *and* $d, d_1, d_2 \in \operatorname{Der}_{f,g}\mathbf{R}$. *Then*

(i) $\bar{d}_0 \in \operatorname{Der}_{f,g}\mathbf{R}$,
(ii) $d0 = 0$,
(iii) $da \leq f(a) \wedge g(a)$,
(iv) $a \leq b$ *implies* $da \leq db$,
(v) $(f(a) + g(a)) \cdot (d1) \leq da$,
(vi) $d1 = 1$ *if and only if* $d = f = g$,
(vii) $(d_1(g(a))) \cdot f(d_2 b) + (d_1(f(b))) \cdot g(d_2 a) \leq (d_1 \circ d_2)(a \cdot b)$,
(viii) *if* $f \circ f = f$ *and* $g \circ g = g$ *then* $d_1 \circ d_2 \in \operatorname{Der}_{f,g}\mathbf{R}$ *if and only if for all* $x, y \in R$ *we have* $(d_1(g(x))) \cdot f(d_2 y) + (d_1(f(y))) \cdot g(d_2 x) \leq ((d_1 \circ d_2)x) \cdot f(y) + g(x) \cdot ((d_1 \circ d_2)y)$.

*Proof.*

(i) This is clear.
(ii) We have $d0 = d(0 \cdot 0) = (d0) \cdot f(0) + g(0) \cdot (d0) = (d0) \cdot 0 + 0 \cdot (d0) = 0$.
(iii) According to (ii) and Lemma 1 (ii) we have

$$(da) \cdot (f(a))' = (da) \cdot f(a') \leq d(a \cdot a') = d0 = 0$$

and hence $(da)\cdot(f(a))' = 0$ whence $da \le f(a)$ according to Lemma 1 (vi). Analogously, according to (ii) and Lemma 1 (ii) we have

$$(da)\cdot(g(a))' = g(a')\cdot(da) \le d(a'\cdot a) = d0 = 0$$

and hence $(da)\cdot(g(a))' = 0$ whence $da \le g(a)$ according to Lemma 1 (vi).

(iv) can be proved exactly as Lemma 4 (iv).

(v) We have

$$(f(a)+g(a))\cdot(d1) = (d1)\cdot f(a)+g(a)\cdot(d1) \le d(1\cdot a)+d(a\cdot 1) = da+da = da.$$

(vi) If $d1 = 1$ then

$$\begin{aligned} f(a) &= (d1)\cdot f(a) \le d(1\cdot a) = da \le f(a),\\ g(a) &= g(a)\cdot(d1) \le d(a\cdot 1) = da \le g(a) \end{aligned}$$

according to (iii) showing $d = f = g$. The converse is trivial.

(vii) We have

$$\begin{aligned} &(d_1(g(a)))\cdot f(d_2b)+(d_1(f(b)))\cdot g(d_2a) \le\\ &\le (d_1(g(a)))\cdot f(d_2b)+(d_1(f(b)))\cdot g(d_2a)+\\ &+((d_1\circ d_2)a)\cdot f(f(b))+g(g(a))\cdot((d_1\circ d_2)b) =\\ &= (d_1(d_2a))\cdot f(f(b))+g(d_2a)\cdot(d_1(f(b)))+\\ &+(d_1(g(a)))\cdot f(d_2b)+g(g(a))\cdot(d_1(d_2b)) =\\ &= d_1((d_2a)\cdot f(b))+d_1(g(a)\cdot(d_2b)) = d_1((d_2a)\cdot f(b)+g(a)\cdot(d_2b)) =\\ &= d_1(d_2(a\cdot b)) = (d_1\circ d_2)(a\cdot b). \end{aligned}$$

(viii) If $f\circ f = f$ and $g\circ g = g$ then according to the proof of (vii) we have

$$\begin{aligned} (d_1\circ d_2)(a\cdot b) = {}&(d_1(g(a)))\cdot f(d_2b)+(d_1(f(b)))\cdot g(d_2a)+\\ &+((d_1\circ d_2)a)\cdot f(b)+g(a)\cdot((d_1\circ d_2)b). \end{aligned}$$

□

For the unary operation $'$, we can prove the following theorem:

**Theorem 8.** *Let* $\mathbf{R} = (R,+,\cdot,',0,1)$ *be a Łukasiewicz semiring,* $f,g \in \operatorname{End}\mathbf{R}$ *and* $d \in \operatorname{Der}_{f,g}\mathbf{R}$. *Then*

(i) $dx' = (dx)'$ *for all* $x \in R$ *if and only if* $d = f = g$,

(ii) $(dx)\cdot(dx') \approx f(x)\cdot(dx') \approx f(x')\cdot(dx) \approx g(x)\cdot(dx') \approx g(x')\cdot(dx) \approx 0$ *and hence* $dx' \le (dx)'$ *for all* $x \in R$,

(iii) *if* $a \in \operatorname{Bool}\mathbf{R}$ *then* $da+da' = d1$.

*Proof.*

(i) If $dx' = (dx)'$ for all $x \in R$ then for all $x \in R$ we have

$$f(x) = (f(x'))' \leq (dx')' = dx \leq f(x),$$
$$g(x) = (g(x'))' \leq (dx')' = dx \leq g(x)$$

according to Lemma 6 (iii) and hence $d = f = g$.

(ii) Due to Lemma 6 (iii) and Lemma 1 (iii) and (ii) we have

$$(dx)\cdot(dx') \leq f(x)\cdot(dx') \leq f(x)\cdot f(x') = f(x\cdot x') = f(0) = 0,$$
$$f(x')\cdot(dx) \leq f(x')\cdot f(x) = f(x\cdot x') = f(0) = 0$$

for all $x \in R$. The proof for $g$ is analogous. By Lemma 1 (vi), $dx' \leq (dx)'$ for all $x \in R$.

(iii) can be proved exactly as Theorem 3 (iii).

□

**Theorem 9.** *Let* $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ *be a Łukasiewicz semiring,* $f, g \in \operatorname{End}\mathbf{R}$ *and* $d \in \operatorname{Der}_{f,g}\mathbf{R}$. *Then*

(i) $(\operatorname{Der}_{f,g}\mathbf{R}, +)$ *is a join-semilattice with the least element* $\bar{d}_0$,

(ii) $(\operatorname{Fix} d) \cap d^{-1}(\{0\}) = \{0\}$.

*Proof.*

(i) It is easy to see that $\operatorname{Der}_{f,g}\mathbf{R}$ is closed with respect to $+$. Hence $(\operatorname{Der}_{f,g}\mathbf{R}, +)$ is a semilattice which is considered as a join-semilattice. We then have $d_1 \leq d_2$ if and only if $d_1 x \leq d_2 x$ for all $x \in R$. Obviously, $\bar{d}_0 \in \operatorname{Der}_{f,g}\mathbf{R}$ and $\bar{d}_0$ is the smallest element of $(\operatorname{Der}_{f,g}\mathbf{R}, \leq)$.

(ii) can be proved exactly as Theorem 4 (iii).

□

It is clear that Proposition 2, Theorem 5 (i) and Theorem 6 remain valid for $(f, g)$-derivations, too.

**Theorem 10.** *Let* $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ *and* $\mathbf{S} = (S, +, \cdot, ', 0, 1)$ *be Łukasiewicz semirings,* $f_1, g_1 \in \operatorname{End}\mathbf{R}$ *and* $f_2, g_2 \in \operatorname{End}\mathbf{S}$. *Then* $(f_1, f_2), (g_1, g_2) \in \operatorname{End}(\mathbf{R}\times\mathbf{S})$ *and*

$$\operatorname{Der}_{(f_1,f_2),(g_1,g_2)}(\mathbf{R}\times\mathbf{S}) = \operatorname{Der}_{f_1,g_1}\mathbf{R}\times\operatorname{Der}_{f_2,g_2}\mathbf{S}.$$

*Proof.* Clearly, $(f_1, f_2), (g_1, g_2) \in \operatorname{End}(\mathbf{R}\times\mathbf{S})$ and

$$\operatorname{Der}_{f_1,g_1}\mathbf{R}\times\operatorname{Der}_{f_2,g_2}\mathbf{S} \subseteq \operatorname{Der}_{(f_1,f_2),(g_1,g_2)}(\mathbf{R}\times\mathbf{S}).$$

In order to prove the converse inclusion, let $d \in \operatorname{Der}_{(f_1,f_2),(g_1,g_2)}(\mathbf{R}\times\mathbf{S})$. Then for each $x \in R$ we have $d(x, 0) \leq (f_1(x), 0)$ according to Lemma 6 (iii) and hence there exists some mapping $d_1$ from $R$ to $R$ satisfying $(d_1 x, 0) = d(x, 0)$ for all $x \in R$.

Analogously, there exists some mapping $d_2$ from $S$ to $S$ satisfying $(0, d_2 y) = d(0, y)$ for all $y \in S$. Now let $a, b \in R$ and $c \in S$. Then

$$\begin{aligned}(d_1(a+b), 0) &= d(a+b, 0) = d((a,0)+(b,0)) = d(a,0)+d(b,0) = \\ &= (d_1 a, 0) + (d_1 b, 0) = (d_1 a + d_1 b, 0),\end{aligned}$$

i.e. $d_1(a+b) = d_1 a + d_1 b$. Moreover,

$$\begin{aligned}(d_1(a \cdot b), 0) &= d(a \cdot b, 0) = d((a,0) \cdot (b,0)) = \\ &= (d(a,0)) \cdot (f_1(b), 0) + (g_1(a), 0) \cdot (d(b,0)) = \\ &= (d_1 a, 0) \cdot (f_1(b), 0) + (g_1(a), 0) \cdot (d_1 b, 0) = \\ &= ((d_1 a) \cdot f_1(b), 0) + (g_1(a) \cdot (d_1 b), 0) = \\ &= ((d_1 a) \cdot f_1(b) + g_1(a) \cdot (d_1 b), 0),\end{aligned}$$

i.e. $d_1(a \cdot b) = (d_1 a) \cdot f_1(b) + g_1(a) \cdot (d_1 b)$. This shows $d_1 \in \mathrm{Der}_{f_1, g_1} \mathbf{R}$. Analogously, $d_2 \in \mathrm{Der}_{f_2, g_2} \mathbf{S}$. Finally,

$$d(a,c) = d((a,0)+(0,c)) = d(a,0) + d(0,c) = (d_1 a, 0) + (0, d_2 c) = (d_1 a, d_2 c)$$

proving $d \in \mathrm{Der}_{f_1, g_1} \mathbf{R} \times \mathrm{Der}_{f_2, g_2} \mathbf{S}$. Hence

$$\mathrm{Der}_{(f_1, f_2), (g_1, g_2)} (\mathbf{R} \times \mathbf{S}) \subseteq \mathrm{Der}_{f_1, g_1} \mathbf{R} \times \mathrm{Der}_{f_2, g_2} \mathbf{S}$$

completing the proof of the theorem. □

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# ON $\alpha_\beta$-CONTRACTIVE AND ADMISSIBLE MAPPINGS

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*Abstract.* In this paper, we introduce the concept of a pair $(f,h)$ called upper class of type II and $\alpha_\beta$-contractive mappings. We obtain that all the corresponding established results of Hussain et al. [7] are immediately consequences of our main result. Our main result generalizes and modifies several existing results in literature. Also, an example is given to support the main result.

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*Keywords:* admissible mappings, contractive mappings, $\alpha_\beta$-contractive, fixed point, $C$-class function

## 1. Introduction and preliminaries

It is well known that Banach contraction principle [2] is one of the most interesting and useful result in nonlinear analysis and whole mathematics in general. This famous result has also very large applications in various fields such as engineering, economic, computer sciences, and many others. This theorem has been extended and generalized by various authors (see, e.g., [7–9, 11, 12]).

In an attempt to generalize this significant principle, many researchers have extended the following result in certain directions.

**Theorem 1** ([3–6, 8])**.** *Let $(X,d)$ be a complete metric space and $T : X \to X$ be a mapping. Assume that there exists a function $\beta : [0,\infty) \to [0,1]$ such that, for any bounded sequence $\{t_n\}$ of positive real numbers, $\beta(t_n) \to 1$ implies $t_n \to 0$ and $d(Tx,Ty) \le \beta(d(x,y))d(x,y)$ for all $x,y \in X$. Then $T$ has a unique fixed point.*

In this paper we introduce a concept of pair $(f,h)$ which we denote as upper class of type II and $\alpha_\beta$-contractive mappings to show the theorems in [7] are immediately consequences of our main approach.

Now, we introduce some definitions and new notations which will be used in the sequel.

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**Definition 1.** We say that $h : [0,+\infty)^3 \to \mathbb{R}$ is a function of subclass of type II if it is continuous and

$$x, y \geq 1 \Rightarrow h(1,1,z) \leq h(x,y,z)$$

*Example* 1. Define $h: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $h(x,y,z) = (z+l)^{xy}, l > 1$;
(b) $h(x,y,z) = (xy+l)^z, l > 1$;
(c) $h(x,y,z) = z$;
(d) $h(x,y,z) = x^m y^n z^p$, $m,n,p \in \mathbb{N}$;
(e) $h(x,y,z) = \frac{x^m+x^n y^p+y^q}{3} z^k$, $m,n,p,q,k \in \mathbb{N}$

for all $x,y,z \in \mathbb{R}^+$. Then $h$ is a function of subclass of type II.

**Definition 2.** Suppose that $f : [0,+\infty)^2 \to \mathbb{R}$ and $h : [0,+\infty)^3 \to \mathbb{R}$. The pair $(f,h)$ is called upper class of type II if $f$ is a continuous function, $h$ a subclass of type II with

$$0 \leq s \leq 1 \Rightarrow f(s,t) \leq f(1,t),$$
$$\text{and } h(1,1,z) \leq f(s,t) \Rightarrow z \leq st.$$

*Example* 2. Define $h: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F}: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $h(x,y,z) = (z+l)^{xy}$, $l > 1$, $\mathcal{F}(s,t) = st+l$;
(b) $h(x,y,z) = (xy+l)^z$, $l > 1$, $\mathcal{F}(s,t) = (1+l)^{st}$;
(c) $h(x,y,z) = z$, $F(s,t) = st$;
(d) $h(x,y,z) = x^m y^n z^p$, $m,n,p \in \mathbb{N}$, $\mathcal{F}(s,t) = s^p t^p$
(e) $h(x,y,z) = \frac{x^m+x^n y^p+y^q}{3} z^k$, $m,n,p,q,k \in \mathbb{N}$, $\mathcal{F}(s,t) = s^k t^k$

for all $x,y,z,s,t \in \mathbb{R}^+$. Then the pair $(\mathcal{F},h)$ is an upper class of type II.

**Definition 3.** Let $(X,d)$ be a metric space and $T : X \to X$ be a mapping. A nonempty subset $F$ of $X$ is called invariant under $T$ if $Tx \in F$, for every $x \in F$.

**Definition 4.** Let $T : X \to X$ be a mapping, $F$ a nonempty subset of $X$ which is invariant under $T$ and $\alpha : F \times F \to [0,+\infty)$. We say that $T$ is an $\alpha_F$-admissible mapping if $\alpha(x,\ y) \geq 1$ implies $\alpha(Tx,\ Ty) \geq 1$, for all $x,\ y\ \in\ F$.

*Remark* 1. A mapping $T$ is called an $\alpha$-admissible mapping (see [12]) if we take $F = X$ in Definition 4.

**Definition 5.** A function $\psi : [0,+\infty) \to [0,+\infty)$ is called altering distance function if the following properties are satisfied:

(1) $\psi$ is continuous and non-decreasing;
(2) $\psi^{-1}(\{0\}) = 0$.

We denote $\Psi$ the set of all altering distance functions.

The following result will be used in the sequel.

**Lemma 1** ([1,9–11]). *Suppose $(X,d)$ is a metric space. Let $\{x_n\}$ be a sequence in $X$ such that $d(x_n,x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that the following sequences tend to $\varepsilon$ when $k \to \infty$:*

$$d(x_{m(k)},x_{n(k)}),\ d(x_{m(k)-1},x_{n(k)+1}),\ d(x_{m(k)-1},x_{n(k)}),$$
$$d(x_{m(k)-1},x_{n(k)-1}),\ d(x_{m(k)+1},x_{n(k)+1}).$$

## 2. Main results

In this section by using the new concept we consider, discuss, improve and generalize the main results from [7]. It is worth mentioning here that in our approach the implication $\beta(t_n) \to 1 \Rightarrow t_n \to 0$ holds also for unbounded sequences $(t_n)$. Therefore, our new results generalize the recent results of [7] in several directions.

**Definition 6.** Let $(X,d)$ be a metric space, $F$ a nonempty subset of $X$, $T : X \to X$ and $\alpha : F \times F \to [0,+\infty)$. A mapping $T$ is said to be $\alpha_\beta$-contractive mapping if there exists a $\beta : [0,+\infty) \to [0,\ 1)$ with the property that $t_n \to 0$ whenever $\beta(t_n) \to 1$ as well as for all $x, y \in F$, the following condition holds:

$$h(\alpha(x,Tx),\alpha(y,Ty),\psi(d(Tx,Ty))) \le f(\beta(d(x,y)),\psi(d(x,y))), \tag{2.1}$$

where the pair $(f,h)$ is a upper class of type II and $\psi \in \Psi$.

**Theorem 2.** *Let $(X,d)$ be a complete metric space and $F$ be a nonempty closed subset of $X$. Suppose that $T : X \to X$ is an $\alpha_F$-admissible mapping and $F$ is invariant under $T$. Further assume that $T$ is an $\alpha_\beta$-admissible contractive mapping. Suppose that there exists $x_0 \in F$ such that $\alpha(x_0,Tx_0) \ge 1$ and one of the following conditions holds,*
*(a) $T$ is continuous.*
*(b) if $\{x_n\}$ is a sequence in $F$ such that $x_n \to z$, $\alpha(x_n,x_{n+1}) \ge 1$, for all $n$, then $\alpha(z,Tz) \ge 1$.*
*Then $T$ has a fixed point.*

*Proof.* Let $x_0 \in F$ such that $\alpha(x_0,Tx_0) \ge 1$. Now, we construct a sequence $\{x_n\}$ in $F$ by $x_n = Tx_{n-1}$, for $n \ge 1$, such that $\alpha(x_n,x_{n+1}) = \alpha(x_n,Tx_n) \ge 1$. Substituting $x = x_{n-1}$ and $y = x_n$ in (2.1), we obtain

$$\begin{aligned} h(1,1,\psi(d(x_n,x_{n+1}))) &\le h(\alpha(x_{n-1},x_n),\alpha(x_n,x_{n+1}),\psi(d(x_n,x_{n+1}))) \\ &\le f(\beta(d(x_{n-1},x_n)),\psi(d(x_{n-1},x_n))) \end{aligned}$$

which implies that

$$\begin{aligned} \psi(d(x_n,x_{n+1})) &\le \beta(d(x_{n-1},x_n))\psi(d(x_{n-1},x_n)) \\ &\le \psi(d(x_{n-1},x_n)). \end{aligned} \tag{2.2}$$

As $\psi \in \Psi$, we have

$$d(x_n,x_{n+1}) \le d(x_{n-1},x_n), \tag{2.3}$$

for every $n \in \mathbb{N}$. Therefore, $\{d(x_n, x_{n+1})\}$ is a decreasing sequence, so there exists some $r \geq 0$, such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r. \tag{2.4}$$

Further from (2.2), we have

$$\frac{\psi(d(x_n, x_{n+1}))}{\psi(d(x_{n-1}, x_n))} \leq \beta(d(x_{n-1}, x_n)) \leq 1.$$

Letting $n \to \infty$ in the above inequality, we have $\lim_{n\to\infty} \beta(d(x_{n-1}, x_n)) = 1$, and this implies that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.5}$$

Now, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence.

By Lemma 1, there exists $\delta > 0$ for which we can find subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \delta. \tag{2.6}$$

Setting $x = x_{m_k-1}$ and $y = x_{n_k-1}$ in (2.1), we obtain

$$\begin{aligned} h\left(1, 1, \psi\left(d(x_{n_k}, x_{m_k})\right)\right) &\leq h\left(\alpha\left(x_{m_k-1}, x_{m_k}\right), \alpha\left(x_{n_k-1}, x_{n_k}\right), \psi\left(d(x_{n_k}, x_{m_k})\right)\right) \\ &\leq f(\beta\left(d\left(x_{m_k-1}, x_{n_k-1}\right)\right), \psi\left(d\left(x_{m_k-1}, x_{n_k-1}\right)\right)), \end{aligned}$$

i.e. $\psi\left(d(x_{n_k}, x_{m_k})\right) \leq \beta\left(d\left(x_{m_k-1}, x_{n_k-1}\right)\right) \psi\left(d\left(x_{m_k-1}, x_{n_k-1}\right)\right)$, which implies that

$$\frac{\psi\left(d(x_{n_k}, x_{m_k})\right)}{\psi\left(d\left(x_{m_k-1}, x_{n_k-1}\right)\right)} \leq \beta\left(d\left(x_{m_k-1}, x_{n_k-1}\right)\right) \leq 1. \tag{2.7}$$

Letting $k \to \infty$ and using (2.6) and (2.7), we obtain

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = 0 \leq \delta, \tag{2.8}$$

which is a contradiction.

This shows that $\{x_n\}$ is a Cauchy sequence and hence it is convergent in the complete set $F$. Hence $x_n \to z \in F$ as $n \to \infty$.

First, we suppose that $T$ is continuous. Therefore, we have

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n = Tz.$$

Next, we suppose that condition (b) holds. Therefore, $\alpha(z, Tz) = 1$. Now, by (2.1), we have

$$\begin{aligned} h(1, 1, \psi(d(Tz, x_{n+1}))) &\leq h(\alpha(z, Tz), \alpha(x_n, Tx_n), \psi(d(Tz, x_{n+1}))) \\ &\leq f(\beta(d(z, x_n)), \psi(d(z, x_n))), \end{aligned}$$

which implies that

$$\psi(d(Tz, x_{n+1})) \leq \beta(d(z, x_n)) \psi(d(z, x_n)).$$

Taking $n \to \infty$ and using the properties of $\psi$ and $\beta$, we have $d(Tz,z) = 0$, that is, $z = Tz$. □

## 3. Some consequences of the main result

If $h(x,y,z)=(z+l)^{xy}$, $l > 1$, $f(x,y) = xy + l$, $\psi(t)=t$, and $F = X$ in Theorem 2, we have Theorem 4 of [7].

**Corollary 1.** *Let $(X,d)$ be a complete metric space and $T : X \to X$ be an $\alpha$-admissible mapping. Assume that there exists a function $\beta : [0,\infty) \to [0,1)$ with the property that $t_n \to 0$ whenever $\beta(t_n) \to 1$, such that*

$$(d(Tx,Ty)+l)^{\alpha(x,Tx)\alpha(y,Ty)} \leq \beta(d(x,y))\, d(x,y) + l$$

*for all $x$, $y \in X$. Suppose that either*
*(a) $T$ is continuous, or*
*(b) if $\{x_n\}$ is a sequence in $X$ such that $x_n \to z$, $\alpha(x_n,x_{n+1}) \geq 1$ for all $n$, then $\alpha(z,Tz) \geq 1$.*
*If there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$, then $T$ has a fixed point.*

If $h(x,y,z)=(xy+l)^z$, $l>1$, $f(x,y) = (1+m)^{xy}$, $m=1$, $\psi(t)=t$, and $F = X$ in Theorem 2, we have Theorem 6 of [7].

**Corollary 2.** *Let $(X,d)$ be a complete metric space and $T : X \to X$ be an $\alpha$-admissible mapping. Assume that there exists a function $\beta : [0,\infty) \to [0,\ 1)$ with the property that $t_n \to 0$ whenever $\beta(t_n) \to 1$, such that*

$$(\alpha(x,Tx)\alpha(y,Ty)+l)^{d(Tx,Ty)} \leq 2^{\beta(d(x,y))d(x,y)}$$

*for all $x$, $y \in X$. Suppose that either*
*(a) $T$ is continuous, or*
*(b) if $\{x_n\}$ is a sequence in $X$ such that $x_n \to z$, $\alpha(x_n,x_{n+1}) \geq 1$ for all $n$, then $\alpha(z,Tz) \geq 1$*
*If there exists $x_0 \in X$ such that $\alpha(x_0,\ Tx_0) \geq 1$, then $T$ has a fixed point.*

If $h(x,y,z) = xyz$, $f(x,y) = xy, \psi(t)=t$, and $F = X$ in Theorem 2, we have Theorem 8 of [7].

**Corollary 3.** *Let $(X,d)$ be a complete metric space and $T : X \to X$ be an $\alpha$-admissible mapping. Assume that there exists a function $\beta : [0,\infty) \to [0,\ 1)$ with the property that $t_n \to 0$ whenever $\beta(t_n) \to 1$, such that*

$$\alpha(x,Tx)\alpha(y,Ty)d(Tx,Ty) \leq \beta(d(x,y))d(x,y)$$

*for all $x$, $y \in X$. Suppose that either*
*(a) $T$ is continuous, or*
*(b) if $\{x_n\}$ is a sequence in $X$ such that $x_n \to z$, $\alpha(x_n,x_{n+1}) \geq 1$ for all $n$, then $\alpha(z,Tz) \geq 1$.*
*If there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$, then $T$ has a fixed point.*

*Remark* 2. Let $h(x,y,z) = xyz$, $f(x,y) = xy$, $\psi(t) = t$, $\alpha(x,y) = 1$, $F = X$ for all $x,y,z \in X$ and $t > 0$. Then we get Theorem 1.

*Example* 3. Let $X = [1,\infty)$ be endowed with a usual metric $d(x,y) = |x-y|$ for all $x,y \in X$ and $T : X \to X$ be defined by

$$T(x) = \begin{cases} \frac{x+14}{8} & 1 \le x \le 4 \\ \frac{x^2}{4}, & x > 4 \end{cases}$$

Define the function $\alpha$, $\beta$, and $\psi$ given by

$$\alpha(x,y) = \begin{cases} 1, & x,y \in [1,4] \\ 0, & \text{otherwise} \end{cases}$$

$$\beta(t) = \frac{1}{1+t}, \ \psi(t) = t$$

Then $T$ is $\alpha$-admissible and we obtain $1 \le y \le x \le 4$

$$\begin{aligned} \alpha(x,Tx)\alpha(y,Ty)d(Tx,Ty) &= d\left(\frac{x+14}{8}, \frac{y+14}{8}\right) \\ &= \frac{1}{8}|x-y| \\ &\le \beta(d(x,y))\psi(d(x,y)) \end{aligned}$$

Hence, $T$ satisfies all the assumptions of Theorem 2 with $h(x,y,z) = xyz$ and $f(s,t) = st$ and thus it has a fixed point (which is $x = 2$).

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# CONVERGENCE ANALYSIS OF FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED NONLOCAL DIFFERENTIAL-DIFFERENCE PROBLEM

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*Abstract.* This study is concerned with a singularly perturbed three-point boundary-value problem with delay. Firstly, bounds on the solution and its derivative of the solution to be used later in the paper are obtained. To solve it numerically, we use an exponentially fitted difference scheme on an equidistant mesh which is established by the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with weight and remainder term in integral form. Then, the stability and convergence analysis of difference scheme is given and it is uniformly convergent in perturbation parameter. Furthermore, numerical results which show the effectiveness of the method are presented.

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*Keywords:* singular perturbation, nonlocal condition, finite difference method, initial layer, uniform convergence

## 1. Introduction

If an ordinary differential equation contains both a perturbation parameter and a delay term, it is called a singularly perturbed differential-difference equation. These equations are a very important role in science and engineering field. For instance, they occur in the study of human pupil light reflex [18], first-exit problems in neurobiology [22], models of physiological processes and diseases [20], optimal control theory [13], optically bistable devices [9] and signal transmission [10], and other models [11].

On the other hand, in recent years many methods have been developed for solving singularly perturbed delay differential problems. In [8, 16] some asymptotic analysis of boundary value problems for second order singularly perturbed differential-difference equations have been considered and some numerical techniques for solving of this type of problems with large and small shifts were considered in [12, 17]. Particularly, reproducing kernel method [12], initial value technique [23], some special finite element method [25] have been used for solving these problems.

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Motivated by the above works, we consider the following model problem:

$$Lu := \varepsilon u''(x) + a(x)u'(x) + b(x)u(x) + c(x)u(x-r) = f(x),\ x \in \Omega, \tag{1.1}$$

subject to the interval and boundary conditions,

$$u(x) = \varphi(x),\ x \in \Omega_0;\ \gamma u(l_1) + u(l) = A,\ l_1 \in \Omega \tag{1.2}$$

where $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 = (0,r]$, $\Omega_2 = (r,l)$, $\bar{\Omega} = [0,l]$, $\Omega_0 = [-r,0]$ and $0 < \varepsilon \ll 1$ is the perturbation parameter, $\gamma > 0$ and $A$ are given constants. $a(x) \geqslant \alpha > 0$, $b(x) \leqslant 0$, $c(x)$, $f(x)$ and $\varphi(x)$ are given sufficiently smooth functions satisfying certain regularity conditions to be specified and $r$ is a constant delay, which is independent of $\varepsilon$.

The special case $r = 0$ and $\varphi(x) = B$ ($B$ is a real constant) of the above problem so called nonlocal boundary value problem of ordinary differential equation, was initiated by [14]. This kind of problems occur in mathematical models of a large number of phenomena in catalytic processes in chemistry and biology, in problems of semiconductors, in problems of hydromechanics, in studying heat transfer problems and some other physical phenomena [1]. In the survey paper [19], nonlocal problems have been extensively studied many researchers. Nonetheless, the existence of positive solutions for the multipoint boundary value problem with delay is studied in [5, 7, 15, 24]. Recently, numerical study of the boundary value problem for linear singularly perturbed differential difference equation with large delay and first type condition is extensively investigated [12, 23, 25]. But, singularly perturbed nonlocal problems with delay are not studied in literature so far.

It is well known that, for small values of $\varepsilon$, standard numerical methods for solving such problems are unstable and do not give accurate results. For that reason, it is important to develop suitable numerical methods for solving these problems, whose accuracy does not depend on the parameter value $\varepsilon$, i.e., methods that are convergent $\varepsilon$-uniformly. These include fitted finite difference methods, finite element methods using special elements such as exponential elements, and methods which use a priori refined or special non-uniform grids which condense in the boundary layers in a special manner. The various approaches to the design and analysis of appropriate numerical methods for singularly perturbed differential equations can be found in [2, 6, 21] and the references therein. The numerical method presented here comprises a fitted difference scheme on a uniform mesh. We have derived this approach on the basis of the method of integral identities with the use of interpolating quadrature rules with the weight and remainder terms in integral form. This results in a local truncation error containing only first order derivatives of exact solution and hence facilitates examination of the convergence. The solution of a singularly perturbed problem of the form (1.1)-(1.2) normally has a boundary layer at $x = 0$.

This paper is organized as follows. In Section 2, we state some important properties of the exact solution. The description of the finite difference discretization

have been given in Section 3. In Section 4, we present the error analysis for the approximate solution. Uniform convergence is proved in the discrete maximum norm. In Section 5, we formulate the iterative algorithm for solving the discrete problem and present numerical results which validate the theoretical analysis computationally. The paper ends with a summary of the main conclusions.

*Notation.* Throughout the paper, $C$ denotes a generic positive constant independent of $\varepsilon$ and the mesh parameter. Some specific, fixed constants of this kind are indicated by subscripting $C$. For any continuous function $g(x)$ denote norms which

$$\|g\|_\infty \equiv \|g\|_{\infty,\bar{\Omega}} = \max_{0\le x\le l} |g(x)|, \|g\|_1 \equiv \|g\|_{1,\Omega} = \int_0^l |g(x)|\,dx,$$

$$\|g\|_{\infty,k} \equiv \|g\|_{\infty,\bar{\Omega}_k}, \quad \|g\|_{1,k} \equiv \|g\|_{1,\Omega_k}, \quad k = 0,1,2.$$

## 2. Properties of Exact Solution

Here we give some properties of the solution of (1.1)-(1.2), which are needed in later sections for the analysis of appropriate numerical solution.

**Lemma 1.** *Let $a(x)$, $b(x)$, $c(x)$, $f(x) \in C(\bar{\Omega})$, $\varphi(x) \in C(\Omega_0)$ and*

$$\lambda := \alpha^{-1}\beta(\|b\|_1 + \|c\|_{1,2}) < 1. \tag{2.1}$$

*Then for the solution $u(x)$ of the problem (1.1)-(1.2) the following estimates hold:*

$$\|u\|_\infty \leqslant C_0, \tag{2.2}$$

$$\left|u'(x)\right| \leqslant C(1 + \frac{1}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}}),\ x \in \bar{\Omega}, \tag{2.3}$$

*where*

$$C_0 = [|\varphi(0)| + |A| + \beta(|\varphi(0)| + \alpha^{-1}\|f\|_1 + \alpha^{-1}\|c\|_{\infty,1}\|\varphi\|_{1,0})](1-\lambda)^{-1},$$
$$\beta = 1 + \gamma l(\gamma l_1 + l)^{-1}.$$

*Proof.* Firstly, we prove (2.2). From (1.1) it follows that

$$u'(x) = u'(0)e^{-\frac{1}{\varepsilon}\int_0^x a(\eta)d\eta} - \frac{1}{\varepsilon}\int_0^x F(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(\eta)d\eta}d\xi \tag{2.4}$$

with

$$F(x) = -f(x) + b(x)u(x) + c(x)u(x-r).$$

Integrating the relation (2.4) over $(0,x)$ we get

$$u(x) = \varphi(0) + u'(0)\int_0^x e^{-\frac{1}{\varepsilon}\int_0^\tau a(\eta)d\eta}d\tau - \frac{1}{\varepsilon}\int_0^x d\tau\int_0^\tau F(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^\tau a(\eta)d\eta}d\xi$$

$$= \varphi(0) + u'(0)\int_0^x e^{-\frac{1}{\varepsilon}\int_0^\tau a(\eta)d\eta} d\tau - \frac{1}{\varepsilon}\int_0^x d\xi F(\xi)\int_\xi^x e^{-\frac{1}{\varepsilon}\int_\xi^\tau a(\eta)d\eta} d\tau.$$

Using now the condition $\gamma u(l_1) + u(l) = A$, we have

$$u'(0) = \frac{A - (1+\gamma)\varphi(0) + \gamma q(l_1) + q(l)}{\gamma p(l_1) + p(l)} \tag{2.5}$$

and so

$$u(x) = \varphi(0) + \frac{A - (1+\gamma)\varphi(0) + \gamma q(l_1) + q(l)}{\gamma p(l_1) + p(l)} p(x) - q(x), \tag{2.6}$$

where

$$p(x) = \int_0^x e^{-\frac{1}{\varepsilon}\int_0^\tau a(\eta)d\eta} d\tau,\ q(x) = \frac{1}{\varepsilon}\int_0^x d\xi F(\xi)\int_\xi^x e^{-\frac{1}{\varepsilon}\int_\xi^\tau a(\eta)d\eta} d\tau.$$

In order to prove (2.2), we will use appropriate Green function as an alternative to (2.6).

$$\begin{aligned} &L_* v := \varepsilon v''(x) + a(x)v'(x) = H(x),\ 0 < x < l, \\ &v(x) = \varphi(x),\ -r \leqslant x \leqslant 0;\ v(l) = A, \end{aligned} \tag{2.7}$$

can be also expressed the solution (2.7) as [3]

$$v(x) = (1 - \frac{p(x)}{p(l)})\varphi(0) + \frac{p(x)}{p(l)}A + \int_0^l G(x,s)H(s)ds \tag{2.8}$$

where

$$G(x,s) = \frac{1}{\varepsilon}\int_s^l e^{-\frac{1}{\varepsilon}\int_s^\tau a(\eta)d\eta} d\tau \times \frac{p(x)}{p(l)} - \frac{1}{\varepsilon}T_0(x-s)\int_s^x e^{-\frac{1}{\varepsilon}\int_s^\tau a(\eta)d\eta} d\tau, \tag{2.9}$$

$$T_0(t) = 1, t \geqslant 0;\ T_0(t) = 0, t < 0,$$

and

$$H(x) = -f(x) + b(x)v(x) + c(x)v(x-r).$$

On the other hand, using transformation

$$u(x) = v(x) - \frac{\gamma x}{\gamma l_1 + l} v(l_1)$$

we can rewrite for the solution of the problem (1.1)-(1.2), such that

$$u(x) = (1 - \frac{p(x)}{p(l)})\varphi(0) + \frac{p(x)}{p(l)}A + \int_0^l G(x,s)F(s)ds - \frac{\gamma x}{\gamma l_1 + l}\int_0^l G(l_1,s)F(s)ds.$$

Also in [3] the authors show that $0\leqslant G(x,s)\leqslant \alpha^{-1}$ for Green's function (2.9), from this point of view

$$\begin{aligned}
|u(x)| &\leqslant |\varphi(0)|+|A|+\int_0^l |G(x,s)|\,|F(s)|\,ds+\frac{\gamma l}{\gamma l_1+l}\int_0^l |G(l_1,s)|\,|F(s)|\,ds\\
&\leqslant |\varphi(0)|+|A|+\max_{x,s\in\bar{\Omega}}|G(x,s)|\int_0^l |F(s)|\,ds\\
&\quad+\max_{l_1,s\in\bar{\Omega}}|G(l_1,s)|\,\frac{\gamma l}{\gamma l_1+l}\int_0^l |F(s)|\,ds\\
&\leqslant |\varphi(0)|+|A|+\alpha^{-1}\beta\int_0^l |F(s)|\,ds\\
&\leqslant |\varphi(0)|+|A|+\alpha^{-1}\beta\,\|f\|_1+\alpha^{-1}\beta\,\|b\|_1\,\|u\|_\infty\\
&\quad+\alpha^{-1}\beta\int_0^l |c(s)u(s-r)|\,ds.
\end{aligned}$$

After, by means of changing integral variable by $t=s-r$, we find that

$$\begin{aligned}
|u(x)| &\leqslant |\varphi(0)|+|A|+\alpha^{-1}\beta\,\|f\|_1+\alpha^{-1}\beta\,\|b\|_1\,\|u\|_\infty\\
&\quad+\alpha^{-1}\beta\int_{-r}^{l-r} |c(r+t)u(t)|\,dt\\
&\leqslant |\varphi(0)|+|A|+\alpha^{-1}\beta\,\|f\|_1+\alpha^{-1}\beta\,\|b\|_1\,\|u\|_\infty\\
&\quad+\alpha^{-1}\beta\int_{-r}^{0} |c(r+t)\varphi(t)|\,dt+\alpha^{-1}\beta\int_0^{l-r} |c(r+t)u(t)|\,dt\\
&\leqslant |\varphi(0)|+|A|+\alpha^{-1}\beta\,\|f\|_1+\alpha^{-1}\beta\,\|b\|_1\,\|u\|_\infty\\
&\quad+\alpha^{-1}\beta\,\|c\|_{\infty,1}\,\|\varphi\|_{1,0}+\alpha^{-1}\beta\,\|u\|_\infty\int_r^l |c(t)|\,dt\\
&\leqslant |\varphi(0)|+|A|+\alpha^{-1}\beta\,\|f\|_1+\alpha^{-1}\beta\,\|b\|_1\,\|u\|_\infty\\
&\quad+\alpha^{-1}\beta\,\|c\|_{\infty,1}\,\|\varphi\|_{1,0}+\alpha^{-1}\beta\,\|c\|_{1,2}\,\|u\|_\infty
\end{aligned}$$

from which (2.2) follows immediately. We have now to prove the estimate (2.3). If we reconsider the terms in (2.5), we obtain the following relations:

$$\begin{aligned}\gamma p(l_1)+p(l) &= \gamma\int_0^{l_1} e^{-\frac{1}{\varepsilon}\int_0^{\tau}a(\eta)d\eta}d\tau+\int_0^{l} e^{-\frac{1}{\varepsilon}\int_0^{\tau}a(\eta)d\eta}d\tau\\ &\geqslant \gamma\int_0^{l_1} e^{-\frac{a^*\tau}{\varepsilon}}d\tau+\int_0^{l} e^{-\frac{a^*\tau}{\varepsilon}}d\tau=\frac{\varepsilon}{a^*}[\gamma(1-e^{-\frac{a^*l_1}{\varepsilon}})+1-e^{-\frac{a^*l}{\varepsilon}}]\\ &\geqslant \frac{\varepsilon}{a^*}[\gamma(1-e^{-a^*l_1})+1-e^{-a^*l}]\equiv c_1\varepsilon,(a^*=\|a\|_\infty),\end{aligned}$$

and

$$\begin{aligned}\gamma|q(l_1)|+|q(l)| &\leqslant \frac{\gamma}{\varepsilon}\int_0^{l_1}d\xi\,|F(\xi)|\int_\xi^{l_1} e^{-\frac{1}{\varepsilon}\int_\xi^{\tau}a(\eta)d\eta}d\tau\\ &\quad+\frac{1}{\varepsilon}\int_0^{l}d\xi\,|F(\xi)|\int_\xi^{l} e^{-\frac{1}{\varepsilon}\int_\xi^{\tau}a(\eta)d\eta}d\tau\\ &\leqslant \frac{\gamma}{\varepsilon}\int_0^{l_1}d\xi\,|F(\xi)|\int_\xi^{l_1} e^{-\frac{\alpha(\tau-\xi)}{\varepsilon}}d\tau+\frac{1}{\varepsilon}\int_0^{l}d\xi\,|F(\xi)|\int_\xi^{l} e^{-\frac{\alpha(\tau-\xi)}{\varepsilon}}d\tau\\ &= \frac{\gamma}{\varepsilon}\int_0^{l_1}d\xi\,|F(\xi)|\,[\alpha^{-1}\varepsilon(1-e^{-\frac{\alpha(l_1-\xi)}{\varepsilon}})]\\ &\quad+\frac{1}{\varepsilon}\int_0^{l}d\xi\,|F(\xi)|\,[\alpha^{-1}\varepsilon(1-e^{-\frac{\alpha(l-\xi)}{\varepsilon}})]\\ &\leqslant \alpha^{-1}(\gamma\int_0^{l_1}|F(\xi)|\,d\xi+\int_0^{l}|F(\xi)|\,d\xi).\end{aligned}$$

For $l_1\leqslant r$ we get

$$\gamma|q(l_1)|+|q(l)|\leqslant \alpha^{-1}\gamma\|f\|_1+\alpha^{-1}\gamma\|b\|_\infty\int_0^{l_1}|u(s)|\,ds$$

$$
\begin{aligned}
&+\alpha^{-1}(1+\gamma)\,\|c\|_\infty \int_{-r}^{0} |\varphi(s)|\,ds + \alpha^{-1}\,\|c\|_\infty \int_{0}^{l-r} |u(s)|\,ds \\
&\leqslant \alpha^{-1}\gamma(\|f\|_1 + \|b\|_\infty C_0 l_1) + \alpha^{-1}(1+\gamma)\,\|c\|_\infty \|\varphi\|_{1,0} \\
&\quad + \alpha^{-1}\,\|c\|_\infty C_0(l-r) \equiv C_1
\end{aligned}
$$

and when $l_1 > r$ we have

$$
\begin{aligned}
\gamma\,|q(l_1)| + |q(l)| &\leqslant \alpha^{-1}\gamma\,\|f\|_1 + \alpha^{-1}\gamma\,\|b\|_\infty \int_0^{l_1} |u(s)|\,ds \\
&\quad + \alpha^{-1}(1+\gamma)\,\|c\|_\infty \int_{-r}^{0} |\varphi(s)|\,ds \\
&\quad + \alpha^{-1}\gamma\,\|c\|_\infty \int_0^{l_1-r} |u(s)|\,ds + \alpha^{-1}\,\|c\|_\infty \int_0^{l-r} |u(s)|\,ds \\
&\leqslant \alpha^{-1}\gamma(\|f\|_1 + \|b\|_\infty C_0 l_1) + \alpha^{-1}(1+\gamma)\,\|c\|_\infty \|\varphi\|_{1,0} \\
&\quad + \alpha^{-1}\,\|c\|_\infty C_0(\gamma(l_1-r) + (l-r)) \equiv C_2.
\end{aligned}
$$

Thus, from (2.5) we are led to

$$
\begin{aligned}
\left|u'(0)\right| &\leqslant \frac{|A| + (1+\gamma)\,|\varphi(0)| + \gamma\,|q(l_1)| + |q(l)|}{\gamma p(l_1) + p(l)} \\
&\leqslant \frac{c_1^{-1}(|A| + (1+\gamma)\,|\varphi(0)| + \max\{C_1, C_2\})}{\varepsilon} \equiv \frac{C_3}{\varepsilon}.
\end{aligned}
$$

Using the preceding in (2.4) and arguing as above we arrive at (2.3). □

## 3. Discrete problem

Henceforth, we denote by $\omega$ a uniform mesh on $\Omega$ :

$$
\omega = \{x_i = ih,\, i = 1, 2, \ldots, N-1;\, h = l/N\}
$$

and $\bar{\omega} = \omega \cup \{x = 0, l\}$. For simplicity, we will suppose that $\frac{r}{l}N = N_0$ and $\frac{l_1}{l}N = N_1$ are integers, i.e., $x_{N_0} = r$ and $x_{N_1} = l_1$. Before describing our numerical method, we introduce some notation for the mesh functions. For any mesh function $g(x)$, we use

$$
g_i = g(x_i),\; g_{\bar{x},i} = (g_i - g_{i-1})/h,\; g_{x,i} = (g_{i+1} - g_i)/h,
$$

$$
g_{\overset{0}{x},i} = (g_{x,i} + g_{\bar{x},i})/2,\; g_{\bar{x}x,i} = (g_{x,i} - g_{\bar{x},i})/h,
$$

$$\|g\|_\infty \equiv \|g\|_{\infty,\bar{\omega}} = \max_{0\leqslant i\leqslant N} |g_i|,\ \|g\|_1 = \|g\|_{1,\omega} = \sum_{i=1}^{N-1} h\,|g_i|.$$

The approach of generating difference method is through the integral identity

$$h^{-1}\int_0^l Lu(x)\psi_i(x)dx = h^{-1}\int_0^l f(x)\psi_i(x)dx, i = 1,2,...,N-1 \tag{3.1}$$

with basis functions

$$\psi_i(x) = \begin{cases} \psi_i^{(1)}(x),\ x_{i-1} < x < x_i, \\ \psi_i^{(2)}(x),\ x_i < x < x_{i+1}, \\ 0,\ x \notin (x_{i-1}, x_{i+1}), \end{cases}$$

where $\psi_i^{(1)}(x)$ and $\psi_i^{(2)}(x)$ are the solutions of the following problems, respectively

$$\varepsilon\psi''(x) - a_i\psi'(x) = 0,\ \ x_{i-1} < x < x_i,\ \psi(x_{i-1}) = 0,\ \ \psi(x_i) = 1,$$

$$\varepsilon\psi''(x) - a_i\psi'(x) = 0,\ \ x_i < x < x_{i+1},\ \psi(x_i) = 1,\ \ \psi(x_{i+1}) = 0.$$

The functions $\psi_i^{(1)}(x)$ and $\psi_i^{(2)}(x)$ can be explicitly expressed as

$$\psi_i^{(1)}(x) = \frac{e^{a_i(x-x_{i-1})/\varepsilon} - 1}{e^{a_i h/\varepsilon} - 1},\ \psi_i^{(2)}(x) = \frac{1 - e^{-a_i(x_{i+1}-x)/\varepsilon}}{1 - e^{-a_i h/\varepsilon}},$$

which, obviously, satisfy

$$h^{-1}\int_{x_{i-1}}^{x_{i+1}} \psi_i(x)dx = 1.$$

Rearranging (3.1), it gives

$$-\varepsilon h^{-1}\int_{x_{i-1}}^{x_{i+1}} \psi_i'(x)u'(x)dx + a_i h^{-1}\int_{x_{i-1}}^{x_{i+1}} \psi_i(x)u'(x)dx + b_i u_i + c_i u_{i-N_0} = f_i - R_i,$$

$$i = 1,2,...,N-1$$

with

$$R_i = R_i^{(1)} + R_i^{(2)} + R_i^{(3)} + R_i^{(4)}, \tag{3.2}$$

$$R_i^{(1)} = h^{-1}\int_{x_{i-1}}^{x_{i+1}} [a(x) - a(x_i)]u'(x)\psi_i(x)dx,$$

$$R_i^{(2)} = h^{-1}\int_{x_{i-1}}^{x_{i+1}} [b(x)u(x) - b(x_i)u(x_i)]\psi_i(x)dx$$

$$= h^{-1} \int_{x_{i-1}}^{x_{i+1}} dx \psi_i(x) \int_{x_{i-1}}^{x_{i+1}} \frac{d}{dx}(b(x)u(x)) K^*_{0,i}(x,\xi) d\xi,$$

$$R_i^{(3)} = h^{-1} \int_{x_{i-1}}^{x_{i+1}} [c(x)u(x-r) - c(x_i)u(x_i - r)]\psi_i(x) dx$$

$$= h^{-1} \int_{x_{i-1}}^{x_{i+1}} dx \psi_i(x) \int_{x_{i-1}}^{x_{i+1}} \frac{d}{dx}(c(x)u(x-r)) K^*_{0,i}(x,\xi) d\xi,$$

$$K^*_{0,i}(x,\xi) = T_0(x-\xi) - T_0(x_i - \xi),\ 1 \leqslant i \leqslant N-1,$$

$$R_i^{(4)} = h^{-1} \int_{x_{i-1}}^{x_{i+1}} [f(x_i) - f(x)]\psi_i(x) dx.$$

To be consistent with [3], we present the following approach for exact solution

$$\ell u_i \equiv \varepsilon \theta_i u_{\bar{x}x,i} + a_i u_{\overset{0}{x},i} + b_i u_i + c_i u_{i-N_0} = f_i - R_i,\ i = 1,2,...,N-1.$$

where

$$\theta_i = \rho_i \coth(\rho_i),\ \rho_i = \frac{a_i h}{2\varepsilon}. \tag{3.3}$$

Herefrom, we propose the following difference scheme for approximating (1.1)-(1.2)

$$\ell y_i \equiv \varepsilon \theta_i y_{\bar{x}x,i} + a_i y_{\overset{0}{x},i} + b_i y_i + c_i y_{i-N_0} = f_i\ ,\ 0 < i < N, \tag{3.4}$$

$$y_i = \varphi_i,\ -N_0 \leqslant i \leqslant 0\ ;\ \gamma y_{N_1} + y_N = A, \tag{3.5}$$

where $\theta$ is given by (3.3).

## 4. Stability bound and convergence

In this section, we show the convergence of our method. Note that the error function $z_i = y_i - u_i$ is the solution of the discrete problem

$$\ell z_i = R_i,\ 0 < i < N, \tag{4.1}$$

$$z_i = 0,\ -N_0 \leqslant i \leqslant 0\ ;\ \gamma z_{N_1} + z_N = 0. \tag{4.2}$$

where the truncation error $R_i$ is given by (3.2).

**Lemma 2.** *If $a(x), b(x), c(x), f(x) \in C^1(\bar{\Omega})$ and $\varphi(x) \in C^1(\Omega_0)$, then for the truncation error $R_i$ we have*

$$\|R\|_1 \leqslant Ch. \tag{4.3}$$

*Proof.* Using the mean value theorem, we get

$$|a(x)-a(x_i)| = \left|a'(\xi)\right| |x-x_i| \leqslant Ch,\ x,\xi \in [x_{i-1},x_{i+1}].$$

Hence

$$\left|R_i^{(1)}\right| \leqslant Ch(h^{-1}\int_{x_{i-1}}^{x_{i+1}} \left|u'(x)\right| \psi_i(x)dx)$$

and taking also into account that $0 \leqslant \psi_i(x) \leqslant 1$, we have

$$\left\|R^{(1)}\right\|_1 \leqslant Ch\sum_{i=1}^{N-1}\int_{x_{i-1}}^{x_{i+1}} \left|u'(x)\right| dx \leqslant 2Ch\int_0^l \left|u'(x)\right| dx.$$

We now use the estimate (2.3) to establish that

$$\left\|R^{(1)}\right\|_1 \leqslant Ch(1+\frac{1}{\varepsilon}\int_0^l e^{-\frac{\alpha x}{\varepsilon}}dx) \leqslant Ch(1+\alpha^{-1}(1-e^{-\frac{\alpha l}{\varepsilon}})) = O(h).$$

For $R_i^{(2)}$, in view of $b \in C^1(\overline{\Omega})$ and using Lemma 1, we obtain

$$\left|R_i^{(2)}\right| \leqslant \int_{x_{i-1}}^{x_{i+1}} \left|\frac{d}{dx}[b(x)u(x)]\right| (\xi)d\xi \leqslant C(1+\int_{x_{i-1}}^{x_{i+1}} \left|u'(\xi)\right| d\xi).$$

Hence

$$\left\|R^{(2)}\right\|_1 \leqslant Ch(1+\sum_{i=1}^{N-1}\int_{x_{i-1}}^{x_{i+1}} \left|u'(\xi)\right| d\xi) \leqslant Ch(1+\int_0^l \left|u'(\xi)\right| d\xi).$$

Using the (2.3), we get

$$\left\|R^{(2)}\right\|_1 \leqslant Ch(1+\frac{1}{\varepsilon}\int_0^l e^{-\frac{\alpha\xi}{\varepsilon}}d\xi) \leqslant Ch(1+\alpha^{-1}(1-e^{-\frac{\alpha l}{\varepsilon}})) = O(h).$$

For $R_i^{(3)}$, in virtue of $c \in C^1(\overline{\Omega})$ and using Lemma 1, we get

$$\left|R_i^{(3)}\right| \leqslant \int_{x_{i-1}}^{x_{i+1}} \left|\frac{d}{dx}[c(x)u(x-r)]\right| (\xi)d\xi \leqslant C(1+\int_{x_{i-1}}^{x_{i+1}} \left|u'(\xi-r)\right| d\xi).$$

Hence

$$\left\|R^{(3)}\right\|_1 \leqslant Ch(1+\sum_{i=1}^{N-1}\int_{x_{i-1}}^{x_{i+1}} \left|u'(\xi-r)\right| d\xi) \leqslant Ch(1+\int_0^l \left|u'(\xi-r)\right| d\xi)$$

and, after replacing $s = \xi - r$ this reduces to

$$\left\| R^{(3)} \right\|_1 \leqslant Ch(1 + \int_{-r}^{l-r} |u'(s)|\, ds) = Ch(1 + \int_{-r}^{0} |\varphi'(s)|\, ds + \int_{0}^{l-r} |u'(s)|\, ds)$$

which yields

$$\left\| R^{(3)} \right\|_1 \leqslant Ch(1 + \|\varphi'\|_{1,0} + \alpha^{-1}(1 - e^{-\frac{\alpha(l-r)}{\varepsilon}})) = O(h).$$

The same estimate is obtained for $R_i^{(4)}$ in the similar manner as above. □

**Lemma 3.** *Let $z_i$ be the solution (4.1)-(4.2) and (2.1) holds true. Then*

$$\|z\|_{\infty,\omega} \leqslant \|R\|_{1,\omega}. \tag{4.4}$$

*Proof.* We here will use the discrete Green's function $G^h(x_i, s_j)$ for the operator

$$L_*^h \overline{z}_i := -\varepsilon\theta_i \overline{z}_{\bar{x}x,i} - a_i \overline{z}_{\underset{x,i}{0}}, \quad 1 \leqslant i \leqslant N-1,$$
$$\overline{z}_0 = 0, \quad \overline{z}_N = 0.$$

As a function of $x_i$ for fixed $s_j$ this function is being defined as

$$L_*^h G^h(x_i, s_j) = \delta^h(x_i, s_j),\ x_i \in \omega,\ s_j \in \omega,$$
$$G^h(0, s_j) = G^h(l, s_j),\ s_j \in \omega,$$

where $\delta^h(x_i, s_j) = h^{-1}\delta_{ij}$ and $\delta_{ij}$ is the Kronecker delta. Using Green's function and below transformation

$$z_i = \overline{z}_i - \frac{\gamma x_i}{\gamma x_{N_1} + x_N} \overline{z}_{N_1},$$

we can write down the following relation for solution of problem (4.1)-(4.2)

$$z_i = \sum_{j=1}^{N-1} hG^h(x_i, s_j)(b_j z_j + c_j z_{j-N_0} - R_j)$$
$$- \frac{\gamma x_i}{\gamma x_{N_1} + x_N} \sum_{j=1}^{N-1} hG^h(x_{N_1}, s_j)(b_j z_j + c_j z_{j-N_0} - R_j),\ x_i \in \omega. \tag{4.5}$$

In the analogous manner as in [4] one can show that $0 \leqslant G^h(x_i, s_j) \leqslant \alpha^{-1}$. Then from (4.5) it follows that

$$\|z\|_{\infty,\omega} \leqslant \alpha^{-1}\{\|z\|_{\infty,\omega} \sum_{j=1}^{N-1} h(|b_j| + |c_j|) + \|R\|_{1,\omega}\}$$
$$+ \frac{\gamma x_N}{\gamma x_{N_1} + x_N} \alpha^{-1}\{\|z\|_{\infty,\omega} \sum_{j=1}^{N-1} h(|b_j| + |c_j|) + \|R\|_{1,\omega}\}$$

$$\leqslant \alpha^{-1}\beta\{\|z\|_{\infty,\omega}(\|b\|_{1,\omega}+\|c\|_{1,\omega})+\|R\|_{1,\omega}\}$$
$$\leqslant \lambda\,\|z\|_{\infty,\omega}+\alpha^{-1}\beta\,\|R\|_{1,\omega},$$

which implies the validity of (4.4).

Now, we give the main result on $\varepsilon$-uniform convergence of the proposed method for solving the problem (1.1)-(1.2). □

**Theorem 1.** *Under the condition of Lemma 1, the solution of (1.1)-(1.2) and the solution (3.4)-(3.5) satisfy the estimate*

$$\|y-u\|_{\infty,\bar{\omega}} \leqslant Ch. \tag{4.6}$$

*Proof.* Combining the previous lemmas, we immediately have (4.6). □

## 5. Algorithm and Numerical Results

In this section, we shall propose the following iterative technique for solving problem (3.4)-(3.5). In addition, we demonstrate the effectiveness of the present method by applying it to three examples of problem (1.1)-(1.2).

$$\varepsilon\theta_i\, y^{(n)}_{\bar{x}x,i}+a_i\, y^{(n)}_{\overset{0}{x},i}+b_i\, y^{(n)}_i = f^{(1)}_i,\ f^{(1)}_i = f_i - c_i\, y_{i-N_0},\ 1\le i\le N_0-1,$$

$$y^{(n)}_i=\varphi_i,\ -N_0\le i\le 0;\ y^{(n)}_{N_0}=\kappa_{n-1},$$

$$\varepsilon\theta_i\, y^{(n)}_{\bar{x}x,i}+a_i\, y^{(n)}_{\overset{0}{x},i}+b_i\, y^{(n)}_i = f^{(2)}_i,\ f^{(2)}_i = f_i - c_i\, y^{(n)}_{i-N_0},\ N_0+1\le i\le N-1,$$

$$y^{(n)}_{N_0}=\kappa_{n-1},\ y^{(n)}_N = A-\gamma\mu_{n-1},$$

where $\kappa_n$ and $\mu_n$ are values of (3.4) for $y_{N_0}$ and $y_{N_1}$ so that

$$\kappa_n = [\varepsilon\theta_{N_0}(y^{(n)}_{N_0+1}+y^{(n)}_{N_0-1})+h^2(a_{N_0}y^{(n)}_{\overset{0}{x},N_0}+c_{N_0}\varphi_0-f_{N_0})](2\varepsilon\theta_{N_0}-b_{N_0}h^2)^{-1},$$

$$\mu_n=\begin{cases}\dfrac{\varepsilon\theta_{N_1}(y^{(n)}_{N_1+1}+y^{(n)}_{N_1-1})+h^2(a_{N_1}y^{(n)}_{\overset{0}{x},N_1}+c_{N_1}\varphi_{N_1-N_0}-f_{N_1})}{2\varepsilon\theta_{N_1}-b_{N_1}h^2}, & N_0\ge N_1,\\[2ex] \dfrac{\varepsilon\theta_{N_1}(y^{(n)}_{N_1+1}+y^{(n)}_{N_1-1})+h^2(a_{N_1}y^{(n)}_{\overset{0}{x},N_1}+c_{N_1}y^{(n)}_{N_1-N_0}-f_{N_1})}{2\varepsilon\theta_{N_1}-b_{N_1}h^2}, & N_0<N_1,\end{cases}$$

$$n=1,2,3,\ldots,$$

$\kappa_0=\mu_0=C_0$, and $C_0$ is given by (2.2).

We consider the following test problems:

*Example* 1. (Case 1: $l_1<r$)

$$\begin{cases}\varepsilon u''(x)+64u'(x)-u(x)+0.25u(x-1)=0.25x-1.25,\ 0<x<\frac{3}{2}\\ u(x)=x,\ -1\leqslant x\leqslant 0\,;\ \frac{1}{4}u(\frac{1}{2})+u(\frac{3}{2})=2,\end{cases}$$

the exact solution is given by

$$u(x)=\begin{cases} 1+d_1e^{m_1x}+d_2e^{m_2x}, & x\in[0,1] \\ d_3e^{m_1(x-1)}+d_4e^{m_2(x-1)}-\frac{59}{4} & \\ -\frac{1}{4\sqrt{d}}(x-1)(\sqrt{d}+d_1e^{m_1(x-1)}-d_2e^{m_2(x-1)}), & x\in\left(1,\frac{3}{2}\right] \end{cases}$$

where

$$d_1=\frac{\frac{133}{8}+\beta_1e^{\frac{m_1}{2}}-(\beta_2-e^{m_2}-\frac{1}{8\sqrt{d}}-\frac{1}{4})e^{\frac{m_2}{2}}}{\sigma_3-\sigma_4+\frac{1}{4}(e^{\frac{m_1}{2}}-e^{\frac{m_2}{2}})},$$

$$d_2=-1-d_1,\ d_3=\alpha_1d_1-\beta_1,\ d_4=-\alpha_2d_1+\beta_2-e^{m_2}.$$

*Example* 2. (Case 2: $l_1=r$)

$$\begin{cases} \varepsilon u''(x)+64u'(x)-u(x)+0.25u(x-1)=0.25x-1.25,\ 0<x<\frac{3}{2} \\ u(x)=x,\ -1\leqslant x\leqslant 0\ ;\ \frac{1}{2}u(1)+u(\frac{3}{2})=2, \end{cases}$$

the exact solution is given by

$$u(x)=\begin{cases} 1+d_1e^{m_1x}+d_2e^{m_2x}, & x\in[0,1] \\ d_3e^{m_1(x-1)}+d_4e^{m_2(x-1)}-\frac{59}{4} & \\ -\frac{1}{4\sqrt{d}}(x-1)(\sqrt{d}+d_1e^{m_1(x-1)}-d_2e^{m_2(x-1)}), & x\in\left(1,\frac{3}{2}\right] \end{cases}$$

where

$$d_1=\frac{\frac{135}{8}+\beta_1e^{\frac{m_1}{2}}-\frac{1}{2}(1-e^{m_2})-(\beta_2-e^{m_2}-\frac{1}{8\sqrt{d}})e^{\frac{m_2}{2}}}{\frac{1}{2}(e^{m_1}-e^{m_2})+\sigma_3-\sigma_4},$$

$$d_2=-1-d_1,\ d_3=\alpha_1d_1-\beta_1,\ d_4=-\alpha_2d_1+\beta_2-e^{m_2}.$$

*Example* 3. (Case 3: $l_1>r$)

$$\begin{cases} \varepsilon u''(x)+64u'(x)-u(x)+0.25u(x-1)=0.25x-1.25,\ 0<x<\frac{3}{2} \\ u(x)=x,\ -1\leqslant x\leqslant 0\ ;\ \frac{1}{2}u(\frac{5}{4})+u(\frac{3}{2})=2, \end{cases}$$

the exact solution is given by

$$u(x)=\begin{cases} 1+d_1e^{m_1x}+d_2e^{m_2x}, & x\in[0,1] \\ d_3e^{m_1(x-1)}+d_4e^{m_2(x-1)}-\frac{59}{4} & \\ -\frac{1}{4\sqrt{d}}(x-1)(\sqrt{d}+d_1e^{m_1(x-1)}-d_2e^{m_2(x-1)}), & x\in\left(1,\frac{3}{2}\right] \end{cases}$$

where

$$d_1=\frac{\frac{777}{32}+\beta_1e^{\frac{m_1}{4}}(\frac{1}{2}+e^{\frac{m_1}{4}})-(\beta_2-e^{m_2}-\frac{1}{8\sqrt{d}})e^{\frac{m_2}{2}}}{\sigma_1-\sigma_2+\sigma_3-\sigma_4}$$
$$-\frac{(\frac{\beta_2}{2}-\frac{e^{m_2}}{2}-\frac{1}{32\sqrt{d}})e^{\frac{m_2}{4}}}{\sigma_1-\sigma_2+\sigma_3-\sigma_4},$$

$$d_2 = -1 - d_1,\ d_3 = \alpha_1 d_1 - \beta_1,\ d_4 = -\alpha_2 d_1 + \beta_2 - e^{m_2},$$

and the other coefficients for all examples are as follows

$$\sigma_k = \begin{cases} (\frac{\alpha_k}{2} + (-1)^k \frac{1}{32\sqrt{d}})e^{\frac{m_k}{4}}, & k = 1,2, \\ (\alpha_{k-2} + (-1)^k \frac{1}{8\sqrt{d}})e^{\frac{m_{k-2}}{2}}, & k = 3,4, \end{cases}$$

$$\alpha_{1,2} = e^{m_{1,2}} + \frac{\varepsilon}{2d},\ \beta_{1,2} = \frac{\varepsilon}{4\sqrt{d}}(63m_{2,1} - \frac{1}{\sqrt{d}} - 1),$$

$$m_1 = \frac{-64+\sqrt{d}}{2\varepsilon},\ m_2 = \frac{-64-\sqrt{d}}{2\varepsilon},\ d = 2^{12} + 4\varepsilon.$$

TABLE 1. Errors and convergence rates for Example 1

| $\varepsilon$ | $N_0 = 64$ | $N_0 = 128$ | $N_0 = 256$ | $N_0 = 512$ | $N_0 = 1024$ |
|---|---|---|---|---|---|
| $2^{-2}$ | 3.53104E-5 | 1.02837E-5 | 2.69286E-6 | 6.83444E-7 | 3.60809E-7 |
| | 1.78 | 1.93 | 1.98 | 1.92 | |
| $2^{-4}$ | 5.75563E-5 | 2.46914E-5 | 8.83647E-6 | 2.57391E-6 | 6.74033E-7 |
| | 1.22 | 1.48 | 1.78 | 1.93 | |
| $2^{-6}$ | 6.37278E-5 | 3.08407E-5 | 1.43934E-5 | 6.17426E-6 | 2.20966E-6 |
| | 1.05 | 1.10 | 1.22 | 1.48 | |
| $2^{-8}$ | 6.52702E-5 | 3.23831E-5 | 1.59358E-5 | 7.71119E-6 | 3.59865E-6 |
| | 1.01 | 1.02 | 1.05 | 1.10 | |
| $2^{-10}$ | 6.56558E-5 | 3.27687E-5 | 1.63214E-5 | 8.09677E-6 | 3.98423E-6 |
| | 1.00 | 1.01 | 1.01 | 1.02 | |
| $2^{-12}$ | 6.57522E-5 | 3.28651E-5 | 1.64178E-5 | 8.19317E-6 | 4.08062E-6 |
| | 1.00 | 1.00 | 1.00 | 1.01 | |
| $2^{-14}$ | 6.57763E-5 | 3.28892E-5 | 1.64419E-5 | 8.21727E-6 | 4.10472E-6 |
| | 1.00 | 1.00 | 1.00 | 1.00 | |
| $2^{-16}$ | 6.57820E-5 | 3.28949E-5 | 1.64475E-5 | 8.22291E-6 | 4.11037E-6 |
| | 1.00 | 1.00 | 1.00 | 1.00 | |
| $2^{-18}$ | 6.57837E-5 | 3.28967E-5 | 1.64493E-5 | 8.22470E-6 | 4.11216E-6 |
| | 1.00 | 1.00 | 1.00 | 1.00 | |
| $2^{-20}$ | 6.57842E-5 | 3.28971E-5 | 1.64498E-5 | 8.22515E-6 | 4.11260E-6 |
| | 1.00 | 1.00 | 1.00 | 1.00 | |
| $e^N$ | 6.57842E-5 | 3.28971E-5 | 1.64498E-5 | 8.22515E-6 | 4.11260E-6 |
| $p^N$ | 1.00 | 1.00 | 1.00 | 1.00 | |

Now, we define the exact error $e_\varepsilon^N$ and the computed parameter-uniform maximum pointwise error $e^N$ as follows:

$$e_\varepsilon^N = \|y - u\|_{\infty,\bar{\omega}},\ e^N = \max_\varepsilon e_\varepsilon^N,$$

where $y$ is the numerical approximation to $u$ for various values of $N$, $\varepsilon$. We also define the computed parameter-uniform rate of convergence to be

$$p^N = \log_2\left(e^N/e^{2N}\right).$$

The values of $\varepsilon$ and $N_0$ ($N = 3N_0/2$) for which we solve the test problems are $\varepsilon = 2^{-2i}, i = 1, 2, ..., 10$; $N_0 = 64, 128, 256, 512, 1024$. The resulting errors $e^N$ and the corresponding numbers $p^N$ after 20 iterations are given in Tables 1-3.

TABLE 2. Errors and convergence rates for Example 2

| $\varepsilon$ | $N_0 = 64$ | $N_0 = 128$ | $N_0 = 256$ | $N_0 = 512$ | $N_0 = 1024$ |
|---|---|---|---|---|---|
| $2^{-2}$ | 2.36474E-5 | 6.88645E-6 | 1.8032E-6 | 4.56462E-7 | 1.34503E-7 |
| | 1.78 | 1.93 | 1.98 | 1.76 | |
| $2^{-4}$ | 3.85540E-5 | 1.65391E-5 | 5.91880E-6 | 1.72400E-6 | 4.51464E-7 |
| | 1.22 | 1.48 | 1.78 | 1.93 | |
| $2^{-6}$ | 4.26883E-5 | 2.06587E-5 | 9.64138E-6 | 4.13580E-6 | 1.48013E-6 |
| | 1.05 | 1.10 | 1.22 | 1.48 | |
| $2^{-8}$ | 4.37216E-5 | 2.16919E-5 | 1.06746E-5 | 5.16533E-6 | 2.41053E-6 |
| | 1.01 | 1.02 | 1.05 | 1.10 | |
| $2^{-10}$ | 4.39799E-5 | 2.19502E-5 | 1.09329E-5 | 5.42362E-6 | 2.66881E-6 |
| | 1.00 | 1.01 | 1.01 | 1.02 | |
| $2^{-12}$ | 4.40444E-5 | 2.20148E-5 | 1.09974E-5 | 5.48819E-6 | 2.73338E-6 |
| | 1.00 | 1.00 | 1.00 | 1.01 | |
| $2^{-14}$ | 4.40606E-5 | 2.20309E-5 | 1.10136E-5 | 5.50433E-6 | 2.74952E-6 |
| | 1.00 | 1.00 | 1.00 | 1.00 | |
| $2^{-16}$ | 4.40643E-5 | 2.20346E-5 | 1.10173E-5 | 5.50805E-6 | 2.75325E-6 |
| | 1.00 | 1.00 | 1.00 | 1.00 | |
| $2^{-18}$ | 4.40655E-5 | 2.20359E-5 | 1.10185E-5 | 5.50930E-6 | 2.75449E-6 |
| | 1.00 | 1.00 | 1.00 | 1.00 | |
| $2^{-20}$ | 4.40659E-5 | 2.20362E-5 | 1.10189E-5 | 5.50961E-6 | 2.75480E-6 |
| | 1.00 | 1.00 | 1.00 | 1.00 | |
| $e^N$ | 4.40659E-5 | 2.20362E-5 | 1.10189E-5 | 5.50961E-6 | 2.75480E-6 |
| $p^N$ | 1.00 | 1.00 | 1.00 | 1.00 | |

## 6. CONCLUSION

In this paper, we have developed a finite difference method for solving the singularly perturbed nonlocal boundary-value problem for a second order delay differential equation. This method was based on an exponentially fitted difference scheme on a uniform mesh. The method was proved to be first order accuracy in the discrete maximum norm, with respect to the perturbation parameter $\varepsilon$. Test problems were

TABLE 3. Errors and convergence rates for Example 3

| $\varepsilon$ | $N_0 = 64$ | $N_0 = 128$ | $N_0 = 256$ | $N_0 = 512$ | $N_0 = 1024$ |
|---|---|---|---|---|---|
| $2^{-2}$ | 3.67457E-5 | 1.07011E-5 | 2.80203E-6 | 8.90948E-7 | 3.19656E-7 |
| | 1.78 | 1.94 | 1.65 | 1.48 | |
| $2^{-4}$ | 5.99030E-5 | 2.57004E-5 | 9.19987E-6 | 2.68102E-6 | 7.89100E-7 |
| | 1.22 | 1.48 | 1.78 | 1.93 | |
| $2^{-6}$ | 6.63273E-5 | 3.21016E-5 | 1.49846E-5 | 6.43084E-6 | 2.30438E-6 |
| | 1.05 | 1.10 | 1.22 | 1.48 | |
| $2^{-8}$ | 6.79328E-5 | 3.37071E-5 | 1.65901E-5 | 8.03061E-6 | 3.75060E-6 |
| | 1.01 | 1.02 | 1.05 | 1.10 | |
| $2^{-10}$ | 6.83341E-5 | 3.41084E-5 | 1.69915E-5 | 8.43195E-6 | 4.15194E-6 |
| | 1.00 | 1.01 | 1.01 | 1.02 | |
| $2^{-12}$ | 6.84344E-5 | 3.42087E-5 | 1.70918E-5 | 8.53229E-6 | 4.25228E-6 |
| | 1.00 | 1.00 | 1.00 | 1.01 | |
| $2^{-14}$ | 6.84595E-5 | 3.42338E-5 | 1.71169E-5 | 8.55737E-6 | 4.27736E-6 |
| | 1.00 | 1.00 | 1.00 | 1.00 | |
| $2^{-16}$ | 6.84656E-5 | 3.42397E-5 | 1.71227E-5 | 8.56318E-6 | 4.28313E-6 |
| | 1.00 | 1.00 | 1.00 | 1.00 | |
| $2^{-18}$ | 6.84673E-5 | 3.42416E-5 | 1.71246E-5 | 8.56511E-6 | 4.28510E-6 |
| | 1.00 | 1.00 | 1.00 | 1.00 | |
| $2^{-20}$ | 6.84677E-5 | 3.42420E-5 | 1.71251E-5 | 8.56558E-6 | 4.28557E-6 |
| | 1.00 | 1.00 | 1.00 | 1.00 | |
| $e^N$ | 6.84677E-5 | 3.42420E-5 | 1.71251E-5 | 8.56558E-6 | 4.28557E-6 |
| $p^N$ | 1.00 | 1.00 | 1.00 | 1.00 | |

addressed for different cases of $l_1$ and $r$. The case of $l_1 = r$ was obtained better results than in other cases. Furthermore, the approximate errors and the rates of convergence for test problems was computed for different values of $\varepsilon$ and $N_0$ in Tables 1-3. Numerical results were carried out to show the efficiency and accuracy of the method. Theoretical results represent undergoing more complicated delay problems.

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# CHARACTERIZATION OF SOME MATRIX CLASSES INVOLVING SOME SETS WITH SPEED

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*Abstract.* The paper introduces the notions of boundedness and convergence with speed for difference sequences, and characterizes certain matrix classes associating the sets of such classes of sequences involving the operator $\Delta$ and two speeds $\lambda = (\lambda_k)$ and $\mu = (\mu_k)$ $(0 < \lambda_k \nearrow \infty, 0 < \mu_k \nearrow \infty)$. The results obtained in this paper should easily extendible to difference sequences of higher orders, and even, in combination with multipliers.

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*Keywords:* Matrix classes, boundedness with speed, convergent with speed, $\Delta$-boundedness, $\Delta$-convergent

## 1. Introduction

While studying the convergent process, it is important to know the speed of convergence of this process. For example, in the theory of approximation, and using numerical methods for solving differential and integral equations, several methods have been worked out for estimating the speed of convergence.

Let, as usual, $m$, $c$, $c_0$ be respectively the spaces of all bounded sequences, of all convergent sequences, of all sequences converging to 0. Throughout this paper indices and summation indices run from 0 to $\infty$ unless otherwise specified.

Let $X, Y$ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix with real and complex entries. If for each $x = (\xi_k) \in X$ the series

$$(Ax)_n = \sum_k a_{nk}\xi_k$$

converges and the sequence $Ax = \{(Ax)_n\}$ belongs to $Y$, we say that the matrix $A$ transforms $X$ into $Y$. By $(X, Y)$, we denote the set of all matrices which transform

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$X$ into $Y$.

A matrix $A$ is said to be regular if $A \in (c,c)$ and $\lim_n (Ax)_n = \lim_k \xi_k$ , for each $x = (\xi_k) \in c$, or in short, we write $A \in (c,c;P)$, where $P$ denotes the preservation of limit.

Following Kangro ([5], [6], [7]) (also see [2]), a convergent sequence $x = (\xi_k)$ with

$$\lim_k \xi_k = \xi \text{ and } v_k = \lambda_k(\xi_k - \xi) \tag{1.1}$$

is called bounded with speed $\lambda$ (shortly, $\lambda$-bounded) if $v_k = O(1)$ and convergent with speed $\lambda$ (shortly, $\lambda$-convergent) if the limit $\lim_k v_k$ exists and is finite.

The set of all $\lambda$-bounded sequences is denoted by $m^\lambda$, and the set of all $\lambda$-convergent sequences by $c^\lambda$. It is not difficult to see that $c^\lambda \subseteq m^\lambda \subseteq c$. In addition to it, for an unbounded sequence $\lambda$ this inclusion is strict. For $\lambda_k = O(1)$, we get $c^\lambda = m^\lambda = c$.

The necessary and sufficient conditions for $A \in (m^\lambda, m^\mu)$, $A \in (c^\lambda, c^\mu)$, and $A \in (c^\lambda, m^\mu)$, were first introduced by Kangro ([5], [6], [7]). The estimation and the comparison of speeds of convergence of series and sequences, based on Kangro's concepts of convergence, boundedness, and summability with speed, have also been studied by Šeletski and Tali ([9], [10]), Stadtmüller and Tali ([11]), and Tammeraid ([12], [13], [14],[15]). For more results on matrix transforms of $m^\lambda$and $c^\lambda$, one can refer to ([5], [7], [8]). An improvement of the $\lambda$-convergence has been studied in ([1]).

In this paper, we shall use the notation $\Delta x$ for the sequence of forward differences:

$$\Delta x_k = x_k - x_{k+1}, k \in N.$$

A sequence $x = (\xi_k)$ is called $\Delta$-convergent if the limit $\lim_k \Delta\xi_k$ exists and is finite.

A $\Delta$-convergent sequence $x = (\xi_k)$ with

$$\lim_k \Delta\xi_k = \varsigma \text{ and } v_k = \lambda_k(\Delta\xi_k - \varsigma) \tag{1.2}$$

is called $\Delta$-bounded with speed $\lambda$ (shorty, $\lambda$-$\Delta$- bounded) if $v_k = O(1)$ and $\Delta$-convergent with speed $\lambda$ (shortly, $\lambda$-$\Delta$-convergent) if the limit $\lim_k v_k$ exists and is finite.

By $m_\Delta$ and $c_\Delta$, we denote the sets of all $\Delta$-bounded sequences and of all $\Delta$-convergent sequences respectively.

The set of all $\lambda$-$\Delta$-bounded sequences is denoted by $m_\Delta^\lambda$ and the set of all $\lambda$-$\Delta$-convergent sequences by $c_\Delta^\lambda$. It is not difficult to see that $c_\Delta^\lambda \subseteq m_\Delta^\lambda \subseteq c_\Delta$. In addition to it, for an unbounded sequence $\lambda$ this inclusion is strict. For $\lambda_k = O(1)$, we get

$c_{\Delta}^{\lambda} = m_{\Delta}^{\lambda} = c_{\Delta}$.

It is easy to see that every convergent sequence is $\Delta$-convergent, but the converse may not be true. For this, let us consider the following example

Let $x = (\xi_k) = (\kappa)$. Then $\Delta\xi_k = \xi_k - \xi_{k+1} = -1$. Thus $(\xi_k)$ is divergent but $\Delta$-convergent.

## 2. Characterization of the matrix classes

We begin this section with few known results that will be required to proof the main results of this paper. Let $e=(1, 1, \dots)$, $e_k = (0, \dots, 0, 1, 0, \dots)$, where 1 is in the $(k+1)^{th}$- position, and $\lambda^{-1}=(\frac{1}{\lambda_k})$.

**Theorem 1** ([3], [4], Silverman-Toeplitz)**.** *$A = (a_{nk})$ is regular, i.e., $A \in (c, c; P)$, if and only if*

$$\sup_{n\geq 0} \sum_k |a_{nk}| < \infty; \tag{2.1}$$

$$\lim_n a_{nk} = \delta_k; \tag{2.2}$$

*and*

$$\lim_n \sum_k a_{nk} = \delta \tag{2.3}$$

*with $\delta_k \equiv 0$ and $\delta \equiv 1$.*

**Theorem 2** ([3], [4])**.** *Let $A = (a_{nk})$ be a matrix method. Then $A \in (c, c)$ if and only if (2.1) holds and the finite limits $\delta_k$ and $\delta$ exist.*

**Theorem 3** ([3], [4])**.** *A method $A = (a_{nk}) \in (c_0, c)$ if and only if conditions (2.1) and (2.2) hold.*

**Theorem 4** ([3], [4])**.** *Let $A = (a_{nk})$ be a matrix method. Then $A \in (m, m) = (c, m) = (c_0, m)$ if and only if condition (2.1) holds.*

**Theorem 5** ([3], [4])**.** *A method $A = (a_{nk}) \in (m, c)$ if and only if conditions (2.1) and (2.2) are satisfied and $\lim_n \sum_k |a_{nk} - \delta_k| = 0$.*
*In this case*

$$\lim_n (Ax)_n = \sum_k \delta_k \xi_k,$$

*for every $x = (\xi_k) \in m$.*

**Theorem 6.** *A method $A = (a_{nk}) \in (m^{\lambda}, m_{\Delta}^{\mu})$ if and only if*

$$\lim_n \Delta a_{nk} = \delta_k^{/}, \tag{2.4}$$

$$Ae \in m_{\Delta}^{\mu}, \tag{2.5}$$

$$\sum_k \frac{|\Delta a_{nk}|}{\lambda_k} = O(1), \tag{2.6}$$

$$\mu_n \sum_k \frac{|\Delta a_{nk} - \delta_k^{/}|}{\lambda_k} = O(1). \tag{2.7}$$

*If $\mu_n = O(1)$ and $\lambda_n \neq O(1)$, then in (2.7), it is necessary to replace $O(1)$ by $o(1)$.*

*Proof.* Necessity: Suppose that $A = (a_{nk}) \in (m^\lambda, m_\Delta^\mu)$. It is obvious that $e_k, e \in m^\lambda$. Hence conditions (2.4) and (2.5) are fulfilled.

Let $x = (\xi_k) \in m^\lambda$, then from (1.1) we have

$\xi_k = \frac{v_k}{\lambda_k} + \xi$ where $\lim_k \xi_k = \xi$, $v_k = O(1)$
it follows that

$$(Ax)_n = \sum_k \frac{a_{nk}}{\lambda_k} v_k + \xi \sum_k a_{nk}$$

and

$$\begin{aligned}
\Delta(Ax)_n &= (Ax)_n - (Ax)_{n+1} \\
&= \sum_k \frac{a_{nk}}{\lambda_k} v_k + \xi \sum_k a_{nk} - \sum_k \frac{a_{(n+1)k}}{\lambda_k} v_k - \xi \sum_k a_{(n+1)k} \\
&= \sum_k \frac{(a_{nk} - a_{(n+1)k})}{\lambda_k} v_k + \xi \sum_k (a_{nk} - a_{(n+1)k}) \\
&= \sum_k \frac{\Delta a_{nk}}{\lambda_k} v_k + \xi \sum_k \Delta a_{nk}.
\end{aligned} \tag{2.8}$$

As $(\sum_k a_{nk}) \in m_\Delta^\mu$, by (2.5), then from (2.8) we can assert that the method

$$A_\lambda = \left(\frac{\Delta a_{nk}}{\lambda_k}\right) \text{ transforms the bounded sequence } (v_k) \text{ into } c.$$

Now we assume that $\lambda_n \neq O(1)$. Then for every sequence $(v_k) \in m$, the sequence $(\frac{v_k}{\lambda_k}) \in c_0$. But for $(\frac{v_k}{\lambda_k})$, there exists a convergent sequence $x = (\xi_k)$ such that $\lim_k \xi_k = \xi$ and $\frac{v_k}{\lambda_k} = (\xi_k - \xi)$. Thus, for every sequence $(v_k) \in m$, there exists a sequence $(\xi_k) \in m^\lambda$ such that $v_k = \lambda_k(\xi_k - \xi)$. Hence $A_\lambda \in (m, c)$. This implies by Theorem 5, the condition (2.6) holds,

$$\lim_n \sum_k \frac{|\Delta a_{nk} - \delta_k^{/}|}{\lambda_\kappa} = 0 \tag{2.9}$$

and $\phi^{/} = \lim_n \Delta(Ax)_n = \sum_k \frac{\delta_k^{/}}{\lambda_\kappa} v_k + \xi \lim_n \sum_k \Delta a_{nk}$.

If $\mu_n \neq O(1)$, then writing

$$\mu_n(\Delta(Ax)_n - \phi^{/}) = \\ = \mu_n \sum_k \frac{\Delta a_{nk} - \delta_k^{/}}{\lambda_k} v_k + \xi \mu_n (\sum_k \Delta a_{nk} - \lim_n \sum_k \Delta a_{nk}). \tag{2.10}$$

By (2.5) we can conclude that the method

$$A_{\lambda,\mu_\Delta} = (\mu_n \frac{\Delta a_{nk} - \delta_k^{/}}{\lambda_k}) \in (m,m).$$

This implies by Theorem 4, condition (2.7) is fulfilled.

If $\mu_n = O(1)$, then in (2.7) it is necessary to replace $O(1)$ by $o(1)$; which is similar to (2.9).

If $\lambda_n = O(1)$, then the proof is similar to the case $\lambda_n \neq O(1)$, but in this case $v_k = o(1)$, and instead of the Theorem 5, it is necessary to use the Theorem 3.

Sufficiency: Conversely assume that the conditions (2.4)-(2.7) are valid. Also, for every $x = (\xi_k) \in m^\lambda$, the relation (2.8) holds and by (2.5), $(\sum_k a_{nk}) \in m_\Delta^\mu$. If $\lambda_n \neq O(1)$ and $\mu_n = O(1)$, then using Theorem 5, we can conclude that the method $A_\lambda \in (m,c)$ by (2.4), (2.6) and (2.9) (in this case, we have (2.9) instead of (2.7)). Thus $A \in (m^\lambda, c_\Delta)$.

If $\lambda_n \neq O(1)$ and $\mu_n \neq O(1)$, then validity of (2.9) follows from the validity of (2.7). In this case also $A_\lambda \in (m,c)$ by (2.4), (2.6) and (2.9), that is $A \in (m^\lambda, c_\Delta)$. Therefore, we can assert that the limit $\phi^{/}$exists and is finite and therefore relation (2.10) is fulfilled for every $x = (\xi_k) \in m^\lambda$. Hence by (2.7) and using Theorem 4, we have $A_{\lambda,\mu_\Delta} \in (m,m)$ and by (2.5), we have $A \in (m^\lambda, m_\Delta^\mu)$. For $\lambda_n = O(1)$, the proof is obvious. □

**Theorem 7.** *A method $A = (a_{nk}) \in (c^\lambda, c_\Delta^\mu)$ if and only if conditions (2.6) and (2.7) are fulfilled and*

$$Ae_k \in c_\Delta^\mu, \tag{2.11}$$

$$Ae \in c_\Delta^\mu, \tag{2.12}$$

$$A\lambda^{-1} \in c_\Delta^\mu. \tag{2.13}$$

*If* $A \in (c^{\lambda}, c_{\Delta}^{\mu})$, *then*

$$\lim_n \mu_n(\Delta(Ax)_n - \phi^{/}) = \sum_k a_k^{\lambda,\mu_\Delta}(v_k - v) + \lim_n \mu_n(\sum_k \Delta a_{nk} - \delta^{/})\xi + \lim_n \mu_n(\sum_k \frac{\Delta a_{nk}}{\lambda_k} - a^{\lambda})v, \tag{2.14}$$

*where*

$$\phi^{/} = \lim_n \Delta(Ax)_n, \ v = \lim_k v_k$$

*and*

$$\delta^{/} = \lim_n \sum_k \Delta a_{nk}, \delta_k^{/} = \lim_n \Delta a_{nk},$$

$$a^{\lambda} = \lim_n \sum_k \frac{\Delta a_{nk}}{\lambda_k}, \ a_k^{\lambda,\mu_\Delta} = \lim_n \mu_n \frac{\Delta a_{nk} - \delta_k^{/}}{\lambda_k}$$

*Proof.* Necessity: Suppose that $A \in (c^{\lambda}, c_{\Delta}^{\mu})$. It is not difficult to see that $e_k$, $e$, $\lambda^{-1} \in c^{\lambda}$ and so the conditions (2.11)-(2.13) hold. For every $x = (\xi_k) \in c^{\lambda}$,the equality (2.8) is satisfied and by (2.12) the limit $\delta^{/}$ exists, so the method $A_{\lambda}$ transforms the convergent sequence $(v_k)$ into $c$. Similar to the proof of necessary part of Theorem 6, it can be easily shown that, for every sequence $(v_k) \in c$, there exists a sequence $x = (\xi_k) \in c^{\lambda}$ such that $v_k = \lambda_k(\xi_k - \xi)$. Hence $A_{\lambda} \in (c, c)$. This means that the finite limits $\delta_k^{/}$ and $a^{\lambda}$ exist and condition (2.6) is fulfilled by virtue of Theorem 2. Using relation (2.8), for every $x \in c^{\lambda}$, we can write

$$\phi^{/} = \lim_n \Delta(Ax)_n = a^{\lambda} v + \sum_k \frac{\delta_k^{/}}{\lambda_k}(v_k - v) + \xi\delta^{/}, \tag{2.15}$$

where $\xi = \lim_k \xi_k$ and $v = \lim_k v_k$.

Now using relations (2.8) and (2.15), we get

$$\mu_n(\Delta(Ax)_n - \phi^{/}) = \mu_n \sum_k \frac{\Delta a_{nk} - \delta_k^{/}}{\lambda_k}(v_k - v) + \mu_n(\sum_k \Delta a_{nk} - \delta^{/})\xi + \mu_n(\sum_k \frac{\Delta a_{nk}}{\lambda_k} - a^{\lambda})v. \tag{2.16}$$

As $n \to \infty$, the finite limits for the last two summands in the right hand side of (2.16) exist by conditions (2.12) and (2.13).This implies that the method $A_{\lambda,\mu_\Delta} \in (c_0, c)$. Thus using Theorem 3, the condition (2.7) is satisfied. Lastly, relation (2.14) holds from (2.16).

Sufficiency: Suppose that (2.6)-(2.7) and (2.11)-(2.13) are fulfilled. We observe that the relation (2.16) holds for every $x = (\xi_k) \in c^\lambda$ and also the finite limits $\delta_k^/, \delta^/, a^\lambda$ exist by (2.11), (2.12) and (2.13) respectively. Since (2.6) also holds, so $A_\lambda \in (c,c)$ by Theorem 2 and therefore for every $x \in c^\lambda$, relations (2.15) and (2.16) hold. Now by conditions (2.12) and (2.13), the finite limits for the last two summands in the right side of (2.16) exist as $n \to \infty$. Finally using conditions (2.7), (2.11) and Theorem 3 we can conclude that the method $A_{\lambda,\mu_\Delta} \in (c_0,c)$. Hence, $A \in (c^\lambda, c_\Delta^\mu)$. □

It is easy to see that conditions (2.4) and (2.6) imply the condition

$$\sum_k \frac{|\delta_k^/|}{\lambda_k} < \infty \tag{2.17}$$

Also conditions (2.7) and (2.17) imply condition (2.6). Therefore, from Theorem 6 and Theorem 7, we get the following corollary:

**Corollary 1.** *The condition (2.6) in Theorem 6 and Theorem 7 can be replaced by the condition (2.17).*

Using Theorem 6 and Corollary 1, we get the following corollary:

**Corollary 2.** *A method $A = (a_{nk}) \in (m^\lambda, c_\Delta)$ if and only if the conditions (2.4), (2.6) and (2.9) are fulfilled and the finite limit $\lim_n \sum_k \Delta a_{nk} = \delta^/$ exists. Also the condition (2.6) can be replaced by the condition (2.17).*

**Theorem 8.** *A method $A = (a_{nk}) \in (c^\lambda, m_\Delta^\mu)$ if and only if the conditions (2.4)-(2.7) are satisfied.*

*Also if $\mu_n = O(1)$ and $\lambda_n \neq O(1)$, then in (2.7), it is necessary to replace $O(1)$ by $o(1)$.*

*Proof.* Necessary Part: Suppose that $A = (a_{nk}) \in (c^\lambda, m_\Delta^\mu)$. It is easy to see that $e_k, e \in c^\lambda$. Hence conditions (2.4) and (2.5) are valid. As equality (2.8) holds for every $x = (\xi_k) \in c^\lambda$, and $(\sum_k a_{nk}) \in m_\Delta^\mu$ by (2.5), then the method $A_\lambda$ transforms the convergent sequence $(v_k)$ into $c$. Similar to the proof of necessary part of Theorem 6, it can be easily shown that for every sequence $(v_k) \in c$, there exists a sequence $x = (\xi_k) \in c^\lambda$ such that $v_k = \lambda_k(\xi_k - \xi)$. Hence $A_\lambda \in (c,c)$. This implies by Theorem 2, the condition (2.6) is satisfied.

Using condition (2.8), for every $x = (\xi_k) \in c^\lambda$, we can write

$$\phi^/ = \lim_n \Delta(Ax)_n = \sum_k \frac{\delta_k^/}{\lambda_\kappa} v_k + \xi \lim_n \sum_k \Delta a_{nk}.$$

If $\mu_n \neq O(1)$, then from relation (2.10) and using condition (2.5) we can assert that the method

$$A_{\lambda,\mu_\Delta} \in (c,m).$$

Therefore using Theorem 4, condition (2.7) is fulfilled.

For $\mu_n = O(1)$, then in (2.7) it is necessary to replace $O(1)$ by $o(1)$; which is equivalent to (2.9).

If $\lambda_n = O(1)$, then the proof is similar to the case $\lambda_n \neq O(1)$, but in this case $(v_k) \in c_0$, and instead of the Theorem 2, it is necessary to use the Theorem 3.

Sufficient Part: It is obvious from the Theorem 6. □

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# APPROXIMATION BY NÖRLUND AND RIESZ TYPE DEFERRED CESÀRO MEANS IN THE SPACE $H_P^{(\omega)}$

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*Abstract.* The deferred Cesàro transformations which have useful properties not possessed by the Cesàro transformation was considered by R.P. Agnew in [1]. In [9], Değer and Küçükaslan introduced a generalization of deferred Cesàro transformations by taking account of some well known transformations such as Woronoi-Nörlund and Riesz, and considered the degree of approximation by the generalized deferred Cesàro means in the space $H(\alpha, p)$, $p \geq 1$, $0 < \alpha \leq 1$ by concerning with some sequence classes. In 2014, Nayak *et al.* studied the rate of convergence problem of Fourier series by deferred Cesàro mean in the space $H_p^{(\omega)}$ introduced by Das *et al.* in [7].

In this study we shall give the degree of approximation by the generalized deferred Cesàro means in the space $H_p^{(\omega)}$. Therefore the results given in [17] are generalized according to the summability method.

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## 1. Introduction

Let $f$ be a $2\pi-$ periodic function and $f \in L_p := L_p(0, 2\pi)$ for $p \geq 1$ and let

$$s_n(f;x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{n} A_k(f;x)$$

partial sum of the first $(n+1)$ terms of the Fourier series of $f \in L_p$ $(p \geq 1)$ at a point $x$.

The degree of approximation of the sums $s_n(f;x)$ in different spaces has been studied by many authors. In [20], Quade investigated approximation properties of the partial sums of Fourier series in $L_p$ norms. Chandra in [5] and Leindler in [13] generalized the results of Quade using the Woronoi-Nörlund and the Riesz means of Fourier series for some sequence classes. A similar problems were studied for more general means in [8] and [15].

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The space

$$H_\alpha = \{f \in C_{2\pi} : |f(x) - f(y)| \leq K|x-y|^\alpha, 0 < \alpha \leq 1\}$$

where $C_{2\pi}$ is space of all $2\pi$-periodic and continuous functions defined on $\mathbb{R}$ with the supremum norm and $K$ is a positive constant, not necessarily the same at each occurrence, is a Banach space (see Prösdorff, [19]) with the norm $\|\cdot\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_C + \sup_{x \neq y} \Delta^\alpha f(x,y) \tag{1.1}$$

where

$$\Delta^\alpha f(x,y) = \frac{|f(x) - f(y)|}{|x-y|^\alpha} \ (x \neq y),$$

by convention $\Delta^0 f(x,y) = 0$ and

$$\|f\|_C = \sup_{x \in [-\pi,\pi]} |f(x)|.$$

The metric generated by the norm (1.1) on $H_\alpha$ is called the Hölder metric. Prösdorff has studied the degree of approximation by Cesàro means in the Hölder metric and proved the following theorem.

**Theorem 1** ([19]). *Let $f \in H_\alpha (0 < \alpha \leq 1)$ and $0 \leq \beta < \alpha \leq 1$. Then*

$$\|\sigma_n(f) - f\|_\beta = O(1) \begin{cases} n^{\beta-\alpha} & ,0 < \alpha < 1; \\ n^{\beta-1} \ln n & ,\alpha = 1 \end{cases}$$

*where $\sigma_n(f)$ is the Cesàro means of the Fourier series of $f$.*

The case $\beta = 0$ in Theorem 1 is due to Alexits [2]. Leindler introduced more general classes than the Hölder classes of $2\pi$-periodic continuous functions, generalized the results of Prösdorff [11]. Chandra obtained a generalization of Theorem 1 by considering the Woronoi-Nörlund transform [4]. In [16] Mohapatra and Chandra considered the problem by a matrix means of the Fourier series of $f \in H_\alpha$.

Further generalizations of the Hölder metric was given in [6] and [7]. In [6], Das *et al.* studied the results regarding the degree of approximation by infinite matrix means involved in the deferred Cesàro means in a generalized Hölder metric.

In [9], the authors considered the degree of approximation to functions in this space with respect to the norm in the space given in [6] by the deferred Woronoi-Nörlund means and the deferred Riesz means of the Fourier series of the functions.

The modulus of continuity of $f \in C_{2\pi}$ is defined by

$$\omega(f,\delta) := \sup_{0<|h|\leq\delta} |f(x+h) - f(x)|.$$

According to this, the class of functions $H^\omega$ is defined by

$$H^\omega := \{f \in C_{2\pi} : \omega(f,\delta) = O(\omega(\delta))\}$$

where $\omega(\delta)$ is a modulus of continuity.

There are numerous papers related to the degree of approximation in $H^{\omega}$ space such as Leindler [11], Mazhar and Totik [14], Das *et al.* [7], Khatri and Mishra [10].

A further generalization of $H^{\omega}$ space has been given by Das, Nath and Ray in [7]. They defined the following notations: If $f \in L_p(0,2\pi)$, $p \geq 1$, then denote

$$H_p^{(\omega)} := \{f \in L_p(0,2\pi), p \geq 1 : A(f,\omega) < \infty\}$$

where $\omega$ is a modulus of continuity and

$$A(f,\omega) := \sup_{t \neq 0} \frac{||f(\cdot + t) - f(\cdot)||_p}{\omega(|t|)}$$

where $\|.\|_p$ will denote $L_p$-norm with respect to $x$ and is defined by

$$\|f\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x|^p \, dx \right\}^{\frac{1}{p}}.$$

The norm in the space $H_p^{(\omega)}$ is defined by

$$||f||_p^{(\omega)} := ||f||_p + A(f,\omega).$$

Given the spaces $H_p^{(\omega)}$ and $H_p^{(v)}$, if $\frac{\omega(t)}{v(t)}$ is nondecreasing , then

$$H_p^{(\omega)} \subseteq H_p^{(v)} \subseteq L_p \quad p \geq 1$$

since

$$||f||_p^{(v)} \leq \max\left(1, \frac{\omega(2\pi)}{v(2\pi)}\right) ||f||_p^{(\omega)}.$$

Especially, the leading studies related to degree of approximation in $H_p^{(\omega)}$ space can be found in [7] and [12].

## 2. Deferred Cesàro mean and its generalization

The concept of deferred Cesàro mean has been given by Agnew as following (see [1]). Let $a = (a_n)$ and $b = (b_n)$ be sequences of nonnegative integers with conditions

$$a_n < b_n \qquad n = 1,2,3,... \tag{2.1}$$

and

$$\lim_{n\to\infty} b_n = +\infty. \tag{2.2}$$

The deferred Cesàro mean, $D_a^b$, determined by $a$ and $b$

$$D_n = D_a^b = \frac{S_{a_n+1} + S_{a_n+2} + \cdots + S_{b_n}}{b_n - a_n} = \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} S_k,$$

where $(S_k)$ is a sequence of real or complex numbers.

Since each $D_a^b$ with conditions (2.1) and (2.2) satisfies the Silverman-Toeplitz conditions, every $D_a^b$ is regular. Note that $D_a^b$ involves, except in case $a_n = 0$ all $n$, means of deferred elements of $(S_n)$. It is also known that $D_{n-1}^n$ is the identity transformation and $D_0^n$ is the $(C,1)$ transformation. The basic properties of $D_a^b$ can be found in [1]. By considering deferred Cesàro means, Değer and Küçükaslan gave the following notations with conditions (2.1) and (2.2) in [9].

Let $(p_n)$ be a positive sequence of real numbers. Then

$$D_a^b N_n(f;x) = \frac{1}{P_0^{b_n-a_n-1}} \sum_{m=a_n+1}^{b_n} p_{b_n-m} S_m(f;x),$$

and

$$D_a^b R_n(f;x) = \frac{1}{P_{a_n+1}^{b_n}} \sum_{m=a_n+1}^{b_n} p_m S_m(f;x)$$

where

$$P_0^{b_n-a_n-1} = \sum_{k=0}^{b_n-a_n-1} p_k \neq 0, \qquad P_{a_n+1}^{b_n} = \sum_{k=a_n+1}^{b_n} p_k \neq 0$$

and

$$S_n(f;x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t)\, D_n(t)\, dt,$$

in which

$$D_n(t) = \frac{\sin(n+\frac{1}{2})t}{2\sin\left(\frac{t}{2}\right)}.$$

These two methods are called *deferred Woronoi-Nörlund means*, $(D_a^b N, p)$, and *deferred Riesz means*, $(D_a^b R, p)$, with respect to $S_m(f;x)$, respectively. In case $b_n = n$ and $a_n = 0$, the methods $D_a^b N_n(f;x)$ and $D_a^b R_n(f;x)$ give us the classicaly known Woronoi-Nörlund and Riesz means, respectively. Provided that $p_n = 1$ for all $n(\geq 0)$, both of them yield deferred Cesàro means

$$D_a^b(f;x) = \frac{1}{b_n - a_n} \sum_{m=a_n+1}^{b_n} S_m(f;x)$$

of $S_m(f,x)$.

In addition to this, if $b_n = n$, $a_n = 0$ and $p_k = 1$ for these two methods, then they coincide with Cesàro method $C_1$. In the event that $a_n = 0$, $(b_n)$ is a strictly increasing sequence of positive integers with $b(0) = 0$ and $p_k = 1$, then they give us Cesàro submethod which is obtained by deleting a set of rows from Cesàro matrix (see [3, 8, 18]).

There exist some inclusions that established relation between deferred method and Cesàro method. Before giving this inclusions without detail, we need to give some definition. If $(a_n)$ and $(b_n)$ satisfy, in addition to (2.1) and (2.2), the condition

$$\frac{a_n}{b_n - a_n} = O(1) \tag{2.3}$$

for all $n$, then $(D_a^b)$ is properly deferred ; such a transformation is called a proper $(D_a^b)$ (see [1]). Therefore we know that

$$(C,1) \subset D(a_n, b_n) \quad \text{if and only if} \quad (D_a^b) \quad \text{is properly deferred,}$$

and

$$D(a_n, n) \subset (C,1).$$

On the other hand,

$$D(a_n, n) \sim (C,1) \quad \text{if and only if} \quad (D_a^b) \quad \text{is properly deferred.}$$

In 2014, Nayak *et al.* has studied the rate of convergence problem of Fourier series by deferred Cesàro mean in the generalized Hölder metric $(H_p^{(w)})$ and gave the following theorem.

**Theorem 2** ([17])**.** *Let $v$ and $w$ be moduli of continuity such that $\frac{w}{v}$ is non-decreasing and $f \in H_p^{(w)}, p \geq 1$ . Let*

$$q_n = (2j+1)p_n + 2j$$

*where $j$ is a positive integer. Then*

$$(i) \parallel D_n(S_n(f;.)) - f(.) \parallel_p^{(v)} = O\left(\frac{1}{p_n+1}\right) + \frac{O(1)}{(p_n+1)^2} \int_{\frac{\pi}{(p_n+1)}}^{\pi} \frac{w(t)}{t^3 v(t)} dt.$$

*If in addition $\frac{w(t)}{tv(t)}$ is non-increasing then*

$$(ii) \parallel D_n(S_n(f;.)) - f(.) \parallel_p^{(v)} = \frac{O(1)}{(p_n+1)^2} \int_{\frac{\pi}{(p_n+1)}}^{\pi} \frac{w(t)}{t^3 v(t)} dt$$

*and a fortiori*

$$(iii) \parallel D_n(S_n(f;.)) - f(.) \parallel_p^{(v)} = O\left(\frac{w(\pi/(p_n+1))}{v(\pi/(p_n+1))}\right).$$

Taking into account of this theorem and the methods given in [9], we shall improve the results in [17] on the degree of approximation by the generalized deferred Cesàro means in the space $H_p^{\omega}$. Therefore the results given in [17] are generalized according to the summability method.

## 3. THEOREMS

In this section we will give two theorems on the degree of approximation by deferred Woronoi-Nörlund means and deferred Riesz means in generalized Hölder metric.

**Theorem 3.** *Let $v$ and $w$ be moduli of continuity such that $\frac{w}{v}$ is nondecreasing and $f \in H_p^{(w)}$, $p \geq 1$. Let $b_n = (2j+1)a_n + 2j$ where $j$ is a positive integer. Moreover let $(p_n)$ be a positive sequence and the conditions*

$$(b_n - a_n)p_{b_n} = O(P_{a_n+1}^{b_n}) \tag{3.1}$$

*and*

$$\sum_{m=a_n+1}^{b_n-1} |\Delta_m(p_m)| = O(|p_{b_n} - p_{a_n+1}|) \tag{3.2}$$

*are satisfied where $\Delta_m(p_m) = p_m - p_{m+1}$. Then*

$$\begin{aligned}&\|D_a^b R_n(f;.) - f(.)\|_p^{(v)}\\ &\quad = O\left(1 + \frac{|p_{b_n} - p_{a_n+1}|}{p_{b_n}}\right)\left\{\frac{1}{a_n+1} + \frac{1}{(a_n+1)^2}\int_{\frac{\pi}{(a_n+1)}}^{\pi} \frac{w(t)}{t^3 v(t)}dt\right\}\end{aligned}$$

*Proof.* By definition of the deferred Riesz means, we have

$$D_a^b R_n(f;.) - f(.) = \frac{1}{P_{a_n+1}^{b_n}} \sum_{m=a_n+1}^{b_n} p_m(S_m(f;.) - f(.)).$$

By elementary methods, we get

$$I := D_a^b R_n(f;.) - f(.) = \frac{1}{P_{a_n+1}^{b_n}} \sum_{m=0}^{b_n-a_n-1} p_{m+a_n+1}(S_{m+a_n+1}(f;.) - f(.)).$$

Using Abel's transformation, we see that

$$\begin{aligned}I &= \frac{1}{P_{a_n+1}^{b_n}} \sum_{m=0}^{b_n-a_n-2} [p_{m+a_n+1} - p_{m+a_n+2}][\sum_{k=0}^{m}(S_{k+a_n+1}(f;.) - f(.))]\\ &\quad + \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=0}^{b_n-a_n-1} [(S_{k+a_n+1}(f;.) - f(.))]p_{b_n} =: I_1 + I_2\end{aligned} \tag{3.3}$$

By considering the second term in right side of the above equality, we write

$$I_2 = \frac{p_{b_n}}{P_{a_n+1}^{b_n}} \frac{b_n - a_n}{b_n - a_n} \sum_{k=a_n+1}^{b_n} (S_k(f;.) - f(.)) = \frac{p_{b_n}(b_n - a_n)}{P_{a_n+1}^{b_n}}(D_a^b(S_n(f;.)) - f(.)).$$

Hence by the condition (3.1) and Theorem 2-(i), we obtain

$$\begin{aligned}\|I_2\|_p^{(v)} &\leq |\frac{p_{b_n}(b_n-a_n)}{P_{a_n+1}^{b_n}}|\|D_a^b(S_n(f;.))-f(.)\|_p^{(v)}\\ &= O(1)\|D_a^b(S_n(f;.))-f(.)\|_p^{(v)}\\ &= O\left(\frac{1}{a_n+1}\right)+\frac{O(1)}{(a_n+1)^2}\int_{\frac{\pi}{(a_n+1)}}^{\pi}\frac{w(t)}{t^3v(t)}dt. \qquad (3.4)\end{aligned}$$

Now let us consider the first term in (3.3). Then we have

$$\begin{aligned}I_1 &= \frac{1}{P_{a_n+1}^{b_n}}\sum_{m=0}^{b_n-a_n-2}[\sum_{k=0}^{m}(S_{k+a_n+1}(f;.)-f(.))][p_{m+a_n+1}-p_{m+a_n+2}]\\ &= \frac{1}{P_{a_n+1}^{b_n}}\sum_{m=a_n+1}^{b_n-1}\Delta_m(p_m)\sum_{k=0}^{m-a_n-1}(S_{k+a_n+1}(f;.)-f(.))\\ &= \frac{1}{P_{a_n+1}^{b_n}}\sum_{m=a_n+1}^{b_n-1}\Delta_m(p_m)\frac{m-a_n}{m-a_n}\sum_{k=a_n+1}^{m}(S_k(f;.)-f(.))\\ &= \frac{1}{P_{a_n+1}^{b_n}}\sum_{m=a_n+1}^{b_n-1}(m-a_n)\Delta_m(p_m)[D_a^m(S_n(f;.))-f(.)] \qquad (3.5)\end{aligned}$$

By considering (3.5) and Theorem 2-(i), we write

$$\begin{aligned}\|I_1\|_p^{(v)} &\leq \frac{1}{P_{a_n+1}^{b_n}}(b_n-a_n)\sum_{m=a_n+1}^{b_n-1}|\Delta_m(p_m)|\|D_a^m(S_n(f;.))-f(.)\|_p^{(v)}\\ &= \frac{(b_n-a_n)}{P_{a_n+1}^{b_n}}\left\{O\left(\frac{1}{a_n+1}\right)+\frac{O(1)}{(a_n+1)^2}\int_{\frac{\pi}{(a_n+1)}}^{\pi}\frac{w(t)}{t^3v(t)}dt\right\}\sum_{m=a_n+1}^{b_n-1}|\Delta_m(p_m)|.\end{aligned}$$

Owing to condition (3.1) and condition (3.2) in the last term, we obtain

$$\|I_1\|_p^{(v)} = \frac{|p_{b_n}-p_{a_{n+1}}|}{p_{b_n}}\left\{O\left(\frac{1}{a_n+1}\right)+\frac{O(1)}{(a_n+1)^2}\int_{\frac{\pi}{(a_n+1)}}^{\pi}\frac{w(t)}{t^3v(t)}dt\right\}. \qquad (3.6)$$

Combining (3.3), (3.4) and (3.6), we get the desired result . Therefore the proof is completed. □

**Corollary 1.** *Under conditions of Theorem 3, if $f \in H_p^{(w)}$ for $p \geq 1$ and $\frac{w(t)}{tv(t)}$ is nonincreasing then*

*(i)* $\| D_a^b R_n(f;.)-f(.) \|_p^{(v)} = O(1)\left(1+\frac{|p_{b_n}-p_{a_n+1}|}{p_{b_n}}\right)\frac{1}{(a_n+1)^2}\int_{\frac{\pi}{(a_n+1)}}^{\pi}\frac{w(t)}{t^3v(t)}dt$

*and furthermore*
*(ii)* $\| D_a^b R_n(f;.) - f(.) \|_p^{(v)} = O(1)\left(1 + \frac{|p_{b_n} - p_{a_n+1}|}{p_{b_n}}\right)\frac{w(\pi/(a_n+1))}{v(\pi/(a_n+1))}$.

Theorem 3 and its corollaries are very important due to the relations between Riesz type deferred method and Riesz method. Let us recall some results establishing these relations given in [9]. Suppose that the sequences $(a_n)$ and $(b_n)$ satisfy the conditions (2.1) and (2.2). If the condition

$$\frac{p_1 + p_2 + \cdots + p_{a_n}}{P_{a_n+1}^{b_n}} = O(1) \tag{3.7}$$

satisfy for $(p_n)$, then we shall say that $(D_a^b R, p)$ is properly deferred and such a transformation is called a proper $(D_a^b R, p)$. We see that if $(p_n) = 1$ for all $n$ with the conditions (2.1) and (2.2), then $(D_a^b R, p)$ and the condition (3.7) are reduced $(D)$ and the condition (2.3), respectively. In [9], we know that "$(R, p) \subset (D_a^b R, p)$ *if and only if* $(D_a^b R, p)$ *is proper*". On the other hand, we have "$(D_a^n R, p) \subset (R, p)$" and "$(D_a^n R, p) \sim (R, p)$ *if and only if* $(D_a^n R, p)$ *is proper*".

**Theorem 4.** *Let $v$ and $w$ be moduli of continuity such that $\frac{w}{v}$ is nondecreasing and $f \in H_p^{(w)}$, $p \geq 1$. Let*

$$b_n = (2j+1)a_n + 2j$$

*where $j$ is a positive integer. Moreover let $(p_n)$ be a positive sequence and the conditions*

$$(b_n - a_n)p_{b_n - a_n - 1} = O(P_0^{b_n - a_n - 1}) \tag{3.8}$$

*and*

$$\sum_{m=a_n+1}^{b_n-1} |\Delta_m(p_{b_n-m})| = O(|p_{b_n-a_n-1} - p_0|) \tag{3.9}$$

*are satisfied. Then*

$$\begin{aligned} &\|D_a^b N_n(f;\cdot) - f(\cdot)\|_p^v \\ &= O\left(\frac{p_0 + |p_{b_n-a_n-1} - p_0|}{p_{b_n-a_n-1}}\right)\left\{\frac{1}{a_n+1} + \frac{1}{(a_n+1)^2}\int_{\frac{\pi}{(a_n+1)}}^{\pi} \frac{w(t)}{t^3 v(t)}dt\right\}. \end{aligned} \tag{3.10}$$

*Proof.* Since

$$J := D_a^b N_n(f;.) - f(.) = \frac{1}{P_0^{b_n-a_n-1}} \sum_{m=0}^{b_n-a_n-1} p_{b_n-m-a_n-1}(S_{m+a_n+1}(f;.) - f(.)),$$

we get that

$$J=\frac{1}{P_0^{b_n-a_n-1}}\sum_{m=0}^{b_n-a_n-2}[\sum_{k=0}^{m}(S_{k+a_n+1}-f(.))][p_{b_n-m-a_n-1}-p_{b_n-m-a_n-2}]$$

$$+\frac{1}{P_0^{b_n-a_n-1}}\sum_{k=0}^{b_n-a_n-1}[(S_{k+a_n+1}(f;.)-f(.)]p_0=:J_1+J_2 \tag{3.11}$$

by Abel's transformation. Let us consider $J_2$. Since

$$J_2=\frac{p_0(b_n-a_n)}{P_0^{b_n-a_n-1}}(D_a^b(S_n(f;.))-f(.)),$$

we have

$$\|J_2\|_p^{(v)}\leq O(1)\frac{p_0}{p_{b_n-a_n-1}}\left\{\frac{1}{a_n+1}+\frac{1}{(a_n+1)^2}\int_{\frac{\pi}{(a_n+1)}}^{\pi}\frac{w(t)}{t^3v(t)}dt\right\} \tag{3.12}$$

by considering (3.8) and Theorem 2-(i). Let us estimate $J_1$. By elementary methods, we know that

$$J_1=\frac{1}{P_0^{b_n-a_n-1}}\sum_{m=a_n+1}^{b_n-1}(p_{b_n-m}-p_{b_n-m-1})\sum_{k=0}^{m-a_n-1}(S_{k+a_n+1}(f;.)-f(.))$$

$$=\frac{1}{P_0^{b_n-a_n-1}}\sum_{m=a_n+1}^{b_n-1}\Delta_m(p_{b_n-m})(m-a_n)[D_a^m(S_n(f;.))-f(.)].$$

Taking into account of (3.8), (3.9) and Theorem 2-(i), we write

$$\|J_1\|_p^{(v)}\leq O\left(\frac{|p_{b_n-a_n-1}-p_0|}{p_{b_n-a_n-1}}\right)\left\{\frac{1}{a_n+1}+\frac{1}{(a_n+1)^2}\int_{\frac{\pi}{(a_n+1)}}^{\pi}\frac{w(t)}{t^3v(t)}dt\right\}. \tag{3.13}$$

Therefore, we get (3.10) by collecting of (3.11)-(3.13). □

**Corollary 2.** *Under conditions of Theorem 4, if $f\in H_p^{(w)}$, $p\geq 1$ and $\frac{w(t)}{tv(t)}$ is nonincreasing then*
*(i)* $\|D_a^bN_n(f;.)-f(.)\|_p^{(v)}=O(1)\frac{1}{(a_n+1)^2}\left(\frac{p_0+|p_{b_n-a_n-1}-p_0|}{p_{b_n-a_n-1}}\right)\int_{\frac{\pi}{(a_n+1)}}^{\pi}\frac{w(t)}{t^3v(t)}dt$
*and furthermore*
*(ii)* $\|D_a^bN_n(f;.)-f(.)\|_p^{(v)}=O(1)(\frac{w(\pi/(a_n+1))}{v(\pi/(a_n+1))})\left(\frac{p_0+|p_{b_n-a_n-1}-p_0|}{p_{b_n-a_n-1}}\right)$.

*Remark* 1. If we take $p_k=1$ in Theorem 3, Theorem 4, Corollary 1 and Corollary 2, then all of results coincide with Theorem 2.

## 4. Conclusion

Several studies have been carried out on the degree of approach in the $H_p^{(w)}$ spaces. The means used in these studies include the results obtained by approaching to functions in this space using the classically known Cesàro, Woronoi-Nörlund and Riesz means. These results depend either directly related to the degree of polynomials, or to the upper and lower limits of the method such as deferred Cesàro means. In this study we have shown that how the speed of the approach is being affected, based on the sequences that determines the Woronoi-Nörlund and Riesz means by considering the deferred type of the Woronoi-Nörlund and Riesz mean introduced in [9]. It is evident that the results generalized previous work on these spaces.

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# SOME NEW PROPERTIES OF UNIVARIATE AND MULTIVARIATE GOTTLIEB POLYNOMIALS

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*Abstract.* The present study deals with some new properties for the Gottlieb polynomials in one and several variables. The results obtained here include various families of multilinear and multilateral generating functions, miscellaneous properties and also some special cases for these polynomials.

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*Keywords:* Gottlieb polynomials, generating function, Pochhammer symbol, recurrence relation

## 1. Introduction

A theoretical connection with the unification of generating functions has great importance in the study of special functions. Steps forward in this directions has been made by some researchers [1, 2, 7, 8].

The (univariate) Gottlieb polynomials were introduced and investigated in 1938, and then have been cited in several articles. These polynomials are defined by [10]

$$\begin{aligned}\varphi_n(x;\lambda) &= e^{-n\lambda}\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}\left(1-e^{\lambda}\right)^k \\ &= e^{-n\lambda}\,{}_2F_1\left(-n,-x;1;1-e^{\lambda}\right),\end{aligned} \tag{1.1}$$

where ${}_2F_1$ denotes Gauss's hypergeometric series whose natural generalization of an arbitrary number of $p$ numerator and $q$ denominator parameters ($p,q \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$) is called and denoted by the generalized hypergeometric series ${}_pF_q$ defined by

$$\begin{aligned}{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta q;\end{matrix}\; z\right] &= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_p)_n}{(\beta_1)_n\ldots(\beta_q)_n}\frac{z^n}{n!} \\ &= {}_pF_q\left(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta q;z\right).\end{aligned}$$

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Here $(\lambda)_\nu$ denotes the Pochhammer symbol defined (in terms of gamma function) by

$$(\lambda)_\nu = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-)$$
$$= \begin{cases} 1, & \text{if } \nu = 0;\ \lambda \in \mathbb{C}\setminus\{0\} \\ \lambda(\lambda+1)...(\lambda+n-1), & \text{if } \nu = n \in \mathbb{N};\ \lambda \in \mathbb{C} \end{cases}$$

and $\mathbb{Z}_0^-$ denotes the set of nonpositive integers and $\Gamma(\lambda)$ is the familiar Gamma function. Furthermore, the hypergeometric function ${}_2F_1$ has the following integral representation [12]:

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt, \tag{1.2}$$

where Re $(c) >$ Re $(b) > 0$, and $|arg(1-z)| < \pi$.

Gottlieb [10] presented following generating functions for his polynomials $\varphi_n(x;\lambda)$ (see also [12, p. 303]), which is denoted by $l_n(x)$ in [10],

$$\sum_{n=0}^{\infty} \varphi_n(x;\lambda)t^n = (1-t)^x\left(1-te^{-\lambda}\right)^{-x-1} \qquad (|t|<1)\ ,$$

$$\sum_{n=0}^{\infty} \frac{(\mu)_n}{n!}\varphi_n(x;\lambda)t^n = \left(1-te^{-\lambda}\right)^{-\mu}\ {}_2F_1\left[\begin{matrix}\mu,-x; \\ 1;\end{matrix}\ \frac{\left(1-e^{-\lambda}\right)t}{1-te^{-\lambda}}\right],$$

$$\sum_{n=0}^{\infty} \varphi_n(x;\lambda)\frac{t^n}{n!} = e^t\ {}_1F_1\left[x+1;1;-\left(1-e^{-\lambda}\right)t\right].$$

In addition to this, we can easily see the following relation for the Gottlieb polynomials [14, p. 449, 20 (i)]:

$$\sum_{n=0}^{\infty} \binom{k+n}{n}\varphi_{k+n}(x;\lambda)t^n = (1-t)^{x-k}\left(1-te^{-\lambda}\right)^{-x-1}\varphi_k\left(x;\ln\left(\frac{e^\lambda-t}{1-t}\right)\right). \tag{1.3}$$

Recently, Khan and Akhlaq [11] introduced and investigated Gottlieb polynomials in two and three variables to give their generating functions. Choi [5], by modifying Khan and Akhlaq's method, showed how to generalize the Gottlieb polynomials in several variables to present two generating functions of the generalized Gottlieb polynomials. Furthermore, he derives $q$-extension of a generalization of Gottlieb polynomials in three variables and gives some formulas deducible from a generalization of these polynomials in several variables (see [4]). Choi introduced a several variable analogue of the Gottlieb polynomials as follows.

**Definition 1.** An extension of the Gottlieb polynomials $\varphi_n(x;\lambda)$ in $m$ variables is defined by [5]

$$\varphi_n^m(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)=\exp(-n\sigma_m)\sum_{r_1=0}^{n}\sum_{r_2=0}^{n-r_1}\sum_{r_3=0}^{n-r_1-r_2}\cdots\sum_{r_m=0}^{n-r_1-r_2-\ldots-r_{m-1}}$$
$$\times\frac{(-n)_{\delta_m}\prod_{j=1}^{m}(-x_j)_{r_j}\prod_{j=1}^{m}\left(1-e^{\lambda_j}\right)^{r_j}}{\prod_{j=1}^{m}r_j!\delta_m!},\tag{1.4}$$

where $n,m\in\mathbb{N}$ and, for convenience,

$$\sigma_m:=\sum_{j=1}^{m}\lambda_j\qquad\text{and}\qquad\delta_m:=\sum_{j=1}^{m}r_j.\tag{1.5}$$

It is noted that the special case $m=1$ of (1.4) reduces immediately to the Gottlieb polynomials in (1.1). Multivariable Gottlieb polynomials defined by (1.4) have the following two generating functions [5]:

$$\sum_{n=0}^{\infty}\varphi_n^m(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)t^n$$
$$=\left(1-te^{-\sigma_m}\right)^{-\left(\sum_{j=1}^{m}x_j\right)-1}\prod_{j=1}^{m}\left(1-te^{\lambda_j-\sigma_m}\right)^{x_j},\tag{1.6}$$

where $m\in\mathbb{N}$ and $\sigma_m$ is given in (1.5).

$$\sum_{n=0}^{\infty}(\mu)_n\varphi_n^m(x_1,\ldots,x_m;\lambda_1,\ldots,\lambda_m)\frac{t^n}{n!}$$
$$=\left(1-te^{-\sigma_m}\right)^{-\mu}F_D^{(m)}\left[\mu,-x_1,\ldots,-x_m;1;\right.$$
$$\left.\frac{t\left(e^{\lambda_1}-1\right)e^{-\sigma_m}}{1-te^{-\sigma_m}},\ldots,\frac{t\left(e^{\lambda_m}-1\right)e^{-\sigma_m}}{1-te^{-\sigma_m}}\right],$$

where $F_D^{(m)}[.]$ denotes one of the Lauricella series in $m$ variables [13, p.33, Eq.(4)] defined by

$$F_D^{(m)}[a,b_1,\ldots,b_m;c;x_1,\ldots,x_m]=\sum_{r_1,r_2,\ldots,r_m=0}^{\infty}\frac{(a)_{\delta_m}(b_1)_{r_1}\ldots(b_m)_{r_m}}{(c)_{\delta_m}}\frac{x_1^{r_1}}{r_1!}\cdots\frac{x_m^{r_m}}{r_m!}$$
$$(\max\{|x_1|,\ldots,|x_m|\}<1)$$

and $\sigma_m$ , $\delta_m$ are given in (1.5).

The main object of this paper is to study different properties of Gottlieb polynomials in one and several variables. Various families of multilinear and multilateral generating functions are derived for these polynomials. Other miscellaneous properties of these multivariable polynomials are also discussed. Some special cases of the results presented in this study are also indicated.

## 2. Generating functions

In this section, firstly, we prove a theorem which gives bilateral generating functions relations for the Gottlieb polynomials defined by (1.1).

**Theorem 1.** *Let*

$$N_{n,m,q}^{\mu,p}(y_1,\dots,y_s;z) := \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_k \Omega_{\mu+pk}(y_1,\dots,y_s) z^k. \tag{2.1}$$

*If*

$$\Lambda_{m,q}[x;\lambda;y_1,\dots,y_s;t] = \sum_{n=0}^{\infty} a_n \varphi_{m+qn}(x;\lambda)\, \Omega_{\mu+pn}(y_1,\dots,y_s) t^n,$$

*then, for every nonnegative integer m, we have*

$$\sum_{n=0}^{\infty} \varphi_{m+n}(x;\lambda)\, N_{n,m,q}^{\mu,p}(y_1,\dots,y_s;z)\, t^n \tag{2.2}$$
$$= (1-t)^{x-m}\left(1-te^{-\lambda}\right)^{-x-1} \Lambda_{m,q}\left[x;\ln\left(\frac{e^{\lambda}-t}{1-t}\right);y_1,\dots,y_s;z\left(\frac{t}{1-t}\right)^q\right].$$

*Proof.* If we denote the left-hand side of (2.2) by $T$ and use (2.1), then we obtain

$$T = \sum_{n=0}^{\infty} \varphi_{m+n}(x;\lambda) \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_k \Omega_{\mu+pk}(y_1,\dots,y_s) z^k t^n.$$

Replacing $n$ by $n+qk$ and then using relation (1.3) we may write

$$\begin{aligned} T &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \varphi_{m+n+qk}(x;\lambda) \binom{m+n+qk}{n} a_k \Omega_{\mu+pk}(y_1,\dots,y_s) z^k t^{n+qk} \\ &= \sum_{k=0}^{\infty} (1-t)^{x-m-qk}\left(1-te^{-\lambda}\right)^{-x-1} \varphi_{m+qk}\left(x;\ln\left(\frac{e^{\lambda}-t}{1-t}\right)\right) a_k \\ &\quad \times \Omega_{\mu+pk}(y_1,\dots,y_s)\left(zt^q\right)^k \\ &= (1-t)^{x-m}\left(1-te^{-\lambda}\right)^{-x-1} \Lambda_{m,q}\left[x;\ln\left(\frac{e^{\lambda}-t}{1-t}\right);y_1,\dots,y_s;z\left(\frac{t}{1-t}\right)^q\right], \end{aligned}$$

which completes the proof. □

Now we give a theorem which is about several families of bilinear and bilateral generating functions for the multivariable Gottlieb polynomials $\varphi_n^m(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m)$.

**Theorem 2.** *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,\dots,y_s)$ of $s$ complex variables $y_1,\dots,y_s$ $(s\in\mathbb{N})$ and of complex order $\mu$, let*

$$\Lambda_{\mu,\nu}(y_1,\dots,y_s;z):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(y_1,\dots,y_s)z^k,$$

*where $a_k\neq 0$, $\mu,\nu\in\mathbb{C}$ and*

$$\begin{aligned}&\Theta_{n,m,p}^{\mu,\nu}(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m;y_1,\dots,y_s;\tau)\\ &:=\sum_{k=0}^{[n/p]}a_k\varphi_{n-pk}^m(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m)\,\Omega_{\mu+\nu k}(y_1,\dots,y_s)\tau^k\end{aligned}\tag{2.3}$$

$(n,p\in\mathbb{N})$. *Then we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}\Theta_{n,m,p}^{\mu,\nu}\left(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m;y_1,\dots,y_s;\frac{\eta}{t^p}\right)t^n\\ &=\left(1-te^{-\sigma_m}\right)^{-\left(\sum_{j=1}^{m}x_j\right)-1}\prod_{j=1}^{m}\left(1-te^{\lambda_j-\sigma_m}\right)^{x_j}\Lambda_{\mu,\nu}(y_1,\dots,y_s;\eta)\end{aligned}\tag{2.4}$$

*provided that each member of* (2.4) *exists.*

*Proof.* If we denote the left-hand side of (2.4) by $S$ and use (2.3), then we obtain

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k\varphi_{n-pk}^m(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m)\,\Omega_{\mu+\nu k}(y_1,\dots,y_s)\eta^k t^{n-pk}.$$

Replacing $n$ by $n+pk$ we may write from the generating function (1.6) that

$$\begin{aligned}S&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\,\varphi_n^m(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m)\,\Omega_{\mu+\nu k}(y_1,\dots,y_s)\eta^k t^n\\ &=\sum_{n=0}^{\infty}\varphi_n^m(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m)\,t^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(y_1,\dots,y_s)\eta^k\\ &=\left(1-te^{-\sigma_m}\right)^{-\left(\sum_{j=1}^{m}x_j\right)-1}\prod_{j=1}^{m}\left(1-te^{\lambda_j-\sigma_m}\right)^{x_j}\Lambda_{\mu,\nu}(y_1,\dots,y_s;\eta),\end{aligned}$$

which completes the proof. □

It is possible to give many applications of our theorems with help of appropriate choices of the multivariable functions $\Omega_{\mu+\psi k}(y_1,\dots,y_s)$, $k\in\mathbb{N}_0$, $s\in\mathbb{N}$.

For example, set $s=m$ and take $\Omega_{\mu+\nu k}(y_1,\dots,y_m)=h_{\mu+\nu k}^{(\beta_1,\dots,\beta_m)}(y_1,\dots,y_m)$ in Theorem 2. Recall that, by $h_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$, we denote the multivariable Lagrange-Hermite polynomials (see, e.g., [2]) generated by

$$\prod_{j=1}^{r}\left\{(1-x_jt^j)^{-\alpha_j}\right\}=\sum_{n=0}^{\infty}h_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)t^n, \tag{2.5}$$

where $|t|<\min\left\{|x_1|^{-1},|x_2|^{-1/2},\dots,|x_r|^{-1/r}\right\}$. We also know that

$$h_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)=\sum_{k_1+2k_2+\dots+rk_r=n}(\alpha_1)_{k_1}\dots(\alpha_r)_{k_r}\frac{x_1^{k_1}}{k_1!}\dots\frac{x_r^{k_r}}{k_r!}.$$

Then, from Theorem 2, we obtain the following result which is a class of bilateral generating functions for the multivariable Lagrange-Hermite polynomials and the multivariable Gottlieb polynomials.

**Corollary 1.** *If*

$$\Lambda_{\mu,\nu}(y_1,\dots,y_m;z):=\sum_{k=0}^{\infty}a_kh_{\mu+\nu k}^{(\beta_1,\dots,\beta_m)}(y_1,\dots,y_m)z^k,\quad (a_k\neq 0,\ \mu,\nu\in\mathbb{C}),$$

*and*

$$\begin{aligned}&\Theta_{n,m,p}^{\mu,\nu}(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m;y_1,\dots,y_m;\tau)\\&:=\sum_{k=0}^{[n/p]}a_k\varphi_{n-pk}^m(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m)h_{\mu+\nu k}^{(\beta_1,\dots,\beta_m)}(y_1,\dots,y_m)\tau^k,\end{aligned}$$

*where $n,p\in\mathbb{N}$, then we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}\Theta_{n,m,p}^{\mu,\nu}\left(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m;y_1,\dots,y_m;\frac{\eta}{t^p}\right)t^n \\&=\left(1-te^{-\sigma_m}\right)^{-\left(\sum_{j=1}^{m}x_j\right)-1}\prod_{j=1}^{m}\left(1-te^{\lambda_j-\sigma_m}\right)^{x_j}\Lambda_{\mu,\nu}(y_1,\dots,y_m;\eta)\end{aligned} \tag{2.6}$$

*provided that each member of* (2.6) *exists.*

*Remark* 1. Using the generating relation (2.5) for the multivariable Lagrange-Hermite polynomials and taking $a_k=1$, $\mu=0$, $\nu=1$, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}\varphi_{n-pk}^m(x_1,\dots,x_m;\lambda_1,\dots,\lambda_m)h_k^{(\beta_1,\dots,\beta_m)}(y_1,\dots,y_m)\eta^kt^{n-pk}$$

$$=\left(1-te^{-\sigma_m}\right)^{-\left(\sum_{j=1}^{m} x_j\right)-1}\prod_{j=1}^{m}\left(1-te^{\lambda_j-\sigma_m}\right)^{x_j}(1-y_j\eta^j)^{-\beta_j},$$

where $\sigma_m := \sum_{j=1}^{m}\lambda_j$ and $|\eta| < \min\left\{|y_1|^{-1}, |y_2|^{-1/2}, \ldots, |y_m|^{-1/m}\right\}$.

Also, choosing $s = m$ and $\Omega_{\mu+\nu k}(y_1, \ldots, y_m) = \varphi^m_{\mu+\nu k}(y_1, \ldots, y_m; \lambda_1, \ldots, \lambda_m)$, $\mu, \nu \in \mathbb{N}_0$, in Theorem 2, we obtain the following class of bilinear generating functions for the multivariable Gottlieb polynomials defined by (1.4).

**Corollary 2.** *If*

$$\Lambda_{\mu,\nu}(y_1, \ldots, y_m; z) := \sum_{k=0}^{\infty} a_k \varphi^m_{\mu+\nu k}(y_1, \ldots, y_m; \lambda_1, \ldots, \lambda_m) z^k, \quad (a_k \neq 0, \mu, \nu \in \mathbb{C}),$$

*and*

$$\begin{aligned}&\Theta^{\mu,\nu}_{n,m,p}(x_1, \ldots, x_m; \lambda_1, \ldots, \lambda_m; y_1, \ldots, y_m; \tau)\\ &:= \sum_{k=0}^{[n/p]} a_k \varphi^m_{n-pk}(x_1, \ldots, x_m; \lambda_1, \ldots, \lambda_m)\varphi^m_{\mu+\nu k}(y_1, \ldots, y_m; \lambda_1, \ldots, \lambda_m)\tau^k,\end{aligned}$$

*where $n, p \in \mathbb{N}$, then we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}\Theta^{\mu,\nu}_{n,m,p}\left(x_1, \ldots, x_m; \lambda_1, \ldots, \lambda_m; y_1, \ldots, y_m; \frac{\eta}{t^p}\right)t^n \qquad (2.7)\\ &=\left(1-te^{-\sigma_m}\right)^{-\left(\sum_{j=1}^{m} x_j\right)-1}\prod_{j=1}^{m}\left(1-te^{\lambda_j-\sigma_m}\right)^{x_j}\Lambda_{\mu,\nu}(y_1, \ldots, y_m; \eta)\end{aligned}$$

*provided that each member of* (2.7) *exists.*

*Remark* 2. Using the generating relation (1.6) for the multivariable Gottlieb polynomials and taking $a_k = 1$, $\mu = 0$, $\nu = 1$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}\varphi^m_{n-pk}(x_1, \ldots, x_m; \lambda_1, \ldots, \lambda_m)\varphi^m_k(y_1, \ldots, y_m; \lambda_1, \ldots, \lambda_m)\eta^k t^{n-pk}\\ &=\left(1-te^{-\sigma_m}\right)^{-\left(\sum_{j=1}^{m}(x_j+y_j)\right)-2}\prod_{j=1}^{m}\left(1-te^{\lambda_j-\sigma_m}\right)^{x_j}\left(1-\eta e^{\lambda_j-\sigma_m}\right)^{y_j},\end{aligned}$$

where $\sigma_m := \sum_{j=1}^{m}\lambda_j$.

Furthermore, for every suitable choice of the coefficients $a_k$ $(k \in \mathbb{N}_0)$, if the multivariable function $\Omega_{\mu+\psi k}(y_1,...,y_s)$ $(s \in \mathbb{N})$ is expressed as an appropriate product of several simpler functions, the assertions of Theorems 1 and 2 can be applied in order to derive various families of multilinear and multilateral generating functions for the univariate and multivariate Gottlieb polynomials.

## 3. Miscellaneous properties

In this section, firstly, we obtain a recurrence relation for the multivariable Gottlieb polynomials. Then, using the relation between Gottlieb and Jacobi polynomials, we derive some properties for the Gottlieb polynomials.

**Theorem 3.** *Multivariable Gottlieb polynomials have the following recurrence relation:*

$$(n+1)\varphi_{n+1}^{m} + e^{\lambda_i - 2\sigma_m} n \varphi_{n-1}^{m} = e^{-\sigma_m}\left(1+n+ne^{\lambda_i}+\sum_{j=1}^{m} x_j\left(1-e^{\lambda_j}\right)\right)\varphi_n^m,$$
$$(\,i=1,...,m\ ;\ n\geq 2)\,.$$

*Proof.* By differentiating each member of the generating function relation (1.6) with respect to t and making some necessary adjustments and then identifying the corresponding coefficients of $t^n$, we arrive at the desired recurrence relation. □

It is known that

$$\varphi_n(x;\lambda) = P_n^{(0,x-n)}\left(2e^{-\lambda}-1\right), \tag{3.1}$$

where the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ are defined by [12]

$$P_n^{(\alpha,\beta)}(z) = \sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}\left(\frac{x+1}{2}\right)^k\left(\frac{x-1}{2}\right)^{n-k}.$$

Equation (3.1) can be used to deduce numerous properties and characteristics of the Gottlieb polynomials from those of the Jacobi polynomials.

In 1966, Izuru Fujiwara studied extended Jacobi polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ which are defined by the Rodrigues formula [9]

$$F_n^{(\alpha,\beta)}(x;a,b,c) = \frac{(-c)^n}{n!}(x-a)^{-\alpha}(b-x)^{-\beta} \times \frac{d^n}{dx^n}\left\{(x-a)^{n+\alpha}(b-x)^{n+\beta}\right\} \qquad (c>0)\,.$$

Szegö [15] showed that $F_n^{(\alpha,\beta)}(x;a,b,c)$ polynomials are a constant multiple of the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ in the form

$$F_n^{(\alpha,\beta)}(x;a,b,c) = \{c(a-b)\}^n P_n^{(\alpha,\beta)}\left(\frac{2(x-a)}{a-b}+1\right)$$

or, equivalently,

$$P_n^{(\alpha,\beta)}(x) = \{c(a-b)\}^{-n} F_n^{(\alpha,\beta)}\left(\frac{1}{2}\{a+b+(a-b)x\};a,b,c\right). \tag{3.2}$$

In addition to this, we know that the following equality [3]:

$$g_n^{(\alpha,\beta)}(x,y) = (y-x)^n P_n^{(-\alpha-n,-\beta-n)}\left(\frac{x+y}{x-y}\right), \tag{3.3}$$

where $g_n^{(\alpha,\beta)}(x,y)$ are known as the Lagrange polynomials generated by [6]

$$(1-xt)^{-\alpha}(1-yt)^{-\beta} = \sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n$$
$$\left(|t| < \min\left\{|x|^{-1},|y|^{-1}\right\}\right).$$

**Theorem 4.** *For the Gottlieb polynomials $\varphi_n(x;\lambda)$, we have*
$(i)$ $\varphi_n(x;\lambda) = \{c(a-b)\}^{-n} F_n^{(0,x-n)}\left(\frac{1}{2}\left\{a+b+(a-b)\left(2e^{-\lambda}-1\right)\right\};a,b,c\right)$
$(ii)$ $\varphi_n\left(x;\ln\frac{a-b}{a}\right) = (b-a)^{-n} g_n^{(-n,-x)}(a,b)$.

*Proof.* $(i)$ Using relations (3.1) and (3.2), we arrived the result.
$(ii)$ Using relations (3.1) and (3.3), we get the desired result. □

**Theorem 5.** *Gottlieb polynomials have the following integral representation:*

$$\varphi_n(x;\lambda) = -\frac{\sin\pi x}{\pi e^{n\lambda}}\int_0^1 t^{-x-1}(1-t)^x\left(1-\left(1-e^{\lambda}\right)t\right)^n dt,$$

*where* $-1 < x < 0$.

*Proof.* If we use definition of Gottlieb polynomials given by (1.1) in (1.2) and make some necessary calculation, we easily arrived this result. □

Finally, using the fact that (see [14])

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{\Gamma(\alpha+\beta+n+1)}\int_0^{\infty} t^{\alpha+\beta+n}e^{-t}L_n^{(\alpha)}\left(\frac{(1-x)t}{2}\right)dt,$$

where $L_n^{(\alpha)}(x)$ are the Laguerre polynomials given by the Rodrigues formula [12]

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+\alpha}\right),$$

and also considering (3.1), we immediately obtain the following result.

**Theorem 6.** *Gottlieb polynomials have the following integral representation:*

$$\varphi_n(x;\lambda) = \frac{1}{x\Gamma(x)} \int_0^\infty t^x e^{-t} L_n\left(\left(1-e^{-\lambda}\right)t\right) dt.$$

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# ON THE SOLUTIONS OF A MULTI-POINT BVP FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH IMPULSES

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*Abstract.* The existence and uniqueness of solutions for a multi-point boundary value problem (BVP) of impulsive fractional differential equations are investigated by means of Schaefer's fixed point theorem and Banach contraction mapping principle. Examples that support the theoretical results are presented.

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*Keywords:* Caputo fractional derivative, impulsive differential equations, fractional differential equations, boundary value problems, Banach contraction mapping principle, Schaefer's fixed point theorem

## 1. Introduction

Even though it is a 300-year-old topic, fractional calculus has come into prominence in the course of time, and nowadays it has an important role in the modeling of real world phenomena based on fractional derivatives [15]. Models arising from fractional order derivatives are excellent instruments for the description of memory and hereditary properties of various materials and processes [15]. Impulsive differential equations, on the other hand, are capable of describing the dynamics of processes in which abrupt changes occur [2, 17]. For instance, according to the switching phenomenon, frequency changes or other sudden noises, the states of the electronic devices are often subject to instantaneous perturbations and experience abrupt changes at certain instants [2, 11], i.e., they exhibit impulsive effects. Therefore, mathematical models with impulsive effects are more accurate to describe the evolutionary processes of the systems. Fractional order and impulsive differential equations have many applications in various fields such as mechanics, electronics, biology, economics, chaos theory, and neuroscience [1, 3–5, 8, 12–14, 19–21].

Various types of impulsive fractional differential equations were considered in the papers [22–27]. Banach, Krasnoselskii's, and Leray-Schauder's fixed point theorems were utilized to show the existence and uniqueness of solutions for a class of fractional order differential switched systems with coupled nonlocal initial and impulsive

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conditions in [24]. The papers [6, 27] were concerned with significant results on formulas for solutions of impulsive fractional differential equations. In [27], a boundary value problem (BVP) of order between 1 and 2 is studied. Fixed point theorems were also used in [6] and [27] to show the existence and uniqueness of solutions. On the other hand, the theory of impulsive fractional differential inclusions was improved in [25] by taking into account the existence of PC-mild solutions of such systems with nonlocal conditions when the linear part is a fractional sectorial operator defined on a separable Banach space. Moreover, existence results based on the Banach and Schauder's fixed point theorems were revealed by Wang and Zhang [26] for fractional impulsive systems with Hadamard derivatives. In the present study, we make use of the Banach and Schaefer's fixed point theorems to deal with the existence and uniqueness of solutions of a BVP of fractional differential equations with impulses. In contrast with [27], in this paper multi-points in the boundary conditions are used and the order of the system is between 2 and 3.

In [10], Liu et al. considered the following nonlinear fractional impulsive differential equation

$$\begin{cases} D_{0^+}^{\beta}\phi_p\left(D_{0^+}^{\alpha}u(t)\right) = f(t,u(t)),\ t \in J', \\ \Delta u(t_k) = I_k(u(t_k)),\ \Delta\phi_p\left(D_{0^+}^{\alpha}u(t_k)\right) = b_k,\ k = 1,2,...,m, \\ u(0) = u_0,\ D_{0^+}^{\alpha}u(0) = u_1, \end{cases}$$

where $\alpha, \beta \in (0,1]$ with $\alpha + \beta > 1$. The existence and uniqueness of solutions were obtained by means of the Schauder's fixed point theorem and Banach contraction mapping principle. Additionally, in [6], the authors considered the following fractional impulsive differential equation

$$\begin{cases} {}^{C}D_t^{q}u(t) = f(t,u(t)),\ t \in J' := J \setminus \{t_1,\dots,t_m\},\ J = [0,1], \\ u(t_k^+) = u(t_k^-) + y_k,\ k = 1,2,\dots,m, \\ u(0) = u_0,\ y_k \in \mathbb{R}, \end{cases}$$

where ${}^{C}D_t^{q}$ is the Caputo fractional derivative of order $q \in (0,1)$, $u_0 \in \mathbb{R}$, and $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Fixed point methods were utilized to obtain some sufficient conditions for the existence of solutions. Moreover, Rehman and Eloe [16] presented a general method for converting an impulsive fractional differential equation to an equivalent integral equation. Using this method, they investigated the existence of solutions for the following four-point impulsive fractional BVP

$$\begin{cases} {}^{C}D^{\alpha}y(x) = f(x,y),\ x \in J, \\ \Delta y(x_k) = I_k(u(t_k)),\ \Delta y'(x_k) = \bar{I}_k(y(x_k)),\ k = 1,2,\dots,m, \\ y(0) = \gamma y(\xi),\ y(1) = \beta y(\eta), \end{cases}$$

where $1 < \alpha < 2$, $\beta$, $\gamma$, $\xi$, $\eta \in \mathbb{R}$, $\xi$, $\eta \in (0,1)$, $J = [0,1]$, and $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Motivated by the above mentioned studies, in this paper, we consider the following $p$-point impulsive fractional BVP

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}u(t) = f(t,u(t)),\ t \in J', \\ \Delta u(t_k) = I_k(u(t_k)),\ \Delta u'(t_k) = \bar{I}_k(u(t_k)), \\ \Delta u''(t_k) = \tilde{I}_k(u(t_k)),\ k = 1,2,\dots,m, \\ au(0) + bu'(0) = \displaystyle\sum_{i=1}^{p-2} c_i u(\xi_i), \\ cu(1) + du'(1) = \displaystyle\sum_{i=1}^{p-2} d_i u(\xi_i),\ u''(0) = 0, \end{cases} \tag{1.1}$$

where ${}^{C}D^{\alpha}_{0^+}$ is the Caputo fractional derivative of order $\alpha \in (2,3)$, $a$, $b$, $c$, and $d$ are real constants, the functions $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, $I_k : \mathbb{R} \to \mathbb{R}$, $\bar{I}_k : \mathbb{R} \to \mathbb{R}$, and $\tilde{I}_k : \mathbb{R} \to \mathbb{R}$ are continuous in all their arguments, $0 < \xi_1 < \xi_2 < \cdots < \xi_{p-2} < 1$, $c_i$, $d_i \in \mathbb{R}$, $i = 1,\dots,p-2$, with $\delta := ac + ad - bc - \sum_{i=1}^{p-2} d_i\xi_i \left(a - \sum_{i=1}^{p-2} c_i\right) + \sum_{i=1}^{p-2} d_i \left(b - \sum_{i=1}^{p-2} c_i\xi_i\right) + c\sum_{i=1}^{p-2} c_i\xi_i \neq 0$, $J = [0,1]$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = 1$, $J' = J \setminus \{t_1, t_2, \dots, t_m\}$, $\Delta u(t_k) = u(t_k^+) - u(t_k)$, and $u(t_k^+) = \lim_{t \to t_k^+} u(t)$ for each $k = 1,2,\dots,m$. The impulse actions $\Delta u'(t_k)$ and $\Delta u''(t_k)$ are defined in a similar way to $\Delta u(t_k)$.

The rest of this paper is organized as follows. In Section 2, we provide some necessary definitions and preliminary lemmas, which are key tools for our main results. In Section 3, we make use of the Schaefer's fixed point theorem and Banach contraction mapping principle to prove the existence and uniqueness of solutions of (1.1). Examples are provided in Section 4 to demonstrate the applications of the main results.

## 2. Preliminaries

The following definitions are needed in the remaining parts of the paper.

**Definition 1** ([9])**.** The Riemann-Liouville fractional integral of order $\alpha$ for a function $g : [0,\infty) \to \mathbb{R}$ is defined as

$$I^{\alpha}_{0^+}g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds,\ t > 0,\ \alpha > 0,$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma$ is the gamma function.

**Definition 2** ([9]). The Riemann-Liouville derivative of order $\alpha$ for a function $g : [0,\infty) \to \mathbb{R}$ can be written as

$$^{L}D^{\alpha}_{0+}g(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{g(s)}{(t-s)^{\alpha+1-n}}ds,\ t>0,$$

where $n = [\alpha]+1$ and $[\alpha]$ is the integer part of $\alpha$.

**Definition 3** ([9]). The Caputo fractional derivative of order $\alpha > 0$ for a continuous function $g : [0,\infty) \to \mathbb{R}$ can be written as

$$^{C}D^{\alpha}_{0+}g(t) = {}^{L}D^{\alpha}_{0+}\left[g(t) - \sum_{i=0}^{n-1}\frac{g^{(i)}(0)}{i!}t^i\right],\ t>0,$$

where $n = [\alpha]+1$ and $[\alpha]$ is the integer part of $\alpha$.

One can confirm that if $g : [0,\infty) \to \mathbb{R}$ is an $n$ times continuously differentiable function, then

$$^{C}D^{\alpha}_{0+}g(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{g^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds = I^{n-\alpha}_{0+}g^{(n)}(t),$$

for $t>0$ and $n-1<\alpha<n$. Moreover, the Caputo derivative of a constant is equal to zero.

We say that a real valued function $u(t)$ defined on $J$ belongs to the space $PC(J,\mathbb{R})$ if it is left-continuous and continuous except, possibly, at the points $t_k$, $k = 1,2,\dots,m$, where it has discontinuities of the first kind. Besides, for a positive integer $\sigma$, we will denote by $PC^{\sigma}(J,\mathbb{R})$ the set of real valued functions whose derivative of order $i$ belongs to the set $PC(J,\mathbb{R})$ for each $i = 0,1,\dots,\sigma$. It is worth noting that the space $PC(J,\mathbb{R})$ is a Banach space endowed with the norm $\|u\| = \sup_{t\in J}|u(t)|$. In the sequel, we will denote $J_0 = [0,t_1]$, $J_1 = (t_1,t_2]$, $\dots$, $J_{m-1} = (t_{m-1},t_m]$, and $J_m = (t_m,1]$. Moreover, let $AC^{\sigma}(J,\mathbb{R})$ be the set of functions $u(t)$ such that $u^{(\sigma-1)} \in PC(J,\mathbb{R})$ and the restriction of $u^{(\sigma-1)}$ on $J_k$, $k = 0,1,\dots,m$, is absolutely continuous.

**Lemma 1** ([16, Corollary 2.5]). *Assume $f : J\times\mathbb{R}\to\mathbb{R}$ is continuous and $n-1<\alpha<n$. Then $u \in \mathcal{AC}^n(J,\mathbb{R})$ satisfies*

$$^{C}D^{\alpha}_{0+}u(t) = f(t,u(t)),\ t\in J,$$

*if and only if $u \in PC^{n-1}(J,\mathbb{R})$ and*

$$u(t) = \sum_{i=0}^{n-1}\frac{u^{(i)}(0)}{i!}t^i + \sum_{i=0}^{n-1}\left(\sum_{0<t_k<t}\frac{1}{i!}\Delta u^{(i)}(t_k)(t-t_k)^i\right) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))ds$$

*for $t\in J$.*

**Theorem 1** ([18, Schaefer's Fixed Point Theorem]). *Let $\mathcal{B}$ be a Banach space and $T : \mathcal{B} \to \mathcal{B}$ be a completely continuous operator. If the set*

$$\mathcal{D}(\mathcal{B}) = \{u \in \mathcal{B} : u = \lambda T u \ \text{ for some } \lambda \in [0,1]\}$$

*is bounded, then $T$ has at least one fixed point.*

**Lemma 2** ([7, $PC$-type Ascoli-Arzela Theorem]). *Let $\Omega \subset PC(J,\mathbb{R})$. Suppose the following conditions are satisfied:*

(i) *$\Omega$ is uniformly bounded;*
(ii) *$\Omega$ is equicontinuous in $J_k$, $k = 0,1,\dots,m$.*

*Then $\Omega$ is a relatively compact subset of $PC(J,\mathbb{R})$.*

## 3. Main results

This section is concerned with the existence as well as uniqueness of solutions of the problem (1.1).

In the following lemma, we provide an equivalent integral equation to problem (1.1).

**Lemma 3.** *A function $u(t)$ is a solution of fractional impulsive BVP (1.1) if and only if it is a solution of the following equation*

$$u(t) = \begin{cases} \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds + \frac{1}{\delta}\sum_{i=1}^{p-2} c_i F_u(\xi_i) \\ \displaystyle\times\left[\left(c+d-\sum_{i=1}^{p-2} d_i\xi_i\right) - \left(c-\sum_{i=1}^{p-2} d_i\right)t\right] \\ \displaystyle+\frac{1}{\delta}\left(\sum_{i=1}^{p-2} d_i F_u(\xi_i) - \mathcal{A}_u\right) \\ \displaystyle\times\left(-b+\sum_{i=1}^{p-2} c_i\xi_i + \left(a-\sum_{i=1}^{p-2} c_i\right)t\right),\ t \in J_0, \\ \displaystyle F_u(t) + \frac{1}{\delta}\sum_{i=1}^{p-2} c_i F_u(\xi_i)\left[\left(c+d-\sum_{i=1}^{p-2} d_i\xi_i\right) - \left(c-\sum_{i=1}^{p-2} d_i\right)t\right] \\ \displaystyle+\frac{1}{\delta}\left(\sum_{i=1}^{p-2} d_i F_u(\xi_i) - \mathcal{A}_u\right)\left(-b+\sum_{i=1}^{p-2} c_i\xi_i + \left(a-\sum_{i=1}^{p-2} c_i\right)t\right), \\ t \in J_k,\ k = 1,2,\dots,m, \end{cases} \tag{3.1}$$

*where*

$$F_u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds + \sum_{0<t_k<t} I_k(u(t_k)) + \sum_{0<t_k<t} \bar{I}_k(u(t_k))(t-t_k) + \sum_{0<t_k<t} \tilde{I}_k(u(t_k))\frac{(t-t_k)^2}{2}, \tag{3.2}$$

*and*

$$\begin{aligned}\mathcal{A}_u = &\int_0^1 \left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right) f(s,u(s))ds \\ &+ c\sum_{k=1}^{m} I_k(u(t_k)) + \sum_{k=1}^{m} \bar{I}_k(u(t_k))\,(c(1-t_k)+d) \\ &+ \frac{1}{2}\sum_{k=1}^{m} \tilde{I}_k(u(t_k))\left(c(1-t_k)^2 + 2d(1-t_k)\right).\end{aligned} \tag{3.3}$$

*Proof.* Suppose that $u(t)$ is solution of (1.1). One can confirm by means of Lemma 1 that

$$\begin{aligned}u(t) = &e_1 + e_2 t + e_3 t^2 + \sum_{0<t_k<t} I_k(u(t_k)) + \sum_{0<t_k<t} \bar{I}_k(u(t_k))(t-t_k) \\ &+ \sum_{0<t_k<t} \tilde{I}_k(u(t_k))\frac{(t-t_k)^2}{2} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds,\end{aligned} \tag{3.4}$$

$$\begin{aligned}u'(t) = &e_2 + 2e_3 t + \sum_{0<t_k<t} \bar{I}_k(u(t_k)) + \sum_{0<t_k<t} \tilde{I}_k(u(t_k))(t-t_k) \\ &+ \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds,\end{aligned} \tag{3.5}$$

and

$$u''(t) = 2e_3 + \sum_{0<t_k<t} \tilde{I}_k(u(t_k)) + \int_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)} f(s,u(s))ds. \tag{3.6}$$

Applying the boundary condition $u''(0) = 0$ in (3.6), we obtain $e_3 = 0$. Now, using the boundary conditions $au(0) + bu'(0) = \sum_{i=1}^{p-2} c_i u(\xi_i)$ and $cu(1) + du'(1) = \sum_{i=1}^{p-2} d_i u(\xi_i)$, it can be verified that

$$\left(a - \sum_{i=1}^{p-2} c_i\right)e_1 + \left(b - \sum_{i=1}^{p-2} c_i\xi_i\right)e_2 = \sum_{i=1}^{p-2} c_i F_u(\xi_i), \tag{3.7}$$

and

$$\left(c-\sum_{i=1}^{p-2} d_i\right)e_1+\left(c+d-\sum_{i=1}^{p-2} d_i\xi_i\right)e_2=\sum_{i=1}^{p-2} d_i F_u(\xi_i)-\mathcal{A}_u. \tag{3.8}$$

Solving the linear system of equations (3.7) and (3.8) for $e_1$ and $e_2$, we get

$$e_1=\frac{1}{\delta}\left[\left(c+d-\sum_{i=1}^{p-2} d_i\xi_i\right)\sum_{i=1}^{p-2} c_i F_u(\xi_i)+\left(-b+\sum_{i=1}^{p-2} c_i\xi_i\right)\left(\sum_{i=1}^{p-2} d_i F_u(\xi_i)-\mathcal{A}_u\right)\right],$$

$$e_2=\frac{1}{\delta}\left[\left(a-\sum_{i=1}^{p-2} c_i\right)\left(\sum_{i=1}^{p-2} d_i F_u(\xi_i)-\mathcal{A}_u\right)-\left(c-\sum_{i=1}^{p-2} d_i\right)\sum_{i=1}^{p-2} c_i F_u(\xi_i)\right].$$

Substituting the values of $e_1$ and $e_2$ in (3.4), we obtain (3.1). Conversely, if $u(t)$ satisfies (3.1), then it is easy to show that $u$ satisfies (1.1). This completes the proof. □

Let us define an operator $T : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ as

$$\begin{aligned} Tu(t) &= F_u(t)+\frac{1}{\delta}\sum_{i=1}^{p-2} c_i F_u(\xi_i)\left[\left(c+d-\sum_{i=1}^{p-2} d_i\xi_i\right)-\left(c-\sum_{i=1}^{p-2} d_i\right)t\right] \\ &\quad+\frac{1}{\delta}\left(\sum_{i=1}^{p-2} d_i F_u(\xi_i)-\mathcal{A}_u\right)\left[-b+\sum_{i=1}^{p-2} c_i\xi_i+\left(a-\sum_{i=1}^{p-2} c_i\right)t\right],\ t\in J, \end{aligned} \tag{3.9}$$

where $F_u(t)$ and $\mathcal{A}_u$ are defined as in (3.2) and (3.3), respectively. Clearly, the fixed points of the operator $T$ are the solutions of the problem (1.1).

For the sake of convenience, we define

$$\rho_1=\sup_{t\in[0,1]}\frac{\left|\left(c+d-\sum_{i=1}^{p-2} d_i\xi_i\right)-\left(c-\sum_{i=1}^{p-2} d_i\right)t\right|}{|\delta|},$$

$$\rho_2=\sup_{t\in[0,1]}\frac{\left|-b+\sum_{i=1}^{p-2} c_i\xi_i+\left(a-\sum_{i=1}^{p-2} c_i\right)t\right|}{|\delta|},$$

$$\kappa_1=\frac{\left|c-\sum_{i=1}^{p-2} d_i\right|}{|\delta|},\qquad \kappa_2=\frac{\left|a-\sum_{i=1}^{p-2} c_i\right|}{|\delta|}.$$

A property of the operator $T$ defined by (3.9) is given in the next lemma.

**Lemma 4.** *The operator $T : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is completely continuous.*

*Proof.* The operator $T$ is continuous in view of the continuity of the functions $f$, $I_k$, $\bar{I}_k$ and $\tilde{I}_k$. Let $\Omega$ be a bounded subset of $PC(J,\mathbb{R})$. There exist positive constants $M_i$, $i=1,2,3,4$, such that $|f(t,u)|\le M_1$, $|I_k(u)|\le M_2$, $\left|\bar{I}_k(u)\right|\le M_3$ and $\left|\tilde{I}_k(u)\right|\le M_4$ for all $u\in\Omega$.

Thus, we have for $u\in\Omega$ that

$$\begin{aligned}
|F_u(\xi_i)| &\le \int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|\,ds+\sum_{0<t_k<\xi_i}|I_k(u(t_k))|\\
&+\sum_{0<t_k<\xi_i}\left|\bar{I}_k(u(t_k))\right|(\xi_i-t_k)+\sum_{0<t_k<\xi_i}\left|\tilde{I}_k(u(t_k))\right|\frac{(\xi_i-t_k)^2}{2}\\
&\le \frac{\xi_i^{\alpha}}{\Gamma(\alpha+1)}M_1+m\left(M_2+\xi_iM_3+\frac{\xi_i^2}{2}M_4\right),
\end{aligned}$$

and

$$\begin{aligned}
|\mathcal{A}_u| &\le \int_0^1\left(\frac{|c|(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{|d|(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right)|f(s,u(s))|\,ds\\
&+|c|\sum_{k=1}^{m}|I_k(u(t_k))|+\sum_{k=1}^{m}\left|\bar{I}_k(u(t_k))\right||c(1-t_k)+d|\\
&+\frac{1}{2}\sum_{k=1}^{m}\left|\tilde{I}_k(u(t_k))\right|\left|c(1-t_k)^2+2d(1-t_k)\right|\\
&\le\left(\frac{|c|}{\Gamma(\alpha+1)}+\frac{|d|}{\Gamma(\alpha)}\right)M_1+m\left(|c|M_2+(|c|+|d|)M_3\right)\\
&+\frac{m}{2}(|c|+2|d|).
\end{aligned}$$

Therefore, one can confirm for $t\in J$ that

$$\begin{aligned}
|Tu(t)| &\le \int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|\,ds+\sum_{0<t_k<t}|I_k(u(t_k))|\\
&+\sum_{0<t_k<t}\left|\bar{I}_k(u(t_k))\right|(t-t_k)+\sum_{0<t_k<t}\left|\tilde{I}_k(u(t_k))\right|\frac{(t-t_k)^2}{2}\\
&+\frac{1}{|\delta|}\sum_{i=1}^{p-2}|c_i||F_u(\xi_i)|\left|\left(c+d-\sum_{i=1}^{p-2}d_i\xi_i\right)-\left(c-\sum_{i=1}^{p-2}d_i\right)t\right|\\
&+\frac{1}{|\delta|}\left(\sum_{i=1}^{p-2}|d_i||F_u(\xi_i)|+|\mathcal{A}_u|\right)\left|-b+\sum_{i=1}^{p-2}c_i\xi_i+\left(a-\sum_{i=1}^{p-2}c_i\right)t\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{M_1}{\Gamma(\alpha+1)} + m\left(M_2 + M_3 + \frac{M_4}{2}\right) \\
&+ \sum_{i=1}^{p-2} (\rho_1 |c_i| + \rho_2 |d_i|) \left[\frac{\xi_i^\alpha}{\Gamma(\alpha+1)} M_1 + m\left(M_2 + \xi_i M_3 + \frac{\xi_i^2}{2} M_4\right)\right] \\
&+ \rho_2 M_1 \left(\frac{|c|}{\Gamma(\alpha+1)} + \frac{|d|}{\Gamma(\alpha)}\right) \\
&+ \rho_2 m \left(|c| M_2 + (|c| + |d|) M_3 + \frac{|c| + 2|d|}{2} M_4\right).
\end{aligned}
$$

Hence, the operator $T$ is uniformly bounded. On the other hand, for any $t \in J_k$, $0 \leq k \leq m$, one can confirm that

$$
\begin{aligned}
|(Tu)'(t)| &\leq \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s))|\, ds + \sum_{0<t_k<t} \left|\bar{I}_k(u(t_k))\right| \\
&+ \sum_{0<t_k<t} \left|\tilde{I}_k(u(t_k))\right| (t-t_k) + \frac{1}{|\delta|} \sum_{i=1}^{p-2} |c_i|\, |F_u(\xi_i)| \left|c - \sum_{i=1}^{p-2} d_i\right| \\
&+ \frac{1}{|\delta|} \left(\sum_{i=1}^{p-2} |d_i|\, |F_u(\xi_i)| + |\mathcal{A}_u|\right) \left|a - \sum_{i=1}^{p-2} c_i\right|.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
|(Tu)'(t)| &\leq \frac{M_1}{\Gamma(\alpha)} + m(M_3 + M_4) \\
&+ \sum_{i=1}^{p-2} (\kappa_1 |c_i| + \kappa_2 |d_i|) \left[\frac{\xi_i^\alpha}{\Gamma(\alpha+1)} M_1 + m\left(M_2 + \xi_i M_3 + \frac{\xi_i^2}{2} M_4\right)\right] \\
&+ \kappa_2 M_1 \left(\frac{|c|}{\Gamma(\alpha+1)} + \frac{|d|}{\Gamma(\alpha)}\right) \\
&+ \kappa_2 m \left(|c| M_2 + (|c| + |d|) M_3 + \frac{|c| + 2|d|}{2} M_4\right).
\end{aligned}
$$

The last inequality implies the equicontinuity of $T(\Omega)$ on $J_k$, $k = 0, 1, \ldots, m$. In accordance with the PC-type Arzela-Ascoli theorem $T(\Omega)$ is relatively compact in $PC(J, \mathbb{R})$. Consequently, the operator $T : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is completely continuous. □

The following conditions are required for the next theorem, which is concerned with the existence of solutions of (1.1).

(C1) There exist nonnegative constants $h$ and $l$ such that

$$|f(t,u)| \leq h + l\,|u|,\ t \in J,\ u \in \mathbb{R};$$

(C2) There exist nonnegative constants $h_k$, $l_k$, $\bar{h}_k$, $\bar{l}_k$, $\tilde{h}_k$, $\tilde{l}_k$ such that for $u \in \mathbb{R}$, $k = 1, 2, \ldots, m$,

$$|I_k(u)| \leq h_k + l_k |u|, \ \left|\bar{I}_k(u)\right| \leq \bar{h}_k + \bar{l}_k |u|, \ \left|\tilde{I}_k(u)\right| \leq \tilde{h}_k + \tilde{l}_k |u|.$$

Let us define

$$\begin{aligned} K^* = &\left(\frac{1+\rho_2|c|}{\Gamma(\alpha+1)} + \frac{\rho_2|d|}{\Gamma(\alpha)} + \sum_{i=1}^{p-2}(\rho_1|c_i| + \rho_2|d_i|)\frac{\xi_i^\alpha}{\Gamma(\alpha+1)}\right) l \\ &+ \sum_{i=1}^{p-2}(\rho_1|c_i| + \rho_2|d_i|) \sum_{0<t_k<\xi_i}\left(l_k + \xi_i \bar{l}_k + \frac{\xi_i^2}{2}\tilde{l}_k\right) \\ &+ \sum_{k=1}^{m}\left((1+|c|\rho_2) l_k + (1+\rho_2(|c|+|d|))\bar{l}_k + \frac{1+\rho_2(|c|+2|d|)}{2}\tilde{l}_k\right). \end{aligned} \tag{3.10}$$

The number $K^*$ defined by (3.10) is used in the following theorem.

**Theorem 2.** *Assume that the conditions* $(C1)$ *and* $(C2)$ *are valid. If* $K^* < 1$, *then the problem* (1.1) *has at least one solution.*

*Proof.* First of all, we know by Lemma 4 that the operator $T : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ defined by equation (3.9) is completely continuous. For $0 \leq \mu \leq 1$, consider the equation

$$u = \mu T u. \tag{3.11}$$

If $u$ is a solution of (3.11), then we have for $t \in J$ that

$$\begin{aligned} |F_u(\xi_i)| \leq &\int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)}(h + l|u(s)|)\,ds + \sum_{0<t_k<\xi_i} h_k + l_k|u(t_k)| \\ &+ \sum_{0<t_k<\xi_i}\left(\bar{h}_k + \bar{l}_k|u(t_k)|\right)(\xi_i - t_k) + \sum_{0<t_k<\xi_i}\left(\tilde{h}_k + \tilde{l}_k|u(t_k)|\right)\frac{(\xi_i - t_k)^2}{2} \\ &\leq \frac{\xi_i^\alpha h}{\Gamma(\alpha+1)} + \sum_{0<t_k<\xi_i}\left(h_k + \xi_i\bar{h}_k + \frac{\xi_i^2}{2}\tilde{h}_k\right) + \frac{\xi_i^\alpha l}{\Gamma(\alpha+1)}\|u\| \\ &\quad + \sum_{0<t_k<\xi_i}\left(l_k + \xi_i\bar{l}_k + \frac{\xi_i^2}{2}\tilde{l}_k\right)\|u\| \end{aligned}$$

and

$$
\begin{aligned}
|\mathcal{A}_u| &\le \int_0^1 \left( \frac{|c|\,(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{|d|\,(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) |f(s,u(s))|\,ds \\
&+ |c| \sum_{k=1}^{m} |I_k(u(t_k))| + \sum_{k=1}^{m} \left| \bar{I}_k(u(t_k)) \right| |c(1-t_k)+d| \\
&+ \frac{1}{2} \sum_{k=1}^{m} \left| \tilde{I}_k(u(t_k)) \right| \left| c(1-t_k)^2 + 2d(1-t_k) \right| \\
&\le \left( \frac{|c|}{\Gamma(\alpha+1)} + \frac{|d|}{\Gamma(\alpha)} \right) (h + l\,\|u\|) \\
&+ \sum_{k=1}^{m} \left( |c|\,h_k + (|c|+|d|)\,\bar{h}_k + \frac{1}{2}(|c|+2|d|)\,\tilde{h}_k \right) \\
&+ \sum_{k=1}^{m} \left( |c|\,l_k + (|c|+|d|)\,\bar{l}_k + \frac{1}{2}(|c|+2|d|)\,\tilde{l}_k \right) \|u\|\,.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&|u(t)| = \mu\,|Tu(t)| \\
&\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (h + l\,|u(s)|)\,ds + \sum_{0<t_k<t} (h_k + l_k\,|u(t_k)|) \\
&+ \sum_{0<t_k<t} \left( \bar{h}_k + \bar{l}_k\,|u(t_k)| \right) (t-t_k) + \sum_{0<t_k<t} \left( \tilde{h}_k + \tilde{l}_k\,|u(t_k)| \right) \frac{(t-t_k)^2}{2} \\
&+ \rho_1 \sum_{i=1}^{p-2} |c_i|\,|F_u(\xi_i)| + \rho_2 \sum_{i=1}^{p-2} |d_i|\,|F_u(\xi_i)| + \rho_2\,|\mathcal{A}_u|
\end{aligned}
$$

and thus,

$$
\begin{aligned}
|u(t)| \leq & \left(\frac{1+\rho_2|c|}{\Gamma(\alpha+1)}+\frac{\rho_2|d|}{\Gamma(\alpha)}+\sum_{i=1}^{p-2}(\rho_1|c_i|+\rho_2|d_i|)\frac{\xi_i^\alpha}{\Gamma(\alpha+1)}\right)h \\
& +\sum_{i=1}^{p-2}(\rho_1|c_i|+\rho_2|d_i|)\sum_{0<t_k<\xi_i}\left(h_k+\xi_i\bar{h}_k+\frac{\xi_i^2}{2}\tilde{h}_k\right) \\
& +\sum_{k=1}^{m}\left((1+\rho_2|c|)h_k+(1+\rho_2(|c|+|d|))\bar{h}_k+\frac{1+\rho_2(|c|+2|d|)}{2}\tilde{h}_k\right) \\
& +\left[\left(\frac{1+\rho_2|c|}{\Gamma(\alpha+1)}+\frac{\rho_2|d|}{\Gamma(\alpha)}+\sum_{i=1}^{p-2}(\rho_1|c_i|+\rho_2|d_i|)\frac{\xi_i^\alpha}{\Gamma(\alpha+1)}\right)l\right. \\
& +\sum_{k=1}^{m}\left((1+\rho_2|c|)l_k+(1+\rho_2(|c|+|d|))\bar{l}_k+\frac{1+\rho_2(|c|+2|d|)}{2}\tilde{l}_k\right) \\
& \left.+\sum_{i=1}^{p-2}(\rho_1|c_i|+\rho_2|d_i|)\sum_{0<t_k<\xi_i}\left(l_k+\xi_i\bar{l}_k+\frac{\xi_i^2}{2}\tilde{l}_k\right)\right]\|u\|.
\end{aligned}
$$

Hence, we have for $t \in J$ that

$$|u(t)| \leq K + K^* \|u\|,$$

where

$$
\begin{aligned}
K = & \left(\frac{1+\rho_2|c|}{\Gamma(\alpha+1)}+\frac{\rho_2|d|}{\Gamma(\alpha)}+\sum_{i=1}^{p-2}(\rho_1|c_i|+\rho_2|d_i|)\frac{\xi_i^\alpha}{\Gamma(\alpha+1)}\right)h \\
& +\sum_{i=1}^{p-2}(\rho_1|c_i|+\rho_2|d_i|)\sum_{0<t_k<\xi_i}\left(h_k+\xi_i\bar{h}_k+\frac{\xi_i^2}{2}\tilde{h}_k\right) \\
& +\sum_{k=1}^{m}\left((1+\rho_2|c|)h_k+(1+\rho_2(|c|+|d|))\bar{h}_k+\frac{1+\rho_2(|c|+2|d|)}{2}\tilde{h}_k\right).
\end{aligned}
$$

Because $K^* < 1$, the last inequality implies that

$$\|u\| \leq \frac{K}{1-K^*}.$$

Therefore, any solution of (3.11) is bounded regardless of the value of $\mu$. According to Theorem 1, the operator $T$ has a fixed point. Consequently, the fractional impulsive multi-point BVP (1.1) has at least one solution. □

Now, we will use the Banach contraction mapping principle to prove the existence and uniqueness of solutions for problem (1.1).

The following conditions are required.

(C3) There exists a positive number $L$ such that $|f(t,u)-f(t,v)| \le L\,|u-v|$ for all $t \in J$ and $u, v \in \mathbb{R}$;

(C4) There exist positive numbers $L_1$, $L_2$, and $L_3$ such that $|I_k(u)-I_k(v)| \le L_1\,|u-v|$, $\left|\bar{I}_k(u)-\bar{I}_k(v)\right| \le L_2\,|u-v|$, and $\left|\tilde{I}_k(u)-\tilde{I}_k(v)\right| \le L_3\,|u-v|$ for all $u$, $v \in \mathbb{R}$, and $k = 1,2,\dots,m$.

**Theorem 3.** *Suppose that the conditions* $(C3)$, $(C4)$ *hold, and*

$$L < \frac{1}{4}\left(\frac{1+\rho_2\,|c|+\sum_{i=1}^{p-2}(\rho_1\,|c_i|+\rho_2\,|d_i|)\,\xi_i^{\alpha}}{\Gamma(\alpha+1)}+\frac{\rho_2\,|d|}{\Gamma(\alpha)}\right)^{-1},$$

$$L_1 < \frac{1}{4m}\left(\sum_{i=1}^{p-2}(\rho_1\,|c_i|+\rho_2\,|d_i|)+1+\rho_2\,|c|\right)^{-1},$$

$$L_2 < \frac{1}{4m}\left(\sum_{i=1}^{p-2}(\rho_1\,|c_i|+\rho_2\,|d_i|)\,\xi_i+1+\rho_2(|c|+|d|)\right)^{-1},$$

$$L_3 < \frac{1}{2m}\left(\sum_{i=1}^{p-2}(\rho_1\,|c_i|+\rho_2\,|d_i|)\,\xi_i^2+1+\rho_2(|c|+2\,|d|)\right)^{-1}.$$

*Then, the problem* (1.1) *has a unique solution.*

*Proof.* Let $u,\ v \in PC(J,\mathbb{R})$. For each $t \in J$, we have

$$\begin{aligned}
&|Tu(t)-Tv(t)| \le |F_u(t)-F_v(t)| \\
&+\frac{1}{|\delta|}\sum_{i=1}^{p-2}|c_i|\,|F_u(\xi_i)-F_v(\xi_i)|\left|\left(c+d-\sum_{i=1}^{p-2}d_i\xi_i\right)-\left(c-\sum_{i=1}^{p-2}d_i\right)t\right| \\
&+\frac{1}{|\delta|}\left(\sum_{i=1}^{p-2}|d_i|\,|F_u(\xi_i)-F_v(\xi_i)|+|\mathcal{A}_u-\mathcal{A}_v|\right) \\
&\times\left|-b+\textstyle\sum_{i=1}^{p-2}c_i\xi_i+\left(a-\sum_{i=1}^{p-2}c_i\right)t\right| \\
&\le\Bigg[\left(\frac{1+\rho_2|c|+\sum_{i=1}^{p-2}(\rho_1|c_i|+\rho_2|d_i|)\xi_i^{\alpha}}{\Gamma(\alpha+1)}+\frac{\rho_2\,|d|}{\Gamma(\alpha)}\right)L \\
&+\left(\sum_{i=1}^{p-2}(\rho_1\,|c_i|+\rho_2\,|d_i|)+1+\rho_2\,|c|\right)mL_1 \\
&+\left(\textstyle\sum_{i=1}^{p-2}(\rho_1\,|c_i|+\rho_2\,|d_i|)\,\xi_i+1+\rho_2(|c|+|d|)\right)mL_2 \\
&+\left(\sum_{i=1}^{p-2}(\rho_1\,|c_i|+\rho_2\,|d_i|)\,\xi_i^2+1+\rho_2(|c|+2\,|d|)\right)\frac{mL_3}{2}\Bigg]\,\|u-v\|\,.
\end{aligned}$$

Since the inequality

$$\begin{aligned}
&\left(\frac{1+\rho_2|c|+\sum_{i=1}^{p-2}(\rho_1|c_i|+\rho_2|d_i|)\xi_i^{\alpha}}{\Gamma(\alpha+1)}+\frac{\rho_2|d|}{\Gamma(\alpha)}\right)L\\
&+\left(\sum_{i=1}^{p-2}(\rho_1|c_i|+\rho_2|d_i|)+1+\rho_2|c|\right)mL_1\\
&+\left(\sum_{i=1}^{p-2}(\rho_1|c_i|+\rho_2|d_i|)\xi_i+1+\rho_2(|c|+|d|)\right)mL_2\\
&+\left(\sum_{i=1}^{p-2}(\rho_1|c_i|+\rho_2|d_i|)\xi_i^2+1+\rho_2(|c|+2|d|)\right)\frac{mL_3}{2}\\
&<1
\end{aligned}$$

holds, $T$ is a contraction mapping. According to the Banach contraction mapping principle, $T$ has a unique fixed point in $PC(J,\mathbb{R})$, which is a solution of the problem (1.1). □

Examples that support the theoretical results are presented in the next section.

## 4. Examples

*Example 1*

Let us take into account the fractional order impulsive BVP

$$\begin{cases}
{}^{C}D_{0+}^{\frac{9}{4}}u(t)=\frac{1}{2}\cos\left(\sqrt{t}\right)+\frac{3}{20}\tanh(u(t)),\ t\in[0,1]\setminus\{t_1,t_2\},\\
\Delta u(t_k)=\frac{(-1)^k}{5}+\frac{1}{40}u(t_k),\ \Delta u'(t_k)=\frac{1}{10}\sin(u(t_k)),\\
\Delta u''(t_k)=\frac{1}{30}\arctan(u(t_k)),\ k=1,2,\\
-3u(0)+2u'(0)=-8u\left(\frac{1}{4}\right)+4u\left(\frac{3}{4}\right),\\
u(1)-2u'(1)=7u\left(\frac{1}{4}\right)-u\left(\frac{3}{4}\right),\ u''(0)=0,
\end{cases}\tag{4.1}$$

where $t_1=\frac{3}{10}$ and $t_2=\frac{4}{5}$.

BVP (4.1) is in the form of (1.1) with $\alpha=\frac{9}{4}$, $m=2$, $p=4$, $a=-3$, $b=2$, $c=1$, $d=-2$ $c_1=-8$, $c_2=4$, $d_1=7$, $d_2=-1$, $\xi_1=\frac{1}{4}$, $\xi_2=\frac{3}{4}$, $f(t,u)=\frac{1}{2}\cos\left(\sqrt{t}\right)+\frac{3}{20}\tanh(u)$, $I_k(u)=\frac{(-1)^k}{5}+\frac{1}{40}u$, $\bar{I}_k(u)=\frac{1}{10}\sin(u)$, and $\tilde{I}_k(u)=\frac{1}{30}\arctan(u)$. It can be verified that $\delta=7$, $\rho_1=\frac{3}{7}$, and $\rho_2=\frac{1}{7}$ for (4.1). Moreover, conditions $(C1)$

and $(C2)$ are satisfied with $h = \frac{1}{2}$, $l = \frac{3}{20}$, $h_1 = h_2 = \frac{1}{5}$, $l_1 = l_2 = \frac{1}{40}$, $\bar{h}_1 = \bar{h}_2 = \tilde{h}_1 = \tilde{h}_2 = 0$, $\bar{l}_1 = \bar{l}_2 = \frac{1}{10}$, $\tilde{l}_1 = \tilde{l}_2 = \frac{1}{30}$. Therefore, according to Theorem 2, BVP (4.1) has at least one solution.

*Example 2*

Consider the fractional order impulsive BVP

$$\begin{cases} {}^{C}D_{0+}^{\frac{5}{2}}u(t) = \sin t + \frac{7}{200}\arctan(u(t)),\ t \in [0,1] \setminus \left\{\frac{3}{4}\right\}, \\ \Delta u\left(\frac{3}{4}\right) = \frac{e^{u\left(\frac{3}{4}\right)}}{50\left(1+e^{u\left(\frac{3}{4}\right)}\right)},\ \Delta u'\left(\frac{3}{4}\right) = \frac{1}{100}u\left(\frac{3}{4}\right), \\ \Delta u''\left(\frac{3}{4}\right) = \frac{1}{20}\cos u\left(\frac{3}{4}\right), \\ u(0) + \frac{1}{2}u'(0) = -u\left(\frac{1}{2}\right) + 8u\left(\frac{7}{8}\right), \\ -u(1) + u'(1) = 4u\left(\frac{1}{2}\right) - 8u\left(\frac{7}{8}\right),\ u''(0) = 0. \end{cases} \tag{4.2}$$

One can confirm that (4.2) is in the form of (1.1) with $\alpha = \frac{5}{2}$, $m = 1$, $t_1 = \frac{3}{4}$, $p = 4$, $a = d = 1$, $b = \frac{1}{2}$, $c = -1 = c_1$, $c_2 = 8$, $d_1 = 4$, $d_2 = -8$, $\xi_1 = \frac{1}{2}$, $\xi_2 = \frac{7}{8}$, $f(t,u) = \sin t + \frac{7}{200}\arctan(u)$, $I_k(u) = \frac{e^u}{50(1+e^u)}$, $\bar{I}_k(u) = \frac{1}{100}u$, and $\tilde{I}_k(u) = \frac{1}{20}\cos u$. One can confirm that $\delta = -12$, $\rho_1 = \frac{5}{12}$, and $\rho_2 = \frac{1}{2}$ for (4.2). Conditions of Theorem 3 are valid for the problem with $L = 0.035$, $L_1 = 0.02$, $L_2 = 0.01$ and $L_3 = 0.05$. Thus, the fractional impulsive BVP (4.2) has a unique solution.

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# REPRESENTATIONS OF RECIPROCALS OF LUCAS SEQUENCES

H. R. HASHIM AND SZ. TENGELY

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*Abstract.* In 1953 Stancliff noted an interesting property of the Fibonacci number $F_{11} = 89$. One has that

$$\frac{1}{89} = \frac{F_0}{10} + \frac{F_1}{10^2} + \frac{F_2}{10^3} + \frac{F_3}{10^4} + \frac{F_4}{10^5} + \frac{F_5}{10^6} + \cdots.$$

De Weger determined a complete list of similar identities in case of the Fibonacci sequence, the solutions are as follows

$$\frac{1}{F_1} = \frac{1}{F_2} = \frac{1}{1} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{2^k}, \qquad \frac{1}{F_5} = \frac{1}{5} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{3^k},$$

$$\frac{1}{F_{10}} = \frac{1}{55} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{8^k}, \qquad \frac{1}{F_{11}} = \frac{1}{89} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{10^k}.$$

In this article we study similar problems in case of general Lucas sequences $U_n(P, Q)$. We deal with equations of the form

$$\frac{1}{U_n(P_2, Q_2)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_1, Q_1)}{x^k},$$

for certain pairs $(P_1, Q_1) \neq (P_2, Q_2)$. We also consider equations of the form

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P, Q)}{x^k} = \sum_{k=1}^{\infty} \frac{R_{k-1}}{y^k},$$

where $R_n$ is a ternary linear recurrence sequence. The proofs are based on results related to Thue equations and elliptic curves.

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*Keywords:* Lucas sequences, Diophantine equations, elliptic curves

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## 1. INTRODUCTION

Let $P$ and $Q$ be non-zero relatively prime integers. The Lucas sequence $\{U_n(P,Q)\}$ is defined by

$$U_0 = 0, U_1 = 1 \text{ and } U_n = PU_{n-1} - QU_{n-2}, \text{ if } n \geq 2.$$

The associated Lucas sequence $\{V_n(P,Q)\}$ is defined by

$$V_0 = 2, V_1 = P \text{ and } V_n = PU_{n-1} - QU_{n-2}, \text{ if } n \geq 2.$$

Terms of Lucas sequences and associated Lucas sequences satisfy the identity

$$V_n^2 - DU_n^2 = 4Q^n, \tag{1.1}$$

where $D = P^2 - 4Q$. In 1953, Stancliff [12] noted an interesting property of the Fibonacci sequence $U_n(1,-1) = F_n$. One has that

$$\frac{1}{F_{11}} = \frac{1}{89} = 0.0112358\ldots = \sum_{k=0}^{\infty} \frac{F_k}{10^{k+1}}.$$

In 1980, Winans [17] studied the related sums

$$\sum_{k=0}^{\infty} \frac{F_{\alpha k}}{10^{k+1}}$$

for certain values of $\alpha$. In 1981 Hudson and Winans [7] characterized all decimal fractions that can be approximated by sums of the type

$$\frac{1}{F_\alpha} \sum_{k=1}^{n} \frac{F_{\alpha k}}{10^{l(k+1)}}, \quad \alpha, l \geq 1.$$

Long [10] obtained a general identity for binary recurrence sequences from which one obtains e.g.

$$\frac{1}{109} = \sum_{k=0}^{\infty} \frac{F_k}{(-10)^{k+1}}, \qquad \frac{1}{10099} = \sum_{k=0}^{\infty} \frac{F_k}{(-100)^{k+1}}.$$

In case of the equation

$$\frac{1}{U_n(P,Q)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P,Q)}{x^k}, \tag{1.2}$$

De Weger [4] determined all $x \geq 2$ in case of $(P,Q) = (1,-1)$. The solutions are as follows

$$\frac{1}{F_1} = \frac{1}{F_2} = \frac{1}{1} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{2^k}, \qquad \frac{1}{F_5} = \frac{1}{5} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{3^k},$$

$$\frac{1}{F_{10}} = \frac{1}{55} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{8^k}, \qquad \frac{1}{F_{11}} = \frac{1}{89} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{10^k}.$$

In 2014 Tengely [15] extended the above result and obtained e.g.

$$\frac{1}{U_{10}} = \frac{1}{416020} = \sum_{k=0}^{\infty} \frac{U_k}{647^{k+1}},$$

where $U_0 = 0, U_1 = 1$ and $U_n = 4U_{n-1} + U_{n-2}, n \geq 2$. Recently Ohtsuka and Nakamura [11] proved that

$$\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & \text{if } n \geq 2 \text{ is even,} \\ F_{n-2} - 1 & \text{if } n \geq 1 \text{ is odd,} \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the floor function. This result has been investigated by several other mathematicians see e.g. [6, 9].

## 2. Auxiliary results

In the proofs we will use the following two results of Köhler [8].

**Lemma 1.** *Let $A, B, a_0, a_1$ be arbitrary complex numbers. Define the sequence $\{a_n\}$ by the recursion $a_{n+1} = Aa_n + Ba_{n-1}$. Then the formula*

$$\sum_{k=0}^{\infty} \frac{a_k}{x^{k+1}} = \frac{a_0 x - Aa_0 + a_1}{x^2 - Ax - B}$$

*holds for all complex $x$ such that $|x|$ is larger than the absolute values of the zeros of $x^2 - Ax - B$.*

**Lemma 2.** *Let arbitrary complex numbers $A_0, A_1, \ldots, A_m, a_0, a_1, \ldots, a_m$ be given. Define the sequence $(a_n)_n$ by the recursion*

$$a_{n+1} = A_0 a_n + A_1 a_{n-1} + \cdots + A_m a_{n-m}$$

*Then for all complex $z$ such that $|z|$ is larger than the absolute values of all zeros of $q(z) = z^{m+1} - A_0 z^m - A_1 z^{m-1} - \cdots - A_m$, the formula*

$$\sum_{k=1}^{\infty} \frac{a_{k-1}}{z^k} = \frac{p(z)}{q(z)}$$

*holds with $p(z) = a_0 z^m + b_1 z^{m-1} + \cdots + b_m$, where $b_k = a_k - \sum_{i=0}^{k-1} A_i a_{k-1-i}$ for $1 \leq k \leq m$.*

## 3. Main results

In this paper we extend the results of [15], we consider the equation

$$\frac{1}{U_n(P_2,Q_2)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_1,Q_1)}{x^k}, \tag{3.1}$$

for certain pairs $(P_1,Q_1) \neq (P_2,Q_2)$. We consider non-degenerate sequences with $1 \leq P \leq 3$ and $Q = \pm 1$. Define the set $S$ as follows

$$\begin{aligned} S =& \{u_1(n) = U_n(1,-1), u_2(n) = U_n(1,1), u_3(n) = U_n(2,-1), u_4(n) = U_n(3,-1), \\ & u_5(n) = U_n(3,1)\}. \end{aligned}$$

**Theorem 1.** *The equation*

$$\frac{1}{u_j(n)} = \sum_{k=1}^{\infty} \frac{u_i(k-1)}{x^k}, \tag{3.2}$$

*has the following solutions with* $1 \leq i,j \leq 5, i \neq j$

$$\begin{aligned} (i,j,n,x) \in \{&(1,2,\{1,2\},2),(1,3,1,2),(1,3,3,3),(1,3,5,6),(1,4,1,2), \\ &(1,4,5,11),(1,4,7,35),(1,5,1,2),(1,5,5,8),(2,1,4,2),(2,1,7,4), \\ &(2,1,8,5),(2,5,2,2),(2,5,4,5),(3,1,3,3),(3,1,9,7),(4,1,4,4), \\ &(4,1,14,21),(4,5,2,4),(4,5,7,21),(5,1,\{1,2\},3),(5,1,5,4), \\ &(5,1,10,9),(5,1,11,11),(5,2,\{1,2\},3),(5,3,1,3),(5,3,3,4), \\ &(5,3,5,7),(5,4,1,3),(5,4,5,12),(5,4,7,36)\}. \end{aligned}$$

We also deal with equations of the form

$$\sum_{k=1}^{\infty} \frac{u_j(k-1)}{x^k} = \sum_{k=1}^{\infty} \frac{R_{k-1}}{y^k}, \tag{3.3}$$

where $R_n$ is a ternary linear recurrence sequence. We provide results in case of the Tribonacci sequence defined by $T_0 = T_1 = 0, T_2 = 1$ and $T_{n+3} = T_{n+2} + T_{n+1} + T_n, n \geq 0$ and Berstel's sequence, that is given by $B_0 = B_1 = 0, B_2 = 1$ and $B_{n+3} = 2B_{n+2} - 4B_{n+1} + 4B_n, n \geq 0$.

**Theorem 2.** *The complete list of solutions of equation (3.3) with* $u_n \in S, R_n \in \{B_n, T_n\}$ *and positive integers* $x, y$ *satisfying conditions of Lemma 1 and 2 is as follows*

| $u_n$ | $R_n$ | $(x,y)$ | $u_n$ | $R_n$ | $(x,y)$ |
|---|---|---|---|---|---|
| $u_1$ | $B_n$ | $\{(25,9)\}$ | $u_1$ | $T_n$ | $\{(2,2)\}$ |
| $u_2$ | $B_n$ | $\{(10,5)\}$ | $u_2$ | $T_n$ | $\{(7,4),(309,46)\}$ |
| $u_3$ | $B_n$ | $\{\}$ | $u_3$ | $T_n$ | $\{(t(t^2-2)+1,t^2-1) : t \geq 2, t \in \mathbb{N}\}$ |
| $u_4$ | $B_n$ | $\{(6,3),(18,7)\}$ | $u_4$ | $T_n$ | $\{\}$ |
| $u_5$ | $B_n$ | $\{(26,9)\}$ | $u_5$ | $T_n$ | $\{\}$ |

## 4. Proofs of the theorems

*Proof of Theorem 1.* Consider equation (3.1), by Lemma 1 we obtain that

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P_1,Q_1)}{x^k} = \frac{1}{x^2 - P_1 x + Q_1}.$$

Hence we have that $U_n(P_2,Q_2) = x^2 - P_1 x + Q_1$. Combining the latter equation with (1.1) we get $V_n(P_2,Q_2)^2 = (P_2^2 - 4Q_2)(x^2 - P_1 x + Q_1)^2 + 4Q_2^n$. The so-called two-cover descent by Bruin and Stoll [3] can be used to prove that a given hyperelliptic curve has no rational points. It is implemented in Magma [2], the procedure is called `TwoCoverDescent`. If it fails and we do not find any rational points on the curve, then we apply the argument by Alekseyev and Tengely [1], that reduces the problem to Thue equations. If we have a rational point on the curve, then using a method by Tzanakis [16] the integral points can be determined. This algorithm is implemented in Magma as `IntegralQuarticPoints`. In this way we collect the possible values of $x$.

| $(P_1,Q_1,P_2,Q_2)$ | $x$ | $(P_1,Q_1,P_2,Q_2)$ | $x$ | $(P_1,Q_1,P_2,Q_2)$ | $x$ |
|---|---|---|---|---|---|
| $(1,-1,1,1)$ | $2$ | $(1,1,1,-1)$ | $2,4,5$ | $(2,-1,1,-1)$ | $3,7$ |
| $(1,-1,2,-1)$ | $2,3,6$ | $(1,1,2,-1)$ | $-$ | $(2,-1,1,1)$ | $-$ |
| $(1,-1,3,-1)$ | $2,11,35$ | $(1,1,3,-1)$ | $2$ | $(2,-1,3,-1)$ | $-$ |
| $(1,-1,3,1)$ | $2,8$ | $(1,1,3,1)$ | $2,5$ | $(2,-1,3,1)$ | $-$ |

| $(P_1,Q_1,P_2,Q_2)$ | $x$ | $(P_1,Q_1,P_2,Q_2)$ | $x$ |
|---|---|---|---|
| $(3,-1,1,-1)$ | $4,21$ | $(3,1,1,-1)$ | $3,4,9,11$ |
| $(3,-1,1,1)$ | $-$ | $(3,1,1,1)$ | $3$ |
| $(3,-1,2,-1)$ | $-$ | $(3,1,2,-1)$ | $3,4,7$ |
| $(3,-1,3,1)$ | $4,21$ | $(3,1,3,-1)$ | $3,12,36$ |

It remains to compute the set of possible values of $n$. We provide details of the computation in case of $(P_1,Q_1,P_2,Q_2) = (3,-1,1,-1)$, following these steps all other equations can be handled. In case of $(P_1,Q_1,P_2,Q_2) = (3,-1,1,-1)$ we have that $x \in \{4,21\}$. If $x = 4$, then we define a matrix $T$ as follows

$$T = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 0 \end{pmatrix}.$$

We have that

$$\frac{1}{4}\left(T^0+T^1+T^2+\cdots+T^{N-1}\right)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}* \\ \sum_{k=1}^{N}\frac{U_{k-1}(3,-1)}{4^k}\end{pmatrix}.$$

It follows that

$$\sum_{k=1}^{N}\frac{U_{k-1}(3,-1)}{4^k}=$$
$$-\frac{2^{-3N-1}}{39}\left(\left(\sqrt{13}+3\right)^N\left(5\sqrt{13}+13\right)+\left(13-5\sqrt{13}\right)\left(-\sqrt{13}+3\right)^N-13\cdot 2^{3N+1}\right),$$

hence we have that

$$\lim_{N\to\infty}\sum_{k=1}^{N}\frac{U_{k-1}(3,-1)}{4^k}=\frac{1}{3}=\frac{1}{U_4(1,-1)}.$$

In this case we obtain that $n=4$. If $x=21$, then

$$T=\begin{pmatrix}3/21 & 1/21\\ 1/21 & 0\end{pmatrix}.$$

In a similar way than in case of $x=4$ we get that

$$\sum_{k=1}^{N}\frac{U_{k-1}(3,-1)}{21^k}=$$
$$\frac{\left(7^N3^N2^{N+1}-\left(\sqrt{13}+3\right)^N\left(3\sqrt{13}+1\right)+\left(3\sqrt{13}-1\right)\left(-\sqrt{13}+3\right)^N\right)2^{-N-1}}{377\cdot 7^N3^N},$$

therefore

$$\lim_{N\to\infty}\sum_{k=1}^{N}\frac{U_{k-1}(3,-1)}{21^k}=\frac{1}{377}=\frac{1}{U_{14}(1,-1)}.$$

The only solution in this case is given by $n=14$. □

*Proof of Theorem 2.* We provide a general argument that works for other sequences as well. Let $a_0=0, a_1=1$ and $a_{n+1}=Aa_n+Ba_{n-1}$. Let $b_0=b_1=0, b_2=1$ and $b_{n+1}=Cb_n+Db_{n-1}+Eb_{n-2}$. Equation (3.3) yields that

$$Y^2=X^3-4CX^2-16DX+16A^2+64B-64E,$$

where $Y=8x-4A$ and $X=4y$. If the cubic polynomial in $X$ is square-free, then we have an elliptic equation and integral points can be determined using the so-called elliptic logarithm method developed by Stroeker and Tzanakis [14] and independently by Gebel, Pethő and Zimmer [5]. There exists a number of software implementations for determining integral points on elliptic curves based on this technique, here we used SageMath [13]. Let us consider the case with $u_2(n), T_n$. We

obtain the elliptic curve $Y^2 = X^3 - 4X^2 - 16X - 112$. Using the SageMath function `integral_points()` we get

$$[(8:4:1),(16:52:1),(29:143:1),(184:2468:1)].$$

From these points we have that $(x,y) \in \{(7,4),(309,46)\}$. As a second example consider the case with $u_4, B_n$. The elliptic curve is given by $Y^2 = X^3 - 8X^2 + 64X - 48$. The list of integral points is

$$[(1:3:1),(4:12:1),(12:36:1),(28:132:1)].$$

Thus we get that $(x,y) \in \{(6,3),(18,7)\}$. Finally let us deal with the special case with $u_3, T_n$. The cubic polynomial is not square-free, it is $(X+4)(X-4)^2$. Therefore we have that $X + 4 = 4y + 4 = u^2$. Hence $y = t^2 - 1$ for some integer $t \geq 2$. It follows that $x = t(t^2-2)+1$. So we obtain infinitely many identities of the form

$$\sum_{k=1}^{\infty} \frac{u_4(k-1)}{(t(t^2-2)+1)^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}}{(t^2-1)^k}.$$

□

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# GENERALIZED CONVEXITY OF THE INVERSE HYPERBOLIC COSINE FUNCTION

YUE HE AND GENDI WANG

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*Abstract.* The generalized convexity of the inverse hyperbolic cosine function related to the hyperbolic metric is investigated in this paper.

2010 *Mathematics Subject Classification:* 33B10; 26D07

*Keywords:* Hölder mean, convexity, concavity, inverse hyperbolic cosine function

## 1. Introduction

The hyperbolic functions and their inverses play an important role in the study of the hyperbolic geometry and quasiconformal mappings [1, 4, 5, 8, 9, 11, 12]. For example, the explicit formulas for the hyperbolic metric in the unit disk $\mathbb{B}^2$ and the upper half plane $\mathbb{H}^2$ are given in terms of the inverse hyperbolic sine and cosine functions, respectively, as follows [5, p.35, p.40]:

$$\rho_{\mathbb{B}^2}(x,y) = 2\operatorname{arsh}\frac{|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}},$$

$$\rho_{\mathbb{H}^2}(x,y) = \operatorname{arch}\left(1+\frac{|x-y|^2}{2\operatorname{Im}x\operatorname{Im}y}\right).$$

In recent papers [9, 11], the authors investigated the properties of hyperbolic Lambert quadrilaterals in the unit disk by studying the inverse hyperbolic tangent and sine functions.

The study of the convexity/concavity with respect to Hölder means, or simply $H_{p,q}$-convexity/concavity, of special functions has attracted attentions of many researchers, see [2, 6, 7, 9–11, 13–17]. In particular, the convexity/concavity of the inverse hyperbolic tangent and sine functions has been studied in [9] and [11], respectively. For the definition of the above so-called generalized convexity/concavity, the reader is referred to Section 2.

In this paper, we continue the work of [9, 11] to study the generalized convexity for the inverse hyperbolic cosine function. Our main result is stated in the following theorem.

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**Theorem 1.** *For $p,q \in \mathbb{R}$, the inverse hyperbolic cosine function* arch *is strictly $H_{p,q}$-convex on $(1,+\infty)$ if and only if $(p,q) \in D_1$, while* arch *is strictly $H_{p,q}$-concave on $(1,+\infty)$ if and only if $(p,q) \in D_2 \cup D_3$, where*

$$D_1 = \{(p,q)|-\infty < p \le 0, 2 \le q < +\infty\},$$
$$D_2 = \{(p,q)|0 \le p \le \frac{2}{3}, -\infty < q \le C(p)\},$$
$$D_3 = \{(p,q)|\frac{2}{3} < p < +\infty, -\infty < q \le 2\},$$

*and $C(p)$ is the same as in Lemma 3(4) with $C(0) = 1$ and $C(\frac{2}{3}) = 2$. In particular, for all $x, y \in (1,+\infty)$, there hold*

$$\operatorname{arch}\sqrt{xy} \le \sqrt{\frac{\operatorname{arch}^2 x + \operatorname{arch}^2 y}{2}} \le \operatorname{arch}\sqrt{\left(\frac{\sqrt[3]{x^2}+\sqrt[3]{y^2}}{2}\right)^3}, \tag{1.1}$$

*with equalities if and only if $x = y$.*

## 2. Preliminaries

For $r,s \in (0,+\infty)$, the *Hölder mean of order $p$* is defined by

$$H_p(r,s) = \left(\frac{r^p+s^p}{2}\right)^{\frac{1}{p}} \text{ for } p \ne 0, \quad H_0(r,s) = \sqrt{rs}.$$

For $p = 1$, we get the arithmetic mean $A = H_1$; for $p = 0$, the geometric mean $G = H_0$; and for $p = -1$, the harmonic mean $H = H_{-1}$. It is well known that $H_p(r,s)$ is continuous and increasing with respect to $p$.

A function $f\colon I \to J$ is called *$H_{p,q}$-convex (concave)* if it satisfies

$$f\Big(H_p(r,s)\Big) \le (\ge) H_q\Big(f(r), f(s)\Big)$$

for all $r,s \in I$, and strictly *$H_{p,q}$-convex (concave)* if the inequality is strict except for $r = s$.

The following monotone form of *l'Hôpital's rule* is of great use in deriving monotonicity properties and obtaining inequalities. See the extensive bibliography of [3].

**Lemma 1** ([1, Theorem 1.25])**.** *For $-\infty < a < b < \infty$, let functions $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, and be differentiable on $(a,b)$. Let $g'(x) \ne 0$ on $(a,b)$. If $f'(x)/g'(x)$ is increasing (deceasing) on $(a,b)$, then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad and \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

*If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.*

We prove the following three lemmas before giving the proof of Theorem 1.

**Lemma 2.** *Let* $r \in (1, +\infty)$.
*(1)The function* $f_1(r) \equiv \frac{\operatorname{arch} r}{\sqrt{r^2-1}}$ *is strictly decreasing with range* $(0, 1)$*;*
*(2)The function* $f_2(r) \equiv 2 + \frac{\operatorname{arch} r + r\sqrt{r^2-1} - 2r^3\sqrt{r^2-1}}{(r^2-1)\left(\operatorname{arch} r + r\sqrt{r^2-1}\right)}$ *is strictly decreasing with range* $(0, \frac{2}{3})$.

*Proof.* (1) Let $f_{11}(r) = \operatorname{arch} r$ and $f_{12}(r) = \sqrt{r^2-1}$, then $f_{11}(1^+) = f_{12}(1^+) = 0$. By differentiation, we have

$$\frac{f_{11}'(r)}{f_{12}'(r)} = \frac{1}{r},$$

which is strictly decreasing. Hence by Lemma 1, the function $f_1$ is strictly decreasing with $f_1(1^+) = 1$ and $f_1(+\infty) = 0$.

(2) Let $f_{21} = \operatorname{arch} r + r\sqrt{r^2-1} - 2r^3\sqrt{r^2-1}$ and $f_{22} = (r^2-1)\left(\operatorname{arch} r + r\sqrt{r^2-1}\right)$, then $f_{21}(1^+) = f_{22}(1^+) = 0$. By differentiation, we have

$$\frac{f_{21}'(r)}{f_{22}'(r)} = \frac{-4}{2 + \frac{f_1(r)}{r}},$$

which is strictly decreasing by (1). Hence by Lemma 1, the function $f_2$ is strictly decreasing with $f_2(1^+) = \frac{2}{3}$ and $f_2(+\infty) = 0$. □

**Lemma 3.** *For* $p \in \mathbb{R}$ *and* $r \in (1, +\infty)$*, define*

$$h_p(r) = 1 + p\sqrt{r^2-1} \cdot \frac{\operatorname{arch} r}{r} + \frac{1}{\sqrt{r^2-1}} \cdot \frac{\operatorname{arch} r}{r}.$$

*(1) If* $p \geq \frac{2}{3}$*, then* $h_p$ *is strictly increasing with range* $(2, +\infty)$.
*(2) If* $p < 0$*, then* $h_p$ *is strictly decreasing with range* $(-\infty, 2)$.
*(3) If* $p = 0$*, then* $h_p$ *is strictly decreasing with range* $(1, 2)$.
*(4) If* $0 < p < \frac{2}{3}$*, then* $h_p$ *is not monotone and the range of* $h_p$ *is* $[C(p), +\infty)$*, where*

$$C(p) = \min_{r \in (1, +\infty)} h_p(r)$$

*with* $1 < C(p) < 2$.

*Proof.* By Lemma 2(1), it is easy to get

$$h_p(1^+) = 2 \quad \text{and} \quad h_p(+\infty) = \begin{cases} +\infty, & p > 0, \\ 1, & p = 0, \\ -\infty, & p < 0. \end{cases}$$

By differentiation, we have

$$h_p'(r) = \frac{1}{r}\left(1 + \frac{\operatorname{arch} r}{r\sqrt{r^2-1}}\right)(p - f_2(r)),$$

where $f_2(r)$ is the same as in Lemma 2(2).

By Lemma 2(2), we have (1)–(3).

(4) If $0 < p < \frac{2}{3}$, since the range of $f_2$ is $(0, \frac{2}{3})$, there exists one and only one point $r_p \in (1, +\infty)$ such that $p = f_2(r_p)$. Then $h_p$ is strictly decreasing on $(1, r_p)$ and increasing on $(r_p, +\infty)$. Since $h_p$ is continuous in $r$, there exists

$$C(p) = \min_{r \in (1,+\infty)} h_p(r)$$

and $1 < C(p) < 2$. □

**Lemma 4.** *Let $p, q \in \mathbb{R}$, $r \in (1, +\infty)$, and $C(p)$ be the same as in Lemma 3(4). Let*

$$g_{p,q}(r) = \frac{\operatorname{arch}^{q-1} r}{r^{p-1}\sqrt{r^2-1}}.$$

*(1) If $p \geq \frac{2}{3}$, then $g_{p,q}$ is strictly decreasing for each $q \leq 2$, and $g_{p,q}$ is not monotone for any $q > 2$.*
*(2) If $p < 0$, then $g_{p,q}$ is strictly increasing for each $q \geq 2$, and $g_{p,q}$ is not monotone for any $q < 2$.*
*(3) If $p = 0$, then $g_{p,q}$ is strictly increasing for each $q \geq 2$, and $g_{p,q}$ is strictly decreasing for each $q \leq 1$, and $g_{p,q}$ is not monotone for any $1 < q < 2$.*
*(4) If $0 < p < \frac{2}{3}$, then $g_{p,q}$ is strictly decreasing for each $q \leq C(p)$, and $g_{p,q}$ is not monotone for any $q > C(p)$.*

*Proof.* By logarithmic differentiation in $r$, we have

$$\frac{g'_{p,q}(r)}{g_{p,q}(r)} = \frac{1}{\sqrt{r^2-1}\cdot \operatorname{arch} r}\left(q - h_p(r)\right),$$

where $h_p(r)$ is the same as in Lemma 3. Hence the results immediately follow from Lemma 3. □

## 3. Proof of main result

We are now in a position to prove Theorem 1.

*Proof of Theorem 1.* Without loss of generality, we may assume that $1 < x \leq y < +\infty$. Let $t = H_p(x, y)$, then $x \leq t \leq y$ and

$$\frac{\partial t}{\partial x} = \frac{1}{2}\left(\frac{x}{t}\right)^{p-1}.$$

The proof is divided into the following four cases.

**Case 1.** $p \neq 0$ and $q \neq 0$.

Define

$$F(x, y) = \operatorname{arch}^q\left(H_p(x, y)\right) - \frac{\operatorname{arch}^q x + \operatorname{arch}^q y}{2}.$$

By differentiation, we have

$$\frac{\partial F}{\partial x} = \frac{q}{2}x^{p-1}\left(\frac{\operatorname{arch}^{q-1} t}{t^{p-1}\sqrt{t^2-1}} - \frac{\operatorname{arch}^{q-1} x}{x^{p-1}\sqrt{x^2-1}}\right) = \frac{q}{2}x^{p-1}\left(g_{p,q}(t) - g_{p,q}(x)\right),$$

where $g_{p,q}$ is defined in Lemma 4.

**Case 1.1** $p \geq \frac{2}{3}$ and $q \leq 2$.

By Lemma 4(1), the function $g_{p,q}$ is strictly decreasing on $(1, +\infty)$.

**Case 1.1.1** If $q > 0$, then $\frac{\partial F}{\partial x} \leq 0$. Hence $F(x, y)$ is strictly decreasing and $F(x, y) \geq F(y, y) = 0$. Namely,

$$\operatorname{arch}(H_p(x, y)) \geq \left(\frac{\operatorname{arch}^q x + \operatorname{arch}^q y}{2}\right)^{\frac{1}{q}} = H_q(\operatorname{arch} x, \operatorname{arch} y),$$

with equality if and only if $x = y$.

**Case 1.1.2** If $q < 0$, then $\frac{\partial F}{\partial x} \geq 0$. Hence $F(x, y)$ is strictly increasing and $F(x, y) \leq F(y, y) = 0$. Namely,

$$\operatorname{arch}(H_p(x, y)) \geq \left(\frac{\operatorname{arch}^q x + \operatorname{arch}^q y}{2}\right)^{\frac{1}{q}} = H_q(\operatorname{arch} x, \operatorname{arch} y),$$

with equality if and only if $x = y$.

In conclusion, arch is strictly $H_{p,q}$-concave on the whole interval $(1, +\infty)$ for $(p, q) \in \{(p, q) | \frac{2}{3} \leq p < +\infty, 0 < q \leq 2\} \cup \{(p, q) | \frac{2}{3} \leq p < +\infty, q < 0\}$.

**Case 1.2** $p \geq \frac{2}{3}$ and $q > 2$.

By Lemma 4(1), the function $g_{p,q}$ is not monotone on $(1, +\infty)$. With an argument similar to Case 1.1, it is easy to see that arch is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on the whole interval $(1, +\infty)$ for $(p, q) \in \{(p, q) | p \geq \frac{2}{3}, q > 2\}$.

**Case 1.3** $p < 0$ and $q \geq 2$.

By Lemma 4(2), the function $g_{p,q}$ is strictly increasing on $(1, +\infty)$ and hence $\frac{\partial F}{\partial x} \geq 0$. Then $F(x, y)$ is strictly increasing and $F(x, y) \leq F(y, y) = 0$. Namely,

$$\operatorname{arch}(H_p(x, y)) \leq \left(\frac{\operatorname{arch}^q x + \operatorname{arch}^q y}{2}\right)^{\frac{1}{q}} = H_q(\operatorname{arch} x, \operatorname{arch} y),$$

with equality if and only if $x = y$.

In conclusion, arch is strictly $H_{p,q}$-convex on the whole interval $(1, +\infty)$ for $(p, q) \in \{(p, q) | p < 0, q \geq 2\}$.

**Case 1.4** $p < 0$ and $q < 2$.

By Lemma 4(2), the function $g_{p,q}$ is not monotone on $(1, +\infty)$. With an argument similar to Case 1.3, it is easy to see that arch is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on the whole interval $(1, +\infty)$ for $(p, q) \in \{(p, q) | p < 0, q < 0\} \cup \{(p, q) | p < 0, 0 < q < 2\}$.

**Case 1.5** $0 < p < \frac{2}{3}$ and $q \leq C(p)$.

By Lemma 4(4), the function $g_{p,q}$ is strictly decreasing on $(1,+\infty)$.

**Case 1.5.1** If $q>0$, then $\frac{\partial F}{\partial x}\le 0$. Hence $F(x,y)$ is strictly decreasing and $F(x,y)\ge F(y,y)=0$. Namely,

$$\operatorname{arch}(H_p(x,y))\ge\left(\frac{\operatorname{arch}^q x+\operatorname{arch}^q y}{2}\right)^{\frac{1}{q}}=H_q(\operatorname{arch}x,\operatorname{arch}y),$$

with equality if and only if $x=y$.

**Case 1.5.2** If $q<0$, then $\frac{\partial F}{\partial x}\ge 0$. Hence $F(x,y)$ is strictly increasing and $F(x,y)\le F(y,y)=0$. Namely,

$$\operatorname{arch}(H_p(x,y))\ge\left(\frac{\operatorname{arch}^q x+\operatorname{arch}^q y}{2}\right)^{\frac{1}{q}}=H_q(\operatorname{arch}x,\operatorname{arch}y),$$

with equality if and only if $x=y$.

In conclusion, arch is strictly $H_{p,q}$-concave on the whole interval $(1,+\infty)$ for $(p,q)\in\{(p,q)|0<p<\frac{2}{3},0<q\le C(p)\}\cup\{(p,q)|0<p<\frac{2}{3},q<0\}$.

**Case 1.6** $0<p<\frac{2}{3}$ and $q>C(p)$.

By Lemma 4(4), the function $g_{p,q}$ is not monotone on $(1,+\infty)$. With an argument similar to Case 1.5, it is easy to see that arch is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on the whole interval $(1,+\infty)$ for $(p,q)\in\{(p,q)|0<p<\frac{2}{3},q>C(p)\}$.

**Case 2.** $p\ne 0$ and $q=0$.

Define

$$F(x,y)=\frac{\operatorname{arch}^2(H_p(x,y))}{\operatorname{arch}x\cdot\operatorname{arch}y}.$$

By logarithmic differentiation, we obtain

$$\frac{1}{F}\frac{\partial F}{\partial x}=x^{p-1}(g_{p,0}(t)-g_{p,0}(x)),$$

where $g_{p,0}$ is defined in Lemma 4.

**Case 2.1** $p\ge\frac{2}{3}$ and $q=0$.

By Lemma 4(1), the function $g_{p,q}$ is strictly decreasing on $(1,+\infty)$ and hence $\frac{\partial F}{\partial x}\le 0$. Then $F(x,y)$ is strictly decreasing and $F(x,y)\ge F(y,y)=1$. Namely,

$$\operatorname{arch}(H_p(x,y))\ge\sqrt{\operatorname{arch}x\cdot\operatorname{arch}y}=H_0(\operatorname{arch}x,\operatorname{arch}y),$$

with equality if and only if $x=y$.

In conclusion, arch is strictly $H_{p,q}$-concave on the whole interval $(1,+\infty)$ for $(p,q)\in\{(p,q)|p\ge\frac{2}{3},q=0\}$.

**Case 2.2** $p<0$ and $q=0$.

By Lemma 4(2), the function $g_{p,q}$ is not monotone on $(1,+\infty)$. With an argument similar to Case 2.1, it is easy to see that arch is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on the whole interval $(1,+\infty)$ for $(p,q)\in\{(p,q)|p<0,q=0)\}$.

**Case 2.3** $0<p<\frac{2}{3}$ and $q=0$.

By Lemma 4(4), the function $g_{p,q}$ is strictly decreasing on $(1,+\infty)$ and hence $\frac{\partial F}{\partial x} \leq 0$. Then $F(x,y)$ is strictly decreasing and $F(x,y) \geq F(y,y) = 1$. Namely,

$$\operatorname{arch}(H_p(x,y)) \geq \sqrt{\operatorname{arch} x \cdot \operatorname{arch} y} = H_0(\operatorname{arch} x, \operatorname{arch} y),$$

with equality if and only if $x = y$.

In conclusion, arch is strictly $H_{p,q}$-concave on the whole interval $(1,+\infty)$ for $(p,q) \in \{(p,q)|0 < p < \frac{2}{3}, q = 0\}$.

**Case 3.** $p = 0$ and $q \neq 0$.

Define

$$F(x,y) = \operatorname{arch}^q(\sqrt{xy}) - \frac{\operatorname{arch}^q x + \operatorname{arch}^q y}{2}.$$

By differentiation, we obtain

$$\frac{\partial F}{\partial x} = \frac{q}{2x}(g_{0,q}(t) - g_{0,q}(x)),$$

where $g_{0,q}$ is defined in Lemma 4.

**Case 3.1** $p = 0$ and $q \geq 2$.

By Lemma 4(3), the function $g_{p,q}$ is strictly increasing on $(1,+\infty)$ and hence $\frac{\partial F}{\partial x} \geq 0$. Then $F(x,y)$ is strictly increasing and $F(x,y) \leq F(y,y) = 0$. Namely,

$$\operatorname{arch}(H_0(x,y)) \leq \left(\frac{\operatorname{arch}^q x + \operatorname{arch}^q y}{2}\right)^{\frac{1}{q}} = H_q(\operatorname{arch} x, \operatorname{arch} y),$$

with equality if and only if $x = y$.

In conclusion, arch is strictly $H_{p,q}$-convex on the whole interval $(1,+\infty)$ for $(p,q) \in \{(p,q)|p = 0, q \geq 2\}$.

**Case 3.2** $p = 0$ and $q \leq 1$.

By Lemma 4(3), the function $g_{p,q}$ is strictly decreasing on $(1,+\infty)$.

**Case 3.2.1** If $0 < q \leq 1$, then $\frac{\partial F}{\partial x} \leq 0$. Hence $F(x,y)$ is strictly decreasing and $F(x,y) \geq F(y,y) = 0$. Namely,

$$\operatorname{arch}(H_0(x,y)) \geq \left(\frac{\operatorname{arch}^q x + \operatorname{arch}^q y}{2}\right)^{\frac{1}{q}} = H_q(\operatorname{arch} x, \operatorname{arch} y),$$

with equality if and only if $x = y$.

**Case 3.2.2** If $q < 0$, then $\frac{\partial F}{\partial x} \geq 0$. Hence $F(x,y)$ is strictly increasing and $F(x,y) \leq F(y,y) = 0$. Namely,

$$\operatorname{arch}(H_0(x,y)) \geq \left(\frac{\operatorname{arch}^q x + \operatorname{arch}^q y}{2}\right)^{\frac{1}{q}} = H_q(\operatorname{arch} x, \operatorname{arch} y),$$

with equality if and only if $x = y$.

In conclusion, arch is strictly $H_{p,q}$-concave on the whole interval $(1,+\infty)$ for $(p,q) \in \{(p,q)|p = 0, 0 < q \leq 1\} \cup \{(p,q)|p = 0, q < 0\}$.

**Case 3.3** $p = 0$ and $1 < q < 2$.

By Lemma 4(3), the function $g_{p,q}$ is not monotone on $(1,+\infty)$. With an argument similar to Case 3.2, it is easy to see that arch is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on the whole interval $(1,+\infty)$ for $(p,q) \in \{(p,q)|p=0,\, 1<q<2)\}$.

**Case 4.** $p=0$ and $q=0$.

By Case 2.3, for all $x, y \in (1,+\infty)$, we have

$$\operatorname{arch}(H_p(x,y)) \geq H_0(\operatorname{arch} x, \operatorname{arch} y) \qquad \text{for} \quad 0<p<\frac{2}{3}.$$

By the continuity of $H_p$ in $p$ and arch in $x$, we have

$$\operatorname{arch}(H_0(x,y)) \geq H_0(\operatorname{arch} x, \operatorname{arch} y),$$

with equality if and only if $x=y$.

In conclusion, arch is strictly $H_{0,0}$-concave on the whole interval $(1,+\infty)$.

By Case 1.1 and Case 3.1, arch is strictly $H_{\frac{2}{3},2}$-concave and strictly $H_{0,2}$-convex on $(1,+\infty)$. Therefore, the inequalities (1.1) hold with equalities if and only if $x=y$.

This completes the proof of Theorem 1. □

Setting $p=1=q$ in Theorem 1, we easily obtain the concavity of arch.

**Corollary 1.** *The inverse hyperbolic cosine function* arch *is strictly concave on* $(1,+\infty)$.

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# NEAR-VECTOR SPACES CONSTRUCTED FROM NEAR DOMAINS

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*Abstract.* In this paper we prove some new results on near-vector spaces and near domains and give a first application of the nearring of quotients with respect to a multiplicative set, namely we construct a new class of near-vector spaces from near domains.

2010 *Mathematics Subject Classification:* 16Y30; 12K05

*Keywords:* nearrings of quotients, near-vector spaces, nearrings

## 1. Introduction

André [1] first generalised the concept of a vector space, i.e., a linear space, to a structure comprising a bit more non-linearity, the so-called near-vector space. In [12] van der Walt showed how to construct an arbitrary finite-dimensional near-vector space, using a finite number of near-fields, all having isomorphic multiplicative semigroups. This construction was used in [7] to characterise all finite dimensional near-vector spaces over $\mathbb{Z}_p$, for $p$ a prime. These results were extended in [8] to all finite dimensional near-vector spaces over arbitrary finite fields. In [6] homogeneous and linear mappings and subspaces were investigated.

Recently, near-vector spaces have been used in several applications, including in secret sharing schemes in cryptography [4] and to construct interesting examples of families of planar nearrings [3]. In addition, they have proved interesting from a model theory perspective too [2].

In this paper we begin with some preliminary material in section 2.1. on near-vector spaces and prove some properties of isomorphisms of near-vector spaces. In section 2.2. we generalise a construction that was first considered in [6] and in section 2.3. we focus on nearrings of quotients, giving some new results that allow for alternative proofs of some of the main known results. In section 2.4. we focus on

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integral nearrings and near domains and prove that if a nearring $N$ is integral, then the nearring of quotients, $N_S$ will be integral and that if $N$ is a near domain and $S$ the set of all cancellable elements of $N$, then $N_S$ will be a nearfield. Finally, as a first application of nearrings of quotients, in section 2.5. we use the results from section 2.4. to construct a new class of near-vector spaces from near domains.

## 2. Results

### 2.1. *Preliminary considerations*

**Definition 1** ([1])**.** A pair $(V, A)$ is called a *near-vector space* if:

(1) $(V, +)$ is a group and $A$ is a set of endomorphisms of $V$;
(2) $A$ contains the endomorphisms 0, *id* and $-id$;
(3) $A^* = A \setminus \{0\}$ is a subgroup of the group $\mathrm{Aut}(V)$;
(4) $A$ acts fixed point free (fpf) on $V$, i.e., for $x \in V, \alpha, \beta \in A$, $x\alpha = x\beta$ implies that $x = 0$ or $\alpha = \beta$;
(5) the quasi-kernel $Q(V)$ of $V$, generates $V$ as a group. Here, $Q(V) = \{x \in V \mid \forall \alpha, \beta \in A, \exists \gamma \in A$ such that $x\alpha + x\beta = x\gamma\}$.

We sometimes refer to $V$ as a *near-vector space over $A$*. The elements of $V$ are called *vectors* and the members of $A$ *scalars*. The action of $A$ on $V$ is called *scalar multiplication*. Note that $-id \in A$ implies that $(V, +)$ is an abelian group. Also, the dimension of the near-vector space, $\dim(V)$, is uniquely determined by the cardinality of an independent generating set for $Q(V)$. See [1] for further details.

In [12] van der Walt proved the following theorem,

**Theorem 1** ([12] Theorem 3.4, p.301)**.** *Let $V$ be a group and let $A := D \cup \{0\}$, where $D$ is a fix point free group of automorphisms of $V$. Then $(V, A)$ is a finite dimensional near-vector space if and only if there exists a finite number of nearfields, $F_1, F_2, \ldots, F_n$, semigroup isomorphisms $\psi_i : A \to F_i$ and a group isomorphism $\Phi : V \to F_1 \oplus F_2 \oplus \cdots \oplus F_n$ such that if*

$$\Phi(v) = (x_1, x_2, \cdots, x_n), \ (x_i \in F_i)$$

*then*

$$\Phi(v\alpha) = (x_1\psi_1(\alpha), x_2\psi_2(\alpha), \cdots, x_n\psi_n(\alpha)),$$

*for all $v \in V$ and $\alpha \in A$.*

According to this theorem we can specify a finite dimensional near-vector space by taking $n$ nearfields $F_1, F_2, \ldots, F_n$ for which there are semigroup isomorphisms $\vartheta_{ij} : (F_j, \cdot) \to (F_i, \cdot)$ with $\vartheta_{ij}\vartheta_{jk} = \vartheta_{ik}$ for $1 \le i, j, k \le n$. We can then take $V := F_1 \oplus F_2 \oplus \cdots \oplus F_n$ as the additive group of the near-vector space and any one of the semigroups $(F_{i_o}, \cdot)$ as the semigroup of endomorphisms by defining

$$(x_1, x_2, \ldots, x_n)\alpha := (x_1\vartheta_{1i_o}(\alpha), x_2\vartheta_{2i_o}(\alpha), \cdots, x_n\vartheta_{ni_o}(\alpha)),$$

for all $x_j \in F_j$ and all $\alpha \in F_{i_o}$.

**Definition 2** ([8] Definition 3.2, p.4)**.** We say that two near-vector spaces $(V_1, A_1)$ and $(V_2, A_2)$ are *isomorphic* (written $(V_1, A_1) \cong (V_2, A_2)$ if there are group isomorphisms $\theta : (V_1, +) \to (V_2, +)$ and $\eta : (A_1^*, \cdot) \to (A_2^*, \cdot)$ such that $\theta(x\alpha) = \theta(x)\eta(\alpha)$ for all $x \in V_1$ and $\alpha \in A_1^*$.

We denote this pair by $(\theta, \eta)$. Isomorphisms map quasi-kernels to quasi-kernels:

**Proposition 1.** *If the near-vector spaces $(V_1, A_1)$ and $(V_2, A_2)$ are isomorphic, then $Q(V_1)$ is mapped to $Q(V_2)$.*

*Proof.* Let $v \in Q(V_1)$. If $v = 0$, then $\theta(v) \in Q(V_2)$, so suppose that $v \neq 0$. Let $\alpha, \beta \in A_2$, then

$$\begin{aligned}\theta(v)\alpha + \theta(v)\beta &= \theta(v)\eta(\alpha_1) + \theta(v)\eta(\beta_1) \text{ for some } \alpha_1, \beta_1 \in A_1,\\ &= \theta(v\alpha_1) + \theta(v\beta_1)\\ &= \theta(v\alpha_1 + v\beta_1)\\ &= \theta(v\gamma_1) \text{ for some } \gamma_1 \in A_1,\\ &= \theta(v)\eta(\gamma_1).\end{aligned}$$

Thus $\theta(v) \in Q(V_2)$. □

It is not difficult to show that

**Lemma 1.** *If $(\theta, \eta)$ is an isomorphism of $(V_1, A_1)$ onto $(V_2, A_2)$, then $\theta$ is uniquely determined on any basis of $(V_1, A_1)$.*

In [1], the concept of regularity is introduced as a central notion. A near-vector space is *regular* if any two vectors of $Q(V)\backslash\{0\}$ are *compatible,* i.e. if for any two vectors $u$ and $v$ of $Q(V)$ there exists a $\lambda \in A\backslash\{0\}$ such that $u + v\lambda \in Q(V)$. Every near-vector space can be uniquely decomposed into a direct sum of regular near-vector spaces $V_j$ $(j \in J)$ ([1], Theorem 4.13, p.12) and there is a unique direct decomposition into maximal regular near-vector spaces, called the *canonical decomposition* of $V$. Thus the theory of near-vector spaces is largely reduced to the theory of regular near-vector spaces.

**Theorem 2.** *If the near-vector spaces $(V_1, A_1)$ and $(V_2, A_2)$ are isomorphic and $(V_1, A_1)$ is regular, $(V_2, A_2)$ will be regular too, i.e. isomorphisms preserve regularity.*

*Proof.* By definition there exist group isomorphisms $\theta : (V_1, +) \to (V_2, +)$ and $\eta : ({A_1}^*, \cdot) \to ({A_2}^*, \cdot)$ such that $\theta(x\alpha) = \theta(x)\eta(\alpha)$ for all $x \in V_1$ and $\alpha \in {A_1}^*$. Let $w_1, w_2 \in Q(V_2)\backslash\{0\}$. Then there exist $v_1, v_2 \in Q(V_1)$ (by Proposition 1), such that $w_1 = \theta(v_1)$ and $w_2 = \theta(v_2)$. Since $V_1$ is regular there exists a $\lambda \in {A_1}^*$ such that $v_1 + v_2\lambda \in Q(V_1)$. Thus

$$\theta(v_1 + v_2\lambda) = \theta(v_1) + \theta(v_2)\eta(\lambda)$$

$$= w_1 + w_2\eta(\lambda) \text{ with } \eta(\lambda) \neq 0.$$

Thus $(V_2, A_2)$ is regular. □

We also have that

**Lemma 2.** *Let $(V_1, A_1)$ and $(V_2, A_2)$ be near-vector spaces and $(\theta, \eta)$ an isomorphism. If $V_1 = \bigoplus_{j\in J} V_j$ is the canonical direct decomposition of $V_1$, then $V_2 = \bigoplus_{j\in J} \theta(V_j)$ is the canonical direct decomposition of $V_2$.*

The proof is similar to that of Lemma 5.5, p.2537 in [6].

2.2. *Near-vector spaces of the form $F^n$ where $F$ is a nearfield*

In [6] van der Walt's characterisation was used to discuss several constructions of near-vector spaces. In particular, the case where we let $V = F^n$, $n \in \mathbb{N}$, $F$ a nearfield and we take all the isomorphisms to be identical, so that

$$(x_1, x_2, \ldots, x_n)\alpha = (x_1\alpha, x_2\alpha, \ldots, x_n\alpha)$$

was considered.

This is the case closest to the normal vector space setting. Let us denote it by $(V, F)$. In fact, when $F$ is a field, $(V, F)$ is a vector space. We will denote the set of all distributive elements of $F$ by $F_d$ (as in [10]), i.e.

$$F_d = \{d \in F \mid d(x + y) = dx + dy \ \forall x, y \in F\}.$$

It is not difficult to check that $F_d$ is a subfield of $F$. Note that $0, 1 \in F_d$. It was shown in [6] that $Q(V) = \cup \mathcal{V}_i$, with $\mathcal{V}_i = (d_1, d_2, \ldots, 1, d_{i+1}, \ldots, d_n)F$, with 1 in position $i$ and $d_i \in F_d$, $i \in \{1, \ldots, n\}$ and that $(V, F)$ is a regular near-vector space (Lemma 3.5., p.2531).

We now generalise,

**Theorem 3.** *Let $F$ be a near-field and $V = F^n$, $n \in \mathbb{N}$ be a near-vector space with the scalar multiplication defined by*

$$(x_1, \ldots, x_n)\alpha = (x_1\zeta(\alpha), \ldots, x_n\zeta(\alpha))$$

*where $\zeta$ is an automorphism of $(F, \cdot)$ and $\alpha \in F$. Then*

$$Q(V) = \{(d_i)\lambda | \lambda \in F, d_i \in F_d \text{ for all } i \in I\},$$

*where $I = \{1, 2, \ldots, n\}$.*

*Proof.* Let $d_i \in F_d$ for $i \in \{1, \ldots, n\}$ and $\alpha, \beta \in F$. We have

$$\begin{aligned}(d_i)\alpha + (d_i)\beta &= (d_i\zeta(\alpha) + d_i\zeta(\beta)) \\ &= (d_i(\zeta(\alpha) + \zeta(\beta))) \text{ since } d_i \text{ is distributive,} \\ &= (d_i\zeta(\gamma)) \text{ with } \gamma = \zeta^{-1}(\zeta(\alpha) + \zeta(\beta)), \\ &= (d_i)\gamma.\end{aligned}$$

Hence $(d_i) \in Q(V)$. Since $Q(V)$ is closed under scalar multiplication we have $Q(V) \supseteq \{(d_i)\lambda | \lambda \in F, d_i \in F_d \text{ for all } i \in I\}$. Now suppose that $(x_i) \in Q(V)$. If $(x_i) = 0$ then $(x_i) \in \{(d_i)\lambda | \lambda \in F, d_i \in F_d \text{ for all } i \in I\}$. Suppose that $(x_i) \neq 0$. Then there is $i_0 \in \{1, \ldots, n\}$ such that $x_{i_0} \neq 0$. Let $d_i = x_i x_{i_0}^{-1}$ for $i \in \{1, \ldots, n\}$. Then $(d_i) = (x_i)\zeta^{-1}(x_{i_0}^{-1})$. Since $(x_i) \in Q(V)$ and $Q(V)$ is closed under scalar multiplication, $(d_i) \in Q(V)$. Then for all $\alpha, \beta \in F$ there is $\gamma \in F$ such that $(d_i)\alpha + (d_i)\beta = (d_i)\gamma$. It follows that $d_i\zeta(\alpha) + d_i\zeta(\beta) = d_i\zeta(\gamma)$ for all $i \in \{1 \ldots, n\}$. Since $\zeta$ is an automorphism, there are $\alpha_1, \beta_1 \in F$ such that $\alpha = \zeta^{-1}(\alpha_1)$ and $\beta = \zeta^{-1}(\beta_1)$. So $d_i\alpha_1 + d_i\beta_1 = d_i\zeta(\gamma)$ for all $i \in \{1 \ldots, n\}$. But $d_{i_0} = 1$. So $\alpha_1 + \beta_1 = \zeta(\gamma)$. Hence $d_i\alpha_1 + d_i\beta_1 = d_i(\alpha_1 + \beta_1)$, and this is verified for all $\alpha_1, \beta_1 \in F$ because $\zeta$ is an automorphism of $(F, \cdot)$. Therefore $d_i$ is distributive and $Q(V) = \{(d_i)\lambda | \lambda \in F, d_i \in F_d \text{ for all } i \in I\}$. □

In fact, if we take $V = F^n$, $n \in \mathbb{N}$ with $F$ a near-field and denote the near-vector space in the above theorem by $(V, F')$, we can show that

**Proposition 2.** *The near-vector spaces $(V, F)$ and $(V, F')$ are isomorphic for the same nearfield $F$.*

*Proof.* Using Definition 2, consider $\theta$ the identity isomorphism and $\eta : (F^*, \cdot) \to (F^*, \cdot)$ the mapping $\eta(\alpha) = \zeta^{-1}(\alpha)$. Since $\zeta$ is an isomorphism, $\zeta^{-1}$ exists and $\eta$ is an isomorphism. Thus $\theta(x\alpha) = \theta(x)\eta(\alpha)$ for all $x \in V$ and $\alpha \in F^*$. □

Finally, since by Theorem 2 isomorphisms preserve regularity, we have that

**Lemma 3.** *$(V, F')$ is a regular near-vector space.*

We end off the section with an example,

*Example* 1. Consider the field $(GF(3^2), +, \cdot)$ with

$$GF(3^2) := \{0, 1, 2, \gamma, 1+\gamma, 2+\gamma, 2\gamma, 1+2\gamma, 2+2\gamma\},$$

where $\gamma$ is a zero of $x^2 + 1 \in \mathbb{Z}_3[x]$.
The operations on $GF(3^2)$ can be defined as follows:

$$+ : (a + b\gamma) + (c + d\gamma) = (a + c)\text{mod}3 \ + ((b + d)\text{mod}3 \ )\gamma,$$

| $\cdot$ | $0$ | $1$ | $2$ | $\gamma$ | $1+\gamma$ | $2+\gamma$ | $2\gamma$ | $1+2\gamma$ | $2+2\gamma$ |
|---|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $1$ | $0$ | $1$ | $2$ | $\gamma$ | $1+\gamma$ | $2+\gamma$ | $2\gamma$ | $1+2\gamma$ | $2+2\gamma$ |
| $2$ | $0$ | $2$ | $1$ | $2\gamma$ | $2+2\gamma$ | $1+2\gamma$ | $\gamma$ | $2+\gamma$ | $1+\gamma$ |
| $\gamma$ | $0$ | $\gamma$ | $2\gamma$ | $2$ | $2+\gamma$ | $2+2\gamma$ | $1$ | $1+\gamma$ | $1+2\gamma$ |
| $1+\gamma$ | $0$ | $1+\gamma$ | $2+2\gamma$ | $2+\gamma$ | $2\gamma$ | $1$ | $1+2\gamma$ | $2$ | $\gamma$ |
| $2+\gamma$ | $0$ | $2+\gamma$ | $1+2\gamma$ | $2+2\gamma$ | $1$ | $\gamma$ | $1+\gamma$ | $2\gamma$ | $2$ |
| $2\gamma$ | $0$ | $2\gamma$ | $\gamma$ | $1$ | $1+2\gamma$ | $1+\gamma$ | $2$ | $2+2\gamma$ | $2+\gamma$ |
| $1+2\gamma$ | $0$ | $1+2\gamma$ | $2+\gamma$ | $1+\gamma$ | $2$ | $2\gamma$ | $2+2\gamma$ | $\gamma$ | $1$ |
| $2+2\gamma$ | $0$ | $2+2\gamma$ | $1+\gamma$ | $1+2\gamma$ | $\gamma$ | $2$ | $2+\gamma$ | $1$ | $2\gamma$ |

In [11], p.257, it is observed that $(GF(3^2), +, \circ)$, with

$$x \circ y := \begin{cases} x \cdot y & \text{if } y \text{ is a square in } (GF(3^2), +, \cdot) \\ x^3 \cdot y & \text{otherwise} \end{cases}$$

is a (right) nearfield, but not a field.

| $\circ$ | $0$ | $1$ | $2$ | $\gamma$ | $1+\gamma$ | $2+\gamma$ | $2\gamma$ | $1+2\gamma$ | $2+2\gamma$ |
|---|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $1$ | $0$ | $1$ | $2$ | $\gamma$ | $1+\gamma$ | $2+\gamma$ | $2\gamma$ | $1+2\gamma$ | $2+2\gamma$ |
| $2$ | $0$ | $2$ | $1$ | $2\gamma$ | $2+2\gamma$ | $1+2\gamma$ | $\gamma$ | $2+\gamma$ | $1+\gamma$ |
| $\gamma$ | $0$ | $\gamma$ | $2\gamma$ | $2$ | $1+2\gamma$ | $1+\gamma$ | $1$ | $2+2\gamma$ | $2+\gamma$ |
| $1+\gamma$ | $0$ | $1+\gamma$ | $2+2\gamma$ | $2+\gamma$ | $2$ | $2\gamma$ | $1+2\gamma$ | $\gamma$ | $1$ |
| $2+\gamma$ | $0$ | $2+\gamma$ | $1+2\gamma$ | $2+2\gamma$ | $\gamma$ | $2$ | $1+\gamma$ | $1$ | $2\gamma$ |
| $2\gamma$ | $0$ | $2\gamma$ | $\gamma$ | $1$ | $2+\gamma$ | $2+2\gamma$ | $2$ | $1+\gamma$ | $1+2\gamma$ |
| $1+2\gamma$ | $0$ | $1+2\gamma$ | $2+\gamma$ | $1+\gamma$ | $2\gamma$ | $1$ | $2+2\gamma$ | $2$ | $\gamma$ |
| $2+2\gamma$ | $0$ | $2+2\gamma$ | $1+\gamma$ | $1+2\gamma$ | $1$ | $\gamma$ | $2+\gamma$ | $2\gamma$ | $2$ |

It is not difficult to see that the the distributive elements of $(GF(3^2), +, \circ)$, denoted by $(GF(3^2), +, \circ)_d$ are the elements $0, 1, 2$. Consider the near-field $F = (GF(3^2, +, \circ)$, put $V = F^3$ with $\alpha \in F$ acting as an endomorphism of $V$ by defining $(x_1, x_2, x_3)\alpha = (x_1\alpha^3, x_2\alpha^3, x_3\alpha^3)$. Thus we have that $Q(V) = \{(d_1, d_2, d_3)\lambda | \lambda \in F, d_i \in \{0, 1, 2\}\}$ and this near-vector space is regular.

### 2.3. *Nearrings of quotients*

The concept of nearrings of quotients was first defined by Maxson [9] and he stated conditions for a nearring to have a nearring of quotients. Graves and Malone [5] subsequently generalised this to the case of nearrings of quotients with respect to a multiplicative set.

We consider the case where $N$ is a non-commutative nearring and begin with the basic results as found in [5, 11]. Corollary 1 and 2 are new results that we use to give alternative proofs to the known results of Theorem 4 and Theorem 5 (See [11]). For more on nearrings we refer the reader to [10, 11].

**Definition 3** ([11] Definition 6.3, p.26)**.** Let $N$ be a nearring and $S$ a subsemigroup of $(N, \cdot)$. A nearring $N_s$ is called a nearring of right quotients of $N$ with respect to $S$ if

(1) $N_s$ is a nearring with identity,
(2) $N$ is embeddable in $N_s$, by a homomorphism, say $h$,
(3) $\forall s \in S$, $h(s)$ is invertible in $(N_s, \cdot)$,
(4) for all $q \in N_s$, there exists $s \in S$ and $n \in N$ such that $q = h(n)h(s)^{-1}$.

We can also define a nearring of left quotients of $N$ with respect to $S$, which has the same definition as above, except property 4 which becomes :

4'. for all $q \in N_s$, there exists $s \in S$ and $n \in N$ such that $q = h(s)^{-1}h(n)$.

It is not difficult to see that any element of $N_s$ can be written as $h(n)h(s)^{-1}$, for $s \in S$ and $n \in N$.

**Definition 4** ([11] Definition 6.4, p.26)**.** Let $N$ be a nearring and $S$ a subsemigroup of $(N,\cdot)$. $N$ is said to fulfill the right Ore condition with respect to $S$, if for all $(n,s) \in N \times S$, there exists $(n_1,s_1) \in N \times S$ such that $ns_1 = sn_1$. Likewise, $N$ is said to fulfill the left Ore condition with respect to $S$ if for all $(n,s) \in N \times S$, there exists $(n_1,s_1) \in N \times S$ such that $s_1 n = n_1 s$.

If $N$ is a nearring and $S$, a subsemigroup of $(N,\cdot)$ and if for all $s \in S$, s is cancellable (both sides) and $N$ satisfies the right Ore condition with respect to $S$, then the relation $\sim$ defined on $N \times S$ by

$$(n,s) \sim (n',s') \text{ if } \exists\,(n_1,s_1) \in N \text{ such that } ss_1 = s'n_1 \text{ implies that } ns_1 = n'n_1$$

is an equivalence relation. Moreover, on the equivalence class of $(n,s) \in N \times S$, denoted by $\frac{n}{s}$, we define the operations "$+$" and "$\cdot$" by

$$\frac{n}{s} + \frac{n'}{s'} = \frac{ns_1 + n'n_1}{ss_1} \quad \text{and}$$
$$\frac{n}{s} \cdot \frac{n'}{s'} = \frac{nn_2}{s's_2},$$

where $(n_1,s_1) \in N \times S$ and $(n_2,s_2) \in N \times S$ fulfill $s'n_1 = ss_1$ and $n's_2 = sn_2$. These operations are well-defined and $(N_s = N \times S/\sim, +, \cdot)$ is a nearring of right quotients of $N$ with respect to $S$.

We prove a new result:

**Corollary 1.** *Let $N_s$ be a nearring of right quotients of $N$ with respect to $S$. Then $h(n)h(s)^{-1} = h(n')h(s')^{-1}$ if and only if $(n,s) \sim (n',s')$ for all $(n,s)$ and $(n',s')$ in $N \times S$.*

*Proof.* Let $(n,s),(n',s') \in N \times S$ such that $(n,s) \sim (n',s')$. Then there exists $(n_1,s_1) \in N \times S$ such that $ss_1 = s'n_1$ implies $ns_1 = n'n_1$. It follows that

$$h(s)h(s_1) = h(s')h(n_1)$$
$$h(n)h(s_1) = h(n')h(n_1).$$

So, since for all $s \in S$, $h(s)$ is invertible in $N_s$,

$$h(n_1) = h(s')^{-1}h(s)h(s_1)$$
$$h(n) = h(n')h(n_1)h(s_1)^{-1}.$$

Therefore

$$\begin{aligned} h(n) &= h(n')h(s')^{-1}h(s)h(s_1)h(s_1)^{-1} \\ &= h(n')h(s')^{-1}h(s). \end{aligned}$$

Thus $h(n)h(s)^{-1} = h(n')h(s')^{-1}$.

Now we show the converse. Let $(n,s),(n',s') \in N \times S$ such that $h(n)h(s)^{-1} = h(n')h(s')^{-1}$. Then there exists $(n_1,s_1) \in N \times S$ such that $h(s')^{-1}h(s) = h(n_1)h(s_1)^{-1}$, by property 4 in the definition of a nearring of right quotients with respect to $S$. So $h(ss_1) = h(s'n_1)$. Since $h$ is a monomorphism, $ss_1 = s'n_1$. Also, since $h(s)h(s_1) = h(s')h(n_1)$, $h(s')^{-1}h(s)h(s_1) = h(n_1)$. So $h(s')^{-1} = h(n_1)h(s_1)^{-1}h(s)^{-1}$. Using the fact that $h(n)h(s)^{-1} = h(n')h(s')^{-1}$, we get $h(n)h(s)^{-1} = h(n')h(n_1)h(s_1)^{-1}h(s)^{-1}$. So $h(n)h(s_1) = h(n')h(n_1)$. Therefore $h(ns_1) = h(n'n_1)$ and $ns_1 = n'n_1$, since $h$ is a monomorphism. Thus $(n,s) \sim (n',s')$. □

The use of this corollary allows us to give an alternate proof to that found in [11], of the following theorem:

**Theorem 4** ([11] Theorem 1.65, p.27)**.** *Let $N$ be a nearring and $S$ a subsemigroup of $(N,\cdot)$. $N$ has a nearring of right quotients with respect to $S$ is equivalent to*

(1) *for all $s \in S$, $s$ is cancellable (on both sides),*
(2) *$N$ satisfies the right Ore condition with respect to $S$.*

*Proof.* Let $S$ be a subsemigroup of $(N,\cdot)$. Suppose that $N$ has a nearring $N_s$ of right quotients with respect to $S$. So $N$ is embeddable in $N_s$, by a homomorphism $h$. Since $S$ is a subsemigroup, $S \neq \varnothing$. Let $s \in S$ and $n,n' \in N$, such that $n's = ns$. Then $h(n')h(s) = h(n)h(s)$. Since $h(s)$ is invertible in $(N_s,\cdot)$, $h(n') = h(n)$. It follows that $n' = n$, since $h$ is a monomorphism. Also if $sn' = sn$, we have $n' = n$. Thus for all $s \in S$, $s$ is cancellable on both sides. Now, let $n \in N$ and $s \in S$. Then $h(s)^{-1}h(n) \in N_s$. So by property 4 of Definition 3, there exists $(s_1,n_1) \in S \times N$, such that $h(s)^{-1}h(n) = h(n_1)h(s_1)^{-1}$. Hence $h(ns_1) = h(sn_1)$, and $ns_1 = sn_1$. Therefore $N$ satisfies the right Ore condition with respect to $S$.

To show the converse, we suppose that $S$ is not empty, for all $s \in S$, $s$ is cancellable (on both sides) and that $N$ satisfies the right Ore condition with respect to $S$. So from the discussion following Definition 4 there exists a nearring of right quotients of $N$ with respect to $S$, namely $(N \times S/\sim,+,\cdot)$. □

*Remark* 1. We note that there is a printing error in the statement of the above theorem in [11](Theorem 1.65, p.27), the left Ore condition should be replaced with the right Ore condition.

*Example* 2. Let us consider the nearring $(M(\mathbb{R}),+,\circ)$. An element $f \in M(\mathbb{R})$ is cancellable if and only if $f$ is bijective. In fact if $f$ is bijective, then $f$ is cancellable. So let us suppose $f$ is cancellable. That implies that for all $g,g' \in M(\mathbb{R})$, $f \circ g = f \circ g'$ implies $g = g'$, also $g \circ f = g' \circ f$ implies $g = g'$. It is not difficult to see that $f \circ g = f \circ g'$ implies $g = g'$, implies that $f$ is injective. Also if $g \circ f = g' \circ f$ implies $g = g'$, then $f$ is surjective.

So now let us consider $N = (\mathbb{R}[x], +, \circ) \subset M(\mathbb{R})$, the set of polynomials defined from $\mathbb{R}^+$, and $S$ the set of all monomials in $N$ with co-domain in $\mathbb{R}^+$ (in other words monomials of the form $ax^n$, with $a \geqslant 0$ and $n \in \mathbb{N}$). Since each monomial is defined on $\mathbb{R}^+$, every element of $S$ is bijective, then cancellable. Also the composition of two monomials is a monomial. Thus $S$ is a subsemigroup of $(N, \circ)$. Moreover for $(f, g) \in N \times S$, $f \circ id = g \circ (g^{-1} \circ f)$. But $(g^{-1} \circ f, id) \in N \times S$, so $N$ satisfies the right Ore condition with respect to $S$. Hence $N$ has a nearring of right quotients with respect to $S$.

The quotient is a set $(N_s, +, \circ)$, where $N_s$ is the set of all summations of all power functions $f(x) = x^\alpha$, with $\alpha \in \mathbb{Q}^+$ and $f$ defined from $\mathbb{R}^+$ to $\mathbb{R}$.

We prove a new corollary:

**Corollary 2.** *Let $N$ be a nearring and $S$ a subsemigroup of $(N, \cdot)$. If $N$ has a nearring of right quotients with respect to $S$, $N_s$, then there exists $(n_1, s_1) \in N \times S$ such that $h(n)h(s)^{-1} + h(n')h(s')^{-1} = h(ns_1 + n'n_1)h(ss_1)^{-1}$, where $ss_1 = s'n_1$ and $h$ is the embedding homomorphism from $N$ to $N_s$. Also, there exists $(n_2, s_2) \in N \times S$ such that $h(n)h(s)^{-1}h(n')h(s')^{-1} = h(nn_2)h(s's_2)^{-1}$, where $n's_2 = sn_2$.*

*Proof.* Since $N$ has a nearring of right quotients with respect to $S$, $N$ satisfies the right Ore condition with respect to $S$. So there exists $(n_1, s_1) \in N \times S$ such that $ss_1 = s'n_1$. Then $h(s')^{-1} = h(n_1)h(s_1)^{-1}h(s)^{-1}$. We have

$$h(ns_1 + n'n_1)h(ss_1)^{-1} = h(n)h(s_1)h(s_1)^{-1}h(s)^{-1} + h(n')h(n_1)h(s_1)^{-1}h(s)^{-1}.$$

Hence $h(ns_1 + n'n_1)h(ss_1)^{-1} = h(n)h(s)^{-1} + h(n')h(s')^{-1}$.
Also, since $(n', s) \in N \times S$, there exists $(n_2, s_2) \in N \times S$ such that $n's_2 = sn_2$, by the right Ore condition. So $h(n_2) = h(s)^{-1}h(n')h(s_2)$.
We have

$$\begin{aligned} h(nn_2)h(s's_2)^{-1} &= h(n)h(n_2)h(s_2)^{-1}h(s')^{-1} \\ &= h(n)h(s)^{-1}h(n')h(s_2)h(s_2)^{-1}h(s')^{-1}. \end{aligned}$$

Hence $h(nn_2)h(s's_2)^{-1} = h(n)h(s)^{-1}h(n')h(s')^{-1}$. □

We will use the above corollary to give an alternate proof of the following result found in [11]:

**Theorem 5** ([11] Theorem 1.66, p.28)**.** *Let $N$ be a nearring. If $N_s$ and $N'_s$ are nearrings of right quotients with respect to $S$, then*

$$N_s \cong N'_s.$$

*Proof.* Since $N_s$ and $N'_s$ are nearrings of right quotients, there exist monomorphisms $h$ and $h'$, from $N$ to $N_s$ and $N$ to $N'_s$, respectively. Let us define the mapping $f$

by

$$f : N_s \to N'_s$$
$$h(n)h(s)^{-1} \mapsto h'(n)h'(s)^{-1}.$$

The mapping $f$ is well-defined. To show this suppose that $h(n)h(s)^{-1} = h(n')h(s')^{-1}$. Then by Corollary 1 $(n,s) \sim (n',s')$. Also by the same Corollary 1 $h'(n)h'(s)^{-1} = h'(n')h'(s')^{-1}$. So $f(h(n)h(s)^{-1}) = f(h(n')h(s')^{-1})$. Thus $f$ is well-defined. Let $h(n)h(s)^{-1}, h(n')h(s')^{-1} \in N_s$. Then by Corollary 2 we have

$$\begin{aligned} f(h(n)h(s)^{-1} + h(n')h(s')^{-1}) &= f(h(ns_1 + n'n_1)h(ss_1)^{-1}) \ (n_1,s_1) \in N \times S \\ \text{fulfilling } s'n_1 = ss_1 & \\ &= h'(ns_1 + n'n_1)h'(ss_1)^{-1} \\ &= h'(n)h'(s)^{-1} + h'(n')h'(s')^{-1} \text{ since } s'n_1 = ss_1 \\ &= f(h(n)h(s)^{-1}) + f(h(n')h(s')^{-1}). \end{aligned}$$

Also by Corollary 2 again, we have

$$\begin{aligned} f(h(n)h(s)^{-1}h(n')h(s')^{-1}) &= f(h(nn_2)h(s's_2)^{-1}), \text{ where } (n_2,s_2) \in N \times S \\ \text{fulfills } n's_2 = sn_2 & \\ &= h'(nn_2)h'(s's_2)^{-1} \\ &= h'(n)h'(s)^{-1}h'(n')h'(s')^{-1}, \text{ since } n's_2 = sn_2 \\ &= f(h(n)h(s)^{-1})f(h(n')h(s')^{-1}). \end{aligned}$$

Thus $f$ is a homomorphism. Now, let $h(n)h(s)^{-1}, h(n')h(s')^{-1} \in N_s$ such that $h'(n)h'(s)^{-1} = h'(n')h'(s')^{-1}$. Then $(s,n) \sim (s',n')$. So $h(n)h(s)^{-1} = h(n')h(s')^{-1}$. Hence $f$ is injective. Let $h'(n)h'(s)^{-1} \in N'_s$. For $(n,s) \in N \times S$, $h(n)h(s)^{-1} \in N_s$ and $f(h(n)h(s)^{-1}) = h'(n)h'(s)^{-1}$. Hence $f$ is a surjection. Therefore $f$ is a bijection. Thus $N_s \cong N'_s$. □

Theorem 5 allow us to speak of the nearring of right quotients, $N_s$, with respect to $S$, for a particular nearring.

2.4. *Integral nearrings and near domains*

Recall that a nearring $(N,+,\cdot)$ is said to be *integral* if it has no non-zero divisors of zero and

**Definition 5.** ([5] Definition 1.4, p.34) A near domain is a nearring $N$ that satisfies the right Ore condition and the left cancellation law.

Then we have the following:

**Theorem 6.** *Let $N$ be a nearring and $S$ a subsemigroup of $(N,\cdot)$. Suppose that $N$ has a nearring of right quotients with respect to $S$, $N_s$. If $N$ is integral, then $N_s$ is integral.*

*Proof.* Let $h(n)h(s)^{-1}, h(n')h(s')^{-1} \in N_s$ with $h$ the monomorphism from $N$ to $N_s$. We have

$$h(n)h(s)^{-1}h(n')h(s')^{-1} = h(nn_2)h(s's_2)^{-1},$$
$$\text{with } sn_2 = n's_2 \text{ for } (n_2,s_2) \in N \times S.$$

Suppose that $h(nn_2)h(s's_2) = 0$. Then $h(nn_2)h(s's_2)^{-1} = h(0)h(t)^{-1}$, for some $t \in S$. This implies that $(nn_2, s's_2) \sim (0,t)$. It follows that there exits $(n_1,s_1) \in N \times S$ such that $nn_2s_2 = 0$. $s_2 \neq 0$ because $h(s_2)$ is invertible. Hence $nn_2 = 0$, since $N$ is integral. Moreover, since $N$ is integral, $n = 0$ or $n_2 = 0$. $n_2 = 0$ implies that $n's_2 = 0$ and so $n' = 0$. Therefore $n = 0$ or $n' = 0$. Thus $h(n)(s)^{-1} = 0$ or $h(n')h(s')^{-1} = 0$. Thus $N_s$ is integral. □

**Proposition 3.** *Let $N$ be a near domain. Then*

(1) $0n = n0 = 0$, *for all* $n \in N$
(2) *$N$ is integral*
(3) *$N$ satisfies the right cancellation law.*

*Proof.* Let $N$ be a near domain and $n, n_1, n_2 \in N$.

(1) It is straightforward to show that $0n = 0$. We have $(n0)(n0) = n(0n)0 = n0$. So $(n0)(n0)0 = (n0)0$. Using the left cancellation law we have $n0 = 0$.
(2) Suppose $n_1n_2 = 0$. If $n_1 \neq 0$ we have $n_1n_2 = n_10$. It follows that $n_2 = 0$ from the left cancellation law.
(3) If $n_1n = n_2n$, with $n \neq 0$, then $(n_1 - n_2)n = 0$. Hence by 2. $n_1 = n_2$.

□

**Corollary 3.** *Let $N$ be a near domain. Let $S$ be the set of all cancellable elements of $N$. Then $N_s$ is a nearfield.*

*Proof.*
Since $N$ is a near domain, every non-zero element is cancellable. So $S = N - \{0\}$. We know from Definition 3 that $h(s)$ is invertible for every $s \in S$. Let $q \in N_s$. Then there exist $(n,s) \in N \times S$ such that $q = h(n)h(s)^{-1}$. Suppose $q \neq 0$. Then $n \neq 0$. So $n \in S$. Hence $h(n)$ is invertible. Therefore $h(n)h(s)^{-1}$ is invertible and its inverse is $h(s)h(n)^{-1}$. □

2.5. *An application to Near-vector spaces*

In this last section we use the results of the previous sections to give a first application of the theory of nearrings of quotients. We construct a new class of near-vector spaces from near domains and completely describe the quasi-kernel.

If $N$ is a nearfield and $S = N - \{0\}$, then $N_s$ is a nearfield. In fact, we can show that $N \simeq N_s$. From the definition of a nearring of quotients we know that there is an embedding map $h$ defined by

$$h : N \to N_s$$
$$n \mapsto \frac{n}{1}$$

with 1 the multiplicative identity of $N$. We just have to show that $h$ is surjective. Let $(n,s) \in N \times S$. We have that $\frac{n}{s} = \frac{ns^{-1}}{1}$. So $h(ns^{-1}) = \frac{n}{s}$. Hence $h$ is an isomorphism and so $N \simeq N_s$. Therefore $q = h(n) \in N_s$ is distributive if and only if $n$ is distributive.

Thus if $N$ is a nearfield then

$$N_s \oplus \ldots \oplus N_s \simeq N \oplus \ldots \oplus N$$

and the study of constructions of the form $N \oplus \ldots \oplus N$ has been discussed in [6].

We now look at the case where $N$ is a near domain, not necessarily a nearfield. Let us consider $N_s$ (with identity 1) with $S$ the set of all cancellable elements. We take $V = N_s \bigoplus \ldots \bigoplus N_s$ with the scalar multiplication defined for $(x_1, \ldots, x_n) \in V$ and $\alpha \in N_s$ by

$$(x_1, \ldots, x_n)\alpha = (x_1\alpha, \ldots, x_n\alpha).$$

We now look at the quasi-kernel $Q(V)$. We know from [6] that the quasi-kernel $Q(V) = \mathcal{V}_1 \cup \ldots \cup \mathcal{V}_n$, where $\mathcal{V}_i = \{(d_1, \ldots, d_{i-1}, 1, d_{i+1}, \ldots, d_n)N_s | d_i \in N_{sd}\}$, with $N_{sd}$ representing the distributive elements of $N_s$. Thus in order to describe $Q(V)$, we need to find the elements of $N_{sd}$.

**Theorem 7.** *$N_{sd} = \{h(n)h(s)^{-1} |$ if $\exists a,b,c \in N$, such that $sa + sb = sc$, then $na + nb = nc$ for $s \in S$, $n \in N\}$*

*Proof.*
Let $q = h(n)h(s)^{-1}, q_1 = h(n_1)h(s_1)^{-1}, q_2 = h(n_2)h(s_2)^{-1} \in N_s$. First suppose that $q$ is distributive and that there are $a,b,c \in N$ such that $sa + sb = sc$. We prove that $na + nb = nc$. We have

$$h(n)h(s)^{-1}(h(sa + sb)) = h(n)h(s)^{-1}(h(sc)).$$

But

$$\begin{aligned} h(n)h(s)^{-1}(h(sa + sb)) &= h(n)h(s)^{-1}(h(s)h(a) + h(s)h(b)) \\ &= h(n)h(a) + h(n)h(b), \text{ since } h(n)h(s)^{-1} \text{ is distributive.} \\ &= h(na + nb), \end{aligned}$$

and

$$h(n)h(s)^{-1}(h(sc)) = h(nc).$$

Hence, since $h$ is injective, $na + nb = nc$. To show the converse, suppose that if there exist $a,b,c \in N$ such that $sa + sb = sc$, then $na + nb = nc$. We have to prove

that $h(n)h(s)^{-1}$ is distributive in $N_s$. So we have to show $q(q_1+q_2)=qq_1+qq_2$. We have

$$\begin{aligned} q(q_1+q_2) &= h(n)h(s)^{-1}h(n_1s^*+n_2n^*)h(s_1s^*)^{-1} \text{ where } s_1s^*=s_2n^* \\ &= h(nn_1^*)h(s_1s^*s_1^*)^{-1} \text{ where } sn_1^*=(n_1s^*+n_2n^*)s_1^*, \end{aligned}$$

and

$$\begin{aligned} qq_1+qq_2 &= h(nn^{*'})h(s_1s^{*'})^{-1}+h(nn_1^{*'})h(s_2s_1^{*'})^{-1} \\ &\quad \text{where } sn^{*'}=n_1s^{*'}, sn_1^{*'}=n_2s_1^{*'} \\ &= h(nn^{*'}s_2^{*'}+nn_1^{*'}n_2^{*'})h(s_1s^{*'}s_2^{*'})^{-1} \text{ where } s_1s^{*'}s_2^{*'}=s_2s_1^{*'}n_2^{*'}. \end{aligned}$$

To show that $h(nn_1^*)h(s_1s^*s_1^*)^{-1}=h(nn^{*'}s_2^{*'}+nn_1^{*'}n_2^{*'})h(s_1s^{*'}s_2^{*'})^{-1}$, we have to find $(n_*^*,s_*^*)\in N\times S$ such that $s_1s^*s_1^*s_*^*=s_1s^{*'}s_2^{*'}n_*^*$ implies $nn_1^*s_*^*=(nn^{*'}s_2^{*'}+nn_1^{*'}n_2^{*'})n_*^*$. Since $N_s$ is a nearring of right quotients with respect to $S$, we have the right Ore condition with respect to $S$. Thus, since $s_1s^*s_1^*\in N, s_1s^{*'}s_2^{*'}\in S$, there exist $(n_*^*,s_*^*)\in N\times S$ such that $s_1s^*s_1^*s_*^*=s_1s^{*'}s_2^{*'}n_*^*$. But $s_1s^*s_1^*s_*^*=s_1s^{*'}s_2^{*'}n_*^*$ implies $s_2n^*s_1^*s_*^*=s_2s_1^{*'}n_2^{*'}n_*^*$, because $s_1s^{*'}s_2^{*'}=s_2s_1^{*'}n_2^{*'}$ and $s_1s^*=s_2n^*$. So we get $s^*s_1^*s_*^*=s^{*'}s_2^{*'}n_*^*$ and $n^*s_1^*s_*^*=s_1^{*'}n_2^{*'}n_*^*$, since $s_1$ and $s_2$ are cancellable. Also we have

$$\begin{aligned} sn^{*'}s_2^{*'}n_*^* &= n_1s^{*'}s_2^{*'}n_*^*, \text{ since } sn^{*'}=n_1s^{*'} \\ &= n_1s^*s_1^*s_*^*, \text{ since } s^*s_1^*s_*^*=s^{*'}s_2^{*'}n_*^* \end{aligned}$$

and

$$\begin{aligned} sn_1^{*'}n_2^{*'}n_*^* &= n_2s_1^{*'}n_2^{*'}n_*^*, \text{ since } sn_1^{*'}=n_2s_1^{*'} \\ &= n_2n^*s_1^*s_*^*, \text{ since } n^*s_1^*s_*^*=s_1^{*'}n_2^{*'}n_*^*. \end{aligned}$$

So we get

$$\begin{aligned} sn^{*'}s_2^{*'}n_*^*+sn_1^{*'}n_2^{*'}n_*^* &= n_1s^*s_1^*s_*^*+n_2n^*s_1^*s_*^* \\ &= (n_1s^*+n_2n^*)s_1^*s_*^* \\ &= sn_1^*s_*^*, \text{ since } sn_1^*=(n_1s^*+n_2n^*)s_1^*. \end{aligned}$$

Hence $s_1s^*s_1^*s_*^*=s_1s^{*'}s_2^{*'}n_*^*$ implies $sn^{*'}s_2^{*'}n_*^*+sn_1^{*'}n_2^{*'}n_*^*=sn_1^*s_*^*$. If we take $a=n^{*'}s_2^{*'}n_*^*, b=n_1^{*'}n_2^{*'}n_*^*, c=n_1^*s_*^*$, from our assumption we have $na+nb=nc$. So $nn^{*'}s_2^{*'}n_*^*+nn_1^{*'}n_2^{*'}n_*^*=nn_1^*s_*^*$. Therefore $h(nn_1^*)h(s_1s^*s_1^*)^{-1}=h(nn^{*'}s_2^{*'}+nn_1^{*'}n_2^{*'})h(s_1s^{*'}s_2^{*'})^{-1}$. Thus $h(n)h(s)^{-1}$ is distributive. □

Using Theorem 7 we can describe the quasi-kernel $Q(V)$ of the near vector space $V$ defined above just by considering the elements of $N$. So we can construct a near vector space over a nearfield from a near domain.

In closing we can now describe $Q(V)$ for the above near-vector space.

**Corollary 4.** *Let us consider the near vector space $V$ defined above, and let $N_1 = \{(n,s) \in N \times N^* |$ if $\exists a,b,c \in N$, such that $sa + sb = sc$, then $na + nb = nc$ for $s \in S$, $n \in N\}$. Then we have*

$$Q = \mathcal{V}_1 \cup \ldots \cup \mathcal{V}_n,$$

*where*

$$\mathcal{V}_i = \{(d_1, \ldots, d_{i-1}, 1, d_{i+1}, \ldots, d_n)N_s | d_i = h(n_i)h(s_i)^{-1}, (n,s) \in N_1\}.$$

Moreover, by Lemma 3.5 [6], this near-vector space is regular.

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# AN IDENTITY WITH DERIVATIONS IN PRIME RINGS

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*Abstract.* Let $R$ be a prime ring with center $Z(R)$, and $d$ a derivation of $R$. Suppose that $(d[x,y]_k)^n - m[x,y]_k \in Z(R)$ for all $x, y \in R$, where $m \neq n, k \geq 1$ are fixed integers. Then $d = 0$ or $R$ satisfies $s_4$, the standard identity in four variables. In the case $(d[x,y]_k)^n - m[x,y]_k = 0$ for all $x, y \in R$, then $d = 0$ or $R$ is commutative.

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## 1. Introduction

In all that follows, unless stated otherwise, $R$ will be an associative ring, $Z(R)$ the center of $R$, $Q$ its Martindale quotient ring and Utumi quotient ring $U$. The center of $Q$, denoted by $C$, is called the extended centroid of $R$ (we refer the reader to [3] for related symbols). For any $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ stand for the commutator $xy - yx$ and anti-commutator $xy + yx$, respectively. Recall that a ring $R$ is prime if for any $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$ and is semiprime if for any $a \in R$, $aRa = \{0\}$ implies $a = 0$. By a derivation on $R$ we mean an additive mapping $d : R \longrightarrow R$ such that $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In particular $d$ is called an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$. For any $x, y \in R$, we set $[x, y]_1 = [x, y] = xy - yx$, and $[x, y]_k = [[x, y]_{k-1}, y]$, where $k > 1$ is an integer. Note that $[x, y]_k = \sum_{i=0}^{k}(-x)^i yx^{k-i}$ and $d([x, y]_k) = [d(x), y]_k + \sum_{i=1}^{k}[[[x, y]_{i-1}, d(y)], y]_{k-i}$.

Many results in literature indicate that global structure of a ring $R$ is often lightly connected to the behavior of additive mappings defined on $R$. The first classic result on this topic is due to Divinsky [7] who proved that a simple Artinian ring is commutative if it has a commuting non-identity automorphism. Over the last few decades, a number of authors have investigated the relationship between the commutativity of

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the ring $R$ and certain specific types of derivations of $R$. In [2], Ashraf and Rehman proved that if $R$ is a prime ring, $I$ a nonzero ideal of $R$ and $d$ is a derivation of $R$ such that $d(x \circ y) = x \circ y$ for all $x, y \in I$, then $R$ is commutative. In [1], Argaç and Inceboz generalized the above result as following: Let $R$ be a prime ring, $I$ a nonzero ideal of $R$ and $n$ a fixed positive integer, if $R$ admits a derivation $d$ with the property $(d(x \circ y))^n = x \circ y$ for all $x, y \in I$, then $R$ is commutative. On the other hand, Daif and Bell [6] showed that if in a semiprime ring $R$ there exists a nonzero ideal $I$ of $R$ and a derivation $d$ such that $d([x, y]) = [x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. In particular, if $I = R$ then $R$ is commutative. Motivated by the above-cited results, our purpose in this article is to obtain some information of the prime ring $R$ involving a central identity $(d[x, y]_k)^n - m[x, y]_k \in Z(R)$ for all $x, y \in R$, where $m, n, k \geq 1$ are fixed integers.

The standard identity $s_4$ in four variables is defined as follows:

$$s_4 = \sum (-1)^\tau X_{\tau(1)} X_{\tau(2)} X_{\tau(3)} X_{\tau(4)}$$

where $(-1)^\tau$ is the sign of a permutation $\tau$ of the symmetric group of degree 4. As is well known, prime rings satisfying $s_4$ can be characterized by the following:

**Fact 1** ([4])**.** Let $R$ be a prime ring with the extended centroid $C$. Then the following are equivalent:

(1) $dim_C RC \leq 4$;

(2) $R$ satisfies $s_4$;

(3) $R$ is commutative or $R$ embeds in $M_2(F)$;

(4) $R$ is algebraic of bounded degree 2 over $C$;

(5) $R$ satisfies $[[x^2, y], [x, y]]$.

## 2. Results

We begin with the following lemmas which are crucial for proving our main results.

**Lemma 1.** *Let $R = M_2(F)$ be the ring of all $2\times 2$ matrices over a field $F$. If $0 \neq a \in R$ such that*

$$([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n = m[x,y]_k$$

*for all $x \in R$ where $m,n,k \geq 1$ are fixed integers, then $a \in Z(R)$.*

*Proof.* Let $a = \sum_{i,j=0}^{2} a_{ij}e_{ij}$ with $a_{ij} \in F$, where $e_{ij}$ is the usual matrix unit with 1 in $(i,j)$-entry and zero elsewhere. Let $x = e_{12}$, $y = e_{11}$. Then $m[x,y]_k = 0$ and $([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n = (-1)^{kn}(-e_{12}a + e_{11}ae_{12})^n$. By assumption we have $(-1)^{kn}(-e_{12}a + e_{11}ae_{12})^n = 0$. Right multiplying by $e_{12}$, it yields that $(-1)^{(k+1)n}a_{21}^n e_{12} = 0$, which implies $a_{21} = 0$. Similarly, we have $a_{12} = 0$. Thus $a$ must be a diagonal matrix. Now set $a = \sum_t a_{tt}e_{tt}$, with $a_{tt} \in F$. Let $\varphi$ be the inner automorphism of $R$ given by $\varphi(x) = (1+e_{ij})a(1-e_{ij})$. Thus $([[a^\varphi,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a^\varphi,y]],y]_{k-i})^n = m[x,y]_k$ for all $x,y \in R$. By above argument,

$$a^\varphi = (1+e_{ij})a(1-e_{ij}) = \Sigma_{i=1}^{k} a_{ii}e_{ii} + (a_{jj} - a_{ii})e_{ij}$$

must be diagonal. Therefore $a_{jj} = a_{ii}$ and so $a \in Z(R)$. □

**Lemma 2.** *Let $R$ be a non-commutative prime ring with center $Z(R)$. If $0 \neq a \in R$ such that*

$$([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n = m[x,y]_k$$

*for all $x,y \in R$, where $n,m,k \geq 1$ are fixed integers, then $a \in Z(R)$.*

*Proof.* By assumption, $R$ satisfies the generalized polynomial identity

$$p(x,y) = ([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n - m[x,y]_k.$$

By Chuang [5], this generalized polynomial identity (GPI) is also satisfied by $U$. If $a \notin C$ then $p(x,y) = 0$ is a nontrivial (GPI) for $U$. In case $C$ is infinite, we have $p(x,y) = 0$ for all $x,y \in U \bigotimes_C \overline{C}$ where $\overline{C}$ is the algebraic closure of $C$. Since both $U$ and $U \bigotimes_C \overline{C}$ are prime and centrally closed [8], we may replace $R$ by $U$ or $U \bigotimes_C \overline{C}$ according to $C$ finite or infinite. Thus we may assume that $R$ is centrally closed over $C$ which is either finite or algebraically closed and $p(x,y) = 0$ for all $x,y \in R$. By Martindale's theorem [13], $R$ is then a primitive ring having nonzero $\mathrm{soc}(R)$ with $C$ as the associated division ring. Hence by Jacobson's theorem [9], $R$ is isomorphic to a dense ring of linear transformations of a vector space $V$ over $C$. If $\dim_C V = k$, then the density of $R$ on $V$ implies that $R \cong M_k(C)$, where $k = \dim_C V$. Since $R$ is noncommutative, $k \geq 2$. If $\dim_C V = 2$, then by Lemma 1 we

have $a \in Z(R)$. Now suppose that $\dim_C V \geq 3$. We want to show that for any $v \in V$, $v$ and $av$ are linearly $C$-dependent. Suppose on contrary that $v$ and $av$ are linearly $C$-independent for some $v \in V$. Since $\dim_C V \geq 3$, there exists $w \in V$ such that $\{v, av, w\}$ are linearly $C$-independent set of vectors. By the density of $R$ on $V$, there exist $x, y \in R$ such that $xv = v, xw = v + w, xav = w; yv = v, yw = 0, yav = v$. Then $0 = p(x,y)v = (-1)^{kn}v$, a contradiction. Therefore $v$, $av$ are linearly $C$-dependent for any $v \in V$. Hence we can write $av = v\alpha_v$ for all $v \in V$ and $\alpha_v \in C$. Then by a standard argument, it is very easy to prove that $\alpha_v$ is independent of the choice of $v \in V$. In fact, Since $dim_C V \geq 3$, then there exist $u, v, w$ which are linearly independent, and so $\alpha_u, \alpha_v, \alpha_w \in C$ such that $au = u\alpha_u$, $av = v\alpha_v$, $aw = w\alpha_w$, that is $a(u+v+w) = u\alpha_u + v\alpha_v + w\alpha_w$. Moreover $a(u+v+w) = (u+v+w)\alpha_{u+v+w}$ for some $\alpha_{u+v+w} \in C$. Then $0 = (\alpha_{u+v+w} - \alpha_u)u + (\alpha_{u+v+w} - \alpha_v)v + (\alpha_{u+v+w} - \alpha_w)w$ and hence $\alpha_u = \alpha_v = \alpha_w = \alpha_{u+v+w}$, that is $\alpha$ does not depend on the choice of $v$. Thus we can write $av = v\alpha$ for all $v \in V$ and $\alpha \in C$ fixed. Now, let $r \in R$, $v \in V$. Since $av = v\alpha$, we have $[a,r]v = (ar)v - (ra)v = a(rv) - r(av) = (rv)\alpha - r(v\alpha) = 0$, that is $[a,r]V = 0$. Hence $[a,r] = 0$ for all $r \in R$, implying $a \in Z(R)$. □

**Theorem 1.** *Let $R$ be a prime ring and $d$ a derivation of $R$. Suppose that $(d[x,y]_k)^n = m[x,y]_k$ for all $x, y \in R$, where $m \neq n, k \geq 1$ are fixed integers. Then $d = 0$ or $R$ is commutative.*

*Proof.* Using the identity $d([x,y]_k) = [d(x), y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1}, d(y)], y]_{k-i}$ and the hypothesis, we have

$$([d(x), y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1}, d(y)], y]_{k-i})^n = m[x,y]_k$$

for all $x, y \in R$. Assume first that $d$ is $Q$-inner, that is, $d(x) = [a,x]$ for all $x \in Q$, where $a$ is a non-central element in $Q$. Then

$$([[a,x], y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1}, [a,y]], y]_{k-i})^n = m[x,y]_k$$

for all $x, y \in U$. Thus by Lemma 2, $a \in Z(R)$ that gives $d = 0$.

Assume next that $d$ is $Q$-outer. Applying Kharchenko's theorem [10], we get $([v,y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1}, u], y]_{k-i})^n = m[x,y]_k$ for all $x, y, u, v \in U$. In particular, for $u = 0$ and $v = x$ we have $([x,y]_k)^n = m[x,y]_k$, for all $x, y \in U$. Note that, this is a polynomial identity and hence there exists a field $F$ such that $R \subseteq M_t(F)$, the ring of $t \times t$ matrices over a field $F$, where $t \geq 1$. By Chuang [5], this generalized polynomial identity (GPI) is also satisfied by $R$ as well. Moreover, $R$ and $M_t(F)$ satisfy the same polynomial identity [11, Lemma 1], that is, $([x,y]_k)^n = m[x,y]_k$ for all $x, y \in M_t(F)$. But by choosing $x = e_{12}$, $y = e_{11}$, we get $0 = ([x,y]_k)^n - m[x,y]_k = (-1)^k(e_{12}^n - me_{12})$. This is a contradiction, ending the proof.

□

**Lemma 3.** *Let $R = M_t(F)$ be the ring of all $t \times t$ matrices over a field $F$ with $t \geq 3$. If $0 \neq a \in R$ and $m, n, k \geq 1$ are fixed integers, such that $([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n - m[x,y]_k \in F \cdot I_t$, for all $x, y \in R$, then $a \in F \cdot I_t$.*

*Proof.* We are given that

$$[([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n - m[x,y]_k, z] = 0$$

for all $x, y, z \in R$. Let $a = (a_{ij})_{t\times t}$. By choosing $x = e_{ij}, y = e_{ii}$ and $z = e_{ik}$ for any $i \neq j \neq k$, we have

$$[(-1)^{kn}[a, e_{ij}]^n - (-1)^k m e_{ij}, e_{ik}] = (-1)^{kn}((e_{ij}a)^n e_{ik} - e_{ik}(ae_{ij})^n) = 0.$$

Thus $a_{ij} = 0$, and so $a$ is a diagonal matrix. Using the same technique in Lemma 1, we get $a \in F \cdot I_t$, proving the lemma. □

**Theorem 2.** *Let $R$ be a prime ring with center $Z(R)$, and $d$ a derivation of $R$. Suppose that $(d[x,y]_k)^n - m[x,y]_k \in Z(R)$ for all $x, y \in R$, where $m \neq n, k \geq 1$ are fixed integers. Then $d = 0$ or $R$ satisfies $s_4$, the standard identity in four variables.*

*Proof.* By assumption $R$ satisfies the generalized differential identity

$$0 = [(d[x,y]_k)^n - m[x,y]_k, w] = [[d(x),y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1}, d(y)],y]_{k-i} - m[x,y]_k, w]$$

for $x, y, w \in R$. By Lee [12], $R$ and $U$ satisfy the same differential identities we may assume that above identity is also satisfied by $U$. Now we consider the following two cases:

*Case 1.* Suppose that $d$ is a $Q$-outer derivation. Then Kharchenko's theorem [10], we have

$$[([v,y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},u],y]_{k-i})^n - m[x,y]_k, w] = 0,$$

for all $x, y, u, v, w \in U$. This is a polynomial identity and hence there exists a field $F$ such that $U \subseteq M_t(F)$ with $t > 1$ and $U$, $M_t(F)$ satisfy the same polynomial identity [11]. If $t \geq 3$ then by choosing $w = e_{13}$, $v = x = e_{12}$, $y = e_{11}$, $u = 0$, we get $0 = [([v,y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},u],y]_{k-i})^n - m[x,y]_k, w] = (-1)^k e_{13}$. This is a contradiction. Thus $t = 2$ and so $R$ satisfies $s_4$ by Fact 1.

*Case 2*. Suppose that $d$ is a $Q$-inner derivation. In this case there exists $a \in Q$ such that $d(x) = [a, x]$ for all $x \in R$. Then we have

$$[([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n - m[x,y]_k, w] = 0,$$

for all $x, y, w \in R$. By localizing $R$ at $Z(R)$ it follows that

$$([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n - m[x,y]_k \in Z(R_Z),$$

for all $x, y \in R_Z$. Since $R$ and $R_Z$ satisfy the same polynomial identities, in order to prove that $R$ satisfies $s_4$, we may assume that $R$ is simple with 1. Hence, $([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n - m[x,y]_k \in Z(R)$ for all $x, y \in R$. Therefore $R$ satisfies a generalized polynomial identity and it is simple with 1, which implies that $Q = RC = R$ and $R$ has a minimal right ideal, whose commuting ring $D$ is a division ring which is finite dimensional over $Z(R)$. However, since $R$ is a simple ring with 1, $R$ must be Artinian, that is, $R = D_s$, the $s \times s$ matrices over $D$, for some $s \geq 1$. By [11] there exists a field $F$ such that $R \subseteq M_t(F)$, the ring of $t \times t$ matrices over field $F$, with $t > 1$, and

$$([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n - m[x,y]_k \in Z(M_t(F)) = F \cdot I_t,$$

for all $x, y \in M_t(F)$. If $t \geq 3$, then by Lemma 3, $a \in F \cdot I_t$ and so $d = 0$. If $t = 2$, then $R$ satisfies $s_4$. □

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# UNIQUENESS OF SOLUTION AND FULLY DISCRETE SCHEME TO NONLINEAR INTEGRO-DIFFERENTIAL AVERAGED MODEL WITH SOURCE TERMS

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*Abstract.* Uniqueness of solution and finite difference scheme of corresponding initial-boundary value problem for one nonlinear partial integro-differential averaged model with source terms are studied. Mentioned model is based on Maxwell system which describes electromagnetic field penetration into a substance. Mixed boundary condition is considered. Large time behavior of solution is fixed too. Convergence of the fully discrete scheme is proved. Wider class of nonlinearity is studied than one has been investigated before.

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*Keywords:* Maxwell equations, system of nonlinear integro-differential equations, uniqueness, asymptotic behavior, finite difference scheme

## 1. Introduction

Investigating many applied problems in nature we are facing to nonlinear integro-differential models which contain derivatives of several variables. Numerous publications deal with the study of integro-differential equations of various kinds (see, for example, the bibliography in [6], [8]). Integro-differential models arise for example at the mathematical simulation of process of a magnetic field penetration into a medium whose electro-conductivity depends on temperature. Numerous works are dedicated to the investigation of Maxwell equations describing above-mentioned process (see, for example, [12], [19], [20] and references therein). In a quasi-stationary case the corresponding system of Maxwell equations [12] can be rewritten in the following form [5]:

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t |rotH|^2\,d\tau\right)rotH\right], \tag{1.1}$$

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where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field and function $a = a(S)$ is defined for $S \in [0, \infty)$.

If vector of magnetic field has the form $H = (0, U, V)$, where $U = U(x,t)$, $V = V(x,t)$, then from (1.1) we get the following system of nonlinear parabolic integro-differential equations:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\frac{\partial U}{\partial x}\right], \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\frac{\partial V}{\partial x}\right], \tag{1.2}$$

where

$$S(x,t) = \int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau. \tag{1.3}$$

Study of the models of type (1.1) have begun in [5]. In that work, in particular, based on Galerkin modified method and compactness arguments as in [17] for nonlinear parabolic equations the theorems of existence of solution of the initial-boundary value problem with first kind boundary conditions for scalar and one-dimensional space case when $a(S) = 1 + S$ and uniqueness for more general cases are proven. One-dimensional scalar variant for the case $a(S) = (1 + S)^p$, $0 < p \leq 1$ is studied in [3]. Investigations for multi-dimensional space cases at first was carried out in [4]. Multidimensional space cases are also discussed in [1], [13]. Asymptotic behavior as $t \to \infty$ of solutions of initial-boundary value problems for (1.1) type models are studied in [8], [9], [10], [15], [23] and in a number of other works as well. In those works main attention is paid to one-dimensional cases. Finite element analogues and Galerkin method algorithm as well as settling of semi-discrete and finite difference schemes for (1.1) type one-dimensional integro-differential models are studied in [2], [7], [8], [9], [15], [18], [22], [23], [24] and in the other works as well for the linear case of diffusion coefficient, i.e. $a(S) = 1 + S$.

Some generalization of the system of type (1.1) is proposed by Prof. G. I. Laptev [14]. In particular, in some physical assumptions, the process of penetration of the magnetic field into a material is modeled by so-called averaged integro-differential model, the (1.2), (1.3) type analog of which have the following form:

$$\frac{\partial U}{\partial t} = a(S)\frac{\partial^2 U}{\partial x^2}, \quad \frac{\partial V}{\partial t} = a(S)\frac{\partial^2 V}{\partial x^2}, \tag{1.4}$$

where

$$S(t) = \int_0^t \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx\, d\tau. \tag{1.5}$$

The literature on the questions of existence, uniqueness, regularity, asymptotic behavior of the solutions and numerical resolution of the initial-boundary value problems to the (1.2), (1.3) and (1.4), (1.5) type models and models like them is very rich (see, for example, [5], [6], [7], [8], [14], [15], [16], [18] and references therein). The

asymptotic behavior and numerical solution of two-dimensional case for the (1.4), (1.5) type averaged integro-differential system is considered for example in [11], [24].

Our aim is to investigate system of nonlinear integro-differential equations (1.4), (1.5) with source terms. Uniqueness and large time behavior of solution of initial-boundary value problem with mixed boundary condition as well as convergence of corresponding fully discrete scheme is studied. In the present note we consider new class of nonlinearity considering more general cases of the diffusion coefficient $a = a(S)$.

## 2. Uniqueness and asymptotic behavior of solution as $t \to \infty$

In the cylinder $[0,1] \times [0,\infty)$ let us consider the following initial-boundary value problem:

$$\begin{aligned} &\frac{\partial U}{\partial t} - a\left(\int_0^t \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx\,d\tau\right) \frac{\partial^2 U}{\partial x^2} + g(U) = f_1(x,t), \\ &\frac{\partial V}{\partial t} - a\left(\int_0^t \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx\,d\tau\right) \frac{\partial^2 V}{\partial x^2} + g(V) = f_2(x,t), \end{aligned} \tag{2.1}$$

$$U(0,t) = V(0,t) = \left.\frac{\partial U(x,t)}{\partial x}\right|_{x=1} = \left.\frac{\partial V(x,t)}{\partial x}\right|_{x=1} = 0, \tag{2.2}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \tag{2.3}$$

where $a = a(S)$, $g$, $f_1$, $f_2$, $U_0$ and $V_0$ are given functions of their arguments.

The following statement takes place.

**Theorem 1.** *If $a = a(S) \geq a_0 = Const > 0$, $a'(S) \geq 0$, $a''(S) \leq 0$, $g$ is monotonically increased function, $U_0, V_0 \in H^1(0,1)$, $U_0(0) = V_0(0) = \left.\frac{dU_0(x)}{dx}\right|_{x=1} = \left.\frac{dV_0(x)}{dx}\right|_{x=1} = 0$, $f_1$, $f_2$, $\frac{\partial f_1}{\partial x}$, $\frac{\partial f_2}{\partial x} \in L_2(Q)$ and problem (2.1) - (2.3) has a solution then it is unique and exponential stabilization of solution as $t \to \infty$ takes place.*

Here we use usual $L_2$ and Sobolev $H^1$ spaces.

To prove the uniqueness of solution we assume that there exist two different $(\overline{U}, \overline{V})$ and $(\overline{\overline{U}}, \overline{\overline{V}})$ solutions of problem (2.1) - (2.3) and introduce the differences $Z = \overline{\overline{U}} - \overline{U}$ and $W = \overline{\overline{V}} - \overline{V}$. To show that $Z = W \equiv 0$ the methodology of proving the convergence theorem, which is given in the next section, monotone growth feature of function $g$ and the following identity is mainly used:

$$\left\{ a\left(\int_0^t \int_0^1 \left[\left(\frac{\partial \overline{\overline{U}}}{\partial x}\right)^2 + \left(\frac{\partial \overline{\overline{V}}}{\partial x}\right)^2\right] dx\,d\tau\right) \frac{\partial \overline{\overline{U}}}{\partial x}\right.$$

$$-a\left(\int_0^t\int_0^1\left[\left(\frac{\partial\overline{U}}{\partial x}\right)^2+\left(\frac{\partial\overline{V}}{\partial x}\right)^2\right]dxd\tau\right)\frac{\partial\overline{U}}{\partial x}\Bigg\}\left(\frac{\partial\overline{\overline{U}}}{\partial x}-\frac{\partial\overline{U}}{\partial x}\right)$$

$$+\Bigg\{a\left(\int_0^t\int_0^1\left[\left(\frac{\partial\overline{\overline{U}}}{\partial x}\right)^2+\left(\frac{\partial\overline{\overline{V}}}{\partial x}\right)^2\right]dxd\tau\right)\frac{\partial\overline{\overline{V}}}{\partial x}$$

$$-a\left(\int_0^t\int_0^1\left[\left(\frac{\partial\overline{U}}{\partial x}\right)^2+\left(\frac{\partial\overline{V}}{\partial x}\right)^2\right]dxd\tau\right)\frac{\partial\overline{V}}{\partial x}\Bigg\}\left(\frac{\partial\overline{\overline{V}}}{\partial x}-\frac{\partial\overline{V}}{\partial x}\right)$$

$$=\int_0^1\frac{d}{d\mu}a\left(\int_0^t\int_0^1\left\{\left[\frac{\partial\overline{U}}{\partial x}+\mu\left(\frac{\partial\overline{\overline{U}}}{\partial x}-\frac{\partial\overline{U}}{\partial x}\right)\right]^2+\left[\frac{\partial\overline{V}}{\partial x}+\mu\left(\frac{\partial\overline{\overline{V}}}{\partial x}-\frac{\partial\overline{U}}{\partial x}\right)\right]^2\right\}dxd\tau\right)$$

$$\times\left[\frac{\partial\overline{U}}{\partial x}+\mu\left(\frac{\partial\overline{\overline{U}}}{\partial x}-\frac{\partial\overline{U}}{\partial x}\right)\right]d\mu\left(\frac{\partial\overline{\overline{U}}}{\partial x}-\frac{\partial\overline{U}}{\partial x}\right)$$

$$+\int_0^1\frac{d}{d\mu}a\left(\int_0^t\int_0^1\left\{\left[\frac{\partial\overline{U}}{\partial x}+\mu\left(\frac{\partial\overline{\overline{U}}}{\partial x}-\frac{\partial\overline{U}}{\partial x}\right)\right]^2+\left[\frac{\partial\overline{V}}{\partial x}+\mu\left(\frac{\partial\overline{\overline{V}}}{\partial x}-\frac{\partial\overline{U}}{\partial x}\right)\right]^2\right\}dxd\tau\right)$$

$$\times\left[\frac{\partial\overline{V}}{\partial x}+\mu\left(\frac{\partial\overline{\overline{V}}}{\partial x}-\frac{\partial\overline{V}}{\partial x}\right)\right]d\mu\left(\frac{\partial\overline{\overline{V}}}{\partial x}-\frac{\partial\overline{V}}{\partial x}\right).$$

For obtaining stabilization of solution stated in the Theorem 1 the method of a-priori estimates based on analogical methodology given in [7] is used and large time behavior of solution is obtained.

## 3. Convergence of the fully discrete scheme

In the rectangle $Q_T=[0,1]\times[0,T]$, where $T$ is a positive constant, let us consider again problem (2.1) - (2.3). On $Q_T$ let us introduce a net with mesh points denoted by $(x_i,t_j)=(ih,j\tau)$, where $i=0,1,...,M$; $j=0,1,...,N$ with $h=1/M$, $\tau=T/N$. The initial line is denoted by $j=0$. The discrete approximation at $(x_i,t_j)$ is designed by $(u_i^j,v_i^j)$ and the exact solution to the problem (2.1) - (2.3) by $(U_i^j,V_i^j)$. We will use the following known notations [21] of forward and backward derivatives:

$$r_{x,i}^j=\frac{r_{i+1}^j-r_i^j}{h},\quad r_{\bar{x},i}^j=\frac{r_i^j-r_{i-1}^j}{h},\quad r_{t,i}^j=\frac{r_i^{j+1}-r_i^j}{\tau}$$

and inner products and norms:

$$(r^j,y^j)=h\sum_{i=1}^{M-1}r_i^jy_i^j,\quad (r^j,y^j]=h\sum_{i=1}^{M}r_i^jy_i^j,$$

$$\|r^j\|=(r^j,r^j)^{1/2},\quad \|r^j]|=(r^j,r^j]^{1/2}.$$

For problem (2.1) - (2.3) let us consider the following finite difference scheme:

$$\begin{aligned}
&\frac{u_i^{j+1}-u_i^j}{\tau}-a\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\left[(u_{\bar{x},\ell}^k)^2+(v_{\bar{x},\ell}^k)^2\right]\right)u_{\bar{x}x,i}^{j+1}+g(u_i^{j+1})=f_{1,i}^j,\\
&\frac{v_i^{j+1}-v_i^j}{\tau}-a\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\left[(u_{\bar{x},\ell}^k)^2+(v_{\bar{x},\ell}^k)^2\right]\right)v_{\bar{x}x,i}^{j+1}+g(v_i^{j+1})=f_{2,i}^j,\\
&\qquad\qquad i=1,2,...,M-1;\ \ j=0,1,...,N-1,
\end{aligned}\tag{3.1}$$

$$u_0^j=v_0^j=u_{\bar{x},M}^j=v_{\bar{x},M}^j=0,\quad j=0,1,...,N,\tag{3.2}$$

$$u_i^0=U_{0,i},\ v_i^0=V_{0,i},\ \ i=0,1,...,M.\tag{3.3}$$

Multiplying equations in (3.1) scalarly by $u_i^{j+1}$ and $v_i^{j+1}$ respectively, it is not difficult to get the inequalities:

$$\|u^n\|^2+\sum_{j=1}^{n}\|u_{\bar{x}}^j]|^2\tau<C,\quad \|v^n\|^2+\sum_{j=1}^{n}\|v_{\bar{x}}^j]|^2\tau<C,\quad n=1,2,...,N,\tag{3.4}$$

where here and below $C$ is a positive constant independent from $\tau$ and $h$.

The a priori estimates (3.4) guarantee the stability of the scheme (3.1) - (3.3). Note, that applying the technique as we prove convergence theorem blow, it is not difficult to prove the uniqueness of the solution of the scheme (3.1) - (3.3) too.

The main statement of the present section can be stated as follows.

**Theorem 2.** *If $a=a(S)\geq a_0=Const>0$, $a'(S)\geq 0$, $a''(S)\leq 0$, $g$ is monotonically increased function and problem (2.1) - (2.3) has a sufficiently smooth solution $(U(x,t),V(x,t))$, then the solution $u^j=(u_1^j,u_2^j,\dots,u_{M-1}^j)$, $v^j=(v_1^j,v_2^j,\dots,v_{M-1}^j)$, $j=1,2,\dots,N$ of the difference scheme (3.1) - (3.3) tends to the solution of continuous problem (2.1) - (2.3) $U^j=(U_1^j,U_2^j,\dots,U_{M-1}^j)$, $V^j=(V_1^j,V_2^j,\dots,V_{M-1}^j)$, $j=1,2,\dots,N$ as $\tau\to 0$, $h\to 0$ and the following estimates are true:*

$$\|u^j-U^j\|\leq C(\tau+h),\quad \|v^j-V^j\|\leq C(\tau+h).\tag{3.5}$$

*Proof.* Introducing the differences $z_i^j = u_i^j - U_i^j$ and $w_i^j = v_i^j - V_i^j$ we get the following relations:

$$\begin{aligned} z_{t,i}^{j+1} - \Bigg\{ a\left(\tau h \sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big[(u_{\bar{x},\ell}^k)^2 + (v_{\bar{x},\ell}^k)^2\Big]\right) u_{\bar{x},i}^{j+1} \\ - a\left(\tau h \sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big[(U_{\bar{x},\ell}^k)^2 + (V_{\bar{x},\ell}^k)^2\Big]\right) U_{\bar{x},i}^{j+1} \Bigg\}_x \\ + g(u_i^{j+1}) - g(U_i^{j+1}) = -\psi_{1,i}^j, \\ w_{t,i}^{j+1} - \Bigg\{ a\left(\tau h \sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big[(u_{\bar{x},\ell}^k)^2 + (v_{\bar{x},\ell}^k)^2\Big]\right) v_{\bar{x},i}^{j+1} \\ - a\left(\tau h \sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big[(U_{\bar{x},\ell}^k)^2 + (V_{\bar{x},\ell}^k)^2\Big]\right) V_{\bar{x},i}^{j+1} \Bigg\}_x \\ + g(v_i^{j+1}) - g(V_i^{j+1}) = -\psi_{2,i}^j, \end{aligned} \tag{3.6}$$

$$z_0^j = w_0^j = z_{\bar{x},M}^j = w_{\bar{x},M}^j = 0, \tag{3.7}$$

$$z_i^0 = w_i^0 = 0, \tag{3.8}$$

where $\psi_{1,i}^j$ and $\psi_{2,i}^j$ are approximation errors of scheme (3.1) and

$$\psi_{k,i}^j = O(\tau + h), \quad k = 1, 2.$$

Multiplying the first equation of system (3.6) scalarly by the grid function $\tau z^{j+1} = \tau(z_1^{j+1}, z_2^{j+1}, \dots, z_{M-1}^{j+1})$ and using the boundary conditions (3.7) we get

$$\begin{aligned} \|z^{j+1}\|^2 - (z^{j+1}, z^j) + \tau h \sum_{i=1}^{M} \Bigg\{ a\left(\tau h \sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big[(u_{\bar{x},\ell}^k)^2 + (v_{\bar{x},\ell}^k)^2\Big]\right) u_{\bar{x},i}^{j+1} \\ - a\left(\tau h \sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big[(U_{\bar{x},\ell}^k)^2 + (V_{\bar{x},\ell}^k)^2\Big]\right) U_{\bar{x},i}^{j+1} \Bigg\} z_{\bar{x},i}^{j+1} \\ + \Big(g(u_i^{j+1}) - g(U_i^{j+1}), u^{j+1} - U^{j+1}\Big) = -\tau(\psi_1^j, z^{j+1}). \end{aligned}$$

Analogously,

$$\|w^{j+1}\|^2-(w^{j+1},w^j)+\tau h\sum_{i=1}^{M}\left\{a\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\left[(u^k_{\bar{x},\ell})^2+(v^k_{\bar{x},\ell})^2\right]\right)v^{j+1}_{\bar{x},i}\right.$$
$$\left.-a\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\left[(U^k_{\bar{x},\ell})^2+(V^k_{\bar{x},\ell})^2\right]\right)V^{j+1}_{\bar{x},i}\right\}w^{j+1}_{\bar{x},i}$$
$$+\left(g(v_i^{j+1})-g(V_i^{j+1}),\,v^{j+1}-V^{j+1}\right)=-\tau(\psi_2^j,w^{j+1}).$$

Adding these two equalities and taking into account monotonicity of the function $g$, from these two equalities we have

$$\begin{aligned}
&\|z^{j+1}\|^2-(z^{j+1},z^j)+\|w^{j+1}\|^2-(w^{j+1},w^j)\\
&+\tau h\sum_{i=1}^{M}\left\{a\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\left[(u^k_{\bar{x},\ell})^2+(v^k_{\bar{x},\ell})^2\right]\right)u^{j+1}_{\bar{x},i}\right.\\
&\left.-a\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\left[(U^k_{\bar{x},\ell})^2+(V^k_{\bar{x},\ell})^2\right]\right)U^{j+1}_{\bar{x},i}\right\}z^{j+1}_{\bar{x},i}\\
&+\tau h\sum_{i=1}^{M}\left\{a\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\left[(u^k_{\bar{x},\ell})^2+(v^k_{\bar{x},\ell})^2\right]\right)v^{j+1}_{\bar{x},i}\right.\\
&\left.-a\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\left[(U^k_{\bar{x},\ell})^2+(V^k_{\bar{x},\ell})^2\right]\right)V^{j+1}_{\bar{x},i}\right\}w^{j+1}_{\bar{x},i}\\
&\le-\tau(\psi_1^j,z^{j+1})-\tau(\psi_2^j,w^{j+1}).
\end{aligned}\tag{3.9}$$

Note that, using the Hadamard formula

$$\varphi(y)-\varphi(z)=\int_0^1\frac{d}{d\mu}\varphi[z+\mu(y-z)]d\mu,$$

below we prove one of the main inequality to estimate terms with nonlinear diffusion coefficient $a(S)$

$$\left\{a\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\left[(u^k_{\bar{x},\ell})^2+(v^k_{\bar{x},\ell})^2\right]\right)u^{j+1}_{\bar{x},i}\right.$$

$$
\begin{aligned}
&-a\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big[(U^k_{\bar{x},\ell})^2+(V^k_{\bar{x},\ell})^2\Big]\right)U^{j+1}_{\bar{x},i}\Big\}\left(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i}\right)\\
&\qquad+\Big\{a\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big[(u^k_{\bar{x},\ell})^2+(v^k_{\bar{x},\ell})^2\Big]\right)v^{j+1}_{\bar{x},i}\\
&-a\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big[(U^k_{\bar{x},\ell})^2+(V^k_{\bar{x},\ell})^2\Big]\right)V^{j+1}_{\bar{x},i}\Big\}\left(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i}\right)
\end{aligned}
$$

$$
\begin{aligned}
=&\int_0^1\Bigg\{\frac{d}{d\mu}a\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big\{\Big[U^k_{\bar{x},\ell}+\mu(u^k_{\bar{x},\ell}-U^k_{\bar{x},\ell})\Big]^2+\Big[V^k_{\bar{x},\ell}+\mu(v^k_{\bar{x},\ell}-V^k_{\bar{x},\ell})\Big]^2\Big\}\right)\\
&\qquad\times\Big[U^{j+1}_{\bar{x},i}+\mu(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i})\Big]\Bigg\}\,d\mu\left(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i}\right)\\
&+\int_0^1\Bigg\{\frac{d}{d\mu}a\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big\{\Big[U^k_{\bar{x},\ell}+\mu(u^k_{\bar{x},\ell}-U^k_{\bar{x},\ell})\Big]^2+\Big[V^k_{\bar{x},\ell}+\mu(v^k_{\bar{x},\ell}-V^k_{\bar{x},\ell})\Big]^2\Big\}\right)\\
&\qquad\times\Big[V^{j+1}_{\bar{x},i}+\mu(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i})\Big]\Bigg\}\,d\mu\left(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i}\right)
\end{aligned}
$$

$$
\begin{aligned}
=&\,2\int_0^1 a'\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big\{\Big[U^k_{\bar{x},\ell}+\mu(u^k_{\bar{x},\ell}-U^k_{\bar{x},\ell})\Big]^2+\Big[V^k_{\bar{x},\ell}+\mu(v^k_{\bar{x},\ell}-V^k_{\bar{x},\ell})\Big]^2\Big\}\right)\\
&\qquad\times\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big\{\Big[U^k_{\bar{x},\ell}+\mu(u^k_{\bar{x},\ell}-U^k_{\bar{x},\ell})\Big]\left(u^k_{\bar{x},\ell}-U^k_{\bar{x},\ell}\right)\\
&\qquad\qquad+\Big[V^k_{\bar{x},\ell}+\mu(v^k_{\bar{x},\ell}-V^k_{\bar{x},\ell})\Big]\left(v^k_{\bar{x},\ell}-V^k_{\bar{x},\ell}\right)\Big\}\\
&\qquad\times\Big[U^{j+1}_{\bar{x},i}+\mu(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i})\Big]\,d\mu\left(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i}\right)\\
&+\int_0^1 a\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big\{\Big[U^k_{\bar{x},\ell}+\mu(u^k_{\bar{x},\ell}-U^k_{\bar{x},i})\Big]^2+\Big[V^k_{\bar{x},\ell}+\mu(v^k_{\bar{x},\ell}-V^k_{\bar{x},\ell})\Big]^2\Big\}\right)\\
&\qquad\times\left(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i}\right)d\mu\left(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i}\right)\\
&+2\int_0^1 a'\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big\{\Big[U^k_{\bar{x},\ell}+\mu(u^k_{\bar{x},\ell}-U^k_{\bar{x},\ell})\Big]^2+\Big[V^k_{\bar{x},\ell}+\mu(v^k_{\bar{x},\ell}-V^k_{\bar{x},\ell})\Big]^2\Big\}\right)\\
&\qquad\times\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big\{\Big[U^k_{\bar{x},\ell}+\mu(u^k_{\bar{x},\ell}-U^k_{\bar{x},\ell})\Big]\left(u^k_{\bar{x},\ell}-U^k_{\bar{x},\ell}\right)\\
&\qquad\qquad+\Big[V^k_{\bar{x},\ell}+\mu(v^k_{\bar{x},\ell}-V^k_{\bar{x},\ell})\Big]\left(v^k_{\bar{x},\ell}-V^k_{\bar{x},\ell}\right)\Big\}
\end{aligned}
$$

$$\times\left[V_{\bar{x},i}^{j+1}+\mu(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1})\right]d\mu\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)$$

$$+\int_0^1 a\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\left\{\left[U_{\bar{x},\ell}^{k}+\mu(u_{\bar{x},\ell}^{k}-U_{\bar{x},\ell}^{k})\right]^2+\left[V_{\bar{x},\ell}^{k}+\mu(v_{\bar{x},\ell}^{k}-V_{\bar{x},\ell}^{k})\right]^2\right\}\right)$$

$$\times\left(v_{\bar{x},\ell}^{j+1}-V_{\bar{x},i}^{j+1}\right)d\mu\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)$$

$$=2\int_0^1 a'\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\left\{\left[U_{\bar{x},\ell}^{k}+\mu(u_{\bar{x},\ell}^{k}-U_{\bar{x},\ell}^{k})\right]^2+\left[V_{\bar{x},\ell}^{k}+\mu(v_{\bar{x},\ell}^{k}-V_{\bar{x},\ell}^{k})\right]^2\right\}\right)$$

$$\times\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\left\{\left[U_{\bar{x},\ell}^{k}+\mu(u_{\bar{x},\ell}^{k}-U_{\bar{x},\ell}^{k})\right]\left(u_{\bar{x},\ell}^{k}-U_{\bar{x},\ell}^{k}\right)\right.$$

$$\left.+\left[V_{\bar{x},\ell}^{k}+\mu(v_{\bar{x},\ell}^{k}-V_{\bar{x},\ell}^{k})\right]\left(v_{\bar{x},\ell}^{k}-V_{\bar{x},\ell}^{k}\right)\right\}$$

$$\times\left\{\left[U_{\bar{x},i}^{j+1}+\mu(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1})\right]\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)\right.$$

$$\left.+\left[V_{\bar{x},i}^{j+1}+\mu(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1})\right]\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)\right\}d\mu$$

$$+\int_0^1 a\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\left\{\left[U_{\bar{x},\ell}^{k}+\mu(u_{\bar{x},\ell}^{k}-U_{\bar{x},i}^{k})\right]^2+\left[V_{\bar{x},\ell}^{k}+\mu(v_{\bar{x},\ell}^{k}-V_{\bar{x},\ell}^{k})\right]^2\right\}\right)$$

$$\times\left[\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)^2+\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)^2\right]d\mu$$

$$=2\int_0^1 a'\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\left\{\left[U_{\bar{x},\ell}^{k}+\mu(u_{\bar{x},\ell}^{k}-U_{\bar{x},\ell}^{k})\right]^2+\left[V_{\bar{x},\ell}^{k}+\mu(v_{\bar{x},\ell}^{k}-V_{\bar{x},\ell}^{k})\right]^2\right\}\right)\xi^{j+1}(\mu)\xi_t^{j}(\mu)\,d\mu$$

$$+\int_0^1 a\left(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\left\{\left[U_{\bar{x},\ell}^{k}+\mu(u_{\bar{x},\ell}^{k}-U_{\bar{x},\ell}^{k})\right]^2+\left[V_{\bar{x},\ell}^{k}+\mu(v_{\bar{x},\ell}^{k}-V_{\bar{x},\ell}^{k})\right]^2\right\}\right)$$

$$\times\left[\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)^2+\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)^2\right]d\mu,$$

where

$$\xi^{j+1}(\mu)=\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\left\{\left[U_{\bar{x},\ell}^{k}+\mu(u_{\bar{x},\ell}^{k}-U_{\bar{x},\ell}^{k})\right]\left(u_{\bar{x},\ell}^{k}-U_{\bar{x},\ell}^{k}\right)\right.$$

$$\left.+\left[V_{\bar{x},\ell}^{k}+\mu(v_{\bar{x},\ell}^{k}-V_{\bar{x},\ell}^{k})\right]\left(v_{\bar{x},\ell}^{k}-V_{\bar{x},\ell}^{k}\right)\right\},$$

$$\xi^{0}(\mu)=0,$$

and therefore,

$$\xi_t^j(\mu) = h\sum_{\ell=1}^{M}\Big[U_{\bar{x},\ell}^{j+1} + \mu(u_{\bar{x},\ell}^{j+1} - U_{\bar{x},\ell}^{j+1})\Big]\Big(u_{\bar{x},\ell}^{j+1} - U_{\bar{x},\ell}^{j+1}\Big)$$

$$+\Big[V_{\bar{x},\ell}^{j+1} + \mu(v_{\bar{x},\ell}^{j+1} - V_{\bar{x},\ell}^{j+1})\Big]\Big(v_{\bar{x},\ell}^{j+1} - V_{\bar{x},\ell}^{j+1}\Big).$$

Introducing the following notation

$$s^{j+1}(\mu) = \tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big\{\Big[U_{\bar{x},\ell}^{k} + \mu(u_{\bar{x},\ell}^{k} - U_{\bar{x},\ell}^{k})\Big]^2 + \Big[V_{\bar{x},\ell}^{k} + \mu(v_{\bar{x},\ell}^{k} - V_{\bar{x},\ell}^{k})\Big]^2\Big\},$$

from the previous equality we have

$$\Bigg\{a\Bigg(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big[(u_{\bar{x},\ell}^{k})^2 + (v_{\bar{x},\ell}^{k})^2\Big]\Bigg)u_{\bar{x},i}^{j+1}$$

$$-a\Bigg(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big[(U_{\bar{x},\ell}^{k})^2 + (V_{\bar{x},\ell}^{k})^2\Big]\Bigg)U_{\bar{x},i}^{j+1}\Bigg\}\Big(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\Big)$$

$$+\Bigg\{a\Bigg(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big[(u_{\bar{x},\ell}^{k})^2 + (v_{\bar{x},\ell}^{k})^2\Big]\Bigg)v_{\bar{x},i}^{j+1}$$

$$-a\Bigg(\tau h\sum_{k=1}^{j+1}\sum_{\ell=1}^{M}\Big[(U_{\bar{x},\ell}^{k})^2 + (V_{\bar{x},\ell}^{k})^2\Big]\Bigg)V_{\bar{x},i}^{j+1}\Bigg\}\Big(v_{\bar{x},i}^{j+1} - V_{\bar{x},i}^{j+1}\Big)$$

$$= 2\int_0^1 a'\Big(s^{j+1}(\mu)\Big)\xi^{j+1}\xi_t^j\,d\mu$$

$$+\int_0^1 a\Big(s^{j+1}(\mu)\Big)\Big[\Big(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\Big)^2 + \Big(v_{\bar{x},i}^{j+1} - V_{\bar{x},i}^{j+1}\Big)^2\Big]d\mu.$$

After substituting this equality in (3.9) we get

$$\begin{aligned}
&\|z^{j+1}\|^2-(z^{j+1},z^j)+\|w^{j+1}\|^2-(w^{j+1},w^j)\\
&+2\tau h\sum_{i=1}^{M}\int_0^1 a'\left(s^{j+1}(\mu)\right)\xi^{j+1}\xi_t^j\,d\mu\\
&+\tau h\sum_{i=1}^{M}\int_0^1 a\left(s^{j+1}(\mu)\right)\left[\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)^2+\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)^2\right]d\mu\\
&\le-\tau(\psi_1^j,z^{j+1})-\tau(\psi_2^j,w^{j+1}).
\end{aligned}\tag{3.10}$$

Taking into account that $a(S)\ge a_0=Const>0$ and relations $s^{j+1}(\mu)\ge 0$,

$$(r^{j+1},r^j)=\frac{1}{2}\|r^{j+1}\|^2+\frac{1}{2}\|r^j\|^2-\frac{1}{2}\|r^{j+1}-r^j\|^2,$$

$$\tau\xi^{j+1}\xi_t^j=\frac{1}{2}\left(\xi^{j+1}\right)^2-\frac{1}{2}\left(\xi^j\right)^2+\frac{\tau^2}{2}\left(\xi_t^j\right)^2$$

from (3.10) we have

$$\begin{aligned}
&\|z^{j+1}\|^2-\frac{1}{2}\|z^{j+1}\|^2-\frac{1}{2}\|z^j\|^2+\frac{1}{2}\|z^{j+1}-z^j\|^2\\
&+\|w^{j+1}\|^2-\frac{1}{2}\|w^{j+1}\|^2-\frac{1}{2}\|w^j\|^2+\frac{1}{2}\|w^{j+1}-w^j\|^2\\
&+h\sum_{i=1}^{M}\int_0^1 a'\left(s^{j+1}(\mu)\right)\left[\left(\xi^{j+1}\right)^2-\left(\xi^j\right)^2\right]d\mu\\
&+\tau^2 h\sum_{i=1}^{M}\int_0^1 a'\left(s^{j+1}(\mu)\right)\left(\xi_t^j\right)^2\,d\mu\\
&+\tau h a_0\sum_{i=1}^{M}\left[\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)^2+\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)^2\right]\\
&\le-\tau(\psi_1^j,z^{j+1})-\tau(\psi_2^j,w^{j+1}).
\end{aligned}\tag{3.11}$$

From (3.11) we arrive at

$$\frac{1}{2}\|z^{j+1}\|^2-\frac{1}{2}\|z^j\|^2+\frac{\tau^2}{2}\|z_t^j\|^2$$

$$+\frac{1}{2}\|w^{j+1}\|^2-\frac{1}{2}\|w^j\|^2+\frac{\tau^2}{2}\|w_t^j\|^2$$

$$+h\sum_{i=1}^{M}\int_0^1 a'\left(s^{j+1}(\mu)\right)\left[\left(\xi^{j+1}\right)^2-\left(\xi^j\right)^2\right]d\mu \tag{3.12}$$

$$+\tau a_0\left(\|z_{\bar{x}}^{j+1}]|^2+\|w_{\bar{x}}^{j+1}]|^2\right)\le\frac{\tau}{2a_0}\left(\|\psi_1^j\|^2+\|\psi_2^j\|^2\right)$$

$$+\frac{\tau a_0}{2}\left(\|z^{j+1}\|^2+\|w^{j+1}\|^2\right).$$

Using discrete analogue of Poincar inequality [21]

$$\|r^{j+1}\|^2\le\|r_{\bar{x}}^{j+1}]|^2,$$

from (3.12) we get

$$\|z^{j+1}\|^2-\|z^j\|^2+\tau^2\|z_t^j\|^2+\|w^{j+1}\|^2-\|w^j\|^2+\tau^2\|w_t^j\|^2$$

$$+2h\sum_{i=1}^{M}\int_0^1 a'\left(s^{j+1}(\mu)\right)\left[\left(\xi^{j+1}\right)^2-\left(\xi^j\right)^2\right]d\mu \tag{3.13}$$

$$+\tau a_0\left(\|z_{\bar{x}}^{j+1}]|^2+\|w_{\bar{x}}^{j+1}]|^2\right)\le\frac{\tau}{a_0}\left(\|\psi_1^j\|^2+\|\psi_2^j\|^2\right).$$

Summing (3.13) from $j=0$ to $j=n-1$ we arrive at

$$\|z^n\|^2+\tau^2\sum_{j=0}^{n-1}\|z_t^j\|^2+\|w^n\|^2+\tau^2\sum_{j=0}^{n-1}\|w_t^j\|^2$$

$$+2h\sum_{j=0}^{n-1}\sum_{i=1}^{M}\int_0^1 a'\left(s^{j+1}(\mu)\right)\left[\left(\xi^{j+1}\right)^2-\left(\xi^j\right)^2\right]d\mu \tag{3.14}$$

$$+\tau a_0\sum_{j=0}^{n-1}\left(\|z_{\bar{x}}^{j+1}]|^2+\|w_{\bar{x}}^{j+1}]|^2\right)\le\frac{\tau}{a_0}\sum_{j=0}^{n-1}\left(\|\psi_1^j\|^2+\|\psi_2^j\|^2\right).$$

Note, that since $s^{j+1}(\mu) \geq s^j(\mu)$, $a'(S) \geq 0$ and $a''(S) \leq 0$, for the second line of last formula we have

$$\sum_{j=0}^{n-1} a'\left(s^{j+1}(\mu)\right)\left[\left(\xi^{j+1}\right)^2 - \left(\xi^j\right)^2\right]$$

$$= a'\left(s^1(\mu)\right)\left(\xi^1\right)^2 - a'\left(s^1(\mu)\right)\left(\xi^0\right)^2$$

$$+a'\left(s^2(\mu)\right)\left(\xi^2\right)^2 - a'\left(s^2(\mu)\right)\left(\xi^1\right)^2$$

$$+\cdots + a'\left(s^n(\mu)\right)\left(\xi^n\right)^2 - a'\left(s^n(\mu)\right)\left(\xi^{n-1}\right)^2$$

$$= a'\left(s^n(\mu)\right)\left(\xi^n\right)^2 + \sum_{j=1}^{n-1}\left[a'\left(s^j(\mu)\right) - a'\left(s^{j+1}(\mu)\right)\right]\left(\xi^j\right)^2 \geq 0.$$

Taking into account the last relation and (3.14) one can deduce

$$\|z^n\|^2 + \|w^n\|^2 + \tau^2 \sum_{j=0}^{n-1} \|z_t^j\|^2 + \tau^2 \sum_{j=0}^{n-1} \|w_t^j\|^2$$
$$+\tau a_0 \sum_{j=0}^{n-1}\left(\|z_{\bar{x}}^{j+1}]|^2 + \|w_{\bar{x}}^{j+1}]|^2\right) \leq \frac{\tau}{a_0}\sum_{j=0}^{n-1}\left(\|\psi_1^j\|^2 + \|\psi_2^j\|^2\right). \tag{3.15}$$

From (3.15) we get (3.5), and thus Theorem 2 is proved. □

## 4. Conclusion

Uniqueness and large time behavior of solution of initial-boundary value problem with mixed boundary conditions as well as finite difference scheme for one nonlinear partial integro-differential averaged model with source terms are investigated. Mentioned model is based on Maxwell system which describes electromagnetic field penetration into a medium whose electro-conductivity depends on temperature. Theorem of uniqueness and stabilization of solution is fixed. Convergence of the fully discrete scheme is proved. Wider class of nonlinearity considering more general cases of the diffusion coefficient $a = a(S)$ is studied.

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# COMMON FIXED POINT THEOREMS WITH AN APPLICATION IN DYNAMIC PROGRAMMING

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*Abstract.* Two common fixed point theorems for a class of contractive mappings are proved in metric spaces. As an application, the existence and uniqueness of solution for a functional equation arising in dynamic programming is given. The results presented in this paper generalize some known results in the literature.

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*Keywords:* common fixed point, class of contractive mappings, diminishing orbital diameters, functional equation, dynamic programming

## 1. Introduction and Preliminaries

Throughout this paper, unless otherwise stated, $(X,d)$ denotes a metric space. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, and for each $t \in \mathbb{R}$, $[t]$ denote the greatest integer not exceeding $t$. Let $\omega$ and $\mathbb{N}$ denote the sets of all nonnegative integers and positive integers, respectively, and

$$\Phi = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is upper semi-continuous and nondecreasing and } \phi(t) < t \text{ for } t > 0\}.$$

Let $f$ be a self mapping of $(X,d)$. For $x, y \in X$ and $A, B \subseteq X$, define

$$O(x,f) = \{f^n x : n \in \omega\}, \quad O(x,y,f) = O(x,f) \cup O(y,f),$$
$$C_f = \{g : g \text{ is a self mapping of } X \text{ and } gf = fg\},$$
$$\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\}, \quad \delta(A) = \delta(A,A).$$

It is easy to see that $\{f^n : n \in \omega\} \subseteq C_f$.

Many authors studied the existence and uniqueness of fixed point and common fixed point for several classes of contractive mappings and families of contractive mappings in metrics spaces, and they used a few fixed point and common fixed point theorems to establish the existence and uniqueness of solution and common solutions for some kinds of functional equations and systems of functional equations arising

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from dynamic programming, for example, see [1–25] and the references therein. Jungck [9] proved some fixed point theorems for $C_f$ in metric spaces. Ohta and Nikaido [21] established two fixed point theorems for contractive mappings which satisfy

$$d(f^k x, f^k y) \le \alpha\delta(O(x,y,f)), \quad \forall x,y \in X, \tag{1.1}$$

where $\alpha \in [0,1)$.

As suggested in Bellman and Lee [1], the basic form of the functional equations in dynamic programming is

$$f(x) = \max_{y\in S} H(x,y,f(T(x,y))), \quad \forall x \in D, \tag{1.2}$$

where $x$ and $y$ denote the state and decision vectors, respectively, $T$ denotes the transformation of the process and $f(x)$ denotes the optimal return function with the initial state $x$.

Motivated by the results in [1–25], in this paper we extend the classes of mappings (1.1) and functional equations (1.2) considered by Ohta and Nikaido [21] and Bellman and Lee [1], respectively, to the following more general classes of contractive mappings and functional equations:

$$d(f^p x, f^q y) \le \varphi(\delta(\cup_{g\in C_f} gO(x,y,f))), \quad \forall x,y \in X, \tag{1.3}$$

where $\varphi \in \Phi$ and $p,q$ are some positive integers, and

$$f(x) = \operatorname*{opt}_{y\in D} \{u(x,y) + H(x,y,f(T(x,y)))\}, \quad \forall x \in S, \tag{1.4}$$

where $u : S\times D \to \mathbb{R}$, $T : S\times D \to S$ and $H : S\times D\times\mathbb{R} \to \mathbb{R}$, the opt denotes max or min. Under certain conditions we study the existence and uniqueness of fixed point, common fixed point and solution for the contractive mapping (1.3), the family of mappings $C_f$ and the functional equation (1.4), respectively, and establish the convergence and error estimates of Picard iterations with respect to the fixed point of the contractive mapping (1.3) and the solution of the functional equation (1.4), respectively. The results presented in this paper extend and improve some known results in [7, 21].

Let us recall the following notation, definitions and lemmas.

**Definition 1** ([2])**.** Let $(X,d)$ be a metric space. A mapping $f : X \to X$ is said to have diminishing orbital diameters in $X$ if

$$\lim_{n\to\infty} \delta(O(f^n x, f)) < \delta(O(x,f)) \quad \text{for all } x \in X \text{ with } \delta(O(x,f)) > 0.$$

**Definition 2** ([6])**.** Let $(X,d)$ be a metric space, $A \subseteq X$ and $A_n \subseteq X$ for $n \in \mathbb{N}$. The sequence $\{A_n\}_{n\in\mathbb{N}}$ is said to converge to $A$ if

(1) each point $a \in A$ is the limit of some convergent sequence $\{a_n : a_n \in A_n$ for each $n \in \mathbb{N}\}$;
(2) for arbitrary $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $A_n \subseteq A_\epsilon$ for $n > k$, where $A_\epsilon$ is the union of all open spheres with centers in $A$ and radius $\epsilon$.

**Lemma 1** ([24]). *Let $\varphi \in \Phi$ and $\varphi^n$ denote the composition of $\varphi$ with itself $n$ times. Then for every $t > 0$, $\varphi(t) < t$ if and only if $\lim_{n\to\infty} \varphi^n(t) = 0$.*

**Lemma 2** ([20]). *Let $f$ a continuous self mapping of a metric space $(X,d)$ such that*

(a) *$f$ has a unique fixed point $u \in X$;*
(b) *$\lim_{n\to\infty} f^n x = u$ for all $x \in X$;*
(c) *there exists an open neighborhood $G$ of $u$ with the property that given any open set $V$ containing $u$ there exists $k \in \mathbb{N}$ such that $f^n G \subseteq V$ for all $n > k$.*

*Then for any $r \in (0,1)$, there exists a metric $d'$, topologically equivalent to $d$, such that $d'(fx, fy) \le rd'(x,y)$ for all $x, y \in X$.*

## 2. Common fixed point theorems for $C_f$

Our main results are as follows.

**Theorem 1.** *Let $(X,d)$ be a bounded complete metric space and $f : X \to X$ satisfy (1.3). Assume that $f$ is continuous. Then*

(i) *$f$ has diminishing orbital diameters in $X$;*
(ii) *$f$ has a unique fixed point $u \in X$, which is also a unique common fixed point of $C_f$;*
(iii) *$d(f^n x, u) \le \varphi^{[n/s]}(\delta(X))$ for all $x \in X$ and $n \in \mathbb{N}$, where $s = \max\{p,q\}$;*
(iv) *$\lim_{n\to\infty} f^n x = u$ for all $x \in X$ and $\{f^n X\}_{n\in\mathbb{N}}$ converges to $\{u\}$;*
(v) *for any $r \in (0,1)$, there exists a metric $d'$, topological equivalent to $d$, such that $d'(fx, fy) \le rd'(x,y)$ for all $x, y \in X$.*

*Proof.* We may assume, without loss of generality, $p \ge q$. For any $x, y \in X$ and $n \ge p$, it follows from (1.3) that

$$\begin{aligned} d(f^n x, f^n y) &= d(f^p f^{n-p} x, f^q f^{n-q} y) \\ &\le \varphi(\delta(\cup_{g\in C_f} gO(f^{n-p}x, f^{n-q}y, f))) \\ &= \varphi(\delta(\cup_{g\in C_f} O(f^{n-p}gx, f^{n-q}gy, f))) \\ &\le \varphi(\delta(O(f^{n-p}X \cup f^{n-q}X))) \\ &= \varphi(\delta(f^{n-p}X)), \end{aligned}$$

which implies that

$$\delta(f^n X) \le \varphi(\delta(f^{n-p}X)), \quad \forall n \ge p. \tag{2.1}$$

It follows from (2.1) and Lemma 1 that

$$\delta(f^{kp}X) \le \varphi(\delta(f^{(k-1)p}X)) \le \cdots \le \varphi^k(\delta(X)) \to 0 \quad \text{as } k \to \infty. \tag{2.2}$$

Because

$$X \supseteq fX \supseteq f^2X \supseteq \cdots \supseteq f^n X \supseteq f^{n+1}X \supseteq \cdots, \quad \forall n \in \omega,$$

by (2.2) we infer that

$$\delta(f^n X) \to 0 \quad \text{as } n \to \infty. \tag{2.3}$$

In view of (2.3), we get that

$$d(f^n x, f^{n+h} x) \le \delta(O(f^n x, f)) \le \delta(f^n X), \quad \forall x \in X, \forall n, h \in \mathbb{N}. \tag{2.4}$$

It follows from (2.3) and (2.4) that $f$ has diminishing orbital diameters and for each $x \in X$, $\{f^n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Thus $\{f^n x\}_{n\in\mathbb{N}}$ converges to some point $u \in X$ by completeness of $X$.

Since $f$ is continuous, it follows that $fu = u$. Suppose that $f$ has a second fixed point $v \in X$. From (2.3) we have

$$d(u,v) \le \delta(f^n X) \to 0 \quad \text{as } n \to \infty,$$

which means that $u = v$, that is, $f$ has a unique fixed point $u$. Let $g \in C_f$. Note that $gu = gfu = fgu$. It follows that $gu$ is a fixed point of $f$. Therefore $gu = u$. It follows from $f \in C_f$ that $u$ is a unique common fixed point of $C_f$. Using (2.1) and (2.3) we conclude that for $x \in X$

$$\begin{aligned} d(f^n x, u) &\le \delta(f^n X, \{u\}) \le \delta(f^n X) \\ &\le \varphi^{[n/p]}(\{\delta(f^i X) : 0 \le i < p\}) \\ &= \varphi^{[n/p]}(\delta(X)), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{2.5}$$

Given $\epsilon > 0$. (2.3) ensures that there exists $k \in \mathbb{N}$ such that $\delta(f^n X) < \frac{\epsilon}{2}$ for $n > k$. Consequently, by (2.4) we deduce that $f^n X \subseteq B(u,\epsilon) = \{x \in X : d(u,x) < \epsilon\}$ for $n > k$. Thus $\{f^n X\}_{n\in\mathbb{N}}$ converges to $\{u\}$.

To show (v), it suffices to show that (c) of Lemma 2 holds. Take $G = X$. For any open set $V$ containing $u$ there exists $\epsilon > 0$ with $B(u,\epsilon) = \{x \in X : d(u,x) < \epsilon\} \subseteq V$. It follows from what we have just proved that $f^n G \subseteq B(u,\epsilon) \subseteq V$ for $n > k$. Hence (c) is satisfied. This completes the proof. □

*Remark* 1. The following example shows that the condition that $f$ be continuous when $p, q > 1$ is necessary in Theorem 1.

*Example* 1. Let $X = [0,1]$ with the usual metric. Define a discontinuous mapping $f : X \to X$ by $f0 = 1$ and $fx = \frac{x}{2}$ for $x \in (0,1]$. Take $p = 2, q = 3$ and $\varphi(t) = \frac{t}{2}$ for $t \ge 0$. It is easy to check that the conditions of Theorem 1 are satisfied except for the continuity assumption. $f$ however has no fixed point in $X$.

*Remark* 2. The following example reveals that the boundedness of $X$ is necessary in Theorem 1.

*Example* 2. Let $X = [1, +\infty)$ with the usual metric. Define a mapping $f : X \to X$ by $fx = 2x$ for $x \in X$. Set $p = 2$, $q = 3$ and $\varphi(t) = \frac{t}{2}$ for $t \ge 0$. It is easily proved that the conditions of Theorem 1 are satisfied except for the boundedness assumption. $f$ however has no fixed point in $X$.

Replacing the boundedness of $X$ by the boundedness of $fX$, as in the proof of Theorem 1, we have

**Theorem 2.** *Let $(X,d)$ be a complete metric space and $f : X \to X$ be a continuous mapping such that $fX$ is bounded and (1.3) holds. Then* (i), (ii), (iv) *and* (v) *in Theorem 1 and the following*

(iii)$'$ $d(f^n x, u) \le \varphi^{[n/s]}(\delta(fX))$ *for all* $x \in X$, $n \in \mathbb{N}$

*hold.*

*Remark* 3. Theorems 1 and 2 extend, improve and unify the corresponding results in [7, 21].

## 3. An application in dynamic programming

Throughout this section, we assume that $X, Y$ are Banach spaces, $S \subseteq X$ is the state space, $D \subseteq Y$ is the decision space, $B(S)$ denotes the set of all bounded real-valued functions on $S$ and $d(f,g) = \sup\{|f(x) - g(x)| : x \in S\}$ for $f, g \in B(S)$. Clearly $(B(S), d)$ is a complete metric space.

**Theorem 3.** *Suppose that the following conditions are satisfied:*

(a) *$u$ and $H$ are bounded;*

(b) *$|H(x,y,g(t)) - H(x,y,h(t))| \le \varphi(\delta(\cup_{m\in C_A} mO(g,h,A)))$ for all $(x,y) \in S \times D$, $g,h \in B(S)$ and $t \in S$, where $\varphi \in \Phi$ and the mapping $A : B(S) \to B(S)$ defined by*

$$Ag(x) = \operatorname*{opt}_{y\in D} \{u(x,y) + H(x,y,g(T(x,y)))\}, \quad \forall (x,g) \in S \times B(S) \tag{3.1}$$

*satisfies*

(c) *For any sequence $\{h_n\}_{n\in\mathbb{N}} \subseteq B(S)$ and $h \in B(S)$,*

$$\lim_{n\to\infty} \sup_{x\in S} |h_n(x) - h(x)| = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \sup_{x\in S} |Ah_n(x) - Ah(x)| = 0.$$

*Then*

(i) *$A$ has diminishing orbital diameters in $B(S)$;*

(ii) *the functional equation* (1.4) *possesses a unique solution $v \in B(S)$, which is both a unique fixed point of $A$ and a unique common fixed point of $C_A$;*

(iii) *$d(A^n x, v) \le \varphi^{[n/s]}(\delta(AB(S)))$ for all $x \in B(S)$, $n \in \mathbb{N}$;*

(iv) *$\lim_{n\to\infty} A^n x = v$ for all $x \in B(S)$ and $\{A^n B(S)\}_{n\in\mathbb{N}}$ converges to $\{v\}$;*

(v) *for any $r \in (0,1)$, there exists a metric $d'$, topological equivalent to $d$, such that $d'(Ax, Ay) \le rd'(x,y)$ for all $x, y \in B(S)$.*

*Proof.* It follows from (a), (c) and (3.1) that $AB(S)$ is bounded and $A$ is continuous. We assume that without loss of generality opt $=$ inf. For any $\varepsilon > 0, x \in S$ and $h, g \in B(S)$, there exist $y, z \in D$ such that

$$Ag(x) > u(x,y) + H(x,y,g(T(x,y))) - \varepsilon, \tag{3.2}$$

$$Ah(x) > u(x,z) + H(x,z,h(T(x,z))) - \varepsilon. \tag{3.3}$$

Also we have

$$Ag(x) \le u(x,z) + H(x,z,g(T(x,z))), \tag{3.4}$$
$$Ah(x) \le u(x,y) + H(x,y,h(T(x,y))). \tag{3.5}$$

From (3.2), (3.5) and (b) we infer that

$$\begin{aligned} Ag(x) - Ah(x) &> H(x,y,g(T(x,y))) - H(x,y,h(T(x,y))) - \varepsilon \\ &\ge -\varphi(\delta(\cup_{m\in C_A} mO(g,h,A))) - \varepsilon. \end{aligned}$$

Similarly, from (3.3) and (3.4) and (b) we know that

$$\begin{aligned} Ag(x) - Ah(x) &< H(x,z,g(T(x,z))) - H(x,z,h(T(x,z))) + \varepsilon \\ &\le \varphi(\delta(\cup_{m\in C_A} mO(g,h,A))) + \varepsilon. \end{aligned}$$

It is easy to see that

$$\begin{aligned} d(Ag,Ah) &= \sup_{x\in S} |Ag(x) - Ah(x)| \\ &\le \varphi(\delta(\cup_{m\in C_A} mO(g,h,A))) + \varepsilon. \end{aligned}$$

Letting $\varepsilon$ tend to zero, we have

$$d(Ag,Ah) \le \varphi(\delta(\cup_{m\in C_A} mO(g,h,A))), \quad \forall g,h \in B(S).$$

Thus Theorem 3 follows from Theorem 2 with $p = q = 1$. Particularly, the unique fixed point $v \in B(S)$ of $A$ is a unique solution of the functional equation (1.4) in $B(S)$. This completes the proof. □

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# CONGRUENCES AND DECOMPOSITIONS OF AG$^{**}$-GROUPOIDS

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*Abstract.* We introduce the concept of completely left inverse AG$^{**}$-groupoids and study some basic congruences and a congruence pair by means of the kernel and trace approach of completely left inverse AG$^{**}$-groupoids. Also, we provide separative and anti-separative decomposition of locally associative AG$^{**}$-groupoids.

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## 1. Introduction

An *Abel-Grassmann's* groupoid (abbreviated as AG-groupoid) or Left Almost Semigroup (briefly LA-semigroup) is a groupoid $S$ satisfying the left invertive law, defined as, $(ab)c = (cb)a$ for all $a, b, c \in S$. Inverse AG-groupoids, their different characterisations and congruences on inverse AG-groupoids using the kernel-normal system and kernel-trace approaches have been studied by many authors which can be found in the literature (see [1–4, 6, 7, 11]).

In this paper, we introduce *completely left inverse* AG$^{**}$-groupoids and investigate a congruence pair consisting a kernel and trace of a congruence of a completely left inverse AG$^{**}$-groupoid. In the second section, some preliminaries and basic results on completely inverse AG$^{**}$-groupoids are mentioned. In Section 3, we introduce completely left inverse AG$^{**}$-groupoids and investigate some basic congruences using the congruence pair. We show that if $\rho$ is a congruence on a completely left inverse AG$^{**}$-groupoid, then $(\ker\rho, \mathrm{tr}\rho)$ is a congruence. In Section 4, we discuss separative and anti-separative decompositions of a locally associative AG$^{**}$-groupoid. Before the proofs of the main results, it is important to recall the basic knowledge and necessary terminology.

## 2. Preliminaries

An AG-groupoid $S$ is regular if $a \in (aS)a$ for all $a \in S$. If for $a \in S$, there exists an element $a^{'}$ such that $a = (aa^{'})a$ and $a^{'} = (a^{'}a)a^{'}$, then we say that $a^{'}$ is inverse

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of $a$. In addition, if inverses commute, that is $a^{'}a = aa^{'}$, then $S$ is called completely regular. If $a \in S$, then

$$V(a) = \{a^{'} \in S : a = (aa^{'})a \text{ and } a^{'} = (a^{'}a)a^{'}\}$$

is called the set of all inverses of $a \in S$. Note that if $a^{'} \in V(a)$ and $b^{'} \in V(b)$, then $a \in V(a')$ and $a^{'}b^{'} \in V(ab)$.

An AG-groupoid $S$ in which every element has a unique inverse is called inverse AG-groupoid. If $a^{-1}$ is the unique inverse of $a \in S$, then a groupoid satisfying the following identities is called a completely inverse AG$^{**}$-groupoid, that is for all $a, b, c \in S$

$$(ab)c = (cb)a,\ a(bc) = b(ac)$$

$$a = (aa^{-1})a,\ \ a^{-1} = (a^{-1}a)a^{-1} \text{ and } aa^{-1} = a^{-1}a.$$

If $S$ is a completely inverse AG$^{**}$-groupoid, then $a^{-1}a \in E_S$, where $E_S$ is the set of idempotents of $S$. If $S$ is a completely inverse AG$^{**}$-groupoid, then $E_S$ is either empty or a semilattice. For any idempotent $e$ in $E_S$, $e^{-1} = e$. Moreover, the set $E_S$ of an AG-groupoid $S$ is a rectangular AG-band, that is for all $e, f \in E_S$, $e = (ef)e$. For futher concepts and results, the reader is referred to [3]. The set of idempotents $E_S$ of an AG-groupoid $S$ is called left (respectively; right) regular AG-band if it satisfies

$$(ef)e = ef \quad \text{(respectively; } (ef)e = fe\text{) for all } e, f \in E_S.$$

Note that if $S$ is an AG$^{**}$-groupoid, then for $e, f \in E_S$

$$ef = (ee)(ff) = (ff)(ee) = fe$$

which shows left and right AG-bands serve the same purpose.

**Lemma 1** ([3]). *Let $S$ be a completely inverse* AG$^{**}$*-groupoid and let $a, b \in S$ such that $ab \in E_S$. Then $ab = ba$.*

**Lemma 2** ([3]). *Completely inverse* AG$^{**}$*-groupoids are idempotent-surjective.*

If $\rho$ is a congruence on a completely inverse AG$^{**}$-groupoid, then $S/\rho$ is also completely inverse AG$^{**}$-groupoid. The natural morphism maps $S$ onto $S/\rho$ by the rule $x \to (x)_\rho$ and by the uniqueness of inverses $(x^{-1})_\rho = (x)_\rho^{-1}$. If $(a, b) \in \rho$, then $(a^{-1}, b^{-1}) \in \rho$ and $(aa^{-1}, bb^{-1}) \in \rho$.

## 3. Congruences in completely left inverse AG$^{**}$-groupoid

In this section, we introduce the notion of completely left inverse AG$^{**}$-groupoids and study certain congruences by means of their kernel and trace for this class of groupoids. The essential part is to describe such congruence in terms of a congruence pair which comprises of a normal subgroupoid and a congruence of a completely left inverse AG$^{**}$-groupoids.

**Definition 1.** A completely inverse AG**-groupoid is called completely left inverse AG**-groupoid if the set $E_S$ of idempotents of $S$ is a left regular AG-band.

**Proposition 1.** *Let $\rho$ be a congruence on a completely inverse* AG**-*groupoid $S$. If $(a,e) \in \rho$ for $e \in E_S$, then $(a,a^{-1}) \in \rho$ and $(a,a^{-1}a) \in \rho$.*

**Lemma 3.** *Let $S$ be a completely left inverse* AG**-*groupoid. If $\rho$ is a congruence on $S$, then $S/\rho$ is a completely left inverse* AG**-*groupoid.*

*Proof.* It is straightforward, and so it is omitted. □

**Definition 2.** A nonempty subset $N$ of a completely left inverse AG**-groupoid $S$ is called normal if
(1) $E_S \subseteq N$,
(2) for every $x \in S$, $x \cdot Nx^{-1} \subseteq N$,
(3) for every $a \in N$, $a^{-1} \in N$.

Let $\rho$ be a congruence on a completely left inverse AG**-groupoid $S$ and $E_S$ be the set of idempotents of $S$. The restriction of $\rho$ on $E_S$, that is $\rho|_{E_S}$ is the trace of $\rho$ denoted by $\mathrm{tr}\rho$. The subset

$$\ker\rho = \{a \in S : (\exists e \in E_S)(a,e) \in \rho\}$$

is the kernel of $\rho$.

**Lemma 4.** *Let $\rho$ be a congruence on a completely left inverse* AG**-*groupoid $S$.*
(1) $\ker\rho$ *is a normal* AG**-*subgroupoid of $S$.*
(2) *For any $a \in S$, $e \in E_S$, if $ea \in \ker\rho$ such that $(e,\ aa^{-1}) \in \mathrm{tr}\rho$, then $a \in \ker\rho$.*
(3) *For any $a \in S$, if $a \in \ker\rho$, then $(a^{-1}a,\ aa^{-1}) \in \mathrm{tr}\rho$.*

*Proof.* (1) Let $\rho$ be a congruence. If $a,b \in \ker\rho$, then $(a,e) \in \rho$, $(b,f) \in \rho$ so that $(ab,ef) \in \rho$ for some $e,f \in E_S$. Hence $ab \in \ker\rho$ and $\ker\rho$ is a subgroupoid of $S$. Obviously, all the idempotents of $S$ lie in $\ker\rho$. Let $a \in \ker\rho$, then $(a,e) \in \rho$ for $e \in E_S$. Therefore for all $x \in S$, $(x^{-1}\cdot ax,\ x^{-1}\cdot ex) \in \rho$. Since $x^{-1}\cdot ex = e\cdot x^{-1}x \in E_S$, thus $x^{-1}\cdot ax \in \ker\rho$. Now if $a \in \ker\rho$, then for $g \in E_S$, $(a,g) \in \rho$. Since $S/\rho$, is left inverse, it is clear that $(a)_\rho^{-1} \in V((a)_\rho)$. Moreover, if $h \in E_S$, then $a^{-1} \in V(h)$ so that $(a^{-1},h) \in \rho$. That is $a^{-1} \in \ker\rho$ for every $a \in \ker\rho$.

(2) If for $a \in S$, $e \in E_S$ and $ea \in \ker\rho$, then there exists $f \in E_S$ such that $(ea,\ f) \in \rho$. Since $(e,\ aa^{-1}) \in \mathrm{tr}\rho$, then $a = aa^{-1}\cdot a \equiv_\rho ea \equiv_\rho f$. Hence $(a,\ f) \in \rho$ and $a \in \ker\rho$.

(3) Let $a \in \ker\rho$. Then $(a,e) \in \rho$ for some $e \in E_S$. By Proposition 1, we have $(a^{-1}a,\ a^{-1}a) \in \rho$. Since $\mathrm{tr}\rho = \rho|_{E_S}$, it follows that $(a^{-1}a, aa^{-1}) \in \mathrm{tr}\rho$. □

**Lemma 5.** *Let $\rho$ be a congruence on a completely left inverse* AG**-*groupoid $S$. If $a^{-1}b \in \ker\rho$, then $ab^{-1} \in \ker\rho$ and for all $a,b \in S$, $((a^{-1}b\cdot ab^{-1}),a^{-1}b) \in \rho$.*

*Proof.* Let $\rho$ be a congruence on $S$. If $a^{-1}b \in \ker\rho$, then $(a^{-1}b,e) \in \rho$ for some $e \in E_S$. Then it is clear that $(ab^{-1})_\rho$ is inverse of $(a^{-1}b)_\rho$ in $S/\rho$ and it follows immediately from the preliminaries and Lemma 3 that $(a^{-1}b)_\rho \in E_{S/\rho}$. Hence $(ab^{-1},f) \in \rho$

for some $f \in E_S$. Thus $ab^{-1} \in \ker\rho$. Moreover, since $S/\rho$ is inverse and $E/\rho$ is a left regular AG-band, we have

$$(a^{-1}b \cdot ab^{-1})_\rho = (e)_\rho (f)_\rho = ((e)_\rho (f)_\rho)(e)_\rho = (e)_\rho = (a^{-1}b)_\rho.$$

□

**Lemma 6.** *Let $\rho$ be a congruence on a completely left inverse* AG$^{**}$*-groupoid $S$ and let $a, b \in S$ and $e \in E_S$. If $(aa^{-1},\ bb^{-1}) \in \mathrm{tr}\rho$ and $ab^{-1} \in \ker\rho$, then*

$$(a \cdot ea^{-1},\ b \cdot eb^{-1}) \in \mathrm{tr}\rho.$$

*Proof.* Let $\rho$ be a congruence on $S$. Let $a, b \in S$ such that $(aa^{-1},\ bb^{-1}) \in \mathrm{tr}\rho$ and $ab^{-1} \in \ker\rho$. Then for all $e \in E_S$, we have

$$\begin{aligned}
a \cdot ea^{-1} &\underset{\rho}{\equiv} a(e(a^{-1}a \cdot a^{-1}) \\
&\equiv a(e(b^{-1}b \cdot a^{-1})) && (\text{since } (aa^{-1}, bb^{-1}) \in \mathrm{tr}\rho) \\
&\equiv a(e(a^{-1}b \cdot b^{-1})) \\
&\equiv a(e((a^{-1}b \cdot (a^{-1}b)^{-1})b^{-1})) && (\text{since } (a^{-1}b)\rho \in E_{S/\rho}) \\
&\equiv a((a^{-1}a \cdot bb^{-1})(eb^{-1})) \\
&\equiv a(b^{-1}b \cdot eb^{-1}) && (\text{since } (aa^{-1},\ bb^{-1}) \in \mathrm{tr}\rho) \\
&\equiv e \cdot ab^{-1} \\
&\equiv e(ab^{-1} \cdot a^{-1}b) && (\text{by Lemma 5}) \\
&\equiv e(aa^{-1} \cdot bb^{-1}) \\
&\underset{\rho}{\equiv} b \cdot eb^{-1} && (\text{since } (aa^{-1},\ bb^{-1}) \in \mathrm{tr}\rho).
\end{aligned}$$

Thus $(a \cdot ea^{-1},\ b \cdot eb^{-1}) \in \mathrm{tr}\rho$. □

**Definition 3.** Let $N$ be normal subgroupoid of a completely left inverse AG$^{**}$-groupoid $S$ and $\tau$ be congruence on a left regular AG-band $E_S$. Then $(N, \tau)$ is a congruence pair of $S$ if for all $a,\ b \in S$ and $e \in E_S$ the following conditions hold.

(1) If $ea \in N$ and $(e,\ a^{-1}a) \in \tau$, then $a \in N$,

(2) If $(aa^{-1},\ bb^{-1}) \in \tau$ and $a^{-1}b \in N$, then $(a \cdot ea^{-1},\ b \cdot eb^{-1}) \in \tau$.

**Theorem 1.** *Let $S$ be a completely left inverse* AG$^{**}$*-groupoid and $(N,\ \tau)$ is congruence pair on $S$. Then the relation*

$$\rho_{(N,\ \tau)} = \{(a, b) \in S \times S :\ (aa^{-1}, bb^{-1}) \in \tau \text{ and } a^{-1}b \in N\}$$

*is a congruence relation.*

*Proof.* Clearly, $(N,\ \tau)$ is reflexive. $\rho_{(N,\ \tau)}$ is symmetric. In fact: $\tau$ is symmetric and by Definition 2 (3), $a^{-1} \in N$ for any $a \in N$. Also, if $(a,\ b),\ (b,\ c) \in \rho$, then $(aa^{-1},\ bb^{-1}) \in \tau$, $(bb^{-1},\ cc^{-1}) \in \tau$ and $a^{-1}b,\ b^{-1}c \in N$. Then by Lemma 1,

$cb^{-1} \in N$. Hence $(aa^{-1},\ cc^{-1}) \in \tau$. Since $(c^{-1}\{(aa^{-1}\cdot bb^{-1})c\}) \in E_S$ and $N$ is normal AG**-subgroupoid, then

$$\begin{aligned}(c^{-1}\{(aa^{-1}\cdot bb^{-1})c\})(a^{-1}c) &= \{c^{-1}(a^{-1}a\cdot c)\}(a^{-1}c) \text{ (since } (a^{-1}a,\ bb^{-1}) \in \tau)\\ &= \{a^{-1}a\cdot c^{-1}c\}(a^{-1}c)\\ &= \{a^{-1}a\cdot b^{-1}b\}(a^{-1}c) \text{ (since } (bb^{-1},\ cc^{-1}) \in \tau)\\ &= (bb^{-1})(a^{-1}c) \text{ (since } (a^{-1}a,\ bb^{-1}) \in \tau)\\ &= (cb^{-1})(a^{-1}b) \in N.\end{aligned}$$

Moreover,

$$\begin{aligned}(a^{-1}c)^{-1}(a^{-1}c) &= c^{-1}(aa^{-1}\cdot c)\\ &= c^{-1}((aa^{-1}\cdot cc^{-1})c)\tau c^{-1}((aa^{-1}\cdot bb^{-1})c)\\ &\quad \text{(since } bb^{-1}\tau cc^{-1}).\end{aligned}$$

Hence by Definition 3 (1), $a^{-1}c \in N$ which implies that $(a,c) \in \rho_{(N,\tau)}$. Thus $\rho_{(N,\ \tau)}$ is equivalence relation.
Let $(a,b) \in \rho_{(N,\ \tau)}$, then $(ac,\ bc) \in \rho_{(N,\ \tau)}$. In fact: if $(aa^{-1},\ bb^{-1}) \in \rho$ and $a^{-1}b \in \ker\rho$, then by Definition 2 (2), $(ac)^{-1}(bc) \in N$. Further, using Definition 3 (2), we have

$$(ac)(ac)^{-1} = (aa^{-1})(cc^{-1}) = (c^{-1}c)(aa^{-1}) = a(c^{-1}c\cdot a^{-1})\tau b(c^{-1}c\cdot b^{-1}).$$

Hence by definition of the relation $\rho_{(N,\ \tau)}$, $(ac,\ bc) \in \rho_{(N,\ \tau)}$.
Similarly, since $\tau$ is a congruence, then $(aa^{-1},\ bb^{-1}) \in \tau$ implies that

$$(ca)(ca)^{-1} = ca\cdot c^{-1}a^{-1} = a^{-1}a\cdot cc^{-1} = c(aa^{-1}\cdot c^{-1})\tau c(bb^{-1}\cdot c^{-1}).$$

It remains to show that $(ca)^{-1}(cb) \in N$. Therefore

$$\begin{aligned}(b^{-1}((c^{-1}c\cdot aa^{-1})b))((ca)^{-1}(cb)) &= ((c^{-1}c\cdot aa^{-1})(b^{-1}b))(c^{-1}a^{-1}\cdot cb)\\ &= ((c^{-1}c\cdot aa^{-1})(b^{-1}b))(c^{-1}c\cdot a^{-1}b)\\ &= ((c^{-1}c)(aa^{-1}\cdot b^{-1}b)(c^{-1}c))(a^{-1}b)\\ &= ((c^{-1}c)(aa^{-1}\cdot b^{-1}b))(a^{-1}b)\\ &\quad \text{(since } E_S \text{ is left regular)}\\ &= (b^{-1}((aa^{-1}\cdot cc^{-1})b))(a^{-1}b) \in N.\end{aligned}$$

Moreover,

$$\begin{aligned}b^{-1}((c^{-1}c\cdot aa^{-1})b) &= ((c^{-1}c\cdot aa^{-1})(c^{-1}c))(b^{-1}b) \quad \text{(since } E_S \text{ is left regular)}\\ &= ((a^{-1}c\cdot ac^{-1})(c^{-1}c))(b^{-1}b)\\ &= ((ac\cdot c^{-1})(a^{-1}c^{-1}\cdot c))(b^{-1}b)\end{aligned}$$

$$= (b(cc^{-1}\cdot a^{-1}))(b^{-1}(c^{-1}c\cdot a)).$$

Thus $(b^{-1}((c^{-1}c\cdot aa^{-1})b))\tau(b(cc^{-1}\cdot a^{-1}))(b^{-1}(c^{-1}c\cdot a))$. Hence by Definition 3 (1) it follows that $a^{-1}(c^{-1}c\cdot b)\in N$. Thus $(ca,\ cb)\in\rho_{(N,\ \tau)}$.

□

**Corollary 1.** *Let $S$ be a completely left inverse* AG$^{**}$*-groupoid and $(N,\ \tau)$ is congruence pair on $S$. Then the relation*

$$\rho_{(N,\ \tau)} = \{(a,b)\in S\times S:\ (aa^{-1},bb^{-1})\in\tau \text{ and } ba^{-1}\in N\}$$

*is a congruence relation.*

**Theorem 2.** *Let $S$ be a completely left inverse* AG$^{**}$*-groupoid. If $\rho$ is a congruence on $S$, then* $(\ker\rho,\ \mathrm{tr}\rho)$ *is a congruence of $S$. Conversely, if $(N,\ \tau)$ is congruence pair on $S$, then the relation*

$$\rho_{(N,\ \tau)} = \{(a,b)\in S\times S:\ (aa^{-1},bb^{-1})\in\tau \text{ and } a^{-1}b\in N\}$$

*is a congruence relation on $S$. Furthermore,*

$$\ker\rho_{(N,\tau)} = N, \mathrm{tr}\rho_{(N,\tau)} = \tau \text{ and } \rho_{(\ker\rho,\mathrm{tr}\rho)} = \rho.$$

*Proof.* The proof of the first part can be followed from Lemma 4, 6 and Theorem 1. We show that $\ker\rho_{(N,\ \tau)} = N$ and $\mathrm{tr}\rho_{(N,\tau)} = \tau$. Let $a\in\ker\rho_{(N,\tau)}$, then for some $e\in E_S$, $(a,e)\in\rho_{(N,\ \tau)}$. It follows that $(ee^{-1},aa^{-1})\in\tau$ and $ea\in N$. Thus by Definition 3 (1), $a\in N$, that is $\ker\rho_{(N,\ \tau)}\subseteq N$. Conversely, suppose that $a\in N$. Then $a^{-1}\in N$. Let $a^{-1}a = e$, it is clear that $(ee^{-1},\ aa^{-1})\in\tau$ and $e^{-1}a = ea = a^{-1}a\cdot a = a\in N$. Thus $(e,a)\in\rho_{(N,\tau)}$. Hence $a\in(e)_{\rho_{(N,\ \tau)}}\subseteq\ker\rho_{(N,\tau)}$. Thus $\ker\rho_{(N,\ \tau)} = N$.

Similarly, we show that $\mathrm{tr}\rho_{(N,\ \tau)}\subseteq\tau$ and $\tau\subseteq\mathrm{tr}\rho_{(N,\ \tau)}$. Let $e,f\in E_S$ such that $(e,f)\in\mathrm{tr}\rho_{(N,\tau)}$. Then since $E_S$ is left regular, therefore $e = (ee^{-1})e = ee^{-1}\equiv_\tau ff^{-1} = (ff^{-1})f = f$ and hence $\mathrm{tr}\rho_{(N,\ \tau)}\subseteq\tau$. Conversely, if $e\equiv_\tau f$, then $ee^{-1} = e\equiv_\tau f = ff^{-1}$ and $e^{-1}f\in E_S\subseteq N$. Thus by definition of $\rho_{(N,\tau)}$, it follows $(e,\ f)\in\rho_{(N,\ \tau)}\cap E_S\times E_S = \mathrm{tr}\rho$. Thus $\mathrm{tr}\rho_{(N,\ \tau)} = \tau$.

Finally, suppose that $(a,b)\in\rho$. Then $(a^{-1}a,a^{-1}b)\in\rho$ so that $a^{-1}b\in\ker\rho$. If $a^{-1}$ is the inverse of $a$ and since $S/\rho$ is completely left inverse, then $(a^{-1})_\rho\in V((a)_\rho) = V((b)_\rho)$. Also, $(b^{-1})_\rho\in V((b)_\rho) = V((a)_\rho)$. It is clear that $(a)_\rho(a^{-1})_\rho = (a)_\rho(b^{-1})_\rho = (b)_\rho(b^{-1})_\rho$ which further implies that
$(aa^{-1},\ bb^{-1})\in\mathrm{tr}\rho$. Thus $(a,\ b)\in\rho_{(\ker\rho,\mathrm{tr}\rho)}$ and $\rho\subseteq\rho_{(\ker\rho,\ \mathrm{tr}\rho)}$. Conversely, let $(a,\ b)\in\rho_{(\ker\rho,\mathrm{tr}\rho)}$. Then $(aa^{-1},\ bb^{-1})\in\mathrm{tr}\rho$ and $a^{-1}b\in\ker\rho$. By Lemma 5, $ab^{-1}\in\ker\rho$ and $(ab^{-1})_\rho\in E_{S/\rho}$. Then there exists $e\in E_S$ such that $(ab^{-1})_\rho = (e)_\rho$, where $(e)_\rho\in E|\rho$. Since $E_{S/\rho}$ is left regular, thus by Lemma 5, we have

$$(ab^{-1})_\rho = (ab^{-1})_\rho((ab^{-1})^{-1})_\rho.$$

Then

$$a\equiv_\rho aa^{-1}\cdot a = bb^{-1}\cdot a \quad (\text{since } (aa^{-1},\ bb^{-1})\in\mathrm{tr}\rho)$$

$$\begin{aligned}
&\equiv_\rho ab^{-1}\cdot b\\
&\equiv_\rho ((ab^{-1})(ab^{-1})^{-1})b\\
&\equiv_\rho (aa^{-1}\cdot b^{-1}b)b\\
&\equiv_\rho b^{-1}b\cdot b \qquad\qquad (\text{since } (aa^{-1},\ bb^{-1})\in \mathrm{tr}\rho)\\
&\equiv_\rho b.
\end{aligned}$$

Hence $\rho_{(\ker\rho,\mathrm{tr}\rho)} = \rho$. This completes the proof. □

## 4. Decompositions of locally associative AG**-groupoids

An AG-groupoid has many characteristics similar to that of a commutative semigroup. Let us consider $x^2y^2 = y^2x^2$, which holds for all $x, y$ in a commutative semigroup. On the other hand one can easily see that it holds in an AG**-groupoid. This simply gives that how an AG**-groupoid has closed connections with commutative algebra. In this section, we generalize the results of Hewitt and Zuckerman for commutative semigroups [5].

An AG-groupoid $S$ is called a locally associative AG-groupoid if $a\cdot aa = aa\cdot a$ for all $a\in S$ [8].

Note that a locally associative AG-groupoid does not necessarily have associative powers. For example, in a locally associative AG-groupoid $S = \{a,b,c\}$, defined by the following table [8]:

| $\cdot$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $c$ | $c$ | $b$ |
| $b$ | $b$ | $b$ | $b$ |
| $c$ | $b$ | $b$ | $b$ |

$$(a\cdot aa)a = b \neq c = a(a\cdot aa).$$

**Definition 4.** A locally associative AG**-groupoid is an AG**-groupoid $S$ satisfying an identity $a\cdot aa = aa\cdot a$ for all $a\in S$.

*Example* 1. Let us consider an AG**-groupoid $S = \{a,b,c,d,e\}$ in the following multiplication table.

| $\cdot$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $e$ | $e$ | $c$ | $e$ |
| $c$ | $a$ | $e$ | $e$ | $b$ | $e$ |
| $d$ | $a$ | $b$ | $c$ | $d$ | $e$ |
| $e$ | $a$ | $e$ | $e$ | $e$ | $e$ |

*It is easy to verify that $S$ is a locally associative* AG**-*groupoid.*

**Proposition 2.** *The following statements hold:*

(1) *Every locally associative* AG$^{**}$*-groupoid has associative powers, that is* $aa^n = a^n a$ *for all* $a \in S$ *and positive integer* $n$ [8].
(2) *In an* AG$^{**}$*-groupoid* $S$, $a^m a^n = a^{m+n}$ *for all* $a \in S$ *and positive integers* $m, n$ [8].
(3) *In a locally associative* AG$^{**}$*-groupoid* $S$, $(a^m)^n = a^{mn}$ *for all* $a \in S$ *and positive integers* $m, n$ [10].
(4) *If* $S$ *is a locally associative* AG$^{**}$*-groupoid and* $a, b \in S$, *then* $(ab)^n = a^n b^n$ *for any* $n \geq 1$ *and* $(ab)^n = b^n a^n$ *for any* $n \geq 2$ [9].
(5) *Let* $S$ *be a locally associative* AG$^{**}$*-groupoid. Then* $a^n = a^{n-1}a = aa^{n-1}$ *for all* $a \in S$ *and* $n > 1$ [10].
(6) *If* $S$ *is a locally associative* AG$^{**}$*-groupoid and* $a, b \in S$, *then* $a^n b^m = b^m a^n$ *for* $m, n > 1$ [8].

Note that $a^{n-1}a = ((((aa)a)a)...a)a$ and $aa^{n-1} = a((((aa)a)a)...a)$.

4.1. *Separative decomposition*

If $S$ is a locally associative AG$^{**}$-groupoid, then $ab^n \cdot c = a \cdot b^n c$ is not generally true for all $a, b, c \in S$, that is $(Sx^n)S \neq S(x^n S)$ for some $x \in S$.

Let us define the relations $\lambda$ and $\mu$ in a locally associative AG$^{**}$-groupoid $S$ as follows:

for all $a, b \in S$, $a\lambda b \iff$ there exists $n \in \mathbb{N}$, such that $a^n \in S(b^n S)$ and $b^n \in S(a^n S)$.

for all $a, b \in S$, $a\mu b \iff$ there exists $n \in \mathbb{N}$, such that $a^n \in (Sb^n)S$ and $b^n \in (Sa^n)S$.

**Theorem 3.** *$\lambda$ is equivalent to $\mu$ on a locally associative* AG$^{**}$*-groupoid* $S$.

*Proof.* Let $a^n \in S(b^n S)$. Then by using Proposition 2(3), we get

$$\begin{aligned} a^{2n} &= (a^n)^2 \in (S \cdot b^n S)^2 = (S \cdot b^n S)(S \cdot b^n S) = (SS)(b^n S \cdot b^n S) \\ &= (SS)(b^n b^n \cdot SS) \\ &= (b^n b^n)(SS \cdot SS) = (SS \cdot SS)(b^n b^n) = (b^n b^n \cdot SS)(SS) \\ &= (SS \cdot b^n b^n)(SS) \\ &\subseteq (Sb^{2n})S. \end{aligned}$$

Similarly, we can show that $b^n \in S(a^n S)$ implies $b^{2n} \in (Sa^{2n})S$.

Conversely, assume that $a^n \in (Sb^n)S$. Then by using Proposition 2(3), we get

$$\begin{aligned} a^{2n} &= (a^n)^2 \in (Sb^n \cdot S)^2 = (Sb^n \cdot S)(Sb^n \cdot S) = (Sb^n \cdot Sb^n)(SS) \\ &= (SS \cdot b^n b^n)(SS) \\ &= (SS)(b^n b^n \cdot SS) \subseteq S(b^{2n} S). \end{aligned}$$

Similarly, we can show that $b^n \in (Sa^n)S$ implies $b^{2n} \in S(a^{2n} S)$. Thus $\lambda$ is equivalent to $\mu$. □

**Theorem 4.** *The relation $\lambda$ on a locally associative* AG$^{**}$*-groupoid $S$ is a congruence relation.*

*Proof.* Clearly $\lambda$ is reflexive and symmetric. For transitivity, let us suppose that $a\lambda b$ and $b\lambda c$, such that $a^n \in S(b^nS)$ and $b^n \in S\cdot c^nS$ for all $a,b,c \in S$ with assumption that $n > 1$. By using Proposition 2(3), we get

$$\begin{aligned} a^n \in S(b^nS) &= b^n(SS) \subseteq (S\cdot c^nS)S = (c^n\cdot SS)S \subseteq (c^nS)S = (SS)c^n \\ &= SS\cdot c^{n-1}c = cc^{n-1}\cdot SS = c^n(SS) = S(c^nS). \end{aligned}$$

Similarly, we can show that $c^n \in S(a^nS)$. Hence $\lambda$ is an equivalence relation. To show that $\lambda$ is compatible, assume that $a\lambda b$ such that for $n > 1$, $a^n \in S(b^nS)$ and $b^n \in S(a^nS)$ for all $a,b \in S$. Let $c \in S$, then

$$\begin{aligned} (ac)^n = a^nc^n \in (S\cdot b^nS)c^n &= (b^n\cdot SS)c^n = (b^{n-1}b\cdot SS)c^n = (SS\cdot bb^{n-1})c^n \\ &= (SS\cdot b^n)c^n = c^nb^n\cdot SS = b^nc^n\cdot SS = S(b^nc^n\cdot S) = S\cdot(bc)^nS. \end{aligned}$$

Similarly, we can show that $(ca)^n \in S((cb)^nS)$. Hence $\lambda$ is a congruence relation on $S$. □

**Definition 5.** A congruence $\sigma$ is said to be separative congruence in $S$, if $ab \equiv_\sigma a^2$ and $ab \equiv_\sigma b^2$ implies that $a \equiv_\sigma b$.

**Theorem 5.** *The relation $\lambda$ on a locally associative* AG$^{**}$*-groupoid $S$ is separative.*

*Proof.* Let $a,b \in S$ such that $ab \equiv_\lambda a^2$ and $ab \equiv_\lambda b^2$. Then for a positive integer $n$,

$$(ab)^n \in S\cdot(a^2)^nS, \quad (a^2)^n \in S\cdot(ab)^nS$$

and

$$(ab)^n \in S\cdot(b^2)^nS, \quad (b^2)^n \in S\cdot(ab)^nS.$$

Now

$$\begin{aligned} a^{2n} &= (a^2)^n \in S\cdot(ab)^nS \in S\cdot(S\cdot(b^2)^nS)S = (S\cdot(b^2)^nS)(SS) \\ &= ((b^2)^n\cdot SS)(SS) \\ &= (SS\cdot SS)(b^nb^n) = (b^nb^n)(SS\cdot SS) = (SS)(b^nb^n\cdot SS) \subseteq S(b^{2n}S). \end{aligned}$$

Similarly we can show that $b^{2n} \in S(a^{2n}S)$. Hence $\lambda$ is separative. □

**Proposition 3.** *If $S$ is a locally associative* AG$^{**}$*-groupoid, then $ab \equiv_\lambda ba$ for all $a,b \in S$, that is $\lambda$ is commutative.*

*Proof.* Let $a,b \in S$ such that $a \equiv_\lambda b$ and $n$ be a positive integer. Then by using Proposition 2(4), we get

$$\begin{aligned} (ab)^n &= a^nb^n \in (S\cdot b^nS)(S\cdot a^nS) = (SS)(b^nS\cdot a^nS) \\ &= (SS)(b^na^n\cdot SS) \subseteq S(ba)^n\cdot S. \end{aligned}$$

Similarly, we can show that $(ba)^n \in S(ab)^n\cdot S$. Hence $ab\lambda ba$. □

**Corollary 2.** *Let $S$ be a locally associative* AG$^{**}$*-groupoid. Then $S/\lambda$ is a separative commutative image of $S$.*

Let us define a relation $\gamma$ on a locally associative AG$^{**}$-groupoid $S$ as follows:
for all $x, y \in S$, $x\gamma y \iff$ there exists $n \in \mathbb{N}$, such that $(xa)^n \in (ya)^n S$ and $(ya)^n \in (xa)^n S$, for some $a \in S$.

**Theorem 6.** *The relation $\gamma$ is a congruence relation on a locally associative* AG$^{**}$*-groupoid $S$.*

*Proof.* Clearly $\gamma$ is reflexive and symmetric. For transitivity let us suppose that $x\gamma y$ and $y\gamma z$, then there exist positive integers $m, n$ such that $(xa)^n \in (ya)^n S$, $(ya)^n \in (xa)^n S$ and $(ya)^m \in (za)^m S$ and $(za)^m \in (ya)^m S$, for some $a \in S$. More specifically, there exists $t_1 \in S$ such that $(xa)^n = (ya)^n t_1$. Assume that $m, n > 1$. Now by using Proposition 2(3) and Proposition 2(4), we get

$$\begin{aligned}(xa)^{mn} &= \big((xa)^n\big)^m = \big((ya)^n t_1\big)^m = \big((ya)^m\big)^n t_1^m \subseteq \big((za)^m S\big)^n S \\ &= (za)^{mn} S^n \cdot S \\ &= (SS^n)(za)^{mn} = (SS^n)\cdot(za)^{mn-1}(za) = (za)(za)^{mn-1}\cdot(S^n S) \\ &\subseteq (za)^{mn} S.\end{aligned}$$

Similarly we can show that $(za)^{mn} \in (xa)^{mn} S$. Hence $\gamma$ is an equivalence relation on $S$.

To show compatibility, let $x\gamma y$, then there exists a positive integer $n$ such that $(xa)^n \in (ya)^n S$ and $(ya)^n \in (xa)^n S$. Hence there exists $t_3 \in S$ such that $(xa)^n = (ya)^n t_3$. Now using Proposition 2(3), Proposition 2(4) and Proposition 2(6) with assumption that $n > 1$, we get

$$\begin{aligned}(xz\cdot a)^{2n} &= \big((xz\cdot a)^n\big)^2 = \big((xz)^n a^n\big)^2 = (x^n z^n \cdot a^n)^2 = (z^n x^n \cdot a^n)^2 \\ &= (a^n x^n \cdot z^n)^2 = (x^n a^n \cdot z^n)^2 = \big((xa)^n z^n\big)^2 = \big((ya)^n t_3 \cdot z^n\big)^2 \\ &= \big((z^n t_3)(ya)^n\big)^2 = (z^{2n} t_3^2)(ya)^{2n} = (z^n z^n \cdot t_3 t_3)(ya)^{2n} \\ &= (t_3 t_3 \cdot z^n z^n)(ya)^{2n} \\ &= (t_3^2 z^{2n})(ya)^{2n} = \big((t_3 z^n)(ya)^n\big)^2 = \big((ya)^n z^n \cdot t_3\big)^2 \\ &= \big((y^n a^n) z^n \cdot t_3\big)^2 \\ &= \big((a^n y^n) z^n \cdot t_3\big)^2 = \big((z^n y^n) a^n \cdot t_3\big)^2 = \big((y^n z^n) a^n \cdot t_3\big)^2 \\ &= \big((yz\cdot a)^n t_3\big)^2 \\ &= (yz\cdot a)^{2n} t_3^2 \in (yz\cdot a)^{2n} S.\end{aligned}$$

Similarly, we can show that $(yz\cdot a)^{2n} \in (xz\cdot a)^{2n} S$. Therefore $xz\gamma yz$. Similarly we can show that $\gamma$ is left compatible. Hence $\gamma$ is a congruence relation on $S$. $\square$

4.2. *Anti-separative decomposition*

In this section, we show that $S/\tau$ is a maximal anti-separative commutative image of a locally associative AG**-groupoid $S$, where $\tau$ is defined as follows:

$a\tau b$ if and only if $ab^n = b^{n+1}$ and $ba^n = a^{n+1}$ for all $a, b \in S$ and a positive integer $n$.

**Lemma 7.** *Let $S$ be a locally associative* AG**-*groupoid. If $ab^m = b^{m+1}$ and $ba^n = a^{n+1}$ for $a, b \in S$ and positive integers $m, n$, then $a\tau b$.*

*Proof.* Without loss of generality let us suppose that $n > m$. Thus by using Proposition 2(2), we get

$$b^{n-m}b^{m+1} = b^{n-m} \cdot ab^m = a \cdot b^{n-m}b^m = ab^n.$$

Hence $a\tau b$. □

**Theorem 7.** *The relation $\tau$ on a locally associative* AG**-*groupoid $S$ is a congruence relation.*

*Proof.* Clearly $\tau$ is reflexive and symmetric. For transitivity, let $a\tau b$ and $b\tau c$, so there exist positive integers $m, n$ such that $ab^n = b^{n+1}$, $ba^n = a^{n+1}$ and $bc^m = c^{m+1}$, $cb^m = b^{m+1}$. Let $k = (n+1)(m+1) - 1$, that is $k = n(m+1) + m$. Now by using Proposition 2(3) and Proposition 2(6), we get

$$\begin{aligned}
ac^k &= ac^{n(m+1)+m} = a \cdot c^{n(m+1)}c^m = a \cdot (c^{m+1})^n c^m = a \cdot (bc^m)^n c^m \\
&= a \cdot (b^n c^{mn})c^m \\
&= b^n c^{mn} \cdot ac^m = b^n a \cdot c^{mn}c^m = c^m c^{mn} \cdot ab^n = c^m c^{mn} \cdot b^{n+1} \\
&= c^{m(n+1)}b^{n+1} \\
&= c^{m(n+1)-1}c \cdot b^n b = bb^n \cdot cc^{m(n+1)-1} = b^{n+1}c^{m(n+1)} = (bc^m)^{n+1} \\
&= (c^{m+1})^{n+1} \\
&= c^{(m+1)(n+1)} = c^{k+1}.
\end{aligned}$$

Similarly, we can show that $ca^k = a^{k+1}$. Thus $\tau$ is an equivalence relation. To show that $\tau$ is compatible, assume that $a\tau b$ such that for some integer $n$,

$$ab^n = b^{n+1} \text{ and } ba^n = a^{n+1}.$$

Let $c \in S$. By using Proposition 2(4), we get

$$(ac)(bc)^n = ac \cdot b^n c^n = ab^n \cdot cc^n = b^{n+1}c^{n+1} = (bc)^{n+1}.$$

Similarly, we can show that $(bc)(ac)^n = (ac)^{n+1}$. Hence $\tau$ is a congruence relation on $S$. □

**Definition 6.** A congruence $\sigma$ is said to be anti-separative congruence in $S$, if $ab \equiv_\sigma a^2$ and $ba \equiv_\sigma b^2$ implies that $a \equiv_\sigma b$.

**Theorem 8.** *The relation $\tau$ is anti-separative.*

*Proof.* Let $a,b \in S$ such that $ab \equiv_\tau a^2$ and $ba \equiv_\tau b^2$. Then by definition of $\tau$ there exist positive integers $m$ and $n$ such that

$$(ab)(a^2)^m = (a^2)^{m+1},\ a^2(ab)^m = (ab)^{m+1},$$

and

$$(ba)(b^2)^n = (b^2)^{n+1},\ b^2(ba)^n = (ba)^{n+1}.$$

Now by using Proposition 2(2) and Proposition 2(3), we get

$$\begin{aligned} ba^{2m+1} &= b\cdot a^{2m}a = a^{2m}\cdot ba = a^m a^m \cdot ba = ab\cdot a^m a^m = ab\cdot a^{2m} \\ &= (ab)(a^2)^m = (a^2)^{m+1} = a^{2m+2}, \end{aligned}$$

and

$$\begin{aligned} ab^{2n+1} &= a\cdot b^{2n}b = b^{2n}\cdot ab = b^n b^n \cdot a = ba\cdot b^n b^n = ba\cdot b^{2n} \\ &= (ba)(b^2)^n = (b^2)^{n+1} = b^{2n+2}. \end{aligned}$$

Thus by using Lemma 7, $a \equiv_\tau b$. Hence $\tau$ is anti-separative. □

**Proposition 4.** *If $S$ is a locally* AG$^{**}$*-groupoid, then $ab \equiv_\tau ba$ for all $a,b \in S$, that is $\tau$ is commutative.*

*Proof.* Let $a,b \in S$ and $n$ be a positive integer. Then by using Proposition 2(6), Proposition 2(2) and Proposition 2(4) with assumption that $n > 1$, we get

$$(ab)(ba)^n = ab\cdot b^n a^n = ab\cdot a^n b^n = aa^n \cdot bb^n = b^n b\cdot a^n a = b^{n+1}a^{n+1} = (ba)^{n+1}.$$

Similarly we can show that $(ba)(ab)^n = (ab)^{n+1}$. Hence $ab \equiv_\tau ba$. □

**Theorem 9.** *Let $S$ be a locally associative* AG$^{**}$*-groupoid. Then $S/\tau$ is a maximal anti-separative commutative image of $S$.*

*Proof.* By Theorem 8, $\tau$ is anti-separative, and hence $S/\tau$ is anti-separative. We now show that $\tau$ is contained in every anti-separative congruence relation $\xi$ on $S$. Let $a \equiv_\tau b$ so that there exists a positive integer $n$ such that

$$ab^n = b^{n+1} \text{ and } ba^n = a^{n+1}.$$

We need to show that $a \equiv_\xi b$, where $\xi$ is an anti-separative congruence on $S$. Let $k$ be a positive integer such that

$$ab^k \equiv_\xi b^{k+1} \text{ and } ba^k \equiv_\xi a^{k+1}.$$

Suppose that $k > 2$. Now by using Proposition 2(2), we get

$$\begin{aligned} (ab^{k-1})^2 &= ab^{k-1}\cdot ab^{k-1} = aa\cdot b^{k-1}b^{k-1} = a^2b^{2k-2}, \\ a^2b^{2k-2} &= aa\cdot b^{k-2}b^k = ab^{k-2}\cdot ab^k \xi ab^{k-2}\cdot b^{k+1} \\ &= ab^{k-2}\cdot b^k b = ab^k \cdot b^{k-2}b = ab^k\cdot b^{k-1}, \end{aligned}$$

and from above, we have

$$a^2b^{2k-2} \equiv_\xi ab^k \cdot b^{k-1} = b^{k-1}b^k \cdot a = b^k b^{k-1} \cdot a = ab^{k-1} \cdot b^k.$$

Thus $(ab^{k-1})^2 \equiv_\xi ab^k \cdot b^{k-1}$. Since $ab^k \equiv_\xi b^{k+1}$ implies that $ab^k \cdot b^{k-1} \equiv_\xi b^{k+1} \cdot b^{k-1}$. Hence $(ab^{k-1})^2 \equiv_\xi (b^k)^2$. It further implies that

$$(ab^{k-1})^2 \equiv_\xi a^2b^{2k-2} = b^{2k-2}a^2 \equiv_\xi (b^k)^2.$$

Thus $ab^{k-1} \equiv_\xi b^k$. Similarly we can show that $ba^{k-1} \equiv_\xi a^k$.

By induction down from $k$, it follows that for $k = 1$, $ab \equiv_\xi b^2$ and $ba \equiv_\xi a^2$. Hence by using anti-separativity and Proposition 4, it follows that $S/\tau$ is a maximal anti-separative commutative image of $S$. □

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# MODULES WHOSE PROJECTION INVARIANT SUBMODULES HAVE PROJECTION INVARIANT CLOSURES

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*Abstract.* We elaborate the class of $PC$-modules in which every projection invariant submodules has projection invariant closures in this study. We provide examples that the class of $PC$-modules does not belong to the system of generalization of extending modules. Moreover, we clarify direct sums and direct summands properties for the former class of modules. It is proved that the aforementioned property is not closed under direct sums. Thereupon, we cope with when the direct sums of $PC$-modules enjoy with the property.

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*Keywords:* complement submodule, extending module, projection invariant submodule

## 1. Introduction

Let $R$ be a ring with unity and $M$ a unitary right $R$-module. A submodule $K$ of $M$ is called a *complement* in $M$ if $K$ has no proper essential extension in $M$. Recall that a module $M$ is called *extending* (or $CS$) if every submodule $N$ of $M$ is essential in a direct summand $M$; or every complement submodule of $M$ is a direct summand of $M$ [4, 15]. A submodule $N$ of $M$ is called a *projection (fully) invariant* if $f(N) \subseteq N$ for all $f^2 = f \in End(M_R)$ ($f \in End(M_R)$). It is well known that every fully invariant submodule is projection invariant (see, [5, page 50]). There are many examples of projection invariant submodules in different algebraic structures. Motivating on this class of submodules, a module $M$ is said to be *$\pi$-extending* [3], if every projection invariant submodule of $M$ is essential in a direct summand of $M$. It is shown that the class of $\pi$-extending modules is closed under direct sums, but not direct summands (see, [3, Example 5.5], [14, Example 4]). Hence it is investigated some special subclasses of $\pi$-extending modules. To this end, a module $M$ is called *strongly $\pi$-extending* [9], if every projection invariant submodule of $M$ is essential in a fully invariant direct summand of $M$. It is proved that the class of strongly $\pi$-extending modules is a proper subclass of $\pi$-extending modules.

Let $N$ be a submodule of $M$. A submodule $T$ of $M$ is called a *closure* (or *essential closure*) of $N$ in $M$, if $N \leq_e T \leq_c M$. It is renowned that, every submodule has a closure (see, [15, Proposition 2.5]). Consequently, we concern with closure

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properties of projection invariant submodules. Thereby, we call a module $M$ is *projection invariant closure module* (denoted, $PC$*-module*), if every projection invariant submodule of $M$ has a projection invariant closure in $M$. We observe indecomposable modules, uniform modules and nonsingular modules are the examples of $PC$-modules. Moreover it is shown that the $PC$ condition is more general than strongly $\pi$-extending property. Although strongly $\pi$-extending property implies $\pi$-extending condition, we provide by counter examples that the classes of $PC$-modules and $\pi$-extending modules are incomparable. Apart from that, we delve into the direct sums and direct summands properties of $PC$-modules. We prove that projection invariant direct summands of $PC$-modules are $PC$-modules. Even so the aforementioned property is not closed under direct sums. Therefore we show when the direct sums of $PC$-modules is $PC$-module. To this end, a module $M$ has *complement sum property, CSP* [7], if the sum of every pair of complements of $M$ is a complement of $M$. Furthermore we get the hang of CSP condition which is not Morita invariant. Additionally, we characterize that the extending and quasi-continuous conditions are equivalent for a module with CSP.

For notation, we use $M_n(R)$ and $R^n$ for the full $n$-by-$n$ matrix ring over $R$ and the direct sum of $n$ copies of $R$ for any positive integer $n$, respectively. For a nonempty subset $X$ of $M$, $X \leq M$, $X \leq_e M$, $X \leq_c M$, $X \leq_d M$ and $X \trianglelefteq_p M$ denote the submodule of $M$, the essential submodule of $M$, the complement submodule of $M$, the direct summand of $M$ and the projection invariant submodule of $M$, respectively. For unknown terminology and notation, see [1, 4, 10, 11].

## 2. Preliminary Results

We locate $PC$-modules with the other renowned classes of modules (e.g., $\pi$-extending, strongly $\pi$-extending) in this section. Observe that indecomposable modules and uniform modules are the examples of $PC$-modules. Furthermore, nonsingular modules have the foregoing property as shown in the first result.

**Lemma 1.** *Every nonsingular module is a $PC$-module.*

*Proof.* Let $M_R$ be a nonsingular module and $X \trianglelefteq_p M$. Then $X \leq_e T \leq_c M$ for some submodule $T$ of $M$. Since $M_R$ is nonsingular and $X \trianglelefteq_p M$, $T$ is a projection invariant submodule of $M$ by [3, Lemma 2.3]. Hence $M$ is a $PC$-module. □

Let $M_{\mathbb{Z}} = \mathbb{Z}/\mathbb{Z}p$ for any prime $p$. It is clear that $M_{\mathbb{Z}}$ is not nonsingular, but it is a $PC$-module. It follows that the converse of above lemma is not true. Moreover, there are examples which show that the $PC$ condition does not imply uniform or indecomposable properties. For example, let $M_{\mathbb{Z}} = (\mathbb{Z} \oplus \mathbb{Z})_{\mathbb{Z}}$. It is renowned that $M_{\mathbb{Z}}$ is nonsingular, and hence it is a $PC$-module by Lemma 1. However $M_{\mathbb{Z}}$ is neither uniform nor indecomposable. The next result identifies the connection between the classes of strongly $\pi$-extending modules and $PC$-modules.

**Lemma 2.** *If $M_R$ is a strongly $\pi$-extending module, then $M_R$ has $PC$ condition. But the converse need not to be true.*

*Proof.* Let $M_R$ be a strongly $\pi$-extending module and $N \trianglelefteq_p M$. Then $N \leq_e D \leq_d M$ for some fully invariant direct summand $D$ of $M$. It follows that $N \leq_e D \leq_c M$ where $D \trianglelefteq_p M$. Therefore $M_R$ is a $PC$-module. On the other hand, for the converse, let $R$ be a domain which is not right Ore. Since $R_R$ is a prime ring, every nonzero ideal of $R_R$ is essential in $R_R$. Thus $R_R$ is a $PC$-module, because $R$ is an indecomposable $R$-module. However $R_R$ is not uniform, so $R$ is not $\pi$-extending and hence it is not strongly $\pi$-extending by [9, Corollary 2.4]. □

It is interpreted that every strongly $\pi$-extending module is $\pi$-extending in [9]. In connection with the above lemma one might ask that are there any implications between $PC$-modules and $\pi$-extending modules? Now, we get across the following counter examples for the former question.

*Example* 1. $(i)$ Consider $M_{\mathbb{Z}} = \prod_{i \in I} A_i$ where $A_i = \mathbb{Z}$ for $i \in I$. Note that the Specker group $M_{\mathbb{Z}}$ is not $\pi$-extending by [5]. On the other hand, $M_{\mathbb{Z}}$ is a nonsingular by [6, Proposition 1.22]. Thereupon $M_{\mathbb{Z}}$ is a PC-module by Lemma 1.

$(ii)$ Let $M_{\mathbb{Z}} = \mathbb{Z} \oplus (\mathbb{Z}/\mathbb{Z}p)$ for any prime $p$. Then $M_{\mathbb{Z}}$ is $\pi$-extending which is not strongly $\pi$-extending by [9, Corollary 2.4 $(iv) \not\Rightarrow (i)$]. Hence $M_{\mathbb{Z}}$ is not a PC-module by Proposition 1(i).

Note that a ring $R$ is called *Abelian* if every idempotent of $R$ is central. The next fact provides the implications between the classes of $\pi$-extending modules, strongly $\pi$-extending modules and $PC$-modules under some additional conditions.

**Proposition 1.** $(i)$ *Let $M_R$ be a PC-module. Then $M_R$ is a $\pi$-extending module if and only if $M_R$ is a strongly $\pi$-extending module.*

$(ii)$ *Let $S = End(M_R)$ be an Abelian ring. If $M_R$ is a $\pi$-extending, then $M_R$ is a PC-module.*

$(iii)$ *If $M_R$ be a PC-module and every projection invariant essentially closed submodule of $M$ is a fully invariant direct summand, then $M_R$ is strongly $\pi$-extending.*

*Proof.* $(i)$ Let $M_R$ be a $\pi$-extending module and $X \trianglelefteq_p M$. Then $X \leq_e K \leq_d M$ for some direct summand $K$ of $M$. Since $M_R$ is a $PC$-module, $K$ is projection invariant in $M$. Hence $K$ is a fully invariant submodule of $M$ by [5, page 50]. Thus $M$ is a strongly $\pi$-extending module. The converse follows by [9, Corollary 2.4].

$(ii)$ Let $X \trianglelefteq_p M$. Then $X \leq_e K \leq_d M$ for some submodule $K$ of $M$. Since $S$ is Abelian, it can be easily seen that every direct summand of $M$ is projection invariant in $M$. Therefore $M$ is a $PC$-module.

$(iii)$ Let $M$ be a $PC$-module and $X \trianglelefteq_p M$. Then there exists a projection invariant submodule $K$ of $M$ such that $X \leq_e K \leq_c M$. By hypothesis, $K$ is a fully invariant direct summand, so $M$ is strongly $\pi$-extending. □

Notice that $M_{\mathbb{Z}} = (\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Q}$ is a $PC$-module for any prime $p$. Indeed, it is clear from [8, Example 2.14], $M_{\mathbb{Z}}$ is a $\pi$-extending module with an Abelian endomorphism ring. Thereby, $M_{\mathbb{Z}}$ is a $PC$-module by Proposition 1$(ii)$. This example and Example 1$(ii)$ show that any submodules of a $PC$-module need not to be a $PC$-module, in general.

One might wonder whether $PC$-condition belongs to the system of generalization of extending modules or not. Hence we supply examples which simplify that there is no implication between the classes of $PC$-modules and extending modules.

*Example* 2. $(i)$ Let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$ be the upper triangular 2-by-2 matrix ring over $\mathbb{Z}$. It is follows that $R_R$ is not extending by [15, Example 3.84]. Since $R_R$ is nonsingular, it is a $PC$-module by Lemma 1.

$(ii)$ Let $S_3$ be the symmetric group on the letter $\{1, 2, 3\}$ and $\mathbb{Z}_3$ the ring of integers modulo 3. Let $R = \mathbb{Z}_3[S_3]$ be the group ring of the group $S_3$ over $\mathbb{Z}_3$. Then $R$ is right self-injective. Thus $R_R$ is extending, and hence it is $\pi$-extending. However $R_R$ is not a strongly $\pi$-extending module by [2, Example 1.1] and [9, Corollary 2.4]. Now assume that $R_R$ is a $PC$-module. Then $R_R$ is not $\pi$-extending by Proposition 1(i), a contradiction. Therefore $R_R$ is not a $PC$-module.

Recall from [12], a module $M$ is called *$UC$-module*, if every submodule of $M$ has a unique closure. Following the idea in [12], it is natural to think of *unique projection invariant closure module* (denoted, $UPC$-module) in which every projection invariant submodule has a unique projection invariant closure.

Observe that every nonsingular and $PC$-module are the examples of $UPC$-modules. However basically the results and their proofs arise out of just simple motifications of the result in [12]. To this end, Example 1$(ii)$ and Example 2$(ii)$ identify that projection invariant submodules need not to have a unique projection invariant closure.

## 3. Direct sums and direct summands

In this section, our main goal is to deal with the direct summand and direct sum properties of $PC$-module.

**Proposition 2.** *Let $M$ be a $PC$-module such that $M = M_1 \oplus M_2$ where $M_1$ and $M_2$ are projection invariant submodules of $M$. Then $M_1$ and $M_2$ are $PC$-modules.*

*Proof.* Let $M = M_1 \oplus M_2$ for some projection invariant $M_1, M_2 \leq M$ and let $X_1$ be a projection invariant submodule of $M_1$. Then $X_1 \oplus M_2$ is a projection invariant submodule of $M$ by [3, Lemma 4.13]. Hence $X_1 \oplus M_2 \leq_e K \leq_c M$ for some projection invariant submodule $K$ of $M$. Since $K \trianglelefteq_p M$, $K = (K \cap M_1) \oplus (K \cap M_2)$ where $K_1 = K \cap M_1 \trianglelefteq_p M_1$ and $K_2 = K \cap M_2 \trianglelefteq_p M_2$ by [5, page 50]. Hence we obtain $X_1 = M_1 \cap (X_1 \oplus M_2) \leq_e K \cap M_1 = K_1$. It is easy to see that $K_1 \leq_c M$. It follows from [10, Proposition 6.24 (1)] that $K_1 \leq_c M_1$. Therefore $M_1$ is a $PC$-module. Similarly, it can be seen that $M_2$ is a $PC$-module. $\square$

Example 1($ii$) examines that the direct sums of $PC$-modules need not to be $PC$-module. To this end, we determine when the class of $PC$-modules is closed under direct sums.

**Proposition 3.** *Let $M = M_1 \oplus M_2$ be an extending module for some $M_1, M_2 \leq M$. If $M_1$ and $M_2$ are PC-modules, then $M$ is a PC-module.*

*Proof.* Let $Y \trianglelefteq_p M$. Then $Y = (Y \cap M_1) \oplus (Y \cap M_2)$ where $Y_1 = Y \cap M_1 \trianglelefteq_p M_1$ and $Y_2 = Y \cap M_2 \trianglelefteq_p M_2$ by [5, page 50]. Hence there exist projection invariant submodules $K_1$ of $M_1$ and $K_2$ of $M_2$ such that $X_1 \leq_e K_1 \leq_c M_1$ and $X_2 \leq_e K_2 \leq_c M_2$. It follows that $M_1$ and $M_2$ are extending by [11, Proposition 2.7]. Thus $K_1$ and $K_2$ are direct summands of $M_1$ and $M_2$, respectively. Consequently $K_1 \oplus K_2$ is a direct summand of $M$, so $K_1 \oplus K_2$ is a complement in $M$. Therefore $X = X_1 \oplus X_2 \leq_e K_1 \oplus K_2 \leq_c M$ such that $K_1 \oplus K_2 \trianglelefteq_p M$. Thereby $M$ is a $PC$-module. □

**Corollary 1.** *Let $M = M_1 \oplus M_2$ be a semisimple (or uniform, or injective) module for some $M_1, M_2 \leq M$. If $M_1$ and $M_2$ are PC-modules, then $M$ is a PC-module.*

*Proof.* It is a consequence of Proposition 3. □

Recall that a module $M_R$ has the *summand sum property, SSP*, if for all $D_1, D_2 \leq_d M_R$, $D_1 + D_2 \leq_d M$. Motivating SSP definition on complement submodules, $M$ has *complement sum property, CSP* [7], if for all $K_1, K_2 \leq_c M_R$, $K_1 + K_2 \leq_c M$. Even though the authors in [7] proved that CSP condition implies SSP, we give the following example which shows that the reverse implication of the former property is not true, in general.

*Example* 3. Let $M$ be the $\mathbb{Z}$-module $\mathbb{Z} \oplus (\mathbb{Z}/\mathbb{Z}p)$ as in Example 1(ii). Note that

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/\mathbb{Z}p) = \mathbb{Z}/\mathbb{Z}p \text{ and } \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/\mathbb{Z}p, \mathbb{Z}) = 0.$$

Thus $M$ has SSP by [15, Exercise 2.41]. On the other hand, we claim $M_{\mathbb{Z}}$ does not have CSP. Assume the contrary. Since $\mathbb{Z}$ and $\mathbb{Z}/\mathbb{Z}p$ are $PC$-modules, $M_{\mathbb{Z}}$ is a $PC$-module by Theorem 3, a contradiction (see, Example 1(ii)). Therefore $M$ does not satisfy CSP.

Now we compose some useful properties of modules with CSP which might help us to consider being Morita invariant property as well as the application to the full matrix rings for the former condition. Let us begin with an easy fact and an example for extending case.

**Lemma 3.** *SSP and CSP conditions are coincide for an extending module.*

*Proof.* It is routine to check. □

*Example* 4. Let $K$ be a field and $R_R = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$. Then $R_R$ is an extending module which does not satisfy $C_3$. Hence $R_R$ does not have SSP. It follows from Lemma 3 that $R_R$ does not have CSP.

**Theorem 1.** *Let $R$ be a ring such that $R = ReR$ and $S = eRe$ for some $e^2 = e \in R$. Then $M_R$ has CSP if and only if the right $S$-module module $Me$ has CSP.*

*Proof.* It is clear from [15, Lemma 2.76 and Proposition 2.77 (ii)]. □

**Corollary 2.** *Let $R$ be a ring such that $R = ReR$ for some $e^2 = e \in R$. Then $R_R$ has CSP if and only if the right $eRe$-module $Re$ satisfies CSP.*

*Proof.* It is clear from Theorem 1. □

**Theorem 2.** *$M_n(R)$ has CSP condition if and only if the free right $R$-module $R^n$ has CSP condition.*

*Proof.* Note that $M_n(R) = M_n(R)eM_n(R)$ where $e$ is the matrix unit with 1 in the $(1,1)$-th position and zero elsewhere. Now apply the Theorem 1 and Corollary 2 to get the theorem. □

As an application of Theorem 2, it can be easily seen that $M_2(\mathbb{Z})$ does not have CSP condition. Indeed, it is renowned that $M_{\mathbb{Z}} = (\mathbb{Z} \oplus \mathbb{Z})_{\mathbb{Z}}$ is an extending module which does not satisfy SSP by [15, Example 2.82]. Hence $M_{\mathbb{Z}}$ does not have CSP by Lemma 3. Thus $M_2(\mathbb{Z})$ does not have CSP by Theorem 2. This example explains that CSP condition is not Morita invariant. Now we proceed our main aim of this section.

**Theorem 3.** *Let $M$ be a right $R$-module with CSP such that $M = M_1 \oplus M_2$ for some $M_1, M_2 \leq M$. If $M_1$ and $M_2$ are $PC$-modules, then $M$ is a PC-module.*

*Proof.* Let $X \trianglelefteq_p M$. Then $X = (X \cap M_1) \oplus (X \cap M_2)$ where $X_1 = X \cap M_1 \trianglelefteq_p M_1$ and $X_2 = X \cap M_2 \trianglelefteq_p M_2$ by [5, page 50]. Hence there exist projection invariant submodules $K_1$ of $M_1$ and $K_2$ of $M_2$ such that $X_1 \leq_e K_1 \leq_c M_1$ and $X_2 \leq_e K_2 \leq_c M_2$. Thus $X \leq_e K_1 \oplus K_2 \leq M$ where $K_1 \oplus K_2 \trianglelefteq_p M$. Note that $K_1 \oplus K_2 \leq_c M$ by CSP condition. Thus $M$ is a $PC$-module. □

Recall from [13], a module $M$ has $(P_n)$ condition if for every submodule $K$ of $M$ such that $T$ is a direct sum $T_1 \oplus \cdots \oplus T_n$ of complements $T_i$ $(1 \leq i \leq n)$ in $M$, every homomorphism $\alpha_1 : T \to M$ can be lifted to a homomorphism $\alpha_2 : M \to M$. In [13], the authors proved that if $M$ satisfies $(P_n)$ then $M$ satisfies $(P_{n-1})$ for all $n \geq 2$. Moreover, they engage a characterization of quasi-continuous modules in terms of $(P_n)$ conditions for every positive integer $n$. Realize that $(P_1)$ does not imply $(P_2)$ (see, [13, Example 10]). The following result spells out $(P_1)$ and $(P_2)$ conditions are equivalent for a module with CSP.

**Proposition 4.** *Let $M$ be a right $R$-module with CSP. Then $M$ has $(P_2)$ condition if and only if $M$ has $(P_1)$ condition.*

*Proof.* Let $M$ be a right $R$-module with CSP. If $M$ has $(P_2)$, then $M$ has $(P_1)$ by [13, page 341]. Conversely, let $M$ has $(P_1)$. Consider $K_1, K_2 \leq_c M$ with $\alpha :$

$K_1 \oplus K_2 \to M$ homomorphism. Since $M$ has CSP, $K_1 \oplus K_2 \leq_c M$. Hence $\alpha : K_1 \oplus K_2 \to M$ can be lifted to a homomorphism $\theta : M \to M$ by the condition of $(P_1)$. Therefore $M$ has $(P_2)$ condition. □

We strengthen the characterization of quasi-continuous modules, which is presented in [13], for a module with CSP condition.

**Theorem 4.** *The followings are equivalent for a module $M_R$ with CSP.*
(1) *$M$ is quasi-continuous.*
(2) *$M$ has $(P_n)$ for every positive integer $n$.*
(3) *$M$ has $(P_n)$ for some integer $n \geq 2$.*
(4) *$M$ has $(P_2)$.*
(5) *$M$ has $(P_1)$.*
(6) *$M$ is extending.*

*Proof.* (1) ⇔ (2) ⇔ (3) ⇔ (4) Clear from [13, Theorem 4].
(4) ⇔ (5) It follows from Proposition 4.
(5) ⇔ (6) Let $M$ has $(P_1)$. Then $M$ has $(P_2)$ by Proposition 4. Hence $M$ is extending from the fact of (4) ⇔ (1). Conversely, let $K \leq_c M$ and $\alpha : K \to M$ be a homomorphism. Since $M$ is extending, $K$ is a direct summand of $M$. Consider $g = \pi\iota$ where $\pi : M \to K$ is projection and $\iota : K \to M$ is inclusion. It is easy to check that $g|_K = \alpha$, hence $M$ has $(P_1)$. □

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# LIGHTLIKE SUBMERSIONS FROM TOTALLY UMBILICAL SEMI-TRANSVERSAL LIGHTLIKE SUBMANIFOLDS

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*Abstract.* We study lightlike submersions from a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold onto an indefinite almost Hermitian manifold. We show that if an indefinite almost Hermitian manifold $B$ admits a lightlike submersion $\phi : M \to B$ from a totally umbilical semi-transversal lightlike submanifold $M$ of an indefinite Kaehler manifold $\bar{M}$ then $B$ is necessarily an indefinite Kaehler manifold. We investigate the condition for a totally umbilical semi-transversal lightlike submanifold $M$ to becomes a product manifold and its fibers become geodesic. Finally, we obtain some characterization theorems related to the sectional curvature of an indefinite Kaehler manifold.

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## 1. Introduction

The study of Riemannian submersions $\phi : M \to B$, from a Riemannian manifold $M$ onto a Riemannian manifold $B$ was initiated by O'Neill [10]. A Riemannian submersion naturally yields a vertical distribution, which is always integrable and a horizontal distribution. On the other hand, for a $CR$-submanifold $M$ of a Kaehler manifold $\bar{M}$ there are two orthogonal complementary distributions $D$ and $D^{\perp}$, such that $D$ is $\bar{J}$-invariant and $D^{\perp}$ is totally real and always integrable (cf. Bejancu [2]), where $\bar{J}$ is almost complex structure of $\bar{M}$. Kobayashi [9] observed the similarity between the total space of a Riemannian submersion and a $CR$-submanifold of a Kaehler manifold in terms of distributions. Then Kobayashi [9] introduced a submersion $\phi : M \to B$, from a $CR$-submanifold $M$ of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold $B$ such that the distributions $D$ and $D^{\perp}$ of the $CR$-submanifold become the horizontal and the vertical distributions respectively, as required by the submersions and $\pi$ restricted to $D$ becomes a complex isometry.

Later, semi-Riemannian submersions were introduced by O'Neill in [11]. As it is known that when $M$ and $B$ are Riemannian manifolds then the fibers are always

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Riemannian manifolds. However, when the manifolds are semi-Riemannian manifolds then the fibers may not be Riemannian (hence semi-Riemannian) manifolds, (see [15]). Therefore in [13], Sahin introduced a screen lightlike submersion from a lightlike manifold onto a semi-Riemannian manifold and in [15], Sahin and Gunduzalp introduced a lightlike submersion from a semi-Riemannian manifold onto a lightlike manifold. It is well-known that semi-Riemannian submersions are of interest in mathematical physics, owing to their applications in the Yang-Mills theory, Kaluza-Klein theory, supergravity and superstring theories [3, 4, 8, 16]. Moreover, the geometry of lightlike submanifolds has potential for applications in mathematical physics, particularly in general relativity (for detail, see [5]) therefore in present paper, we study lightlike submersions from a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold onto an almost Hermitian manifold.

## 2. Lightlike submanifolds

Let $(\bar{M},\bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index $q$ such that $m,n \geq 1$, $1 \leq q \leq m+n-1$ and $(M,g)$ be an $m$-dimensional submanifold of $\bar{M}$ and $g$ be the induced metric of $\bar{g}$ on $M$. If $\bar{g}$ is degenerate on the tangent bundle $TM$ of $M$ then $M$ is called a lightlike submanifold of $\bar{M}$, (see [5]). For a degenerate metric $g$ on $M$, $TM^{\perp}$ is a degenerate $n$-dimensional subspace of $T_x\bar{M}$. Thus both $T_xM$ and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $Rad(T_xM) = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping $Rad(TM) : x \in M \longrightarrow Rad(T_xM)$, defines a smooth distribution on $M$ of rank $r > 0$ then the submanifold $M$ of $\bar{M}$ is called an $r$-lightlike submanifold and $Rad(TM)$ is called the radical distribution on $M$.

Screen distribution $S(TM)$ is a semi-Riemannian complementary distribution of $Rad(TM)$ in $TM$, that is, $TM = Rad(TM) \perp S(TM)$ and $S(TM^{\perp})$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to $TM$ in $T\bar{M}\mid_M$ and to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ respectively. Then $T\bar{M}\mid_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp})$.

**Theorem 1** ([5]). *Let $(M,g,S(TM),S(TM^{\perp}))$ be an $r$-lightlike submanifold of a semi-Riemannian manifold $(\bar{M},\bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and a basis of $ltr(TM)\mid_{\mathcal{U}}$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_{\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of $M$ such that*

$$\bar{g}(N_i,\xi_j) = \delta_{ij}, \quad \bar{g}(N_i,N_j) = 0, \textit{for any} \quad i,j \in \{1,2,..,r\}, \tag{2.1}$$

*where $\{\xi_1,...,\xi_r\}$ is a lightlike basis of $Rad(TM)$.*

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ then for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{2.2}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here $\nabla$ is a torsion-free linear connection on $M$, $h$ is a symmetric bilinear form on $\Gamma(TM)$ which is called the second fundamental form, $A_U$ is a linear operator on $M$ and known as a shape operator.

Considering the projection morphisms $L$ and $S$ of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, then (2.2) becomes

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \quad \bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2.3}$$

where $h^l(X,Y) = L(h(X,Y)), h^s(X,Y) = S(h(X,Y)), D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$. As $h^l$ and $h^s$ are $ltr(TM)$-valued and $S(TM^{\perp})$-valued respectively, therefore they are called as the lightlike second fundamental form and the screen second fundamental form on $M$. In particular

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \quad \bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{2.4}$$

where $X \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Using (2.3) and (2.4), we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y). \tag{2.5}$$

Let $\bar{R}$ and $R$ be the curvature tensors of $\bar{\nabla}$ and $\nabla$, respectively then by straightforward calculations (see [5]), we have

$$\begin{aligned}\bar{R}(X,Y)Z = {} & R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y \\ & - A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) \\ & + D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z)) + (\nabla_X h^s)(Y,Z) \\ & - (\nabla_Y h^s)(X,Z) + D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z)).\end{aligned} \tag{2.6}$$

## 3. Semi-transversal lightlike submanifolds

Let $(\bar{M}, \bar{J}, \bar{g})$ be an indefinite almost Hermitian manifold and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ with respect to the indefinite metric $\bar{g}$. Then $\bar{M}$ is called an indefinite Kaehler manifold [1] if the almost complex structure $\bar{J}$ is parallel with respect to $\bar{\nabla}$, that is $(\bar{\nabla}_X \bar{J})Y = 0$, for any $X, Y \in \Gamma(T\bar{M})$.

**Definition 1** ([12])**.** Let $M$ be a lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ then $M$ is called a semi-transversal lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(i) $Rad(TM)$ is transversal with respect to $\bar{J}$.

(ii) There exists a real non-null distribution $D \subset S(TM)$ such that $S(TM) = D \oplus D^{\perp}$, $\bar{J}(D) = D$, $\bar{J}D^{\perp} \subset S(TM^{\perp})$, where $D^{\perp}$ is orthogonal complementary to $D$ in $S(TM)$.

Then tangent bundle of a semi-transversal lightlike submanifold is decomposed as $TM = D \perp D'$, where $D' = D^{\perp} \perp Rad(TM)$. We say $M$ is a proper semi-transversal lightlike submanifold if $D \neq \{0\}$ and $D^{\perp} \neq \{0\}$. Therefore $dim(Rad(TM)) \geq 2$ and for a proper $M$, $dim(D) \geq 2s, s > 1$, $dim(D^{\perp}) \geq 1$ and $dim(Rad(TM)) = dim(ltr(TM))$. Thus $dim(M) \geq 5$ and $dim(\bar{M}) \geq 8$. Next, we give example of semi-transversal lightlike submanifolds.

*Example* 1. Let $M$ be a 5-dimensional submanifold of $(R_2^{10}, \bar{g})$ given by $x_1 = u_1 cosh\theta$, $x_2 = u_2 cosh\theta$, $x_3 = u_1 sinh\theta$, $x_4 = u_2 sinh\theta$, $x_5 = u_3$, $x_6 = \sqrt{1-u_3^2}$, $x_7 = u_4$, $x_8 = u_8$, $x_9 = u_2$, $x_{10} = u_1$, where $\bar{g}$ is of signature $(-,-,+,+,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8, \partial x_9, \partial x_{10}\}$. Then $TM$ is spanned by $Z_1 = cosh\theta \partial x_1 + sinh\theta \partial x_3 + \partial x_{10}$, $Z_2 = cosh\theta \partial x_2 + sinh\theta \partial x_4 + \partial x_9$, $Z_3 = x_6 \partial x_5 - x_5 \partial x_6$, $Z_4 = \partial x_7$, $Z_5 = \partial x_8$. Clearly $M$ is a 2-lightlike submanifold with $Rad(TM) = span\{Z_1, Z_2\}$ and the lightlike transversal bundle is spanned by

$$N_1 = \frac{1}{2}(-cosh\theta \partial x_1 - sinh\theta \partial x_3 + \partial x_{10}),\ N_2 = -\frac{1}{2}(cosh\theta \partial x_2 + sinh\theta \partial x_4 - \partial x_9),$$

and $\bar{J}Z_1 = -2N_2$ and $\bar{J}Z_2 = 2N_1$. Hence $\bar{J}(Rad(TM)) = ltr(TM)$. Since $\bar{J}Z_4 = Z_5$ then $D = span\{Z_4, Z_5\}$ which is an invariant distribution on $M$. By direct calculations, the transversal screen bundle $S(TM^{\perp}))$ is spanned by

$$W_1 = sinh\theta \partial x_1 + cosh\theta \partial x_3,\quad W_2 = sinh\theta \partial x_2 + cosh\theta \partial x_4,\quad W_3 = x_6 \partial x_6 + x_5 \partial x_5.$$

Thus $\bar{J}W_3 = -Z_3$. Hence $D^{\perp} = span\{Z_3\}$ is an anti-invariant distribution on $M$ and $span\{W_1, W_2\}$ is invariant and $span\{W_3\}$ is anti-invariant subbundles of $S(TM^{\perp})$ respectively. Thus it enables us to choose $S(TM) = span\{Z_3, Z_4, Z_5\}$. Hence $M$ is a proper semi-transversal lightlike submanifold.

Let $M$ be a semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Let $Q$, $P_1$, $P_2$ and $P$ be the projection morphisms from $TM$ on $D$, $Rad(TM)$, $D^{\perp}$ and $D'$ respectively. Then for any $X \in \Gamma(TM)$, we put

$$X = QX + P_1X + P_2X. \tag{3.1}$$

Applying $\bar{J}$ to (3.1), we obtain $\bar{J}X = \bar{J}QX + \bar{J}P_1X + \bar{J}P_2X$, can be written as $\bar{J}X = TQX + wP_1X + wP_2X$. Put $wP_1 = w_1$ and $wP_2 = w_2$, then we have

$$\bar{J}X = TX + w_1X + w_2X, \tag{3.2}$$

where $TX \in \Gamma(D), w_1X \in \Gamma(ltr(TM))$ and $w_2X \in \Gamma(\bar{J}D^{\perp}) \subset S(TM^{\perp})$. Similarly, for any $V \in \Gamma(S(TM^{\perp}))$, we can write

$$\bar{J}V = EV + FV, \tag{3.3}$$

where $EV \in \Gamma(D^{\perp})$ and $FV \in \Gamma(\mu)$, where $\mu$ is a complementary bundle of $\bar{J}D^{\perp}$ in $S(TM^{\perp})$. Differentiating (3.2) and using (2.3), (2.4) and (3.3), for any $X \in \Gamma(TM)$, we have the following lemma.

**Lemma 1.** *Let $M$ be a semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then we have*

$$(\nabla_X T)Y = A_{w_1 Y}X + A_{w_2 Y}X + \bar{J}h^l(X,Y) + Eh^s(X,Y), \tag{3.4}$$

$$(\nabla_X w_1)Y = -h^l(X,TY) - D^l(X,w_2 Y), \tag{3.5}$$

$$(\nabla_X w_2)Y = Fh^s(X,Y) - h^s(X,TY) - D^s(X,w_1 Y), \textit{where} \tag{3.6}$$

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \quad (\nabla_X w_1)Y = \nabla^l_X w_1 Y - w_1 \nabla_X Y, \tag{3.7}$$

$$(\nabla_X w_2)Y = \nabla^s_X w_2 Y - w_2 \nabla_X Y. \tag{3.8}$$

**Definition 2** ([6])**.** A lightlike submanifold $(M,g)$ of a semi-Riemannian manifold $(\bar{M},\bar{g})$ is said to be a totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on $M$, called the transversal curvature vector field of $M$, such that $h(X,Y) = H\bar{g}(X,Y)$, for $X,Y \in \Gamma(TM)$. Using (2.3), clearly $M$ is a totally umbilical, if and only if, for $X,Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$, on each coordinate neighborhood $\mathcal{U}$ there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^{\perp}))$ such that

$$h^l(X,Y) = H^l g(X,Y), \quad h^s(X,Y) = H^s g(X,Y), \quad D^l(X,W) = 0. \tag{3.9}$$

**Lemma 2.** *Let $M$ be a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ then the distribution $D'$ defines a totally geodesic foliation in $M$.*

*Proof.* Let $X,Y \in \Gamma(D')$ then using (3.4) and (3.7), we obtain $T\nabla_X Y = -A_{w_1 Y}X - A_{w_2 Y}X - \bar{J}h^l(X,Y) - Eh^s(X,Y)$. On taking inner product both sides with $Z \in \Gamma(D)$, we further obtain

$$\begin{aligned} g(T\nabla_X Y, Z) &= \bar{g}(\bar{\nabla}_X w_1 Y, Z) + \bar{g}(\bar{\nabla}_X w_2 Y, Z) = -\bar{g}(\bar{J}Y, \bar{\nabla}_X Z) \\ &= \bar{g}(Y, \bar{\nabla}_X \bar{J}Z) = g(Y, \nabla_X Z'), \end{aligned} \tag{3.10}$$

where $Z' = \bar{J}Z \in \Gamma(D)$. Since $M$ is a totally umbilical lightlike submanifold then for any $X \in \Gamma(D')$ and $Z \in \Gamma(D)$, with (3.5) and (3.7), we have $w_1 \nabla_X Z = h^l(X,TZ) = H^l g(X,TZ) = 0$ and using (3.6) and (3.8), we have $w_2 \nabla_X Z = -Fh^s(X,Z) + h^s(X,TZ) = -FH^s g(X,Z) + H^s g(X,TZ) = 0$, these facts imply that $\nabla_X Z \in \Gamma(D)$, for any $X \in \Gamma(D')$ and $Z \in \Gamma(D)$. Therefore (3.10) implies that $g(T\nabla_X Y, Z) = 0$, then the non degeneracy of the distribution $D$ implies that $T\nabla_X Y = 0$. Hence the result follows. □

**Theorem 2** ([12])**.** *Let $M$ be a semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribution $D'$ is integrable, if and only if $A_{wZ}V = A_{wV}Z$, for any $Z,V \in \Gamma(D')$.*

**Theorem 3.** *Let $M$ be a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ then the distribution $D'$ is integrable.*

*Proof.* Let $X, Y \in \Gamma(D')$ then using (3.4) and (3.7) with the Lemma 2, we get $A_{wY}X = -\bar{J}h^l(X,Y) - Eh^s(X,Y)$ this implies that $A_{wY}X \in \Gamma(D')$ and moreover the symmetric property of the second fundamental form $h$ gives that $A_{wY}X = A_{wX}Y$. Hence by virtue of the Theorem 2, the result follows. □

## 4. Semi-transversal lightlike submersions

Let $\phi : M \to B$ be a mapping from a Riemannian manifold $M$ onto a Riemannian manifold $B$ then it is said to be a Riemannian submersion if it satisfies the following axioms:

A1. $\phi$ has maximal rank. This implies that for each $b \in B$, $\phi^{-1}(b)$ is a submanifold of $M$, known as *fiber*, of dimension $dimM - dimB$. A vector field tangent to the fibers is called vertical vector field and orthogonal to fibers is called horizontal vector field.

A2. $\phi_*$ preserves the lengths of horizontal vectors.

The Riemannian submersions were introduced by O'Neill in [10] and since then plenty of work on this subject matter has been done (for detail, see [7, 14] and many references therein). In the study of submersions, the vertical distribution $\mathcal{V}$ of $M$ is defined by $\mathcal{V}_p = ker\ d\phi_p, p \in M$, which is always integrable and the orthogonal complementary distribution to $\mathcal{V}$ is defined by $\mathcal{H}_p = (ker\ d\phi_p)^{\perp}$, denoted by $\mathcal{H}$ and called a horizontal distribution. Therefore the tangent bundle $TM$ of $M$ has the following decomposition $TM = \mathcal{V} \oplus \mathcal{H}$.

Since the vertical distribution of the Riemannian submersion $\phi : M \to B$ and the totally real distribution $D^{\perp}$ of the $CR$-submanifold $M$ of a Kaehler manifold are always integrable. Therefore Kobayashi [9] introduced the submersion $\phi : M \to B$ from a $CR$-submanifold $M$ of a Kaehler manifold onto an almost Hermitian manifold $B$ such that the distributions $D$ and $D^{\perp}$ of the $CR$-submanifold become the horizontal and the vertical distributions respectively, required by the submersion and $\phi$ restricted to $D$ becomes a complex isometry.

We have seen that for a Riemannian submersion, the tangent bundle of the source manifold splits into horizontal and vertical part. On the other hand, the tangent bundle of a lightlike submanifold splits into screen and radical part and these natural splitting of the tangent bundle plays an important role in the study of lightlike submanifolds. Therefore Sahin [13] introduced screen lightlike submersion between a lightlike manifold and a semi-Riemannian manifold. Further in [15], Sahin and Gunduzalp introduced the idea of a lightlike submersion from a semi-Riemannian manifold onto a lightlike manifold.

From Theorem 3, we know that for a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold the distribution $D'$ is integrable. Then

a totally umbilical semi-transversal lightlike submanifold meets our requirements to define a submersion on it analogous to a submersion of a $CR$-submanifold. Significant applications of semi-Riemannian submersions in physics and the growing importance of lightlike submanifolds and hypersurfaces in mathematical physics, especially in relativity (see [5]), motivated us to work on this subject matter.

**Definition 3.** Let $(M, g_M, D)$ be a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ and $(B, g_B)$ be an indefinite almost Hermitian manifold. Then we say that a smooth mapping $\phi : (M, g_M, D) \to (B, g_B)$ is a lightlike submersion if

(a) at every $p \in M, \mathcal{V}_p = ker(d\phi)_p = D'$.
(b) at each point $p \in M$, the differential $d\phi_p$ restricts to an isometry of the horizontal space $\mathcal{H}_p = D_p$ onto $T_{\phi(p)}B$, that is, $g_D(X, Y) = g_B(d\phi(X), d\phi(Y))$, for every vector fields $X, Y \in \Gamma(D)$.

Obviously from the definition, the restriction of the differential $d\phi_p$ to the distribution $\mathcal{H}_p = D_p$ maps that space isomorphically onto $T_{\phi(p)}B$. Then for any tangent vector $\widetilde{X} \in T_{\phi(p)}B$, we say that the tangent vector $X \in D_p$ is a horizontal lift of $\widetilde{X}$ as for submersions. If $\widetilde{X}$ is a vector field on an open subset $U$ of $B$ then the horizontal lift of $\widetilde{X}$ is the vector field $X \in \Gamma(D)$ on $\phi^{-1}(U)$ such that $d\phi(X) = \widetilde{X} o \phi$ and the vector field $X$ is called a *basic vector field*. Now, we give example of lightlike submersions.

*Example* 2. Let $M$ be a 5-dimensional semi-transversal lightlike submanifold of $R_2^{10}$ as in Example (1) and $B = R_1^2$ be an indefinite almost Hermitian manifold. Let the metrics be defined as $g_M = -(dx_1)^2 - (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2$ and $g_B = -(dy_1)^2 + (dy_2)^2$, where $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}$ and $y_1, y_2$ be the canonical co-ordinates of $R_2^{10}$ and $R^2$, respectively. We define a map $\phi : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) \in R_2^{10} \mapsto (x_7, x_8) \in R_1^2$. Then the kernel of $d\phi$ is

$$ker(d\phi) = D' = span\{Z_1 = cosh\theta\partial x_1 + sinh\theta\partial x_3 + \partial x_{10},$$
$$Z_2 = cosh\theta\partial x_2 + sinh\theta\partial x_4 + \partial x_9, Z_3 = x_6\partial x_5 - x_5\partial x_6\},$$

where $d\phi(Z_1) = 0, d\phi(Z_2) = 0$ and $d\phi(Z_3) = 0$. By direct computation, we obtain $D = span\{Z_4 = \partial x_7, Z_5 = \partial x_8\}$, where $d\phi(Z_4) = \partial y_1, d\phi(Z_5) = \partial y_2$. Then it follows that $g_M(Z_4, Z_4) = g_B(d\phi(Z_4), d\phi(Z_4)) = 1$ and $g_M(Z_5, Z_5) = g_B(d\phi(Z_5), d\phi(Z_5)) = -1$. Hence $\phi$ is a semi-transversal lightlike submersion.

**Theorem 4.** *Let $\phi : M \to B$ be a lightlike submersion from a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ onto an indefinite almost Hermitian manifold $B$. If $X$ and $Y$ are basic vectors $\phi$-related to $\widetilde{X}, \widetilde{Y}$ respectively, then*

(i) $g_M(X, Y) = g_B(\widetilde{X}, \widetilde{Y}) o \phi$.

(ii) $[X,Y]^{\mathcal{H}}$ *is the basic vector field and* $\phi$*-related to* $[\widetilde{X},\widetilde{Y}]$.
(iii) $(\nabla^M_X Y)^{\mathcal{H}}$ *is the basic vector field and* $\phi$*-related to* $(\nabla^B_{\widetilde{X}}\widetilde{Y})$.
(iv) *For any vertical vector field* $V$, $[X,V]$ *is vertical.*

*Proof.* Let $X$ and $Y$ be basic vector fields of $M$ then $(i)$ follows immediately from part (b) of the Definition 3. Since $P$ and $Q$ be the projections from $TM$ on the distributions $D'$ and $D$ of a semi-transversal lightlike submanifold of indefinite Kaehler manifold respectively, then $[X,Y] = P[X,Y] + Q[X,Y]$. Therefore the horizontal part $Q[X,Y]$ of $[X,Y]$ is a basic vector field and corresponds to $[\widetilde{X},\widetilde{Y}]$, that is, $d\phi(Q[X,Y]) = [d\phi(X), d\phi(Y)]$. Next, from the Koszul's formula, we have

$$\begin{aligned} 2g_M(\nabla_X Y, Z) &= X(g_M(Y,Z)) + Y(g_M(Z,X)) - Z(g_M(X,Y)) \\ &\quad - g_M(X,[Y,Z]) + g_M(Y,[Z,X]) + g_M(Z,[X,Y]) \end{aligned} \tag{4.1}$$

for any $X,Y,Z \in \Gamma(D)$. Consider $X,Y$ and $Z$ are the horizontal lifts of the vector fields $\widetilde{X},\widetilde{Y}$ and $\widetilde{Z}$ respectively, then $X(g_M(Y,Z)) = \widetilde{X}(g_B(\widetilde{Y},\widetilde{Z}))o\phi$ and $g_M(Z,[X,Y]) = g_B(\widetilde{Z},[\widetilde{X},\widetilde{Y}])o\phi$ then from (4.1), we have

$$\begin{aligned} 2g_M(\nabla^M_X Y, Z) &= \widetilde{X}(g_B(\widetilde{Y},\widetilde{Z}))o\phi + \widetilde{Y}(g_B(\widetilde{Z},\widetilde{X}))o\phi - \widetilde{Z}(g_B(\widetilde{X},\widetilde{Y}))o\phi \\ &\quad - g_B(\widetilde{X},[\widetilde{Y},\widetilde{Z}])o\phi + g_B(\widetilde{Y},[\widetilde{Z},\widetilde{X}])o\phi + g_B(\widetilde{Z},[\widetilde{X},\widetilde{Y}])o\phi \\ &= 2g_B(\nabla^B_{\widetilde{X}}\widetilde{Y},\widetilde{Z}). \end{aligned} \tag{4.2}$$

Thus from (4.2), (iii) follows, since $\phi$ is surjective and $\widetilde{Z}$ is arbitrarily chosen. Finally, let $V \in \Gamma(D')$ then $[X,V]$ is $\phi$-related to $[\widetilde{X},0]$, hence $(iv)$ follows and this completes the proof of the theorem. □

Let $\nabla^B$ be the covariant differentiation on $B$ then we define the corresponding operator $\widetilde{\nabla}^B$ for basic vector fields of $B$ by assuming $\widetilde{\nabla}^B_X Y = (\nabla^M_X Y)^{\mathcal{H}}$, for any basic vector fields $X$ and $Y$. Thus from (iii) the Theorem 4, $\widetilde{\nabla}^B_X Y$ is a basic vector field and $d\phi(\nabla^M_X Y)^{\mathcal{H}} = d\phi(\widetilde{\nabla}^B_X Y) = \nabla^B_{\widetilde{X}}\widetilde{Y}$. Thus we define the tensor fields $C_1$ and $C_2$, using (3.1) as

$$\nabla^M_X Y = \widetilde{\nabla}^B_X Y + C_1(X,Y) + C_2(X,Y), \tag{4.3}$$

for any $X,Y \in \Gamma(D)$, where $C_1(X,Y)$ and $C_2(X,Y)$ denote the vertical parts of $\nabla^M_X Y$. It is easy to check that $C_1$ and $C_2$ are bilinear maps from $D \times D \to Rad(TM)$ and $D \times D \to D^{\perp}$ respectively.

**Theorem 5.** *Let* $\phi : M \to B$ *be a lightlike submersion of a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold* $\bar{M}$ *onto an indefinite almost Hermitian manifold* $B$ *then for any basic vector fields* $X$ *and* $Y$*, we have*

(i) *the tensor fields* $C_1$ *and* $C_2$ *are skew-symmetric, that is,* $C_1(X,Y) = -C_1(Y,X)$ *and* $C_2(X,Y) = -C_2(Y,X)$*;*

(ii) $P_1[X,Y] = 2C_1(X,Y)$ *and* $P_2[X,Y] = 2C_2(X,Y)$,

*Proof.* (i) Let $Z \in \Gamma(D^{\perp})$ be any vertical vector field then for any basic vector field $X \in \Gamma(D)$, we have

$$\begin{aligned} 0 &= Z(g(X,X)) = 2\bar{g}(\bar{\nabla}_Z X, X) = 2g(\nabla^M_X Z - [X,Z], X) = -2\bar{g}(Z, \bar{\nabla}_X X) \\ &= -2g(Z, \widetilde{\nabla}^B_X X + C_1(X,X) + C_2(X,X)) = -2g(Z, C_2(X,X)), \end{aligned}$$

then the non degeneracy of the distribution $D^{\perp}$ implies that $C_2(X,X) = 0$, that is $C_2$ is skew-symmetric. Similarly, let $\bar{J}N \in \Gamma(Rad(TM))$ be a vertical vector field where $N \in \Gamma(ltr(TM))$, we have

$$\begin{aligned} 0 &= \bar{J}N(g(X,X)) = -2\bar{g}(\bar{\nabla}_N X, X) = -2g(\nabla^M_X N - [X,N], X) \\ &= 2g(N, \widetilde{\nabla}^B_X X + C_1(X,X) + C_2(X,X)) = 2g(N, C_1(X,X)), \end{aligned}$$

then using (2.1), we obtain $C_1(X,X) = 0$, that is $C_1$ is skew-symmetric.
(ii) For basic vector fields $X, Y \in \Gamma(D)$, we have $[X,Y] = \nabla^M_X Y - \nabla^M_Y X$, using (3.1), (4.3) and skew-symmetric property of $C_1$ and $C_2$, result follows. □

Next for a basic vector field $X$ and a vertical vector field $Z$, using (3.1), we define the tensor field $T$ as

$$\nabla^M_X Z = (\nabla^M_X Z)^{\mathcal{H}} + (\nabla^M_X Z)^{\mathcal{V}} = T_X Z + (\nabla^M_X Z)^{\mathcal{V}}, \tag{4.4}$$

where $T$ is a bilinear map from $D \times D' \to D$. Since $[X,Z] = \nabla^M_X Z - \nabla^M_Z X$ and $[X,Z]$ is vertical therefore

$$Q(\nabla^M_X Z) = Q(\nabla^M_Z X) = T_X Z, \quad (\nabla^M_X Z)^{\mathcal{V}} = (\nabla^M_Z X)^{\mathcal{V}}. \tag{4.5}$$

Let $X$ and $Y$ be basic vector fields and $Z$ be a vertical vector field such that $Z \in \Gamma(D^{\perp})$ then using (4.3), the tensor fields $T$ and $C_2$ are related by

$$g(T_X Z, Y) = \bar{g}(\bar{\nabla}_X Z, Y) = -g(Z, \nabla_X Y) = -g(Z, C_2(X,Y)), \tag{4.6}$$

and if $Z \in \Gamma(Rad(TM))$ then

$$g(T_X Z, Y) = -\bar{g}(Z, h^l(X,Y)). \tag{4.7}$$

**Theorem 6.** *Let $\phi : M \to B$ be a lightlike submersion of a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ onto an indefinite almost Hermitian manifold $B$ then $B$ is also an indefinite Kaehler manifold. Moreover if $\bar{H}$ and $H^B$ denote the holomorphic sectional curvatures of $\bar{M}$ and $B$, respectively then for any unit basic vector $X \in \Gamma(\mathcal{H})$ of $M$, we have*

$$\bar{R}^{\bar{M}}(X, \bar{J}X, X, \bar{J}X) = R^B(\widetilde{X}, \bar{J}\widetilde{X}, \widetilde{X}, \bar{J}\widetilde{X}) + 4\|H^s\|^2.$$

*Proof.* Let $X, Y \in \Gamma(D)$ be basic vector fields then using (2.3) and (4.3), we have

$$\bar{\nabla}_X Y = \widetilde{\nabla}^B_X Y + C_1(X,Y) + C_2(X,Y) + h^l(X,Y) + h^s(X,Y). \tag{4.8}$$

On applying $\bar{J}$ on both sides of (4.8), we obtain

$$\bar{J}\bar{\nabla}_X Y = \bar{J}\widetilde{\nabla}^B_X Y + \bar{J}C_1(X,Y) + \bar{J}C_2(X,Y) + \bar{J}h^l(X,Y) + Eh^s(X,Y) + Fh^s(X,Y), \tag{4.9}$$

on replacing $Y$ by $\bar{J}Y$ in (4.8), we have

$$\bar{\nabla}_X \bar{J}Y = \widetilde{\nabla}^B_X \bar{J}Y + C_1(X,\bar{J}Y) + C_2(X,\bar{J}Y) + h^l(X,\bar{J}Y) + h^s(X,\bar{J}Y). \tag{4.10}$$

Since $\bar{M}$ is a Kaehler manifold therefore $\bar{\nabla}_X \bar{J}Y = \bar{J}\bar{\nabla}_X Y$, then equating (4.9) and (4.10), we obtain

$$\widetilde{\nabla}^B_X \bar{J}Y = \bar{J}\widetilde{\nabla}^B_X Y \in \Gamma(\mathcal{H}), \tag{4.11}$$

$$C_1(X,\bar{J}Y) = \bar{J}h^l(X,Y) \in \Gamma(Rad(TM)), \tag{4.12}$$

$$C_2(X,\bar{J}Y) = Eh^s(X,Y) \in \Gamma(D^\perp), \tag{4.13}$$

$$h^s(X,\bar{J}Y) = \bar{J}C_2(X,Y) + Fh^s(X,Y) \in \Gamma(S(TM^\perp)), \tag{4.14}$$

$$h^l(X,\bar{J}Y) = \bar{J}C_1(X,Y) \in \Gamma(ltr(TM)). \tag{4.15}$$

From (4.11), we see that almost complex structure $\bar{J}$ of $B$ is parallel and hence $B$ is also an indefinite Kaehler manifold.

From (3.3), it is clear that $U \in \Gamma(\bar{J}D^\perp) \subset S(TM^\perp)$, if and only if, $FU = 0$ then $\bar{J}U = EU$ and $U \in \Gamma(\mu = (\bar{J}D^\perp)^\perp) \subset S(TM^\perp)$, if and only if, $EU = 0$ then $\bar{J}U = FU$. Therefore from (4.13), (4.14) and skew-symmetric property of $C_2$, we obtain $C_2(X,\bar{J}Y) = C_2(Y,\bar{J}X)$, $C_2(\bar{J}X,Y) = C_2(\bar{J}Y,X)$, $C_2(\bar{J}X,\bar{J}Y) = C_2(X,Y)$ and $h^s(X,\bar{J}Y) + h^s(Y,\bar{J}X) = 2Fh^s(X,Y)$. On the other hand, since $M$ is a totally umbilical semi-transversal lightlike submanifold then we have $h^s(X,\bar{J}Y) + h^s(Y,\bar{J}X) = g(X,\bar{J}Y)H^s + g(Y,\bar{J}X)H^s = 0$. Therefore $Fh^s(X,Y) = 0$ and this implies that $h^s(X,Y) \in \Gamma(\bar{J}D^\perp)$, for any $X,Y \in \Gamma(D)$. By virtue of totally umbilical property of $M$, we also have $h^s(\bar{J}X,\bar{J}Y) = h^s(X,Y)$. Similarly using (4.12) and (4.15), we obtain $C_1(X,\bar{J}Y) = C_1(Y,\bar{J}X)$, $C_1(\bar{J}X,Y) = C_1(\bar{J}Y,X)$, $C_1(\bar{J}X,\bar{J}Y) = C_1(X,Y)$ and $h^l(\bar{J}X,\bar{J}Y) = h^l(X,Y)$, $h^l(\bar{J}X,Y) + h^l(X,\bar{J}Y) = 0$. Now, for any $X,Y,Z \in \Gamma(D)$, using (4.3) and (4.4), we have

$$\nabla_X\nabla_Y Z = \widetilde{\nabla}^B_X\widetilde{\nabla}^B_Y Z + T_X C_1(Y,Z) + T_X C_2(Y,Z) + vertical, \tag{4.16}$$

$$\nabla_Y\nabla_X Z = \widetilde{\nabla}^B_Y\widetilde{\nabla}^B_X Z + T_Y C_1(X,Z) + T_Y C_2(X,Z) + vertical, \tag{4.17}$$

$$\nabla_{[X,Y]}Z = \widetilde{\nabla}^B_{Q[X,Y]}Z + 2T_Z C_1(X,Y) + 2T_Z C_2(X,Y) + vertical. \tag{4.18}$$

Further using (4.16)-(4.18), we obtain

$$R^M(X,Y)Z = (R^B(\widetilde{X},\widetilde{Y})\widetilde{Z})^* + T_X C_1(Y,Z) + T_X C_2(Y,Z) - T_Y C_1(X,Z) - T_Y C_2(X,Z) - 2T_Z C_1(X,Y) - 2T_Z C_2(X,Y) + vertical, \tag{4.19}$$

where $(R^B(\widetilde{X},\widetilde{Y})\widetilde{Z})^*$ denotes the basic vector field of $M$ corresponding to $R^B(\widetilde{X},\widetilde{Y})\widetilde{Z}$. Using (4.19) in (2.6), we obtain

$$\begin{aligned}\bar{R}^{\bar{M}}(X,Y)Z = {} & (R^B(\widetilde{X},\widetilde{Y})\widetilde{Z})^* + T_X C_1(Y,Z) + T_X C_2(Y,Z) - T_Y C_1(X,Z) \\ & - T_Y C_2(X,Z) - 2T_Z C_1(X,Y) - 2T_Z C_2(X,Y) + A_{h^l(X,Z)}Y \\ & - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y - A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) \\ & - (\nabla_Y h^l)(X,Z) + D^l(X,h^s(Y,Z)) - D^l(Y,h^s(X,Z)) \\ & + (\nabla_X h^s)(Y,Z) - (\nabla_Y h^s)(X,Z) + D^s(X,h^l(Y,Z)) \\ & - D^s(Y,h^l(X,Z)) + vertical.\end{aligned}$$

Now, for basic vector field $W \in \Gamma(D)$ with (2.4), (2.5), (4.4)-(4.7), we obtain

$$\begin{aligned}\bar{R}^{\bar{M}}(X,Y,Z,W) = {} & R^B(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W}) - \bar{g}(C_1(Y,Z),h^l(X,W)) \\ & - g(C_2(Y,Z),C_2(X,W)) + \bar{g}(C_1(X,Z),h^l(Y,W)) \\ & + g(C_2(X,Z),C_2(Y,W)) + 2\bar{g}(C_1(X,Y),h^l(Z,W)) \\ & + 2g(C_2(X,Y),C_2(Z,W)) + g(A_{h^l(X,Z)}Y,W) \\ & - g(A_{h^l(Y,Z)}X,W) + \bar{g}(h^s(X,Z),h^s(Y,W)) \\ & - \bar{g}(h^s(Y,Z),h^s(X,W)). \end{aligned} \tag{4.20}$$

Now, using (2.4) and (4.3), we have $g(A_{h^l(X,Z)}Y,W) = \bar{g}(h^l(X,Z),\bar{\nabla}_Y W) = \bar{g}(h^l(X,Z),C_1(Y,W))$ and similarly $g(A_{h^l(Y,Z)}X,W) = \bar{g}(h^l(Y,Z),C_1(X,W))$. Using these expressions with (4.15) in (4.20), we obtain

$$\begin{aligned}\bar{R}^{\bar{M}}(X,Y,Z,W) = {} & R^B(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W}) + \bar{g}(\bar{J}h^l(Y,\bar{J}Z),h^l(X,W)) \\ & - g(C_2(Y,Z),C_2(X,W)) - \bar{g}(\bar{J}h^l(X,\bar{J}Z),h^l(Y,W)) \\ & + g(C_2(X,Z),C_2(Y,W)) - 2\bar{g}(\bar{J}h^l(X,\bar{J}Y),h^l(Z,W)) \\ & + 2g(C_2(X,Y),C_2(Z,W)) - \bar{g}(\bar{J}h^l(Y,\bar{J}W),h^l(X,Z)) \\ & + \bar{g}(\bar{J}h^l(X,\bar{J}W),h^l(Y,Z)) + \bar{g}(h^s(X,Z),h^s(Y,W)) \\ & - \bar{g}(h^s(Y,Z),h^s(X,W)). \end{aligned} \tag{4.21}$$

To compare holomorphic sectional curvature of $\bar{M}$ with that of $B$, set $Y = \bar{J}X$, $Z = X$ and $W = \bar{J}X$ in (4.21) and then using the hypothesis that $M$ is a totally umbilical semi-transversal lightlike submanifold, we obtain $\bar{R}^{\bar{M}}(X,\bar{J}X,X,\bar{J}X) = R^B(\widetilde{X},\bar{J}\widetilde{X},\widetilde{X},\bar{J}\widetilde{X}) + \|C_2(X,X)\|^2 + 3\|C_2(X,\bar{J}X)\|^2 + \|h^s(X,X)\|^2$. Since $Fh^s(X,Y) = 0$ therefore (4.14) implies $\|h^s(X,X)\|^2 = \|C_2(X,\bar{J}X)\|^2$ and by virtue of the totally umbilical property of $M$, (4.14) implies that $C_2(X,X) = -\bar{J}h^s(X,\bar{J}X) = -\bar{J}(H^S g(X,\bar{J}X)) = 0$. Thus the holomorphic sectional curvature of $\bar{M}$ is given

as

$$\begin{aligned}\bar{R}^{\bar{M}}(X,\bar{J}X,X,\bar{J}X) &= R^B(\widetilde{X},\bar{J}\widetilde{X},\widetilde{X},\bar{J}\widetilde{X})+4\|C_2(X,\bar{J}X)\|^2\\ &= R^B(\widetilde{X},\bar{J}\widetilde{X},\widetilde{X},\bar{J}\widetilde{X})+4\|h^s(X,X)\|^2\\ &= R^B(\widetilde{X},\bar{J}\widetilde{X},\widetilde{X},\bar{J}\widetilde{X})+4\|H^s\|^2.\end{aligned}$$

This completes the proof. □

**Theorem 7.** *Let $\phi : M \to B$ be a lightlike submersion of a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ onto an indefinite almost Hermitian manifold $B$. If the distribution $D$ is integrable, then $M$ is a lightlike product manifold.*

*Proof.* Let the distribution $D$ be an integrable therefore $P_1[X,Y]=0$ and $P_2[X,Y]=0$, for any $X,Y \in \Gamma(D)$, where $P_1$ and $P_2$ are the projection morphisms from $TM$ to $Rad(TM)$ and $D^{\perp}$, respectively. Therefore using the Theorem 5, we have $C_1(X,Y)=0$ and $C_2(X,Y)=0$. Hence using (4.3), we obtain that $\nabla^M_X Y \in \Gamma(D)$, for any $X,Y \in \Gamma(D)$, consequently the distribution $D$ defines a totally geodesic foliation in $M$. Moreover, from the Lemma 2, the distribution $D'$ also defines a totally geodesic foliation in $M$. Thus using the De Rham's theorem, $M$ is a product manifold $M_1 \times M_2$, where $M_1$ and $M_2$ are the leaves of the distributions of $D$ and $D'$. □

**Theorem 8.** *Let $\phi : M \to B$ be a lightlike submersion of a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ onto an indefinite almost Hermitian manifold $B$ such that $\bar{J}(D^{\perp}) = S(TM^{\perp})$. Then the fibers are totally geodesic submanifolds of $M$.*

*Proof.* Let $U,V \in \Gamma(D')$ and then define

$$\nabla^M_U V = \hat{\nabla}_U V + L(U,V), \tag{4.22}$$

where $\hat{\nabla}_U V = (\nabla^M_U V)^{\mathcal{V}}$ and $L(U,V) = (\nabla^M_U V)^{\mathcal{H}}$. Since the distribution $D'$ is integrable always, then $L(U,V) = L(V,U)$. Now, using the Kaehlerian property of $\bar{M}$, we have $\bar{\nabla}_U \bar{J}V = \bar{J}\bar{\nabla}_U V$, since $\bar{J}(D^{\perp}) = S(TM^{\perp})$, then

$$-A_{\bar{J}V}U + \nabla^t_U \bar{J}V = \bar{J}\hat{\nabla}_U V + \bar{J}L(U,V) + \bar{J}h(U,V).$$

On comparing the horizontal and vertical components both sides, we get

$$\mathcal{H}(A_{\bar{J}V}U) = -\bar{J}L(U,V), \quad \mathcal{V}(A_{\bar{J}V}U) = -\bar{J}h(U,V). \tag{4.23}$$

From (4.22), it is clear that the fibers are totally geodesic submanifolds of $M$, if and only if, $L(U,V)=0$ or using $(4.23)_1$, if and only if, $A_{\bar{J}V}U \in \Gamma(D')$, for any $U,V \in \Gamma(D')$. Now, particularly choose $V \in D^{\perp}$ then using the hypothesis of this theorem $\bar{J}V \in \Gamma(S(TM^{\perp}))$. Let $Y \in \Gamma(D)$ then using (2.5) with the fact that M is a totally umbilical lightlike submanifold, we obtain $g(A_{\bar{J}V}U,Y) = \bar{g}(h^s(U,Y),\bar{J}V) = g(U,Y)\bar{g}(H^s,\bar{J}V) = 0$. Similarly, let $V \in \Gamma(Rad(TM))$ then $g(A_{\bar{J}V}U,Y) = \bar{g}(\bar{J}V,\bar{\nabla}_U Y) = -\bar{g}(V,h^l(U,\bar{J}Y)) = -g(U,\bar{J}Y)\bar{g}(V,H^l) = 0$. Thus $A_{\bar{J}V}U \in \Gamma(D')$ and the assertion follows. □

**Theorem 9.** *Let $\phi : M \to B$ be a lightlike submersion of a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ onto an indefinite almost Hermitian manifold $B$. Then the sectional curvature of $\bar{M}$ and of the fiber are related by*

$$\bar{K}(U \wedge V) = \hat{K}(U \wedge V) + g(A_{h^l(U,U)}V, V) - g(A_{h^l(V,U)}U, V) + g([A_{\bar{J}V}, A_{\bar{J}U}]U, V),$$

*for any orthonormal vector fields $U, V \in \Gamma(D^{\perp})$.*

*Proof.* Let $\nabla$ and $\hat{\nabla}$ be the connections of semi-transversal lightlike submanifold $M$ and its fiber, respectively. Let $R$ and $\hat{R}$ be the curvature tensors of $\nabla$ and $\hat{\nabla}$, respectively then for any $U, V \in \Gamma(D^{\perp})$, using (4.22) we have

$$R(U,V)U = \nabla_U(\hat{\nabla}_V U + L(V,U)) - \nabla_V(\hat{\nabla}_U U + L(U,U)) - (\hat{\nabla}_{[U,V]}U + L([U,V],U)),$$

this further implies that

$$R(U,V,U,V) = g(\nabla_U \hat{\nabla}_V U, V) + g(\nabla_U L(V,U), V) - g(\nabla_V \hat{\nabla}_U U, V) - g(\nabla_V L(U,U), V) - g(\hat{\nabla}_{[U,V]}U, V).$$

Again using (4.22), it leads to

$$R(U,V,U,V) = \hat{R}(U,V,U,V) + g(\nabla_U L(V,U), V) - g(\nabla_V L(U,U), V). \tag{4.24}$$

Now, using the fact that $M$ is totally umbilical lightlike submanifold, we get

$$\begin{aligned} g(\nabla_U L(V,W), F) &= g(\bar{\nabla}_U L(V,W) - g(h^l(U, L(V,W)), F) \\ &= -g(L(V,W), \nabla_U F) = -g(L(V,W), L(U,F)), \end{aligned}$$

for any $U, V, W, F \in \Gamma(D^{\perp})$ therefore (4.24) becomes

$$R(U,V,U,V) = \hat{R}(U,V,U,V) - g(L(U,V), L(U,V)) + g(L(U,U), L(V,V)). \tag{4.25}$$

Using (2.5), (2.6) and $M$ is totally umbilical lightlike submanifold, we have

$$\begin{aligned} \bar{R}(U,V,U,V) = R(U,V,U,V) &+ g(A_{h^l(U,U)}V, V) - g(A_{h^l(V,U)}U, V) \\ &+ \bar{g}(h^s(V,V), h^s(U,U)) - \bar{g}(h^s(U,V), h^s(V,U)). \end{aligned}$$

Further using (4.23), (4.25) and the fact $L(U,V) = L(V,U)$, we obtain

$$\begin{aligned} \bar{R}(U,V,U,V) = \hat{R}(U,V,U,V) &- g(\mathcal{H}(A_{\bar{J}U}V), \mathcal{H}(A_{\bar{J}U}V)) \\ &+ g(\mathcal{H}(A_{\bar{J}U}U), \mathcal{H}(A_{\bar{J}V}V)) + g(A_{h^l(U,U)}V, V) \\ &- g(A_{h^l(V,U)}U, V) + g(\mathcal{V}(A_{\bar{J}V}V), \mathcal{V}(A_{\bar{J}U}U)) \\ &- g(\mathcal{V}(A_{\bar{J}V}U), \mathcal{V}(A_{\bar{J}V}U)). \end{aligned}$$

Since $U, V \in \Gamma(D^{\perp})$ and let $X \in \Gamma(D)$ then using (2.3), we get $g(A_{\bar{J}U}V, X) = 0$, which further implies that $A_{\bar{J}U}V \in \Gamma(D^{\perp})$ and $A_{\bar{J}U}V = A_{\bar{J}V}U$, then

$$\bar{R}(U,V,U,V) = \hat{R}(U,V,U,V) - g(A_{\bar{J}U}V, A_{\bar{J}U}V) + g(A_{\bar{J}U}U, A_{\bar{J}V}V) + g(A_{h^l(U,U)}V, V) - g(A_{h^l(V,U)}U, V). \tag{4.26}$$

Now, let $W \in \Gamma(S(TM^{\perp}))$ then for $U, V \in \Gamma(D^{\perp})$, using (2.5), we have $g(A_W U, V) = g(U, A_W V)$. Using this fact with $A_{\bar{J}U}V \in \Gamma(D^{\perp})$, we get

$$\begin{aligned} g(A_{\bar{J}U}V, A_{\bar{J}U}V) - g(A_{\bar{J}U}U, A_{\bar{J}V}V) &= g(A_{\bar{J}V}U, A_{\bar{J}U}V) - g(A_{\bar{J}U}U, A_{\bar{J}V}V) \\ &= g(A_{\bar{J}U}A_{\bar{J}V}U, V) - g(A_{\bar{J}V}A_{\bar{J}U}U, V) \\ &= -g([A_{\bar{J}V}, A_{\bar{J}U}]U, V). \end{aligned} \tag{4.27}$$

On using (4.27) in (4.26), the assertion follows. □

Now we define O'Neill's tensors [10] for a lightlike submersion. Let $\nabla$ be a connection of $M$ then tensors $\mathcal{T}$ and $\mathcal{A}$ of type $(1,2)$ are given by

$$\mathcal{T}_X Y = \mathcal{H}\nabla_{\mathcal{V}X}\mathcal{V}Y + \mathcal{V}\nabla_{\mathcal{V}X}\mathcal{H}Y, \quad \mathcal{A}_X Y = \mathcal{H}\nabla_{\mathcal{H}X}\mathcal{V}Y + \mathcal{V}\nabla_{\mathcal{H}X}\mathcal{H}Y. \tag{4.28}$$

Using (4.28), we have the following lemma.

**Lemma 3.** *Let $\phi : M \to B$ be a lightlike submersion of a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ onto an indefinite almost Hermitian manifold $B$. Then we have the following:*

(i) $\nabla_U V = \mathcal{T}_U V + \mathcal{V}\nabla_U V$.
(ii) $\nabla_V X = \mathcal{H}\nabla_V X + \mathcal{T}_V X$.
(iii) $\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V$.
(iv) $\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y$,

*for any $X, Y \in \mathcal{H}$ and $U, V \in \mathcal{V}$.*

**Theorem 10.** *Let $\phi : M \to B$ be a lightlike submersion of a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ onto an indefinite almost Hermitian manifold $B$ such that $\bar{J}(D^{\perp}) = S(TM^{\perp})$. Then $\bar{K}(X \wedge V) = \|H^s\|^2 - \|T_X V\|^2$, for any unit vector fields $X \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$.*

*Proof.* Let $X \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$ then using the Theorem 5 and Lemma 3 with (4.3), we obtain

$$\begin{aligned} g(R(V,X)X, V) &= g(\nabla_V \mathcal{H}(\nabla_X X), V) - g(\nabla_X \mathcal{H}(\nabla_V X), V) \\ &\quad - g(\nabla_X \mathcal{T}_V X, V) + g(\mathcal{T}_{[X,V]}X, V). \end{aligned}$$

It should be noted that $g(\nabla_V \mathcal{H}(\nabla_X X), V) = -g(\mathcal{H}(\nabla_X X), \nabla_V V)$, and similarly $g(\nabla_X \mathcal{H}(\nabla_V X), V) = -g(\mathcal{H}(\nabla_V X), \nabla_X V)$. Therefore we have

$$\begin{aligned} g(R(V,X)X, V) &= -g(\mathcal{H}(\nabla_X X), \nabla_V V) + g(\mathcal{H}(\nabla_V X), \nabla_X V) \\ &\quad - g(\nabla_X \mathcal{T}_V X, V) + g(\mathcal{T}_{[X,V]}X, V). \end{aligned} \tag{4.29}$$

Since $\bar{J}(D^{\perp}) = S(TM^{\perp})$ then using the Theorem 8, we have $L(U,V) = 0$, for $U, V \in \Gamma(D^{\perp})$. Hence using the definition of $\mathcal{T}$ with (2.3) and (4.22), we get

$$g(\mathcal{T}_V X, U) = -g(\mathcal{T}_V U, X) = -g(L(V,U), X) = 0. \tag{4.30}$$

Now, using (4.22), we have

$$g(\mathcal{H}(\nabla_X X), \nabla_V V) = g(\mathcal{H}(\nabla_X X), L(V,V)) = 0. \tag{4.31}$$

Since $M$ is a totally umbilical then using (4.30), we obtain

$$\begin{aligned} g(\nabla_X \mathcal{T}_V X, V) &= -g(\mathcal{T}_V X, \bar{\nabla}_X V) = -g(\mathcal{T}_V X, \mathcal{V}(\nabla_X V)) \\ &= g(L(V, \mathcal{V}(\nabla_X V)), X) = 0. \end{aligned} \tag{4.32}$$

Since for a vertical vector field $V$, $[X,V]$ is always vertical therefore again using (4.30), we have

$$g(\mathcal{T}_{[X,V]} X, V) = -g(L([X,V], V), X) = 0. \tag{4.33}$$

Using (4.6) and (4.31)-(4.33) in (4.29), we obtain

$$g(R(V,X)X, V) = g(T_X V, T_X V). \tag{4.34}$$

Since $M$ is a totally umbilical then using (2.6) and (4.34), we get

$$\begin{aligned} \bar{R}(X,V,X,V) &= -g(T_X V, T_X V) + g(h^l(X,X), \nabla_V V) \\ &\quad + g(h^s(X,X), h^s(V,V)). \end{aligned} \tag{4.35}$$

Now, using Kaehlerian property of $\bar{M}$, we have $\bar{\nabla}_V \bar{J}\xi = \bar{J}\bar{\nabla}_V \xi$, for $V \in \Gamma(D^{\perp})$ and $\xi \in \Gamma(Rad(TM))$. Using the Lemma 3 with (2.4) and then comparing the horizontal components of resulting equation, we obtain

$$A_{\bar{J}\xi} V = -\bar{J}\mathcal{T}_V \xi. \tag{4.36}$$

Since $M$ is semi-transversal lightlike submanifold then for $\xi \in \Gamma(Rad(TM))$, $\bar{J}\xi \in \Gamma(ltr(TM))$ and using (4.28) for any $U, V \in \mathcal{V}$, $\mathcal{T}_U V = \mathcal{H}\nabla_{\mathcal{V}U}\mathcal{V}V \in \mathcal{H}$. Therefore (4.36) implies that $A_{\bar{J}\xi} V \in \mathcal{H}$ or $A_N V \in \mathcal{H}$. Then for $V \in \Gamma(D^{\perp})$ and $N \in \Gamma(ltr(TM))$, we have $g(\nabla_V V, N) = -g(V, \bar{\nabla}_V N) = g(V, A_N V) = 0$. This implies that $\nabla_V V$ has no component in $Rad(TM)$. Using this fact in (4.35) with (3.9), the assertion follows. □

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# A GENERALIZATION OF NIL-CLEAN RINGS

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*Abstract.* The conditions that allow an element of an associative, unital, not necessarily commutative ring $R$, to be represented as a sum of (commuting) idempotents and one nilpotent element are analyzed. Some applications to group rings are also presented.

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## 1. Introduction

An element $a$ in an associative unital ring $R$ is called clean if it can be represented as a sum $a = e + u$, where $e$ is an idempotent element and $u$ is a unit. This notion was introduced by Nicholson in [8]. If one can find such elements $e$ and $u$ such that $a = e + u$ and $eu = ue$, the element $a$ is called strongly clean. The ring $R$ itself is called (strongly) clean if every element in $R$ is (strongly) clean.

Many families of clean rings were investigated in previous decades. In recent years, a particular attention has been paid to the nil-clean rings and its relatives. A nil-clean ring (see [5]) is a ring in which every element is nil-clean, which means that every element can be written as a sum of an idempotent element and a nilpotent one. Analogously, we have a notion of strongly nil-clean elements (and rings). For some of the results concerning this class of rings and some of the related classes of rings, the reader may wish to consult also [1, 3, 7, 10].

A class of strongly 2-nil-clean rings was introduced in [4]. Namely, an element $a$ in a ring $R$ is called strongly 2-nil-clean if it can be represented in the form $a = e + f + n$, where $e$ and $f$ are idempotents, $n$ is a nilpotent element and they all commute with each other.

In this paper we analyze elements of a ring which can be written as a sum of finitely many idempotents and one nilpotent element which are pairwise commutative. If the number of idempotents which appear in this sum is $s$, we call these elements strongly

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$s$-nil-clean. It turns out that if every element in a ring is strongly $s$-nil-clean for some $s$, this ring has finite characteristic and every element in this ring is strongly $(p-1)$-nil-clean, where $p$ is the largest prime dividing the characteristic of this ring (see Theorem 1). These rings are naturally called strongly $(p-1)$-nil-clean rings and they are all strongly clean (see Corollary 3 and the discussion preceding it). There are many examples of strongly $(p-1)$-nil-clean rings. Theorem 2 shows that in every commutative ring $R$ of finite characteristic $k$, elements which are strongly $s$-nil-clean for some $s$, form a subring which is $(p-1)$-nil clean (where $p$ is the largest prime dividing $k$) and if this ring contains idempotents or nilpotents not belonging to $\mathbb{Z}_k$ (which is necessarily contained in $R$) we have a non-trivial example of such a ring. Proposition 5 provides examples of finite commutative local rings which are $(p-1)$-nil-clean.

The plan of the paper is as follows. In Section 2, we analyze sums of idempotents and one nilpotent element and derive our main criteria for strongly $(p-1)$-nil-clean elements in a ring. Section 3 deals with some structure theorems. It is shown that in analyzing strongly $(p-1)$-nil-clean rings, we may reduce this analysis to the case when $p$ is a nilpotent element in a ring under investigation. We also show in this section that strongly $(p-1)$-nil-clean rings are strongly $\pi$-regular and, consequently, strongly clean.

Section 4 deals with the investigation of group rings $RG$ where $R$ is a $(p-1)$-nil-clean commutative ring and $G$ is a group. For example, we show that, for the ring $RG$ to be strongly $(p-1)$-nil-clean, when the characteristic of $R$ is of the form $p^s$, for a prime integer $p$, it is necessary that the order of any element of a group $G$ is of the form $dp^k$, for some $d \mid (p-1)$ and $k \geq 0$ (see Lemma 3). In the case of a commutative group $G$ this condition is also sufficient (see Theorem 4).

We emphasize that we work in associative, unital rings which need not be commutative. When a ring is commutative, we drop the adjective "strongly" since it is unnecessary. We use the same symbol $k$ to denote the integer $k$ and to denote the ring element $k1_R$. It will always be clear what we mean. Finally, we denote the field with $p$ elements by $\mathbb{Z}_p$, the Jacobson radical by $J(R)$ and the set of nilpotent elements by $N(R)$.

## 2. Basic results

A ring in which every element is a sum of certain number of idempotents and one nilpotent element, that commute with each other, is a generalization of strongly nil-clean rings and strongly 2-nil-clean rings. In view of this, we introduce the following definition.

**Definition 1.** An element $a$ of a ring $R$ is $s$-nil-clean if it can be written in the following form:

$$a = e_1 + \cdots + e_s + n, \tag{2.1}$$

where elements $e_1, \dots, e_s$ are idempotents and $n$ is nilpotent. If an element $a$ can be written in the form (2.1) so that elements in this sum are pairwise commutative, we say that this element is strongly $s$-nil-clean. If every element in R is (strongly) $s$-nil clean, we say that $R$ is a (strongly) $s$-nil-clean ring.

Of course, if $a$ is (strongly) $s$-nil-clean and $s < t$, $a$ is also $t$-nil-clean — we simply add $t - s$ zeroes to the presentation of $a$ as a (strongly) $s$-nil-clean element.

*Remark* 1. It is clear that, if $f : R \to S$ is a ring homomorphism and an element $a \in R$ is (strongly) $s$-nil-clean, then $f(a) \in S$ is (strongly) $s$-nil-clean. Similarly, an element $a = (a_1, \dots, a_l) \in R_1 \times \cdots \times R_l$ is (strongly) $s$-nil-clean iff $a_i$ is (strongly) $s$-nil-clean for all $i$. So, homomorphic images and finite direct products of (strongly) $s$-nil-clean rings are itself (strongly) $s$-nil-clean. Also, a subring of a strongly $s$-nil-clean ring has the same property, as we shall see later. However, this does not hold for $s$-nil-clean rings. Namely, if we have a ring which is $s$-nil-clean, but it is not strongly $s$-nil-clean, it is enough to take an element $a$ which is not strongly $s$-nil-clean and look at the subring generated by this element. This subring is commutative, so it cannot be $s$-nil-clean – if it were, this element would also be strongly $s$-nil-clean.

We begin our analysis with a useful result concerning sum of several idempotents and one nilpotent element.

**Proposition 1.** *Let $R$ be a ring and suppose that element $a \in R$ is strongly $s$-nil clean. Then $a(a-1)\cdots(a-s)$ is nilpotent.*

*Proof.* Let $a = e_1 + e_2 + \cdots + e_s + n$, where $e_1, e_2, \dots, e_s$ are idempotents and $n$ is nilpotent that commute with each other. Observe that

$$1 = ((1-e_1)+e_1)((1-e_2)+e_2)\cdots((1-e_s)+e_s).$$

After multiplication, we get a sum of products of the form

$$e_{i_1}\cdots e_{i_k}(1-e_{j_1})\cdots(1-e_{j_{s-k}}),$$

where $1 \le k \le s$, $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{s-k}$ and $\{i_1, \dots, i_k, j_1, \dots, j_{s-k}\} = \{1, \dots, s\}$. Next, we get that

$$\begin{aligned}
&(k-a)e_{i_1}\cdots e_{i_k}(1-e_{j_1})\cdots(1-e_{j_{s-k}})\\
&= ((1-e_{i_1})+\cdots+(1-e_{i_k})-e_{j_1}-\cdots-e_{j_{s-k}}-n)e_{i_1}\cdots e_{i_k}(1-e_{j_1})\cdots(1-e_{j_{s-k}})\\
&= -ne_{i_1}\cdots e_{i_k}(1-e_{j_1})\cdots(1-e_{j_{s-k}}).
\end{aligned}$$

This follows from the fact that $(1-e)e = e - e^2 = 0$, for an idempotent $e$. Since $n$ is nilpotent, so is this product. Thus, when 1 is multiplied by $a(a-1)\cdots(a-s)$, we get a sum of nilpotent elements that commute with each other. Therefore, $a(a-1)\cdots(a-s)$ is nilpotent. □

The following corollary is simple, but important.

**Corollary 1.** *a) If a ring $R$ is such that $-1$ is strongly $s$-nil-clean for some $s \geq 1$, then this ring has finite characteristic.*
*b) If* $\mathrm{char}(R) = k$, *then $-1$ is strongly $(p-1)$-nil-clean, where $p$ is the largest prime dividing $k$.*

*Proof.* a) From the previous proposition we conclude that $(-1)(-2)\cdots(-(s+1)) = (-1)^{s+1}(s+1)!$ is nilpotent, so $((s+1)!)^m = 0$ for some $m \geq 1$ and the characteristic of the ring $R$ is not 0.
b) It is enough to show that $-1$ is $(p-1)$-nil-clean in the ring $\mathbb{Z}_k$, which is contained in $R$. If $k = p_1^{\alpha_1}\cdots p_l^{\alpha_l}$ is the prime factorization of $k$, where $p_1 < \cdots < p_l = p$, then $\mathbb{Z}_k \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_l^{\alpha_l}}$. Since $-1 \mapsto (-1,\ldots,-1)$ under this isomorphism, this reduces the proof to show that $-1$ is $(p-1)$-nil-clean in $\mathbb{Z}_{p_i^{\alpha_i}}$. This is clear since $-1 = p_i^{\alpha_i} - 1 = \underbrace{1+\cdots+1}_{p_i-1} + p_i(p_i^{\alpha_i-1}-1)$, and $p_i(p_i^{\alpha_i-1}-1)$ is nilpotent in $\mathbb{Z}_{p_i^{\alpha_i}}$. □

*Example* 1. The ring $\mathbb{Z}_k$ is (strongly) $(p-1)$-nil-clean, where $p$ is the largest prime integer dividing $k$. Namely, using the notation from the previous corollary, this reduces to show that $\mathbb{Z}_{p_i{}^{\alpha_i}}$ is $(p-1)$-nil-clean. Since any element in $\mathbb{Z}_{p_i^{\alpha_i}}$ can be written in the form $\underbrace{1+\cdots+1}_{s} + p_i t$, for some $s \in \{0,\ldots,p_i-1\}$ and $t \in \{0,\ldots,p_i^{\alpha_i-1}-1\}$, and since $p_i$ is nilpotent in $\mathbb{Z}_{p_i^{\alpha_i}}$, we are done.

**Lemma 1.** *If* $k = \mathrm{char}(R) = p_1^{\alpha_1}\cdots p_l^{\alpha_l}$ *is the prime factorization of the characteristic of the ring $R$, then $R \cong R_1 \times \cdots \times R_l$, where $R_i = R/p_i^{\alpha_i}R$. In particular,* $\mathrm{char}(R_i) = p_i^{\alpha_i}$.

*Proof.* This follows easily from the Chinese remainder theorem taking into account the fact that elements $p_i^{\alpha_i}$ are central. □

Since the products of the form $a(a-1)\cdots(a-s)$ are important for our investigation, we introduce the symbol $(a)_k := a(a-1)\cdots(a-(k-1))$ (falling factorial, as is known in combinatorics), where $k$ is a positive integer.

We have the following corollary.

**Corollary 2.** *Let $R$ be a ring. Suppose that element $a \in R$ is strongly $s$-nil clean and $k(<s)$ is nilpotent. Then $(a)_k$ is a nilpotent element.*

*Proof.* Clearly

$$(a)_s = a(a-1)\cdots(a-(s-1)) = a^{t_0}(a-1)^{t_1}\cdots(a-(k-1))^{t_{k-1}} + kq(a),$$

for some non-negative integers $t_i$, such that $\sum_{i=0}^{k-1} t_i = s$ and polynomial $q(X) \in \mathbb{Z}[X]$. Taking into account that $k$ is nilpotent, the result follows. □

In order to see what the fact that $(a)_s$ is nilpotent implies, we start with a simple lemma.

**Lemma 2.** *Let $p$ be a prime integer and $r, m \geq 1$ arbitrary positive integers. Then the element $x$ in the ring $\mathbb{Z}_{p^r}[X]/\langle X^m(X-1)^m \cdots (X-(p-1))^m\rangle$ is (strongly) $(p-1)$-nil-clean, where $x$ is the class of $X$ in this quotient ring.*

*Proof.* Ideals $\langle X-i\rangle$ and $\langle X-j\rangle$ are coprime in $\mathbb{Z}_{p^r}[X]$, for all $0 \leq i < j < p$, since $(X-i)-(X-j) = j-i$ and $j-i$ is invertible in the ring $\mathbb{Z}_{p^r}[X]$. So, $\langle (X-i)^m\rangle, \langle (X-j)^m\rangle$ are coprime as well. By applying the Chinese remainder theorem we obtain the isomorphism

$$\mathbb{Z}_{p^r}[X]/\langle X^m \cdots (X-(p-1))^m\rangle \cong \mathbb{Z}_{p^r}[X]/\langle X^m\rangle \times \cdots \times \mathbb{Z}_{p^r}[X]/\langle (X-(p-1))^m\rangle,$$

such that $x \mapsto (x, \ldots, x)$. Thus, it is enough to show that $x$ has the desired presentation in all factors and since $x = \underbrace{1+\cdots+1}_{i} + (x-i)$ in the factor $\mathbb{Z}_{p^r}[X]/\langle (X-i)^m\rangle$, this is true. □

**Proposition 2.** *Let $R$ be a ring. Suppose that the element $p$ is nilpotent, where $p$ is a prime integer, and let $a \in R$ be such that $(a)_p$ is nilpotent. Then $a$ is strongly $(p-1)$-nil-clean.*

*Proof.* Consider the homomorphism $f: \mathbb{Z}[X] \to R$, given by $f(X) = a$. An immediate consequence of the fact that $(a(a-1)\cdots(a-(p-1)))^m = 0$ and that $p^r = 0$ in $R$, for some $m, r \geq 1$, is the existence of an induced homomorphism

$$\bar{f}: \mathbb{Z}_{p^r}[X]/\langle X^m(X-1)^m \cdots (X-(p-1))^m\rangle \to R, \text{ such that } x \mapsto a.$$

Since $x$ is strongly $(p-1)$-nil-clean, so is its image $a$. □

**Proposition 3.** *Let $R$ be a ring of characteristic $k (> 0)$. If $p$ is the largest prime dividing $k$ and $a \in R$ is such that $(a)_s$ is nilpotent for some $s \geq 1$, then $a$ is strongly $(p-1)$-nil-clean.*

*Proof.* Under isomorphism $R \cong R_1 \times \cdots \times R_l$, where $R_i = R/p_i^{\alpha_i}R$ and $k = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, implied by Lemma 1, the element $a$ goes to $(a_1, \ldots, a_l)$. Also, $a$ is strongly $(p-1)$-nil-clean iff $a_i$ is such for all $i$. However, from the fact that $(a)_s$ is nilpotent, it follows that $(a_i)_s \in R_i$ is nilpotent for all $i$. From this, it easily follows that $(a_i)_{p_i}$ is also nilpotent for all $i$. Namely, if $s < p_i$, this follows since $(a_i)_{p_i} = (a_i)_s(a_i - s)_{p_i - s}$ and if $s > p_i$, it follows from Corollary 2. Since $p_i$ is nilpotent in $R_i$, Proposition 2 gives that $a_i \in R_i$ is strongly $(p_i - 1)$-nil-clean for all $i$. From the fact that $p_i \leq p$ for all $i$, it follows that these elements $a_i$ are all $(p-1)$-nil-clean and so is $a$. □

**Theorem 1.** *Let $R$ be a ring such that every element $a \in R$ is strongly $s$-nil-clean for some $s$. Then $R$ has finite characteristic and $R$ is strongly $(p-1)$-nil-clean, where $p$ is the largest prime dividing* $\mathrm{char}(R)$.

*Proof.* Corollary 1 shows that $R$ has finite characteristic and from Proposition 1 and Proposition 3 it follows that every element is strongly $(p-1)$-nil-clean, where $p$ is the largest prime dividing this characteristic. □

Theorem 1 shows that one needs only to investigate strongly $(p-1)$-nil-clean rings, where $p$ is a prime integer. For example, the class of all strongly 3-nil-clean rings is the same as the class of strongly 2-nil-clean rings, and the class of strongly 9-nil-clean-rings is the same as the class of strongly 6-nil-clean rings. Namely, if a ring is, say, strongly 9-nil-clean, then $(a)_{10}$ is nilpotent for all $a \in R$. So, this is true for $a = 10$. Consequently, $10! = (10)_{10}$ is nilpotent, so $\mathrm{char}(R) \mid (10!)^m$, for some $m \geq 1$. We conclude that the largest prime dividing $\mathrm{char}(R)$ is at most 7 (it may be smaller), so our ring is strongly 6-nil-clean.

**Proposition 4.** *A subring of a strongly $(p-1)$-nil-clean ring is also strongly $(p-1)$-nil-clean.*

*Proof.* Let $S$ be a subring of a strongly $(p-1)$-nil-clean ring $R$ and let $a \in S$. Since $a \in R$, $a$ is strongly $(p-1)$-nil-clean, and according to Proposition 1 element $(a)_p$ is nilpotent. $R$ is of finite characteristic, say $k$, which means that $\mathrm{char}(S) = k$, with $p$ being the largest prime dividing $k$. When we apply Proposition 3, we get that $a$ is strongly $(p-1)$-nil-clean in $S$. □

**Theorem 2.** *Let $R$ be a commutative ring of finite characteristic $k$ and $p$ the largest prime dividing $k$. Let $S = \{a \in R : a \text{ is } s\text{-nil-clean for some } s\}$. Then $S$ is the largest subring of $R$ which is $(p-1)$-nil-clean.*

*Proof.* Since $k = 0$ in $R$, we have: $-1 = \underbrace{1 + \cdots + 1}_{k-1}$, so $-1$ is $(k-1)$-nil-clean and $-1 \in S$. Also, if $a, b \in S$, then $a = e_1 + \cdots + e_s + n$, $b = f_1 + \cdots + f_t + n'$ and we get $ab = \sum_{i,j} e_i f_j + N$, where $N$ is nilpotent and all $e_i f_j$ are idempotents ($R$ is a commutative ring). So, $ab \in S$. Similarly, $a + b = e_1 + \cdots + e_s + f_1 + \cdots + f_t + n + n' \in S$. Finally, since $a - b = a + (-1)b$ we conclude that $a - b \in S$ as well. So, $S$ is a ring in which every element is $s$-nil-clean for some $s$. From Theorem 1, we conclude that $S$ is actually $(p-1)$-nil-clean ring. It is clear that $S$ is the largest such subring. □

*Remark* 2. From the Theorem 2 it is clear that, in order to show that a commutative ring of finite characteristic $k$ is $(p-1)$-nil-clean, it is enough to check only its ring generators over $\mathbb{Z}_k$. For example, the ring $\mathbb{Z}_{p^r}[X]/\langle X^m(X-1)^m \cdots (X-(p-1))^m\rangle$, appearing in Lemma 2, is $(p-1)$-nil-clean, since it is generated by $x$ and this element is $(p-1)$-nil-clean.

The following proposition provides us with a lot of examples of commutative $(p-1)$-nil-clean rings.

**Proposition 5.** *Let $R$ be a finite commutative local ring, $M$ its maximal ideal and $R/M \cong \mathbb{Z}_p$. Then $R$ is $(p-1)$-nil-clean.*

*Proof.* We know that every element in $M$ is nilpotent. If $x \in R$, then $x + M = s + M$, for some $s \in \{0, \dots, p-1\}$. So, $x = \underbrace{1 + \cdots + 1}_{s} + m$, where $m$ is nilpotent. □

*Remark* 3. In the case of non-commutative rings, the set of all elements which are strongly $s$-nil-clean for some $s$ do not necessarily form a subring. Such examples will be given in Section 3 and Section 4. However, a simple application of Zorn's lemma shows that there exist maximal subrings which are strongly $(p-1)$-nil-clean.

The following proposition gives another characterization of strongly $(p-1)$-nil-clean elements.

**Proposition 6.** *Let $R$ be a ring of finite characteristic $k$, $p$ the largest prime dividing $k$ and $a \in R$. The following conditions are equivalent.*

(1) *$(a)_p$ is nilpotent.*
(2) *$a$ is strongly $(p-1)$-nil-clean.*
(3) *$a = b + n$, where $b \in R$ is such that $(b)_p = 0$, $n$ is nilpotent and $bn = nb$.*

*Proof.* (1) $\Longrightarrow$ (2). This is contained in Proposition 3.
(2) $\Longrightarrow$ (3). Let $a = e_1 + \cdots + e_{p-1} + n$ be a $(p-1)$-nil-clean decomposition of $a$. Take $b := e_1 + \cdots + e_{p-1}$. The proof of Proposition 1 shows that $(b)_p = 0$.
(3) $\Longrightarrow$ (1). Assume that $a = b + n$. So, we have

$$(a)_p = (b+n)((b-1)+n)\cdots((b-p+1)+n).$$

Therefore, since $n$ and $b$ commute, $(a)_p = (b)_p + nq(n,b) = nq(n,b)$, for some polynomial $q(X,Y) \in \mathbb{Z}[X,Y]$. Since $n$ is nilpotent, so is $(a)_p$. □

## 3. Structure theorems

The purpose of this section is to discuss the structure of (strongly) $(p-1)$-nil-clean rings, for prime number $p$.

The following proposition sums up the discussion from the previous section.

**Proposition 7.** *Suppose that* $\mathrm{char}(R) = k = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, *where $p_1 < \cdots < p_l = p$. Then $R$ is strongly $(p-1)$-nil-clean if and only if $R_i$ is strongly $(p_i - 1)$-nil-clean, where $R_i = R/p_i^{\alpha_i} R$ and $1 \le i \le l$.*

This shows that in investigation of strongly $(p-1)$-nil-clean rings, for $p$ prime, we can reduce our analysis to the case when $p$ is nilpotent (equivalently, when the characteristic of the ring is a power of a prime).

Let us recall that a ring $R$ is called strongly $\pi$-regular if for every element $a \in R$ there exists $n \ge 1$ and $x \in R$ such that $a^n = a^{n+1}x$.

**Theorem 3.** *Every strongly $(p-1)$-nil-clean ring is strongly $\pi$-regular.*

*Proof.* It is enough to consider the case when $p$ is a nilpotent element. Then $(p-1)!$ is invertible. Let $a \in R$. Since $(a)_p$ is nilpotent, we have that $((a)_p)^s = 0$ for some $s$. But

$$0 = ((a)_p)^s = (a(a-1)\cdots(a-(p-1)))^s = a^s((p-1)!)^s + a^{s+1}y,$$

for some $y \in R$. Since $(p-1)!$ is invertible, we get that $a^s = a^{s+1}x$, for some $x \in R$ and we are done. □

It is well known that strongly $\pi$-regular ring is strongly clean (see [2, Proposition 2.6], [9, Theorem 1], [5, Corollary 2.4]). Also, Jacobson radical of a strongly $\pi$-regular ring is nil and commutative strongly $\pi$-regular rings have Krull dimension 0 (see [2]). So, we have the following corollary.

**Corollary 3.** *Every strongly $(p-1)$-nil-clean ring is strongly clean.*

The following proposition is rather useful.

**Proposition 8.** *Let $R$ be a ring, $a \in R$ and let $p$ be a nilpotent element, where $p$ is prime. Then $a^p - a$ is nilpotent if and only if $(a)_p$ is nilpotent.*

*Proof.* It is well-known that $X^p - X = (X)_p$ in $\mathbb{Z}_p[X]$. So, $a^p - a - (a)_p = pr(a)$, for some polynomial $r(X) \in \mathbb{Z}[X]$. From this fact, the proof follows immediately. □

For future reference, we formulate the following corollary which directly follows from Proposition 6 and Proposition 8.

**Corollary 4.** *Let $R$ be a ring. If $p$ is nilpotent, then $R$ is a strongly $(p-1)$-nil-clean ring if and only if $a^p - a$ is nilpotent for every $a \in R$.*

Let us proceed with some of the special properties of $(p-1)$-nil-clean rings.

**Proposition 9.** *Let $R$ be a ring and let $I$ be any nil ideal of $R$. Then $R$ is $(p-1)$-nil-clean if and only if $R/I$ is $(p-1)$-nil-clean.*

*Proof.* ($\Longrightarrow$) As observed before, this is trivial since $R/I$ is a homomorphic image of $R$.
($\Longleftarrow$) Suppose that $R/I$ is $(p-1)$-nil-clean. Let $x \in R$. Then $x+I$ is $(p-1)$-nil-clean. Thus, $x + I = (x_1 + I) + (x_2 + I) + \cdots + (x_{p-1} + I) + (y + I)$, where $x_i + I$ are idempotents, $1 \le i \le p-1$, and $y + I$ is nilpotent. It is well known that idempotents lift modulo nil ideals (see, e.g. [6, Theorem 21.28]) so there are idempotents $e_i$ such that $x_i + I = e_i + I$. So, $x - e_1 - e_2 - \cdots - e_{p-1} - y \in I$, i.e., $x = e_1 + e_2 + \cdots + e_{p-1} + y + n$, for some $n \in I$. Element $y + n$ is nilpotent. Indeed, since $y + I$ is nilpotent, $y^k \in I$ for some $k \in \mathbb{N}$. Every element different from $y^k$ in the sum one gets in the expansion of $(y+n)^k$, is in $I$ and we can conclude that $(y+n)^k \in I$, so $y + n$ is nilpotent. Therefore $R$ is $(p-1)$-nil-clean. □

An analogous result holds for the strongly $(p-1)$-nil-clean rings.

**Proposition 10.** *Let $R$ be a ring and let $I$ be any nil ideal of $R$. Then $R$ is strongly $(p-1)$-nil-clean if and only if $R/I$ is strongly $(p-1)$-nil-clean.*

*Proof.* ($\Longrightarrow$) Again, this is trivial since $R/I$ is a homomorphic image of $R$. ($\Longleftarrow$) Let $a \in R$. Since $R/I$ is strongly $(p-1)$-nil-clean, one has $((a+I)_p)^k = I$, for some $k \in \mathbb{N}$. Consequently, $((a)_p)^k \in I$. As $I$ is nil ideal, $((a)_p)^k \in N(R)$. So, $(a)_p$ is nilpotent. Since $(p)_p$ is also nilpotent, the characteristic $k$ of $R$ is finite. The characteristic $l$ of $R/I$ has the property that $p$ is the largest prime dividing this characteristic, but this also holds for $k$. Namely, $l \in I$ and therefore $l$ is nilpotent in $R$. So we have that $k \mid l^s$ for some $s$, and also $l \mid k$. It follows that the sets of primes dividing $k$ and $l$ are the same. Now the result follows from Proposition 6. □

The following corollary follows directly from the fact that $J(R)$ is nil for a strongly $(p-1)$-nil-clean ring and Proposition 10.

**Corollary 5.** *A ring $R$ is strongly $(p-1)$-nil-clean if and only if $J(R)$ is nil and $R/J(R)$ is strongly $(p-1)$-nil-clean.*

## 4. Group rings

Let us recall the notion of a group ring. Let $G$ be a group, written multiplicatively, and let $R$ be a commutative ring. The group ring of $G$ over $R$, denoted by $RG$, is a free $R$-module with generating set $G$, i.e.:

$$RG = \bigoplus_{g \in G} Rg.$$

So, elements of $RG$ are formal finite sums of the form $\sum_i r_i g_i$, with $r_i \in R$, $g_i \in G$, while the multiplication is implied by multiplication in $G$. The identity of this ring is $1_R e$, where $1_R$ is the identity in $R$ and $e$ is the neutral element of $G$. We denote the identity simply by 1.

Our main interest here is focused on strongly $(p-1)$-nil-clean group rings $RG$. It is obvious that if $RG$ is strongly $(p-1)$-nil-clean, so is $R$. Since we assume that the coefficient ring $R$ is commutative, we refrain from using adjective "strongly" when referring to $R$, we use it only for $RG$ when appropriate. We begin by discussing rings $R$, such that $\mathrm{char}(R)$ is a power of a prime.

**Lemma 3.** *Let $R$ be a $(p-1)$-nil-clean ring such that* $\mathrm{char}(R) = p^s$, *for a prime $p$ and some $s \geq 1$ and let $G$ be a group. For the list of conditions:*

(1) *$RG$ is strongly $(p-1)$-nil-clean;*
(2) *For each $g \in G$, the element $g^{p-1} - 1$ is nilpotent;*
(3) *For each $g \in G$, there exists $k \geq 0$ and $d \mid (p-1)$ such that $\omega(g) = dp^k$,*

*the following holds:* (1) $\Longrightarrow$ (2) *and* (2) $\Longleftrightarrow$ (3). *Here, $\omega(g)$ denotes the order of $g$ in $G$.*

*Proof.* (1) $\Longrightarrow$ (2). Let $g \in G$. Since $RG$ is strongly $(p-1)$-nil-clean, from Corollary 4 it follows that $g^p - g$ is nilpotent. Since $g$ is invertible, we get that $g^{p-1} - 1$ is nilpotent.
(3) $\Longrightarrow$ (2). Let $g \in G$. Then $\omega(g) = dp^k$, where $k \geq 0$ and $d \mid (p-1)$. Let $p-1 = ds$. Since $\omega(g^d) = p^k$ and $\gcd(s, p^k) = 1$, we have that $\omega(g^{p-1}) = \omega((g^d)^s) = p^k$. Therefore,

$$(g^{p-1}-1)^{p^k} = \underbrace{(g^{p-1})^{p^k} - 1}_{0} + p \cdot u, \text{ for some } u \in RG.$$

Since $p$ is nilpotent, the element $g^{p-1} - 1$ is nilpotent as well.
(2) $\Longrightarrow$ (3). Let $g \in G$. The order of $g$ cannot be infinite — in that case, it would not be possible for $g^{p-1} - 1$ to be nilpotent. Namely, the element $g^{(p-1)s}$ in the sum $(g^{p-1}-1)^s = g^{(p-1)s} + \cdots + (-1)^s$ cannot be cancelled out.

So, let us suppose that $\omega(g) = tp^k$, for some $k \geq 0$ and $t$ such that $p \nmid t$ and $\gcd(t, p-1) = d \neq t$. Also, let $t = dt_1$, $p-1 = dz_1$ and $h = g^{p-1}$. Since $\omega(g^d) = t_1 p^k$ and $\gcd(z_1, t_1 p^k) = 1$, it follows that

$$\omega(h) = \omega(g^{p-1}) = \omega((g^d)^{z_1}) = t_1 p^k.$$

Since $h-1$ is nilpotent, $h^{p^k} - 1$ is nilpotent as well. Let $h_1 = h^{p^k}$. Then $\omega(h_1) = t_1$. Let us focus on the polynomial $f(X) = (X-1)^{t_1} - (X^{t_1} - 1)$, which is clearly divisible by $X - 1$:

$$f(X) = (X-1)((X-1)^{t_1-1} - (X^{t_1-1} + \cdots + X + 1)) = (X-1) f_1.$$

This follows from the fact that $t_1 \neq 1$ (which also implies that $h_1 - 1 \neq 0$). We can see that

$$f(X) = (X-1)(-t_1 + (X-1)q(X)),$$

for some polynomial $q \in \mathbb{Z}[X]$, since $f_1(1) = -t_1$. We can conclude that

$$f(h_1) = (h_1-1)^{t_1} - \underbrace{(h_1^{t_1} - 1)}_{0} = (h_1-1)(-t_1 + (h_1-1)q(h_1)).$$

We know that $h_1 - 1$ is nilpotent. As $p \nmid t_1$ and $p$ is nilpotent, element $t_1$ is invertible in $R$. So

$$(h_1-1)^{t_1} = u(h_1-1),$$

for an invertible $u \in RG$. So

$$(h_1-1)((h_1-1)^{t_1-1} - u) = 0,$$

and since $(h_1-1)^{t_1-1} - u$ is invertible, we have that $h_1 - 1 = 0$. That is a contradiction.

It is easy to check that the proof of 2 $\Longleftrightarrow$ 3 is valid, although shorter, even for $p = 2$. $\square$

In the previous lemma, $G$ was an arbitrary group. If we add commutativity, we actually get equivalence (1) $\iff$ (2).

**Theorem 4.** *Let $R$ be a $(p-1)$-nil-clean ring such that* $\mathrm{char}(R) = p^s$*, for a prime $p$ and some $s \geq 1$ and let $G$ be an Abelian group. Then $RG$ is $(p-1)$-nil-clean iff $g^{p-1} - 1$ is nilpotent for every $g \in G$.*

*Proof.* We only need to prove the "if" part. It follows directly from the Remark following Theorem 2 since elements of the group $G$ form a generating set for $RG$ over $R$. □

*Example* 2. The previous theorem does not hold for non-commutative groups. Let us take $R = \mathbb{Z}_5$ and $G = \mathbb{D}_4$, the dihedral group of order 8 generated by elements $s$ and $r$ such that $s^2 = 1 = r^4$, $sr = r^3 s$. In this group, $g^4 = 1$ for all $g \in G$, so the condition that $g^4 - 1$ is nilpotent is trivially satisfied. However, direct computation shows that

$$\begin{aligned}(s+sr)_5 &= 2s + 2r + 3sr^2 + 3r^3\\ ((s+sr)_5)^8 &= 3 + 2r^2\\ (3+2r^2)^2 &= 3 + 2r^2,\end{aligned}$$

so $(s+sr)_5$ is not nilpotent and the group ring $\mathbb{Z}_5\mathbb{D}_4$ is not strongly 4-nil-clean.

Let us concentrate now on the general case.

**Proposition 11.** *Let $R$ be a $(p-1)$-nil-clean ring and* $\mathrm{char}(R) = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$*, so that $l > 1$, $p_1 < \cdots < p_l = p$.*

(1) *If $G$ is an elementary Abelian 2-group, then $RG$ is strongly $(p-1)$-nil-clean.*
(2) *If $G$ is an elementary Abelian group in which every element has order $q$ and $q \mid \gcd(p_1 - 1, \ldots, p_l - 1)$, then $RG$ is strongly $(p-1)$-nil-clean.*

*Proof.* (1) As we know, $R \cong R_1 \times \cdots \times R_l$, where $p_i$ is nilpotent in $R_i$, and consequently $RG \cong R_1 G \times \cdots \times R_l G$. So, all the rings $R_i$ are strongly $(p-1)$-nil-clean, and since $p_i$ is nilpotent in $R_i$, $R_i$ is actually strongly $(p_i - 1)$-nil-clean. We will use Theorem 4. Since $2 \mid (p_i - 1)$ for all $i \geq 2$ and $g^2 = 1$ for all $g \in G$, we have that $g^{p_i - 1} - 1 = 0$ for all $g \in G$. If $p_1 > 2$, the same holds for $p_1$. If $p_1 = 2$, then $(g-1)^2 = g^2 - 2g + 1 = 2(1-g)$. Since 2 is nilpotent in $R_1 G$, $g - 1$ is also nilpotent in $R_1 G$. In this case also, from $G$ being Abelian, we can conclude that $R_i G$ is $(p_i - 1)$-nil-clean. Therefore, $RG$ is $(p-1)$-nil-clean.
(2) Similarly, from $g^q = 1$, and $q \mid \gcd(p_1 - 1, \ldots, p_l - 1)$, we conclude that $g^{p_i - 1} - 1 = 0$ for all $g \in G$, and proceed as in (1). □

**Theorem 5.** *Let* $\mathrm{char}(R) = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$*, where $l > 1$, $p_1 < \cdots < p_l = p$ and let $G$ be a group. Suppose that $RG$ is strongly $(p-1)$-nil-clean ring.*

(1) *For all $g \in G$ the following holds: $\omega(g) \mid \gcd(p_2 - 1, \ldots, p_l - 1)$ and $\omega(g) = d_1 p_1^s$, such that $d_1 \mid (p_1 - 1)$ and $s \geq 0$.*

(2) *If there exists $h \in G$ such that $\omega(h) \geq p_1$, then $p_1 \mid (p_i - 1)$, for $2 \leq i \leq l$.*
(3) *If for all $g \in G$, $\omega(g) < p_1$, then $\omega(g) \mid \gcd(p_1 - 1, \ldots, p_l - 1)$.*
(4) *If $p_1 = 2$, then every element in $G$ has order $2^s$ for some $s \geq 0$.*
(5) *If $p_1 = 2$ and $p_i \equiv 3 \pmod 4$ for at least one $i \geq 2$, then $G$ is an elementary Abelian 2-group.*

*Proof.* (1) As before, $RG \cong R_1G \times \cdots \times R_lG$, where $p_i$ is nilpotent in $R_iG$. Lemma 3 shows that $g^{p_i - 1} - 1$ is nilpotent in $R_iG$. It follows that $\omega(g) = d_i p_i^{k_i}$ for some $k_i \geq 0$ and $d_i \mid (p_i - 1)$. So,

$$d_1 p_1^{k_1} = d_2 p_2^{k_2} = \cdots = d_l p_l^{k_l}.$$

Since $d_1 < p_1$, it is clear that $d_1 p_1^{k_1}$ cannot be divisible by any prime greater than $p_1$. So $k_i = 0$ for $i \geq 2$. Therefore,

$$\omega(g) = d_1 p_1^{k_1} = d_2 = \cdots = d_l, \tag{4.1}$$

for all $g \in G$, where $d_i \mid (p_i - 1)$. Hence, $\omega(g) \mid (p_2 - 1), \ldots, \omega(g) \mid (p_l - 1)$, and we are done.
(2) If $h \in G$ is such that $\omega(h) \geq p_1$, we have $\omega(h) = d_1 p_1^{k_1} = d_2 = \cdots = d_l$, where $k_1 \geq 1$. Since $d_i \mid (p_i - 1)$, it follows that $p_1 \mid (p_i - 1)$.
(3) Under this assumption, we get that $k_1 = 0$ in 4.1, hence for all $g \in G$

$$\omega(g) = d_1 = d_2 = \cdots = d_l.$$

We conclude that $\omega(g) \mid \gcd(p_1 - 1, \ldots, p_l - 1)$.
(4) The fourth assertion follows easily. Namely, in this case $p_1 = 2$, so $\omega(g) = d_1 2^{k_1}$, where $d_1 \mid (2 - 1)$. So, $\omega(g) = 2^{k_1}$.
(5) It is enough to show that there are no elements of order 4 in $G$. If it were, then for an element $g \in G$, we would have equalities

$$4 = d_2 = \cdots = d_l,$$

where $d_i \mid (p_i - 1)$, for $2 \leq i \leq l$. This would imply that $4 \mid (p_i - 1)$, that is $p_i \equiv 1 \pmod 4$, for $2 \leq i \leq l$, which is a contradiction. Hence, we can conclude that $G$ is an elementary Abelian 2-group. □

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# EFFICIENT APPROXIMATION FOR COUNTING OF FORMAL CONCEPTS GENERATED FROM FORMAL CONTEXT

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*Abstract.* The number of formal concepts generated from the input context is an important parameter in the cost functions of concept formation algorithms. The calculation of concept count for any arbitrary context is a hard, NP-complete problem and only rough approximation methods can be found in the literature to solve this problem. This paper introduces an efficient numerical approximation algorithm for contexts where attribute probabilities are independent from the objects instances. The preconditions required by the approximation method are usually met in the FCA applications, thus the proposed method provides an efficient tool for practical complexity analysis, too.

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## 1. Introduction

In the development of Formal Concept Analysis (FCA) applications, a key issue is the cost efficiency of concept set and concept lattice management. The cost function [12] [14] usually contains the following key parameters:

- $N$ : number of objects
- $M$: number of attributes
- $L$: the average number of attributes related to an arbitrary object (context density)
- $C$: total number of concepts generated.

Although the value $C$ is a function of the context, parameter $C$ is considered as an independent base parameter. The reason of this simplification is that the relationship to calculate $C$ is too complex, no simple analytical description is known.

This paper introduces a probabilistic model and a related efficient numerical approximation method to determine the count of generated concepts. The probabilistic model is based on the model presented in [9], where the goal of the work was to determine the significance of the generated formal concepts. Unlike the original paper, our model is aimed at an efficient approximation of the total number of concepts.

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The proposed method can be used among others in complexity analysis of FCA algorithms in many application areas.

The FCA provides tools to manage and to investigate the concept set generated from an input formal context. A formal context is defined as a triplet $< \mathcal{G}, \mathcal{M}, \mathcal{I} >$, where $\mathcal{I}$ is a binary relation between $\mathcal{G}$ (set of objects) and $\mathcal{M}$ (set of attributes). The property $(g,m) \in \mathcal{I}$ is met if and only if the attribute $m$ is true for the object $g$. Two derivation operators are introduced as mappings between the powersets of $\mathcal{G}$ and $\mathcal{M}$. For $A \subseteq \mathcal{G}, B \subseteq \mathcal{M}$:

$$f(A) = A^I = \{m \in \mathcal{M} | \forall g \in A : (g,m) \in I)\},$$

$$g(B) = B^I = \{g \in \mathcal{G} | \forall m \in B : (g,m) \in I)\}.$$

For a context $< \mathcal{G}, \mathcal{M}, \mathcal{I} >$, a formal concept is defined as a pair $(A, B)$, where $A \subseteq \mathcal{G}$, $B \subseteq \mathcal{M}$, $A = B^I$, $B = A^I$ are met. The composition of these derivations are closure operators, $A \mapsto A^{II}, A \subseteq \mathcal{G}$ and respectively $B \mapsto B^{II}, B \subseteq \mathcal{M}$. Regarding the derivation operator, the components of a formal concept satisfy the $A = A^{II}$, $B = B^{II}$ conditions, too. The $A$ component is called the extent of the concept, while $B$ is the intent part.

On the set of formal concepts $\mathcal{C}$ generated from the context $< \mathcal{G}, \mathcal{M}, \mathcal{I} >$, a partial ordering relation is defined in the following way:

$$(A_1, B_1) \leq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2.$$

It can be shown that $A_1 \subseteq A_2$ if and only if $B_2 \subseteq B_1$. The obtained partially ordered set $(\mathcal{C}, \leq)$ is in fact a complete lattice, called the concept lattice of the context $< \mathcal{G}, \mathcal{M}, \mathcal{I} >$.

The size of $\mathcal{C}$ is a key factor in the cost analysis of FCA algorithms. Due to the complex relationship between the size of $\mathcal{C}$ and the context parameters, there is no simple and efficient approximation to determine the total number of concepts. The first related result was presented by Ganter and Wille in [7] showing that the size of $\mathcal{C}$ may increase exponentially in the parameters $N$ and $M$. Beside the parameters $N$ and $M$, also the density of the context plays an important role in the complexity analysis. A context with large values of $N$ and $M$, but having a sparse $\mathcal{I}$, yields a small concept set. One of the first analytical results in counting the concepts can be found in [16] presented by Schutt. The paper provides the following upper approximation for the count value:

$$C \leq 3/2 \cdot 2^{\sqrt{|I|+1}} - 1.$$

Later, Kuznetsov has been proved in [10] that calculation algorithm for the total number of concepts belongs to the NP-complete problem class.

A more sharper theoretical upper bound was shown by Prisner in [15] and by Albano and Chornomaz in [1]. They have investigated a special type of contexts, the contranomial scale free contexts. For a given set $\mathcal{S}$, the context $< \mathcal{S}, \mathcal{S}, \neq >$ is a contranomial scale context. If the set $\mathcal{S}$ contains $k$ elements then the context belongs

to the class $N^c(k)$ . For any $N^c(k)$-free context, the upper bound for the concept number is given with

$$C \leq (|G||M|)^{k-1} + 1.$$

According to the literature, there are only few proposals to provide a precise approximation method. One important result is presented in [4], where a sampling approach was used to estimate the concept count. The sampling method traverses the concept lattice by random walk and it works with a series of increasing sub-contexts. The candidate concepts are checked whether they are contained in other subcontexts already tested or not. The main drawback of the proposed algorithm is the high calculation cost as the number of candidate concepts for sampling is very high.

Due to these efficiency problems, our method uses a different approach. We take a simplified probability model where the attribute occurence probabilities are independent from the objects. This independence model was used also in [9] and [6], where the goal was to calculate the relevance of the discovered concepts.

The applied data matrix model is based on a fixed matrix marginal approach presented also in [8]. In the approach, the sums of the columns (the probabilities of the attributes) are fixed. The article presents also a novel algorithm calculating the concept probability index, but the cost of the proposed algorithm is too high for large practical data contexts. The concept probability index can be used also in the fuzzy FCA models [5] to provide an uncertainty level for knowledge engineering.

An interesting generalization of the probability model can be found in [3] to determine the basic level of concepts generated by FCA. Basic level concepts are those concepts which are used to refer to objects of our everyday life. The basic level can be seen as a compromise between the accuracy of classification at a maximally general level and the predictive power of a maximally specific level [13]. Using the example given in [3], when we refer to a particular dog then we usually say 'It is a dog.' rather than 'This is a German Shepherd.' or 'This is a mammal.'. The elements of the basic level concept sets are characterized by the fact that they have significantly larger cohesion than its upper neighbors and they have only a slightly smaller cohesion than its lower neighbors. A similar research was presented also in [11] to calculate concept interestingness using the concept probability, concept stability and concept robustness as the main components of the interestingness measure.

## 2. Concept probability model

A basic assumption in our model is that the input context is generated randomly. The probability that object $i$ is linked to attribute $j$ is denoted by

$$p_{ij}.$$

We assume that

$$\forall j, i_1, i_2 : p_{i_1,j} = p_{i_2,j}.$$

The elements of the context matrix are either 1 (the attribute is true) or 0. A concept corresponds to a special sub-matrix of the context matrix. An example is shown in Figure 1, where $K$ denotes the context matrix, $A$ is the concept subcontext, while $B, C, D, E$ are the complementary sub-contexts of $A$. Although the concept subcontext is a single rectangle in the example, the subcontexts are usually fragmented.

A sub-context $A$ corresponds to a formal concept if and only if:

- all elements in $A$ are set to 1
- for each column in sub-contexts $B$, $D$, one of the elements is equal to 0
- for each row in sub-contexts $C, E$, one of the elements is equal to 0.

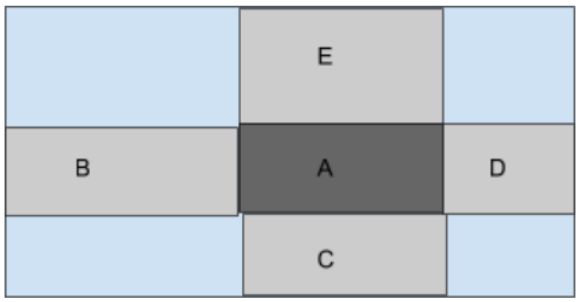


FIGURE 1. Example context and sub-contexts

We introduce a random variable $\xi_A$ for every candidate sub-contexts $A$, where the value is set to 1 if the $A$ belongs to a concept in the current experiment. Otherwise, the value is equal to 0. The mean value of $\xi_A$ shows the probability that $A$ belongs to a concept. Next, we take a new random variable $\xi$ which is equal to the sum of the candidate level random variables. The total number of concepts in the context is estimated with the expected value of $\xi$.

The calculation of the mean value for $\xi_A$ is based on the following considerations. First, we know that all matrix elements in $A$ must be equal to 1 an the corresponding probability is equal to

$$\prod_{(i,j)\in A} p_{i,j}.$$

The probability that every column in the region $B \cup D$ contains at least one element with value 0:

$$\prod_{o\in B\cup D} (1 - \prod_{(i,j)\in o} p_{i,j}),$$

where $o$ denotes a column. The corresponding probability that every row in the region $C \cup E$ contains at least one element with value 0 can be given with

$$\prod_{s\in C\cup E} (1 - \prod_{(i,j)\in s} p_{i,j}).$$

where $s$ is a row element. Based on these considerations, the mean value of $\xi$, i.e. the sum over the set of all possible candidate sub-contexts is equal to:

$$C = \sum_{A \subset K} \prod_{(i,j) \in A} p_{i,j} \prod_{o \in B \cup D} (1 - \prod_{(i,j) \in o} p_{i,j}) \prod_{s \in C \cup E} (1 - \prod_{(i,j) \in s} p_{i,j}). \tag{2.1}$$

This expression shows our base formula for the approximation of concept count.

### 2.1. *Uniform distribution*

In this case, we assume that the probability for every object-attribute pair is the same:

$$\forall i, j : p_{i,j} = p.$$

The corresponding formula for the approximation concept count can be transformed into the following simple formula:

$$C = \sum_{n=1}^{N} \sum_{m=1}^{M} \binom{N}{n} \binom{M}{m} p^{n \cdot m} (1 - p^n)^{M-m} (1 - p^m)^{N-n}. \tag{2.2}$$

This formula can be implemented with the following R code:

```
cptcnt = function(N,M,P) {
  val = 0;
  for (Nx in 1:N) {
    for (My in 1:M) {
      val = val + getval(N,M,Nx,My,P)
    }
  }
  return (val);
}

getval = function (N, M, Nx, My,P) {
  c = choose(N , Nx)*choose(M , My)*(P^(Nx*My))*
    (1 - P^Nx)^(M - My)*(1 - P^My)^(N - Nx);
  return (c);
}
```

To demonstrate the accuracy of the presented model, Figure 2 shows the result of a comparison test, where the theoretical calculation is compared with the experimental measurement. The figure shows the dependency of the calculated and measured concept counts ($C$) in dependency form attribute probability ($P$). For the eperimental measurement, we used our Java implementation of the InClose [2] algorithm. The test is based on random generation of the context using the following parameter settings: $N$=15, $M$=10; $P$=0.1..1.0 and the length of the runs to calculate the mean

value is 5. The dashed line corresponds to the measured values. From the viewpoint of the practice, the result shows a good estimation accuracy.

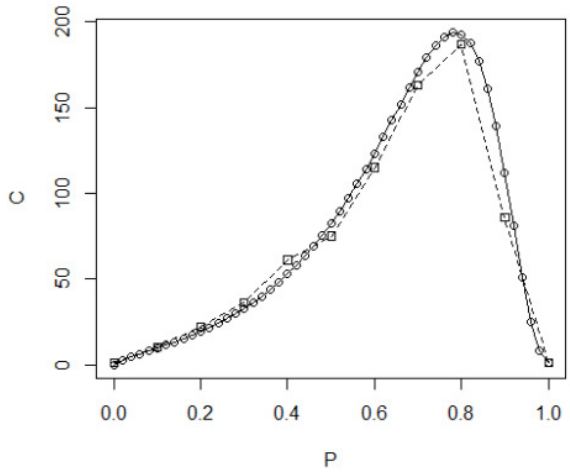


FIGURE 2. Accuracy test of the approximation formula

### 2.2. *Not uniform distribution*

Next, we turn to the general case, when different attributes may have different probability values. On the other hand, according to our base condition, this probability is independent from the single objects:

$$\forall j, i_1, i_2 : p_{i_1,j} = p_{i_2,j}.$$

In this case, the general formula can be transformed into the following form:

$$C = \sum_{n=1}^{N} \binom{N}{n} \sum_{Y \subset T} \prod_{j \in Y} p_j^n \prod_{j \in B \cup D} (1 - p_j^n)(1 - \prod_{j \in Y} p_j)^{N-n}. \tag{2.3}$$

In the expression, the symbol $Y$ denotes an arbitrary subset of the attributes. In the following figures, the results of some comparison tests can be observed. In the tests, the results of the theoretical calculations are compared with the experimental measurements. Figure 3 is related to the parameter set ($N$=100; $M$=8,10,12,14,16; $P$=0.1-0.3). As the comparisons show the theoretical model provides a very good approximation.

## 3. SAMPLING-BASED COST REDUCTION

Although, the approximation algorithm presented in the previous section, provides a good accuracy, it has a significant weakness: it has a very high execution cost. The

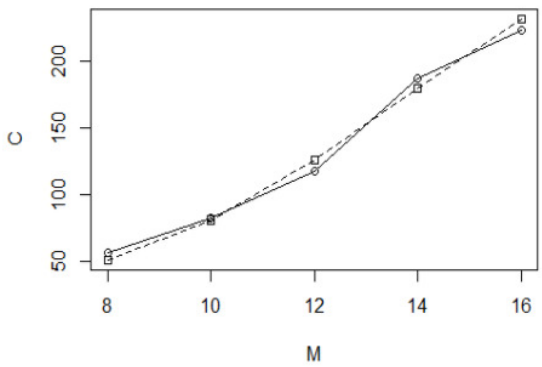


FIGURE 3. Accuracy test of the approximation formula

algorithm requires a large amount of time for smaller problems too. For example, taking the context with parameterset ($N$=100,$M$=16, $P$=0.1..0.3), the runtime is over 967 seconds. Figure 4 shows the corresponding dependency between the execution time and the size of the attribute set ($M$). In order to apply the approximation algorithms to real-life size problems, the base algorithm must be updated to an optimized version.

The full enumeration of the candidates implemented in the baseline version of the approximation algorithm is not suitable to handle larger contexts. The aim of the optimization is to eliminate some candidates in the evaluation process. The initial formula containing full enumeration is equal to

$$C = \sum_{n=1}^{N} \binom{N}{n} \sum_{Y \subset T} \prod_{j \in Y} p_j^n \prod_{j \in B \cup D} (1 - p_j^n)(1 - \prod_{j \in Y} p_j)^{N-n}. \tag{3.1}$$

This formula processes all candidates ordered by the sub-context size. Our analysis shows that the candidates with different size values usually have very different probability weights. Considering all the candidates with size parameter $(n,m)$, the corresponding sub-total is given with

$$C_{n,m} = \binom{N}{n} \sum_{Y \subset T, |Y|=m} \prod_{j \in Y} p_j^n \prod_{j \in B \cup D} (1 - p_j^n)(1 - \prod_{j \in Y} p_j)^{N-n}. \tag{3.2}$$

Investigating the $cnt_{n,m}$ values for different $(n,m)$ parameters, we can see that there are dominating $(n,m)$ pairs where the values are significantly higher than the values on the complement area. Fig F5 shows the distribution for the context ($N$=30, $M$=14, $P_1$=0.2,$P_2$=0.6). The $x$ and $y$ axes correspond to $n$ and $m$, while the $z$ axis denotes the count value. In this example, the dominance area involves only small $(n,m)$ values.

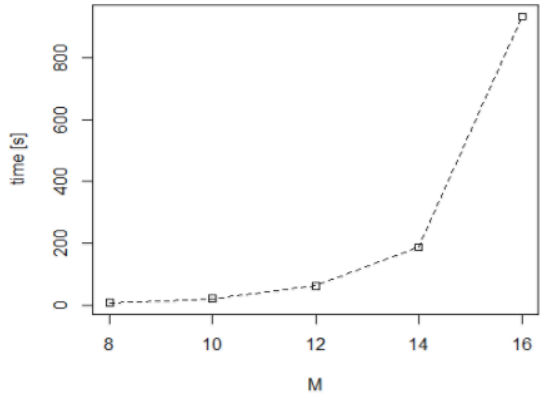


FIGURE 4. Execution cost function

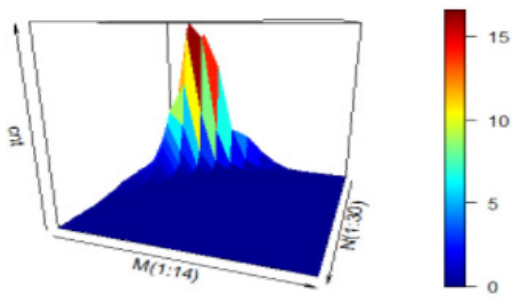


FIGURE 5. The $C_{n,m}$ distribution

The position of the dominance area depends on the attribute probabilities of the input context. The Figures 6-9 show the maximum positions for different $P$ values taking uniform attribute probability distribution. The parameter of the input context is ($N$=40,$M$=14,$P$=(0.2,0.6,0.8,0.96)).

An important observation is that the dominance zone for the not very dense contexts is always near the origo position. In the case of not uniform attribute probability distribution, the dominance zone is an area near the corresponding uniform dominance positions. This case is shown in Figure 10, where the input context is generated with the parameterset( $N$=10,$M$=14,$P$=0.6-0.8).

Based on the presented properties of the dominance zones, the implemented optimization applies the following steps:

- For the calculation of a $C_{i,j}$ component, a sampling technique is applied instead of full enumeration of the corresponding candidates.

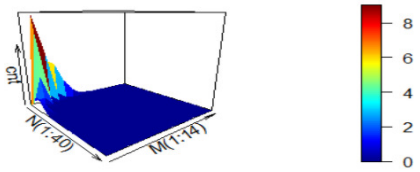


FIGURE 6. The $C_{n,m}$ distribution ($P = 0.2$)

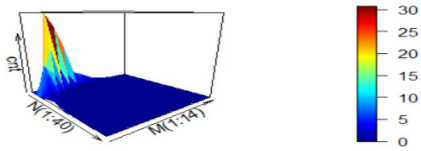


FIGURE 7. The $C_{n,m}$ distribution ($P = 0.6$)

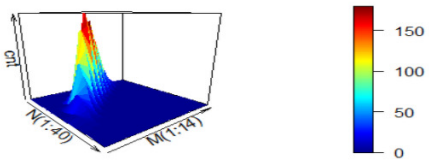


FIGURE 8. The $C_{n,m}$ distribution ($P = 0.8$)

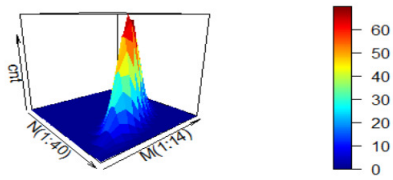


FIGURE 9. The $C_{n,m}$ distribution ($P = 0.96$)

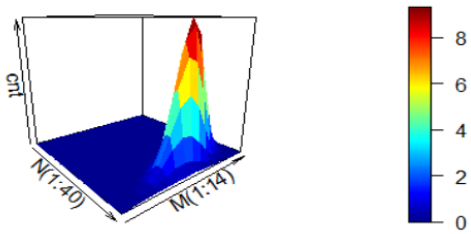


FIGURE 10. The $C_{n,m}$ distribution

- The enumeration of the candidates is restricted only to the dominance zones, the candidates outside the dominance zone are eliminated.

The sampling process applies the method of simple sampling without replacement approach. Using this method, the corresponding confidence interval can be given with

$$\triangle \bar{x_n} = t \frac{S_n}{\sqrt{n}} \sqrt{(1 - \frac{n}{N})},$$

where $n$: size of the sample, $N$: size of the population, $\bar{x_n}$: mean value, $S_n$: standard deviation, $t$: t-score value. Using this formula and the desired t-score value, we can

determine an optimum sample size with the following formula:

$$n_o = \frac{N}{1 + \frac{N \Delta \bar{x}}{t^2 S_n^2}}.$$

Our proposed algorithm uses this formula to determine the length of the sampling. To determine the required apriori values, we use a pre-sampling phase with a moderate fixed sample size. The estimation of the deviation value is based on the result of this pre-sampling phase.

According to our experiences, the initial pre-sampling phase generates usually an approximation value significantly higher than the real deviation. To optimize this process, we have implemented a mechanism to stop the sampling process if the deviation of the last time window is below of the threshold. Thus there are two termination criteria in the sampling process :

- the length of the sampling is equal to the calculated;
- the deviation of the last time window is below a the threshold.

The second cost reduction method restricts the full enumeration on the parameter space $(n,m)$ to the dominance region. In general case, this reduction is performed with the following algorithm:

- build up a coarse grid on the $(n,m)$ parameter space
- calculate $C_{n,m}$ for each node of the grid
- determine $(n_0,m_0) = argmax_{(m,n)}\{cnt_{n,m}\}$
- select a dominance factor $1 > \alpha > 0$
- process all elements $(n,m)$ which are connected to $(n_0,m_0)$.

A connectivity relationship is used to determine the dominance zone as a maximal cluster. The relationship is defined on the usual way. Two elements $(n_s,m_s)$, $(n_l,m_l)$ are connected if

- $C_{n_s,m_s} \geq \alpha \cdot C_{n_0,m_0}, C_{n_l,m_l} \geq \alpha \cdot C_{n_0,m_0}$
- there exists a sequence of neighboring connected elements:$(n_1 = n_s, m_1 = m_s), (n_2,m_2), (n_2,m_3), ..., (n_i = n_e, m_i = m_e)$.

The method merges only those elements $(n,m)$ into the dominance zone where $C_{n,m} \geq \alpha \cdot C_{n_0,m_0}$. The exploration starts at the element $(n_0,m_0)$. At a given position it will test all the neighboring elements. The method implements a greedy algorithms and it terminates if no new element with high $C_{n,m}$ value can be discovered. The sampling process will be executed for all elements of the discovered dominance zone.

In the case of sparse contexts, the general algorithm can be reduced to a faster variant. In this case, the dominance zone is located near the origo position. In the practical applications, the contexts usually are sparse contexts, otherwise we would manage exponential large set of concepts. For the sparse context the dominance zone is explored in this way:

- initially, we take the element (0,0) and calculate $C_{0,0}$

- a nested loop on the elements is started, where both the object and attribute indexes start at value 0. Both loops terminate if the increase of the accumulated $cnt$ value is below a threshold. If the increase of the accumulated value is very low, then the element last tested is outside the dominance value.

The proposed method is implemented as an algorithm returning the expected mean value and the corresponding deviation value. The deviation is estimated with the following approach. At a given element $(n,m)$, we determine first the standard deviation of the sample values:

$$s = \frac{\sqrt{\sum_i^l \frac{(f_i - \bar{f})^2}{l}}}{\sqrt{l}}.$$

The standard deviation related to $cnt_{n,m}$ is equal to

$$s_{n,m} = \binom{M}{m}\binom{N}{n} s.$$

Considering the whole element space, the total deviation can be calculated with

$$S = \sqrt{\sum_{(m,n)} s_{m,n}^2}.$$

## 4. Test results

Based on the performed tests, we can say that the proposed algorithm provides a unique and fast approximation tool to determine the expected number of concepts for contexts where the attribute probabilities are independent from each others and from the object instances. Some typical test results are shown in Table 1. The meaning of the columns is the following: $C_e$: measured average, $C_a$: value by base approximation; $C_{ao}$: value by optimized base approximation; $t_e$: time for concept enumeration, $t_a$: time of base approximation; $time_{ao}$: time of optimized base approximation; Prisner: the value of the Priosner approximationIn the table some values are left blank, because they could not be calculated due to high execution cost or to high memory demand. The test results show that the baseline upper approximation provide a very inaccurate values, they cannot be used for practical cost estimations.

Table 2 shows some values related to the approximation of the standard deviation $C_e$: measured average, $ad_a$: measured deviation, $C_{ao}$: calculated average, $sd_{ao}$: calculated deviation)

The approximation algorithm, can be used also for extreme parameter values where the available concept enumeration methods would require extreme large execution time. The initial approximation algorithm with evaluation of all components can be used only for small sized contexts ($N < 3000, M < 30$). The best concept set enumeration algorithms can process also larger contexts, in our test environment, the threshold value is about ($C < 5000000$). On the other hand, the proposed optimized

TABLE 1. Test results on cost efficiency of the approximation method

| $N$ | $M$ | $P$ | $C_e$ | $C_{ao}$ | $t_e$ | $t_{ao}$ | Prisner |
|---|---|---|---|---|---|---|---|
| 3000 | 50 | 0.1 | 23889 | 23460 | 1.42 | 0.002 | $10^{25}$ |
| 1000 | 30 | 0.1 | 1910 | 1850 | 0.11 | 0.001 | $10^{13}$ |
| 3000 | 100 | 0.1 | 183210 | 189700 | 7.26 | 0.002 | $10^{43}$ |
| 10000 | 100 | 0.1 | 901331 | 906413 | 35.9 | 0.002 | $10^{60}$ |
| 100000 | 1000 | 0.1 | - | 4.71e9 | - | 0.002 | $10^{120}$ |
| 100000 | 5000 | 0.01 | - | 8.1e6 | - | 0.003 | $10^{90}$ |
| 1000000 | 2000 | 0.05 | - | 1.77e12 | - | 0.004 | $10^{140}$ |
| 100 | 12 | 0.1-0.3 | 120 | 120 | 0.02 | 0.1 | $10^{6}$ |
| 100 | 16 | 0.1-0.3 | 231 | 223 | 0.02 | 0.1 | $10^{7}$ |
| 200 | 13 | 0.1-0.2 | 220 | 162 | 0.03 | 0.1 | $10^{8}$ |
| 1000 | 13 | 0.1-0.2 | 461 | 422 | 0.06 | 0.2 | $10^{10}$ |
| 10000 | 13 | 0.1-0.2 | 1446 | 1336 | 0.18 | 0.25 | $10^{12}$ |
| 10000 | 30 | 0.1-0.2 | 49954 | 49367 | 1.55 | 0.28 | $10^{20}$ |

TABLE 2. Test results on standard deviation of the approximation method

| $N$ | $M$ | $P$ | $C_e$ | $sd_e$ | $C_{ao}$ | $sd_{ao}$ |
|---|---|---|---|---|---|---|
| 1000 | 100 | 0.1-0.2 | 238000 | 12707 | 247583 | 12870 |
| 5000 | 100 | 0.1-0.2 | 3056012 | 168400 | 3029816 | 144998 |

method can be applied for larger contexts too ($N < 1000000000, M < 10000, C < 1e25$) with a maximal execution time of 5 seconds. This execution time data shows that the algorithm is very efficient and it can be used for larger complexity analysis too. Next figures presents some complexity functions for larger contexts, Figure 11: ($N = 10000..100000, M = 200, P = 0.1$), Figure 12:($N = 10000, M = 100..1000, P = 0.1$) and Figure 13: ($N = 10000, M = 200, P = 0.02..0.24$).

## 5. CONCLUSIONS

The calculation of the concept count for any arbitrary context is a hard, NP-complete problem and only rough approximation methods can be found in the literature to solve this problem. The paper proposes a novel algorithmic approach to approximate the total number of concepts where the attribute probabilities are independent from each others and from the single objects. The algorithm is very efficient especially for rare contexts which are mainly used in the practical FCA applications. The proposed algorithm provides a better practical approximation with a significantly better execution cost than the baseline approximation [16], [15]. The method can be used among others also for complexity analysis of large scale FCA problems.

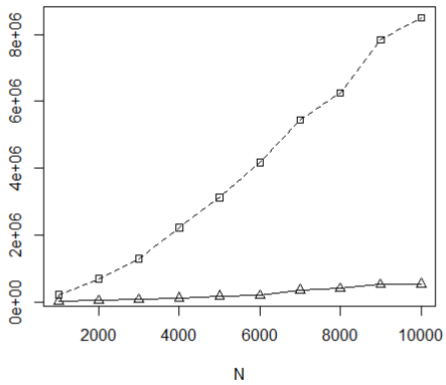


FIGURE 11. Mean and deviation of concept count

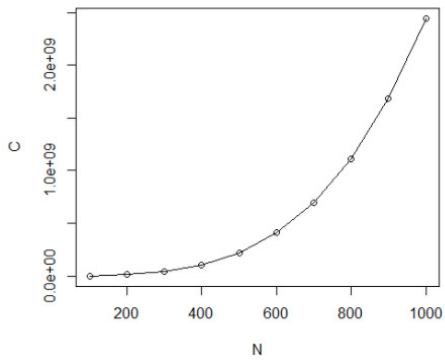


FIGURE 12. Mean and deviation of concept count

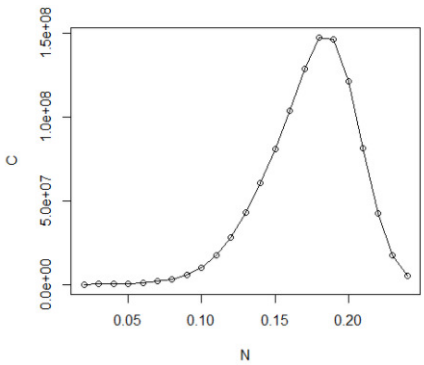


FIGURE 13. Mean and deviation of concept count

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# REFINEMENT OF HERMITE–HADAMARD TYPE INEQUALITIES FOR *S*–CONVEX FUNCTIONS

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*Abstract.* We give refinement of few inequalities from [13], related of the left–hand side of the Hermite–Hadamard type inequalities for the class of mappings whose second derivatives at certain powers are $s$-convex in the second sense.

2010 *Mathematics Subject Classification:* 26A51; 26D07; 26D10; 26D15

*Keywords:* Hermite–Hadamard type inequality, $s$-convex function, generalized mean inequality

## 1. Preliminaries

It is a well known fact that convexity and its generalizations play an important role in different parts of mathematics and science, mainly in optimization theory. Also, modern analysis directly or indirectly involves the applications of convexity. Due to its applications and significant importance, the concept of convexity has been extended and generalized in several directions. For example, this concept introduced the classes of $s$-convex functions, Godunova-Levin functions, $P$-functions, all included in class of $h$-convex functions (for more details see [6, 17] and literature therein).

These are reasons of the topmost motivations for examining the properties of generalized convex functions. In this note we will deal with the concepts of $s$-convexity, which were introduced by [1].

**Definition 1.** Let $s$ be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to \mathbb{R}$ is said to be $s$-convex (in the second sense), if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

for all $x, y \in [0,\infty)$ and $\lambda \in [0,1]$.

If $-f$ is $s$-convex, then $f$ is $s$-concave.

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It can be easily seen that convexity means just $s$-convexity when $s = 1$, so this concept is a generalization of convexity. It is an interesting fact that the set of nonnegative $s$-convex functions strictly flares as $s$ strictly decreases. In addition, $s$-convex functions have a good relationship with algebraic operations just like the convex ones. For example, the sum of two $s$-convex functions is $s$-convex and by multiplying an $s$-convex function with a non-negative scalar we get an $s$-convex function again (see [7] and [3]). Also, there is a nice correspondence between convex and $s$-convex functions. Namely, the class of $s$-convex functions belongs to the class of locally $s$-Hölder functions (see [2] and [12]), the class of 1-Hölder functions coincides with the class of locally Lipschitz functions and it is a known fact that a convex function is a locally Lipschitz one if it is locally bounded from above at a point of an open domain (see [10]).

Many interesting inequalities are proved for convex functions in the literature. For example, extensively studied result is Hermite-Hadamard's inequality. Hermite (1883) and Hadamard (1896) independently have shown that the convex functions are related to an integral inequality, and this inequality is known as Hermite-Hadamard inequality.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. The following double inequality is Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Both inequalities hold in the reversed direction if $f$ is concave. The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a,b] \to \mathbb{R}$. For further generalizations and new inequalities related to Hermite-Hadamard inequality interested readers are referred to [4, 5, 8, 9, 11, 15, 16].

The main aim of this paper is to establish new inequalities of Hermite-Hadamard type for the class of functions whose second derivatives at certain powers are $s$-convex functions in the second sense. For this class of functions, Sarikaya and Kiris got estimates of the expression $\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|$ (see [13]). Our results represent improvement of the results given in [13].

## 2. Main results

In this section we will present some estimates of the expression on the left–hand side of Hermite–Hadamard inequality.

**Theorem 1.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on $I^\circ$, $a, b \in I$ with $a < b$. If $|f''|^q$ is $s$-convex in second sense on $[a,b]$, for some fixed $s \in (0,1]$*

*and $q > 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\cdot\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}, \tag{2.1}$$

*where $\frac{1}{p}+\frac{1}{q}=1$.*

*Proof.* If $\alpha \le \beta$ and $A, B > 0$, we have $\left(\frac{A^\alpha+B^\alpha}{2}\right)^{\frac{1}{\alpha}} \le \left(\frac{A^\beta+B^\beta}{2}\right)^{\frac{1}{\beta}}$ (power means inequality). Specially, for $A, B \ge 0$ and $q \ge 1$, we have $\left(\frac{A^{\frac{1}{q}}+B^{\frac{1}{q}}}{2}\right)^q \le \frac{A+B}{2}$ (the case $AB = 0$ is trivial). For $A = |f''(a)|^q + (2^{s+1}-1)|f''(b)|^q$ and $B = (2^{s+1}-1)|f''(a)|^q + |f''(b)|^q$, according to Theorem 4 in [13], we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
&\quad\le \frac{(b-a)^2}{2^{s+4}}\cdot\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}\cdot\left[\left(\frac{|f''(a)|^q+(2^{s+1}-1)|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\frac{(2^{s+1}-1)|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\right] \\
&\quad= \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{2^{-s}}{s+1}\right)^{\frac{1}{q}}\cdot\frac{1}{2}\cdot\left[\left(|f''(a)|^q+(2^{s+1}-1)|f''(b)|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left((2^{s+1}-1)|f''(a)|^q+|f''(b)|^q\right)^{\frac{1}{q}}\right] \\
&\quad\le \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{2^{-s}}{s+1}\right)^{\frac{1}{q}}\cdot\left\{\frac{1}{2}\cdot\left[\left(|f''(a)|^q+(2^{s+1}-1)|f''(b)|^q\right)\right.\right. \\
&\qquad\left.\left.+\left((2^{s+1}-1)|f''(a)|^q+|f''(b)|^q\right)\right]\right\}^{\frac{1}{q}} \\
&\quad= \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{2^{-s}}{s+1}\right)^{\frac{1}{q}}\cdot\left[2^s\left(|f''(a)|^q+|f''(b)|^q\right)\right]^{\frac{1}{q}} \\
&\quad= \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\cdot\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

□

*Remark* 1. If we take $s = 1$ in Theorem 1, then inequality (2.1) becomes inequality obtained in Theorem 4 in [14].

**Theorem 2.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on $I^\circ$, $a, b \in I$ with $a < b$. If $|f''|^q$ is $s$-convex in second sense on $[a, b]$, for some fixed $s \in (0, 1]$*

*and $q > 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\cdot\left(|f''(a)|+|f''(b)|\right), \quad (2.2)$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* Let $g(x) = (1+x)^q - x^q - 1$ for $x \geq 0$ and $q > 1$. Since $g'(x) = q[(1+x)^{q-1} - x^{q-1}] \geq 0$ and $g(0) = 0$, it follows that $g$ is nondecreasing on $[0,\infty)$, so we have $A^q + B^q \leq (A+B)^q$ for $A, B \geq 0$ (the case $AB = 0$ is trivial). By Theorem 1 it follows

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
&\quad\leq \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\cdot\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}} \\
&\quad= \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\cdot\left(|f''(a)|^q+|f''(b)|^q\right)^{\frac{1}{q}} \\
&\quad\leq \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\cdot\left(|f''(a)|+|f''(b)|\right).
\end{aligned}$$

□

*Remark* 2. Note that the expression occurring on the right-hand side of the inequality proven in Theorem 4 in [13] is equal to the product of $2^{\frac{s}{p}}$ and the expression (2.2), so the inequality from Theorem 2 is refinement of that inequality. Also, if we take $s = 1$ in Theorem 2, then the right–hand side of inequality (2.2) becomes $R = \dfrac{(b-a)^2}{2^{\frac{1}{q}+3}(2p+1)^{\frac{1}{p}}}\cdot\left(|f''(a)|+|f''(b)|\right)$ and the right–hand side of inequality from Corollary 1 in [13] is $\dfrac{(b-a)^2}{2^{\frac{2}{q}+2}(2p+1)^{\frac{1}{p}}}\cdot\left(|f''(a)|+|f''(b)|\right) = 2^{\frac{1}{p}}\cdot R \geq R$, so we get refinement of this result, too.

## 3. Applications to special means

Of course, since the results of Section 2 improve the results from Section 2 in [13], results from Section 3 in [13] which are consequences of those results also can be improved. We will present some of them and we will present few new applications to special means.

Let $g : I \to I_1 \subseteq [0,\infty)$ be a convex function on $I$. Then $g^s$ is $s$-convex on $I$, $0 < s < 1$. We will use this in proofs of our propositions, that will be presented in further text.

For arbitrary positive real numbers $a,b$, we consider the applications of our theorems to the following special means:

(a) The arithmetic mean: $A = A(a,b) := \dfrac{a+b}{2}$;

(b) The geometric mean: $G = G(a,b) := \sqrt{ab}$;

(c) The harmonic mean: $H = H(a,b) := \dfrac{2ab}{a+b}$;

(d) The logarithmic mean: $L = L(a,b) := \begin{cases} a, & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a}, & \text{if } a \neq b \end{cases}$;

(e) The $p$–logarithmic mean: $L_p = L_p(a,b) := \begin{cases} a, & \text{if } a = b \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if } a \neq b \end{cases}$, $p \in \mathbb{R} \setminus \{-1,0\}$;

(f) The identric mean:

$$I = I(a,b) := \begin{cases} a, & \text{if } a = b \\ \frac{1}{e} \cdot \left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } a \neq b \end{cases}.$$

It is well known that $L_p$ is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequalities

$$H \leq G \leq L \leq I \leq A.$$

The following propositions hold:

**Proposition 1.** *Let $0 < a < b$, $s \in (0,1]$ and $p > 1$. Then we have*

$$\left|L_{s+1}^{s+1}(a,b) - A^{s+1}(a,b)\right| \leq \frac{s(b-a)^2}{4} \cdot \left(\frac{s+1}{2p+1}\right)^{\frac{1}{p}} \cdot A(a^{s-1}, b^{s-1}).$$

*Proof.* The proof is immediate from Theorem 2 applied for function $f : [a,b] \to \mathbb{R}$, $f(x) = x^{s+1}$, $s \in (0,1]$. □

*Remark* 3. Note that the left–hand side of the inequality in previous proposition does not depend on $p$, so we can conclude that

$$\left|L_{s+1}^{s+1}(a,b) - A^{s+1}(a,b)\right| \leq \inf_{p>1} \left(\frac{s+1}{2p+1}\right)^{\frac{1}{p}} \cdot \frac{s(b-a)^2}{4} \cdot A(a^{s-1}, b^{s-1})$$

for any $s \in (0,1]$, $p > 1$, $a < b$. If $g(p) = \ln\left(\frac{s+1}{2p+1}\right)^{\frac{1}{p}} = \frac{1}{p} \ln \frac{s+1}{2p+1}$ for $p \geq 1$ and $s \in (0,1]$, then we have $g'(p) = -\frac{1}{p^2}\left(\ln \frac{s+1}{2p+1} + \frac{2p}{2p+1}\right)$. If $h(p) = \ln \frac{s+1}{2p+1} + \frac{2p}{2p+1}$ for $p \geq 1$ and $s \in (0,1]$, then we have $h'(p) = -\frac{4p}{(2p+1)^2}$, so $h(p)$ decreases. Since $\lim\limits_{p\to\infty} h(p) = -\infty$ and $\lim\limits_{p\to 1+} h(p) = \ln \frac{s+1}{3} + \frac{2}{3}$ we have:

(a) if $0 < s \leq 3e^{-\frac{2}{3}} - 1 \approx 0.54$, then $h(p) \leq 0$ for $p \in [1,\infty)$, so $g'(p) = -\frac{1}{p^2} h(p) \geq 0$ for $p \in [1,\infty)$. It follows that $\inf\limits_{p>1} \left(\frac{s+1}{2p+1}\right)^{\frac{1}{p}} = \frac{s+1}{3}$, so, if

$p > 1$, $s \in (0, 3e^{-\frac{2}{3}} - 1]$ we have

$$\left|L_{s+1}^{s+1}(a,b) - A^{s+1}(a,b)\right| \leq \frac{s(s+1)}{12} \cdot (b-a)^2 \cdot A(a^{s-1}, b^{s-1});$$

(b) if $3e^{-\frac{2}{3}} - 1 < s \leq 1$, then $h(p) > 0$ for $p \in [1, p_0(s))$ and $h(p) < 0$ for $p \in (p_0(s), \infty)$, where $p_0(s)$ is an unique solution of equation $\ln \frac{s+1}{2p+1} + \frac{2p}{2p+1} = 0$. It follows that $g(p) = -\frac{1}{p^2} h(p)$ decreases on $(1, p_0(s))$ and increases on $(p_0(s), \infty)$, so for $p > 1$ and $s \in (3e^{-\frac{2}{3}} - 1, 1]$ we have

$$\left|L_{s+1}^{s+1}(a,b) - A^{s+1}(a,b)\right| \leq \frac{s(b-a)^2}{4} \cdot \left(\frac{s+1}{2p_0(s)+1}\right)^{\frac{1}{p_0(s)}} \cdot A(a^{s-1}, b^{s-1}).$$

**Proposition 2.** *Let* $0 < a < b$, $s \in (0,1]$ *and* $p > 1$. *Then we have*

$$\left|L_{s+1}^{s+1}(a,b) - A^{s+1}(a,b)\right| \leq \frac{s(b-a)^2}{4} \cdot \left(\frac{s+1}{4p+2}\right)^{\frac{1}{p}} \cdot \left[A\left(a^{q(s-1)}, b^{q(s-1)}\right)\right]^{\frac{1}{q}},$$

*where* $\frac{1}{p} + \frac{1}{q} = 1$.

*Proof.* The proof is immediate from Theorem 1 applied for function $f : [a,b] \to \mathbb{R}$, $f(x) = x^{s+1}$, $s \in (0,1]$. □

*Remark* 4. Under conditions of Proposition 2, for $s = 1$ we derive

$$\left|L_2^2(a,b) - A^2(a,b)\right| \leq \frac{(b-a)^2}{4} \cdot \left(\frac{1}{2p+1}\right)^{\frac{1}{p}},$$

which is refinement of the inequality derived in part 1(b) of Section 3 in [13] (note that there is a misprint on this place in [13], the right–hand side of inequality obtained there should be $\frac{(b-a)^2}{4} \cdot \left(\frac{4}{2p+1}\right)^{\frac{1}{p}}$). Also, the left–hand side of the inequality in derived inequality does not depend on $p$, so we have $\left|L_2^2(a,b) - A^2(a,b)\right| \leq \frac{(b-a)^2}{4} \cdot \inf\limits_{p>1} \left(\frac{1}{2p+1}\right)^{\frac{1}{p}}$. If $g(p) = \ln\left(\frac{1}{2p+1}\right)^{\frac{1}{p}} = \frac{1}{p} \ln \frac{1}{2p+1}$ for $p \geq 1$, then we have $g'(p) = -\frac{1}{p^2}\left(\ln \frac{1}{2p+1} + \frac{2p}{2p+1}\right)$. Since it holds $\ln x < x - 1$ for $x > 1$, it follows that $\ln \frac{1}{2p+1} + \frac{2p}{2p+1} < \frac{1}{2p+1} - 1 + \frac{2p}{2p+1} = 0$, so we have $g'(p) > 0$ for $p > 1$. It follows that $g(p)$ increases on $[1, \infty)$. Therefore $\inf\limits_{p>1} e^{g(p)} = e^{g(1)} = \frac{1}{3}$, so we have $\left|L_2^2(a,b) - A^2(a,b)\right| \leq \frac{(b-a)^2}{12}$. This is the result derived in part 1(a) of Section 3 in [13], and since $L_2^2(a,b) - A^2(a,b) = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(a-b)^2}{12}$ for $a \neq b$, we can conclude that inequality derived in Proposition 2 is sharp.

**Proposition 3.** *Let* $0 < a < b$ *and* $p > 1$. *Then we have*

$$\left|\frac{b^2 \ln b - a^2 \ln a}{b-a} - A(a,b)\left[1 + \ln A^2(a,b)\right]\right| \leq \frac{(b-a)^2}{2(4p+2)^{\frac{1}{p}}} \cdot \left[A\left(a^{-q}, b^{-q}\right)\right]^{\frac{1}{q}},$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* The proof is immediate from Proposition 2, after multiplying both side of the inequality with $\frac{2}{s}$ and passing to the limit $s\to 0+$. □

**Proposition 4.** *Let* $0<a<b$, $s\in(0,1]$ *and* $p>1$. *Then we have*

$$\left|L_{-s}^{-s}(a,b)-A^{-s}(a,b)\right|\le\frac{s(b-a)^2}{4}\cdot\left(\frac{s+1}{2p+1}\right)^{\frac{1}{p}}\cdot A(a^{-s-2},b^{-s-2}).$$

*Proof.* The proof is immediate from Theorem 2 applied for function $f:[a,b]\to\mathbb{R}$, $f(x)=\dfrac{1}{x^s}$, $s\in(0,1]$. □

**Proposition 5.** *Let* $0<a<b$, $s\in(0,1]$ *and* $p>1$. *Then we have*

$$\left|\ln\frac{I(a,b)}{A(a,b)}\right|\le\frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\cdot\frac{\left[A(a^{2q},b^{2q})\right]^{\frac{1}{q}}}{a^2b^2},$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* The proof is immediate from Theorem 1 applied for function $f:[a,b]\to\mathbb{R}$, $f(x)=\ln x$. □

For instance, if $s=1$ then we have

$$\left|\ln\frac{I(a,b)}{A(a,b)}\right|\le\frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\frac{\left[A(a^{2q},b^{2q})\right]^{\frac{1}{q}}}{a^2b^2}.$$

**Proposition 6.** *Let* $0<a<b$, $s\in(0,1]$ *and* $p>1$. *Then we have*

$$\left|\ln\frac{I(a,b)}{A(a,b)}\right|\le\frac{(b-a)^2}{4(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\cdot\frac{A(a^2,b^2)}{a^2b^2},$$

$$\textit{i.e.}\quad\left|\ln\frac{I(a,b)}{A(a,b)}\right|\le\frac{(b-a)^2}{4(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\cdot\frac{1}{H(a^2,b^2)},$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* The proof is immediate from Theorem 2 applied for function $f:[a,b]\to\mathbb{R}$, $f(x)=\ln x$. □

**Proposition 7.** *Let* $0<\alpha<\beta$, $s\in(0,1]$ *and* $p>1$. *Then we have*

$$\left|L_s^s(\alpha,\beta)-G^{s+1}(\alpha,\beta)\right|\le\frac{(s+1)\cdot\ln^2\frac{\beta}{\alpha}}{4}\cdot\left(\frac{s+1}{2p+1}\right)^{\frac{1}{p}}\cdot A(\alpha^{s+1},\beta^{s+1}).$$

*Proof.* The proof is immediate from Theorem 2 applied for function $f:[a,b]\to\mathbb{R}$, $f(x)=e^{(s+1)x}$, $s\in(0,1]$ and $\alpha=e^a$, $\beta=e^b$. □

For instance, if $s = 1$ then we have

$$\left|L_1(\alpha,\beta) - G^2(\alpha,\beta)\right| \leq \frac{\ln^2 \frac{\beta}{\alpha}}{2^{\frac{1}{q}}(2p+1)^{\frac{1}{p}}} \cdot A(\alpha^2,\beta^2),$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

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# IMPROVEMENT OF FRACTIONAL HERMITE-HADAMARD TYPE INEQUALITY FOR CONVEX FUNCTIONS

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*Abstract.* In this paper, it is proved that fractional Hermite-Hadamard inequality and fractional Hermite-Hadamard-Fejér inequality are just results of Hermite-Hadamard-Fejér inequality. After this, a new fractional Hermite-Hadamard inequality which is not a result of Hermite-Hadamard-Fejér inequality and better than given in [9] by Sarıkaya et al. is obtained. Also, a new equality is proved and some new fractional midpoint type inequalities are given. Our results generalizes the results given in [5] by Kırmacı.

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*Keywords:* convex functions, Hermite-Hadamard inequalities, Riemann-Liouville fractional integrals, midpoint type inequalities

## 1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality [2, 3].

The most well-known inequalities related to the integral mean of a convex function $f$ are the Hermite Hadamard inequality or its weighted versions, the so-called Hermite-Hadamard-Fejér inequality.

In [1], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1):

**Theorem 1.** *Let $f : [a,b] \to \mathbb{R}$ be convex function. Then, the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{1.2}$$

*holds, where $g : [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$ (i.e. $g(x) = g(a+b-x)$ for all $x \in [a,b]$).*

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In [5], Kırmacı used the following equality to obtain midpoint type inequalities and some applications:

**Lemma 1.** *Let $a,b \in I$ with $a < b$ and $f : I^\circ \to \mathbb{R}$ is a differentiable mapping ($I^\circ$ the interior of $I$). If $f' \in L[a,b]$, then we have*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right) \\ &= (b-a)\int_0^{1/2} t f'(ta+(1-t)b)\,dt + \int_{1/2}^1 (t-1) f'(ta+(1-t)b)\,dt. \end{aligned} \tag{1.3}$$

Following definitions of the left and right side Riemann-Liouville fractional integrals are well known in the literature.

**Definition 1.** Let $a,b \in \mathbb{R}$ with $a < b$ and $f \in L[a,b]$. The left and right Riemann-Liouville fractional integrals $R^\theta_{a+} f$ and $R^\theta_{b-} f$ of order $\theta > 0$ are defined by

$$R^\theta_{a+} f(x) = \frac{1}{\Gamma(\theta)}\int_a^x (x-t)^{\theta-1} f(t)dt,\ x > a$$

and

$$R^\theta_{b-} f(x) = \frac{1}{\Gamma(\theta)}\int_x^b (t-x)^{\theta-1} f(t)dt,\ x < b$$

respectively, where $\Gamma(\theta)$ is the Gamma function defined by $\Gamma(\theta) = \int_0^\infty e^{-t} t^{\theta-1} dt$ (see [6, page 69] and [10, page 4]).

In [9], Sarıkaya et al. proved the following fractional Hermite-Hadamard type inequality:

**Theorem 2.** *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L[a,b]$. If $f$ is a convex function on $[a,b]$, then the following inequalities for fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\theta+1)}{2(b-a)^\theta}\left[R^\theta_{a+} f(b) + R^\theta_{b-} f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{1.4}$$

*with $\theta > 0$.*

*Remark* 1. In Theorem 2, it is not necessary supposing that $f$ be a positive function and $a,b$ are positive real numbers. From the Definition 1, it is clear that $a,b$ are any real numbers such as $a < b$.

In [4], İşcan proved the following fractional Hermite-Hadamard-Fejér type inequality:

**Theorem 3.** *Let $f:[a,b]\to\mathbb{R}$ be a convex function with $a<b$ and $f\in L[a,b]$. If $g:[a,b]\to\mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\left[R_{a+}^{\theta}g(b)+R_{b-}^{\theta}g(a)\right] &\leq \left[R_{a+}^{\theta}(fg)(b)+R_{b-}^{\theta}(fg)(a)\right] \\ &\leq \frac{f(a)+f(b)}{2}\left[R_{a+}^{\theta}g(b)+R_{b-}^{\theta}g(a)\right] \end{aligned} \tag{1.5}$$

*with $\theta>0$.*

In [7], Kunt et al. proved the following left Riemann-Liouville fractional Hermite-Hadamard type inequality and next equality:

**Theorem 4.** *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a convex function. If $f\in L[a,b]$, then the following inequality for the left Riemann-Liouville fractional integral holds:*

$$f\left(\frac{\theta a+b}{\theta+1}\right)\leq\frac{\Gamma(\theta+1)}{(b-a)^{\theta}}R_{a+}^{\theta}f(b)\leq\frac{\theta f(a)+f(b)}{\theta+1} \tag{1.6}$$

*with $\theta>0$.*

**Lemma 2.** *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on $(a,b)$. If $f'\in L[a,b]$, then the following equality for the left Riemann-Liouville fractional integrals holds:*

$$\begin{aligned} &\frac{\Gamma(\theta+1)}{(b-a)^{\theta}}R_{a+}^{\theta}f(b)-f\left(\frac{\theta a+b}{\theta+1}\right) \\ &\qquad=(b-a)\left[\begin{array}{c}\int_0^{\frac{\theta}{\theta+1}}t^{\theta}f'(ta+(1-t)b)\,dt \\ +\int_{\frac{\theta}{\theta+1}}^{1}\left(t^{\theta}-1\right)f'(ta+(1-t)b)\,dt\end{array}\right] \end{aligned} \tag{1.7}$$

*with $\theta>0$.*

In [8], Kunt et al. proved the following right Riemann-Liouville fractional Hermite-Hadamard type inequality and next equality:

**Theorem 5.** *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a convex function. If $f\in L[a,b]$, then the following inequality for the right Riemann-Liouville fractional integral holds:*

$$f\left(\frac{a+\theta b}{\theta+1}\right)\leq\frac{\Gamma(\theta+1)}{(b-a)^{\theta}}R_{b-}^{\theta}f(a)\leq\frac{f(a)+\theta f(b)}{\theta+1} \tag{1.8}$$

*with $\theta>0$.*

**Lemma 3.** *Let $a,b \in \mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on $(a,b)$. If $f' \in L[a,b]$, then the following equality for the right Riemann-Liouville fractional integrals holds:*

$$\frac{\Gamma(\theta+1)}{(b-a)^{\theta}} R_{b-}^{\theta} f(a) - f\left(\frac{a+\theta b}{\theta+1}\right) = (b-a)\left[\begin{array}{c} \int_0^{\frac{\theta}{\theta+1}} -t^{\theta} f'(tb+(1-t)a)\,dt \\ +\int_{\frac{\theta}{\theta+1}}^{1}\left(1-t^{\theta}\right) f'(tb+(1-t)a)\,dt \end{array}\right] \tag{1.9}$$

*with $\theta>0$.*

In our studies we noticed that fractional Hermite-Hadamard type inequality given in Theorem 2 and fractional Hermite-Hadamard-Fejér type inequality given in Theorem 3 are just result of Hermite-Hadamard-Fejér inequality (given in Theorem 1), with a special selection of the weighted function. This show how strong the Hermite-Hadamard-Fejér inequality is. However, we will prove new fractional Hermite-Hadamard type inequality which is not a result of Theorem 1. Also, we will have new fractional midpoint type inequalities.

## 2. Results of Hermite-Hadamard-Fejér inequality

**Proposition 1.** *Theorem 2 is a result of Theorem 1.*

*Proof.* In Theorem 1, let we choose $g(x)=(x-a)^{\theta-1}+(b-x)^{\theta-1}$ for $\theta>0$, $a,b\in\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ (It is clear $g(x)$ nonnegative, integrable and symmetric to $\frac{a+b}{2}$). Computing the following integrals, we have

$$\int_a^b g(x)dx = \int_a^b (x-a)^{\theta-1}+(b-x)^{\theta-1}\,dx = \frac{2(b-a)^{\theta}}{\theta}, \tag{2.1}$$

$$\begin{aligned}\int_a^b f(x)g(x)dx &= \int_a^b \left[(x-a)^{\theta-1}+(b-x)^{\theta-1}\right] f(x)dx \\ &= \int_a^b (x-a)^{\theta-1} f(x)dx + \int_a^b (b-x)^{\theta-1} f(x)dx \\ &= \Gamma(\theta)\left[R_{a+}^{\theta} f(b) + R_{b-}^{\theta} f(a)\right].\end{aligned} \tag{2.2}$$

Combining (1.2), (2.1) and (2.2) we have (1.4). This completes the proof. □

**Proposition 2.** *Theorem 3 is a result of Theorem 1.*

*Proof.* In Theorem 1, let we choose $w(x)=\left[(x-a)^{\theta-1}+(b-x)^{\theta-1}\right]g(x)$ for $\theta>0$, $a,b\in\mathbb{R}$, $g:[a,b]\to\mathbb{R}$ and $g(x)$ nonnegative, integrable and symmetric to

$\frac{a+b}{2}$ (It is clear $w(x)$ nonnegative, integrable and symmetric to $\frac{a+b}{2}$). Computing the following integrals, we have

$$\int_a^b w(x)\,dx = \int_a^b \left[(x-a)^{\theta-1} + (b-x)^{\theta-1}\right] g(x)\,dx \tag{2.3}$$

$$= \int_a^b (x-a)^{\theta-1} g(x)\,dx + \int_a^b (b-x)^{\theta-1} g(x)\,dx$$

$$= \Gamma(\theta)\left[R_{a+}^{\theta} g(b) + R_{b-}^{\theta} g(a)\right],$$

$$\int_a^b f(x)w(x)dx = \int_a^b \left[(x-a)^{\theta-1} + (b-x)^{\theta-1}\right] f(x)g(x)\,dx \tag{2.4}$$

$$= \int_a^b (x-a)^{\theta-1} f(x)g(x)\,dx + \int_a^b (b-x)^{\theta-1} f(x)g(x)\,dx$$

$$= \Gamma(\theta)\left[R_{a+}^{\theta}(fg)(b) + R_{b-}^{\theta}(fg)(a)\right].$$

Combining (1.2), (2.3) and (2.4) we have (1.5). This completes the proof. □

*Remark* 2. Theorem 4 and Theorem 5 are not results of Theorem 1.

## 3. Improvement of Fractional Hermite-Hadamard Type Inequality

We will use Theorem 4 and Theorem 5 to have new fractional Hermite-Hadamard type inequality better than (1.4).

**Theorem 6.** *Let $a,b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a convex function. If $f \in L[a,b]$, then the following inequality for fractional integral holds:*

$$\frac{f\left(\frac{\theta a+b}{\theta+1}\right) + f\left(\frac{a+\theta b}{\theta+1}\right)}{2} \le \frac{\Gamma(\theta+1)}{2(b-a)^{\theta}}\left[R_{a+}^{\theta} f(b) + R_{b-}^{\theta} f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{3.1}$$

*with $\theta > 0$.*

*Proof.* If (1.6) and (1.8) gather side by side and dividing into 2, it is hold the desired result. □

*Remark* 3. Since, $f$ is a convex function on $[a,b]$, it is clear $f\left(\frac{a+b}{2}\right) \le \frac{f\left(\frac{\theta a+b}{\theta+1}\right)+f\left(\frac{a+\theta b}{\theta+1}\right)}{2}$ for $\theta > 0$. It means that

(1) Theorem 6 is better than Theorem 2,
(2) In Theorem 6 if one takes $\theta = 1$, one has (1.1),
(3) Theorem 6 is not a result of Theorem 1.

## 4. New fractional midpoint type inequalities

We will now prove an equality to have new fractional midpoint type inequalities.

**Lemma 4.** *Let $a,b \in \mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on $(a,b)$. If $f' \in L[a,b]$, then the following equality for the fractional integrals holds:*

$$\frac{\Gamma(\theta+1)}{2(b-a)^{\theta}}\left[R^{\theta}_{a+}f(b)+R^{\theta}_{b-}f(a)\right]-\frac{f\left(\frac{\theta a+b}{\theta+1}\right)+f\left(\frac{a+\theta b}{\theta+1}\right)}{2} \tag{4.1}$$

$$=\frac{b-a}{2}\left[\begin{array}{c}\int_0^{\frac{\theta}{\theta+1}} t^{\theta} f'(ta+(1-t)b)\,dt \\ +\int_{\frac{\theta}{\theta+1}}^{1}\left(t^{\theta}-1\right) f'(ta+(1-t)b)\,dt \\ +\int_0^{\frac{\theta}{\theta+1}} -t^{\theta} f'(tb+(1-t)a)\,dt \\ +\int_{\frac{\theta}{\theta+1}}^{1}\left(1-t^{\theta}\right) f'(tb+(1-t)a)\,dt\end{array}\right]$$

*Proof.* If (1.7) and (1.9) gather side by side and dividing into 2, it is hold the desired result. □

**Corollary 1.** *In Lemma 4, if one takes $\theta=1$, one has Lemma 1.*

**Theorem 7.** *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on $(a,b)$. If $|f'|$ is convex on $[a,b]$, then the following fractional midpoint type inequality holds:*

$$\left|\frac{\Gamma(\theta+1)}{2(b-a)^{\theta}}\left[R^{\theta}_{a+}f(b)+R^{\theta}_{b-}f(a)\right]-\frac{f\left(\frac{\theta a+b}{\theta+1}\right)+f\left(\frac{a+\theta b}{\theta+1}\right)}{2}\right| \tag{4.2}$$

$$\leq (b-a)\frac{\theta^{\theta+1}}{(\theta+1)^{\theta+2}}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$

*with $\theta>0$.*

*Proof.* Using Lemma 4 and the convexity of $|f'|$, we have

$$\left|\frac{\Gamma(\theta+1)}{2(b-a)^{\theta}}\left[R^{\theta}_{a+}f(b)+R^{\theta}_{b-}f(a)\right]-\frac{f\left(\frac{\theta a+b}{\theta+1}\right)+f\left(\frac{a+\theta b}{\theta+1}\right)}{2}\right|$$

$$\frac{b-a}{2}\left[\begin{array}{c}\int_0^{\frac{\theta}{\theta+1}} t^{\theta}\left|f'(ta+(1-t)b)\right|dt \\ +\int_{\frac{\theta}{\theta+1}}^{1}\left(1-t^{\theta}\right)\left|f'(ta+(1-t)b)\right|dt \\ +\int_0^{\frac{\theta}{\theta+1}} t^{\theta}\left|f'(tb+(1-t)a)\right|dt \\ +\int_{\frac{\theta}{\theta+1}}^{1}\left(1-t^{\theta}\right)\left|f'(tb+(1-t)a)\right|dt\end{array}\right]$$

$$\leq \frac{b-a}{2}\left[\begin{array}{c} \int_0^{\frac{\theta}{\theta+1}} t^\theta \left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt \\ +\int_{\frac{\theta}{\theta+1}}^1 \left(1-t^\theta\right)\left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt \\ +\int_0^{\frac{\theta}{\theta+1}} t^\theta \left[t\left|f'(b)\right|+(1-t)\left|f'(a)\right|\right]dt \\ +\int_{\frac{\theta}{\theta+1}}^1 \left(1-t^\theta\right)\left[t\left|f'(b)\right|+(1-t)\left|f'(a)\right|\right]dt \end{array}\right]$$

$$= \frac{b-a}{2}\left[\begin{array}{c} \int_0^{\frac{\theta}{\theta+1}} t^\theta \left[\left|f'(a)\right|+\left|f'(b)\right|\right]dt \\ +\int_{\frac{\theta}{\theta+1}}^1 \left(1-t^\theta\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]dt \end{array}\right]$$

$$= \frac{b-a}{2}\left[\int_0^{\frac{\theta}{\theta+1}} t^\theta dt+\int_{\frac{\theta}{\theta+1}}^1 \left(1-t^\theta\right)dt\right]\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$

$$= (b-a)\frac{\theta^{\theta+1}}{(\theta+1)^{\theta+2}}\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

This completes the proof. □

**Corollary 2.** *In Theorem 7, if one takes $\theta = 1$, one has [5, Theorem 2.2].*

**Theorem 8.** *Let $a,b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function on $(a,b)$. If $|f'|^q$ is convex on $[a,b]$ for $q \geq 1$, then the following fractional midpoint type inequality holds:*

$$\left|\frac{\Gamma(\theta+1)}{2(b-a)^\theta}\left[R_{a+}^\theta f(b)+R_{b-}^\theta f(a)\right]-\frac{f\left(\frac{\theta a+b}{\theta+1}\right)+f\left(\frac{a+\theta b}{\theta+1}\right)}{2}\right| \tag{4.3}$$

$$\leq \frac{b-a}{2}\frac{\theta^{\theta+1}}{(\theta+1)^{\theta+2}}\left[\begin{array}{c} \left(\frac{\theta}{\theta+2}\left|f'(a)\right|^q+\frac{2}{\theta+2}\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(\frac{(\theta+1)^\theta+2\theta^{\theta+1}}{2\theta^\theta(\theta+2)}\left|f'(a)\right|^q+\frac{4\theta^\theta-(\theta+1)^\theta}{2\theta^\theta(\theta+2)}\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(\frac{\theta}{\theta+2}\left|f'(b)\right|^q+\frac{2}{\theta+2}\left|f'(a)\right|^q\right)^{\frac{1}{q}} \\ +\left(\frac{(\theta+1)^\theta+2\theta^{\theta+1}}{2\theta^\theta(\theta+2)}\left|f'(b)\right|^q+\frac{4\theta^\theta-(\theta+1)^\theta}{2\theta^\theta(\theta+2)}\left|f'(a)\right|^q\right)^{\frac{1}{q}} \end{array}\right]$$

*with $\theta > 0$.*

*Proof.* Using Lemma 4, power mean inequality and the convexity of $|f'|^q$, we have

$$\left|\frac{\Gamma(\theta+1)}{2(b-a)^\theta}\left[R_{a+}^\theta f(b)+R_{b-}^\theta f(a)\right]-\frac{f\left(\frac{\theta a+b}{\theta+1}\right)+f\left(\frac{a+\theta b}{\theta+1}\right)}{2}\right|$$

$$\leq \frac{b-a}{2}\left[\begin{array}{c} \int_0^{\frac{\theta}{\theta+1}} t^\theta \left|f'(ta+(1-t)b)\right| dt \\ +\int_{\frac{\theta}{\theta+1}}^1 \left(1-t^\theta\right)\left|f'(ta+(1-t)b)\right| dt \\ +\int_0^{\frac{\theta}{\theta+1}} t^\theta \left|f'(tb+(1-t)a)\right| dt \\ +\int_{\frac{\theta}{\theta+1}}^1 \left(1-t^\theta\right)\left|f'(tb+(1-t)a)\right| dt \end{array}\right]$$

$$\leq \frac{b-a}{2}\left[\begin{array}{c} \left(\int_0^{\frac{\theta}{\theta+1}} t^\theta dt\right)^{1-\frac{1}{q}} \left(\int_0^{\frac{\theta}{\theta+1}} t^\theta \left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\theta}{\theta+1}}^1 \left(1-t^\theta\right) dt\right)^{1-\frac{1}{q}} \left(\int_{\frac{\theta}{\theta+1}}^1 \left(1-t^\theta\right)\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_0^{\frac{\theta}{\theta+1}} t^\theta dt\right)^{1-\frac{1}{q}} \left(\int_0^{\frac{\theta}{\theta+1}} t^\theta \left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\theta}{\theta+1}}^1 \left(1-t^\theta\right) dt\right)^{1-\frac{1}{q}} \left(\int_{\frac{\theta}{\theta+1}}^1 \left(1-t^\theta\right)\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \end{array}\right]$$

$$\leq \frac{b-a}{2}\left(\frac{\theta^{\theta+1}}{(\theta+1)^{\theta+2}}\right)^{1-\frac{1}{q}}\left[\begin{array}{c} \left(\int_0^{\frac{\theta}{\theta+1}} t^\theta \left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right] dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\theta}{\theta+1}}^1 \left(1-t^\theta\right)\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right] dt\right)^{\frac{1}{q}} \\ +\left(\int_0^{\frac{\theta}{\theta+1}} t^\theta \left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right] dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\theta}{\theta+1}}^1 \left(1-t^\theta\right)\left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right] dt\right)^{\frac{1}{q}} \end{array}\right]$$

$$\leq \frac{b-a}{2}\frac{\theta^{\theta+1}}{(\theta+1)^{\theta+2}}\left[\begin{array}{c} \left(\frac{\theta}{\theta+2}\left|f'(a)\right|^q+\frac{2}{\theta+2}\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(\frac{(\theta+1)^\theta+2\theta^{\theta+1}}{2\theta^\theta(\theta+2)}\left|f'(a)\right|^q+\frac{4\theta^\theta-(\theta+1)^\theta}{2\theta^\theta(\theta+2)}\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(\frac{\theta}{\theta+2}\left|f'(b)\right|^q+\frac{2}{\theta+2}\left|f'(a)\right|^q\right)^{\frac{1}{q}} \\ +\left(\frac{(\theta+1)^\theta+2\theta^{\theta+1}}{2\theta^\theta(\theta+2)}\left|f'(b)\right|^q+\frac{4\theta^\theta-(\theta+1)^\theta}{2\theta^\theta(\theta+2)}\left|f'(a)\right|^q\right)^{\frac{1}{q}} \end{array}\right].$$

This completes the proof. □

**Corollary 3.** *In Theorem 8, if one takes $\theta = 1$, one has the following midpoint type inequality,*

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{8}\left[\begin{array}{c} \left(\frac{1}{3}\left|f'(a)\right|^q+\frac{2}{3}\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(\frac{2}{3}\left|f'(a)\right|^q+\frac{1}{3}\left|f'(b)\right|^q\right)^{\frac{1}{q}} \end{array}\right]. \tag{4.4}$$

**Theorem 9.** *Let $a,b \in \mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on $(a,b)$. If $|f'|^q$ is convex on $[a,b]$ for $q>1$, then the following fractional midpoint*

*type inequality holds:*

$$\left|\frac{\Gamma(\theta+1)}{2(b-a)^{\theta}}\left[R_{a+}^{\theta}f(b)+R_{b-}^{\theta}f(a)\right]-\frac{f\left(\frac{\theta a+b}{\theta+1}\right)+f\left(\frac{a+\theta b}{\theta+1}\right)}{2}\right| \tag{4.5}$$

$$\leq\frac{b-a}{2}\left[\begin{array}{c}\left(\int_0^{\frac{\theta}{\theta+1}}t^{\theta p}dt\right)^{\frac{1}{p}}\left(\frac{\theta^2}{2(\theta+1)^2}|f'(a)|^q+\frac{\theta^2+2\theta}{2(\theta+1)^2}|f'(b)|^q\right)^{\frac{1}{q}}\\+\left(\int_{\frac{\theta}{\theta+1}}^1(1-t^{\theta})^p dt\right)^{\frac{1}{p}}\left(\frac{2\theta+1}{2(\theta+1)^2}|f'(a)|^q+\frac{1}{2(\theta+1)^2}|f'(b)|^q\right)^{\frac{1}{q}}\\+\left(\int_0^{\frac{\theta}{\theta+1}}t^{\theta p}dt\right)^{\frac{1}{p}}\left(\frac{\theta^2}{2(\theta+1)^2}|f'(b)|^q+\frac{\theta^2+2\theta}{2(\theta+1)^2}|f'(a)|^q\right)^{\frac{1}{q}}\\+\left(\int_{\frac{\theta}{\theta+1}}^1(1-t^{\theta})^p dt\right)^{\frac{1}{p}}\left(\frac{2\theta+1}{2(\theta+1)^2}|f'(b)|^q+\frac{1}{2(\theta+1)^2}|f'(a)|^q\right)^{\frac{1}{q}}\end{array}\right]$$

*with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $\theta>0$.

*Proof.* Using Lemma 4, Holder inequality and the convexity of $|f'|^q$, we have

$$\left|\frac{\Gamma(\theta+1)}{2(b-a)^{\theta}}\left[R_{a+}^{\theta}f(b)+R_{b-}^{\theta}f(a)\right]-\frac{f\left(\frac{\theta a+b}{\theta+1}\right)+f\left(\frac{a+\theta b}{\theta+1}\right)}{2}\right|$$

$$\leq\frac{b-a}{2}\left[\begin{array}{c}\int_0^{\frac{\theta}{\theta+1}}t^{\theta}|f'(ta+(1-t)b)|dt\\+\int_{\frac{\theta}{\theta+1}}^1(1-t^{\theta})|f'(ta+(1-t)b)|dt\\+\int_0^{\frac{\theta}{\theta+1}}t^{\theta}|f'(tb+(1-t)a)|dt\\+\int_{\frac{\theta}{\theta+1}}^1(1-t^{\theta})|f'(tb+(1-t)a)|dt\end{array}\right]$$

$$\leq\frac{b-a}{2}\left[\begin{array}{c}\left(\int_0^{\frac{\theta}{\theta+1}}t^{\theta p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\theta}{\theta+1}}|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\\+\left(\int_{\frac{\theta}{\theta+1}}^1(1-t^{\theta})^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{\theta}{\theta+1}}^1|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\\+\left(\int_0^{\frac{\theta}{\theta+1}}t^{\theta p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\theta}{\theta+1}}|f'(tb+(1-t)a)|^q dt\right)^{\frac{1}{q}}\\+\left(\int_{\frac{\theta}{\theta+1}}^1(1-t^{\theta})^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{\theta}{\theta+1}}^1|f'(tb+(1-t)a)|^q dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq\frac{b-a}{2}\left[\begin{array}{c}\left(\int_0^{\frac{\theta}{\theta+1}}t^{\theta p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\theta}{\theta+1}}\left[t|f'(a)|^q+(1-t)|f'(b)|^q\right]dt\right)^{\frac{1}{q}}\\+\left(\int_{\frac{\theta}{\theta+1}}^1(1-t^{\theta})^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{\theta}{\theta+1}}^1\left[t|f'(a)|^q+(1-t)|f'(b)|^q\right]dt\right)^{\frac{1}{q}}\\+\left(\int_0^{\frac{\theta}{\theta+1}}t^{\theta p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\theta}{\theta+1}}\left[t|f'(b)|^q+(1-t)|f'(a)|^q\right]dt\right)^{\frac{1}{q}}\\+\left(\int_{\frac{\theta}{\theta+1}}^1(1-t^{\theta})^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{\theta}{\theta+1}}^1\left[t|f'(b)|^q+(1-t)|f'(a)|^q\right]dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq \frac{b-a}{2}\left[\begin{array}{c}\left(\int_0^{\frac{\theta}{\theta+1}} t^{\theta p}dt\right)^{\frac{1}{p}}\left(\frac{\theta^2}{2(\theta+1)^2}\left|f'(a)\right|^q+\frac{\theta^2+2\theta}{2(\theta+1)^2}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\\ +\left(\int_{\frac{\theta}{\theta+1}}^1 \left(1-t^{\theta}\right)^p dt\right)^{\frac{1}{p}}\left(\frac{2\theta+1}{2(\theta+1)^2}\left|f'(a)\right|^q+\frac{1}{2(\theta+1)^2}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\\ +\left(\int_0^{\frac{\theta}{\theta+1}} t^{\theta p}dt\right)^{\frac{1}{p}}\left(\frac{\theta^2}{2(\theta+1)^2}\left|f'(b)\right|^q+\frac{\theta^2+2\theta}{2(\theta+1)^2}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\\ +\left(\int_{\frac{\theta}{\theta+1}}^1 \left(1-t^{\theta}\right)^p dt\right)^{\frac{1}{p}}\left(\frac{2\theta+1}{2(\theta+1)^2}\left|f'(b)\right|^q+\frac{1}{2(\theta+1)^2}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\end{array}\right].$$

This completes the proof. □

**Corollary 4.** *In Theorem 9, if one takes $\theta = 1$, one has [5, Theorem 2.3].*

## 5. Competing interests

The authors declare that they have no competing interests.

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# DETERMINANTS CONTAINING POWERS OF POLYNOMIAL SEQUENCES

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*Abstract.* We derive identities for the determinants of matrices whose entries are (rising) powers of (products of) polynomials that satisfy a recurrence relation. In particular, these results cover the cases for Fibonacci polynomials, Lucas polynomials and certain orthogonal polynomials. These identities naturally generalize the determinant identities obtained by Alfred, Carlitz, Prodinger, Tangboonduangjit and Thanatipanonda.

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*Keywords:* Fibonacci polynomial, Lucas polynomial, Chebyshev polynomial, determinant identity

## 1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence. Let $m \geq 1$ and $n$ be any nonnegative integer. Let $[F^m_{n+i+j}]_{0\leq i,j\leq m}$ be the $(m+1)\times(m+1)$ matrix with entries $F_{n+i+j}$, $0 \leq i, j \leq m$. In 1966, Carlitz [4] derived the following determinant identity for the matrix $[F^m_{n+i+j}]_{0\leq i,j\leq m}$:

$$\det([F^m_{n+i+j}]_{0\leq i,j\leq m}) = (-1)^{(n+1)\binom{m+1}{2}}(F_1^m F_2^{m-1}\cdots F_m)^2 \prod_{i=0}^{m} \binom{m}{i}. \tag{1.1}$$

This identity is related to the problems posted by Alfred [1, p. 48] in 1963 and Parker [9, p. 303] in 1964 respectively. Let $s$ and $k$ be any integers. Tangboonduangjit and Thanatipanonda [12] generalized the determinant identity (1.1) as follows:

$$\det([F^m_{s+k(n+i+j)}]_{0\leq i,j\leq m}) = (-1)^{(s+kn+1)\binom{m+1}{2}}(F_k^m F_{2k}^{m-1}\cdots F_{mk})^2 \prod_{i=0}^{m} \binom{m}{i}. \tag{1.2}$$

Let $F_n^{\langle m\rangle}$ be the *rising powers* of the Fibonacci numbers defined by

$$F_n^{\langle m\rangle} := F_n F_{n+1}\cdots F_{n+m-1}.$$

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Prodinger [10] obtained the following determinant identity for the matrix $[F^{\langle m\rangle}_{n+i+j}]_{0\le i,j\le m}$:

$$\det([F^{\langle m\rangle}_{n+i+j}]_{0\le i,j\le m}) = (-1)^{n\binom{m+1}{2}+\binom{m+2}{3}}(F_1F_2\cdots F_m)^{m+1}. \tag{1.3}$$

Tangboonduangjit and Thanatipanonda [11] generalized the determinant identity (1.3) as follows:

$$\det([F^{\langle m\rangle}_{n+i+j}]_{0\le i,j\le d-1}) = (-1)^{n\binom{d}{2}+\binom{d+1}{3}}\prod_{i=1}^{d-1}(F_iF_{m+1-i})^{d-i}\cdot\prod_{i=d-1}^{2(d-1)}F^{\langle m+1-d\rangle}_{n+i}. \tag{1.4}$$

where $d\ge 2$. It is worthwhile to note that Tangboonduangjit and Thanatipanonda [11, 12] derived the determinant identities more generally, for matrices whose entries include (rising) powers of terms that satisfy a second-order linear recurrence relation with constant coefficients. By using analogous techniques in determinant calculus, we derive determinant identities for matrices whose entries are (rising) powers of polynomials that satisfy certain recurrence relations. As corollaries, we provide determinant identities for matrices whose entries are (rising) powers of Fibonacci polynomials, Lucas polynomials and certain orthogonal polynomials. As an application, we obtain new identities in the case of Fibonacci numbers. For example, for $n\ge 1$, by Corollary 2, we get

$$\det\left(\left[\frac{1}{F_{n+i+j}}\right]_{0\le i,j\le m}\right) = \frac{(-1)^{n\binom{m+1}{2}}\prod_{i=0}^{m}F_{i+1}^{2(m-i)}}{\prod_{0\le i,j\le m}F_{n+i+j}}. \tag{1.5}$$

## 2. Main results

**Definition 1.** Let $p,q,r,a,b$ and $c$ be any real numbers. Let $\mathbb{Z}_{\ge 0}=\{0,1,2,\ldots\}$. The sequence of polynomials in variable $x$,

$$\mathcal{P}(x)(p,q,r;a,b,c) := (P_n)_{n\in\mathbb{Z}},$$

is defined by

$$P_0 := p,\quad P_1 := qx+r,\quad P_{n+2} := (ax+b)P_{n+1}+cP_n,\ \text{for } n\in\mathbb{Z}_{\ge 0}.$$

For $n<0$, $P_n$ is defined by

$$P_n := -\frac{ax+b}{c}P_{n+1}+\frac{1}{c}P_{n+2}.$$

The discriminant $\Delta_{\mathcal{P}}$ is defined by

$$\Delta_{\mathcal{P}} := (q^2-apq)x^2+(2qr-apr-bpq)x+(r^2-bpr-cp^2).$$

**Theorem 1.** *Let $\mathcal{P}(x)(p_1,q_1,r_1;a,b,c) = (P_n)_{n\in\mathbb{Z}}$, $\mathcal{Q}(x)(p_2,q_2,r_2;a,b,c) = (Q_n)_{n\in\mathbb{Z}}$ and $\mathcal{U}(x)(0,0,1;a,b,c) = (U_n)_{n\in\mathbb{Z}}$ be the sequences of polynomials defined by real numbers $p_1,p_2,q_1,q_2,r_1,r_2,a,b,c$ where $c \neq 0$. Then*

$$P_{s+i}Q_{s+j} - P_sQ_{s+i+j} = (-c)^s(P_1Q_j - P_0Q_{j+1})U_i$$

*for all integers $s,i,j$.*

*Proof.* We prove it by induction on $i$. It is trivial for $i = 0$. If $i = 1$, we have

$$\begin{aligned}\begin{pmatrix} P_{s+1} & Q_{s+j+1} \\ P_s & Q_{s+j}\end{pmatrix} &= \begin{pmatrix} ax+b & c \\ 1 & 0\end{pmatrix}\begin{pmatrix} P_s & Q_{s+j} \\ P_{s-1} & Q_{s+j-1}\end{pmatrix} = \\ &= \begin{pmatrix} ax+b & c \\ 1 & 0\end{pmatrix}^s\begin{pmatrix} P_1 & Q_{j+1} \\ P_0 & Q_j\end{pmatrix}, \end{aligned} \tag{2.1}$$

$$\begin{aligned}\begin{pmatrix} P_{s+1} & Q_{s+j+1} \\ P_s & Q_{s+j}\end{pmatrix} &= \begin{pmatrix} ax+b & c \\ 1 & 0\end{pmatrix}^{-1}\begin{pmatrix} P_{s+2} & Q_{s+j+2} \\ P_{s+1} & Q_{s+j+1}\end{pmatrix} = \\ &= \begin{pmatrix} ax+b & c \\ 1 & 0\end{pmatrix}^s\begin{pmatrix} P_1 & Q_{j+1} \\ P_0 & Q_j\end{pmatrix} \end{aligned} \tag{2.2}$$

for $s \geq 0$ and $s < 0$ respectively. We take the determinants on both sides of (2.1) and (2.2) to get

$$\begin{aligned} P_{s+1}Q_{s+j} - P_sQ_{s+j+1} &= (-c)^s(P_1Q_j - P_0Q_{j+1}) \\ &= (-c)^s(P_1Q_j - P_0Q_{j+1})U_1. \end{aligned} \tag{2.3}$$

For $i > 1$, we assume that the identity is true for $i-1$ and $i-2$. We have

$$\begin{aligned} &P_{s+i}Q_{s+j} - P_sQ_{s+i+j} \\ &= \begin{vmatrix} P_{s+i} & Q_{s+i+j} \\ P_s & Q_{s+j}\end{vmatrix} \\ &= \begin{vmatrix} (ax+b)P_{s+i-1} + cP_{s+i-2} & (ax+b)Q_{s+i+j-1} + cQ_{s+i+j-2} \\ P_s & Q_{s+j}\end{vmatrix} \\ &= (ax+b)\begin{vmatrix} P_{s+i-1} & Q_{s+j+(i-1)} \\ P_s & Q_{s+j}\end{vmatrix} + c\begin{vmatrix} P_{s+i-2} & Q_{s+j+(i-2)} \\ P_s & Q_{s+j}\end{vmatrix} \\ &= (ax+b)(-c)^s(P_1Q_j - P_0Q_{j+1})U_{i-1} + c(-c)^s(P_1Q_j - P_0Q_{j+1})U_{i-2} \\ &= (-c)^s(P_1Q_j - P_0Q_{j+1})((ax+b)U_{i-1} + cU_{i-2}) \\ &= (-c)^s(P_1Q_j - P_0Q_{j+1})U_i. \end{aligned}$$

For $i < 0$, we assume that the identity is true for $i+1$ and $i+2$. We have

$$\begin{aligned} &P_{s+i}Q_{s+j} - P_sQ_{s+i+j} = \begin{vmatrix} P_{s+i} & Q_{s+i+j} \\ P_s & Q_{s+j}\end{vmatrix} \\ &= \begin{vmatrix} -\frac{ax+b}{c}P_{s+i+1} + \frac{1}{c}P_{s+i+2} & -\frac{ax+b}{c}Q_{s+i+j+1} + \frac{1}{c}Q_{s+i+j+2} \\ P_s & Q_{s+j}\end{vmatrix} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{ax+b}{c}\begin{vmatrix} P_{s+i+1} & Q_{s+i+j+1} \\ P_s & Q_{s+j} \end{vmatrix} + \frac{1}{c}\begin{vmatrix} P_{s+i+2} & Q_{s+i+j+2} \\ P_s & Q_{s+j} \end{vmatrix} \\
&= -\frac{ax+b}{c}(-c)^s(P_1Q_j - P_0Q_{j+1})U_{i+1} + \frac{1}{c}(-c)^s(P_1Q_j - P_0Q_{j+1})U_{i+2} \\
&= (-c)^s(P_1Q_j - P_0Q_{j+1})\Big(-\frac{ax+b}{c}U_{i+1} + \frac{1}{c}U_{i+2}\Big) \\
&= (-c)^s(P_1Q_j - P_0Q_{j+1})U_i.
\end{aligned}
$$

□

*Remark* 1. We recover the *generalized Catalan Identity* by Melham and Shannon [8] (see also Tangboonduangjit and Thanatipanonda [12, Proposition 1]) by substituting $x = 1$ in Theorem 1.

**Corollary 1.** *Let $\mathcal{P}(x)(p,q,r;a,b,c) = (P_n)_{n\in\mathbb{Z}}$ and $\mathcal{U}(x)(0,0,1;a,b,c) = (U_n)_{n\in\mathbb{Z}}$ be the sequences of polynomials defined by real numbers $p,q,r,a,b,c$ where $c \neq 0$. Then*

$$P_jP_1 - P_0P_{j+1} = \Delta_{\mathcal{P}}\cdot U_j, \tag{2.4}$$

$$P_{s+i}P_{s+j} - P_sP_{s+i+j} = (-c)^s\Delta_{\mathcal{P}}\cdot U_iU_j \tag{2.5}$$

*where $\Delta_{\mathcal{P}}$ is the discriminant of $\mathcal{P}(x)(p,q,r;a,b,c)$.*

*Proof.* By setting $s = 0, j = 1, p_1 = p_2 = p, q_1 = q_2 = q, r_1 = r_2 = r$ in Theorem 1, we get

$$P_iP_1 - P_0P_{i+1} = (P_1P_1 - P_0P_2)U_i. \tag{2.6}$$

We note that $P_0 = p$, $P_1 = qx + r$ and $P_2 = (ax+b)P_1 + cP_0 = (ax+b)(qx+r) + cp$. Hence, we obtain (2.4) by simplifying (2.6). On the other hand, by setting $p_1 = p_2 = p, q_1 = q_2 = q, r_1 = r_2 = r$ in Theorem 1, we get

$$P_{s+i}P_{s+j} - P_sP_{s+i+j} = (-c)^s(P_1P_j - P_0P_{j+1})U_i = (-c)^s\Delta_{\mathcal{P}}\cdot U_iU_j$$

in which the last equality is based on (2.4). □

**Lemma 1.** *Let $m \geq 1$. Let $B_i, D_i$ be polynomials in variable $x$, $A_i, C_i$ be rational functions in variable $x$, for $0 \leq i \leq m$. Let $[(A_jB_i + C_jD_i)^m]_{0\leq i,j\leq m}$ be the $(m+1)\times(m+1)$ matrix with entries $(A_jB_i + C_jD_i)^m, 0 \leq i, j \leq m$. Then we have the following determinant identity:*

$$
\begin{aligned}
&\det([(A_jB_i + C_jD_i)^m]_{0\leq i,j\leq m}) = \\
&= \prod_{0\leq i<j\leq m}\Big((B_iD_j - B_jD_i)(A_iC_j - A_jC_i)\Big)\cdot\prod_{i=0}^{m}\binom{m}{i}.
\end{aligned} \tag{2.7}
$$

*Proof.* We invoke the following result by Krattenthaler [7, Lemma 10] (see also Tangboonduangjit and Thanatipanonda [12, Lemma 3]:

$$\det([(c_j d_i + 1)^m]_{0\le i,j\le m}) = \prod_{0\le i<j\le m} \Big((d_i - d_j)(c_i - c_j)\Big) \cdot \prod_{i=0}^{m} \binom{m}{i} \tag{2.8}$$

where $c_j, d_i$ are real numbers for $0 \le i, j \le m$. First, we prove the lemma for polynomials $A_i, B_i, C_i, D_i$ for all $0 \le i \le m$. For the values of $x$ such that $C_j \neq 0$ and $D_i \neq 0$ for $0 \le i, j \le m$, let

$$c_j = \frac{A_j}{C_j}, \quad d_i = \frac{B_i}{D_i}, \text{ for } 0 \le i, j \le m.$$

We note that

$$\begin{aligned}\det([(c_j d_i + 1)^m]_{0\le i,j\le m}) &= \det\Big(\Big[\Big(\frac{A_j B_i + C_j D_i}{C_j D_i}\Big)^m\Big]_{0\le i,j\le m}\Big) \\ &= \Big(\prod_{0\le i\le m} \frac{1}{(C_i D_i)^m}\Big) \cdot \det([(A_j B_i + C_j D_i)^m]_{0\le i,j\le m}). \end{aligned} \tag{2.9}$$

Also, we have

$$\begin{aligned}\prod_{0\le i<j\le m} \Big((d_i - d_j)(c_i - c_j)\Big) &= \prod_{0\le i<j\le m} \Big(\Big(\frac{B_i D_j - B_j D_i}{D_i D_j}\Big) \cdot \Big(\frac{A_i C_j - A_j C_i}{C_i C_j}\Big)\Big) \\ &= \prod_{0\le i\le m} \Big(\frac{1}{(C_i D_i)^m}\Big) \cdot \prod_{0\le i<j\le m} \Big((B_i D_j - B_j D_i)(A_i C_j - A_j C_i)\Big). \end{aligned} \tag{2.10}$$

By (2.8), (2.9), (2.10), we get (2.7) as desired.

Based on the facts that there are only a finite number of roots for $C_j, D_i$ where $0 \le i, j \le m$ and the determinant of a matrix with polynomial entries is a continuous function in $x$, the equality (2.7) still holds true for the values of $x$ such that $C_j = 0$ or $D_i = 0$ for some $i$ or $j$.

Next, we assume that $A_i$ and $C_i$ are rational functions for all $0 \le i \le m$. We write $A_i$ and $C_i$ as follows:

$$A_i = \frac{E_i}{F_i}, \quad C_i = \frac{G_i}{H_i} \text{ for } 0 \le i \le m,$$

where $E_i, F_i, G_i, H_i$ are all polynomials for $0 \le i \le m$. For the values of $x$ such that $F_i \neq 0$ and $H_i \neq 0$ for all $0 \le i \le m$, we get

$$\begin{aligned}
&\det([(A_j B_i + C_j D_i)^m]_{0\le i,j\le m}) = \det\Big(\Big[\Big(\frac{H_j E_j B_i + D_i F_j G_j}{F_j H_j}\Big)^m\Big]_{0\le i,j\le m}\Big)\\
&= \Big(\prod_{0\le i\le m} \frac{1}{(F_i H_i)^m}\Big)\cdot \det([((H_j E_j)B_i + (G_j F_j)D_i)^m]_{0\le i,j\le m})\\
&= \Big(\prod_{0\le i\le m} \frac{1}{(F_i H_i)^m}\Big)\Big(\prod_{i=0}^{m}\binom{m}{i}\Big)\cdot\\
&\qquad \cdot\Big(\prod_{0\le i<j\le m}(B_i D_j - B_j D_i)(H_i E_i G_j F_j - H_j E_j G_i F_i)\Big)\\
&= \Big(\prod_{i=0}^{m}\binom{m}{i}\Big)\Big(\prod_{0\le i<j\le m}(B_i D_j - B_j D_i)\Big(\frac{E_i}{F_i}\cdot\frac{G_j}{H_j} - \frac{E_j}{F_j}\cdot\frac{G_i}{H_i}\Big)\Big)\\
&= \Big(\prod_{i=0}^{m}\binom{m}{i}\Big)\Big(\prod_{0\le i<j\le m}(B_i D_j - B_j D_i)\big(A_i C_j - A_j C_i\big)\Big).
\end{aligned}$$

For the values of $x$ such that $F_i = 0$ or $H_i = 0$ for some $i$, the equality still holds true as the determinant of a matrix with polynomial entries is a continuous function. □

**Theorem 2.** *Let $s, k, n$ be any integers, $m \ge 1$. Let $\mathcal{P}(x)(p,q,r;a,b,c) = (P_n)_{n\in\mathbb{Z}}$ and $\mathcal{U}(x)(0,0,1;a,b,c) = (U_n)_{n\in\mathbb{Z}}$ be the sequences of polynomials defined by real numbers $p,q,r,a,b,c$ where $c \neq 0$. The determinant of the matrix $[P^m_{s+k(n+i+j)}]_{0\le i,j\le m}$ is given by*

$$\begin{aligned}
&\det([P^m_{s+k(n+i+j)}]_{0\le i,j\le m})\\
&= (-1)^{(s+kn+1)\binom{m+1}{2}}\Delta_{\mathcal{P}}^{\binom{m+1}{2}}\cdot c^{(s+kn)\binom{m+1}{2}+2k\binom{m+1}{3}}\prod_{i=0}^{m}\Big(\binom{m}{i}U_{k(i+1)}^{2(m-i)}\Big)
\end{aligned}$$

*where $\Delta_{\mathcal{P}}$ is the discriminant of $\mathcal{P}(x)(p,q,r;a,b,c)$.*

*Proof.* By substituting $s = -km, i = kj', j = s' + k(n+m+i')$ into (2.5) in Corollary 1 and then replacing $s', i', j'$ by $s, i, j$ respectively, we get

$$\begin{aligned}
&P_{k(j-m)}P_{s+k(n+i)} - P_{-km}P_{s+k(n+i+j)} = (-c)^{-km}\Delta_{\mathcal{P}}\cdot U_{kj}U_{s+k(n+m+i)},\\
&P_{s+k(n+i+j)} = \frac{P_{k(j-m)}}{P_{-km}}\cdot P_{s+k(n+i)} + \frac{-(-c)^{-km}\Delta_{\mathcal{P}}\cdot U_{kj}}{P_{-km}}\cdot U_{s+k(n+m+i)}.
\end{aligned} \tag{2.11}$$

By substituting $s = s' + k(n+m+i'), i = k(j'-i'), j = -km$ into Theorem 1 and then replacing $s', i', j'$ by $s, i, j$ respectively, we get

$$\begin{aligned} &P_{s+k(n+i)}U_{s+k(n+m+j)} - P_{s+k(n+j)}U_{s+k(n+m+i)} \\ &= (-c)^{s+k(n+m+i)}(U_1 P_{-km} - U_0 P_{-km+1})U_{k(j-i)} \\ &= (-c)^{s+k(n+m+i)} P_{-km} U_{k(j-i)}. \end{aligned} \tag{2.12}$$

By substituting $s = ki', i = k(j'-i'), j = -km$ into Theorem 1 and then replacing $i', j'$ by $i, j$ respectively, we get

$$P_{k(i-m)}U_{kj} - P_{k(j-m)}U_{ki} = (-c)^{ki} P_{-km} U_{k(j-i)}. \tag{2.13}$$

By (2.11), we get

$$\begin{aligned} &\det([P^m_{s+k(n+i+j)}]_{0\le i,j\le m}) \\ &= \det\left(\left[\left(\frac{P_{k(j-m)}}{P_{-km}} \cdot P_{s+k(n+i)} + \frac{-(-c)^{-km}\Delta_{\mathcal{P}} \cdot U_{kj}}{P_{-km}} \cdot U_{s+k(n+m+i)}\right)^m\right]_{0\le i,j\le m}\right). \end{aligned} \tag{2.14}$$

By (2.7), the term in (2.14) becomes

$$\begin{aligned} &\prod_{i=0}^{m}\binom{m}{i} \cdot \prod_{0\le i<j\le m}\Big((P_{s+k(n+i)}U_{s+k(n+m+j)} - P_{s+k(n+j)}U_{s+k(n+m+i)})\Big) \cdot \\ &\qquad \left(\frac{-(-c)^{-km}\Delta_{\mathcal{P}}}{P^2_{-km}}\big(P_{k(i-m)}U_{kj} - P_{k(j-m)}U_{ki}\big)\right)\Big). \end{aligned} \tag{2.15}$$

By (2.12), (2.13), the term in (2.15) becomes

$$\prod_{i=0}^{m}\binom{m}{i} \cdot \prod_{0\le i<j\le m}\Big((-1)^{s+kn+1}c^{s+k(n+2i)}\Delta_{\mathcal{P}} \cdot U^2_{k(j-i)}\Big). \tag{2.16}$$

As a consequence, we get the desired result by standard counting arguments. □

*Remark* 2. We recover Theorem 5 in the work of Tangboonduangjit and Thanatipanonda [12] by substituting $x = 1$ in Theorem 2.

Next, we look at other determinant identities.

**Lemma 2.** *Let $m \ge 1$. Let $B_i, D_i$ be polynomials in variable $x$ for $0 \le i \le m$. Let $A_j, C_j$ be rational functions in variable $x$ for $i \in \mathbb{Z}$. Let $(d_i)_{1\le i\le r}$ and $(e_i)_{1\le i\le r}$ be sequences of integers. Then*

$$\det\left(\left[\prod_{f=j+1}^{m}(A_{d_f}B_i + C_{d_f}D_i) \cdot \prod_{g=1}^{j}(A_{e_g}B_i + C_{e_g}D_i)\right]_{0\le i,j\le m}\right)$$

$$= \prod_{0\le i<j\le m} (B_i D_j - B_j D_i) \cdot \prod_{1\le i\le j\le m} (C_{e_i} A_{d_j} - A_{e_i} C_{d_j}).$$

*Proof.* By the factorization method of Krattenthaler [7, Section 4], it is plain to get the following identity:

$$\det\Big(\Big[\prod_{f=j+1}^{m} (X_i + F_f) \cdot \prod_{g=1}^{j} (X_i + G_g)\Big]_{0\le i,j\le m}\Big) = \prod_{0\le i<j\le m} (X_j - X_i) \cdot \prod_{1\le i\le j\le m} (F_j - G_i) \quad (2.17)$$

where $X_i$ for $0 \le i \le m$, $D_j, E_j$ for $1 \le j \le m$ are some indeterminates. For the values of $x$ such that $D_i \ne 0$ and $A_j \ne 0$ for $0 \le i \le m$ and $j \in \mathbb{Z}$, let

$$X_i = \frac{B_i}{D_i}, \quad F_j = \frac{C_{d_j}}{A_{d_j}}, \quad G_j = \frac{C_{e_j}}{A_{e_j}}$$

for $0 \le i \le m$ and $1 \le j \le m$. By similar reasoning as in the proof of Lemma 1, we get the desired result by clearing the denominators on both sides of (2.17). For the values of $x$ which are the roots of $D_i$ or $A_j$ for some $i$ or $j$, the equality still holds true based on the fact that the determinant of a matrix with polynomial entries is a continuous function. □

**Theorem 3.** *Let $s, k, n$ be any integers, $m \ge 1$. Let $\mathcal{P}(x)(p,q,r;a,b,c) = (P_n)_{n\in\mathbb{Z}}$ and $\mathcal{U}(x)(0,0,1;a,b,c) = (U_n)_{n\in\mathbb{Z}}$ be the sequences of polynomials defined by real numbers $p,q,r,a,b,c$ where $c \ne 0$. Let $(d_i)_{1\le i\le m}$ and $(e_i)_{1\le i\le m}$ be sequences of integers. Then*

$$\det\Big(\Big[\prod_{f=j+1}^{m} P_{s+k(n+i+d_f)} \prod_{g=1}^{j} P_{s+k(n+i+e_g)}\Big]_{0\le i,j\le m}\Big)$$
$$= (-\Delta_{\mathcal{P}})^{\binom{m+1}{2}} (-c)^{(s+kn)\binom{m+1}{2}+k\binom{m+1}{3}} \prod_{l=1}^{m} U_{kl}^{m+1-l} \cdot \prod_{1\le i\le j\le m} (-c)^{kd_j} U_{k(e_i-d_j)}$$

*where $\Delta_{\mathcal{P}}$ is the discriminant of $\mathcal{P}(x)(p,q,r;a,b,c)$.*

*Proof.* By (2.11), Lemma 2 and Corollary 1, the theorem can be proved in the same way as in the proof the Theorem 2. □

**Lemma 3.** *Let $m \ge 1$. Let $A_i, B_i$ are polynomials in variable $x$ for $0 \le i \le m$. Let $C_i, D_i$ be rational functions in variable $x$ for $0 \le i \le m$. Then,*

$$\det\Big(\Big[\frac{1}{A_i D_j + B_i C_j}\Big]_{0\le i,j\le m}\Big) = \frac{\prod_{0\le i<j\le m}(A_i B_j - A_j B_i)(C_i D_j - D_i C_j)}{\prod_{0\le i,j\le m}(A_i D_j + B_i C_j)}$$

*provided that the denominators on both sides of the identity are nonzero.*

*Proof.* First, we invoke a result of Krattenthaler [7, Theorem 12]. That is,

$$\det\left(\left[\frac{1}{x_i+y_j}\right]_{0\le i,j\le m}\right)=\frac{\prod_{0\le i<j\le m}(x_i-x_j)(y_i-y_j)}{\prod_{0\le i,j\le m}(x_i+y_j)} \tag{2.18}$$

where $x_i$ and $y_i$ are indeterminates for $0\le i,j\le m$. We first assume that $A_i, B_i, C_i, D_i$ are all polynomials for all $0\le i\le m$. For the values of $x$ such that $B_i, D_i$ are nonzero for all $0\le i\le m$, let

$$x_i=\frac{A_i}{B_i},\quad y_i=\frac{C_i}{D_i}\text{ for } 0\le i\le m.$$

By similar reasoning as shown in the proof of Lemma 1, we get the desired result by some algebraic simplification for the cases where $A_i, B_i, C_i, D_i$ are polynomials for all $0\le i\le m$.

We extend the proof to the cases where $C_i$ and $D_i$ are rational functions by the same arguments as in the proof of Lemma 1, based on the fact that the determinant of a matrix with rational functions as entries is a continuous function provided that the denominators on both sides of the identity are nonzero. □

**Theorem 4.** *Let $s,k,n$ be any integers, $m\ge 1$. Let $\mathcal{P}(x)(p,q,r;a,b,c)=(P_n)_{n\in\mathbb{Z}}$ and $\mathcal{U}(x)(0,0,1;a,b,c)=(U_n)_{n\in\mathbb{Z}}$ be the sequences of polynomials defined by real numbers $p,q,r,a,b,c$ where $c\neq 0$. The determinant of the matrix $[1/P_{s+k(n+i+j)}]_{0\le i,j\le m}$ is given by*

$$\det\left(\left[\frac{1}{P_{s+k(n+i+j)}}\right]_{0\le i,j\le m}\right)=\frac{c^{(s+kn)\binom{m+1}{2}+2k\binom{m+1}{3}}\Delta_{\mathcal{P}}^{\binom{m+1}{2}}\prod_{i=0}^{m}U_{k(i+1)}^{2(m-i)}}{(-1)^{(s+kn)\binom{m+1}{2}}\prod_{0\le i,j\le m}P_{s+k(n+i+j)}}$$

*where $\Delta_{\mathcal{P}}$ is the discriminant of $\mathcal{P}(x)(p,q,r;a,b,c)$, provided that the denominators on both sides of the identity are nonzero.*

*Proof.* The proof is essentially the same as the proof of Theorem 2 by applying (2.11), (2.12), (2.13) to Lemma 3 and some standard counting arguments. □

Let $A$ be a $m\times m$ matrix. Let $A_k(i,j)$ be the determinant of the $k\times k$ submatrix of $A$ whose first entry is at the position of the $i^{th}$-row and the $j^{th}$-column of $A$.

**Lemma 4.** *Let $A$ be a $m\times m$ matrix whose entries are rational functions in variable $x$. Then*

$$A_m(1,1)A_{m-2}(2,2)=A_{m-1}(1,1)A_{m-1}(2,2)-A_{m-1}(2,1)A_{m-1}(1,2).$$

*Proof.* We invoke the Desnanot-Jacobi identity [3] for a matrix $A$ of size $m\times m$ with indeterminates as entries.

$$A_m(1,1)A_{m-2}(2,2)=A_{m-1}(1,1)A_{m-1}(2,2)-A_{m-1}(2,1)A_{m-1}(1,2).$$

To extend this result to the case where the matrix $A$ has rational functions as entries, we simply use the same strategy as in the proof of Lemma 1. □

Let $m \geq 1$. The *rising powers* of a sequence of polynomials $\mathcal{P}(x)(p,q,r;a,b,c) = (P_n)_{n\in\mathbb{Z}}$ is denoted by $P_n^{\langle m\rangle}$, which is defined by

$$P_n^{\langle m\rangle} := P_n P_{n+1} \cdots P_{n+m-1}.$$

**Theorem 5.** *Let $n$ be any integer. Let $m \geq 1$ and $d \geq 1$. Let $\mathcal{P}(x)(p,q,r;a,b,c) = (P_n)_{n\in\mathbb{Z}}$ and $\mathcal{U}(x)(0,0,1;a,b,c) = (U_n)_{n\in\mathbb{Z}}$ be the sequences of polynomials defined by real numbers $p,q,r,a,b,c$ where $c \neq 0$. Then*

$$\begin{aligned}&\det([P_{n+i+j}^{\langle m\rangle}]_{0\leq i,j\leq d-1})\\ &\quad = (-1)^{n\binom{d}{2}+\binom{d+1}{3}} c^{(n+d-2)\binom{d}{2}} \Delta_{\mathcal{P}}^{\binom{d}{2}} \cdot \prod_{i=1}^{d-1} \left(U_i U_{r+1-i}\right)^{d-i} \cdot \prod_{i=d-1}^{2(d-1)} P_{n+1}^{\langle m+1-d\rangle}\end{aligned}$$

*where $\Delta_{\mathcal{P}}$ is the discriminant of $\mathcal{P}(x)(p,q,r;a,b,c)$.*

*Proof.* The proof is based on induction on $d$, Lemma 4 and Theorem 1. It is essentially identical to the proof of Theorem 2.1 in the work of Tangboonduangjit and Thanatipanonda [11] and hence we skip it. □

If we set $p = q = b = 0$ and $r = a = c = 1$, then we get the sequence of Fibonacci polynomials in $\mathcal{P}(x)(0,0,1;1,0,1) = (F_n(x))_{n\in\mathbb{Z}}$ where the sequence $(F_n(x))_{n\in\mathbb{Z}}$ is defined by

$$F_0(x) \equiv 0, \quad F_1(x) \equiv 1, \quad F_{n+2}(x) = xF_{n+1}(x) + F_n(x).$$

We recover the Fibonacci numbers and Pell numbers by evaluating $F_n(x)$ at $x = 1$ and $x = 2$ respectively. We note that $\Delta_{\mathcal{P}} = 1$ and $\mathcal{U}(x)(0,0,1;1,0,1) = (F_n(x))_{n\in\mathbb{Z}}$. By Theorem 2, Theorem 3, Theorem 4 and Theorem 5, we get the following corollary:

**Corollary 2.** *Let $m \geq 1$ and $d \geq 1$. Let $s,k,n$ be any integers. Let $(d_i)_{1\leq i\leq m}$ and $(e_i)_{1\leq i\leq m}$ be sequences of integers. Then*

$$\begin{aligned}&\det([(F_{s+k(n+i+j)}(x))^m]_{0\leq i,j\leq m}) = (-1)^{(s+kn+1)\binom{m+1}{2}}\cdot\\ &\quad \cdot \prod_{i=0}^{m} \binom{m}{i} (F_{(i+1)k}(x))^{2(m-i)},\\ &\det\left(\left[\prod_{f=j+1}^{m} F_{s+k(n+i+d_f)}(x) \prod_{g=1}^{j} F_{s+k(n+i+e_g)}(x)\right]_{0\leq i,j\leq m}\right)\\ &= (-1)^{(s+kn+1)\binom{m+1}{2}+k\binom{m+1}{3}} \prod_{l=1}^{m} (F_{kl}(x))^{m+1-l} \cdot \prod_{1\leq i\leq j\leq m} (-1)^{kd_j} F_{k(e_i-d_j)}(x),\end{aligned}$$

$$\det\left(\left[\frac{1}{F_{s+k(n+i+j)}(x)}\right]_{0\le i,j\le m}\right)=\frac{(-1)^{(s+kn)\binom{m+1}{2}}\prod_{i=0}^{m}(F_{k(i+1)}(x))^{2(m-i)}}{\prod_{0\le i,j\le m}F_{s+k(n+i+j)}(x)},$$

$$\det([(F_{n+i+j}(x))^{\langle m\rangle}]_{0\le i,j\le d-1})$$
$$=(-1)^{n\binom{d}{2}+\binom{d+1}{3}}\cdot\prod_{i=1}^{d-1}\left(F_i(x)F_{m+1-i}(x)\right)^{d-i}\cdot\prod_{i=d-1}^{2(d-1)}(F_{n+1}(x))^{\langle m+1-d\rangle}.$$

*Remark* 3. We recover the identities (1.2) and (1.4) by setting $x=1$ in the first identity and the last identity in Corollary 2 respectively.

*Remark* 4. We recover the results shown by Alfred [2] by setting $x=1, s=0, k=1, n=0$ and $d_i\equiv 0$, $e_i\equiv 1$ for all $1\le i,j\le m$ in the second identity in Corollary 2.

*Remark* 5. We get the identity (1.5) by setting $x=1$, $s=0$, $k=n=1$ in the third identity in Corollary 2.

If we set $p=2$, $q=a=c=1$ and $r=b=0$, then we get the sequence of Lucas polynomials in $\mathcal{P}(x)(2,1,0;1,0,1)=(L_n(x))_{n\in\mathbb{Z}}$ where the sequence $(L_n(x))_{n\in\mathbb{Z}}$ is defined by

$$L_0(x)\equiv 2,\quad L_1(x)\equiv x,\quad L_{n+2}(x)=xL_{n+1}(x)+L_n(x).$$

We recover the Lucas numbers by evaluating $L_n(x)$ at $x=1$. We note that $\Delta_{\mathcal{P}}=(-x^2-4)$ and $\mathcal{U}(x)(0,0,1;1,0,1)=(F_n(x))_{n\in\mathbb{Z}}$. By Theorem 2, Theorem 3, Theorem 4 and Theorem 5, we get the following corollary:

**Corollary 3.** *Let $m\ge 1$ and $d\ge 1$. Let $s,k,n$ be any integers. Let $(d_i)_{1\le i\le m}$ and $(e_i)_{1\le i\le m}$ be sequences of integers.*

$$\det([(L_{s+k(n+i+j)}(x))^m]_{0\le i,j\le m})$$
$$=(-1)^{(s+kn)\binom{m+1}{2}}(x^2+4)^{\binom{m+1}{2}}\prod_{i=0}^{m}\binom{m}{i}(F_{(i+1)k}(x))^{2(m-i)},$$

$$\det\left(\left[\prod_{f=j+1}^{m}L_{s+k(n+i+d_f)}(x)\prod_{g=1}^{j}L_{s+k(n+i+e_g)}(x)\right]_{0\le i,j\le m}\right)$$
$$=(-1)^{(s+kn)\binom{m+1}{2}+k\binom{m+1}{3}}(x^2+4)^{\binom{m+1}{2}}\prod_{l=1}^{m}(F_{kl}(x))^{m+1-l}\cdot$$
$$\cdot\prod_{1\le i\le j\le m}(-1)^{kd_j}F_{k(e_i-d_j)}(x),$$

$$\det\left(\left[\frac{1}{L_{s+k(n+i+j)}(x)}\right]_{0\le i,j\le m}\right)$$

$$= \frac{(-1)^{(s+kn+1)\binom{m+1}{2}}(x^2+4)^{\binom{m+1}{2}}}{\prod_{0\le i,j\le m} L_{s+k(n+i+j)}(x)} \prod_{i=0}^{m} (F_{k(i+1)}(x))^{2(m-i)}$$

$$\det([(L_{n+i+j}(x))^{\langle m\rangle}]_{0\le i,j\le d-1})$$

$$= (-1)^{(n+1)\binom{d}{2}+\binom{d+1}{3}}(x^2+4)^{\binom{d}{2}} \cdot \prod_{i=1}^{d-1} \left(F_i(x)F_{m+1-i}(x)\right)^{d-i} \cdot$$

$$\cdot \prod_{i=d-1}^{2(d-1)} (L_{n+1}(x))^{\langle m+1-d\rangle}.$$

If we set $p=q=1$, $a=2$, $c=-1$ and $r=b=0$, then we get the sequence of Chebyshev polynomials of the first kind in $\mathcal{P}(x)(1,1,0;2,0,-1)=(T_n(x))_{n\in\mathbb{Z}}$ where the sequence $(T_n(x))_{n\in\mathbb{Z}}$ is defined by

$$T_0(x)\equiv 1, \quad T_1(x)\equiv x, \quad T_{n+2}(x)=2xT_{n+1}(x)-T_n(x).$$

We note that $\Delta_{\mathcal{P}}=(-x^2+1)$.

If we set $p=1$, $a=q=2$, $c=-1$ and $r=b=0$, then we get the sequence of Chebyshev polynomials of the second kind in $\mathcal{P}(x)(1,2,0;2,0,-1)=(S_n(x))_{n\in\mathbb{Z}}$ where the sequence $(S_n(x))_{n\in\mathbb{Z}}$ is defined by

$$S_0(x)\equiv 1, \quad S_1(x)\equiv 2x, \quad S_{n+2}(x)=2xS_{n+1}(x)-S_n(x).$$

We note that $\Delta_{\mathcal{P}}=(-2x^2+1)$.

We note that

$$\mathcal{U}(x)(0,0,1;2,0,-1)=(U_n(x))_{n\in\mathbb{Z}}$$

where

$$U_n(x)=S_{n-1}(x) \text{ for } n\in\mathbb{Z}.$$

We get two corollaries by Theorem 2, Theorem 3, Theorem 4 and Theorem 5.

**Corollary 4.** *Let $m\ge 1$ and $d\ge 1$. Let $s,k,n$ be any integers. Let $(d_i)_{1\le i\le m}$ and $(e_i)_{1\le i\le m}$ be sequences of integers. Then*

$$\det([(T_{s+k(n+i+j)}(x))^m]_{0\le i,j\le m}) = (x^2-1)^{\binom{m+1}{2}} \prod_{i=0}^{m} \binom{m}{i} (S_{(i+1)k-1}(x))^{2(m-i)},$$

$$\det\left(\left[\prod_{f=j+1}^{m} T_{s+k(n+i+d_f)}(x) \prod_{g=1}^{j} T_{s+k(n+i+e_g)}(x)\right]_{0\le i,j\le m}\right)$$

$$= (x^2-1)^{\binom{m+1}{2}} \prod_{l=1}^{m} (S_{kl-1}(x))^{m+1-l} \prod_{1\le i\le j\le m} S_{k(e_i-d_j)-1}(x),$$

$$\det\Big(\Big[\frac{1}{T_{s+k(n+i+j)}(x)}\Big]_{0\le i,j\le m}\Big) = \frac{(-x^2+1)^{\binom{m+1}{2}}\prod_{i=0}^{m}(S_{k(i+1)-1}(x))^{2(m-i)}}{\prod_{0\le i,j\le m}T_{s+k(n+i+j)}(x)},$$

$$\det([(T_{n+i+j}(x))^{\langle m\rangle}]_{0\le i,j\le d-1})$$
$$= (-1)^{d\binom{d}{2}+\binom{d+1}{3}}(-x^2+1)^{\binom{d}{2}}\prod_{i=1}^{d-1}\big(S_{i-1}(x)S_{m-i}(x)\big)^{d-i}\cdot\prod_{i=d-1}^{2(d-1)}(T_{n+1}(x))^{\langle m+1-d\rangle}.$$

**Corollary 5.** *Let $m\ge 1$ and $d\ge 1$. Let $s,k,n$ be any integers. Let $(d_i)_{1\le i\le m}$ and $(e_i)_{1\le i\le m}$ be sequences of integers. Then*

$$\det([(S_{s+k(n+i+j)}(x))^m]_{0\le i,j\le m}) = (2x^2-1)^{\binom{m+1}{2}}\cdot\prod_{i=0}^{m}\binom{m}{i}(S_{(i+1)k-1}(x))^{2(m-i)},$$

$$\det\Big(\Big[\prod_{f=j+1}^{m}S_{s+k(n+i+d_f)}(x)\prod_{g=1}^{j}S_{s+k(n+i+e_g)}(x)\Big]_{0\le i,j\le m}\Big)$$
$$= (2x^2-1)^{\binom{m+1}{2}}\prod_{l=1}^{m}(S_{kl-1}(x))^{m+1-l}\prod_{1\le i\le j\le m}S_{k(e_i-d_j)-1}(x),$$

$$\det\Big(\Big[\frac{1}{S_{s+k(n+i+j)}(x)}\Big]_{0\le i,j\le m}\Big) = \frac{(-2x^2+1)^{\binom{m+1}{2}}\prod_{i=0}^{m}(S_{k(i+1)-1}(x))^{2(m-i)}}{\prod_{0\le i,j\le m}S_{s+k(n+i+j)}(x)},$$

$$\det([(S_{n+i+j}(x))^{\langle m\rangle}]_{0\le i,j\le d-1})$$
$$= (-1)^{d\binom{d}{2}+\binom{d+1}{3}}(-2x^2+1)^{\binom{d}{2}}\prod_{i=1}^{d-1}\big(S_{i-1}(x)S_{m-i}(x)\big)^{d-i}\cdot\prod_{i=d-1}^{2(d-1)}(S_{n+1}(x))^{\langle m+1-d\rangle}.$$

By Favard's theorem [6] (see also the standard reference textbook by Chihara [5, Chapter 2]), the sequence $\mathcal{P}(x)(1,q,r;1,b,c) = (P_n)_{n\in\mathbb{Z}_{\ge 0}}$ forms a sequence of orthogonal polynomials (with respect to certain linear functional) for $q\ne 0$ and $c\ne 0$. By Theorem 2, Theorem 3, Theorem 4 and Theorem 5, we state some determinant identities for matrices containing (powers of) such orthogonal polynomials.

**Corollary 6.** *Let $n\ge 0$, $m\ge 1$ and $d\ge 1$. Let $(P_n)_{n\in\mathbb{Z}_{\ge 0}}$ be a sequence of orthogonal polynomials of the form:*

$$P_0\equiv 1,\quad P_1\equiv qx+r,\quad P_{n+2}=(x+b)P_{n+1}+cP_n$$

*where $c\ne 0$, $q\ne 0$ and $r,b$ are any real numbers. Then*

$$\det([P^m_{n+i+j}]_{0\le i,j\le m})$$
$$= (-1)^{(n+1)\binom{m+1}{2}}\cdot\Delta^{\binom{m+1}{2}}\cdot c^{n\binom{m+1}{2}+2\binom{m+1}{3}}\cdot\prod_{i=0}^{m}\Big(\binom{m}{i}U_{i+1}^{2(m-i)}\Big),$$

$$\det\Big(\Big[\prod_{f=j+1}^{m} P_{n+i+d_f}\prod_{g=1}^{j} P_{n+i+e_g}\Big]_{0\le i,j\le m}\Big)$$
$$=(-\Delta)^{\binom{m+1}{2}}(-c)^{n\binom{m+1}{2}+\binom{m+1}{3}}\prod_{l=1}^{m}U_l^{m+1-l}\cdot\prod_{1\le i\le j\le m}(-c)^{d_j}U_{e_i-d_j},$$
$$\det\Big(\Big[\frac{1}{P_{s+k(n+i+j)}}\Big]_{0\le i,j\le m}\Big)$$
$$=\frac{(-1)^{(s+kn)\binom{m+1}{2}}c^{(s+kn)\binom{m+1}{2}+2k\binom{m+1}{3}}\Delta^{\binom{m+1}{2}}\prod_{i=0}^{m}U_{k(i+1)}^{2(m-i)}}{\prod_{0\le i,j\le m}P_{s+k(n+i+j)}},$$
$$\det([P_{n+i+j}^{\langle m\rangle}]_{0\le i,j\le d-1})$$
$$=(-1)^{n\binom{d}{2}+\binom{d+1}{3}}c^{(n+d-2)\binom{d}{2}}\Delta^{\binom{d}{2}}\cdot\prod_{i=1}^{d-1}\big(U_iU_{m+1-i}\big)^{d-i}\cdot\prod_{i=d-1}^{2(d-1)}P_{n+1}^{\langle m+1-d\rangle}.$$

*where* $\Delta=(q^2-q)x^2+(qr-r-bq)x+(r^2-br-c)$ *and* $(U_n)_{n\in\mathbb{Z}_{\ge 0}}$ *is the sequence of orthogonal polynomials defined by*

$$U_0\equiv 0,\quad U_1\equiv 1,\quad U_{n+2}=(x+b)U_{n+1}+cU_n.$$

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# NUMERICAL SOLUTION OF LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEMS FOR SWITCHING SYSTEMS

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*Abstract.* In this paper we discuss the approach for optimal switching control problem with unknown switching points. The case with unknown switching point is more general and generalizes the results existing in the literature. By using suitable transformation, the main problem is reduced into a problem with known interval and further the unknown boundary of the integral in the minimization functional is reduced to the known one. This fact is illustrated by an example. The reduced problem is solved numerically by using the Gradient Projection Method Algorithm.

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*Keywords:* optimal control, switching system, numerical solution, finite approximation

## 1. Introduction

There are many articles dedicated to the Linear-Quadratic Optimal Control Problems (LQOCPs) for switching systems. These problems are particular kind of hybrid systems. Examples of switching systems can be found in the area of engineering, chemical processes, automotive systems and military services. The published results in the literature can be mainly classified into two categories; one is theoretical [3,4,6,7,9,10,16–19,22] and practical [2,8,11,13–15,20,21]. The very earliest result which is proved a maximum principe for hybrid system for autonomous switching system is in [20]. More theoretical results of the maximum principle are obtained by Picolli in [18] and Sussman in [19] which are correspondingly is called hybrid maximum principle and maximum principle for the hybrid system in the case of the minimization functional is non-smooth. In [6, 22] switching systems are investigated by using dynamical programming approach to derive Hamilton-Jacobi-Belmann equations. But there are some practical results for the switching optimal control problem which has significant applications to real-world problems. In [5] conceptual algorithms were given for general hybrid optimal control problems. In [13], for a class of discrete-time hybrid system an algorithm is given by using constrained differential programming approach by author. An application to power train control can be found in [1]. Some heuristically oriented methods have been reported in [12], which used algorithms pruning the search trees in discrete-time LQR (Linear-Quadratic Regulator)

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control of switched linear system. An efficient algorithm, called the Time-Optimal Switching (TOS) algorithm, is proposed for the time-optimal switching control of nonlinear systems with a single control output is considered by Kaya and Noakes in [8]. In [8], firstly, a switching control is found using the STC (Switching Time Computation) method to get from an initial point to a target point with a given number of switchings. Then by means of constrained optimization techniques, the cost being considered as a summation of the arc times, a minimum-time switching control solution is obtained.

The rest of this paper is organized as follows: The problem formulation and certain definitions are given in Section 2, the transformation for the given problem and related theorems are described in Section 3, "Gradient Projection Method Algorithm" for this problem is given in Section 4, numerical results on the example are given in Section 5. Finally, in Section 6 the conclusion of the paper is presented.

## 2. Problem formulation

In Kurina and Zhou [10], the authors studied the following minimizing optimal control problem:

*Problem I*: Minimizing the functional

$$J(u,t_1) = \frac{1}{2}\langle C_1x_1(t_1) - C_2x_2(t_1), F\,(C_1x_1(t_1) - C_2x_2(t_1))\rangle + \sum_{j=1}^{2}\int_{t_{j-1}}^{t_j} (\langle x_j(t), W_j(t)x_j(t)\rangle + \langle u_j(t), R_j(t)u_j(t)\rangle)dt \tag{2.1}$$

where, $u = (u_1, u_2)$, with respect to the trajectories of the system

$$\dot{x}_j(t) = A_j(t)x_j(t) + B_j(t)u_j(t), \quad t_{j-1} \le t \le t_j, \ j = 1,2 \tag{2.2}$$

with the following boundaries: $x_1(0) = x^0$, $x_2(T) = x^T$.

Here, $0 = t_0 < t_1 < t_2 = T$, the values $t_0, t_2$ are fixed, $t_1$ is not fixed, $x_j(t) \in X_j, u_j(t) \in U_j, A_j(t), W_j \in L(X_j), B_j(t) \in L(U_j, X_j), R_j(t) \in L(U_j)$ for all $t \in [t_{j-1}, t_j], j = 1,2; C_1 \in L(X_1, Y), C_2 \in L(X_2, Y), F \in L(Y), X_j, U_j, Y$ are real finite dimensional Euclidean spaces, the operators $F, W_j(t) \ge 0, R_j(t) > 0$ for all $t \in [t_{j-1}, t_j]; x^0 \in X_1, x^T \in X_2$ are given and symmetric, the operators $F, C_1, C_2$ are independent of $t$, but the other operators depend continually on t in the corresponding segment $[t_{j-1}, t_j], j = 1,2, <.,.>$ means an inner product in the appropriate spaces.

*Remark* 1. In [10], it is assumed that the intermediate point $t_1$ is fixed. For this, the minimization functional has the form:

$$J(u) = \frac{1}{2}\langle C_1x_1(t_1) - C_2x_2(t_1), F\,(C_1x_1(t_1) - C_2x_2(t_1))\rangle$$

$$+\sum_{j=1}^{2}\int_{t_{j-1}}^{t_j}(\langle x_j(t),W_j(t)x_j(t)\rangle+\langle u_j(t),R_j(t)u_j(t)\rangle)dt \tag{2.3}$$

i,e., in [9, 10], the minimization functional is not depend from the switching point $t_1$, because $t_1$ is fixed. For this reason in the papers the minimizing functional is written in the form $J(u)$. But in the presented paper, we consider a more general case. It is considered that the point $t_1$ is unknown, the minimizing functional has the form as in (2.1), i.e., $J(u,t_1)$. Let us make the following substitution $u(t)=(u_1(t),u_2(t))$ and $x(t)=(x_1(t),x_2(t))$.

**Definition 1.** The triple $w=(t_1,u(t),x(t))$ is called admissible, if it satisfies all constraints of *Problem I* (about the constraints see [10]).

**Definition 2.** The triple $w^0=(t_1,u(t),x(t))$ is called optimal control, if $J(w^0)\leq J(w)$ for all admissible process $w$.

## 3. Transformation

Let us take following transformation. Assume a new parameter $x_{n+1}$ such us satisfies following differential equation with initial condition in the interval $[t_0,t_2]$ and $\frac{dx_{n+1}(t)}{dt}=0$ with initial condition $x_{n+1}(0)=t_1$. It means that $x_{n+1}$ is constant in $[t_0,t_2]$. Next, a new independent time variable $\tau$ is introduced as:

$$t=\begin{cases} t_0+(x_{n+1}-t_0)\tau, & 0\leq\tau<1\\ x_{n+1}+(t_2-x_{n+1})(\tau-1), & 1\leq\tau\leq 2\end{cases} \tag{3.1}$$

then we can write

$$dt=\begin{cases}(x_{n+1}-t_0)d\tau, & 0\leq\tau<1\\ (t_2-x_{n+1})d\tau, & 1\leq\tau\leq 2.\end{cases} \tag{3.2}$$

Clearly, (3.1) is a linear mapping with $t:\tau\to[t_0,t_1]$ when $\tau\in[0,1)$ and $t:\tau\to[t_1,t_2]$ when $\tau\in[1,2]$. In fact, $\tau=0$ corresponds to $t=t_0$, $\tau=1$ corresponds to $t=t_1$, and $\tau=2$ to $t=t_2$. By using relation (3.1) it is easy to introduce the inverse mapping $\tau=\frac{t-t_0}{x_{n+1}-t_0}$, for $0\leq\tau\leq 1$ and $\tau=\frac{t-x_{n+1}}{t_2-x_{n+1}}$, for $1\leq\tau\leq 2$. By introducing $x_{n+1}$, $\tau$ and certain substitutions $y_i(\tau)=x_i(t(\tau))$, $v_i(\tau)=u_i(t(\tau))$, $i=1,2$ and using relation (3.2) the main problem is transcribed into the following equivalent form.

*Problem II*:

$$subsystem(1):\begin{cases}\frac{dy_1(\tau)}{d\tau} &= (x_{n+1}-t_0)(A_1(\tau)y_1(\tau)+B_1(\tau)v_1(\tau))\\ \frac{dx_{n+1}}{d\tau} &= 0\\ x_{n+1}(0) &= t_1\end{cases} \tag{3.3}$$

in the interval $\tau \in [0,1)$ and

$$subsystem(2): \begin{cases} \frac{dy_2(\tau)}{d\tau} & = & (t_2 - x_{n+1})(A_2(\tau)y_2(\tau) + B_2(\tau)v_2(\tau)) \\ \frac{dx_{n+1}}{d\tau} & = & 0 \\ x_{n+1}(0) & = & t_1 \end{cases} \tag{3.4}$$

in the interval $\tau \in [1,2]$ and the minimizing functional takes the form

$$\begin{aligned} \tilde{J}(v, x_{n+1}) = & \frac{1}{2}\langle C_1 y_1(1) - C_2 y_2(1), F(C_1 y_1(1)) - C_2 y_2(1))\rangle \\ & + \int_0^1 (x_{n+1} - t_0)(\langle y_1(\tau), W_1(\tau) y_1(\tau)\rangle + \langle v_1(\tau), R_1(\tau) v_1(\tau)\rangle) d\tau \\ & + \int_1^2 (t_2 - x_{n+1})(\langle y_2(\tau), W_2(t) y_2(\tau)\rangle + \langle v_2(\tau), R_2(\tau) v_2(\tau)\rangle) d\tau. \end{aligned} \tag{3.5}$$

After this transformation we reduce *Problem I* to *Problem II*. In *Problem II*, the state trajectory is $y(\tau) = (y_1(\tau), y_2(\tau))$ and the control tuple is $v(\tau) = (v_1(\tau), v_2(\tau), x_{n+1})$, $0 \le \tau \le 2$.

Since $x_{n+1}$ is an unknown constant (parameter) in the interval $[0,2]$ (see (3.3) and (3.4)), after the transformation, the dimension of *Problem II* will be the same as the dimension of *Problem I*.

**Theorem 1.** *There is a one-to-one corresponding between the admissible process $(t_1, x(t), u(t))$ for Problem I and the admissible process $(y(\tau), v(\tau))$ for Problem II.*

*Proof.* By using transformation from the admissible process $(t_1, x(t), u(t))$, we obtained admissible process $(y(\tau), v(\tau))$. Let us prove inverse opinion; if $(y(\tau), v(\tau)$ is an admissible process (where $v(\tau) = (v_1(\tau), v_2(\tau))$ in problem (3.3)-(3.4), then by using relation (3.1) we can say, if we take $\tau = 0$ then $t = t_0, \tau = 1$ then $t = x_{n+1}$ (in fact $x_{n+1}(0) = t_1$), and for $\tau = 2$ then $t = t_2$. It means we obtained intervals $[t_0, t_1]$ and $[t_1, t_2]$. From relation (3.1), we have $\tau = \frac{t - t_0}{x_{n+1} - t_0}$, $0 \le \tau \le 1$ and $\tau = \frac{t - x_{n+1}}{t_2 - x_{n+1}}$, $1 \le \tau \le 2$. Then, introducing the notions $x_1(t) = y_1(\tau(t))$ and $x_2(t) = y_2(\tau(t))$ we obtain $\dot{x}_1 = \dot{y}_1(\tau(t))(\frac{1}{x_{n+1} - t_0})$ and $\dot{x}_2 = \dot{y}_2(\tau(t))(\frac{1}{t_2 - x_{n+1}})$ by using the chain rule. If we consider this in (3.3) and (3.4), we can come to the point that $(t_1, x(t), u(t))$ is the admissible process for the equations (2.1) and (2.2). □

**Theorem 2.** *This corresponding mapping between the admissible processes $(t_1, x(t), u(t))$ and $(y(t), v(t))$ for the equations (2.2), (3.3) and (3.4) preserves the value of the cost functionals (2.1) and (3.5).*

*Proof.* In fact, assume that process $(t_1^0, x^0(t), u^0(t))$ is an optimal control for *Problem I*. Let us take process $(y^0(\tau), v^0(\tau))$, which is obtained from the optimal

process $(t_1^0, x^0(t), u^0(t))$ of the above mentioned transformation. Assume that $(y^0(\tau), v^0(\tau))$, is not an optimal process and there exists another optimal process $(\tilde{y}(\tau), \tilde{v}(\tau))$ with $\tilde{J}(\tilde{y}(\tau), \tilde{v}(\tau)) \leq J(y^0(\tau), v^0(\tau))$. Take the corresponding admissible process, which is obtained by the inverse transformation from the process $(\tilde{x}_{n+1}, \tilde{y}(\tau), \tilde{v}(\tau))$ and denote it by $(t_1, u(t), x(t))$. Then, it is clear that the cost $J(t_1, u(t), x(t)) = \tilde{J}(\tilde{y}(\tau), \tilde{v}(\tau)) \leq \tilde{J}(y^0(\tau), v^0(\tau) = \tilde{J}(t_1^0, x^0(t), u^0(t))$. But it contradicts to the optimality of the process $(t_1^0, x^0(t), u^0(t))$ in Definition 2. The inverse opinion can be proved in the same way. □

Using the theorems, it is straightforward to affirm the following Corollary.

**Corollary 1.** *If the process $(t_1^0, x^0(t), u^0(t))$ gives minimum for Problem I, then the process $(y^0(\tau), v^0(\tau))$, which is obtained after transformation, gives minimum value for Problem II, and vice versa.*

## 4. Gradient projection method algorithm

We have three optimized arguments: First one is the scalar argument $t_1 \in [t_0, t_f]$, the second one is a first control function $v_1(t)$ for $t \in [t_0, t_{mid}]$ and the last one is a second control function $v_2(t)$, for $t \in [t_{mid}, t_f]$. That is $x = (t_1, v_1(t), v_2(t))$ with the cost function $J(t_1, v_1(t), v_2(t))$ and with the only constraint put on $t_1 : t_0 \leq t_1 \leq t_f$.

In the present form, the above admissible process arguments represent an infinite-dimensional optimization problem. By applying the "parametrization technique", we can reduce the initial infinite-dimensional optimization problem to a finite-dimensional optimization problem. The usefulness of this procedure is that for solution to a finite-dimensional optimization problem there exists a sufficiently powerful arsenal of methods and algorithms.

To convert the problem into a finite-dimensional optimization problem we apply the following parametrization technique: Let's partition the sections $[t_0, t_{mid}]$ and $[t_{mid}, t_f]$ into finite number of sub-segments:

$$[t_0, t_{mid}] = \bigcup_{i=1}^{N} [a_i, b_i) \text{and} [t_{mid}, t_f] = \bigcup_{j=1}^{M} [c_j, d_j).$$

Instead of the functions $v_1(t)$ and $v_2(t)$ we consider their piecewise constant approximations:

$v_1(t) = u_1^i =$ constant, if $t \in [a_i, b_i)$, $i = 1, 2, ..., N$;

$v_2(t) = u_2^j =$ constant, if $t \in [c_j, d_j)$, $j = 1, 2, ..., M$;

Thus, instead of the admissible process arguments we obtain a finite-dimensional optimization problem:

$t_1$, $u_1^i$, $u_2^i$ with the cost function: $J(t_1; u_1^1, u_1^2, ..., u_1^N; u_2^1, u_2^2, ..., u_2^M)$.

To solve the above finite-dimensional optimization problem we propose to use first-order optimization techniques, i.e. gradient-based methods, e.g. gradient projection procedure. Here are the steps of this procedure:

1) As an initial guess we choose some values for the optimized arguments of the cost function:

$x^0 = (t_1^0, u_1^{1^0}, u_1^{2^0}, ..., u_1^{N^0}; u_2^{1^0}, u_2^{2^0}, ..., u_2^{M^0})$ so that the constraint is satisfied.

2) Then the considered procedure is an ordinary gradient method

$$x^{k+1} = x^k - \alpha_k . \nabla f(x_k), \tag{4.1}$$

where $\nabla f(x_k)$ is the gradient of the cost functional at the point $x_k$ ; $\alpha_k$ is the step in the direction of the anti-gradient.

3) If after completing the next iteration of (4.1) we trespass the allowable boundaries for the argument $x_1^{k+1}$, which in our case is $t_1^{k+1}$, we put it back into $[t_0, t_f]$ according to the following formula:

$$t_1^{k+1} = \begin{cases} 0, & t_1^{k+1} < 0 \\ 2, & t_1^{k+1} > 2 \end{cases}$$

4) We repeat steps 2-3 for new $k := k+1$ until some exit criterion is satisfied. Possible exit criterions:

- $\|\nabla f(x_k)\| \le \epsilon_1$ • $|x^{k+1} - x^k| < \epsilon_3$ • $|f(x^{k+1}) - f(x^k)| < \epsilon_2$

## 5. Example

In this paper, inspired by [9], we consider the switching point $t_1$ as non fixed. Then, we will try to reduce the unknown switching case to the known switching case, after which all the procedure in [10] can be used. Consider the following problem of minimizing the functional,

$$\begin{aligned} J(x, u_1, u_2, t_1) = \frac{1}{2}[(x_{11}(t_1) + x_{21}(t_1))^2 + \int_0^{t_1} (x_{11}^2(t) + 2x_{11}(t)x_{12}(t) \\ + 3x_{12}^2(t) + u_1^2(t))dt \ + \int_{t_1}^{2} (x_{21}^2(t) + 8x_{22}^2(t) \\ + u_2^2(t))dt] \end{aligned} \tag{5.1}$$

with respect the trajectories of the systems

$$subsystem(1): \begin{cases} \dot{x}_{11}(t) - x_{11}(t) & = & 0 \\ x_{12}(t) + u_1(t) & = & 0 \quad \text{for } t \in [0, t_1) \\ x_{11}(0) & = & -1, \end{cases} \tag{5.2}$$

$$subsystem(2): \begin{cases} \dot{x}_{21}(t) & = & 0 \\ x_{22}(t) - u_2(t) & = & 0 \quad \text{for } t \in [t_1, 2] \\ x_{21}(2) & = & 1. \end{cases} \tag{5.3}$$

We will use transformation (3.1) which is reduced problem (5.2), (5.3) to the new problem without unknown switching point. For this aim, take new variable $\dot{x}_{n+1}(t) = 0,\ x_{n+1}(0) = t_1$. From this differential equation, it is clear $x_{n+1} = t_1$ is unknown constant in $[0,2]$. Take also the state trajectories $y_{i,j}(\tau) = x_{i,j}(t(\tau))$, and controls $v_i(\tau) = u_i(t(\tau))$ where $i,j = 1,2$. Let us also use interval transformation in (3.1) with $t_0 = 0$ and $t_2 = 2$. Then we can come the point that, if $\tau = 0$ then $t = 0$, if $\tau = 1$ then $t = x_{n+1} = t_1$, and, if $\tau = 2$ then $t = 2$. If we use all these transformations, then the minimizing functional and the state equations will take the following form:

$$J(v) = \frac{1}{2}[(y_{11}(1) + y_{21}(1))^2 + t_1 \int_0^1 (y_{11}^2(\tau) + 2y_{11}(\tau)y_{21}(\tau) + 3y_{12}^2 + v_1^2(\tau))d\tau + (2-t_1)\int_1^2 (y_{21}^2(\tau) + 8y_{22}^2(\tau) + v_2^2(\tau))d\tau] \tag{5.4}$$

where, $v = (v_1, v_2)$, and state equations takes the form

$$subsystem(1): \begin{cases} \dot{y}_{11}(t) - t_1 y_{11}(t) &= 0 \\ y_{12}(t) + v_1(t) &= 0 \quad \text{for } \tau \in [0,1) \\ y_{11}(0) &= -1, \end{cases} \tag{5.5}$$

$$subsystem(2): \begin{cases} \dot{y}_{21}(t) &= 0 \\ y_{22}(t) - v_2(t) &= 0 \quad \text{for } \tau \in [1,2] \\ y_{21}(2) &= 1. \end{cases} \tag{5.6}$$

If we solve (5.5) with respect to the states $y_{11}(t)$ and $y_{12}(t)$ and handle (5.6) with respect to the states $y_{21}(t)$ and $y_{22}(t)$, then putting these in (5.4), the functional cost gets the form:

$$J(t_1, v_1, v_2) = \frac{1}{2}[(1-\exp(t_1))^2 + t_1 \int_0^1 (\exp(2t_1\tau) + 2\exp(t_1\tau)v_1(\tau) + 4v_1^2(\tau))d\tau + (2-t_1)\int_1^2 (1 + 9v_2^2(\tau))d\tau]. \tag{5.7}$$

To solve (5.7) by finite-optimization techniques first we transform the functional into finite-dimensional problem as follows:

$$J(t_1, w_1, w_2) = \frac{1}{2}[(1-\exp(t_1))^2 + t_1 \sum_{i=1}^{N} \int_0^1 (\exp(2t_1\tau) + 2\exp(t_1\tau)w_1^i(\tau) + 4(w_1^i)^2(\tau))d\tau + (2-t_1)\sum_{j=1}^{M} \int_1^2 (1 + 9(w_2^j)^2(\tau))d\tau] \tag{5.8}$$

where, $v_1(t) = w_1^i =$ constant, if $t \in [0,1)$; $v_2(t) = w_2^j =$ constant, if $t \in [1,2]$. Then, by using "Gradient Projection Method" we can obtain the following optimal

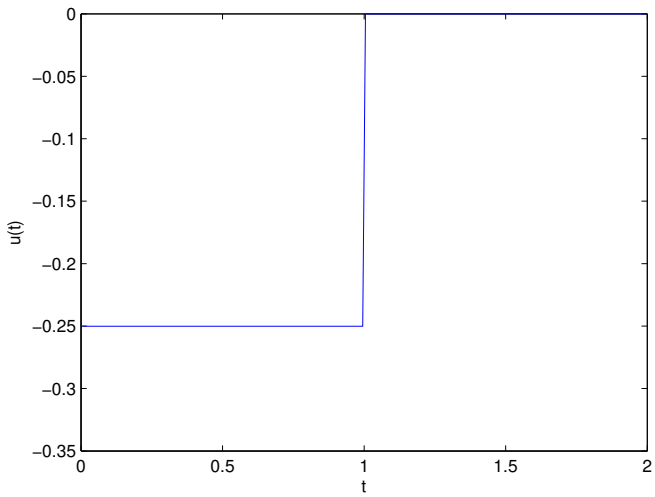


FIGURE 1. Optimal Control Input

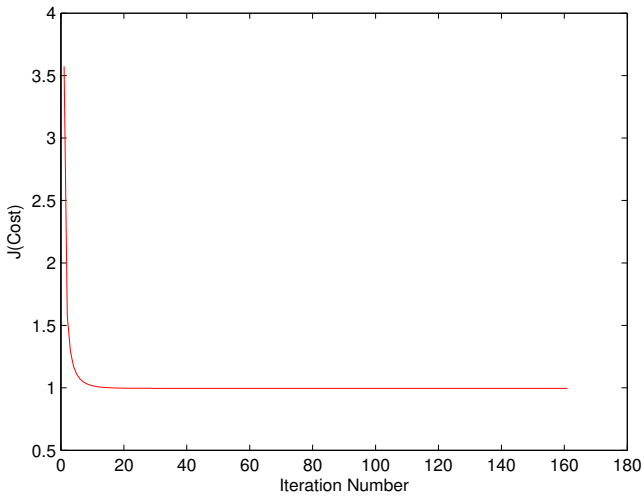


FIGURE 2. Optimal Cost

control input and state variable histories numerically (Figures 1 and 2). By applying Gradient Algorithm for the initial nominal $t_1 = 1.0$, after 160 iterations we find that the optimal switching time $t_1^* = 0.0653$ and the optimal cost $J^* = 0.9958$. The

computation takes about 0.7387 seconds of CPU time using C Sharp as programming language on an Intel(R)Core(TM)i7-3720QM 2.60 GHz PC with 8GB of RAM.

## 6. Conclusion

In this paper we obtained the approach for optimal switching control problem with unknown switching points which described in [9, 10]. At that case, switching point $t_1$ admitted as unknown and unfixed point in the known interval for state equations and unknown boundary of the integral. Moreover, the cost functional components was transcribed by the linear transformation and the system was solved by using Gradient Projection Method numerically.

It is also possible to say for the future works that if there are $K$ numbers of switchings, then it is no difficulty in applying the previous method to the problems with several subsystems. If there exist non fixed switchings, $t_0, t_1, t_2, ..., t_K$ and $T = 0$ with $0 = t_0 < t_1 < t_2 <, ..., < t_K < T = 0$, then we can transcribe the problem into an equivalent problem by introducing $K$ new state variables $x_{n+1}, x_{n+2}, ..., x_{n+K}$ which correspond to the switching instants $t_1, t_2, ..., t_K$ and satisfy the following equations: $\frac{dx_{n+i}}{d\tau} = 0, \quad x_{n+i}(0) = t_i, \quad \tau \in [1,2], \quad i = 1, 2, ..., K.$

The new independent time variable $\tau$ has a linear relationships with $t$ where $\tau = 0$ corresponds to $t = t_0, \quad \tau = 1$ corresponds to $t = t_1, ...,$ $\tau = K + 1$ corresponds to $t = t_T$.

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# ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SECOND ORDER DIFFERENCE EQUATIONS WITH DEVIATING ARGUMENT

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*Abstract.* We consider nonlinear second order difference equations with deviating argument of the form

$$\Delta(r_n \Delta x_n) = a_n f(x_{n+\sigma-1}, x_{n+\sigma-2}, \dots, x_{n+\sigma-m}) + b_n.$$

We present sufficient conditions for the existence of solutions with prescribed asymptotic behavior. Moreover, we study the asymptotic behavior of solutions. We use $o(n^s)$, for a given nonpositive real $s$, as a measure of approximation.

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*Keywords:* difference equation, prescribed asymptotic behavior, bounded solution, convergent solution, quasidifference, deviating argument

## 1. Introduction

In this paper we consider the nonlinear second order difference equation with deviating argument of the form

$$\Delta(r_n \Delta x_n) = a_n f(x_{n+\sigma-1}, x_{n+\sigma-2}, \dots, x_{n+\sigma-m}) + b_n \tag{E}$$

$$r_n, a_n, b_n \in \mathbb{R}, \quad r_n > 0, \quad \sigma \in \mathbb{Z}, \quad m \in \mathbb{N}, \quad f : \mathbb{R}^m \to \mathbb{R}.$$

Here $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ denote the set of positive integers, the set of all integers, and the set of real numbers, respectively. By a solution of (E) we mean a sequence $x : \mathbb{N} \to \mathbb{R}$ satisfying (E) for large $n$.

An important issue in the asymptotic theory of ordinary and delay differential equations is constructing sufficient conditions which ensure the existence of solutions with prescribed asymptotic behavior. From this point of view many authors studied second order differential equations with deviating argument of the form

$$\left(rx'\right)'(t) = a(t) f(x(\sigma_0(t))) \tag{1.1}$$

or

$$x''(t) = f(t, x(\sigma_1(t)), \dots, x(\sigma_m(t))) \tag{1.2}$$

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where $r : \mathbb{R} \to (0, +\infty)$, $\sigma_i(t) \to \infty$ as $t \to \infty$ for $i = 0, \dots, m$, see e.g. [1], [2], [3], [11], and the references therein. Since difference equations can be treated as discretization of differential equations, the existence of solutions with prescribed asymptotic behavior of difference equations was studied in literature too, see for example, in [4], [12], [13] and the references therein.

In papers [4–9], the first author presented a new theory of the study of asymptotic properties of solutions of difference equations of the form

$$\Delta^2 x_n = a_n f(x_{\sigma(n)}) + b_n$$

in which $\mathrm{o}(n^s)$, for $s \le 0$, is used a measure of approximation. In the paper [10], we extend some of results on difference equations of Sturm-Liouville type of the form

$$\Delta(r_n \Delta x_n) = a_n f(x_{\sigma(n)}) + b_n.$$

This paper is a continuation of these investigations. Our main goal is to present sufficient conditions for the existence of a solution $x$ of equation (E) such that

$$x_n = c(r_1^{-1} + \cdots + r_{n-1}^{-1}) + d + \mathrm{o}(n^s), \tag{1.3}$$

where $c, d \in \mathbb{R}$ and $s \in (-\infty, 0]$. We give also sufficient conditions for a given solution $x$ of equation (E) to have an asymptotic property (1.3).

The paper is organized as follows. In Section 2, we introduce notation and present some preliminary lemmas. Next, Section 3 is devoted to our first main result Theorem 1, some consequences of it and the example which proves that one of assumptions in main theorem is essential. In Section 4 we prove our second main result and some corollaries from it. Moreover, this section includes the example which proves that one of assumptions of main theorem is not "too" strong.

## 2. Preliminaries

The space of all sequences $x : \mathbb{N} \to \mathbb{R}$ we denote by $\mathbb{R}^{\mathbb{N}}$. If $x, y$ in $\mathbb{R}^{\mathbb{N}}$, then $xy$ and $|x|$ denotes the sequences defined by $(xy)_n = x_n y_n$ and $|x|_n = |x_n|$, respectively. Moreover,

$$\|x\| = \sup\{|x_n| : n \in \mathbb{N}\}.$$

For any sequence $x \in \mathbb{R}^{\mathbb{N}}$ we define a sequence $x^* : \mathbb{N} \to \mathbb{R}^m$ by

$$x_n^* = \begin{cases} (0, 0, \dots, 0) & \text{for } n < m - \sigma + 1 \\ (x_{n+\sigma-1}, x_{n+\sigma-2}, \dots, x_{n+\sigma-m}) & \text{for } n \ge m - \sigma + 1. \end{cases}$$

We use the symbol $d_m$ to denote the max-metric on $\mathbb{R}^m$ defined by

$$d_m(u, v) = \max\{|u_1 - v_1|, \dots, |u_m - v_m|\}.$$

Moreover, $\overline{\mathrm{B}}(u, \alpha)$ denotes the closed ball of radius $\alpha$ centered at a point $u \in \mathbb{R}^m$. We say that a function $g : \mathbb{R}^m \to \mathbb{R}$ is bounded at infinity if there exists a real number $\lambda$ such that $g$ is bounded on the set

$$[\lambda, \infty) \times \cdots \times [\lambda, \infty) = [\lambda, \infty)^m.$$

In the same way, the boundedness at minus infinity can be defined.

**Lemma 1.** *If $y \in \mathbb{R}^{\mathbb{N}}$ and $\Delta(r_n \Delta y_n) = 0$, then there exist real constants $c, d$ such that*

$$y_n = c \sum_{j=1}^{n-1} \frac{1}{r_j} + d \tag{2.1}$$

*for any $n$. If $c, d \in \mathbb{R}$ and $y \in \mathbb{R}^{\mathbb{N}}$ is defined by (2.1), then $\Delta(r_n \Delta y_n) = 0$.*

*Proof.* We leave an easy proof of this lemma to the reader. □

**Lemma 2** ([10, Lemma 3])**.** *Assume $s \in (-\infty, 0]$, $t \in [s, \infty)$, $r_n^{-1} = \mathrm{O}(n^t)$, $a \in \mathbb{R}^{\mathbb{N}}$, and*

$$\sum_{n=1}^{\infty} n^{1+t-s} |a_n| < \infty, \quad \textit{then} \quad \sum_{n=1}^{\infty} \frac{1}{n^s r_n} \sum_{j=n}^{\infty} |a_j| < \infty.$$

**Lemma 3.** *Assume $s \in (-\infty, 0]$, $t \in [s, \infty)$, $r_n^{-1} = \mathrm{O}(n^t)$, $a \in \mathbb{R}^{\mathbb{N}}$, and at least one of the following conditions is satisfied*

$$(a)\ \liminf_{n\to\infty} n \left( \frac{|a_n|}{|a_{n+1}|} - 1 \right) > 2 + t - s, \qquad (b)\ \liminf_{n\to\infty} n \log \frac{|a_n|}{|a_{n+1}|} > 2 + t - s.$$

*Then*

$$\sum_{n=1}^{\infty} \frac{1}{n^s r_n} \sum_{j=n}^{\infty} |a_j| < \infty.$$

*Proof.* Using [7, Lemma 6.3] in case (a), or [7, Lemma 6.4] in case (b) we obtain

$$\sum_{n=1}^{\infty} n^{1+t-s} |a_n| < \infty.$$

By Lemma 2 we get the result. □

**Lemma 4.** *Assume $\lambda \in \mathbb{R}$, $r_n^{-1} = \mathrm{O}(n^\lambda)$, $a \in \mathbb{R}^{\mathbb{N}}$, and at least one of the following conditions is satisfied*

$$(a)\ \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1, \qquad (b)\ \limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1.$$

*Then*

$$\sum_{n=1}^{\infty} \frac{1}{n^s r_n} \sum_{j=n}^{\infty} |a_j| < \infty$$

*for any $s \in (-\infty, 0]$.*

*Proof.* Let $s \in (-\infty, 0]$. Choose a real number $t$ such that $t > \max(s, \lambda)$. Then $r_n^{-1} = \mathrm{O}(n^t)$ and, using the ratio test in case (a), or the root test in case (b) we get

$$\sum_{n=1}^{\infty} n^{1+t-s} |a_n| < \infty.$$

Hence the assertion is a consequence of Lemma 2. □

**Lemma 5** ([10, Lemma 5])**.** *If*

$$\sum_{k=1}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} |u_i| < \infty,$$

*then*

$$\sum_{k=1}^{\infty} |u_k| \sum_{i=1}^{k} \frac{1}{r_i} < \infty \quad \text{and} \quad \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{i=k}^{\infty} |u_i| \le \sum_{k=n}^{\infty} |u_k| \sum_{i=1}^{k} \frac{1}{r_i}$$

*for any* $n \in \mathbb{N}$.

**Lemma 6** ([4, Lemma 4.7])**.** *Assume* $y, \rho : \mathbb{N} \to \mathbb{R}$, *and* $\lim_{n\to\infty} \rho_n = 0$. *In the set* $X = \{x \in \mathbb{R}^{\mathbb{N}} : |x - y| \le |\rho|\}$ *we define a metric by the formula*

$$d(x, z) = \|x - z\|. \tag{2.2}$$

*Then any continuous map* $H : X \to X$ *has a fixed point.*

## 3. Solutions with prescribed asymptotic behavior

In this section we establish various conditions under which for a given solution $y$ of the equation $\Delta(r_n \Delta y_n) = 0$ and a given nonpositive real $s$ there exists a solution $x$ of (E) such that $x_n = y_n + \mathrm{o}(n^s)$.

**Theorem 1.** *Assume* $s \in (-\infty, 0]$, $y$ *is a solution of the equation* $\Delta(r_n \Delta y_n) = 0$,

$$\sum_{k=1}^{\infty} \frac{1}{k^s r_k} \sum_{j=k}^{\infty} (|a_j| + |b_j|) < \infty, \quad q \in \mathbb{N}, \quad \alpha \in (0, \infty), \quad U = \bigcup_{n=q}^{\infty} \overline{\mathrm{B}}(y_n^*, \alpha),$$

*and* $f$ *is continuous and bounded on* $U$. *Then there exists a solution* $x$ *of* (E) *such that* $x_n = y_n + \mathrm{o}(n^s)$.

*Proof.* For $n \in \mathbb{N}$ and $x \in \mathbb{R}^{\mathbb{N}}$ let

$$F(x)(n) = a_n f(x_n^*) + b_n. \tag{3.1}$$

There exists $L > 0$, such that

$$|f(u)| \le L \tag{3.2}$$

for any $u \in U$. Since $s \le 0$ we have

$$\sum_{k=1}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} (L|a_j| + |b_j|) < \infty.$$

Let

$$Y = \{x \in \mathbb{R}^{\mathbb{N}} : |x - y| \le \alpha\}, \quad \rho \in \mathbb{R}^{\mathbb{N}}, \quad \rho_n = \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} (L|a_j| + |b_j|).$$

If $x \in Y$, then $x_n^* \in U$ for large $n$. Hence the sequence $(f(x_n^*))$ is bounded for any $x \in Y$. Define sequences $w, g$ by

$$w_j = L|a_j| + |b_j|, \qquad g_n = \sum_{k=n}^{\infty} \frac{1}{k^s r_k} \sum_{j=k}^{\infty} w_j.$$

Then

$$n^{-s} \rho_n = n^{-s} \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} w_j = \sum_{k=n}^{\infty} \frac{1}{n^s r_k} \sum_{j=k}^{\infty} w_j \le \sum_{k=n}^{\infty} \frac{1}{k^s r_k} \sum_{j=k}^{\infty} w_j = g_n. \tag{3.3}$$

Using the assumption

$$\sum_{k=1}^{\infty} \frac{1}{k^s r_k} \sum_{j=k}^{\infty} (|a_j| + |b_j|) < \infty$$

we get $g_n = \mathrm{o}(1)$. Hence, by (3.3),

$$\rho_n = n^s g_n = n^s \mathrm{o}(1) = \mathrm{o}(n^s).$$

Therefore there exists an index $p$ such that

$$\rho_n \le \alpha \quad \text{and} \quad n + \sigma - m \ge q \quad \text{for} \quad n \ge p.$$

Let

$$X = \{x \in \mathbb{R}^{\mathbb{N}} : |x - y| \le \rho \text{ and } x_n = y_n \text{ for } n < p\},$$

$$H : Y \to \mathbb{R}^{\mathbb{N}}, \qquad H(x)(n) = \begin{cases} y_n & \text{for } n < p \\ y_n + \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} F(x)(j) & \text{for } n \ge p. \end{cases}$$

Note that $X \subset Y$. If $x \in X$, then for $n \ge p$ we have

$$|H(x)(n) - y_n| = \left| \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} F(x)(j) \right| \le \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} |F(x)(j)| \le \rho_n.$$

Therefore $HX \subset X$. Let $x \in X$, and $\varepsilon > 0$. By Lemma 5, we have

$$\sum_{k=1}^{\infty} |a_k| \sum_{i=1}^{k} \frac{1}{r_i} < \infty.$$

Choose an index $q \geq p$ and a positive constant $\gamma$ such that

$$L \sum_{k=q}^{\infty} |a_k| \sum_{i=1}^{k} \frac{1}{r_i} < \varepsilon \quad \text{and} \quad \gamma \sum_{k=p}^{q} |a_k| \sum_{i=1}^{k} \frac{1}{r_i} < \varepsilon.$$

Let

$$C = \bigcup_{n=1}^{q} \overline{\mathrm{B}}(y_n^*, \alpha).$$

Since $C$ is a compact subset of $\mathbb{R}^m$, $f$ is uniformly continuous on $C$. Choose a positive $\delta$ such that if $u_1, u_2 \in C$ and $d_m(u_1, u_2) < \delta$, then

$$|f(u_1) - f(u_2)| < \gamma.$$

Choose $z \in X$ such that $\|x - z\| < \delta$. Then

$$\begin{aligned} \|Hx - Hz\| &= \sup_{n \geq p} \left| \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} (F(x)(j) - F(z)(j)) \right| \\ &\leq \sum_{k=p}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} |F(x)(j) - F(z)(j)| \leq \sum_{k=p}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} |a_j| |f(x_j^*) - f(z_j^*)|. \end{aligned}$$

By Lemma 5

$$\sum_{k=p}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} |a_j| |f(x_j^*) - f(z_j^*)| \leq \sum_{k=p}^{\infty} |a_k| |f(x_k^*) - f(z_k^*)| \sum_{i=1}^{k} \frac{1}{r_i}.$$

Hence

$$\|Hx - Hz\| \leq \gamma \sum_{k=p}^{q} |a_k| \sum_{i=1}^{k} \frac{1}{r_i} + 2L \sum_{k=q}^{\infty} |a_k| \sum_{i=1}^{k} \frac{1}{r_i} < 3\varepsilon.$$

Therefore the map $H : X \to X$ is continuous with respect to the metric defined by (2.2). By Lemma 6 there exists a point $x \in X$ such that $x = Hx$. Then for $n \geq p$ we have

$$x_n = y_n + \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} F(x)(j).$$

Hence

$$\Delta(r_n \Delta x_n) = \Delta(r_n \Delta y_n) + \Delta\left( r_n \Delta \left( \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} F(x)(j) \right) \right)$$

for $n \geq p$. Define a sequence $G$ by

$$G_n = \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} F(x)(j).$$

Then

$$r_n \Delta G_n = r_n \left( -\frac{1}{r_n} \sum_{j=n}^{\infty} F(x)(j) \right),$$

$$\Delta(r_n \Delta G_n) = F(x)(n) = a_n f(x_n^*) + b_n = a_n f(x_{n+\sigma-1}, x_{n+\sigma-2}, \ldots, x_{n+\sigma-m}) + b_n$$

for large $n$. Therefore $x$ is a solution of (E). Since $x \in X$ and $\rho_n = \mathrm{o}(n^s)$, we have $x_n = y_n + \mathrm{o}(n^s)$. □

**Corollary 1.** *Assume $f$ is continuous, $s \in (-\infty, 0]$, and*

$$\sum_{k=1}^{\infty} \frac{1}{k^s r_k} \sum_{j=k}^{\infty} (|a_j| + |b_j|) < \infty.$$

*Then for any bounded solution $y$ of the equation $\Delta(r_n \Delta y_n) = 0$ there exists a solution $x$ of (E) such that $x_n = y_n + \mathrm{o}(n^s)$.*

*Proof.* Assume $y$ is a bounded solution of the equation $\Delta(r_n \Delta y_n) = 0$. Then the set

$$U = \bigcup_{n=1}^{\infty} \overline{\mathrm{B}}(y_n^*, 1)$$

is bounded. Hence $f$ is continuous and bounded on $U$. By Theorem 1 there exists a solution $x$ of (E) such that $x_n = y_n + \mathrm{o}(n^s)$. □

**Corollary 2.** *Assume $f$ is continuous, $s \in (-\infty, 0]$, and*

$$\sum_{k=1}^{\infty} \frac{1}{k^s r_k} \sum_{j=k}^{\infty} (|a_j| + |b_j|) < \infty.$$

*Then for any real constant $d$ there exists a solution $x$ of (E) such that $x_n = d + \mathrm{o}(n^s)$.*

*Proof.* Any constant sequence is a bounded solution of the equation $\Delta(r_n \Delta y_n) = 0$. Hence the assertion is a consequence of Corollary 1. □

**Corollary 3.** *Assume $f$ is continuous, $s \in (-\infty, 0]$,*

$$\sum_{k=1}^{\infty} \frac{1}{r_k} < \infty, \quad \textit{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^s r_k} \sum_{j=k}^{\infty} (|a_j| + |b_j|) < \infty.$$

*Then for any $c, d \in \mathbb{R}$ there exists a solution $x$ of (E) such that*

$$x_n = c\sum_{j=1}^{n-1}\frac{1}{r_j} + d + \mathrm{o}(n^s).$$

*Proof.* Define a sequence $y$ by

$$y_n = c\sum_{j=1}^{n-1}\frac{1}{r_j} + d.$$

By Lemma 1, $y$ is a solution of the equation $\Delta(r_n\Delta y_n) = 0$. By assumption, the sequence $y$ is bounded. Using Corollary 1 we get the result. □

**Corollary 4.** *Assume $f$ is continuous and bounded, $s \in (-\infty, 0]$, and*

$$\sum_{k=1}^{\infty}\frac{1}{k^s r_k}\sum_{j=k}^{\infty}(|a_j| + |b_j|) < \infty.$$

*Then for any solution $y$ of the equation $\Delta(r_n\Delta y_n) = 0$ there exists a solution $x$ of (E) such that $x_n = y_n + \mathrm{o}(n^s)$.*

*Proof.* This corollary is an immediate consequence of Theorem 1. □

**Corollary 5.** *Assume $f$ is continuous, $s \in (-\infty, 0]$, $t \in [s, \infty)$, $r_n^{-1} = \mathrm{O}(n^t)$, and*

$$\sum_{n=1}^{\infty} n^{1+t-s}(|a_n| + |b_n|) < \infty.$$

*Then for any bounded solution $y$ of the equation $\Delta(r_n\Delta y_n) = 0$ there exists a solution $x$ of (E) such that $x_n = y_n + \mathrm{o}(n^s)$.*

*Proof.* This corollary is a consequence of Lemma 2 and Corollary 1. □

**Corollary 6.** *Assume $f$ is continuous, $s \in (-\infty, 0]$, $t \in [s, \infty)$, $r_n^{-1} = \mathrm{O}(n^t)$, and*

$$\liminf_{n\to\infty} n\left(\frac{|a_n| + |b_n|}{|a_{n+1}| + |b_{n+1}|} - 1\right) > 2 + t - s.$$

*Then for any bounded solution $y$ of the equation $\Delta(r_n\Delta y_n) = 0$ there exists a solution $x$ of (E) such that $x_n = y_n + \mathrm{o}(n^s)$.*

*Proof.* This corollary is a consequence of Lemma 3 and Corollary 5. □

**Corollary 7.** *Assume $f$ is continuous and bounded at infinity, $\lambda \in \mathbb{R}$,*

$$\frac{1}{r_n} = \mathrm{O}(n^\lambda) \quad \textit{and} \quad \limsup_{n\to\infty}\frac{|a_{n+1}| + |b_{n+1}|}{|a_n| + |b_n|} < 1.$$

*Then for any positive $c$, any real $d$ and any $s \in (-\infty, 0]$ there exists a solution $x$ of* (E) *such that*

$$x_n = c\sum_{j=1}^{n-1}\frac{1}{r_j} + d + \mathrm{o}(n^s).$$

*Proof.* Let $c \in (0,\infty)$, $d \in \mathbb{R}$, $s \in (-\infty, 0]$. Define a sequence $y$ by

$$y_n = c\sum_{j=1}^{n-1}\frac{1}{r_j} + d.$$

By Lemma 1, $y$ is a solution of the equation $\Delta(r_n \Delta y_n) = 0$. By Lemma 4,

$$\sum_{k=1}^{\infty}\frac{1}{k^s r_k}\sum_{j=k}^{\infty}(|a_j| + |b_j|) < \infty.$$

If the series

$$\sum_{n=1}^{\infty}\frac{1}{r_n} \tag{3.4}$$

is convergent then the sequence $y$ is bounded and, using Corollary 1 we get the result. Assume the series (3.4) is divergent. Since $c > 0$, we have $y_n \to \infty$. There exists a real number $\alpha$ such that $f$ is bounded on $[\alpha,\infty)^m$. There exists an index $q$ such that $y_{n+\sigma-m} > \alpha + 1$ for any $n \geq q$. Let

$$U = \bigcup_{n=q}^{\infty}\overline{\mathrm{B}}(y_n^*, 1).$$

Then $U \subset [\alpha,\infty)^m$. Hence $f$ is continuous and bounded on $U$. Now, using Theorem 1 we obtain the result. □

Now we present an example that proves the assumption that the function $f$ is bounded on some "neighborhood" of $(y_n^*)$ such that $(y_n)$ solves $\Delta(r_n \Delta y_n)$ =0 in Theorem 1, is essential.

*Example* 1. Assume $m = 2$,

$$r_n = n^{-1}, \quad a_n = 2^{-n}, \quad b_n = 0, \quad \sigma = 0, \quad s = 0, \quad f(x,y) = x^3 + \exp(2y).$$

Then equation (E) takes the form

$$\Delta(n^{-1}\Delta x_n) = 2^{-n}\left(x_{n-1}^3 + \exp(2x_{n-2})\right). \tag{3.5}$$

Let

$$y_n = \sum_{k=1}^{n-1}\frac{1}{r_k} = \sum_{k=1}^{n-1}k = \frac{n(n-1)}{2}.$$

Then $f$ is continuous and not bounded on $\bigcup_{n=q}^{\infty} \overline{\mathrm{B}}(y_n^*, \alpha)$ for any $q \in \mathbb{N}$ and $\alpha > 0$. Moreover,

$$\sum_{k=1}^{\infty} \frac{1}{k^s r_k} \sum_{j=k}^{\infty} (|a_j| + |b_j|) = \sum_{k=1}^{\infty} k \sum_{j=k}^{\infty} \frac{1}{2^j} = 2 \sum_{k=1}^{\infty} \frac{k}{2^k} < \infty.$$

Assume $x$ is a solution of (3.5) such that

$$x_n = y_n + z_n,$$

for large $n$ and $z_n = \mathrm{o}(n^s) = \mathrm{o}(1)$. Since $\Delta(\frac{1}{n}\Delta y_n) = 0$ for large $n$, we have

$$\Delta(\tfrac{1}{n}\Delta z_n) = \Delta(\tfrac{1}{n}\Delta x_n) = 2^{-n}\left(x_{n-1}^3 + \exp(2x_{n-2})\right) > 0$$

for large $n$. Therefore, the sequence $\frac{1}{n}\Delta z_n$ is eventually increasing, and there exists the limit

$$\lambda = \lim_{n\to\infty} \tfrac{1}{n}\Delta z_n > -\infty.$$

If $\lambda < \infty$, then the sequence $\frac{1}{n}\Delta z_n$ is convergent in $\mathbb{R}$. Hence the series

$$\sum_{n=1}^{\infty} \Delta(\tfrac{1}{n}\Delta x_n) = \sum_{n=1}^{\infty} \Delta(\tfrac{1}{n}\Delta z_n)$$

is convergent. On the other hand for large $n$

$$\Delta(\tfrac{1}{n}\Delta x_n) = 2^{-n}\left(x_{n-1}^3 + \exp(2x_{n-2})\right) \geq 2^{-n}\exp(2x_{n-2}).$$

Since

$$x_n \geq \frac{y_n}{2}$$

for large $n$, we get that

$$\Delta(\tfrac{1}{n}\Delta z_n) > 2^{-n}\exp((n-2)(n-3)).$$

Hence $\lambda = \infty$. Therefore $\frac{1}{n}\Delta z_n > 1$ for large $n$ and we get

$$\sum_{n=1}^{\infty} \Delta z_n \geq \sum_{n=1}^{\infty} n = \infty.$$

On the other hand, since $z_n \to 0$, the series $\sum_{n=1}^{\infty} \Delta z_n$ is convergent. $\square$

## 4. Asymptotic behavior of solutions

In this section we present sufficient conditions for a given solution $x$ of equation (E) to have an asymptotic property $x_n = y_n + \mathrm{o}(n^s)$, where $y$ is a solution of the equation $\Delta(r_n \Delta y_n) = 0$ and $s \in (-\infty, 0]$.

**Theorem 2.** *Assume $x$ is a solution of (E) such that the sequence $(f(x_n^*))$ is bounded,*

$$s \in (-\infty, 0], \quad \textit{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^s r_k} \sum_{j=k}^{\infty} (|a_j| + |b_j|) < \infty.$$

*Then there exists a solution $y$ of the equation $\Delta(r_n \Delta y_n) = 0$ such that*

$$x_n = y_n + \mathrm{o}(n^s).$$

*Proof.* Define a sequence $u$ by

$$u_n = \Delta(r_n \Delta x_n) = a_n f(x_n^*) + b_n.$$

Since the sequence $(f(x_n^*))$ is bounded, we have

$$\sum_{k=1}^{\infty} \frac{1}{k^s r_k} \sum_{j=k}^{\infty} |u_j| < \infty.$$

Define sequences $w, y, z$ by

$$w_n = \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} u_j, \quad y_n = x_n - w_n, \quad z_n = \sum_{k=n}^{\infty} \frac{1}{k^s r_k} \sum_{j=k}^{\infty} u_j.$$

Then

$$n^{-s}|w_n| \le n^{-s} \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} |u_j| = \sum_{k=n}^{\infty} \frac{1}{n^s r_k} \sum_{j=k}^{\infty} |u_j| \le \sum_{k=n}^{\infty} \frac{1}{k^s r_k} \sum_{j=k}^{\infty} |u_j| = \mathrm{o}(1).$$

Hence $w_n = \mathrm{o}(n^s)$. Moreover

$$\Delta(r_n \Delta w_n) = \Delta\left(r_n \Delta\left(\sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} u_j\right)\right) = -\Delta\left(r_n\left(\frac{1}{r_n} \sum_{j=n}^{\infty} u_j\right)\right) = u_n$$

and we obtain

$$\Delta(r_n \Delta y_n) = \Delta(r_n \Delta x_n) - \Delta(r_n \Delta w_n) = u_n - u_n = 0, \quad x_n = y_n + w_n = y_n + \mathrm{o}(n^s).$$

□

**Corollary 8.** *Assume $f$ is locally bounded, $s \in (-\infty, 0]$, and*

$$\sum_{k=1}^{\infty} \frac{1}{k^s r_k} \sum_{j=k}^{\infty} (|a_j| + |b_j|) < \infty.$$

*Then for any bounded solution $x$ of* (E) *there exists a solution $y$ of the equation*

$$\Delta(r_n \Delta y_n) = 0$$

*such that* $x_n = y_n + \mathrm{o}(n^s)$.

*Proof.* Assume $x$ is a bounded solution of (E). Then the sequence $x^*$ is also bounded. Since $f$ is locally bounded, the sequence $(f(x_n^*))$ is bounded. Hence the result follows from Theorem 2. □

**Corollary 9.** *Assume $f$ is bounded, $s \in (-\infty, 0]$, and*

$$\sum_{k=1}^{\infty} \frac{1}{k^s r_k} \sum_{j=k}^{\infty} (|a_j| + |b_j|) < \infty.$$

*Then for any solution $x$ of* (E) *there exists a solution $y$ of the equation*

$$\Delta(r_n \Delta y_n) = 0$$

*such that* $x_n = y_n + \mathrm{o}(n^s)$.

*Proof.* The assertion is an immediate consequence of Theorem 2. □

**Corollary 10.** *Assume $f$ is bounded and*

$$\sum_{k=1}^{\infty} \left( \frac{1}{r_k} + |a_k| + |b_k| \right) < \infty.$$

*Then any solution $x$ of* (E) *is convergent.*

*Proof.* Let $s = 0$ and let $x$ be a solution of (E). By assumption the series $\sum_{k=1}^{\infty} 1/r_k$ is convergent and the sequence $u$ defined by $u_k = \sum_{j=k}^{\infty} (|a_j| + |b_j|)$ is convergent to zero. Hence

$$\sum_{k=1}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} (|a_j| + |b_j|) < \infty. \tag{4.1}$$

Therefore, by Corollary 9 and Lemma 1, there exist real constants $c, d$ such that

$$x_n = c \sum_{j=1}^{n-1} \frac{1}{r_j} + d + \mathrm{o}(n^s).$$

Since $s = 0$ and $\sum_{k=1}^{\infty} 1/r_k < \infty$, we get $\lim_{n\to\infty} x_n = c \sum_{k=1}^{\infty} 1/r_k + d$. □

**Corollary 11.** *Assume $f$ is locally bounded and*

$$\sum_{k=1}^{\infty} \left( \frac{1}{r_k} + |a_k| + |b_k| \right) < \infty.$$

*Then any bounded solution $x$ of* (E) *is convergent.*

*Proof.* Let $s = 0$ and let $x$ be a bounded solution of (E). As in the proof of Corollary 10 we obtain (4.1). Using Corollary 8 and Lemma 1 we get the result. □

**Corollary 12.** *Assume $f$ is locally bounded, $s \in (-\infty, 0]$, $t \in [s, \infty)$, $r_n^{-1} = \mathrm{O}(n^t)$, and*

$$\sum_{n=1}^{\infty} n^{1+t-s}(|a_n| + |b_n|) < \infty.$$

*Then for any bounded solution $x$ of (E) there exists a solution $y$ of the equation $\Delta(r_n \Delta y_n) = 0$ such that $x_n = y_n + \mathrm{o}(n^s)$.*

*Proof.* The assertion is a consequence of Lemma 2 and Corollary 8. □

**Corollary 13.** *Assume $f$ is locally bounded, $s \in (-\infty, 0]$, $t \in [s, \infty)$, $r_n^{-1} = \mathrm{O}(n^t)$,*

$$\sum_{n=1}^{\infty} \frac{1}{r_n} = \infty, \quad \textit{and} \quad \sum_{n=1}^{\infty} n^{1+t-s}(|a_n| + |b_n|) < \infty.$$

*Then for any bounded solution $x$ of (E) there exists a real constant $d$ such that*

$$x_n = d + \mathrm{o}(n^s).$$

*Proof.* Assume $x$ be a bounded solution of (E). By Lemma 2 and Corollary 8, there exist $c, d \in \mathbb{R}$ such that

$$x_n = c \sum_{j=1}^{n-1} \frac{1}{r_j} + d + \mathrm{o}(n^s).$$

Since $x$ is bounded and $\sum_{n=1}^{\infty} 1/r_n = \infty$, we have $c = 0$. □

**Corollary 14.** *Assume $f$ is locally bounded,*

$$\frac{1}{r_n} = \mathrm{O}(1), \quad \textit{and} \quad \sum_{n=1}^{\infty} n(|a_n| + |b_n|) < \infty.$$

*Then any bounded solution of (E) is convergent.*

*Proof.* Let $t = s = 0$ and let $x$ be a bounded solution of (E). By Corollary 12 there exists a solution $y$ of the equation $\Delta(r_n \Delta y_n) = 0$ such that $x_n = y_n + \mathrm{o}(1)$. Then $y$ is a bounded sequence. By Lemma 1 any bounded solution $y$ of the equation $\Delta(r_n \Delta y_n) = 0$ is convergent. Hence $x$ is convergent. □

**Corollary 15.** *Assume $f$ is bounded at infinity, $s \in (-\infty, 0]$, $t \in [s, \infty)$, $r_n^{-1} = \mathrm{O}(n^t)$,*

$$\liminf_{n\to\infty} n \log \frac{|a_n| + |b_n|}{|a_{n+1}| + |b_{n+1}|} > 2 + t - s,$$

*and $x$ is a solution of (E) such that $\lim_{n\to\infty} x_n = \infty$. Then there exists a solution $y$ of the equation $\Delta(r_n \Delta y_n) = 0$ such that $x_n = y_n + \mathrm{o}(n^s)$.*

*Proof.* Since $f$ is bounded at infinity and $\lim_{n\to\infty} x_n = \infty$, the sequence $(f(x_n^*))$ is bounded. Using Lemma 3 and Theorem 2 we get the result. □

**Corollary 16.** *Assume $f$ is locally bounded, $\lambda \in \mathbb{R}$, $r_n^{-1} = \mathrm{O}(n^\lambda)$,*

$$\limsup_{n\to\infty} \frac{|a_{n+1}| + |b_{n+1}|}{|a_n| + |b_n|} < 1,$$

*and $x$ is a bounded solution of (E). Then for any $s \in (-\infty, 0]$ there exists a solution $y$ of the equation $\Delta(r_n \Delta y_n) = 0$ such that $x_n = y_n + \mathrm{o}(n^s)$.*

*Proof.* This assertion is a consequence of Lemma 4 and Corollary 8. □

Now we present an example that proves the assumption

$$\sum_{k=1}^{\infty} \frac{1}{k^s r_k} \sum_{j=k}^{\infty} |a_j| < \infty.$$

is not enough in Theorem 2.

*Example* 2. Assume $m = 2$, $s = 0$, $\sigma = 0$,

$$r_n = n^2, \quad a_n = \frac{1}{n^2}, \quad b_n = 2n + 1 - \frac{2}{n^2}, \quad f(x,y) = \frac{x}{|x|+1} + \frac{y+3}{|y|+2}.$$

Then equation (E) takes the form

$$\Delta(n^2 \Delta x_n) = \frac{1}{n^2}\left(\frac{x_{n-1}}{|x_{n-1}|+1} + \frac{x_{n-2}+3}{|x_{n-2}|+2}\right) + 2n + 1 - \frac{2}{n^2}. \tag{4.2}$$

Notice that $f$ is bounded and

$$\sum_{k=1}^{\infty} \frac{1}{k^s r_k} \sum_{j=k}^{\infty} |a_j| = \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=k}^{\infty} \frac{1}{j^2} < \infty.$$

Moreover, the sequence $x_n = n$, is a solution of (4.2). On the other hand, any solution of the equation $\Delta(n^2 \Delta y_n) = 0$ is of the form

$$y_n = c \sum_{k=1}^{n-1} \frac{1}{r_k} + d = c \sum_{k=1}^{n-1} \frac{1}{k^2} + d$$

for some $c, d \in \mathbb{R}$. Hence any solution of $\Delta(n^2 \Delta y_n) = 0$ is convergent, which means that $x$ cannot be approximated by any solution of the equation $\Delta(n^2 \Delta y_n) = 0$.

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# ON THE CONSTRUCTION OF SOLUTIONS OF GENERAL LINEAR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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*Abstract.* For the linear boundary value problem

$$x'(t) = p(x)(t) + q(t), \quad l(x) = c_0$$

on the closed interval $I \subset R$, where $p : C(I, R^n) \to L(I, R^n)$ is a strongly bounded linear operator, $l : C(I, R^n) \to R^n$ is the bounded linear functional, $q \in L(I, R^n)$ and $c_0 \in R^n$, we describe a method of construction of its solution using successive approximations by the sequence of the solutions of simple boundary value problems. We prove the conditions which guarantee the convergence of the above mentioned sequences in general and special cases, we prove the stability of the convergence in some sense. Also, for illustration, we solve some typical problems in MAPLE.

2010 *Mathematics Subject Classification:* 34K06; 34K13; 34B15

*Keywords:* system of functional differential equations, general boundary value problems, argument deviation, construction of solutions, successive approximations

## 1. STATEMENT OF THE MAIN RESULTS

### 1.1. *Introduction*

Consider the system of linear functional differential equations on interval $I = [a, b]$

$$x'(t) = p(x)(t) + q(t), \tag{1.1}$$

with the linear boundary value conditions

$$l(x) = c_0, \tag{1.2}$$

where $p : C(I; R^n) \to L(I; R^n)$ is a linear strongly bounded operator (i.e., $p$ is a linear operator and there is Lebesgue integrable function $\eta : I \to R_+$ such that $\|p(x)(t)\| \le \eta(t)\|x\|_C$ for almost all $t \in I$, $x \in C(I; R^n)$), $l : C(I; R^n) \to R^n$ is a linear bounded functional (i.e., $l$ is linear functional and there is the constant $\alpha \in R_+$ such that $\|l(x)\| \le \alpha\|x\|$ for all $x \in C(I; R^n)$), $q \in L(I; R^n)$ and $c_0 \in R^n$.

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The a solution of the problem (1.1), (1.2) is understood to be an absolutely continuous vector valued function $x : I \to R^n$ which satisfies the system (1.1) almost everywhere on $I$, and satisfies the conditions (1.2).

Papers dealing with boundary value problems contain numerous interesting and, in a sense, optimal conditions ensuring the unique solvability and correctness of various boundary value problems for linear functional differential equations and systems, both in regular and singular cases (see, e.g., [1–14, 21, 22] and references therein). However, unlike systems of ordinary differential equations, in the case of boundary value problems for more general systems of functional differential equations, there are still almost no methods for approximate construction of their solutions. We can mention in this relation the parametrization methods (see, e.g., [18, 24]) using the ideas close to [17, 19, 20, 23] in the ordinary case.

This lack is dealt with in the present paper, which gives a construction that can be considered as a generalization of a number of previously published. A special attention should be paid to work [12], to which this paper is a follow-up. The work [11], using the Fredholm problem (1.1), (1.2), provide general sufficient and, in the cases where $p$ is a Volterra type operator, also necessary and sufficient (in a sense, unimprovable) conditions for the unique solvability of the boundary value problems (1.1), (1.2). A number of consequences for specific kinds of systems (1.1) and special cases of boundary conditions (1.2) are given. The works mentioned also contain statements on the continuous dependence of solutions of problems in question on small changes in right-hand side terms of systems (1.1) and boundary conditions (1.2) (the so-called correctness of problems (1.1), (1.2)).

### 1.2. *Notation and definitions*

The following notations are used throughout the paper:

$R = ]-\infty, +\infty[$, $R_+ = [0, \infty[$; $\chi_I$ is a characteristic function of interval $I$, i.e.,

$$\chi_I(t) = \begin{cases} 1 & \text{for } t \in I, \\ 0 & \text{for } t \notin I; \end{cases}$$

$R^n$ is the space of $n$-dimensional column vectors $x = (x_i)_{i=1}^n$ with elements $x_i \in R$ $(i = 1, \dots, n)$, and norm

$$\|x\| = \sum_{i=1}^n |x_i|;$$

$R^{n\times n}$ is the space of $n \times n$-matrices $X = (x_{ik})_{i,k=1}^n$ with elements $x_{ik} \in R$ $(i, k = 1, \dots, n,)$ and norm

$$\|X\| = \sum_{i,k=1}^n |x_{ik}|;$$

$$R_+^n = \{(x_i)_{i=1}^n \in R^n : x_i \geq 0 \ (i = 1, \dots, n)\};$$

$$R_+^{n\times n} = \left\{(x_{ik})_{i,k=1}^n \in R^{n\times n} : \ x_{ik} \geq 0 (i,k = 1,\dots,n)\right\};$$

if $x, y \in R^n$ and $X, Y \in R^{n\times n}$, then

$$x \leq y \Leftrightarrow y - x \in R_+^n, \ X \leq Y \Leftrightarrow Y - X \in R_+^{n\times n};$$

if $x = (x_i)_{i=1}^n \in R^n$ and $X = (x_{ik})_{i,k=1}^n \in R^{n\times n}$, then

$$|x| = (|x_i|)_{i=1}^n, \ |X| = (|x_{ik}|)_{i,k=1}^n;$$

$\det(X)$ is the determinant of matrix $X$;
$X^{-1}$ is the inverse matrix to $X$;
$r(X)$ is the spectral radius of matrix $X$;
$E$ is the unit matrix;
$\Theta$ is the zero matrix;
$C(I;R^n)$ is the space of continuous vector functions[1] $x : I \to R^n$ with norm

$$\|x\|_C = \max\left\{\|x(t)\| : t \in I\right\};$$

if $x = (x_i)_{i=1}^n \in C(I;R^n)$, then

$$|x|_C = (\|x_i\|_C)_{i=1}^n;$$

$\tilde{C}(I;R^n)$ is the space of absolutely continuous vector functions $x : I \to R^n$ with norm

$$\|x\|_{\tilde{C}} = \|x\|_C + \|x'\|_L;$$

$C(I;R^{n\times n})$ is the space of continuous matrix functions $X \to R^{n\times n}$;
$L(I;R^n)$ is the space of vector valued functions $x : I \to R^n$ with the Lebesgue integrable elements, with norm

$$\|x\|_L = \int_a^b \|x(t)\|\,dt;$$

if $x = (x_i)_{i=1}^n \in L(I;R^n)$, then

$$|x|_L = (\|x_i\|_L)_{i=1}^n;$$

$L(I;R^{n\times n})$ is the space of Lebesgue integrable matrix functions $X : I \to R^{n\times n}$;

If $g : C(I;R^n) \to L(I;R^n)$ is a linear operator, then $|g| : C(I;R^n) \to L(I;R^n)$ denotes a non-negative operator such that

$$|g(x)| \leq |g|(|x|) \quad \text{for all } x \in C(I;R^n).$$

If $Z \in C(I;R^{n\times n})$ is a matrix function with columns $z_1,\dots,z_n$, and $g : C(I;R^n) \to L(I;R^n)$ is a linear operator, respectively $g : C(I;R^n) \to R^n$ is a linear functional, then $g(Z)$ denote a matrix function, respectively the constant matrix with columns $g(z_1),\dots,g(z_n)$.

[1] A vector (matrix) function is said to be continuous, integrable, etc., if such are its elements.

Let us say that $p : C(I;R^n) \to L(I;R^n)$ is the Volterra operator with respect to $t_0 \in I$, if for any $t \in I$ and $x \in C(I,R^n)$ where $x(s) = 0$ for $s \in I_{t_0,t}$ the equality $p(x)(s) = 0$ holds for almost all $s \in I_{t_0,t}$, where

$$I_{t_0,t} = \begin{cases} [t_0, t] & \text{if } t_0 \le t \\ [t, t_0] & \text{if } t < t_0 \end{cases}.$$

Special cases of system frequently occurring in practice (1.1) are: a linear system of ordinary differential equations

$$x'(t) = P(t)x(t) + q(t), \tag{1.3}$$

a linear system of ordinary differential equations with a deviated argument

$$x'(t) = P(t)x(\tau(t)) + q(t), \tag{1.4}$$

respectively with multiple argument deviations

$$x'(t) = \sum_{i=1}^{\bar{s}} P_i(t)x(\tau_i(t)) + q(t), \tag{1.5}$$

where $P, P_i \in L(I, R^{n\times n})$, $\tau, \tau_i : I \to R$ are measurable functions ($i = 1,\dots,\bar{s}$) and if $\tau(t) \le t$, $\tau_i(t) \le t$ for almost all $t \in I$, then $\tau$ and $\tau_i$, are so-called delayed arguments, and $q \in L(I, R^n)$.

Systems (1.3), (1.4) and (1.5) are special cases of linear systems (1.1) with a Volterra operator.

For example special cases of conditions (1.2) are: the initial condition

$$x(t_0) = c_0, \tag{1.6}$$

the multi-point boundary condition

$$\sum_{j=1}^{s} A_j x(t_j) = c_0, \tag{1.7}$$

the Cauchy type condition

$$x(\bar{t}_0) = l_0(x) + c_0, \tag{1.8}$$

and the periodic boundary condition

$$x(b) - x(a) = 0, \tag{1.9}$$

where $\bar{t}_0, t_j \in I (j = 1,\dots,s)$, $l_0 : C(I,R^n) \to R^n$ is a linear bounded functional, $A_j \in R^{n\times n}$ $(j = 1,\dots,s)$ and $c_0 \in R^n$.

For the construction of solution of the problem (1.1), (1.2) and description of its properties we use the method and notifications from work [13]:

Let $x \in C(I;R^n)$, $k, m \in N$ and $t_0 \in I$, then we use following notations:

$$p^0(x)(t) = p^{k,0}(x)(t) = x(t), \quad p^k(x)(t) = \int_{t_0}^{t} p(p^{k-1}(x))(s)ds,$$

$$\Lambda_k = l\left(\sum_{i=1}^{k} p^{i-1}(E)\right), \quad \hat{q}(t) = \int_{t_0}^{t} q(s)ds,$$

and if $\Lambda_k$ is regular, then

$$p^{k,m}(x)(t) = p^m(x)(t) - \left[\sum_{i=1}^{m} p^{i-1}(E)(t)\right] \Lambda_k^{-1} l\left(p^k(x)\right),$$

and

$$D^{k,m}(\hat{q}, c_0)(t) = \left[\sum_{i=1}^{k} p^{i-1}(E)(t)\right] \Lambda_k^{-1}\left[c_0 - l\left(\sum_{i=1}^{k} p^{i-1}(\hat{q})\right) + \sum_{i=1}^{m} p^{i-1}(\hat{q})(t)\right].$$

1.3. *Main results*

The following fundamental propositions about the solvability of a general linear boundary value problem (1.1), (1.2) is proved in [13, Theorem 1.1.1]:

**Proposition 1.** *The boundary value problem* (1.1), (1.2) *is uniquely solvable if the corresponding homogeneous problem*

$$\begin{aligned} x'(t) &= p(x)(t), \\ l(x) &= 0 \end{aligned}$$

*has only the trivial solution.*

**Proposition 2.** *Let $k, m \in N$, $m_0 \in N \cup \{0\}$ and $A \in R_+^{n\times n}$ be such that*

$$r(A) < 1, \tag{1.10}$$

*$\Lambda_k$ is regular, and for all the solutions of homogeneous boundary value problem corresponding to the nonhomogeneous problem* (1.1), (1.2), *the inequality*

$$|p^{k,m}(x)|_C \le A|p^{k,m_0}(x)|_C \tag{1.11}$$

*holds. Then problem* (1.1), (1.2) *is uniquely solvable.*

It is clear that the solution of system (1.1) is also the solution of the systems

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} p(x)(s)ds + \int_{t_0}^{t} q(s)ds = \dots \\ &= p^k(x)(t) + \left[\sum_{i=1}^{k} p^{i-1}(E)(t)\right] x(t_0) + \left[\sum_{i=1}^{k} p^{i-1}(\hat{q})(t)\right] \end{aligned}$$

for any $k \in N$ and arbitrary value $x(t_0)$. If for our $k \in N$ the matrix $\Lambda_k$ is regular, then from the boundary condition (1.2) it follows that

$$c_0 = l\left(p^k(x)\right) + l\left(\sum_{i=1}^{k} p^{i-1}(E)\right)x(t_0) + l\left(\sum_{i=1}^{k} p^{i-1}(\hat{q})\right)$$

and then

$$x(t_0) = \Lambda_k^{-1}\left[c_0 - l\left(p^k(x)\right) - l\left(\sum_{i=1}^{k} p^{i-1}(\hat{q})\right)\right]. \tag{1.12}$$

Therefore, the solution of the problem (1.1), (1.2) is also the solution of equation

$$x(t) = p^{k,m}(x)(t) + D^{k,m}(\hat{q}, c_0)(t). \tag{1.13}$$

and vice versa.

Let now $x_0 \in C(I, R^n)$, and for arbitrary $\nu \in N$ the vector-function $x_\nu$ be defined by equality

$$x_\nu(t) = p^{k,m}(x_{\nu-1})(t) + D^{k,m}(\hat{q}, c_0)(t). \tag{1.14}$$

Then the next theorem is true.

**Theorem 1.** *Let $k, m \in N, m_0 \in N \cup \{0\}$, and $A \in R_+^{n\times n}$ be such that the matrix $\Lambda_k$ is regular,*

$$r(A) < 1, \qquad r\big(A||p^{k,m_0}|(E)|_C\big) < 1, \tag{1.15}$$

*and for arbitrary $x \in C(I, R^n)$ the inequality*

$$|p^{k,m}(x)|_C \le A|p^{k,m_0}(x)|_C \tag{1.16}$$

*holds. Then the problem (1.1), (1.2) has unique solution $x$, and*

$$x = \lim_{\nu\to\infty} x_\nu \quad \textit{uniformly on } I$$

*where functions $x_\nu$ are defined by the equality (1.14) and $x_0 \in C(I; R^n)$ is arbitrary.*

*Proof.* From the conditions of our theorem and Proposition 2 it follows that there is a unique solution $x$ of problem (1.1), (1.2). Now, let us assume that $x_\nu$ ($\nu \in N$) are vector functions defined by the equation (1.14) for arbitrary $x_0 \in C(I; R^n)$. Then from (1.13), (1.14), (1.16), and the positivity of the operator $|p^{k,m_0}|$, we obtain:

$$\begin{aligned}
|x - x_\nu|_C &= |p^{k,m}(x - x_{\nu-1})|_C \\
&\le A|p^{k,m_0}(x - x_{\nu-1})|_C \le A||p^{k,m_0}|(E)|_C|x - x_{\nu-1}|_C \\
&\le (A||p^{k,m_0}|(E)|_C)^2|x - x_{\nu-2}|_C \le \dots \\
&\le (A||p^{k,m_0}|(E)|_C)^\nu|x - x_0|_C.
\end{aligned}$$

From conditions (1.15) it follows that

$$\lim_{\nu\to\infty}\left(A||p^{k,m_0}|(E)|_C\right)^{\nu} = \Theta,$$

and from the last inequality, it is obvious that $x_\nu \to x$ uniformly on $I$, which prove our theorem. □

Let us $p : C(I, R^n) \to L(I, R^n)$ the Volterra operator. Then, as it is proved in [13], the equality $p^{k,m_0}(x) \equiv x$ holds, and the conditions of Proposition 2 are sufficient and necessary for the unique solvability of problem (1.1), (1.2).

**Theorem 2.** *Let $p$ be the Volterra operator with respect to $t_0 \in I$. Then the boundary value problem (1.1), (1.2) is uniquely solvable if and only if there are $k, m \in N$ and $A \in R_+^{n\times n}$ such that the matrix $\Lambda_k$ is regular,*

$$r(A) < 1, \tag{1.17}$$

*and for all $x \in C(I, R^n)$ the inequality*

$$|p^{k,m}(x)|_C \le A|x|_C \tag{1.18}$$

*holds. Moreover, for the solution $x$ of the boundary value problem (1.1), (1.2), the representation $x = \lim_{\nu\to\infty} x_\nu$ holds, where functions $x_\nu$ ($\nu \in N$) are defined by equality (1.14), and $x_0 \in C(I; R^n)$ is arbitrary.*

*Proof.* The first part of our theorem, the necessity and sufficiency of its conditions for unique solvability of problem (1.1), (1.2), when $p$ is the Volterra operator, follows from the theorems proved in [13].

On the other hand, from our assumptions it is clear that $p^{k,m_0}(x) \equiv x$ and then $|p^{k,m_0}|(E) = E$. Therefore, the assertion of the second part of our theorem concerning the construction of the solution follows from the second part of Theorem 1. □

*Remark* 1. As problems (1.1), (1.2), and (1.13) are equivalent for such $k \in N$ that matrix $\Lambda_k$ is regular, problem (1.1), (1.2) is also equivalent to problem (1.1), (1.12).

The functional on the right-hand side of equality (1.12) is composed from continuous functionals and operators and it is continuous with respect to $x$. Therefore, in view of the unique solvability of problem (1.1), (1.2), we can construct the sequence $\{x_\nu\}_{\nu=1}^{\infty}$ of successive approximations for the solution of problem (1.1), (1.2) using the method of iterations, where

$$x_\nu'(t) = p(x_{\nu-1})(t) + q(t) \tag{1.19}$$

and

$$x_\nu(t_0) = \Lambda_k^{-1}\left[c_0 - l(p^k(x_{\nu-1})) - l\left(\sum_{i=1}^{k} p^{i-1}(\hat{q})\right)\right], \tag{1.20}$$

if $x_0 \in C(I, R^n)$ is arbitrary.

*Remark* 2. The above-mentioned method of successive approximations is in a certain sense stable. Let $x$ be the solution of problem (1.1), (1.2), and $\bar{x}_\nu$ be defined by equalities:

$$\bar{x}_\nu(t) = p^{k,m}(\bar{x}_{\nu-1})(t) + D^{k,m}(\hat{q} + \hat{\bar{q}}_\nu, c_0 + \bar{c}_\nu)(t),$$

where $\bar{q}_\nu \in L(I, R^n)$, $\bar{c}_\nu \in R^n$ and

$$\lim_{\nu\to\infty}\left(\int_a^b |\bar{q}_\nu(t)|dt + |\bar{c}_\nu|\right) = 0,$$

then

$$\begin{aligned}|x - \bar{x}_\nu|_C &\le |p^{k,m}(x - \bar{x}_\nu)|_C + |D^{k,m}(\hat{\bar{q}}_\nu, \bar{c}_\nu)|_c \\ &\le A^\nu |x - x_0|_C + |D^{k,m}(\hat{\bar{q}}_\nu, \bar{c}_\nu)|_c.\end{aligned}$$

From the last inequality, given our assumptions about $\bar{q}_\nu$ and $\bar{c}_\nu$, and the facts that $\lim_{\nu\to\infty} D^{k,m}(\hat{\bar{q}}_\nu, \bar{c}_\nu) = 0$ (uniformly on $I$), and $A^\nu \to \Theta$ if $\nu \to \infty$, it follows that the sequence $\bar{x}_\nu$ converges to function $x$ uniformly on $I$. As the problems under consideration are equivalent, problem (1.19), (1.20) is also stable in this sense.

*Remark* 3. If we introduce notation $T(x) = p^{k,m}(x_{\nu-1})(t) + D^{k,m}(\hat{q}, c_0)$, then $T : C(I, R^n) \to C(I, R^n)$ is a linear operator, and from inequality

$$\begin{aligned}|T(x) - T(y)|_C = |p^{k,m}(x - y)|_C &\le \\ \le A||p^{k,m_0}|(E)|_C |x - y|_C,\end{aligned}$$

where $r\left(A||p^{k,m_0}|(E)|_C\right) < 1$, it follows that operator $T$ is a contraction.

It is clear that unique solvability of problem (1.1),(1.2), and possibility of construction of this solution by the method of successive approximations, follows from the Banach's fixed-point theorem.

The theorems stated above imply the following corollaries. First, if we assume that $l(x) = x(t_0)$, where $t_0 \in I$ is arbitrary, we obtain the conditions of unique solvability and construction of the solution of the initial problem for a of linear functional-differential equations.

**Corollary 1.** *Let $p$ be the Volterra operator. Then problem* (1.1), (1.6) *is uniquely solvable if there are $m \in N$, and $A \in R_+^{n\times}$, such that $r(A) < 1$, and for arbitrary $x \in C(I, R^n)$ inequality*

$$| p^m(x) |_C \le A | x |_C$$

*holds. Moreover, for solution $x$ of problem* (1.1), (1.6) *representation $x = \lim_{\nu\to\infty} x_\nu$ holds, where $x_\nu$ ($\nu \in N$) are the solutions of problem* (1.19),

$$x_\nu(t_0) = c_0 \tag{1.21}$$

*and $x_0 \in C(I, R^n)$ is arbitrary.*

If we assume that $p(x)(t) = P(t)x(\tau(t))$, and $l(x) = \sum_{j=1}^{s} A_j x(t_j)$, where $\tau : I \to R$ is a measurable function, $\tau(t) \le t$ for $t \in I$, $A_j \in R^{n\times n}$ and $t_j \in I$ if $j = 1,\dots,s$ we obtain the conditions of unique solvability and construction of the solution of a multi-point problem for the system of linear differential equations with deviated argument.

**Corollary 2.** *Let $\tau : I \to R$ be a measurable function, $\tau(t) \le t$ for almost all $t \in I$, $u : ]-\infty, a] \to R^n$ be a continuous bounded function, $det\ \Lambda_1 \ne 0$, $x(t) = u(t)$ for $t \le a$, and*

$$r\left(\int_a^b \bar{P}(t)ds + \sum_1^s |\ \Lambda_1 A_j\ | \int_a^{t_j} \bar{P}(t)dt\right) < 1,$$

*where $\Lambda_1 = \sum_{j=1}^{s} A_j$, and $\bar{P}(t) = \chi_I(\tau(t))|P(t)|$. Then problem (1.4), (1.7) is uniquely solvable.*

*Moreover, for solution $x$ of problem (1.4), (1.7) the representation $x = \lim_{\nu\to\infty} x_\nu$ holds (uniformly on $I$), where $x_\nu$ ($\nu \in N$) are solutions of the problem*

$$x_\nu'(t) = \chi_I(\tau(t))P(t)x(\tau_0(t)) + (1-\chi_I(\tau(t)))P(t)u(\tau(t)) + q(t),$$

$$x_\nu(t_0) = \Lambda_1^{-1}\left[c_0 - \sum_{j=1}^{s} A_j \int_{t_0}^{t_j} [\chi_I(\tau(t))P(t)x_{\nu-1}(t) - q(t))\,dt\right],$$

*where*

$$\tau_0(t) = \begin{cases} \tau(t) & \text{if } \tau(t) \ge a, \\ a & \text{if } \tau(t) < a, \end{cases}$$

*and $x_0 \in C(I, R^n)$ is arbitrary.*

*Remark* 4. An analogous proposition can be obtained also for the system with deviating arguments (1.5).

*Remark* 5. From Corollary 2, a criterion of unique solvability and description of successive approximations for the initial problem for system (1.4) with delay (see also Corollary 1) also follows. On the other hand Corollary 2 does not include a criterion of unique solvability and description of successive approximations for the periodic problem, because if $l(x) = x(b) - x(a)$, then $\Lambda_1 = E - E = 0$ and therefore $\det \Lambda_1 = 0$.

We will obtain a criterion of unique solvability and description of successive approximations for the periodic problem for the systems of linear differential equations with deviated arguments (1.4) and (1.5) assuming that $l(x) = x(b) - x(a)$ and $k = 2$.

**Corollary 3.** *Let $i = 1,\dots,\bar{s}$, the measurable functions $\tau_i : I \to R$ be such that $\tau_i(t) \le t$ for almost all $t \in I$,*

$$\tau_{0i}(t) = \begin{cases} \tau_i(t) & \text{if } \tau_i(t) \ge a \\ a & \text{if } \tau_i(t) < a, \end{cases}$$

*$u : ]-\infty, a] \to R^n$ be a continuous bounded function, and $P_i \in L(I, R^{n\times n})$. Then if*

$$\det \Lambda_2 \neq 0$$

*and*

$$r\left(\int_a^b \bar{P}(t)dt + \mid \Lambda_2^{-1} \mid \int_a^b \bar{P}(t)dt\right) < 1,$$

*where*

$$\Lambda_2 = \int_a^b \sum_{i=1}^{\bar{s}} \chi(\tau_i(t)) P_i(t)dt, \qquad \bar{P}(t) = \sum_{i=1}^{\bar{s}} \chi_I(\tau_i(t)) \mid P_i(t) \mid,$$

*system (1.5) has unique solution $x$ under the periodic condition*

$$x(b) - x(a) = c_0, \quad x(t) = u(t) \quad \text{for} \quad t \in R \setminus I,$$

*and the representation $x = \lim_{\nu\to\infty} x_\nu$ holds, where $x_\nu$ ($\nu \in N$) are solutions of the*

$$\begin{aligned} x_\nu'(t) = &\sum_{i=1}^{\bar{s}} \chi_I(\tau_i(t)) P_i(t) x_{\nu-1}(\tau_{0i}(t)) + \\ &+ \sum_{i=1}^{\bar{s}} (1 - \chi_I(\tau_i(t)))\, P_i(t) u(t) + q(t), \\ x_\nu(b) = &\Lambda_2^{-1}\left[c_0 - \int_a^b \left(\sum_{i=1}^{\bar{s}} \chi_I(\tau_i(t)) P_i(t) x_{\nu-1}(t) - q(t)\right) dt\right] \end{aligned}$$

*for arbitrary $x_0 \in C(I, R^n)$.*

If we assume that $l(x) = x(t_0) - l_0(x)$, we will obtain conditions of unique solvability and construction of the solution of Cauchy-Nicoletti type problems (1.1), (1.8) for arbitrary systems with the Volterra operator, with deviated arguments (1.4) and (1.5).

In the general case, we have

$$\Lambda_k = E - \sum_{i=1}^{k} l_0(p^{i-1}(E)), \tag{1.22}$$

and then from (1.12) we obtain

$$x(t_0) = \Lambda_k^{-1}\left[c_0 + l_0(p^k(x)) + \sum_{i=1}^{k} l_0(p^{i-1}(\hat{q}))\right]. \tag{1.23}$$

**Corollary 4.** *Let $k, m \in N$, $m_0 \in N \cup \{0\}$, $A \in R_+^{n\times n}$ be such that the matrix $\Lambda_k$ from (1.22) be regular, and condition (1.17) holds, $u : ]-\infty, a] \to R^n$ be a continuous bounded function, and for arbitrary $x \in C(I, R^n)$ inequality (1.18) holds.*

*Then problem* (1.1), (1.8) *has unique solution* $x$ *and the representation* $x = \lim_{\nu\to\infty} x_\nu$ *holds, where* $x_\nu$ $(\nu \in N)$ *are solutions of the problem* (1.19),

$$x_\nu(t_0) = \Lambda_k^{-1}\left[c_0 + l_0(p^k(x_{\nu-1}) + \sum_{i=1}^{k} l_0(p^{i-1})(\hat{q})\right]$$

*for arbitrary* $x_0 \in C(I, R^n)$.

In this corollary if we assume that $l_0 = 0$, or $t_0 = a$, $l_0(x) = x(b)$ or $t_0 = b$, $l_0(x) = x(a)$, we will obtain a criterion of unique solvability and construction of successive approximations for the periodic problems for the mentioned above systems.

Now, if we assume that $l_0(x) \equiv 0$, i,e., $x(t_0) = c_0$, then $\Lambda_k = E$ $(k \in N)$, $l_0(x) = x(b)$, $\bar{t}_0 = a$, i.e., $x(a) = x(b)$, and therefore

$$\Lambda_1 = 0,$$
$$\Lambda_2 = -\int_a^b p(E)(t)dt,$$
$$\Lambda_k = -\sum_{i=1}^{k}\int_a^b p(p^{i-1}(E))(t)dt \quad \text{for} \quad k \geq 3,$$

and

$$x(t_0) = \Lambda_k^{-1}\left[c_0 + \int_a^b p(p^{k-1}(x))(t)dt + \sum_{i=2}^{k}\int_a^b p(p^{i-2}(\hat{q}))(t)dt + \int_a^b q(s)ds\right].$$

Also, if $l_0(x) = x(a)$, $\bar{t}_0 = b$, i.e., $x(b) = x(a)$, then

$$\Lambda_1 = 0,$$
$$\Lambda_2 = \int_a^b p(E)(t)dt,$$
$$\Lambda_k = \sum_{i=1}^{k}\int_a^b p(p^{i-1}(E))(t)dt \quad \text{for} \quad k \geq 3,$$

and

$$x(b) = \Lambda_k^{-1}\left[c_0 - \int_a^b p(p^{k-1}(x))(t)dt - \sum_{i=2}^{k}\int_a^b p(p^{i-2}(\hat{q}))(t)dt - \int_a^b q(s)ds\right].$$

## 2. Examples

For the illustration of calculations of (converging to solution) approximations of solutions of a given problem, only such approximations are used, the graphs of which

are differentiable under given conditions. On Figures 1 and 2 shown below, the function of "history" $u$ is marked by a dotted line $\cdots$, the "starting function" $x_0$ of the iterative process is marked by a dashed line $---$, approximations of solutions are marked by a dot-and-dash line $-\cdot-\cdot$, and the last calculated approximation of solution $x_\nu$ is marked by a solid line —— . Analogous problems are solved by MAPLE in the works [3, 15, 16].

2.1.

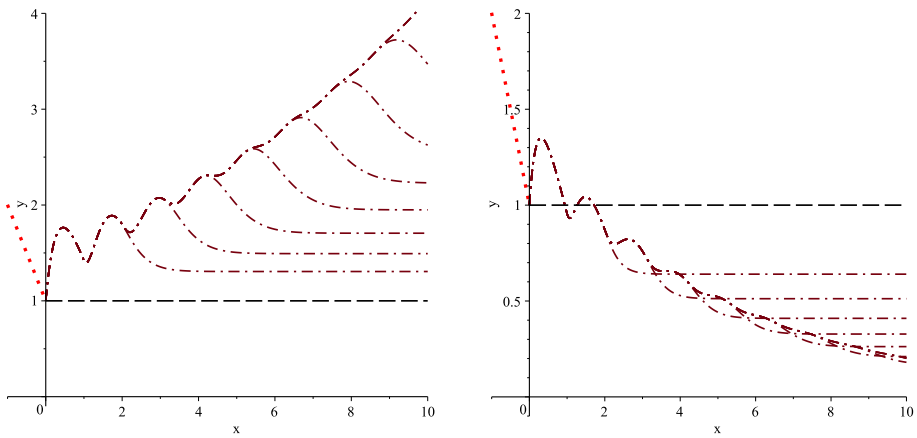
Let us solve the problem

$$x'(t) = \lambda x(t) + \mu x(t-1), \qquad x(0) = 1, \tag{2.1}$$

on interval $I = [0, 10]$, where $u(t) = -t + 1$ for $t \in [-1, 0]$. The problem is solved by local methods in [2].

By choosing $\lambda = -3,5$, $\mu = 4$, $x_0(t) = 1$ and $\nu = 9$ we obtain the 9th approximation of the solution of the problem, see Figure 1a.

By choosing $\lambda = 0,5$, $\mu = -1$, $x_0(t) = 1$ and $\nu = 10$ we obtain the 10th approximation of the solution to the problem, see Figure 1b.




**(a)** **(b)**

FIGURE 1. Solution of problem (2.1)

2.2.

Let us solve the problem

$$\begin{aligned} x'(t) &= -x(t-1), \\ x(0) &= 1, \end{aligned} \tag{2.2}$$

on interval $I = [0, 3]$, where $x(t) = 1$ for $t \in [-1, 0]$. The problem is solved by the local methods in [2].

By choosing $x_0(t) = 1$, we obtain the 3rd approximation of solution of the problem, see Figure 2a.

By choosing $x_0(t) = 0$, we obtain the 3th approximation of solution to the problem, see Figure 2b, variant 2.

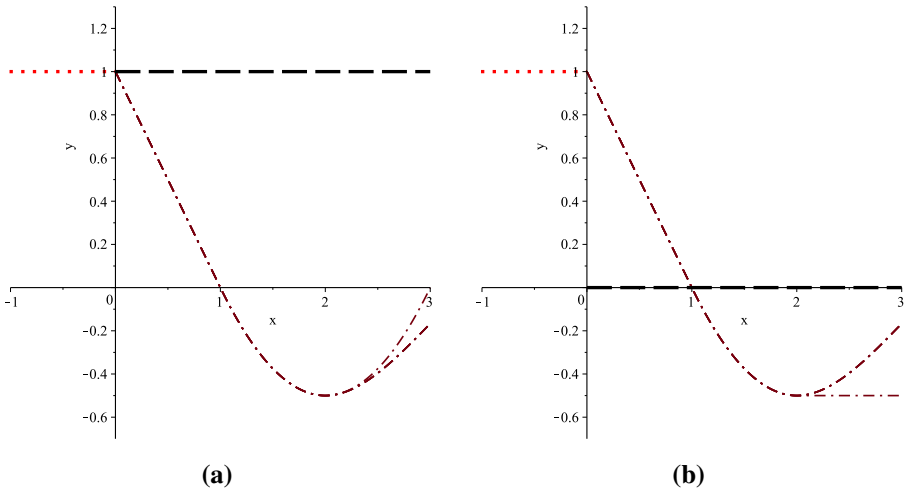



FIGURE 2. Solution of problem (2.2)

The next example is a generalization of the problem studied by Jan Tinbergen, a Dutch physicist and economist.

2.3.

Let us solve the problem

$$f'(t) = -a(t)f(t-\delta(t)), \quad t \in [0,T], \tag{2.3}$$

$$f(t) = h(t), \quad t \in [\Delta_m, 0], \tag{2.4}$$

where functions $a$ and $\delta$ are (for the sake of simplification) continuous on interval $[0,T]$, $\delta(t) > 0$ for $t \in [0,T]$, $\Delta_m = \min_{t\in[0,T]}(t-\delta(t))$, and $h$ is a continuous function on interval $[\Delta_m, 0]$. If $\delta(t) \equiv \Delta \geq 0$ (i.e., is a constant), equation (2.3) will be an equation with a constant delay and $\Delta_m = -\Delta$. Moreover, if $a(t) \equiv a > 0$ i.e., (2.3) is a linear differential equation with constant coefficients and constant delay, studied by Jan Tinbergen. In this case we construct 6th ($\nu = 6$) approximation.

In order to demonstrate possibilities of a new approach to solution of the original problem, assume that the "historical development" prior to time $t = 0$ can be simulated by the function $f = 2\sin t + 1$ and that the length of delay will vary. For that reason, the paper [25] considered a situation in which the delay $\Delta(t)$ is not constant, but it is a function of time $t$. For illustration, we worked on the assumption that the

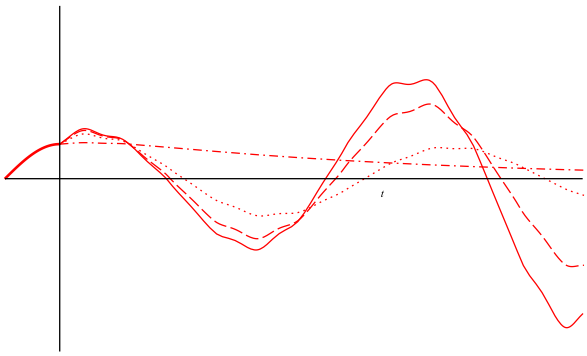



FIGURE 3. Solution to Tinbergen's model for a fluctuating delay

delay can be expressed by the function $\Delta(t) = 2,5 + 0,5\sin(5t)$. The parameter $a$ was chosen in accordance with Tinbergen's findings.

## ACKNOWLEDGEMENT

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# SOME INTERESTING CONGRUENCES FOR BALLOT NUMBERS

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*Abstract.* In this paper, we determine the sums $\sum_{k=0}^{n-1}\binom{2k+d}{k}/x^k$, $\sum_{k=0}^{n-1}k\binom{2k+d}{k}/x^k$ and some congruences can be obtained by using them. For example, for an odd prime $p\neq 5$,

$$\sum_{k=0}^{(p-1)/2}(-1)^k k\binom{2k+d}{k}\equiv\frac{(-1)^d}{5}\left(\frac{5}{p}\right)\left(F_{d+1-\left(\frac{5}{p}\right)}-(d+1)L_{d+1-\left(\frac{5}{p}\right)}\right)\pmod{p},$$

and for an odd prime $p$,

$$\sum_{k=0}^{(p-1)/2}\frac{k}{(-4)^k}B(k,d)\equiv-(-2)^{d-1}\left(\frac{2}{p}\right)dP_{d+1-\left(\frac{2}{p}\right)}\pmod{p},$$

where $d\in\{0,\ldots,(p-1)/2\}$, $F_n$ is the $n$th Fibonacci number, $L_n$ is the $n$th Lucas number and $P_n$ is the $n$th Pell number. $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

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*Keywords:* congruences, binomial coefficients, Ballot numbers

## 1. Introduction

The some elementary combinatorial properties of the Catalan and Ballot numbers are given in [2], [4] and [3]. In [1], [7], É. Lucas and N.J. Fine gave how to compute binomial coefficients modulo a prime. Lucas Theorem is given as follows:

If $p$ is a prime, $n$, $m$, $n_0$ and $m_0$ are non-negative integers, and $n_0$, $m_0$ are both less than $p$, then

$$\binom{np+n_0}{mp+m_0}\equiv\binom{n}{m}\binom{n_0}{m_0}\pmod{p}. \tag{1.1}$$

The Catalan numbers are given by

$$C_n=\frac{1}{n+1}\binom{2n}{n}=\binom{2n}{n}-\binom{2n}{n+1},\quad n\in\mathbb{N}.$$

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In [2], the Catalan numbers are special cases of the Ballot numbers

$$B(n,k)=\frac{k}{2n+k}\binom{2n+k}{n}.$$

In [13], Z.W. Sun and R. Tauraso obtained $\sum_{k=0}^{p^a-1}\binom{2k}{k+d}/m^k$ and $\sum_{k=0}^{p-1}\binom{2k}{k+d}/km^{k-1}$ mod $p$ for all $d=0,1,...,p^a$, where $m$ is any integer not divisible by $p$. For example, they showed that if $p\neq 2,5$, then

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}\equiv -5\frac{F_{p-\left(\frac{p}{5}\right)}}{p}\pmod{p}.$$

In [8], Z.W. Sun determined $\sum_{k=0}^{p^a-1}\binom{2k}{k+d}/m^k\ mod\ p^2$ for $d=0,1$; for example,

$$\sum_{k=0}^{p^a-1}\binom{2k}{k+d}/m^k\equiv\left(\frac{m^2-4m}{p^a}\right)+\left(\frac{m^2-4m}{p^{a-1}}\right)u_{p-\left(\frac{m^2-4m}{p}\right)}\pmod{p^2},$$

where an odd prime $p$ and $a,m\in\mathbb{Z}$ with $a>0$, $p\nmid m$.

In [11], Z.W. Sun used Lucas quotients in order to obtain $\sum_{k=0}^{(p-1)/2}\binom{2k}{k}/m^k$ modulo $p^2$ for any integer $m\not\equiv 0 \pmod{p}$; especially, he determined the following congruence:

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{16^k}\equiv\left(\frac{3}{p}\right)\pmod{p^2}.$$

In [9], Z.W. Sun gave the following congruence:

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)4^k}\equiv(-1)^{(p+1)/2}q_p(2)\pmod{p^2},$$

where an odd prime $p$ and $q_p(2)$ is a Fermat quotient.

In [6], S. Koparal and N. Ömür presented congruences involving central binomial coefficients and harmonic numbers. For example, for an odd prime $p$,

$$\sum_{k=0}^{(p-1)/2}(-1)^k\binom{2k}{k}H_{k-1}\equiv\frac{2^p}{p}\left(2F_{p+1}-5^{(p-1)/2}-1\right)\pmod{p},$$

where $H_n$ is the $n$th harmonic number.

In [5], K.H. Pilehrood et all gave that for a prime $p \neq 2,5$,

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k} F_{2k+1}}{(2k+1)\,16^k} \equiv (-1)^{(p+1)/2} \frac{F_p - \left(\frac{p}{5}\right)}{p} \pmod{p^2}.$$

In this paper, we determine the sums $\sum_{k=0}^{n-1} \binom{2k+d}{k}/x^k$, $\sum_{k=0}^{n-1} k\binom{2k+d}{k}/x^k$ and some congruences can be obtained by using them.

Two sequences $\{u_n(x)\}$ and $\{v_n(x)\}$ of polynomials are defined by for $n > 0$

$$u_{n+1}(x) = xu_n(x) - u_{n-1}(x) \text{ and } v_{n+1}(x) = xv_n(x) - v_{n-1}(x),$$

where $u_0(x) = 0,\ u_1(x) = 1$ and $v_0(x) = 2,\ v_1(x) = x$, respectively. The characteristic equation $y^2 - xy + 1 = 0$ of the sequences $\{u_n(x)\}$ and $\{v_n(x)\}$ has two roots

$$\alpha(x) = \frac{x + \sqrt{\Delta}}{2} \text{ and } \beta(x) = \frac{x - \sqrt{\Delta}}{2},$$

where $\Delta = x^2 - 4$. The Binet formulas of the sequences $\{u_n(x)\}$ and $\{v_n(x)\}$ are as follows:

$$u_n(x) = \frac{(\alpha(x))^n - (\beta(x))^n}{\alpha(x) - \beta(x)} \text{ and } v_n(x) = (\alpha(x))^n + (\beta(x))^n,$$

respectively. Clearly, for any $n \in \mathbb{N}$,

$$xu_n(x) + v_n(x) = 2u_{n+1}(x), \tag{1.2}$$

$$u_n(x) + \frac{x+2}{\Delta} v_n(x) = \frac{2}{\Delta}(v_{n+1}(x) + v_n(x)), \tag{1.3}$$

$$\Delta u_n(x) = v_{n+1}(x) - v_{n-1}(x). \tag{1.4}$$

It is seen that

$$(-1)^{n-1} u_n(-3) = F_{2n} \text{ and } (-1)^n v_n(-3) = L_{2n}, \tag{1.5}$$

$$(-1)^{n-1} u_n(-6) = \frac{1}{2} P_{2n} \text{ and } (-1)^n v_n(-6) = Q_{2n},$$

where $F_n$ and $L_n$ are the $n$th Fibonacci number and $n$th Lucas number, and $P_n$ and $Q_n$ are the $n$th Pell number and the $n$th Pell-Lucas number, respectively.

## 2. Some congruences related to Ballot numbers

In this section, we will investigate some congruences with the combinatorial identities. Now, we give the following lemmas for further use.

**Lemma 1.** *Let $r,s \in \mathbb{Z}$ and $D = r^2 - 4s$. Suppose that $p$ is an odd prime with $p \nmid sD$. Then*

$$\left(\frac{r \pm \sqrt{D}}{2}\right)^{p-\left(\frac{D}{p}\right)} \equiv s^{\left(1-\left(\frac{D}{p}\right)\right)/2} \pmod{p}, \tag{2.1}$$

*where $\frac{r\pm\sqrt{D}}{2}$ are roots of the equation $y^2 - ry + s = 0$ [10].*

**Lemma 2.** *For any $n \in \mathbb{N}$, we have*

$$\frac{d}{dx}\left(u_n(x-2)\right) = \frac{1}{\Delta}\left((n+1)\,v_n(x-2) - 2u_{n+1}(x-2)\right),$$

*where $\Delta$ as before.*

*Proof.* By differentiating both sides of $\alpha(x-2) = \frac{x-2+\sqrt{\Delta}}{2}$, we write

$$\frac{d}{dx}\left(\alpha(x-2)\right) = \frac{x-2+\sqrt{\Delta}}{2\sqrt{\Delta}} = \frac{\alpha(x-2)}{\sqrt{\Delta}}.$$

Similarly, it is clearly seen that

$$\frac{d}{dx}\left(\beta(x-2)\right) = \frac{-\beta(x-2)}{\sqrt{\Delta}} \text{ and } \frac{d}{dx}\left(\sqrt{\Delta}\right) = \frac{2\alpha(x-2)}{\sqrt{\Delta}} - 1 = \frac{2\beta(x-2)}{\sqrt{\Delta}} + 1.$$

Thus

$$\begin{aligned}
&\frac{d}{dx}\left(u_n(x-2)\right) = \\
&= \frac{d}{dx}\left(\frac{\alpha^n(x-2) - \beta^n(x-2)}{\sqrt{\Delta}}\right) \\
&= \frac{1}{\Delta}\left(\left(n\frac{\alpha^n(x-2)}{\sqrt{\Delta}} + n\frac{\beta^n(x-2)}{\sqrt{\Delta}}\right)\sqrt{\Delta}\right. \\
&\quad \left. -\left(\frac{2\alpha(x-2)}{\sqrt{\Delta}} - 1\right)\alpha^n(x-2) + \left(\frac{2\beta(x-2)}{\sqrt{\Delta}} + 1\right)\beta^n(x-2)\right) \\
&= \frac{1}{\Delta}\left(n\alpha^n(x-2) + n\beta^n(x-2) - 2\left(\frac{\alpha^{n+1}(x-2) - \beta^{n+1}(x-2)}{\sqrt{\Delta}}\right)\right. \\
&\quad \left. +\alpha^n(x-2) + \beta^n(x-2)\right) \\
&= \frac{1}{\Delta}\left(n v_n(x-2) - 2u_{n+1}(x-2) + v_n(x-2)\right) \\
&= \frac{1}{\Delta}\left((n+1)\,v_n(x-2) - 2u_{n+1}(x-2)\right).
\end{aligned}$$

This concludes the proof. □

**Theorem 1.** *For any $n$, $d \in \mathbb{Z}^+$, we have*

$$\sum_{k=0}^{n-1}\binom{2k+d}{k}x^{n-1-k} = \sum_{k=0}^{n+d-1}\binom{2n+d}{k}u_{n+d-k}(x-2) - x^{n+\lfloor d/2\rfloor}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(x-2)+u_{\lfloor d/2\rfloor}(x-2)\right), \tag{2.2}$$

*where* $\varepsilon = \left(1-(-1)^d\right)/2$.

*Proof.* To prove (2.2), we shall apply induction method on $n$.

For $n = 1$, we must show that for $d \in \mathbb{Z}^+$,

$$\sum_{k=0}^{d}\binom{d+2}{k}u_{d+1-k}(x-2) = = 1 + x^{\lfloor d/2\rfloor+1}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(x-2)+u_{\lfloor d/2\rfloor}(x-2)\right). \tag{2.3}$$

We have

$$\begin{aligned}
&\sum_{k=0}^{d}\binom{d+2}{k}u_{d+1-k}(x-2) = \\
&= 1+\sum_{k=0}^{d+2}\binom{d+2}{k}u_{d+1-k}(x-2) = 1+\sum_{k=0}^{d+2}\binom{d+2}{k}u_{k-1}(x-2) \\
&= 1+\frac{1}{\alpha(x-2)-\beta(x-2)}\left\{\frac{1}{\alpha(x-2)}\sum_{k=0}^{d+2}\binom{d+2}{k}(\alpha(x-2))^k\right. \\
&\quad \left.-\frac{1}{\beta(x-2)}\sum_{k=0}^{d+2}\binom{d+2}{k}(\beta(x-2))^k\right\} \\
&= 1+\frac{1}{\alpha(x-2)-\beta(x-2)}\left\{\frac{1}{\alpha(x-2)}(1+\alpha(x-2))^{d+2}\right. \\
&\quad \left.-\frac{1}{\beta(x-2)}(1+\beta(x-2))^{d+2}\right\}.
\end{aligned}$$

Using the identity $\alpha(x-2)\beta(x-2) = 1$ , we get

$$\begin{aligned}
&\sum_{k=0}^{d}\binom{d+2}{k}u_{d+1-k}(x-2) = \\
&= 1+\frac{1}{\alpha(x-2)-\beta(x-2)}\left\{(\beta(x-2))(1+\alpha(x-2))^{d+2}\right. \\
&\quad \left.-(\alpha(x-2))(1+\beta(x-2))^{d+2}\right\}.
\end{aligned}$$

From $d = \lfloor d/2 \rfloor + \lfloor d/2 \rfloor + \varepsilon$, we rewrite

$$\begin{aligned}
&\sum_{k=0}^{d} \binom{d+2}{k} u_{d+1-k}(x-2) = \\
&= 1 + \frac{1}{\alpha(x-2) - \beta(x-2)} \\
&\quad \times \Big\{ (\beta(x-2))(1+\alpha(x-2))^{\varepsilon}(1+\alpha(x-2))^{\lfloor d/2 \rfloor + 1}(1+\alpha(x-2))^{\lfloor d/2 \rfloor + 1} \\
&\quad - (\alpha(x-2))(1+\beta(x-2))^{\varepsilon}(1+\beta(x-2))^{\lfloor d/2 \rfloor + 1}(1+\beta(x-2))^{\lfloor d/2 \rfloor + 1} \Big\}.
\end{aligned}$$

Using the identity $(1+\alpha(x-2))(1+\beta(x-2)) = x$, we get

$$\begin{aligned}
&\sum_{k=0}^{d} \binom{d+2}{k} u_{d+1-k}(x-2) = \\
&= 1 + \frac{1}{\alpha(x-2) - \beta(x-2)} \\
&\quad \times \Big\{ (\beta(x-2) + \varepsilon)(1+\alpha(x-2))^{\lfloor d/2 \rfloor + 1}(1+\alpha(x-2))^{\lfloor d/2 \rfloor + 1} \\
&\quad - (\alpha(x-2) + \varepsilon)(1+\beta(x-2))^{\lfloor d/2 \rfloor + 1}(1+\beta(x-2))^{\lfloor d/2 \rfloor + 1} \Big\} \\
&= 1 + \frac{x^{\lfloor d/2 \rfloor + 1}}{\alpha(x-2) - \beta(x-2)} \Big\{ (\alpha(x-2))^{\lfloor d/2 \rfloor} - (\beta(x-2))^{\lfloor d/2 \rfloor} \\
&\quad + \varepsilon(\alpha(x-2))^{\lfloor d/2 \rfloor + 1} - \varepsilon(\beta(x-2))^{\lfloor d/2 \rfloor + 1} \Big\} \\
&= 1 + x^{\lfloor d/2 \rfloor + 1}\left( u_{\lfloor d/2 \rfloor}(x-2) + \varepsilon u_{\lfloor d/2 \rfloor + 1}(x-2) \right).
\end{aligned}$$

So, (2.2) holds for $n = 1$.

We assume that the result is true for some integer $n \geq 1$.

We must show that for $n+1$, (2.2) holds. By the induction hypothesis, for any $d \in \mathbb{Z}^+$, we write

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{2k+d}{k} x^{n-k} = \\
&= \binom{2n+d}{n} + x \sum_{k=0}^{n-1} \binom{2k+d}{k} x^{n-1-k} \\
&= \binom{2n+d}{n} + x \sum_{k=0}^{n+d-1} \binom{2n+d}{k} u_{n+d-k}(x-2) \\
&\quad - x^{n+1+\lfloor d/2 \rfloor}\left( \varepsilon u_{\lfloor d/2 \rfloor + 1}(x-2) + u_{\lfloor d/2 \rfloor}(x-2) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \binom{2n+d}{n} + x\sum_{k=0}^{n+d-1}\binom{2n+d}{k}u_{n+d-k}(x-2)\\
&\quad -2\sum_{k=0}^{n+d-1}\binom{2n+d}{k}u_{n+d-k}(x-2)+2\sum_{k=0}^{n+d-1}\binom{2n+d}{k}u_{n+d-k}(x-2)\\
&\quad -\sum_{k=0}^{n+d-1}\binom{2n+d}{k}u_{n+d-1-k}(x-2)+\sum_{k=0}^{n+d-1}\binom{2n+d}{k}u_{n+d-1-k}(x-2)\\
&\quad -x^{n+\lfloor d/2\rfloor+1}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(x-2)+u_{\lfloor d/2\rfloor}(x-2)\right)\\
&= (x-2)\sum_{k=0}^{n+d}\binom{2n+d}{k}u_{n+d-k}(x-2)-\sum_{k=0}^{n+d}\binom{2n+d}{k}u_{n+d-1-k}(x-2)\\
&\quad +2\sum_{k=0}^{n+d-1}\binom{2n+d}{k}u_{n+d-k}(x-2)+\sum_{k=0}^{n+d-1}\binom{2n+d}{k}u_{n+d-1-k}(x-2)\\
&\quad -x^{n+\lfloor d/2\rfloor+1}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(x-2)+u_{\lfloor d/2\rfloor}(x-2)\right).
\end{aligned}
$$

Since $(x-2)\,u_n(x-2)=u_{n+1}(x-2)+u_{n-1}(x-2)$, we have

$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{2k+d}{k}x^{n-k}=\\
&= \sum_{k=0}^{n+d}\binom{2n+d}{k}u_{n+d+1-k}(x-2)+2\sum_{k=-1}^{n+d-1}\binom{2n+d}{k}u_{n+d-k}(x-2)\\
&\quad +\sum_{k=-2}^{n+d-1}\binom{2n+d}{k}u_{n+d-1-k}(x-2)\\
&\quad -x^{n+\lfloor d/2\rfloor+1}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(x-2)+u_{\lfloor d/2\rfloor}(x-2)\right)\\
&= \sum_{k=0}^{n+d}\binom{2n+d}{k}u_{n+d+1-k}(x-2)+2\sum_{k=0}^{n+d}\binom{2n+d}{k-1}u_{n+d+1-k}(x-2)\\
&\quad +\sum_{k=0}^{n+d+1}\binom{2n+d}{k-2}u_{n+d+1-k}(x-2)\\
&\quad -x^{n+\lfloor d/2\rfloor+1}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(x-2)+u_{\lfloor d/2\rfloor}(x-2)\right),
\end{aligned}
$$

and hence

$$\sum_{k=0}^{n}\binom{2k+d}{k}x^{n-k}=$$
$$=\sum_{k=0}^{n+d}\left(\binom{2n+d}{k}+2\binom{2n+d}{k-1}+\binom{2n+d}{k-2}\right)u_{n+d+1-k}(x-2)$$
$$-x^{n+\lfloor d/2\rfloor+1}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(x-2)+u_{\lfloor d/2\rfloor}(x-2)\right).$$

By the binomial identity $\binom{2n+d}{k}+2\binom{2n+d}{k-1}+\binom{2n+d}{k-2}=\binom{2n+d+2}{k}$, we get

$$\sum_{k=0}^{n}\binom{2k+d}{k}x^{n-k}=\sum_{k=0}^{n+d}\binom{2(n+1)+d}{k}u_{n+d+1-k}(x-2)$$
$$-x^{n+\lfloor d/2\rfloor+1}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(x-2)+u_{\lfloor d/2\rfloor}(x-2)\right).$$

Hence the result is true for all integers $n\geq 0$. □

As a result of Theorem 1, we may give the following congruence.

**Corollary 1.** *Let $p$ be an odd prime. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\binom{2k+d}{k}\equiv m^{\lfloor d/2\rfloor-(p-1)/2} \tag{2.4}$$
$$\times\left\{(1-\varepsilon)u_{(p-1)/2+\lfloor d/2\rfloor}(m-2)+((m-1)\varepsilon+1)u_{(p+1)/2+\lfloor d/2\rfloor}(m-2)\right\}$$
$$-m^{\lfloor d/2\rfloor+1}(\varepsilon u_{\lfloor d/2\rfloor+1}(m-2)+u_{\lfloor d/2\rfloor}(m-2))\pmod{p},$$

*where $\varepsilon$, $\Delta$ as before, $d\in\{0,1,\ldots,(p-1)/2\}$ and $m\in\mathbb{Z}$ with $p\nmid m\Delta$.*

*Proof.* Substituting $n=(p+1)/2$ and $x=m$ in (2.2), we write

$$\sum_{k=0}^{(p-1)/2}\binom{2k+d}{k}m^{(p-1)/2-k}=\sum_{k=0}^{(p-1)/2+d}\binom{p+d+1}{k}u_{(p+1)/2+d-k}(m-2)$$
$$-m^{(p+1)/2+\lfloor d/2\rfloor}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(m-2)+u_{\lfloor d/2\rfloor}(m-2)\right).$$

By the congruence in (1.1), it is easily seen that

$$m^{(p-1)/2}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k+d}{k}}{m^k}\equiv$$
$$\equiv\sum_{k=0}^{d+1}\binom{d+1}{k}u_{(p+1)/2+d-k}(m-2)$$
$$-m^{\lfloor d/2\rfloor+(p+1)/2}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(m-2)+u_{\lfloor d/2\rfloor}(m-2)\right)$$

$$
\begin{aligned}
&= \sum_{k=0}^{d+1} \binom{d+1}{k} u_{(p-1)/2+k}(m-2) \\
&\quad - m^{\lfloor d/2 \rfloor + (p+1)/2} \left( \varepsilon u_{\lfloor d/2 \rfloor + 1}(m-2) + u_{\lfloor d/2 \rfloor}(m-2) \right) \\
&= \frac{(\alpha(m-2))^{(p-1)/2}}{\alpha(m-2) - \beta(m-2)} \sum_{k=0}^{d+1} \binom{d+1}{k} (\alpha(m-2))^k \\
&\quad - \frac{(\beta(m-2))^{(p-1)/2}}{\alpha(m-2) - \beta(m-2)} \sum_{k=0}^{d+1} \binom{d+1}{k} (\beta(m-2))^k \\
&\quad - m^{(p+1)/2 + \lfloor d/2 \rfloor} \left( \varepsilon u_{\lfloor d/2 \rfloor + 1}(m-2) + u_{\lfloor d/2 \rfloor}(m-2) \right) \pmod{p}.
\end{aligned}
$$

By Binomial theorem, we have

$$
\begin{aligned}
&m^{(p-1)/2} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k+d}{k}}{m^k} \equiv \\
&\equiv \frac{(\alpha(m-2))^{(p-1)/2} (1+\alpha(m-2))^{d+1}}{\alpha(m-2) - \beta(m-2)} - \frac{(\beta(m-2))^{(p-1)/2} (1+\beta(m-2))^{d+1}}{\alpha(m-2) - \beta(m-2)} \\
&\quad - m^{(p+1)/2 + \lfloor d/2 \rfloor} \left( \varepsilon u_{\lfloor d/2 \rfloor + 1}(m-2) + u_{\lfloor d/2 \rfloor}(m-2) \right) \pmod{p}.
\end{aligned}
$$

From $d = \lfloor d/2 \rfloor + \lfloor d/2 \rfloor + \varepsilon$, then

$$
\begin{aligned}
&\sum_{k=0}^{(p-1)/2} \frac{\binom{2k+d}{k}}{m^k} \equiv \\
&\equiv \frac{1}{\alpha(m-2) - \beta(m-2)} \times \\
&\quad \times \Big\{ (\alpha(m-2))^{(p-1)/2} (1+\alpha(m-2))^{\lfloor d/2 \rfloor + \varepsilon + 1} (1+\alpha(m-2))^{\lfloor d/2 \rfloor} \\
&\quad - (\beta(m-2))^{(p-1)/2} (1+\beta(m-2))^{\lfloor d/2 \rfloor + \varepsilon + 1} (1+\beta(m-2))^{\lfloor d/2 \rfloor} \Big\} \\
&\quad - m^{(p+1)/2 + \lfloor d/2 \rfloor} \left( \varepsilon u_{\lfloor d/2 \rfloor + 1}(m-2) + u_{\lfloor d/2 \rfloor}(m-2) \right) \pmod{p}.
\end{aligned}
$$

Using the equalities of $\alpha(m-2)\beta(m-2) = 1$ and $(1+\alpha(m-2))(1+\beta(m-2)) = m$, we have the proof. □

For example, when $d = 0$ and $m \in \mathbb{Z}$ with $p \nmid m$

$$
\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{m^k} \equiv \left( \frac{m(m-4)}{p} \right) \pmod{p} \text{ [12]},
$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{1}{(-4)^k}\binom{2k+d}{k} \equiv (-2)^d \left(\frac{2}{p}\right) P_{d-\left(\frac{2}{p}\right)} \pmod{p},$$

where $d \in \{0,1,\ldots,(p-1)/2\}$.

**Theorem 2.** *For any $n$, $d \in \mathbb{Z}^+$, we have*

$$\begin{aligned}\sum_{k=0}^{n-1} k\binom{2k+d}{k}x^{n-1-k} &= \sum_{k=0}^{n+d-1}\binom{2n+d}{k}\Big((n-1)\,u_{n+d-k}(x-2) \\ &-\frac{x}{\Delta}\left((n+d+1-k)\,v_{n+d-k}(x-2)-2u_{n+d+1-k}(x-2)\right)\Big) \\ &+(\lfloor d/2\rfloor+1)\,x^{n+\lfloor d/2\rfloor}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(x-2)+u_{\lfloor d/2\rfloor}(x-2)\right) \\ &+\frac{x^{n+\lfloor d/2\rfloor+1}}{\Delta}\Big(\varepsilon\left((\lfloor d/2\rfloor+2)\,v_{\lfloor d/2\rfloor+1}(x-2)-2u_{\lfloor d/2\rfloor+2}(x-2)\right) \\ &+(\lfloor d/2\rfloor+1)\,v_{\lfloor d/2\rfloor}(x-2)-2u_{\lfloor d/2\rfloor+1}(x-2)\Big),\end{aligned} \tag{2.5}$$

*where $\varepsilon$, $\Delta$ as before.*

*Proof.* To prove equality (2.5), we take the derivative of (2.2) with respect to x:

$$\begin{aligned}&\sum_{k=0}^{n-1}\binom{2k+d}{k}(n-1-k)\,x^{n-2-k} = \\ &= \sum_{k=0}^{n+d-1}\binom{2n+d}{k}\frac{1}{\Delta}\left((n+d+1-k)\,v_{n+d-k}(x-2)\ -2u_{n+d+1-k}(x-2)\right) \\ &-(n+\lfloor d/2\rfloor)\,x^{n+\lfloor d/2\rfloor-1}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(x-2)+u_{\lfloor d/2\rfloor}(x-2)\right) \\ &-\frac{x^{n+\lfloor d/2\rfloor}}{\Delta}\Big(\varepsilon\left((\lfloor d/2\rfloor+2)\,v_{\lfloor d/2\rfloor+1}(x-2)-2u_{\lfloor d/2\rfloor+2}(x-2)\right) \\ &+(\lfloor d/2\rfloor+1)\,v_{\lfloor d/2\rfloor}(x-2)-2u_{\lfloor d/2\rfloor+1}(x-2)\Big),\end{aligned}$$

and then

$$\begin{aligned}&\sum_{k=0}^{n-1} k\binom{2k+d}{k}x^{n-2-k} = \\ &= \frac{n-1}{x}\sum_{k=0}^{n-1}\binom{2k+d}{k}x^{n-1-k} \\ &-\frac{1}{\Delta}\sum_{k=0}^{n+d-1}\binom{2n+d}{k}\left((n+d+1-k)\,v_{n+d-k}(x-2)-2u_{n+d+1-k}(x-2)\right)\end{aligned} \tag{2.6}$$

$$+(n+\lfloor d/2\rfloor)\,x^{n+\lfloor d/2\rfloor-1}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(x-2)+u_{\lfloor d/2\rfloor}(x-2)\right)$$
$$+\frac{x^{n+\lfloor d/2\rfloor}}{\Delta}\left(\varepsilon\left((\lfloor d/2\rfloor+2)\,v_{\lfloor d/2\rfloor+1}(x-2)-2u_{\lfloor d/2\rfloor+2}(x-2)\right)\right.$$
$$\left.(\lfloor d/2\rfloor+1)\,v_{\lfloor d/2\rfloor}(x-2)-2u_{\lfloor d/2\rfloor+1}(x-2)\right).$$

Multiplying both sides of (2.6) with $x$ and using Lemma 1, the proof is the complete. □

Now, from Theorem 2, we have the following congruence:

**Corollary 2.** *Let $p$ be an odd prime. Then*

$$\sum_{k=0}^{(p-1)/2} k\frac{\binom{2k+d}{k}}{m^k}\equiv \tag{2.7}$$
$$\equiv-\frac{m^{\lfloor d/2\rfloor+1-(p-1)/2}}{\Delta}\times \tag{2.8}$$
$$\times\left((d+2)\left(v_{(p+1)/2+\lfloor d/2\rfloor}(m-2)+\varepsilon v_{(p+1)/2+\lfloor d/2\rfloor+1}(m-2)\right)\right.$$
$$\left.-2\left((1-\varepsilon)\,u_{(p+1)/2+\lfloor d/2\rfloor}(m-2)+((m-1)\,\varepsilon+1)\,u_{(p+1)/2+\lfloor d/2\rfloor+1}(m-2)\right)\right)$$
$$+\frac{m^{\lfloor d/2\rfloor+\varepsilon+1}}{\Delta}\left(2(\lfloor d/2\rfloor+1)\left(v_{\lfloor d/2\rfloor+1}(m-2)+(1-\varepsilon)\,v_{\lfloor d/2\rfloor}(m-2)\right)\right.$$
$$\left.-(2-\varepsilon)\,m u_{\lfloor d/2\rfloor+1}(m-2)\right)\qquad(\mathrm{mod}\ p),$$

*where $\varepsilon$, $\Delta$ as before, $d\in\{0,1,\ldots,(p-1)/2\}$ and $m\in\mathbb{Z}$ with $p\nmid m\Delta$.*

*Proof.* Substituting $n=(p+1)/2$ and $x=m$ in (2.5), we write

$$\sum_{k=0}^{(p-1)/2} k\binom{2k+d}{k}m^{(p-1)/2-k}=$$
$$=\sum_{k=0}^{(p-1)/2+d}\binom{p+d+1}{k}\left(\frac{p-1}{2}u_{(p+1)/2+d-k}(m-2)\right.$$
$$\left.-\frac{m}{\Delta}\left(\left(\frac{p+1}{2}+d+1-k\right)v_{(p+1)/2+d-k}(m-2)-2u_{(p+1)/2+d+1-k}(m-2)\right)\right)$$
$$+(\lfloor d/2\rfloor+1)\,m^{(p+1)/2+\lfloor d/2\rfloor}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(m-2)+u_{\lfloor d/2\rfloor}(m-2)\right)$$
$$+\frac{m^{(p+1)/2+\lfloor d/2\rfloor+1}}{\Delta}\left(\varepsilon\left((\lfloor d/2\rfloor+2)\,v_{\lfloor d/2\rfloor+1}(m-2)-2u_{\lfloor d/2\rfloor+2}(m-2)\right)\right.$$
$$\left.+(\lfloor d/2\rfloor+1)\,v_{\lfloor d/2\rfloor}(m-2)-2u_{\lfloor d/2\rfloor+1}(m-2)\right).$$

By the congruence in (1.1), it is easily seen that

$$
\begin{aligned}
&m^{(p-1)/2}\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k+d}{k}}{m^k}\equiv\\
&\equiv\sum_{k=0}^{d+1}\binom{d+1}{k}\Bigg(-\frac{1}{2}u_{(p+1)/2+d-k}(m-2)\\
&\quad-\frac{m}{\Delta}\left(\left(\frac{3}{2}+d-k\right)v_{(p+1)/2+d-k}(m-2)-2u_{(p+1)/2+d+1-k}(m-2)\right)\Bigg)\\
&\quad+(\lfloor d/2\rfloor+1)m^{(p+1)/2+\lfloor d/2\rfloor}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(m-2)+u_{\lfloor d/2\rfloor}(m-2)\right)\\
&\quad+\frac{m^{(p+1)/2+\lfloor d/2\rfloor+1}}{\Delta}\left(\varepsilon\left((\lfloor d/2\rfloor+2)v_{\lfloor d/2\rfloor+1}(m-2)-2u_{\lfloor d/2\rfloor+2}(m-2)\right)\right.\\
&\quad\left.+(\lfloor d/2\rfloor+1)v_{\lfloor d/2\rfloor}(m-2)-2u_{\lfloor d/2\rfloor+1}(m-2)\right)\\
&=-\frac{1}{2}\sum_{k=0}^{d+1}\binom{d+1}{k}u_{(p-1)/2+k}(m-2)+\frac{2m}{\Delta}\sum_{k=0}^{d+1}\binom{d+1}{k}u_{(p+1)/2+k}(m-2)\\
&\quad-\frac{m}{\Delta}\sum_{k=0}^{d+1}\binom{d+1}{k}\left(k+\frac{1}{2}\right)v_{(p-1)/2+k}(m-2)\\
&\quad+(\lfloor d/2\rfloor+1)m^{(p+1)/2+\lfloor d/2\rfloor}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(m-2)+u_{\lfloor d/2\rfloor}(m-2)\right)\\
&\quad+\frac{m^{(p+1)/2+\lfloor d/2\rfloor+1}}{\Delta}\left(\varepsilon\left((\lfloor d/2\rfloor+2)v_{\lfloor d/2\rfloor+1}(m-2)-2u_{\lfloor d/2\rfloor+2}(m-2)\right)\right.\\
&\quad\left.+(\lfloor d/2\rfloor+1)v_{\lfloor d/2\rfloor}(m-2)-2u_{\lfloor d/2\rfloor+1}(m-2)\right)\qquad(\mathrm{mod}\ p).
\end{aligned}
$$

From the Binet formulae of the sequences $\{u_n(m-2)\}$ and $\{v_n(m-2)\}$ and Binomial theorem, we have

$$
\begin{aligned}
&m^{(p-1)/2}\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k+d}{k}}{m^k}\equiv\\
&\equiv-\frac{1}{2}m^{\lfloor d/2\rfloor}\left((1-\varepsilon)u_{(p-1)/2+\lfloor d/2\rfloor}(m-2)+((m-1)\varepsilon+1)u_{(p+1)/2+\lfloor d/2\rfloor}(m-2)\right)\\
&\quad+\frac{2}{\Delta}m^{\lfloor d/2\rfloor+1}\left((1-\varepsilon)u_{(p+1)/2+\lfloor d/2\rfloor}(m-2)+((m-1)\varepsilon+1)u_{(p+1)/2+\lfloor d/2\rfloor+1}(m-2)\right)\\
&\quad-\frac{m^{\lfloor d/2\rfloor+1}}{\Delta}(d+1)\left(v_{(p+1)/2+\lfloor d/2\rfloor}(m-2)+\varepsilon v_{(p+1)/2+\lfloor d/2\rfloor+1}(m-2)\right)\\
&\quad-\frac{m^{\lfloor d/2\rfloor+1}}{2\Delta}\left((1-\varepsilon)v_{(p-1)/2+\lfloor d/2\rfloor}(m-2)+((m-1)\varepsilon+1)v_{(p+1)/2+\lfloor d/2\rfloor}(m-2)\right)\\
&\quad+(\lfloor d/2\rfloor+1)m^{(p+1)/2+\lfloor d/2\rfloor}\left(\varepsilon u_{\lfloor d/2\rfloor+1}(m-2)+u_{\lfloor d/2\rfloor}(m-2)\right)
\end{aligned}
$$

$$+\frac{m^{(p+1)/2+\lfloor d/2\rfloor+1}}{\Delta}\left(\varepsilon\left((\lfloor d/2\rfloor+2)\,v_{\lfloor d/2\rfloor+1}(m-2)-2u_{\lfloor d/2\rfloor+2}(m-2)\right)\right.$$
$$\left.+(\lfloor d/2\rfloor+1)\,v_{\lfloor d/2\rfloor}(m-2)-2u_{\lfloor d/2\rfloor+1}(m-2)\right) \pmod{p}.$$

From (1.2), (1.3) and (1.4), we obtained the desired result. □

For example, for an odd prime $p \neq 5$

$$\sum_{k=0}^{(p-1)/2}(-1)^k k\binom{2k+d}{k} \equiv$$
$$\equiv \frac{(-1)^d}{5}\left(\frac{5}{p}\right)\left(F_{d+1-\left(\frac{5}{p}\right)}-(d+1)\,L_{d+1-\left(\frac{5}{p}\right)}\right) \pmod{p},$$

and for an odd prime $p$,

$$\sum_{k=0}^{(p-1)/2}\frac{k}{(-4)^k}\binom{2k+d}{k} \equiv$$
$$\equiv (-2)^{d-2}\left(\frac{2}{p}\right)\left(P_{d+1-\left(\frac{2}{p}\right)}-\frac{d+1}{2}Q_{d+1-\left(\frac{2}{p}\right)}\right) \pmod{p},$$

where $d \in \{0,1,\ldots,(p-1)/2\}$.

From Corollary 1 and Corollary 2, clearly the congruences are given as follows:

**Corollary 3.** *Let $p$ be an odd prime and $d \in \{0,1,2,\ldots,(p-1)/2\}$. For $m \in \mathbb{Z}$ with $p \nmid m\Delta$, then*

$$\sum_{k=0}^{(p-1)/2}\frac{B(k,d)}{m^k} \equiv$$
$$\equiv m^{\lfloor (d+1)/2\rfloor-(p-1)/2-\varepsilon}\left\{u_{(p-1)/2+\lfloor (d+1)/2\rfloor-\varepsilon}(m-2)-u_{(p+1)/2+\lfloor (d+1)/2\rfloor}(m-2)\right\}$$
$$-m^{\lfloor (d+1)/2\rfloor}\left(u_{\lfloor (d-1)/2\rfloor}(m-2)-u_{\lfloor (d-1)/2\rfloor+2-\varepsilon}(m-2)\right)-\delta_{0,d} \pmod{p},$$

*and*

$$\sum_{k=0}^{(p-1)/2}\frac{k}{m^k}B(k,d) \equiv$$
$$\equiv \frac{m^{\lfloor (d+1)/2\rfloor}}{\Delta}d\left\{m^{1-\varepsilon}\left(v_{\lfloor (d+1)/2\rfloor-\varepsilon}(m-2)-v_{\lfloor (d+1)/2\rfloor+1}(m-2)\right)\right.$$
$$\left.-m^{-(p-1)/2}\left(v_{(p+1)/2+\lfloor (d-1)/2\rfloor}(m-2)-v_{(p+1)/2+\lfloor (d+1)/2\rfloor+1-\varepsilon}(m-2)\right)\right\} \pmod{p},$$

*where $\varepsilon, \Delta$ as before and $\delta_{i,j}$ is the Kronecker delta.*

*Proof.* Using the binomial identities $\binom{n}{k}=\frac{n}{k}\binom{n-1}{k-1}$ and $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$, we get

$$\sum_{k=0}^{(p-1)/2}\frac{B(k,d)}{m^k}=\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\frac{d}{2k+d}\binom{2k+d}{k}=\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\frac{d}{k}\binom{2k+d-1}{k-1}.$$

For $k,d\in\mathbb{Z}^+$, it is known that

$$\binom{2k+d-1}{k}=\binom{2k+d-1}{k-1}+\frac{d}{k}\binom{2k+d-1}{k-1}.$$

So, we have

$$\begin{aligned}
&\sum_{k=0}^{(p-1)/2}\frac{B(k,d)}{m^k}\\
&=\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\left(\binom{2k+d-1}{k}-\binom{2k+d-1}{k-1}\right)\\
&=\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\binom{2k+d-1}{k}-\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\binom{2k+d-1}{k-1}\\
&=\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\binom{2k+d-1}{k}-\sum_{k=-1}^{(p-3)/2}\frac{1}{m^{k+1}}\binom{2k+d+1}{k}\\
&=\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\binom{2k+d-1}{k}-\frac{1}{m}\sum_{k=0}^{(p-3)/2}\frac{1}{m^k}\binom{2k+d+1}{k}-\binom{d-1}{-1}\\
&=\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\binom{2k+d-1}{k}-\frac{1}{m}\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\binom{2k+d+1}{k}-\delta_{0,d}\\
&\quad+\frac{1}{m^{(p+1)/2}}\binom{p+d}{(p-1)/2}.
\end{aligned}$$

Since $\binom{p+d}{(p-1)/2}\equiv 0 \pmod{p}$ for $d\in\{0,1,2,\ldots,(p-1)/2\}$, from (1.1), we get

$$\begin{aligned}
&\sum_{k=0}^{(p-1)/2}\frac{B(k,d)}{m^k}\equiv\\
&\equiv\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\binom{2k+d-1}{k}-\frac{1}{m}\sum_{k=0}^{(p-1)/2}\frac{1}{m^k}\binom{2k+d+1}{k}-\delta_{0,d}\pmod{p}.
\end{aligned}$$

So, taking $d-1$ and $d+1$ instead of $d$ in (2.4), respectively, this concludes the proof. □

For example, for an odd prime $p \neq 5$

$$\sum_{k=0}^{(p-1)/2} (-1)^k B(k,d) \equiv (-1)^{d+1} \left(\frac{5}{p}\right) L_{d-\left(\frac{5}{p}\right)} - \delta_{0,d} \pmod{p},$$

and for an odd prime $p$,

$$\sum_{k=0}^{(p-1)/2} \frac{k}{(-4)^k} B(k,d) \equiv -(-2)^{d-1} \left(\frac{2}{p}\right) dP_{d+1-\left(\frac{2}{p}\right)} \pmod{p},$$

where $d \in \{0,1,2,\ldots,(p-1)/2\}$.

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# ON A CLASS OF UNIVALENT FUNCTIONS DEFINED BY SĂLĂGEAN INTEGRO-DIFFERENTIAL OPERATOR

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*Abstract.* In this paper we consider the $\mathscr{L}^n : \mathcal{A} \to \mathcal{A}$, $\mathscr{L}^n f(z) = (1-\lambda)\mathscr{D}^n f(z) + \lambda I^n f(z)$ linear operator, where $\mathscr{D}^n$ is the Sălăgean differential operator and $I^n$ is the Sălăgean integral operator. We study several differential subordinations generated by $\mathscr{L}^n$. We introduce a class of holomorphic functions $L_n^m(\beta)$, and obtain some subordination results.

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*Keywords:* analytic functions, convex function, Sălăgean integro-differential operator, differential operator, differential subordination, dominant, best dominant

## 1. Preliminaries

Let $U$ be the unit disk in the complex plane:

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $\mathcal{H}(U)$ be the space of holomorphic functions in $U$ and let

$$\mathcal{A}_m = \left\{f \in \mathcal{H}(U) : \ f(z) = z + a_{m+1}z^{m+1} + \cdots, z \in U\right\}$$

with $\mathcal{A}_1 = \mathcal{A}$. For $a \in \mathbb{C}$ and $m \in \mathbb{N}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N} = \{1, 2, \ldots\}$ let

$$\mathcal{H}[a,m] = \left\{f \in \mathcal{H}(U) : \ f(z) = a + a_m z^m + a_{m+1}z^{m+1} + \cdots, z \in U\right\}.$$

Denote by

$$K = \left\{f \in \mathcal{A} : \Re\frac{zf''(z)}{f'(z)} + 1 > 0, z \in U\right\}$$

the class of normalized convex functions in $U$.

**Definition 1** ([5], def. 3.5.1)**.** Let $f$ and $g$ be analytic functions in $U$. We say that the function $f$ is subordinate to the function $g$, if there exists a function $w$, which is analytic in $U$ and $w(0) = 0; |w(z)| < 1; z \in U$, such that $f(z) = g(w(z))$; $\forall z \in U$. We denote by $\prec$ the subordination relation. If $g$ is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

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Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ be a function and let $h$ be univalent in $U$. If $p$ is analytic in $U$ and satisfies the (second-order) differential subordination

$(i)$ $\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z)$, $(z \in U)$

then $p$ is called a solution of the differential subordination. The univalent function $q$ is called a dominant of the solution of the differential subordination, or more simply a dominant, if $p \prec q$ for all $p$ satisfying $(i)$. A dominant $\widetilde{q}$, which satisfies $\widetilde{q} \prec q$ for all dominants $q$ of $(i)$ is said to be the best dominant of $(i)$. The best dominant is unique up to a rotation of $U$. In order to prove the original results we use the following lemmas.

**Lemma 1** (Hallenbeck and Ruscheweyh, [2]). *Let $h$ be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $\Re\gamma \geq 0$. If $p \in \mathcal{H}[a,n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z), \quad z \in U$$

*then*

$$p(z) \prec q(z) \prec h(z), z \in U$$

*where*

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1}dt, \quad z \in U.$$

**Lemma 2** (Miller and Mocanu, [3]). *Let $q$ be a convex function in $U$ and let*

$$h(z) = q(z) + n\alpha zq'(z), \quad z \in U$$

*where $\alpha > 0$ and $n$ is a positive integer. If*

$$p(z) = q(0) + p_n z^n + p_{n+1}z^{n+1} + \cdots, \quad z \in U$$

*is holomorphic in $U$ and*

$$p(z) + n\alpha zp'(z) \prec h(z), \quad z \in U$$

*then*

$$p(z) \prec q(z)$$

*and this result is sharp.*

**Definition 2** ([8]). For $f \in \mathcal{A}, n \in \mathbb{N}_0$, the Sălăgean differential operator $\mathscr{D}^n$ is defined by $\mathscr{D}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathscr{D}^0 f(z) = f(z),$$

$$\cdots$$

$$\mathscr{D}^{n+1} f(z) = z\left(\mathscr{D}^n f(z)\right)', z \in U$$

*Remark* 1. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$\mathscr{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, z \in U.$$

**Definition 3** ([8])**.** For $f \in \mathcal{A}, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \ldots\}$, the operator $I^n$ is defined by

$$I^0 f(z) = f(z),$$

$$\ldots$$

$$I^n f(z) = I\left(I^{n-1} f(z)\right), z \in U$$

*Remark* 2. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k,$$

$z \in U$, $(n \in \mathbb{N}_0)$ and $z\left(I^n f(z)\right)' = I^{n-1} f(z)$.

**Definition 4.** Let $\lambda \geq 0, n \in \mathbb{N}$. Denote by $\mathscr{L}^n$ the operator given by $\mathscr{L}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathscr{L}^n f(z) = (1-\lambda)\mathscr{D}^n f(z) + \lambda I^n f(z), z \in U.$$

*Remark* 3. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$\mathscr{L}^n f(z) = z + \sum_{k=2}^{\infty} \left[k^n (1-\lambda) + \lambda \frac{1}{k^n}\right] a_k z^k, z \in U. \tag{1.1}$$

## 2. Main results

**Theorem 1.** *Let $q$ be a convex function, $q(0) = 1$ and let $h$ be the function*

$$h(z) = q(z) + zq'(z), z \in U.$$

*If $f \in \mathcal{A}$, $\lambda \geq 0$, $n \in \mathbb{N}$ and satisfies the differential subordination*

$$\left[\mathscr{L}^n f(z)\right]' \prec h(z), \ \ z \in U \tag{2.1}$$

*then*

$$\frac{\mathscr{L}^n f(z)}{z} \prec q(z), \ \ z \in U$$

*and this result is sharp.*

*Proof.* Let

$$p(z)=\frac{\mathscr{L}^n f(z)}{z}=\frac{z+\sum_{k=2}^{\infty}\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]a_k z^k}{z}=1+p_n z^n+p_{n+1}z^{n+1}+\cdots \tag{2.2}$$

$z\in U$. From (2.2) we have $p\in\mathscr{H}[1,1]$. Let

$$\mathscr{L}^n f(z)=zp(z), z\in U. \tag{2.3}$$

Differentiating (2.3), we obtain

$$\left[\mathscr{L}^n f(z)\right]'=p(z)+zp'(z), z\in U. \tag{2.4}$$

Then (2.1) becomes

$$p(z)+zp'(z)\prec h(z), z\in U. \tag{2.5}$$

By using Lemma 2, we have

$$p(z)\prec q(z), z\in U,$$

i.e.

$$\frac{\mathscr{L}^n f(z)}{z}\prec q(z),\quad z\in U.$$

□

*Remark* 4. If $\lambda=0$ we get Theorem 4 from Oros [6] and for $\lambda=1$ we get Theorem 4 from Bălăeţi [1].

*Example* 1. For $\lambda=0$, $n=1$, $f\in\mathscr{A}$ we deduce that

$$f'(z)+zf''(z)\prec\frac{1}{(1-z)^2},\quad z\in U$$

implies

$$f'(z)\prec\frac{1}{1-z},\quad z\in U.$$

*Example* 2. For $\lambda=1$, $n=1$, $f\in\mathscr{A}$ we deduce that

$$\frac{f(z)}{z}\prec\frac{1}{(1-z)^2},\quad z\in U$$

implies

$$\frac{\int_0^z f(t)t^{-1}dt}{z}\prec\frac{1}{1-z},\quad z\in U.$$

**Theorem 2.** *Let $q$ be a convex function, $q(0)=1$ and let $h$ be the function*

$$h(z)=q(z)+zq'(z), z\in U.$$

*If $f\in\mathcal{A}$, $\lambda\geq 0$, $n\in\mathbb{N}$ and satisfies the differential subordination*

$$\left(\frac{z\mathcal{L}^{n+1}f(z)}{\mathcal{L}^{n}f(z)}\right)'\prec h(z),\ \ z\in U \tag{2.6}$$

*then*

$$\frac{\mathcal{L}^{n+1}f(z)}{\mathcal{L}^{n}f(z)}\prec q(z),\ \ z\in U$$

*and this result is sharp.*

*Proof.* Let

$$p(z)=\frac{\mathcal{L}^{n+1}f(z)}{\mathcal{L}^{n}f(z)}=\frac{z+\sum_{k=2}^{\infty}\left[k^{n+1}(1-\lambda)+\lambda\frac{1}{k^{n+1}}\right]a_kz^k}{z+\sum_{k=2}^{\infty}\left[k^{n}(1-\lambda)+\lambda\frac{1}{k^{n}}\right]a_kz^k}.$$

We have $p'(z)=\frac{\left(\mathcal{L}^{n+1}f(z)\right)'}{\mathcal{L}^{n}f(z)}-p(z)\frac{\left(\mathcal{L}^{n}f(z)\right)'}{\mathcal{L}^{n}f(z)}$ and
$p(z)+zp'(z)=\left(\frac{z\mathcal{L}^{n+1}f(z)}{\mathcal{L}^{n}f(z)}\right)'$.
Relation (2.6) becomes

$$p(z)+zp'(z)\prec h(z)=q(z)+zq'(z),\ \ z\in U.$$

By using Lemma 2 we have

$$p(z)\prec q(z)\ \ i.e.\ \ \frac{\mathcal{L}^{n+1}f(z)}{\mathcal{L}^{n}f(z)}\prec q(z),\ \ z\in U.$$

□

**Theorem 3.** *Let $q$ be a convex function, $q(0)=1$ and let $h$ be the function*

$$h(z)=q(z)+zq'(z), z\in U.$$

*If $f\in\mathcal{A}$, $\lambda\geq 0$, $n\in\mathbb{N}$ and satisfies the differential subordination*

$$\left(\mathcal{L}^{n+1}f(z)\right)'+\lambda\left[\left(I^{n-1}f(z)\right)'-\left(I^{n+1}f(z)\right)'\right]\prec h(z),\ \ z\in U \tag{2.7}$$

*then*

$$\left[\mathcal{L}^{n}f(z)\right]'\prec q(z),\ \ z\in U$$

*and this result is sharp.*

*Proof.* By using the properties of operator $\mathscr{L}^n$, we obtain

$$\mathscr{L}^{n+1} f(z) = (1-\lambda)\mathscr{D}^{n+1} f(z) + \lambda I^{n+1} f(z), z \in U. \tag{2.8}$$

Then (2.7) becomes

$$\left[(1-\lambda)\mathscr{D}^{n+1} f(z) + \lambda I^{n+1} f(z)\right]' + \lambda\left[\left(I^{n-1} f(z)\right)' - \left(I^{n+1} f(z)\right)'\right] \prec h(z), \quad z \in U. \tag{2.9}$$

After computation we get

$$(1-\lambda)\left[\mathscr{D}^{n+1} f(z)\right]' + \lambda\left[I^{n-1} f(z)\right]' \prec h(z)$$

or equivalently

$$(1-\lambda)\left[z\left(\mathscr{D}^n f(z)\right)'\right]' + \lambda\left[z\left(I^n f(z)\right)'\right]' \prec h(z).$$

The above relation is equivalent to

$$(1-\lambda)\left[\left(\mathscr{D}^n f(z)\right)' + z\left(\mathscr{D}^n f(z)\right)''\right] + \lambda\left[\left(I^n f(z)\right)' + z\left(I^n f(z)\right)''\right] \prec h(z)$$

or

$$\left[\mathscr{L}^n f(z)\right]' + z\left[\mathscr{L}^n f(z)\right]'' \prec h(z), \quad z \in U. \tag{2.10}$$

Let

$$p(z) = (1-\lambda)\left[\mathscr{D}^n f(z)\right]' + \lambda\left[I^n f(z)\right]' = \left[\mathscr{L}^n f(z)\right]', \quad z \in U \tag{2.11}$$

$$= (1-\lambda)\left[z + \sum_{k=2}^{\infty} k^n a_k z^k\right]' + \lambda\left[z + \sum_{k=2}^{\infty} \frac{1}{k^n} a_k z^k\right]' =$$

$$= (1-\lambda)\left[1 + \sum_{k=2}^{\infty} k^{n+1} a_k z^{k-1}\right] + \lambda\left[1 + \sum_{k=2}^{\infty} \frac{1}{k^{n-1}} a_k z^{k-1}\right] =$$

$$= 1 + \sum_{k=2}^{\infty}\left[k^{n+1}(1-\lambda) + \lambda\frac{1}{k^{n-1}}\right] a_k z^{k-1} = 1 + p_1 z + p_2 z^2 + \cdots$$

In view of (2.11), we deduce that $p \in \mathscr{H}[1,1]$. Using the notation in (2.11), the (2.10) differential subordination becomes

$$p(z) + zp'(z) \prec h(z) = q(z) + zq'(z), \quad z \in U.$$

By using Lemma 2 we have

$$p(z) \prec q(z) \ \ i.e. \ \ \left[\mathscr{L}^n f(z)\right]' \prec q(z), \quad z \in U.$$

□

*Remark* 5. If $\lambda = 0$ we get Theorem 2 from Oros [6] and for $\lambda = 1$ we get Theorem 2 from Bălăeţi [1].

*Example* 3. For $\lambda = 0$, $n = 1$, $f \in \mathcal{A}$ we deduce that

$$f'(z) + 3zf''(z) + z^2 f'''(z) \prec 1 + 2z, \quad z \in U$$

implies

$$f'(z) + zf''(z) \prec 1 + z, \quad z \in U.$$

**Theorem 4.** *Let $h \in \mathcal{H}(U)$ such that $h(0) = 1$ and*

$$\Re\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \quad z \in U.$$

*If $f \in \mathcal{A}$ satisfies the differential subordination*

$$\left(\mathcal{L}^{n+1} f(z)\right)' + \lambda\left[\left(I^{n-1} f(z)\right)' - \left(I^{n+1} f(z)\right)'\right] \prec h(z), \quad z \in U \tag{2.12}$$

*then*

$$\left[\mathcal{L}^n f(z)\right]' \prec q(z), \quad z \in U$$

*where $q$ is given by $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function $q$ is convex and is the best dominant.*

*Proof.* If we use the differential subordination technique we can see that the function $g$ is convex.[3], p. 66 By using (2.11) we obtain

$$\left(\mathcal{L}^{n+1} f(z)\right)' + \lambda\left[\left(I^{n-1} f(z)\right)' - \left(I^{n+1} f(z)\right)'\right] = p(z) + zp'(z), \quad z \in U$$

Then (2.12) becomes

$$p(z) + zp'(z) \prec h(z), \quad z \in U.$$

Since $p \in \mathcal{H}[1,1]$, we deduce that $p(z) \prec q(z)$, i.e.

$$\left[\mathcal{L}^n f(z)\right]' \prec q(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U$$

and $q$ is the best dominant. □

*Remark* 6. If $\lambda = 0$ we get Theorem 3 from Oros [6].

*Example* 4. For $\lambda = 0$, $n = 0$, $h(z) = \frac{1+z}{1-z}$ we deduce that

$$f'(z) + zf''(z) \prec \frac{1+z}{1-z}, \quad z \in U,$$

implies

$$f'(z) \prec 1 - \frac{2}{z} ln\,(1-z), \quad z \in U.$$

**Theorem 5.** *Let $h \in \mathcal{H}(U)$ such that $h(0) = 1$ and*

$$\Re\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \quad z \in U.$$

*If $f \in \mathcal{A}$ satisfies the differential subordination*

$$\left[\mathcal{L}^n f(z)\right]' \prec h(z), \quad z \in U \tag{2.13}$$

*then*

$$\frac{\mathcal{L}^n f(z)}{z} \prec q(z), \quad z \in U$$

*where $q$ is given by $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function $q$ is convex and is the best dominant.*

*Proof.* If we use the differential subordination technique we can see that the function $g$ is convex. [3], p. 66. Differentiating both sides in (2.2) we obtain

$$\left[\mathcal{L}^n f(z)\right]' = p(z) + zp'(z), \quad z \in U$$

Then (2.13) becomes

$$p(z) + zp'(z) \prec h(z), \quad z \in U.$$

Since $p \in \mathcal{H}[1,1]$, we deduce that $p(z) \prec q(z)$, i.e.

$$\frac{\mathcal{L}^n f(z)}{z} \prec q(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U$$

and $q$ is the best dominant. □

*Remark* 7. If $\lambda = 0$ we get Theorem 5 from Oros [6] and for $\lambda = 1$ we get Theorem 5 from Bălăeţi [1].

*Example* 5. For $\lambda = 0$, $n = 1$, $h(z) = \frac{1}{(1+z)^2}$ we deduce that

$$f'(z) \prec \frac{1}{(1+z)^2}, \quad z \in U,$$

implies

$$\frac{f(z)}{z} \prec \frac{1}{1+z}, \quad z \in U.$$

We get the same result as [4].

**Definition 5** ([7], [9], [1], [6])**.** If $0 \le \beta < 1$ and $n \in \mathbb{N}$, we let $L_n^m(\beta)$ stand for the class of functions $f \in \mathcal{A}_m$, which satisfy the inequality

$$\Re\left[\mathcal{L}^n f(z)\right]' > \beta, \quad (z \in U).$$

*Remark* 8. For $n = 0$ we obtain $\Re f'(z) > \beta$.

**Theorem 6.** *The set $L_n^m(\beta)$ is convex.*

*Proof.* Let the function

$$f_i(z) = z + \sum_{k=2}^{\infty} a_{k_i} z^k, \quad i = 1,2 \quad z \in U$$

be in the class $L_n^m(\beta)$. It is sufficient to show that the function

$$h(z) = \mu_1 f_1(z) + \mu_2 f_2(z)$$

with $\mu_1, \mu_2 \geq 0$ and $\mu_1 + \mu_2 = 1$ is in $L_n(\beta)$. Since

$$h(z) = z + \sum_{k=2}^{\infty} \left(\mu_1 a_{k_1} + \mu_2 a_{k_2}\right) z^k, \quad z \in U$$

then

$$\mathscr{L}^n h(z) = z + \sum_{k=2}^{\infty} \left[k^n (1-\lambda) + \lambda \frac{1}{k^n}\right] \left(\mu_1 a_{k_1} + \mu_2 a_{k_2}\right) z^k, \quad z \in U. \tag{2.14}$$

Differentiating (2.14), we get

$$\left[\mathscr{L}^n h(z)\right]' = 1 + \sum_{k=2}^{\infty} \left[k^{n+1} (1-\lambda) + \lambda \frac{1}{k^{n-1}}\right] \left(\mu_1 a_{k_1} + \mu_2 a_{k_2}\right) z^{k-1}.$$

Hence

$$\Re\left[\mathscr{L}^n h(z)\right]' = 1 + \Re\left\{\mu_1 \sum_{k=2}^{\infty} \left[k^{n+1} (1-\lambda) + \lambda \frac{1}{k^{n-1}}\right] a_{k_1} z^{k-1}\right\} +$$

$$+ \Re\left\{\mu_2 \sum_{k=2}^{\infty} \left[k^{n+1} (1-\lambda) + \lambda \frac{1}{k^{n-1}}\right] a_{k_2} z^{k-1}\right\}. \tag{2.15}$$

Since $f_1, f_2 \in L_n^m(\beta)$, we obtain

$$\Re\left\{\mu_i \sum_{k=2}^{\infty} \left[k^{n+1} (1-\lambda) + \lambda \frac{1}{k^{n-1}}\right] a_{k_i} z^{k-1}\right\} > \mu_i (\beta - 1), \quad i = 1,2. \tag{2.16}$$

Using (2.16) we get from (2.15)

$$\Re\left[\mathscr{L}^n h(z)\right]' > 1 + \mu_1 (\beta - 1) + \mu_2 (\beta - 1),$$

and since $\mu_1 + \mu_2 = 1$, we deduce

$$\Re\left[\mathscr{L}^n h(z)\right]' > \beta, \quad (z \in U)$$

i.e. $L_n^m(\beta)$ is convex. □

**Theorem 7.** *If* $0 \leq \beta < 1$ *and* $m, n \in \mathbb{N}$ *then we have*

$$L_n^m(\beta) \subset L_{n+1}^m(\delta),$$

*where* $\delta(\beta, m) = 2\beta - 1 + 2(1-\beta)\frac{1}{m}\sigma\left(\frac{1}{m}\right)$ *and* $\sigma(x) = \int_0^z \frac{t^{x-1}}{1+t}dt$. *The result is sharp.*

*Proof.* Assume that $f \in L_n^m(\beta)$. Let $\mathscr{L}^n f(z) = zp(z), z \in U$. Differentiating, we obtain

$$\left[\mathscr{L}^n f(z)\right]' = p(z) + zp'(z), z \in U.$$

Since $f \in L_n^m(\beta)$, from Definition 5 we have

$$\Re\left(p(z) + zp'(z)\right) > \beta, z \in U$$

which is equivalent to

$$p(z) + zp'(z) \prec \frac{1 + (2\beta - 1)z}{1+z} \equiv h(z), z \in U$$

By using Lemma 1, we have:

$$p(z) \prec q(z) \prec h(z), z \in U,$$

where

$$q(z) = \frac{1}{mz^{\frac{1}{m}}}\int_0^z \frac{1 + (2\beta - 1)t}{1+t} t^{\frac{1}{m}-1}dt =$$

$$= \frac{1}{mz^{\frac{1}{m}}}\int_0^z \left[2\beta - 1 + 2(1-\beta)\frac{1}{1+t}\right] t^{\frac{1}{m}-1}dt =$$

$$= \frac{1}{mz^{\frac{1}{m}}}\int_0^z (2\beta - 1)t^{\frac{1}{m}-1}dt + \frac{2(1-\beta)}{mz^{\frac{1}{m}}}\int_0^z \frac{t^{\frac{1}{m}-1}}{1+t}dt =$$

$$= 2\beta - 1 + 2(1-\beta)\frac{1}{m}\sigma\left(\frac{1}{m}\right)\frac{1}{z^{\frac{1}{m}}}, z \in U.$$

The function $q$ is convex and is the best dominant. From $p(z) \prec q(z)$ follows that

$$\Re p(z) > \Re q(1) = \delta(\beta, m) = 2\beta - 1 + 2(1-\beta)\frac{1}{m}\sigma\left(\frac{1}{m}\right),$$

from which we deduce that $L_n^m(\beta) \subset L_{n+1}^m(\delta)$. □

*Remark* 9. If $\lambda = 0$ we get Theorem 1 from Oros [6] and for $\lambda = 1$ we get Theorem 1 from Bălăeţi [1].

**Theorem 8.** *Let $q$ be a convex function in $U$ with $q(0)=1$ and let*

$$h(z)=q(z)+\frac{1}{c+2}zq'(z), z\in U,$$

*where $c$ is a complex number, with $\Re c>-2$.*
*If $f\in L_n^m(\beta)$ and $F=I_c(f)$, where*

$$F(z)=I_c(f)(z)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t)dt,\ \ \Re c>-2, \tag{2.17}$$

*then*

$$\left[\mathscr{L}^n f(z)\right]'\prec h(z),\ \ z\in U, \tag{2.18}$$

*implies*

$$\left[\mathscr{L}^n F(z)\right]'\prec q(z),\ \ z\in U,$$

*and this result is sharp.*

*Proof.* From (2.17), we have

$$z^{c+1}F(z)=(c+2)\int_0^z t^c f(t)dt,\ \ \Re c>-2,\ \ z\in U. \tag{2.19}$$

Differentiating, with respect to z, we obtain

$$(c+1)F(z)+zF'(z)=(c+2)f(z),\ \ z\in U$$

and

$$(c+1)\mathscr{L}^n F(z)+z\left[\mathscr{L}^n F(z)\right]'=(c+2)\mathscr{L}^n f(z),\ \ z\in U. \tag{2.20}$$

Differentiating (2.20), we obtain

$$\left[\mathscr{L}^n F(z)\right]'+\frac{z}{c+2}\left[\mathscr{L}^n F(z)\right]''=\left[\mathscr{L}^n f(z)\right]',\ \ z\in U. \tag{2.21}$$

Using (2.21), the differential subordination (2.18) becomes

$$\left[\mathscr{L}^n F(z)\right]'+\frac{1}{c+2}z\left[\mathscr{L}^n F(z)\right]''\prec h(z)=q(z)+\frac{1}{c+2}zq'(z),\ \ z\in U. \tag{2.22}$$

Let

$$p(z)=\left[\mathscr{L}^n F(z)\right]'=\left\{z+\sum_{k=2}^{\infty}\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]a_k z^k\right\}'= \tag{2.23}$$

$$=1+p_1z+p_2z^2+\cdots,\ \ z\in U,\ \ p\in\mathscr{H}[1,1].$$

Replacing (2.23) in (2.22) we obtain

$$p(z)+\frac{1}{c+2}zp'(z)\prec h(z)=q(z)+\frac{1}{c+2}zq'(z),\ \ z\in U,$$

Using Lemma 1, we obtain $p(z)\prec q(z)$ i.e.

$$\left[\mathscr{L}^n F(z)\right]'\prec q(z),\ \ z\in U,$$

and q is the best dominant. □

*Remark* 10. If $\lambda = 0$ we get Theorem 2.2 from Tăut et alii [9].

*Example* 6. If we take $c = 1+2i$ and $q(z) = \dfrac{1+z}{1-z}$ then
$h(z) = \dfrac{\left(1-z^2\right)(3+2i)+2z}{(3+2i)(1-z)^2}$.
From Theorem 8 we deduce

$$\left[\mathscr{L}^n f(z)\right]' \prec \frac{\left(1-z^2\right)(3+2i)+2z}{(3+2i)(1-z)^2}, \quad z \in U,$$

implies

$$\left[\mathscr{L}^n F(z)\right]' \prec \frac{1+z}{1-z}, \quad z \in U,$$

where $F$ is given by (2.17).

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# ADDITIVE $(\alpha,\beta)$-FUNCTIONAL EQUATIONS AND LINEAR MAPPINGS

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*Abstract.* In this paper, we investigate the additive $(\alpha,\beta)$-functional equation

$$f(x)+\overline{\alpha}f(\alpha y)+f(z)=\beta^{-1}f(\beta(x+y+z)) \tag{0.1}$$

for all complex numbers $\alpha$ with $|\alpha|=1$ and for a fixed nonzero complex number $\beta$.

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of the additive $(\alpha,\beta)$-functional equation (0.1) in complex Banach spaces.

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*Keywords:* Hyers-Ulam stability, additive $(\alpha,\beta)$-functional equation, $\mathbb{C}$-linear mapping, fixed point method, direct method

## 1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [19] concerning the stability of group homomorphisms.

The functional equation $f(x+y)=f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [10] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [18] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. See [2, 6, 8, 12, 15, 16, 20] for more information on functional equations.

We recall a fundamental result in fixed point theory.

**Theorem 1** ([3, 7])**.** *Let $(X,d)$ be a complete generalized metric space and let $J:X\to X$ be a strictly contractive mapping with Lipschitz constant $\alpha<1$. Then for*

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*each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

*for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;*

(3) *$y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [11] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 5, 17]).

In Section 2, we solve the additive $(\alpha, \beta)$-functional equation (0.1) in vector spaces and prove the Hyers-Ulam stability of the additive $(\alpha, \beta)$-functional equation (0.1) in Banach spaces by using the fixed point method.

In Section 3, we prove the Hyers-Ulam stability of the additive $(\alpha, \beta)$-functional equation (0.1) in Banach spaces by using the direct method.

Throughout this paper, assume that $X$ is a complex normed space and that $Y$ is a complex Banach space. Let $\beta$ be a fixed nonzero complex number.

## 2. Additive $(\alpha, \beta)$-functional equation (0.1) in complex Banach spaces $I$

We solve the additive $(\alpha, \beta)$-functional equation (0.1) in complex vector spaces.

**Lemma 1.** *Let $X$ and $Y$ be complex vector spaces. If a mapping $f : X \to Y$ satisfies*

$$f(x) + \overline{\alpha} f(\alpha y) + f(z) = \beta^{-1} f(\beta(x + y + z)) \tag{2.1}$$

*for all $x, y, z \in X$ and all $\alpha \in \mathbb{T} := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$, then $f : X \to Y$ is $\mathbb{C}$-linear.*

*Proof.* Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = z = 0$ in (2.1), we get $(2 + \overline{\alpha}) f(0) = \beta^{-1} f(0)$ for all $\alpha \in \mathbb{T}$. So $f(0) = 0$.

Letting $\alpha = 1$, $y = -x$ and $z = 0$ in (2.1), we get $f(x) + f(-x) = 0$ and so $f(-x) = -f(x)$ for all $x \in X$.

Letting $\alpha = 1$ and $z = -x - y$ in (2.1), we get

$$f(x) + f(y) - f(x + y) = f(x) + f(y) + f(-x - y) = 0$$

and so

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$.

Letting $z=0$ and $y=-x$ in (2.1), we get $f(x)+\overline{\alpha}f(-\alpha x)=0$ and so $f(\alpha x)=\alpha f(x)$ for all $x\in X$ and all $\alpha\in\mathbb{T}$. By the same reasoning as in the proof of [14, Theorem 2.1], the mapping $f:X\to Y$ is $\mathbb{C}$-linear. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive $(\alpha,\beta)$-functional equation (2.1) in complex Banach spaces.

**Theorem 2.** *Let $\varphi:X^3\to[0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)\le\frac{L}{2}\varphi(x,y,z) \tag{2.2}$$

*for all $x,y,z\in X$. Let $f:X\to Y$ be a mapping satisfying $f(0)=0$ and*

$$\left\|f(x)+\overline{\alpha}f(\alpha y)+f(z)-\beta^{-1}f(\beta(x+y+z))\right\|\le\varphi(x,y,z) \tag{2.3}$$

*for all $x,y,z\in X$ and all $\alpha\in\mathbb{T}$. Then there exists a unique $\mathbb{C}$-linear mapping $A:X\to Y$ such that*

$$\|f(x)-A(x)\|\le\frac{L}{2(1-L)}(\varphi(-x,-x,2x)+2\varphi(-x,x,0)) \tag{2.4}$$

*for all $x\in X$.*

*Proof.* Let $\alpha=1$.

Replacing $x$ by $-x$ and letting $y=-x$ and $z=2x$ in (2.3), we get

$$\|2f(-x)+f(2x)\|\le\varphi(-x,-x,2x) \tag{2.5}$$

for all $x\in X$.

Replacing $x$ by $-x$ and letting $y=x$ and $z=0$ in (2.3), we get

$$\|f(-x)+f(x)\|\le\varphi(-x,x,0) \tag{2.6}$$

for all $x\in X$.

It follows from (2.5) and (2.6) that

$$\|f(2x)-2f(x)\|\le\varphi(-x,-x,2x)+2\varphi(-x,x,0) \tag{2.7}$$

for all $x\in X$.

Consider the set

$$S:=\{h:X\to Y,\ \ h(0)=0\}$$

and introduce the generalized metric on $S$:

$$\begin{aligned}&d(g,h)\\&=\inf\{\mu\in\mathbb{R}_+:\|g(x)-h(x)\|\le\mu(\varphi(-x,-x,2x)+2\varphi(-x,x,0)),\ \ \forall x\in X\},\end{aligned}$$

where, as usual, $\inf\phi=+\infty$. It is easy to show that $(S,d)$ is complete (see [13]).

Now we consider the linear mapping $J:S\to S$ such that

$$Jg(x):=2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon(\varphi(-x,-x,2x) + 2\varphi(-x,x,0))$$

for all $x \in X$. Hence

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| \\ &\le 2\varepsilon\left(\varphi\left(-\frac{x}{2},-\frac{x}{2},x\right) + 2\varphi\left(-\frac{x}{2},\frac{x}{2},0\right)\right) \\ &\le 2\varepsilon\frac{L}{2}(\varphi(-x,-x,2x) + 2\varphi(-x,x,0)) \\ &\le L\varepsilon(\varphi(-x,-x,2x) + 2\varphi(-x,x,0))\end{aligned}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.7) that

$$\begin{aligned}\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| &\le \varphi\left(-\frac{x}{2},-\frac{x}{2},x\right) + 2\varphi\left(-\frac{x}{2},\frac{x}{2},0\right) \\ &\le \frac{L}{2}(\varphi(-x,-x,2x) + 2\varphi(-x,x,0))\end{aligned}$$

for all $x \in X$. So $d(f, Jf) \le \frac{L}{2}$.

By Theorem 1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) $A$ is a fixed point of $J$, i.e.,

$$A(x) = 2A\left(\frac{x}{2}\right) \tag{2.8}$$

for all $x \in X$. The mapping $A$ is a unique fixed point of $J$ in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that $A$ is a unique mapping satisfying (2.8) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x) - A(x)\| \le \mu(\varphi(-x,-x,2x) + 2\varphi(-x,x,0))$$

for all $x \in X$;

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f,A) \le \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x) - A(x)\| \le \frac{L}{2(1-L)}(\varphi(-x,-x,2x) + 2\varphi(-x,x,0))$$

for all $x \in X$.

It follows from (2.2) and (2.3) that

$$\begin{aligned}
&\left\| A(x) + \overline{\alpha} A(\alpha y) + A(z) - \beta^{-1} A(\beta(x+y+z)) \right\| \\
&= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x}{2^n}\right) + \overline{\alpha} f\left(\frac{\alpha y}{2^n}\right) + f\left(\frac{z}{2^n}\right) - \beta^{-1} f\left(\beta\left(\frac{x+y+z}{2^n}\right)\right) \right\| \\
&\le \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0
\end{aligned}$$

for all $x, y, z \in X$ and all $\alpha \in \mathbb{T}$. So

$$A(x) + \overline{\alpha} A(\alpha y) + A(z) - \beta^{-1} A(\beta(x+y+z)) = 0$$

for all $x, y, z \in X$ and all $\alpha \in \mathbb{T}$. By Lemma 1, the mapping $A : X \to Y$ is $\mathbb{C}$-linear. □

**Corollary 1.** *Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying*

$$\left\| f(x) + \overline{\alpha} f(\alpha y) + f(z) - \beta^{-1} f(\beta(x+y+z)) \right\| \le \theta(\|x\|^r + \|y\|^r + \|z\|^r) \quad (2.9)$$

*for all $x, y, z \in X$ and all $\alpha \in \mathbb{T}$. Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2^r + 6}{2^r - 2} \theta \|x\|^r$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 2 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Then we can choose $L = 2^{1-r}$ and we get the desired result. □

**Theorem 3.** *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

*for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{1}{2(1-L)} (\varphi(-x, -x, 2x) + 2\varphi(-x, x, 0))$$

*for all $x \in X$.*

*Proof.* It follows from (2.7) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \le \frac{1}{2} (\varphi(-x, -x, 2x) + 2\varphi(-x, x, 0))$$

for all $x \in X$.

Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2. □

**Corollary 2.** *Let $r < 1$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (2.9). Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{6+2^r}{2-2^r}\theta\|x\|^r$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 3 by taking $\varphi(x,y,z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x,y,z \in X$. Then we can choose $L = 2^{r-1}$ and we get desired result. □

## 3. Additive $(\alpha,\beta)$-functional equation (0.1) in complex Banach spaces $II$

In this section, using the direct method, we prove the Hyers-Ulam stability of the additive $(\alpha,\beta)$-functional equation (2.1) in complex Banach spaces.

**Theorem 4.** *Let $\varphi : X^3 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x,y,z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty,$$

$$\left\| f(x) + \overline{\alpha} f(\alpha y) + f(z) - \beta^{-1} f(\beta(x+y+z)) \right\| \le \varphi(x,y,z) \tag{3.1}$$

*for all $x,y,z \in X$ and all $\alpha \in \mathbb{T}$. Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{1}{2}(\Psi(-x,-x,2x) + 2\Psi(-x,x,0)) \tag{3.2}$$

*for all $x \in X$.*

*Proof.* Let $\alpha = 1$.

It follows from (2.7) that

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \le \varphi\left(-\frac{x}{2}, -\frac{x}{2}, x\right) + 2\varphi\left(-\frac{x}{2}, \frac{x}{2}, 0\right)$$

for all $x \in X$. Hence

$$\left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| \le \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\|$$

$$\leq \sum_{j=l}^{m-1}\left(2^j\varphi\left(-\frac{x}{2^{j+1}},-\frac{x}{2^{j+1}},\frac{x}{2^j}\right)+2^{j+1}\varphi\left(-\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},0\right)\right) \tag{3.3}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.3) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since $Y$ is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.3), we get (3.2).

Now, let $T : X \to Y$ be another additive mapping satisfying (3.2). Then we have

$$\begin{aligned}\|A(x)-T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right)-2^q T\left(\frac{x}{2^q}\right)\right\| \\ &\leq \left\|2^q A\left(\frac{x}{2^q}\right)-2^q f\left(\frac{x}{2^q}\right)\right\|+\left\|2^q T\left(\frac{x}{2^q}\right)-2^q f\left(\frac{x}{2^q}\right)\right\| \\ &\leq 2^q\Psi\left(-\frac{x}{2^q},-\frac{x}{2^q},\frac{2x}{2^q}\right)+2^{q+1}\Psi\left(-\frac{x}{2^q},\frac{x}{2^q},0\right),\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $A$.

The rest of the proof is similar to the proof of Theorem 2. □

**Corollary 3.** *Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying (2.9). Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that*

$$\|f(x)-A(x)\| \leq \frac{2^r+6}{2^r-2}\theta\|x\|^r$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 4 by taking $\varphi(x,y,z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x,y,z \in X$. □

**Theorem 5.** *Let $\varphi : X^3 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (3.1) and*

$$\Psi(x,y,z) := \sum_{j=0}^{\infty}\frac{1}{2^j}\varphi(2^j x,2^j y,2^j z) < \infty$$

*for all $x,y,z \in X$. Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that*

$$\|f(x)-A(x)\| \leq \frac{1}{2}(\Psi(-x,-x,2x)+2\Psi(-x,x,0)) \tag{3.4}$$

*for all $x \in X$.*

*Proof.* It follows from (2.7) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq (\varphi(-x,-x,2x) + 2\varphi(-x,x,0))$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f\left(2^j x\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x\right) \right\| \\ &\leq \sum_{j=l}^{m-1} \left( \frac{1}{2^{j+1}} \varphi(-2^j x, -2^j x, 2^{j+1} x) + \frac{1}{2^j} \varphi(-2^j x, 2^j x, 0) \right) \end{aligned} \tag{3.5}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.5) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.5), we get (3.4).

The rest of the proof is similar to the proofs of Theorems 2 and 4. □

**Corollary 4.** *Let $r < 1$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (2.9). Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that*

$$\| f(x) - A(x) \| \leq \frac{6 + 2^r}{2 - 2^r} \theta \|x\|^r$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 5 by taking $\varphi(x,y,z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. □

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# TO A QUESTION ON THE EXISTENCE OF REGULAR MOSAICS ON A HYPERBOLIC PLANE OF POSITIVE CURVATURE

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*Abstract.* A hyperbolic plane $\hat{H}$ of positive curvature can be realized on the exterior domain with respect to an oval curve in a projective plane. The plane $\hat{H}$ is the Cayley – Klein model of a two-dimensional de Sitter space. It is proved that there is no regular mosaic on the plane $\hat{H}$.

2010 *Mathematics Subject Classification:* 51M10; 51M20; 52C20

*Keywords:* hyperbolic plane of positive curvature, generalized polygon, regular mosaic, hyperbolic regular mosaic

## 1. Introduction

### 1.1. *The hyperbolic plane $\hat{H}$ of positive curvature*

Let $H^2$ be an extended hyperbolic plane with absolute oval curve $\gamma$ [41, §5.1.1]. Every line on the plane $H^2$ belongs to one of three types depending on its position with respect to the absolute. Lines intersecting the absolute in two real (imaginary conjugate) points are called *hyperbolic* (*elliptic*). Any isotropic line on $H^2$ tangent to the absolute is called *parabolic*.

In the Cayley – Klein model the interior domain with respect to the oval curve $\gamma$ is a complete Lobachevskii plane $\Lambda^2$ which is also called a Bolyai – Lobachevskian plane. This plane is the hyperbolic plane of negative curvature. On the set of all exterior points of the plane $H^2$ with respect to the absolute curve $\gamma$ one can construct different geometries. If we consider hyperbolic, elliptic, and parabolic lines as lines in our geometry, we obtain a *hyperbolic plane $\hat{H}$ of positive curvature* [31, 32], [40, §4.1.1]. Geometry of the plane $\hat{H}$ of curvature $1/\rho^2$ can be realized on the sphere of real radius $\rho$ in the pseudo-Euclidean space $\mathbb{R}^3_1$ with antipodal points identified. Therefore the plane $\hat{H}$ is the projective model of a two-dimensional de Sitter space [10, 14]. The number $\rho$ is called the *curvature radius* of $\hat{H}$. The plane $\hat{H}$ is homeomorphic to the Möbius band without boundary. The group $G$ of projective automorphisms of the absolute oval curve $\gamma$ is the fundamental group of transformations for $\hat{H}$, $H^2$, and the Lobachevskii plane $\Lambda^2$.

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1.2. *Problem statement*

Let $\mathbb{K}$ be the Euclidean plane $\mathbb{R}^2$, the elliptic plane $\mathbb{E}^2$ or the Lobachevskii plane $\Lambda^2$.

A *regular mosaic* on the plane $\mathbb{K}$ consists of $q$ congruent regular $p$-gons which fit together round a common vertex without gaps and overlappings. The Schläfli symbol $\{p,q\}$ denotes the type of the regular mosaic. Each regular mosaic of the type $\{p,q\}$ determines the full group $[p,q]$ of its symmetries and generates the regular normal partition of the plane $\mathbb{K}$ [11], [12, §5.1].

Denote the measure of the interior angle at a vertex of a regular $p$-gon by $\varphi$. On the Euclidean plane we have $\varphi = \pi(1-2/p)$. For the measure $\varphi$ on the plane $\mathbb{E}^2$ (or $\Lambda^2$) the inequality $\varphi > \pi(1-2/p)$ (or $\varphi < \pi(1-2/p)$ respectively) holds. Therefore the positive integers $p$ and $q$ from the Schläfli symbol of a regular mosaic in the plane $\mathbb{K}$ satisfy one of the following conditions:

(i) $(p-2)(q-2) < 4$ if $\mathbb{K}$ is the elliptic plane $\mathbb{E}^2$;
(ii) $(p-2)(q-2) = 4$ if $\mathbb{K}$ is the Euclidean plane $\mathbb{R}^2$;
(iii) $(p-2)(q-2) > 4$ if $\mathbb{K}$ is the Lobachevskii plane $\Lambda^2$.

The conditions (i) – (iii) allow us to find all regular mosaics of the planes $\mathbb{E}^2$, $\mathbb{R}^2$, and $\Lambda^2$. According to the inequality (i) all "spherical mosaics" are devided into seven types: $\{2,q\}$, $\{q,2\}$, $\{3,3\}$, $\{3,4\}$, $\{4,3\}$, $\{3,5\}$, $\{5,3\}$, where both of the Schläfli symbols $\{2,q\}$ and $\{q,2\}$ determine infinite series of regular mosaics. Owing to the inequality (ii) there exist only three types of regular mosaics, $\{3,6\}$, $\{4,4\}$, $\{6,3\}$, on the Euclidean plane $\mathbb{R}^2$. The inequality (iii) does not impose restrictions for a types number of the "hyperbolic mosaics". This number is infinite.

Regular mosaics (in other terminology, regular honeycombs) first were mentioned in the works by Klein (1879) and Stringham (1880). Schlegel (1883) and Schläfli (1855) investigated these objects in classical spaces of constant curvature, that is, Euclidean, elliptic and Lobachevskii spaces (see review in [11, 12]). The fundamental work of Poincare [28] has given base for the research of Fuchsian groups, that is, discrete transformations groups of the Lobachevskii plane $\Lambda^2$, and has generated thereby a particular interest to the "hyperbolic mosaics". This subject still attracts a lot of attention (see, for instance, [3, 4, 16, 26, 27]). There are also other geometrical objects closely connected with Fuchsian groups, in particular, Coxeter groups of the plane $\Lambda^2$ (see, for instance, [8, 18, 20]).

The development of geometry on the ideal domain of the Lobachevskii plane $\Lambda^2$ has set new goals. In academic discussion new issues have been raised. One of them can be formulated from general point of view: *whether the geometry of the plane $\hat{H}$ can make a contribution to the theory of Fuchsian groups?* The research of this issue has led to the partitions construction on the plane $\hat{H}$. At first the series of so-called *simple partitions* have been constructed [29, 32]. A simple partition of the plane $\hat{H}$ or a figure on $\hat{H}$ is monohedral, that is, all its cells are congruent. A cell of a simple

partition is simple 4-contour with the border consisting of four parabolic segments. In [29] the *star* of a regular odd-dimensional $n$-contour is studied. The star consists of $n$ congruent simple 4-contours adjoining on whole edges. The component 4-contours of the star locate out along an elliptic line. Three vertices of each component 4-contour lie in a hypercycle. Such packing is a bit similar to packing of cells in a regular mosaic. But unlike a mosaic it is impossible to use the star as an independent cell of partition as the star is a topological equivalent of the Möbius band.

In Fig. 1 the star of a regular 9-contour is shown. The component 4-contours $Q_1$, $\ldots$, $Q_9$ of the star locate out along the elliptic line $l$. The hypercycle $\omega$ contains three vertices from each component 4-contour of the star.

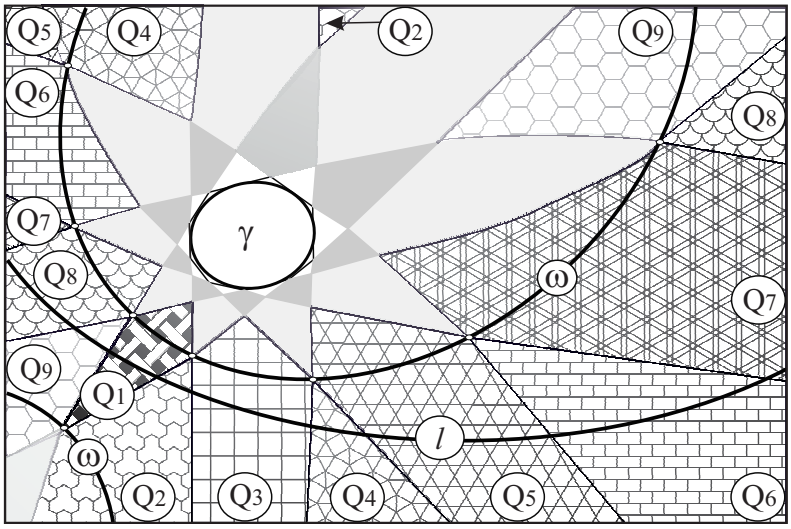


FIGURE 1. The absolute curve $\gamma$ and the star of a regular 9-contour. The component 4-contours $Q_1, \ldots, Q_9$ of the star.

We have also constructed an infinite *tiling* which may be considered as an remote analogue of a mosaic in the following sense: there exist both a simple partition of the tiling and a partition of the plane $\hat{H}$ by the tiling. Since a simple partition on the plane $\hat{H}$ is not normal [32, Theorem 5.1.1], the tiling significantly differs from a mosaic. In [34] the first normal *fan partitions* of the plane $\hat{H}$ are constructed. A fan partition is formed by special packing of congruent fans. Cells of a fan partition are also simple 4-contours, but not all cells are congruent. By Theorem 5.1.1 from [32] there is no regular mosaic consisting of simple 4-contours. Here we put more general question on the existence of regular mosaics. In Theorem 2 we prove that there is no regular mosaic on the plane $\hat{H}$. The simple proof of this fundamental fact is based on the theorem about the polygon area in the plane $\hat{H}$, see Theorem 3.1 in [38]. Since various stages of the proof of Theorem 3.1 have been published in three papers, we

provide its short sketch in adjuvant Section 2. Besides, in Section 2 we give strict definitions of the notions used in this paper.

## 2. Preliminaries

### 2.1. *Angles of the plane $\hat{H}$*

There are three types of the pencils of lines on the plane $\hat{H}$. A pencil of lines on $\hat{H}$ is *hyperbolic* (*elliptic*) if its centre is an exterior (interior) point with respect to the absolute. The centre of a *parabolic* pencil lies on the absolute. Lines of a hyperbolic (elliptic) pencil are said to be *intersecting* (*hyperparallel*) on $\hat{H}$. Lines of a parabolic pencil are said to be *parallel*. Different variants of location of a pair of lines with respect to the absolute define fifteen types of angles of the plane $\hat{H}$ [30], [31, §4.3].

All types of angles of the plane $\hat{H}$ are presented in Table 1. We denote the types of pencils of lines by following symbols: $H$ — hyperbolic, $E$ — elliptic, $P$ — parabolic. For the types of lines in the plane $\hat{H}$ we use the following designations: $h$ — hyperbolic, $e$ — elliptic, $p$ — parabolic.

Table 1. The types and the measures of angles on the plane $\hat{H}$

| Type of an angle (short designation) | Measure $\upsilon$ ($\tilde{\upsilon}$) of an angle | Type of a pencil | Types of lines | |
|---|---|---|---|---|
| | | | $a$ | $b$ |
| Valiana | — | $H$ | $p$ | $p$ |
| Semicovaliana | — | | | |
| Hyperbolic flag | — | $H$ | $p$ | $h$ |
| Hyperbolic pseudoflag | — | | | |
| Parabolic flag | — | $P$ | | |
| Elliptic flag | — | $H$ | $p$ | $e$ |
| Elliptic pseudoflag | — | | | |
| Half-plane | $\upsilon \in [0;\pi]$ | $E$ | $h$ | $h$ |
| Hyperbolic angle | $\upsilon \in \mathbb{R}_+$ or $\upsilon_a \in \mathbb{R}_-$ | $H$ | | |
| Hyperbolic pseudoangle | $\tilde{\upsilon} = i\,\pi + \upsilon,\ \upsilon \in \mathbb{R}_+$ | | | |
| Strip | — | $P$ | | |
| Pseudostrip | — | | | |
| Quasiangle | $\tilde{\upsilon} = \varepsilon\,\upsilon + i\,\pi/2,$ $\upsilon \in \mathbb{R}_+,\ \varepsilon = 1;-1;0$ | $H$ | $h$ | $e$ |
| Elliptic angle | $\upsilon \in \mathbb{R}_+$ | $H$ | $e$ | $e$ |
| Elliptic pseudoangle | $\tilde{\upsilon} = i\,\pi - \upsilon,\ \upsilon \in \mathbb{R}_+$ | | | |

Angles of six types between non-parallel and non-parabolic lines in the plane $\hat{H}$ are measurable by means of the absolute. Measurable angles of five types possess

vertices from the proper domain of the plane $\hat{H}$. Elliptic and hyperbolic angles, elliptic and hyperbolic pseudoangles, and quasiangles belong to such types. An elliptic angle has real positive measure while hyperbolic one has both positive measure and negative agreed measure. The measure of an elliptic pseudoangle is a number $i\pi - \upsilon$, and the measure of a hyperbolic pseudoangle is a number $i\pi + \upsilon$, where $\upsilon \in \mathbb{R}_+$.

Only non-parabolic lines of different types can be orthogonal on the plane $\hat{H}$. A line $a$ is orthogonal to a line $b$ if it contains a pole of the line $b$ with respect to the absolute. Two orthogonal lines form two adjacent *right quasiangles*. The measure of right quasiangle equals $i\frac{\pi}{2}$.

Assume that a hyperbolic line $a$ and an elliptic line $b$ form adjacent quasiangles with a common vertex $K$. Let $a'$ be an orthogonal line to the line $a$ and let $K \in a'$. The quasiangle between the lines $a$ and $b$ is called *hyperbolic* (or *elliptic*) if it contains (or, respectively, does not contain) the line $a'$. The measure of a hyperbolic (elliptic) quasiangle is a number $i\frac{\pi}{2} + \varepsilon\upsilon$, where $\upsilon \in \mathbb{R}_+$ and $\varepsilon = 1$ ($\varepsilon = -1$) [31, §4.5].

### 2.2. *Polygons of the plane $\hat{H}$*

Let $\Phi$ be a figure of the plane $H^2$. If $\Phi \cap \hat{H} = \Phi$ ($\Phi \cap \hat{H} \neq \Phi$), then the figure $\Phi$ is called a *finite* (*infinite*) *figure* on $\hat{H}$.

In this paper we assume that the absolute curve $\gamma$ is added to the extended hyperbolic plane $H^2$. Such gluing the components $\hat{H}$ and $\Lambda^2$ of the plane $H^2$ by means of the curve $\gamma$ allow us to simplify the introduction of basic objects. Let $\overline{H}^2 = \gamma \cup H^2$.

On the plane $\overline{H}^2$ there are two types of simple closed curves. A simple closed curve is called *one-sided* if its removal from $\overline{H}^2$ does not break connectivity of the plane $\overline{H}^2$. A simple closed curve is called *two-sided* if it divides the plane $\overline{H}^2$ into two connected components.

Let $F$ be a simple closed two-sided broken line on the plane $\overline{H}^2$ and let $F \cap \hat{H} \neq \varnothing$. Then $F$ divides the plane $\overline{H}^2$ into two connected components. One of these components is topological equivalent to an open disk. Denote it by $o$. The second component is topological equivalent to the Möbius band without boundary. Denote it by $\mu$. The figure $F_o = F \cup o$ ($F_\mu = F \cup \mu$) is called a *generalized polygon* (*generalized Möbius polygon*) of the plane $\hat{H}$. The generalized polygon $F_o$ is also called a *generalized polyhedral* or a *n-hedral* depending on the number $n$ of its vertices.

Vertices and segments of the broken line $F$ are called respectively *vertices* and *edges* of the polygon with *boundary* $F$. A polygon vertex is called *proper* or *ideal* if it belongs to the plane $\hat{H}$ or to the plane $\Lambda^2$ respectively. A polygon vertex is called *absolute* if it lies on the absolute curve $\gamma$. A polygon vertex is called *true* in case the polygon edges containing this vertex do not lie on the same line.

Let $F$ be a generalized polygon or a generalized Möbius polygon of the plane $\hat{H}$ and let all vertices of the polygon $F$ be proper and true. Assume that lines $b$ and $d$

contain edges of the polygon $F$ with a common vertex $A$. The lines $b$, $d$ and a polar line $a$ of the point $A$ with respect to the absolute $\gamma$ divide the plane $\overline{H}^2$ into four generalized trihedrals. We call them the *basic quarters* of the vertex $A$ and denote by $\eta_j$, $j = 1,2,3,4$.

Assume that an angle $\eta$ of the plane $\overline{H}^2$ between the lines $b$ and $d$ contains the basic quarter $\eta_j$ of the vertex $A$. We define a *type* of the basic quarter $\eta_j$ as the type of the angle $\eta$. If the angle $\eta$ is measurable, we call its measure a *measure* of the basic quarter $\eta_j$. Let the figures of four sets

$$\left(\eta_1^1,\dots,\eta_1^m\right),\ \left(\eta_2^1,\dots,\eta_2^n\right),\ \left(\eta_3^1,\dots,\eta_3^r\right),\ \left(\eta_4^1,\dots,\eta_4^s\right), \tag{2.1}$$

where $m$, $n$, $r$, and $s$ are positive integers, be connected components of the figures $F\cap\eta_1$, $F\cap\eta_2$, $F\cap\eta_3$, and $F\cap\eta_4$ respectively. The set of figures (2.1) contains one or three figures possessing the point $A$ and a nonempty interior. Denote the quantity (sum) of these figures by $Q_A$ ($F_A$).

Let us consider all possibilities.

(1) Assume that $Q_A = 1$. Then one of the basic quarters of the vertex $A$ contains the figure $F_A$. We call this quarter the *interior angle* of the polygon $F$ at the vertex $A$.

(2) Assume that $Q_A = 3$. Then only one of the basic quarters of the vertex $A$ does not contain components of the figure $F_A$. Denote it by $\eta_4$. We call the sum of the quarters $\eta_1$, $\eta_2$, $\eta_3$ the *interior angle* of the polygon $F$ at the vertex $A$. If the basic quarters $\eta_1$, $\eta_2$, $\eta_3$ are measurable, we define the *measure* of the interior angle of the polygon $F$ at the vertex $A$ as the sum of the measures of these quarters.

In Fig. 2, a (b) the interior angle $\eta_1$ ($\eta_1\cup\eta_2\cup\eta_3$) of the polygon $F$ at the vertex $A$ is shown.

Under the condition $Q_A = 1$ (or $Q_A = 3$) the interior angle of the polygon $F$ at the vertex $A$ is convex (or, respectively, non-convex).

*Remark* 1. The sum of the measures of adjacent angles with a vertex at a proper point of the plane $\hat{H}$ is equal to $i\pi$. Therefore the sum of the measures of all basic quarters of a polygon vertex is equal to $2i\pi$. This means that the measure of a full angle round proper point in the plane $\hat{H}$ equals $2i\pi$.

In classical geometries of the planes of constant curvature $\mathbb{E}^2$, $\Lambda^2$, and $\mathbb{R}^2$ the notion "full angle round a point" is closely connected with the notion "rotation round a point". In the geometry of the plane $\hat{H}$ these notions significantly differ. Each line on the plane $\mathbb{E}^2$, $\Lambda^2$, or $\mathbb{R}^2$ under the rotation round an arbitrary proper point can cover the full plane and return to its initial location. Each non-parabolic line on the plane $\hat{H}$ under the rotation round an arbitrary proper point infinitely moves away from its initial location. Staying of the same type, the line under such rotation go to the parabolic line containing the centre of rotation. In this regard, the notion "full

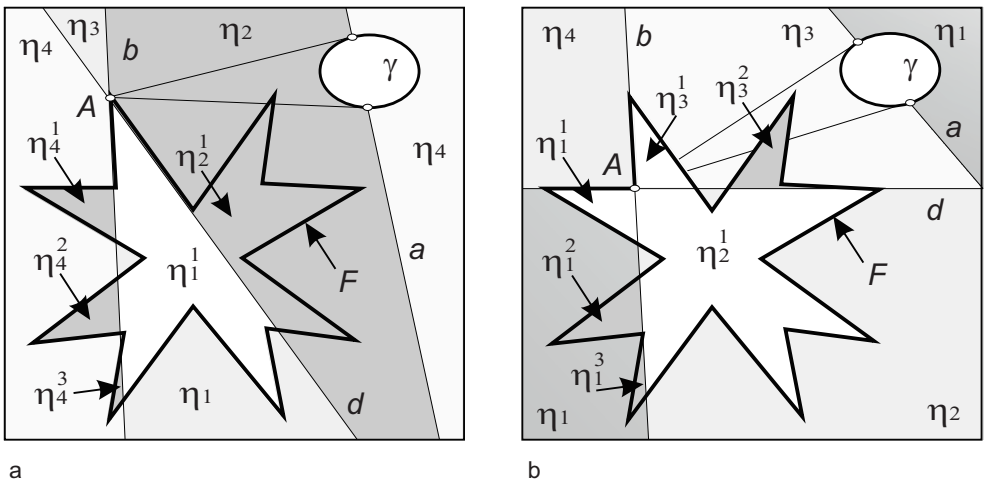


FIGURE 2. The absolute curve $\gamma$ and the interior angle $\eta_1$ ($\eta_1 \cup \eta_2 \cup \eta_3$) of the polygon $F$ at the vertex $A$ (a) ((b)).

angle round a point" in the geometry of the plane $\hat{H}$ possess conditional character. We apply this notion in the sense of the packing of figures round a point, that is, ignoring the rotation.

2.3. *Theorem on the area of a generalized polygon of the plane $\hat{H}$*

The foundation of the volumes theory, in particular, of the areas theory, in the hyperbolic space was given in classical works of the founders of non-Euclidean geometry, Lobachevsky and Bolyai (see, for instance, [6, 25]). Later this theory was developed in works of many authors (see, for instance, [5, 7, 9, 19, 21, 42, 43]). Profound survey on this topic is presented in papers [2, 21, 44].

It is known that on the plane $\Lambda^2$ of curvature $\left(-1/\rho^2\right)$ the area of a triangle can be expressed through the measures $\hat{A}$, $\hat{B}$, and $\hat{C}$ of interior angles at vertices by the formula

$$S = \rho^2 \left(\pi - \hat{A} - \hat{B} - \hat{C}\right). \tag{2.2}$$

This formula for the first time was obtained by Lambert as a corollary of the hypothesis of the acute angle [24]. It is natural that considering projective model of the hyperbolic plane, some authors wondered about the truth of formula (2.2) on the extended hyperbolic plane $H^2$ for the generalized triangles with vertices both on interior, and on exterior domain of the plane with respect to the absolute (see, for instance, [5, 7, 43]). Unfortunately, the mistakes made by authors resulted in incorrect conclusions. Analyzing the known works, we emphasize two main problems.

First, we note incorrect definitions of angles between lines in the plane $H^2$ (and $\hat{H}$). This problem is multilayered. Let us dwell on two important points in brief. Despite careful elaboration of this topic by leaders of geometry (see, for instance, [1,17]), some authors do not notice conceptual distinctions between the notions "angle" and "measure of an angle". Besides, some authors overlook the following fact. The plane $\hat{H}$ is homeomorphic to the Möbius band without boundary. Therefore adjacent angles between lines on $\hat{H}$ can be topologically distinct (see [31, §4.3]). In this regard, the introduction of the basic notions in the plane $\hat{H}$ geometry demands extreme accuracy.

The second main problem consists in incorrect adaptation of Laguerre formulae [23] to the calculation of the measures of angles with vertices in the exterior domain with respect to the absolute of the plane $H^2$. Choosing a multiplier in the Laguerre formulae, authors recede from the general logic stated by Klein in [22]. We discuss this problem in [37].

Trying to improve the situation, we come back to basis of non-Euclidean geometry in its projective understanding. Basing on classical works [1,15,17,22], we carefully define all basic notions of the plane $\hat{H}$ geometry (see [30–32]). Only after this we go to the calculation of the figures areas[1].

The foundation of the areas theory of the plane $\hat{H}$ is set in works [33,36,38,39]. The notion of the area is introduced such that the finite figures of the plane $\hat{H}$ possess the real positive areas. Moreover, for each figure $F$ of the plane $H^2$ the following equality holds

$$S_{\Lambda^2}(F) = i S_{\hat{H}}(F), \tag{2.3}$$

where $S_{\Lambda^2}(F)$ or $S_{\hat{H}}(F)$ is the area of the figure $F$ calculated in geometry of the plane $\Lambda^2$ or $\hat{H}$ respectively (see [33,36]).

In this paper due to the theorem on the area of a generalized polygon without parabolic edges (see [38, Theorem 3.1]) we obtain the simple proof of the fundamental fact of the plane $\hat{H}$ geometry. Let us formulate this theorem.

**Theorem 1.** *Let $F$ be a generalized polygon without parabolic edges on the plane $\hat{H}$ of curvature radius $\rho$, $\rho \in \mathbb{R}_+$. Denote the measures of interior angles of the polygon $F$ at vertices $A_1, A_2, \dots, A_n$ by $\hat{A}_1, \hat{A}_2, \dots, \hat{A}_n$ respectively. Then the area $S_{\hat{H}}(F)$ of the polygon $F$ can be expressed by the formula*

$$S_{\hat{H}}(F) = \rho^2 \left( \sum_{j=1}^{n} \tau_j \hat{A}_j - i\pi(n-2) \right), \tag{2.4}$$

[1] To create the theory of volumes on the ideal domain $\hat{H}^3$ of a hyperbolic space $H^3$ it is necessary to pass a similar way. We begin this way in works [35,37]. Notice that Gauss emphasized complexity of the volumes calculation in the hyperbolic space by the term "die Dschungel" (jungle). The volumes calculation in the space $\hat{H}^3$ is a much more complicated problem, because this space contains lines and planes of different topological types.

$$\textit{where}\ \ \tau_j = \begin{cases} 1 & \textit{if } A_j \textit{ is a proper or absolute vertex of the polygon } F, \\ i & \textit{if } A_j \textit{ is an ideal vertex of the polygon } F. \end{cases}$$

Note that known formula (2.2) is a special case of formula (2.4). Indeed, by formula (2.4) for an arbitrary triangle $ABC$ in the plane $\Lambda^2$ with the measures $\hat{A}_1 = \hat{A}$, $\hat{A}_2 = \hat{B}$, and $\hat{A}_3 = \hat{C}$ of interior angles at vertices $A$, $B$, and $C$ respectively, we have $\tau_1 = \tau_2 = \tau_3 = i$ and

$$S_{\hat{H}}(ABC) = \rho^2 \left( i\,\hat{A} + i\,\hat{B} + i\,\hat{C} - i\,\pi \right).$$

From here via equality (2.3) we find the area of the triangle $ABC$ in the plane $\Lambda^2$ geometry:

$$S_{\Lambda^2}(ABC) = i\,S_{\hat{H}}(ABC) = \rho^2 \left( \pi - \hat{A} - \hat{B} - \hat{C} \right).$$

Thus for an arbitrary triangle in the plane $\Lambda^2$ formula (2.2) holds.

We prove Theorem 1 in three stages. At first, in paper [33], we choose an orthogonal hypercyclic coordinate system on the plane $\hat{H}$ and prove the following formula for the calculation of the area of a rectangular trihedral with an elliptic (hyperbolic) cathetus of length $a$ ($b$)

$$S = \rho^2 \ln \frac{\sinh \frac{b}{\rho} + \cosh \frac{b}{\rho} \sin \frac{a}{\rho}}{\sin \frac{a}{\rho} + \cos \frac{a}{\rho} \sinh \frac{b}{\rho}}. \tag{2.5}$$

After that, in [36], via formula (2.5) we prove the following formula for a finite trihedral of the plane $\hat{H}$ with the measures $\hat{A}_1$, $\hat{A}_2$, and $\hat{A}_3$ of interior angles

$$S = \rho^2 \left( \hat{A}_1 + \hat{A}_2 + \hat{A}_3 - i\,\pi \right). \tag{2.6}$$

In article [38], we generalize formula (2.6). For this end we show that on the plane $\hat{H}$ there are 48 types of generalized trihedrals with non-parabolic edges. Hereafter for the generalized trihedral $A_1A_2A_3$ of each type we prove the formula

$$S = \rho^2 \left( \tau_1 \hat{A}_1 + \tau_2 \hat{A}_2 + \tau_3 \hat{A}_3 - i\,\pi \right), \tag{2.7}$$

where $\tau_j = 1$ ($\tau_j = i$) in case the point $A_j$ is the proper or absolute (ideal) vertex of the generalized trihedral $A_1A_2A_3$.

Finally, for a generalized polygon without parabolic edges after its triangulation and applying formula (2.7) to every cell of partition we obtain formula (2.4).

## 3. Main theorem

In this section we prove a theorem which reveals significant differences between the plane $\hat{H}$ geometry and the classical geometries of planes with constant curvature.

**Theorem 2.** *There is no regular mosaic on the plane $\hat{H}$.*

*Proof.* **I.** At the first stage we prove the following assertion.

$\Delta$: *If there is a regular mosaic of the plane $\hat{H}$, then interior angles of its cells are right quasiangles.*

Assume that there is a regular mosaic $\mathfrak{M}$ of the type $\{p,q\}$ on the plane $\hat{H}$. Let a proper point $A$ of the plane $\hat{H}$ be a common vertex of $q$ congruent regular $p$-hedrals $F_1, \dots, F_q$ from the mosaic $\mathfrak{M}$. Denote the interior angle of the $p$-hedral $F_j$ at the vertex $A$ by $\alpha_j$, where $j = 1,\dots,q$. Let $k_1$ and $k_2$ be the parabolic lines passing through the point $A$. The lines $k_1$, $k_2$ contain four rays $k_{11}$, $k_{12}$, $k_{21}$, $k_{22}$ with a common origin $A$. The type of each angle $\alpha_j$ is unambiguously determined by its location with respect to these rays. Since the angles $\alpha_1, \dots, \alpha_q$ are congruent, they are equally located with respect to the rays from the set $k_{11}$, $k_{12}$, $k_{21}$, $k_{22}$. In the sense of the task each angle $\alpha_j$ is convex and is not straight angle. Hence each angle $\alpha_j$ contains no more than two rays from the set $k_{11}$, $k_{12}$, $k_{21}$, $k_{22}$. Moreover, if the angle $\alpha_j$ contains two rays from this set, then the sides of the angle $\alpha_j$ contain two these rays. It is clear that the ray $k_{11}$ lies on the side or on the interior domain of some angle from the set $\alpha_1, \dots, \alpha_q$. Denote this angle by $\alpha_1$. Then the side or, respectively, the interior domain of each of the angles $\alpha_2, \dots, \alpha_q$ contains a ray from the set $k_{12}$, $k_{21}$, $k_{22}$. Consequently, the quantity of angles in the set $\alpha_1, \dots, \alpha_q$ is equal to the quantity of rays in the set $k_{11}$, $k_{12}$, $k_{21}$, $k_{22}$. Therefore $q = 4$ and a priori only two following cases are possible.

(1) Each of the rays $k_{11}$, $k_{12}$, $k_{21}$, $k_{22}$ lies on a common side of two adjacent angles from the set $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ (Fig. 3, a).
(2) Each of the rays $k_{11}$, $k_{12}$, $k_{21}$, $k_{22}$ divides one of the angles $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$ (Fig. 3, b).

In the first case the angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are not congruent. Indeed, by the condition (1) the sides of the angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ lie on the parabolic lines $k_1 = k_{11} \cup k_{12}$ and $k_2 = k_{21} \cup k_{22}$. Hence two angles from the set $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$ belong to the valiana of the point $A$ (see the angles $\alpha_1$ and $\alpha_3$ in Fig. 3, a). Two other angles belong to the covaliana of this point. Thus under the condition (1) the mosaic $\mathfrak{M}$ is not regular. This contradicts our assumption.

The second case does not lead to contradictions. Let us continue our reasonings, accepting the condition (2).

The sides of each angle $\alpha_j$ divide the pair of the parabolic lines $k_1$, $k_2$. Hence they are non-parabolic lines of different types. This means that each angle $\alpha_j$ is a quasiangle. Since the quasiangles $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$ are congruent, they are right quasiangles. The assertion $\Delta$ is proved.

**II.** Now we check the existence of a polygon which forms a regular mosaic on the plane $\hat{H}$. Considering the assertion $\Delta$, we choose the incomplete list of properties of such polygon, and we prove that this list contains incompatible requirements.

Assume that there is a polygon $F$ satisfying the following conditions.

(1) All interior angles of the polygon $F$ are right quasiangles.

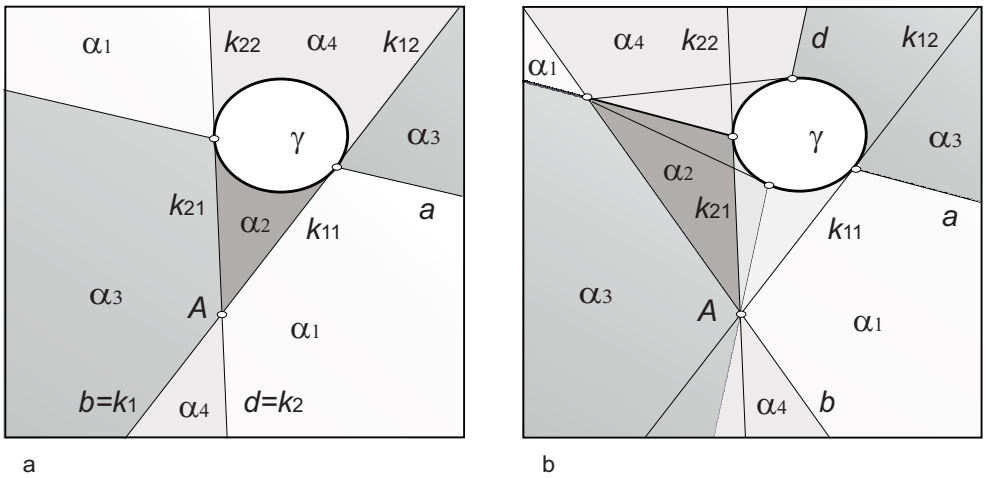


FIGURE 3. The angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ at the vertex $A$ of the regular mosaic $\mathfrak{M}$. The rays $k_{11}, k_{12}, k_{21}, k_{22}$ lie on the sides of the angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ (a). The rays $k_{11}, k_{12}, k_{21}, k_{22}$ divide the angles $\alpha_1, \alpha_3, \alpha_2, \alpha_4$, respectively (b).

(2) The polygon $F$ is topological equivalent to a disk.
(3) The polygon $F$ is finite figure of the plane $\hat{H}$.

Denote the vertices of the polygon $F$ by $A_1, \ldots, A_p$. Under the condition (1) the measures $\hat{A}_1, \ldots, \hat{A}_p$ of the interior angles of the polygon $F$ at the vertices respectively $A_1, \ldots, A_p$ equal $i\pi/2$.

Due to the requirement (2) the polygon $F$ satisfies the Theorem 1 conditions. By the condition (3) in the formulation of Theorem 1 for each vertex of the polygon $F$ we have $\tau_j = 1$. Moreover, the area $S_{\hat{H}}(F)$ of the polygon $F$ is a real positive number. Formula (2.4) yields

$$S_{\hat{H}}(F) = \rho^2 \left( i\frac{\pi}{2} p - i\pi(p-2) \right) = \frac{i\pi\rho^2}{2}(4-p). \tag{3.1}$$

The expression (3.1) implies $S_{\hat{H}}(F) \in \mathbb{C}$ and $\mathrm{Re}\left(S_{\hat{H}}(F)\right) = 0$. These facts contradict the notion of the area of a finite figure in the plane $\hat{H}$. Hence the conditions (1)–(3) are incompatible. So there is no polygon forming the regular mosaic on the plane $\hat{H}$. This yields the assertion of the theorem. The theorem is proved. □

## 4. COMPARATIVE ANALYSIS OF METHODS AND RESULTS

In Theorem 2 we proved that there is no regular mosaic on the plane $\hat{H}$. In the scientific discussion of this result the following question was raised: how the assertion about regular mosaics depends on the choice of the definition of angles on the

plane $\hat{H}$. In Subsection 2.2 we have discussed the main problems of the definition of objects in hyperbolic geometry in this regard. Taking into account the prevalence of these problems, we show here, with specific examples, how they impede the solution of the questions about polygons and, in general, are slowing down the development of the plane $\hat{H}$ geometry.

While sharing the views expressed by Klein, we accept the following statement: *Geometry is the totality of figures properties wich are invariant in transformations of some group.* According to this statement we define geometrical figures with an accuracy to their congruence, understanding the congruence of geometrical figures as the possibility of their coincidence under some transformation of the space containing these figures. The congruence of geometrical figures differs from equality of the measures of these figures if they are measurable. Indeed, let us consider two segments of the length $a$ in the plane $\hat{H}$, where $a \in (0, \pi\rho)$. Assume that one of them is elliptic, and the second is hyperbolic. Having equal lengths, these segments are not congruent because any transformation of the group $G$ retains the line type. Now let us consider an elliptic angle and half-plane of the plane $\hat{H}$. Such geometrical figures can have equal measures from interval $(0, \pi)$ (see Table 1 and [31, §4.5]), but these figures are not congruent. The examples given show us that the information about the measure of a geometrical figure can't unambiguously characterize this figure. Therefore such information can't serve as the definition of a geometrical object.

Let us address, for example, Definition 2.1 of generalized angles on the hyperbolic plane from paper [13]. To reduce reasonings, we shall consider only objects of the plane $\hat{H}$. In our terms, in the case 1 (b) of Definition 2.1 two hyperbolic lines with the common point on exterior domain with respect to the absolute are presented. Such lines determine three objects on the plane $\hat{H}$. These objects are as follows: two congruent hyperbolic angles and hyperbolic pseudoangle (see angles $\alpha_1$, $\alpha_2$, and, respectively, $\alpha_3$ in Fig. 4, a). A hyperbolic angle and a hyperbolic pseudoangle are topologically distinct. In article [13] these objects are not distinguishable and geometrically are not defined. It is worth noticing that the phrase "their angle is the length" from Definition 2.1 is incorrect because an angle is a geometrical figure, and the length is a number.

In the case 2 of Definition 2.1 two elliptic lines are presented. Such lines determine elliptic angle and elliptic pseudoangle on the plane $\hat{H}$ (see angles $\alpha_1$ and $\alpha_2$ in Fig. 4, b). These topologically different objects are not distinguishable and geometrically are not defined in paper [13] too. This means that at the using of Definition 2.1 the questions about polygons angles in the plane $\hat{H}$ will remain unresolved.

Formally Formula (6) from [13] yields the correct result only for the hyperbolic cosine of the measures of hyperbolic or elliptic angles and hyperbolic or elliptic pseudoangles. But Formula (6) does not give the actual values of the measures of these objects. Moreover, Remark 2.2 conclusively confuses the reader. The hyperbolic measures of adjacent or, in terms of paper [13], complementary angles except

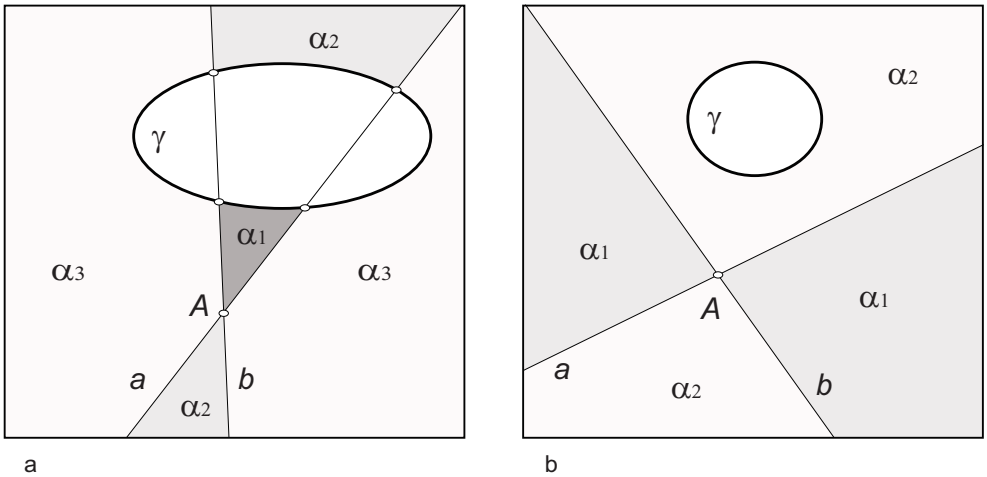


FIGURE 4. The hyperbolic angles $\alpha_1$, $\alpha_2$ and the hyperbolic pseudoangle $\alpha_3$ with the hyperbolic sides $a$, $b$ and the vertex $A$ (a). The elliptic angle $\alpha_1$ and the elliptic pseudoangle $\alpha_2$ with the elliptic sides $a$, $b$ and the vertex $A$ (b).

quasiangles are differ in their nature. The measure of a hyperbolic or elliptic angle is real, the measure of its adjacent pseudoangle is complex (see Table 1). The sum of the measures of adjacent angles is equal to $i\pi$ (see [31, §4.5]). Therefore the using of real instead of complex measures is inadmissible. We notice also that the both signs $\pm$ in Formula (6) is required because of the homogeneity of the coordinates used in this definition.

Thus Definition 2.1 from [13] is incomplete. On its basis it is impossible to investigate the polygons of the plane $\hat{H}$, in particular, it is impossible to prove Theorem 1 and to solve the question on the existence of regular mosaics.

The following full scheme of the introduction of geometrical objects was formed in classic geometrical works. We believe that this scheme will be useful to researchers in the field of non-Euclidean geometries.

(1) The geometrical definition of the class of congruent objects.
Definitions can be, for example, genetic, constructive, ”through a sort and specific differences”, by means of the classification.
(2) The indications and analytical conditions of the belonging of an arbitrary object to the entered class.
(3) The proof of the existence of an object from the entered class.
(4) The solution of the question about the possibility of the measurement for the entered objects.
(5) The geometrical definition of the measure of the entered objects.
(6) The determination of the admissible values of the entered measure.

(7) The analytical expression of the entered measure in the choosen coordinate system.

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# THE SEVENTH INTERNATIONAL WORKSHOP – 2018 "CONSTRUCTIVE METHODS FOR NON-LINEAR BOUNDARY VALUE PROBLEMS"

MIKLÓS RONTÓ

*Received 02 October, 2018*

In the period from 5 July to 8 July, 2018, the Seventh International Workshop–2018 *Constructive Methods for Non-Linear Boundary Value Problems* had taken place in Miskolc, Hungary. The list of participants and the program can be found on the workshop web site:

http://mat76.mat.uni-miskolc.hu/bvpworkshop/about

The workshop was organized by the Institute of Mathematics of the University of Miskolc and Institute of Mathematics of the Academy of Sciences of Czech Republic in cooperation with the Regional Committee of the Hungarian Academy of Sciences in Miskolc and Junior Mathematical Society in Miskolc.

The scope of the conference covered various topics related to the modern theory of boundary value problems in a broad sense, including initial value and periodic problems for functional differential equations, differential variational inequalities, certain singular problems, the method of lower and upper functions, stability, limit cycles, problems with state-dependent impulses, positive solutions of boundary value problems, oscillation theory, theory of abstract Kurzweil-Stieltjes integrals, differential-algebraic systems, vector valued metric approach in the theory of differential equations and inclusions, boundary value problems with integral and nonlocal conditions, asymptotic decomposition, numerical-analytic methods for boundary value problems, discontinuous differential equations with delayed and advanced arguments, equations with $p$-Laplacian.

The workshop was attended by more than 55 researchers from Austria, Brazil, Chile, Czech Republic, Georgia, Israel, Mexico, Poland, Romania, Russia, Ukraine, and Hungary, who held 20 and 30-minute lectures.

The Organizing and Program Committee of the workshop consisting of I. Astashova (Russia), J. Diblík (Czech Republic), A. Domoshnitsky (Israel), R. Hakl (Czech Republic), I. Kiguradze (Georgia), P. Körtesi (Hungary), N. Partsvania (Georgia), M. Perestyuk (Ukraine), N. Pop (Romania) I. Rachůnková (Czech Republic), A. Rontó (Czech Republic), A. Samoilenko (Ukraine), I. Szántó (Chile), J. Szigeti (Hungary), M. Tvrdý (Czech Republic), I. Piller (Hungary) and the author of this

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- Faculty of Mechanical Engineering and Informatics, University of Miskolc
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This meeting was a continuation of the sixt previous workshops held in Miskolc in 2000, 2003 and 2015 [4], in Sárospatak in 2006 [1], in Eger in 2009 [2], and in Tokaj in 2012 [3].

We believe that the workshop has been successful and hope that the fruitful tradition of this kind of "compact" international meetings oriented at exchange of ideas of researchers working in closely related fields will continue in future.

FIGURE 1. The participants of the workshop in front of the university main building

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# NONOSCILLATION OF EVEN ORDER EULER TYPE HALF-LINEAR DIFFERENCE EQUATIONS

VOJTĚCH RŮŽIČKA

*Received 24 February, 2017*

*Abstract.* We establish nonoscillation criteria for the even order half-linear difference equation of Euler type

$$\sum_{l=0}^{n}(-1)^{n-l}\beta_{n-l}\Delta^{n-l}\left(k^{(\alpha-lp)}\Phi\left(\Delta^{n-l}x_{k+l}\right)\right)=0,\quad \beta_n:=1,$$

where $\Phi(t):=|t|^{p-1}\operatorname{sgn}t$, $p\in(1,\infty)$, $n\in\mathbb{N}$, $k^{(\beta)}$ denotes the falling factorial power (for $\beta\in\mathbb{R}$) and $\alpha,\beta_0,\beta_1,\dots,\beta_{n-1}$ are real constants. For the two-term equation

$$(-1)^n\Delta^n\left(k^{(\alpha)}\Phi\left(\Delta^n x_k\right)\right)+\beta_0 k^{(\alpha-np)}\Phi(x_{k+n})=0$$

we establish the constant $\gamma_{n,p,\alpha}$ such that the two-term equation is nonoscillatory if $\beta_0>-\gamma_{n,p,\alpha}$. The criteria are derived using the variational technique and they are further extended via the theory of regularly varying sequences.

2010 *Mathematics Subject Classification:* 39A12; 39A21

*Keywords:* Euler half-linear difference equation, higher order half-linear difference equation, nonoscillation criterion, variational principle, energy functional

## 1. Introduction

We consider the $2n$-th order half-linear difference equation

$$\sum_{l=0}^{n}(-1)^{n-l}\Delta^{n-l}\left(r_k^{[n-l]}\Phi\left(\Delta^{n-l}x_{k+l}\right)\right)=0, \tag{1.1}$$

where $n\in\mathbb{N}$, $\Phi(t):=|t|^{p-1}\operatorname{sgn}t$ is the odd power function, the real number $p$ is such that $p>1$, $\left\{r_k^{[j]}\right\}_{k=1}^{\infty}$ is real-valued sequence for every $j\in\{0,1,\dots,n\}$ and $r_k^{[n]}\neq 0$ for $k\in\mathbb{N}$. The phrase "half-linear" reflects the fact that the solution space

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is homogenous, but not additive. Further we consider the energy functional

$$\mathcal{F}_n(\{y_k\};N,\infty) := \sum_{k=N}^{\infty}\left[\sum_{l=0}^{n} r_k^{[n-l]}\left|\Delta^{n-l}y_{k+l}\right|^p\right]$$

*associated* with equation (1.1), where $N \in \mathbb{N}$ and a sequence $\{y_k\}$ is from the set $\mathcal{D}_n(N)$ (definition of this set will be recalled later). We focus on a special cases of (1.1), namely on Euler type equation (1.2) and its extension (3.27). Consider the Euler type half-linear difference equation

$$\sum_{l=0}^{n}(-1)^{n-l}\beta_{n-l}\Delta^{n-l}\left(k^{(\alpha-lp)}\Phi\left(\Delta^{n-l}x_{k+l}\right)\right)=0, \quad \beta_n := 1, \tag{1.2}$$

where $\beta_0,\beta_1,\dots,\beta_{n-1}$ are real numbers and $\alpha \in \mathbb{R}\setminus\{p-1,2p-1,\dots,np-1\}$. For $k \in \mathbb{N}$ and $\beta \in \mathbb{R}$ the symbol $k^{(\beta)}$ denotes so-called *falling factorial power* (see [14, Definition 2.3]), which can be expressed as

$$k^{(\beta)} = \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}$$

for $k \in \mathbb{N}\setminus\{\beta-i \mid i \in \mathbb{N}\}$ and $\beta \in \mathbb{R}$, where $\Gamma$ denotes the Gamma function defined for $t \in \mathbb{R}\setminus\{0,-1,-2,\dots\}$. Recall that for $t \in (0,\infty)$ we have

$$\Gamma(t) := \int_0^{\infty} \mathrm{e}^{-s}s^{t-1}\,\mathrm{d}s.$$

Furthermore, recall that for sequences $\{a_k\}$ and $\{b_k\}$ of non-zero real numbers, we write $a_k \sim b_k$ as $k \to \infty$ and say that the sequences $\{a_k\}$ and $\{b_k\}$ are *asymptotically equivalent*, if $\lim_{k\to\infty} a_k/b_k = 1$. Now, from Stirling's formula

$$\lim_{t\to\infty}\frac{\Gamma(t+1)}{\left(\frac{t}{\mathrm{e}}\right)^t\sqrt{2\pi t}} = 1$$

we get the known relation $\frac{\Gamma(k+\beta)}{\Gamma(k)} \sim k^{\beta}$ as $k \to \infty$ (for $\beta \in \mathbb{R}$), hence,

$$k^{\beta} \sim k^{(\beta)} \quad \text{as} \quad k \to \infty. \tag{1.3}$$

In this article, we focus on getting conditions (for the coeficients $\beta_0,\beta_1,\dots,\beta_{n-1}$) which guarantee the nonoscillation of equation (1.2). We use the variational technique which is at disposal for general equation (1.1) by the results of article [3]. The main result of [3] is formulated (in a slightly different form) in Theorem 3 in this paper.

Our motivation comes mainly from the results for the continuous version of equation (1.2), i.e., for the differential equation

$$\sum_{l=0}^{n}(-1)^{n-l}\beta_{n-l}\left(t^{\alpha-lp}\Phi\left(x^{(n-l)}\right)\right)^{(n-l)}=0, \quad \beta_n := 1, \tag{1.4}$$

where $\alpha \in \mathbb{R} \setminus \{p-1, 2p-1, \dots, np-1\}$. Denote

$$\gamma_{n,p,\alpha} := \prod_{i=1}^{n} \left( \frac{|ip-1-\alpha|}{p} \right)^p .$$

Consider also the special cases

$$\gamma_{p,\alpha} := \gamma_{1,p,\alpha} = \left( \frac{|p-1-\alpha|}{p} \right)^p \quad \text{and} \quad \gamma_{n,2,\alpha} = \frac{1}{4^n} \prod_{i=1}^{n} (2i-1-\alpha)^2.$$

The following results are known criteria for special cases of equation (1.4). In [7, Theorem 1.4.4] it is shown that the second order equation

$$-\left(t^\alpha \Phi\left(x'\right)\right)' + \frac{\gamma}{t^{p-\alpha}} \Phi(x) = 0 \tag{1.5}$$

is nonoscillatory if and only if $\gamma + \gamma_{1,p,\alpha} \geq 0$ (for $\alpha = 0$ see the older result in [9]). For equation (1.5) the number $-\gamma_{1,p,\alpha}$ is the *critical constant*, i.e., the constant which is the "borderline" (as for the parameter $\gamma$) between oscillation and nonoscillation of equation (1.5). For the two-term $2n$-th order equation

$$(-1)^n \left(t^\alpha \Phi\left(x^{(n)}\right)\right)^{(n)} + \gamma t^{\alpha-np} \Phi\left(x\right) = 0 \tag{1.6}$$

we have so far only the following implication. Equation (1.6) is nonoscillatory if $\gamma + \gamma_{n,p,\alpha} > 0$ (for general $\alpha$ see [4, Theorem 3.2], for $\alpha = 0$ see the older result in [7, Theorem 9.4.5]).

If $p = 2$ then $\Phi(t) = |t| \operatorname{sgn} t = t$. Therefore, equation (1.4) with $p = 2$ is the linear differential equation. As a special case we get the two-term linear equation

$$(-1)^n \left(t^\alpha x^{(n)}\right)^{(n)} + \gamma t^{\alpha-np} x = 0. \tag{1.7}$$

In [13, page 132] (for $\alpha = 0$ see [11, pages 97-98]) it is shown that equation (1.7) is nonoscillatory if and only if $\gamma + \gamma_{n,2,\alpha} \geq 0$, i.e., the number $-\gamma_{n,2,\alpha}$ is the critical constant for equation (1.7).

The variational technique is at disposal (see [7, Theorem 9.4.4]) also for the continuous version of equation (1.1), i.e., for the differential equation

$$\sum_{l=0}^{n} (-1)^{n-l} \left(r_{n-l}(t) \Phi\left(x^{(n-l)}\right)\right)^{(n-l)} = 0. \tag{1.8}$$

In [4], we use the variational principle together with the Wirtinger inequality, which enables us to show positivity of the energy functional associated with equation (1.8). In the discrete case (in this article), to show positivity of the energy functional associated with equation (1.2) we use inequalities obtained by using Lemma 5. This approach is different from the one in the continuous case, because we do not use any discrete Wirtinger type inequality.

In the linear discrete case it is known the following nonoscillation criterion (see [8, Theorem 9]). The two-term linear difference equation

$$(-1)^n \Delta^n \left(k^{(\alpha)} \Delta^n x_k\right) + \frac{\gamma}{k^{(2n-\alpha)}} x_{k+n} = 0 \tag{1.9}$$

is nonoscillatory if $\gamma_{n,2,\alpha} + \gamma > 0$. In (1.9) if we take $k^{(\alpha-2n)}$ instead of $1\big/k^{(2n-\alpha)}$ then the proof from [8] of this criterion still works (by using (1.3)) with the same result. Note that the constant $-\gamma_{n,2,\alpha}$ is optimal (critical) for a slightly different type of Euler linear difference equation, namely for equation (4.5) with $-\gamma$ instead of $-\gamma_{n,2,\alpha}$ (see [10, Corollary 4.2]).

This paper is organized as follows. In the second section we rewrite equation (1.1) into a difference system and then we define the concept of generalized zero for the difference system and for equation (1.1) respectively. Further we define the concept of nonoscillation of equation (1.1) and we give two variational lemmas. The second section also contains two nonoscillation criteria, which plays important role in our later proofs. The end of the section is devoted to recalling basic concepts from the theory of regularly varying sequences. Section 3 presents two new nonoscillation criteria for equation (1.2) and is supplemented by remarks on a generalization via the concept of regularly varying sequence.

## 2. Preliminaries

In order to define the concept of nonoscillation for general half-linear equation (1.1), we need to define the concept of generalized zero for this equation. Further, in order to define the concept of generalized zero for equation (1.1), we transform equation (1.1) into a Hamiltonian type difference system.

Similar observations as in the previous paragraph hold also for the continuous case, i.e. for equation (1.8) (instead of the concept of generalized zero we get the concept of zero point of multiplicity $n$ from the transformation of (1.8) into a Hamiltonian type differential system; see [5]).

The following paragraphs (which lead as to the definition of generalized zero) are modeled according to the article [3]. Let $\{x_k\}$ be a solution of equation (1.1). Set

$$\begin{aligned} u_k^{[i]} &= \Delta^{i-1} x_{k+n-i}, \\ v_k^{[n]} &= r_k^{[n]} \Phi\left(\Delta^n x_k\right), \\ v_k^{[n-j]} &= -\Delta v_k^{[n-j+1]} + r_k^{[n-j]} \Phi\left(\Delta^{n-j} x_{k+j}\right) \end{aligned} \tag{2.1}$$

for $k \in \mathbb{N}$, $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n-1$. Denote the column vectors

$$u_k = \left(u_k^{[i]}\right)_{i=1}^n \quad \text{and} \quad v_k = \left(v_k^{[i]}\right)_{i=1}^n$$

for $k \in \mathbb{N}$, and note that the number $v_k^{[i]}$ can be expressed as

$$v_k^{[i]} = \sum_{l=0}^{n-i} (-1)^{n-i-l} \Delta^{n-i-l} \left( r_k^{[n-l]} \Phi \left( \Delta^{n-l} x_{k+l} \right) \right)$$

for $k \in \mathbb{N}$ and $i = 1, 2, \ldots, n$. Then the sequence $\{(u_k, v_k)\}$ is a solution of the Hamiltonian type difference system

$$\Delta u_k = A u_{k+1} + B_k \Phi^{-1}(v_k), \quad \Delta v_k = C_k \Phi(u_{k+1}) - A^T v_k, \tag{2.2}$$

where $\{B_k\}$ and $\{C_k\}$ are square matrix sequences of order $n$ such that

$$B_k = \operatorname{diag}\left\{0, 0, \ldots, 0, 1 \Big/ \Phi^{-1}\left(r_k^{[n]}\right)\right\} \quad \text{and} \quad C_k = \operatorname{diag}\left\{r_k^{[0]}, r_k^{[1]}, \ldots, r_k^{[n-1]}\right\};$$

and the matrix

$$A = \left(a_{ij}\right)_{i,j=1}^n \quad \text{with} \quad a_{ij} = \begin{cases} 1 & \text{for } j = i+1, \\ 0 & \text{elsewhere.} \end{cases}$$

For vector $a = (a_i)_{i=1}^n$, denote $\Phi(a) := (\Phi(a_i))_{i=1}^n$ and $\Phi^{-1}(a) := \left(\Phi^{-1}(a_i)\right)_{i=1}^n$, where $\Phi^{-1}(t) := |t|^{q-2} t$ is the inverse function of $\Phi(t)$. The constant $q$ is the conjugate number of $p$, i.e., $q := \frac{p}{p-1}$.

Now, we consider the general matrix difference system

$$\Delta u_k = A_k u_{k+1} + B_k \Phi^{-1}(v_k), \quad \Delta v_k = C_k \Phi(u_{k+1}) - A_k^T v_k, \tag{2.3}$$

where $\{B_k\}$ and $\{C_k\}$ are symmetric matrix sequences and $\{I - A_k\}$ is an invertible matrix sequence (symbol $I$ denotes the identity matrix). Let $m \in \mathbb{N}$, then we say that an interval $(m, m+1]$ contains the generalized zero of a solution $\{(u_k, v_k)\}$ of system (2.3) if

$$u_m \neq 0, \quad u_{m+1} \in \operatorname{Im}(I - A_m)^{-1} B_m \quad \text{and} \quad u_m^T B_m^\dagger (I - A_m) u_{m+1} \leq 0,$$

where $B_m^\dagger$ denotes the Moore-Penrose pseudoinverse of matrix $B_m$ and Im denotes the image.

In order to define generalized zero for equation (1.1) we proceed as follows. Let $m \in \mathbb{N}$ and $m \geq n$. We say that a nontrivial solution $\{x_k\}$ of equation (1.1) has a generalized zero in the interval $(m, m+1]$ if the solution $\{(u_k^*, v_k^*)\}_{k=n}^\infty$ of corresponding system (2.2) has a generalized zero in $(m, m+1]$, where $(u_k^*, v_k^*) := (u_{k-n+1}, v_{k-n+1})$ for every $k \in \mathbb{N}$ such that $k \geq n$; and the sequences $\{u_k\}$ and $\{v_k\}$ are given by relations (2.1). The shift ensures that our definition will be the same as the one in article [3] (the same shift is used for the linear case in [1, Remark 5 (ii)]), where it is considered equation (1.1) with $x_{k+1-n+l}$ instead of $x_{k+l}$ (such equation is equivalent to our equation (1.1)).

Now we rewrite this procedure explicitly in terms of equation (1.1). For a real vector $d = (d_i)_{i=1}^n$, we have $d \in \operatorname{Im}(I - A_m)^{-1} B_m$ if and only if there exists the vector $c = (c_i)_{i=1}^n$ such that

$$d = (I - A)^{-1} B_m c \text{ or equivalently (by a direct computation) } d_i = c_n \Big/ \Phi^{-1}\left(r_m^{[n]}\right)$$

for $i = 1, 2, \ldots, n$, i.e., the vector $d$ has equal components. Let $\{x_k\}$ be a nontrivial solution of (1.1). From the definition of $\{u_k^*\}$ and from the definition of $\{u_k\}$ we have

$$u_{m+1}^* = u_{(m+1)-n+1} = \left(\Delta^{i-1} x_{(m+1-n+1)+n-i}\right)_{i=1}^n = \left(\Delta^{i-1} x_{m-(i-2)}\right)_{i=1}^n.$$

If

$$u_{m+1}^* \in \operatorname{Im}(I - A_m)^{-1} B_m,$$

then $u_{m+1}^*$ has equal components, i.e., we have

$$\left(x_{m+1}, \Delta x_m, \Delta^2 x_{m-1}, \ldots, \Delta^{n-1} x_{m-(n-2)}\right)^T = (x_{m+1}, x_{m+1}, x_{m+1}, \ldots, x_{m+1})^T.$$

Hence, $x_{m-(n-2)} = x_{m-(n-3)} = \ldots = x_m = 0$. Next, for an arbitrary $i \in \{1, 2, \ldots, n\}$ we have

$$\Delta^{i-1} x_{m-(i-1)} = \sum_{l=1}^{i} (-1)^{i-l} \binom{i-1}{l-1} x_{m-(i-l)},$$

which is the $i$-th component of $u_m^*$. For $i \in \{1, 2, \ldots, n-1\}$ we have $x_{m-(i-1)} = x_{m-(i-2)} = \ldots = x_m = 0$, hence,

$$u_m^* = \left(0, 0, \ldots, 0, (-1)^{n-1} x_{m-(n-1)}\right)^T.$$

Therefore the relation $u_m^* \neq 0$ is equivalent with the relation $x_{m-(n-1)} \neq 0$. Finally, it can be shown that

$$B_m^\dagger (I - A) = \operatorname{diag}\left\{0, 0, \ldots, 0, \Phi^{-1}\left(r_m^{[n]}\right)\right\}.$$

Hence,

$$u_m^{*T} B_m^\dagger (I - A) u_{m+1}^* = (-1)^{n-1} \Phi^{-1}\left(r_m^{[n]}\right) x_{m-(n-1)} x_{m+1}.$$

**Definition 1.** Let $m \in \mathbb{N}$ and $m \geq n$. We say that a nontrivial solution $\{x_k\}$ of equation (1.1) has a *generalized zero* in the interval $(m, m+1]$ if $x_{m-(n-2)} = x_{m-(n-3)} = \ldots = x_m = 0$ (for $n > 1$),

$$x_{m-(n-1)} \neq 0 \quad \text{and} \quad (-1)^{n-1} r_m^{[n]} x_{m-(n-1)} x_{m+1} \leq 0. \tag{2.4}$$

Note that in relation (2.4) the constant $r_m^{[n]}$ appears, but only the sign of $r_m^{[n]}$ is important. This definition agrees with Hartman's one in [12] and it also matches the definition in [1] for the linear case.

**Definition 2.** We say that equation (1.1) is *nonoscillatory* (at infinity) if there exists $N \in \mathbb{N}$ such that $N \geq n$ and no nontrivial solution of equation (1.1) has two or more generalized zeros in $(N, \infty)$. Otherwise, equation (1.1) is called *oscillatory*.

Note that Definition 1 and Definition 2 agree with the definitions in [15] for the second order equation (equation (1.1) with $n = 1$).

Before we formulate variational lemmas, we make another note on the linear case (equation (1.1) with $p = 2$). System (2.3) with $p = 2$ reduces to the general linear Hamiltonian system. For linear Hamiltonian systems we have the Reid type roundabout theorem (see [1]) which guarantees equivalence between nonoscillation of equation (1.1) with $p = 2$, positivity of the energy functional associated with equation (1.1) with $p = 2$ and solvability of the so-called Riccati matrix equation associated with equation (1.1) with $p = 2$.

Similar remarks as in the previous paragraph hold also in the continuous case, i.e., for the equation

$$\sum_{l=0}^{n} (-1)^{n-l} \left( r_{n-l}(t) x^{(n-l)} \right)^{(n-l)} = 0$$

(see [18]).

Nonoscillation of an equation is equivalent to positivity of its energy functional also for equations (1.1) and (1.8) if $n = 1$, i.e., for the second order half-linear equations (the proof for (1.1) with $n = 1$ is given in [15]).

Next, we formulate the variational lemma for the second order equation

$$-\Delta \left( r_k^{[1]} \Phi\left(\Delta x_k\right) \right) + r_k^{[0]} \Phi\left(x_{k+1}\right) = 0, \tag{2.5}$$

which is a special case of equation (1.1). Denote

$$\begin{aligned} \mathcal{D}_n(N) := \{ \{y_k\}_{k=1}^{\infty} \mid y_k = 0 \text{ for } k \leq N + n - 1, \\ \exists m \in \mathbb{N} \text{ such that } m > N + n - 1 \text{ and } y_k = 0 \text{ for } k \geq m \} \end{aligned}$$

for $N \in \mathbb{N}$. Note that $\mathcal{D}_n(N) = \mathcal{D}_1(N + n - 1)$ for $N \in \mathbb{N}$, and $\mathcal{D}_n(N_2) \subseteq \mathcal{D}_n(N_1)$ for $N_1, N_2 \in \mathbb{N}$ such that $N_1 \leq N_2$.

**Lemma 1** ([15])**.** *Equation* (2.5) *is nonoscillatory if and only if there exists* $N \in \mathbb{N}$ *such that*

$$\mathcal{F}_1(\{y_k\}; N, \infty) = \sum_{k=N}^{\infty} \left[ r_k^{[1]} |\Delta y_k|^p + r_k^{[0]} |y_{k+1}|^p \right]$$

*is positive for every nontrivial sequence* $\{y_k\} \in \mathcal{D}_1(N)$.

For second order equation (2.5) we have the following two nonoscillation criteria, which will be applied to equation (1.2) with $n = 1$.

**Theorem 1** (O. Došlý, P. Řehák [6]). *Suppose that* $\sum^{\infty} r_k^{[0]}$ *is convergent,* $r_k^{[1]} > 0$ *for large* $k$, $\sum^{\infty} \left(r_k^{[1]}\right)^{1-q} = \infty$ *and*

$$\lim_{k\to\infty} \frac{\left(r_k^{[1]}\right)^{1-q}}{\sum^{k-1} \left(r_j^{[1]}\right)^{1-q}} = 0. \tag{2.6}$$

*Denote*

$$\mathcal{A}_k := \left(\sum^{k-1} \left(r_j^{[1]}\right)^{1-q}\right)^{p-1} \left(\sum_{j=k}^{\infty} r_j^{[0]}\right).$$

*If*

$$\liminf_{k\to\infty} \mathcal{A}_k > -\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1} \quad \textit{and} \quad \limsup_{k\to\infty} \mathcal{A}_k < \frac{2p-1}{p}\left(\frac{p-1}{p}\right)^{p-1},$$

*then equation* (2.5) *is nonoscillatory.*

Note that if $r_k^{[1]} > 0$ for large $k$, $\sum^{\infty} \left(r_k^{[1]}\right)^{1-q} = \infty$ and $\sum^{\infty} r_k^{[0]} = -\infty$, then equation (2.5) is oscillatory (see [15, Theorem 4] or [7, Theorem 8.2.14]).

Further note that if $r_k^{[0]} \le 0$ for large $k$, then the constant $-\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}$ is critical. Indeed, the condition

$$\limsup_{k\to\infty} \mathcal{A}_k < -\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}$$

with assumptions $r_k^{[1]} > 0$ for large $k$, $r_k^{[0]} \le 0$ for large $k$,

$$\sum^{\infty} \left(r_k^{[1]}\right)^{1-q} = \infty, \quad \sum^{\infty} r_k^{[0]} \text{ is convergent} \quad \text{and} \quad (2.6)$$

implies that equation (2.5) is oscillatory (see [7, Theorem 8.2.15]).

**Theorem 2** (O. Došlý, P. Řehák [6]). *Suppose that* $r_k^{[1]} > 0$ *for large* $k$, $\sum^{\infty} \left(r_k^{[1]}\right)^{1-q} < \infty$ *and*

$$\lim_{k\to\infty} \frac{\left(r_k^{[1]}\right)^{1-q}}{\sum_{j=k}^{\infty} \left(r_j^{[1]}\right)^{1-q}} = 0. \tag{2.7}$$

*Denote*

$$\mathcal{B}_k := \left(\sum_{j=k}^{\infty} \left(r_j^{[1]}\right)^{1-q}\right)^{p-1} \left(\sum^{k-1} r_j^{[0]}\right).$$

*If*

$$\liminf_{k\to\infty} \mathcal{B}_k > -\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1} \quad \text{and} \quad \limsup_{k\to\infty} \mathcal{B}_k < \frac{2p-1}{p}\left(\frac{p-1}{p}\right)^{p-1},$$

*then equation* (2.5) *is nonoscillatory.*

In case of general (even order) equation (1.1), the variational relation presented later in Lemma 2 is obtained from the following theorem.

**Theorem 3** (O. Došlý [3])**.** *Let $N_0 \in \mathbb{N}$ be such that $N_0 \geq n$ and let $N_1 \in \mathbb{N}$ be such that $N_1 \geq N_0 + n + 1$. If the interval $(N_0, N_1 + 1]$ contains two generalized zeros of a solution $\{x_k\}$ of equation* (1.1)*, then there exists a nontrivial sequence $\{y_k\} \in \mathcal{D}_1(N_0)$ such that $y_k = 0$ for $k \geq N_1 - n + 2$ and*

$$\mathcal{F}_n(\{y_k\}; N_0 - n + 1, N_1 - n + 1) = \sum_{k=N_0-n+1}^{N_1-n+1} \left[\sum_{l=0}^{n} r_k^{[n-l]} \left|\Delta^{n-l} y_{k+l}\right|^p\right] \leq 0.$$

Theorem 3 can be reformulated as follows. If we set $N = N_0 - n + 1$ and $X = N_1 - n + 1$, then the condition $N_0 \geq n$ means that $N \in \mathbb{N}$ and the condition $N_1 \geq N_0 + n + 1$ is reduced to $X \geq N + n + 1$. If we rewrite Theorem 3 in terms of such $N$ and $X$ then we can easily obtain the following variational lemma (in Definition 2 we take $N + n - 1$ (instead of $N$), which is obviously greater than or equal to $n$).

**Lemma 2.** *Equation* (1.1) *is nonoscillatory if there exists $N \in \mathbb{N}$ such that*

$$\mathcal{F}_n(\{y_k\}; N, \infty) = \sum_{k=N}^{\infty} \left[\sum_{l=0}^{n} r_k^{[n-l]} \left|\Delta^{n-l} y_{k+l}\right|^p\right]$$

*is positive for any nontrivial sequence $\{y_k\} \in \mathcal{D}_n(N)$.*

Next we recall the definition of regularly varying sequences and some of their selected properties (see [2, 17]).

**Definition 3.** Let $\vartheta \in \mathbb{R}$. A positive sequence $\{a_k\}$ is said to be *regularly varying (at infinity) of index $\vartheta$*, if

$$\lim_{k\to\infty} \frac{a_{[\lambda k]}}{a_k} = \lambda^{\vartheta}$$

for every positive real $\lambda$, where $[t]$ denotes the integer part of $t$. The set of all regularly varying sequences of index $\vartheta$ is denoted by $\mathcal{RV}(\vartheta)$. Further $\mathcal{RV} := \bigcup_{\vartheta\in\mathbb{R}} \mathcal{RV}(\vartheta)$ and $\mathcal{SV} := \mathcal{RV}(0)$. Sequences from the set $\mathcal{SV}$ are called *slowly varying*.

**Lemma 3.** *The following statements hold.*

(a) *A sequence $\{a_k\}$ belongs to $\mathcal{RV}(\vartheta)$ if and only if there exists $\{L_k\} \in \mathcal{SV}$ such that $a_k = k^{\vartheta} L_k$ for $k \in \mathbb{N}$.*
(b) *If $\{L_k\} \in \mathcal{SV}$ and $\{K_k\}$ is such that $K_k \sim L_k$ as $k \to \infty$, then $\{K_k\} \in \mathcal{SV}$.*

(c) *A sequence* $\{a_k\} \in \mathcal{RV}(\vartheta)$ *if and only if there exists* $\{L_k\} \in \mathcal{SV}$ *such that* $a_k = k^{(\vartheta)} L_k$ *for* $k \in \mathbb{N}$.

(d) *If* $\{a_k\} \in \mathcal{RV}(\vartheta)$, *then* $\{b_k\} \in \mathcal{RV}(\vartheta\beta)$ *for every* $\beta \in \mathbb{R}$, *where* $b_k = a_k^{\beta}$ *for* $k \in \mathbb{N}$.

(e) *Let* $\{a_k\} \in \mathcal{RV}(\vartheta_1)$ *and* $\{b_k\} \in \mathcal{RV}(\vartheta_2)$. *Then* $\{a_k b_k\} \in \mathcal{RV}(\vartheta_1 + \vartheta_2)$.

Now, we give a few examples of the slowly varying sequences. Trivial examples are positive constant sequences. A typical example is the sequence $\{\ln k\}$. Further the sequences $\{K_k\}$, $\{L_k\}$ and $\{M_k\}$ are slowly varying, where

$$K_k = \prod_{i=1}^{n} (\ln_i k)^{\mu_i} \text{ for } k \in \mathbb{N}, \text{ where } \ln_1 k := \ln k, \ln_{i+1} k := \ln(\ln_i k)$$
$$\text{and } \mu_i \in \mathbb{R} \text{ for } i \in \mathbb{N};$$
$$L_k = \exp\left\{ \prod_{i=1}^{n} (\ln_i k)^{\nu_i} \right\} \text{ for } k \in \mathbb{N}, \text{ where } \nu_i \in (0,1) \text{ for } i \in \{1,2,\dots,n\};$$
$$M_k = \frac{\ln \Gamma(k)}{k} \text{ for } k \in \mathbb{N}.$$

The next statement is very important for the extension of our results (Theorem 5 and Theorem 6) from the next section via the theory of regularly varying sequences.

**Theorem 4** (Karamata type theorem [2, 17])**.** *Let* $\{L_k\} \in \mathcal{SV}$. *Then*

$$\sum_{j=1}^{k-1} j^{\vartheta} L_j \sim \frac{k^{\vartheta+1}}{\vartheta+1} L_k \quad \text{as} \quad k \to \infty$$

*for every real* $\vartheta$ *such that* $\vartheta > -1$; *and*

$$\sum_{j=k}^{\infty} j^{\vartheta} L_j \sim -\frac{k^{\vartheta+1}}{\vartheta+1} L_k \quad \text{as} \quad k \to \infty$$

*for every real* $\vartheta$ *such that* $\vartheta < -1$.

## 3. Nonoscillation criteria

Consider equation (1.2) with $n = 1$ and $\beta_0 = \gamma$, i.e., the equation

$$-\Delta\left(k^{(\alpha)} \Phi(\Delta x_k)\right) + \gamma k^{(\alpha-p)} \Phi(x_{k+1}) = 0. \tag{3.1}$$

**Lemma 4.** *Let* $\alpha \neq p-1$ *and* $\gamma + \gamma_{p,\alpha} > 0$. *Then equation* (3.1) *is nonoscillatory.*

*Proof.* If $\gamma \geq 0$, then nonoscillation of equation (3.1) immediately follows from Lemma 1. Alternatively, we can use the Sturm comparison theorem (see [6, 15]) for equation (2.5). By this theorem, from nonoscillation of the equation $-\Delta\left(k^{(\alpha)} \Phi(\Delta x_k)\right) = 0$ it follows nonoscillation of equation (3.1) for $\gamma \geq 0$.

Let $\gamma \in \left(-\gamma_{p,\alpha}, 0\right)$. We often use relation (1.3) without referring to it. We prove the cases $\alpha < p-1$ and $\alpha > p-1$ separately. In the both cases we have $\left(k^{(\alpha)}\right)^{1-q} \sim k^{\alpha(1-q)}$ as $k \to \infty$, because

$$\lim_{k\to\infty} \frac{\left(k^{(\alpha)}\right)^{1-q}}{k^{\alpha(1-q)}} = \lim_{k\to\infty} \left(\frac{k^{(\alpha)}}{k^{\alpha}}\right)^{1-q} = 1.$$

Furthermore we have $k^{\alpha(1-q)} \sim k^{(\alpha(1-q))}$ for $k \to \infty$. Hence, $\left(k^{(\alpha)}\right)^{1-q} \sim k^{(\alpha(1-q))}$ for $k \to \infty$. It holds that $\alpha(1-q) = -\frac{\alpha}{p-1}$, hence, $\alpha(1-q) > -1$ for $\alpha < p-1$ and $\alpha(1-q) < -1$ for $\alpha > p-1$.

Let $\alpha < p-1$. We verify the assumptions of Theorem 1 for equation (3.1). We have

$$\lim_{k\to\infty} \frac{\left(k^{(\alpha)}\right)^{1-q}}{\sum^{k-1}\left(j^{(\alpha)}\right)^{1-q}} = \lim_{k\to\infty} \frac{k^{(\alpha(1-q))}}{\sum^{k-1} j^{\alpha(1-q)}} = \alpha(1-q) \lim_{k\to\infty} \frac{k^{(\alpha(1-q)-1)}}{k^{\alpha(1-q)}} = 0. \tag{3.2}$$

Indeed, from the discrete l'Hospital rule we get $\sum^{k-1}\left(j^{(\alpha)}\right)^{1-q} \sim \sum^{k-1} j^{\alpha(1-q)}$ as $k \to \infty$, because $\lim_{k\to\infty}\sum^{k-1} j^{\alpha(1-q)} = \infty$ for $\alpha(1-q) > -1$.

Further, from the limit comparison test,

$$\sum^{\infty}\left(k^{(\alpha)}\right)^{1-q} = \infty \quad \text{and} \quad \sum^{\infty} \gamma k^{(\alpha-p)} \quad \text{is convergent.} \tag{3.3}$$

Now we compute $\lim_{k\to\infty} \mathcal{A}_k$. We have

$$\sum_{j=k}^{\infty} \gamma j^{(\alpha-p)} = -\gamma \frac{k^{(\alpha-p+1)}}{\alpha-p+1} \quad \text{and} \quad \sum^{k-1}\left(j^{(\alpha)}\right)^{1-q} \sim \frac{k^{(\alpha(1-q)+1)}}{\alpha(1-q)+1} \text{ as } k \to \infty,$$

where the latter relation is obtained using the discrete l'Hospital rule. Hence,

$$\lim_{k\to\infty} \mathcal{A}_k = \lim_{k\to\infty} \left[\left(\frac{k^{(\alpha(1-q)+1)}}{\alpha(1-q)+1}\right)^{p-1}\right](-\gamma)\frac{k^{(\alpha-p+1)}}{\alpha-p+1} = \frac{\gamma(p-1)^{p-1}}{(p-1-\alpha)^p}. \tag{3.4}$$

The inequalities

$$\frac{\gamma(p-1)^{p-1}}{(p-1-\alpha)^p} > -\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1} \quad \text{and} \quad \frac{\gamma(p-1)^{p-1}}{(p-1-\alpha)^p} < \frac{2p-1}{p}\left(\frac{p-1}{p}\right)^{p-1}$$

are equivalent (for $\alpha < p-1$) with the inequalities $\gamma > -\gamma_{p,\alpha}$ and $\gamma < (2p-1)\gamma_{p,\alpha}$, respectively, and the inequality $\gamma < (2p-1)\gamma_{p,\alpha}$ holds for an arbitrary $\gamma \in (-\infty, 0)$. Hence, by Theorem 1, equation (3.1) is nonoscillatory for $\gamma \in \left(-\gamma_{p,\alpha}, 0\right)$ and $\alpha < p-1$.

Let $\alpha > p-1$. Similarly as in the previous case, we verify the assumptions of Theorem 2 for equation (3.1). By the limit comparison test we get

$$\sum^{\infty}\left(k^{(\alpha)}\right)^{1-q} < \infty,$$

and by the l'Hospital rule we have

$$\lim_{k\to\infty}\frac{\left(k^{(\alpha)}\right)^{1-q}}{\sum_{j=k}^{\infty}\left(j^{(\alpha)}\right)^{1-q}} = \lim_{k\to\infty}\frac{k^{(\alpha(1-q))}}{\sum_{j=k}^{\infty} j^{\alpha(1-q)}} = -\alpha(1-q)\lim_{k\to\infty}\frac{k^{(\alpha(1-q)-1)}}{k^{\alpha(1-q)}} = 0.$$

Now we compute $\lim_{k\to\infty}\mathcal{B}_k$. By the l'Hospital rule, we can verify that

$$\sum_{j=k}^{\infty}\left(j^{(\alpha)}\right)^{1-q} \sim -\frac{k^{(\alpha(1-q)+1)}}{\alpha(1-q)+1} \quad \text{and} \quad \sum^{k-1}\gamma j^{(\alpha-p)} \sim \gamma\frac{k^{(\alpha-p+1)}}{\alpha-p+1}$$

as $k\to\infty$. Hence,

$$\lim_{k\to\infty}\mathcal{B}_k = \lim_{k\to\infty}\left[\left(\frac{-k^{(\alpha(1-q)+1)}}{\alpha(1-q)+1}\right)^{p-1}\right]\gamma\frac{k^{(\alpha-p+1)}}{\alpha-p+1} = \frac{\gamma(p-1)^{p-1}}{(\alpha-p+1)^{p}}.$$

The inequalities

$$\frac{\gamma(p-1)^{p-1}}{(\alpha-p+1)^{p}} > -\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1} \quad \text{and} \quad \frac{\gamma(p-1)^{p-1}}{(\alpha-p+1)^{p}} < \frac{2p-1}{p}\left(\frac{p-1}{p}\right)^{p-1}$$

are equivalent (for $\alpha > p-1$) with the inequalities $\gamma > -\gamma_{p,\alpha}$ and $\gamma < (2p-1)\gamma_{p,\alpha}$, respectively. Hence, by Theorem 2, equation (3.1) is nonoscillatory for $\gamma \in \left(-\gamma_{p,\alpha}, 0\right)$ and $\alpha > p-1$. The proof is completed. □

Note that we can prove also the following oscillation complement of Lemma 4 (see the text after Theorem 1). If $\alpha \neq p-1$ and $\gamma + \gamma_{p,\alpha} < 0$, then equation (3.1) is oscillatory. We will not present the proof of this result in details since we do not need it.

*Remark* 1. The previous lemma can be generalized in the following sense. Consider the equation

$$-\Delta\left(f_k\Phi\left(\Delta x_k\right)\right) + \gamma g_k\Phi\left(x_{k+1}\right) = 0, \tag{3.5}$$

where $\{f_k\} \in \mathcal{RV}(\alpha)$, $\{g_k\} \in \mathcal{RV}(\alpha-p)$, $\alpha \neq p-1$, $\gamma \in \mathbb{R}$ and the sequences $\{f_k\}$ and $\{g_k\}$ take the forms

$$f_k = k^{\alpha}K_k,\ k\in\mathbb{N} \quad \text{and} \quad g_k = k^{\alpha-p}L_k,\ k\in\mathbb{N} \tag{3.6}$$

where $\{K_k\},\{L_k\} \in \mathcal{SV}$ (see Lemma 3). We show that if $\gamma + \gamma_{p,\alpha} > 0$ and $K_k \sim L_k$ as $k\to\infty$, then equation (3.5) is nonoscillatory. We use the proof of Lemma 4 with the following modifications. The paragraphs with relations (3.2), (3.3), (3.4) are replaced by the paragraphs with relations (3.7), (3.8), (3.9) below, respectively.

Let $\gamma \in \left(-\gamma_{p,\alpha}, 0\right)$ and $\alpha < p-1$. It holds that

$$\lim_{k\to\infty} \frac{k^{\alpha(1-q)} K_k^{1-q}}{\sum^{k-1} j^{\alpha(1-q)} K_j^{1-q}} = (\alpha(1-q)+1) \lim_{k\to\infty} \frac{k^{\alpha(1-q)} K_k^{1-q}}{k^{\alpha(1-q)+1} K_k^{1-q}} = 0. \tag{3.7}$$

Indeed, we have $\left\{K_k^{1-q}\right\} \in \mathcal{SV}$ and $\alpha(1-q) > -1$ for $\alpha < p-1$, hence, by Theorem 4, we get $\sum^{k-1} j^{\alpha(1-q)} K_j^{1-q} \sim (\alpha(1-q)+1)^{-1} k^{\alpha(1-q)+1} K_k^{1-q}$ as $k \to \infty$.

Further, by Theorem 4,

$$\sum^{\infty} k^{\alpha(1-q)} K_k^{1-q} = \infty, \quad \sum^{\infty} \gamma k^{\alpha-p} L_k \quad \text{is convergent} \tag{3.8}$$

and $\sum_{j=k}^{\infty} \gamma j^{\alpha-p} L_j \sim -\gamma \frac{k^{\alpha-p+1}}{\alpha-p+1} L_k$ as $k \to \infty$ for $\alpha - p < -1$.

Now we compute $\lim_{k\to\infty} \mathcal{A}_k$. We have

$$\lim_{k\to\infty} \mathcal{A}_k = \lim_{k\to\infty} \left[ \left( \frac{k^{\alpha(1-q)+1} K_k^{1-q}}{\alpha(1-q)+1} \right)^{p-1} \right] (-\gamma) \frac{k^{\alpha-p+1}}{\alpha-p+1} L_k = \frac{\gamma(p-1)^{p-1}}{(p-1-\alpha)^p}. \tag{3.9}$$

Similarly in the case $\gamma \in \left(-\gamma_{p,\alpha}, 0\right)$ and $\alpha > p-1$.

Now we formulate the lemma, which help us to estimate the summands from the energy functional associated with equation (1.2).

**Lemma 5.** *Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R} \setminus \{p-1, 2p-1, \dots, mp-1\}$ and $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{m-1}$ be arbitrary positive real numbers. Then there exists $N \in \mathbb{N}$ such that*

$$\sum_{k=N}^{\infty} \left[ k^{(\alpha-jp)} \left| \Delta^{m-j} y_{k+j} \right|^p + \left(\varepsilon_j - \gamma_{p,\alpha-jp}\right) k^{(\alpha-(j+1)p)} \left| \Delta^{m-j-1} y_{k+j+1} \right|^p \right] \tag{3.10}$$

*is positive for every nontrivial sequence $\{y_k\} \in \mathcal{D}_m(N)$ and for every $j \in \{0, 1, \dots, m-1\}$.*

*Proof.* First, consider the equation

$$-\Delta \left( k^{(\beta)} \Phi\left(\Delta x_k\right) \right) + \gamma k^{(\beta-p)} \Phi\left(x_{k+1}\right) = 0, \tag{3.11}$$

where $\beta \in \mathbb{R}$, $p \in (1, \infty)$ and $\gamma \in \mathbb{R}$.

By Lemma 4, equation (3.11) is nonoscillatory if $\gamma + \gamma_{p,\beta} > 0$ and $\beta \neq p-1$. Choose an arbitrary $\varepsilon \in (0, \infty)$, $j \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{R} \setminus \{p(j+1)-1\}$. Set $\beta = \alpha - jp$ and $\gamma = \varepsilon - \gamma_{p,\alpha-jp}$, then equation (3.11) becomes the equation

$$-\Delta \left( k^{(\alpha-jp)} \Phi\left(\Delta x_k\right) \right) + (\varepsilon - \gamma_{p,\alpha-jp}) k^{(\alpha-(j+1)p)} \Phi\left(x_{k+1}\right) = 0 \tag{3.12}$$

and we have

$$\gamma + \gamma_{p,\beta} = \varepsilon - \gamma_{p,\alpha-jp} + \gamma_{p,\alpha-jp} = \varepsilon > 0 \quad \text{and} \quad \beta = \alpha - jp \neq p-1.$$

Hence, by Lemma 4, equation (3.12) is nonoscillatory.

Now, let $m \in \mathbb{N}$, $\alpha \in \mathbb{R} \setminus \{p-1, 2p-1, \dots, mp-1\}$ and $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{m-1}$ be positive real numbers. Equation (3.11) with $\beta = \alpha - jp$ and $\gamma = \varepsilon_j - \gamma_{p,\alpha-jp}$, i.e., the equation

$$-\Delta\left(k^{(\alpha-jp)}\Phi\left(\Delta x_k\right)\right) + (\varepsilon_j - \gamma_{p,\alpha-jp})k^{(\alpha-(j+1)p)}\Phi\left(x_{k+1}\right) = 0$$

is nonoscillatory for any $j \in \{0, 1, \dots, m-1\}$. By Lemma 1, for every $j \in \{0, 1, \dots, m-1\}$ there exists $N_j \in \mathbb{N}$ such that the energy functional

$$\sum_{k=N_j}^{\infty}\left[k^{(\alpha-jp)}|\Delta z_k|^p + \left(\varepsilon_j - \gamma_{p,\alpha-jp}\right)k^{(\alpha-(j+1)p)}|z_{k+1}|^p\right]$$

is positive for every nontrivial $\{z_k\} \in \mathcal{D}_1(N_j)$.

Denote $N = \max\{N_0, N_1, \dots, N_{m-1}\}$, then we have $\mathcal{D}_1(N) \subseteq \mathcal{D}_1(N_j)$ for $j = 0, 1, \dots, m-1$. Note that for $\{z_k\} \in \mathcal{D}_1(N)$ we have $z_k = 0$ for $k \in \{1, 2, \dots, N\}$. Hence,

$$\sum_{k=N_j}^{N-1}\left[k^{(\alpha-jp)}|\Delta z_k|^p + \left(\varepsilon_j - \gamma_{p,\alpha-jp}\right)k^{(\alpha-(j+1)p)}|z_{k+1}|^p\right] = 0$$

for every $\{z_k\} \in \mathcal{D}_1(N)$ and for every $j \in \{0, 1, \dots, m-1\}$ such that $N_j \le N-1$, i.e.,

$$\sum_{k=N}^{\infty}\left[k^{(\alpha-jp)}|\Delta z_k|^p + \left(\varepsilon_j - \gamma_{p,\alpha-jp}\right)k^{(\alpha-(j+1)p)}|z_{k+1}|^p\right] > 0$$

for every nontrivial $\{z_k\} \in \mathcal{D}_1(N)$ and for every $j \in \{0, 1, \dots, m-1\}$.

Choose an arbitrary $j \in \{0, 1, \dots, m-1\}$ and an arbitrary nontrivial $\{y_k\} \in \mathcal{D}_m(N)$. Note that $y_1 = y_2 = \dots = y_{N+j} = \dots = y_{N+m-1} = 0$. Then the sequence $\{z_k\}$, defined by the relation $z_k = \Delta^{m-j-1}y_{k+j}$ for $k \in \mathbb{N}$, is nontrivial and it belongs to the set $\mathcal{D}_1(N)$. Hence,

$$\sum_{k=N}^{\infty}\left[k^{(\alpha-jp)}\left|\Delta^{m-j}y_{k+j}\right|^p + \left(\varepsilon_j - \gamma_{p,\alpha-jp}\right)k^{(\alpha-(j+1)p)}\left|\Delta^{m-j-1}y_{k+j+1}\right|^p\right] > 0$$

for every nontrivial $\{y_k\} \in \mathcal{D}_m(N)$ and for every $j \in \{0, 1, \dots, m-1\}$. □

*Remark* 2. Due to Remark 1, Lemma 5 can be generalized in the following way. Expression (3.10) is replaced by

$$\sum_{k=N}^{\infty}\left[k^{(\alpha-jp)}L_k^{[m-j]}\left|\Delta^{m-j}y_{k+j}\right|^p\right.$$

$$+\left(\varepsilon_j-\gamma_{p,\alpha-jp}\right)k^{(\alpha-(j+1)p)}L_k^{[m-(j+1)]}\left|\Delta^{m-j-1}y_{k+j+1}\right|^p\Bigg],$$

where $\left\{L_k^{[i]}\right\}\in\mathcal{SV}$ for $i=0,1,\dots,m$ and $L_k^{[m-j]}\sim L_k^{[m-(j+1)]}$ as $k\to\infty$ for $j=0,1,\dots,m-1$.

In the proof, equation (3.11) is replaced by

$$-\Delta\left(k^{(\beta)}K_k\Phi\left(\Delta x_k\right)\right)+\gamma k^{(\beta-p)}L_k\Phi\left(x_{k+1}\right)=0, \tag{3.13}$$

where $\beta\in\mathbb{R}$, $p\in(1,\infty)$, $\gamma\in\mathbb{R}$ and the sequences $\{K_k\}$ and $\{L_k\}$ are from the set $\mathcal{SV}$ such that $K_k\sim L_k$ as $k\to\infty$. Then it is easy to rewrite the rest of the proof. Note that equation (3.13) is equation (3.5), where the sequences

$$\left\{\frac{k^{(\alpha)}}{k^{\alpha}}K_k\right\}_{n=1}^{\infty}\quad\text{and}\quad\left\{\frac{k^{(\alpha-p)}}{k^{\alpha-p}}L_k\right\}_{n=1}^{\infty}$$

are slowly varying components of $\{f_k\}$ and $\{g_k\}$ (see Lemma 3), respectively.

Consider equation (1.2) with $\beta_1=\beta_2=\ldots=\beta_{n-1}=0$ and $\beta_0=\gamma$, i.e., the two-term equation

$$(-1)^n\Delta^n\left(k^{(\alpha)}\Phi\left(\Delta^n x_k\right)\right)+\gamma k^{(\alpha-np)}\Phi(x_{k+n})=0. \tag{3.14}$$

**Theorem 5.** *If*

$$\gamma+\gamma_{n,p,\alpha}>0$$

*and* $\alpha\in\mathbb{R}\setminus\{p-1,2p-1,\dots,np-1\}$, *then equation* (3.14) *is nonoscillatory.*

*Proof.* By Lemma 2 it is sufficient to prove that there exists $N\in\mathbb{N}$ such that

$$\sum_{k=N}^{\infty}\left[k^{(\alpha)}\left|\Delta^n y_k\right|^p+\gamma k^{(\alpha-np)}\left|y_{k+n}\right|^p\right]>0 \tag{3.15}$$

for every nontrivial $\{y_k\}\in\mathcal{D}_n(N)$.

To prove inequality (3.15) we use inequalities obtained via Lemma 5. Therefore, first we determine a set of $n-1$ positive real numbers. Recall that we have

$$\gamma_{p,\alpha}=\left(\frac{|p-1-\alpha|}{p}\right)^p,\quad \gamma_{p,\alpha-lp}=\left(\frac{|(l+1)p-1-\alpha|}{p}\right)^p\text{ and}$$

$$\gamma_{n,p,\alpha}=\prod_{l=0}^{n-1}\gamma_{p,\alpha-lp}$$

for $l=0,1,\dots,n-1$. Let $\varepsilon\in(0,\infty)$ be such that

$$\varepsilon<\min\left\{\gamma_{n,p,\alpha},\gamma+\gamma_{n,p,\alpha}\right\}. \tag{3.16}$$

Define real numbers $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{2n-2}$ by the recurrence relations

$$\varepsilon_{2l-1} = \frac{\varepsilon_{2l+1}}{2\gamma_{p,\alpha-lp}}, \qquad \varepsilon_{2l} = \frac{\varepsilon_{2l+1}\gamma_{p,\alpha-lp}}{2\gamma_{l+1,p,\alpha} - \varepsilon_{2l+1}},$$
$$\varepsilon_{2(n-1)-1} = \frac{\varepsilon}{2\gamma_{p,\alpha-(n-1)p}}, \qquad \varepsilon_{2(n-1)} = \frac{\varepsilon\gamma_{p,\alpha-(n-1)p}}{2\gamma_{n,p,\alpha} - \varepsilon}$$

for $l = 1, 2, \dots, n-2$. Condition (3.16) guarantees that the inequalities

$$\gamma_{p,\alpha-lp} > \varepsilon_{2l} > 0, \quad \gamma_{l,p,\alpha} > \varepsilon_{2l-1} > 0$$

hold for $l = 1, 2, \dots, n-1$. Indeed, we have

$$\begin{aligned}
\gamma_{n,p,\alpha} > \varepsilon > 0 \quad &\text{implies} \quad \gamma_{p,\alpha-(n-1)p} > \varepsilon_{2(n-1)} > 0 \quad \text{and} \\
&\qquad\qquad\quad \gamma_{n-1,p,\alpha} > \varepsilon_{2(n-1)-1} > 0; \\
\gamma_{n-1,p,\alpha} > \varepsilon_{2n-3} > 0 \quad &\text{implies} \quad \gamma_{p,\alpha-(n-2)p} > \varepsilon_{2(n-2)} > 0 \quad \text{and} \\
&\qquad\qquad\quad \gamma_{n-2,p,\alpha} > \varepsilon_{2(n-2)-1} > 0; \\
&\qquad\qquad \vdots \\
\gamma_{3,p,\alpha} > \varepsilon_5 > 0 \quad &\text{implies} \quad \gamma_{p,\alpha-2p} > \varepsilon_4 > 0 \quad \text{and} \\
&\qquad\qquad\quad \gamma_{2,p,\alpha} > \varepsilon_3 > 0; \\
\gamma_{2,p,\alpha} > \varepsilon_3 > 0 \quad &\text{implies} \quad \gamma_{p,\alpha-p} > \varepsilon_2 > 0 \quad \text{and} \\
&\qquad\qquad\quad \gamma_{1,p,\alpha} = \gamma_{p,\alpha} > \varepsilon_1 > 0.
\end{aligned}$$

Now we use Lemma 5. We have $\alpha \in \mathbb{R} \setminus \{p-1, 2p-1, \dots, np-1\}$. Denote $H = \{\varepsilon_1\} \cup \{\varepsilon_{2i} \mid i = 1, 2, \dots, n-1\}$. By Lemma 5, for the elements of the set $H$ there exists $N \in \mathbb{N}$ such that for every nontrivial $\{y_k\} \in \mathcal{D}_n(N)$ we have

$$\sum_{k=N}^{\infty} \left[ k^{(\alpha)} |\Delta^n y_k|^p + (\varepsilon_1 - \gamma_{p,\alpha}) k^{(\alpha-p)} \left|\Delta^{n-1} y_{k+1}\right|^p \right] > 0 \tag{3.17}$$

and

$$\sum_{k=N}^{\infty} \left[ k^{(\alpha-jp)} \left|\Delta^{n-j} y_{k+j}\right|^p + (\varepsilon_{2j} - \gamma_{p,\alpha-jp}) k^{(\alpha-(j+1)p)} \left|\Delta^{n-j-1} y_{k+j+1}\right|^p \right] > 0 \tag{3.18}$$

for $j = 1, 2, \dots, n-1$.

By a direct computation we can easily verify that for $l = 1, 2, \dots, n-2$ we have

$$(\gamma_{l,p,\alpha} - \varepsilon_{2l-1})(\gamma_{p,\alpha-lp} - \varepsilon_{2l}) = \gamma_{l+1,p,\alpha} - \varepsilon_{2(l+1)-1} \tag{3.19}$$

and

$$(\gamma_{n-1,p,\alpha} - \varepsilon_{2n-3})(\gamma_{p,\alpha-(n-1)p} - \varepsilon_{2n-2}) = \gamma_{n,p,\alpha} - \varepsilon. \tag{3.20}$$

Now we are ready to prove inequality (3.15). Among others, we use the relations

$$\gamma_{l,p,\alpha} > \varepsilon_{2l-1} \quad \text{and} \quad \gamma_{n,p,\alpha} - \varepsilon > -\gamma \tag{3.21}$$

for $l = 1, 2, \dots, n-1$. We have

$$\begin{aligned}
\sum_{k=N}^{\infty} k^{(\alpha)} |\Delta^n y_k|^p &\overset{(3.17)}{>} (\gamma_{p,\alpha} - \varepsilon_1) \sum_{k=N}^{\infty} k^{(\alpha-p)} \left|\Delta^{n-1} y_{k+1}\right|^p \\
&\overset{(3.18),(3.21)}{>} (\gamma_{p,\alpha} - \varepsilon_1)(\gamma_{p,\alpha-p} - \varepsilon_2) \sum_{k=N}^{\infty} k^{(\alpha-2p)} \left|\Delta^{n-2} y_{k+2}\right|^p \\
&\overset{(3.19)}{=} (\gamma_{2,p,\alpha} - \varepsilon_3) \sum_{k=N}^{\infty} k^{(\alpha-2p)} \left|\Delta^{n-2} y_{k+2}\right|^p \\
&\overset{(3.18),(3.21)}{>} (\gamma_{2,p,\alpha} - \varepsilon_3)(\gamma_{p,\alpha-2p} - \varepsilon_4) \sum_{k=N}^{\infty} k^{(\alpha-3p)} \left|\Delta^{n-3} y_{k+3}\right|^p \\
&\overset{(3.19)}{=} (\gamma_{3,p,\alpha} - \varepsilon_5) \sum_{k=N}^{\infty} k^{(\alpha-3p)} \left|\Delta^{n-3} y_{k+3}\right|^p \\
&\ \ \vdots \\
&\overset{(3.18),(3.21)}{>} (\gamma_{n-1,p,\alpha} - \varepsilon_{2n-3})(\gamma_{p,\alpha-(n-1)p} - \varepsilon_{2n-2}) \sum_{k=N}^{\infty} k^{(\alpha-np)} |y_{k+n}|^p \\
&\overset{(3.20)}{=} (\gamma_{n,p,\alpha} - \varepsilon) \sum_{k=N}^{\infty} k^{(\alpha-np)} |y_{k+n}|^p \\
&\overset{(3.21)}{>} -\gamma \sum_{k=N}^{\infty} k^{(\alpha-np)} |y_{k+n}|^p
\end{aligned}$$

for every nontrivial $\{y_k\} \in \mathcal{D}_n(N)$. □

Note that the constant $-\gamma_{n,p,\alpha}$ is optimal in the case $n = 1$ of equation (3.14) (see the paragraph below the proof of Lemma 4).

Consider equation (1.2), i.e., the full-term equation

$$\begin{aligned}
&(-1)^n \Delta^n \left(k^{(\alpha)} \Phi\left(\Delta^n x_k\right)\right) + (-1)^{n-1} \beta_{n-1} \Delta^{n-1} \left(k^{(\alpha-p)} \Phi\left(\Delta^{n-1} x_{k+1}\right)\right) + \dots \\
&\dots - \beta_1 \Delta \left(k^{(\alpha-(n-1)p)} \Phi\left(\Delta x_{k+n-1}\right)\right) + \beta_0 k^{(\alpha-np)} \Phi\left(x_{k+n}\right) = 0.
\end{aligned}$$

The technique of the previous proof can be also used to obtain a criterion for equation (1.2). In fact, the criterion for the two-term equation is a special case of the criterion for the full-term equation.

The following notation greatly simplifies the formulation of the next theorem. Denote

$$\gamma(1) := \gamma_{p,\alpha} + \beta_{n-1} \quad \text{and} \quad \gamma(k+1) := \gamma(k)\gamma_{p,\alpha-kp} + \beta_{n-1-k}$$

for $k = 1, 2, \dots, n-1$. Then

$$\begin{aligned}
\gamma(2) &= \gamma_{p,\alpha}\gamma_{p,\alpha-p} + \beta_{n-1}\gamma_{p,\alpha-p} + \beta_{n-2},\\
\gamma(3) &= \gamma_{p,\alpha}\gamma_{p,\alpha-p}\gamma_{p,\alpha-2p} + \beta_{n-1}\gamma_{p,\alpha-p}\gamma_{p,\alpha-2p} + \beta_{n-2}\gamma_{p,\alpha-2p} + \beta_{n-3},\\
&\vdots\\
\gamma(n) &= \gamma_{n,p,\alpha} + \sum_{k=1}^{n-1}\left[\prod_{l=k}^{n-1}\gamma_{p,\alpha-lp}\right]\beta_{n-k} + \beta_0.
\end{aligned}$$

**Theorem 6.** *If*

$$\gamma(k) > 0$$

*for every $k \in \{1, 2, \dots, n\}$ and $\alpha \in \mathbb{R} \setminus \{p-1, 2p-1, \dots, np-1\}$, then equation (1.2) is nonoscillatory.*

*Proof.* By Lemma 2 it is sufficient to prove that there exists $N \in \mathbb{N}$ such that the energy functional

$$\begin{aligned}
&\widetilde{\mathcal{F}}_n(\{y_k\}, N, \infty)\\
&:= \sum_{k=N}^{\infty}\left[k^{(\alpha)}|\Delta^n y_k|^p + \beta_{n-1}k^{(\alpha-p)}\left|\Delta^{n-1}y_{k+1}\right|^p + \dots + \beta_0 k^{(\alpha-np)}|y_{k+n}|^p\right]
\end{aligned} \tag{3.22}$$

associated with equation (1.2) is positive for every nontrivial $\{y_k\} \in \mathcal{D}_n(N)$.

Similarly as in the proof of Theorem 5 we use Lemma 5 here. Let $\varepsilon \in (0, \infty)$ be such that

$$\varepsilon < \gamma(n) \quad \text{and} \quad \varepsilon < 2^{n-l-1}\gamma(l)\prod_{j=l}^{n-1}\gamma_{p,\alpha-jp} \tag{3.23}$$

for every $l = 1, 2, \dots, n-1$. Define real numbers $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{2n-2}$ by the relations

$$\varepsilon_{2l-1} = \frac{\varepsilon}{2^{n-l}\prod_{j=l}^{n-1}\gamma_{p,\alpha-jp}} \quad \text{and} \quad \varepsilon_{2l} = \frac{\varepsilon\gamma_{p,\alpha-lp}}{2^{n-l}\gamma(l)\left[\prod_{j=l}^{n-1}\gamma_{p,\alpha-jp}\right] - \varepsilon}$$

for $l = 1, 2, \dots, n-1$. Note that if $\beta_1 = \beta_2 = \dots = \beta_{n-1} = 0$, then the constants $\varepsilon_{2l-1}$ and $\varepsilon_{2l}$ are the same as in the previous proof for each $l \in \{1, 2, \dots, n-1\}$. From conditions (3.23) we have the inequalities

$$\gamma_{p,\alpha-lp} > \varepsilon_{2l} > 0 \quad \text{and} \quad \gamma(l) > \varepsilon_{2l-1} > 0$$

for $l = 1, 2, \dots, n-1$.

Denote $H = \{\varepsilon_1\} \cup \{\varepsilon_{2i} \,|\, i = 1, 2, \dots, n-1\}$. By Lemma 5, for the elements of the set $H$ there exists $N \in \mathbb{N}$ such that the relations (3.17) and (3.18) hold for every nontrivial $\{y_k\} \in \mathcal{D}_n(N)$ and for every $j = 1, 2, \dots, n-1$.

By direct computation we can easily verify that for $l = 1,2,\dots,n-2$ we have

$$(\gamma(l)-\varepsilon_{2l-1})\left(\gamma_{p,\alpha-lp}-\varepsilon_{2l}\right) = [\gamma(l+1)-\beta_{n-1-l}]-\varepsilon_{2(l+1)-1} \tag{3.24}$$

and

$$(\gamma(n-1)-\varepsilon_{2n-3})\left(\gamma_{p,\alpha-(n-1)p}-\varepsilon_{2n-2}\right) = [\gamma(n)-\beta_0]-\varepsilon. \tag{3.25}$$

Now we prove positivity of functional (3.22). Among others, we use the relations

$$\gamma(l) > \varepsilon_{2l-1} \quad \text{and} \quad \gamma(n)-\varepsilon > 0 \tag{3.26}$$

for $l = 1,2,\dots,n-1$. We have

$$\begin{aligned}
&\sum_{k=N}^{\infty}\left[k^{(\alpha)}|\Delta^n y_k|^p + \beta_{n-1}k^{(\alpha-p)}\left|\Delta^{n-1}y_{k+1}\right|^p\right] \\
&\quad \overset{(3.17)}{>} \left[\left(\gamma_{p,\alpha}-\varepsilon_1\right)+\beta_{n-1}\right]\sum_{k=N}^{\infty}k^{(\alpha-p)}\left|\Delta^{n-1}y_{k+1}\right|^p \\
&\quad = (\gamma(1)-\varepsilon_1)\sum_{k=N}^{\infty}k^{(\alpha-p)}\left|\Delta^{n-1}y_{k+1}\right|^p \\
&\quad \overset{(3.18),(3.26)}{>} (\gamma(1)-\varepsilon_1)\left(\gamma_{p,\alpha-p}-\varepsilon_2\right)\sum_{k=N}^{\infty}k^{(\alpha-2p)}\left|\Delta^{n-2}y_{k+2}\right|^p \\
&\quad \overset{(3.24)}{=} (\gamma(2)-\beta_{n-2}-\varepsilon_3)\sum_{k=N}^{\infty}k^{(\alpha-2p)}\left|\Delta^{n-2}y_{k+2}\right|^p.
\end{aligned}$$

for every nontrivial $\{y_k\} \in \mathcal{D}_n(N)$. Therefore,

$$\begin{aligned}
&\sum_{k=N}^{\infty}\left[k^{(\alpha)}|\Delta^n y_k|^p + \beta_{n-1}k^{(\alpha-p)}\left|\Delta^{n-1}y_{k+1}\right|^p + \beta_{n-2}k^{(\alpha-2p)}\left|\Delta^{n-2}y_{k+2}\right|^p\right] \\
&\quad > [(\gamma(2)-\beta_{n-2}-\varepsilon_3)+\beta_{n-2}]\sum_{k=N}^{\infty}k^{(\alpha-2p)}\left|\Delta^{n-2}y_{k+2}\right|^p \\
&\quad \overset{(3.18),(3.26)}{>} (\gamma(2)-\varepsilon_3)\left(\gamma_{p,\alpha-2p}-\varepsilon_4\right)\sum_{k=N}^{\infty}k^{(\alpha-3p)}\left|\Delta^{n-3}y_{k+3}\right|^p \\
&\quad \overset{(3.24)}{=} (\gamma(3)-\beta_{n-3}-\varepsilon_5)\sum_{k=N}^{\infty}k^{(\alpha-3p)}\left|\Delta^{n-3}y_{k+3}\right|^p.
\end{aligned}$$

for every nontrivial $\{y_k\} \in \mathcal{D}_n(N)$. Continuing similarly step by step, we obtain

$$\sum_{k=N}^{\infty}\Big[k^{(\alpha)}|\Delta^n y_k|^p + \dots + \beta_2 k^{(\alpha-(n-2)p)}\left|\Delta^2 y_{k+n-2}\right|^p$$

$$
\begin{aligned}
&+\beta_1 k^{(\alpha-(n-1)p)}|\Delta y_{k+n-1}|^p\Big] \\
&> [(\gamma(n-1)-\beta_1-\varepsilon_{2n-3})+\beta_1]\sum_{k=N}^{\infty} k^{(\alpha-(n-1)p)}|\Delta y_{k+n-1}|^p \\
&\overset{(3.18),(3.26)}{>} (\gamma(n-1)-\varepsilon_{2n-3})\left(\gamma_{p,\alpha-(n-1)p}-\varepsilon_{2n-2}\right)\sum_{k=N}^{\infty} k^{(\alpha-np)}|y_{k+n}|^p \\
&\overset{(3.25)}{=} (\gamma(n)-\beta_0-\varepsilon)\sum_{k=N}^{\infty} k^{(\alpha-np)}|y_{k+n}|^p .
\end{aligned}
$$

for every nontrivial $\{y_k\}\in\mathcal{D}_n(N)$. Hence, the functional $\widetilde{\mathcal{F}}_n(\{y_k\},N,\infty)$ is greater than the expression

$$
[(\gamma(n)-\beta_0-\varepsilon)+\beta_0]\sum_{k=N}^{\infty} k^{(\alpha-np)}|y_{k+n}|^p ,
$$

which is positive (see (3.26)) for every nontrivial $\{y_k\}\in\mathcal{D}_n(N)$, i.e., $\widetilde{\mathcal{F}}_n(\{y_k\},N,\infty)$ is positive for every nontrivial sequence $\{y_k\}\in\mathcal{D}_n(N)$. □

*Remark* 3. In view of Remark 1 and Remark 2, Theorem 5 and Theorem 6 can be generalized for the equation

$$
\sum_{l=0}^{n}(-1)^{n-l}\beta_{n-l}\Delta^{n-l}\left(f_k^{[n-l]}\Phi\left(\Delta^{n-l}x_{k+l}\right)\right)=0, \quad \beta_n:=1, \tag{3.27}
$$

where $\beta_0,\beta_1,\dots,\beta_{n-1}$ are real numbers, $p\in(1,\infty)$, $\alpha\in\mathbb{R}\setminus\{p-1,2p-1,\dots,np-1\}$ and for every $l\in\{0,1,\dots,n\}$ the sequence $\left\{f_k^{[n-l]}\right\}\in\mathcal{RV}(\alpha-lp)$ and it has the form

$$
f_k^{[n-l]}=k^{(\alpha-lp)}L_k^{[n-l]},\ k\in\mathbb{N},
$$

where $\left\{L_k^{[n-l]}\right\}\in\mathcal{SV}$.

Now we can formulate the following nonoscillation criterion. If

$$
L_k^{[0]}\sim L_k^{[1]}\sim\dots\sim L_k^{[n]} \quad \text{as} \quad k\to\infty
$$

and $\gamma(k)>0$ for every $k\in\{1,2,\dots,n\}$, then equation (3.27) is nonoscillatory. In particular, if $L_k^{[0]}\sim L_k^{[n]}$ as $k\to\infty$ and $\gamma+\gamma_{n,p,\alpha}>0$, then equation (3.27) with $\beta_1=\beta_2=\ldots=\beta_{n-1}=0$ and $\beta_0=\gamma$, i.e., the two-term equation

$$
(-1)^n\Delta^n\left(f_k^{[n]}\Phi\left(\Delta^n x_k\right)\right)+\gamma f_k^{[0]}\Phi\left(x_{k+n}\right)=0 \tag{3.28}
$$

is nonoscillatory.

As far as we know our method of the proof is new even in the linear case ($p=2$). Moreover, the above criterion is new in the linear case. Here it is worthy of

note that equation (4.5) with $-\gamma$ instead of $\gamma_{n,2,\alpha}$ is a special case of (3.28). Thus, equation (3.28) has the higher order special (linear type) case for which the constant $-\gamma_{n,p,\alpha}$ is critical.

Note that if we wanted to give a proof directly for (3.28), then (according to Remark 2) we would have the following possibilities for replacing inequalities (3.17) and (3.18). We can take arbitrary sequences

$$\left\{f_k^{[1]}\right\}, \left\{f_k^{[2]}\right\}, \dots, \left\{f_k^{[n-1]}\right\}$$

such that for every $l \in \{1,2,\dots,n-1\}$ we have

$$\left\{f_k^{[n-l]}\right\} \in \mathcal{RV}(\alpha - lp)$$

and $f_k^{[n-l]}$ has a slowly varying component asymptotically equivalent to $\left\{L_k^{[n]}\right\}$. Then, inequality (3.17) is replaced by inequality (3.29); and the inequalities contained in (3.18) (for each $j \in \{1,2,\dots,n-1\}$ we have one inequality) are replaced by the inequalities contained in (3.30) and by inequality (3.31), where

$$\sum_{k=N}^{\infty}\left[k^{(\alpha)}L_k^{[n]}|\Delta^n y_k|^p + \left(\varepsilon_1 - \gamma_{p,\alpha}\right) f_k^{[n-1]}\left|\Delta^{n-1}y_{k+1}\right|^p\right] > 0, \tag{3.29}$$

$$\sum_{k=N}^{\infty}\left[f_k^{[n-j]}\left|\Delta^{n-j}y_{k+j}\right|^p + \left(\varepsilon_{2j} - \gamma_{p,\alpha-jp}\right) f_k^{[n-(j+1)]}\left|\Delta^{n-j-1}y_{k+j+1}\right|^p\right] > 0 \tag{3.30}$$

for $j = 1,2,\dots,n-2$, and

$$\sum_{k=N}^{\infty}\left[f_k^{[1]}|\Delta y_{k+n-1}|^p + \left(\varepsilon_{2(n-1)} - \gamma_{p,\alpha-(n-1)p}\right)k^{(\alpha-np)}L_k^{[0]}|y_{k+n}|^p\right] > 0. \tag{3.31}$$

*Example* 1. Consider the second order equation

$$-\Delta\left(f_k\Phi\left(\Delta x_k\right)\right) + \beta_0 g_k \Phi\left(x_{k+1}\right) = 0, \tag{3.32}$$

where $\{f_k\} \in \mathcal{RV}(-2)$ and $\{g_k\} \in \mathcal{RV}(-7)$ have asymptotically equivalent slowly varying components, i.e., equation (3.27) with $n = 1$, $\alpha = -2$ and $p = 5$. By Remark 3, equation (3.32) is nonoscillatory if $\beta_0 > -8$. Indeed, we have $\gamma_{1,5,-2} = \gamma_{5,-2} = \left(\frac{|5-1-(-2)|}{3}\right)^3 = 8$. For more specific example we can take, in addition, $f_k = \frac{\ln k}{k^2}$ and $g_k = \frac{\ln k}{k^{(7)}} + \frac{1}{k^7 \ln k}$ for $k \in \mathbb{N}$.

*Example* 2. If $n = 3$ then the assumptions of Theorem 6 read as

$$\gamma_{p,\alpha} + \beta_2 > 0,$$
$$\gamma_{p,\alpha}\gamma_{p,\alpha-p} + \beta_2\gamma_{p,\alpha-p} + \beta_1 > 0,$$

$$\gamma_{p,\alpha}\gamma_{p,\alpha-p}\gamma_{p,\alpha-2p}+\beta_2\gamma_{p,\alpha-p}\gamma_{p,\alpha-2p}+\beta_1\gamma_{p,\alpha-2p}+\beta_0>0$$

and $\alpha\in\mathbb{R}\setminus\{p-1,2p-1,3p-1\}$. Hence, $\beta_2\in(-\gamma_{p,\alpha},\infty)$. Assume that $\beta_2=(\varepsilon_1-1)\gamma_{p,\alpha}$ for some $\varepsilon_1\in(0,1)$. Then from the second assumption we have $\beta_1\in(-\varepsilon_1\gamma_{p,\alpha}\gamma_{p,\alpha-p},\infty)$. Note that if $\beta_2=0$, then $\beta_1\in(-\gamma_{2,p,\alpha},\infty)$. Choose $\beta_1=\varepsilon_1(\varepsilon_2-1)\gamma_{p,\alpha}\gamma_{p,\alpha-p}$ for some $\varepsilon_2\in(0,1)$. Then from the third assumption we have $\beta_0\in(-\varepsilon_1\varepsilon_2\gamma_{3,p,\alpha},\infty)$. Note that if $\beta_1=\beta_2=0$, then $\beta_0\in(-\gamma_{3,p,\alpha},\infty)$.

Consider the sixth order equation

$$\begin{aligned}-\Delta^3\left(a_k\Phi\left(\Delta^3x_k\right)\right)+8(\varepsilon_1-1)\Delta^2\left(b_k\Phi\left(\Delta^2x_{k+1}\right)\right)\\ -192\varepsilon_1(\varepsilon_2-1)\Delta\left(c_k\Phi\left(\Delta x_{k+2}\right)\right)+\beta_0d_k\Phi\left(x_{k+3}\right)=0,\end{aligned}\tag{3.33}$$

where $\varepsilon_1,\varepsilon_2\in(0,1)$ and $\{a_k\}\in\mathcal{RV}(-4)$, $\{b_k\}\in\mathcal{RV}(-7)$, $\{c_k\}\in\mathcal{RV}(-10)$ and $\{d_k\}\in\mathcal{RV}(-13)$ have asymptotically equivalent slowly varying components, i.e., equation (3.27) with $n=3$, $\alpha=-2$, $p=3$, $\beta_2=8(\varepsilon_1-1)$ and $\beta_1=192\varepsilon_1(\varepsilon_2-1)$.

Then

$$\begin{aligned}\gamma_{3,-4}=8,\ \gamma_{3,-7}=27,\ \gamma_{3,-4}\gamma_{3,-7}=192,\ \gamma_{3,-10}=64,\\ \gamma_{3,-4}\gamma_{3,-7}\gamma_{3,-10}=\gamma_{3,3,-4}=13824,\end{aligned}$$

and equation (3.33) is nonoscillatory if $\beta_0\in(-13824\varepsilon_1\varepsilon_2,\infty)$.

## 4. Open problems

(i) In the article [16, Theorem 2.2 and Theorem 3.1], which is considered in a more general setting of dynamic equations on time scales, one can found the following generalization of Theorem 1 and Theorem 2. Assume that the limits

$$\lim_{k\to\infty}\frac{\left(r_k^{[1]}\right)^{1-q}}{\sum_{j=a}^{k-1}\left(r_j^{[1]}\right)^{1-q}}=M\quad\text{and}\quad\lim_{k\to\infty}\frac{\left(r_k^{[1]}\right)^{1-q}}{\sum_{j=k}^{\infty}\left(r_j^{[1]}\right)^{1-q}}=N\tag{4.1}$$

exist (less restrictive compare to conditions (2.6) and (2.7)). Denote

$$\gamma_D(M):=\begin{cases}\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1} & \text{if } M=0,\\ \left(\frac{(M+1)^{\frac{p-1}{p}}-1}{M}\right)^{\alpha}\frac{M}{(M+1)^{p-1}-1} & \text{if } M\in(0,\infty),\\ 0 & \text{if } M=\infty\end{cases}\tag{4.2}$$

and

$$\gamma_C(N):=\begin{cases}\left(\frac{p-1}{p}\right)^{p}=q^{-p} & \text{if } N=0,\\ (1-N)\left(\frac{1-(1-N)^{\frac{p-1}{p}}}{N}\right)^{p} & \text{if } N\in(0,1),\\ 0 & \text{if } N=1.\end{cases}\tag{4.3}$$

Let $\left\{\sum_{j=k}^{\infty} r_j^{[0]}\right\}_{k=1}^{\infty}$ exist and be nonpositive and nontrivial for large $k$, $r_k^{[1]} > 0$ for large $k$, $\sum_{j=a}^{\infty}\left(r_j^{[1]}\right)^{1-q} = \infty$, and $M$ be given by (4.1). Then equation (2.5) is nonoscillatory provided

$$\liminf_{k\to\infty}\left(\sum_{j=a}^{k-1}\left(r_j^{[1]}\right)^{1-q}\right)^{p-1}\left(\sum_{j=k}^{\infty} r_j^{[0]}\right) > -\gamma_D(M).$$

Let $r_k^{[1]} > 0$ for large $k$, $\sum_{j=a}^{\infty}\left(r_j^{[1]}\right)^{1-q} < \infty$,

$$\left\{\sum_{j=k}^{\infty}\left[\sum_{i=j+1}^{\infty}\left(r_i^{[1]}\right)^{1-q}\right]^{p} r_j^{[0]}\right\}_{k=1}^{\infty}$$

exist and be nonpositive for large $k$, and $N$ given by (4.1). Then equation (2.5) is nonoscillatory provided

$$\liminf_{k\to\infty}\left(\sum_{j=k}^{\infty}\left(r_j^{[1]}\right)^{1-q}\right)^{-1}\left(\sum_{j=k}^{\infty}\left[\sum_{i=j+1}^{\infty}\left(r_i^{[1]}\right)^{1-q}\right]^{p} r_j^{[0]}\right) > -\gamma_C(N). \tag{4.4}$$

If $M = 0$, then the constant $-\gamma_D(0)$ is the same as the estimate for $\liminf_{k\to\infty}\mathcal{A}_k$ in Theorem 1. If $N = 0$, $r_k^{[1]} = k^{(\alpha)}$, $r_k^{[0]} = \gamma k^{(\alpha-p)}$ for $k \in \mathbb{N}$, $\gamma < 0$ and $\alpha > p-1$, then the left side of inequality (4.4) takes the form

$$\liminf_{k\to\infty}\left(\sum_{j=k}^{\infty}\left(j^{(\alpha)}\right)^{1-q}\right)^{-1}\left(\sum_{j=k}^{\infty}\left[\sum_{i=j+1}^{\infty}\left(i^{(\alpha)}\right)^{1-q}\right]^{p}\gamma j^{(\alpha-p)}\right) = \frac{\gamma(p-1)^p}{(\alpha-p+1)^p}$$

and it holds that

$$\frac{\gamma(p-1)^p}{(\alpha-p+1)^p} > -\gamma_C(0) \quad \text{if and only if} \quad \gamma > -\gamma_{p,\alpha}.$$

Hence, for $N = 0$ and $\alpha > p-1$ we obtained the same result as in the proof of Lemma 4 in the case $\alpha > p-1$, where Theorem 2 is used. Therefore, we believe that for $M \neq 0$ and $N \neq 0$ our criteria can be extended via the results from [16].

(ii) All our results are obtained under the restriction $\alpha \in \mathbb{R}\setminus\{p-1, 2p-1, \dots, np-1\}$, which is the condition required in our technique. We can try to deal with the cases $\alpha = ip-1$, $i = 1, 2, \dots, n$ in the further research.

(iii) Note that Theorem 1 and Theorem 2 do not require existence of the limits

$$\lim_{k\to\infty}\mathcal{A}_k \quad \text{and} \quad \lim_{k\to\infty}\mathcal{B}_k,$$

but in this article we work only with the coeficients of (3.1), for which these limits exist. Thus, other extension of our results may involve such sequences $\left\{r_k^{[0]}\right\}$ and $\left\{r_k^{[1]}\right\}$ that the limits $\lim_{k\to\infty}\mathcal{A}_k$ and $\lim_{k\to\infty}\mathcal{B}_k$ do not exist.

(iv) Oscillation behavior of two-term equation (3.14) in the critical case $\gamma = -\gamma_{n,p,\alpha}$ is still unknown. We believe that we will be able to prove a half-linear extension of the following result. In [10] it is proved that the linear two-term equation

$$(-1)^n \Delta^n \left(k^{(\alpha)} \Delta^n x_k\right) - \gamma_{n,2,\alpha}(k+n-2c(\alpha))^{(\alpha-2n)} x_{k+n} = 0 \tag{4.5}$$

with

$$c(\alpha) := n\chi_{(2n-1,\infty)}(\alpha) + \sum_{l=1}^{n-1} l\chi_{(2l-1,2l+1)}(\alpha) \quad \text{and} \quad \alpha \in \mathbb{R}\setminus\{1,3,5,\dots,2n-1\}$$

is nonoscillatory (for $t \in \mathbb{R}$ the symbol $\chi_M(t)$ denotes the indicator function for the set $M$ of real numbers).

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# ABOUT GLOBAL SOLUTION OF NONHOMOGENEOUS NEUTRAL PARTIAL DIFFERENTIAL EQUATION WITH DEVIATING ARGUMENT IN THE TIME VARIABLE

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*Abstract.* One class of nonhomogeneous neutral partial differential equations with deviating argument in the time variable is investigated. We find conditions under which it is possible to construct global solutions of these equations. We describe the structure of these solutions and their construction algorithm and we prove the theorem about substantiation of this method.

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*Keywords:* global solution, deviating argument, a complete normed space, contraction mapping principle

## 1. Introduction

Very often neutral differential equations are used in mathematical models of many problems in natural science to describe processes the velocity of which at the given time depends on the state and velocity at the previous moments. For example, such equations arise in the simulation of solid state body motion controllers in systems with feedback, in the transmission of lossless current, in the study of self-oscillations in a long circuit with a tunnel diode, in some problems of control theory, etc.

We should mention recent works on finding solutions for such equations and studying their properties, including articles J. Banaś and I.J. Cabrera [1], D.J. Khusainov, J. Diblik, E.I. Kuzmich [6], W.G. El-Sayed [3], S. Karimi Vanani, A. Aminataei [14].

It's important to study the conditions of existence of the global solutions for neutral differential equations. Works [13, 15, 16] are devoted to solve this problem. In some of these papers, the theorems of uniqueness and continuity of solutions are obtained on a small segment at first, and then the method of steps is applied. The theorems in [9] allow us to reduce the question of a continuity (or uniqueness) of solutions of a neutral differential-functional equation to the study of behavior of solutions of scalar differential inequalities.

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Some works should be noted, related to the existence and uniqueness of the solutions of boundary and initial boundary-value problems for neutral equations, in particular, there are works of G.A. Kamenskij, A.D. Myshkis, A.L. Skubachevskij [4], M.E. Drakhlin [2], O.I. Kiun [7], N.G. Kazakova [5], L.A. Minazhdinova [8].

## 2. The main research results

In this paper an algorithm for constructing a global solution for a nonhomogeneous neutral partial differential equation with deviating argument in the time variable is described, and the conditions for its existence are given.

Consider the equation

$$u_t(x,t) = p(t)u_{xx}(x,t+\mu) + r(t)u_t(x,t+\mu) + q(x,t),\ (x,t) \in Q, \tag{2.1}$$

with zero boundary conditions

$$u(0,t) = u(l,t) = 0,\ t \in \mathbb{R}, \tag{2.2}$$

where $Q = \{(x,t) : 0 < x < l, t \in \mathbb{R}\}$, $p(t)$, $r(t)$ are continuous functions in $\mathbb{R}$, and the deviation of $\mu$ is small enough.

### 2.1. *Determining the structure of solution for problem* (2.1), (2.2)

Previously in [11], it has been found that the corresponding homogeneous problem

$$u_t(x,t) = p(t)u_{xx}(x,t+\mu) + r(t)u_t(x,t+\mu)$$

has its eigenfunctions and eigenvalues respectively

$$X_k(x) = sin\frac{k\pi x}{l},\ \lambda_k = (\frac{\pi k}{l})^2,\ k = 1,...,n,$$

with some $n \geq 1$.

Consider the nonhomogeneous equation (2.1). We construct a global solution $u(x,t)$ of the problem (2.1), (2.2) as a sum

$$u(x,t) = \sum_{k=1}^{n} X_k(x)T_k(t). \tag{2.3}$$

Obviously, the boundary conditions are satisfied.

It is assumed that the function $q(x,t)$ can be represented as a sum of the first $n$ terms of the Fourier series:

$$q(x,t) = \sum_{k=1}^{n} q_k(t)sin\frac{k\pi x}{l}, \qquad q_k(t) = \frac{2}{l}\int_0^l q(\xi,t)sin\frac{k\pi\xi}{l}d\xi. \tag{2.4}$$

Putting (2.3) in equation (2.1), and taking into account (2.4), we have:

$$\sum_{k=1}^{n}\left[T_k'(t) - r(t)T_k'(t+\mu) + \lambda_k p(t)T_k(t+\mu) - q_k(t)\right]sin\frac{k\pi x}{l} = 0.$$

This equation is satisfied if all the expansion coefficients equal to zero, that is

$$T_k'(t) - r(t)T_k'(t+\mu) + \lambda_k p(t)T_k(t+\mu) - q_k(t) = 0,\ k = 1,\dots,n. \tag{2.5}$$

In work [12] it was shown, that for this equation we can construct corresponding equation without deviating argument, all the solutions of which will be global solutions of the equation (2.5). This equation has a form

$$T_k'(t) + \bar{p}_k(t)T_k(t) - \bar{q}_k(t) = 0,\ k = 1,\dots,n. \tag{2.6}$$

We can define $\bar{p}(t)$ and $\bar{q}(t)$ according to the algorithm defined in [10]. The general solution of equation (2.6) is determined by Cauchy formula

$$T_k(t) = T_{k,0}e^{-\int_{t_0}^{t}\bar{p}_k(s)ds} + \int_{t_0}^{t}\bar{q}_k(\tau)e^{-\int_{\tau}^{t}\bar{p}_k(s)ds}d\tau,$$

where

$$T_{k,0} = T_k(t_0),\ \ t, t_0 \in \mathbb{R}.$$

This function satisfies equation (2.5) if

$$\begin{aligned}
T_k'(t) &= -\bar{p}_k(t)\Big[T_{k,0}e^{-\int_{t_0}^{t}\bar{p}_k(s)ds} + \int_{t_0}^{t}\bar{q}_k(\tau)e^{-\int_{\tau}^{t}\bar{p}_k(s)ds}d\tau\Big] + \bar{q}_k(t)\\
&= r(t)\Big[-\bar{p}_k(t+\mu)\big[T_{k,0}e^{-\int_{t_0}^{t+\mu}\bar{p}_k(s)ds} + \int_{t_0}^{t+\mu}\bar{q}_k(\tau)e^{-\int_{\tau}^{t+\mu}\bar{p}_k(s)ds}d\tau\big] + \bar{q}_k(t+\mu)\Big]\\
&\quad -\lambda_k p(t)\Big[T_{k,0}e^{-\int_{t_0}^{t+\mu}\bar{p}_k(s)ds} + \int_{t_0}^{t+\mu}\bar{q}_k(\tau)e^{-\int_{\tau}^{t+\mu}\bar{p}_k(s)ds}d\tau\Big] + q_k(t),\ \ t \in \mathbb{R}.
\end{aligned} \tag{2.7}$$

Putting in (2.7) $T_{k,0} = 0$, we have

$$\begin{aligned}
&-\bar{p}_k(t)\Big[\int_{t_0}^{t}\bar{q}_k(\tau)e^{-\int_{\tau}^{t}\bar{p}_k(s)ds}d\tau\Big] + \bar{q}_k(t)\\
&= -\big(r(t)\bar{p}_k(t+\mu) + \lambda_k p(t)\big)\Big[\int_{t_0}^{t+\mu}\bar{q}_k(\tau)e^{-\int_{\tau}^{t+\mu}\bar{p}_k(s)ds}d\tau\Big]\\
&\quad + r(t)\bar{q}_k(t+\mu) + q_k(t),\ \ t \in \mathbb{R}.
\end{aligned} \tag{2.8}$$

Taking into account (2.7) and (2.8), we obtain

$$\bar{p}_k(t)e^{-\int_{t_0}^{t}\bar{p}_k(s)ds} = \big(r(t)\bar{p}_k(t+\mu) + \lambda_k p(t)\big)e^{-\int_{t_0}^{t+\mu}\bar{p}_k(s)ds},$$

where we receive

$$\bar{p}_k(t) = \big(r(t)\bar{p}_k(t+\mu) + \lambda_k p(t)\big)e^{\int\limits_{t+\mu}^{t} \bar{p}_k(s)ds}, \; k \le n, \; t \in \mathbb{R}. \tag{2.9}$$

Substituting (2.9) in (2.8):

$$-\big(r(t)\bar{p}_k(t+\mu) + \lambda_k p(t)\big)\int\limits_{t_0}^{t} \bar{q}_k(\tau)e^{\int\limits_{t+\mu}^{\tau} \bar{p}_k(s)ds} d\tau + \bar{q}_k(t)$$

$$= -\big(r(t)\bar{p}_k(t+\mu) + \lambda_k p(t)\big)\int\limits_{t_0}^{t+\mu} \bar{q}_k(\tau)e^{\int\limits_{t+\mu}^{\tau} \bar{p}_k(s)ds} d\tau + r(t)\bar{q}_k(t+\mu) + q_k(t).$$

$$\bar{q}_k(t) = q_k(t) + r(t)\bar{q}_k(t+\mu) + \big(r(t)\bar{p}_k(t+\mu) + \lambda_k p(t)\big)\int\limits_{t+\mu}^{t} \bar{q}_k(\tau)e^{\int\limits_{t+\mu}^{\tau} \bar{p}_k(s)ds} d\tau, k \le n, \; t \in \mathbb{R}. \tag{2.10}$$

Then the solution $T_k(t)$ of equation (2.6) can be written as

$$T_k(t) = c_k e^{-\int\limits_0^t \bar{p}_k(s)ds} + \int\limits_0^t \bar{q}_k(\tau)e^{-\int\limits_\tau^t \bar{p}_k(s)ds} d\tau, \tag{2.11}$$

where $c_k$ are arbitrary constants, $t_0 = 0$. And we find $\bar{p}_k(t)$ and $\bar{q}_k(t)$ using equations (2.9), (2.10).

So, if all the solutions of equation (2.6) are global solutions of equation (2.5), then function $\bar{p}$ satisfies equation (2.9), and function $\bar{q}$ satisfies equation (2.10). The existence of solutions for equations (2.9) and (2.10) is a necessary and sufficient condition to ensure that all solutions for equation (2.6) are global solutions for equation (2.5).

We found conditions under which the solution of equation (2.6) is the global solution of equation (2.5). The following theorem is the main result of our research.

**Theorem 1.** *Assume functions $p$, $r$ and $q$ satisfy the conditions imposed above, moreover*

$$|p(t)| < \beta, \;\; |r(t)| < \gamma, \;\; \beta, \gamma = const, \;\; t \in \mathbb{R},$$

*and also the following inequality holds*

$$\gamma + (\gamma + \lambda_n \beta |\mu|)e < 1, \;\; \gamma < \frac{1}{e}, \tag{2.12}$$

*where $n$ is an integer part of the number $\frac{l}{\pi}\sqrt{\frac{1-\gamma e}{e\beta|\mu|}}$.*

*Then the global solution exists in the form (2.3) for the problem (2.1), (2.2).*

*Proof of Theorem 1.* Consider the equation (2.9). The solution of this equation is a continuous function. We find conditions under which this equation has a unique solution, using the contraction mapping principle.

Let's define an operator $(S\bar{p}_k)(t)$ for function $\bar{p}_k$, which is continuous in $\mathbb{R}$.

$$(S\bar{p}_k)(t) = (r(t)\bar{p}_k(t+\mu) + \lambda_k p(t))e^{\int_{t+\mu}^{t} \bar{p}_k(s)ds}.$$

We search solution in the space $C(m)$. Let $\|\bar{p}_k\|_0 = \sup_{t\in\mathbb{R}} |\bar{p}_k(t)| \le m$. Then the next estimation for the $(S\bar{p}_k)(t)$ is true

$$\|S\bar{p}_k\|_0 = \sup_{t\in\mathbb{R}} \left|(r(t)\bar{p}_k(t+\mu) + \lambda_k p(t))e^{\int_{t+\mu}^{t} \bar{p}(s)ds}\right| \le (\gamma m + \lambda\beta)e^{|\mu|m}.$$

If inequality

$$(\gamma m + \lambda_k\beta)e^{|\mu|m} \le m \tag{2.13}$$

is satisfied, then the operator $(S\bar{p}_k)(t)$ reflects the space $C(m)$ to itself.

We estimate the difference $(S\bar{p}_{k,1})(t) - (S\bar{p}_{k,2})(t)$:

$$\begin{aligned}
|(S\bar{p}_{k,1})(t) - (S\bar{p}_{k,2})(t)| &= |r(t)(\bar{p}_{k,1}(t+\mu)e^{\int_{t+\mu}^{t} \bar{p}_{k,1}(s)ds} \\
&\quad - \bar{p}_{k,2}(t+\mu)e^{\int_{t+\mu}^{t} \bar{p}_{k,2}(s)ds}) \\
&\quad + \lambda_k p(t)(e^{\int_{t+\mu}^{t} \bar{p}_{k,1}(s)ds} - e^{\int_{t+\mu}^{t} \bar{p}_{k,2}(s)ds})| \\
&= |r(t)(\bar{p}_{k,1}(t+\mu)(e^{\int_{t+\mu}^{t} \bar{p}_{k,1}(s)ds} - 1) + \bar{p}_{k,1}(t+\mu) \\
&\quad - \bar{p}_{k,2}(t+\mu)(e^{\int_{t+\mu}^{t} \bar{p}_{k,2}(s)ds} - 1) - \bar{p}_{k,2}(t+\mu)) \\
&\quad + \lambda_k p(t)(e^{\int_{t+\mu}^{t} \bar{p}_{k,1}(s)ds} - e^{\int_{t+\mu}^{t} \bar{p}_{k,2}(s)ds})| \\
&\le (\gamma + \lambda_k\beta|\mu|)e^{|\mu|m}\|\bar{p}_{k,1} - \bar{p}_{k,2}\|_0.
\end{aligned}$$

Thus, when the inequality

$$(\gamma + \lambda_k\beta|\mu|)e^{|\mu|m} < 1 \tag{2.14}$$

is satisfied, then $(S\bar{p}_k)(t)$ is the contraction operator in the space $C(m)$.

We suppose conditions (2.13) and (2.14) are justified simultaneously. Let the following estimation be true

$$(\gamma + \lambda_k\beta|\mu|)e < 1. \tag{2.15}$$

Then the equation

$$(\gamma m + \lambda_k\beta)e^{|\mu|m} = m$$

has the solutions, for which the next condition is valid

$$\frac{(\gamma + \lambda_k\beta|\mu|)m}{\gamma m + \lambda_k\beta} < 1.$$

But the operator $(S\bar{p}_k)(t)$ is a contraction operator and reflects the space $C(m)$ to itself only for

$$m < \frac{1}{|\mu|}.$$

Thus, if this condition is satisfied, then the space $C(m)$ is a complete normed space and the operator $S$ has a single fixed point in this space, so the solution of equation (2.9) exists and it is unique.

Consider the equation (2.10). Without loss of generality, we assume that $|q(x,t)| \leq \frac{1}{2}$.

We define the operator $S_1$ in the space $C(M)$ of the functions $\bar{q}_k$, which are defined and continuous in $\mathbb{R}$, such that

$$\|\bar{q}_k\|_0 = \sup_{t\in\mathbb{R}} |\bar{q}_k(t)| \leq M.$$

$(S_1\bar{q}_k)(t)$ is continuous function in $\mathbb{R}$, and the next estimates are carried

$$\|S_1\bar{q}_k\|_0 \leq 1 + \gamma M + (\gamma m + \lambda_k\beta)|\mu| M e^{|\mu|m},$$

$$|(S\bar{q}_{k,1})(t) - (S\bar{q}_{k,2})(t)| \leq \gamma + (\gamma m + \lambda_k\beta)|\mu| e^{|\mu|m} \|\bar{q}_{k,1} - \bar{q}_{k,2}\|_0,$$

where $\bar{q}_{k,1}, \bar{q}_{k,2}$ are arbitrary functions in $C(M)$.

The condition $\gamma + (\gamma + \lambda_k\beta|\mu|)e < 1$ is justified due to the (2.12), so if

$$M \geq \frac{1}{1 - \gamma - (\gamma m + \lambda_k\beta)|\mu| e^{|\mu|m}}$$

then operator $S_1$ is a contraction operator and reflects the space $C(M)$ to itself. The space $C(M)$ is a complete normed space. And it's enough that the operator $S_1$ has a single fixed point in the space $C(M)$. This point is a single solution of equation (2.10) in $C(M)$.

Let us find a range of values for $k$. Since $\lambda_k = (\frac{\pi k}{l})^2$, we can find an approximate estimate for parameter $k$ from condition (2.15).

$$k < \frac{l}{\pi}\sqrt{\frac{1-\gamma e}{e\beta|\mu|}}. \tag{2.16}$$

Hence, $k \in [1,n]$, where $n$ is an integer part of $\frac{l}{\pi}\sqrt{\frac{1-\gamma e}{e\beta|\mu|}}$.

Therefore, the global solution of equation (2.5) has a form (2.11), where $\bar{p}_k(t)$ and $\bar{q}_k(t)$ are the solutions of equations (2.9), (2.10), and $k$ is determined by the formula (2.16). □

2.2. *Solving the equations* (2.9), (2.10)

We solve the equations (2.9), (2.10) for $\bar{p}_k(t)$ and $\bar{q}_k(t)$ by the method of successive approximations.

For initial approximation we take such:

$$\bar{p}_k^{(0)}(t) = \lambda_k p(t), \;\; \bar{q}_k^{(0)}(t) = q(t).$$

Then

$$\bar{p}_k^{(\eta)}(t) = (r(t)\bar{p}_k^{(\eta-1)}(t+\mu) + \lambda_k p(t))e^{\int_{t+\mu}^{t} \bar{p}_k^{(\eta-1)}(\tau)d\tau}, \;\; \eta = 1,2,...,$$
$$\bar{q}_k^{(\nu)}(t) = q_k(t) + r(t)\bar{q}_k^{(\nu-1)}(t+\mu) + (r(t)\bar{p}_k^{(\eta)}(t+\mu) + \lambda_k p(t))$$
$$\times \int_{t+\mu}^{t} \bar{q}_k^{(\nu-1)}(\tau)e^{\int_{t+\mu}^{\tau} \bar{p}_k^{(\eta)}(s)ds} d\tau, \quad \nu = 1,2,..., \;\; k = \overline{1,n}, \;\; t \in \mathbb{R}.$$

These sequences are uniformly convergent in $\mathbb{R}$ due to the conditions of the theorem, so

$$\bar{p}_k = \lim_{\eta\to\infty} \bar{p}_k^{(\eta)}, \bar{q}_k = \lim_{\nu\to\infty} \bar{q}_k^{(\nu)}.$$

Let us write the first approximation for $\bar{p}_k$ in explicit form

$$\bar{p}_k^{(1)}(t) = (r(t)p(t+\mu) + p(t))\lambda_k e^{\lambda_k \int_{t+\mu}^{t} p(s)ds}.$$

Taking into account conditions of Theorem 1, we can obtain the following estimate:

$$|\bar{p}_k^{(1)}(t) - \bar{p}_k^{(0)}(t)| \le \lambda_k \beta((\gamma+1)e^{|\mu|\beta\lambda_k} - 1).$$

According to estimates by the method of successive approximations such an assessment takes place for each next approximation

$$|\bar{p}_k^{(j)} - \bar{p}_k| \le \frac{\alpha^j |\bar{p}_k^{(1)} - \bar{p}_k^{(0)}|}{1-\alpha}.$$

In our case $\alpha = (\gamma + \lambda_k \beta|\mu|)e$, then we obtain for $j = \eta$:

$$|\bar{p}_k^{(\eta)} - \bar{p}_k| \le \frac{\alpha^\eta \lambda_k \beta \left|(\gamma+1)e^{|\mu|\beta\lambda_k} - 1\right|}{1-\alpha}.$$

Let us write the first approximation for $\bar{q}_k$ in explicit form

$$\bar{q}_k^{(1)}(t) = q(t) + r(t)q(t+\mu) + (r(t)\bar{p}_k^{(\eta)}(t+\mu) + \lambda_k p(t))$$
$$\times \int_{t+\mu}^{t} q(\tau)e^{\int_{t+\mu}^{\tau} \bar{p}_k^{(\eta)}(s)ds} d\tau.$$

Therefore, the following estimate is true:

$$|\bar{q}_k^{(1)}(t) - \bar{q}_k^{(0)}(t)| \le \gamma + (\gamma + \lambda_k \beta|\mu|)e.$$

Given this, we get

$$|\bar{q}_k^{(\nu)} - \bar{q}_k| \leq \frac{\alpha^\nu |\bar{q}_k^{(1)} - \bar{q}_k^{(0)}|}{1-\alpha} \leq \frac{\alpha^\nu(\gamma + (\gamma + \lambda_k \beta |\mu|)e)}{1-\alpha}, \quad k = \overline{1,n}, \quad t \in \mathbb{R}.$$

So, we can write an approximate solution for the equation (2.6) in a form (2.11) by finding an approximation for each of $\bar{p}_k(t)$ and $\bar{q}_k(t)$.

Thus, the solution of the problem (2.1), (2.2) takes the form

$$u(x,t) = \sum_{k=1}^{n} (c_k e^{-\int_0^t \bar{p}_k(s)ds} + \int_0^t \bar{q}_k(\tau) e^{-\int_\tau^t \bar{p}_k(s)ds} d\tau) sin \frac{k\pi x}{l}.$$

## Conclusions

So, in our research we found conditions under which there is a global solution of the problem (2.1), (2.2) and constructed it.

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# UPPER BOUNDS ON THE DIAMETER FOR FINSLER MANIFOLDS WITH WEIGHTED RICCI CURVATURE

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*Abstract.* In this paper we obtain some Cheeger-Gromov-Taylor type compactness theorems for a forward complete and connected Finsler manifold of dimensional $n \geq 2$ via weighted Ricci curvatures. The proofs are based on the index form of a minimal unit speed geodesic segment, Bochner-Weitzenböck formula and Hessian comparison theorem.

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## 1. Introduction and Main Theorems

In [8], Myers obtained a compactness theorem in Riemannian manifolds. The theorem of Myers concludes that if $\mathrm{Ric} \geq (n-1)K > 0$, then $\mathrm{diam}(M) \leq \pi/\sqrt{K}$. Later, Cheeger-Gromov-Taylor [3] proved that if there exist $p \in M$ and $r_0, \nu > 0$ such that

$$\mathrm{Ric} \geq (n-1)\frac{(\frac{1}{4}+\nu^2)}{r^2} \tag{1.1}$$

holds for all $r(x) \geq r_0 > 0$ where $r$ is distance function defined with respect to a fixed point $p \in M$, i.e., $r(x) = d(x,p)$, then $M$ is compact and the diameter is bounded from above by $\mathrm{diam}_p(M) < r_0 e^{\pi/\nu}$. By using Bakry-Emery Ricci tensor, $\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}\, f$, Soylu [12] attained a generalization of Cheeger-Gromov-Taylor's compactness theorem.

For $m$-Bakry-Emery Ricci tensor, Wang [14] proved that, if the following inequality

$$\mathrm{Ric}_{f,m} = \mathrm{Ric} + \mathrm{Hess}\, f - \frac{\mathrm{d}f \otimes \mathrm{d}f}{m-n} \geq -(m-1)\frac{K_0}{(1+r)^2} \tag{1.2}$$

holds for all $x \in M$, where $K_0 < -\frac{1}{4}$ and $r$ is distance function defined with respect to a fixed point $p \in M$, then $M$ is compact and the diameter has the upper bound $\mathrm{diam}(M) < 2(e^{2\pi/\overline{K}} - 1)$, where $\overline{K} = \sqrt{-K_0 - \frac{1}{4}}$.

We can find various kinds of generalizations of the Myers theorem in [4, 6, 7, 13, 15].

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Finsler geometry is a natural generalization of Riemannian geometry. The validity of the Myers compactness theorem for Finsler manifolds was shown by Shen [11] without any modification. Later, using the weighted Ricci curvature $\mathrm{Ric}_N := \mathrm{Ric} + \dot{S} - \frac{S^2}{N-n} \geq K > 0$, $N \in (n, \infty)$, Ohta [9] obtained a compactness theorem and gave an upper bound for the diameter of $n$-dimensional Finsler manifolds as $\mathrm{diam}(M) \leq \pi\sqrt{(N-1)/K}$. In [16], Wu establish a generalized Myers theorem under line integral curvature bound for Finsler manifolds. In [2], Anastasiei extended to Finsler manifolds the compactness theorems of Ambrose and Galloway (see [1] and [5], respectively). Yin [18] acquired two Myers-type compactness theorems for a Finsler manifold with a positive weighted Ricci curvature bound and an advisable condition on the distortion or the $S$-curvature.

Throughout this paper, $(M, F)$ is a connected forward complete $n$-dimensional smooth Finsler manifold, $r(x) = d(x, p)$ is the forward distance function from $p \in M$ and $d\mu$ is an arbitrary positive $\mathcal{C}^\infty$-measure on $M$. Here, there is no canonical measure like the volume measure in Riemannian geometry. Thus we begin with an arbitrary measure on $M$.

We are now ready to give our main results.

**Theorem 1.** *Let $(M, F, d\mu)$ be a forward complete and connected Finsler manifold of dimension $n$ with arbitrary volume form and let $r$ be the distance function $r(x) = d(x, p)$ with respect to a fixed point $p \in M$. Assume that the weighted Ricci curvature*

$$\mathrm{Ric}_\infty := \mathrm{Ric} + \dot{S} \geq (n-1)\frac{H}{r^2}, \tag{1.3}$$

*and the distortion $|\tau| \leq (n-1)k$ for all $x \in M$ such that $r(x) \geq r_0 > 0$, where the constants $k$ and $H$ satisfy the inequalities $k \geq 0$ and $H > 1/4$. Then $M$ is compact and the diameter from the point $p \in M$ satisfies*

$$\mathrm{diam}_p(M) \leq r_0 \exp\left(\frac{2}{4H-1}\sqrt{32k^2 + (4H-1)\pi^2 + 16k\sqrt{4k^2 + (4H-1)H\pi^2}}\right). \tag{1.4}$$

The distortion $\tau$ is a smooth function on $M$ when $M$ is a Riemannian manifold. Therefore the diameter estimate (1.4) of Theorem 1 coincides with the diameter estimate of Theorem 1.1 in [12].

**Theorem 2.** *Let $(M, F, d\mu)$ be a forward complete and connected Finsler manifold of dimension $n$ with arbitrary volume form and let $r$ be the distance function $r(x) = d(x, p)$ with respect to a fixed point $p \in M$. Assume that the weighted Ricci curvature*

$$\mathrm{Ric}_N := \mathrm{Ric}_\infty - \frac{S^2}{N-n} \geq (N-1)\frac{H}{r^2} \tag{1.5}$$

*for all $N \in (n, \infty)$ and $r(x) \geq r_0 > 0$, where $H > 1/4$. Then $M$ is compact and the diameter from the point $p \in M$ satisfies*

$$\mathrm{diam}_p(M) \leq r_0 e^{2\pi/\sqrt{4H-1}}. \tag{1.6}$$

The diameter estimate (1.6) obtained in the above theorem coincides with the result of Cheeger-Gromov-Taylor in [3] obtained for the original Ricci tensor in the Riemannian manifolds.

**Theorem 3.** *Let $(M, F, d\mu)$ be a forward complete and connected Finsler manifold of dimension $n$ with arbitrary volume form and let $r$ be the distance function $r(x) = d(x, p)$ with respect to a fixed point $p \in M$. Suppose that the weighted Ricci curvature*

$$\mathrm{Ric}_N := \mathrm{Ric}_\infty - \frac{S^2}{N-n} \geq (N-1)\frac{H}{(1+r)^2} \tag{1.7}$$

*for all $x \in M$ and $N \in (n, \infty)$, where $H > 1/4$. Then $M$ is compact and the diameter satisfies*

$$\mathrm{diam}(M) \leq (1+\lambda)(e^{2\pi/\sqrt{4H-1}} - 1), \tag{1.8}$$

*where $\lambda$ is the reversibility.*

We review below some basic informations about the Finsler manifolds to be used in the proofs of main theorems.

## 2. A brief review of Finsler geometry

Let $(M, F)$ be a Finsler $n$-manifold with Finsler metric $F : TM \to [0, \infty)$. Let $\pi : TM \to M$ be the natural projection and $(x, y)$ be a point of $TM$ such that $x \in M$ and $y \in T_xM$. A *Finsler metric* is a $\mathcal{C}^\infty$-Finsler structure of $M$ with the following properties:

1. $F$ is $\mathcal{C}^\infty$ on $TM \setminus 0$ (Regularity),
2. $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$ (Positive homogeneity),
3. The $n \times n$ Hessian matrix
$$g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$$
is positive-definite at every point of $TM \setminus 0$ (Strong convexity).

The *Chern curvature* $R^V$ for vectors fields $X, Y, Z \in T_xM \setminus 0$ is defined by

$$R^V(X, Y)Z := \nabla^V_X \nabla^V_Y Z - \nabla^V_Y \nabla^V_X Z - \nabla^V_{[X,Y]} Z, \tag{2.1}$$

and the *flag curvature* is defined as follows:

$$K(V, W) := \frac{g_V(R^V(V, W)W, V)}{g_V(V, V)g_V(W, W) - g_V(V, W)^2}, \tag{2.2}$$

where $V, W \in T_xM \setminus 0$ are linearly independent vectors. Then the *Ricci curvature* of $V$ (as the trace of the flag curvature) is defined by

$$\mathrm{Ric}(V) := \sum_{i=1}^{n-1} K(V, E_i), \tag{2.3}$$

where $\{E_1, E_2, ..., E_{n-1}, V/F(V)\}$ is an orthonormal basis of $T_xM$ with respect to $g_V$.

Let $d\mu = \sigma(x)dx^1dx^2...dx^n$ be the volume form on $M$. For a vector $V \in T_xM \setminus 0$,

$$\tau(x,V) := \ln \frac{\sqrt{\det(g_{ij}(x,V))}}{\sigma(x)} \tag{2.4}$$

is a scalar function on $T_xM \setminus 0$ which is called the *distortion* of $(M, F, d\mu)$. We say that the distortion $\tau$ is a $\mathcal{C}^\infty$-function, if $M$ is a Riemannian manifold. Setting

$$S(x,V) := \frac{d}{dt}\big(\tau(\gamma(t), \dot\gamma(t))\big)|_{t=0}, \tag{2.5}$$

where $\gamma$ is the geodesic with $\gamma(0) = x$, $\dot\gamma(0) = V$. $S(x, \lambda V) = \lambda S(x,V)$ for all $\lambda > 0$. $S$ is a scalar function on $T_xM \setminus 0$ which is called the *$S$-curvature*. From the definition, it seems that the $S$-curvature measures the rate of change in the distortion along geodesics in the direction $V \in T_xM$.

For all $N \in (n, \infty)$, we define the *weighted Ricci curvature* of $(M, F, d\mu)$ as follows (see [9]):

$$\begin{cases} \mathrm{Ric}_N(V) := \mathrm{Ric}(V) + \dot S(V) - \frac{S(V)^2}{N-n}, \\ \mathrm{Ric}_\infty(V) := \mathrm{Ric}(V) + \dot S(V), \\ \mathrm{Ric}_n(V) := \begin{cases} \mathrm{Ric} + \dot S(V), & \text{if } S(V) = 0 \\ -\infty & \text{otherwise.} \end{cases} \end{cases}$$

Also $\mathrm{Ric}_N(cV) := c^2\mathrm{Ric}_N(V)$ for $c > 0$.

We say that $(M, F)$ is *forward complete* if each geodesic $\gamma : [0, \ell] \to M$ is extended to a geodesic on $[0, \infty)$, in other words, if exponential map is defined on whole $TM$. Then the Hopf-Rinow theorem gives that every pair of points in $M$ can be joined by a minimal geodesic.

The *Legendre transformation* $\mathfrak{L} : TM \to T^*M$ is defined by

$$\mathfrak{L}(W) := \begin{cases} g_W(W, .), & W \neq 0, \\ 0 & W = 0. \end{cases}$$

For a smooth function $h : M \to \mathbb{R}$, the *gradient vector* of $h$ at $x \in M$ is defined as $\nabla h(x) := \mathfrak{L}^{-1}(dh)$.

Given a smooth vector field $Z = Z^i \partial/\partial x^i$ on $M$, the *divergence* of $Z$ with respect to an arbitrary volume form $d\mu = e^\varphi dx^1dx^2...dx^n$ is defined by

$$\mathrm{div} Z := \sum_{i=1}^n \left( \frac{\partial Z^i}{\partial x^i} + Z^i \frac{\partial \varphi}{\partial x^i} \right). \tag{2.6}$$

Then we define the *Finsler-Laplacian* of $h$ by $\Delta h := \mathrm{div}(\nabla h) = \mathrm{div}(\mathfrak{L}^{-1}(dh))$.

The following lemma is useful to prove Theorem 3 (see [17]).

**Lemma 1.** *Let $(M, F, d\mu)$ be a Finsler $n$-manifold, and $h : M \to \mathbb{R}$ a smooth function on $M$. Then on $U = \{x \in M : \nabla h|_x \neq 0\}$ we have*

$$\Delta h = \sum_i H(h)(E_i, E_i) - S(\nabla h) := \mathrm{tr}_{\nabla h} H(h) - S(\nabla h), \tag{2.7}$$

*where $E_1, E_2, ..., E_n$ is a local $g_{\nabla h}$-orthonormal frame on $U$.*

Finally, define *reversibility* $\lambda := \lambda(M, F)$ as follows:

$$\lambda := \sup_{x \in M, y \in TM \setminus 0} \frac{F(x, -y)}{F(x, y)}. \tag{2.8}$$

Obviously, $\lambda \in [1, \infty]$, and $\lambda = 1$ if and only if $(M, F)$ is reversible.

## 3. The proofs of the theorems

Let $(M, F, d\mu)$ be a Finsler manifold of dimensional $n$ and $r(x) = d(x, p)$ be a distance function with respect to a fixed point $p \in M$. It is well known that $r$ is only smooth on $M - \big(C_p \cup \{p\}\big)$ where $C_p$ is the cut locus of the point $p \in M$. We assume that $\gamma$ is a minimal unit speed geodesic segment. We have $\nabla r = \dot{\gamma}$ in the adapted coordinates with respect to the $r$, and also have $F(\nabla r) = 1$ (see [11]). On the other hand, using the Finsler metric we obtain a weighted Riemannian metric $g_{\nabla r}$. Thus we can apply the Riemannian calculation for $g_{\nabla r}$ (on $M - \big(C_p \cup \{p\}\big)$).

In order to prove the Theorem 1 and Theorem 2, we use the index form of a minimal unit speed geodesic, and to prove Theorem 3, we use Bochner-Weitzenböck formula and Hessian comparison theorem in Finsler geometry.

*Proof of Theorem 1.* Let $q \in M$ be a point and let $\sigma$ be a minimal unit speed geodesic segment from $p$ to $q$ of length $\ell$ such that $\sigma(0) = p$, $\sigma(\ell) = q$ and $\ell > r_0 > 0$. Since the inequality $\ell > r_0$ holds, $\ell$ can be parametrized by $\mu > 0$ such that

$$\ell = r_0 e^{\mu\pi} > r_0. \tag{3.1}$$

By virtue of any subsegment of a minimal unit speed geodesic segment is also a minimal unit speed geodesic segment, we have the minimal unit speed geodesic segment $\gamma$ defined by $\gamma(t) = \sigma|_{[r_0, \ell]}(t)$ where $\gamma : [r_0, \ell] \to M$ and $\gamma(r_0) = \sigma(r_0) = \tilde{q}$, $\gamma(\ell) = \sigma(\ell) = q$. Let $\{E_1 = \dot{\gamma}, E_2, \ldots, E_n\}$ be a parallel $g_{\nabla r}$-orthonormal frame along $\gamma$ and let $f \in \mathcal{C}^\infty([r_0, \ell])$ be a real-valued smooth function such that $f(r_0) = f(\ell) = 0$. Then we have

$$\mathrm{I}(fE_i, fE_i) = \int_{r_0}^{\ell} \Big( g_{\nabla r}(\dot{f} E_i, \dot{f} E_i) - g_{\nabla r}(R^{\nabla r}(fE_i, \nabla r)\nabla r, fE_i) \Big) dt. \tag{3.2}$$

It is obvious that (3.2) yields, by $g_{\nabla r}(R^{\nabla r}(\nabla r,\nabla r)\nabla r,\nabla r)=0$ and the assumption (1.3) given in Theorem 1,

$$\begin{aligned}\sum_{i=2}^{n}\mathrm{I}(fE_i,fE_i)&=\int_{r_0}^{\ell}\Big((n-1)\dot f^2-f^2\mathrm{Ric}(\nabla r)\Big)dt\\&=\int_{r_0}^{\ell}\Big((n-1)\dot f^2-f^2\mathrm{Ric}_\infty(\nabla r)+f^2\dot S(\nabla r)\Big)dt\\&\le\int_{r_0}^{\ell}\Big((n-1)\big(\dot f^2-\frac{Hf^2}{r^2}\big)+f^2\dot S(\nabla r)\Big)dt.\end{aligned}\tag{3.3}$$

Here, the term $f^2\dot S(\nabla r)$ equals to

$$\begin{aligned}f^2\dot S(\nabla r)&=-2f\dot f S(\nabla r)+\frac{d}{dt}\big(f^2S(\nabla r))\big)=-2f\dot f\frac{d\tau}{dt}+\frac{d}{dt}\big(f^2S(\nabla r))\\&=2\tau\frac{d}{dt}(f\dot f)-2\frac{d}{dt}(\tau f\dot f)+\frac{d}{dt}\big(f^2S(\nabla r))\big).\end{aligned}\tag{3.4}$$

Integrating both sides of (3.4) and using the assumption $|\tau|\le(n-1)k$, we obtain

$$\int_{r_0}^{\ell}\big(f^2\dot S(\nabla r)\big)dt=2\int_{r_0}^{\ell}\tau\frac{d}{dt}(f\dot f)dt\le 2(n-1)k\int_{r_0}^{\ell}\Big|\frac{d}{dt}(f\dot f)\Big|dt,\tag{3.5}$$

because of $f(r_0)=f(\ell)=0$. By use of (3.5), the inequality (3.3) becomes

$$\sum_{i=2}^{n}\mathrm{I}(fE_i,fE_i)\le\int_{r_0}^{\ell}(n-1)\Big(\dot f^2-\frac{Hf^2}{r^2}\Big)dt+2(n-1)k\int_{r_0}^{\ell}\Big|\frac{d}{dt}(f\dot f)\Big|dt.\tag{3.6}$$

Set

$$f(t)=\mu r_0\sqrt{r(\gamma(t))}\sin\Big(\frac{1}{\mu}\ln\frac{r(\gamma(t))}{r_0}\Big).\tag{3.7}$$

Therefore we have

$$\begin{aligned}\frac{1}{r_0^2(n-1)}\sum_{i=2}^{n}\mathrm{I}(fE_i,fE_i)\le&-\frac14\int_{r_0}^{\ell}\frac{(4H-1)\mu^2}{r}\sin^2\Big(\frac{1}{\mu}\ln\frac{r}{r_0}\Big)dr\\&+\int_{r_0}^{\ell}\frac1r\left(\cos^2\Big(\frac{1}{\mu}\ln\frac{r}{r_0}\Big)+\frac{\mu}{2}\sin\Big(\frac{2}{\mu}\ln\frac{r}{r_0}\Big)\right)dr\\&+2k\int_{r_0}^{\ell}\frac1r\Big|\frac{\mu}{2}\sin\Big(\frac{2}{\mu}\ln\frac{r}{r_0}\Big)+\cos\Big(\frac{2}{\mu}\ln\frac{r}{r_0}\Big)\Big|dr.\end{aligned}\tag{3.8}$$

In (3.8), considering the change variable $u=\ln\frac{r}{r_0}$, by $\ell=r_0e^{\mu\pi}$, we obtain

$$\frac{1}{r_0^2(n-1)}\sum_{i=2}^{n}\mathrm{I}(fE_i,fE_i)\le-\frac14\int_0^{\mu\pi}(4H-1)\mu^2\sin^2\Big(\frac{1}{\mu}u\Big)du$$

$$+\int_0^{\mu\pi}\left(\cos^2(\frac{1}{\mu}u)+\frac{\mu}{2}\sin(\frac{2}{\mu}u)\right)du$$
$$+2k\int_0^{\mu\pi}\left|\frac{\mu}{2}\sin(\frac{2}{\mu}u)+\cos(\frac{2}{\mu}u)\right|du, \tag{3.9}$$

from which

$$\frac{1}{r_0^2(n-1)}\sum_{i=2}^{n}\mathrm{I}(fE_i,fE_i)\le\frac{\mu}{8}\big(4\pi-(4H-1)\pi\mu^2+16k\sqrt{\mu^2+4}\big). \tag{3.10}$$

In the right hand side of (3.10), if the inequality

$$4\pi-(4H-1)\pi\mu^2+16k\sqrt{\mu^2+4}<0 \tag{3.11}$$

holds, then the index form $I$ is not positive semi-definite. This is a contradiction. Hence, we must take

$$4\pi-(4H-1)\pi\mu^2+16k\sqrt{\mu^2+4}\ge 0. \tag{3.12}$$

Thus

$$\mu\le\frac{2}{(4H-1)\pi}\sqrt{32k^2+(4H-1)\pi^2+16k\sqrt{4k^2+(4H-1)H\pi^2}}. \tag{3.13}$$

Using the parametrization $\ell=r_0e^{\mu\pi}$ given in (3.1), we find

$$\ell=r_0e^{\mu\pi}\le r_0\exp\left(\frac{2}{4H-1}\sqrt{32k^2+(4H-1)\pi^2+16k\sqrt{4k^2+(4H-1)H\pi^2}}\right). \tag{3.14}$$

Thus, $M$ is compact and the diameter of $M$ has the upper bound (1.4).

*Proof of Theorem 2.* By similar arguments given in the proof of Theorem 1, we have

$$\sum_{i=2}^{n}\mathrm{I}(fE_i,fE_i)=\int_{r_0}^{\ell}\Big((n-1)\dot f^2-f^2\mathrm{Ric}(\nabla r)\Big)dt. \tag{3.15}$$

Using the assumption (1.5) in the above integral expression, we get

$$\sum_{i=2}^{n}\mathrm{I}(fE_i,fE_i)\le\int_{r_0}^{\ell}\left((n-1)\dot f^2-(N-1)\frac{Hf^2}{r^2}\right)dt$$
$$+\int_{r_0}^{\ell}\left(f^2\dot S(\nabla r)-f^2\frac{(S(\nabla r))^2}{N-n}\right)dt. \tag{3.16}$$

In the inequality (3.16), the term $f^2\dot S(\nabla r)$ equals to

$$f^2\dot S(\nabla r)=-2f\dot f S(\nabla r)+\frac{d}{dt}(f^2S(\nabla r)). \tag{3.17}$$

Integrating both sides of (3.17), we obtain

$$\int_{r_0}^{\ell} f^2 \dot{S}(\nabla r)dt = \int_{r_0}^{\ell} -2f\dot{f}S(\nabla r)dt, \tag{3.18}$$

by $f(r_0) = f(\ell) = 0$. If we take $P = -\dot{f}$ and $T = fS(\nabla r)$, then the Cauchy-Schwarz inequality

$$\int_{r_0}^{\ell} PT\,dt = \int_{r_0}^{\ell} -f\dot{f}S(\nabla r)dt \leq \Big(\int_{r_0}^{\ell} \dot{f}^2 dt\Big)^{1/2}\Big(\int_{r_0}^{\ell} f^2(S(\nabla r))^2 dt\Big)^{1/2}. \tag{3.19}$$

Because of the facts

$$A = (N-n)\int_{r_0}^{\ell} \dot{f}^2 dt \geq 0 \text{ and } B = \frac{1}{N-n}\int_{r_0}^{\ell} f^2(S(\nabla r))^2 dt \geq 0, \tag{3.20}$$

where $N \in (n,\infty)$, we have the inequality $\sqrt{AB} \leq \frac{1}{2}(A+B)$, i.e.,

$$\Big(\int_{r_0}^{\ell} \dot{f}^2 dt\Big)^{1/2}\Big(\int_{r_0}^{\ell} f^2(S(\nabla r))^2 dt\Big)^{1/2} \leq \int_{r_0}^{\ell} \frac{1}{2}(N-n)\dot{f}^2 dt + \int_{r_0}^{\ell} f^2 \frac{(S(\nabla r))^2}{2(N-n)} dt. \tag{3.21}$$

Using (3.21) in (3.19), we find

$$\int_{r_0}^{\ell} -f\dot{f}S(\nabla r)dt \leq \int_{r_0}^{\ell}\Big(\frac{1}{2}(N-n)\dot{f}^2 + f^2\frac{(S(\nabla r))^2}{2(N-n)}\Big)dt. \tag{3.22}$$

Therefore we have

$$\int_{r_0}^{\ell} f^2\dot{S}(\nabla r)dt = \int_{r_0}^{\ell} -2f\dot{f}S(\nabla r)dt \leq \int_{r_0}^{\ell}\Big((N-n)\dot{f}^2 + f^2\frac{(S(\nabla r))^2}{N-n}\Big)dt. \tag{3.23}$$

Inserting (3.23) into (3.16), we obtain

$$\sum_{i=2}^{n} \mathrm{I}(fE_i, fE_i) \leq (N-1)\int_{r_0}^{\ell}\Big(\dot{f}^2 - \frac{Hf^2}{r^2}\Big)dt. \tag{3.24}$$

In the inequality (3.24), let us consider the choice

$$f(t) = \mu r_0\sqrt{r(\gamma(t))}\sin\Big(\frac{1}{\mu}\ln\frac{r(\gamma(t))}{r_0}\Big). \tag{3.25}$$

Thereby the inequality (3.24) yields

$$\frac{1}{N-1}\sum_{i=2}^{n}\mathrm{I}(fE_i, fE_i) \leq \int_{r_0}^{\ell}\frac{r_0^2}{r}\Big(\cos^2\Big(\frac{1}{\mu}\ln\frac{r}{r_0}\Big) + \frac{\mu}{2}\sin\Big(\frac{2}{\mu}\ln\frac{r}{r_0}\Big)\Big)dr$$

$$-\frac{1}{4}\int_{r_0}^{\ell}\frac{r_0^2}{r}(4H-1)\mu^2\sin^2(\frac{1}{\mu}\ln\frac{r}{r_0})dr. \tag{3.26}$$

In the above inequality, considering the change variable $u=\ln\frac{r}{r_0}$, , by use of $\ell = r_0e^{\mu\pi}$, we get

$$\frac{1}{N-1}\sum_{i=2}^{n}\mathrm{I}(fE_i,fE_i)\leq\int_0^{\mu\pi}r_0^2\left(\cos^2(\frac{1}{\mu}u)+\frac{\mu}{2}\sin(\frac{2}{\mu}u)\right)du$$
$$-\frac{1}{4}\int_0^{\mu\pi}r_0^2(4H-1)\mu^2\sin^2(\frac{1}{\mu}u)du, \tag{3.27}$$

which implies

$$\frac{1}{N-1}\sum_{i=2}^{n}\mathrm{I}(fE_i,fE_i)\leq r_0^2\frac{\mu\pi}{8}\left(4-(4H-1)\mu^2\right). \tag{3.28}$$

In the right hand side of (3.28), if the inequality

$$4-(4H-1)\mu^2<0 \tag{3.29}$$

holds, then we conclude that the index form $I$ is not positive semi-definite. But, since $\gamma$ is minimal geodesic, this is a contradiction. Hence, we must take

$$4-(4H-1)\mu^2\geq 0. \tag{3.30}$$

Thus we obtain

$$\mu\leq\frac{2}{\sqrt{4H-1}}. \tag{3.31}$$

Using the parametrization $\ell = r_0e^{\mu\pi}$, we find

$$\ell = r_0e^{\mu\pi}\leq r_0e^{2\pi/\sqrt{4H-1}}. \tag{3.32}$$

Thus, $M$ is compact and the diameter of $M$ has the upper bound (1.6).

*Proof of Theorem 3.* We know that $r(x)=d(x,p)$ is a distance function from a fixed point $p\in M$ and it is smooth on $M-\left(C_p\cup\{p\}\right)$. Also it satisfies $F(\nabla r)=1$. In Finsler geometry, recall that the Bochner-Weitzenböck formula [10] for a smooth function $u\in\mathcal{C}^\infty(M)$

$$0=\Delta^{\nabla u}\left(\frac{F(\nabla u)^2}{2}\right)=\mathrm{Ric}_\infty(\nabla u)+D(\Delta u)(\nabla u)+\|\nabla^2u\|^2_{HS(\nabla u)}. \tag{3.33}$$

From the Bochner formula applied to distance function $r$ and by Lemma 1, we have, on $M-\left(C_p\cup\{p\}\right)$,

$$0=\mathrm{Ric}_\infty(\nabla r)+D(\Delta r)(\nabla r)+\|\nabla^2r\|^2_{HS(\nabla r)}$$
$$\geq\mathrm{Ric}_\infty(\nabla r)+g_{\nabla r}(\nabla^{\nabla r}\Delta r,\nabla r)+\frac{(\Delta r+S(\nabla r))^2}{n-1}. \tag{3.34}$$

By virtue of the inequality $(a \mp b)^2 \geq \frac{1}{\beta+1}a^2 - \frac{1}{\beta}b^2$ holding for all real numbers $a, b$ and positive real number $\beta$, we have

$$\frac{\Big(\Delta r + S(\nabla r)\Big)^2}{n-1} \geq \frac{(\Delta r)^2}{(n-1)(\beta+1)} - \frac{(S(\nabla r))^2}{(n-1)\beta}. \tag{3.35}$$

In the case where $N > n$, taking $\beta = \frac{N-n}{n-1} > 0$, (3.34) yields

$$0 \geq \mathrm{Ric}_\infty(\nabla r) + g_{\nabla r}(\nabla^{\nabla r}\Delta r, \nabla r) + \frac{(\Delta r)^2}{N-1} - \frac{(S(\nabla r))^2}{N-n}. \tag{3.36}$$

Applying the assumption (1.7) given in Theorem 3 to (3.36), we find

$$0 \geq \partial_r(\Delta r) + \frac{(\Delta r)^2}{N-1} + (N-1)\frac{H}{(1+r)^2}. \tag{3.37}$$

The above inequality can be rewritten as

$$0 \geq \partial_r\Big(\frac{\Delta r}{N-1}\Big) + \left(\frac{\Delta r}{N-1}\right)^2 + \frac{H}{(1+r)^2}. \tag{3.38}$$

We know from the Hessian comparison theorem in [17], if there is a local vector field $X$ on an open set $U$ of $p \in M$ with $g_{\nabla r}(X,X) = 1$, $g_{\nabla r}(\nabla r, X) = 0$, then $H(r)(X,X) \sim \frac{1}{r}$ as $r \to 0^+$. Hence, using the Lemma 1, we have

$$\lim_{r\to 0^+} r\Big(\frac{1}{N-1}\Delta r\Big) = \lim_{r\to 0^+} r\left(\frac{1}{N-1}\Big(\mathrm{tr}_{\nabla r}H(r) - S(\nabla r)\Big)\right) = \frac{n-1}{N-1} < 1, \tag{3.39}$$

where $N > n$. By (3.38) and (3.39), we obtain, on $M - \big(C_p \cup \{p\}\big)$,

$$\frac{1}{N-1}\Delta r \leq \frac{1}{2(1+r)}\Big(1 + \sqrt{4H-1}\cot\Big(\frac{\sqrt{4H-1}}{2}\ln(1+r)\Big)\Big), \tag{3.40}$$

where $H > 1/4$. Indeed, the function

$$Y(r) = \frac{1}{2(1+r)}\Big(1 + \sqrt{4H-1}\cot\Big(\frac{\sqrt{4H-1}}{2}\ln(1+r)\Big)\Big) \tag{3.41}$$

is a solution of the Riccati differential equation

$$Y'(r) + (Y(r))^2 + \frac{H}{(1+r)^2} = 0. \tag{3.42}$$

Because of $\lim_{r\to 0^+} rY(r) = 1$ and (3.39), we have

$$\lim_{r\to 0^+} r\Big(\frac{1}{N-1}\Delta r\Big) \leq \lim_{r\to 0^+} rY(r). \tag{3.43}$$

Thus, for a sufficiently small positive constant $\varepsilon \in (0,T)$ the inequality

$$\frac{1}{N-1}\Delta r(\varepsilon) \leq Y(\varepsilon) \tag{3.44}$$

is ensured. In that case, the Riccati comparison theorem gives the inequality

$$\frac{1}{N-1}\Delta r(t) \leq Y(t) \tag{3.45}$$

for every $t \in [\varepsilon, T)$.

Let $q \in M$ be any point, and let $\sigma$ be a minimal unit speed geodesic segment from $p$ to $q$. Suppose that the inequality

$$d(p,q) > e^{2\pi/\sqrt{4H-1}} - 1 \tag{3.46}$$

is satisfied. Then, since $\sigma$ is a minimal unit speed geodesic segment from $p$ to $q$, we have the fact that the point $\sigma(e^{2\pi/\sqrt{4H-1}} - 1)$ is outside the cut locus of $p \in M$, i.e.,

$$\sigma(e^{2\pi/\sqrt{4H-1}} - 1) \in M - \big(C_p \cup \{p\}\big). \tag{3.47}$$

Therefore the distance function $r$ is smooth at this point. Namely, at this point, left hand side of (3.40) is a constant. However, the right side of (3.40) tends to $-\infty$ as $r \to (e^{2\pi/\sqrt{4H-1}} - 1)^-$, i.e.,

$$\lim_{r \to (e^{2\pi/\sqrt{4H-1}}-1)^-} \frac{1}{2(1+r)}\Big(1 + \sqrt{4H-1}\cot\big(\frac{\sqrt{4H-1}}{2}\ln(1+r)\big)\Big) = -\infty. \tag{3.48}$$

This is a contradiction. Hence, (3.46) does not hold. It must be

$$d(p,q) \leq e^{2\pi/\sqrt{4H-1}} - 1. \tag{3.49}$$

Therefore $M$ is compact. Let $\lambda$ be the reversibility. For any points $p', q' \in M$, due to the triangle inequality and the inequality (3.49), we obtain

$$d(p',q') \leq d(p',p) + d(p,q') \leq \lambda d(p,p') + d(p,q'), \tag{3.50}$$

and so

$$d(p',q') \leq (1+\lambda)(e^{2\pi/\sqrt{4H-1}} - 1). \tag{3.51}$$

This completes the proof of theorem.

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# AN EXTRAGRADIENT-LIKE PARALLEL METHOD FOR PSEUDOMONOTONE EQUILIBRIUM PROBLEMS AND SEMIGROUP OF NONEXPANSIVE MAPPINGS

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*Abstract.* In this paper we propose a parallel iterative hybrid methods for finding a common element of the solution sets of a finite family of pseudomonotone equilibrium problems and the fixed points set of a semigroup-nonexpensive mappings in Hilbert spaces. Under mild conditions, we obtain the strong convergence of the proposed iterative process. Some numerical experiments are given to verify the efficiency the proposed algorithm.

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## 1. Introduction

Throughout the paper we suppose that $\mathcal{H}$ is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$, $C$ is a closed convex subset in $\mathcal{H}$ and that $f_i : C\times C \to \mathbb{R}$, $i = 1,2,\dots,N$, $T(s) : C \to C$, $s \geq 0$. Conditions for $f_i$, $i = 1,2,\dots,N$ and $T(s)$ will be detailed later. We are interested in a solution method for the system defined as

$$\text{Find } x^* \in C : \qquad f_i(x^*, y) \geq 0 \quad \forall\ y \in C,\ i = 1,2,\dots,N, \tag{1.1}$$

$$x^* = T(s)(x^*) \quad \forall s \geq 0. \tag{1.2}$$

The problem (1.1) can be considered as a system of equilibrium problems, which was first introduced by Blum and Oettli in [2]: Given $f : C\times C \to \mathcal{R}$,

$$\text{find } x^* \in C \text{ such that } f(x^*, y) \geq 0 \ \ \forall y \in C. \tag{1.3}$$

The equilibrium problems play an important role in optimization and nonlinear analysis. It is well known that many problems, such as variational inequalities, Nash equilibrium problems, saddle point problems, complementarity problems can be formulated as special cases of equilibrium problems. There are different methods for

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solving (1.3), see, for example, [1–4], [5], [8], [12–16], [20], [21, 22], and the references cited therein. Among them, we are interested in *Extragradient method* introduced in [9, 20] due to its simplicity and efficiency:

$$\begin{cases} x_0 \in C, \\ y_n = \operatorname*{argmin}_{y \in C} \left\{ \lambda f(x_n, y) + \frac{1}{2}\|y - x_n\|^2 \right\}, \\ x_{n+1} = \operatorname*{argmin}_{y \in C} \left\{ \lambda f(y_n, y) + \frac{1}{2}\|y - x_n\|^2 \right\}. \end{cases} \tag{1.4}$$

Under assumptions that the bifunctions $f$ is pseudomonone and Lipschitz-type continuous, the authors proved that the sequence $\{x_n\}$ generated by (1.4) weakly converges to a solution of (1.3).

It is well known that the problems of finding common fixed points of nonexpansive mappings and of nonexpansive semigroups is an important problem in fixed point theory and applications; in particular, in image recovery, convex feasibility problem, and signal processing problem (see e.g. [7], [14]). Iterative approximation methods for these problems in Hilbert or Banach spaces have been studied extensively by many authors; see, for example, [6], [17], [23], [24, 25], and the references therein. Finding a common element of the set of fixed points of nonexpansive mappings or a semigroup of nonexpansive mappings and the set of solutions to a equilibrium problem has been studied extensively in the literature; see, for example, [4], [5], [12], [17], [18], [21, 22], [25] and the references therein. The common approach in these papers is to use a proximal point algorithm for handling the equilibrium problem. For monotone equilibrium problems the subproblems needed to solve in the proximal point method are strongly monotone, and therefore they have a unique solution that can be approximated by available methods. However, for pseudomonone problems the subproblems, in general, may have nonconvex solution set due to the fact that the regularized bifunctions do not inherit any pseudomonotoniciy property from the original one.

In this article, motivated by [4, 11] and inspired by [9, 20], we propose a parallel iterative method for finding a common element of the solution sets of a finite family of pseudomonotone equilibrium problems and the set of fixed points of a nonexpansive semigroup in Hilbert spaces. The main point here is that we combine a parallel splitting-up technique and the extragradient procedure rather than a proximal point algorithm for dealing with a finite family of pseudomonotone equilibrium problems and Mann's iterative algorithms for finding fixed points of nonexpansive mappings. We obtain the strong convergence of iterative processes.

The paper is organized as follows: In Section 2, we collect some definitions and results needed for further investigation. We describe a novel parallel hybrid iterative

method in the Section 3. The convergence analysis for the proposed method is detailed in Section 4. Some special cases and illustrative examples are provided in the last section.

## 2. PRELIMINARIES

In this section, we recall some definitions and results that will be used in the sequel. In what follows by $x_n \rightharpoonup x$ we mean that the sequence $\{x_n\}$ converges to $x$ in the weak topology. Let $C$ be a nonempty closed convex of a Hilbert space $\mathcal{H}$. We recall that mapping $T : C \to C$ is said to be *nonexpansive* on $C$ if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for al } x, y \in C.$$

Let $F(T)$ denote the set of fixed points of $T$. A family $\{T(s) : s \in \mathbb{R}_+\}$ of mappings from $C$ into itself is called a *nonexpansive semigroup* on $C$ if it satisfies the following conditions:

(i) for each $s \in \mathbb{R}_+$, $T(s)$ is a nonexpansive mapping on $C$;
(ii) $T(0)x = x$ for all $x \in C$;
(iii) $T(s_1 + s_2) = T(s_1) \circ T(s_2)$ for all $s_1, s_2 \in \mathbb{R}_+$;
(iv) for each $x \in C$, the mapping $T(\cdot)x$ from $\mathbb{R}_+$ into $C$ is continuous.

Let $\mathcal{F} = \bigcap_{s \geq 0} F(T(s))$ be the set of all common fixed points of $\{T(s) : s \in \mathbb{R}_+\}$. We know that $\mathcal{F}$ is nonempty if $C$ is bounded (see [3]).

We begin with the following properties of nonexpansive mappings.

**Lemma 1** ([10]). *Let $C$ be a closed convex subset of a Hilbert space $\mathcal{H}$ and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \neq \varnothing$. If a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup z$ and $x_n - Sx_n \to 0$, then $z = Sz$.*

**Lemma 2** ([23]). *Let $C$ be a nonempty bounded closed convex subset of $\mathcal{H}$ and let $\{T(s) : s \in \mathbb{R}_+\}$ be a nonexpansive semigroup on $C$. Then, for any $h \geq 0$*

$$\lim_{s \to \infty} \sup_{y \in C} \left\| T(h)\Big(\frac{1}{s}\int_0^s T(t)y\,dt\Big) - \frac{1}{s}\int_0^s T(t)y\,dt \right\| = 0$$

Since $C$ is a nonempty closed and convex subset of $H$, for every $x \in H$, there exists a unique element $P_C x$, defined by

$$P_C x = \arg\min\{\|y - x\| : y \in C\}.$$

The mapping $P_C : H \to C$ is called the metric (orthogonal) projection of $H$ onto $C$. It is also known that $P_C$ is firmly nonexpansive, or 1-inverse strongly monotone (1-ism), i.e.,

$$\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2 \quad \forall x, y \in H. \tag{2.1}$$

Besides, we have

$$\|x - P_C y\|^2 + \|P_C y - y\|^2 \leq \|x - y\|^2 \quad \forall x \in C,\ \forall y \in H. \tag{2.2}$$

Moreover, $z = P_C x$ if only if

$$\langle x - z, z - y \rangle \geq 0 \quad \text{for all } y \in C. \tag{2.3}$$

The bifunction $f : C \times C \to \mathbb{R}$ is called *monotone* on $C$ if

$$f(x,y) + f(y,x) \leq 0 \text{ for all } x, y \in C;$$

*pseudomonotone* on $C$ if

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0 \text{ for all } x, y \in C.$$

It is obvious that any monotone bifunction is a pseudomonotone one, but not vice versa.

Throughout this paper we consider bifunctions with the following properties:

$(C1)$ $f$ is pseudomonotone, i.e., for all $x, y \in C$,

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0;$$

$(C2)$ $f$ is Lipschitz-type continuous, i.e., there exist two positive constants $c_1, c_2$ such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2 \text{ for all } x, y, z \in C;$$

$(C3)$ $f$ is weakly continuous on $C \times C$;

$(C4)$ $f(x,.)$ is convex, subdifferentiable on $C$ and $f(x,x) = 0$ for every $x \in C$.

The following statements will be needed in the next section.

**Lemma 3** ([1]). *If the bifunction $f$ satisfies Assumptions $(C1)-(C4)$, then the solution set of equilibrium problems:*

$$\textit{Find } x^* \in C : \ f(x^*, y) \geq 0 \ \textit{ for all } y \in C$$

*is weakly closed and convex.*

**Lemma 4** ([8]). *Let $C$ be a convex subset of a real Hilbert space $H$ and $g : C \to \mathbb{R}$ be a convex and subdifferentiable function on $C$. Then, $x^*$ is a solution to the following convex problem*

$$\min\{g(x) : x \in C\}$$

*if only if $0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(.)$ denotes the subdifferential of $g$ and $N_C(x^*)$ is the normal cone of $C$ at $x^*$.*

It is also known that $\mathcal{H}$ satisfies Opial's condition. See the following definition in [19].

**Definition 1** ([19]). A Banach space $X$ is said to satisfy Opial's condition if whenever $\{x_n\}$ is a sequence in $X$ which converges weakly to $x$, as $k \to \infty$, then

$$\lim_{n\to\infty} \sup \|x_n - x\| < \lim_{n\to\infty} \sup \|x_n - y\| \quad \forall y \in X, \text{ with } x \neq y.$$

## 3. MAIN RESULTS

In this section, based on the hybrid method in mathematical programming, projection, extragradient method and Mann's iteration, we propose a parallel iterative method for finding a common element of the solution sets of a finite family of equilibrium problems with pseudomonotone bifunctions $\{f_i\}_{i=1}^N$ and the set of fixed points of nonexpansive semigroup mappings $\{T(s): s \geq 0\}$ in a Hilbert space $\mathcal{H}$.
We denote by $Sol(C, f_i)$ the set of (1.1), $i = 1, 2, \ldots, N$,. In what follows, we assume that the solution set

$$\Omega = \mathcal{F} \bigcap \left( \bigcap_{i=1}^{N} Sol(C, f_i) \right)$$

is nonempty and each bifunction $f_i, (i = 1, \ldots, N)$ satisfies all the conditions $(C1)-(C4)$. Observe that we can choose the same Lipschitz coefficients $\{c_1, c_2\}$ for all bifunctions $f_i, i = 1, 2, \ldots, N$. Indeed, condition $(C2)$ implies that

$$f_i(x,z) - f_i(x,y) - f_i(y,z) \leq c_{1i}\|x-y\|^2 + c_{2i}\|y-z\|^2 \leq c_1\|x-y\|^2 + c_2\|y-z\|^2,$$

where $c_1 = \max_{1 \leq i \leq N} c_{1i}$ and $c_2 = \max_{1 \leq i \leq N} c_{2i}$. Hence,

$$f_i(x,y) + f_i(y,z) \geq f_i(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2.$$

Further, since $\Omega \neq \emptyset$, by Lemma 3, the sets $Sol(C, f_i), i = 1, \ldots, N$ are nonempty, closed and convex, hence the solution set $\Omega$ is a nonempty closed and convex subset of $C$. Thus given any fixed element $x_0 \in C$ there exists a unique element $\bar{x} := P_\Omega x_0$.

**Algorithm 1.** *Choose positive number* $0 < \rho < \min\left(\frac{1}{2c_1}, \frac{1}{2c_2}\right)$ *and the positive sequences* $\{\mu_n\} \subset [a, 1]$ *for some* $a \in (0, 1)$.
*Seek a starting point* $x_0 \in C$ *and set* $n := 0$.
*Step 1.*

- *Solve the strongly convex programs*

$$y_n^i = argmin\{\rho f_i(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\},$$
$$z_n^i = argmin\{\rho f_i(y_n^i, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\}, \ i = 1, \ldots, N;$$

- *Find positive integer*

$$i_n = \operatorname*{argmax}_{1 \leq i \leq N}\{\|z_n^i - x_n\|\},$$

*and set* $z_n := z_n^{i_n}$;

*Step 2. Compute* $u_n = (1 - \mu_n)x_n + \mu_n T_n z_n$, *where* $T_n$ *is defined as*

$$T_n x := \frac{1}{s_n}\int_0^{s_n} T(s)x\,ds, \ \forall x \in C \ with \ \lim_{n \to +\infty} s_n = +\infty;$$

$$x_{n+1} = P_{(H_n \cap W_n)} x_0, \text{ where}$$
$$H_n = \{z \in \mathcal{H} : \|u_n - z\| \le \|x_n - z\|\},$$
$$W_n = \{z \in \mathcal{H} : \langle x_n - z, x_0 - x_n \rangle \ge 0\}.$$

*Increase $n$ by* 1 *and go back to Step* 1.

We now state and prove the convergence of the proposed iteration method.

## 4. Convergence results

In this section, we show the strong convergence of the sequences $\{x_n\}$ and $\{u_n\}$ defined by Algorithm 1 to the common element in a real Hilbert space.

For establishing the strong convergence of $\{x_n\}$ and $\{u_n\}$ in Algorithm 1, we need the following result (see [20]).

**Lemma 5.** *Suppose that $x^* \in Sol(C, f_i)$ and $x_n$, $y_n^i$, $z_n^i$, $i = 1, \dots, N$, are as in Step 1 of Algorithm 1. Then*

$$\|z_n^i - x^*\|^2 \le \|x_n - x^*\|^2 - (1 - 2\rho c_1)\|y_n^i - x_n\|^2 - (1 - 2\rho c_2)\|y_n^i - z_n^i\|^2.$$

**Theorem 1.** *Let $C$ be a nonempty closed convex subset in a real Hilbert space $\mathcal{H}$, $\{T(s) : s \in \mathbb{R}_+\}$ be a nonexpansive semigroup on $C$, $f_i$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions $(C_1) - (C_4)$. Suppose that $\Omega \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by the Algorithm 1, where $\{\mu_n\} \subset [a, 1]$ for some $a \in (0, 1)$. Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to an element $p^* = P_\Omega x_0$.*

*Proof.* It is obvious that $H_n$ and $W_n$ are closed and convex for every $n \ge 0$. So that the $\{x_n\}$ is well defined for every $n \ge 0$. Moreover, it is easy seen that $T_n$ is nonexpensive for all $n \ge 0$.

Now we divide the proof into several steps.

**Step 1.** *Claim that $\Omega \subset H_n \cap W_n$ for every $n \ge 0$.*

Indeed, for each $x^* \in \Omega$, by Lemma 5, we have

$$\|z_n^i - x^*\| \le \|x_n - x^*\| \text{ for all } n \ge 0.$$

From the definition of $i_n$, we have

$$\|z_n - x^*\| \le \|x_n - x^*\| \text{ for all } n \ge 0. \tag{4.1}$$

From the convexity of $\|\cdot\|^2$, the nonexpansiveness of $T_n$ and (4.1) it follows that

$$\begin{aligned}
\|u_n - x^*\|^2 &= \big\|(1 - \mu_n)(x_n - x^*) + \mu_n(T_n z_n - x^*)\big\|^2 \\
&\le (1 - \mu_n)\|x_n - x^*\|^2 + \mu_n\|T_n z_n - T_n x^*\|^2 \\
&\le (1 - \mu_n)\|x_n - x^*\|^2 + \mu_n\|z_n - x^*\|^2 \\
&\le (1 - \mu_n)\|x_n - x^*\|^2 + \mu_n\|x_n - x^*\|^2 \\
&= \|x_n - x^*\|^2 \quad \forall n \ge 0,
\end{aligned} \tag{4.2}$$

which implies $x^* \in H_n$. Hence $\Omega \subset H_n$ for all $n \geq 0$.

Next we show $\Omega \subset W_n$ for all $n \geq 0$. Indeed, when $n = 0$, we have $x_0 \in C$ and $W_0 = H$. Consequently, $\Omega \subset H_0 \cap W_0$. By induction, suppose $\Omega \subset H_m \cap W_m$ for some $m \geq 0$. We have to prove that $\Omega \subset H_{m+1} \cap W_{m+1}$. Since $x_{m+1} = P_{H_m \cap W_m} x_0$, by (2.3), for every $z \in \Omega \subset H_m \cap W_m$, it holds that

$$\langle x_{m+1} - z, x_0 - x_{m+1} \rangle \geq 0,$$

which means that $z \in W_{m+1}$. Note that $\Omega \subset H_n$ for all $n \geq 0$, we can conclude that $\Omega \subset H_n \cap W_n$ for all $n \geq 0$.

**Step 2.** *Claim that for all $i = 1, \ldots, N$, we have*

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|x_n - z_n^i\| = \lim_{n\to\infty} \|x_n - y_n^i\| = 0.$$

Indeed, from $x_n = P_{W_n} x_0$ and (2.2), it follows that, for every $u \in \Omega \subset W_n$, we get

$$\|x_n - x_0\|^2 \leq \|u - x_0\|^2 - \|u - x_n\|^2 \leq \|u - x_0\|^2. \tag{4.3}$$

This implies that the sequence $\{x_n\}$ is bounded. From (4.2) and (4.1), it follows that the sequences $\{u_n\}$ and $\{z_n\}$ are also bounded.

Observing that $x_{n+1} = P_{H_n \cap W_n} x_0 \in W_n, x_n = P_{W_n} x_0$, from (2.2) we have

$$\|x_n - x_0\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_0\|^2. \tag{4.4}$$

Thus, the sequence $\{\|x_n - x_0\|\}$ is nondecreasing, hence there exists the limit of the sequence $\{\|x_n - x_0\|\}$. From (4.4) we obtain

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.$$

Letting $n \to \infty$, we find

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.5}$$

Since $x_{n+1} \in H_n$, it follows that $\|u_n - x_{n+1}\| \leq \|x_{n+1} - x_n\|$. Thus

$$\|u_n - x_n\| \leq \|u_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\|.$$

The last inequality together with (4.5) implies that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0 \tag{4.6}$$

Moreover, from (4.2), Lemma 5 and the definition of $i_n$ for any fixed $x^* \in \Omega$, we have

$$\begin{aligned}\|u_n - x^*\|^2 &\leq (1-\mu_n)\|x_n - x^*\|^2 + \mu_n \|z_n - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - \mu_n \Big[(1-2\rho c_1)\|y_n^{i_n} - x_n\|^2 + (1-2\rho c_2)\|y_n^{i_n} - z_n\|^2\Big].\end{aligned}$$

Therefore

$$\begin{aligned}&a\big[(1-2\rho c_1)\|y_n^{i_n} - x_n\|^2 + (1-2\rho c_2)\|y_n^{i_n} - z_n\|^2\big] \\ &\leq \mu_n\big[(1-2\rho c_1)\|y_n^{i_n} - x_n\|^2 + (1-2\rho c_2)\|y_n^{i_n} - z_n\|^2\big]\end{aligned}$$

$$\begin{aligned}
&\leq \|x_n - x^*\|^2 - \|u_n - x^*\|^2 \\
&= \left(\|x_n - x^*\| - \|u_n - x^*\|\right)\left(\|x_n - x^*\| + \|u_n - x^*\|\right) \\
&\leq \|x_n - u_n\|\left(\|x_n - x^*\| + \|u_n - x^*\|\right). \qquad (4.7)
\end{aligned}$$

Using the last inequality together with (4.6) and taking into account the boundedness of two sequences $\{u_n\}$ and $\{x_n\}$ as well as the conditions of $\{\mu_n\}$, $\rho$, we come to the relations

$$\lim_{n\to\infty}\left\|y_n^{i_n} - x_n\right\| = \lim_{n\to\infty}\left\|y_n^{i_n} - z_n\right\| = 0. \qquad (4.8)$$

From $\|z_n - x_n\| \leq \|z_n - y_n^{i_n}\| + \|y_n^{i_n} - x_n\|$ and (4.8), we obtain

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \qquad (4.9)$$

By the definition of $i_n$, we get

$$\lim_{n\to\infty}\left\|z_n^i - x_n\right\| = 0 \qquad (4.10)$$

for all $i = 1, \ldots, N$. From Lemma 5 and (4.10), we obtain

$$\lim_{n\to\infty}\left\|y_n^i - x_n\right\| = 0$$

for all $i = 1, \ldots, N$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to some element $p$.

**Step 3.** *Claim that* $p \in \Omega$.

From (4.6) and (4.9), we obtain also that $\{u_{n_k}\}$ and $\{z_{n_k}\}$ converges weakly to $p$. Since $\{u_n\} \subset C$ and $C$ is a closed convex subset in $\mathcal{H}$, we have $p \in C$.

Now, we prove that $p \in \Omega$. To this end, first we show that $p \in \bigcap_{i=1}^{N} Sol(C, f_i)$. Noting that

$$y_n^i = \operatorname{argmin}\{\rho f_i(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : \ y \in C\},$$

by Lemma 4, we obtain

$$0 \in \partial_2\left\{\rho f_i(x_n, y) + \frac{1}{2}\|x_n - y\|^2\right\}(y_n^i) + N_C(y_n^i).$$

Therefore, there exists $w \in \partial_2 f_i(x_n, y_n^i)$ and $\bar{w} \in N_C(y_n^i)$ such that

$$\rho w + x_n - y_n^i + \bar{w} = 0. \qquad (4.11)$$

Since $\bar{w} \in N_C(y_n^i)$, $\left\langle \bar{w}, y - y_n^i\right\rangle \leq 0$ for all $y \in C$. This together with (4.11) implies that

$$\rho\left\langle w, y - y_n^i\right\rangle \geq \left\langle y_n^i - x_n, y - y_n^i\right\rangle \qquad (4.12)$$

for all $y \in C$. Since $w \in \partial_2 f_i(x_n, y_n^i)$,

$$f_i(x_n, y) - f_i(x_n, y_n^i) \geq \left\langle w, y - y_n^i \right\rangle \text{ for all } y \in C. \tag{4.13}$$

From (4.12) and (4.13), we get

$$\rho\left(f_i(x_n, y) - f_i(x_n, y_n^i)\right) \geq \left\langle y_n^i - x_n, y - y_n^i \right\rangle \text{ for all } y \in C. \tag{4.14}$$

Since $x_{n_k} \rightharpoonup p$ and $\|x_n - y_n^i\| \to 0$ as $n \to \infty$, we have $y_{n_k}^i \rightharpoonup p$. Letting $n = n_k$ in (4.14), passing to the limit as $k \to \infty$ and using assumptions $(C3)$, we conclude that $f_i(p, y) \geq 0$ for all $y \in C$, $i = 1, 2, \ldots, N$. Thus, $p \in \bigcap_{i=1}^{N} Sol(C, f_i)$.

Now, we prove that $p = T(h)p$ for all $h > 0$. First, we obtain from Step 3 of the algorithm that

$$\begin{aligned}
a\|u_n - T_n u_n\| &\leq \mu_n \|u_n - T_n u_n\| \\
&\leq \mu_n \left(\|u_n - T_n z_n\| + \|T_n z_n - T_n u_n\|\right) \\
&\leq \|\mu_n u_n - \mu_n T_n z_n\| + \mu_n \|T_n z_n - T_n u_n\| \\
&= \|\mu_n u_n + (1-\mu_n)x_n - u_n\| + \mu_n \|T_n z_n - T_n u_n\| \\
&\leq (1-\mu_n)\|x_n - u_n\| + \mu_n \|z_n - u_n\| \\
&\leq \|u_n - x_n\| + \mu_n \|z_n - x_n\|
\end{aligned} \tag{4.15}$$

Taking into account $\|u_n - x_n\| \to 0$ and $\|z_n - x_n\| \to 0$, it follows that

$$\lim_{n\to\infty} \|u_n - T_n u_n\| = 0. \tag{4.16}$$

Note that

$$\begin{aligned}
\|T(h)u_n - u_n\| &\leq \left\|T(h)u_n - T(h)\left(\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right)\right\| \\
&\quad + \left\|T(h)\left(\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right) - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| \\
&\quad + \left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - u_n\right\| \\
&\leq 2\left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - u_n\right\| \\
&\quad + \left\|T(h)\left(\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right) - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\|.
\end{aligned} \tag{4.17}$$

Since the sequence $\{u_n\}$ is bounded, we can apply Lemma 2 to get

$$\lim_{n\to\infty} \left\|T(h)\left(\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right) - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| = 0, \tag{4.18}$$

for every $h \in (0,\infty)$ and therefore, by (4.16), (4.17) and (4.18), we obtain

$$\lim_{n\to\infty} \|T(h)u_n - u_n\| = 0$$

for each $h > 0$ from which we have by Lemma 1 that $p$ is a fixed point of $T(h)$ for all $h > 0$. Hence $p \in \mathcal{F}$.

**Step 4.** *The sequence $\{x_n\}$ converges strongly to $p^* := P_\Omega x_0$.*

Indeed, from $p^* \in \Omega$ and (4.3), we obtain

$$\|x_n - x_0\| \le \|p^* - x_0\| \quad \forall n \ge 0.$$

The last inequality together with $x_{n_k} \rightharpoonup p$ and the weak lower semicontinuity of the norm $\|.\|$ imply that

$$\|p - x_0\| \le \liminf_{k\to\infty} \|x_{n_k} - x_0\| \le \limsup_{k\to\infty} \|x_{n_k} - x_0\| \le \|p^* - x_0\|.$$

On the other hand, since $p^* := P_\Omega x_0$, we have $\|p - x_0\| \ge \|p^* - x_0\|$. Hence, $p^* = p$ and $\lim_{k\to\infty} \|x_{n_k} - x_0\| = \|p^* - x_0\|$. Taking into account $x_{n_k} \rightharpoonup p^*$, we have $x_{n_k} \to p^*$. Finally, suppose that $\{x_{n_j}\}$ is an another weakly convergent subsequence of $\{x_n\}$. By a similar argument as above, we conclude that $\{x_{n_j}\}$ converges strongly to $p^* := P_\Omega x_0$. Therefore, the sequence $\{x_n\}$ generated by the Algorithm 1 converges strongly to $P_\Omega x_0$. Then the strong convergence of the sequences $\{u^n\}$ to $p$ is followed from (4.6). The proof is now completed. □

## 5. Special cases and illustrative examples

If $N = 1$, Algorithm 1 reduces to the following one for finding a common element in the solution set of pseudomonotone equilibrium problems and the set of the fixed points of a nonexpansive semigroup in Hilbert spaces.

**Corollary 1.** *Let $C$ be a nonempty closed convex subset in a real Hilbert space $\mathcal{H}$, $\{T(s) : s \in \mathbb{R}_+\}$ be a nonexpansive semigroup on $C$, $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions $(C_1)-(C_4)$. Suppose that $\Omega = \mathcal{F} \cap Sol(f,C) \neq \varnothing$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{aligned}
x^0 &\in C \text{ chosen arbitrarily},\\
y_n &= argmin\{\rho f(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : \ y \in C\},\\
z_n &= argmin\{\rho f(y_n, y) + \frac{1}{2}\|x_n - y\|^2 : \ y \in C\},\\
u_n &= (1-\mu_n)x_n + \mu_n T_n z_n,\\
H_n &= \{z \in \mathcal{H} : \ \|u_n - z\| \le \|x_n - z\|\},\\
W_n &= \{z \in \mathcal{H} : \ \langle x_n - z, x_0 - x_n\rangle \ge 0\},\\
x_{n+1} &= P_{(H_n \cap W_n)} x_0,
\end{aligned}$$

*where* $\{\mu_n\} \subset [a,1]$, $a \in (0,1)$. *Then,* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to an element* $p^* \in \Omega$.

*Proof.* Taking $N = 1$ in Theorem 1, we get the desired conclusion easily.

Now putting $f_i(x,y) = 0$, $i = 1,2,\ldots,N$, for all $x, y \in C$, we obtain the following result for finding a common fixed point of a nonexpansive semigroup $\{T(s) : s \in \mathbb{R}_+\}$ on $C$. □

**Corollary 2.** *Let* $C$ *be a nonempty closed convex subset in a real Hilbert space* $\mathcal{H}, \{T(s) : s \in \mathbb{R}_+\}$ *be a nonexpansive semigroup on* $C$ *such that* $\mathcal{F} \neq \varnothing$. *Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by*

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily}, \\ u_n &= (1-\mu_n)x_n + \mu_n T_n x_n, \\ H_n &= \{z \in \mathcal{H} : \|u_n - z\| \le \|x_n - z\|\}, \\ W_n &= \{z \in \mathcal{H} : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{(H_n \cap W_n)} x_0, \end{aligned}$$

*where* $\{\mu_n\} \subset [a,1]$, $a \in (0,1)$. *Then,* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to an element* $p^* \in \mathcal{F}$.

If $T(s)x = x$ for all $s > 0$ and $x \in C$, Algorithm 1 reduces to the following one for finding a common element in the solution-set of a finite family of pseudomonotone equilibrium problems in Hilbert spaces.

**Corollary 3.** *Let* $C$ *be a nonempty closed convex subset in a real Hilbert space* $\mathcal{H}$, $f_i$ *be bifunctions from* $C \times C$ *to* $\mathbb{R}$ *satisfying conditions* $(C_1)-(C_4)$. *Suppose that* $\Omega = \bigcap_{i=1}^{N} Sol(f_i, C) \neq \varnothing$. *Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by*

$$\begin{aligned} x^0 &\in C \text{ chosen arbitrarily}, \\ y_n^i &= argmin\{\rho f_i(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\},\ i = 1,\ldots,N, \\ z_n^i &= argmin\{\rho f_i(y_n^i, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\},\ i = 1,\ldots,N, \\ i_n &= \operatorname*{argmax}_{1 \le i \le N}\{\|z_n^i - x_n\|\},\ \ z_n := z_n^{i_n}, \\ u_n &= (1-\mu_n)x_n + \mu_n z_n, \\ H_n &= \{z \in \mathcal{H} : \|u_n - z\| \le \|x_n - z\|\}, \\ W_n &= \{z \in \mathcal{H} : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{(H_n \cap W_n)} x_0. \end{aligned}$$

*where* $\{\mu_n\} \subset [a,1]$ *for some* $a \in (0,1)$. *Then,* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to an element* $p \in \Omega$.

To illustrate the proposed algorithm, we consider the following examples. The computer used in these experiments had an Intel Boxed Core CPU Q9400 6M Cache, 2.66 GHz, 1333 MHz FSB and 4 GB of memory. The language was MATLAB 2010b.

*Example* 1. Let $\mathcal{H} = \mathbb{R}^k$ with the inner product $\langle x, y\rangle := x_1 y_1 + \cdots + x_k y_k$ for all $x = (x_1, x_2, \cdots, x_k), y = (y_1, y_2, \cdots, y_k) \in \mathcal{H}$. Let $C := [-1, 1]^k$ be a $k$-dimensional box in $\mathcal{H}$. For all $x, y \in C$ and for each $i \in \{1, 2, \ldots, N\}$, we define the operator $f_i$ by

$$f_i(x, y) := \sum_{j=1}^{k} \alpha_{ij}(y_j^2 - x_j^2)$$

where $\alpha_{ij} \in (0, 1)$ are randomly generated. An elementary computation shows that conditions $(C_1) - (C_4)$ are satisfied for all $f_i$, $i = 1, 2, \ldots, N$. To define a nonexpansive semigroup let us consider the matrix

$$T(s) = \begin{pmatrix} e^{-s} & 0 & 0 & \cdots & 0 \\ 0 & e^{-s} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \ s \in \mathbb{R},$$

and let

$$\begin{aligned} T(s)x &= \begin{pmatrix} e^{-s} & 0 & 0 & \cdots & 0 \\ 0 & e^{-s} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} x \\ &= \begin{pmatrix} e^{-s} & 0 & 0 & \cdots & 0 \\ 0 & e^{-s} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}. \end{aligned}$$

It is easy to verify that $\{T(s) : s \geq 0\}$ is a nonexpansive semigroup on $C$ and that the common solution-set is $\Omega = \mathcal{F} \bigcap \big( \bigcap_{i=1}^{N} Sol(C, f_i)\big) = \{(0, 0, 0, \ldots, 0)^T\}$.
We apply Algorithm 1 to solve problem (1.1)-(1.2). Note that the mapping $T_n$ in Algorithm 1 can be found in a closed form:

$$T_n z = (z_1 \frac{1 - e^{-s_n}}{s_n}, z_2 \frac{1 - e^{-s_n}}{s_n}, z_3, \ldots, z_k)^T.$$

We choose the parameters as follows:

- $\rho = 10$;

- $\mu_n = 0.9 \ \ \forall n \geq 1$;
- $s_n = n \ \ \forall n \geq 1$;
- the stopping rule: $\|x_n - x^*\| \leq 5.10^{-3}$, where $x^* = (0,0,0,\ldots,0)^T$ is the unique solution of problem (1.1)-(1.2).

First, we test Algorithm 1 with $k = 6$, $N = 3$, $x_0 = (1,1,1,1,1)^T$. The results are presented in Table 1. The approximate solution is obtained after 198 iterations.

TABLE 1. Iterations of Algorithm 1 in Example 1 with starting point $x_0 = (1,1,1,1,1,1)^T$

| Iter(n) | $x_n^1$ | $x_n^2$ | $x_n^3$ | $x_n^4$ | $x_n^5$ | $x_n^6$ | $\|x_n - x^*\|$ |
|---|---|---|---|---|---|---|---|
| 0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 2.4495 |
| 1 | 0.5810 | 0.5720 | 0.6221 | 0.5845 | 0.5796 | 0.6328 | 1.4593 |
| 2 | 0.3339 | 0.3348 | 0.3715 | 0.3507 | 0.3485 | 0.3787 | 0.8657 |
| 3 | 0.2029 | 0.1925 | 0.2287 | 0.1993 | 0.1949 | 0.2360 | 0.5137 |
| 4 | 0.0879 | 0.1278 | 0.1129 | 0.1499 | 0.1588 | 0.1103 | 0.3109 |
| 5 | 0.3335 | 0.0252 | 0.2648 | -0.0727 | -0.1500 | 0.3029 | 0.5491 |
| 6 | 0.1932 | 0.0471 | 0.1795 | -0.0013 | -0.0393 | 0.2039 | 0.3390 |
| 7 | 0.1034 | 0.0715 | 0.1194 | 0.0610 | 0.0514 | 0.1309 | 0.2314 |
| ... | ... | ... | ... | ... | ... | ... | ... |
| 198 | 0.0017 | 0.0017 | 0.0022 | 0.0017 | 0.0024 | 0.0023 | 0.0049 |

Next, we test our algorithm with different choices of $k$, $N$ and $x_0$. The results are presented in Table 2.

TABLE 2. Performance of Algorithm 1 in Example 1 with different $k$, $N$ and $x_0$

| | $x_0 = (1,1,1,\cdots,1)^T$ | | $x_0 = (-1,-1,-1,\cdots,-1)^T$ | |
|---|---|---|---|---|
| | CPU times | Iter. | CPU times | Iter. |
| k=6, N=3 | 32.0823 | 135 | 32.7314 | 140 |
| k=6, N=6 | 85.4715 | 171 | 110.8062 | 241 |
| k=10, N=6 | 244.5807 | 423 | 231.3017 | 393 |
| k=20, N=3 | 364.0974 | 889 | 351.2896 | 862 |

*Example* 2. Consider problem (1.1)-(1.2) with $N = 2$

$$\text{find } x^* \in \Omega := EP(C, f_1) \cap EP(C, f_2) \cap F, \tag{5.1}$$

where

$$f_1 : \mathcal{R}^3 \times \mathcal{R}^3 \to \mathcal{R}, \ \ f_1(x,y) = \|y\|^4 - \|x\|^4 \ \ \forall x, y \in \mathcal{R}^3,$$

$$f_2 : \mathcal{R}^3 \times \mathcal{R}^3 \to \mathcal{R}, \quad f_2(x,y) = \langle Ax + By + q, y - x\rangle \quad \forall x, y \in \mathcal{R}^3,$$
$$A = \begin{pmatrix} 3 & 0 & 4 \\ 2 & 6 & 3 \\ 3 & 6 & 8 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 2 & 7 & 6 \end{pmatrix}, q = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix},$$

F is the set of fixed points of the nonexpansive semigroup $T(t)$ defined by

$$T(t) : \mathcal{R}^3 \to \mathcal{R}^3, \quad T(t)x = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \forall x \in \mathcal{R}^3, \quad t > 0.$$

The feasible set is $C := [0,1]^3 \subset \mathcal{R}^3$. It is easy seen that $f_1$, $f_2$ is pseudomonotone and all the conditions of Theorem 1 are satisfied. Moreover, we can check that $\Omega = \{x^*\}$ where $x^* = (0,0,0)^T$ is the unique solution of problem (5.1). We apply Algorithm 1 to problem (5.1). Note that if we choose $s_n = n \quad \forall n \geq 1$, the mappings $T_n$ can be expressed in the form

$$T_n(x) = \frac{1}{n} \begin{pmatrix} x_1 \sin n + x_2(\cos n - 1) \\ x_1(1 - \cos n) + x_2 \sin n \\ n x_3 \end{pmatrix}.$$

Now, we compute the Lipschitz constants of $f_i$. It is easy seen that $f_1$ is Lipschitz-type continuous with any constants $c_1, c_2 > 0$. For $f_2$, we have

$$\begin{aligned} f_2(x,y) + f_2(y,z) - f_2(x,z) &= \langle Ax + By + q, y - x\rangle + \langle Ay + Bz + q, z - y\rangle \\ &\quad - \langle Ax + Bz + q, z - x\rangle \\ &= \langle A(y-x), z - y\rangle + \langle B(y-z), y - x\rangle \\ &\geq -\frac{(\|A\| + \|B\|)}{2}\|x - y\|^2 - \frac{(\|A\| + \|B\|)}{2}\|y - z\|^2. \end{aligned}$$

Hence, $\frac{1}{2c_1} = \frac{1}{2c_2} = \frac{1}{\|A\|+\|B\|} = 0.0418$. The parameters in Algorithm 1 are chosen as follows

- $\rho = 0.04$;
- $\mu = 0.5$;
- $s_n = n \quad \forall n \geq 1$;
- $x_0 = (1,1,1)^T$;
- the stopping rule: $\|x_n - x^*\| \leq 5.10^{-3}$.

The results are presented in Table 3. The approximate solution is obtained after 137 iterations.

## CONCLUSION

We have proposed a parallel iterative method for finding a common element in the solution sets of a finite family of pseudomonotone equilibrium problems and the set of fixed points for a semigroup nonexpansive mappings. For handling a finite family

TABLE 3. Iterations of Algorithm 1 in Example 2.

| Iter(n) | $x_n^1$ | $x_n^2$ | $x_n^3$ | $\|x_n - x^*\|$ |
|---|---|---|---|---|
| 0 | 1.0000 | 1.0000 | 1.0000 | 1.7321 |
| 1 | 0.8401 | 0.9462 | 0.8360 | 1.5166 |
| 2 | 0.5758 | 0.8754 | 0.7388 | 1.2820 |
| 3 | 0.4044 | 0.6031 | 0.6616 | 0.9823 |
| 4 | 0.3552 | 0.3659 | 0.5562 | 0.7546 |
| 5 | 0.2393 | 0.3172 | 0.4439 | 0.5957 |
| 6 | 0.3476 | 0.1376 | 0.3436 | 0.5078 |
| 7 | 0.1764 | 0.2634 | 0.3484 | 0.4711 |
| ... | ... | ... | ... | ... |
| 137 | -0.0020 | 0.0010 | 0.0010 | 0.0024 |

of pseudomonotone equilibrium we have used a parallel splitting-up technique and the extragrandient with the Lipschitz-type continuous. The strong convergence of the proposed method has been established by using cutting planes.

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# ON THE CESÀRO SUMMABILITY FOR FUNCTIONS OF TWO VARIABLES

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*Abstract.* For a continuous function $f(T,S)$ on $\mathbb{R}^2_+ = [0,\infty)\times[0,\infty)$, we define its integral on $\mathbb{R}^2_+$ by

$$F(T,S) = \int_0^T \int_0^S f(t,s)dtds,$$

and its $(C,\alpha,\beta)$ mean by

$$\sigma_{\alpha,\beta}(T,S) = \int_0^T \int_0^S \left(1-\frac{t}{T}\right)^{\alpha}\left(1-\frac{s}{S}\right)^{\beta} f(t,s)dtds,$$

where $\alpha > -1$, and $\beta > -1$. We say that $\int_0^\infty \int_0^\infty f(t,s)dtds$ is $(C,\alpha,\beta)$ integrable to $L$ if $\lim_{T,S\to\infty}\sigma_{\alpha,\beta}(T,S) = L$ exists.

We prove that if $\lim_{T,S\to\infty}\sigma_{\alpha,\beta}(T,S) = L$ exists for some $\alpha > -1$ and $\beta > -1$, then $\lim_{T,S\to\infty}\sigma_{\alpha+h,\beta+k}(T,S) = L$ exists for all $h > 0$ and $k > 0$.

Next, we prove that if $\int_0^\infty \int_0^\infty f(t,s)dtds$ is $(C,1,1)$ integrable to $L$ and

$$T\int_0^S f(T,s)ds = O(1)$$

and

$$S\int_0^T f(t,S)ds = O(1)$$

then $\lim_{T,S\to\infty} F(T,S) = L$ exists.

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*Keywords:* $(C,\alpha,\beta)$ integrability, improper double integral, convergence in Pringsheim's sense, Cesàro integrability $(C,1,1)$, $(C,1,0)$ and $(C,0,1)$, Tauberian conditions and theorems

## 1. Introduction

Let $f(t)$ be a continuous function on $[0,\infty)$. The improper integral $\int_0^\infty f(t)dt$ is said to be $(C,\alpha)$ integrable to $L$ for some $\alpha > -1$ if the limit

$$\lim_{T\to\infty}\int_0^T \left(1-\frac{t}{T}\right)^{\alpha} f(t)dt = L$$

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exists. For all $\alpha,\beta \in \mathbb{R}$ with $-1<\alpha<\beta$, the $(C,\alpha)$ integrability implies the $(C,\beta)$ integrability. This implication is a classical result in the summability theory [1, p. 106]. The converse of this implication may be true by adding some suitable condition on the $(C,\beta)$ integrability of the improper integral $\int_0^\infty f(t)dt$. Any theorem which states that convergence of the improper integral follows from the $(C,\alpha)$ integrability of the improper integral and a Tauberian condition is said to be a Tauberian theorem.

As a special case, Laforgia [7] obtained a sufficient condition under which convergence of the improper integral follows from $(C,1)$ integrability of the improper integral. Móricz and Németh [9] established some one-sided and two-sided bounded Tauberian conditions for real or complex valued functions. Recently, Çanak and Totur ([2, 3]) have proved the generalized Littlewood theorem and Hardy-Littlewood type Tauberian theorems for the $(C,1)$ integrability of a continuous function on $[0,\infty)$ by using the concept of the general control modulo analogous to the one defined by Dik [6]. Çanak and Totur [4] have also given alternative proofs of some classical type Tauberian theorems for the $(C,1)$ integrability of a continuous function on $[0,\infty)$. Çanak and Totur [5] generalized the results of Laforgia [7] for the $(C,\alpha)$ integrability of functions by weighted mean methods.

For a continuous function $f(T,S)$ on $\mathbb{R}_+^2=[0,\infty)\times[0,\infty)$, we define its integral on $\mathbb{R}_+^2$ by

$$F(T,S)=\int_0^T\int_0^S f(t,s)dtds, \tag{1.1}$$

and its $(C,\alpha,\beta)$ mean by

$$\sigma_{\alpha,\beta}(T,S)=\int_0^T\int_0^S\left(1-\frac{t}{T}\right)^\alpha\left(1-\frac{s}{S}\right)^\beta f(t,s)dtds,$$

where $\alpha>-1$ and $\beta>-1$. An improper integral

$$\int_0^\infty\int_0^\infty f(t,s)dtds \tag{1.2}$$

is said to be $(C,\alpha,\beta)$ integrable to $L$ if

$$\lim_{T,S\to\infty}\sigma_{\alpha,\beta}(T,S)=L \tag{1.3}$$

The $(C,0,0)$ integrability is the convergence of the improper integral (1.2).

However, there are some $(C,\alpha,\beta)$ integrable functions which fail to converge as improper integrals. Adding some Tauberian condition, one may get the converse.

In this paper, we prove that the $(C,\alpha,\beta)$ integrability of (1.2) where $\alpha>-1$ and $\beta>-1$ implies the $(C,\alpha+h,\beta+k)$ integrability of (1.2) for all $h>0$ and $k>0$. As a corollary to this result, we show that if (1.2) converges to $L$, then (1.2) is $(C,h,k)$ integrable to $L$ for all $h>0$ and $k>0$. But, the converse of this implication might be true under some conditions imposed on the function. Furthermore, we give conditions under which (1.2) follows from the $(C,1,1)$ integrability of (1.2). It will be shown as

a corollary of our first result in this paper that convergence of the improper integral (1.2) implies the existence of the limit $\lim_{T,S\to\infty}\sigma_{h,k}(T,S)$ for all $h>0$ and $k>0$.

## 2. Results

The following theorem shows that $(C,\alpha,\beta)$ integrability of (1.2), where $\alpha>-1$ and $\beta>-1$, implies $(C,\alpha+h,\beta+k)$ integrability of (1.2), where all $h>0$ and $k>0$.

**Theorem 1.** *If (1.2) is $(C,\alpha,\beta)$ integrable to $L$ for some $\alpha>-1$ and $\beta>-1$, then it is $(C,\alpha+h,\beta+k)$ integrable to $L$ for all $h>0$ and $k>0$.*

*Proof.* Consider

$$\int_0^T\int_0^S \varphi(t,s;T,S)\sigma_{\alpha,\beta}(T,S)\,dt\,ds, \tag{2.1}$$

where

$$\varphi(t,s;T,S)=\frac{1}{B(\alpha+1,h)}\frac{1}{B(\beta+1,k)}\frac{1}{T}\left(\frac{t}{T}\right)^{\alpha}\left(1-\frac{t}{T}\right)^{h-1}\frac{1}{S}\left(\frac{s}{S}\right)^{\beta}\left(1-\frac{s}{S}\right)^{k-1} \tag{2.2}$$

where $B$ denotes the Beta function defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\quad x>0,y>0.$$

Letting $u=\frac{t}{T}$ and $v=\frac{s}{S}$, we have

$$\int_0^T\int_0^S \varphi(t,s;T,S)\,dt\,ds=1. \tag{2.3}$$

We need to prove that

$$\lim_{T,S\to\infty}\int_0^T\int_0^S \varphi(t,s;T,S)\sigma_{\alpha,\beta}(T,S)\,dt\,ds=L. \tag{2.4}$$

Since

$$\lim_{T,S\to\infty}\sigma_{\alpha,\beta}(T,S)=L \tag{2.5}$$

by the hypothesis, there exists a value $T_\varepsilon$ for any given $\varepsilon>0$ such that

$$|\sigma_{\alpha,\beta}(T,S)-L|<\varepsilon,\quad T\geq T_\varepsilon,\ S\geq S_\varepsilon. \tag{2.6}$$

It follows from (2.3) that

$$\int_0^T\int_0^S \varphi(t,s;T,S)\sigma_{\alpha,\beta}(T,S)\,dt\,ds-L$$
$$=\int_0^T\int_0^S \varphi(t,s;T,S)[\sigma_{\alpha,\beta}(T,S)-L]\,dt\,ds. \tag{2.7}$$

To prove (2.4), it suffices to show that

$$\left|\int_0^T\int_0^S \varphi(t,s;T,S)\sigma_{\alpha,\beta}(T,S)dtds - L\right| < 4\varepsilon, \tag{2.8}$$

provided that $T$ and $S$ are large enough.

We notice that by the hypothesis, the function $\sigma_{\alpha,\beta}(T,S)$ is bounded on $\mathbb{R}_+^2$, that is,

$$|\sigma_{\alpha,\beta}(T,S) - L| < K, \quad 0 \le T,S < \infty,$$

for some constant $K$. Using (2.3) and (2.6), we obtain, by (2.7),

$$\begin{aligned}
&\left|\int_0^T\int_0^S \varphi(t,s;T,S)[\sigma_{\alpha,\beta}(T,S) - L]dtds\right| \\
&\le \int_0^{T_\epsilon}\int_0^{S_\epsilon} \varphi(t,s;T,S)|\sigma_{\alpha,\beta}(T,S) - L|dtds \\
&\quad + \int_0^{T_\epsilon}\int_{S_\epsilon}^{S} \varphi(t,s;T,S)|\sigma_{\alpha,\beta}(T,S) - L|dtds \\
&\quad + \int_{T_\epsilon}^{T}\int_0^{S_\epsilon} \varphi(t,s;T,S)|\sigma_{\alpha,\beta}(T,S) - L|dtds \\
&\quad + \epsilon\int_{T_\epsilon}^{T}\int_{S_\epsilon}^{S} \varphi(t,s;T,S)dtds \\
&\le K\int_0^{T_\epsilon}\int_0^{S_\epsilon} \varphi(t,s;T,S)dtds + K\int_0^{T_\epsilon}\int_{S_\epsilon}^{S} \varphi(t,s;T,S)dtds \\
&\quad + K\int_{T_\epsilon}^{T}\int_0^{S_\epsilon} \varphi(t,s;T,S)dtds + \epsilon\int_0^{T}\int_0^{S} \varphi(t,s;T,S)dtds \\
&= K\int_0^{T_\epsilon}\int_0^{S_\epsilon} \varphi(t,s;T,S)dtds + K\int_0^{T_\epsilon}\int_{S_\epsilon}^{S} \varphi(t,s;T,S)dtds \\
&\quad + K\int_{T_\epsilon}^{T}\int_0^{S_\epsilon} \varphi(t,s;T,S)dtds + \epsilon
\end{aligned}$$

By the substitution $u = \frac{t}{T}$, $v = \frac{s}{S}$, we have

$$\begin{aligned}
\int_0^{T_\epsilon}\int_0^{S_\epsilon} \varphi(t,s;T,S)dtds &= \frac{1}{B(\alpha+1,h)}\int_0^{T_\varepsilon}\frac{1}{T}\left(\frac{t}{T}\right)^{\alpha}\left(1-\frac{t}{T}\right)^{h-1}dt \\
&\quad \times \frac{1}{B(\beta+1,k)}\int_0^{S_\epsilon}\frac{1}{S}\left(\frac{s}{S}\right)^{\beta}\left(1-\frac{s}{S}\right)^{k-1}ds
\end{aligned}$$

$$= \frac{1}{B(\alpha+1,h)} \int_0^{T_\varepsilon/T} u^\alpha (1-u)^{h-1} du$$
$$\times \frac{1}{B(\beta+1,k)} \int_0^{S_\varepsilon/S} v^\beta (1-v)^{k-1} dv$$

which tends to zero when $T, S \to \infty$ for any fixed $T_\varepsilon$ and $S_\varepsilon$. Thus, there exist some $\widehat{T_\varepsilon^1}$ and $\widehat{S_\varepsilon^1}$ such that

$$K \int_0^{T_\epsilon} \int_0^{S_\epsilon} \varphi(t,s;T,S) dt ds < \varepsilon, \quad T \geq \widehat{T_\varepsilon^1}, \ S \geq \widehat{S_\varepsilon^1}.$$

By the substitution $u = \frac{t}{T}$, $v = \frac{s}{S}$, we have

$$\int_0^{T_\epsilon} \int_{S_\epsilon}^{S} \varphi(t,s;T,S) dt ds = \frac{1}{B(\alpha+1,h)} \int_0^{T_\varepsilon} \frac{1}{T} \left(\frac{t}{T}\right)^\alpha \left(1-\frac{t}{T}\right)^{h-1} dt$$
$$\times \frac{1}{B(\beta+1,k)} \int_{S_\epsilon}^{S} \frac{1}{S} \left(\frac{s}{S}\right)^\beta \left(1-\frac{s}{S}\right)^{k-1} ds$$
$$= \frac{1}{B(\alpha+1,h)} \int_0^{T_\varepsilon/T} u^\alpha (1-u)^{h-1} du$$
$$\times \frac{1}{B(\beta+1,k)} \int_{S_\varepsilon/S}^{1} v^\beta (1-v)^{k-1} dv$$

which tends to zero when $T, S \to \infty$ for any fixed $T_\varepsilon$ and $S_\varepsilon$ (Note that $\frac{1}{B(\beta+1,k)} \int_{S_\varepsilon/S}^1 v^\beta (1-v)^{k-1} dv$ tends to 1 as $S \to \infty$ ). Thus, there exist some $\widehat{T_\varepsilon^2}$ and $\widehat{S_\varepsilon^2}$ such that

$$K \int_0^{T_\epsilon} \int_{S_\epsilon}^{S} \varphi(t,s;T,S) dt ds < \varepsilon, \quad T \geq \widehat{T_\varepsilon^2}, \ S \geq \widehat{S_\varepsilon^2}.$$

Similarly, the integral

$$\int_{T_\epsilon}^{T} \int_0^{S_\epsilon} \varphi(t,s;T,S) dt ds$$

tends to to zero when $T, S \to \infty$ for any fixed $T_\varepsilon$ and $S_\varepsilon$ (Note that $\frac{1}{B(\alpha+1,h)} \int_{T_\varepsilon/T}^1 u^\alpha (1-u)^{k-1} dv$ tends to 1 as $S \to \infty$ ). Thus, there exist some $\widehat{T_\varepsilon^3}$ and $\widehat{S_\varepsilon^3}$ such that

$$K \int_{T_\epsilon}^{T} \int_0^{S_\epsilon} \varphi(t,s;T,S) dt ds < \varepsilon, \quad T \geq \widehat{T_\varepsilon^3}, \ S \geq \widehat{S_\varepsilon^3}.$$

Hence, we have (2.8) for $T \geq \max\{T_\epsilon, \widehat{T_\varepsilon}^1, \widehat{T_\varepsilon}^2, \widehat{T_\varepsilon}^3\}$, $S \geq \max\{S_\epsilon, \widehat{S_\varepsilon}^1, \widehat{S_\varepsilon}^2, \widehat{S_\varepsilon}^3\}$ and this proves (2.4). We obtain

$$\begin{aligned}
&\int_0^T \int_0^S \varphi(t,s;T,S)\sigma_{\alpha,\beta}(t,s)dtds \\
&= \int_0^T \int_0^S \varphi(t,s;T,S)\left(\int_0^t \int_0^s \left(1-\frac{u}{t}\right)^\alpha \left(1-\frac{v}{s}\right)^\beta f(u,v)dudv\right)dtds \\
&= \int_0^T \int_0^S f(u,v)\left(\int_u^T \int_v^S \varphi(t,s;T,S)\left(1-\frac{u}{t}\right)^\alpha \left(1-\frac{v}{s}\right)^\beta dtds\right)dudv \\
&= \int_0^T \int_0^S f(u,v)I(u,v;T,S)dudv,
\end{aligned}$$

where

$$I(u,v;T,S) = \int_u^T \int_v^S \varphi(t,s;T,S)\left(1-\frac{u}{t}\right)^\alpha \left(1-\frac{v}{s}\right)^\beta dtds.$$

Here, we write $I(u,v;T,S)$ as

$$\begin{aligned}
I(u,v;T,S) &= \int_u^T \int_v^S \varphi(t,s;T,S)\left(1-\frac{u}{t}\right)^\alpha \left(1-\frac{v}{s}\right)^\beta dtds \\
&= \left(\frac{1}{B(\alpha+1,h)}\int_u^T \frac{1}{T}\left(\frac{t}{T}\right)^\alpha \left(1-\frac{t}{T}\right)^{h-1}\left(1-\frac{u}{t}\right)^\alpha dt\right) \\
&\quad \times \left(\frac{1}{B(\beta+1,k)}\int_v^S \frac{1}{S}\left(\frac{s}{S}\right)^\beta \left(1-\frac{s}{S}\right)^{k-1}\left(1-\frac{v}{s}\right)^\beta ds\right) \\
&= \left(\frac{1}{B(\alpha+1,h)}\left(\frac{1}{T}\right)^{\alpha+1}\int_u^T \left(1-\frac{t}{T}\right)^{h-1}(t-u)^\alpha dt\right) \\
&\quad \times \left(\frac{1}{B(\beta+1,k)}\left(\frac{1}{S}\right)^{\beta+1}\int_v^S \left(1-\frac{s}{S}\right)^{k-1}(s-v)^\beta ds\right) \\
&= I_1(u,T)I_2(v,S),
\end{aligned}$$

where

$$I_1(u,T) = \frac{1}{B(\alpha+1,h)}\left(\frac{1}{T}\right)^{\alpha+1}\int_u^T \left(1-\frac{t}{T}\right)^{h-1}(t-u)^\alpha dt,$$

and

$$I_2(v,S) = \frac{1}{B(\beta+1,k)}\left(\frac{1}{S}\right)^{\beta+1}\int_v^S \left(1-\frac{s}{S}\right)^{k-1}(s-v)^\beta ds.$$

Substituting $t = T-(T-u)x$ in $I_1(u,T)$, we have

$$\begin{aligned} I_1(u,T) &= \frac{1}{B(\alpha+1,h)}\left(1-\frac{u}{T}\right)^{h-1}\left(1-\frac{u}{T}\right)^{\alpha+1}\int_0^1 x^{h-1}(1-x)^{\alpha}dx \\ &= \left(1-\frac{u}{T}\right)^{\alpha+h}, \end{aligned}$$

and similarly we have

$$I_2(v,S) = \left(1-\frac{v}{S}\right)^{\beta+k}.$$

These show that

$$\begin{aligned} &\int_0^T\int_0^S \varphi(t,s;T,S)\sigma_{\alpha,\beta}(t,s)dtds \\ &= \int_0^T\int_0^S \left(1-\frac{u}{T}\right)^{\alpha+h}\left(1-\frac{v}{S}\right)^{\beta+k} f(u,v)dudv = \sigma_{\alpha+h,\beta+k}(T,S) \end{aligned}$$

This completes the proof of Theorem 1. □

## 3. The case $\alpha = 1$, $\beta = 0$ or $\alpha = 0$, $\beta = 1$

Similar to the $(C,1,1)$ integrability, one can improve the theory of the $(C,1,0)$ or the $(C,0,1)$ integrability. Since this theory is similar to the theory of integrability of functions of one variable, we only present it without detailed proofs.

**Definition 1.** Let $f(T,S)$ be a continuous function on $\mathbb{R}_+^2$ and $F(T,S)$ be defined as in (1.1). We define $(C,1,0)$ and $(C,0,1)$ means of (1.1) by

$$\sigma_{1,0}(T,S) := \int_0^T\int_0^S \left(1-\frac{t}{T}\right) f(t,s)dtds \tag{3.1}$$

and

$$\sigma_{0,1}(T,S) := \int_0^T\int_0^S \left(1-\frac{s}{S}\right) f(t,s)dtds, \tag{3.2}$$

respectively. We say that (1.2) is $(C,1,0)$ and $(C,0,1)$ integrable on $\mathbb{R}_+^2$ if

$$\lim_{T,S\to\infty} \sigma_{1,0}(T,S) \tag{3.3}$$

and

$$\lim_{T,S\to\infty} \sigma_{0,1}(T,S) \tag{3.4}$$

exist and are finite, respectively.

The $(C,1,0)$ and $(C,0,1)$ summability methods are regular method. Namely, if (1.2) converges to $L$, then (1.2) is both $(C,1,0)$ and $(C,0,1)$ integrable to $L$.

**Theorem 2.** *If (1.2) is $(C,1,0)$ integrable to $L$ and*

$$T\int_0^S f(T,s)ds = O(1) \tag{3.5}$$

*then (1.2) converges to $L$.*

**Theorem 3.** *If (1.2) is $(C,0,1)$ integrable to $L$*

$$S\int_0^T f(t,S)dt = O(1) \tag{3.6}$$

*then (1.2) converges to $L$.*

Since the proofs of Theorem 2 and Theorem 3 can be obtained with similar steps as in Theorem 3.2 in [7], we omit the proofs.

## 4. The case $\alpha = 1$, $\beta = 1$

**Definition 2.** Let $f(T,S)$ be a continuous function on $\mathbb{R}^2_+$. We say that (1.2) is $(C,1,1)$ integrable on $\mathbb{R}^2_+$, if

$$\lim_{T,S\to\infty}\int_0^T\int_0^S\left(1-\frac{t}{T}\right)\left(1-\frac{s}{S}\right)f(t,s)dtds \tag{4.1}$$

exists and is finite.

As a result of Theorem 1, we have the following corollary.

**Corollary 1.** *If (1.2) converges to $L$, then (1.2) is $(C,1,1)$ integrable to $L$.*

*Proof.* Take $\alpha = \beta = 0$ and $h = k = 1$ in Theorem 1. □

That the converse of Corollary 1 is not true in general is shown by the following examples.

*Example* 1. The integral $\int_0^\infty\int_0^\infty \cos t\cos s\,dtds$ converges to zero, in $(C,1,1)$ sense.

By (4.1), we have

$$\begin{aligned}&\int_0^T\int_0^S\left(1-\frac{t}{T}\right)\left(1-\frac{s}{S}\right)\cos t\cos s dtds\\ &=\int_0^T\left(1-\frac{t}{T}\right)\cos t dt\int_0^S\left(1-\frac{s}{S}\right)\cos s ds=\frac{(1-\cos T)(1-\cos S)}{TS},\end{aligned}$$

which tends to zero as $T,S\to\infty$.

*Example* 2. The integral $\int_0^\infty\int_0^\infty \sin t\sin s\,dtds$ converges to 1, in $(C,1,1)$ sense.

By (4.1), we have

$$\int_0^T \int_0^S \left(1-\frac{t}{T}\right)\left(1-\frac{s}{S}\right)\sin t \sin s\,dt\,ds$$
$$= \int_0^T \left(1-\frac{t}{T}\right)\sin t\,dt \int_0^S \left(1-\frac{s}{S}\right)\sin s\,ds = \frac{(T-\sin T)(S-\sin S)}{TS},$$

which tends to 1 as $T,S \to \infty$.

A convolution theorem for $(C,1,1)$ integrability is given by the following theorem.

**Theorem 4.** *Let the integrals*

$$\int_0^\infty \int_0^\infty f(t,s)\,dt\,ds, \quad \int_0^\infty \int_0^\infty g(t,s)\,dt\,ds$$

*be convergent in* $(C,0,0)$ *sense, to* $L_1$ *and* $L_2$*, respectively. Then*

$$h(t,s) = \int_0^t \int_0^s f(t-u,s-v)g(t,s)\,du\,dv \tag{4.2}$$

*converges in* $(C,1,1)$ *sense to* $L_1L_2$.

*Proof.* We need to show that

$$\lim_{T,S\to\infty} \int_0^T \int_0^S \left(1-\frac{t}{T}\right)\left(1-\frac{s}{S}\right)h(t,s)\,dt\,ds = L_1L_2 \tag{4.3}$$

We define $F$ and $G$ by

$$F(t,s) = \int_0^t \int_0^s f(t,s)\,dt\,ds \tag{4.4}$$

and

$$G(t,s) = \int_0^t \int_0^s g(t,s)\,dt\,ds \tag{4.5}$$

By (4.2), we get

$$\int_0^T \int_0^S \left(1-\frac{t}{T}\right)\left(1-\frac{s}{S}\right)h(t,s)\,dt\,ds$$
$$= \int_0^T \int_0^S \left(1-\frac{t}{T}\right)\left(1-\frac{s}{S}\right)\left(\int_0^t \int_0^s f(t-u,s-v)g(t,s)\,du\,dv\right)dt\,ds$$
$$= \int_0^T \int_0^S g(t,s)\,du\,dv \left(\int_u^T \int_v^S \left(1-\frac{t}{T}\right)\left(1-\frac{s}{S}\right)f(t-u,s-v)\,dt\,ds\right)$$

The substitutions $t-u=\omega$ and $s-v=\delta$ and the subsequent integration by parts give

$$\int_0^T \int_0^S g(t,s)\,du\,dv \int_u^T \int_v^S \left(1-\frac{t}{T}\right)\left(1-\frac{s}{S}\right)f(t-u,s-v)\,dt\,ds$$

$$= \int_0^T \int_0^S g(u,v)dudv \int_0^{T-u} \int_0^{S-v} \left(1-\frac{u+\omega}{T}\right)\left(1-\frac{v+\delta}{S}\right) f(\omega,\delta)d\omega d\delta$$
$$= \frac{1}{TS}\int_0^T \int_0^S G(u,v)F(T-u,S-v)dudv$$

Since $F(T,S) \to L_1$ and $G(T,S) \to L_2$ as $T \to \infty$ and $S \to \infty$, we have that for some $T_\epsilon$ and $S_\epsilon$

$$|F(T,S)-L_1| \le \epsilon \tag{4.6}$$

and

$$|G(T,S)-L_2| \le \epsilon \tag{4.7}$$

for $T \ge T_\epsilon$ and $S \ge S_\epsilon$. Then we have

$$\begin{aligned}
&\left|\frac{1}{TS}\int_0^T \int_0^S G(u,v)F(T-u,S-v)dudv - L_1L_2\right| \\
&= \left|\frac{1}{TS}\int_0^T \int_0^S (G(u,v)-L_2)\,F(T-u,S-v)dudv \right. \\
&\quad \left. + \int_0^T \int_0^S L_2\,(F(T-u,S-v)-L_1)\,dudv\right| \\
&\le \frac{1}{TS}\int_0^{T_\epsilon} \int_0^{S_\epsilon} |L_2||F(T-u,S-v)-L_1|\,dudv \\
&\quad + \frac{1}{TS}\int_0^{T_\epsilon} \int_{S_\epsilon}^{S} |L_2||F(T-u,S-v)-L_1|\,dudv \\
&\quad + \frac{1}{TS}\int_{T_\epsilon}^{T} \int_0^{S_\epsilon} |L_2||F(T-u,S-v)-L_1|\,dudv \\
&\quad + \frac{1}{TS}\int_{T_\epsilon}^{T} \int_{S_\epsilon}^{S} |L_2||F(T-u,S-v)-L_1|\,dudv
\end{aligned}$$

Since $F(T,S) \to L_1$ and $G(T,S) \to L_2$ as $T \to \infty$ and $S \to \infty$, there exist some constants $N_1$ and $N_2$ such that

$$|F(T,S)-L_1| \le N_1 \tag{4.8}$$

and

$$|G(T,S)-L_2| \le N_2 \tag{4.9}$$

for all $T$ and $S$. If we use (4.6), (4.7), (4.8), and (4.9), and then letting $T$ and $S$ tend to $\infty$ independently, we have the desired result. □

By the next theorem we give a sufficient condition under which $(C,1,1)$ integrability of (1.2) follows from $(C,0,0)$ integrability of (1.2).

**Theorem 5.** *If (1.2) is $(C,1,1)$ integrable to $L$ and*

$$T\int_0^S f(T,s)ds = O(1) \tag{4.10}$$

*and*

$$S\int_0^T f(t,S)dt = O(1) \tag{4.11}$$

*then (1.2) converges to $L$.*

*Proof.* Let (1.2) be $(C,1,1)$ integrable to $L$; that is,

$$G(T,S) := \int_0^T\int_0^S \left(1-\frac{t}{T}\right)\left(1-\frac{s}{S}\right) f(t,s)dtds \to L,\ T,S\to\infty \tag{4.12}$$

We rewrite $G(T,S)$ as

$$G(T,S) = \int_0^T \left(1-\frac{s}{S}\right)\frac{\partial}{\partial t}G_1(T,s)dt, \tag{4.13}$$

where

$$G_1(T,S) := \int_0^T\int_0^S \left(1-\frac{t}{T}\right) f(t,s)dtds. \tag{4.14}$$

It follows from (4.12), (4.13), and (4.14) that $\frac{\partial}{\partial S}G_1(T,S)$ is $(C,0,1)$ integrable to $L$.

By (4.12), we have

$$G(T,S) = G_1(T,S) - H_1(T,S), \tag{4.15}$$

where

$$H_1(T,S) = \frac{1}{S}\int_0^S s\frac{\partial}{\partial s}G_1(T,s)dt. \tag{4.16}$$

We have to show that $H_1(T,S)\to 0$ as $T,S\to\infty$.

By (4.13), we find

$$\frac{\partial}{\partial S}G(T,S) = \frac{1}{S^2}\int_0^S s\frac{\partial}{\partial s}G_1(T,s)ds = \frac{H_1(T,S)}{T} \tag{4.17}$$

We also have

$$\begin{aligned}\int_{S_1}^{S_2}\frac{\partial}{\partial S}G(T,S)dS &= G(T,S_1)-G(T,S_2)\\ &= \int_{S_1}^{S_2}\frac{H_1(T,S)}{S}dT\\ &= \int_{\log S_1}^{\log S_2} H_1(T,e^v)dv\\ &= \int_{\log S_1}^{\log S_2} R(T,v)dv.\end{aligned}$$

Here, we used the substitution $S = e^v$ and $R(T,v) = H_1(T,e^v)$. We need to show that $\lim_{v\to\infty} R(T,v) = 0$. By the simple calculation, we have

$$\frac{\partial}{\partial v} R(T,v) = e^v \frac{\partial}{\partial v} H_1(T,e^v) = S\frac{\partial}{\partial S} H_1(T,S). \tag{4.18}$$

By (4.16), we get

$$SH_1(T,S) = \int_0^S s\frac{\partial}{\partial s} G_1(T,s)ds. \tag{4.19}$$

Differentiation the both sides of (4.19) with respect to $T$ gives

$$H_1(T,S) + S\frac{\partial}{\partial S} H_1(T,S) = S\frac{\partial}{\partial S} G_1(T,S). \tag{4.20}$$

Taking the $(C,1,0)$ mean of the both sides of (4.11) we have

$$S\frac{\partial}{\partial S} G_1(T,S) = O(1), \tag{4.21}$$

which implies that

$$S\frac{\partial}{\partial S} H_1(T,S) = O(1) \tag{4.22}$$

by (4.18).

We can easily obtain $H_1(T,S) \to 0$ as $T,S \to \infty$ by following the steps of Theorem 3.2 in [7]. It follows from (4.12) and (4.15) that $G_1(T,S) \to L$ as $T,S \to \infty$.

Since $G_1(T,S) \to L$ as $T,S \to \infty$ and the condition (4.10), we have $\lim_{T,S\to\infty} F(T,S) = L$ by Theorem 2.

□

*Remark* 1. Analogous Tauberian results were proved in [8] for double improper integrals with a different perspective.

## 5. Conclusion

In this paper, we extended the classical Tauberian theorems given for the $(C,\alpha)$ integrability of the improper integrals of functions of one variable to those of the $(C,\alpha,\beta)$ integrability improper integrals of functions of two by using the methods employed in Laforgia [7]. The analogous results for the functions of several variables can be obtained by the similar techniques.

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# INVARIANTS UNDER DECOMPOSITION OF THE CONJUGATION IN THE MOD 2 DUAL LEIBNIZ-HOPF ALGEBRA

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*Abstract.* The Leibniz-Hopf algebra is the free associative algebra on one generator, $S^n$, in each positive degree, with coproduct $\Delta(S^n) = \sum S^j \otimes S^{n-j}$. Let $\mathcal{C}$ and $\mathcal{R}$ denote coarsening and reversing operations on the mod 2 dual Leibniz-Hopf algebra. We consider decomposition of the Hopf algebra conjugation $\chi = \mathcal{C} \circ \mathcal{R}$ in this dual Hopf algebra and calculate bases for the fixed points of the operations $\mathcal{C}$ and $\mathcal{R}$.

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*Keywords:* Hopf algebra, Leibniz-Hopf algebra, antipode, Steenrod algebra, quasisymmetric functions

## 1. Introduction

The Leibniz-Hopf algebra $\mathcal{F}$ is the free associative $\mathbf{Z}$-algebra on generators $S^1, S^2, S^3 \ldots$ with the graded cocommutative Hopf algebra structure determined by $\Delta(S^n) = \sum_{i+j=n} S^i \otimes S^j$ (where $S^0$ denotes the unit 1). $\mathcal{F}$ is isomoprhic to the 'ring of noncommutative symmetric functions' [7] and has been studied in [9–11]. The graded dual Hopf algebra, $\mathcal{F}^* = \oplus_n \mathrm{Hom}(\mathcal{F}_n, \mathbf{Z})$ (where $\mathcal{F}_n$ denotes the degree $n$ part of $\mathcal{F}$), is the ring of quasi-symmetric functions with the outer coproduct, which has been studied in [2, 6, 8–12]. This algebra was the subject of the Ditters conjecture [1,5,11] which makes it important in combinatorics. The mod 2 reduction $\mathcal{F} \otimes \mathbf{Z}/2$ also has a connection with topology, since the mod 2 Steenrod algebra is naturally defined as a quotient of $\mathcal{F} \otimes \mathbf{Z}/2$ by the Adem relations [13]. From now on we denote $\mathcal{F} \otimes \mathbf{Z}/2$ by $\mathcal{F}_2$.

As $\mathcal{F}_2$ is the free $\mathbf{Z}/2$-algebra on $S^1, S^2, \ldots$, a basis for $\mathcal{F}_2$ is given by all words $S^{j_1} S^{j_2} \cdots S^{j_l}$. We denote the corresponding dual basis for $\mathcal{F}_2^* = \oplus_n \mathrm{Hom}(\mathcal{F}_n, \mathbf{Z}/2)$ by $\{S_{j_1, j_2, \ldots, j_l}\}$. Since $\mathcal{F}$ is the cocommutative graded Hopf algebra, $\mathcal{F}^*$ is a commutative graded Hopf algebra with a unique conjugation operation, $\chi$, which satisfies

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$\chi^2 = 1$(where 1 denotes the identity homomorphism). A formula for $\chi$ was introduced by Ehrenborg [6, Proposition 3.4] and for $\mathcal{F}_2^*$ we can simplify it to:

$$\chi(S_{j_1,j_2,\ldots,j_l}) = \sum S_{i_1,\ldots,i_k}$$

summed over all coarsenings $i_1,\ldots,i_k$ of the *reversed* word $j_l,\ldots,j_1$, i.e., all words $i_1,\ldots,i_k$ that admit $j_l,\ldots,j_1$ as a refinement. As an example,

$$\chi(S_{5,2,1}) = S_{1,2,5} + S_{3,5} + S_{1,7} + S_8.$$

The length of a word is the number of its letters so $S_{2,1,2,6}$ has length 4. A word of length $n$ has $n-1$ commas and, hence, $2^{n-1}$ coarsenings.

Using conjugation invariants in $\mathcal{F}_2^*$ is an algebraic tool to understand the conjugation invariants in the mod 2 dual Steenrod algebra [14, Section 5]. Motivated by this in [4], the author and Crossley calculated a vector space basis for the invariants of $\mathcal{F}_2^*$ under the operation $\chi$. In [3] the analogous question was considered for $\mathcal{F}_2$. In [16, Section 6] the author and Kaji introduced an explicit correspondence between the invariant elements of $\mathcal{F}_2$ and of $\mathcal{F}_2^*$ under the Hopf algebra conjugation operation. See also [15] for the relationship between the conjugation invariants in $\mathcal{F}_2$ and the conjugation invariants in the mod 2 Steenrod algebra.

The conjugation operation $\chi$ comprises both coarsening and reversing operations. In this paper we calculate vector space bases for the invariants under coarsening and reversing operations. We also investigate relations between those spaces with the space of conjugation invariants in $\mathcal{F}_2^*$.

## 2. Terminology and results

Given a word $S_{i_1,i_2,\ldots,i_k}$, define its image under the map $\mathcal{C}$ to be

$$\mathcal{C}(S_{i_1,i_2,\ldots,i_k}) = \sum S_{l_1,\ldots,l_n} \tag{2.1}$$

summed over all coarsenings $l_1,\ldots,l_n$ of $i_1,\ldots,i_k$, and define its image under the map $\mathcal{R}$ to be $\mathcal{R}(S_{i_1,i_2,\ldots,i_k}) = S_{i_k,\ldots,i_2,i_1}$. Here, $\mathcal{C}$ and $\mathcal{R}$ denote coarsening and reversing operations on $\mathcal{F}_2^*$. It is clear that $\mathcal{R}^2 = 1$. Both $\mathcal{C}$ and $\mathcal{R}$ are homomorphisms on $\mathcal{F}_2^*$, and by definition we have: $\chi = \mathcal{C} \circ \mathcal{R}$. Moreover, one can see that $\mathcal{C}$ and $\mathcal{R}$ commute, i.e., $\mathcal{C} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{C}$. This together with the property that $\chi^2 = 1$ implies that $\mathcal{C}^2 = 1$.

An element $x \in \mathcal{F}_2^*$ is an invariant under $\mathcal{C}$ if, and only if, $\mathcal{C}(x) = x$, i.e., $(\mathcal{C} - 1)(x) = 0$. Hence, $\mathrm{Ker}(\mathcal{C} - 1)$ is a subspace of $\mathcal{F}_2^*$, which is formed by the invariants under $\mathcal{C}$ in $\mathcal{F}_2^*$. Similarly, $\mathrm{Ker}(\mathcal{R} - 1)$ is a subspace of $\mathcal{F}_2^*$, which is formed by the invariants under $\mathcal{R}$ in $\mathcal{F}_2^*$. We now introduce a new terminology. A word $S_{i_1,i_2,\ldots,i_k}$ is said to be uniterminal if its last letter is equal to 1, i.e., $i_k = 1$. We denote this word by UT.

As an example, in the degree 4 part of $\mathcal{F}_2^*$ the UTs are: $S_{3,1}, S_{2,1,1}, S_{1,2,1}, S_{1,1,1,1}$. We also recall terminologies from [4]. A word $S_{i_1,i_2,\ldots,i_k}$ is a palindrome if $i_1 = i_k$,

$i_2 = i_{k-1}$, etc. The non-palindromes form obvious pairs: a non-palindrome $S_{i_1,\dots,i_k}$ pairs with $S_{i_k,\dots,i_1}$. In each pair, one term will be higher in lexicographic ordering, and one lower. We call the first an HNP (higher non-palindrome) and the second an LNP. For example, $S_{5,4,1,3}$ is an HNP, $S_{4,1,2,4}$ is an LNP.

**Theorem 1.** *In the mod* 2 *dual Leibniz-Hopf algebra invariants under* $\mathcal{C}$ *comprise exactly the image of* $\mathcal{C}-1$*, i.e.,* $\mathrm{Ker}(\mathcal{C}-1) = \mathrm{Im}(\mathcal{C}-1)$.

**Theorem 2.** *In degree* $n$ *part of mod* 2 *dual Leibniz-Hopf algebra the subspace* $\mathrm{Im}(\mathcal{R}-1)$ *has a basis consisting of the* $(\mathcal{R}-1)$*-images of all HNPs.*

*Consequently the dimension of* $\mathrm{Im}(\mathcal{R}-1)$ *in degree* $n$ *is* $2^{n-2} - 2^{(n-2)/2}$ *if* $n$ *is even, and* $2^{n-2} - 2^{(n-3)/2}$ *if* $n$ *is odd.*

The dimension results in the last statement are obtained by simple combinatorial calculations. (See [4, Section 3] and [3, Section 2] for details). Furthermore, Theorem 2 together with the rank and nullity theorem gives the result.

**Theorem 3.** *Let* $\mathrm{Ker}(\mathcal{R}-1)_n$ *denote the subspace of degree* $n$ *invariants under* $\mathcal{R}$ *in the mod* 2 *dual Leibniz-Hopf algebra. Then*

$$\dim \mathrm{Ker}(\mathcal{R}-1)_n = \begin{cases} 2^{n-2} + 2^{(n-2)/2}, \text{ if } n \text{ is even,} \\ 2^{n-2} + 2^{(n-3)/2}, \text{ if } n \text{ is odd.} \end{cases}$$

**Theorem 4.** *The space of invariants under* $\mathcal{R}$*,* $\mathrm{Ker}(\mathcal{R}-1)$*, has a basis consisting of the following:*

(1) *All the palindromes; and*
(2) *The* $(\mathcal{R}-1)$*-images of all HNPs.*

**Proposition 1.** *In the mod* 2 *dual Leibniz-Hopf algebra we have:*

(1) $\mathrm{Ker}(\chi-1) \cap \mathrm{Ker}(\mathcal{C}-1) = \mathrm{Ker}(\chi-1) \cap \mathrm{Ker}(\mathcal{C}-1) \cap \mathrm{Ker}(\mathcal{R}-1)$.
(2) $\mathrm{Ker}(\chi-1) \cap \mathrm{Ker}(\mathcal{R}-1) = \mathrm{Ker}(\chi-1) \cap \mathrm{Ker}(\mathcal{R}-1) \cap \mathrm{Ker}(\mathcal{C}-1)$.
(3) $\mathrm{Ker}(\mathcal{C}-1) \cap \mathrm{Ker}(\mathcal{R}-1) = \mathrm{Ker}(\mathcal{C}-1) \cap \mathrm{Ker}(\mathcal{R}-1) \cap \mathrm{Ker}(\chi-1)$.

## 3. Proof of Theorem 1

We first give the following auxiliary results.

**Theorem 5.** *The image of* $(\mathcal{C}-1)$ *on* $\mathcal{F}_2^*$ *has a basis consisting of the* $(\mathcal{C}-1)$*-images of all UTs.*

**Lemma 1.** *Let* $S_{i_1,\dots,i_k}$ *be a UT. Among the summands of longest length in* $(\mathcal{C}-1)(S_{i_1,\dots,i_k})$ *there is a summand* $S_{i_1,\dots,i_{k-2},i_{k-1}+i_k}$*, and this summand does not occur as a longest length summand in the* $\mathcal{C}-1$ *image of any other UT.*

*Proof.* Let $S_{i_1,\dots,i_k}$ be a $k$ length UT, then the longest summands in $(\mathcal{C}-1)(S_{i_1,\dots,i_k})$ are of length $k-1$, and $S_{i_1,\dots,i_{k-2},i_{k-1}+i_k}$ is one of them. We shall show that $S_{i_1,\dots,i_{k-2},i_{k-1}+i_k}$ cannot arise as a longest summand in the $(\mathcal{C}-1)$ image of any

other UT, say $S_{j_1,\dots,j_l}$. In $(\mathcal{C}-1)(S_{j_1,\dots,j_l})$, similarly the longest summands are of length $l-1$, namely,

$$S_{j_1+j_2,j_3,\dots,j_l}, S_{j_1,j_2+j_3,\dots,j_l}, \dots, S_{j_1,\dots,j_{l-2}+j_{l-1},j_l}, S_{j_1,\dots,j_{l-2},j_{l-1}+j_l}. \tag{3.1}$$

We see that all the terms in (3.1) are UTs except the last term $S_{j_1,\dots,j_{l-2},j_{l-1}+j_l}$ $(j_{l-1}+j_l>1)$. For this term to equal $S_{i_1,\dots,i_{k-2},i_{k-1}+i_k}$, we must have $l=k$, $,j_1=i_1, j_2=i_2,\dots,j_{l-2}=i_{k-2}$, and $j_{l-1}+j_l=i_{k-1}+i_k$. By definition, $j_l=i_k=1$ which implies that $S_{j_1,\dots,j_l}=S_{i_1,\dots,i_k}$. On the other hand, none of the UT terms in (3.1) can equal to a non-UT term $S_{i_1,\dots,i_{k-2},i_{k-1}+i_k}$. This completes the proof. □

*Proof of Theorem 5.* Suppose that we have a sum of UTs whose image under $\mathcal{C}-1$ is 0. Order these summands so that shorter terms come before longer terms. Let $S_{i_1,\dots,i_k}$ be the last summand with respect to this ordering, then it has the term $S_{i_1,\dots,i_{k-2},i_{k-1}+i_k}$ as longest summand in its image under $\mathcal{C}-1$. Lemma 1 tells us that having length $k-1$, this term cannot arise in the $(\mathcal{C}-1)$-image of any shorter term or of any other UT of length $k$. Hence, in the $(\mathcal{C}-1)$-image of the sum the term $S_{i_1,\dots,i_{k-2},i_{k-1}+i_k}$ cannot be cancelled, so this image cannot be zero. This contradicts the hypothesis showing that $S_{i_1,\dots,i_k}$ cannot be a UT. This implies that the sum is itself zero. Hence, the UTs have linearly independent images under $\mathcal{C}-1$.

The number of UTs in degree $n$ is $2^{n-2}$, since each UT has 1 as the last term and the remaining terms can be any word of degree $n-1$, of which there are $2^{n-2}$. Hence, in $n$ degrees, the UTs form a set of $2^{n-2}$ elements. Thus, the above linear independence establishes that $\dim \operatorname{Im}(\mathcal{C}-1)\geq 2^{n-2}$. On the other hand, $\operatorname{Im}(\mathcal{C}-1)\subset \operatorname{Ker}(\mathcal{C}-1)$, since $\mathcal{C}^2=1$, so in each degree we have $\dim \operatorname{Im}(\mathcal{C}-1)\leq \frac{1}{2}\dim(\mathcal{F}_2^*)$. In all degrees, this means $\dim \operatorname{Im}(\mathcal{C}-1)\leq \frac{1}{2}2^{n-1}=2^{n-2}$. Consequently, $\dim \operatorname{Im}(\mathcal{C}-1)=2^{n-2}$. We know the $(\mathcal{C}-1)$-images of UTs are linearly independent and the number of UTs matches $\dim \operatorname{Im}(C-1)$. Hence, the $(\mathcal{C}-1)$-images of UTs must be a basis for $\operatorname{Im}(\mathcal{C}-1)$. Moreover, this shows that $\operatorname{Ker}(\mathcal{C}-1)=\operatorname{Im}(\mathcal{C}-1)$. □

This completes the proof of Theorem 1 giving a basis for $\operatorname{Ker}(\mathcal{C}-1)$.

## 4. Proof of Theorem 4

We first deal with Theorem 2 (Theorem 3 was proved in Section 2).

*Proof of Theorem 2.* Let $S_{i_1,\dots,i_k}$ be an HNP, then it is clear that the longest summands in $(\mathcal{R}-1)(S_{i_1,\dots,i_k})$ are $S_{i_1,\dots,i_k}$ and its reverse. It is clear that these summands cannot arise in the $(\mathcal{R}-1)$-image of any other HNP. Using this fact together with the argument in the proof of Theorem 5 shows that the $(\mathcal{R}-1)$-images of all HNPs are linearly independent. What is left to show that no other terms contribute anything further to the image. i.e., the $(\mathcal{R}-1)$-image of every palindrome and LNP can be expressed in terms of the $(\mathcal{R}-1)$-images of HNPs. Let $S_{i_1,\dots,i_k}$ be a palindrome, then $(\mathcal{R}-1)(S_{i_1,\dots,i_k})=0$. On the other hand, if $S_{i_1,\dots,i_k}$ be an LNP, then

$(\mathcal{R}-1)(S_{i_1,\ldots,i_k}) = (\mathcal{R}-1)(S_{i_k,\ldots,i_1})$, where $S_{i_k,\ldots,i_1}$ is a HNP. Hence, the image of $(\mathcal{R}-1)$ on $\mathcal{F}_2^*$ is as stated. □

**Lemma 2.** *In the mod* 2 *dual Leibniz-Hopf algebra all the palindromes and the* $(\mathcal{R}-1)$*-images of all HNPs are linearly independent.*

*Proof.* Suppose that there are distinct palindromes $p_1,\ldots,p_k$ and there are distinct HNPs $h_1,\ldots,h_r$ such that

$$p_1+\cdots+p_k = (\mathcal{R}-1)(h_1)+\cdots+(\mathcal{R}-1)(h_r). \tag{4.1}$$

The longest summands (i.e., the maximal-length summands) on the left of Eq. (4.1) are all palindromes and these summands cannot cancel, since $p_1,\ldots,p_k$ are different. On the other hand, the longest summands in $(\mathcal{R}-1)(h_j)$ are $h_j$ and its reverse, both of which are non-palindromes. Again, none of these summands can cancel since $h_1,\ldots,h_r$ are all different HNPs. Thus the maximal-length summands on the right of Eq.(4.1) are non-palindromes. This contradiction establishes that Eq.(4.1) can only hold if both sides are 0. Hence all the palindromes and the $(\mathcal{R}-1)$-images of all HNPs are linearly independent. □

By Lemma 2 all the palindromes and the $(\mathcal{R}-1)$-images of all HNPs are linearly independent. We will complete the proof of Theorem 4 by using the dimension argument that we used for the proof of Theorem 5. Elementary combinatorial calculations shows us that the number of palindromes in degree $n$ is $2^{n/2}$ if $n$ is even, and $2^{(n-1)/2}$ if $n$ is odd. (See [4, Section 3] for details). By Theorem 2 we also know the number of HNPs in even and odd degrees. From this we can see the number of palindromes and HNPs in each degree matches the dimension given in Theorem 3. Hence, all the palindromes and the $(\mathcal{R}-1)$-images of all HNPs form a basis.

*Proof of Proposition 1.* We prove (1), the proofs of (2) and (3) being similar. Let $x \in \mathrm{Ker}(\chi-1)\cap\mathrm{Ker}(\mathcal{C}-1)$, then $\chi(x)=x$ and $\mathcal{C}(x)=x$. This implies that $\chi(x)=x=(\mathcal{R}\circ\mathcal{C})(x)=\mathcal{R}(x)$ from which we can deduce $\mathrm{Ker}(\chi-1)\cap\mathrm{Ker}(\mathcal{C}-1)\subset\mathrm{Ker}(\mathcal{R}-1)$. This completes the proof.

□

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# DISTANCE $k$-DOMINATION IN SOME CYCLE RELATED GRAPHS

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*Abstract.* In this paper we determine distance $k-$domination number of graph obtained by duplication of vertices altogether by edges in cycle $C_n$, splitting graph of cycle $C_n$ as well as graph obtained by duplication of edges altogether by vertices in cycle $C_n$.

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*Keywords:* dominating set, domination number, distance domination, splitting graph

## 1. Introduction

Graph $G = (V(G), E(G))$, we mean simple, finite, connected and undirected graph. The open neighbourhood $N(v)$ of $v \in V(G)$ is the set of all vertices adjacent to $v$. That is, $N(v) = \{u \in V(G)/uv \in E(G)\}$. The closed neighbourhood of $v \in V(G)$ is the set $N[v] = N(v) \cup \{v\}$. The distance $d(u, v)$ between two vertices $u$ and $v$ is the length of the shortest $uv-$path in $G$, if exists, otherwise $d(u, v) = \infty$. The open $k-$neighbourhood $N_k(v)$ of a vertex $v \in V(G)$ is the set of all vertices of $G$ which are different from $v$ and at a distance at most $k$ from $v$ in $G$. That is $N_k(v) = \{u \in V(G)/d(u, v) \leq k\}$. The closed $k-$neighbourhood set is defined as $N_k[v] = N_k(v) \cup \{v\}$. It is obvious that $N(v) = N_1(v)$. A set $D \subseteq V(G)$ is called a dominating set if every vertex in $V(G) - D$ is adjacent to at least one vertex in $D$. For terminology and notation not defined here we follow West [13] and Haynes *et al.* [3]. The concept of distance dominating set was initiated by Slater [7] with special reference to communication network, while the term distance $k-$dominating set was given by Henning *et al.* [5]. For an integer $k \geq 1$, $D \subseteq V(G)$ is a distance $k-$dominating set of $G$, if every vertex in $V(G) - D$ is within the distance $k$ from some vertex $v \in D$. That is, $N_k[D] = V(G)$. The minimum cardinality among all the distance $k-$dominating sets of $G$ is called the distance $k-$domination number of $G$ and it is denoted by $\gamma_k(G)$. It is obvious that $\gamma(G) = \gamma_1(G)$. A distance $k-$dominating set of cardinality $\gamma_k(G)$ is called a $\gamma_k-$set. Many reserchers have explored the concept of distance $k-$domination in graphs. The distance domination number for cartesian products of two paths has been investigated by Klobucar [6]

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while the distance domination in the context of spanning tree of the graph is discussed by Griggs and Hutchinson [2]. The bounds on the distance two-domination number and the classes of graphs attaining these bounds are reported in Sridharan *et al.* [8]. Tian and Xu [9] have established upper bound for distance $k-$domination for connected graph $G$ and show that $\gamma_k(G) \leq \left\lfloor \frac{n-\Delta+k-1}{k} \right\rfloor$. The same authors in [10] have studied average distance and distance domination number and established an upper bound of average distance in terms of distance domination. Fischermann and Volkmann [1] have characterized the graphs whose distance $n-$domination number is equal to half of their number of vertices, when the diameter is greater or equal to $2n-1$. Vaidya and Kothari [12] have investigated distance $k-$domination number of total graph, shadow graph and middle graph of path $P_n$. The same authors in [11] have investigated distance $k-$domination number for the graphs obtained by graph operations on some standard graphs. For more bibliographic references on distance $k-$domination, the readers are advised to refer a survey article by Henning [4].

## 2. Results

**Proposition 1** ([4]). *Let $k \geq 1$ and $D$ be a distance $k$-dominating set of a graph $G$. Then $D$ is a minimal distance $k$-dominating set of $G$ if and only if each $d \in D$ has at least one of the following two properties hold.*
*(1) There exist a vertex $v \in V(G) - D$ such that $N_k(v) \cap D = \{d\}$.*
*(2) The vertex $d$ is at distance at least $k+1$ from every other vertex $d$ of $D$ in $G$.*

**Definition 1.** Duplication of a vertex $v$ by a new edge $e = v'v''$ of graph $G$ produces a new graph $G'$ such that $N(v') \cap N(v'') = \{v\}$.

**Theorem 1.** *If $G$ is a graph obtained by duplication of vertices altogether by edges in cycle $C_n$ $(n \leq 2k-1)$ then $\gamma_k(G) = 1$.*

*Proof.* Let $G$ be a graph obtained by duplication of vertices $v_1, v_2, \ldots, v_n$ by edges $u_{2i-1}u_{2i}$ $(1 \leq i \leq n)$ in cycle $C_n$. Then $D = \left\{ v_{\left\lfloor \frac{n}{2} \right\rfloor} \right\}$ is distance $k$-dominating set of $G$ as $n \leq 2k-1$. Hence $\gamma_k(G) = 1$. □

**Theorem 2.** *If $G$ is a graph obtained by duplication of vertices altogether by edges in cycle $C_n$ $(n > 2k-1)$ then*

$$\gamma_k(G) = \begin{cases} \left\lfloor \dfrac{n}{2k-1} \right\rfloor + 1, & \text{for } n \equiv 1, 2, \ldots, 2k-2 \,(\text{mod } 2k-1) \\ \dfrac{n}{2k-1} & \text{for } n \equiv 0 \,(\text{ mod } 2k-1) \end{cases}$$

*Proof.* Let $G$ be a graph obtained by duplication of vertices $v_1, v_2, \ldots, v_n$ by edges $u_{2i-1}u_{2i}$ $(1 \leq i \leq n)$ in cycle $C_n$. One can observe that $v_i$'s dominate more vertices than $u_i$'s. Here vertices from $v_{n-(k-1)}$ to $v_{k+1}$, $u_{2n-(2k-3)}$ to $u_{2n}$ and $u_1$

to $u_{2k}$ are dominated by a vertex $v_1$ at a distance $k$. Also $d(v_1, u_{2k}) = k$, and $d(u_{2n-(2k-3)}, v_1) = k$. Hence $2k-1$ consecutive vertices from $u_i$'s dominated by only one vertex $v_1$. Hence

$$\gamma_k(G) \geq \left\lfloor \frac{n}{2k-1} \right\rfloor \tag{2.1}$$

Now depending upon the number of vertices of $C_n$, consider the following subsets,
For $n \equiv 1, 2, \ldots, 2k-2 \,(\mathrm{mod}\ 2k-1)$

$$D = \left\{ v_{1+(2k-1)j} / 0 \leq j \leq \left\lfloor \frac{n}{2k-1} \right\rfloor \right\}, |D| = \left\lfloor \frac{n}{2k-1} \right\rfloor + 1,$$

for $n \equiv 0 \,(\mathrm{mod}\ 2k-1)$

$$D = \left\{ v_{1+(2k-1)j} / 0 \leq j < \frac{n}{2k-1} \right\}, |D| = \frac{n}{2k-1}.$$

We claim that each $D$ is a distance $k$-dominating set as
For $j \neq 0$,

$$d(v_{1+(2k-1)j}, v_{i+(2k-1)j}) \leq k, \text{ where } -k+1 \leq i \leq k+1, i \neq 1$$

$$d(v_{1+(2k-1)j}, u_{i+(2k-1)j}) \leq k, \text{ where } (3-2k) + (2k-1)j \leq i \leq 2k + (2k-1)j$$

and for $j = 0$,

$$d(v_1, v_i) \leq k, \text{ where } n-k+1 \leq i \leq n, 2 \leq i \leq k+1$$

$$d(v_1, u_i) \leq k, \text{ where } 1 \leq i \leq 2k, 2n-(2k-3) \leq i \leq 2n$$

Therefore for $j \neq 0$,

$$\begin{aligned} N_k(v_{1+(2k-1)j}) = \{ & v_{2+(2k-1)j}, v_{3+(2k-1)j}, \ldots, v_{(k+1)+(2k-1)j}, \ldots, v_{(2k-1)j}, \\ & v_{(2k-1)j-1}, \ldots, v_{(2k-1)j-(k-1)}, u_{1+2(2k-1)j}, u_{2(1+(2k-1)j)}, \\ & \ldots, u_{2k+2((2k-1)j)}, u_{(3-2k)+2((2k-1)j)}, u_{(4-2k)+2((2k-1)j)}, \\ & \ldots, u_{2(2k-1)j} \}. \end{aligned}$$

While for $j = 0$,

$$\begin{aligned} N_k(v_1) = \{ & v_2, v_3, \ldots, v_{k+1}, v_{n-(k-1)}, v_{n-k}, \ldots, v_n, u_1, u_2, \ldots, u_{2k}, u_{2n}, \\ & u_{2n-1}, \ldots, u_{2n-(2k-3)} \}. \end{aligned}$$

Then $N_k[D] = N_k[v_1] \cup N_k[v_{1+(2k-1)j}] = V(G)$.
For some $j = j_1$, $v_{1+(2k-1)j_1} \in D$ and for some $j = j_2$, $v_{1+(2k-1)j_2} \in D$,

$$d(v_{1+(2k-1)j_1}, v_{1+(2k-1)j_2}) = (j_2 - j_1)(2k-1) \geq k+1.$$

Which implies that every vertex $d$ of $D$ is at a distance $k+1$ apart from every other vertex of $D$ in $G$. Thus by Proposition 1 above defined $D$ is a minimal distance

$k-$dominating set of $G$ and by expression 2.1 it is also of minimum cardinality for $n > 2k-1$. Hence

$$\gamma_k(G) = \begin{cases} \left\lfloor \dfrac{n}{2k-1} \right\rfloor + 1, & \text{for } n \equiv 1, 2, ..., 2k-2 \,(\text{mod } 2k-1) \\ \dfrac{n}{2k-1} & \text{for } n \equiv 0 \,(\text{mod } 2k-1) \end{cases}$$

□

**Illustration 1.** *Distance* 3*−dominating set in graph G obtained by duplication of vertices in cycle* $C_{22}$ *altogether by edges is shown by solid vertices in* FIGURE *1.*

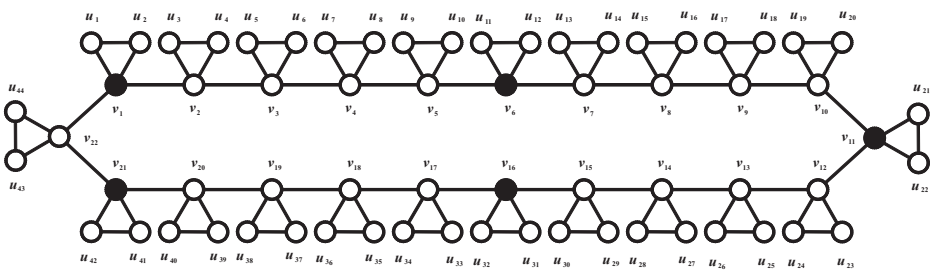


FIGURE 1.

**Definition 2.** For a graph $G$, the splitting graph $S'(G)$ of graph $G$ is obtained by adding a new vertex $v'$ corresponding to each vertex $v$ of $G$ such that $N(v) = N(v')$.

**Theorem 3.** *If* $n \leq 2k+1$, $k \neq 1$, *then* $\gamma_k(S'(C_n)) = 1$.

*Proof.* Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle $C_n$ and $u_1, u_2, \ldots, u_n$ be the vertices corresponding to $v_1, v_2, \ldots, v_n$ which are added to obtain $S'(C_n)$. Then $D = \left\{ v_{\left\lceil \frac{n}{2} \right\rceil} \right\}$ is distance $k$-dominating set of $S'(C_n)$ as $n \leq 2k+1$. Hence $\gamma_k(S'(C_n)) = 1$. □

**Theorem 4.** *If* $n > 2k+1$ *then*

$$\gamma_k(S'(C_n)) = \begin{cases} \left\lfloor \dfrac{n}{2k+1} \right\rfloor + 1, & \text{for } n \equiv 1, 2, ..., 2k \,(\text{mod } 2k+1) \\ \dfrac{n}{2k+1} & \text{for } n \equiv 0 \,(\text{mod } 2k+1) \end{cases}$$

*Proof.* Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle $C_n$ and $u_1, u_2, \ldots, u_n$ be the vertices corresponding to $v_1, v_2, \ldots, v_n$ which are added to obtain $S'(C_n)$. One can observe that $v_i$'s dominate more vertices than $u_i$'s at a distance $k$. In graph $S'(C_n)$

vertices $v_{n-(k-1)}$ to $v_{k+1}$ and $u_{n-(k-1)}$ to $u_{n+k}$ are dominated by a vertex $v_1$ at a distance $k$. Also $d(v_{n-(k-1)}, v_{k+1}) = 2k+1$, and $d(u_{n-(k-1)}, u_{k+1}) = 2k+1$. Hence $2k+1$ consecutive vertices from $v_i$'s and $(2k+1)$ consecutive vertices from $u_i$'s dominated by only one vertex at a distance $k$. Which implies that

$$\gamma_k\left(S'(C_n)\right) \geq \left\lfloor \frac{n}{2k+1} \right\rfloor \tag{2.2}$$

Now depending upon the number of vertices of $C_n$, consider the following subsets,
For $n \equiv 1, 2, \ldots, 2k \pmod{2k+1}$

$$D = \left\{ v_{1+(2k+1)j} / 0 \leq j \leq \left\lfloor \frac{n}{2k+1} \right\rfloor \right\}, |D| = \left\lfloor \frac{n}{2k+1} \right\rfloor + 1,$$

for $n \equiv 0 \pmod{2k+1}$

$$D = \left\{ v_{1+(2k+1)j} / 0 \leq j < \frac{n}{2k+1} \right\}, |D| = \frac{n}{2k+1}.$$

We claim that each $D$ is a distance $k$-dominating set as
for $j \neq 0$,

$$d(v_{1+(2k+1)j}, v_{i+(2k+1)j}) \leq k, \text{ where } -k+1 \leq i \leq k+1, i \neq 1$$

$$d(v_{1+(2k+1)j}, u_{i+(2k+1)j}) \leq k, \text{ where } -k+1 \leq i \leq k+1$$

while for $j = 0$

$$d(v_1, v_i) \leq k, \text{ where } n-k+1 \leq i \leq n, 1 \leq i \leq k+1$$

$$d(v_1, u_i) \leq k, \text{ where } 1 \leq i \leq k+1, n-k+1 \leq i \leq n$$

Therefore $j \neq 0$,

$$\begin{aligned} N_k(v_{1+(2k+1)j}) = \{ & v_{2+(2k+1)j}, v_{3+(2k+1)j}, \ldots, v_{(k+1)+(2k+1)j}, v_{(2k+1)j}, \\ & v_{(2k+1)j-1}, \ldots, v_{(2k+1)j-(k-1)}, u_{1+(2k+1)j}, u_{2+(2k+1)j}, \ldots, \\ & u_{(k+1)+(2k+1)j}, u_{(2k+1)j}, u_{(2k+1)j-1}, \ldots, u_{(2k+1)j-(k-1)} \}. \end{aligned}$$

While for $j = 0$,

$$\begin{aligned} N_k(v_1) = \{ & v_2, v_3, \ldots, v_{k+1}, v_n, v_{n-1}, v_{n-(-k+1)}, u_1, u_2, \ldots, u_{k+1}, u_{n-k+1}, \\ & u_{n-k+2}, \ldots, u_n \}. \end{aligned}$$

Then $N_k[D] = N_k[v_1] \cup N_k[v_{1+(2k+1)j}] = V(S'(C_n))$.
For some $j = j_1$, $v_{1+(2k+1)j_1} \in D$ and for some $j = j_2$, $v_{1+(2k+1)j_2} \in D$,

$$d(v_{1+(2k+1)j_1}, v_{1+(2k+1)j_2}) = (j_2 - j_1)(2k+1) \geq k+1$$

This implies that every vertex $d$ of $D$ is at a distance $k+1$ apart from every other vertex of $D$ in $S'(C_n)$. Thus by Proposition 1 above defined $D$ is a minimal distance

$k-$dominating set of $G$ and by expression 2.2 it is also of minimum cardinality for $n > 2k-1$. Hence

$$\gamma_k(S'(C_n)) = \begin{cases} \left\lfloor \dfrac{n}{2k+1} \right\rfloor + 1, & \text{for } n \equiv 1, 2, \ldots, 2k \pmod{2k+1} \\ \dfrac{n}{2k+1} & \text{for } n \equiv 0 \pmod{2k+1} \end{cases}$$

□

**Illustration 2.** *Distance* 4*−dominating set in* $S'(C_{20})$ *is shown by solid vertices in* FIGURE 2.

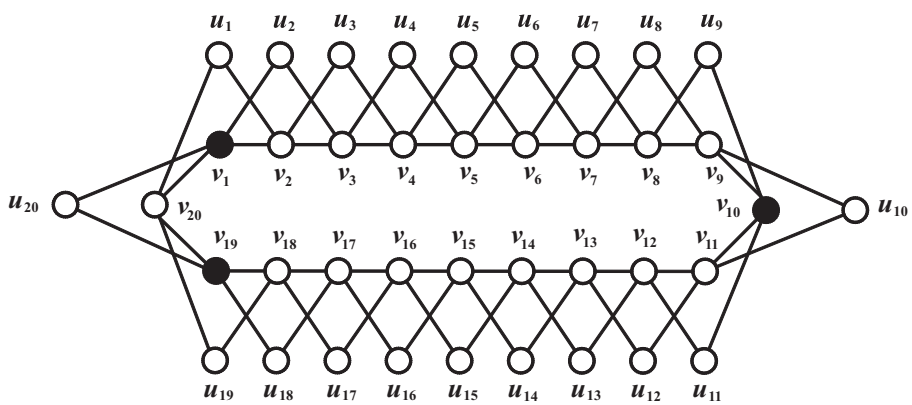


FIGURE 2.

**Definition 3.** Duplication of an edge $e = uv$ by a new vertex $w$ in a graph $G$ produces a new graph $G'$ such that $N(w) = \{u, v\}$.

**Theorem 5.** *If $G$ is a graph obtained by duplication of edges altogether by vertices in cycle $C_n$*
*$(n \leq 2k)$ then $\gamma_k(G) = 1$.*

*Proof.* Let $G$ be a graph obtained by duplication of edges $v_i v_{i+1}$ altogether by vertices $u_i$, $(1 \leq i < n)$ in cycle $C_n$. Then $\left\{ v_{\left\lfloor \frac{n}{2} \right\rfloor} \right\}$ is distance $k$-dominating set of $G$ as $n \leq 2k$. Hence $\gamma_k(G) = 1$. □

**Theorem 6.** *If G is a graph obtained by duplication of edges altogether by vertices in cycle $C_n$ $(n > 2k)$ then*

$$\gamma_k(G) = \begin{cases} \left\lfloor \frac{n}{2k} \right\rfloor + 1, & \text{for } n \equiv 1, 2, ..., 2k-1 \,(\text{mod } 2k) \\ \frac{n}{2k} & \text{for } n \equiv 0 (\text{mod } 2k) \end{cases}$$

*Proof.* Let G be a graph obtained by duplication of edges $v_i v_{i+1}$ altogether by vertices $u_i$ $(1 \le i < n)$ in cycle $C_n$. One can observe that $v_i$'s dominate more vertices than $u_i$'s at a distance $k$. In graph $G$ vertices $v_{n-k}$ to $v_{k+1}$ and $u_{n-(k-1)}$ to $u_k$ are dominated by a vertex $v_1$ at a distance $k$. Also $d(u_{n-(k-1)}, v_1) = k$ and $d(v_1, u_k) = k$. Hence $2k$ consecutive vertices of $u_i$'s are dominated by only one vertex at a distance $k$. Hence

$$\gamma_k(G) \ge \left\lfloor \frac{n}{2k} \right\rfloor \tag{2.3}$$

Now depending upon the number of vertices of $C_n$, consider the following subsets.
For $n \equiv 1, 2, \ldots, 2k-1$ (mod $2k$)

$$D = \left\{ v_{1+(2k)j} / 0 \le j \le \left\lfloor \frac{n}{2k} \right\rfloor \right\}, |D| = \left\lfloor \frac{n}{2k} \right\rfloor + 1,$$

for $n \equiv 0$ (mod $2k$)

$$D = \left\{ v_{1+(2k)j} / 0 \le j < \frac{n}{2k} \right\}, |D| = \frac{n}{2k}.$$

Now we claim that each $D$ is a distance $k$-dominating set as
for $j \ne 0$,

$$d(v_{1+2kj}, v_{i+2kj}) \le k, \text{ where } -k+1 \le i \le k+1, i \ne 1,$$

$$d(v_{1+2kj}, u_{i+2kj}) \le k, \text{ where } -k+1 \le i \le k$$

while for $j = 0$,

$$d(v_1, v_i) \le k, \text{ where } n-k+1 \le i \le n, 2 \le i \le k+1$$

$$d(v_1, u_i) \le k, \text{ where } n-k+1 \le i \le n, 1 \le i \le k$$

Therefore for $j \ne 0$,

$$N_k(v_{1+2kj}) = \{v_{2+2kj}, v_{3+2kj}, \ldots, v_{(k+1)+2kj}, v_{2kj}, v_{2kj-1}, \ldots, v_{2kj-(k-1)}, u_{1+2kj}, u_{2+2kj}, \ldots, u_{k+2kj}, u_{2kj}, u_{2kj-1}, \ldots, u_{2kj-(k-1)}\}.$$

While for $j = 0$,

$$N_k(v_1) = \{v_2, v_3, \ldots, v_{k+1}, v_{n-k+1}, v_{n-k+2}, \ldots, v_n, u_1, u_2, \ldots, u_k, u_{n-k+1}, u_{n-k+2}, \ldots, u_n\}.$$

This implies that $N_k[D] = N_k[v_1] \cup N_k[v_{1+(2k)j}] = V(G)$.
For some $j = j_1$, $v_{1+(2k)j_1} \in D$ and for some $j = j_2$, $v_{1+(2k)j_2} \in D$,

$$d(v_{1+(2k)j_1}, v_{1+(2k)j_2}) = (j_2 - j_1)2k \geq k+1$$

Which implies that every vertex $d$ of $D$ is at a distance $k+1$ apart from every other vertex of $D$ in $G$. Thus by Proposition 1 above defined $D$ is a minimal distance $k-$dominating set of $G$ and by expression 2.3 it is also of minimum cardinality for $n > 2k$. Hence

$$\gamma_k(G) = \begin{cases} \left\lfloor \dfrac{n}{2k} \right\rfloor + 1, & \text{for } n \equiv 1, 2, \ldots, 2k-1 \,(\text{mod } 2k) \\ \dfrac{n}{2k} & \text{for } n \equiv 0 \,(\text{mod } 2k) \end{cases}$$

□

**Illustration 3.** *Distance* $2-$*dominating set in graph* $G$ *obtained by duplication of edges in* $C_{18}$ *altogether by vertices is shown by solid vertices in* FIGURE 3.

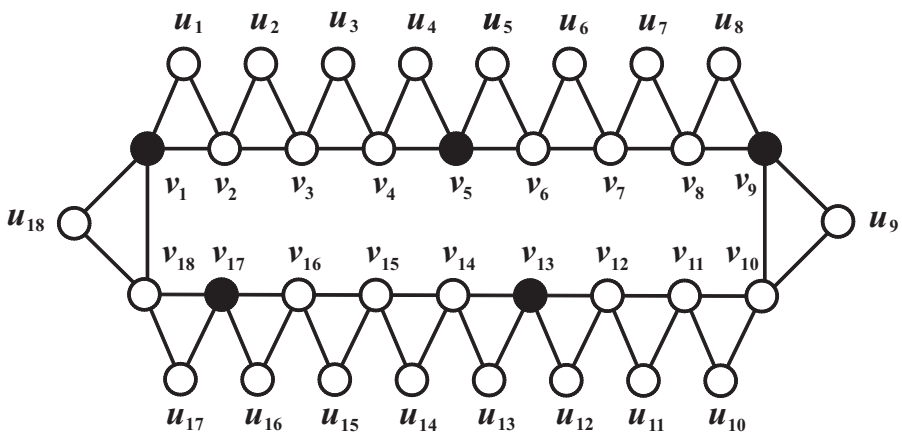


FIGURE 3.

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# ON INVESTIGATION OF SOME NON-LINEAR INTEGRAL BOUNDARY VALUE PROBLEM

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*Abstract.* We suggest a constructive approach for the solvability analysis and approximate solution of certain types of partially solved Lipschitzian differential systems with mixed two-point and integral non-linear boundary conditions. The practical application of the suggested technique is shown on a numerical example.

2010 *Mathematics Subject Classification:* 34B15

*Keywords:* implicit differential systems, non-linear integral boundary conditions, parametrization technique, successive approximations

## 1. Problem setting

This article uses the approach proposed in [2], [5], [4] in the case of the following non-linear boundary value problem with mixed two-point and integral restrictions

$$\frac{dx(t)}{dt} = f\left(t, x(t), \frac{dx(t)}{dt}\right), t \in [a,b], \tag{1.1}$$

$$g\left(x(a), x(b), \int_a^b h(s, x(s))ds\right) = d. \tag{1.2}$$

We suppose that $f : [a,b] \times D \times D_1 \to \mathbb{R}^n$ is a continuous function defined on a bounded sets $D \subset \mathbb{R}^n$ , $D^1 \subset \mathbb{R}^n$ (domain $D := D_\rho$ will be concretized later, see (1.8), $D^1$ is given ) and $d \in \mathbb{R}^n$ is a given vector. Moreover $f$, $g$: $D \times D \times D_2 \to \mathbb{R}^n$ and $h : [a,b] \times D \to \mathbb{R}^n$ are Lipschitzian in the following form

$$|f(t,u,v) - f(t,\widetilde{u},\widetilde{v})| \le K_1 |u - \widetilde{u}| + K_2 |v - \widetilde{v}|, \tag{1.3}$$

$$|g(u,w,p) - g(\widetilde{u},\widetilde{w},\widetilde{p})| \le K_3 |u - \widetilde{u}| + K_4 |w - \widetilde{w}| + K_5 |p - \widetilde{p}| \tag{1.4}$$

$$|h(t,u) - h(t,\widetilde{u})| \le K_6 |u - \widetilde{u}| \tag{1.5}$$

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for any $t \in [a,b]$ fixed, all $\{u,\widetilde{u}\} \subset D$, $\{v,\widetilde{v}\} \subset D^1$, $\{w,\widetilde{w}\} \subset D$, $\{p,\widetilde{p}\} \subset D_2$, where

$$D_2 := \left\{ \int_a^b h(t,x(t))dt : t \in [a,b],\ x \in D \right\}$$

and $K_1 - K_5$ are non-negative square matrices of dimension $n$. The inequalities between vectors are understood componentwise. A similar convention is adopted for the "absolute value", "max", "min" operations. The symbol $I_n$ stands for the unit matrix of dimension $n$, $r(K)$ denotes a spectral radius of a square matrix $K$.

By the solution of the problem (1.1), (1.2) we understand a continuously differentiable function with property (1.2) satisfying (1.1) on $[a,b]$.

We fix certain bounded sets $D_a \subset \mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$ and focus on the solutions $x$ of the given problem with property $x(a) \in D_a$ and $x(b) \in D_b$. Instead of the non-local boundary value problem (1.1), (1.2), we consider the parameterized family of two-point "model -type " problems with simple separated conditions

$$\frac{dx(t)}{dt} = f\left(t, x(t), \frac{dx(t)}{dt}\right),\ t \in [a,b], \tag{1.6}$$

$$x(a) = z,\ x(b) = \eta, \tag{1.7}$$

where $z = (z_1, z_2, ..., z_n)$, $\eta = (\eta_1, \eta_2, ..., \eta_n)$ are considered as parameters.

If $z \in \mathbb{R}^n$ and $\rho$ is a vector with non-negative components, $O(z,\rho) := \{\xi \in \mathbb{R}^n : |\xi - z| \le \rho\}$ stands for the componentwise $\rho$ -neighbourhood of $z$. For given two bounded connected sets $D_a \subset \mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$, introduce the set

$$D_{a,b} := (1-\theta)z + \theta\eta,\ z \in D_a,\ \eta \in D_b,\ \theta \in [0,1]$$

and its componentwise $\rho-$neighbourhood by putting

$$D = D_\rho := O(D_{a,b},\rho) = \bigcup_{\xi \in D_{a,b}} O(\xi,\rho) \tag{1.8}$$

We suppose that

$$r(K_2) < 1,\ r(Q) < 1, \tag{1.9}$$

where

$$Q := \frac{3(b-a)}{10} K, \tag{1.10}$$

$$K = K_1 + K_2\ [I_n - K_2]^{-1} K_1 = [I_n - K_2]^{-1} K_1.$$

On the base of function $f : [a,b] \times D \times D^1 \to \mathbb{R}^n$ we introduce the vector

$$\delta_{[a,b],D,D^1}(f) := \frac{1}{2}\left[\max_{(t,x,y)\in[a,b]\times D\times D^1} f(t,x,y) - \min_{(t,x,y)\in[a,b]\times D\times D^1} f(t,x,y)\right] \tag{1.11}$$

and suppose that the $\rho-$neighbourhood in (1.8) is such that

$$\rho \geq \frac{b-a}{2}\delta_{[a,b],D,D^1}(f). \tag{1.12}$$

## 2. Main statements

The investigation of the solutions of parameterized problem (1.6) and (1.7) is connected with the properties of the following special sequence of functions well posed on the interval $t \in [a,b]$

$$x_0(t,z,\eta) = z + \frac{t-a}{b-a}[\eta - z] = \left[1 - \frac{t-a}{b-a}\right] z + \frac{t-a}{b-a}\eta,\ t \in [a,b], \tag{2.1}$$

$$x_{m+1}(t,z,\eta) = z + \int_a^t f\left(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{ds}\right) ds -$$

$$-\frac{t-a}{b-a}\int_a^b f\left(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{ds}\right) ds + \frac{t-a}{b-a}[\eta - z],\ t \in [a,b], \tag{2.2}$$

$m = 0,1,2,\ldots,$

**Theorem 1.** *Let assumptions (1.3)-(1.5) and (1.9) hold. Then, for all fixed $(z,\eta) \in D_a \times D_b$ :*

*1. The functions of the sequence (2.2) are continuously differentiable functions on the interval $t \in [a,b]$, have values in the domain $D = D_\rho$ and satisfy the two-point separated boundary conditions (1.7).*

*2. The sequence of functions (2.2) in $t \in [a,b]$ converges uniformly as $m \to \infty$ to the limit function*

$$x_\infty(t,z,\eta) = \lim_{m\to\infty} x_m(t,z,\eta), \tag{2.3}$$

*satisfying the two-point separated boundary conditions (1.7).*

*3. The limit function $x_\infty(t,z,\eta)$ is the unique continuously differentiable solution of the integral equation*

$$x(t) = z + \int_a^t f\left(s, x(s), \frac{dx(s)}{ds}\right) ds - \frac{t-a}{b-a}\int_a^b f\left(s, x(s), \frac{dx(s)}{ds}\right) ds +$$

$$+ \frac{t-a}{b-a}[\eta - z], \tag{2.4}$$

*i.e. it is the solution of the Cauchy problem for the modified system of integro-differential equations :*

$$\frac{dx}{dt} = f\left(t, x, \frac{dx(t)}{dt}\right) + \frac{1}{b-a}\Delta(z,\eta),\ x(a) = z \tag{2.5}$$

*where* $\Delta(z,\eta) : D_a \times D_b \to \mathbb{R}^n$ *is a mapping given by formula*

$$\Delta(z,\eta) := [\eta - z] - \int_a^b f\left(s, x_\infty(s,z,\eta), \frac{dx_\infty(s,z,\eta)}{ds}\right) ds. \tag{2.6}$$

*4.The following error estimate holds:*

$$|x_\infty(t,z,\eta) - x_m(t,z,\eta)| \leqslant$$

$$\leqslant \frac{10}{9}\alpha_1(t,a,b-a) Q^m (1_n - Q)^{-1} \delta_{[a,b],D,D^1}(f), \tag{2.7}$$

*for any* $t \in [a,b]$ *and* $m \geq 0$, *where* $\delta_{[a,b],D,D^1}(f)$ *is given in (1.11) and*

$$\alpha_1(t,a,b-a) = 2(t-a)\left(1 - \frac{t-a}{b-a}\right), \ \alpha_1(t,a,b-a) \leq \frac{b-a}{2}. \tag{2.8}$$

*Proof.* The validity of this statement can be established similarly to Theorem 1 in [4]. □

**Theorem 2.** *Under the assumption of Theorem 1, the limit function* $x_\infty(t,z,\eta) : [a,b] \times D_a \times D_b \to \mathbb{R}^n$ *defined by (2.3) is a continuously differentiable solution of the original BVP (1.1), (1.2) if and only if the pair of vectors* $(z,\eta)$ *satisfies the system of* $2n$ *determining algebraic equations*

$$\begin{cases} \Delta(z,\eta) = \eta - z - \int_a^b f\left(s, x_\infty(s,z,\eta), \frac{dx_\infty(s,z,\eta)}{ds}\right) ds = 0, \\ \Lambda(z,\eta) = g\left(x_\infty(a,z,\eta), x_\infty(b,z,\eta), \int_a^b h(s, x_\infty(s,z,\eta)) ds\right) - d = 0. \end{cases} \tag{2.9}$$

Note, that similarly as in [3], the solvability of the determining system (2.9) on the base of (1.3)-(1.5) and (1.9) can be established by studying its $m-$th approximate versions:

$$\begin{cases} \Delta_m(z,\eta) = \eta - z - \int_a^b f\left(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{ds}\right) ds = 0, \\ \Lambda_m(z,\eta) = g\left(x_m(a,z,\eta), x_m(b,z,\eta), \int_a^b h(s, x_m(s,z,\eta)) ds\right) - d = 0. \end{cases} \tag{2.10}$$

where $m$ is fixed.

**Lemma 1.** *Under the assumptions of Theorem 1, for the exact and approximate determining functions defined by (2.9) and (2.10) for any* $(z,\eta) \in D_a \times D_b$ *and* $m \geq 1$

*hold the following estimates:*

$$|\Delta(z,\eta)-\Delta_m(z,\eta)| \le \frac{10(b-a)^2}{27} K Q^m (1_n - Q)^{-1} \delta_{[a,b],D,D_1}(f), \tag{2.11}$$

$$|\Lambda(z,\eta)-\Lambda_m(z,\eta)| \le \frac{5(b-a)}{9}[K_3+K_4+ \\ +(b-a)K_5K_6]\ Q^m(1_n-Q)^{-1}\delta_{[a,b],D,D_1}(f), \tag{2.12}$$

*where the matrix $Q$ and the vector $\delta_{[a,b],D,D_1}(f)$ are given respectively in (1.10) and (1.11).*

*Proof.* Let us fix an arbitrary $(z,\eta) \in D_a \times D_b$. Direct computation gives that

$$\int_a^b \alpha_1(t,a,b-a)dt = \frac{(b-a)^2}{3}.$$

On the base of (1.1) and (1.3), when $u \ne \widetilde{u}$, we have

$$|f(t,u,v)-f(t,\widetilde{u},\widetilde{v})| \le K|u-\widetilde{u}|,$$

where matrix $K$ is given in (1.10). Taking into account (2.7) we obtain

$$|\Delta(z,\eta)-\Delta_m(z,\eta)| =$$

$$= \left| \int_a^b f\left(s, x_\infty(s,z,\eta), \frac{dx_\infty(s,z,\eta)}{dt}\right) ds - \int_a^b f\left(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{dt}\right) ds \right| \le$$

$$\le K \int_a^b \frac{10}{9}\alpha_1(s,a,b) Q^m (1_n-Q)^{-1}\delta_{[a,b],D,D_1}(f)ds =$$

$$= \frac{10(b-a)^2}{27} K Q^m (1_n-Q)^{-1}\delta_{[a,b],D,D_1}(f),$$

which proves (2.11).

From (2.9) and (2.10) using the Lipschitz conditions (1.4), (1.5) and estimates (2.7), (2.8), we obtain

$$|\Lambda(z,\eta)-\Lambda_m(z,\eta)| = \left| g\left(x_\infty(a,z,\eta), x_\infty(b,z,\eta), \int_a^b h(s,x_\infty(b,z,\eta))ds\right) - \right.$$

$$\left. - g\left(x_m(a,z,\eta), x_m(b,z,\eta), \int_a^b h(s,x_m(s,z,\eta))ds\right)\right| \le$$

$$\le K_3|x_\infty(a,z,\eta)-x_m(a,z,\eta)| + K_4|x_\infty(b,z,\eta)-x_m(b,z,\eta)| +$$

$$+(b-a)\,K_5K_6\,|x_\infty(t,z,\eta)-x_m(t,z,\eta)|\leq$$
$$\leq\frac{5(b-a)}{9}\left[K_3+K_4+(b-a)\,K_5K_6\right]\;Q^m\,(1_n-Q)^{-1}\,\delta_{[a,b],D,D_1}(f),$$

i.e. (2.12) holds also. □

Based on both exact and approximate determining systems (2.9) and (2.10) let us introduce the mappings $H:D_a\times D_b\to\mathbb{R}^{2n}$ and $H_m:D_a\times D_b\to\mathbb{R}^{2n}$ by setting

$$H(z,\eta)=\begin{bmatrix}[\eta-z]-\int\limits_a^b f\left(s,x_\infty(a,z,\eta),x_\infty(b,z,\eta),\int\limits_a^b h(s,x_\infty(s,z,\eta))\right)ds,\\ g\left(x_\infty(a,z,\eta),x_\infty(b,z,\eta),\int\limits_a^b h(s,x_\infty(b,z,\eta))ds\right)-d,\end{bmatrix}\tag{2.13}$$

$$H_m(z,\eta)=\begin{bmatrix}[\eta-z]-\int\limits_a^b f\left(s,x_m(a,z,\eta),x_m(b,z,\eta),\int\limits_a^b h(s,x_m(s,z,\eta))\right)ds,\\ g\left(x_m(a,z,\eta),x_m(b,z,\eta),\int\limits_a^b h(s,x_m(b,z,\eta))ds\right)-d,\end{bmatrix}\tag{2.14}$$

$(z,\eta)\in D_a\times D_b$. We see from Theorem 2 that the critical points of the vector field $H$ of the form (2.13) determine solutions of the non-linear boundary value problem (1.1)-(1.2). The next statement establishes a similar result based upon properties of vector field $H_m$ explicity known from (2.14).

**Theorem 3.** *Assume that the conditions of Lemma 1 hold. Moreover, one can specify an $m\geq 1$ and a set*

$$\Gamma:=D_1\times D_2\subset\mathbb{R}^{2n},$$

*where $D_1\subset D_a, D_2\subset D_b$ are certain bounded open sets such that the mapping $H_m$ satisfies the relation*

$$|H_m(z,\eta)|\rhd_{\partial\Gamma}\begin{bmatrix}\frac{10(b-a)^2}{27}KQ^m\,(1_n-Q)^{-1}\,\delta_{[a,b],D,D_1}(f)\\ \frac{5(b-a)}{9}\left[K_3+K_4+(b-a)\,K_5K_6\right]\;Q^m\,(1_n-Q)^{-1}\,\delta_{[a,b],D,D_1}(f)\end{bmatrix}\tag{2.15}$$

*on the boundary $\partial\Gamma$ of the set $\Omega$. If, in addition*

$$\deg(H_m,\Omega,0)\neq 0,\tag{2.16}$$

*then there exists a pair $(z^*,\eta^*)\in D_1\times D_2$ for which the function*

$$x^*(\cdot):=x_\infty\left(\cdot,z^*,\eta^*\right)$$

*is a solution of the non-linear boundary value problem (1.1)-(1.2).*

In (2.15) the binary relation $\rhd_{\partial\Gamma}$ is defined in [1] as a kind of strict inequality for vector functions and it means that at every point on the boundary $\partial\Gamma$ at least one of the components of the vector $|H_m(z,\eta)|$ is greater than the corresponding component of the vector on the right-hand side . The degree in (2.16) is the Brouwer degree because all the vectors fields are finite-dimensional. Likewise, all the terms on the right-hand side of (2.15) are computed explicitly e.g. by using computer algebra system.

*Proof.* The proof can be carried out similarly as in Theorem 4 from [3]. □

## 3. Example

Let us apply the approach described above to the system of differential equations

$$\begin{cases} \frac{dx_1(t)}{dt}=\frac{1}{2}x_2^2(t)-t\frac{dx_2(t)}{dt}x_1(t)+\frac{1}{32}t^3-\frac{1}{32}t^2+\frac{9}{40}t \\ \frac{dx_2(t)}{dt}=\frac{1}{2}\frac{dx_1(t)}{dt}x_1(t)-t^2x_2(t)+\frac{15}{64}t^3+\frac{1}{80}t+\frac{1}{4} \end{cases}, \quad t\in[0,1],$$

considered with non-linear boundary conditions

$$x_1(0)\,x_2(1)+\left[\int_0^1 x_1(s)ds\right]^2=-\frac{311}{14400},$$

$$x_1(1)x_2(0)-\int_0^1 x_2(s)ds=-\frac{1}{8}.$$

Introduce the vector of parameters $z=col(z_1,z_2)$, $\eta=col(\eta_1,\eta_2)$. Let us consider the following choice of the subsets $D_a$, $D_b$ and $D^1$ :

$$D_a=D_b=\{(x_1,x_2):-0.1\le x_1\le 0.2,\ -0.2\le x_2\le 0.3\},$$

$$D^1=\left\{\left(\frac{dx_1}{dt},\frac{dx_2}{dt}\right):-0.1\le\frac{dx_1}{dt}\le 0.3,\ -0.1\le\frac{dx_2}{dt}\le 0.3\right\}.$$

In this case $D_{a,b}=D_a=D_b$. For $\rho=col(\rho_1,\rho_2)$ involved in (1.12), we choose the vector $\rho=col(0.4;0.4)$. Then, in view of (2.13) the sets (1.8) and $D_2$ takes the form:

$$D=D_\rho=\{(x_1,x_2):-0.5\le x_1\le 0.6,\ -0.6\le x_2\le 0.7\}$$

and

$$D_2=\{(x_1,x_2):0.25\le x_1\le 0.36,\ -0.6\le x_2\le 0.7\}.$$

A direct computation shows that the conditions (1.3)-(1.5) hold with

$$K_1=\begin{bmatrix}0.3 & 0.7\\ 0.15 & 1\end{bmatrix}, K_2=\begin{bmatrix}0 & 0.6\\ 0.3 & 0\end{bmatrix},\ K_3=\begin{bmatrix}0.3 & 0\\ 0 & 0.2\end{bmatrix},$$

$$K_4=\begin{bmatrix}0 & 0.2\\ 0.3 & 0\end{bmatrix}, K_5=\begin{bmatrix}1.2 & 0\\ 0 & 1\end{bmatrix},\ K_6=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

TABLE 1.

| m | $z_1$ | $z_2$ | $\eta_1$ | $\eta_2$ |
|---|---|---|---|---|
| 0 | -0.089643967 | -0.0002812586 | 0.03176891 | 0.25026338 |
| 1 | -0.0994489263 | 0.00051937347 | 0.0255001973 | 0.2504687527 |
| 4 | -0.0999998827 | $7.744981\cdot 10^{-8}$ | 0.02500007591 | 0.25000011 |
| 6 | -0.1000000004 | $-2.263731\cdot 10^{-10}$ | 0.02499999973 | 0.2499999996 |
| Exact | -0.1 | 0 | 0.025 | 0.25 |

and therefore $r(K_2) = \sqrt{0.18} < 1$, and in (1.10) the matrix

$$K = \begin{bmatrix} 0.4756097561 & 1.585365854 \\ 0..2926829268 & 1.475609756 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.07134146342 & 0.2378048781 \\ 0.04390243902 & 0.2213414634 \end{bmatrix}, \; r(Q) = 0.273090089272152 < 1.$$

Furthermore, in view of (1.11)

$$\delta_{[a,b],D,D^1}(f) := \frac{1}{2}\left[\max_{(t,x,y)\in[a,b]\times D\times D^1} f(t,x,y) - \min_{(t,x,y)\in[a,b]\times D\times D^1} f(t,x,y)\right] =$$

$$= \begin{bmatrix} 0.31 \\ 0.7325 \end{bmatrix},$$

$$\rho = \begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix} \geq \frac{b-a}{2}\delta_{[a,b],D,D^1}(f) = \begin{bmatrix} 0.155 \\ 0.36625 \end{bmatrix}.$$

We thus see that all conditions of Theorem 1 are fulfilled, and the sequence of functions (2.2) for this example is uniformly convergent.

Applying Maple 14, we can carried out the calculations.

It is easy to check that

$$x_1^*(t) = \frac{t^2}{8} - \frac{1}{10}, \quad x_2^*(t) = \frac{t}{4}$$

is an exact continuously differentiable solution of the problem (1.1), (1.2). For a different number of approximations $m$ we obtain from (2.10) the following numerical values for the introduced parameters, which are presented in Table 3.

On the Figure 1 one can see the graphs of the exact solution (solid line) and its zero ($\diamond$) and sixth approximation ($\times$) for the first and second coordinates.

The error of the sixth approximation ($m = 6$) for the first and second components:

$$\max_{t\in[0,1]} \left|x_1^*(t) - x_{61}(t)\right| \leq 1\cdot 10^{-9}, \quad \max_{t\in[0,1]} \left|x_2^*(t) - x_{62}(t)\right| \leq 5\cdot 10^{-9}.$$

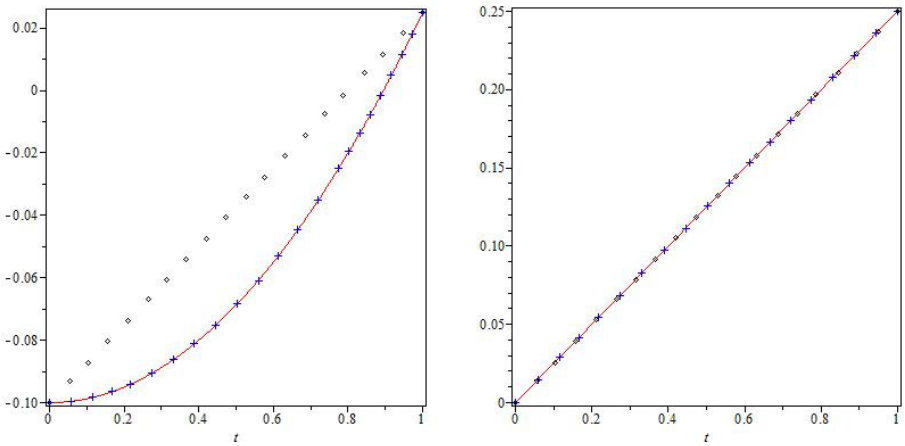


FIGURE 1.

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