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A NEW NUMERICAL TECHNIQUE FOR SOLVING FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

OMER ACAN AND DUMITRU BALEANU

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Abstract. We propose conformable Adomian decomposition method (CADM) for fractional partial differential equations (FPDEs). This method is a new Adomian decomposition method (ADM) based on conformable derivative operator (CDO) to solve FPDEs. At the same time, conformable reduced differential transform method (CRDTM) for FPDEs is briefly given and a numerical comparison is made between this method and the newly introduced CADM. In applied science, CADM can be used as an alternative method to obtain approximate and analytical solutions for FPDEs as CRDTM. In this study, linear and non-linear three problems are solved by these two methods. In these methods, the obtained solutions take the form of a convergent series with easily computable algorithms. For the applications, the obtained results by these methods are compared to each other and with the exact solutions. When applied to FPDEs, it is seen that CADM approach produces easy, fast and reliable solutions as CRDTM.

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1. INTRODUCTION

Fractional differential equations have a substantial contributions in fields, e.g. optics, biology, physics, chemistry, mathematics, fluids mechanics, applied mathematics, and engineering [18, 26, 40–42]. We recall that finding an analytical solutions to these problems is not always possible [9–11, 20, 24, 27–29, 37–39]. As a result, it becomes crucial to manage these problems appropriately and solve them or develop the required solutions. ADM, which is introduced [4–6] in the 1980's, is one of the important mathematical methods used to solve many problems in real world. Since then, a number of studies have been conducted on ADM such as linear and non-linear, homogeneous and non-homogeneous operator equations which including fractional or non-fractional ODEs, PDEs, integral equations, integro-differential equations, etc. (see [12, 13, 15, 16, 25, 30, 32–36] and references therein). A new derivative called CDO was suggested [1, 7, 22]. By the help of it, the behaviors of many problems were investigated and some solutions techniques were applied

[1–3, 7, 8, 14, 17, 19, 21, 23, 31]. This new subject gives academicians an opportunity to study further in many engineering, physical and applied mathematics problems.

The aim of this study is to introduce CADM by using CDO and ADM for the first time in the literature. This method can be used to solve many linear and non-linear FPDEs. We will briefly mentioned CRDTM to compare our CADM with it. The problems will be solved both by the CRDTM and the first proposed CADM. The obtained solutions by these methods will be compared. Thus, in section 2, we present some basic definitions and important properties of CDO. Next, in section 3, we propose CADM. In sections 4, we introduce CRDTM to compare with our method. In section 5, we give applications of CADM and CRDTM. We give the conclusion in the final section.

2. BASIC DEFINITIONS

Definition 1. Given a function $f_1 : [0, \infty) \rightarrow \mathbb{R}$. Then the CDO of f_1 order α is defined by [1, 7, 22]:

$$(T_\alpha f_1)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f_1(t + \varepsilon t^{1-\alpha}) - f_1(t)}{\varepsilon}$$

for all $t > 0, \alpha \in (0, 1]$.

Lemma 1 ([1, 7, 22]). Let f_1, g_1 be α and β -differentiable at a point $t > 0$ for α . Then

- (i) $T_\alpha(af_1 + bg_1) = a(T_\alpha f_1) + b(T_\alpha g_1)$ for all $a, b \in \mathbb{R}$ and $\alpha \in (0, 1]$,
- (ii) $T_\alpha(f_1(t)) = 0$, for constant function $f_1(t) = \lambda, \alpha \in (0, 1]$,
- (iii) $T_\alpha(f_1 g_1) = f_1(T_\alpha g_1) + g_1(T_\alpha f_1), \alpha \in (0, 1]$,
- (iv) $T_\alpha(f_1/g_1) = \frac{g_1(T_\alpha f_1) - f_1(T_\alpha g_1)}{g_1^2}, \alpha \in (0, 1]$,
- (v) If f_1 is n times differentiable at t , then $T_\alpha(f_1(t)) = t^{[\alpha]-\alpha} f_1^{([\alpha])}(t), \alpha \in (n, n+1]$. Where $[\alpha]$ is the smallest integer greater than or equal to α .

Lemma 2. [1] Suppose that f_1 is infinitely α -differentiable function for $\alpha \in (0, 1]$ at a neighborhood of a point t_0 . Then f_1 has the conformable power series expansion:

$$f_1(t) = \sum_{k=0}^{\infty} \frac{\left({}_t T_\alpha^{(k)} f_1\right)(t_0) (t-t_0)^{\alpha k}}{\alpha^k k!}, \quad t_0 < t < t_0 + R^{1/\alpha}, R > 0.$$

Here $\left({}_t T_\alpha^{(k)} f_1\right)(t_0)$ denotes the application of the conformable derivative for k times.

3. CONFORMABLE ADOMIAN DECOMPOSITION METHOD

We will briefly introduce CADM for FPDES in this section. We write the non-linear FPDES in the standard operator form

$$L_\alpha(u(x, t)) + R(u(x, t)) + N(u(x, t)) = g(x, t) \quad (3.1)$$

where $L_\alpha = {}_\alpha T$ is a linear operator with conformable derivative of order α ($n < \alpha \leq n+1$), N is a non-linear operator, R is the other part of the linear operator and $g(x, t)$ is a non-homogeneous term. If the linear operator in eq. (3.1) is applied to Lemma 1, the following equation is obtained:

$$t^{[\alpha]-\alpha} \frac{\partial^{[\alpha]}}{\partial t^{[\alpha]}} u(x, t) + R(u(x, t)) N(u(x, t)) = g(x, t). \quad (3.2)$$

Applying $L_\alpha^{-1} = \int_0^t \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} \frac{1}{\xi_n^{[\alpha]-\alpha}} (\cdot) d\xi_n d\xi_{n-1} \cdots d\xi_1$, ($n < \alpha \leq n+1$) the inverse of operator, to both sides of (3.2), it is obtained as

$$L_\alpha^{-1} L_\alpha(u(x, t)) = L_\alpha^{-1} g(x, t) - L_\alpha^{-1} R(u(x, t)) - L_\alpha^{-1} N(u(x, t)). \quad (3.3)$$

The general solution of the given equation is decomposed into the sum

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t). \quad (3.4)$$

The non-linear part $N(u)$ can be decomposed into the infinite polynomial series obtained by

$$N(u) = \sum_{n=0}^{\infty} A_n, (u_0, u_1, \dots, u_n), \quad (3.5)$$

where A_n is the so-called Adomian polynomials (APs). These APs can be calculated for all types of non-linearity by the help of algorithms built by Adomian [5, 6, 12, 15, 25, 32]. u and $N(u)$, respectively, is obtained as

$$u = \sum_{i=0}^{\infty} \lambda^i u_i, N(u) = N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right) = \sum_{i=0}^{\infty} \lambda^i A_i \quad (3.6)$$

where λ is the convenience parameter. From (3.6), APs A_n are obtained as

$$n! A_n = \frac{d^n}{d\lambda^n} \left[N\left(\sum_{n=0}^{\infty} \lambda^n u_n\right) \right]_{\lambda=0}.$$

These APs can be calculated easily with the following Maple code:

Substituting (3.4) and (3.5) into (3.3), it is obtained

$$\sum_{n=0}^{\infty} u_n = \theta + L_\alpha^{-1} g - L_\alpha^{-1} R\left(\sum_{n=0}^{\infty} u_n\right) - L_\alpha^{-1} \left(\sum_{n=0}^{\infty} A_n\right). \quad (3.7)$$


```

restart:
ord:=m:    #Order (Enter a number for m.)
NF:=N(u(x,t))  #Nonlinear Function
u[t]:=sum(u[b]*t^b,b=0..ord):
NF[t]:=subs(u(x,t)=u[t],NF):
s:=expand(NF[t],t):
dt:=unapply(s,t):
for i from 0 to ord do
A[i]:=((D@@i)(dt)(0)/i!):
print(AA[i],A[i]): #Adomian Polinoms
od:

```

where $\theta = u(x, 0)$ is initial condition (IC). From (3.7), the iterates are defined by the following recursive formulas:

$$\begin{aligned}
 u_0 &= \theta + L_\alpha^{-1} g, \\
 u_1 &= -L_\alpha^{-1} R u_0 - L_\alpha^{-1} A_0, \\
 &\vdots \\
 u_{n+1} &= -L_\alpha^{-1} R u_n - L_\alpha^{-1} A_n, \quad n \geq 0.
 \end{aligned} \tag{3.8}$$

Therefore, from (3.8), the approximate solution of (3.1) is obtained by

$$\tilde{u}_m(x, t) = \sum_{n=0}^m u_n(x, t). \tag{3.9}$$

Hence, from (3.9), the exact solution of (3.1) can be obtained as

$$u(x, t) = \lim_{m \rightarrow \infty} \tilde{u}_m(x, t).$$

4. CONFORMABLE REDUCED DIFFERENTIAL TRANSFORM METHOD

In this section, it is given basic definitions and properties of CRDTM for FPDEs [3].

Definition 2. Assume $u(x, t)$ is analytic function and differentiated continuously with respect to time t and space x in the its domain. the conformable reduced differential transformed (CRDT) of $u(x, t)$ is defined as [3]

$$U_k^\alpha(x) = \frac{1}{\alpha^k k!} \left[\left({}_t T_\alpha^{(k)} u \right) \right]_{t=t_0}$$

where some $0 < \alpha \leq 1$, α is describing the order of CDO,

${}_t T_\alpha^{(k)} u = \underbrace{({}_t T_\alpha {}_t T_\alpha \cdots {}_t T_\alpha)}_{k \text{ times}} u(x, t)$ and $U_k^\alpha(x)$ is the CRDT function.

Definition 3. Let $U_k^\alpha(x)$ be the CRDT of $u(x, t)$. Inverse CRDT of $U_k^\alpha(x)$ is defined as [3]

$$u(x, t) = \sum_{k=0}^{\infty} U_k^\alpha(x) (t - t_0)^{\alpha k} = \sum_{k=0}^{\infty} \frac{1}{\alpha^k k!} \left[{}_t T_\alpha^{(k)} u \right]_{t=t_0} (t - t_0)^{\alpha k}$$

CRDT of ICs for integer order derivatives are defined as [3]

$$U_k^\alpha(x) = \begin{cases} \frac{1}{(\alpha k)!} \left[\frac{\partial^{\alpha k}}{\partial t^{\alpha k}} u(x, t) \right]_{t=t_0} & \text{if } \alpha k \in \mathbb{Z}^+ \\ 0 & \text{if } \alpha k \notin \mathbb{Z}^+ \end{cases} \quad \text{for } k = 0, 1, 2, \dots, \left(\frac{n}{\alpha} - 1 \right)$$

where n is the order of CDO of PDE.

By consideration of

$$U_0^\alpha(x) = f(x)$$

as transformation of IC

$$u(x, 0) = f(x).$$

A straightforward iterative calculations gives the $U_k^\alpha(x)$ values for $k = 1, 2, 3, \dots, n$. Then the set of $\{U_k^\alpha(x)\}_{k=0}^n$ gives the approximate result as:

$$\tilde{u}_n(x, t) = \sum_{k=0}^n U_k^\alpha(x) t^{k\alpha},$$

where n is approximate result order. The exact solution can be obtained as:

$$u(x, t) = \lim_{n \rightarrow \infty} \tilde{u}_n(x, t)$$

The fundamental operations of CRDTM that can be deduced from Definition 2 and Definition 3 are listed in Table 1 [3].

5. NUMERICAL CONSIDERATION

To illustrate the effectiveness of the given CADM and CRDTM, three examples are considered in this section. All the results are calculated by software MAPLE.

Example 1. Firstly, consider the linear time and space fractional diffusion equation:

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x, t) = \frac{\partial^{2\beta}}{\partial x^{2\beta}} u(x, t) \quad t > 0, x \in R, 0 < \alpha, \beta \leq 1 \quad (5.1)$$

with the IC

$$u(x, 0) = \sin\left(\frac{x^\beta}{\beta}\right). \quad (5.2)$$

Exact result of the problem (5.1) in conformable sense is

$$u(x, t) = \sin\left(\frac{x^\beta}{\beta}\right) e^{-\frac{t^\alpha}{\alpha}}.$$

TABLE 1. Basic operations of the CRDTM [3].

Original function	Transformed function
$u(x, t)$	$U_k^\alpha(x) = \frac{1}{\alpha^k k!} \left[\left({}_t T_\alpha^{(k)} u \right) \right]_{t=t_0}$
$u(x, t) = av(x, t) \pm bw(x, t)$	$U_k^\alpha(x) = aV_k^\alpha(x) \pm bW_k^\alpha(x)$
$u(x, t) = v(x, t)w(x, t)$	$U_k^\alpha(x) = \sum_{s=0}^k V_s^\alpha(x) W_{k-s}^\alpha(x)$
$u(x, t) = {}_t T_\alpha v(x, t)$	$U_k^\alpha(x) = \alpha(k+1) V_{k+1}^\alpha(x)$
$u(x, t) = x^m(t-t_0)^n$	$U_k^\alpha(x) = x^m \delta(k - \frac{n}{\alpha})$

Solution by CADM: Solve this problem by using CADM. Let $L_\alpha = T_\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ be a linear operator, then the operator form of (5.1) is as follows

$$T_\alpha u(x, t) = \frac{\partial^{2\beta} u(x, t)}{\partial x^{2\beta}} \quad t > 0, x \in R, 0 < \alpha, \beta \leq 1 \quad (5.3)$$

By the help of Lemma 1, eq. (5.3) can be written as

$$t^{1-\beta} \frac{\partial u(x, t)}{\partial t} = \frac{\partial^{2\beta} u(x, t)}{\partial x^{2\beta}} \quad t > 0, x \in R, 0 < \alpha, \beta \leq 1. \quad (5.4)$$

If $L_\alpha^{-1} = \int_0^t \frac{1}{\xi^{1-\alpha}} (.) d\xi$, which is the inverse of L_α , is applied to both sides of eq. (5.4), we get

$$u(x, t) = u(x, 0) - L_\alpha^{-1} \left(\frac{\partial^{2\beta}}{\partial x^{2\beta}} u(x, t) \right).$$

According to (3.8) and the IC (5.2), we can write

$$\begin{aligned} u_0 &= \sin\left(\frac{x^\beta}{\beta}\right), \\ u_1 &= -L_\alpha^{-1} \left(\frac{\partial^2}{\partial x^2} u_0 \right), \\ &\vdots \\ u_{n+1} &= -L_\alpha^{-1} \left(\frac{\partial^2}{\partial x^2} u_n \right), n \geq 0. \end{aligned} \quad (5.5)$$

From (5.5), we conclude the terms of decomposition series as:

$$\begin{aligned} u_0 &= \sin\left(\frac{x^\beta}{\beta}\right), \quad u_1 = -\sin\left(\frac{x^\beta}{\beta}\right) \frac{t^\alpha}{\alpha}, \\ u_2 &= \sin\left(\frac{x^\beta}{\beta}\right) \frac{t^{2\alpha}}{2\alpha^2}, \dots, u_n = (-1)^n \sin\left(\frac{x^\beta}{\beta}\right) \frac{t^{n\alpha}}{n!\alpha^n}, \dots \end{aligned} \quad (5.6)$$

Thus, by using (5.6), the approximate solution of (5.1) obtained by CADM is

$$\tilde{u}_m(x, t) = \sum_{n=0}^m u_n(x, t) = \sum_{n=0}^m (-1)^n \sin\left(\frac{x^\beta}{\beta}\right) \frac{t^{n\alpha}}{n!\alpha^n}. \quad (5.7)$$

From (5.7) we obtain

$$u(x, t) = \lim_{m \rightarrow \infty} \tilde{u}_m(x, t) = \sin\left(\frac{x^\beta}{\beta}\right) e^{-\frac{t^\alpha}{\alpha}}. \quad (5.8)$$

This analytical approximate solution (5.8) is the exact solution.

Solution by CRDTM: Now solve this problem by using CRDTM. By taking the CRDT of (5.1), it can be obtained that

$$\alpha(k+1)U_{k+1}^\alpha(x) = \frac{\partial^{2\beta}}{\partial x^{2\beta}} U_k^\alpha(x) \quad (5.9)$$

where $U_k^\alpha(x)$ is the CRDT function. From the IC (5.2) we write

$$U_0^\alpha(x) = \sin\left(\frac{x^\beta}{\beta}\right) \quad (5.10)$$

Substituting (5.10) into (5.9), it can be obtained the following $U_k^\alpha(x)$ values

$$\begin{aligned} U_1^\alpha(x) &= -\sin\left(\frac{x^\beta}{\beta}\right) \frac{1}{\alpha}, \quad U_2^\alpha(x) = \sin\left(\frac{x^\beta}{\beta}\right) \frac{1}{2!\alpha^2}, \\ U_3^\alpha(x) &= -\sin\left(\frac{x^\beta}{\beta}\right) \frac{1}{3!\alpha^3}, \dots, U_n^\alpha(x) = \sin\left(\frac{x^\beta}{\beta}\right) \frac{(-1)^n}{n!\alpha^n}, \dots \end{aligned}$$

Then, the set of values $\{U_k^\alpha(x)\}_{k=0}^n$ gives the following approximate result

$$\tilde{u}_n(x, t) = \sum_{k=0}^n U_k^\alpha(x) t^{k\alpha} = \sum_{k=0}^n \sin\left(\frac{x^\beta}{\beta}\right) \frac{(-1)^k}{k!\alpha^k} t^{k\alpha}. \quad (5.11)$$

From (5.11) we obtain

$$u(x, t) = \lim_{n \rightarrow \infty} \tilde{u}_n(x, t) = \sin\left(\frac{x^\beta}{\beta}\right) e^{-\frac{t^\alpha}{\alpha}}. \quad (5.12)$$

This approximate solution (5.12) is the exact solution.

Remark 1. If take $\alpha = \beta = 1$ in the problem (5.1), then Example 1 is reduced to standard diffusion equation

$$\frac{\partial}{\partial t} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t) \quad t > 0, x \in R$$

with IC

$$u(x, 0) = \sin(x)$$

and our analytical approximate solutions (5.8) and (5.12) imply

$$u(x, t) = \sin(x)e^{-t}$$

and this solutions is the exact result of the standard problem in the literature.

The Aproximate solutions obtained by both CADM and CRDTM give us the exact solution.

Example 2. Secondly, let us consider the non-linear time and space fractional gas dynamics equation:

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x, t) + \frac{1}{2} \frac{\partial^\beta}{\partial x^\beta} u^2(x, t) - u(x, t) (1 - u(x, t)) = 0, \quad 0 \leq \alpha, \beta \leq 1 \quad (5.13)$$

subject to IC

$$u(x, 0) = e^{-\frac{x^\beta}{\beta}}. \quad (5.14)$$

The exact solutions of (5.13) in conformable sense is

$$u(x, t) = e^{\frac{t^\alpha}{\alpha} - \frac{x^\beta}{\beta}}.$$

Solution by CADM: Solve the problem by using CADM. Let $L_\alpha = T_\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ be a linear operator, then the operator form of (5.13) is as follows

$$T_\alpha u(x, t) = -\frac{1}{2} \frac{\partial^\beta}{\partial x^\beta} u^2(x, t) + u(x, t) (1 - u(x, t)), \quad 0 \leq \alpha, \beta \leq 1. \quad (5.15)$$

By the help of Lemma 1, eq. (5.15) can be written as

$$t^{1-\alpha} \frac{\partial u(x, t)}{\partial t} = u(x, t) - u(x, t) \frac{\partial^\beta}{\partial x^\beta} u(x, t) - u^2(x, t), \quad 0 \leq \alpha, \beta \leq 1. \quad (5.16)$$

If $L_\alpha^{-1} = \int_0^t \frac{1}{\xi^{1-\alpha}} (.) d\xi$, which is the inverse of L_α , is applied to both sides of eq. (5.16), we get

$$u(x, t) = u(x, 0) + L_\alpha^{-1} (u(x, t)) - L_\alpha^{-1} \left(u(x, t) \frac{\partial^\beta}{\partial x^\beta} u(x, t) + u^2(x, t) \right).$$

According to (3.8) and IC (5.14), we can write the following recursive relations:

$$\begin{aligned} u_0 &= e^{-\frac{x^\beta}{\beta}} \\ u_1 &= L_\alpha^{-1}(u_0) - L_\alpha^{-1}(A_0) \\ &\vdots \\ u_{n+1} &= L_\alpha^{-1}(u_n) - L_\alpha^{-1}(A_n), n \geq 0. \end{aligned} \quad (5.17)$$

where A_n 's are APs. By using the Maple code above, for the non-linear term

$$N(u(x)) = u(x, t) \frac{\partial}{\partial x} u(x, t) + u^2(x, t),$$

the APs can be obtain as:

$$\begin{aligned} A_0 &= u_0^2 + u_0 \frac{\partial^\beta}{\partial x^\beta} u_0 \\ A_1 &= 2u_0 u_1 + u_0 \frac{\partial^\beta}{\partial x^\beta} u_1 + u_1 \frac{\partial^\beta}{\partial x^\beta} u_0 \\ A_2 &= u_1^2 + 2u_0 u_2 + u_0 \frac{\partial^\beta}{\partial x^\beta} u_2 + u_1 \frac{\partial^\beta}{\partial x^\beta} u_1 + u_2 \frac{\partial^\beta}{\partial x^\beta} u_0 \\ A_3 &= 2u_1 u_2 + 2u_0 u_3 + u_0 \frac{\partial^\beta}{\partial x^\beta} u_3 + u_1 \frac{\partial^\beta}{\partial x^\beta} u_2 + u_2 \frac{\partial^\beta}{\partial x^\beta} u_1 + u_3 \frac{\partial^\beta}{\partial x^\beta} u_0 \\ &\vdots \end{aligned} \quad (5.18)$$

From (5.17) and (5.18), we conclude the terms of decomposition series as:

$$u_0 = e^{-\frac{x^\beta}{\beta}}, u_1 = e^{-\frac{x^\beta}{\beta}} \frac{t^\alpha}{\alpha}, u_2 = e^{-\frac{x^\beta}{\beta}} \frac{t^{2\alpha}}{2\alpha^2}, \dots, u_n = e^{-\frac{x^\beta}{\beta}} \frac{t^{n\alpha}}{n!\alpha^n}, \dots \quad (5.19)$$

Thus, From (5.19), the approximate solution of (5.13) obtained by CADM is

$$\tilde{u}_m(x, t) = \sum_{n=0}^m u_n(x, t) = \sum_{n=0}^m e^{-\frac{x^\beta}{\beta}} \frac{t^{n\alpha}}{n!\alpha^n}. \quad (5.20)$$

From (5.20) we obtain

$$u(x, t) = \lim_{m \rightarrow \infty} \tilde{u}_m(x, t) = e^{\frac{t^\alpha}{\alpha} - \frac{x^\beta}{\beta}}. \quad (5.21)$$

This analytical approximate solution (5.21) is the exact solution.

Solution by CRDTM: Now solve this problem by using CRDTM. By taking the CRDT of (5.13), it can be obtained that

$$\alpha(k+1)U_{k+1}^\alpha(x) = - \sum_{r=0}^k U_{k-r}^\alpha(x) \frac{\partial^\beta}{\partial x^\beta} U_r^\alpha(x) + U_k^\alpha(x) - \sum_{r=0}^k U_{k-r}^\alpha(x) U_r^\alpha(x) \quad (5.22)$$

where $U_k^\alpha(x)$ is the CRDT function. From the IC (5.14) we write

$$U_0^\alpha(x) = e^{-\frac{x^\beta}{\beta}} \quad (5.23)$$

Substituting (5.23) into (5.22), it can be obtained the following $U_k^\alpha(x)$ values

$$U_1^\alpha(x) = e^{-\frac{x^\beta}{\beta}} \frac{1}{\alpha}, \quad U_2^\alpha(x) = e^{-\frac{x^\beta}{\beta}} \frac{1}{2!\alpha^2}, \dots, U_n^\alpha(x) = e^{-\frac{x^\beta}{\beta}} \frac{1}{n!\alpha^n}, \dots$$

Then, the set of values $\{U_k^\alpha(x)\}_{k=0}^n$ gives the following approximate result

$$\tilde{u}_n(x, t) = \sum_{k=0}^n U_k^\alpha(x) t^{k\alpha} = \sum_{k=0}^n e^{-\frac{x^\beta}{\beta}} \frac{1}{k!\alpha^k} t^{k\alpha}. \quad (5.24)$$

From (5.24) we obtain

$$u(x, t) = \lim_{n \rightarrow \infty} \tilde{u}_n(x, t) = e^{\frac{t^\alpha}{\alpha} - \frac{x^\beta}{\beta}}. \quad (5.25)$$

This approximate solution (5.25) is the exact solution.

Remark 2. If take $\alpha = \beta = 1$ in the problem (5.13), then Example 2 is reduced to standard gas dynamics equation

$$\frac{\partial}{\partial t} u(x, t) + \frac{1}{2} \frac{\partial}{\partial x} u^2(x, t) - u(x, t) (1 - u(x, t)) = 0$$

with IC

$$u(x, 0) = e^{-x}$$

our analytical approximate solutions (5.25) and (5.21) imply

$$u(x, t) = e^{t-x}$$

and this solution is the exact result of the standard problem in the literature.

The approximate solutions obtained by both CADM and CRDTM give us the existed exact solution.

Example 3. Finally, let us consider the non-linear time and space FPDE:

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x, t) + (1 + u(x, t)) \frac{\partial^\alpha}{\partial x^\alpha} u(x, t) = 0, \quad 0 \leq \alpha \leq 1 \quad (5.26)$$

subject to IC

$$u(x, 0) = \frac{x^\alpha - \alpha}{2\alpha}. \quad (5.27)$$

The exact solutions of (5.26) in conformable sense is

$$u(x, t) = \frac{x^\alpha - t^\alpha - \alpha}{t^\alpha - 2\alpha}.$$

Solution by CADM: Solve the problem by using CADM. Let $L_\alpha = T_\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ be a linear operator, then the operator form of (5.26) is as follows

$$T_\alpha u(x, t) = -(1 + u(x, t)) \frac{\partial^\alpha}{\partial x^\alpha} u(x, t), \quad 0 \leq \alpha \leq 1. \quad (5.28)$$

By the help of Lemma 1, eq. (5.28) can be written as

$$t^{1-\alpha} \frac{\partial u(x, t)}{\partial t} = -\frac{\partial^\alpha}{\partial x^\alpha} u(x, t) - u(x, t) \frac{\partial^\alpha}{\partial x^\alpha} u(x, t), 0 \leq \alpha \leq 1. \quad (5.29)$$

If $L_\alpha^{-1} = \int_0^t \frac{1}{\xi^{1-\alpha}} (\cdot) d\xi$, which is the inverse of L_α , is applied to both sides of eq. (5.29), we get

$$u(x, t) = u(x, 0) - L_\alpha^{-1} \left(\frac{\partial^\alpha}{\partial x^\alpha} u(x, t) \right) - L_\alpha^{-1} \left(u(x, t) \frac{\partial^\alpha}{\partial x^\alpha} u(x, t) \right).$$

According to 3.8 and IC (5.27), we can write the following recursive relations:

$$\begin{aligned} u_0 &= \frac{x^\alpha - \alpha}{2\alpha} \\ u_1 &= L_\alpha^{-1}(u_0) - L_\alpha^{-1}(A_0) \\ &\vdots \\ u_{n+1} &= L_\alpha^{-1}(u_n) - L_\alpha^{-1}(A_n), n \geq 0. \end{aligned} \quad (5.30)$$

where A_n 's are APs. By using the Maple code above, for the non-linear term

$$N(u(x)) = u(x, t) \frac{\partial}{\partial x} u(x, t) + u^2(x, t),$$

the APs can be obtain as:

$$\begin{aligned} A_0 &= u_0 \frac{\partial^\alpha}{\partial x^\alpha} u_0 \\ A_1 &= u_0 \frac{\partial^\alpha}{\partial x^\alpha} u_1 + u_1 \frac{\partial^\alpha}{\partial x^\alpha} u_0 \\ A_2 &= u_0 \frac{\partial^\alpha}{\partial x^\alpha} u_2 + u_1 \frac{\partial^\alpha}{\partial x^\alpha} u_1 + u_2 \frac{\partial^\alpha}{\partial x^\alpha} u_0 \\ A_3 &= u_0 \frac{\partial^\alpha}{\partial x^\alpha} u_3 + u_1 \frac{\partial^\alpha}{\partial x^\alpha} u_2 + u_2 \frac{\partial^\alpha}{\partial x^\alpha} u_1 + u_3 \frac{\partial^\alpha}{\partial x^\alpha} u_0 \\ &\vdots \end{aligned} \quad (5.31)$$

From (5.30) and (5.31), we conclude the terms of decomposition series as:

$$\begin{aligned} u_0 &= \frac{x^\alpha - \alpha}{2\alpha}, u_1 = -\frac{x^\alpha + \alpha}{(2\alpha)^2} t^\alpha, u_2 = \frac{x^\alpha + \alpha}{(2\alpha)^3} t^{2\alpha}, \\ u_3 &= -\frac{x^\alpha + \alpha}{(2\alpha)^4} t^{3\alpha}, \dots, u_n = (-1)^n \frac{x^\alpha + \alpha}{(2\alpha)^{n+1}} t^{n\alpha}, \dots \end{aligned} \quad (5.32)$$

Thus, from (5.32), the approximate solution of (5.26) obtained by CADM is

$$\tilde{u}_m(x, t) = \sum_{n=0}^m u_n(x, t) = \frac{x^\alpha - \alpha}{2\alpha} + \sum_{n=1}^m (-1)^n \frac{x^\alpha + \alpha}{(2\alpha)^{n+1}} t^{n\alpha}. \quad (5.33)$$

Solution by CRDTM: Now solve this problem by using CRDTM. By taking the CRDT of (5.26), it can be obtained that

$$\alpha(k+1)U_{k+1}(x) = -U_k(x) - \sum_{r=0}^k U_{k-r}(x) \frac{\partial}{\partial x} U_r(x) \quad (5.34)$$

where $U_k^\alpha(x)$ is the CRDT function. From the IC (5.27) we write

$$U_0^\alpha(x) = \frac{x^\alpha - \alpha}{2\alpha} \quad (5.35)$$

Substituting (5.35) into (5.34), it can be obtained the following $U_k^\alpha(x)$ values

$$U_1^\alpha(x) = -\frac{x^\alpha + \alpha}{(2\alpha)^2}, \quad U_2^\alpha(x) = \frac{x^\alpha + \alpha}{(2\alpha)^3}, \dots, U_n^\alpha(x) = (-1)^n \frac{x^\alpha + \alpha}{(2\alpha)^{n+1}}, \dots$$

Then, the set of values $\{U_k^\alpha(x)\}_{k=0}^n$ gives the following approximate result

$$\tilde{u}_m(x, t) = \sum_{k=0}^m U_k^\alpha(x) t^{k\alpha} = \frac{x^\alpha - \alpha}{2\alpha} + \sum_{n=1}^m (-1)^n \frac{x^\alpha + \alpha}{(2\alpha)^{n+1}} t^{n\alpha}. \quad (5.36)$$

Now, we compare the seventh iteration CADM and CRDTM solutions with the exact solution on the graphs for some α values. These comparisons can be seen in fig. 1 and fig. 2.

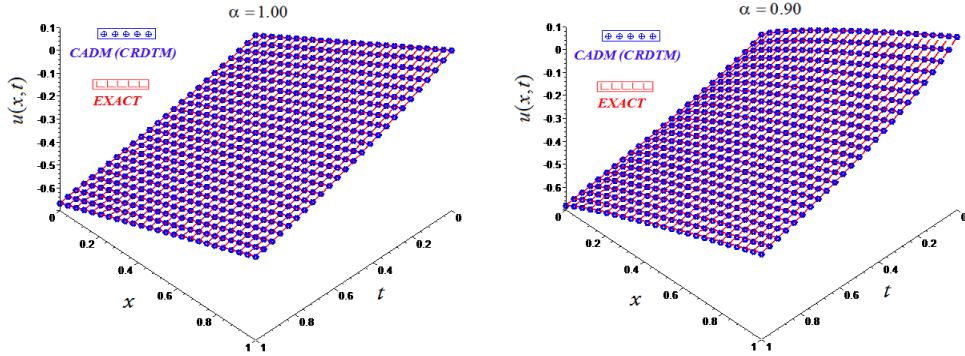


FIGURE 1. Comparison of seventh iteration approximate results of CADM (CRDTM) with the exact solutions for eq. (5.26).

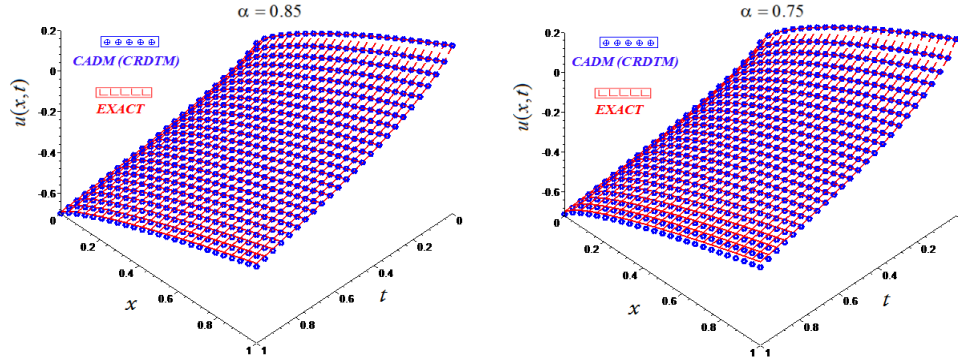


FIGURE 2. Comparison of seventh iteration approximate results of CADM (CRDTM) with the exact solutions for eq. (5.26).

6. CONCLUSION

The fundamental goal of this article is to construct the approximate solutions of FPDEs. The goal has been achieved by using CADM for the first time and it is compared with CRDTM. CADM and CRDTM are applied to different linear and non-linear conformable time and space FPDEs. And also the approximate analytical solutions obtained by CADM and CRDTM are compared to each other and with the exact solutions. CADM and CRDTM offer solutions with easily computable components as convergent series. Approximate solutions obtained by CADM are exactly same as the solutions obtained by CRDTM for time and space FPDEs. The CADM gives quantitatively reliable results as CRDTM, and also it requires less computational work than existing other methods. As a result, in recent years, FDEs emerging as models in fields such as mathematics, physics, chemistry, biology and engineering makes it necessary to investigate the methods of solutions and we hope that this study is an improvement in this direction.

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Authors’ addresses

Omer Acan

Siirt University, Department of Mathematics, Art and Science Faculty, 56100 Siirt, Turkey

E-mail address: omeracan@yahoo.com

Dumitru Baleanu

Institute of Space Sciences, Magurele-Bucharest, Romania

Current address: Cankaya University, Faculty of Art and Science, Department of Mathematics and Computer Sciences, Ankara, Turkey

E-mail address: dumitru@cankaya.edu.tr



ON THE MONOTONICITY OF q -SCHURER-STANCU TYPE POLYNOMIALS

ANA MARIA ACU, CARMEN VIOLETA MURARU,
AND VOICHIȚA ADRIANA RADU

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Abstract. Some properties of monotonicity and convexity of the q -Schurer-Stancu operators are considered. The paper contains also numerical examples based on Matlab algorithms, which verify these properties.

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Keywords: generalized Schurer-Stancu operators, q -integers, monotonicity, convexity

1. PRELIMINARIES

In the last decades, the application of q -calculus represents one of the most interesting areas of research in approximation theory. Lupaş [12] introduced in 1987 a q -type of the Bernstein operators and in 1997 another generalization of these operators based on q -integers was introduced by Phillips [16]. Their approximation properties were studied by Videnskii [18], N. Mahmudov [13], T. Acar and A. Aral [1] and O. Dalmanoglu [9, 10]. In time, many authors have been studying new classes of q -generalized operators ([2–4, 6, 7, 17]).

Before proceeding further, we mention some basic definitions and notations from q -calculus. For any fixed real number $q > 0$, the q -integer $[k]_q$, for $k \in \mathbb{N}$ is defined as

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1, \\ k, & q = 1. \end{cases}$$

The q -factorial integer and the q -binomial coefficients are :

$$[k]_q! = \begin{cases} [k]_q [k-1]_q \dots [1]_q, & k = 1, 2, \dots \\ 1, & k = 0, \end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}, \quad (n \geq k \geq 0).$$

The q -analogue of $(x - a)_q^n$ is the polynomial

$$(x - a)_q^n = \begin{cases} 1, & \text{if } n = 0, \\ (x - a)(x - qa) \dots (x - q^{n-1}a), & \text{if } n \geq 1. \end{cases}$$

Let p be a non-negative integer and let α, β be some real parameters satisfying the conditions $0 \leq \alpha \leq \beta$. In 2003, D. Bărbosu [8] introduced for any $f \in C[0, 1 + p]$ and $x \in [0, 1]$ the Schurer-Stancu operators as follows

$$S_{m,p}^{(\alpha,\beta)}(f, q, x) = \sum_{k=0}^{m+p} p_{m,k}(x) f\left(\frac{k + \alpha}{m + \beta}\right),$$

where $p_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k}$.

Recently, P.N. Agrawal, V. Gupta and A.S. Kumar [5] introduced the class of q -Schurer-Stancu operators. For any $m \in \mathbb{N}$, p a fixed non negative integer number and α, β some real parameters satisfying the conditions $0 \leq \alpha \leq \beta$, they constructed the class of generalized q -Schurer-Stancu operators

$$\tilde{S}_{m,p}^{(\alpha,\beta)} : C[0, 1 + p] \rightarrow C[0, 1],$$

as follows

$$\tilde{S}_{m,p}^{(\alpha,\beta)}(f, q, x) = \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) f\left(\frac{[k]_q + \alpha}{[m]_q + \beta}\right), \quad x \in [0, 1], \quad (1.1)$$

where $\tilde{p}_{m,k}(x) = \left[\begin{matrix} m+p \\ k \end{matrix} \right]_q x^k (1-x)_q^{m+p-k}$.

If $\alpha = \beta = 0$ the above operators reduce to the Bernstein-Schurer operators introduced by Muraru in [14].

Lemma 1 ([5]). *For the operators defined in (1.1) the following properties hold*

1. $\tilde{S}_{m,p}^{(\alpha,\beta)}(e_0, q, x) = 1$,
2. $\tilde{S}_{m,p}^{(\alpha,\beta)}(e_1, q, x) = \frac{\alpha}{[m]_q + \beta} + \frac{[m+p]_q}{[m]_q + \beta} x$,
- 3.

$$\begin{aligned} & \tilde{S}_{m,p}^{(\alpha,\beta)}(e_2, q, x) \\ &= \frac{\alpha^2}{([m]_q + \beta)^2} + \frac{[m+p]_q^2}{([m]_q + \beta)^2} x^2 + \frac{2\alpha[m+p]_q x}{([m]_q + \beta)^2} + \frac{[m+p]_q x(1-x)}{([m]_q + \beta)^2}. \end{aligned}$$

The next result is based on Popoviciu's technique and it is expressed in terms of the first order modulus of continuity.

Theorem 1 ([5]). *If $f \in C[0, 1 + p]$ and $q \in (0, 1)$ then*

$$\left\| \tilde{S}_{m,p}^{(\alpha,\beta)}(f, q, x) - f(x) \right\| \leq \frac{5}{4} \omega_f(\delta_m) \quad (1.2)$$

holds, where

$$\delta_m = \frac{1}{[m]_q + \beta} \sqrt{[m + p]_q + 4(q^m[p]_q + \alpha - \beta)^2}.$$

2. MONOTONICITY OF THE q -SCHURER-STANCU OPERATORS

Oruc and Philips [15] showed that for a convex function f on $[0, 1]$, the q -Bernstein polynomials are monotonic decreasing. In this section we will prove a similar result for q -Schurer-Stancu operators.

Theorem 2. *Let f be a convex and increasing function on $[0, p + 1]$. Then, for $0 < q \leq 1$ and $\beta \leq \frac{[p]_q}{q^p}$,*

$$\tilde{S}_{m-1,p}^{(\alpha,\beta)}(f, q, x) \geq \tilde{S}_{m,p}^{(\alpha,\beta)}(f, q, x), \quad (2.1)$$

for $0 \leq x \leq 1$ and $m \geq 2$.

Proof. For $0 < q < 1$ we have

$$\begin{aligned} & \prod_{s=0}^{m+p-1} (1 - q^s x)^{-1} \left[\tilde{S}_{m-1,p}^{(\alpha,\beta)}(f, q, x) - \tilde{S}_{m,p}^{(\alpha,\beta)}(f, q, x) \right] \\ &= \sum_{k=0}^{m+p-1} \left[\begin{matrix} m+p-1 \\ k \end{matrix} \right]_q x^k \prod_{s=m+p-k-1}^{m+p-1} (1 - q^s x)^{-1} f\left(\frac{[k]_q + \alpha}{[m-1]_q + \beta}\right) \\ & \quad - \sum_{k=0}^{m+p} \left[\begin{matrix} m+p \\ k \end{matrix} \right]_q x^k \prod_{s=m+p-k}^{m+p-1} (1 - q^s x)^{-1} f\left(\frac{[k]_q + \alpha}{[m]_q + \beta}\right). \end{aligned}$$

Denote

$$\psi_k(x) = x^k \prod_{s=m+p-k}^{m+p-1} (1 - q^s x)^{-1} \quad (2.2)$$

and using the following relation

$$x^k \prod_{s=m+p-k-1}^{m+p-1} (1 - q^s x)^{-1} = \psi_k(x) + q^{m+p-k-1} \psi_{k+1}(x)$$

we find

$$\prod_{s=0}^{m+p-1} (1 - q^s x)^{-1} \left[\tilde{S}_{m-1,p}^{(\alpha,\beta)}(f, q, x) - \tilde{S}_{m,p}^{(\alpha,\beta)}(f, q, x) \right]$$

$$\begin{aligned}
&= \sum_{k=0}^{m+p-1} f\left(\frac{[k]_q + \alpha}{[m-1]_q + \beta}\right) \left[\begin{matrix} m+p-1 \\ k \end{matrix} \right]_q \left\{ \psi_k(x) + q^{m+p-k-1} \psi_{k+1}(x) \right\} \\
&- \sum_{k=0}^{m+p} f\left(\frac{[k]_q + \alpha}{[m]_q + \beta}\right) \left[\begin{matrix} m+p \\ k \end{matrix} \right]_q \psi_k(x) = \sum_{k=0}^{m+p-1} f\left(\frac{[k]_q + \alpha}{[m-1]_q + \beta}\right) \left[\begin{matrix} m+p-1 \\ k \end{matrix} \right]_q \psi_k(x) \\
&\quad + \sum_{k=1}^{m+p} q^{m+p-k} f\left(\frac{[k-1]_q + \alpha}{[m-1]_q + \beta}\right) \left[\begin{matrix} m+p-1 \\ k-1 \end{matrix} \right]_q \psi_k(x) \\
&- \sum_{k=0}^{m+p} f\left(\frac{[k]_q + \alpha}{[m]_q + \beta}\right) \left[\begin{matrix} m+p \\ k \end{matrix} \right]_q \psi_k(x) = \sum_{k=1}^{m+p-1} \left\{ f\left(\frac{[k]_q + \alpha}{[m-1]_q + \beta}\right) \left[\begin{matrix} m+p-1 \\ k \end{matrix} \right]_q \right. \\
&\quad \left. + q^{m+p-k} f\left(\frac{[k-1]_q + \alpha}{[m-1]_q + \beta}\right) \left[\begin{matrix} m+p-1 \\ k-1 \end{matrix} \right]_q - f\left(\frac{[k]_q + \alpha}{[m]_q + \beta}\right) \left[\begin{matrix} m+p \\ k \end{matrix} \right]_q \right\} \psi_k(x) \\
&\quad + \left\{ f\left(\frac{[m+p-1]_q + \alpha}{[m-1]_q + \beta}\right) - f\left(\frac{[m+p]_q + \alpha}{[m]_q + \beta}\right) \right\} \psi_{m+p}(x) \\
&\quad + \left\{ f\left(\frac{\alpha}{[m-1]_q + \beta}\right) - f\left(\frac{\alpha}{[m]_q + \beta}\right) \right\} \psi_0(x) \\
&= \sum_{k=1}^{m+p-1} \left[\begin{matrix} m+p \\ k \end{matrix} \right]_q a_k \psi_k(x) + \left\{ f\left(\frac{[m+p-1]_q + \alpha}{[m-1]_q + \beta}\right) - f\left(\frac{[m+p]_q + \alpha}{[m]_q + \beta}\right) \right\} \psi_{m+p}(x) \\
&\quad + \left\{ f\left(\frac{\alpha}{[m-1]_q + \beta}\right) - f\left(\frac{\alpha}{[m]_q + \beta}\right) \right\} \psi_0(x),
\end{aligned}$$

where

$$\begin{aligned}
a_k &= f\left(\frac{[k]_q + \alpha}{[m-1]_q + \beta}\right) \frac{[m+p-k]_q}{[m+p]_q} + q^{m+p-k} f\left(\frac{[k-1]_q + \alpha}{[m-1]_q + \beta}\right) \frac{[k]_q}{[m+p]_q} \\
&\quad - f\left(\frac{[k]_q + \alpha}{[m]_q + \beta}\right).
\end{aligned}$$

From (2.2) it is clear that each $\psi_k(x)$ is non-negative on $[0, 1]$ for $0 \leq q \leq 1$ and thus, it suffices to show that each a_k is non-negative.

Since f is convex on $[0, p+1]$, for any t_0, t_1 such that $0 \leq t_0 < t_1 \leq p+1$ and any λ , $0 < \lambda < 1$, we have

$$f(\lambda t_0 + (1-\lambda)t_1) \leq \lambda f(t_0) + (1-\lambda)f(t_1). \quad (2.3)$$

Let $t_0 = \frac{[k-1]_q + \alpha}{[m-1]_q + \beta}$, $t_1 = \frac{[k]_q + \alpha}{[m-1]_q + \beta}$ and $\lambda = q^{m+p-k} \frac{[k]_q}{[m+p]_q}$. Then $0 \leq t_0 < t_1 \leq p+1$ and $0 < \lambda < 1$ for $1 \leq k \leq m+p-1$. If we replace them in the relation (2.3), it follows

$$q^{m+p-k} \frac{[k]_q}{[m+p]_q} f\left(\frac{[k-1]_q + \alpha}{[m-1]_q + \beta}\right) + \frac{[m+p-k]_q}{[m+p]_q} f\left(\frac{[k]_q + \alpha}{[m-1]_q + \beta}\right)$$

$$\geq f \left(q^{m+p-k} \frac{[k]_q}{[m+p]_q} \cdot \frac{[k-1]_q + \alpha}{[m-1]_q + \beta} + \frac{[m+p-k]_q}{[m+p]_q} \cdot \frac{[k]_q + \alpha}{[m-1]_q + \beta} \right).$$

Using the inequality $[k]_q([k-1]_q + \alpha) \geq ([k]_q + \alpha)[k-1]_q$ and f increasing function, it follows

$$\begin{aligned} & q^{m+p-k} \frac{[k]_q}{[m+p]_q} f \left(\frac{[k-1]_q + \alpha}{[m-1]_q + \beta} \right) + \frac{[m+p-k]_q}{[m+p]_q} f \left(\frac{[k]_q + \alpha}{[m-1]_q + \beta} \right) \quad (2.4) \\ & \geq f \left(\frac{[k]_q + \alpha}{[m+p]_q} \cdot \frac{q^{m+p-k} [k-1]_q + [m+p-k]_q}{[m-1]_q + \beta} \right) = f \left(\frac{[k]_q + \alpha}{[m+p]_q} \cdot \frac{[m+p-1]_q}{[m-1]_q + \beta} \right). \end{aligned}$$

Since f is increasing on $[0, p+1]$ and

$$\begin{aligned} a_k &= f \left(\frac{[k]_q + \alpha}{[m-1]_q + \beta} \right) \frac{[m+p-k]_q}{[m+p]_q} + q^{m+p-k} f \left(\frac{[k-1]_q + \alpha}{[m-1]_q + \beta} \right) \frac{[k]_q}{[m+p]_q} \\ &- f \left(\frac{[k]_q + \alpha}{[m+p]_q} \cdot \frac{[m+p-1]_q}{[m-1]_q + \beta} \right) + \left\{ f \left(\frac{[k]_q + \alpha}{[m+p]_q} \cdot \frac{[m+p-1]_q}{[m-1]_q + \beta} \right) - f \left(\frac{[k]_q + \alpha}{[m]_q + \beta} \right) \right\}, \end{aligned}$$

from the inequality (2.4) we obtain $a_k \geq 0$, $k = \overline{1, m+p-1}$.

Therefore $\tilde{S}_{m-1,p}^{(\alpha,\beta)}(f, q, x) \geq \tilde{S}_{m,p}^{(\alpha,\beta)}(f, q, x)$.

For $q = 1$ and $0 \leq x < 1$ in a similar way the property (2.1) is verified.

For $q = 1$ and $x = 1$ we have

$$\tilde{S}_{m-1,p}^{(\alpha,\beta)}(f, 1, 1) - \tilde{S}_{m,p}^{(\alpha,\beta)}(f, 1, 1) = f \left(\frac{m+p-1+\alpha}{m-1+\beta} \right) - f \left(\frac{m+p+\alpha}{m+\beta} \right) \geq 0.$$

□

Theorem 3. If f is convex, then for all $m \geq 1$ and $0 < q \leq 1$ it follows

- i) $\tilde{S}_{m,p}^{(\alpha,\beta)}(f, q, x) \geq f(x)$, for $x \in [0, 1]$, f increasing on $[0, 1]$ and $\beta = \alpha + \varepsilon$, $\varepsilon \in [0, q^m[p]_q]$;
- ii) $\tilde{S}_{m,p}^{(\alpha,\beta)}(f, q, x) \geq f(x)$, for $x \in \left[0, \frac{\alpha}{\beta - q^m[p]_q}\right]$, f increasing on $[0, 1]$ and $\beta > \alpha + q^m[p]_q$;
- iii) $\tilde{S}_{m,p}^{(\alpha,\beta)}(f, q, x) \geq f(x)$, for $x \in \left(\frac{\alpha}{\beta - q^m[p]_q}, 1\right]$, f decreasing on $[0, 1]$ and $\beta > \alpha + q^m[p]_q$.

Proof. We consider the knots $x_k = \frac{[k]_q + \alpha}{[m]_q + \beta}$, $0 \leq k \leq m+p$. From Lemma 1 it follows

$$\sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) = 1, \quad \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x)x_k = \frac{\alpha}{[m]_q + \beta} + \frac{[m+p]_q}{[m]_q + \beta}x.$$

Using the convexity of function f we have

$$\begin{aligned}\tilde{S}_{m,p}^{(\alpha,\beta)}(f,q,x) &= \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) f(x_k) \geq f\left(\sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) x_k\right) \\ &= f\left(\frac{\alpha}{[m]_q + \beta} + \frac{[m+p]_q}{[m]_q + \beta} x\right).\end{aligned}$$

The following inequalities hold

- a) $\frac{\alpha}{[m]_q + \beta} + \frac{[m+p]_q}{[m]_q + \beta} x \geq x$ for $\beta = \alpha + \epsilon$, $\epsilon \in [0, q^m[p]]$, $x \in [0, 1]$;
- b) $\frac{\alpha}{[m]_q + \beta} + \frac{[m+p]_q}{[m]_q + \beta} x \geq x$ for $\beta > \alpha + q^m[p]_q$, $x \in \left[0, \frac{\alpha}{\beta - q^m[p]_q}\right]$;
- c) $\frac{\alpha}{[m]_q + \beta} + \frac{[m+p]_q}{[m]_q + \beta} x \leq x$ for $\beta > \alpha + q^m[p]_q$, $x \in \left[\frac{\alpha}{\beta - q^m[p]_q}, 1\right]$.

The theorem is proved using the monotony of function f and the inequalities a)-c). $\square$

3. NUMERICAL EXAMPLE

Davis [11] proved that for any convex function f , the classical Bernstein polynomial is convex and the sequence of Bernstein polynomials is monotonic decreasing. Oruc and Philips [15] extend these results for the Bernstein operators in q -calculus for $0 < q \leq 1$. In this section we will verify numerically these properties for the q -Schurer-Stancu operators.

TABLE 1. The q -Schurer-Stancu operators

x	$\tilde{S}_{30,6}^{(3,5)}(f,q,x)$	$\tilde{S}_{90,6}^{(3,5)}(f,q,x)$
0	0.029115231597413	0.026565252311249
0.1	0.089811163243826	0.083383934253406
0.2	0.191687642176340	0.179475155093384
0.3	0.347058215579779	0.326749960898829
0.4	0.570050753689721	0.538887425606676
0.5	0.876786656660820	0.831510780091276
0.6	1.285573442130916	1.222376845885256
0.7	1.817111513724159	1.731579581814080
0.8	2.494715950321931	2.381768592612226
0.9	3.344554197167635	3.198383491142964
1	4.395900582819147	4.209905050209471

In Table 1 are calculated the values of the q -Schurer-Stancu operators $\tilde{S}_{30,6}^{(3,5)}(f, q, x)$ and $\tilde{S}_{90,6}^{(3,5)}(f, q, x)$ for $f(x) = x^3 e^{x+1}$ and $q = 0.9$. Also, in the Figure 1 are given the graphics of these operators.

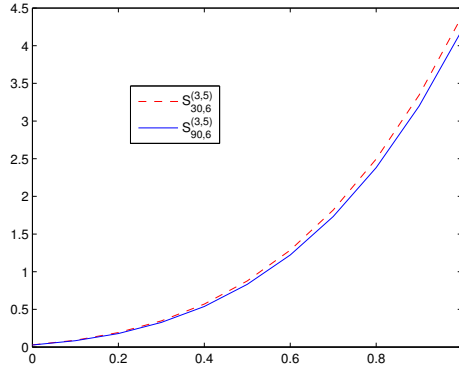


FIGURE 1. The monotonicity of the q -Schurer-Stancu operators

In the next part of this section we will give some numerical examples which verify the inequalities proved in Theorem 3.

Example 1. If $n = 50$, $p = 5$, $\alpha = 3$, $\beta = 3.0211$, $q = 0.9$, $f(x) = x^3 e^{x+1}$, it follows $\tilde{S}_{m,p}^{(\alpha,\beta)}(f, q, x) \geq f(x)$ for all $x \in [0, 1]$.

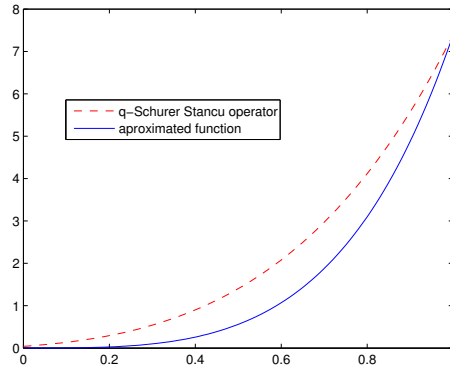


FIGURE 2. The q -Schurer-Stancu operators for increasing function and $\beta = \alpha + \epsilon$

Example 2. If $n = 50$, $p = 5$, $\alpha = 3$, $q = 0.9$, $\beta = 8.0211$, $f(x) = x^3 e^{x+1}$, it follows $\tilde{S}_{m,p}^{(\alpha,\beta)}(f, q, x) \geq f(x)$ for all $x \in [0, 0.375]$.

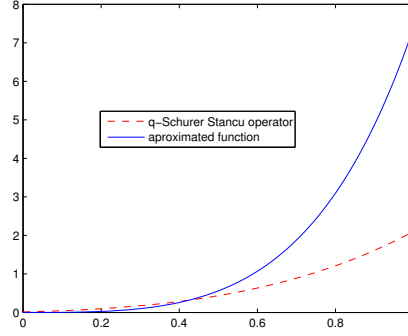


FIGURE 3. The q -Schurer-Stancu operators for increasing function and $\beta > \alpha + q^m[p]_q$

Example 3. If $n = 50$, $p = 5$, $\alpha = 3$, $q = 0.9$, $\beta = 8.0211$, $f(x) = e^{-x^2}$, it follows $\tilde{S}_{m,p}^{(\alpha,\beta)}(f, q, x) \geq f(x)$ for all $x \in [0.375, 1]$.

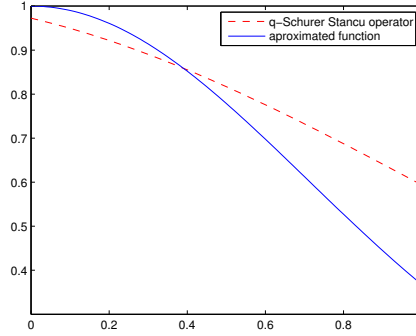


FIGURE 4. The q -Schurer-Stancu operators for decreasing function

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*Authors' addresses***Ana Maria Acu**

Lucian Blaga University of Sibiu, Department of Mathematics and Informatics, Str. Dr. I. Ratiu,
No.5-7, RO-550012 Sibiu, Romania

E-mail address: acuana77@yahoo.com

Carmen Violeta Muraru

Vasile Alecsandri University of Bacău, Department of Mathematics, Informatics and Educational
Sciences, Calea Mărășești 158, RO-600115 Bacău, Romania

E-mail address: carmen_7419@yahoo.com, cmuraru@ub.ro

Voichița Adriana Radu

Babes-Bolyai University, FSEGA, Department of Statistics-Forecasts-Mathematics, Str. Teodor Mi-
hali, No.58-60, RO-400591 Cluj Napoca, Romania

E-mail address: voichita.radu@econ.ubbcluj.ro, voichita.radu@gmail.com



EXISTENCE OF TWO SYMMETRIC SOLUTIONS FOR NEUMANN PROBLEMS

GHASEM A. AFROUZI, MAHNAZ BAGHERI, AND ARMIN HADJIAN

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Abstract. In this paper, we investigate the existence of at least two distinct cylindrically symmetric weak solutions for some elliptic problems involving a p -Laplace operator, subject to Neumann boundary conditions in a strip-like domain of the Euclidean space.

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1. INTRODUCTION

Let $\mathcal{O} \subset \mathbb{R}^m$ be a bounded domain with smooth boundary and $\Omega := \mathcal{O} \times \mathbb{R}^n$ be a strip-like domain. Define the space of cylindrically symmetric functions by

$$W_c^{1,p}(\Omega) := \{u \in W^{1,p}(\Omega) : u(x, \cdot) \text{ is radially symmetric for all } x \in \mathcal{O}\}.$$

In this space, Molica Bisci and Rădulescu in [7, Theorem 2.1] studied the existence of at least three cylindrically symmetric solutions for the following elliptic Neumann problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda \alpha(x, y) f(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where ν denotes the outward unit normal to $\partial\Omega$, $p > m + n$ is a real number, λ is a positive real parameter and $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. Moreover, $\alpha \in L^1(\Omega)$ is a non-negative cylindrically symmetric function and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

In this paper, our goal is to obtain the existence of at least two distinct cylindrically symmetric weak solutions for problem (1.1) under suitable conditions on α and f .

We denote by c_p the best embedding constant of $W_c^{1,p}(\Omega)$ into $L^\infty(\Omega)$, i.e.,

$$c_p := \sup_{u \in W_c^{1,p}(\Omega)} \frac{\|u\|_{L^\infty(\Omega)}}{\|u\|_{W_c^{1,p}(\Omega)}}, \quad (1.2)$$

where

$$\|u\|_{L^\infty} := \operatorname{esssup}_{(x,y) \in \Omega} |u(x,y)|;$$

see [4, Theorem 2.2]. Further, Let $\alpha \in L^1(\Omega)$ is a non-negative cylindrically symmetric function such that

$$\alpha_0 := \inf_{(x,y) \in \Omega} \alpha(x,y) > 0,$$

and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying the following condition:

$$(f_1) \quad |f(t)| \leq a_1 + a_2|t|^{s-1}, \quad \forall t \in \mathbb{R},$$

for some non-negative constants a_1, a_2 and $s > p$. We put $F(\xi) := \int_0^\xi f(t)dt$, for every $\xi \in \mathbb{R}$. Moreover, we introduce the functional $I_\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ associated with problem (1.1),

$$I_\lambda(u) := \frac{1}{p} \left(\int_\Omega |\nabla u(x,y)|^p dx dy + \int_\Omega |u(x,y)|^p dx dy \right) - \lambda \int_\Omega \alpha(x,y) F(u(x,y)) dx dy.$$

Fixing the real parameter λ , a function $u \in W^{1,p}(\Omega)$ is said to be a weak solution of (1.1) if for all $v \in W^{1,p}(\Omega)$,

$$\begin{aligned} & \int_\Omega |\nabla u(x,y)|^{p-2} \nabla u(x,y) \cdot \nabla v(x,y) dx dy + \int_\Omega |u(x,y)|^{p-2} u(x,y) v(x,y) dx dy \\ &= \lambda \int_\Omega \alpha(x,y) f(u(x,y)) v(x,y) dx dy. \end{aligned}$$

Hence, the critical points of I_λ are exactly the weak solutions of problem (1.1).

Definition 1. A Gâteaux differentiable function I satisfies the Palais-Smale condition (in short (PS)-condition) if any sequence $\{u_n\}$ such that

- (a) $\{I_\lambda(u_n)\}$ is bounded,
- (b) $\|I'_\lambda(u_n)\|_{X^*} \rightarrow 0$, as $n \rightarrow \infty$,

has a convergent subsequence.

We shall prove our results applying the following critical point theorem, which is a more precise version of Ricceri's variational principle [12, Theorem 2.5]. We point out that Ricceri's variational principle generalizes the celebrated three critical point theorem of Pucci and Serrin [9, 10] and is an useful result that gives alternatives for the multiplicity of critical points of certain functions depending on a parameter.

Theorem 1 (see [2, Theorem 3.2]). *Let X be a real Banach space and let $\Phi, \Psi : X \rightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{u \in \Phi^{-1}[-\infty, r]} \Psi(u) < +\infty$ and assume that, for each*

$$\lambda \in \left[0, \frac{r}{\sup_{u \in \Phi^{-1}[-\infty, r]} \Psi(u)} \right],$$

the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies (PS)-condition and it is unbounded from below. Then, for each $\lambda \in]0, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}[$, the functional I_λ admits two distinct critical points.

For completeness, we refer the interested reader to the recent papers [3, 6] where Ricceri's variational principle has been developed on studying nonlinear Neumann problems. See also [1, 5].

2. MAIN RESULTS

In this section we establish the main abstract result of this paper. We recall that c_p is the constant of the continuous embedding $W_c^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$; see (1.2).

Theorem 2. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying condition (f_1) . Moreover, assume that*

(f_2) there exist two constants $\eta > p$ and $L > 0$ such that

$$0 < \eta F(t) \leq t f(t), \quad |t| \geq L.$$

Then, for each $\lambda \in]0, \lambda^[$, problem (1.1) admits at least two distinct cylindrically symmetric weak solutions, where*

$$\lambda^* := \frac{s}{(s a_1 c_p p^{1/p} + a_2 c_p^s p^{s/p}) \|\alpha\|_{L^1}}.$$

Proof. Our aim is to apply Theorem 1 to problem (1.1) in the case $r = 1$ to the Banach space $X := W_c^{1,p}(\Omega)$ endowed with the norm

$$\|u\|_{W^{1,p}} := \left(\int_{\Omega} |\nabla u(x, y)|^p dx dy + \int_{\Omega} |u(x, y)|^p dx dy \right)^{1/p}.$$

For every $u \in X$ we set

$$\Phi(u) := \frac{\|u\|_{W^{1,p}}^p}{p}, \quad \Psi(u) := \int_{\Omega} \alpha(x, y) F(u(x, y)) dx dy.$$

Clearly Φ and Ψ are continuously Gâteaux differentiable and

$$\Phi'(u)(v) :=$$

$$\int_{\Omega} |\nabla u(x, y)|^{p-2} \nabla u(x, y) \cdot \nabla v(x, y) dx dy + \int_{\Omega} |u(x, y)|^{p-2} u(x, y) v(x, y) dx dy,$$

and

$$\Psi'(u)(v) := \int_{\Omega} \alpha(x, y) f(u(x, y)) v(x, y) dx dy,$$

for every $v \in X$. Moreover, Φ' admits a continuous inverse on X^* and Ψ' is a compact operator.

Now we prove that $I_\lambda := \Phi - \lambda\Psi$ satisfies (PS)-condition for every $\lambda > 0$. Namely, we will prove that any sequence $\{u_n\} \subset X$ satisfying

$$m := \sup_n I_\lambda(u_n) < +\infty, \quad \lim_{n \rightarrow +\infty} \|I'_\lambda(u_n)\|_{X^*} = 0,$$

contains a convergent subsequence. From above, we can actually assume that

$$\left| \frac{1}{\eta} \langle I'_\lambda(u_n), u_n \rangle \right| \leq \|u_n\|_{W_{1,p}}.$$

For n large enough, we have

$$\begin{aligned} m \geq I_\lambda(u_n) &= \frac{1}{p} \left(\int_\Omega |\nabla u_n(x, y)|^p dx dy + \int_\Omega |u_n(x, y)|^p dx dy \right) \\ &\quad - \lambda \int_\Omega \alpha(x, y) F(u_n(x, y)) dx dy, \end{aligned}$$

then

$$\begin{aligned} I_\lambda(u_n) &\geq \frac{1}{p} \left(\int_\Omega |\nabla u_n(x, y)|^p dx dy + \int_\Omega |u_n(x, y)|^p dx dy \right) \\ &\quad - \frac{\lambda}{\eta} \int_\Omega \alpha(x, y) f(u_n(x, y)) u_n(x, y) dx dy \\ &= \left(\frac{1}{p} - \frac{1}{\eta} \right) \left(\int_\Omega |\nabla u_n(x, y)|^p dx dy + \int_\Omega |u_n(x, y)|^p dx dy \right) \\ &\quad + \frac{1}{\eta} \left(\int_\Omega |\nabla u_n(x, y)|^p dx dy + \int_\Omega |u_n(x, y)|^p dx dy \right) \\ &\quad - \lambda \int_\Omega \alpha(x, y) f(u_n(x, y)) u_n(x, y) dx dy \\ &= \left(\frac{1}{p} - \frac{1}{\eta} \right) \|u_n\|_{W_{1,p}}^p + \frac{1}{\eta} \langle I'_\lambda(u_n), u_n \rangle. \end{aligned}$$

Thus,

$$m + \|u_n\|_{W_{1,p}} \geq I_\lambda(u_n) - \frac{1}{\eta} \langle I'_\lambda(u_n), u_n \rangle \geq \left(\frac{1}{p} - \frac{1}{\eta} \right) \|u_n\|_{W_{1,p}}^p.$$

Consequently, $\{\|u_n\|\}$ is bounded. By the Eberlian-Smulyan theorem, without loss of generality, we assume that $u_n \rightharpoonup u$. Then $\Psi'(u_n) \rightarrow \Psi'(u)$ because of compactness. Since $I'_\lambda(u_n) = \Phi'(u_n) - \lambda\Psi'(u_n) \rightarrow 0$, then $\Phi'(u_n) \rightarrow \lambda\Psi'(u)$. Since Φ' has a continuous inverse, then $u_n \rightarrow u$ and so I_λ satisfies (PS)-condition.

From (f₂), there is a positive constant C such that

$$F(t) \geq C|t|^\eta \tag{2.1}$$

for all $|t| > L$. In fact, setting $b := \min_{|\xi|=L} F(\xi)$ and

$$\varphi_t(\beta) := F(\beta t), \quad \forall \beta > 0, \tag{2.2}$$

by (f₂), for every $|t| > L$ one has

$$0 < \eta \varphi_t(\beta) = \eta F(\beta t) \leq \beta t \cdot f(\beta t) = \beta \varphi'_t(\beta), \quad \forall \beta > \frac{L}{|t|}.$$

Therefore,

$$\int_{L/|t|}^1 \frac{\varphi'_t(\beta)}{\varphi_t(\beta)} d\beta \geq \int_{L/|t|}^1 \frac{\eta}{\beta} d\beta.$$

Then

$$\varphi_t(1) \geq \varphi_t\left(\frac{L}{|t|}\right) \frac{|t|^\eta}{L^\eta}.$$

Taking into account of (2.2), we obtain

$$F(t) \geq F\left(\frac{L}{|t|}t\right) \frac{|t|^\eta}{L^\eta} \geq b \frac{|t|^\eta}{L^\eta} \geq C |t|^\eta,$$

where $C > 0$ is a constant. Thus, (2.1) is proved.

Fixed $u_0 \in X \setminus \{0\}$, for each $t > 1$ one has

$$I_\lambda(tu_0) \leq \frac{1}{p} t^p \|u_0\|_{W^{1,p}}^p - \lambda \alpha_0 C t^\eta \int_{\Omega} |u_0(x, y)|^\eta dx dy.$$

Since $\eta > p$, this condition guarantees that I_λ is unbounded from below. Fixed $\lambda \in]0, \lambda^*[$, from definition of Φ it follows that

$$\|u\|_{W^{1,p}} < p^{1/p}, \quad (2.3)$$

for each $u \in X$ such that $u \in \Phi^{-1}(]-\infty, 1])$. Moreover, (f₁), the compact embedding $X \hookrightarrow L^\infty(\Omega)$ and (2.3) imply that, for each $u \in \Phi^{-1}(]-\infty, 1])$, we have

$$\begin{aligned} \Psi(u) &\leq \int_{\Omega} \alpha(x, y) (a_1 |u(x, y)| + \frac{a_2}{s} |u(x, y)|^s) dx dy \\ &\leq (a_1 \|u\|_{L^\infty} + \frac{a_2}{s} \|u\|_{L^\infty}^s) \|\alpha\|_{L^1} \\ &\leq (a_1 c_p \|u\|_{W^{1,p}} + \frac{a_2 c_p^s}{s} \|u\|_{W^{1,p}}^s) \|\alpha\|_{L^1} \\ &< (a_1 c_p p^{1/p} + \frac{a_2}{s} c_p^s p^{s/p}) \|\alpha\|_{L^1}, \end{aligned}$$

and so,

$$\sup_{u \in \Phi^{-1}(]-\infty, 1])} \Psi(u) \leq (a_1 c_p p^{1/p} + \frac{a_2}{s} c_p^s p^{s/p}) \|\alpha\|_{L^1} = \frac{1}{\lambda^*} < \frac{1}{\lambda} \quad (2.4)$$

From (2.4) one has

$$\lambda \in]0, \lambda^*[\subseteq \left[0, \frac{1}{\sup_{u \in \Phi^{-1}(]-\infty, 1])} \Psi(u)} \right[.$$

Hence, Theorem 1.2 assures the existence of at least two distinct critical points for problem (1.1). Also, it is proved in [7, proof of Theorem 2.1] that I_λ is an invariant functional with respect to the action of the compact group of linear isometries of $\mathbb{R}^n$. Thus, we can apply the principle of symmetric criticality (see [8]) to the smooth and isometric invariant functional I_λ and deduce that problem (1.1) admits at least two distinct cylindrically symmetric weak solutions. The proof is complete. $\square$

Remark 1. We observe that, if f is non-negative and $f(0) \neq 0$, then Theorem 2 ensures the existence of two positive cylindrically symmetric weak solutions for problem (1.1) (see, e.g., [11, Theorem 11.1]).

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*Authors' addresses***Ghasem A. Afrouzi**

Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

E-mail address: afrouzi@umz.ac.ir

Mahnaz Bagheri

Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

E-mail address: m.bagheri@yahoo.com

Armin Hadjian

Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, P.O. Box 1339, Bojnord 94531, Iran

E-mail address: hadjian83@gmail.com, a.hadjian@ub.ac.ir



ON THE BARR EXACTNESS PROPERTY OF $\mathbf{BMod}/R$

H. GÜLSÜN AKAY AND U. EGE ARSLAN

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Abstract. In this work, it is shown that the category $\mathbf{BMod}/R$ of braided crossed modules over a fixed commutative algebra R is an exact category in the sense of Barr.

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1. INTRODUCTION

Exact categories for additive categories were introduced by Quillen [9]. Barr defined exact categories for non-additive categories [1]. He introduced exact categories in order to define a good notion of non-abelian cohomology.

We will mention regular categories closely related by exact categories. There exist in the literature many different definitions of regular categories, which are all equivalent under the assumptions that finite limits and coequalisers exist. We will recall the following definition from [7], but a weaker version of it is given in [1].

A category C is called regular if it satisfies the following three properties:

- i: C is finitely complete,
- ii: If $f : X \longrightarrow Y$ is a morphism in C , and

$$\begin{array}{ccc} Z & \xrightarrow{p_0} & X \\ p_1 \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback (then $Z \rightrightarrows X$ is called the kernel pair of f), the coequaliser of p_0, p_1 exists;

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iii: If $f : X \longrightarrow Y$ is a morphism in C , and

$$\begin{array}{ccc} W & \longrightarrow & X \\ g \downarrow & & \downarrow f \\ Z & \longrightarrow & Y \end{array}$$

is a pullback, and if f is a regular epimorphism, then g is a regular epimorphism as well.

If a regular category additionally has the property that every equivalence relation is effective that is every equivalence relation is a kernel pair, then it is called a Barr exact category.

Examples of exact categories are:

- (1) The category *Set* of sets.
- (2) The category of non empty sets.
- (3) Any abelian category.
- (4) Every partially ordered set considered as a category.
- (5) For any small category C , the functor category (C^{op}, Set) .

Brown and Gilbert introduced in [3] the notion of braided regular crossed module of groupoids and groups as an algebraic model for homotopy 3-types equivalent to Conduché's 2-crossed module. The reduced case of braided regular crossed module is called a braided crossed module of groups. Braided crossed modules in the category of commutative algebras were defined by Ulualan in [10].

The purpose of this paper is to answer the question whether the category of braided crossed modules of commutative algebras is exact. We prove that the category of braided crossed modules of commutative algebras is an exact category.

Throughout this paper k will be a fixed commutative ring with $0 \neq 1$. All k -algebras will be commutative and associative.

2. BRAIDED CROSSED MODULES

In order to give the notion of braided crossed modules of commutative algebras, we will recall the concept of crossed modules of commutative algebras. Crossed modules of groups originate in algebraic topology and more particularly in homotopy theory. Mac Lane and Whitehead showed in [6] that crossed modules of groups modelled homotopy 2-types (3-types in their notation). The commutative algebra case of crossed modules is contained in the paper of Lichtenbaum and Schlessinger [5] and also in the work of Gerstenhaber [4] under different names. Some categorical results and Koszul complex link are also given by Porter [8].

Definition 1 ([8]). A crossed module of commutative algebras, (C, R, ∂) , is an R -algebra, C , together with an R -algebra morphism $\partial : C \longrightarrow R$ such that for all

$c, c' \in C$

$$\partial(c) \cdot c' = cc'$$

where R is a k -algebra. A morphism of crossed modules from (C, R, ∂) to (C', R', ∂') is a pair of k -algebra morphisms, $\phi : C \rightarrow C'$ and $\psi : R \rightarrow R'$ such that

$$(i) \partial' \phi = \psi \partial \quad \text{and} \quad (ii) \phi(r \cdot c) = \psi(r) \cdot \phi(c)$$

for all $r \in R$ and $c \in C$. We thus get the category XMod of crossed modules.

There is, for a fixed algebra R , a subcategory XMod/R of the category of crossed modules, which has as objects those crossed modules with R as the “base”, i.e., all (C, R, ∂) for this fixed R , and having as morphisms from (C, R, ∂_1) to (C', R, ∂_2) just those (f_1, f_0) in XMod in which $f_0 : R \rightarrow R$ is the identity homomorphism on R .

Definition 2 ([10]). A braided crossed module of commutative algebras $\partial : C \rightarrow R$ is a crossed module with the braiding function $\{-, -\} : R \times R \rightarrow C$ satisfying the following axioms:

$$\text{BCM1)} \partial\{r, r'\} = rr',$$

$$\text{BCM2)} \{\partial c, \partial c'\} = cc',$$

$$\text{BCM3)} \{\partial c, r\} = r \cdot c \text{ and } \{r, \partial c\} = r \cdot c,$$

$$\text{BCM4)} \{rr', r''\} - \{r, r'r''\} = 0,$$

for all $r, r', r'' \in R$ and $c, c' \in C$.

We denote such a braided crossed module of commutative algebras by $\{C, R, \partial\}$.

If $\{C, R, \partial_1\}$ and $\{C', R', \partial_2\}$ are braided crossed modules, a morphism

$$(f_1, f_0) : \{C, R, \partial_1\} \rightarrow \{C', R', \partial_2\},$$

of braided crossed modules is given by a morphism of crossed modules such that

$$\{-, -\}(f_0 \times f_0) = f_1 \{-, -\}.$$

We thus get the category $\text{BXM}(\text{MOD})$ of braided crossed modules of commutative algebras.

In the case of a morphism (f_1, f_0) between braided crossed modules with the same base R , i.e. where f_0 is the identity on R with $f_1 \{-, -\} = \{-, -\}$, then we say that f_1 is a morphism of braided crossed R -modules, $\partial_1 : C \rightarrow R$ is a braided crossed R -module and we use $\{C, \partial_1\}$ instead of $\{C, R, \partial_1\}$. This gives a subcategory $\text{BXM}(\text{MOD})/R$ of $\text{BXM}(\text{MOD})$. Our results are obtained for this subcategory.

Several well known examples of crossed modules give rise to braided crossed modules as follows.

Examples of braided crossed modules

- (1) Any identity map of k -algebras $\partial : X \rightarrow X$ is a braided crossed module with $\{x, y\} = xy$.
- (2) If C is a k -algebra and C^2 is an ideal generated by $\{c_1 c_2 \mid c_1, c_2 \in C\}$. Then $\partial : C^2 \rightarrow C$ is a braided crossed module with $\{c_1, c_2\} = c_1 c_2$, for $c_1, c_2 \in C$.

(3) Any R -module M can be considered as an R -algebra with zero multiplication and hence the zero morphism $0 : M \longrightarrow R$ is a braided crossed module with $\{r, r'\} = 0$.

(4) Let $\{C, R, \partial_1\}$ and $\{C', R', \partial_2\}$ be two braided crossed modules, then

$$\{C \times C', R \times R', \partial\}$$

is a braided crossed module.

3. THE BARR EXACTNESS PROPERTY OF BRAIDED CROSSED MODULES

Our aim now is to obtain that $\mathbf{BMod}/R$ is an exact category. For this purpose, we have to prove some statements in this section.

Proposition 1. *In $\mathbf{BMod}/R$ every pair of morphisms with common domain and codomain has an equaliser.*

Proof. Let $f, g : \{C, \partial\} \longrightarrow \{D, \delta\}$ be two morphisms of braided crossed R -modules. Let E denotes the set $E = \{c \in C \mid f(c) = g(c)\}$. It can be easily checked that $\{E, \varepsilon\}$ has the structure of a braided crossed R -module, and the inclusion $u : \{E, \varepsilon\} \longrightarrow \{C, \partial\}$ is a morphism of braided crossed R -modules and clearly $fu = gu$.

Suppose that there exist a braided crossed R -module, $\{E', \varepsilon'\}$ and a morphism $u' : \{E', \varepsilon'\} \longrightarrow \{C, \partial\}$ of braided crossed R -modules such that $fu' = gu'$. Then for all $x \in E'$, $f(u'(x)) = g(u'(x))$, and hence $u'(x) \in E$. Thus, we have $\alpha : \{E', \varepsilon'\} \longrightarrow \{E, \varepsilon\}$ by $\alpha(x) = u'(x)$ from which we get

$$\varepsilon\alpha(x) = \varepsilon u'(x) = \partial u'(x) = \varepsilon'(x),$$

$$\alpha(r \cdot x) = u'(r \cdot x) = r \cdot u'(x) = r \cdot \alpha(x),$$

for all $x \in E', r \in R$. Since u, u' are braided crossed R -module morphisms, it is clear that $\alpha\{r, r'\} = \{r, r'\}$ for all $r, r' \in R$.

Let $\alpha' : E' \longrightarrow E$ be a morphism of braided crossed R -module such that $u\alpha' = u'$. Since $\alpha(x) = u'(x) = u\alpha'(x) = \alpha'(x)$ for all $x \in E'$, we have that α is the unique morphism which makes the diagram

$$\begin{array}{ccccc} \{E, \varepsilon\} & \xrightarrow{u} & \{C, \partial\} & \xrightleftharpoons[g]{f} & \{D, \delta\} \\ \uparrow \alpha & & \nearrow u' & & \\ \{E', \varepsilon'\} & & & & \end{array}$$

commutative. Hence u is the equaliser of (f, g) , as required. $\square$

Proposition 2. *$\mathbf{BMod}/R$ has finite products.*

Proof. Let $\{C, \partial\}$ and $\{D, \delta\}$ be braided crossed R -modules. The product $\{C \sqcap D, \tau\}$ is the pullback over the terminal object $\{R, i_R\}$,

$$\begin{array}{ccc} \{C \sqcap D, \tau\} & \xrightarrow{\partial'} & \{C, \partial\} \\ \delta' \downarrow & & \downarrow \partial \\ \{D, \delta\} & \xrightarrow{\delta} & \{R, i_R\} \end{array}$$

where $C \sqcap D = \{(c, d) \mid \partial(c) = \delta(d)\}$ and $\tau : C \sqcap D \rightarrow R$ is defined by $\tau = \partial\partial' = \delta\delta'$. Then by induction, $\mathbf{BMod}/R$ has finite products. $\square$

Proposition 3. $\mathbf{BMod}/R$ is finitely complete.

Proof. Follows from Propositions 1 and 2. $\square$

Proposition 4. In $\mathbf{BMod}/R$ every morphism has a kernel pair, and the kernel pair has a coequaliser.

Proof. Let $\{A, \partial\}$ and $\{B, \beta\}$ be two braided crossed R -modules. Let $f : \{A, \partial\} \rightarrow \{B, \beta\}$ be a morphism of braided crossed R -modules. Then (A, f) is a crossed B -module, where B acts on A via β and the homomorphism $\alpha : A \times_B A \rightarrow B$ defined by $\alpha(a, a') = f(a) = f(a')$ is a crossed B -module, where

$$A \times_B A = \{(a, a') \mid f(a) = f(a')\}$$

is a B -algebra:

$$\alpha(b \cdot (a, a')) = f(b \cdot a) = b \cdot f(a) = b \cdot \alpha(a, a'),$$

$$\alpha(a, a') \cdot (a_1, a'_1) = (f(a) \cdot a_1, f(a') \cdot a'_1) = (a, a') (a_1, a'_1),$$

for all $(a, a'), (a_1, a'_1) \in A \times_B A, b \in B$.

We can define $\partial' : A \times_B A \rightarrow R$ by $\partial'(a, a') = \partial(a) = \partial(a')$, since $\beta\alpha = \beta f = \partial$. It is easily checked that $(A \times_B A, R, \partial')$ is a crossed module:

$$\partial'(r \cdot (a, a')) = \partial(r \cdot a) = r \cdot \partial(a) = r \cdot \partial'(a, a'),$$

$$\begin{aligned} \partial'(a, a') \cdot (a_1, a'_1) &= \partial(a) \cdot (a_1, a'_1) \\ &= (\partial(a) \cdot a_1, \partial(a) \cdot a'_1) \\ &= (\partial(a) \cdot a_1, \partial(a') \cdot a'_1) \\ &= (aa_1, a'a'_1) \\ &= (a, a') (a_1, a'_1), \end{aligned}$$

for all $(a, a'), (a_1, a'_1) \in A \times_B A, r \in R$.

Below we will show that $\{A \times_B A, \partial'\}$ is a braided crossed R -module with the braiding map

$$\{-, -\} : R \times R \longrightarrow A \times_B A$$

defined by $\{r, r'\} = (\{r, r'\}, \{r, r'\})$,

BCM1)

$$\partial' \{r, r'\} = \partial \{r, r'\} = rr',$$

BCM2)

$$\begin{aligned} \{\partial'(a, a'), \partial'(b, b')\} &= (\{\partial a, \partial b\}, \{\partial a', \partial b'\}) \\ &= (ab, a'b') \\ &= (a, a')(b, b'), \end{aligned}$$

BCM3)

$$\begin{aligned} \{\partial'(a, a'), r\} &= (\{\partial a, r\}, \{\partial a', r\}) \\ &= (r \cdot a, r \cdot a') \\ &= r \cdot (a, a') \\ \{r, \partial'(a, a')\} &= (\{r, \partial a\}, \{r, \partial a'\}) \\ &= (r \cdot a, r \cdot a') \\ &= r \cdot (a, a'), \end{aligned}$$

BCM4)

$$\begin{aligned} \{rr', r''\} - \{r, r'r''\} &= (\{rr', r''\}, \{rr', r''\}) - (\{r, r'r''\}, \{r, r'r''\}) \\ &= (\{rr', r''\} - \{r, r'r''\}, \{rr', r''\} - \{r, r'r''\}) \\ &= (0, 0), \end{aligned}$$

for all $(a, a'), (b, b') \in A \times_B A, r, r', r'' \in R$.

The following diagram

$$\begin{array}{ccccc} A \times_B A & \xrightleftharpoons[p_2]{p_1} & A & \xrightarrow{f} & B \\ \downarrow \partial' & & \downarrow \partial & & \downarrow \beta \\ R & \xlongequal{\quad} & R & \xlongequal{\quad} & R \end{array}$$

commutes and the morphisms p_1 and p_2 above are morphisms of braided crossed R -modules. This construction satisfies universal property: Let $\{E, \delta\}$ be a braided

crossed R -module and $p'_1, p'_2 : \{E, \delta\} \longrightarrow \{A, \partial\}$ be any morphisms of braided crossed R -modules with $f p'_1 = f p'_2$, then there exist a unique morphism

$$h : \{E, \delta\} \longrightarrow \{A \times_B A, \partial'\}$$

given by $h(e) = (p'_1(e), p'_2(e))$, for all $e \in E$, which makes the diagram

$$\begin{array}{ccccc}
 \{E, \delta\} & & & & \\
 \swarrow p'_1 & \searrow h & & & \\
 & \{A \times_B A, \partial'\} & \xrightarrow{p_1} & \{A, \partial\} & \\
 \downarrow p'_2 & \downarrow p_2 & & \downarrow f & \\
 & \{A, \partial\} & \xrightarrow{f} & \{B, \beta\} &
 \end{array}$$

commutative.

Then (p_1, p_2) is the kernel pair of the morphism f .

Now we will show that the pair (p_1, p_2) has a coequaliser. Let I be an ideal of A generated by all the elements of the form $p_1(x) - p_2(x)$, for all $x = (a, a') \in A \times_B A$. We will define the braided crossed R -module $\delta : A/I \longrightarrow R$ by $\delta(a + I) = \partial(a)$, $\{r, r'\} = \{r, r'\} + I$ for $a \in A, r, r' \in R$. Since $I \subseteq \text{Ker } \partial$, δ is well defined. By the definition of $\{A/I, \delta\}$, the morphism $q : \{A, \partial\} \longrightarrow \{A/I, \delta\}$ is the induced projection and it is a morphism of braided crossed R -modules, i.e., the diagram

$$\begin{array}{ccccc}
 A \times_B A & \xrightleftharpoons[p_2]{p_1} & A & \xrightarrow{q} & A/I \\
 & \searrow \partial' & \downarrow \partial & \swarrow \delta & \\
 & & R & &
 \end{array}$$

commutes. Suppose there exist a braided crossed R -module $\{A', \alpha'\}$ and a morphism of braided crossed R -modules $q' : \{A, \partial\} \longrightarrow \{A', \alpha'\}$ such that $q' p_1 = q' p_2$, then there exists a unique morphism $\varphi : \{A/I, \alpha\} \longrightarrow \{A', \alpha'\}$ defined by $\varphi(a + I) = q'(a)$, satisfying $\varphi q = q'$, i.e. q is the universal among all the morphisms $q' :$

$\{A, \partial\} \longrightarrow \{A', \alpha'\}$ for any braided crossed R -module $\{A', \alpha'\}$. So we get the following commutative diagram:

$$\begin{array}{ccccc}
 A \times_B A & \xrightarrow[p_2]{p_1} & A & \xrightarrow{q} & A/I \\
 & \searrow \partial' & \downarrow \partial & \swarrow q' & \downarrow \varphi \\
 & & R & \xleftarrow[\delta']{\delta} & A'
 \end{array}$$

Then q is the coequaliser of the pair (p_1, p_2) .

Therefore in $\mathbf{BMod}/R$, the kernel pair of every morphism exists and has a coequaliser. $\square$

Proposition 5. *In $\mathbf{BMod}/R$ every regular epimorphisms are stable under pullback.*

Proof. Let $\Phi : \{C, \mu\} \longrightarrow \{D, \sigma\}$ be a regular epimorphism in $\mathbf{BMod}/R$, which means that $C \longrightarrow D$ is a regular epimorphism of k -algebras. Let $\eta : \{G, \theta\} \longrightarrow \{D, \sigma\}$ be any morphism in $\mathbf{BMod}/R$. The pullback of Φ along η is $\Phi^* : \{C \times_D G, \rho\} \longrightarrow \{G, \theta\}$ given by $\Phi^*(c, g) = g$ where $C \times_D G = \{(c, g) \mid \Phi(c) = \eta(g)\}$ and $\rho : C \times_D G \longrightarrow R$ is a braided crossed R -module defined by $\rho(c, g) = \mu(c) = \theta(g)$ with $\{r, r'\} = (\{r, r'\}, \{r, r'\})$, for all $r, r' \in R, c \in C, g \in G$. Since in the category of k -algebras the regular epimorphisms are characterised as the surjective homomorphisms, these are closed in this way under pullback. Thus Φ^* is a surjective homomorphism.

$$\begin{array}{ccc}
 \{C \times_D G, \rho\} & \xrightarrow{\quad} & \{C, \mu\} \\
 \Phi^* \downarrow & & \downarrow \Phi \\
 \{G, \theta\} & \xrightarrow{\eta} & \{D, \sigma\}
 \end{array}$$

We claim that the surjective morphism Φ^* is a regular epimorphism, that is Φ^* is the coequaliser of a pair of morphisms. Define

$$E = \{(x, y) \in (C \times_D G) \times (C \times_D G) \mid \Phi^*(x) = \Phi^*(y)\}$$

and let $\{E, \alpha\} \xrightarrow[p]{q} \{C \times_D G, \rho\}$ be the first and second projections. Since G is isomorphic to the quotient of $C \times_D G$, Φ^* is the coequaliser of p and q . Thus we get that if Φ is regular epimorphism, so is Φ^* , as required. $\square$

Theorem 1. $\mathbf{BMod}/R$ is regular.

Proof. The proof is a direct consequence of Propositions 3, 4 and 5. $\square$

Now we recall the definition of equivalence relation from [2].

Definition 3. Let A be a category with finite limits. If A is an object, a subobject $(d^0, d^1) : E \longrightarrow A \times A$ is called an equivalence relation if it is

ER1. Reflexive: There is an arrow $r : A \longrightarrow E$ such that $d^0 r = d^1 r = i_A$;

ER2. Symmetric: There is an arrow $s : E \longrightarrow E$ such that $d^0 s = d^1$ and $d^1 s = d^0$;

ER3. Transitive: If

$$\begin{array}{ccc} T & \xrightarrow{q_1} & E \\ q_2 \downarrow & & \downarrow d^0 \\ E & \xrightarrow{d^1} & A \end{array}$$

is a pullback, there is an arrow $t : T \longrightarrow E$ such that $d^1 t = d^1 q_1$ and $d^0 t = d^0 q_2$.

Proposition 6. Every equivalence relation

$$\{E, \partial\} \xrightleftharpoons[v]{u} \{A, \alpha\}$$

in the category of braided crossed R -modules is effective.

Proof. Let A/E be the set of all equivalence classes $[a]$ with respect to E , i.e., $[a] = \{b \in A \mid (a, b) \in E\}$. A/E has the structure of an R -algebra. $\bar{\alpha} : A/E \longrightarrow R$ induced by α is well defined since $\alpha u = \alpha v$. We can form the braided crossed R -module $\bar{\alpha}$ with braiding map $\{r, r'\} = [\{r, r'\}]$ and get the following diagram,

$$\begin{array}{ccccc} E & \xrightleftharpoons[v]{u} & A & \xrightarrow{q} & A/E \\ & \searrow \partial & \downarrow \alpha & \swarrow \bar{\alpha} & \\ & & R & & \end{array}$$

of morphisms of braided crossed R -modules, where q is the projection onto A/E . By the definition of an equivalence relation on A , we have $(u(x), v(x)) \in E$, for $x \in E$. Since

$$E \subseteq A \times_R A = \{(a, a') \mid \alpha(a) = \alpha(a')\},$$

we get $\alpha u(x) = \alpha v(x)$, thus $(0, u(x) - v(x)) \in E$, therefore $qu = qv$. Suppose that there exist a braided crossed R -module $\{D, \omega\}$ with $u', v' : \{D, \omega\} \longrightarrow \{A, \alpha\}$ such that $qu' = qv'$, so $[u'(d)] = [v'(d)]$, i.e. $(u'(d), v'(d)) \in E$, and therefore there

exists a unique morphism $\theta : \{D, \omega\} \longrightarrow \{E, \partial\}$, such that the diagram

$$\begin{array}{ccccc}
 \{D, \omega\} & & & & \\
 \downarrow v' & \searrow \theta & \xrightarrow{u'} & & \\
 & \{E, \partial\} & \xrightarrow{u} & \{A, \alpha\} & \\
 & \downarrow v & & \downarrow q & \\
 & \{A, \alpha\} & \xrightarrow{q} & \{A/E, \bar{\alpha}\} &
 \end{array}$$

commutes.

Thus (u, v) is the kernel pair of a morphism $q : \{A, \alpha\} \longrightarrow \{A/E, \bar{\alpha}\}$ in the category of braided crossed R -modules. $\square$

Theorem 2. *The category $\mathbf{BMod}/R$ of braided crossed R -modules is a Barr exact category.*

Proof. The conditions of exact category for $\mathbf{BMod}/R$ are satisfied by Theorem 1 and Proposition 6, which completes the proof. $\square$

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*Authors' addresses***H. Gülsün Akay**

Eskişehir Osmangazi University, Department of Mathematics and Computer Science, 26480 Eskişehir,
Turkey

E-mail address: hgulsun@ogu.edu.tr

U. Ege Arslan

Eskişehir Osmangazi University, Department of Mathematics and Computer Science, 26480 Eskişehir,
Turkey

E-mail address: uege@ogu.edu.tr



IMPACT OF MIGRATION ON EPIDEMIOLOGICAL DYNAMICS WITH SATURATED INCIDENCE RATE

ALI AL-QAHTANI, SHABAN ALY, AND FATMA HUSSEIN

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Abstract. Recently an SIS epidemic reaction-diffusion model with Neumann (or no-flux) boundary condition has been proposed and studied by several authors to understand the dynamics of disease transmission in a spatially heterogeneous environment in which the individuals are subject to a random movement. In this paper an SIS epidemiological model with saturated incidence rate is proposed to describe the dynamics of disease spread among identical patches due to population migration. First the stability conditions for the endemic equilibrium for the corresponding kinetic system and reaction-diffusion system without diffusion are analyzed and proved. Moreover, we prove that at a critical value of the bifurcation parameter the positive endemic equilibrium becomes linear non-constant stationary solutions only when diffusion also plays a role in the reaction-diffusion system, which shows that the strong effects of diffusion on the Turing instability. Numerical simulations are provided to illustrate and extend the theoretical results.

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1. THE MODEL

Infectious diseases have tremendous influence on human life and will bring huge panic and disaster to mankind once out of control. Every year millions of human beings suffer from or die of various infectious diseases. In order to predict the spreading of infectious diseases, many epidemic models have been proposed and analyzed in recent years ([6, 10, 19, 20, 22]). In epidemiology, mathematical models have been an important method in analysing the spread and control of infectious diseases qualitatively and quantitatively. More recently, many studies have provided that spatial epidemic model is an appropriate tool for investigating fundamental mechanism of complex spatiotemporal epidemic dynamics ([3, 11]). In these studies, reaction diffusion equations have been intensively used to describe spatiotemporal dynamics and pattern formation in the spatial epidemic model, starting with the pioneer work of Turing. The Turing instability ([18]) has been extensively investigated for biological and chemical processes. Moreover, pattern formation from the Turing instability in

nonlinear complex systems is actively investigated in the fields such as social networks, molecular computing, embryology and elsewhere in biology and chemistry ([4, 5, 9, 12]).

Continuous models, usually in the form of nonlinear ordinary differential equations (NODE), have formed a large part of the traditional mathematical epidemiology literature. In such models, the classical assumptions are that the total population is divided into any number of classes according to their epidemiological status, and that the transmission of the infection in the population is modelled by incidence terms. Some infectious diseases do not confer immunity. Such infections do not have a recovered state and individuals become susceptible again after infection. Diseases such as tuberculosis, meningitis, sexually transmitted diseases or bacterial infections and gonorrhea exhibit this phenomenon ([4, 9, 12]). This type of disease can be modelled by the *SIS* type. Many forms are possible for the incidence term in epidemic models, the most common are the simple mass action and standard incidence terms. Li et al. ([8]) studied an *SIS* model with bilinear incidence rate βSI and treatment. In this paper, we consider saturation incidence rate and assume the force of infection is in this version $\frac{\beta S^n I^m}{1+cS}$ which is saturated with the susceptible. In this paper we demonstrate how a mathematical model can describe epidemiological phenomena and how can we use such a model to analyze endemic states and help eradicate disease. A model was set up using a system of nonlinear ordinary differential equations and show how the Turing phenomenon looks like in this ODE setting.

We consider a *SIS* type of disease transmission. The population is divided into two classes: susceptible individuals and infectious individuals. Susceptible individuals become infective after contact with infective individuals. Infective individuals return to susceptible class when they are recovered i.e. no immunity is conferred by going through the disease.

However, in nature, the tendency of the susceptible would be to keep away from the infected for the reason that the susceptible have ability to recognize the infected group and move away from them (cf. [7, 14–16]). Diseases such as tuberculosis, meningitis, bacterial infections and sexual diseases can be easily transmitted from one country (regions, cities or patch) to other countries (regions, cities or patch)(cf. [1, 2, 17]). Thus, it is important to consider the impact of migration on spread of a disease. Our interest is to study a *SIS* epidemiological model in patchy space in which the per capita migration rate of infective specie is influenced only by its own density (self-migration), and the per capita migration rate of susceptible specie is influenced not only by its own density but also to the density of the other one (cross-migration).

To formulate the *SIS* epidemic diffusion model in the following subsections, some assumptions and parameters are specified as follows.

The susceptible individuals and the infected individuals in city k at time t are denoted as $S_k(t)$ and $I_k(t)$, respectively ($k \in \{1, 2\}$; $t \in \mathbb{R}_0^+$). $1 \leq m, n \in \mathbb{N}$ are

constants describing the incidence rate of the disease. a and b are the natural birth rate and the natural death rate, respectively. γ is the recovery rate. α is the mortality caused by the disease. β is the disease transmission coefficient. c is the saturation factor that measures the inhibitory effect. $d_i > 0$ are the diffusion coefficients ($i \in \{1; 2\}$). $\rho \in C^1$ are positive functions modeling the cross-diffusion effect.

Hence, the ordinary differential equations (ODE) for the *SIS* epidemic diffusion model can be formulated as

$$\begin{aligned}\dot{S}_1(t) &= -\frac{\beta S_1^n I_1^m}{1 + c S_1} - b S_1 + \gamma I_1 + a(S_1 + I_1) + d_1(\rho(I_2)S_2 - \rho(I_1)S_1), \\ \dot{I}_1(t) &= \frac{\beta S_1^n I_1^m}{1 + c S_1} - (\alpha + b + \gamma)I_1 + d_2(I_2 - I_1), \\ \dot{S}_2(t) &= -\frac{\beta S_2^n I_2^m}{1 + c S_2} - b S_2 + \gamma I_2 + a(S_2 + I_2) + d_1(\rho(I_1)S_1 - \rho(I_2)S_2), \\ \dot{I}_2(t) &= \frac{\beta S_2^n I_2^m}{1 + c S_2} - (\alpha + b + \gamma)I_2 + d_2(I_1 - I_2).\end{aligned}\tag{1.1}$$

ODEs (1.1) describes the following dynamics of epidemic diffusion among the population groups. The per capita migration rate of infective specie is influenced only by its own density (self-migration), and the per capita migration rate of susceptible specie is influenced not only by its own density but also to the density of the other one (cross-migration).

2. MODEL ANALYSIS

In this section we analyze a simple model of the temporal behavior of an infectious disease which is not extended in space ($d_1 = d_2 = 0$),

$$\begin{aligned}\dot{S}_1(t) &= -\frac{\beta S_1^n I_1^m}{1 + c S_1} - b S_1 + \gamma I_1 + a(S_1 + I_1), \\ \dot{I}_1(t) &= \frac{\beta S_1^n I_1^m}{1 + c S_1} - (\alpha + b + \gamma)I_1, \\ \dot{S}_2(t) &= -\frac{\beta S_2^n I_2^m}{1 + c S_2} - b S_2 + \gamma I_2 + a(S_2 + I_2), \\ \dot{I}_2(t) &= \frac{\beta S_2^n I_2^m}{1 + c S_2} - (\alpha + b + \gamma)I_2.\end{aligned}\tag{2.1}$$

Define the new variable $N(t) = S(t) + I(t)$. In each patches, summing the two equations in (2) provides the equation for the total population $\dot{N}(t) = (a - b)N - \alpha I$.

Thus, the total population size may vary in time, as traditionally assumed. In the absence of a mortality rate caused by the disease, and assuming a greater birth rate than a mortality rate, i.e., $\alpha = 0$ and $a > b$, the population will grow indefinitely. Therefore, we could think of the disease as regulating the growth of the population. For this reason, we will assume from here on that $\alpha > a - b > 0$. These facts are known substantially from paper of Carrero et al (cf. [3]).

In this case the unique positive equilibrium of (2.1) is

- $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_1$ where $m = 1$ and

$$I = g_1(S) = \frac{(1 + cS)(\alpha + b + \gamma)}{\beta S}, \quad I = g_2(S) = \frac{a - b}{\alpha + b - a} \cdot S$$

resp.

- $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_m$ where $1 < m \in \mathbb{N}$ are the intersection of the curves

$$I = h_1(S) = \left(\frac{(\alpha + b + \gamma)(1 + cS)}{\beta S^m} \right)^{\frac{1}{m-1}}, \quad I = h_2(S) = \frac{a - b}{\alpha + b - a} \cdot S.$$

The Jacobian matrix of system (2.1) linearized at $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_1$ is

$$J(n, 1) = \begin{pmatrix} R & -(\alpha + b - a) & 0 & 0 \\ (a - b)T(n - \frac{c\bar{S}}{1 + c\bar{S}}) & 0 & 0 & 0 \\ 0 & 0 & R & -(\alpha + b - a) \\ 0 & 0 & (a - b)T(n - \frac{c\bar{S}}{1 + c\bar{S}}) & 0 \end{pmatrix} \quad (2.2)$$

where $T := \frac{\alpha + b + \gamma}{\alpha + b - a}$ and $R := (a - b)(1 - nT + \frac{c\bar{S}}{1 + c\bar{S}} \cdot T)$.

The equilibrium $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_1$ is locally asymptotically stable if the zeroes of the characteristic polynomial

$$P_4(\lambda, n, 1) = (P_2(\lambda, n, 1))^2, \quad (2.3)$$

have negative real part, where

$$P_2(\lambda, n, 1) = \lambda^2 - (a - b)(1 - nT + \frac{c\bar{S}}{1 + c\bar{S}} \cdot T)\lambda + (a - b)(\alpha + b + \gamma)(n - \frac{c\bar{S}}{1 + c\bar{S}}).$$

Because

$$\det(J(n, 1)) = (a - b)(\alpha + b + \gamma)(n - \frac{c\bar{S}}{1 + c\bar{S}}) > 0,$$

the zeroes of the characteristic polynomial (2.3) have negative real part, and the equilibrium point $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_1$ is locally asymptotically stable if $n > n^*$ and unstable if $n < n^*$, where

$$n^* = \frac{\alpha + b - a}{\alpha + b + \gamma} + \frac{c\bar{S}}{1 + c\bar{S}}. \quad (2.4)$$

The Jacobian matrix of system (2.1) linearized at $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_m$ is

$$J(n, m) = \begin{pmatrix} J_{11} & J_{12} & 0 & 0 \\ J_{21} & J_{22} & 0 & 0 \\ 0 & 0 & J_{11} & J_{12} \\ 0 & 0 & J_{21} & J_{22} \end{pmatrix}. \quad (2.5)$$

where

$$\begin{aligned} J_{11} &= (a-b)(1-nT + \frac{c\bar{S}}{1+c\bar{S}} \cdot T), & J_{12} &= -m(\alpha+b+\gamma) + a + \gamma \\ J_{21} &= (a-b)T(n - \frac{c\bar{S}}{1+c\bar{S}}), & J_{22} &= (m-1)(\alpha+b+\gamma). \end{aligned}$$

The characteristic polynomial of $J(n, m)$ is

$$Q_{kinetic}(\lambda, n, m) = (Q_2(\lambda, n, m))^2$$

where

$$Q_2(\lambda, n, m) = \lambda^2 - (J_{11} + J_{22})\lambda + J_{11} \cdot J_{22} - J_{12} \cdot J_{21}.$$

Because

$$\begin{aligned} J_{11} + J_{22} &= (a-b)(1-nT + \frac{c\bar{S}}{1+c\bar{S}} \cdot T) + (m-1)(\alpha+b+\gamma), \\ J_{11} \cdot J_{22} - J_{12} \cdot J_{21} &= (a-b)(1-nT + \frac{c\bar{S}}{1+c\bar{S}} \cdot T) \cdot (m-1)(\alpha+b+\gamma) \\ &\quad + [m(\alpha+b+\gamma) - a - \gamma] \cdot (a-b)T(n - \frac{c\bar{S}}{1+c\bar{S}}) \\ &= (a-b)(\alpha+b+\gamma) \left[(m-1) + (n - \frac{c\bar{S}}{1+c\bar{S}}) \right] > 0 \end{aligned}$$

the equilibrium point $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_m$ is asymptotically stable if $n > n^*$ (see Figure 1) and unstable if $n < n^*$, where

$$n^* = (m-1) \frac{\alpha+b-a}{a-b} + \frac{\alpha+b-a}{\alpha+b+\gamma} + \frac{c\bar{S}}{1+c\bar{S}}. \quad (2.6)$$

Note: The assumption comes from the fact that the total population is described by $N'(t) = (a-b)N - \alpha I$. Therefore, in order for the disease to have a regulatory effect on the population it is biologically reasonable to assume $\alpha > a-b > 0$.

3. THE IMPACT OF A SELF-MIGRATION

In this Section, we model the spatial spread as a diffusive process, where both classes have diffusion coefficient. The migration rate of each individuals

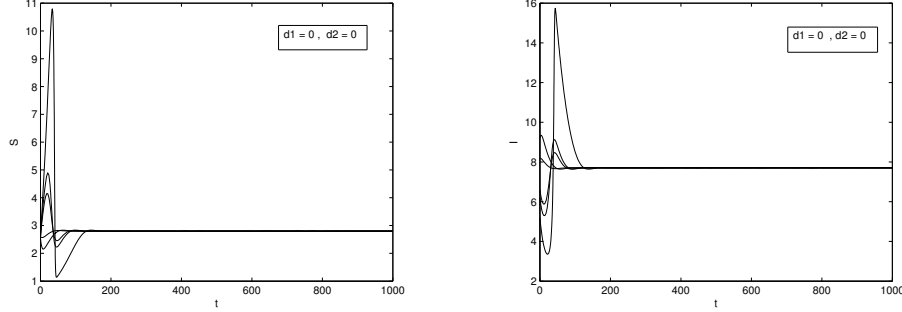


FIGURE 1. The unique endemic solution without diffusion ($d_1 = d_2 = 0$) at $a = 0.05$, $b = 0.006$, $\alpha = 0.06$, $\beta = 0.0056$, $\gamma = 0.04$, $c = 0.05$, $m = 2$, $n = 1$; is $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_m = (2.8013, 7.7036, 2.8013, 7.7036)$ which is stable. (Figure produced by applying MATLAB.)

is influenced only by its own density. The model can be written as:

$$\begin{aligned}
 \dot{S}_1(t) &= -\frac{\beta S_1^n I_1^m}{1 + c S_1} - b S_1 + \gamma I_1 + a(S_1 + I_1) + d_1(S_2 - S_1), \\
 \dot{I}_1(t) &= \frac{\beta S_1^n I_1^m}{1 + c S_1} - (\alpha + b + \gamma) I_1 + d_2(I_2 - I_1), \\
 \dot{S}_2(t) &= -\frac{\beta S_2^n I_2^m}{1 + c S_2} - b S_2 + \gamma I_2 + a(S_2 + I_2) + d_1(S_1 - S_2), \\
 \dot{I}_2(t) &= \frac{\beta S_2^n I_2^m}{1 + c S_2} - (\alpha + b + \gamma) I_2 + d_2(I_1 - I_2).
 \end{aligned} \tag{3.1}$$

Definition 1. The equilibrium $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_m$ of (1.1) is said to be diffusion-ally (Turing) unstable if it is an asymptotically stable of (2.1) but it is unstable with respect to (3.1) (cf. [4-8]).

The Jacobian matrix of the system (3.1) linearized at the endemic equilibria $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_m$ given by

$$J(d_1, d_2) = \begin{pmatrix} J_{11} - d_1 & J_{12} & d_1 & 0 \\ J_{21} & J_{22} - d_2 & 0 & d_2 \\ d_1 & 0 & J_{11} - d_1 & J_{12} \\ 0 & d_2 & J_{21} & J_{22} - d_2 \end{pmatrix}.$$

In order to have Turing instability of the system (3.1), the characteristic polynomial

$$\det(J(d_1, d_2) - \lambda I) = Q_2(\lambda, n, m) [\lambda^2 - \text{trace}(J(d_1, d_2)) + \det(J(d_1, d_2))], \quad (3.2)$$

must have at least one eigenvalue with positive real part.

The equilibrium $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_m$ is diffusionally unstable if and only if the following criteria holds:

- $Q_2(\lambda, n, m)$ has roots with negative real parts;
- $\det(J(d_1, d_2)) < 0$

since in this case $\text{trace}(J(d_1, d_2)) = \text{trace}(j(n, m)) - 2(d_1 + d_2) < 0$. This can prove the following Theorem.

Theorem 1. Assume that $n > n^*$ holds. Turing instability occurs if

$$d_1 > d_{1crit} = \frac{\det(J(n, m)) - 2d_2 \left[(a - b)(1 - nT + \frac{c}{1+c\bar{S}} \cdot T) \right]}{2(m-1)(\alpha + b + \gamma) - 4d_2} \quad (3.3)$$

Proof. Since

$$\begin{aligned} \det(J(d_1, d_2)) &= \det(J(n, m)) - 2d_2 \left[(a - b)(1 - nT + \frac{c\bar{S}}{1+c\bar{S}} \cdot T) \right] \\ &\quad - 2d_1(m-1)(\alpha + b + \gamma) + 4d_1d_2, \end{aligned}$$

therefore (3.3) implies Turing instability. $\square$

We apply our analytical approach to the following example and we are looking for conditions which imply Turing instability (diffusion driven instability).

Example 1. Trying to prepare an example comparable to that of [1], we choose $a = 0.05$, $b = 0.006$, $\alpha = 0.06$, $\beta = 0.0056$, $\gamma = 0.04$, $c = 0.05$, $m = 2$, $n = 1$, $d_2 = 0.05$.

The unique endemic equilibrium is $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_m = (2.8013, 7.7036, 2.8013, 7.7036)$. We consider d_1 as a bifurcation parameter. In this case at $d_{1crit} \cong 2.4974$, we have four eigenvalues $\lambda_i (i = 1, 2, 3, 4)$ such that $\Re(\lambda_i) < 0, (i = 1, 2, 3)$ and $\lambda_4 = 0$.

Thus,

- if $d_1 < d_{1crit}$ then $\Re(\lambda_i) < 0, (i = 1, 2, 3, 4)$ and $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_m$ is asymptotically stable;
- if $d_1 > d_{1crit}$ then $\Re(\lambda_i) < 0, (i = 1, 2, 3)$, $\lambda_4 > 0$, and $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_m$ is unstable.

Therefore, as d_1 is increases and passes through $d_1 = d_{1crit}$ then the spatially homogeneous equilibrium loses its stability (see Figure 2). Numerical calculations show that two new spatially non-constant equilibria emerge (see Figure 2).

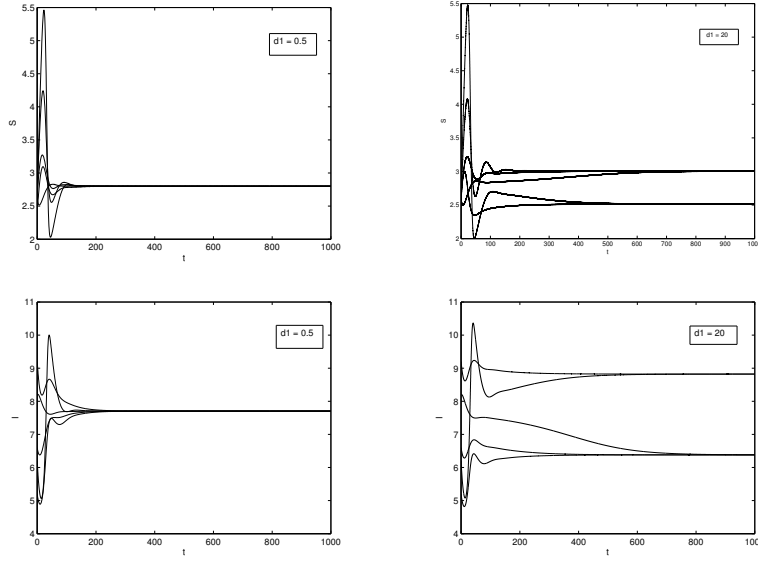


FIGURE 2. Left figures: The solutions S and I before bifurcation at $d_1 = 0.5$; the solution is stable. Right figures: The solutions S and I after bifurcation at $d_1 = 20$; the solutions loss its stability by Turing bifurcation. (Figure produced by applying MATLAB.)

4. THE IMPACT OF A CROSS-MIGRATION

In this Section, we model the spatial spread as a diffusive process, where both classes have diffusion coefficient. Rate of infected species is influenced only by its own density, i.e. there is no response to the density of the other one, but the susceptible species is influenced not only by its own but also by the other one's density (cf. [13, 21]). The Jacobian matrix of system (1.1) at $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_m$ can be written as:

$$J(d_1, d_2, \rho) = \begin{pmatrix} J_{11} - d_1\rho & J_{12} - d_1\rho'\bar{S} & d_1\rho & d_1\rho'\bar{S} \\ J_{21} & J_{22} - d_2 & 0 & d_2 \\ d_1\rho & d_1\rho'\bar{S} & J_{11} - d_1\rho & J_{12} - d_1\rho'\bar{S} \\ 0 & d_2 & J_{21} & J_{22} - d_2 \end{pmatrix}$$

where ρ and ρ' are to be taken at $\bar{S}$.

$$\det(J(d_1, d_2, \rho) - \lambda I) = \begin{vmatrix} J_{11} - d_1\rho - \lambda & J_{12} - d_1\rho'\bar{S} & d_1\rho & d_1\rho'\bar{S} \\ J_{21} & J_{22} - d_2 - \lambda & 0 & d_2 \\ d_1\rho & d_1\rho'\bar{S} & J_{11} - d_1\rho - \lambda & J_{12} - d_1\rho'\bar{S} \\ 0 & d_2 & J_{21} & J_{22} - d_2 - \lambda \end{vmatrix}. \quad (4.1)$$

Using the properties of determinant $C'_1 = C_3 + C_1, C'_2 = C_4 + C_2$ and $R'_3 = R_3 - R_1, R'_4 = R_4 - R_2$, we get

$$\det(J(d_1, d_2, \rho) - \lambda I) = \begin{vmatrix} J_{11} - \lambda & J_{12} & d_1\rho & d_1\rho'\bar{S} \\ J_{21} & J_{22} - \lambda & 0 & d_2 \\ 0 & 0 & J_{11} - 2d_1\rho - \lambda & J_{12} - 2d_1\rho'\bar{S} \\ 0 & 0 & J_{21} & J_{22} - 2d_2 - \lambda \end{vmatrix}. \quad (4.2)$$

The characteristic polynomial is

$$Q_{cross}(d_1, d_2) = Q_2(\lambda, n, m) [\lambda^2 - \text{trace}(J(d_1, d_2, \rho))\lambda + \det(J(d_1, d_2, \rho))]. \quad (4.3)$$

We know that $Q_2(\lambda, n, m)$ has two roots with negative real parts and $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_m$ is diffusionally unstable if and only if the following criteria holds:

- $Q_2(\lambda, n, m)$ has roots with negative real parts;
- $\det(J(d_1, d_2, \rho)) < 0$.

Thus can prove the following Theorem.

Theorem 2. Assume that $n > n^*$ holds. Turing instability occurs if

$$d_1 > d_{1_{crit}} = \frac{\det(J(n, m)) - 2d_2 \left[(a - b)(1 - nT + \frac{c\bar{S}}{1+c\bar{S}} \cdot T) \right]}{2\rho[(m - 1)(\alpha + b + \gamma) - 2d_2] - 2\rho'\bar{S}(a - b)T(n - \frac{c\bar{S}}{1+c\bar{S}})} \quad (4.4)$$

Proof. Since

$$\text{trace}(J(d_1, d_2, \rho)) = \text{trace}(J(n, m)) - 2(d_1\rho + d_2) < 0,$$

and

$$\begin{aligned}
\det(J(d_1, d_2, \rho)) &= \\
&= \det(J(n, m)) + 2d_2(2d_1\rho - J_{11}) - 2d_1\rho[J_{22} - \frac{\rho'}{\rho}\bar{S}J_{21}], \\
&= \det(J(n, m)) + 2d_2[2d_1\rho - (a - b)(1 - nT + \frac{c\bar{S}}{1 + c\bar{S}} \cdot T)] \\
&\quad - 2d_1\rho[(m - 1)(\alpha + b + \gamma) - \frac{\rho'}{\rho}\bar{S}(a - b)T(n - \frac{c\bar{S}}{1 + c\bar{S}})].
\end{aligned}$$

therefore (4.4) implies Turing instability.

The equilibrium $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_m$ of system (4.4) is asymptotically stable if $\frac{\rho'}{\rho}$ is sufficiently large; if $\frac{\rho'}{\rho}$ is sufficiently small and either ρ or d_1 are sufficiently big then $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_m$ loses its stability by a Turing bifurcation. $\square$

We illustrate the results by the following example of migration function and we are looking for conditions which imply Turing instability (diffusion driven instability).

Example 2. Trying to prepare an example comparable to that of [1], we choose $a = 0.05$, $b = 0.006$, $\alpha = 0.06$, $\beta = 0.0056$, $\gamma = 0.04$, $c = 0.05$, $m = 2$, $n = 1$, $d_2 = 0.05$, $\rho(I(t)) = h \exp(\frac{I(t)}{h})$.

The unique endemic equilibrium is $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_m = (2.8013, 7.7036, 2.8013, 7.7036)$. We consider h as a bifurcation parameter. In this case at h_{crit} , we have four eigenvalues λ_i ($i = 1, 2, 3, 4$) such that $\Re(\lambda_i) < 0$, ($i = 1, 2, 3$) and $\lambda_4 = 0$. Thus,

- if $h < h_{crit}$ then $\Re(\lambda_i) < 0$, ($i = 1, 2, 3, 4$) and $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_m$ is asymptotically stable;
- if $h > h_{crit}$ then $\Re(\lambda_i) < 0$, ($i = 1, 2, 3$), $\lambda_4 > 0$, and $(\bar{S}, \bar{I}, \bar{S}, \bar{I})_m$ is unstable.

Analytical studies shown that a cross-diffusion response can stabilize an unstable equilibrium of standard system (self-diffusion) and destabilize a stable equilibrium of standard system. Numerical studies shown that if the bifurcation parameter is increased through a critical value of the bifurcation parameter h the system undergoes a Turing bifurcation and the spatially homogeneous endemic equilibrium loses its stability and two new stable equilibria emerge. We conclude that the cross migration response is an important factor that should not be ignored when pattern emerges (see Figure 3).

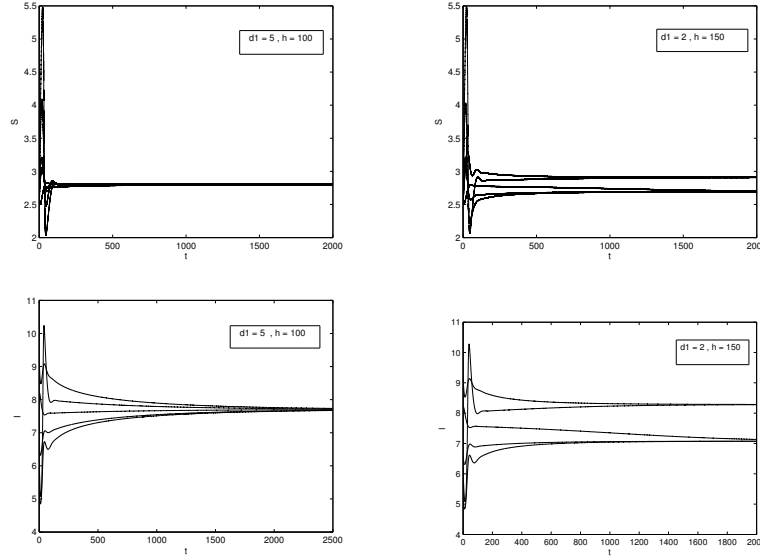


FIGURE 3. Left figures: The solutions S and I before bifurcation at $d_1 = 5$ and $h = 100$; the unstable solution with self diffusion stable is stable with cross diffusion. Right figures: The solutions S and I after bifurcation at $d_1 = 2$ and $h = 150$; the solutions loss its stability by Turing bifurcation. (Figure produced by applying MATLAB.)

It should be noted that, after the bifurcation, the sum of the stable equilibrium values of species S at the two patches (and, similarly, that of species I) is equal to the double of its spatially homogeneous equilibrium value $\bar{S}$ (resp., $\bar{I}$).

5. CONCLUSIONS

In this paper, we have developed a theoretical framework for studying the phenomenon of pattern formation in a SIS (Susceptible - Infective - Susceptible) epidemic model with saturated incidence rate in order to stimulate the dynamics of disease transmission under the influence of a population migration among identical patches. Applying a stability analysis and suitable numerical simulations, we investigate the Turing parameter space and the Turing bifurcation diagram, this case is a well-known phenomenon of cross-diffusion driven instability. Numerical calculations show that a two new equilibria with non equals compounds emerge and these equilibria are asymptotically stable, so that this is a pitchfork bifurcation.

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Authors’ addresses

Ali Al-Qahtani

Current address: King Khalid University, Faculty of Science, Department of Mathematics, Applied Mathematics Group, Abha 9004, Saudi Arabia

Shaban Aly

Al-Azhar University, Faculty of Science, Mathematics Department, Assiut 71511, Egypt

Current address: King Khalid University, Faculty of Science, Department of Mathematics, Applied Mathematics Group, Abha 9004, Saudi Arabia

E-mail address: shhaly70@yahoo.com

Fatma Hussien

Department of Mathematics, Faculty of Science, Assiut University, Egypt



PURELY BAER MODULES AND PURELY RICKART MODULES

S. EBRAHIMI ATANI, M. KHORAMDEL, AND S. DOLATI PISHHESARI

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Abstract. We introduce the notions of purely Baer and purely Rickart modules. We provide several characterizations and investigate properties of each of these concepts. We provide new characterizations of several well-known classes of rings in terms of purely Baer and purely Rickart modules. It is shown that R is a von Neumann regular ring iff every right R -module is purely Baer (purely Rickart). Also, we prove R is left semihereditary iff every (finitely generated) free right R -module is purely Baer. Examples illustrating the results are presented.

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Keywords: Baer modules, purely Baer modules, Rickart modules, purely Rickart modules

1. INTRODUCTION

The notions of Rickart and Baer rings have their roots in functional analysis, with close links to C^* -algebras and von Neumann algebras. Kaplansky introduced Baer rings to abstract various properties of AW^* -algebras and von Neumann algebras and complete $*$ -regular rings in [11]. Motivated by Kaplansky's work on Baer rings, the notion of Rickart rings appeared in Maeda [15] and was further studied by Hattori [9] and other authors. A ring R is called *Baer* (resp. *right Rickart* (or p.p.)) if the right annihilator of any nonempty subset (resp. any single element) of R is generated by an idempotent, as a right ideal of R . It is well known that Baer rings and Rickart rings play an important role in providing a rich supply of idempotents and hence in the structure theory for rings.

Recently, the notions of Baer and Rickart rings and their generalizations were extended and studied in a general module-theoretic setting [1, 13, 14, 16–18].

A module is called *extending* if every its submodule is essential in a direct summand. In [6], the notion of extending generalized to purely extending by replacing “direct summand” with “pure submodule”. In [4], basic characterizations of purely extending modules are given.

Motivated by the notions of Baer and purely extending modules, we introduce the notion of purely Baer modules. In Section 3, we investigate purely Baer modules and give some results related to them. It is shown that a direct summand of a purely Baer module is purely Baer. Our focus, in this section, is on the question: When are (free)

right R -modules over a ring R purely Baer? We obtain characterizations of well-known classes of rings, in terms of free purely Baer modules over them. We show that the class of rings for which every (resp. free) module is purely Baer, is precisely that of (resp. left semihereditary) von Neumann regular rings. As an application of this, we prove that a commutative domain R is Prüfer if and only if every free R -module is purely Baer. Some characterizations of right purely Baer rings are given. It is known that the definition of Baer rings is left-right symmetric. However we show that the definition of purely Baer rings is asymmetric.

In Section 4, the notion of purely Rickart modules is introduced. A characterization of von Neumann regular rings in terms of purely Rickart modules is given. It is shown that every free right R -module is purely Rickart if and only if every right ideal of R is flat. We provide some characterizations of right purely Rickart rings. Also we show that the definition of purely Rickart rings is left-right symmetric, however the definition of Rickart rings is asymmetric.

2. PRELIMINARIES

Throughout this paper, R is an associative ring with identity and all modules are unitary. By M_R (resp. ${}_R M$), we denote a right (resp. left) R -module and $S = \text{End}(M_R)$ denotes the endomorphism ring of M_R . M^I (resp. $M^{(I)}$) stands for the direct product (resp. direct sum) of copies of M indexed by a set I . Let M be a right R -module, I a subset of R and X a subset of $\text{End}(M_R)$. We write $r_M(X) = \{m \in M : xm = 0 \text{ for all } x \in X\}$ and $r_R(I)$ (resp. $l_R(I)$) = the right (resp. left) annihilator I in R . Let M be a module over a ring R . For submodules N and K of M , $N \leq K$ denotes N is a submodule of K . What follows, by $\leq^\oplus$, $\leq^{ess}$ and $E(M)$ we denote, respectively, a module direct summand, an essential submodule and the injective hull of M . For a ring R , $\text{Mat}_n(R)$ denotes the ring of $n \times n$ matrices over R .

In the following, we recall some known notions and facts needed in the sequel.

Definition 1. (1) A (short) exact sequence $\xi : 0 \rightarrow K \xrightarrow{\varphi} N \rightarrow M \rightarrow 0$ of right R -modules is said to be *pure* (exact) if $\xi \otimes_R T$ is an exact sequence (of abelian groups) for any left R -module T . In this case, we say that $\varphi(K)$ is a *pure* submodule of N (see [12] and [19]). It is clear that every direct summand is a pure submodule.

(2) A module M over a ring R will be called *finitely presented* if there exists an exact sequence $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ of R -modules, where F is free and both F and K are finitely generated (see [12] and [19]).

(3) A right module M over a ring R is called *divisible* provided $Mx = M$ for all regular elements $x \in R$ (see [8]).

(4) A right R -module M is called *pure injective*, if M is injective with respect to every pure exact sequence of right R -modules. A ring R is called *right pure semisimple* if every right R -module is pure injective (see [22]).

(5) A module M_R is called *FP-injective* (or absolutely pure) if, for any finitely generated submodule K of a free right R -module F , every homomorphism $K \rightarrow M$ extends to a map $F \rightarrow M$ (see [22]).

(6) A module M_R is called *regular*, provided that each of its submodule is pure (see [5]).

(7) A right R -module M is said to be *Baer* (resp. *Rickart*), if for any left ideal I of $\text{End}(M)$ (resp. $\phi \in \text{End}(M)$), $r_M(I)$ (resp. $r_M(\phi)$) is a direct summand of M (see [17], [18], [13] and [14]).

(8) A module M is called *purely extending* if every submodule of M is essential in a pure submodule of M . Equivalently, every closed submodule is a pure submodule (see [4]).

(9) A module is called *torsionless* if it can be embedded in a direct product of copies of the base ring (see [12]).

We refer to [12] and [22] for the undefined notions in this article.

Proposition 1 ([12, Corollary 4.86]). *Let $\xi : 0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ be an exact sequence of R -modules with F flat. Then M is flat if and only if ξ is pure.*

Proposition 2 ([21, Lemma 2.2]). *Let R be a ring and $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ an exact sequence of right R -modules with P projective. Then the following statements are equivalent:*

- (1) M is flat;
- (2) Given any $x \in K$, there exists a homomorphism $g : P \rightarrow K$ such that $g(x) = x$;
- (3) Given any $x_1, \dots, x_n$ in K , there exists a homomorphism $g : P \rightarrow K$ such that $g(x_i) = x_i$ for $i = 1, 2, \dots, n$.

Proposition 3 ([2, Theorem 4.1]). *For any ring R , the following statements are equivalent:*

- (1) R is left semi-hereditary;
- (2) Every right ideal of R is flat, and the direct product of an arbitrary family of copies of R is flat as a right R -module;
- (2) Every torsion-less right R -module is flat.

3. PURELY BAER MODULES

In this section, we investigate connections between purely Baer modules and various existing concepts and obtain some of their useful properties. Examples of purely Baer modules include semisimple modules, regular modules, Baer modules, nonsingular purely extending modules and modules over von Neumann regular rings.

Definition 2. Let M be a right R -module and $S = \text{End}_R(M)$. Then M is called a *purely Baer* module, if $r_M(I)$ is a pure submodule of M for each left ideal I of S .

A ring R is called (resp. left) *right purely Baer*, if (resp. $l_R(I)$) $r_R(I)$ is a pure (resp. left) right ideal of R for each (resp. right) left ideal I of R .

First, we characterize right purely Baer rings in terms of cyclic torsionless modules.

Theorem 1. *Let R be a ring. Then R is right purely Baer if and only if every cyclic torsionless right R -module is flat.*

Proof. It is known that a cyclic right R -module R/I is torsionless if and only if $I = r_R(X)$ for some subset X of R . Assume that R is right purely Baer. Let M be a cyclic torsionless right R -module. As $M \cong R/I$ for some right ideal I of R , $I = r_R(X)$ for some $X \subseteq R$. Since R is right purely Baer, $I = r_R(J)$ is pure in R_R , where J is a left ideal generated by X . Thus R/I is flat by Proposition 1 and so M is flat.

Conversely, assume that every cyclic torsionless right R -module is flat. Let I be a left ideal of R . Then $R/r_R(I)$ is torsionless and so is flat. Hence $r_R(I)$ is pure by Proposition 1. Thus R is right purely Baer. $\square$

In the following, it is shown that direct summands of a purely Baer module are purely Baer.

Proposition 4. *Let M be a purely Baer module with $S = \text{End}(M)$. Then every direct summand of M is purely Baer.*

Proof. Let M be a purely Baer module and $N \leq^\oplus M$ say $M = N \oplus K$ for some $K \leq M$. Let J be a left ideal of $S' = \text{End}(N)$. Set $I = \{\psi \oplus 0|_K : \psi \in J\}$. Then $r_M(SI)$ is a pure submodule of M , because M is purely Baer. As $K \subseteq r_M(SI)$, $r_M(SI) = r_M(SI) \cap N \oplus K$. An inspection shows that $r_M(SI) \cap N = r_N(J)$. Since $r_N(J) \leq^\oplus r_M(SI)$, $r_N(J)$ is pure in $r_M(SI)$. Therefore by transitivity of the pure submodules property, $r_N(J)$ is pure in M . Hence $r_N(J)$ is pure in N . Therefore N is purely Baer. $\square$

In the following, it is shown that the notions of purely Baer modules and Baer modules coincide for finitely generated flat modules over a right noetherian ring and torsion free injective modules over a semiprime right Goldie ring.

Theorem 2. (1) *Let R be a right noetherian ring and M a finitely generated flat right R -module. Then M is purely Baer if and only if it is Baer.*

(2) *Let M be a torsion free injective right R -module over a semiprime right Goldie ring R . Then M is purely Baer if and only if it is Baer.*

(3) *Let R be a right pure semisimple ring. Then a right R -module M is purely Baer if and only if M is Baer.*

Proof. (1) Let M be a purely Baer module and I a left ideal of $S = \text{End}(M_R)$. Then $r_M(I)$ is pure in M . Hence by Proposition 1, $M/r_M(I)$ is flat. As $M/r_M(I)$ is finitely generated and R is right noetherian, $M/r_M(I)$ is finitely presented by [12, Theorem 4.29]. Hence $M/r_M(I)$ is projective by [12, Theorem 4.30]. This implies that $r_M(I) \leq^\oplus M$. Hence M is a Baer module. The converse is clear.

(2) Let M be a purely Baer module and I a left ideal of $S = \text{End}(M_R)$. Then $r_M(I)$ is pure in M . Hence by [22, 34.8], for each regular element $x \in R$,

$$r_M(I)x = r_M(I) \cap Mx = r_M(I) \cap M = r_M(I),$$

because M is divisible. Hence $r_M(I)$ is divisible. Therefore [7, Theorem 7.11] implies that $r_M(I)$ is injective. Hence $r_M(I) \leq^\oplus M$. So M is Baer. The converse is clear.

(3) If R is right pure semisimple, then every pure exact sequence splits. Hence M is purely Baer if and only if it is Baer. $\square$

Theorem 3. *Let M be a nonsingular right R -module. If M is purely extending, then M is a purely Baer module.*

Proof. Let M be a purely extending module and I a left ideal of $\text{End}_R(M)$. Then $r_M(I) \leq^{ess} T$ for some pure submodule T of M . Let $t \in T$. Then $J = \{r \in R : tr \in r_M(I)\}$ is an essential right ideal of R_R . Thus $tJ \subseteq r_M(I)$ and so for each $f \in I$, $f(tJ) = f(t)J = 0$. As M is nonsingular, $f(t) = 0$ for each $f \in I$. Therefore $r_M(I) = T$ is a pure submodule of M and it implies that M is a purely Baer module. $\square$

The next example shows that the converse of Theorem 3 is not true.

Example 1. [4] Let

$$R = \begin{pmatrix} F & 0 & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix},$$

the F -subalgebra of the ring of 3×3 matrices over a field F . This ring is left and right artinian hereditary (so it is left and right semihereditary). Hence R is left and right nonsingular. Hence by Theorem 5, R is right purely Baer. We show R is not purely extending. Let I be a closed right ideal of R . Then I is finitely generated (because R is noetherian) and so R/I is finitely presented. If I is pure, then R/I is flat by Proposition 1. As R/I is finitely presented, R/I is projective by [12, Theorem 4.30]. Hence $I \leq^\oplus R_R$. This implies that if R_R is purely extending, then R_R is extending. However R is not right extending (see [4] and [3, Example 5.5]).

Next, we characterize the class of rings R for which every right R -module is purely Baer as precisely that of the von Neumann regular rings.

Theorem 4. *The following are equivalent for a ring R .*

- (1) *Every right R -module is purely Baer;*
- (2) *Every purely extending right R -module is purely Baer;*
- (3) *Every extending right R -module is purely Baer;*
- (4) *Every injective right R -module is purely Baer;*
- (5) *Every right R -module is FP-injective;*
- (6) *R is von Neumann regular.*

Proof. (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (5) Let M be an arbitrary right R -module. Then by (4), $E(M) \oplus E(E(M)/M)$ is purely Baer. Let $\psi : E(M) \rightarrow E(E(M)/M)$ be defined by $\psi(x) = x + M$ for each $x \in E(M)$. We can extend ψ to an endomorphism $\bar{\psi}$ of $E(M) \oplus E(E(M)/M)$ and $\text{Ker}(\bar{\psi}) = M$. Hence M is pure in $E(M) \oplus E(E(M)/M)$. This implies that M is pure in $E(M)$. Hence M is FP-injective by [22, 35.8].

(5) $\Rightarrow$ (6) By [22, 37.6], R is a von Neumann regular ring.

(6) $\Rightarrow$ (1) By [20, Proposition 3.18], every right R -module is flat. Let M be a right R -module and I a left ideal of $\text{End}(M)$. By (6), M and $M/r_M(I)$ are flat modules. This implies that $r_M(I)$ is pure in M by Proposition 1. $\square$

The following examples exhibit purely Baer modules which are not Baer.

Example 2. Let

$$R = \left(\begin{array}{cc} \prod_{i=1}^{\infty} F_i & \oplus_{i=1}^{\infty} F_i \\ \oplus_{i=1}^{\infty} F_i & < \oplus_{i=1}^{\infty} F_i, 1 > \end{array} \right),$$

where $F_i = F$ is a field for each $i \in \mathbb{N}$ and $< \oplus_{i=1}^{\infty} F_i, 1 >$ is the F -algebra generated by $\oplus_{i=1}^{\infty} F_i$ and 1. Then R is a von Neumann regular ring. Hence by Theorem 4, R_R is a purely Baer module. However, it is not a Baer module by [13, Example 2.19].

Example 3. Let F be a field and V be an infinite dimensional vector space over F . Set $J = \{x \in \text{End}_F(V) \mid \dim_F(xV) < \infty\}$ and $R = F + J$. By [7, Example 6.19], R is regular. Therefore $M = R \oplus R/J$ is purely Baer by Theorem 4. However, M is not Baer. If M is Baer, then M is Rickart and so $J \leq^{\oplus} R$ by [13, Proposition 2.24], a contradiction (because $J = \text{Soc}(R_R)$ and J is essential in R_R (see [7])).

Proposition 5. *Let P be a finitely generated projective right R -module. Then P is a purely Baer module if and only if $S = \text{End}(P)$ is a right purely Baer ring.*

Proof. Let $P = x_1 R + \dots + x_n R$ ($x_i \in P$) be a purely Baer projective module and I be a left ideal of S . We show $r_S(I)$ is a pure right ideal of S . Let $f \in r_S(I)$. Then $f(x_i) \in r_P(I)$ for each $1 \leq i \leq n$. As P is purely Baer, $r_P(I)$ is a pure submodule. Hence $P/r_P(I)$ is flat by Proposition 1 and so by Proposition 2, there exists a homomorphism $\psi : P \rightarrow r_P(I)$ such that $\psi(f(x_i)) = f(x_i)$ for each $1 \leq i \leq n$. Hence $\psi f = f$. We can take ψ as an endomorphism of P . Since $\psi(P) \subseteq r_P(I)$, $\psi \in r_S(I)$. Now let $\phi : S \rightarrow r_S(I)$ be defined by $\phi(h) = \psi h$ for each $h \in S$. Then $\phi(f) = \psi f = f$. Therefore by Proposition 2, $S/r_S(I)$ is a flat right S -module. Hence by Proposition 1, $r_S(I)$ is a pure right ideal of S . Thus S is right purely Baer.

Conversely, assume that P is a finitely generated projective right R -module and $S = \text{End}(P_R)$ a right purely Baer ring. We show P is purely Baer. Since P is finitely generated and projective, $P \leq^{\oplus} R^n$ for some positive integer n . As $\text{End}(R^n) \cong \text{Mat}_n(R)$, there exists $E^2 = E \in \text{Mat}_n(R)$ such that $P = ER^n$. Hence $S = E\text{Mat}_n(R)E$.

Let I be a left ideal of S and $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \in r_P(I)$. Then an inspection shows that

$$B = \begin{pmatrix} b_1 & b_1 & \cdots & b_1 \\ b_2 & b_2 & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ b_n & b_n & \cdots & b_n \end{pmatrix} \in r_S(I).$$

As $b \in P$, $Eb = b$. Thus $EB = B$. This implies that $B \in S$, because $Bh = EBh \in ER^n$ for each $h \in R^n$. Since $r_S(I)$ is a pure right ideal in S , $S/r_S(I)$ is a flat right S -module by Proposition 1. Hence by Proposition 2, there exists a homomorphism $\theta : S \rightarrow r_S(I)$ such that $\theta(B) = B$. We can take θ as a right endomorphism of S . Hence $\theta(B) = \theta(1)B = B$, where 1 is identity element of S and $\theta(1) \in r_S(I)$.

Let $\theta(1) = A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$. As $AB = B$, $\sum_{j=1}^n a_{ij}b_j = b_i$ for each

$1 \leq i \leq n$. Hence $Ab = b$. As $A \in r_S(I)$, we can define $\beta : P \rightarrow r_P(I)$ by $f(p) = Ap$ for each $p \in P$. Since $A \in r_S(I)$, $Ap \in r_P(I)$ for each $p \in P$. Also $Ab = b$ implies that $\beta(b) = Ab = b$. Therefore by Proposition 2, $P/r_P(I)$ is a flat right R -module and so by Proposition 1, $r_P(I)$ is pure in P . Hence P is purely Baer. $\square$

Next, we characterize rings R for which every (finitely generated) free right R -module is purely Baer. We show that these are precisely the left semihereditary rings.

Theorem 5. *The following are equivalent for a ring R :*

- (1) *Every free right R -module is purely Baer;*
- (2) *Every projective right R -module is purely Baer;*
- (3) *Every finitely generated free right R -module is purely Baer;*
- (4) *Every finitely generated projective right R -module is purely Baer;*
- (5) *Every finitely generated torsionless right R -module is flat;*
- (6) *Every torsionless right R -module is flat;*
- (7) *R is a left semihereditary ring.*

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are clear from Proposition 4.

(2) $\Rightarrow$ (3) is clear.

(4) $\Rightarrow$ (5) Let M be a finitely generated torsionless right R -module. Hence $M \subseteq R^I$ for some index set I . As M is finitely generated, there exists an epimorphism $\psi : F \rightarrow M$, where F is a finitely generated free right R -module. By (4), F is purely Baer. It can be shown that $\text{Ker}(\psi) = \bigcap_{i \in I} \text{Ker}(\pi_i \psi)$, where π_i is the canonical projection from R^I onto its i th coordinates. We can take $\pi_i \psi$ as an endomorphism

of F . Hence $\text{Ker}(\psi) = r_M(J)$, where J is a left ideal of $S = \text{End}(F)$ generated by the set $\{\pi_i \psi\}_{i \in I}$. Since F is purely Baer, $\text{Ker}(\psi)$ is a pure submodule of F . Therefore $M \cong F/\text{Ker}(\psi)$ is flat by Proposition 1.

(5) $\Rightarrow$ (6) As every submodule of a torsionless right R is torsionless, every finitely generated submodule of a torsionless right R -module is flat. Hence every torsionless right R -module is flat (it is known that, a module is flat provided that every of its finitely generated submodule is flat).

(6) $\Rightarrow$ (7) is from Proposition 3.

(7) $\Rightarrow$ (1) Let F be a free right R -module and I a left ideal of $S = \text{End}(F)$. As for each $\psi \in S$, $F/\text{Ker}(\psi) \cong \text{Im}(\psi)$, $F/\text{Ker}(\psi)$ is a torsionless module. Hence $\prod_{\psi \in I} F/\text{Ker}(\psi)$ is torsionless, because direct product of torsionless modules is torsionless. Let $\phi : F/\bigcap_{\psi \in I} \text{Ker}(\psi) \rightarrow \prod_{\psi \in I} F/\text{Ker}(\psi)$ be defined by $\phi(x + \bigcap_{\psi \in I} \text{Ker}(\psi)) = (x + \text{Ker}(\psi))_{\psi \in I}$ for each $x \in F$. It is clear that ϕ is a monomorphism. Thus $F/\bigcap_{\psi \in I} \text{Ker}(\psi)$ is torsionless. Therefore by Proposition 3, $F/\bigcap_{\psi \in I} \text{Ker}(\psi)$ is flat. Hence $r_F(I) = \bigcap_{\psi \in I} \text{Ker}(\psi)$ is pure in F by Proposition 1. Thus F is a purely Baer module. $\square$

Corollary 1. *The following are equivalent for a ring R .*

- (1) R is left semihereditary;
- (2) For each $n > 1$, $\text{Mat}_n(R)$ is a right purely Baer ring.

Proof. By Theorem 5, R is left semihereditary if and only if R^n is purely Baer right R -module if and only if $\text{Mat}_n(R)$ is right purely Baer for each n , by Proposition 5. $\square$

Remark 1. By Theorem 5, every finitely generated free right R -module is purely Baer if and only if every free right R -module is purely Baer. But it is not true for the class of Baer modules. Let R be a commutative Prüfer domain which is not a Dedekind domain (hereditary). Then every finitely generated free right R -module is Baer by [18, Theorem 3.9]. However there is a free right R -module that is not Baer, by [18, Theorem 3.3].

The following example gives an example of a module M such that every submodule of M is purely Baer.

Example 4. Let

$$R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}.$$

By [12, Example 2.33], R is right hereditary and left semihereditary. Let $M = (eR)^n$, where e is an arbitrary idempotent of R and $n \in \mathbb{N}$. Since R is right noetherian and right hereditary, every submodule of M is finitely generated and projective. Therefore by Theorem 5, every submodule of M is purely Baer. Also, for each submodule N of M , $\text{End}(N)$ is a right purely Baer ring, by Proposition 5.

The next example shows that the definition of purely Baer rings is not left-right symmetric. As we know, the definition of Baer rings is left-right symmetric.

Example 5. Let

$$T = \begin{pmatrix} S/I & S/I \\ 0 & S \end{pmatrix},$$

where S is a von Neumann ring and I a right ideal of S which is not a direct summand of S_S . Then by [12, Example 2.34], T is left semihereditary but not right semihereditary. By Corollary 1, there exists positive integer n such that $\text{Mat}_n(T)$ is a right purely Baer ring which is not left purely Baer ring.

As a consequence of Theorem 5, we provide a characterization of Prüfer domains in terms of the purely Baer property for (finitely generated) free (projective) modules.

Theorem 6. *For any commutative domain R , the following are equivalent:*

- (1) *Every free R -module is purely extending;*
- (2) *Every free R -module is purely Baer;*
- (3) *Every finitely generated free R -module is purely Baer;*
- (4) *R is a Prüfer domain.*

Proof. (1) $\Rightarrow$ (2) Since R is nonsingular, every free R -module is nonsingular. Therefore every free R -module is purely Baer by Theorem 3.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4) It is known that a commutative domain is semihereditary if and only if it is a Prüfer domain by [12, Theorem 4.69]. Hence the result is clear by Theorem 5.

(4) $\Rightarrow$ (1) [4, Proposition 2.1] $\square$

4. PURELY RICKART MODULES

Motivated by the definitions of Rickart modules, we introduce the notions of purely Rickart modules and relatively purely Rickart modules and collect some basic properties of these classes of modules. Examples of purely Rickart modules include semisimple modules, Rickart modules, purely Baer modules and modules over regular rings. We begin with the key definition of this section.

Definition 3. Let M be a right R -module and $S = \text{End}(M_R)$. Then M is called a *purely Rickart module*, if $r_M(\psi) = \text{Ker}(\psi)$ is a pure submodule of M for each $\psi \in S$.

A ring R is called (resp. left) *right purely Rickart*, if (resp. $l_R(a)$) $r_R(a)$ is a pure (resp. left) right ideal of R for each $a \in R$.

It is known that the definition of Rickart rings is asymmetric. However the following shows that the definition of purely Rickart rings is symmetric. Also purely Rickart rings are known as P.F rings [10].

Theorem 7. *The following are equivalent for a ring R :*

- (1) R is right purely Rickart;
- (2) Every principle right ideal of R is flat;
- (3) Every principle left ideal of R is flat;
- (4) R is left purely Rickart.

Proof. (1) $\Leftrightarrow$ (2) and (3) $\Rightarrow$ (4) Let $a \in R$. Then (resp. Ra) aR is flat as a (resp. left) right R -module if and only if (resp. $l_R(a)$) $r_R(a)$ is a pure (resp. left) right ideal of R by Proposition 1.

(2) $\Leftrightarrow$ (3) By [10]. □

One can easily show that under the following conditions, the notions of Rickart module and purely Rickart module coincide, as in the proof of Theorem 2.

Proposition 6. (1) *Let R be a right noetherian ring and M a finitely generated flat right R -module. Then M is purely Rickart if and only if it is Rickart.*

(2) *Let M be a torsion free injective right R -module over a semiprime right Goldie ring R . Then M is purely Rickart if and only if it is Rickart.*

(3) *Let R be a right pure semisimple ring and M a right R -module. Then M is purely Rickart if and only if it is Rickart.*

Definition 4. A module M is called N -purely Rickart, if $\text{Ker}(\phi)$ is a pure submodule of M for every homomorphism $\phi : M \rightarrow N$.

In view of the above definition, a right R -module M is purely Rickart if and only if M is M -purely Rickart.

The following propositions are useful to prove our main theorems.

Proposition 7. (1) *Let M and N be right R -modules. Then M is any N -purely Rickart if and only if for any direct summand $K \leq^\oplus M$ and any submodule $L \leq N$, K is L -purely Rickart.*

(2) *The following are equivalent for a right R -module M :*

- (i) M is purely Rickart;
 - (ii) For any submodule $N \leq M$ and any direct summand $K \leq^\oplus M$, K is N -purely Rickart;
 - (iii) For any $K, N \leq^\oplus M$, and $\phi \in \text{Hom}_R(M, N)$, $\text{Ker}(\phi|_K)$ is pure in K .
- (3) *Every direct summand of a purely Rickart module is a purely Rickart module.*

Proof. (1) Suppose that M is any N -purely Rickart. Let $K = eM$ for some idempotent element e of $\text{End}(M_R)$ and $\phi : K \rightarrow L$ be a homomorphism. Then ϕe is a homomorphism from M to N . Hence $\text{Ker}(\phi e)$ is pure in M . As $\text{Ker}(\phi e) = \text{Ker}(\phi) \oplus (1 - e)M$, $\text{Ker}(\phi)$ is pure in $\text{Ker}(\phi e)$. This implies that $\text{Ker}(\phi)$ is pure in M and so is in N . Thus K is L -purely Rickart. The converse is clear.

(2) and (3) are clear from (1). □

We now characterize the von Neumann regular rings in terms of purely Rickart modules.

Theorem 8. *The following are equivalent for a ring R :*

- (1) *Every right R -module is purely Baer;*
- (2) *Every right R -module is purely Rickart;*
- (3) *Every pure injective right R -module is injective;*
- (4) *R is von Neumann regular.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) Let M be a pure injective right R -module. By (2), $E(M) \oplus E(E(M)/M)$ is purely Rickart. It implies that $E(M)$ is $E(E(M)/M)$ -purely Rickart by Proposition 7. Therefore $\text{Ker}(\pi) = M$ is pure in $E(M)$, where $\pi : E(M) \rightarrow E(E(M)/M)$ is natural homomorphism. Since M is pure injective, the pure exact sequence $0 \rightarrow M \rightarrow E(M) \rightarrow E(M)/M \rightarrow 0$ splits. Hence $M \leq^\oplus E(M)$. Thus M is injective.

(3) $\Rightarrow$ (4) By [22, 37.6]

(4) $\Rightarrow$ (1) By Theorem 4. □

It is clear that every Rickart module is purely Rickart. The following examples exhibit purely Rickart modules which are not Rickart.

Example 6. Let $R = \prod_{i=1}^{\infty} F_i$, where $F_i = F$ is a field for each $i \in \mathbb{N}$ and $I = \bigoplus_{i=1}^{\infty} F_i$. Then by Theorem 8, $M = R \oplus R/I$ is a purely Rickart right R -module, which is not a Rickart module (if M is a Rickart module then $I \leq^\oplus R$ by [13, Proposition 2.24] which is a contradiction).

Example 7. Let $S = \prod_{i=1}^{\infty} \mathbb{Z}_2$. Consider

$$R = \{(a_n)_{n=1}^{\infty} \in S \mid a_n \text{ is eventually constant}\},$$

a subring of S . Then R is a von Neumann regular ring. Hence by Theorem 8, $R \oplus E(R)$ is a purely Rickart module which is not a Rickart module by [14, Example 2.18].

In the next theorem, we characterize the rings R for which every free R -module is purely Rickart.

Theorem 9. *The following are equivalent for a ring R :*

- (1) *Every free right R -module is purely Rickart;*
- (2) *Every projective right R -module is purely Rickart;*
- (3) *Every finitely generated free right R -module is purely Rickart;*
- (4) *Every finitely generated projective right R -module is purely Rickart;*
- (5) *For each $n > 1$, $\text{Mat}_n(R)$ is a right purely Rickart ring;*
- (6) *Every right ideal of R is flat;*
- (7) *Every submodule of a flat right R -module is flat.*

Since condition (6) is left-right symmetric, the left-handed versions of (1), (2), (3), (4), (5) and (7) also hold.

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are clear by Proposition 7.

(2) $\Rightarrow$ (3) is clear.

(5) $\Rightarrow$ (6) Assume that $\text{Mat}_n(R)$ is a right purely Rickart ring for any $n > 1$. Let $I = a_1 R + \dots + a_n R$ be a finitely generated right ideal of R and $A \in \text{Mat}_n(R)$ with first row $(a_1, a_2, \dots, a_n)$, and zero elsewhere. Then $r_{\text{Mat}_n(R)}(\text{Mat}_n(R)A)$ is a pure right ideal of $\text{Mat}_n(R)$. Hence $A\text{Mat}_n(R)$ is a flat right $\text{Mat}_n(R)$ -module by Proposition 1. Since R and $\text{Mat}_n(R)$ are Morita equivalent and flatness is preserved across Morita equivalences, $(A\text{Mat}_n(R))_R \cong I_R^n$ implies that I is a flat right R -module. Hence every finitely generated right ideal of R is flat. So every right ideal of R is flat.

(6) $\Rightarrow$ (5) Let $A \in \text{Mat}_n(R)$. As $\text{Mat}_n(R)$ is a flat right R -module, $A\text{Mat}_n(R)$ is a flat right R -module. Hence $A\text{Mat}_n(R)$ is a flat right $\text{Mat}_n(R)$ -module by a similar argument as in above. Hence $\text{Mat}_n(R)$ is right purely Rickart by Theorem 7.

(4) $\Rightarrow$ (6) Let I be a finitely generated right ideal of R . Then I is homomorphic image of a finitely generated free right R -module F . Hence there exists an epimorphism $\varphi : F \rightarrow I$. We can take φ as an endomorphism F . Hence $r_F(\varphi)$ is a pure submodule of F . Thus by Proposition 1, $F/r_F(\varphi) \cong I$ is a flat R -module. Hence every finitely generated right ideal of R is flat. So every right ideal of R is flat.

(6) $\Rightarrow$ (7) By [12, Lemma 4.66], every submodule of a flat right R -module is flat.

(7) $\Rightarrow$ (1) Let F be a free right R -module and $\psi \in \text{End}(F)$. Then $M/\text{Ker}(\psi) \cong \text{Im}(\psi) \leq F$ implies that $M/\text{Ker}(\psi)$ is flat by (7). Hence $\text{Ker}(\psi)$ is a pure submodule of F . Hence F is a purely Rickart module. $\square$

Remark 2. By Theorem 9, every finitely generated free right R -module is purely Rickart if and only if every free right R -module is purely Rickart. However, the similar result for the class of Rickart modules does not hold true. Let R be right semihereditary ring which is not hereditary. Then by [14, Theorem 3.6], every finitely generated free right R -module is Rickart. However there is a free right R -module that is not Rickart, by [14, Theorem 3.5].

The next example shows that the class of purely Rickart modules properly contains the class of purely Baer modules.

Example 8. Let

$$R = \begin{pmatrix} S & S/I \\ 0 & S/I \end{pmatrix},$$

where S is a von Neumann ring and I a right ideal of S which is not a direct summand of S_S . Then by [12, Example 2.34], R is right semihereditary but not left semihereditary. Since R is right semihereditary, every right ideal of R is flat. Hence by Theorem 9, every free right R -module is purely Rickart. As R is not left semihereditary, there is a free right R -module which is not purely Baer by Theorem 5. Hence there exists a free R -module F which is purely Rickart but not purely Baer.

The following example proves the existence of a module M such that M^n is a purely Rickart (purely Baer) module, but M^{n+1} is not purely Rickart (purely Baer). The following example is due to Jøndrup (see [10], [14, Example 3.15] and [18, Example 3.17]).

Example 9. Let n be any natural number, K be any commutative field, and let R be the K -algebra on the $2(n+1)$ generators X_i, Y_i ($i = 1, \dots, n+1$) with the defining relation

$$\sum_{i=1}^{n+1} X_i Y_i = 0.$$

As R^n is a Baer module by [18, Example 3.17], R^n is purely Rickart (purely Baer). But R^{n+1} is not purely Rickart (purely Baer). Otherwise, if R^{n+1} is purely Rickart (purely Baer), then one can show that every $(n+1)$ generated ideal of R is flat, as in the proof of Theorem 9 ((4) $\Rightarrow$ (6)). However, it is proved that there exists an $(n+1)$ generated ideal which is not flat (see [10]). Hence R^{n+1} is not purely Rickart (purely Baer).

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Authors' addresses

S. Ebrahimi Atani

University of Guilan, Department of Mathematics, P.O.Box 1914, Rasht, Iran

E-mail address: ebrahimi@guilan.ac.ir

M. Khoramdel

University of Guilan, Department of Mathematics, P.O.Box 1914, Rasht, Iran

E-mail address: mehdikhoramdel@gmail.com

S. Dolati Pishhesari

University of Guilan, Department of Mathematics, P.O.Box 1914, Rasht, Iran

E-mail address: saboura-dolati@yahoo.com



SOME NEW CLASSES OF CONVEX FUNCTIONS AND INEQUALITIES

MUHAMMAD UZAIR AWAN, MUHAMMAD ASLAM NOOR,
KHALIDA INAYAT NOOR, AND AWAIS GUL KHAN

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Abstract. In this article, we introduce some new class of convex functions involving two arbitrary auxiliary functions $h_1, h_2 : I \rightarrow \mathbb{R}$, which are called (h_1, h_2) -convex functions. We derive some new integral inequalities for these classes of convex functions. We also discuss some special cases which can be deduced from our main results. Results obtained in this paper may be viewed as a significant refinement and improvement of the previously known results. The ideas and techniques of this work may be a starting point for future research.

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1. INTRODUCTION

A set $\mathcal{C} \subset \mathbb{R}$ is said to be a convex set, if

$$tx + (1-t)y \in \mathcal{C}, \quad \forall x, y \in \mathcal{C}, t \in [0, 1].$$

A function $f : \mathcal{C} \rightarrow \mathbb{R}$ is said to be a convex function in the classical sense, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \quad \forall x, y \in \mathcal{C}, t \in [0, 1].$$

In recent years several new generalizations of classical convexity have been given, for example see [1–3, 8, 17, 18]. Varosanec [18] introduced the notion of h -convexity which along with classical convex functions generalizes several other class of convex functions. The formal definition of h -convex functions is given as:

Definition 1 ([18]). Let $h : (0, 1) \subseteq J \rightarrow \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. We say that $f : \mathcal{C} \rightarrow \mathbb{R}$ is an h -convex function, if f is non-negative and for all $x, y \in \mathcal{C}$, $t \in (0, 1)$, we have

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y).$$

Corresponding author: Muhammad Uzair Awan.

For the different choices of the auxiliary function $h(\cdot)$, we have different other classes of convex functions such as: Breckner type of s -convex functions [1], Godunova-Levin-Dragomir type of s -convex functions [4], Godunova-Levin type of functions [8] and P -functions [7]. Since the appearance of this definition many researchers shown their special interest in studying this class of convex functions. Sarikaya et al. [15] has improved the Hermite-Hadamard's inequality for this class of convex functions. Recent trend of the research in this field has shown that theory of convexity and theory of inequalities have a close relationship. Many inequalities can be obtained via convex functions and naturally they can be extended for generalizations of convex functions, see [5, 6, 10–12, 14, 16].

Motivated by the research going on in this field, we introduce the notion of so-called (h_1, h_2) -convex functions, which is the main motivation of this paper. These classes involves two auxiliary functions namely $h_1, h_2 : J \rightarrow \mathbb{R}$. We show that these classes include several new and known classes of convex functions as special cases. We also derive some new estimates for Hermite-Hadamard type of inequalities and for the integral $\int_a^b (u-a)^\alpha (b-u)^\beta f(u) du$ via (h_1, h_2) -convex functions. Some new and known special cases which can be deduced from our main results, are also discussed.

2. PRELIMINARIES

We now define the new classes of convex functions involving two arbitrary functions.

Definition 2. Let $h_1, h_2 : (0, 1) \subseteq J \rightarrow \mathbb{R}$ be two real functions, $h_1, h_2 \not\equiv 0$. We say that $f : \mathcal{C} \rightarrow \mathbb{R}$ is an (h_1, h_2) -convex function, if

$$f(tx + (1-t)y) \leq h_1(t)h_2(1-t)f(x) + h_1(1-t)h_2(t)f(y), \quad \forall x, y \in \mathcal{C}, t \in (0, 1).$$

We now discuss several special cases.

I. If $h_2(t) = 1$, then Definition 2 reduces to the definition for h -convex functions [18].

II. If $h_1(t) = 1 = h_2(t)$, then Definition 2 reduces to the definition for P -functions [7].

III. If $h_1(t) = t^s$ and $h_2(t) = t^s$ in Definition 2, then we have the class of s -convex functions of third kind.

Definition 3. Let $s \in [0, 1]$ be a real number. We say that $f : \mathcal{C} \rightarrow \mathbb{R}$ is an s - tgs -convex functions, if

$$f(tx + (1-t)y) \leq t^s(1-t)^s[f(x) + f(y)], \quad \forall x, y \in \mathcal{C}, t \in [0, 1].$$

Note that if we take $s = 1$ in Definition 3, then we have the definition of tgs -convex functions [17].

IV. If $h_1(t) = t^{-s}$ and $h_2(t) = t^{-s}$ in Definition 2, then we have the class of s -Godunova-Levin-Dragomir tgs -convex functions.

Definition 4. Let $s \in (0, 1]$ be a real number. We say that $f : \mathcal{C} \rightarrow \mathbb{R}$ is an s -Godunova-Levin-Dragomir tgs -convex function, if

$$f(tx + (1-t)y) \leq \frac{1}{t^s} \frac{1}{(1-t)^s} [f(x) + f(y)], \quad \forall x, y \in \mathcal{C}, t \in [0, 1].$$

Note that if we take $s = 1$ in Definition 4, then we have the definition of Godunova-Levin type of tgs -convex functions, which appears to be a new definition.

V. If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$ in Definition 2, then we have a new class of convex functions which is called as Breckner type of (s_1, s_2) -convex functions.

Definition 5. Let $s_1, s_2 \in (0, 1]$ be two real numbers. We say that $f : \mathcal{C} \rightarrow \mathbb{R}$ is an (s_1, s_2) -convex function, if

$$f(tx + (1-t)y) \leq t^{s_1} (1-t)^{s_2} f(x) + (1-t)^{s_1} t^{s_2} f(y), \quad \forall x, y \in \mathcal{C}, t \in [0, 1].$$

VI. If $h_1(t) = t^{-s_1}$ and $h_2(t) = t^{-s_2}$ in Definition 2, then we have a new class of convex functions which is called as Godunova-Levin-Dragomir type of (s_1, s_2) -convex functions.

Definition 6. Let $s_1, s_2 \in (0, 1]$ be two real numbers. We say that $f : \mathcal{C} \rightarrow \mathbb{R}$ is an (s_1, s_2) -convex function, if

$$f(tx + (1-t)y) \leq \frac{1}{t^{s_1} (1-t)^{s_2}} f(x) + \frac{1}{(1-t)^{s_1} t^{s_2}} f(y), \quad \forall x, y \in \mathcal{C}, t \in [0, 1].$$

It is clear that these new classes of convex functions are quite general and include several new and previously known classes of convex functions as special cases.

The following results will be helpful in deriving our main results in this paper.

Lemma 1 ([13]). Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f \in \mathcal{L}[a, b]$, then

$$\int_a^b (u-a)^\alpha (b-u)^\beta f(u) du = (b-a)^{\alpha+\beta+1} \int_0^1 t^\alpha (1-t)^\beta f((1-t)a + tb) dt.$$

Lemma 2 ([9]). Let $\mathcal{I}^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$, $a, b \in \mathcal{I}^\circ$ with $a < b$ where $\mathcal{I}^\circ$ is the interior of $\mathcal{I}$. If $f^{(n)}$ exists on $\mathcal{I}^\circ$ and $f^{(n)} \in \mathcal{L}[a, b]$, then for $n \geq 1$, we have

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a)$$

$$= \frac{(b-a)^n}{2n!} \int_0^1 t^{n-1} (n-2t) f^{(n)}(ta + (1-t)b) dt.$$

3. MAIN RESULTS

In this section we derive our main results.

Theorem 1. *Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a (h_1, h_2) -convex function. If $f \in \mathcal{L}[a, b]$, then for $h_1(\frac{1}{2})h_2(\frac{1}{2}) \neq 0$, we have*

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq [f(a) + f(b)] \int_0^1 h_1(t)h_2(1-t) dt.$$

Proof. Since f is an (h_1, h_2) -convex function, for $x = ta + (1-t)b$, $y = (1-t)a + tb$ and $t = \frac{1}{2}$, we have

$$f\left(\frac{a+b}{2}\right) \leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)[f(ta + (1-t)b) + f((1-t)a + tb)].$$

Integrating both sides of the above inequality with respect to t on $[0, 1]$, we have

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx. \quad (3.1)$$

Also

$$f(ta + (1-t)b) \leq h_1(t)h_2(1-t)f(x) + h_1(1-t)h_2(t)f(y).$$

Integrating both sides of the above inequality with respect to t on $[0, 1]$, we have

$$\frac{1}{b-a} \int_a^b f(x) dx \leq [f(a) + f(b)] \int_0^1 h_1(t)h_2(1-t) dt. \quad (3.2)$$

On summation of inequalities (3.1) and (3.2) the proof is complete. $\square$

We now discuss a new special case of Theorem 1.

If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$ in Theorem 1, then we have a result for Brecker type of (s_1, s_2) -convex functions.

Corollary 1. *Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a Brecker type of (s_1, s_2) -convex function. If $f \in \mathcal{L}[a, b]$, then for $s_1, s_2 \in [0, 1]$ we have*

$$\frac{1}{2^{1-s_1-s_2}} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq [f(a) + f(b)] \mathbb{B}(s_1 + 1, s_2 + 1).$$

If $h_1(t) = t^{-s_1}$ and $h_2 = t^{-s_2}$ in Theorem 1, then we have a result for Godunova-Levin-Dragomir type of (s_1, s_2) -convex functions.

Corollary 2. *Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a Godunova-Levin-Dragomir type of (s_1, s_2) -convex function. If $f \in \mathcal{L}[a, b]$, then for $s_1, s_2 \in [0, 1]$, we have*

$$\frac{1}{2^{1+s_1+s_2}} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq [f(a) + f(b)] \mathbb{B}(1-s_1, 1-s_2).$$

Our next result is a lower bound for Hermite-Hadamard's inequality via product of two (h_1, h_2) -convex functions.

Theorem 2. *Let $f, g : \mathcal{I} = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be two (h_1, h_2) -convex functions such that $h_1^2(\frac{1}{2})h_2^2(\frac{1}{2}) \neq 0$. If $fg \in \mathcal{L}[a, b]$, then, we have*

$$\begin{aligned} & \frac{1}{2h_1^2(\frac{1}{2})h_2^2(\frac{1}{2})} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ & - \left[M(a, b) \int_0^1 [h_1(t)h_2(t)h_1(1-t)h_2(1-t)] dt + N(a, b) \int_0^1 h_1^2(t)h_2^2(1-t) dt \right] \\ & \leq \frac{1}{b-a} \int_a^b f(x)g(x) dx. \end{aligned}$$

where

$$M(a, b) = f(a)g(a) + f(b)g(b), \quad (3.3)$$

and

$$N(a, b) = f(a)g(b) + f(b)g(a), \quad (3.4)$$

respectively.

Proof. Since f and g are (h_1, h_2) -convex functions, so

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ & \leq h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) [f(ta + (1-t)b) + f((1-t)a + tb)] \\ & \quad \times h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) [g(ta + (1-t)b) + g((1-t)a + tb)] \\ & = h_1^2\left(\frac{1}{2}\right) h_2^2\left(\frac{1}{2}\right) [f(ta + (1-t)b)g(ta + (1-t)b) \\ & \quad + f((1-t)a + tb)g((1-t)a + tb) + f(ta + (1-t)b)g((1-t)a + tb) \\ & \quad + f((1-t)a + tb)g(ta + (1-t)b)] \end{aligned}$$

$$\begin{aligned}
&\leq h_1^2\left(\frac{1}{2}\right)h_2^2\left(\frac{1}{2}\right)[f(ta+(1-t)b)g(ta+(1-t)b) \\
&\quad + f((1-t)a+tb)g((1-t)a+tb) \\
&\quad + [2h_1(t)h_2(t)h_1(1-t)h_2(1-t)][f(a)g(a)+f(b)g(b)] \\
&\quad + [h_1^2(1-t)h_2^2(t)+h_1^2(t)h_2^2(1-t)][f(a)g(b)+f(b)g(a)]].
\end{aligned}$$

Integrating the above inequality with respect to t on $[0, 1]$, we have

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
&\leq 2h_1^2\left(\frac{1}{2}\right)h_2^2\left(\frac{1}{2}\right)\left[\frac{1}{b-a}\int_0^1 f(x)g(x)dx\right. \\
&\quad \left.+ M(a,b)\int_0^1 [h_1(t)h_2(t)h_1(1-t)h_2(1-t)]dt + N(a,b)\int_0^1 h_1^2(t)h_2^2(1-t)dt\right].
\end{aligned}$$

This implies

$$\begin{aligned}
&\frac{1}{2h_1^2\left(\frac{1}{2}\right)h_2^2\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
&\leq \frac{1}{b-a}\int_0^1 f(x)g(x)dx \\
&\quad + M(a,b)\int_0^1 [h_1(t)h_2(t)h_1(1-t)h_2(1-t)]dt + N(a,b)\int_0^1 h_1^2(t)h_2^2(1-t)dt.
\end{aligned}$$

This completes the proof. $\square$

Next we discuss a new special case of Theorem 2.

If $h_1(t) = t^{s_1}$ and $h_2 = t^{s_2}$ in Theorem 2, then we have a result for Breckner type of (s_1, s_2) -convex functions.

Corollary 3. *Let $f, g : \mathcal{J} = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be two (s_1, s_2) -convex functions such that $s_1, s_2 \in [0, 1]$. If $fg \in \mathcal{L}[a, b]$, then we have*

$$\begin{aligned}
&\frac{1}{2^{1-2s_1-2s_2}}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
&\quad - \left[M(a,b)\mathbb{B}(s_1+s_2+1, s_1+s_2+1) + N(a,b)\mathbb{B}(2s_1+1, 2s_2+1) \right]
\end{aligned}$$

$$\leq \frac{1}{b-a} \int_0^1 f(x)g(x)dx$$

where $M(a, b)$ and $N(a, b)$ are given in (3.3) and (3.4) respectively.

If $h_1(t) = t^{-s_1}$ and $h_2 = t^{-s_2}$ in Theorem 2, then we have a result for Godunova-Levin-Dragomir type of (s_1, s_2) -convex functions.

Corollary 4. Let $f, g : \mathcal{I} = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be two Godunova-Levin-Dragomir type of (s_1, s_2) -convex functions such that $s_1, s_2 \in [0, 1]$. If $fg \in \mathcal{L}[a, b]$, then we have

$$\begin{aligned} & \frac{1}{2^{1+2s_1+2s_2}} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ & - \left[M(a, b) \mathbb{B}(1-s_1-s_2, 1-s_1-s_2) + N(a, b) \mathbb{B}(1-2s_1, 1-2s_2) \right] \\ & \leq \frac{1}{b-a} \int_0^1 f(x)g(x)dx \end{aligned}$$

where $M(a, b)$ and $N(a, b)$ are given in (3.3) and (3.4) respectively.

Our next result is the extension of the upper bound of Hermite-Hadamard type inequality via product of two (h_1, h_2) -convex functions.

Theorem 3. Let $f, g : \mathcal{I} = [a, b] \rightarrow \mathbb{R}$ be two (h_1, h_2) -convex functions. If $fg \in \mathcal{L}[a, b]$, then, we have

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x)g(x)dx \\ & \leq M(a, b) \int_0^1 h_1^2(t)h_2^2(1-t)dt + N(a, b) \int_0^1 h_1(t)h_2(1-t)h_1(1-t)h_2(t)dt, \end{aligned}$$

where $M(a, b)$ and $N(a, b)$ are given by (3.3) and (3.4) respectively.

Proof. Since f and g are (h_1, h_2) -convex functions, then

$$f(ta + (1-t)b) \leq h_1(t)h_2(1-t)f(a) + h_1(1-t)h_2(t)f(b),$$

and

$$g(ta + (1-t)b) \leq h_1(t)h_2(1-t)g(a) + h_1(1-t)h_2(t)g(b).$$

Multiplying both sides of the above inequality and then integrating it with respect to t on $[0, 1]$, we have

$$\begin{aligned} & \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b)dt \\ & \leq [f(a)g(a) + f(b)g(b)] \int_0^1 h_1^2(t)h_2^2(1-t)dt \\ & \quad + [f(a)g(b) + f(b)g(a)] \int_0^1 h_1(t)h_2(1-t)h_1(1-t)h_2(t)dt. \end{aligned}$$

This implies

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x)g(x)dx \\ & \leq M(a, b) \int_0^1 h_1^2(t)h_2^2(1-t)dt + N(a, b) \int_0^1 h_1(t)h_2(1-t)h_1(1-t)h_2(t)dt. \end{aligned}$$

This completes the proof. $\square$

The next result is a special case of Theorem 3.

If $h_1(t) = t^{s_1}$ and $h_2 = t^{s_2}$ in Theorem 3, then we have a result for Breckner type of (s_1, s_2) -convex functions.

Corollary 5. *Let $f, g : \mathcal{I} = [a, b] \rightarrow \mathbb{R}$ be two Breckner type of (s_1, s_2) -convex functions where $s_1, s_2 \in [0, 1]$. If $fg \in \mathcal{L}[a, b]$, then we have*

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x)g(x)dx \\ & \leq M(a, b)\mathbb{B}(2s_1 + 1, 2s_2 + 1) + N(a, b)\mathbb{B}(s_1 + s_2 + 1, s_1 + s_2 + 1), \end{aligned}$$

where $M(a, b)$ and $N(a, b)$ are given by (3.3) and (3.4) respectively.

If $h_1(t) = t^{-s_1}$ and $h_2 = t^{-s_2}$ in Theorem 3, then we have a result for Godunova-Levin-Dragomir type of (s_1, s_2) -convex functions.

Corollary 6. Let $f, g : \mathcal{I} = [a, b] \rightarrow \mathbb{R}$ be two Godunova-Levin-Dragomir type of (s_1, s_2) -convex functions where $s_1, s_2 \in [0, 1]$. If $fg \in \mathcal{L}[a, b]$, then we have

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x)g(x)dx \\ & \leq M(a, b)\mathbb{B}(1-2s_1, 1-2s_2) + N(a, b)\mathbb{B}(1-s_1-s_2, 1-s_1-s_2), \end{aligned}$$

where $M(a, b)$ and $N(a, b)$ are given by (3.3) and (3.4) respectively.

Theorem 4. Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f \in \mathcal{L}[a, b]$. If f is an (h_1, h_2) -convex function, then we have

$$\int_a^b (u-a)^\alpha (b-u)^\beta f(u)du \leq (b-a)^{\alpha+\beta+1} [\psi_1(t)f(a) + \psi_2(t)f(b)],$$

where

$$\psi_1(t) := \int_0^1 t^\alpha (1-t)^\beta h_1(1-t)h_2(t)dt, \quad (3.5)$$

and

$$\psi_2(t) := \int_0^1 t^\alpha (1-t)^\beta h_1(t)h_2(1-t)dt, \quad (3.6)$$

respectively.

Proof. Using Lemma 1 and the fact that f is an (h_1, h_2) -convex function, we have

$$\begin{aligned} & \int_a^b (u-a)^\alpha (b-u)^\beta f(u)du \\ & = (b-a)^{\alpha+\beta+1} \int_0^1 t^\alpha (1-t)^\beta f((1-t)a + tb)dt \\ & \leq (b-a)^{\alpha+\beta+1} \int_0^1 t^\alpha (1-t)^\beta [h_1(1-t)h_2(t)f(a) + h_1(t)h_2(1-t)f(b)]dt \\ & = (b-a)^{\alpha+\beta+1} [\psi_1(t)f(a) + \psi_2(t)f(b)]. \end{aligned}$$

This completes the proof. $\square$

If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$ in Theorem 4, then we have

Corollary 7. Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f \in \mathcal{L}[a, b]$. If f is a Breckner type of (s_1, s_2) -convex function, where $s_1, s_2 \in [0, 1]$, then we have

$$\int_a^b (u-a)^\alpha (b-u)^\beta f(u) du \leq (b-a)^{\alpha+\beta+1} [\psi_1'(t) f(a) + \psi_2'(t) f(b)],$$

where

$$\psi_1'(t) := \mathbb{B}(\alpha + s_2 + 1, \beta + s_1 + 1), \quad (3.7)$$

and

$$\psi_2'(t) := \mathbb{B}(\alpha + s_1 + 1, \beta + s_2 + 1), \quad (3.8)$$

respectively.

If $h_1(t) = t^{-s_1}$ and $h_2(t) = t^{-s_2}$ in Theorem 4, then we have

Corollary 8. Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f \in \mathcal{L}[a, b]$. If f is a Godunova-Levin-Dragomir type of (s_1, s_2) -convex function, where $s_1, s_2 \in [0, 1]$, then we have

$$\int_a^b (u-a)^\alpha (b-u)^\beta f(u) du \leq (b-a)^{\alpha+\beta+1} [\psi_1''(t) f(a) + \psi_2''(t) f(b)],$$

where

$$\psi_1''(t) := \mathbb{B}(\alpha - s_2 + 1, \beta - s_1 + 1), \quad (3.9)$$

and

$$\psi_2''(t) := \mathbb{B}(\alpha - s_1 + 1, \beta - s_2 + 1), \quad (3.10)$$

respectively.

Theorem 5. Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f \in \mathcal{L}[a, b]$. If $|f|^{\frac{r}{r-1}}$ is a (h_1, h_2) -convex function, then we have

$$\begin{aligned} & \int_a^b (u-a)^\alpha (b-u)^\beta f(u) du \\ & \leq (b-a)^{\alpha+\beta+1} \mathbb{B}(r\alpha + 1, r\beta + 1) \left[\left\{ |f(a)|^{\frac{r}{r-1}} + |f(b)|^{\frac{r}{r-1}} \right\} \int_0^1 h_1(t) h_2(1-t) dt \right]^{\frac{r-1}{r}}. \end{aligned}$$

Proof. Using Lemma 1, Holder's inequality and the fact that $|f|^{\frac{r}{r-1}}$ is an (h_1, h_2) -convex function, then

$$\begin{aligned}
& \int_a^b (u-a)^\alpha (b-u)^\beta f(u) du \\
& \leq (b-a)^{\alpha+\beta+1} \left[\int_0^1 t^{r\alpha} (1-t)^{r\beta} dt \right]^{\frac{1}{r}} \left[\int_0^1 |f((1-t)a + tb)|^{\frac{r}{r-1}} dt \right]^{\frac{r-1}{r}} \\
& \leq (b-a)^{\alpha+\beta+1} \mathbb{B}(r\alpha+1, r\beta+1) \\
& \quad \left[\int_0^1 \left\{ h_1(1-t)h_2(t)|f(a)|^{\frac{r}{r-1}} + h_1(t)h_2(1-t)|f(b)|^{\frac{r}{r-1}} \right\} dt \right]^{\frac{r-1}{r}} \\
& \leq (b-a)^{\alpha+\beta+1} \mathbb{B}(r\alpha+1, r\beta+1) \\
& \quad \left[\left\{ |f(a)|^{\frac{r}{r-1}} + |f(b)|^{\frac{r}{r-1}} \right\} \int_0^1 h_1(t)h_2(1-t) dt \right]^{\frac{r-1}{r}}.
\end{aligned}$$

This completes the proof. $\square$

If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$ in Theorem 5, then we have

Corollary 9. Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f \in \mathcal{L}[a, b]$. If $|f|^{\frac{r}{r-1}}$ is a Breckner type of (s_1, s_2) -convex function, where $s_1, s_2 \in [0, 1]$, then we have

$$\begin{aligned}
& \int_a^b (u-a)^\alpha (b-u)^\beta f(u) du \\
& \leq (b-a)^{\alpha+\beta+1} \mathbb{B}(r\alpha+1, r\beta+1) \left[\left\{ |f(a)|^{\frac{r}{r-1}} + |f(b)|^{\frac{r}{r-1}} \right\} \mathbb{B}(\alpha+s_1+1, \beta+s_2+1) \right]^{\frac{r-1}{r}}.
\end{aligned}$$

If $h_1(t) = t^{-s_1}$ and $h_2(t) = t^{-s_2}$ in Theorem 5, then we have

Corollary 10. Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f \in \mathcal{L}[a, b]$. If $|f|^{\frac{r}{r-1}}$ is a Godunova-Levin-Dragomir type of (s_1, s_2) -convex function, where $s_1, s_2 \in [0, 1]$, then we have

$$\begin{aligned}
& \int_a^b (u-a)^\alpha (b-u)^\beta f(u) du \\
& \leq (b-a)^{\alpha+\beta+1} \mathbb{B}(1-r\alpha, 1-r\beta) \left[\left\{ |f(a)|^{\frac{r}{r-1}} + |f(b)|^{\frac{r}{r-1}} \right\} \mathbb{B}(1-\alpha-s_1, 1-\beta-s_2) \right]^{\frac{r-1}{r}}.
\end{aligned}$$

Theorem 6. Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f \in \mathcal{L}[a, b]$. If $|f|^r$ is an (h_1, h_2) -convex function, then we have

$$\begin{aligned} & \int_a^b (u-a)^\alpha (b-u)^\beta f(u) du \\ & \leq (b-a)^{\alpha+\beta+1} [\mathbb{B}(\alpha+1, \beta+1)]^{\frac{r-1}{r}} [\psi_1(t)|f(a)|^r + \psi_2(t)|f(b)|^r]^{\frac{1}{r}}, \end{aligned}$$

where $\psi_1(t)$ and $\psi_2(t)$ are given by (3.5) and (3.6) respectively.

Proof. Using Lemma 1, Holder's inequality and the fact that $|f|^r$ is an (h_1, h_2) -convex function, then

$$\begin{aligned} & \int_a^b (u-a)^\alpha (b-u)^\beta f(u) du \\ & \leq (b-a)^{\alpha+\beta+1} \left[\int_0^1 (1-t)^\alpha t^\beta dt \right]^{\frac{r-1}{r}} \left[\int_0^1 t^\alpha (1-t)^\beta |f((1-t)a + tb)|^r dt \right]^{\frac{1}{r}} \\ & \leq (b-a)^{\alpha+\beta+1} [\mathbb{B}(\alpha+1, \beta+1)]^{\frac{r-1}{r}} \\ & \quad \times \left[\int_0^1 t^\alpha (1-t)^\beta [h_1(1-t)h_2(t)|f(a)|^r + h_1(t)h_2(1-t)|f(b)|^r] dt \right]^{\frac{1}{r}} \\ & = (b-a)^{\alpha+\beta+1} [\mathbb{B}(\alpha+1, \beta+1)]^{\frac{r-1}{r}} [\psi_1(t)|f(a)|^r + \psi_2(t)|f(b)|^r]^{\frac{1}{r}}. \end{aligned}$$

This completes the proof. $\square$

If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$ in Theorem 6, then we have

Corollary 11. Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f \in \mathcal{L}[a, b]$. If $|f|^r$ is a Breckner type of (s_1, s_2) -convex function, where $s_1, s_2 \in [0, 1]$, then we have

$$\begin{aligned} & \int_a^b (u-a)^\alpha (b-u)^\beta f(u) du \\ & \leq (b-a)^{\alpha+\beta+1} [\mathbb{B}(\alpha+1, \beta+1)]^{\frac{r-1}{r}} [\psi'_1(t)|f(a)|^r + \psi'_2(t)|f(b)|^r]^{\frac{1}{r}}, \end{aligned}$$

where $\psi'_1(t)$ and $\psi'_2(t)$ are given by (3.7) and (3.8) respectively.

If $h_1(t) = t^{-s_1}$ and $h_2(t) = t^{-s_2}$ in Theorem 6, then we have

Corollary 12. Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f \in \mathcal{L}[a, b]$. If $|f|^r$ is a Godunova-Levin-Dragomir type of (s_1, s_2) -convex function, where $s_1, s_2 \in [0, 1]$, then we have

$$\begin{aligned} & \int_a^b (u-a)^\alpha (b-u)^\beta f(u) du \\ & \leq (b-a)^{\alpha+\beta+1} [\mathbb{B}(1-\alpha, 1-\beta)]^{\frac{r-1}{r}} [\psi_1''(t)|f(a)|^r + \psi_2''(t)|f(b)|^r]^{\frac{1}{r}}, \end{aligned}$$

where $\psi_1''(t)$ and $\psi_2''(t)$ are given by (3.9) and (3.10) respectively.

Now using Lemma 2, we derive some new inequalities for n -times differentiable (h_1, h_2) -convex functions.

Theorem 7. Let $f : \mathcal{I}^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$, $a, b \in \mathcal{I}^\circ$ with $a < b$ where $\mathcal{I}^\circ$ is the interior of $\mathcal{I}$. Also suppose $f^{(n)}$ exists on $\mathcal{I}^\circ$ and $f^{(n)} \in \mathcal{L}[a, b]$. If $|f^{(n)}|^q$ is an (h_1, h_2) -convex function, then for $n, q \geq 1$, we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right| \\ & \leq \frac{(b-a)^n}{2n!} \left(\frac{n-1}{n+1} \right)^{1-\frac{1}{q}} (\theta_1(t)|f^{(n)}(a)|^q + \theta_2(t)|f^{(n)}(b)|^q)^{\frac{1}{q}}, \end{aligned}$$

where

$$\theta_1(t) := \int_0^1 t^{n-1} (n-2t) h_1(t) h_2(1-t) dt, \quad (3.11)$$

and

$$\theta_2(t) := \int_0^1 t^{n-1} (n-2t) h_1(1-t) h_2(t) dt, \quad (3.12)$$

respectively.

Proof. Using Lemma 2, property of modulus, power means inequality and the fact that $|f^{(n)}|^q$ is an (h_1, h_2) -convex function, we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right| \\ & = \left| \frac{(b-a)^n}{2n!} \int_0^1 t^{n-1} (n-2t) f^{(n)}(ta + (1-t)b) dt \right| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{(b-a)^n}{2n!} \int_0^1 t^{n-1} (n-2t) |f^{(n)}(ta + (1-t)b)| dt \\
&\leq \frac{(b-a)^n}{2n!} \left(\int_0^1 t^{n-1} (n-2t) dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{n-1} (n-2t) |f^{(n)}(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \\
&\leq \frac{(b-a)^n}{2n!} \left(\frac{n-1}{n+1} \right)^{1-\frac{1}{q}} \\
&\quad \times \left(\int_0^1 t^{n-1} (n-2t) [h_1(t)h_2(1-t)|f^n(a)|^q + h_1(1-t)h_2(t)|f^n(b)|^q] dt \right)^{\frac{1}{q}} \\
&= \frac{(b-a)^n}{2n!} \left(\frac{n-1}{n+1} \right)^{1-\frac{1}{q}} (\theta_1(t)|f^n(a)|^q + \theta_2(t)|f^n(b)|^q)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. $\square$

If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$ in Theorem 7, then we have

Corollary 13. Let $f : \mathcal{I}^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$, $a, b \in \mathcal{I}^\circ$ with $a < b$ where $\mathcal{I}^\circ$ is the interior of $\mathcal{I}^\circ$. Also suppose $f^{(n)}$ exists on $\mathcal{I}^\circ$ and $f^{(n)} \in \mathcal{L}[a, b]$. If $|f^{(n)}|^q$ is a Breckner type of (s_1, s_2) -convex function, then for $s_1, s_2 \in [0, 1]$ and $n, q \geq 1$, we have

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right| \\
&\leq \frac{(b-a)^n}{2n!} \left(\frac{n-1}{n+1} \right)^{1-\frac{1}{q}} (\theta'_1(t)|f^n(a)|^q + \theta'_2(t)|f^n(b)|^q)^{\frac{1}{q}},
\end{aligned}$$

where

$$\theta'_1(t) := n\mathbb{B}(n + s_1, s_2 + 1) - 2\mathbb{B}(n + s_1 + 1, s_2 + 1),$$

and

$$\theta'_2(t) := n\mathbb{B}(n + s_2, s_1 + 1) - 2\mathbb{B}(n + s_2 + 1, s_1 + 1),$$

respectively.

If $h_1(t) = t^{-s_1}$ and $h_2(t) = t^{-s_2}$ in Theorem 7, then we have

Corollary 14. Let $f : \mathcal{I}^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$, $a, b \in \mathcal{I}^\circ$ with $a < b$ where $\mathcal{I}^\circ$ is the interior of $\mathcal{I}^\circ$. Also suppose $f^{(n)}$ exists on $\mathcal{I}^\circ$ and $f^{(n)} \in \mathcal{L}[a, b]$. If $|f^{(n)}|^q$ is a Godunova-Levin-Dragomir type of (s_1, s_2) -convex function, then for $s_1, s_2 \in [0, 1]$ and $n, q \geq 1$,

we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right| \leq \frac{(b-a)^n}{2n!} \left(\frac{n-1}{n+1} \right)^{1-\frac{1}{q}} (\theta_1''(t) |f^n(a)|^q + \theta_2''(t) |f^n(b)|^q)^{\frac{1}{q}},$$

where

$$\theta_1''(t) := n\mathbb{B}(n-s_1, 1-s_2) - 2\mathbb{B}(n-s_1+1, 1-s_2),$$

and

$$\theta_2''(t) := n\mathbb{B}(n-s_2, 1-s_1) - 2\mathbb{B}(n-s_2+1, 1-s_2),$$

respectively.

Theorem 8. Let $f : \mathcal{I}^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$, $a, b \in \mathcal{I}^\circ$ with $a < b$ where $\mathcal{I}^\circ$ is the interior of $\mathcal{I}$. Also suppose $f^{(n)}$ exists on $\mathcal{I}^\circ$ and $f^{(n)} \in \mathcal{L}[a, b]$. If $|f^{(n)}|^P$ is a (h_1, h_2) -convex function, then for $n, q \geq 1$, we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right| \leq \frac{(b-a)^n}{2n(n-1)^{\frac{1}{q}}} (\phi_1(t) |f^n(a)|^q + \phi_2(t) |f^n(b)|^q)^{\frac{1}{q}},$$

where

$$\phi_1(t) := \int_0^1 t^{q(n-1)} (n-2t) h_1(t) h_2(1-t) dt, \quad (3.13)$$

and

$$\phi_2(t) := \int_0^1 t^{q(n-1)} (n-2t) h_1(1-t) h_2(t) dt, \quad (3.14)$$

respectively.

Proof. Using Lemma 2, property of modulus, Holder's inequality and the fact that $|f^{(n)}|^q$ is an (h_1, h_2) -convex function, we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right|$$

$$\begin{aligned}
&= \left| \frac{(b-a)^n}{2n!} \int_0^1 t^{n-1} (n-2t) f^{(n)}(ta + (1-t)b) dt \right| \\
&\leq \frac{(b-a)^n}{2n!} \int_0^1 t^{n-1} (n-2t) |f^{(n)}(ta + (1-t)b)| dt \\
&\leq \frac{(b-a)^n}{2n!} \left(\int_0^1 (n-2t) dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{q(n-1)} (n-2t) |f^{(n)}(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \\
&\leq \frac{(b-a)^n}{2n!} (n-1)^{1-\frac{1}{q}} \\
&\quad \times \left(\int_0^1 t^{q(n-1)} (n-2t) [h_1(t)h_2(1-t)|f^n(a)|^q + h_1(1-t)h_2(t)|f^n(b)|^q] dt \right)^{\frac{1}{q}} \\
&= \frac{(b-a)^n}{2n(n-1)^{\frac{1}{q}}} (\phi_1(t)|f^n(a)|^q + \phi_2(t)|f^n(b)|^q)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. $\square$

If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$ in Theorem 8, then we have

Corollary 15. Let $f : \mathcal{I}^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$, $a, b \in \mathcal{I}^\circ$ with $a < b$ where $\mathcal{I}^\circ$ is the interior of $\mathcal{I}$. Also suppose $f^{(n)}$ exists on $\mathcal{I}^\circ$ and $f^{(n)} \in \mathcal{L}[a, b]$. If $|f^{(n)}|^P$ is a Breckner type of (s_1, s_2) -convex function, then for $s_1, s_2 \in [0, 1]$ and $n, q \geq 1$, we have

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right| \\
&\leq \frac{(b-a)^n}{2n(n-1)^{\frac{1}{q}}} (\phi'_1(t)|f^n(a)|^q + \phi'_2(t)|f^n(b)|^q)^{\frac{1}{q}},
\end{aligned}$$

where

$$\phi'_1(t) := n\mathbb{B}(nq + s_1 - q + 1, s_2 + 1) - 2\mathbb{B}(nq + s_1 - q + 2, s_2 + 1),$$

and

$$\phi'_2(t) := n\mathbb{B}(nq + s_2 - q + 1, s_1 + 1) - 2\mathbb{B}(nq + s_2 - q + 2, s_1 + 1),$$

respectively.

If $h_1(t) = t^{-s_1}$ and $h_2(t) = t^{-s_2}$ in Theorem 8, then we have

Corollary 16. Let $f : \mathcal{I}^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$, $a, b \in \mathcal{I}^\circ$ with $a < b$ where $\mathcal{I}^\circ$ is the interior of $\mathcal{I}$. Also suppose $f^{(n)}$ exists on $\mathcal{I}^\circ$ and $f^{(n)} \in \mathcal{L}[a, b]$. If $|f^{(n)}|^P$ is a Godunova-Levin-Dragomir type of (s_1, s_2) -convex function, then for $s_1, s_2 \in [0, 1]$ and $n, q \geq 1$, we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right| \leq \frac{(b-a)^n}{2n(n-1)^{\frac{1}{q}}} (\phi_1''(t) |f^n(a)|^q + \phi_2''(t) |f^n(b)|^q)^{\frac{1}{q}},$$

where

$$\phi_1''(t) := n\mathbb{B}(nq - s_1 - q + 1, 1 - s_2) - 2\mathbb{B}(nq - s_1 - q + 2, 1 - s_2),$$

and

$$\phi_2''(t) := n\mathbb{B}(nq - s_2 - q + 1, 1 - s_1) - 2\mathbb{B}(nq - s_2 - q + 2, 1 - s_1),$$

respectively.

CONCLUSION

In this paper, we have introduced a new extension of convex functions which is called as (h_1, h_2) -convex functions. We have noticed that it contains some new and known classes of convex functions among one of those is h -convex functions. We have derived several new generalizations of Hermite-Hadamard type inequalities via (h_1, h_2) -convex functions. We have also discussed some of new special cases which can be easily deduced from our main results. It is expected that the interested readers may further explore the property of (h_1, h_2) -convexity of functions.

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Authors' addresses

Muhammad Uzair Awan

Department of Mathematics, GC University, Faisalabad, Pakistan

E-mail address: awan.uzair@gmail.com

Muhammad Aslam Noor

Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan

E-mail address: noormaslam@gmail.com

Khalida Inayat Noor

Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan

E-mail address: khalidanoor@hotmail.com

Awais Gul Khan

Department of Mathematics, GC University, Faisalabad, Pakistan

E-mail address: awaisgul Khan@gmail.com



BIBO STABILITY OF DISCRETE CONTROL SYSTEMS WITH SEVERAL TIME DELAYS

ESSAM AWWAD, ISTVÁN GYŐRI, AND FERENC HARTUNG

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Abstract. This paper investigates the bounded input bounded output (BIBO) stability in a class of control system of nonlinear difference equations with several time delays. The proofs are based on our studies on the boundedness of the solutions of a general class of nonlinear Volterra difference equations with delays.

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Keywords: boundedness, Volterra difference equations, bounded input bounded output (BIBO) stability, difference equations with delays

1. INTRODUCTION

Time delays play an important role in control systems, since a delay naturally appears when a system wants to measure or react to information. Stability or stabilization of a system is one of the central question which is investigated in control theory [10–12]. Because of its simplicity, the bounded input bounded output (BIBO) stability of control systems is widely investigated. The sufficient conditions for BIBO stability of control systems without delays are obtained in [18, 19] by using Liapunov function techniques. More recently many researchers have focused their interest on the BIBO stability of nonlinear discrete and continuous feedback control systems with or without delays [1, 2, 5–9, 13–15, 17].

In this paper we consider a class of discrete control systems with multiple time delays. We search for delayed feedback controls such that the corresponding closed loop system be BIBO stable. We rewrite the closed loop system as an equivalent nonlinear Volterra difference equation (VDE) with delays. The BIBO stability results are based on our theorem which formulate sufficient conditions for the boundedness of the solutions of delayed VDEs. The results presented in this manuscript extend the methods introduced in [1] for nonlinear differential equations with a single delay and boundedness of ordinary VDEs presented in [3].

The structure of the manuscript is the following. Section 2 contains the precise problem statement, the definitions of BIBO stability and local BIBO stability, and we

rewrite our closed loop control equation as an equivalent VDE. Section 3 formulates sufficient conditions for the boundedness of a general class of nonlinear VDEs with multiple delays. Section 4 contains our BIBO stability results for cases when the nonlinearity has a sub-linear, linear or super-linear estimates.

In the rest of this section we introduce some notations which will be used throughout this paper. $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}^d$ and $\mathbb{R}^{d \times d}$ denote the set of real numbers, nonnegative real numbers, d -dimensional real column vectors and $d \times d$ -dimensional real matrices, respectively. The maximum norm on $\mathbb{R}^d$ is denoted by $\|\cdot\|$, i.e., $\|x\| := \max_{1 \leq i \leq d} |x_i|$, where $x = (x_1, \dots, x_d)^T$. The matrix norm on $\mathbb{R}^{d \times d}$ generated by the maximum vector norm will be denoted by $\|\cdot\|$, as well. Let $\mathbb{Z}_+$ and $\mathbb{N}$ be the set of nonnegative and positive integers, respectively. $L^\infty(\mathbb{Z}_+, \mathbb{R}^d)$ will denote the set of bounded sequences $r : \mathbb{Z}_+ \rightarrow \mathbb{R}^d$ with norm $\|r\|_\infty := \sup_{n \in \mathbb{Z}_+} \|r(n)\|$. Let $\tau > 0$ be a fixed integer, $S([- \tau, 0], \mathbb{R}^d)$ denotes the set of finite sequences

$$\left\{ \psi : \{-\tau, -\tau + 1, \dots, 0\} \rightarrow \mathbb{R}^d \right\}$$

and $\|\psi\|_\tau := \max_{-\tau \leq n \leq 0} \|\psi(n)\|$. For a given sequence x and an integer n the forward difference operator is defined by $\Delta x(n) := x(n+1) - x(n)$.

2. PROBLEM STATEMENT

In this paper we consider the nonlinear discrete control system with several delays

$$\begin{aligned} \Delta x(n) &= g(n, x(n - \sigma_1(n)), \dots, x(n - \sigma_\ell(n))) + u(n), \quad n \in \mathbb{Z}_+, \\ y(n) &= Cx(n), \quad n \in \mathbb{Z}_+. \end{aligned} \quad (2.1)$$

Here $x(n) \in \mathbb{R}^d$ is the state vector, $u(n) \in \mathbb{R}^d$ is the input vector and $y(n) \in \mathbb{R}^{d_1}$ is the output vector of the system (2.1), $C \in \mathbb{R}^{d_1 \times d}$ is a constant matrix, $\sigma_i : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$, $i = 1, \dots, \ell$ are bounded delay functions, and the nonlinear function $g : \mathbb{Z}_+ \times \underbrace{\mathbb{R}^d \times \dots \times \mathbb{R}^d}_\ell \rightarrow \mathbb{R}^d$ satisfies

$$\|g(n, x^{(1)}, \dots, x^{(\ell)})\| \leq b(n) \varphi \left(\max_{1 \leq m \leq \ell} \|x^{(m)}\| \right), \quad n \in \mathbb{Z}_+, \quad x^{(1)}, \dots, x^{(\ell)} \in \mathbb{R}^d, \quad (2.2)$$

where $b(n) > 0$ for all $n \in \mathbb{Z}_+$, and $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a monotone nondecreasing mapping.

Our general problem (2.1) satisfying condition (2.2) includes, e.g., linear control systems

$$\Delta x(n) = A_1(n)x(n - \sigma_1(n)) + \dots + A_\ell(n)x(n - \sigma_\ell(n)) + u(n), \quad n \in \mathbb{Z}_+,$$

and nonlinear control systems of the form

$$\Delta x_i(n) = \sum_{j=1}^d a_{ij}(n) x_j^p(n - \sigma_j(n)) + u_i(n), \quad n \in \mathbb{Z}_+, i = 1, \dots, d,$$

where $x(n) = (x_1(n), \dots, x_d(n))^T$, $u(n) = (u_1(n), \dots, u_d(n))^T$, $p > 0$; or a polynomial difference system

$$\Delta x_i(n) = \sum_{j=1}^{\ell} a_{ij}(n) x_1^{q_{ij1}}(n - \sigma_j(n)) \cdots x_d^{q_{ijd}}(n - \sigma_j(n)) + u_i(n)$$

for $n \in \mathbb{Z}_+$, $i = 1, \dots, d$, where $q_{ijk} \in \mathbb{R}_+$ for $i, k = 1, \dots, d$ and $j = 1, \dots, \ell$; or the scalar nonautonomous control system of the form

$$\Delta x(n) = \frac{a(n)x^p(n - \sigma_1(n))}{b(n) + x^q(n - \sigma_2(n))} + u(n), \quad n \in \mathbb{Z}_+,$$

where $p, q > 0$. In all the above cases assumption (2.2) holds under natural conditions with $\varphi(t) = t^p$ with some $p > 0$.

We assume that the uncontrolled system, i.e., (2.1) with $u \equiv 0$ has unbounded solutions. Our goal is to find a positive diagonal matrix D and a positive integer k such that the delayed feedback law of the form

$$u(n) = -Dx(n - k) + r(n) \quad (2.3)$$

guarantees that the closed-loop delayed system

$$\begin{aligned} \Delta x(n) &= g(n, x(n - \sigma_1(n)), \dots, x(n - \sigma_\ell(n))) - Dx(n - k) + r(n), \quad n \in \mathbb{Z}_+, \\ y(n) &= Cx(n), \quad n \in \mathbb{Z}_+, \\ x(n) &= \psi(n), \quad n \in \{-\tau, -\tau + 1, \dots, 0\} \end{aligned} \quad (2.4)$$

is BIBO stable. Here $r(n)$ is the reference input, $D = \text{diag}(\lambda_1, \dots, \lambda_d)$, $\lambda_i > 0$ for $i = 1, \dots, d$, $\psi \in S([- \tau, 0], \mathbb{R}^d)$ is the initial sequence associated to the equation where

$$\tau := \max \left\{ \max_{1 \leq j \leq \ell} \|\sigma_j\|_\infty, k \right\}, \quad (2.5)$$

The assumed diagonal form of the feedback law (2.3) is one of the simplest possible choice. In its implementation it is important to know how large delay can be. In Theorem 2 and 3 we give sufficient conditions on how to select the feedback gain D and the time delay k to guarantee the boundedness of the solutions. Our conditions (see (4.1) and (4.13) below) show that the larger the delay the smaller gain can guarantee the boundedness of the solution.

Following [16], we introduce the next definition of BIBO stability.

Definition 1. The closed loop system (2.4) is said to be BIBO stable if there exist positive constants θ_1 and $\theta_2 = \theta_2(\|\psi\|_\tau)$ such that every solution of the system (2.4) satisfies

$$\|y(n)\| \leq \theta_1 \|r\|_\infty + \theta_2, \quad n \in \mathbb{Z}_+$$

for every reference input $r \in L^\infty(\mathbb{Z}_+, \mathbb{R}^d)$.

Later we need the notion of local BIBO stability (see similar definition in [1] for the continuous case).

Definition 2. The closed loop system (2.4) is said to be locally BIBO stable if there exist positive constants δ_1 , δ_2 and θ satisfying

$$\|y(n)\| \leq \theta, \quad n \in \mathbb{Z}_+$$

provided that $\|\psi\|_\tau < \delta_1$ and $\|r\|_\infty < \delta_2$.

Our approach is the following. We associate the linear system

$$\Delta z(n) = -Dz(n-k), \quad n \in \mathbb{Z}_+ \quad (2.6)$$

with the constant delay $k \in \mathbb{N}$ and the initial condition

$$z(n) = \psi(n), \quad -k \leq n \leq 0 \quad (2.7)$$

to (2.4). Then the state equation in (2.4) can be considered as the nonlinear perturbation of (2.6), and by the variation of constants formula (see, e.g., Lemma 4 in [4]) we get

$$x(n) = z(n) + \sum_{j=0}^{n-1} W(n-j-1)[g(j, x(j-\sigma_1(j)), \dots, x(j-\sigma_\ell(j))) + r(j)] \quad (2.8)$$

for $n \in \mathbb{Z}_+$, where $z(n)$ is the solution of (2.6)-(2.7) and W is the fundamental matrix solution of (2.6), i.e., the solution of the IVP

$$\begin{aligned} \Delta W(n) &= -DW(n-k), \quad n \in \mathbb{Z}_+, \\ W(n) &= \begin{cases} \mathbf{0}, & -k \leq n \leq -1, \\ \mathbf{I}, & n = 0. \end{cases} \end{aligned} \quad (2.9)$$

Here $\mathbf{I} \in \mathbb{R}^{d \times d}$ is the identity matrix and $\mathbf{0} \in \mathbb{R}^{d \times d}$ is the zero matrix. Since D is a diagonal matrix, it is easy to see that $W(n)$ is a diagonal matrix too for all $n \in \mathbb{Z}_+$.

We can rewrite the equation (2.8) as a VDE

$$\begin{aligned} x(n+1) &= z(n+1) + \sum_{j=0}^n W(n-j)g(j, x(j-\sigma_1(j)), \dots, x(j-\sigma_\ell(j))) \\ &\quad + \sum_{j=0}^n W(n-j)r(j), \quad n \in \mathbb{Z}_+, \end{aligned}$$

and so it is equivalent to

$$x(n+1) = \sum_{j=0}^n f(n, j, x(j-\sigma_1(j)), \dots, x(j-\sigma_\ell(j))) + h(n), \quad n \in \mathbb{Z}_+, \quad (2.10)$$

where

$$h(n) := z(n+1) + \sum_{j=0}^n W(n-j)r(j), \quad (2.11)$$

and

$$f(n, j, x^{(1)}, \dots, x^{(\ell)}) := W(n-j)g(j, x^{(1)}, \dots, x^{(\ell)}) \quad (2.12)$$

for $0 \leq j \leq n$, $x^{(i)} \in \mathbb{R}^d$, $1 \leq i \leq \ell$. The equation (2.10) is a nonlinear VDE with several delay functions.

3. BOUNDEDNESS OF THE SOLUTIONS OF VDES WITH DELAYS

In this section we give a general result for the boundedness of the solutions of nonlinear VDEs with multiple delays which is a natural extensions of the results presented in [3] for nonlinear VDEs without delays.

We consider the nonlinear VDE with several delays

$$x(n+1) = \sum_{j=0}^n f(n, j, x(j-\sigma_1(j)), \dots, x(j-\sigma_\ell(j))) + h(n), \quad n \in \mathbb{Z}_+, \quad (3.1)$$

with the associated initial condition

$$x(n) = \psi(n), \quad -\tau \leq n \leq 0, \quad (3.2)$$

where τ is a positive integer constant. We assume the following conditions.

(B1) For any fixed $0 \leq j \leq n$ and $j, n \in \mathbb{Z}_+$

$$f(n, j, \cdot, \dots, \cdot) : \underbrace{\mathbb{R}^d \times \dots \times \mathbb{R}^d}_\ell \rightarrow \mathbb{R}^d.$$

(B2) For any $0 \leq j \leq n$ and $1 \leq i \leq d$ there exists an $a_i(n, j) \in \mathbb{R}_+$ such that

$$|f_i(n, j, x^{(1)}, \dots, x^{(\ell)})| \leq a_i(n, j) \varphi \left(\max_{1 \leq m \leq \ell} \|x^{(m)}\| \right) \quad (3.3)$$

holds for $x^{(1)}, \dots, x^{(\ell)} \in \mathbb{R}^d$ with a monotone non-decreasing mapping $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, where $f = (f_1, \dots, f_d)^T$.

(B3) $h(n) = (h_1(n), \dots, h_d(n))^T \in \mathbb{R}^d$ for $n \in \mathbb{Z}_+$.

(B4) $\sigma_i : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ satisfies $|\sigma_i(n)| \leq \tau$ for $n \in \mathbb{Z}_+$ and $i = 1, \dots, \ell$.

(B5) $\psi \in S([-\tau, 0], \mathbb{R}^d)$.

Clearly, problem (3.1)-(3.2) has a unique solution under the above conditions. The next result formulates sufficient conditions implying the boundedness of the solutions.

Theorem 1. *Let ψ be fixed, (B1)-(B5) are satisfied and let $x(n; \psi)$ be the solution of (3.1)-(3.2). Suppose there exist $N \in \mathbb{Z}_+$, $\mu \in \mathbb{R}_+$ and $v \geq \mu$ such that for $i = 1, \dots, d$*

$$\sum_{j=0}^N a_i(N, j)\varphi(\mu) + |h_i(N)| \leq v, \quad (3.4)$$

$$\sum_{j=0}^N a_i(n, j)\varphi(\mu) + \sum_{j=N+1}^n a_i(n, j)\varphi(v) + |h_i(n)| \leq v, \quad n \geq N+1 \quad (3.5)$$

and

$$\|x(n; \psi)\| \leq \mu, \quad n \in \{-\tau, \dots, N\}. \quad (3.6)$$

Then the solution is bounded by v , i.e.

$$\|x(n; \psi)\| \leq v, \quad n \geq -\tau. \quad (3.7)$$

Proof. Consider the solution $x(n) = x(n; \psi)$, $n \in \mathbb{Z}_+$ of (3.1) with the initial condition (3.2), and let μ and N be such that (3.6) holds. Then, by using (B2), (3.4), (3.6) and the monotonicity of φ , we have for $i = 1, \dots, d$

$$\begin{aligned} |x_i(N+1)| &\leq \sum_{j=0}^N |f_i(N, j, x(j-\sigma_1(j)), \dots, x(j-\sigma_\ell(j)))| + |h_i(N)| \\ &\leq \sum_{j=0}^N a_i(N, j)\varphi\left(\max_{-\tau \leq m \leq N} \|x(m)\|\right) + |h_i(N)| \\ &\leq \sum_{j=0}^N a_i(N, j)\varphi(\mu) + |h_i(N)| \\ &\leq v, \end{aligned}$$

Therefore $\|x(N+1)\| \leq v$, so (3.7) holds for $n = N+1$.

Now we show that (3.7) holds for any $n \geq N+1$. Assume, for the sake of contradiction, that there exists $n_0 \geq N+1$ and $i_0 \in \{1, \dots, d\}$ such that

$$|x_{i_0}(n_0+1)| = |x_{i_0}(n_0+1; \psi)| > v, \quad (3.8)$$

and

$$|x_i(n)| = |x_i(n; \psi)| \leq v, \quad N+1 \leq n \leq n_0, \quad i = 1, \dots, d. \quad (3.9)$$

Hence, from equation (3.1), we get

$$\begin{aligned}
|x_{i_0}(n_0 + 1)| &\leq \sum_{j=0}^N |f_{i_0}(n_0, j, x(j - \sigma_1(j)), \dots, x(j - \sigma_\ell(j)))| \\
&\quad + \sum_{j=N+1}^{n_0} |f_{i_0}(n_0, j, x(j - \sigma_1(j)), \dots, x(j - \sigma_\ell(j)))| + |h_{i_0}(n_0)| \\
&\leq \sum_{j=0}^N a_{i_0}(n_0, j) \varphi(\max_{-\tau \leq m \leq N} \|x(m)\|) \\
&\quad + \sum_{j=N+1}^{n_0} a_{i_0}(n_0, j) \varphi(\max_{-\tau \leq m \leq j} \|x(m)\|) + |h_{i_0}(n_0)|.
\end{aligned}$$

Since φ is a monotone non-decreasing mapping, (3.5), (3.6) and (3.9) yield

$$|x_{i_0}(n_0 + 1)| \leq \sum_{j=0}^N a_{i_0}(n_0, j) \varphi(\mu) + \sum_{j=N+1}^{n_0} a_{i_0}(n_0, j) \varphi(v) + |h_{i_0}(n_0)| \leq v.$$

This contradicts to our hypothesis (3.8), so inequality (3.7) holds. $\square$

4. MAIN RESULTS

Our main goal in this section is to formulate sufficient conditions which grantee the BIBO stability of the closed loop system (2.4). We will assume that function φ in (2.2) is a power function. Our first result is given for the case when g in (2.2) has a sub-linear estimate, i.e., when $\varphi(t) = t^p$, with $0 < p < 1$ in (2.2).

Theorem 2. *Let $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a function which satisfies inequality (2.2) with $\varphi(t) = t^p$, $0 < p < 1$, $t \geq 0$. The feedback control system (2.4) with $D = \text{diag}(\lambda_1, \dots, \lambda_d)$ and $k \in \mathbb{N}$ is BIBO stable if*

$$\|b\|_\infty := \sup_{n \in \mathbb{Z}_+} b(n) < \infty \quad \text{and} \quad 0 < \lambda_i < 2 \cos \frac{k\pi}{2k+1}, \quad i = 1, \dots, d \quad (4.1)$$

hold.

Proof. Let $\psi = (\psi_1, \dots, \psi_d)^T \in S([-\tau, 0], \mathbb{R}^d)$, and $z(n) = (z_1(n), \dots, z_d(n))^T$ be the solution of the IVP (2.6)-(2.7). Then, for $i = 1, \dots, d$, z_i is the solution of the IVP

$$\Delta z_i(n) = -\lambda_i z_i(n-k), \quad n \in \mathbb{Z}_+ \quad (4.2)$$

with initial condition

$$z_i(n) = \psi_i(n), \quad -k \leq n \leq 0. \quad (4.3)$$

It is known (see, e.g., [4]) that condition (4.1) yields that there exists a positive constant M and $\rho \in (0, 1)$ such that

$$|z_i(n)| \leq M \|\psi\|_\tau \rho^n, \quad n \in \mathbb{Z}_+, \quad i = 1, \dots, d, \quad (4.4)$$

where $\|\psi\|_\tau := \max_{-\tau \leq j \leq 0} \|\psi(j)\|$. Hence every solution of (4.2) tends to zero as $n \rightarrow \infty$, and

$$\|z\|_\infty := \sup_{n \in \mathbb{Z}_+} \|z(n)\| \leq M \|\psi\|_\tau < \infty. \quad (4.5)$$

Let $W(n) = \text{diag}(w_1(n), \dots, w_d(n))$ be the solution of (2.9). Relation (4.4) yields $\lim_{n \rightarrow \infty} w_i(n) = 0$ for $i = 1, \dots, d$, and

$$\eta := \max_{0 \leq i \leq d} \sum_{n=0}^{\infty} |w_i(n)| < \infty. \quad (4.6)$$

From (2.8), for all $n \in \mathbb{Z}_+$ and $i = 1, \dots, d$, we have

$$\begin{aligned} x_i(n+1) &= z_i(n+1) \\ &+ \sum_{j=0}^n w_i(n-j) [g_i(j, x(j-\sigma_1(j)), \dots, x(j-\sigma_\ell(j))) + r_i(j)], \end{aligned} \quad (4.7)$$

where $x(n) = (x_1(n), \dots, x_d(n))^T$, $g = (g_1, \dots, g_d)^T$ and $r = (r_1, \dots, r_d)^T$. Therefore (2.11) and (2.12) imply

$$f_i(n, j, x^{(1)}, \dots, x^{(\ell)}) = w_i(n-j) g_i(j, x^{(1)}, \dots, x^{(\ell)})$$

and

$$h_i(n) = z_i(n+1) + \sum_{j=0}^n w_i(n-j) r_i(j).$$

Hence, by (2.2),

$$\begin{aligned} |f_i(n, j, x^{(1)}, \dots, x^{(\ell)})| &\leq |w_i(n-j)| |g_i(j, x^{(1)}, \dots, x^{(\ell)})| \\ &\leq |w_i(n-j)| b(j) \varphi \left(\max_{-\tau \leq m \leq \ell} \|x^{(m)}\| \right), \end{aligned}$$

so the conditions (B1)-(B5) hold with $a_i(n, j) := |w_i(n-j)| b(j)$, $0 \leq j \leq n$.

By (4.1), (4.5), (4.6) and the definition of the infinity norm, we obtain

$$\begin{aligned} \gamma &:= \max_{1 \leq i \leq d} \sup_{n \in \mathbb{Z}_+} |h_i(n)| \\ &\leq \max_{1 \leq i \leq d} \sup_{n \in \mathbb{Z}_+} |z_i(n)| + \max_{1 \leq i \leq d} \sup_{n \in \mathbb{Z}_+} \sum_{j=0}^n |w_i(n-j)| \|r(j)\| \end{aligned}$$

$$\leq \sup_{n \in \mathbb{Z}_+} \|z(n)\| + \|r\|_\infty \max_{1 \leq i \leq d} \sum_{j=0}^{\infty} |w_i(j)| \quad (4.8)$$

$$= \|z\|_\infty + \eta \|r\|_\infty < \infty. \quad (4.9)$$

By conditions (4.1) and (4.6) we get

$$\begin{aligned} \alpha &:= \max_{1 \leq i \leq d} \sup_{n \in \mathbb{Z}_+} \sum_{j=0}^n a_i(n, j) \\ &= \max_{1 \leq i \leq d} \sup_{n \in \mathbb{Z}_+} \sum_{j=0}^n |w_i(n-j)| b(j) \\ &\leq \|b\|_\infty \max_{1 \leq i \leq d} \sum_{j=0}^{\infty} |w_i(j)| \\ &= \eta \|b\|_\infty < \infty. \end{aligned} \quad (4.10)$$

Now we show that the inequalities (3.4) and (3.5) are satisfied with

$$\varphi(t) = t^p, \quad t \geq 0, \quad N = 0, \quad \mu := \|\psi\|_\tau := \max_{-\tau \leq n \leq 0} \|x(n)\| \quad (4.11)$$

and

$$v := \max \left(2(\eta \|b\|_\infty \|\psi\|_\tau^p + \|z\|_\infty + \eta \|r\|_\infty), (2\eta \|b\|_\infty)^{\frac{1}{1-p}}, \|\psi\|_\tau \right). \quad (4.12)$$

By using (4.9) and (4.10), it is clear that for $i = 1, \dots, d$

$$a_i(0, 0) \|\psi\|_\tau^p + |h_i(0)| \leq \eta \|b\|_\infty \|\psi\|_\tau^p + \|z\|_\infty + \eta \|r\|_\infty \leq v,$$

therefore (3.4) holds with (4.11) and (4.12). We have $v \geq (2\eta \|b\|_\infty)^{\frac{1}{1-p}}$, and so (4.10) and the definition of α yield

$$v^{p-1} \alpha \leq \frac{\alpha}{2\eta \|b\|_\infty} \leq \frac{1}{2}.$$

Similarly, using $v \geq 2(\eta \|b\|_\infty \|\psi\|_\tau^p + \|z\|_\infty + \eta \|r\|_\infty)$ and the inequalities (4.9) and (4.10), we obtain

$$\frac{1}{v} (\alpha \|\psi\|_\tau^p + \gamma) \leq \frac{\alpha \|\psi\|_\tau^p + \gamma}{2(\eta \|b\|_\infty \|\psi\|_\tau^p + \|z\|_\infty + \eta \|r\|_\infty)} \leq \frac{1}{2}.$$

Thus

$$v^{p-1} \alpha + \frac{1}{v} (\alpha \|\psi\|_\tau^p + \gamma) \leq 1,$$

hence for all $n \geq 1$, we have for $i = 1, \dots, d$

$$a_i(n, 0) \varphi(\|\psi\|_\tau) + \sum_{j=1}^n a_i(n, j) \varphi(v) + |h_i(n)| \leq \alpha \|\psi\|_\tau^p + \alpha v^p + \gamma \leq v,$$

consequently, (3.6) holds with (4.11) and (4.12). Then all the conditions of Theorem 1 are satisfied, therefore the solution x of the closed loop system (2.4) is bounded by v for $n \geq -\tau$, i.e.,

$$\begin{aligned} \|x(n)\| &\leq v = \max \left(2(\eta\|b\|_\infty\|\psi\|_\tau^p + \|z\|_\infty + \eta\|r\|_\infty), (2\eta\|b\|_\infty)^{\frac{1}{1-p}}, \|\psi\|_\tau \right) \\ &\leq 2\eta\|r\|_\infty + \max \left(2(\eta\|b\|_\infty\|\psi\|_\tau^p + \|z\|_\infty), (2\eta\|b\|_\infty)^{\frac{1}{1-p}}, \|\psi\|_\tau \right) \end{aligned}$$

for $n \geq -\tau$. Then

$$\|y(n)\| \leq \|C\|\|x(n)\| \leq \theta_1\|r\|_\infty + \theta_2, \quad n \in \mathbb{Z}_+,$$

where $\theta_1 := 2\eta\|C\|$ and

$$\theta_2 := \|C\| \max \left(2(\eta\|b\|_\infty\|\psi\|_\tau^p + \|z\|_\infty), (2\eta\|b\|_\infty)^{\frac{1}{1-p}}, \|\psi\|_\tau \right).$$

Hence, by Definition 1, the closed loop system (2.4) is BIBO stable. $\square$

It is easy to see that for $k = 1$ the last inequality of (4.1) gives the upper bound $\lambda_i < 1$, and as $k \rightarrow \infty$, the upper bound of λ_i in condition (4.1) tends monotonically to 0. Therefore large delay allows only small gain in the control law.

In the following theorem a sufficient condition is given for the BIBO stability in the case of a linear estimate of the function g .

Theorem 3. *Let $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a continuous function which satisfies inequality (2.2) with $\varphi(t) = t$, $t \geq 0$. The closed loop system (2.4) with $D = \text{diag}(\lambda_1, \dots, \lambda_d)$ and $k \in \mathbb{N}$ is BIBO stable if*

$$\|b\|_\infty < \frac{1}{\eta} \quad \text{and} \quad 0 < \lambda_i < 2 \cos \frac{k\pi}{2k+1}, \quad i = 1, \dots, d \quad (4.13)$$

hold, where η is defined by (4.6).

Proof. As in the proof of Theorem 2, we rewrite (2.4) in the form of (4.7), and define the functions f_i , a_i and h_i for $i = 1, \dots, d$. Then the conditions (B1)-(B5) are satisfied.

Next we show that the inequalities (3.4) and (3.5) are satisfied with

$$\varphi(t) = t, \quad N = 0, \quad \mu := \|\psi\|_\tau \quad \text{and} \quad v := \max \left(\frac{\eta\|r\|_\infty + M\|\psi\|_\tau}{1 - \eta\|b\|_\infty}, \|\psi\|_\tau \right), \quad (4.14)$$

where the positive constant M is defined in (4.4), $\|\psi\|_\tau := \sup_{-\tau \leq n \leq 0} \|x(n)\|$. Since $a_i(n, j) := w_i(n-j)b(j)$, $0 \leq j \leq n$, we have

$$\sum_{j=0}^n a_i(n, j) = \sum_{j=0}^n |w_i(n-j)b(j)|$$

$$\begin{aligned}
&\leq \max_{1 \leq i \leq d} \sup_{n \in \mathbb{Z}_+} \sum_{j=0}^n |w_i(n-j)|b(j) \\
&\leq \|b\|_\infty \max_{1 \leq i \leq d} \sum_{n=0}^{\infty} |w_i(n)| \\
&= \eta \|b\|_\infty
\end{aligned} \tag{4.15}$$

$$< 1. \tag{4.16}$$

By (4.4), (4.8), (4.13), (4.14) and (4.16), we have for $n \in \mathbb{Z}_+$, $i = 1, \dots, d$

$$\begin{aligned}
v &\geq \frac{\eta \|r\|_\infty + M \|\psi\|_\tau}{1 - \eta \|b\|_\infty} \\
&\geq \frac{\|r\|_\infty \sum_{j=0}^n |w_i(n-j)| + |z_i(n+1)|}{1 - \sum_{j=0}^n |w_i(n-j)|b(j)} \\
&\geq \frac{\sum_{j=0}^n |w_i(n-j)| |r_i(j)| + |z_i(n+1)|}{1 - \sum_{j=0}^n |w_i(n-j)|b(j)}.
\end{aligned}$$

Since $h_i(n) = \sum_{j=0}^n w_i(n-j)r(j) + z_i(n+1)$, it follows

$$v \geq \frac{|h_i(n)|}{1 - \sum_{j=0}^n |w_i(n-j)|b(j)}, \quad n \in \mathbb{Z}_+.$$

Therefore

$$v \sum_{j=0}^n |w_i(n-j)|b(j) + |h_i(n)| \leq v, \quad n \in \mathbb{Z}_+, \quad i = 1, \dots, d.$$

Hence the above inequality and $v \geq \|\psi\|_\tau$ yield for $n = 0$ and $i = 1, \dots, d$

$$a_i(0,0)\|\psi\|_\tau + |h_i(0)| \leq v|w_i(0)|b(0) + |h_i(0)| \leq v,$$

and so (3.4) is satisfied with (4.14). Similarly, for $n \in \mathbb{N}$ and $i = 1, \dots, d$

$$v|w_i(n)|b(0) + v \sum_{j=1}^n |w_i(n-j)|b(j) + |h_i(n)| \leq v.$$

Therefore

$$a_i(n, 0)\|\psi\|_\tau + v \sum_{j=1}^n a_i(n, j) + |h_i(n)| \leq v, \quad n \in \mathbb{N}, \quad i = 1, \dots, d,$$

consequently, (3.5) is satisfied with (4.14). Then all the conditions of Theorem 1 hold with (4.14), therefore the solution x of the closed loop system (2.4) is bounded by v , i.e.,

$$\|x(n)\| \leq v, \quad n \in \mathbb{Z}_+.$$

Hence

$$\begin{aligned} \|y(n)\| &\leq \|C\| \|x(n)\| \\ &\leq \|C\| v \\ &= \|C\| \max\left(\frac{\eta \|r\|_\infty + M \|\psi\|_\tau}{1 - \eta \|b\|_\infty}, \|\psi\|_\tau\right) \\ &\leq \theta_1 \|r\|_\infty + \theta_2, \end{aligned}$$

where

$$\theta_1 := \frac{\|C\| \eta}{1 - \eta \|b\|_\infty} \quad \text{and} \quad \theta_2 := \|C\| \max\left(\frac{M \|\psi\|_\tau}{1 - \eta \|b\|_\infty}, \|\psi\|_\tau\right).$$

Then, by Definition 1, the feedback control system (2.4) is BIBO stable. $\square$

Corollary 1. *Let $g : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a continuous function which satisfies inequality (2.2) with $\varphi(t) = t$, $t \geq 0$. The closed loop system (2.4) with $D = \text{diag}(\lambda_1, \dots, \lambda_d)$ and $k \in \mathbb{N}$ is BIBO stable if*

$$\|b\|_\infty < \lambda_i \leq \frac{k^k}{(k+1)^{k+1}}, \quad i = 1, \dots, d \quad (4.17)$$

hold.

Proof. Under our condition (4.17) and from Lemma 4 in [4] we get that the fundamental solution w_i of (4.2)-(4.3) is positive and

$$\sum_{j=0}^{\infty} w_i(j) = \frac{1}{\lambda_i}, \quad i = 1, \dots, d.$$

Therefore

$$\eta = \max\left(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_d}\right),$$

and hence $\eta \|b\|_\infty < 1$. The proof is similar to the proof of Theorem 3 and it is omitted. $\square$

In the next theorem it is shown that in the super-linear case there exist positive diagonal gain D and positive delay k such that the solutions of the closed loop system are bounded for small initial functions and small reference inputs, i.e., the system is locally BIBO stable.

Theorem 4. *Let $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a continuous function which satisfies inequality (2.2) with $\varphi(t) = t^p$, $p > 1$, $t \geq 0$. Then the solution x of the feedback control system (2.4) is locally BIBO stable if (4.1) holds.*

Proof. Suppose $\lambda_1, \dots, \lambda_d$ are fixed satisfying (4.1), $D = \text{diag}(\lambda_1, \dots, \lambda_d)$, and let z be the solution of the IVP (2.6)-(2.7), and η be defined by (4.6). Let $\|\psi\|_\tau \leq \delta_1$ and $\|r\|_\infty \leq \delta_2$, where δ_1, δ_2 will be specified later. From (4.5) and (4.9) we have

$$\|z\|_\infty \leq M \|\psi\|_\tau \leq M \delta_1$$

and

$$\gamma := \max_{1 \leq i \leq d} \sup_{n \in \mathbb{Z}_+} |h_i(n)| \leq \|z\|_\infty + \eta \|r\|_\infty \leq M \delta_1 + \eta \delta_2 < \infty,$$

and from (4.1) and (4.15) it follows

$$\alpha := \max_{1 \leq i \leq d} \sup_{n \in \mathbb{Z}_+} \sum_{j=0}^n a_i(n, j) \leq \eta \|b\|_\infty < \infty.$$

Since $p > 1$ and η and $\|b\|_\infty$ are positive and finite, we select the positive constants δ_1 and δ_2 so that

$$\alpha \delta_1^p + M \delta_1 + \eta \delta_2 \leq \frac{1}{2} \left(\frac{1}{2\eta \|b\|_\infty} \right)^{\frac{1}{p-1}} \quad \text{and} \quad 0 < \delta_1 \leq \left(\frac{1}{2\eta \|b\|_\infty} \right)^{\frac{1}{p-1}} \quad (4.18)$$

hold.

Next we show that the inequalities (3.4) and (3.5) are satisfied with

$$\varphi(t) = t^p, \quad N = 0, \quad \mu := \|\psi\|_\tau \quad \text{and} \quad v := \left(\frac{1}{2\eta \|b\|_\infty} \right)^{\frac{1}{p-1}}. \quad (4.19)$$

We note that the definitions of μ, δ_1 and the second part of (4.18) yield $\mu \leq v$. Using the definition of $v, p > 1$ and (4.18) we get

$$v \geq \frac{1}{2} \left(\frac{1}{2\eta \|b\|_\infty} \right)^{\frac{1}{p-1}} \geq \alpha \delta_1^p + M \delta_1 + \eta \delta_2 \geq \alpha \|\psi\|_\tau^p + \gamma \geq a_i(0, 0) \|\psi\|_\tau^p + |h_i(0)|$$

for $i = 1, \dots, d$, hence the condition (3.4) holds with (4.19).

Similarly, the definition of $v, p > 1$ and (4.18) yield

$$v - \alpha v^p \geq v - \eta \|b\|_\infty v^p = \frac{1}{2} \left(\frac{1}{2\eta \|b\|_\infty} \right)^{\frac{1}{p-1}} \geq \alpha \delta_1^p + M \delta_1 + \eta \delta_2 \geq \alpha \|\psi\|_\tau^p + \gamma.$$

Then the definitions of α and γ imply

$$a_i(n, 0) \|\psi\|_{\tau}^p + \sum_{j=1}^n a_i(n, j) v^p + |h_i(n)| \leq \alpha \|\psi\|_{\tau}^p + \alpha v^p + \gamma \leq v, \quad n \in \mathbb{N},$$

therefore the condition (3.5) holds with (4.19).

Therefore the conditions of Theorem 1 are satisfied with (4.19), so the solution of the closed loop system (2.4) is bounded by v , i.e.,

$$\|x(n)\| < v = \left(\frac{1}{p\eta\|b\|_{\infty}} \right)^{\frac{1}{p-1}}, \quad n \in \mathbb{Z}_+.$$

Hence

$$\|y(n)\| \leq \|C\| \|x(n)\| \leq \theta, \quad n \in \mathbb{Z}_+,$$

where

$$\theta := \|C\| \left(\frac{1}{p\eta\|b\|_{\infty}} \right)^{\frac{1}{p-1}}.$$

By Definition 2 the closed loop system (2.4) is locally BIBO stable. $\square$

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Authors’ addresses

Essam Awwad

Department of Mathematics, Faculty of Science, Benha University, Egypt

E-mail address: esam_mh@yahoo.com

István Győri

Department of Mathematics, University of Pannonia, Hungary

E-mail address: gyori@almos.uni-panon.hu

Ferenc Hartung

Department of Mathematics, University of Pannonia, Hungary

E-mail address: hartung.ferenc@uni-pannon.hu



A NADLER-TYPE FIXED POINT THEOREM IN DISLOCATED SPACES AND APPLICATIONS

H. AYDI, A. FELHI, ERDAL KARAPINAR, AND S. SAHMIM

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Abstract. In this paper, we introduce the concept of a Hausdorff dislocated metric. We initiate the study of fixed point theory for multi-valued mappings on dislocated metric space using the Hausdorff dislocated metric and we prove a generalization of the well known Nadler's fixed point theorem. Moreover, we provide some examples and we give an application of our main result.

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1. INTRODUCTION AND PRELIMINARIES

Let (X, d) be a metric space and $CB(X)$ denotes the collection of all nonempty closed and bounded subsets of X . For $A, B \in CB(X)$, define

$$H(A, B) := \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\},$$

where $d(x, A) := \inf\{d(x, a) : a \in A\}$ is the distance of a point x to the set A . It is known that H is a metric on $CB(X)$, called the Hausdorff metric induced by the metric d .

Definition 1. Let X be any nonempty set. An element x in X is said to be a fixed point of a multi-valued mapping $T : X \rightarrow 2^X$ if $x \in Tx$, where 2^X denotes the collection of all nonempty subsets of X .

We recall that a multi-valued mapping $T : X \rightarrow CB(X)$ is said to be a contraction if

$$H(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$ and for some k in $[0, 1)$.

The study of fixed points for multi-valued contractions using the Hausdorff metric was initiated by Nadler [18] who proved the following theorem.

Theorem 1 ([18]). Let (X, d) be a complete metric space and $T : X \rightarrow CB(X)$ be a contraction mapping. Then, there exists $x \in X$ such that $x \in Tx$.

The notion of dislocated metric space was introduced by Hitzler and Seda [12] (see also [11]). Later, Amini-Harandi [9] re-discovered the notion of dislocated metric under the name of "metric-like". In this paper, the author [9] presented some fixed point results in the class of dislocated metric spaces. Very recently, Karapinar and Salimi [19] established some fixed point theorems for cyclic contractions. For more fixed point results on dislocated metric spaces, see e.g. [1–3, 7, 8, 13, 15, 16, 20–23].

Definition 2. Let X be a nonempty set. A function $\sigma : X \times X \rightarrow [0, \infty)$ is said to be a dislocated metric (or a metric-like) on X if for any $x, y, z \in X$, the following conditions hold:

- (σ_1) $\sigma(x, y) = 0 \implies x = y$;
- (σ_2) $\sigma(x, y) = \sigma(y, x)$;
- (σ_3) $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$.

The pair (X, σ) is then called a dislocated metric (metric-like) space.

It is known that a partial metric [17] is also a dislocated metric. So, a trivial example of a dislocated metric space is the pair $([0, \infty), \sigma)$, where $\sigma : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is defined as $\sigma(x, y) = \max\{x, y\}$.

In the sequel, $\mathbb{R}_0^+$ represents the set of all nonnegative reals. In the following example, we give a dislocated metric which is neither a metric nor a partial metric.

Example 1 ([6]). Take $X = \{1, 2, 3\}$ and consider the dislocated metric $\sigma : X^2 \rightarrow \mathbb{R}_0^+$ given by

$$\begin{aligned} \sigma(1, 1) &= 0, \quad \sigma(2, 2) = 1, \quad \sigma(3, 3) = \frac{2}{3}, \\ \sigma(1, 2) &= \sigma(2, 1) = \frac{9}{10}, \quad \sigma(2, 3) = \sigma(3, 2) = \frac{4}{5}, \\ \sigma(1, 3) &= \sigma(3, 1) = \frac{7}{10}. \end{aligned}$$

Since $\sigma(2, 2) \neq 0$, σ is not a metric and since $\sigma(2, 2) > \sigma(1, 2)$, σ is not a partial metric [17].

Each dislocated metric σ on X generates a T_0 topology τ_σ on X which has as a base the family open σ -balls $\{B_\sigma(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_\sigma(x, \varepsilon) = \{y \in X : |\sigma(x, y) - \sigma(x, x)| < \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

Observe that a sequence $\{x_n\}$ in a dislocated metric space (X, σ) converges to a point $x \in X$, with respect to τ_σ , if and only if $\sigma(x, x) = \lim_{n \rightarrow \infty} \sigma(x, x_n)$.

Definition 3. Let (X, σ) be a dislocated metric space.

- (a) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n, m \rightarrow \infty} \sigma(x_n, x_m)$ exists and is finite.

- (b) (X, σ) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_σ to a point $x \in X$ such that $\lim_{n \rightarrow \infty} \sigma(x, x_n) = \sigma(x, x) = \lim_{n, m \rightarrow \infty} \sigma(x_n, x_m)$.

We need in the sequel the following trivial inequality

$$\sigma(x, x) \leq 2\sigma(x, y) \quad \text{for all } x, y \in X. \quad (1.1)$$

In this paper, we introduce a new concept called a Hausdorff dislocated metric. Using this concept, we establish a fixed point result for multi-valued mappings involving a generalized contraction. We derive many interesting corollaries on existing known results in the literature. Our obtained results are supported by some examples and an application to an integral equation.

2. HAUSDORFF DISLOCATED METRIC

Let (X, σ) be a dislocated metric space. Let $CB^\sigma(X)$ be the family of all nonempty, closed and bounded subsets in the dislocated metric space (X, σ) , induced by the dislocated metric σ . Note that the boundedness is given as follows: A is a bounded subset in (X, σ) if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_\sigma(x_0, M)$, that is,

$$|\sigma(x_0, a) - \sigma(x_0, x_0)| < M.$$

The Closedness is taken in (X, τ_σ) (where τ_σ is the topology induced by σ). Let $\bar{A}$ be the closure of A with respect to the dislocated metric σ . We have

Definition 4.

$$\begin{aligned} a \in \bar{A} &\iff B_\sigma(a, \varepsilon) \cap A \neq \emptyset \quad \text{for all } \varepsilon > 0 \\ &\iff \text{there exists } x_n \in A, \quad x_n \rightarrow a \text{ in } (X, \sigma). \end{aligned}$$

If $A \in CB^\sigma(X)$, then $\bar{A} = A$.

For $A, B \in CB^\sigma(X)$ and $x \in X$, define

$$\begin{aligned} \sigma(x, A) &= \inf\{\sigma(x, a), a \in A\}, \quad \delta_\sigma(A, B) = \sup\{\sigma(a, B) : a \in A\} \quad \text{and} \\ \delta_\sigma(B, A) &= \sup\{\sigma(b, A) : b \in B\}. \end{aligned}$$

Lemma 1. *Let (X, σ) be a dislocated metric space and A be any nonempty set in (X, σ) , then*

$$\text{if } \sigma(a, A) = 0, \quad \text{then } a \in \bar{A}. \quad (2.1)$$

Also, if $\{x_n\}$ is a sequence in (X, σ) that is τ_σ -convergent to $x \in X$, then

$$\lim_{n \rightarrow \infty} |\sigma(x_n, A) - \sigma(x, A)| = \sigma(x, x). \quad (2.2)$$

Proof. If $\sigma(a, A) = 0$, so $\inf_{x \in A} \sigma(a, x) = 0$, that is, for all $\varepsilon > 0$, there exists $x \in A$ such that $\sigma(a, x) < \varepsilon$. Hence, for all $n \geq 1$, there exists $x_n \in A$ such that

$$\sigma(a, x_n) < \frac{1}{n}.$$

Thus, $\lim_{n \rightarrow \infty} \sigma(a, x_n) = 0$. By (1.1), we have $\sigma(a, a) \leq 2\sigma(a, x_n)$, $\forall n$. Then, $\sigma(a, a) \leq 2 \lim_{n \rightarrow \infty} \sigma(a, x_n) = 0$. Finally, we obtain $\lim_{n \rightarrow \infty} \sigma(a, x_n) = \sigma(a, a) = 0$, which means that $\{x_n\}$ converges to a in (X, σ) . By Definition 4, $a \in \bar{A}$.

The equality from (2.2) follows from the inequality

$$|\sigma(x_n - A) - \sigma(x, A)| = \sigma(x_n, x).$$

□

Remark 1. It was shown in Remark 2.1 from [4] that if A is a subset of a partial metric space (X, p) and $x \in X$, then

$$x \in \bar{A} \iff p(x, A) = p(x, x).$$

We show by an example that this property is not longer true in a dislocated metric space.

Example 2. Let $X = \{0, 1\}$ and $\sigma : X \times X \rightarrow \mathbb{R}_0^+$ be defined by

$$\sigma(0, 0) = 2 \quad \text{and} \quad \sigma(x, y) = 1 \text{ if } (x, y) \neq (0, 0).$$

Then, (X, σ) is a dislocated metric space. Note that σ is not a partial metric on X because $\sigma(0, 0) \geq \sigma(1, 0)$.

We have $0 \in \bar{X} (= X)$, but $\sigma(0, X) = \min\{\sigma(0, 0), \sigma(0, 1)\} = 1 \neq \sigma(0, 0)$

Let (X, σ) be a dislocated metric space. For $A, B \in CB^\sigma(X)$, define

$$H_\sigma(A, B) = \max\{\delta_\sigma(A, B), \delta_\sigma(B, A)\}.$$

Now, we shall study some properties of $H_\sigma : CB^\sigma(X) \times CB^\sigma(X) \rightarrow [0, \infty)$.

Proposition 1. *Let (X, σ) be a dislocated metric space. For all $A, B, C \in CB^\sigma(X)$, we have the following:*

- (i) : $H_\sigma(A, A) = \delta_\sigma(A, A) = \sup\{\sigma(a, A) : a \in A\}$;
- (ii) : $H_\sigma(A, B) = H_\sigma(B, A)$;
- (iii) : $H_\sigma(A, B) = 0$ implies that $A = B$;
- (iv) : $H_\sigma(A, B) \leq H_\sigma(A, C) + H_\sigma(C, B)$.

Proof. (i) and (ii) are clear.

(iii) Suppose that $H_\sigma(A, B) = 0$. Then,

$$\sup_{a \in A} \sigma(a, B) = 0.$$

Mention that $\sup_{a \in A} \sigma(a, B) = 0$, implies $\forall a \in A, \sigma(a, B) = 0$. Then, by lemma 1, $a \in \overline{B} = B$. As a is arbitrary in A , we conclude that $A \subset B$. Similarly, $H_\sigma(B, A) = 0$ implies $B \subset A$.

(iv) Let $a \in A, b \in B$ and $c \in C$. As

$$\sigma(a, b) \leq \sigma(a, c) + \sigma(c, b),$$

so we have

$$\sigma(a, B) \leq \sigma(a, c) + \sigma(c, B) \leq \sigma(a, c) + \delta_\sigma(C, B) \leq \sigma(a, C) + \delta_\sigma(C, B),$$

since c is an arbitrary element of C . As a is an arbitrary element of A , it follows

$$\delta_\sigma(A, B) \leq \delta_\sigma(A, C) + \delta_\sigma(C, B) \leq H_\sigma(A, C) + H_\sigma(C, B).$$

Similarly, due to symmetry of H_σ , we have

$$\delta_\sigma(B, A) \leq H_\sigma(A, C) + H_\sigma(C, B).$$

Combining the two above inequalities, we get (iv). □

Remark 2. The converse of assertion (iii) from Proposition 1 is not true in general as it is clear from the following example.

Example 3. Let $X = \{0, 1\}$ be endowed with the dislocated metric $\sigma : X \times X \rightarrow [0, \infty)$ defined by

$$\sigma(1, 1) = 2 \quad \text{and} \quad \sigma(0, 0) = \sigma(0, 1) = \sigma(1, 0) = 1.$$

Note that σ is not a partial metric since $\sigma(1, 1) > \sigma(1, 0)$. From (i) of Proposition 1, we have

$$\begin{aligned} H_\sigma(X, X) &= \delta_\sigma(X, X) = \sup\{\sigma(x, X), x \in \{0, 1\}\} \\ &= \max\{\sigma(0, \{0, 1\}), \sigma(1, \{0, 1\})\} = 1 \neq 0. \end{aligned}$$

In view of Proposition 1, we call the mapping $H_\sigma : CB^\sigma(X) \times CB^\sigma(X) \rightarrow [0, +\infty)$, a Hausdorff dislocated metric induced by σ .

Remark 3. It is easy to show that any Hausdorff metric is a Hausdorff dislocated metric. The converse is not true (see Example 3).

3. FIXED POINT OF MULTI-VALUED CONTRACTION MAPPINGS

We start with the following simple useful lemma. One may find its analogous for the partial metric case in [5].

Lemma 2. Let $A, B \in CB^\sigma(X)$ and $a \in A$. Then, for all $\varepsilon > 0$, there exists a point $b \in B$ such that $\sigma(a, b) \leq H_\sigma(A, B) + \varepsilon$.

The inequality from Lemma 2 also appears in Nadler's paper [18]. Now, we state and prove our main result.

Theorem 2. *Let (X, σ) be a complete dislocated metric space. If $T : X \rightarrow CB^\sigma(X)$ is a multi-valued mapping such that for all $x, y \in X$, we have*

$$H_\sigma(Tx, Ty) \leq k M(x, y) \quad (3.1)$$

where $k \in [0, 1)$ and

$$M(x, y) = \max \left\{ \sigma(x, y), \sigma(x, Tx), \sigma(y, Ty), \frac{1}{4} (\sigma(x, Ty) + \sigma(y, Tx)) \right\}.$$

Then, T has a fixed point.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. Clearly, if $\sigma(x_0, x_1) = 0$, then $x_0 = x_1$ and x_0 is a fixed point of T . Assume $\sigma(x_0, x_1) > 0$. Since $Tx_0, Tx_1 \in CB^\sigma(X)$ and $x_1 \in Tx_0$, Lemma 2 implies the existence of a point $x_2 \in Tx_1$ such that

$$\sigma(x_2, x_1) \leq H_\sigma(Tx_1, Tx_0) + \frac{1-k}{2} M(x_1, x_0). \quad (3.2)$$

If $\sigma(x_2, x_1) = 0$, then $x_2 = x_1$ and x_1 is a fixed point of T . Assuming $\sigma(x_2, x_1) > 0$, then, by Lemma 2, there is a point $x_3 \in Tx_2$ such that

$$\sigma(x_3, x_2) \leq H_\sigma(Tx_2, Tx_1) + \frac{1-k}{2} M(x_2, x_1). \quad (3.3)$$

Continuing in this fashion, we complete a sequence $(x_n) \subset X$ such that $x_{n+1} \in Tx_n$ and $\sigma(x_n, x_{n+1}) > 0$ with

$$\sigma(x_{n+1}, x_n) \leq H_\sigma(Tx_n, Tx_{n-1}) + \frac{1-k}{2} M(x_n, x_{n-1}).$$

Then, we get

$$\begin{aligned} & \sigma(x_{n+1}, x_n) \\ & \leq k M(x_n, x_{n-1}) + \frac{1-k}{2} M(x_n, x_{n-1}) \\ & = \frac{1+k}{2} M(x_n, x_{n-1}) \\ & \leq \frac{1+k}{2} \max \{ \sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1}), \frac{1}{4} [\sigma(x_n, x_n) + \sigma(x_{n-1}, x_{n+1})] \}. \end{aligned}$$

By a triangular inequality, we get

$$\begin{aligned} \frac{1}{4} (\sigma(x_n, x_n) + \sigma(x_{n-1}, x_{n+1})) & \leq \frac{1}{4} (3\sigma(x_n, x_{n-1}) + \sigma(x_{n+1}, x_n)) \\ & \leq \max \{ \sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1}) \}. \end{aligned}$$

Then

$$\sigma(x_n, x_{n+1}) \leq \frac{1+k}{2} \max \{ \sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1}) \}.$$

Now, if $\sigma(x_n, x_{n+1}) > \sigma(x_{n-1}, x_n)$, then we have

$$\sigma(x_n, x_{n+1}) \leq \frac{1+k}{2} \sigma(x_n, x_{n+1}) < \sigma(x_n, x_{n+1}),$$

which is a contradiction. So, for all $n \geq 1$, $\sigma(x_n, x_{n+1}) \leq \sigma(x_n, x_{n-1})$. Finally, we get

$$\sigma(x_n, x_{n+1}) \leq \frac{1+k}{2} \sigma(x_{n-1}, x_n), \quad \forall n \geq 1.$$

Moreover, by induction, one finds

$$\sigma(x_n, x_{n+1}) \leq \left(\frac{1+k}{2}\right)^n \sigma(x_0, x_1), \quad \forall n \geq 1.$$

Since $k \in [0, 1)$, we have $\sum_{n \geq 0} \left(\frac{1+k}{2}\right)^n < \infty$. So, for all $p \geq 0$, we have

$$\begin{aligned} \sigma(x_n, x_{n+p}) &\leq \sigma(x_n, x_{n+1}) + \sigma(x_{n+1}, x_{n+2}) + \dots + \sigma(x_{n+p-1}, x_{n+p}) \\ &\leq \sum_{i=n}^{n+p-1} \left(\frac{1+k}{2}\right)^i \sigma(x_0, x_1) \\ &\leq \sum_{i=n}^{\infty} \left(\frac{1+k}{2}\right)^i \sigma(x_0, x_1) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \tag{3.4}$$

Thus, by symmetry of σ , we obtain

$$\lim_{n, m \rightarrow \infty} \sigma(x_n, x_m) = 0. \tag{3.5}$$

This yields that the sequence $\{x_n\}$ is Cauchy. Since (X, σ) is complete, the sequence $\{x_n\}$ converges to a point $x^* \in X$, i.e.,

$$\lim_{n \rightarrow \infty} \sigma(x_n, x^*) = \sigma(x^*, x^*) = \lim_{n, m \rightarrow \infty} \sigma(x_n, x_m) = 0. \tag{3.6}$$

We have $\sigma(x^*, Tx^*) \leq \sigma(x^*, x_{n+1}) + \sigma(x_{n+1}, Tx^*)$.

Since $x_{n+1} \in Tx_n$, it follows

$$\begin{aligned} \sigma(x^*, Tx^*) &\leq \sigma(x^*, x_{n+1}) + \delta_\sigma(Tx_n, Tx^*) \\ &\leq \sigma(x^*, x_{n+1}) + H_\sigma(Tx_n, Tx^*) \\ &\leq \sigma(x^*, x_{n+1}) + kM(x_n, x^*), \end{aligned}$$

where

$$M(x_n, x^*)$$

$$= \max \left\{ \sigma(x_n, x^*), \sigma(x_n, Tx_n), \sigma(x^*, Tx^*), \frac{1}{4} (\sigma(x_n, Tx^*) + \sigma(x^*, Tx_n)) \right\}.$$

We have

$$\begin{aligned} \sigma(x_n, Tx_n) &\leq \sigma(x_n, x_{n+1}), \\ \sigma(x^*, Tx_n) &\leq \sigma(x^*, x_{n+1}). \end{aligned}$$

When passing to limit, it should be mentioned that, by Lemma 1 and (3.6),

$$\sigma(x^*, Tx_n) \rightarrow \sigma(x^*, Tx^*).$$

Again, by taking $n \rightarrow \infty$ and using (3.6), we obtain

$$\begin{aligned} \sigma(x^*, Tx^*) &\leq k \max \left\{ \sigma(x^*, Tx^*), \frac{1}{4} \sigma(x^*, Tx^*) \right\} \\ &= k \sigma(x^*, Tx^*). \end{aligned}$$

Since, $k \in [0, 1)$, we have $\sigma(x^*, Tx^*) = 0$. Finally, by lemma 1, we have $x^* \in \overline{Tx^*} = Tx^*$. Then, x^* is a fixed point of T . $\square$

As consequences of our main result, we may state the following immediate corollaries.

Corollary 1 (Hardy-Rogers type [10]). *Let (X, σ) be a complete dislocated metric space. If $T : X \rightarrow CB^\sigma(X)$ is a multi-valued mapping such that for all $x, y \in X$, we have*

$$H_\sigma(Tx, Ty) \leq a\sigma(x, y) + b\sigma(x, Tx) + c\sigma(y, Ty) + d[\sigma(x, Ty) + \sigma(y, Tx)] \quad (3.7)$$

where $a, b, c, d \in [0, 1)$ such that $a + b + c + 4d < 1$. Then, T has a fixed point.

Corollary 2 (Kannan type [14]). *Let (X, σ) be a complete dislocated metric space. If $T : X \rightarrow CB^\sigma(X)$ is a multi-valued mapping such that for all $x, y \in X$, we have*

$$H_\sigma(Tx, Ty) \leq a\sigma(x, y) + b\sigma(x, Tx) + c\sigma(y, Ty) \quad (3.8)$$

where $a, b, c \in [0, 1)$ such that $a + b + c < 1$. Then, T has a fixed point.

Corollary 3. *Let (X, σ) be a complete dislocated metric space. If $T : X \rightarrow CB^\sigma(X)$ is a multi-valued mapping such that for all $x, y \in X$, we have*

$$H_\sigma(Tx, Ty) \leq k \sigma(x, y) \quad (3.9)$$

where $k \in [0, 1)$. Then, T has a fixed point.

Corollary 4 ([4]). *Let (X, σ) be a complete partial metric space. If $T : X \rightarrow CB^\sigma(X)$ is a multi-valued mapping such that for all $x, y \in X$, we have*

$$H_\sigma(Tx, Ty) \leq k \sigma(x, y) \quad (3.10)$$

where $k \in [0, 1)$. Then, T has a fixed point.

Corollary 5 ([18]). *Let (X, σ) be a complete metric space. If $T : X \rightarrow CB^\sigma(X)$ is a multi-valued mapping such that for all $x, y \in X$, we have*

$$H_\sigma(Tx, Ty) \leq k \sigma(x, y) \quad (3.11)$$

where $k \in [0, 1)$. Then, T has a fixed point.

Corollary 6. *Let (X, σ) be a complete dislocated metric space. If $T : X \rightarrow X$ is a single-valued mapping such that for all $x, y \in X$, we have*

$$\begin{aligned} & \sigma(Tx, Ty) \\ & \leq k \max \left\{ \sigma(x, y), \sigma(x, Tx), \sigma(y, Ty), \frac{1}{4} (\sigma(x, Ty) + \sigma(y, Tx)) \right\} \end{aligned} \quad (3.12)$$

where $k \in [0, 1)$. Then, T has a fixed point $x \in X$, that is, $Tx = x$.

4. EXAMPLES AND AN APPLICATION

First, we give the following illustrative examples where the main result of Aydi et al. [4] is not applicable..

Example 4. Let $X = \{0, 1, 2\}$ and $\sigma : X \times X \rightarrow [0, \infty)$ defined by

$$\begin{aligned} \sigma(0, 0) = \sigma(1, 1) = 0, \quad \sigma(2, 2) &= \frac{23}{48} \\ \sigma(0, 1) = \sigma(1, 0) &= \frac{1}{3}, \quad \sigma(0, 2) = \sigma(2, 0) = \frac{11}{24} \quad \text{and} \quad \sigma(1, 2) = \sigma(2, 1) = \frac{1}{2}. \end{aligned}$$

Then, (X, σ) is a complete dislocated metric space. Note that σ is not a partial metric on X because $\sigma(2, 2) \geq \sigma(2, 0)$.

Define the map $T : X \rightarrow CB^\sigma(X)$ by

$$T0 = T1 = \{0\}, \quad T2 = \{0, 1\}$$

Note that it easy that Tx is bounded and is closed for all $x \in X$ in the dislocated metric space (X, σ) .

We shall show that

$$H_\sigma(Tx, Ty) \leq \frac{8}{11} M(x, y), \quad \forall x, y \in X.$$

For this, we distinguish the following cases:

case1 : $x, y \in \{0, 1\}$. We have

$$H_\sigma(Tx, Ty) = \sigma(0, 0) = 0 \leq \frac{8}{11} \sigma(x, y) \leq \frac{8}{11} M(x, y).$$

case2 : $x \in \{0, 1\}, y = 2$. We have

$$\begin{aligned} H_\sigma(Tx, Ty) &= H_\sigma(\{0\}, \{0, 1\}) = \max\{\sigma(0, \{0, 1\}), \max\{\sigma(0, 0), \sigma(0, 1)\}\} \\ &= \max\{\min\{\sigma(0, 0), \sigma(0, 1)\}, \frac{1}{3}\} = \frac{1}{3} \end{aligned}$$

$$\leq \frac{8}{11}\sigma(x, y) \leq \frac{8}{11}M(x, y).$$

case3 : $x = y = 2$. We have

$$\begin{aligned} H_\sigma(Tx, Ty) &= H_\sigma(\{0, 1\}, \{0, 1\}) = \max\{\sigma(0, \{0, 1\}), \sigma(1, \{0, 1\})\} \\ &= \min\{\sigma(0, 1), \sigma(1, 1)\} = 0 \leq \frac{8}{11}\sigma(2, 2) \leq \frac{8}{11}M(2, 2). \end{aligned}$$

Thus, all the required hypotheses of Theorem 2 are satisfied. Then, T has a fixed point. Here, $x = 0$ is the unique fixed point of T .

Example 5. Let $X = \{0, 1, 2\}$ and $\sigma : X \times X \rightarrow [0, \infty)$ defined by

$$\sigma(0, 0) = 0, \sigma(1, 1) = 3, \sigma(2, 2) = 1$$

$$\sigma(0, 1) = \sigma(1, 0) = 7, \quad \sigma(0, 2) = \sigma(2, 0) = 3 \quad \text{and} \quad \sigma(1, 2) = \sigma(2, 1) = 4.$$

Then, (X, σ) is a complete dislocated metric space. Note that σ is not a partial metric on X because $\sigma(0, 1) \geq \sigma(2, 0) + \sigma(2, 1) - \sigma(2, 2)$.

Define the map $T : X \rightarrow CB^\sigma(X)$ by

$$T0 = T2 = \{0\} \quad \text{and} \quad T1 = \{0, 2\}.$$

Note that Tx is bounded and is closed for all $x \in X$ in the dislocated metric space (X, σ) .

We shall show that

$$H_\sigma(Tx, Ty) \leq \frac{3}{4}M(x, y), \quad \forall x, y \in X.$$

For this, we consider the following cases:

case1 : $x, y \in \{0, 2\}$. We have

$$H_\sigma(Tx, Ty) = \sigma(0, 0) = 0 \leq \frac{3}{4}M(x, y).$$

case2 : $x \in \{0, 2\}, y = 1$. We have

$$\begin{aligned} H_\sigma(Tx, Ty) &= H_\sigma(\{0\}, \{0, 2\}) = \max\{\sigma(0, \{0, 2\}), \max\{\sigma(0, 0), \sigma(0, 2)\}\} \\ &= \max\{0, 3\} = 3 \leq \frac{3}{4}\sigma(x, y) \leq \frac{3}{4}M(x, y). \end{aligned}$$

case3 : $x = y = 1$. We have

$$\begin{aligned} H_\sigma(Tx, Ty) &= H_\sigma(\{0, 2\}, \{0, 2\}) = \max\{\sigma(0, \{0, 2\}), \sigma(2, \{0, 2\})\} \\ &= \min\{\sigma(0, 2), \sigma(2, 2)\} = 1 \leq \frac{3}{4}\sigma(1, 1) \leq \frac{3}{4}M(1, 1). \end{aligned}$$

Therefore, all the required hypotheses of Theorem 2 are satisfied. Here, $x = 0$ is the unique fixed point of T .

Example 6. Let $X = [0, 1]$ and $\sigma : X \times X \rightarrow [0, \infty)$ defined by

$$\sigma(x, y) = x + y, \quad \forall x, y \in X$$

Then, (X, σ) is a complete dislocated metric space. Note that σ is not a partial metric on X because $\sigma(x, x) > \sigma(x, y)$ for all $x > y$. σ is not also a metric on X since $\sigma(1, 1) = 2$.

Define the map $T : X \rightarrow CB^\sigma(X)$ by

$$Tx = \{0, \frac{x^2}{1+x}\}, \quad \forall x \in X$$

It is easy that Tx is bounded and is closed for all $x \in X$ in the dislocated metric space (X, σ) .

We shall show that

$$H_\sigma(Tx, Ty) \leq \frac{1}{2}M(x, y), \quad \forall x, y \in X.$$

For this, we consider the following cases:

case1 : $x = y$. We have

$$\begin{aligned} H_\sigma(Tx, Ty) &= \max\{\sigma(0, Tx), \sigma(\frac{x^2}{1+x}, Tx)\} \\ &= \max\{\min\{\sigma(0, 0), \sigma(0, \frac{x^2}{1+x})\}, \min\{\sigma(0, \frac{x^2}{1+x}), \sigma(\frac{x^2}{1+x}, \frac{x^2}{1+x})\}\} \\ &= \max\{0, \frac{x^2}{1+x}\} = \frac{x^2}{1+x} \leq x = \frac{1}{2}\sigma(x, x) \leq \frac{1}{2}M(x, y). \end{aligned}$$

case2 : $x \neq y$. Since σ is symmetric, we suppose $x > y$. We have

$$\begin{aligned} &H_\sigma(Tx, Ty) \\ &= H_\sigma(\{0, \frac{x^2}{1+x}\}, \{0, \frac{y^2}{1+y}\}) \\ &= \sup\{\max\{\sigma(0, \{0, \frac{y^2}{1+y}\}), \sigma(\frac{x^2}{1+x}, \{0, \frac{y^2}{1+y}\})\}, \\ &\max\{\sigma(0, \{0, \frac{x^2}{1+x}\}), \sigma(\frac{y^2}{1+y}, \{0, \frac{x^2}{1+x}\})\}\} \\ &= \max\{\sigma(\frac{x^2}{1+x}, \{0, \frac{y^2}{1+y}\}), \sigma(\frac{y^2}{1+y}, \{0, \frac{x^2}{1+x}\})\} \\ &= \max\{\frac{x^2}{1+x}, \frac{y^2}{1+y}\} = \frac{x^2}{1+x} \leq \frac{1}{2}x \leq \frac{1}{2}(x+y) = \frac{1}{2}\sigma(x, y) \leq \frac{1}{2}M(x, y). \end{aligned}$$

Thus, all the required hypotheses of Theorem 2 are satisfied. Here, $x = 0$ is the unique fixed point of T .

Now, we provide an application on the research of a solution of an integral equation. For instance, using Corollary 6, we will prove the existence of a solution of the following integral equation.

$$x(t) = \int_a^b K(t, x(s)) ds, \quad (4.1)$$

where $K : [a, b] \times \mathbb{R} \rightarrow [0, \infty)$ is a continuous nonnegative function.

Throughout this part, let $X = C([a, b], [0, \infty))$ be the set of real nonnegative continuous functions defined on $[a, b]$. Take the dislocated metric $\sigma : X \times X \rightarrow [0, \infty)$ defined by

$$\sigma(x, y) = \|x\|_\infty + \|y\|_\infty = \max_{s \in [a, b]} x(s) + \max_{s \in [a, b]} y(s) \quad \text{for all } x, y \in X.$$

Mention that σ is not partial metric on X . But, it is easy that (X, d) is a complete dislocated metric space.

Now, take the operator $T : X \rightarrow X$ defined by

$$Tx(t) = \int_a^b K(t, x(s)) ds. \quad (4.2)$$

Mention that (4.1) has a solution if and only if the operator T has a fixed point.

The main result is

Theorem 3. *Assume that there exists $\lambda \in (0, 1)$, such that for every $s \in [a, b]$ and $u \in X$, we have*

$$K(s, u(s)) \leq \frac{\lambda}{b-a} u(s).$$

Then, T has a fixed point in X .

Proof. For all $x \in X$

$$\begin{aligned} |T(x)(t)| &\leq \int_a^b |K(t, s, x(s))| ds \\ &\leq \frac{\lambda}{b-a} \int_a^b x(s) ds \leq \lambda \|x\|_\infty. \end{aligned}$$

It follows that for all $x, y \in X$

$$\sigma(Tx, Ty) \leq \lambda \sigma(x, y) \leq \lambda M(x, y). \quad (4.3)$$

Therefore, all the hypotheses of Corollary 6 are satisfied. Consequently, T has a fixed point, that is, (4.1) has a solution $x \in X$. □

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Authors' addresses

H. Aydi

University of Dammam Department of Mathematics. College of Education of Jubail, P.O: 12020 ., 31961 Industrial Jubail, Saudi Arabia, and Department of Medical Research, China Medical University Hospital, China Medical University, Taichung , Taiwan

E-mail address: hmaydi@uod.edu.sa

A. Felhi

KF University, College of Sciences. Department of Mathematics, Al-hasa, Saudi Arabia

Current address: Université de Carthage, Institut Préparatoire de Bizerte, Jarzouna, Tunisie

E-mail address: afelhi@kfu.edu.sa

Erdal Karapınar

Atilim University, Department of Mathematics 06836, İncek Ankara, Turkey

E-mail address: erdalkarapinar@yahoo.com

S. Sahmim

KF University, College of Sciences. Department of Mathematicst, Al-hasa, Saudi Arabia

Current address: Université de Carthage, Laboratoire D'Ingénierie Mathématiques, Ecole Polytechniques de Tunisie, P.B.743-2078, La Marsa, Tunisie

E-mail address: ssahmim@kfu.edu.sa



ON STRONGLY PRIME SUBMODULES

ABDULRASOOL AZIZI

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Abstract. Let R be a commutative ring with identity and M an R -module. A proper submodule N of M is called strongly prime [resp. strongly semiprime], if $((N + Rx) : M)y \subseteq N$ [resp. $((N + Rx) : M)x \subseteq N$] for $x, y \in M$ implies that $x \in N$ or $y \in N$ [resp. $x \in N$]. Strongly prime and strongly semiprime submodules are studied, in this paper.

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1. INTRODUCTION

Throughout this paper all rings are commutative with identity and all modules are unitary. Also we consider R to be a ring and M a unitary R -module. If N is a submodule of M , then $(N : M) = \{t \in R \mid tM \subseteq N\}$.

Recall from [12] that a proper submodule N of M is said to be *strongly prime* [resp. *strongly semiprime*], if $((N + Rx) : M)y \subseteq N$ [resp. $((N + Rx) : M)x \subseteq N$] for $x, y \in M$ implies that $x \in N$ or $y \in N$ [resp. $x \in N$].

If N is a strongly prime [resp. strongly semiprime] submodule of M and $(N : M) = I$, then we say that N is an I -strongly prime [resp. I -strongly semiprime] submodule of M .

A proper submodule N of M is *prime* if the condition $ra \in N$, $r \in R$ and $a \in M$ implies that either $a \in N$ or $rM \subseteq N$. In this case, if $P = (N : M)$, we say that N is a P -prime submodule of M , and it is easy to see that P is a prime ideal of R (see, for example, [2–8, 10, 11, 13]).

In [11], another notion of strongly prime submodules was introduced. It is easy to see that for modules over commutative rings with identity, the strongly prime submodules introduced in [11] coincide with the prime submodules. However, the strongly prime submodules in this paper are special cases of prime submodules (see (1)(2)).

In [12, Proposition 1.1], it is shown that:

- (1) Every strongly prime submodule is strongly semiprime and also prime.

(2) Every maximal submodule is strongly prime.

A characterization of strongly prime submodules is given in (1), and there we prove that the converse of (1) is also correct, see also (2). The converse of (2) is studied in (3), (1), and (6).

Prime submodules have been the subject of numerous publications in the past. So the relations between prime submodules and strongly prime submodules will be helpful in connecting the previous papers on the prime notion for modules to this new notion of prime. Especially there are a few papers studying the form of the elements of the intersection of prime submodules (for example, [3, 7, 10, 13]). This is a motivation for studying the equality of the intersection of prime submodules with the intersection of strongly prime submodules in this paper (see (7) and (3)).

The Generalized Principal Ideal Theorem (GPIT) states that if R is a Noetherian ring and P is a minimal prime ideal over an ideal I generated by n elements of R , then $ht\ P \leq n$. The module version of GPIT related to prime submodules has been studied in [6] and slightly in [5]. In [12, Theorem 2.3], it is proved that the module version of GPIT related to strongly prime submodules is true for every Noetherian flat module. This shows that the behavior of height of strongly prime submodules is closer to prime ideals in comparing with prime submodules, and strongly prime submodules behave better than prime submodules for studying the dimension theory in modules.

In [12, Theorem 1.7], it is claimed that any intersection of strongly prime submodules is a strongly semiprime submodule. We show that this claim is incorrect (see (1) and (4)).

The existence of strongly prime submodules are studied in (1), (4), (3), (9). The results (2) and (5) are devoted to studying the existence of strongly semiprime submodules. In (3) and (3) we show that strongly prime submodules exist rarely in injective modules, although (9) shows that they are found frequently in projective modules. We apply the notion of strongly prime submodules and show that if there exists a non-zero module over a primary ring that is both projective and injective, then the ring has only one prime ideal (see (6)).

2. STRONGLY PRIME AND STRONGLY SEMIPRIME SUBMODULES

Let I be an ideal of R . A submodule N of M is called I -maximal, if $(N : M) = I$ and the submodule N is maximal with respect to this property, that is, if L is a submodule of M containing N with $(L : M) = I$, then $L = N$ (see [2] and [10, p. 1810]).

One can easily see that if N is I -maximal and I is a prime [resp. maximal] ideal, then N is a prime [resp. maximal] submodule (see [10, Lemma 3.2]).

The following result is a generalization of [12, Proposition 1.1]. This result also shows that strongly prime submodules are exactly I -maximal submodules, where I is a prime ideal of R .

Proposition 1. *Let N be a submodule of M . Then the following are equivalent:*

- (1) *N is a strongly prime submodule of M .*
- (2) *N is a strongly semiprime submodule and also a prime submodule of M .*
- (3) *N is a strongly semiprime submodule of M and $(N : M)$ is a prime ideal.*
- (4) *N is $(N : M)$ -maximal and $(N : M)$ is a prime ideal of R .*

Proof. ((a) $\implies$ (b)). See [12, Proposition 1.1].

((b) $\implies$ (c)). Note that for every prime submodule N of M , the ideal $(N : M)$ is prime.

((c) $\implies$ (d)). To show that N is $(N : M)$ -maximal, suppose that L is a submodule of M containing N with $(L : M) = (N : M)$. Let x be an arbitrary element of L . As $N \subseteq (N + Rx) \subseteq L$, we have $(N : M) \subseteq ((N + Rx) : M) \subseteq (L : M) = (N : M)$, that is, $((N + Rx) : M) = (N : M)$. Then $((N + Rx) : M)x = (N : M)x \subseteq N$, and since N is a strongly semiprime submodule, $x \in N$. This shows that $L = N$, which completes the proof.

((d) $\implies$ (a)). Let $P = (N : M)$. Since N is P -maximal and P is a prime ideal, by [10, Lemma 3.2], N is a prime submodule; so if $((N + Rx) : M)y \subseteq N$, for $x \in M$ and $y \in M \setminus N$, then $((N + Rx) : M) \subseteq (N : M)$ and evidently $(N : M) \subseteq ((N + Rx) : M)$. Hence $((N + Rx) : M) = (N : M) = P$ and since N is P -maximal, $x \in (N + Rx) = N$, and so $x \in N$. $\square$

Example 1. Let P be a maximal ideal of R and F a free R module with $\text{rank } F > 1$.

- (1) F contains a P -strongly prime submodule.
- (2) F contains a P -prime submodule that is not strongly prime.

Proof. Let $F = \bigoplus_{i \in \alpha} R$, and assume that $i_0 \in \alpha$.

(1) We show that $N = P \oplus (\bigoplus_{i \in \alpha, i \neq i_0} R)$ is a P -strongly prime submodule of F . Obviously $PF \subseteq \bigoplus_{i \in \alpha} P \subseteq N$, that is $P \subseteq (N : F)$, and as P is a maximal ideal of R , we have $(N : F) = P$. Now let L be a submodule of F containing N with $(L : F) = (N : F)$. Since $(L : F) = P$ is a maximal ideal, L is a P -prime submodule of F . If $N \neq L$, then let $\{x_i\}_{i \in \alpha} \in L \setminus N$, and so $x_{i_0} \notin P$. Obviously $\{(1 - \delta_{i_0 i})x_i\}_{i \in \alpha} \in N \subseteq L$, and thus $x_{i_0}\{\delta_{i_0 i}\}_{i \in \alpha} = \{x_i\}_{i \in \alpha} - \{(1 - \delta_{i_0 i})x_i\}_{i \in \alpha} \in L$, and since L is a P -prime submodule of M , we have $e_{i_0} = \{\delta_{i_0 i}\}_{i \in \alpha} \in L$. Also evidently for each $j \in \alpha$, $j \neq i_0$, we have $e_j = \{\delta_{ji}\}_{i \in \alpha} \in N \subseteq L$. Therefore $L = F$, which is a contradiction. This shows that N is a P -strongly prime submodule of F .

(2) By [4, Corollary 2.9(i)] if M is a flat module and $PM \neq M$, then PM is a P -prime submodule of M . So the submodule PF is a P -prime submodule of F and note that $PF \subset N$ and $(N : F) = P$, hence PF is not a strongly prime submodule of F . $\square$

Recall that an ideal I of R is semiprime if $I = \sqrt{I}$. Strongly semiprime submodules of a module are characterized in the following.

Proposition 2. *A submodule N of M is strongly semiprime if and only if $(N : M)$ is a semiprime ideal, and N is $(N : M)$ -maximal.*

Proof. ($\implies$). To show that $(N : M)$ is a semiprime ideal, on the contrary, let $r \in \sqrt{(N : M)} \setminus (N : M)$. Suppose that $1 < n$ is the smallest positive integer with $r^n \in (N : M)$, and let $z \in r^{n-1}M \setminus N$. Then $Rz \subseteq r^{n-1}M \subseteq rM$, and so $((N + Rz) : M) \subseteq ((N + rM) : M)$. Hence:

$$\begin{aligned} ((N + Rz) : M)z &\subseteq ((N + rM) : M)r^{n-1}M \\ &= r^{n-1}((N + rM) : M)M \subseteq r^{n-1}(N + rM) \subseteq N, \end{aligned}$$

and as N is a strongly semiprime submodule, $z \in N$, which is impossible.

To show that N is $(N : M)$ -maximal, suppose that L is a submodule of M containing N with $(L : M) = (N : M)$. Let x be an arbitrary element of L . Then $((N + Rx) : M) \subseteq ((L + Rx) : M) = (L : M) = (N : M)$. Then $((N + Rx) : M)x \subseteq (N : M)x \subseteq N$, and so $x \in N$, because N is a strongly semiprime submodule. This shows that $L = N$, and thus N is $(N : M)$ -maximal.

($\impliedby$) Suppose $((N + Rx) : M)x \subseteq N$, where $x \in M$. Let $r_1, r_2 \in ((N + Rx) : M)$. Then $r_1r_2M \subseteq r_1(N + Rx) \subseteq N + Rr_1x \subseteq N + ((N + Rx) : M)x \subseteq N$, and so $r_1r_2 \in (N : M)$. Consequently $((N + Rx) : M)^2 \subseteq (N : M)$, and so $((N + Rx) : M) \subseteq \sqrt{(N : M)} = (N : M) \subseteq ((N + Rx) : M)$, which implies that $N + Rx = N$, by our assumption. Thus $x \in N$. $\square$

The following corollary shows that if $\{N_i\}_{i \in \alpha}$ is a family of distinct P -strongly prime submodules of M with $|\alpha| > 1$, then $\cap_{i \in \alpha} N_i$ is never a strongly semiprime submodule. This corollary rejects [12, Theorem 1.7], which claims that any intersection of strongly prime submodules is a strongly semiprime submodule.

Corollary 1. *Let $\{N_i\}_{i \in \alpha}$ be a family of P -strongly prime submodules of M . Then $\cap_{i \in \alpha} N_i$ is a strongly semiprime submodule if and only if $N_i = N_j$ for all $i, j \in \alpha$.*

Proof. ($\implies$). Let $B = \cap_{i \in \alpha} N_i$. Note that $(B : M) = \cap_{i \in \alpha} (N_i : M) = P = (N_j : M)$ for each $j \in \alpha$. Since B is strongly semiprime, by (2) it is $(B : M)$ -maximal and $B \subseteq N_j$ with $(B : M) = (N_j : M)$, so $N_j = B$ for all $j \in \alpha$, which completes the proof.

($\impliedby$). Is evident. $\square$

Corollary 2. *For a ring R , the following are equivalent:*

- (1) *$\text{Spec}(R)$ is a chain.*
- (2) *Every strongly semiprime submodule of any R -module M is strongly prime.*

Proof. ((1) $\implies$ (2)). Every two prime ideals of R are comparable, so $\sqrt{(N : M)}$ is an intersection of a chain of prime ideals, thus it is a prime ideal. By (2), $(N : M) = \sqrt{(N : M)}$, thus it is a prime ideal. Now by (1), N is a strongly prime submodule.

((2) $\implies$ (1)). Let P and Q be two prime ideals of R . Then $\sqrt{P \cap Q} = \sqrt{P} \cap \sqrt{Q} = P \cap Q$, that is, $P \cap Q$ is a semiprime ideal, and so by our assumption it is a prime ideal. Now if $P \not\subseteq Q$ and $Q \not\subseteq P$, then consider $p \in P \setminus Q$ and $q \in Q \setminus P$. Thus $pq \in P \cap Q$, but $p \notin P \cap Q$, and $q \notin P \cap Q$. $\square$

Recall that a commutative ring is primary if its zero ideal is primary (see [1, Chapter 4]). The set of zero-divisors of R is $\mathcal{Z}(R) = \{r \in R \mid \exists 0 \neq s \in R, rs = 0\}$. Also

$$\mathcal{N}(R) = \{r \in R \mid \exists n \in \mathbb{N}, r^n = 0\}.$$

A characterization of primary rings is given in the following lemma.

Lemma 1. *The following are equivalent:*

- (1) $\mathcal{Z}(R) \subseteq \mathcal{N}(R)$.
- (2) $\mathcal{Z}(R) = \mathcal{N}(R)$, and it is a prime ideal.
- (3) R is a primary ring.

Proof. ((1) $\implies$ (2)). We know that $\mathcal{Z}(R)$ is a union of prime ideals, and $\mathcal{N}(R)$ is the intersection of all prime ideals. Let $\mathcal{Z}(R) = \cup_{i \in \alpha} P_i$, where each P_i is a prime ideal, and let $i_0 \in \alpha$. Then $\cup_{i \in \alpha} P_i = \mathcal{Z}(R) \subseteq \mathcal{N}(R) \subseteq P_{i_0} \subseteq \cup_{i \in \alpha} P_i$.

((2) $\implies$ (3) and ((3) $\implies$ (1)) are obvious. $\square$

Proposition 3. *Let R be a primary ring, M a divisible R -module and N a submodule of M . Then the following are equivalent:*

- (1) N is a maximal submodule.
- (2) N is a strongly prime submodule.
- (3) N is a strongly semiprime submodule.
- (4) N is a maximal submodule and $(N : M)$ is the unique prime ideal of R .

Proof. ((1) $\implies$ (2) $\implies$ (3))) see [12, Proposition 1.1].

((4) $\implies$ (1)). Is obvious.

((3) $\implies$ (4)). Let N be a strongly semiprime submodule of M and L a proper submodule of M containing N . We show that $(N : M) = (L : M)$.

On the contrary, let $r \in (L : M) \setminus (N : M)$. By (2), $(N : M)$ is a semiprime ideal, so $\mathcal{Z}(R) \subseteq \mathcal{N}(R) \subseteq \sqrt{(N : M)} = (N : M)$; and hence $r \in R \setminus \mathcal{Z}(R)$. Since M is divisible, $M = rM \subseteq L \subseteq M$, which is a contradiction. Thus $(N : M) = (L : M)$, and by (2), N is $(N : M)$ -maximal, therefore $N = L$, and so N is a maximal submodule of M . This also implies that $(N : M)$ is a maximal ideal.

If $t \in (N : M) \setminus \mathcal{Z}(R)$, then as M is divisible, $M = tM \subseteq N$, which is impossible. Hence $(N : M) \subseteq \mathcal{Z}(R) \subseteq \mathcal{N}(R) \subseteq P$ for each prime ideal P of R , and as $(N : M)$ is maximal, $(N : M) = P$, the ring R has a unique prime ideal $(N : M)$. $\square$

Note 4. The following corollary shows that in a divisible module over an integral domain (particularly in a vector space), even the intersection of two distinct maximal (strongly prime) submodules is not strongly semiprime, because it is not a maximal submodule. The following result is also a generalization of [12, Proposition 1.3].

Corollary 3. *Let R be an integral domain and N a submodule of M . Then the following are equivalent:*

- (1) M is an injective module and N is a maximal submodule.
- (2) M is a divisible module and N is a strongly prime submodule.
- (3) M is a divisible module and N is a strongly semiprime submodule.
- (4) N is a maximal submodule and R is a field.

Proof. ((1) $\implies$ (2) $\implies$ (3)) and ((4) $\implies$ (1)) are obvious.

((3) $\implies$ (4)). The proof is given by (3). $\square$

Example 2. Let $\mathbb{R}$ be the field of real numbers and consider $M = \mathbb{R}^2$. By (3), the strongly prime submodules of M are exactly the maximal submodules of M , and we know that the maximal submodules of M are exactly the lines passing through the origin. Hence although M is a Noetherian module over a Noetherian ring, it has infinitely many minimal strongly prime submodules. Note that according to [11, Theorem 4.2], a Noetherian module over a Noetherian ring has finitely many minimal prime submodules.

For an ideal I of R , a primary decomposition $I = \cap_{i=1}^n Q_i$ will be called coprime, if either $n = 1$ or Q_i 's are coprime; equivalently when $\sqrt{Q_i}$'s are coprime. When this is the case, because of the Second Uniqueness Theorem (see [1, Theorem 4.10]), I has a unique minimal primary decomposition. For example, in any semilocal ring R , for each $n \in \mathbb{N}$, the ideal $(\mathcal{J}(R))^n$ has a coprime primary decomposition. Also evidently each ideal of an Atrinian ring or a Noetherian domain of dimension one has a coprime primary decomposition. Let $S = K[x_1, x_2, x_3, \dots]$, where K is a field and $x_1, x_2, x_3, \dots$ are independent indeterminates and let $I = S(x_1 - c_1)^{n_1}(x_1 - c_2)^{n_2}(x_1 - c_3)^{n_3} \dots (x_1 - c_m)^{n_m}$, where $c_1, c_2, c_3, \dots, c_m$ are distinct element of K , and $n_1, n_2, n_3, \dots, n_m \in \mathbb{N}$. Then the ideal I of S has a coprime primary decomposition. Indeed $\cap_{i=1}^m S(x_1 - c_i)^{n_i}$ is a coprime primary decomposition of I . Thus in the non-Noetherian infinite dimensional ring $R = S/I$, the zero ideal has a coprime primary decomposition.

Theorem 1. *Let the zero ideal of R have a coprime primary decomposition. If M is a divisible R -module, then every strongly prime submodule of M is maximal.*

Proof. If the zero ideal of R is primary, then the proof is given by (3). Now suppose that $0 = \cap_{i=1}^n Q_i$ is a coprime primary decomposition of the zero ideal. Consider the natural ring homomorphism $\varphi : R \rightarrow (R/Q_1) \times (R/Q_2) \times \dots \times (R/Q_n)$, $\varphi(r) = (r + Q_1, r + Q_2, \dots, r + Q_n)$. Since $0 = \cap_{i=1}^n Q_i$ and Q_i 's are coprime, by [1, Proposition 1.10], φ is a ring isomorphism. Hence without loss of generality, we can assume that $R = R_1 \times R_2 \times \dots \times R_n$, where each R_i is a primary ring. Put $M_1 = (1, 0, \dots, 0)M$, $M_2 = (0, 1, 0, \dots, 0)M, \dots, M_n = (0, 0, \dots, 0, 1)M$. One can easily see that M_1 is an R_1 -module with the multiplication $r_1((1, 0, \dots, 0)m) = (1, 0, \dots, 0)((r_1, 0, \dots, 0)m)$ for each $r_1 \in R_1$ and $m \in M$. Similarly each M_i is an R_i -module and $M \cong M_1 \times M_2 \times \dots \times M_n$ as R -modules. So we

can assume that $M = M_1 \times M_2 \times \cdots \times M_n$ is a divisible $R_1 \times R_2 \times \cdots \times R_n$ -module, and each M_i is an R_i -module. Evidently for each $r_1 \in R_1$, we have:

- (1) $r_1 \in \mathcal{Z}(R_1)$ if and only if $(r_1, 1, 1, \dots, 1) \in \mathcal{Z}(R)$.
- (2) $r_1 M_1 = M_1$ if and only if $(r_1, 1, 1, \dots, 1)M = M$.

Therefore as M is a divisible R -module, M_1 is a divisible R_1 -module. Similarly M_i is a divisible R_i -module, for each $i > 1$.

Let T be a submodule of M . It is easy to see that:

- (3) $T = T_1 \times T_2 \times \cdots \times T_n$, where each T_i is a submodule of M_i as an R_i -module.
- (4) $(T : M) = (T_1 : M_1) \times (T_2 : M_2) \times \cdots \times (T_n : M_n)$.

Let N be a strongly prime submodule of M . Then by (3), $N = N_1 \times N_2 \times \cdots \times N_n$, where each N_i is a submodule of M_i as an R_i -module. We show that there exists a $1 \leq j \leq n$ such that N_j is a strongly prime submodule of M_j and $N_i = M_i$ for each $i \neq j$.

By [9, p. 6, Exercise 1.2], the prime ideals of R are of the form $R_1 \times R_2 \times \cdots \times R_{j-1} \times P_j \times R_{j+1} \times \cdots \times R_n$, where P_j is a prime ideal of R_j . Thus as by (4) and (1), $(N : M) = (N_1 : M_1) \times (N_2 : M_2) \times \cdots \times (N_n : M_n)$ is a prime ideal of R , there exists a $1 \leq j \leq n$ such that $(N_j : M_j)$ is a prime ideal of R_j and $(N_i : M_i) = R_i$ for each $i \neq j$. Hence $N_i = M_i$ for each $i \neq j$. Now we show that N_j is a strongly prime submodule of M_j .

Let L_j be a submodule of M_j containing N_j with $(N_j : M_j) = (L_j : M_j)$. Consider $L = M_1 \times M_2 \times \cdots \times M_{j-1} \times L_j \times M_{j+1} \times \cdots \times M_n$. By (4), $(N : M) = (L : M)$, and since N is a strongly prime submodule of M and $N \subseteq L$, by (1), we have $N = L$, which implies that $N_j = L_j$. Thus again by (1), N_j is a strongly prime submodule of M_j . Now as R_j is a primary ring, by (3), N_j is a maximal submodule of M_j . Hence by (3), $N = M_1 \times M_2 \times \cdots \times M_{j-1} \times N_j \times M_{j+1} \times \cdots \times M_n$ is a maximal submodule of M . $\square$

The set of all prime [resp. strongly prime] submodules of an R -module M is denoted by $\text{Spec}(M)$ or $\text{Spec}_R(M)$ [resp. $\text{SSpec}(M)$ or $\text{SSpec}_R(M)$]. Evidently if M is finitely generated, then $\text{SSpec}_R(M) \neq \emptyset$.

Proposition 5. *Let the zero ideal of R have a coprime primary decomposition and $\dim R > 0$. Then there does not exist any non-zero finitely generated divisible R -module. Particularly there does not exist any non-zero finitely generated injective R -module.*

Proof. Let M be a non-zero finitely generated divisible R -module. According to the proof of (1), we can assume that $R = R_1 \times R_2 \times \cdots \times R_n$, where each R_i is a primary ring, and $M = M_1 \times M_2 \times \cdots \times M_n$ is an $R_1 \times R_2 \times \cdots \times R_n$ -module, where each M_i is a non-zero finitely generated divisible R_i -module, and so $\text{SSpec}_{R_i}(M_i) \neq \emptyset$. Then (3) implies that R_i has a unique prime ideal P_i , and hence the prime ideals of R are exactly $\hat{P}_i = R_1 \times R_2 \times \cdots \times R_{i-1} \times P_i \times R_{i+1} \times \cdots \times R_n$, for $1 \leq i \leq n$.

Evidently $\widehat{P}_i \not\subseteq \widehat{P}_j$, for any $i \neq j$. This shows that $\dim R = 0$, which is a contradiction. $\square$

Let $S = R \setminus \mathcal{Z}(R)$. Then S is a multiplicatively closed subset of R , and the ring R_S is called the total quotient ring of R . Since S contains no zero divisors, the natural map $R \rightarrow R_S$ is injective, so R_S is a ring extension of R , and evidently R_S is an R -module.

Lemma 2. *Let N be a submodule of M with $(N : M) = P$.*

- (1) *N is a maximal submodule if and only if N is a strongly prime submodule and P is a maximal ideal of R .*
- (2) *If P is a maximal ideal, then there exists a P -strongly prime (maximal) submodule of M containing N .*

Proof. (1) The proof is evident by (1).

(2) Note that M/N is a vector space over the field R/P , so it has a maximal subspace, say $\mathcal{M}/N$. Then $\mathcal{M}$ is a maximal submodule of M containing N , and as $P = (N : M) \subseteq (\mathcal{M} : M)$, we have $P = (\mathcal{M} : M)$. $\square$

Corollary 4. *Let the zero ideal of R have a coprime primary decomposition, and let K be the total quotient ring of R . Then the following are equivalent:*

- (1) $\dim R = 0$.
- (2) *There exists a P -strong prime R -submodule of K , for each minimal prime ideals P of R .*
- (3) $K = R$.

Proof. Evidently K is a divisible R -module. Let $0 = \cap_{i=1}^n Q_i$ be a coprime primary decomposition of the zero ideal and $\sqrt{Q_i} = P_i$. Then the minimal prime ideals of R are exactly $P_1, P_2, \dots, P_n$, also $\mathcal{Z}(R) = \cup_{i=1}^n P_i$.

((a) $\implies$ (b)). As $\dim R = 0$, the ideals $P_1, P_2, \dots, P_n$ are all of the maximal ideals of R . Obviously each P_i is a proper submodule of K as an R -module. We show that $(P_i : K) = P_i$, and so by (2)(2), there exists a P_i -strongly prime R -submodule of K , which completes the proof.

Let $a/s \in K$, where $a \in R$ and $s \in R \setminus \mathcal{Z}(R)$. As $s \notin \mathcal{Z}(R) = \cup_{i=1}^n P_i$, it is a unit element of R , and hence for each $p_i \in P_i$, we have $p_i(a/s) = (s^{-1}p_i a)/1 \in P_i$. Consequently $P_i K \subseteq P_i$, that is $P_i \subseteq (P_i : K)$, and since P_i is a maximal ideal, $P_i = (P_i : K)$.

((b) $\implies$ (c)). By our assumption there exists a P_i -strongly prime submodule N_i of K as an R -module, and (1) implies that N_i is a maximal submodule of K . Hence for each $1 \leq i \leq n$, the ideal P_i is a maximal ideal R , and so the ideals $P_1, P_2, \dots, P_n$, are all of the maximal ideals of R . Now let $b/t \in K$, where $b \in R$ and $t \in R \setminus \mathcal{Z}(R)$. As $t \notin \mathcal{Z}(R) = \cup_{i=1}^n P_i$, it is a unit element of R , and hence $b/t = (t^{-1}b)/1 \in R$. Consequently $K = R$.

((c) $\implies$ (a)). Since $K = R$, the R -module K is a non-zero finitely generated divisible module, and so by (5), $\dim R = 0$. $\square$

Recall from [13] that an R -module M is said to be *special*, if for each maximal ideal $\mathfrak{M}$ of R , every $a \in \mathfrak{M}$ and any $m \in M$, there exist $c \in R \setminus \mathfrak{M}$ and $k \in \mathbb{N}$ such that $ca^k m = 0$. *Semi-simple modules* (direct sum of simple modules), *locally Artinian modules* (modules in which every cyclic submodule is Artinian) and *semi-Artinian modules* (modules of which every homomorphic image has a nonzero simple submodule) are special (see [13, Section 3]).

Proposition 6.

- (1) *Every strongly prime submodule of a special module is a maximal submodule.*
- (2) *$\dim R = 0$ if and only if every strongly prime submodule of any R -module is maximal.*

Proof. (1) Let N be a strongly prime submodule of M and $\mathfrak{M}$ a maximal ideal of R containing $(N : M)$. Consider $m \in M \setminus N$, and $a \in \mathfrak{M}$. Since M is special, there exist $c \in R \setminus \mathfrak{M}$ and $k \in \mathbb{N}$ such that $ca^k m = 0$. Then $ca^k m \in N$ and as N is a prime submodule and $c \notin (N : M)$ and $m \notin N$, we have $a \in (N : M)$, and so $(N : M) = \mathfrak{M}$. This shows that $(N : M)$ is a maximal ideal of R , and so by (2)(1), N is a maximal submodule of M .

(2) Let $\dim R = 0$. By [13, Theorem 3.5], every module over a zero dimensional ring is special, so the proof is given by part (1).

For the 'if part' it is enough to consider R as an R -module. $\square$

Example 3.

- (1) Consider the $\mathbb{Z}$ -module $M = \mathbb{Z}/n\mathbb{Z}$, where n is a positive integer. Evidently each semiprime ideal of $\mathbb{Z}$ is an ideal generated by a square free integer. It is easy to see that strongly semiprime submodules of M are of the form $k\mathbb{Z}/n\mathbb{Z}$, where k is a square free integer with $k|n$.
- (2) More generally let R be a Dedekind domain and consider the cyclic R -module $M = R/I$, where I is a proper ideal of R . If the prime factorization of I is $P_1^{\alpha_1} P_2^{\alpha_2} \dots P_m^{\alpha_m}$, then the strongly semiprime submodules of M are $P_1^{\beta_1} P_2^{\beta_2} \dots P_m^{\beta_m} / I$, where each β_i is either 0 or 1, and at least for one $1 \leq i \leq m$, $\beta_i = 1$.
- (3) Each strongly prime submodule of the $\mathbb{Z}$ -module $M = \mathbb{Z}/n\mathbb{Z}$ is of the form $p\mathbb{Z}/n\mathbb{Z}$, where p is a prime factor of n , so they are exactly the maximal submodules of M .

The following result guarantees the existence of an I -strongly semiprime submodule in a finitely generated module M , for any semiprime ideal I containing $\text{Ann } M$.

Theorem 2. *Let B be a submodule of a finitely generated R -module M . If $(B : M) \subseteq I$, where I is a proper semiprime ideal of R , then there exists an I -strongly semiprime submodule of M containing B .*

Proof. Let $\mathcal{T} = \{C \leq M \mid B \subseteq C, (C : M) \subseteq I\}$. By Zorn's Lemma $\mathcal{T}$ has a maximal element, say N . We prove that N is an I -strongly semiprime submodule of M .

According to [9, Ex. 2.2., p. 13], if M is finitely generated and I an ideal of R , then $\sqrt{(IM : M)} = \sqrt{\text{Ann}(M) + I}$. Hence:

$$\begin{aligned} ((N + IM) : M) &= (I \frac{M}{N} : \frac{M}{N}) \subseteq \sqrt{(I \frac{M}{N} : \frac{M}{N})} = \sqrt{\text{Ann}(\frac{M}{N}) + I} \\ &= \sqrt{(N : M) + I} = I. \end{aligned}$$

So $N + IM \in \mathcal{T}$, and since N is a maximal element of $\mathcal{T}$, we have $N = N + IM$. Hence $I \subseteq ((N + IM) : M) = (N : M) \subseteq I$, and thus $(N : M) = I$.

Now suppose that L is a submodule of M containing N with $(L : M) = (N : M)$. Then as N is a maximal element of $\mathcal{T}$, evidently $L = N$. Hence by (2), N is a strongly semiprime submodule of M . $\square$

Corollary 5. *Let B be a submodule of a finitely generated R -module M . Then there exists a $\sqrt{(B : M)}$ -strongly semiprime submodule of M containing B .*

The following result is a generalization of [5, Lemma 4] or [10, Theorem 3.3].

Lemma 3. *Let B be a submodule of M and P a prime ideal of R . Then B is contained in a P -strongly prime submodule, if one of the following holds:*

- (1) M is a finitely generated R -module and $(B : M) \subseteq P$.
- (2) M is a special R -module and B is a P -prime submodule of M .

Proof. (1) By (2), there exists a P -strongly semiprime submodule N of M containing B , and since $(N : M) = P$ is a prime ideal, (1) implies that N is a P -strongly prime submodule of M .

(2) Follow the proof of (6)(1) to see that if B is a P -prime submodule of a special module, then P is a maximal ideal. Now by (2)(2), B is contained in a P -strongly prime and containing B . $\square$

Lemma 4. [2, Theorem 2.8] *If M is a finitely generated module, then every P -prime submodule of M is an intersection of P -maximal submodules of M .*

Let B be a proper submodule of an R -module M . The intersection of all prime submodules of M containing B is denoted by $\text{rad}(B)$ or $\text{rad}_R(B)$. If M has no prime submodules containing B , then we consider $\text{rad}(B) = M$. Also according to [12], the intersection of all strongly prime submodules of M containing B is denoted by $\text{s-rad}(B)$ or $\text{s-rad}_R(B)$. If M has no strongly prime submodules containing B , then we say $\text{s-rad}(B) = M$.

The following result shows that $\text{s-rad}(B)$ coincides with $\text{rad}(B)$, for finitely generated or special modules.

Proposition 7. $s\text{-rad}(B) = \text{rad}(B)$, for every submodule B of M , if M is a finitely generated or a special module.

Proof. By (3), $\text{rad}(B) = M$ if and only if $s\text{-rad}(B) = M$. Now suppose that $\text{rad}(B) \neq M$. As each strongly prime submodule is a prime submodule, $\text{rad}(B) \subseteq s\text{-rad}(B)$.

First suppose that M is a finitely generated module. For proving the converse inclusion, note that by (4), every P -prime submodule of M is an intersection of P -maximal submodules of M and by (1), the P -maximal submodules of M are exactly the P -strongly prime submodules of M . Hence every P -prime submodule of M is an intersection of strongly prime submodules. This shows that $s\text{-rad}(B) \subseteq \text{rad}(B)$, when M is finitely generated.

Now assume that M is a special module. From the proof of (6), for any P -prime submodule N of M , the ideal P is maximal.

It is well-known that in a vector space the zero subspace is an intersection of maximal subspaces, so for the vector space M/N over the field R/P , we have $N/N = \bigcap_{i \in \alpha} \bar{M}_i$, where each $\bar{M}_i$ is a maximal subspace of M/N . Then for each $i \in \alpha$, $\bar{M}_i = N_i/N$, where N_i is a maximal submodule of M containing N . Hence $N = \bigcap_{i \in \alpha} N_i$, that is, every prime submodule of M is an intersection of strongly prime submodules, and this completes the proof. $\square$

Example 4. By (3), the $\mathbb{Z}$ -module $\mathbb{Q}$ of all rational numbers has no strongly prime submodules, although the zero submodule is a prime submodule. Hence $s\text{-rad}(0) = \mathbb{Q}$ and $\text{rad}(0) = 0$, that is, $s\text{-rad}(0) \neq \text{rad}(0)$. This example and also the following result show that (7) does not hold in general.

Theorem 3. For a ring R , the following are equivalent:

- (1) $\dim R = 0$.
- (2) $s\text{-rad}(B) = \text{rad}(B)$, for every R -module M and any $B \leq M$.
- (3) $s\text{-rad}(0) = \text{rad}(0)$, for every R -module M .

Proof. ((i) $\implies$ (ii)) is given by (7), and ((ii) $\implies$ (iii)) is obvious.

((iii) $\implies$ (i)). Let P be a prime ideal of R . We show that P is a maximal ideal of R . First we prove that $s\text{-rad}_{R/P}(0) = \text{rad}_{R/P}(0)$, for every R/P -module M' .

Evidently, we can consider M' as an R -module, by the natural pull back homomorphism $R \rightarrow R/P$, that is, we define $rx = (r + P)x$, for each $r \in R$ and $x \in M'$. Let N' be a prime submodule of M' as an R/P -module. It is easy to check that:

- (1) (*) N'/N' is a prime submodule of M'/N' as an R -module.
- (2) (**) If for $N' \subseteq N \leq M'$, N/N' is a strongly prime submodule of M'/N' as an R -module, then N is a strongly prime submodule of M' as an R/P -module.

By our assumption and (*), N'/N' is an intersection of strongly semiprime submodules of M'/N' as an R -module. Now by (**), N' is an intersection of strongly

semiprime submodules of M' as an R/P -module. Consequently $s\text{-rad}_{R/P}(0) = \text{rad}_{R/P}(0)$, for every R/P -module module.

Now suppose that K is the quotient field of R/P . It is easy to see that for the R/P -module K , we have $\text{Spec}_{R/P}(K) = \{0\}$, and so $s\text{-rad}_{R/P}(0) = \text{rad}_{R/P}(0) = 0$ for the R/P -module K . Thus by (4), R/P is a field. $\square$

Let S be a multiplicatively closed subset of R . For any $N \leq M_S$, we consider $N^c = \{x \in M \mid x/1 \in N\}$.

Lemma 5. [8, Proposition 1] *Let M be an R -module and S a multiplicatively closed subset of R .*

- (1) *If N is a P -prime submodule of M with $P \cap S = \emptyset$, then N_S is a P_S -prime submodule of M_S as an R_S -module and $(N_S)^c = N$.*
- (2) *If T is a Q -prime submodule of M_S as an R_S -module, then T^c is a Q^c -prime submodule of M , $(T^c)_S = T$ and $Q^c \cap S = \emptyset$.*

The following result is a generalization of [12, Theorem 1.5 and Corollary 1.6].

Proposition 8. *Let M be an R -module, S a multiplicatively closed subset of R , and N a P -strongly prime submodule of M with $P \cap S = \emptyset$.*

- (1) *N_S is a P_S -strongly prime submodule of M_S as an R_S -module and $(N_S)^c = N$.*
- (2) *If P is a maximal element of $\{I \mid I \text{ is an ideal of } R \text{ with } I \cap S = \emptyset\}$, then N_S is a maximal submodule of M_S .*
- (3) *If M is a finitely generated module, then N_S is a maximal submodule of M_S if and only if P is a maximal element of the set $\{I \mid I \text{ is an ideal of } R \text{ with } I \cap S = \emptyset\}$.*
- (4) *Let T be a Q -strongly prime submodule of M_S as an R_S -module, where M is finitely generated or a special module, or Q^c is a maximal ideal of R . Then T^c is a Q^c -strongly prime submodule of M .*

Proof. (i) As N is a strongly prime submodule, it is a prime submodule, and so according to (5)(i), $N_S \neq M_S$ and $(N_S)^c = N$. Then N_S is a strongly prime submodule of M_S , by [12, Theorem 1.5].

(ii) Let $\mathfrak{M}$ be a maximal ideal of R_S containing P_S . Then $P = (P_S)^c \subseteq \mathfrak{M}^c$ and $\mathfrak{M}^c \cap S = \emptyset$. Now since P is a maximal element of the set $\{I \mid I \text{ is an ideal of } R \text{ with } I \cap S = \emptyset\}$, we have $P = \mathfrak{M}^c$, and so $P_S = (\mathfrak{M}^c)_S = \mathfrak{M}$. Thus P_S is a maximal ideal of R_S , and by part (i), N_S is a P_S -strongly prime submodule, so by (2)(1), N_S is a maximal submodule of M_S .

(iii) Let N_S be a maximal submodule of M_S , and suppose

$$\mathcal{A} = \{I \supseteq P \mid I \text{ is an ideal of } R \text{ with } I \cap S = \emptyset\}.$$

By Zorn's lemma $\mathcal{A}$ has a maximal element, say $\mathfrak{M}'$. By (3), N is contained in an $\mathfrak{M}'$ -strongly prime submodule N' of M . Part (i) implies that N'_S is a strongly prime

submodule of M_S containing N_S , and since N_S is a maximal submodule, $N_S = N'_S$, and so $N = (N_S)^c = (N'_S)^c = N'$. Thus $P = (N : M) = (N' : M) = \mathfrak{M}'$, which implies that P is a maximal element of $\{I \mid I \text{ is an ideal of } R \text{ with } I \cap S = \emptyset\}$.

(iv) Since T is a Q -prime submodule, (5),(ii), implies that T^c is a Q^c -prime submodule of M .

If M is finitely generated or a special module, then according to (3), T^c is contained in a Q^c -strongly prime submodule L of M . Then $T = (T^c)_S \subseteq L_S$ and by (5)(i), $(L_S : M_S) = Q$. According to (1), T is Q -maximal, hence $T = L_S$ and by (5)(i), $T^c = (L_S)^c = L$. Then T^c is a Q^c -strongly prime submodule of M .

Now let Q^c be a maximal ideal of R . If T^c is contained in a submodule U of M with $(T^c : M) = (U : M)$, then as $Q^c = (U : M)$ is a maximal ideal, U is a prime submodule and (5)(i) implies that $(U_S : M_S) = Q$. Evidently $T = (T^c)_S \subseteq U_S$ and $Q = (T : M_S) = (U_S : M_S)$, and as T is a strongly prime submodule, we have $T = U_S$, which implies that $T^c = U$. This shows that T^c is a Q^c -strongly prime submodule of M . $\square$

The following example shows that the contraction of a strongly prime submodule is not necessarily strongly prime, in general.

Example 5. Let P be the zero ideal of $\mathbb{Z}$. Then $\mathbb{Z}_P = \mathbb{Q}$ and $\mathbb{Q}_P = \mathbb{Q}$. So $\text{SSpec}_{\mathbb{Z}_P}(\mathbb{Q}_P) = \{0\}$, although $0^c = 0 \notin \text{SSpec}_{\mathbb{Z}}(\mathbb{Q})$, because $\text{SSpec}_{\mathbb{Z}}(\mathbb{Q}) = \emptyset$, by (3).

From (3) we can conclude that injective modules over a primary ring with more than one prime ideal or particularly over an integral domain which is not field, have no strongly prime submodules. The following result shows the opposite of this for projective modules, the dual of injective modules.

Proposition 9. *Let M be a non-zero a projective module, and P a prime ideal of R . Then M has a strongly prime submodule containing PM if and only if $PM \neq M$.*

Proof. Let $PM \neq M$. By [4, Corollary 2.9(i)] if M is a flat module and $PM \neq M$, then PM is a P -prime submodule of M . Now by (5)(i), $(PM)_P$ is a P_P -prime submodule of M_P . Particularly this shows that $M_P \neq 0$.

Now suppose that $\mathfrak{M}$ is a maximal ideal of R containing P . By $(M_{\mathfrak{M}})_{P_{\mathfrak{M}}} \cong M_P$ and $M_P \neq 0$, we have $M_{\mathfrak{M}} \neq 0$. We show that $M_{\mathfrak{M}}$ has a maximal submodule. Since $M_{\mathfrak{M}}$ is a non-zero projective module over the local ring $R_{\mathfrak{M}}$, it is a non-zero free $R_{\mathfrak{M}}$ -module. Hence we can assume that $M_{\mathfrak{M}}$ is a direct sum of copies of $R_{\mathfrak{M}}$.

If $M_{\mathfrak{M}} = R_{\mathfrak{M}}$, then obviously $M_{\mathfrak{M}}$ has a maximal submodule (ideal in this case). Now let $M_{\mathfrak{M}} = \bigoplus_{i \in \alpha} R_{\mathfrak{M}}$, where $|\alpha| > 1$, and assume that $i_0 \in \alpha$. Consider $T = \mathfrak{M}_{\mathfrak{M}} \oplus (\bigoplus_{i \in \alpha, i \neq i_0} R_{\mathfrak{M}})$. Obviously $\mathfrak{M}_{\mathfrak{M}} M_{\mathfrak{M}} \subseteq \bigoplus_{i \in \alpha} \mathfrak{M}_{\mathfrak{M}} \subseteq T$, that is $\mathfrak{M}_{\mathfrak{M}} \subseteq (T : M_{\mathfrak{M}})$, and as $\mathfrak{M}_{\mathfrak{M}}$ is a maximal ideal of $R_{\mathfrak{M}}$, we have $(T : M_{\mathfrak{M}}) = \mathfrak{M}_{\mathfrak{M}}$. Thus by (2)(2), T is contained in an $\mathfrak{M}_{\mathfrak{M}}$ -strongly prime submodule $\mathcal{M}$ of $M_{\mathfrak{M}}$. Therefore as $(\mathfrak{M}_{\mathfrak{M}})^c = \mathfrak{M}$ is a maximal ideal of R , by (8)(iv), $\mathcal{M}^c$ is an $\mathfrak{M}$ -strongly prime submodule of M containing PM .

The 'only if' part is obvious. $\square$

Recall that a ring R is semi simple, if and only if all R -modules are projective, if and only if all R -modules are injective (see [14, Proposition 4.5]).

Corollary 6. *Let the zero ideal of R have a coprime primary decomposition. If there exists a non-zero R -module that is both projective and injective, then R has finitely many prime ideals and $\dim R = 0$.*

Proof. Let M be a non-zero R -module that is both projective and injective. By the proof of (1), we can assume that $R = R_1 \times R_2 \times \cdots \times R_n$, where each R_i is a primary ring, and $M = M_1 \times M_2 \times \cdots \times M_n$, where each M_i is a non-zero divisible R_i -module. We show that M_1 is a projective R_1 -module.

Let $f_1 : A_1 \rightarrow B_1$ be a surjective R_1 -module homomorphism, and let $g_1 : M_1 \rightarrow B_1$ be an R_1 -module homomorphism. For each $2 \leq i \leq n$, consider $A_i = B_i = 0$, and let $A = A_1 \times A_2 \times \cdots \times A_n$ to $B = B_1 \times B_2 \times \cdots \times B_n$. Define $f : A \rightarrow B$, $f(a_1, a_2, a_3, \dots, a_n) = (f_1(a_1), 0, 0, \dots, 0)$ and $g : M \rightarrow B$, $g(a_1, a_2, \dots, a_n) = (g_1(a_1), 0, 0, \dots, 0)$. Obviously f is a surjective R -module homomorphism and g is an R -module homomorphism.

As M is a projective R -module, there exists an R -module homomorphism $h : M \rightarrow A$ with $f \circ h = g$. Consider $\pi_1 : A \rightarrow A_1$, $\pi_1(a_1, a_2, \dots, a_n) = a_1$, and $\ell_1 : M_1 \rightarrow M$, $\ell_1(a_1) = (a_1, 0, \dots, 0)$, and let $h_1 : M_1 \rightarrow A_1$, $h_1 = \pi_1 \circ h \circ \ell_1$. One can easily see that h_1 is an R_1 -module homomorphism with $f_1 \circ h_1 = g_1$. This shows that M_1 is a projective R_1 -modules. Similarly M_i is a projective R_i -modules, for each $2 \leq i \leq n$.

As $M_i \neq 0$ for each $1 \leq i \leq n$, we have $M_{P_i} \neq 0$, for some prime ideal P_i of R_i , and by the proof of (9), $\text{SSpec}_{R_i}(M_i) \neq \emptyset$. Now as each M_i is a divisible R_i -module, by the proof of (5), $\dim R = 0$ and each R_i has a unique prime ideal, and therefore R has finitely many prime ideals. $\square$

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Author's address

Abdulrasool Azizi

Shiraz University, Department of Mathematics, College of Sciences, Shiraz 71457-44776, Iran

E-mail address: aazizi@shirazu.ac.ir a-azizi@yahoo.com



GENERALIZED SUPPLEMENTED LATTICES

ÇİĞDEM BIÇER, CELİL NEBIYEV, AND ALI PANCAR

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Abstract. In this work, we define (amply) generalized supplemented lattices and investigate some properties of these lattices. In this paper, all lattices are complete modular lattices with the smallest element 0 and the greatest element 1. Let L be a lattice, $1 = a_1 \vee a_2 \vee \dots \vee a_n$ and the quotient sublattices $a_1/0, a_2/0, \dots, a_n/0$ be generalized supplemented, then L is generalized supplemented. If L is an amply generalized supplemented lattice, then for every $a \in L$, the quotient sublattice $1/a$ is amply generalized supplemented.

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1. INTRODUCTION

Throughout this paper, all lattices are complete modular lattices with the smallest element 0 and the greatest element 1. Let L be a lattice, $a, b \in L$ and $a \leq b$. A sublattice $\{x \in L \mid a \leq x \leq b\}$ is called a *quotient sublattice*, denoted by b/a . An element a' of a lattice L is called a *complement* of a if $a \wedge a' = 0$ and $a \vee a' = 1$. A lattice L is said to be *complemented* if each element in L has at least one complement. An element c of L is said to be *compact* if for every subset X of L such that $c \leq \vee X$ there is a finite $F \subseteq X$ such that $c \leq \vee F$. A lattice L is said to be *compactly generated* if each of its elements is a join of compact elements. An element a of L is said to be *small* or *superfluous* if $a \vee b \neq 1$ holds for every $b \neq 1$ and denoted by $a \ll L$. The meet of all the maximal elements ($\neq 1$) of a lattice L is called the *radical* of L and denoted by $r(L)$. An element c of L is called a *supplement* of b in L if it is minimal for $b \vee c = 1$. a is a supplement of b in lattice L if and only if $a \vee b = 1$ and $a \wedge b \ll a/0$. L is called a *supplemented lattice* if every element of L has a supplement in L . We say that an element b of L *lies above* an element a of L if $a \leq b$ and $b \ll 1/a$. L is said to be *hollow* if every element ($\neq 1$) is superfluous in L , and L is said to be *local* if L has a greatest element ($\neq 1$). An element a of L is called a *weak supplement* of b in L if $a \vee b = 1$ and $a \wedge b \ll L$. L is called a *weakly supplemented lattice*, if every element of L has a weak supplement in L . An element $a \in L$ has *ample supplements* in L if for every $b \in L$ with $a \vee b = 1$, a has a supplement b' in L with $b' \leq b$. L is called an *amply supplemented lattice*, if every

element of L has ample supplements in L . It is clear that every supplemented lattice is weakly supplemented and every amply supplemented lattice is supplemented.

More information about (amply) supplemented lattices are in [1, 2] and [3]. More results about (amply) supplemented modules are in [4, 6] and [7]. The definitions of (amply) generalized supplemented modules and some properties of them are in [5]. We generalize some properties of (amply) generalized supplemented modules.

In this paper, we constitute relationships between generalized supplemented quotient sublattices and generalized supplemented lattices by Corollary 1, Theorem 3 and Corollary 2. We also constitute relationships between amply generalized supplemented quotient sublattices and amply generalized supplemented lattices by Proposition 3 and Proposition 5.

2. GENERALIZED SUPPLEMENTED LATTICES

In this part, generalized supplement elements and generalized supplemented lattices are defined and some properties of them are given.

Definition 1. Let L be a lattice and $a \in L$. An element $b \in L$ is called a *generalized supplement* or *Rad-supplement* of a in L if $a \vee b = 1$ and $a \wedge b \leq r(b/0)$. In this fact, a has a generalized supplement (or Rad-supplement) in L . L is said to be *generalized supplemented* or *Rad-supplemented* if every element of L has a generalized supplement or Rad-supplement in L .

It is clear that every supplement element is generalized supplement in a lattice, also supplemented lattices are generalized supplemented but a generalized supplemented lattice need not to be supplemented. We give an example about this in the next part.

Lemma 1 ([3], Theorem 1.5). *Let L be a lattice and $a, b \in L$. Then the quotient sublattices $(a \vee b)/b$ and $a/(a \wedge b)$ are isomorphic.*

Lemma 2. *Let L be a lattice and c be a generalized supplement of b in L .*

(a) *If $a \leq b$ and $a \vee c = 1$ then c is a generalized supplement of a in L .*

(b) $r(c/0) = c \wedge r(L)$.

Proof. (a) Since c is a generalized supplement of b in L , $b \vee c = 1$ and $b \wedge c \leq r(c/0)$. Since $a \leq b$, $a \wedge c \leq b \wedge c \leq r(c/0)$. Thus c is a generalized supplement of a in L .

(b) $r(c/0) \leq c \wedge r(L)$ is clear. Let x be a maximal element ($\neq c$) in $c/0$. Since $\frac{1}{b \vee x} = \frac{c \vee b}{b \vee x} = \frac{c \vee b \vee x}{b \vee x} \cong \frac{c}{c \wedge (b \vee x)} = \frac{c}{(c \wedge b) \vee x} = \frac{c}{x}$, $b \vee x$ is a maximal element ($\neq 1$) of L . Hence $r(L) \leq b \vee x$ and $c \wedge r(L) \leq c \wedge (b \vee x) = (c \wedge b) \vee x = x$. Since $c \wedge r(L) \leq x$ for every maximal element ($\neq c$) in $c/0$, $c \wedge r(L) \leq r(c/0)$ and $r(c/0) = c \wedge r(L)$. $\square$

Theorem 1. *Let L be a lattice and b be a maximal element ($\neq 1$) in L . If c is a generalized supplement of b in L then $b \wedge c = r(c/0)$ and $r(c/0)$ is the unique maximal element ($\neq c$) of $c/0$.*

Proof. Since b is maximal ($\neq 1$) in L , by Lemma 1, $b \wedge c$ is a maximal element ($\neq c$) in $c/0$. Because of definition of the radical, $r(c/0) \leq b \wedge c$, and c being generalized supplement of b in L , $b \wedge c \leq r(c/0)$. Hence $b \wedge c = r(c/0)$ is the unique maximal element in $c/0$. $\square$

Theorem 2. *Let L be a lattice and $1 \neq b \in L$. If c is a generalized supplement of b in L and $r(L) \ll L$, then there exists a maximal element m ($\neq 1$) in L such that $b \leq m$.*

Proof. If $b \leq r(L)$, then it is clear. Let $b \not\leq r(L)$. Then $r(c/0) = c \wedge r(L) \neq c$ by Lemma 2 (b). Hence there exists an element $x \leq c$ such that x is a maximal element ($\neq c$) in $c/0$. Let $m = b \vee x$. By Lemma 1 and modularity $\frac{1}{m} = \frac{1}{b \vee x} = \frac{b \vee c}{b \vee x} \cong \frac{c}{c \wedge (b \vee x)} = \frac{c}{(c \wedge b) \vee x} = \frac{c}{x}$ and so m is a maximal element ($\neq 1$) in L such that $b \leq m$. $\square$

Lemma 3 ([3], Lemma 7.8 (i)). *Let L be a lattice. Then $a \vee r(L) \leq r(1/a)$ for every $a \in L$.*

Lemma 4 ([3], Exercise 7.3). *Let L be a lattice and $a \in L$. Then $r(a/0) \leq r(L)$.*

Lemma 5. *Let L be a lattice and c be a generalized supplement of b in L . Then for $a \leq b$, $a \vee c$ is a generalized supplement of b in $1/a$.*

Proof. Since c is a generalized supplement of b in L , $b \vee c = 1$ and $b \wedge c \leq r(c/0)$. Then $b \vee (a \vee c) = 1$ and by Lemma 3 and Lemma 4, $(a \vee c) \wedge b = a \vee (b \wedge c) \leq a \vee r(c/0) \leq a \vee r((a \vee c)/0) \leq r((a \vee c)/a)$. Hence $a \vee c$ is a generalized supplement of b in $1/a$. $\square$

Corollary 1. *Let L be a lattice. If L is generalized supplemented and $a \in L$, then $1/a$ is also generalized supplemented.*

Proof. Clear from Lemma 5. $\square$

Lemma 6 ([3], Lemma 12.3). *Let L be a lattice. Then for every $a, b, c \in L$, $[(b \vee c) \wedge a] \leq [b \wedge (a \vee c)] \vee [c \wedge (a \vee b)]$.*

Lemma 7. *Let L be a lattice and $a, b, c \in L$. If $a/0$ is generalized supplemented and $a \vee b$ has a generalized supplement in L , then b has a generalized supplement in L .*

Proof. Let c be a generalized supplement of $a \vee b$ in L and d be a generalized supplement of $(b \vee c) \wedge a$ in $a/0$. Clearly, $1 = a \vee b \vee c = b \vee c \vee d$, $(b \vee c) \wedge d \leq r(d/0)$, $b \wedge (c \vee d) \leq [c \wedge (b \vee d)] \vee [d \wedge (b \vee c)] \leq r(c/0) \vee r(d/0) \leq r((c \vee d)/0)$. Hence $c \vee d$ is a generalized supplement of b in L . $\square$

Theorem 3. *Let L be a lattice $a, b \in L$ and $a \vee b = 1$. If $a/0$ and $b/0$ are generalized supplemented, then L is generalized supplemented.*

Proof. Let c be arbitrary in L . $a \vee (b \vee c) = 1$ has a generalized supplement 0 in L . Since $a/0$ is generalized supplemented, by Lemma 7, $b \vee c$ has a generalized supplement in L . And since $b/0$ is generalized supplemented, c has a generalized supplement in L . So, L is generalized supplemented. $\square$

Corollary 2. *If $1 = a_1 \vee a_2 \vee \dots \vee a_n$ and the quotient sublattices $a_1/0, a_2/0, \dots, a_n/0$ be generalized supplemented, then L is generalized supplemented.*

If $a < b$ and $a \leq c < b$ implies that $c = a$, then a is said to be *covered* by b . If 0 is covered by a for some element a of L , then a is called an *atom*. A lattice L is said to be *semitatomic* if 1 is a join of atoms in L . [3]

Lemma 8 ([3], Theorem 6.7). *Every modular compactly generated complemented lattice is semiatomic.*

Proposition 1. *Let L be a modular compactly generated lattice. If L is generalized supplemented, then the quotient sublattice $1/r(L)$ is semiatomic.*

Proof. Clearly, $1/r(L)$ is a modular compactly generated lattice. Let $a \in 1/r(L)$. Since L is generalized supplemented, a has a generalized supplement b in L . Then we have $a \vee (b \vee r(L)) = 1$ and $a \wedge (b \vee r(L)) = r(L)$. Hence $1/r(L)$ is complemented and by Lemma 8, $1/r(L)$ is semiatomic. $\square$

Proposition 2. *Let L be a lattice. If L is generalized supplemented and $r(L) \ll L$, then L is weakly supplemented.*

Proof. Clear. $\square$

3. AMPLY GENERALIZED SUPPLEMENTED LATTICES

In this section, we define amply generalized supplemented or amply Rad-supplemented lattices and we give some properties of them.

Definition 2. Let L be a lattice. We say an element a of L has *ample generalized supplements* in L if for every b of L with $a \vee b = 1$, a has a generalized supplement b' in L with $b' \leq b$. L is called *amply generalized supplemented* or *amply Rad-supplemented* if every element of L has ample generalized supplements in L .

It is clear that every amply generalized supplemented lattice is generalized supplemented. But the converse is not always true (see Example 1). Every amply supplemented modular lattice is amply generalized supplemented.

Proposition 3. *Let L be a lattice. If L is amply generalized supplemented, then the quotient sublattice $1/a$ is also amply generalized supplemented for every element a of L .*

Proof. Let $a \in L$ and $x, y \in 1/a$ with $x \vee y = 1$. Since L is amply generalized supplemented, there exists an element y' of L such that $y' \leq y, x \vee y' = 1$ and $x \wedge$

$y' \leq r(y'/0)$. Then $x \wedge (a \vee y') = a \vee (x \wedge y') \leq a \vee r(y'/0) \leq r((a \vee y')/a)$. Hence $a \vee y'$ is a generalized supplement of x in $1/a$ with $a \vee y' \leq y$. $\square$

Proposition 4. *Let L be a lattice, $a, b \in L$ and $a \vee b = 1$. If a and b have ample generalized supplements in L , then $a \wedge b$ has ample generalized supplements in L .*

Proof. Let $(a \wedge b) \vee c = 1$ with $c \in L$. We clearly see that $b \vee (a \wedge c) = 1$ and $a \vee (b \wedge c) = 1$. Since $a \vee (b \wedge c) = 1$ and a has ample generalized supplements in L , then there exists an element x of L such that $x \leq b \wedge c$, $a \vee x = 1$ and $a \wedge x \leq r(x/0)$. Since $b \vee (a \wedge c) = 1$ and b has ample generalized supplements in L , then there exists an element y of L such that $y \leq a \wedge c$, $b \vee y = 1$ and $b \wedge y \leq r(y/0)$. Since $x \leq b$ and $a \vee x = 1$, $b = b \wedge 1 = b \wedge (a \vee x) = (a \wedge b) \vee x$. Similarly, we see that $a = (a \wedge b) \vee y$. Then $1 = a \vee b = [(a \wedge b) \vee y] \vee [(a \wedge b) \vee x] = (a \wedge b) \vee x \vee y$. Since $a \wedge x \leq r(x/0)$ and $b \wedge y \leq r(y/0)$, then $(a \wedge b) \wedge (x \vee y) = (a \wedge x) \vee (b \wedge y) \leq r(x/0) \vee r(y/0) \leq r((x \vee y)/0)$. Hence $x \vee y$ is a generalized supplement of $a \wedge b$ in L with $x \vee y \leq c$. $\square$

Proposition 5. *Let L be a lattice and a be a supplement element in L . If L is amply generalized supplemented, then the quotient sublattice $a/0$ is amply generalized supplemented.*

Proof. Clear. $\square$

Theorem 4. *Let L be a lattice. If L is amply generalized supplemented, then for every $a \in L$, there exist $x, y \in L$ such that x is a generalized supplement in L , $a = x \vee y$ and $y \leq r(L)$.*

Proof. Let b be a generalized supplement of a in L and x be a generalized supplement of b in L with $x \leq a$. Since b is a generalized supplement of a in L , $a \vee b = 1$ and $a \wedge b \leq r(b/0)$. Since x is a generalized supplement of b in L , $x \vee b = 1$ and $x \wedge b \leq r(x/0)$. Let $y = a \wedge b$. This case $y = a \wedge b \leq r(b/0) \leq r(L)$. Since $x \vee b = 1$ and $x \leq a$, $a = a \wedge 1 = a \wedge (x \vee b) = x \vee (a \wedge b) = x \vee y$. $\square$

Lemma 9. *Let L be a lattice and $a, b \in L$ such that $a \leq b$. Then the following assertions are equivalent.*

(a) b lies above a in L .

(b) $a \vee x = 1$ for every element $x \in L$ with $b \vee x = 1$.

Proof. (a) $\Rightarrow$ (b) Let b lies above a in L . Then $b \ll 1/a$. Let $b \vee x = 1$ with $x \in L$. Then $b \vee a \vee x = 1$, and since $a \vee x \in 1/a$ and $b \ll 1/a$, then $a \vee x = 1$.

(b) $\Rightarrow$ (a) Let $b \vee x = 1$ with $x \in 1/a$. By hypothesis, $a \vee x = 1$ and since $x \leq a$, $x = a \vee x = 1$. Thus b lies above a . $\square$

Lemma 10. *Let L be a lattice. If every element of L lies above an element x in L such that $x/0$ is generalized supplemented, then L is amply generalized supplemented.*

Proof. Let $a \vee b = 1$. By hypothesis, a lies above an element x in L such that $x/0$ is generalized supplemented. Since $a \ll 1/x$, by Lemma 9, $b \vee x = 1$. Let y be a generalized supplement of $b \wedge x$ in $x/0$. This case $1 = b \vee x = b \vee y$ and $b \wedge y \leq r(y/0)$. Hence y is a generalized supplement of b in L with $y \leq a$. $\square$

Corollary 3. *Let L be a lattice. If $a/0$ is generalized supplemented for every $a \in L$, then L is amply generalized supplemented.*

Proof. Clear from Lemma 10, because every element a of L lies above a in L . $\square$

We can prove this Corollary directly as follows:

Let $a, b \in L$ with $a \vee b = 1$. Since $a \wedge b \in b/0$, by assumption, there is a generalized supplement c of $a \wedge b$ in $b/0$. That is, $c \in b/0$, $(a \wedge b) \vee c = b$ and $a \wedge c = a \wedge b \wedge c \leq r(c/0)$. Since $a \vee b = 1$ and $(a \wedge b) \vee c = b$, $1 = a \vee b = a \vee (a \wedge b) \vee c = a \vee c$. Hence c is a generalized supplement of a in L with $c \leq b$. Thus L is amply generalized supplemented.

Example 1. Let Ω be family of all the submodules of $\mathbb{Z}$ -module $\mathbb{Q}$. Ω is a lattice with $\subseteq$. This case, for $K, L \in \Omega$, $K \vee L = \sup\{K, L\} = K + L$, $K \wedge L = \inf\{K, L\} = K \cap L$. Ω is generalized supplemented but not supplemented [7]. Since $\mathbb{Z}$ is a Dedekind domain, $\mathbb{Q}$ is a quotient field of $\mathbb{Z}$ and $\mathbb{Z}$ is not local, Ω is not amply generalized supplemented.

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*Authors' addresses***Çiğdem Biçer**

Ondokuz Mayıs University, Department of Mathematics, Kurupelit, Atakum Samsun, Turkey

E-mail address: cigdem.bicer184@hotmail.com

Celil Nebiyev

Ondokuz Mayıs University, Department of Mathematics, Kurupelit, Atakum Samsun, Turkey

E-mail address: cnebiyev@omu.edu.tr

Ali Pancar

Ondokuz Mayıs University, Department of Mathematics, Kurupelit, Atakum Samsun, Turkey

E-mail address: apancar@omu.edu.tr



COEFFICIENT ESTIMATES FOR A CLASS OF ANALYTIC BI-UNIVALENT FUNCTIONS RELATED TO PSEUDO-STARLIKE FUNCTIONS

SERAP BULUT

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Abstract. In this paper we introduce and investigate an interesting subclass $\mathcal{LB}_{\Sigma}^{h,p}(\lambda)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. For functions belonging to the class $\mathcal{LB}_{\Sigma}^{h,p}(\lambda)$, we obtain estimates on the first two Taylor-Maclaurin coefficients a_2 and a_3 . The results presented in this paper would generalize and improve some recent work of Joshi et al. [5].

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1.1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem [4] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Thus every function $f \in \mathcal{A}$ has an inverse f^{-1} , which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$, in the sense that f^{-1} has a univalent analytic continuation to $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief history and interesting examples of functions in the class Σ , see [6] (see also [2]). In fact, the aforecited work of Srivastava *et al.* [6] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Xu *et al.* [7, 8].

Recently, Babalola [1] defined the class $\mathcal{L}_\lambda(\beta)$ of λ -pseudo-starlike functions of order β as follows:

Suppose $0 \leq \beta < 1$ and $\lambda \geq 1$ is real. A function $f \in \mathcal{A}$ given by (1.1) belongs to the class $\mathcal{L}_\lambda(\beta)$ of λ -pseudo-starlike functions of order β in the unit disk $\mathbb{U}$ if and only if

$$\Re \left(\frac{z (f'(z))^\lambda}{f(z)} \right) > \beta \quad (z \in \mathbb{U}).$$

Babalola [1] proved that all pseudo-starlike functions are Bazilevič of type $1 - 1/\lambda$, order $\beta^{1/\lambda}$ and univalent in $\mathbb{U}$.

Motivated by this definition, Joshi *et al.* [5] introduced the following two subclasses of the bi-univalent function class Σ and obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients a_2 and a_3 of functions in each of these subclasses.

Definition 1 ([5]). A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{LB}_\Sigma^\lambda(\alpha)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \left| \arg \left(\frac{z (f'(z))^\lambda}{f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U}) \quad (1.3)$$

and

$$\left| \arg \left(\frac{w (g'(w))^\lambda}{g(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (w \in \mathbb{U}), \quad (1.4)$$

where $0 < \alpha \leq 1$, $\lambda \geq 1$ and the function $g = f^{-1}$ is given by (1.2).

We call $\mathcal{LB}_\Sigma^\lambda(\alpha)$ the class of strongly λ -bi-pseudo-starlike functions of order α . Also for $\lambda = 1$, we get $\mathcal{LB}_\Sigma^1(\alpha) = \mathcal{S}_\Sigma^*[\alpha]$ the class of strongly bi-starlike functions of order α , introduced and studied by Brannan and Taha [2].

Theorem 1 ([5]). Let $f(z)$ given by (1.1) be in the class $\mathcal{LB}_\Sigma^\lambda(\alpha)$ ($0 < \alpha \leq 1$, $\lambda \geq 1$). Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{(2\lambda - 1)(2\lambda - 1 + \alpha)}} \quad (1.5)$$

and

$$|a_3| \leq \frac{4\alpha^2}{(2\lambda-1)^2} + \frac{2\alpha}{3\lambda-1}. \quad (1.6)$$

Definition 2 ([5]). A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{LB}_\Sigma(\lambda, \beta)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \Re \left(\frac{z(f'(z))^\lambda}{f(z)} \right) > \beta \quad (z \in \mathbb{U}) \quad (1.7)$$

and

$$\Re \left(\frac{w(g'(w))^\lambda}{g(w)} \right) > \beta \quad (w \in \mathbb{U}), \quad (1.8)$$

where $0 \leq \beta < 1$, $\lambda \geq 1$ and the function $g = f^{-1}$ is defined by (1.2).

We call $\mathcal{LB}_\Sigma(\lambda, \beta)$ the class of λ -bi-pseudo-starlike functions of order β . Also for $\lambda = 1$, we get $\mathcal{LB}_\Sigma(1, \beta) = \mathcal{S}_\Sigma^*(\beta)$ the class of bi-starlike functions of order β , introduced and studied by Brannan and Taha [2].

Theorem 2 ([5]). Let $f(z)$ given by (1.1) be in the class $\mathcal{LB}_\Sigma(\lambda, \beta)$ ($0 \leq \beta < 1$, $\lambda \geq 1$). Then

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{\lambda(2\lambda-1)}} \quad (1.9)$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(2\lambda-1)^2} + \frac{2(1-\beta)}{3\lambda-1}. \quad (1.10)$$

Here, in our present paper, inspiring by some of the aforecited works (especially [5]), we introduce the following subclass of the analytic function class $\mathcal{A}$, analogously to the definition given by Xu et al. [7].

Definition 3. Let the functions $h, p : \mathbb{U} \rightarrow \mathbb{C}$ be so constrained that

$$\min \{ \Re(h(z)), \Re(p(z)) \} > 0 \quad (z \in \mathbb{U}) \quad \text{and} \quad h(0) = p(0) = 1.$$

Also let the function f defined by (1.1) be in the analytic function class $\mathcal{A}$. We say that $f \in \mathcal{LB}_\Sigma^{h,p}(\lambda)$ ($\lambda \geq 1$) if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \frac{z(f'(z))^\lambda}{f(z)} \in h(\mathbb{U}) \quad (z \in \mathbb{U}) \quad (1.11)$$

and

$$\frac{w(g'(w))^\lambda}{g(w)} \in p(\mathbb{U}) \quad (w \in \mathbb{U}), \quad (1.12)$$

where the function $g = f^{-1}$ is defined by (1.2).

Setting $\lambda = 1$ in Definition 3, we get the class $\mathcal{LB}_{\Sigma}^{h,p}(1) = \mathcal{B}_{\Sigma}^{h,p}$ introduced and studied by Bulut [3].

Remark 1. There are many choices of the functions h and p which would provide interesting subclasses of the analytic function class $\mathcal{A}$. For example, if we let

$$h(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad \text{and} \quad p(z) = \left(\frac{1-z}{1+z}\right)^{\alpha} \quad (0 < \alpha \leq 1, z \in \mathbb{U})$$

or

$$h(z) = \frac{1+(1-2\beta)z}{1-z} \quad \text{and} \quad p(z) = \frac{1-(1-2\beta)z}{1+z} \quad (0 \leq \beta < 1, z \in \mathbb{U}),$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 3. If $f \in \mathcal{LB}_{\Sigma}^{h,p}(\lambda)$, then

$$f \in \Sigma \quad \text{and} \quad \left| \arg \left(\frac{z(f'(z))^{\lambda}}{f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, \lambda \geq 1, z \in \mathbb{U})$$

and

$$\left| \arg \left(\frac{w(g'(w))^{\lambda}}{g(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, \lambda \geq 1, w \in \mathbb{U})$$

or

$$f \in \Sigma \quad \text{and} \quad \Re \left(\frac{z(f'(z))^{\lambda}}{f(z)} \right) > \beta \quad (0 \leq \beta < 1, \lambda \geq 1, z \in \mathbb{U})$$

and

$$\Re \left(\frac{w(g'(w))^{\lambda}}{g(w)} \right) > \beta \quad (0 \leq \beta < 1, \lambda \geq 1, w \in \mathbb{U}),$$

where the function $g = f^{-1}$ is defined by (1.2). This means that

$$f \in \mathcal{LB}_{\Sigma}^{\lambda}(\alpha) \quad (0 < \alpha \leq 1, \lambda \geq 1)$$

or

$$f \in \mathcal{LB}_{\Sigma}(\lambda, \beta) \quad (0 \leq \beta < 1, \lambda \geq 1).$$

Motivated and stimulated especially by the work of Joshi *et al.* [5], we propose to investigate the bi-univalent function class $\mathcal{LB}_{\Sigma}^{h,p}(\lambda)$ introduced in Definition 3 here and derive coefficient estimates on the first two Taylor-Maclaurin coefficients a_2 and a_3 for a function $f \in \mathcal{LB}_{\Sigma}^{h,p}(\lambda)$ given by (1.1). Our results for the bi-univalent function class $\mathcal{LB}_{\Sigma}^{h,p}(\lambda)$ would generalize and improve the related work of Joshi *et al.* [5].

2. A SET OF GENERAL COEFFICIENT ESTIMATES

In this section we state and prove our general results involving the bi-univalent function class $\mathcal{LB}_{\Sigma}^{h,p}(\lambda)$ given by Definition 3.

Theorem 3. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class $\mathcal{LB}_{\Sigma}^{h,p}(\lambda)$. Then*

$$|a_2| \leq \min \left\{ \sqrt{\frac{|h'(0)|^2 + |p'(0)|^2}{2(2\lambda - 1)^2}}, \sqrt{\frac{|h''(0)| + |p''(0)|}{4\lambda(2\lambda - 1)}} \right\} \quad (2.1)$$

and

$$|a_3| \leq \min \{\gamma, \delta\}, \quad (2.2)$$

where

$$\gamma = \frac{|h'(0)|^2 + |p'(0)|^2}{2(2\lambda - 1)^2} + \frac{|h''(0)| + |p''(0)|}{4(3\lambda - 1)}$$

and

$$\delta = \frac{\begin{cases} (2\lambda^2 + 2\lambda - 1)|h''(0)| - (2\lambda^2 - 4\lambda + 1)|p''(0)| & , \quad 1 \leq \lambda < 1 + \frac{\sqrt{2}}{2} \\ (2\lambda^2 + 2\lambda - 1)|h''(0)| + (2\lambda^2 - 4\lambda + 1)|p''(0)| & , \quad \lambda \geq 1 + \frac{\sqrt{2}}{2} \end{cases}}{4\lambda(2\lambda - 1)(3\lambda - 1)}.$$

Proof. First of all, we write the argument inequalities in (1.11) and (1.12) in their equivalent forms as follows:

$$\frac{z(f'(z))^\lambda}{f(z)} = h(z) \quad (z \in \mathbb{U}),$$

and

$$\frac{w(g'(w))^\lambda}{g(w)} = p(w) \quad (w \in \mathbb{U}),$$

respectively, where $h(z)$ and $p(w)$ satisfy the conditions of Definition 3. Furthermore, the functions $h(z)$ and $p(w)$ have the following Taylor-Maclaurin series expansions:

$$h(z) = 1 + h_1 z + h_2 z^2 + \dots$$

and

$$p(w) = 1 + p_1 w + p_2 w^2 + \dots,$$

respectively. Now, upon equating the coefficients of $\frac{z(f'(z))^\lambda}{f(z)}$ with those of $h(z)$ and the coefficients of $\frac{w(g'(w))^\lambda}{g(w)}$ with those of $p(w)$, we get

$$(2\lambda - 1)a_2 = h_1, \quad (2.3)$$

$$(2\lambda^2 - 4\lambda + 1)a_2^2 + (3\lambda - 1)a_3 = h_2, \quad (2.4)$$

$$-(2\lambda - 1)a_2 = p_1 \quad (2.5)$$

and

$$(2\lambda^2 + 2\lambda - 1)a_2^2 - (3\lambda - 1)a_3 = p_2. \quad (2.6)$$

From (2.3) and (2.5), we obtain

$$h_1 = -p_1 \quad (2.7)$$

and

$$2(2\lambda - 1)^2 a_2^2 = h_1^2 + p_1^2. \quad (2.8)$$

Also, from (2.4) and (2.6), we find that

$$2\lambda(2\lambda - 1)a_2^2 = h_2 + p_2. \quad (2.9)$$

Therefore, we find from the equations (2.8) and (2.9) that

$$|a_2|^2 \leq \frac{|h'(0)|^2 + |p'(0)|^2}{2(2\lambda - 1)^2}$$

and

$$|a_2|^2 \leq \frac{|h''(0)| + |p''(0)|}{4\lambda(2\lambda - 1)},$$

respectively. So we get the desired estimate on the coefficient $|a_2|$ as asserted in (2.1).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (2.6) from (2.4). We thus get

$$2(3\lambda - 1)a_3 - 2(3\lambda - 1)a_2^2 = h_2 - p_2. \quad (2.10)$$

Upon substituting the value of a_2^2 from (2.8) into (2.10), it follows that

$$a_3 = \frac{h_1^2 + p_1^2}{2(2\lambda - 1)^2} + \frac{h_2 - p_2}{2(3\lambda - 1)}.$$

We thus find that

$$|a_3| \leq \frac{|h'(0)|^2 + |p'(0)|^2}{2(2\lambda - 1)^2} + \frac{|h''(0)| + |p''(0)|}{4(3\lambda - 1)}.$$

On the other hand, upon substituting the value of a_2^2 from (2.9) into (2.10), it follows that

$$a_3 = \frac{(2\lambda^2 + 2\lambda - 1)h_2 + (-2\lambda^2 + 4\lambda - 1)p_2}{2\lambda(2\lambda - 1)(3\lambda - 1)}.$$

We thus obtain

$$|a_3| \leq \frac{\begin{cases} (2\lambda^2 + 2\lambda - 1)|h''(0)| - (2\lambda^2 - 4\lambda + 1)|p''(0)| & , 1 \leq \lambda < 1 + \frac{\sqrt{2}}{2} \\ (2\lambda^2 + 2\lambda - 1)|h''(0)| + (2\lambda^2 - 4\lambda + 1)|p''(0)| & , \lambda \geq 1 + \frac{\sqrt{2}}{2} \end{cases}}{4\lambda(2\lambda - 1)(3\lambda - 1)}.$$

This evidently completes the proof of Theorem 3. $\square$

3. COROLLARIES AND CONSEQUENCES

Setting $\lambda = 1$ in Theorem 3, we get following consequence.

Corollary 1 ([3]). *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class $\mathcal{B}_{\Sigma}^{h,p}$. Then*

$$|a_2| \leq \min \left\{ \sqrt{\frac{|h'(0)|^2 + |p'(0)|^2}{2}}, \sqrt{\frac{|h''(0)| + |p''(0)|}{4}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{|h'(0)|^2 + |p'(0)|^2}{2} + \frac{|h''(0)| + |p''(0)|}{8}, \frac{3|h''(0)| + |p''(0)|}{8} \right\}.$$

If we set

$$h(z) = \left(\frac{1+z}{1-z} \right)^{\alpha} \quad \text{and} \quad p(z) = \left(\frac{1-z}{1+z} \right)^{\alpha} \quad (0 < \alpha \leq 1, z \in \mathbb{U})$$

in Theorem 3, we can readily deduce Corollary 2.

Corollary 2. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class $\mathcal{LB}_{\Sigma}^{\lambda}(\alpha)$ ($0 < \alpha \leq 1, \lambda \geq 1$). Then*

$$|a_2| \leq \sqrt{\frac{2\alpha^2}{\lambda(2\lambda-1)}}$$

and

$$|a_3| \leq \begin{cases} \frac{2\alpha^2}{\lambda(2\lambda-1)} & , \quad 1 \leq \lambda < 1 + \frac{\sqrt{2}}{2} \\ \frac{2\alpha^2}{3\lambda-1} & \lambda \geq 1 + \frac{\sqrt{2}}{2} \end{cases}.$$

Remark 2. Corollary 2 is an improvement of Theorem 1.

If we set

$$h(z) = \frac{1 + (1-2\beta)z}{1-z} \quad \text{and} \quad p(z) = \frac{1 - (1-2\beta)z}{1+z} \quad (0 \leq \beta < 1, z \in \mathbb{U})$$

in Theorem 3, we can readily deduce Corollary 3.

Corollary 3. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class $\mathcal{LB}_{\Sigma}(\lambda, \beta)$ ($0 \leq \beta < 1, \lambda \geq 1$). Then*

$$|a_2| \leq \begin{cases} \sqrt{\frac{1-\beta}{\lambda(2\lambda-1)}} & 0 \leq \beta \leq \frac{2\lambda+1}{4\lambda} \\ \frac{2(1-\beta)}{(2\lambda-1)} & \frac{2\lambda+1}{4\lambda} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq \begin{cases} \min \left\{ \frac{4(1-\beta)^2}{(2\lambda-1)^2} + \frac{1-\beta}{3\lambda-1}, \frac{1-\beta}{\lambda(2\lambda-1)} \right\} & , \quad 1 \leq \lambda < 1 + \frac{\sqrt{2}}{2} \\ \frac{1-\beta}{3\lambda-1} & \lambda \geq 1 + \frac{\sqrt{2}}{2} \end{cases}.$$

Remark 3. Corollary 3 is an improvement of Theorem 2.

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Author’s address

Serap Bulut

Kocaeli University, Faculty of Aviation and Space Sciences, Arslanbey Campus, 41285 Kartepe-Kocaeli, Turkey

E-mail address: serap.bulut@kocaeli.edu.tr



A NOTE ON MERENTES–RIESZ BOUNDED VARIATION SPACES

RENÉ ERLÍN CASTILLO, HUMBERTO RAFEIRO, AND EDUARD TROUSSELOT

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Abstract. In this paper we introduce a function space with some generalization of bounded variation and study some of its properties, like embeddings, decompositions and others.

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1. INTRODUCTION

Around two centuries ago C. Jordan introduced the notion of a function of bounded variation and established the relation between these functions and monotonic ones when he was studying convergence of Fourier series. Later on the concept of bounded variation was generalized in various directions by many mathematicians, such as, L. Ambrosio, R. Caccioppoli, L. Cesari, E. Conway, G. Dal Maso, E. de Giorgi, S. Hudjaev, J. Musielak, O. Oleinik, W. Orlicz, F. Riesz, J. Smoller, L. Tonelli, A. Vol’pert, N. Wiener, among many others. It is noteworthy to mention that many of these generalizations were motivated by problems in such areas as calculus of variations, convergence of Fourier series, geometric measure theory, mathematical physics, etc. For many applications of functions of bounded variation in mathematical physics see the monograph [6]. For a thorough exposition regarding bounded variation spaces and related topics, see the recent monograph [1].

In 1992 N. Merentes [4] generalized the concept of bounded p -variation in the sense of Riesz defining the notion of bounded $(p, 2)$ -variation. We say that a function $f : [a, b] \rightarrow \mathbb{R}$ has bounded $(p, 2)$ -variation in $[a, b]$ if the number

$$\begin{aligned} V_{p,2}^R(f) &= V_{p,2}^R(f, [a, b]) \\ &= \sup_{\Pi} \sum_{j=1}^{n-1} \left| \frac{f(b_j) - f(d_j)}{b_j - d_j} - \frac{f(c_j) - f(a_j)}{c_j - a_j} \right|^p \frac{1}{(b_j - a_j)^{p-1}} \end{aligned}$$

is finite, where $p \geq 1$ and the supremum is taken on the set of all block partitions of $[a, b]$.

The set of all bounded $(p, 2)$ -variation functions is denoted by $RV_{(p,2)}([a, b])$ which has an algebra structure. Moreover, it was shown that all functions that have bounded $(p, 2)$ -variation also has bounded second-variation. In this work we generalize this concept to obtain the bounded $(p, 2, \alpha)$ -variation functions in the sense of Riesz and obtain some characterizations of this new space.

2. PRELIMINARIES

Before introducing the bounded variation space we will need some auxiliary results.

Definition 1. A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be an α -Lipschitz function if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leq M |\alpha(x) - \alpha(y)|$$

for all $x, y \in (a, b)$ with $x \neq y$. We define the space $\alpha\text{-Lip}[a, b]$ as the space of all α -Lipschitz functions. This space is normable, via the norm

$$\|f\|_{\alpha\text{-Lip}} := |f(a)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|\alpha(x) - \alpha(y)|}.$$

We now introduce the concepts of α absolutely continuous function and α -derivative.

Definition 2. A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be *absolutely continuous with respect to α* if, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that if $\{(a_j, b_j)\}_{j=1}^n$ are disjoint open subintervals of $[a, b]$, then

$$\sum_{j=1}^n |\alpha(b_j) - \alpha(a_j)| < \delta \quad \text{implies} \quad \sum_{j=1}^n |f(b_j) - f(a_j)| < \varepsilon.$$

All functions in $\alpha\text{-AC}[a, b]$ are bounded and form an algebra of functions under pointwise defined standard operations.

Definition 3. Suppose f and α are real-valued functions defined on the same open interval I and let $x_0 \in I$. We say that f is α -differentiable at x_0 if the following limit exists

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{\alpha(x) - \alpha(x_0)}.$$

If the limit exists we denote its value by $f'_\alpha(x_0)$, which we call the α -derivative of f at x_0 .

3. $\text{RV}_{(p,2,\alpha)}[a,b]$ IS A NORMED SPACE

We want to recall the so-called *Popoviciu variation* (introduced in 1933 by T. Popoviciu in [5]) for a partition $\Pi = \{a = x_1 < x_2 < \dots < x_m = b\}$ and a function $f : [a, b] \rightarrow \mathbb{R}$ is given by

$$\text{Var}_{k,1}(f, \Pi, [a, b]) = \sum_{j=1}^{m-k+1} |f[x_j, \dots, x_{j+k-1}] - f[x_{j-1}, \dots, x_{j+k-2}]|$$

where $f[\cdot, \dots, \cdot]$ is defined recursively in the following way:

$$\begin{aligned} f[x_0] &:= f(x_0), \\ f[x_0, x_1] &:= \frac{f[x_1] - f[x_0]}{x_1 - x_0} \\ f[x_0, x_1, x_2] &:= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \\ &\dots\dots \\ f[x_0, x_1, \dots, x_k] &:= \frac{f[x_1, x_2, \dots, x_k] - f[x_0, x_1, \dots, x_{k-1}]}{x_k - x_0}. \end{aligned}$$

In the following we will consider a block partition Π of the interval $[a, b]$. It will be taken in the following way

$$\begin{aligned} \Pi &= \{a = x_{1,1} < x_{1,2} \leq x_{1,3} < x_{1,4} = x_{2,1} < x_{2,2} \leq x_{2,3} < x_{2,4} \\ &= x_{3,1} < \dots < x_{n-1,4} = x_{n,1} < x_{n,2} \leq x_{n,3} < x_{n,4} = b\}, \end{aligned} \quad (3.1)$$

in place of the regular partition.

Definition 4. Let f be a real-valued function defined on $[a, b]$, Π be a block partition of $[a, b]$ and α be an increasing function. Let

$$\sigma_{(p,2,\alpha)}^R(f, \Pi) = \sum_{j=1}^n \frac{|f_\alpha[x_{j,4}, x_{j,3}] - f_\alpha[x_{j,2}, x_{j,1}]|^p}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}}$$

where

$$f_\alpha[a, b] = \frac{f(b) - f(a)}{\alpha(b) - \alpha(a)}$$

and

$$\text{V}_{(p,2,\alpha)}^R(f, [a, b]) = \text{V}_{(p,2,\alpha)}^R(f) = \sup_{\Pi} \sigma_{(p,2,\alpha)}^R(f, \Pi),$$

where the supremum is taken on all block partitions of $[a, b]$. To the number $\text{V}_{(p,2,\alpha)}^R(f, [a, b])$ we call the *Riesz $(p, 2, \alpha)$ -variation of the function f in $[a, b]$* . If $\text{V}_{(p,2,\alpha)}^R(f, [a, b]) < \infty$, then we say that f has *bounded Riesz $(p, 2, \alpha)$ -variation*. The set of all functions is denoted by $\text{RV}_{(p,2,\alpha)}([a, b])$.

Remark 1. When $p = 1$ we observe that $BV_{(2,\alpha)}([a, b]) = RV_{(1,2,\alpha)}([a, b])$.

In the following result we will show that if f has Riesz bounded $(p, 2, \alpha)$ -variation, then f has bounded α -second variation.

Theorem 1. *We have $RV_{(p,2,\alpha)}([a, b]) \hookrightarrow BV_{(2,\alpha)}([a, b])$, with*

$$V_{(2,\alpha)}(f, [a, b]) \leq (\alpha(b) - \alpha(a))^{\frac{p-1}{p}} V_{(p,2,\alpha)}^R(f, [a, b]). \quad (3.2)$$

Proof. Let Π be a block partition of type (3.1). Using the Hölder inequality we obtain

$$\begin{aligned} & \sum_{j=1}^n \frac{|f_\alpha[x_{j,4}, x_{j,3}] - f_\alpha[x_{j,2}, x_{j,1}]|}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{\frac{p-1}{p}}} |\alpha(x_{j,4}) - \alpha(x_{j,1})|^{\frac{p-1}{p}} \\ & \leq \left(\sum_{j=1}^n \frac{|f_\alpha[x_{j,4}, x_{j,3}] - f_\alpha[x_{j,2}, x_{j,1}]|^p}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |\alpha(x_{j,4}) - \alpha(x_{j,1})| \right)^{\frac{p-1}{p}} \\ & \leq \left(V_{(p,2,\alpha)}^R(f, [a, b]) \right)^{\frac{1}{p}} (\alpha(b) - \alpha(a))^{\frac{p-1}{p}}. \end{aligned}$$

Since the obtained inequality holds for all block partitions we obtain the desired inequality (3.2). $\square$

Remark 2. We know (see [3]) that if $f \in BV_{(2,\alpha)}([a, b])$, then there exists the right and left α -derivative $f'_{\alpha+}(x_0)$ and $f'_{\alpha-}(x_0)$ on each $x_0 \in (a, b)$ and $f'_{\alpha+}(a)$ and $f'_{\alpha-}(b)$. The last result allow us to conclude that this is also true if $f \in RV_{(p,2,\alpha)}([a, b])$. In particular, there exists $f'_{\alpha+}(a)$ which we write as $f'_\alpha(a)$.

Lemma 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a function such that $f \in RV_{(p,2,\alpha)}([a, b])$ and $V_{(p,2,\alpha)}^R(f, [a, b]) = 0$. Then there exists $\lambda, \mu \in \mathbb{R}$ such that $f(x) = \lambda\alpha(x) + \mu$ for all $x \in [a, b]$.*

Proof. Since $V_{(p,2,\alpha)}^R(f, [a, b]) = 0$ we have that $\sigma_{(p,2,\alpha)}^R(f, \Pi) = 0$ for all block partitions Π of $[a, b]$. Let us consider the particular partition given by $\Pi_0 = \{a < x \leq x < b\}$,

$$\sigma_{(p,2,\alpha)}^R(f, \Pi_0) = \left| \frac{f(b) - f(x)}{\alpha(b) - \alpha(x)} - \frac{f(x) - f(a)}{\alpha(x) - \alpha(a)} \right|^p \frac{1}{|\alpha(b) - \alpha(a)|^{p-1}} = 0,$$

where

$$\left| \frac{f(b) - f(x)}{\alpha(b) - \alpha(x)} - \frac{f(x) - f(a)}{\alpha(x) - \alpha(a)} \right|^p = 0,$$

Direct calculations show that

$$f(x) = \frac{f(b) - f(a)}{\alpha(b) - \alpha(a)} \alpha(x) + \frac{f(a)\alpha(b) - f(b)\alpha(a)}{\alpha(b) - \alpha(a)}$$

and now taking λ and μ in the following manner

$$\lambda = \frac{f(b) - f(a)}{\alpha(b) - \alpha(a)}, \quad \text{and} \quad \mu = \frac{f(a)\alpha(b) - f(b)\alpha(a)}{\alpha(b) - \alpha(a)}$$

we have the desired result. $\square$

Remark 3. The set $\text{RV}_{(p,2,\alpha)}([a,b])$ can be equipped with a linear space structure considering the operator $f \mapsto \|f\|_{\text{RV}_{(p,2,\alpha)}([a,b])}$ defined in the space $\text{RV}_{(p,2,\alpha)}([a,b])$ given in the following way:

$$\|f\|_{\text{RV}_{(p,2,\alpha)}([a,b])} := |f(a)| + |f'_\alpha(a)| + \left(V_{(p,2,\alpha)}^R(f, [a,b]) \right)^{\frac{1}{p}}, \quad (3.3)$$

for $f \in \text{RV}_{(p,2,\alpha)}([a,b])$.

Theorem 2. The operator $f \mapsto \|f\|_{\text{RV}_{(p,2,\alpha)}([a,b])}$ is a norm in the space $\text{RV}_{(p,2,\alpha)}([a,b])$.

Proof. Let us take f such that $\|f\|_{\text{RV}_{(p,2,\alpha)}([a,b])} = 0$. From (3.3) this means that $|f(a)| = 0$, $|f'_\alpha(a)| = 0$ and $V_{(p,2,\alpha)}^R(f) = 0$. Since $V_{(p,2,\alpha)}^R(f) = 0$ from Lemma 1 we deduce that

$$f(x) = \lambda\alpha(x) + \mu, \quad \lambda, \mu \in \mathbb{R}.$$

Since $f'_\alpha(x) = \lambda$ for $x \in [a,b]$ we conclude that f'_α is a constant function which is null at $x = a$ and thus $\lambda = 0$, from which we get that $f \equiv 0$ since $f(a) = 0$.

It is straightforward to see that for $\lambda \in \mathbb{R}$ we have that

$$V_{(p,2,\alpha)}^R(\lambda f) = |\lambda|^p V_{(p,2,\alpha)}^R(f)$$

and now by the definition of the operator (3.3) implies the homogeneity of the operator under consideration.

Let us now prove the triangle inequality. We now introduce the following notation

$$\Delta_\alpha^j(f) = f_\alpha[x_{j,4}, x_{j,3}] - f_\alpha[x_{j,2}, x_{j,1}].$$

Let $f, g \in \text{RV}_{(p,2,\alpha)}([a,b])$ and Π be a block partition of $[a,b]$ as in (3.1), then

$$\begin{aligned} \sigma_{(p,2,\alpha)}^R(f+g, \Pi) &= \sum_{j=1}^n \frac{|\Delta_\alpha^j(f+g)|^p}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} \\ &= \sum_{j=1}^n \frac{|\Delta_\alpha^j(f) + \Delta_\alpha^j(g)|^p}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} \\ &\leq \sum_{j=1}^n \frac{|\Delta_\alpha^j(f) + \Delta_\alpha^j(g)|^{p-1}}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} |\Delta_\alpha^j(f)| \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^n \frac{|\Delta_{\alpha}^j(f) + \Delta_{\alpha}^j(g)|^{p-1}}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} |\Delta_{\alpha}^j(g)| \\
& \leq \sum_{j=1}^n \frac{|\Delta_{\alpha}^j(f) + \Delta_{\alpha}^j(g)|^{p-1}}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{\frac{(p-1)^2}{p}}} \frac{|\Delta_{\alpha}^j(f)|}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{\frac{p-1}{p}}} \\
& \quad + \sum_{j=1}^n \frac{|\Delta_{\alpha}^j(f) + \Delta_{\alpha}^j(g)|^{p-1}}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{\frac{(p-1)^2}{p}}} \frac{|\Delta_{\alpha}^j(g)|}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{\frac{p-1}{p}}} \\
& \leq \left[\sum_{j=1}^n \frac{|\Delta_{\alpha}^j(f) + \Delta_{\alpha}^j(g)|^p}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} \right]^{\frac{p-1}{p}} \left[\sum_{j=1}^n \frac{|\Delta_{\alpha}^j(f)|^p}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} \right]^{\frac{1}{p}} \\
& \quad + \left[\sum_{j=1}^n \frac{|\Delta_{\alpha}^j(f) + \Delta_{\alpha}^j(g)|^p}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} \right]^{\frac{p-1}{p}} \left[\sum_{j=1}^n \frac{|\Delta_{\alpha}^j(g)|^p}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} \right]^{\frac{1}{p}}.
\end{aligned}$$

We therefore have

$$\left(\sigma_{(p,2,\alpha)}^R(f+g, \Pi) \right)^{\frac{1}{p}} \leq \left(\sigma_{(p,2,\alpha)}^R(f, \Pi) \right)^{\frac{1}{p}} + \left(\sigma_{(p,2,\alpha)}^R(g, \Pi) \right)^{\frac{1}{p}}$$

and since this is true for all partitions, we have

$$\mathbf{V}_{(p,2,\alpha)}^R(f+g, [a,b]) \leq \mathbf{V}_{(p,2,\alpha)}^R(f, [a,b]) + \mathbf{V}_{(p,2,\alpha)}^R(g, [a,b]),$$

and now it easily follows the triangle inequality. $\square$

4. EMBEDDING WITH $\mathbf{RV}_{(p,2,\alpha)}([a,b])$

We will show that if $p < q$, then there exists an embedding between the spaces $\mathbf{RV}_{(p,2,\alpha)}([a,b])$ and $\mathbf{RV}_{(q,2,\alpha)}([a,b])$. We will need this fact to show completeness of $\mathbf{RV}_{(p,2,\alpha)}([a,b])$.

Theorem 3. *If $1 < q < p < \infty$, then $\mathbf{RV}_{(p,2,\alpha)}([a,b]) \hookrightarrow \mathbf{RV}_{(q,2,\alpha)}([a,b])$, with*

$$\|f\|_{\mathbf{RV}_{(q,2,\alpha)}([a,b])} \leq \max \left\{ 1, (\alpha(b) - \alpha(a))^{\frac{1}{q} - \frac{1}{p}} \right\} \|f\|_{\mathbf{RV}_{(p,2,\alpha)}([a,b])},$$

for $f \in \mathbf{RV}_{(p,2,\alpha)}([a,b])$.

Proof. Let $f \in \mathbf{RV}_{(p,2,\alpha)}([a,b])$ and Π be a block partition of $[a,b]$ as in (3.1). Let us consider

$$\begin{aligned}
\sigma_{(q,2,\alpha)}^R(f, \Pi) &= \sum_{j=1}^n \frac{|\Delta_{\alpha}^j(f)|^q}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{q-1}} \\
&= \sum_{j=1}^n \frac{|\Delta_{\alpha}^j(f)|^q}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{\frac{(p-1)q}{p}}} |\alpha(x_{j,4}) - \alpha(x_{j,1})|^{\frac{p-q}{p}}.
\end{aligned}$$

Applying the Hölder inequality we obtain

$$\begin{aligned} \sigma_{(q,2,\alpha)}^R(f, \Pi) &\leq \left[\sum_{j=1}^n \frac{|\Delta_{\alpha}^j(f)|^p}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} \right]^{\frac{q}{p}} \left[\sum_{j=1}^n |\alpha(x_{j,4}) - \alpha(x_{j,1})| \right]^{\frac{p-q}{p}} \\ &= \left[\sigma_{(p,2,\alpha)}^R(f, \Pi) \right]^{\frac{q}{p}} (\alpha(b) - \alpha(a))^{\frac{p-q}{p}}, \end{aligned}$$

from which

$$\left[V_{(q,2,\alpha)}^R(f, [a, b]) \right]^{\frac{1}{q}} \leq (\alpha(b) - \alpha(a))^{\frac{1}{q} - \frac{1}{p}} \left[V_{(p,2,\alpha)}^R(f, [a, b]) \right]^{\frac{1}{p}}$$

whence

$$\|f \mid \text{RV}_{(q,2,\alpha)}([a, b])\| \leq \max \left\{ 1, (\alpha(b) - \alpha(a))^{\frac{1}{q} - \frac{1}{p}} \right\} \|f \mid \text{RV}_{(p,2,\alpha)}([a, b])\|,$$

for $f \in \text{RV}_{(p,2,\alpha)}([a, b])$. □

Remark 4. The proof of the above theorem remains valid if $q = 1$.

Corollary 1. For $p \geq 1$ we have $\text{RV}_{(p,2,\alpha)}([a, b]) \hookrightarrow \text{BV}_{(2,\alpha)}([a, b])$, with

$$\|f \mid \text{BV}_{(2,\alpha)}([a, b])\| \leq \max \left\{ 1, (\alpha(b) - \alpha(a))^{\frac{p-1}{p}} \right\} \|f \mid \text{RV}_{(p,2,\alpha)}([a, b])\|.$$

The following corollary follows from the previous results and from results from [2].

Corollary 2. If $1 < q < p < \infty$, then

$$\begin{aligned} \text{RV}_{(p,2,\alpha)}[a, b] &\hookrightarrow \text{RV}_{(q,2,\alpha)}[a, b] \hookrightarrow V_{(2,\alpha)}[a, b] \hookrightarrow \alpha\text{-Lip}[a, b] \hookrightarrow \\ &\hookrightarrow \text{RV}_{(p,\alpha)}[a, b] \hookrightarrow \text{RV}_{(q,\alpha)}[a, b] \hookrightarrow \alpha\text{-AC}[a, b] \hookrightarrow V[a, b] \hookrightarrow B[a, b]. \end{aligned}$$

5. $\text{RV}_{(p,2,\alpha)}([a, b])$ IS A BANACH SPACE

Theorem 4. The space $(\text{RV}_{(p,2,\alpha)}([a, b]), \|\cdot \mid \text{RV}_{(p,2,\alpha)}([a, b])\|)$ is a Banach space.

Proof. Let $(f_k)_{k \in \mathbb{N}}$ be a Cauchy sequence in $\text{RV}_{(p,2,\alpha)}([a, b])$. Given $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\|f_q - f_r \mid \text{RV}_{(p,2,\alpha)}([a, b])\| < \varepsilon$$

if $q, r > N_\varepsilon$, from which we get the following system of inequalities

$$\begin{cases} |f_q(a) - f_r(a)| < \varepsilon, \\ |(f_q)'_\alpha(a) - (f_r)'_\alpha(a)| < \varepsilon, \\ V_{(p,2,\alpha)}^R(f_q - f_r) < \varepsilon^p. \end{cases}$$

From Corollary 2 we conclude that $(f_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in $\alpha\text{-Lip}[a, b]$, thus we have

$$\|f_q - f_r\|_{\alpha\text{-Lip}[a, b]} < \varepsilon K$$

with $K = \max \left\{ 1, (\alpha(b) - \alpha(a))^{\frac{p-1}{p}} \right\}$, hence

$$|f_q(a) - f_r(a)| + \sup_{x \neq y} \left| \frac{(f_q - f_r)(x) - (f_q - f_r)(y)}{\alpha(x) - \alpha(y)} \right| < \varepsilon K$$

and so

$$|f_q(x) - f_r(x)| < K\varepsilon(1 + \alpha(b) - \alpha(a)), \quad x \in [a, b].$$

This tells us that for each $x \in [a, b]$ $(f_k(x))_{k \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is a complete space, we can define $f : [a, b] \rightarrow \mathbb{R}$ as $x \mapsto f(x) := \lim_{k \rightarrow \infty} f_k(x)$.

We are about to prove that:

- (i) $f \in \text{RV}_{(p,2,\alpha)}([a, b])$, and
- (ii) $(f_k)_{k \in \mathbb{N}}$ converges to f in the $\text{RV}_{(p,2,\alpha)}([a, b])$ -norm.

(i) Let Π be a block partition of $[a, b]$ as in (3.1). Since $(f_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in $\text{RV}_{(p,2,\alpha)}([a, b])$ the norm sequence $(\|f_k\|_{\text{RV}_{(p,2,\alpha)}([a, b])})_{k \in \mathbb{N}}$ is bounded, that is, there exists $M > 0$ such that $\|f\|_{\text{RV}_{(p,2,\alpha)}([a, b])} \leq M$ for $k \in \mathbb{N}$. From this fact we have

$$\begin{aligned} & \left[\sum_{j=1}^n \left| \frac{f(x_{j,4}) - f(x_{j,3})}{\alpha(x_{j,4}) - \alpha(x_{j,3})} - \frac{f(x_{j,2}) - f(x_{j,1})}{\alpha(x_{j,2}) - \alpha(x_{j,1})} \right|^p \frac{1}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} \right]^{\frac{1}{p}} \\ &= \lim_{k \rightarrow \infty} \left[\sum_{j=1}^n \left| \frac{f_k(x_{j,4}) - f_k(x_{j,3})}{\alpha(x_{j,4}) - \alpha(x_{j,3})} - \frac{f_k(x_{j,2}) - f_k(x_{j,1})}{\alpha(x_{j,2}) - \alpha(x_{j,1})} \right|^p \frac{1}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} \right]^{\frac{1}{p}} \\ &\leq \lim_{k \rightarrow \infty} \left[V_{(p,2,\alpha)}^R(f_k, [a, b]) \right]^{\frac{1}{p}} \\ &\leq \lim_{k \rightarrow \infty} \|f_k\|_{\text{RV}_{(p,2,\alpha)}([a, b])} \\ &\leq M \end{aligned}$$

for all partitions Π , then $(V_{(p,2,\alpha)}^R(f))^{\frac{1}{p}} \leq M$ and thus $f \in \text{RV}_{(p,2,\alpha)}([a, b])$. Using embedding we may observe that $f'_\alpha(a)$ exists.

(ii) One more time let us consider Π to be a block partition of $[a, b]$ as in (3.1). Let $q, r > N_\varepsilon$, then

$$\begin{aligned} & \sum_{j=1}^n \left| \frac{(f_q - f_r)(x_{j,4}) - (f_q - f_r)(x_{j,3})}{\alpha(x_{j,4}) - \alpha(x_{j,3})} - \frac{(f_q - f_r)(x_{j,2}) - (f_q - f_r)(x_{j,1})}{\alpha(x_{j,2}) - \alpha(x_{j,1})} \right|^p \\ & \quad \times \frac{1}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} \\ & \leq V_{(p,2,\alpha)}^R((f_q - f_r), [a, b]) < \varepsilon^p. \end{aligned}$$

Letting $r \rightarrow \infty$ the above expression becomes

$$\begin{aligned} & \sum_{j=1}^n \left| \frac{(f_q - f)(x_{j,4}) - (f_q - f)(x_{j,3})}{\alpha(x_{j,4}) - \alpha(x_{j,3})} - \frac{(f_q - f)(x_{j,2}) - (f_q - f)(x_{j,1})}{\alpha(x_{j,2}) - \alpha(x_{j,1})} \right|^p \\ & \quad \times \frac{1}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} < \varepsilon^p. \end{aligned}$$

This holds for any partition of $[a, b]$, therefore $V_{(p,2,\alpha)}^R(f_q - f) < \varepsilon^p$ when $q > N_\varepsilon$.

Let $h \in \mathbb{R}^+$ be such that $a < a + h < s < t \leq b$, then

$$\begin{aligned} & \left| \frac{(f_q - f)(t) - (f_q - f)(s)}{\alpha(t) - \alpha(s)} - \frac{(f_q - f)(a + h) - (f_q - f)(a)}{\alpha(a + h) - \alpha(a)} \right|^p \frac{1}{|\alpha(t) - \alpha(s)|^{p-1}} \\ & \leq V_{(p,2,\alpha)}^R(f_q - f) < \varepsilon^p \end{aligned}$$

if $q > N_\varepsilon$. Letting $h \rightarrow 0$ we have

$$\left| \frac{(f_q - f)(t) - (f_q - f)(s)}{\alpha(t) - \alpha(s)} - (f_q - f)'_\alpha(a) \right|^p \frac{1}{|\alpha(t) - \alpha(s)|^{p-1}} \leq \varepsilon^p$$

from which

$$\left| \frac{(f_q - f)(t) - (f_q - f)(s)}{\alpha(t) - \alpha(s)} - (f_q)'_\alpha(a) + (f)'_\alpha(a) \right|^p \leq \varepsilon^p |\alpha(t) - \alpha(s)|^{p-1}.$$

Since $f_q, f \in \text{RV}_{(p,2,\alpha)}([a, b]) \hookrightarrow \alpha\text{-Lip}[a, b]$, we have

$$\begin{aligned} |(f_q)'_\alpha(a) - (f)'_\alpha(a)| & \leq \varepsilon |\alpha(b) - \alpha(a)|^{\frac{p-1}{p}} + \left| \frac{(f_q - f)(t) - (f_q - f)(s)}{\alpha(t) - \alpha(s)} \right| \\ & \leq \varepsilon |\alpha(b) - \alpha(a)|^{\frac{p-1}{p}} + \|f_q - f\|_{\alpha\text{-Lip}[a, b]} \\ & \leq \varepsilon |\alpha(b) - \alpha(a)|^{\frac{p-1}{p}} + \varepsilon K = \tilde{K} \varepsilon \end{aligned}$$

if $q > N_\varepsilon$, where

$$\tilde{K} = \max \left\{ K, (\alpha(b) - \alpha(a))^{\frac{p-1}{p}} \right\}$$

since $\|f_q - f\|_{\alpha\text{-Lip}[a,b]} = \lim_{r \rightarrow \infty} \|f_q - f_r\|_{\alpha\text{-Lip}[a,b]} < K\varepsilon$. Finally, for $q > N_\varepsilon$ we obtain

$$|f_q(a) - f(a)| + |(f_q)'_\alpha(a) - (f)'_\alpha(a)| + \left[V_{(p,2,\alpha)}^R(f_q - f, [a, b]) \right]^{\frac{1}{q}} \leq (\tilde{K} + 2)\varepsilon,$$

in other words

$$\|f_q - f\|_{\text{RV}_{(p,2,\alpha)}([a,b])} \leq (\tilde{K} + 2)\varepsilon$$

if $q > N_\varepsilon$ which means that $(f_k)_{k \in \mathbb{N}}$ converges to $f \in \text{RV}_{(p,2,\alpha)}([a,b])$ in the norm $\|\cdot\|_{\text{RV}_{(p,2,\alpha)}([a,b])}$. $\square$

6. $\text{RV}_{(p,2,\alpha)}([a,b])$ IS A BANACH ALGEBRA

We are going to show that $\text{RV}_{(p,2,\alpha)}([a,b])$ is closed under the multiplication of functions.

Theorem 5. *Let $f, g \in \text{RV}_{(p,2,\alpha)}([a,b])$. Then $fg \in \text{RV}_{(p,2,\alpha)}([a,b])$.*

Proof. Let Π be a block partition of $[a,b]$ as in (3.1). Let us consider

$$\begin{aligned} \sigma_{(p,2,\alpha)}^R(f \cdot g, \Pi) &= \sum_{j=1}^n \frac{|(fg)_\alpha[x_{j,4}, x_{j,3}] - (fg)_\alpha[x_{j,2}, x_{j,1}]|^p}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} \\ &= \sum_{j=1}^n \frac{|(fg)_\alpha[x_{j,4}, x_{j,3}] - (fg)_\alpha[x_{j,2}, x_{j,1}]|^{p-1}}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} \\ &\quad \times |(fg)_\alpha[x_{j,4}, x_{j,3}] - (fg)_\alpha[x_{j,2}, x_{j,1}]|. \end{aligned} \quad (6.1)$$

Observe that the term $|(fg)_\alpha[x_{j,4}, x_{j,3}] - (fg)_\alpha[x_{j,2}, x_{j,1}]| =: T$ can be written as

$$\begin{aligned} T &= |f(x_{j,4})g_\alpha[x_{j,4}, x_{j,3}] + g(x_{j,3})f_\alpha[x_{j,4}, x_{j,3}] \\ &\quad - f(x_{j,2})g_\alpha[x_{j,2}, x_{j,1}] - g(x_{j,1})f_\alpha[x_{j,2}, x_{j,1}]| \end{aligned}$$

and now adding and subtracting appropriate terms and grouping the terms we obtain

$$\begin{aligned} T &= |f(x_{j,4})(g_\alpha[x_{j,4}, x_{j,3}] - g_\alpha[x_{j,2}, x_{j,1}]) + g(x_{j,3})(f_\alpha[x_{j,4}, x_{j,3}] - f_\alpha[x_{j,2}, x_{j,1}]) \\ &\quad + (f(x_{j,4}) - f(x_{j,2}))g_\alpha[x_{j,2}, x_{j,1}] + (g(x_{j,3}) - g(x_{j,1}))f_\alpha[x_{j,2}, x_{j,1}]|. \end{aligned} \quad (6.2)$$

From Corollary 2 we have

$$\text{RV}_{(p,2,\alpha)}([a,b]) \hookrightarrow \alpha\text{-Lip}[a,b] \quad \text{and} \quad \text{RV}_{(p,2,\alpha)}([a,b]) \hookrightarrow \mathcal{B}[a,b].$$

This let us write

$$|f(x_{j,4})| \leq \|f\|_\infty, \quad |g(x_{j,3})| \leq \|g\|_\infty$$

and

$$\frac{f(\xi) - f(\eta)}{\alpha(\xi) - \alpha(\eta)} \leq \text{Lip}_\alpha(f), \quad \frac{g(\xi) - g(\eta)}{\alpha(\xi) - \alpha(\eta)} \leq \text{Lip}_\alpha(g),$$

for all $\xi, \eta \in [a, b]$. Using these estimates we obtain

$$\begin{aligned} & |f(x_{j,4}) - f(x_{j,2})| |g_\alpha[x_{j,2}, x_{j,1}]| + |g(x_{j,3} - g(x_{j,1}))| |f_\alpha[x_{j,2}, x_{j,1}]| \\ &= \frac{|f(x_{j,4}) - f(x_{j,2})|}{|\alpha(x_{j,4}) - \alpha(x_{j,2})|} \cdot \frac{|g(x_{j,2}) - g(x_{j,1})|}{|\alpha(x_{j,2}) - \alpha(x_{j,1})|} |\alpha(x_{j,4}) - \alpha(x_{j,1})| \\ &+ \frac{|f(x_{j,2}) - f(x_{j,1})|}{|\alpha(x_{j,2}) - \alpha(x_{j,1})|} \cdot \frac{|g(x_{j,3}) - g(x_{j,1})|}{|\alpha(x_{j,3}) - \alpha(x_{j,1})|} |\alpha(x_{j,4}) - \alpha(x_{j,1})| \\ &\leq (\text{Lip}_\alpha f)(\text{Lip}_\alpha g) (\alpha(x_{j,4}) - \alpha(x_{j,2}) + \alpha(x_{j,3}) - \alpha(x_{j,1})) \end{aligned}$$

replacing all this into (6.2) we have

$$\begin{aligned} & |(fg)_\alpha[x_{j,4}, x_{j,3}] - (fg)_\alpha[x_{j,2}, x_{j,1}]| \\ &\leq \|f\|_\infty |g_\alpha[x_{j,4}, x_{j,3}] - g_\alpha[x_{j,2}, x_{j,1}]| + \|g\|_\infty |f_\alpha[x_{j,4}, x_{j,3}] - f_\alpha[x_{j,2}, x_{j,1}]| \\ &+ (\text{Lip}_\alpha f)(\text{Lip}_\alpha g) (\alpha(x_{j,4}) - \alpha(x_{j,2}) + \alpha(x_{j,3}) - \alpha(x_{j,1})). \end{aligned}$$

Now replacing this last estimate into (6.1), separating summations and fixing exponents to apply the Hölder inequality we have

$$\begin{aligned} \sigma_{(p,2,\alpha)}^R(fg, \Pi) &\leq \|f\|_\infty \sum_{j=1}^n \frac{|(fg)_\alpha[x_{j,4}, x_{j,3}] - (fg)_\alpha[x_{j,2}, x_{j,1}]|^{p-1}}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{\frac{(p-1)^2}{p}}} \\ &\times \frac{|g_\alpha[x_{j,4}, x_{j,3}] - g_\alpha[x_{j,2}, x_{j,1}]|}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{\frac{p-1}{p}}} \\ &+ \|g\|_\infty \sum_{j=1}^n \frac{|(fg)_\alpha[x_{j,4}, x_{j,3}] - (fg)_\alpha[x_{j,2}, x_{j,1}]|^{p-1}}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{\frac{(p-1)^2}{p}}} \\ &\times \frac{|f_\alpha[x_{j,4}, x_{j,3}] - f_\alpha[x_{j,2}, x_{j,1}]|}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{\frac{p-1}{p}}} \\ &+ (\text{Lip}_\alpha f)(\text{Lip}_\alpha g) \sum_{j=1}^n \frac{|(fg)_\alpha[x_{j,4}, x_{j,3}] - (fg)_\alpha[x_{j,2}, x_{j,1}]|^{p-1}}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{\frac{(p-1)^2}{p}}} \\ &\times \frac{|\alpha(x_{j,4}) - \alpha(x_{j,2}) + \alpha(x_{j,3}) - \alpha(x_{j,1})|}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{\frac{p-1}{p}}}. \end{aligned}$$

Applying in each summation the Hölder inequality we obtain

$$\begin{aligned} \sigma_{(p,2,\alpha)}^R(fg, \Pi) &\leq \|f\|_\infty \left[\sigma_{(p,2,\alpha)}^R(fg, \Pi) \right]^{\frac{p-1}{p}} \left[\sigma_{(p,2,\alpha)}^R(g, \Pi) \right]^{\frac{1}{p}} \\ &+ \|g\|_\infty \left[\sigma_{(p,2,\alpha)}^R(fg, \Pi) \right]^{\frac{p-1}{p}} \left[\sigma_{(p,2,\alpha)}^R(f, \Pi) \right]^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned}
& + (\text{Lip}_\alpha f)(\text{Lip}_\alpha g) \left[\sigma_{(p,2,\alpha)}^R(fg, \Pi) \right]^{\frac{p-1}{p}} \\
& \times \left[\sum_{j=1}^n \frac{|\alpha(x_{j,4}) - \alpha(x_{j,2}) + \alpha(x_{j,3}) - \alpha(x_{j,1})|^p}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} \right]^{\frac{1}{p}}.
\end{aligned}$$

Simplifying we have

$$\begin{aligned}
\left[\sigma_{(p,2,\alpha)}^R(fg, \Pi) \right]^{\frac{1}{p}} & \leq \|f\|_\infty \left[\sigma_{(p,2,\alpha)}^R(g, \Pi) \right]^{\frac{1}{p}} + \|g\|_\infty \left[\sigma_{(p,2,\alpha)}^R(f, \Pi) \right]^{\frac{1}{p}} \\
& + (\text{Lip}_\alpha f)(\text{Lip}_\alpha g) \left[\sum_{j=1}^n \frac{|\alpha(x_{j,4}) - \alpha(x_{j,2}) + \alpha(x_{j,3}) - \alpha(x_{j,1})|^p}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} \right]^{\frac{1}{p}} \quad (6.3)
\end{aligned}$$

Simplifying the last bracket in (6.3), for $j = 1, \dots, n$, we can observe that

$$\begin{aligned}
& \frac{|\alpha(x_{j,4}) - \alpha(x_{j,2}) + \alpha(x_{j,3}) - \alpha(x_{j,1})|^p}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} \\
& = \left| \frac{\alpha(x_{j,4}) - \alpha(x_{j,2}) + \alpha(x_{j,3}) - \alpha(x_{j,1})}{\alpha(x_{j,4}) - \alpha(x_{j,1})} \right|^{p-1} |\alpha(x_{j,4}) - \alpha(x_{j,2}) + \alpha(x_{j,3}) - \alpha(x_{j,1})| \\
& = \left| 1 + \frac{\alpha(x_{j,3}) - \alpha(x_{j,2})}{\alpha(x_{j,4}) - \alpha(x_{j,1})} \right|^{p-1} |\alpha(x_{j,4}) - \alpha(x_{j,2}) + \alpha(x_{j,3}) - \alpha(x_{j,1})|. \quad (6.4)
\end{aligned}$$

Since $x_{j,1} < x_{j,2} \leq x_{j,3} < x_{j,4}$ we have

$$\begin{aligned}
& \alpha(x_{j,3}) - \alpha(x_{j,2}) < \alpha(x_{j,4}) - \alpha(x_{j,1}); \\
& \alpha(x_{j,4}) - \alpha(x_{j,2}) > 0; \\
& \alpha(x_{j,3}) - \alpha(x_{j,1}) > 0.
\end{aligned}$$

Substituting this into (6.4)

$$\begin{aligned}
& \frac{|\alpha(x_{j,4}) - \alpha(x_{j,2}) + \alpha(x_{j,3}) - \alpha(x_{j,1})|^p}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} \\
& \leq 2^{p-1} (\alpha(x_{j,4}) - \alpha(x_{j,2}) + \alpha(x_{j,3}) - \alpha(x_{j,1}))
\end{aligned}$$

and now summing we obtain

$$\sum_{j=1}^n \frac{|\alpha(x_{j,4}) - \alpha(x_{j,2}) + \alpha(x_{j,3}) - \alpha(x_{j,1})|^p}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|^{p-1}} \leq 2^p (\alpha(b) - \alpha(a)).$$

Substituting this inequality into (6.3) we have

$$\begin{aligned}
\left[\sigma_{(p,2,\alpha)}^R(fg, \Pi) \right]^{\frac{1}{p}} & \leq \|f\|_\infty \left[\sigma_{(p,2,\alpha)}^R(g, [a, b]) \right]^{\frac{1}{p}} + \|g\|_\infty \left[\sigma_{(p,2,\alpha)}^R(f, [a, b]) \right]^{\frac{1}{p}} \\
& + 2(\alpha(b) - \alpha(a))^{\frac{1}{p}} (\text{Lip}_\alpha f)(\text{Lip}_\alpha g),
\end{aligned}$$

this last inequality holds for all partition Π of $[a, b]$, then

$$\begin{aligned} \left[V_{(p,2,\alpha)}^R(fg, [a, b]) \right]^{\frac{1}{p}} &\leq \|f\|_{\infty} \left[V_{(p,2,\alpha)}^R(g, [a, b]) \right]^{\frac{1}{p}} \\ &\quad + \|g\|_{\infty} \left[V_{(p,2,\alpha)}^R(f, [a, b]) \right]^{\frac{1}{p}} \\ &\quad + 2(\alpha(b) - \alpha(a))^{\frac{1}{p}} (\text{Lip}_{\alpha} f)(\text{Lip}_{\alpha} g) < \infty, \end{aligned} \quad (6.5)$$

from which it follows that $fg \in \text{RV}_{(p,2,\alpha)}([a, b])$. $\square$

We now obtain the following corollary.

Corollary 3. *Let $f, g \in \text{RV}_{(p,2,\alpha)}([a, b])$, then*

$$\begin{aligned} \|fg \mid \text{RV}_{(p,2,\alpha)}([a, b])\| &\leq P \|f \mid \text{RV}_{(p,2,\alpha)}([a, b])\| \|g \mid \text{RV}_{(p,2,\alpha)}([a, b])\| \\ &\quad + \|f\|_{\infty} \|g \mid \text{RV}_{(p,2,\alpha)}([a, b])\| \\ &\quad + \|g\|_{\infty} \|f \mid \text{RV}_{(p,2,\alpha)}([a, b])\|. \end{aligned}$$

with $P = 2 \max \left\{ (\alpha(b) - \alpha(a))^{\frac{1}{p}}, (\alpha(b) - \alpha(a))^{\frac{2p-2}{p}} \right\}$.

Proof. Note that

$$\begin{aligned} \text{Lip}_{\alpha}(f) &\leq \|f \mid \alpha\text{-Lip}[a, b]\| \leq \|f \mid V_{(2,\alpha)}[a, b]\| \\ &\leq \max \left\{ 1, (\alpha(b) - \alpha(a))^{\frac{p-1}{p}} \right\} \|f \mid \text{RV}_{(p,2,\alpha)}([a, b])\| \end{aligned}$$

and similarly to g . Then (6.5) can be written as

$$\begin{aligned} \left[V_{(p,2,\alpha)}^R(fg, [a, b]) \right]^{\frac{1}{p}} &\leq \|f\|_{\infty} \left[V_{(p,2,\alpha)}^R(g, [a, b]) \right]^{\frac{1}{p}} + \|g\|_{\infty} \left[V_{(p,2,\alpha)}^R(f, [a, b]) \right]^{\frac{1}{p}} \\ &\quad + 2(\alpha(b) - \alpha(a))^{\frac{1}{p}} \left[\max \left\{ 1, (\alpha(b) - \alpha(a))^{\frac{p-1}{p}} \right\} \right]^2 \\ &\quad \times \|f \mid \text{RV}_{(p,2,\alpha)}([a, b])\| \|g \mid \text{RV}_{(p,2,\alpha)}([a, b])\|. \end{aligned}$$

$\square$

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Authors’ addresses

René Erlín Castillo

Universidad Nacional de Colombia, Colombia

E-mail address: recastillo@unal.edu.co

Humberto Rafeiro

Pontificia Universidad Javeriana, Colombia

E-mail address: silva-h@javeriana.edu.co

Eduard Trousselot

Universidad de Oriente, Venezuela

E-mail address: eddycharles2007@gmail.com



QUASI-SEMI-HOMOMORPHISMS AND GENERALIZED PROXIMITY RELATIONS BETWEEN BOOLEAN ALGEBRAS

SERGIO A. CELANI

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Abstract. In this paper we shall define the notion of quasi-semi-homomorphisms between Boolean algebras, as a generalization of the quasi-modal operators introduced in [3], of the notion of meet-homomorphism studied in [12] and [11], and the notion of precontact or proximity relation defined in [8]. We will prove that the class of Boolean algebras with quasi-semi-homomorphism is a category, denoted by **BoQS**. We shall prove that this category is equivalent to the category **StQB** of Stone spaces where the morphisms are binary relations, called quasi-Boolean relations, satisfying additional conditions. This duality extends the duality for meet-homomorphism given by P. R. Halmos in [12] and the duality for quasi-modal operators proved in [3].

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Keywords: proximity relations, precontact algebras, quasi-modal operator, quasi-semi-homomorphisms, Stone spaces, quasi-Boolean relations

1. INTRODUCTION

Recall that a *modal algebra* is a Boolean algebra A with an operator $\Box : A \rightarrow A$ such that $\Box 1 = 1$, and $\Box(a \wedge b) = \Box a \wedge \Box b$, for all $a, b \in A$. It is well known that the variety of modal algebras is the algebraic semantic of normal modal logics [10, 16]. Modal algebras are dual objects of descriptive general frames, also called modal spaces, i.e., Stone spaces with a relation verifying certain conditions (see [10], and [16]). P. R. Halmos define in [12] the notion of meet-homomorphism (or hemihomomorphism) between Boolean algebras. Recall that a meet-homomorphism between two Boolean algebras A and B , is a function $h : A \rightarrow B$ such that $h(1) = 1$, and $h(a \wedge b) = h(a) \wedge h(b)$, for all $a, b \in A$. If $A = B$, then h is a modal operator [10, 16]. Let X and Y be the Stone spaces of A and B , respectively. As it follows from [12] and [11], a meet-homomorphism $h : A \rightarrow B$ is dually characterized by means of a relation $R \subseteq Y \times X$ such that $R(y)$ is a closed subset of X , for each $y \in Y$, and $h_R(U) = \{y \in Y : R(y) \subseteq U\}$ is a clopen subset of Y , for each clopen $U \subseteq X$. These relations are called Boolean relations in [12], or Boolean correspondences in [11] (see also [16]).

In [3], the notions of quasi-modal operator and quasi-modal algebra were introduced as a generalization of the notion of modal operator and modal algebra, respectively. A quasi-modal operator in a Boolean algebra A is a map Δ that sends each element $a \in A$ to an ideal Δa of A , and satisfies analogous conditions with the modal operator $\Box$ of modal algebras. A quasi-modal algebra is a pair $\langle A, \Delta \rangle$ where A is a Boolean algebra and Δ is a quasi-modal operator. We note that a quasi-modal operator is not an operation, but has many similar properties to modal operators.

In this paper we shall introduce maps between a Boolean algebra A and the set of all ideals of another Boolean algebra B satisfying analogous conditions with the meet-homomorphism between Boolean algebras [12]. We call these maps *quasi-semi-homomorphisms*. One of the main objectives of this paper is to study this class of maps, and their topological representation.

As we will explain below, the quasi-modal operators are closely connected with the proximity or precontact relations defined between Boolean algebras. We recall that a proximity relation defined on a set X is a binary relation $\delta \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ satisfying certain conditions (see Definition 2). If $U, V \in \mathcal{P}(X)$, then the intuitive meaning of a proximity relation δ is that $U\delta V$ holds, when U is close to V in some sense. A proximity or precontact space, also called a nearness space, is a pair $\langle X, \delta \rangle$, where X is a set and δ is a proximity relation. Since $\mathcal{P}(X)$ is a Boolean algebra, we can introduced an abstract definition of proximity relation in the class of Boolean algebras (see [15] and [4]). In the literature, there exist many classes of Boolean algebras endowed with some type of proximity relations. As examples, we can mention the Boolean contact algebras defined in [9], or the Boolean connection algebras defined in [17]. For other versions of Boolean algebras endowed relations see [5], [8], [7], [19], and [18]. In [8] the notions of *proximity relation* on a Boolean algebra and the *proximity Boolean algebras* were defined as an abstract version of proximity spaces [15]. This class of structures is the most general class of Boolean algebras endowed with a proximity relation. We note that the notion of proximity Boolean algebras is equivalent to the notion of precontact algebras [8].

There exists a strong connection between proximity relations defined in a Boolean algebra and quasi-modal operator. Given a proximity relation δ in a Boolean algebra A , we can prove that the set $\Delta_\delta b = \{a \in A : (a, \neg b) \notin \delta\}$ is an ideal of A . So, we have a map Δ_δ that send elements to ideals of the algebra A . As we shall see, this map is a quasi-modal operator. Conversely, if we have a quasi-modal operator Δ defined in a Boolean algebra A , then the relation $a\delta_\Delta b$ defined by $a \notin \Delta \neg b$, is a proximity relation on A (for the details see Theorem 1). Thus, we have that the notions of proximity relation and quasi-modal operator are interdefinable. Moreover, since the notion of quasi-semi-homomorphism is a generalization of the notion of quasi-modal operator, and this last is equivalent to the notion of proximity relation, we will get that it is possible to introduce a generalization of the notion of proximity relation.

The paper is organized as follows. In Section 2 we start recalling some basic definitions and results on Stone duality for Boolean algebras. In Section 3 we shall introduce the notion of quasi-semi-homomorphism and the notion of generalized proximity relation. Also, we shall prove that the notions of quasi-semi-homomorphism and generalized proximity relation are equivalent, and as consequence of this fact, we have that the notions of quasi-modal operator and proximity relation are equivalent. This fact has strong consequences, because it puts the proximity relations very close to the modal operators. We shall see that the class of Boolean algebras with the quasi-semi-homomorphism form a category denoted by **BoQS**. In Section 4, we shall introduce the notion of generalized quasi-Boolean relation between Stone spaces, and we shall prove some properties. We shall prove that the class of Stone spaces with the generalized quasi-Boolean relations form a category, symbolized by **StQB**. In Section 5 we shall prove that the categories **StQB** and **BoQS** are dually equivalent. As an application of this duality we will prove a generalization of the result that assert that the Boolean homomorphisms are the minimal elements in the set of all join-homomorphisms between two Boolean algebras (see [11]). In this last section we prove that the minimal elements in the set of all quasi-Boolean relations defined between two Stone spaces is a Boolean relation.

2. PRELIMINARIES

We assume that the reader is familiar with basic concepts of Boolean algebras and topological duality (see [1] or [13]).

We recall that a subset of a topological space X is *clopen* if it is both closed and open, and that X is *zero-dimensional* if the set of clopen subsets of X forms a basis for the topology. We shall denote by $\mathcal{O}(X)$ ($\mathcal{C}(X)$) the set of all open subsets (closed subsets) of X . The closure of a subset Z is denoted by $\text{cl}(Z)$. We shall denote by $\text{Clo}(X)$ the set of all clopen subsets of X . Clearly the notions of Hausdorff and T_0 coincide in the realm of zero-dimensional spaces. A *Stone space* X is *zero-dimensional, compact and Hausdorff* topological space. We note that a Stone space is *totally disconnected*, i.e., given distinct points $x, y \in X$, there is $U \in \text{Clo}(X)$ of X such that $x \in U$ and $y \notin U$. If X is a Stone space, then $\text{Clo}(X)$ is a Boolean algebra under the set theoretical operations.

If $A = \langle A, \vee, \wedge, \neg, 0, 1 \rangle$ is a Boolean algebra, by $\text{Ul}(A)$ we shall denote the set of all ultrafilters (or proper maximal filters) of A while by $\text{Id}(A)$ and $\text{Fi}(A)$ we shall denote the families of all ideals and filters of A , respectively.

Let X be a Stone space. The map $\varepsilon_X : X \rightarrow \text{Ul}(\text{Clo}(X))$ given by

$$\varepsilon_X(x) = \{U \in \text{Clo}(X) : x \in U\}$$

is a bijective and continuous function. Let A be a Boolean algebra and let

$$\beta_A : A \rightarrow \mathcal{P}(\text{Ul}(A))$$

the Stone map defined by $\beta_A(a) = \{P \in \text{Ul}(A) : a \in P\}$. Sometimes we will write β instead of β_A . With each Boolean algebra A we can associate a Stone space $\langle \text{Ul}(A), \tau_A \rangle$ whose points are the elements of $\text{Ul}(A)$ and the topology τ_A is determined by the clopen basis $\beta[A] = \{\beta_A(a) : a \in A\}$. If misunderstanding is excluded, we write $\text{Ul}(A)$ instead of $\langle \text{Ul}(A), \tau_A \rangle$. Thus, if X is a Stone space, then $X \cong \text{Ul}(\text{Clo}(X))$, and if A is a Boolean algebra, then $A \cong \text{Clo}(\text{Ul}(A))$.

If A is a Boolean algebra, then there exists a duality between ideals (filters) of A and open (closed) sets of $\text{Ul}(A)$. More precisely, for $I \in \text{Id}(A)$ and $F \in \text{Fi}(A)$. The value of the function $\varphi_A[I] = \{P \in \text{Ul}(A) : I \cap P \neq \emptyset\}$ is an open of $\text{Ul}(A)$, and thus φ_A is an one-to-one mapping between $\text{Id}(A)$ and the set of $\mathcal{O}(\text{Ul}(A))$ of all open subset of $\text{Ul}(A)$. The function ψ_A defined by $\psi_A[F] = \{P \in \text{Ul}(A) : F \subseteq P\}$, is a one-to-one mapping between $\text{Fi}(A)$ and the set $\mathcal{C}(\text{Ul}(A))$ of all closed subset of $\text{Ul}(A)$. We note that $\varphi_A[I] = \bigcup \{\beta(a) : a \in I\}$. If we denote by $\bar{\wedge}$ and by $\vee$ the meet and the join in the set $\text{Id}(A)$, respectively, then $\varphi_A[I_1 \vee I_2] = \varphi_A[I_1] \cup \varphi_A[I_2]$, and $\varphi_A[I_1 \bar{\wedge} I_2] = \varphi_A[I_1] \cap \varphi_A[I_2]$ (see [13] and [16] for further information on Boolean duality).

Let A be a Boolean algebra. The filter (ideal) generated by a subset $Y \subseteq A$ is denoted by $F(Y)$ ($I(Y)$). If $Y = \{a\}$, then we write $F(a) = [a]$ ($I(a) = (a)$). The set complement of a subset $Y \subseteq A$ will be denoted by Y^c or $A - Y$.

3. QUASI-SEMI-HOMOMORPHISMS

In this section we introduce the main notion of this paper. We define the notion of quasi-semi-homomorphism as a generalization of the notion of quasi-modal operator [2, 3] and the notion of semi-homomorphism between Boolean algebras [11, 12].

Definition 1. Let A and B be two Boolean algebras. A *quasi-semi-homomorphism* is a function $\Delta : A \rightarrow \text{Id}(B)$ such that it verifies the following conditions for all $a, b \in A$:

$$\text{Q1 } \Delta(a \wedge b) = \Delta a \cap \Delta b,$$

$$\text{Q2 } \Delta 1 = B.$$

In the following $QS[A, B]$ stands for the set of all quasi-semi-homomorphism defined between A and B . If $\Delta_1, \Delta_2 \in QS[A, B]$ we define $\Delta_1 \leq \Delta_2$ by $\Delta_1(a) \subseteq \Delta_2(a)$, for all $a \in A$. This gives an order relation in $QS[A, B]$. We note that when $A = B$, the elements of $QS[A, A] = QS[A]$ are called quasi-modal operators in [3]. A pair $\langle A, \Delta \rangle$, where $\Delta \in QS[A]$ is called a *quasi-modal algebra*.

If $\Delta \in QS[A, B]$, then Δ is monotonic, because if $a \leq b$, then $a = a \wedge b$, and so $\Delta a = \Delta(a \wedge b) = \Delta a \cap \Delta b$, i.e., $\Delta a \subseteq \Delta b$.

Example 1. Let A be a Boolean algebra. The map $I_A : A \rightarrow \text{Id}(A)$ given by $I_A(a) = [a]$, for each $a \in A$, is clearly a quasi-semi-homomorphism.

Example 2. Let A and B two Boolean algebras. We recall first that a meet-homomorphisms or *meet-homomorphism* [11] [12], is a function $h : A \rightarrow B$ such that

$h(1) = 1$, and $h(a \wedge b) = h(a) \wedge h(b)$, for all $a, b \in A$. The function h can be extended to a map $\Delta_h : A \rightarrow \text{Id}(B)$ of the following form. Put $\Delta_h(a) = (h(a)]$, for each $a \in A$. It is clear that Δ_h verifies the equalities $\Delta_h(a \wedge b) = \Delta_h(a) \cap \Delta_h(b)$ and $\Delta_h(1) = B$. Thus, Δ_h is a quasi-semi-homomorphism. An element $\Delta \in QS[A, B]$ is called a *principal* quasi-semi-homomorphism if Δa is principal ideal, for each $a \in A$. In other words, for each $a \in A$, there exists $b \in B$ such that $\Delta a = (b]$. It is clear that if Δ is principal, then the map $h_\Delta : A \rightarrow B$ defined by $h(a) = b$ iff $\Delta a = (b]$ is a meet-hemimorphism. Thus, the class of principal quasi-semi-homomorphisms is equivalent to the class of meet-homomorphisms.

Recall that when $A = B$ and $\square : A \rightarrow A$ is a meet-homomorphism, the pair $\langle A, \square \rangle$ is called a modal algebra [16]. So, the class of modal algebras can be identified with the class of pairs $\langle A, \Delta \rangle$ where A is a Boolean algebra and Δ is a principal quasi-semi-homomorphism.

The following example is fundamental in the representation theory of quasi-semi-homomorphisms.

Example 3. Let X and Y be two sets. Let R be a relation between X and Y . Define a function $\bar{\Delta}_R : \mathcal{P}(Y) \rightarrow \text{Id}(\mathcal{P}(X))$, as $\bar{\Delta}_R(U) = (\Delta_R(U)]$, where $\Delta_R(U) = \{x \in X : R(x) \subseteq U\}$, with $U \in \mathcal{P}(Y)$. Then it is easy to see that $\bar{\Delta}_R \in QS[\mathcal{P}(Y), \mathcal{P}(X)]$.

Let A and B be two Boolean algebras. For each $\Delta \in QS[A, B]$, we define the dual quasi-semi-homomorphism $\nabla : A \rightarrow \text{Fi}(B)$ by $\nabla a = \neg \Delta \neg a$, where $\neg \Delta x = \{\neg y : y \in \Delta x\}$. We note that $c \in \nabla(a \vee b) = \neg \Delta \neg(a \vee b) = \neg \Delta(\neg a \wedge \neg b)$ iff $\neg c \in \Delta(\neg a \wedge \neg b) = \Delta \neg a \cap \Delta \neg b$ iff $\neg c \in \Delta \neg a$ and $\neg c \in \Delta \neg b$ iff $c \in \nabla a$ and $c \in \nabla b$ iff $c \in \nabla a \cap \nabla b$. Thus the map ∇ verifies the following conditions:

$$\text{Q3 } \nabla(a \vee b) = \nabla a \cap \nabla b,$$

$$\text{Q4 } \nabla 0 = B.$$

Now we introduce a notion that generalizes the notion of proximity relation (also called precontact relation) defined in a Boolean algebra [8] [7] [14].

Definition 2. Let A and B be two Boolean algebras. A *generalized precontact or generalized proximity relation* between A and B is a relation $\delta \subseteq A \times B$ such that

$$\text{P1 } \text{ If } a\delta b, \text{ then } a \neq 0 \text{ and } b \neq 0.$$

$$\text{P2 } a\delta(b \vee c) \text{ iff } a\delta b \text{ or } a\delta c.$$

$$\text{P3 } (a \vee b)\delta c \text{ iff } a\delta c \text{ or } b\delta c.$$

When $A = B$, a generalized precontact relation δ is called a *proximity or precontact relation*, and the pair $\langle A, \delta \rangle$ is called a *proximity or precontact algebra* [6–8]. An important example of proximity relations are the proximity spaces. There are many other notions of proximity, and we suggest the reader consults the fundamental text by Naimpally and Warrack [15] for more examples, or the paper [18].

Theorem 1. *Let A and B be two Boolean algebras. There exists a bijective correspondence between quasi-semi-homomorphisms between A and B and the generalized proximity relations defined between A and B .*

Proof. Let A and B be two Boolean algebras. If $\delta \subseteq A \times B$ is a generalized proximity relation, then we prove that the subset of A

$$\Delta_\delta b = \{a \in A : (a, \neg b) \notin \delta\},$$

is an ideal of A . Let $b \in B$. As $(0, \neg b) \notin \delta$, we have that $0 \in \Delta_\delta b$. Let $a_1, a_2 \in A$. Suppose that $a_1 \leq a_2$ and $a_2 \in \Delta_\delta b$. Then $(a_2, \neg b) \notin \delta$. As $a_2 = a_1 \vee a_2$, by condition P3 of Definition 2 we have that $(a_2, \neg b) = (a_1 \vee a_2, \neg b) \notin \delta$ iff $(a_1, \neg b) \notin \delta$ and $(a_2, \neg b) \notin \delta$. Thus, $(a_1, \neg b) \notin \delta$, i.e., $a_1 \in \Delta_\delta b$. Suppose that $a_1 \in \Delta_\delta b$ and $a_2 \in \Delta_\delta b$. Then, $(a_1, \neg b) \notin \delta$ and $(a_2, \neg b) \notin \delta$. Again, by condition P3 of Definition 2 we have that $(a_1 \vee a_2, \neg b) \notin \delta$, i.e., $a_1 \vee a_2 \in \Delta_\delta b$. Thus, $\Delta_\delta b \in \text{Id}(A)$, for each $b \in B$. Then the map $\Delta_\delta : B \rightarrow \text{Id}(A)$ is well defined.

Let $b_1, b_2 \in B$ and $a \in A$. Then by condition P2 of Definition 2 we have the following equivalences:

$$\begin{aligned} a \in \Delta_\delta(b_1 \wedge b_2) & \text{ iff } (a, \neg(b_1 \wedge b_2)) = (a, \neg b_1 \vee \neg b_2) \notin \delta \\ & \text{ iff } (a, \neg b_1) \notin \delta \text{ and } (a, \neg b_2) \notin \delta \\ & \text{ iff } a \in \Delta_\delta b_1 \cap \Delta_\delta b_2. \end{aligned}$$

By condition P1 of Definition 2 we have that $(b, \neg 1) = (b, 0) \notin \delta$, for all $b \in B$. Thus, $1 \in \Delta_\delta(b)$, for all $b \in B$.

Conversely. Let $\Delta : A \rightarrow \text{Id}(B)$ be a quasi-semi-homomorphism. Define the relation

$$\delta_\Delta = \{(a, b) \in A \times B : a \notin \Delta \neg b\}.$$

Let $(a, b) \in \delta_\Delta$. If $a = 0$, then $0 \notin \Delta \neg b$, which is a contradiction because $\Delta \neg b$ is an ideal. If $b = 0$, then $a \notin \Delta \neg 0 = \Delta 1 = A$, which is a contradiction. Thus, $a \neq 0$ and $b \neq 0$.

Let $a, b \in A$, and $c \in B$. Taking into account that $\Delta \neg c$ is an ideal of B , we get the following equivalences: $(a \vee b, c) \in \delta_\Delta$ iff $a \vee b \notin \Delta \neg c$ iff $a \notin \Delta \neg c$ or $b \notin \Delta \neg c$ iff $(a, c) \in \delta_\Delta$ or $(b, c) \in \delta_\Delta$.

Let $a \in A$ and $b, c \in B$. Then $(a, b \vee c) \in \delta_\Delta$ iff

$$a \notin \Delta \neg(b \vee c) = \Delta(\neg b \wedge \neg c) = \Delta \neg b \cap \Delta \neg c$$

iff $a \notin \Delta \neg b$ or $a \notin \Delta \neg c$ iff $(a, b) \in \delta_\Delta$ or $(a, c) \in \delta_\Delta$. Thus, δ_Δ is a generalized proximity relation between A and B . $\square$

By Theorem 1 we have that the notions of proximity relations and quasi-modal operators are interdefinable.

Definition 3. Let $\Delta \in QS[A, B]$. For each $C \subseteq A$ and for each $D \subseteq B$ define

$$(1) \quad \Delta C = I\left(\bigcup_{c \in C} \Delta c\right),$$

- (2) $\nabla C = F(\bigcup_{c \in C} \nabla c)$,
- (3) $\Delta^{-1}(D) = \{a \in A : \Delta a \cap D \neq \emptyset\}$,
- (4) $\nabla^{-1}(D) = \{a \in A : \nabla a \subseteq D\}$.
- (5) If $D = [a]$, we write $\Delta^{-1}(a)$ instead of $\Delta^{-1}([a])$.

In the following lemma we summarize some properties well known in the theory of Boolean algebras with proximity relations (see [8, 14, 19]). For completeness we will give some proofs.

Lemma 1. *Let $\Delta \in QS[A, B]$.*

- (1) Then $\Delta^{-1}(F) = \bigcup_{a \in F} \Delta^{-1}(a) \in \text{Fi}(A)$ for each $F \in \text{Fi}(B)$. Moreover, this union is directed.
- (2) If $P \in \text{Ul}(B)$, then $\nabla^{-1}(P)^c \in \text{Id}(A)$.
- (3) $\Delta I = \bigcup_{a \in I} \Delta a$ for each $I \in \text{Id}(A)$. Moreover, this union is directed.
- (4) $\Delta(I_1 \cap I_2) = \Delta(I_1) \cap \Delta(I_2)$ for all $I_1, I_2 \in \text{Id}(A)$.

Proof. (1) Let $F \in \text{Fi}(B)$. It is easy to see $\Delta^{-1}(F) \in \text{Fi}(A)$. Let $a \in A$. Then

$$\begin{aligned}
 \Delta a \cap F \neq \emptyset & \text{ iff } \exists b \in F (b \in \Delta a) \\
 & \text{ iff } \exists b \in F ([b] \cap \Delta a \neq \emptyset) \\
 & \text{ iff } \exists b \in F (a \in \Delta^{-1}([b]) = \Delta^{-1}(b)) \\
 & \text{ iff } \exists b \in F (a \in \bigcup_{b \in F} \Delta^{-1}(b)).
 \end{aligned}$$

In order to see that this union is directed suppose that $a, b \in F$. Then it is easy to see that $\Delta^{-1}(a) \cup \Delta^{-1}(b) \subseteq \Delta^{-1}(a \vee b)$, and as $a \vee b \in F$, we get that this union is directed.

(2) We prove that $\nabla^{-1}(P)^c \in \text{Id}(A)$, when $P \in \text{Ul}(B)$. Let $a \leq b$ and $a \in \nabla^{-1}(P)^c$. Then $\nabla a \not\subseteq P$, and as $\nabla b \subseteq \nabla a$, because ∇ is anti-monotonic, we have that $b \notin \nabla^{-1}(P)^c$. Thus $\nabla^{-1}(P)^c$ is decreasing. Let $a, b \in \nabla^{-1}(P)^c$. Then $\nabla a \not\subseteq P$ and $\nabla b \not\subseteq P$. Then there exist $p_1 \in \nabla a - P$ and $p_2 \in \nabla b - P$. So, $p_1 \vee p_2 \in \nabla a \vee \nabla b$, and as P is prime, $p_1 \vee p_2 \notin P$. Then, $p_1 \vee p_2 \in \nabla^{-1}(P)^c$. It is clear that $0 \in \nabla^{-1}(P)^c$, because $\nabla 0 = B$. Thus, $\nabla^{-1}(P)^c$ is an ideal of A .

(3) Let $I \in \text{Id}(A)$. We prove that $I(\bigcup_{c \in C} \Delta c) = \bigcup_{a \in I} \Delta a$. It is clear that $\bigcup_{a \in I} \Delta a \subseteq I(\bigcup_{c \in C} \Delta c)$. Let $c \in I(\bigcup_{c \in C} \Delta c)$. Then there exists $a_i \in I$, and there exists $x_i \in \Delta a_i$, with $1 \leq i \leq n$, such that $c \leq x_1 \vee \dots \vee x_n$. Since $\Delta a_i \subseteq \Delta(a_1 \vee \dots \vee a_n)$, for $1 \leq i \leq n$, then $x_1 \vee \dots \vee x_n \in \Delta(a_1 \vee \dots \vee a_n)$. As $a_1 \vee \dots \vee a_n \in I$, and $c \in \Delta(a_1 \vee \dots \vee a_n)$, we get that $c \in \bigcup_{a \in I} \Delta a$. Thus the union is directed.

(4). Let $I_1, I_2 \in \text{Id}(A)$. As Δ is monotonic, $\Delta(I_1 \cap I_2) \subseteq \Delta(I_1) \cap \Delta(I_2)$. Let $c \in \Delta(I_1) \cap \Delta(I_2)$. Then by item (3), there exist $a \in I_1$ and $b \in I_2$ such that $c \in \Delta a \cap \Delta b = \Delta(a \wedge b)$. As $a \wedge b \in I_1 \cap I_2$, we get that $c \in \Delta(I_1 \cap I_2)$. $\square$

Let A, B and C be Boolean algebras. Let $\Delta_1 \in QS[A, B]$ and $\Delta_2 \in QS[B, C]$. We define the composition of Δ_2 with Δ_1 . Recall that for each subset D of B ,

we can consider an ideal $\Delta_2(D) = \bigvee \{\Delta_2 b : b \in D\}$. Then, as for each $a \in A$, we consider the ideal $\Delta_2[\Delta_1(a)] \in \text{Id}(C)$. Then, define the composition of Δ_2 with Δ_1 , in symbols $\Delta_2 \circ \Delta_1$, as

$$(\Delta_2 \circ \Delta_1)(a) = \Delta_2[\Delta_1(a)],$$

for each $a \in A$. We need to prove that $\Delta_2 \circ \Delta_1 \in QS[A, C]$. In the following result we use the quasi-semi-homomorphism defined in Example 1.

Lemma 2. *Let A , B and C be Boolean algebras. Let $\Delta_1 \in QS[A, B]$ and $\Delta_2 \in QS[B, C]$. Then:*

- (1) $\Delta_2 \circ \Delta_1 \in QS[A, C]$.
- (2) $\Delta_1 \circ I_A = \Delta_1$ and $I_B \circ \Delta_2 = \Delta_2$.

Proof. (1) By (4) of Lemma 1 we get that

$$\begin{aligned} (\Delta_2 \circ \Delta_1)(a \wedge b) &= \Delta_2[\Delta_1(a \wedge b)] &= \Delta_2[\Delta_1(a) \cap \Delta_1(b)] \\ &= \Delta_2[\Delta_1 a] \cap \Delta_2[\Delta_1 b] &= (\Delta_2 \circ \Delta_1)(a) \cap (\Delta_2 \circ \Delta_1)(b). \end{aligned}$$

Moreover, $(\Delta_2 \circ \Delta_1)(1) = \Delta_2[\Delta_1 1] = \Delta_2[B] = A$. Thus, $\Delta_2 \circ \Delta_1$ is a quasi-semi-homomorphism.

(2) Let $a \in A$. Then $(\Delta_1 \circ I_A)(a) = \Delta_1[I_A a] = \Delta_1[a] = \Delta_1 a$. The proof of the identity $I_B \circ \Delta_2 = \Delta_2$ is similar. $\square$

Thus we can conclude that we have a category, denoted by **BoQS**, whose objects are Boolean algebras and whose morphism are quasi-semi-homomorphisms. In the next section we will prove that the category **BoQS** is dually equivalent to a category whose objects are Stone spaces, and whose morphism are a particular class of binary relations between Stone spaces.

In the following result we will characterize the isomorphisms (or iso-arrow) in the category **BoQS**. This result will be needed later.

Lemma 3. *Let A and B be Boolean algebras and $\Delta \in QS[A, B]$. Then the following conditions are equivalent:*

- (1) Δ is an iso-arrow in the category **BoQS**.
- (2) There exists an one to one and onto function $h : A \rightarrow B$ such that $\Delta a = (h(a))$, for each $a \in A$.

Proof. (1) $\Rightarrow$ (2) Since Δ is an iso-arrow in the category **BoQS**, there exists $\Pi \in QS[B, A]$ such that $\Delta \circ \Pi = I_B$ and $\Pi \circ \Delta = I_A$, where I_A and I_B are the quasi-semi-homomorphisms defined in Example 1. Let $a \in A$. Then $(\Pi \circ \Delta)(a) = I_A(a) = (a)$. As $(\Pi \circ \Delta)(a) = \Pi[\Delta a] = \bigcup \{\Pi b : b \in \Delta a\}$, there exists $b \in \Delta a$ such that $\Pi b = (a)$. We prove that b is unique. Suppose that there are $b_1, b_2 \in B$ such that $\Pi b_1 = \Pi b_2$. As $\Delta \circ \Pi = I_B$, we get

$$(b_1) = (\Delta \circ \Pi)(b_1) = \Delta[\Pi b_1] = \Delta[\Pi b_2] = (\Delta \circ \Pi)(b_2) = (b_2).$$

So, $b_1 = b_2$. Then for each $a \in A$ there exists a unique $b \in B$ such that $\Pi b = (a]$. So, we can consider the function $h : A \rightarrow B$ defined by:

$$h(a) = b \text{ iff } \Pi b = (a],$$

for each $a \in A$. We note that

$$\Pi(h(a)) = (a]. \quad (3.1)$$

Similarly we can prove that there exists a function $k : B \rightarrow A$ such that

$$k(b) = a \text{ iff } \Delta a = (b],$$

for each $b \in B$. Also, we note that

$$\Delta(k(b)) = (b]. \quad (3.2)$$

We prove that $k \circ h = Id_A$ and $h \circ k = Id_B$. Let $a \in A$. Then as $\Pi \circ \Delta = I_A$ we get that:

$$\begin{aligned} ((k \circ h)(a)) &= \Pi[\Delta(k(h(a)))] = \Pi[\Delta(k \circ h)(a)] \stackrel{(3.2)}{=} \Pi(h(a)) \\ &= \Pi h(a) \stackrel{(3.1)}{=} (a] \end{aligned}$$

(2) $\Rightarrow$ (1) Assume that there exists an one to one and onto function $h : A \rightarrow B$ such that $\Delta a = (h(a)]$, for each $a \in A$. So, there exists an one to one and onto function $g : B \rightarrow A$ such that $(g \circ h)(a) = a$ for all $a \in A$, and $(h \circ g)(b) = b$ for all $b \in B$. Consider the quasi-semi-homomorphism $\Pi : B \rightarrow \text{Id}(A)$ defined by $\Pi(b) = (g(b)]$. Then we prove that $\Delta \circ \Pi = I_B$ and $\Pi \circ \Delta = I_A$. We prove that $(\Delta \circ \Pi)(b) = (b]$. Let $b, d \in B$ such that $d \in (\Delta \circ \Pi)(b) = \Delta[\Pi b] = \bigcup \{\Delta c : c \in \Pi b = (g(b)]\}$. So, there exists $c \in A$ and $d \in B$ such that $c \leq g(b)$ and $d \in \Delta c = (h(c)]$. So, $d \leq h(c)$, and thus $d \leq h(c) \leq h(g(b)) = b$, i.e., $c \in (b]$. So, $(\Delta \circ \Pi)(b) \subseteq (b]$. The other inclusion it is left to the reader. Thus, $\Delta \circ \Pi = I_B$. Similarly we can prove that $\Pi \circ \Delta = I_A$. Therefore, Δ is an iso-arrow in the category **BoQS**. $\square$

4. GENERALIZED QUASI-BOOLEAN RELATIONS

Let X and Y be two topological spaces. Let $R \subseteq X \times Y$ be a relation. We shall say that R is *upper-semi-continuous* (u.s.c) if $\Delta_R(O) = \{x \in X : R(x) \subseteq O\}$ is an open subset of X for every open subset O of Y . We note that $\Delta_R(O)$ is open for each open O of Y iff $\nabla_R(C) = \{x \in X : R(x) \cap C \neq \emptyset\}$ is a closed of X for each closed C of Y . We shall say that R is *point-compact* (*point-closed*) if $R(x)$ is a compact (closed) subset of Y , for each $x \in X$. Clearly, if Y is a compact space, a relation $R \subseteq X \times Y$ is point-compact iff it is point-closed.

Lemma 4. *Let X and Y be two topological space. Suppose that Y is zero-dimensional space. Let R be a point-compact relation. Then the following conditions are equivalent:*

- (1) R is upper-semi-continuous,

(2) $\Delta_R(U) \in \mathcal{O}(X)$, for each $U \in \text{Clo}(Y)$.

Proof. If R is upper-semi-continuous, then $\Delta_R(U) \in \text{Clo}(X)$, for each $U \in \text{Clo}(Y)$, because $\text{Clo}(Y) \subseteq \mathcal{O}(Y)$.

Conversely. Assume that $\Delta_R(U) \in \mathcal{O}(X)$, for each $U \in \text{Clo}(Y)$. Let $O \in \mathcal{O}(Y)$. As Y is zero-dimensional, $\text{Clo}(Y)$ is a basis, so $O = \bigcup \{U_i \in \text{Clo}(Y) : U_i \subseteq O\}$. Since Δ_R is monotonic,

$$\bigcup \{\Delta_R(U_i) : U_i \subseteq O\} \subseteq \Delta_R(\bigcup \{U_i \in \text{Clo}(Y) : U_i \subseteq O\}) = \Delta_R(O).$$

We prove the other inclusion. Let $x \in \Delta_R(\bigcup \{U_i \in \text{Clo}(Y) : U_i \subseteq O\})$, i.e., $R(x) \subseteq \bigcup \{U_i \in \text{Clo}(Y) : U_i \subseteq O\}$. As $R(x)$ is a compact subset of Y , there exists a finite family $\{U_i \subseteq O : 1 \leq i \leq n\}$ such that $R(x) \subseteq U_1 \cup \dots \cup U_n = U \subseteq O$, i.e., $x \in \Delta_R(U)$. Thus, $\bigcup \{\Delta_R(U_i) : U_i \subseteq O\} = \Delta_R(O)$. Consequently $\Delta_R(O)$ is an open subset of X , because $\Delta_R(U_i) \in \text{Clo}(X)$, for each $U_i \in \text{Clo}(Y)$. $\square$

Remark 1. By the previous Lemma, when X and Y are Stone spaces, and $R \subseteq X \times Y$, we have that the following conditions are equivalent:

- (1) R is a point-compact relation and $\Delta_R(O) \in \mathcal{O}(X)$, for each $O \in \mathcal{O}(Y)$.
- (2) R is a point-closed relation and $\Delta_R(U) \in \mathcal{O}(X)$, for each $U \in \text{Clo}(Y)$.

Definition 4. Let X and Y be two Stone spaces. We shall say that a binary relation $R \subseteq X \times Y$ is a *quasi-Boolean relation* if

- (1) R is a point-closed relation,
- (2) $\Delta_R(U) \in \mathcal{O}(X)$, for each $U \in \text{Clo}(Y)$.

If $\Delta_R(U) \in \text{Clo}(X)$, for each $U \in \text{Clo}(Y)$, then R is called a *Boolean relation* [12], also called a Boolean correspondence in [11]. It is clear that every Boolean relation is a quasi-Boolean relation.

Remark 2. Let X be a Stone space. A pair $\langle X, R \rangle$, where R is a quasi-Boolean relation defined in X is called a *quasi-modal space*. The quasi-modal spaces are the dual objects of the quasi-modal algebras (see [3] and [2]). If R is a Boolean relation, then the pair $\langle X, R \rangle$ is called a *modal space or descriptive general frame* [10, 16]. The modal spaces are the dual of the modal algebras, i.e., pairs $\langle A, \Box \rangle$, where A is a Boolean algebra and $\Box$ is a modal operator.

Given a Stone space X , the map $\varepsilon_X : X \rightarrow \text{Ul}(\text{Clo}(X))$ defined by

$$\varepsilon_X(x) = \{U \in \text{Clo}(X) : x \in U\}$$

, is a bijective and continuous function. Thus, for each $P \in \text{Ul}(\text{Clo}(X))$ there exists a unique $x \in X$ such that $\varepsilon_X(x) = P$.

Let X and Y be two Stone spaces. Let $R \subseteq X \times Y$ be a relation. For each $x \in X$ we can consider the set

$$\Delta_R^{-1}(\varepsilon_X(x)) = \{U \in \text{Clo}(X_2) : x \in \Delta_R(U)\} = \{U \in \text{Clo}(X_2) : R(x) \subseteq U\}.$$

We define the relation $R_{\Delta_R} \subseteq \text{Ul}(\text{Clo}(X)) \times \text{Ul}(\text{Clo}(Y))$, as follows:

$$(\varepsilon_X(x), \varepsilon_Y(y)) \in R_{\Delta_R} \text{ iff } \Delta_R^{-1}(\varepsilon_X(x)) \subseteq \varepsilon_Y(y).$$

In the following Lemma we shall give an equivalent condition to the condition (1) of Definition 4.

Lemma 5. *Let X_1 and X_2 be two Stone spaces. Let $R \subseteq X_1 \times X_2$ be a relation. Suppose that $\Delta_R(U)$ is an open subset of X_1 , for each $U \in \text{Clo}(X_2)$. Then the following conditions are equivalent*

- (1) $R(x)$ is a closed subset of X_2 , for each $x \in X_1$,
- (2) $(x, y) \in R$ iff $(\varepsilon_1(x), \varepsilon_2(y)) \in R_{\Delta_R}$.

Proof. (1) implies (2). Let $x, y \in X_1$. It is clear that if $(x, y) \in R$ then $(\varepsilon_1(x), \varepsilon_2(y)) \in R_{\Delta_R}$. Suppose that $y \notin R(x)$. As $R(x)$ is a closed subset of X_2 , there exists $U \in \text{Clo}(X_2)$ such that $y \notin U$ and $R(x) \subseteq U$. So, $x \in \Delta_R(U)$. Then $U \in \Delta_R^{-1}(\varepsilon_1(x))$ and $U \not\subseteq \varepsilon_2(y)$, i.e., $(\varepsilon_1(x), \varepsilon_2(y)) \notin R_{\Delta_R}$.

(2) implies (1). We prove that $\text{cl}(R(x)) = R(x)$. Suppose that there exists $y \in \text{cl}(R(x))$ but $y \notin R(x)$. Then $(\varepsilon_1(x), \varepsilon_2(y)) \notin R_{\Delta_R}$, i.e., there exists $U \in \text{Clo}(X_2)$ such that $U \in \Delta_R^{-1}(\varepsilon_1(x))$ and $U \not\subseteq \varepsilon_2(y)$. Then $x \in \Delta_R(U)$ and $y \notin U$, i.e., $R(x) \subseteq U$ and $y \notin U$. So, $y \notin \text{cl}(R(x))$, which is a contradiction. Thus, $\text{cl}(R(x)) \subseteq R(x)$, and consequently $R(x)$ is a closed subset of X_2 . $\square$

Let X and Y be two Stone spaces. By Lemma 5 we have that a relation $R \subseteq X_1 \times X_2$ is a quasi-Boolean relation iff R satisfies the following conditions:

- (1) $(x, y) \in R$ iff $(\varepsilon_1(x), \varepsilon_2(y)) \in R_{\Delta_R}$,
- (2) $\Delta_R(U) \in \mathcal{O}(X)$, for each $U \in \text{Clo}(Y)$.

We denote by $QB[X, Y]$ the set of all quasi-Boolean relations between two Stone spaces X and Y .

Lemma 6. *Let X and Y be Stone spaces. Let $R \in QB[X, Y]$. Then $R[C]$ is a closed subset of Y for each closed subset C of X .*

Proof. Let C be a closed subset of X . We note that $R[C] = \bigcup \{R(x) : x \in C\}$. It suffices to prove that for any $y \notin R[C]$ there exists $U \in \text{Clo}(Y)$ such that $R[C] \subseteq U$ and $y \notin U$. Take $y \notin R[C]$. Then $y \notin R(x)$ for each $x \in C$. As R is point-closed, for each $x \in C$ there exists $U_x \in \text{Clo}(Y)$ such that $R(x) \subseteq U_x$ and $y \notin U_x$. So, $x \in \Delta_R(U_x)$, for each $x \in C$. Thus, $C \subseteq \bigcup \{\Delta_R(U_x) : x \in X\}$, and as C is compact, there exists $x_1, \dots, x_n \in C$ such that

$$C \subseteq \Delta_R(U_{x_1}) \cup \dots \cup \Delta_R(U_{x_n}) \subseteq \Delta_R(U_{x_1} \cup \dots \cup U_{x_n}) = \Delta_R(U),$$

i.e., $R[C] \subseteq U$. Therefore there exists $U \in \text{Clo}(Y)$ such that $y \notin U$ and $R[C] \subseteq U$. $\square$

Lemma 7. *If $R \in QB[X, Y]$, then $\bar{\Delta}_R \in QS[\text{Clo}(Y), \text{Clo}(X)]$.*

Proof. If $R \in QB[X, Y]$, then by Example 3 it is clear that $\bar{\Delta}_R : \text{Clo}(Y) \rightarrow \text{Id}(\text{Clo}(X))$ is generalized quasi-semi-homomorphism. Thus, $\bar{\Delta}_R \in QS[\text{Clo}(Y), \text{Clo}(X)]$. $\square$

Let X, Y and Z be Stone spaces. Let $R \in QB[X, Y]$, and $S \in QB[Y, Z]$. The composition of R with S is the relation

$$R \circ S = \{(x, z) \in X \times Z : \exists y \in Y [(x, y) \in R \text{ and } (y, z) \in S]\}.$$

We note that $(R \circ S)(x) = S[R(x)] = \bigcup \{S(y) : y \in R(x)\}$.

Lemma 8. *The composition of quasi-Boolean relations is a quasi-Boolean relation.*

Proof. Let X, Y and Z be Stone spaces. Let $R \in QB[X, Y]$ and $S \in QB[Y, Z]$. We prove that $R \circ S \subseteq X \times Z$ is point-closed. Let $x \in X$. As $R(x)$ is a closed subset of Y , by Lemma 6, we get that $S[R(x)] = (S \circ R)(x)$ is a closed subset of Z .

We prove that $(\Delta_R \circ \Delta_S)(U) = \Delta_{R \circ S}(U)$, for each $U \in \text{Clo}(Z)$. Let $x \in (\Delta_R \circ \Delta_S)(U) = \Delta_R(\Delta_S(U))$, i.e., $R(x) \subseteq \Delta_S(U)$. Let $z \in (R \circ S)(x) = S[R(x)] = \bigcup \{S(y) : y \in R(x)\}$. Then there exists $y \in R(x)$ such that $z \in S(y)$. As $R(x) \subseteq \Delta_S(U)$, $y \in \Delta_S(U)$, i.e., $S(y) \subseteq U$. So, $z \in U$. Thus, $(\Delta_R \circ \Delta_S)(U) \subseteq \Delta_{R \circ S}(U)$.

Let $x \in \Delta_{R \circ S}(U)$. Then $(R \circ S)(x) = S[R(x)] = \bigcup \{S(y) : y \in R(x)\} \subseteq U$, i.e., $S(y) \subseteq U$, for all $y \in R(x)$. So, $y \in \Delta_S(U)$ for all $y \in R(x)$, i.e., $R(x) \subseteq \Delta_S(U)$. Then $x \in \Delta_R(\Delta_S(U)) = (\Delta_R \circ \Delta_S)(U)$. Thus, $\Delta_{R \circ S}(U) \subseteq (\Delta_R \circ \Delta_S)(U)$. $\square$

Let $f : X \rightarrow Y$ be a function between two Stone Spaces. Consider the relation $f^* \subseteq X \times Y$ defined by

$$f^* = \{(x, y) \in X \times Y : f(x) = y\}.$$

Lemma 9. *Let X and Y be two Stone spaces. If $f : X \rightarrow Y$ is a function such that $f^{-1}(U)$ is an open subset of X for each $U \in \text{Clo}(Y)$, then $f^* \in QB[X, Y]$.*

Proof. It is clear that $\Delta_{f^*}(U) = \{x : f^*(x) \subseteq U\} = \{x : x \subseteq f^{-1}(U)\} = f^{-1}(U)$. Thus, $\Delta_{f^*}(U)$ is an open subset of X for each $U \in \text{Clo}(Y)$. Also, as Y is a Stone Space, we have $f^*(x)$ is a closed subset of Y , for each $x \in X$. Thus, $f^* \in QB[X, Y]$. $\square$

Using the previous lemma we obtain the following result.

Corollary 1. *Let X be a Stone space. Consider the $\varepsilon_X : X \rightarrow \text{Ul}(\text{Clo}(X))$. Then the relation $\varepsilon_X^* \subseteq X \times \text{Ul}(\text{Clo}(X))$ given by $(x, P) \in \varepsilon_X^*$ iff $\varepsilon_X(x) = P$ is a generalized quasi-Boolean relation.*

By Lemma 8 we conclude that the Stone spaces with generalized quasi-Boolean relations is a category, denoted by **StQB** where the identity morphism is the identity map Id_X , where X is a Stone space. The careful reader may have realized that the notation of composition of relations reverses the order of the actual composition in

the category. We have decided to preserve this usual notation, instead of giving a new one, in order to make the paper more readable.

In the following result we characterized the isomorphisms (or iso-arrow) in the category **StQB**.

Lemma 10. *Let X and Y be Stone spaces and $R \in QB[X, Y]$. Then the following conditions are equivalent:*

- (1) R is an iso-arrow in the category **StQB**.
- (2) There exists an one-to-one and onto function $f : X \rightarrow Y$ such that $R = f^*$, satisfying the condition $f^{-1}(U)$ is an open set for each $U \in \text{Clo}(Y)$. i.e., f is a continuous function between the Stone spaces X and Y .

Proof. (1) $\Rightarrow$ (2) Let $S \in QB[Y, X]$ such that $R \circ S = Id_X^*$ and $R \circ S = Id_Y^*$, where Id_Y^* and Id_X^* are quasi-Boolean relations corresponding to the functions Id_X and Id_Y , respectively (see Lemma 9). Then for every $x \in X$, $(R \circ S)(x) = S(R(x)) = Id_X^*(x) = \{x\}$. Using the fact that $S \circ Id_X^* = S$, and $x \in S(R(x))$, then there exists $y \in R(x)$ such that $S(y) = \{x\}$. We prove that y is unique. Suppose that there are $y_1, y_2 \in Y$ such that $S(y_1) = S(y_2)$. As $S \circ R = Id_Y^*$, we get

$$\{y_1\} = (S \circ R)(y_1) = R(S(y_1)) = R(S(y_2)) = (S \circ R)(y_2) = \{y_2\}.$$

Thus, $y_1 = y_2$. So we conclude that for each $x \in X$ there exists a unique $y \in Y$ such that $S(y) = \{x\}$. Let us denote by $f : X \rightarrow Y$ the function defined by

$$f(x) = y \text{ iff } S(y) = \{x\},$$

for each $x \in X$. Similarly we can prove that there exists a function $g : Y \rightarrow X$ such that $R(g(y)) = \{y\}$, for each $y \in Y$. As $Id_Y^* \circ S = S$, we get that $(Id_Y^* \circ S)(y) = S(Id_Y^*(y)) = S(\{y\}) = S(y)$, for each $y \in Y$. Then for every $x \in X$ we have that $\{g(f(x))\} = (R \circ S)(g(f(x))) = S(R(g(f(x)))) = S(\{f(x)\}) = \{x\}$. So, $g(f(x)) = x$, for each $x \in X$, i.e., $g \circ f$ is the identity function on X .

Changing the roles of f and g , we obtain that $f \circ g$ is the identity function on Y . We conclude that f is a one to one map from X onto Y and g is its inverse. Observe that $R(x) = R(g(f(x))) = \{f(x)\}$. Then $R = f^*$. Similarly, we have that $S = g^*$.

Consider now $U \in \text{Clo}(Y)$. Since R is a quasi-Boolean relation, we have that

$$\begin{aligned} \Delta_R(U) &= \{x \in X : R(x) \subseteq U\} = \{x \in X : f^*(x) \subseteq U\} \\ &= \{x \in X : \{f(x)\} \subseteq U\} = \{x \in X : f(x) \in U\} \\ &= f^{-1}(U). \end{aligned}$$

So, $f^{-1}(U)$ is an open subset of X .

The direction (2) $\Rightarrow$ (1) follows straightforward from Lemma 9 and the definition of iso-arrow. $\square$

5. CATEGORICAL DUALITY

In this section we prove that there exists a category, whose object are Boolean algebras, and whose morphism are quasi-Boolean relations. In order to complete the duality we need to see how to define quasi-Boolean relation from each quasi-semi-homomorphism between two Boolean algebras.

Let $\Delta \in QS[A, B]$. We define a relation $R_\Delta \subseteq \text{Ul}(B) \times \text{Ul}(A)$ by

$$\begin{aligned} (P, Q) \in R_\Delta &\Leftrightarrow \forall a \in A : \Delta a \cap P \neq \emptyset \text{ then } a \in Q \\ &\Leftrightarrow \Delta^{-1}(P) \subseteq Q. \end{aligned}$$

We note that when $A = B$, the relation R_Δ is the relation used in [3] in the representation of quasi-modal algebras.

We give now a equivalent characterization for the relation R_Δ . We recall that, given $\Delta \in QS[A, B]$, the generalized proximity relation $\delta_\Delta \subseteq B \times A$ is defined as $(b, a) \in \delta_\Delta$ iff $b \notin \Delta \neg a$.

Lemma 11. *Let A and B be two Boolean algebras. Let $\Delta \in QS[A, B]$. Let $(P, Q) \in \text{Ul}(B) \times \text{Ul}(A)$. Then the following conditions are equivalent:*

- (1) $(Q, P) \in R_\Delta$,
- (2) $Q \times P \subseteq \delta_\Delta$

Proof. Let $(P, Q) \in \text{Ul}(B) \times \text{Ul}(A)$. Assume that $(Q, P) \in R_\Delta$. Let $(q, p) \in Q \times P$. If $(q, p) \notin \delta_\Delta$, then $q \in \Delta \neg p \cap Q$, i.e., $\neg p \in \Delta^{-1}(Q) \subseteq P$. So, $\neg p \wedge p = 0 \in P$, which is a contradiction. Thus, $Q \times P \subseteq \delta_\Delta$.

Assume that $Q \times P \subseteq \delta_\Delta$. Let $\Delta a \cap Q \neq \emptyset$. Then there exist $q \in \Delta a$ and $q \in Q$. Suppose that $a \notin P$. Then $\neg a \in P$. So, $(q, \neg a) \in Q \times P \subseteq \delta_\Delta$, i.e., $q \notin \Delta \neg \neg a = \Delta a$, which is a contradiction. Thus, $(Q, P) \in R_\Delta$. $\square$

Remark 3. When $A = B$, the relation given in (2) is the definition used in [14] for the topological representation of some extensions of proximity Boolean algebras.

Lemma 12. *Let $\Delta \in QS[A, B]$. Let $P \in \text{Ul}(B)$ and $I \in \text{Id}(A)$. Then*

$$\Delta I \cap P = \emptyset \Leftrightarrow \exists Q \in \text{Ul}(A) [\Delta^{-1}(P) \subseteq Q \text{ and } I \cap Q = \emptyset].$$

Proof. Let $P \in \text{Ul}(B)$ and $I \in \text{Id}(A)$. We note that $\Delta I \cap P = \emptyset$ iff $I \cap \Delta^{-1}(P) = \emptyset$. Indeed. Suppose that $\Delta I \cap P = \emptyset$ and suppose that there exists $a \in I \cap \Delta^{-1}(P)$. Then $\Delta a \cap P \neq \emptyset$, i.e., there exist $p \in P$ and $p \in \Delta a$. As $a \in I$, we get that $p \in \Delta I \cap P$, which is a contradiction. Thus $I \cap \Delta^{-1}(P) = \emptyset$. The other direction is similar and left to the reader.

Assume that $I \cap \Delta^{-1}(P) = \emptyset$. Consider the family

$$\mathcal{F} = \{H \in \text{Fi}(A) : I \cap H = \emptyset \text{ and } \Delta^{-1}(P) \subseteq H\}.$$

As $\Delta^{-1}(P)$ is a filter of A and $\Delta^{-1}(P) \in \mathcal{F}$, then $\mathcal{F} \neq \emptyset$. By Zorn's Lemma, we can take a maximal $Q \in \mathcal{F}$. It remains to show that Q is an ultrafilter of A . Let

$a \in A$. Take the filters $F_a = F(Q \cup \{a\})$ and $F_{\neg a} = F(Q \cup \{\neg a\})$. If $a, \neg a \notin Q$, then $F_a, F_{\neg a} \notin \mathcal{F}$. So, $F_a \cap I \neq \emptyset$ and $F_{\neg a} \cap I \neq \emptyset$. Then there exists $q_1, q_2 \in Q$ such that $q_1 \wedge a \in I$ and $q_2 \wedge \neg a \in I$. Take $q = q_1 \wedge q_2$. As I is an ideal of A , we have that $(q \wedge a) \vee (q \wedge \neg a) = q \wedge (a \vee \neg a) = q \wedge 1 = q \in I$, which is a contradiction. Thus, $Q \in \text{Ul}(A)$. So, $\Delta^{-1}(P) \subseteq Q$ and $Q \cap I = \emptyset$. The other direction it is easy and left to the reader. $\square$

Theorem 2. Let $\Delta \in QS[A, B]$. Let $a \in A$ and $P \in \text{Ul}(B)$. Then

- (1) $a \in \Delta^{-1}(P) \Leftrightarrow \forall Q \in \text{Ul}(A) : \Delta^{-1}(P) \subseteq Q \text{ then } a \in Q$,
- (2) $a \in \nabla^{-1}(P) \Leftrightarrow \exists Q \in \text{Ul}(A) : Q \subseteq \nabla^{-1}(P) \text{ and } a \in Q$.

Proof. We prove (1). The proof of (2) follows by duality. Assume that $a \notin \Delta^{-1}(P)$, i.e., $\Delta a \cap P = \emptyset$. By Lemma 12 we get that there exists $Q \in \text{Ul}(A)$ such that $\Delta^{-1}(P) \subseteq Q$ and $a \notin Q$. The other direction is immediate. $\square$

Recall that if I is an ideal of a Boolean algebra B , then

$$\varphi_B[I] = \{P \in \text{Ul}(B) : I \cap P \neq \emptyset\}$$

is an open subset of the Stone space of B .

Theorem 3. Let A and B two Boolean algebras. Let $\Delta \in QS[A, B]$. Then

- (1) $\varphi_B[\Delta a] = \Delta_{R_\Delta}(\beta_A(a))$, for all $a \in A$.
- (2) $R_\Delta \in \text{QB}[\text{Ul}(B), \text{Ul}(A)]$.

Proof. (1) Let $a \in A$. Let $P \in \Delta_{R_\Delta}(\beta_A(a))$. Then $R_\Delta(P) \subseteq \beta_A(a)$. If $P \notin \varphi_B[\Delta a]$, then $\Delta a \cap P = \emptyset$. So, there exists $Q \in R_\Delta(P)$ such that $a \notin Q$. Then, $R_\Delta(P) \not\subseteq \beta_A(a)$, which is a contradiction. Thus, $P \in \varphi_B[\Delta a]$. The other inclusion is easy and left to the reader. Thus, $\Delta_{R_\Delta}(\beta_A(a))$ is an open subset.

(2) By Theorem 2 we deduce that $R_\Delta(P) = \bigcap \{\beta_A(a) : \Delta a \cap P \neq \emptyset\}$. Therefore, $R_\Delta(P)$ is a closed subset for each $P \in \text{Ul}(A)$, i.e., R_Δ is point-closed. $\square$

We recall that if $\Delta_1 \in QS[A, B]$ and $\Delta_2 \in QS[B, C]$, then $\Delta_2 \circ \Delta_1 \in QS[A, C]$. Thus, $R_{\Delta_2 \circ \Delta_1} \subseteq \text{Ul}(C) \times \text{Ul}(A)$.

Lemma 13. Let A, B and C be Boolean algebras. Let $\Delta_1 \in QS[A, B]$ and $\Delta_2 \in QS[B, C]$. Then $R_{\Delta_2 \circ \Delta_1} = R_{\Delta_2} \circ R_{\Delta_1}$.

Proof. Let $(P, Q) \in \text{Ul}(C) \times \text{Ul}(B)$ such that $(P, Q) \in R_{\Delta_2 \circ \Delta_1}$. Then $(\Delta_2 \circ \Delta_1)^{-1}(P) \subseteq Q$, i.e., for all $a \in A$ such that $(\Delta_2 \circ \Delta_1)(a) \cap P \neq \emptyset$, then $a \in Q$. We note that as $Q^c = A - Q$ is an ideal, we have that $\Delta_1(Q^c)$ is an ideal. We prove that

$$\Delta_2^{-1}(P) \cap \Delta_1(Q^c) = \emptyset.$$

Otherwise there exists $a \notin Q$ and $b \in B$ such that $\Delta_2 b \cap P \neq \emptyset$ and $b \in \Delta_1 a$. So, there exists $c \in \Delta_2 b \cap P$. Then, $c \in (\Delta_2 \circ \Delta_1)(a) \cap P$, i.e., $a \in (\Delta_2 \circ \Delta_1)^{-1}(P)$. Thus, $a \in Q$, which is a contradiction. Thus, there exists $D \in \text{Ul}(B)$ such that

$\Delta_2^{-1}(P) \subseteq D$ and $\Delta_1^{-1}(D) \subseteq Q$, i.e., $(P, D) \in R_{\Delta_2}$ and $(D, Q) \in R_{\Delta_1}$. Therefore, $(P, Q) \in R_{\Delta_2 \circ \Delta_1}$.

To prove the other inclusion, let $(P, Q) \in R_{\Delta_2 \circ \Delta_1}$. Then there exists $D \in \mathbf{Ul}(B)$ such that $\Delta_2^{-1}(P) \subseteq D$ and $\Delta_1^{-1}(D) \subseteq Q$. Let $a \in A$ such that $(\Delta_2 \circ \Delta_1)(a) \cap P = \Delta_2[\Delta_1(a)] \cap P \neq \emptyset$. Then there exists $b \in B$ and there exists $c \in C$ such that $b \in \Delta_1 a$ and $c \in \Delta_2 b \cap P$. So, $b \in \Delta_2^{-1}(P)$. So, $b \in \Delta_1 a \cap D$, and consequently $a \in Q$. Thus, $(\Delta_2 \circ \Delta_1)^{-1}(P) \subseteq Q$, i.e., $(P, Q) \in R_{\Delta_2 \circ \Delta_1}$. $\square$

Define a contravariant functor $\alpha : \mathbf{BoQS} \rightarrow \mathbf{StQB}$ by

$$\alpha(A) = \langle \mathbf{Ul}(A), \tau_A \rangle \quad \text{if } A \text{ is a Boolean algebra}$$

$$\alpha(\Delta) = R_\Delta \quad \text{if } \Delta \in QS[A, B].$$

Define a contravariant functor $\eta : \mathbf{StQB} \rightarrow \mathbf{BoQS}$ as

$$\eta(X) = \text{Clo}(X) \quad \text{if } X \text{ is a Boolean space}$$

$$\eta(R) = \Delta_R \quad \text{if } R \in QB[X, Y].$$

Since for each Boolean algebra A the map $\beta_A : A \rightarrow \text{Clo}(\mathbf{Ul}(A))$ is an isomorphism in $\mathbf{BoQS}$, we get that Theorem 3 means that the composite functor $\eta \circ \alpha$ is naturally equivalent to the identity functor, the natural equivalence being given by the isomorphisms β_A . On the other hand, since for each Stone space X , the map ε_X is a homeomorphism from X onto $X(\text{Clo}(X))$, it follows that the relation ε_X^* defined by

$$(x, P) \in \varepsilon_X^* \text{ iff } \varepsilon_X(x) = P$$

is a quasi-Boolean relation, and by Lemma 5 we have that ε_X^* is an isomorphism in $\mathbf{StQB}$. It is easy to see ε_X^* is a natural equivalence from the composite functor $\eta \circ \alpha$ to the identity functor from in $\mathbf{StQB}$, i.e., $R_{\Delta_R} \circ \varepsilon_X^* = R \circ \varepsilon_Y^*$ for $R \in QS[X, Y]$. Similarly, it is easy to see that β_A is a natural equivalence between the identity functor in $\mathbf{BoQS}$ and $\alpha \circ \eta$. Thus, we have the following result.

Theorem 4. *The contravariant functors η and α and the natural equivalences ε and β define a dual equivalence between the category of Boolean algebras with quasi-semi-homomorphisms and the category of Stone spaces with quasi-Boolean relations.*

As an application of the above duality we prove a generalization of the result that asserts that the Boolean homomorphisms are the minimal elements in the set of all join-homomorphisms between two Boolean algebras (see [11]). Now we prove that the minimal elements in the set of all quasi-Boolean relations defined between two Stone spaces is a Boolean relation.

Let A and B be two Boolean algebras. Let X and Y be the Stone spaces of A and B , respectively. Let $QS[X, Y]$ the set of all quasi-Boolean relations defined between X and Y endowed with the order given by the inclusion between relations. Let Δ_1

and $\Delta_2 \in QS[A, B]$ and let R_{Δ_1} and $R_{\Delta_2} \in QS[X, Y]$ the associated quasi-Boolean relations. It is clear that $\Delta_1 \leq \Delta_2$ if and only if $R_{\Delta_1} \subseteq R_{\Delta_2}$.

Theorem 5. *Let X and Y be two Stone spaces. An element of $QS[X, Y]$ is minimal if and only if it is a Boolean relation.*

Proof. Let $R \subseteq X \times Y$ be a minimal element $QS[X, Y]$. We prove that R is a Boolean relation. As R is point-closed, we have to see that $\Delta_R(U)$ is a closed subset of X , for each $U \in \text{Clo}(Y)$. Let $x \in \text{cl}(\Delta_R(U))$. Suppose that $x \notin \Delta_R(U)$. Then $R(x) \not\subseteq U$. So there exists $y \in R(x)$ such that $y \notin U$. Define the relation R_U as:

$$R_U(z) = R(z) \cap U^c,$$

for each $z \in X$. It is clear that $R_U(z)$ is a closed subset for each $z \in X$. Thus R_U is point-closed. Moreover, for $V \in \text{Clo}(Y)$ we have that

$$\begin{aligned} \Delta_{R_U}(V) &= \{z \in X : R_U(z) \subseteq V\} = \{z \in X : R(z) \cap U^c \subseteq V\} \\ &= \{z \in X : R(z) \subseteq U \cup V\} = \{z \in X : z \in \Delta_R(U \cup V)\} \\ &= \Delta_R(U \cup V). \end{aligned}$$

Since R is a quasi-Boolean relation, $\Delta_R(U \cup V)$ is an open subset of X . Then R_U is a quasi-Boolean relation. It is clear that $R_U \subset R$. Thus R is not minimal element in $QS[X, Y]$, which is a contradiction. Therefore, $\text{cl}(\Delta_R(U)) = \Delta_R(U)$, i.e., $\Delta_R(U)$ is a closed subset of X . Consequently, R is a Boolean relation. $\square$

FINAL REMARKS

In this paper we have proved a generalization of the Halmos's duality [12] [11], and the duality given in [3] for quasi-modal algebras.

There are several possibilities to extend the results given in this work. One possibility is to consider local Boolean algebras with a special class of morphisms. We recall that a local Boolean algebra is a pair of the form $\langle A, I \rangle$, where A is a Boolean algebra and I is an ideal of A , such that $[I] = A$. A local homomorphism between two local algebras $\langle A, I \rangle$ and $\langle B, J \rangle$ is a Boolean homomorphism $h : A \rightarrow B$ satisfying the following condition:

(LH): For each $b \in J$ there exists $a \in I$ such that $b \leq h(a)$, i.e., $J \subseteq (h[I])$.

A meaningful extension of the Stone duality is given by Geogi Dimov in [6]. In this paper it is shown that the category of local Boolean algebras with local homomorphism is dually equivalent to the category of Boolean spaces (= zero-dimensional locally compact Hausdorff spaces) with continuous maps. In a future work we shall study local Boolean algebras with meet-homomorphisms satisfying the condition **(LH)**, and the representation theory by means of Stone spaces with a relation satisfying certain conditions.

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Author's address

Sergio A. Celani

Departamento de Matemática, Facultad de Ciencias Exactas, UNCPBA, 7000 Tandil, Argentina

E-mail address: `scelani@exa.unicen.edu.ar`



THE INCOMPLETE SRIVASTAVA'S TRIPLE HYPERGEOMETRIC FUNCTIONS γ_A^H AND Γ_A^H

JUNESANG CHOI AND RAKESH K. PARMAR

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Abstract. Motivated mainly by certain interesting recent extensions of the generalized hypergeometric function [15], the second Appell function [6] and Srivastava's triple hypergeometric functions [9], we introduce here the family of incomplete Srivastava's triple hypergeometric functions γ_A^H and Γ_A^H . We then systematically investigate several properties of each of these incomplete Srivastava's triple hypergeometric functions including, for example, their various integral representations, transformation formula, reduction formula, derivative formula and recurrence relations. Various (known or new) special cases and consequences of the results presented here are also considered.

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

The familiar *incomplete Gamma functions* $\gamma(s, x)$ and $\Gamma(s, x)$ defined by

$$\gamma(s, x) := \int_0^x t^{s-1} e^{-t} dt \quad (\Re(s) > 0; x \geq 0) \quad (1.1)$$

and

$$\Gamma(s, x) := \int_x^\infty t^{s-1} e^{-t} dt \quad (x \geq 0; \Re(s) > 0 \text{ when } x = 0), \quad (1.2)$$

respectively, satisfy the following decomposition formula:

$$\gamma(s, x) + \Gamma(s, x) := \Gamma(s) \quad (\Re(s) > 0). \quad (1.3)$$

Each of these functions plays an important role in the study of the analytic solutions of a variety of problems in diverse areas of science and engineering (see, e.g., [1, 4, 7, 16, 17, 23]).

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}^-$ and $\mathbb{C}$ denote the sets of positive integers, negative integers and complex numbers, respectively,

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad \text{and} \quad \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}.$$

Moreover, the parameter $x \geq 0$ used above in (1.1) and (1.2) and elsewhere in this paper is independent of $\Re(z)$ of the complex number $z \in \mathbb{C}$.

Recently, Srivastava *et al.* [15] introduced and studied in a rather systematic manner the following two families of generalized incomplete hypergeometric functions:

$${}_p\gamma_q \left[\begin{matrix} (\alpha_1, x), \alpha_2, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1; x)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \quad (1.4)$$

and

$${}_p\Gamma_q \left[\begin{matrix} (\alpha_1, x), \alpha_2, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{[\alpha_1; x]_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}, \quad (1.5)$$

where, in terms of the incomplete Gamma functions $\gamma(s, x)$ and $\Gamma(s, x)$ defined by (1.1) and (1.2), respectively, the *incomplete* Pochhammer symbols $(\lambda; x)_v$ and $[\lambda; x]_v$ ($\lambda; v \in \mathbb{C}; x \geq 0$) are defined as follows:

$$(\lambda; x)_v := \frac{\gamma(\lambda + v, x)}{\Gamma(\lambda)} \quad (\lambda, v \in \mathbb{C}; x \geq 0) \quad (1.6)$$

and

$$[\lambda; x]_v := \frac{\Gamma(\lambda + v, x)}{\Gamma(\lambda)} \quad (\lambda, v \in \mathbb{C}; x \geq 0), \quad (1.7)$$

so that, obviously, these incomplete Pochhammer symbols $(\lambda; x)_v$ and $[\lambda; x]_v$ satisfy the following decomposition relation:

$$(\lambda; x)_v + [\lambda; x]_v := (\lambda)_v \quad (\lambda; v \in \mathbb{C}; x \geq 0). \quad (1.8)$$

Here, and in what follows, $(\lambda)_v$ ($\lambda, v \in \mathbb{C}$) denotes the Pochhammer symbol (or the shifted factorial) which is defined (in general) by

$$(\lambda)_v := \frac{\Gamma(\lambda + v)}{\Gamma(\lambda)} = \begin{cases} 1 & (v = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \cdots (\lambda + v - 1) & (v = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases} \quad (1.9)$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ -quotient exists (see, for details, [17, p. 21 *et seq.*]).

As already observed by Srivastava *et al.* [15], the definitions (1.4) and (1.5) readily yield the following decomposition formula:

$$\begin{aligned} {}_p\gamma_q \left[\begin{matrix} (\alpha_1, x), \alpha_2, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] + {}_p\Gamma_q \left[\begin{matrix} (\alpha_1, x), \alpha_2, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] \\ = {}_pF_q \left[\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] \end{aligned} \quad (1.10)$$

for the familiar generalized hypergeometric function ${}_pF_q$.

More recently, Çetinkaya [6] introduced and studied various properties of the following two families of the incomplete second Appell hypergeometric functions γ_2 and Γ_2 :

$$\gamma_2[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2] = \sum_{m,p=0}^{\infty} \frac{(\alpha; x)_{m+p} (\beta_1)_m (\beta_2)_p}{(\gamma_1)_m (\gamma_2)_p} \frac{x_1^m}{m!} \frac{x_2^p}{p!} \quad (1.11)$$

and

$$\Gamma_2[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2] = \sum_{m,p=0}^{\infty} \frac{[\alpha; x]_{m+p} (\beta_1)_m (\beta_2)_p}{(\gamma_1)_m (\gamma_2)_p} \frac{x_1^m}{m!} \frac{x_2^p}{p!}. \quad (1.12)$$

Very recently, Choi *et al.* [9] introduced and studied various properties of the following two families of the incomplete Srivastava's triple hypergeometric functions γ_B^H and Γ_B^H as follows:

$$\begin{aligned} &\gamma_B^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x_1, x_2, x_3] \\ &= \sum_{m,n,p=0}^{\infty} \frac{(\alpha; x)_{m+n+p} (\beta_1)_{m+n} (\beta_2)_{n+p}}{(\gamma_1)_m (\gamma_2)_n (\gamma_3)_p} \frac{x_1^m}{m!} \frac{x_2^n}{n!} \frac{x_3^p}{p!} \end{aligned} \quad (1.13)$$

and

$$\begin{aligned} &\Gamma_B^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x_1, x_2, x_3] \\ &= \sum_{m,n,p=0}^{\infty} \frac{[\alpha; x]_{m+n+p} (\beta_1)_{m+n} (\beta_2)_{n+p}}{(\gamma_1)_m (\gamma_2)_n (\gamma_3)_p} \frac{x_1^m}{m!} \frac{x_2^n}{n!} \frac{x_3^p}{p!} \end{aligned} \quad (1.14)$$

$$(x \geq 0; |x_1| < r, |x_2| < s, |x_3| < t, r + s + t + 2\sqrt{rst} = 1 \text{ when } x = 0).$$

In a sequel to the aforementioned work by Srivastava *et al.* [15], Çetinkaya [6] and Choi *et al.* [9] and motivated essentially by these families of incomplete hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$, incomplete second Appell functions γ_2 and Γ_2 and incomplete Srivastava's triple hypergeometric functions γ_B^H and Γ_B^H (see, for details, [6, 9, 15] and the references cited therein), we aim here at systematically investigating the family of the incomplete Srivastava's triple hypergeometric functions γ_A^H and Γ_A^H to present various representations and formulas, for example, various definite and semi-definite integral representations involving the Laguerre polynomials, Bessel and modified Bessel functions, transformation formula, reduction formula, derivative formula and recurrence relations. For various other investigations involving generalizations of the hypergeometric function ${}_pF_q$, which were motivated essentially by the pioneering work of Srivastava *et al.* [15], the interested reader may refer to recent papers on the subject (see, for example, [8, 19–22] and the references cited in each of these papers).

2. THE INCOMPLETE SRIVASTAVA'S TRIPLE HYPERGEOMETRIC FUNCTIONS

In terms of the incomplete Pochhammer symbol $(\lambda; x)_v$ and $[\lambda; x]_v$ defined by (1.6) and (1.7), we introduce the following incomplete Srivastava's triple hypergeometric functions γ_A^H and Γ_A^H : For $\alpha, \beta_1, \beta_2 \in \mathbb{C}$ and $\gamma_1, \gamma_2 \in \mathbb{C} \setminus \mathbb{Z}_0^-$,

$$\begin{aligned} & \gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3] \\ &= \sum_{m,n,p=0}^{\infty} \frac{(\alpha; x)_{m+p} (\beta_1)_{m+n} (\beta_2)_{n+p}}{(\gamma_1)_m (\gamma_2)_{n+p}} \frac{x_1^m x_2^n x_3^p}{m! n! p!} \end{aligned} \quad (2.1)$$

$$(x \geq 0; |x_1| < r, |x_2| < s, |x_3| < t, r + s + t = 1 + st \text{ when } x = 0)$$

and

$$\begin{aligned} & \Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3] \\ &= \sum_{m,n,p=0}^{\infty} \frac{[\alpha; x]_{m+p} (\beta_1)_{m+n} (\beta_2)_{n+p}}{(\gamma_1)_m (\gamma_2)_{n+p}} \frac{x_1^m x_2^n x_3^p}{m! n! p!} \end{aligned} \quad (2.2)$$

$$(x \geq 0; |x_1| < r, |x_2| < s, |x_3| < t, r + s + t = 1 + st \text{ when } x = 0).$$

In view of (1.8), these incomplete Srivastava's triple hypergeometric functions satisfy the following decomposition formula:

$$\begin{aligned} & \gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3] + \Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3] \\ &= H_A[\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3], \end{aligned} \quad (2.3)$$

where H_A is the familiar Srivastava's triple hypergeometric functions (see, for details, [10–14, 17]).

Remark 1. It is interesting to note that the special cases of (2.1) and (2.2) when $x_2 = 0$ reduce to the known incomplete second Appell hypergeometric functions (1.11) and (1.12). Also, the special cases of (2.1) and (2.2) when $x_2 = 0$ and $x_3 = 0$ or $x_1 = 0$ are seen to yield the known incomplete families of Gauss hypergeometric functions [15].

In view of the formula (2.3), it is sufficient to discuss properties and characteristics of one of the incomplete Srivastava's triple hypergeometric functions γ_A^H and Γ_A^H .

3. INTEGRAL REPRESENTATIONS OF Γ_A^H

In this section, we apply (1.2) and (1.7) to present certain integral representations of the incomplete Srivastava's triple hypergeometric functions Γ_A^H . We also obtain various integral representations involving Laguerre polynomial, Bessel and modified Bessel functions.

Theorem 1. The following integral representation for Γ_A^H in (2.2) holds true:

$$\begin{aligned} \Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3] &= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)} \\ &\times \int_x^\infty \int_0^\infty e^{-s-t} t^{\alpha-1} s^{\beta_1-1} {}_0F_1(-; \gamma_1; x_1 st) {}_1F_1(\beta_2; \gamma_2; x_2 s + x_3 t) dt ds \\ &(x \geq 0; \max\{\Re(x_2), \Re(x_3)\} < 1, \min\{\Re(\alpha), \Re(\beta_1)\} > 0 \text{ when } x = 0). \end{aligned} \quad (3.1)$$

Proof of Theorem 1. Using the integral representations of the incomplete Pochhammer symbol $[\alpha; x]_{m+p}$ by considering (1.2) and (1.7), the classical Pochhammer symbol $(\beta_1)_{m+n}$ and using the elementary series identity [18, p. 52, Eq. 1.6(2)]:

$$\sum_{m_1, m_2=0}^{\infty} \Omega(m_1 + m_2) \frac{x_1^{m_1} x_2^{m_2}}{m_1! m_2!} = \sum_{m=0}^{\infty} \Omega(m) \frac{(x_1 + x_2)^m}{m!}, \quad (3.2)$$

in (2.2), we are led to the desired result. $\square$

Theorem 2. The following triple integral representation for Γ_A^H in (2.2) holds true:

$$\begin{aligned} \Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3] &= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)} \\ &\times \int_x^\infty \int_0^\infty \int_0^\infty e^{-s-t} t^{\alpha-1} s^{\beta_1-1} u^{\beta_2-1} \\ &\times {}_0F_1(-; \gamma_1; x_1 st) {}_0F_1(-; \gamma_2; x_2 us + x_3 ut) dt ds du \\ &(x \geq 0; \min\{\Re(\alpha), \Re(\beta_1), \Re(\beta_2)\} > 0 \text{ when } x = 0). \end{aligned} \quad (3.3)$$

Proof of Theorem 2. Using the elementary integral representation [2, p. 678, Eq.(4)]:

$${}_1F_1(\lambda; \mu; z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-t} {}_0F_1(-; \mu; zt) dt \quad (\Re(\lambda) > 0) \quad (3.4)$$

in (3.1), we are led to the desired integral representation. $\square$

The Laguerre polynomial $L_n^{(\alpha)}(x)$ of order (index) α and degree n in x , Bessel function $J_\nu(z)$ and the modified Bessel function $I_\nu(z)$ are expressible in terms of hypergeometric functions as follows (see, e.g., [2]; see also [1, p. 265, Eq. (3.2)] and [5, 23]):

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1(-n; \alpha+1; x), \quad (3.5)$$

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} {}_0F_1\left(\text{---}; \nu+1; -\frac{1}{4}z^2\right) \quad (\nu \in \mathbb{C} \setminus \mathbb{Z}^-) \quad (3.6)$$

and

$$I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} {}_0F_1\left(\text{---}; \nu+1; \frac{1}{4}z^2\right) \quad (\nu \in \mathbb{C} \setminus \mathbb{Z}^-). \quad (3.7)$$

Now, applying the relationships (3.5) to (3.1), (3.6) and (3.7) to (3.1), and (3.5), (3.6) and (3.7) to (3.1), respectively, we can deduce certain interesting integral representations for the incomplete Srivastava's triple hypergeometric function in (2.2) asserted by Corollaries 1, 2 and 3 below. Their proofs are omitted.

Corollary 1. *The following integral representation for Γ_A^H in (2.2) holds true:*

$$\begin{aligned} \Gamma_A^H[(\alpha, x), \beta_1, -m; \gamma_1, \gamma_2 + 1; x_1, x_2, x_3] &= \frac{m!}{(\gamma_2 + 1)_m \Gamma(\alpha) \Gamma(\beta_1)} \\ &\times \int_x^\infty \int_0^\infty e^{-s-t} t^{\alpha-1} s^{\beta_1-1} {}_0F_1(-; \gamma_1; x_1 st) L_m^{(\gamma_2)}(x_2 s + x_3 t) dt ds. \end{aligned} \quad (3.8)$$

Corollary 2. *Each of the following double integral representations holds true:*

$$\begin{aligned} \Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1 + 1, \gamma_2; -x_1, x_2, x_3] &= \frac{\Gamma(\gamma_1 + 1) x_1^{-\frac{\gamma_1}{2}}}{\Gamma(\alpha) \Gamma(\beta_1)} \\ &\times \int_x^\infty \int_0^\infty e^{-s-t} t^{\alpha-\frac{\gamma_1}{2}-1} s^{\beta_1-\frac{\gamma_1}{2}-1} J_{\gamma_1}(2\sqrt{x_1 st}) {}_1F_1(\beta_2; \gamma_2; x_2 s + x_3 t) dt ds \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} \Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1 + 1, \gamma_2; x_1, x_2, x_3] &= \frac{\Gamma(\gamma_1 + 1) x_1^{-\frac{\gamma_1}{2}}}{\Gamma(\alpha) \Gamma(\beta_1)} \\ &\times \int_x^\infty \int_0^\infty e^{-s-t} t^{\alpha-\frac{\gamma_1}{2}-1} s^{\beta_1-\frac{\gamma_1}{2}-1} I_{\gamma_1}(2\sqrt{x_1 st}) {}_1F_1(\beta_2; \gamma_2; x_2 s + x_3 t) dt ds, \end{aligned} \quad (3.10)$$

provided that the involved integrals are convergent.

Corollary 3. *Each of the following double integral representations holds true:*

$$\begin{aligned} \Gamma_A^H[(\alpha, x), \beta_1, -m; \gamma_1 + 1, \gamma_2 + 1; -x_1, x_2, x_3] &= \frac{m! \Gamma(\gamma_1 + 1) x_1^{-\frac{\gamma_1}{2}}}{(\gamma_2 + 1)_m \Gamma(\alpha) \Gamma(\beta_1)} \\ &\times \int_x^\infty \int_0^\infty e^{-s-t} t^{\alpha-\frac{\gamma_1}{2}-1} s^{\beta_1-\frac{\gamma_1}{2}-1} J_{\gamma_1}(2\sqrt{x_1 st}) L_m^{(\gamma_2)}(x_2 s + x_3 t) dt ds \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} \Gamma_A^H[(\alpha, x), \beta_1, -m; \gamma_1 + 1, \gamma_2 + 1; x_1, x_2, x_3] &= \frac{m! \Gamma(\gamma_1 + 1) x_1^{-\frac{\gamma_1}{2}}}{(\gamma_2 + 1)_m \Gamma(\alpha) \Gamma(\beta_1)} \\ &\times \int_x^\infty \int_0^\infty e^{-s-t} t^{\alpha-\frac{\gamma_1}{2}-1} s^{\beta_1-\frac{\gamma_1}{2}-1} I_{\gamma_1}(2\sqrt{x_1 st}) L_m^{(\gamma_2)}(x_2 s + x_3 t) dt ds, \end{aligned} \quad (3.12)$$

provided that the involved integrals are convergent.

4. TRANSFORMATION AND REDUCTION FORMULA OF Γ_A^H

In this section, we present a transformation formula and a reduction formula for the incomplete Srivastava's triple hypergeometric functions Γ_A^H .

Theorem 3. *The following transformation formula for Γ_A^H holds true:*

$$\begin{aligned} \Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3] &= (1-x_2)^{-\beta_2} (1-x_3)^{-\alpha} \\ &\times \Gamma_A^H\left[(\alpha, x(1-x_3)), \beta_1, \gamma_2 - \beta_2; \gamma_1, \gamma_2; \frac{x_1}{(1-x_2)(1-x_3)}, \frac{x_2}{x_2-1}, \frac{x_3}{x_3-1}\right]. \end{aligned} \quad (4.1)$$

Proof of Theorem 3. If we first apply Kummer's transformation formula (see, e.g., [2, p. 125, Eq. (2)]):

$${}_1F_1(\alpha; \beta; z) = e^z {}_1F_1(\beta - \alpha; \beta; -z) \quad (4.2)$$

to (3.1), we find that

$$\begin{aligned} \Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3] &= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)} \int_x^\infty \int_0^\infty e^{-s(1-x_2)-t(1-x_3)} t^{\alpha-1} s^{\beta_1-1} \\ &\times {}_0F_1(-; \gamma_1; x_1 st) {}_1F_1(\gamma_2 - \beta_2; \gamma_1; -x_2 s - x_3 t) dt ds. \end{aligned} \quad (4.3)$$

The substitution $t(1-x_3) = u$, $s(1-x_2) = v$ in (4.3), leads to

$$\begin{aligned} \Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3] &= \frac{(1-x_2)^{-\beta_1} (1-x_3)^{-\alpha}}{\Gamma(\alpha)\Gamma(\beta_1)} \\ &\times \int_{x(1-x_3)}^\infty \int_0^\infty e^{-u-v} u^{\alpha-1} v^{\beta_1-1} {}_0F_1\left(-; \gamma_1; \frac{x_1 uv}{(1-x_2)(1-x_3)}\right) \\ &\times {}_1F_1\left(\gamma_2 - \beta_2; \gamma_1; \frac{x_2 v}{(x_2-1)} + \frac{x_3 u}{(x_3-1)}\right) du dv. \end{aligned} \quad (4.4)$$

which, in view of (3.1), is easily seen to be the same as the right-hand side of (4.1). $\square$

Theorem 4. *The following reduction formula for Γ_A^H holds true:*

$$\begin{aligned} \Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \beta_2; x_1, x_2, x_3] &= (1-x_2)^{-\beta_1} (1-x_3)^{-\alpha} \\ &\times {}_2F_1\left[(\alpha, x(1-x_3)), \beta_1; \frac{x_1}{\gamma_1}; \frac{x_1}{(1-x_2)(1-x_3)}\right]. \end{aligned} \quad (4.5)$$

Proof of Theorem 4. Setting $\gamma_2 = \beta_2$ in the integral representation (3.1), we have

$$\begin{aligned} \Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \beta_2; x_1, x_2, x_3] &= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)} \int_x^\infty \int_0^\infty e^{-s(1-x_2)-t(1-x_3)} t^{\alpha-1} s^{\beta_1-1} {}_0F_1(-; \gamma_1; x_1 st) dt ds. \end{aligned} \quad (4.6)$$

Setting $t(1-x_3) = u$, $s(1-x_2) = v$ and using (3.4) in (4.6), we obtain

$$\begin{aligned} \Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \beta_2; x_1, x_2, x_3] &= \frac{(1-x_2)^{-\beta_1} (1-x_3)^{-\alpha}}{\Gamma(\alpha)} \\ &\times \int_{x(1-x_3)}^{\infty} e^{-u} u^{\alpha-1} {}_1F_1\left(\beta_1; \gamma_1; \frac{x_1 u}{(1-x_2)(1-x_3)}\right) du. \end{aligned} \quad (4.7)$$

Finally, using the known result in Srivastava *et al.* [15, p. 665, Eq. (3.6)]:

$${}_2F_1\left[\begin{matrix} (a, x); b \\ c; z \end{matrix}\right] = \frac{1}{\Gamma(a)} \int_x^{\infty} e^{-t} t^{a-1} {}_1F_1(b; c; zt) dt$$

in (4.7), we are led to the desired result (4.5). $\square$

5. DERIVATIVE FORMULA AND RECURRENCE RELATIONS OF Γ_A^H

Differentiating, partially, both sides of (2.2) with respect to x_1, x_2 and x_3, m, n and p times, respectively, we obtain a derivative formula for the incomplete Srivastava's triple hypergeometric function Γ_A^H given in the following theorem.

Theorem 5. *The following derivative formula for Γ_A^H holds true:*

$$\begin{aligned} \frac{\partial^{m+n+p}}{\partial x_1^m \partial x_2^n \partial x_3^p} \Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3] &= \frac{(\alpha)_{m+p} (\beta_1)_{m+n} (\beta_2)_{n+p}}{(\gamma_1)_m (\gamma_2)_{n+p}} \\ &\times \Gamma_A^H[(\alpha + m + p, x), \beta_1 + m + n, \beta_2 + n + p; \gamma_1 + m, \gamma_2 + n + p; x_1, x_2, x_3]. \end{aligned} \quad (5.1)$$

Next we give recurrence relations for the incomplete Srivastava triple hypergeometric function Γ_A^H .

Theorem 6. *The following recurrence relation for Γ_A^H holds true:*

$$\begin{aligned} \Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3] &= \Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1 - 1, \gamma_2; x_1, x_2, x_3] \\ &+ \frac{\alpha \beta_1 x_1}{\gamma_1 (1 - \gamma_1)} \Gamma_A^H[(\alpha + 1, x), \beta_1 + 1, \beta_2; \gamma_1 + 1, \gamma_2; x_1, x_2, x_3] \end{aligned} \quad (5.2)$$

Proof of Theorem 6. Using the well-known contiguous relation for the function ${}_0F_1$ (see [3, p. 12]):

$${}_0F_1(-; \gamma - 1; x) - {}_0F_1(-; \gamma; x) - \frac{x}{\gamma(\gamma - 1)} {}_0F_1(-; \gamma + 1; x) = 0$$

in the integral representation (3.1), we are led to the desired result. $\square$

Theorem 7. *The following recurrence relation for Γ_A^H holds true:*

$$\begin{aligned} &(\gamma_2 - \beta_2 - 1) \Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2] \\ &= (\gamma_2 - 1) \Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2 - 1] - \beta_2 \Gamma_A^H[(\alpha, x), \beta_1, \beta_2 + 1; \gamma_1, \gamma_2], \end{aligned} \quad (5.3)$$

where the variables which are not explicitly mentioned are assumed to be unchanged in value.

Proof of Theorem 7. Using the well-known contiguous relation for the function ${}_1F_1$ (see [2, p. 124, Eq.(6)]):

$$(c - b - 1) {}_1F_1(b; c; x) = (c - 1) {}_1F_1(b; c - 1; x) - b {}_1F_1(b + 1; c; x)$$

in the integral representation (3.1), we are led to the desired result. $\square$

6. CONCLUDING REMARKS AND OBSERVATIONS

In our present investigation, with the help of the incomplete Pochhammer symbols $(\lambda; x)_\nu$ and $[\lambda; x]_\nu$, we have introduced the incomplete Srivastava triple hypergeometric function Γ_A^H , whose special cases when $x_2 = 0$ reduces to the incomplete Appell functions of two variables (see [6]) and when $x_2 = 0$, $x_3 = 0$ or $x_1 = 0$ reduces to the incomplete Gauss hypergeometric function (see [15]), respectively, and investigated their diverse properties such mainly as integral representations, derivative formula, reduction formula and recurrence relation. The special cases of the results presented here when $x = 0$ would reduce to the corresponding well-known results for the Srivastava's triple hypergeometric function H_A (see, for details, [10–14, 17]).

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Authors' addresses

Junesang Choi

Department of Mathematics, Dongguk University, Gyeongju 780-714, Republic of Korea

E-mail address: junesang@mail.dongguk.ac.kr

Rakesh K. Parmar

Department of Mathematics, Government College of Engineering and Technology, Bikaner-334004, Rajasthan, India

E-mail address: rakeshparmar27@gmail.com



ON THE RECIPROCAL SUMS OF SQUARE OF GENERALIZED BI-PERIODIC FIBONACCI NUMBERS

GINKYU CHOI AND YOUNSEOK CHOO

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Abstract. Recently Basbük and Yazlık [1] proved identities related to the reciprocal sum of generalized bi-periodic Fibonacci numbers starting from 0 and 1, and raised an open question whether we can obtain similar results for the reciprocal sum of m^{th} power ($m \geq 2$) of the same numbers. In this paper we derive identities for the reciprocal sum of square of generalized bi-periodic Fibonacci numbers with arbitrary initial conditions.

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1. INTRODUCTION

Throughout this paper we use the notation $\{G_n\}_{n=0}^{\infty} = S(G_0, G_1, a, b)$ to denote the generalized bi-periodic Fibonacci numbers $\{G_n\}_{n=0}^{\infty}$ generated from the recurrence relation [4]

$$G_n = \begin{cases} aG_{n-1} + G_{n-2}, & \text{if } n \in \mathbb{N}_e; \\ bG_{n-1} + G_{n-2}, & \text{if } n \in \mathbb{N}_o, \end{cases} \quad (n \geq 2),$$

with initial conditions G_0 and G_1 , where G_0, G_1, a and b are real numbers, and $\mathbb{N}_e$ ($\mathbb{N}_o$, respectively) denotes the set of positive even (odd, respectively) integers.

Recently Ohtsuka and Nakamura [8] found interesting properties of the Fibonacci numbers $\{F_n\}_{n=0}^{\infty} = S(0, 1, 1, 1)$ and proved (1.1) and (1.2) below, where $\lfloor \cdot \rfloor$ indicates the floor function.

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_n - F_{n-1}, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ F_n - F_{n-1} - 1, & \text{if } n \geq 3 \text{ and } n \in \mathbb{N}_o, \end{cases} \quad (1.1)$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ F_{n-1}F_n, & \text{if } n \geq 3 \text{ and } n \in \mathbb{N}_o. \end{cases} \quad (1.2)$$

The work of Ohtsuka and Nakamura was generalized by several authors [1, 2, 5–7]. In particular, Basbük and Yazlik [1] considered the reciprocal sum of generalized bi-periodic Fibonacci numbers $\{G_n\}_{n=0}^\infty = S(0, 1, a, b)$ and proved the following theorem.

Theorem 1. *Let a and b positive integers. Then, for $\{G_n\}_{n=0}^\infty = S(0, 1, a, b)$, we have*

$$\left[\left(\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{\psi(k)}}{G_k} \right)^{-1} \right] = \begin{cases} G_n - G_{n-1}, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ G_n - G_{n-1} - 1, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o, \end{cases} \quad (1.3)$$

where

$$\psi(k) = \xi(k+1) - \xi(n+1) - (-1)^n \left\lfloor \frac{k-n}{2} \right\rfloor,$$

and $\xi(n)$ is the parity function such that

$$\xi(n) = \begin{cases} 0, & \text{if } n \in \{0\} \cup \mathbb{N}_e; \\ 1, & \text{if } n \in \mathbb{N}_o. \end{cases}$$

In [1], Basbük and Yazlik raised an open question whether we can obtain similar results for the reciprocal sum of m^{th} power ($m \geq 2$) of the same numbers.

In this paper we derive identities for the reciprocal sum of square of generalized bi-periodic Fibonacci numbers $\{G_n\}_{n=0}^\infty = S(G_0, G_1, a, b)$, where G_0 is a nonnegative integer and G_1 is a positive integer.

2. MAIN RESULTS

Lemma 1 below will be used to prove our main results.

Lemma 1. *For $\{G_n\}_{n=0}^\infty = S(G_0, G_1, a, b)$, (a)-(c) below hold:*

- (a) $G_n G_{n+1} - G_{n-1} G_{n+2} = (-1)^n (b G_0^2 + a b G_0 G_1 - a G_1^2)$.
- (b) $a^{\xi(n+1)} b^{\xi(n)} G_{n-1} G_{n+1} - a^{\xi(n)} b^{\xi(n+1)} G_n^2 = (-1)^n (a G_1^2 - a b G_0 G_1 - b G_0^2)$.
- (c) $G_{n+1} G_{n+2} - G_{n-1} G_n = a^{\xi(n)} b^{\xi(n+1)} G_n^2 + a^{\xi(n+1)} b^{\xi(n)} G_{n+1}^2$.

Proof. (a) and (b) are special cases of [3, Theorem 2.2]. Since

$$G_n = a^{\xi(n-1)} b^{\xi(n)} G_{n-1} + G_{n-2},$$

then (c) follows from the identity

$$\begin{aligned} G_n G_{n+1} &= (G_{n+2} - a^{\xi(n+1)} b^{\xi(n)} G_{n+1}) G_{n+1} \\ &= G_n (a^{\xi(n)} b^{\xi(n+1)} G_n + G_{n-1}). \end{aligned}$$

□

The main results of this paper are stated in Theorem 2. For the ease of presentation, we use the following notation for $\{G_n\}_{n=0}^\infty = S(G_0, G_1, a, b)$

$$\Phi(G) := b^2 G_0^2 + ab^2 G_0 G_1 - ab G_1^2.$$

Theorem 2. Let G_0 be a nonnegative integer and let G_1, a and b positive integers. Then, for $\{G_n\}_{n=0}^\infty = S(G_0, G_1, a, b)$, (a) and (b) below hold:

(a) If

$$\frac{\Phi(G)}{ab+2} \notin \mathbb{Z},$$

define

$$g := \left\lfloor \frac{\Phi(G)}{ab+2} \right\rfloor + \Delta,$$

where

$$\Delta = \begin{cases} 1, & \text{if } \Phi(G) > 0; \\ 0, & \text{if } \Phi(G) < 0. \end{cases}$$

(i) If $\Phi(G) > 0$, then there exist positive integers n_0 and n_1 such that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} bG_{n-1}G_n + g - 1, & \text{if } n \geq n_0 \text{ and } n \in \mathbb{N}_e; \\ bG_{n-1}G_n - g, & \text{if } n \geq n_1 \text{ and } n \in \mathbb{N}_o. \end{cases} \quad (2.1)$$

(ii) If $\Phi(G) < 0$, then there exist positive integers n_2 and n_3 such that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} bG_{n-1}G_n + g, & \text{if } n \geq n_2 \text{ and } n \in \mathbb{N}_e; \\ bG_{n-1}G_n - g - 1, & \text{if } n \geq n_3 \text{ and } n \in \mathbb{N}_o. \end{cases} \quad (2.2)$$

(b) If

$$\frac{\Phi(G)}{ab+2} \in \mathbb{Z},$$

then there exist positive integers n_4 and n_5 such that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} bG_{n-1}G_n + \hat{g}, & \text{if } n \geq n_4 \text{ and } n \in \mathbb{N}_e; \\ bG_{n-1}G_n - \hat{g}, & \text{if } n \geq n_5 \text{ and } n \in \mathbb{N}_o, \end{cases} \quad (2.3)$$

where

$$\hat{g} := \frac{\Phi(G)}{ab+2}.$$

Proof. (a) To prove (2.1), assume that $\Phi(G) > 0$. Then

$$\Phi(G) - g(ab+2) < 0.$$

Firstly, consider

$$\begin{aligned}
X_1 &= \frac{1}{bG_{n-1}G_n + (-1)^n g} - \frac{1}{bG_{n+1}G_{n+2} + (-1)^n g} \\
&\quad - \frac{\left(\frac{a}{b}\right)^{1-\xi(n)}}{G_n^2} - \frac{\left(\frac{a}{b}\right)^{1-\xi(n+1)}}{G_{n+1}^2} \\
&= \frac{Y_1}{(bG_{n-1}G_n + (-1)^n g)(bG_{n+1}G_{n+2} + (-1)^n g)G_n^2 G_{n+1}^2},
\end{aligned}$$

where, by Lemma 1(c)

$$Y_1 = \left\{ \left(\frac{a}{b}\right)^{1-\xi(n+1)} G_n^2 + \left(\frac{a}{b}\right)^{1-\xi(n)} G_{n+1}^2 \right\} \hat{Y}_1,$$

with

$$\begin{aligned}
\hat{Y}_1 &= b^2(G_n^2 G_{n+1}^2 - G_{n-1}G_n G_{n+1}G_{n+2}) \\
&\quad - (-1)^n g b(G_{n-1}G_n + G_{n+1}G_{n+2}) - g^2.
\end{aligned}$$

By Lemma 2.1(a),(b), we have

$$\begin{aligned}
G_n^2 G_{n+1}^2 - G_{n-1}G_n G_{n+1}G_{n+2} \\
&= (G_n G_{n+1} - G_{n-1}G_{n+2})G_n G_{n+1} \\
&= (-1)^n (bG_0^2 + abG_0G_1 - aG_1^2)G_n G_{n+1},
\end{aligned}$$

and

$$\begin{aligned}
G_{n-1}G_n + G_{n+1}G_{n+2} \\
&= (G_{n+1} - a^{\xi(n)}b^{\xi(n+1)}G_n)G_n + G_{n+1}(a^{\xi(n+1)}b^{\xi(n)}G_{n+1} + G_n) \\
&= (ab + 2)G_n G_{n+1} + a^{\xi(n+1)}b^{\xi(n)}G_{n-1}G_{n+1} - a^{\xi(n)}b^{\xi(n+1)}G_n^2 \\
&= (ab + 2)G_n G_{n+1} + (-1)^n (aG_1^2 - abG_0G_1 - bG_0^2).
\end{aligned}$$

Then

$$\begin{aligned}
\hat{Y}_1 &= (-1)^n b^2 (bG_0^2 + abG_0G_1 - aG_1^2)G_n G_{n+1} \\
&\quad - (-1)^n g b \left\{ (ab + 2)G_n G_{n+1} + (-1)^n (aG_1^2 - abG_0G_1 - bG_0^2) \right\} - g^2 \\
&= (-1)^n b G_n G_{n+1} \left\{ \Phi(G) - g(ab + 2) \right\} + g \Phi(G) - g^2.
\end{aligned}$$

If $n \in \mathbb{N}_e$, then there exists a positive integer m_0 such that, for $n \geq m_0$, $X_1 < 0$, and

$$\frac{1}{bG_{n-1}G_n + (-1)^n g} - \frac{1}{bG_{n+1}G_{n+2} + (-1)^n g} < \frac{\left(\frac{a}{b}\right)^{1-\xi(n)}}{G_n^2} + \frac{\left(\frac{a}{b}\right)^{1-\xi(n+1)}}{G_{n+1}^2}.$$

Repeatedly applying the above inequality, we have

$$\frac{1}{bG_{n-1}G_n + (-1)^n g} < \sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2}, \text{ if } n \geq m_0 \text{ and } n \in \mathbb{N}_e. \quad (2.4)$$

Similarly, we obtain, for some positive integer m_1 ,

$$\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2} < \frac{1}{bG_{n-1}G_n + (-1)^n g}, \text{ if } n \geq m_1 \text{ and } n \in \mathbb{N}_o. \quad (2.5)$$

Next, consider

$$\begin{aligned} X_2 &= \frac{1}{bG_{n-1}G_n + (-1)^n g - 1} - \frac{1}{bG_nG_{n+1} + (-1)^{n+1} g - 1} - \frac{\left(\frac{a}{b}\right)^{1-\xi(n)}}{G_n^2} \\ &= \frac{Y_2}{(bG_{n-1}G_n + (-1)^n g - 1)(bG_nG_{n+1} + (-1)^{n+1} g - 1)G_n^2}, \end{aligned}$$

where

$$\begin{aligned} Y_2 &= bG_n^3G_{n+1} - ba^{1-\xi(n)}b^{\xi(n)}G_{n-1}G_n^2G_{n+1} - bG_{n-1}G_n^3 \\ &\quad - (-1)^n g(2G_n^2 - a^{1-\xi(n)}b^{\xi(n)}G_{n-1}G_n + a^{1-\xi(n)}b^{\xi(n)}G_nG_{n+1}) \\ &\quad + a^{1-\xi(n)}b^{\xi(n)}(G_{n-1}G_n + G_nG_{n+1}) + a^{1-\xi(n)}b^{\xi(n)-1}(g^2 - 1). \end{aligned}$$

Using Lemma 1(a), we have

$$\begin{aligned} &bG_n^3G_{n+1} - ba^{1-\xi(n)}b^{\xi(n)}G_{n-1}G_n^2G_{n+1} - bG_{n-1}G_n^3 \\ &= bG_n^2G_{n+1}(G_n - a^{1-\xi(n)}b^{\xi(n)}G_{n-1}) - bG_{n-1}G_n^3 \\ &= bG_n^2(G_{n-2}G_{n+1} - G_{n-1}G_n) \\ &= (-1)^n bG_n^2(bG_0^2 + abG_0G_1 - aG_1^2), \end{aligned}$$

and

$$\begin{aligned} &2G_n^2 - a^{1-\xi(n)}b^{\xi(n)}G_{n-1}G_n + a^{1-\xi(n)}b^{\xi(n)}G_nG_{n+1} \\ &= 2G_n^2 - a^{1-\xi(n)}b^{\xi(n)}G_{n-1}G_n + a^{1-\xi(n)}b^{\xi(n)}G_n(a^{1-\xi(n+1)}b^{\xi(n+1)}G_n + G_{n-1}) \\ &= (ab + 2)G_n^2. \end{aligned}$$

Hence we obtain

$$\begin{aligned} Y_2 &= (-1)^n G_n^2 \left\{ \Phi(G) - g(ab + 2) \right\} \\ &\quad + a^{1-\xi(n)}b^{\xi(n)}(G_{n-1}G_n + G_nG_{n+1}) + a^{1-\xi(n)}b^{\xi(n)-1}(g^2 - 1). \end{aligned}$$

If $n \in \mathbb{N}_e$, then there exists a positive integer m_2 such that, for $n \geq m_2$, $X_2 > 0$, and

$$\frac{\left(\frac{a}{b}\right)^{1-\xi(n)}}{G_n^2} < \frac{1}{bG_{n-1}G_n + (-1)^n g - 1} - \frac{1}{bG_nG_{n+1} + (-1)^{n+1} g - 1}.$$

Repeatedly applying the above inequality, we have

$$\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2} < \frac{1}{bG_{n-1}G_n + (-1)^n g - 1}, \text{ if } n \geq m_2 \text{ and } n \in \mathbb{N}_e. \quad (2.6)$$

Similarly, consider

$$\begin{aligned} X_3 &= \frac{1}{bG_{n-1}G_n + (-1)^n g + 1} - \frac{1}{bG_nG_{n+1} + (-1)^{n+1} g + 1} - \frac{\left(\frac{a}{b}\right)^{1-\xi(n)}}{G_n^2} \\ &= \frac{Y_3}{(bG_{n-1}G_n + (-1)^n g + 1)(bG_nG_{n+1} + (-1)^{n+1} g + 1)G_n^2}, \end{aligned}$$

where

$$\begin{aligned} Y_3 &= Y_2 - 2a^{1-\xi(n)}b^{\xi(n)}(G_{n-1}G_n + G_nG_{n+1}) \\ &= (-1)^n G_n^2 \left\{ \Phi(G) - g(ab + 2) \right\} \\ &\quad - a^{1-\xi(n)}b^{\xi(n)}(G_{n-1}G_n + G_nG_{n+1}) + a^{1-\xi(n)}b^{\xi(n)-1}(g^2 - 1). \end{aligned}$$

If $n \in \mathbb{N}_o$, then there exists a positive integer m_3 such that, for $n \geq m_3$, $X_3 < 0$, and

$$\frac{1}{bG_{n-1}G_n + (-1)^n g + 1} - \frac{1}{bG_nG_{n+1} + (-1)^{n+1} g + 1} < \frac{\left(\frac{a}{b}\right)^{1-\xi(n)}}{G_n^2},$$

from which we have

$$\frac{1}{bG_{n-1}G_n + (-1)^n g + 1} < \sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2}, \text{ if } n \geq m_3 \text{ and } n \in \mathbb{N}_o. \quad (2.7)$$

Then (2.1) follows from (2.4), (2.5), (2.6) and (2.7).

Now suppose that $\Phi(G) < 0$. In this case, we have

$$\Phi(G) - g(ab + 2) > 0,$$

and (2.4), (2.5), (2.6) and (2.7) are respectively modified as

$$\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2} < \frac{1}{bG_{n-1}G_n + (-1)^n g}, \text{ if } n \geq m_4 \text{ and } n \in \mathbb{N}_e, \quad (2.8)$$

$$\frac{1}{bG_{n-1}G_n + (-1)^n g} < \sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2}, \text{ if } n \geq m_5 \text{ and } n \in \mathbb{N}_o, \quad (2.9)$$

$$\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2} < \frac{1}{bG_{n-1}G_n + (-1)^n g - 1}, \text{ if } n \geq m_6 \text{ and } n \in \mathbb{N}_o, \quad (2.10)$$

and

$$\frac{1}{bG_{n-1}G_n + (-1)^n g + 1} < \sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2}, \text{ if } n \geq m_7 \text{ and } n \in \mathbb{N}_e. \quad (2.11)$$

Then, (2.2) easily follows and the proof of (a) is completed.

(b) Suppose that

$$\frac{\Phi(G)}{ab+2} \in \mathbb{Z}.$$

We recall the proof of (a). Replacing g by $\hat{g}$, we have

$$\hat{Y}_1 = \hat{g}\Phi(G) - \hat{g}^2 = (ab+1)\hat{g}^2 > 0.$$

Hence there exist positive integers m_8 and m_9 such that $X_1 > 0$ if $n \geq m_8$ and $n \in \mathbb{N}_e$ or if $n \geq m_9$ and $n \in \mathbb{N}_o$. Hence we obtain

$$\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2} < \frac{1}{bG_{n-1}G_n + (-1)^n \hat{g}}, \text{ if } n \geq m_8 \text{ } (n \in \mathbb{N}_e) \text{ or if } n \geq m_9 \text{ } (n \in \mathbb{N}_o). \quad (2.12)$$

Similarly, there exist positive integers m_{10} and m_{11} such that $X_3 < 0$ if $n \geq m_{10}$ and $n \in \mathbb{N}_e$ or if $n \geq m_{11}$ and $n \in \mathbb{N}_o$, from which we have

$$\frac{1}{bG_{n-1}G_n + (-1)^n \hat{g} + 1} < \sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2}, \text{ if } n \geq m_{10} \text{ } (n \in \mathbb{N}_e) \text{ or if } n \geq m_{11} \text{ } (n \in \mathbb{N}_o). \quad (2.13)$$

Then, (2.3) follows from (2.12) and (2.13), and (b) is also proved. $\square$

Example 1. For $\{G_n\}_{n=0}^{\infty} = S(2, 1, 2, 1)$, we have $\Phi(G) = 6$ and

$$g = \left\lfloor \frac{6}{4} \right\rfloor + 1 = 2.$$

Then, from (2.1), we have

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{2^{1-\xi(k)}}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} G_{n-1}G_n + 1, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ G_{n-1}G_n - 2, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases}$$

Example 2. Consider $\{G_n\}_{n=0}^\infty = S(0, 1, a, b)$ with a and b positive integers. In this case, we have $\Phi(G) = -ab < 0$ and

$$g = \left\lfloor \frac{-ab}{ab+2} \right\rfloor = -1.$$

Then, from (2.2), we obtain

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{\left(\frac{a}{b}\right)^{1-\xi(k)}}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} bG_{n-1}G_n - 1, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ bG_{n-1}G_n, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases}$$

Example 3. For $\{G_n\}_{n=0}^\infty = S(2, 1, 4, 2)$, we have $\Phi(G) = 40$, $\hat{g} = 4$. Then, from (2.3), we have

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{2^{1-\xi(k)}}{G_k^2} \right)^{-1} \right\rfloor = \begin{cases} 2G_{n-1}G_n + 4, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ 2G_{n-1}G_n - 4, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases}$$

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*Authors' addresses***Ginkyu Choi**

Hongik University, Department of Electronic and Electrical Engineering, 2639 Sejong-Ro, 30016 Sejong, Republic of Korea

E-mail address: gkchoi@hongik.ac.kr

Younseok Choo

Hongik University, Department of Electronic and Electrical Engineering, 2639 Sejong-Ro, 30016 Sejong, Republic of Korea

E-mail address: yschoo@hongik.ac.kr



RATE OF CONVERGENCE OF q - ANALOGUE OF A CLASS OF NEW BERNSTEIN TYPE OPERATORS

SHEETAL DESHWAL, ANA MARIA ACU, AND P. N. AGRAWAL

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Abstract. Sharma [32] introduced a q -analogue of a new sequence of classical Bernstein type operators defined by Deo et al. [14] for functions defined in the interval $[0, \frac{n}{n+1}]$. The purpose of this paper is to study the rate of convergence of these operators with the aid of the modulus of continuity and a Lipschitz type space. Subsequently, we define the bivariate case of these operators and discuss the approximation properties by means of the complete and partial modulus of continuity, Lipschitz class and the Peetre's K - functional. Some numerical results which show the rate of convergence of these operators to certain functions using Maple algorithms are given. Lastly, we construct the associated GBS operators and study the approximation of Bögel continuous and Bögel differentiable functions. The comparison of convergence of the bivariate operator and its GBS type operator is made considering numerical examples.

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1. INTRODUCTION

For $f \in C[0, 1]$, Bernstein [8] constructed a sequence of polynomials

$$B_n(f; x) = \sum_{k=0}^n b_{n,k}(x) f\left(\frac{k}{n}\right),$$

where

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad n = 1, 2, \dots, \quad x \in [0, 1],$$

and proved that the sequence $B_n(f; x)$ converges to $f(x)$, as $n \rightarrow \infty$, uniformly in $x \in [0, 1]$. These polynomials are called Bernstein polynomials and possess many remarkable properties. For $f \in L_1[0, 1]$, the space of Lebesgue integrable functions in $[0, 1]$, Durrmeyer [18] introduced an integral modification of Bernstein operators

as

$$D_n(f; x) = (n+1) \sum_{k=0}^n b_{nk}(x) \int_0^1 b_{nk}(t) f(t) dt,$$

which was extensively studied by Derriennic [15].

In recent years, the applications of q -calculus in the area of approximation theory is one of the main areas of research (see [1], [2], [13]). Also, the reader should consult the monographies of A. Aral et al. [5], Gupta et al. [23] and G. Tachev et al. [24]. In 1987, for $f \in C[0, 1]$ and $0 < q < 1$, Lupas [27] introduced a q -analogue of Bernstein polynomials. After a decade, another q -generalization of Bernstein polynomials was introduced by Phillips [31]. The q -analogue of Bernstein polynomials due to Phillips was studied by several researchers e.g Ostrovska [29, 30], Kim [26] and Wang [38] etc. In 2005, Derriennic [16] introduced the q -analogue of Bernstein-Durrmeyer polynomials with Jacobi weights and studied some approximation properties. Later, Gupta [21] introduced the q -analogue of the Bernstein-Durrmeyer operators which was investigated later by Finta and Gupta ([20], [22]) and several other researchers. Dalmanoglu [13] introduced the Kantorovich type modification of q -Bernstein polynomials and established some approximation results. Muraru [28] introduced Bernstein-Schurer polynomials based on q -integers and established the rate of convergence in terms of the modulus of continuity. Agrawal et al. [2] considered the Stancu variant of these operators and obtained some local and global direct results. Later, Agrawal et al. [4] proposed the Durrmeyer type modification of these operators and discussed some local direct results and the rate of convergence of the modified limit q -Bernstein-Schurer type operators.

Deo et al. [14] introduced a new sequence of Bernstein type operators V_n as

$$(V_n f)(x) = \sum_{k=0}^n p_{n,k}(x) f\left(\frac{k}{n}\right), \quad (1.1)$$

where

$$p_{n,k}(x) = \left(1 + \frac{1}{n}\right)^n \binom{n}{k} x^k \left(\frac{n}{n+1} - x\right)^{n-k}, \quad x \in \left[0, \frac{n}{n+1}\right].$$

In the same paper, to approximate Lebesgue integrable functions on the interval $[0, 1]$, the authors defined an integral modification of the operators (1.1) as

$$(L_n f)(x) = \frac{(n+1)^2}{n} \sum_{k=0}^n p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k}(t) f(t) dt,$$

and studied some approximation properties.

From [25, page 69], the definite q -integral in the interval $[0, a]$, $a > 0$ is defined as:

$$\int_0^a f(x) d_q x = (1-q)a \sum_{j=0}^{\infty} q^j f(q^j a).$$

Later, Sharma [32] investigated the q -analogue of these operators given by

$$(L_{n,q} f)(x) = \frac{[n+1]_q^2}{[n]_q} \sum_{k=0}^n q^{-k} p_{n,k,q}(x) \int_0^{\frac{[n]_q}{[n+1]_q}} p_{n,k,q}(qt) f(t) d_q t, \quad (1.2)$$

where

$$p_{n,k,q}(x) := \frac{[n+1]_q^n}{[n]_q^n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \left(\frac{[n]_q}{[n+1]_q} - x \right)_q^{n-k}, \quad x \in \left[0, \frac{[n]_q}{[n+1]_q} \right].$$

In the present paper, first we obtain the order of approximation of the operators defined by (1.2) by means of the modulus of continuity and the Lipschitz class. Then we proceed to define the bivariate generalization of these operators and investigate their rate of convergence with the help of the moduli of continuity and the K- functional. Lastly, we introduce the associated GBS operators and discuss their degree of approximation by means of the mixed modulus of smoothness.

2. DIRECT RESULTS

Lemma 1 ([32]). *For the operators given by (1.2), the following equalities hold:*

- i) $(L_{n,q} 1)(x) = 1$;
- ii) $(L_{n,q} t)(x) = \frac{[n]_q}{[n+1]_q [n+2]_q} + \frac{q[n]_q x}{[n+2]_q}$;
- iii) $(L_{n,q} t^2)(x) = \frac{[n]_q}{[n+2]_q [n+3]_q} \left\{ [n-1]_q q^4 x^2 + \frac{q^3 x [n]_q + q(1+2q)x[n]_q}{[n+1]_q} + \frac{[2]_q [n]_q}{[n+1]_q^2} \right\}$.

Consequently,

- i) $(L_{n,q}(t-x))(x) = \frac{[n]_q}{[n+1]_q [n+2]_q} + \frac{q[n]_q - [n+2]_q}{[n+2]_q} x$;
- ii) $(L_{n,q}(t-x)^2)(x) = \frac{[2]_q [n]_q^2}{[n+1]_q^2 [n+2]_q [n+3]_q} + \left(\frac{q^3 [n]_q^2 + q(1+2q)[n]_q^2 - 2[n]_q [n+3]_q}{[n+1]_q [n+2]_q [n+3]_q} \right) x + \left(\frac{q^4 [n-1]_q [n]_q - 2q[n]_q [n+3]_q + [n+2]_q [n+3]_q}{[n+2]_q [n+3]_q} \right) x^2$.

Lemma 2. For the operators given by (1.2), the following equality holds

$$\begin{aligned} (L_{n,q}(t-x)^4)(x) = & \frac{1}{[n+1]_q^4 [n+2]_q [n+3]_q [n+4]_q [n+5]_q} \cdot \left\{ x^4 q^{14} (1-q)^4 [n]_q^8 \right. \\ & + q^9 x^3 (1-q)^2 (8xq^6 - 11xq^5 - q^4 + 3xq^4 + 4q^3 - 4xq^3 + 6q^2 - 6xq^2 + 4q - 4xq + 1) [n]_q^7 \\ & - q^5 x^2 (-1 - 3q + q^{12} x^3 + 3q^{11} x^3 + 3q^{10} x^3 + 12xq^2 - 36xq^8 + 4xq + 15xq^3 + 8xq^6 \\ & + 6xq^5 + 4xq^4 - 19q^5 x^2 - 17q^6 x^2 + 74q^{10} x^2 + 53q^8 x^2 - 26q^{11} x^2 - 74q^9 x^2 - 8q^3 x^2 \\ & \left. - 6q^2 x^2 + 9q^4 x^2 + 2xq^9 - 3xq^{10} - 7q^2 + 4q^8 - 7q^3 + 3q^6 + 6q^5 - 2q^4 + 2q^7) [n]_q^6 + \mathcal{O}([n]_q^5) \right\}. \end{aligned}$$

In what follows, let $(q_n)_n, 0 < q_n < 1$ be a sequence satisfying the following condition

$$\lim_{n \rightarrow \infty} q_n = 1.$$

Remark 1. Let $(q_n)_n, 0 < q_n < 1$ be a sequence satisfying the following conditions

$$\lim_{n \rightarrow \infty} q_n = 1, \quad \lim_{n \rightarrow \infty} q_n^n = c, \quad c \in [0, 1).$$

By simple computations, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} [n]_{q_n} (L_{n,q_n}(t-x))(x) &= 1 - (c+1)x, \\ \lim_{n \rightarrow \infty} [n]_{q_n} (L_{n,q_n}(t-x)^2)(x) &= 2x(1-x), \\ \lim_{n \rightarrow \infty} [n]_{q_n}^2 (L_{n,q_n}(t-x)^4)(x) &= x^2(1-x)(7x^2 - 7x + 5). \end{aligned}$$

3. MAIN RESULTS

Next we assume that $C[0, \frac{[n]_q}{[n+1]_q}]$ is the class of all real valued continuous functions on $[0, \frac{[n]_q}{[n+1]_q}]$ endowed with the norm $\| \cdot \|$ defined as

$$\|f\| = \sup_{x \in [0, \frac{[n]_q}{[n+1]_q}]} |f(x)|.$$

Lemma 3. For each $f \in C[0, \frac{[n]_q}{[n+1]_q}]$, we have $\|L_{n,q}(f)\| \leq \|f\|$.

Proof. Using the definition (1.2) and Lemma 1, the proof of this lemma easily follows. Hence, the details are omitted. $\square$

For $f \in C[0, \frac{[n]_q}{[n+1]_q}]$, the Steklov mean is defined as

$$f_h(x) = \frac{4}{h^2} \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} [2f(x+u+v) - f(x+2(u+v))] du dv. \quad (3.1)$$

The first and second order modulus of continuity are respectively defined as

$$\omega(f, \delta) = \sup_{x, u, v \in [0, \frac{[n]_q}{[n+1]_q}], |u-v| \leq \delta} |f(x+u) - f(x+v)|$$

and

$$\omega_2(f, \delta) = \sup_{x, u, v \in \left[0, \frac{[n]_q}{[n+1]_q}\right], |u-v| \leq \delta} |f(x+2u) - 2f(x+u+v) + f(x+2v)|, \quad \delta > 0.$$

Lemma 4. *The Steklov mean $f_h(x)$ satisfies the following properties:*

- i) $\|f_h - f\| \leq \omega_2(f, h)$,
- ii) *If f is continuous, then $f'_h, f''_h \in C\left[0, \frac{[n]_q}{[n+1]_q}\right]$ and*

$$\|f'_h\|_{C\left[0, \frac{[n]_q}{[n+1]_q}\right]} \leq \frac{5}{h} \omega(f, h), \quad \|f''_h\|_{C\left[0, \frac{[n]_q}{[n+1]_q}\right]} \leq \frac{9}{h^2} \omega_2(f, h).$$

Theorem 1. *Let $f \in C\left[0, \frac{[n]_{qn}}{[n+1]_{qn}}\right]$. Then for each $x \in \left[0, \frac{[n]_{qn}}{[n+1]_{qn}}\right]$, we have*

$$\begin{aligned} |(L_{n,qn} f)(x) - f(x)| &\leq 5\omega\left(f, \frac{1}{\sqrt{[n+2]_{qn}}}\right) \left(\frac{1+2x}{\sqrt{[n+2]_{qn}}}\right) + \omega_2\left(f, \frac{1}{\sqrt{[n+2]_{qn}}}\right) \\ &\left[2 + \frac{9}{2} \left\{ \left(\frac{q_n^4 [n-1]_{qn} [n]_{qn} - 2q_n [n]_{qn} [n+3]_{qn} + [n+2]_{qn} [n+3]_{qn}}{[n+3]_{qn}} \right) x^2 \right. \right. \\ &\left. \left. + \left(\frac{q_n^3 [n]_{qn}^2 + q_n (1+2q_n) [n]_{qn}^2 - 2[n]_{qn} [n+3]_{qn}}{[n+1]_{qn} [n+3]_{qn}} \right) x + \frac{[2]_{qn} [n]_{qn}^2}{[n+1]_{qn}^2 [n+3]_{qn}} \right\} \right]. \end{aligned}$$

Proof. Using the Steklov mean f_h defined by (3.1), we may write

$$\begin{aligned} |(L_{n,qn} f)(x) - f(x)| &\leq \\ &\leq |(L_{n,qn} (f - f_h))(x)| + |(L_{n,qn} (f_h - f_h(x)))(x)| + |f_h(x) - f(x)|. \end{aligned} \quad (3.2)$$

Using Lemma 3 and Lemma 4, we have

$$|(L_{n,qn} (f - f_h))(x)| \leq \|f - f_h\| \leq \omega_2(f, h).$$

Now, by Taylor's expansion, we have

$$\begin{aligned} f_h(t) &= f_h(x) + (t-x)f'_h(x) + \int_x^t (t-u)f''_h(u)du \\ |(L_{n,qn} (f_h - f_h(x)))(x)| &\leq |(L_{n,qn} ((t-x)f'_h(x)))(x)| \\ &\quad + \left| L_{n,qn} \left(\int_x^t (t-u)f''_h(u)du \right) (x) \right| \\ &\leq \|f'_h\| |(L_{n,qn} (t-x))(x)| \\ &\quad + \|f''_h\| |(L_{n,qn} \left(\int_x^t |t-u| du \right))(x)| \\ &= \|f'_h\| |(L_{n,qn} (t-x))(x)| + \frac{1}{2} \|f''_h\| |(L_{n,qn} (t-x)^2)(x)|. \end{aligned}$$

Applying Lemma 4, Lemma 1 and choosing $h = \sqrt{\frac{1}{[n+2]_{q_n}}}$, we get the required result. $\square$

Let us assume that $\delta_{n,q_n}(x) = \sqrt{L_{n,q_n}((t-x)^2)(x)}$.

Theorem 2. *If f has a continuous derivative f' and $\omega(f', \delta)$ is the modulus of continuity of f' on $[0, \frac{[n]_{q_n}}{[n+1]_{q_n}}]$, then*

$$|(L_{n,q_n} f)(x) - f(x)| \leq M |\mu_{n,q_n}(x)| + \omega(f', \delta_{n,q_n}^2(x)) \left(1 + \delta_{n,q_n}(x)\right),$$

where M is a positive constant such that $|f'(x)| \leq M, x \in \left[0, \frac{[n]_{q_n}}{[n+1]_{q_n}}\right]$ and

$$\mu_{n,q_n}(x) = \left(\frac{q_n[n]_{q_n} - [n+2]_{q_n}}{[n+2]_{q_n}} x + \frac{[n]_{q_n}}{[n+1]_{q_n}[n+2]_{q_n}} \right). \quad (3.3)$$

Proof. On applying the mean value theorem, we get

$$\begin{aligned} f(t) - f(x) &= (t-x)f'(\xi) \\ &= (t-x)f'(x) + (t-x)(f'(\xi) - f'(x)), \end{aligned}$$

where ξ lies between x and t . Using the definition (1.2) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |(L_{n,q_n} f)(x) - f(x)| &\leq |f'(x)| |(L_{n,q_n}(t-x))(x)| + \frac{[n+1]_{q_n}^2}{[n]_{q_n}} \sum_{k=0}^n q_n^{-k} p_{n,k,q_n}(x) \\ &\quad \times \int_0^{\frac{[n]_{q_n}}{[n+1]_{q_n}}} p_{n,k,q_n}(q_n t) |t-x| |f'(\xi) - f'(x)| d_{q_n} t \\ &\leq M |\mu_{n,q_n}(x)| + \frac{[n+1]_{q_n}^2}{[n]_{q_n}} \sum_{k=0}^n q_n^{-k} p_{n,k,q_n}(x) \\ &\quad \times \int_0^{\frac{[n]_{q_n}}{[n+1]_{q_n}}} p_{n,k,q_n}(q_n t) \omega(f', \delta) \left(\frac{|t-x|}{\delta} + 1 \right) |t-x| d_{q_n} t \\ &\leq M |\mu_{n,q_n}(x)| + \frac{\omega(f', \delta)}{\delta} (L_{n,q_n}(t-x)^2)(x) \\ &\quad + \omega(f', \delta) \sqrt{(L_{n,q_n}(t-x)^2)(x)}. \end{aligned}$$

On choosing $\delta := \delta_{n,q_n}^2(x)$, we get the desired result. $\square$

Szász [36], considered the Lipschitz type space defined as:

$$Lip_M^*(\xi) := \left\{ f \in C[0, \infty) : |f(t) - f(x)| \leq M_f \frac{|t - x|^\xi}{(t + x)^{\frac{\xi}{2}}}; \text{ where } M_f \right. \\ \left. \text{is a constant which depends on } f, t \in [0, \infty), x \in (0, \infty) \right\},$$

where $0 < \xi \leq 1$ and $r \in (0, 1]$, to establish the uniform convergence of the Szász operators for functions in this space.

For $r \in (0, 1]$ and $M > 0$, we define an analogue of this space in our case as follows:

$$Lip_M^*(r) := \left\{ f \in C \left[0, \frac{[n]_q}{[n+1]_q} \right] : |f(t) - f(x)| \leq M \frac{|t - x|^r}{(t + x)^{r/2}}; \right. \\ \left. x \in \left(0, \frac{[n]_{q_n}}{[n+1]_{q_n}} \right], t \in \left[0, \frac{[n]_{q_n}}{[n+1]_{q_n}} \right] \right\}.$$

We observe that we get only pointwise approximation due to the presence of x in the error estimate of $(L_{n,q_n} f)(x) - f(x)$, while in the case of Szász operators [36], it turns out that this x gets cancelled leading to the uniform convergence of the operators.

Theorem 3. Let $f \in Lip_M^*(r)$. Then, for all $x \in \left(0, \frac{[n]_{q_n}}{[n+1]_{q_n}} \right]$ we have

$$|(L_{n,q_n} f)(x) - f(x)| \leq M \left(\frac{\delta_{n,q_n}^2(x)}{x} \right)^{r/2}.$$

Proof. Let $r \in (0, 1]$, applying the Hölder's inequality for integration and then for summation with $u = 2/r$ and $v = 2/(2-r)$, and Lemma 1, we have,

$$\begin{aligned} & |(L_{n,q_n} f)(x) - f(x)| \\ & \leq \frac{[n+1]_{q_n}^2}{[n]_{q_n}} \sum_{k=0}^n q_n^{-k} p_{n,k,q_n}(x) \int_0^{\frac{[n]_{q_n}}{[n+1]_{q_n}}} p_{n,k,q_n}(q_n t) |f(t) - f(x)| d_{q_n} t \\ & \leq \frac{[n+1]_{q_n}^2}{[n]_{q_n}} \sum_{k=0}^n q_n^{-k} p_{n,k,q_n}(x) \left\{ \left(\int_0^{\frac{[n]_{q_n}}{[n+1]_{q_n}}} |f(t) - f(x)|^u p_{n,k,q_n}(q_n t) d_{q_n} t \right)^{\frac{1}{u}} \right. \\ & \quad \times \left. \left(\int_0^{\frac{[n]_{q_n}}{[n+1]_{q_n}}} p_{n,k,q_n}(q_n t) d_{q_n} t \right)^{\frac{1}{v}} \right\} \\ & \leq \left(\sum_{k=0}^n \frac{[n+1]_{q_n}^2}{[n]_{q_n}} q_n^{-k} p_{n,k,q_n}(x) \int_0^{\frac{[n]_{q_n}}{[n+1]_{q_n}}} |f(t) - f(x)|^u p_{n,k,q_n}(q_n t) d_{q_n} t \right)^{1/u} \end{aligned}$$

$$\begin{aligned}
& \times \left(\sum_{k=0}^n \frac{[n+1]_{q_n}^2}{[n]_{q_n}} q_n^{-k} p_{n,k,q_n}(x) \int_0^{\frac{[n]_{q_n}}{[n+1]_{q_n}}} p_{n,k,q_n}(q_n t) d_{q_n} t \right)^{1/v} \\
& \leq \frac{M}{(x)^{r/2}} \left(\sum_{k=0}^n \frac{[n+1]_{q_n}^2}{[n]_{q_n}} q_n^{-k} p_{n,k,q_n}(x) \int_0^{\frac{[n]_{q_n}}{[n+1]_{q_n}}} (t-x)^2 p_{n,k,q_n}(q_n t) d_{q_n} t \right)^{1/u} \\
& = M \left\{ \frac{\delta_{n,q_n}^2(x)}{x} \right\}^{r/2}.
\end{aligned}$$

Hence, the proof is completed. $\square$

4. THE CONSTRUCTION OF OPERATORS FOR THE BIVARIATE CASE

In this section, we introduce the bivariate case of the generalized Durrmeyer type operators (1.2). Let $I_j = [0, \frac{[n_j]_{q_{n_j}}}{[n_j+1]_{q_{n_j}}}]$ $j = 1, 2$. In what follows, let (q_{n_j}) , $j = 1, 2$, be sequences in $(0, 1)$ such that $\lim_{n_j \rightarrow \infty} q_{n_j} = 1$. For $I = I_1 \times I_2$, let $C(I)$ denote the space of all real valued continuous functions on I with the norm $\|f\|_{C(I)} = \sup_{(x,y) \in I} |f(x, y)|$. For $f \in C(I)$, the bivariate case of the operators given by (1.2) is defined as:

$$\begin{aligned}
& (L_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) = \\
& \frac{[n_1+1]_{q_{n_1}}^2}{[n_1]_{q_{n_1}}} \frac{[n_2+1]_{q_{n_2}}^2}{[n_2]_{q_{n_2}}} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} q_{n_1}^{-k_1} q_{n_2}^{-k_2} p_{n_1, k_1, q_{n_1}}(x) p_{n_2, k_2, q_{n_2}}(y) \\
& \times \int_{t=0}^{\frac{[n_1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}}} \int_{s=0}^{\frac{[n_2]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}}} p_{n_1, k_1, q_{n_1}}(q_{n_1} t) p_{n_2, k_2, q_{n_2}}(q_{n_2} s) f(t, s) d_{q_{n_1}} t d_{q_{n_2}} s.
\end{aligned} \tag{4.1}$$

Lemma 5. Let $e_{ij}(x, y) = x^i y^j$, $(i, j) \in \mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\}$ with $i + j \leq 2$. For the operators given by (4.1), there hold the following equalities:

$$\begin{aligned}
& \text{i) } (L_{n_1, n_2, q_{n_1}, q_{n_2}} e_{00})(x, y) = 1; \\
& \text{ii) } (L_{n_1, n_2, q_{n_1}, q_{n_2}} e_{10})(x, y) = \frac{[n_1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}[n_1+2]_{q_{n_1}}} + \frac{q_{n_1}[n_1]_{q_{n_1}}x}{[n_1+2]_{q_{n_1}}}; \\
& \text{iii) } (L_{n_1, n_2, q_{n_1}, q_{n_2}} e_{01})(x, y) = \frac{[n_2]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}[n_2+2]_{q_{n_2}}} + \frac{q_{n_2}[n_2]_{q_{n_2}}y}{[n_2+2]_{q_{n_2}}}; \\
& \text{iv) } (L_{n_1, n_2, q_{n_1}, q_{n_2}} e_{20})(x, y) = \frac{[n_1]_{q_{n_1}}}{[n_1+2]_{q_{n_1}}[n_1+3]_{q_{n_1}}} \left\{ [n_1-1]_{q_{n_1}} q_{n_1}^4 x^2 \right. \\
& \quad \left. + \frac{q_{n_1}^3 x [n_1]_{q_{n_1}} + q_{n_1} (1+2q_{n_1}) x [n_1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}} + \frac{(1+q_{n_1})[n_1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}^2} \right\};
\end{aligned}$$

$$\begin{aligned} \text{v) } (L_{n_1, n_2, q_{n_1}, q_{n_2}} e_{02})(x, y) &= \frac{[n_2]_{q_{n_2}}}{[n_2 + 2]_{q_{n_2}} [n_2 + 3]_{q_{n_2}}} \left\{ [n_2 - 1]_{q_{n_2}} q_{n_2}^4 y^2 \right. \\ &\quad \left. + \frac{q_{n_2}^3 y [n_2]_{q_{n_2}} + q_{n_2} (1 + 2q_{n_2}) y [n_2]_{q_{n_2}}}{[n_2 + 1]_{q_{n_2}}} + \frac{(1 + q_{n_2}) [n_2]_{q_{n_2}}}{[n_2 + 1]_{q_{n_2}}^2} \right\}. \end{aligned}$$

Next, we state Korovkin type theorem, given by Volkov [37]. With the help of this theorem we study the convergence of the sequence $(L_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y)$ to the function $f(x, y)$.

Theorem 4 ([37]). *Let $J_1, J_2 \subset \mathbb{R}$ be compact intervals of the real line and let $\{E_{m,n} f\}$ be a sequence of linear positive operators applying the space $C(J_1 \times J_2)$ into itself. Suppose that the following relations*

- i) $E_{m,n}(1; x, y) = 1 + a_{m,n}(x, y),$
- ii) $E_{m,n}(t; x, y) = x + b_{m,n}(x, y),$
- iii) $E_{m,n}(s; x, y) = y + c_{m,n}(x, y),$
- iv) $E_{m,n}(t^2 + s^2; x, y) = x^2 + y^2 + d_{m,n}(x, y),$

hold, for each $(x, y) \in J_1 \times J_2$.

If the sequence $\{a_{m,n}(x, y)\}, \{b_{m,n}(x, y)\}, \{c_{m,n}(x, y)\}, \{d_{m,n}(x, y)\}$ converge to zero uniformly on $J_1 \times J_2$, then the sequence $\{E_{m,n} f\}$ converges to f , uniformly on $J_1 \times J_2$, for each $f \in C(J_1 \times J_2)$.

Remark 2. In view of Theorem 4 and Lemma 5, it easily follows that for each $f \in C(I)$,

$$\lim_{n_1, n_2 \rightarrow \infty} (L_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) = f(x, y)$$

uniformly on I .

In the following we give some numerical results which show the rate of convergence of the operator $L_{n_1, n_2, q_{n_1}, q_{n_2}} f$ to certain functions using Maple algorithms.

Example 1. Let us consider $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(x, y) = x^2 y^2 + 2x^2 y - 3y^2$. The convergence of the operator $L_{n_1, n_2, q_{n_1}, q_{n_2}} f$ to the function f is illustrated in Figure 1 and Figure 2, respectively for $n_1 = n_2 = 10, q_{n_1} = q_{n_2} = 0.5$ and $n_1 = n_2 = 100, q_{n_1} = q_{n_2} = 0.9$, respectively. We remark that as the values of n_1 and n_2 increase, the error in the approximation of the function by the operator becomes smaller.

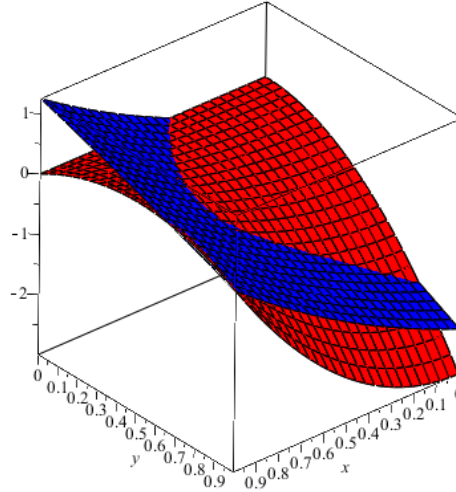


FIGURE 1. The convergence of $(L_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y)$ to $f(x, y)$, for $q_{n_1} = q_{n_2} = 0.5$ (red f , blue $L_{n_1, n_2, q_{n_1}, q_{n_2}}$)

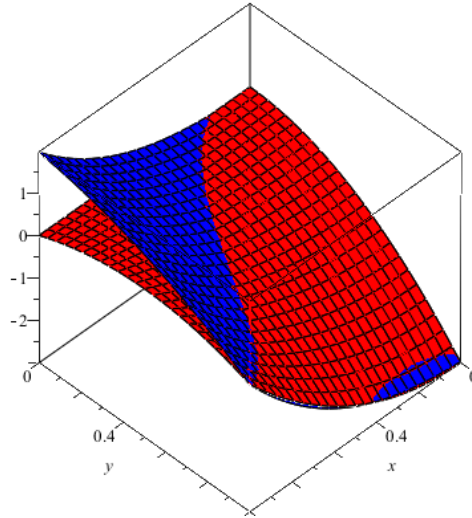


FIGURE 2. The convergence of $(L_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y)$ to $f(x, y)$, for $q_{n_1} = q_{n_2} = 0.9$ (red f , blue $L_{n_1, n_2, q_{n_1}, q_{n_2}}$)

If $J_1, J_2 \subset \mathbb{R}$ are compact intervals and $f \in \mathbb{R}^{J_1 \times J_2}$, the modulus of continuity $\omega(\delta_1, \delta_2)$, for any $\delta_1 > 0, \delta_2 > 0$ is defined as

$$\omega(\delta_1, \delta_2) = \sup_{(x_1, y_1), (x_2, y_2) \in J_1 \times J_2} \left\{ |f(x_1, y_1) - f(x_2, y_2)| : |x_1 - x_2| \leq \delta_1, |y_1 - y_2| \leq \delta_2 \right\}.$$

In what follows, let

$$\delta_{n_1, q_{n_1}}(x) = \left((L_{n_1, n_2, q_{n_1}, q_{n_2}}(t-x)^2)(x, y) \right)^{1/2} = \left((L_{n_1, q_{n_1}}(t-x)^2)(x) \right)^{1/2},$$

and

$$\delta_{n_2, q_{n_2}}(y) = \left((L_{n_1, n_2, q_{n_1}, q_{n_2}}(s-y)^2)(x, y) \right)^{1/2} = \left((L_{n_2, q_{n_2}}(s-y)^2)(y) \right)^{1/2}.$$

Next, we recall the following Shisha-Mond theorem [33].

Theorem 5 ([33]). *Let $J_1, J_2 \subset \mathbb{R}$ be compact intervals, $B(J_1 \times J_2) = \{f \in \mathbb{R}^{J_1 \times J_2} : f \text{ is bounded on } J_1 \times J_2\}$ and let $E : C(J_1 \times J_2) \rightarrow B(J_1 \times J_2)$ be a linear positive operator. For each $f \in C(J_1 \times J_2)$, $(x, y) \in J_1 \times J_2$ and any $\delta_1 > 0, \delta_2 > 0$, the following inequality*

$$\begin{aligned} |E(f; x, y) - f(x, y)| &\leq |f(x, y)| |E(1; x, y) - 1| \\ &+ \left\{ E(1; x, y) + \delta_1^{-1} \sqrt{E(1; x, y) E((t-x)^2; x, y)} + \delta_2^{-1} \sqrt{E(1; x, y) E((s-y)^2; x, y)} \right. \\ &\left. + \delta_1^{-1} \delta_2^{-1} E(1; x, y) \sqrt{E((t-x)^2; x, y) E((s-y)^2; x, y)} \right\} \omega(\delta_1, \delta_2) \end{aligned}$$

holds.

Theorem 6. *Let $f \in C(I)$ and $(x, y) \in I$. Then the operator $L_{n_1, n_2, q_{n_1}, q_{n_2}}$ satisfies the following inequality*

$$|(L_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y)| \leq 4\omega(\delta_{n_1, q_{n_1}}(x), \delta_{n_2, q_{n_2}}(y)).$$

Proof. Applying Theorem 5 and Lemma 5, and choosing $\delta_1 = \delta_{n_1, q_{n_1}}(x)$ and $\delta_2 = \delta_{n_2, q_{n_2}}(y)$ we get

$$\begin{aligned} |(L_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y)| &\leq \left\{ 1 + \delta_{n_1, q_{n_1}}^{-1}(x) \sqrt{(L_{n_1, n_2, q_{n_1}, q_{n_2}}(t-x)^2)(x, y)} \right. \\ &+ \delta_{n_2, q_{n_2}}^{-1}(y) \sqrt{(L_{n_1, n_2, q_{n_1}, q_{n_2}}(s-y)^2)(x, y)} \\ &+ \delta_{n_1, q_{n_1}}^{-1}(x) \delta_{n_2, q_{n_2}}^{-1}(y) \sqrt{(L_{n_1, n_2, q_{n_1}, q_{n_2}}(t-x)^2)(x, y) (L_{n_1, n_2, q_{n_1}, q_{n_2}}(s-y)^2)(x, y)} \left. \right\} \\ &\times \omega(\delta_{n_1, q_{n_1}}(x); \delta_{n_2, q_{n_2}}(y)) \end{aligned}$$

Hence, we get the required result. $\square$

For $f \in C(I)$ and $\delta > 0$, the first order complete modulus of continuity for the bivariate case is defined as:

$$\bar{\omega}(f; \delta) = \sup \left\{ |f(t, s) - f(x, y)| : (t, s), (x, y) \in I \text{ and } \sqrt{(t-x)^2 + (s-y)^2} \leq \delta \right\}.$$

Further, the partial moduli of continuity with respect to x and y are given by

$$\omega_1(f; \delta) = \sup \left\{ |f(x_1, y) - f(x_2, y)| : y \in I_2 \text{ and } |x_1 - x_2| \leq \delta \right\},$$

and

$$\omega_2(f; \delta) = \sup \left\{ |f(x, y_1) - f(x, y_2)| : x \in I_1 \text{ and } |y_1 - y_2| \leq \delta \right\}.$$

Theorem 7. For $f \in C(I)$, there holds the inequality

$$|(L_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y)| \leq 2(\omega_1(f; \delta_{n_1, q_{n_1}}(x)) + \omega_2(f; \delta_{n_2, q_{n_2}}(y))).$$

Proof. Using the definition of partial moduli of continuity, Lemma 5 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |(L_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y)| &\leq (L_{n_1, n_2, q_{n_1}, q_{n_2}} |f(t, s) - f(x, y)|)(x, y) \\ &\leq (L_{n_1, n_2, q_{n_1}, q_{n_2}} |f(t, s) - f(t, y)|)(x, y) + (L_{n_1, n_2, q_{n_1}, q_{n_2}} |f(t, y) - f(x, y)|)(x, y) \\ &\leq \omega_2(f; \delta_{n_2, q_{n_2}}(y)) \left[1 + \frac{1}{\delta_{n_2, q_{n_2}}(y)} (L_{n_1, n_2, q_{n_1}, q_{n_2}} |s - y|)(x, y) \right] \\ &\quad + \omega_1(f; \delta_{n_1, q_{n_1}}(x)) \left[1 + \frac{1}{\delta_{n_1, q_{n_1}}(x)} (L_{n_1, n_2, q_{n_1}, q_{n_2}} |t - x|)(x, y) \right] \\ &\leq \omega_2(f; \delta_{n_2, q_{n_2}}(y)) \left[1 + \frac{1}{\delta_{n_2, q_{n_2}}(y)} \left((L_{n_1, n_2, q_{n_1}, q_{n_2}} (s - y)^2)(x, y) \right)^{1/2} \right] \\ &\quad + \omega_1(f; \delta_{n_1, q_{n_1}}(x)) \left[1 + \frac{1}{\delta_{n_1, q_{n_1}}(x)} \left((L_{n_1, n_2, q_{n_1}, q_{n_2}} (t - x)^2)(x, y) \right)^{1/2} \right], \end{aligned}$$

from which the required result is immediate. $\square$

4.1. Degree of approximation

In this section, let us assume that $\lim_{n_j \rightarrow \infty} q_{n_j} = 1, j = 1, 2$. We study the degree of approximation for the bivariate operators (4.1) by means of the Lipschitz class.

For $0 < \alpha \leq 1$ and $0 < \beta \leq 1$, we define the Lipschitz class $Lip_M(\alpha, \beta)$ for the bivariate case as follows:

$$|f(t, s) - f(x, y)| \leq M |t - x|^\alpha |s - y|^\beta, \text{ for every } (t, s), (x, y) \in I.$$

Theorem 8. Let $f \in Lip_M(\alpha, \beta)$. Then, we have

$$|(L_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y)| \leq M \delta_{n_1, q_{n_1}}^\alpha(x) \delta_{n_2, q_{n_2}}^\beta(y).$$

Proof. By our hypothesis, we may write

$$|(L_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y)| \leq (L_{n_1, n_2, q_{n_1}, q_{n_2}} |f(t, s) - f(x, y)|)(x, y)$$

$$\begin{aligned} &\leq M(L_{n_1, n_2, q_{n_1}, q_{n_2}} |t-x|^\alpha |s-y|^\beta)(x, y) \\ &= M(L_{n_1, n_2, q_{n_1}, q_{n_2}} |t-x|^\alpha)(x) (L_{n_1, n_2, q_{n_1}, q_{n_2}} |s-y|^\beta)(y). \end{aligned}$$

Now, using the Hölder's inequality with $u_1 = \frac{2}{\alpha}$, $v_1 = \frac{2}{2-\alpha}$ and $u_2 = \frac{2}{\beta}$, $v_2 = \frac{2}{2-\beta}$, and Lemma 5, we have

$$\begin{aligned} &|(L_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y)| \\ &\leq M \{(L_{n_1, n_2, q_{n_1}, q_{n_2}} (t-x)^2)(x)\}^{\alpha/2} (L_{n_1, n_2, q_{n_1}, q_{n_2}} 1)(x) \\ &\quad \times \{(L_{n_1, n_2, q_{n_1}, q_{n_2}} (s-y)^2)(y)\}^{\beta/2} (L_{n_1, n_2, q_{n_1}, q_{n_2}} 1)(y) \\ &= M \delta_{n_1, q_{n_1}}^\alpha(x) \delta_{n_2, q_{n_2}}^\beta(y). \end{aligned}$$

Thus, we get the desired result. $\square$

Let $C^1(I)$ be the space of functions $f(x, y)$ whose first order partial derivatives are continuous on I .

Theorem 9. Let $f \in C^1(I)$. Then, we have

$$|(L_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y)| \leq \|f'_x\|_{C(I)} \delta_{n_1, q_{n_1}}(x) + \|f'_y\|_{C(I)} \delta_{n_2, q_{n_2}}(y).$$

Proof. For $(t, s) \in I$, we have

$$f(t, s) - f(x, y) = \int_x^t f'_u(u, s) du + \int_y^s f'_v(x, v) dv.$$

Applying the operator $(L_{n_1, n_2, q_{n_1}, q_{n_2}} \cdot)(x, y)$ on both sides of above equality and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} &|(L_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y)| \leq \left(L_{n_1, n_2, q_{n_1}, q_{n_2}} \left| \int_x^t f'_u(u, s) du \right| \right)(x, y) \\ &\quad + \left(L_{n_1, n_2, q_{n_1}, q_{n_2}} \left| \int_y^s f'_v(x, v) dv \right| \right)(x, y) \\ &\leq \|f'_x\|_{C(I)} (L_{n_1, q_{n_1}} |t-x|)(x) + \|f'_y\|_{C(I)} (L_{n_2, q_{n_2}} |s-y|)(y) \\ &\leq \|f'_x\|_{C(I)} \left((L_{n_1, q_{n_1}} (t-x)^2)(x) \right)^{1/2} + \|f'_y\|_{C(I)} \left((L_{n_2, q_{n_2}} (s-y)^2)(y) \right)^{1/2}. \end{aligned}$$

Hence, we get the required result. $\square$

Let $C^2(I)$ denote the space of functions $f(x, y)$ whose second order partial derivatives are continuous on I , endowed with the norm

$$\|f\|_{C^2(I)} = \|f\|_{C(I)} + \sum_{i=1}^2 \left(\left\| \frac{\partial^i f}{\partial x^i} \right\|_{C(I)} + \left\| \frac{\partial^i f}{\partial y^i} \right\|_{C(I)} \right).$$

The Peetre's K -functional of the function $f \in C(I)$ is defined by

$$\mathcal{K}(f; \delta) = \inf_{g \in C^2(I)} \{ \|f - g\|_{C(I)} + \delta \|g\|_{C^2(I)} \}, \delta > 0.$$

Also, from [12, pp.192] it is known that

$$\mathcal{K}(f; \delta) \leq M \left\{ \bar{\omega}_2(f; \sqrt{\delta}) + \min(1, \delta) \|f\|_{C(I)} \right\}, \quad (4.2)$$

holds for all $\delta > 0$. The constant M in the above inequality is independent of δ and f and $\bar{\omega}_2(f; \sqrt{\delta})$ is the second order modulus of continuity for the bivariate case.

Theorem 10. *Let $f \in C(I)$. Then for all $n_1, n_2 \in \mathbb{N}$ and each $(x, y) \in I$ there exists a constant $\mathcal{C} > 0$ such that*

$$\begin{aligned} |(L_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y)| &\leq \mathcal{C} \left\{ \bar{\omega}_2 \left(f; \frac{1}{2} \sqrt{C_{n_1, n_2, q_{n_1}, q_{n_2}}(x, y)} \right) \right. \\ &\quad \left. + \min \left(1, \frac{C_{n_1, n_2, q_{n_1}, q_{n_2}}(x, y)}{4} \right) \|f\|_{C(I)} \right\} + \omega(f, \sqrt{\psi_{n_1, n_2, q_{n_1}, q_{n_2}}(x, y)}), \end{aligned}$$

where

$$\begin{aligned} \psi_{n_1, n_2, q_{n_1}, q_{n_2}}(x, y) &= \left(\frac{[n_1]_{q_{n_1}} (1 + q_{n_1} x [n_1 + 1]_{q_{n_1}})}{[n_1 + 1]_{q_{n_1}} [n_1 + 2]_{q_{n_1}}} - x \right)^2 \\ &\quad + \left(\frac{[n_2]_{q_{n_2}} (1 + q_{n_2} y [n_2 + 1]_{q_{n_2}})}{[n_2 + 1]_{q_{n_2}} [n_2 + 2]_{q_{n_2}}} - y \right)^2, \end{aligned}$$

and

$$C_{n_1, n_2, q_{n_1}, q_{n_2}}(x, y) = \delta_{n_1, q_{n_1}}^2(x) + \delta_{n_2, q_{n_2}}^2(y) + \psi_{n_1, n_2, q_{n_1}, q_{n_2}}(x, y).$$

Proof. First, we define the auxiliary operator

$$\begin{aligned} (L_{n_1, n_2, q_{n_1}, q_{n_2}}^* f)(x, y) &= f(x, y) + (L_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) \\ &\quad - f \left(\frac{[n_1]_{q_{n_1}} (1 + q_{n_1} x [n_1 + 1]_{q_{n_1}})}{[n_1 + 1]_{q_{n_1}} [n_1 + 2]_{q_{n_1}}}, \frac{[n_2]_{q_{n_2}} (1 + q_{n_2} y [n_2 + 1]_{q_{n_2}})}{[n_2 + 1]_{q_{n_2}} [n_2 + 2]_{q_{n_2}}} \right). \end{aligned}$$

Applying Lemma 4 it is obvious that

$$(L_{n_1, n_2, q_{n_1}, q_{n_2}}^* 1)(x, y) = 1, (L_{n_1, n_2, q_{n_1}, q_{n_2}}^* t)(x, y) = x, (L_{n_1, n_2, q_{n_1}, q_{n_2}}^* s)(x, y) = y,$$

and hence

$$(L_{n_1, n_2, q_{n_1}, q_{n_2}}^* (t - x))(x, y) = 0 = (L_{n_1, n_2, q_{n_1}, q_{n_2}}^* (s - y))(x, y).$$

Further, for $f \in C(I)$, we have (see also Lemma 3)

$$\|L_{n_1, n_2, q_{n_1}, q_{n_2}}^* f\|_{C(I)} \leq 3 \|f\|_{C(I)}.$$

Let $g \in C^2(I)$ be arbitrary. By Taylor's theorem, we have

$$\begin{aligned} g(t, s) - g(x, y) &= \frac{\partial g(x, y)}{\partial x}(t - x) + \int_x^t (t - u) \frac{\partial^2 g(u, y)}{\partial u^2} du \\ &\quad + \frac{\partial g(x, y)}{\partial y}(s - y) + \int_y^s (s - v) \frac{\partial^2 g(x, v)}{\partial v^2} dv. \end{aligned}$$

Applying the operator $L_{n_1, n_2, q_{n_1}, q_{n_2}}^*$ on both sides of the above equation, we get

$$\begin{aligned} (L_{n_1, n_2, q_{n_1}, q_{n_2}}^* g)(x, y) - g(x, y) &= \left(L_{n_1, n_2, q_{n_1}, q_{n_2}}^* \left(\int_x^t (t - u) \frac{\partial^2 g(u, y)}{\partial u^2} du \right) \right)(x, y) \\ &\quad + \left(L_{n_1, n_2, q_{n_1}, q_{n_2}}^* \left(\int_y^s (s - v) \frac{\partial^2 g(x, v)}{\partial v^2} dv \right) \right)(x, y). \end{aligned}$$

For simplicity, we set

$$z_1 = \frac{[n_1]_{q_{n_1}}}{[n_1 + 1]_{q_{n_1}} [n_1 + 2]_{q_{n_1}}} \left(1 + q_{n_1} [n_1 + 1]_{q_{n_1}} x \right)$$

and

$$z_2 = \frac{[n_2]_{q_{n_2}}}{[n_2 + 1]_{q_{n_2}} [n_2 + 2]_{q_{n_2}}} \left(1 + q_{n_2} [n_2 + 1]_{q_{n_2}} y \right).$$

Then,

$$\begin{aligned} & |(L_{n_1, n_2, q_{n_1}, q_{n_2}}^* g(t, s))(x, y) - g(x, y)| \\ & \leq L_{n_1, n_2, q_{n_1}, q_{n_2}} \left(\left| \int_x^t (t - u) \frac{\partial^2 g(u, y)}{\partial u^2} du \right| \right)(x, y) + \left| \int_x^{z_1} \left| (z_1 - u) \right| \left| \frac{\partial^2 g(u, y)}{\partial u^2} \right| du \right| \\ & + \left| \int_y^{z_2} \left| (z_2 - v) \right| \left| \frac{\partial^2 g(x, v)}{\partial v^2} \right| dv \right| + L_{n_1, n_2, q_{n_1}, q_{n_2}} \left(\left| \int_y^s (s - v) \frac{\partial^2 g(x, v)}{\partial v^2} dv \right| \right)(x, y) \\ & \leq \left\{ (L_{n_1, n_2, q_{n_1}, q_{n_2}}(t - x)^2)(x, y) + (z_1 - x)^2 \right\} \|g\|_{C^2(I)} \\ & + \left\{ (L_{n_1, n_2, q_{n_1}, q_{n_2}}(s - y)^2)(x, y) + (z_2 - y)^2 \right\} \|g\|_{C^2(I)} \\ & = \left(\delta_{n_1, q_{n_1}}^2(x) + \delta_{n_2, q_{n_2}}^2(y) + \psi_{n_1, n_2, q_{n_1}, q_{n_2}}(x, y) \right) \|g\|_{C^2(I)} \\ & = C_{n_1, n_2, q_{n_1}, q_{n_2}}(x, y) \|g\|_{C^2(I)}. \end{aligned}$$

Hence, we have

$$\begin{aligned} & |(L_{n_1, n_2, q_{n_1}, q_{n_2}} f(t, s))(x, y) - f(x, y)| \leq |(L_{n_1, n_2, q_{n_1}, q_{n_2}}^* \{f - g\})(x, y)| \\ & + |g(x, y) - f(x, y)| + |(L_{n_1, n_2, q_{n_1}, q_{n_2}}^* g)(x, y) - g(x, y)| + |f(z_1, z_2) - f(x, y)| \end{aligned}$$

$$\leq \left(4\|f - g\| + C_{n_1, n_2, q_{n_1}, q_{n_2}}(x, y)\|g\|_{C^2(I)} \right) + \omega\left(f; \sqrt{\psi_{n_1, n_2, q_{n_1}, q_{n_2}}(x, y)}\right).$$

Taking the infimum on the right side over all $g \in C^2(I)$, we get

$$\begin{aligned} |(L_{n_1, n_2, q_{n_1}, q_{n_2}} f(t, s))(x, y) - f(x, y)| &\leq 4K_2 \left(f, \frac{C_{n_1, n_2, q_{n_1}, q_{n_2}}(x, y)}{4} \right) \\ &\quad + \omega\left(f; \sqrt{\psi_{n_1, n_2, q_{n_1}, q_{n_2}}(x, y)}\right). \end{aligned}$$

Now, using the relation (4.2), we get the desired result. $\square$

5. CONSTRUCTION OF GBS OPERATORS OF q -BERNSTEIN-SCHURER-DURMEYER TYPE

Bögel was the first person who proposed the concepts of B -continuous and B -differentiable functions in his papers [9] and [10]. After that Dobrescu and Matei [17] showed that any B -continuous function on a bounded interval can be uniformly approximated by the boolean sum of bivariate generalization of Bernstein polynomials. For detailed history of the work in this direction we refer the reader to some of the papers in this direction (cf. [3], [6], [7], [19], [34] and [35] etc.).

Next, we recall some definitions which will be used in this section.

A function $f : I \rightarrow \mathbb{R}$ is called a B -continuous (Bögel continuous) at $(x_0, y_0) \in I$ if

$$\lim_{(x, y) \rightarrow (x_0, y_0)} \Delta f[(x, y); (x_0, y_0)] = 0,$$

where $\Delta f[(x, y); (x_0, y_0)]$ denotes the mixed difference defined by

$$\Delta f[(x, y); (x_0, y_0)] = f(x, y) - f(x, y_0) - f(x_0, y) + f(x_0, y_0).$$

The space of all B -continuous functions is denoted by $C_b(I)$.

A function $f : I \rightarrow \mathbb{R}$ is called a B -differentiable (Bögel differentiable) function at $(x_0, y_0) \in I$ if the limit

$$\lim_{(x, y) \rightarrow (x_0, y_0)} \frac{\Delta f[(x, y); (x_0, y_0)]}{(x - x_0)(y - y_0)}$$

exists finitely.

The limit is called the B -differential of f at the point (x_0, y_0) and is denoted by $D_B f(x_0, y_0)$ and the space of all B -differentiable functions is denoted by $D_b(I)$ i.e. $D_b(I) = \{f : I \rightarrow \mathbb{R} | D_B f \text{ exists}\}$. The function $f : I \rightarrow \mathbb{R}$ is called B -bounded on I if there exists $M > 0$ such that $|\Delta f[(t, s); (x, y)]| \leq M$, for every $(x, y), (t, s) \in I$.

Throughout this paper $B_b(I)$, denotes all B -bounded functions on $I \rightarrow \mathbb{R}$ equipped with the norm

$$\|f\|_B = \sup_{(x, y), (t, s) \in I} |\Delta f[(t, s); (x, y)]|.$$

The mixed modulus of smoothness of $f \in C_b(I)$ is a function $\omega_{mixed} : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ defined as

$$\omega_{mixed}(f; \delta_1, \delta_2) := \sup \{ |\Delta f[(t, s); (x, y)]| : |x - t| < \delta_1, |y - s| < \delta_2 \},$$

for all $(x, y), (t, s) \in I$ and for any $(\delta_1, \delta_2) \in (0, \infty) \times (0, \infty)$.

Let us assume that $\lim_{n_j \rightarrow \infty} q_{n_j} = 1$, $j = 1, 2$. For $f \in C_b(I)$, the GBS operator $T_{n_1, n_2, q_{n_1}, q_{n_2}}$ associated to $L_{n_1, n_2, q_{n_1}, q_{n_2}}$ is defined as:

$$\begin{aligned} & (T_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) \\ &= \frac{[n_1 + 1]_{q_{n_1}}^2 [n_2 + 1]_{q_{n_2}}^2}{[n_1]_{q_{n_1}} [n_2]_{q_{n_2}}} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} q_{n_1}^{-k_1} q_{n_2}^{-k_2} p_{n_1, k_1, q_{n_1}}(x) p_{n_2, k_2, q_{n_2}}(y) \times \\ & \int_{t=0}^{\frac{[n_1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}}} \int_{s=0}^{\frac{[n_2]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}}} p_{n_1, k_1, q_{n_1}}(q_{n_1} t) p_{n_2, k_2, q_{n_2}}(q_{n_2} s) [f(x, s) + f(t, y) - f(t, s)] dq_{n_1} t dq_{n_2} s. \end{aligned} \quad (5.1)$$

Example 2. For $n_1 = n_2 = 10$, $q_{n_1} = \frac{n_1}{n_1 + 1}$, $q_{n_2} = 1 - \frac{1}{n_2}$, the comparison of convergence of the function $L_{n_1, n_2, q_{n_1}, q_{n_2}} f$ (green) and its GBS type function $T_{n_1, n_2, q_{n_1}, q_{n_2}} f$ (yellow) to $f(x, y) = x^2 y^2 + 2x^2 y - 3y^2$ (red) is illustrated in Figure 3. We remark that the operator $T_{n_1, n_2, q_{n_1}, q_{n_2}}$ gives a better approximation than the operator $L_{n_1, n_2, q_{n_1}, q_{n_2}}$. In the Table 1 we computed the error of approximation for $L_{n_1, n_2, q_{n_1}, q_{n_2}}$ and $T_{n_1, n_2, q_{n_1}, q_{n_2}}$ at certain points.

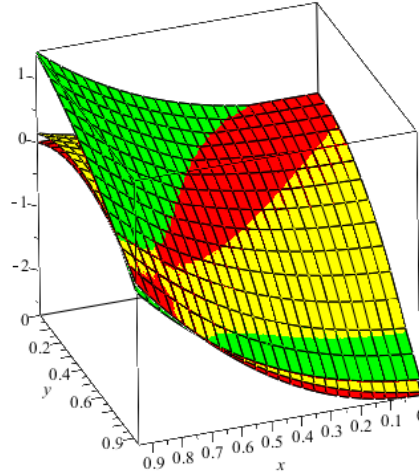


FIGURE 3. The convergence of $L_{n_1, n_2, q_{n_1}, q_{n_2}} f$ and $T_{n_1, n_2, q_{n_1}, q_{n_2}} f$ to f (red f , green $L_{n_1, n_2, q_{n_1}, q_{n_2}} f$ and yellow $T_{n_1, n_2, q_{n_1}, q_{n_2}} f$)

TABLE 1. Error of approximation for $L_{n_1, n_2, q_{n_1}, q_{n_2}}$ and $T_{n_1, n_2, q_{n_1}, q_{n_2}}$

x	y	$ (L_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y) $	$ (T_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y) $
0.9	0.1	1.0104585870	0.0811695820
0.8	0.1	0.7228289936	0.0026123070
0.7	0.1	0.4853680787	0.0472137716
0.6	0.1	0.2931805555	0.0732162679
0.9	0.2	0.7971775100	0.0721465470
0.8	0.2	0.5423273555	0.0022888760
0.7	0.2	0.3335672866	0.0406177704
0.6	0.2	0.1660020136	0.0614810095
0.4	0.4	0.1556621432	0.0204867702
0.5	0.5	0.1116013761	0.0147209150
0.6	0.6	0.0665209683	0.0077961258
0.2	0.9	0.1925811440	0.1020977850
0.1	0.9	0.1899526260	0.0854533490

Theorem 11. For every $f \in C_b(I)$, at each point $(x, y) \in I$, we have

$$|(T_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y)| \leq 4\omega_{mixed}(f; \delta_{n_1, q_{n_1}}(x), \delta_{n_2, q_{n_2}}(y)).$$

Proof. By the definition of $\omega_{mixed}(f; \delta_{n_1}, \delta_{n_2})$ and using the elementary inequality

$$\omega_{mixed}(f; \lambda_1 \delta_{n_1}, \lambda_2 \delta_{n_2}) \leq (1 + \lambda_1)(1 + \lambda_2) \omega_{mixed}(f; \delta_{n_1}, \delta_{n_2}),$$

where $\lambda_1, \lambda_2 > 0$ and $\delta_{n_1}, \delta_{n_2} > 0$, we may write

$$\begin{aligned} |\Delta f[(t, s); (x, y)]| &\leq \omega_{mixed}(f; |t - x|, |s - y|) \\ &\leq \left(1 + \frac{|t - x|}{\delta_1}\right) \left(1 + \frac{|s - y|}{\delta_2}\right) \omega_{mixed}(f; \delta_1, \delta_2), \end{aligned} \quad (5.2)$$

for every $(x, y), (t, s) \in I$ and for any $\delta_1, \delta_2 > 0$. From the definition of $\Delta f[(t, s); (x, y)]$ we get

$$f(x, s) + f(t, y) - f(t, s) = f(x, y) - \Delta f[(t, s); (x, y)].$$

On applying the linear positive operator $(L_{n_1, n_2, q_{n_1}, q_{n_2}})(x, y)$ to this equality and taking into account the definition of operator $T_{n_1, n_2, q_{n_1}, q_{n_2}}$ given by (5.1) we can write

$$\begin{aligned} &(T_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) \\ &= f(x, y) (L_{n_1, n_2, q_{n_1}, q_{n_2}} e_{00})(x, y) - (L_{n_1, n_2, q_{n_1}, q_{n_2}} \Delta f[(t, s); (x, y)])(x, y). \end{aligned}$$

Since $(L_{n_1, n_2, q_{n_1}, q_{n_2}} e_{00})(x, y) = 1$, considering the inequality (5.2) and applying the Cauchy-Schwarz inequality we obtain,

$$\begin{aligned} |(T_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y)| &\leq (L_{n_1, n_2, q_{n_1}, q_{n_2}} |\Delta f[(t, s); (x, y)]|)(x, y) \\ &\leq \left((L_{n_1, n_2, q_{n_1}, q_{n_2}} e_{00})(x, y) + \delta_{n_1}^{-1} \sqrt{(L_{n_1, n_2, q_{n_1}, q_{n_2}} (t - x)^2)(x, y)} \right. \\ &\quad \left. + \delta_{n_2}^{-1} \sqrt{(L_{n_1, n_2, q_{n_1}, q_{n_2}} (s - y)^2)(x, y)} + \delta_{n_1}^{-1} \delta_{n_2}^{-1} \sqrt{(L_{n_1, n_2, q_{n_1}, q_{n_2}} (t - x)^2)(x, y)} \right. \\ &\quad \left. \times \sqrt{(L_{n_1, n_2, q_{n_1}, q_{n_2}} (s - y)^2)(x, y)} \right) \omega_{mixed}(f; \delta_{n_1}, \delta_{n_2}), \end{aligned}$$

from which the desired result is immediate, on choosing $\delta_{n_1} = \delta_{n_1, q_{n_1}}(x)$ and $\delta_{n_2} = \delta_{n_2, q_{n_2}}(y)$. $\square$

For $f \in C_b(I)$, the Lipschitz class $Lip_M(\xi, \gamma)$ with $M > 0$ and $\xi, \gamma \in (0, 1]$ is defined as

$$\begin{aligned} Lip_M(\xi, \gamma) &= \left\{ f \in C_b(I) : |\Delta f[(t, s); (x, y)]| \leq M |t - x|^\xi |s - y|^\gamma, \right. \\ &\quad \left. \text{for } (t, s), (x, y) \in I \right\}. \end{aligned}$$

Our next theorem gives the degree of approximation for the operators $T_{n_1, n_2, q_{n_1}, q_{n_2}}$ by means of the Lipschitz class of Bögél continuous functions.

Theorem 12. Let $f \in Lip_M(\xi, \gamma)$ then we have

$$\left| (T_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y) \right| \leq M \delta_{n_1, q_{n_1}}^\xi(x) \delta_{n_2, q_{n_2}}^\gamma(y),$$

for $M > 0$, $\xi, \gamma \in (0, 1]$.

Proof. By the definition of the operator $(T_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y)$ and Lemma 4, we may write

$$\begin{aligned} (T_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) &= (L_{n_1, n_2, q_{n_1}, q_{n_2}} (f(x, s) + f(t, y) - f(t, s)))(x, y) \\ &= (L_{n_1, n_2, q_{n_1}, q_{n_2}} \{f(x, y) - \Delta f[(t, s); (x, y)]\})(x, y) \\ &= f(x, y) (L_{n_1, n_2, q_{n_1}, q_{n_2}} e_{00})(x, y) \\ &\quad - (L_{n_1, n_2, q_{n_1}, q_{n_2}} (\Delta f[(t, s); (x, y)]))(x, y). \end{aligned}$$

By the hypothesis, we get

$$\begin{aligned} \left| (T_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y) \right| &\leq (L_{n_1, n_2, q_{n_1}, q_{n_2}} |\Delta f[(t, s); (x, y)]|)(x, y) \\ &\leq M \left(L_{n_1, n_2, q_{n_1}, q_{n_2}} |t - x|^\xi |s - y|^\gamma \right)(x, y) \\ &= M (L_{n_1, q_{n_1}} |t - x|^\xi)(x) (L_{n_2, q_{n_2}} |s - y|^\gamma)(y). \end{aligned}$$

Now, using the Hölder's inequality with $u_1 = 2/\xi$, $v_1 = 2/(2 - \xi)$ and $u_2 = 2/\gamma$, $v_2 = 2/(2 - \gamma)$, we have

$$\begin{aligned} \left| (L_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y) \right| &\leq M \left(\left(L_{n_1, q_{n_1}} (t - x)^2 \right)(x) \right)^{\xi/2} \left((L_{n_1, q_{n_1}} e_0)(x) \right)^{(2-\xi)/2} \\ &\quad \times \left(\left(L_{n_2, q_{n_2}} (s - y)^2 \right)(y) \right)^{\gamma/2} \left((L_{n_2, q_{n_2}} e_0)(y) \right)^{(2-\gamma)/2}. \end{aligned}$$

Now, applying Lemma 1, we obtain the degree of local approximation for B -continuous functions belonging to $Lip_M(\xi, \gamma)$. $\square$

Theorem 13. Let the function $f \in D_b(I)$ with $D_B f \in B(I)$ and $\lim_{n_j \rightarrow \infty} q_{n_j}^{n_j} = a_j \in [0, 1]$, $j = 1, 2$. Then, there exists $M > 0$ such that for each $(x, y) \in I$, we have

$$\begin{aligned} &\left| (T_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y) \right| \\ &\leq \frac{M}{[n_1]_{q_{n_1}}^{1/2} [n_2]_{q_{n_2}}^{1/2}} \left(\|D_B f\|_\infty + \omega_{mixed}(D_B f; [n_1]_{q_{n_1}}^{-1/2}, [n_2]_{q_{n_2}}^{-1/2}) \right). \end{aligned}$$

Proof. We need the following mean value theorem for B -differentiable functions [11, Proposition 17c]

$$\Delta f[(t, s); (x, y)] = (t - x)(s - y) D_B f(\xi, \eta), \quad f \in D_b(I),$$

with ξ between x and t and η between y and s .

It is also evident that

$$D_B f(\xi, \eta) = \Delta D_B f(\xi, \eta) + D_B f(\xi, y) + D_B f(x, \eta) - D_B f(x, y).$$

Since $D_B f \in B(I)$, by above relations, we can write

$$\begin{aligned} & |(L_{n_1, n_2, q_{n_1}, q_{n_2}} \Delta f[(t, s); (x, y)])(x, y)| \\ &= |(L_{n_1, n_2, q_{n_1}, q_{n_2}} (t-x)(s-y) D_B f(\xi, \eta))(x, y)| \\ &\leq (L_{n_1, n_2, q_{n_1}, q_{n_2}} |t-x||s-y| |\Delta D_B f(\xi, \eta)|)(x, y) \\ &+ (L_{n_1, n_2, q_{n_1}, q_{n_2}} |t-x||s-y| (|D_B f(\xi, y)| \\ &+ |D_B f(x, \eta)| + |D_B f(x, y)|))(x, y) \\ &\leq (L_{n_1, n_2, q_{n_1}, q_{n_2}} (|t-x||s-y| \omega_{mixed}(D_B f; |\xi-x|, |\eta-y|) \\ &+ 3 \|D_B f\|_\infty (L_{n_1, n_2, q_{n_1}, q_{n_2}} |t-x||s-y|))(x, y)). \end{aligned}$$

Using the property (5.2) of ω_{mixed} , and applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} & |(T_{n_1, n_2, q_{n_1}, q_{n_2}} f)(x, y) - f(x, y)| = |(L_{n_1, n_2, q_{n_1}, q_{n_2}} \Delta f[(t, s); (x, y)])(x, y)| \\ &\leq 3 \|D_B f\|_\infty \sqrt{(L_{n_1, n_2, q_{n_1}, q_{n_2}} (t-x)^2 (s-y)^2)(x, y)} \\ &+ \left(L_{n_1, n_2, q_{n_1}, q_{n_2}} |t-x||s-y| (x, y) + \delta_{n_1}^{-1} (L_{n_1, n_2, q_{n_1}, q_{n_2}} (t-x)^2 |s-y|)(x, y) \right. \\ &+ \delta_{n_2}^{-1} (L_{n_1, n_2, q_{n_1}, q_{n_2}} |t-x|(s-y)^2)(x, y) \\ &+ \delta_{n_1}^{-1} \delta_{n_2}^{-1} (L_{n_1, n_2, q_{n_1}, q_{n_2}} (t-x)^2 (s-y)^2)(x, y) \left. \right) \omega_{mixed}(D_B f; \delta_{n_1}, \delta_{n_2}) \\ &\leq 3 \|D_B f\|_\infty \sqrt{(L_{n_1, n_2, q_{n_1}, q_{n_2}} ((t-x)^2 (s-y)^2)(x, y)} \\ &+ \left(\sqrt{(L_{n_1, n_2, q_{n_1}, q_{n_2}} (t-x)^2 (s-y)^2)(x, y)} \right. \\ &+ \delta_{n_1}^{-1} \sqrt{(L_{n_1, n_2, q_{n_1}, q_{n_2}} (t-x)^4 (s-y)^2)(x, y)} \\ &+ \delta_{n_2}^{-1} \sqrt{(L_{n_1, n_2, q_{n_1}, q_{n_2}} (t-x)^2 (s-y)^4)(x, y)} \\ &+ \delta_{n_1}^{-1} \delta_{n_2}^{-1} (L_{n_1, n_2, q_{n_1}, q_{n_2}} (t-x)^2 (s-y)^2)(x, y) \left. \right) \omega_{mixed}(D_B f; \delta_{n_1}, \delta_{n_2}). \end{aligned} \tag{5.3}$$

We observe that for $(x, y), (t, s) \in I^2$ and $i, j \in \{1, 2\}$

$$\begin{aligned} & (L_{n_1, n_2, q_{n_1}, q_{n_2}} (t-x)^{2i} (s-y)^{2j})(x, y) \\ &= (L_{n_1, q_{n_1}} (t-x)^{2i})(x) (L_{n_2, q_{n_2}} (s-y)^{2j})(y). \end{aligned} \tag{5.4}$$

If (q_{n_j}) , $j = 1, 2$, are sequences in $(0, 1)$ such that $q_{n_j} \rightarrow 1$ and $q_{n_j}^{n_j} \rightarrow a_j$ ($0 \leq a_j < 1$), as $n_j \rightarrow \infty$, using Remark 1, it follows

$$(L_{n_j, q_{n_j}}(t-x)^2)(x) \leq \frac{M_1}{[n_j]_{q_{n_j}}} \quad (5.5)$$

$$(L_{n_j, q_{n_j}}(t-x)^4)(x) \leq \frac{M_2}{[n_j]_{q_{n_j}}^2}, \quad (5.6)$$

for some constants $M_1, M_2 > 0$.

Let $\delta_{n_1} = \frac{1}{[n_1]_{q_{n_1}}^{1/2}}$, and $\delta_{n_2} = \frac{1}{[n_2]_{q_{n_2}}^{1/2}}$.

Then, combining (5.3)-(5.6), we obtain the desired result. $\square$

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Authors' addresses

Sheetal Deshwal

Indian Institute of Technology Roorkee, Department of Mathematics, Roorkee-247667, India

E-mail address: sheetald1990@gmail.com

Ana Maria Acu

Lucian Blaga University of Sibiu, Department of Mathematics and Informatics, Str. Dr. I. Ratiu, No.5-7, RO-550012 Sibiu, Romania

E-mail address: acuana77@yahoo.com

P. N. Agrawal

Indian Institute of Technology Roorkee, Department of Mathematics, Roorkee-247667, India

E-mail address: pna_iitr@yahoo.co.in



ON THE DEDEKIND SUMS AND ITS NEW RECIPROCITY FORMULA

XIAOYING DU AND LEI ZHANG

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Abstract. The main purpose of this paper is using the analytic method and the properties of Dirichlet L -functions to study the computational problem of one kind Dedekind sums, and give a new reciprocity formula for it.

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Keywords: Dedekind sums, computational problem, reciprocity formula, analytic method

1. INTRODUCTION

Let q be a natural number and h an integer prime to q . The classical Dedekind sums

$$S(h, q) = \sum_{a=1}^q \left(\left(\frac{a}{q} \right) \right) \left(\left(\frac{ah}{q} \right) \right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer,} \end{cases}$$

describes the behaviour of the logarithm of the eta-function (see [6, 7]) under modular transformations. Many authors have studied the arithmetical properties of $S(h, q)$ and obtained many interesting results, some of them can be found in [10–12]. For example, Conrey et al [3] studied the mean value distribution of $S(h, k)$, and proved the asymptotic formula

$$\sum_{h=1}^k |S(h, k)|^{2m} = f_m(k) \left(\frac{k}{12} \right)^{2m} + O \left(\left(k^{\frac{9}{5}} + k^{2m-1+\frac{1}{m+1}} \right) \cdot \ln^3 k \right), \quad (1.1)$$

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where $\sum'_h$ denotes the summation over all h such that $(k, h) = 1$, and

$$\sum_{m=1}^{\infty} \frac{f_m(n)}{n^s} = 2 \cdot \frac{\zeta^2(2m)}{\zeta(4m)} \cdot \frac{\zeta(s+4m-1)}{\zeta^2(s+2m)} \cdot \zeta(s).$$

C. Jia [5] improved the error term in (1) to $O(k^{2m-1} \ln^3 k)$, if $m \geq 2$.

Walum [9] obtained an identity between the mean square value of $S(h, p)$ and the fourth power mean of Dirichlet L -functions. That is, he proved the following:

$$\sum_{h=1}^{p-1} |S(h, p)|^2 = \frac{p^2}{\pi^4(p-1)} \cdot \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^4,$$

where p be an odd prime.

Perhaps the most famous property of Dedekind sums is the reciprocity formula (see references [2, 4] and [6]):

$$S(h, k) + S(k, h) = \frac{h^2 + k^2 + 1}{12hk} - \frac{1}{4} \quad (1.2)$$

for all $(h, k) = 1$, $h > 0$ and $k > 0$.

An interesting three term version of (1.2) was also discovered by H. Rademacher and E. Grosswald [7].

In this paper, as a note of [8], we use the analytic method and the properties of Dirichlet L -functions to study the computational problem of one kind Dedekind sums, and give a new reciprocity formula for $S(k, h)$. That is, we will prove the following:

Theorem 1. *Let h and k are two positive odd numbers with $(k, h) = 1$. Then we have the identity*

$$S(2 \cdot \bar{k}, h) + S(2 \cdot \bar{h}, k) = \frac{h^2 + k^2 + 4}{24hk} - \frac{1}{4},$$

where integers $\bar{k}$ and $\bar{h}$ satisfying the congruence equation $k \cdot \bar{k} \equiv 1 \pmod{h}$ and $h \cdot \bar{h} \equiv 1 \pmod{k}$.

We prove this result by the analytic method and the properties of Dirichlet L -functions, which is distinct from other methods of proving reciprocity formula of Dedekind sums. Whether there exists a direct elementary method to prove this identity is an interesting problem.

2. PRELIMINARIES

To complete the proof of our theorem, we need to prove several lemmas. Hereinafter, we shall use some properties of Dirichlet L -functions, all of these can be found in reference [1], so they will not be repeated here.

Lemma 1. *Let $q > 2$ be an integer, then for any integer a with $(a, q) = 1$, we have the identity*

$$S(a, q) = \frac{1}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a) |L(1, \chi)|^2,$$

where $L(1, \chi)$ denotes the Dirichlet L -function corresponding to the character $\chi \bmod d$.

Proof. See Lemma 2 of [8]. $\square$

Lemma 2. *Let a and q are two positive odd numbers with $(a, q) = 1$. Then we have the identity*

$$S(a, 2q) + S(2 \cdot a, q) + S(\bar{2} \cdot a, q) = 3S(a, q),$$

where $\bar{2} \cdot 2 \equiv 1 \pmod{q}$.

Proof. Let χ_2^0 denotes the principal character mod 2. Then for any non-principal character $\chi \bmod d$ with $(2, d) = 1$, note that the identity

$$\begin{aligned} |L(1, \chi \chi_2^0)|^2 &= \left| \prod_p \left(1 - \frac{\chi(p) \chi_2^0(p)}{p} \right)^{-1} \right|^2 = \left| \prod_{p>2} \left(1 - \frac{\chi(p)}{p} \right)^{-1} \right|^2 \\ &= \left| 1 - \frac{\chi(2)}{2} \right|^2 \cdot \left| \prod_p \left(1 - \frac{\chi(p)}{p} \right)^{-1} \right|^2 = \left| 1 - \frac{\chi(2)}{2} \right|^2 \cdot |L(1, \chi)|^2 \\ &= \left(\frac{5}{4} - \frac{\chi(2)}{2} - \frac{\bar{\chi}(2)}{2} \right) \cdot |L(1, \chi)|^2, \end{aligned}$$

from Lemma 1 and the properties of Euler function $\phi(n)$ we have

$$\begin{aligned} S(a, 2q) &= \frac{1}{2q\pi^2} \cdot \sum_{d|2q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a) |L(1, \chi)|^2 \\ &= \frac{1}{2q\pi^2} \cdot \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a) |L(1, \chi)|^2 \\ &\quad + \frac{1}{2q\pi^2} \cdot \sum_{d|q} \frac{4d^2}{\phi(2d)} \sum_{\substack{\chi \bmod 2d \\ \chi(-1)=-1}} \chi(a) |L(1, \chi)|^2 \\ &= \frac{1}{2} \cdot S(a, q) + \frac{2}{\pi^2 q} \cdot \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a) \chi_2^0(a) |L(1, \chi \chi_2^0)|^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \cdot S(a, q) \\
&\quad + \frac{2}{\pi^2 q} \cdot \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a) |L(1, \chi)|^2 \cdot \left(\frac{5}{4} - \frac{\chi(2)}{2} - \frac{\bar{\chi}(2)}{2} \right) \\
&= \frac{1}{2} \cdot S(a, q) + \frac{5}{2} \cdot \frac{1}{\pi^2 q} \cdot \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a) |L(1, \chi)|^2 \\
&\quad - \frac{1}{\pi^2 q} \cdot \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(2a) |L(1, \chi)|^2 \\
&\quad - \frac{1}{\pi^2 q} \cdot \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(\bar{2}a) |L(1, \chi)|^2 \\
&= \frac{1}{2} \cdot S(a, q) + \frac{5}{2} \cdot S(a, q) - S(2 \cdot a, q) - S(\bar{2} \cdot a, q) \\
&= 3S(a, q) - S(2 \cdot a, q) - S(\bar{2} \cdot a, q)
\end{aligned}$$

or

$$S(a, 2q) + S(2 \cdot a, q) + S(\bar{2} \cdot a, q) = 3S(a, q).$$

This proves Lemma 2. $\square$

3. PROOF OF THEOREM

In this section, we shall complete the proof of our theorem. For any positive odd numbers h and k with $(k, h) = 1$, applying Lemma 2 repeatedly we have

$$S(k, 2h) + S(2 \cdot k, h) + S(\bar{2} \cdot k, h) = 3S(k, h) \quad (3.1)$$

and

$$S(h, 2k) + S(2 \cdot h, k) + S(\bar{2} \cdot h, k) = 3S(h, k). \quad (3.2)$$

Adding (3.1) and (3.2), then applying reciprocity formula (1.2) we have

$$\begin{aligned}
&S(k, 2h) + S(2h, k) + S(2 \cdot k, h) + S(h, 2k) + S(\bar{2} \cdot k, h) + S(\bar{2} \cdot h, k) \\
&\quad = 3S(k, h) + 3S(h, k)
\end{aligned}$$

or

$$\begin{aligned}
&\frac{4h^2 + k^2 + 1}{24kh} - \frac{1}{4} + \frac{4k^2 + h^2 + 1}{24kh} - \frac{1}{4} + S(\bar{2} \cdot k, h) + S(\bar{2} \cdot h, k) \\
&\quad = 3 \left(\frac{h^2 + k^2 + 1}{12hk} - \frac{1}{4} \right)
\end{aligned}$$

or

$$S(\bar{2} \cdot k, h) + S(\bar{2} \cdot h, k) = \frac{h^2 + k^2 + 4}{24hk} - \frac{1}{4}. \quad (3.3)$$

Note that if positive integers n and q satisfying $(n, q) = 1$, then $S(n, q) = S(\bar{n}, q)$, where $\bar{n}$ satisfying the congruence equation $n \cdot \bar{n} \equiv 1 \pmod{q}$.

Combining this property and (3.3) we may immediately deduce the identity

$$S(2 \cdot \bar{k}, h) + S(2 \cdot \bar{h}, k) = \frac{h^2 + k^2 + 4}{24hk} - \frac{1}{4}.$$

This completes the proof of our theorem.

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Authors' addresses

Xiaoying Du

School of Mathematics, Jinzhong University, Jinzhong 030619, Shanxi, P. R. China

E-mail address: duxiaoying83@163.com

Lei Zhang

School of Mathematics, Jinzhong University, Jinzhong 030619, Shanxi, P. R. China

E-mail address: zhanglei84052501@163.com



STABILITY OF TRIVIAL TORUS OF NONLINEAR MULTIFREQUENCY SYSTEMS VIA SIGN-DEFINITE QUADRATIC FORMS

PETRO FEKETA, OLENA A. KAPUSTIAN, AND MYKOLA M. PERESTYUK

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Abstract. This paper studies stability properties of the trivial invariant torus of a class of non-linear extensions of dynamical systems on torus. Two theorems on exponential stability and instability of the invariant torus in terms of quadratic forms that are sign-definite in nonwandering set of the dynamical system have been proven.

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1. INTRODUCTION

Fundamental results of the qualitative theory of multifrequency oscillations, in particular, the problems of the existence and stability of invariant sets of dynamical systems defined in the direct product of m -dimensional torus and n -dimensional Euclidean space, have been developed by A. M. Samoilenko and summarized in [12]. In [6], the stability properties of invariant tori have been studied in terms of sign-definite quadratic forms. In this paper, we establish new conditions for exponential stability and instability of the trivial invariant torus of nonlinear extension of dynamical system on torus which are formulated in terms of quadratic forms that are sign-definite not on the entire surface of the torus, but in nonwandering set [7] of dynamical system on torus only. The corresponding results for linear extensions of dynamical systems on torus have been obtained in [1, 3, 8–11].

2. MOTIVATION AND PROBLEM STATEMENT

We consider a system of ordinary differential equations defined in the direct product of m -dimensional torus $\mathcal{T}_m$ and n -dimensional Euclidean space $\mathbb{R}^n$

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = P(\varphi, x)x, \quad (2.1)$$

where $\varphi = (\varphi_1, \dots, \varphi_m)^T \in \mathcal{T}_m$, $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, function P is continuous in $\mathcal{T}_m \times \mathbb{R}^n$ and for every $x \in \mathbb{R}^n$ $P(\cdot, x)$, $a(\cdot) \in C(\mathcal{T}_m)$; $C(\mathcal{T}_m)$ is a space of continuous 2π -periodic with respect to each of the component φ_v , $v = 1, \dots, m$ functions defined on $\mathcal{T}_m$.

We assume that the following conditions hold:

$$\exists M > 0 \text{ such that } \forall (\varphi, x) \in \mathcal{T}_m \times \mathbb{R}^n \quad \|P(\varphi, x)\| \leq M; \quad (2.2)$$

$$\forall r > 0 \exists L = L(r) > 0 \text{ such that } \forall x', x'', \|x'\| \leq r, \|x''\| \leq r, \forall \varphi \in \mathcal{T}_m \\ \|P(\varphi, x'') - P(\varphi, x')\| \leq L \|x'' - x'\|; \quad (2.3)$$

$$\exists A > 0 \quad \forall \varphi', \varphi'' \in \mathcal{T}_m \quad \|a(\varphi'') - a(\varphi')\| \leq A \|\varphi'' - \varphi'\|. \quad (2.4)$$

Condition (2.4) guarantees that the system

$$\frac{d\varphi}{dt} = a(\varphi) \quad (2.5)$$

generates a dynamical system on $\mathcal{T}_m$, which we will denote by $\varphi_t(\varphi)$.

Definition 1 ([7]). A point $\varphi \in \mathcal{T}_m$ is called a nonwandering point of dynamical system (2.5) if there exist a neighbourhood $U(\varphi)$ and a moment of time $T = T(\varphi) > 0$ such that

$$U(\varphi) \cap \varphi_t(U(\varphi)) = \emptyset \quad \forall t \geq T.$$

Let us denote by Ω a set of all nonwandering points of dynamical system (2.5). Since $\mathcal{T}_m$ is a compact set, the set Ω is nonempty, invariant, and compact subset of $\mathcal{T}_m$ [12]. Additionally, the following holds:

Lemma 1 ([7]). For any $\varepsilon > 0$ there exist $T(\varepsilon) > 0$ and $N(\varepsilon) > 0$ such that for any $\varphi \notin \Omega$ the corresponding trajectory $\varphi_t(\varphi)$ spends only a finite time that is bounded by $T(\varepsilon)$ outside the ε -neighbourhood of the set Ω , and leaves this set not more than $N(\varepsilon)$ times.

Definition 2 ([12]). Trivial invariant torus

$$x = 0, \quad \varphi \in \mathcal{T}_m$$

of the system (2.1) is called exponentially stable if there exist constants $K > 0$, $\gamma > 0$, and $\delta > 0$ such that for all $\varphi \in \mathcal{T}_m$ and for all $x^0 \in \mathbb{R}^n$, $\|x^0\| \leq \delta$ it holds that

$$\forall t \geq 0 \quad \|x(t, \varphi, x^0)\| \leq K \|x^0\| e^{-\gamma t}, \quad (2.6)$$

where $x(t, \varphi, x^0)$ is a solution to the Cauchy problem

$$\frac{dx}{dt} = P(\varphi_t(\varphi), x)x, \quad x(0) = x^0. \quad (2.7)$$

In [5], the conditions for the exponential stability of the trivial invariant torus of the system (2.1) have been established in terms of the properties of function $\varphi \mapsto P(\varphi, 0)$ in the nonwandering set Ω of dynamical system (2.5):

Lemma 2 ([5]). *Let*

$$\forall \varphi \in \Omega \quad \lambda(\varphi, 0) < 0, \quad (2.8)$$

where $\lambda(\varphi, x)$ is the largest eigenvalue of the matrix

$$\hat{P}(\varphi, x) = \frac{1}{2} \left(P(\varphi, x) + P^T(\varphi, x) \right).$$

Then, the trivial invariant torus of system (2.1) is exponentially stable.

The following example demonstrates the case when the trivial invariant torus is exponentially stable (this will be proven in Theorem 1), however the condition (2.8) does not hold.

Example 1. Consider a system defined in $\mathcal{T}_1 \times \mathbb{R}^2$

$$\frac{d\varphi}{dt} = -\sin^2\left(\frac{\varphi}{2}\right), \quad (2.9)$$

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = \begin{pmatrix} \sin(\varphi + x_1 + x_2)x_1 & -x_2 \\ x_1 & -\sin(x_1 - x_2 - \varphi)x_2 \end{pmatrix}. \quad (2.10)$$

Dynamical system on torus $\mathcal{T}_1$ that are generated by (2.9) has a nonwandering set

$$\Omega = \{\varphi = 0\}.$$

However, the matrix

$$\hat{P}(0, \bar{0}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

does not fulfill condition (2.8).

The main results of this paper are theorems on exponential stability and instability of the trivial torus of the systems of a class (2.9) with conditions given in terms of quadratic forms that are sign-definite in nonwandering set Ω of dynamical system on torus only.

3. MAIN RESULTS

For any $\varphi \in \mathcal{T}_m$, $x \in \mathbb{R}^n$ let us denote

$$\begin{aligned} \hat{S}(\varphi, x) &= \frac{\partial S(\varphi, x)}{\partial \varphi} a(\varphi) + \frac{\partial S(\varphi, x)}{\partial x} (P(\varphi, x)x) \\ &\quad + S(\varphi, x)P(\varphi, x) + P^T(\varphi, x)S(\varphi, x), \end{aligned} \quad (3.1)$$

where $S = S(\varphi, x)$ is a symmetric matrix of a class $C^1(\mathcal{T}_m \times \mathbb{R}^n)$.

Theorem 1. *Let there exist a symmetric matrix $S = S(\varphi, x)$ of the class $C^1(\mathcal{T}_m \times \mathbb{R}^n)$ such that*

$$\forall \varphi \in \Omega \quad S(\varphi, 0) > 0, \quad \hat{S}(\varphi, 0) < 0. \quad (3.2)$$

Then, the trivial torus of system (2.1) is exponentially stable.

Proof. Due to the conditions (3.2) and continuous dependence of the polynomial roots on the coefficients [4], we get that for some $r > 0$, $\gamma > 0$ the following inequalities hold

$$\forall \varphi \in O_r(\Omega) \quad \forall x \in \mathbb{R}^n, \|x\| < r \quad S(\varphi, x) \geq \gamma E, \quad \hat{S}(\varphi, x) \leq -\gamma E. \quad (3.3)$$

Also, there exists a constant $C = C(r) > 0$ such that for all $\varphi \in T_m$ and for all $x \in \mathbb{R}^n$, $\|x\| \leq r$

$$\|S(\varphi, x)\| + \|\hat{S}(\varphi, x)\| \leq C. \quad (3.4)$$

Denote by

$$V(\varphi, x) = (S(\varphi, x)x, x). \quad (3.5)$$

Let

$$\varphi \in O_r(\Omega) \quad \text{and} \quad \forall s \geq 0 \quad \varphi_s(\varphi) \in O_r(\Omega).$$

Then, for the solution to (2.7) $x(t) = x(t, \varphi, x^0)$ with $\|x^0\| < r$ the following estimate $\|x(t)\| < r$ holds for $t \in [0, T)$, $0 < T \leq +\infty$. Hence, from (3.3),

$$\gamma \|x(t)\|^2 \leq V(\varphi_t(\varphi), x(t)) \leq C \|x(t)\|^2,$$

$$\frac{d}{dt} V(\varphi_t(\varphi), x(t)) \leq -\gamma \|x(t)\|^2$$

hold for $t \in [0, T)$. From the last inequalities we get that

$$V(\varphi_t(\varphi), x(t)) \leq V(\varphi, x^0) e^{-\frac{\gamma}{C} t}.$$

Hence, there exist constants $K_1 > 0$ and $\gamma_1 > 0$ such that $\forall t \in [0, T)$

$$\|x(t)\| \leq K_1 \|x^0\| e^{-\gamma_1 t}. \quad (3.6)$$

For $\|x^0\| < \frac{r}{K_1}$ we obtain that $T = +\infty$ and the inequality (3.6) holds for all $t \geq 0$.

Now, let $\varphi \in O_r(\Omega)$, but there exists $t_1 > 0$ such that

$$\forall t \in [0, t_1) \quad \varphi_t(\varphi) \in O_r(\Omega), \quad \varphi_{t_1}(\varphi) \notin O_r(\Omega).$$

From Lemma 1, there exist

$$\begin{aligned} N(\varphi, r) &\leq N(r), \quad \{\tau_i(\varphi, r)\}_{i=1}^{N(\varphi, r)+1}, \quad \{t_i(\varphi, r)\}_{i=1}^{N(\varphi, r)} \\ &\sum_{i=1}^{N(\varphi, r)+1} \tau_i(\varphi, r) =: T(\varphi, r) \leq T(r) \end{aligned}$$

such that

$$\begin{aligned} &\varphi_t(\varphi) \in O_r(\Omega) \\ &\forall t \in (0, t_1) \cup \bigcup_{k=1}^{N(\varphi, r)-1} \left(\sum_{i=1}^k (\tau_i + t_i), \sum_{i=1}^k (\tau_i + t_{i+1}) \right) \cup \left(\sum_{i=1}^{N(\varphi, r)} (\tau_i + t_i), +\infty \right). \end{aligned} \quad (3.7)$$

Then, for $t \in [0, t_1]$

$$\|x(t)\| \leq K_1 \|x^0\| e^{-\gamma_1 t} < r \quad \text{if} \quad \|x^0\| < \frac{r}{K_1}.$$

For $t \in [t_1, t_1 + \tau_1]$, from (2.2) and Wazewski inequality [2],

$$\begin{aligned} \|x(t)\| &\leq \|x(t_1)\| e^{M(t-t_1)} \\ &\leq K_1 \|x^0\| e^{(\gamma_1+M)\tau_1} e^{-\gamma_1 t} \\ &< r \quad \text{if} \quad \|x^0\| < \frac{r}{K_1 e^{(\gamma_1+M)\tau_1}}. \end{aligned}$$

For $t \in [t_1 + \tau_1, t_1 + \tau_1 + t_2]$,

$$\|x(t)\| \leq K_1^2 \|x^0\| e^{(\gamma_1+M)\tau_1} e^{-\gamma_1 t} < r \quad \text{if} \quad \|x^0\| < \frac{r}{K_1^2 e^{(\gamma_1+M)\tau_1}}.$$

Continuing this process, due to (3.7), finally we get:

$$\text{for } K := K_1^{N(r)} e^{(\gamma+M)T(r)}, \quad \delta := \frac{r}{K_1^{N(r)} e^{(\gamma+M)T(r)}}, \quad \|x^0\| < \delta,$$

$$\forall t \geq 0 \quad \|x(t)\| \leq K \|x^0\| e^{-\gamma_1 t}. \quad (3.8)$$

Now, let us consider the case of $\varphi \notin O_r(\Omega)$. In this case,

$$\forall \tau_0 \in (0, T(r)) \quad \varphi_{\tau_0}(\varphi) \in O_r(\Omega)$$

and

$$\forall t \in [0, \tau_0] \quad \|x(t)\| \leq \|x^0\| e^{(M+\gamma_1)T(r)} e^{-\gamma_1 t}.$$

Then, for

$$\hat{K} = K e^{(\gamma+M)T(r)}, \quad \hat{\delta} = \frac{\delta}{e^{(\gamma+M)T(r)}}$$

we obtain the required estimate (2.6). This completes the proof. $\square$

Example 2 (revisited). Let us illustrate the usage of Theorem 1 for the system (2.9), (2.10). Let

$$S = S(\varphi, x) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} > 0$$

Then,

$$\hat{S}(0, \bar{0}) = \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix} < 0$$

which guarantee exponential stability of the trivial invariant torus.

The following theorem provides sufficient conditions for instability of the trivial torus of system (2.1) in terms of sign-definite on the set Ω quadratic forms.

Theorem 2. *Let there exist a symmetric matrix $S = S(\varphi, x)$ of the class $C^1(\mathcal{T}_m \times \mathbb{R}^n)$ such that for the matrix (3.1) and for the quadratic form (3.5) the following conditions hold:*

$$\forall \varphi \in \Omega \quad \hat{S}(\varphi, 0) > 0, \quad (3.9)$$

$$\forall \delta > 0 \exists x_0 \in \mathbb{R}^n, \|x_0\| < \delta, \exists \varphi_0 \in \Omega \quad \text{such that} \quad V(\varphi_0, x_0) > 0. \quad (3.10)$$

Then, the trivial torus of system (2.1) is unstable.

Proof. From (3.9) and (3.3), $\exists \beta > 0$ and $\exists r > 0$ such that

$$\forall \varphi \in \Omega \quad \forall x \in \mathbb{R}^n, \|x\| < r \quad \hat{S}(\varphi, x) \geq \beta E.$$

Let for arbitrary $\delta > 0$ ($\delta < r$), the corresponding x_0, φ_0 are from the condition (3.10), and $x(t) = x(t, \varphi_0, x_0)$ is the solution to the Cauchy problem (2.7). Next, we show that $\exists t_1 > 0$ such that $\|x(t_1)\| = r$. This will be sufficient to prove instability.

Suppose the opposite: Let

$$\forall t \geq 0 \quad \|x(t)\| < r.$$

Then, for the function $v(t) = V(\varphi_t(\varphi_0), x(t))$ we have:

$$\frac{d}{dt} v(t) \geq \beta \|x(t)\|^2. \quad (3.11)$$

From the condition (3.10), $v(0) = V(\varphi_0, x_0) = \alpha > 0$. Hence,

$$\forall t \geq 0 \quad v(t) \geq \alpha. \quad (3.12)$$

Then, $\exists \varepsilon > 0$ such that $\forall t \geq 0 \quad \|x(t)\| \geq \varepsilon$. Really, if it is not so then there exists a sequence $t_k \rightarrow \infty$ such that $\|x(t_k)\| \rightarrow 0$. Finally,

$$v(t_k) = (S(\varphi_{t_k}(\varphi_0), x(t_k))x(t_k), x(t_k)) \leq C \|x(t_k)\|^2 \rightarrow 0,$$

where constant $C > 0$ is from (3.4), which contradicts (3.12).

Then, from (3.11):

$$v(t) \geq \alpha + \varepsilon^2 t \quad \forall t \geq 0.$$

Hence, $\forall t \geq 0$

$$\|x(t)\|^2 \geq \frac{1}{C} (\alpha + \varepsilon^2 t).$$

The latter estimate implies the existence of $t_1 > 0$ such that $\|x(t_1)\| = r$. This completes the proof. $\square$

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*Authors’ addresses***Petro Feketa**

University of Kaiserslautern, Department of Mechanical and Process Engineering, Gottlieb-Daimler-Straße, Postfach 3049, 67663 Kaiserslautern, Germany
E-mail address: petro.feketa@mv.uni-kl.de

Olena A. Kapustian

Taras Shevchenko National University of Kyiv, Faculty of Computer Science and Cybernetics, Volodymyrska Street 64, 01601 Kyiv, Ukraine
E-mail address: olena.kap@gmail.com

Mykola M. Perestyuk

Taras Shevchenko National University of Kyiv, Faculty of Mechanics and Mathematics, Volodymyrska Street 64, 01601 Kyiv, Ukraine
E-mail address: perestyuknn@gmail.com



A MEAN VALUE THEOREM FOR N -SIMPLE FUNCTIONALS IN THE SENSE OF POPOVICIU

IOAN GAVREA, MIRCEA IVAN, AND VICUTA NEAGOS

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Abstract. We establish some mean value theorems involving n -simple functionals in the sense of Popoviciu. In particular, we obtain the Kowalewski mean value formula.

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1. PRELIMINARIES AND AUXILIARY RESULTS

Throughout the paper $n \geq 1$ denotes an integer. Let $[a, b]$ be a real interval and $a \leq x_0 < \dots < x_n \leq b$.

1.1. The divided difference

In most books on numerical analysis, the divided differences of a function $f: \{x_0, \dots, x_n\} \rightarrow \mathbb{R}$ are defined recursively:

$$[x_i; f] := f(x_i), \quad i = 0, \dots, n,$$

$$[x_0, \dots, x_k; f] = \frac{[x_1, \dots, x_k; f] - [x_0, \dots, x_{k-1}; f]}{x_k - x_0}, \quad k = 1, \dots, n,$$

but can be written also in terms of determinants,

$$[x_0, \dots, x_n; f] = \frac{1}{V(x_0, \dots, x_n)} \begin{vmatrix} 1 & x_0 & \dots & x_0^{n-1} & f(x_0) \\ 1 & x_1 & \dots & x_1^{n-1} & f(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & \dots & x_n^{n-1} & f(x_n) \end{vmatrix},$$

where $V(x_0, \dots, x_n)$ denotes the Vandermonde determinant

$$\begin{vmatrix} 1 & x_0 & \dots & x_0^n \\ 1 & x_1 & \dots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^n \end{vmatrix}.$$

1.2. Convexity of high order

The standard definition of n -convex functions is that through divided differences (see, e.g., [6]). A function $f: [a, b] \rightarrow \mathbb{R}$ is said to be n -convex if

$$[x_0, \dots, x_n; f] > 0,$$

for any pairwise distinct points $x_0, \dots, x_n \in [a, b]$.

1.3. Simple form functionals

Let $e_i: [a, b] \rightarrow \mathbb{R}$, $e_i(x) = x^i$, $i = 0, 1, \dots, n$. The definition and the theorem rewritten below belong to Popoviciu.

Definition 1 ([7, 9]). A linear functional $A: C[a, b] \rightarrow \mathbb{R}$ is said to be n -simple in the sense of Popoviciu if:

- (i) $A(e_0) = A(e_1) = \dots = A(e_{n-1}) = 0$,
- (ii) $A(f) \neq 0$ for any n -convex function $f \in C[a, b]$.

In particular, $A(e_n) \neq 0$.

A connection between divided differences and n -simple functionals is supplied by the following celebrated result of Popoviciu.

Theorem 1 ([7, 9, Popoviciu]). A linear functional $A: C[a, b] \rightarrow \mathbb{R}$ is n -simple if and only if for any $f \in C[a, b]$, there exist distinct points $\xi_0, \dots, \xi_n \in (a, b)$ such that

$$A(f) = A(e_n) [\xi_0, \dots, \xi_n; f]. \quad (1.1)$$

1.4. The finite difference operator

The finite difference with step h ($0 < h \leq (b-a)/n$) of the function $f: [a, b] \rightarrow \mathbb{R}$ at the point $x \in [a, b)$ is defined by

$$\Delta_h^n f(x) = \sum_{k=0}^n (-1)^k \binom{n}{k} f(x + (n-k)h).$$

Recall that finite difference can be written as a divided difference on equidistant points:

$$\Delta_h^n f(x) = n! h^n [x, x+h, \dots, x+nh; f].$$

Let f possess a derivative of order n on the interval (a, b) and $x, x+nh \in [a, b]$.

By using the Lagrange mean value theorem, one can prove, step by step, that there exist $c_k \in (x, x+nh)$, $k = 1, 2, \dots, n$, such that:

$$\Delta_h^n f(x) = h \Delta_h^{n-1} f'(c_1) = \dots = h^k \Delta_h^{n-k} f^{(k)}(c_k) = \dots = h^n \Delta_h^0 f^{(n)}(c_n), \quad (1.2)$$

where $\Delta_h^0 f^{(n)}(c_n) = f^{(n)}(c_n)$

One connection between the divided difference and the n -th derivative of a function is given by a classical result of Cauchy (see, e.g., [11, Theorem 2.10]):

Proposition 1 (Cauchy). *Let $f \in C^n[a, b]$ and $x_0, \dots, x_n$ be pairwise distinct points in $[a, b]$. Then there exists at least one point $\xi \in [a, b]$ such that*

$$[x_0, \dots, x_n; f] = \frac{f^{(n)}(\xi)}{n!}.$$

A much deeper mean value property for divided differences is provided by the following theorem of Popoviciu.

Proposition 2 ([8, Popoviciu (1954)]). *If the function f is continuous on an interval $[a, b]$ containing the distinct points $x_0, \dots, x_n$, then there exists $c \in (a, b)$ and $h > 0$ such that*

$$[x_0, \dots, x_n; f] = \frac{\Delta_h^n f(c)}{n! h^n}. \quad (1.3)$$

The following result is a generalization of Cauchy's mean-value theorem:

Theorem 2 ([3, Kowalewski (1932), p. 16] and [10, Raikov (1939)]). *If $f, g: [a, b] \rightarrow \mathbb{R}$ are continuous and n times differentiable on (a, b) , and $g^{(n)}(x) \neq 0$ for any $x \in (a, b)$, then there exists a point $c \in (a, b)$ such that*

$$\frac{[x_0, \dots, x_n; f]}{[x_0, \dots, x_n; g]} = \frac{f^{(n)}(c)}{g^{(n)}(c)}.$$

For the purpose of simplicity, and without loss of generality, we will drop out the condition $g^{(n)}(x) \neq 0$ for any $x \in (a, b)$ and consider the following form of the previous equation,

$$[x_0, \dots, x_n; f] g^{(n)}(c) = [x_0, \dots, x_n; g] f^{(n)}(c). \quad (1.4)$$

2. MAIN RESULTS

We give a simple elementary self-contained proof of Kowalewski's mean value formula (1.4) (see also [4]). Furthermore, we generalize (1.4) in two directions. Firstly, the divided differences are replaced with an n -simple functional. Secondly, we take a combination of forward differences and derivatives in place of the n -th derivatives in (1.4).

Consider the auxiliary function $H : [a, b] \rightarrow \mathbb{R}$,

$$H(x) = \begin{vmatrix} 1 & x_0 & \cdots & x_0^{n-1} & f(x_0) & g(x_0) \\ 1 & x_1 & \cdots & x_1^{n-1} & f(x_1) & g(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} & f(x_n) & g(x_n) \\ 1 & x & \cdots & x^{n-1} & f(x) & g(x) \end{vmatrix}.$$

It is obvious that $H(x_i) = 0, i = 0, 1, \dots, n$. By using the Generalized Rolle Theorem, we deduce that there exists a point $c \in (x_0, x_n)$ such that $H^{(n)}(c) = 0$, i.e.,

$$0 = \begin{vmatrix} 1 & x_0 & \cdots & x_0^{n-1} & f(x_0) & g(x_0) \\ 1 & x_1 & \cdots & x_1^{n-1} & f(x_1) & g(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} & f(x_n) & g(x_n) \\ 0 & 0 & \cdots & 0 & f^{(n)}(c) & g^{(n)}(c) \end{vmatrix}.$$

By expanding the previous determinant in terms of the last row, and dividing the result by the Vandermonde determinant $V(x_0, \dots, x_n)$, the proof is complete. $\square$

The following result is a generalization of Kowalewski's mean value formula (1.4).

Theorem 3. *Let $A: C[a, b] \rightarrow \mathbb{R}$ be an n -simple functional. and suppose that $f, g: [a, b] \rightarrow \mathbb{R}$ are continuous and possess derivatives of suitable order on (a, b) . Then there exist points $c_k \in (a, b), k = 0, \dots, n$, such that*

$$A(f) \Delta_h^{n-k} g^{(k)}(c_k) = A(g) \Delta_h^{n-k} f^{(k)}(c_k). \quad (2.1)$$

In the particular cases of $k = n$, and $k = 0$, Eq. (2.1) gives:

$$A(f) g^{(n)}(c_n) = A(g) f^{(n)}(c_n), \quad (2.2)$$

$$A(f) \Delta_h^n g(c_0) = A(g) \Delta_h^n f(c_0). \quad (2.3)$$

Proof. We consider the auxiliary function $h: [a, b] \rightarrow \mathbb{R}$,

$$h = A(f) g - A(g) f.$$

It follows that $A(h) = 0$. Since $A(e_n) \neq 0$, using (1.1) we deduce that there exist distinct points $\xi_0, \dots, \xi_n \in (a, b)$ such that

$$[\xi_0, \dots, \xi_n; h] = 0.$$

By using Popoviciu's mean value formula (1.3), we deduce that there exists a point $c \in (a, b)$ such that

$$\Delta_h^n h(c) = 0.$$

Next, by using (1.2), we obtain that there exist $c_k \in (a, b)$ such that

$$\Delta_h^{n-k} h^{(k)}(c_k) = 0, \quad k = 1, 2, \dots, n,$$

and the proof is complete. $\square$

Remark 1. Since the functional $A(f) := [x_0, \dots, x_n; f]$ is obviously of simple form, from Theorem 3, Eq. (2.2), we deduce the Kowalevski result (1.4).

A variant of the Kowalevski result (1.4) in the case of non-differentiable functions, i.e.,

$$[x_0, \dots, x_n; f] \Delta_h^n g(c_0) = [x_0, \dots, x_n; g] \Delta_h^n f(c_0),$$

follows from (2.3).

We end the paper with an application of Peano's representation formula for continuous linear functionals. For $t \in [a, b]$, let $\varphi_t: [a, b] \rightarrow \mathbb{R}$,

$$\varphi_t(x) := (x - t)_+^{n-1} := \left(\frac{x - t + |x - t|}{2} \right)^{n-1}, \quad x \in [a, b].$$

If $A: C^n[a, b] \rightarrow \mathbb{R}$ is a continuous linear functional with the property that

$$A(e_i) = 0, \quad i = 0, \dots, n-1,$$

then the following representation formula (see, e.g., [1, 2, 5]),

$$A(f) = \frac{1}{(n-1)!} \int_0^1 A(\varphi_t) f^{(n)}(t) dt, \quad (2.4)$$

holds true.

Remark 2. Let $A: C[a, b] \rightarrow \mathbb{R}$ be a continuous n -simple functional and suppose that $f, g \in C^n[a, b]$. Then there exist points $c_k \in (a, b)$, $k = 0, \dots, n$, such that

$$\begin{aligned} & \Delta_h^{n-k} g^{(k)}(c_k) \int_0^1 A((\cdot - t)_+^{n-1}) f^{(n)}(t) dt \\ &= \Delta_h^{n-k} f^{(k)}(c_k) \int_0^1 A((\cdot - t)_+^{n-1}) g^{(n)}(t) dt. \end{aligned}$$

Proof. In (2.1) we replace A by its Peano representation (2.4). □

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Authors' addresses

Ioan Gavrea

Technical University of Cluj Napoca, Department of Mathematics, Str. Memorandumului nr. 28, 400114 Cluj-Napoca, Romania

E-mail address: ioan.gavrea@math.utcluj.ro

Mircea Ivan

Technical University of Cluj Napoca, Department of Mathematics, Str. Memorandumului nr. 28, 400114 Cluj-Napoca, Romania

E-mail address: mircea.ivan@math.utcluj.ro

Vicuta Neagos

Technical University of Cluj Napoca, Department of Mathematics, Str. Memorandumului nr. 28, 400114 Cluj-Napoca, Romania

E-mail address: vicuta.neagos@math.utcluj.ro



APPROXIMATE CONTROLLABILITY OF IMPULSIVE NON-LOCAL NON-LINEAR FRACTIONAL DYNAMICAL SYSTEMS AND OPTIMAL CONTROL

SARRA GUECHI, AMAR DEBBOUCHE, AND DELFIM F. M. TORRES

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Abstract. We establish existence, approximate controllability and optimal control of a class of impulsive non-local non-linear fractional dynamical systems in Banach spaces. We use fractional calculus, sectorial operators and Krasnoselskii fixed point theorems for the main results. Approximate controllability results are discussed with respect to the inhomogeneous non-linear part. Moreover, we prove existence results of optimal triplets of corresponding fractional control systems with Bolza cost functionals.

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1. INTRODUCTION

We are concerned with an impulsive non-local non-linear fractional control dynamical system of form

$$\begin{cases} {}^C D_t^q x(t) = Ax(t) + f(t, x(t), (Hx)(t)) + Bu(t), & t \in (0, b] \setminus \{t_1, t_2, \dots, t_m\}, \\ x(0) + g(x) = x_0 \in X, & \Delta x(t_i) = I_i(x(t_i^-)) + Dv(t_i^-), \quad i = 1, 2, \dots, m, \end{cases} \quad (1.1)$$

where ${}^C D_t^q$ is the Caputo fractional derivative of order $0 < q < 1$, the state $x(\cdot)$ takes its values in a Banach space X with norm $\|\cdot\|$, and $x_0 \in X$. Let $A : D(A) \subset X \rightarrow X$ be a sectorial operator of type (M, θ, q, μ) on X , $H : I \times I \times X \rightarrow X$ represents a Volterra-type operator such that $(Hx)(t) = \int_0^t h(t, s, x(s))ds$, the control functions $u(\cdot)$ and $v(\cdot)$ are given in $L^2(I, U)$, U is a Banach space, B and D are bounded linear operators from U into X . Here, one has $I = [0, b]$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = b$, $I_i : X \rightarrow X$ are impulsive functions that characterize the jump of the solutions at impulse points t_i , the non-linear term $f : I \times X \times X \rightarrow X$, the non-local function $g : PC(I, X) \rightarrow X$, with PC defined later, $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$, where $x(t_i^+)$ and $x(t_i^-)$ are the right and left limits of x at the point t_i , respectively.

Derivatives and integrals of arbitrary order, the main objects of Fractional Calculus (FC), have kept the interest of many scientists in recent years, since they provide an excellent tool to describe hereditary properties of various materials and processes. During the past decades, FC and its applications have gained a lot of importance, due to successful results in modelling several complex phenomena in numerous seemingly diverse and widespread fields of science and engineering, such as heat conduction, diffusion, propagation of waves, radiative transfer, kinetic theory of gases, diffraction problems and water waves, radiation, continuum mechanics, geophysics, electricity and magnetism, as well as in mathematical economics, communication theory, population genetics, queuing theory and medicine. For details on the theory and applications of FC see [9]. For recent developments in non-local and impulsive fractional differential problems see [1, 2, 8, 10] and references therein.

The problem of controllability is one of the most important qualitative aspects of dynamical systems in control theory. It consists to show the existence of a control function that steers the solution of the system from its initial state to a final state, where the initial and final states may vary over the entire space. This concept plays a major role in finite-dimensional control theory, so that it is natural to try to generalize it to infinite dimensions [14]. Moreover, exact controllability for semi-linear fractional order systems, when the non-linear term is independent of the control function, is proved by assuming that the controllability operator has an induced inverse on a quotient space. However, if the semi-group associated with the system is compact, then the controllability operator is also compact and hence the induced inverse does not exist because the state space is infinite dimensional [17]. Thus, the concept of exact controllability is too strong and has limited applicability, while approximate controllability is a weaker concept completely adequate in applications.

On the other hand, control systems are often based on the principle of feedback, where the signal to be controlled is compared to a desired reference, and the discrepancy is used to compute a corrective control action. Fractional optimal control of a distributed system is an optimal control problem for which the system dynamics is defined with fractional differential equations. Recently, attention has been paid to prove existence, approximate controllability and/or optimal control for different classes of fractional differential equations [4–7].

In [11], optimal control of non-instantaneous impulsive differential equations is studied. Qin et al. investigate approximate controllability and optimal control of fractional dynamical systems of order $1 < q < 2$ in Banach spaces [13]. Debbouche and Antonov established approximate controllability of semi-linear Hilfer fractional differential inclusions with impulsive control inclusion conditions in Banach spaces [3]. Motivated by the above works, here we construct an impulsive non-local non-linear fractional control dynamical system and prove new sufficient conditions to treat the questions of approximate controllability and optimal control.

The paper is organized as follows. In Section 2, we recall some facts from fractional calculus, q -resolvent families, and useful versions of fixed point techniques that are used for obtaining our main results. In Section 3, we form appropriate sufficient conditions and prove existence results for the fractional control system (1.1). In Section 4, we investigate the question of approximate controllability. We end with Section 5, where we obtain optimal controls corresponding to fractional control systems with a Bolza cost functional.

2. PRELIMINARIES

Here we present some preliminaries from fractional calculus [9], operator theory [12] and fixed point techniques [1], which are used throughout the work to obtain the desired results.

Definition 1. The left-sided Riemann–Liouville fractional integral of order $\alpha > 0$, with lower limit a , for a function $f : [a, +\infty) \rightarrow \mathbb{R}$, is defined as

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

provided the right side is point-wise defined on $[a, +\infty)$, where $\Gamma(\cdot)$ is the Euler gamma function. If $a = 0$, then we can write $I_{0+}^{\alpha} f(t) = (g_{\alpha} * f)(t)$, where

$$g_{\alpha}(t) := \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

and $*$ denotes convolution of functions. Moreover, $\lim_{\alpha \rightarrow 0} g_{\alpha}(t) = \delta(t)$, with δ the delta Dirac function.

Definition 2. The left-sided Riemann–Liouville fractional derivative of order $\alpha > 0$, $n-1 \leq \alpha < n$, $n \in \mathbb{N}$, for a function $f : [a, +\infty) \rightarrow \mathbb{R}$, is defined by

$${}^L D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > a,$$

where function f has absolutely continuous derivatives up to order $n-1$.

Definition 3. The left-sided Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, for a function $f : [a, +\infty) \rightarrow \mathbb{R}$, is defined by

$${}^C D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I_{a+}^{n-\alpha} f^{(n)}(t), \quad t > a,$$

where function f has absolutely continuous derivatives up to order $n-1$.

Throughout the paper, by $PC(I, X)$ we denote the space of X -valued bounded functions on I with the uniform norm $\|x\|_{PC} = \sup\{\|x(t)\|, t \in I\}$ such that $x(t_i^+)$ exists for any $i = 0, \dots, m$ and $x(t)$ is continuous on $(t_i, t_{i+1}]$, $i = 0, \dots, m$, $t_0 = 0$ and $t_{m+1} = b$.

Definition 4 (See [16]). Let $A : D \subseteq X \rightarrow X$ be a closed and linear operator. We say that A is *sectorial* of type (M, θ, q, μ) , if there exists $\mu \in \mathbb{R}$, $0 < \theta < \frac{\pi}{2}$ and $M > 0$ such that the q -resolvent of A exists outside the sector

$$\mu + S_\theta = \{\mu + \lambda^q : \lambda \in \mathbb{C}, |\text{Arg}(-\lambda^q)| < \theta\}$$

and

$$\|(\lambda^q I - A)^{-1}\| \leq \frac{M}{|\lambda^q - \mu|}, \lambda^q \notin \mu + S_\theta.$$

Remark 1. If A is a sectorial operator of type (M, θ, q, μ) , then it is not difficult to see that A is the infinitesimal generator of a q -resolvent family $T_q(t)_{t \geq 0}$ in a Banach space, where $T_q(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} R(\lambda^q, A) d\lambda$.

Definition 5 (Motivated by [3, 16]). A state function $x \in PC(I, X)$ is called a mild solution of (1.1) if it satisfies the following integral equations:

$$x(t) = S_q(t)(x_0 - g(x)) + \int_0^t T_q(t-s)(f(s, x(s), (Hx)(s)) + Bu(s)) ds$$

if $t \in [0, t_1]$, and

$$\begin{aligned} x(t) = S_q(t - t_i)[x(t_i^-) + I_i(x(t_i^-)) + Dv(t_i^-)] \\ + \int_{t_i}^t T_q(t-s)[f(s, x(s), (Hx)(s)) + Bu(s)] ds \end{aligned}$$

if $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$, where

$$S_q(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} \lambda^{q-1} R(\lambda^q, A) d\lambda \quad \text{and} \quad T_q(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} R(\lambda^q, A) d\lambda$$

with c being a suitable path such that $\lambda^q \notin \mu + S_\theta$ for $\lambda \in c$.

Let $x_{t_k}(x(0), \Delta x(t_{k-1}); u, v)$, $k = 1, \dots, m+1$, be the state value of (1.1) at time t_k , corresponding to the non-local initial value $x(0)$, the impulsive values $\Delta x(t_{k-1}) = x(t_{k-1}^+) - x(t_{k-1}^-)$ and the controls u and v . For every $x(0)$ and $\Delta x(t_{k-1}) \in X$, we introduce the set

$$\mathfrak{R}(t_k, x(0), \Delta x(t_{k-1})) = \{x_{t_k}(x(0), \Delta x(t_{k-1}); u, v) : u(\cdot), v(\cdot) \in L^2(I, U)\},$$

which is called the *reachable set* of system (1.1) at time t_k (if $k = m+1$, then t_k is the terminal time). Its closure in X is denoted by $\overline{\mathfrak{R}(t_k, x(0), \Delta x(t_{k-1}))}$.

Definition 6. The impulsive control system (1.1) is said to be approximately controllable on I if $\overline{\mathfrak{R}(t_k, x(0), \Delta x(t_{k-1}))} = X$, that is, given an arbitrary $\epsilon > 0$, it is possible to steer from the points $x(0)$ and $\Delta x(t_{k-1})$ at time t_k all points in the state space X within a distance ϵ .

Consider the linear impulsive fractional control system

$$\begin{cases} {}^C D_t^q x(t) = Ax(t) + Bu(t), \\ x(0) = x_0 \in X, \\ \Delta x(t_i) = Dv(t_i^-), \quad i = 1, \dots, m. \end{cases} \quad (2.1)$$

Approximate controllability for the linear impulsive control fractional system (2.1) is a natural generalization of the notion of approximate controllability of a linear first-order control system ($q = 1$ and $t_i = D = 0$, $i = 1, 2, \dots, m$, i.e., $t \in [t_m, t_{m+1}] = [0, b]$). The controllability operators associated with (2.1) are

$$\begin{aligned} \Psi_{t_{k-1},1}^{t_k} &= \int_{t_{k-1}}^{t_k} T_q(t_k - s) B B^* T_q^*(t_k - s) ds, \quad k = 1, \dots, m+1, \\ \Psi_{t_{k-1},2}^{t_k} &= S_q(t_k - t_{k-1}) D D^* S_q^*(t_k - t_{k-1}), \quad k = 2, \dots, m+1, \end{aligned} \quad (2.2)$$

where $T_q^*(\cdot)$, $S_q^*(\cdot)$, B^* and D^* denote the adjoints of $T_q(\cdot)$, $S_q(\cdot)$, B and D , respectively. Moreover, for $\lambda > 0$, we consider the relevant operator

$$\mathcal{R}(\lambda, \Psi_{t_{k-1},i}^{t_k}) = \left(\lambda I + \Psi_{t_{k-1},i}^{t_k} \right)^{-1}, \quad i = 1, 2. \quad (2.3)$$

It is easy to verify that $\Psi_{t_{k-1},1}^{t_k}$ and $\Psi_{t_{k-1},2}^{t_k}$ are linear bounded operators.

Lemma 1 (See [3]). *The linear impulsive control fractional system (2.1) is approximately controllable on I if and only if $\lambda \mathcal{R}(\lambda, \Psi_{t_{k-1},i}^{t_k}) \rightarrow 0$ as $\lambda \rightarrow 0^+$, $i = 1, 2$, in the strong operator topology.*

Lemma 2 (Krasnoselskii theorem [15]). *Let X be a Banach space and E be a bounded, closed, and convex subset of X . Let Q_1, Q_2 be maps of E into X such that $Q_1 x + Q_2 y \in E$ for every $x, y \in E$. If Q_1 is a contraction and Q_2 is compact and continuous, then equation $Q_1 x + Q_2 x = x$ has a solution on E .*

3. EXISTENCE OF A MILD SOLUTION

We prove existence for system (1.1). Define $K_i^* = \sup_{t \in I} \int_{t_{i-1}}^{t_i} m(t, s) ds < \infty$, $i = 1, \dots, m+1$. For any $r > 0$, let $\Omega_r := \{x \in PC(I, X) \mid \|x\| \leq r\}$. We make the following assumptions:

- (H₁) The operators $S_q(t)_{t \geq 0}$ and $T_q(t)_{t \geq 0}$, generated by A , are bounded and compact, such that $\sup_{t \in I} \|S_q(t)\| \leq M$ and $\sup_{t \in I} \|T_q(t)\| \leq M$.
- (H₂) The non-linearity $f : I \times X \times X \rightarrow X$ is continuous and compact; there exist functions $\mu_i \in L^\infty(I, \mathbb{R}^+)$, $i = 1, 2, 3$, and positive constants α_1 and α_2 such that $\|f(t, x, y)\| \leq \mu_1(t) + \mu_2(t)\|x\| + \mu_3(t)\|y\|$ and $\|f(t, x, Hx) - f(t, y, Hy)\| = \alpha_1\|x - y\| + \alpha_2\|Hx - Hy\|$.
- (H₃) Function $g : PC(I, X) \rightarrow X$ is completely continuous and there exists a positive constant β such that $\|g(x) - g(y)\| \leq \beta\|x - y\|$, $x, y \in X$.

- (H₄) Associated with $h : \Delta \times X \rightarrow X$, there exists $m(t, s) \in PC(\Delta, \mathbb{R}^+)$ such that $\|h(t, s, x(s))\| \leq m(t, s)\|x\|$ for each $(t, s) \in \Delta$ and $x, y \in X$, where $\Delta = \{(t, s) \in \mathbb{R}^2 | t_i \leq s, t \leq t_{i+1}, i = 0, \dots, m\}$.
- (H₅) For every $x_1, x_2, x \in X$ and $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$, I_i are continuous and compact and there exist positive constants d_i, e_i such that

$$\|I_i(x_1(t_i^-)) - I_i(x_2(t_i^-))\| \leq d_i \sup_{t \in (t_i, t_{i+1}]} \|x_1(t) - x_2(t)\|$$

$$\text{and } \|I_i(x(t_i^-))\| \leq e_i \sup_{t \in (t_i, t_{i+1}]} \|x(t)\|.$$

Theorem 1. *Let $x_0 \in X$. If conditions (H₁)–(H₅) hold, then the impulsive non-local fractional control system (1.1) has a fixed point on I provided $M\beta < 1$ and $M(1 + d_i) < 1$, $i = 1, \dots, m$, that is, (1.1) has at least one mild solution on $t \in [0, b] \setminus \{t_1, \dots, t_m\}$.*

Proof. Define the operators Q_1 and Q_2 on Ω_r as follows:

$$(Q_1 x)(t) = \begin{cases} S_q(t)(x_0 - g(x)), & t \in [0, t_1] \\ S_q(t - t_i)[x(t_i^-) + I_i(x(t_i^-)) + Dv(t_i^-)], & t \in (t_i, t_{i+1}], \end{cases}$$

$$(Q_2 x)(t) = \begin{cases} \int_0^t T_q(t-s)(f(s, x(s), (Hx)(s)) + Bu(s))ds, & t \in [0, t_1], \\ \int_{t_i}^t T_q(t-s)(f(s, x(s), (Hx)(s)) + Bu(s))ds, & t \in (t_i, t_{i+1}], \end{cases}$$

$i = 1, \dots, m$. We take the controls

$$\begin{aligned} u &= B^* T_q^*(t_k - t) \mathcal{R}(\lambda, \Psi_{t_{k-1}, 1}^{t_k}) P_1^k(x(\cdot)), \\ v &= D^* S_q^*(t_k - t_{k-1}) \mathcal{R}(\lambda, \Psi_{t_{k-1}, 2}^{t_k}) P_2^k(x(\cdot)), \end{aligned} \quad (3.1)$$

where

$$\begin{aligned} P_1^k(x(\cdot)) &= \begin{cases} x_1 - S_q(t_1)(x_0 - g(x)) \\ \quad - \int_0^{t_1} T_q(t_1 - s) f(s, x(s), (Hx)(s)) ds, & k = 1, \\ x_k - S_q(t_k - t_{k-1})[x(t_{k-1}^-) + I_{k-1}(x(t_{k-1}^-))] \\ \quad - \int_{t_{k-1}}^{t_k} T_q(t_k - s) f(s, x(s), (Hx)(s)) ds, & k = 2, \dots, m+1, \end{cases} \\ P_2^k(x(\cdot)) &= \begin{cases} x_k - S_q(t_k - t_{k-1})[x(t_{k-1}^-) + I_{k-1}(x(t_{k-1}^-))] \\ \quad - \int_{t_{k-1}}^{t_k} T_q(t_k - s) f(s, x(s), (Hx)(s)) ds, & k = 2, \dots, m+1. \end{cases} \end{aligned}$$

For any $\lambda > 0$, we shall show that $Q_1 + Q_2$ has a fixed point on Ω_r , which is a solution of system (1.1). According to (3.1), together with (2.2) and (2.3), we have

$$\|u(t)\| \leq \frac{1}{\lambda} M \|B\| \|P_1(x(\cdot))\| \text{ and } \|v(t)\| \leq \frac{1}{\lambda} M \|D\| \|P_2(x(\cdot))\|. \quad (3.2)$$

Using assumptions (H_1) – (H_5) , we get

$$\begin{aligned}
\|P_1^1(x(\cdot))\| &\leq \|x_1\| + \|S_q(t_1)\| \|x_0 - g(x)\| \\
&\quad + \int_0^{t_1} \|T_q(t_1 - s)\| \|f(s, x(s), (Hx)(s))\| ds \\
&\leq \|x_1\| + M(\|x_0\| + \|g(x)\|) \\
&\quad + Mt_1 (\|\mu_1\|_{L^\infty(I, \mathbb{R}^+)} + r\|\mu_2\|_{L^\infty(I, \mathbb{R}^+)} + K_1^* r \|\mu_3\|_{L^\infty(I, \mathbb{R}^+)}) \\
&\leq \|x_1\| + M\|x_0\| + M\beta\|x\| + M\|g(0)\| \\
&\quad + Mt_1 (\|\mu_1\|_{L^\infty(I, \mathbb{R}^+)} + r\|\mu_2\|_{L^\infty(I, \mathbb{R}^+)} + K_1^* r \|\mu_3\|_{L^\infty(I, \mathbb{R}^+)})
\end{aligned}$$

and, for $k = 2, \dots, m+1$,

$$\begin{aligned}
\|P_1^k(x(\cdot))\| &\leq \|x_k\| + \|S_q(t_k - t_{k-1})\| [\|x(t_{k-1}^-)\| + \|I_{k-1}(x(t_{k-1}^-))\|] \\
&\quad + \int_{t_{k-1}}^{t_k} \|T_q(t_k - s)\| \|f(s, x(s), (Hx)(s))\| ds \\
&\leq \|x_k\| + M(\|x(t_{k-1}^-)\| + e_i\|x\|) + M(t_k - t_{k-1}) \\
&\quad \times (\|\mu_1\|_{L^\infty(I, \mathbb{R}^+)} + r\|\mu_2\|_{L^\infty(I, \mathbb{R}^+)} + K_k^* r \|\mu_3\|_{L^\infty(I, \mathbb{R}^+)}) \\
&\leq \|x_k\| + M(\|x(t_{k-1}^-)\| + r e_i) + M(t_k - t_{k-1}) \\
&\quad \times (\|\mu_1\|_{L^\infty(I, \mathbb{R}^+)} + r\|\mu_2\|_{L^\infty(I, \mathbb{R}^+)} + K_k^* r \|\mu_3\|_{L^\infty(I, \mathbb{R}^+)}) .
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\|P_2^k(x(\cdot))\| &\leq \|x_k\| + M(\|x(t_{k-1}^-)\| + r e_{k-1}) \\
&\quad + M(t_k - t_{k-1}) (\|\mu_1\|_{L^\infty(I, \mathbb{R}^+)} + r\|\mu_2\|_{L^\infty(I, \mathbb{R}^+)} + K_k^* r \|\mu_3\|_{L^\infty(I, \mathbb{R}^+)}) ,
\end{aligned}$$

$k = 2, \dots, m+1$. For any $x \in \Omega_r$, we obtain

$$\begin{aligned}
&\|(Q_1x)(t) + (Q_2x)(t)\| \\
&\leq M(\|x_0\| + \|g(x)\|) \\
&\quad + Mt_1 (\|\mu_1\|_{L^\infty(I, \mathbb{R}^+)} + r\|\mu_2\|_{L^\infty(I, \mathbb{R}^+)} + K_1^* r \|\mu_3\|_{L^\infty(I, \mathbb{R}^+)} + \|B\|\|u\|) \\
&\leq M(\|x_0\| + \beta r + \|g(0)\|) \\
&\quad + Mt_1 (\|\mu_1\|_{L^\infty(I, \mathbb{R}^+)} + r\|\mu_2\|_{L^\infty(I, \mathbb{R}^+)} + K_1^* r \|\mu_3\|_{L^\infty(I, \mathbb{R}^+)} + \|B\|\|u\|)
\end{aligned}$$

for $t \in [0, t_1]$, and

$$\begin{aligned}
&\|(Q_1x)(t) + (Q_2x)(t)\| \\
&\leq M(\|x(t_{k-1}^-)\| + e_{k-1}\|x\| + \|D\|\|v(t_{k-1}^-)\|) + M(t_k - t_{k-1}) \\
&\quad \times (\|\mu_1\|_{L^\infty(I, \mathbb{R}^+)} + r\|\mu_2\|_{L^\infty(I, \mathbb{R}^+)} + K_k^* r \|\mu_3\|_{L^\infty(I, \mathbb{R}^+)} + \|B\|\|u\|) \\
&\leq M(\|x(t_{k-1}^-)\| + e_{k-1}r + \|D\|\|v(t_{k-1}^-)\|) + M(t_k - t_{k-1}) \\
&\quad \times (\|\mu_1\|_{L^\infty(I, \mathbb{R}^+)} + r\|\mu_2\|_{L^\infty(I, \mathbb{R}^+)} + K_k^* r \|\mu_3\|_{L^\infty(I, \mathbb{R}^+)} + \|B\|\|u\|)
\end{aligned}$$

for $t \in (t_{k-1}, t_k]$. By the inequalities (3.2), we can find $\xi_1, \xi_2 > 0$ such that

$$\|(Q_1x)(t) + (Q_2x)(t)\| \leq \begin{cases} \xi_1, t \in [0, t_1], \\ \xi_2, t \in (t_{k-1}, t_k], k = 2, \dots, m+1. \end{cases}$$

Hence, $Q_1x + Q_2x$ is bounded. Now, let $x, y \in \Omega_r$. We have

$$\|(Q_1x)(t) - (Q_1y)(t)\| \leq \|S_q(t)\| \|g(x) - g(y)\| \leq M\beta \|x - y\|$$

for $t \in [0, t_1]$ and

$$\begin{aligned} & \|(Q_1x)(t) - (Q_1y)(t)\| \\ & \leq \|S_q(t - t_{k-1})\| [\|x(t_{k-1}^-) - y(t_{k-1}^-)\| + \|I_{k-1}(x(t_{k-1}^-)) - I_{k-1}(y(t_{k-1}^-))\|] \\ & \leq M [\|x(t_{k-1}^-) - y(t_{k-1}^-)\| + d_{k-1} \|x - y\|] \end{aligned}$$

for $t \in (t_{k-1}, t_k]$, $k = 2, \dots, m+1$. Since $M\beta < 1$ and $M(1 + d_{k-1}) < 1$, $k = 2, \dots, m+1$, it follows that Q_1 is a contraction mapping. Let $\{x_n\}$ be a sequence in Ω_r such that $x_n \rightarrow x \in \Omega_r$. Since f and g are continuous, i.e., for all $\epsilon > 0$, there exists a positive integer n_0 , such that for $n > n_0$ $\|f(s, x_n(s), (Hx_n)(s)) - f(s, x(s), (Hx)(s))\| \leq \epsilon$ and $\|g(x_n) - g(x)\| \leq \epsilon$, the continuity of $I_i(x)$ on $(t_i, t_{i+1}]$ gives $\|I_i(x_n(t_i^-)) - I_i(x(t_i^-))\| \leq \epsilon$, $i = 1, \dots, m$. Now, for all $t \in [0, t_1]$,

$$\begin{aligned} & \|(Q_2x_n)(t) - (Q_2x)(t)\| \\ & \leq \int_0^{t_1} \|T_q(t - \tau)\| \|BB^*T_q^*(t_1 - \tau)\mathcal{R}(\lambda, \Psi_{t_0,1}^{t_1})\| [\|S_q(t_1)(g(x_n) - g(x))\| \\ & \quad + \int_0^{t_1} \|T_q(t_1 - s)\| \|f(s, x_n(s), (Hx_n)(s)) - f(s, x(s), (Hx)(s))\| ds] d\tau \\ & \quad + \int_0^{t_1} \|T_q(t - s)\| \|f(s, x_n(s), (Hx_n)(s)) - f(s, x(s), (Hx)(s))\| ds \\ & \leq \frac{\epsilon}{\lambda} M^3 \|B\|^2 t_1 (2t_1 + 1). \end{aligned}$$

Moreover, for all $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$, one has

$$\begin{aligned} & \|(Q_2x_n)(t) - (Q_2x)(t)\| \\ & \leq \int_{t_i}^t \|T_q(t - \tau)\| \|BB^*T_q^*(t_{i+1} - \tau)\mathcal{R}(\lambda, \Psi_{t_i,1}^{t_{i+1}})\| \\ & \quad \times \left[\|S_q(t_{i+1} - t_i)[x_n(t_i^-) - x(t_i^-) + I_i(x_n(t_i^-)) - I_i(x(t_i^-))]\| \right. \\ & \quad \left. + \int_{t_i}^{t_{i+1}} \|T_q(t_{i+1} - s)\| \|f(s, x_n(s), (Hx_n)(s)) - f(s, x(s), (Hx)(s))\| ds \right] d\tau \\ & \quad + \int_{t_i}^t \|T_q(t - s)\| \|f(s, x_n(s), (Hx_n)(s)) - f(s, x(s), (Hx)(s))\| ds \end{aligned}$$

$$\leq \frac{2\epsilon}{\lambda} M^3 \|B\|^2 (t_{i+1} - t_i)(t_{i+1} - t_i + 1).$$

Therefore, Q_2 is continuous. Next, we prove the compactness of Q_2 . For that, we first show that the set $\{(Q_2x)(t) : x \in \Omega_r\}$ is relatively compact in $PC(I, X)$. By the assumptions of our theorem, we have

$$\begin{aligned} & \| (Q_2x)(t) \| \\ & \leq M t_1 (\|\mu_1\|_{L^\infty(I, \mathbb{R}^+)} + r \|\mu_2\|_{L^\infty(I, \mathbb{R}^+)} + K_1^* r \|\mu_3\|_{L^\infty(I, \mathbb{R}^+)} + \|B\| \|u\|), \end{aligned}$$

for $t \in [0, t_1]$, and

$$\begin{aligned} & \| (Q_2x)(t) \| \leq M(t_k - t_{k-1}) \\ & \quad \times (\|\mu_1\|_{L^\infty(I, \mathbb{R}^+)} + r \|\mu_2\|_{L^\infty(I, \mathbb{R}^+)} + K_k^* r \|\mu_3\|_{L^\infty(I, \mathbb{R}^+)} + \|B\| \|u\|), \end{aligned}$$

for $t \in (t_{k-1}, t_k]$, which gives the uniformly boundedness of $\{(Q_2x)(t) : x \in \Omega_r\}$. We now show that $Q_2(\Omega_r)$ is equicontinuous. Functions $\{(Q_2x)(t) : x \in \Omega_r\}$ are equicontinuous at $t = 0$. For any $x \in \Omega_r$, if $0 < r_1 < r_2 \leq t_1$, then

$$\begin{aligned} & \| (Q_2x)(r_2) - (Q_2x)(r_1) \| \\ & \leq \int_0^{r_1} \|T_q(r_2 - s) - T_q(r_1 - s)\| [\|Bu(s)\| + \|f(s, x(s), (Hx)(s))\|] ds \\ & \quad + \int_{r_1}^{r_2} \|T_q(r_2 - s)\| [\|Bu(s)\| + \|f(s, x(s), (Hx)(s))\|] ds \\ & \leq [r_1 \|T_q(r_2 - s) - T_q(r_1 - s)\| + M(r_2 - r_1)] \\ & \quad \times (\|B\| \|u\| + \|\mu_1\|_{L^\infty(I, \mathbb{R}^+)} + r \|\mu_2\|_{L^\infty(I, \mathbb{R}^+)} + K_1^* r \|\mu_3\|_{L^\infty(I, \mathbb{R}^+)}). \end{aligned}$$

Similarly, if $t_i < r_1 < r_2 \leq t_{i+1}$, then

$$\begin{aligned} & \| (Q_2x)(r_2) - (Q_2x)(r_1) \| \\ & \leq \int_{t_i}^{r_1} \|T_q(r_2 - s) - T_q(r_1 - s)\| [\|Bu(s)\| + \|f(s, x(s), (Hx)(s))\|] ds \\ & \quad + \int_{r_1}^{r_2} \|T_q(r_2 - s)\| [\|Bu(s)\| + \|f(s, x(s), (Hx)(s))\|] ds \\ & \leq [(r_1 - t_i) \|T_q(r_2 - s) - T_q(r_1 - s)\| + M(r_2 - r_1)] \\ & \quad \times (\|B\| \|u\| + \|\mu_1\|_{L^\infty(I, \mathbb{R}^+)} + r \|\mu_2\|_{L^\infty(I, \mathbb{R}^+)} + K_{i+1}^* r \|\mu_3\|_{L^\infty(I, \mathbb{R}^+)}). \end{aligned}$$

From (H_1) , it follows the continuity of operator $T_q(\cdot)$ in the uniform operator topology. Thus, the right hand side of the above inequality tends to zero as $r_2 \rightarrow r_1$. Therefore, $\{(Q_2x)(t) : x \in \Omega_r\}$ is a family of equicontinuous functions. According to the infinite dimensional version of the Ascoli–Arzela theorem, it remains to prove that, for any $t \in [0, b] \setminus \{t_1, \dots, t_m\}$, the set $V(t) := \{(Q_2x)(t) : x \in \Omega_r\}$ is relatively compact in $PC(I, X)$. The case $t = 0$ is trivial: $V(0) = \{(Q_2x)(0) : x(\cdot) \in \Omega_r\}$ is

compact in $PC(I, X)$. Let $t \in (0, t_1]$ be a fixed real number and h be a given real number satisfying $0 < h < t_1$. Define $V_h(t) = \{(Q_2^h x)(t) : x \in \Omega_r\}$,

$$\begin{aligned} (Q_2^h x)(t) &= \int_0^{t-h} T_q(t-s)Bu(s)ds + \int_0^{t-h} T_q(t-s)f(s, x(s), (Hx)(s))ds \\ &= T_q(h) \int_0^{t-h} T_q(t-s-h)Bu(s)ds \\ &\quad + T_q(h) \int_0^{t-h} T_q(t-s-h)f(s, x(s), (Hx)(s))ds \\ &= T_q(h)y_1(t, h). \end{aligned}$$

We use same arguments, we fix $t \in (t_i, t_{i+1}]$, and let h be a given real number satisfying $t_i < h < t_{i+1}$, we define $V_h(t) = \{(Q_2^h x)(t) : x \in \Omega_r\}$,

$$\begin{aligned} (Q_2^h x)(t) &= \int_{t_i}^{t-h} T_q(t-s)Bu(s)ds + \int_{t_i}^{t-h} T_q(t-s)f(s, x(s), (Hx)(s))ds \\ &= T_q(h) \int_{t_i}^{t-h} T_q(t-s-h)Bu(s)ds \\ &\quad + T_q(h) \int_{t_i}^{t-h} T_q(t-s-h)f(s, x(s), (Hx)(s))ds \\ &= T_q(h)y_2(t, h). \end{aligned}$$

The compactness of $T_q(h)$ in $PC(I, X)$, together with the boundedness of both $y_1(t, h)$ and $y_2(t, h)$ on Ω_r , give the relative compactness of the set $V_h(t)$ in $PC(I, X)$. Moreover, for all $t \in [0, t_1]$,

$$\begin{aligned} &\|(Q_2 x)(t) - (Q_2^h x)(t)\| \\ &\leq \int_{t-h}^t T_q(t-s)Bu(s)ds + \int_{t-h}^t T_q(t-s)f(s, x(s), (Hx)(s))ds \\ &\leq hM (\|B\|\|u\| + \|\mu_1\|_{L^\infty(I, \mathbb{R}^+)} + r\|\mu_2\|_{L^\infty(I, \mathbb{R}^+)} + K_1^* r\|\mu_3\|_{L^\infty(I, \mathbb{R}^+)}) . \end{aligned}$$

Also, for all $t \in (t_i, t_{i+1}]$,

$$\begin{aligned} &\|(Q_2 x)(t) - (Q_2^h x)(t)\| \\ &\leq \int_{t-h}^t T_q(t-s)Bu(s)ds + \int_{t-h}^t T_q(t-s)f(s, x(s), (Hx)(s))ds \\ &\leq hM (\|B\|\|u\| + \|\mu_1\|_{L^\infty(I, \mathbb{R}^+)} + r\|\mu_2\|_{L^\infty(I, \mathbb{R}^+)} + K_{i+1}^* r\|\mu_3\|_{L^\infty(I, \mathbb{R}^+)}) . \end{aligned}$$

Choose h small enough. It implies that there are relatively compact sets arbitrarily close to the set $V(t)$ for each $t \in [0, b] \setminus \{t_1, \dots, t_m\}$. Then, $V(t)$, $t \in [0, b] \setminus \{t_1, \dots, t_m\}$, is relatively compact in $PC(I, X)$. Since it is compact at $t = 0$, we

have the relatively compactness of $V(t)$ in $PC(I, X)$ for all $t \in [0, b] \setminus \{t_1, \dots, t_m\}$. Hence, by the Arzela–Ascoli theorem, we conclude that Q_2 is compact. From Lemma 2, we ensure that the control system (1.1) has at least one mild solution on $t \in [0, b] \setminus \{t_1, \dots, t_m\}$. $\square$

4. APPROXIMATE CONTROLLABILITY

In this section, with help of the obtained existence theorem of mild solutions, we show an approximate controllability result for system (1.1).

Theorem 2. *If (H_1) – (H_5) are satisfied and $\lambda \mathcal{R}(\lambda, \Psi_{t_{k-1}, i}^{t_k}) \rightarrow 0$ in the strong operator topology as $\lambda \rightarrow 0^+$, $i = 1, 2$, then the impulsive non-local fractional control system (1.1) is approximately controllable on $t \in [0, b] \setminus \{t_1, \dots, t_m\}$.*

Proof. According to Theorem 1, $Q_1^\lambda + Q_2^\lambda$ has a fixed point in Ω_r for any $\lambda > 0$. This implies that there exists $\bar{x}^\lambda \in (Q_1^\lambda + Q_2^\lambda)(\bar{x}^\lambda)$ such that

$$\bar{x}^\lambda(t) = \begin{cases} S_q(t)(x_0 - \bar{g}^\lambda(\bar{x}^\lambda)) \\ \quad + \int_0^t T_q(t-s)[\bar{f}^\lambda(s, \bar{x}^\lambda(s), (H\bar{x}^\lambda)(s)) + B\bar{u}^\lambda(s)]ds, & t \in [0, t_1], \\ S_q(t-t_{k-1})[\bar{x}^\lambda(t_{k-1}^-) + \overline{I_{k-1}}^\lambda(\bar{x}^\lambda(t_{k-1}^-)) + D\bar{v}^\lambda(t_{k-1}^-)] \\ \quad + \int_{t_{k-1}}^t T_q(t-s)[\bar{f}^\lambda(s, \bar{x}^\lambda(s), (H\bar{x}^\lambda)(s)) + B\bar{u}^\lambda(s)]ds, & t \in (t_{k-1}, t_k], \end{cases}$$

where for $t \in [0, t_1]$ we have

$$\begin{aligned} \bar{u}^\lambda = B^* T_q^*(t_1 - t) \mathcal{R}(\lambda, \Psi_{0,1}^{t_1}) & \left[x_1 - S_q(t_1)(x_0 - \bar{g}^\lambda(\bar{x}^\lambda)) \right. \\ & \left. - \int_0^{t_1} T_q(t_1 - s) \bar{f}^\lambda(s, \bar{x}^\lambda(s), (H\bar{x}^\lambda)(s)) ds \right] \end{aligned}$$

while for $k = 2, \dots, m+1$

$$\begin{aligned} \bar{u}^\lambda &= B^* T_q^*(t_k - t) \mathcal{R}(\lambda, \Psi_{t_{k-1},1}^{t_k}) \left[x_k - S_q(t_k - t_{k-1})[\bar{x}^\lambda(t_{k-1}^-) + \overline{I_{k-1}}^\lambda(\bar{x}^\lambda(t_{k-1}^-))] \right. \\ & \quad \left. - \int_{t_{k-1}}^{t_k} T_q(t_k - s) \bar{f}^\lambda(s, \bar{x}^\lambda(s), (H\bar{x}^\lambda)(s)) ds \right] \end{aligned}$$

and

$$\begin{aligned} \bar{v}^\lambda &= D^* S_q^*(t_k - t_{k-1}) \mathcal{R}(\lambda, \Psi_{t_{k-1},2}^{t_k}) \\ & \quad \times \left[x_k - S_q(t_k - t_{k-1})[\bar{x}^\lambda(t_{k-1}^-) + \overline{I_{k-1}}^\lambda(\bar{x}^\lambda(t_{k-1}^-))] \right] \end{aligned}$$

$$- \int_{t_{k-1}}^{t_k} T_q(t_k - s) \bar{f}^\lambda(s, \bar{x}^\lambda(s), (H\bar{x}^\lambda)(s)) ds \Big].$$

Furthermore,

$$\begin{aligned} \bar{x}^\lambda(t_1) &= S_q(t_1)(x_0 - \bar{g}^\lambda(\bar{x}^\lambda)) \\ &\quad + \int_0^{t_1} T_q(t_1 - s) [\bar{f}^\lambda(s, \bar{x}^\lambda(s), (H\bar{x}^\lambda)(s)) + B\bar{u}^\lambda(s)] ds, \\ \bar{x}^\lambda(t_k) &= S_q(t_k - t_{k-1})[\bar{x}^\lambda(t_{k-1}^-) + \overline{I_{k-1}}^\lambda(\bar{x}^\lambda(t_{k-1}^-)) + D\bar{v}^\lambda(t_{k-1}^-)] \\ &\quad + \int_{t_{k-1}}^{t_k} T_q(t_k - s) [\bar{f}^\lambda(s, \bar{x}^\lambda(s), (H\bar{x}^\lambda)(s)) + B\bar{u}^\lambda(s)] ds, \end{aligned}$$

$k = 2, \dots, m+1$, with

$$\begin{aligned} x_{t_1} - \bar{x}^\lambda(t_1) &= x_1 - \Psi_{0,1}^{t_1} \mathcal{R}(\lambda, \Psi_{0,1}^{t_1}) \Big\{ x_1 - S_q(t_1)(x_0 - \bar{g}^\lambda(\bar{x}^\lambda)) \\ &\quad - \int_0^{t_1} T_q(t_1 - s) \bar{f}^\lambda(s, \bar{x}^\lambda(s), (H\bar{x}^\lambda)(s)) ds \Big\} \\ &\quad - S_q(t_1)(x_0 - \bar{g}^\lambda(\bar{x}^\lambda)) - \int_0^{t_1} T_q(t_1 - s) \bar{f}^\lambda(s, \bar{x}^\lambda(s), (H\bar{x}^\lambda)(s)) ds, \\ x_{t_k} - \bar{x}^\lambda(t_k) &= x_k - \Psi_{k-1,2}^{t_k} \mathcal{R}(\lambda, \Psi_{k-1,2}^{t_k}) \\ &\quad \times \Big\{ x_k - S_q(t_k - t_{k-1}) \Big[\bar{x}^\lambda(t_{k-1}^-) + \overline{I_{k-1}}^\lambda(\bar{x}^\lambda(t_{k-1}^-)) \Big] \\ &\quad - \int_{t_{k-1}}^{t_k} T_q(t_k - s) \bar{f}^\lambda(s, \bar{x}^\lambda(s), (H\bar{x}^\lambda)(s)) ds \Big\} \\ &\quad - S_q(t_k - t_{k-1}) \Big[\bar{x}^\lambda(t_{k-1}^-) + \overline{I_{k-1}}^\lambda(\bar{x}^\lambda(t_{k-1}^-)) \Big] \\ &\quad - \int_{t_{k-1}}^{t_k} T_q(t_k - s) \bar{f}^\lambda(s, \bar{x}^\lambda(s), (H\bar{x}^\lambda)(s)) ds \\ &\quad - \Psi_{k-1,1}^{t_k} \mathcal{R}(\lambda, \Psi_{k-1,1}^{t_k}) \Big\{ x_k - S_q(t_k - t_{k-1}) \Big[\bar{x}^\lambda(t_{k-1}^-) + \overline{I_{k-1}}^\lambda(\bar{x}^\lambda(t_{k-1}^-)) \Big] \\ &\quad - \int_{t_{k-1}}^{t_k} T_q(t_k - s) \bar{f}^\lambda(s, \bar{x}^\lambda(s), (H\bar{x}^\lambda)(s)) ds \Big\}, \quad k = 2, \dots, m+1. \end{aligned}$$

From (2.3) we have $I - \Psi_{t_{k-1},i}^{t_k} \mathcal{R}(\lambda, \Psi_{t_{k-1},i}^{t_k}) = \lambda \mathcal{R}(\lambda, \Psi_{t_{k-1},i}^{t_k})$, $i = 1, 2$, and

$$x_{t_1} - \bar{x}^\lambda(t_1) = \lambda \mathcal{R}(\lambda, \Psi_{0,1}^{t_1}) \Big\{ x_1 - S_q(t_1)(x_0 - \bar{g}^\lambda(\bar{x}^\lambda))$$

$$- \int_0^{t_1} T_q(t_1-s) \bar{f}^\lambda(s, \bar{x}^\lambda(s), (H\bar{x}^\lambda)(s)) ds \Big\}, \quad (4.1)$$

$$\begin{aligned} x_{t_k} - \bar{x}^\lambda(t_k) = & \lambda \left[\mathcal{R}(\lambda, \Psi_{k-1,1}^{t_k}) + \mathcal{R}(\lambda, \Psi_{k-1,2}^{t_k}) \right] \\ & \times \left\{ x_k - S_q(t_k - t_{k-1}) \left[\bar{x}^\lambda(t_{k-1}^-) + \overline{I_{k-1}}^\lambda(\bar{x}^\lambda(t_{k-1}^-)) \right] \right. \\ & \left. - \int_{t_{k-1}}^{t_k} T_q(t_k-s) \bar{f}^\lambda(s, \bar{x}^\lambda(s), (H\bar{x}^\lambda)(s)) ds \right\}, \quad k = 2, \dots, m+1. \end{aligned} \quad (4.2)$$

Since compactness of both $S_q(t)_{t>0}$ and $T_q(t)_{t>0}$ hold, and also boundedness of $\bar{f}^\lambda$, $\bar{g}^\lambda$ and $\overline{I_{k-1}}^\lambda$, we can use on (4.1)–(4.2) the fact that $\lambda \mathcal{R}(\lambda, \Psi_{t_{k-1},i}^{t_k}) \rightarrow 0$ in the strong operator topology as $\lambda \rightarrow 0^+$, $i = 1, 2$. This gives $\|x_{t_k} - \bar{x}^\lambda(t_k)\|_\alpha \rightarrow 0$ as $\lambda \rightarrow 0^+$, $i = 1, 2$. Hence, the impulsive non-local fractional control system (1.1) is approximately controllable on $t \in [0, b] \setminus \{t_1, \dots, t_m\}$. $\square$

5. OPTIMALITY

Let Y be a separable reflexive Banach space and $w_f(Y)$ represent a class of non-empty, closed and convex subsets of Y . The multifunction $w : I \rightarrow w_f(Y)$ is measurable and $w(\cdot) \subset E$, where E is a bounded set of Y . We give the admissible control set as follows:

$$U_{ad} = \{(u, v) \in L^1(E) \times L^1(E) | u(t), v(t) \in w(t) \text{ a.e.}\} \neq \emptyset.$$

Consider the following impulsive nonlocal fractional control system:

$$\begin{cases} {}^C D_t^q x(t) = Ax(t) + f(t, x(t), (Hx)(t)) + \mathcal{B}u(t), t \in (0, b] \setminus \{t_1, t_2, \dots, t_m\}, \\ x(0) + g(x) = x_0 \in X, \\ \Delta x(t_i) = I_i(x(t_i^-)) + \mathcal{D}v(t_i^-), i = 1, 2, \dots, m, \quad (u, v) \in U_{ad}, \end{cases} \quad (5.1)$$

where $\mathcal{B}, \mathcal{D} \in L^\infty(I, L(Y, X))$. It is clear that $\mathcal{B}u, \mathcal{D}v \in L^1(I, X)$ for all $(u, v) \in U_{ad}$. Let $x^{u,v}$ be a mild solution of system (5.1) corresponding to controls $(u, v) \in U_{ad}$. We consider the Bolza problem (BP): find an optimal triplet $(x^0, u^0, v^0) \in PC(I, X) \times U_{ad}$ such that $\mathcal{J}(x^0, u^0, v^0) \leq \mathcal{J}(x^{u,v}, u, v)$, for all $(u, v) \in U_{ad}$, where

$$\mathcal{J}(x^{u,v}, u, v) = \sum_{i=1}^{m+1} \left[\Phi(x^{u,v}(t_i)) + \int_{t_{i-1}}^{t_i} \mathcal{L}(t, x^{u,v}(t), u(t), v(t)) dt \right],$$

$i = 1, \dots, m+1$. The following extra assumptions are needed:

- (H₆) The functional $\mathcal{L} : I \times X \times Y^2 \rightarrow \mathbb{R} \cup \{\infty\}$ is Borel measurable.
- (H₇) $\mathcal{L}(t, \cdot, \cdot, \cdot)$ is sequentially lower semi-continuous on $X \times Y^2$, a.e. on I .
- (H₈) $\mathcal{L}(t, \cdot, \cdot)$ is convex on Y^2 for each $x \in X$ and almost all $t \in I$.

(H₉) There is a non-negative function $\varphi \in L^\infty(I, \mathbb{R})$ and $c_1, c_2, c_3 \geq 0$ such that $\mathcal{L}(t, x, u, v) \geq \varphi(t) + c_1 \|x\| + c_2 \|u\|_Y^p + c_3 \|v\|_Y^p$.

(H₁₀) The functional $\Phi : X \rightarrow \mathbb{R}$ is continuous and non-negative.

Theorem 3. *If (H₆)–(H₁₀) hold together with the assumptions of Theorem 1, then the Bolza problem (BP) admits at least one optimal triplet on $PC \times U_{ad}$.*

Proof. Assume $\inf \{ \mathcal{J}(x^{u,v}, u, v) | (u, v) \in U_{ad} \} = \delta < +\infty$. From (H₆)–(H₁₀),

$$\begin{aligned} & \mathcal{J}(x^{u,v}, u, v) \\ & \geq \sum_{i=1}^{m+1} \left[\Phi(x^{u,v}(t_i)) + \int_{t_{i-1}}^{t_i} \{ \varphi(t) + c_1 \|x(t)\| + c_2 \|u(t)\|_Y^p + c_3 \|v(t)\|_Y^p \} dt \right] \\ & \geq -\eta > -\infty, \quad i = 1, \dots, m+1. \end{aligned}$$

Here, η is a positive constant, i.e., $\delta \geq -\eta > -\infty$. By the definition of infimum, there exists a minimizing sequence of feasible triplets $\{(x^n, u^n, v^n)\} \subset \mathcal{A}_{ad}$, where $\mathcal{A}_{ad} \equiv \{(x, u, v) | x \text{ is a mild solution of system (5.1) corresponding to } (u, v) \in U_{ad}\}$, such that $\mathcal{J}(x^n, u^n, v^n) \rightarrow \delta$ as $m \rightarrow +\infty$. As $\{(u^n, v^n)\} \subseteq U_{ad}$ and $\{u^n, v^n\}$ is bounded in $L^1(I, Y)$, then there exists a subsequence, still denoted by $\{(u^n, v^n)\}$, and $u^0, v^0 \in L^1(I, Y)$, such that $(u^n, v^n) \xrightarrow{\text{weakly}} (u^0, v^0)$ in $L^1(I, Y) \times L^1(I, Y)$. Since the admissible control set U_{ad} is convex and closed, by Marzur lemma, we have $(u^0, v^0) \in U_{ad}$. Suppose that x^n is a mild solution of system (5.1), corresponding to u^n and v^n , that satisfies

$$x^n(t) = S_q(t)(x_0 - g(x^n)) + \int_0^t T_q(t-s)(f(s, x^n(s), (Hx^n)(s)) + \mathcal{B}u^n(s))ds,$$

for $t \in [0, t_1]$, and

$$\begin{aligned} x^n(t) &= S_q(t - t_i)[x^n(t_i^-) + I_i(x^n(t_i^-)) + \mathcal{D}v^n(t_i^-)] \\ &\quad + \int_{t_i}^t T_q(t-s)[f(s, x^n(s), (Hx^n)(s)) + \mathcal{B}u^n(s)]ds \end{aligned}$$

for $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$. From (H₂), the non-linear function f is bounded and continuous. Then, there exists a subsequence (with the same notation)

$\{f(s, x^n, (Hx^n)(s))\}$ and $f(s, x^0, (Hx^0)(s)) \in L^1(I, X)$ such that $f(s, x^n, (Hx^n)(s))$ converges weakly to $f(s, x^0, (Hx^0)(s))$. Also, the same arguments on (H₃) and (H₅) yield other weak convergences of $g(x^n)$ and $I_i(x^n)$ to $g(x^0)$ and $I_i(x^0)$, respectively. Let us denote

$$(P_1x)(t) = S_q(t)g(x) + \int_0^t T_q(t-s)(f(s, x(s), (Hx)(s)) + \mathcal{B}u(s))ds, \quad t \in [0, t_1],$$

$$(P_2x)(t) = S_q(t - t_i)[x(t_i^-) + I_i(x(t_i^-)) + \mathcal{D}v(t_i^-)]$$

$$+ \int_{t_i}^t T_q(t-s)[f(s, x(s), (Hx)(s)) + \mathcal{B}u(s)]ds, \quad t \in (t_i, t_{i+1}], \quad i = 1, \dots, m.$$

Obviously, $(P_1x)(t)$ and $(P_2x)(t)$ are strongly continuous operators. Thus, $(P_1x^n)(t)$ and $(P_2x^n)(t)$ strongly converge to $(P_1x)(t)$ and $(P_2x)(t)$, respectively. Next, we consider the system

$$x^0(t) = S_q(t)(x_0 - g(x^0)) + \int_0^t T_q(t-s)(f(s, x^0(s), (Hx^0)(s)) + \mathcal{B}u^0(s))ds,$$

$t \in [0, t_1]$, and

$$\begin{aligned} x^0(t) = & S_q(t - t_i)[x^0(t_i^-) + I_i(x^0(t_i^-)) + \mathcal{D}v^0(t_i^-)] \\ & + \int_{t_i}^t T_q(t-s)[f(s, x^0(s), (Hx^0)(s)) + \mathcal{B}u^0(s)]ds, \end{aligned}$$

$t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$. It is not difficult to check that $\|x^n(t) - x^0(t)\| \rightarrow 0$ as $n \rightarrow \infty$. Therefore, we can infer that x^n strongly converges to x^0 in $PC(I, X)$ as $n \rightarrow \infty$. From assumptions (H_6) – (H_{10}) and Balder's theorem, we get

$$\begin{aligned} \eta &= \lim_{n \rightarrow \infty} \sum_{i=1}^{m+1} \left[\Phi(x^n(t_i)) + \int_{t_{i-1}}^{t_i} \mathcal{L}(t, x^n(t), u^n(t), v^n(t))dt \right] \\ &\geq \sum_{i=1}^{m+1} \left[\Phi(x^0(t_i)) + \int_{t_{i-1}}^{t_i} \mathcal{L}(t, x^0(t), u^0(t), v^0(t))dt \right] \\ &= \mathcal{J}(x^0, u^0, v^0) \geq \eta, \quad i = 1, \dots, m+1, \end{aligned}$$

which implies that $\mathcal{J}$ attains its minimum at $(x^0, u^0, v^0) \in PC(I, X) \times U_{ad}$. $\square$

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*Authors' addresses***Sarra Guechi**

Guelma University, Department of Mathematics, 24000 Guelma, Algeria

E-mail address: guechi.sara@yahoo.fr

Amar Debbouche

Guelma University, Department of Mathematics, 24000 Guelma, Algeria

E-mail address: amar-debbouche@yahoo.fr

Delfim F. M. Torres

University of Aveiro, DMat, CIDMA, 3810-193 Aveiro, Portugal

E-mail address: delfim@ua.pt



QUASIPOLARITY OF SPECIAL MORITA CONTEXT RINGS

O. GÜRGÜN, S. HALICIOGLU, AND A. HARMANCI

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Abstract. In this paper, we introduce a new concept of Morita context rings which we call special Morita context rings. We determine the conditions under which this kind of rings are quasipolar. We use special Morita context rings to extend some results of quasipolar rings. Then many of the main results of quasipolar rings are special cases of our results for this general setting. Several basic characterizations and properties of these rings are given.

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1. INTRODUCTION

Throughout this paper all rings are associative with identity and modules are unital. The *commutant* and *second commutant* of $a \in R$ are defined by $\text{comm}(a) = \{x \in R \mid xa = ax\}$, $\text{comm}^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in \text{comm}(a)\}$, respectively. $U(R)$, $J(R)$ and $\text{Nil}(R)$ will denote the set of all invertible elements, the Jacobson radical of R and the set of all nilpotent elements, respectively. An element a in a ring R is called *quasinilpotent* if $1 - ax \in U(R)$ for any $x \in \text{comm}(a)$. The set of all quasinilpotent elements of R will be denoted by $QN(R)$ ([9]). Set $J^\#(R) = \{x \in R \mid \exists n \in \mathbb{N} \text{ such that } x^n \in J(R)\}$. Obviously, $J(R) \subseteq J^\#(R) \subseteq QN(R)$ and $\text{Nil}(R) \subseteq J^\#(R) \subseteq QN(R)$.

The notion of a quasipolar ring was introduced by Harte in his 1991 study on quasinilpotent in rings. An element $a \in R$ is called *quasipolar* provided that there exists an idempotent $p \in R$ such that $p \in \text{comm}^2(a)$, $a + p \in U(R)$ and $ap \in QN(R)$. A ring R is *quasipolar* in case every element in R is quasipolar. Any idempotent p satisfying the above conditions is called a *spectral idempotent* of a . Koliha [11] introduced the concept of a generalized Drazin inverse in a complex Banach algebra. An element a of R is *generalized Drazin invertible* [12] in case there is an element $b \in R$ satisfying $ab^2 = b$, $b \in \text{comm}^2(a)$ and $a^2b - a \in QN(R)$. Such b , if it exists, is unique; it is called a *generalized Drazin inverse* of a and will be denoted by a^g . Koliha and Patricio [12] proved any quasipolar element $a \in R$ has a unique spectral

idempotent denoted by a^π , and a is quasipolar if and only if a is generalized Drazin invertible. Quasipolar rings have been studied by several authors [2, 4, 7, 12, 22].

Recall some definitions. A ring is called *strongly clean* if every element is the sum of an idempotent and a unit which commute (see [16]). Following [1], an element $a \in R$ is said to be *strongly π -regular* if $a^n \in a^{n+1}R \cap Ra^{n+1}$ for some $n \in \mathbb{N}$. An element $a \in R$ is called *pseudopolar* if there exists $p^2 = p \in \text{comm}^2(a)$ such that $a + p \in U(R)$ and $ap \in J^\#(R)$ ([21]). An element a of R is (*pseudo*) *Drazin invertible* ([21]) [5] in case there is an element $b \in R$ satisfying $ab^2 = b$, $b \in \text{comm}^2(a)$ and $(a^2b - a \in J^\#(R))$ $a^2b - a \in \text{Nil}(R)$. Such b , if it exists, is unique; it is called a (*pseudo*) *Drazin inverse* of a and will be denoted by (a^{pD}) a^D . In 1958, Drazin showed that a is strongly π -regular if and only if a has a Drazin inverse. Wang and Chen proved that a is pseudopolar if and only if a is pseudo Drazin invertible. By definitions, we conclude that any strongly π -regular element is pseudopolar, any pseudopolar element is quasipolar and any quasipolar element is strongly clean.

Morita contexts appeared as a key ingredient in the work of Morita that described equivalences between full categories of modules over rings with identities. Morita context rings form a very large class of rings generalizing matrix rings. One of the fundamental results in this direction says that the categories of left modules over the rings A and B are equivalent if and only if there exists a strict Morita context connecting A and B . Other applications, though not stated in an explicit form, can be found in various places. A Morita context (A, B, M, N, ψ, ϕ) consists of two rings A and B , two bimodules ${}_A M_B$, ${}_B N_A$ and a pair of bimodule homomorphisms

$$\psi : M \otimes_B N \rightarrow A \text{ and } \phi : N \otimes_A M \rightarrow B$$

which satisfy the following associativity:

$$\psi(m \otimes n)m' = m\phi(n \otimes m') \text{ and } \phi(n \otimes m)n' = n\psi(m \otimes n') \quad (1.1)$$

for any $n, n' \in N$, $m, m' \in M$. These conditions insure that the set of generalized matrices $\begin{bmatrix} a & m \\ n & b \end{bmatrix}; a \in A, b \in B, m \in M, n \in N$ will form a ring, called the ring

of the *Morita context*. A Morita context $\begin{bmatrix} A & M \\ N & B \end{bmatrix}$ with $A = B = R$, $M = N$, and $\psi = \phi = \varphi$ is called a *special Morita context*. Throughout $\mathcal{S}$ denotes the ring of a special Morita context $[R, M, \varphi]$.

The main purpose of this paper is to study quasipolarity of special Morita contexts which are a natural generalization of (generalized) full matrix rings ($K_s(R)$ or $M_2(R; s)$) $M_2(R)$. One of the motivations to study the concept of special Morita context rings is to construct nontrivial examples for quasipolar rings (see Examples 1, 2, 3). In the present paper we use special Morita context rings to extend some results of quasipolar rings (e.g. [2, Theorem 2.10], [10, Theorem 22], [19, Theorem 15, Theorem 18 and Theorem 22] and [21, Theorem 1.4]).

This paper is organized as follows. In Section 2, the forms of the Jacobson radical and the center of special Morita contexts are determined and some properties of this class of rings are investigated. Furthermore, we define (uniquely) weakly quasipolar rings and determine the relation between quasipolar rings and pseudopolar rings (or equivalently, uniquely strongly π -rad clean rings). In particular, we show that R is quasipolar if and only if R is uniquely weakly quasipolar (Theorem 2). This gives an affirmative answer to the question in [12, Remark 4.8] and [11, Lemma 2.4]. In Section 3, criteria are obtained for a single element of $\mathcal{S}$ to be quasipolar for a local ring R . As a result, we see that (in Proposition 5) $\mathcal{S}$ is quasipolar if and only if $\mathcal{S}$ is weakly quasipolar where M is uniquely bleached and R is local. In Section 4, we determine when a special Morita contexts over a commutative local ring is quasipolar. It is shown that $\mathcal{S}$ is quasipolar if and only if $x^2 - x + w = 0$ is solvable for every $w \in \text{Im}\varphi$ where R is commutative local and $\text{Im}\varphi \subseteq J(R)$. This extend and improve many known results such as [10, Corollaries 11 and 12] and [19, Theorem 18]. In particular, we prove that if R is commutative local and $\text{Im}\varphi$ is nilpotent, then $\mathcal{S}$ is quasipolar. This yields the main result of [19] (see Example 1). Several equivalent conditions on quasipolar special Morita context rings over a (commutative) local ring are obtained.

In this paper, the ring of integers modulo n is denoted by $\mathbb{Z}_n$, and we write $M_n(R)$ and $C(R)$ for the rings of all $n \times n$ matrices over the ring R and the set of all central elements of R , respectively. For elements a, b in a ring R , we use the notation $a \sim b$ to mean that a is similar to b , that is, $b = u^{-1}au$ for some $u \in U(R)$. We set $(M : J(R)) = \{m \in M \mid \varphi(m \otimes n) \in J(R) \text{ for all } n \in M\}$ and $(J(R) : M) = \{m \in M \mid \varphi(n \otimes m) \in J(R) \text{ for all } n \in M\}$.

2. PRELIMINARY RESULTS

In this section, we decide the forms of the Jacobson radical and the center of special Morita contexts. We introduce (uniquely) weakly quasipolar rings and investigate relations between quasipolar rings and pseudopolar rings (or equivalently, uniquely strongly π -rad clean rings). We prove that R is quasipolar if and only if R is uniquely weakly quasipolar (Theorem 2). As a result we obtain an affirmative answer to the question in [12, Remark 4.8] and [11, Lemma 2.4].

The Jacobson radical formula of a Morita context was determined by Sands in his 1973 study on Radicals and Morita contexts (see [17]):

Theorem 1. *Let R be a ring. Then*

$$J(\mathcal{S}) = \begin{bmatrix} J(R) & (M : J(R)) \\ (M : J(R)) & J(R) \end{bmatrix} = \begin{bmatrix} J(R) & (J(R) : M) \\ (J(R) : M) & J(R) \end{bmatrix}.$$

Corollary 1. *Let R be a ring and $\text{Im}\varphi \subseteq J(R)$. Then*

$$J(\mathcal{S}) = \begin{bmatrix} J(R) & M \\ M & J(R) \end{bmatrix}.$$

Proposition 1. *Let R be a ring and $\text{Im}\varphi \subseteq J(R)$. If $J(R)$ is nilpotent, then $J(\mathcal{S})$ is nilpotent.*

Proof. Assume that $[J(R)]^t = 0$ for some $t \in \mathbb{N}$. By direct calculation one sees that

$$[J(\mathcal{S})]^t \subseteq \begin{bmatrix} [J(R)]^t + \text{Im}\varphi & M \\ M & [J(R)]^t + \text{Im}\varphi \end{bmatrix} \subseteq \begin{bmatrix} \text{Im}\varphi & M \\ M & \text{Im}\varphi \end{bmatrix} =: T.$$

Further, we have

$$T^2 = \begin{bmatrix} \text{Im}\varphi & \text{Im}\varphi \cdot M \\ \text{Im}\varphi \cdot M & \text{Im}\varphi \end{bmatrix}, \text{ and so } T^{2t} = \begin{bmatrix} [\text{Im}\varphi]^t & [\text{Im}\varphi]^t \cdot M \\ [\text{Im}\varphi]^t \cdot M & [\text{Im}\varphi]^t \end{bmatrix} = 0.$$

Therefore $[J(\mathcal{S})]^{2t^2} = 0$, that is, $J(\mathcal{S})$ is nilpotent, as desired. $\square$

The following lemma is verified by direct calculation.

Lemma 1. *Let R be a ring. Then*

$$C(\mathcal{S}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in C(R) \text{ and } am = mb, bm = ma \text{ for all } m \in M \right\}.$$

Lemma 2. *Let R be a ring and $\text{Im}\varphi \subseteq J(R)$. Then $\begin{bmatrix} u & m \\ n & v \end{bmatrix}$ is invertible in $\mathcal{S}$ if and only if $u, v \in U(R)$.*

Proof. Since units always lift modulo the Jacobson radical, it is clear by Corollary 1. $\square$

Lemma 3. *Let R be a local ring and let $\alpha^2 = \alpha \in \mathcal{S}$. Then there exist $\beta, \gamma \in U(\mathcal{S})$ such that $\beta\alpha\gamma = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$.*

Proof. Write $\alpha = \begin{bmatrix} e & m \\ n & f \end{bmatrix}$ where $e, f \in R$ and $m, n \in M$. Since $\alpha^2 = \alpha$, we have

$$e = e^2 + \varphi(m \otimes n), \quad m = em + mf, \quad n = ne + fn, \quad f = f^2 + \varphi(n \otimes m).$$

If $e \in U(R)$, then we see that

$$\begin{bmatrix} 1 & 0 \\ -ne^{-1} & 1 \end{bmatrix} \begin{bmatrix} e & m \\ n & f \end{bmatrix} \begin{bmatrix} 1 & -e^{-1}m \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & f - \varphi(ne^{-1} \otimes m) \end{bmatrix}.$$

Similarly, if $f \in U(R)$, then there exist $\beta, \gamma \in U(\mathcal{S})$ such that $\beta\alpha\gamma = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$.

Now assume that $e, f \in J(R)$. Then $\alpha \in J(\mathcal{S})$, and so it is zero. We complete the proof. $\square$

In view of [18, Theorem 4], we have the following result.

Corollary 2. *Let R be a local ring and let $\alpha^2 = \alpha \in \mathcal{S}$. Then there exists a $\beta \in U(\mathcal{S})$ such that $\beta\alpha\beta^{-1} = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$.*

Let M be an R - R -bimodule and let $a \in R$. $l_a : M \rightarrow M$ and $r_a : M \rightarrow M$ denote, respectively, the abelian group endomorphisms of M given by $l_a(m) = am$ and $r_a(m) = ma$ for all $m \in M$.

Lemma 4. *Let $\alpha = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}, X = \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{S}$. Then $X \in \text{comm}(\alpha)$ if and only if $a \in \text{comm}(x)$, $b \in \text{comm}(y)$, $m \in \ker(l_x - r_y)$ and $n \in \ker(l_y - r_x)$.*

Proof. It is straightforward. $\square$

Corollary 3. *Let $\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{S}$ and $\beta = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{S}$. Then*

$$\text{comm}(\alpha) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in R \right\} = \text{comm}(\beta).$$

Lemma 5. *Let R be a ring. Then $\begin{bmatrix} a & m \\ n & b \end{bmatrix} \mapsto \begin{bmatrix} b & n \\ m & a \end{bmatrix}$ is an automorphism of $\mathcal{S}$.*

Remark 1. If R is isomorphic to a ring S by f , then $a \in R$ is quasipolar if and only if $f(a)$ is quasipolar in S .

Lemma 6. *Let R be a local ring. Then $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in QN(\mathcal{S})$ if and only if $a, b \in J(R)$ if and only if $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in J(\mathcal{S})$.*

Proof. If R is a local ring, then $J(R) = QN(R)$, and so the proof is clear. $\square$

Wang and Chen prove that ([21, Theorem 1.4]) if there exists $p^2 = p \in \text{comm}(a)$ such that $a + p \in U(R)$ and $a^k p \in J(R)$ for some $k \in \mathbb{N}$, then p is unique if and only if $p \in \text{comm}^2(a)$. We extend this result as the following.

Theorem 2. *Let R be a ring and assume that $p^2 = p \in \text{comm}(a)$, $a + p \in U(R)$ and $ap \in R^{qnil}$ for some $a, p \in R$. Then $p \in \text{comm}^2(a)$ if and only if p is unique.*

Proof. If $p \in \text{comm}^2(a)$, then p is unique by [12, Proposition 2.3]. For the converse, suppose p is unique and let $xa = ax$ for some $x \in R$. Set $c := xp - pxp$, $b = (a + p)^{-1}(1 - p)$. Then $ab^2 = b$, $1 - p = ab = ba$, $p + c =: e$ is an idempotent, $ep = e$ and $pe = p$. We show that $a + e \in U(R)$ and $ae \in QN(R)$. Since $(a + e)(b + p) = 1 + ap + px(1 - p)(b + p) = 1 + pa + pxb = 1 + p(a + xb)$ is

invertible if and only if $1 + (a + xb)p = 1 + ap$ is invertible, and $ap \in QN(R)$, $a + e$ is a right invertible element. Further, as $(b + p)(a + e) = 1 + ap + px - pxp = 1 + ap + pxab = 1 + ap(1 + xb)$ is invertible if and only if $1 + (1 + xb)ap = 1 + ap$ is invertible, $a + e$ is a left invertible element. Hence $a + e$ is invertible. Let $yae = aey$ for some $y \in R$. We prove that $1 + aey \in U(R)$. We know that $1 + aey \in U(R)$ if and only if $1 + aye = 1 + aye p \in U(R)$ if and only if $1 + paye = 1 + apye \in U(R)$ because $ep = e$. Furthermore, $pyeap = pyepa = pyea = pyae$ and $appye = apye = apeye = paeye = pyaee = pyae$, we conclude that $1 + apye \in U(R)$ as $pa \in QN(R)$. Thus $e = p + c = p$, and so $c = 0$ because p is unique. That is, we have $xp = pxp$. Analogously, it can be shown that $px = pxp$. Therefore $xp = px$, and so $p \in comm^2(a)$. The proof is completed. $\square$

Definition 1. Let R be a ring. Then $a \in R$ is called *weakly quasipolar* providing that there exists an idempotent $e \in comm(a)$ such that $a + e \in U(R)$ and $ae \in QN(R)$. If this representation is unique, then $a \in R$ is called *uniquely weakly quasipolar*. A ring R is called *(uniquely) weakly quasipolar* if any element of R is (uniquely) weakly quasipolar.

Remark 2. Lemma 2.4 in [11], Koliha proved that every weakly quasipolar element is quasipolar in a Banach algebra. But it is not true in a ring. Let $R = \mathbb{Z}_2[t_1, t_2, \dots]_{(t_1)}$ be a ring of polynomials in countably many indeterminates, localized at the prime ideal (t_1) (for details see [5, Example 2.4.3]). Let σ be the map which satisfies $\sigma(t_i) = t_i + 1$. Then $R[[x; \sigma]]$ is the skew formal power series local ring over R . Consider $A = \begin{bmatrix} t_2 & 0 \\ 0 & t_1 \end{bmatrix} \in M_2(R[[x; \sigma]])$ where $t_1 \in J(R[[x; \sigma]])$, $t_2 \in U(R[[x; \sigma]])$.

We directly see that A is weakly quasipolar with $E_1 = \begin{bmatrix} 0 & x \\ 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore A is not quasipolar.

By Theorem 2, we have the following result.

Corollary 4. Let R be a ring. The following are equivalent for $a \in R$.

- (1) a is uniquely weakly quasipolar.
- (2) a is quasipolar.

3. NONCOMMUTATIVE CASES

The goal of this section is to investigate quasipolarity of special Morita contexts over local rings. We begin with the following definition.

Definition 2. We say that a bimodule ${}_R M_R$ is called *(bleached) cobleached* provided that for any $a \in J(R)$, $b \in U(R)$, both $l_a - r_b$ and $l_b - r_a$ are (surjective) injective on M . Further, a bimodule ${}_R M_R$ will be called *uniquely bleached* if, for every $j \in J(R)$ and $u \in U(R)$, $l_a - r_b$ and $l_b - r_a$ are surjective as well as injective.

Huang, Tang and Zhou [10, Corollary 6] proved that if $K_s(R)$ is quasipolar, then R is cobleached where R is local. Our next result shows that the latter assumption is superfluous.

Theorem 3. *Let R be a ring. If $\mathcal{S}$ is quasipolar, then M is cobleached.*

Proof. Let $u \in U(R)$, $j \in J(R)$, and let $(l_u - r_j)(m) = 0$ where $m \in M$. Consider $\alpha = \begin{bmatrix} u & 0 \\ 0 & j \end{bmatrix} \in \mathcal{S}$. Take $E = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $E\alpha = \alpha E$, $\alpha + E$ is invertible and αE is quasipotent. Since α is quasipolar, by Theorem 2, E must be unique and $E \in \text{comm}^2(\alpha)$. Note that $\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \in \text{comm}(\alpha)$. So E commutes with $\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$, which implies $m = 0$. As required. $\square$

Proposition 2. *Let R be a local ring. Then $\alpha \in \mathcal{S}$ is strongly clean if and only if either $\alpha \in U(\mathcal{S})$, or $I_2 - \alpha \in U(\mathcal{S})$, or α is similar to $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.*

Proof. “ $\Rightarrow$ ” Assume that $\alpha = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ is strongly clean. Then there exists an idempotent $E \in \mathcal{S}$ such that $\alpha - E \in U(\mathcal{S})$ and $E \in \text{comm}(\alpha)$. By Corollary 2, there exists $\beta \in U(\mathcal{S})$ such that $\beta^{-1}E\beta = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$. Since R is local and $E^2 = E$, we have $e^2 = e, f^2 = f \in \{0, 1\}$. If $E = 0$, then $\alpha \in U(\mathcal{S})$. If $E = I_2$, then $I_2 - \alpha \in U(\mathcal{S})$. It follows that α is similar to $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ where $a, b \in R$ because $E \in \text{comm}(\alpha)$.

“ $\Leftarrow$ ” If $\alpha \in U(\mathcal{S})$ or $I_2 - \alpha \in U(\mathcal{S})$, then it is strongly clean. Hence we can assume that α is similar to $\beta = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Since R is local, there exist $e^2 = e, f^2 = f \in R$ such that $a - e, b - f \in U(R)$, $e \in \text{comm}(a)$ and $f \in \text{comm}(b)$. Therefore $\beta - F \in U(\mathcal{S})$ and $\beta F = F\beta$ where $F = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$. That is, β is strongly clean. It is well known that if x is strongly clean and $x \sim y$, then y is strongly clean. Thus α is strongly clean, as asserted. $\square$

Proposition 3. *Let R be a local ring. Then the following statements are equivalent for $\alpha = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in \mathcal{S}$.*

- (1) α is quasipolar.
- (2) Both $l_x - r_y$ and $r_x - l_y$ are injective.
- (3) $\text{comm}(\alpha) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in R \right\}$.

Proof. (1) $\Rightarrow$ (2) Suppose that $(l_x - r_y)(m) = 0$ for some $m \in M$ and $\alpha + E \in U(\mathcal{S})$ where $E = E^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in comm^2(\alpha)$ and $\alpha E \in QN(\mathcal{S})$. By Lemma 4, we get $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \in comm(\alpha)$. Then $b = 0 = c$, $a^2 = a$ and $d^2 = d$ because $E^2 = E \in comm^2(\alpha)$. Since R is local, $a, d \in \{0, 1\}$. In view of Corollary 3, $comm(E)$ is the set of all diagonal matrices in $\mathcal{S}$. We conclude that $m = 0$; that is, $l_x - r_y$ is injective. Similarly, we see that $r_x - l_y$ is injective. So holds (2).

(2) $\Rightarrow$ (1) If $x, y \in U(R)$ or $x, y \in J(R)$, then we easily see that α is quasipolar in $\mathcal{S}$. Let $x \in J(R)$ and $y \in U(R)$. Write $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $E^2 = E \in comm(\alpha)$ and $\alpha + E \in U(\mathcal{S})$. Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in comm(\alpha)$. According to Lemma 4, $b \in ker(l_x - r_y)$ and $c \in ker(r_x - l_y)$. By (2), we have $b = 0 = c$. Hence we get $E \in comm^2(\alpha)$ by Corollary 3. As $\alpha E = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \in J(\mathcal{S})$ by Theorem 1, we have $\alpha E \in QN(\mathcal{S})$. That is, α is quasipolar in $\mathcal{S}$.

(2) $\Rightarrow$ (3) Let $X = \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in comm(\alpha)$. By Lemma 4, we have $m \in ker(l_x - r_y)$ and $n \in ker(r_x - l_y)$. By assumption, we get $m = n = 0$, and so $comm(\alpha) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in R \right\}$.

(3) $\Rightarrow$ (2) Suppose that $(l_x - r_y)(n) = xn - ny = 0$ for some $n \in M$. This gives that $X = \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix} \in comm(\alpha)$. By (3), we have $n = 0$ and so $l_x - r_y$ is injective. Similarly, it can be proved that $r_x - l_y$ is injective. Therefore we complete the proof. $\square$

Remark 3. As is well known, if ab is quasipolar and $(ab)^{gD} = c$, then so is ba and $(ba)^{gD} = bc^2a$ [15] (or see [6]). In particular, if a is quasipolar and $a^{gD} = c$, then $u^{-1}au = b$ is quasipolar for any $u \in U(R)$ and $b^{gD} = u^{-1}cu$.

Proposition 4. Let R be a local ring. Then $\alpha \in \mathcal{S}$ is quasipolar if and only if $\alpha \in U(\mathcal{S})$, or $\alpha \in QN(\mathcal{S})$, or α is similar to $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ where $l_a - r_b, l_b - r_a$ are injective.

Proof. “ $\Rightarrow$ ” Assume that $\alpha = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ is quasipolar with spectral idempotent $E = \begin{bmatrix} p & m \\ n & q \end{bmatrix}$. Then, by Corollary 2, there exists $\beta \in U(\mathcal{S})$ such that $\beta^{-1}E\beta = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$. Since R is local and $E^2 = E$, we have $e^2 = e, f^2 = f \in \{0, 1\}$. If $E = 0$,

then $\alpha \in U(\mathcal{S})$. If $E = I_2$, then $\alpha \in QN(\mathcal{S})$. It follows that α is similar to $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ where $a, b \in R$ because $E \in comm^2(\alpha)$. In view of Proposition 3, $l_a - r_b, l_b - r_a$ are injective, as asserted.

“ $\Leftarrow$ ” It is clear from Proposition 3 and Remark 3. $\square$

Remark 4. Clearly, if $a \in U(R) \cup QN(R)$, then it is weakly quasipolar. In this case, we say that a is *trivial weakly quasipolar*. If a is not in $U(R) \cap QN(R)$, we say that a is *non-trivial weakly quasipolar*.

Theorem 4. Let R be a local ring. Assume that α is non-trivial weakly quasipolar with $E^2 = E \in \mathcal{S}$. The following are equivalent.

- (1) E is unique.
- (2) $E \in comm^2(\alpha)$.
- (3) α is similar to $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ where $l_a - r_b, l_b - r_a$ are injective.

Proof. By Theorem 2, we conclude that (1) $\Leftrightarrow$ (2).

(2) $\Rightarrow$ (3) By assumption, we get α is quasipolar. Then α is similar to $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ where $l_a - r_b, l_b - r_a$ are injective by Proposition 4.

(3) $\Rightarrow$ (1) In view of Proposition 4, α is quasipolar, and so E is unique. $\square$

Theorem 5. Let R be a local ring. For any $u \in U(R)$ and $j \in J(R)$, the following are equivalent.

- (1) $r_u - l_j$ is injective and $l_u - r_j$ is an isomorphism.
- (2) For each $m \in M$, $\alpha = \begin{bmatrix} u & m \\ 0 & j \end{bmatrix}$ is quasipolar in $\mathcal{S}$ with $\alpha^\pi = \begin{bmatrix} 0 & n \\ 0 & 1 \end{bmatrix}$ for some $n \in M$.

Proof. (1) $\Rightarrow$ (2) Let $m \in M$. Since $l_u - r_j$ is an isomorphism, there exists $n \in M$ such that $un - nj = -m$. Write $E = \begin{bmatrix} 0 & n \\ 0 & 1 \end{bmatrix}$. This gives $E \in comm(\alpha)$. As

$$\begin{aligned} (\alpha + E) & \left(\begin{bmatrix} u^{-1} & -u^{-1}(m+n)(j+1)^{-1} \\ 0 & (j+1)^{-1} \end{bmatrix} \right) \\ & = \left(\begin{bmatrix} u^{-1} & -u^{-1}(m+n)(j+1)^{-1} \\ 0 & (j+1)^{-1} \end{bmatrix} \right), \end{aligned}$$

$(\alpha + E) = I_2$, we have $\alpha + E \in U(\mathcal{S})$. We show that $E \in comm^2(\alpha)$. Let $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \in comm(\alpha)$. This implies that $z = 0, tj = jt, ux = xu$ and $uy - yj =$

$xm - mt$ because $r_u - l_j$ is injective. We conclude that

$$\begin{aligned}
 u(y + xn - nt) - (y + xn - nt)j &= uy - yj + uxn - unt - xnj + ntj \\
 &= uy - yj + xun - unt - xnj + njt \\
 &= xm - mt + xun - unt - xnj + njt \\
 &= xm - mt + x(un - nj) + (nj - un)t \\
 &= xm - mt + x(-m) + mt \\
 &= 0.
 \end{aligned}$$

Therefore $XE = EX$ because $l_u - r_j$ is injective. Finally, we show that $\alpha E \in QN(\mathcal{S})$. Assume $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in comm(\alpha E)$. Then $I_2 + \alpha EA \in U(\mathcal{S})$ because

$$\alpha E = \begin{bmatrix} 0 & nj \\ 0 & j \end{bmatrix}, \text{ as desired.}$$

(2) $\Rightarrow$ (1) If $m = 0$, then $l_u - r_j, r_u - l_j$ are injective. It remains to prove that $l_u - r_j$ is surjective. That is, for each $m \in M$, there exists $n \in M$ such that $un - nj = m$. Let $m \in M$. We write $\alpha = \begin{bmatrix} u & m \\ 0 & j \end{bmatrix}$. By (2), there exists an idempotent $E = \begin{bmatrix} 0 & n \\ 0 & 1 \end{bmatrix}$ such that $E \in comm^2(\alpha)$, $\alpha + E \in U(\mathcal{S})$, and $\alpha E \in QN(\mathcal{S})$. Since $\alpha E = E\alpha$, we get $u(-n) - (-n)j = m$. $\square$

By a similar method of the proof of Theorem 5, we can derive the following.

Theorem 6. *Let R be a local ring. For any $u \in U(R)$ and $j \in J(R)$, the following are equivalent.*

- (1) $l_u - r_j$ is injective and $r_u - l_j$ is an isomorphism.
- (2) For any $m \in M$, $\alpha = \begin{bmatrix} u & 0 \\ m & j \end{bmatrix}$ is quasipolar in $\mathcal{S}$ with $\alpha^\pi = \begin{bmatrix} 0 & 0 \\ n & 1 \end{bmatrix}$ for some $n \in M$.

According to Theorems 5 and 6, the following result is immediate.

Corollary 5. *Let R be a local ring. The following statements are equivalent.*

- (1) M is uniquely bleached.
- (2) $\begin{bmatrix} R & M \\ 0 & R \end{bmatrix}$ or $\begin{bmatrix} R & 0 \\ M & R \end{bmatrix}$ is quasipolar.

Proposition 5. *Let R be a local ring and M is cobleached. Then the following are equivalent.*

- (1) $\alpha \in \mathcal{S}$ is quasipolar.
- (2) $\alpha \in \mathcal{S}$ is weakly quasipolar.

Proof. (1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (1) Assume that $E^2 = E \in \text{comm}(\alpha)$, $\alpha + E \in U(R)$ and $\alpha E \in QN(\mathcal{S})$. If E is zero or identity in $\mathcal{S}$, then α is quasipolar. Hence, we deduce that E is similar to the diagonal matrix by Corollary 2. Thus, we have α is similar to the diagonal matrix. Since M is cobleached, we get that α is quasipolar by Theorem 4. $\square$

4. COMMUTATIVE CASES

Let R be a commutative ring and $\alpha = \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{S}$. In this section, we assume that φ satisfying symmetry; that is, $\varphi(m \otimes n) = \varphi(n \otimes m)$ for all $m, n \in M$. We define $\det(\alpha) = ab - \varphi(m \otimes n)$ and $\text{tr}(\alpha) = a + b$, and $r\alpha = \begin{bmatrix} ra & rm \\ rn & rb \end{bmatrix}$ for $r \in R$. Note that if R is commutative and M is right R -module, then M is left R -module where $rm := mr$ for all $r \in R, m \in M$. Therefore, M is uniquely bleached.

Lemma 7. *Let R be a commutative ring and let $\alpha, \beta \in \mathcal{S}$. The following hold:*

- (1) $\det(\alpha\beta) = \det(\alpha)\det(\beta)$.
- (2) $\alpha \in U(\mathcal{S})$ if and only if $\det(\alpha) \in U(R)$.
- (3) If $\alpha \sim \beta$, then $\det(\alpha) = \det(\beta)$ and $\text{tr}(\alpha) = \text{tr}(\beta)$.
- (4) $\alpha^2 - \text{tr}(\alpha)\alpha + \det(\alpha)I_2 = 0$.
- (5) $\det(I_2 + \alpha) = 1 + \text{tr}(\alpha) + \det(\alpha)$.

Proof. It is straightforward. $\square$

Theorem 7. *Let R be a commutative ring and let $\alpha \in \mathcal{S}$. Then $\det(\alpha), \text{tr}(\alpha) \in J(R)$ if and only if α is quasiniipotent in $\mathcal{S}$.*

Proof. Assume that $\det(\alpha), \text{tr}(\alpha) \in J(R)$ and let $\alpha = \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{S}$. By Lemma 7(4), we have $\alpha^2 = \text{tr}(\alpha)\alpha - \det(\alpha)I_2 \in J(\mathcal{S})$, and so $\alpha \in QN(\mathcal{S})$. Conversely, suppose that $\alpha \in QN(\mathcal{S})$. To prove that $\det(\alpha) \in J(R)$, let $y \in R$. It is easy to check that $\beta\alpha = \alpha\beta = \det(\alpha)I_2$ where $\beta = \begin{bmatrix} b & -m \\ -n & a \end{bmatrix} \in \mathcal{S}$. Since $\alpha \in QN(\mathcal{S})$, we have that $I_2 + y\beta\alpha \in U(\mathcal{S})$, and so $1 + y\det(\alpha) \in U(R)$. Therefore, we get $\det(\alpha) \in J(R)$. Let $x \in R$. Then we have $I_2 + x\alpha \in U(\mathcal{S})$, and so $\det(I_2 + x\alpha) = 1 + x\text{tr}(\alpha) + x^2\det(\alpha) \in U(R)$ by Lemma 7(2). This gives that $1 + x\text{tr}(\alpha) \in U(R)$; thus, $\text{tr}(\alpha) \in J(R)$. The proof is completed. $\square$

Corollary 6. *Let R be a commutative ring. Then the following statements are equivalent.*

- (1) $\alpha \in QN(\mathcal{S})$.
- (2) $\det(\alpha), \text{tr}(\alpha) \in J(R)$.
- (3) $\alpha^2 \in J(\mathcal{S})$.
- (4) $\alpha^k \in J(\mathcal{S})$ for some $k \geq 2$.

Remark 5. Let R be a commutative ring. Then we conclude that

$$QN(\mathcal{S}) = J^\#(\mathcal{S}) = \{\alpha \in \mathcal{S} \mid \alpha^2 \in J(\mathcal{S})\}.$$

Recall that an element a in a ring R is called *strongly π -rad clean* if there exists an idempotent $e \in \text{comm}(a)$ such that $a - e \in U(R)$ and $(ae)^k \in J(R)$ for some integer k .

Corollary 7. Let R be a commutative local ring. The following are equivalent.

- (1) α is pseudopolar in $\mathcal{S}$.
- (2) α is quasipolar in $\mathcal{S}$.
- (3) α is weakly quasipolar in $\mathcal{S}$.
- (4) α is strongly π -rad clean in $\mathcal{S}$.

Proof. (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (3) are clear. By Remark 5, we deduce that (3) $\Rightarrow$ (4) and (2) $\Rightarrow$ (1). According to Proposition 5, we have (3) $\Rightarrow$ (2). $\square$

Theorem 8. Let R be a commutative local ring and let $\alpha \in \mathcal{S}$. Then the following statements are equivalent.

- (1) α is quasipolar in $\mathcal{S}$.
- (2) $\alpha \in U(\mathcal{S})$, or $\alpha \in QN(\mathcal{S})$, or $x^2 - \text{tr}(\alpha)x + \det(\alpha) = 0$ is solvable.

Proof. (1) $\Rightarrow$ (2) Let $\alpha \in \mathcal{S}$ be quasipolar. We may assume that $\alpha \notin U(\mathcal{S})$ and $\alpha \notin QN(\mathcal{S})$. Then α is similar to $\beta = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ where $a \in U(R)$, $b \in J(R)$ by Proposition 4. According to Lemma 7(3), $\text{tr}(\alpha) = \text{tr}(\beta)$ and $\det(\alpha) = \det(\beta)$. This gives $x^2 - \text{tr}(\alpha)x + \det(\alpha) = x^2 - \text{tr}(\beta)x + \det(\beta)$. Since $a^2 - \text{tr}(\beta)a + \det(\beta) = 0$, the equation $x^2 - \text{tr}(\alpha)x + \det(\alpha) = 0$ is solvable in R .

(2) $\Rightarrow$ (1) If α is invertible or quasinilpotent in $\mathcal{S}$, then α is quasipolar. Let $\alpha = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathcal{S}$ and suppose $x^2 - \text{tr}(\alpha)x + \det(\alpha) = 0$ has roots $a, b \in R$. Since $\alpha \notin U(\mathcal{S}) \cap QN(\mathcal{S})$, we get $\det(\alpha) = a_{11}a_{22} - \varphi(a_{12} \otimes a_{21}) = ab \in J(R)$ and $\text{tr}(\alpha) = a_{11} + a_{22} = a + b \in U(R)$ by Theorem 7. So one of a, b must be in $U(R)$ and the other must be in $J(R)$. By Lemma 5 and Remark 1, we can assume that $a \in U(R)$, $b \in J(R)$ and $a_{11} \in U(R)$. Let $\beta = \begin{bmatrix} 1 & 0 \\ a_{21}(a - a_{22})^{-1} & 1 \end{bmatrix}$. By Lemma 7(2), $\beta \in U(\mathcal{S})$ and easy calculation shows that $\beta^{-1}\alpha\beta$ is an formal triangular matrix in $\begin{bmatrix} R & M \\ 0 & R \end{bmatrix}$. Therefore α is quasipolar from Corollary 5 and Remark 3. We complete the proof. $\square$

Now we extend [19, Theorem 15] as follows.

Theorem 9. Let R be a commutative local ring and let $\alpha \in \mathcal{S}$. Then the following statements are equivalent.

- (1) α is non-trivial strongly clean in $\mathcal{S}$.
 (2) $\det(\alpha) \in J(R)$, $\text{tr}(\alpha) \in 1 + J(R)$ and $x^2 - \text{tr}(\alpha)x + \det(\alpha) = 0$ is solvable.

Proof. (1) $\Rightarrow$ (2) It is clear by Proposition 2 and Lemma 7.

(2) $\Rightarrow$ (1) Since $\det(\alpha) \in J(R)$ and $\text{tr}(\alpha) \in 1 + J(R)$, we deduce that $\alpha \notin QN(\mathcal{S}) \cap U(\mathcal{S})$. By Theorem 8, α is quasipolar, and so α is strongly clean. $\square$

Remark 6. According to Theorem 9, $\alpha \in \mathcal{S}$ is non-trivial strongly clean if and only if it is non-trivial quasipolar in $\mathcal{S}$ where R is commutative local.

Theorem 10. Let R be a commutative local ring. Then the following are equivalent.

- (1) $\mathcal{S}$ is quasipolar.
 (2) For every $\alpha \in \mathcal{S}$ with $\det(\alpha) \in J(R)$, one of the following holds:
 (i) $\text{tr}(\alpha) \in J(R)$,
 (ii) $x^2 - \text{tr}(\alpha)x + \det(\alpha) = 0$ is solvable in R .

Proof. (1) $\Rightarrow$ (2) Suppose that $\alpha \in \mathcal{S}$ with $\det(\alpha) \in J(R)$. By (1), there exists an idempotent $e \in \mathcal{S}$ such that $e \in \text{comm}^2(\alpha)$ and $\alpha + e \in U(\mathcal{S})$ and $ea \in QN(\mathcal{S})$. If $e = I_2$, then $\alpha \in QN(\mathcal{S})$ and so $\text{tr}(\alpha) \in J(R)$ by Theorem 7. So we can assume that $\alpha \notin QN(\mathcal{S})$. Since $\det(\alpha) \in J(R)$, $\alpha \notin U(\mathcal{S})$ by Lemma 7(2). According to Theorem 8, the equation $x^2 - \text{tr}(\alpha)x + \det(\alpha) = 0$ is solvable in R .

(2) $\Rightarrow$ (1) Let $\alpha \in \mathcal{S}$. If $\det(\alpha) \in U(R)$, then $\alpha \in U(\mathcal{S})$ and so α is quasipolar. Let $\det(\alpha) \in J(R)$. If $\text{tr}(\alpha) \in J(R)$, then α is quasipolar by Theorem 7. Hence we assume that $\text{tr}(\alpha) \in U(R)$. This gives $\alpha \notin U(\mathcal{S})$ and $\alpha \notin QN(\mathcal{S})$. By Theorem 8, α is quasipolar and so $\mathcal{S}$ is quasipolar. $\square$

Lemma 8. Let R be a ring and let $u \in U(R) \cap C(R)$. Then $a \in R$ is quasipolar ring if and only if ua is quasipolar in R .

Proof. “ $\Rightarrow$ ” It follows from [3, Lemma 2].

“ $\Leftarrow$ ” Assume that $au = s + q$ where s is strongly regular, $s \in \text{comm}^2(au)$, $q \in QN(R)$ and $sq = qs = 0$ by [7, Corollary 2.17]. Then we get $a = u^{-1}s + u^{-1}q$. It can be shown that $u^{-1}s \in \text{comm}^2(a)$ and $u^{-1}su^{-1}q = u^{-1}qu^{-1}s = 0$ because $u \in C(R)$. Since s is strongly regular, there exists $t \in \text{comm}^2(s)$ such that $s = s^2t$. Multiplying by u^{-1} yields $u^{-1}s = u^{-1}s^2t = [u^{-1}s]^2ut$. This gives that $u^{-1}s$ is strongly regular in R . To prove $u^{-1}q \in QN(R)$, let $xu^{-1}q = u^{-1}qx$. As u is central, we deduce that $u^{-1}xq = qu^{-1}x = u^{-1}qx$, and so $1 + u^{-1}qx \in U(R)$ because $q \in QN(R)$, as asserted. $\square$

Theorem 11. Let R be a commutative local ring. The following are equivalent.

- (1) $\mathcal{S}$ is pseudopolar.
 (2) $\mathcal{S}$ is quasipolar.
 (3) $\mathcal{S}$ is weakly quasipolar.
 (4) $\mathcal{S}$ is strongly π -rad clean.

(5) $\mathcal{S}$ is strongly clean.

Proof. It suffices to show that (5) $\Rightarrow$ (2). Let $\alpha = \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{S}$. We can further assume that $\alpha \notin U(\mathcal{S}) \cap QN(\mathcal{S})$. Then we have $\det(\alpha) \in J(R)$ and $\text{tr}(\alpha) \in U(R)$.

Case I. Let $\varphi(m \otimes n) \in J(R)$. Then we may assume $a \in U(R)$ and $b \in J(R)$ by Lemma 5. We write $a^{-1}\alpha = \begin{bmatrix} 1 & p \\ q & j \end{bmatrix}$ where $a^{-1}n = q$, $a^{-1}m = p$ and $a^{-1}b = j$. Note that $a^{-1}I_2$ is central invertible in $\mathcal{S}$ by Lemma 1. Since $j \in J(R)$ and $j + (-\varphi(p \otimes q)) =: w \in j + \text{Im}\varphi$, we have $\det(a^{-1}\alpha) \in J(R)$ and $\det(I_2 - a^{-1}\alpha) \in J(R)$. This implies $a^{-1}\alpha$ is non-trivial strongly clean. Thus $x^2 - \text{tr}(a^{-1}\alpha)x + \det(a^{-1}\alpha) = 0$ is solvable by Theorem 9, and so $a^{-1}\alpha$ is quasipolar, and so α is quasipolar by Lemma 8.

Case II. Let $\varphi(m \otimes n) =: u \in U(R)$. Now we prove that $t^2 - t - w = 0$ is solvable for all $w \in J(R)$. Let $w \in J(R)$ and write $\beta = \begin{bmatrix} 0 & u^{-1}m \\ nw & 1 \end{bmatrix}$. Since $\det(\beta) = 0 - \varphi(u^{-1}m \otimes nw) = -u^{-1}w\varphi(m \otimes n) = -u^{-1}wu = -w \in J(R)$ and $\det(I_2 - \beta) = 1 - \text{tr}(\beta) + \det(\beta) = -w \in J(R)$, β is a non-trivial strongly clean element, and so $t^2 - t - w = 0$ is solvable by Theorem 9. Hence $t^2 - t + \frac{\det(\alpha)}{\text{tr}(\alpha)^2} = 0$ is solvable. This gives that $x^2 - \text{tr}(\alpha)x + \det(\alpha) = 0$ is solvable. Consequently, α is quasipolar. We complete the proof. $\square$

Remark 7. According to the proof Theorem 11, we deduce that if $\mathcal{S}$ is strongly clean and $\text{Im}\varphi \cap U(R) \neq \emptyset$, then $x^2 - x - w = 0$ is solvable for all $w \in J(R)$.

Theorem 12. Let R be a commutative local ring with $\text{Im}\varphi \subseteq J(R)$ and let $\alpha \in \mathcal{S}$. Then the following are equivalent.

- (1) $\mathcal{S}$ is quasipolar.
- (2) $x^2 - (1 + j)x + w = 0$ is solvable for any $j \in J(R)$ and $w \in j + \text{Im}\varphi$.
- (3) $x^2 - x + w = 0$ is solvable for any $w \in \text{Im}\varphi$.

Proof. (1) $\Rightarrow$ (2) Let $\alpha = \begin{bmatrix} 1 & -m \\ n & j \end{bmatrix}$ where $j \in J(R)$, $w = j + \varphi(m \otimes n)$. By (1), α is quasipolar. Since $\det(\alpha) = j + \varphi(m \otimes n) \in J(R)$ and $\text{tr}(\alpha) = 1 + j \in U(R)$, we have that $x^2 - (1 + j)x + w = 0$ is solvable by Theorem 10(2).

(2) $\Rightarrow$ (1) Let $\alpha = \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{S}$. If $a, b \in J(R)$ or $a, b \in U(R)$, then $\alpha \in J(\mathcal{S})$ or $\alpha \in U(\mathcal{S})$, and so α is quasipolar. Hence, by Lemma 5, we may assume that $a \in U(R)$ and $b \in J(R)$. Then we conclude that $a^{-1}\alpha = \begin{bmatrix} 1 & p \\ q & j \end{bmatrix}$ where $a^{-1}n = q$, $a^{-1}m = p$ and $a^{-1}b = j$. Since $j \in J(R)$ and $j + (-\varphi(p \otimes q)) =: w \in j + \text{Im}\varphi$, by assumption, we get $x^2 - (1 + j)x + w = 0$ is solvable. This gives that $a^{-1}\alpha$ is quasipolar, and so α is quasipolar by Lemma 8.

(1) $\Rightarrow$ (3) Similar to the proof of (1) $\Rightarrow$ (2).

(3) $\Rightarrow$ (1) Let $\alpha = \begin{bmatrix} a & m_1 \\ n_1 & b \end{bmatrix} \in \mathcal{S}$. If $a, b \in J(R)$ or $a, b \in U(R)$, then $\alpha \in J(\mathcal{S})$ or $\alpha \in U(\mathcal{S})$, and so α is quasipolar. Hence, by Lemma 5, we may assume that $a \in U(R)$ and $b \in J(R)$. Then we conclude that $a^{-1}\alpha = \begin{bmatrix} 1 & m_2 \\ n_2 & j \end{bmatrix} =: \beta$ where $a^{-1}n_1 = n_2$, $a^{-1}m_1 = m_2$ and $a^{-1}b = j$. It is well known that β is strongly clean if and only if $I_2 - \beta$ is strongly clean. Hence, we now consider $I_2 - \beta = \begin{bmatrix} 0 & -m_2 \\ -n_2 & 1-j \end{bmatrix}$. We directly see that $(1-j)^{-1}(I_2 - \beta) = \begin{bmatrix} 0 & m \\ n & 1 \end{bmatrix}$ where $m = -(1-j)^{-1}m_2$ and $n = -(1-j)^{-1}n_2$. By (3), we have that $x^2 - x - \varphi(m \otimes n) = 0$ is solvable, and so $(1-j)^{-1}(I_2 - \beta)$ is (non-trivial) quasipolar. In view of Lemma 8, $I_2 - \beta$ is (non-trivial) quasipolar; hence β is (non-trivial) quasipolar by Remark 6. Therefore, by Lemma 8, we deduce that α is quasipolar. $\square$

The following theorem is a generalization of Theorem 22 in [19].

Theorem 13. *Let R be a commutative local ring. If $\text{Im}\varphi$ is a nilpotent ideal of R , then $\mathcal{S}$ is quasipolar.*

Proof. We can assume that $(\text{Im}\varphi)^t = 0$ for some $t \in \mathbb{N}$. In this case, we also note that $\text{Im}\varphi \subseteq J(R)$. By Theorem 12, it suffices to prove that $x^2 - x + w = 0$ is solvable for any $w \in \text{Im}\varphi$. Let $w = \varphi(m \otimes n)$ and $f_0(x) = x^2 - x + w = 0$. We set

$$W_k = \{w^k x^2 - ux + w = 0 \in R[x] \mid u \in 1 + J(R), w \in J(R)\}$$

for any $k \geq 0$. Then we have $f_0(w + wx) = wf_1(x)$ where $f_1(x) = wx^2 + x + w \in W_1$. Further, we see that $f_1(-w + wx) = wf_2(x)$ where $f_2(x) = w^2x^2 - (2w^2 - 1)x + w^2 \in W_2$. By iteration of this process, we get $f_{k-1}(w_k + wx) = wf_k(x)$ for all $k = 1, \dots, t$. Therefore,

$$\begin{aligned} & f_t(a_t) \in R, \\ \Rightarrow & f_{t-1}(a_{t-1}) \in wR, \\ \Rightarrow & f_{t-2}(a_{t-2}) \in w^2R, \\ & \vdots \\ \Rightarrow & f_1(a_1) \in w^{t-1}R, \\ \Rightarrow & f_0(a_0) \in w^tR = 0. \end{aligned}$$

So we see that $x^2 - x + w = 0$ is solvable. We complete the proof. $\square$

We wind up the paper with some examples of special Morita contexts and quasipolar rings. In spite of the fact that it is difficult to find a pair of bimodule homomorphisms which satisfy the associativity rule, we supply the following examples.

Example 1. Let $\Gamma = [M, N, A, B, \psi, \phi]$ be a Morita context rings. Take $M = N = R = A = B$ and $\psi = \phi =: \varphi$ is defined by $a \otimes b \mapsto sab$ (or $a \otimes b \mapsto s^2ab$) where

s is central in R . Then $\mathcal{S}$ is a generalized matrix ring over R denoted by $K_s(R)$ (or $M_2(R; s^2)$) (for more details see [8, 10, 13, 14, 19, 20]). It is easy to see that $\mathcal{S}$ is isomorphic to $M_2(R)$ when $s = 1$. Therefore, we see that the special Morita context rings are a natural generalization of generalized matrix ring.

Example 2. Let R be a local ring with $\text{Im}\varphi = 0$. Then $\mathcal{S}$ is quasipolar if and only if $\mathcal{S}$ is strongly clean and M is cobleached (if and only if M is uniquely bleached).

Proof. “ $\Rightarrow$ ” It is clear.

“ $\Leftarrow$ ” Assume that $\mathcal{S}$ is strongly clean and M is cobleached. Firstly, we prove that M is bleached. Let $m \in M$, $u \in U(R)$ and $j \in J(R)$. Consider $\beta = \begin{bmatrix} u & m \\ 0 & j \end{bmatrix} \in \mathcal{S}$. By assumption, β is strongly clean. That is, there exists an idempotent E such that $\beta - E$ is invertible and $E \in \text{comm}(\beta)$. Take $E = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$. Since $E^2 = E$ and $\beta E = E\beta$, we directly see that $x^2 = x$, $t^2 = t$, $xu = ux$, $tj = jt$ and $uy + mt = xm + yj$, and hence $x = 0$, $t = 1$ and $uy - yj = -m$ because R is local and $\beta - E \in U(\mathcal{S})$. Thus M is uniquely bleached. Let $\alpha = \begin{bmatrix} a & m \\ n & b \end{bmatrix}$. It is enough to show that α is weakly quasipolar by Proposition 5. Without loss of generality, we may assume that $a \in U(R)$ and $b \in J(R)$. As M is bleached, there exist $x, y \in M$ such that $ux - xj = -m$ and $jy - yu = n$. Write $E = \begin{bmatrix} 0 & x \\ y & 1 \end{bmatrix}$. This gives that $E^2 = E$, $E \in \text{comm}(\alpha)$ and $\alpha E \in J(\mathcal{S})$, and so α is weakly quasipolar. $\square$

Example 3. We consider the special Morita context $\mathcal{S} = \begin{bmatrix} R & M \\ M & R \end{bmatrix}$ where $R = \mathbb{Z}_4$, $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\varphi : M \otimes M \rightarrow \mathbb{Z}_4$ is defined by $\varphi((\bar{a}, \bar{b}) \otimes (\bar{c}, \bar{d})) = \overline{2(ac + bd)}$. This implies that φ satisfy the associativity conditions and $\text{Im}\varphi = \{\bar{0}, \bar{4}\}$, and so $\text{Im}\varphi$ is nilpotent ideal in R . Hence, by Theorem 13, $\mathcal{S} = [R, M, \varphi]$ is quasipolar.

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Authors' addresses

O. Gürgün

Ankara University, Department of Mathematics, 06100 Ankara, Turkey
E-mail address: orhangurgun@gmail.com

S. Halicioglu

Ankara University, Department of Mathematics, 06100 Ankara, Turkey
E-mail address: halici@ankara.edu.tr

A. Harmanci

Hacettepe University, Department of Mathematics, 06800 Ankara, Turkey
E-mail address: harmanci@hacettepe.edu.tr



MONOUNARY ALGEBRAS WITH EASY DIRECT LIMITS

EMÍLIA HALUŠKOVÁ

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Abstract. Let $\mathcal{A}$ be an algebra such that exactly algebras isomorphic to a retract of $\mathcal{A}$ can be constructed from $\mathcal{A}$ by direct limits. Then $\mathcal{A}$ is said to be an algebra with easy direct limits. We will prove that if $\mathcal{A}$ is a monounary, then $\mathcal{A}$ is countable and the number of retracts of $\mathcal{A}$ is not equal to $\aleph_0$. Further, we will see that the number of non-isomorphic monounary algebras with easy direct limits is $2^{\aleph_0}$.

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1. INTRODUCTION

The importance of the notion of retract is well known and commonly appreciated in mathematics. There are many papers dealing with retracts of algebraic structures, see e.g. [7], [10] for non-complete review. Actual results concerning monounary algebras can be found in [1],[9] and [8].

The algebraic construction of direct limit is a well-known method of building up new algebras from given ones. It helps to make progress in particular topics, recently see e.g. [11], [12]. Furthermore it is interesting to investigate this construction generally, cf.[6], [15], [14].

Let $\mathcal{A}$ be an algebra. We denote by $\underline{\mathbf{L}}\mathcal{A}$ the class of all isomorphic copies of direct limits which can be obtained from $\mathcal{A}$ and we denote by $\mathbf{R}\mathcal{A}$ the set of all retracts of $\mathcal{A}$. Then $\mathbf{R}\mathcal{A} \subseteq \underline{\mathbf{L}}\mathcal{A}$. We will say that $\mathcal{A}$ is an algebra with easy direct limits if every algebra from $\underline{\mathbf{L}}\mathcal{A}$ is isomorphic to a retract of $\mathcal{A}$.

The aim of this paper is to investigate algebras with easy direct limits. If $\mathcal{A}$ is finite, then $\mathcal{A}$ is an algebra with easy direct limits, cf.[5]. Some infinite algebras with easy direct limits can be found in [4].

We will prove results concerning monounary algebras. Properties of monounary algebras with easy direct limits run from section 4. A monounary algebra with easy direct limits is countable. It contains a line or a cycle in every component. There

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enter at most finitely many arrows to every point of it. The number of retracts of it is never equal to $\aleph_0$.

The section 5 contains a description of all monounary algebras with easy direct limits which have a bijective basic operation. This description implies that the number of non-isomorphic algebras with easy direct limits is equal to $2^{\aleph_0}$, see Thm.3.

2. PRELIMINARIES

The cardinality of a set A is denoted by $\|A\|$. The set of all positive integers is denoted by $\mathbb{N}$ and the set of all non-negative integers is denoted by $\mathbb{N}_0$.

We deal with monounary algebras. The fundamental operation is denoted by f . Short remarks about algebras of arbitrary type are before Lemma 6 and after Example 1.

For monounary terminology see e.g. [10], [13].

Let $\mathcal{A} = (A, f)$ be a monounary algebra. A subset B of A is termed as a *chain* of $\mathcal{A}$ if for every $a, b \in B$ there is $n \in \mathbb{N}_0$ either $f^n(a) = b$ or $f^n(b) = a$. If A is a chain of $\mathcal{A}$, then we will say that $\mathcal{A}$ is a *basic algebra*.

Basic monounary algebras were introduced in [7] and will be used in the proof of Cor 3.

If $\mathcal{A}$ is connected and contains no cycle, then it is said to be a *line*, if for each $a \in A$ there exists a unique $b \in A$ with $f(b) = a$. A line is a basic algebra.

Denote

$$\begin{aligned}\downarrow b &= \{d \in A : f^k(d) = b \text{ for some } k \in \mathbb{N}_0\}, \\ f^{-1}(b) &= \{d \in A : f(d) = b\}, \\ \uparrow b &= \{f^k(b) : k \in \mathbb{N}_0\}.\end{aligned}$$

We remind the notion of *degree* $s(a)$ of an element $a \in A$. Let us define by $A^{(\infty)}$ the set of all elements $a \in A$ such that the set $\downarrow a$ contains an infinite chain of $\mathcal{A}$. Further, we put $A^{(0)} = \{a \in A : f^{-1}(a) = \emptyset\}$. Now we define a set $A^{(\lambda)} \subseteq A$ for each ordinal λ by induction. Assume that we have defined $A^{(\alpha)}$ for each ordinal $\alpha < \lambda$. Then we put

$$A^{(\lambda)} = \{a \in A \setminus \bigcup_{\alpha < \lambda} A^{(\alpha)} : f^{-1}(a) \subseteq \bigcup_{\alpha < \lambda} A^{(\alpha)}\}.$$

The sets $A^{(\lambda)}$ are pairwise disjoint. For each $a \in A$, either $a \in A^{(\infty)}$ or there is an ordinal λ with $a \in A^{(\lambda)}$. In the former case we put $s(a) = \infty$, in the latter we set $s(a) = \lambda$.

Let $\mathcal{B} = (B, f)$ be a subalgebra of $\mathcal{A}$. Then $\mathcal{B}$ is said to be a *retract* of $\mathcal{A}$ if there exists an endomorphism h of $\mathcal{A}$ such that $h(A) = B$ and $h^2(a) = h(a)$ for every $a \in A$.

We will say that $\mathcal{B}$ is a *basic retract* of $\mathcal{A}$, if $\mathcal{B}$ is a retract of $\mathcal{A}$ and $\mathcal{B}$ is a basic algebra.

The notion of a direct limit we apply by [2], §21.

Let $\langle P, \leq \rangle$ be a directed partially ordered set. For each $p \in P$, let $\mathcal{A}_p = (A_p, f)$ be a monounary algebra. Assume that if $p, q \in P$, $p \neq q$, then $A_p \cap A_q = \emptyset$. Suppose that for each pair of elements p and q in P with $p < q$, we have a homomorphism φ_{pq} of $\mathcal{A}_p$ into $\mathcal{A}_q$ such that $p < q < s$ implies that $\varphi_{ps} = \varphi_{pq} \circ \varphi_{qs}$. For each $p \in P$, suppose that φ_{pp} is the identity on A_p . The family $\{P, \mathcal{A}_p, \varphi_{pq}\}$ is said to be *direct*.

Assume that $p, q \in P$ and $a \in A_p$, $b \in A_q$. Put $a \equiv b$ if there exists $s \in P$ with $p \leq s$, $q \leq s$ such that $\varphi_{ps}(a) = \varphi_{qs}(b)$.

For each $a \in \bigcup_{p \in P} A_p$ put $\bar{a} = \{b \in \bigcup_{p \in P} A_p : a \equiv b\}$.

Denote $\bar{A} = \{\bar{a} : a \in \bigcup_{p \in P} A_p\}$. Define $f(\bar{a}) = \overline{f(a)}$.

Then the monounary algebra $\bar{\mathcal{A}} = (\bar{A}, f)$ is said to be a *direct limit* of the direct family $\{P, \mathcal{A}_p, \varphi_{pq}\}$. We express this situation as follows

$$\{P, \mathcal{A}_p, \varphi_{pq}\} \longrightarrow \bar{\mathcal{A}}. \quad (2.1)$$

Note that in the category theory this construction corresponds to (directed) colimit.

Let (1) be valid. The following lemmas follow from the definition of the direct limit.

Lemma 1. *Let $p \in P$ and $a, b \in A_p$. If a, b are in one component of $\mathcal{A}_p$, then $\bar{a}, \bar{b}$ are in one component of $\bar{\mathcal{A}}$.*

Lemma 2. *Let $p \in P$ and $a \in A_p$. If a is a cyclic element of $\mathcal{A}_p$, then $\bar{a}$ is a cyclic element of $\bar{\mathcal{A}}$.*

Lemma 3. *Let $u \in \bar{A}$ belong to an n -element cycle of $\bar{\mathcal{A}}$. Then there exist $p \in P$ and $a \in A_p \cap u$ such that a belongs to an n -element cycle of $\mathcal{A}_p$. Moreover, for all $q \geq p$ the point $\varphi_{pq}(a)$ belongs to an n -element cycle of $\mathcal{A}_q$.*

Lemma 4. *Let $u_1, \dots, u_k \in \bar{A}$. Then there exist $p \in P$ and $a_1, \dots, a_k \in A_p$ such that $a_1 \in u_1, \dots, a_k \in u_k$.*

Lemma 5. *Let the operation f be injective (surjective) on A_p for all $p \in P$. Then the operation f is injective (surjective) on $\bar{A}$.*

Let $\mathcal{A}$ be an algebra. The class $\underline{\mathbf{L}} \mathcal{A}$ is created by all direct limits of families in which only algebras isomorphic to $\mathcal{A}$ occur.

The following lemma will be repeatedly used in this paper. It holds for algebras of arbitrary type and it is proved as Lemma 2 of [4]:

Lemma 6. *If $\mathcal{A}' \in \mathbf{RA}$ and $\mathcal{B} \in \underline{\mathbf{L}} \mathcal{A}'$, then $\mathcal{B} \in \underline{\mathbf{L}} \mathcal{A}$.*

Let I be a nonempty set. For each $i \in I$ let $\mathcal{B}_i = (B_i, f)$ be a monounary algebra. We denote by $\sum_{i \in I} \mathcal{B}_i$ a monounary algebra which is a disjoint union of algebras $\mathcal{B}_i, i \in I$.

Let $\mathcal{A} = \sum_{i \in I} \mathcal{B}_i$ and $\mathcal{B}_i$ be connected for all $i \in I$. Let $i \in I$. If $\mathcal{B}_i$ contains a cycle of length k for some $k \in \mathbb{N}$, then we denote by $\mathcal{C}_i$ a cycle of length k . Else we denote by $\mathcal{C}_i$ a line. Put $\mathcal{A}^\diamond = \sum_{i \in I} \mathcal{C}_i$.

The following lemma is proved as Lemma 4 in [3].

Lemma 7. $\mathcal{A}^\diamond \in \underline{\mathbf{L}}\mathcal{A}$

Corollary 1. Let $\mathcal{A} = (A, f)$ be a monounary algebra. Then there exists $\mathcal{B} \in \underline{\mathbf{L}}\mathcal{A}$ such that the operation of $\mathcal{B}$ is bijective.

3. AUXILIARY RESULTS

In this section let $\mathcal{A} = (A, f)$ be a monounary algebra. We will prove five preliminary lemmas about algebras in $\underline{\mathbf{L}}\mathcal{A}$.

Lemma 8. Let $\mathcal{D} = (D, f)$ be a component of $\mathcal{A} = (A, f)$ and ν be an infinite cardinal number. If $\mathcal{A}$ has ν components isomorphic to $\mathcal{D}$, then $\underline{\mathbf{L}}\mathcal{A}$ contains an algebra with more than ν components isomorphic to $\mathcal{D}$.

Proof. Let μ be the cardinality of D . Denote by $\mathcal{B} = (B, f)$ a subalgebra of $\mathcal{A}$ which is created by ν components isomorphic to $\mathcal{D}$. Take $\kappa > \nu$ and K the set of cardinality κ . Let $i \in K$. Let B_i be a set of the cardinality $\nu \cdot \mu$, $B_i \cap B_j = \emptyset$ for $i \neq j$. Let f be defined on B_i in a such way that $(B_i, f) \cong \mathcal{B}$.

Take P the set of all finite subsets of K . Consider $\langle P, \subseteq \rangle$. Let $p \in P$. Put

$$A_p = \{(d, p) : d \in \bigcup_{j \in p} B_j \cup (A \setminus B)\},$$

$$f(d, p) = (f(d), p),$$

$$\mathcal{A}_p = (A_p, f).$$

We have $(\bigcup_{j \in p} B_j, f) \cong \mathcal{B}$ since p is finite. Therefore $\mathcal{A}_p \cong \mathcal{A}$.

Let $p, q \in P, p \subseteq q$ and $\varphi_{pq}(x, p) = (x, q)$ for every $(x, p) \in A_p$. Obviously $\{P, \mathcal{A}_p, \varphi_{pq}\}$ is a direct family.

Let $\overline{\mathcal{A}} = (\overline{A}, f)$ be a direct limit of this family. Then $\overline{\mathcal{A}} \in \underline{\mathbf{L}}\mathcal{A}$. If $i, j \in K, i \neq j$ and $d_i \in B_i, d_j \in B_j$, then $\overline{(d_i, \{i\})}$ and $\overline{(d_j, \{j\})}$ are in isomorphic but different components of $\overline{\mathcal{A}}$. Therefore $\overline{\mathcal{A}}$ contains at least $\kappa > \nu$ components isomorphic to $\mathcal{D}$. $\square$

Lemma 9. Let κ be an infinite cardinal. Let $a \in A$ and $D \subseteq A$ be such that

- (1) $D \subseteq f^{-1}(a)$;
- (2) $(\downarrow d_1 \cup \uparrow a, f) \cong (\downarrow d_2 \cup \uparrow a, f)$ for all $d_1, d_2 \in D$;

(3) D is infinite.

Then there exists $\overline{\mathcal{A}} = (\overline{A}, f) \in \underline{\mathbf{L}}\mathcal{A}$ such that $\|\overline{A}\| > \kappa$.

Proof. Denote $B = \bigcup_{d \in D} \downarrow d$ and $v = \|B\|$. Let M be a set of cardinality greater than κ . For $m \in M$ let B_m be a set of cardinality of v and D_m be such that $D_m \subseteq B_m$ and $\|D_m\| = \|D\|$. Assume that $B_m \cap A = \emptyset$ and $B_m \cap B_{m'} = \emptyset$ for $m \neq m'$. Further, let f be defined on B_m in a such way that $f(t) = a$ for $t \in D_m$ and $(B_m \cup \uparrow a, f) \cong (B \cup \uparrow a, f)$.

Let P be the set of all finite subsets of M . Let $p \in P$. Put

$$A_p = \{(d, p) : d \in \bigcup_{m \in p} B_m \cup (A \setminus B)\}, \quad f(d, p) = (f(d), p).$$

Let $\mathcal{A}_p = (A_p, f)$. Since p is finite, we have that $\mathcal{A}_p \cong \mathcal{A}$.

Let $p, q \in P, p \subseteq q$. We define $\varphi_{pq}(x, p) = (x, q)$ for every $(x, p) \in A_p$. We have that φ_{pq} is a homomorphism from $\mathcal{A}_p$ into $\mathcal{A}_q$. Thus $\{P, \mathcal{A}_p, \varphi_{pq}\}$ is a direct family. Let $\overline{\mathcal{A}} = (\overline{A}, f)$ be a direct limit of this family.

Suppose that $v \in \overline{A}$. Let $(d, p) \in v \cap A_p$. If $d \in A \setminus B$, then $v = \{(d, q) : q \in P\}$. If $d \in B_m$ for some $m \in p$, then $v = \{(d, q) : q \in \{s \in P : m \in s\}\} = \overline{(d, \{m\})}$. Further, $\overline{(d, \{m\})} \neq \overline{(t, \{n\})}$ for every $d, t \in \bigcup_{m \in M} D_m, d \neq t$ and $m, n \in M$.

Take

$$V = \{\overline{(t, \{m\})} : t \in D_m, m \in M\}.$$

We obtain

$$\|\overline{A}\| \geq \|V\| = \sum_{m \in M} \|D_m\| = \|M\| \cdot \|D\| \geq \|M\| > \kappa.$$

□

Lemma 10. Let $a \in A$ be such that the set $K = \{d \in f^{-1}(a) : s(d) = \infty\}$ is infinite. Then there exists $\overline{\mathcal{A}} = (\overline{A}, f) \in \underline{\mathbf{L}}\mathcal{A}$ such that $\|\overline{A}\| > \|A\|$.

Proof. Let $\mathcal{A}' = (A', f) \in \mathbf{RA}$ be such that $a \in A', f^{-1}(a) \cap A' = K$ and f is injective on $\downarrow d$ for all $d \in K$. The algebra $\mathcal{A}'$ satisfies assumptions of Lemma 9 for the element a and the set K . Take $\kappa = \|A\|$. Then there exists $\overline{\mathcal{A}} = (\overline{A}, f) \in \underline{\mathbf{L}}\mathcal{A}'$ such that $\|\overline{A}\| > \|A\|$. We obtain $\overline{\mathcal{A}} \in \underline{\mathbf{L}}\mathcal{A}$ according to Lemma 6. □

Lemma 11. Let $k \in \mathbb{N}$ and $a \in A$ be such that the set $\{d \in f^{-1}(a) : s(d) \leq k\}$ is infinite. Then there exists $\overline{\mathcal{A}} = (\overline{A}, f) \in \underline{\mathbf{L}}\mathcal{A}$ such that $\|\overline{A}\| > \|A\|$.

Proof. Denote $L = \{d \in f^{-1}(a) : s(d) \leq k\}$. The elements of L have degrees from the set $\{0, 1, \dots, k\}$. The infiniteness of L implies that there exists $i \in \{0, 1, \dots, k\}$ such that the set L contains $\|L\|$ elements of degree i .

There exists $\mathcal{A}' = (A', f) \in \mathbf{RA}$ such that $a \in A', f^{-1}(a) \cap A' = f^{-1}(a)$ and f is injective on $\downarrow d \cap A'$ for all $d \in L$ such that $s(d) = i$.

The algebra $\mathcal{A}'$ satisfies assumptions of Lemma 9 for the element a and the set $D = \{d \in f^{-1}(a) : s(d) = i\}$. Take $\kappa = \|A\|$. Then there exists $\overline{\mathcal{A}} = (\overline{A}, f) \in \underline{\mathbf{L}} \mathcal{A}'$ such that $\|\overline{A}\| > \|A\|$. We obtain $\overline{\mathcal{A}} \in \underline{\mathbf{L}} \mathcal{A}$ according to Lemma 6. $\square$

Lemma 12. *Let $a \in A$ and the sequence $\{d_k, k \in \mathbb{N}\}$ be such that $d_k \in f^{-1}(a)$, $s(d_k) \in \mathbb{N}$ and $s(d_k) < s(d_{k+1})$ for every $k \in \mathbb{N}$. Then there exist $\overline{\mathcal{A}} = (\overline{A}, f) \in \underline{\mathbf{L}} \mathcal{A}$ and $u \in \overline{A}$ such that the set $\{v \in f^{-1}(u) : s(v) = \infty\}$ is infinite*

Proof. Let $\mathcal{A}' = (A', f) \in \mathbf{RA}$ be such that $a \in A'$, $f^{-1}(a) \cap A' = f^{-1}(a)$ and f is injective on $\downarrow d_k \cap A'$ for all $k \in \mathbb{N}$. Put

$$D = \bigcup_{k \in \mathbb{N}} (\downarrow d_k \cap A').$$

Let $a' \in D$. Then $f^n(a') = d_k$ for some uniquely determined $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$, since f is injective on $\downarrow d_k \cap A'$. We will write $f^{-n}(d_k) = a'$.

Let $p \in \mathbb{N}$. Put $A_p = \{(d, p) : d \in A'\}$, $f(d, p) = (f(d), p)$, $\mathcal{A}_p = (A_p, f)$. Let $\varphi_{p,p+1}$ be a mapping from A_p into A_{p+1} such that $\varphi_{p,p+1}(d, p) = (d, p+1)$ for $d \in A' \setminus D$ and $\varphi_{p,p+1}(f^{-n}(d_k), p) = (f^{-n}(d_{k+1}), p+1)$. This mapping is the homomorphism from $\mathcal{A}_p$ into $\mathcal{A}_{p+1}$. Consider a common linear order on $\mathbb{N}$. We obtained a direct family $\{\mathbb{N}, \mathcal{A}_p, \varphi_{pq}\}$. Let $\overline{\mathcal{A}} = (\overline{A}, f)$ be a direct limit of this family.

Take $u = \overline{(a, 1)}$. Then $u = \{(a, p) : p \in \mathbb{N}\}$ and

$$f^{-1}(u) = \{\overline{(d_i, 1)}, \overline{(d_1, i)} : i \in \mathbb{N}\} \cup \{\overline{(d, 1)} : d \in f^{-1}(a) \setminus \{d_k, k \in \mathbb{N}\}\}.$$

Further, if $i \in \mathbb{N}$, then

$$\begin{aligned} \downarrow \overline{(d_1, i)} &= \{\overline{(d_1, i)}, \overline{(f^{-1}(d_1), i)}, \dots, \overline{(f^{-s(d_1)}(d_1), i)}, \overline{(f^{-s(d_1)-1}(d_2), i+1)}, \\ &\quad \dots, \overline{(f^{-s(d_2)}(d_2), i+1)}, \overline{(f^{-s(d_2)-1}(d_3), i+2)}, \dots\} \end{aligned}$$

and f is injective on $\downarrow \overline{(d_1, i)}$. Thus $s(\overline{(d_1, i)}) = \infty$. The set $\{\overline{(d_1, i)} : i \in \mathbb{N}\}$ is infinite, because $\varphi_{i,k}(d_1, i) = (d_{1+k-i}, k)$. $\square$

4. THE MAIN RESULT

Several properties of monounary algebras with easy direct limits are proved in this section. They do not give a full description of all monounary algebras with easy direct limits but allow to observe some interesting cardinal characteristics, see Cor 2 and Cor. 3. Prop. 1 can be generalized to universal algebras and this generalization is formulated in the end of this section. Other assertions are closely connected to an unary fundamental operation.

We will suppose that $\mathcal{A} = (A, f)$ is an algebra with easy direct limits in propositions 1 - 3.

Proposition 1. *There exists $\mathcal{B} = (B, f) \in \mathbf{RA}$ such that f is bijective on B .*

Proof. It follows from Cor. 1. $\square$

Proposition 2. *The number of components of $\mathcal{A}$ without a cycle is finite and the number of components of $\mathcal{A}$ which contain an n -element cycle is finite for every $n \in \mathbb{N}$.*

Proof. Let $\mathcal{A} = \sum_{k \in K} \mathcal{B}_k$, where $\mathcal{B}_k$ are connected for all $k \in K$.

Let $n \in \mathbb{N}$. Suppose that $\mathcal{A}$ contains infinitely many n -element cycles. Denote

$$K' = \{k \in K : \mathcal{B}_k \text{ has a cycle of length } n\}.$$

Let $k \in K$. Consider an algebra $\mathcal{D}_k$ such that $\mathcal{D}_k = \mathcal{B}_k$ if $k \notin K'$ and $\mathcal{D}_k$ a cycle of length n if $k \in K'$. Take $\mathcal{D} = \sum_{k \in K} \mathcal{D}_k$.

We have $\mathcal{D} \in \mathbf{RA}$. There exists $\mathcal{E} \in \underline{\mathbf{L}}\mathcal{D}$ such that $\mathcal{E}$ contains more cycles of length n than $\mathcal{D}$ in view of Lemma 8. According to the number of n -element cycles of $\mathcal{D}$ we obtain that $\mathcal{E}$ is not isomorphic to a subalgebra of $\mathcal{A}$. That means $\mathcal{E}$ is not isomorphic to a retract of $\mathcal{A}$. In view of Lemma 6 we have $\mathcal{E} \in \underline{\mathbf{L}}\mathcal{A}$.

Analogously we prove that $\mathcal{A}$ is not an algebra with easy direct limits if $\mathcal{A}$ contains infinitely many components with a line.

Finally, suppose that $\mathcal{A}$ contains infinitely many components which contain no cycle and no line. In view of Lemma 7 there exists an algebra of $\underline{\mathbf{L}}\mathcal{A}$ such that it contains an infinitely many components with a line. Therefore $\mathcal{A}$ is not an algebra with easy direct limits. $\square$

Proposition 3. *The set $f^{-1}(a)$ is finite for every $a \in A$.*

Proof. We denote $K_a = \{b \in f^{-1}(a) : s(b) = \infty\}$ for every $a \in A$.

Assume that there exists $a \in A$ such that K_a is infinite. We use Lemma 10 to obtain an algebra $\overline{\mathcal{A}} = (\overline{A}, f) \in \underline{\mathbf{L}}\mathcal{A}$ which has the cardinality greater than the set A . Thus $\overline{\mathcal{A}}$ is not isomorphic to a subalgebra of $\mathcal{A}$ and $\mathcal{A}$ is not an algebra with easy direct limits, a contradiction.

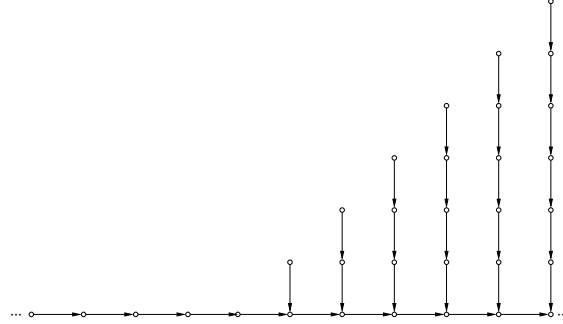
Now suppose that $a \in A$ is such that the set $f^{-1}(a)$ is infinite. Then $\{c \in f^{-1}(a) : s(c) \neq \infty\}$ is infinite. The assumption of Lemma 12 fails to hold, otherwise there is $\overline{\mathcal{A}} \in \underline{\mathbf{L}}\mathcal{A}$ such that $\overline{\mathcal{A}}$ is not a retract of $\mathcal{A}$, which is a contradiction. Therefore there exists $b \in f^{-1}(a)$ with $s(b) = \max\{s(c) : c \in f^{-1}(a) : s(c) \neq \infty\}$. This implies that either $s(a) = \infty$ or $\infty \neq s(a) = s(b) + 1$.

If $s(a) \notin \mathbb{N} \cup \{\infty\}$, then $s(b) \geq \omega$ (where ω is the least infinite ordinal number), b fulfils the assumption of Lemma 12 and this yields a contradiction too.

Further, let $s(a) \in \mathbb{N} \cup \{\infty\}$. Then $s(b) \in \mathbb{N}$ and $\{c \in f^{-1}(a) : s(c) \neq \infty\} = \{c \in f^{-1}(a) : s(c) \in \mathbb{N}\} = \{c \in f^{-1}(a) : s(c) \leq s(b)\}$. According to Lemma 11 again there exists $\overline{\mathcal{A}} \in \underline{\mathbf{L}}\mathcal{A}$ such that $\overline{\mathcal{A}}$ is not a retract of $\mathcal{A}$, which is a contradiction. $\square$

Theorem 1. *If $\mathcal{A} = (A, f)$ is an algebra with easy direct limits, then*

- (i) *every component of $\mathcal{A}$ without a cycle contains a line,*

FIGURE 1. The algebra $\mathcal{A}$

- (ii) the number of components of $\mathcal{A}$ without a cycle is finite,
- (iii) the number of components of $\mathcal{A}$ which contain an n -element cycle is finite for every $n \in \mathbb{N}$,
- (iv) the set $f^{-1}(a)$ is finite for every $a \in A$.

Proof. Properties (ii) and (iii) are valid according to Prop. 2. Therefore the number of lines of the algebra $\mathcal{A}^\diamond$ is finite. This yields that $\mathcal{A}^\diamond$ is isomorphic to a retract of $\mathcal{A}$ according to Lemma 7. Thus (i) is satisfied. Property (iv) is proved in Prop. 3. $\square$

It is easy to see that conditions (i) - (iv) are independent.

Corollary 2. *If $\mathcal{A} = (A, f)$ is an algebra with easy direct limits, then A is countable.*

Proof. The number of components of $\mathcal{A}$ is countable according to (ii) and (iii). The number of elements of an arbitrary component of $\mathcal{A}$ is countable according to (iv). $\square$

Corollary 3. *Let $\mathcal{A} = (A, f)$ be an algebra with easy direct limits. Then $\|\mathbf{RA}\| \neq \aleph_0$.*

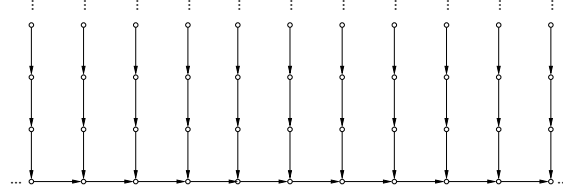
Proof. We have $\|\mathbf{RA}\| \leq 2^{\aleph_0}$ since A is countable.

Every component of $\mathcal{A}$ contains a cycle or a line in view of Thm. 1 (i). Thus every component of $\mathcal{A}$ contains a basic retract. Therefore there is no component of $\mathcal{A}$ with the number of retracts equal to $\aleph_0$ according to Thm.5.9 of [7]. It implies $\|\mathbf{RA}\| \neq \aleph_0$ according to Thm.5.10 of [7]. $\square$

Example 1. Let

$$A = \{(0, 0), (a, 0), (-a, 0), (a, b) : a, b \in \mathbb{N}, b \leq a\}.$$

Let $f(a, b) = (a, b - 1)$ for $b > 1$ and $f(a, 0) = (a + 1, 0)$. Then $\mathcal{A} = (A, f)$ has properties (i)-(iv) and it is not an algebra with easy direct limits.

FIGURE 2. The algebra $\overline{\mathcal{A}}$

We hint why $\mathcal{A}$, see Fig.1, is not an algebra with easy direct limits. Consider $d \in A$ such that

1. $s(d) = \infty$,
2. f is injective on $\downarrow d$,
3. there is $e \in A \setminus \{d\}$ such that $f(e) = f(d)$.

The algebra $\mathcal{A}$ contains exactly one point $d \in A$ which satisfies the conditions 1. - 3, namely $d = (0, 0)$.

For $n \in \mathbb{N}$ let $A_n = \{(a, b, n) : (a, b) \in A\}$ and $f(a, b, n) = (f(a, b), n)$. Put $\mathcal{A}_n = (A_n, f)$. Further, let $\varphi_{n, n+1}(a, b, n) = (a + 1, b, n + 1)$. Then $\{\mathbb{N}, \mathcal{A}_n, \varphi_{nm}\}$ is a direct family, $\mathcal{A}_n \cong \mathcal{A}$. Let $\{\mathbb{N}, \mathcal{A}_n, \varphi_{nm}\} \longrightarrow \overline{\mathcal{A}}$, see Fig.2.

For every $a > 0$ we have as on Fig.3

$$\begin{aligned} f(\overline{(a, a, n+1)}) &= \overline{f(a, a, n+1)} = \overline{(f(a, a), n+1)} = \\ &= \overline{(a, a-1, n+1)} = \overline{\varphi_{n, n+1}(a-1, a-1, n)} = \overline{(a-1, a-1, n)}. \end{aligned}$$

Further,

$$\overline{(1, 1, n)} \neq \overline{(0, 0, n)} \text{ and } f(\overline{(1, 1, n)}) = \overline{(0, 0, n-1)} = f(\overline{(0, 0, n)}).$$

The elements $\overline{(1, 1, n)}$ of $\overline{\mathcal{A}}$ for every $n \in \mathbb{N}$ are pairwise different elements which satisfy conditions 1. - 3.

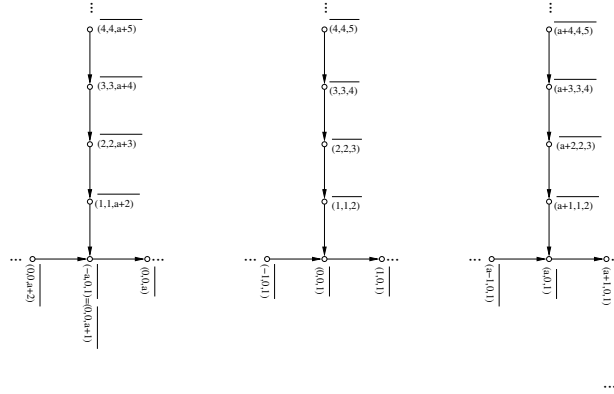
Remark 1. Corollary 1 and Proposition 1 can be generalized for universal algebras by the following way:

Let $\mathcal{A}$ be an algebra of type F and f be a unary term operation over F such that f is an endomorphism of the algebra $\mathcal{A}$. Then

- (1) there exists $\mathcal{B} = (B, F) \in \underline{\mathbf{L}}\mathcal{A}$ such that f is bijective on B ,
- (2) if $\mathcal{A}$ is an algebra with easy direct limits, then
there exists $\mathcal{B} = (B, F) \in \mathbf{R}\mathcal{A}$ such that f is bijective on B .

5. ALGEBRAS WITH A BIJECTIVE OPERATION

Let A be a nonempty set and f be a bijective operation from A into A . We will prove that $\mathcal{A} = (A, f)$ is an algebra with easy direct limits if and only if the number of pairwise isomorphic components of $\mathcal{A}$ is finite.

FIGURE 3. A description of $\overline{\mathcal{A}}$, $a \in \mathbb{N}$

Example 2. Let $\mathcal{A} = (A, f)$ be monounary algebra such that every component of $\mathcal{A}$ is a cycle of the length 2^n for some $n \in \mathbb{N}$ and for every $n \in \mathbb{N}$ there is exactly one cycle of the length 2^n . Then $\mathcal{A}$ is an algebra with easy direct limits such that A is countable and $\mathbf{R}\mathcal{A}$ consists of uncountable many non-isomorphic algebras. ($\mathcal{A}$ is an algebra with easy direct limits according to the fact that every algebra of $\underline{\mathbf{L}}\mathcal{A}$ contains a 2-element cycle and by Lemmas 2 and 3.)

The following assertion is obvious.

Lemma 13. *Let $\mathcal{A} = (A, f)$ be a monounary algebra. The following conditions are equivalent:*

- (i) *the operation f is bijective;*
- (ii) *$\text{card } f^{-1}(a) = 1$ for every $a \in A$;*
- (iii) *if $\mathcal{B}$ is a component of $\mathcal{A}$, then $\mathcal{B}$ is a cycle or a line.*

Theorem 2. *Let $\mathcal{A} = (A, f)$ be a monounary algebra with a bijective operation. Then the following conditions are equivalent:*

- (a) *$\mathcal{A}$ is an algebra with easy direct limits.*
- (b) *The number of pairwise isomorphic components of $\mathcal{A}$ is finite.*
- (c) *Conditions (i) - (iv) from Thm.1 are satisfied.*

Proof. The condition (a) implies (b) according to Lemma 8.

Conditions (i) and (iv) from Thm.1 are satisfied since $\mathcal{A}$ has a bijective operation. Thus (b) is equivalent to (c).

Suppose that (b) is satisfied. Let (1) be valid and $\mathcal{A}_p \cong \mathcal{A}$. Then the operation f of $\overline{\mathcal{A}}$ is bijective by Lemma 5.

Let $\mathcal{B}_1, \dots, \mathcal{B}_k$ be pairwise isomorphic components of $\overline{\mathcal{A}}$. Take u_i an element of $\mathcal{B}_i$ for $i = 1, \dots, k$. According to Lemma 4 choose $p \in P$ and $a_1, \dots, a_k \in A_p$ such

that $a_1 \in u_1, \dots, a_k \in u_k$. Lemma 1 yields that $a_1, \dots, a_k$ are in different components of $\mathcal{A}_p$.

Let $\mathcal{B}_1$ be a cycle of length n . Then there exists $q \in P$ such that $q \geq p$ and $f^n(\varphi_{pq}(a_i)) = \varphi_{pq}(a_i)$ according to Lemma 3 and the definition of a direct limit. The element $\varphi_{pq}(a_i)$ belongs to an n -element cycle, since $\overline{\varphi_{pq}(a_i)} = u_i$ and u_i belongs to an n -element cycle. In view of Lemma 1 elements $\varphi_{pq}(a_1), \dots, \varphi_{pq}(a_k)$ are in different components of $\mathcal{A}_q$. Conclude that $\mathcal{A}$ contains k cycles of the length n as its components.

Now let $\mathcal{B}_1$ be a line. Let $i \in \{1, \dots, k\}$. We get that a_i is in no cycle according to the definition of a direct limit. Therefore the component of $\mathcal{A}_p$, which possesses a_i , is a line, since the operation of $\mathcal{A}_p$ is bijective. Conclude that $\mathcal{A}$ contains k lines as its components.

We have proved that $\overline{\mathcal{A}}$ consists of at most such number of isomorphic components as $\mathcal{A}$.

Let $\mathcal{A}$ be an algebra without a cycle. Then every component of $\mathcal{A}$ is a line and $\overline{\mathcal{A}}$ is a retract of $\mathcal{A}$.

Suppose that $\mathcal{A}$ has a cycle. Let us order lengths of cycles of the algebra $\mathcal{A}$ according to the divisibility and let minimal elements of this ordered set be $\{k_i, i \in I\}$. We remark that the set I is nonempty and it is at most countable. Let $i \in I$. If a belongs to a k_i -element cycle of $\mathcal{A}$, then every endomorphism of $\mathcal{A}$ maps a into a cycle of length k_i . Thus $\overline{a}$ belongs to a cycle of length k_i according to the direct limit definition. Therefore $\overline{\mathcal{A}}$ contains a cycle of length k_i for every $i \in I$. That means that $\overline{\mathcal{A}}$ is a retract of $\mathcal{A}$. □

Theorem 3. *The number of all non-isomorphic monounary algebras with easy direct limits is equal to $2^{\aleph_0}$.*

Proof. The number of all non-isomorphic monounary algebras with easy direct limits which have a bijective fundamental operation is equal to $2^{\aleph_0}$ according to the previous theorem. Therefore the assertion is valid in view of Cor. 2. □

Proposition 4. *The number of monounary algebras with easy direct limits which have $2^{\aleph_0}$ retracts is $2^{\aleph_0}$.*

Proof. Let $K \subseteq \mathbb{N}$. Take an algebra $\mathcal{A}_K$ such that

- (1) every component of $\mathcal{A}_K$ is a cycle,
- (2) if $k \in K$, then $\mathcal{A}_K$ contains exactly 3 cycles of length k ,
- (3) if $k \notin K$, then $\mathcal{A}_K$ contains exactly 2 cycles of length k .

□

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Author’s address

Emília Halušková

Mathematical Institute, Slovak Academy of Sciences, Grešáková 6, 040 01 Košice, Slovakia

E-mail address: ehaluska@saske.sk



CONVERGENCE OF CESÀRO MEANS WITH VARYING PARAMETERS OF WALSH-FOURIER SERIES

ANAS AHMAD ABU JOUDEH AND GYÖRGY GÁT

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Abstract. In 2007 Akhobadze [1] (see also [2]) introduced the notion of Cesàro means of Fourier series with variable parameters. In the present paper we prove the almost everywhere convergence of the the Cesàro (C, α_n) means of integrable functions $\sigma_n^{\alpha_n} f \rightarrow f$, where $\mathbb{N}_{\alpha, K} \ni n \rightarrow \infty$ for $f \in L^1(I)$, where I is the Walsh group for every sequence $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$. This theorem for constant sequences α that is, $\alpha \equiv \alpha_1$ was proved by Fine [3].

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1. INTRODUCTION AND MAIN RESULTS

We follow the standard notions of dyadic analysis introduced by the mathematicians F. Schipp, P. Simon, W. R. Wade (see e.g. [9]) and others. Denote by $\mathbb{N} := \{0, 1, \dots\}$, $\mathbb{P} := \mathbb{N} \setminus \{0\}$, the set of natural numbers, the set of positive integers and $I := [0, 1)$ the unit interval. Denote by $\lambda(B) = |B|$ the Lebesgue measure of the set B ($B \subset I$). Denote by $L^p(I)$ the usual Lebesgue spaces and $\|\cdot\|_p$ the corresponding norms ($1 \leq p \leq \infty$). Set

$$\mathcal{J} := \left\{ \left[\frac{p}{2^n}, \frac{p+1}{2^n} \right) : p, n \in \mathbb{N} \right\}$$

the set of dyadic intervals and for given $x \in I$ and let $I_n(x)$ denote the interval $I_n(x) \in \mathcal{J}$ of length 2^{-n} which contains x ($n \in \mathbb{N}$). Also use the notion $I_n := I_n(0)$ ($n \in \mathbb{N}$). Let

$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)}$$

be the dyadic expansion of $x \in I$, where $x_n = 0$ or 1 and if x is a dyadic rational number ($x \in \{\frac{p}{2^n} : p, n \in \mathbb{N}\}$) we choose the expansion which terminates in 0's. The

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notion of the Hardy space $H(I)$ is introduced in the following way [9]. A function $a \in L^\infty(I)$ is called an atom, if either $a = 1$ or a has the following properties: $\text{supp } a \subseteq I_a$, $\|a\|_\infty \leq |I_a|^{-1}$, $\int_I a = 0$, for some $I_a \in \mathcal{J}$. We say that the function f belongs to H , if f can be represented as $f = \sum_{i=0}^\infty \lambda_i a_i$, where a_i 's are atoms and for the coefficients (λ_i) the inequality $\sum_{i=0}^\infty |\lambda_i| < \infty$ is true. It is known that H is a Banach space with respect to the norm

$$\|f\|_H := \inf \sum_{i=0}^\infty |\lambda_i|,$$

where the infimum is taken over all decompositions $f = \sum_{i=0}^\infty \lambda_i a_i \in H$.

Set the definition of the n th ($n \in \mathbb{N}$) Walsh-Paley function at point $x \in I$ as:

$$\omega_n(x) := \prod_{j=0}^\infty (-1)^{x_j n_j},$$

where $\mathbb{N} \ni n = \sum_{j=0}^\infty n_j 2^j$ ($n_j \in \{0, 1\}$ ($j \in \mathbb{N}$)). It is known (see [8] or [10]) that the system $(\omega_n, n \in \mathbb{N})$ is the character system of $(I, +)$, where the group operation $+$ is the so-called dyadic or logical addition on I . That is, for any $x, y \in I$

$$x + y := \sum_{n=0}^\infty |x_n - y_n| 2^{-(n+1)}.$$

Denote by

$$\hat{f}(n) := \int_I f \omega_n d\lambda, \quad D_n := \sum_{k=0}^{n-1} \omega_k, \quad K_n^1 := \frac{1}{n+1} \sum_{k=0}^n D_k$$

the Fourier coefficients, the Dirichlet and the Fejér or $(C, 1)$ kernels, respectively. It is also known that the Fejér or $(C, 1)$ means of f is

$$\begin{aligned} \sigma_n^1 f(y) &:= \frac{1}{n+1} \sum_{k=0}^n S_k f(y) = \int_I f(x) K_n^1(y+x) d\lambda(x) \\ &= \frac{1}{n+1} \sum_{k=0}^n \int_I f(x) D_k(y+x) d\lambda(x), \quad (n \in \mathbb{N}, y \in I). \end{aligned}$$

It is known [9] that for $n \in \mathbb{N}, x \in I$ it holds

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n \\ 0, & \text{if } x \notin I_n \end{cases}$$

and also that

$$D_n(x) = \omega_n(x) \sum_{k=1}^\infty D_{2^k}(x) n_k (-1)^{x_k},$$

where $n = \sum_{i=1}^{\infty} n_i 2^i$, $n_i = \{0, 1\}$ ($i \in \mathbb{N}$).

Denote by $K_n^{\alpha_n}$ the kernel of the summability method (C, α_n) and call it the (C, α_n) kernel or the Cesàro kernel for $\alpha_n \in \mathbb{R} \setminus \{-1, -2, \dots\}$

$$K_n^{\alpha_n} = \frac{1}{A_n^{\alpha_n}} \sum_{k=0}^n A_{n-k}^{\alpha_n-1} D_k,$$

where

$$A_k^{\alpha_n} = \frac{(\alpha_n + 1)(\alpha_n + 2) \dots (\alpha_n + n)}{k!}.$$

It is known [12] that $A_n^{\alpha_n} = \sum_{k=0}^n A_k^{\alpha_n-1}$, $A_k^{\alpha_n} - A_{k+1}^{\alpha_n} = -\frac{\alpha_n A_k^{\alpha_n}}{k+1}$. The (C, α_n) Cesàro means of integrable function f is

$$\sigma_n^{\alpha_n} f(y) := \frac{1}{A_n^{\alpha_n}} \sum_{k=0}^n A_{n-k}^{\alpha_n-1} S_k f(y) = \int_I f(x) K_n^{\alpha_n}(y+x) d\lambda(x).$$

In [3] Fine proved the almost everywhere convergence $\sigma_n^{\alpha_n} f \rightarrow f$ for all integrable function f with constant sequence $\alpha_n = \alpha_1 > 0$. On the rate of convergence of Cesàro means in this constant case see the paper of Fridli [4]. For the two-dimensional situation see the paper of Goginava [7].

Comment 1. With respect to other locally constant orthonormal systems for instance it was a question of Taibleson [8] open for a long time, that does the Fejér-Lebesgue theorem, that is the a.e. convergence $\sigma_n^1 f \rightarrow f$ hold for all integrable function f with respect to the character system of the group of 2-adic integers. In 1997 Gát answered [1] this question in the affirmative. Zheng and Gát generalized this result [9,2] for more general orthonormal systems.

Set two variable function $P(n, \alpha) := \sum_{i=0}^{\infty} n_i 2^{i\alpha}$ for $n \in \mathbb{N}, \alpha \in \mathbb{R}$. For instance $P(n, 1) = n$. Also set for sequences $\alpha = (\alpha_n)$ and positive reals K the subset of natural numbers

$$\mathbb{N}_{\alpha, K} := \left\{ n \in \mathbb{N} : \frac{P(n, \alpha_n)}{n^{\alpha_n}} \leq K \right\}.$$

We can easily remark that for a sequence α such that $1 > \alpha_n \geq \alpha_0 > 0$ we have $\mathbb{N}_{\alpha, K} = \mathbb{N}$ for some K depending only on α_0 . We also remark that $2^n \in \mathbb{N}_{\alpha, K}$ for every $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$ and $K \geq 1$.

In this paper C denotes an absolute constant and C_K another one which may depend only on K . The introduction of (C, α_n) means due to Akhobadze investigated [1] the behavior of the L^1 -norm convergence of $\sigma_n^{\alpha_n} f \rightarrow f$ for the trigonometric system. In this paper it is also supposed that $1 > \alpha_n > 0$ for all n .

The main aim of this paper is to prove :

Theorem 1. Suppose that $1 > \alpha_n > 0$. Let $f \in L^1(I)$. Then we have the almost everywhere convergence $\sigma_n^{\alpha_n} f \rightarrow f$ provided that $\mathbb{N}_{\alpha, K} \ni n \rightarrow \infty$.

The method we use to prove Theorem 1 is to investigate the maximal operator $\sigma_*^\alpha f := \sup_{n \in \mathbb{N}_{\alpha,K}} |\sigma_n^{\alpha_n} f|$. We also prove that this operator is a kind of type (H, L) and of type (L^p, L^p) for all $1 < p \leq \infty$. That is,

Theorem 2. *Suppose that $1 > \alpha_n > 0$. Let $|f| \in H(I)$. Then we have*

$$\|\sigma_*^\alpha f\|_1 \leq C_K \| |f| \|_H.$$

Moreover, the operator σ_*^α is of type (L^p, L^p) for all $1 < p \leq \infty$. That is,

$$\|\sigma_*^\alpha f\|_p \leq C_{K,p} \|f\|_p$$

for all $1 < p \leq \infty$.

For the sequence $\alpha_n = 1$ Theorem 2 is due to Fujii [5]. For the more general but constant sequence $\alpha_n = \alpha_1$ see Weisz [11].

Basically, in order to prove Theorem 1 we verify that the maximal operator $\sigma_*^\alpha f$ ($\alpha = (\alpha_n)$) is of weak type (L^1, L^1) . The way we get this, the investigation of kernel functions, and its maximal function on the unit interval I by making a hole around zero and some quasi locality issues (for the notion of quasi-locality see [9]). To have the proof of Theorem 2 is the standard way. We need several Lemmas in the next section.

2. PROOFS

Lemma 1. *For $j, n \in \mathbb{N}$, $j < 2^n$ we have*

$$D_{2^n-j}(x) = D_{2^n}(x) - \omega_{2^n-1}(x) D_j(x).$$

Proof.

$$D_{2^n}(x) = \sum_{k=0}^{2^n-1} \omega_k(x) = \sum_{k=0}^{2^n-j-1} \omega_k(x) + \sum_{k=2^n-j}^{2^n-1} \omega_k(x) = D_{2^n-j} + \sum_{k=2^n-j}^{2^n-1} \omega_k(x).$$

We have to prove :

$$\sum_{k=2^n-j}^{2^n-1} \omega_k(x) = \omega_{2^n-1}(x) D_j(x).$$

For $k < j$, $k = k_{n-1}2^{n-1} + \dots + k_12^1 + k_0$ we have

$$\begin{aligned} & \omega_{2^n-1}(x) \omega_k \\ &= \omega_{2^{n-1}+\dots+2^1+2^0}(x) \omega_{k_{n-1}2^{n-1}+\dots+k_0}(x) \\ &= \omega_{(1+k_{n-1}(\bmod 2))2^{n-1}+\dots+(1+k_0(\bmod 2))2^0}(x) \\ &= \omega_{(1-k_{n-1}(\bmod 2))2^{n-1}+\dots+(1-k_0(\bmod 2))2^0}(x) \\ &= \omega_{2^{n-1}+2^{n-2}+\dots+2^0-(k_{n-1}2^{n-1}+\dots+k_0)}(x) = \omega_{2^n-1-k}(x). \end{aligned}$$

Thus,

$$\omega_{2^n-1}(x)D_j(x) = \omega_{2^n-1}(x) \sum_{k=0}^{j-1} \omega_k(x) = \sum_{k=0}^{j-1} \omega_{2^n-1-k}(x) = \sum_{k=2^n-j}^{2^n-1} \omega_k(x).$$

This completes the proof of Lemma 1. $\square$

Introduce the following notations: for $a, n, j \in \mathbb{N}$ let $n_{(j)} := \sum_{i=0}^{j-1} n_i 2^i$, that is, $n_{(0)} = 0, n_{(1)} = n_0$ and for $2^B \leq n < 2^{B+1}$, let $|n| := B, n = n_{(B+1)}$. Moreover, introduce the following functions and operators for $n \in \mathbb{N}$ and $1 > \alpha_n > 0$

$$\begin{aligned} T_n^{\alpha_a} &:= \frac{1}{A_n^{\alpha_a}} \sum_{j=0}^{2^{|n|}-1} A_{n-j}^{\alpha_a-1} D_j, \\ \tilde{T}_n^{\alpha_a} &:= \frac{1}{A_n^{\alpha_a}} D_{2^B} \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \\ &\quad + (1-\alpha_a) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} |K_j^1| + A_n^{\alpha_a-1} 2^B |K_{2^B-1}^1|, \\ t_n^{\alpha_a} f(y) &:= \int_I f(x) T_n^{\alpha_a}(y+x) d\lambda(x), \\ \tilde{t}_n^{\alpha_a} f(y) &:= \int_I f(x) \tilde{T}_n^{\alpha_a}(y+x) d\lambda(x). \end{aligned}$$

Now, we need to prove the next Lemma which means that maximal operator $\sup_{n,a} |\tilde{t}_n^{\alpha_a}|$ is quasi-local. This lemma together with the next one are the most important tools in the proof of the main results of this paper.

Lemma 2. *Let $1 > \alpha_a > 0$, $f \in L^1(I)$ such that $\text{supp } f \subset I_k(u)$, $\int_{I_k(u)} f d\lambda = 0$ for some dyadic interval $I_k(u)$. Then we have*

$$\int_{I \setminus I_k(u)} \sup_{n,a \in \mathbb{N}} |\tilde{t}_n^{\alpha_a} f| d\lambda \leq C \|f\|_1.$$

Moreover, $|T_n^{\alpha_a}| \leq \tilde{T}_n^{\alpha_a}$.

Proof. It is easy to have that for $n < 2^k$ and $x \in I_k(u)$ we have $\tilde{T}_n^{\alpha_a}(y+x) = \tilde{T}_n^{\alpha_a}(y+u)$ and

$$\begin{aligned} &\int_{I_k(u)} f(x) \tilde{T}_n^{\alpha_a}(y+x) d\lambda(x) \\ &= \tilde{T}_n^{\alpha_a}(y+u) \int_{I_k(u)} f(x) d\lambda(x) = 0. \end{aligned}$$

Therefore,

$$\int_{I \setminus I_k(u)} \sup_{n, a \in \mathbb{N}} \tilde{t}_n^{\alpha_a} f d\lambda = \int_{I \setminus I_k(u)} \sup_{n \geq 2^k, a \in \mathbb{N}} \tilde{t}_n^{\alpha_a} f d\lambda.$$

Recall that $B = |n|$. Then

$$\begin{aligned} A_n^{\alpha_a} T_n^{\alpha_a} &= \sum_{j=0}^{2^B-1} A_{2^B+n_{(B)}-j}^{\alpha_a-1} D_j \\ &= \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_{2^B-j} \end{aligned}$$

By Lemma 1 we have

$$\begin{aligned} A_n^{\alpha_a} T_n^{\alpha_a} &= D_{2^B} \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} - \omega_{2^B-1} \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} D_j. \end{aligned}$$

It is easy to have that $\frac{1}{A_n^{\alpha_a}} D_{2^B}(z) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} = 0$, for any $z \in I \setminus I_k$. This holds because $D_{2^B}(z) = 0$ for $B = |n| \geq k$ and $z \in I \setminus I_k$. By the help of the Abel transform we get:

$$\begin{aligned}
& \left| \sum_{j=0}^{2^B-1} A_{n(B)+j}^{\alpha_a-1} D_j \right| \\
&= \left| \sum_{j=0}^{2^B-1} (A_{n(B)+j}^{\alpha_a-1} - A_{n(B)+j+1}^{\alpha_a-1}) \sum_{i=0}^j D_i + A_{n(B)+2^B}^{\alpha_a-1} \sum_{i=0}^{2^B-1} D_i \right| \\
&= \left| (1-\alpha_a) \sum_{j=0}^{2^B-1} A_{n(B)+j}^{\alpha_a-1} \frac{j+1}{n(B)+j+1} K_j^1 + A_n^{\alpha_a-1} 2^B K_{2^B-1}^1 \right| \\
&= \left| (1-\alpha_a) \sum_{j=0}^{2^k-1} A_{n(B)+j}^{\alpha_a-1} \frac{j+1}{n(B)+j+1} K_j^1 + (1-\alpha_a) \sum_{j=2^k}^{2^B-1} A_{n(B)+j}^{\alpha_a-1} \frac{j+1}{n(B)+j+1} K_j^1 \right. \\
&\quad \left. + A_n^{\alpha_a-1} 2^B K_{2^B-1}^1 \right| \\
&\leq (1-\alpha_a) \sum_{j=0}^{2^k-1} A_{n(B)+j}^{\alpha_a-1} \frac{j+1}{n(B)+j+1} |K_j^1| \\
&\quad + (1-\alpha_a) \sum_{j=2^k}^{2^B-1} A_{n(B)+j}^{\alpha_a-1} \frac{j+1}{n(B)+j+1} |K_j^1| + A_n^{\alpha_a-1} 2^B |K_{2^B-1}^1| \\
&=: I + II + III.
\end{aligned}$$

These equalities above immediately proves inequality $|T_n^{\alpha_a}| \leq \tilde{T}_n^{\alpha_a}$.

Since for any $j < 2^k$ we have that the Fejér kernel $K_j^1(y+x)$ depends (with respect to x) only on coordinates $x_0, \dots, x_{k-1}$, then $\int_{I_k(u)} f(x) |K_j^1(y+x)| d\lambda(x) = |K_j^1(y+u)| \int_{I_k(u)} f(x) d\lambda(x) = 0$ gives $\int_{I_k(u)} f(x) I(y+x) d\lambda(x) = 0$.

On the other hand,

$$\begin{aligned}
II &= (1-\alpha_a) \sum_{j=2^k}^{2^B-1} A_{n(B)+j}^{\alpha_a-1} \frac{j+1}{n(B)+j+1} |K_j^1| \\
&\leq \sup_{j \geq 2^k} |K_j^1| (1-\alpha_a) \sum_{j=0}^n A_j^{\alpha_a-1} = A_n^{\alpha_a} (1-\alpha_a) \sup_{j \geq 2^k} |K_j^1|.
\end{aligned}$$

This by Lemma 3 in [6] gives

$$\int_{I \setminus I_k} \sup_{n \geq 2^k, a \in \mathbb{N}} \frac{1}{A_n^{\alpha_a}} II d\lambda \leq \int_{I \setminus I_k} \sup_{j \geq 2^k} |K_j^1| d\lambda \leq C.$$

The situation with III is similar. Namely,

$$\frac{A_n^{\alpha_a-1}n}{A_n^{\alpha_a}} = \frac{\alpha_a \cdot n}{(\alpha_a + n)} \leq \alpha_a < 1.$$

So, just as in the case of II we apply Lemma 3 in [6]

$$\int_{I \setminus I_k} \sup_{n \geq 2^k, a \in \mathbb{N}} \frac{1}{A_n^{\alpha_a}} III d\lambda \leq \int_{I \setminus I_k} \sup_{n \geq 2^k, a \in \mathbb{N}} |K_{2^{|n|-1}}^1| d\lambda \leq C.$$

Therefore, substituting $z = x + y \in I \setminus I_k$ (since $x \in I_k(u)$ and $y \in I \setminus I_k(u)$)

$$\begin{aligned} & \int_{I \setminus I_k(u)} \sup_{n \geq 2^k, a \in \mathbb{N}} \tilde{t}_n^{\alpha_a} f d\lambda \\ &= \int_{I \setminus I_k(u)} \sup_{n \geq 2^k, a \in \mathbb{N}} \left| \int_{I_k(u)} f(x) \tilde{T}_n^{\alpha_a}(y+x) d\lambda(x) \right| d\lambda(y) \\ &\leq \int_{I \setminus I_k(u)} \int_{I_k(u)} |f(x)| \sup_{n \geq 2^k, a \in \mathbb{N}} \frac{1}{A_n^{\alpha_a}} (II(y+x) + III(y+x)) d\lambda(x) \\ &= \int_{I_k(u)} |f(x)| \int_{I \setminus I_k} \sup_{n \geq 2^k, a \in \mathbb{N}} \frac{1}{A_n^{\alpha_a}} (II(z) + III(z)) d\lambda(z) d\lambda(x) \\ &\leq C \int_{I_k(u)} |f(x)| d\lambda(x). \end{aligned}$$

This completes the proof of Lemma 2. $\square$

A straightforward corollary of this lemma is:

Corollary 1. *Let $1 > \alpha_a > 0$. Then we have $\|T_n^{\alpha_a}\|_1 \leq \|\tilde{T}_n^{\alpha_a}\|_1 \leq C$, $\|t_n^{\alpha_a} f\|_1$, $\|\tilde{t}_n^{\alpha_a} f\|_1 \leq C \|f\|_1$ and $\|t_n^{\alpha_a} g\|_\infty$, $\|\tilde{t}_n^{\alpha_a} g\|_\infty \leq C \|g\|_\infty$ for all natural numbers a, n , where C is some absolute constant and $f \in L^1$, $g \in L^\infty$. That is, operators $\tilde{t}_n^{\alpha_a}, t_n^{\alpha_a}$ is of type (L^1, L^1) and (L^∞, L^∞) (uniformly in n).*

Proof. The proof is a straightforward consequence of Lemma 2 and an easy estimation below. Let $B = |n|$. Then

$$\begin{aligned} \|A_n^{\alpha_a} \tilde{T}_n^{\alpha_a}\|_1 &\leq \|D_{2^B}\|_1 \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \\ &+ (1-\alpha_a) \sum_{j=0}^{2^B-1} A_{n_{(B)}+j}^{\alpha_a-1} \frac{j+1}{n_{(B)}+j+1} \|K_j^1\|_1 + A_n^{\alpha_a-1} 2^B \|K_{2^B-1}^1\|_1. \end{aligned}$$

Then by $\|D_{2^B}\|_1 = 1$, $\|K_j^1\|_1 \leq C$ we complete the proof of Corollary 1. $\square$

Lemma 3. *Let n, N be any natural numbers and $0 < \alpha < 1$. Then we have*

$$\frac{A_n^\alpha}{A_N^\alpha} \leq 2 \left(\frac{n+1}{N} \right)^\alpha.$$

Proof. By definition we have

$$\frac{A_n^\alpha}{A_N^\alpha} = \left(1 - \frac{\alpha}{n+1+\alpha} \right) \cdots \left(1 - \frac{\alpha}{N+\alpha} \right) \leq \left(1 - \frac{\alpha}{n+2} \right) \cdots \left(1 - \frac{\alpha}{N+1} \right).$$

It is well-known that

$$\left(1 - \frac{\alpha}{i(n+1)+1} \right) \cdots \left(1 - \frac{\alpha}{(i+1)(n+1)} \right) \leq \left(1 - \frac{\alpha}{(i+1)(n+1)} \right)^{n+1} \leq (e^{-1})^{\frac{\alpha}{i+1}}$$

for every $n \in \mathbb{N}$. This gives

$$\begin{aligned} \left(1 - \frac{\alpha}{n+2} \right) \cdots \left(1 - \frac{\alpha}{N+1} \right) &\leq (e^{-1})^{\alpha \sum_{i=2}^{\lfloor \frac{N}{n+1} \rfloor} \frac{1}{i}} \\ &\leq (e^{-1})^{\alpha \log_e \lfloor \frac{N}{n+1} \rfloor - 1 + c} \\ &\leq 2 (e^{-1})^{\alpha \log_e \left(\frac{N}{n+1} \right)} = 2 \left(\frac{n+1}{N} \right)^\alpha, \end{aligned}$$

where $c \approx 0.5772$ is the Euler-Mascheroni constant. This completes the proof of Lemma 3. $\square$

Recall that the two variable function $P(n, \alpha) = \sum_{i=0}^{\infty} n_i 2^{i\alpha}$ for $n \in \mathbb{N}, \alpha \in \mathbb{R}$ and $K \in \mathbb{R}$ determines the set of natural numbers

$$\mathbb{N}_{\alpha, K} = \left\{ n \in \mathbb{N} : \frac{P(n, \alpha_n)}{n^{\alpha_n}} \leq K \right\}.$$

Let $n = 2^{h_s} + \cdots + 2^{h_0}$, where $h_s > \cdots > h_0 \geq 0$ are integers. That is, $|n| = h_s$. Let $n^{(j)} := 2^{h_j} + \cdots + 2^{h_0}$. This means $n = n^{(s)}$. Define the following kernel function and operators

$$\tilde{K}_n^{\alpha_n} := \tilde{T}_{n^{(s)}}^{\alpha_n} + \sum_{l=0}^s \left(\frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} D_{2^{h_l}} + \frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} \tilde{T}_{n^{l-1}}^{\alpha_n} \right)$$

and

$$\tilde{\sigma}_n^{\alpha_n} f := f * \tilde{K}_n^{\alpha_n}, \quad \tilde{\sigma}_*^{\alpha} f := \sup_{n \in \mathbb{N}_{\alpha, K}} |f * \tilde{K}_n^{\alpha}|.$$

In the sequel we prove that maximal operator $\tilde{\sigma}_*^{\alpha} f$ is quasi-local. This is the very base of the proof of the main results of this paper. That is, Theorem 1 and Theorem 2.

Lemma 4. Let $1 > \alpha_n > 0$, $f \in L^1(I)$ such that $\text{supp } f \subset I_k(u)$, $\int_{I_k(u)} f d\lambda = 0$ for some dyadic interval $I_k(u)$. Then we have

$$\int_{I \setminus I_k(u)} \tilde{\sigma}_*^\alpha f d\lambda \leq C_K \|f\|_1,$$

where constant C_K can depend only on K .

Proof. Recall that $n = 2^{h_s} + \dots + 2^{h_0}$, where $h_s > \dots > h_0 \geq 0$ are integers. That is, $|n| = h_s$. Let $n^{(j)} := 2^{h_j} + \dots + 2^{h_0}$. This means $n = n^{(s)}$. Use also the notation

$$\begin{aligned} & \tilde{K}_{n^{(s)}}^{\alpha_n} \\ &= \tilde{T}_{n^{(s)}}^{\alpha_n} + \sum_{l=0}^s \left(\frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} D_{2^{h_l}} + \frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} \tilde{T}_{n^{(l-1)}}^{\alpha_n} \right) \\ &=: G_1 + G_2 + G_3. \end{aligned}$$

Since $n^{(l-1)} < 2^{h_{l-1}+1}$, then by Lemma 3 we have

$$\frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} \leq 2 \left(\frac{n^{(l-1)} + 1}{n^{(s)}} \right)^{\alpha_n} \leq 2 \frac{2^{\alpha_n(h_{l-1}+1)}}{2^{\alpha_n h_s}} \leq C \frac{2^{h_{l-1}\alpha_n}}{n^{\alpha_n}}.$$

That is, by the above written we also have

$$\begin{aligned} & \int_{I \setminus I_k(u)} \sup_{n \in \mathbb{N}} \left| \int_{I_k(u)} f(x) G_2(y+x) d\lambda(x) \right| d\lambda(y) \\ & \int_{I \setminus I_k(u)} \sup_{n \in \mathbb{N}} \sum_{l=0}^s \frac{2^{h_{l-1}\alpha_n}}{n^{\alpha_n}} \left| \int_{I_k(u)} f(x) D_{2^{h_l}}(y+x) d\lambda(x) \right| d\lambda(y) = 0 \end{aligned}$$

since $f * D_{2^h} = 0$ for $h \leq k$ because of the $\mathcal{A}_k$ measurability of D_{2^h} and $\int f = 0$. Besides, for $h > k$ $D_{2^h}(y+x) = 0$ ($y+x \notin I_k$).

As a result of these estimations above and by the help of Lemma 2, that is the quasi-locality of operator $\tilde{t}_*^\alpha = \sup_{n,a \in \mathbb{N}} |\tilde{t}_n^{\alpha_a}|$ we conclude

$$\begin{aligned} & \int_{I \setminus I_k(u)} \sup_{n \in \mathbb{N}} \left| \int_{I_k(u)} f(x) (G_1(y+x) + G_3(y+x)) d\lambda(x) \right| d\lambda(y) \\ & \leq C_K \int_{I \setminus I_k(u)} \sup_{n,a \in \mathbb{N}} \left| \int_{I_k(u)} f(x) \tilde{T}_n^{\alpha_a}(y+x) d\lambda(x) \right| d\lambda(y) \\ & \leq C_K \|f\|_1. \end{aligned}$$

This completes the proof of Lemma 4. $\square$

Lemma 5. The operator $\tilde{\sigma}_*^\alpha$ is of type (L^∞, L^∞) ($\tilde{\sigma}_*^\alpha f := \sup_{n \in \mathbb{N}_{\alpha,K}} |\tilde{\sigma}_n^{\alpha_n} f|$).

Proof. By the help of the method of Lemma 4 and by Corollary 1 we have

$$\begin{aligned} \|\tilde{K}_n^{\alpha_n}\|_1 &= \|\tilde{K}_{n^{(s)}}^{\alpha_n}\|_1 \leq \|\tilde{T}_{n^{(s)}}^{\alpha_n}\|_1 + \sum_{l=0}^s \left(\frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} \|D_{2^{h_l}}\|_1 + \frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} \|\tilde{T}_{n^{l-1}}^{\alpha_n}\|_1 \right) \\ &\leq C + C \sum_{l=0}^s \frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} \leq C_K \end{aligned}$$

because $n \in \mathbb{N}_{\alpha, K}$. Hence $\tilde{\sigma}_*^\alpha$ is of type (L^∞, L^∞) (with constant C_K). This completes the proof of Lemma 5. $\square$

Now, we can prove the main tool in order to have Theorem 1 for operator $\sigma_*^\alpha f := \sup_{n \in \mathbb{N}_{\alpha, K}} |\sigma_n^{\alpha_n} f|$.

Lemma 6. *The operators $\tilde{\sigma}_*^\alpha$ and σ_*^α are of weak type (L^1, L^1) .*

Proof. First, we prove Lemma 6 for operator $\tilde{\sigma}_*^\alpha$. We apply the Calderon-Zygmund decomposition lemma [9]. That is, let $f \in L^1$ and $\|f\|_1 < \delta$. Then there is a decomposition:

$$f = f_0 + \sum_{j=1}^{\infty} f_j$$

such that $\|f_0\|_\infty \leq C\delta$, $\|f_0\|_1 \leq C\|f\|_1$ and $I^j = I_{k_j}(u^j)$ are disjoint dyadic intervals for which

$$\text{supp } f_j \subset I^j, \int_{I^j} f_j d\lambda = 0, |F| \leq \frac{C\|f_1\|}{\delta}$$

($u^j \in I$, $k_j \in N$, $j \in P$), where $F = \cup_{i=1}^{\infty} I^j$. By the σ -sublinearity of the maximal operator with an appropriate constant C_K we have

$$\lambda(\tilde{\sigma}_*^\alpha f > 2C_K\delta) \leq \lambda(\tilde{\sigma}_*^\alpha f_0 > C_K\delta) + \lambda(\tilde{\sigma}_*^\alpha (\sum_{i=1}^{\infty} f_i) > C_K\delta) := I + II.$$

Since by Lemma 5 $\|\tilde{\sigma}_*^\alpha f_0\|_\infty \leq C_K\|f_0\|_\infty \leq C_K\delta$ then we have $I = 0$. So,

$$\begin{aligned} \lambda(\tilde{\sigma}_*^\alpha (\sum_{i=1}^{\infty} f_i) > C_K\delta) &\leq |F| + \lambda(\bar{F} \cap \{\tilde{\sigma}_*^\alpha (\sum_{i=1}^{\infty} f_i) > C_K\delta\}) \\ &\leq \frac{C_K\|f\|_1}{\delta} + \frac{C_K}{\delta} \sum_{i=1}^{\infty} \int_{I \setminus I^j} \tilde{\sigma}_*^\alpha f_j d\lambda =: \frac{C_K\|f\|_1}{\delta} + \frac{C_K}{\delta} \sum_{i=1}^{\infty} III_j, \end{aligned}$$

where

$$\begin{aligned} III_j &:= \int_{I \setminus I^j} \tilde{\sigma}_*^\alpha f_j d\lambda \\ &\leq \int_{I \setminus I_{k_j}(u^j)} \sup_{n \in N_{\alpha, K}} \left| \int_{I_{k_j}(u^j)} f_j(x) \tilde{K}_n^{\alpha_n}(y+x) d\lambda(x) \right| d\lambda(y). \end{aligned}$$

The forthcoming estimation of III_j is given by the help Lemma 4

$$III_j \leq C_K \|f_j\|_1.$$

That is, operator $\tilde{\sigma}_*^\alpha$ is of weak type (L^1, L^1) . Next, we prove the estimation

$$|K_n^{\alpha_n}| \leq \tilde{K}_n^{\alpha_n}. \quad (1)$$

To prove (1) recall again that $n = 2^{h_s} + \dots + 2^{h_0}$, where $h_s > \dots > h_0 \geq 0$ are integers. Since $n = 2^{h_s} + n^{(s-1)}$, then we have

$$\begin{aligned} \sum_{j=2^{h_s}}^{2^{h_s} + n^{(s-1)}} A_{n^{(s-1)} + 2^{h_s} - j}^{\alpha_n - 1} D_j &= \sum_{k=0}^{n^{(s-1)}} A_{n^{(s-1)} - k}^{\alpha_n - 1} D_{2^{h_s} + k} \\ &= D_{2^{h_s}} \sum_{k=0}^{n^{(s-1)}} A_{n^{(s-1)} - k}^{\alpha_n - 1} + \omega_{2^{h_s}} \sum_{k=0}^{n^{(s-1)}} A_{n^{(s-1)} - k}^{\alpha_n - 1} D_k \\ &= D_{2^{h_s}} A_{n^{(s-1)}}^{\alpha_n} + \omega_{2^{h_s}} A_{n^{(s-1)}}^{\alpha_n} K_{n^{(s-1)}}^{\alpha_n}. \end{aligned}$$

So, by the help of the equalities above we get

$$K_{n^{(s)}}^{\alpha_n} = T_{n^{(s)}}^{\alpha_n} + \frac{A_{n^{(s-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} \left(D_{2^{h_s}} + \omega_{2^{h_s}} K_{n^{(s-1)}}^{\alpha_n} \right).$$

Apply this last formula recursively and Lemma 2. (Note that $n^{(-1)} = 0, T_0^{\alpha_n} = K_0^{\alpha_n} = 0, A_0^{\alpha_n} = 1$.)

$$\begin{aligned} |K_n^{\alpha_n}| &= |K_{n^{(s)}}^{\alpha_n}| \leq |T_{n^{(s)}}^{\alpha_n}| + \sum_{l=0}^s \left(\prod_{j=l}^s \frac{A_{n^{(j-1)}}^{\alpha_n}}{A_{n^{(j)}}^{\alpha_n}} D_{2^{h_l}} + \prod_{j=l}^s \frac{A_{n^{(j-1)}}^{\alpha_n}}{A_{n^{(j)}}^{\alpha_n}} |T_{n^{l-1}}^{\alpha_n}| \right) \\ &= |T_{n^{(s)}}^{\alpha_n}| + \sum_{l=0}^s \left(\frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} D_{2^{h_l}} + \frac{A_{n^{(l-1)}}^{\alpha_n}}{A_{n^{(s)}}^{\alpha_n}} |T_{n^{l-1}}^{\alpha_n}| \right) \\ &\leq \tilde{K}_{n^{(s)}}^{\alpha_n} = \tilde{K}_n^{\alpha_n}. \end{aligned}$$

This completes the proof of inequality (1). This inequality gives that the operator σ_*^α is also of weak type (L^1, L^1) since

$$\lambda(\sigma_*^\alpha f > 2C_K\delta) \leq \lambda(\tilde{\sigma}_*^\alpha |f| > 2C_K\delta) \leq C_K \frac{\|f\|_1}{\delta} = C_K \frac{\|f\|_1}{\delta}.$$

This completes the proof of Lemma 6. $\square$

Proof of Theorem 1. Let $P \in \mathbf{P}$ be a polynomial where $P(x) = \sum_{i=0}^{2^k-1} c_i \omega_i$. Then for all natural number $n \geq 2^k$, $n \in \mathbb{N}_{\alpha,K}$ we have that $S_n P \equiv P$. Consequently, the statement $\sigma_n^{\alpha_n} P \rightarrow P$ holds everywhere (of course not only for restricted $n \in \mathbb{N}_{\alpha,K}$). Now, let $\epsilon, \delta > 0$, $f \in L^1$. Let $P \in \mathbf{P}$ be a polynomial such that $\|f - P\|_1 < \delta$. Then

$$\begin{aligned} & \lambda\left(\overline{\lim}_{n \in \mathbb{N}_{\alpha,K}} |\sigma_n^{\alpha_n} f - f| > \epsilon\right) \\ & \leq \lambda\left(\overline{\lim}_{n \in \mathbb{N}_{\alpha,K}} |\sigma_n^{\alpha_n} (f - P)| > \frac{\epsilon}{3}\right) + \lambda\left(\overline{\lim}_{n \in \mathbb{N}_{\alpha,K}} |\sigma_n^{\alpha_n} P - P| > \frac{\epsilon}{3}\right) \\ & \quad + \lambda\left(\overline{\lim}_{n \in \mathbb{N}_{\alpha,K}} |P - f| > \frac{\epsilon}{3}\right) \\ & \leq \lambda\left(\sup_{n \in \mathbb{N}_{\alpha,K}} |\sigma_n^{\alpha_n} (f - P)| > \frac{\epsilon}{3}\right) + 0 + \frac{3}{\epsilon} \|P - f\|_1 \leq C_K \|P - f\|_1 \frac{3}{\epsilon} \leq \frac{C_K}{\epsilon} \delta \end{aligned}$$

because σ_*^α is of weak type (L^1, L^1) (with any fixed $K > 0$). This holds for all $\delta > 0$. That is, for an arbitrary $\epsilon > 0$ we have

$$\lambda\left(\overline{\lim}_{n \in \mathbb{N}_{\alpha,K}} |\sigma_n^{\alpha_n} f - f| > \epsilon\right) = 0$$

and consequently we also have

$$\lambda\left(\overline{\lim}_{n \in \mathbb{N}_{\alpha,K}} |\sigma_n^{\alpha_n} f - f| > 0\right) = 0.$$

This finally gives

$$\begin{aligned} & \overline{\lim}_{n \in \mathbb{N}_{\alpha,K}} |\sigma_n^{\alpha_n} f - f| = 0 \text{ a.e.}, \\ & \sigma_n^{\alpha_n} f \rightarrow f \text{ a.e.} \quad (n \in \mathbb{N}_{\alpha,K}). \end{aligned}$$

This completes the proof of Theorem 1. $\square$

Proof of Theorem 2. Inequality (1), Lemma 5 and Lemma 6 by the interpolation theorem of Marcinkiewicz [9] give that the operator σ_*^α is of type (L^p, L^p) for all $1 < p \leq \infty$. In the sequel we prove that operator $\tilde{\sigma}_*^\alpha f = \sup_{n \in \mathbb{N}_{\alpha,K}} |f * \tilde{K}_n^\alpha|$ is of type (H, L) .

Let a be an atom ($a \neq 1$ can be supposed), $\text{supp } a \subset I_k(x)$, $\|a\|_\infty \leq 2^k$ for some $k \in \mathbb{N}$ and $x \in I$. Then, $n < 2^k$, $n \in \mathbb{N}_{\alpha,K}$ implies $a * \tilde{K}_n^\alpha = 0$ because $\tilde{K}_n^\alpha$ is $\mathcal{A}_k$ measurable for $n < 2^k$ and $\int_{I_k(x)} a(t) d\lambda(t) = 0$. That is,

$$\tilde{\sigma}_*^\alpha a = \sup_{\mathbb{N}_{\alpha,K} \ni n \geq 2^k} |\tilde{\sigma}_n^{\alpha_n} f|.$$

By the help Lemma 4 we have

$$\begin{aligned}
 \int_{I \setminus I_k(x)} \tilde{\sigma}_*^\alpha a \, d\lambda &= \int_{I \setminus I_k(x)} \sup_{\mathbb{N}_{\alpha, K} \ni n \geq 2^k} \left| \int_{I_k(x)} a(y) \tilde{K}_n^{\alpha_n}(z+y) d\lambda(y) \right| d\lambda(z) \\
 &\leq C_K \int_{I_k(x)} |a(y)| d\lambda(y) \\
 &\leq C_K \|a\|_1 \\
 &\leq C_K.
 \end{aligned}$$

Since the operator $\tilde{\sigma}_*^\alpha$ is of type (L^2, L^2) (i.e. $\|\tilde{\sigma}_*^\alpha f\|_2 \leq C_K \|f\|_2$ for all $f \in L^2(I)$), then we have

$$\begin{aligned}
 \|\tilde{\sigma}_*^\alpha a\|_1 &= \int_{I \setminus I_k(x)} \tilde{\sigma}_*^\alpha a + \int_{I_k(x)} \tilde{\sigma}_*^\alpha a \\
 &\leq C_K + |I_k(x)|^{\frac{1}{2}} \|\tilde{\sigma}_*^\alpha a\|_2 \\
 &\leq C_K + C_K 2^{\frac{-k}{2}} \|a\|_2 \\
 &\leq C_K + C_K 2^{\frac{-k}{2}} 2^{\frac{k}{2}} \\
 &\leq C_K.
 \end{aligned}$$

That is $\|\tilde{\sigma}_*^\alpha a\|_1 \leq C_K$ and consequently the σ -sublinearity of $\tilde{\sigma}_*^\alpha$ gives

$$\begin{aligned}
 \|\tilde{\sigma}_*^\alpha f\|_1 &\leq \sum_{i=0}^{\infty} |\lambda_i| \|\tilde{\sigma}_*^\alpha a_i\|_1 \\
 &\leq C_K \sum_{i=0}^{\infty} |\lambda_i| \\
 &\leq C_K \|f\|_H
 \end{aligned}$$

for all $\sum_{i=0}^{\infty} \lambda_i a_i \in H$. That is, the operator $\tilde{\sigma}_*^\alpha$ is of type (H, L) . This by inequality (1) and then by $\|\sigma_*^\alpha f\|_1 \leq \|\tilde{\sigma}_*^\alpha |f|\|_1 \leq C_K \|f\|_H$ completes the proof of Theorem 2. $\square$

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Authors' addresses

Anas Ahmad Abu Joudeh

Institute of Mathematics, University of Debrecen, H-4010 Debrecen, Pf. 12, Hungary

E-mail address: anas.abujoudeh@mailbox.unideb.edu.hu, mr.anas-judeh@yahoo.com

György Gát

Institute of Mathematics, University of Debrecen, H-4010 Debrecen, Pf. 12, Hungary

E-mail address: gat.gyorgy@science.unideb.hu



BLENDING TYPE APPROXIMATION BY GENERALIZED BERNSTEIN-DURRMEYER TYPE OPERATORS

ARUN KAJLA AND TUNCER ACAR

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Abstract. In this article we construct a Durrmeyer modification of the operators introduced by Chen et al. in [10] based on a non-negative real parameter. We establish local approximation, global approximation, Voronovskaja type asymptotic theorem. The rate of convergence for differentiable functions whose derivatives are of bounded variation is also obtained.

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1. INTRODUCTION

The approximation theory by linear positive operators investigates how the functions can be best approximated by simpler functions. The most famous basic result for convergence of linear positive operators is due to Weierstrass who introduced an important theorem named Weierstrass' approximation theorem. At last in 1912 Bernstein introduced the most famous algebraic polynomials $B_n(f; x)$ in approximation theory in order to give a constructive proof of Weierstrass' theorem, which are given by

$$B_n(f; x) = \sum_{k=0}^n p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0, 1],$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and he proved that if $f \in C[0, 1]$ then $B_n(f; x)$ converges uniformly to $f(x)$ in $[0, 1]$.

The Bernstein operators have been used in many branches of mathematics and computer science. Due to their useful structure, Bernstein polynomials and their modifications have been intensively studied. Among other papers, we refer the readers to [3, 7, 9, 13, 23, 25].

For $f \in C[0, 1]$, Chen et al. in [10] introduced a generalization of the Bernstein operators based on a non-negative parameter α ($0 \leq \alpha \leq 1$) as follows:

$$T_n^{(\alpha)}(f; x) = \sum_{k=0}^n p_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right), \quad x \in [0, 1] \quad (1.1)$$

where

$$p_{n,k}^{(\alpha)}(x) = \left[\binom{n-2}{k} (1-\alpha)x + \binom{n-2}{k-2} (1-\alpha)(1-x) + \binom{n}{k} \alpha x(1-x) \right] x^{k-1} (1-x)^{n-k-1} \quad (1.2)$$

and $n \geq 2$. They proved the rate of convergence, Voronovskaja type asymptotic formula and shape preserving properties for these operators. For the special case, $\alpha = 1$, these operators reduce the well-known Bernstein operators.

The Durrmeyer type modification of the operators is a method to approximate the Lebesgue integrable functions. For this aim, many researchers have studied in this direction. Gupta and Rassias [18] introduced the Lupas-Durrmeyer type operators based on Polya distribution and established asymptotic approximation, local and global results. Goyal et al. [14] considered a one parameter family of Baskakov-Szász type operators and studied quantitative convergence theorems for these operators. Gupta et al. [16] introduced hybrid operators involving inverse Polya-Eggenberger distribution and studied degree of approximation of these operators which include global approximation and uniform convergence. Very recently, Acu and Gupta [15] defined a summation-integral type operators depending on two parameters and discussed some approximation results e.g. local approximation, Voronovskaja type asymptotic theorem and weighted approximation of these operators. In the literature survey, several authors have studied the approximation behavior of mixed hybrid operators (cf. [1, 2, 4–6, 8, 17, 19–21]).

Inspired by their work, for $f \in C[0, 1]$ we define the following Durrmeyer type modification of the operators (1.1) as:

$$D_n^{(\alpha)}(f; x) = (n+1) \sum_{k=0}^n p_{n,k}^{(\alpha)}(x) \int_0^1 p_{n,k}(t) f(t) dt, \quad x \in [0, 1]. \quad (1.3)$$

The purpose of this paper is to study the Voronovskaja type theorem, local approximation, pointwise estimates and global approximation results for these operators (1.3). The rate of convergence for differentiable functions whose derivatives are of bounded variation is also obtained.

2. BASIC RESULTS

In what follows let $\|\cdot\|$ denote the uniform norm on $C[0, 1]$.

Let $e_i = t^i, i \in \mathbb{N} \cup \{0\}$. By simple computation, we get

$$\int_0^1 p_{n,k}(t) t^i dt = \frac{n!(k+i)!}{k!(n+i+1)!}. \quad (2.1)$$

In order to prove the main results, we will show some lemmas in this section. We need the following auxiliary results.

Lemma 1 ([10]). *For the operators $T_n^{(\alpha)}(f; x)$, we have*

$$\begin{aligned} \text{(i)} \quad & T_n^{(\alpha)}(e_0; x) = 1; \\ \text{(ii)} \quad & T_n^{(\alpha)}(e_1; x) = x; \\ \text{(iii)} \quad & T_n^{(\alpha)}(e_2; x) = x^2 + \frac{(n+2(1-\alpha))}{n^2} x(1-x); \\ \text{(iv)} \quad & T_n^{(\alpha)}(e_3; x) = x^3 + \frac{3(n+2(1-\alpha))}{n^2} x^2(1-x) \\ & + \frac{(n+6(1-\alpha))}{n^3} x(1-x)(1-2x); \\ \text{(v)} \quad & T_n^{(\alpha)}(e_4; x) = x^4 + \frac{6(n+2(1-\alpha))}{n^2} x^3(1-x) \\ & + \frac{4(n+6(1-\alpha))}{n^3} x^2(1-x)(1-2x) \\ & + \frac{((3n(n-2) + 12(n-6)(1-\alpha))x(1-x) + (n+14(1-\alpha)))}{n^4} x(1-x). \end{aligned}$$

Lemma 2. *For the operators $D_n^{(\alpha)}(f; x)$, we have*

$$\begin{aligned} \text{(i)} \quad & D_n^{(\alpha)}(e_0; x) = 1; \\ \text{(ii)} \quad & D_n^{(\alpha)}(e_1; x) = x + \frac{1-2x}{(n+2)}; \\ \text{(iii)} \quad & D_n^{(\alpha)}(e_2; x) = x^2 + \frac{2x^2(\alpha-3n-4)}{(n+2)(n+3)} + \frac{2x(2n-\alpha+1)}{(n+2)(n+3)} + \frac{2}{(n+2)(n+3)}; \\ \text{(iv)} \quad & D_n^{(\alpha)}(e_3; x) = x^3 + \frac{6x^3(-n(5+2n-\alpha)-2(1+\alpha))}{(n+2)(n+3)(n+4)} \\ & + \frac{3x^2(n(3n-2\alpha-1)+10(\alpha-1))}{(n+2)(n+3)(n+4)} + \frac{18x(n-\alpha+1)}{(n+2)(n+3)(n+4)} \\ & + \frac{6}{(n+2)(n+3)(n+4)}; \\ \text{(v)} \quad & D_n^{(\alpha)}(e_4; x) = x^4 + \frac{x^4(-4(n+3)(16+n(3+5n))+12\alpha(n-3)(n-2))}{(n+2)(n+3)(n+4)(n+5)} \\ & + \frac{4x^3(n-2)(n(4n-3\alpha-1)+33(\alpha-1))}{(n+2)(n+3)(n+4)(n+5)} \end{aligned}$$

$$\begin{aligned}
& + \frac{24x^2(n+3n^2+14(\alpha-1)-4n\alpha)}{(n+2)(n+3)(n+4)(n+5)} + \frac{48x(2n-3\alpha+3)}{(n+2)(n+3)(n+4)(n+5)} \\
& + \frac{24}{(n+2)(n+3)(n+4)(n+5)}.
\end{aligned}$$

Proof. This lemma follows easily applying Lemma 1 and relation (2.1). Hence the details are omitted. $\square$

Lemma 3. From Lemma 2, we get

$$\begin{aligned}
\text{(i)} \quad D_n^{(\alpha)}(t-x; x) &= \frac{1-2x}{n+2}; \\
\text{(ii)} \quad D_n^{(\alpha)}((t-x)^2; x) &= \frac{2x(1-x)(n-\alpha-2)}{(n+2)(n+3)} + \frac{2}{(n+2)(n+3)}; \\
\text{(iii)} \quad D_n^{(\alpha)}((t-x)^4; x) &= \frac{12x^3(x-2)(n(n-2\alpha-19)+46\alpha-36)}{(n+2)(n+3)(n+4)(n+5)} \\
& + \frac{12x^2(n(n-2\alpha-25)+58\alpha-38)}{(n+2)(n+3)(n+4)(n+5)} + \frac{24x(3n-6\alpha+1)}{(n+2)(n+3)(n+4)(n+5)} \\
& + \frac{24}{(n+2)(n+3)(n+4)(n+5)}.
\end{aligned}$$

Lemma 4. For $f \in C[0, 1]$, we have $\|D_n^{(\alpha)}(f; \cdot)\| \leq \|f\|$.

Proof. From definition (1.3) and Lemma 2, we have

$$\begin{aligned}
\|D_n^{(\alpha)}(f; \cdot)\| &\leq (n+1) \sum_{k=0}^n p_{n,k}^{(\alpha)}(x) \int_0^1 p_{n,k}(t) |f(t)| dt \\
&\leq \|f\| D_n^{(\alpha)}(e_0; x) = \|f\|.
\end{aligned}$$

$\square$

Lemma 5. For $n \in \mathbb{N}$, we have

$$D_n^{(\alpha)}((t-x)^2; x) \leq \frac{2\gamma_n^2(x)}{(n+2)},$$

where $\gamma_n^2(x) = \varphi^2(x) + \frac{1}{(n+2)}$ and $\varphi^2(x) = x(1-x)$.

Proof. This result is obtained by straightforward computation, but the details are omitted. $\square$

Remark 1. Let $\Theta_n^{\alpha,m} := D_n^{(\alpha)}((t-x)^m; x)$, $m = 1, 2, 4$ be the central moments of $D_n^{(\alpha)}$, we get

$$\begin{aligned}
\lim_{n \rightarrow \infty} n \Theta_n^{\alpha,1}(x) &= 1-2x, \\
\lim_{n \rightarrow \infty} n \Theta_n^{\alpha,2}(x) &= 2x(1-x),
\end{aligned}$$

$$\lim_{n \rightarrow \infty} n^2 \Theta_n^{\alpha,4}(x) = 12x^2(1-x)^2.$$

3. BASIC CONVERGENCE THEOREM

Theorem 1. Suppose that $f \in C[0, 1]$ and $\alpha \in [0, 1]$. Then $\lim_{n \rightarrow \infty} D_n^{(\alpha)}(f; x) = f(x)$, uniformly in $[0, 1]$.

Proof. Applying Lemma 2, $D_n^{(\alpha)}(e_0; x) = 1$, $D_n^{(\alpha)}(e_1; x) \rightarrow x$, $D_n^{(\alpha)}(e_2; x) \rightarrow x^2$ as $n \rightarrow \infty$, uniformly in $[0, 1]$. By the well-known Bohman-Korovkin theorem, it follows that $D_n^{(\alpha)}(f; x) \rightarrow f(x)$ as $n \rightarrow \infty$, uniformly in $[0, 1]$. $\square$

4. LOCAL APPROXIMATION

The K -functional is given by :

$$K_2(f, \delta) = \inf\{\|f - g\| + \delta\|g''\| : g \in W^2\} \quad (\delta > 0),$$

where $W^2 = \{g : g'' \in C[0, 1]\}$ and $\|\cdot\|$ is the uniform norm on $C[0, 1]$. By [11] there exists a positive constant $M > 0$ such that

$$K_2(f, \delta) \leq M\omega_2(f, \sqrt{\delta}), \quad (4.1)$$

where the second order modulus of continuity for $f \in C[0, 1]$ is defined as

$$\omega_2(f, \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x, x+2h \in [0, 1]} |f(x+2h) - 2f(x+h) + f(x)|.$$

We define the usual modulus of continuity for $f \in C[0, 1]$ as

$$\omega(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x, x+h \in [0, 1]} |f(x+h) - f(x)|.$$

Theorem 2. For the operators $D_n^{(\alpha)}$, there exists a constant $M > 0$ such that

$$|D_n^{(\alpha)}(f; x) - f(x)| \leq M\omega_2\left(f, (n+2)^{-1/2}\gamma_n(x)\right) + \omega\left(f, \left|\frac{1-2x}{n+2}\right|\right),$$

where $f \in C[0, 1]$, $\alpha \in [0, 1]$, $\gamma_n^2(x) = \varphi^2(x) + \frac{1}{(n+2)}$ and $x \in [0, 1]$.

Proof. We define the auxiliary operators as follows:

$$\overline{D}_n^{(\alpha)}(f; x) = D_n^{(\alpha)}(f; x) + f(x) - f\left(\frac{nx+1}{n+2}\right).$$

Then, we can easily check that

$$\overline{D}_n^{(\alpha)}(1; x) = 1 \quad \text{and} \quad \overline{D}_n^{(\alpha)}(t; x) = x.$$

Let $g \in W^2$ and $t \in [0, 1]$. By Taylor's expansion we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Applying the operator $\overline{D}_n^{(\alpha)}$ on both sides of the above relation, we may write

$$\begin{aligned}\overline{D}_n^{(\alpha)}(g; x) &= g(x) + \overline{D}_n^{(\alpha)}\left(\int_x^t (t-u)g''(u)du\right) \\ &= g(x) + D_n^{(\alpha)}\left(\int_x^t (t-u)g''(u)du, x\right) \\ &\quad - \int_x^{\frac{nx+1}{(n+2)}} \left(\frac{nx+1}{n+2} - u\right) g''(u)du.\end{aligned}$$

Hence

$$\begin{aligned}|\overline{D}_n^{(\alpha)}(g; x) - g(x)| &\leq D_n^{(\alpha)}\left(\left|\int_x^t |t-u||g''(u)|du\right|, x\right) \\ &\quad + \left|\int_x^{\frac{nx+1}{(n+2)}} \left|\frac{nx+1}{n+2} - u\right| |g''(u)|du\right| \\ &\leq \left\{D_n^{(\alpha)}((t-x)^2; x) + \left(\frac{nx+1}{n+2} - x\right)^2\right\} \|g''\| \\ &= \left\{D_n^{(\alpha)}((t-x)^2; x) + \left(\frac{1-2x}{n+2}\right)^2\right\} \|g''\|. \quad (4.2)\end{aligned}$$

From Lemma 5, we have

$$\begin{aligned}D_n^{(\alpha)}((t-x)^2; x) + \left(\frac{1-2x}{n+2}\right)^2 &\leq \frac{2}{(n+2)} \gamma_n^2(x) + \left(\frac{1-2x}{n+2}\right)^2 \\ &\leq \frac{2}{(n+2)} \gamma_n^2(x) + \frac{1}{(n+2)^2} \\ &\leq \frac{3}{(n+2)} \gamma_n^2(x).\end{aligned} \quad (4.3)$$

Thus, by (4.2) we have

$$|\overline{D}_n^{(\alpha)}(g; x) - g(x)| \leq \frac{3}{(n+2)} \gamma_n^2(x) \|g''\|, \quad (4.4)$$

where $x \in [0, 1]$. Furthermore, by Lemma 4, we have

$$|\overline{D}_n^{(\alpha)}(f; x)| \leq 3\|f\|, \quad (4.5)$$

for all $f \in C[0, 1]$ and $x \in [0, 1]$.

Now, for $f \in C[0, 1]$ and $g \in W^2$, using (4.4) and (4.5) we obtain that

$$|D_n^{(\alpha)}(f; x) - f(x)| \leq \left| \overline{D}_n^{(\alpha)}(f; x) - f(x) + f\left(\frac{nx+1}{n+2}\right) - f(x) \right|$$

$$\begin{aligned} &\leq |\overline{D}_n^{(\alpha)}(f-g;x)| + |\overline{D}_n^{(\alpha)}(g;x) - g(x)| + |g(x) - f(x)| \\ &\quad + \left| f\left(\frac{nx+1}{n+2}\right) - f(x) \right| \\ &\leq 4\|f-g\| + \frac{3}{(n+2)}\gamma_n^2(x)\|g''\| + \omega\left(f, \left|\frac{1-2x}{n+2}\right|\right). \end{aligned}$$

Taking the infimum on the right hand side over all $g \in W^2$, we get

$$|D_n^{(\alpha)}(f;x) - f(x)| \leq 4K_2\left(f, \frac{1}{(n+2)}\gamma_n^2(x)\right) + \omega\left(f, \left|\frac{1-2x}{n+2}\right|\right).$$

Now considering the relation (4.1), we obtain

$$|D_n^{(\alpha)}(f;x) - f(x)| \leq M\omega_2\left(f, (n+2)^{-1/2}\gamma_n(x)\right) + \omega\left(f, \left|\frac{1-2x}{n+2}\right|\right),$$

which completes the proof. $\square$

Let $a_1 \geq 0$, $a_2 > 0$ and let us now consider the Lipschitz-type space [24]:

$$Lip_M^*(\rho) := \left\{ f \in C[0,1] : |f(t) - f(x)| \leq M \frac{|t-x|^\rho}{(t+a_1x^2+a_2x)^{\frac{\rho}{2}}}; x, t \in (0,1) \right\},$$

where $\rho \in (0,1]$.

Theorem 3. Let $f \in Lip_M^*(\rho)$. Then, for all $x \in (0,1]$, we have

$$|D_n^{(\alpha)}(f;x) - f(x)| \leq M \left(\frac{\Theta_n^{\alpha,2}(x)}{a_1x^2+a_2x} \right)^{\frac{\rho}{2}},$$

where $\Theta_n^{\alpha,2}(x) = D_n^{(\alpha)}((t-x)^2; x)$.

Proof. First, we show the result for the case $\rho = 1$. We may write

$$\begin{aligned} |D_n^{(\alpha)}(f;x) - f(x)| &\leq (n+1) \sum_{k=0}^n p_{n,k}^{(\alpha)}(x) \int_0^1 |f(t) - f(x)| dt \\ &\leq M(n+1) \sum_{k=0}^n p_{n,k}^{(\alpha)}(x) \int_0^1 p_{n,k}(t) \frac{|t-x|}{\sqrt{(t+a_1x^2+a_2x)}} dt. \end{aligned}$$

Using the fact that $\frac{1}{\sqrt{(t+a_1x^2+a_2x)}} < \frac{1}{\sqrt{(a_1x^2+a_2x)}}$ and the Cauchy-Schwarz inequality, we have

$$|D_n^{(\alpha)}(f;x) - f(x)| \leq \frac{M(n+1)}{\sqrt{(a_1x^2+a_2x)}} \sum_{k=0}^n p_{n,k}^{(\alpha)}(x) \int_0^1 p_{n,k}(t) |t-x| dt$$

$$= \frac{M}{\sqrt{(a_1x^2 + a_2x)}} D_n^{(\alpha)}(|t-x|; x) \leq M \left(\sqrt{\frac{\Theta_n^{\alpha,2}(x)}{a_1x^2 + a_2x}} \right),$$

hence the result is obtained for $\rho = 1$. Now, we prove the theorem for the case $0 < \rho < 1$. By the Hölder's inequality with $p = \frac{1}{\rho}$ and $q = \frac{1}{1-\rho}$, we get

$$\begin{aligned} |D_n^{(\alpha)}(f; x) - f(x)| &\leq (n+1) \sum_{k=0}^n p_{n,k}^{(\alpha)}(x) \int_0^1 p_{n,k}(t) |f(t) - f(x)| dt \\ &\leq \left\{ \sum_{k=0}^n p_{n,k}^{(\alpha)}(x) \left((n+1) \int_0^1 p_{n,k}(t) |f(t) - f(x)| dt \right)^{\frac{1}{\rho}} \right\}^{\rho} \\ &\leq \left\{ (n+1) \sum_{k=0}^n p_{n,k}^{(\alpha)}(x) \int_0^1 p_{n,k}(t) |f(t) - f(x)|^{\frac{1}{\rho}} dt \right\}^{\rho} \\ &\leq M \left\{ (n+1) \sum_{k=0}^n p_{n,k}^{(\alpha)}(x) \int_0^1 p_{n,k}(t) \frac{|t-x|}{\sqrt{(t+a_1x^2+a_2x)}} dt \right\}^{\rho} \\ &\leq \frac{M}{(a_1x^2 + a_2x)^{\frac{\rho}{2}}} \left\{ (n+1) \sum_{k=0}^n p_{n,k}^{(\alpha)}(x) \int_0^1 p_{n,k}(t) |t-x| dt \right\}^{\rho} \\ &\leq \frac{M}{(a_1x^2 + a_2x)^{\frac{\rho}{2}}} (D_n^{(\alpha)}(|t-x|; x))^{\rho} \leq M \left(\frac{\Theta_n^{\alpha,2}(x)}{a_1x^2 + a_2x} \right)^{\frac{\rho}{2}}. \end{aligned}$$

Thus, the proof is completed. $\square$

Next, we study the local direct estimate of the operators defined in (1.3) applying the Lipschitz-type maximal function of order ρ given by Lenze [22] as

$$\widetilde{\omega}_{\rho}(f, x) = \sup_{t \neq x, t \in [0,1]} \frac{|f(t) - f(x)|}{|t-x|^{\rho}}, \quad x \in [0,1] \text{ and } \rho \in (0,1]. \quad (4.6)$$

Theorem 4. Let $f \in C[0,1]$ and $0 < \rho \leq 1$. Then, for all $x \in [0,1]$, we have

$$|D_n^{(\alpha)}(f; x) - f(x)| \leq \widetilde{\omega}_{\rho}(f, x) \left\{ \frac{2}{(n+2)} \gamma_n^2(x) \right\}^{\frac{\rho}{2}}.$$

Proof. In view of (4.6), we have

$$|f(t) - f(x)| \leq \widetilde{\omega}_{\rho}(f, x) |t-x|^{\rho}$$

and

$$|D_n^{(\alpha)}(f; x) - f(x)| \leq D_n^{(\alpha)}(|f(t) - f(x)|; x) \leq \widetilde{\omega}_{\rho}(f, x) D_n^{(\alpha)}(|t-x|^{\rho}; x).$$

Now, applying the Hölder's inequality with $p = \frac{2}{\rho}$ and $\frac{1}{q} = 1 - \frac{1}{p}$, we have

$$\begin{aligned} |D_n^{(\alpha)}(f; x) - f(x)| &\leq \widetilde{\omega}_\rho(f, x) D_n^{(\alpha)}((t-x)^2; x)^{\frac{\rho}{2}} \\ &\leq \widetilde{\omega}_\rho(f, x) \left\{ \frac{2}{(n+2)} \gamma_n^2(x) \right\}^{\frac{\rho}{2}}. \end{aligned}$$

□

5. GLOBAL APPROXIMATION

Let $f \in C[0, 1]$ and $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0, 1]$. The second order Ditzian-Totik modulus of smoothness and corresponding K -functional are given by, respectively,

$$\begin{aligned} \omega_2^\varphi(f, \sqrt{\delta}) &= \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \pm h\varphi(x) \in [0, 1]} |f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x))|, \\ \tilde{K}_{2, \varphi(x)}(f, \delta) &= \inf\{ \|f - g\| + \delta \|\varphi^2 g''\| + \delta^2 \|g''\| : g \in W^2(\varphi) \}, (\delta > 0), \end{aligned}$$

where $W^2(\varphi) = \{g \in C[0, 1] : g' \in AC_{loc}[0, 1], \varphi^2 g'' \in C[0, 1]\}$ and $g' \in AC_{loc}[0, 1]$ means that g is differentiable and g' is absolutely continuous on every closed interval $[a, b] \subset (0, 1)$. It is known ([12], Theorem 1.3.1) that there exists a positive constant $M > 0$, such that

$$\tilde{K}_{2, \varphi(x)}(f, \delta) \leq M \omega_2^\varphi(f, \sqrt{\delta}). \quad (5.1)$$

Also, the Ditzian-Totik modulus of first order is given by

$$\overrightarrow{\omega}_\psi(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x \pm \frac{h}{2}\psi(x) \in [0, 1]} \left| f\left(x + \frac{h}{2}\psi(x)\right) - f\left(x - \frac{h}{2}\psi(x)\right) \right|,$$

where $\psi : [0, 1] \rightarrow \mathbb{R}$ is an admissible step-weight function.

Now we state our next main result.

Theorem 5. *Let $f \in C[0, 1]$ and $0 \leq \alpha \leq 1$. Then, for $x \in [0, 1]$,*

$$\|D_n^{(\alpha)} f - f\| \leq M \omega_2^\varphi(f, (n+2)^{-1/2}) + \overrightarrow{\omega}_\psi(f, (n+2)^{-1}) + \omega(f; (n+2)^{-1}),$$

where $\varphi^2(x) = x(1-x)$ and $\psi(x) = \begin{cases} 1-2x & x \in [0, 1/2] \\ 2x-1 & x \in [1/2, 1] \end{cases}$.

Proof. We consider the auxiliary operators as follows:

$$\overline{D}_n^{(\alpha)}(f; x) = D_n^{(\alpha)}(f; x) + f(x) - f\left(\frac{nx+1}{n+2}\right).$$

Let $g \in W^2(\varphi)$ then by using Taylor's expansion of g , on proceeding as in the proof of Theorem 2, we obtain that

$$\begin{aligned} |\overline{D}_n^{(\alpha)}(g; x) - g(x)| &\leq D_n^{(\alpha)} \left(\left| \int_x^t |t-u| |g''(u)| du \right|, x \right) \\ &\quad + \int_x^{\frac{nx+1}{n+2}} \left| \frac{nx+1}{n+2} - u \right| |g''(u)| du. \end{aligned} \quad (5.2)$$

Setting $u = \beta x + (1-\beta)t$, $\beta \in [0, 1]$, and also applying the concavity of γ_n^2 , we have

$$\frac{|t-u|}{\gamma_n^2(u)} = \frac{\beta |t-x|}{\gamma_n^2(\beta x + (1-\beta)t)} \leq \frac{\beta |t-x|}{\gamma_n^2(x)\beta + \gamma_n^2(t)(1-\beta)} \leq \frac{|t-x|}{\gamma_n^2(x)}. \quad (5.3)$$

Thus, inequality (5.2), in view of (5.3) leads us to

$$\begin{aligned} |\overline{D}_n^{(\alpha)}(g; x) - g(x)| &\leq D_n^{(\alpha)} \left(\left| \int_x^t \frac{|t-u|}{\gamma_n^2(u)} du \right|, x \right) \|\gamma_n^2 g''\| \\ &\quad + \left(\int_x^{\frac{nx+1}{n+2}} \frac{\left| \frac{nx+1}{n+2} - u \right|}{\gamma_n^2(u)} du \right) \|\gamma_n^2 g''\|. \\ &\leq \frac{1}{\gamma_n^2(x)} \|\gamma_n^2 g''\| \left[D_n^{(\alpha)}((t-x)^2; x) + \left(\frac{1-2x}{n+2} \right)^2 \right]. \end{aligned} \quad (5.4)$$

Now, using inequality (4.3), we get

$$\begin{aligned} |\overline{D}_n^{(\alpha)}(g; x) - g(x)| &\leq \frac{3}{(n+2)} \|\gamma_n^2 g''\| \\ &\leq \frac{3}{(n+2)} \left(\|\varphi^2 g''\| + \frac{1}{(n+2)} \|g''\| \right). \end{aligned}$$

Applying (4.5) and (5.4), we have for $f \in C[0, 1]$,

$$\begin{aligned} |D_n^{(\alpha)}(f; x) - f(x)| &\leq |\overline{D}_n^{(\alpha)}(f-g; x)| + |\overline{D}_n^{(\alpha)}(g; x) - g(x)| + |g(x) - f(x)| \\ &\quad + \left| f\left(\frac{nx+1}{n+2}\right) - f(x) \right| \\ &\leq 4\|f-g\| + \frac{3}{(n+2)} \|\varphi^2 g''\| + \frac{3}{(n+2)^2} \|g''\| \\ &\quad + \left| f\left(\frac{nx+1}{n+2}\right) - f(x) \right| \end{aligned}$$

Taking the infimum on the right hand side over all $g \in W^2(\varphi)$, we get

$$|D_n^{(\alpha)}(f; x) - f(x)| \leq 4\tilde{K}_{2,\varphi} \left(f, \frac{1}{n+2} \right) + \left| f\left(\frac{nx+1}{n+2}\right) - f(x) \right|. \quad (5.5)$$

Also,

$$\begin{aligned} \left| f\left(\frac{nx+1}{n+2}\right) - f(x) \right| &= \left| f\left(x + \frac{1-2x}{n+2}\right) - f(x) \right| \\ &\leq \left| f\left(x + \frac{(1-2x)}{n+2}\right) - f\left(x - \frac{(1-2x)}{n+2}\right) \right| \end{aligned} \quad (5.6)$$

$$\begin{aligned} &+ \left| f\left(x - \frac{(1-2x)}{n+2}\right) - f(x) \right| \\ &\leq \overrightarrow{\omega}_\psi(f, (n+2)^{-1}) + \omega(f; (n+2)^{-1}). \end{aligned} \quad (5.7)$$

Hence, combining (5.1), (5.5) and (5.6), the desired relation is immediate. $\square$

6. POINTWISE CONVERGENCE OF $D_n^{(\alpha)}$

Now we present a Voronovskaja type asymptotic formula for the operators $D_n^{(\alpha)}$.

Theorem 6. *Let $f \in C[0, 1]$ and $\alpha \in [0, 1]$. If f', f'' exists at a point $x \in [0, 1]$ then*

$$\lim_{n \rightarrow \infty} n \left(D_n^{(\alpha)}(f; x) - f(x) \right) = (1-2x)f'(x) + x(1-x)f''(x), \quad (6.1)$$

Further, if $f'' \in C[0, 1]$ then (6.1) holds uniformly in $[0, 1]$.

Proof. By Taylor's formula, we can write

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x) + \zeta(t, x)(t-x)^2, \quad (6.2)$$

where $\zeta(t, x) \rightarrow 0$ as $t \rightarrow x$ and is a continuous function on $[0, 1]$. Operating $D_n^{(\alpha)}$ to (6.2) and Remark 1, we get

$$\begin{aligned} D_n^{(\alpha)}(f; x) - f(x) &= f'(x)D_n^{(\alpha)}((t-x); x) + \frac{1}{2}f''(x)D_n^{(\alpha)}((t-x)^2; x) \\ &\quad + D_n^{(\alpha)}(\zeta(t, x)(t-x)^2; x), \\ \lim_{n \rightarrow \infty} n \left(D_n^{(\alpha)}(f; x) - f(x) \right) &= (1-2x)f'(x) + x(1-x)f''(x) \\ &\quad + \lim_{n \rightarrow \infty} n D_n^{(\alpha)}(\zeta(t, x)(t-x)^2; x). \end{aligned}$$

Since $\zeta(t, x) \rightarrow 0$ as $t \rightarrow x$, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that $|\zeta(t, x)| < \epsilon$ whenever $|t-x| < \delta$. For $|t-x| \geq \delta$, we have $|\zeta(t, x)| \leq M \frac{(t-x)^2}{\delta^2}$, for some $M > 0$. Let $\chi_\delta(t)$ denote the characteristic function of the interval $(x-\delta, x+\delta)$. From Lemma 3, we get

$$\begin{aligned} |D_n^{(\alpha)}(\zeta(t, x)(t-x)^2; x)| &\leq D_n^{(\alpha)}(|\zeta(t, x)|(t-x)^2 \chi_\delta(t); x) \\ &\quad + D_n^{(\alpha)}(|\zeta(t, x)|(t-x)^2 (1-\chi_\delta(t)); x) \end{aligned}$$

$$\begin{aligned}
&< \epsilon D_n^{(\alpha)}((t-x)^2; x) + \frac{M}{\delta^2} D_n^{(\alpha)}((t-x)^4; x) \\
&= \epsilon O\left(\frac{1}{n}\right) + O\left(\frac{1}{n^2}\right),
\end{aligned}$$

which implies that $\lim_{n \rightarrow \infty} n D_n^{*(1/n)}(\zeta(t, x)(t-x)^2; x) = 0$, due to the arbitrariness of $\epsilon > 0$. This proves the first assertion of the theorem.

To prove the uniformity assertion, due to the uniform continuity of f in $[0, 1]$, the δ in the above proof can be chosen independent of x and all the other estimates hold uniformly in $x \in [0, 1]$. $\square$

7. RATE OF CONVERGENCE

$DBV[0, 1]$ denotes the class of all absolutely continuous functions f defined on $[0, 1]$, having on $[0, 1]$ a derivative f' equivalent to a function of bounded variation on $[0, 1]$. We notice that the functions $f \in DBV[0, 1]$ possess a representation

$$f(x) = \int_0^x g(t)dt + f(0)$$

where $g \in BV[0, 1]$, i.e., g is a function of bounded variation on $[0, 1]$.

The operators $D_n^{(\alpha)}(f; x)$ also admit the integral representation

$$D_n^{(\alpha)}(f; x) = \int_0^1 N_n^{(\alpha)}(x, t) f(t) dt, \quad x \in [0, 1], \quad (7.1)$$

where the kernel $N_n^{(\alpha)}(x, t)$ is given by

$$N_n^{(\alpha)}(x, t) = (n+1) \sum_{k=0}^n p_{n,k}^{(\alpha)}(x) p_{n,k}(t).$$

Lemma 6. For a fixed $x \in (0, 1)$ and sufficiently large n , we have

- (i) $\vartheta_n(x, y) = \int_0^y N_n^{(\alpha)}(x, t) dt \leq \frac{2}{(n+2)} \frac{\gamma_n^2(x)}{(x-y)^2}, \quad 0 \leq y < x,$
- (ii) $1 - \vartheta_n(x, z) = \int_z^1 N_n^{(\alpha)}(x, t) dt \leq \frac{2}{(n+2)} \frac{\gamma_n^2(x)}{(z-x)^2}, \quad x < z < 1.$

Proof. (i) Using Lemma 5 we get

$$\begin{aligned}
\vartheta_n(x, y) &= \int_0^y N_n^{(\alpha)}(x, t) dt \leq \int_0^y \left(\frac{x-t}{x-y} \right)^2 N_n^{(\alpha)}(x, t) dt \\
&= D_n^{(\alpha)}((t-x)^2; x)(x-y)^{-2} \leq \frac{2}{(n+2)} \frac{\gamma_n^2(x)}{(x-y)^2}.
\end{aligned}$$

The proof of (ii) is similar hence the details are omitted. $\square$

Theorem 7. Let $f \in DBV[0, 1]$. Then for every $x \in (0, 1)$ and sufficiently large n , we have

$$\begin{aligned} |D_n^{(\alpha)}(f; x) - f(x)| &\leq \left| \frac{(1-2x)}{n+2} \right| \frac{|f'(x+) + f'(x-)|}{2} \\ &\quad + \sqrt{\frac{2}{(n+2)}} \gamma_n(x) \frac{|f'(x+) - f'(x-)|}{2} \\ &\quad + \frac{2\gamma_n^2(x)}{(n+2)} x^{-1} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-(x/k)}^x (f'_x) + \frac{x}{\sqrt{n}} \bigvee_{x-(x/\sqrt{n})}^x (f'_x) \\ &\quad + \frac{2\gamma_n^2(x)}{(1-x)(n+2)} \sum_{k=1}^{[\sqrt{n}]x+((1-x)/k)} \bigvee_x (f'_x) \\ &\quad + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+((1-x)/\sqrt{n})} (f'_x), \end{aligned}$$

where $\bigvee_a^b(f'_x)$ denotes the total variation of f'_x on $[a, b]$ and f'_x is defined by

$$f'_x(t) = \begin{cases} f'(t) - f'(x-), & 0 \leq t < x \\ 0, & t = x \\ f'(t) - f'(x+), & x < t \leq 1. \end{cases} \quad (7.2)$$

Proof. Since $D_n^{(\alpha)}(e_0; x) = 1$, by applying (7.1), for every $x \in (0, 1)$ we obtain

$$\begin{aligned} D_n^{(\alpha)}(f; x) - f(x) &= \int_0^1 N_n^{(\alpha)}(x, t) (f(t) - f(x)) dt \\ &= \int_0^1 N_n^{(\alpha)}(x, t) \int_x^t f'(u) du dt. \end{aligned} \quad (7.3)$$

For any $f \in DBV[0, 1]$, by (7.2) we may write

$$\begin{aligned} f'(u) &= f'_x(u) + \frac{1}{2}(f'(x+) + f'(x-)) + \frac{1}{2}(f'(x+) - f'(x-)) \operatorname{sgn}(u - x) \\ &\quad + \delta_x(u) [f'(u) - \frac{1}{2}(f'(x+) + f'(x-))], \end{aligned} \quad (7.4)$$

where

$$\delta_x(u) = \begin{cases} 1, & u = x \\ 0, & u \neq x. \end{cases}$$

Obviously,

$$\int_0^1 \left(\int_x^t \left(f'(u) - \frac{1}{2}(f'(x+) + f'(x-)) \right) \delta_x(u) du \right) N_n^{(\alpha)}(x, t) dt = 0.$$

By (7.1) and simple calculations we have

$$\begin{aligned} \int_0^1 \left(\int_x^t \frac{1}{2} (f'(x+) + f'(x-)) du \right) N_n^{(\alpha)}(x, t) dt \\ = \frac{1}{2} (f'(x+) + f'(x-)) \int_0^1 (t-x) N_n^{(\alpha)}(x, t) dt \\ = \frac{1}{2} (f'(x+) + f'(x-)) D_n^{(\alpha)}((t-x); x) \end{aligned}$$

and

$$\begin{aligned} \left| \int_0^1 N_n^{(\alpha)}(x, t) \left(\int_x^t \frac{1}{2} (f'(x+) - f'(x-)) \operatorname{sgn}(u-x) du \right) dt \right| \\ \leq \frac{1}{2} |f'(x+) - f'(x-)| \int_0^1 |t-x| N_n^{(\alpha)}(x, t) dt \\ \leq \frac{1}{2} |f'(x+) - f'(x-)| D_n^{(\alpha)}(|t-x|; x) \\ \leq \frac{1}{2} |f'(x+) - f'(x-)| \left(D_n^{(\alpha)}((t-x)^2; x) \right)^{1/2}. \end{aligned}$$

Considering Lemmas 3 and 5 and using (7.3), (7.4) we obtain the following estimate

$$\begin{aligned} |D_n^{(\alpha)}(f; x) - f(x)| &\leq \frac{1}{2} |f'(x+) + f'(x-)| \left| \frac{1-2x}{(n+2)} \right| \\ &\quad + \frac{1}{2} |f'(x+) - f'(x-)| \sqrt{\frac{2}{(n+2)}} \gamma_n(x) \\ &\quad + \left| \int_0^x \left(\int_x^t f'_x(u) du \right) N_n^{(\alpha)}(x, t) dt \right. \\ &\quad \left. + \int_x^1 \left(\int_x^t f'_x(u) du \right) N_n^{(\alpha)}(x, t) dt \right|. \end{aligned} \quad (7.5)$$

Let

$$\begin{aligned} F_n^{(\alpha)}(f'_x, x) &= \int_0^x \left(\int_x^t f'_x(u) du \right) N_n^{(\alpha)}(x, t) dt, \\ G_n^{(\alpha)}(f'_x, x) &= \int_x^1 \left(\int_x^t f'_x(u) du \right) N_n^{(\alpha)}(x, t) dt. \end{aligned}$$

To complete the proof, it is sufficient to estimate the terms $F_n^{(\alpha)}(f'_x, x)$ and $G_n^{(\alpha)}(f'_x, x)$. Since $\int_a^b d_t \vartheta_n(x, t) \leq 1$ for all $[a, b] \subseteq [0, 1]$, using integration by parts

and applying Lemma 6 with $y = x - (x/\sqrt{n})$, we have

$$\begin{aligned}
 |F_n^{(\alpha)}(f'_x, x)| &= \left| \int_0^x \left(\int_x^t f'_x(u) du \right) d_t \vartheta_n(x, t) \right| = \left| \int_0^x \vartheta_n(x, t) f'_x(t) dt \right| \\
 &\leq \left(\int_0^y + \int_y^x \right) |f'_x(t)| |\vartheta_n(x, t)| dt \\
 &\leq \frac{2\gamma_n^2(x)}{(n+2)} \int_0^y \bigvee_t^x (f'_x)(x-t)^{-2} dt + \int_y^x \bigvee_t^x (f'_x) dt \\
 &\leq \frac{2\gamma_n^2(x)}{(n+2)} \int_0^y \bigvee_t^x (f'_x)(x-t)^{-2} dt + \frac{x}{\sqrt{n}} \bigvee_{x-(x/\sqrt{n})}^x (f'_x).
 \end{aligned}$$

By the substitution of $u = x/(x-t)$, we obtain

$$\begin{aligned}
 \frac{2\gamma_n^2(x)}{(n+2)} \int_0^{x-(x/\sqrt{n})} (x-t)^{-2} \bigvee_t^x (f'_x) dt &= \frac{2\gamma_n^2(x)}{(n+2)} x^{-1} \int_1^{\sqrt{n}} \bigvee_{x-(x/u)}^x (f'_x) du \\
 &\leq \frac{2\gamma_n^2(x)}{(n+2)} x^{-1} \sum_{k=1}^{[\sqrt{n}]} \int_k^{k+1} \bigvee_{x-(x/k)}^x (f'_x) du \\
 &\leq \frac{2\gamma_n^2(x)}{(n+2)} x^{-1} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-(x/k)}^x (f'_x).
 \end{aligned}$$

Thus,

$$|F_n^{(\alpha)}(f'_x, x)| \leq \frac{2\gamma_n^2(x)}{(n+2)} x^{-1} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-(x/k)}^x (f'_x) + \frac{x}{\sqrt{n}} \bigvee_{x-(x/\sqrt{n})}^x (f'_x). \quad (7.6)$$

Using integration by parts and applying Lemma 6 with $z = x + ((1-x)/\sqrt{n})$, we have

$$\begin{aligned}
 |G_n^{(\alpha)}(f'_x, x)| &= \left| \int_x^1 \left(\int_x^t f'_x(u) du \right) N_n^{(\alpha)}(x, t) dt \right| \\
 &= \left| \int_x^z \left(\int_x^t f'_x(u) du \right) d_t (1 - \vartheta_n(x, t)) \right. \\
 &\quad \left. + \int_z^1 \left(\int_x^t f'_x(u) du \right) d_t (1 - \vartheta_n(x, t)) \right| \\
 &= \left| \left[\int_x^t f'_x(u) (1 - \vartheta_n(x, t)) du \right]_x^z - \int_x^z f'_x(t) (1 - \vartheta_n(x, t)) dt \right|
 \end{aligned}$$

$$\begin{aligned}
& + \int_z^1 \left(\int_x^t f'_x(u) du \right) d_t (1 - \vartheta_n(x, t)) \Big| \\
& = \left| \int_x^z f'_x(u) du (1 - \vartheta_n(x, z)) - \int_x^z f'_x(t) (1 - \vartheta_n(x, t)) dt \right. \\
& \quad \left. + \left[\int_x^t f'_x(u) du (1 - \vartheta_n(x, t)) \right]_z^1 - \int_z^1 f'_x(t) (1 - \vartheta_n(x, t)) dt \right| \\
& = \left| \int_x^z f'_x(t) (1 - \vartheta_n(x, t)) dt + \int_z^1 f'_x(t) (1 - \vartheta_n(x, t)) dt \right| \\
& \leq \frac{2\gamma_n^2(x)}{(n+2)} \int_z^1 \bigvee_x (f'_x)(t-x)^{-2} dt + \int_x^z \bigvee_x (f'_x) dt \\
& = \frac{2\gamma_n^2(x)}{(n+2)} \int_{x+((1-x)/\sqrt{n})}^1 \bigvee_x (f'_x)(t-x)^{-2} dt \\
& \quad + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+((1-x)/\sqrt{n})} (f'_x).
\end{aligned}$$

By the substitution of $v = (1-x)/(t-x)$, we get

$$|G_n^{(\alpha)}(f'_x, x)| \leq \frac{2\gamma_n^2(x)}{(n+2)} \int_1^{\sqrt{n}^{x+((1-x)/v)}} \bigvee_x (f'_x)(1-x)^{-1} dv \quad (7.7)$$

$$\begin{aligned}
& + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+((1-x)/\sqrt{n})} (f'_x) \\
& \leq \frac{2\gamma_n^2(x)}{(1-x)(n+2)} \sum_{k=1}^{[\sqrt{n}]} \int_k^{k+1} \bigvee_x^{x+((1-x)/v)} (f'_x) dv \quad (7.8)
\end{aligned}$$

$$\begin{aligned}
& + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+((1-x)/\sqrt{n})} (f'_x) \\
& = \frac{2\gamma_n^2(x)}{(1-x)(n+2)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+((1-x)/k)} (f'_x) \quad (7.9)
\end{aligned}$$

$$+ \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+((1-x)/\sqrt{n})} (f'_x). \quad (7.10)$$

Combining the estimates (7.5)-(7.7), we get the required result. $\square$

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Authors' addresses

Arun Kajla

Department of Mathematics, Central University of Haryana, Haryana-123031, India

E-mail address: rachitkajla47@gmail.com

Tuncer Acar

Department of Mathematics, Faculty of Science, Selcuk University, Selcuklu, Konya, 42003, Turkey

E-mail address: tunceracar@ymail.com



UNIFORM NUMERICAL APPROXIMATION FOR PARAMETER DEPENDENT SINGULARLY PERTURBED PROBLEM WITH INTEGRAL BOUNDARY CONDITION

MUSTAFA KUDU, ILHAME AMIRALI, AND GABIL M. AMIRALIYEV

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Abstract. In this paper, a parameter-uniform numerical method for a parameterized singularly perturbed ordinary differential equation containing integral boundary condition is studied. Asymptotic estimates on the solution and its derivatives are derived. A numerical algorithm based on upwind finite difference operator and an appropriate piecewise uniform mesh is constructed. Parameter-uniform error estimate for the numerical solution is established. Numerical results are presented, which illustrate the theoretical results.

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1. INTRODUCTION

In this paper, we consider the following parameterized singular perturbation problem with integral boundary condition arising in many scientific applications [16, 23](see also references therein):

$$\varepsilon u' + f(t, u, \lambda) = 0, \quad t \in \Omega = (0, T], \quad T > 0, \quad (1.1)$$

$$u(0) + \int_0^T c(s)u(s)ds = A, \quad (1.2)$$

$$u(T) = B, \quad (1.3)$$

where $\varepsilon \in (0, 1]$ is the perturbation parameter, λ is known as the control parameter, A and B are given constants. The functions $c(t) \geq 0$ and $f(t, u, \lambda)$ are assumed to be sufficiently continuously differentiable for our purpose in $\overline{\Omega} = \Omega \cup \{t = 0\}$ and $\Omega \times \mathbb{R}^2$ respectively and moreover

$$0 < \alpha \leq \frac{\partial f}{\partial u} \leq a^* < \infty,$$

$$0 < m_1 \leq \left| \frac{\partial f}{\partial \lambda} \right| \leq M_1 < \infty.$$

By a solution of (1.1)-(1.3) we mean $\{u(t), \lambda\} \in C^1[0, T] \times \mathbb{R}$ for which problem (1.1)-(1.3) is satisfied.

Singularly perturbed differential equations are typically characterized by a small parameter ε multiplying some or all of the highest order terms in the differential equation as normally boundary layers occur in their solutions. These equations play an important role in today's advanced scientific computations. Many mathematical models starting from fluid dynamics to the problems in mathematical biology are modelled by singularly perturbed problems. Typical examples include high Reynold's number flow in the fluid dynamics, heat transport problem etc. For more details on singular perturbation, one can refer to the books [10, 12, 19, 21] and the references therein. The numerical analysis of singular perturbation cases has always been far from trivial because of the boundary layer behavior of the solution. Such problem undergo rapid changes within very thin layers near the boundary or inside the problem domain [19, 21]. It is well known that standard numerical methods for solving such problems are unstable and fail to give accurate results when the perturbation parameter is small. Therefore, it is important to develop suitable numerical methods to these problems, whose accuracy does not depend on the parameter value, i.e. methods that are convergence ε -uniformly. For the various approaches on the numerical solution of differential equations with steep gradients and continuous solutions we may refer to the studies [8, 10–12]. Parameterized boundary value problems have been considered by many researchers for many years. Such problems arise in physical chemistry and physics, describing the exothermic and isothermal chemical reactions, the steady-state temperature distributions, the oscillation of a mass attached by two springs lead to a differential equation with a parameter [18, 22]. An overview of some existence and uniqueness results and applications of parameterized equations may be obtained, for example, in [13, 16, 18, 22](see, also references therein). In [18, 22], the authors have also been considered some approximating aspects of this kind of problems. But in the above-mentioned papers, algorithms are only concerned with the regular cases (i.e., when the boundary layers are absent). In recent years, many researchers presented the numerical methods for the singular perturbation cases of parameterized problems. Uniform convergent finite-difference schemes for solving parameterized singularly perturbed two-point boundary value problems have been considered in [2, 3, 9, 17, 24, 25](see, also references therein). In [2, 3, 17] authors used boundary layer technique for solving analogous problem. A methodology based on the homotopy analysis technique to approximate the analytic solution was investigated in [24, 25]. Also it is well known that nonlinear differential equations with integral boundary conditions have been used in description of many phenomena in the applied sciences, e.g., heat conduction, chemical engineering, underground water flow and so on [6, 15, 20]. Therefore, boundary value problems involving integral

boundary conditions have been studied by many authors [1, 4, 5, 7, 11, 14, 16, 23] (see, also references therein). Some approximating aspects of this kind of problems in the regular cases, i.e, in absence of layers, were investigated in [4, 11, 14, 16, 23]. In recent years, many researchers considered the singularly perturbed case for these problems. In [1, 5, 7] authors develop a finite difference scheme on Shishkin mesh for problem with integral boundary conditions and proved that the method is nearly first order convergent except for a logarithmic factor. A hybrid scheme, which is second order convergent on Shishkin mesh was discussed in [7]. For the numerical methods, concerning to second order singularly perturbed differential equations with integral boundary conditions can be seen e.g., [5]. In this paper, as far as we know the numerical solution of the singularly perturbed boundary value problem containing both control parameter and integral condition is first being considered. For the numerical solution of such problems, requires specific approach in constructing of the appropriate difference scheme and examining the error analysis. The scheme is constructed by the method of integral identities with the use of appropriate quadrature rules with the remainder terms in integral form. We show that the proposed scheme is uniformly convergent in the discrete maximum norm accuracy of $O(N^{-1} \ln N)$ on Shishkin meshes. First, the asymptotic estimates for the continuous solution are given in Section 2, which are needed in later sections for the analysis of appropriate numerical solution. In Section 3, we describe the finite discretization and give the difference scheme on a piecewise uniform grid. In Section 4, the convergence analysis is carried out. Finally, in Section 5 presents some numerical results to confirm the theoretical analysis. Henceforth, C and c denote the generic positive constants independent of both the perturbation parameter ε and mesh parameter N . Such subscripted constants are also independent of ε and mesh parameter, but whose values are fixed.

2. ASYMPTOTIC BEHAVIOR OF THE EXACT SOLUTION

In this section, we give a priori estimates for the solution and its derivatives of the problem (1.1)-(1.3), which indicate the asymptotic behavior of the solution and its first derivative in respect to perturbation parameter. These estimates are unimprovable in terms of the view of behavior in ε and will be used in order to analyse the numerical solution. We also denote $\|g\|_\infty = \max_{[0,T]} |g(t)|$ for any $g \in C[0, T]$.

Lemma 2.1 *The solution $\{u(t), \lambda\}$ of the problem (1.1)-(1.3) satisfies the following bounds:*

$$|\lambda| \leq c_0, \quad (2.1)$$

$$\|u\|_\infty \leq c_1, \quad (2.2)$$

where

$$c_0 = m_1^{-1} \left\{ \frac{\alpha |A|}{e^{\alpha T} - 1} + \frac{|B| a^* (1 - \|c\|_\infty T)}{m_1 (e^{a^* T} - 1)} + \|F\|_\infty \right\},$$

$$c_1 = |u(0)| + \alpha^{-1}(\|F\|_\infty + |\lambda| M_1) = |u(0)| + \alpha^{-1}(\|F\|_\infty + c_0 M_1),$$

$$|u'(t)| \leq C \left(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha t}{\varepsilon}}\right), \quad t \in [0, T], \quad (2.3)$$

provided $a \in C^1[0, T]$ and $\left|\frac{\partial f}{\partial t}\right| \leq C$ for $t \in [0, T]$ and $|u| \leq c_1, |\lambda| \leq c_0$.

Proof. The quasilinear equation (1.1) can be written as

$$\varepsilon u' + a(t)u = F(t) + \lambda b(t), \quad t \in [0, T], \quad (2.4)$$

where

$$a(t) = \frac{\partial f}{\partial u}(t, \tilde{u}, \tilde{\lambda}),$$

$$b(t) = -\frac{\partial f}{\partial \lambda}(t, \tilde{u}, \tilde{\lambda}),$$

$\tilde{u} = \gamma u, \tilde{\lambda} = \gamma \lambda$ ($0 < \gamma < 1$)—intermediate values.

Integrating (2.4), (1.3) we have

$$u(t) = B e^{\frac{1}{\varepsilon} \int_t^T a(\xi) d\xi} - \frac{1}{\varepsilon} \int_t^T F(\xi) e^{\frac{1}{\varepsilon} \int_t^\xi a(\eta) d\eta} d\xi + \frac{\lambda}{\varepsilon} \int_t^T b(\xi) e^{\frac{1}{\varepsilon} \int_t^\xi a(\eta) d\eta} d\xi,$$

from which, after using the integral boundary condition (1.2), it follows that,

$$\begin{aligned} & B e^{\frac{1}{\varepsilon} \int_0^T a(\xi) d\xi} - \frac{1}{\varepsilon} \int_0^T F(\xi) e^{\frac{1}{\varepsilon} \int_0^\xi a(\eta) d\eta} d\xi + \frac{\lambda}{\varepsilon} \int_0^T b(\xi) e^{\frac{1}{\varepsilon} \int_0^\xi a(\eta) d\eta} d\xi \\ & + B \int_0^T c(s) e^{\frac{1}{\varepsilon} \int_s^T a(\xi) d\xi} ds - \frac{1}{\varepsilon} \int_0^T c(s) \left[\int_s^T F(\xi) e^{\frac{1}{\varepsilon} \int_s^\xi a(\eta) d\eta} d\xi \right] ds \\ & + \frac{\lambda}{\varepsilon} \int_0^T c(s) \left[\int_s^T b(\xi) e^{\frac{1}{\varepsilon} \int_s^\xi a(\eta) d\eta} d\xi \right] ds = A \end{aligned}$$

and

$$\begin{aligned} \lambda = & \frac{A}{\frac{1}{\varepsilon} \int_0^T b(\xi) e^{\frac{1}{\varepsilon} \int_0^\xi a(\eta) d\eta} d\xi + \frac{1}{\varepsilon} \int_0^T b(\xi) \left[\int_s^T c(s) e^{\frac{1}{\varepsilon} \int_s^\xi a(\eta) d\eta} ds \right] d\xi} \\ & - \frac{B(e^{\frac{1}{\varepsilon} \int_0^T a(\xi) d\xi} + \int_0^T c(s) e^{\frac{1}{\varepsilon} \int_s^T a(\xi) d\xi} ds)}{\frac{1}{\varepsilon} \int_0^T b(\xi) e^{\frac{1}{\varepsilon} \int_0^\xi a(\eta) d\eta} d\xi + \frac{1}{\varepsilon} \int_0^T b(\xi) \left[\int_s^T c(s) e^{\frac{1}{\varepsilon} \int_s^\xi a(\eta) d\eta} ds \right] d\xi} \\ & + \frac{\frac{1}{\varepsilon} \int_0^T F(\xi) e^{\frac{1}{\varepsilon} \int_0^\xi a(\eta) d\eta} d\xi + \frac{1}{\varepsilon} \int_0^T F(\xi) \left[\int_s^T c(s) e^{\frac{1}{\varepsilon} \int_s^\xi a(\eta) d\eta} ds \right] d\xi}{\frac{1}{\varepsilon} \int_0^T b(\xi) e^{\frac{1}{\varepsilon} \int_0^\xi a(\eta) d\eta} d\xi + \frac{1}{\varepsilon} \int_0^T b(\xi) \left[\int_s^T c(s) e^{\frac{1}{\varepsilon} \int_s^\xi a(\eta) d\eta} ds \right] d\xi}. \end{aligned} \quad (2.5)$$

As $c(t) \geq 0$, then after applying the mean value theorem for integrals, we deduce that,

$$\left| \frac{\frac{1}{\varepsilon} \int_0^T F(\xi) e^{\frac{1}{\varepsilon} \int_0^\xi a(\eta) d\eta} d\xi + \frac{1}{\varepsilon} \int_0^T F(\xi) [\int_s^T c(s) e^{\frac{1}{\varepsilon} \int_s^\xi a(\eta) d\eta} ds] d\xi}{\frac{1}{\varepsilon} \int_0^T b(\xi) e^{\frac{1}{\varepsilon} \int_0^\xi a(\eta) d\eta} d\xi + \frac{1}{\varepsilon} \int_0^T b(\xi) [\int_s^T c(s) e^{\frac{1}{\varepsilon} \int_s^\xi a(\eta) d\eta} ds] d\xi} \right| \leq m_1^{-1} \|F\|_\infty \quad (2.6)$$

and

$$\begin{aligned} & \left| \frac{B(e^{\frac{1}{\varepsilon} \int_0^T a(\xi) d\xi} + \int_0^T c(s) e^{\frac{1}{\varepsilon} \int_s^T a(\xi) d\xi} ds)}{\frac{1}{\varepsilon} \int_0^T b(\xi) e^{\frac{1}{\varepsilon} \int_0^\xi a(\eta) d\eta} d\xi + \frac{1}{\varepsilon} \int_0^T b(\xi) [\int_s^T c(s) e^{\frac{1}{\varepsilon} \int_s^\xi a(\eta) d\eta} ds] d\xi} \right| \\ & \leq \frac{|B| (1 + \|c\|_\infty T)}{m_1 \varepsilon^{-1} \int_0^T e^{\frac{1}{\varepsilon} \int_s^T a(\eta) d\eta} d\xi} \\ & \leq \frac{|B| (1 + \|c\|_\infty T)}{m_1 (a^*)^{-1} (1 - e^{-\frac{a^* T}{\varepsilon}})} \leq \frac{|B| (1 + \|c\|_\infty T)}{m_1 (a^*)^{-1} (1 - e^{-a^* T})}, (\varepsilon \leq 1). \end{aligned} \quad (2.7)$$

Also, for the first term in right side of (2.5) for $\varepsilon \leq 1$ values, we get

$$\begin{aligned} & \left| \frac{A}{\frac{1}{\varepsilon} \int_0^T b(\xi) e^{\frac{1}{\varepsilon} \int_0^\xi a(\eta) d\eta} d\xi + \frac{1}{\varepsilon} \int_0^T b(\xi) [\int_s^T c(s) e^{\frac{1}{\varepsilon} \int_s^\xi a(\eta) d\eta} ds] d\xi} \right| \\ & \leq \frac{|A|}{\frac{1}{\varepsilon} \int_0^T b(\xi) e^{\frac{1}{\varepsilon} \int_0^\xi a(\eta) d\eta} d\xi} \\ & \leq \frac{|A|}{m_1 \alpha^{-1} (e^{\frac{\alpha T}{\varepsilon}} - 1)} \leq \frac{|A|}{m_1 \alpha^{-1} (e^{\alpha T} - 1)} \leq \frac{\alpha |A|}{m_1 (e^{\alpha T} - 1)}. \end{aligned} \quad (2.8)$$

The relation (2.5), by taking into consideration here (2.6)-(2.8), immediately leads to (2.1).

Now, integrating (2.4), we have

$$u(t) = u(0) e^{-\frac{1}{\varepsilon} \int_0^t a(\eta) d\eta} + \frac{1}{\varepsilon} \int_0^t \Phi(\xi) e^{-\frac{1}{\varepsilon} \int_\xi^t a(\eta) d\eta} d\xi; \quad \Phi(s) = F(s) - \lambda b(s),$$

from which, by setting the integral boundary condition (1.2), we get

$$u(0) = \frac{A - \frac{1}{\varepsilon} \int_0^T c(s) [\int_0^s \Phi(\xi) e^{-\frac{1}{\varepsilon} \int_\xi^s a(\eta) d\eta} d\xi] ds}{1 + \int_0^T c(s) e^{-\frac{1}{\varepsilon} \int_0^s a(\xi) d\xi} ds}.$$

Since $c(t)$ is nonnegative, then

$$\begin{aligned}
 |u(0)| &= \left| \frac{A - \frac{1}{\varepsilon} \int_0^T c(s) \left[\int_0^s \Phi(\xi) e^{-\frac{1}{\varepsilon} \int_\xi^s a(\eta) d\eta} d\xi \right] ds}{1 + \int_0^T c(s) e^{-\frac{1}{\varepsilon} \int_0^s a(\xi) d\xi} ds} \right| \\
 &\leq |A| + \frac{1}{\varepsilon} \int_0^T c(s) \left[\int_0^s |\Phi(\xi)| e^{-\frac{1}{\varepsilon} \int_\xi^s a(\eta) d\eta} d\xi \right] ds \\
 &\leq |A| + \frac{1}{\varepsilon} \|c\|_\infty (\|F\|_\infty + M_1 c_0) \alpha^{-1} \varepsilon \int_0^T (1 - e^{-\frac{\alpha s}{\varepsilon}}) ds \\
 &\leq |A| + \alpha^{-1} \|c\|_\infty T (\|F\|_\infty + M_1 c_0). \tag{2.9}
 \end{aligned}$$

Next, by virtue of maximum principle we have

$$\begin{aligned}
 \|u\|_\infty &\leq |u(0)| + \alpha^{-1} \|F - b\lambda\|_\infty \\
 &\leq |u(0)| + \alpha^{-1} (\|F\|_\infty + |\lambda| M_1),
 \end{aligned}$$

which, after taking into account (2.1) and (2.9) leads to (2.2).

To prove (2.3), first we estimate $u'(0)$:

$$|u'(0)| \leq \frac{|F(0) - a(0)u(0) - b(0)\lambda|}{\varepsilon} \leq \frac{C}{\varepsilon}.$$

Differentiating, now the equation (2.4), we have

$$\varepsilon v'' + p(t)v' = g(t),$$

with

$$v = u', \quad p(t) = \frac{\partial f}{\partial u}(t, u(t), \lambda) \quad \text{and} \quad g(t) = \frac{\partial f}{\partial t}(t, u(t), \lambda).$$

So

$$v(t) = v(0) e^{-\frac{1}{\varepsilon} \int_0^t a(s) ds} + \frac{1}{\varepsilon} \int_0^t g(s) e^{-\frac{1}{\varepsilon} \int_s^t a(\xi) d\xi} ds.$$

Since $p(t) \geq \alpha > 0$ and $|g(t)| \leq C$, for $v(t)$ we then obtain

$$\begin{aligned}
 |v(t)| &\leq \frac{C}{\varepsilon} e^{-\frac{\alpha t}{\varepsilon}} + \frac{C}{\varepsilon} \int_0^t e^{-\frac{\alpha(t-s)}{\varepsilon}} ds \\
 &\leq \frac{C}{\varepsilon} e^{-\frac{\alpha t}{\varepsilon}} + C(1 - e^{-\frac{\alpha t}{\varepsilon}})
 \end{aligned}$$

which implies validity of (2.3). □

3. DISCRETE PROBLEM

Let ω_N be any non-uniform mesh on Ω :

$$\omega_N = \{0 < t_1 < t_2 < \dots < t_{N-1} < t_N = T\}$$

and $\bar{\omega}_N = \omega_N \cup \{t = 0\}$. For each $i \geq 1$, we set the step size $h_i = t_i - t_{i-1}$. To simplify the notation we set $g_i = g(t_i)$ for any function $g(t)$, while g_i^N denotes an approximation of $g(t)$ at t_i .

For any mesh function $\{w_i\}$ defined on $\bar{\omega}_N$ we use

$$w_{\bar{t},i} = (w_i - w_{i-1})/h_i,$$

$$\|w\|_\infty \equiv \|w\|_{\infty, \bar{\omega}_N} := \max_{0 \leq i \leq N} |w_i|.$$

To obtain approximation for (1.1) we integrate (1.1) over (t_{i-1}, t_i) :

$$\varepsilon u_{\bar{t},i} + h_i^{-1} \int_{t_{i-1}}^{t_i} f(t, u(t), \lambda) dt = 0, \quad 1 \leq i \leq N,$$

which yields the relation

$$\varepsilon u_{\bar{t},i} + f(t_i, u_i, \lambda) + R_i = 0, \quad 1 \leq i \leq N, \quad (3.1)$$

with local truncation error

$$R_i = -h_i^{-1} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) \frac{d}{dt} f(t, u(t), \lambda) dt. \quad (3.2)$$

To define an approximation for the boundary condition (1.2), here we use the composite right-side rectangle rule:

$$u(0) + \int_0^T c(s)u(s)ds = u_0 + \sum_{i=1}^N h_i c_i u_i + r$$

with remainder term

$$r = -\sum_{i=1}^N \int_{t_{i-1}}^{t_i} (t - t_{i-1}) \frac{d}{dt} (c(t)u(t)) dt. \quad (3.3)$$

Consequently

$$u_0 + \sum_{i=1}^N h_i c_i u_i + r = A. \quad (3.4)$$

Neglecting R_i and r in (3.1) and (3.4), we propose the following difference scheme for approximating (1.1)-(1.3):

$$\varepsilon u_{\bar{t},i}^N + f(t_i, u_i^N, \lambda^N) = 0, \quad 1 \leq i \leq N, \quad (3.5)$$

$$u_0^N + \sum_{i=1}^N h_i c_i u_i^N = A, \quad (3.6)$$

$$u_N^N = B. \quad (3.7)$$

The difference scheme (3.5)-(3.7), in order to be ε -uniform convergent, we will use the Shishkin mesh. For an even number N , the piecewise uniform mesh takes $N/2$ points in the interval $[0, \sigma]$ and also $N/2$ points in the interval $[\sigma, T]$, where the transition point σ , which separates the fine and coarse portions of the mesh, is obtained by taking

$$\sigma = \min \left\{ \frac{T}{2}, \alpha^{-1} \varepsilon \ln \varepsilon \right\}.$$

In practice one usually has $\sigma \leq T$, so the mesh is fine on $[0, \sigma]$ and coarse on $[\sigma, T]$. Hence, if we denote by $h^{(1)}$ and $h^{(2)}$ the step size in $[0, \sigma]$ and $[\sigma, T]$, respectively, we have

$$h^{(1)} = 2\sigma N^{-1}, \quad h^{(2)} = 2(T - \sigma)N^{-1}, \\ h^{(1)} \leq TN^{-1}, TN^{-1} \leq h^{(2)} < 2TN^{-1}, \quad h^{(1)} + h^{(2)} = 2TN^{-1},$$

so

$$\bar{\omega}_N = \begin{cases} t_i = ih^{(1)}, & \text{for } i = 0, 1, \dots, N/2; \quad h^{(1)} = 2\sigma/N, \\ t_i = \sigma + (i - N/2)h^{(2)}, & \text{for } i = N/2 + 1, \dots, N; \quad h^{(2)} = 2(T - \sigma)/N. \end{cases}$$

In the rest of the paper we only consider this mesh.

4. UNIFORM ERROR ESTIMATES

To investigate the convergence of the method, note that the error functions $z_i^N = u_i^N - u_i$, $0 \leq i \leq N$, $\mu^N = \lambda^N - \lambda$ are the solution of the discrete problem

$$\varepsilon z_{\bar{t},i}^N + f(t_i, u_i^N, \lambda^N) - f(t_i, u_i, \lambda) = R_i, \quad 1 \leq i \leq N, \quad (4.1)$$

$$z_0^N + \sum_{i=1}^N h_i c_i z_i^N - r = 0, \quad (4.2)$$

$$z_N^N = 0. \quad (4.3)$$

where the truncation errors R_i and r are given by (3.2) and (3.3), respectively.

Lemma 4.1. The solution of the first order difference equation

$$y_i = q_i y_{i-1} + \varphi_i, \quad 1 \leq i \leq N$$

can be expressed in the following forms:

$$y_i = y_0 Q_i + \sum_{k=1}^i \varphi_k Q_{i-k} \quad (4.4)$$

or

$$y_i = y_N Q_{N-i}^{-1} - \sum_{k=i+1}^N \varphi_k Q_{k-i}^{-1} \quad (4.5)$$

where

$$Q_{i-k} = \begin{cases} 1, & k = i, \\ \prod_{\ell=k+1}^i q_\ell, & 1 \leq k \leq i-1. \end{cases}$$

The relations (4.4) and (4.5) can be easily verified by induction in i .

Lemma 4.2. Under the above assumptions of Section 1 and Lemma 2.1, for the error functions R and r , the following estimates hold:

$$\|R\|_{\infty, \omega_N} \leq C N^{-1} \ln N, \quad (4.6)$$

$$|r| \leq C N^{-1} \ln N. \quad (4.7)$$

Proof. From explicit expression (3.2) for R_i , on an arbitrary mesh we have

$$\begin{aligned} |R_i| &\leq h_i^{-1} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) \left| \frac{\partial f}{\partial t}(t, u(t), \lambda) + \frac{\partial f}{\partial u}(t, u(t), \lambda) u'(t) \right| dt \\ &\leq C h_i^{-1} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) (1 + |u'(t)|) dt, \quad 1 \leq i \leq N. \end{aligned}$$

This inequality together with (2.3) enables us to write

$$|R_i| \leq C \left\{ h_i^{-1} + h_i^{-1} \varepsilon^{-1} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) e^{-\alpha t/\varepsilon} dt \right\}, \quad 1 \leq i \leq N, \quad (4.8)$$

in which

$$h_i = \begin{cases} h^{(1)}, & 1 \leq i \leq N/2, \\ h^{(2)}, & N/2 + 1 \leq i \leq N. \end{cases}$$

We consider first the case $\sigma = T/2$ and so $T/2 \leq \alpha^{-1} \varepsilon \ln N$ and $h^{(1)} = h^{(2)} = T N^{-1}$. Hereby, since

$$h_i^{-1} \varepsilon^{-1} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) e^{-\alpha t/\varepsilon} dt \leq \varepsilon^{-1} h^{(1)} \leq \frac{2 \ln N}{\alpha T} \frac{T}{N} = 2 \alpha^{-1} N^{-1} \ln N,$$

it follows from (4.8) that

$$|R_i| \leq C N^{-1} \ln N, \quad 1 \leq i \leq N. \quad (4.9)$$

We now consider the case $\sigma = \alpha^{-1} \varepsilon \ln N$ and estimate R_i on $[0, \sigma]$ and $[\sigma, T]$ separately. In the layer region $[0, \sigma]$, inequality (4.8) reduces to

$$|R_i| \leq C (1 + \varepsilon^{-1}) h^{(1)} = C (1 + \varepsilon^{-1}) \frac{\alpha^{-1} \varepsilon \ln N}{N/2}, \quad 1 \leq i \leq N/2.$$

Hence

$$|R_i| \leq C N^{-1} \ln N, \quad 1 \leq i \leq N/2. \quad (4.10)$$

It remains to estimate R_i for $N/2 + 1 \leq i \leq N$. In this case we are able to write (4.8) as

$$|R_i| \leq C \left\{ h^{(2)} + \alpha^{-1} \left(e^{-\frac{\alpha t_{i-1}}{\varepsilon}} - e^{-\frac{\alpha t_i}{\varepsilon}} \right) \right\}, \quad N/2 + 1 \leq i \leq N. \quad (4.11)$$

Since $t_i = \alpha^{-1} \varepsilon \ln N + (i - N/2)h^{(2)}$ it follows that:

$$e^{-\frac{\alpha t_{i-1}}{\varepsilon}} - e^{-\frac{\alpha t_i}{\varepsilon}} = \frac{1}{N} e^{-\frac{\alpha(i-1-\frac{N}{2})h^{(2)}}{\varepsilon}} \left(1 - e^{-\frac{\alpha h^{(2)}}{\varepsilon}} \right) < N^{-1}$$

and this together with (4.11) to give the bound

$$|R_i| \leq C N^{-1}. \quad (4.12)$$

The inequalities (4.9), (4.10) and (4.12) finish the proof of (4.6).

Finally, we estimate the remainder term r . From the explicit expression (3.3) we obtain

$$|r| \leq \sum_{i=1}^N \int_{t_{i-1}}^{t_i} c(t) |t - t_{i-1}| |u'(t)| dt, \quad 1 \leq i \leq N,$$

This inequality together with (2.3) enable use to write

$$|r| \leq \|c\|_{\infty} C \sum_{i=1}^N h_i \int_{t_{i-1}}^{t_i} \left(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha t}{\varepsilon}} \right) dt, \quad 1 \leq i \leq N. \quad (4.13)$$

From (4.13), the validity of (4.7) follows:

$$\begin{aligned} |r| &\leq C \sum_{i=1}^{N/2} h^{(1)} \int_{t_{i-1}}^{t_i} \left(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha t}{\varepsilon}} \right) dt + C \sum_{i=N/2+1}^N h^{(2)} \int_{t_{i-1}}^{t_i} \left(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha t}{\varepsilon}} \right) dt, \\ &\leq C h^{(1)} \int_0^T \left(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha t}{\varepsilon}} \right) dt + C h^{(2)} \int_0^T \left(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha t}{\varepsilon}} \right) dt, \\ &\leq C \left(h^{(1)} + h^{(2)} \right) \leq C N^{-1} \ln N. \end{aligned}$$

□

Lemma 4.3. For the solution of (4.1)-(4.3), the following estimates hold

$$|\mu^N| \leq C |r|, \quad (4.14)$$

$$|z_0^N| \leq |r| + \|c\|_{\infty} T B_N \left(|\mu^N| M_1 + \|R\|_{\infty} \right), \quad (4.15)$$

$$|z_i^N| \leq |z_0^N| + \alpha^{-1} (M_1 |\mu^N| + \|R\|_{\infty}), \quad 1 \leq i \leq N-1, \quad (4.16)$$

where

$$B_N = \sum_{\ell=1}^N \frac{h_\ell}{\varepsilon + a_\ell h_\ell} Q_{N-\ell},$$

$$Q_{N-\ell} = \begin{cases} 1, & \text{for } \ell = N, \\ \prod_{s=\ell+1}^N \frac{\varepsilon}{\varepsilon + a_s h_s}, & \text{for } 1 \leq \ell \leq N-1. \end{cases}$$

Proof. The equation (4.1) can be rewritten as

$$\varepsilon z_{t,i}^N + a_i z_i^N = b_i \mu^N + R_i, \quad 1 \leq i \leq N-1, \quad (4.17)$$

with

$$a_i = \frac{\partial f}{\partial u} \left(t_i, u_i + \gamma z_i^N, \lambda + \gamma \mu^N \right),$$

$$b_i = -\frac{\partial f}{\partial \lambda} \left(t_i, u_i + \gamma z_i^N, \lambda + \gamma \mu^N \right), \quad 0 < \gamma < 1.$$

From (4.17) we have

$$z_i^N = \frac{\varepsilon}{\varepsilon + a_i h_i} z_{i-1}^N + \mu^N \frac{h_i b_i}{\varepsilon + a_i h_i} + \frac{h_i R_i}{\varepsilon + a_i h_i}. \quad (4.18)$$

Solving the first-order difference equation with respect to z_i^N by using (4.5) and setting the boundary condition (4.3), we get

$$z_i^N = -\mu^N \sum_{k=i+1}^N \frac{h_k b_k}{\varepsilon + a_k h_k} Q_{k-i}^{-1} - \sum_{k=i+1}^N \frac{h_k R_k}{\varepsilon + a_k h_k} Q_{k-i}^{-1}. \quad (4.19)$$

Taking into consideration in (4.19) the integral boundary condition (4.2), we have

$$\mu^N = \frac{r}{\sum_{k=1}^N \frac{h_k b_k}{\varepsilon + a_k h_k} Q_k^{-1} + \sum_{k=1}^N h_k c_k \sum_{s=k+1}^N \frac{h_s b_s}{\varepsilon + a_s h_s} Q_{s-k}^{-1}} + \frac{\sum_{k=1}^N \frac{h_k R_k}{\varepsilon + a_k h_k} Q_k^{-1} + \sum_{k=1}^N h_k c_k \sum_{s=k+1}^N \frac{h_s R_s}{\varepsilon + a_s h_s} Q_{s-k}^{-1}}{\sum_{k=1}^N \frac{h_k b_k}{\varepsilon + a_k h_k} Q_k^{-1} + \sum_{k=1}^N h_k c_k \sum_{s=k+1}^N \frac{h_s b_s}{\varepsilon + a_s h_s} Q_{s-k}^{-1}}. \quad (4.20)$$

Now, we estimate separately the terms on the right-hand side of equality (4.20). For the first term, we have

$$\left| \frac{r}{\sum_{k=1}^N \frac{h_k b_k}{\varepsilon + a_k h_k} Q_k^{-1} + \sum_{k=1}^N h_k c_k \sum_{s=k+1}^N \frac{h_s b_s}{\varepsilon + a_s h_s} Q_{s-k}^{-1}} \right|$$

$$\leq \frac{|r|}{m_1 \sum_{k=1}^N \frac{h_k}{\varepsilon + a_k h_k} Q_k^{-1}} \leq \frac{a^* |r|}{m_1 \rho_* \sum_{k=1}^N (1 + \rho_*)^{k-1}} = \frac{a^* |r|}{m_1 [(1 + \rho_*)^N - 1]},$$

here $\rho_k = a_k h_k / \varepsilon$ and $\rho_* = \min \rho_k$. Therefore, it is not hard to see that

$$\left| \frac{r}{\sum_{k=1}^N \frac{h_k b_k}{\varepsilon + a_k h_k} Q_k^{-1} + \sum_{k=1}^N h_k c_k \sum_{s=k+1}^N \frac{h_s b_s}{\varepsilon + a_s h_s} Q_{s-k}^{-1}} \right| \leq C |r| \quad (4.21)$$

Next, evidently

$$\left| \frac{\sum_{k=1}^N \frac{h_k R_k}{\varepsilon + a_k h_k} Q_k^{-1} + \sum_{k=1}^N h_k c_k \sum_{s=k+1}^N \frac{h_s R_s}{\varepsilon + a_s h_s} Q_{s-k}^{-1}}{\sum_{k=1}^N \frac{h_k b_k}{\varepsilon + a_k h_k} Q_k^{-1} + \sum_{k=1}^N h_k c_k \sum_{s=k+1}^N \frac{h_s b_s}{\varepsilon + a_s h_s} Q_{s-k}^{-1}} \right| \leq m_1^{-1} \|R\|_{\infty}. \quad (4.22)$$

After taking into consideration (4.21) and (4.22) in (4.20), we arrive at (4.14).

Now, we need to estimate z_0 . From (4.18), by using (4.4) we have

$$z_i^N = z_0^N Q_i + \mu^N \sum_{k=1}^i \frac{h_k b_k}{\varepsilon + a_k h_k} Q_{i-k} + \sum_{k=1}^i \frac{h_k R_k}{\varepsilon + a_k h_k} Q_{i-k}.$$

From here, by virtue of (4.2) it follows that

$$z_0^N = \frac{r - \sum_{k=1}^N h_k c_k \left(\mu^N \sum_{\ell=1}^k \frac{h_{\ell} b_{\ell}}{\varepsilon + a_{\ell} h_{\ell}} Q_{k-\ell} + \sum_{\ell=1}^k \frac{h_{\ell} R_{\ell}}{\varepsilon + a_{\ell} h_{\ell}} Q_{k-\ell} \right)}{1 + \sum_{k=1}^N h_k c_k Q_k}.$$

Thereby

$$\begin{aligned} |z_0^N| &\leq |r| + \|c\|_{\infty} T \left\{ \mu^N \left| \sum_{\ell=1}^N \frac{h_{\ell} |b_{\ell}|}{\varepsilon + a_{\ell} h_{\ell}} Q_{N-\ell} + \sum_{\ell=1}^N \frac{h_{\ell} |R_{\ell}|}{\varepsilon + a_{\ell} h_{\ell}} Q_{N-\ell} \right| \right\}, \\ &\leq |r| + \|c\|_{\infty} T \left\{ (M_1 |\mu^N| + \|R\|_{\infty}) \sum_{\ell=1}^N \frac{h_{\ell}}{\varepsilon + a_{\ell} h_{\ell}} Q_{N-\ell} \right\}, \end{aligned}$$

which implies validity of (4.15).

Finally, an applying the maximum principle for the difference operator $L^N z_i^N := \varepsilon z_{i,i}^N + a_i z_i^N$, $1 \leq i \leq N$, to Eq. (4.17) immediately leads to (4.16). $\square$

Combining the two previous lemmas gives us the following convergence result.

Theorem 4.1. Let $\{u(t), \lambda\}$ and $\{u_i^N, \lambda^N\}$ be the exact solution and discrete solution on $\bar{\omega}_N$ respectively. Then the following estimates hold

$$\begin{aligned} |\lambda - \lambda^N| &\leq C N^{-1} \ln N, \\ \|u - u^N\|_{\infty, \bar{\omega}_N} &\leq C N^{-1} \ln N. \end{aligned}$$

Proof. This follows immediately by combining the previous lemmas. $\square$

5. ALGORITHM AND NUMERICAL RESULTS

Here, we consider a test problem to show the applicability and efficiency of the method described in this paper.

a) We solve the nonlinear problem (3.5)-(3.7) using the following quasilinearization technique:

$$\begin{aligned}\lambda^{(n)} &= \lambda^{(n-1)} - \frac{\left(B - u_{N-1}^{(n-1)} - u_{i-1}^{(n)}\right) \rho_N^{-1} + f\left(T, B, \lambda^{(n-1)}\right)}{\partial f / \partial \lambda\left(T, B, \lambda^{(n)}\right)}, \\ u_0^{(n)} &= A - c_N h_N B - \sum_{i=1}^{N-1} h_i b_i u_i^{(n-1)}, \\ u_i^{(n)} &= u_i^{(n-1)} - \frac{\left(u_i^{(n-1)} - u_{i-1}^{(n)}\right) \rho_i^{-1} + f\left(t_i, u_i^{(n-1)}, \lambda^{(n)}\right)}{\partial f / \partial u\left(t_i, u_i^{(n-1)}, \lambda^{(n)}\right) + \rho_i^{-1}}, \quad n = 1, 2, \dots\end{aligned}$$

where, $\rho_i = h_i / \varepsilon$; $\lambda^{(0)}$ and $u_i^{(0)}$ ($1 \leq i \leq N-1$) are the initial iterations given.

b) Consider the test problem:

$$\begin{aligned}\varepsilon u' + 2u - e^{-u} + t^2 + \lambda + \tanh(\lambda + t) &= 0, \quad 0 < t < 1, \\ u(0) + \frac{1}{4} \int_0^1 e^{-s} u(s) ds &= 1, \\ u(1) &= 0.\end{aligned}$$

The exact solution of our test problem is not available. Therefore we use the double mesh principle to estimate the errors and to compute the experimental rates of convergence. The error estimates obtained in this way are denoted by

$$e_u^{\varepsilon, N} = \max_{\omega_N} \left| u^{\varepsilon, N} - \tilde{u}^{\varepsilon, 2N} \right|, \quad e_\lambda^{\varepsilon, N} = \left| \lambda^{\varepsilon, N} - \tilde{\lambda}^{\varepsilon, 2N} \right|,$$

where $\left\{ \tilde{u}^{\varepsilon, 2N}, \tilde{\lambda}^{\varepsilon, 2N} \right\}$ is the approximate solution on the mesh

$$\tilde{\omega}_{2N} = \{t_{i/2} : i = 0, 1, \dots, 2N\}$$

with $t_{i+1/2} = (t_i + t_{i+1})/2$ for $i = 0, 1, \dots, N-1$. The corresponding rates of convergence are calculated by

$$p_u^{\varepsilon, N} = \ln(e_u^{\varepsilon, N} / e_u^{\varepsilon, 2N}) / \ln 2$$

for u , and

$$p_\lambda^{\varepsilon, N} = \ln(e_\lambda^{\varepsilon, N} / e_\lambda^{\varepsilon, 2N}) / \ln 2$$

for λ . The ε -uniform errors p_u^N , p_λ^N are estimated from

$$e_u^N = \max_\varepsilon e_u^{\varepsilon, N}, \quad e_\lambda^N = \max_\varepsilon e_\lambda^{\varepsilon, N}.$$

The corresponding ε -uniform convergence rates are

$$p_u^N = \ln(e_u^N / e_u^{2N}) / \ln 2, \quad p_\lambda^N = \ln(e_\lambda^N / e_\lambda^{2N}) / \ln 2.$$

In the computations in this section we take $\alpha = 2$. The initial guess in the iteration process is taken as $u_i^{(0)} = 1 - t_i^2$, $\lambda^{(0)} = -0.4$ and the stopping criterion is

$$\max_i |u_i^{(n)} - u_i^{(n-1)}| \leq 10^{-5}, \quad |\lambda^{(n)} - \lambda^{(n-1)}| \leq 10^{-5}.$$

The values of ε and N for which we solve the test problem are $\varepsilon = 2^{-i}$, $i = 2, 4, \dots, 16$; $N = 64, 128, 256, 512, 1024$. Some results of numerical experiment are displayed in Tables 1 and 2. The numerical results are the clear illustration of the error estimates.

TABLE 1. Errors $e_u^{\varepsilon, N}$ computed ε -uniform errors e_u^N and convergence rates $p_u^{\varepsilon, N}$ on ω_N .

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-2}	0.00047526	0.00025293	0.00133677	0.00069196	0.00035325
	0.91	0.92	0.95	0.97	
2^{-4}	0.00477384	0.00254057	0.00133344	0.00069070	0.00035261
	0.91	0.93	0.95	0.97	
2^{-6}	0.00357052	0.00203655	0.00115359	0.00064445	0.00035261
	0.81	0.82	0.84	0.87	
2^{-8}	0.00354583	0.00203655	0.00115359	0.00064445	0.00035261
	0.80	0.82	0.84	0.87	
2^{-10}	0.00349702	0.00203655	0.00115359	0.00064445	0.00035261
	0.78	0.82	0.84	0.87	
2^{-12}	0.00349702	0.00203655	0.00115359	0.00064445	0.00035261
	0.78	0.82	0.84	0.87	
2^{-14}	0.00349702	0.00203655	0.00115359	0.00064445	0.00035261
	0.78	0.82	0.84	0.87	
2^{-16}	0.00349702	0.00203655	0.00115359	0.00064445	0.00035261
	0.78	0.82	0.84	0.87	
$e_u^{\varepsilon, N}$	0.00349702	0.00203655	0.00133677	0.00069196	0.00035261
$p_u^{\varepsilon, N}$	0.78	0.82	0.84	0.97	

TABLE 2. Errors $e_{\lambda}^{\varepsilon,N}$ computed ε -uniform errors e_{λ}^N and convergence rates $p_{\lambda}^{\varepsilon,N}$ on ω_N .

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-2}	0.03255711 0.83	0.01831433 0.87	0.01002063 0.91	0.00533283 0.95	0.00276045
2^{-4}	0.03134362 0.77	0.01838045 0.81	0.01048388 0.85	0.00581630 0.89	0.00313856
2^{-6}	0.03134362 0.77	0.01838045 0.81	0.01048388 0.85	0.00581630 0.89	0.00313856
2^{-8}	0.03134362 0.77	0.01838045 0.81	0.01048388 0.85	0.00581630 0.89	0.00313856
2^{-10}	0.03134362 0.77	0.01838045 0.81	0.01048388 0.85	0.00581630 0.89	0.00313856
2^{-12}	0.03134362 0.77	0.01838045 0.81	0.01048388 0.85	0.00581630 0.89	0.00313856
2^{-14}	0.03134362 0.77	0.01838045 0.81	0.01048388 0.85	0.00581630 0.89	0.00313856
2^{-16}	0.03134362 0.77	0.01838045 0.81	0.01048388 0.85	0.00581630 0.89	0.00313856
$e_{\lambda}^{\varepsilon,N}$	0.03134362	0.01838045	0.01048388	0.00581630	0.00313856
$p_{\lambda}^{\varepsilon,N}$	0.77	0.81	0.85	0.89	

6. CONCLUSION

A parameterized singular perturbation problem with integral boundary condition is considered. The difference scheme is constructed by the method of integral identities with the use of interpolating quadrature rules with the weight and remainder terms in integral form. The numerical method presented here comprises a backward difference operator on a non-uniform mesh for the equation and composite rectangle rule for the integral condition. It is shown that the method displays uniform convergence with respect to the perturbation parameter. Numerical results confirm our theoretical analysis. The main lines for the analysis of the uniform convergence carried out here can be used for the study of more complicated nonlinear singularly perturbed analogous type problems.

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Authors' addresses

Mustafa Kudu

Department of Mathematics, Faculty of Arts and Sciences, Erzincan University, 24100, Erzincan, Turkey

E-mail address: muskud28@yahoo.com

Ilhame Amirali

Department of Mathematics, Faculty of Arts and Sciences, Düzce University, 81620, Düzce, Turkey

E-mail address: ailhame@gmail.com

Gabil M. Amiraliyev

Department of Mathematics, Faculty of Arts and Sciences, Erzincan University, 24100, Erzincan, Turkey

E-mail address: gabilamirali@yahoo.com



SOME RESULTS FOR q -POLY-BERNOULLI POLYNOMIALS WITH A PARAMETER

M. MECHACHA, M. A. BOUTICHE, M. RAHMANI, AND D. BEHLOUL

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Abstract. The main object of this paper is to investigate a new class of the generalized q -poly-Bernoulli numbers and polynomials with a parameter. We give explicit formulas and a recursive method for the calculation of the q -poly-Bernoulli numbers and polynomials. As a consequence, we derive a method for the calculation of the special values at negative integral points of the Arakawa–Kaneko zeta function also known as generalized Hurwitz zeta function.

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1. INTRODUCTION

Let q be an indeterminate with $0 \leq q < 1$. The q -analogue of x is defined by

$$[x]_q = \frac{1 - q^x}{1 - q}$$

with $[0]_q = 0$ and $\lim_{q \rightarrow 1} [x]_q = x$. Recently Komatsu in [12] introduced and studied a new family of polynomials, called q -poly-Bernoulli polynomials $B_{n,\rho,q}^{(k)}(z)$ with a real parameter ρ which are defined by the following generating function:

$$F_{q,\rho}(t; z) := \frac{\rho}{1 - e^{-\rho t}} \text{Li}_{k,q} \left(\frac{1 - e^{-\rho t}}{\rho} \right) e^{-tz} = \sum_{n=0}^{\infty} B_{n,\rho,q}^{(k)}(z) \frac{t^n}{n!}, \quad (1.1)$$

$$(n \geq 0; k \in \mathbb{Z}; \rho \neq 0)$$

where $\text{Li}_{k,q}(z)$ is the q -polylogarithm function [11] defined by

$$\text{Li}_{k,q}(z) = \sum_{n=1}^{\infty} \frac{z^n}{[n]_q^k}.$$

Clearly, we have

$$\lim_{q \rightarrow 1} B_{n,\rho,q}^{(k)}(z) = B_{n,\rho}^{(k)}(z),$$

which is the poly-Bernoulli polynomial with a ρ parameter [7], and

$$\lim_{q \rightarrow 1} \text{Li}_{k,q}(z) = \text{Li}_k(z),$$

which is the ordinary polylogarithm function, defined by

$$\text{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}. \quad (1.2)$$

In addition, when $z = 0$, $B_{n,\rho}^{(k)}(0) = B_{n,\rho}^{(k)}$ is the poly-Bernoulli number with a ρ parameter. When $z = 0$ and $\rho = 1$, $B_{n,1}^{(k)}(0) = B_n^{(k)}$ is the poly-Bernoulli number [1-3, 10] defined by

$$\frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}, \quad (1.3)$$

In this paper, we propose to investigate a new class of the generalized q -poly-Bernoulli numbers and polynomials with a parameter which we call (m, q) -poly-Bernoulli polynomials with a parameter ρ . We establish several properties of these polynomials. The study of (m, q) -poly-Bernoulli polynomials with a parameter yields an interesting algorithm for calculating $\mathbb{B}_{n,m}^{(k)}(z; \rho, q)$. As an application, we derive a recursive method for the calculation of the special values at negative integral points of the Arakawa-Kaneko zeta function.

We first recall some basic definitions and some results [8, 16] that will be useful in the rest of the paper. The (signed) Stirling numbers $s(n, i)$ of the first kind are the coefficients in the following expansion:

$$x(x-1)\cdots(x-n+1) = \sum_{i=0}^n s(n, i) x^i, \quad n \geq 1$$

and satisfy the recurrence relation given by

$$s(n+1, i) = s(n, i-1) - ns(n, i) \quad (1 \leq i \leq n). \quad (1.4)$$

The Stirling numbers of the second kind, denoted $S(n, i)$ are the coefficients in the expansion

$$x^n = \sum_{i=0}^n S(n, i) x(x-1)\cdots(x-i+1), \quad n \geq 1.$$

These numbers count the number of ways to partition a set of n elements into exactly i nonempty subsets.

The exponential generating functions for $s(n, i)$ and $S(n, i)$ are given by

$$\sum_{n=i}^{\infty} s(n, i) \frac{z^n}{n!} = \frac{1}{i!} [\ln(1+z)]^i$$

and

$$\sum_{n=i}^{\infty} S(n, i) \frac{z^n}{n!} = \frac{1}{i!} (e^z - 1)^i,$$

respectively.

The weighted Stirling numbers $\mathcal{S}_n^i(x)$ of the second kind are defined by (see [5,6])

$$\begin{aligned} \mathcal{S}_n^i(x) &= \frac{1}{i!} \Delta^i x^n \\ &= \frac{1}{i!} \sum_{j=0}^i (-1)^{i-j} \binom{i}{j} (x+j)^n, \end{aligned}$$

where Δ denotes the forward difference operator. The exponential generating function of $\mathcal{S}_n^i(x)$ is given by

$$\sum_{n=i}^{\infty} \mathcal{S}_n^i(x) \frac{z^n}{n!} = \frac{1}{i!} e^{xz} (e^z - 1)^i \quad (1.5)$$

and weighted Stirling numbers $\mathcal{S}_n^i(x)$ satisfy the following recurrence relation:

$$\mathcal{S}_{n+1}^i(x) = \mathcal{S}_n^{i-1}(x) + (x+i) \mathcal{S}_n^i(x) \quad (1 \leq i \leq n).$$

In particular, we have for nonnegative integer r

$$\mathcal{S}_n^i(0) = S(n, i) \text{ and } \mathcal{S}_n^i(r) = \left\{ \begin{matrix} n+r \\ i+r \end{matrix} \right\}_r.$$

where $\left\{ \begin{matrix} n \\ i \end{matrix} \right\}_r$ denotes the r -Stirling numbers of the second kind [4].

2. THE (m, q) -POLY-BERNOULLI NUMBERS WITH A PARAMETER ρ

In order to compute $B_{n,\rho,q}^{(k)} := B_{n,\rho,q}^{(k)}(0)$, we define (m, q) -poly-Bernoulli numbers $\mathbb{B}_{n,m}^{(k)}(\rho, q)$ with a parameter ρ in terms of m -Stirling numbers of the second kind by:

$$\mathbb{B}_{n,m}^{(k)}(\rho, q) = \frac{(-\rho)^n [m+1]_q^k}{m!} \sum_{i=0}^n \frac{(m+i)! \mathcal{S}_n^i(m)}{(-\rho)^i [m+i+1]_q^k}, \quad m \geq 0 \quad (2.1)$$

with $\mathbb{B}_{0,m}^{(k)}(\rho, q) = 1$ and $\mathbb{B}_{n,0}^{(k)}(\rho, q) = B_{n,\rho,q}^{(k)}$.

By direct computation from (2.1), we find

$$\begin{aligned} \mathbb{B}_{0,m}^{(k)}(\rho, q) &= 1, \\ \mathbb{B}_{1,m}^{(k)}(\rho, q) &= (m+1) \left(\frac{q^{m+1} - 1}{q^{m+2} - 1} \right)^k - m, \end{aligned}$$

$$\begin{aligned}\mathbb{B}_{2,m}^{(k)}(\rho, q) &= (m+1)(m+2) \left(\frac{q^{m+1}-1}{q^{m+3}-1} \right)^k \\ &\quad - (2m^2 + 3m + 1) \left(\frac{q^{m+1}-1}{q^{m+2}-1} \right)^k + m^2.\end{aligned}$$

The following Theorem gives us a relation between the (m, q) -poly-Bernoulli numbers $\mathbb{B}_{n,m}^{(k)}(\rho, q)$ and q -poly-Bernoulli numbers $B_n^{(k)}(\rho, q) := B_{n,\rho,q}^{(k)}$.

Theorem 1. For $m \geq 0$, we have

$$\mathbb{B}_{n,m}^{(k)}(\rho, q) = \frac{(-\rho)^m [m+1]_q^k}{m!} \sum_{i=0}^m s(m, i) \frac{B_{n+i}^{(k)}(\rho, q)}{(-\rho)^i}. \quad (2.2)$$

Proof. The explicit formula (2.2) can be derived from a known result in [14, p. 681, Corollary 1] for the Stirling transform upon specializing the initial sequence

$$a_{0,m} = \frac{m!}{(-\rho)^m [m+1]_q^k}.$$

□

The next Theorem contains the exponential generating function for (m, q) -poly-Bernoulli numbers with a parameter ρ .

Theorem 2. The exponential generating function for $\mathbb{B}_{n,m}^{(k)}(\rho, q)$ is given by

$$\sum_{n=0}^{\infty} \mathbb{B}_{n,m}^{(k)}(\rho, q) \frac{t^n}{n!} = \frac{(-\rho)^{m+n} [m+1]_q^k}{m!} e^{mt} \left(e^{-t} \frac{d}{dt} \right)^m F_{q,\rho}(t; z).$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{B}_{n,m}^{(k)}(\rho, q) \frac{t^n}{n!} &= \frac{(-\rho)^m [m+1]_q^k}{m!} \sum_{i=0}^m s(m, i) \sum_{n=0}^{\infty} \frac{B_{n+i}^{(k)}(\rho, q)}{(-\rho)^i} \frac{t^n}{n!} \\ &= \frac{(-\rho)^{m+n} [m+1]_q^k}{m!} \sum_{i=0}^m s(m, i) \frac{d^i}{dt^i} \left(\frac{\rho}{1-e^t} \text{Li}_{k,q} \left(\frac{1-e^t}{\rho} \right) \right).\end{aligned}$$

Since [13]

$$\sum_{i=0}^m s(m, i) \left(\frac{d}{dt} \right)^i = e^{mt} \left(e^{-t} \frac{d}{dt} \right)^m,$$

we get the desired result. □

Next, we propose an algorithm, which is based on a three-term recurrence relation, for calculating the (m, q) -poly-Bernoulli numbers $\mathbb{B}_{n,m}^{(k)}(\rho, q)$ with a parameter ρ .

Theorem 3. For every integer k , the $\mathbb{B}_{n,m}^{(k)}(\rho, q)$ satisfies the following three-term recurrence relation:

$$\mathbb{B}_{n+1,m}^{(k)}(\rho, q) = (m+1) \left(\frac{q^{m+1}-1}{q^{m+2}-1} \right)^k \mathbb{B}_{n,m+1}^{(k)}(\rho, q) - \rho m \mathbb{B}_{n,m}^{(k)}(\rho, q) \quad (2.3)$$

with the initial sequence given by $\mathbb{B}_{0,m}^{(k)}(\rho, q) = 1$.

Proof. From (2.2) and (1.4), we have

$$\mathbb{B}_{n,m+1}^{(k)}(\rho, q) = \frac{(-\rho)^{m+1} [m+2]_q^k}{(m+1)!} \sum_{i=0}^{m+1} (s(m, i-1) - m s(m, i)) \frac{B_{n+i}^{(k)}(\rho, q)}{(-\rho)^i}.$$

After some simplifications, we find that

$$\mathbb{B}_{n,m+1}^{(k)}(\rho, q) = \frac{1}{[m+1]_q^k} \frac{(-\rho)[m+2]_q^k}{m+1} \left(\frac{1}{(-\rho)} \mathbb{B}_{n+1,m}^{(k)}(\rho, q) - m \mathbb{B}_{n,m}^{(k)}(\rho, q) \right).$$

This evidently equivalent to (2.3). $\square$

Remark 1. If we set $\rho = 1, k = 1$ and $q \rightarrow 1$, in (2.3), we get

$$B_{n+1,m} = \frac{(m+1)^2}{(m+2)} B_{n,m+1} - m B_{n,m}, \quad (2.4)$$

an algorithm for the classical Bernoulli numbers with $B_1 = \frac{1}{2}$. See [15] for the case $B_1 = -\frac{1}{2}$.

3. THE (m, q) -POLY-BERNOULLI POLYNOMIALS WITH A PARAMETER ρ

For $m \geq 0$, let us consider the (m, q) -poly-Bernoulli polynomials with a parameter $\mathbb{B}_{n,m}^{(k)}(z; \rho, q)$ as follows:

$$\mathbb{B}_{n,m}^{(k)}(z; \rho, q) = \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} \mathbb{B}_{i,m}^{(k)}(\rho, q) z^{n-i}. \quad (3.1)$$

It is easy to show that the generating function of $\mathbb{B}_{n,m}^{(k)}(z; \rho, q)$ is given by

$$\begin{aligned} \sum_{n \geq 0} \mathbb{B}_{n,m}^{(k)}(z; \rho, q) \frac{t^n}{n!} &= e^{-zt} \sum_{n \geq 0} \mathbb{B}_{n,m}^{(k)}(\rho, q) \frac{t^n}{n!} \\ &= \frac{1}{m!} (-\rho)^{m+n} [m+1]_q^k e^{(m-z)t} \left(e^{-t} \frac{d}{dt} \right)^m F_{q,\rho}(t; z). \end{aligned}$$

Next, we show an explicit formula about $\mathbb{B}_{n,m}^{(k)}(z; \rho, q)$.

Theorem 4. *The following formula holds true*

$$\mathbb{B}_{n,m}^{(k)}(z; \rho, q) = \frac{[m+1]_q^k}{m!} \sum_{i=0}^n (-\rho)^{n-i} \frac{(m+i)!}{[m+i+1]_q^k} \mathcal{S}_n^i \left(\frac{z}{\rho} + m \right).$$

Proof. From (3.1), we have

$$\begin{aligned} \mathbb{B}_{n,m}^{(k)}(z; \rho, q) &= \frac{[m+1]_q^k}{m!} \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} \sum_{j=0}^i \frac{(-\rho)^{i-j} (m+j)! \mathcal{S}_i^j(m)}{[m+j+1]_q^k} z^{n-i} \\ &= \frac{[m+1]_q^k}{m!} \sum_{j=0}^n \frac{(-\rho)^{n-j} (m+j)!}{[m+j+1]_q^k} \sum_{i=0}^n \binom{n}{i} \mathcal{S}_i^j(m) \left(\frac{z}{\rho} \right)^{n-i}. \end{aligned}$$

By using the relation

$$\mathcal{S}_n^i(x+y) = \sum_{l=0}^n \binom{n}{l} \mathcal{S}_l^i(x) y^{n-l},$$

we obtain

$$\mathbb{B}_{n,m}^{(k)}(z; \rho, q) = \frac{[m+1]_q^k (-\rho)^n}{m!} \sum_{j=0}^n \frac{(m+j)!}{(-\rho)^j [m+j+1]_q^k} \mathcal{S}_n^j \left(\frac{z}{\rho} + m \right),$$

we arrive at the desired result. $\square$

Theorem 5. *The polynomials $\mathbb{B}_{n,m}^{(k)}(z; \rho, q)$ satisfy the following three-term recurrence relation:*

$$\begin{aligned} \mathbb{B}_{n+1,m}^{(k)}(z; \rho, q) &= (m+1) \left(\frac{q^{m+1}-1}{q^{m+2}-1} \right)^k \mathbb{B}_{n,m+1}^{(k)}(z; \rho, q) \\ &\quad + (z - \rho m) \mathbb{B}_{n,m}^{(k)}(z; \rho, q), \end{aligned} \quad (3.2)$$

with the initial sequence given by

$$\mathbb{B}_{0,m}^{(k)}(z; \rho, q) = 1.$$

Proof. From (3.1), we get

$$\begin{aligned} \frac{d}{dz} \mathbb{B}_{n,m}^{(k)}(z; \rho, q) &= \sum_{i=0}^n (n-i) (-1)^{n-i} \binom{n}{i} \mathbb{B}_{i,m}^{(k)}(\rho, q) z^{n-i-1} \\ &= n \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} \mathbb{B}_{i,m}^{(k)}(\rho, q) z^{n-i-1} \\ &\quad - n \sum_{i=1}^n (-1)^{n-i} \binom{n-1}{i-1} \mathbb{B}_{i,m}^{(k)}(\rho, q) z^{n-i-1}. \end{aligned}$$

Then

$$z \frac{d}{dz} \mathbb{B}_{n,m}^{(k)}(z; \rho, q) = n \mathbb{B}_{n,m}^{(k)}(z; \rho, q) - n \sum_{i=0}^{n-1} (-1)^{n-i-1} \binom{n-1}{i} \mathbb{B}_{i+1,m}^{(k)}(\rho, q) z^{n-i-1}.$$

Now, using (2.3), we have

$$z \frac{d}{dz} \mathbb{B}_{n,m}^{(k)}(z; \rho, q) = n \mathbb{B}_{n,m}^{(k)}(z; \rho, q) + n \rho m \sum_{i=0}^{n-1} \binom{n-1}{i} \mathbb{B}_{i,m}^{(k)}(\rho, q) (-z)^{n-i-1} - n(m+1) \left(\frac{q^{m+1}-1}{q^{m+2}-1} \right)^k \sum_{i=0}^{n-1} \binom{n-1}{i} \mathbb{B}_{i,m+1}^{(k)}(\rho, q) (-z)^{n-i-1},$$

which, after simplification, yields

$$z \mathbb{B}_{n-1,m}^{(k)}(z; \rho, q) = \mathbb{B}_{n,m}^{(k)}(z; \rho, q) - (m+1) \left(\frac{q^{m+1}-1}{q^{m+2}-1} \right)^k \mathbb{B}_{n-1,m+1}^{(k)}(z; \rho, q) + \rho m \mathbb{B}_{n-1,m}^{(k)}(z; \rho, q),$$

which is obviously equivalent to (3.2) and the proof is complete. $\square$

As consequence of Theorem 5, one can deduce a three-term recurrence relation for (m, q) -poly-Bernoulli polynomials with a parameter ρ and negative upper indices $\mathbb{B}_{n,m}^{(-k)}(z; \rho, q)$.

Corollary 1. *The $\mathbb{B}_{n,m}^{(-k)}(z; \rho, q)$ satisfies the following three-term recurrence relation:*

$$\mathbb{B}_{n+1,m}^{(-k)}(z; \rho, q) = (m+1) \left(\frac{q^{m+2}-1}{q^{m+1}-1} \right)^k \mathbb{B}_{n,m+1}^{(-k)}(z; \rho, q) + (z - \rho m) \mathbb{B}_{n,m}^{(-k)}(z; \rho, q),$$

with the initial sequence given by

$$\mathbb{B}_{0,m}^{(-k)}(z; \rho, q) = 1.$$

The next result gives a method for the calculation of the special values at negative integral points of the Arakawa–Kaneko zeta function. Recall that the Arakawa–Kaneko zeta function $\xi_k(s, x)$, for $s \in \mathbb{C}$, $x > 0$, $k \in \mathbb{Z}$, is defined by [9]

$$\xi_k(s, x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \frac{\text{Li}_k(1-e^{-t})}{1-e^{-t}} e^{-xt} dt$$

It is well-known that the special values at negative integral points are given in terms of poly-Bernoulli polynomials

$$\xi_k(-n, x) = (-1)^n B_n^{(k)}(x), n \geq 0.$$

We now present the following algorithm for $\xi_k(-n, x)$. We start with the sequence $\mathcal{K}_{0,m} = 1$ as the first row of the matrix $(\mathcal{K}_{n,m})_{n,m \geq 0}$. Each entry is determined recursively by

$$\mathcal{K}_{n+1,m}(k, x) = \frac{(m+1)^{k+1}}{(m+2)^k} \mathcal{K}_{n,m+1}(k, x) + (x-m) \mathcal{K}_{n,m}(k, x).$$

Then

$$\xi_k(-n, x) = (-1)^n \mathcal{K}_{n,0}(k, x)$$

where $\mathcal{K}_{n,0}(k, x)$ are the first column of the matrix $(\mathcal{K}_{n,m})_{n,m \geq 0}$.

4. CONCLUSION

In our present research, we have investigated a new class of the generalized q -poly-Bernoulli numbers and polynomials with a parameter. As a consequence, we derive a method for the calculation of the special values at negative integral points of the Arakawa-Kaneko zeta function. We have also given a recursive method for the calculation of q -poly-Bernoulli numbers and polynomials with parameter.

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Authors’ addresses

M. Mechacha

USTHB, Faculty of Mathematics, BP 32, El Alia, 16111 Bab Ezzouar, Algiers, Algeria

E-mail address: mmechacha@usthb.dz

M. A. Boutiche

USTHB, Faculty of Mathematics, BP 32, El Alia, 16111 Bab Ezzouar, Algiers, Algeria

E-mail address: mboutiche@usthb.dz

M. Rahmani

USTHB, Faculty of Mathematics, BP 32, El Alia, 16111 Bab Ezzouar, Algiers, Algeria

E-mail address: mrahmani@usthb.dz

D. Behloul

USTHB, Department of Computer Science, BP 32, El Alia, 16111 Bab Ezzouar, Algiers, Algeria

E-mail address: behloul.djilali@gmail.com



INFINITE SYSTEMS OF INTEGRAL EQUATIONS IN FRÉCHET SPACES USING THE NOTION OF L-FUNCTIONS

MOHSEN HOSSEINZADEH MOGHADDAM, REZA ALLAHYARI,
MOHSEN ERFANIAN OMIDVAR, AND MAHMOUD HASSANI

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Abstract. The aim of this paper is to prove some fixed point theorems for L-functions with the help of measure of noncompactness and the Tychonoff fixed point theorem. Also, we prove some existence theorems for a general infinite system of integral equations. As an application, we study the problem of the existence of solutions for infinite systems of nonlinear integral equations of Hammerstein type in two variables. The results obtained extend several ones. Finally, an example is presented to guarantee our results.

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Keywords: measure of noncompactness, Fréchet spaces, Tychonoff fixed point theorem, infinite systems of equations

1. INTRODUCTION AND PRELIMINARIES

There are three major branches of fixed point theory in functional analysis, and each branch has its celebrated theorems. In the topological branch of fixed point theory, the main three theorems are Brouwer's fixed point theorem, its infinite dimensional version, Schauder's fixed point theorem, and Tychonoff fixed point theorem on locally convex spaces which in each of them compactness plays an essential role.

Theorem 1 (Schauder [1]). *Let C be a closed and convex subset of a Banach space E . Then every compact and continuous map $F : C \rightarrow C$ has at least one fixed point.*

Theorem 2 (Tychonoff fixed point theorem [1]). *Let E be a Hausdorff locally convex linear topological space, C a convex subset of E and $F : C \rightarrow E$ a continuous mapping such that*

$$F(C) \subseteq A \subseteq C$$

with A compact. Then F has at least one fixed point.

The concept of measure of noncompactness together with the well-known Darbo's fixed point theorem have played a basic role in nonlinear functional analysis, especially in topological fixed point theory. Up to now, several papers have been published

on the existence and behavior of solutions of nonlinear differential and integral equations, using the technique of measure of noncompactness together with the Darbo's fixed point theorem. Some of these works are noted in [3, 4, 7, 8, 10, 12–16, 18, 19]. Recently, the theory of infinite systems of integral or differential equations can also be used in solving some problems for differential and integral equations (see [5, 11, 22–25]).

On the other hand, In 1969, Meir and Keeler [21] proved the following very interesting fixed-point theorem, which is a generalization of the Banach contraction principle [6].

Definition 1 ([21]). Let (X, d) be a metric space. Then a mapping T on X is said to be a Meir-Keeler contraction (MKC, for short) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \implies d(Tx, Ty) < \varepsilon, \quad (1.1)$$

for all $x, y \in X$.

Theorem 3 (Meir and Keeler [21]). *Let (X, d) be a complete metric space. If $T : X \longrightarrow X$ is a Meir-Keeler contraction, then T has a unique fixed point.*

Next, Lim [20] and Suzuki [26] introduced the notion of L-functions and characterized Meir-Keeler contractions in metric spaces. Moreover, Aghajani et al. [2] introduce the notion of Meir-Keeler condensing operator on a Banach space, a characterization using strictly L-functions and provide a few generalizations of Darbo's fixed point theorem. In this paper, we state and prove some fixed point theorems for L-functions on Fréchet spaces with the help of measure of noncompactness and the Tychonoff fixed point theorem, which is an extension of the results [21], Lim [20], Suzuki [26] and Darbo [13]. Then, we prove some existence theorems for a general infinite system of integral equations. Finally, using the obtained results, we are going to study the existence of continuous solutions of the infinite system of nonlinear integral equations of Hammerstein type

$$\begin{aligned} & x_n(t, s) \\ & = f_n\left(t, s, x_1(t, s), \dots, x_n(t, s), \int_0^\infty \int_0^\infty k_n(t, s, v, w) h_n(v, w, (x_j(v, w))_{j=1}^\infty) dv dw\right), \end{aligned} \quad (1.2)$$

$t, s \in \mathbb{R}_+$, $n \in \mathbb{N}$. The functions f_n , k_n and h_n ($n \in \mathbb{N}$) are continuous and satisfy some certain conditions, specified later. Furthermore, an example is presented to guarantee our results.

Now, we recall some basic facts concerning measures of noncompactness. Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, +\infty)$. Let $(E, \|\cdot\|)$ be a real Banach space with zero element 0. Let $\overline{B}(x, r)$ denote the closed ball centered at x with radius r . The symbol $\overline{B}_r$ stands for the ball $\overline{B}(0, r)$. For X , a nonempty subset of E , we denote by $\overline{X}$ and $\text{Conv}X$ the closure and the closed convex hull of X , respectively. Moreover, let us denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

A topological vector space (TVS) is a vector space X over the field $\mathbb{R}$ which is endowed with a topology such that the maps $(x, y) \rightarrow x + y$ and $(\alpha, x) \rightarrow \alpha x$ are continuous from $X \times X$ and $\mathbb{R} \times X$ to X . A topological vector space is called locally convex if there is a basic for the topology consisting of convex sets (that is, sets A such that if $x, y \in A$ then $tx + (1-t)y \in A$ for $0 < t < 1$).

Definition 2 (Definition 1.11 in [9]). A Fréchet space is a complete linear metric space or equivalently, a complete total paranormed space. In other words, a locally convex space is called a Fréchet space if it is metrizable and the underlying metric space is complete.

Definition 3 ([5]). Let $\mathcal{M}$ be a class of subsets of a Fréchet space E , we say $\mathcal{M}$ is an admissible set if $\mathfrak{N}_E \cap \mathcal{M} \neq \emptyset$ and if $X \in \mathcal{M}$, then $\text{Conv}(X), \overline{X} \in \mathcal{M}$.

Definition 4 ([5]). Let $\mathcal{M}$ be an admissible subset of a Fréchet space E , we say that $\mu : \mathcal{M} \rightarrow \mathbb{R}_+$ be a measure of noncompactness on Fréchet space E if it satisfies the following conditions:

- (1°) The family $\ker \mu = \{X \in \mathcal{M} : \mu(X) = 0\}$ is nonempty and $\ker \mu \subseteq \mathfrak{N}_E$.
- (2°) $X \subset Y \implies \mu(X) \leq \mu(Y)$.
- (3°) $\mu(\overline{X}) = \mu(X)$.
- (4°) $\mu(\text{Conv} X) = \mu(X)$.
- (5°) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda \mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
- (6°) If $\{X_n\}$ is a sequence of closed sets from $\mathcal{M}$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \dots$, and if $\lim_{n \rightarrow \infty} \mu(X_n) = 0$, then $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

Definition 5 (Lim [20]). A function φ from $\mathbb{R}_+$ into itself is called an L-function if $\varphi(0) = 0$, $\varphi(s) > 0$ for $s \in (0, +\infty)$, and for every $s \in (0, +\infty)$ there exists $\delta > 0$ such that $\varphi(t) \leq s$, for all $t \in [s, s + \delta]$.

Definition 6 ([2]). A function θ from $\mathbb{R}_+$ into itself is called a strictly L-function if $\theta(0) = 0$, $\theta(s) > 0$ for $s \in (0, +\infty)$, and for every $s \in (0, +\infty)$ there exists $\delta > 0$ such that $\theta(t) < s$, for all $t \in [s, s + \delta]$.

2. FIXED POINT THEOREMS

In this section, we present some fixed point theorems on a Fréchet space.

Definition 7. Let Ω be a nonempty subset of a Fréchet space E , $\mathcal{M}$ an admissible set such that $\Omega \in \mathcal{M}$ and $\mu : \mathcal{M} \rightarrow \mathbb{R}_+$ be a measure of noncompactness on E . We say that an operator $F : \Omega \rightarrow \Omega$ is a Meir-Keeler condensing operator if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq \mu(X) < \varepsilon + \delta \implies \mu(FX) < \varepsilon, \quad (2.1)$$

and $F(X) \in \mathcal{M}$ for any nonempty subset $X \in \mathcal{M}$.

Now, we are ready to state our first main theorem for this section.

Theorem 4. *Let Ω be a nonempty, closed and convex subset of a Fréchet space E , $\mathcal{M}$ an admissible set such that $\Omega \in \mathcal{M}$ and $\mu : \mathcal{M} \rightarrow \mathbb{R}_+$ be a measure of non-compactness on E . Let $F : \Omega \rightarrow \Omega$ be a continuous and Meir-Keeler condensing operator, then F has at least one fixed point.*

Proof. By induction, we obtain a sequence $\{\Omega_n\}$ such that $\Omega_0 = \Omega$ and $\Omega_n = \text{Conv}(F\Omega_{n-1})$, $n \geq 1$. It is obvious that $\Omega_n \in \mathcal{M}$ and $\Omega_{n+1} \subseteq \Omega_n$ for all $n \in \mathbb{N}$. If there exists an integer $N \geq 0$ such that $\mu(\Omega_N) = 0$, then Ω_N is compact. Thus, Tychonoff fixed point theorem implies that F has a fixed point. Now assume that $\mu(\Omega_n) \neq 0$ for $n \geq 0$. Define $\varepsilon_n = \mu(\Omega_n)$. Since $\Omega_{n+1} \subseteq \Omega_n$ and by (2°) so we have $\{\varepsilon_n\}$ is a positive decreasing sequence of real numbers and there exists $\gamma \geq 0$ such that $\varepsilon_n \rightarrow \gamma$ as $n \rightarrow \infty$. We claim that $\gamma = 0$. Suppose, on the contrary, that $\gamma > 0$, then there exists n_0 such that $n > n_0$ implies $\gamma \leq \varepsilon_n < \gamma + \delta(\gamma)$, therefore by the definition of Meir-Keeler condensing operator, $\varepsilon_{n+1} < \gamma$ which is a contradiction. Therefore, $\gamma = 0$, that is, $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Since the sequence (Ω_n) is nested, in view of axiom (6°) of Definition 4 we deduce that the set $\Omega_\infty = \bigcap_{n=1}^{\infty} \Omega_n$ is nonempty, closed and convex subset of the set Ω . Moreover, the set Ω_∞ is invariant under the operator F and belongs to $\text{Ker}\mu$. Thus, applying Tychonoff fixed point theorem, F has a fixed point. $\square$

Lemma 1. *Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an increasing and right continuous function. Then the following conditions are equivalent:*

- (a) $\varphi(t) < t$ for any $t > 0$ and $\varphi(0) = 0$.
- (b) φ is an L-function.
- (c) φ is a strictly L-function.

Proof. Let φ satisfy condition (a). Since $\varphi(t) < t$ for any $t > 0$ and φ a right continuous function, so for any $s > 0$ there exist $\delta_s > 0$ such that

$$|\varphi(t) - \varphi(s)| < s - \varphi(s)$$

for all $t \in [s, s + \delta_s]$. Thus, $\varphi(t) < s$ for all $t \in [s, s + \delta_s]$ and φ satisfies condition (b). Now, assume that φ satisfies condition (b). Thus, for any $s > 0$ there exists $\delta > 0$ such that $\varphi(t) \leq s$ for all $t \in [s, s + \delta]$. Since φ is an increasing function, so $\varphi(t) < s$ for all $t \in [s, s + \frac{\delta}{2}]$ and φ satisfies condition (c).

It remains to be shown that condition (c) implies condition (a). Because φ is a strictly L-function, it implies that $\varphi(s) < s$ for all $s > 0$ and therefore the proof is obvious. $\square$

Now, we formulate and prove a fixed point theorem using strictly L-functions as an application of Theorem 4.

Theorem 5. *Let Ω be a nonempty, closed and convex subset of a Fréchet space E , $\mathcal{M}$ an admissible set such that $\Omega \in \mathcal{M}$ and $\mu : \mathcal{M} \rightarrow \mathbb{R}_+$ a measure of noncompactness on E . Then T is a Meir-Keeler condensing operator if and only if there exists a (increasing, right continuous) strictly L-function ϑ such that*

$$\mu(TX) \leq \vartheta(\mu(X)), \quad (2.2)$$

for each $X \subseteq \Omega$ and $X \in \mathcal{M}$.

Proof. Let $\varepsilon > 0$ be given. By the assumption, there exists $\delta > 0$ such that $\vartheta(t) < \varepsilon$ if $\varepsilon \leq t < \varepsilon + \delta$. If X is a subset of Ω such that

$$\varepsilon \leq \mu(X) < \varepsilon + \delta(\varepsilon),$$

thus,

$$\mu(T(X)) \leq \vartheta(\mu(X)) < \varepsilon$$

and T is a Meir-Keeler condensing operator. For the necessity part, assume that T is a Meir-Keeler condensing. From the definition of Meir-Keeler condensing, we can define a function $\alpha : (0, \infty) \rightarrow (0, \infty)$, such that

$$\varepsilon \leq \mu(X) < \varepsilon + 2\alpha(\varepsilon) \implies \mu(T(X)) < \varepsilon, \quad (2.3)$$

for $\varepsilon \in (0, \infty)$. Using such α , we define a nondecreasing function $\beta : (0, \infty) \rightarrow [0, \infty)$, by

$$\beta(t) = \inf\{\varepsilon : t \leq \varepsilon + \alpha(\varepsilon)\}$$

for $t \in (0, \infty)$. Since $t \leq t + \alpha(t)$, we note that $\beta(t) \leq t$ for $t \in (0, \infty)$. Define a function ϕ_1 from $[0, \infty)$ into itself by

$$\phi_1(t) = \begin{cases} 0 & \text{if } t = 0, \\ \beta(t) & \text{if } t > 0 \text{ and } \min\{\varepsilon > 0 : t \leq \varepsilon + \alpha(\varepsilon)\} \text{ exists,} \\ \frac{\beta(t)+t}{2} & \text{otherwise.} \end{cases} \quad (2.4)$$

Similar to Proposition 1 in [20], ϕ_1 be an L-function. Now, we define ϕ_2 , ϕ_3 and ϑ from $[0, \infty)$ into itself by

$$\phi_2(t) = \sup\{\phi_1(s) : s \leq t\},$$

$$\phi_3(t) = \inf\{\phi_2(s) : s > t\}$$

and

$$\vartheta(t) = \frac{\phi_3(t) + t}{2}$$

for $t \in [0, \infty)$. Then we have

$$0 \leq \phi_1(t) \leq \phi_2(t) \leq \phi_3(t) \leq \vartheta(t) \leq t$$

for all $t \in (0, \infty)$, ϕ_2 is a nondecreasing L-function, ϕ_3 is a nondecreasing, right continuous L-function and ϑ is an increasing, right continuous L-function. Therefore, by Lemma 1 we have ϑ is a increasing and right continuous strictly L-function. Fix

$X \in \mathcal{M}$ such that $\mu(X) \neq 0$. From the definition of β , there exists $\varepsilon \leq \mu(X)$ such that $\varepsilon \leq \mu(X) \leq \varepsilon + \alpha(\varepsilon)$. Thus,

$$\mu(TX) < \varepsilon \leq \beta(\mu(X)) \leq \vartheta(\mu(X))$$

holds. $\square$

Remark 1. If there exists a strictly L-function ϑ such that T satisfies in condition (2.2) then there exists an increasing and right continuous strictly L-function ϑ' such that T satisfies in condition (2.2).

3. INFINITE SYSTEMS OF CONDENSING OPERATORS

In this section, we prove some existence theorems for a general infinite system of equations involving condensing operators.

Let (E_i, d_i) be a Fréchet space for all $i \in \mathbb{N}$, $d(x, y) = \sup \left\{ \frac{1}{2^i} \min\{1, d_i(x_i, y_i)\} : i \in \mathbb{N} \right\}$, $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots) \in \prod_{i \in \mathbb{N}} E_i$. Then $(\prod_{i \in \mathbb{N}} E_i, d)$ is a Fréchet space.

Theorem 6 (Tychonoff's theorem[17]). *Let $\{(X_i, \tau_i) : i \in \mathbb{N}\}$ be any family of topological spaces. Then $\prod_{i \in \mathbb{N}} (X_i, \tau_i)$ is compact if and only if each (X_i, τ_i) is compact.*

Remark 2. We use the notation $\mathbb{R}^\omega$ which denotes the countable Cartesian product of $\mathbb{R}_+$ with itself, and l^∞ consists of all bounded sequences of scalars.

Now we are ready to state and prove the main results of this section.

Theorem 7. *Suppose μ_i be a measure of noncompactness on Fréchet spaces E_i for all $i \in \mathbb{N}$. Moreover assume that the function $F : l^\infty \rightarrow \mathbb{R}_+$ is convex, non-decreasing and $F((x_i)_{i=1}^\infty) = 0$ if and only if $x_i = 0$ for all $i \in \mathbb{N}$. If we define*

$$\mathcal{M} = \left\{ X \subseteq \prod_{i=1}^\infty E_i : \sup_i \{\mu_i(\pi_i(X))\} < \infty \right\},$$

where $\pi_i(X)$ denotes the natural projections of $\prod_{i=1}^\infty E_i$ into E_i and

$\mu : \mathcal{M} \rightarrow \mathbb{R}_+$ be defined by

$$\mu(X) = F\left(\left(\mu_i(\pi_i(X))\right)_{i=1}^\infty\right), \quad (3.1)$$

then $\mathcal{M}$ is an admissible set and μ is a measure of noncompactness on Fréchet space

$$E = \prod_{i=1}^\infty E_i.$$

Proof. It can be easily verified that $\mathcal{M}$ is an admissible set. Now, we prove μ is a measure of noncompactness on $E = \prod_{i=1}^{\infty} E_i$. For proving the condition (2°), suppose that $X \subset Y$. Since μ_i be a measure of noncompactness, we have $\mu_i(\pi_i(X)) \leq \mu_i(\pi_i(Y))$ for all $i \in \mathbb{N}$, and using F is nondecreasing we imply that

$$\mu(X) = F\left((\mu_i(\pi_i(X)))_{i=1}^{\infty}\right) \leq \mu(Y) = F\left((\mu_i(\pi_i(Y)))_{i=1}^{\infty}\right),$$

which shows that the condition (2°) is valid. The properties (3°)-(5°) are simple consequences of

$$\pi_i(\lambda U + (1-\lambda)V) = \lambda \pi_i(U) + (1-\lambda) \pi_i(V), \quad (3.2)$$

$$\pi_i(\text{Conv} X) = \text{Conv} \pi_i(X),$$

$$\pi_i(X) \subseteq \pi_i(\overline{X}) \subseteq \overline{\pi_i(X)}.$$

Now we show (1°). If $\mu(X) = 0$ for $X \in \mathcal{M}$ then $\mu_i(\pi_i(X)) = 0$ for each $i \in \mathbb{N}$. Hence, by virtue of (1°) of Definition 4 for measure of noncompactness μ_i , $\pi_i(X)$ is relatively compact for all $i \in \mathbb{N}$ and by Theorem 6, $\prod_{i=1}^{\infty} \pi_i(X)$ is relatively compact.

Thus, $X \subseteq \prod_{i=1}^{\infty} \pi_i(X)$ is relatively compact. Finally, it suffices to show (6°). suppose that $\{X_n\}$ is a sequence of closed sets from $\mathcal{M}$ such that $X_{n+1} \subseteq X_n$ for $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \mu(X_n) = 0$. So we have

$$\lim_{n \rightarrow \infty} F\left((\mu_i(\pi_i(X_n)))_{i=1}^{\infty}\right) = 0. \quad (3.3)$$

Since $X_{n+1} \subseteq X_n$, $\mu_i(\pi_i(X_{n+1})) \leq \mu_i(\pi_i(X_n))$ and $\mu_i(\pi_i(X_n)) \geq 0$ for all $i \in \mathbb{N}$. Thus, there is an $r_i \geq 0$ so that

$$\lim_{n \rightarrow \infty} \mu_i(\pi_i(X_n)) = r_i.$$

Because F is continuous, then

$$\lim_{n \rightarrow \infty} F\left((\mu_i(\pi_i(X_n)))_{i=1}^{\infty}\right) = F\left(\lim_{n \rightarrow \infty} (\mu_i(\pi_i(X_n)))_{i=1}^{\infty}\right) = F((r_i)_{i=1}^{\infty}).$$

On the other hand, using (3.3) we have $F((r_i)_{i=1}^{\infty}) = 0$. By assumption of the theorem we imply that $r_i = 0$ for all $i \in \mathbb{N}$, and so $\lim_{n \rightarrow \infty} \mu_i(\pi_i(X_n)) = 0$. By (6°) of

definition of measure of noncompactness on E_n we have $X_i^{\infty} = \bigcap_{n=1}^{\infty} \pi_i(X_n) \neq \emptyset$

for all $i \in \mathbb{N}$. Therefore we get $\prod_{i=1}^{\infty} X_i^{\infty} \subseteq X_{\infty}$ and $X_{\infty} \neq \emptyset$. This completes the proof. $\square$

Example 1. Let μ_n ($n \in \mathbb{N}$) be measures of noncompactness on Fréchet spaces E_n , respectively. Considering $F_1((x_n)_{n=1}^\infty) = \sup_{n \in \mathbb{N}} b_n x_n$ and $F_2((x_n)_{n=1}^\infty) = \sum_{n=1}^\infty a_n x_n$ (the functions F_1, F_2 are defined on l^∞) such that $a_n, b_n \in \mathbb{R}_+$, $\sum_{n=1}^\infty a_n < \infty$ and $\{b_n\}$ be a bounded sequence, then all the conditions of Theorem 7 are satisfied. Therefore, $\tilde{\mu}_1 := \sup_{n \in \mathbb{N}} b_n \mu(X_n)$ and $\tilde{\mu}_2 := \sum_{n=1}^\infty a_n \mu(X_n)$ define measures of noncompactness in the Fréchet space $E = \prod_{i=1}^\infty E_i$ where X_n ($n \in \mathbb{N}$), denotes the natural projections of X into E_n .

Theorem 8. Let Ω_i ($i \in \mathbb{N}$) be a nonempty, convex and closed subset of a Fréchet space E_i , μ_i an arbitrary measure of noncompactness on E_i and $\sup_i \{\mu_i(\Omega_i)\} < \infty$.

Let $F_i : \prod_{i=1}^\infty \Omega_i \longrightarrow \Omega_i$ ($i = 1, 2, \dots$) be a continuous operator such that

$$a_i \mu_i(F_i(\prod_{j=1}^\infty X_j)) \leq \varphi(\sup_j \{a_j \mu_j(X_j)\}) \quad (3.4)$$

for any subset X_i of Ω_i ($i \in \mathbb{N}$) where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a strictly L -function and $\{a_i\}$ is a bounded sequence of positive real numbers. Then there exist $(x_j^*)_{j=1}^\infty \in \prod_{j=1}^\infty \Omega_j$ such that

$$F_i((x_j^*)_{j=1}^\infty) = x_i^* \quad (3.5)$$

for all $i \in \mathbb{N}$.

Proof. Let us consider $\tilde{F} : \prod_{i=1}^\infty \Omega_i \longrightarrow \prod_{i=1}^\infty \Omega_i$ in the following way:

$$\tilde{F}((x_j)_{j=1}^\infty) = (F_1((x_j)_{j=1}^\infty), F_2((x_j)_{j=1}^\infty), \dots, F_i((x_j)_{j=1}^\infty), \dots)$$

for all $(x_j)_{j=1}^\infty \in \prod_{i=1}^\infty \Omega_i$. It is obvious that $\tilde{F}$ is continuous. It suffices to show that the hypothesis (2.2) of Theorem 5 holds where μ is defined by Example 1. Take an

arbitrary nonempty subset X of $\prod_{i=1}^{\infty} \Omega_i$. Now, by (2°) and (3.4) we obtain

$$\begin{aligned} \mu(\widetilde{F}(X)) &\leq \mu\left(\prod_{i=1}^{\infty} F_i\left(\prod_{j=1}^{\infty} \pi_j(X)\right)\right) \\ &= \sup_i a_i \mu_i\left(F_i\left(\prod_{j=1}^{\infty} \pi_j(X)\right)\right) \\ &\leq \sup_i \varphi\left(\sup_j a_j \mu_j(\pi_j(X))\right) \\ &\leq \sup_i \varphi(\mu(X)) \\ &\leq \varphi(\mu(X)). \end{aligned} \quad (3.6)$$

Thus, using Theorem 5, $\widetilde{F}$ is a continuous and Meir-Keeler condensing operator. Now applying Theorem 4, $\widetilde{F}$ has a fixed point and there exist $(x_j^*)_{j=1}^{\infty} \in \prod_{j=1}^{\infty} \Omega_j$ such that

$$(x_j^*)_{j=1}^{\infty} = \widetilde{F}((x_j^*)_{j=1}^{\infty}) = (F_1((x_j^*)_{j=1}^{\infty}), F_2((x_j^*)_{j=1}^{\infty}), \dots, F_j((x_j^*)_{j=1}^{\infty}), \dots)$$

and that (3.5) holds. $\square$

4. EXISTENCE OF A SOLUTION FOR A INFINITE SYSTEM OF INTEGRAL EQUATIONS

In the following section, we will work in the classical Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ consisting of all real functions defined, bounded and continuous on $\mathbb{R}_+ \times \mathbb{R}_+$ equipped with the standard norm

$$\|x\| = \sup\{|x(t, s)| : t, s \geq 0\}.$$

Now, we present the definition of a special measure of noncompactness in $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ which will be needed in the sequel.

To do this, let X be a fixed nonempty and bounded subset of $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ and fix a positive number T . For $x \in X$ and $\epsilon > 0$, denote by $\omega^T(x, \epsilon)$ the modulus of the continuity of function x on the interval $[0, T]$, i.e.,

$$\omega^T(x, \epsilon) = \sup\{|x(t, s) - x(u, v)| : t, s, u, v \in [0, T], |t - u| \leq \epsilon, |s - v| \leq \epsilon\}.$$

Further, let us put

$$\begin{aligned} \omega^T(X, \epsilon) &= \sup\{\omega^T(x, \epsilon) : x \in X\}, \\ \omega_0^T(X) &= \lim_{\epsilon \rightarrow 0} \omega^T(X, \epsilon) \end{aligned}$$

and

$$\omega_0(X) = \lim_{T \rightarrow \infty} \omega_0^T(X).$$

Moreover, for two fixed numbers $t, s \in \mathbb{R}_+$ let us define the function μ on the family $\mathfrak{M}_{BC(\mathbb{R}_+ \times \mathbb{R}_+)}$ by the following formula

$$\mu(X) = \omega_0(X) + \limsup_{\|(t,s)\| \rightarrow \infty} \text{diam} X(t, s),$$

where $\|(t, s)\| = \max(t, s)$ and $X(t, s) = \{x(t, s) : x \in X\}$. Similar to [8], it can be shown that the function μ is a measure of noncompactness in the space $BC(\mathbb{R}_+ \times \mathbb{R}_+)$.

As an application of Theorem 8 we prove the existence of solutions for the infinite system of integral equations of Hammerstein type in two variables (1.2).

Theorem 9. Assume that the following conditions are satisfied:

- (a₁) $f_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ ($n \in \mathbb{N}$) is continuous. Moreover, there exists a nondecreasing, right continuity and concave strictly L -function φ such that

$$|f_n(t, s, x_1, \dots, x_n, u) - f_n(t, s, y_1, \dots, y_n, v)| \leq \varphi\left(\max_{1 \leq i \leq n} |x_i - y_i|\right) + |u - v|; \quad (4.1)$$

- (a₂) $M := \sup\{|f_n(t, s, 0, \dots, 0)| : t, s \in \mathbb{R}_+, n \in \mathbb{N}\} < \infty$;

- (a₃) $k_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ are continuous functions for all $n \in \mathbb{N}$;

- (a₄) $h_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^\omega \rightarrow \mathbb{R}$ ($n \in \mathbb{N}$) is continuous and there exist a continuous function $a_n : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and a continuous and nondecreasing function $b_n : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$|h_n(t, s, (x_j)_{j=1}^\infty)| \leq a_n(t, s) b_n\left(\sup_{1 \leq j < \infty} |x_j|\right)$$

for all $t, s \in \mathbb{R}_+$ and $(x_j)_{j=1}^\infty \in \mathbb{R}^\omega$ with $\sup_{1 \leq j < \infty} |x_j| < \infty$. Also the function

$(v, w) \rightarrow a_n(v, w) k_n(v, w, t, s)$ is integrable over $\mathbb{R}_+ \times \mathbb{R}_+$ for any fixed $t, s \in \mathbb{R}_+$ and $n \in \mathbb{N}$.

- (a₅) there exists a positive constant D such that

$$D = \sup\left\{\int_0^\infty \int_0^\infty a_n(v, w) |k_n(t, s, v, w)| dv dw : t, s \in \mathbb{R}_+, n \in \mathbb{N}\right\} < \infty,$$

and

$$\lim_{\|(t,s)\| \rightarrow \infty} \int_0^\infty \int_0^\infty a_n(v, w) |k_n(t, s, v, w)| dv dw = 0; \quad (4.2)$$

- (a₆) the following equalities are hold:

$$\begin{aligned} \lim_{T \rightarrow \infty} \left\{ \sup\left\{\int_T^\infty \int_0^T a_n(v, w) |k_n(t, s, v, w)| dv dw : t, s \in \mathbb{R}_+\right\} \right\} &= 0, \\ \lim_{T \rightarrow \infty} \left\{ \sup\left\{\int_0^\infty \int_T^\infty a_n(v, w) |k_n(t, s, v, w)| dv dw : t, s \in \mathbb{R}_+\right\} \right\} &= 0, \end{aligned} \quad (4.3)$$

for all $n \in \mathbb{N}$;
 (a₇) there exists a positive solution r_0 of the inequality

$$\varphi(r) + M + Db_n(r) \leq r$$

for all $n \in \mathbb{N}$.

Then the infinite system of equations (1.2) has at least one solution in the space $(BC(\mathbb{R}_+ \times \mathbb{R}_+))^\omega$.

Proof. Let us fix arbitrarily $n \in \mathbb{N}$. Define $H_n : (BC(\mathbb{R}_+ \times \mathbb{R}_+))^\omega \longrightarrow BC(\mathbb{R}_+ \times \mathbb{R}_+)$ by

$$H_n((x_j)_{j=1}^\infty)(t, s) = f_n(t, s, x_1(t, s), \dots, x_n(t, s), \int_0^\infty \int_0^\infty k_n(t, s, v, w) h_n(v, w, (x_j(v, w))_{j=1}^\infty) dv dw). \quad (4.4)$$

In light of (4.4) and assumptions (a₁)-(a₇), f_n is continuous, k_n is continuous, h_n is continuous and x_i for $i \in \mathbb{N}$ are continuous. On the other hand, integral of continuous function is continuous. Therefore, we infer that the function $H_n((x_j)_{j=1}^\infty)$ is continuous for arbitrarily $(x_j)_{j=1}^\infty \in (BC(\mathbb{R}_+ \times \mathbb{R}_+))^\omega$ because it is the composition of continuous functions. Moreover, in view of our assumptions, for arbitrarily fixed $(x_j)_{j=1}^\infty \in (BC(\mathbb{R}_+ \times \mathbb{R}_+))^\omega$ and $t, s \in \mathbb{R}_+$, we obtain

$$\begin{aligned} & |H_n((x_j)_{j=1}^\infty)(t, s)| \\ & \leq \left| f_n(t, s, x_1(t, s), \dots, x_n(t, s), \int_0^\infty \int_0^\infty k_n(t, s, v, w) h_n(v, w, (x_j(v, w))_{j=1}^\infty) dv dw) \right. \\ & \quad \left. - f_n(t, s, 0, \dots, 0) \right| + |f_n(t, s, 0, \dots, 0)| \\ & \leq \varphi(\max_{1 \leq i \leq n} |x_i(t, s)|) + |f_n(t, s, 0, \dots, 0)| \\ & \quad + \int_0^\infty \int_0^\infty |k_n(t, s, v, w)| |a_n(v, w)| b_n(|(x_j(v, w))_{j=1}^\infty|) \\ & \leq \varphi(\max_{1 \leq i \leq n} |x_i(t, s)|) + M + Db_n(\sup_{1 \leq j < \infty} |x_j(v, w)|) \\ & \leq \varphi(\max_{1 \leq i \leq n} \|x_i\|) + M + Db_n(\sup_{1 \leq j < \infty} \|x_j\|). \end{aligned} \quad (4.5)$$

Thus,

$$\|H_n(x)\| \leq \varphi(\max_{1 \leq i \leq n} \|x_i\|) + M + Db_n(\sup_{1 \leq j < \infty} \|x_j\|) \quad (4.6)$$

and $H_n((x_j)_{j=1}^\infty) \in BC(\mathbb{R}_+ \times \mathbb{R}_+)$ for any $(x_j)_{j=1}^\infty \in (BC(\mathbb{R}_+ \times \mathbb{R}_+))^\omega$. By (4.6) and using (a₇), the function H_n maps $(\bar{B}_{r_0})^\omega$ into $\bar{B}_{r_0}$.

Now we claim that H_n is a continuous function on $(\bar{B}_{r_0})^\omega$ for all $n \in \mathbb{N}$. To do this, let us fix $0 < \varepsilon < \frac{1}{2^n}$ and take arbitrary $x = ((x_j)_{j=1}^\infty), y = ((y_j)_{j=1}^\infty) \in (BC(\mathbb{R}_+ \times \mathbb{R}_+))^\omega$

$\mathbb{R}_+))^\omega$ such that $d(x, y) = \sup \left\{ \frac{1}{2^i} \min\{1, \|x_i - y_i\|\} : i \in \mathbb{N} \right\} < \varepsilon$. Then, for $t, s, v, w \in \mathbb{R}_+$, we get

$$\begin{aligned} & \left| H_n((x_j)_{j=1}^\infty)(t, s) - H_n((y_j)_{j=1}^\infty)(t, s) \right| \\ & \leq \left| f_n\left(t, s, x_1(t, s), \dots, x_n(t, s), \int_0^\infty \int_0^\infty k_n(t, s, v, w) h_n(v, w, (x_j(v, w))_{j=1}^\infty) dv dw\right) \right. \\ & \quad \left. - f_n\left(t, s, y_1(t, s), \dots, y_n(t, s), \int_0^\infty \int_0^\infty k_n(t, s, v, w) h_n(v, w, (y_j(v, w))_{j=1}^\infty) dv dw\right) \right| \\ & \leq \varphi\left(\max_{1 \leq i \leq n} |x_i(t, s) - y_i(t, s)|\right) \\ & \quad + \left| \int_0^\infty \int_0^\infty k_n(t, s, v, w) [h_n(v, w, (x_j(v, w))_{j=1}^\infty) - h_n(v, w, (y_j(v, w))_{j=1}^\infty)] dv dw \right|. \end{aligned}$$

On the other hand, assumption (a5) ensure that there exists a positive number T such that for $\max(t, s) > T$ we have

$$\begin{aligned} \left| H_n((x_j)_{j=1}^\infty)(t, s) - H_n((y_j)_{j=1}^\infty)(t, s) \right| & \leq \varphi\left(\max_{1 \leq i \leq n} |x_i(t, s) - y_i(t, s)|\right) \\ & \quad + 2b_n(r_0) \int_0^\infty \int_0^\infty |k_n(t, s, v, w)| a_n(v, w) dv dw \\ & \leq \varphi(2^n \varepsilon) + b_n(r_0) \varepsilon. \end{aligned}$$

Suppose that $t, s \in [0, T]$. By applying the assumptions, we infer that

$$\begin{aligned} & \left| H_n((x_j)_{j=1}^\infty)(t, s) - H_n((y_j)_{j=1}^\infty)(t, s) \right| \\ & \leq \varphi\left(\max_{1 \leq i \leq n} |x_i(t, s) - y_i(t, s)|\right) \\ & \quad + \left| \int_0^\infty \int_0^\infty k_n(t, s, v, w) [h_n(v, w, (x_j(v, w))_{j=1}^\infty) - h_n(v, w, (y_j(v, w))_{j=1}^\infty)] dv dw \right| \\ & \leq \varphi(2^n \varepsilon) + \left\{ \int_0^\infty \left\{ \int_0^T |k_n(t, s, v, w)| |h_n(v, w, (x_j(v, w))_{j=1}^\infty) - h_n(v, w, (y_j(v, w))_{j=1}^\infty)| dv \right. \right. \\ & \quad \left. \left. + \int_T^\infty |k_n(t, s, v, w)| [|h_n(v, w, (x_j(v, w))_{j=1}^\infty| + h_n(v, w, (y_j(v, w))_{j=1}^\infty)] dv \right\} dw \right\} \\ & \leq \varphi(2^n \varepsilon) + \int_0^T \int_0^T |k_n(t, s, v, w)| |h_n(v, w, (x_j(v, w))_{j=1}^\infty) - h_n(v, w, (y_j(v, w))_{j=1}^\infty)| dv dw \\ & \quad + \int_0^T \int_T^\infty |k_n(t, s, v, w)| [|h_n(v, w, (x_j(v, w))_{j=1}^\infty| + |h_n(v, w, (y_j(v, w))_{j=1}^\infty)|] dv dw \\ & \quad + \int_T^\infty \int_0^\infty |k_n(t, s, v, w)| [|h_n(v, w, (x_j(v, w))_{j=1}^\infty| + |h_n(v, w, (y_j(v, w))_{j=1}^\infty)|] dv dw \\ & \leq \varphi(2^n \varepsilon) + K_T^n \omega_{r_0}^T(h_n, \varepsilon) + 2b_n(r_0) \int_0^T \int_T^\infty a_n(v, w) |k_n(t, s, v, w)| dv dw \\ & \quad + 2b_n(r_0) \int_T^\infty \int_0^\infty a_n(v, w) |k_n(t, s, v, w)| dv dw, \end{aligned}$$

where

$$K_T^n = \sup\{k_n(t, s, v, w) : t, s, v, w \in [0, T]\}$$

$$\begin{aligned} \omega_{r_0}^T(h_n, \varepsilon) &= \sup\{|h_n(v, w, (x_j)_{j=1}^\infty) - h_n(v, w, (y_j)_{j=1}^\infty)| \\ &\quad : v, w \in [0, T], x_i, y_i \in [-r_0, r_0], |x_i - y_i| \leq \varepsilon\}. \end{aligned}$$

From the continuity of the function h_n on the compact set $[0, T] \times [0, T] \times [-r_0, r_0]^\omega$ (by using Tychonoff's theorem), we have $\omega_{r_0}^T(h_n, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and in view of assumption (a_6) we can choose T in such a way that three last terms of the above estimate are sufficiently small. Thus H_n is a continuous function on $(BC(\mathbb{R}_+ \times \mathbb{R}_+))^\omega$.

Now we assert that H_n satisfies all the conditions of Theorem 8. Let X_i be nonempty and bounded subsets of $\bar{B}_{r_0}$ for all $i \in \mathbb{N}$ such that $\sup_i(\mu(X_i)) < \infty$, and assume that $T > 0$ and $\varepsilon > 0$ are arbitrary constants. Let $t_1, t_2, s_1, s_2 \in [0, T]$ such that $|t_2 - t_1| \leq \varepsilon$, $|s_2 - s_1| \leq \varepsilon$ and $x_i \in X_i$ for all $i \in \mathbb{N}$. Then, by the assumptions we have:

$$\begin{aligned} & \left| H_n((x_i)_{i=1}^\infty)(t_2, s_2) - H_n((x_i)_{i=1}^\infty)(t_1, s_1) \right| \\ & \leq \left| f_n(t_2, s_2, x_1(t_2, s_2), \dots, x_n(t_2, s_2), \int_0^\infty \int_0^\infty k_n(t_2, s_2, v, w) h_n(v, w, (x_i(v, w))_{i=1}^\infty) dv dw) \right. \\ & \quad \left. - f_n(t_1, s_1, x_1(t_1, s_1), \dots, x_n(t_1, s_1), \int_0^\infty \int_0^\infty k_n(t_1, s_1, v, w) h_n(v, w, (x_i(v, w))_{i=1}^\infty) dv dw) \right| \\ & \leq \left| f_n(t_2, s_2, x_1(t_2, s_2), \dots, x_n(t_2, s_2), \int_0^\infty \int_0^\infty k_n(t_2, s_2, v, w) h_n(v, w, (x_i(v, w))_{i=1}^\infty) dv dw) \right. \\ & \quad \left. - f_n(t_2, s_2, x_1(t_1, s_1), \dots, x_n(t_1, s_1), \int_0^\infty \int_0^\infty k_n(t_2, s_2, v, w) h_n(v, w, (x_i(v, w))_{i=1}^\infty) dv dw) \right| \\ & \quad + \left| f_n(t_2, s_2, x_1(t_1, s_1), \dots, x_n(t_1, s_1), \int_0^\infty \int_0^\infty k_n(t_2, s_2, v, w) h_n(v, w, (x_i(v, w))_{i=1}^\infty) dv dw) \right. \\ & \quad \left. - f_n(t_1, s_1, x_1(t_1, s_1), \dots, x_n(t_1, s_1), \int_0^\infty \int_0^\infty k_n(t_2, s_2, v, w) h_n(v, w, (x_i(v, w))_{i=1}^\infty) dv dw) \right| \\ & \quad + \left| f_n(t_1, s_1, x_1(t_1, s_1), \dots, x_n(t_1, s_1), \int_0^\infty \int_0^\infty k_n(t_2, s_2, v, w) h_n(v, w, (x_i(v, w))_{i=1}^\infty) dv dw) \right. \\ & \quad \left. - f_n(t_1, s_1, x_1(t_1, s_1), \dots, x_n(t_1, s_1), \int_0^\infty \int_0^\infty k_n(t_1, s_1, v, w) h_n(v, w, (x_i(v, w))_{i=1}^\infty) dv dw) \right| \\ & \leq \varphi(\max_{1 \leq j \leq n} |x_j(t_2, s_2) - x_j(t_1, s_1)|) + \omega_{r_0, D_1}^T(f_n, \varepsilon) \\ & \quad + \left| \int_0^\infty \int_0^\infty [k_n(t_2, s_2, v, w) - k_n(t_1, s_1, v, w)] h_n(v, w, (x_i(v, w))_{i=1}^\infty) dv dw \right| \\ & \leq \varphi(\max_{1 \leq j \leq n} |x_j(t_2, s_2) - x_j(t_1, s_1)|) + \omega_{r_0, D_1}^T(f_n, \varepsilon) \\ & \quad + \int_0^T \int_0^T |k_n(t_2, s_2, v, w) - k_n(t_1, s_1, v, w)| |h_n(v, w, (x_i(v, w))_{i=1}^\infty)| dv dw \\ & \quad + \int_0^T \int_T^\infty |k_n(t_2, s_2, v, w) - k_n(t_1, s_1, v, w)| |h_n(v, w, (x_i(v, w))_{i=1}^\infty)| dv dw \\ & \quad + \int_T^\infty \int_0^\infty |k_n(t_2, s_2, v, w) - k_n(t_1, s_1, v, w)| |h_n(v, w, (x_i(v, w))_{i=1}^\infty)| dv dw \end{aligned}$$

$$\begin{aligned}
&\leq \varphi(\max_{1 \leq j \leq n} \omega^T(x_j, \varepsilon)) + \omega_{r_0, D_1}^T(f_n, \varepsilon) + T^2 U_{r_0}^T \omega^T(k_n, \varepsilon) \\
&\quad + b_n(r_0) \int_0^T \int_T^\infty [|k_n(t_2, s_2, v, w)| + |k_n(t_1, s_1, v, w)|] a_n(v, w) dv dw \\
&\quad + b_n(r_0) \int_T^\infty \int_0^\infty [|k_n(t_2, s_2, v, w)| + |k_n(t_1, s_1, v, w)|] a_n(v, w) dv dw
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
D_1 &= b_n(r_0) D (\text{see assumption } (a_5)) \\
\omega_{r_0, D_1}^T(f_n, \varepsilon) &= \sup\{|f_n(t_2, s_2, x_1, \dots, x_n, y) - f_n(t_1, s_1, x_1, \dots, x_n, y)| : t_1, s_1, t_2, s_2 \in [0, T], \\
&\quad |t_2 - t_1| \leq \varepsilon, |s_2 - s_1| \leq \varepsilon, |x_i| \leq r_0, |y| \leq D_1\}, \\
\omega^T(x_i, \varepsilon) &= \sup\{|x_i(t_2, s_2) - x_i(t_1, s_1)| : t_1, s_1, t_2, s_2 \in [0, T], |t_2 - t_1| \leq \varepsilon, |s_2 - s_1| \leq \varepsilon\}, \\
\omega^T(k_n, \varepsilon) &= \sup\{|k_n(t_2, s_2, v, w) - k_n(t_1, s_1, v, w)| : \\
&\quad t_1, s_1, t_2, s_2, v, w \in [0, T], |t_2 - t_1| \leq \varepsilon, |s_2 - s_1| \leq \varepsilon\}, \\
U_{r_0}^T &= \sup\{|h_n(v, w, (x_j)_{j=1}^\infty)| : v, w \in [0, T], x_i \in [-r_0, r_0]\}.
\end{aligned}$$

Since x_i was an arbitrary element of X_i for all $i \in \mathbb{N}$ in (4.7), we obtain

$$\begin{aligned}
\omega^T(H_n(\prod_{i=1}^\infty X_i)) &\leq \varphi(\max_{1 \leq i \leq n} \omega^T(x_i, \varepsilon)) + \omega_{r_0, D_1}^T(f_n, \varepsilon) + T^2 U_{r_0}^T \omega^T(k_n, \varepsilon) \\
&\quad + b_n(r_0) \int_0^T \int_T^\infty [|k_n(t_2, s_2, v, w)| + |k_n(t_1, s_1, v, w)|] a_n(v, w) dv dw \\
&\quad + b_n(r_0) \int_T^\infty \int_0^\infty [|k_n(t_2, s_2, v, w)| + |k_n(t_1, s_1, v, w)|] a_n(v, w) dv dw,
\end{aligned}$$

and by the uniform continuity of f_n and k_n on the compact sets $[0, T] \times [0, T] \times [-r_0, r_0]^n \times [-D_1, D_1]$ and $[0, T] \times [0, T] \times [0, T] \times [0, T]$ respectively, we have $\omega_{r_0, D_1}^T(f_n, \varepsilon) \rightarrow 0$ and $\omega^T(k_n, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Therefore, we obtain

$$\begin{aligned}
\omega^T(H_n(\prod_{i=1}^\infty X_i)) &\leq \varphi(\max_{1 \leq i \leq n} \omega^T(x_i, \varepsilon)) \\
&\quad + b_n(r_0) \int_0^T \int_T^\infty [|k_n(t_2, s_2, v, w)| + |k_n(t_1, s_1, v, w)|] a_n(v, w) dv dw \\
&\quad + b_n(r_0) \int_T^\infty \int_0^\infty [|k_n(t_2, s_2, v, w)| + |k_n(t_1, s_1, v, w)|] a_n(v, w) dv dw.
\end{aligned}$$

Now taking $T \rightarrow \infty$ and by using assumption (a_6) we get

$$\omega_0(H_n(\prod_{i=1}^\infty X_i)) \leq \varphi(\max_{1 \leq i \leq n} \omega_0(X_i)). \tag{4.8}$$

On the other hand, for all $x_i, y_i \in X_i$ ($i \in \mathbb{N}$) and $t, s \in \mathbb{R}_+$ we get

$$\begin{aligned} & \left| H_n((x_i)_{i=1}^\infty)(t, s) - H_n((y_i)_{i=1}^\infty)(t, s) \right| \\ & \leq \varphi(\max_{1 \leq i \leq n} |x_i(t, s) - y_i(t, s)|) \\ & \quad + \left| \int_0^\infty \int_0^\infty k_n(t, s, v, w) [h_n(v, w, (x_i(v, w))_{i=1}^\infty) - h_n(v, w, (y_i(v, w))_{i=1}^\infty)] dv dw \right| \\ & \leq \varphi(\max_{1 \leq i \leq n} |x_i(t, s) - y_i(t, s)|) + 2b_n(r_0) \int_0^\infty \int_0^\infty a_n(v, w) |k_n(t, s, v, w)| dv dw. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} & \text{diam}(H_n(\prod_{i=1}^\infty X_i)(t, s)) \\ & \leq \varphi(\max_{1 \leq i \leq n} \text{diam}(X_i(t, s))) + 2b_n(r_0) \int_0^\infty \int_0^\infty a_n(v, w) |k_n(t, s, v, w)| dv dw. \end{aligned} \quad (4.9)$$

Taking $\|(t, s)\| \rightarrow \infty$ in the inequality (4.9), then using (a₅) we have

$$\lim_{\|(t, s)\| \rightarrow \infty} \text{diam}(H_n(\prod_{i=1}^\infty X_i)(t, s)) \leq \varphi(\max_{1 \leq i \leq n} \lim_{\|(t, s)\| \rightarrow \infty} \text{diam}(X_i(t, s))). \quad (4.10)$$

Further, combining (4.8) and (4.10) we get

$$\begin{aligned} \mu(H_n(\prod_{i=1}^\infty X_i)) &= \omega_0(H_n(\prod_{i=1}^\infty X_i)) + \lim_{\|(t, s)\| \rightarrow \infty} \text{diam} H_n(\prod_{i=1}^\infty X_i)(t, s) \\ &\leq \varphi(\max_{1 \leq i \leq n} \omega_0(X_i)) + \varphi(\max_{1 \leq i \leq n} \lim_{\|(t, s)\| \rightarrow \infty} \text{diam}(X_i(t, s))) \\ &\leq \varphi[\max_{1 \leq i \leq n} (\omega_0(X_i) + \lim_{\|(t, s)\| \rightarrow \infty} \text{diam}(X_i(t, s)))] \\ &\leq 2\varphi[\max_{1 \leq i \leq n} \frac{1}{2}(\omega_0(X_i) + \lim_{\|(t, s)\| \rightarrow \infty} \text{diam}(X_i(t, s)))]. \end{aligned}$$

Let us consider the measure of noncompactness $\mu_1 = \frac{1}{2}\mu$, thus we get

$$\mu_1(H_n(\prod_{i=1}^\infty X_i)) \leq \varphi(\sup_i \mu_1(X_i)).$$

Now by using Theorem 8, there exist $(x_i)_{i=1}^\infty \in (BC(\mathbb{R}_+ \times \mathbb{R}_+))^\omega$ such that

$$\begin{aligned} & x_n(t, s) \\ &= f_n(t, s, x_1(t, s), \dots, x_n(t, s), \int_0^\infty \int_0^\infty k_n(t, s, v, w) h_n(v, w, (x_i(v, w))_{i=1}^\infty) dv dw) \end{aligned}$$

and the proof is complete. $\square$

Example 2. Consider the following system of functional integral equations

$$x_n(t, s) = \frac{t^{2n} s^{2n}}{t^{4n} s^{4n} + 1} + \tanh\left(\frac{1}{n} \sum_{i=1}^n |x_i(t, s)|\right) + \int_0^\infty \int_0^\infty \frac{e^{-nw} (e^{-nv} - 1)}{2^n [t^n s^n + 1]} \sum_{i=1}^\infty e^{-iv} {}^{2n-1}\sqrt{\tanh x_i(v, w)} dv dw \quad (4.11)$$

Eq. (4.11) is a special case of Eq. (1.2) where

$$\begin{aligned} f_n(t, s, x_1, \dots, x_n, z) &= \frac{t^{2n} s^{2n}}{t^{4n} s^{4n} + 1} + \tanh\left(\frac{1}{n} \sum_{i=1}^n |x_i|\right) + z, \\ k_n(t, s, v, w) &= \frac{e^{-nw} (e^{-nv} - 1)}{2^n [t^n s^n + 1]}, \\ h_n(t, s, (x_j)_{j=1}^\infty) &= \sum_{i=1}^\infty e^{-it} {}^{2n-1}\sqrt{\tanh x_i}, \\ a_n(t, s) &= \frac{1}{e^t - 1}, \quad b_n(r) = {}^{2n-1}\sqrt{\tanh r}. \end{aligned}$$

Suppose that $t, s \in \mathbb{R}_+$. Now, by taking $\varphi(t) = \tanh(t)$ we have

$$\begin{aligned} &|f_n(t, s, x_1, \dots, x_n, u) - f_n(t, s, y_1, \dots, y_n, v)| \\ &\leq \left| \tanh\left(\frac{1}{n} \sum_{i=1}^n |x_i|\right) - \tanh\left(\frac{1}{n} \sum_{i=1}^n |y_i|\right) \right| + |u - v| \\ &\leq \tanh\left(\frac{1}{n} \sum_{i=1}^n |x_i - y_i|\right) + |u - v| \\ &\leq \tanh\left(\max_{1 \leq i \leq n} |x_i - y_i|\right) + |u - v| \\ &= \varphi\left(\max_{1 \leq i \leq n} |x_i - y_i|\right) + |u - v|. \end{aligned} \quad (4.12)$$

Thus, we infer that condition (a_1) holds. On the other hand

$$\begin{aligned} M &= \sup\{|f_n(t, s, 0, \dots, 0)| : t, s \in \mathbb{R}_+, n \in \mathbb{N}\} \\ &= \sup\left\{\frac{t^{2n} s^{2n}}{t^{4n} s^{4n} + 1} : t, s \in \mathbb{R}_+, n \in \mathbb{N}\right\} = \frac{1}{2} \end{aligned}$$

which shows the condition (a_2) is valid. Also, assumption (a_3) clearly is evident. In order to show that assumption (a_4) is satisfied, let us assume that $\sup_{1 \leq i < \infty} |x_i| < \infty$,

so we have

$$|h_n(t, s, (x_j)_{j=1}^\infty)| = \sum_{i=1}^\infty e^{-it} {}^{2n-1}\sqrt{\tanh x_i} \quad (4.13)$$

$$\begin{aligned} &\leq \frac{1}{e^t - 1} \sqrt[2n-1]{\tanh\left(\sup_{1 \leq i < \infty} |x_i|\right)} \\ &= a_n(t, s) b_n\left(\sup_{1 \leq i < \infty} |x_i|\right). \end{aligned}$$

On the other hand, the function $(v, w) \rightarrow a_n(v, w)k_n(v, w, t, s)$ is integrable over $\mathbb{R}_+ \times \mathbb{R}_+$ for any fixed $t, s \in \mathbb{R}_+$ and $n \in \mathbb{N}$. Thus, condition (a_4) is valid. Further, we get:

$$\begin{aligned} D &= \sup\left\{\frac{1}{2^n[t^n s^n + 1]} \int_0^\infty \int_0^\infty \left|\frac{e^{-nw}(e^{-nv} - 1)}{e^v - 1}\right| dv dw : t, s \in \mathbb{R}_+, n \in \mathbb{N}\right\} \\ &= \sup\left\{\frac{1}{2^n[t^n s^n + 1]} \int_0^\infty \int_0^\infty |e^{-nw}(e^{-v} + e^{-2v} + \dots + e^{-nv})| dv dw : t, s \in \mathbb{R}_+, n \in \mathbb{N}\right\} \\ &= \sup\left\{\frac{1}{2^n[t^n s^n + 1]n} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right) : t, s \in \mathbb{R}_+, n \in \mathbb{N}\right\} \quad (4.14) \\ &= \frac{1}{2} < \infty \end{aligned}$$

and

$$\lim_{\|t, s\| \rightarrow \infty} \frac{1}{2^n[t^n s^n + 1]} \int_0^\infty \int_0^\infty \frac{e^{-nw}(e^{-nv} - 1)}{e^v - 1} dv dw = 0.$$

This implies that the condition (a_5) holds. Moreover, for arbitrarily fixed $T > 0$ we obtain:

$$\int_T^\infty \int_0^T a_n(v, w) |k_n(t, s, v, w)| dv dw \leq \frac{e^{-nT}}{2^n n} \left(n - e^{-T} - \frac{e^{-2T}}{2} - \dots - \frac{e^{-nT}}{n}\right)$$

and

$$\int_0^\infty \int_T^\infty a_n(v, w) |k_n(t, s, v, w)| dv dw \leq \frac{1}{2^n n} \left(e^{-T} + \frac{e^{-2T}}{2} + \dots + \frac{e^{-nT}}{n}\right).$$

From the above estimate, we infer that condition (a_6) is valid. It is easy to see that each number $r \geq 4$ satisfies the inequality in condition (a_7) , i.e.,

$$\varphi(r) + M + D b_n(r) = \tanh(r) + \frac{1}{2} + \frac{1}{2} \sqrt[2n-1]{r} \leq r.$$

Thus, as the number r_0 we can take $r_0 = 4$. Consequently, all the conditions of Theorem 9 are satisfied. Hence the system of functional integral equations (4.11) has at least one solution which belong to the space $(BC(\mathbb{R}_+ \times \mathbb{R}_+))^\omega$.

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Authors' addresses

Mohsen Hosseinzadeh Moghaddam

Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran.

E-mail address: mohsen11124@gmail.com

Reza Allahyari

Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran. (Corresponding author)

E-mail address: rezaallahyari@mshdiau.ac.ir

Mohsen Erfanian Omidvar

Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran.

E-mail address: erfanian@mshdiau.ac.ir

Mahmoud Hassani

Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran.

E-mail address: hassani@mshdiau.ac.ir



MORE ON LIE DERIVATIONS OF A GENERALIZED MATRIX ALGEBRA

A.H. MOKHTARI AND H.R. EBRAHIMI VISHKI

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Abstract. Motivated by the elaborate work of Cheung [Linear Multilinear Algebra, **51** (2003), 299-310], we apply the construction of a Lie derivation on a generalized matrix algebra to give a characterization for a Lie derivation on a generalized matrix algebra to be proper. Our approach not only provides a direct proof for some known results in the theory, but also it presents several sufficient conditions assuring the properness of Lie derivations on certain generalized matrix algebras.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ be a unital algebra (over a commutative unital ring $\mathbf{R}$) and $\mathcal{M}$ be an $\mathcal{A}$ –module. A linear mapping $\mathcal{D} : \mathcal{A} \rightarrow \mathcal{M}$ is said to be a derivation if

$$\mathcal{D}(ab) = \mathcal{D}(a)b + a\mathcal{D}(b) \quad (a, b \in \mathcal{A}).$$

A linear mapping $\mathcal{L} : \mathcal{A} \rightarrow \mathcal{M}$ is called a Lie derivation if

$$\mathcal{L}([a, b]) = [\mathcal{L}(a), b] + [a, \mathcal{L}(b)] \quad (a, b \in \mathcal{A}),$$

where $[a, b] = ab - ba$. Every derivation is trivially a Lie derivation. If $\mathcal{D} : \mathcal{A} \rightarrow \mathcal{A}$ is a derivation and $\tau : \mathcal{A} \rightarrow Z(\mathcal{A})$ ($:=$ the center of $\mathcal{A}$) is a linear map then $\mathcal{D} + \tau$ is a Lie derivation if and only if τ vanishes at commutators (i.e. $\tau([a, b]) = 0$, for all $a, b \in \mathcal{A}$). Lie derivations of this form are called proper Lie derivations. Therefore a Lie derivation $\mathcal{L}$ is proper if and only if $\mathcal{L} = \mathcal{D} + \tau$ for some derivation $\mathcal{D}$ and a linear center valued map τ on $\mathcal{A}$ (i.e. $\tau(\mathcal{A}) \subseteq Z(\mathcal{A})$). The fundamental question is that under what conditions a Lie derivation on an algebra is proper. We say that an algebra $\mathcal{A}$ has Lie derivation property if every Lie derivation from $\mathcal{A}$ into itself is proper.

In this paper, we study the Lie derivation property for the general matrix algebras. First we briefly introduce a general matrix algebra. A Morita context $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \Phi_{\mathcal{M}\mathcal{N}}, \Psi_{\mathcal{N}\mathcal{M}})$ consists of two unital algebras $\mathcal{A}, \mathcal{B}$, an $(\mathcal{A}, \mathcal{B})$ –module $\mathcal{M}$,

a $(\mathcal{B}, \mathcal{A})$ -module $\mathcal{N}$, and two module homomorphisms $\Phi_{\mathcal{M}\mathcal{N}} : \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} \longrightarrow \mathcal{A}$ and $\Psi_{\mathcal{N}\mathcal{M}} : \mathcal{N} \otimes_{\mathcal{A}} \mathcal{M} \longrightarrow \mathcal{B}$ satisfying the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} \otimes_{\mathcal{A}} \mathcal{M} & \xrightarrow{\Phi_{\mathcal{M}\mathcal{N}} \otimes I_{\mathcal{M}}} & \mathcal{A} \otimes_{\mathcal{A}} \mathcal{M} \\ \downarrow I_{\mathcal{M}} \otimes \Psi_{\mathcal{N}\mathcal{M}} & & \downarrow \cong \\ \mathcal{M} \otimes_{\mathcal{B}} \mathcal{B} & \xrightarrow{\cong} & \mathcal{M} \end{array}$$

and

$$\begin{array}{ccc} \mathcal{N} \otimes_{\mathcal{A}} \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} & \xrightarrow{\Psi_{\mathcal{N}\mathcal{M}} \otimes I_{\mathcal{N}}} & \mathcal{B} \otimes_{\mathcal{B}} \mathcal{N} \\ \downarrow I_{\mathcal{N}} \otimes \Phi_{\mathcal{M}\mathcal{N}} & & \downarrow \cong \\ \mathcal{N} \otimes_{\mathcal{A}} \mathcal{A} & \xrightarrow{\cong} & \mathcal{N}. \end{array}$$

For a Morita context $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \Phi_{\mathcal{M}\mathcal{N}}, \Psi_{\mathcal{N}\mathcal{M}})$, the set

$$\mathcal{G} = \left(\begin{array}{cc} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{array} \right) = \left\{ \left(\begin{array}{cc} a & m \\ n & b \end{array} \right) \middle| a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}, b \in \mathcal{B} \right\}$$

forms an algebra under the usual matrix operations, where at least one of two modules $\mathcal{M}$ and $\mathcal{N}$ is distinct from zero. The algebra $\mathcal{G}$ is called a generalized matrix algebra. In the case where $\mathcal{N} = 0$, $\mathcal{G}$ becomes the so-called triangular algebra $\text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ whose Lie derivations are extensively investigated by Cheung [4]. Generalized matrix algebras were first introduced by Sands [12], where he studied various radicals of algebras occurring in Morita contexts.

A direct verification reveals that the center $Z(\mathcal{G})$ of $\mathcal{G}$ is

$$Z(\mathcal{G}) = \{a \oplus b \mid a \in Z(\mathcal{A}), b \in Z(\mathcal{B}), am = mb, na = bn \text{ for all } m \in \mathcal{M}, n \in \mathcal{N}\},$$

where $a \oplus b = \left(\begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) \in \mathcal{G}$. We also define two natural projections $\pi_{\mathcal{A}} : \mathcal{G} \longrightarrow \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathcal{G} \longrightarrow \mathcal{B}$ by

$$\pi_{\mathcal{A}} : \left(\begin{array}{cc} a & m \\ n & b \end{array} \right) \mapsto a \quad \text{and} \quad \pi_{\mathcal{B}} : \left(\begin{array}{cc} a & m \\ n & b \end{array} \right) \mapsto b.$$

It is clear that, $\pi_{\mathcal{A}}(Z(\mathcal{G})) \subseteq Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathcal{G})) \subseteq Z(\mathcal{B})$. Further, similar to [4, Proposition 3] one can show that, if $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$ -module (i.e. $a\mathcal{M} = \{0\}$ necessities $a = 0$ and $\mathcal{M}b = \{0\}$ necessities $b = 0$) then there exists a unique algebra isomorphism

$$\varphi : \pi_{\mathcal{A}}(Z(\mathcal{G})) \longrightarrow \pi_{\mathcal{B}}(Z(\mathcal{G}))$$

such that $am = m\varphi(a)$ and $\varphi(a)n = na$ for all $m \in \mathcal{M}$, $n \in \mathcal{N}$; or equivalently, $a \oplus \varphi(a) \in Z(\mathcal{G})$ for all $a \in \mathcal{A}$.

Martindale [9] was the first one who investigated the Lie derivation property of certain primitive rings. Cheung [3] initiated the study of various mappings on triangular algebras; in particular, he studied the Lie derivation property for triangular algebras in [4]. Following his seminal work [4], Lie derivations on a wide variety of algebras have been studied by many authors (see [1, 2, 5–8, 10, 11, 13, 14] and the references therein). The main result of Cheung [4] has recently extended by Du and Wang [5] for a generalized matrix algebra. They have shown that [5, Theorem 1] a general matrix algebra $\mathcal{G}$ with $\mathcal{M}$ faithful enjoys the Lie derivation property if $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A})$, $\pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$, and either $\mathcal{A}$ or $\mathcal{B}$ does not contain nonzero central ideals. This result has been developed for Lie n -derivation in [13, Theorem 1], (see also [14, Theorem 2.1]).

In this paper, we use the construction of Lie derivations of a generalized matrix algebra $\mathcal{G}$ (Proposition 1) to give a criterion for properness of Lie derivations on $\mathcal{G}$ (see Theorem 1 and Corollary 1). We then deduce not only, as a byproduct, the aforementioned result of Du and Wang (Theorem 2), but also we provide some alternative sufficient conditions ensuring the Lie derivation property for $\mathcal{G}$ (Theorems 3, 4). We then come to our main result, Theorem 5, collecting some sufficient conditions assuring the Lie derivation property for a generalized matrix algebra. In the last section, we include some applications of our results to some special generalized matrix algebras such as: triangular algebras, the full matrix algebras and the algebras of operators on a Banach space.

2. PROPER LIE DERIVATIONS

From now on, we assume that the modules $\mathcal{M}$ and $\mathcal{N}$ appeared in the generalized matrix algebra $\mathcal{G}$ are 2-torsion free; ($\mathcal{M}$ is called 2-torsion free if $2m = 0$ implies $m = 0$ for any $m \in \mathcal{M}$).

We commence with the following result providing the construction of (Lie) derivations of a generalized matrix algebra. It needs a standard argument, however Li and Wei [6, Propositions 4.1, 4.2] gave a complete proof for the first two parts of it.

Proposition 1. *Let $\mathcal{G} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{pmatrix}$ be a generalized matrix algebra. Then*

- *A linear mapping $\mathcal{L}$ on $\mathcal{G}$ is a Lie derivation if and only if it has the presentation*

$$\mathcal{L} \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} P(a) - mn_0 - m_0n + h_{\mathcal{B}}(b) & am_0 - m_0b + f(m) \\ n_0a - bn_0 + g(n) & h_{\mathcal{A}}(a) + n_0m + nm_0 + Q(b) \end{pmatrix}, \quad (2.1)$$

for some $m_0 \in \mathcal{M}$, $n_0 \in \mathcal{N}$ and some linear maps $P : \mathcal{A} \rightarrow \mathcal{A}$, $Q : \mathcal{B} \rightarrow \mathcal{B}$, $f : \mathcal{M} \rightarrow \mathcal{M}$, $g : \mathcal{N} \rightarrow \mathcal{N}$, $h_{\mathcal{B}} : \mathcal{B} \rightarrow Z(\mathcal{A})$ and $h_{\mathcal{A}} : \mathcal{A} \rightarrow Z(\mathcal{B})$ satisfying the following conditions:

- P, Q are Lie derivations;
- $h_{\mathcal{B}}([b, b']) = 0$, $h_{\mathcal{A}}([a, a']) = 0$;
- $f(am) = P(a)m - mh_{\mathcal{A}}(a) + af(m)$, $f(mb) = mQ(b) - h_{\mathcal{B}}(b)m + f(m)b$;

- (d) $g(na) = nP(a) - h_{\mathcal{A}}(a)n + g(n)a$, $g(bn) = Q(b)n - nh_{\mathcal{B}}(b) + bg(n)$;
 (e) $P(mn) - h_{\mathcal{B}}(nm) = mg(n) + f(m)n$, $Q(nm) - h_{\mathcal{A}}(mn) = g(n)m + nf(m)$;

for all $a, a' \in \mathcal{A}, b, b' \in \mathcal{B}, m \in \mathcal{M}$ and $n \in \mathcal{N}$.

• A linear mapping $\mathcal{D}$ on $\mathcal{G}$ is a derivation if and only if it has the presentation

$$\mathcal{D} \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} P'(a) - mn_0 - m_0n & am_0 - m_0b + f'(m) \\ n_0a - bn_0 + g'(n) & n_0m + nm_0 + Q'(b) \end{pmatrix}, \quad (2.2)$$

where $m_0 \in \mathcal{M}, n_0 \in \mathcal{N}, P' : \mathcal{A} \rightarrow \mathcal{A}, Q' : \mathcal{B} \rightarrow \mathcal{B}, f' : \mathcal{M} \rightarrow \mathcal{M}$ and $g' : \mathcal{N} \rightarrow \mathcal{N}$ are linear maps satisfying the following conditions:

- (a') P', Q' are derivations;
 (b') $f'(am) = P'(a)m + af'(m)$, $f'(mb) = mQ'(b) + f'(m)b$;
 (c') $g'(na) = nP'(a) + g'(n)a$, $g'(bn) = Q'(b)n + bg'(n)$;
 (d') $P'(mn) = mg'(n) + f'(m)n$, $Q'(nm) = g'(n)m + nf'(m)$;

for all $a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M}$ and $n \in \mathcal{N}$.

• A linear mapping τ on $\mathcal{G}$ is center valued and vanishes at commutators (i.e. $\tau(\mathcal{G}) \subseteq \mathcal{Z}$ and $\tau([\mathcal{G}, \mathcal{G}]) = 0$) if and only if τ has the presentation

$$\tau \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} \ell_{\mathcal{A}}(a) + h_{\mathcal{B}}(b) & 0 \\ 0 & h_{\mathcal{A}}(a) + \ell_{\mathcal{B}}(b) \end{pmatrix}, \quad (2.3)$$

where $\ell_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{Z}(\mathcal{A}), h_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{Z}(\mathcal{A}), h_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{Z}(\mathcal{B}), \ell_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{Z}(\mathcal{B})$ are linear maps vanishing at commutators, having the following properties:

- (a'') $\ell_{\mathcal{A}}(a) \oplus h_{\mathcal{A}}(a) \in \mathcal{Z}(\mathcal{G})$ and $h_{\mathcal{B}}(b) \oplus \ell_{\mathcal{B}}(b) \in \mathcal{Z}(\mathcal{G})$, for all $a \in \mathcal{A}, b \in \mathcal{B}$;
 (b'') $\ell_{\mathcal{A}}(mn) = h_{\mathcal{B}}(nm)$ and $h_{\mathcal{A}}(mn) = \ell_{\mathcal{B}}(nm)$, for all $m \in \mathcal{M}, n \in \mathcal{N}$.

Following the method of Cheung [4, Theorem 6] in the next theorem, we give a necessary and sufficient condition for a Lie derivation on a generalized matrix algebra $\mathcal{G}$ to be proper.

Theorem 1. Let $\mathcal{G}$ be a generalized matrix algebra. A Lie derivation $\mathcal{L}$ on $\mathcal{G}$ of the form presented in (2.1) is proper if and only if there exist linear mappings $\ell_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{Z}(\mathcal{A})$ and $\ell_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{Z}(\mathcal{B})$ satisfying the following conditions:

- (A) $P - \ell_{\mathcal{A}}$ and $Q - \ell_{\mathcal{B}}$ are derivations on $\mathcal{A}$ and $\mathcal{B}$, respectively;
 (B) $\ell_{\mathcal{A}}(a) \oplus h_{\mathcal{A}}(a) \in \mathcal{Z}(\mathcal{G})$ and $h_{\mathcal{B}}(b) \oplus \ell_{\mathcal{B}}(b) \in \mathcal{Z}(\mathcal{G})$, for all $a \in \mathcal{A}, b \in \mathcal{B}$;
 (C) $\ell_{\mathcal{A}}(mn) = h_{\mathcal{B}}(nm)$ and $\ell_{\mathcal{B}}(nm) = h_{\mathcal{A}}(mn)$, for all $m \in \mathcal{M}, n \in \mathcal{N}$.

Proof. Employing the characterization of $\mathcal{L}$ as in Proposition 1, for the sufficiency, we define $\mathcal{D}$ and τ by

$$\mathcal{D} \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} (P - \ell_{\mathcal{A}})(a) - mn_0 - m_0n & am_0 - m_0b + f(m) \\ n_0a - bn_0 + g(n) & n_0m + nm_0 + (Q - \ell_{\mathcal{B}})(b) \end{pmatrix}$$

and

$$\tau \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} \ell_{\mathcal{A}}(a) + h_{\mathcal{B}}(b) & 0 \\ 0 & h_{\mathcal{A}}(a) + \ell_{\mathcal{B}}(b) \end{pmatrix}.$$

Then a direct verification reveals that $\mathcal{D}$ is a derivation, τ is center valued and $\mathcal{L} = \mathcal{D} + \tau$.

For the converse, suppose that $\mathcal{L}$ is of the form $\mathcal{D} + \tau$, where $\mathcal{D}$ is a derivation and τ maps into $Z(\mathcal{G})$. Applying the presentations (2.1), (2.2) for $\mathcal{L}$ and $\mathcal{D}$, respectively, we get $\tau = \mathcal{L} - \mathcal{D}$ as

$$\tau \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} (P - P')(a) + h_{\mathcal{B}}(b) & 0 \\ 0 & h_{\mathcal{A}}(a) + (Q - Q')(b) \end{pmatrix}.$$

By setting $\ell_{\mathcal{A}} = P - P'$, $\ell_{\mathcal{B}} = Q - Q'$, a direct verification shows that $\ell_{\mathcal{A}}, \ell_{\mathcal{B}}$ are our desired maps satisfying the required conditions. $\square$

Remark 1. It is worthwhile mentioning that in the case where $\mathcal{M}$ is faithful as an $(\mathcal{A}, \mathcal{B})$ -module then;

- (i) In Proposition 1, the condition (b) becomes redundant as it can be followed by (a) and (c). Indeed, for $a, a' \in \mathcal{A}, m \in \mathcal{M}$, from (c) we get

$$f([a, a']m) = P([a, a'])m - mh_{\mathcal{A}}([a, a']) + [a, a']f(m). \quad (2.4)$$

On the other hand, employing (c) and then (a), we have

$$\begin{aligned} f([a, a']m) &= f(aa'm - a'am) \\ &= P(a)a'm - a'mh_{\mathcal{A}}(a) + af(a'm) - (P(a')am - amh_{\mathcal{A}}(a') + a'f(am)) \\ &= P(a)a'm - a'mh_{\mathcal{A}}(a) + a(P(a')m - mh_{\mathcal{A}}(a') + a'f(m)) \\ &\quad - (P(a')am - amh_{\mathcal{A}}(a') + a'(P(a)m - mh_{\mathcal{A}}(a) + af(m))) \\ &= [P(a), a']m + [a, P(a')]m + [a, a']f(m) \\ &= P([a, a'])m + [a, a']f(m). \end{aligned} \quad (2.5)$$

Comparing the equations (2.4) and (2.5) reveals that $mh_{\mathcal{A}}([a, a']) = 0$, for every $m \in \mathcal{M}$, and the faithfulness of $\mathcal{M}$ (as a right $\mathcal{B}$ -module) implies that $h_{\mathcal{A}}([a, a']) = 0$, as claimed.

- (ii) In Proposition 1, the condition (a') can also be dropped as it can be derived from (b') by a similar argument as in (i), (see [4, Page 303]).
- (iii) In Theorem 1, the same reason as in (ii) shows that the condition (A) in Theorem 1, stating that $P - \ell_{\mathcal{A}}$ and $Q - \ell_{\mathcal{B}}$ are derivations, is superfluous.

As a consequence of Theorem 1, we have the following criterion characterizing Lie derivation property for a generalized matrix algebra $\mathcal{G}$.

Corollary 1. *Let $\mathcal{G}$ be a generalized matrix algebra and let $\mathcal{L}$ be a Lie derivation on $\mathcal{G}$ of the form presented in (2.1). If $\mathcal{L}$ is proper, then*

- (A') $h_{\mathcal{A}}(\mathcal{A}) \subseteq \pi_{\mathcal{B}}(Z(\mathcal{G}))$, $h_{\mathcal{B}}(\mathcal{B}) \subseteq \pi_{\mathcal{A}}(Z(\mathcal{G}))$, and
 (B') $h_{\mathcal{B}}(nm) \oplus h_{\mathcal{A}}(mn) \in Z(\mathcal{G})$, for all $m \in \mathcal{M}, n \in \mathcal{N}$.

The converse also holds in the case when $\mathcal{M}$ is faithful.

Proof. The necessity follows trivially from Theorem 1. For the sufficiency suppose that $\mathcal{M}$ is faithful. Let $\varphi : \pi_{\mathcal{A}}(Z(\mathcal{G})) \rightarrow \pi_{\mathcal{B}}(Z(\mathcal{G}))$ be the isomorphism satisfying $a \oplus \varphi(a) \in Z(\mathcal{G})$ for all $a \in \mathcal{A}$, whose existence guaranteed by the faithfulness of $\mathcal{M}$. By virtue of (A'), we define $\ell_{\mathcal{A}} : \mathcal{A} \rightarrow Z(\mathcal{A})$ and $\ell_{\mathcal{B}} : \mathcal{B} \rightarrow Z(\mathcal{B})$ by $\ell_{\mathcal{A}} = \varphi^{-1} \circ h_{\mathcal{A}}$ and $\ell_{\mathcal{B}} = \varphi \circ h_{\mathcal{B}}$. It is now obvious that $\ell_{\mathcal{A}}(a) \oplus h_{\mathcal{A}}(a) \in Z(\mathcal{G})$ and $h_{\mathcal{B}}(b) \oplus \ell_{\mathcal{B}}(b) \in Z(\mathcal{G})$, for all $a \in \mathcal{A}, b \in \mathcal{B}$. Further, (B') follows that

$$\ell_{\mathcal{A}}(mn) = \varphi^{-1}(h_{\mathcal{A}}(mn)) = h_{\mathcal{B}}(nm) \text{ and } \ell_{\mathcal{B}}(nm) = \varphi(h_{\mathcal{B}}(nm)) = h_{\mathcal{A}}(mn).$$

Now Theorem 1 together with Remark 1 (iii) confirm that $\mathcal{L}$ is proper, as required. $\square$

3. SOME SUFFICIENT CONDITIONS

We commence with the following result which employs Corollary 1 to give a proof for a modification of the main result of Du and Wang [5]. See also [13, Corollary 1] and examine [14, Theorem 2.1] for $n = 2$.

Theorem 2 ([5, Theorem 1]). *Let $\mathcal{G}$ be a generalized matrix algebra with faithful $\mathcal{M}$. Then $\mathcal{G}$ has Lie derivation property if*

- (i) $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A}), \pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$, and
- (ii) either $\mathcal{A}$ or $\mathcal{B}$ does not contain nonzero central ideals.

Proof. Let $\mathcal{L}$ be a Lie derivation of the form presented in (2.1). From Corollary 1, as (i) implies (A'), we only need to show that $h_{\mathcal{B}}(nm) \oplus h_{\mathcal{A}}(mn) \in Z(\mathcal{G})$ for all $m \in \mathcal{M}, n \in \mathcal{N}$. Without loss of generality suppose that $\mathcal{A}$ has no nonzero central ideals. Set

$$\gamma(a, b) = \ell_{\mathcal{A}}(a) + h_{\mathcal{B}}(b) \quad (a \in \mathcal{A}, b \in \mathcal{B}),$$

where, as in the proof of Corollary 1, $\ell_{\mathcal{A}} = \varphi^{-1} \circ h_{\mathcal{A}}$. Then $p_{\mathcal{A}} = P - \ell_{\mathcal{A}}$ is a derivation. Now Proposition 1 (e) implies that

$$p_{\mathcal{A}}(mn) = mg(n) + f(m)n - \gamma(mn, -nm) \quad (m \in \mathcal{M}, n \in \mathcal{N}),$$

so for each $a \in \mathcal{A}$, $p_{\mathcal{A}}(amn) - amg(n) - f(am)n = -\gamma(amn, -nam)$. By the latter identity and the fact that $p_{\mathcal{A}}$ is a derivation, we get

$$p_{\mathcal{A}}(a)mn + ap_{\mathcal{A}}(mn) - amg(n) - p_{\mathcal{A}}(a)mn - af(m)n = -\gamma(amn, -nam).$$

These relations follow that $a\gamma(mn, -nm) = \gamma(amn, -nam)$, for $a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}$.

Therefore for any two elements $m \in \mathcal{M}$ and $n \in \mathcal{N}$, the set $\mathcal{A}\gamma(mn, -nm)$ is a central ideal of $\mathcal{A}$. Thus $\ell_{\mathcal{A}}(mn) - h_{\mathcal{B}}(nm) = \gamma(mn, -nm) = 0$ and so $h_{\mathcal{B}}(nm) \oplus h_{\mathcal{A}}(mn) = \ell_{\mathcal{A}}(mn) \oplus h_{\mathcal{A}}(mn) \in Z(\mathcal{G})$, as claimed. $\square$

For an algebra $\mathcal{A}$ satisfying condition (ii) of Theorem 2, one can directly show that every noncommutative unital prime algebra with a nontrivial idempotent does not contain nonzero central ideals. In particular, $B(X)$, the algebra of operators on

a Banach space X when the dimension X is greater than 1, does not contain central ideal. The full matrix algebra $M_n(A)$, $n \geq 2$, has also no nonzero central ideal, [5, Lemma 1]. In [5, Theorem 2], they also showed that in a generalized matrix algebra $\mathcal{G}$ with loyal $\mathcal{M}$ if $\mathcal{A}$ is noncommutative, then $\mathcal{A}$ has no central ideals.

In the next result, we provide some new sufficient conditions assuring the Lie derivation property for $\mathcal{G}$. We say that an algebra $\mathcal{A}$ is a domain if it has no zero divisors if $aa' = 0$ implies $a = 0$ or $a' = 0$ for any two elements $a, a' \in \mathcal{A}$.

Theorem 3. *Let $\mathcal{G}$ be a generalized matrix algebra with faithful $\mathcal{M}$. Then $\mathcal{G}$ has Lie derivation property if*

- (i) $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A})$, $\pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$, and
- (ii) $\mathcal{A}$ and $\mathcal{B}$ are domain.

Proof. Let $\mathcal{L}$ be a Lie derivation of the form presented in (2.1). From Corollary 1, we only need to show that $h_{\mathcal{B}}(nm) \oplus h_{\mathcal{A}}(mn) \in Z(\mathcal{G})$.

Let $m \in \mathcal{M}, n \in \mathcal{N}$. Using the identities in (c) of Proposition 1 for $a = mn$ and $b = nm$, we get

$$P(mn)m - mh_{\mathcal{A}}(mn) + mnf(m) = mQ(nm) - h_{\mathcal{B}}(nm)m + f(m)nm. \quad (3.1)$$

Multiplying by m the identities in (e) of Proposition 1, we get

$$\begin{aligned} P(mn)m - h_{\mathcal{B}}(nm)m &= mg(n)m + f(m)nm, \\ mQ(nm) - mh_{\mathcal{A}}(mn) &= mg(n)m + mnf(m). \end{aligned} \quad (3.2)$$

Combining the equations in (3.1) and (3.2), we get $2(h_{\mathcal{B}}(nm)m - mh_{\mathcal{A}}(mn)) = 0$, so the 2-torsion freeness of $\mathcal{M}$ implies that

$$h_{\mathcal{B}}(nm)m = mh_{\mathcal{A}}(mn). \quad (3.3)$$

The faithfulness of $\mathcal{M}$ guarantees the existence of an isomorphism φ from $\pi_{\mathcal{A}}(Z(\mathcal{G}))$ to $\pi_{\mathcal{B}}(Z(\mathcal{G}))$ satisfying $a \oplus \varphi(a) \in Z(\mathcal{G})$ for all $a \in \mathcal{A}$. As $\pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$ we define $\ell_{\mathcal{A}} : \mathcal{A} \rightarrow Z(\mathcal{A})$ and $\ell_{\mathcal{B}} : \mathcal{B} \rightarrow Z(\mathcal{B})$ by $\ell_{\mathcal{A}} = \varphi^{-1} \circ h_{\mathcal{A}}$ and $\ell_{\mathcal{B}} = \varphi \circ h_{\mathcal{B}}$, respectively. It follows that $\ell_{\mathcal{A}}(a) \oplus h_{\mathcal{A}}(a) \in Z(\mathcal{G})$ and $h_{\mathcal{B}}(b) \oplus \ell_{\mathcal{B}}(b) \in Z(\mathcal{G})$ for all $a \in \mathcal{A}, b \in \mathcal{B}$. By (3.3), we then have

$$(h_{\mathcal{B}}(nm) - \ell_{\mathcal{A}}(mn))m = 0; \quad (3.4)$$

or equivalently,

$$m(\ell_{\mathcal{B}}(nm) - h_{\mathcal{A}}(mn)) = 0. \quad (3.5)$$

If $\mathcal{N}m = 0$ and $m\mathcal{N} = 0$, then trivially $h_{\mathcal{B}}(nm) - \ell_{\mathcal{A}}(mn) = 0$, otherwise either $m\mathcal{N} \neq 0$ or $\mathcal{N}m \neq 0$. Take $0 \neq n_0 \in \mathcal{N}$. If $mn_0 \neq 0$, then as $\mathcal{A}$ is a domain, (3.4) implies that $h_{\mathcal{B}}(nm) - \ell_{\mathcal{A}}(mn) = 0$; so

$$h_{\mathcal{B}}(nm) \oplus h_{\mathcal{A}}(mn) = \ell_{\mathcal{A}}(mn) \oplus h_{\mathcal{A}}(mn) \in Z(\mathcal{G}).$$

If $n_0m \neq 0$, then as $\mathcal{B}$ is a domain, (3.5) implies that $\ell_{\mathcal{B}}(nm) - h_{\mathcal{A}}(mn) = 0$; so

$$h_{\mathcal{B}}(nm) \oplus h_{\mathcal{A}}(mn) = h_{\mathcal{B}}(nm) \oplus \ell_{\mathcal{B}}(nm) \in Z(\mathcal{G}).$$

We thus have $h_{\mathcal{B}}(nm) \oplus h_{\mathcal{A}}(mn) \in Z(\mathcal{G})$ for all $m \in \mathcal{M}$ and $n \in \mathcal{N}$, as required. $\square$

Here we impose a condition on $\mathcal{M}$ which is slightly stronger than the faithfulness of $\mathcal{M}$. We say $\mathcal{M}$ as an $(\mathcal{A}, \mathcal{B})$ -module is strongly faithful if $\mathcal{M}$ satisfies in one of the following conditions:

$\mathcal{M}$ is faithful as a right $\mathcal{B}$ -module and $am = 0$ implies $a = 0$ or $m = 0$ for any $a \in \mathcal{A}, m \in \mathcal{M}$.

$\mathcal{M}$ is faithful as a left $\mathcal{A}$ -module and $mb = 0$ implies $m = 0$ or $b = 0$ for any $m \in \mathcal{M}, b \in \mathcal{B}$.

It is evident that if $\mathcal{M}$ is strongly faithful, then $\mathcal{M}$ is faithful and either $\mathcal{A}$ or $\mathcal{B}$ has no zero divisors. As an immediate consequence of the above theorem, we have the following corollary.

Theorem 4. *A generalized matrix algebra $\mathcal{G}$ has Lie derivation property, if*

- (i) $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A}), \pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$, and
- (ii) $\mathcal{M}$ is strongly faithful.

Proof. Following the proof of Theorem 3, the equations (3.4) and (3.5) together with strong faithfulness of $\mathcal{M}$ ensure that $h_{\mathcal{B}}(nm) - \ell_{\mathcal{A}}(mn) = 0$, so $h_{\mathcal{B}}(nm) \oplus h_{\mathcal{A}}(mn) = \ell_{\mathcal{A}}(mn) \oplus h_{\mathcal{A}}(mn) \in Z(\mathcal{G})$. The conclusion now follows from Corollary 1. $\square$

We remark that, to the best of our knowledge, we do not know when one can withdraw the strong faithfulness in Theorem 4.

Similar to what Cheung has introduced in [4, Section 3], we also introduce a critical subalgebra $\mathcal{W}_{\mathcal{A}}$ of an algebra $\mathcal{A}$. With the same notations as in Theorem 1, suppose that P is a Lie derivation on $\mathcal{A}$ and $\ell_{\mathcal{A}} : \mathcal{A} \rightarrow Z(\mathcal{A})$ is a linear map such that $P - \ell_{\mathcal{A}}$ is a derivation on $\mathcal{A}$. Set

$$\mathcal{V}_{\mathcal{A}} = \{a \in \mathcal{A} : \ell_{\mathcal{A}}(a) \oplus h_{\mathcal{A}}(a) \in Z(\mathcal{G})\}.$$

In other words, $\mathcal{V}_{\mathcal{A}}$ consists those elements $a \in \mathcal{A}$ such that

$$\ell_{\mathcal{A}}(a)m = mh_{\mathcal{A}}(a), \quad n\ell_{\mathcal{A}}(a) = h_{\mathcal{A}}(a)n, \text{ for all } m \in \mathcal{M}, n \in \mathcal{N}.$$

It is also easy to verify that $\mathcal{V}_{\mathcal{A}} \subseteq h_{\mathcal{A}}^{-1}(\pi_{\mathcal{B}}(Z(\mathcal{G})))$, with equality holds in the case where $\mathcal{M}$ is faithful. With some modifications in the proof of [4, Proposition 10], one can show that $\mathcal{V}_{\mathcal{A}}$ is a subalgebra of $\mathcal{A}$ containing all commutators and idempotents. More properties of $\mathcal{V}_{\mathcal{A}}$ are investigated in [10].

We denote by $\mathcal{W}_{\mathcal{A}}$ the smallest subalgebra of $\mathcal{A}$ contains all commutators and idempotents. We are especially dealing with those algebras satisfying $\mathcal{W}_{\mathcal{A}} = \mathcal{A}$. If $\mathcal{W}_{\mathcal{A}} = \mathcal{A}$, then trivially $h_{\mathcal{A}}(\mathcal{A}) \subseteq \pi_{\mathcal{B}}(Z(\mathcal{G}))$, or equivalently, $\pi_{\mathcal{B}}(\mathcal{L}(\mathcal{A})) \subseteq \pi_{\mathcal{B}}(Z(\mathcal{G}))$. Some examples of algebras satisfying $\mathcal{W}_{\mathcal{A}} = \mathcal{A}$ are: the full matrix algebra $\mathcal{A} = M_n(A), n \geq 2$, where A is a unital algebra, and also every simple unital algebra $\mathcal{A}$ with a nontrivial idempotent.

Motivated by the Cheung's idea [4], regarding the latter observations and some suitable combinations of the various assertions in the results 2, 3 and 4, we apply Theorem 1 and Corollary 1 to arrive the main result of the paper providing several sufficient conditions ensuring the Lie derivation property for a generalized matrix algebra $\mathcal{G}$, which is a generalization of [4, Theorem 11].

Theorem 5. *A generalized matrix algebra $\mathcal{G}$ has Lie derivation property, if the following three conditions hold:*

- (I) $\pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$ and $\mathcal{M}$ is a faithful left $\mathcal{A}$ -module, or $\mathcal{A} = \mathcal{W}_{\mathcal{A}}$ and $\mathcal{M}$ is a faithful left $\mathcal{A}$ -module, or $\mathcal{A}$ has Lie derivation property and $\mathcal{A} = \mathcal{W}_{\mathcal{A}}$.
- (II) $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A})$ and $\mathcal{M}$ is a faithful right $\mathcal{B}$ -module, or $\mathcal{B} = \mathcal{W}_{\mathcal{B}}$ and $\mathcal{M}$ is a faithful right $\mathcal{B}$ -module, or $\mathcal{B}$ has Lie derivation property and $\mathcal{B} = \mathcal{W}_{\mathcal{B}}$.
- (III) One of the following assertions holds:
 - (i) Either $\mathcal{A}$ or $\mathcal{B}$ does not contain nonzero central ideals.
 - (ii) $\mathcal{A}$ and $\mathcal{B}$ are domain.
 - (iii) Either $\mathcal{M}$ or $\mathcal{N}$ is strongly faithful.

4. SOME APPLICATIONS

The main examples of generalized matrix algebras are: a triangular algebra $\text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$, a unital algebra with a nontrivial idempotent, the algebra $B(X)$ of operators on a Banach space X and the full matrix algebra $M_n(\mathcal{A})$ on a unital algebra $\mathcal{A}$.

Lie derivations on trivial generalized matrix algebras and $\text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$

We say $\mathcal{G}$ is a trivial generalized matrix algebra when $\mathcal{M}\mathcal{N} = 0$ and $\mathcal{N}\mathcal{M} = 0$. The main example of trivial generalized matrix algebra is the so-called triangular algebra $\text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ whose Lie derivation property has been extensively investigated by Cheung [3, 4]. As an immediate consequence of Corollary 1 and Theorem 5, we get the following corollary which generalizes [4, Theorem 11] to trivial generalized matrix algebras.

Corollary 2. *Let $\mathcal{G}$ be a trivial generalized matrix algebra and let $\mathcal{L}$ be a Lie derivation on $\mathcal{G}$ of the form presented in (2.1). If $\mathcal{L}$ is proper then $h_{\mathcal{A}}(\mathcal{A}) \subseteq \pi_{\mathcal{B}}(Z(\mathcal{G}))$ and $h_{\mathcal{B}}(\mathcal{B}) \subseteq \pi_{\mathcal{A}}(Z(\mathcal{G}))$. The converse is also hold in the case when $\mathcal{M}$ is faithful.*

In particular, a trivial generalized matrix algebra $\mathcal{G}$ has Lie derivation property, if the following two conditions hold:

- (I) $\pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$ and $\mathcal{M}$ is a faithful left $\mathcal{A}$ -module, or $\mathcal{A} = \mathcal{W}_{\mathcal{A}}$ and $\mathcal{M}$ is a faithful left $\mathcal{A}$ -module, or $\mathcal{A}$ has Lie derivation property and $\mathcal{A} = \mathcal{W}_{\mathcal{A}}$.
- (II) $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A})$ and $\mathcal{M}$ is a faithful right $\mathcal{B}$ -module, or $\mathcal{B} = \mathcal{W}_{\mathcal{B}}$ and $\mathcal{M}$ is a faithful right $\mathcal{B}$ -module, or $\mathcal{B}$ has Lie derivation property and $\mathcal{B} = \mathcal{W}_{\mathcal{B}}$.

The following example which has been adapted from [1, Example 3.8] presents a trivial generalized matrix algebra (which is not a triangular algebra) without the Lie derivation property.

Example 1. Let $\mathcal{M}$ be a commutative unital algebra of dimension 3 (on the commutative unital ring R) with base $\{1, a_0, b_0\}$ such that $a_0^2 = b_0^2 = a_0b_0 = b_0a_0 = 0$. Set $\mathcal{N} = \mathcal{M}$ and let $\mathcal{A}$ and $\mathcal{B}$ be the subalgebras of $\mathcal{M}$ generated by $\{1, a_0\}$ and $\{1, b_0\}$, respectively. Then $\mathcal{M}\mathcal{N} = 0 = \mathcal{N}\mathcal{M}$, so the generalized matrix algebra $\mathcal{G} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{pmatrix}$ is trivial, and a direct calculation reveals that the map $\mathcal{L} : \mathcal{G} \rightarrow \mathcal{G}$ defined as

$$\mathcal{L} \begin{pmatrix} r_1 + r_2a_0 & s_1 + s_2a_0 + s_3b_0 \\ t_1 + t_2a_0 + t_3b_0 & u_1 + u_2b_0 \end{pmatrix} = \begin{pmatrix} u_2a_0 & -s_3a_0 - s_2b_0 \\ -t_3a_0 - t_2b_0 & r_2b_0 \end{pmatrix},$$

(where the coefficients are taken from the ring R) is a non-proper Lie derivation. It is worthwhile mentioning that $Z(\mathcal{G}) = R \cdot 1_{\mathcal{G}}$ and

$$\pi_{\mathcal{A}}(Z(\mathcal{G})) = R \cdot 1_{\mathcal{A}} \neq Z(\mathcal{A}), \pi_{\mathcal{B}}(Z(\mathcal{G})) = R \cdot 1_{\mathcal{B}} \neq Z(\mathcal{B}).$$

Furthermore, it is also easy to verify that $\mathcal{A} \neq \mathcal{W}_{\mathcal{A}}, \mathcal{B} \neq \mathcal{W}_{\mathcal{B}}$. Therefore, none of the conditions (I) and (II) of Corollary 2 is hold. Nevertheless, $\mathcal{A}$ and $\mathcal{B}$ have Lie derivation property. It should be also noted that $h_{\mathcal{A}}(\mathcal{A}) \not\subseteq \pi_{\mathcal{B}}(Z(\mathcal{G})), h_{\mathcal{B}}(\mathcal{B}) \not\subseteq \pi_{\mathcal{A}}(Z(\mathcal{G}))$; as, $h_{\mathcal{A}}(a_0) = b_0$ and $h_{\mathcal{B}}(b_0) = a_0$.

This example can be compared to that was given by Cheung in [4, Example 8]. They clarify the same discipline, however, $\mathcal{G}$ is not a triangular algebra.

Lie derivations on unital algebras with a nontrivial idempotent

In the following, we investigate the Lie derivation property for some unital algebras with a nontrivial idempotent. Let $\mathcal{A}$ be a unital algebra with a nontrivial idempotent p and $q = 1 - p$. Then $\mathcal{A}$ enjoys the Peirce decomposition $\mathcal{G} = \begin{pmatrix} p\mathcal{A}q & p\mathcal{A}q \\ q\mathcal{A}p & q\mathcal{A}q \end{pmatrix}$, as a generalized matrix algebra. Applying Theorem 5 for this generalized matrix algebra $\mathcal{G}$, we obtain the following result which partly improves the case $n = 2$ of a result given by Wang [14, Theorem 2.1] and Benkovič [1, Theorem 5.3].

Corollary 3. *Let $\mathcal{A}$ be a 2-torsion free unital algebra with a nontrivial idempotent p satisfying*

$$pap \cdot p\mathcal{A}q = 0 \text{ implies } pap = 0, \quad \text{and} \quad p\mathcal{A}q \cdot qaq = 0 \text{ implies } qaq = 0, \quad (4.1)$$

for any $a \in \mathcal{A}$, where $q = 1 - p$. Then $\mathcal{A}$ has Lie derivation property, if the following three conditions hold:

- (I) $Z(q\mathcal{A}q) = Z(\mathcal{A})q$, or $p\mathcal{A}p = \mathcal{W}_{p\mathcal{A}p}$, or $p\mathcal{A}p$ has Lie derivation property and $p\mathcal{A}p = \mathcal{W}_{p\mathcal{A}p}$.

- (II) $Z(p\mathcal{A}p) = Z(\mathcal{A})p$, or $q\mathcal{A}q = \mathcal{W}_{q\mathcal{A}q}$, or $q\mathcal{A}q$ has Lie derivation property and $q\mathcal{A}q = \mathcal{W}_{q\mathcal{A}q}$.
- (III) One of the following assertions holds:
- (i) Either $p\mathcal{A}p$ or $q\mathcal{A}q$ does not contain nonzero central ideals.
 - (ii) $p\mathcal{A}p$ and $q\mathcal{A}q$ are domain.
 - (iii) Either $p\mathcal{A}q$ or $q\mathcal{A}p$ is strongly faithful.

As a consequence of the above corollary, we bring the following result showing that the algebra $B(X)$ of bounded operators on a Banach space X enjoys Lie derivation property. The same result has already proved by Lu and Jing [7] for Lie derivable map at zero and idempotents by a completely different method. See also [8] for the properness of nonlinear derivations on $B(X)$.

Corollary 4. *Let X be a Banach space of dimension greater than 2. Then $B(X)$ has Lie derivation property.*

Proof. Set $\mathcal{A} = B(X)$. Consider a nonzero element $x_0 \in X$ and $f_0 \in X^*$ such that $f_0(x_0) = 1$, then $p = x_0 \otimes f_0$ defined by $y \mapsto f_0(y)x_0$ is a nontrivial idempotent. A direct verification reveals that $\mathcal{A}$ satisfies the implications (4.1). Indeed, if $pTp \cdot p\mathcal{A}q = \{0\}$ for some $T \in \mathcal{A}$, then choose a nonzero element $y \in q(X)$ such that $q(y) = y$. Let $x \in X$, there exists an operator $S \in B(X)$ such that $S(y) = x$ (e.g. $S := x \otimes g$ for some $g \in X^*$ with $g(y) = 1$). We then get $pTp(x) = pTp(S(q(y))) = pTp \cdot pSq(y) = 0$. Further, it can be readily verified that $Z(\mathcal{A}) = \mathbb{C}I_X = \mathbb{C}(p + q)$, $Z(p\mathcal{A}p) = \mathbb{C}p$ and $Z(q\mathcal{A}q) = \mathbb{C}q$. In particular, $Z(p\mathcal{A}p) = Z(\mathcal{A})p$, $Z(q\mathcal{A}q) = Z(\mathcal{A})q$. These also imply that neither $p\mathcal{A}p$ nor $q\mathcal{A}q$ has no central ideals. By Corollary 3 $\mathcal{A} = B(X)$ has Lie derivation property. $\square$

We conclude this section with an application of Corollary 3 to the full matrix algebra $\mathcal{A} = M_n(A)$, $n \geq 2$, where A is a unital algebra. Then $p = e_{11}$ is a nontrivial idempotent and $q = e_{11} + \cdots e_{nn}$. It is easy to verify that $p\mathcal{A}p = A$, $q\mathcal{A}q = M_{n-1}(A)$. As $Z(\mathcal{A}) = Z(A)1_{\mathcal{A}}$, we get $Z(p\mathcal{A}p) = Z(\mathcal{A})p$, $Z(q\mathcal{A}q) = Z(\mathcal{A})q$, so both conditions (I) and (II) in Corollary 3 are fulfilled. Further, in the case where $n \geq 3$, the algebra $q\mathcal{A}q = M_{n-1}(A)$ does not contain nonzero central ideals, so $\mathcal{A} = M_n(A)$ also satisfies assumption (i) of the aforementioned corollary. Therefore $M_n(A)$ has Lie derivation property for $n \geq 3$. This is an adaptation of [5, Corollary 1]. For $n = 2$, the same result was treated in [1, Corollary 5.7].

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Authors' addresses

A.H. Mokhtari

Technical Faculty of Ferdows, University of Birjand, Iran

E-mail address: a.mokhtari@birjand.ac.ir

H.R. Ebrahimi Vishki

Department of Pure Mathematics and Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran

E-mail address: vishki@um.ac.ir



GENERALISED DISTANCES OF SEQUENCES I: B -DISTANCES

BENEDEK NAGY

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Abstract. In this paper, we investigate the B -distances of infinite sequences. For this purpose we use generalized neighbourhood sequences. The general neighbourhood sequences were introduced for measuring distances in digital geometry ($\mathbb{Z}^n$). We extend their application to sequences, and present an algorithm which provides a shortest path between two sequences. We also present a formula to calculate the B -distance of any two sequences for a neighbourhood sequence B .

We also investigate the concept of k -convergent sequences for $k \in \mathbb{N}$, that concept is generally weaker than the convergence. We will use the term k -sequence which is a kind of generalization of the concept of 0-sequence. We also show some connection between the B -distances of sequences and the properties of their difference sequences.

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1. INTRODUCTION

The theory of sequences is widely used in mathematical analysis, calculus and in various applications. A good old textbook in this topic is Knopp's book [8].

One of the first used distance among finite sequences was the Hamming-distance [1, 3]: the H -distance of two same-length sequences over a finite alphabet is the number of places where they differ. We can extend this definition to infinite sequences also, over infinite alphabets ($\mathbb{Z}$, or $\mathbb{R}$). Allowing infinite distances this extension is natural.

There are other possibilities to measure the distances of finite and infinite sequences which contain numbers. The supremum norm is used for example in [9]. We will call this metric as sup-distance. It is another possibility to use the inf-distance, which is not a widely used distance function (due to some unpleasant properties).

In this paper, we investigate distances with integer values, they are based on various neighbourhood sequences (B) and neighbouring relations among the sequences.

The theory of neighbourhood sequences comes from digital geometry. In digital geometry the discrete space is used, i.e., points can have only integer co-ordinates.

Two different points in $\mathbb{Z}^m$ are k -neighbours ($k, m \in \mathbb{N} \cup \{\infty\}, k \leq m$), if their corresponding elements are equal up to at most k exceptions, and the difference of the exceptional values are at most 1. While in digital geometry the elements of $\mathbb{Z}^n$ or $\mathbb{Z}^\infty$ were called points, we will call the elements of $\mathbb{R}^\infty$ sequences. Indeed, we can use the same neighbourhood criteria for sequences.

After fixing the value of k , we may define the distance of two sequences as the number of steps of the shortest path between these sequences, where a step means a movement from a sequence to one of its k -neighbours. One can check that by this definition we get a generalized metric on $\mathbb{R}^\infty$, for each $k \in \{1, 2, \dots\}$, and that these generalized metrics are different for the separate values of k .

To obtain these distances we fixed k in the beginning, in other words, we used the same k in each step for walking from a sequence p to a sequence q in $\mathbb{R}^\infty$. The situation is more complicated if we may change the value of k after every step. A sequence $(b_i)_{i=1}^\infty$ is called a neighbourhood sequence over the set of the sequences, if $b_i \in \mathbb{N} \cup \{\infty\}$ ($i \in \mathbb{N}$). The concept of distances based on neighbourhood relations comes from [6, 13]. The periodic neighbourhood sequences were introduced in [5, 6, 14], while the general notion in [7, 10]. (We mention that the sequences in [5, 14] were called “neighbourhood sequences” while in [7] “generalized neighbourhood sequences”, but for simplicity we use the above definition.) Moreover in [7, 10, 11] the authors investigated the ∞ -dimensional space $\mathbb{Z}^\infty$.

By the help of an neighbourhood sequence $(b_i)_{i=1}^\infty$ we may define the distance of sequences p, q in the following way. We take the length of a shortest path from p to q , but at the i -th step, now, we may move from a sequence to another if and only if they are b_i -neighbours. Certainly, this notion is a generalization of the original one, as we may choose $b_i = k$ for each $i \in \mathbb{N}$, with any $k \in \{1, 2, \dots\}$. It is obvious, that these so-called B -distances have only non-negative integer values based on their definitions.

In [10] we have presented an algorithm which provides a shortest path between digital points, both in case of finite and infinite dimensional spaces. As we mentioned, the neighbourhood sequence $(b_i)_{i=1}^\infty$ with $b_i = k$ ($i \in \mathbb{N}$) generates a generalized metric on $\mathbb{R}^\infty$ ($m \in \mathbb{N}$) for any k . However, it is easy to find neighbourhood sequences, even periodic ones, such that the distances with respect to these neighbourhood sequences do not provide (generalized) metrics on the set of sequences. In [10] we have proved a necessary and sufficient condition for distance functions based on neighbourhood sequences to define a generalized metric in $\mathbb{Z}^\infty$. The purpose of this paper is to generalize these concepts from $\mathbb{Z}^\infty$ to $\mathbb{R}^\infty$.

The structure of this paper is as follows. In the next section, we give our notation and we define basic concepts. In the third section, we present an algorithm to solve the shortest path problem between any two sequences. In section four, we describe some properties of B -distances. We compare the B -distances with other type of distances. We also present a necessary and sufficient condition to a B -distance to

give a generalized metric over the set (i.e., the space) of sequences. A formula to calculate the B -distance between two arbitrary sequence with a given neighbourhood sequence B is also derived.

Although some of the concepts corresponding to sequences and distances are well-known we briefly recall them; we believe that this general overview will be helpful to understand the subsequent parts of the paper. Some of the results may seem to be simple to follow, but we have decided to include them here to have a self-contained paper and to show a relatively complete picture about this field.

2. NOTATIONS AND DEFINITIONS

First, we recall and define some basic concepts about the sequences. Although some of the concepts are well known from the literature mentioned earlier, we believe that due to various relations of these concepts to our topic it is worth to recall them in details. We start with the simplest concepts and we are going to the direction of more complex concepts. Let us start with a notation. Throughout the paper $\mathbb{R}^\infty$ will denote the set of all sequences. A sequence $p = (p(i))_{i=1}^\infty$ is periodic if there is a value $l \in \mathbb{N}$ such that $p(i) = p(i + l)$ for each element of p (l is called the period of p).

Definition 1. A sequence $p = (p(i))_{i=1}^\infty$ – where $p(i) \in \mathbb{R}$ for all i – is convergent if there exists $x \in \mathbb{R}$ such that for all $\varepsilon > 0$ there is an $n(\varepsilon)$ such that for all $n \in \mathbb{N}$ if $n > n(\varepsilon)$, then $|p(n) - x| < \varepsilon$. We say that x is the limit of the sequence p . If a sequence convergent and its limit is 0 then we call it 0-sequence.

In this paper, we investigate more general types of convergence: k -convergences.

Definition 2. A sequence $p \in \mathbb{R}^\infty$ is k -convergent (for a fixed non-negative value k), if there exists $n \in \mathbb{N}$ such that for all i, j with $i > n$ and $j > n$ we have $|p(i) - p(j)| \leq k$.

Now we extend the definition of 0-sequence to k -sequence in the following way.

Definition 3. A sequence $p \in \mathbb{R}^\infty$ is k -sequence (for a fixed positive value of k) if there exists a natural number n such that for all $i \in \mathbb{N}$ if $i > n$ then $|p(i)| < k$.

The next statements are evident about the relation of convergence, k -convergence and k -sequence.

Proposition 1.

- 1.1 If a sequence p is k -convergent, then it is k' -convergent for all $k' > k$.*
- 1.2 The sequence p is convergent if and only if it is k -convergent for all $k > 0$.*
- 1.3 The sequence p is 0-convergent if and only if its tail is constant. (In this case it is convergent, also.)*
- 1.4 If the sequence p is a k -sequence for some $k > 0$, then it is $2k$ -convergent. Moreover, if $p(i) \geq 0$ for all $i \in \mathbb{N}$, then p is also k -convergent.*

1.5 For each k -convergent sequence p there is a value j such that, p is j -sequence.

1.6 The (elementwise) sum and the (elementwise) difference of a k -convergent sequence p and a j -convergent sequence q are $k + j$ -convergent.

Lemma 1. If the sequences p and q are convergent, then their sum- and their difference-sequence are k -convergent for some k .

Proof. It is a simple consequence of the facts in Proposition 1 (and the previous definitions). $\square$

For periodic sequences we have the following facts.

Proposition 2.

2.1 Each periodic sequence p with period l is k -convergent for $k \geq \max_{1 \leq i, j \leq l} (p(i) - p(j))$.

2.2 Each periodic sequence p with period l is k -sequence for $k \geq \max_{1 \leq i \leq l} (|p(i)|)$.

From here we will use the terms k -convergence and k -sequence with arbitrary non-negative integer values of k , however our definition works for all (not necessary integer) non-negative value of k .

Now, we give some basic ideas about our distance functions.

Definition 4. A function $d : \mathbb{R}^\infty \times \mathbb{R}^\infty \rightarrow \mathbb{R} \cup \{\infty\}$ is called a generalized metric on $\mathbb{R}^\infty$, if it satisfies the following conditions:

- a) $\forall p, q \in \mathbb{R}^\infty$: $d(p, q) \geq 0$, and $d(p, q) = 0$ if and only if $p = q$ (positive definiteness),
- b) $\forall p, q \in \mathbb{R}^\infty$: $d(p, q) = d(q, p)$, (symmetry)
- c) $\forall p, q, r \in \mathbb{R}^\infty$: $d(p, q) + d(q, r) \geq d(p, r)$ (triangle inequality).

Moreover, if for every possible pair of $p, q \in \mathbb{R}^\infty$ the distance $d(p, q)$ is finite, then it is a metric.

If instead of point a) we have only

- a') $\forall p, q \in \mathbb{R}^\infty$: $d(p, q) \geq 0$, and $d(p, p) = 0$,

then the function $d : \mathbb{R}^\infty \times \mathbb{R}^\infty \rightarrow \mathbb{R} \cup \{\infty\}$ is a semi-metric on $\mathbb{R}^\infty$.

For measuring distances of sequences, usually the so-called supremum norm is used [9]. We can use also the Hamming-distance; it is finite only when the two sequences differ in finitely many places. The H -distance is one of the first discrete distances, it results always only non-negative integer values.

Definition 5. The sup-distance of the sequence p and q is given by

$$d(p, q; \text{sup}) = \sup(|p(i) - q(i)|).$$

The Hamming-distance of p and q is

$$d(p, q; H) = \sum_{p(i) \neq q(i)} 1.$$

The inf-distances of p and q is

$$d(p, q; inf) = inf(|p(i) - q(i)|).$$

The discrete metric over the set of sequences is the following:

$$d(p, q; disc) = \begin{cases} 0, & \text{if } p = q \\ 1, & \text{if } p \neq q. \end{cases}$$

It is well-known that $d(p, q; sup)$ and $d(p, q; H)$ are generalized metrics over $\mathbb{R}^\infty$. For the inf-distance one can prove that the properties b) and a') of Definition 4 hold only, therefore it is an unusual distance. The discrete metric is the simplest metric, but it is not practical for applications.

One of our most important investigations is introducing the neighbourhood relation among sequences.

Definition 6. Let p and q be two sequences in $\mathbb{R}^\infty$. Let k be a non-negative integer. The sequences p and q are k -neighbours, if the following two conditions hold:

- $|p(i) - q(i)| \leq 1$ for all $i \in \mathbb{N}$, and
- $\sum_{i \in \mathbb{N}, p(i) \neq q(i)} 1 \leq k$.

Definition 7. The infinite sequence $B = (b_i)_{i=1}^\infty$ ($b_i \in \mathbb{N} \cup \{\infty\}$) is called a neighbourhood sequence. If for some $l \in \mathbb{N}$, $b_i = b_{i+l}$ holds for every $i \in \mathbb{N}$, then B is called periodic (with period l).

For investigating distances of sequences, we will use their difference sequences in the following way.

Notation 1. Let p and q be two sequences. Put $w(i) = |p(i) - q(i)|$ for all i , and $w = (w(i))_{i=1}^\infty$. The sequence w is called the (absolute) difference of p and q .

The up-integer-difference-sequence (uids) u of p and q is defined by the top (i.e., ceiling) of the elements of their absolute difference as $u(i) = \lceil w(i) \rceil = \lceil |p(i) - q(i)| \rceil$, where $\lceil x \rceil$ is the upper integer part of the real number x , i.e. $\lceil x \rceil = \inf\{k \mid k \in \mathbb{Z}, k \geq x\}$.

Definition 8. Let p and q be two sequences and $B = (b_i)_{i=1}^\infty$ be an neighbourhood sequence. A finite sequence of sequences $\Pi(p, q; B)$ of the form $p = p_0, p_1, \dots, p_m = q$, where $p_{i-1}, p_i \in \mathbb{R}^\infty$ are b_i -neighbours for $1 \leq i \leq m$, is called a B -path from p to q . We write $m = |\Pi(p, q; B)|$ for the length of the path.

Remark 1. It is possible that there are no B -paths between two sequences. For example, if the set $\{|p(i) - q(i)| : i \in \mathbb{N}\}$ –, i.e. the set of elements of the difference sequence of them – is unbounded, then there are no neighbourhood sequence B , for which a B -path would exist between the sequences $p = (p(i))_{i=1}^{\infty}$ and $q = (q(i))_{i=1}^{\infty}$.

In the next section we will prove a necessary and sufficient condition for the existence of B -path between two sequences.

Now, we are ready to define the B -distance of any two sequences.

Definition 9. Let $p, q \in \mathbb{R}^{\infty}$ and B be an neighbourhood sequence. If there is no B -path between p and q , then we put $d(p, q; B) = \infty$. Otherwise, denote by $\Pi^*(p, q; B)$ a shortest path (i.e., a B -path with minimal length) from p to q , and set $d(p, q; B) = |\Pi^*(p, q; B)|$. We call $d(p, q; B)$ the B -distance of the sequences p and q .

It is evident that using the definition of B -distance above, it is positive definite (point a) of Definition 4) for any neighbourhood sequence B .

Definition 10. Let B_1 and B_2 be two neighbourhood sequences. We say that B_1 is faster than B_2 , if

$$d(p, q; B_1) \leq d(p, q; B_2) \quad \text{for all } p, q \in \mathbb{R}^{\infty}.$$

We denote this relation by $B_1 \sqsubseteq^* B_2$.

Originally, the relation $\sqsubseteq^*$ was introduced by Das [4] in the two dimensional digital space, and by Fazekas *et al.* in [7] for higher dimensions. We will use it for infinite sequences ($\mathbb{R}^{\infty}$).

For later use we need to introduce some further notations.

Definition 11. Let $m \in \mathbb{N}$ and $B = (b_i)_{i=1}^{\infty}$ an neighbourhood sequence. Put

$$b_i^{(m)} = \min(b_i, m) \text{ and } B^{(m)} = (b_i^{(m)})_{i=1}^{\infty}.$$

The sequence $B^{(m)}$ is called the m -limited sequence of B . Denote by $f_k(i)$ the i -th subsums of the k -limited sequence of B , i.e., put

$$f_k(i) = \begin{cases} \sum_{j=1}^i b_j^{(k)}, & \text{if } i \geq 1, \\ 0, & \text{if } i = 0. \end{cases}$$

Definition 12. Let $B = (b_i)_{i=1}^{\infty}$ be an neighbourhood sequence. The sequence $B(j) = (b_i)_{i=j}^{\infty}$ is called the j -shifted neighbourhood sequence of B .

3. MINIMAL PATH

In this section, we give an algorithm which provides a shortest path between arbitrary two sequences, if such a path exists. As we mentioned in Remark 1, it is possible that there is no path between two given sequences with a given neighbourhood sequence. The following lemma provides a criterion for the existence of a path between two sequences.

Lemma 2. *A finite B -path exists between the sequences p and q , i.e. their distance is finite, if and only if the difference sequence of them (w) has a supremum and the neighbourhood sequence B contains the symbol ∞ , at least k times, where k is the maximal element of the uids (u) of the sequences p and q , which occurs infinitely many times.*

Proof. First, we prove the case when the distance is infinite.

If the difference sequence w of p and q has not got a supremum (like the sequence of natural numbers $(1, 2, 3, \dots)$), then it is impossible to reach q from p (or from p to q) in finitely many steps, because if we want to take number m steps from p , for every $m \in \mathbb{N}$, we can find an element i of the sequence w , such that $w(i) > m$.

If the sequence B contains the symbol ∞ less times than k , then after the step with b_i , for which $\forall j \geq i, b_j < \infty$, we have infinite number of non-zero values in the difference sequence of q and the reached sequence. Thus, it is impossible to reach the sequence q in finitely many steps, with steps which are changing only finite number of elements.

Now, we are proving the sufficiency of the conditions to have a finite distance:

If the uids u of the sequences (p, q) has a largest element, then we denote this value by m . Let the sequence B contain the symbol ∞ at least k times. Then after the step with $b_j = \infty$, where b_j is the k -th value ∞ in B , we have only finite number of non-zero elements in the difference sequence of q and the reached sequence r . Forasmuch m was the largest element of the w , then after the j -th step, each element of the difference sequence of q and r is less or equal to m . Thus, we have finitely many finite numbers in the difference of q and r , then the sum of them is also finite. Therefore, we need only finitely many steps to reach q from p . (Our proof is same from q to p .) $\square$

Remark 2. If the neighbourhood sequence B is periodic, and it contains the element ∞ , then B contains ∞ at infinitely many positions.

The following algorithm provides one of the shortest B -paths between two arbitrary sequences, if such a path exists. This algorithm is based on the algorithm in [10], which works in finite and infinite dimensional digital spaces. (For algorithm works in finite dimensional digital space with only periodic neighbourhood sequences see [5].)

Algorithm 1

Input: An neighbourhood sequence $B = (b_i)_{i=1}^{\infty}$ and $p, q \in \mathbb{R}^{\infty}$, such that $d(p, q; B) < \infty$.

- step 1. Let $w^{(0)}$ be the absolute difference of p and q , $t(i) = \text{sgn}(p(i) - q(i))$, ($i \in \mathbb{N}$), and put $j = 0$ and $\Pi = (p)$.
- step 2. If $w^{(j)}(i) = 0$ for every i then goto step 8, else set $j = j + 1$.
- step 3. Put $w^{(j)} = w^{(j-1)}$.
- step 4. If b_j is finite, then select the largest b_j entries of $w^{(j)}$. If b_j is infinite, then select all the entries of $w^{(j)}$.
- step 5. For each selected $w^{(j)}(i)$ if $w^{(j)}(i) \geq 1$, let $w^{(j)}(i) = w^{(j-1)}(i) - 1$ else let $w^{(j)}(i) = 0$.
- step 6. Append to the path Π the sequence x_j defined by $x_j(i) = q(i) + w^{(j)}(i)t(i)$ for all i .
- step 7. Goto 2.
- step 8. Output Π as a minimal B -path between p and q , and j as the length of this path.

Lemma 3. *The B -distance of the sequences p and q is invariant to the shift, i.e. for arbitrary $r \in \mathbb{R}^{\infty}$ the following statement is true:*

$$d(p, q; B) = d((p(i) + r(i))_{i=1}^{\infty}, (q(i) + r(i))_{i=1}^{\infty}; B).$$

Proof. If $d(p, q; B)$ is finite, then let $\Pi^*(p, q; B) : (p = p_0), p_1, \dots, (p_m = q)$ be a shortest path from p to q .

It is easy to check that $\Pi(p', q'; B)$ with sequences $p'_j = (p_j(i) + r(i))_{i=1}^{\infty}$ ($0 \leq j \leq m$) is also a shortest B -path between $p' = (p(i) + r(i))_{i=1}^{\infty}$ and $q' = ((q(i) + r(i))_{i=1}^{\infty})$.

Otherwise, suppose that $d(p, q; B) = \infty$. Our aim to show that $d(p', q'; B) = \infty$ also with a shift by the sequence r . Contrary suppose that it is not. Then, we have a shortest path between p' and q' . Now it is finite distance therefore using the previous part of the proof we have the same distance between p'' and q'' where we use a shift by an arbitrary r' . We choose r' as the sequence $(-r(i))_{i=1}^{\infty}$, therefore $p'' = p$ and $q'' = q$. But it is a contradiction, because a distance is finite or infinite, but it is not both of them. $\square$

Observe that the algorithm is a greedy algorithm. The following theorem is about the correctness of our algorithm. We use the term step as step of the algorithm, however some steps are complex.

Theorem 1. *Algorithm 1 terminates after finitely many steps and provides a B -path with minimal length between the sequences p and q .*

Proof. We will deduce the statement to the case of sequences over $\mathbb{Z}^\infty$, for which similar theorem was proved in [10].

Using Lemma 3, without loss of generality, we can assume that the sequence $p = (0)_{i=1}^\infty$ and q is an arbitrary element of $\mathbb{R}^\infty$.

It is evident that our distance is based on the number of steps, independently how much is the change of the values in a step (i.e., they are 1, or may be less than 1). Substituting the values which changes less than 1 via the values are changing exactly 1, we get the same distance.

Therefore, the distance of arbitrary two sequences can be calculated by

$$d(p, q; B) = d(o, u; B),$$

where $o = (0)_{i=1}^\infty$ and u is the uids of p and q .

The sequences o and u contain only integer values, therefore the correctness of the algorithm follows from the correctness of the algorithm working on sequences in $\mathbb{Z}^\infty$ [10]. $\square$

Lemma 4. *The B -distance is symmetric.*

Proof. Our proof is based on the algorithm. In the case when at step 6 we append the sequence $x_j(i) = p(i) - w^{(j)}(i)t(i)$ for all i we get a shortest path from $q = x_0$ to p , which has the same length as the shortest path from p to q . In the case of infinite distance: if there is no shortest path between two sequences, then it does not exist in any directions. $\square$

Consequence 1. The B -distance of two sequences p and q depends only on the neighbourhood sequence B and the up-integer-difference-sequence u of p and q .

Using Consequence 1 we can adopt some properties from [10], where only the sequences with integer values was studied. In the next section we analyse some properties of the B -distances.

4. PROPERTIES OF B -DISTANCES OF SEQUENCES

In this section, first we state the relation between the sup-distance and the B -distances.

Proposition 3. *For any two $p, q \in \mathbb{R}^\infty$,*

$$d(p, q; (\infty)_{i=1}^\infty) = \lceil d(p, q; \text{sup}) \rceil.$$

Remark 3. Using a variation of Algorithm 1 and the definition of B -distances with $B = (\infty)_{i=1}^\infty$ we get exactly the same value for this, (∞) -distance, as for the sup-distance. The difference should be at the last step by reaching q . Increasing the path length by $\text{sup}(w^{(j-1)}(i))$ instead of 1 in step 2 of the algorithm the provided minimal path-length will be exactly the same as the sup-distance of p and q .

Proposition 4. *Using the neighbourhood sequence $(1)_{i=1}^{\infty}$ the distance (we will use the notation (1)-distance) of any two sequences p and q is at least the number of elements in which they differ, i.e.*

$$d(p, q; (1)_{i=1}^{\infty}) \geq \sum_{p(i) \neq q(i)} 1 = d(p, q; H).$$

Moreover, we have the following statement among the values of B -distances.

Lemma 5. *For any two sequence p and q the distances are in the following relation:*

$$\sum \lceil u(i) \rceil = d(p, q; (1)_{i=1}^{\infty}) \geq d(p, q; B) \geq d(p, q; (\infty)_{i=1}^{\infty}) = \max(\lceil u(i) \rceil),$$

where B is an arbitrary neighbourhood sequence.

Proof. It is easy to show by using the algorithm. $\square$

Using neighbourhood sequences for calculating distances between sequences we have wide a variety of distances between the distances using $(1)_{i=1}^{\infty}$ and $(\infty)_{i=1}^{\infty}$. These two distances are the ceiling (i.e., top) of the distances L_1 and L_{∞} , respectively (see [2]). Although the L -distances (also known as Minkowski distances) were originally introduced for points, we can define and use them for infinite sequences also: $d(p, q; L_1) = \sum w(i)$ and $d(p, q; L_{\infty}) = \sup(w(i))$, the distances L_j is defined by $(\sum (w(i))^j)^{\frac{1}{j}}$, the distance L_2 is the usual Euclidean distance. The distance L_{∞} is given by the limit of distances L_i with $i \rightarrow \infty$ and it is the same as the sup-distance. These L -distances are not pleasant in many cases because for infinite sequences the sum $\sum_i (w(i))^j$ is usually infinite. Hence calculating the L -distances we have some difficulties. (In [12] we return to this problem.) For these reasons we recommend the B -distances which have more pleasant properties in this point of view.

The previous lemma is about the values of B -distances. We will study more precisely the relation ‘faster’ of neighbourhood sequences later.

Lemma 6. *The following statements are equivalent.*

- *The B -distance of $p, q \in \mathbb{R}^{\infty}$ is finite for every neighbourhood sequence B .*
- *The sequences p and q have the same tail (i.e. $\exists j, \forall k > j, p(k) = q(k)$).*
- *The difference sequence w of p and q is 0-convergent and 0-sequence.*

Proof. It is evident from the definitions that a sequence has only finitely many non-zero elements if and only if it is 0-convergent and 0-sequence. (And the difference sequence of the sequences p and q has this property if and only if they have the same tail.) Otherwise, from Lemma 5 we know that the (1)-distance is the greatest among the B -distances. Moreover the (1)-distance of two sequences – for which their difference sequence has only finitely many non-zero element – is finite. Therefore every B -distance of them is finite, thus the equivalence of the statements follows. $\square$

Moreover, we generalize the fact above in the following theorem.

Theorem 2. *The B -distance of two arbitrary sequences p and q is finite with the neighbourhood sequence B including the symbol ∞ at least k times if and only if the difference w of p and q is a k -sequence. (And their B -distance is infinite if the number of ∞ is k in B , and the difference w is not a k -sequence.)*

Proof. If the difference sequence w of p and q is a k -sequence, then it does not contain greater values than k infinitely many times (its tail does not contain such elements). Obviously, applying the algorithm to obtain a shortest path, after the k -th step by the elements ∞ , w contains only finitely many non-zero elements, and their sum is finite, hence each of them is finite. Therefore, after the step by the k -th ∞ , we need only finitely many steps to reach q . Thus, it is proven that if B contains at least k -times the symbol ∞ , then the B -distance of p and q is finite when their difference is a k -sequence. In the other way around, if w is not a k -sequence, then it must contain infinitely many elements which are not less than k . In this case, after the last (k -th) step with the symbol ∞ , the sum $\sum_{w^{(j)}(i) > 0} 1$ is infinite, therefore we cannot decrease it to 0 with finitely many ‘finite’-steps. Therefore, in that case, the B -distance of p and q is infinite. $\square$

Now we present a formula to calculate the B -distance of any two sequences. Using Consequence 1 we adopt the B -distance on $\mathbb{Z}^\infty$, which can be found in [11]. For this calculation we will use the sequence v which has the same elements as the uids u , sorting by non-decreasing order (i.e., the multiset of elements u is the same as the multiset of elements of v and $v(i) \geq v(j)$ for $i < j$). We also use the j -limited sequence $B^{(j)}$ of the neighbourhood sequence B from Definition 11.

Proposition 5. *The B -distance of any two sequences p and q is given by*

$$d(p, q; B) = \max_{i \in \mathbb{N}} \{d^{(i)}(p, q)\},$$

where

$$d^{(i)}(p, q) = \max \left\{ h \mid \sum_{k=1}^i v(k) > \sum_{k=1}^{h-1} b_k^{(i)} \right\}.$$

The following lemma is very useful if we would like to decide numerically whether an neighbourhood sequence is faster or not than another one. Here the subsums of neighbourhood sequences of Definition 11 are also used.

Lemma 7. *Let B_1 and B_2 be two neighbourhood sequences. Then,*

$$d(p, q; B_1) \leq d(p, q; B_2), \quad \text{for all } p, q \in \mathbb{R}^\infty,$$

if and only if

$$f_k^{(1)}(i) \geq f_k^{(2)}(i), \quad \text{for all } i, k \in \mathbb{N},$$

where $f_k^{(1)}(i)$ and $f_k^{(2)}(i)$ correspond to B_1 and B_2 , respectively.

Proof. This statement is given by a simple calculation from the previous proposition. In Proposition 5 we use the values $f_i(h-1)$ on the right hand side of the inequalities of the calculation of the values $d^{(i)}(p, q)$. Substituting it to the form for calculating B -distance, one can easily check the statement of this lemma. $\square$

Remark 4. As a simple consequence of the previous lemma we obtain that if B_1 and B_2 are two neighbourhood sequences with $B_1 \sqsupseteq^* B_2$, then for every $i \in \mathbb{N}$ among the first i elements of B_1 there are at least as many ∞ symbols as among the first i elements of B_2 .

Based on the previous lemma we study the relation ‘faster’ among the neighbourhood sequences.

Lemma 8. *The faster relation is not a complete ordering, i.e., there are neighbourhood sequences which are non-comparable.*

Proof. Let $B_1 = (\infty, 1, 1, 1, \dots)$ (where each element is 1 after the first) and $B_2 = (10, 10, \infty, \infty, \dots)$ (each element is ∞ after the second). Let $p = (\frac{1}{i})_{i=1}^\infty$ and $q = (\frac{2}{i})_{i=1}^\infty$; then $d(p, q, B_1) = 1$ and $d(p, q, B_2) = 3$. Let $r = (\max(\frac{6}{i}, \frac{1}{2}))_{i=1}^\infty$, then $d(r, q, B_2) = 4$ and $d(r, q, B_1) = 6$. $\square$

Remark 5. It is clear that $\sqsupseteq^*$ is a reflexive, antisymmetric, transitive relation on the set of neighbourhood sequences, i.e., it is a partial order. However, it does not form a lattice. (It can be proven in the same way as in [7].)

Now, we want to know whether a B -distance is a generalized metric above the set of sequences $\mathbb{R}^\infty$. The distance based on an arbitrary neighbourhood sequence, in general, does not satisfy the conditions of a (generalized) metric. However, in geometry those distances are the most useful (e.g., have more practical interest), which have this property. In the next part of this section we give a necessary and sufficient condition for a distance based on a neighbourhood sequence to be a (generalized) metric (and we continue this analysis in [12], too).

Lemma 9. *Let p and q be arbitrary sequences in $\mathbb{R}^\infty$, and let the neighbourhood sequence B_1 be faster than the neighbourhood sequence B_2 (i.e., $B_1 \sqsupseteq^* B_2$). If there is no B_1 -path between p and q , then there does not exist any B_2 -path between them.*

Proof. From Definition 10, if $d(p, q; B_1) = \infty$ and B_1 is faster than B_2 , then $d(p, q; B_2) = \infty$. $\square$

We show a property in which the distances based neighbourhood sequences differ in $\mathbb{Z}^\infty$ and in $\mathbb{R}^\infty$.

Proposition 6. *Let $p, q, r \in \mathbb{R}^\infty$ such a way that $p(i) < r(i) < q(i)$. If p, q, r contain only integer elements, then the following statements hold (with their difference-sequences):*

$$u_{pr}(i) + u_{rq}(i) = u_{pq}(i),$$

because

$$w_{pr}(i) + w_{rq}(i) = w_{pq}(i),$$

and $w_{pr}(i) = u_{pr}(i)$, $w_{rq}(i) = u_{rq}(i)$ and $w_{pq}(i) = u_{pq}(i)$ for all i , since

$$(p(i) - r(i)) + (r(i) - q(i)) = p(i) - q(i).$$

For non integer valued sequences it is easy to show a counterexample.

The next theorem is the extension of the results about metrical properties of [10] (for previous results, see [5], concerning periodic neighbourhood sequences in finite dimension), to the arbitrary sequences. To formulate our result we need only the simple concepts of the faster relation and the shifted sequence (similarly to the case of $\mathbb{Z}^\infty$, as it is shown in [10]). We have to care only the triangle inequality, which is a statement for three sequences and we showed in Proposition 6 that the case of $\mathbb{Z}^\infty$ is not the same as $\mathbb{R}^\infty$. (There is no problem with the symmetry by Lemma 4) Based on the proof of the case $\mathbb{Z}^\infty$ using the properties of uids sequences the next theorem can be established.

Theorem 3. *The distance function based on an neighbourhood sequence B generates a generalized metric space on the set $\mathbb{R}^\infty$, if and only if $B(i)$ is faster than B for all $i \in \mathbb{N}$.*

(This theorem is proven in a more general form in the second part of the paper, see [12].)

By Lemma 7 one can decide, whether an neighbourhood sequence give a generalized metric space or not:

Proposition 7. *The B -distance is a generalized metric if and only if*

$$d(p, q; B(i)) \leq d(p, q; B), \quad \text{for all } p, q \in \mathbb{R}^\infty \text{ and } i \in \mathbb{N}, \text{ i.e.,}$$

$$\sum_{j=1}^m b_{i+j}^k \geq \sum_{j=1}^m b_j^k, \quad \text{for all } i, k, m \in \mathbb{N}.$$

By using the definition of the k -limited sequence it is equivalent to the condition

$$\sum_{j=1+i}^{m+i} \min(b_j, k) \geq \sum_{j=1}^m \min(b_j, k), \quad \text{for all } i, k, m \in \mathbb{N}.$$

Now we study the structure of the set of neighbourhood sequences based on another kind of partial ordering (see [7]).

Definition 13. For $B = (b_i)_{i=1}^\infty, B' = (b'_i)_{i=1}^\infty$ we write $B \sqsupseteq B'$ if and only if $b_i \geq b'_i$ for every $i \in \mathbb{N}$.

It turns out that $\sqsupseteq$ has much more pleasant properties than $\sqsupseteq^*$. Actually, this relation is proper part of the relation ‘faster’ (and in a certain sense, it is much stricter).

Remark 6. Let B be an neighbourhood sequence. Then for the limited neighbourhood sequences of B the following relations hold: $B \sqsupseteq B^k \sqsupseteq B^m$ for every $k, m \in \mathbb{N}$ with $k > m$.

We will use this new partial ordering relation to get lattice on the set of neighbourhood sequences, as the next statement shows.

Theorem 4. *The set of neighbourhood sequences is a distributive lattice using the relation $\sqsupseteq$, with greatest lower bound $(1)_{i=1}^\infty$ and least upper bound $(\infty)_{i=1}^\infty$. Moreover, this lattice is complete.*

Proof. The proof goes in the same way as in [7] for ∞ D neighbourhood sequences in $\mathbb{Z}^\infty$. $\square$

See [7] for other results using the relation $\sqsupseteq$ and $\sqsupseteq^*$ among special subsets of the neighbourhood sequences. Their properties are independent of their using conditions, therefore we do not give more details on them here. Instead of it, we are going to show another type of extensions of these distances in [12] as a continuation of this paper.

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Author's address

Benedek Nagy

Eastern Mediterranean University, Faculty of Arts and Sciences, Department of Mathematics, Famagusta, North Cyprus, Mersin-10, Turkey

E-mail address: nbenedek.inf@gmail.com



VARIOUS SHADOWING PROPERTIES AND THEIR EQUIVALENT FOR EXPANSIVE ITERATED FUNCTION SYSTEMS

MEHDI FATEHI NIA

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Abstract. In this paper we introduce expansive *iterated function systems*, (**IFS**) on a compact metric space then various shadowing properties and their equivalence are considered for expansive **IFS**.

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1. INTRODUCTION

The notion of shadowing plays an important role in dynamical systems, specially; in stability theory [1, 11, 12]. Various shadowing properties for expansive maps and their equivalence have been studied by Lee and Sakai [10, 14]. More precisely, they prove the following theorems:

Theorem 1 ([14]). *Let f be an expansive homeomorphism on a compact metric space $(X; d)$. Then the following conditions are mutually equivalent:*

- (a) *f has the shadowing property,*
- (b) *f has the continuous shadowing property,*
- (c) *there is a compatible metric D for X such that f has the Lipschitz shadowing property with respect to D ,*
- (d) *f has the limit shadowing property,*
- (e) *there is a compatible metric D for X such that f has the strong shadowing property with respect to D .*

Theorem 2 ([10]). *Let f be a positively expansive map on a compact metrizable space X . Then the following conditions are mutually equivalent:*

- (a) *f is an open map,*
- (b) *f has the shadowing property,*
- (c) *there is a metric such that f has the Lipschitz shadowing property,*
- (d) *there is a metric such that f has the s -limit shadowing property,*
- (e) *there is a metric such that f has the strong shadowing property.*

In the other hand, iterated function systems(**IFS**), are used for the construction of deterministic fractals and have found numerous applications, in particular to image compression and image processing [2]. Important notions in dynamics like attractors, minimality, transitivity, and shadowing can be extended to IFS (see [3, 4, 7–9]). The authors defined the shadowing property for a parameterized iterated function system and prove that if a parameterized IFS is uniformly expanding (or contracting), then it has the shadowing property[9].

In this paper we present an approach to shadowing property for iterated function systems. At first, we introduce *expansive iterated function systems* on a compact metric space. Then continuous shadowing, limit shadowing and Lipschitz shadowing properties are defined for an **IFS**, $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ where Λ is a nonempty finite set and $f_\lambda : X \rightarrow X$ is homeomorphism, for all $\lambda \in \Lambda$. Theorems 3 and 4 are the main result of the present work. Actually in these theorems we prove that the limit shadowing property, the Lipschitz shadowing property are all equivalent to the shadowing property for expansive **IFS** on a compact metric space. The method is essentially the same as that used in [10, 13, 14]. Finally, we introduce the strong expansive **IFS** and show that for a strong expansive **IFS** the continuous shadowing property and the shadowing property are equivalent.

2. PRELIMINARIES

In this section, we give some definitions and notations as well as some preliminary results that are needed in the sequel. Let (X, d) be a complete metric space. Let us recall that an *Iterated Function System(IFS)*

$\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is any family of continuous mappings $f_\lambda : X \rightarrow X$, $\lambda \in \Lambda$, where Λ is a finite nonempty set (see[9]).

Let $\Lambda^{\mathbb{Z}}$ denote the set of all infinite sequences $\{\lambda_i\}_{i \in \mathbb{Z}}$ of symbols belonging to Λ . A typical element of $\Lambda^{\mathbb{Z}}$ can be denoted as $\sigma = \{\dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots\}$ and we use the shorted notation

$$\mathcal{F}_{\sigma_0} = id,$$

$$\mathcal{F}_{\sigma_n} = f_{\lambda_{n-1}} \circ f_{\lambda_{n-2}} \circ \dots \circ f_{\lambda_0},$$

$$\mathcal{F}_{\sigma_{-n}} = f_{\lambda_{-n}}^{-1} \circ f_{\lambda_{-(n-1)}}^{-1} \circ \dots \circ f_{\lambda_{-1}}^{-1}.$$

Please note that if f_λ is a homeomorphism map for all $\lambda \in \Lambda$, then for every $n \in \mathbb{Z}$ and $\sigma \in \Lambda^{\mathbb{Z}}$, $\mathcal{F}_{\sigma_n}$ is a homeomorphism map on X .

A sequence $\{x_n\}_{n \in \mathbb{Z}}$ in X is called an orbit of the **IFS** $\mathcal{F}$ if there exist $\sigma \in \Lambda^{\mathbb{Z}}$ such that $x_{n+1} = f_{\lambda_n}(x_n)$, for each $\lambda_n \in \sigma$.

The **IFS** $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is *uniformly expanding* if there exists

$$\beta = \inf_{\lambda \in \Lambda} \inf_{x \neq y} \frac{d(f_\lambda(x), f_\lambda(y))}{d(x, y)}$$

and this number called also the *expanding ratio*, is greater than one [9].

We say that $\mathcal{F}$ is expansive if there exist a $e > 0$ such that for every arbitrary $\sigma \in \Lambda^{\mathbb{Z}}$, $d(\mathcal{F}_{\sigma_i}(x), \mathcal{F}_{\sigma_i}(y)) < e$, for all $i \in \mathbb{Z}$, implies that $x = y$.

Remark 1. Let $\mathcal{F}$ be an uniformly expanding **IFS** and $\beta > 1$ is it's expanding ratio number. Suppose that for every $\sigma \in \Lambda^{\mathbb{Z}}$, $\mathcal{F}_{\sigma_n}$ and $z \in X$, $\{\mathcal{F}_{\sigma_n}(z)\}_{n \in \mathbb{Z}}$, is an infinite set.

Consider the number $1 < \gamma < \beta$, since Λ is a finite set there exists at most a finite set of points $x, y \in X$ such that $\frac{d(f_{\lambda}(x), f_{\lambda}(y))}{d(x, y)} \leq \gamma$, for some $\lambda \in \Lambda$. Now, Let $\sigma = \{\dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots\}$ be an element of $\Lambda^{\mathbb{Z}}$ and consider distinct points $x, y \in X$. Suppose that $d(\mathcal{F}_{\sigma_i}(x), \mathcal{F}_{\sigma_i}(y)) < 1$ for all $i \in \mathbb{Z}$. Since every orbit is an infinite set and f_{λ_i} is injective, for all $i \in \mathbb{Z}$, then there exists $k > 0$ such that $\frac{d(f_{\lambda_{k+i}}(\mathcal{F}_{\sigma_{k+i-1}}(x)), f_{\lambda_{k+i}}(\mathcal{F}_{\sigma_{k+i-1}}(y)))}{d(\mathcal{F}_{\sigma_{k+i-1}}(x), \mathcal{F}_{\sigma_{k+i-1}}(y))} > \gamma$, for all $i \geq 0$. So, $\gamma^i d(\mathcal{F}_{\sigma_k}(x), \mathcal{F}_{\sigma_k}(y)) < d(\mathcal{F}_{\sigma_{k+i}}(x), \mathcal{F}_{\sigma_{k+i}}(y)) < 1$, for all $i > 0$, and consequently $x = y$. Then uniformly expanding implies the expansivity.

Also following example shows that expansivity does not imply uniformly expanding.

Example 1. Let Σ denote the set of all bi-infinite sequence $x = (\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots)$ where $x_n = 0$ or 1 . The set Σ becomes a compact metric space if we define the distance between two points x, y by $\rho(x, y) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{2^{|i|}}$. This is well known that the shift map $g : \Sigma \rightarrow \Sigma$ defined by $(\sigma(x))_i = x_{i+1}$ is a homeomorphism and so by compactness of Σ is not uniformly expanding. If $x \neq y$ are two point in Σ then, for some $k \in \mathbb{Z}$, $x_k \neq y_k$ and hence $\rho(g^k(x), g^k(y)) \geq 1$, this implies that $g : \Sigma \rightarrow \Sigma$ is an expansive homeomorphism. Let $f : \Sigma \rightarrow \Sigma$ be a map defined by $(f(x))_i = x_{i+2}$, using the above arguments, we can prove that if $x \neq y$ are two point in Σ then, for some $k \in \mathbb{Z}$, $\rho(f^k(x), f^k(y)) \geq \frac{1}{2}$. Now consider the IFS $\mathcal{F} = \{\Sigma; f_i \mid i \in \{1, 2\}\}$, where $f_1 = g$ and $f_2 = f$. If $x \neq y$ are two point in Σ then, for every $\sigma \in \Lambda^{\mathbb{Z}}$, there exists $k \in \mathbb{Z}$ such that $\rho(\mathcal{F}_{\sigma_k}(x), \mathcal{F}_{\sigma_k}(y)) \geq \frac{1}{2}$. So the IFS $\mathcal{F} = \{\Sigma; f_i \mid i \in \{1, 2\}\}$ is an expansive IFS contains homeomorphism functions and is not uniformly expanding.

Given $\delta > 0$, a sequence $\{x_i\}_{i \in \mathbb{Z}}$ in X is called a δ -pseudo orbit of $\mathcal{F}$ if there exist $\sigma \in \Lambda^{\mathbb{Z}}$ such that for every $\lambda_i \in \sigma$, we have $d(x_{i+1}, f_{\lambda_i}(x_i)) < \delta$.

One says that the **IFS** $\mathcal{F}$ has the *shadowing property* if, given $\epsilon > 0$, there exists $\delta > 0$ such that for any δ -pseudo orbit $\{x_i\}_{i \in \mathbb{Z}}$ there exist an orbit $\{y_i\}_{i \in \mathbb{Z}}$, satisfying the inequality $d(x_i, y_i) \leq \epsilon$ for all $i \in \mathbb{Z}$. In this case one says that the $\{y_i\}_{i \in \mathbb{Z}}$ or the point y_0 , ϵ - shadows the δ -pseudo orbit $\{x_i\}_{i \in \mathbb{Z}}$ [9].

Please note that if Λ is a set with one member then the **IFS** $\mathcal{F}$ is an ordinary discrete dynamical system. In this case the shadowing property for $\mathcal{F}$ is ordinary shadowing

property for a discrete dynamical system.

The point y_0 in definition of shadowing property related to the sequence σ which makes the orbit $\{y_i\}_{i \in \mathbb{Z}}$ is unique.

Remark 2. Suppose that the **IFS** $\mathcal{F}$ satisfied shadowing property and is expansive with expansive constant $e > 0$. Consider $0 < \epsilon < \frac{e}{3}$ and choose $\delta > 0$ corresponding to ϵ in the definition of shadowing property. Let $\{x_i\}_{i \in \mathbb{Z}}$ be a δ -pseudo orbit for **IFS**, there exist a sequences $\sigma = \{\dots, \mu_{-1}, \mu_0, \mu_1, \dots\}$ and $\{y_i\}_{i \in \mathbb{Z}}$ such that $y_{i+1} = f_{\mu_i}(y_i)$ and $d(x_i, y_i) < \epsilon$, for all $i \in \mathbb{Z}$. If $\{z_i\}_{i \in \mathbb{Z}}$ be another sequence such that $z_{i+1} = f_{\mu_i}(z_i)$ and $d(x_i, z_i) < \epsilon$, for all $i \in \mathbb{Z}$, then for all $i \in \mathbb{Z}$:

$$d(\mathcal{F}_{\sigma_i}(y_0), \mathcal{F}_{\sigma_i}(z_0)) = d(y_i, z_i) \leq d(y_i, x_i) + d(x_i, z_i) < 2\epsilon < e.$$

Thus by expansivity of the **IFS** $\mathcal{F}$ we have that $y_0 = z_0$.

We say that $\mathcal{F}$ has the *Lipschitz shadowing property* if there are $L > 0$ and $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ and any ϵ -pseudo orbit $\{x_i\}_{i \in \mathbb{Z}}$ of $\mathcal{F}$ there exist $y \in X$ and $\sigma \in \Lambda^{\mathbb{Z}}$ such that $d(\mathcal{F}_{\sigma_i}(y), x_i) < L\epsilon$, for all $i \in \mathbb{Z}$.

We say that $\mathcal{F}$ has the *limit shadowing property* if: for any sequence $\{x_i\}_{i \in \mathbb{Z}}$ of points in X , if $\lim_{i \rightarrow \pm\infty} d(f_{\lambda_i}(x_i), x_{i+1}) = 0$, for some

$\sigma = \{\dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots\} \in \Lambda^{\mathbb{Z}}$ then there is an orbit $\{y_i\}_{i \in \mathbb{Z}}$ such that $\lim_{i \rightarrow \pm\infty} d(x_i, y_i) = 0$

Let $X^{\mathbb{Z}}$ be the set of all sequences $\{x_i\}_{i \in \mathbb{Z}}$ of points in X and let $\tilde{d}$ be the metric on $X^{\mathbb{Z}}$ defined by

$$\tilde{d}(\{x_i\}_{i \in \mathbb{Z}}; \{y_i\}_{i \in \mathbb{Z}}) = \sup_{i \in \mathbb{Z}} \frac{d(x_i, y_i)}{2^{|i|}},$$

for $\{x_i\}_{i \in \mathbb{Z}}, \{y_i\}_{i \in \mathbb{Z}} \in X^{\mathbb{Z}}$. Let $p(\mathcal{F}, \delta)$ be the set of all δ -pseudo-orbits ($\delta > 0$) of $\mathcal{F}$ with the subspace topology of $X^{\mathbb{Z}}$ [14].

We say that $\mathcal{F}$ has the *continuous shadowing property* if for every $\epsilon > 0$, there are a $\delta > 0$ and a continuous map $r : p(\mathcal{F}, \delta) \rightarrow X$ such that

$d(\mathcal{F}_{\sigma_i}(r(x)), x_i) < \epsilon$, where $\sigma = \{\dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots\}$, $x = \{x_i\}_{i \in \mathbb{Z}}$ and $d(f_{\lambda_i}(x_i), x_{i+1}) < \delta$, for all $i \in \mathbb{Z}$.

3. RESULTS

By Theorem 2, Sakai showed that any positively expansive open map has the shadowing property. In this section we introduce open **IFS** and show that for an expansive **IFS**, the openness; shadowing property and Lipschitz shadowing property are equivalent.

$\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is said to be an open **IFS** if f_λ is an open map, for all $\lambda \in \Lambda$.

Definition 1 ([10]). Let $f : X \rightarrow X$ be a continuous map on a compact metric space. We say that f expands small distances if there exist constants $\delta_0 > 0$ and

$\alpha > 1$ such that $0 < d(x, y) < \delta_0$ ($x, y \in X$) implies
 $d(f(x), f(y)) > \alpha d(x, y)$.

We say that $\mathcal{F}$ expands small distance, if there are constants $\delta_0 > 0$ and $\alpha > 1$ such that $d(f_\lambda(x), f_\lambda(y)) > \alpha d(x, y)$ whenever $0 < d(x, y) < \delta_0$ ($\lambda \in \Lambda$)

Remark 3. Suppose that $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is an expansive **IFS**, then f_λ is an expansive function and by Lemma 1. of [10] expand small distance. Let in the proof of Lemma 1. of [10]

$V_n = \{(x, y) \in X \times X : d(\mathcal{F}_{\sigma_i}(x), \mathcal{F}_{\sigma_i}(y)) \leq c, \text{ for all } \sigma \in \Lambda^{\mathbb{Z}} \text{ and all } |i| < n\}$.
 So, $\mathcal{F}$ expand small distance.

To prove Theorem 3, we need the following lemma.

Lemma 1. Suppose that $\mathcal{F}$ expands small distance with related constants $\alpha > 1$ and $\delta_0 > 0$, then the following are equivalent:

i) $\mathcal{F}$ is an open IFS.

ii) There exists $0 < \delta_1 < \frac{\delta_0}{2}$ such that if $d(f_\lambda(x), y) < \alpha \delta_1$ then

$B_{\delta_1}(x) \cap f_\lambda^{-1}(y) \neq \emptyset$, for all $\lambda \in \Lambda$, where $B_{\delta_1}(x)$ is the neighborhood of x with radius δ_1 .

Proof. Since every f_λ expands small distance then by Lemma 2. of [10], for every $\lambda \in \Lambda$ the following are equivalent:

i) f_λ is an open map.

ii) there exists $0 < \delta_\lambda < \frac{\delta_0}{2}$ such that if $d(f_\lambda(x), y) < \alpha \delta_\lambda$ then

$B_{\delta_\lambda}(x) \cap f_\lambda^{-1}(y) \neq \emptyset$.

Because of the proof of Lemma 1 in [5] for every $\lambda \in \Lambda$ there exist infinitely $0 < \delta < \delta_\lambda$ such that $d(f_\lambda(x), y) < \alpha \delta$ implies $B_{\delta_\lambda}(x) \cap f_\lambda^{-1}(y) \neq \emptyset$. So this sufficient to take

$\delta_1 = \min\{\delta_\lambda : \lambda \in \Lambda\}$. □

Corollary 1. Let the IFS $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ expands small distance with related constants $\alpha > 1$ and $\delta_0 > 0$, and satisfied condition ii in Lemma 1. Then for all $0 < \epsilon \leq \delta_1$, $\lambda \in \Lambda$ and $x, y \in X$, $d(f_\lambda(x), y) < \epsilon$ implies that $B_{\frac{\epsilon}{\alpha}}(x) \cap f_\lambda^{-1}(y) \neq \emptyset$.

Proof. Let $d(f_\lambda(x), y) < \epsilon$, since $\epsilon \leq \delta_1$ by Lemma 1. we have that $B_{\delta_1}(x) \cap f_\lambda^{-1}(y) \neq \emptyset$. Take $z \in B_{\delta_1}(x) \cap f_\lambda^{-1}(y)$. This is clear that $y = f_\lambda(z)$ and $d(x, z) < \delta_1$. Since $\mathcal{F}$ expands small distances,

$$\alpha d(x, z) < d(f_\lambda(x), f_\lambda(z)) = d(f_\lambda(x), y) < \epsilon.$$

Thus $d(x, z) < \frac{\epsilon}{\alpha}$ and hence $B_{\frac{\epsilon}{\alpha}}(x) \cap f_\lambda^{-1}(y) \neq \emptyset$. □

Theorem 3. Suppose that $\mathcal{F}$ expands small distance with related constants $\alpha > 1$ and $\delta_0 > 0$, the following conditions are equivalent:

i) $\mathcal{F}$ is an open IFS.

ii) $\mathcal{F}$ has the shadowing property.

iii) $\mathcal{F}$ has the Lipschitz shadowing property.

Proof. (iii $\Rightarrow$ ii) By definitions of the shadowing and Lipschitz shadowing properties this is clear that the Lipschitz shadowing property implies the shadowing property.

(i $\Rightarrow$ iii) Let $L = \frac{2\alpha}{\alpha-1} = 2 \sum_{k=0}^{\infty} \alpha^{-k} > 1$ and fix any $0 < \epsilon < \frac{\delta_1}{L}$, where δ_1 and α be as in Lemma 1. Suppose that $\{x_i\}_{i \in \mathbb{Z}}$ is an ϵ -pseudo orbit for $\mathcal{F}$; there is $\sigma = \{\dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots\} \in \Lambda^{\mathbb{Z}}$ such that $d(f_{\lambda_i}(x_i), x_{i+1}) < \epsilon$ for all $i \in \mathbb{Z}$.

Pick any $i \geq 1$ and put $\alpha_j = \sum_{k=0}^{j-1} \alpha^{-k}$ for $j \geq 1$. Since $d(f_{\lambda_i}(x_i), x_{i+1}) < \epsilon$ then, by Lemma 1 and Corollary 1, there exists $y_{i-1}^{(i)} \in B_{\frac{\epsilon}{\alpha}}(x_{i-1})$ such that $f_{\lambda_{i-1}}(y_{i-1}^{(i)}) = x_i$. Thus

$$\begin{aligned} d(f_{\lambda_{i-2}}(x_{i-2}), y_{i-1}^{(i)}) &\leq d(f_{\lambda_{i-2}}(x_{i-2}), x_{i-1}) + d(x_{i-1}, y_{i-1}^{(i)}) \\ &< \epsilon + \frac{\epsilon}{\alpha} = \epsilon(1 + \frac{1}{\alpha}) < \epsilon L. \end{aligned}$$

Hence there exists $y_{i-2}^{(i)} \in B_{\alpha_2 \frac{\epsilon}{\alpha}}(x_{i-2})$ such that $f_{\lambda_{i-2}}(y_{i-2}^{(i)}) = y_{i-1}^{(i)}$ and so

$$\begin{aligned} d(f_{\lambda_{i-2}}(x_{i-2}), y_{i-2}^{(i)}) &\leq d(f_{\lambda_{i-2}}(x_{i-2}), x_{i-1}) + d(x_{i-1}, y_{i-1}^{(i)}) \\ &< \epsilon + \alpha_2 \frac{\epsilon}{\alpha} < \alpha_3 \epsilon < \epsilon L. \end{aligned}$$

Because of Lemma 1 and Corollary 1 there exists $y_{i-3}^{(i)} \in B_{\alpha_3 \frac{\epsilon}{\alpha}}(x_{i-3})$ such that

$f_{\lambda_{i-3}}(y_{i-3}^{(i)}) = y_{i-2}^{(i)}$. Thus $d(f_{\lambda_{i-3}}(x_{i-3}), y_{i-3}^{(i)}) < \alpha_4 \epsilon < \epsilon L$.

Repeating the process, we can find:

$y_0^{(i)} \in B_{\alpha_i \frac{\epsilon}{\alpha}}(x_0)$ such that $f_{\lambda_0}(y_0^{(i)}) = y_1^{(i)}$,

$y_{-1}^{(i)} \in B_{\alpha_{i+1} \frac{\epsilon}{\alpha}}(x_0)$ such that $f_{\lambda_{-1}}(y_{-1}^{(i)}) = y_0^{(i)}$,

$\vdots$

$y_{-i}^{(i)} \in B_{\alpha_{2i} \frac{\epsilon}{\alpha}}(x_0)$ such that $f_{\lambda_{-i}}(y_{-i}^{(i)}) = y_{-i+1}^{(i)}$.

Since X is compact, if we let $y_k = \lim_{i \rightarrow \infty} y_k^{(i)}$, then $f_{\lambda_k}(y_k) = y_{k+1}$ and $d(y_k, x_k) < \epsilon L$, for all $k \in \mathbb{Z}$. Therefore $\mathcal{F}$ has the Lipschitz shadowing property.

(ii $\Rightarrow$ i). Since $\mathcal{F}$ has the shadowing property, there exist $0 < \delta < \frac{\delta_0}{2}$ such that every $\delta\alpha$ -pseudo orbit of $\mathcal{F}$ is δ_0 -shadowed by some point. Now, fix $v \in \Lambda$. Consider $x, y \in X$ such that $d(f_v(x), y) < \delta\alpha$ and define a $\delta\alpha$ -pseudo orbit of $\mathcal{F}$ by $x_0 = x$ and $x_i = f_v^{i-1}(y)$ ($i \in \mathbb{Z}$). Then there exists $z \in X$ and $\sigma = \{\dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots\} \in \Lambda^{\mathbb{Z}}$ such that $d(f_{\sigma_i}(z), x_i) < \delta_0$, for all

$i \in \mathbb{Z}$. Less of generality; by proof of Theorem 2.2. in [9] and this fact that $x_{i+1} = f_v(x_i)$ ($i \in \mathbb{Z}$), we can assume that $\lambda_i = v$ for all $i \geq 0$. Then $\alpha^{i-1} d(f_v(z), y) \leq d(f_v^i(z), f_v^{i-1}(y)) \leq \delta_0$ for all $i \geq 0$, so $f_v(z) = y$. This implies that $z = f_v^{-1}(y)$

and $d(x, z) < \delta_0$, then

$d(x, z) < \frac{d(f_v(x), f_v(z))}{\alpha} = \frac{(f_v(x), y)}{\alpha} < \frac{\delta\alpha}{\alpha}$. Hence $B_\delta(x) \cap f_v^{-1}(y) \neq \emptyset$. So, by Lemma 1 $\mathcal{F}$ is an open IFS. $\square$

The next theorem is one of the main results of this paper and demonstrates that for an expansive **IFS**, the limit shadowing property and the shadowing property are equivalent.

Theorem 4. *Let X be a compact metric space and $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ be an expansive IFS on $\mathbb{Z}$. The following conditions are equivalent:*

- i) $\mathcal{F}$ has the shadowing property,*
- ii) there is a compatible metric D for X such that $\mathcal{F}$ has the limit shadowing property with respect to D .*

Proof. By definitions the assertion $(ii \Rightarrow i)$ is clear.

To prove $(i \Rightarrow ii)$, at first we have the following lemmas.

Lemma 2. *There is a compatible metric D on X and $K \geq 1$ such that*

$$\begin{cases} D(f_\lambda(x), f_\lambda(y)) \leq KD(x, y), \\ D(f_\lambda^{-1}(x), f_\lambda^{-1}(y)) \leq KD(x, y) \end{cases}$$

for any $x, y \in X$ and $\lambda \in \Lambda$.

Proof. Since $\mathcal{F}$ is expansive then f_λ is expansive, for every $\lambda \in \Lambda$. So, by [10] (page 3) for every $\lambda \in \Lambda$ there exists $K_\lambda > 1$ such that

$$\begin{cases} D(f_\lambda(x), f_\lambda(y)) \leq K_\lambda D(x, y), \\ D(f_\lambda^{-1}(x), f_\lambda^{-1}(y)) \leq K_\lambda D(x, y) \end{cases}$$

for any $x, y \in X$. Take $K = \max\{K_\lambda : \lambda \in \Lambda\}$, the proof is complete. $\square$

To prove $(i \Rightarrow ii)$ we need to define the local stable set and the local unstable set for an **IFS**.

Let $\epsilon > 0$, $\sigma \in \Lambda^{\mathbb{Z}}$ and x be an arbitrary point of X then

$$W_{\epsilon_0}^s(x, \sigma) = \{y; d(\mathcal{F}_{\sigma_n}(x), \mathcal{F}_{\sigma_n}(y)) \leq \epsilon, \forall n \geq 0\},$$

$$W_{\epsilon_0}^u(x, \sigma) = \{y; d(\mathcal{F}_{\sigma_{-n}}(x), \mathcal{F}_{\sigma_{-n}}(y)) \leq \epsilon, \forall n > 0\}$$

is said to be the local stable set and the local unstable set of x respect to $\sigma \in \Lambda^{\mathbb{Z}}$.

Lemma 3. *There exist constants $\epsilon_0 > 0$ and $\eta < 1$ such that*

$$\begin{cases} D(\mathcal{F}_{\sigma_i}(x), \mathcal{F}_{\sigma_i}(y)) \leq \eta^i D(x, y) & \text{if } y \in W_{\epsilon_0}^s(x, \sigma), \\ D(\mathcal{F}_{\sigma_{-i}}(x), \mathcal{F}_{\sigma_{-i}}(y)) \leq \eta^i D(x, y) & \text{if } y \in W_{\epsilon_0}^u(x, \sigma) \end{cases}$$

Proof. Since for every $\lambda \in \Lambda$, f_λ is an expansive map, To proof the lemma this is sufficient to in Lemma 1 of [13] we assume that

$$W_n = \{(x, y) \in X \times X : d(\mathcal{F}_{\sigma_i}(x), \mathcal{F}_{\sigma_i}(y)) \leq c, \text{ for all } |i| < n\}.$$

The rest of proof is similar to [13]. $\square$

($i \Rightarrow ii$) Let D be the compatible metric for X by the above lemmas.

First, notice that if $\{z_i\}_{i \in \mathbb{Z}}$ is an ϵ -pseudo orbit of $\mathcal{F}$, ($\epsilon \leq \frac{\epsilon_0}{2L}$) i.e.

$D(f_{\lambda_i}(z_i), z_{i+1}) < \epsilon$, for some $\sigma = \{\dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots\} \in \Lambda^{\mathbb{Z}}$. Then by Theorem 3, there exists $y \in X$ such that $D(\mathcal{F}_{\sigma_i}(y), z_i) < L\epsilon$ for all $i \in \mathbb{Z}$.

Now, suppose that $\{x_i\}_{i \in \mathbb{Z}}$ is a sequence of points in X such that

$\lim_{i \rightarrow \pm\infty} D(f_{\lambda_i}(x_i), x_{i+1}) = 0$, for some $\sigma = \{\dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots\} \in \Lambda^{\mathbb{Z}}$. For any $\delta > 0$ ($\delta < \frac{\epsilon_0}{2L}$), there exists $I_\delta > 0$ such that $|i| > I_\delta$ implies that $D(f_{\lambda_i}(x_i), x_{i+1}) < \delta$. Note that

$$\{\dots, f_{\lambda_{I_\delta-2}}^{-1}(f_{\lambda_{I_\delta-1}}^{-1}(x_{I_\delta})), f_{\lambda_{I_\delta-1}}^{-1}(x_{I_\delta}), x_{I_\delta}, x_{I_\delta+1}, \dots\}$$

is a δ -pseudo orbit of $\mathcal{F}$, and by Theorem 3 there exists $y_\delta \in X$ such that

$D(\mathcal{F}_{\sigma_i}(y_\delta), x_i) < L\delta$ for all $i \geq I_\delta$. By the same way, there exists $z_\delta \in X$, Such that $D(\mathcal{F}_{\sigma_{-i}}(z_\delta), x_{-i}) < L\delta$ for all $i \geq I_\delta$. Thus:

$$D(\mathcal{F}_{\sigma_i}(y), \mathcal{F}_{\sigma_i}(y_\delta)) \leq D(\mathcal{F}_{\sigma_i}(y), x_i) + D(x_i, \mathcal{F}_{\sigma_i}(y_\delta)) < \epsilon_0$$

for all $i \geq I_\delta$. This implies that $\mathcal{F}_{\sigma_{I_\delta}}(y_\delta) \in W_{\epsilon_0}^s(\mathcal{F}_{\sigma_{I_\delta}}(y))$. So that, by Lemma 3, $D(\mathcal{F}_{\sigma_i}(y_\delta), \mathcal{F}_{\sigma_i}(y)) \leq \eta^{i-I_\delta} D(\mathcal{F}_{\sigma_{I_\delta}}(y_\delta), \mathcal{F}_{\sigma_{I_\delta}}(y))$ for all $i \geq I_\delta$. Mimicking the procedure, we have $D(\mathcal{F}_{\sigma_{-i}}(y_\delta), \mathcal{F}_{\sigma_{-i}}(y)) \leq \eta^{i-I_\delta} D(\mathcal{F}_{\sigma_{-I_\delta}}(y_\delta), \mathcal{F}_{\sigma_{-I_\delta}}(y))$ for all $i \geq I_\delta$. Take $J_\delta > I_\delta$ such that $\epsilon_0 \eta^{i-I_\delta} < \delta$ if $i \geq J_\delta$. Since

$$D(\mathcal{F}_{\sigma_i}(y), x_i) \leq D(\mathcal{F}_{\sigma_i}(y), \mathcal{F}_{\sigma_i}(y_\delta)) + D(\mathcal{F}_{\sigma_i}(y_\delta), x_i)$$

and

$$D(\mathcal{F}_{\sigma_{-i}}(y), x_{-i}) \leq D(\mathcal{F}_{\sigma_{-i}}(y), \mathcal{F}_{\sigma_{-i}}(y_\delta)) + D(\mathcal{F}_{\sigma_{-i}}(y_\delta), x_{-i}).$$

It is easy to see that

$\max\{D(\mathcal{F}_{\sigma_i}(y), x_i), D(\mathcal{F}_{\sigma_{-i}}(y), x_{-i})\} < (L+1)\delta$. Thus

$\lim_{i \rightarrow \pm\infty} D(\mathcal{F}_{\sigma_i}(y), x_i) = 0$. □

Fix $\delta > 0$ and $\lambda \in \Lambda$. Suppose that $\{x_i\}_{i \in \mathbb{Z}_+}$ is a δ -pseudo orbit for $\mathcal{F}$ and consider $\{y_i\}_{i \in \mathbb{Z}}$ as the following:

$$y_i = \begin{cases} x_i & \text{if } i \geq 0, \\ f_\lambda^i(x_0) & \text{if } i < 0. \end{cases}$$

So, $\{y_i\}_{i \in \mathbb{Z}}$ is a δ -pseudo orbit for $\mathcal{F}$. Then shadowing properties on $\mathbb{Z}$ implies the shadowing properties on $\mathbb{Z}_+$.

By Theorems 3, 4 and Theorem 3.2. of [6] we have the following corollary.

Corollary 2. *Let X be a compact metric space. If $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is an expansive IFS with the limit (Lipschitz) shadowing property (on $\mathbb{Z}_+$), then so is $\mathcal{F}^{-1} = \{X; g_\lambda | \lambda \in \Lambda\}$ where $f_\lambda : X \rightarrow X$ is homeomorphism and $g_\lambda = f_\lambda^{-1}$ for all $\lambda \in \Lambda$.*

By Theorems 3, 4 and Theorem 3.5. of [6] we have the following corollary.

Corollary 3. *Let Λ be a finite set, $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is an IFS and let $k > 0$ be an integer. Set $\mathcal{F}^k = \{g_\mu | \mu \in \Pi\} = \{f_{\lambda_k} \circ \dots \circ f_{\lambda_1} | \lambda_1, \dots, \lambda_k \in \Lambda\}$. If $\mathcal{F}$ has the limit (Lipschitz) shadowing property (on $\mathbb{Z}_+$), then so does $\mathcal{F}^k$.*

We say that $\mathcal{F}$ is strongly expansive if there exist a metric d for X and a constant $e \geq 0$ such that for every two arbitrary $\sigma, \mu \in \Lambda^{\mathbb{Z}}$, $d(\mathcal{F}_{\sigma_i}(x), \mathcal{F}_{\mu_i}(y)) < e$, for all $i \in \mathbb{Z}$, implies that $x = y$.

To prove Theorem 5, we need the following lemma.

Lemma 4. *Suppose that $\mathcal{F}$ is an expansive IFS with expansive constant e , α is an arbitrary positive number and $\sigma, \mu \in \Lambda^{\mathbb{Z}}$. For all $x, y \in X$ there exists an integer $N = N(e, \alpha) > 0$ such that if $d(\mathcal{F}_{\sigma_i}(x), \mathcal{F}_{\mu_i}(y)) \leq e$ for all $|i| \leq N$, then $d(x, y) < \alpha$.*

Proof. We will give a proof by contradiction. Suppose that for each $n \geq 1$, there exist x_n and y_n with $d(\mathcal{F}_{\sigma_i}(x_n), \mathcal{F}_{\mu_i}(y_n)) \leq e$ for all $|i| \leq n$ and $d(x_n, y_n) \geq \alpha$. Since X is a compact metric space, we may assume that $x_n \rightarrow x$ and $y_n \rightarrow y$ and hence that $\mathcal{F}_{\sigma_i}(x_n) \rightarrow \mathcal{F}_{\sigma_i}(x)$ $\mathcal{F}_{\mu_i}(y_n) \rightarrow \mathcal{F}_{\mu_i}(y)$ for every $|i| \geq 1$. Then $d(\mathcal{F}_{\sigma_i}(x), \mathcal{F}_{\mu_i}(y)) \leq e$ for all $i \in \mathbb{Z}$ and $d(x, y) \geq \alpha$, which is a contradiction. $\square$

Theorem 5. *Let X be a compact metric space and $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ be an strongly expansive IFS on $\mathbb{Z}$. The following conditions are equivalent:*

- i) $\mathcal{F}$ has the shadowing property,*
- ii) $\mathcal{F}$ has the continuous shadowing property.*

Proof. By definitions, this is clear that continuous shadowing property implies the shadowing property.

(i $\Rightarrow$ ii). Let $e > 0$ be an expansive constant of $\mathcal{F}$. For $0 < \epsilon < \frac{e}{3}$, let $\delta = \delta(\epsilon) < \epsilon$ be as in the shadowing property of $\mathcal{F}$. It is easy to see that for any δ -pseudo orbit $\{x_i\}_{i \in \mathbb{Z}}$ of $\mathcal{F}$ by Remark 2 there exists $\sigma = \{\dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots\} \in \Lambda^{\mathbb{Z}}$ and a unique $y \in X$ satisfying $d(\mathcal{F}_{\sigma_i}(y), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. So we define $r : P(\mathcal{F}, \delta) \rightarrow X$ by $r(\{x_i\}_{i \in \mathbb{Z}}) = y$. To show that the map r is continuous, choose an arbitrary constant $\alpha > 0$, by Lemma 4 there exists an integer $N = N(e, \alpha) > 0$ such that if $d(\mathcal{F}_{\sigma_i}(y), \mathcal{F}_{\mu_i}(y')) \leq e$ for $\sigma, \mu \in \Lambda^{\mathbb{Z}}$ and all $|i| \leq N$, then $d(y, y') < \alpha$. Pick $\beta > 0$ such that $2^N \beta < \frac{e}{3}$. Let $\{x_i\}_{i \in \mathbb{Z}}, \{x'_i\}_{i \in \mathbb{Z}} \in P(\mathcal{F}, \delta)$ be given two δ -pseudo orbit of $\mathcal{F}$ with related sequences $\sigma, \mu \in \Lambda^{\mathbb{Z}}$ and let $r(\{x_i\}_{i \in \mathbb{Z}}) = y$ and $r(\{x'_i\}_{i \in \mathbb{Z}}) = y'$. if $d(\{x_i\}_{i \in \mathbb{Z}}, \{x'_i\}_{i \in \mathbb{Z}}) < \beta$, then we see that $d(x_i, x'_i) < \frac{\epsilon}{2}$ for all $|i| < N$, so that $d(\mathcal{F}_{\sigma_i}(y), \mathcal{F}_{\mu_i}(y')) \leq d(\mathcal{F}_{\sigma_i}(y), x_i) + d(x_i, x'_i) + d(\mathcal{F}_{\mu_i}(y'), x'_i) < e$ for all $|i| \leq N$. Thus $d(y, y') < \alpha$ by the choice of N , and the conclusion is obtained. $\square$

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Author's address

Mehdi Fatehi Nia

Yazd University, Department of Mathematics, 89195-741 Yazd, Iran

E-mail address: fatehiniam@yazd.ac.ir



A CORRECTION TO APPROXIMATION OF GENERALIZED HOMOMORPHISMS IN QUASI-BANACH ALGEBRAS

ISMAIL NIKOUFAR

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Abstract. Eshaghi et. al [Approximation of generalized homomorphisms in quasi-Banach algebras, An. St. Univ. Ovidius Constanta, 17(2), (2009), 203–214] defined the notion of generalized homomorphisms in quasi-Banach algebras. They investigated generalized homomorphisms from quasi-Banach algebras to p -Banach algebras and proved the generalized Hyers–Ulam–Rassias stability. In this paper, we show that their results only hold for Banach algebras and then we correct and prove the results for p -Banach algebras.

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1. INTRODUCTION AND PRELIMINARIES

We know that a $\mathbb{C}$ -linear mapping $f : A \rightarrow B$ is called a homomorphism if $f(xy) = f(x)f(y)$ for all $x, y \in A$ so that every homomorphism is a generalized homomorphism, but the converse is false, in general.

Eshaghi et. al [1] investigated generalized homomorphisms from quasi-Banach algebras to p -Banach algebras and proved the generalized Hyers–Ulam–Rassias stability. In this paper, we verify that the results presented in [1] hold for Banach algebras. Then, we correct their results and prove the results for p -Banach algebras. We remark that the presenting results in this paper hold for p -Banach algebras, where $0 < p \leq 1$, in general. The stability problems of several functional equations have been extensively investigated by a number of authors in p -Banach algebras and there are many interesting results concerning this problem (see [2, 3, 5] and references therein).

Let X be a real linear space. A quasi-norm is a real-valued function on X satisfying the following conditions:

- (i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$,
- (ii) $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$,
- (iii) there is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasi-normed space if $\|\cdot\|$ is a quasi-norm on X . The smallest possible K is called the modulus of concavity of $\|\cdot\|$. A quasi-Banach space is a complete quasi-normed space. Indeed, by a quasi-Banach space we mean a quasi-normed space in which every $\|\cdot\|$ -Cauchy sequence in X converges. This class includes Banach spaces and the most significant class of quasi-Banach spaces which are not Banach spaces. A quasi-norm $\|\cdot\|$ is called a p -norm ($0 < p \leq 1$) if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p \quad (1.1)$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p -Banach space.

Let $(A, \|\cdot\|)$ be a quasi-normed space. The quasi-normed space $(A, \|\cdot\|)$ is called a quasi-normed algebra if A is an algebra and there is a constant $K > 0$ such that $\|xy\| \leq K\|x\|\|y\|$ for all $x, y \in A$. A quasi-Banach algebra is a complete quasi-normed algebra. If the quasi-norm $\|\cdot\|$ is a p -norm, then the quasi-Banach algebra is called a p -Banach algebra. Eshaghi et. al [1] defined the notion of generalized homomorphisms in quasi-Banach algebra as follows:

Definition 1. Let A be a quasi-Banach algebra with quasi-norm $\|\cdot\|_A$ and let B be a p -Banach algebra with p -norm $\|\cdot\|_B$. A $\mathbb{C}$ -linear mapping $f : A \rightarrow B$ is called a generalized homomorphism if there exists a homomorphism $h : A \rightarrow B$ such that $f(xy) = f(x)h(y)$ for all $x, y \in A$.

Then, they investigated generalized homomorphisms from quasi-Banach algebras to p -Banach algebras associated with the following functional equation

$$rf\left(\frac{x+y}{r}\right) = f(x) + f(y)$$

and proved the generalized Hyers–Ulam–Rassias stability and superstability of generalized homomorphisms in quasi-Banach algebras. In this paper, we prove that their results only hold for Banach algebras and then we correct their results and confirm the results for p -Banach algebras.

2. MAIN PROBLEMS

Following [1] throughout this paper, assume that A is a quasi-Banach algebra with quasi-norm $\|\cdot\|_A$ and that B is a p -Banach algebra with p -norm $\|\cdot\|_B$. In addition, we assume r to be a constant positive integer.

We will use the following lemma in this section.

Lemma 1 ([4]). Let X and Y be linear spaces and let $f : X \rightarrow Y$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in X$ and $\mu \in \mathbb{T}^1 := \{z \in \mathbb{C} : |z| = 1\}$. Then the mapping f is $\mathbb{C}$ -linear.

The following two theorems proved in [1, Theorems 2.2 and 2.5]. In these theorems, the authors want to prove the generalized Hyers–Ulam–Rassias stability of generalized homomorphisms from quasi-Banach algebras to p -Banach algebras.

Theorem 1. Suppose $f : A \rightarrow B$ is a mapping with $f(0) = 0$ for which there exist a mapping $g : A \rightarrow B$ with $g(0) = 0$, $g(1) = 1$ and a function $\varphi : A^4 \rightarrow \mathbb{R}^+$ such that

$$\|rf(\frac{\mu a + \mu b + cd}{r}) - \mu f(a) - \mu f(b) - f(c)g(d)\|_B \leq \varphi(a, b, c, d), \quad (2.1)$$

$$\|g(\mu ab + \mu cd) - \mu g(a)g(b) - \mu g(c)g(d)\|_B \leq \varphi(a, b, c, d), \quad (2.2)$$

and

$$\tilde{\varphi}(a, b, c, d) := \sum_{i=0}^{\infty} \frac{\varphi(2^i a, 2^i b, 2^i c, 2^i d)}{2^i} < \infty$$

for all $a, b, c, d \in A$ and all $\mu \in \mathbb{T}^1$. Then, there exists a unique generalized homomorphism $h : A \rightarrow B$ such that

$$\|f(a) - h(a)\|_B \leq \frac{1}{2} \tilde{\varphi}(a, a, 0, 0)$$

for all $a \in A$.

Theorem 2. Suppose $f : A \rightarrow B$ is a mapping with $f(0) = 0$ for which there exist a mapping $g : A \rightarrow B$ with $g(0) = 0$, $g(1) = 1$ and a function $\varphi : A^4 \rightarrow \mathbb{R}^+$ satisfying the inequalities (2.1), (2.2) and

$$\tilde{\varphi}(a, b, c, d) := \sum_{i=1}^{\infty} 2^i \varphi(\frac{a}{2^i}, \frac{b}{2^i}, \frac{c}{2^i}, \frac{d}{2^i}) < \infty$$

for all $a, b, c, d \in A$ and all $\mu \in \mathbb{T}^1$. Then, there exists a unique generalized homomorphism $h : A \rightarrow B$ such that

$$\|f(a) - h(a)\|_B \leq \frac{1}{2} \tilde{\varphi}(a, a, 0, 0)$$

for all $a \in A$.

In the following theorems we correct the results and prove the generalized Hyers–Ulam–Rassias stability of generalized homomorphisms from quasi–Banach algebras to p –Banach algebras.

Theorem 3. Suppose $f : A \rightarrow B$ is a mapping with $f(0) = 0$ for which there exist a mapping $g : A \rightarrow B$ with $g(0) = 0$, $g(1) = 1$ and a function $\varphi : A^4 \rightarrow \mathbb{R}^+$ such that

$$\|rf(\frac{\mu a + \mu b + cd}{r}) - \mu f(a) - \mu f(b) - f(c)g(d)\|_B \leq \varphi(a, b, c, d), \quad (2.3)$$

$$\|g(\mu ab + \mu cd) - \mu g(a)g(b) - \mu g(c)g(d)\|_B \leq \varphi(a, b, c, d), \quad (2.4)$$

and

$$\tilde{\varphi}(a, b, c, d) := \left(\sum_{i=0}^{\infty} \frac{\varphi(2^i a, 2^i b, 2^i c, 2^i d)^p}{2^{ip}} \right)^{\frac{1}{p}} < \infty \quad (2.5)$$

for all $a, b, c, d \in A$ and all $\mu \in \mathbb{T}^1$. Then, there exists a unique generalized homomorphism $h : A \rightarrow B$ such that

$$\|f(a) - h(a)\|_B \leq \frac{1}{2} \tilde{\varphi}(a, a, 0, 0) \quad (2.6)$$

for all $a \in A$.

Proof. Setting $a = b, c = d = 0$ and $r = \mu = 1$ in (2.3) and dividing both sides of the resulting inequality by 2, we obtain

$$\left\| \frac{f(2a)}{2} - f(a) \right\|_B \leq \frac{\varphi(a, a, 0, 0)}{2} \quad (2.7)$$

for all $a \in A$. Then, we have

$$\left\| \frac{f(2a)}{2} - f(a) \right\|_B^p \leq \frac{\varphi(a, a, 0, 0)^p}{2^p} \quad (2.8)$$

for all $a \in A$. Replacing a in (2.7) by $2a$ and dividing both sides of the resulting inequality by 2, we get

$$\left\| \frac{f(2^2 a)}{2^2} - \frac{f(2a)}{2} \right\|_B \leq \frac{\varphi(2a, 2a, 0, 0)}{2^2} \quad (2.9)$$

for all $a \in A$. Then, we have

$$\left\| \frac{f(2^2 a)}{2^2} - \frac{f(2a)}{2} \right\|_B^p \leq \frac{\varphi(2a, 2a, 0, 0)^p}{2^{2p}} \quad (2.10)$$

for all $a \in A$. Applying (2.8), (2.10), and (1.1) we detect that

$$\left\| \frac{f(2^2 a)}{2^2} - f(a) \right\|_B^p \leq \frac{\varphi(a, a, 0, 0)^p}{2^p} + \frac{\varphi(2a, 2a, 0, 0)^p}{2^{2p}} \quad (2.11)$$

for all $a \in A$. By induction on n we conclude that

$$\left\| \frac{f(2^n a)}{2^n} - f(a) \right\|_B^p \leq \frac{1}{2^p} \sum_{i=0}^{n-1} \frac{\varphi(2^i a, 2^i a, 0, 0)^p}{2^{ip}} \quad (2.12)$$

for all $a \in A$ and all non-negative integers n . Consequently,

$$\left\| \frac{f(2^{n+m} a)}{2^{n+m}} - \frac{f(2^m a)}{2^m} \right\|_B \leq \frac{1}{2} \left(\sum_{i=m}^{n+m-1} \frac{\varphi(2^i a, 2^i a, 0, 0)^p}{2^{ip}} \right)^{\frac{1}{p}} \quad (2.13)$$

for all non-negative integers n and m with $n \geq m$ and all $a \in A$. It follows from (2.5) and (2.13) that the sequence $\{\frac{f(2^n a)}{2^n}\}$ is Cauchy in B for all $a \in A$. Since B is a p -Banach algebras, this sequence is convergent in B for all $a \in A$. Define the mapping

$$h(a) := \lim_{n \rightarrow \infty} \frac{f(2^n a)}{2^n}. \quad (2.14)$$

Setting $c = d = 0, r = 1$ and replacing a, b by $2^n a, 2^n b$, respectively, in (2.3) and dividing both sides of (2.3) by 2^n and taking the limit as $n \rightarrow \infty$ we deduce

$$h(\mu a + \mu b) = \mu h(a) + \mu h(b) \quad (2.15)$$

for all $a, b \in A$ and $\mu \in \mathbb{T}^1$. So the mapping h is $\mathbb{C}$ -linear by Lemma 1. Note that inequality (2.6) follows from (2.12) and (2.14). To show that h is unique, let k be another $\mathbb{C}$ -linear mapping satisfying (2.6). From (2.6) we conclude that

$$\begin{aligned} \|h(a) - k(a)\|_B^p &= \frac{1}{2^{np}} \|h(2^n a) - k(2^n a)\|_B^p \\ &\leq \frac{1}{2^{np}} (\|h(2^n a) - f(2^n a)\|_B^p + \|f(2^n a) - k(2^n a)\|_B^p) \\ &\leq \frac{1}{2^{np}} \frac{2}{2^p} \tilde{\varphi}(2^n a, 2^n a, 0, 0)^p \\ &= \frac{2}{2^p} \sum_{i=n}^{\infty} \frac{\varphi(2^i a, 2^i a, 0, 0)^p}{2^{ip}} \end{aligned}$$

for all $a \in A$. The right hand side tends to zero as $n \rightarrow \infty$. The rest of the proof is similar to that of [1, Theorem 2.2] and we omit it. $\square$

Theorem 4. Suppose $f : A \rightarrow B$ is a mapping with $f(0) = 0$ for which there exist a mapping $g : A \rightarrow B$ with $g(0) = 0, g(1) = 1$ and a function $\varphi : A^4 \rightarrow \mathbb{R}^+$ satisfying the inequality (2.3) and (2.4) and

$$\tilde{\varphi}(a, b, c, d) := \left(\sum_{i=1}^{\infty} 2^{ip} \varphi\left(\frac{a}{2^i}, \frac{b}{2^i}, \frac{c}{2^i}, \frac{d}{2^i}\right)^p \right)^{\frac{1}{p}} < \infty \quad (2.16)$$

for all $a, b, c, d \in A$ and all $\mu \in \mathbb{T}^1$. Then, there exists a unique generalized homomorphism $h : A \rightarrow B$ such that

$$\|f(a) - h(a)\|_B \leq \frac{1}{2} \tilde{\varphi}(a, a, 0, 0) \quad (2.17)$$

for all $a \in A$.

Proof. Setting $a = b, c = d = 0$ and $r = \mu = 1$ in (2.3), we obtain

$$\|f(2a) - 2f(a)\|_B \leq \varphi(a, a, 0, 0) \quad (2.18)$$

for all $a \in A$. Replacing a in (2.18) by $\frac{a}{2}$, we get

$$\|f(a) - 2f\left(\frac{a}{2}\right)\|_B \leq \varphi\left(\frac{a}{2}, \frac{a}{2}, 0, 0\right) \quad (2.19)$$

for all $a \in A$. Replacing a in (2.19) by $\frac{a}{2}$ and multiplying both sides of the resulting inequality by 2, we detect

$$\|2f\left(\frac{a}{2}\right) - 2^2 f\left(\frac{a}{2^2}\right)\|_B \leq 2\varphi\left(\frac{a}{2^2}, \frac{a}{2^2}, 0, 0\right) \quad (2.20)$$

for all $a \in A$. Using (2.19), (2.20), and (1.1) we conclude that

$$\|f(a) - 2^2 f(\frac{a}{2^2})\|_B^p \leq \varphi(\frac{a}{2}, \frac{a}{2}, 0, 0)^p + 2^p \varphi(\frac{a}{2^2}, \frac{a}{2^2}, 0, 0)^p \quad (2.21)$$

for all $a \in A$. By induction on n we deduce that

$$\|f(a) - 2^n f(\frac{a}{2^n})\|_B^p \leq \frac{1}{2^p} \sum_{i=1}^n 2^{ip} \varphi(\frac{a}{2^i}, \frac{a}{2^i}, 0, 0)^p \quad (2.22)$$

for all $a \in A$ and all non-negative integers n . Hence,

$$\|2^m f(\frac{a}{2^m}) - 2^{n+m} f(\frac{a}{2^{n+m}})\|_B^p \leq \frac{1}{2^p} \sum_{i=m+1}^{m+n} 2^{ip} \varphi(\frac{a}{2^i}, \frac{a}{2^i}, 0, 0)^p \quad (2.23)$$

for all non-negative integers n and m with $n \geq m$ and all $a \in A$. It follows from (2.16) and (2.23) that the sequence $\{2^n f(\frac{a}{2^n})\}$ is Cauchy in B for all $a \in A$ so that this sequence is convergent in B . Define the mapping

$$h(a) := \lim_{n \rightarrow \infty} 2^n f(\frac{a}{2^n}). \quad (2.24)$$

The rest of the proof is similar to that of Theorem 3 and we omit it. $\square$

Corollary 1. Suppose $f : A \rightarrow B$ is a mapping with $f(0) = 0$ for which there exist constants $\epsilon > 0, \alpha \neq 1$ and a mapping $g : A \rightarrow B$ with $g(0) = 0, g(1) = 1$ such that

$$\begin{aligned} \|rf(\frac{\mu a + \mu b + cd}{r}) - \mu f(a) - \mu f(b) - f(c)g(d)\|_B \\ \leq \epsilon(\|a\|_A^\alpha + \|b\|_A^\alpha + \|c\|_A^\alpha + \|d\|_A^\alpha), \end{aligned}$$

$\|g(\mu ab + \mu cd) - \mu g(a)g(b) - \mu g(c)g(d)\|_B \leq \epsilon(\|a\|_A^\alpha + \|b\|_A^\alpha + \|c\|_A^\alpha + \|d\|_A^\alpha)$ for all $a, b, c, d \in A$ and all $\mu \in \mathbb{T}^1$. Then, there exists a unique generalized homomorphism $h : A \rightarrow B$ such that

$$\|f(a) - h(a)\|_B \leq \frac{\epsilon}{|1 - 2^{p(\alpha-1)}|^{\frac{1}{p}}} \|a\|_A^\alpha. \quad (2.25)$$

for all $a \in A$.

Proof. Define $\varphi(a, b, c, d) := \epsilon(\|a\|_A^\alpha + \|b\|_A^\alpha + \|c\|_A^\alpha + \|d\|_A^\alpha)$. If $0 \leq \alpha < 1$, then Theorem 3 entails that

$$\tilde{\varphi}(a, a, 0, 0) = \frac{2\epsilon}{(1 - 2^{p(\alpha-1)})^{\frac{1}{p}}} \|a\|_A^\alpha.$$

Consequently,

$$\|f(a) - h(a)\|_B \leq \frac{\epsilon}{(1 - 2^{p(\alpha-1)})^{\frac{1}{p}}} \|a\|_A^\alpha. \quad (2.26)$$

If $\alpha > 1$, then by applying Theorem 4 we find that

$$\tilde{\varphi}(a, a, 0, 0) = \frac{2\epsilon}{(2^{p(\alpha-1)} - 1)^{\frac{1}{p}}} \|a\|_A^\alpha.$$

Hence,

$$\|f(a) - h(a)\|_B \leq \frac{\epsilon}{(2^{p(\alpha-1)} - 1)^{\frac{1}{p}}} \|a\|_A^\alpha. \quad (2.27)$$

From inequalities (2.26) and (2.27) we conclude inequality (2.25). $\square$

3. CONCLUSIONS

We conclude that

- (i) in Theorems 3, 4, if we take $p = 1$, then we obtain [1, Theorems 2.2, 2.5], respectively,
- (ii) in Corollary 1, if we take $p = 1$, then we deduce [1, Corollary 2.3],
- (iii) in Corollary 1, if we take $p = 1$, $\epsilon = \frac{\delta}{2}$, and $\alpha = 0$, then we recover [1, Corollary 2.4].

Indeed, the results presented in [1] hold for 1-Banach algebras. We know that 1-Banach algebras exactly coincide with Banach algebras and so the results of [1] only hold for Banach algebras. Our presented results in this paper hold for p -Banach algebras where $0 < p \leq 1$.

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Author's address

Ismail Nikoufar

Department of Mathematics, Payame Noor University, P.O. BOX 19395-3697 Tehran, Iran

E-mail address: `nikoufar@pnu.ac.ir`



ON STRONGLY STARLIKE FUNCTIONS OF ORDER (α, β)

MAMORU NUNOKAWA AND JANUSZ SOKÓŁ

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Abstract. We consider the class $\mathcal{SS}^*(\alpha, \beta)$ of analytic functions which satisfy the condition $-\pi\beta/2 < \arg\{zf'(z)/f(z)\} < \pi\alpha/2$ for all z in the unit disc $\mathbb{E}$ on the complex plane, where $0 \leq \alpha < 1$ and $0 \leq \beta < 1$. For $\alpha = \beta$ the class $\mathcal{SS}^*(\alpha, \beta)$ is equal to the well known class $\mathcal{SS}^*(\beta)$ of strongly starlike functions of order β . In this work we derive a sufficient condition for analytic function to be in the class $\mathcal{SS}^*(\alpha, \beta)$ strongly starlike functions of order (α, β) .

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of analytic functions f in the unit disc $\mathbb{E} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$ with the normalization $f(0) = 0$, $f'(0) = 1$. A function $f \in \mathcal{A}$ is said to be starlike of order δ if

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \delta \quad (z \in \mathbb{E}), \quad (1.1)$$

for some $0 \leq \delta < 1$, Robertson [8]. We denote by $\mathcal{S}^*(\delta)$ the class of functions starlike of order δ . We say that a function $f \in \mathcal{A}$ is strongly starlike of order β if and only if

$$\left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2} \beta \quad (z \in \mathbb{E}),$$

for some β ($0 < \beta \leq 1$). Let $\mathcal{SS}^*(\beta)$ denote the class of strongly starlike functions of order β . The class $\mathcal{SS}^*(\beta)$ was introduced independently by Stankiewicz [9, 10] and by Brannan and Kirvan [1]. In [11] Takahashi and Nunokawa defined the following subclass of $\mathcal{A}$:

$$\mathcal{SS}^*(\alpha, \beta) = \left\{ f \in \mathcal{A} : \frac{-\pi\beta}{2} < \arg \frac{zf'(z)}{f(z)} < \frac{\pi\alpha}{2}, z \in \mathbb{E} \right\},$$

for some $0 < \alpha \leq 1$, and for some $0 < \beta \leq 1$. We recall here the fact, that in [2] and in [3] a similar class was studied, see also [4, p.141]. Note that $\mathcal{SS}^*(\min\{\alpha, \beta\}) \subset$

$\mathcal{S}\mathcal{S}^*(\alpha, \beta) \subset \mathcal{S}\mathcal{S}^*(\max\{\alpha, \beta\})$. Of course for $\alpha = \beta$ the class $\mathcal{S}\mathcal{S}^*(\alpha, \beta)$ becomes the class $\mathcal{S}\mathcal{S}^*(\beta)$. It is easily seen that $\mathcal{S}\mathcal{S}^*(\alpha, \beta) \subset \mathcal{S}^*(0)$.

2. PRELIMINARY RESULTS

Lemma 1 ([5]). *Let $w(z) = a + w_n z^n + w_{n+1} z^{n+1} + \dots$ be analytic in $\mathbb{E}$ with $w(z) \not\equiv a$ and $n \geq 1$. If $z_0 = r_0 e^{i\theta}$, $0 < r_0 < 1$ and*

$$|w(z_0)| = \max_{|z| \leq r_0} |w(z)|,$$

then it follows that

$$\frac{z_0 w'(z_0)}{w(z_0)} = m,$$

where

$$m \geq n \frac{|w(z_0) - a|^2}{|w(z_0)|^2 - |a|^2} \geq n \frac{|w(z_0)| - |a|}{|w(z_0)| + |a|}.$$

Theorem 1. *Let p be analytic in $\mathbb{E}$ with $p(0) = 1$ and $p(z) \neq 0$. If there exist two points $z_1 \in \mathbb{E}$ and $z_2 \in \mathbb{E}$ such that $|z_1| = |z_2| = r < 1$ and for $z \in \mathbb{E}_r = \{z : |z| < r\}$*

$$-\frac{\pi\beta}{2} = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi\alpha}{2}, \quad (2.1)$$

with some $0 < \alpha \leq 2$, $0 < \beta \leq 2$, then we have

$$\frac{z_1 p'(z_1)}{p(z_1)} = -im \left(\frac{\alpha + \beta}{2} \right) \frac{1 + s^2}{2s} \quad (2.2)$$

and

$$\frac{z_2 p'(z_2)}{p(z_2)} = im \left(\frac{\alpha + \beta}{2} \right) \frac{1 + t^2}{2t}, \quad (2.3)$$

where

$$m \geq \frac{1 - |a|}{1 + |a|}, \quad a = i \tan \frac{\pi}{4} \left(\frac{\alpha - \beta}{\alpha + \beta} \right)$$

and where

$$s = |p(z_1)|^{2/(\alpha+\beta)}, \quad t = |p(z_2)|^{2/(\alpha+\beta)}.$$

Proof. The assumption (2.1) says that the domain $p(\mathbb{E}_r)$ lies in a sector between two rays $\arg\{w\} = -\pi\beta/2$ and $\arg\{w\} = \pi\alpha/2$ and it contacts with the rays at $p(z_1)$ and at $p(z_2)$. The idea of this proof is that we transform this sector into the unit disc and then we will use the Lemma 1. We restrict our considerations to proving (2.3), the proof of (2.2) runs analogously as that of (2.3). The function

$$q(z) = A \{p(z)\}^B, \quad (z \in \mathbb{E}_r), \quad (2.4)$$

where

$$A = \exp \left\{ -i \frac{\pi(\alpha - \beta)}{2(\alpha + \beta)} \right\}, \quad B = \frac{2}{\alpha + \beta},$$

maps $\mathbb{E}_r$ onto the set $q(\mathbb{E}_r)$ on the right half-plane $\Re\{\omega\} > 0$. The boundary $\partial q(\mathbb{E}_r)$ is tangent to the imaginary axis at $q(z_1)$ and at $q(z_2)$ because $\partial p(\mathbb{E}_r)$ is tangent to the sector $-\pi\beta/2 < \arg w < \pi\alpha/2$ at $p(z_1)$ and at $p(z_2)$. Moreover, $q(z_1)$ lies on the negative imaginary axis, while $q(z_2)$ lies on the positive imaginary axis,

$$\arg\{q(z_1)\} = \arg\left\{A\{p(z_1)\}^B\right\} = -\frac{\pi(\alpha-\beta)}{2(\alpha+\beta)} - \frac{\beta\pi}{2} \frac{2}{\alpha+\beta} = -\frac{\pi}{2} \quad (2.5)$$

and

$$\arg\{q(z_2)\} = \arg\left\{A\{p(z_2)\}^B\right\} = -\frac{\pi(\alpha-\beta)}{2(\alpha+\beta)} + \frac{\alpha\pi}{2} \frac{2}{\alpha+\beta} = \frac{\pi}{2}, \quad (2.6)$$

with

$$|\arg\{q(z)\}| = \left|\arg\left\{A\{p(z)\}^B\right\}\right| < \frac{\pi}{2} \quad (z \in \mathbb{E}_r).$$

This shows that

$$\Re\{q(z)\} = \Re\left\{A\{p(z)\}^B\right\} > 0 \quad (z \in \mathbb{E}_r).$$

Therefore, the function

$$\phi(z) = \frac{1-q(z)}{1+q(z)} \quad (z \in \mathbb{E}_r) \quad (2.7)$$

maps the $\mathbb{E}_r$ into the unit $\mathbb{E}$ disc and satisfies

$$\max_{|z| \leq r} |\phi(z)| = |\phi(z_1)| = |\phi(z_2)| = 1. \quad (2.8)$$

By logarithmic differentiation of (2.7), we have

$$\frac{z\phi'(z)}{\phi(z)} = \frac{zp'(z)}{p(z)} \frac{2ABp^B(z)}{1-A^2p^{2B}(z)}.$$

For the case $\arg\{p(z_2)\} = \alpha\pi/2$, from (2.6), we can put

$$Ap^B(z_2) = it, \quad 0 < t.$$

Applying Lemma 1 and (2.8), we have

$$\begin{aligned} \frac{z_2\phi'(z_2)}{\phi(z_2)} &= -\frac{z_2p'(z_2)}{p(z_2)} \frac{2ABp^B(z_2)}{1-A^2p^{2B}(z_2)} \\ &= -\frac{z_2p'(z_2)}{p(z_2)} \frac{4it}{\alpha+\beta} \frac{1}{1+t^2} \\ &= m \end{aligned} \quad (2.9)$$

where

$$m \geq n \frac{|\phi(z_2) - \phi(0)|^2}{|\phi(z_2)|^2 - |\phi(0)|^2} \geq \frac{1 - |\phi(0)|}{1 + |\phi(0)|} = \frac{1 - |a|}{1 + |a|}$$

and

$$a = i \tan \frac{\pi}{4} \left(\frac{\alpha - \beta}{\alpha + \beta} \right).$$

From (2.9), we obtain (2.3) and for the case $\arg\{p(z_1)\} = -\alpha\pi/2$, applying the same method as the above and from (2.5) and (2.8), we obtain (2.2). It completes the proof of Theorem 1. $\square$

The above result is a stronger form of Theorem 1 in [7]. For $\alpha = \beta$ Theorem 1 becomes Nunokawa's lemma [6].

3. MAIN THEOREM

Our main result is contained in:

Theorem 2. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathbb{E}$. Assume that $0 < \alpha \leq 1$, $0 < \beta \leq 1$ and

$$-\frac{\pi\beta}{2} - \mathfrak{B}(\alpha, \beta) < \arg \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \frac{\pi\alpha}{2} + \mathfrak{A}(\alpha, \beta) \quad z \in \mathbb{E}, \quad (3.1)$$

where $\mathfrak{A}(1, 1) = 0$, $\mathfrak{B}(1, 1) = 0$, while for $\alpha + \beta < 2$

$$\mathfrak{B}(\alpha, \beta) = \tan^{-1} \frac{m(\alpha + \beta)g(t_0)\cos(\pi\beta/2)}{2 + m(\alpha + \beta)g(t_0)\sin(\pi\beta/2)},$$

$$\mathfrak{A}(\alpha, \beta) = \tan^{-1} \frac{m(\alpha + \beta)g(t_0)\cos(\pi\alpha/2)}{2 + m(\alpha + \beta)g(t_0)\sin(\pi\alpha/2)},$$

and where,

$$m \geq \frac{1 - |a|}{1 + |a|}, \quad a = i \tan \left(\frac{\alpha - \beta}{\alpha + \beta} \right),$$

$$g(t) = \frac{1 + t^2}{2t(2 + \alpha + \beta)/2}, \quad t_0 = \sqrt{\frac{2 + \alpha + \beta}{2 - \alpha - \beta}}.$$

Then we have

$$-\frac{\pi\beta}{2} < \arg \left\{ \frac{zf'(z)}{f(z)} \right\} < \frac{\pi\alpha}{2} \quad z \in \mathbb{E}, \quad (3.2)$$

Proof. Let us define the function

$$p(z) = \frac{zf'(z)}{f(z)}, \quad p(0) = 1,$$

then it follows that

$$p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)}.$$

If there exists two points $z_1, z_2 \in \mathbb{E}$, $|z_1| = |z_2| = r$ such that

$$-\frac{\pi\beta}{2} = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi\alpha}{2},$$

with some $0 < \alpha \leq 1$, $0 < \beta \leq 1$, then for all $z \in \mathbb{E}_r$, then from Theorem 1

$$\frac{z_1 p'(z_1)}{p(z_1)} = -im \left(\frac{\alpha + \beta}{2} \right) \frac{1 + s^2}{2s}$$

and

$$\frac{z_2 p'(z_2)}{p(z_2)} = im \left(\frac{\alpha + \beta}{2} \right) \frac{1 + t^2}{2t},$$

where

$$m \geq \frac{1 - |a|}{1 + |a|}, \quad a = i \tan \frac{\pi}{4} \left(\frac{\alpha - \beta}{\alpha + \beta} \right)$$

and where

$$s = |p(z_1)|^{2/(\alpha + \beta)}, \quad t = |p(z_2)|^{2/(\alpha + \beta)}.$$

For the case $\arg \{p(z_2)\} = \alpha\pi/2$, from Theorem 1 we have

$$\begin{aligned} \arg \left\{ 1 + \frac{z_2 f''(z_2)}{f'(z_2)} \right\} &= \arg \left\{ p(z_2) + \frac{z_2 p'(z_2)}{p(z_2)} \right\} \\ &= \arg \{p(z_2)\} + \arg \left\{ 1 + \frac{z_2 p'(z_2)}{p(z_2)} \frac{1}{p(z_2)} \right\} \\ &= \frac{\alpha\pi}{2} + \arg \left\{ 1 + im \left(\frac{\alpha + \beta}{2} \right) \left(\frac{1 + t^2}{2t} \right) \frac{1}{t^{(\alpha + \beta)/2} e^{i\pi\alpha/2}} \right\} \\ &= \frac{\alpha\pi}{2} + \arg \left\{ 1 + m \left(\frac{\alpha + \beta}{2} \right) \left(\frac{1 + t^2}{2t^{(2 + \alpha + \beta)/2}} \right) e^{i\pi(1 - \alpha)/2} \right\}. \end{aligned}$$

Hence, if $\alpha = 1$ then $\arg \{1 + z_2 f''(z_2)/f'(z_2)\} = \alpha\pi/2$, which contradicts (3.1) because in this case $\mathfrak{A} = 0$. For $0 < \alpha < 1$, it easily confirm that

$$\arg \left\{ 1 + m \left(\frac{\alpha + \beta}{2} \right) \left(\frac{1 + t^2}{2t^{(2 + \alpha + \beta)/2}} \right) e^{i\pi(1 - \alpha)/2} \right\}.$$

takes its minimum value when the function

$$g(t) = \frac{1 + t^2}{2t^{(2 + \alpha + \beta)/2}}, \quad t > 0,$$

takes its minimum value at $t_0 = \sqrt{(2 + \alpha + \beta)/(2 - \alpha - \beta)}$ and so, we have

$$\begin{aligned} &\arg \left\{ 1 + \frac{z_2 f''(z_2)}{f'(z_2)} \right\} \\ &\geq \frac{\alpha\pi}{2} + \tan^{-1} \frac{m(\alpha + \beta)g(t_0) \sin(\pi(1 - \alpha)/2)}{2 + m(\alpha + \beta)g(t_0) \cos(\pi(1 - \alpha)/2)} \end{aligned}$$

This contradicts (3.1) and for the case $\arg \{p(z_1)\} = \beta\pi/2$, applying the same method as the above we obtain for $\beta = 1$ a contradiction with (3.1), while for $0 < \beta < 1$ we

have

$$\begin{aligned} & \arg \left\{ 1 + \frac{z_1 f''(z_1)}{f'(z_1)} \right\} \\ & \leq -\frac{\beta\pi}{2} - \tan^{-1} \frac{m(\alpha + \beta)g(t_0) \sin(\pi(1 - \beta)/2)}{2 + m(\alpha + \beta)g(t_0) \cos(\pi(1 - \beta)/2)}. \end{aligned}$$

This also contradicts (3.1) and therefore, it completes the proof. $\square$

If $\alpha = \beta$ in the above theorem then we get the following corollary.

Corollary 1. Assume that

$$-\frac{\pi}{4} - \tan^{-1} \frac{\sqrt{2}}{2\sqrt[4]{27} + \sqrt{2}} < \arg \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} < \frac{\pi}{4} + \tan^{-1} \frac{\sqrt{2}}{2\sqrt[4]{27} + \sqrt{2}} \quad z \in \mathbb{E}.$$

Then we have

$$-\frac{\pi}{4} < \arg \left\{ \frac{z f'(z)}{f(z)} \right\} < \frac{\pi}{4} \quad z \in \mathbb{E},$$

this means that f is strongly starlike of order $1/2$.

Note that

$$\tan^{-1} \frac{\sqrt{2}}{2\sqrt[4]{27} + \sqrt{2}} = 0.23 \dots$$

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*Authors' addresses***Mamoru Nunokawa**

University of Gunma, Hoshikuki-cho 798-8, Chuou-Ward, Chiba, 260-0808, Japan

E-mail address: mamoru_nuno@doctor.nifty.jp

Janusz Sokół

University of Rzeszów, Faculty of Mathematics and Natural Sciences, ul. Prof. Pigonia 1, 35-310 Rzeszów, Poland

E-mail address: jsokol@ur.edu.pl



OSCILLATORY PROPERTIES OF CERTAIN SYSTEM OF NON-LINEAR ORDINARY DIFFERENTIAL EQUATIONS

ZDENĚK OPLUŠTIL

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Abstract. We consider the two-dimensional system of non-linear differential equations

$$u' = g(t)|v|^{\frac{1}{\alpha}} \operatorname{sgn} v, \quad v' = -p(t)|u|^{\alpha} \operatorname{sgn} u,$$

where $\alpha > 0$, $g : [0, +\infty[\rightarrow [0, +\infty[$, and $p : [0, +\infty[\rightarrow \mathbb{R}$ are locally integrable functions. In the case $\int^{+\infty} g(s)ds = +\infty$, the considered system has been widely studied in particular cases such linear systems as well as second order linear and half-linear differential equations. However, the case $\int^{+\infty} g(s)ds < +\infty$ has not been studied in detail in the existing literature. Moreover, we allow that the coefficient g can have zero points in any neighbourhood of infinity and consequently, considered system can not be rewritten as the second order linear or half-linear differential equation in this case. In the paper, new oscillation criteria are established in the case $\int^{+\infty} g(s)ds < +\infty$ and without restricted assumption function p preserves its sign (which is usually considered).

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1. INTRODUCTION

On the half-line $\mathbb{R}_+ = [0, +\infty[$, we consider the two-dimensional system of non-linear ordinary differential equations

$$\begin{aligned} u' &= g(t)|v|^{\frac{1}{\alpha}} \operatorname{sgn} v, \\ v' &= -p(t)|u|^{\alpha} \operatorname{sgn} u, \end{aligned} \tag{1.1}$$

where $\alpha > 0$ and $p, g : \mathbb{R}_+ \rightarrow \mathbb{R}$ are locally Lebesgue integrable functions.

Under a solution of system (1.1) on the interval $J \subseteq [0, +\infty[$ we understand a vector function (u, v) , where functions $u, v : J \rightarrow \mathbb{R}$ are absolutely continuous on every compact interval contained in J and satisfy equalities (1.1) almost everywhere in J .

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It is known (see [7]) that all non-extendable solutions of system (1.1) are defined on the whole interval $[0, +\infty[$. Therefore, when we are speaking about a solution of system (1.1), we assume without loss of generality that it is defined on $[0, +\infty[$.

Definition 1. A solution (u, v) of system (1.1) is called *non-trivial* if $u \not\equiv 0$ on any neighborhood of $+\infty$. We say that a non-trivial solution (u, v) of system (1.1) is *oscillatory* if the function u has a sequence of zeros tending to infinity, and *non-oscillatory* otherwise.

In [7, Theorem 1.1], it is proved that a certain analogue of Sturm's theorem holds for system (1.1), if the additional assumption

$$g(t) \geq 0 \quad \text{for a. e. } t \geq 0 \quad (1.2)$$

is satisfied. Especially, under assumption (1.2), if system (1.1) has an oscillatory solution, then any other its non-trivial solution is also oscillatory. Moreover, if (u, v) is an oscillatory solution to system (1.1) then also the function v oscillates.

On the contrary, if $g \equiv 0$ on some neighborhood of $+\infty$, then all non-trivial solutions of system (1.1) are non-oscillatory. Consequently, it is natural to assume that inequality (1.2) is satisfied and

$$\text{meas}\{\tau \geq t : g(\tau) > 0\} > 0 \quad \text{for } t \geq 0. \quad (1.3)$$

Definition 2. We say that system (1.1) is *oscillatory* if all its non-trivial solutions are oscillatory.

It is clear that the half-linear equation

$$(r(t)|u'|^{q-1}\text{sgn}u')' + p(t)|u|^{q-1}\text{sgn}u = 0 \quad (1.4)$$

is a particular case of system (1.1). Indeed, this equation is usually studied under the assumptions that $q > 1$, p, r are continuous functions on $[0, +\infty[$, and $r(t) > 0$ for $t \geq a$. If the function u is a solution of equation (1.4) (i.e., $u \in C^1$ and $r|u'|^{q-1}\text{sgn}u' \in C^1$), then the vector function $(u, r|u'|^{q-1}\text{sgn}u')$ is a solution of system (1.1) with $g(t) := r^{\frac{1}{1-q}}(t)$ and $\alpha := q - 1$.

In the case $\int_0^{+\infty} r^{\frac{1}{1-q}}(s)ds = +\infty$ (i.e. $\int_0^{+\infty} g(s)ds = +\infty$ in (1.1)), there are many interesting results in the existing literature (see, e.g., [2, 3, 5, 6, 8]). However, in the case $\int_0^{+\infty} r^{\frac{1}{1-q}}(s)ds < +\infty$ (i.e. $\int_0^{+\infty} g(s)ds < +\infty$ in (1.1)), as far as we know, only a few results are known. Namely, some Hille and Nehari type oscillations criteria are presented in the papers [4, 9], where together with $\int_0^{+\infty} r^{\frac{1}{1-q}}(s)ds < +\infty$ is assumed that

$$p(t) \geq 0 \quad \text{for } t \geq 0. \quad (1.5)$$

In addition, the coefficient g can have zero points in any neighbourhood of infinity and consequently, considered system can not be rewritten as the half-linear differential equation (1.4) in general.

In this paper we "remove" the assumption (1.5) and we obtain oscillation criteria, which complement and generalize above mentioned results. Therefore, throughout the paper, we assume that the coefficient g is integrable on $[0, +\infty[$, i.e.,

$$\int_0^{+\infty} g(s)ds < +\infty. \quad (1.6)$$

Let

$$f(t) := \int_t^{+\infty} g(s)ds \quad \text{for } t \geq 0.$$

In view of assumptions (1.2), (1.3), and (1.6), we have

$$\lim_{t \rightarrow +\infty} f(t) = 0 \quad (1.7)$$

and

$$f(t) > 0 \quad \text{for } t \geq 0. \quad (1.8)$$

For any $\lambda > \alpha$, we put

$$c_\alpha(t; \lambda) := (\lambda - \alpha) f^{\lambda-\alpha}(t) \int_0^t \frac{g(s)}{f^{\lambda-\alpha+1}(s)} \left(\int_0^s f^\lambda(\xi) p(\xi) d\xi \right) ds \quad \text{for } t \geq 0. \quad (1.9)$$

The following statement was established in [1].

Theorem 1 ([1, Corollary 2.11 (with $\nu = 1 - \alpha + \lambda$)]). *Let conditions (1.2), (1.3), and (1.6) hold, $\lambda > \alpha$, and either*

$$\lim_{t \rightarrow +\infty} c_\alpha(t; \lambda) = +\infty,$$

or

$$-\infty < \liminf_{t \rightarrow +\infty} c_\alpha(t; \lambda) < \limsup_{t \rightarrow +\infty} c_\alpha(t; \lambda).$$

Then system (1.1) is oscillatory.

It is obvious that two cases are not covered by Theorem 1, namely, the function $c_\alpha(\cdot; \lambda)$ has a finite limit and $\liminf_{t \rightarrow +\infty} c_\alpha(t; \lambda) = -\infty$. We are interested in the first case when there exists finite limit of the function $c_\alpha(\cdot; \lambda)$, i.e., if

$$\lim_{t \rightarrow +\infty} c_\alpha(t; \lambda) =: c_\alpha^*(\lambda) \in \mathbb{R}. \quad (1.10)$$

2. MAIN RESULTS

This section contains formulations of all the results of the paper. Their proofs are presented in detail in Section 3.

Theorem 2. *Let $\lambda > \alpha$ and (1.10) hold. Let, moreover, the inequality*

$$\limsup_{t \rightarrow +\infty} \frac{-1}{f^{\lambda-\alpha}(t) \ln f(t)} (c_\alpha^*(\lambda) - c_\alpha(t; \lambda)) > \left(\frac{\alpha}{1+\alpha} \right)^{1+\alpha} \quad (2.1)$$

be satisfied. Then system (1.1) is oscillatory.

As we presented above, known oscillation criteria for the equation (1.4) (with integrable function $r^{\frac{1}{1-q}}(t)$) are established in [4, 9] under sign assumption on its coefficient $p(t)$. We show an example of the system (1.1), where we can use the oscillatory criteria from the Theorem 2, but we can not apply results from above mentioned papers.

Example 1. Consider the system (1.1), where $\alpha = 2$

$$g(t) := \frac{1}{(t+1)^2}, \quad \text{and} \quad p(t) := (t+1)^4 \cos t \quad \text{for } t \geq 0.$$

Obviously, the function p and its integral

$$\begin{aligned} \int_0^t p(s) ds &= (t^4 + 4t^3 - 6t^2 - 20t + 13) \sin t \\ &\quad + (4t^3 + 12t^2 - 12t - 20) \cos t + 20 \quad \text{for } t \geq 0 \end{aligned}$$

change their signs in any neighbourhood of $+\infty$. Therefore neither of results stated in the papers [4, 9] can be used.

On the other hand, from (1.9) we obtain

$$\begin{aligned} c_2(t; 4) &= \frac{2}{(1+t)^2} \int_0^t (1+s) \left(\int_0^s \cos \xi d\xi \right) ds \\ &= \frac{2}{(1+t)^2} (1 + \sin t - t \cos t - \cos t) \quad \text{for } t \geq 0. \end{aligned}$$

Hence, the function $c_2(\cdot, 4)$ has the finite limit

$$c_\alpha^*(4) = \lim_{t \rightarrow +\infty} c_2(t; 4) = 0,$$

i.e., (1.10) holds. Moreover,

$$\limsup_{t \rightarrow +\infty} \frac{(t+1)^2}{\ln(t+1)} (c_\alpha^*(4) - c_2(t; 4)) = \limsup_{t \rightarrow +\infty} \frac{2(\cos t + t \cos t - \sin t - 1)}{\ln(t+1)} = +\infty.$$

Consequently, condition (2.1) is fulfilled with $\alpha = 2$ and according to Theorem 2, system (1.1) is oscillatory.

Introduce the following notations. For any $\lambda \in]\alpha, +\infty[$ and $\mu \in [0, \alpha[$, we put

$$Q(t; \alpha, \lambda) := \frac{1}{f^{\lambda-\alpha}(t)} \left(c_\alpha^*(\lambda) - \int_0^t p(s) f^\lambda(s) ds \right) \quad \text{for } t \geq 0, \quad (2.2)$$

$$H(t; \alpha, \mu) := f^{\alpha-\mu}(t) \int_0^t p(s) f^\mu(s) ds \quad \text{for } t \geq 0, \quad (2.3)$$

where the number $c_\alpha^*(\lambda)$ is given by (1.10). Moreover, let denote

$$\begin{aligned} Q_*(\alpha, \lambda) &:= \liminf_{t \rightarrow +\infty} Q(t; \alpha, \lambda), & H_*(\alpha, \mu) &:= \liminf_{t \rightarrow +\infty} H(t; \alpha, \mu), \\ Q^*(\alpha, \lambda) &:= \limsup_{t \rightarrow +\infty} Q(t; \alpha, \lambda), & H^*(\alpha, \mu) &:= \limsup_{t \rightarrow +\infty} H(t; \alpha, \mu). \end{aligned} \quad (2.4)$$

Theorem 2 yields the following statements.

Corollary 1. Let $\lambda > \alpha$, (1.10) hold, and $Q_*(\alpha, \lambda) > -\infty$. Let, moreover,

$$\limsup_{t \rightarrow +\infty} \frac{-1}{\ln f(t)} \int_0^t f^\alpha(s) p(s) ds > \left(\frac{\alpha}{\alpha + 1} \right)^{\alpha+1} \quad (2.5)$$

Then system (1.1) is oscillatory.

Corollary 2. Let $\mu \in [0, \alpha[, \lambda \in]\alpha, +\infty[$, and (1.10) hold. Let, moreover,

$$\liminf_{t \rightarrow +\infty} (Q(t; \alpha, \lambda) + H(t; \alpha, \mu)) > \frac{(\lambda - \mu)}{(\lambda - \alpha)(\alpha - \mu)} \left(\frac{\alpha}{1 + \alpha} \right)^{1+\alpha}. \quad (2.6)$$

Then system (1.1) is oscillatory.

Corollary 3. Let $\mu \in [0, \alpha[, \lambda \in]\alpha, +\infty[$, and (1.10) hold. Let, moreover, either

$$Q_*(\alpha, \lambda) > \frac{1}{\lambda - \alpha} \left(\frac{\alpha}{1 + \alpha} \right)^{1+\alpha}, \quad (2.7)$$

or

$$H_*(\alpha, \mu) > \frac{1}{\alpha - \mu} \left(\frac{\alpha}{1 + \alpha} \right)^{1+\alpha}. \quad (2.8)$$

Then system (1.1) is oscillatory.

Remark 1. Corollary 3 generalizes result presented in [4]. Indeed, in [4, Theorem 3.1] is established condition (2.7) with $\lambda = \alpha + 1$, but there is an additional assumption $p(t) \geq 0$ for $t \geq a$. We proved oscillation criteria without this additional restrictions, so they can be apply, even if the function $p(t)$ changes its sign (see the following example).

Example 2. Consider the system (1.1), where $\alpha = 2$,

$$g(t) := \frac{1}{(t+1)^2}, \text{ and } p(t) := (t+1) \left(\cos(\ln(t+1)) + \sin(\ln(t+1)) + \frac{4}{3} \right) \text{ for } t \geq 0.$$

Obviously, the function p changes its sign in any neighbourhood of $+\infty$, i.e. we can not use the criteria presented in [4].

On the other hand from (1.9) we obtain

$$\begin{aligned} c_2(t; 3) &= \frac{1}{(1+t)} \int_0^t \left(\int_0^s \frac{\cos(\ln(\xi+1) + \sin(\ln(\xi+1) + \frac{4}{3}))}{(\xi+1)^2} d\xi \right) ds \\ &= \frac{7t - 4\ln(1+t) - 3\sin(\ln(1+t))}{3(1+t)} \quad \text{for } t \geq 0. \end{aligned}$$

Therefore, the function $c_2(\cdot, 3)$ has the finite limit

$$c_2^*(3) = \lim_{t \rightarrow +\infty} c_2(t; 3) = \frac{7}{3},$$

i.e., (1.10) holds. In view of (2.2) we have

$$\begin{aligned} Q(t; 2, 3) &= (t+1) \left(\frac{7}{3} - \int_0^t \frac{\cos(\ln(1+s) + \sin(\ln(1+s) + \frac{4}{3}))}{(1+s)^2} ds \right) \\ &= \cos(\ln(t+1)) + \frac{4}{3} \quad \text{for } t \geq 0. \end{aligned}$$

Hence

$$Q_*(\alpha, \lambda) = \liminf_{t \rightarrow +\infty} Q(t; 2, 3) = \frac{1}{3} > \frac{8}{27} = Q_*(\alpha, \lambda) > \frac{1}{\lambda - \alpha} \left(\frac{\alpha}{1 + \alpha} \right)^{1+\alpha}$$

with $\alpha = 2$ and $\lambda = 3$. Consequently, condition (2.7) is satisfied and according to Corollary 3, system (1.1) is oscillatory.

Theorem 3. Let $\mu \in [0, \alpha[, \lambda \in]\alpha, +\infty[,$ and (1.10) hold. Let, moreover,

$$\limsup_{t \rightarrow +\infty} (Q(t; \alpha, \lambda) + H(t; \alpha, \mu)) > \frac{1}{\lambda - \alpha} \left(\frac{\lambda}{1 + \alpha} \right)^{1+\alpha} + \frac{1}{\alpha - \mu} \left(\frac{\mu}{1 + \alpha} \right)^{1+\alpha}. \quad (2.9)$$

Then system (1.1) is oscillatory.

Remark 2. Let us notice that if $p \geq 0$ for $t \geq 0$ then conditions $Q^*(\alpha, \lambda) > 1$ or $H^*(\alpha, \mu) > 1$ guarantees that (2.9) with $\lambda = \alpha + 1$ and $\mu = 0$ is satisfied. Consequently, one can see Theorem 3 generalizes criteria established in [4, Theorem 3.5].

Now we provide two statements complementing Corollary 3 in a certain sense.

Theorem 4. Let $\mu \in [0, \alpha[, \lambda \in]\alpha, +\infty[,$ and (1.10) hold. Let, moreover, the inequalities

$$\frac{\alpha}{\lambda - \alpha} \left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}} \right) \leq Q_*(\alpha, \lambda) \leq \frac{1}{\lambda - \alpha} \left(\frac{\alpha}{\alpha + 1} \right)^{\alpha+1} \quad (2.10)$$

and

$$H^*(\alpha, \mu) > \frac{1}{\alpha - \mu} \left(\frac{\mu}{1 + \alpha} \right)^{1+\alpha} + \gamma - A(\alpha, \lambda) \quad (2.11)$$

be satisfied, where

$$\gamma := \left(\frac{\lambda}{1 + \alpha} \right)^\alpha \quad (2.12)$$

and $A(\alpha, \lambda)$ is the smallest root of the equation

$$\alpha |x - \gamma|^{\frac{1+\alpha}{\alpha}} + \alpha x + (\lambda - \alpha) Q_*(\alpha, \lambda) - \alpha \gamma = 0. \quad (2.13)$$

Then system (1.1) is oscillatory.

Theorem 5. Let $\mu \in [0, \alpha[$, $\lambda \in]\alpha, +\infty[$, and (1.10) hold. Let, moreover, the inequalities

$$\left(\frac{\mu}{1 + \alpha} \right)^\alpha \frac{\alpha(1 + \alpha - \mu)}{(\alpha - \mu)(1 + \alpha)} \leq H_*(\alpha, \mu) \leq \frac{1}{\alpha - \mu} \left(\frac{\alpha}{1 + \alpha} \right)^{1+\alpha} \quad (2.14)$$

and

$$Q^*(\alpha, \lambda) > B(\alpha, \mu) + \frac{1}{\lambda - \alpha} \left(\frac{\lambda}{1 + \alpha} \right)^{1+\alpha} \quad (2.15)$$

be satisfied, where $B(\alpha, \mu)$ is the greatest root of the equation

$$\alpha |x|^{\frac{1+\alpha}{\alpha}} + \alpha x + (\alpha - \mu) H_*(\alpha, \mu) = 0. \quad (2.16)$$

Then system (1.1) is oscillatory.

Finally, we present an assertion, when both conditions (2.10) and (2.14) are fulfilled. In this case, we can obtain sharper results than those in Theorems 4 and 5.

Theorem 6. Let $\mu \in [0, \alpha[$, $\lambda \in]\alpha, +\infty[$, and (1.10) hold. Let, moreover, conditions (2.10) and (2.14) be satisfied and

$$\limsup_{t \rightarrow +\infty} (Q(t; \alpha, \lambda) + H(t; \alpha, \mu)) > B(\alpha, \mu) - A(\alpha, \lambda) + Q_*(\alpha, \lambda) + H_*(\alpha, \mu) + \gamma, \quad (2.17)$$

where the number γ is defined by (2.12), $A(\alpha, \lambda)$ is the smallest root of equation (2.13), and $B(\alpha, \mu)$ is the greatest root of equation (2.16). Then system (1.1) is oscillatory.

3. PROOFS OF MAIN RESULTS

We first formulate auxiliary lemmas, which we need to prove main statements.

Lemma 1 ([1, Lemma 3.1]). *Let $\alpha > 0$ and $\omega \geq 0$. Then the inequality*

$$\omega|x| - \alpha|x|^{\frac{1+\alpha}{\alpha}} \leq \left(\frac{\omega}{1+\alpha}\right)^{1+\alpha}$$

is satisfied for all $x \in \mathbb{R}$.

Lemma 2 ([1, Lemma 3.2]). *Let $\alpha > 0$. Then*

$$\alpha|x+y|^{\frac{1+\alpha}{\alpha}} \geq \alpha|y|^{\frac{1+\alpha}{\alpha}} + (1+\alpha)x|y|^{\frac{1}{\alpha}} \operatorname{sgn} y \quad \text{for } x, y \in \mathbb{R}.$$

Lemma 3. *Let $\liminf_{t \rightarrow +\infty} c_\alpha(t; \lambda) > -\infty$ and (u, v) is a solution of system (1.1) satisfying*

$$u(t) \neq 0 \quad \text{for } t \geq t_u \quad (3.1)$$

with $t_u > 0$. Then

$$\int_{t_u}^{+\infty} g(s) f^{\lambda-1-\alpha}(s) h(s) ds < +\infty, \quad (3.2)$$

where

$$h(t) := \alpha|f^\alpha(t)\rho(t) - \gamma|^{\frac{1+\alpha}{\alpha}} + (1+\alpha)f^\alpha(t)\rho(t)\gamma^{\frac{1}{\alpha}} - \alpha\gamma^{\frac{1+\alpha}{\alpha}} \quad \text{for } t \geq t_u \quad (3.3)$$

the number γ is defined by (2.12) and

$$\rho(t) := \frac{v(t)}{|u(t)|^\alpha \operatorname{sgn} u(t)} + \frac{1}{f^\alpha(t)} \left(\frac{\lambda}{1+\alpha}\right)^\alpha \quad \text{for } t \geq t_u. \quad (3.4)$$

Proof. Let notice that according to Lemma 2, we have

$$h(t) \geq 0 \quad \text{for } t \geq t_u. \quad (3.5)$$

From the proof [1, Corollary 2.11] one can see that

$$\liminf_{t \rightarrow +\infty} f^{n(\lambda-\alpha)}(t) \int_0^t \left[\frac{1}{f^{(\lambda-\alpha)}(t)} - \frac{1}{f^{(\lambda-\alpha)}(s)} \right]^n f^\lambda(s) p(s) ds > -\infty,$$

where $n \in \mathbb{N}$, $n > \max\{1, \alpha\}$. Therefore from [1, Lemma 4.1] we obtain (3.2). $\square$

Proof of Theorem 2. Assume on the contrary that system (1.1) is not oscillatory. Then system (1.1) has non-oscillatory solution (u, v) , i.e. there exists $t_u > 0$ such that (3.1) holds. Now we can define the function ρ by (3.4) and from (1.1) we derive that

$$\rho'(t) = -p(t) - \alpha g(t) \left| \rho(t) - \frac{\gamma}{f^\alpha(t)} \right|^{\frac{1+\alpha}{\alpha}} + \alpha \gamma \frac{g(t)}{f^{1+\alpha}(t)} \quad \text{for a. e. } t \geq t_u, \quad (3.6)$$

where the number γ is defined by (2.12). Multiplaying the last equality by $f^\lambda(t)$ and integrating it from t_u to t , we obtain

$$\begin{aligned} \int_{t_u}^t f^\lambda(s) \rho'(s) ds &= -\alpha \int_{t_u}^t g(s) f^{\lambda-1-\alpha}(s) |\rho(s) f^\alpha(s) - \gamma|^{\frac{1+\alpha}{\alpha}} ds \\ &\quad + \alpha \gamma \int_{t_u}^t g(s) f^{\lambda-1-\alpha}(s) ds - \int_{t_u}^t f^\lambda(s) p(s) ds \quad \text{for } t \geq t_u. \end{aligned}$$

Integrating the left-hand side of the last equality by parts, we get

$$\begin{aligned} f^\lambda(t) \rho(t) &= \left(\alpha \gamma - \alpha \gamma^{\frac{1+\alpha}{\alpha}} \right) \int_{t_u}^t g(s) f^{\lambda-1-\alpha}(s) ds - \int_{t_u}^t f^\lambda(s) p(s) ds \\ &\quad + f^\lambda(t_u) \rho(t_u) - \int_{t_u}^t g(s) f^{\lambda-1-\alpha}(s) h(s) ds \quad \text{for } t \geq t_u, \end{aligned}$$

where the function h is defined in (3.3). Thus,

$$\begin{aligned} f^\lambda(t) \rho(t) &= \delta(t_u) - \int_0^t f^\lambda(s) p(s) ds - \int_{t_u}^t g(s) f^{\lambda-1-\alpha}(s) h(s) ds \\ &\quad - \frac{\alpha \left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}} \right)}{\lambda - \alpha} f^{\lambda-\alpha}(t) \quad \text{for } t \geq t_u, \end{aligned} \quad (3.7)$$

where

$$\delta(t_u) := f^\lambda(t_u) \rho(t_u) + \int_0^{t_u} f^\lambda(s) p(s) ds + \frac{\alpha \left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}} \right)}{\lambda - \alpha} f^{\lambda-\alpha}(t_u).$$

On the other hand one can verify (see Lemma 3 and the proof of [1, Lemma 4.2]) that the finite limit of function $c_\alpha(\cdot; \lambda)$ is

$$\begin{aligned} c_\alpha^*(\lambda) &= f^\lambda(t_u) \rho(t_u) + \int_0^{t_u} f^\lambda(s) p(s) ds + \frac{\alpha \left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}} \right)}{\lambda - \alpha} f^{\lambda-\alpha}(t_u) \\ &\quad - \int_{t_u}^{+\infty} g(s) f^{\lambda-1-\alpha}(s) h(s) ds. \end{aligned} \quad (3.8)$$

Hence, by virtue of relation (3.2), it follows from (3.7) that

$$\begin{aligned} f^\lambda(t) \rho(t) &= c_\alpha^*(\lambda) - \int_0^t f^\lambda(s) p(s) ds + \int_t^{+\infty} g(s) f^{\lambda-1-\alpha}(s) h(s) ds \\ &\quad - \frac{\alpha \left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}} \right)}{\lambda - \alpha} f^{\lambda-\alpha}(t) \quad \text{for } t \geq t_u. \end{aligned} \quad (3.9)$$

Multiplaying of the last equality by $g(t)f^{\alpha-1-\lambda}(t)$ and integrating it from t_u to t , we obtain

$$\begin{aligned} \int_{t_u}^t g(s)f^{\alpha-1}(s)\rho(s)ds &= - \int_{t_u}^t \frac{g(s)}{f^{1+\lambda-\alpha}(s)} \left(\int_0^s f^\lambda(\xi)p(\xi)d\xi \right) ds \\ &\quad + c_\alpha^*(\lambda) \int_{t_u}^t \frac{g(s)}{f^{1+\lambda-\alpha}(s)} ds - \frac{\alpha}{\lambda-\alpha} \left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}} \right) \int_{t_u}^t \frac{g(s)}{f(s)} ds \\ &\quad + \int_{t_u}^t \frac{g(s)}{f^{1+\lambda-\alpha}(s)} \left(\int_s^{+\infty} g(\xi)f^{\lambda-1-\alpha}(\xi)h(\xi)d\xi \right) ds \quad \text{for } t \geq t_u. \end{aligned} \quad (3.10)$$

One can see

$$\begin{aligned} &\int_{t_u}^t \frac{g(s)}{f^{1+\lambda-\alpha}(s)} \left(\int_s^{+\infty} g(\xi)f^{\lambda-1-\alpha}(\xi)h(\xi)d\xi \right) ds \\ &= \frac{1}{(\lambda-\alpha)f^{\lambda-\alpha}(t)} \int_t^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds + \frac{1}{\lambda-\alpha} \int_{t_u}^t \frac{g(s)}{f(s)} h(s)ds \\ &\quad - \frac{1}{(\lambda-\alpha)f^{\lambda-\alpha}(t_u)} \int_{t_u}^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds \quad \text{for } t \geq t_u. \end{aligned}$$

By virtue of the latter equality, (3.10) yields that

$$\begin{aligned} \int_{t_u}^t g(s)f^{\alpha-1}(s)\rho(s)ds &= \frac{c_\alpha^*(\lambda)}{\lambda-\alpha} \left[\frac{1}{f^{\lambda-\alpha}(t)} - \frac{1}{f^{\lambda-\alpha}(t_u)} \right] - \frac{1}{(\lambda-\alpha)f^{\lambda-\alpha}(t)} c_\alpha(t; \lambda) \\ &\quad + \frac{1}{(\lambda-\alpha)f^{\lambda-\alpha}(t_u)} c_\alpha(t_u; \lambda) + \frac{1}{(\lambda-\alpha)f^{\lambda-\alpha}(t)} \int_t^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds \\ &\quad - \frac{1}{(\lambda-\alpha)f^{\lambda-\alpha}(t_u)} \int_{t_u}^{+\infty} g(s)f^{\lambda-1-\alpha}(s)h(s)ds + \frac{1}{\lambda-\alpha} \int_{t_u}^t \frac{g(s)}{f(s)} h(s)ds \\ &\quad - \frac{\alpha}{\lambda-\alpha} \left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}} \right) \int_{t_u}^t \frac{g(s)}{f(s)} ds \quad \text{for } t \geq t_u \end{aligned}$$

Hence,

$$\begin{aligned}
& \frac{1}{f^{\lambda-\alpha}(t)} (c_{\alpha}^*(\lambda) - c_{\alpha}(t; \lambda)) \\
&= \int_{t_u}^t \frac{g(s)}{f(s)} \left[(\lambda - \alpha) f^{\alpha}(s) \rho(s) - h(s) + \alpha \left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}} \right) \right] ds \\
&+ \frac{1}{f^{\lambda-\alpha}(t_u)} \left[c_{\alpha}^*(\lambda) - c_{\alpha}(t_u; \lambda) + \int_{t_u}^{+\infty} g(s) f^{\lambda-1-\alpha}(s) h(s) ds \right] \\
&- \frac{1}{f^{\lambda-\alpha}(t)} \int_t^{+\infty} g(s) f^{\lambda-1-\alpha}(s) h(s) ds \quad \text{for } t \geq t_u.
\end{aligned} \tag{3.11}$$

On the other hand, (2.12), (3.3), and Lemma 1 with $\omega := \alpha$ yield the estimate

$$\begin{aligned}
& (\lambda - \alpha) f^{\alpha}(s) \rho(s) - h(s) + \alpha \left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}} \right) \\
&= -\alpha (f^{\alpha}(s) \rho(s) - \gamma) - \alpha |f^{\alpha}(s) \rho(s) - \gamma|^{\frac{1+\alpha}{\alpha}} \\
&\leq \left(\frac{\alpha}{1+\alpha} \right)^{1+\alpha} \quad \text{for } s \geq t_u.
\end{aligned} \tag{3.12}$$

Moreover, in view of (1.2), (1.8), and (3.5), it is clear that

$$\frac{1}{f^{\lambda-\alpha}(t)} \int_t^{+\infty} g(s) f^{\lambda-1-\alpha}(s) h(s) ds \geq 0 \quad \text{for } t \geq t_u.$$

Therefore, according to the last inequality and (3.12), it follows from (3.11) that

$$\begin{aligned}
& \frac{1}{f^{\lambda-\alpha}(t)} [c_{\alpha}^*(\lambda) - c_{\alpha}(t; \lambda)] \leq - \left(\frac{\alpha}{1+\alpha} \right)^{1+\alpha} \ln \frac{f(t)}{f(t_u)} \\
&+ \frac{1}{f^{\lambda-\alpha}(t_u)} \left[c_{\alpha}^*(\lambda) - c_{\alpha}(t_u; \lambda) + \int_{t_u}^{+\infty} g(s) f^{\lambda-1-\alpha}(s) h(s) ds \right] \quad \text{for } t \geq t_u.
\end{aligned}$$

Consequently, in view of (1.7), we get

$$\limsup_{t \rightarrow +\infty} \frac{-1}{f^{\lambda-\alpha}(t) \ln f(t)} [c_{\alpha}^*(\lambda) - c_{\alpha}(t; \lambda)] \leq \left(\frac{\alpha}{1+\alpha} \right)^{1+\alpha},$$

which contradicts (2.1). □

Proof of Corollary 1. It is clear that

$$\begin{aligned} c_\alpha(t; \lambda) &= (\lambda - \alpha) f^{\lambda - \alpha}(t) \int_0^t \frac{g(s)}{f^{1 + \lambda - \alpha}(s)} \left(\int_0^s f^\lambda(\xi) p(\xi) d\xi \right) ds \\ &= \int_0^t f^\lambda(s) p(s) ds - f^{\lambda - \alpha}(t) \int_0^t f^\alpha(s) p(s) ds. \end{aligned}$$

Hence, by virtue of the definition (2.2), we have

$$\frac{-1}{f^{\lambda - \alpha}(t) \ln f(t)} [c_\alpha^*(\lambda) - c_\alpha(t; \lambda)] = \frac{-Q(t; \alpha, \lambda)}{\ln f(t)} - \frac{1}{\ln f(t)} \int_0^t f^\alpha(s) p(s) ds$$

for $t \geq 0$. Now, the last equality, (1.7), (2.5), and the assumption $Q_*(\alpha, \lambda) > -\infty$ guarantee the validity of condition (2.1) and thus, the assertion of the corollary follows from Theorem 2. $\square$

Proof of Corollary 2. It is not difficult to verify that

$$\begin{aligned} \frac{1}{f^{\lambda - \alpha}(t) \ln f(t)} (c_\alpha^*(\lambda) - c_\alpha(t; \lambda)) &= \\ \frac{\lambda - \alpha}{\ln f(t)} \int_0^t \frac{g(s)}{f(s)} Q(s; \alpha, \lambda) ds + \frac{c_\alpha^*(\lambda)}{f^{\lambda - \alpha}(0) \ln f(t)} &\quad \text{for } t \geq 0 \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} Q(t; \alpha, \lambda) + H(t; \alpha, \mu) &= \\ (\lambda - \mu) f^{\alpha - \mu}(t) \int_0^t g(s) f^{\mu - \alpha - 1}(s) Q(s; \alpha, \lambda) ds + \frac{c_\alpha^*(\lambda)}{f^{\lambda - \mu}(0)} f^{\alpha - \mu}(t) &\quad (3.14) \end{aligned}$$

for $t \geq 0$, where functions $Q(\cdot; \alpha, \lambda)$ and $H(\cdot; \alpha, \mu)$ are defined by (2.2) and (2.3). Moreover, it is easy to show that

$$\begin{aligned} \int_0^t \frac{g(s)}{f(s)} Q(s; \alpha, \lambda) ds &= f^{\alpha - \mu}(t) \int_0^t g(s) f^{\mu - \alpha - 1}(s) Q(s; \alpha, \lambda) ds \\ &+ (\alpha - \mu) \int_0^t g(s) f^{\alpha - \mu - 1}(s) \left(\int_0^s g(\xi) f^{\mu - \alpha - 1}(\xi) Q(\xi; \alpha, \lambda) d\xi \right) ds \end{aligned} \quad (3.15)$$

for $t \geq 0$. On the other hand, in view of (2.6), relation (3.14) yields

$$\liminf_{t \rightarrow +\infty} f^{\alpha-\mu}(t) \int_0^t g(s) f^{\mu-\alpha-1}(s) Q(s; \alpha, \lambda) ds > \frac{1}{(\lambda - \alpha)(\alpha - \mu)} \left(\frac{\alpha}{\alpha + 1} \right)^{\alpha+1}.$$

Consequently, in view of (1.7), it follows from (3.15) that

$$\liminf_{t \rightarrow +\infty} \frac{-1}{\ln f(t)} \int_0^t \frac{g(s)}{f(s)} Q(s; \alpha, \lambda) ds > \frac{1}{\lambda - \alpha} \left(\frac{\alpha}{\alpha + 1} \right)^{\alpha+1}.$$

The last inequality, by virtue of (1.7) and (3.13), yields the validity of condition (2.1). Therefore, the assertion of the corollary follows from Theorem 2. $\square$

Proof of Corollary 3. First let assumption (2.7) hold. Then it follows from (3.13) that condition (2.1) is satisfied. Therefore, by virtue of Theorem 2 system (1.1) is oscillatory.

Now suppose that assumption (2.8) is fulfilled. It is not difficult to verify that

$$\int_0^t f^\alpha(s) p(s) ds = H(t; \alpha, \mu) + (\alpha - \mu) \int_0^t \frac{g(s)}{f(s)} H(s; \alpha, \mu) ds \quad \text{for } t \geq 0.$$

Hence, in view of (2.8), we get

$$\liminf_{t \rightarrow +\infty} \frac{-1}{\ln f(t)} \int_0^t f^\alpha(s) p(s) ds > \left(\frac{\alpha}{\alpha + 1} \right)^{\alpha+1}. \quad (3.16)$$

On the other hand, one can see

$$c'_\alpha(t; \lambda) = \frac{(\lambda - \alpha)g(t)}{f^{\alpha-\lambda+1}(t)} \int_0^t f^\alpha(s) p(s) ds \quad \text{for a. e. } t \geq 0,$$

where function $c_\alpha(\cdot; \lambda)$ is given by (1.9). By integrating the last equality from τ to t , we obtain

$$c_\alpha(\tau; \lambda) - c_\alpha(t; \lambda) = (\lambda - \alpha) \int_t^\tau \frac{g(s)}{f^{\alpha-\lambda+1}(s)} \left(\int_0^s f^\alpha(\xi) p(\xi) d\xi \right) ds \quad \tau \geq t \geq 0$$

and therefore, by virtue of assumption (1.10) and condition (3.16), we have

$$\begin{aligned} c_\alpha^*(\lambda) - c_\alpha(t; \lambda) &= \\ &= (\lambda - \alpha) \int_t^{+\infty} \frac{g(s) \ln f(s)}{f^{\alpha-\lambda+1}(s)} \left(\frac{1}{\ln f(s)} \int_0^s f^\alpha(\xi) p(\xi) d\xi \right) ds \quad \text{for } t \geq 0. \end{aligned} \quad (3.17)$$

In view of (1.7) and (3.16), there exist $\varepsilon > 0$ and $t_\varepsilon > 0$ such that

$$f(t) < 1, \quad \frac{-1}{\ln f(t)} \int_0^t f^\alpha(s) p(s) ds \geq \left(\frac{\alpha}{\alpha + 1} \right)^{\alpha+1} + \varepsilon \quad \text{for } t \geq t_\varepsilon.$$

Hence, (3.17) yields that

$$c_{\alpha}^*(\lambda) - c_{\alpha}(t; \lambda) \geq -(\lambda - \alpha) \left(\left(\frac{\alpha}{\alpha + 1} \right)^{\alpha+1} + \varepsilon \right) \int_t^{+\infty} \frac{g(s) \ln f(s)}{f^{\alpha-\lambda+1}(s)} \quad \text{for } t \geq t_{\varepsilon}.$$

By virtue of $\varepsilon > 0$ and (1.7), from the last relation we derive inequality (2.1). Consequently, the assertion of the corollary follows from Theorem 2. $\square$

Proof of Theorem 3. Assume on the contrary that system (1.1) is not oscillatory, i.e., there exists a solution (u, v) of system (1.1) satisfying relation (3.1) with $t_u > 0$. Analogously to the proof of Theorem 2 one can show that equality (3.6) and (3.9) hold, where the number γ , and the functions h, ρ are given by (2.12), (3.3), and (3.4).

Multiplaying of (3.6) by $f^{\mu}(t)$ and integrating it from t_u to t , we get

$$\begin{aligned} \int_{t_u}^t f^{\mu}(s) \rho'(s) ds &= - \int_{t_u}^t f^{\mu}(s) p(s) ds - \alpha \int_{t_u}^t g(s) f^{\mu-\alpha-1}(s) |\rho(s) f^{\alpha}(s) - \gamma|^{\frac{1+\alpha}{\alpha}} ds \\ &\quad + \alpha \gamma \int_{t_u}^t g(s) f^{\mu-\alpha-1}(s) ds \quad \text{for } t \geq t_u. \end{aligned}$$

Now we integrate the left-hand side of the last equality by parts and we obtain

$$\begin{aligned} f^{\alpha}(t) \rho(t) &= \\ &+ f^{\alpha-\mu}(t) \int_{t_u}^t g(s) f^{\mu-\alpha-1}(s) \left[-\mu f^{\alpha}(s) \rho(s) - \alpha |\rho(s) f^{\alpha}(s) - \gamma|^{\frac{1+\alpha}{\alpha}} \right] ds \\ &+ \delta(t_u) f^{\alpha-\mu}(t) - H(t; \alpha, \mu) + \frac{\alpha \gamma}{\alpha - \mu} \quad \text{for } t \geq t_u, \end{aligned} \quad (3.18)$$

where the function $H(\cdot; \alpha, \mu)$ is defined by (2.3) and

$$\delta(t_u) := f^{\mu}(t_u) \rho(t_u) + \int_0^{t_u} f^{\mu}(s) p(s) ds - \frac{\alpha \gamma}{(\alpha - \mu)} \frac{1}{f^{\alpha-\mu}(t_u)}. \quad (3.19)$$

On the other hand, multiplaying of (3.9) by $f^{\alpha-\lambda}(t)$, we get

$$\begin{aligned} f^{\alpha}(t) \rho(t) &= Q(t; \alpha, \lambda) + \frac{1}{f^{\lambda-\alpha}(t)} \int_t^{+\infty} g(s) f^{\lambda-1-\alpha}(s) h(s) ds \\ &\quad - \frac{\alpha \left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}} \right)}{\lambda - \alpha} \quad \text{for } t \geq t_u, \end{aligned}$$

where the function $Q(\cdot; \alpha, \lambda)$ is defined by (2.2). Hence, by virtue of (3.18), one can see

$$\begin{aligned} Q(t; \alpha, \lambda) + H(t; \alpha, \mu) &= -\frac{1}{f^{\lambda-\alpha}(t)} \int_t^{+\infty} g(s) f^{\lambda-1-\alpha}(s) h(s) ds \\ &+ f^{\alpha-\mu}(t) \int_{t_u}^t g(s) f^{\mu-\alpha-1}(s) \left[-\mu f^\alpha(s) \rho(s) - \alpha |\rho(s) f^\alpha(s) - \gamma|^{\frac{1+\alpha}{\alpha}} \right] ds \quad (3.20) \\ &+ \frac{\alpha}{\lambda-\alpha} \left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}} \right) + \frac{\alpha \gamma}{\alpha-\mu} + \delta(t_u) f^{\alpha-\mu}(t) \quad \text{for } t \geq t_u. \end{aligned}$$

Moreover, it follows Lemma 1 with $\omega := \mu \geq 0$ that

$$-\mu (f^\alpha(t) \rho(t) - \gamma) - \alpha |\rho(t) f^\alpha(t) - \gamma|^{\frac{1+\alpha}{\alpha}} \leq \left(\frac{\mu}{1+\alpha} \right)^{1+\alpha} \quad \text{for } t \geq t_u. \quad (3.21)$$

Therefore, using (2.12), (3.5), and (3.21) in relation (3.20) yields

$$\begin{aligned} Q(t; \alpha, \lambda) + H(t; \alpha, \mu) &\leq \frac{1}{\lambda-\alpha} \left(\frac{\lambda}{1+\alpha} \right)^{1+\alpha} + \frac{1}{\alpha-\mu} \left(\frac{\mu}{1+\alpha} \right)^{1+\alpha} \\ &+ \delta_1(t_u) f^{\alpha-\mu}(t) \quad \text{for } t \geq t_u, \end{aligned} \quad (3.22)$$

where

$$\delta_1(t_u) := \delta(t_u) + \left[\mu \gamma - \left(\frac{\mu}{1+\alpha} \right)^{1+\alpha} \right] \frac{1}{(\alpha-\mu) f^{\alpha-\mu}(t_u)}. \quad (3.23)$$

Consequently, by virtue of (1.7) relation (3.22) contradicts with assumption (2.9). $\square$

Proof of Theorem 4. Assume on the contrary that system (1.1) is not oscillatory. Then there exists a solution (u, v) of system (1.1) satisfying relation (3.1) with $t_u > 0$. Analogously to the proof of Theorem 3 one can show that relation (3.18) holds, where the numbers $\gamma, \delta(t_u)$ and the functions $H(\cdot; \alpha, \mu), \rho$ are given by (2.12), (3.19), (2.3), and (3.4). Moreover using inequality (3.21) in (3.18) yields

$$f^\alpha(t) \rho(t) \leq \delta_1(t_u) f^{\alpha-\mu}(t) - H(t; \alpha, \mu) + \frac{1}{\alpha-\mu} \left(\frac{\mu}{1+\alpha} \right)^{1+\alpha} + \gamma \quad \text{for } t \geq t_u, \quad (3.24)$$

where $\delta_1(t_u)$ is given by (3.23).

Now we show that the estimate

$$\liminf_{t \rightarrow +\infty} f^\alpha(t) \rho(t) \geq A(\alpha, \lambda) \quad (3.25)$$

holds, where $A(\alpha, \lambda)$ denotes the smallest root of equation (2.13). Similarly, as in the proof of Theorem 2 we can derive that equality (3.8) holds. Multiplying (3.8) by

$f^{\alpha-\lambda}(t)$, we obtain

$$f^\alpha(t)\rho(t) = Q(t; \alpha, \lambda) + \frac{1}{f^{\lambda-\alpha}(t)} \int_t^{+\infty} g(s) f^{\lambda-1-\alpha}(s) h(s) ds - \frac{\alpha \left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}} \right)}{\lambda - \alpha} \quad \text{for } t \geq t_u. \quad (3.26)$$

Let denote

$$m := \liminf_{t \rightarrow +\infty} f^\alpha(t)\rho(t). \quad (3.27)$$

Observe that, if $m = +\infty$, then (3.25) is fulfilled. Therefore, we assume

$$m < +\infty.$$

In view of (2.10), (3.5), and (3.27), it follows from relation (3.26)

$$m \geq Q_*(\alpha, \lambda) - \frac{\alpha}{\lambda - \alpha} \left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}} \right) \geq 0, \quad (3.28)$$

where $Q_*(\alpha, \lambda)$ is given by (2.4).

First suppose that $Q_*(\alpha, \lambda) = \frac{\alpha}{\lambda - \alpha} (\gamma - \gamma^{\frac{1+\alpha}{\alpha}})$, then it is clear that 0 is a root of equation (2.13). Moreover, in view of Lemma 2 and the assumption $\lambda > \alpha$, one can derive that the function $x \mapsto \alpha|x - \gamma|^{\frac{1+\alpha}{\alpha}} + \alpha x - \alpha \gamma^{\frac{1+\alpha}{\alpha}}$ is positive on $]-\infty, 0[$. Consequently, by virtue of notation (3.27) and relation (3.28), desired estimate (3.25) holds.

Now suppose that $Q_*(\alpha, \lambda) > \frac{\alpha}{\lambda - \alpha} (\gamma - \gamma^{\frac{1+\alpha}{\alpha}})$. Let

$\varepsilon \in]0, Q_*(\alpha, \lambda) - \frac{\alpha}{\lambda - \alpha} (\gamma - \gamma^{\frac{1+\alpha}{\alpha}})[$ be arbitrary. According to (3.28), it is clear that

$$m > \varepsilon. \quad (3.29)$$

In view of (2.4) and (3.27), there exists $t_\varepsilon \geq t_u$ such that

$$f^\alpha(t)\rho(t) \geq m - \varepsilon \quad \text{and} \quad Q(t; \alpha, \lambda) \geq Q_*(\alpha, \lambda) - \varepsilon \quad \text{for } t \geq t_\varepsilon. \quad (3.30)$$

Then (3.26) yields that

$$f^\alpha(t)\rho(t) \geq Q_*(\alpha, \lambda) - \varepsilon + \frac{1}{f^{\lambda-\alpha}(t)} \int_t^{+\infty} g(s) f^{\lambda-1-\alpha}(s) h(s) ds - \frac{\alpha \left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}} \right)}{\lambda - \alpha} \quad \text{for } t \geq t_\varepsilon. \quad (3.31)$$

On the other hand, one can see that the function $x \mapsto \alpha|x - \gamma|^{\frac{1+\alpha}{\alpha}} + (1 + \alpha)x\gamma^{\frac{1}{\alpha}} - \alpha\gamma^{\frac{1+\alpha}{\alpha}}$ is non-decreasing on $[0, +\infty[$. Therefore, by using (3.5), (3.29), and (3.30) in (3.31), we obtain that

$$f^\alpha(t)\rho(t) \geq Q_*(\alpha, \lambda) - \varepsilon + \frac{\alpha|(m - \varepsilon) - \gamma|^{\frac{1+\alpha}{\alpha}} - \alpha\gamma + \lambda(m - \varepsilon)}{\lambda - \alpha} \quad \text{for } t \geq t_\varepsilon,$$

which implies

$$m \geq Q_*(\alpha, \lambda) - \varepsilon + \frac{\alpha|(m - \varepsilon) - \gamma|^{\frac{1+\alpha}{\alpha}} - \alpha\gamma + \lambda(m - \varepsilon)}{\lambda - \alpha}.$$

Since ε was arbitrary, the latter relation leads to the inequality

$$\alpha|m - \gamma|^{\frac{1+\alpha}{\alpha}} + \alpha m + Q_*(\alpha, \lambda)(\lambda - \alpha) - \alpha\gamma \leq 0. \quad (3.32)$$

One can show that the function $y : x \mapsto \alpha|x - \gamma|^{\frac{1+\alpha}{\alpha}} + \alpha x + Q_*(\alpha, \lambda)(\lambda - \alpha) - \alpha\gamma$ is decreasing on $] -\infty, \gamma - (\frac{\alpha}{1+\alpha})^\alpha]$ and increasing on $[\gamma - (\frac{\alpha}{1+\alpha})^\alpha, +\infty[$. Moreover, in view of assumption (2.10), the function y is non-positive at the point $\gamma - (\frac{\alpha}{1+\alpha})^\alpha$, which together with (3.27), and (3.32) yields desired estimate (3.25).

Let now $\tilde{\varepsilon} > 0$ be arbitrary. In view of (3.25) there exists $t_\varepsilon \geq t_u$ such that

$$f^\alpha(t)\rho(t) \geq A(\alpha, \lambda) - \tilde{\varepsilon} \quad \text{for } t \geq t_\varepsilon,$$

Hence, it follows from (3.24) that

$$H(t; \alpha, \mu) \leq \delta_1(t_u) f^{\alpha-\mu}(t) - A(\alpha, \lambda) + \tilde{\varepsilon} + \frac{1}{\alpha - \mu} \left(\frac{\mu}{1 + \alpha} \right)^{1+\alpha} + \gamma \quad \text{for } t \geq t_\varepsilon.$$

Since $\tilde{\varepsilon}$ was arbitrary, in view of (1.7) and (2.4), from the latter inequality we obtain

$$H^*(\alpha, \mu) \leq \frac{1}{\alpha - \mu} \left(\frac{\mu}{1 + \alpha} \right)^{1+\alpha} + \gamma - A(\alpha, \lambda, \gamma),$$

which contradicts assumption (2.11). $\square$

Proof of Theorem 5. Assume on the contrary that system (1.1) is not oscillatory, i.e., there exists a solution (u, v) of system (1.1) satisfying relation (3.1) with $t_u > 0$.

First we show that the estimate

$$\limsup_{t \rightarrow +\infty} (f^\alpha(t)\rho(t) - \gamma) \leq B(\alpha, \mu), \quad (3.33)$$

holds, where the number γ , the function ρ are defined by (2.12), (3.4), and $B(\alpha, \mu)$ is the greatest root of equation (2.16). Let denote

$$M := \limsup_{t \rightarrow +\infty} (f^\alpha(t)\rho(t) - \gamma). \quad (3.34)$$

It is clear that, if $M = -\infty$ then (3.33) holds. Therefore, we suppose that

$$M > -\infty.$$

Analogously to the proof of Theorem 4, we can derive inequality (3.24), where the function $H(t; \alpha, \mu)$ is defined by (2.3). Then, in view of (1.7) and (3.34), it follows from inequality (3.24) that

$$M \leq -H_*(\alpha, \mu) + \frac{1}{\alpha - \mu} \left(\frac{\mu}{1 + \alpha} \right)^{1+\alpha}. \quad (3.35)$$

First we assume that $H_*(\alpha, \mu) = \left(\frac{\mu}{1+\alpha}\right)^\alpha \frac{\alpha(1+\alpha-\mu)}{(\alpha-\mu)(1+\alpha)}$. Then it is not difficult to verify that $-\left(\frac{\mu}{1+\alpha}\right)^\alpha$ is a root of the equation (2.16) and the function $x \mapsto \alpha|x|^{\frac{1+\alpha}{\alpha}} + \alpha x + (\alpha - \mu)H_*(\alpha, \mu)$ is positive on $]-\left(\frac{\mu}{1+\alpha}\right)^\alpha, +\infty[$. Therefore, (3.34) and (3.35) yields that (3.33) is satisfied.

Now we suppose

$$H_*(\alpha, \mu) > \left(\frac{\mu}{1+\alpha}\right)^\alpha \frac{\alpha(1+\alpha-\mu)}{(\alpha-\mu)(1+\alpha)}.$$

Hence, it follows from (3.35)

$$M < -\left(\frac{\mu}{1+\alpha}\right)^\alpha.$$

On the other hand, similarly as in the proof of the Theorem 3 we can derive equality (3.18). Let $\varepsilon \in]0, -\left(\frac{\mu}{1+\alpha}\right)^\alpha - M[$ be arbitrary and choose $t_\varepsilon \geq t_u$ such that

$$f^\alpha(t)\rho(t) - \gamma \leq M + \varepsilon, \quad H(t; \alpha, \mu) \geq H_*(\alpha, \mu) - \varepsilon \quad \text{for } t \geq t_\varepsilon. \quad (3.36)$$

One can see that the function $x \mapsto \mu x + \alpha|x|^{\frac{1+\alpha}{\alpha}}$ is non-increasing on interval $]-\infty, -\left(\frac{\mu}{1+\alpha}\right)^\alpha]$ and thus, using relations (3.36) and $M + \varepsilon < -\left(\frac{\mu}{1+\alpha}\right)^\alpha$ in (3.18), we get

$$\begin{aligned} f^\alpha(t)\rho(t) &\leq \delta_2(t_u) f^{\alpha-\mu}(t) - H_*(\alpha, \mu) + \varepsilon + \frac{\alpha\gamma}{\alpha-\mu} - \frac{\mu\gamma}{\alpha-\mu} \\ &\quad + f^{\alpha-\mu}(t) \int_{t_u}^t g(s) f^{\mu-\alpha-1}(s) \left[-\mu(M + \varepsilon) - \alpha|M + \varepsilon|^{\frac{1+\alpha}{\alpha}} \right] ds \quad \text{for } t \geq t_\varepsilon, \end{aligned}$$

where

$$\delta_2(t_u) := f^\mu(t_u)\rho(t_u) + \int_0^{t_u} f^\mu(s)p(s)ds - \gamma \frac{1}{f^{\alpha-\mu}(t_u)}.$$

Hence,

$$\begin{aligned} f^\alpha(t)\rho(t) - \gamma &\leq \delta_3(t_u) f^{\alpha-\mu}(t) - H_*(\alpha, \mu) + \varepsilon \\ &\quad - \frac{\mu(M + \varepsilon) + \alpha|M + \varepsilon|^{\frac{1+\alpha}{\alpha}}}{\alpha - \mu} \quad \text{for } t \geq t_\varepsilon, \end{aligned}$$

where

$$\delta_3(t_u) := \delta_2(t_u) + \frac{\mu(M + \varepsilon) + \alpha|M + \varepsilon|^{\frac{1+\alpha}{\alpha}}}{(\alpha - \mu)} \frac{1}{f^{\alpha-\mu}(t_u)},$$

which, by virtue of the assumption $\alpha > \mu$, condition (1.7) and notation (3.34), yields that

$$M \leq -H_*(\alpha, \mu) + \varepsilon - \frac{\mu(M + \varepsilon) + \alpha|M + \varepsilon|^{\frac{1+\alpha}{\alpha}}}{\alpha - \mu}.$$

Because of ε was choosen arbitrary, from the latter inequality follows

$$\alpha|M|^{\frac{1+\alpha}{\alpha}} + \alpha M + (\alpha - \mu)H_*(\alpha, \mu) \leq 0. \quad (3.37)$$

One can easily verify that the function $y : x \mapsto \alpha|x|^{\frac{1+\alpha}{\alpha}} + \alpha x + H_*(\alpha, \mu)(\alpha - \mu)$ is decreasing on $] -\infty, -(\frac{\alpha}{1+\alpha})^\alpha]$ and increasing on $[-(\frac{\alpha}{1+\alpha})^\alpha, +\infty[$. Moreover, in view of assumption (2.14), the function y is non-positive at the point $-(\frac{\alpha}{1+\alpha})^\alpha$, which together with (3.34), and (3.37) yields desired estimate (3.33).

Let now $\tilde{\varepsilon} > 0$ be arbitrary. Then, by virtue of (3.33), there exists $t_\varepsilon \geq t_u$ such that

$$f^\alpha(t)\rho(t) - \gamma \leq B(\alpha, \mu) + \tilde{\varepsilon} \quad \text{for } t \geq t_\varepsilon.$$

Observe that, analogously to the proof of Theorem 4 we can derive relation (3.26). Then, in view of the last inequality, (1.2), (1.8) and (3.5), it follows from (3.26)

$$Q(t; \alpha, \lambda) \leq B(\alpha, \mu) + \tilde{\varepsilon} + \gamma + \frac{\alpha}{\lambda - \alpha} \left(\gamma - \gamma^{\frac{1+\alpha}{\alpha}} \right) \quad \text{for } t \geq t_\varepsilon.$$

Since $\tilde{\varepsilon}$ was arbitrary, from the last inequality and (2.4), we obtain

$$Q^*(\alpha, \lambda) \leq B(\alpha, \mu) + \frac{\gamma^{\frac{1+\alpha}{\alpha}}}{\lambda - \alpha},$$

which is in contradiction with (2.15). $\square$

Proof of Theorem 6. Suppose on the contrary that system (1.1) is not oscillatory. Then there exists a solution (u, v) of system (1.1) satisfying relation (3.1) with $t_u > 0$. Put

$$\tilde{m} := A(\alpha, \lambda), \quad \tilde{M} := B(\alpha, \mu), \quad (3.38)$$

i.e., $\tilde{m}$ denotes the smallest root of equation (2.13) and $\tilde{M}$ is the greatest root of equation (2.16). Analogously to the proofs of Theorems 4 and 5 we can derive that estimates (3.25) and (3.33) hold. Consequently, by virtue of (3.38), we have

$$\liminf_{t \rightarrow +\infty} f^\alpha(t)\rho(t) \geq \tilde{m}, \quad \limsup_{t \rightarrow +\infty} (f^\alpha(t)\rho(t) - \gamma) \leq \tilde{M}, \quad (3.39)$$

where the number γ and the function ρ are defined by (2.12) and (3.4).

On the other hand, in view of (2.10), one can show that the function $y : x \mapsto \alpha|x - \gamma|^{\frac{1+\alpha}{\alpha}} + \alpha x + Q_*(\alpha, \lambda)(\lambda - \alpha) - \alpha\gamma$ is positive on $] -\infty, 0[$ and there exists $\tilde{x} \in [0, +\infty[$ such that $y(\tilde{x}) \leq 0$, which yields that $\tilde{m} \geq 0$. Moreover, by virtue of (2.14), one can easily verify that the function $z : x \mapsto \alpha|x|^{\frac{1+\alpha}{\alpha}} + \alpha x + (\alpha - \mu)H_*(\alpha, \mu)$ is positive on $] -(\frac{\mu}{1+\alpha})^\alpha, +\infty[$ and there exists $\tilde{x} \leq -(\frac{\mu}{1+\alpha})^\alpha$ such that $z(\tilde{x}) \leq 0$. Consequently, we have $\tilde{M} \leq -(\frac{\mu}{1+\alpha})^\alpha$.

We first assume that $\tilde{m} > 0$ and $\tilde{M} < -(\frac{\mu}{1+\alpha})^\alpha$. Let $\varepsilon \in]0, \min\{\tilde{m}, -(\frac{\mu}{1+\alpha})^\alpha - \tilde{M}\}[$ be arbitrary. Then, it follows from (3.39) that, there exists $t_\varepsilon \geq t_u$ such that

$$f^\alpha(t)\rho(t) \geq \tilde{m} - \varepsilon, \quad f^\alpha(t)\rho(t) - \gamma \leq \tilde{M} + \varepsilon \quad \text{for } t \geq t_\varepsilon. \quad (3.40)$$

Observe that, the function $x \mapsto \alpha|x - \gamma|^{\frac{1+\alpha}{\alpha}} + (1+\alpha)x\gamma^{\frac{1}{\alpha}}$ is non-decreasing on $[0, +\infty[$ and thus, in view of (3.3) and (3.40), we obtain

$$\begin{aligned} \frac{1}{f^{\lambda-\alpha}(t)} \int_t^{+\infty} g(s) f^{\lambda-1-\alpha}(s) h(s) ds \geq \\ + \frac{\alpha|\widetilde{m} - \varepsilon - \gamma|^{\frac{1+\alpha}{\alpha}} + \lambda(\widetilde{m} - \varepsilon) - \alpha\gamma^{\frac{1+\alpha}{\alpha}}}{\lambda - \alpha} \quad \text{for } t \geq t_\varepsilon. \end{aligned} \quad (3.41)$$

Moreover, the function $x \mapsto -\mu x - \alpha|x|^{\frac{1+\alpha}{\alpha}}$ is non-decreasing on $] -\infty, -(\frac{\mu}{1+\alpha})^\alpha[$. Therefore, by virtue of (3.40), we obtain

$$\begin{aligned} f^{\alpha-\mu}(t) \int_{t_\varepsilon}^t g(s) f^{\mu-\alpha-1}(s) \left[-\mu f^\alpha(s) \rho(s) - \alpha|\rho(s) f^\alpha(s) - \gamma|^{\frac{1+\alpha}{\alpha}} \right] ds \\ \leq \frac{-\mu(\widetilde{M} + \varepsilon) - \alpha|\widetilde{M} + \varepsilon|^{\frac{1+\alpha}{\alpha}} - \mu\gamma}{\alpha - \mu} \quad \text{for } t \geq t_\varepsilon. \end{aligned} \quad (3.42)$$

On the other hand, analogously to the proof of Theorem 3 we can derive relation (3.20), where the number $\delta(t_u)$ and the function h are defined by (3.19) and (3.3). Now it follows from (3.20), (3.41), and (3.42)

$$\begin{aligned} Q(t; \alpha, \lambda) + H(t; \alpha, \mu) \leq \widetilde{M} + \varepsilon + H_*(\alpha, \mu) - (\widetilde{m} - \varepsilon) + Q_*(\alpha, \lambda) + \gamma \\ - \frac{\alpha(\widetilde{M} + \varepsilon) + \alpha|\widetilde{M} + \varepsilon|^{\frac{1+\alpha}{\alpha}} + (\alpha - \mu)H_*(\alpha, \mu)}{\alpha - \mu} \\ - \frac{\alpha|\widetilde{m} - \varepsilon - \gamma|^{\frac{1+\alpha}{\alpha}} + \alpha(\widetilde{m} - \varepsilon) + (\lambda - \alpha)Q_*(\alpha, \lambda) - \alpha\gamma}{\lambda - \alpha} \\ + \delta(t_\varepsilon) f^{\alpha-\mu}(t) \quad \text{for } t \geq t_\varepsilon, \end{aligned} \quad (3.43)$$

where

$$\delta(t_\varepsilon) := \delta(t_u) + \int_{t_u}^{t_\varepsilon} g(s) f^{\mu-\alpha-1}(s) \left[-\mu f^\alpha(s) \rho(s) - \alpha|\rho(s) f^\alpha(s) - \gamma|^{\frac{1+\alpha}{\alpha}} \right] ds.$$

Since ε was arbitrary, in view of (1.7) and (3.38), inequality (3.43) yields that

$$\begin{aligned} \limsup_{t \rightarrow +\infty} (Q(t; \alpha, \lambda) + H(t; \alpha, \mu)) \leq B(\alpha, \mu) - A(\alpha, \lambda, \gamma) \\ + Q_*(\alpha, \lambda) + H_*(\alpha, \mu) + \gamma, \end{aligned} \quad (3.44)$$

which contradicts assumption (2.17).

If $\widetilde{m} = 0$ then, in view of (3.5), it is clear that

$$\frac{1}{f^{\lambda-\alpha}(t)} \int_t^{+\infty} g(s) f^{\lambda-1-\alpha}(s) h(s) ds \geq 0 = \frac{\alpha|\widetilde{m} - \gamma|^{\frac{1+\alpha}{\alpha}} + \lambda\widetilde{m} - \alpha\gamma^{\frac{1+\alpha}{\alpha}}}{\alpha - \lambda} \quad (3.45)$$

for $t \geq t_u$. On the other hand, if $\widetilde{M} = -\left(\frac{\mu}{1+\alpha}\right)^\alpha$ then, using Lemma 1 with $\omega := \mu$, one can show that

$$\begin{aligned} & f^{\alpha-\mu}(t) \int_{t_u}^t g(s) f^{\mu-\alpha-1}(s) \left[-\mu f^\alpha(s) \rho(s) - \alpha |\rho(s) f^\alpha(s) - \gamma|^{\frac{1+\alpha}{\alpha}} \right] ds \\ & \leq \frac{\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \mu\gamma}{\alpha - \mu} - \frac{f^{\alpha-\mu}(t)}{f^{\alpha-\mu}(t_u)} \left[\frac{\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \mu\gamma}{\alpha - \mu} \right] \\ & = \frac{-\mu\widetilde{M} - \alpha|\widetilde{M}|^{\frac{1+\alpha}{\alpha}} - \mu\gamma}{\alpha - \mu} - \frac{f^{\alpha-\mu}(t)}{f^{\alpha-\mu}(t_u)} \left[\frac{\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha} - \mu\gamma}{\alpha - \mu} \right] \quad \text{for } t \geq t_u. \end{aligned} \quad (3.46)$$

Consequently, if $\widetilde{m} = 0$ (resp. $\widetilde{M} = -\left(\frac{\mu}{1+\alpha}\right)^\alpha$), then we derive the inequality (3.44) from (3.20) similarly as above, but we use (3.45) instead of (3.41) (resp. (3.46) instead of (3.42)). $\square$

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Author's address

Zdeněk Opluštil

Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology, Technická 2, 616 69 Brno, Czech Republic



CHARACTERIZATIONS OF STRONG WELL-POSEDNESS FOR A CLASS OF MULTI-VALUED VARIATIONAL INEQUALITIES

M. OVEISIHA

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Abstract. In this paper, by using the limiting subdifferential we consider the well-posedness for multi-valued variational inequalities and give some equivalence formulations for them. Moreover, we show that the strong well-posedness for a multi-valued variational inequality is equivalent to the existence and uniqueness of its solution.

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1. INTRODUCTION

The classical concept of well-posedness for a minimization problem, which has been known as the Tykhonov well-posedness, was introduced by Tykhonov [7] in 1966. A minimization problem is Tykhonov well-posed if it has a unique solution and every minimizing sequence of the problem converges to the unique solution. In the last decades, various concepts of well-posedness such as α -well-posedness, Hadamard well-posedness, Levitin-Polyak well-posedness and well-posedness by perturbations have been presented and studied for optimization problems, see [3, 4, 10, 12] and references therein. The concept of well-posedness for hemivariational inequality was first introduced by Goeleven and Motreanu [2] to provide some conditions guaranteeing the existence and uniqueness of a solution for a hemivariational inequality. Later, Xiao and Huang [10] considered a concept of well-posedness for a variational-hemivariational inequality and obtained the equivalence of well-posedness between the variational-hemivariational inequality and the corresponding inclusion problem. Very recently, Xiao et al. [9] established two kinds of conditions under which the strong and weak well-posedness for the hemivariational inequality are equivalent to the existence and uniqueness of its solutions, respectively.

In this paper, by using the limiting subdifferential we extend the concept of well-posedness to a class of multi-valued variational inequality which include as a special

case the classical variational and hemivariational inequalities. Moreover, we establish some equivalence results for them. The paper is organized as follows: Section 2 prepares briefly some preliminary notions and results used in sequel. In Section 3, we show that the strong well-posedness for a multi-valued variational inequality is equivalent to the existence and uniqueness of its solution. Also, a metric characterization for the strong well-posedness of multi-valued variational inequality is obtained.

2. NOTATIONS AND PRELIMINARIES

Let X be a Banach space and X^* its topological dual space. The norm in X and X^* will be denoted by $\|\cdot\|$. We denote $\langle \cdot, \cdot \rangle$, $[x, y]$ and $]x, y[$ the dual pair between X and X^* , the line segment for $x, y \in X$, and the interior of $[x, y]$, respectively. Now, we recall some concepts of subdifferentials that we need in the next section.

Definition 1 ([6]). Let X be a normed vector space, Ω be a nonempty subset of X , $x \in \Omega$ and $\varepsilon \geq 0$. The set of ε -normals to Ω at x is

$$\widehat{N}_\varepsilon(x; \Omega) := \{x^* \in X^* \mid \limsup_{u \rightarrow x} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \leq \varepsilon\}.$$

Assume that $\bar{x} \in \Omega$, the limiting normal cone to Ω at $\bar{x}$ is

$$N(\bar{x}; \Omega) := \limsup_{x \rightarrow \bar{x}, \varepsilon \downarrow 0} \widehat{N}_\varepsilon(x; \Omega).$$

Let $J : X \rightarrow \bar{\mathbb{R}}$ be finite at $\bar{x} \in X$; the limiting subdifferential of J at $\bar{x}$ is defined as follows

$$\partial_M J(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, J(\bar{x})); \text{epi} J)\}.$$

Remark 1 ([6]). The set-valued mapping $x \mapsto \partial_M J(x)$ has closed graph for locally Lipschitz functions.

Definition 2. Let $T : X \rightarrow 2^{X^*}$ be a set-valued mapping. T is said to be relaxed invariant monotone with respect to η if there exists a constant α such that for any $x, y \in X$ and any $u \in T(x), v \in T(y)$, one has

$$\langle v, \eta(x, y) \rangle + \langle u, \eta(y, x) \rangle \leq -\alpha(\|\eta(x, y)\|^2 + \|\eta(y, x)\|^2).$$

Remark 2. (1) When $T : X \rightarrow X^*$ is a single-valued operator, we obtain the definition of a relaxed invariant monotone operator.

(2) If $\alpha = 0$, then the Definition 2 reduces to the definition of an invariant monotone map.

Definition 3 ([11]). A mapping $T : X \rightarrow X^*$ is said to be hemicontinuous if for any $x_1, x_2 \in X$, the function $t \mapsto \langle T(x_1 + tx_2), x_2 \rangle$ from $[0, 1]$ into $]-\infty, +\infty[$ is continuous at 0_+ .

Condition C ([5]). Let $\eta : X \times X \rightarrow X$. Then, for any $x, y \in X, \lambda \in [0, 1]$

$$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y), \quad \eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y).$$

Remark 3. By some computation, we can see that if Condition C holds, then for any $x_1, x_2 \in X$ and $\lambda \in [0, 1]$

$$\eta(x_1 + \lambda\eta(x_2, x_1), x_1) = \lambda\eta(x_2, x_1).$$

Now, suppose that $J : X \rightarrow \mathbb{R}$, $\eta : X \times X \rightarrow X$, $A : X \rightarrow X^*$ is a mapping and $f \in X^*$ is some given element. Consider the following multi-valued variational-like inequality associated with (A, f, J) :

$MVLI(A, f, J)$: Find $\bar{x} \in X$ such that for any $x \in X$, there exists $\xi \in \partial_M J(\bar{x})$, that

$$\langle A\bar{x} - f + \xi, \eta(x, \bar{x}) \rangle \geq 0.$$

Definition 4. A sequence $\{x_n\} \subset X$ is said to be an approximating sequence for the $MVLI(A, f, J)$, if there exists $\{\epsilon_n\}$ with $\epsilon_n \downarrow 0$ such that for any $x \in X$ there exists $x_n^* \in \partial_M J(x_n)$, that

$$\langle Ax_n - f + x_n^*, \eta(x, x_n) \rangle \geq -\epsilon_n \|\eta(x, x_n)\|.$$

Definition 5. The multi-valued variational-like inequality $MVLI(A, f, J)$ is said to be strongly well-posed if it has a unique solution $\bar{x}$ on X and for every approximating sequence $\{x_n\}$, $\eta(\bar{x}, x_n)$ converges strongly to 0.

3. MAIN RESULTS

In this section, we establish some conditions under which the well-posedness for the multi-valued variational-like inequality is equivalent to the existence and uniqueness of its solution. Theorems in this section extend theorems in [8, 9] from hemivariational inequalities with Clarke's subdifferential which is convex to multi-valued variational-like inequalities with limiting subdifferential which is not necessarily convex.

Theorem 1. Assume that operator $A : X \rightarrow X^*$ is hemicontinuous and relaxed invariant monotone with constant c and J is locally Lipschitz such that $\partial_M J$ satisfies relaxed invariant monotonicity condition with constant α . Consider the following assertions:

- (i) $\bar{x}$ is a solution of the $MVLI(A, f, J)$.
- (ii) $\bar{x}$ is a solution of the following associated multi-valued variational-like inequality:

$AMVLI(A, f, J)$: Find $\bar{x} \in X$ such that for any $x \in X$ there exists $x^* \in \partial_M J(x)$ such that $\langle Ax - f + x^*, \eta(x, \bar{x}) \rangle \geq 0$.

If $c + \alpha \geq 0$ and η is skew, then (i) $\Rightarrow$ (ii). If η satisfies Condition C , then (ii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii). Let $\bar{x} \in X$ be a solution of $MVLI(A, f, J)$. Hence, for any $x \in X$ there exists $\xi \in \partial_M J(\bar{x})$ such that

$$\langle A\bar{x} - f + \xi, \eta(x, \bar{x}) \rangle \geq 0. \quad (3.1)$$

By the relaxed invariant monotonicity of $\partial_M J$, for any $x \in X$ and $x^* \in \partial_M J(x)$, one has

$$\langle x^*, \eta(\bar{x}, x) \rangle + \langle \xi, \eta(x, \bar{x}) \rangle \leq -\alpha(\|\eta(x, \bar{x})\|^2 + \|\eta(\bar{x}, x)\|^2). \quad (3.2)$$

It follows from relaxed invariant monotonicity of the operator A , (3.1) and (3.2), that

$$\begin{aligned} \langle Ax - f + x^*, \eta(x, \bar{x}) \rangle &\geq \langle Ax + x^*, \eta(x, \bar{x}) \rangle - \langle A\bar{x} + \xi, \eta(x, \bar{x}) \rangle \\ &= -[\langle Ax, \eta(\bar{x}, x) \rangle + \langle A\bar{x}, \eta(x, \bar{x}) \rangle + \langle x^*, \eta(\bar{x}, x) \rangle + \langle \xi, \eta(x, \bar{x}) \rangle] \\ &\geq (c + \alpha)(\|\eta(x, \bar{x})\|^2 + \|\eta(\bar{x}, x)\|^2), \end{aligned}$$

which shows that $\bar{x}$ is a solution of $AMVLI(A, f, J)$.

(ii) $\Rightarrow$ (i). Conversely, let $\bar{x}$ be a solution to the $AMVLI(A, f, J)$. Hence, for any $x \in X$ there exists $x^* \in \partial_M J(x)$ such that

$$\langle Ax - f + x^*, \eta(x, \bar{x}) \rangle \geq 0. \quad (3.3)$$

For any $z \in X$ and $t \in [0, 1]$, set $x(t) = \bar{x} + t\eta(z, \bar{x})$ in inequality (3.3), we have

$$\langle A(\bar{x} + t\eta(z, \bar{x})) - f + x_t^*, \eta(\bar{x} + t\eta(z, \bar{x}), \bar{x}) \rangle \geq 0,$$

that $x_t^* \in \partial_M J(x(t))$. It follows Condition C, that

$$\langle A(\bar{x} + t\eta(z, \bar{x})) - f + x_t^*, \eta(z, \bar{x}) \rangle \geq 0.$$

Since J is locally Lipschitz, we deduce that $\partial_M J$ is locally bounded (Corollary 1.81 in [6]). Hence, there exists a neighborhood of $\bar{x}$ and a constant $\ell > 0$ such that for each z in this neighborhood and $x^* \in \partial_M J(z)$, we have $\|x^*\| \leq \ell$. Since $x(t) \rightarrow \bar{x}$ when $t \rightarrow 0$ for t to be sufficiently small $\|x_t^*\| \leq \ell$, without loss of generality we may assume that $x_t^* \rightarrow x^*$ in weak*-topology. Now, hemicontinuity of the operator A on X implies that

$$\langle A\bar{x} - f + x^*, \eta(z, \bar{x}) \rangle \geq 0.$$

By the arbitrariness of $z \in X$, we conclude that $\bar{x}$ is a solution of $MVLI(A, f, J)$. $\square$

Proposition 1. Let $C^* \subset X^*$ be nonempty, closed, convex and bounded, $\varphi : X \rightarrow \mathbb{R}$ be proper, convex and lower semi-continuous and $y \in X$ be arbitrary. Assume that η is continuous and affine with respect to the first argument and for each $x \in X$, there exists $x^*(x) \in C^*$ such that

$$\langle x^*(x), \eta(x, y) \rangle \geq \varphi(y) - \varphi(y + \eta(x, y)).$$

Then, there exists $y^* \in C^*$ such that

$$\langle y^*, \eta(x, y) \rangle \geq \varphi(y) - \varphi(y + \eta(x, y)), \quad \forall x \in X.$$

Proof. With some minor modification in the proof of Proposition 3.3 in [1], we can deduce the proof. $\square$

Theorem 2. Let $A : X \rightarrow X^*$ be relaxed invariant monotone with constant c and J a l.s.c. function that its limiting subdifferential satisfies relaxed invariant monotonicity condition with constant α . If $\alpha + c > 0$ and η is continuous and affine with respect to the second argument and skew, then $MVLI(A, f, J)$ is strongly well-posed if and only if it has a unique solution on X .

Proof. The necessity is obvious. For the sufficiency, suppose that $MVLI(A, f, J)$ has a unique solution $\bar{x}$. Since $\bar{x}$ is the unique solution of $MVLI(A, f, J)$, for any $x \in X$, there exists $x^* \in \partial_M J(\bar{x})$ such that

$$\langle A\bar{x} - f + x^*, \eta(x, \bar{x}) \rangle \geq 0. \quad (3.4)$$

Suppose that $\{x_n\}$ is an approximating sequence for $MVLI(A, f, J)$. It follows that there exists $\epsilon_n \downarrow 0$ such that for any $x \in X$, there exists $\xi_n(x) \in \partial_M J(x_n)$ that

$$\langle Ax_n - f + \xi_n(x), \eta(x, x_n) \rangle \geq -\epsilon_n \|\eta(x, x_n)\|.$$

Now, consider the nonempty, convex and bounded set $\text{co}\{Ax_n - f + \xi_n \mid \xi_n \in \partial_M J(x_n)\}$. Hence, it follows from Proposition 1 with $\varphi(x) = \epsilon_n \|x - x_n\|$ that there exists ξ_n which is independent on x , such that

$$\langle Ax_n - f + \xi_n, \eta(x, x_n) \rangle \geq -\epsilon_n \|\eta(x, x_n)\|, \quad \forall x \in X.$$

By choosing ξ_n , we can set $\xi_n = \sum_{i=1}^m \lambda_i \xi_n^i$ that $m \in \mathbb{N}$, $\sum_{i=1}^m \lambda_i = 1$ and $\xi_n^i \in \partial_M J(x_n)$. Hence,

$$\sum_{i=1}^m \lambda_i \langle Ax_n - f + \xi_n^i, \eta(x, x_n) \rangle \geq -\epsilon_n \|\eta(x, x_n)\|, \quad \forall x \in X.$$

Now, set $x = \bar{x}$ in above inequality, yields

$$\sum_{i=1}^m \lambda_i \langle Ax_n - f + \xi_n^i, \eta(\bar{x}, x_n) \rangle \geq -\epsilon_n \|\eta(\bar{x}, x_n)\|.$$

Hence, it follows from relaxed invariant monotonicity of the operator A , relaxed invariant monotonicity of the $\partial_M J$, the skewness of η and above inequality that

$$\begin{aligned} -\epsilon_n \|\eta(\bar{x}, x_n)\| &\leq \sum_{i=1}^m \lambda_i \langle Ax_n - f + \xi_n^i, \eta(\bar{x}, x_n) \rangle \\ &\leq \sum_{i=1}^m \lambda_i [\langle Ax_n + \xi_n^i, \eta(\bar{x}, x_n) \rangle - \langle A\bar{x} + \xi_n^i, \eta(\bar{x}, x_n) \rangle] \\ &= \sum_{i=1}^m \lambda_i [-\langle A\bar{x} - Ax_n + \xi_n^i - \xi_n^i, \eta(\bar{x}, x_n) \rangle] \\ &\leq -\sum_{i=1}^m 2\lambda_i (c + \alpha) \|\eta(\bar{x}, x_n)\|^2 = -2(c + \alpha) \|\eta(\bar{x}, x_n)\|^2, \end{aligned}$$

where $\zeta_n^i \in \partial_M J(\bar{x})$ is obtained from (3.4) by setting $x = x_n$. Since $c + \alpha > 0$, it follows that

$$\|\eta(\bar{x}, x_n)\| \leq \frac{\epsilon_n}{2(c + \alpha)}.$$

Taking limit at both sides of the above inequality, implies that $\eta(\bar{x}, x_n)$ converges strongly to 0. $\square$

Example 1. Let $X = \mathbb{R}$, $f = 0$, A be the identity map and J be defined as

$$J(x) = \begin{cases} x^2 + 2x & \text{if } x > 0, \\ x^2 - x & \text{if } x \leq 0. \end{cases}$$

The limiting subdifferential of J is

$$\partial_M J(x) = \begin{cases} 2x + 2 & \text{if } x > 0, \\ [-1, 2] & \text{if } x = 0, \\ 2x - 1 & \text{if } x < 0. \end{cases}$$

Let η be defined as $\eta(x, y) := k(x - y)$, such that $0 < k \leq 1$. Then by some computation we can see that $\partial_M J, A$ are relaxed invariant monotone with constants $\alpha = 1$, $c = \frac{1}{2}$, respectively. Hence, all assumptions of Theorem 2 are fulfilled and $\bar{x} = 0$ is a unique solution of (MVL I) and therefore it is strongly well-posed.

For any $\epsilon > 0$, consider the following two sets:

$$\begin{aligned} \Omega(\epsilon) &= \{\bar{x} : \forall x \in X, \exists x^* \in \partial_M J(\bar{x}) \text{ s.t. } \langle A\bar{x} - f + x^*, \eta(x, \bar{x}) \rangle \geq -\epsilon \|\eta(x, \bar{x})\|\}, \\ \Psi(\epsilon) &= \{\bar{x} : \forall x \in X, \exists x^* \in \partial_M J(\bar{x}) \text{ s.t. } \langle -Ax, \eta(\bar{x}, x) \rangle + \langle -f + x^*, \eta(x, \bar{x}) \rangle \geq \\ &\quad -\epsilon \|\eta(x, \bar{x})\|\}. \end{aligned}$$

Lemma 1. Suppose that $A : X \rightarrow X^*$ is invariant monotone and hemicontinuous. Then $\Omega(\epsilon) = \Psi(\epsilon)$ for all $\epsilon > 0$.

Proof. Taking into account the invariant monotonicity of mapping A , it is easy to obtain that $\Omega(\epsilon) \subset \Psi(\epsilon)$. For the other side suppose that $\bar{x} \in \Psi(\epsilon)$. Then, for any $x \in X$, there exists $x^* \in \partial_M J(\bar{x})$ such that

$$\langle -Ax, \eta(\bar{x}, x) \rangle + \langle -f + x^*, \eta(x, \bar{x}) \rangle \geq -\epsilon \|\eta(x, \bar{x})\|. \quad (3.5)$$

Set $x = \bar{x} + t\eta(z, \bar{x})$ in (3.5), that $z \in X$ and $t \in [0, 1]$, yields

$$\begin{aligned} \langle -A(\bar{x} + t\eta(z, \bar{x})), \eta(\bar{x}, \bar{x} + t\eta(z, \bar{x})) \rangle + \langle -f + x^*, \eta(\bar{x} + t\eta(z, \bar{x}), \bar{x}) \rangle \geq \\ -\epsilon \|\eta(\bar{x} + t\eta(z, \bar{x}), \bar{x})\|. \end{aligned}$$

By using Condition C, we obtain

$$\langle -A(\bar{x} + t\eta(z, \bar{x})), -\eta(z, \bar{x}) \rangle + \langle -f + x^*, \eta(z, \bar{x}) \rangle \geq -\epsilon \|\eta(z, \bar{x})\|.$$

Now, it follows from the hemicontinuity of mapping A that

$$\langle A\bar{x} - f + x^*, \eta(z, \bar{x}) \rangle \geq -\epsilon \|\eta(z, \bar{x})\|,$$

which shows that $\bar{x} \in \Omega(\epsilon)$. This completes the proof. $\square$

Lemma 2. *Suppose that $A : X \rightarrow X^*$ is a hemicontinuous mapping. If J is locally Lipschitz and η is continuous with respect to the second argument, then $\Omega(\epsilon)$ is closed in X for all $\epsilon > 0$.*

Proof. Let $\{x_n\} \subset \Omega(\epsilon)$ be a sequence such that $x_n \rightarrow \bar{x}$ in X . Then for any $x \in X$, there exists $x_n^* \in \partial_M J(x_n)$ such that

$$\langle Ax_n - f + x_n^*, \eta(x, x_n) \rangle \geq -\epsilon \|\eta(x, x_n)\|. \quad (3.6)$$

Since J is locally Lipschitz, there exists a subsequence of x_n^* that convergent to a $x^* \in \partial_M J(\bar{x})$ in weak*-topology. Consider (3.6) with this subsequence, taking limit at both sides of it and using this fact that A is hemicontinuous and η is continuous with respect to the second argument, we obtain

$$\langle A\bar{x} - f + x^*, \eta(x, \bar{x}) \rangle \geq -\epsilon \|\eta(x, \bar{x})\|,$$

which implies that $\bar{x} \in \Omega(\epsilon)$. This completes the proof. $\square$

Theorem 3. *Suppose that $A : X \rightarrow X^*$ is hemicontinuous and invariant monotone with respect to η that η is continuous with respect to the second argument, satisfies Condition C and skew. Then $MVLI(A, f, J)$ is strongly well-posed if and only if*

$$\Omega(\epsilon) \neq \emptyset, \forall \epsilon > 0 \quad \text{and} \quad \text{diam}(\Omega(\epsilon)) \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 0.$$

Proof. “Necessity” follows similarly from the first part of Theorem 3.1 in [8]. Hence, we prove the “Sufficiency”. Suppose that $\{x_n\} \subset X$ is an approximating sequence for $MVLI(A, f, J)$. Then there exist a nonnegative sequence $\epsilon_n \rightarrow 0$ such that for any $x \in X$, there exists $\xi_n \in \partial_M J(x_n)$ such that

$$\langle Ax_n - f + \xi_n, \eta(x, x_n) \rangle \geq -\epsilon_n \|\eta(x, x_n)\|, \quad (3.7)$$

it means that $x_n \in \Omega(\epsilon_n)$. It follows from $\text{diam}(\Omega(\epsilon)) \rightarrow 0$, that $\{x_n\}$ is a cauchy sequence and so converges strongly to some point $\bar{x} \in X$. Since J is locally lipschitz, there exists a subsequence of ξ_n (e.g. $\{\xi_{n_i}\}$) that is convergent to a $\xi \in \partial_M J(\bar{x})$ in weak*-topology. Taking limit at the both side of (3.7) and using this fact that A is monotone and η is skew and continuous with respect to the second argument, we obtain

$$\begin{aligned} \langle Ax - f + \xi, \eta(x, \bar{x}) \rangle &= \lim_{i \rightarrow \infty} \langle Ax - f + \xi_{n_i}, \eta(x, x_{n_i}) \rangle \\ &\geq \lim_{i \rightarrow \infty} \langle Ax_{n_i} - f + \xi_{n_i}, \eta(x, x_{n_i}) \rangle \\ &\geq \lim_{i \rightarrow \infty} -\epsilon_{n_i} \|\eta(x, x_{n_i})\| = 0. \end{aligned}$$

Now, by using Theorem 1, $\bar{x}$ is a solution of $MVLI(A, f, J)$. Since, $\text{diam}(\Omega(\epsilon)) \rightarrow 0$ when $\epsilon \rightarrow 0$, $MVLI(A, f, J)$ has a unique solution. This completes the proof. $\square$

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Author's address

M. Oveisiha

Department of Mathematics, Faculty of Science, Imam Khomeini International University, P.O. Box 34149-16818, Qazvin, Iran

E-mail address: oveisiha@sci.ikiu.ac.ir



CANONICAL ALMOST GEODESIC MAPPINGS OF TYPE $\pi_2(0, F)$, $\theta \in \{1, 2\}$ BETWEEN GENERALIZED PARABOLIC KÄHLER MANIFOLDS

MILOŠ Z. PETROVIĆ

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Abstract. We introduce a generalized parabolic Kähler manifold and consider special canonical almost geodesic mappings of type $\pi_2(0, F)$, $\theta \in \{1, 2\}$ between generalized Riemannian manifolds and between introduced generalized parabolic Kähler manifolds, particularly. Some invariant geometric objects with respect to these mappings are examined.

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1. INTRODUCTION

The use of a non-symmetric affine connection became especially interesting after the works of A. Einstein [3] on the Unified Field Theory. In 1951, L.P. Eisenhart [4] introduced a generalized Riemannian space as a differentiable manifold equipped with a non-symmetric basic tensor. Eisenhart's generalized Riemannian space is a particular case of a non-symmetric affine connection space. Some significant contributions to the study of geometry of non-symmetric affine connection spaces were made by E. Brinis, F. Graif, M. Prvanović [17] and S.M. Minčić [11–14].

Geodesic lines play an important role in modeling of various physical processes. A mapping between two manifolds with linear connection, which preserves geodesics is called a geodesic mapping. Generalizing the notions of geodesic lines and geodesic mappings, Sinyukov [18] introduced the concept of almost geodesic lines and almost geodesic mappings of affine connected spaces without torsion. He indicated three types of almost geodesic mappings of manifolds without torsion, π_1 , π_2 and π_3 .

The theory of geodesic and almost geodesic mappings of affine connected and Riemannian spaces is an active field of differential geometry, see for instance [1, 2, 5–10, 21, 23].

Almost geodesic mappings of type $\pi_2(e)$, $e = \pm 1$, from spaces with affine connection onto Riemannian spaces are considered in [10, 23], while the paper [5] is dedicated to canonical almost geodesic mappings of type $\pi_2(e = 0)$ between Riemannian spaces with an almost affinor structure, and between parabolic Kählerian spaces, particularly. Several papers are devoted to almost geodesic mappings of type $\pi_2(e = \pm 1)$, $\theta \in \{1, 2\}$ and its special cases $\pi_{2,\theta}(e = \pm 1, F)$, $\theta \in \{1, 2\}$ between manifolds with non-symmetric affine connection, see [15, 19, 21]. In the papers [16, 22] some invariant geometric objects with respect to special almost geodesic mappings of type π_1 and π_3 , respectively, are examined, by considering equitorsion mappings. In [15] we presented systems of differential equations of Cauchy type for almost geodesic mappings of the second type of manifolds with non-symmetric linear connection, also we found some invariant geometric object of almost geodesic mappings of type $\pi_{2,\theta}(e = -1, F)$, $\theta \in \{1, 2\}$ under some assumptions.

In the present paper, we extend and improve results from [5]. We consider canonical almost geodesic mappings of type $\pi_{2,\theta}(0, F)$, $\theta \in \{1, 2\}$ between generalized Riemannian manifolds. Also, we introduce a generalized parabolic Kähler manifold and consider canonical almost geodesic mappings of type $\pi_{2,\theta}(0, F)$, $\theta \in \{1, 2\}$ between such manifolds. The wider class of metrics enables us to find more invariant geometric objects than in the classical (symmetric) case [5].

2. SPECIAL CANONICAL ALMOST GEODESIC MAPPINGS OF GENERALIZED RIEMANNIAN MANIFOLDS

In the sense of Eisenhart (see [4]) a generalized Riemannian space is a differentiable manifold M equipped with a metric g , which is generally non-symmetric. Therefore, the metric g can be described as follows

$$g(X, Y) = \underline{g}(X, Y) + \underset{\vee}{g}(X, Y), \text{ for all } X, Y \in T_p(M).$$

Here $\underline{g}$ denotes the symmetric part of the metric g and $\underset{\vee}{g}$ denotes the skew-symmetric part of g , i.e.

$$\underline{g}(X, Y) = \frac{1}{2}(g(X, Y) + g(Y, X)) \text{ and } \underset{\vee}{g}(X, Y) = \frac{1}{2}(g(X, Y) - g(Y, X)),$$

where $X, Y \in T_p(M)$ and $T_p(M)$ is the tangent vector space of M at $p \in M$.

The non-symmetric linear connection ${}^1\nabla$ of the generalized Riemannian manifold with the metric g is explicitly defined by

$$g({}^1\nabla_X Y, Z) = \frac{1}{2}(Xg(Y, Z) + Yg(Z, X) - Zg(Y, X)), \quad X, Y, Z \in T_p(M). \quad (2.1)$$

Let us denote by ∇ the Levi-Civita connection corresponding to the symmetric metric $\underline{g}$. This connection is the symmetric part of non-symmetric linear connection ${}^1\nabla$, i.e.

$$\nabla_X Y = \frac{1}{2}({}^1\nabla_X Y + {}^1\nabla_Y X), X, Y \in T_p(M).$$

Also, it is well known that on the manifold M with non-symmetric linear connection ${}^1\nabla$ can be defined another non-symmetric linear connection ${}^2\nabla$ in the following way

$${}^2\nabla_X Y = {}^1\nabla_Y X + [X, Y], X, Y \in \mathfrak{X}(M),$$

where as usual $\mathfrak{X}(M)$ denotes the set of smooth vector fields on M and $[\cdot, \cdot]$ denotes the Lie bracket [17].

M.S. Stanković in [19] introduced two kinds of almost geodesic lines, as follows. Let $c : I \rightarrow M$ be a curve on a manifold M with non-symmetric linear connection ${}^1\nabla$, satisfying the regularity condition $c'(t) \neq 0, t \in I$. Denote by $\xi(t) = (c(t), c'(t))$ the tangent vector field along c , and let us put

$$\xi_1 = {}^\theta\nabla_\xi \xi, \quad \xi_2 = {}^\theta\nabla_\xi \xi_1, \quad \theta \in \{1, 2\}.$$

If the vector fields ξ and ξ_1 are independent at any point (hence the (local) curve c is not a geodesic one) we can put ${}^\theta D = \text{span}(\xi, \xi_1)$, $\theta \in \{1, 2\}$. The curve c is an almost geodesic line of the kind θ ($\theta \in \{1, 2\}$) if and only if $\xi_2 \in {}^\theta D$. In [15] we gave an equivalent definition of almost geodesic lines of manifolds with non-symmetric linear connection, it is Definition 1.

Definition 1 ([15]). Let $c : I \rightarrow M$ be a curve on a manifold M with non-symmetric linear connection satisfying the regularity condition $c'(t) \neq 0$ and let $\xi(t) = (c(t), c'(t))$ be the tangent vector field along c . The curve c is called an *almost geodesic of the kind θ* ($\theta \in \{1, 2\}$) if there exist vector fields X_1 and X_2 satisfying ${}^\theta\nabla_\xi X_i = a_i^j X_j$ for some differentiable functions $a_i^j : I \rightarrow \mathbb{R}$ and differentiable real functions $b^i(t)$ along c such that $\xi = b^1 X_1 + b^2 X_2$ holds.

Definition 2 ([15, 19]). A diffeomorphism $f : M \rightarrow \overline{M}$ of n -dimensional manifolds with non-symmetric linear connection is called an *almost geodesic mapping of the kind θ* ($\theta = 1, 2$) if any geodesic line of the manifold M turns into an almost geodesic line of the kind θ of the manifold $\overline{M}$.

Let M and $\overline{M}$ be two generalized Riemannian manifolds of dimension $n > 2$ with the metrics g and $\overline{g}$, respectively. We can consider these manifolds in the *common coordinate system with respect to the diffeomorphism $f : M \rightarrow \overline{M}$* . In this coordinate system the corresponding points $p \in M$ and $f(p) \in \overline{M}$ have the same coordinates. Therefore, we can suppose $M \equiv \overline{M}$ and for $\theta \in \{1, 2\}$ we can put

$${}^\theta P = {}^\theta \overline{\nabla} - {}^\theta \nabla,$$

where ${}^\theta P$ is the tensor field of type $(1, 2)$, called the *deformation tensor field of linear connections* ${}^\theta \bar{\nabla}$ and ${}^\theta \nabla$ with respect to the mapping f .

In what follows we will use $\sum_{CS(\cdot, \cdot, \cdot)}$ to denote the cyclic sum on arguments in brackets, for instance for an arbitrary tensor field A we have

$$\sum_{CS(X, Y, Z)} A(X, Y, Z) = A(X, Y, Z) + A(Y, Z, X) + A(Z, X, Y).$$

A diffeomorphism $f : M \rightarrow \bar{M}$ is an *almost geodesic mapping of the kind* θ , $\theta \in \{1, 2\}$ if and only if [15]

$$P_\theta(X_1, X_2, X_3) \wedge {}^\theta P(X_4, X_5) \wedge X_6 = 0, \quad X_i \in \mathfrak{X}(M), i = 1, \dots, 6,$$

where ${}^\theta P$ is the deformation tensor field of connections ${}^\theta \bar{\nabla}$ and ${}^\theta \nabla$, with respect to the diffeomorphism f , and P_1, P_2 are tensor fields of type $(1, 3)$, defined by

$$P_1(X, Y, Z) = \sum_{CS(X, Y, Z)} {}^1 \nabla_Z {}^1 P(X, Y) + {}^1 P({}^1 P(X, Y), Z), \quad X, Y, Z \in \mathfrak{X}(M)$$

and

$$P_2(X, Y, Z) = \sum_{CS(X, Y, Z)} {}^2 \nabla_Z {}^2 P(X, Y) + {}^2 P(Z, {}^2 P(X, Y)), \quad X, Y, Z \in \mathfrak{X}(M).$$

Basic equations of canonical almost geodesic mappings of type $\pi_2^\theta(e = 0)$, $\theta \in \{1, 2\}$ between generalized Riemannian manifolds are given by

$${}^\theta P(X, Y) = \sum_{CS(X, Y)} \varphi(X) F Y + (-1)^{(\theta-1)} K(X, Y), \quad (2.2)$$

$$\sum_{CS(X, Y)} \left({}^\theta \nabla_Y F X - (-1)^\theta K(F Y, X) \right) = \sum_{CS(X, Y)} (\mu(X) F Y - \mu(F X) Y), \quad (2.3)$$

where $X, Y \in \mathfrak{X}(M)$, φ is a 1-form, K is an anti-symmetric tensor field of type $(1, 2)$ defined by

$$K(X, Y) = \frac{1}{2} ({}^1 P(X, Y) - {}^1 P(Y, X)) = \frac{1}{2} ({}^2 P(Y, X) - {}^2 P(X, Y)),$$

and F is a tensor field of type $(1, 1)$ satisfying

$$F^2 = 0.$$

If the affinor structure F satisfies an additional condition

$$\text{Tr}(F) = F_p^p = 0,$$

then we denote by $\pi_2^\theta(0, F)$, $\theta \in \{1, 2\}$.

A canonical almost geodesic mapping $f : M \rightarrow \overline{M}$ of type $\pi_2(0, F)$, $\theta \in \{1, 2\}$ has the *property of reciprocity* if its inverse mapping $f^{-1} : \overline{M} \rightarrow M$ is a canonical almost geodesic mapping of type $\pi_2(0, F)$. Since the deformation tensor fields 1P and ${}^1\overline{P}$ of linear connections ${}^1\nabla$ and ${}^1\overline{\nabla}$ with respect to the mappings f and f^{-1} , respectively, satisfy the relation

$${}^1\overline{P}(X, Y) = -{}^1P(X, Y), \quad X, Y \in \mathfrak{X}(M),$$

without loss of generality we can suppose

$$\overline{\varphi} = -\varphi, \quad \overline{F} = F, \quad \overline{K} = -K,$$

or in components

$$\overline{\varphi}_i = -\varphi_i, \quad \overline{F}_i^h = F_i^h, \quad \overline{K}_{ij}^h = -K_{ij}^h. \quad (2.4)$$

Almost geodesic mappings of manifolds with non-symmetric linear connection, which satisfy the property of reciprocity are investigated in [15, 19, 21, 22]. A necessary and sufficient condition for an almost geodesic mapping $f : M \rightarrow \overline{M}$ of type π_2 , $\theta \in \{1, 2\}$ to have the property of reciprocity is expressed by

$$F^2 = \alpha I + \beta F,$$

where I is the identity matrix and α, β are some scalar functions.

2.1. Invariants

We use traditional tensor calculus approach “by components”. In local coordinates, with respect to a local chart (U, φ) , $\varphi = (x^1, \dots, x^n)$, we have

$${}^1\nabla_i \frac{\partial}{\partial x^j} = {}^1\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^h \frac{\partial}{\partial x^h}, \quad {}^2\nabla_i \frac{\partial}{\partial x^j} = {}^2\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ji}^h \frac{\partial}{\partial x^h},$$

and

$$\nabla_i \frac{\partial}{\partial x^j} = \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^h \frac{\partial}{\partial x^h},$$

where ij signifies a symmetrization with division and the functions Γ_{ij}^h are generalized Christoffel symbols.

A. Einstein [3] used two kinds of covariant differentiation of a tensor a_j^i :

$$a_{j|_1}^i = a_{j,m}^i + \Gamma_{pm}^i a_j^p - \Gamma_{jm}^p a_p^i, \quad a_{j|_2}^i = a_{j,m}^i + \Gamma_{mp}^i a_j^p - \Gamma_{mj}^p a_p^i,$$

where $a_{j,m}^i$ denotes the partial derivative of a tensor a_j^i with respect to x^m .

S.M. Minčič [11] has used two more kinds of covariant differentiation of tensors:

$$a_{j|_3}^i = a_{j,m}^i + \Gamma_{pm}^i a_j^p - \Gamma_{mj}^p a_p^i, \quad a_{j|_4}^i = a_{j,m}^i + \Gamma_{mp}^i a_j^p - \Gamma_{jm}^p a_p^i.$$

Also, we can consider covariant differentiation with respect to the Levi-Civita connection ∇ , that is

$$\nabla_m a_j^i \equiv a_{j;m}^i = a_{j,m}^i + \Gamma_{pm}^i a_j^p - \Gamma_{jm}^p a_p^i,$$

where Γ_{pm}^i is the symmetric part of Γ_{pm}^i .

Let us denote by $\underset{\theta}{|}$ and $\underset{\theta}{||}$, $\theta = 1, \dots, 4$, the covariant derivatives with respect to the generalized Christoffel symbols Γ_{ij}^h and $\bar{\Gamma}_{ij}^h$, respectively.

In local coordinates the basic equation (2.2) reads

$$\bar{\Gamma}_{ij}^h - \Gamma_{ij}^h = \varphi_i F_j^h + K_{ij}^h, \quad (2.5)$$

where φ_i is the covariant vector corresponding to the linear form φ , while F_i^h and K_{ij}^h are components of tensor fields F and K , respectively.

By using covariant differentiation of the first kind and the equation (2.5) we obtain

$$F_{i||j}^h = F_{i|j}^h + \varphi_p F_i^p F_j^h + K_{pj}^h F_i^p - K_{ij}^p F_p^h. \quad (2.6)$$

After contracting the relation (2.6) over the indices j and h and by using (2.4) we get

$$\bar{F}_{i||\alpha}^\alpha + \frac{1}{2} \bar{K}_{p\alpha}^\alpha \bar{F}_i^p - \frac{1}{2} \bar{K}_{i\alpha}^p \bar{F}_p^\alpha = F_{i|\alpha}^\alpha + \frac{1}{2} K_{p\alpha}^\alpha F_i^p - \frac{1}{2} K_{i\alpha}^p F_p^\alpha, \quad (2.7)$$

i.e. the tensor A_i defined by

$$A_i = F_{i|\alpha}^\alpha + \frac{1}{2} K_{p\alpha}^\alpha F_i^p - \frac{1}{2} K_{i\alpha}^p F_p^\alpha, \quad (2.8)$$

is invariant with respect to the mapping f .

Analogously, by using covariant differentiation of the kind θ ($\theta = 2, 3, 4$) we can prove that the tensors A_i , $\theta = 2, 3, 4$ given by

$$\begin{aligned} A_i &= F_{i|_2}^\alpha + \frac{1}{2} K_{\alpha p}^\alpha F_i^p - \frac{1}{2} K_{\alpha i}^p F_p^\alpha, \\ A_i &= F_{i|_3}^\alpha + \frac{1}{2} K_{p\alpha}^\alpha F_i^p - \frac{1}{2} K_{\alpha i}^p F_p^\alpha, \\ A_i &= F_{i|_4}^\alpha + \frac{1}{2} K_{\alpha p}^\alpha F_i^p - \frac{1}{2} K_{i\alpha}^p F_p^\alpha, \end{aligned} \quad (2.9)$$

are invariant with respect to the mapping f .

In the nontrivial case, when $F_i^h \neq 0$, which is of particular importance for us, there exists a $(1, 1)$ tensor $\overset{*}{F}_i^h \neq 0$ such that $\overset{*}{F}_\beta^\alpha F_\alpha^\beta = n$. After contracting (2.6) with $\overset{*}{F}_h^j$

we find

$$n\varphi_p F_i^p = (F_{i||\beta}^\alpha - F_{i|\beta}^\alpha) F_\alpha^\beta - K_{p\beta}^\alpha F_\alpha^\beta F_i^p + K_{i\beta}^p F_\alpha^\beta F_p^\alpha. \quad (2.10)$$

From (2.4) we have

$$F_i^h = \overline{F}_i^h, \quad (2.11)$$

so we can conclude

$${}^*F_i^h = \overline{{}^*F}_i^h. \quad (2.12)$$

Also, the condition (2.4) ensures the relation

$$K_{ij}^h = \frac{1}{2}(K_{ij}^h - \overline{K}_{ij}^h). \quad (2.13)$$

Now, from (2.6) by using (2.10)–(2.13) we obtain

$$B_{ij}^h = \overline{B}_{ij}^h,$$

where the tensor B_{ij}^h is defined by

$$B_{ij}^h = F_{i||j}^h - \frac{1}{n}(F_{i||\beta}^\alpha + \frac{1}{2}K_{\gamma\beta}^\alpha F_i^\gamma - \frac{1}{2}K_{i\beta}^\gamma F_\gamma^\alpha) F_\alpha^\beta F_j^h + \frac{1}{2}K_{\gamma j}^h F_i^\gamma - \frac{1}{2}K_{ij}^\gamma F_\gamma^h, \quad (2.14)$$

and the tensor $\overline{B}_{ij}^h$ is defined by

$$\overline{B}_{ij}^h = \overline{F}_{i||j}^h - \frac{1}{n}(\overline{F}_{i||\beta}^\alpha + \frac{1}{2}\overline{K}_{\gamma\beta}^\alpha \overline{F}_i^\gamma - \frac{1}{2}\overline{K}_{i\beta}^\gamma \overline{F}_\gamma^\alpha) \overline{F}_\alpha^\beta \overline{F}_j^h + \frac{1}{2}\overline{K}_{\gamma j}^h \overline{F}_i^\gamma - \frac{1}{2}\overline{K}_{ij}^\gamma \overline{F}_\gamma^h.$$

Analogously, we can prove that the tensors B_{ij}^h , $\theta = 2, 3, 4$, defined by

$$\begin{aligned} B_{2ij}^h &= F_{i||j}^h - \frac{1}{n}(F_{i||\beta}^\alpha + \frac{1}{2}K_{\beta\gamma}^\alpha F_i^\gamma - \frac{1}{2}K_{\beta i}^\gamma F_\gamma^\alpha) F_\alpha^\beta F_j^h + \frac{1}{2}K_{j\gamma}^h F_i^\gamma - \frac{1}{2}K_{ji}^\gamma F_\gamma^h, \\ B_{3ij}^h &= F_{i||j}^h - \frac{1}{n}(F_{i||\beta}^\alpha + \frac{1}{2}K_{\gamma\beta}^\alpha F_i^\gamma - \frac{1}{2}K_{\beta i}^\gamma F_\gamma^\alpha) F_\alpha^\beta F_j^h + \frac{1}{2}K_{\gamma j}^h F_i^\gamma - \frac{1}{2}K_{ji}^\gamma F_\gamma^h, \\ B_{4ij}^h &= F_{i||j}^h - \frac{1}{n}(F_{i||\beta}^\alpha + \frac{1}{2}K_{\beta\gamma}^\alpha F_i^\gamma - \frac{1}{2}K_{i\beta}^\gamma F_\gamma^\alpha) F_\alpha^\beta F_j^h + \frac{1}{2}K_{j\gamma}^h F_i^\gamma - \frac{1}{2}K_{ij}^\gamma F_\gamma^h, \end{aligned} \quad (2.15)$$

are also invariant with respect to the mapping f .

The previous discussion generalize Theorem 1 from [5] to the case of generalized Riemannian manifolds. Namely, the tensors A_{ij}^h , $\theta = 1, \dots, 4$, given by (2.8) and (2.9) are generalizations of the tensor $A_i = F_{i;\alpha}^\alpha$, while the tensors B_{ij}^h , $\theta = 1, \dots, 4$,

given by (2.14) and (2.15) are generalizations of the tensor B_{ij}^h given by

$$B_{ij}^h = F_{i;j}^h - \frac{1}{n} F_{i;\beta}^\alpha F_\alpha^*{}^\beta F_j^h, \quad (2.16)$$

where $(;)$ denotes covariant differentiation with respect to the Levi-Civita connection.

When $(;)$ denotes covariant differentiation with respect to the symmetric part ∇ of non-symmetric linear connection ${}^1\nabla$, it is obvious that the tensors $A_i = F_{i;\alpha}^\alpha$ and B_{ij}^h are invariant with respect to the mapping f of generalized Riemannian manifolds.

3. SPECIAL CANONICAL ALMOST GEODESIC MAPPINGS OF GENERALIZED PARABOLIC KÄHLER MANIFOLDS

We use Eisenhart's idea of generalized Riemannian spaces to generalize the notion of a parabolic Kähler manifold. Namely, we consider a parabolic Kähler manifold with a non-symmetric metric. M.S. Stanković et al. [20] have already considered similar generalization for classical (elliptic) Kähler manifolds. They assumed that the affinor F is covariantly constant with respect to both of connections ${}^1\nabla$ and ${}^2\nabla$. We use weaker condition, by assuming that the affinor F is covariantly constant with respect to the symmetric part of non-symmetric linear connection ${}^1\nabla$.

Definition 3. A generalized Riemannian manifold M with a metric g is called a *generalized parabolic Kähler manifold* if there exists a $(1, 1)$ tensor field F on M such that

$$F^2 = 0, \quad \nabla F = 0, \quad g(X, Y) = \omega g(X, FY), \quad \omega = \pm 1, \text{ for all } X, Y \in T_p(M),$$

where ∇ denotes the Levi-Civita connection corresponding to the symmetric part $\underline{g}$ of metric g .

In what follows we consider only generalized parabolic Kähler manifolds for which $\omega = 1$ in Definition 3. Let M and $\overline{M}$ be two generalized parabolic Kähler manifolds of dimension $n > 2$, with the metrics g and $\overline{g}$, respectively and the affinor structure F . As in the case of usual parabolic Kähler manifolds, the conditions

$$F^2 = 0 \text{ and } \text{Tr}(F) = F_p^p = 0$$

are satisfied.

The non-symmetric linear connection ${}^1\nabla$, defined by (2.1), can be represented as follows

$${}^1\nabla_X Y = \nabla_X Y + \frac{1}{2} {}^1T(X, Y), \quad (3.1)$$

where ∇ denotes the symmetric part of non-symmetric connection ${}^1\nabla$ and 1T is the torsion tensor field of connection ${}^1\nabla$.

For an anti-symmetric tensor field K given by

$$K(X, Y) = \frac{1}{2} ({}^1\overline{T}(X, Y) - {}^1T(X, Y)), \quad (3.2)$$

according to (3.1) we have

$$\begin{aligned} {}^1\nabla_Y FX + K(Y, FX) &= \nabla_Y FX + \frac{1}{2} {}^1T(Y, FX) + \frac{1}{2} {}^1\bar{T}(Y, FX) - \frac{1}{2} {}^1T(Y, FX) \\ &= \nabla_Y FX + \frac{1}{2} {}^1\bar{T}(Y, FX). \end{aligned}$$

Analogously, we can prove the relation

$${}^2\nabla_Y FX - K(Y, FX) = \nabla_Y FX + \frac{1}{2} {}^2\bar{T}(Y, FX).$$

Therefore the basic equations (2.2) and (2.3) in the case of canonical almost geodesic mappings of type $\pi_2(0, F)$, $\theta \in \{1, 2\}$ (with a priori defined affinor F) between generalized parabolic Kähler manifolds have the following form

$$\begin{aligned} {}^\theta P(X, Y) &= \sum_{CS(X, Y)} \varphi(X) FY + (-1)^{(\theta-1)} K(X, Y), \\ \frac{1}{2} \sum_{CS(X, Y)} {}^\theta \bar{T}(Y, FX) &= \sum_{CS(X, Y)} (\mu(X) FY - \mu(FX) Y), \end{aligned}$$

where $X, Y \in \mathcal{X}(M)$, φ is a 1-form and K is the anti-symmetric tensor field of type (1, 2) given by (3.2).

It is well known that the affinor structure F is locally integrable if and only if on a manifold exists a symmetric linear connection ∇ such that $\nabla F = 0$. Therefore, the affinor structure F of a generalized parabolic Kähler manifold is locally integrable.

This fact enables us to consider another affinor structure F^* such that [5]

$$F_\alpha^h F_i^*{}^\alpha + F_\alpha^*{}^h F_i^\alpha = \delta_i^h \quad (3.3)$$

holds on each local chart U of a generalized parabolic Kähler manifold.

In [5] it was proved that the geometric object

$$\Gamma_{ij}^h - \frac{1}{n+1} F_{(i}^h \Gamma_{j)\beta}^\alpha F_\alpha^*{}^\beta \quad (3.4)$$

is invariant with respect to the canonical almost geodesic mapping of type $\pi_2(e=0)$ between parabolic Kähler manifolds. In what follows we give some generalizations of the geometric object given by (3.4), to the case of a canonical almost geodesic mapping of type $\pi_2(0, F)$, $\theta \in \{1, 2\}$ between generalized parabolic Kähler manifolds.

Theorem 1. *Let $f : M \rightarrow \overline{M}$ be a canonical almost geodesic mapping of type $\pi_2(0, F)$, $\theta \in \{1, 2\}$ between generalized parabolic Kähler manifolds M and $\overline{M}$.*

Then the geometric objects C_{ij}^h , $\theta = 1, \dots, 4$, given by

$$C_{1ij}^h = \Gamma_{ij}^h - \left[\frac{1}{n+1} \left(\Gamma_{iq}^p F_p^q + \frac{1}{n} [F_{p|1\beta}^\alpha F_\alpha^\beta + \frac{1}{2} K_{\gamma\beta}^\alpha F_\alpha^\beta F_p^\gamma - \frac{1}{2} K_{p\beta}^\gamma F_\gamma^\alpha F_\alpha^\beta] F_i^p + \frac{1}{2} K_{iq}^p F_p^q \right) F_j^h \right]_{(ij)} + \frac{1}{2} K_{ij}^h, \quad (3.5)$$

$$C_{2ij}^h = \Gamma_{ij}^h - \left[\frac{1}{n+1} \left(\Gamma_{iq}^p F_p^q + \frac{1}{n} [F_{p|2\beta}^\alpha F_\alpha^\beta + \frac{1}{2} K_{\beta\gamma}^\alpha F_\alpha^\beta F_p^\gamma - \frac{1}{2} K_{\beta p}^\gamma F_\gamma^\alpha F_\alpha^\beta] F_i^p + \frac{1}{2} K_{iq}^p F_p^q \right) F_j^h \right]_{(ij)} + \frac{1}{2} K_{ij}^h, \quad (3.6)$$

$$C_{3ij}^h = \Gamma_{ij}^h - \left[\frac{1}{n+1} \left(\Gamma_{iq}^p F_p^q + \frac{1}{n} [F_{p|3\beta}^\alpha F_\alpha^\beta + \frac{1}{2} K_{\gamma\beta}^\alpha F_\alpha^\beta F_p^\gamma - \frac{1}{2} K_{\beta p}^\gamma F_\gamma^\alpha F_\alpha^\beta] F_i^p + \frac{1}{2} K_{iq}^p F_p^q \right) F_j^h \right]_{(ij)} + \frac{1}{2} K_{ij}^h, \quad (3.7)$$

$$C_{4ij}^h = \Gamma_{ij}^h - \left[\frac{1}{n+1} \left(\Gamma_{iq}^p F_p^q + \frac{1}{n} [F_{p|4\beta}^\alpha F_\alpha^\beta + \frac{1}{2} K_{\beta\gamma}^\alpha F_\alpha^\beta F_p^\gamma - \frac{1}{2} K_{p\beta}^\gamma F_\gamma^\alpha F_\alpha^\beta] F_i^p + \frac{1}{2} K_{iq}^p F_p^q \right) F_j^h \right]_{(ij)} + \frac{1}{2} K_{ij}^h, \quad (3.8)$$

are invariant with respect to the mapping f .

Proof. Contracting the basic equation (2.5) with F_h^j we obtain

$$\begin{aligned} (\bar{\Gamma}_{iq}^p - \Gamma_{iq}^p) F_p^q &= \varphi_i F_q^p F_p^q + \varphi_q F_p^q F_i^p + K_{iq}^p F_p^q \\ &= n\varphi_i + \varphi_q (F_p^q F_i^p + F_p^q F_i^p - F_p^q F_i^p) + K_{iq}^p F_p^q \\ &\stackrel{(3.3)}{=} n\varphi_i + \varphi_q \delta_i^q - \varphi_q F_p^q F_i^p + K_{iq}^p F_p^q. \end{aligned}$$

Therefore,

$$\begin{aligned} (n+1)\varphi_i &= (\bar{\Gamma}_{iq}^p - \Gamma_{iq}^p) F_p^q + \varphi_q F_p^q F_i^p - K_{iq}^p F_p^q \\ &\stackrel{(3.3)}{=} (\bar{\Gamma}_{iq}^p - \Gamma_{iq}^p) F_p^q + \frac{1}{n} \left[(F_{p|1\beta}^\alpha - F_{p|1\beta}^\alpha) F_\alpha^\beta - K_{\gamma\beta}^\alpha F_\alpha^\beta F_p^\gamma \right. \\ &\quad \left. + K_{p\beta}^\gamma F_\gamma^\alpha F_\alpha^\beta \right] F_i^p - K_{iq}^p F_p^q. \end{aligned} \quad (3.9)$$

Now, after changing (3.9) into the basic equation (2.5), we get

$$\begin{aligned} \bar{\Gamma}_{ij}^h = \Gamma_{ij}^h + \frac{1}{n+1} & \left[\left((\bar{\Gamma}_{iq}^p - \Gamma_{iq}^p) F_p^q + \frac{1}{n} [(F_{p||\beta}^\alpha - F_{p||\beta}^\alpha) F_\alpha^\beta - K_{\gamma\beta}^\alpha F_\alpha^\beta F_p^\gamma \right. \right. \\ & \left. \left. + K_{p\beta}^\gamma F_\gamma^\alpha F_\alpha^\beta] F_i^p - K_{iq}^p F_p^q \right) F_j^h \right]_{(ij)} + K_{ij}^h. \end{aligned}$$

From the previous equation, by using (2.10)–(2.13), we obtain the following relation

$$\begin{aligned} \bar{\Gamma}_{ij}^h - \left[\frac{1}{n+1} & \left(\bar{\Gamma}_{iq}^p \bar{F}_p^q + \frac{1}{n} [\bar{F}_{p||\beta}^\alpha \bar{F}_\alpha^\beta + \frac{1}{2} \bar{K}_{\gamma\beta}^\alpha \bar{F}_\alpha^\beta \bar{F}_p^\gamma - \frac{1}{2} \bar{K}_{p\beta}^\gamma \bar{F}_\gamma^\alpha \bar{F}_\alpha^\beta] \bar{F}_i^p \right. \right. \\ & \left. \left. + \frac{1}{2} \bar{K}_{iq}^p \bar{F}_p^q \right) \bar{F}_j^h \right]_{(ij)} + \frac{1}{2} \bar{K}_{ij}^h \\ = \Gamma_{ij}^h - \left[\frac{1}{n+1} & \left(\Gamma_{iq}^p F_p^q + \frac{1}{n} [F_{p||\beta}^\alpha F_\alpha^\beta + \frac{1}{2} K_{\gamma\beta}^\alpha F_\alpha^\beta F_p^\gamma - \frac{1}{2} K_{p\beta}^\gamma F_\gamma^\alpha F_\alpha^\beta] F_i^p \right. \right. \\ & \left. \left. + \frac{1}{2} K_{iq}^p F_p^q \right) F_j^h \right]_{(ij)} + \frac{1}{2} K_{ij}^h, \end{aligned}$$

which proves that the geometric object C_{ij}^h defined by (3.5) is invariant with respect to the mapping f .

In a similar manner one can conclude that the geometric objects C_{ij}^h , $\theta = 2, 3, 4$, determined by (3.6)–(3.8) are invariant with respect to the mapping f . $\square$

When we consider a mapping between two affine connected manifolds with torsion, we can consider the so called equitorsion mapping, it is a mapping which preserves the torsion tensor.

Definition 4 ([16, 22]). An almost geodesic mapping $f : M \rightarrow \bar{M}$ of affine connected manifolds M and $\bar{M}$ with the torsion tensors T_{ij}^h and $\bar{T}_{ij}^h$, respectively, is an *equitorsion almost geodesic mapping* if the following condition holds

$$T_{ij}^h = \bar{T}_{ij}^h.$$

Equation (3.2) in local coordinates reads $K_{ij}^h = \frac{1}{2}(\bar{T}_{ij}^h - T_{ij}^h)$. Therefore the geometric objects C_{ij}^h , $\theta = 1, \dots, 4$, given by (3.5)–(3.8), with respect to an equitorsion canonical almost geodesic mapping of type $\pi_2(0, F)$, $\theta \in \{1, 2\}$ between generalized parabolic Kähler manifolds take the following forms

$$C_{ij}^h = \Gamma_{ij}^h - \left[\frac{1}{n+1} \left(\Gamma_{iq}^p \bar{F}_p^q + \frac{1}{n} F_{p||\beta}^\alpha \bar{F}_\alpha^\beta F_i^p \right) \right]_{(ij)}, \quad \theta = 1, \dots, 4. \quad (3.10)$$

Note that the geometric objects given by (3.5)–(3.8) and (3.10) are not tensors, since the generalized Christoffel symbols Γ_{ij}^h are not tensors (see [14], p. 10).

The geometric object

$$C_{ij}^h = \Gamma_{ij}^h - \frac{1}{n+1} F_{(i}^h \Gamma_{j)\beta}^\alpha F_\alpha^\beta, \quad (3.11)$$

where Γ_{ij}^h is the symmetric part of Γ_{ij}^h , is evidently invariant with respect to the canonical almost geodesic mapping of type $\pi_2(0, F)$, $\theta \in \{1, 2\}$ between generalized parabolic Kähler manifolds. This geometric object is a tensor as well as the geometric object given by (3.4).

Remark 1. The geometric objects, given by (3.5)–(3.8), (3.10) and (3.11) are generalizations of the tensor, given by (3.4).

4. CONCLUSION

Invariant geometric objects of canonical almost geodesic mappings of type $\pi_2(0, F)$, $\theta \in \{1, 2\}$ are examined. Since the available literature does not contain any results about invariants of almost geodesic mappings of type $\pi_2(e)$, $\theta \in \{1, 2\}$ for $e = 0$, this paper somehow fills the gap in the theory of almost geodesic mappings of manifolds with non-symmetric affine connection.

A generalized parabolic Kähler manifold is introduced and some results concerning invariant geometric objects of canonical almost geodesic mappings of type $\pi_2(e = 0)$, between parabolic Kähler manifolds are extended. This fact opens up possibilities for further extension of results from usual parabolic Kähler manifolds to generalized parabolic Kähler manifolds.

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*Author's address***Miloš Z. Petrović**

University of Niš, Department of Mathematics, Višegradska 33, 18000 Niš, Serbia

E-mail address: milos.petrovic@pmf.edu.rs



ON A POWER OF n -ARY RELATIONAL SYSTEMS CARRIED BY STRONG HOMOMORPHISMS

N. PHROMMARAT

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Abstract. First, we introduce and discuss a new operation of product of n -ary relational systems which lies, as for generality, between their direct product and the direct product of their reflexive hulls. Then we study the power of n -ary relational systems obtained as the subsystem of their direct power carried by strong homomorphisms. We show that, with respect to the new operation of product, the power of n -ary relational systems studied satisfies weak forms of the first and second exponential laws and, with respect to the direct product and direct sum, it satisfies the third exponential law.

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1. INTRODUCTION

In his pioneering papers [2] and [3], G. Birkhoff introduced and studied the operation of a cardinal power of partially ordered sets and showed the validity of the first, second and third exponential laws for the operation, i.e.,

$(\mathbf{A}^{\mathbf{B}})^{\mathbf{C}} \cong \mathbf{A}^{\mathbf{B} \times \mathbf{C}}$ - the first exponential law,

$\prod_{i \in I} \mathbf{A}_i^{\mathbf{B}} \cong (\prod_{i \in I} \mathbf{A}_i)^{\mathbf{B}}$ - the second exponential law,

$\prod_{i \in I} \mathbf{A}^{\mathbf{B}_i} \cong \mathbf{A}^{\sum_{i \in I} \mathbf{B}_i}$ (if $\mathbf{B}_i, i \in I$, are pair-wise disjoint) - the third exponential law.

Birkhoff's arithmetic of ordered sets has been generalized by several authors, see, e.g., [5] and [9]. The concept of a cardinal power was extended to relational systems in [7]. In this note, we will continue the study of the arithmetic of relational system.

In [8], M. Novotný and J. Šlapal introduced and studied new operations of product and power of relational systems which satisfy the first exponential law. The new product was defined by combining the direct (i.e., cardinal) operations of sum and product, and the new power was defined as the intersection of the direct power and another power of relational systems. In [1], František Bednařík and Josef Šlapal

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studied the subsystem of the cardinal power of relational systems carried by the set of strong homomorphisms.

In the present paper, we will introduce a new operation of product of n -ary relational systems which generalizes the concept of combined product in [8]. Moreover, we will study the subsystem of the direct power carried by the set of strong homomorphisms. We will prove the validity of the third exponential law and weak forms of the first and second exponential laws.

2. COMBINED PRODUCT OF n -ARY RELATIONAL SYSTEMS

Throughout this paper, n denotes a positive integer. By an n -ary relation p on a set A we mean a subset $p \subseteq A^n$. The pair (A, p) is called an n -ary relational system.

Let $\mathbf{A} = (A, p), \mathbf{B} = (B, q)$ be a pair of n -ary relational systems. A mapping $f : B \rightarrow A$ is said to be a *homomorphism* of $\mathbf{B}$ into $\mathbf{A}$ if $(x_1, \dots, x_n) \in q$ implies $(f(x_1), \dots, f(x_n)) \in p$. A homomorphism f of $\mathbf{B} = (B, q)$ into $\mathbf{A} = (A, p)$ is said to be a *strong homomorphism* of $\mathbf{B}$ into $\mathbf{A}$ if, for every $b_1, \dots, b_n \in B$, $(f(b_1), \dots, f(b_n)) \in p$ implies $(b_1, \dots, b_n) \in q$. We denote by $Hom(\mathbf{B}, \mathbf{A})$ the set of all homomorphisms of $\mathbf{B}$ into $\mathbf{A}$ and by $SHom(\mathbf{B}, \mathbf{A})$ the set of all strong homomorphisms of $\mathbf{B}$ into $\mathbf{A}$.

If $f : B \rightarrow A$ is a bijection and both $f : B \rightarrow A$ and $f^{-1} : A \rightarrow B$ are homomorphisms, then f is called an *isomorphism* of $\mathbf{B}$ onto $\mathbf{A}$. If there exists an isomorphism of $\mathbf{B}$ onto $\mathbf{A}$, we say that $\mathbf{B}$ and $\mathbf{A}$ are *isomorphic*, in symbols $B \cong A$. An n -ary relational system $\mathbf{A} = (A, p)$ is said to be a *subsystem* of an n -ary relational system $\mathbf{B} = (B, q)$ if $A \subseteq B$ and $p = q \cap A^n$. Given a pair of n -ary relational systems $\mathbf{A}, \mathbf{B}$, we write $\mathbf{B} \preceq \mathbf{A}$ if $\mathbf{B}$ can be *embedded* into $\mathbf{A}$, i.e., if there is a relational subsystem $\mathbf{A}'$ of $\mathbf{A}$ such that $\mathbf{B} \cong \mathbf{A}'$.

The *direct product* of a family $\mathbf{A}_i = (A_i, p_i), i \in I$, of n -ary relational systems is the n -ary relational system $\prod_{i \in I} \mathbf{A}_i = (\prod_{i \in I} A_i, q)$ where $\prod_{i \in I} A_i$ is the cartesian product of sets and, for any $f_1, \dots, f_n \in \prod_{i \in I} A_i, (f_1, \dots, f_n) \in q$ if and only if $(f_1(i), \dots, f_n(i)) \in p_i$ for each $i \in I$. If the set I is finite, say $I = \{1, \dots, m\}$, then we write $\mathbf{A}_1 \times \dots \times \mathbf{A}_m$ instead of $\prod_{i \in I} \mathbf{A}_i$. If $\mathbf{A}_i = \mathbf{A}$ for each $i \in I$, then we write $\mathbf{A}^I$ instead of $\prod_{i \in I} \mathbf{A}_i$.

The *direct sum* of a family $\mathbf{A}_i = (A_i, p_i), i \in I$, of n -ary relational systems is the n -ary relational system $\sum_{i \in I} \mathbf{A}_i = (\bigcup_{i \in I} A_i, \bigcup_{i \in I} p_i)$. If the set I is finite, say $I = \{1, \dots, m\}$, then we write $\mathbf{A}_1 \uplus \dots \uplus \mathbf{A}_m$ instead of $\sum_{i \in I} \mathbf{A}_i$.

An n -ary relational system $\mathbf{A} = (A, p)$ is said to be *reflexive* provided that $(a_1, a_2, \dots, a_n) \in p$ whenever $a_1 = a_2 = \dots = a_n$. Let $\mathbf{A} = (A, p)$ be an n -ary relational system. We denote by $\bar{p}$ the n -ary relational system on G such that, for every $a_1, \dots, a_n \in A, (a_1, \dots, a_n) \in \bar{p}$ if and only if $a_1 = \dots = a_n$ or $(a_1, \dots, a_n) \in p$. The n -ary relational systems $(A, \bar{p})$ is called the *reflexive hull* of $\mathbf{A}$ and is denoted by $\bar{\mathbf{A}}$.

Let $\mathbf{A}_i = (A_i, p_i), i \in I$, be a family of n -ary relational systems. The *combined product* of the family $\mathbf{A}_i, i \in I$, is the n -ary relational system $\bigotimes_{i \in I} \mathbf{A}_i = (\prod_{i \in I} A_i, r)$ given by $\bigotimes_{i \in I} \mathbf{A}_i = \sum_{i \in I} \prod_{j \in I} \mathbf{A}_{ij}$ where

$$\mathbf{A}_{ij} = \begin{cases} \bar{\mathbf{A}}_j & \text{if } i = j, \\ \mathbf{A}_j & \text{if } i \neq j. \end{cases}$$

Thus, for any $(a_1^i; i \in I), \dots, (a_n^i; i \in I) \in \prod_{i \in I} \mathbf{A}_i$ we have $(a_1^i; i \in I), \dots, (a_n^i; i \in I) \in r$ if and only if there exists a subset $J \subseteq I$, $\text{card } J \leq 1$, such that $(a_1^i, \dots, a_n^i) \in p_i$ for every $i \in I \setminus J$ and $a_1^i = a_2^i = \dots = a_n^i$ for every $i \in J$.

If the set I is finite, say $I = \{1, \dots, m\}$, we write $\mathbf{A}_1 \otimes \dots \otimes \mathbf{A}_m$ instead of $\bigotimes_{i \in I} \mathbf{A}_i$. We then clearly have $\mathbf{A}_1 \otimes \dots \otimes \mathbf{A}_m = (\mathbf{A}_{11} \times \mathbf{A}_{12} \times \dots \times \mathbf{A}_{1m}) \uplus (\mathbf{A}_{21} \times \mathbf{A}_{22} \times \dots \times \mathbf{A}_{2m}) \uplus \dots \uplus (\mathbf{A}_{m1} \times \mathbf{A}_{m2} \times \dots \times \mathbf{A}_{mm}) = (\bar{\mathbf{A}}_1 \times \mathbf{A}_2 \times \dots \times \mathbf{A}_m) \uplus (\mathbf{A}_1 \times \bar{\mathbf{A}}_2 \times \dots \times \mathbf{A}_m) \uplus \dots \uplus (\mathbf{A}_1 \times \mathbf{A}_2 \times \dots \times \bar{\mathbf{A}}_m)$.

In particular, if $I = \{1, 2\}$, then, for any $(a_1, b_1), \dots, (a_n, b_n) \in \mathbf{A}_1 \times \mathbf{A}_2$, $((a_1, b_1), \dots, (a_n, b_n)) \in r$ if and only if one of the following three conditions is satisfied:

- (i) $(a_1, \dots, a_n) \in p_1$ and $(b_1, \dots, b_n) \in p_2$,
- (ii) $a_1 = a_2 = \dots = a_n$ and $(b_1, \dots, b_n) \in p_2$,
- (iii) $(a_1, \dots, a_n) \in p_1$ and $b_1 = b_2 = \dots = b_n$.

Example 1. Let $\mathbf{A} = (A, p), \mathbf{B} = (B, q)$ be binary relational systems where $A = \{1, 2\}, B = \{a, b\}, p = \{(1, 2)\}$ and $q = \{(a, b)\}$. Then $\mathbf{A} \otimes \mathbf{B} = (A \times B, r)$ where $r = \{((1, a), (2, b)), ((1, a), (1, b)), ((2, a), (2, b)), ((1, a), (2, a)), ((1, b), (2, b))\}$.

Example 2. Let $\mathbf{A} = (A, p), \mathbf{B} = (B, q)$ be binary relational systems where $A = \{1, 2\}, B = \{a\}, p = \{(1, 2)\}$ and $q = \{(a, a)\}$. Then $\mathbf{A} \otimes \mathbf{B} = (A \times B, r)$ where $r = \{((1, a), (2, a)), ((1, a), (1, a)), ((2, a), (2, a))\}$. Thus, $\mathbf{A} \otimes \mathbf{B}$ is reflexive.

Remark 1. Let $\mathbf{A}_i, i \in I$, be a family of n -ary relational systems. If $\mathbf{A}_i$ is reflexive for every $i \in I$, then $\prod_{i \in I} \mathbf{A}_i = \bigotimes_{i \in I} \mathbf{A}_i$. If at most one of the n -ary relational systems $\mathbf{A}_i, i \in I$, is not reflexive, then $\bigotimes_{i \in I} \mathbf{G}_i$ is reflexive.

First, we will show that the combined product of n -ary relational systems distributes over their direct sum.

Proposition 1. Let $\mathbf{A} = (A, p)$ and $\mathbf{B}_i = (B_i, q_i), i \in I$, be n -ary relational systems. Then $\mathbf{A} \otimes \sum_{i \in I} \mathbf{B}_i = \sum_{i \in I} (\mathbf{A} \otimes \mathbf{B}_i)$.

Proof. Let $\sum_{i \in I} \mathbf{B}_i = (\bigcup_{i \in I} B_i, s), \mathbf{A} \otimes \mathbf{B}_i = (A \times B_i, r_i)$ for each $i \in I$, $\mathbf{A} \otimes \sum_{i \in I} \mathbf{B}_i = (A \times \bigcup_{i \in I} B_i, u)$ and $\sum_{i \in I} (\mathbf{A} \otimes \mathbf{B}_i) = (\bigcup_{i \in I} (A \times B_i), v)$. We will show that, for every

$$(a_1, b_1), \dots, (a_n, b_n) \in A \times \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \times B_i), ((a_1, b_1), \dots, (a_n, b_n)) \in u$$

if and only if $((a_1, b_1), \dots, (a_n, b_n)) \in v$. It is easy to see that the following five conditions are equivalent:

- (1) $((a_1, b_1), \dots, (a_n, b_n)) \in u$;
- (2) one of the following three cases occurs:
 - (i) $(a_1, \dots, a_n) \in p$ and $(b_1, \dots, b_n) \in s$,
 - (ii) $a_1 = a_2 = \dots = a_n$ and $(b_1, \dots, b_n) \in s$,
 - (iii) $(a_1, \dots, a_n) \in p$ and $b_1 = b_2 = \dots = b_n$;
- (3) one of the following three cases occurs:
 - (i) $(a_1, \dots, a_n) \in p$ and $(b_1, \dots, b_n) \in q_i$ for some $i \in I$,
 - (ii) $a_1 = a_2 = \dots = a_n$ and $(b_1, \dots, b_n) \in q_i$ for some $i \in I$,
 - (iii) $(a_1, \dots, a_n) \in p$ and $b_1 = b_2 = \dots = b_n$;
- (4) $((a_1, b_1), \dots, (a_n, b_n)) \in r_i$ for some $i \in I$;
- (5) $((a_1, b_1), \dots, (a_n, b_n)) \in v$.

This proves the statement. □

3. POWER OF RELATIONAL SYSTEMS WITH RESPECT TO STRONG HOMOMORPHISMS

Definition 1. Let $\mathbf{B} = (B, q)$ and $\mathbf{A} = (A, p)$ be n -ary relational systems. The *direct power* of $\mathbf{A}$ and $\mathbf{B}$ is the n -ary relational system $(Hom(\mathbf{B}, \mathbf{A}), r)$ where, for any $f_1, \dots, f_n \in Hom(\mathbf{B}, \mathbf{A})$, $(f_1, \dots, f_n) \in r$ if and only if $(f_1(x), \dots, f_n(x)) \in p$ for each $x \in B$. The subsystem $(SHom(\mathbf{B}, \mathbf{A}), r \cap (SHom(\mathbf{B}, \mathbf{A}))^n)$ of the direct power $(Hom(\mathbf{B}, \mathbf{A}), r)$ of $\mathbf{A}$ and $\mathbf{B}$ is called the *direct power of $\mathbf{A}$ and $\mathbf{B}$ with respect to strong homomorphisms*.

For any pair of n -ary relational systems $\mathbf{A}, \mathbf{B}$, we denote by $\mathbf{A}^{\mathbf{B}}$ and $\mathbf{A}^{\diamond \mathbf{B}}$ the direct power and the direct power with respect to strong homomorphisms, respectively, of $\mathbf{A}$ and $\mathbf{B}$.

It is easy to see that the direct power $\mathbf{A}^{\mathbf{B}}$ of n -ary relational systems is reflexive whenever $\mathbf{A}$ is reflexive.

Example 3. Let $\mathbf{A} = (A, p), \mathbf{B} = (B, q)$ be binary relational systems where $A = \{1, 2\}, B = \{a\}, p = A \times A$ and $q = \{(a, a)\}$. Assume that $f : A \rightarrow B$ is a map given by $f(1) = f(2) = a$. Then, $Hom(\mathbf{A}, \mathbf{B}) = SHom(\mathbf{A}, \mathbf{B}) = \{f\}$ and $\mathbf{A}^{\diamond \mathbf{B}} = (SHom(\mathbf{B}, \mathbf{A}), s)$ where $s = \{(f, f)\}$.

Example 4. Let $\mathbf{A} = (A, p), \mathbf{B} = (B, q)$ be binary relational systems where $A = \{1, 2, 3\}, B = \{a, b\}, p = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3)\}$ and $q = B \times B$. Then, every map of A into B is a homomorphism of A into B . Assume that $f, g : A \rightarrow B$ are maps given by $f(1) = f(2) = a, f(3) = b$ and $g(1) = g(3) = a, g(2) = b$. Then, $SHom(\mathbf{A}, \mathbf{B}) = \{f, g\}$ and $\mathbf{A}^{\diamond \mathbf{B}} = (SHom(\mathbf{B}, \mathbf{A}), s)$ where $s = SHom(\mathbf{A}, \mathbf{B}) \times SHom(\mathbf{A}, \mathbf{B})$.

We will show now that, for the combined product of n -ary relational systems and their power with respect to strong homomorphisms, weak forms of the first and second exponential laws hold.

Theorem 1. *Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be n -ary relational systems. Then*

$$\mathbf{A}^{\diamond(\mathbf{B} \otimes \mathbf{C})} \preceq (\mathbf{A}^{\diamond \mathbf{B}})^{\diamond \mathbf{C}}.$$

Proof. Let $\mathbf{A} = (A, q), \mathbf{B} = (B, p), \mathbf{C} = (C, s), \mathbf{A}^{\diamond \mathbf{B}} = (SHom(\mathbf{B}, \mathbf{A}), r), \mathbf{B} \otimes \mathbf{A} = (B \times A, v), (\mathbf{A}^{\diamond \mathbf{B}})^{\diamond \mathbf{C}} = (SHom(\mathbf{C}, SHom(\mathbf{B}, \mathbf{A})), t)$ and $\mathbf{A}^{\diamond(\mathbf{B} \otimes \mathbf{C})} = (SHom(\mathbf{B} \otimes \mathbf{C}, \mathbf{A}), u)$ be n -ary relational systems.

We define a map $\alpha : SHom(\mathbf{B} \otimes \mathbf{C}, \mathbf{A}) \rightarrow SHom(\mathbf{C}, SHom(\mathbf{B}, \mathbf{A}))$ by $\alpha(h)(c)(b) = h(b, c)$ whenever $h \in SHom(\mathbf{B} \otimes \mathbf{C}, \mathbf{A}), c \in C$ and $b \in B$.

Let $h \in SHom(\mathbf{B} \otimes \mathbf{C}, \mathbf{A}), c \in C$ and $(b_1, \dots, b_n) \in q$. Consequently, $((b_1, c), \dots, (b_n, c)) \in v$. Since h is a homomorphism, we have $(h(b_1, c), \dots, h(b_n, c)) \in p$. Thus, $(\alpha(h)(c)(b_1), \dots, \alpha(h)(c)(b_n)) = (h(b_1, c), \dots, h(b_n, c)) \in p$. Therefore, $\alpha(h)(c) \in Hom(\mathbf{B}, \mathbf{A})$ for each $c \in C$.

Let $b_1, \dots, b_n \in B$ and $(\alpha(h)(c)(b_1), \dots, \alpha(h)(c)(b_n)) \in p$. Then $(\alpha(h)(c)(b_1), \dots, \alpha(h)(c)(b_n)) = (h(b_1, c), \dots, h(b_n, c)) \in p$. Since h is a strong homomorphism, we get $((b_1, c), \dots, (b_n, c)) \in v$. Thus, $(b_1, \dots, b_n) \in q$. Then, $\alpha(h)(c) \in SHom(\mathbf{B}, \mathbf{A})$ for each $c \in C$.

Next, let $(c_1, \dots, c_n) \in s$. Then $((b, c_1), \dots, (b, c_n)) \in v$. Since h is a homomorphism, we get $(h(b, c_1), \dots, h(b, c_n)) \in p$. So $(\alpha(h)(b)(c_1), \dots, \alpha(h)(b)(c_n)) = (h(b, c_1), \dots, h(b, c_n)) \in p$. Consequently, $(\alpha(h)(c_1), \dots, \alpha(h)(c_n)) \in r$. Hence, $\alpha(h) \in Hom(\mathbf{C}, Hom(\mathbf{B}, \mathbf{A}))$.

Let $c_1, \dots, c_n \in C$ and $(\alpha(h)(c_1), \dots, \alpha(h)(c_n)) \in r$. Then $(\alpha(h)(c_1)b, \dots, \alpha(h)(c_n)b) \in p$, hence $(\alpha(h)(b)(c_1), \dots, \alpha(h)(b)(c_n)) = (h(b, c_1), \dots, h(b, c_n)) \in p$. Now, as h is a strong homomorphism, it follows that $((b, c_1), \dots, (b, c_n)) \in v$. Hence $(c_1, \dots, c_n) \in s$. Therefore, $\alpha(h) \in SHom(\mathbf{C}, Hom(\mathbf{B}, \mathbf{A}))$.

We will show that α is a homomorphism. Let $(h_1, \dots, h_n) \in u, (c_1, \dots, c_n) \in s$ and $(b_1, \dots, b_n) \in q$. Then we have $((b_1, c_1), \dots, (b_n, c_n)) \in v$ and, consequently, $(h_1(b_1, c_1), \dots, h_n(b_n, c_n)) \in p$. Therefore, $(\alpha(h_1)(c_1)(b_1), \dots, \alpha(h_n)(c_n)(b_n)) = (h_1(b_1, c_1), \dots, h_n(b_n, c_n)) \in p$. Hence, $(\alpha(h_1)(c_1), \dots, \alpha(h_n)(c_n)) \in r$, so that $(\alpha(h_1), \dots, \alpha(h_n)) \in t$. Thus, α is a homomorphism.

Finally, we will show that α is an injection. Let $\alpha(h_1) = \alpha(h_2)$ for a pair $h_1, h_2 \in SHom(\mathbf{B} \otimes \mathbf{C}, \mathbf{A})$. Then $\alpha(h_1)c = \alpha(h_2)c$ for every $c \in C$ and we have $\alpha(h_1)(c)b = \alpha(h_2)(c)b$ for every $c \in C$ and $b \in B$. Hence $h_1(b, c) = h_2(b, c)$ for every $(b, c) \in B \times C$. Thus, $h_1 = h_2$ and we have shown that α is an injection.

Therefore α is an embedding of $\mathbf{A}^{\diamond(\mathbf{B} \otimes \mathbf{C})}$ into $(\mathbf{A}^{\diamond \mathbf{B}})^{\diamond \mathbf{C}}$ and the proof is complete. $\square$

Theorem 2. Let $\mathbf{A}_i = (A_i, p_i), i \in I$, be a family of n -ary relational systems and $\mathbf{B}$ be an n -ary relational system. Then

$$\bigotimes_{i \in I} (\mathbf{A}_i^{\diamond \mathbf{B}}) \preceq (\bigotimes_{i \in I} \mathbf{A}_i)^{\diamond \mathbf{B}}.$$

Proof. Let $\mathbf{A}_i = (A_i, p_i)$ for every $i \in I$, $\mathbf{B} = (B, q)$, $\bigotimes_{i \in I} \mathbf{A}_i = (\prod_{i \in I} A_i, r)$, $\mathbf{A}_i^{\diamond \mathbf{B}} = (SHom(\mathbf{B}, \mathbf{A}_i), u_i)$ for every $i \in I$, $\bigotimes_{i \in I} \mathbf{A}_i^{\diamond \mathbf{B}} = (\prod_{i \in I} SHom(\mathbf{B}, \mathbf{A}_i), s)$ and $(\bigotimes_{i \in I} \mathbf{A}_i)^{\diamond \mathbf{B}} = (Hom(\mathbf{B}, \bigotimes_{i \in I} \mathbf{A}_i), t)$.

We define a map $\alpha: \prod_{i \in I} SHom(\mathbf{B}, \mathbf{A}_i) \rightarrow (\prod_{i \in I} A_i)^H$ by $\alpha(f^i; i \in I)(b) = (f^i(b); i \in I)$ for each $b \in B$.

Let $(f^i; i \in I) \in \prod_{i \in I} SHom(\mathbf{B}, \mathbf{A}_i)$ and $(b_1, \dots, b_n) \in q$. Since $f^i \in Hom(\mathbf{B}, \mathbf{A}_i)$, we have $(f^i(b_1), \dots, f^i(b_n)) \in p_i$ for every $i \in I$. Then $((f^i(b_1); i \in I), \dots, (f^i(b_n); i \in I)) \in r$ and we have $(\alpha(f^i; i \in I)(b_1), \dots, \alpha(f^i; i \in I)(b_n)) \in r$. Therefore, $\alpha(f^i; i \in I) \in Hom(\mathbf{B}, \bigotimes_{i \in I} \mathbf{A}_i)$.

Let $(f^i; i \in I) \in \prod_{i \in I} SHom(\mathbf{B}, \mathbf{A}_i)$ and

$$(\alpha(f^i; i \in I)(b_1), \dots, \alpha(f^i; i \in I)(b_n)) \in r.$$

Consequently, $((f^i(b_1); i \in I), \dots, (f^i(b_n); i \in I)) \in r$. Therefore, $(f^i(b_1), \dots, f^i(b_n)) \in p_i$ for every $i \in I$. As f^i is a strong homomorphism for every $i \in I$, we have $(b_1, \dots, b_n) \in q$. We have shown that α maps $\prod_{i \in I} SHom(\mathbf{B}, \mathbf{A}_i)$ into $SHom(\mathbf{B}, \bigotimes_{i \in I} \mathbf{A}_i)$.

We will show that α is a homomorphism. Let $(f_1^i; i \in I), \dots, (f_n^i; i \in I) \in \prod_{i \in I} SHom(\mathbf{B}, \mathbf{A}_i)$ and $((f_1^i; i \in I), \dots, (f_n^i; i \in I)) \in s$. Let $J \subseteq I$, $\text{card } J \leq 1$. Consequently, $(f_1^i, \dots, f_n^i) \in u_i$ for every $i \in I \setminus J$ and $f_1^i = f_2^i = \dots = f_n^i$ for every $i \in J$. Hence, $(f_1^i(b), \dots, f_n^i(b)) \in p_i$ for every $i \in I \setminus J$ and every $b \in B$ and $f_1^i(b) = f_2^i(b) = \dots = f_n^i(b)$ for every $i \in J$ and every $b \in B$. Hence, $((f_1^i(b); i \in I), \dots, (f_n^i(b); i \in I)) \in r$ for every $b \in B$. Therefore, $(\alpha(f_1^i; i \in I)(b), \dots, \alpha(f_n^i; i \in I)(b)) \in r$ for every $b \in B$ and then $(\alpha(f_1^i; i \in I), \dots, \alpha(f_n^i; i \in I)) \in t$. Thus, α is a homomorphism of $\bigotimes_{i \in I} (\mathbf{A}_i^{\diamond \mathbf{B}})$ into $(\bigotimes_{i \in I} \mathbf{A}_i)^{\diamond \mathbf{B}}$.

Suppose that $\alpha(f^i; i \in I) = \alpha(g^i; i \in I)$ where $(f^i; i \in I), (g^i; i \in I) \in \prod_{i \in I} SHom(\mathbf{B}, \mathbf{A}_i)$. Then $(f^i(b); i \in I) = \alpha(f^i; i \in I)(b) = \alpha(g^i; i \in I)(b) = (g^i(b); i \in I)$ for every $b \in B$. Therefore, $f^i(b) = g^i(b)$ for every $i \in I$ and every $b \in B$. Hence, $f^i = g^i$ for every $i \in I$. Thus, $\alpha: \prod_{i \in I} SHom(\mathbf{B}, \mathbf{A}_i) \rightarrow SHom(\mathbf{B}, \bigotimes_{i \in I} \mathbf{A}_i)$ is an injection. We have shown that $\bigotimes_{i \in I} (\mathbf{A}_i^{\diamond \mathbf{B}}) \preceq (\bigotimes_{i \in I} \mathbf{A}_i)^{\diamond \mathbf{B}}$. $\square$

Next, we will show that, for the powers of n -ary relational systems with respect to strong homomorphisms, the third exponential law holds.

Theorem 3. Let $\mathbf{A}$ be an n -ary relational system and let $\mathbf{B}_i, i \in I$, be a family of pair-wise disjoint n -ary relational systems. Then

$$\prod_{i \in I} \mathbf{A}^{\diamond \mathbf{B}_i} \cong \mathbf{A}^{\diamond \sum_{i \in I} \mathbf{B}_i}.$$

Proof. Let $\mathbf{A} = (A, p)$, $\mathbf{B}_i = (B_i, q_i)$ for every $i \in I$, $\sum_{i \in I} \mathbf{B}_i = (\sum_{i \in I} B_i, v)$, $\mathbf{A}^{\diamond \mathbf{B}_i} = (SHom(\mathbf{B}_i, \mathbf{A}), r_i)$ for every $i \in I$, $\prod_{i \in I} \mathbf{A}^{\diamond \mathbf{B}_i} = (\prod_{i \in I} SHom(\mathbf{B}_i, \mathbf{A}), t)$ and $\mathbf{A}^{\diamond \sum_{i \in I} \mathbf{B}_i} = (SHom(\sum_{i \in I} \mathbf{B}_i, \mathbf{A}), u)$.

We define a map $\varphi : \prod_{i \in I} SHom(\mathbf{B}_i, \mathbf{A}) \rightarrow SHom(\sum_{i \in I} \mathbf{B}_i, \mathbf{A})$ by $\varphi(f^i; i \in I) = h$ where $h(t) = f^i(t)$ whenever $t \in H_i$ (and $i \in I$). Let $(b_1, \dots, b_n) \in q_i$ for any $i \in I$. Since f_i is a homomorphism, we have $(\varphi(f^i; i \in I)b_1, \dots, (\varphi(f^i; i \in I)b_n)) = (h(b_1), \dots, (h(b_n))) = (f^i(b_1), \dots, (f^i(b_n))) \in r$ for any $i \in I$. Then $\varphi(f^i; i \in I) \in Hom(\sum_{i \in I} \mathbf{B}_i, A)$.

Let $(\varphi(f^i; i \in I)b_1, \dots, (\varphi(f^i; i \in I)b_n))$ for every $b_1, \dots, b_n \in B$. Thus, $((f^i(b_1), \dots, (f^i(b_n))) = (h(b_1), \dots, (h(b_n))) = \varphi(f^i; i \in I)b_1, \dots, (\varphi(f^i; i \in I)b_n) \in r$ for any $i \in I$. Since f^i is a strong homomorphism, we have $(b_1, \dots, b_n) \in q_i$ for any $i \in I$. Therefore, $\varphi(f^i; i \in I) \in SHom(\sum_{i \in I} \mathbf{B}_i, A)$.

Further, we define a map $\alpha : SHom(\sum_{i \in I} \mathbf{B}_i, \mathbf{A}) \rightarrow \prod_{i \in I} SHom(\mathbf{B}_i, \mathbf{A})$ by $\alpha(h) = (f^i; i \in I)$ whenever $h \in SHom(\sum_{i \in I} \mathbf{B}_i, \mathbf{A})$ where $f^i = h|_{B_i}$ for every $i \in I$. It is easy to see that f_i is a strong homomorphism of $\mathbf{B}_i$ into $\mathbf{A}$ for every $i \in I$. It follows that $\alpha(h) \in \prod_{i \in I} SHom(\mathbf{B}_i, \mathbf{A})$.

We will show that both φ and α are homomorphisms. Let $(f_1^i; i \in I), \dots, (f_n^i; i \in I) \in \prod_{i \in I} SHom(\mathbf{B}_i, \mathbf{A})$ and $((f_1^i; i \in I), \dots, (f_n^i; i \in I)) \in T$. Consequently, $(f_1^i, \dots, f_n^i) \in r_i$ for each $i \in I$. Then

$$(f_1^i(b), \dots, f_n^i(b)) = (h_1(b), \dots, h_n(b)) \in p$$

for each $i \in I$ and $b \in \sum_{i \in I} B_i$. Therefore, $(\varphi(f_1^i; i \in I), \dots, \varphi(f_n^i; i \in I)) \in u$. Thus, φ is a homomorphism.

Let $h_1, \dots, h_n \in SHom(\sum_{i \in I} \mathbf{B}_i, \mathbf{A})$ and $(h_1, \dots, h_n) \in U$. Then $(h_1|_{B_i}(b), \dots, h_n|_{B_i}(b)) = (f_1^i(b), \dots, f_n^i(b)) \in p$ for any $i \in I$ and $b \in \sum_{i \in I} B_i$. Therefore, $(f_1^i, \dots, f_n^i) \in r_i$. Thus, $((f_1^i; i \in I), \dots, (f_n^i; i \in I)) \in T$ and then $(\alpha(h_1), \dots, \alpha(h_n)) \in t$.

It follows that φ is an isomorphism with $\varphi^{-1} = \alpha$. $\square$

Remark 2. An n -ary hyperalgebra is an $(n+1)$ -ary relational system $\mathbf{G} = (G, p)$ such that, for any $x_1, \dots, x_n \in G$, there exists $y \in G$ with $(x_1, \dots, x_n, y) \in p$ (cf. [10]). The $(n+1)$ -ary relation p on $\mathbf{G}$ may then be considered to be a map $p : G^n \rightarrow \exp G$ and it is called an n -ary hyperoperation on G ; we write $y \in p(x_1, \dots, x_n)$ instead of $(x_1, \dots, x_n, y) \in p$. If such y is unique whenever $x_1, \dots, x_n \in G$, then the $(n+1)$ -ary relational system $\mathbf{G}$ is nothing but an n -ary algebra (cf. [6]). The $(n+1)$ -ary relation p may then be considered to be a map $p : G^n \rightarrow G$ and it is called an n -ary operation on G ; we write $y = p(x_1, \dots, x_n)$ instead of $(x_1, \dots, x_n, y) \in p$. Powers of hyperalgebras were studied in [10] and those of algebras in [4]. Of course, if

$\mathbf{A}, \mathbf{B}$ are n -ary algebras (hyperalgebras), then the power $\mathbf{A}^{\diamond \mathbf{B}}$ need not be an n -ary algebra (hyperalgebra). So, it is an open problem to find conditions under which the three exponential laws or at least their weak forms are satisfied for n -ary algebras (hyperalgebras).

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Author’s address

N. Phrommarat

Department of Mathematics, Faculty of Science and Technology, Chiang Rai Rajabhat University,
57100 Chiang Rai, Thailand

E-mail address: nitimachaisansuk@gmail.com



ON THE CONHARMONIC CURVATURE TENSOR OF KENMOTSU MANIFOLDS WITH GENERALIZED TANAKA-WEBSTER CONNECTION

D. G. PRAKASHA AND B. S. HADIMANI

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Abstract. In this paper, we study a generalized Tanaka-Webster connection on a Kenmotsu manifold. We study the conharmonic curvature tensor with respect to the generalized Tanaka-Webster connection $\tilde{\nabla}$ and also characterize conharmonically flat and locally ϕ -conharmonically symmetric Kenmotsu manifold with respect to the connection $\tilde{\nabla}$. Besides these we also classify Kenmotsu manifolds which satisfy $\tilde{K} \cdot \tilde{R} = 0$ and $\tilde{P} \cdot \tilde{K} = 0$, where $\tilde{K}$ and $\tilde{P}$ are the conharmonic curvature tensor, the projective curvature tensor and Riemannian curvature tensor, respectively with respect to the connection $\tilde{\nabla}$.

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1. INTRODUCTION

In [15], Tanno classified almost contact metric manifold M whose automorphism group attains the maximum dimension. For such a manifold, the sectional curvature of plane section containing ξ is a constant, say c . (1) If $c > 0$, M is a homogeneous Sasakian manifold of constant ϕ -sectional curvature. (2) If $c = 0$, M is global Riemannian product of a line or circle with a Kähler manifold of constant holomorphic sectional curvature. (3) If $c < 0$, M is warped product space $\mathbb{R} \times_f \mathbb{C}^n$. In 1972, Kenmotsu [9] characterized the differential geometric properties of manifolds of class (3); the structure so obtained is now known as Kenmotsu structure. A Kenmotsu structure is not Sasakian (see [9]). Kenmotsu manifolds have also been studied in several papers [5, 8, 11] and the references therein.

On the other hand, Tanaka-Webster connection is canonical affine connection defined on a non-degenerate pseudo-Hermitian CR-manifold (see [14, 17]). The generalized Tanaka-Webster connection for contact metric manifolds by the canonical connection was first studied by Tanno [16]. This connection coincides with the Tanaka-Webster

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connection if the associated CR-structure is integrable. Acet et al [10] studied a Kenmotsu manifold with respect to the generalized Tanaka-Webster connection.

The paper is organized as follows: After preliminaries, in Section 3, we give a brief account of information regarding the generalized Tanaka-Webster connection $\tilde{\nabla}$ on Kenmotsu manifolds and obtain some results. In Section 4, we study a conharmonically flat Kenmotsu manifold with respect to the connection $\tilde{\nabla}$. Section 5 deals with the study of locally ϕ -conharmonically symmetric Kenmotsu manifold with respect to the connection $\tilde{\nabla}$. Sections 6 and 7 are devoted to the study of Kenmotsu manifolds with respect to the connection $\tilde{\nabla}$ satisfying the conditions $\tilde{K}(\xi, X) \cdot R = 0$ and $\tilde{P}(\xi, X) \cdot \tilde{R} = 0$, respectively.

2. PRELIMINARIES

An $n(= 2m + 1)$ -dimensional differentiable manifold M is called an almost contact Riemannian manifold if either its structural group can be reduced to $U(n) \times \{I\}$ or equivalently, there is an almost contact structure (ϕ, ξ, η) consisting of a $(1,1)$ tensor field ϕ , a vector field ξ , and 1-form η -satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad (2.1)$$

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \cdot \phi = 0. \quad (2.2)$$

Let g be Riemannian metric compatible with (ϕ, ξ, η) , that is

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (2.3)$$

or equivalently,

$$g(X, \phi Y) = -g(\phi X, Y) \text{ and } g(X, \xi) = \eta(X) \quad (2.4)$$

for any vector fields X, Y on M [2]. If moreover,

$$(\nabla_X \phi)Y = -\eta(Y)\phi X - g(X, \phi Y)\xi, \quad (2.5)$$

$$\nabla_X \xi = X - \eta(X)\xi, \quad (2.6)$$

where ∇ denotes the Riemannian connection of g hold, then (M, ϕ, ξ, η, g) is called an almost Kenmotsu manifold. An almost Kenmotsu manifold becomes a Kenmotsu manifold if

$$g(X, \phi Y) = d\eta(X, Y). \quad (2.7)$$

In a Kenmotsu manifold M , the following relation holds [9]:

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \quad (2.8)$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (2.9)$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \quad (2.10)$$

$$S(X, \xi) = -(n-1)\eta(X), \quad (2.11)$$

where R is the Riemannian curvature tensor and S is Ricci tensor defined by $S(X, Y) = g(QX, Y)$, where Q is Ricci operator.

A Kenmotsu manifold M is said to be an η -Einstein manifold if its Ricci tensor S of the form

$$S = ag + b\eta \otimes \eta, \quad (2.12)$$

for some smooth functions a and b .

Lemma 1 ([8]). *Any η -Einstein Kenmotsu manifold of dimension ≥ 5 with $b = \text{constant}$ is Einstein.*

3. GENERALIZED TANAKA-WEBSTER CONNECTION ON A KENMOTSU MANIFOLD

In the following, we consider the generalized Tanaka-Webster connection $\tilde{\nabla}$ for a Riemannian manifold M defined by

$$\tilde{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)Y \cdot \xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y, \quad (3.1)$$

for all vector fields X and Y , where ∇ is Levi-Civita connection on M .

If we use (2.6) and (2.8) in (3.1), we obtain

$$\tilde{\nabla}_X Y = \nabla_X Y + g(X, Y)\xi - \eta(Y)X + \eta(X)\phi Y, \quad (3.2)$$

for all vector fields X and Y . We call the connection $\tilde{\nabla}$ defined by (3.2) on a Kenmotsu manifold, *the generalized Tanaka-Webster connection on a Kenmotsu manifold*.

Let M be an n -dimensional Kenmotsu manifold. The curvature tensor $\tilde{R}$ of M with respect to the connection $\tilde{\nabla}$ is defined by

$$\tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X, Y]}Z. \quad (3.3)$$

Then, in a Kenmotsu manifold, we have

$$\tilde{R}(X, Y)Z = R(X, Y)Z + g(Y, Z)X - g(X, Z)Y, \quad (3.4)$$

where $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z$, is the curvature tensor of M with respect to the connection ∇ .

The Ricci tensor $\tilde{S}$ and the scalar curvature $\tilde{r}$ of the Kenmotsu manifold M with respect to the connection $\tilde{\nabla}$ is given by

$$\tilde{S}(X, Y) = \sum_{i=1}^n g(\tilde{R}(e_i, X)Y, e_i) = S(X, Y) + (n-1)g(X, Y) \quad (3.5)$$

and

$$\tilde{r} = \sum_{i=1}^n \tilde{S}(e_i, e_i) = r + n(n-1), \quad (3.6)$$

where $\tilde{r}$ and r are the scalar curvatures of the connection $\tilde{\nabla}$ and ∇ , respectively. So with the above background, we obtain the following theorem:

Theorem 1. *For a Kenmotsu manifold M with generalized Tanaka-Webster connection $\tilde{\nabla}$,*

- a) *the curvature tensor $\tilde{R}$ is given by (3.3),*
- b) *the Ricci tensor $\tilde{S}$ is given by (3.5),*
- c) $\tilde{R}(X, Y)Z + \tilde{R}(Y, Z)X + \tilde{R}(Z, X)Y = 0,$
- d) $\tilde{R}(X, Y, Z, W) + \tilde{R}(X, Y, W, Z) = 0,$
- e) $\tilde{R}(X, Y, Z, W) + \tilde{R}(Y, X, Z, W) = 0,$
- f) $\tilde{R}(X, Y, Z, W) - \tilde{R}(Z, W, X, Y) = 0,$
- g) $\tilde{R}(X, Y)\xi = \tilde{R}(\xi, X)Y = \tilde{R}(\xi, X)\xi = 0,$
- h) $\tilde{S}(X, \xi) = 0,$
- i) $\tilde{r} = r + n(n-1),$
- j) *The Ricci tensor $\tilde{S}$ is symmetric.*

Now we begin with the following:

Corollary 1. *If a Kenmotsu manifold is Ricci-flat with respect to generalized Tanaka-webster connection, then it is an Einstein manifold.*

Proof: The Proof follows immediately from (3.5).

Theorem 2. *Let M be a Kenmotsu manifold. If the curvature tensor $\tilde{R}$ of the generalized Tanaka-Webster connection $\tilde{\nabla}$ vanishes, then M is locally isomorphic to the hyperbolic space $H^n(-1)$.*

Proof: Let the curvature tensor $\tilde{R}$ of the connection $\tilde{\nabla}$ vanishes. That is, $\tilde{R} = 0$. In view of Eq.(3.4), we have

$$R(X, Y)Z = g(X, Z)Y - g(Y, Z)X. \quad (3.7)$$

This gives

$$R(X, Y, Z, W) = -[g(Y, Z)g(X, W) - g((X, Z)g(Y, W))]. \quad (3.8)$$

This shows that, M is of constant negative curvature -1.

A space form is said to be hyperbolic if and only if the sectional curvature is negative [3]. Thus, M is locally isometric to the hyperbolic space $H^n(-1)$.

Theorem 3. *Let M be a Kenmotsu manifold. If M is of constant curvature c with respect to the connection $\tilde{\nabla}$, then M is of constant curvature $c - 1$ with respect to the connection ∇ .*

Proof: Let M be of constant curvature c . Then from (3.3) we have

$$\tilde{R}(X, Y, Z, W) = c[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \quad (3.9)$$

By taking account of (3.9) in (3.4), it follows that

$$R(X, Y, Z, W) = (c - 1)[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \quad (3.10)$$

This shows that, M is of constant scalar curvature $c - 1$. This completes the proof.

The notion of a conharmonic curvature tensor was first studied by Ishii [7]. A rank four tensor K' that remains invariant under conharmonic transformation for an n -dimensional Riemannian Manifold M , is given by

$$\begin{aligned} K'(X, Y, Z, W) = & R'(X, Y, Z, W) \\ & - \frac{1}{2n-1} [g(Y, Z)S(X, W) - g(X, Z)S(Y, W) \\ & + S(Y, Z)g(X, W) - S(X, Z)g(Y, W)], \end{aligned} \quad (3.11)$$

where R' denotes the Riemannian curvature tensor type $(0, 4)$ and K' denotes the conharmonic curvature tensor of type $(0, 4)$ defined by

$$R'(X, Y, Z, W) = g(R(X, Y)Z, W), \quad (3.12)$$

$$K'(X, Y, Z, W) = g(K(X, Y)Z, W). \quad (3.13)$$

where R is the Riemannian tensor of type $(0, 3)$, K is the conharmonic curvature tensor of type $(0, 3)$ and S denotes the Ricci tensor of type $(0, 2)$. The curvature tensor defined by Eq.(3.11) is known as conharmonic curvature tensor [7]. A manifold whose conharmonic curvature vanishes at every point of the manifold is called conharmonically flat manifold. Thus this tensor represents the deviation of the manifold from conharmonic flatness. It satisfies all the symmetry properties of the Riemannian curvature tensor R . There are many physical applications of tensor K . For example, we refer the readers to see [1]. A conharmonic curvature tensor on a Kenmotsu manifold has been studied in [4].

Analogous to the conharmonic curvature tensor K with respect to Levi-Civita connection ∇ , we give the conharmonic curvature tensor $\tilde{K}$ with respect to generalized Tanaka-Webster connection $\tilde{\nabla}$.

In a Kenmotsu manifold M of dimension $n > 2$, the conharmonic curvature tensor $\tilde{K}$ with respect to the Tanaka-Webster connection $\tilde{\nabla}$ is given by

$$\begin{aligned} \tilde{K}(X, Y)Z = & \tilde{R}(X, Y)Z - \frac{1}{(n-2)} [\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y \\ & + g(Y, Z)\tilde{Q}X - g(X, Z)\tilde{Q}Y] \end{aligned} \quad (3.14)$$

for all vector fields X, Y and Z on M , where $\tilde{R}$, $\tilde{S}$ and $\tilde{Q}$ are the Riemannian curvature tensor, Ricci tensor and Ricci operator, respectively with respect to the connection $\tilde{\nabla}$.

Using (3.4) and (3.5) in (3.14), we get

$$\tilde{K}(X, Y)Z = K(X, Y)Z + [g(Y, Z)X - g(X, Z)Y] \quad (3.15)$$

or

$$\tilde{K}(X, Y)Z = R(X, Y)Z - \frac{n}{n-2} [g(Y, Z)X - g(X, Z)Y]$$

$$\begin{aligned}
& -\frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y \\
& + g(Y, Z)QX - g(X, Z)QY].
\end{aligned} \tag{3.16}$$

In a Kenmotsu manifold, using (2.10) and (2.11), the equation (3.16) gives

$$\begin{aligned}
\widetilde{K}(\xi, Y)Z &= \frac{(n-3)}{(n-2)}[\eta(Z)Y - g(Y, Z)\xi] \\
&+ \frac{1}{(n-2)}[\eta(Z)QY - S(Y, Z)\xi] = -\widetilde{K}(Y, \xi)Z
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
\eta(\widetilde{K}(X, Y)Z) &= \frac{(n-3)}{(n-2)}[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)] \\
&+ \frac{1}{(n-2)}[S(X, Z)\eta(Y) - S(Y, Z)\eta(X)].
\end{aligned} \tag{3.18}$$

4. CONHARMONICALLY FLAT KENMOTSU MANIFOLD WITH RESPECT TO THE CONNECTION $\widetilde{\nabla}$

A conharmonic curvature tensor K with respect to Levi-Civita connection ∇ is said to be flat if it vanishes identically (that is, $K = 0$) with respect to the connection ∇ . A conharmonically flat Kenmotsu manifold with respect to the semi-symmetric metric connection has been studied in [12].

Assume that, M is conharmonically flat Kenmotsu manifold with respect to the connection ∇ . That is, $\widetilde{K} = 0$. Then from (3.14), we have

$$\begin{aligned}
\widetilde{R}(X, Y)Z &= \frac{1}{n-2}[g(Y, Z)\widetilde{Q}X - g(X, Z)\widetilde{Q}Y \\
&+ \widetilde{S}(Y, Z)Y - \widetilde{S}(X, Z)X].
\end{aligned} \tag{4.1}$$

This gives,

$$K(X, Y)Z = [g(Y, Z)X - g(X, Z)Y] \tag{4.2}$$

or equivalently,

$$\begin{aligned}
R(Y, X)Z &= \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\
&+ \frac{n}{(n-2)}[g(Y, Z)X - g(X, Z)Y].
\end{aligned} \tag{4.3}$$

Taking inner product with W in (4.3), then

$$\begin{aligned}
& R'(X, Y, Z, W) \\
&= \frac{1}{(n-2)}[S(Y, Z)g(X, W) - S(X, Z)g(Y, W) + g(Y, Z)S(X, W)
\end{aligned}$$

$$-g(X, Z)S(Y, W)] + \frac{n}{n-2}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \quad (4.4)$$

where $R'(X, Y, Z, W) = g(R(X, Y)Z, W)$.

Putting $X = W = \xi$ in (4.4) and using (2.10) and (2.11), we have

$$S(Y, Z) = -(n-1)g(Y, Z). \quad (4.5)$$

Thus, we can state the following:

Theorem 4. *A conharmonically flat Kenmotsu manifold $M(n > 2)$ with respect to generalized Tanaka-Webster connection $\tilde{\nabla}$ is an Einstein manifold.*

In view of (4.5), (4.3) reduces to

$$R(Y, X)Z = -[g(Y, Z)X - g(X, Z)Y]. \quad (4.6)$$

That is, M is locally isometric to the locally hyperbolic space $H^n(-1)$.

On the other hand, If M is locally isometric to the hyperbolic space $H^n(-1)$. Then (4.6) holds.

By the virtue of (4.6), (3.2) gives

$$\tilde{R}(X, Y)Z = 0. \quad (4.7)$$

Similarly, by taking account of (4.5) in (3.5), we have

$$\tilde{S}(Y, Z) = 0. \quad (4.8)$$

Using (4.7) and (4.8) in (3.14) we obtain

$$\tilde{K}(X, Y)Z = 0. \quad (4.9)$$

Thus, M is conharmonically flat with respect to the connection $\tilde{\nabla}$. This leads to the following:

Theorem 5. *An n -dimensional Kenmotsu manifolds M ($n > 2$) is conharmonically flat with respect to the generalized Tanaka-Webster connection $\tilde{\nabla}$ if and only if it is locally isometric to the hyperbolic sphere $H^n(-1)$.*

5. LOCALLY ϕ -CONHARMONICALLY SYMMETRIC KENMOTSU MANIFOLDS WITH RESPECT TO THE CONNECTION $\tilde{\nabla}$

The notion of locally ϕ -symmetry was first studied by Takahashi [13] on a Sasakian manifold. In this section we consider a locally ϕ -conharmonically symmetric Kenmotsu manifolds with respect to the connection $\tilde{\nabla}$.

Definition 1. An Kenmotsu manifold M is said to be locally ϕ -conharmonically symmetric with respect to the generalized Tanaka-Webster connection $\tilde{\nabla}$ if the conharmonic curvature tensor $\tilde{K}$ with respect to the connection $\tilde{\nabla}$ satisfies

$$\phi^2((\tilde{\nabla}_W \tilde{K})(X, Y)Z) = 0, \quad (5.1)$$

where X, Y, Z and W are horizontal vector fields on M .

From (3.2), we have that

$$\begin{aligned}(\tilde{\nabla}_W \tilde{K})(X, Y)Z &= (\nabla_W \tilde{K})(X, Y)Z + g(W, \tilde{K}(X, Y)Z)\xi \\ &\quad - \eta(\tilde{K}(X, Y)Z)W + \eta(W)\phi \tilde{K}(X, Y)Z.\end{aligned}\quad (5.2)$$

Now, Differentiating (3.15) in the direction of W , we get

$$(\nabla_W \tilde{K})(X, Y)Z = (\nabla_W K)(X, Y)Z. \quad (5.3)$$

Then, using (4.3) and (5.3) in (5.2), we have

$$\begin{aligned}&(\tilde{\nabla}_W \tilde{K})(X, Y)Z \\ &= (\nabla_W K)(X, Y)Z + g(W, \tilde{K}(X, Y)Z)\xi \\ &\quad + \eta(W)\phi \tilde{K}(X, Y)Z - \frac{(n-3)}{(n-2)}[g(X, Z)\eta(Y)W - g(Y, Z)\eta(X)W] \\ &\quad - \frac{1}{(n-2)}[S(X, Z)\eta(Y)W - S(Y, Z)\eta(X)W].\end{aligned}\quad (5.4)$$

Applying ϕ^2 on both sides of (5.4) and using (2.1) and (2.2), we obtain

$$\begin{aligned}&\phi^2((\tilde{\nabla}_W \tilde{K})(X, Y)Z) \\ &= \phi^2((\nabla_W K)(X, Y)Z) - \eta(W)\phi \tilde{K}(X, Y)Z \\ &\quad + \frac{(n-3)}{(n-2)}[g(X, Z)\eta(Y)W - g(Y, Z)\eta(X)W] \\ &\quad + \frac{1}{(n-2)}[S(X, Z)\eta(Y)W - S(Y, Z)\eta(X)W] \\ &\quad - \frac{(n-3)}{(n-2)}[g(X, Z)\eta(Y)\eta(W)\xi - g(Y, Z)\eta(X)\eta(W)\xi] \\ &\quad - \frac{1}{(n-2)}[S(X, Z)\eta(Y)\eta(W)\xi - S(Y, Z)\eta(X)\eta(W)\xi].\end{aligned}\quad (5.5)$$

Now, if X, Y, W are horizontal vector fields, then the above equation reduces to

$$\phi^2((\tilde{\nabla}_W \tilde{K})(X, Y)Z) = \phi^2(\nabla_W K)(X, Y)Z. \quad (5.6)$$

This shows that, M is locally conharmonically ϕ -symmetric with respect to the connection $\tilde{\nabla}$ if and only if it is so with respect to the connection ∇ .

Hence we state the following:

Theorem 6. *A Kenmotsu manifold $M(n > 3)$ is locally ϕ -conharmonically symmetric with respect to generalized Tanaka-Webster connection $\tilde{\nabla}$ if and only if it is so with respect to the Levi-Civita connection ∇ .*

6. KENMOTSU MANIFOLD WITH RESPECT TO THE CONNECTION $\widetilde{\nabla}$ SATISFYING $\widetilde{K}(\xi, X) \cdot \widetilde{R} = 0$

In this section consider a Kenmotsu manifold M satisfying the condition

$$\widetilde{K}(\xi, X) \cdot \widetilde{R} = 0. \quad (6.1)$$

We define $\widetilde{K}(\xi, X) \cdot \widetilde{R}$ by

$$\begin{aligned} & (\widetilde{K}(\xi, X) \cdot \widetilde{R})(Y, Z)W \\ &= \widetilde{K}(\xi, X) \cdot \widetilde{R}(Y, Z)W - \widetilde{R}(\widetilde{K}(\xi, X)Y, Z)W \\ & \quad - \widetilde{R}(Y, \widetilde{K}(\xi, X)Z)W - \widetilde{R}(Y, Z)\widetilde{K}(\xi, X)W. \end{aligned} \quad (6.2)$$

From (6.1) and (6.2), we have

$$\begin{aligned} & (\widetilde{K}(\xi, X) \cdot \widetilde{R})(\xi, Z)W - \widetilde{R}(\widetilde{K}(\xi, X)\xi, Z)W \\ & - \widetilde{R}(\xi, \widetilde{K}(\xi, X)Z)W - \widetilde{R}(\xi, Z)\widetilde{K}(\xi, X)W = 0. \end{aligned} \quad (6.3)$$

Using the property(g) of Theorem 3.1 in (6.3), we obtain

$$\widetilde{R}(\widetilde{K}(\xi, X)\xi, Z)W = 0, \quad (6.4)$$

which on using (3.17), gives

$$\frac{(n-3)}{(n-2)}\widetilde{R}(X, Z)W + \frac{1}{(n-2)}\widetilde{R}(QX, Z)W = 0. \quad (6.5)$$

Taking inner-product of (6.5) with U and using (3.4) we get

$$\begin{aligned} & \frac{(n-3)}{(n-2)}[R(X, Z, W, U) + g(Z, W)g(X, U) - g(X, W)g(Z, U)] \\ & + \frac{1}{(n-2)}[R(QX, Z)W, U) + g(Z, W)S(X, U) - S(X, W)g(Z, U)] \\ & = 0. \end{aligned} \quad (6.6)$$

Let $\{e_i\} (1 \leq i \leq n)$ an orthonormal basis of the tangent space at any point of the manifold. Putting $Z = W = e_i$ in (6.6) and taking summation over i , $1 \leq i \leq n$, we get

$$S^2(X, U) = -2S(X, U) - (n-1)(n-3)g(X, U). \quad (6.7)$$

This leads to the following:

Proposition 1. *In an n -dimensional ($n > 3$) Kenmotsu manifold M with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ if the condition $\widetilde{K}(\xi, X) \cdot R = 0$ holds on M , then the equation (6.7) is satisfied on M .*

Now we need the following:

Lemma 2 ([6]). *Let A be symmetric $(0, 2)$ -tensor at a point X of a Semi-Riemannian manifold (M^n, g) , $(n > 1)$, and let $T = g \bar{\wedge} A$ be the Kulkarni-Nomizu product of g and A , Then, the relation.*

$$T \cdot T = \alpha Q(g, T), \quad \alpha \in \mathbb{R} \quad (6.8)$$

is satisfied at x if and only if the condition

$$A^2 = \alpha A + \lambda g, \quad \lambda \in \mathbb{R} \quad (6.9)$$

holds at x .

From Proposition 1 and Lemma 2, we have the following:

Corollary 2. *Let M be an n -dimensional $(n > 3)$ Kenmotsu manifold with respect to the generalized Tanaka-Webster connection $\tilde{\nabla}$ satisfying the condition $\tilde{K}(\xi, X) \cdot \tilde{R} = 0$. Then $T \cdot T = \alpha Q(g, T)$, where $T = g \bar{\wedge} A$ and $\alpha = -2$.*

7. KENMOTSU MANIFOLDS WITH RESPECT TO THE CONNECTION $\tilde{\nabla}$ SATISFYING $\tilde{P}(\xi, X) \cdot \tilde{K} = 0$

We consider a Kenmotsu manifold M satisfying the condition

$$\tilde{P}(\xi, X) \cdot \tilde{K} = 0, \quad (7.1)$$

where $\tilde{P}$ is the projective curvature tensor with respect to the connection $\tilde{\nabla}$ given by

$$\tilde{P}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{(n-1)}[\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y]. \quad (7.2)$$

Now we define $\tilde{P} \cdot \tilde{K}$ by

$$\begin{aligned} & (\tilde{P}(\xi, X) \cdot \tilde{K})(Y, Z)W \\ &= \tilde{P}(\xi, X)\tilde{K}(Y, Z)W - \tilde{K}(\tilde{P}(\xi, X)Y, Z)W \\ & \quad - \tilde{K}(Y, \tilde{P}(\xi, X)Z)W - \tilde{K}(Y, Z)\tilde{P}(\xi, X)W. \end{aligned} \quad (7.3)$$

From (7.1) and (7.3) we have

$$(\tilde{P}(\xi, X) \cdot \tilde{K})(Y, Z)W = 0. \quad (7.4)$$

For a Kenmotsu manifold M , we obtain from (7.2) that

$$\tilde{P}(\xi, Y)Z = -\frac{1}{(n-1)}S(Y, Z)\xi - g(Y, Z)\xi. \quad (7.5)$$

Taking the inner product with U in (7.4) and using (7.5) we obtain

$$\begin{aligned} & g(X, \tilde{K}(Y, Z)W)\eta(U) - g(X, Y)g(\tilde{K}(\xi, Z)W, U) \\ & - g((X, Z)g(\tilde{K}(Y, \xi)W, U) - g(X, W)g(\tilde{K}(Y, Z)\xi, Y) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(n-1)} [S(X, \tilde{K}(Y, Z)W)\eta(Y) - S(X, Y)g(\tilde{K}(\xi, V)W, U) \\
& - S(X, Z)g(\tilde{K}(Y, \xi)W, U) - S(X, W)g(\tilde{K}(Y, Z)\xi, U)] = 0.
\end{aligned} \quad (7.6)$$

Taking $Z = \xi$ in (7.6), we have

$$\begin{aligned}
& g(X, \tilde{K}(Y, \xi)W)\eta(U) - g(X, W)g(\tilde{K}(Y, \xi)\xi, U) \\
& + \frac{1}{(n-1)} [S(X, \tilde{K}(Y, \xi)W)\eta(U) - S(X, W)g(\tilde{K}(Y, \xi)\xi, U)] = 0.
\end{aligned} \quad (7.7)$$

Using (3.17) in (7.7), we get

$$\begin{aligned}
& \frac{(n-3)}{(n-2)} [g(X, Y)\eta(U)\eta(W) + \frac{1}{(n-1)} S(X, Y)\eta(U)\eta(W) \\
& + g(X, W)\eta(U)\eta(Y) - g(X, W)g(U, Y) \\
& + \frac{1}{(n-1)} S(X, W)\eta(U)\eta(Y) - \frac{1}{(n-1)} S(X, W)g(U, Y)] \\
& + \frac{1}{(n-2)} [S(X, Y)\eta(U)\eta(W) + \frac{1}{(n-1)} S(QX, Y)\eta(U)\eta(W) \\
& - S(U, Y)g(X, W) - (n-1)g(X, W)\eta(U)\eta(Y) \\
& - \frac{1}{n-1} S(X, W)S(U, Y) - S(X, W)\eta(U)\eta(Y)] = 0,
\end{aligned} \quad (7.8)$$

where $S(QX, Y) = S^2(X, Y)$.

Let $\{e_i\}$ ($1 \leq i \leq n$) be orthonormal basis of the tangent space at any point. Then the sum for ($1 \leq i \leq n$) of the relation (7.8) for $X = W = e_i$ gives

$$\begin{aligned}
& (r + n(n-1)) \left[\frac{-1}{(n-2)} S(U, Y) - \left(\frac{n-3}{n-2} \right) g(U, Y) + \frac{-2}{n-2} \eta(U)\eta(Y) \right] \\
& = 0.
\end{aligned} \quad (7.9)$$

This implies, either $r + n(n-1) = 0$ or

$$S(Y, U) = -(n-3)g(Y, U) - 2\eta(Y)\eta(U). \quad (7.10)$$

If $r + n(n-1) = 0$, then from (3.6) we have $\tilde{r} = 0$.

Next, if the equation (7.10) holds, then in the view of Lemma 1, M is an Einstein manifold. Hence, we state that the following:

Theorem 7. *Let M be an n -dimensional ($n > 3$) Kenmotsu manifold with respect to the generalized Tanaka-Webster connection $\tilde{\nabla}$ satisfying the condition $\tilde{P}(\xi, X) \cdot \tilde{K} = 0$. Then either $\tilde{r} = 0$, that is, the scalar curvature with respect to the connection $\tilde{\nabla}$ vanishes or M is an Einstein manifold.*

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Authors' addresses

D. G. Prakasha

Department of Mathematics, Karnatak University, Dharwad - 580003, India

E-mail address: prakashadg@email.com, dgprakasha@kud.ac.in

B. S. Hadimani

Department of Mathematics, Karnatak University, Dharwad - 580003, India.

Current address: Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal - 576104, India.

E-mail address: balachandrash@gmail.com, bs.hadimani@manipal.edu



ON K -CIRCULANT MATRICES INVOLVING THE FIBONACCI NUMBERS

BILJANA RADIČIĆ

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Abstract. Let k be a nonzero complex number. In this paper we consider a k -circulant matrix whose first row is $(F_1, F_2, \dots, F_n)$, where F_n is the n^{th} Fibonacci number, and investigate the eigenvalues and Euclidean (or Frobenius) norm of that matrix. Also, the upper and lower bounds for the spectral norm of the Hadamard inverse of that matrix are obtained.

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1. INTRODUCTION

In this paper, k is a nonzero complex number and $\mathbb{C}^{n \times n}$ denotes the set of all complex matrices of order n . Any n^{th} root of k and any primitive n^{th} root of unity are denoted by ψ and ω , respectively. Symbols $\lambda_j, j = \overline{0, n-1}$, $|C|$, $\|C\|_E$, $\|C\|_2$ and $C^{\circ-1}$ stand for the eigenvalues, the determinant, the Euclidean norm, the spectral norm and the Hadamard inverse of $C \in \mathbb{C}^{n \times n}$, respectively. Namely, for $C = [c_{i,j}] \in \mathbb{C}^{n \times n}$,

$\|C\|_E = (\sum_{i,j=1}^n |c_{i,j}|^2)^{\frac{1}{2}}$, $\|C\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i(C^*C)}$, where C^* is the conjugate transpose of C , and $C^{\circ-1} = [c_{i,j}^{-1}]$.

Definition 1. A matrix C of order n with the first row $(c_0, c_1, c_2, \dots, c_{n-1})$ is called a k -circulant matrix if C has the following form:

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ kc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ kc_{n-2} & kc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ kc_2 & kc_3 & kc_4 & \cdots & c_0 & c_1 \\ kc_1 & kc_2 & kc_3 & \cdots & kc_{n-1} & c_0 \end{bmatrix}. \quad (1.1)$$

We shall write $C = \text{circ}_n\{c_0, c_1, c_2, \dots, c_{n-1}\}$ if a matrix C has the form (1.1). The designation for the order of a matrix can be omitted if the dimension of a matrix is known. Circulant matrices are k -circulant matrices for $k = 1$ and skew circulant matrices are k -circulant matrices for $k = -1$.

The Fibonacci numbers $\{F_n\}$ satisfy the following recursive relation:

$$F_n = F_{n-2} + F_{n-1}, \quad n \geq 2, \quad (1.2)$$

with initial conditions $F_0 = 0$ and $F_1 = 1$.

Let α and β be the roots of the equation $x^2 - x - 1 = 0$ i.e.

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}, \quad \alpha\beta = -1, \quad \alpha + \beta = 1 \quad \text{and} \quad \alpha - \beta = \sqrt{5}. \quad (1.3)$$

Binet's formula for the Fibonacci numbers is:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right). \quad (1.4)$$

Let us mention that the Lucas numbers $\{L_n\}$ satisfy the same recursive relation (as the Fibonacci numbers) but with initial conditions $L_0 = 2$ and $L_1 = 1$ and

$$L_n = \alpha^n + \beta^n = \left(\frac{1 + \sqrt{5}}{2} \right)^n + \left(\frac{1 - \sqrt{5}}{2} \right)^n \quad (1.5)$$

is Binet's formula for the Lucas numbers.

The following identities hold for the Fibonacci numbers:

$$\sum_{i=1}^n F_i = F_{n+2} - 1 \quad \text{and} \quad \sum_{i=1}^n F_i^2 = F_n F_{n+1} \left[= \frac{L_{2n+1} - (-1)^n}{5} \right]. \quad (1.6)$$

More information about these numbers can be found in [2, 3, 8–10, 12, 14, 15].

In [4, 13] the authors investigated the determinants and inverses of circulant (of skew circulant) matrices whose first rows are $(F_1, F_2, \dots, F_n)$ and $(L_1, L_2, \dots, L_n)$. The paper [5] is devoted to k -circulant matrices with the Fibonacci and Lucas numbers, and an upper bound estimation of the spectral norm for such matrices was given in that paper.

The motivation for this paper is the paper [17] in which the authors investigated k -circulant matrices with the generalized r -Horadam numbers $\{H_{r,n}\}$ which are defined as follows:

$$H_{r,n+2} = f(r)H_{r,n+1} + g(r)H_{r,n}, \quad n \geq 0,$$

where $r \in \mathbb{R}^+$, $H_{r,0} = a$, $H_{r,1} = b$, $a, b \in \mathbb{R}$ and $f^2(r) + 4g(r) > 0$, and presented the upper and lower bounds for the spectral norms of such matrices.

Theorem 1 (Theorem 5. [17]). Let $H = \text{circ}_k(H_{r,0}, H_{r,1}, \dots, H_{r,n-1})$.

a) If $|k| \geq 1$, then

$$\sqrt{\sum_{i=0}^{n-1} H_{r,i}^2} \leq \|H\|_2 \leq \sqrt{\left(a^2(1-|k|^2) + |k|^2 \sum_{i=0}^{n-1} H_{r,i}^2\right) \left(1-a^2 + \sum_{i=0}^{n-1} H_{r,i}^2\right)} \quad (1.7)$$

b) If $|k| < 1$, then

$$|k| \sqrt{\sum_{i=0}^{n-1} H_{r,i}^2} \leq \|H\|_2 \leq \sqrt{n \sum_{i=0}^{n-1} H_{r,i}^2}. \quad (1.8)$$

Also, in [17], the formulae for the eigenvalues and determinant of a k -circulant matrix with the generalized r -Horadam numbers were derived.

Theorem 2 (Theorem 7. [17]). Let $H = \text{circ}_k(H_{r,0}, H_{r,1}, \dots, H_{r,n-1})$. Then the eigenvalues of H are:

$$\lambda_j = \frac{kH_{r,n} + (g(r)kH_{r,n-1} - b + af(r))\psi\omega^{-j} - H_{r,0}}{g(r)(\psi\omega^{-j})^2 + f(r)\psi\omega^{-j} - 1}, \quad j = \overline{0, n-1}. \quad (1.9)$$

Theorem 3 (Theorem 8. [17]). Let $H = \text{circ}_k(H_{r,0}, H_{r,1}, \dots, H_{r,n-1})$. Then the determinant of H is:

$$|H| = \frac{(H_{r,0} - kH_{r,n})^n - (g(r)kH_{r,n-1} - b + af(r))^n k}{(1 - k\alpha^n)(1 - k\beta^n)}, \quad (1.10)$$

where α and β are the roots of the equation $x^2 - f(r)x - g(r) = 0$.

Let us also mention that, in [16], the authors considered circulant matrices with the generalized r -Horadam numbers and obtained the determinants and inverses of such matrices.

In the present paper, we consider the matrix

$$F = \text{circ}_k(F_1, F_2, \dots, F_n) \quad (1.11)$$

and improve the result in relation to the eigenvalues of (1.11) which can be obtained from (1.9) because the authors did not consider the case when the denominator is equal to zero. Also, we determine the Euclidean norm of (1.11) and derive the upper and lower bounds for the spectral norm of the Hadamard inverse of (1.11). The results are presented in the next section.

2. RESULTS

Let us recall that ψ is any n^{th} root of k and ω is any primitive n^{th} root of unity. Also, throughout this section, α and β are the roots of the equation $x^2 - x - 1 = 0$. In order to obtain the eigenvalues of (1.11), we need the following lemma.

Lemma 1 (Lemma 4. [1]). *The eigenvalues of $C = circ\{c_0, c_1, c_2, \dots, c_{n-1}\}$ are:*

$$\lambda_j = \sum_{i=0}^{n-1} c_i (\psi \omega^{-j})^i, \quad j = \overline{0, n-1}. \quad (2.1)$$

Moreover, in this case:

$$c_i = \frac{1}{n} \sum_{j=0}^{n-1} \lambda_j (\psi \omega^{-j})^{-i}, \quad i = \overline{0, n-1}. \quad (2.2)$$

Theorem 4. *Let F be the matrix as in (1.11). Then the eigenvalues of F are given by the following formulae:*

1) If $\psi \omega^{-j} = \frac{1}{\alpha}$, then

$$\lambda_j = \frac{1}{\sqrt{5}} \left[n\alpha + \frac{1 - (-1)^n \beta^{2n}}{\sqrt{5}} \right], \quad (2.3)$$

2) If $\psi \omega^{-j} = \frac{1}{\beta}$, then

$$\lambda_j = \frac{1}{\sqrt{5}} \left[\frac{1 - (-1)^n \alpha^{2n}}{\sqrt{5}} - n\beta \right], \quad (2.4)$$

3) If $\psi \omega^{-j} \neq \frac{1}{\alpha}$ and $\psi \omega^{-j} \neq \frac{1}{\beta}$, then

$$\lambda_j = \frac{kF_{n+1} - 1 + kF_n \psi \omega^{-j}}{(\psi \omega^{-j})^2 + \psi \omega^{-j} - 1}. \quad (2.5)$$

Proof. Based on Lemma 1 and (1.4), it follows:

1) Suppose that $\psi \omega^{-j} = \frac{1}{\alpha}$. Then,

$$\begin{aligned} \lambda_j &= \sum_{i=0}^{n-1} F_{i+1} (\psi \omega^{-j})^i = \frac{1}{\sqrt{5}} \sum_{i=0}^{n-1} \left[\alpha^{i+1} - \beta^{i+1} \right] \left(\frac{1}{\alpha} \right)^i \\ &= \frac{1}{\sqrt{5}} \left[\alpha \sum_{i=0}^{n-1} 1 - \beta \sum_{i=0}^{n-1} \left(\frac{\beta}{\alpha} \right)^i \right] = \frac{1}{\sqrt{5}} \left[n\alpha - \beta \frac{1 - \left(\frac{\beta}{\alpha} \right)^n}{1 - \frac{\beta}{\alpha}} \right] \\ &= \frac{1}{\sqrt{5}} \left[n\alpha + \frac{1 - (-1)^n \beta^{2n}}{\sqrt{5}} \right], \end{aligned}$$

2) Suppose that $\psi\omega^{-j} = \frac{1}{\beta}$. Then,

$$\begin{aligned}\lambda_j &= \sum_{i=0}^{n-1} F_{i+1}(\psi\omega^{-j})^i = \frac{1}{\sqrt{5}} \sum_{i=0}^{n-1} [\alpha^{i+1} - \beta^{i+1}] \left(\frac{1}{\beta}\right)^i \\ &= \frac{1}{\sqrt{5}} \left[\alpha \sum_{i=0}^{n-1} \left(\frac{\alpha}{\beta}\right)^i - \beta \sum_{i=0}^{n-1} 1 \right] = \frac{1}{\sqrt{5}} \left[\alpha \frac{1 - (\frac{\alpha}{\beta})^n}{1 - \frac{\alpha}{\beta}} - n\beta \right] \\ &= \frac{1}{\sqrt{5}} \left[\frac{1 - (-1)^n \alpha^{2n}}{\sqrt{5}} - n\beta \right],\end{aligned}$$

3) Suppose that $\psi\omega^{-j} \neq \frac{1}{\alpha}$ and $\psi\omega^{-j} \neq \frac{1}{\beta}$. Then, λ_j follows from (1.9). □

The previously obtained result will be illustrated by the following example.

Example 1. Let

$$F = \text{circ}\{_{9-4\sqrt{5}}(1, 1, 2, 3, 5, 8)\}$$

i.e.

$$F = \begin{bmatrix} 1 & 1 & 2 & 3 & 5 & 8 \\ 8(9-4\sqrt{5}) & 1 & 1 & 2 & 3 & 5 \\ 5(9-4\sqrt{5}) & 8(9-4\sqrt{5}) & 1 & 1 & 2 & 3 \\ 3(9-4\sqrt{5}) & 5(9-4\sqrt{5}) & 8(9-4\sqrt{5}) & 1 & 1 & 2 \\ 2(9-4\sqrt{5}) & 3(9-4\sqrt{5}) & 5(9-4\sqrt{5}) & 8(9-4\sqrt{5}) & 1 & 1 \\ 9-4\sqrt{5} & 2(9-4\sqrt{5}) & 3(9-4\sqrt{5}) & 5(9-4\sqrt{5}) & 8(9-4\sqrt{5}) & 1 \end{bmatrix}.$$

Since $n = 6$ and $k = 9 - 4\sqrt{5}$ i.e. $\psi = -\beta$ and $\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, based on Theorem 4, it follows that

★: $\psi\omega^0 = \frac{1}{\alpha}$, so λ_0 is obtained based on 1) of Theorem 4: $\lambda_0 = -29 + 15\sqrt{5}$;

★: $\psi\omega^{-j} \neq \frac{1}{\alpha}$ and $\psi\omega^{-j} \neq \frac{1}{\beta}$, for $j = \overline{1, 5}$, so λ_j , for $j = \overline{1, 5}$, are obtained based on 3) of Theorem 4: $\lambda_{1,5} = -\frac{1}{2} \left[51 - 23\sqrt{5} \pm i\sqrt{3}(29 - 13\sqrt{5}) \right]$, $\lambda_{2,4} = 7 - 3\sqrt{5} \mp i\sqrt{3}(29 - 13\sqrt{5})$, $\lambda_3 = 72 - 32\sqrt{5}$

Bearing in mind that $|F| = \prod_{j=0}^{n-1} \lambda_j$, it follows that

$$|F| = -238300041216 - 106571018240\sqrt{5}.$$

Let us remark, in relation to the previous example, that the determinant of $F = \text{circ}\{9_{-4}\sqrt{5}(1, 1, 2, 3, 5, 8)\}$ is not possible to obtain using the result of Theorem 3.

The next theorem is devoted to determining the Euclidean norm of (1.11). The following formula will be needed.

For all x ,

$$\sum_{i=1}^{n-1} ix^i = \frac{x - nx^n + (n-1)x^{n+1}}{(1-x)^2}. \quad (2.6)$$

Theorem 5. *Let F be the matrix as in (1.11). Then the Euclidean norm of F is:*

$$\|F\|_E = \sqrt{\frac{1}{5} \left\{ n[L_{2n+1} - (-1)^n] + (|k|^2 - 1) \left[-L_{2n} + (n-1)L_{2n+1} + \frac{5}{2} + \frac{1-2n}{2}(-1)^n \right] \right\}}. \quad (2.7)$$

Proof. From the definition of the Euclidean norm, using (1.4), (1.5), (1.6) and (2.6), we obtain:

$$\begin{aligned} & (\|F\|_E)^2 \\ &= \sum_{i,j=1}^n |f_{i,j}|^2 \\ &= nF_1^2 + [(n-1) + |k|^2]F_2^2 + [(n-2) + 2|k|^2]F_3^2 + \cdots + [1 + (n-1)|k|^2]F_n^2 \\ &= \sum_{i=0}^{n-1} (n-i)F_{i+1}^2 + |k|^2 \sum_{i=1}^{n-1} iF_{i+1}^2 \\ &= n \sum_{i=0}^{n-1} F_{i+1}^2 + (|k|^2 - 1) \sum_{i=1}^{n-1} iF_{i+1}^2 \\ &= \frac{1}{5} \left[n[L_{2n+1} - (-1)^n] + (|k|^2 - 1) \sum_{i=1}^{n-1} i \left[\alpha^{2i+2} - 2(\alpha\beta)^{i+1} + \beta^{2i+2} \right] \right] \\ &= \frac{n}{5} [L_{2n+1} - (-1)^n] + \frac{|k|^2 - 1}{5} \left(\alpha^2 \frac{\alpha^2 - n\alpha^{2n} + (n-1)\alpha^{2n+2}}{\alpha^2} \right. \\ &\quad \left. + 2 \frac{-1 - n(-1)^n + (n-1)(-1)^{n+1}}{4} + \beta^2 \frac{\beta^2 - n\beta^{2n} + (n-1)\beta^{2n+2}}{\beta^2} \right) \\ &= \frac{n}{5} [L_{2n+1} - (-1)^n] + \frac{|k|^2 - 1}{5} (-n\alpha^{2n} + (n-1)\alpha^{2n+2} + \frac{5}{2} - \frac{n}{2}(-1)^n \\ &\quad - \frac{n-1}{2}(-1)^n - n\beta^{2n} + (n-1)\beta^{2n+2}) \\ &= \frac{n}{5} [L_{2n+1} - (-1)^n] + \frac{|k|^2 - 1}{5} \left[-nL_{2n} + (n-1)L_{2n+2} + \frac{5}{2} + \frac{1-2n}{2}(-1)^n \right] \end{aligned}$$

$$= \frac{n}{5} [L_{2n+1} - (-1)^n] + \frac{|k|^2 - 1}{5} \left[-L_{2n} + (n-1)L_{2n+1} + \frac{5}{2} + \frac{1-2n}{2}(-1)^n \right].$$

Therefore,

$$\|F\|_E = \sqrt{\frac{1}{5} \{n [L_{2n+1} - (-1)^n] + (|k|^2 - 1) [-L_{2n} + (n-1)L_{2n+1} + \frac{5}{2} + \frac{1-2n}{2}(-1)^n]\}}.$$

□

The upper and lower bounds for the spectral norm of the Hadamard inverse of (1.11) will be given by the following theorem. We use the well - known inequalities

$$\frac{\|C\|_E}{\sqrt{n}} \leq \|C\|_2 \leq \|C\|_E, \quad (2.8)$$

which hold for any matrix C of order n , and the following lemma.

Lemma 2 ([7]). *Let $M = [m_{i,j}]$ and $N = [n_{i,j}]$ be matrices of order $m \times n$. Then*

$$\|M \circ N\|_2 \leq r_1(M) \circ c_1(N), \quad (2.9)$$

where $M \circ N = [m_{i,j} n_{i,j}]$ is the Hadamard product (or Schur product),

$$r_1(M) = \max_{1 \leq i \leq m} \sqrt{\sum_{j=1}^n |m_{i,j}|^2} \quad \text{and} \quad c_1(N) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^m |n_{i,j}|^2}.$$

More information about the Hadamard product can be found in [6, 11].

Theorem 6. *Let F be the matrix as in (1.11).*

1) *If $|k| \geq 1$, then*

$$\sqrt{\frac{5n}{L_{2n+1} - (-1)^n}} \leq \|F^{\circ-1}\|_2 \leq \sqrt{n(1 + (n-1)|k|^2)}, \quad (2.10)$$

2) *If $|k| < 1$, then*

$$|k| \sqrt{\frac{5n}{L_{2n+1} - (-1)^n}} \leq \|F^{\circ-1}\|_2 \leq n. \quad (2.11)$$

Proof. From the definition of the Euclidean norm, it follows that

$$\|F^{\circ-1}\|_E^2 = \sum_{i=0}^{n-1} (n-i) \frac{1}{F_{i+1}^2} + |k|^2 \sum_{i=1}^{n-1} i \frac{1}{F_{i+1}^2}. \quad (2.12)$$

1) If $|k| \geq 1$, then

$$\begin{aligned} \|F^{\circ-1}\|_E^2 &\geq \sum_{i=0}^{n-1} (n-i) \frac{1}{F_{i+1}^2} + \sum_{i=1}^{n-1} i \frac{1}{F_{i+1}^2} = n \sum_{i=0}^{n-1} \frac{1}{F_{i+1}^2} \\ &= n \sum_{i=1}^n \frac{1}{F_i^2} \geq n \sum_{i=1}^n \frac{1}{F_n^2} = \left(\frac{n}{F_n}\right)^2 \geq \frac{n^2}{F_n F_{n+1}} \\ &= \frac{5n^2}{L_{2n+1} - (-1)^n}. \end{aligned}$$

Therefore,

$$\frac{\|F^{\circ-1}\|_E}{\sqrt{n}} \geq \sqrt{\frac{5n}{L_{2n+1} - (-1)^n}}.$$

We conclude from (2.8) that

$$\|F^{\circ-1}\|_2 \geq \sqrt{\frac{5n}{L_{2n+1} - (-1)^n}}.$$

Now, we shall obtain the upper bound for the spectral norm of $F^{\circ-1}$. Let R and S be the following matrices:

$$R = \begin{bmatrix} \frac{1}{F_1} & \frac{1}{F_2} & \frac{1}{F_3} & \cdots & \frac{1}{F_n} \\ k & \frac{1}{F_1} & \frac{1}{F_2} & \cdots & \frac{1}{F_{n-1}} \\ k & k & \frac{1}{F_1} & \cdots & \frac{1}{F_{n-2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k & k & k & \cdots & \frac{1}{F_1} \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{F_n} & 1 & 1 & \cdots & 1 \\ \frac{1}{F_{n-1}} & \frac{1}{F_n} & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{F_2} & \frac{1}{F_3} & \frac{1}{F_4} & \cdots & 1 \end{bmatrix}.$$

Then,

$$r_1(R) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^n |r_{i,j}|^2} = \sqrt{1 + (n-1)|k|^2}$$

and

$$c_1(S) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^n |s_{i,j}|^2} = \sqrt{n}.$$

Since $F^{\circ-1} = R \circ S$, based on Lemma 2, we can write

$$\|F^{\circ-1}\|_2 \leq r_1(R) \circ c_1(S) = \sqrt{n(1 + (n-1)|k|^2)}.$$

2) If $|k| < 1$, then

$$\begin{aligned}\|F^{\circ-1}\|_E^2 &\geq \sum_{i=0}^{n-1} (n-i) |k|^2 \frac{1}{F_{i+1}^2} + \sum_{i=1}^{n-1} i |k|^2 \frac{1}{F_{i+1}^2} = n |k|^2 \sum_{i=0}^{n-1} \frac{1}{F_{i+1}^2} \\ &= n |k|^2 \sum_{i=1}^n \frac{1}{F_i^2} \geq n |k|^2 \sum_{i=1}^n \frac{1}{F_n^2} = |k|^2 \left(\frac{n}{F_n}\right)^2 \geq |k|^2 \frac{n^2}{F_n F_{n+1}} \\ &= |k|^2 \frac{5n^2}{L_{2n+1} - (-1)^n}.\end{aligned}$$

Therefore,

$$\frac{\|F^{\circ-1}\|_E}{\sqrt{n}} \geq |k| \sqrt{\frac{5n}{L_{2n+1} - (-1)^n}}.$$

We conclude from (2.8) that

$$\|F^{\circ-1}\|_2 \geq |k| \sqrt{\frac{5n}{L_{2n+1} - (-1)^n}}.$$

Now, we shall obtain the upper bound for the spectral norm of $F^{\circ-1}$. Let Q and W be the following matrices:

$$Q = \begin{bmatrix} \frac{1}{F_1} & 1 & 1 & \cdots & 1 \\ \frac{k}{F_n} & \frac{1}{F_1} & 1 & \cdots & 1 \\ \frac{k}{F_{n-1}} & \frac{k}{F_n} & \frac{1}{F_1} & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{k}{F_2} & \frac{k}{F_3} & \frac{k}{F_4} & \cdots & \frac{1}{F_1} \end{bmatrix} \text{ and } W = \begin{bmatrix} 1 & \frac{1}{F_2} & \frac{1}{F_3} & \cdots & \frac{1}{F_n} \\ 1 & 1 & \frac{1}{F_2} & \cdots & \frac{1}{F_{n-1}} \\ 1 & 1 & 1 & \cdots & \frac{1}{F_{n-2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Then,

$$r_1(Q) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^n |q_{i,j}|^2} = \sqrt{n}$$

and

$$c_1(W) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^n |w_{i,j}|^2} = \sqrt{n}.$$

Since $F^{\circ-1} = Q \circ W$, based on Lemma 2, we can write

$$\|F^{\circ-1}\|_2 \leq r_1(Q) \circ c_1(W) = n.$$

□

3. CONCLUSION

In this paper, we investigated the eigenvalues, the Euclidean norm and the upper and lower bounds for the spectral norm of the Hadamard inverse of

$$F = \text{circ}_k\{F_1, F_2, \dots, F_n\},$$

where F_n is the n^{th} Fibonacci number and k is a nonzero complex number.

From the fact that *the eigenvalues of an upper triangular matrix are the diagonal entries*, the eigenvalues of a semicirculant matrix (i.e. a k -circulant matrix for $k = 0$) with the first row $(F_1, F_2, \dots, F_n)$ are: $\lambda_j = 1$, ($j = \overline{0, n-1}$). The Euclidean norm of such (semicirculant) matrix can be obtained from (2.7) i.e. in (2.7) k can be equal to 0. Semicirculant matrices are not Hadamard invertible.

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Author's address

Biljana Radićić

University of Belgrade, Serbia

E-mail address: radicic.biljana@yahoo.com



QUANTITATIVE ESTIMATES FOR SOME MODIFIED BERNSTEIN-STANCU OPERATORS

VOICHIȚA ADRIANA RADU

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Abstract. In the papers [11, 12] starting with the Bernstein operators, some Stancu type operators are constructed

$$(C_n f)(x) = \sum_{k=0}^n \frac{k!}{n^k} \binom{n}{k} \mathbf{m}_{k,n} \left[0, \frac{1}{n}, \dots, \frac{k}{n}; f \right] x^k, \quad f \in C[0, 1],$$

where Y is the linear space of all functions $[0, 1] \rightarrow \mathbb{R}$ and the real numbers $(m_{k,n})_{k=0}^\infty$ are selected in order to preserve some important properties of Bernstein operators.

For $\mathbf{m}_{j,n} = \frac{(a_n)_j}{j!}$, $a_n \in (0, 1]$ we obtained Bernstein-Stancu operators

$$(\bar{C}_n f)(x) = \sum_{k=0}^n \frac{(a_n)_k}{n^k} \binom{n}{k} \left[0, \frac{1}{n}, \dots, \frac{k}{n}; f \right] x^k.$$

The aim of this paper is to give some estimates for this operators using moduli of smoothness of first and second order.

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1. INTRODUCTION

First of all, we recall some notions and operators which will be used in the paper.

Let Π be the algebra of polynomials with real coefficients and Π_n be the linear space of all real polynomials of degree $\leq n$.

For $k \in \mathbb{N}$, $z \in \mathbb{C}$ let $(z)_0 = 1$ and $(z)_k = z(z+1)\dots(z+k-1)$.

For $n \in \mathbb{N}$, let $B_n : Y \rightarrow \Pi_n$ be the Bernstein operators, defined for any $f \in C[0, 1]$ by

$$(B_n f)(x) = \sum_{k=0}^n b_{n,k}(x) f\left(\frac{k}{n}\right)$$

where

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

are the Bernstein fundamental polynomials.

For $g : [0, 1] \rightarrow \mathbb{R}$ the Stancu operators $S_k^{} : g \rightarrow S_k^{} g$, $k \in \mathbb{N}$ are defined as $(S_0^{} g)(x) = g(0)$ and for $k \in \{1, 2, \dots\}$

$$(S_k^{} g)(x) = \frac{1}{b_k} \sum_{j=0}^k \binom{k}{j} (bx)_j (b-bx)_{k-j} \cdot g\left(\frac{j}{k}\right), \quad x \in [0, 1]$$

where $b \in [0, 1]$ is a real parameter (see [13, 24, 25]).

2. THE CONSTRUCTION OF THE MODIFIED BERNSTEIN-STANCU OPERATORS

Approximation theory has been used in the theory of approximation of continuous functions by means of sequences of positive linear operators and still there remains a very active area of research. There are many approximating operators that their Korovkin type approximation properties and rates of convergence are investigated.

We list some of the mathematicians that relate their names to this fields of constructing and studding approximation properties of the linear and positive operators: A. Lupaş [20], O. Agratini [5], D. Bărbosu [8], [9], I. Gavrea, H.H. Gonska and D.P. Kacso [16], [17], O. Dogru [14], U. Abel, M. Ivan, R. Păltănea [1] and Y. Kageyama [18].

A new direction of generalization of the linear and positive operators are q-calculus as we can see in the pioneering works of A. Lupaş [19] and G.M. Philips [23]. Some of the most recent appearances in this direction are the papers of O. Agratini [6], P.N. Agrawal, Z. Finta and A. Sathish Kumar [15], [7], G. Nowak and V. Gupta [21], A.M. Acu, C.V. Muraru, D. Bărbosu and D.F. Sofonea [2], [4] and [3].

In [11] was constructed a new class of linear and positive operators starting with the derivatives of the Bernstein operators.

For $j \in \{0, 1, \dots, n\}$

$$\frac{1}{j!} \frac{d^j (B_n f)(x)}{dx^j} = \binom{n}{j} \frac{j!}{n^j} \sum_{k=0}^{n-j} b_{n-j,k}(x) \left[\frac{k}{n}, \frac{k+1}{n}, \dots, \frac{k+j}{n}; f \right],$$

the following formula holds

$$(B_n f)(x) = \sum_{k=0}^n \frac{k!}{n^k} \binom{n}{k} \left[0, \frac{1}{n}, \dots, \frac{k}{n}; f \right] x^k. \quad (2.1)$$

Starting with (2.1), we investigated the modifications

$$C_n : Y \rightarrow \Pi_n$$

$$(C_n f)(x) = \sum_{k=0}^n \frac{k!}{n^k} \binom{n}{k} \mathbf{m}_{k,n} \left[0, \frac{1}{n}, \dots, \frac{k}{n}; f \right] x^k, \quad f \in Y, \quad (2.2)$$

where the real numbers $(\mathbf{m}_{k,n})_{k=0}^\infty$ are selected in order to preserve some important properties of Bernstein operators.

Observe that from (2.2)

$$\begin{cases} C_n e_0 = \mathbf{m}_{0,n} \\ C_n e_1 = \mathbf{m}_{1,n} e_1 \\ C_n e_2 = \mathbf{m}_{2,n} e_2 + \frac{e_1}{n} (\mathbf{m}_{1,n} - \mathbf{m}_{2,n} e_1) \\ (C_n \Omega_{2,x})(x) = (\mathbf{m}_{2,n} - 2\mathbf{m}_{1,n} + \mathbf{m}_{0,n}) x^2 + \frac{x}{n} (\mathbf{m}_{1,n} - \mathbf{m}_{2,n} x) \end{cases}$$

where $e_j(t) = t^j$ and $\Omega_{2,x} = (t-x)^2$.

In the following we shall consider that $\mathbf{m}_{0,n} = 1$ and $\lim_{n \rightarrow \infty} \mathbf{m}_{1,n} = 1$.

We also consider that

$$\mathbf{m}_{j,n} = \frac{(a_n)_j}{j!}, \quad a_n \in (0, 1]$$

Then the operator C_n from (2.2), denoted further by $\overline{C}_n$, becomes

$$(\overline{C}_n f)(x) = \sum_{k=0}^n \frac{(a_n)_k}{n^k} \binom{n}{k} \left[0, \frac{1}{n}, \dots, \frac{k}{n}; f \right] x^k, \quad f \in Y. \quad (2.3)$$

$\overline{C}_n$ are called **Bernstein-Stancu operators**, when $a_n \in (0, 1)$ (see [11], Definition 11).

3. A SUMMING UP OF THE APPROXIMATION PROPERTIES OF $\overline{C}_n$ OPERATOR

First, in [12], it is demonstrated that $\overline{C}_n$ is a linear and positive operator that transform any polynomial of degree $s \leq n$ into a polynomial of degree s and preserves the convexity of order j , if $j, n \in \mathbb{N}^*, 0 \leq j \leq n-2$.

Also, the operator $\overline{C}_n$ from (2.3) may be written in the Bernstein basis in the form ([11], Theorem 10)

$$(\overline{C}_n f)(x) = \sum_{k=0}^n b_{n,k}(x) \overline{C}_{k,n}[f]$$

with

$$\overline{C}_{k,n}[f] = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} f\left(\frac{j}{n}\right) (a_n)_j (1-a_n)_{k-j}.$$

In the same time $\overline{C}_n$ can be written using Stancu functionals.

Observe that

$$\overline{C}_0 f = \overline{C}_{0,0}[f] := f(0)$$

and

$$(S_k^{<1>} g)(a_n) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (a_n)_j (1-a_n)_{k-j} g\left(\frac{j}{k} \cdot \frac{k}{n}\right), \quad k \geq 1.$$

Therefore,

$$\overline{C}_{k,n}[f] = \left(S_k^{<1>} g_{n,k}^{<f>}\right)(a_n)$$

with

$$g_{n,k}^{<f>}(t) = f\left(t \frac{k}{n}\right), \quad k \geq 1.$$

In order to provide the convergence theorem, the following identities hold true ([11], Lemma 8):

$$\begin{cases} (\overline{C}_n e_0)(x) = 1 \\ (\overline{C}_n e_1)(x) = a_n x = x - (1-a_n)x \\ (\overline{C}_n e_2)(x) = x^2 + \frac{x(1-x)}{n} a_n + \frac{1-a_n}{2} \left(\frac{a_n}{n} - (2+a_n)\right) x^2. \end{cases}$$

Also, if $f \in Y$, $\Omega_{2,x} = (t-x)^2$ and $a_n \in (0, 1]$, then

$$(\overline{C}_n \Omega_{2,x})(x) = \frac{x(1-x)}{n} a_n + x^2(1-a_n) \left(\frac{2-a_n}{2} + \frac{a_n}{2n}\right).$$

Moreover

$$|(\overline{C}_n \Omega_{2,x})(x)| \leq \frac{a_n}{4n} + (1-a_n), \quad \forall x \in [0, 1]. \quad (3.1)$$

Applying the Bohman - Korovkin theorem and the above assertions ([12], Theorem 5) we can see that the sequence $\{\overline{C}_n f\}_{n \geq 1}$ converges to f , uniformly on $[0, 1]$ for any $f \in Y$.

The asymptotic behavior of the sequence $(\overline{C}_n)_{n=1}^\infty$ on a certain subspaces of $C[-1, 1]$ is given in the following proposition ([11], Theorem 15) and it was demonstrated applying a version of a general proposition given by R. G. Mamedov:

Theorem 1 ([11], Theorem 15). *Suppose $x_0 \in [0, 1]$ and $f''(x_0)$ exists. If $a_n \in (0, 1)$, $\lim_{n \rightarrow \infty} a_n = 1$ and exists $L := \lim_{n \rightarrow \infty} n(1-a_n)$, then*

$$\lim_{n \rightarrow \infty} n[f(x_0) - (\overline{C}_n f)(x_0)] = -\frac{x(1-x)}{2} f''(x_0) + \left[x_0 f'(x_0) - \frac{x_0^2}{4} f''(x_0) \right] L.$$

In order to obtain an overview of the approximation properties of this operator, we add here some more properties.

Lemma 1. If $f \in C[0, 1]$, then $\|\overline{C}_n f\| \leq \|f\|$, where $\|\cdot\|$ is the uniform norm on $C[0, 1]$.

Proof. Using the fact that the operator is linear and positive we have the identity

$$\|\overline{C}_n\| = \|\overline{C}_n e_0\|.$$

And from the above property $(\overline{C}_n e_0)(x) = e_0$ then $\|\overline{C}_n\| = 1$. $\square$

Lemma 2. The operator $\overline{C}_n f$ can be represented by the following expansion

$$(\overline{C}_n f)(x) = f(0) + \sum_{k=1}^n \frac{(a_n)_k x^k}{k!} \binom{n}{k} (\Delta_{n-1}^k f)(0),$$

where $\Delta_{n-1}^k f(0)$ is the finite difference of order k , with the step n^{-1} and the starting point 0 of the function f , that is

$$\Delta_{n-1}^k f(0) = \sum_{j=0}^k (-1)^j \binom{k}{j} f\left(\frac{k-j}{n}\right).$$

Proof. By making use of the following relation between divided differences and finite differences

$$[x_0, x_0 + h, \dots, x_0 + kh; f] = \frac{1}{k! \cdot h^k} \cdot \Delta_h^k f(x_0)$$

we obtain

$$\left[0, \frac{1}{n}, \dots, \frac{k}{n}; f\right] = \frac{n^k}{k!} \Delta_{n-1}^k f(0).$$

Replacing it in (2.3) we are led to the desired formula. $\square$

Remark 1. If we set $x = 0$ in the expansion formula from above, then we find

$$(\overline{C}_n f)(0) = f(0).$$

Naturally follows the fact that the polynomial (2.3) is interpolating at the end 0 of the interval $[0, 1]$ i.e. $(\overline{C}_n f)(0) = f(0)$.

4. ESTIMATES FOR $\overline{C}_n$ IN TERMS OF MODULI OF SMOOTHNESS

The main tools to estimating the degree of approximation by positive linear functionals and operators are the moduli of smoothness of first and second order, given by (see, for example, [10])

$$\omega(f; \delta) := \sup \{|f(x) - f(t)| : x, t \in I, |x - t| \leq \delta\}$$

$$\omega_2(f; \delta) := \sup \left\{ \left| f(x) - 2f\left(\frac{x+t}{2}\right) + f(t) \right| : x, t \in I, |x - t| \leq 2\delta \right\}$$

for $f \in C(I)$ and $\delta \geq 0$.

For $f \in C[a, b]$, a useful modification represents the least concave majorant of $\omega(f; \cdot)$ given by

$$\tilde{\omega}(f; \epsilon) = \begin{cases} \sup_{0 \leq x \leq \epsilon \leq y \leq b-a, x \neq y} \frac{(\epsilon-x)\omega(f, y) + (y-\epsilon)\omega(f, x)}{y-x} & \text{if } 0 \leq \epsilon \leq b-a \\ \tilde{\omega}(f; \epsilon) = \omega(f; b-a) & \text{if } \epsilon > b-a \end{cases}$$

First, using the result obtained by O. Shisha and B. Mond in 1968, we recall from [12] the estimation for $\overline{C}_n$ using the modulus of continuity.

Theorem 2 ([12], Theorem 9). *If $a_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} a_n = 1$ and $L := \lim_{n \rightarrow \infty} n(1 - a_n)$ exists, then for any $f \in Y$, $x \in [0, 1]$ and $n \geq 1$, the Bernstein-Stancu operators (2.3) verify*

$$|(\overline{C}_n f)(x) - f(x)| \leq \left(1 + \frac{1}{\delta} \sqrt{\frac{a_n}{4n} + (1 - a_n)}\right) \omega(f; \delta).$$

Corollary 1 ([12], Corollary 10). *If $a_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} a_n = 1$ and $L := \lim_{n \rightarrow \infty} n(1 - a_n)$ exists, then for any $f \in Y$, $x \in [0, 1]$ and $n \geq 1$, the Bernstein-Stancu operators (2.3) verify*

$$|(\overline{C}_n f)(x) - f(x)| \leq \left(1 + \sqrt{\frac{a_n}{4} + n(1 - a_n)}\right) \omega(f; \frac{1}{\sqrt{n}}).$$

Corollary 2 ([12], Corollary 11). *If $a_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} a_n = 1$ and $L := \lim_{n \rightarrow \infty} n(1 - a_n)$ exists, then for any $f \in Y$, $x \in [0, 1]$, $n \geq 1$ and*

$$\delta = \sqrt{\frac{a_n}{4n} + (1 - a_n)}$$

then Bernstein-Stancu operators verify

$$|(\overline{C}_n f)(x) - f(x)| \leq 2\omega(f; \delta).$$

Theorem 3 ([12], Theorem 12). *If $a_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} a_n = 1$ and $L := \lim_{n \rightarrow \infty} n(1 - a_n)$ exists, then for any $f \in Y$, $x \in [0, 1]$ and $n \geq 1$, the Bernstein-Stancu operators (2.3) verify*

$$|(\overline{C}_n f)(x) - f(x)| \leq |a_n x - x| \cdot |f'(x)| + 2\delta\omega(f'; \delta),$$

where

$$\delta = \sqrt{\frac{a_n}{4n} + (1 - a_n)}.$$

The main results of this paper are direct estimates via ω , $\tilde{\omega}$ and ω_2 .

Let $K = [a, b]$ and $K' \subseteq K$ be also compact and let $L : C(K) \rightarrow C(K')$ be a positive linear operator. The following result was obtained by H.H. Gonska in [17].

Theorem 4. *For the linear and positive operator L that reproduce the constant functions, the following inequality holds:*

$$|L(f; x) - f(x)| \leq \max \left\{ 1, \frac{1}{h} \cdot L(|e_1 - x|; x) \right\} \cdot \tilde{\omega}(f; h),$$

for all $f \in C(K)$, $x \in K'$ and $h > 0$.

In our case, we have

Theorem 5. *If $a_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} a_n = 1$ and $L := \lim_{n \rightarrow \infty} n(1 - a_n)$ exists, then for any $f \in Y$, $x \in [0, 1]$ and $n \geq 1$, for Bernstein-Stancu operators (2.3), the following inequality holds:*

$$|(\overline{C_n} f)(x) - f(x)| \leq \tilde{\omega} \left(f; \sqrt{\frac{a_n}{4n} + (1 - a_n)} \right).$$

Proof. Using the Gonska's result and the inequality (3.1) we obtain

$$|(\overline{C_n} f)(x) - f(x)| \leq \max \left\{ 1, \frac{1}{h} \cdot \sqrt{\frac{a_n}{4n} + (1 - a_n)} \right\} \cdot \tilde{\omega}(f; h),$$

and putting $h = \sqrt{\frac{a_n}{4n} + (1 - a_n)}$ leads to the desired result. $\square$

Due to the fact that ω_2 annihilates linear functions, it is advantageous to measure the degree of approximation by means of this modulus of smoothness.

Further, we recall the following results given by R. Păltănea in [22].

Theorem 6. *For any $f \in C(K)$ all $x \in C(K')$ and $0 < h < \frac{1}{2} \text{length}(K)$ we have*

$$\begin{aligned} |L(f; x) - f(x)| &\leq |L(e_0; x) - 1| \cdot |f(x)| + |L(e_1 - x; x)| \cdot \frac{1}{h} \omega_1(f, h) \\ &\quad + \left(L(e_0; x) + \frac{1}{2} \cdot \frac{1}{h^2} L((e_1 - x)^2; x) \right) \omega_2(f, h). \end{aligned}$$

Thus we can state

Theorem 7. *If $a_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} a_n = 1$ and $L := \lim_{n \rightarrow \infty} n(1 - a_n)$ exists, then for any $f \in Y$, $x \in [0, 1]$, $n \geq 1$ and $0 < h \leq \frac{1}{2}$ we have:*

$$|(\overline{C_n} f)(x) - f(x)| \leq \frac{1}{h} (1 - a_n) \omega(f, h) + \left(1 + \frac{1}{2} \cdot \frac{1}{h^2} \left(\frac{a_n}{4n} + (1 - a_n) \right) \right) \omega_2(f, h)$$

Proof. The assertion follows from the Păltănea's theorem using inequality (3.1). $\square$

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Author's address

Voichița Adriana Radu

Babes-Bolyai University, FSEGA, Department of Statistics-Forecasts-Mathematics, Str. Teodor Mihali, No.58-60, RO-400591 Cluj Napoca, Romania

E-mail address: voichita.radu@econ.ubbcluj.ro, voichita.radu@gmail.com



INVERSE AND FACTORIZATION OF TRIANGULAR TOEPLITZ MATRICES

ADEM ŞAHİN

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Abstract. In this paper, we present a new approach for finding the inverse of some triangular Toeplitz matrices using the generalized Fibonacci polynomials and give a factorization of these matrices. We also give a new proof of Trudi's formula using our result.

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1. INTRODUCTION

Let L_n be the lower triangular Toeplitz matrix:

$$L_n = \begin{bmatrix} t_0 & 0 & \cdots & 0 \\ t_1 & t_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ t_{n-1} & \cdots & t_1 & t_0 \end{bmatrix}$$

where $(t_i)_{i=0,1,2,\dots,n-1}$ are real and/or complex numbers. In the matrix theory, there is quite an interest in the theory and applications of triangular Toeplitz matrices. There are a number of studies focusing on the linear algebra of the triangular Toeplitz matrices. For example, in [2] the authors discussed the linear algebra of the Pascal matrix, in [8] the authors examined the linear algebra of the k -Fibonacci matrix and the symmetric k -Fibonacci matrix, in [6] the authors studied the Pell matrix. In [9], Lee *et al.* defined $n \times n$ -Fibonacci matrix and obtained the inverse matrix of the Fibonacci matrix. The Fibonacci matrix $F_n = [f_{i,j}]_{i,j=1,2,\dots,n}$ and the inverse matrix of F_n as follows:

$$F_n = [f_{i,j}] = \begin{cases} f_{i-j+1}, & \text{for } i - j + 1 \geq 0, \\ 0, & \text{for } i - j + 1 < 0, \end{cases}$$

and

$$F_n^{-1} = [f_{i,j}^t] = \begin{cases} 1, & \text{for } i = j, \\ -1, & \text{for } i - 2 \leq j \leq i - 1, \\ 0, & \text{otherwise,} \end{cases}$$

where f_n is n th Fibonacci numbers.

The inverse of Toeplitz matrices was first studied by Trench [18] in 1964 and by Gohberg and Semencul [4] in 1972. In the last decades some papers related to computing the inverse of a nonsingular Toeplitz matrix and the lower triangular Toeplitz matrix were presented, etc. [1, 3, 5, 7, 11, 16, 17, 19, 21]. In [16] Merca derived the inverse of triangular Toeplitz matrix using symmetric functions.

In this paper, we obtain the inverse of $n \times n$ lower triangular Toeplitz matrix T_n as follows:

$$T_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -t_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -t_{n-1} & \cdots & -t_1 & 1 \end{bmatrix} \quad (1.1)$$

where $(t_i)_{i=1,2,\dots,n-1}$ are the elements of a ring.

To achieve this goal, we use the generalized Fibonacci polynomials which are general form of a large number of recurrent relation numbers and polynomials. MacHenry [12, 13] defined generalized Fibonacci polynomials $(F_{k,n}(t))$, where t_i ($1 \leq i \leq k$) are constant coefficients of the core polynomial

$$P(x; t_1, t_2, \dots, t_k) = x^k - t_1 x^{k-1} - \cdots - t_k,$$

which is denoted by the vector $t = (t_1, t_2, \dots, t_k)$. $F_{k,n}(t)$ is defined inductively by

$$\begin{aligned} F_{k,n}(t) &= 0, \quad n < 0; \\ F_{k,0}(t) &= 1, \\ F_{k,n+1}(t) &= t_1 F_{k,n}(t) + \cdots + t_k F_{k,n-k+1}(t). \end{aligned} \quad (1.2)$$

In addition, in [14] the authors obtained $F_{k,n}(t)$ ($n, k \in \mathbb{N}, n \geq 1$) as,

$$F_{k,n}(t) = \sum_{b \vdash n} \binom{|b|}{b_1, \dots, b_n} t_1^{b_1} \cdots t_n^{b_n}. \quad (1.3)$$

Throughout this paper, the notations $b \vdash n$ and $|b|$ are used instead of $\sum_{j=1}^n j b_j = n$ and $\sum_{j=1}^n b_j$, respectively.

Corollary 1 ([10]). Let $F_{k,n}(t)$ be the generalized Fibonacci polynomials and $H_{-(k,n)}$ be the $n \times n$ lower Hessenberg matrix such that

$$H_{-(k,n)} = \begin{bmatrix} t_1 & -1 & 0 & \cdots & 0 \\ t_2 & t_1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{k-1} & t_{k-2} & t_{k-3} & \cdots & -1 \\ t_k & t_{k-1} & t_{k-2} & \cdots & t_1 \end{bmatrix}.$$

Then,

$$\det H_{-(k,n)} = F_{k,n}(t).$$

2. THE INVERSE OF $N \times N$ LOWER TRIANGULAR TOEPLITZ MATRIX

In this section, we obtain the inverse of matrix T_n (1.1). This result was obtained in [16] as a result of the study on symmetry between complete symmetric functions and elementary symmetric functions. We present new approach for this using definition of generalized Fibonacci polynomials.

Theorem 1. Let T_n be the $n \times n$ lower triangular Toeplitz matrix in (1.1) and $F_{k,n}(t)$ be the generalized Fibonacci polynomials defined in (1.2), then

$$(T_n)^{-1} = [t_{i,j}] = \begin{cases} F_{k,i-j}(t), & \text{for } i - j > 0, \\ 1, & \text{for } i - j = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof.

$$A_n = [a_{i,j}] = \begin{cases} F_{k,i-j}(t), & \text{for } i - j > 0, \\ 1, & \text{for } i - j = 0, \\ 0, & \text{otherwise.} \end{cases}$$

and R_i is i th row vector of A_n , C_i is i th column vector of T_n .

If we show the equation $A_n T_n = I = [e_{i,j}]$, the proof is completed. It is obvious that $e_{ij} = 0$ for $i < j$ and $e_{ii} = \langle R_i, C_i \rangle = 1$ from the definitions of A_n and T_n . Now we obtain e_{ij} for $i > j$;

$$e_{ij} = \langle R_i, C_j \rangle = F_{k,i-j+1}(t) - t_1 F_{k,i-j}(t) - t_2 F_{k,i-j}(t) - \cdots - t_k F_{k,0}(t).$$

From the definition of generalized Fibonacci polynomials and previous equation, we obtain $e_{ij} = 0$ for $i > j$, which ends the proof. $\square$

Example 1. We obtain the inverse of matrix T_5 using Theorem 1;

$$(T_5)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -t_1 & 1 & 0 & 0 & 0 \\ -t_2 & -t_1 & 1 & 0 & 0 \\ -t_3 & -t_2 & -t_1 & 1 & 0 \\ -t_4 & -t_3 & -t_2 & -t_1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ F_{k,1}(t) & 1 & 0 & 0 & 0 \\ F_{k,2}(t) & F_{k,1}(t) & 1 & 0 & 0 \\ F_{k,3}(t) & F_{k,2}(t) & F_{k,1}(t) & 1 & 0 \\ F_{k,4}(t) & F_{k,3}(t) & F_{k,2}(t) & F_{k,1}(t) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & 0 & 0 & 0 & 0 & 0 \\ t_1 & & & 1 & 0 & 0 & 0 & 0 \\ t_2 + t_1^2 & & & t_1 & 1 & 0 & 0 & 0 \\ t_3 + 2t_1t_2 + t_1^3 & & & t_2 + t_1^2 & t_1 & 1 & 0 & 0 \\ t_4 + 2t_1t_3 + t_2^2 + t_1^4 + 3t_1^2t_2 & & & t_3 + 2t_1t_2 + t_1^3 & t_2 + t_1^2 & t_1 & 1 & 0 \end{bmatrix}.$$

Example 2. We obtain " $s_{8,3}$ " for matrix $S_n = [s_{i,j}]_{n \times n}$ which is the inverse of matrix $R_n = [r_{ij}]_{n \times n} = \begin{cases} 1, & i = j, \\ 0, & i < j, \\ j - i, & i > j, \end{cases}$ using Theorem 1 and equation (1.3);

$$s_{8,3} = F_{k,5}(1, 2, 3, 4, 5) = \sum_{b \vdash 5} \binom{|b|}{b_1, \dots, b_5} 1^{b_1} \dots 5^{b_5} = 55.$$

Since $F_{k,n}(t)$ is the general form of the Fibonacci type numbers and polynomials, the results that we obtained are applicable for many polynomials and sequences, such as generalized order- k Fibonacci, Pell and Jacobsthal numbers, generalized bivariate Fibonacci p -polynomials, bivariate Jacobsthal p -polynomials, Chebyshev polynomial of the second kind and bivariate Pell p -polynomials etc.[see Table 1]

The generalized bivariate Fibonacci p -polynomials [20] are, for $n > p$,

$$F_{p,n}(x, y) = xF_{p,n-1}(x, y) + yF_{p,n-p-1}(x, y), \quad (2.1)$$

with boundary conditions for $n = 1, 2, \dots, p$, $F_{p,0}(x, y) = 0$, $F_{p,n}(x, y) = x^{n-1}$.

Corollary 2. Let $F_{p,n}(x, y)$ be the generalized bivariate Fibonacci p -polynomials defined in (2.1) and $T_{n,p}$ is $n \times n$ lower triangular Toeplitz matrix as

$$T_{n,p} = [c_{i,j}] = \begin{cases} 1, & \text{for } i - j = 0, \\ -x, & \text{for } i - j = 1, \\ -y, & \text{for } i - j = p + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$(T_{n,p})^{-1} = [a_{i,j}] = \begin{cases} F_{p,i-j}(x, y), & \text{for } i - j > 0, \\ 1, & \text{for } i - j = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. It is obvious that $F_{k-1,n}(x, y) = F_{k,n}(t)$ for $t_1 = x, t_i = 0 (2 \leq i \leq (k - 1)), t_k = y$. So using Theorem 1, we obtain the required result. $\square$

TABLE 1. [20] Cognate polynomial sequences.

x	y	p	$F_{p,n}(x, y)$
x	y	1	bivariate Fibonacci polynomials $F_n(x, y)$
x	1	p	Fibonacci p -polynomials $F_{p,n}(x)$
x	1	1	Fibonacci polynomials $f_n(x)$
1	1	p	Fibonacci p -numbers $F_p(n)$
1	1	1	Fibonacci numbers F_n
$2x$	y	p	bivariate Pell p -polynomials $F_{p,n}(2x, y)$
$2x$	y	1	bivariate Pell polynomials $F_n(2x, y)$
$2x$	1	p	Pell p -polynomials $P_{p,n}(x)$
$2x$	1	1	Pell polynomials $P_n(x)$
2	1	1	Pell numbers P_n
$2x$	-1	1	second kind Chebysev polynomials $U_{n-1}(x)$
x	$2y$	p	bivariate Jacobsthal p -polynomials $F_{p,n}(x, 2y)$
x	$2y$	1	bivariate Jacobsthal polynomials $F_n(x, 2y)$
1	$2y$	1	Jacobsthal Polynomials $J_n(y)$
1	2	1	Jacobsthal Numbers J_n

Corollary 3. Let U_n be the $n \times n$ lower triangular Toeplitz matrix as

$$U_n = [u_{i,j}] = \begin{cases} \sum_{m=0}^{\lfloor \frac{i-j-1}{2} \rfloor} \binom{i-j}{2m+1} x^{i-j-2m} (x^2-1)^m, & \text{for } i-j > 0, \\ 1, & \text{for } i-j = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$(U_n)^{-1} = \begin{cases} 1, & \text{for } i-j = 0, i-j = 2, \\ -2x, & \text{for } i-j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. It is obvious in the Table 1 that $F_{1,n}(x, y) = U_{n-1}(x)$ and

$$U_n(x) = \sum_{m=0}^{\lfloor \frac{i-j}{2} \rfloor} \binom{i-j+1}{2m+1} x^{i-j-2m} (x^2-1)^m$$

from the definition of the second kind of Chebyshev polynomials. So using Corollary 2, we obtain the required result. $\square$

3. FACTORIZATIONS OF LOWER TRIANGULAR TOEPLITZ MATRIX

The set of all square matrices of order n is denoted by P_n . A matrix $P \in P_n$ of the form

$$P = \begin{bmatrix} P_{11} & 0 & \cdots & 0 \\ 0 & P_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & P_{kk} \end{bmatrix}$$

in which $P_{ii} \in P_{n_i}$, $i \in \{1, 2, \dots, k\}$, and $\sum_{i=1}^k n_i = n$, is called block diagonal. Notationally, such a matrix is often indicated as $P = P_{11} \oplus P_{22} \oplus \cdots \oplus P_{kk}$; this is called the direct sum of the matrices $P_{11}, P_{22}, \dots, P_{kk}$.

In [9], Lee *et al.* gave the factorization of Fibonacci matrix. Now we consider the factorization of lower triangular Toeplitz matrix. We define the matrices C_n and $\widetilde{T}_n$ by

$$C_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -F_{k,1}(t) & & & \\ \vdots & & I_{n+2} & \\ -F_{k,n+2}(t) & & & \end{bmatrix} \text{ and } \widetilde{T}_n = [1] \oplus T_n. \quad (3.1)$$

Theorem 2. Let T_n be the $n \times n$ lower triangular Toeplitz matrix in (1.1) and matrices $C_n, \widetilde{T}_n$ defined in (3.1). Then $(\widetilde{T}_{k-1})(C_{k-3}) = T_n$ for $k \geq 3$.

Proof. For $k = 3$, we have $(\widetilde{T}_2)(C_0) = T_3$. Letting $k > 3$ and applying the definition of generalized Fibonacci polynomials, the proof complete. $\square$

Theorem 3. Let T_n be the $n \times n$ lower triangular Toeplitz matrix in (1.1), the matrix C_n defined in (3.1) and I_n be the $n \times n$ identity matrix. Then $T_n = (I_{n-2} \oplus (C_{-1})) \dots (I_1 \oplus (C_{n-4}))(C_{n-3})$.

Proof. The proof is an immediate consequence of Theorem 2. $\square$

Example 3. We obtain the factorization of the matrix T_5 using Theorem 3;

$$T_5 = (I_3 \oplus C_{-1})(I_2 \oplus C_0)(I_1 \oplus C_1)C_2,$$

i.e.

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -t_1 & 1 & 0 & 0 & 0 \\ -t_2 & -t_1 & 1 & 0 & 0 \\ -t_3 & -t_2 & -t_1 & 1 & 0 \\ -t_4 & -t_3 & -t_2 & -t_1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -t_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -t_1 & 1 & 0 \\ 0 & 0 & -(t_1^2 + t_2) & 0 & 1 \end{bmatrix} \\ & \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -t_1 & 1 & 0 & 0 \\ 0 & -(t_1^2 + t_2) & 0 & 1 & 0 \\ 0 & -(t_3 + 2t_1t_2 + t_1^3) & 0 & 0 & 1 \end{bmatrix} \\ & \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -t_1 & 1 & 0 & 0 & 0 \\ -(t_1^2 + t_2) & 0 & 1 & 0 & 0 \\ -(t_3 + 2t_1t_2 + t_1^3) & 0 & 0 & 1 & 0 \\ -(t_4 + 2t_1t_3 + t_2^2 + t_1^4 + 3t_1^2t_2) & 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Lemma 1. Let C_n be the $(n+3) \times (n+3)$ Hessenberg matrix in 3.1, then

$$(C_n)^{-1} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ F_{k,1}(t) & & & \\ \vdots & & I_{n+2} & \\ F_{k,n+2}(t) & & & \end{bmatrix}$$

Proof. The proof is obvious from the matrix product. $\square$

Corollary 4. Let T_n be the $n \times n$ lower triangular Toeplitz matrix in (1.1), the matrix C_n defined in (3.1) and I_n be the $n \times n$ identity matrix. Then $(T_n)^{-1} = (C_{n-3})^{-1}(I_1 \oplus (C_{n-4}))^{-1} \dots (I_{n-2} \oplus (C_{-1}))^{-1}$.

Proof. The proof follows by a simple calculation using the previous lemma and the equation $(I_k \oplus (C_{n-k-3}))^{-1} = I_k \oplus (C_{n-k-3})^{-1}$. $\square$

4. A NEW PROOF OF TRUDI'S FORMULA

Merca [15] gave a proof of the Trudi's formula. We give a different proof of this identity using our results.

Theorem 4 (Trudi's formula [15]). Let m be a positive integer. Then

$$\det \begin{bmatrix} a_1 & a_0 & \cdots & 0 \\ a_2 & a_1 & \cdots & \\ \vdots & \ddots & \ddots & \vdots \\ a_{m-1} & a_{m-2} & \cdots & a_1 & a_0 \\ a_m & a_{m-1} & \cdots & a_2 & a_1 \end{bmatrix} = \sum_{(k_1, k_2, \dots, k_m)} \binom{k_1 + \cdots + k_m}{k_1, \dots, k_m} (-a_0)^{m-k_1-\dots-k_m} a_1^{k_1} a_2^{k_2} \cdots a_m^{k_m}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + mt_m = m$.

Proof. Using the properties of determinants of Hessenberg matrices, we can write

$$\det \begin{bmatrix} a_1 & a_0 & 0 & 0 \\ a_2 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_0 \\ a_m & \cdots & a_2 & a_1 \end{bmatrix} = \det \begin{bmatrix} a_1 & -1 & 0 & \cdots & 0 \\ -a_0 a_2 & a_1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ (-1)^{m-2} a_0^{m-2} a_{m-1} & (-1)^{m-3} a_0^{m-3} a_{m-2} & \ddots & a_1 & -1 \\ (-1)^{m-1} a_0^{m-1} a_m & (-1)^{m-2} a_0^{m-2} a_{m-1} & & -a_0 a_2 & a_1 \end{bmatrix}.$$

And if we take

$$t_1 = a_1, t_2 = -a_0 a_2, \dots, t_m = (-1)^{m-1} a_0^{m-1} a_m$$

in equation 1.3 and Corollary 1, we obtain

$$\begin{aligned}
& \det \begin{bmatrix} a_1 & -1 & 0 & \cdots & 0 \\ -a_0 a_2 & a_1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ (-1)^{m-2} a_0^{m-2} a_{m-1} & (-1)^{m-3} a_0^{m-3} a_{m-2} & \ddots & a_1 & -1 \\ (-1)^{m-1} a_0^{m-1} a_m & (-1)^{m-2} a_0^{m-2} a_{m-1} & & -a_0 a_2 & a_1 \end{bmatrix} \\
&= \sum_{(k_1, k_2, \dots, k_m)} \binom{k_1 + \cdots + k_m}{k_1, \dots, k_m} a_1^{k_1} (-a_0 a_2)^{k_2} \cdots ((-a_0)^{m-1} a_m)^{k_m}
\end{aligned}$$

Finally, when we make the necessary calculations, equation

$$\begin{aligned}
& \sum_{(k_1, k_2, \dots, k_m)} \binom{k_1 + \cdots + k_m}{k_1, \dots, k_m} a_1^{k_1} (-a_0 a_2)^{k_2} \cdots ((-a_0)^{m-1} a_m)^{k_m} \\
&= \sum_{(k_1, k_2, \dots, k_m)} \binom{k_1 + \cdots + k_m}{k_1, \dots, k_m} (-a_0)^{m-k_1-\cdots-k_m} a_1^{k_1} a_2^{k_2} \cdots a_m^{k_m}
\end{aligned}$$

is obtained. $\square$

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Author's address

Adem Şahin

Gaziosmanpaşa University, Faculty of Education, 60250, Tokat, Turkey

E-mail address: adem.sahin@gop.edu.tr, hessenberg.sahin@gmail.com



APPLICATIONS OF MEASURE OF NONCOMPACTNESS TO COUPLED FIXED POINTS AND SYSTEMS OF INTEGRAL EQUATIONS

A. SAMADI

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Abstract. In this paper, some existence theorems involving generalized contractive conditions with respect to a measure are proved. By applying our results, we study some coupled fixed point theorems, and discuss the existence of solutions for a class of the system of integral equations. Finally, an example is included to show the efficiency of our results.

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1. INTRODUCTION

Integral equations are used naturally in applied problems, such as in a lot of problems in physics and engineering. Also, especially integral equations have been linked in many applications in the kinetic theory of gases, the theory of radioactive transfer, see for example [9, 11, 12]. The existence theorems for nonlinear integral equations have been studied in many papers with the help of the technique of measures of noncompactness which was initiated by Kuratowski [10]. The Kuratowski measure of noncompactness has attracted the interest of mathematicians working in the study of functional equations, ordinary and partial differential equations and many other fields. In fact, since measures of noncompactness are functions suitable for measuring the degree of noncompactness of a given set, they are very useful tools in the wide area of functional analysis such as the metric fixed point theory and the theory of operator equations in Banach spaces (see [3, 13, 14]). In this paper, first we recall some essential concepts and results that will be used later. Then, we give some new fixed point theorems applying the technique of measure of noncompactness. In the third section, we apply our results to a coupled fixed point. Finally in order to indicate the applicability of our results, we study the problem of the existence of solutions for a class of system of integral equations.

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Now we recall some notations, definitions and theorems which will be needed further on.

Throughout this paper we assume that E is a real Banach space with norm $\|\cdot\|$ and zero element θ . Let X be a nonempty subset of E . The closure and the closed convex hull of X will be denoted by $\overline{X}$ and $\text{Conv}(X)$, respectively. Moreover, let us denote by M_E the family of all nonempty and bounded subsets of E and by N_E its subfamily consisting of all relatively compact sets.

The following definition of measure of noncompactness will be used in our results.

Definition 1 ([6]). A mapping $\mu : M_E \rightarrow [0, \infty)$ is called a measure of noncompactness if it satisfies the following conditions:

- (1) The family $\text{Ker}\mu = \{X \in M_E : \mu(X) = 0\}$ is nonempty and $\text{Ker}\mu \subseteq N_E$.
- (2) $X \subseteq Y \implies \mu(X) \leq \mu(Y)$.
- (3) $\mu(\overline{X}) = \mu(X)$.
- (4) $\mu(\text{Conv}(X)) = \mu(X)$.
- (5) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
- (6) If $\{X_n\}$ is a sequence of closed sets from M_E such that $X_{n+1} \subseteq X_n$ for $n = 1, 2, \dots$ and $\lim_{n \rightarrow \infty} \mu(X_n) = 0$, then $\bigcap_{n=1}^{\infty} X_n$ is nonempty.

Theorem 1 (Schauder [3]). Let U be a nonempty, bounded, closed and convex subset of a Banach space E . Then every continuous and compact map $F : U \rightarrow U$ has at least one fixed point in U .

Theorem 2 (Darbo[8]). Let Q be a nonempty, closed, bounded and convex subset of a Banach space E and $F : Q \rightarrow Q$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset X of Q . Then F has a fixed point in Q .

The following definitions, theorems and examples will be used further on.

Definition 2 ([7]). An element $(x, y) \in X \times X$ is called coupled fixed point of the mapping $F : X \times X \rightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.

Theorem 3 ([6]). Suppose $\mu_1, \mu_2, \dots, \mu_n$ are measures of noncompactness in $E_1, E_2, \dots, E_n$ respectively. Moreover assume that the function $F : [0, \infty)^n \rightarrow [0, \infty)$ is convex and

$F(x_1, x_2, \dots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, \dots, n$. Then

$$\mu(X) = F(\mu_1(X_1), \mu_2(X_2), \dots, \mu_n(X_n))$$

defines a measure of noncompactness in $E_1 \times E_2 \times \dots \times E_n$ where X_i denote the natural projection of X into E_i for $i = 1, 2, \dots, n$.

Example 1 ([2]). Let μ be a measure of noncompactness in the Banach space E and $F(x, y) = x + y$ for $(x, y) \in [0, \infty)^2$. Then F has all the properties in Theorem 3. Hence $\bar{\mu}(X) = \mu(X_1) + \mu(X_2)$ is a measure of noncompactness in the space $E \times E$ where $X_i, i = 1, 2$ denote the natural projections of X into E .

Now, inspired by Definition 2.4 of [1], the following definition is introduced which is basic for our main results.

Definition 3. Let Θ denote the class of those functions $\phi : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ which satisfy the following conditions

- (δ_1) $\phi(t_1, t_2) = \phi$ is increasing in t_1 and t_2 .
- (δ_2) $t_{n+1} < \phi(t_n, t_n)$ implies that $t_{n+1} < t_n$ for each positive sequence $\{t_n\}$.
- (δ_3) $\phi(u, u) \leq u$ for each $u \in [0, \infty)$.

Example 2. Let $\phi : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ defined by

$$\phi(t_1, t_2) = at_1 + bt_2$$

where $a + b = 1$ and $b \neq 1$. Then $\phi \in \Theta$

Definition 4 ([4]). Let $F : (0, \infty) \rightarrow \mathbb{R}$ and $\theta : (0, \infty) \rightarrow (0, \infty)$ be two mappings. Throughout the paper, let Δ be the set of all pairs (θ, F) satisfying the following:

- (Δ_1) $\theta(t_n) \rightarrow 0$ for each strictly decreasing sequence $\{t_n\}$;
- (Δ_2) F is strictly increasing function;
- (Δ_3) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$;
- (Δ_4) If $\{t_n\}$ be a decreasing sequence such that $t_n \rightarrow 0$ and $\theta(t_n) < F(t_n) - F(t_{n+1})$, then we have $\sum_{n=1}^{\infty} t_n < \infty$.

Example 3 ([4]). Let $F(t) = \ln(t)$ and $\theta(t) = -\ln(\alpha(t))$ for each $t \in (0, \infty)$, where $\alpha : (0, \infty) \rightarrow (0, 1)$ satisfies $\limsup_{s \rightarrow t^+} \alpha(s) < 1$, for all $t \in (0, \infty)$. Then $(\theta, F) \in \Delta$.

2. SOME FIXED POINT RESULTS VIA A NEW GENERALIZED CONTRACTIVE CONDITION

Now inspired by the existing contractive condition in [1], the main result of this paper is stated.

Theorem 4. Let C be a nonempty bounded, closed and convex subset of a Banach space E . Assume $T : C \rightarrow C$ is a continuous operator satisfying

$$\theta(\psi(\mu(T(X)))) + f(\psi(\mu(T(X)))) \leq f(\phi(\psi(\mu(X)), \psi(\mu(T(X)))) \quad (2.1)$$

for all nonempty subset X of C , where μ is an arbitrary measure of noncompactness defined in E , $\psi : [0, \infty) \rightarrow [0, \infty)$ is nondecreasing such that $\psi(t) = 0$ if and only if $t = 0$, $\phi \in \Theta$ and $(\theta, f) \in \Delta$. Then T has a fixed point in C .

Proof. Define a sequence $\{C_n\}$ by letting $C_0 = C$ and $C_n = \text{Conv}(TC_{n-1})$, $n \geq 1$. If there exists an integer $N \geq 0$ such that $\mu(C_N) = 0$, then C_N is relatively compact and Theorem 1 implies that T has a fixed point. So we assume that $\mu(C_n) > 0$ for each $n \in \mathbb{N}$. By our assumptions, we get

$$\begin{aligned}\theta(\psi(\mu(C_{n+1}))) + f(\psi(\mu(C_{n+1}))) &\leq f(\phi(\psi(\mu(C_n)), \psi(\mu(C_n)))) \\ &\leq f(\psi(\mu(C_n))).\end{aligned}\quad (2.2)$$

Consequently, by (Δ_2) , we have

$$\psi(\mu(C_{n+1})) < \psi(\mu(C_n)).$$

Since the sequence $\{\psi(\mu(C_n))\}$ is non-increasing sequence, there exists $t \geq 0$ such that $\lim_{n \rightarrow \infty} \psi(\mu(C_n)) = t$. Now we show that $t = 0$. On the contrary, assume that $t > 0$. From (2.2) we have

$$\sum_{i=1}^{\infty} \theta(\psi(\mu(C_{i+2}))) \leq f(\psi(\mu(C_2))) - f(\psi(\mu(C_{n+1}))). \quad (2.3)$$

Keeping in mind our assumptions we have $\theta(\psi(\mu(C_n))) \rightarrow 0$. So, we infer that $\sum_{i=1}^{i=n-1} \theta(\psi(\mu(C_{i+2}))) = \infty$ and consequently $\lim_{n \rightarrow \infty} f(\psi(\mu(C_{n+1}))) = -\infty$. So, by Δ_3 we get $\psi(\mu(C_n)) \rightarrow 0$ which is a contradiction. Hence, $\psi(\mu(C_n)) \rightarrow 0$ as $n \rightarrow \infty$. Now we prove that $\mu(C_n) \rightarrow 0$. Since $\{\psi(\mu(C_n))\}$ is a decreasing sequence and ψ is nondecreasing, we obtain that $\{\mu(C_n)\}$ is a decreasing sequence of positive numbers. Consequently there exists $r \geq 0$ such that $\lim_{n \rightarrow \infty} \mu(C_n) = r^+$. Since ψ is nondecreasing, we arrive that

$$\psi(r) \leq \psi(\mu(C_n)). \quad (2.4)$$

Letting $n \rightarrow \infty$ in (2.4), we have $\psi(r) \leq 0$. So $r = 0$ which implies that $\mu(C_n) \rightarrow 0$. On the other hand, since $C_{n+1} \subseteq C_n$ and $\mu(C_n) \rightarrow 0$, from condition (6) of Definition 1 we obtain that $C_\infty = \bigcap_{n=1}^{\infty} C_n$ is nonempty, closed, convex and $C_\infty \subseteq C$. Moreover, taking in to account our assumptions we infer that C_∞ is invariant under the operator T and $C_\infty \in \text{Ker } \mu$. Consequently, from Theorem 1 we deduce that T has a fixed point. $\square$

Corollary 1. *Let C be a nonempty bounded, closed and convex subset of the Banach space E . Assume $T : C \rightarrow C$ is a continuous operator satisfying*

$$\theta(\mu(T(X))) + f(\mu(T(X))) \leq f(\mu(X)) \quad (2.5)$$

for all nonempty subset X of C , where μ is an arbitrary measure of noncompactness defined in E and $(\theta, f) \in \Delta$. Then T has at least one fixed point in C .

Proof. Obviously, (2.5) is a special case of (2.1) with $\psi(t) = t$ and $\phi(t_1, t_2) = t_1$. Hence, the application of Theorem 4 completes the proof. $\square$

Corollary 2. *Let C be a nonempty bounded, closed and convex subset of a Banach space E . Assume $T : C \rightarrow C$ is a continuous operator satisfying*

$$\mu(T(X)) \leq \alpha(\mu(T(X)))\mu(X) \quad (2.6)$$

for all nonempty subset X of C , where μ is an arbitrary measure of noncompactness defined in E and $\alpha : (0, \infty) \rightarrow [0, 1)$ with $\limsup_{t \rightarrow r^+} \alpha(t) < 1$ for all $r \geq 0$. Then T has a fixed point in C .

Proof. By applying Corollary 1 with $f(t) = \ln(t)$ and $\theta(t) = \ln(\alpha(t))$, the proof will be completed. $\square$

Corollary 3. Let C be a nonempty bounded, closed and convex subset of the Banach space E . Assume $T : C \rightarrow C$ is a continuous operator satisfying

$$\mu(T(X)) \leq \varphi(\mu(T(X)))\mu(X) \quad (2.7)$$

for all nonempty subset X of C , where μ is an arbitrary measure of noncompactness defined in E and $\varphi : (0, \infty) \rightarrow [0, 1)$ is a non-decreasing function. Then T has a fixed point in C .

Proof. Since φ is non-decreasing function, we have $\limsup_{t \rightarrow r^+} \varphi(t) < 1$ for all $r \geq 0$.

Thus, Corollary 2 completes the proof. $\square$

Theorem 5. Let C be a nonempty bounded, closed and convex subset of a Banach space E . Assume $T : C \rightarrow C$ is a continuous operator satisfying

$$\theta(\mu(X)) + f(\mu(T(X))) \leq f(\mu(X)) \quad (2.8)$$

for all nonempty subset X of C , where μ is an arbitrary measure of noncompactness defined in E and $(\theta, f) \in \Delta$. Then T has a fixed point in C .

Proof. Similar to the proof of Theorem 4, we can construct the sequence $\{C_n\}$ such that

$$\theta(\mu(C_n)) + f(\mu(C_{n+1})) \leq f(\mu(C_n)), \quad (2.9)$$

which yields that $\mu(C_{n+1}) < \mu(C_n)$. So there exists $r \geq 0$ such that $\mu(C_n) \rightarrow r$. On the other hand, from (2.9) we have

$$\sum_{i=1}^{n-1} \theta(\mu(C_{i+1})) < f(\mu(C_2)) - f(\mu(C_{n+1})).$$

Now by using the technique in Theorem 4, we have $\mu(C_n) \rightarrow 0$. Therefore, taking into account that $C_{n+1} \subseteq C_n$, from condition (6) of Definition 1 we conclude that $C_\infty = \bigcap_{n=1}^\infty C_n$ is nonempty, closed, convex and $C_\infty \subseteq C$. Moreover, the set C_∞ is invariant under the operator T and belong to $\text{Ker} \mu$. Consequently, from Theorem 1 we deduce that T has a fixed point. $\square$

Corollary 4. Let C be a nonempty bounded, closed and convex subset of the Banach space E . Assume $T : C \rightarrow C$ is a continuous operator satisfying

$$\mu(T(X)) \leq \alpha(\mu(X))\mu(X) \quad (2.10)$$

for all nonempty subset X of C , where μ is an arbitrary measure of noncompactness defined in E and $\alpha : (0, \infty) \rightarrow [0, 1)$ with $\limsup_{t \rightarrow r^+} \alpha(t) < 1$ for all $r \geq 0$. Then T has a fixed point in C .

Proof. Taking $f(t) = \ln(t)$ and $\theta(t) = \ln(\alpha(t))$ in Theorem 5, the result follows. $\square$

Corollary 5. Let C be a nonempty bounded, closed and convex subset of the Banach space E . Assume $T : C \rightarrow C$ is a continuous operator satisfying

$$\mu(T(X)) \leq \varphi(\mu(X))\mu(X) \quad (2.11)$$

for all nonempty subset X of C , where μ is an arbitrary measure of noncompactness defined in E and $\varphi : [0, \infty) \rightarrow [0, 1)$ is a non-decreasing function. Then T has a fixed point in C .

Proof. Since φ is non-decreasing, so $\limsup_{t \rightarrow r^+} \varphi(t) < 1$ for all $r \geq 0$. Consequently, applying Corollary 4 with $\varphi = \alpha$, we have the result. $\square$

3. COUPLED FIXED POINT RESULTS

In this section, as an application of Theorem 4 we study the existence of coupled fixed point to a special class of operators. Let Ψ denote all functions $\psi : [0, \infty) \rightarrow [0, \infty)$ such that

- (1) ψ is nondecreasing and $\psi(t) = 0$ if and only if $t = 0$,
- (2) $\psi(t + s) \leq \psi(t) + \psi(s)$ for all $t, s \geq 0$.

Theorem 6. Let C be a nonempty bounded, closed and convex subset of the Banach space E . Assume $T : C \times C \rightarrow C$ is a continuous operator satisfying

$$\begin{aligned} \theta(\psi(\mu(T(X_1 \times X_2)))) &+ f(\psi(\mu(T(X_1 \times X_2)))) \\ &\leq \frac{1}{2} f(\phi(\psi(\mu(X_1) + \mu(X_2)), \psi(\mu(X_1) + \mu(X_2)))) \end{aligned} \quad (3.1)$$

for all nonempty subsets $X_1, X_2 \subset C$, where μ is an arbitrary measure of noncompactness defined in E , $\psi \in \Psi$, $\phi \in \Theta$ and $(\theta, f) \in \Delta$ such that

$$\theta(t + s) \leq \theta(t) + \theta(s) \quad \text{and} \quad f(t + s) \leq f(t) + f(s).$$

Then T has at least a coupled fixed point.

Proof. We consider a mapping $\bar{T} : C \times C \rightarrow C \times C$ defined by $\bar{T}(x, y) = (T(x, y), T(y, x))$. Since T is continuous, the continuity of $\bar{T}$ is followed. From example 1, we know that $\bar{\mu}(X) = \mu(X_1) + \mu(X_2)$ defines a measure of noncompactness on $E \times E$ for any $X_1, X_2 \subseteq C$ where $X_i = 1, 2$ indicate the natural projection of X into E . Let

$X \subseteq C \times C$ be a nonempty subset. Then, due to (3.1) and condition (2) of Definition 1 we infer that

$$\begin{aligned}
 & \theta(\psi(\bar{\mu}(\bar{T}(X)))) + f(\psi(\bar{\mu}(\bar{T}(X)))) \\
 & \leq \theta(\psi(\bar{\mu}(T(X_1 \times X_2) \times T(X_2 \times X_1)))) \\
 & + f(\psi(\bar{\mu}(T(X_1 \times X_2) \times T(X_2 \times X_1)))) \\
 & = \theta(\psi(\mu(T(X_1 \times X_2)) + \mu(T(X_2 \times X_1)))) \\
 & + f(\psi(\mu(T(X_1 \times X_2)) + \mu(T(X_2 \times X_1)))) \quad (3.2) \\
 & \leq \theta(\psi(\mu(T(X_1 \times X_2)))) + \theta(\psi(\mu(T(X_2 \times X_1)))) \\
 & + f(\psi(\mu(T(X_1 \times X_2)))) + f(\psi(\mu(T(X_2 \times X_1)))) \\
 & \leq f(\phi(\psi(\mu(X_1) + \mu(X_2)), \psi(\mu(X_1) + \mu(X_2)))) \\
 & = f(\phi(\psi(\bar{\mu}(X)), \psi(\bar{\mu}(X)))).
 \end{aligned}$$

So all the conditions of Theorem 4 hold true and $\bar{T}$ has a fixed point. Hence, T has a coupled fixed point. $\square$

Corollary 6. Let C be a nonempty bounded, closed and convex subset of a Banach space E . Assume $T : C \times C \rightarrow C$ is a continuous operator satisfying

$$\theta(\mu(T(X_1 \times X_2))) + f(\mu(T(X_1 \times X_2))) \leq f(\mu(X_1) + \mu(X_2)) \quad (3.3)$$

for all nonempty subsets $X_1, X_2 \subset C$, where μ is an arbitrary measure of noncompactness defined in E and $(\theta, f) \in \Delta$ such that $\theta(t + s) \leq \theta(t) + \theta(s)$ and $f(t + s) \leq f(t) + f(s)$. Then T has at least a coupled fixed point.

Proof. Take $\phi(t_1, t_2) = t_1$ and $\psi = I$ in Theorem 6. $\square$

Corollary 7. Let C be a nonempty bounded, closed and convex subset of the Banach space E . Assume $T : C \times C \rightarrow C$ is a continuous operator satisfying

$$\mu(T(X_1 \times X_2)) \leq \alpha(\mu(T(X_1 \times X_2)))(\mu(X_1) + \mu(X_2)) \quad (3.4)$$

for all nonempty subsets $X_1, X_2 \subset C$, where μ is an arbitrary measure of noncompactness defined in E and $\alpha : (0, \infty) \rightarrow [0, 1)$ with $\limsup_{t \rightarrow r^+} \alpha(t) < 1$ for all $r \geq 0$.

Then T has at least a coupled fixed point.

Proof. Let $\theta(t) = \ln(\alpha(t))$ and $f(t) = \ln(t)$. So from (3.4) we have

$$\theta(\mu(T(X_1 \times X_2))) + f(\mu(T(X_1 \times X_2))) \leq f(\mu(X_1) + \mu(X_2))$$

for all nonempty subsets $X_1, X_2 \subseteq C$. Now, Corollary 6 guarantees that T has a coupled fixed point. $\square$

4. SOLVABILITY OF SYSTEMS OF INTEGRAL EQUATIONS

This section is devoted to the study of the existence of solutions for the systems of integral equations

$$\begin{aligned} x(t) &= F\left(t, h(t, x(\alpha(t)), y(\alpha(t))), \right. \\ &\quad \left. (Tx)(t) \int_0^{\eta(t)} \varphi(t, s) g(t, s, x(\gamma(s)), y(\gamma(s))) ds \right) \\ y(t) &= F\left(t, h(t, y(\alpha(t)), x(\alpha(t))), \right. \\ &\quad \left. (Ty)(t) \int_0^{\eta(t)} \varphi(t, s) g(t, s, y(\gamma(s)), x(\gamma(s))) ds \right), \end{aligned} \quad (4.1)$$

in the space $BC(R_+) \times BC(R_+)$ consisting of all bounded and continuous real functions on R_+ . For $x \in BC(R_+)$ the norm of x is defined by $\|x\| = \sup\{|x(t)| : t \geq 0\}$. Now, we recall the definition of measure of noncompactness in the space $BC(R_+)$ which was introduced by Banas in [5]. Fix a nonempty bounded subset X of $BC(R_+)$ and a positive number $K > 0$. For $x \in X$ and $\varepsilon > 0$ put

$$\begin{aligned} \omega^K(x, \varepsilon) &= \sup\{|x(t) - y(t)|; t, s \in [0, K], |t - s| \leq \varepsilon\}, \\ \omega^K(X, \varepsilon) &= \sup\{\omega^K(x, \varepsilon); x \in X\}, \\ \omega_0^K(X) &= \lim_{\varepsilon \rightarrow 0} \omega^K(X, \varepsilon), \\ \omega_0(X) &= \lim_{K \rightarrow \infty} \omega_0^K(X). \end{aligned}$$

Furthermore, for a fixed number $t \in R^+$, let us define the following equation:

$$\begin{aligned} X(t) &= \{x(t); x \in X\}, \\ diam X(t) &= \sup\{|x(t) - y(t)|; x, y \in X\}. \end{aligned}$$

Finally, let

$$\mu(X) = \omega_0(X) + \limsup_{t \rightarrow \infty} diam X(t). \quad (4.2)$$

Banas [5] proved that the above function is a measure of noncompactness in the space $BC(R_+)$. Now, the existence of solutions for the integral equations (4.1) is studied under the following assumptions.

- (1) $\gamma, \alpha, \eta : R_+ \rightarrow R_+$ are continuous functions and $\alpha(t) \rightarrow \infty$ as $t \rightarrow \infty$.
- (2) The functions $F : R_+ \times R \times R \rightarrow R$ and $h : R_+ \times R \times R \rightarrow R$ are continuous functions and there exist positive real number $\tau > 0$ such that

$$\begin{aligned} |F(t, x_1, x_2) - F(t, y_1, y_2)| &\leq e^{-\tau} (|x_1 - y_1| + |x_2 - y_2|), \\ |h(t, x_1, x_2) - h(t, y_1, y_2)| &\leq e^{-\tau} (|x_1 - y_1| + |x_2 - y_2|), \end{aligned}$$

for $t \in R_+$ and $x_1, x_2, y_1, y_2 \in R$.

- (3) The functions $t \rightarrow F(t, 0, 0)$ and $t \rightarrow h(t, 0, 0)$ are bounded on R_+ i.e $M_1, M_2 < \infty$ where $M_1 = \sup\{F(t, 0, 0); t \geq 0\}$, $M_2 = \sup\{h(t, 0, 0); t \geq 0\}$.
 (4) $T : BC(R_+) \rightarrow BC(R_+)$ is a continuous operator such that

$$| (Tx)(t_1) - (Tx)(t_2) | \leq \psi_1(|x(t_2) - x(t_1)|), \quad | (Tx)(t) | \leq a + b \|x\|,$$

$$| (Tx)(t) - (Tu)(t) | \leq \psi_1(|x(t) - u(t)|).$$

for $x \in BC(R_+)$ and $t_2, t_1 \in R_+$, where $\psi_1 : R_+ \rightarrow R_+$ is continuous and nondecreasing with $\psi_1(0) = 0$ and a, b are positive real numbers.

- (5) $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous on $\mathbb{R}^+ \times \mathbb{R}^+$ and the function $t \rightarrow \varphi(t, s)$ is nondecreasing for each $s \in \mathbb{R}^+$.
 (6) There exist continuous functions

$$a, b : R_+ \rightarrow R_+ \quad g : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

such that

$$\lim_{t \rightarrow \infty} a(t) \int_0^{\eta(t)} b(s) ds = 0 \quad | \varphi(t, s) g(t, s, x, y) | \leq a(t) b(s),$$

for all $t, s \in R^+$ and $x, y \in R$ such that $s \leq t$.

- (7) There exists a positive solution r_0 of the inequality

$$2re^{-2\tau} + M_2e^{-\tau} + e^{-\tau}(a + br)q + M_1 \leq r \quad (4.3)$$

where $q = \sup\{a(t) \int_0^{\eta(t)} b(s) ds; t \geq 0\}$.

Theorem 7. Under assumptions (1) – (7), Eq (4.1) has at least one solution in the space $BC(R_+) \times BC(R_+)$.

Proof. Let us consider the operator G on the space $BC(R_+) \times BC(R_+)$ by

$$\begin{aligned} G(x, y)(t) &= F\left(t, h(t, x(\alpha(t)), y(\alpha(t))), \right. \\ &\quad \left. (Tx)(t) \int_0^{\eta(t)} \varphi(t, s) g(t, s, x(\gamma(s)), y(\gamma(s))) ds \right). \end{aligned} \quad (4.4)$$

We know that the space $BC(R_+) \times BC(R_+)$ is equipped with the norm

$$\|(x, y)\| = \|x\|_\infty + \|y\|_\infty.$$

On the other hand, obviously G is continuous. Taking into account our assumptions, we infer that

$$\begin{aligned}
 |G(x, y)(t)| &\leq e^{-\tau}(|h(t, x(\alpha(t)), y(\alpha(t))) - h(t, 0, 0)| \\
 &\quad + |h(t, 0, 0)| + |(Tx)(t) \int_0^{\eta(t)} \varphi(t, s) u(t, s, x(\gamma(s)), y(\gamma(s))) ds|) \\
 &\quad + |F(t, 0, 0, 0, 0)| \leq e^{-\tau}(e^{-\tau}(|x(\alpha(t))| + |y(\alpha(t))|) \\
 &\quad + M_2 + (a + b \|x\|)a(t) \int_0^{\eta(t)} b(s) ds) + M_1 \\
 &\leq e^{-2\tau}(\|x\| + \|y\|) + e^{-\tau}M_2 + e^{-\tau}(a + b \|x\|)q + M_1.
 \end{aligned} \tag{4.5}$$

Consequently, from (4.5) and condition (7) we infer that $G(\overline{B_{r_0}} \times \overline{B_{r_0}}) \subseteq \overline{B_{r_0}}$. Now, we indicate that G is a continuous on $\overline{B_{r_0}} \times \overline{B_{r_0}}$. To do this, let $\varepsilon > 0$ be an arbitrary fixed number and $(x, y), (u, v) \in \overline{B_{r_0}} \times \overline{B_{r_0}}$ such that $\|(x, y) - (u, v)\|_{\overline{B_{r_0}} \times \overline{B_{r_0}}} < \frac{\varepsilon}{2}$. Then, we have

$$\begin{aligned}
 &|G(x, y)(t) - G(u, v)(t)| \\
 &\leq e^{-\tau}(|h(t, x(\alpha(t)), y(\alpha(t))) - h(t, u(\alpha(t)), v(\alpha(t)))|) \\
 &\quad + e^{-\tau}(|(Tx)(t) \int_0^{\eta(t)} \varphi(t, s) u(t, s, x(\gamma(s)), y(\gamma(s))) ds \\
 &\quad - (Tu)(t) \int_0^{\eta(t)} \varphi(t, s) g(t, s, u(\gamma(s)), v(\gamma(s))) ds|) \\
 &\leq e^{-2\tau}(|x(\alpha(t)) - u(\alpha(t))| + |y(\alpha(t)) - v(\alpha(t))|) \\
 &\quad + e^{-\tau} |(Tx)(t) \int_0^{\eta(t)} \varphi(t, s) (g(t, s, x(\gamma(s)), y(\gamma(s))) \\
 &\quad - g(t, s, u(\gamma(s)), v(\gamma(s)))) ds| \\
 &\quad + e^{-\tau} |(Tx)(t) - (Tu)(t)| \times \\
 &\quad |\int_0^{\eta(t)} \varphi(t, s) g(t, s, u(\gamma(s)), v(\gamma(s))) ds|.
 \end{aligned} \tag{4.6}$$

Thus, from (4.6) we have

$$\begin{aligned}
 & | G(x, y)(t) - G(u, v)(t) | \\
 & \leq e^{-2\tau} (\| x - u \| + \| y - v \|) \\
 & + e^{-\tau} (a + b \| x \|) | \int_0^{\eta(t)} \varphi(t, s) (g(t, s, x(\gamma(s)), y(\gamma(s))) \\
 & - g(t, s, u(\gamma(s)), v(\gamma(s)))) ds | \\
 & + e^{-\tau} \| Tx - Tu \| a(t) \int_0^{\eta(t)} b(s) ds.
 \end{aligned} \tag{4.7}$$

Furthermore, by Condition (6) there exists $T > 0$ such that

$$a(t) \int_0^{\eta(t)} b(s) ds < \frac{\varepsilon}{2}. \tag{4.8}$$

for all $t > T$.

Hence, by combining the inequalities (4.7) and (4.8), we deduce that

$$\begin{aligned}
 | G(x, y)(t) - G(u, v)(t) | & \leq e^{-2\tau} \varepsilon + 2e^{-\tau} (a + b \| x \|) \varepsilon \\
 & + \| (Tx) - (Tu) \| e^{-\tau} \varepsilon.
 \end{aligned} \tag{4.9}$$

Now we define the equality $\omega^T(g, \varepsilon)$ as follows:

$$\begin{aligned}
 \omega^T(g, \varepsilon) = \sup \{ & | g(t, s, x, y) - g(t, s, u, v) | : t \in [0, T], s \in [0, \eta_T]; x, y, u \\
 & , v \in [-r_0, r_0], \| (x, y) - (u, v) \|_{BC(R_+) \times BC(R_+)} < \varepsilon \},
 \end{aligned}$$

where

$$\eta_T = \sup \{ \eta(t); t \in [0, T] \}.$$

On the other hand from (4.6) for an arbitrary fixed $t \in [0, T]$ we have

$$\begin{aligned}
 | G(x, y)(t) - G(u, v)(t) | & \leq e^{-2\tau} (\| x - u \| + \| y - v \|) \\
 & + e^{-\tau} (a + b \| x \|) | \int_0^{\eta(t)} \varphi(t, s) \omega^T(g, \varepsilon) ds | \\
 & + e^{-\tau} \| x - u \| a(t) \int_0^{\eta(t)} b(s) ds.
 \end{aligned} \tag{4.10}$$

By applying the continuity of g on $[0, T] \times [0, \eta_T] \times [-r_0, r_0] \times [-r_0, r_0]$, we have $\omega^T(g, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence, due to (4.9) and (4.10) we conclude that G is continuous. Now, let $T, \varepsilon \in R_+$ and X_1, X_2 are arbitrary nonempty subsets of $\overline{B_{r_0}}$. Assume $t_1, t_2 \in [0, T]$ such that $| t_2 - t_1 | \leq \varepsilon$ and $\eta(t_1) \leq \eta(t_2)$. In view of our assumptions,

for $(x, y) \in X_1 \times X_2$ we get

$$\begin{aligned}
& | G(x, y)(t_2) - G(x, y)(t_1) | \\
& \leq | G(t_2, h(t_2, x(\alpha(t_2))), y(\alpha(t_2))), \\
& (Tx)(t_2) \int_0^{\eta(t_2)} \varphi(t_2, s) g(t_2, s, x(\gamma(s)), y(\gamma(s))) ds \\
& - G(t_1, h(t_1, x(\alpha(t_1))), y(\alpha(t_1))), (Tx)(t_1) \times \\
& \int_0^{\eta(t_1)} \varphi(t_1, s) g(t_1, s, x(\gamma(s)), y(\gamma(s))) ds | \quad (4.11) \\
& \leq e^{-\tau} (| h(t_2, x(\alpha(t_2))), y(\alpha(t_2))) - h(t_2, x(\alpha(t_1)), y(\alpha(t_1))) | \\
& + | h(t_2, x(\alpha(t_1))), y(\alpha(t_1))) - h(t_1, x(\alpha(t_1)), y(\alpha(t_1))) | \\
& + e^{-\tau} | (Tx)(t_2) \int_0^{\eta(t_2)} \varphi(t_2, s) g(t_2, s, x(\gamma(s)), y(\gamma(s))) ds \\
& - (Tx)(t_1) \int_0^{\eta(t_1)} \varphi(t_1, s) g(t_1, s, x(\gamma(s)), y(\gamma(s))) ds |).
\end{aligned}$$

Thus, from (4.11) we get

$$\begin{aligned}
& | G(x, y)(t_2) - G(x, y)(t_1) | \\
& \leq e^{-2\tau} (| x(\alpha(t_2)) - x(\alpha(t_1)) | + | y(\alpha(t_2)) - y(\alpha(t_1)) |) + e^{-\tau} \omega_{r_0}^T(h, \varepsilon) \\
& + e^{-\tau} (| (Tx)(t_2) \int_0^{\eta(t_2)} \varphi(t_2, s) g(t_2, s, x(\gamma(s)), y(\gamma(s))) ds \\
& - (Tx)(t_1) \int_0^{\eta(t_1)} \varphi(t_1, s) g(t_1, s, x(\gamma(s)), y(\gamma(s))) ds |) \quad (4.12) \\
& \leq e^{-2\tau} (\omega^T(x, \omega^T(\alpha, \varepsilon)) + \omega^T(y, \omega^T(\alpha, \varepsilon))) + e^{-\tau} \omega_{r_0}^T(h, \varepsilon) \\
& + e^{-\tau} (| (Tx)(t_2) \int_0^{\eta(t_2)} \varphi(t_2, s) g(t_2, s, x(\gamma(s)), y(\gamma(s))) ds \\
& - (Tx)(t_1) \int_0^{\eta(t_1)} \varphi(t_1, s) g(t_1, s, x(\gamma(s)), y(\gamma(s))) ds |).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& | (Tx)(t_2) \int_0^{\eta(t_2)} \varphi(t_2, s) g(t_2, s, x(\gamma(s)), y(\gamma(s))) ds \\
& - (Tx)(t_1) \int_0^{\eta(t_1)} \varphi(t_1, s) g(t_1, s, x(\gamma(s)), y(\gamma(s))) ds | \\
& \leq | (Tx)(t_2) \int_0^{\eta(t_2)} \varphi(t_2, s) g(t_2, s, x(\gamma(s)), y(\gamma(s))) ds \\
& - (Tx)(t_1) \int_0^{\eta(t_2)} \varphi(t_2, s) g(t_2, s, x(\gamma(s)), y(\gamma(s))) ds | \\
& + | (Tx)(t_1) \int_0^{\eta(t_2)} \varphi(t_2, s) g(t_2, s, x(\gamma(s)), y(\gamma(s))) ds \\
& - (Tx)(t_1) \int_0^{\eta(t_2)} \varphi(t_1, s) g(t_2, s, x(\gamma(s)), y(\gamma(s))) ds | \\
& + | (Tx)(t_1) \int_0^{\eta(t_2)} \varphi(t_1, s) g(t_2, s, x(\gamma(s)), y(\gamma(s))) ds \\
& - (Tx)(t_1) \int_0^{\eta(t_1)} \varphi(t_1, s) g(t_1, s, x(\gamma(s)), y(\gamma(s))) ds | \\
& \leq | (Tx)(t_2) - (Tx)(t_1) | \int_0^{\eta(t_2)} \varphi(t_2, s) g(t_2, s, x(\gamma(s)), y(\gamma(s))) ds | \\
& + | (Tx)(t_1) | \int_0^{\eta(t_2)} | \varphi(t_2, s) - \varphi(t_1, s) | g(t_2, s, x(\gamma(s)), y(\gamma(s))) ds | \\
& + | (Tx)(t_1) | \int_{\eta(t_1)}^{\eta(t_2)} | \varphi(t_1, s) | g(t_2, s, x(\gamma(s)), y(\gamma(s))) \\
& - g(t_1, s, x(\gamma(s)), y(\gamma(s))) ds \leq \psi_1(\omega(x, \varepsilon) \parallel \varphi \parallel a(t_2) \int_0^{\eta(t_2)} b(s) ds \\
& + (a + b \parallel x \parallel) \omega_\varphi(\varepsilon, \cdot) a(t_2) \int_0^{\eta(t_2)} b(s) ds \\
& + (a + b \parallel x \parallel) \parallel \varphi \parallel \omega_{r_0}^T(g, \varepsilon) \omega^T(\eta, \varepsilon).
\end{aligned} \tag{4.13}$$

Now, take into consideration (4.12) and (4.13) we have

$$\begin{aligned}
 |G(x, y)(t_2) - G(x, y)(t_1)| &\leq e^{-2\tau}(\omega^T(x, \omega^T(\alpha, \varepsilon)) + \omega^T(y, \omega^T(\alpha, \varepsilon))) \\
 &\quad + e^{-\tau}\omega_{r_0}^T(h, \varepsilon) + e^{-\tau}(\psi_1(\omega(x, \varepsilon)) \times \\
 &\quad \|\varphi\| a(t_2) \int_0^{\eta(t_2)} b(s)ds) + e^{-\tau}(a + b\|x\|) \times \\
 &\quad \omega_\varphi(\varepsilon, \cdot) \times a(t_2) \int_0^{\eta(t_2)} b(s)ds \\
 &\quad + e^{-\tau}(a + b\|x\|)\|\varphi\| \times \\
 &\quad \omega_{r_0}^T(g, \varepsilon)\omega^T(\eta, \varepsilon).
 \end{aligned} \tag{4.14}$$

where

$$\begin{aligned}
 \omega_{r_0}^T(g, \varepsilon) &= \sup\{|g(t_1, s, x, y) - g(t_2, s, x, y)| : t_1, t_2 \in [0, T], \\
 &\quad |t_1 - t_2| < \varepsilon, s \in [0, \eta_T], x, y \in [-r_0, r_0]\}, \\
 \omega^T(x, \omega^T(\alpha, \varepsilon)) &= \sup\{|x(t_1) - x(t_2)| : t_1, t_2 \in [0, T], |t_1 - t_2| \leq \omega^T(\alpha, \varepsilon)\}, \\
 \omega_{r_0}^T(h, \varepsilon) &= \sup\{|h(t_2, x, y) - h(t_1, x, y)| : t_1, t_2 \in [0, T], |t_1 - t_2| \leq \varepsilon \\
 &\quad , x, y \in [-r_0, r_0]\}, \\
 \omega_\varphi(\varepsilon, \cdot) &= \sup\{|\varphi(t, s) - \varphi(\acute{t}, s)| : t, \acute{t} \in [0, T], |t - \acute{t}| \leq \varepsilon\}, \\
 \omega^T(\alpha, \varepsilon) &= \sup\{|\alpha(t_2) - \alpha(t_1)| : t_2, t_1 \in [0, T], |t_2 - t_1| < \varepsilon\}.
 \end{aligned} \tag{4.15}$$

Moreover, in the light of the uniform continuity of the functions g , h and φ on $[0, T] \times [0, \eta_T] \times [-r_0, r_0] \times [-r_0, r_0]$, $[0, T] \times [-r_0, r_0] \times [-r_0, r_0]$ and $[0, T] \times [0, T]$, we have $\omega_{r_0}^T(g, \varepsilon) \rightarrow 0$, $\omega_{r_0}^T(h, \varepsilon) \rightarrow 0$ and $\omega_\varphi(\varepsilon, \cdot) \rightarrow 0$. Also because of the uniform continuity of α, η on $[0, T]$ we have $\omega^T(\alpha, \varepsilon) \rightarrow 0$, $\omega^T(\eta, \varepsilon) \rightarrow 0$. Now, this remarks and the inequalities in (4.14) imply that

$$\omega_0^T(G(X_1 \times X_2), \varepsilon) \leq e^{-2\tau}(\omega_0^T(X_1) + \omega_0^T(X_2)), \tag{4.16}$$

and hence

$$\omega_0(G(X_1 \times X_2)) \leq e^{-2\tau}(\omega_0(X_1) + \omega_0(X_2)). \tag{4.17}$$

Now, arbitrary elements $(x, y), (u, v) \in X_1 \times X_2$ are chosen so that for $t \in R^+$, we have

$$\begin{aligned} |G(x, y)(t) - G(u, v)(t)| &\leq e^{-\tau}(|x(\alpha(t)) - u(\alpha(t))| + |y(\alpha(t)) - v(\alpha(t))|) \\ &\quad + e^{-\tau}(|(Tx)(t) - (Tu)(t)| + |\int_0^{\eta(t)} \varphi(t, s)g(t, s, x(\gamma(s)), y(\gamma(s)))ds \\ &\quad - \int_0^{\eta(t)} \varphi(t, s)g(t, s, u(\gamma(s)), v(\gamma(s)))ds|) \\ &\leq e^{-\tau}(\text{diam}X_1(\alpha(t)) + \text{diam}X_2(\alpha(t))) \\ &\quad + e^{-\tau}((a + b \|x\|) + (c + d \|u\|))a(t) \int_0^{\eta(t)} b(s)ds. \end{aligned} \quad (4.18)$$

Now, using (4.18) and the notion of diameter of a set, we have

$$\begin{aligned} \text{diam}G(X_1 \times X_2)(t) &\leq e^{-\tau}(\text{diam}X_1(\alpha(t)) + \text{diam}X_2(\alpha(t))) \\ &\quad + e^{-\tau}((a + b \|x\|) + (c + d \|x\|))a(t) \int_0^{\eta(t)} b(s)ds, \end{aligned} \quad (4.19)$$

and hence

$$\begin{aligned} \limsup_{t \rightarrow \infty} \text{diam}G(X_1 \times X_2)(t) &\leq e^{-\tau}(\limsup_{t \rightarrow \infty} \text{diam}X_1(\alpha(t)) \\ &\quad + \limsup_{t \rightarrow \infty} \text{diam}X_2(\alpha(t))). \end{aligned} \quad (4.20)$$

Combining (4.17), (4.20) and (4.2) we get

$$\mu(G(X_1 \times X_2)) \leq e^{-\tau}(\mu(X_1) + \mu(X_2)) \quad (4.21)$$

By passing to logarithms, we earn

$$\ln(\mu(G(X_1 \times X_2))) \leq \ln(e^{-\tau}(\mu(X_1) + \mu(X_2))).$$

Consequently,

$$\tau + \ln(\mu(G(X_1 \times X_2))) \leq \ln(\mu(X_1) + \mu(X_2)).$$

Then all conditions of Corollary 6 hold true with $F(t) = \ln(t)$ and $\theta(t) = \tau$ for all $t \in R_+$. Consequently, from Corollary 6 G has a coupled fixed point in the space $BC(R_+) \times BC(R_+)$. $\square$

Example 4. Now, we will study the following system of integral equations

$$\begin{cases} x(t) = e^{-t-\tau} \cos\left(\frac{e^{-t-\tau}}{(1+|x|+|y|)} + \cos(\frac{1}{1+|x(t)|}) \int_0^t e^s \arctan(\frac{e^{-3t+s}}{8+|x|+|y|})ds\right) \\ y(t) = e^{-t-\tau} \cos\left(\frac{e^{-t-\tau}}{(1+|x|+|y|)} + \cos(\frac{1}{1+|y(t)|}) \int_0^t e^s \arctan(\frac{e^{-3t+s}}{8+|y|+|x|})ds\right). \end{cases} \quad (4.22)$$

This system is a special case of the system of integral equations (4.1) with

$$F(t, x, y) = e^{-t-\tau} \cos(x + y), \quad h(t, x, y) = \frac{e^{-t-\tau}}{1 + |x| + |y|},$$

$$(Tx)(t) = \cos\left(\frac{1}{1 + |x(t)|}\right), \quad g(t, s, x, y) = \arctan\left(\frac{e^{-3t+s}}{8 + |y| + |x|}\right),$$

$$\varphi(t, s) = e^t, \quad \alpha(t) = \eta(t) = \gamma(t) = t.$$

It is easily seen that α, η, γ satisfy the assumption (1). Further, the function $F(t, 0, 0) = e^{-\tau-t}$ is bounded with $M_1 = e^{-\tau}$. Also, the function $|h(t, 0, 0)| = e^{-t-\tau}$ is bounded with $M_2 = e^{-\tau}$. Since $F(t, x, y) = e^{-t-\tau} \cos(x + y)$ and $h(t, x, y) = \frac{e^{-t-\tau}}{1 + |x| + |y|}$, then, for all $t \in R_+$ and $x_1, x_2, y_1, y_2 \in R$, we have

$$|F(t, x_1, y_1) - F(t, x_2, y_2)| \leq e^{-\tau} (|x_1 - x_2| + |y_1 - y_2|),$$

$$|h(t, x_1, y_1) - h(t, x_2, y_2)| \leq e^{-\tau} (|x_1 - x_2| + |y_1 - y_2|).$$

Consequently, F and h satisfy the assumption (2). In this example $(Tx)(t) = \cos(\frac{1}{1 + |x(t)|})$ verifies assumption (4) with $a = 1, b = 0$ and $\psi_1 = t$. Moreover, assumption (5) holds with $\varphi(t, s) = e^t$. On the other hand, for all $t, s \in R_+$ and $x, y \in R$ with $s \leq t$, we get

$$|\varphi(t, s)g(t, s, x, y)| \leq e^{-2t+s}.$$

Thus, assumption (6) holds with $a(t) = e^{-2t}$ and $b(s) = e^s$. Consequently, the existent inequality in assumption (7) has the form

$$2re^{-\tau} + e^{-2\tau} + e^{-\tau} \leq r.$$

It is easily seen that the last inequality have a positive solution. Consequently, all the conditions of Theorem 7 are satisfied and Theorem 7 guarantees that the system of integral equation (4.22) has at least on solution in the space $BC(R_+) \times BC(R_+)$.

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Author's address

A. Samadi

Department of Mathematics, Miyaneh Branch, Islamic Azad University, Miyaneh, Iran

E-mail address: samadiayub@m-iau.ac.ir



A NOVEL INTEGRABILITY ANALYSIS OF A GENERALIZED RIEMANN TYPE HYDRODYNAMIC HIERARCHY

ANATOLIY M. SAMOILENKO, YAREMA A. PRYKARPATSKYY,
DENIS BLACKMORE, AND ANATOLIY K. PRYKARPATSKI

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Abstract. The complete integrability of a generalized Riemann type hydrodynamic hierarchy is studied by means of a novel combination of symplectic and differential-algebraic tools. A compatible pair of polynomial Poissonian structures, a Lax representation and a related infinite hierarchy of conservation laws are constructed. The current investigation provides an interesting glimpse of what is apparently a far wider range of applications.

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1. INTRODUCTION

Since the Riemann classical works on two-dimensional hydrodynamic type equations and their invariants during the last decades there has been achieved great progress [7, 8] in studies of their analytical properties, in particular, in stating the existence of the Poissonian representations and infinite hierarchies of conservation laws. Important results in studying the existence of hierarchies of both local and non-local conservation laws for a general type of nonlinear differential equations were obtained in [15, 16] by means of differential-geometric methods, which in the two-dimensional case also makes it possible to construct the corresponding Lax type representations. Recently new mathematical approaches, based on differential-algebraic and differential geometric methods and techniques, were applied in works [5, 13] for studying the Lax type integrability of nonlinear differential equations of Korteweg-de Vries and Riemann type. In particular, a great deal of analytical studies [4, 5, 11] were devoted to finding the corresponding Lax-type representations of the infinite Riemann type hydrodynamical hierarchy, suggested recently (by M. Pavlov and D. Holm [9]) in the form

$$D_t^N u = 0, \quad (1.1)$$

where the differentiation $D_t := \partial/\partial t + u\partial/\partial x$, $N \in \mathbb{N}$, $(x, t)^T \in \mathbb{R}^2$ and $u \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R})$. It was found that the related dynamical system

$$D_t u_1 = u_2, \dots, D_t u_j = u_{j+1}, \dots, D_t u_N = 0, \quad (1.2)$$

defined on a 2π -periodic infinite-dimensional smooth functional manifold $M^N \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^N)$, possesses [11, 13] for an arbitrary integer $N \in \mathbb{N}$ a suitable Lax type representation

$$D_x f = l_N[u; \lambda] f, \quad D_t f = q_N(\lambda) f \quad (1.3)$$

with $\lambda \in \mathbb{C}$ being a complex spectral parameter and $f \in L_\infty(\mathbb{R}; \mathbb{C}^N)$ and matrices $l_N[u; \lambda], q_N(\lambda) \in \text{End } \mathbb{C}^2$. Here, by definition, $u_1 := u \in C^\infty(\mathbb{R}^2; \mathbb{R})$ and the differentiations

$$D_t := \partial/\partial t + u_1 D_x, \quad D_x := \partial/\partial x \quad (1.4)$$

satisfy on the manifold M^N the following commutation relationship:

$$[D_x, D_t] = u_{1,x} D_x. \quad (1.5)$$

In particular, for the cases $N = 2, 3$ and $N = 4$ the following exact matrix polynomial in $\lambda \in \mathbb{C}$ expressions

$$l_2[u; \lambda] = \begin{pmatrix} \lambda u_{1,x} & u_{2,x} \\ -2\lambda^2 & -\lambda u_{1,x} \end{pmatrix}, \quad q_2(\lambda) = \begin{pmatrix} 0 & 0 \\ -\lambda & 0 \end{pmatrix}, \quad (1.6)$$

$$l_3[u; \lambda] = \begin{pmatrix} \lambda^2 u_{1,x} & -\lambda u_{2,x} & u_{3,x} \\ 3\lambda^3 & -2\lambda^2 u_{1,x} & \lambda u_{3,x} \\ 6\lambda^4 r_3^{(1)}[u] & -3\lambda^3 & \lambda^2 u_{1,x} \end{pmatrix}, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix},$$

$$l_4[u; \lambda] = \begin{pmatrix} -\lambda^3 u_{1,x} & \lambda^2 u_{2,x} & -\lambda u_{3,x} & u_{4,x} \\ -4\lambda^4 & 3\lambda^3 u_{1,x} & -2\lambda^2 u_{2,x} & \lambda u_{3,x} \\ -10\lambda^5 r_4^{(1)}[u] & 6\lambda^4 & -3\lambda^3 u_{1,x} & \lambda^2 u_{2,x} \\ -20\lambda^6 r_4^{(2)}[u] & 10\lambda^5 r_4^{(1)}[u] & -4\lambda^4 & \lambda^3 u_{1,x} \end{pmatrix},$$

$$q_4(\lambda) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix},$$

were presented in exact form.

In the present work we will be interested in studying the complete integrability of a new important dispersionless Riemann type hydrodynamic hierarchy

$$D_t^{N-1} u = \bar{z}_x^s, \quad D_t \bar{z} = 0 \quad (1.7)$$

on a 2π -periodic functional manifold $\bar{M}^N \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^N)$, where $s, N \in \mathbb{N}$ are arbitrary natural numbers, the vector $(u, D_t u, D_t^2 u, \dots, D_t^{N-1} u, \bar{z})^\top \in \bar{M}^N$, the differentiations $D_x := \partial/\partial x$, $D_t := \partial/\partial t + u\partial/\partial x$ satisfy as above the Lie-algebraic commutator relationship (1.5) and $t \in \mathbb{R}$ is an evolution parameter. This system can be considered at $s = 2$ and $N = 3$ as a nontrivial generalization of the dispersionless Riemann hydrodynamic system (1.1), extensively studied by means of different mathematical tools in [4, 5, 10, 11, 13, 14]. For the case $s = 2$ and $N = 2$ it is well known [1, 12] that the system (1.7) is a smooth Lax integrable bi-Hamiltonian flow on the 2π -periodic functional manifold $\bar{M}_2$, whose Lax representation is given by the compatible linear system

$$D_x f = \begin{pmatrix} \bar{z}_x & 0 \\ -\lambda(u + u_x/\bar{z}_x) & -\bar{z}_{xx}/\bar{z}_x \end{pmatrix} f, \quad D_t f = \begin{pmatrix} 0 & 0 \\ -\lambda\bar{z}_x & u_x \end{pmatrix} f, \quad (1.8)$$

where $f \in C^\infty(\mathbb{R}^2; \mathbb{R}^2)$ and $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter.

At $s = 2$ and $N = 3$ dynamical system (1.7) is equivalent to that on a 2π -periodic functional manifold $\bar{M}_3 \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^3)$ for a vector $(u, v, \bar{z})^\top \in \bar{M}_3$:

$$D_t u = v, \quad D_t v = \bar{z}_x^2, \quad D_t \bar{z} = 0. \quad (1.9)$$

The latter can be easily rewritten by means of the change of variables $z := \bar{z}_x^2$ as that on a 2π -periodic functional manifold $M_3 \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^3)$ for a vector $(u, v, z)^\top \in M_3$

$$D_t u = v, \quad D_t v = z, \quad D_t z = -2zu_x, \quad (1.10)$$

or in the form of the flow

$$\begin{pmatrix} du/dt \\ dv/dt \\ dz/dt \end{pmatrix} = K[u, v, z] := \begin{pmatrix} v - uu_x \\ z - uv_x \\ -2u_x z - uz_x \end{pmatrix}, \quad (1.11)$$

defining a standard smooth dynamical system on the infinite-dimensional functional manifold M_3 , where $K : M_3 \rightarrow T(M_3)$ is the corresponding vector field on M_3 .

Concerning the new dynamical system (1.11) we succeeded in proving the following result based on the symplectic gradient-holonomic and differential algebraic tools.

Proposition 1. *The Riemann type hydrodynamic flow (1.11) is a bi-Hamiltonian dynamical system on the functional manifold M_3 with respect to two compatible Poissonian structures $\vartheta, \eta : T^*(M_3) \rightarrow T(M_3)$*

$$\vartheta := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2z^{1/2}D_x z^{1/2} \end{pmatrix}, \quad \eta := \begin{pmatrix} \partial^{-1} & u_x \partial^{-1} & 0 \\ \partial^{-1}u_x & v_x \partial^{-1} + \partial^{-1}v_x & \partial^{-1}z_x - 2z \\ 0 & z_x \partial^{-1} + 2z & 0 \end{pmatrix}, \quad (1.12)$$

possessing an infinite hierarchy of mutually commuting conservation laws and a non-autonomous Lax representation of the form

$$\begin{aligned} D_t f &= \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & 0 \\ 0 & -\lambda\sqrt{z} & u_x \end{pmatrix} f, \\ D_x f &= \begin{pmatrix} \lambda^2 u \sqrt{z} & \lambda v \sqrt{z} & z \\ -\lambda^3 t u \sqrt{z} & -\lambda^2 t v \sqrt{z} & -\lambda t z \\ \lambda^4 (t u v - u^2) - & -\lambda v_x / \sqrt{z} + & -\lambda^2 \sqrt{z} (u - t v) - \\ -\lambda^2 u_x / \sqrt{z} & + \lambda^3 (t v^2 - u v) & -z_x / 2z \end{pmatrix} f, \end{aligned} \quad (1.13)$$

where $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter and $f \in C^\infty(\mathbb{R}^2; \mathbb{R}^3)$.

2. SYMPLECTIC GRADIENT-HOLONOMIC INTEGRABILITY ANALYSIS

Our first steps in proving Proposition 1 are fashioned using the symplectic gradient-holonomic method, which takes us a long way towards the desired result.

2.1. Poissonian structure analysis on the functional manifold M_3

By employing the symplectic gradient-holonomic approach [1, 6, 12] to studying the integrability of smooth nonlinear dynamical systems on functional manifolds, one can find a set of conservation laws for (1.11) by constructing solutions $\varphi := \varphi[u, v, z] \in T^*(M_3)$ to the functional Lax gradient equation:

$$d\varphi/dt + K'^*, \varphi = \text{grad} \mathcal{L}, \quad (2.1)$$

where $\varphi' = \varphi'^*$, $\mathcal{L} \in \mathcal{D}(M_3)$ is a suitable Lagrangian functional (in the space of smooth functionals on M_3) and the linear operator $K'^*, : T^*(M_3) \rightarrow T^*(M_3)$ is the adjoint with respect to the standard convolution $(\cdot, \cdot)$ on $T^*(M_3) \times T(M_3)$, of the Fréchet-derivative of a nonlinear mapping $K : M_3 \rightarrow T(M_3)$; namely,

$$K'^*, = \begin{pmatrix} u D_x & -v_x & z_x + 2z D_x \\ 1 & u_x + u D_x & 0 \\ 0 & 1 & -u_x + u D_x \end{pmatrix}. \quad (2.2)$$

The Lax gradient equation (2.1) can be, owing to (1.11), rewritten as

$$D_t \varphi + k[u, v, z] \varphi = \text{grad} \mathcal{L}, \quad (2.3)$$

where the matrix operator

$$k[u, v, z] := \begin{pmatrix} 0 & -v_x & z_x + 2z D_x \\ 1 & u_x & 0 \\ 0 & 1 & -u_x \end{pmatrix}. \quad (2.4)$$

The first vector elements

$$\varphi_\partial[u, v, z] = (z - u v_x, -v + u u_x, u), \mathcal{L}_\partial = 0 \quad (2.5)$$

$$\begin{aligned}\varphi_\eta[u, v, z] &= (v_x, -u_x, -1)^\top, \mathcal{L}_\eta = 0, \\ \varphi_0[u, v, z] &= (-(u_x z^{-1/2})_x, (z^{-1/2})_x, (v_x/2 - u_x^2/4)z^{-3/2})^\top, \mathcal{L}_0 = 0,\end{aligned}$$

as can be easily checked, are solutions of the functional equation (2.3). From an application of the standard Volterra homotopy formula

$$H := \int_0^1 d\mu(\varphi[\mu u, \mu v, \mu z], (u, v, z)^\top), \quad (2.6)$$

one finds the conservation laws for (1.11); namely,

$$\begin{aligned}H_\eta &= \frac{1}{2} \int_0^{2\pi} dx (2uz - v^2 - u^2 v_x), \\ H_\vartheta &:= \int_0^{2\pi} dx (uv_x/2 - vu_x/2 - z), \quad H_0 := \frac{1}{2} \int_0^{2\pi} dx (u_x^2 - 2v_x)z^{-1/2}.\end{aligned} \quad (2.7)$$

It is now quite easy, making use of the conservation laws (2.7), to construct a Poissonian structure $\vartheta : T^*(M_3) \rightarrow T(M_3)$ for the dynamical system (1.11). If we use the representations

$$\begin{aligned}H_\vartheta &= \int_0^{2\pi} dx (uv_x/2 - vu_x/2 - z) := (\psi_\vartheta, (u_x, v_x, z_x)^\top), \\ \psi_\vartheta &:= (-v/2, u/2, z^{-1/2} D_x^{-1} z^{1/2}/2)^\top,\end{aligned} \quad (2.8)$$

it follows that the vector $\psi_\vartheta \in T^*(M_3)$ satisfies the Lax gradient equation (2.3):

$$D_t \psi_\vartheta + k[u, v, z] \psi_\vartheta = \text{grad } \mathcal{L}_\vartheta, \quad (2.9)$$

where the Lagrangian function $\mathcal{L}_\vartheta = (\psi_\vartheta, K) - H_\vartheta$. Thus, based on the inverse co-symplectic functional expression

$$\vartheta^{-1} := \psi'_\vartheta - \psi'^{*}_\vartheta = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & z^{-1/2} D_x^{-1} z^{-1/2}/2 \end{pmatrix} \quad (2.10)$$

one readily obtains the linear co-symplectic operator on the manifold M_3 :

$$\vartheta := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2z^{1/2} D_x z^{1/2} \end{pmatrix}, \quad (2.11)$$

which is the corresponding Poissonian operator for the dynamical system (1.11). It is also important to observe that the dynamical system (1.11) is a Hamiltonian flow on the functional manifold M_3 with respect to the Poissonian structure (2.11).

$$K[u, v, z] = -\vartheta \text{ grad } H_\eta. \quad (2.12)$$

2.2. Poissonian structure analysis on $\bar{M}_3$

In what follows, we shall find it convenient to construct other Poissonian structures for dynamical system (1.9) on the manifold $\bar{M}_3$, rewritten in the equivalent form

$$\frac{d}{dt} \begin{pmatrix} u \\ v \\ \bar{z} \end{pmatrix} = \bar{K}[u, v, \bar{z}] := \begin{pmatrix} v - uu_x \\ \bar{z}_x^2 - uv_x \\ 0 \end{pmatrix}, \quad (2.13)$$

where $\bar{K} : \bar{M}_3 \rightarrow T(\bar{M}_3)$ is the corresponding vector field on $\bar{M}_3$. To proceed, we need to obtain additional solutions to the Lax gradient equation (2.3) on the functional manifold $\bar{M}_3$

$$D_t \bar{\psi} + \bar{k}[u, v, \bar{z}] \bar{\psi} = \text{grad } \bar{\mathcal{L}}, \quad (2.14)$$

where the matrix operator is

$$\bar{k}[u, v, \bar{z}] := \begin{pmatrix} 0 & -v_x & -\bar{z}_x \\ 1 & u_x & 0 \\ 0 & -2\partial \bar{z}_x & u_x \end{pmatrix}, \quad (2.15)$$

and which we may rewrite in the componentwise form

$$\begin{aligned} D_t \bar{\psi}^{(1)} &= v_x \bar{\psi}^{(2)} + \bar{z}_x \bar{\psi}^{(3)} + \delta \bar{\mathcal{L}} / \delta u, \\ D_t \bar{\psi}^{(2)} &= -\bar{\psi}^{(1)} - u_x \bar{\psi}^{(2)} + \delta \bar{\mathcal{L}} / \delta v, \\ D_t \bar{\psi}^{(3)} &= 2(\bar{z}_x \bar{\psi}^{(2)})_x - u_x \bar{\psi}^{(3)} + \delta \bar{\mathcal{L}} / \delta \bar{z}, \end{aligned} \quad (2.16)$$

where the vector $\bar{\psi} := (\bar{\psi}^{(1)}, \bar{\psi}^{(2)}, \bar{\psi}^{(3)})^\top \in T^*(\bar{M}_3)$. As a simple consequence of (2.16), one obtains the following system of differential relationships:

$$\begin{aligned} D_t^3 \tilde{\psi}^{(2)} &= -2\bar{z}_x^2 \tilde{\psi}_x^{(2)} + D_t^2 \partial^{-1} (\delta \bar{\mathcal{L}} / \delta v) - \\ &\quad - \partial^{-1} \langle \text{grad } \bar{\mathcal{L}}, (u_x, v_x, \bar{z}_x)^\top \rangle, \\ D_t \tilde{\psi}^{(2)} &= -\tilde{\psi}^{(1)} + \partial^{-1} (\delta \bar{\mathcal{L}} / \delta v), \\ D_t \tilde{\psi}^{(3)} &= 2\bar{z}_x \tilde{\psi}_x^{(2)} + \partial^{-1} (\delta \bar{\mathcal{L}} / \delta \bar{z}). \end{aligned} \quad (2.17)$$

Here we have defined $(\bar{\psi}^{(1)}, \bar{\psi}^{(2)}, \bar{\psi}^{(3)})^\top := (\tilde{\psi}_x^{(1)}, \tilde{\psi}_x^{(2)}, \tilde{\psi}_x^{(3)})^\top$ and made use of the commutator relationship for differentiations D_t and D_x :

$$[D_t, \alpha^{-1} D_x] = 0, \quad (2.18)$$

which holds for the function $\alpha := 1/\bar{z}_x$, where $D_t \bar{z} = 0$. It therefore follows that after solving the first equation of system (2.17), one can recursively solve the remaining two equations. In particular, it is easy to see that the three vector elements

$$\begin{aligned} \tilde{\psi}_0 &= (-v, u, -2\bar{z}_x)^\top, \quad \bar{\mathcal{L}}_0 = 0; \\ \tilde{\psi}_\theta &= (-u_x/\bar{z}_x, 1/\bar{z}_x, (u_x^2 - 2v_x)/(2\bar{z}_x^2))^\top, \quad \bar{\mathcal{L}}_\theta = 0; \\ \tilde{\psi}_\eta &= (u/2, -x/2, \partial^{-1}[(2v_x - u_x^2)/(2\bar{z}_x)])^\top, \quad \bar{\mathcal{L}}_\eta = (D_x \tilde{\psi}_\eta, \bar{K}) - H_\partial, \end{aligned} \quad (2.19)$$

are solutions of the system (2.17). The first two elements of (2.19) lead to the Volterra symmetric vectors $\bar{\psi}_0 = D_x \tilde{\psi}_0, \bar{\psi}_\theta = D_x \tilde{\psi}_\theta \in T^*(\bar{M}_3) : \bar{\psi}'_0 = \bar{\psi}'_0{}^*, \bar{\psi}'_\theta = \bar{\psi}'_\theta{}^*$ entailing the trivial conservation laws $(\bar{\psi}_0, \bar{K}) = 0 = (\bar{\psi}_\theta, \bar{K})$. The third element of (2.19) gives rise to the Volterra asymmetric vector $\bar{\psi}_\eta := D_x \tilde{\psi}_\eta : \bar{\psi}'_\eta \neq \bar{\psi}'_\eta{}^*$, entailing the following inverse co-symplectic functional expression:

$$\bar{\eta}^{-1} := \bar{\psi}'_\eta - \bar{\psi}'_\eta{}^* = \begin{pmatrix} \partial & 0 & -\partial \frac{u_x}{\bar{z}_x} \\ 0 & 0 & \partial \frac{1}{\bar{z}_x} \\ -\frac{u_x}{\bar{z}_x} \partial & \frac{1}{\bar{z}_x} \partial & \frac{u_x}{2\bar{z}_x} \partial \frac{u_x}{\bar{z}_x} - \frac{v_x}{\bar{z}_x} \partial \frac{1}{\bar{z}_x} - \frac{1}{\bar{z}_x} \partial \frac{v_x}{\bar{z}_x} \end{pmatrix}. \quad (2.20)$$

Correspondingly, the Poissonian operator $\bar{\eta} : T^*(\bar{M}_3) \rightarrow T(\bar{M}_3)$ is

$$\bar{\eta} = \begin{pmatrix} \partial^{-1} & u_x \partial^{-1} & 0 \\ \partial^{-1} u_x & v_x \partial^{-1} + \partial^{-1} v_x & \partial^{-1} \bar{z}_x \\ 0 & \bar{z}_x \partial^{-1} & 0 \end{pmatrix}, \quad (2.21)$$

subject to which the following Hamiltonian representation

$$\bar{K}[u, v, \bar{z}] = -\bar{\eta} \text{grad } H_\eta|_{z=\bar{z}_x^2} \quad (2.22)$$

holds on the manifold $\bar{M}_3$.

2.3. Hamiltonian integrability analysis

Next, we return to our integrability analysis of the dynamical system (1.11) on the functional manifold M_3 . It is easy to recalculate the form of the Poissonian operator (2.21) on the manifold $\bar{M}_3$ to that acting on the manifold M_3 , giving rise to the second Hamiltonian representation of (1.11):

$$K[u, v, z] = -\eta \text{grad } H_\eta, \quad (2.23)$$

where $\eta : T^*(M_3) \rightarrow T(M_3)$ is the corresponding Poissonian operator. As a next important point, the Poissonian operators (2.11) and (2.21) are compatible [1–3, 12] on the manifold $\bar{M}_3$; that is, the operator pencil $(\vartheta + \lambda \eta) : T^*(M_3) \rightarrow T(M_3)$ is also Poissonian for arbitrary $\lambda \in \mathbb{R}$. As a consequence, any operator of the form

$$\vartheta_n := \vartheta(\vartheta^{-1} \eta)^n \quad (2.24)$$

for all $n \in \mathbb{Z}$ is Poissonian on the manifold M_3 . Using now the homotopy formula (2.6) and recursion property of the Poissonian pair (2.12) and (2.21), it is easy to construct the related infinite hierarchy of mutually commuting conservation laws

$$\gamma_j = \int_0^1 d\mu (\text{grad } \gamma_j[\mu u, \mu v, \mu z], (u, v, z)^\top), \quad (2.25)$$

$$\text{grad } \gamma_j[u, v, z] := \Lambda^j \text{grad } H_\eta,$$

for the dynamical system (1.11), where $j \in \mathbb{Z}_+$ and $\Lambda := \vartheta^{-1}\eta : T^*(M_3) \rightarrow T^*(M_3)$ is the corresponding recursion operator, which satisfies the so called associated Lax commutator relationship

$$d\Lambda/dt = [\Lambda, K'^*]. \quad (2.26)$$

In the course of above analysis and observations, we have proved the following result.

Proposition 2. *The Riemann hydrodynamic system (1.11) is a bi-Hamiltonian dynamical system on the functional manifold M_3 with respect to the compatible Poissonian structures $\vartheta, \eta : T^*(M_3) \rightarrow T(M_3)$*

$$\vartheta := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2z^{1/2}D_x z^{1/2} \end{pmatrix}, \eta := \begin{pmatrix} \partial^{-1} & u_x \partial^{-1} & 0 \\ \partial^{-1}u_x & v_x \partial^{-1} + \partial^{-1}v_x & \partial^{-1}z_x - 2z \\ 0 & z_x \partial^{-1} + 2z & 0 \end{pmatrix} \quad (2.27)$$

and possesses an infinite hierarchy of mutually commuting conservation laws (2.25).

Concerning the existence of an additional infinite and parametrically $\mathbb{R} \ni \lambda$ -ordered hierarchy of conservation laws for the dynamical system (1.11), it is instructive to consider the dispersive nonlinear dynamical system

$$\begin{pmatrix} du/d\tau \\ dv/d\tau \\ dz/d\tau \end{pmatrix} = -\vartheta \operatorname{grad} H_0[u, v, z] := \begin{pmatrix} -(z^{-1/2})_x \\ -(u_x z^{-1/2})_x \\ z^{1/2}(\frac{u_x^2 - 2v_x}{2z})_x \end{pmatrix} = \tilde{K}[u, v, z]. \quad (2.28)$$

By solving the corresponding Lax equation

$$d\tilde{\varphi}/dt + \tilde{K}'^* \tilde{\varphi} = 0 \quad (2.29)$$

for an element $\tilde{\varphi} \in T^*(M_3)$ in a suitably chosen asymptotic form, one can construct an infinite ordered hierarchy of conservation laws for (1.11), which we will not delve into here. This hierarchy and the existence of an infinite and parametrically $\mathbb{R} \ni \lambda$ -ordered hierarchy of conservation laws for the Riemann type dynamical system (1.11) provided compelling indications that it is completely integrable in the sense of Lax on the functional manifold M_3 . We shall complete our integrability analysis in the next section using rather powerful differential-algebraic tools that were devised recently in [11, 13, 14].

3. DIFFERENTIAL-ALGEBRAIC INTEGRABILITY ANALYSIS: $N = 3$

Consider a polynomial differential ring $\mathcal{K}\{u\} \subset \mathcal{K} := \mathbb{R}\{\{x, t\}\}$ generated by a fixed functional variable $u \in \mathbb{R}\{\{x, t\}\}$ and invariant with respect to two differentiations $D_x := \partial/\partial x$ and $D_t := \partial/\partial t + u\partial/\partial x$ that satisfy the Lie-algebraic commutator relationship (1.5)

$$[D_x, D_t] = u_x D_x \quad (3.1)$$

together with the constraint (1.10) expressed in the differential-algebraic functional form

$$D_t^3 u = -2D_t^2 u D_x u.$$

Since the Lax representation for the dynamical system (1.11) can be interpreted [1, 13] as the existence of a finite-dimensional invariant ideal $\mathcal{I}\{u\} \subset \mathcal{K}\{u\}$ realizing the corresponding finite-dimensional representation of the the Lie-algebraic commutator relationship (3.1), this ideal can be constructed as

$$\mathcal{I}\{u\} := \{\lambda^2 u f_1 + \lambda v f_2 + z^{1/2} f_3 \in \mathcal{K}\{u\} : f_j \in \mathcal{K}, 1 \leq j \leq 3, \lambda \in \mathbb{R}\}, \quad (3.2)$$

where $v = D_t u, z = D_t^2 u$ and $\lambda \in \mathbb{R}$ is an arbitrary real parameter. To find finite-dimensional representations of the D_x - and D_t -differentiations, it is necessary [13] first to find the D_t -invariant kernel $\ker D_t \subset \mathcal{I}\{u\}$ and next to check its invariance with respect to the D_x -differentiation. It is easy to show that

$$\ker D_t = \{f \in \mathcal{K}^3\{u\} : D_t f = q(\lambda) f, \lambda \in \mathbb{R}\}, \quad (3.3)$$

where the matrix $q(\lambda) := q[u, v, z; \lambda] \in \text{End } \mathcal{K}\{u\}^3$ is given as

$$q(\lambda) = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & 0 \\ 0 & -\lambda\sqrt{z} & u_x \end{pmatrix}. \quad (3.4)$$

To obtain the corresponding representation of the D_x -differentiation in the space $\mathcal{K}^3$, it suffices to find a matrix $l(\lambda) := l[u, v, z; \lambda] \in \text{End } \mathcal{K}\{u\}^3$ such that

$$D_x f = l(\lambda) f \quad (3.5)$$

for $f \in \mathcal{K}\{u\}^3$ and the related ideal

$$\mathcal{R}\{u\} := \{ \langle g, f \rangle_{\mathcal{K}^3} : f \in \ker D_t \subset \mathcal{K}^3\{u\}, g \in \mathcal{K}^3 \} \quad (3.6)$$

is D_x -invariant with respect to the matrix differentiation representation (3.5). Straight-forward calculations using this invariance condition then yield the following matrix

$$l(\lambda) = \begin{pmatrix} \lambda^2 u \sqrt{z} & \lambda v \sqrt{z} & z \\ -\lambda^3 t u \sqrt{z} & -\lambda^2 t v \sqrt{z} & -\lambda t z \\ \lambda^4 (t u v - u^2) - & -\lambda v_x / \sqrt{z} + & -\lambda^2 \sqrt{z} (u - t v) - \\ -\lambda^2 u_x / \sqrt{z} & + \lambda^3 (t v^2 - u v) & -z_x / 2z \end{pmatrix} \quad (3.7)$$

entering the linear equation (3.5). Thus, the following proposition is proved.

Proposition 3. *The generalized Riemann type dynamical system (1.11) is a bi-Hamiltonian integrable flow possessing a non-autonomous Lax representation of the form*

$$D_t f = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & 0 \\ 0 & -\lambda\sqrt{z} & u_x \end{pmatrix} f, \quad (3.8)$$

$$D_x f = \begin{pmatrix} \lambda^2 u \sqrt{z} & \lambda v \sqrt{z} & z \\ -\lambda^3 t u \sqrt{z} & -\lambda^2 t v \sqrt{z} & -\lambda t z \\ \lambda^4 (t u v - u^2) - & -\lambda v_x / \sqrt{z} + & -\lambda^2 \sqrt{z} (u - t v) - \\ -\lambda^2 u_x / \sqrt{z} & +\lambda^3 (t v^2 - u v) & -z_x / 2z \end{pmatrix} f,$$

where $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter and $f \in C^\infty(\mathbb{R}^2; \mathbb{R}^3)$.

Remark 1. Simple analogs of the above differential-algebraic calculations for the case $N = 2$ lead readily to the corresponding Riemann type hydrodynamic system

$$D_t u = \bar{z}_x^2, \quad D_t \bar{z} = 0 \quad (3.9)$$

on the functional manifold $\bar{M}_2$, which possesses the following matrix Lax representation:

$$D_t f = \begin{pmatrix} 0 & 0 \\ -\lambda \bar{z}_x & u_x \end{pmatrix}, \quad D_x f = \begin{pmatrix} \bar{z}_x & 0 \\ -\lambda(u + u_x / \bar{z}_x) & -\bar{z}_{xx} / \bar{z}_x \end{pmatrix} f, \quad (3.10)$$

where $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter and $f \in C^\infty(\mathbb{R}^2; \mathbb{R}^2)$.

As one can readily see, these differential-algebraic results provide a direct proof of Proposition 1 describing the integrability of system (1.11) for $N = 3$. The matrices (3.7) are not of standard form since they depend explicitly on the temporal evolution parameter $t \in \mathbb{R}$. Nonetheless, the matrices (3.4) and (3.7) satisfy for all $\lambda \in \mathbb{R}$ the well-known Zakharov–Shabat type compatibility condition

$$D_t l(\lambda) = [q(\lambda), l(\lambda)] + D_x l(\lambda) - u_x l(\lambda), \quad (3.11)$$

which follows from the Lax type relationships (3.3) and (3.5)

$$D_t f = q(\lambda) f, \quad D_x f = l(\lambda) f \quad (3.12)$$

and the commutator condition (3.1). Moreover, taking into account that the dynamical system (1.11) has a compatible Poissonian pair (2.11) and (2.21) depending only on the variables $(u, v, z)^\top \in M_3$ and not depending on the temporal variable $t \in \mathbb{R}$, one can certainly assume that it also possesses a standard autonomous Lax representation, which can possibly be found by means of a suitable gauge transformation of (3.12). We plan to pursue this line of analysis in a forthcoming paper.

4. CONCLUDING REMARKS

A new nonlinear Hamiltonian dynamical system representing a Riemann type hydrodynamic equation (1.7) in two and three dimensions proves to be a very interesting example of a Lax integrable dynamical system, as we have proved here. In particular, the integrability prerequisites of this dynamical system, such as compatible Poissonian structures, an infinite hierarchy of conservation laws and related Lax representation have been constructed by means of both the symplectic gradient-holonomic approach [1, 6, 12] and innovative differential-algebraic tools devised recently [4, 13]

for analyzing the integrability of a special infinite hierarchy of Riemann type hydrodynamic systems. It is also quite clear from recent research in this area and our work in this paper that the dynamical system (1.7) is a Lax integrable bi-Hamiltonian flow for arbitrary integers $N \in \mathbb{N}$; this is perhaps most readily verified by means of the differential-algebraic approach, which was devised and successfully applied here for the cases $N = 2$ and 3 .

We have seen in the course of this investigation that perhaps the most important lesson that one can derive from this approach is the following: If an analysis of a given nonlinear Hamiltonian dynamical system via the gradient-holonomic method indicates (but does not necessarily prove) that the system is Lax integrable, then its Lax representation can often be shown to exist and then successfully derived by means of a suitably constructed invariant differential ideal $\mathcal{I}\{u\}$ of the ring $\mathcal{K}\{u\}$ in accordance with the differential-algebraic approach developed here. Consequently, when it comes to applying this lesson to the investigation of other nonlinear dynamical systems, it is natural to start with systems that are known to be Lax integrable and then to try to identify and characterize those algebraic structures responsible for the existence of a related finite-dimensional matrix representation for the basic D_x - and D_t -differentiations in a vector space $\mathcal{K}^p$ for some finite $p \in \mathbb{Z}_+$.

It seems plausible that if one could do this for several classes of Lax integrable dynamical systems, certain patterns in the algebraic structures may be detected that can be used to assemble a more extensive array of symplectic and differential-algebraic tools capable of resolving the question of complete integrability for many other types of nonlinear Hamiltonian dynamical systems. Moreover, if the integrability is established in this manner, the approach should also serve as a means of constructing associated artifacts of the integrability such as Lax representations and hierarchies of mutually commuting invariants. As a particular differential-algebraic problem of interest concerning these matrix representations, one can seek to develop a scheme for the effective construction of functional generators of the corresponding invariant finite-dimensional ideals $\mathcal{I}\{u\} \subset \mathcal{K}\{u\}$ under given differential-algebraic constraints imposed on the D_x - and D_t -differentiations.

We have demonstrated here that an approach combining the gradient-holonomic method with some recently devised differential-algebraic techniques can be a very effective and efficient way of investigating integrability for a particular class of infinite-dimensional Hamiltonian dynamical systems (generalized Riemann hydrodynamical systems). But a closer look at the specific details of the approach reveals, we believe, that this combination of methods can be adapted to perform effective integrability investigations of a much wider range of dynamical systems - a goal that we intend to pursue in the near future.

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Authors' addresses

Anatolij M. Samoilenko

The Institute of Mathematics of NAS of Ukraine, and the Department of Applied Mathematics at the University of Agriculture in Krakow, Poland

Yarema A. Prykarpatskyy

The Institute of Mathematics of NAS of Ukraine, and the Department of Applied Mathematics at the University of Agriculture in Krakow, Poland

E-mail address: yarpry@gmail.com

Denis Blackmore

Department of Mathematical Sciences at the New Jersey Institute of Technology (NJIT), Newark NJ 07102, USA

E-mail address: deblac@m.njit.edu

Anatolij K. Prykarpatski

The Institute of Mathematics at the Cracow University of Technology, Poland

E-mail address: pryk.anat@ua.fm



SHARP PARTIAL CLOSURE OPERATOR

BRANIMIR ŠEŠELJA, ANNA SLIVKOVÁ, AND ANDREJA TEPAVČEVIĆ

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Abstract. As an improvement of existing relationships among collections of sets, closure operators and posets, a particular, so called sharp partial closure operator (SPCO) is introduced. It is proved that there is always a unique SPCO corresponding to a given partial closure system. Moreover, an SPCO has the greatest domain among all partial operators corresponding to a given system. If it is a function, an SCPO is a classical closure operator. Dealing with partial closure systems, we introduce principal ones, corresponding to principal ideals of a poset and accordingly, we define principal SPCO's. Finally, we prove a representation theorem for posets in terms of principal SPCO's and principal partial closure systems.

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1. INTRODUCTION

Connection among closure systems (Moore's families), closure operators and complete lattices is a well known topic in basics of order theory and lattices.

There is an analogue relationship among partial closure (centralized) systems, partial closure operators and posets. Still, the analogy is not full, since the correspondence among partial closure systems and partial closure operators is not unique, as in the case of lattices.

Closure operators and systems appear as a well known basic tool in the research of ordered sets, topology, universal algebra, logic, ... Among numerous relevant books, we mention [1, 11] as related to our work. Some particular papers dealing more closely with various aspects of closure systems and connections to ordered structures are given in References. Namely, papers [3, 4] give surveys on closure systems over finite sets, their properties and properties of the corresponding lattices. In [2], the lattice of particular completions of a finite poset is analyzed. Completions of a poset are also a subject of [8]. Extensive research of generalizations of closure systems and related posets, together with the corresponding properties is done by M. Ern  (e.g., [5, 6]). In [7], the lattice of all Dedekind-MacNeille completions of posets with

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the fixed join-irreducibles is investigated. The number of closure systems on particular cardinalities is investigated in [9], and in [10], the lattice itself of such systems is described. Apart from implicit analysis connected with the corresponding collections of sets, there is not much research of closure operators which are not functions. In [12] a partial closure operator is defined as a special case of the definition in this work: there it is defined on downward-closed subsets (order-ideals) of a lattice (in our work we deal generally with arbitrary sets). In the mentioned paper, partial closure operators are used in developing the semantic foundations of concurrent constraint computing. Our present work is based on our paper [13], moreover it is an extension of this previous one.

In this paper, we present an improvement of existing relationships among collections of sets, closure operators and posets. In order to make the correspondence among partial closure systems and partial closure operators unique, we introduce a particular, so called sharp partial closure operator (SPCO). This is a partial operation on the power set, fulfilling closure like axioms, plus additional one, called sharpness. We prove that there is always a unique SPCO corresponding to a given partial closure system. Moreover, an SPCO has the greatest domain among all partial operators corresponding to a given system. In addition, if an SCPO is a function, then it is a classical closure operator. Dealing with collections of subsets as a counterpart of operators, we analyze partial closure, or centralized systems (point closures). Among these we introduce so called principal ones, corresponding to principal ideals of a poset. Accordingly, we define principal SPCO's. Finally, we prove a representation theorem for posets in terms of principal SPCO's and principal partial closure systems.

By these results we establish bijective correspondences among posets, principal SPCO's and principal partial closure operators.

2. PRELIMINARIES

We start with well known notions and basic properties of closure systems and closure operators, pointing to our notation.

As it is known, a **closure system** (Moore's family) $\mathcal{F}$ on a nonempty set S is a collection of subsets of S , which is closed under arbitrary set intersections.

A **closure operator** on a nonempty set S is a unary operation $X \mapsto \overline{X}$ on the power set $P(S)$, which for all $X, Y \subseteq S$ fulfils properties

$$X \subseteq \overline{X}, \quad X \subseteq Y \text{ implies } \overline{X} \subseteq \overline{Y}, \quad \overline{\overline{X}} = \overline{X}.$$

As usual, if $X \subseteq S$ and $\overline{X} = X$, then X is a **closed** set. The family of closed sets $\mathcal{F}_C$ is the **range** of a closure operator.

Recall that the range of a closure operator on S is a closure system on the same set. On the other hand, if $\mathcal{F}$ is a closure system on S , then the map $x \mapsto \bigcap \{Y \in \mathcal{F} \mid$

$x \in Y\}$, for $x \in S$, is a closure operator on S . This correspondence among closure systems and corresponding operators is unique.

A closure system is a complete lattice under inclusion, and as a converse, the collection of principal ideals of a lattice is a closure system, which is, when equipped by inclusion, order isomorphic with the lattice itself. Still, the closure system of principal ideals is not the only closure system isomorphic to a given lattice.

3. PARTIAL CLOSURE OPERATORS AND SYSTEMS

Our aim is to establish a particular relationship among collections of sets, operators and posets. This relationship should be analogue (as much as possible) to the one among closure operators, closure systems and complete lattices. We use the relevant known results in this field, and the basic definitions and properties are those given in [13]. Still, our present approach brings some new requirements, which enable essential improvements of the mentioned relationship.

For a nonempty set S , let $C : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ be a partial mapping satisfying:

P_{C1} : If $C(X)$ is defined, then $X \subseteq C(X)$.

P_{C2} : If $C(X)$ and $C(Y)$ are defined, then $X \subseteq Y$ implies $C(X) \subseteq C(Y)$.

P_{C3} : If $C(X)$ is defined, then $C(C(X))$ is also defined and $C(C(X)) = C(X)$.

P_{C4} : $C(\{x\})$ is defined for every $x \in S$.

As defined in [13], a partial mapping C fulfilling properties P_{C1} – P_{C4} is a **partial closure operator** on S .

As usual, if $X \subseteq S$ and $C(X) = X$, then we call X a **closed** set. The family of closed sets $\mathcal{F}_C$ is called the **range** of a partial closure operator C . The **exact domain** of a partial closure operator C on S is denoted by $Dom(C)$:

$$Dom(C) := \{X \mid X \subseteq S \text{ and } C(X) \text{ is defined}\}.$$

Let C be a partial closure operator on S . If $C(X)$ is defined, then it is straightforward to check that, equivalently to the same property of a closure operator (which is not partial),

$$C(X) = \bigcap \{Y \in \mathcal{F}_C \mid X \subseteq Y\}. \quad (3.1)$$

We say that a partial closure operator C on S is **sharp**, if it satisfies the condition:

P_{C5} : Let $B \subseteq S$. If $\bigcap \{X \in \mathcal{F}_C \mid B \subseteq X\} \in \mathcal{F}_C$, then $C(B)$ is defined and

$$C(B) = \bigcap \{X \in \mathcal{F}_C \mid B \subseteq X\} \quad (\text{sharpness}).$$

We also say that a partial operator on S , fulfilling properties P_{C1} – P_{C5} is an **SPCO** on S .

Notice that if in P_{C5} there does not exist a set $X \in \mathcal{F}_C$ such that $B \subseteq X$, then straightforwardly $C(B)$ is not defined (because of $B \subseteq C(B)$).

Observe also that a closure operator C on S (i.e., an operator which is a function) trivially fulfils condition Pc_5 , which reduces to the condition (3.1).

Remark 1. By (3.1) the converse implication in the condition Pc_5 is always valid.

A partial closure operator C on a set S is **complete**, if it satisfies:

Pc_6 : If $\{X_i \mid i \in I\}$ is a chain and $C(X_i)$ is defined for every $i \in I$, then also $C(\bigcup X_i)$ is defined.

In addition, C is **algebraic** if it is complete and satisfies the following:

Pc_7 : If $C(X)$ is defined, then

$$\{C(Y) \mid Y \subseteq X, C(Y) \text{ is defined and } Y \text{ is finite}\} \text{ is a directed set,} \quad (3.2)$$

$$\text{and } C(X) = \bigcup \{C(Y) \mid Y \subseteq X, C(Y) \text{ is defined and } Y \text{ is finite}\}. \quad (3.3)$$

We note that the condition Pc_5 can not be derived from the conditions Pc_1 – Pc_4 , as shown by the following example.

Example 1. Let C be a partial mapping defined on $\{a, b, c\}$ with

$$C : \begin{pmatrix} \{a\} & \{b\} & \{c\} & \{a, b, c\} \\ \{a\} & \{b\} & \{a, b, c\} & \{a, b, c\} \end{pmatrix}.$$

It is straightforward to check that C satisfies conditions Pc_1 – Pc_4 , but the property Pc_5 does not hold because $C(\{a, b\})$ is not defined.

From the same example, it follows that Pc_5 can neither be derived from the above conditions, to which Pc_6 and Pc_7 are added.

Further, neither of the conditions Pc_6 and Pc_7 can be derived from Pc_1 – Pc_5 , as shown by the following example.

Example 2. Let C be a partial mapping defined on $\mathbb{N}$ by

$$C(X) = \begin{cases} X, & \text{if } X \text{ is a finite subset of } \mathbb{N}; \\ E_1, & \text{if } X \text{ is an infinite subset of } E_1; \end{cases}$$

where E is the set of all even natural numbers and $E_1 = E \cup \{1\}$. This is a sharp partial closure operator, but it is not complete. Indeed, consider the family $\{X_i \mid i \in \mathbb{N}\}$, where $X_i = \{1, 2, \dots, i\}$. This family is a chain and $C(X_i)$ is defined for every $i \in \mathbb{N}$, but $C(\bigcup_{i \in \mathbb{N}} X_i) = C(\mathbb{N})$ is not defined.

The constructed example does not satisfy Pc_7 either. Indeed, $C(E) = E_1$, but there does not exist a finite subset of even numbers that contains 1, hence we cannot represent $C(E)$ as the union of closures of all finite subsets of E .

Next we deal with the set counterpart of partial closure operators.

A **partial closure system** (in the literature known also as a *centralized system*, e. g. [5, 6]) is a family $\mathcal{F}$ of subsets of a nonempty set S satisfying:

$$\text{For every } x \in S, \bigcap \{X \in \mathcal{F} \mid x \in X\} \in \mathcal{F}. \quad (3.4)$$

We say that the set $\bigcap(X \in \mathcal{F} \mid x \in X)$ is a **centralized intersection** for $x \in S$. Observe that from the above definition it follows that a partial closure system $\mathcal{F}$ on S also satisfies $\bigcup \mathcal{F} = S$.

The following is a refinement of a theorem from [13].

Theorem 1. *The range of a partial closure operator on a set S is a partial closure system.*

Conversely, for every partial closure system $\mathcal{F}$ on S , there is a unique sharp partial closure operator on S such that its range is $\mathcal{F}$.

Proof. The first part of this theorem was proved in [13], still we repeat it here for the sake of completeness.

Let C be a partial closure operator on a set S and $\mathcal{F}_C$ be its range.

For $x \in S$, let $\mathcal{F}_x := \{X \in \mathcal{F}_C \mid x \in X\}$. We need to show that $\bigcap \mathcal{F}_x \in \mathcal{F}_C$. If $X \in \mathcal{F}_x$, then $X = C(X)$ and $x \in X$, therefore $\{x\} \subseteq X$ and $C(\{x\}) \subseteq C(X) = X$, hence $C(\{x\}) \subseteq \bigcap \mathcal{F}_x$. Since we have $C(\{x\}) \in \mathcal{F}_x$, it follows that $C(\{x\}) = \bigcap \mathcal{F}_x$ and condition (3.4) holds.

For the other direction, let $\mathcal{F}$ be a partial closure system on a set S . We define partial mapping $C : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ as follows:

$$C(X) := \bigcap \{Y \in \mathcal{F} \mid X \subseteq Y\}$$

if intersection on the right-hand side is in $\mathcal{F}$, otherwise $C(X)$ is not defined.

If, for some $X \subseteq S$, the closure $C(X)$ is defined, then it is easy to see that C has properties Pc_1 – Pc_3 . The property Pc_4 holds because, for $x \in S$, $C(\{x\})$ is defined by (3.4). Hence, C is a partial closure operator on S . Now we show that C is sharp, i.e., that also Pc_5 holds. Let $B \subseteq S$ and assume that

$$\bigcap \{X \in \mathcal{F}_C \mid B \subseteq X\} \in \mathcal{F}_C.$$

Then, by the definition of C , this partial operator fulfills Pc_5 and the range of C is $\mathcal{F}$.

It remains to show that the SPCO defined in this way is the unique partial mapping with the range $\mathcal{F}$ satisfying properties Pc_1 – Pc_5 . Assume that there exists another partial mapping $K : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ satisfying Pc_1 – Pc_5 and that the range of K is also $\mathcal{F}$. We prove that $\mathcal{F}_C = \mathcal{F}_K$. Namely, we show that for $X \subseteq S$, $K(X)$ is defined if and only if $C(X)$ is defined. If $K(X)$ is defined, by Remark 1, $\bigcap \{Y \in \mathcal{F}_K \mid X \subseteq Y\} \in \mathcal{F}_K$ and $K(X) = \bigcap \{Y \in \mathcal{F}_K \mid X \subseteq Y\}$. Since $\mathcal{F}_K = \mathcal{F}_C$, $\bigcap \{Y \in \mathcal{F}_K \mid X \subseteq Y\} = \bigcap \{Y \in \mathcal{F}_C \mid X \subseteq Y\}$ and hence $\bigcap \{Y \in \mathcal{F}_C \mid X \subseteq Y\} \in \mathcal{F}_C$ and by the condition Pc_5 , $C(X)$ is defined and $C(X) = K(X)$. If we suppose that $C(X)$ is defined, similarly we obtain that $K(X)$ is defined and $C(X) = K(X)$.

defined, we need to prove that $\bigcap \{Y \in \mathcal{F} \mid X \subseteq Y\}$ belongs to $\mathcal{F}$. Since the range of operator K is $\mathcal{F}$, for every $Y \in \mathcal{F}$ there exists $U \in \mathcal{P}(S)$ such that $K(U) = Y$. Therefore, we need to show that such that $C(Y)$ is defined (that is, $C(Y) \in \mathcal{F}$) there

exists $U \subseteq S$ such that $C(Y) = K(U)$, we have $\{C(Y) \mid X \subseteq C(Y)\} = \{K(U) \mid X \subseteq K(U)\}$ and then $K(U) = K(X)$.

S , and they agree on these subsets. This completes the proof. $\square$

Example 3. Let C_s be a partial mapping defined on $\{a, b, c\}$ with

$$C_s : \begin{pmatrix} \{a\} & \{b\} & \{c\} & \{a, b\} & \{a, c\} & \{b, c\} & \{a, b, c\} \\ \{a\} & \{b\} & \{a, b, c\} & \{a, b, c\} & \{a, b, c\} & \{a, b, c\} & \{a, b, c\} \end{pmatrix}.$$

This partial mapping is an SPCO on the set $\{a, b, c\}$. Note that the range $\mathcal{F}_{C_s}$ here is equal to the range $\mathcal{F}_C$ of the partial closure operator from Example 1. This implies that there is no 1-1 correspondence between partial closure operators and partial closure systems. However, as proven in Theorem 1, there is a bijective correspondence between SPCO's and partial closure systems.

By the above, it is clear that for a given partial closure system $\mathcal{F}$ on S , there is a collection of partial closure operators on S whose range is $\mathcal{F}$, among which, by Theorem 1, precisely one is sharp. In addition, the latter is maximal in the following sense.

Proposition 1. *Let $\mathcal{F}$ be a partial closure system on S . The sharp partial closure operator has the greatest domain among all partial closure operators whose range is $\mathcal{F}$. In addition, if D is a partial closure operator and C the sharp closure operator with the same domain, then $C(A) = D(A)$, for all $A \subseteq S$ for which D is defined.*

Proof. Let D be an arbitrary partial closure operator whose range is $\mathcal{F}$, and let C be the sharp one with the same range $\mathcal{F}$. Now, if $A \subseteq S$ and $D(A)$ is defined, i.e., $A \in \text{Dom}(D)$, then $C(D(A)) = D(A)$, since the ranges of C and D coincide by assumption.

We have that

$$D(A) = \bigcap (X \in \mathcal{F} \mid A \subseteq X) \in \mathcal{F}.$$

By the P_{C5} , it directly follows that $C(A)$ is defined and $C(A) = \bigcap (X \in \mathcal{F} \mid A \subseteq X)$. Hence, $C(A) = D(A)$. $\square$

The sharp partial closure operator is a natural generalization of the closure operator, as follows.

Theorem 2. *If the range $\mathcal{F}$ of a sharp partial closure operator C on a set S forms a complete lattice with respect to set inclusion, then C is a function. Conversely, if C is a closure operator on S , then it is sharp.*

Proof. Let $X \subseteq S$. We have $X \subseteq \bigcup \{C(\{x\}) \mid x \in X\}$, and since the range $\mathcal{F}$ is a complete lattice, the supremum of the collection $\{C(\{x\}) \mid x \in X\}$ exists and contains its union, which implies that $\bigvee \{C(\{x\}) \mid x \in X\} \in \mathcal{F}$. If $X \subseteq Y$ for a set Y such that $Y \in \mathcal{F}$, then $\bigvee \{C(\{x\}) \mid x \in X\} \subseteq Y$. Indeed, for every $x \in X$, $C(\{x\}) \subseteq Y$. Hence,

$\bigcap \{Y \in \mathcal{F} \mid X \subseteq Y\} = \bigvee \{C(\{x\}) \mid x \in X\}$. By PC_5 we have that $C(X)$ is defined, so C is a function and $C(X) = \bigvee \{C(\{x\}) \mid x \in X\}$.

Suppose now that C is a closure operator. Then its range forms a complete lattice with respect to a set inclusion [1]. Let $B \subseteq S$. The closure $C(B)$ is defined because C is a function, and it is obvious that it satisfies PC_5 . $\square$

As shown in paper [13], a completion of a partial closure system to a closure system is equivalent to Dedekind MacNeille completion. Here we present a completion of any nonempty collection of subset of S to a partial closure system. Clearly, by adding all singletons of S , we get a partial closure system, but then the existing centralized intersections may not be preserved. Therefore, we introduce another completion, as follows.

For an arbitrary nonempty collection $\mathcal{F}$ of subsets of a set S , we define an extension $\widehat{\mathcal{F}} \subseteq \mathcal{P}(S)$ as follows:

$$\widehat{\mathcal{F}} := \mathcal{F} \cup \left\{ \bigcap_{x \in Y} Y \in \mathcal{F} \mid x \in S \right\}.$$

Example 4. Let

$S = \{a, b, c, d, e, f, g\}$ and

$\mathcal{F} = \{\{b\}, \{c\}, \{e\}, \{a, b, c\}, \{b, c, d, e, f\}, \{e, f, g\}\}.$

Then $\widehat{\mathcal{F}} = \mathcal{F} \cup \{\{e, f\}\}.$

The following is a straightforward consequence of the definition of $\widehat{\mathcal{F}}$.

Proposition 2. *For an arbitrary nonempty collection $\mathcal{F}$ of a set S , the extension $\widehat{\mathcal{F}}$ is a partial closure system on S which preserves all intersections and centralized intersections existing in $\mathcal{F}$.*

Recall that the collection of all principal ideals of a complete lattice L is a closure system which is, when ordered by inclusion, order isomorphic with L under the mapping $i(x) = \downarrow x$, $x \in L$. In addition, this closure system consists of closed sets under the corresponding closure operator.

However, it is clear that not every closure system is isomorphic with a collection of all principal ideals of a complete lattice L .

The analogue statement is true for posets and related partial closure operators and partial closure systems.

In the following we introduce a special type of partial closure systems which are isomorphic to collections of all principal ideals in posets.

We say that a partial closure system $\mathcal{F}$ on a nonempty set S is **principal** if $\emptyset \notin \mathcal{F}$ and for every $X \in \mathcal{F}$ we have

$$\left| X \setminus \bigcup \{Y \in \mathcal{F} \mid Y \subsetneq X\} \right| = 1. \quad (3.5)$$

Our main motivation for the above definition, as already mentioned, are principal ideals in a poset.

Proposition 3. *Let $(S, \leq)$ be a poset. Then the family $\{\downarrow x \mid x \in S\}$ of principal ideals is a principal partial closure system.*

Proof. It is easy to see that condition (3.4) holds and that $\emptyset \notin \{\downarrow x \mid x \in S\} = \mathcal{F}$. Let us show that for every $\downarrow x \in \mathcal{F}$ we have $|\downarrow x \setminus \bigcup\{\downarrow y \in \mathcal{F} \mid \downarrow y \subsetneq \downarrow x\}| = 1$.

Obviously, $x \in \downarrow x \setminus \bigcup\{\downarrow y \in \mathcal{F} \mid \downarrow y \subsetneq \downarrow x\}$. Suppose that there is element an $z \neq x$ such that $z \in \downarrow x \setminus \bigcup\{\downarrow y \in \mathcal{F} \mid \downarrow y \subsetneq \downarrow x\}$. It follows that $z < x$, therefore $\downarrow z \in \{\downarrow y \in \mathcal{F} \mid \downarrow y \subsetneq \downarrow x\}$, which is a contradiction with $z \notin \bigcup\{\downarrow y \in \mathcal{F} \mid \downarrow y \subsetneq \downarrow x\}$. $\square$

Let $\mathcal{F}$ be a principal partial closure system on a set S . In order to prove the opposite connection of principal partial closure systems and principal ideals in a poset, we introduce a mapping:

$G : \mathcal{F} \rightarrow S$ defined by

$$G(X) = x, \text{ where } x \in X \setminus \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X\}. \quad (3.6)$$

The mapping is well defined by the definition of the principal partial closure system.

Proposition 4. *If $\mathcal{F}$ is a principal partial closure system on a set S then the mapping $G : \mathcal{F} \rightarrow S$ defined by (3.6) is a bijection.*

Proof. First, let $X_1, X_2 \in \mathcal{F}$ such that $G(X_1) = G(X_2)$. Therefore, there exists $x \in S$ such that $\{x\} = X_1 \setminus \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X_1\} = X_2 \setminus \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X_2\}$. Since $\mathcal{F}$ is a partial closure operator, a set $T = \bigcap\{Z \in \mathcal{F} \mid x \in Z\}$ is in $\mathcal{F}$. Hence, $T \subseteq X_1 \cap X_2$. Since $x \in T$, we have that $T \notin \{Y \in \mathcal{F} \mid Y \subsetneq X_1\}$. By $T \subseteq X_1 \cap X_2 \subseteq X_1$, it follows that $T = X_1$. Similarly, we have $T = X_2$ and then $X_1 = X_2$, which implies that the mapping G is injective.

Now, let $x \in S$ and denote $X_x = \bigcap\{X \in \mathcal{F} \mid x \in X\}$. Since $\mathcal{F}$ is a partial closure system, we have $X_x \in \mathcal{F}$, and we shall show that $G(X_x) = x$. We have $x \in X_x$ and $x \notin \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X_x\}$ because X_x is the smallest set (with respect to set inclusion) in $\mathcal{F}$ that contains x . Since $|X_x \setminus \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X_x\}| = 1$, it follows that $\{x\} = X_x \setminus \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X_x\}$. Hence G is also a surjective mapping. $\square$

Using the introduced bijection G , an order on S can be naturally induced by the set inclusion in a principal partial closure system $\mathcal{F}$ on S , as follows: for all $x, y \in S$,

$$x \leq y \text{ if and only if } G^{-1}(x) \subseteq G^{-1}(y). \quad (3.7)$$

It is straightforward to check that $\leq$ is an order on S . Therefore, as a consequence of Proposition 4, we get the following.

Corollary 1. *Let $\mathcal{F}$ be a principal partial closure system on a set S , and $\leq$ the order on S , defined by (3.7). Then, the function G defined by (3.6) is an order isomorphism from $(\mathcal{F}, \subseteq)$ to $(S, \leq)$. In addition, the collection of principal ideals in $(S, \leq)$ is $\mathcal{F}$.*

Proof. The function G is a bijection by Proposition 4, which is, by the definition of $\leq$ on S , compatible with the corresponding orders. In other words, if $X, Y \in \mathcal{F}$, we have that $X \subseteq Y$ if and only if $G(X) \leq G(Y)$. To prove that subsets in $\mathcal{F}$ are principal ideals, for $x \in S$, we use the denotation from Proposition 4, $G^{-1}(x) = X_x$. We will prove that $\downarrow x = X_x$. If $y \leq x$, then $G^{-1}(y) \subseteq G^{-1}(x)$ and since $y \in G^{-1}(y)$, we have that $y \in G^{-1}(x)$. On the other hand, suppose that $y \in X_x$. Then, $X_y \subseteq X_x$ and hence, $y \leq x$. by the definition of $\leq$ on S , if $y \in X$, then either $y = x$ or $y \in X \setminus \{x\}$. Therefore, $X = \downarrow x$ with respect to the order $\leq$. Since G is a bijection, all the elements from $\mathcal{F}$ are in the form X_x for $x \in X$, so all of them coincides with the principal ideals of $(S, \leq)$. $\square$

We can also start from a poset, and via principal ideals we get a partial closure system, which induces the starting order, as follows.

Corollary 2. *Let $(S, \leq)$ be a poset and $\mathcal{F}$ a partial closure system consisting of its principal ideals. Then, the order on S defined by (3.7) coincides with $\leq$.*

Proof. By Proposition 3, principal ideals make a principal partial closure system. The function G defined by (3.6) associates to every principal ideal its generator, and by (3.7), inclusion among principal ideals induces the existing order $\leq$ from the poset. $\square$

Finally, we introduce a partial closure operator which corresponds to a principal partial closure system.

A partial closure operator C on S is **principal** if it satisfies

Pc_8 : If $X = C(X)$, then there exists unique $x \in X$ such that $x \notin \bigcup \{Y \in \mathcal{F}_C \mid Y \subsetneq X\}$.

It is easy to see that the axioms Pc_5 and Pc_8 are independent.

A connection among these notions can be explained as follows.

The range of a principal closure operator is a principal partial closure system and the sharp partial closure operator obtained from a principal partial closure system, as defined in Theorem 1, is principal.

Obviously, the empty set can not be closed under a principal partial closure operator. As an additional property, we prove that the range of a principal partial closure operator consists of closures of singletons.

Proposition 5. *Let C be a principal partial closure operator on S . If $X \in \mathcal{F}_C$, then there exists $x \in X$ such that $C(\{x\}) = X$.*

Proof. If X is a closed set, then by Pc_8 there exists a unique $x \in X$ such that $x \notin \bigcup \{Y \in \mathcal{F}_C \mid Y \subsetneq X\}$. From $x \in C(\{x\}) \subseteq C(X) = X$ it follows that $C(\{x\}) = X$. $\square$

The following is a *Representation theorem* of posets by SPCO's and by the corresponding partial closure systems.

Theorem 3. *Let $(S, \leq)$ be a poset. The partial mapping $C : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ defined by*

$$C(X) = \downarrow(\bigvee X), \text{ if there exists } \bigvee X,$$

otherwise not defined, is a principal SPCO. The corresponding partial closure system is principal and it is isomorphic with S .

Proof. It is straightforward to check that C is a partial closure operator. In order to prove that it is sharp, suppose that $B \subseteq S$ and that

$$\bigcap \{X \in \mathcal{F}_C \mid B \subseteq X\} \in \mathcal{F}_C.$$

Then, there is a set $Z \subseteq S$, such that $\bigcap \{X \in \mathcal{F}_C \mid B \subseteq X\} = \downarrow(\bigvee Z)$. Consequently, for every $b \in B$, $b \leq \downarrow(\bigvee Z)$. Suppose there is another upper bound of B , say x . Then $B \subseteq \downarrow x$ and $C(\downarrow x) = \downarrow x$. Hence, $\downarrow(\bigvee Z) \subseteq \downarrow x$ and $\bigvee Z \leq x$. Therefore, $C(B) = \downarrow(\bigvee Z) = C(Z)$.

It is easy to see that C is principal by the definition.

Closed elements are principal ideals of S , hence the corresponding partial closure system is isomorphic with S . $\square$

To sum up, we have bijective correspondences among:

- posets
- principal sharp partial closure operators
- principal partial closure systems.

Indeed, correspondences are witnessed by Theorem 3; they are bijective by Theorem 1, Propositions 3, 4 and Corollaries 1, 2.

In particular, if we deal with posets which are complete lattices, then the bijective correspondence already exists among closure systems and closure operators. As mentioned, every closure operator fulfils the sharpness property. Still, to every lattice there correspond more closure operators and systems. If the closure operators and systems are principal, then we get bijective correspondences as for posets.

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Authors’ addresses

Branimir  selja

University of Novi Sad, Faculty of Sciences, Department of Mathematics and Informatics, Faculty of Sciences, Trg Dositeja Obradovi a 4, Novi Sad 21000, Serbia,
E-mail address: seselja@dmi.uns.ac.rs

Anna Slivkov 

University of Novi Sad, Faculty of Sciences, Department of Mathematics and Informatics, Faculty of Sciences, Trg Dositeja Obradovi a 4, Novi Sad 21000, Serbia,
E-mail address: anna.slivkova@dmi.uns.ac.rs

Andreja Tepav evi 

University of Novi Sad, Faculty of Sciences, Department of Mathematics and Informatics, Faculty of Sciences, Trg Dositeja Obradovi a 4, Novi Sad 21000, Serbia,
E-mail address: andreja@dmi.uns.ac.rs



ON WEAK ARMENDARIZ MODULES OVER COMMUTATIVE RINGS

M. SHABANI AND A. YOUSEFIAN DARANI

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Abstract. Let M be a module over a commutative ring R . In this paper we generalize some annihilator conditions on rings to modules. Denote by $Nil(M)$ the set of all nilpotent elements of M . M is said to be weak Armendariz if $f(x)m(x) = 0$, where $f(x) = \sum_{i=0}^n a_i x^i \in R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^k m_j x^j \in M[x] \setminus \{0\}$, then $a_i m_j \in Nil(M)$ for each $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, k$. We prove that the class of these modules are closed under direct sum, finite product and localization. We also prove that if $\frac{M}{N}$ is weak Armendariz, then so is M . Furthermore, we show that if D -module M is torsion, for a domain D , then M is weak Armendariz if and only if $T(M)$ is weak Armendariz, where $T(M)$ is the torsion submodule of M .

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1. INTRODUCTION

Throughout this paper all rings are considered to be commutative with a nonzero identity and all modules are unitary unless otherwise stated. Let R be a ring, M an R -module and N a submodule of M . We denote by $(N :_R M)$ the set of all r in R such that $rM \subseteq N$. The annihilator of M denoted by $ann_R(M)$ is $(0 :_R M)$. An R -module M is called *faithful* if $ann_R(M) = 0$. $R[x]$ denotes the polynomial ring over the ring R and $M[x]$ the polynomial module over the module M .

An R -module M is called a *multiplication module* if every submodule N of M has the form IM for some ideal of R . In this case $N = (N :_R M)M$, since $I \subseteq (N :_R M)$ and so $N = IM \subseteq (N :_R M)M \subseteq N$. According to [6], a submodule N of M is called *pure* if $IN = N \cap IM$ for every ideal I of R . Pure submodules of multiplication modules are multiplication, too. A submodule N of M is idempotent if $N = (N :_R M)N$ (See [1]). A submodule N of M is *prime* whenever $rm \in N$, for some $r \in R$ and $m \in M$ implies that $m \in N$ or $r \in (N :_R M)$. In this case $P := (N :_R M)$ is a prime ideal of R and N is called a *P*-prime submodule of M .

We recall that an ideal I of R is *nilpotent* if $I^k = 0$ for positive integer k and an element r of R is nilpotent if $r^k = 0$ for some $k \in \mathbb{N}$. Also we denote by $Nil(R)$

the set of all nilpotent elements of R . According to [2], a submodule N of M is called *nilpotent* if $(N :_R M)^k N = 0$ for some positive integer k . We say that $m \in M$ is *nilpotent* if Rm is a nilpotent submodule of M . By this definition, clearly the zero submodule of M is nilpotent and hence the zero element of M is nilpotent. $Nil(M)$ denotes the set of all nilpotent elements of M . $Nil(M)$ is not necessarily a submodule of M , but if M is faithful, then $Nil(M)$ is a submodule of M , by [2, Theorem 6]. Moreover, if M is a faithful multiplication R -module, then $Nil(M) = Nil(R)M = \cap P$, where P runs over all prime submodules of M . By [2, Proposition 4(2)], if I is a nilpotent ideal of R , then IM is nilpotent in M and the converse is true if M is faithful and also by this Proposition, if $K \subseteq N$ and N is nilpotent in M , then $\frac{N}{K}$ is nilpotent in $\frac{M}{K}$ as an R -module and the converse is true if K is nilpotent in M and M is faithful. By [2, Theorem 3], if N is a direct summand in multiplication module M , then N is multiplication and idempotent.

Let R be an associative ring with identity. R is said to be *Armendariz* if $f(x)g(x) = 0$, where $f(x) = \sum_{i=0}^m a_i x^i, g(x) = \sum_{j=0}^n b_j x^j \in R[x] \setminus \{0\}$, then $a_i b_j = 0$, for each $0 \leq i \leq m$ and $0 \leq j \leq n$ (See [5]).

An R -module M is called *Armendariz* if $f(x)m(x) = 0$, where $f(x) = \sum_{i=0}^n a_i x^i \in R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^k m_j x^j \in M[x]$ implies that $a_i m_j = 0$ for every $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, k$ (See [5]). A ring R is called *weak Armendariz* if for given $f(x) = \sum_{i=0}^n a_i x^i, g(x) = \sum_{j=0}^n b_j x^j \in R[x] \setminus \{0\}$, $f(x)g(x) = 0$ implies that $a_i b_j \in Nil(R)$ for each $0 \leq i \leq n$ and $0 \leq j \leq n$ (the converse is obviously true). It is obvious that Armendariz rings are weak Armendariz and that subrings of (weak) Armendariz rings are still (weak) Armendariz (See [3]).

In this paper we generalize the concept of weak Armendariz rings defined on associative rings to modules over commutative rings and introduce weak Armendariz modules. An R -module M is called *weak Armendarize* if $f(x)m(x) = 0$ implies that $a_i m_j \in Nil(M)$, for each $0 \leq i \leq n$ and $0 \leq j \leq k$, where $f(x) = \sum_{i=0}^n a_i x^i \in R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^k m_j x^j \in M[x]$. Note that commutative rings are always weak Armendariz.

2. WEAK ARMENDARIZ MODULES

In this section, we study some properties of weak Armendariz modules and investigate the relations between weak Armendariz rings and weak Armendariz modules.

Definition 1. Let R be a ring. An R -module M is called *weak Armendarize* if $f(x)m(x) = 0$ implies that $a_i m_j \in Nil(M)$, for each $0 \leq i \leq n$ and $0 \leq j \leq k$, where $f(x) = \sum_{i=0}^n a_i x^i \in R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^k m_j x^j \in M[x] \setminus \{0\}$.

Proposition 1. Let R be a ring. Then every Armendariz R -module M is weak Armendariz and the converse is true if M has no nonzero nilpotent elements.

Corollary 1. *Let M be a finitely generated faithful multiplication R -module. Then the $\frac{R}{\text{Nil}(R)}$ -module $\frac{M}{\text{Nil}(M)}$ is Armendariz if and only if it is weak Armendariz.*

Proof. First note that $\text{Nil}(M)$ is a submodule of M and $\text{Nil}(M) = \text{Nil}(R)M$, by [2, Theorem 6]. So the $\frac{R}{\text{Nil}(R)}$ -module $\frac{M}{\text{Nil}(M)}$ is well-defined. According to [2, Corollary 7], $\frac{M}{\text{Nil}(M)}$ has no nonzero nilpotent elements. Now the result follows from Proposition 1. $\square$

Proposition 2. *The direct limit of every direct system of weak Armendariz modules is weak Armendariz.*

Proof. Let I be a directed set and $\{M_i; i \in I\}$ a direct system of weak Armendariz R -modules. Set $M = \varinjlim_{i \in I} M_i$. We know that $M = \bigcup_{i \in I} M_i = \sum_{i \in I} M_i$, by [7, Example 5.32]. Let $m(x) = \sum_{j=0}^n m_j x^j \in M[x]$ and $f(x) = \sum_{k=0}^m a_k x^k \in R[x]$, with $f(x)m(x) = 0$. Hence there exists $t \in I$ such that $m_j \in M_t$, for each $j = 0, 1, \dots, n$. So $m(x) \in M_t[x]$. Since M_t is weak Armendariz, $a_k m_j \in \text{Nil}(M_t)$ and hence there exists positive integer n_{kj} such that $(R(a_k m_j) :_R M_t)^{n_{kj}} R(a_k m_j) = 0$, for every $k = 0, 1, \dots, m$ and $j = 0, 1, \dots, n$. But $(R(a_k m_j) :_R M) \subseteq (R(a_k m_j) :_R M_t)$. Therefore, $(R(a_k m_j) :_R M)^{n_{kj}} R(a_k m_j) = 0$, for each $k = 0, 1, \dots, m$ and $j = 0, 1, \dots, n$. Hence M is a weak Armendariz R -module. $\square$

Corollary 2. *If every finitely generated submodule of an R -module M is weak Armendariz, then M is weak Armendariz.*

Proof. We know every module is a direct limit of its finitely generated submodules, by [7, Example 5.32]. Now the result is trivial by Proposition 2. $\square$

Definition 2. A ring R is said to be *strongly weak Armendariz* if every R -module is weak Armendariz.

Corollary 3. *A ring R is strongly weak Armendariz if and only if every finitely generated R -module is weak Armendariz.*

Proof. $\Rightarrow$ It is trivial.

$\Leftarrow$ We know that every R -module M is a direct limit of its finitely generated submodules. Now the result follows from the assumption and Proposition 2. $\square$

Proposition 3. *A ring R is weak Armendariz if and only if it is weak Armendariz as an R -module.*

Proof. We know the R -module R is multiplication. So $\text{Nil}(R) = \text{Nil}(R)R$, by [2, Theorem 6]. Thus the result is obvious. $\square$

Proposition 4. *Every pure submodule of a weak Armendariz multiplication module is weak Armendariz.*

Proof. Let N be a pure submodule of the weak Armendariz multiplication module M and $f(x)m(x) = 0$ for $f(x) = \sum_{i=0}^n a_i x^i \in R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^k m_j x^j \in N[x] \setminus \{0\}$. Clearly, N is a multiplication submodule of M . Since $m(x) \in M[x]$ and M is weak Armendariz, $a_i m_j \in \text{Nil}_R(M)$. By [2, Proposition 4], $(R(a_i m_j) :_R M)$ is a nilpotent ideal of R and so $(R(a_i m_j) :_R M) \subseteq \text{Nil}(R)$, for each $0 \leq i \leq n$ and $0 \leq j \leq k$. By [2, Theorem 6], $\text{Nil}_R(M) = \text{Nil}(R)M$ and $\text{Nil}_R(N) = \text{Nil}(R)N$, since M and N are multiplication. Since N is a pure submodule of M and $R(a_i m_j) = (R(a_i m_j) :_R M)M$, $a_i m_j \in N \cap (R(a_i m_j) :_R M)M = (R(a_i m_j) :_R M)N \subseteq \text{Nil}(R)N = \text{Nil}_R(N)$, for each $0 \leq i \leq n$ and $0 \leq j \leq k$. Thus N is weak Armendariz. $\square$

Corollary 4. *A multiplication R -module M is weak Armendariz if and only if every finitely generated pure submodule of M is weak Armendariz.*

Proof. We know that every module is a direct limit of its finitely generated submodules, by [7, Example 5.32]. Now the result follows from Proposition 2 and 4. $\square$

Proposition 5. *Let M be a multiplication R -module. Then the following statements are equivalent:*

- (1) M is weak Armendariz;
- (2) eM and $(1-e)M$ are weak Armendariz R -modules for every idempotent element e of R .
- (3) eM and $(1-e)M$ are weak Armendariz R -modules for some idempotent element e of R .

Proof. $1 \Rightarrow 2$: Let e be an idempotent element of R . So $M = eM \oplus (1-e)M$. It means that eM and $(1-e)M$ are direct summands in M and so both of them are multiplication and idempotent submodules of M , by [2, Theorem 3]. Therefore, eM and $(1-e)M$ are pure submodules of M and hence they are weak Armendariz, by Proposition 4.

$2 \Rightarrow 3$: It is trivial.

$3 \Rightarrow 1$: Assume that there exists an idempotent element e of R such that eM and $(1-e)M$ are weak Armendariz R -modules. Let $f(x) = \sum_{i=0}^m a_i x^i \in R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^n m_j x^j \in M[x] \setminus \{0\}$, with $f(x)m(x) = 0$. Put $f_1(x) = ef(x)$, $f_2(x) = (1-e)f(x)$, $m_1(x) = em(x)$ and $m_2(x) = (1-e)m(x)$. Then we have $0 = f(x)m(x) = f_1(x)m_1(x) = f_2(x)m_2(x)$. By assumption, $ea_i m_j \in \text{Nil}(eM)$ and $(1-e)a_i m_j \in \text{Nil}((1-e)M)$, for every $i = 0, 1, \dots, m$ and $j = 0, 1, \dots, n$.

So there exist positive integers k_{ij} and t_{ij} such that

$$(R(ea_i m_j) :_R eM)^{k_{ij}} R(ea_i m_j) = 0$$

and

$$(R((1-e)a_i m_j) :_R (1-e)M)^{t_{ij}} R((1-e)a_i m_j) = 0$$

for each $0 \leq i \leq m$ and $0 \leq j \leq n$. Set $k = \max\{k_{ij}, t_{ij}; 0 \leq i \leq m, 0 \leq j \leq n\}$. Note that $m = em + (1-e)m$, for each element $m \in M$, and so $Rm = R(em) + R((1-e)m)$. Hence, we have

$$\begin{aligned} (R(a_im_j) :_R M)^k R(a_im_j) &= (R(a_im_j) :_R M)^k [R(ea_im_j) + R((1-e)a_im_j)] \\ &= (R(a_im_j) :_R M)^k R(ea_im_j) \\ &\quad + (R(a_im_j) :_R M)^k R((1-e)a_im_j) \\ &\subseteq (R(ea_im_j) :_R eM)^k R(ea_im_j) \\ &\quad + (R((1-e)a_im_j) :_R (1-e)M)^k R((1-e)a_im_j) \\ &= 0 \end{aligned}$$

So $a_im_j \in Nil_R(M)$, for each $0 \leq i \leq m$ and $0 \leq j \leq n$. Therefore, M is weak Armendariz. $\square$

Theorem 1. *Every finite direct product of weak Armendariz modules is weak Armendariz.*

Proof. Suppose that $M_1, M_2, \dots, M_k$ are weak Armendariz R -modules. Set $M = \prod_{t=1}^k M_t$. Clearly, M is an R -module. Let $f(x)m(x) = 0$, for some $m(x) = \sum_{i=0}^n m_i x^i \in M[x] \setminus \{0\}$ and $f(x) = \sum_{j=0}^p a_j x^j \in R[x] \setminus \{0\}$, where $m_i = (m_{i1}, \dots, m_{ik}) \in M$.

Define $m_t(x) = \sum_{i=0}^n m_{it} x^i \in M_t[x]$, for each $1 \leq t \leq k$. From $f(x)m(x) = 0$ we have $a_0 m_0 = a_1 m_0 + a_0 m_1 = \dots = a_p m_n = 0$. This implies that

$$\begin{aligned} a_0 m_{01} &= \dots = a_0 m_{0k} = 0 \\ a_1 m_{01} + a_0 m_{11} &= \dots = a_1 m_{0k} + a_0 m_{1k} = 0 \\ &\vdots \\ a_p m_{n1} &= \dots = a_p m_{nk} = 0 \end{aligned}$$

This means that $f(x)m_t(x) = 0$ in $M_t[x]$, for every $1 \leq t \leq k$. Since each M_t is weak Armendariz module, we have $a_j m_{it} \in Nil_R(M_t)$ and so there exists positive integer n_{ijt} , for each $i = 0, 1, \dots, n$, $j = 0, 1, \dots, p$ and $0 \leq t \leq k$, such that $(R(a_j m_{it}) : M_t)^{n_{ijt}} R(a_j m_{it}) = 0$ in M_t . If we put $n_{ij} = \max\{n_{ijt}; 1 \leq t \leq k\}$, then

$$\begin{aligned} (R(a_j m_i) : M)^{n_{ij}} R(a_j m_i) &\subseteq \left(\prod_{t=1}^k R(a_j m_{it}) : \prod_{t=1}^k M_t \right)^{n_{ij}} \left(\prod_{t=1}^k R(a_j m_{it}) \right) \\ &= \left(\bigcap_{t=1}^k (R(a_j m_{it}) : M_t) \right)^{n_{ij}} \left(\prod_{t=1}^k R(a_j m_{it}) \right) \end{aligned}$$

$$\begin{aligned}
&\subseteq \bigcap_{t=1}^k ((R(a_j m_{it}) : M_t)^{n_{ij}}) \prod_{t=1}^k (R(a_j m_{it})) \\
&= \prod_{t=1}^k \bigcap_{i=1}^k ((R(a_j m_{it}) : M_t)^{n_{ij}} R(a_j m_{it})) \\
&= 0
\end{aligned}$$

So $a_j m_i \in \text{Nil}_R(M)$, for each $0 \leq i \leq n$ and $0 \leq j \leq p$. Thus M is a weak Armendariz R -module. $\square$

Proposition 6. Let R be a ring and N a nilpotent submodule of a faithful R -module M . If $\frac{M}{N}$ is a weak Armendariz R -module, then M is weak Armendariz.

Proof. Let $f(x) = \sum_{i=0}^n a_i x^i$ and $m(x) = \sum_{j=0}^k m_j x^j$ are polynomials in $R[x] \setminus \{0\}$ and $M[x] \setminus \{0\}$, respectively, such that $f(x)m(x) = 0$. Set $\bar{m}(x) = \sum_{j=0}^k \bar{m}_j x^j$, where $\bar{m}_j$ denotes the coset of m_j in $\frac{M}{N}$. So $f(x)\bar{m}(x) = 0$ in the R -module $\frac{M}{N}$. Since $\frac{M}{N}$ is weak Armendariz, $a_i \bar{m}_j \in \text{Nil}(\frac{M}{N})$, for every $0 \leq i \leq n$ and $0 \leq j \leq k$. Thus $R(a_i \bar{m}_j) = \frac{(a_i m_j)R + N}{N}$ is a nilpotent submodule of $\frac{M}{N}$. But N is nilpotent in M , so $R(a_i m_j) + N$ is nilpotent in M and also $R(a_i m_j) = (R(a_i m_j) + N) \cap R(a_i m_j)$ is nilpotent in M , by [2, Proposition 4]. This means that $a_i m_j \in \text{Nil}(M)$, for each $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, k$ and therefore M is weak Armendariz. $\square$

Proposition 7. Let R be a ring and N a submodule of an R -module M such that $\frac{M}{N}$ is a faithful weak Armendariz R -module. Then M is weak Armendariz.

Proof. Let $f(x) = \sum_{i=0}^m a_i x^i \in R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^n m_j x^j \in M[x] \setminus \{0\}$, with $f(x)m(x) = 0$. Set $\bar{m}(x) = \sum_{j=0}^n \bar{m}_j x^j = \sum_{j=0}^n (m_j + N)x^j$. Hence $\bar{m}(x) \in \frac{M}{N}[x]$ and $f(x)\bar{m}(x) = \bar{0} = N$. By assumption and [2, Proposition 4], $a_i \bar{m}_j \in \text{Nil}_R(\frac{M}{N})$ and so $(R(a_i \bar{m}_j) :_R \frac{M}{N})$ is a nilpotent ideal of R , for each $0 \leq i \leq m$ and $0 \leq j \leq n$, since $\frac{M}{N}$ is a faithful R -module. But $(R(a_i m_j) :_R M) \subseteq (R(a_i \bar{m}_j) :_R \frac{M}{N})$. Hence, $(R(a_i m_j) :_R M)$ is a nilpotent ideal of R and so $a_i m_j \in \text{Nil}_R(M)$, for each $0 \leq i \leq m$ and $0 \leq j \leq n$. Thus, M is a weak Armendariz R -module. $\square$

Proposition 8. Let R be a ring and N a nilpotent submodule of a faithful R -module M such that $N \subseteq \text{Nil}(M)$. If M is weak Armendariz, then the R -module $\frac{M}{N}$ is weak Armendariz.

Proof. Suppose that M is a weak Armendariz R -module. Let $f(x) = \sum_{i=0}^n a_i x^i \in R[x] \setminus \{0\}$ and $\bar{m}(x) = \sum_{j=0}^k \bar{m}_j x^j \in (\frac{M}{N})[x] \setminus \{0\}$ satisfy $f(x)\bar{m}(x) = 0$, where $\bar{m}_j = m_j + N$, for each $j = 0, 1, \dots, k$ and for some $m_j \in M$. Therefore, $a_i m_j \in N \subseteq$

$Nil(M)$ and hence $R(a_i m_j)$ is a nilpotent submodule of M , for every $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, k$. According to [2, Proposition 4], $R(a_i m_j) + N$ is nilpotent in M and $\frac{R(a_i m_j) + N}{N}$ is nilpotent in $\frac{M}{N}$. On the other hand, $\frac{R(a_i m_j) + N}{N} = R(a_i \overline{m_j})$. So $a_i \overline{m_j} \in Nil(\frac{M}{N})$, for every $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, k$. Thus, $\frac{M}{N}$ is a weak Armendariz R -module. $\square$

Proposition 9. *Let R be a ring and $\{M_i\}_{i \in I}$ a family of faithful weak Armendariz R -modules. Then the direct sum $M = \coprod_{i \in I} M_i$ is weak Armendariz.*

Proof. Let $f(x) = \sum_{j=0}^n a_j x^j \in R[x]$ and $m(x) = \sum_{k=0}^t m_k x^k \in M[x] \setminus \{0\}$, with $f(x)m(x) = 0$. We can assume that $m_k = (m_{ik})_{i \in I} \in \coprod_{i \in I} M_i$. Set $m_i(x) = \sum_{k=0}^t m_{ik} x^k$. It is clear that $f(x)m_i(x) = 0$, for each $i \in I$. Since each M_i is weak Armendariz, $a_j m_{ik} \in Nil(M_i)$, for every $i \in I$, $j = 0, 1, \dots, n$ and $k = 0, 1, \dots, t$. Hence, $(R(a_j m_{ik}) : M_i)$ is a nilpotent ideal of R , by [2, Proposition 4].

On the other hand, $I_k = \{i \in I; m_{ik} \neq 0\}$ is a finite set and also $R(a_j m_k) \subseteq \coprod_{i \in I} R(a_j m_{ik})$, for each $k = 0, 1, \dots, t$. So we have

$$\begin{aligned} (R(a_j m_k) : \coprod_{i \in I} M_i) &\subseteq (\coprod_{i \in I} R(a_j m_{ik}) : \coprod_{i \in I} M_i) \\ &\subseteq (\coprod_{i \in I_k} R(a_j m_{ik}) : \coprod_{i \in I_k} M_i) \\ &= \bigcap_{i \in I_k} (R(a_j m_{ik}) : M_i) \end{aligned}$$

for each $k = 0, 1, \dots, t$ and $j = 0, \dots, n$. Finite intersection of nilpotent ideals is nilpotent. So $a_j m_k \in Nil(M)$, for every $0 \leq j \leq n$ and $0 \leq k \leq t$. Therefore, M is a weak Armendariz R -module. $\square$

We recall that if M is an R -module and S a multiplicatively closed subset of R , then $S^{-1}M$ has an $S^{-1}R$ -module structure. Also, M is called S -torsion free if $ms \neq 0$, for every $m \in M$ and $s \in S$.

Lemma 1. *Let R be a ring, S a multiplicatively closed subset of R and M a finitely generated R -module. Let $\frac{m}{s} \in S^{-1}M$, for $m \in M$ and $s \in S$. Then $m \in Nil_R(M)$ if and only if $\frac{m}{s} \in Nil_{S^{-1}R}(S^{-1}M)$.*

Proof. Let $m \in M$ and $k \in \mathbb{N}$. Since M is finitely generated, we have

$$\begin{aligned} S^{-1}((Rm : M)^k Rm) &= S^{-1}((Rm :_R M)^k) S^{-1}(Rm) \\ &= (S^{-1}(Rm :_R M))^k S^{-1}(Rm) \\ &= ((S^{-1}(Rm) :_{S^{-1}R} S^{-1}M)^k S^{-1}(Rm)) \end{aligned}$$

Also, $S^{-1}(Rm) = S^{-1}R(\frac{m}{s})$, for each $s \in S$. So the result is obvious. $\square$

Proposition 10. *Let R be a ring, S a multiplicatively closed subset of R and M a finitely generated S -torsion free R -module. Then M is weak Armendariz if and only if $S^{-1}M$ is weak Armendariz as an $S^{-1}R$ -module.*

Proof. Let M be a weak Armendariz R -module and $f(x)m(x) = 0$, where $f(x) = \sum_{i=0}^m \alpha_i x^i \in S^{-1}R[x]$ and $m(x) = \sum_{j=0}^n \beta_j x^j \in S^{-1}M[x]$.

We can assume that $\alpha_i = \frac{a_i}{u_i}$ and $\beta_j = \frac{m_j}{v_j}$, for some $a_i \in R$, $m_j \in M$ and $u_i, v_j \in S$. Setting $u = u_1 u_2 \dots u_m$ and $v = v_1 v_2 \dots v_n$, we have

$$\begin{aligned} 0 &= f(x)m(x) = \sum_{i=0}^m \sum_{j=0}^n \alpha_i \beta_j x^{i+j} \\ &= \sum_{i=0}^m \sum_{j=0}^n \frac{a_i m_j}{uv} x^{i+j} \end{aligned}$$

Hence, $\frac{a_i m_j}{uv} = 0$ and so there exists $s_{ij} \in S$ such that $s_{ij}(a_i m_j) = 0$, for each $0 \leq i \leq m$ and $0 \leq j \leq n$. Set $f_1(x) = \sum_{i=0}^m a_i x^i$ and $m_1(x) = \sum_{j=0}^n m_j x^j$. Clearly, $f_1(x) \in R[x] \setminus \{0\}$, $m_1(x) \in M[x] \setminus \{0\}$ and $f_1(x)m_1(x) = 0$. Since M is a weak Armendariz R -module, $a_i m_j \in \text{Nil}_R(M)$, for each $0 \leq i \leq m$ and $0 \leq j \leq n$. Now, by Lemma 10, $\frac{(a_i m_j)}{uv} = (a_i m_j)(uv)^{-1} \in \text{Nil}_{S^{-1}R}(S^{-1}M)$ and therefore $S^{-1}M$ is weak Armendariz.

Now suppose that $S^{-1}M$ is a weak Armendariz $S^{-1}R$ -module. Let $g(x) = \sum_{i=0}^m b_i x^i \in R[x] \setminus \{0\}$ and $n(x) = \sum_{j=0}^k n_j x^j \in M[x] \setminus \{0\}$ with $g(x)n(x) = 0$. But $g(x) \in S^{-1}R[x] \setminus \{0\}$ and $n(x) \in S^{-1}M[x] \setminus \{0\}$. So $\frac{b_i n_j}{1} = b_i n_j \in \text{Nil}_{S^{-1}R}(S^{-1}M)$, for each $0 \leq i \leq m$ and $0 \leq j \leq k$. Hence, $b_i n_j \in \text{Nil}_R(M)$, by Lemma 10. Thus, M is a weak Armendariz R -module. $\square$

We recall that $R[x, x^{-1}]$ denotes the Laurent polynomial ring over R . For an R -module M , let $M[x, x^{-1}] = \{\sum_i k^n m_i x^i; n, k \in \mathbb{Z}, m_i \in M\}$. $M[x, x^{-1}]$ is an $R[x, x^{-1}]$ -module under the addition operation and the following scalar product operation. For $m(x) = \sum_i m_i x^i \in M[x, x^{-1}]$ and $f(x) = \sum_j a_j x^j \in R[x, x^{-1}]$, then $f(x)m(x) = \sum_k (\sum_{i+j=k} a_j m_i) x^k$ (see [4]).

Corollary 5. *Let M be a finitely generated R -module. Then $M[x]$ is a weak Armendariz $R[x]$ -module if and only if $M[x, x^{-1}]$ is a weak Armendariz $R[x, x^{-1}]$ -module.*

Proof. Set $S = \{1, x, x^2, \dots\}$. Then S is a multiplicatively closed subset of $R[x]$. Also $R[x, x^{-1}] = S^{-1}R[x]$ and $M[x, x^{-1}] = S^{-1}M[x]$. Now the result follows from Proposition 10. $\square$

Corollary 6. *Let R be a ring and S a multiplicatively closed subset of R . Then the localization $S^{-1}R$ is strongly weak Armendariz if and only if R is strongly weak Armendariz.*

For a commutative domain R and R -module M , the torsion submodule of M is defined by $T(M) = \{x \in M \mid ann_R(x) \neq 0\}$. In this case, $T(M)$ is a submodule of M , called the torsion part of M . An R -module M is called a *torsion* module if $T(M) = M$.

Proposition 11. *Let D be a domain and M a D -module. If the D -module $T(M)$ is weak Armendariz, then M is also weak Armendariz, too.*

Proof. Let $T(M)$ be weak Armendariz and $f(x)m(x) = 0$, where $f(x) = \sum_{i=0}^n a_i x^i \in D[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^n m_j x^j \in M[x]$. Since $f(x) \neq 0$, we may assume that $a_0 \neq 0$. Also we have $a_0 m_0 = a_0 m_1 + a_1 m_0 = a_0 m_2 + a_1 m_1 + a_2 m_0 = \dots = a_k m_n = 0$. Multiplying $a_0 m_1 + a_1 m_0 = 0$ by a_0 implies that $a_0^2 m_1 = 0$. Similarly, multiplying $a_0 m_2 + a_1 m_1 + a_2 m_0 = 0$ by a_0^2 implies that $a_0^3 m_2 = 0$. Continuing this process, we have $a_0^{j+1} m_j = 0$ and so $m_j \in T(M)$, for every $j = 0, 1, \dots, n$. Hence $m(x) \in T(M)[x]$. Since $T(M)$ is weak Armendariz, $a_i m_j \in Nil(T(M))$, for every $i = 0, 1, \dots, k$ and $j = 0, 1, \dots, n$. So there exists a positive integer n_{ij} , for every $i = 0, 1, \dots, k$ and $j = 0, 1, \dots, n$, such that $(D(a_i m_j) :_D T(M))^{n_{ij}} D(a_i m_j) = 0$. But $(D(a_i m_j) :_D M) \subseteq (D(a_i m_j) :_D T(M))$. Therefore, $(D(a_i m_j) :_D M)^{n_{ij}} D(a_i m_j) = 0$, for each $0 \leq i \leq k$ and $0 \leq j \leq n$. Thus $a_i m_j \in Nil(M)$ and M is a weak Armendariz D -module. $\square$

Theorem 2. *Let R be an integral domain and Q the quotient field of R . If M is a weak Armendariz Q -module, then M is weak Armendariz as an R -module and the converse is true if M is a finitely generated R -module.*

Proof. First note that if $M = 0$, then the result is trivial. Now assume that $M \neq 0$. $\Rightarrow$: Suppose that M is a weak Armendariz Q -module. Let $f(x) = \sum_{i=0}^n a_i x^i \in R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^k m_j x^j$ belongs to an $R[x]$ -module $M[x]$ with $f(x)m(x) = 0$. By assumption, $a_i m_j \in Nil_Q(M)$, for every $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, k$. M is a faithful Q -module, since $M \neq 0$. So the ideal $(Q(a_i m_j) :_Q M)$ is a nilpotent ideal of Q , by [2, Proposition 4]. But $(R(a_i m_j) :_R M) \subseteq (Q(a_i m_j) :_Q M)$. Thus $(R(a_i m_j) :_R M)$ is a nilpotent ideal of R and so $a_i m_j \in Nil_R(M)$, for every $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, k$, by [2, Proposition 4]. Therefore, M is a weak Armendariz R -module.

$\Leftarrow$: Now, suppose that M is a weak Armendariz R -module. Let $f(x)m(x) = 0$, where $f(x) = \sum_{i=0}^n \alpha_i x^i \in Q[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^k m_j x^j$ belongs to the $Q[x]$ -module $M[x]$. We may assume that $\alpha_i = s_i^{-1} b_i$, for each $i = 0, 1, \dots, n$ and for some $b_i \in R$ and regular element $s_i \in R$. Set $s = s_0 s_1 \dots s_n$. So we can assume that $\alpha_i = s^{-1} a_i$, for each $i = 0, 1, \dots, n$ and for some $a_i \in R$. Then we have

$$0 = f(x)m(x) = \sum_{i=0}^n \sum_{j=0}^k s^{-1} a_i m_j x^{i+j} = s^{-1} f'(x)m(x)$$

where $f'(x) = \sum_{i=0}^n a_i x^i \in R[x] \setminus \{0\}$. Therefore, $f'(x)m(x) = 0$. Since M is a weak Armendariz R -module, $a_i m_j \in \text{Nil}_R(M)$. M is a faithful finitely generated R -module, hence the ideal $(R(a_i m_j) :_R M)$ is a nilpotent ideal of R and also $Q(R(a_i m_j) :_R M) = (Q(a_i m_j) :_Q M)$, for each $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, k$. So $a_i m_j \in \text{Nil}_Q(M)$, for every $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, k$. Thus M is a weak Armendariz Q -module. $\square$

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Authors' addresses

M. Shabani

University of Mohaghegh Ardabili, Department of Mathematics and Applications, P. O. Box 179, Ardabil, Iran

E-mail address: m.shabani@uma.ac.ir

A. Yousefian Darani

University of Mohaghegh Ardabili, Department of Mathematics and Applications, P. O. Box 179, Ardabil, Iran

E-mail address: yousefian@uma.ac.ir, youseffian@gmail.com



A SHORT NOTE ON NESTED SUMS

NIR SHVALB AND SHLOMI HACHOEN

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Abstract. We express a set of nested sums as a 1-index weighted sum using Stars and Bars Theorem.

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1. INTRODUCTION

Nested sums appear in various fields of research. Here, we consider the following summation:

$$\sum_{a_1=0}^N \sum_{a_2=0}^{a_1} \sum_{a_3=0}^{a_2} \cdots \sum_{a_n=0}^{a_{n-1}} f(\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n) \quad (1.1)$$

where $f(m)$ may be any map defined for $0 \leq m \leq 2N$. Under certain constraints on the coefficients α_i , $i \in \{1, \dots, n\}$, we simplify the expression to the form $\sum_{\gamma} f(\gamma) P(\gamma)$

where $P(\gamma)$ is the number of occurrences of γ within the sum. Surprisingly, the authors could not find any formula concerning such expressions. Consider for example the sum:

$$S = \sum_{a_1=0}^N \sum_{a_2=0}^{a_1} \sum_{a_3=0}^{a_2} \sum_{a_4=0}^{a_3} f(2a_3 - a_4),$$

applying Theorem 1 below, simplifies the expression to:

$$S = \sum_{\gamma=0}^{2N} \sum_{\phi=\phi_{\min}(\gamma)}^{\phi_{\max}(\gamma)} \binom{N-\gamma+\phi+2}{2} f(\gamma)$$

where, $\phi_{\min}(\gamma) = \max\{0, \gamma - N\}$ and $\phi_{\max}(\gamma) = \min\{N, \lfloor \gamma/2 \rfloor\}$.

2. DISCUSSION

Denote $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)^T$ and $\underline{a} = (a_1, \dots, a_n)$ and $\gamma = \underline{\alpha} \cdot \underline{a}$ we set:

$$\begin{aligned} x_1 &= N - a_1, \\ x_2 &= a_1 - a_2, \\ &\vdots \\ x_n &= a_{n-1} - a_n, \\ x_{n+1} &= a_n \end{aligned}$$

so

$$N = x_1 + x_2 + \dots + x_{n+1} \quad (2.1)$$

where $0 \leq x_i \leq N, i \in \{1, \dots, n+1\}$. This implies that:

$$\gamma = N\beta_1 - x_1\beta_1 - x_2\beta_2 \dots - x_n\beta_n$$

where $\beta_j = \sum_{i=j}^n \alpha_i$.

Substituting into Eq. 2.1 we get the *Frobenius Equations* (cf. [2]):

$$N\beta_1 - \gamma = x_1\beta_1 + \dots + x_n\beta_n \quad (2.2)$$

The number of integer solutions to this equation yields the following:

Theorem 1. *Consider the sum:*

$$S = \sum_{a_1=0}^N \sum_{a_2=0}^{a_1} \sum_{a_3=0}^{a_2} \dots \sum_{a_n=0}^{a_{n-1}} f(\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n)$$

Denote by $\mathfrak{J}^+$ the set of indices of coefficients $\beta_j = \sum_{i=j}^n \alpha_i$ that are equal to +1, by

$\mathfrak{J}^0$ the set of coefficients' indices that vanish and by $\mathfrak{J}^-$ the set of coefficients' indices that are equal -1. Moreover we assume that $\beta_1 = 1$. Then

$$S = \sum_{\gamma=0}^{2N} P(\gamma, \underline{\alpha}, N) f(\gamma)$$

where the occurrence function is:

$$P(\gamma, \underline{\alpha}, N) = \begin{cases} \binom{N-\gamma+\|\mathfrak{J}^+\|-1}{\|\mathfrak{J}^+\|-1} & \text{if } \|\mathfrak{J}^-\| = 0 \\ \sum_{\varphi=\varphi_{\min}(\gamma)}^{\varphi_{\max}(\gamma)} \binom{\varphi+\|\mathfrak{J}^-\|-1}{\|\mathfrak{J}^-\|-1} \binom{N-\gamma+\varphi+\|\mathfrak{J}^+\|-1}{\|\mathfrak{J}^+\|-1} & \text{otherwise} \end{cases}$$

and

$$\varphi_{min}(\gamma) = \max\{0, \gamma - N\}, \quad \varphi_{max}(\gamma) = \min\{N\|\mathcal{J}^-\|, \lfloor \frac{\gamma}{2} \rfloor\}.$$

Proof. Recall the "Stars and Bars Theorem" by W. Feller [1]: "For any pair of positive integers n and k , the number of k -tuples of non-negative integers whose sum is n is equal to $\binom{n+k-1}{k-1}$ ".

So in the case where $\|\mathcal{J}^-\| = 0$, the number of solutions for Eq. 2.2 takes the desired form. (Here $\|\cdot\|$ denotes the cardinality of a set). In the general case we have $\|\mathcal{J}^+\|, \|\mathcal{J}^-\|, \|\mathcal{J}^0\| > 0$, and Eq. 2.2 takes the form:

$$N - \gamma + \sum_{k \in \mathcal{J}^-} x_k = \sum_{i \in \mathcal{J}^+} x_i \quad (2.3)$$

Equation $\varphi = \sum_{k \in \mathcal{J}^-} x_k$ have $\binom{\varphi + \|\mathcal{J}^-\| - 1}{\|\mathcal{J}^-\| - 1}$ roots. The number of solutions to the equation $N - \gamma + \varphi = \sum_{i \in \mathcal{J}^+} x_i$ (thinking of the left hand side as a constant value) is $\binom{N - \gamma + \varphi + \|\mathcal{J}^+\| - 1}{\|\mathcal{J}^+\| - 1}$. Summing over φ yields:

$$P(\gamma, \underline{\alpha}, N) = \sum_{\varphi = \varphi_{min}}^{\varphi_{max}} \binom{\varphi + \|\mathcal{J}^-\| - 1}{\|\mathcal{J}^-\| - 1} \binom{N - \gamma + \varphi + \|\mathcal{J}^+\| - 1}{\|\mathcal{J}^+\| - 1} \quad (2.4)$$

To produce the upper and lower bound of φ for the summation above note that $N \geq x_i \geq 0$ for all x_i 's so $N\|\mathcal{J}^-\| \geq \varphi \geq 0$. Eq. 2.3 along with the fact that $\beta_1 = 1$ (since it must be positive by definition) then implies $\varphi \leq N(\|\mathcal{J}^+\| - 1) + \gamma$. Substituting Eq. 2.1 into Eq. 2.2 yields:

$$\begin{aligned} \gamma &= (\beta_1 - \beta_2)x_2 + (\beta_1 - \beta_3)x_3 + \cdots + (\beta_1 - \beta_n)x_n + \beta_1 x_{n+1} \\ &= 2 \sum_{k \in \mathcal{J}^-} x_k + \sum_{k \in \mathcal{J}^+} x_k x_j + x_{n+1} \end{aligned} \quad (2.5)$$

where we used:

$$\beta_1 - \beta_i = \begin{cases} 0 & i \in \mathcal{J}^+ \\ 1 & i \in \mathcal{J}^0 \\ 2 & i \in \mathcal{J}^- \end{cases}$$

Note that the maximal γ occurs when $\sum_{k \in \mathcal{J}^-} x_k = N$ so that $\sum_{k \in \mathcal{J}^+} x_k = 0$ (see 2.1). Taking the two last terms of Eq. 2.5 to zero, it yields $\varphi \leq \frac{\gamma}{2}$. Since $\binom{a}{b}$ vanishes when $b > a$ we also require $\varphi \geq \gamma - N$. Finally, we obtain:

$$\varphi_{min}(\gamma) = \max\{0, \gamma - N\}, \quad \varphi_{max}(\gamma) = \min\{N\|\mathcal{J}^-\|, \lfloor \frac{\gamma}{2} \rfloor\}$$

□

Example 1. Consider the case where n is odd and $\|\mathcal{J}^-\| = 0$ (e.g. γ is the alternating sum: $a_1 - a_2 + a_3 - \dots + a_n$). Let $f(k, n, p)$ be the probability mass binomial function $\binom{n}{k} p^k (1-p)^{n-k}$. Applying Theorem 1 yields :

$$\begin{aligned} \sum_{a_1=0}^N \sum_{a_2=0}^{a_1} \dots \sum_{a_n=0}^{a_{n-1}} f(\|\mathcal{J}^+\|, N - \gamma, p) \\ = \frac{p^{\|\mathcal{J}^+\| - 1}}{(\|\mathcal{J}^+\|!)^2} \left[(1-p)^{-N - \|\mathcal{J}^+\| - 1} - (1-p)^{N - \|\mathcal{J}^+\|} \right] \end{aligned}$$

where we used the identity $\binom{a-b}{b} \binom{a}{b} = (b!)^{-2}$.

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Authors' addresses

Nir Shvalb

Ariel University, Faculty of Engineering, Ariel, 40700, Ariel, Israel
E-mail address: nirsh@ariel.ac.il

Shlomi Hacoen

Ariel University, Faculty of Engineering, Ariel, 40700, Ariel, Israel
E-mail address: hacohenshlomi@gmail.com



GLOBAL ASYMPTOTIC STABILITY OF NONLINEAR PERIODIC IMPULSIVE EQUATIONS

V.I. SLYN'KO AND CEMIL TUNÇ

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Abstract. Pseudo-linear impulsive differential equations in a Banach space are considered. It is assumed that the conditions of a small change in the operator coefficients of the equation are satisfied. Using the method of "frozen" coefficients and the methods of commutator calculus, the problem of global asymptotic stability of a pseudo-linear impulsive differential equation is reduced to the problem of estimating the evolution operator for linear impulsive differential equation with constant operator coefficients. The obtained results are applied for stability study of a nonlinear system of ordinary impulsive differential equations. Lyapunov's direct method is used for estimating the fundamental matrix of the corresponding system of impulsive differential equations with constant coefficients. The stability conditions are formulated in terms of the solvability of certain linear matrix inequalities.

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Keywords: commutator calculus, Lyapunov's direct method, Baker-Campbell-Hausdorff formula, impulsive systems, global stability, Magnus series

The stability of solutions of the impulsive differential equations has been studied in many papers [3, 4, 7, 8, 11]. In the monographs, [7, 11], the direct Lyapunov's method for nonlinear systems of impulsive differential equations is developed, and some results related to the first method of Lyapunov are presented.

In [8], the direct Lyapunov's method is developed using the second derivatives of the auxiliary function, while in [3] the higher derivatives of the Lyapunov function are used.

The problems of the global existence of solutions and the stability of abstract differential equations were also considered in the papers [13].

An actual and little studied problem of the theory of stability for solutions of impulsive differential equations is to find or to estimate the stability region, in particular, the establishment of the conditions for global asymptotic stability of the equilibrium position for this class of systems. Some results in this direction have been obtained

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in [2] for pseudolinear impulsive systems, under additional monotonicity conditions for these systems with respect to a certain cone in the phase space.

In the present paper, based on the methods of "frozen" coefficients [10] and the direct Lyapunov's method, in combination with the methods of commutator calculus, the problem of global stability of the zero solution of a nonlinear periodic impulsive differential equation in Banach space is investigated.

1. AUXILIARY RESULTS

In this section, we give some basic algebraic concepts that will be used later (See [9]). Let us describe the construction of the free associative algebra $\mathfrak{A}$ over the field of real numbers $\mathbb{R}$ with two generators x, y . A word over the alphabet (x, y) is a finite sequence of symbols from the alphabet, and for the sequence $\underbrace{z \dots z}_n$, where z is the

generator, we use the notation z^n , $n \in \mathbb{N}$, $z^0 = 1$. The basis of the algebra $\mathfrak{A}_0$ consists of all words over the alphabet (x, y) . The empty word is identified with the element $1 \in \mathbb{R}$, therefore $\mathbb{R} \subset \mathfrak{A}_0$ and the field $\mathbb{R}$ is the center of algebra $\mathfrak{A}_0$. Consequently, $\mathfrak{A}_0$ consists of all possible linear combinations of words with coefficients from $\mathbb{R}$. Free associative algebra $\mathfrak{A}$ is defined as the completion of the algebra $\mathfrak{A}_0$ in some special topology and consists of all formal infinite series with coefficients from the field $\mathbb{R}$. Commutator of two elements $a \in \mathfrak{A}$, $b \in \mathfrak{A}$ which is defined by the formula

$$[a, b] = ab - ba$$

introduces into $\mathfrak{A}$ the structure of the Lie algebra. The commutation operator ad_a , $a \in \mathfrak{A}$ is defined as a linear mapping $\mathfrak{A} \rightarrow \mathfrak{A}$,

$$\text{ad}_a(y) = [a, y], \quad y \in \mathfrak{A}.$$

Let $f(x, y) \in \mathfrak{A}$, $z \in \mathfrak{A}$, $\lambda \in \mathbb{R}$. Then the polarization identity

$$f(x + \lambda z, y) = f(x, y) + \lambda f_1(x, y, z) + \lambda^2 f_2(x, y, z) + \dots$$

defines the derivative of Hausdorff $\left(z \frac{\partial}{\partial x}\right) f \stackrel{df}{=} f_1(x, y, z)$.

We define recursively the following Lie elements of the algebra $\mathfrak{A}$

$$\{y, x^0\} = y, \quad \{y, x^{l+1}\} = [\{y, x^l\}, x], \quad l \in \mathbb{N}.$$

It is easy to see that

$$\text{ad}_x^l(y) = (-1)^l \{y, x^l\}.$$

If $p(x) = \sum_{k=0}^{\infty} p_k x^k$ is a series from the generator x , then

$$\{y, p(x)\} \stackrel{df}{=} \sum_{k=0}^{\infty} p_k \{y, x^k\}.$$

We also note that if $p(x) = \sum_{k=0}^{\infty} p_k x^k$, $q(x) = \sum_{k=0}^{\infty} q_k x^k$, then

$$\{y, p(x)q(x)\} = \{\{y, p(x)\}, q(x)\}. \quad (1.1)$$

Similarly, for an arbitrary finite sequence $x_1, \dots, x_n$ of the elements from $\mathfrak{A}$, we define recursively an element $\{x_1, \dots, x_{n-1}, x_n\}$ as

$$\{x_1, \dots, x_{n-1}, x_n\} = [\{x_1, \dots, x_{n-1}\}, x_n].$$

Moreover, we have

$$e^{-x} y e^x = \{y, e^x\} \quad (1.2)$$

and the following identities of F. Hausdorff are valid

$$e^{-x} \left(y \frac{\partial}{\partial x} \right) e^x = \left\{ y, \frac{e^x - 1}{x} \right\}, \quad \left(\left(y \frac{\partial}{\partial x} \right) e^x \right) e^{-x} = \left\{ y, \frac{1 - e^{-x}}{x} \right\}. \quad (1.3)$$

Note that the identities (1.1) — (1.3) have a formal character.

If X is a Banach space and $L(X)$ is a Banach algebra of bounded linear operators acting in X , then (1.1) is satisfied in the common part of the domain of convergence of the series $p(x)$, $q(x)$, and (1.2), (1.3) are satisfied for all $x \in L(X)$, $y \in L(X)$.

Consider the operator differential equation

$$\frac{dU(t, s)}{dt} = A(t)U(t, s), \quad U(s, s) = 1, \quad t \geq s, \quad (1.4)$$

where $U \in C^1(\mathbb{R} \times \mathbb{R}; L(X))$, $A \in C(\mathbb{R}; L(X))$.

In paper [9] W. Magnus has established the conditions under which the operator $U(t)$ can be represented as

$$U(t, s) = \exp \hat{\Omega}(t, s), \quad t \geq s.$$

In addition, the function $\hat{\Omega}(t, s)$ is a solution of the Cauchy problem for a nonlinear differential equation (W. Magnus equation)

$$\frac{d\hat{\Omega}(t, s)}{dt} = \left\{ A(t), \frac{\hat{\Omega}(t, s)}{1 - e^{-\hat{\Omega}(t, s)}} \right\} = \sum_{n=0}^{\infty} \beta_n \{A(t), \hat{\Omega}^n(t, s)\}, \quad \hat{\Omega}(s, s) = 0. \quad (1.5)$$

Here $\beta_1 = 1/2$, $\beta_n = 0$ for $n = 3, 5, \dots$, $\beta_{2n} = \frac{(-1)^{n-1} B_{2n}}{(2n)!}$, for $n \in \mathbb{N}$ and $\beta_0 = 1$, B_{2n} are Bernoulli numbers [5].

Integration of this Cauchy problem (1.5) leads to the formal series of W. Magnus. This series is a continual generalization of the classical Baker-Campbell-Hausdorff formula. It is well known that the series on the right-hand side of this formula does not always converge. Therefore, in the present paper, we will use the following representation of the solution of the Cauchy problem (1.4)

$$U(t, s) = (I + F(t, s)) e^{\int_s^t A(\tau) d\tau},$$

where $F(t, s)$ is the function defined below, I is identity operator.

Next we will derive a differential operator equation for the function $F(t, s)$. Applying the chain rule for the derivative of a composite function [9], we obtain

$$\frac{dF(t, s)}{dt} = \frac{dU(t, s)}{dt} e^{-\hat{A}(t, s)} - U(t, s) \left(A(t) \frac{\partial}{\partial X} \right) e^X \Big|_{X=-\hat{A}(t, s)},$$

where $\hat{A}(t, s) = \int_s^t A(\tau) d\tau$.

Hence, taking into account the Hausdorff identity, we get

$$\begin{aligned} \frac{dF(t, s)}{dt} &= A(t)U(t, s)e^{-\hat{A}(t, s)} + U(t, s)e^{-\int_s^t A(\tau) d\tau} \left\{ A(t), \frac{e^{-\hat{A}(t, s)} - 1}{\hat{A}(t, s)} \right\} \\ &= A(t)(F(t, s) + I) - (F(t, s) + I)(A(t) - \Psi(t, s)) \\ &= [A(t), F(t, s)] + F(t, s)\Psi(t, s) + \Psi(t, s), \end{aligned}$$

where

$$\Psi(t, s) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+1)!} \{A(t), \hat{A}^k(t, s)\}.$$

Thus, the function $F(t, s)$ is a solution of the Cauchy problem

$$\frac{dF(t, s)}{dt} = [A(t), F(t, s)] + F(t, s)\Psi(t, s) + \Psi(t, s), \quad F(s, s) = 0.$$

For further discussion, it is necessary to recall the notion of the logarithmic operator norm [1]. For a bounded linear operator $A \in L(X)$, the logarithmic operator norm $\Lambda(A)$ is defined as follows

$$\Lambda(A) = \lim_{\varepsilon \rightarrow 0+} \frac{\|I + \varepsilon A\| - 1}{\varepsilon},$$

where ε is real numbers.

2. PROBLEM STATEMENT

Consider the impulsive differential equation

$$\begin{aligned} \frac{dx(t)}{dt} &= A(t, x(t))x(t), \quad t \neq n\theta, \\ \Delta x(t) &= B(x(t))x(t), \quad t = n\theta, \end{aligned} \tag{2.1}$$

where $x \in X$, $A \in C^{0,1}(\mathbb{R} \times X; L(X))$, $\theta > 0$, $\Delta x(t) = x(t+0) - x(t)$, $x(t+0) = \lim_{s \rightarrow t, s > t} x(s)$, $B \in C^1(X; L(X))$ and $A(t, x)$ is a θ -periodic function of t .

Let $x(t; x_0)$ be a solution of the Cauchy problem (2.1) with initial condition $x(0+0, x_0) = x_0$. In this case, as usual, it is assumed that $x(t-0; x_0) = x(t, x_0)$.

The aim of this paper is to obtain sufficient conditions for the global asymptotic stability of the solution $x = 0$ of the impulsive differential equation (2.1).

Definition 1. The solution $x = 0$ of (2.1) is said to be globally asymptotically stable if for any positive number ε there exists a positive number $\delta = \delta(\varepsilon)$ such that the inequality $\|x_0\| < \delta$ implies the estimate $\|x(t; x_0)\| < \varepsilon$ for all $t \geq 0$ and for all $x_0 \in X$

$$\lim_{t \rightarrow \infty} \|x(t, x_0)\| = 0.$$

We will study the global asymptotic stability of the solution $x = 0$ under the following additional assumptions.

Assumption 1. We assume that for the differential equation (2.1) the following conditions are satisfied:

- (1) There exist positive constants a and b such that for all $(t, x) \in \mathbb{R} \times X$ the inequalities

$$\sup_{(t,x) \in [0, \theta] \times X} \|A(t, x)\| \leq a, \quad \sup_{(t,x) \in [0, \theta] \times X} \|A'_x(t, x)\| \leq b$$

are fulfilled, where $A'_x \in L(X, L(X))$ is a Frechet derivative of x [6].

- (2) There exist functions $\alpha \in C([0, \theta]; \mathbb{R})$, $\gamma \in C([0, \theta] \times [0, \theta]; \mathbb{R}_+)$ such that for all $t \in [0, \theta]$ the following inequalities hold

$$\sup_{x \in X} \Lambda(A(t, x)) \leq \alpha(t), \quad \sup_{x \in X} \|A'_x(t, x)(A(\tau, x)x)\| \leq \gamma(t, \tau).$$

- (3) There exists function $\beta \in C(\mathbb{R}; \mathbb{R}_+)$ such that for all $t \in [0, \theta]$ the following estimate is valid

$$\sup_{x \in X} \|B'_x(x)(A(t, x)x)\| \leq \beta(t).$$

3. MAIN RESULT

Consider the nonlinear impulsive differential equation (2.1). Let $T \in (0, \theta]$, $t_k = kh$, $k = 0, 1, \dots, s$, $h = \frac{T}{s}$. We define the function $x_h(t; x_0)$ inductively by the formulas

$$x_h(0; x_0) = x_0, \quad x_h(t; x_0) = e^{(t-kh)A(t_k, x_k)} x_k, \quad t \in (kh, (k+1)h],$$

where $x_k = x_h(kh; x_0)$.

Using the continuity of the operator-valued function $A(t, x)$ and the integral inequality of Gronwall-Bellman, we can prove the validity of the following auxiliary assertions (Lemma 1 and Lemma 2), similar to how it was done in [12].

Lemma 1. Assume that for the impulsive differential equation (2.1) the condition (1) of Assumption 1 holds. Then uniformly over $t \in [0, \theta]$, we get

$$\lim_{h \rightarrow 0+} \|x(t; x_0) - x_h(t; x_0)\| = 0.$$

Along with the impulsive differential equation (2.1), we consider the linear impulsive differential equation

$$\begin{aligned}\frac{dy(t)}{dt} &= A(t, x_0)y(t), \quad t \neq n\theta, \quad y(0+0) = x_0, \\ \Delta y(t) &= B(x_0)y(t), \quad t = n\theta,\end{aligned}\quad (3.1)$$

where $y(t) \in X$, $x_0 \in X$. Similarly, we define a continuous function for $t \in (kh, (k+1)h]$ by the formula

$$y_h(t, x_0) = e^{A(kh, x_0)(t-kh)} e^{A((k-1)h, x_0)h} \dots e^{A(0, x_0)h} x_0.$$

Similarly, we can prove the following statement.

Lemma 2. *Assume that for the impulsive differential equation (2.1) the condition (1) of Assumption 1 holds. Then uniformly over $t \in (0, \theta]$ we have*

$$\lim_{h \rightarrow 0+} \|y(t; x_0) - y_h(t; x_0)\|_X = 0.$$

The following statement establishes an estimate of the error in the "freezing" of the coefficients of the equation (2.1) and is a generalization of Lemma 5.3 from the monograph [12], in which the autonomous case is considered.

Lemma 3 (cf. [2]). *If the conditions of Assumption 1 are satisfied, then the following inequalities hold*

$$\begin{aligned}\|x(\theta; x_0) - y(\theta; x_0)\| &\leq e^{\int_0^\theta \alpha(s) ds} \left(e^{\int_0^\theta \int_0^s \gamma(s, \eta) ds d\eta} - 1 \right) \|x_0\|, \\ \|B(x(\theta; x_0)) - B(x_0)\| &\leq \int_0^\theta \beta(s) ds.\end{aligned}$$

Along with the original impulsive differential equation (2.1), we consider the linear differential equation

$$\begin{aligned}\frac{dz(t)}{dt} &= A_0(x_0)z(t), \quad t \neq n\theta, \\ \Delta z(t) &= (B(x_0) + \hat{B}(x_0) + B(x_0)\hat{B}(x_0))z(t), \quad t = n\theta,\end{aligned}\quad (3.2)$$

where $z \in X$, and the linear operators $A_0(x_0)$ and $\hat{B}_0(x_0)$ are defined as follows

$$A_0(x_0) = \frac{1}{\theta} \int_0^\theta A(\tau, x_0) d\tau, \quad \hat{A}(t, x_0) = \int_0^t A(\tau, x_0) d\tau, \quad \hat{B}(x_0) = F(\theta, x_0),$$

$$\frac{dF(t, x_0)}{dt} = [A(t, x_0), F(t, x_0)] + F(t, x_0)\Psi(t, x_0) + \Psi(t, x_0), \quad F(0, x_0) = 0, \quad (3.3)$$

where

$$\Psi(t, x_0) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+1)!} \{A(t, x_0), \hat{A}^k(t, x_0)\}.$$

Define a monodromy operator of the linear impulsive differential equation (3.2) by

$$U(x_0) = (I + B(x_0))(I + \hat{B}(x_0))e^{\theta A_0(x_0)}. \quad (3.4)$$

The following statement is the main result of this paper.

Theorem 1. Assume that the conditions of Assumption 1 are valid for impulsive differential equation (2.1) and there exist positive constants ζ_i , $i = 1, 2, 3$ such that

$$\sup_{x_0 \in X} \|U(x_0)\| \leq \zeta_1, \quad \sup_{x_0 \in X} \|(I + \hat{B}(x_0))e^{\theta A_0(x_0)}\| \leq \zeta_2, \quad \sup_{x_0 \in X} \|I + B(x_0)\| \leq \zeta_3.$$

Let

$$\zeta_4 = e^{\int_0^\theta \alpha(s) ds} \left(e^{\int_0^\theta \int_0^s \gamma(s, \eta) ds d\eta} - 1 \right), \quad \zeta_5 = \int_0^\theta \beta(s) ds.$$

Then the fulfillment of the inequality

$$\zeta_1 + \zeta_4 \zeta_3 + \zeta_5 \zeta_2 + \zeta_5 \zeta_4 < 1$$

guarantees the global asymptotic stability of the solution $x = 0$ of the differential equation (2.1).

Proof. We obtain

$$\begin{aligned} x(\theta + 0; x_0) - y(\theta + 0; x_0) &= (I + B(x_0))(x(\theta; x_0) - y(\theta; x_0)) \\ &+ (B(x(\theta; x_0)) - B(x_0))y(\theta; x_0) + (B(x(\theta; x_0)) - B(x_0))(x(\theta; x_0) - y(\theta; x_0)). \end{aligned}$$

From the assertion of Lemma 1 and the conditions of Theorem 1, it follows that

$$\begin{aligned} \|x(\theta + 0; x_0) - y(\theta + 0; x_0)\| &\leq \|I + B(x_0)\| \|x(\theta; x_0) - y(\theta; x_0)\| \\ &+ \|(B(x(\theta; x_0)) - B(x_0))y(\theta; x_0)\| + \|B(x(\theta; x_0)) - B(x_0)\| \|x(\theta; x_0) - y(\theta; x_0)\| \\ &\leq \zeta_4 \zeta_3 \|x_0\| + \zeta_5 \|y(\theta; x_0)\| + \zeta_5 \zeta_4 \|x_0\|. \end{aligned}$$

Consider that

$$y(\theta; x_0) = (I + \hat{B}(x_0))e^{\theta A_0(x_0)} x_0,$$

$$y(\theta + 0; x_0) = (I + B(x_0))(I + \hat{B}(x_0))e^{\theta A_0(x_0)} x_0 = U(x_0)x_0.$$

Then, we get

$$\begin{aligned} \|x(\theta + 0; x_0)\| &\leq \|U(x_0)\| \|x_0\| + \zeta_4 \zeta_3 \|x_0\| + \zeta_5 \zeta_2 \|x_0\| + \zeta_5 \zeta_4 \\ &\leq (\zeta_1 + \zeta_4 \zeta_3 + \zeta_5 \zeta_2 + \zeta_5 \zeta_4) \|x_0\| := q \|x_0\|. \end{aligned}$$

Due to the periodicity of the differential equation (2.1), we obtain

$$\|x(n\theta + 0; x_0)\| \leq q^n \|x_0\|$$

from which it follows the global asymptotic stability of the solution $x = 0$ of the impulsive differential equation (2.1). Thus Theorem 1 is proved. $\square$

Thus, the study of the global stability of the solution $x = 0$ for the nonlinear differential equation (2.1) is reduced to estimating the solutions of the linear differential equation with constant operator coefficients (3.2) and solving the auxiliary differential equation (3.3) to obtain the operator $\hat{B}(x_0)$. As well as for solving of the W. Magnus equation, one can use the method of successive approximations to solve this auxiliary equation.

Let us define recursively a sequence

$$\begin{aligned} F_0(t, x_0) &\equiv 0, \\ F_{m+1}(t, x_0) &= \int_0^t \left([A(s, x_0), F_m(s, x_0)] + F_m(s, x_0)\Psi(s, x_0) + \Psi(s, x_0) \right) ds. \end{aligned} \quad (3.5)$$

Then, assuming that

$$\begin{aligned} \sup_{x_0 \in X} \max_{s \in [0, \theta]} (\| \text{ad}_{A(s, x_0)} \| + \|\Psi(s, x_0)\|) &\leq \rho_0, \\ \sup_{x_0 \in X} \max_{s \in [0, \theta]} \|\Psi(s, x_0)\| &\leq \rho_1, \end{aligned}$$

where $\rho_i, i = 1, 2$ are positive constants, we obtain

$$\|F(t, x_0) - F_m(t, x_0)\| \leq \epsilon_m \rho_1 \theta,$$

$$\text{where } \epsilon_m = e^{\theta \rho_0} - \sum_{k=0}^{m-1} \frac{(\theta \rho_0)^k}{k!}.$$

Consider a linear impulsive differential equation of the form

$$\begin{aligned} \frac{dz(t)}{dt} &= A_0(x_0)z(t), \quad t \neq n\theta, \\ \Delta z(t) &= (B(x_0) + \hat{B}_m(x_0) + B(x_0)\hat{B}_m(x_0))z(t), \quad t = n\theta, \end{aligned} \quad (3.6)$$

where $\hat{B}_m(x_0) = F_m(\theta, x_0)$, and let $U_m(x_0)$ be the monodromy operator of this equation

$$U_m(x_0) = (I + B(x_0))(I + \hat{B}_m(x_0))e^{\theta A_0(x_0)}.$$

Theorem 1 implies the following statement.

Corollary 1. Assume that for the impulsive differential equation (2.1) the conditions of Assumption 1 are satisfied and there exist positive constants $\zeta'_i, i = 1, 2$ such

that

$$\sup_{x_0 \in X} \|U_m(x_0)\| \leq \zeta'_1, \quad \sup_{x_0 \in X} \|(I + \hat{B}_m(x_0))e^{\theta A_0(x_0)}\| \leq \zeta'_2$$

and a constant α_0 such that $\sup_{x_0 \in X} \Lambda(A_0(x_0)) \leq \alpha_0$.

If

$$\zeta'_1 + \zeta_4 \zeta_3 + \zeta_5 \zeta'_2 + \zeta_4 \zeta_5 + \zeta_6(\zeta_3 + \zeta_5)e^{\alpha_0 \theta} < 1,$$

where $\zeta_6 = \theta \rho_1 \epsilon_m$, then the solution $x = 0$ of the impulsive differential equation (2.1) is globally asymptotically stable.

4. EXAMPLE

Consider a system of impulsive differential equations

$$\begin{aligned} \frac{dx(t)}{dt} &= \psi(x(t))A(t)x(t), \quad t \neq n\theta, \\ \Delta x(t) &= \psi(x(t))Bx(t), \quad t = n\theta, \end{aligned} \quad (4.1)$$

where $x \in \mathbb{R}^m$, $\psi \in C^1(\mathbb{R}^m, \mathbb{R})$, $A \in C(\mathbb{R}; L(\mathbb{R}^m))$, $A(t + \theta) = A(t)$, $B \in L(\mathbb{R}^m)$.

Assume that a nonlinear function $\psi(x)$ satisfies the following condition. There exist positive constants σ , ψ_m , ψ_M such that

$$0 < \psi_m \leq \psi(x) \leq \psi_M < +\infty, \quad \sup_{x \in \mathbb{R}^n} \|\nabla \psi(x)\| \|x\| \psi(x) \leq \sigma. \quad (4.2)$$

Denote

$$A_0 = \frac{1}{\theta} \int_0^\theta A(\tau) d\tau, \quad \hat{A}(t) = \int_0^t A(\tau) d\tau, \quad \Psi_m(t) = \{A(t), \hat{A}^m(t)\}.$$

The corresponding linear differential equation with "frozen" coefficients has the form

$$\begin{aligned} \frac{dy(t)}{dt} &= \psi(x_0)A_0 y(t), \quad t \neq n\theta, \\ \Delta y(t) &= (\psi(x_0)B + \hat{B}(x_0) + \psi(x_0)B\hat{B}(x_0))y(t), \quad t = n\theta, \end{aligned} \quad (4.3)$$

where $y \in \mathbb{R}^n$, $\hat{B}(x_0) = F(\theta, x_0)$, $F(t, x_0)$ is a solution of the Cauchy problem

$$\frac{dF(t, x_0)}{dt} = \psi(x_0)[A(t), F(t, x_0)] + F(t, x_0)\Psi(t, x_0) + \Psi(t, x_0), \quad F(0, x_0) = 0, \quad (4.4)$$

$$\Psi(t, x_0) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \psi^{k+1}(x_0)}{(k+1)!} \{A(t), \hat{A}^k(t)\}.$$

We represent the solution of the Cauchy problem (4.4) in the integral form

$$F(t, x_0) = \int_0^t \Phi(t, s) \left(\frac{\psi(x_0)}{2} [A^T(s) + A(s), F(s, x_0)] + F(s, x_0) \Psi(s, x_0) + \Psi(s, x_0) \right) \Phi^T(t, s) ds, \quad (4.5)$$

where operator $\Phi(t, s)$ is a solution of the Cauchy problem

$$\frac{d\Phi(t, s)}{dt} = \frac{1}{2}(A(t) - A^T(t))\Phi(t, s), \quad \Phi(s, s) = I.$$

Since $\|\Phi(t, s)\| = 1$, then from (4.4) it follows the estimate

$$\begin{aligned} & \|F(t, x_0)\| \\ & \leq \int_0^t \left(\frac{\psi_M}{2} \|\text{ad}_{A^T(s)+A(s)}\| + \|\Psi(s, x_0)\| \right) \|F(s, x_0)\| ds + \int_0^\theta \|\Psi(s, x_0)\| ds. \end{aligned}$$

Applying the integral Gronwall–Bellman inequality, we obtain the estimate

$$\|F(\theta, x_0)\| \leq \int_0^\theta \|\Psi(s, x_0)\| ds \exp \left(\int_0^\theta \left(\frac{\psi_M}{2} \|\text{ad}_{A^T(s)+A(s)}\| + \|\Psi(s, x_0)\| \right) ds \right).$$

Since

$$\|\Psi(t, x_0)\| \leq \sum_{k=1}^{\infty} \frac{\|\Psi_k(t)\| \psi_M^{k+1}}{(k+1)!},$$

then we obtain

$$\begin{aligned} \|\hat{B}(x_0)\| & \leq \int_0^\theta \sum_{k=1}^{\infty} \frac{\|\Psi_k(s)\| \psi_M^{k+1}}{(k+1)!} ds \\ & \times \exp \left(\int_0^\theta \left(\frac{\psi_M}{2} \|\text{ad}_{A^T(s)+A(s)}\| + \sum_{k=1}^{\infty} \frac{\|\Psi_k(s)\| \psi_M^{k+1}}{(k+1)!} \right) ds \right) := \eta. \end{aligned}$$

Next, we consider the question of estimating the monodromy matrix of the system of equations (4.3). Assume the linear operator A_0 satisfies the Routh–Hurwitz conditions, i.e. $\max_{\lambda \in \sigma(A_0)} \text{Re } \lambda < 0$. Then, the Lyapunov matrix equation

$$A_0^T X + X A_0 = -Q,$$

where Q is a symmetric positive-definite linear matrix, has a unique solution X – a symmetric positive-definite operator.

Consider the Lyapunov function $v(y) = y^T X y$. Then

$$\left. \frac{dv}{dt} \right|_{(4.3)} \leq -\psi(x_0) y^T Q y \leq -\frac{\psi_m \lambda_m(Q)}{\lambda_M(X)} v(y(t)), \quad t \neq n\theta. \quad (4.6)$$

Denote

$$\begin{aligned} \bar{\psi} &= \frac{\psi_m + \psi_M}{2}, \quad R = -(B^T X + X B + \bar{\psi} B^T X B), \\ \eta_0 &= \frac{\psi_M - \psi_m}{2} \|B^T X + X B\| + \frac{(\psi_M - \psi_m)(3\psi_M + \psi_m)}{4} \|B^T X B\|, \\ \eta_1 &= \|X - \bar{\psi} R\| + \eta_0, \quad \vartheta = \frac{\eta_0 + \eta(2 + \eta)\eta_1}{\lambda_m(X)}. \end{aligned}$$

Therefore for $t = n\theta$ we obtain

$$v(y(t + 0)) \leq (1 - \bar{\psi} \Lambda(R, X) + \vartheta) v(y(t)), \quad (4.7)$$

where

$$\Lambda(R, X) = \begin{cases} \frac{\lambda_m(R)}{\lambda_M(X)}, & \lambda_m(R) \geq 0, \\ \frac{\lambda_m(R)}{\lambda_m(X)}, & \lambda_m(R) < 0. \end{cases}$$

From (4.6) and (4.7) it follows that

$$\|U(x_0)\| \leq \sqrt{\frac{(1 - \bar{\psi} \Lambda(R, X) + \vartheta) \lambda_M(X)}{\lambda_m(X)}} e^{-\theta \frac{\psi_m \lambda_m(Q)}{2\lambda_M(X)}} := \zeta_1.$$

Similarly, we obtain

$$\|(I + \hat{B}(x_0)) e^{\psi(x_0)\theta A_0}\| \leq (1 + \eta) \sqrt{\frac{\lambda_M(X)}{\lambda_m(X)}} e^{-\theta \frac{\psi_m \lambda_m(Q)}{2\lambda_M(X)}} := \zeta_2.$$

Taking into account the assumption (4.2), we get

$$\begin{aligned} \zeta_3 &= \|I + \bar{\psi} B\| + \frac{\psi_M - \psi_m}{2} \|B\|, \\ \zeta_4 &= e^{\psi^* \int_0^\theta \Lambda(A(s)) ds} (e^{\sigma \int_0^\theta \int_0^\tau \|A(\tau)\| \|A(s)\| d\tau ds} - 1), \quad \zeta_5 = \sigma \|B\| \int_0^\theta \|A(s)\| ds, \end{aligned} \quad (4.8)$$

where

$$\psi^* = \begin{cases} \psi_M, & \text{for } \int_0^\theta \Lambda(A(s)) ds > 0, \\ \psi_m, & \text{for } \int_0^\theta \Lambda(A(s)) ds \leq 0. \end{cases}$$

Theorem 1 implies the following statement.

Proposition 1. Assume that the condition (4.2) is satisfied for the system (4.1) and $\max_{\lambda \in \sigma(A_0)} \operatorname{Re} \lambda < 0$. If the following inequality is valid

$$\zeta_1 + \zeta_4 \zeta_3 + \zeta_5 \zeta_2 + \zeta_4 \zeta_5 < 1,$$

then the solution $x = 0$ of the system (4.1) is globally asymptotically stable.

Next we will consider the case when, in general, the matrix A_0 does not satisfy the Routh–Hurwitz condition. Assume that for a given positive-definite symmetric matrix G there exists a positive-definite matrix X satisfying the conditions

$$\begin{aligned} \theta(A_0^T X + X A_0) + B^T X + X B + \bar{\psi} B^T X B &= -G, \\ A_0^{T^2} X + 2A_0^T X A_0 + X A_0^2 &\geq 0. \end{aligned} \quad (4.9)$$

Consider the Lyapunov function $v(y) = y^T X y$. Then we get

$$v(x_0) - v(y(\theta; x_0)) = -\theta \frac{dv(y(\theta; x_0))}{dt} \Big|_{(4.1)} + \frac{1}{2} \frac{d^2 v(y(c; x_0))}{dt^2} \Big|_{(4.1)} \theta^2,$$

where $c \in (0, \theta)$.

We have

$$\begin{aligned} \frac{dv(y(\theta; x_0))}{dt} \Big|_{(4.1)} &= -\psi(x_0) y^T(\theta; x_0) Q y(\theta; x_0), \\ \frac{d^2 v(y(c; x_0))}{dt^2} \Big|_{(4.1)} &= \psi^2(x_0) y^T(c; x_0) (A_0^{T^2} X + 2A_0^T X A_0 + X A_0^2) y(c; x_0) \geq 0, \end{aligned}$$

where $Q = -(A_0^T X + X A_0)$.

Consequently, we obtain

$$\begin{aligned} v(y(\theta; x_0)) - v(x_0) &\leq -\theta \bar{\psi} y^T(\theta; x_0) Q y(\theta; x_0) + \theta \frac{\psi_M - \psi_m}{2} \|Q\| \|y(\theta; x_0)\|^2, \\ v(y(\theta + 0; x_0)) - v(y(\theta; x_0)) &\leq \bar{\psi} y^T(\theta; x_0) (B^T X + X B + \bar{\psi} B^T X B) y(\theta; x_0) \\ &\quad + \vartheta v(y(\theta; x_0)). \end{aligned}$$

This implies that

$$\begin{aligned} v(y(\theta + 0; x_0)) - v(x_0) &\leq -\bar{\psi} y^T(\theta; x_0) G y(\theta; x_0) + \vartheta_1 v(y(\theta; x_0)) \\ &\leq \left(-\frac{\bar{\psi} \lambda_m(G)}{\lambda_M(X)} + \vartheta_1 \right) v(y(\theta; x_0)), \end{aligned}$$

where $\vartheta_1 = \vartheta + \frac{(\psi_M - \psi_m) \|Q\| \theta}{2 \lambda_m(X)}$.

Let $\delta = \frac{\bar{\psi} \lambda_m(G)}{\lambda_M(X)} - \vartheta_1 > 0$. Then

$$\frac{dv}{dt} \Big|_{(4.1)} = -\psi(x_0) y^T Q y \geq -\psi(x_0) \lambda_M(Q) \|y\|^2 \geq -\Lambda^*(Q, X) v(y),$$

where

$$\Lambda^*(Q, X) = \begin{cases} \frac{\lambda_M(Q)\psi_M}{\lambda_m(X)}, & \text{for } \lambda_M(Q) > 0, \\ \frac{\lambda_M(Q)\psi_m}{\lambda_M(X)}, & \text{for } \lambda_M(Q) \leq 0. \end{cases}$$

Therefore, we obtain

$$v(y(\theta; x_0)) \geq v(x_0)e^{-\theta\Lambda^*(Q, X)}.$$

Thus, we get

$$v(y(\theta + 0; x_0)) - v(x_0) \leq -\delta v(y(\theta; x_0)) \leq -\delta e^{-\theta\Lambda^*(Q, X)} v(x_0).$$

From this it follows the estimate

$$\|U(x_0)\| \leq \sqrt{\frac{\lambda_M(X)}{\lambda_m(X)}(1 - \delta e^{-\theta\Lambda^*(Q, X)})} := \tilde{\zeta}_1.$$

Similarly, we obtain

$$\|(I + \hat{B}(x_0))e^{\psi(x_0)\theta A_0}\| \leq (1 + \eta) \sqrt{\frac{\lambda_M(X)}{\lambda_m(X)}} e^{-\psi^{**}\theta\Lambda(Q, X)/2} := \tilde{\zeta}_2,$$

where

$$\psi^{**} = \begin{cases} \psi_M, & \text{for } \Lambda(Q, X) < 0, \\ \psi_m, & \text{for } \Lambda(Q, X) \geq 0. \end{cases}$$

The constants $\zeta_3, \zeta_4, \zeta_5$ are determined by the formulas (4.8), and the matrix R by the formula

$$R = -(B^T X + XB + \bar{\psi} B^T XB).$$

An immediate consequence of Theorem 1 is the global stability conditions for the solution $x = 0$ of the system of nonlinear impulsive differential equations (4.1).

Proposition 2. Assume that for system (4.1) the conditions (4.2) are fulfilled and for a given symmetric positive-definite matrix G there exists a symmetric positive-definite matrix X satisfying conditions (4.9). If

$$\frac{\bar{\psi}\lambda_m(G)}{\lambda_M(X)} - \vartheta_1 > 0, \quad \tilde{\zeta}_1 + \zeta_4\zeta_3 + \zeta_5\tilde{\zeta}_2 + \zeta_4\zeta_5 < 1,$$

then the solution $x = 0$ of system (4.1) is globally asymptotically stable.

5. NUMERICAL EXAMPLE

Let us consider the linear impulsive system (4.1) with matrices

$$A(t) = A_0 + A_1 \cos \omega t + A_2 \sin \omega t, \quad B = \begin{pmatrix} -0.5 & 0 \\ 0 & 0.25 \end{pmatrix},$$

where $\omega = \frac{2\pi}{\theta}$ and

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let $\psi_M = 0.205$, $\psi_m = 0.195$, $\sigma = 0.004$, $\theta = 0.365625$. Next we will study the asymptotic stability of considered system based on Proposition 2. Let $X = I$. Then we obtain $\eta \leq 0.01027$, $\eta_0 \leq 0.00551$, $\eta_1 \leq 1.10801$, $\vartheta_1 \leq 0.03203$, $\lambda_m(G) = 0.21875$, $\delta \geq 0.01172$, $\zeta_1 \leq 0.99495$, $\zeta_3 \leq 1.0525$, $\zeta_4 \leq 0.00193$, $\zeta_5 \leq 0.00180$.

Hence, if the inequality $\zeta_1 + \zeta_4\zeta_3 + \zeta_5\zeta_2 + \zeta_4\zeta_5 < 1$, $\delta > 0$ is fulfilled, then the hypotheses of Proposition 2 are satisfied and the linear impulsive system (4.1) is asymptotically stable for the particular case.

We note that the matrix A_0 does not satisfy the Routh-Hurwitz conditions, and also the condition $r_\sigma(I + B) < 1$ does not hold for the matrix B . Therefore, the construction of the Lyapunov function for system (4.1) in this particular case is difficult.

6. DISCUSSION OF RESULTS

First of all, we note that the problem of global asymptotic stability of the zero solution for a nonlinear impulsive differential equation has some specificity in comparison with a similar problem for differential equations without impulsive action and it is more complicated. For example, if we consider the linear system of differential equations (4.1) in the case when there is no impulsive action, i.e. $B = 0$, then the solution $x = 0$ of the system (4.1) will be globally asymptotically stable if $\psi(x) > 0$.

If $B \neq 0$, then from the asymptotic stability of the linear system

$$\begin{aligned} \frac{dx(t)}{dt} &= A(t)x(t), \quad t \neq n\theta, \\ \Delta x(t) &= Bx(t), \quad t = n\theta, \end{aligned}$$

generally speaking, does not follow the global asymptotic stability of the equilibrium position $x = 0$ of the system (4.1). Let us consider a one-dimensional impulsive differential equation

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t), \quad t \neq n\theta, \\ \Delta x(t) &= -\gamma x(t), \quad t = n\theta, \end{aligned} \tag{6.1}$$

where $x \in \mathbb{R}$, $\gamma \in (0, 1)$.

Obviously, the inequality $e^\theta(1 - \gamma) < 1$ guarantees (global) asymptotic stability of the linear impulsive differential equation (6.1). Let us consider a nonlinear impulsive differential equation

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t)(1 + x^2(t)), \quad t \neq n\theta, \\ \Delta x(t) &= -\gamma x(t)(1 + x^2(t)), \quad t = n\theta. \end{aligned} \tag{6.2}$$

The zero solution of this equation is locally asymptotically stable (under the condition $e^{\theta}(1-\gamma) < 1$). Let us prove that the solution $x = 0$ is not globally asymptotically stable.

Let $x(t, x_0)$ be a solution of the Cauchy problem (6.2), $x(0+, x_0) = x_0$, $\omega^+(x_0)$ is a length of the right maximal interval of existence of the solution of the corresponding ordinary differential equation without impulsive action. It is easy to see that $\omega^+(x_0) < \frac{1}{2x_0^2}$. Therefore, for all $x_0 > \frac{1}{\sqrt{\theta}}$, the inequality $\omega^+(x_0) < \frac{\theta}{2}$ is fulfilled. Hence, such a solution $x(t, x_0)$ goes to infinity until the moment of the first impulse action. Therefore, this solution is not infinitely-prolongable, and, as a result, the solution $x = 0$ is not globally asymptotically stable.

The proposed method of investigating the global asymptotic stability reduces the problem of stability of the solution of the linear operator equations and makes it possible to establish the global asymptotic stability under various assumptions about the dynamic properties of the continuous and discrete components of the impulsive system. The idea of applying the methods of commutator calculus in problems of stability theory of the nonlinear impulsive equations in a Banach space is new and opens up new possibilities for the development of the Lyapunov's direct method for the stability study of solutions of nonautonomous differential equations.

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Authors' addresses

V.I. Slyn'ko

S.P. Timoshenko Institute of Mechanics of NAS of Ukraine, Stability of Processes Department, 3 Nesterov st., 03680, MSP, Kiev 57, Ukraine

E-mail address: vitstab@ukr.net

Cemil Tunç

Van Yuzuncu Yil University, Department of Mathematics, Faculty of Sciences, 65080-Kampus, Van, Turkey

E-mail address: cemtunc@yahoo.com



BOUNDARY VALUE PROBLEMS FOR BAGLEY–TORVIK FRACTIONAL DIFFERENTIAL EQUATIONS AT RESONANCE

SVATOSLAV STANĚK

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Abstract. We investigate the nonlocal fractional boundary value problem $u'' = A^c D^\alpha u + f(t, u, {}^c D^\mu u, u')$, $u'(0) = u'(T)$, $\Lambda(u) = 0$, at resonance. Here, $\alpha \in (1, 2)$, $\mu \in (0, 1)$, f and $\Lambda: C[0, T] \rightarrow \mathbb{R}$ are continuous. We introduce a "three-component" operator $\mathcal{S}$ which first component is related to the fractional differential equation and remaining ones to the boundary conditions. Solutions of the problem are given by fixed points of $\mathcal{S}$. The existence of fixed points of $\mathcal{S}$ is proved by the Leray–Schauder degree method.

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1. INTRODUCTION

Let $T > 0$ be given and $J = [0, T]$. Denote by $\mathcal{A}$ the set of (generally nonlinear) functionals $\Lambda: C(J) \rightarrow \mathbb{R}$ which are

- (a) continuous, $\Lambda(0) = 0$,
- (b) increasing, that is, $x, y \in C(J)$, $x(t) < y(t)$ on $J \Rightarrow \Lambda(x) < \Lambda(y)$.

Remark 1. Let $\Lambda \in \mathcal{A}$ be linear. Then it follows from property (b) of Λ that Λ takes bounded sets into bounded sets. Hence Λ is a linear bounded functional.

Example 1. Let $p \in C(J)$ be positive, $n \in \mathbb{N}$, $0 \leq t_0 < t_1 < \dots < t_n \leq T$, and $a_k > 0$, $k = 0, 1, \dots, n$. Then the functionals

$$\Lambda_1(x) = \max\{x(t): t \in J\}, \quad \Lambda_2(x) = \min\{x(t): t \in J\},$$

$$\Lambda_3(x) = \int_0^T p(s)(x(s))^{2n-1} ds, \quad \Lambda_4(x) = \sum_{k=0}^n a_k x(t_k)$$

and their linear combinations with positive coefficients belong to the set $\mathcal{A}$.

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We discuss the fractional boundary value problem

$$u''(t) = A {}^c D^\alpha u(t) + f(t, u(t), {}^c D^\mu u(t), u'(t)), \quad (1.1)$$

$$u'(0) = u'(T), \quad \Lambda(u) = 0, \quad \Lambda \in \mathcal{A}. \quad (1.2)$$

Here, ${}^c D$ denotes the Caputo fractional derivative, $A \in \mathbb{R}$, $\alpha \in (1, 2)$, $\mu \in (0, 1)$, and the function f satisfies the condition:

(H) there exists $\Delta > 0$ such that $f \in C(\mathcal{D} \times [-\Delta, \Delta])$, where

$$\mathcal{D} = J \times [-\Delta T, \Delta T] \times [-\Delta K, \Delta K], \quad K = \frac{T^{1-\mu}}{\Gamma(2-\mu)},$$

and

$$f(t, x, y, -\Delta) \leq 0, \quad f(t, x, y, \Delta) \geq 0 \quad \text{for } (t, x, y) \in \mathcal{D}.$$

Equation 1.1 is the fractional differential equation of the Bagley–Torvik type. Its special case is the equation $u'' = A {}^c D^{3/2} u + au + \varphi(t)$. This equation with ${}^c D^{3/2}$ replaced by the Riemann–Liouville fractional derivative $D^{3/2}$ is called the Bagley–Torvik equation. Torvik and Bagley [22] used this equation in modelling the motion of a rigid plate immersing in a Newtonian fluid. Analytical and numerical solutions of the problem

$$u'' = A D^{3/2} u + au + \varphi(t), \quad u(0) = 0, \quad u'(0) = 0,$$

are given in [13, 16, 18], while for the problem

$$u'' = A {}^c D^\alpha u + au + \varphi(t), \quad u(0) = u_0, \quad u'(0) = u_1,$$

in [5, 6, 8, 11, 23]. The existence results for solutions of the generalized Bagley–Torvik equation (1.1) satisfying the boundary conditions $u'(0) = 0$, $u(T) + au'(T) = 0$ are given in [20]. Here, f is a Carathéodory function.

Definition 1. We say that $u \in C^2(J)$ is a solution of problem (1.1), (1.2) if u satisfies the boundary conditions (1.2) and (1.1) holds for $t \in J$.

We recall that the Riemann–Liouville fractional integral I^γ of order $\gamma > 0$ of a function $x: J \rightarrow \mathbb{R}$ is defined as [10, 13, 16]

$$I^\gamma x(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} x(s) \, ds,$$

and the Caputo fractional derivative ${}^c D^\gamma x$ of order $\gamma > 0$, $\gamma \notin \mathbb{N}$, of a function $x: J \rightarrow \mathbb{R}$ is given by the formula [10, 13]

$${}^c D^\gamma x(t) = \frac{d^n}{dt^n} \int_0^t \frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)} \left(x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} s^k \right) ds,$$

where $n = [\gamma] + 1$, $[\gamma]$ means the integral part of γ and Γ is the Euler gamma function. If $x \in C^n(J)$ and $n - 1 < \gamma < n$, then

$${}^c D^\gamma x(t) = \int_0^t \frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)} x^{(n)}(s) ds = I^{n-\gamma} x^{(n)}(t).$$

In particular, if $x \in C^2(J)$ and $\alpha \in (1, 2)$, $\mu \in (0, 1)$, then

$$\begin{aligned} {}^c D^\alpha x(t) &= \int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)} x''(s) ds, \quad t \in J, \\ {}^c D^\alpha x(t) &= \frac{d}{dt} \int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)} (x'(s) - x'(0)) ds = {}^c D^{\alpha-1} x'(t), \quad t \in J, \\ {}^c D^\mu x(t) &= \frac{d}{dt} \int_0^t \frac{(t-s)^{-\mu}}{\Gamma(1-\mu)} (x(s) - x(0)) ds = I^{1-\mu} x'(t), \quad t \in J. \end{aligned}$$

It is well known [10, 13] that $I^\gamma: C(J) \rightarrow C(J)$ for $\gamma \in (0, 1)$. Therefore, if $x \in C^2(J)$, then ${}^c D^\alpha x, {}^c D^\mu x \in C(J)$ for $\alpha \in (1, 2)$ and $\mu \in (0, 1)$.

We will show that problem (1.1), (1.2) is at resonance. The linear function $x(t) = at + b$ is a solution of the problem $u'' - A {}^c D^\alpha u = 0$, $u'(0) = u'(T)$, for each $a, b \in \mathbb{R}$. Let us consider the set of all functions $at + b$ which are solutions of the equation $\Lambda(at + b) = 0$, where Λ is from (1.2).

If Λ is linear, then $b = -\frac{a\Lambda(t)}{\Lambda(1)}$. Hence $\left\{a \left(t - \frac{\Lambda(t)}{\Lambda(1)}\right) : a \in \mathbb{R}\right\}$ is the set of solutions to problem $u'' - A {}^c D^\alpha u = 0$, (1.2). This set is a one-dimensional linear subspace of $C^2(J)$.

Let Λ be nonlinear. If $a = 0$, then $b = 0$. Let $a \in \mathbb{R} \setminus \{0\}$. By our Lemma 1 (for $\lambda = 1$), there exists $\xi_a \in J$ such that $a\xi_a + b = 0$. Hence $b = -a\xi_a$ and the equality $\Lambda(a(t - \xi_a)) = 0$ is true. ξ_a is determined uniquely. If this is not true, then there exists $\rho_a \in J$, $\rho_a \neq \xi_a$, such that $\Lambda(a(t - \rho_a)) = 0$. Since $a(t - \xi_a) \neq a(t - \rho_a)$ for all $t \in J$, and therefore either $a(t - \xi_a) < a(t - \rho_a)$ or $a(t - \xi_a) > a(t - \rho_a)$ on J , it follows from property (b) of Λ that $\Lambda(a(t - \xi_a)) \neq \Lambda(a(t - \rho_a))$, which is impossible. Consequently, $u = 0$ and $\{a(t - \xi_a) : a \in \mathbb{R} \setminus \{0\}\}$ is the set of solutions to the problem $u'' - A {}^c D^\alpha u = 0$, (1.2). In contrast to previous case, this set is not a one-dimensional linear subspace of $C^2(J)$.

In order to show the solvability of problem (1.1), (1.2), we have to overcome troubles that derivatives are of fractional order, the problem is at resonance and finally that Λ in the boundary conditions (1.2) is generally a nonlinear functional. To this end, an auxiliary "three-component" operator $\mathcal{F}$ is introduced. Its first component is related to equation (1.1) and remaining ones to the boundary conditions (1.2). Solutions of (1.1), (1.2) are given by fixed points of $\mathcal{F}$. The existence of fixed points of $\mathcal{F}$ is proved by means of the Leray-Schauder degree method [7].

In the literature, see [1–4, 12, 14, 19] and references therein, existence results for fractional boundary value problems at resonance are usually proved by using the coincidence degree theory due to Mawhin [15].

Our main result is as follows.

Theorem 1. *Let (H) hold and let $A > 0$. Then problem (1.1), (1.2) has at least one solution.*

The paper is organized as follows. In Section 2 we state the results which are used in the next sections. Section 3 is devoted to auxiliary boundary value problems. To this end operators $\mathcal{Q}$, $\mathcal{S}$, $\mathcal{K}_\lambda$ and $\mathcal{H}_\lambda$ are introduced and their properties are given. In Section 4 Theorem 1 is proved. An example demonstrates our results.

Throughout the paper $\alpha \in (1, 2)$, $\mu \in (0, 1)$, $K = \frac{T^{1-\mu}}{\Gamma(2-\mu)}$ and $\|x\| = \max\{|x(t)| : t \in J\}$ is the norm in $C(J)$.

2. PRELIMINARIES

This section contains the results that we will need in the next sections.

Lemma 1. *Let $\Lambda \in \mathcal{A}$ and let the equality*

$$\Lambda(x) + (\lambda - 1)\Lambda(-x) = 0$$

hold for some $x \in C(J)$ and $\lambda \in [0, 1]$. Then there exists $\xi \in J$ such that $x(\xi) = 0$.

Proof. Assume that the statement is not true. Then either $x > 0$ or $x < 0$ on J . If $x > 0$ on J , then $\Lambda(x) > 0$, $\Lambda(-x) < 0$, and therefore $\Lambda(x) + (\lambda - 1)\Lambda(-x) > 0$, which is impossible. Similarly, $x < 0$ on J leads to a contradiction. $\square$

The following maximal principle follows immediately from [17, Lemma 2.1] and [9, Lemma 2.7] and its proof.

Lemma 2 (Maximum principle). *Let $t_0 \in (0, T]$, $x \in C^1[0, t_0]$, $x(t) \leq x(t_0)$ for $t \in [0, t_0]$, $x(0) < x(t_0)$ and $x'(t_0) = 0$. Let $\gamma \in (0, 1)$. Then*

$${}^c D^\gamma x(t)|_{t=t_0} > 0.$$

Corollary 1. *Let $t_0 \in (0, T]$, $x \in C^1[0, t_0]$, $x(t) \geq x(t_0)$ for $t \in [0, t_0]$, $x(0) > x(t_0)$ and $x'(t_0) = 0$. Let $\gamma \in (0, 1)$. Then*

$${}^c D^\gamma x(t)|_{t=t_0} < 0.$$

Lemma 3 ([21]). *Let $r \in C(J)$ and $\gamma \in (0, 1)$. Then the initial value problem*

$$x'(t) = A {}^c D^\gamma x(t) + r(t), \quad x(0) = a, \quad A, a \in \mathbb{R},$$

has the unique solution

$$x(t) = a + \int_0^t r(s) ds + A \int_0^t \left(\int_0^s (s-\xi)^{-\gamma} E_{1-\gamma, 1-\gamma} (A(s-\xi)^{1-\gamma}) r(\xi) d\xi \right) ds,$$

where

$$E_{1-\gamma, 1-\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma((k+1)(1-\gamma))}, \quad z \in \mathbb{R},$$

is the classical Mittag-Leffler function.

Lemma 4 ([24, Lemma 2.2]). *Let $\rho \in (0, 1)$ and let $E_{\rho, \rho}$ be the Mittag-Leffler function. Then*

$$E_{\rho, \rho}(z) > 0, \quad E'_{\rho, \rho}(z) > 0 \quad \text{for } z \in \mathbb{R}.$$

We also need the following result.

Lemma 5. *Let $h \in C(J)$ and $A, c_1, c_2 \in \mathbb{R}$. Then the initial value problem*

$$u''(t) = A {}^c D^\alpha u(t) + h(t), \quad u(0) = c_2, \quad u'(0) = c_1, \quad (2.1)$$

has the unique solution

$$\begin{aligned} u(t) = & c_1 t + c_2 + \int_0^t (t-s) h(s) \, ds \\ & + A \int_0^t (t-s) \left(\int_0^s (s-\xi)^{1-\alpha} E_{2-\alpha, 2-\alpha} (A(s-\xi)^{2-\alpha}) h(\xi) \, d\xi \right) ds. \end{aligned} \quad (2.2)$$

Proof. Since ${}^c D^\alpha x(t) = {}^c D^{\alpha-1} x'(t)$ for $t \in J$ and $x \in C^2(J)$, the equation of (2.1) can be written as

$$u''(t) = A {}^c D^{\alpha-1} u'(t) + h(t). \quad (2.3)$$

Hence, by Lemma 3 (for $r = h$ and with x and γ replaced by u' and $\alpha - 1$),

$$\begin{aligned} u'(t) = & c_1 + \int_0^t h(s) \, ds \\ & + A \int_0^t \left(\int_0^s (s-\xi)^{1-\alpha} E_{2-\alpha, 2-\alpha} (A(s-\xi)^{2-\alpha}) h(\xi) \, d\xi \right) ds, \end{aligned}$$

where $u'(0) = c_1$. Consequently, $u(t) = c_2 + \int_0^t u'(s) \, ds$ is the unique solution of problem (2.1) and (2.2) follows. $\square$

3. OPERATORS

In this section auxiliary operators are introduced and their properties are proved. The most important of these operators is an operator $\mathcal{S}$ by which the solvability of problem (1.1), (1.2) is proved in Section 4.

Let

$$\chi_1(x) = \begin{cases} \Delta T & \text{for } x > \Delta T, \\ x & \text{for } |x| \leq \Delta T, \\ -\Delta T & \text{for } x < -\Delta T, \end{cases} \quad \chi_2(y) = \begin{cases} \Delta K & \text{for } y > \Delta K, \\ y & \text{for } |y| \leq \Delta K, \\ -\Delta K & \text{for } y < -\Delta K, \end{cases}$$

where Δ and K are from (H). Let

$$\tilde{f}(t, x, y, z) = f(t, \chi_1(x), \chi_2(y), z) \quad \text{for } (t, x, y, z) \in J \times \mathbb{R}^2 \times [-\Delta, \Delta]$$

and

$$f^*(t, x, y, z) = \begin{cases} \tilde{f}(t, x, y, \Delta) + \frac{z - \Delta}{z} & \text{if } z > \Delta, \\ \tilde{f}(t, x, y, z) & \text{if } |z| \leq \Delta, \\ \tilde{f}(t, x, y, -\Delta) - \frac{z + \Delta}{z} & \text{if } z < -\Delta. \end{cases}$$

Under condition (H), $f^* \in C(J \times \mathbb{R}^3)$,

$$\left. \begin{aligned} f^*(t, x, y, -\Delta) &\leq 0, \quad f^*(t, x, y, \Delta) \geq 0 \quad \text{for } (t, x, y) \in J \times \mathbb{R}^2, \\ f^*(t, x, y, z) &< 0 \quad \text{for } (t, x, y, z) \in J \times \mathbb{R}^2 \times (-\infty, -\Delta), \\ f^*(t, x, y, z) &> 0 \quad \text{for } (t, x, y, z) \in J \times \mathbb{R}^2 \times (\Delta, \infty), \end{aligned} \right\} \quad (3.1)$$

and

$$|f^*(t, x, y, z)| \leq E \quad \text{for } (t, x, y, z) \in J \times \mathbb{R}^3, \quad (3.2)$$

where

$$E = 1 + \max \left\{ |f(t, x, y, z)| : (t, x, y) \in \mathcal{D}, z \in [-\Delta, \Delta] \right\}.$$

Consider the fractional differential equation

$$u''(t) = A {}^c D^\alpha u(t) + f^*(t, u(t), {}^c D^\mu u(t), u'(t)) \quad (3.3)$$

associated to equation (1.1). Keeping in mind Lemma 5 define operators $\mathcal{Q}: C^1(J) \rightarrow C(J)$ and $\mathcal{S}: C^1(J) \times \mathbb{R}^2 \rightarrow C^1(J) \times \mathbb{R}^2$ by the formulae

$$\begin{aligned} (\mathcal{Q}x)(t) &= f^*(t, x(t), {}^c D^\mu x(t), x'(t)) \\ &\quad + A \int_0^t (t-s)^{1-\alpha} E_{2-\alpha, 2-\alpha} (A(t-s)^{2-\alpha}) f^*(s, x(s), {}^c D^\mu x(s), x'(s)) ds, \\ \mathcal{S}(x, c_1, c_2) &= \left(c_1 t + c_2 + \int_0^t (t-s)(\mathcal{Q}x)(s) ds, c_1 + \int_0^T (\mathcal{Q}x)(s) ds, c_2 + \Lambda(x) \right), \end{aligned}$$

where Λ is from (1.2).

Lemma 6. *Let (H) hold. If (x, c_1, c_2) is a fixed point of the operator $\mathcal{S}$, then x is a solution of problem (3.3), (1.2) and $x'(0) = c_1$, $x(0) = c_2$.*

Proof. Let (x, c_1, c_2) be a fixed point of the operator $\mathcal{S}$, that is, $\mathcal{S}(x, c_1, c_2) = (x, c_1, c_2)$. Then

$$x(t) = c_1 t + c_2 + \int_0^t (t-s)(\mathcal{Q}x)(s) ds, \quad t \in J, \quad (3.4)$$

$$\int_0^T (\mathcal{Q}x)(s) ds = 0, \quad (3.5)$$

$$\Lambda(x) = 0. \quad (3.6)$$

It follows from (3.4) and Lemma 5 (for $h(t) = f^*(t, x(t), {}^c D^\mu x(t), x'(t))$) that $x(0) = c_2$, $x'(0) = c_1$ and x is a solution of (3.3).

Since (cf. (3.4))

$$x'(t) = c_1 + \int_0^t (\mathcal{Q}x)(s) \, ds, \quad t \in J,$$

we conclude from (3.5) that $x'(T) = c_1$. Hence $x'(0) = x'(T)$. The last equality together with (3.6) give that x satisfies the boundary conditions (1.2). Consequently, x is a solution of problem (3.3), (1.2) and $x'(0) = c_1$, $x(0) = c_2$. $\square$

In order to prove that the operator $\mathcal{S}$ admits a fixed point, for $\lambda \in [0, 1]$, we first introduce an operator $\mathcal{K}_\lambda: C^1(J) \times \mathbb{R}^2 \rightarrow C^1(J) \times \mathbb{R}^2$,

$$\begin{aligned} & \mathcal{K}_\lambda(x, c_1, c_2) \\ &= \left(c_1 t + c_2, c_1 + (1 - \lambda)x'(0) + \lambda \int_0^T (\mathcal{Q}x)(s) \, ds, c_2 + \Lambda(x) + (\lambda - 1)\Lambda(-x) \right). \end{aligned}$$

Let

$$\begin{aligned} \Omega = & \left\{ (x, c_1, c_2) \in C^1(J) \times \mathbb{R}^2 \right. \\ & \left. : \|x\| < \Delta T + 1, \|x'\| < \Delta + 1, |c_1| < \Delta + 1, |c_2| < \Delta T + 1 \right\}, \end{aligned} \quad (3.7)$$

where Δ is from (H).

Lemma 7. *Let (H) hold and let $A > 0$. Then*

$$\deg(\mathcal{I} - \mathcal{K}_1, \Omega, 0) \neq 0, \quad (3.8)$$

where "deg" stands for the Leray-Schauder degree and $\mathcal{I}$ is the identity operator on $C^1(J) \times \mathbb{R}^2$.

Proof. Let $M: [0, 1] \times C^1(J) \times \mathbb{R} \rightarrow C^1(J) \times \mathbb{R}$, $M(\lambda, x, c_1, c_2) = \mathcal{K}_\lambda(x, c_1, c_2)$. Since $f^* \in C(J \times \mathbb{R}^3)$, we conclude from Lemma 4 that $\mathcal{Q}$ is a continuous operator. As Λ is continuous and takes bounded sets into bounded sets, it is easy to prove that M is a completely continuous operator.

Due to

$$\mathcal{K}_0(-x, -c_1, -c_2) = -\mathcal{K}_0(x, c_1, c_2) \quad \text{for } (x, c_1, c_2) \in C^1(J) \times \mathbb{R}^2,$$

$\mathcal{K}_0$ is an odd operator.

Assume that $M(\lambda_0, x, c_1, c_2) = (x, c_1, c_2)$ for some $(x, c_1, c_2) \in C^1(J) \times \mathbb{R}^2$ and $\lambda_0 \in [0, 1]$. Then

$$x(t) = c_1 t + c_2, \quad t \in J, \quad (3.9)$$

$$(1 - \lambda_0)x'(0) + \lambda_0 \int_0^T (\mathcal{Q}x)(s) \, ds = 0, \quad (3.10)$$

$$\Lambda(x) + (\lambda_0 - 1)\Lambda(-x) = 0. \quad (3.11)$$

Lemma 1 together with (3.11) give $x(\xi) = 0$ for some $\xi \in J$. Hence (cf. (3.9)) $c_1\xi + c_2 = 0$, and therefore $x(t) = c_1(t - \xi)$ on J .

We now prove that

$$|c_1| \leq \Delta. \quad (3.12)$$

Let $c_1 > \Delta$. Then $x' = c_1 > \Delta$ on J , and therefore $f^*(t, x(t), {}^cD^\mu x(t), x'(t)) > 0$ for $t \in J$ by (3.1). This fact together with $A > 0$ and Lemma 4 imply $(\mathcal{Q}x)(t) > 0$ on J . Hence $(1 - \lambda_0)c_1 + \lambda_0 \int_0^T (\mathcal{Q}x)(s) ds > 0$, which contradicts (3.10). Therefore $c_1 \leq \Delta$. Similarly, if $c_1 < -\Delta$, then we have $f^*(t, x(t), {}^cD^\mu x(t), x'(t)) < 0$ and $(\mathcal{Q}x)(t) < 0$ for $t \in J$, which again contradicts (3.10). Hence (3.12) is true.

Consequently, $|x(t)| = |c_1(t - \xi)| \leq \Delta T$, $|x'(t)| = |c_1| \leq \Delta$, $|{}^cD^\mu x(t)| = |I^{1-\mu} x'(t)| \leq \Delta K$ on J and $|c_2| = |x(0)| \leq \Delta T$. As a result,

$$M(\lambda, x, c_1, c_2) \neq (x, c_1, c_2) \quad \text{for } (x, c_1, c_2) \in \partial\Omega \text{ and } \lambda \in [0, 1].$$

Hence, by the Borsuk antipodal theorem and the homotopy property, the relations

$$\deg(\mathcal{I} - \mathcal{K}_0, \Omega, 0) \neq 0,$$

$$\deg(\mathcal{I} - \mathcal{K}_0, \Omega, 0) = \deg(\mathcal{I} - \mathcal{K}_1, \Omega, 0)$$

hold. Combining these relations we obtain (3.8). $\square$

Finally, let for $\lambda \in [0, 1]$ an operator $\mathcal{H}_\lambda: C^1(J) \times \mathbb{R}^2 \rightarrow C^1(J) \times \mathbb{R}^2$ be defined as

$$\mathcal{H}_\lambda(x, c_1, c_2) = \left(c_1 t + c_2 + \lambda \int_0^t (t-s)(\mathcal{Q}x)(s) ds, c_1 + \int_0^T (\mathcal{Q}x)(s) ds, c_2 + \Lambda(x) \right).$$

Then, for $(x, c_1, c_2) \in C^1(J) \times \mathbb{R}^2$,

$$\mathcal{H}_0(x, c_1, c_2) = \mathcal{K}_1(x, c_1, c_2), \quad (3.13)$$

$$\mathcal{H}_1(x, c_1, c_2) = \mathcal{S}(x, c_1, c_2). \quad (3.14)$$

Lemma 8. *Let (H) hold. Let $V: [0, 1] \times C^1(J) \times \mathbb{R} \rightarrow C^1(J) \times \mathbb{R}$ and $V(\lambda, x, c_1, c_2) = \mathcal{H}_\lambda(x, c_1, c_2)$. Then V is a completely continuous operator.*

Proof. We first prove that V is continuous. To this end let $\{x_n\} \subset C^1(J)$, $\{c_{n,i}\} \subset \mathbb{R}$, $i = 1, 2$, $\{\lambda_n\} \subset [0, 1]$ be convergent sequences and let $\lim_{n \rightarrow \infty} x_n = x$ in $C^1(J)$, $\lim_{n \rightarrow \infty} c_{n,i} = c_i$, $\lim_{n \rightarrow \infty} \lambda_n = \lambda$ in $\mathbb{R}$, where $x \in C^1(J)$, $c_i, \lambda \in \mathbb{R}$, $i = 1, 2$. Then $\lim_{n \rightarrow \infty} f^*(t, x_n(t), {}^cD^\mu x_n(t), x'_n(t)) = f^*(t, x(t), {}^cD^\mu x(t), x'(t))$ uniformly on J . This together with Lemma 4 imply that $\lim_{n \rightarrow \infty} (\mathcal{Q}x_n)(t) = (\mathcal{Q}x)(t)$ uniformly on J . Hence

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(c_{n,1}t + c_{n,2} + \lambda_n \int_0^t (t-s)(\mathcal{Q}x_n)(s) ds \right) &= c_1t + c_2 + \lambda \int_0^t (t-s)(\mathcal{Q}x)(s) ds, \\ \lim_{n \rightarrow \infty} \left(c_{n,1} + \lambda_n \int_0^T (\mathcal{Q}x_n)(s) ds \right) &= c_1 + \lambda \int_0^T (\mathcal{Q}x)(s) ds \end{aligned}$$

uniformly on J . Besides,

$$\begin{aligned}\lim_{n \rightarrow \infty} \left(c_{n,1} + \int_0^T (\mathcal{Q}x_n)(s) \, ds \right) &= c_1 + \int_0^T (\mathcal{Q}x)(s) \, ds, \\ \lim_{n \rightarrow \infty} (c_{n,2} + \Lambda(x_n)) &= c_2 + \Lambda(x).\end{aligned}$$

Consequently, V is a continuous operator.

Let $\Phi \subset C^1(J) \times \mathbb{R}^2$ be bounded and let $\|x\| \leq L$, $\|x'\| \leq L$, $|c_1| \leq L$, $|c_2| \leq L$ for $(x, c_1, c_2) \in \Phi$, where L is a positive constant. Let $W = E_{2-\alpha, 2-\alpha}(|A|T^{2-\alpha})$. Then, by (3.2) and Lemma 4, the relation

$$\begin{aligned}|(\mathcal{Q}x)(t)| &\leq E + |A|E \int_0^t (t-s)^{1-\alpha} E_{2-\alpha, 2-\alpha}(A(t-s)^{2-\alpha}) \, ds \\ &\leq E + |A|EW \int_0^t (t-s)^{1-\alpha} \, ds \leq E + |A|EW \frac{T^{2-\alpha}}{2-\alpha} = H\end{aligned}$$

holds for $t \in J$ and $(x, c_1, c_2) \in \Phi$. Hence

$$\begin{aligned}\left| c_1 t + c_2 + \lambda \int_0^t (t-s)(\mathcal{Q}x)(s) \, ds \right| &\leq L(T+1) + \frac{HT^2}{2}, \\ \left| c_1 + \lambda \int_0^T (\mathcal{Q}x)(s) \, ds \right| &\leq L + HT, \\ |c_2 + \Lambda(x)| &\leq L + \max\{|\Lambda(-L)|, \Lambda(L)\}\end{aligned}$$

for $t \in J$, $(x, c_1, c_2) \in \Phi$ and $\lambda \in [0, 1]$, and therefore the set $V([0, 1] \times \Phi) = \{V(\lambda, x, c_1, c_2) : \lambda \in [0, 1], (x, c_1, c_2) \in \Phi\}$ is bounded in $C^1(J) \times \mathbb{R}^2$. In view of $\|\mathcal{Q}x\| \leq H$ we see that the set $\left\{ c_1 + \lambda \int_0^t (\mathcal{Q}x)(s) \, ds : (x, c_1, c_2) \in \Phi, \lambda \in [0, 1] \right\}$ is equicontinuous on J .

Hence the Arzelà–Ascoli theorem and the Bolzano–Weierstrass compactness theorem in $\mathbb{R}$ guarantee that the set $V([0, 1] \times \Phi)$ is relatively compact in $C^1(J) \times \mathbb{R}^2$. Consequently, V is completely continuous. $\square$

4. THE PROOF OF THEOREM 1 AND AN EXAMPLE

Proof. Suppose that $(x, c_1, c_2) \in C^1(J) \times \mathbb{R}^2$ is a fixed point of $\mathcal{H}_\lambda$ for some $\lambda \in [0, 1]$, that is, $\mathcal{H}_\lambda(x, c_1, c_2) = (x, c_1, c_2)$. If $\lambda = 0$, then it follows from the proof of Lemma 7 (cf. (3.13)) that $(x, c_1, c_2) \in \Omega$, where Ω is given in (3.7). Let $\lambda \in (0, 1]$. Then

$$x(t) = c_1 t + c_2 + \lambda \int_0^t (t-s)(\mathcal{Q}x)(s) \, ds, \quad t \in J, \quad (4.1)$$

$$\int_0^T (\mathcal{Q}x)(s) \, ds = 0, \quad (4.2)$$

$$\Lambda(x) = 0. \quad (4.3)$$

Hence

$$x'(t) = c_1 + \lambda \int_0^t (\mathcal{Q}x)(s) ds, \quad t \in J, \quad (4.4)$$

so $x'(0) = c_1$, and, by (4.2), $x'(T) = c_1 + \lambda \int_0^T (\mathcal{Q}x)(s) ds = c_1$. Consequently,

$$x'(0) = x'(T). \quad (4.5)$$

Suppose that $c_1 > \Delta$, where Δ is from (H). Then $f^*(0, x(0), 0, c_1) > 0$ by (3.1), and therefore $f^*(t, x(t), {}^cD^\mu x(t), x'(t)) > 0$ on a right neighbourhood of $t = 0$. If there is some $\xi \in (0, T]$ such that $f^*(t, x(t), {}^cD^\mu x(t), x'(t)) > 0$ on $[0, \xi]$ and $f^*(\xi, x(\xi), {}^cD^\mu x(t)|_{t=\xi}, x'(\xi)) = 0$, then $(\mathcal{Q}x)(t) > 0$ on $[0, \xi]$ because $A > 0$, which gives $x'(t) > c_1$ for $t \in (0, \xi]$. Hence $f^*(t, x(t), {}^cD^\mu x(t), x'(t)) > 0$ on $[0, \xi]$, contrary to $f^*(\xi, x(\xi), {}^cD^\mu x(t)|_{t=\xi}, x'(\xi)) = 0$. Consequently,

$$f^*(t, x(t), {}^cD^\mu x(t), x'(t)) > 0, \quad (\mathcal{Q}x)(t) > 0, \quad t \in J.$$

Thus $x'(T) > c_1 = x'(0)$, which contradicts (4.5). Hence $c_1 \leq \Delta$. Similarly, we can prove that $c_1 \geq -\Delta$. To summarize, $|c_1| \leq \Delta$.

Suppose that $\max\{x'(t) : t \in J\} = x'(\xi) > \Delta$. Then $\xi \in (0, T)$ and $x'(\xi) - x'(0) > 0$. By (4.4), $x \in C^2(J)$ and $x'' = \lambda \mathcal{Q}x$. Hence $x''(\xi) = 0$ and by Lemma 5 and (2.3) (for $h(t) = \lambda f^*(t, x(t), {}^cD^\mu x(t), x'(t))$) the equality

$$x''(t) = A {}^cD^{\alpha-1} x'(t) + \lambda f^*(t, x(t), {}^cD^\mu x(t), x'(t)), \quad t \in J,$$

holds. Lemma 2 (for $t_0 = \xi$, $\gamma = \alpha - 1$ and x replaced by x') shows that ${}^cD^{\alpha-1} x'(t)|_{t=\xi} > 0$. Hence

$$x''(\xi) = A {}^cD^{\alpha-1} x'(t)|_{t=\xi} + \lambda f^*(\xi, x(\xi), {}^cD^\mu x(t)|_{t=\xi}, x'(\xi)) > 0,$$

which is impossible. Hence $x'(t) \leq \Delta$ for $t \in J$. Similarly, by Corollary 1, we can prove that $x' \geq -\Delta$ on J . Consequently,

$$|x'(t)| \leq \Delta, \quad t \in J.$$

Next, it follows from (4.3) and Lemma 1 that $x(\tau) = 0$ for some $\tau \in J$. Therefore $|x(t)| = \left| \int_\tau^t x'(s) ds \right| \leq \Delta |t - \tau| \leq \Delta T$, $|{}^cD^\mu x(t)| = |I^{1-\mu} x'(t)| \leq \Delta K$. As $c_1 = x'(0)$ and $c_2 = x(0)$, we have proved

$$\|x\| \leq \Delta T, \quad \|{}^cD^\mu x\| \leq \Delta K, \quad \|x'\| \leq \Delta, \quad |c_1| \leq \Delta T, \quad |c_2| \leq \Delta, \quad (4.6)$$

which implies $V(\lambda, x, c_1, c_2) \neq (x, c_1, c_2)$ for $(x, c_1, c_2) \in \partial\Omega$ and $\lambda \in [0, 1]$, where V is from Lemma 8. Combinig Lemma 8 with the homotopy property we have

$$\deg(\mathcal{I} - \mathcal{H}_0, \Omega, 0) = \deg(\mathcal{I} - \mathcal{H}_1, \Omega, 0).$$

This equality together with (3.8) and (3.13) give

$$\deg(\mathcal{I} - \mathcal{H}_1, \Omega, 0) \neq 0.$$

Hence there exists a fixed point (x, c_1, c_2) of $\mathcal{H}_1$. Lemma 6 and (3.14) guarantee that x is a fixed point of problem (3.3), (1.2) and $c_1 = x'(0)$, $c_2 = x(0)$. Due to (4.6), $f^*(t, x(t), {}^c D^\mu x(t), x'(t)) = f(t, x(t), {}^c D^\mu x(t), x'(t))$ for $t \in J$, and therefore x is a solution of problem (1.1), (1.2). $\square$

Example 2. Let $p \in C(J \times \mathbb{R}^3)$ be bounded, $a, b, c \in C(J)$, $c > 0$ on J , and $n \in \mathbb{N}$, $\beta, \gamma \in (0, 2n - 1)$. Then the function

$$f(t, x, y, z) = p(t, x, y, z) + a(t)|x|^{\beta-1}x + b(t)|y|^\gamma + c(t)z^{2n-1}$$

satisfies condition (H). Really, let $|p(t, x, y, z)| \leq L$ for $(t, x, y, z) \in J \times \mathbb{R}^3$ and $c_* = \min\{c(t): t \in J\}$. Since

$$\lim_{v \rightarrow \infty} \left(L v^{1-2n} + \|a\| T^\beta v^{\beta+1-2n} + \|b\| K^\gamma v^{\gamma+1-2n} \right) = 0, \quad K = \frac{T^{1-\mu}}{\Gamma(2-\mu)},$$

there exists $\Delta > 0$ such that

$$L \Delta^{1-2n} + \|a\| T^\beta \Delta^{\beta+1-2n} + \|b\| K^\gamma \Delta^{\gamma+1-2n} \leq c_*.$$

Hence $L + \|a\|(\Delta T)^\beta + \|b\|(\Delta K)^\gamma \leq c_* \Delta^{2n-1}$, and therefore for $(t, x, y) \in \mathcal{D}$, where $\mathcal{D}$ is from (H), the inequalities

$$f(t, x, y, \Delta) \geq -L - \|a\|(\Delta T)^\beta - \|b\|(\Delta K)^\gamma + c_* \Delta^{2n-1} \geq 0,$$

$$f(t, x, y, -\Delta) \leq L + \|a\|(\Delta T)^\beta + \|b\|(\Delta K)^\gamma - c_* \Delta^{2n-1} \leq 0$$

hold. Theorem 1 gives that the equation

$$\begin{aligned} u'' &= A {}^c D^\alpha u + p(t, u, {}^c D^\mu u, u') \\ &\quad + a(t)|u|^{\beta-1}u + b(t){}^c D^\mu |u|^\gamma + c(t)(u')^{2n-1}, \quad A > 0, \end{aligned} \quad (4.7)$$

has at least one solution u satisfying the boundary conditions (1.2) and $\|u\| \leq \Delta T$, $\|{}^c D^\mu u\| \leq \Delta K$, $\|u'\| \leq \Delta$.

In particular, there exists a solution of (4.7) satisfying the boundary conditions

$$\min\{u(t): t \in J\} = 0, \quad u'(0) = u'(T),$$

that is, u is a nonnegative solution of the problem.

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Author's address

Svatoslav Staněk

Department of Mathematical Analysis, Faculty of Science

Current address: Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic

E-mail address: svatoslav.stanek@upol.cz



PROFESSOR MIKLÓS RONTÓ. ON THE OCASSION OF HIS 75TH BIRTHDAY

JENŐ SZIGETI

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March 31th, 2018, has marked the 75th birthday of Miklós Rontó, Professor, Doctor of Physical and Mathematical Sciences (1986), Doctor of Mathematical Sciences (1994), now Professor Emeritus at the University of Miskolc, Hungary, an outstanding expert in the theory of boundary value problems for ordinary differential equations, who devoted more than 53 years of his life to education and research in mathematics. The purpose of this note is to review some particular moments, dates, facts, and thoughts which could furnish a picture of Miklós as we have known him for many years.

Miklós Rontó was born in the town of Beregszász (Hungary) in 1943. Having graduated from a secondary school in Beregszász in 1959, he received admission to

the Uzhgorod (Ungvár) University and studied there at the Faculty of Physics and Mathematics.

After his university studies, in 1964, he received a Diploma in Mathematics and Computational Science and an official assignment to the Institute of Cybernetics of the Ukrainian Academy of Sciences, which he had already been affiliated to during the last two university years since 1963.

For 30 years had he been working in research institutes of the Ukrainian Academy of Sciences at various positions. In 1963–1971, he was a research staff member at the Institute of Cybernetics of the Ukrainian Academy of Sciences in Kiev (now V. M. Glushkov Institute of Cybernetics of NAS of Ukraine). Since 1971 till 1989, he had worked at the Institute of Electrodynamics and Institute of Problems of Modelling in Energetics (now Pukhov Institute for Modelling in Energy Engineering, named after his founder and director until 1981, a prominent scientist, Academician Georgy E. Pukhov) as a researcher and, later, as a leading researcher.

In 1971, in the Institute of Mathematics of the Ukrainian Academy of Sciences in Kiev, he defended the dissertation entitled “Finding periodic solutions using collocation methods” and received the degree of a Candidate of Physical and Mathematical Sciences. The dissertation and the papers related to it concern the investigation of periodic solutions of non-linear ordinary differential equations. He developed and substantiated a new version of the collocation method, which can be rather easily and efficiently used for finding approximate solutions in the case of systems of ordinary differential equations involving non-linearities of a general form.

In 1986, again in the Kiev Institute of Mathematics, he successfully defended the dissertation “Constructive numerical-analytic methods for the investigation of boundary value problems with parameters” and received the degree of a Doctor of Physical and Mathematical Sciences. In relation to this topic, he succeeded to develop a new direction of research concerning the so-called numerical-analytic methods based on successive approximations, their construction and rigorous substantiation. For wide classes of ordinary differential and delay differential equations, the developed methods are well applicable both to the qualitative analysis (study of existence of solutions) and approximate solution of the problems in question (explicit construction of higher order approximations).

Since 1989 till 1994, Miklós Rontó had been a head of the Laboratory of Differential and Integral Equations at the Institute of Mathematics in Kiev. Meanwhile, since 1986 till 1992, he had been a professor at the Chair of Differential and Integral Equations of the Kiev Shevchenko University.

For many years, Miklós had been combining the theoretical research with the work on purely applied problems, which influenced significantly his scientific career, thus giving him an opportunity to see the problems from completely different viewpoints. More than a decade of his connection with the Institute of Mathematics in Kiev has been particularly fruitful, where Miklós had been carrying out his research in

the world-famous Kiev scientific school on differential equations associated with the names of Krylov, Bogolyubov, Mitropolsky, and Samoilenko.

Miklós's official connection with the University of Miskolc started in 1992, when he became a visiting professor and a year later, in 1993, a full professor at the Chair of Analysis. In 1994, he received in Budapest a diploma of Doctor of Mathematical Sciences.

Since 1997 till 2008, Miklós had been the head of the Department of Analysis of the University of Miskolc, and, for a certain period till 2007, also the director of the Institute of Mathematics.

The results of Miklós's research and educational activities have been a significant contribution to our department. His great devotion and energy brought about a new impulse in the research on differential equations carried out in our university, and as a teacher, he has always been most liked by his students for the excellent quality of lectures, a very pleasant disposition, and the ability to explain and to interest. Under his supervision, 10 students obtained the degree of Candidate of Science and 2 the PhD degree.

Since 2007 till 2011, Miklós Rontó had also been a professor of the Mathematics Department at the Eszterházy Károly High School in Eger (now Eszterházy Károly University).

The database of the Hungarian Academy of Sciences MTMT.hu (Magyar Tudományos Művek Tára,

<https://vm.mtmt.hu/www/index.php?AuthorID=10003243>)

currently lists 240 publications by Miklós Rontó. **MathSciNet** lists above 100 works by M. Rontó, their average Thomson Reuters IF is about 15. In various databases one can find about 1200 citations of M. Rontó's works. The list of journals where his works are published includes, in particular, *Abstract and Applied Analysis*, *Annales Univ. Sci. Budapest. Sec. Math.*, *Acta Electrotechnica et Informatica*, *Applied Mathematics and Computation*, *Archivum Mathematicum* (Brno), *Boundary value Problems*, *Carpathian J. Math.*, *Computers & Mathematics with Applications*, *Doklady Akad. Nauk SSSR*, *Electronic Journal of Qualitative Theory of Differential Equations*, *Georgian Math. J.*, *Mathematica Pannonica*, *Nonlin. Analysis TMA*, *Nonlinear Oscillations*, *Periodica Mathematica Hungarica*, *Publ. Math. Debrecen*, *Ukrainian Math. J.*

Among Miklós Rontó's publications, there are three monographs [5, 7, 8] written in Russian jointly with his teacher, Academician A. M. Samoilenko, and translated to English [4, 6]. He is also an author of a book chapter [1] and two university textbooks in Hungarian [2, 3]. Miklós's papers and books form a significant contribution to the mathematical science and education.

Miklós Rontó is the holder of a number of prestigious honours and awards:

1996: *Ukrainian State Award* in the field of science and technology

1997–1999: *Three-year Széchenyi Professorship* for research and education

- 2000:** Miklós Rontó is included in *Outstanding People of the 20th Century*, 2nd ed. (Cambridge, England), in honour of an outstanding contribution to mathematical research and education
- 2003:** *Doctor Honoris Causa*, Uzhgorod (Ungvár) National University
- 2003–2004:** *Charles Simonyi Professorship*
- 2007:** *Academic Award* of the Hungarian Academy of Sciences
- 2011:** *Pro Universitate* award of the University of Miskolc
- 2013:** *Professor Emeritus* title, University of Miskolc
- 2018:** *Albert Szent-Györgyi Prize*, Hungarian Ministry of Human Resources

Apart of the research and education, Miklós maintains many administrative and organisational activities. Already for more than 20 years, he has been the head of the research seminar of the Institute of Mathematics of our university. Since 2000, he is the founder and editor-in-chief of the journal *Miskolc Mathematical Notes* (Thomson Reuters IF 2016/17: 0.388). He is also a member of the editorial boards of the journals *Nonlinear Oscillations*, *Annales Mathematicae et Informaticae*, *Scientific Transactions of the Uzhgorod National University*, *Mathematical Series*.

Miklós Rontó is the founder and principal organiser of the series of international workshops with the common name “Constructive Methods for Non-linear Boundary Value Problems” held in Miskolc (2000, 2003, 2015, 2018), Sárospatak (2006), Eger (2009), Tokaj (2012).

We know well that, on his way to professional success, Miklós has always felt the warmth of his family, his beloved wife Valentina and son András with his wife Natalia and children Nicolas and Michael. Valentina has been his true friend, aide, and support already for 50 years. Interestingly enough, his wife, son, and daughter-in-law all have a degree in differential equations.

It is now more than 50 years that Miklós Rontó has been working in mathematical research and education. Let me, in the name of our colleagues, thank him for cooperation and friendship and cordially wish him long life, good health, and family happiness, new nice ideas and achievements.

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Author's address

Jenő Szigeti

Institute of Mathematics, University of Miskolc, Hungary

E-mail address: matjeno@uni-miskolc.hu



NEURAL NETWORKS WITH DISTRIBUTED DELAYS AND HÖLDER CONTINUOUS ACTIVATION FUNCTIONS

NASSER-EDDINE TATAR

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Abstract. We consider a system which arises in Neural Network Theory with distributed delays involving Hölder continuous activation functions. We prove some results on global exponential stability of the system. This extends the previous works where activation functions were assumed to be Lipschitz continuous.

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Keywords: neural network, distributed delay, exponential stability, Hölder continuity, non-Lipschitz continuity

1. INTRODUCTION

In this work we investigate the following system

$$x'_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^m b_{ij} \int_{-\infty}^t K_{ij}(t-s) f_j(x_j(s)) ds + c_i(t), \quad i = 1, \dots, m \quad (1.1)$$

for $t > 0$, with $x_j(t) = x_{0j}(t)$, $t \in (-\infty, 0]$ where $x_{0j}(t)$ are given continuous functions and b_{ij} are real constants. The functions $a_i(t)$ (nonnegative), $c_i(t)$, $i = 1, \dots, m$ are continuous functions and f_j are the activation functions. The functions $K_{ij}(t)$ are assumed to be continuous and integrable.

This system arises in (Artificial) Neural Network theory [7, 8] in which there is a growing interest these last two decades. Unlike the conventional machines (based on von Neumann architecture) which use a single processor, the Neural Network consists of a large number of processors (arranged in layers) and in general a larger number of interconnections between them. A processor receives inputs from the preceding layer, process it and then pass it to the subsequent layer. There are various applications in: engineering, science, biology, finance, medicine and geology. Neural Networks are used to understand (complex) phenomena in these fields. They are able

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to classify and recognize patterns. They are also used to solve mathematical programming problems. Their advantages over the traditional computers are the ability to perform huge parallel computations, to classify certain products and materials and to predict certain phenomena.

There are many papers in the literature dealing with the local or global asymptotic stability of the system. In particular, we are witnessing a lot of interest in exponential stability of the (unique) equilibrium of the system for this problem and other similar ones [2, 3, 15–18, 21, 23, 25, 26, 28] to cite but a few. A lot of efforts are devoted in improving the set of conditions on the different coefficients involved in the system as well as the class of activation functions. For the coefficients the main conditions turn around a kind of dominance of the "dissipation" coefficients a_i on the other coefficients. As for the activation functions the first researchers have dealt with specific explicit ones. Then they moved to the assumptions: monotonicity, differentiability and boundedness. These conditions were later weakened to a Lipschitz condition. After that not much has been done for continuous but not Lipschitz continuous activation functions compared to the Lipschitz case or even the discontinuous case. Non-Lipschitz continuous activation functions arise in many fields [10], see also [6] for an application involving Hölder continuous functions. For a slightly weaker condition or partial Lipschitz condition we refer the reader to [3, 4, 23, 25].

Hölder continuous activation functions were studied by Forti *et al.* in [4] for the problem

$$x' = Bx + Tg(x) + I.$$

The authors assumed that each component of g is bounded and is a non-decreasing piecewise continuous function. The matrix $-T$ is assumed Lyapunov Diagonally Stable. An exponential stability result and a finite time convergence result is proved there and extended to a larger class of non-Lipschitz continuous activation functions under a stronger condition. In addition to that the authors investigated discontinuous activation functions using the theory of Filippov. In fact discontinuous activation functions have been treated in several other papers (see [1, 5, 9, 12–14, 19, 20, 22, 24]) first under some boundedness and monotonicity conditions. Later these conditions were dropped under some other conditions.

Here we consider variable coefficients and activation functions f_j that are Hölder continuous *i.e.*

$$|f_j(x) - f_j(y)| \leq L_j |x - y|^{\alpha_j}, \quad 0 < \alpha_j < 1 \quad (1.2)$$

for some positive constants L_j , $j = 1, \dots, m$. Clearly, Hölder continuous functions are not necessarily Lipschitz continuous. We prove global exponential stability of the system. This is achieved with the help of a Gronwall-type inequality due to E. H. Yang [27], some appropriate estimates and some Lyapunov-type functionals.

The local existence is standard whereas the global existence may be derived using (2.2) and the Gronwall-type Lemma 1 below.

In the next section we state and prove our results.

2. EXPONENTIAL STABILITY

In this section it is proved that the system is globally asymptotically stable in an exponential manner when the activation functions (all or some of them) are only Hölder continuous.

Definition 1. We say that the system (1.1) is globally asymptotically stable if for any two solutions $x_i(t)$ and $\bar{x}_i(t)$ we have

$$\lim_{t \rightarrow \infty} \sum_{i=1}^m |x_i(t) - \bar{x}_i(t)| = 0.$$

It is said to be exponentially asymptotically stable if there exist two positive constants M and η such that

$$\sum_{i=1}^m |x_i(t) - \bar{x}_i(t)| \leq M e^{-\eta t}, \quad t > 0.$$

In case the coefficients in problem (1.1) are constant and there exists a unique equilibrium x_i^* , $i = 1, \dots, m$, that is a solution of

$$0 = -a_i x_i^* + \sum_{j=1}^m b_{ij} f_j(x_j^*) \int_0^\infty K_{ij}(s) ds + c_i, \quad i = 1, \dots, m. \quad (2.1)$$

then we obtain exponential stability of this equilibrium.

We denote by $y_i(t) = x_i(t) - \bar{x}_i(t)$, $y(t) = \sum_{i=1}^m |y_i(t)|$, $a(t) := \min_{1 \leq i \leq m} \{a_i(t)\}$,

$$\phi_1(t) := a(t) - \sum_{i,j=1}^m \frac{L_j |b_{ij}|}{p_j} \mu_j(t) \int_0^\infty |K_{ij}(s)| e^{\int_t^{t+s} a(\sigma) d\sigma} ds, \quad t \geq 0$$

and

$$\chi_1(t) := \sum_{i,j=1}^m \frac{L_j |b_{ij}|}{q_j} \int_{-\infty}^t |K_{ij}(t-s)| \mu_j^{-q_j/p_j}(s) ds, \quad t \geq 0$$

for some positive continuous functions $\mu_j(t)$, $p_j = 1/\alpha_j$ and $q_j = 1/(1 - \alpha_j)$, $j = 1, \dots, m$.

Theorem 1. Assume that f_j , $j = 1, \dots, m$ are Hölder continuous and $K_{ij}(t)$, $i, j = 1, \dots, m$ are continuous and integrable functions. Assume further that $a_i(t)$, $b_{ij}(t)$, L_j , $i, j = 1, \dots, m$ and the continuous functions $\mu_i(t) > 0$ are such that $\phi_1(t) \geq 0$ (not identically zero), $\int_0^t \phi_1(\sigma) d\sigma \rightarrow \infty$ as $t \rightarrow \infty$ and

$$\int_0^t \chi_1(s) e^{\int_0^s \phi_1(\sigma) d\sigma} ds$$

is at most of polynomial growth $P(t)$. Then, the system (1.1) globally asymptotically stable at the rate $P(t) e^{-\int_0^t \phi_1(s) ds}$, that is

$$y(t) \leq P(t) e^{-\int_0^t \phi_1(s) ds}, \quad t > 0.$$

Proof. From the system (1.1), the assumptions (1.2), (2.2) and the adopted notation we get

$$D^+ |y_i(t)| \leq -a_i(t) |y_i(t)| + \sum_{j=1}^m L_j |b_{ij}| \int_{-\infty}^t |K_{ij}(t-s)| |y_j(s)|^{\alpha_j} ds, \quad t > 0,$$

for $i = 1, \dots, m$, where D^+ denotes the right Dini derivative. Therefore,

$$D^+ y(t) \leq -\min_{1 \leq i \leq m} \{a_i(t)\} y(t) + \sum_{i,j=1}^m L_j |b_{ij}| \int_{-\infty}^t |K_{ij}(t-s)| |y_j(s)|^{\alpha_j} ds, \quad t > 0$$

Hence, as we denote $a(t) := \min_{1 \leq i \leq m} \{a_i(t)\}$, we get

$$D^+ y(t) \leq -a(t) y(t) + \sum_{i,j=1}^m L_j |b_{ij}| \int_{-\infty}^t |K_{ij}(t-s)| |y_j(s)|^{\alpha_j} ds, \quad t > 0. \quad (2.2)$$

We estimate $|y(t)|^{\alpha_i}$, $0 < \alpha_i < 1$, $i = 1, \dots, m$ using Young's inequality as follows

$$|y_j(t)|^{\alpha_j} \leq \frac{\mu_j(t)}{p_j} |y_j(t)| + \frac{\mu_j^{-q_j/p_j}(t)}{q_j}, \quad \mu_j(t) > 0, \quad \frac{1}{p_j} + \frac{1}{q_j} = 1, \quad j = 1, \dots, m, \quad t \geq 0 \quad (2.3)$$

with $p_j = 1/\alpha_j$ and $q_j = 1/(1 - \alpha_j)$, $j = 1, \dots, m$. Plugging these relations (2.3) in (2.2) we find

$$\begin{aligned} D^+ y(t) &\leq -a(t) y(t) \\ &\quad + \sum_{i,j=1}^m L_j |b_{ij}| \int_{-\infty}^t |K_{ij}(t-s)| \left[\frac{\mu_j(s)}{p_j} |y_j(s)| + \frac{\mu_j^{-q_j/p_j}(s)}{q_j} \right] ds \\ &\leq -a(t) y(t) + \sum_{i,j=1}^m \frac{L_j |b_{ij}|}{p_j} \int_{-\infty}^t |K_{ij}(t-s)| \mu_j(s) |y_j(s)| ds \\ &\quad + \sum_{i,j=1}^m \frac{L_j |b_{ij}|}{q_j} \int_{-\infty}^t |K_{ij}(t-s)| \mu_j^{-q_j/p_j}(s) ds, \quad t > 0. \end{aligned} \quad (2.4)$$

Now we introduce the following functional

$$V_1(t) := y(t) + \Phi_1(t), \quad t \geq 0 \quad (2.5)$$

where

$$\Phi_1(t) := \sum_{i,j=1}^m \frac{L_j |b_{ij}|}{p_j} e^{-\int_0^t a(s) ds} \int_0^\infty |K_{ij}(s)| \int_{t-s}^t e^{\int_0^{\tau+s} a(\sigma) d\sigma} \mu_j(\tau) |y_j(\tau)| d\tau ds.$$

This functional $\Phi_1(t)$ will help us get rid of the delayed terms. Indeed, by differentiation it is easy to see that

$$\begin{aligned}
 \Phi_1'(t) &:= -a(t)\Phi_1(t) + \sum_{i,j=1}^m \frac{L_j |b_{ij}|}{p_j} e^{-\int_0^t a(s)ds} \\
 &\quad \times \int_0^\infty |K_{ij}(s)| \left[e^{\int_0^{t+s} a(\sigma)d\sigma} \mu_j(t) |y_j(t)| - e^{\int_0^t a(\sigma)d\sigma} \mu_j(t-s) |y_j(t-s)| \right] ds \\
 &= -a(t)\Phi_1(t) + \sum_{j=1}^m \left[\sum_{i=1}^m \frac{L_j |b_{ij}|}{p_j} \int_0^\infty |K_{ij}(s)| e^{\int_t^{t+s} a(\sigma)d\sigma} ds \right] \mu_j(t) |y_j(t)| \\
 &\quad - \sum_{i,j=1}^m \frac{L_j |b_{ij}|}{p_j} \int_0^\infty |K_{ij}(s)| \mu_j(t-s) |y_j(t-s)| ds, \quad t > 0
 \end{aligned} \tag{2.6}$$

where the last term is equal to the second term in the right hand side of (2.4) with an opposite sign. So summing up (2.4) and (2.6) we get

$$\begin{aligned}
 D^+ V_1(t) &\leq -a(t)y(t) - a(t)\Phi_1(t) \\
 &\quad + \sum_{i,j=1}^m \frac{L_j |b_{ij}|}{p_j} \int_{-\infty}^t |K_{ij}(t-s)| \mu_j(s) |y_j(s)| ds \\
 &\quad + \sum_{i,j=1}^m \frac{L_j |b_{ij}|}{q_j} \int_{-\infty}^t |K_{ij}(t-s)| \mu_j^{-q_j/p_j}(s) ds \\
 &\quad + \sum_{j=1}^m \left[\sum_{i=1}^m \frac{L_j |b_{ij}|}{p_j} \int_0^\infty |K_{ij}(s)| e^{\int_t^{t+s} a(\sigma)d\sigma} ds \right] \mu_j(t) |y_j(t)| \\
 &\quad - \sum_{i,j=1}^m \frac{L_j |b_{ij}|}{p_j} \int_0^\infty |K_{ij}(s)| \mu_j(t-s) |y_j(t-s)| ds \\
 &\leq - \left[a(t) - \sum_{i,j=1}^m \frac{L_j |b_{ij}|}{p_j} \mu_j(t) \int_0^\infty |K_{ij}(s)| e^{\int_t^{t+s} a(\sigma)d\sigma} ds \right] V_1(t) \\
 &\quad + \sum_{i,j=1}^m \frac{L_j |b_{ij}|}{q_j} \int_{-\infty}^t |K_{ij}(t-s)| \mu_j^{-q_j/p_j}(s) ds, \quad t > 0.
 \end{aligned} \tag{2.7}$$

With our notation we can rewrite (2.7) as

$$D^+ V_1(t) \leq -\phi_1(t)V_1(t) + \chi_1(t), \quad t > 0.$$

We derive that

$$D^+ \left\{ V_1(t) e^{\int_0^t \phi_1(s)ds} \right\} \leq \chi_1(t) e^{\int_0^t \phi_1(s)ds}, \quad t > 0.$$

A standard comparison theorem (see Theorem 1.4.1 in [11]) implies that

$$V_1(t)e^{\int_0^t \phi_1(s)ds} \leq V_1(0) + \int_0^t \chi_1(s)e^{\int_0^s \phi_1(\sigma)d\sigma} ds, \quad t > 0$$

or

$$V_1(t) \leq V_1(0)e^{-\int_0^t \phi_1(s)ds} + e^{-\int_0^t \phi_1(s)ds} \int_0^t \chi_1(s)e^{\int_0^s \phi_1(\sigma)d\sigma} ds, \quad t > 0.$$

The assumption that

$$\int_0^t \chi_1(s)e^{\int_0^s \phi_1(\sigma)d\sigma} ds$$

is at most of polynomial growth implies that $V_1(t)$ that and therefore $y(t)$ is decaying exponentially to zero. $\square$

The following lemma due to E. H. Yang [27] will be needed in our next result.

Lemma 1. *Let $u(t)$ and $f_i(t)$ $i = 1, \dots, n$ be non-negative continuous functions on an interval $J = [0, T)$, $0 < T \leq \infty$ and $a(t)$ a non-decreasing function such that $a(t) \geq 1$ for $t \in J$. If*

$$u(t) \leq a(t) + \sum_{i=1}^n \int_0^t f_i(s)(u(s))^{r_i} ds, \quad t \in J, \quad 0 < r_i < 1$$

then

$$u(t) \leq a(t) \prod_{i=1}^n G_i(t), \quad t \in J$$

where

$$G_i(t) := \left[1 + (1 - r_i) \prod_{k=1}^{i-1} G_k(t) \int_0^t f_i(s) ds \right]^{\frac{1}{2-r_i}}, \quad i = 1, \dots, n$$

and $\prod_{i=1}^0 G_i(t) = 1, t \in J$.

In our present situation we will use

$$G_i(t) := \left[1 + (1 - \alpha_i) \prod_{k=1}^{i-1} G_k(t) \int_0^t \tilde{b}_i(s) ds \right]^{\frac{1}{2-\alpha_i}}, \quad i = 1, \dots, m$$

where

$$\tilde{b}_i(t) := \left[\sum_{j=1}^m L_j |b_{ij}| \int_0^\infty |K_{ij}(s)| e^{\int_0^{s+t} a(\sigma)d\sigma} ds \right] e^{-\alpha_j \int_0^t a(\sigma)d\sigma}.$$

Theorem 2. *Assume that f_j are Hölder continuous i.e. satisfy the relations (1.2), $j = 1, \dots, m$, $K_{ij}(t)$, $i, j = 1, \dots, m$ are continuous and integrable functions,*

$\prod_{i=1}^m G_i(t)$ grows at most as a polynomial and $\int_0^t a(s)ds \rightarrow \infty$ as $t \rightarrow \infty$. Then, we have

$$y(t) \leq [1 + y(0)] \prod_{i=1}^m G_i(t) \exp\left(-\int_0^t a(s)ds\right), \quad t > 0.$$

Proof. We start from the inequality (see (2.2))

$$D^+ y(t) \leq -a(t)y(t) + \sum_{i,j=1}^m L_j |b_{ij}| \int_{-\infty}^t |K_{ij}(t-s)| |y_j(s)|^{\alpha_j} ds, \quad t > 0. \quad (2.8)$$

Consider the new functional

$$V_2(t) := y(t) + \Phi_2(t), \quad t \geq 0 \quad (2.9)$$

where

$$\Phi_2(t) := \sum_{i,j=1}^m L_j |b_{ij}| e^{-\int_0^t a(s)ds} \int_0^\infty |K_{ij}(s)| \int_{t-s}^t e^{\int_0^{s+\tau} a(\sigma)d\sigma} |y_j(\tau)|^{\alpha_j} d\tau. \quad (2.10)$$

A differentiation of $\Phi_2(t)$ in (2.10) gives

$$\begin{aligned} \Phi_2'(t) &= -a(t)\Phi_2(t) + \sum_{i,j=1}^m L_j |b_{ij}| e^{-\int_0^t a(s)ds} \int_0^\infty |K_{ij}(s)| e^{\int_0^{s+t} a(\sigma)d\sigma} ds |y_j(t)|^{\alpha_j} \\ &\quad - \sum_{i,j=1}^m L_j |b_{ij}| \int_0^\infty |K_{ij}(s)| |y_j(t-s)|^{\alpha_j} ds, \quad t \geq 0 \end{aligned} \quad (2.11)$$

and from (2.8)-(2.11) we see that

$$\begin{aligned} D^+ V_2(t) &\leq -a(t)y(t) - a(t)\Phi_2(t) + \sum_{i,j=1}^m L_j |b_{ij}| \int_{-\infty}^t |K_{ij}(t-s)| |y_j(s)|^{\alpha_j} ds \\ &\quad + \sum_{i,j=1}^m L_j |b_{ij}| e^{-\int_0^t a(s)ds} \int_0^\infty |K_{ij}(s)| e^{\int_0^{s+t} a(\sigma)d\sigma} ds |y_j(t)|^{\alpha_j} \\ &\quad - \sum_{i,j=1}^m L_j |b_{ij}| \int_0^\infty |K_{ij}(s)| |y_j(t-s)|^{\alpha_j} ds \\ &\leq -a(t)V_2(t) \\ &\quad + \sum_{j=1}^m \left[\sum_{i=1}^m L_j |b_{ij}| e^{-\int_0^t a(s)ds} \int_0^\infty |K_{ij}(s)| e^{\int_0^{s+t} a(\sigma)d\sigma} ds \right] V_2^{\alpha_j}(t), \quad t > 0. \end{aligned}$$

Thus

$$D^+ \left[V_2(t) \exp \int_0^t a(s)ds \right]$$

$$\begin{aligned}
&\leq \sum_{j=1}^m \left[\sum_{i=1}^m L_j |b_{ij}| \int_0^\infty |K_{ij}(s)| e^{\int_0^{s+t} a(\sigma) d\sigma} ds \right] V_2^{\alpha_j}(t) \\
&\leq \sum_{j=1}^m \left[\sum_{i=1}^m L_j |b_{ij}| \int_0^\infty |K_{ij}(s)| e^{\int_0^{s+t} a(\sigma) d\sigma} ds \right] \left[\exp \left(-\alpha_j \int_0^t a(s) ds \right) \right] \\
&\quad \times \left[V_2(t) \exp \int_0^t a(s) ds \right]^{\alpha_j}, \quad t > 0
\end{aligned}$$

and by comparison and integration

$$\begin{aligned}
V_2(t) \exp \int_0^t a(s) ds &\leq V_2(0) + \int_0^t \left\{ \sum_{j=1}^m \left[\sum_{i=1}^m L_j |b_{ij}| \int_0^\infty |K_{ij}(\tau)| e^{\int_0^{\tau+s} a(\sigma) d\sigma} d\tau \right] \right. \\
&\quad \left. e^{-\alpha_j \int_0^s a(\sigma) d\sigma} \left[V_2(s) \exp \int_0^s a(\sigma) d\sigma \right]^{\alpha_j} \right\} ds, \quad t > 0.
\end{aligned}$$

In short

$$\tilde{V}_2(t) \leq V_2(0) + \sum_{j=1}^m \int_0^t \tilde{b}_j(s) [\tilde{V}_2(s)]^{\alpha_j} ds, \quad t > 0$$

where $\tilde{V}_2(t) := V_2(t) \exp \int_0^t a(s) ds$ and

$$\tilde{b}_i(t) := \left[\sum_{i=1}^m L_j |b_{ij}| \int_0^\infty |K_{ij}(\tau)| e^{\int_0^{\tau+t} a(\sigma) d\sigma} d\tau \right] e^{-\alpha_j \int_0^t a(\sigma) d\sigma}, \quad i = 1, \dots, m.$$

Therefore we can apply Lemma 1 to obtain

$$\tilde{V}_2(t) \leq [1 + V_2(0)] \prod_{j=1}^m G_j(t)$$

where

$$G_j(t) := \left[1 + (1 - \alpha_j) \prod_{k=1}^{j-1} G_k(t) \int_0^t \tilde{b}_j(s) ds \right]^{\frac{1}{2-\alpha_j}}, \quad j = 1, \dots, m$$

and $\prod_{i=1}^0 G_i(t) = 1, t \geq 0$. The proof is complete. $\square$

Remark 1. The coefficients b_{ij} being assumed constants is only a technical assumption to avoid duplication of an undesirable term in the derivative of $\Phi_2(t)$.

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Author's address

Nasser-eddine Tatar

King Fahd University of Petroleum and Minerals, Department of Mathematics and Statistics, Dhahran, 31261 City, Saudi Arabia

E-mail address: tatarn@kfupm.edu.sa



ON THE ORDER OF APPEARANCE OF THE DIFFERENCE OF TWO LUCAS NUMBERS

PAVEL TROJOVSKÝ

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Abstract. Let F_n be the n th Fibonacci number and let L_n be the n th Lucas number. The order of appearance $z(n)$ of a natural number n is defined as the smallest natural number k such that n divides F_k . For instance, $z(L_n) = 2n$, for all $n > 2$. In this paper, among other things, we prove that

$$z(L_m - L_n) = \frac{5F_p}{p} \cdot \frac{m^2 - n^2}{4},$$

for all distinct positive integers $m \equiv n \pmod{4}$, with $\gcd(m, n) = p > 2$ prime.

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1. INTRODUCTION

Let $(F_n)_{n \geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. These numbers are well-known for possessing amazing properties (consult [4] together with its very extensive annotated bibliography for additional references and history). We cannot go very far in the lore of Fibonacci numbers without encountering its companion Lucas sequence $(L_n)_{n \geq 0}$ which follows the same recursive pattern as the Fibonacci numbers, but with initial values $L_0 = 2$ and $L_1 = 1$.

The study of the divisibility properties of Fibonacci numbers has always been a popular area of research. Let n be a positive integer number, the *order (or rank) of appearance* of n in the Fibonacci sequence, denoted by $z(n)$, is defined as the smallest positive integer k , such that $n \mid F_k$ (some authors also call it *order of apparition*, or *Fibonacci entry point*). There are several results about $z(n)$ in the literature. For instance, in 1975, J. Sallé [13] proved that $z(n) \leq 2n$, for all positive integers n . This is the sharpest upper bound for $z(n)$, since for example, $z(6) = 12$ and $z(30) = 60$ (see [10] for related results). In the case of a prime number p , one has the better upper bound $z(p) \leq p + 1$, which is a consequence of the known congruence $F_{p - (\frac{p}{5})} \equiv 0$

$(\text{mod } p)$, for $p \neq 2$, where $(\frac{a}{q})$ denotes the Legendre symbol of a with respect to a prime $q > 2$.

We remark that there is no a closed formula for $z(n)$, but by using computational methods and several calculations, one can see patterns of $z(n)$ for some positive integers n . For example, with this computational approach, Marques [7–9] found explicit formulas for the order of appearance of integers related to Fibonacci and Lucas number, such as $C_{mk}/C_k, C_m \pm 1, C_n C_{n+1} C_{n+2}$ and C_n^k , where $(C_n)_n = (F_n)_n$ or $(L_n)_n$.

In this paper, we study the order of appearance of $L_m - L_n$. Again, we used Mathematica to search for patterns for $z(L_m - L_n)$. We were surprised to find out that in some cases, these values are related to the sequences

$$1, 2, 5, 13, 89, 233, 1597, 4181, 28657, 514229, \dots$$

and

$$3, 4, 11, 29, 199, 521, 3571, 9349, 64079, 1149851, \dots$$

A search in the On-Line Encyclopedia of Integer Sequences [14] is enough to conclude that, surprisingly, these sequences are the first few Fibonacci and Lucas numbers with prime indexes, respectively. In fact, this interpretation is confirmed by exhaustive calculations. More precisely, our main results are the following.

Theorem 1. *Let m and $n > 1$ be positive distinct integers, such that $m \equiv n \pmod{4}$. We have*

(i) *If $\gcd(m, n) = 1$, then*

$$z(L_m - L_n) = \frac{5(m^2 - n^2)}{4}.$$

(ii) *If $\gcd(m, n) = p$ is prime, then*

$$z(L_m - L_n) = \frac{5F_p}{p} \cdot \frac{m^2 - n^2}{4}.$$

2. AUXILIARY RESULTS

Before proceeding further, we recall some facts on Fibonacci and Lucas numbers for the convenience of the reader.

Lemma 1. *We have*

- (a) $F_n \mid F_m$ if and only if $n \mid m$.
- (b) $L_n \mid F_m$ if and only if $n \mid m$ and m/n is even.
- (c) $5F_n \mid F_{5n}$, for all integer n .
- (d) If $d = \gcd(m, n)$, then $\gcd(F_m, F_n) = F_d$,

$$\gcd(L_m, L_n) = \begin{cases} L_d, & \text{if } v_2(m) = v_2(n); \\ 1 \text{ or } 2, & \text{otherwise} \end{cases}$$

$$\text{and } \gcd(F_m, L_n) = \begin{cases} L_d, & \text{if } m/d \text{ is even and } n/d \text{ is odd;} \\ 1 \text{ or } 2, & \text{otherwise.} \end{cases}$$

- (e) $F_{p-(\frac{5}{p})} \equiv 0 \pmod{p}$, for all primes p . In particular, $\gcd(F_p, p) = 1$, if $p \neq 5$.
- (f) $L_p \equiv 1 \pmod{p}$, for all primes p . In particular, $\gcd(L_p, p) = 1$.

Here, as usual, $(\frac{a}{q})$ denotes the Legendre symbol of a with respect to a prime $q > 2$.

Proof of these assertions can be found in [4]. We refer the reader to [1, 3, 6, 11] for more details and additional bibliography.

The second lemma is a consequence of the previous one

Lemma 2 (Cf. Lemma 2.2 of [7]). *We have*

- (a) *If $F_n \mid m$, then $n \mid z(m)$.*
 (b) *If $L_n \mid m$, then $2n \mid z(m)$.*
 (c) *If $n \mid F_m$, then $z(n) \mid m$.*

Lemma 3. *For all primes $p \neq 5$, we have that $\gcd(z(p), p) = 1$.*

Proof. By combining Lemma 1 (e) together with Lemma 2 (c), we conclude that $z(p) \mid p - (\frac{5}{p})$. Thus, when $p \neq 5$, one has that $(\frac{5}{p}) = \pm 1$ and so $z(p)$ divides $p - 1$ or $p + 1$. This yields that $z(p) = p + 1$ or $z(p) \leq p - 1$ and in any case $\gcd(z(p), p) = 1$. $\square$

The next lemma will be very useful in the proof of our theorems. First, we recall the identities

$$\begin{aligned} F_a L_b &= F_{a+b} + (-1)^b F_{a-b}, \\ 5F_a F_b &= L_{a+b} - (-1)^b L_{a-b}, \\ L_a L_b &= L_{a+b} + (-1)^b L_{a-b} \end{aligned} \tag{2.1}$$

for all integers a and b . These identities can be easily deduced from Binet's formulas:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n, \text{ for } n \geq 0,$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$.

Lemma 4. *Let m and n be positive integers, such that $m \equiv n \pmod{4}$. Then*

- (a) $F_{(m-n)/2} L_{(m+n)/2} = F_m - F_n$.
 (b) $F_{(m+n)/2} L_{(m-n)/2} = F_m + F_n$.
 (c) $L_{(m-n)/2} L_{(m+n)/2} = L_m + L_n$.
 (d) $5F_{(m+n)/2} F_{(m-n)/2} = L_m - L_n$.

Proof. It suffices to use suitable choices of a and b in the identities (2.1). For example, if $a = (m-n)/2$ and $b = (m+n)/2$, then $a+b = m$ and $a-b = -n$. By (2.1), we have

$$F_{(m-n)/2} L_{(m+n)/2} = F_m + (-1)^{\frac{m+n}{2}+n+1} F_n,$$

where we used that $F_{-n} = (-1)^{n+1} F_n$. The proof of (a) is complete, since $(m+n)/2 + n + 1$ is odd. In fact, this follows from

$$m \equiv n \pmod{4} \Rightarrow \frac{m+n}{2} \equiv n \pmod{2} \Rightarrow \frac{m+n}{2} + n + 1 \equiv 2n + 1 \pmod{2}.$$

The other items can be proved in the same way. $\square$

The p -adic order (or valuation) of r , $v_p(r)$, is the exponent of the highest power of a prime p which divides r . Throughout the paper, we shall use the known facts that $v_p(ab) = v_p(a) + v_p(b)$ and that $a \mid b$ if and only if $v_p(a) \leq v_p(b)$, for all primes p .

We remark that the p -adic order of Fibonacci and Lucas numbers was completely characterized, see [2, 5, 12]. For instance, from the main results of Lengyel [5], we extract the following two results.

Lemma 5. *For $n \geq 1$, we have*

$$v_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod{3}; \\ 1, & \text{if } n \equiv 3 \pmod{6}; \\ 3, & \text{if } n \equiv 6 \pmod{12}; \\ v_2(n) + 2, & \text{if } n \equiv 0 \pmod{12}, \end{cases}$$

$v_5(F_n) = v_5(n)$, and if p is prime $\neq 2$ or 5 , then

$$v_p(F_n) = \begin{cases} v_p(n) + v_p(F_{z(p)}), & \text{if } n \equiv 0 \pmod{z(p)}; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 6. *Let $k(p)$ be the period modulo p of the Fibonacci sequence. For all primes $p \neq 5$, we have*

$$v_2(L_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod{3}; \\ 2, & \text{if } n \equiv 3 \pmod{6}; \\ 1, & \text{if } n \equiv 0 \pmod{6} \end{cases}$$

$$v_p(L_n) = \begin{cases} v_p(n) + v_p(F_{z(p)}), & \text{if } k(p) \neq 4z(p) \text{ and } n \equiv \frac{z(p)}{2} \pmod{z(p)}; \\ 0, & \text{otherwise.} \end{cases}$$

With all of the above tools in hand, we now move to the proof of the theorem.

3. THE PROOF OF THEOREM 1

3.1. Item (i)

By Lemma 4 (d), we have

$$L_m - L_n = 5F_{\frac{m+n}{2}} F_{\frac{m-n}{2}}.$$

Since $F_{(m \pm n)/2}$ divides $L_m - L_n$, then $(m \pm n)/2 \mid z(L_m - L_n)$. Note that $(m - n)/2$ and $(m + n)/2$ are coprime which yields

$$\frac{m^2 - n^2}{4} \mid z(L_m - L_n). \quad (3.1)$$

Also, $F_{(m \pm n)/2} \mid F_{(m^2 - n^2)/4}$ and since

$$\gcd(F_{(m+n)/2}, F_{(m-n)/2}) = F_{\gcd((m+n)/2, (m-n)/2)} = F_1 = 1$$

we conclude that $F_{(m+n)/2} F_{(m-n)/2} \mid F_{(m^2 - n^2)/4}$. Thus

$$L_m - L_n = 5F_{\frac{m+n}{2}} F_{\frac{m-n}{2}} \mid 5F_{\frac{m^2 - n^2}{4}} \mid F_{5\left(\frac{m^2 - n^2}{4}\right)},$$

where we used Lemma 1 (c). Therefore

$$z(L_m - L_n) \mid 5 \cdot \left(\frac{m^2 - n^2}{4} \right). \quad (3.2)$$

The combination between (3.1) and (3.2) yields

$$z(L_m - L_n) \in \left\{ \frac{m^2 - n^2}{4}, 5 \cdot \left(\frac{m^2 - n^2}{4} \right) \right\}.$$

Now, it suffices to prove that $5F_{(m+n)/2} F_{(m-n)/2} \nmid F_{(m^2 - n^2)/4}$. In fact, this follows because

$$\begin{aligned} \nu_5(5F_{(m+n)/2} F_{(m-n)/2}) &= 1 + \nu_5((m^2 - n^2)/4) \\ &> \nu_5((m^2 - n^2)/4) = \nu_5(F_{(m^2 - n^2)/4}). \end{aligned}$$

This completes the proof. $\square$

3.2. Item (ii)

Proceeding as before, we can easily deduce that

$$\frac{m^2 - n^2}{4p} \mid z(L_m - L_n). \quad (3.3)$$

Here we used that $\gcd((m - n)/2, (m + n)/2) = p$ for all prime $p = \gcd(m, n)$.

Now, we claim that $L_m - L_n$ divides $F_{5F_p(m^2 - n^2)/4p}$, or equivalently, that

$$\nu_q(5F_{(m+n)/2} F_{(m-n)/2}) \leq \nu_q(F_{5F_p(m^2 - n^2)/4p}),$$

holds for all primes q . Let us split the proof into four cases:

Case 1. $q = 5$. In this case, it follows directly that

$$\nu_5(5F_{(m+n)/2} F_{(m-n)/2}) = 1 + \nu_5((m^2 - n^2)/4)$$

while

$$\begin{aligned} \nu_5(F_{5F_p(m^2 - n^2)/4p}) &= 1 + \nu_5((m^2 - n^2)/4) + \nu_5(F_p) - \nu_5(p) \\ &= 1 + \nu_5((m^2 - n^2)/4), \end{aligned}$$

where we used that $v_5(F_p) = v_5(p)$.

Case 2. $q = p \neq 2, 5$. We have

$$v_p(5F_{(m+n)/2}F_{(m-n)/2}) = v_p((m + \delta n)/2) + v_p(F_{z(p)}),$$

where $\delta \in \{-1, 1\}$ and we used that p cannot divide both $F_{(m+n)/2}$ and $F_{(m-n)/2}$, otherwise $z(p)$ would divide p which is an absurdity by Lemma 3. On the other hand, we have

$$v_p(F_{5F_p(m^2-n^2)/4p}) = v_p((m + \delta n)/2) + v_p((m - \delta n)/2) - 1 + v_p(F_{z(p)}).$$

The result follows because $v_p((m - \delta n)/2) \geq 1$, since $p = \gcd(m, n) > 2$.

Case 3. $q \neq 2, 5, p$. In this case, we have

$$v_q(5F_{(m+n)/2}F_{(m-n)/2}) = v_q((m^2 - n^2)/4) + \epsilon v_q(F_{z(q)}),$$

where ϵ is 1 or 2 according to the value of $v_q((m^2 - n^2)/4)$ is 1 or not, respectively. For the other valuation, we obtain

$$v_q(F_{5F_p(m^2-n^2)/4p}) = v_q(F_p) + v_q((m^2 - n^2)/4) + v_q(F_{z(q)}).$$

If $\epsilon = 1$, the conclusion is immediate. For $\epsilon = 2$, then $z(q) \mid p$ and so $z(q) = p$. Therefore $v_q(F_p) = v_q(F_{z(q)})$ and the result follows.

Case 4. $q = 2$. Since $v_2(5F_{(m+n)/2}F_{(m-n)/2}) = v_2(F_{(m+n)/2}) + v_2(F_{(m-n)/2})$, we need to consider two subcases:

Subcase 4.1 $p \neq 3$. In this case, 3 divides only one among $(m - n)/2$ and $(m + n)/2$. So, we have $v_2(5F_{(m+n)/2}F_{(m-n)/2}) = 1$, when $3 \mid (m + n)/2$ (because $(m + n)/2 \equiv 3 \pmod{6}$) and when $3 \mid (m - n)/2$ one obtains $v_2(5F_{(m+n)/2}F_{(m-n)/2}) = 3$ or $v_2(m - n) + 1$ depending on $(m - n)/4$ is odd or even, respectively. However, it is easy to infer that

$$v_2(F_{5F_p(m^2-n^2)/4p}) = \begin{cases} 1, & \text{if } 3 \mid (m + n)/2; \\ 3, & \text{if } 24 \nmid (m - n)/2; \\ v_2(m - n) + 1, & \text{if } 24 \mid (m - n)/2, \end{cases}$$

yielding the desired inequality.

Subcase 4.2 $p = 3$. In this case, we get

$$v_2(5F_{(m+n)/2}F_{(m-n)/2}) = \begin{cases} 4, & \text{if } (m - n)/4 \text{ is odd;} \\ v_2(m - n) + 2, & \text{if } (m - n)/4 \text{ is even.} \end{cases}$$

Now, note that $3 \mid m \pm n$ and $8 \mid (m^2 - n^2)$ and then $12 \mid 5(m^2 - n^2)/3$. Hence, by Lemma 5,

$$v_2(F_{5(m^2-n^2)/3}) = v_2((m^2 - n^2)/3) + 2 = v_2(m - n) + 3 > v_2(m - n) + 2 \geq 4.$$

The result follows.

Summarizing, we proved that $L_m - L_n$ divides $F_{5F_p(m^2-n^2)/4p}$ and so, by Lemma 2 (c), we obtain

$$z(L_m - L_n) \mid 5F_p(m^2 - n^2)/4p. \quad (3.4)$$

By (3.3) and (3.4), one concludes that

$$z(L_m - L_n) = t(m^2 - n^2)/4p,$$

where t divides $5F_p$, that is, $5F_p = st$, for some positive integer s . Thus, in order to complete the proof of theorem, it suffices to prove that $s = 1$. Suppose, to derive a contradiction, that $s > 1$ and let q be a prime factor of s . Note that $q \neq p$ unless $p = 5$.

- $p = 5$ (and then $q = 5$). In this case, $25 = ts$ and so $t = 1, 5$ or 25 . Note that

$$v_5(5F_{(m+n)/2}F_{(m-n)/2}) = 1 + v_5(m^2 - n^2) > v_5(m^2 - n^2) = v_5(F_{(m^2-n^2)/4}).$$

Thus $5F_{(m+n)/2}F_{(m-n)/2} \nmid F_{(m^2-n^2)/4}$ and the result follows.

- $p \neq 5$ (and so $q \neq 5$). Since $5F_p = ts$, then $q \mid F_p$ yielding $z(q) = p$. Therefore

$$v_q(5F_{(m+n)/2}F_{(m-n)/2}) = v_q((m + \delta n)/2) + 2v_q(F_{z(q)}),$$

where we used that q divides both $F_{(m+n)/2}$ and $F_{(m-n)/2}$, here $\delta \in \{0, 1\}$. On the other hand,

$$\begin{aligned} v_q(5F_{(m+n)/2}F_{(m-n)/2}) &= v_q(F_p) + v_q((m + \delta n)/2) - v_q(s) + v_q(F_{z(q)}) \\ &= v_q((m + \delta n)/2) - v_q(s) + 2v_q(F_{z(q)}) \\ &< v_q(5F_{(m+n)/2}F_{(m-n)/2}), \end{aligned}$$

since $q \mid s$ and we used that $v_q(F_p) = v_q(F_{z(q)})$. The proof of theorem is complete. $\square$

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Author's address

Pavel Trojovský

Faculty of Science University of Hradec Králové, Department of Mathematics, Rokitsanského 62, Hradec Králové 50003, Czech Republic

E-mail address: pavel.trojovsky@uhk.cz



SIMPSON TYPE QUANTUM INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS

M. TUNÇ, E. GÖV, AND S. BALGEÇTI

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Abstract. In this paper we establish some new Simpson type quantum integral inequalities for convex functions. Moreover, we obtain some inequalities for special means.

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1. INTRODUCTION

A function $f : J \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex on J if the inequality

$$f(tu + (1-t)v) \leq tf(u) + (1-t)f(v) \quad (1.1)$$

holds for all $u, v \in J$ and $t \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

Convex functions play an important role in mathematical inequalities. The most famous inequality have been used with convex functions is Hermite-Hadamard, which is stated as follows:

Let $f : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $u, v \in J$ with $u < v$. Then the following double inequalities hold:

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v f(x) dx \leq \frac{f(u) + f(v)}{2}. \quad (1.2)$$

In recent years quantum calculus has been actively studied. There are numerous applications in many mathematical areas like special functions, integral transforms, quantum mechanics, information technology and mathematical inequalities. At present q analogous of many inequalities have been established. In the view of these developments q-convexity and convexity of q analogous of the inequalities has also been considered, see [3–5, 7, 9–11].

The inequality given below is well known in the literature as Simpson's inequality:

$$\frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b-a)^4 \quad (1.3)$$

where the mapping $f : [a, b] \rightarrow \mathbb{R}$ is assumed to be four times continuously differentiable on the interval and $f^{(4)}$ to be bounded on (a, b) , that is,

$$\|f^{(4)}\|_{\infty} = \sup_{t \in (a, b)} |f^{(4)}| < \infty.$$

Inequality (1.3) have been studied by many authors. For more details see [1, 2, 6, 8]. The aim of this paper is to establish q analogues of Simpson type inequalities based on convexity. The consequences of Simpson type inequalities for convex functions are given as special cases when $q \rightarrow 1$.

2. PRELIMINARIES

In this section, we recall some previously known concepts and basic results.

Let $J = [a, b] \subset \mathbb{R}$ be an interval and $0 < q < 1$ be a constant. We define q -derivative of a function $f : J \rightarrow \mathbb{R}$ at a point $x \in J$ on $[a, b]$ as follows.

Definition 1. Assume $f : J \rightarrow \mathbb{R}$ is a continuous function and let $x \in J$. Then the expression

$${}_a D_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a, \quad {}_a D_q f(a) = \lim_{x \rightarrow a} {}_a D_q f(x), \quad (2.1)$$

is called the q -derivative on J of function f at x .

Also if $a = 0$ in (2.1), then ${}_0 D_q f(a) = D_q f$, where D_q is the q -derivative of the function $f(x)$ defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

For more details, see [7].

Lemma 1 ([10]). Let $\alpha \in \mathbb{R}$, then we have

$${}_a D_q (x-a)^{\alpha} = \left(\frac{1-q^{\alpha}}{1-q} \right) (x-a)^{\alpha-1}. \quad (2.2)$$

Definition 2. Let $f : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Then q -integral on J is defined as

$$\int_a^x f(t) {}_a d_q t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \quad (2.3)$$

for $x \in J$. If $a = 0$ in (2.2), then we have the classical q -integral [7].

Moreover, if $v \in (a, x)$ then the definite q -integral on J is defined by

$$\int_v^x f(t) {}_a d_q t = \int_a^x f(t) {}_a d_q t - \int_a^v f(t) {}_a d_q t$$

$$\begin{aligned}
 &= (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \\
 &\quad - (1-q)(v-a) \sum_{n=0}^{\infty} q^n f(q^n v + (1-q^n)a).
 \end{aligned}$$

Lemma 2 ([11]). For $\alpha \in \mathbb{R} \setminus \{-1\}$, the following formula holds:

$$\int_a^x (t-a)^\alpha {}_a d_q t = \left(\frac{1-q}{1-q^{\alpha+1}} \right) (x-a)^{\alpha+1}. \quad (2.4)$$

3. RESULTS

We begin with the following lemma.

Lemma 3. Let $f : J \rightarrow \mathbb{R}$ be a continuous function and $0 < q < 1$. If ${}_a D_q f$ is an integrable function on J° (the interior of J), then the following inequality holds:

$$\begin{aligned}
 &\frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \\
 &= (b-a) \int_0^1 p(t) {}_a D_q f((1-t)a + tb) {}_0 d_q t
 \end{aligned} \quad (3.1)$$

where

$$p(t) = \begin{cases} qt - \frac{1}{6} & , \quad t \in \left[0, \frac{1}{2}\right) \\ qt - \frac{5}{6} & , \quad t \in \left[\frac{1}{2}, 1\right) \end{cases}.$$

Proof. From Definition 1 and Definition 2, we have

$$\begin{aligned}
 &\int_0^{\frac{1}{2}} \left(qt - \frac{1}{6} \right) {}_a D_q f((1-t)a + tb) {}_0 d_q t \\
 &= \int_0^{\frac{1}{2}} qt {}_a D_q f((1-t)a + tb) {}_0 d_q t - \frac{1}{6} \int_0^{\frac{1}{2}} {}_a D_q f((1-t)a + tb) {}_0 d_q t \\
 &= \int_0^{\frac{1}{2}} q \frac{f((1-t)a + tb) - f((1-qt)a + qtb)}{(1-q)(b-a)} {}_0 d_q t \\
 &\quad - \frac{1}{6} \int_0^{\frac{1}{2}} \frac{f((1-t)a + tb) - f((1-qt)a + qtb)}{(1-q)(b-a)t} {}_0 d_q t \\
 &= \frac{1}{2} \sum_{n=0}^{\infty} q^{n+1} \frac{f\left(\left(1 - \frac{1}{2}q^n\right)a + \frac{1}{2}q^n b\right)}{(b-a)} - \frac{1}{2} \sum_{n=0}^{\infty} q^{n+1} \frac{f\left(\left(1 - \frac{1}{2}q^{n+1}\right)a + \frac{1}{2}q^{n+1}b\right)}{(b-a)} \\
 &\quad - \frac{1}{6} \sum_{n=0}^{\infty} \frac{f\left(\left(1 - \frac{1}{2}q^n\right)a + \frac{1}{2}q^n b\right)}{(b-a)} + \frac{1}{6} \sum_{n=0}^{\infty} \frac{f\left(\left(1 - \frac{1}{2}q^{n+1}\right)a + \frac{1}{2}q^{n+1}b\right)}{(b-a)}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}q \sum_{n=0}^{\infty} q^n \frac{f\left(\left(1-\frac{1}{2}q^n\right)a + \frac{1}{2}q^n b\right)}{(b-a)} - \frac{1}{2} \sum_{n=1}^{\infty} q^n \frac{f\left(\left(1-\frac{1}{2}q^n\right)a + \frac{1}{2}q^n b\right)}{(b-a)} \\
&\quad - \frac{1}{6} \frac{1}{b-a} \sum_{n=0}^{\infty} f\left(\left(1-\frac{1}{2}q^n\right)a + \frac{1}{2}q^n b\right) \\
&\quad + \frac{1}{6} \frac{1}{b-a} \sum_{n=1}^{\infty} f\left(\left(1-\frac{1}{2}q^n\right)a + \frac{1}{2}q^n b\right) \\
&= -\frac{1}{b-a} (1-q) \frac{1}{2} \sum_{n=0}^{\infty} q^n f\left(\left(1-\frac{1}{2}q^n\right)a + \frac{1}{2}q^n b\right) \\
&\quad + \frac{1}{2} \frac{1}{b-a} f\left(\frac{a+b}{2}\right) - \frac{1}{6} \frac{1}{b-a} \left\{ f\left(\frac{a+b}{2}\right) - f(a) \right\} \\
&= -\frac{1}{b-a} \int_0^{\frac{1}{2}} f((1-t)a + tb) {}_0d_q t + \frac{1}{3(b-a)} f\left(\frac{a+b}{2}\right) + \frac{1}{6(b-a)} f(a).
\end{aligned}$$

For the second part of the integral, we have

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 \left(qt - \frac{5}{6}\right) {}_aD_q f((1-t)a + tb) {}_0d_q t \\
&= \int_0^1 \left(qt - \frac{5}{6}\right) {}_aD_q f((1-t)a + tb) {}_0d_q t \\
&\quad - \int_0^{\frac{1}{2}} \left(qt - \frac{5}{6}\right) {}_aD_q f((1-t)a + tb) {}_0d_q t
\end{aligned}$$

and similarly we obtain

$$\begin{aligned}
&\int_0^1 \left(qt - \frac{5}{6}\right) {}_aD_q f((1-t)a + tb) {}_0d_q t \\
&= -\frac{1}{b-a} \int_0^1 f((1-t)a + tb) {}_0d_q t + \frac{1}{6(b-a)} f(b) + \frac{5}{6(b-a)} f(a)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \left(qt - \frac{5}{6}\right) {}_aD_q f((1-t)a + tb) {}_0d_q t \\
&= -\frac{1}{b-a} \int_0^{\frac{1}{2}} f((1-t)a + tb) {}_0d_q t - \frac{1}{3(b-a)} f\left(\frac{a+b}{2}\right) + \frac{5}{6(b-a)} f(a).
\end{aligned}$$

Thus, we have

$$\int_0^1 p(t) {}_aD_q f((1-t)a + tb) {}_0d_q t$$

$$\begin{aligned}
&= -\frac{1}{b-a} \int_0^1 f((1-t)a + tb) {}_0d_q t + \frac{2}{3(b-a)} f\left(\frac{a+b}{2}\right) \\
&\quad + \frac{1}{6(b-a)} \{f(a) + f(b)\} \\
&= -\frac{1}{(b-a)^2} \int_a^b f(x) {}_ad_q x + \frac{2}{3(b-a)} f\left(\frac{a+b}{2}\right) + \frac{1}{6(b-a)} \{f(a) + f(b)\}
\end{aligned}$$

We complete the proof. $\square$

Remark 1. If $q \rightarrow 1$, then (3.1) reduces to

$$\begin{aligned}
&\frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \\
&= (b-a) \int_0^1 p(t) f'(tb + (1-t)a) dt,
\end{aligned}$$

where

$$p(t) = \begin{cases} t - \frac{1}{6}, & t \in [0, \frac{1}{2}) \\ t - \frac{5}{6}, & t \in [\frac{1}{2}, 1) \end{cases}.$$

See also [1, Lemma 1].

Lemma 4. Let $0 < q < 1$ be a constant. Then,

$$\int_0^{\frac{1}{2}} (1-t) \left| qt - \frac{1}{6} \right| {}_0d_q t = \frac{1}{216} \frac{36q^3 + 12q^2 + 12q + 1}{q^3 + 2q^2 + 2q + 1}. \quad (3.2)$$

Proof. By computing directly and using (2.4), we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}} (1-t) \left| qt - \frac{1}{6} \right| {}_0d_q t \\
&= \int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| {}_0d_q t - \int_0^{\frac{1}{2}} t \left| qt - \frac{1}{6} \right| {}_0d_q t \\
&= \int_0^{\frac{1}{6q}} \left(\frac{1}{6} - qt \right) {}_0d_q t + \int_{\frac{1}{6q}}^{\frac{1}{2}} \left(qt - \frac{1}{6} \right) {}_0d_q t \\
&\quad - \left(\int_0^{\frac{1}{6q}} t \left(\frac{1}{6} - qt \right) {}_0d_q t + \int_{\frac{1}{6q}}^{\frac{1}{2}} t \left(qt - \frac{1}{6} \right) {}_0d_q t \right) \\
&= \int_0^{\frac{1}{6q}} \left(\frac{1}{6} - qt \right) {}_0d_q t + \int_0^{\frac{1}{2}} \left(qt - \frac{1}{6} \right) {}_0d_q t - \int_0^{\frac{1}{6q}} \left(qt - \frac{1}{6} \right) {}_0d_q t \\
&\quad - \left(\int_0^{\frac{1}{6q}} t \left(\frac{1}{6} - qt \right) {}_0d_q t + \int_0^{\frac{1}{2}} t \left(qt - \frac{1}{6} \right) {}_0d_q t - \int_0^{\frac{1}{6q}} t \left(qt - \frac{1}{6} \right) {}_0d_q t \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{36} \frac{6q-1}{q+1} - \frac{1}{216} \frac{18q^2+18q-7}{q^3+2q^2+2q+1} \\
&= \frac{1}{216} \frac{36q^3+12q^2+12q+1}{q^3+2q^2+2q+1}.
\end{aligned}$$

□

Lemma 5. Let $0 < q < 1$ be a constant. Then,

$$\int_{\frac{1}{2}}^1 (1-t) \left| qt - \frac{5}{6} \right| {}_0d_q t = \frac{1}{216} \frac{12q^2+12q+5}{q^3+2q^2+2q+1}. \quad (3.3)$$

Proof. By computing directly and using (2.4), we have

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 (1-t) \left| qt - \frac{5}{6} \right| {}_0d_q t \\
&= \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| {}_0d_q t - \int_{\frac{1}{2}}^1 t \left| qt - \frac{5}{6} \right| {}_0d_q t \\
&= \int_{\frac{1}{2}}^{\frac{5}{6q}} \left(\frac{5}{6} - qt \right) {}_0d_q t + \int_{\frac{5}{6q}}^1 \left(qt - \frac{5}{6} \right) {}_0d_q t \\
&\quad - \left(\int_{\frac{1}{2}}^{\frac{5}{6q}} t \left(\frac{5}{6} - qt \right) {}_0d_q t + \int_{\frac{5}{6q}}^1 t \left(qt - \frac{5}{6} \right) {}_0d_q t \right) \\
&= \frac{5}{36(q+1)} - \frac{1}{216} \frac{18q^2+18q+25}{q^3+2q^2+2q+1} \\
&= \frac{1}{216} \frac{12q^2+12q+5}{q^3+2q^2+2q+1}.
\end{aligned}$$

□

Theorem 1. Let $f : J \rightarrow \mathbb{R}$ be a continuous function and $0 < q < 1$. If $|{}_aD_q f|$ is convex and integrable function on J° , then the following inequality holds:

$$\begin{aligned}
&\frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \\
&\leq \frac{(b-a)}{12} \left[\frac{2q^2+2q+1}{q^3+2q^2+2q+1} |{}_aD_q f(b)| + \frac{1}{3} \frac{6q^3+4q^2+4q+1}{q^3+2q^2+2q+1} |{}_aD_q f(a)| \right].
\end{aligned} \quad (3.4)$$

Proof. Using Lemma 3 and the convexity of $|{}_aD_q f|$ on J° , we have

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \right|$$

$$\begin{aligned}
&= (b-a) \left| \int_0^{\frac{1}{2}} \left(qt - \frac{1}{6} \right) {}_a D_q f (tb + (1-t)a) {}_0 d_q t \right. \\
&\quad \left. + \int_{\frac{1}{2}}^1 \left(qt - \frac{5}{6} \right) {}_a D_q f (tb + (1-t)a) {}_0 d_q t \right| \\
&\leq (b-a) \left(\int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| |{}_a D_q f (tb + (1-t)a)| {}_0 d_q t \right. \\
&\quad \left. + \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| |{}_a D_q f (tb + (1-t)a)| {}_0 d_q t \right) \\
&\leq (b-a) \int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| (t |{}_a D_q f (b)| + (1-t) |{}_a D_q f (a)|) {}_0 d_q t \\
&\quad + (b-a) \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| (t |{}_a D_q f (b)| + (1-t) |{}_a D_q f (a)|) {}_0 d_q t \\
&= |{}_a D_q f (b)| (b-a) \int_0^{\frac{1}{2}} t \left| qt - \frac{1}{6} \right| {}_0 d_q t + |{}_a D_q f (a)| (b-a) \int_0^{\frac{1}{2}} (1-t) \left| qt - \frac{1}{6} \right| {}_0 d_q t \\
&\quad + |{}_a D_q f (b)| (b-a) \int_{\frac{1}{2}}^1 t \left| qt - \frac{5}{6} \right| {}_0 d_q t + |{}_a D_q f (a)| (b-a) \int_{\frac{1}{2}}^1 (1-t) \left| qt - \frac{5}{6} \right| {}_0 d_q t
\end{aligned}$$

Applying Lemma 4 and Lemma 5, we have

$$\begin{aligned}
&\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \right| \\
&\leq \frac{(b-a)}{216} \frac{18q^2 + 18q - 7}{q^3 + 2q^2 + 2q + 1} |{}_a D_q f (b)| + \frac{(b-a)}{216} \frac{36q^3 + 12q^2 + 12q + 1}{q^3 + 2q^2 + 2q + 1} |{}_a D_q f (a)| \\
&\quad + \frac{(b-a)}{216} \frac{18q^2 + 18q + 25}{q^3 + 2q^2 + 2q + 1} |{}_a D_q f (b)| + \frac{(b-a)}{216} \frac{12q^2 + 12q + 5}{q^3 + 2q^2 + 2q + 1} |{}_a D_q f (a)| \\
&= \frac{(b-a)}{12} \frac{2q^2 + 2q + 1}{q^3 + 2q^2 + 2q + 1} |{}_a D_q f (b)| + \frac{(b-a)}{36} \frac{6q^3 + 4q^2 + 4q + 1}{q^3 + 2q^2 + 2q + 1} |{}_a D_q f (a)|.
\end{aligned}$$

The proof is complete. $\square$

Remark 2. If $q \rightarrow 1$, then (3.4) reduces to

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{5(b-a)}{72} [f'(a) + f'(b)]$$

See also [1, Corollary 1].

Corollary 1. In Theorem 1, if $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$, then we have

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \right|$$

$$\leq \frac{(b-a)}{12} \left[\frac{2q^2 + 2q + 1}{q^3 + 2q^2 + 2q + 1} |{}_a D_q f(b)| + \frac{1}{3} \frac{6q^3 + 4q^2 + 4q + 1}{q^3 + 2q^2 + 2q + 1} |{}_a D_q f(a)| \right].$$

This inequality can be considered a product of midpoint q -Hadamard type inequality.

Remark 3. In Corollary 1, if $q \rightarrow 1$, then we obtain

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{5(b-a)}{72} [f'(a) + f'(b)].$$

See also [1, Corollary 3].

Theorem 2. Let $f : J = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a q -differentiable function on J° with ${}_a D_q$ be continuous and integrable on J where $0 < q < 1$. If $|{}_a D_q f|^r$ is convex function where $p, r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \right| \\ & \leq \frac{(b-a)}{2^{\frac{1}{r}}} \left(\frac{(1-q)}{6^{p+1} q (1-q^{p+1})} \right)^{\frac{1}{p}} \\ & \quad \left\{ \left(1 + (3q-1)^{p+1} \right)^{\frac{1}{p}} \left(|{}_a D_q f(a)|^r + \left| {}_a D_q f\left(\frac{a+b}{2}\right) \right|^r \right)^{\frac{1}{r}} \right. \\ & \quad \left. + \left[(5-3q)^{p+1} + (6q-5)^{p+1} \right]^{\frac{1}{p}} \left(|{}_a D_q f(b)|^r + \left| {}_a D_q f\left(\frac{a+b}{2}\right) \right|^r \right)^{\frac{1}{r}} \right\}. \end{aligned} \quad (3.5)$$

Proof. From Lemma 3, using the well known Hölder integral inequality, we have

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \right| \\ & \leq (b-a) \left(\int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| |{}_a D_q f(tb + (1-t)a)| {}_0 d_q t \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| |{}_a D_q f(tb + (1-t)a)| {}_0 d_q t \right) \\ & \leq (b-a) \left(\int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right|^p {}_0 d_q t \right)^{1/p} \left(\int_0^{\frac{1}{2}} |{}_a D_q f(tb + (1-t)a)|^r {}_0 d_q t \right)^{1/r} \\ & \quad + (b-a) \left(\int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right|^p {}_0 d_q t \right)^{1/p} \left(\int_{\frac{1}{2}}^1 |{}_a D_q f(tb + (1-t)a)|^r {}_0 d_q t \right)^{1/r}. \end{aligned}$$

From (2.4), it is easy to see that

$$\begin{aligned} \int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right|^p {}_0d_q t &= \int_0^{\frac{1}{6q}} \left(\frac{1}{6} - qt \right)^p {}_0d_q t + \int_{\frac{1}{6q}}^{\frac{1}{2}} \left(qt - \frac{1}{6} \right)^p {}_0d_q t \\ &= (-1)^{p+1} q^p \int_{\frac{1}{6q}}^0 \left(t - \frac{1}{6q} \right)^p {}_0d_q t + q^p \int_{\frac{1}{6q}}^{\frac{1}{2}} \left(t - \frac{1}{6q} \right)^p {}_0d_q t \\ &= q^p \left(\frac{1-q}{1-q^{p+1}} \left(\frac{1}{6q} \right)^{p+1} \right) + q^p \left(\frac{1-q}{1-q^{p+1}} \left(\frac{1}{2} - \frac{1}{6q} \right)^{p+1} \right) \\ &= \frac{(1 + (3q-1)^{p+1})(1-q)}{6^{p+1}q(1-q^{p+1})}, \end{aligned}$$

analogously

$$\begin{aligned} \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right|^p {}_0d_q t &= \int_{\frac{1}{2}}^{\frac{5}{6q}} \left(\frac{1}{6} - qt \right)^p {}_0d_q t + \int_{\frac{5}{6q}}^1 \left(qt - \frac{1}{6} \right)^p {}_0d_q t \\ &= (-1)^{p+1} q^p \int_{\frac{5}{6q}}^{\frac{1}{2}} \left(t - \frac{5}{6q} \right)^p {}_0d_q t + q^p \int_{\frac{5}{6q}}^1 \left(t - \frac{5}{6q} \right)^p {}_0d_q t \\ &= \frac{[(5-3q)^{p+1} + (6q-5)^{p+1}](1-q)}{6^{p+1}q(1-q^{p+1})}. \end{aligned}$$

Since $|{}_aD_q f|$ is convex by (1.2), we have

$$\int_0^{\frac{1}{2}} |{}_aD_q f(tb + (1-t)a)|^r {}_0d_q t \leq \frac{|{}_aD_q f(a)|^r + |{}_aD_q f\left(\frac{a+b}{2}\right)|^r}{2}$$

and

$$\int_{\frac{1}{2}}^1 |{}_aD_q f(tb + (1-t)a)|^r {}_0d_q t \leq \frac{|{}_aD_q f(b)|^r + |{}_aD_q f\left(\frac{a+b}{2}\right)|^r}{2}.$$

So, we obtain

$$\begin{aligned} &\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \right| \\ &\leq (b-a) \left(\frac{(1 + (3q-1)^{p+1})(1-q)}{6^{p+1}q(1-q^{p+1})} \right)^{\frac{1}{p}} \left(\frac{|{}_aD_q f(a)|^r + |{}_aD_q f\left(\frac{a+b}{2}\right)|^r}{2} \right)^{\frac{1}{r}} \\ &\quad + (b-a) \left(\frac{[(5-3q)^{p+1} + (6q-5)^{p+1}](1-q)}{6^{p+1}q(1-q^{p+1})} \right)^{\frac{1}{p}} \left(\frac{|{}_aD_q f(b)|^r + |{}_aD_q f\left(\frac{a+b}{2}\right)|^r}{2} \right)^{\frac{1}{r}}. \end{aligned}$$

The proof is completed. □

Remark 4. If $q \rightarrow 1$, then (3.5) reduces to

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{2^{\frac{1}{r}}} \left(\frac{1+2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \left[\left(|f'(a)|^r + \left| f'\left(\frac{a+b}{2}\right) \right|^r \right)^{1/r} \right. \\ & \quad \left. + \left(\left| f'\left(\frac{a+b}{2}\right) \right|^r + |f'(b)|^r \right)^{1/r} \right]. \end{aligned}$$

See also [1, Corollary 4].

Corollary 2. In Theorem 2, if $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$, then we have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \right| \\ & \leq \frac{(b-a)}{2^{\frac{1}{r}}} \left(\frac{(1-q)}{6^{p+1}q(1-q^{p+1})} \right)^{\frac{1}{p}} \\ & \quad \left\{ \left(1 + (3q-1)^{p+1} \right)^{\frac{1}{p}} \left(|{}_a D_q f(a)|^r + \left| {}_a D_q f\left(\frac{a+b}{2}\right) \right|^r \right)^{\frac{1}{r}} \right. \\ & \quad \left. + \left[(5-3q)^{p+1} + (6q-5)^{p+1} \right]^{\frac{1}{p}} \left(|{}_a D_q f(b)|^r + \left| {}_a D_q f\left(\frac{a+b}{2}\right) \right|^r \right)^{\frac{1}{r}} \right\}. \end{aligned}$$

Remark 5. In Corollary 2, if $q \rightarrow 1$, then we have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{2^{\frac{1}{r}}} \left(\frac{1+2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \left[\left(|f'(a)|^r + \left| f'\left(\frac{a+b}{2}\right) \right|^r \right)^{1/r} \right. \\ & \quad \left. + \left(\left| f'\left(\frac{a+b}{2}\right) \right|^r + |f'(b)|^r \right)^{1/r} \right]. \end{aligned}$$

See also [1, Corollary 6] and take $s = 1$.

Theorem 3. Let $f : J = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a q -differentiable function on J° with ${}_a D_q f$ be continuous and integrable on J where $0 < q < 1$. If $|{}_a D_q f|^r$ is convex

function, then the following inequality holds:

$$\begin{aligned}
 & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \right| \\
 & \leq (b-a) \left(\frac{1}{216} \right)^{\frac{1}{r}} \left(\frac{6q-1}{36(q+1)} \right)^{1-\frac{1}{r}} \\
 & \quad \times \left(|{}_a D_q f(b)|^r \frac{18q^2+18q-7}{q^3+2q^2+2q+1} + |{}_a D_q f(a)|^r \frac{36q^3+12q^2+12q+1}{q^3+2q^2+2q+1} \right)^{1/r} \\
 & \quad + (b-a) \left(\frac{1}{216} \right)^{\frac{1}{r}} \left(\frac{5}{36(q+1)} \right)^{1-\frac{1}{r}} \\
 & \quad \times \left(|{}_a D_q f(b)|^r \frac{18q^2+18q+25}{q^3+2q^2+2q+1} + |{}_a D_q f(a)|^r \frac{12q^2+12q+5}{q^3+2q^2+2q+1} \right)^{1/r}.
 \end{aligned} \tag{3.6}$$

Proof. From Lemma 3 and using the well known power mean integral inequality and convexity of $|{}_a D_q f|^r$, we have

$$\begin{aligned}
 & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \right| \\
 & \leq (b-a) \left[\int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| |{}_a D_q f(tb + (1-t)a)| {}_0 d_q t \right. \\
 & \quad \left. + \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| |{}_a D_q f(tb + (1-t)a)| {}_0 d_q t \right] \\
 & \leq (b-a) \left(\int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| {}_0 d_q t \right)^{1-\frac{1}{r}} \left(\int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| |{}_a D_q f(tb + (1-t)a)|^r {}_0 d_q t \right)^{1/r} \\
 & \quad + (b-a) \left(\int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| {}_0 d_q t \right)^{1-\frac{1}{r}} \left(\int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| |{}_a D_q f(tb + (1-t)a)|^r {}_0 d_q t \right)^{1/r} \\
 & \leq (b-a) \left(\int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| {}_0 d_q t \right)^{1-\frac{1}{r}} \\
 & \quad \times \left(|{}_a D_q f(b)|^r \int_0^{\frac{1}{2}} t \left| qt - \frac{1}{6} \right| {}_0 d_q t + |{}_a D_q f(a)|^r \int_0^{\frac{1}{2}} (1-t) \left| qt - \frac{1}{6} \right| {}_0 d_q t \right)^{1/r} \\
 & \quad + (b-a) \left(\int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| {}_0 d_q t \right)^{1-\frac{1}{r}}
 \end{aligned}$$

$$\times \left(|{}_a D_q f(b)|^r \int_{\frac{1}{2}}^1 t \left| qt - \frac{5}{6} \right| {}_0 d_q t + |{}_a D_q f(a)|^r \int_{\frac{1}{2}}^1 (1-t) \left| qt - \frac{5}{6} \right| {}_0 d_q t \right)^{1/r}.$$

From Lemmas 4 and 5, we have

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \right| \\ & \leq (b-a) \left(\frac{1}{36} \frac{6q-1}{q+1} \right)^{1-\frac{1}{r}} \\ & \quad \times \left(|{}_a D_q f(b)|^r \frac{1}{216} \frac{18q^2+18q-7}{q^3+2q^2+2q+1} + |{}_a D_q f(a)|^r \frac{1}{216} \frac{36q^3+12q^2+12q+1}{q^3+2q^2+2q+1} \right)^{1/r} \\ & \quad + (b-a) \left(\frac{5}{36(q+1)} \right)^{1-\frac{1}{r}} \\ & \quad \times \left(|{}_a D_q f(b)|^r \frac{1}{216} \frac{18q^2+18q+25}{q^3+2q^2+2q+1} + |{}_a D_q f(a)|^r \frac{1}{216} \frac{12q^2+12q+5}{q^3+2q^2+2q+1} \right)^{1/r}. \end{aligned}$$

The proof is completed. $\square$

Remark 6. If $q \rightarrow 1$, then (3.6) reduces to

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{216^{\frac{1}{r}}} \left(\frac{5}{72} \right)^{1-\frac{1}{r}} \left\{ \left(\frac{29}{6} |f'(b)|^r + \frac{61}{6} |f'(a)|^r \right)^{1/r} \right. \\ & \quad \left. + \left(\frac{29}{6} |f'(a)|^r + \frac{61}{6} |f'(b)|^r \right)^{1/r} \right\}. \end{aligned}$$

See also [1, Theorem 7] and take $s = 1$.

Corollary 3. In Theorem 2, if $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$, then we have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \right| \\ & \leq (b-a) \left(\frac{1}{216} \right)^{\frac{1}{r}} \left(\frac{6q-1}{36(q+1)} \right)^{1-\frac{1}{r}} \\ & \quad \times \left(|{}_a D_q f(b)|^r \frac{18q^2+18q-7}{q^3+2q^2+2q+1} + |{}_a D_q f(a)|^r \frac{36q^3+12q^2+12q+1}{q^3+2q^2+2q+1} \right)^{1/r} \\ & \quad + (b-a) \left(\frac{1}{216} \right)^{\frac{1}{r}} \left(\frac{5}{36(q+1)} \right)^{1-\frac{1}{r}} \end{aligned}$$

$$\times \left(|{}_a D_q f(b)|^r \frac{18q^2 + 18q + 25}{q^3 + 2q^2 + 2q + 1} + |{}_a D_q f(a)|^r \frac{12q^2 + 12q + 5}{q^3 + 2q^2 + 2q + 1} \right)^{1/r}.$$

Corollary 4. In Corollary 3, if $q \rightarrow 1$, then we have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{216^{\frac{1}{r}}} \left(\frac{5}{72}\right)^{1-\frac{1}{r}} \left\{ \left(\frac{29}{6} |f'(b)|^r + \frac{61}{6} |f'(a)|^r\right)^{1/r} \right. \\ & \quad \left. + \left(\frac{29}{6} |f'(a)|^r + \frac{61}{6} |f'(b)|^r\right)^{1/r} \right\}. \end{aligned}$$

4. APPLICATIONS

For arbitrary real numbers, we consider the following means:

$$\text{The arithmetic mean : } A(a, b) = \frac{a+b}{2},$$

$$\text{The generalized log-mean : } L_p(a, b) = \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}}$$

where $p \in \mathbb{R} \setminus \{-1, 0\}$, $a, b \in \mathbb{R}$, $a \neq b$.

We derive some new inequalities for the above means in the following.

Proposition 1. Let $0 < a < b$, $n \in \mathbb{N}$, $0 < q < 1$, then

$$\begin{aligned} & \left| \frac{1}{3} A(a^n, b^n) + \frac{2}{3} A^n(a, b) - \frac{(n+1)(1-q)}{1-q^{n+1}} L_n^n(a, b) \right| \\ & \leq \frac{(b-a)}{12} \left[\frac{2q^2 + 2q + 1}{q^3 + 2q^2 + 2q + 1} \frac{b^n - (qb + (1-q)a)^n}{(b-a)(1-q)} + \frac{1}{3} \frac{6q^3 + 4q^2 + 4q + 1}{q^3 + 2q^2 + 2q + 1} n a^{n-1} \right]. \end{aligned} \quad (4.1)$$

Proof. The proof is obvious from Theorem 1 applied $f(x) = x^n$. $\square$

Corollary 5. Let $0 < a < b$, $n \in \mathbb{N}$, $q \rightarrow 1$, then (4.1) reduces to

$$\left| \frac{1}{3} A(a^n, b^n) + \frac{2}{3} A^n(a, b) - L_n^n(a, b) \right| \leq \frac{5(b-a)}{72} n [b^{n-1} + a^{n-1}].$$

Remark 7. If $q \rightarrow 1$ and $n = 1$ then (4.1) reduces to

$$|A(a, b) - L(a, b)| \leq \frac{5}{72} (b-a).$$

See also [1, Page 13].

Proposition 2. Let $0 < a < b$, $n \in \mathbb{N}$, $0 < q < 1$, then

$$\begin{aligned}
 & \left| \frac{1}{3} A(a^n, b^n) + \frac{2}{3} A^n(a, b) - \frac{(n+1)(1-q)}{1-q^{n+1}} L_n^n(a, b) \right| \\
 & \leq \frac{(b-a)}{2^{1/r}} \left(\frac{(1-q)}{6^{p+1}q(1-q^{p+1})} \right)^{\frac{1}{p}} \\
 & \quad \times \left\{ \left(1 + (3q-1)^{p+1} \right)^{\frac{1}{p}} \left(|na^{n-1}|^r + \left| \frac{\left(\frac{a+b}{2}\right)^n - \left(\left(\frac{b-a}{2}\right)q + a\right)^n}{\left(\frac{b-a}{2}\right)(1-q)} \right|^r \right)^{\frac{1}{r}} \right. \\
 & \quad \left. + \left((5-3q)^{p+1} + (6q-5)^{p+1} \right)^{\frac{1}{p}} \right. \\
 & \quad \left. \times \left(\left| \frac{b^n - (qb + (1-q)a)^n}{(b-a)(1-q)} \right|^r + \left| \frac{\left(\frac{a+b}{2}\right)^n - \left(\left(\frac{b-a}{2}\right)q + a\right)^n}{\left(\frac{b-a}{2}\right)(1-q)} \right|^r \right)^{\frac{1}{r}} \right\}.
 \end{aligned} \tag{4.2}$$

Proof. The proof is obvious from Theorem 2 applied $f(x) = x^n$. $\square$

Corollary 6. Let $0 < a < b$, $n \in \mathbb{N}$, $q \rightarrow 1$, then (4.2) reduces to

$$\begin{aligned}
 & \left| \frac{1}{3} A(a^n, b^n) + \frac{2}{3} A^n(a, b) - L_n^n(b, a) \right| \\
 & \leq \frac{(b-a)}{2^{1/r}} n \left(\frac{1}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} (1+2^{p+1})^{1/p} \\
 & \quad \times \left\{ \left(|a^{n-1}|^r + \left| \left(\frac{a+b}{2} \right)^{n-1} \right|^r \right)^{\frac{1}{r}} + \left(|b^{n-1}|^r + \left| \left(\frac{a+b}{2} \right)^{n-1} \right|^r \right)^{\frac{1}{r}} \right\}.
 \end{aligned}$$

Proposition 3. Let $0 < a < b$, $n \in \mathbb{N}$, $0 < q < 1$, then

$$\begin{aligned}
 & \left| \frac{1}{3} A(a^n, b^n) + \frac{2}{3} A^n(a, b) - \frac{(n+1)(1-q)}{1-q^{n+1}} L_n^n(a, b) \right| \\
 & \leq (b-a) \left(\frac{1}{216} \right)^{\frac{1}{r}} \left(\frac{6q-1}{36(q+1)} \right)^{1-\frac{1}{r}} \\
 & \quad \times \left(\left| \frac{b^n - (qb + (1-q)a)^n}{(b-a)(1-q)} \right|^r \frac{18q^2 + 18q - 7}{q^3 + 2q^2 + 2q + 1} + |na^{n-1}|^r \frac{36q^3 + 12q^2 + 12q + 1}{q^3 + 2q^2 + 2q + 1} \right)^{1/r} \\
 & \quad + (b-a) \left(\frac{1}{216} \right)^{\frac{1}{r}} \left(\frac{5}{36(q+1)} \right)^{1-\frac{1}{r}}
 \end{aligned} \tag{4.3}$$

$$\times \left(\left| \frac{b^n - (qb + (1-q)a)^n}{(b-a)(1-q)} \right|^r \frac{18q^2 + 18q + 25}{q^3 + 2q^2 + 2q + 1} + |na^{n-1}|^r \frac{12q^2 + 12q + 5}{q^3 + 2q^2 + 2q + 1} \right)^{1/r}.$$

Proof. The proof is obvious from Theorem 3 applied $f(x) = x^n$. $\square$

Corollary 7. Let $0 < a < b$, $n \in \mathbb{N}$, $q \rightarrow 1$, then (4.3) reduces to

$$\begin{aligned} & \left| \frac{1}{3} A(a^n, b^n) + \frac{2}{3} A^n(a, b) - L_n^n(b, a) \right| \\ & \leq \frac{n(b-a)}{(216)^{\frac{1}{r}}} \left(\frac{5}{72} \right)^{1-\frac{1}{r}} \\ & \times \left\{ \left(\frac{29}{6} |b^{n-1}|^r + \frac{61}{6} |a^{n-1}|^r \right)^{1/r} + \left(\frac{61}{6} |b^{n-1}|^r + \frac{29}{6} |a^{n-1}|^r \right)^{1/r} \right\}. \end{aligned}$$

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*Authors' addresses***M. Tunç**

Mustafa Kemal University, Mathematic Department, 31000 Hatay, Turkey

E-mail address: mevluttttunc@gmail.com

E. Göv

Mustafa Kemal University, Mathematic Department, 31000 Hatay, Turkey

E-mail address: esordulu@gmail.com

S. Balgeçti

Mustafa Kemal University, Mathematic Department, 31000 Hatay, Turkey

E-mail address: sevilbalgecti@gmail.com



WEYL PROJECTIVE OBJECTS $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$ FOR EQUITORSION GEODESIC MAPPINGS

NENAD O. VESIĆ

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Abstract. In this paper, invariants of geodesic mappings of non-symmetric affine connection manifolds are studied. It is obtained new generalizations of the Weyl projective tensor of these manifolds. At the end of this paper, generalized invariants of a geodesic mapping between special three dimensional generalized Riemannian manifolds $\mathbb{G}\mathbb{R}_3$ and $\mathbb{G}\mathbb{R}_3$ are obtained.

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1. INTRODUCTION AND MOTIVATION

Many research papers and monographs [1–19, 21–27] are focused on development of the affine connection spaces theory, on the theory of mappings between these spaces, on the theory of invariants of these mappings and on the applications of them. N. S. Sinyukov [21], J. Mikeš with his research group [9–11, 22], G. Hall [6, 7], G. P. Pokhariyal [16, 17], U. P. Singh [20] and many other researchers have given significant contributions to the development of the concept of symmetric affine connection spaces.

Consider an N -dimensional manifold $\mathcal{M}_N$ on which a non-symmetric affine connection ∇ is defined. If $\mathfrak{X}(\mathcal{M}_N)$ is Lie algebra of smooth vector fields and $X, Y \in \mathfrak{X}(\mathcal{M}_N)$, then the mapping

$$\nabla : \mathfrak{X}(\mathcal{M}_N) \times \mathfrak{X}(\mathcal{M}_N) \rightarrow \mathfrak{X}(\mathcal{M}_N)$$

defines the **non-symmetric connection** on $\mathcal{M}_N$ if $\nabla_X Y \neq \nabla_Y X + [X, Y]$. The following identities are satisfied:

$$\begin{aligned} \nabla_{Y_1+Y_2} X &= \nabla_{Y_1} X + \nabla_{Y_2} X, & \nabla_{fY} X &= f \nabla_Y X, \\ \nabla_Y (X_1 + X_2) &= \nabla_Y X_1 + \nabla_Y X_2, & \nabla_Y (fX) &= Yf \cdot X + f \nabla_Y X, \end{aligned}$$

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for $X, Y, X_1, X_2, Y_1, Y_2 \in \mathfrak{X}(\mathcal{M}_N)$, $f \in \mathcal{F}(\mathcal{M}_N)$, where $\mathcal{F}(\mathcal{M}_N)$ is an algebra of smooth real functions on $\mathcal{M}_N$.

Definition 1 ([4, 25, 27]). An N -dimensional differentiable manifold $\mathcal{M}_N$ endowed with a non-symmetric affine connection $\nabla_Y X$ is **the non-symmetric manifold** $\mathbb{GA}_N$.

Remark 1 ([4, 8]). A differentiable manifold endowed with a non-symmetric metric $G(X, Y)$ (the basic tensor $G(X, Y)$) is the generalized Riemannian manifold $\mathbb{GR}_N$. Affine connection coefficients of the manifold $\mathbb{GR}_N$ are Christoffel symbols of the second kind of this manifold.

Let ∇ be a non-symmetric affine connection. With regard to local chart $(x^1, \dots, x^N)$, in the base of $(\partial_i)_{i=1}^N = (\partial/\partial x^i)_{i=1}^N$, we obtain that is

$$\nabla_{\partial_k} \partial_j = \Gamma_{jk}^\alpha \partial_\alpha.$$

for coefficients Γ_{jk}^i non-symmetric by indices j and k .

In the whole paper, we shall use the capital Latin letters $X, Y, Z, \dots$ to denote smooth vector fields on a smooth manifold $\mathcal{M}_N$.

Let $\mathbb{GA}_N = (\mathcal{M}_N, \nabla)$ be a non-symmetric affine connection manifold. Because of the non-symmetry $\nabla_Y X + [X, Y] \neq \nabla_X Y$, the symmetric and anti-symmetric part of the affine connection coefficients $\nabla_Y X$ are respectively defined as:

$$\begin{aligned} \widetilde{\nabla}_Y X &= \frac{1}{2}(\nabla_Y X + \nabla_X Y - [X, Y]), \\ T(X, Y) &= \frac{1}{2}(\nabla_Y X - \nabla_X Y + [X, Y]). \end{aligned} \quad (1.1)$$

It is evident

$$\widetilde{\nabla}_Y X - \widetilde{\nabla}_X Y = [X, Y]. \quad (1.2)$$

The anti-symmetric part $T(X, Y)$ of the affine connection ∇ is **the torsion tensor**. The coordinates of the affine connection coefficients $\nabla_Y X$ are Γ_{jk}^i . The coordinates of the symmetric and the anti-symmetric parts of the coefficients Γ_{jk}^i are:

$$\widetilde{S}_{jk}^i = \frac{1}{2}(\Gamma_{jk}^i + \Gamma_{kj}^i) \quad \text{and} \quad T_{jk}^i = \frac{1}{2}(\Gamma_{jk}^i - \Gamma_{kj}^i),$$

An N -dimensional manifold $\mathcal{M}_N$ endowed with the symmetric affine connection $\widetilde{\nabla}_Y X$ is called **the associated (symmetric affine connection) manifold** $\mathbb{A}_N$ of the manifold $\mathbb{GA}_N$. Based on the symmetry of affine connection coefficients $\widetilde{\nabla}_Y X$ by X and Y , it exists only one type of covariant differentiation with respect to the affine connection of the manifold $\mathbb{A}_N$. For this reason, it exists only one curvature tensor of this manifold:

$$R(X; Y, Z) = \widetilde{\nabla}_Z \widetilde{\nabla}_Y X - \widetilde{\nabla}_Y \widetilde{\nabla}_Z X + \widetilde{\nabla}_{[Y, Z]} X. \quad (1.3)$$

A. Einstein [1–3] was the first who applied a non-symmetric affine connection in the research about gravity. He worked on the Unified Field Theory (Non-symmetric Gravitational Theory - NGT). In NGT, the basic tensor $G(X, Y)$ is a non-symmetric tensor of the type $(0, 2)$. The symmetric part

$$g(X, Y) = \frac{1}{2}(G(X, Y) + G(Y, X))$$

of the tensor $G(X, Y)$ is related to gravitation. The anti-symmetric part

$$F(X, Y) = \frac{1}{2}(G(X, Y) - G(Y, X)) \quad (= G(X, Y) - g(X, Y))$$

of the basic tensor $G(X, Y)$ is related to electromagnetism. Unlike Riemannian and generalized Riemannian manifolds, where affine connection coefficients are functions of the basic tensors $g(X, Y)$ and $G(X, Y)$, in Einstein's works affine connection coefficients of manifolds at NGT satisfy the *Einstein metricity condition*:

$$ZG(X, Y) - G(\nabla_Z X, Y) - G(X, \nabla_Y Z) = 0. \quad (1.4)$$

The coordinate form of this condition is

$$\frac{\partial G_{ij}}{\partial x^k} - \Gamma_{ik}^\alpha G_{\alpha j} - \Gamma_{kj}^\alpha G_{i\alpha} = 0, \quad (1.5)$$

for affine connection coefficients Γ_{jk}^i of the manifold at NGT.

Based on the Einstein's metricity condition, it exists two types ∇^+ and ∇^- of covariant differentiation (see [5]). For example, for a tensor A of the type $(1, 1)$ they are:

$$(\nabla_Y^+ A)(X) = (\tilde{\nabla}_Y A)(X) + T(AX, Y) - A(T(X, Y)), \quad (1.6)$$

$$(\nabla_Y^- A)(X) = (\tilde{\nabla}_Y A)(X) - T(AX, Y) + A(T(X, Y)). \quad (1.7)$$

Following the excellent Eisenhart's book [4] and motivated by different results from the theory of symmetric affine connection manifolds, S. Minčić [12–14] has obtained different significant results in the theory of non-symmetric affine connection manifolds. A geometry of these connections has been studied by F. Graiff [5], M. Prvanović [19] and many others.

From the differences

$$\nabla_Z^+ \nabla_Y^+ X - \nabla_Y^+ \nabla_Z^+ X, \quad \nabla_Z^- \nabla_Y^- X - \nabla_Y^- \nabla_Z^- X, \quad \nabla_Z^- \nabla_Y^+ X - \nabla_Y^+ \nabla_Z^- X,$$

one obtains the curvature tensors:

$$\begin{aligned} R_1(X; Y, Z) &= R(X; Y, Z) + (\tilde{\nabla}_Z T)(X, Y) - (\tilde{\nabla}_Y T)(X, Z) \\ &\quad + T(T(X, Y), Z) - T(T(X, Z), Y), \end{aligned} \quad (1.8)$$

$$\begin{aligned} R_2(X; Y, Z) &= R(X; Y, Z) - (\tilde{\nabla}_Z T)(X, Y) + (\tilde{\nabla}_Y T)(X, Z) \\ &\quad + T(T(X, Y), Z) - T(T(X, Z), Y), \end{aligned} \quad (1.9)$$

$$\begin{aligned} R_3(X; Y, Z) = R(X; Y, Z) + (\widetilde{\nabla}_Z T)(X, Y) + (\widetilde{\nabla}_Y T)(X, Z) \\ - T(T(X, Y), Z) + T(T(X, Z), Y) - 2T(T(Y, Z), X), \end{aligned} \quad (1.10)$$

where $R(X; Y, Z)$ is given by the formula (1.3). As one can see, these curvature tensors are introduced by using an algebraic approach to the curvature. Some of the geometric meaning of these curvature tensors has been pointed out in [18].

1.1. Geodesic mappings between affine connection manifolds

Geodesic mappings between affine connection manifolds have an important role in applications of differential geometry. Definitions and a lot of properties of mappings between symmetric affine connection manifolds are presented in books [9–11, 21]. Many other authors have continued this research (see [16, 17, 22]).

A diffeomorphism $f : \mathbb{G}\mathbb{A}_N \rightarrow \mathbb{G}\overline{\mathbb{A}}_N$ is called a geodesic mapping of $\mathbb{G}\mathbb{A}_N$ onto $\mathbb{G}\overline{\mathbb{A}}_N$ if any geodesic curve in $\mathbb{G}\mathbb{A}_N$ it maps onto a geodesic curve in $\mathbb{G}\overline{\mathbb{A}}_N$. The affine connections ∇ and $\overline{\nabla}$ of the manifolds $\mathbb{G}\mathbb{A}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$ satisfy the equation

$$\overline{\nabla}_Y X = \nabla_Y X + \psi(X)Y + \psi(Y)X + \xi(X, Y) \quad (1.11)$$

for a 1-form ψ and an anti-symmetric tensor ξ of the type (0,2).

Invariants of geodesic mappings of a non-symmetric affine connection manifold $\mathbb{G}\mathbb{A}_N$ and some their properties are obtained in [25–27]. The main goal of this paper is to obtain some other generalizations of invariants of geodesic mappings defined on the manifold $\mathbb{G}\mathbb{A}_N$.

Having a geodesic mapping of two general affine connection manifolds, we cannot find a generalization of Weyl tensor as an invariant of geodesic mapping in general case. For this reason, one assumes that $\xi(X, Y) = 0$ holds, that is, $\overline{T}(X, Y) = T(X, Y)$. These mappings are **the equitorsion geodesic mappings**.

2. INVARIANTS OF EQUITORSION GEODESIC MAPPINGS

Let $f : \mathbb{G}\mathbb{A}_N \rightarrow \mathbb{G}\overline{\mathbb{A}}_N$ be an equitorsion geodesic mapping of a non-symmetric affine connection manifold $\mathbb{G}\mathbb{A}_N$. The basic equation of this mapping is (see [25–27])

$$\overline{\nabla}_Y X = \nabla_Y X + \psi(X)Y + \psi(Y)X, \quad (2.1)$$

for a 1-form ψ . The symmetric parts $\widetilde{\nabla}_Y X$ and $\widetilde{\overline{\nabla}}_Y X$ of the affine connection coefficients $\nabla_Y X$ and $\overline{\nabla}_Y X$ satisfy the equation

$$\widetilde{\overline{\nabla}}_Y X = \widetilde{\nabla}_Y X + \frac{1}{N+1} \left((\text{Tr}\{U \rightarrow \nabla_U X\})Y + (\text{Tr}\{U \rightarrow \nabla_U Y\})X \right), \quad (2.2)$$

The curvature tensors $\bar{R}(X; Y, Z)$ and $R(X; Y, Z)$ of the associated manifolds $\bar{\mathbb{A}}_N$ and $\mathbb{A}_N$ satisfy the relation

$$\begin{aligned} \bar{R}(X; Y, Z) = & R(X; Y, Z) + \frac{1}{N+1} (Ric(Y, Z) - Ric(Z, Y))X \\ & - \frac{1}{N+1} (\bar{Ric}(Y, Z) - \bar{Ric}(Z, Y))X \\ & + \frac{1}{N^2-1} \left((NRic(X, Z) + Ric(Z, X))Y \right. \\ & \quad \left. - (NRic(Y, Z) + Ric(Z, Y))X \right) \\ & - \frac{1}{N^2-1} \left((N\bar{Ric}(X, Z) + \bar{Ric}(Z, X))Y \right. \\ & \quad \left. - (N\bar{Ric}(Y, Z) + \bar{Ric}(Z, Y))X \right), \end{aligned} \quad (2.3)$$

for the Ricci-curvature tensors $Ric(X, Y) = \text{Tr}\{U \rightarrow R(X; Y, U)\}$ and $\bar{Ric}(X, Y) = \text{Tr}\{U \rightarrow \bar{R}(X; Y, U)\}$ of the manifolds $\mathbb{A}_N$ and $\bar{\mathbb{A}}_N$.

From the equation (2.3), one obtains that the Weyl projective tensor

$$\begin{aligned} W(X, Y, Z) = & R(X; Y, Z) + \frac{1}{N+1} (Ric(Y, Z) - Ric(Z, Y))X \\ & + \frac{1}{N^2-1} \left((NRic(X, Z) + Ric(Z, X))Y \right. \\ & \quad \left. - (NRic(X, Y) + Ric(Y, X))Z \right) \end{aligned} \quad (2.4)$$

is an invariant of the mapping f . The coordinates of the Weyl projective tensor W are

$$W_{jmn}^i = R_{jmn}^i + \frac{1}{N+1} \delta_j^i R_{[mn]} + \frac{N}{N^2-1} \delta_{[m}^i R_{jn]} + \frac{1}{N^2-1} \delta_{[m}^i R_{n]j}. \quad (2.5)$$

2.1. Weyl projective tensors $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$ of geodesic mappings

From the equation (1.8), we obtain the following equation:

$$\begin{aligned} Ric(X, Y) = & Ric_1(X, Y) \\ & - \text{Tr}\{U \rightarrow (\tilde{\nabla}_U T)(X, Y)\} + \text{Tr}\{U \rightarrow (\tilde{\nabla}_Y T)(X, U)\} \\ & - \text{Tr}\{U \rightarrow T(T(X, Y), U)\} + \text{Tr}\{U \rightarrow T(T(X, U), Y)\}, \end{aligned} \quad (2.6)$$

for Ricci-curvature tensor $Ric(X, Y) = \text{Tr}\{U \rightarrow R(X; Y, U)\}$.

The similar equations can be derived for Ricci-curvature tensors $Ric_2(X, Y)$ and $Ric_3(X, Y)$. In the same manner, we obtain the corresponding correlations between

the Ricci-curvature tensors $\overline{Ricc}(X, Y)$ and $\overline{Ricc}_\theta(X, Y)$, $\theta = 1, 2, 3$ of the manifolds $\overline{\mathbb{A}}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$. Because $\overline{T}(X, Y) = T(X, Y)$, it holds

$$\begin{aligned} (\widetilde{\nabla}_Z T)(X, Y) &= (\widetilde{\nabla}_Z T)(X, Y) - \widetilde{\nabla}_{T(X, Y)} Z + T(\widetilde{\nabla}_Z X, Y) + T(X, \widetilde{\nabla}_Z Y) \\ &\quad + \widetilde{\nabla}_{\overline{T}(X, Y)} Z - \overline{T}(\widetilde{\nabla}_Z X, Y) - \overline{T}(X, \widetilde{\nabla}_Z Y). \end{aligned} \quad (2.7)$$

From the expressions (2.3, 2.6, 2.7) involved in the correlation

$$\begin{aligned} \overline{R}_1(X; Y, Z) &= R_1(X; Y, Z) + \frac{1}{N+1} (Ricc(Y, Z) - Ricc(Z, Y) - \overline{Ricc}(Y, Z) + \overline{Ricc}(Z, Y)) X \\ &\quad + \frac{1}{N^2-1} ((N Ricc(X, Z) + Ricc(Z, X)) Y - (N Ricc(X, Y) + Ricc(Y, X)) Z) \\ &\quad - \frac{1}{N^2-1} ((N \overline{Ricc}(X, Z) + \overline{Ricc}(Z, X)) Y - (N \overline{Ricc}(X, Y) + \overline{Ricc}(Y, X)) Z) \\ &\quad + (\widetilde{\nabla}_Z T)(X, Y) - (\widetilde{\nabla}_Z T)(X, Y) - (\widetilde{\nabla}_Y T)(X, Z) + (\widetilde{\nabla}_Y T)(X, Z), \end{aligned}$$

we obtain

$$\overline{\mathcal{W}}_1(X, Y, Z) = \mathcal{W}_1(X, Y, Z)$$

for

$$\begin{aligned} \mathcal{W}_1(X, Y, Z) &= R_1(X; Y, Z) + \frac{1}{N+1} (Ricc_1(Y, Z) - Ricc_1(Z, Y)) X \\ &\quad + \frac{1}{N^2-1} ((N Ricc_1(X, Z) + Ricc_1(Z, X)) Y \\ &\quad \quad \quad - (N Ricc_1(X, Y) + Ricc_1(Y, X)) Z) \\ &\quad - \widetilde{\nabla}_{T(X, Y)} Z + T(\widetilde{\nabla}_Z X, Y) + T(X, \widetilde{\nabla}_Z Y) \\ &\quad + \widetilde{\nabla}_{T(X, Z)} Y - T(\widetilde{\nabla}_Y X, Z) - T(X, \widetilde{\nabla}_Y Z) \\ &\quad + \frac{2}{N+1} \text{Tr} \{U \rightarrow T(\widetilde{\nabla}_U Y, Z) - T(\widetilde{\nabla}_U Z, Y)\} X \\ &\quad + \frac{1}{N+1} \text{Tr} \{U \rightarrow \widetilde{\nabla}_{T(U, Z)} Y - T(U, Y) Z X\} \\ &\quad - \frac{1}{N-1} \text{Tr} \{U \rightarrow T(\widetilde{\nabla}_Z X Y - \widetilde{\nabla}_Y X Z, U)\} \end{aligned} \quad (2.8)$$

and the corresponding $\overline{\mathcal{W}}_1(X, Y, Z)$. In the same manner, we obtain

$$\overline{\mathcal{W}}_2(X, Y, Z) = \mathcal{W}_2(X, Y, Z) \quad \text{and} \quad \overline{\mathcal{W}}_3(X, Y, Z) = \mathcal{W}_3(X, Y, Z)$$

for

$$\begin{aligned}
\mathcal{W}_2(X, Y, Z) = & \frac{R(X; Y, Z)}{2} + \frac{1}{N+1} (Ric_2(Y, Z) - Ric_2(Z, Y))X \\
& + \frac{1}{N^2-1} \left((N Ric_2(X, Z) + Ric_2(Z, X))Y \right. \\
& \quad \left. - (N Ric_2(X, Y) + Ric_2(Y, X))Z \right) \\
& + \widetilde{\nabla}_{T(X, Y)}Z - T(\widetilde{\nabla}_Z X, Y) - T(X, \widetilde{\nabla}_Z Y) \\
& - \widetilde{\nabla}_{T(X, Z)}Y + T(\widetilde{\nabla}_Y X, Z) + T(X, \widetilde{\nabla}_Y Z) \\
& - \frac{2}{N+1} \text{Tr}\{U \rightarrow T(\widetilde{\nabla}_U Y, Z) - T(\widetilde{\nabla}_U Z, Y)\}X \\
& - \frac{1}{N+1} \text{Tr}\{U \rightarrow \widetilde{\nabla}_{T(U, Z)}Y - T(U, Y)Z\}X \\
& + \frac{1}{N-1} \text{Tr}\{U \rightarrow T(\widetilde{\nabla}_Z XY - \widetilde{\nabla}_Y XZ, U)\},
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
\mathcal{W}_3(X, Y, Z) = & \frac{R(X; Y, Z)}{3} + \frac{1}{N+1} (Ric_3(Y, Z) - Ric_3(Z, Y))X \\
& + \frac{1}{N^2-1} \left((N Ric_3(X, Z) + Ric_3(Z, X))Y \right. \\
& \quad \left. - (N Ric_3(X, Y) + Ric_3(Y, X))Z \right) \\
& - \widetilde{\nabla}_{T(X, Y)}Z + T(\widetilde{\nabla}_Z X, Y) + T(X, \widetilde{\nabla}_Z Y) \\
& - \widetilde{\nabla}_{T(X, Z)}Y + T(\widetilde{\nabla}_Y X, Z) + T(X, \widetilde{\nabla}_Y Z) \\
& + \frac{2}{N+1} \text{Tr}\{U \rightarrow T(\widetilde{\nabla}_U Y, Z) - T(\widetilde{\nabla}_U Z, Y)\}X \\
& + \frac{1}{N+1} \text{Tr}\{U \rightarrow \widetilde{\nabla}_{T(U, Z)}Y - T(U, Y)Z\}X \\
& + \frac{1}{N-1} \text{Tr}\{U \rightarrow T(\widetilde{\nabla}_Z XY - \widetilde{\nabla}_Y XZ, U)\}.
\end{aligned} \tag{2.10}$$

Theorem 1. Let $f : \mathbb{G}\mathbb{A}_N \rightarrow \mathbb{G}\overline{\mathbb{A}}_N$ be an equitorsion geodesic mapping. The geometrical objects (2.8, 2.9, 2.10) are invariants of the mapping f . $\square$

Corollary 1. The invariants $\mathcal{W}_1(X, Y, Z)$, $\mathcal{W}_2(X, Y, Z)$, $\mathcal{W}_3(X, Y, Z)$ and the Weyl projective tensor $W(X, Y, Z)$ satisfy the equations:

$$\begin{aligned}
\mathcal{W}_1(X, Y, Z) = & W(X, Y, Z) + ZT(X, Y) - YT(X, Z) \\
& + T(T(X, Y), Z) - T(T(X, Z), Y),
\end{aligned} \tag{2.11}$$

$$\begin{aligned} \mathcal{W}_2(X, Y, Z) &= W(X, Y, Z) - ZT(X, Y) + YT(X, Z) \\ &\quad + T(T(X, Y), Z) - T(T(X, Z), Y), \end{aligned} \quad (2.12)$$

$$\begin{aligned} \mathcal{W}_3(X, Y, Z) &= W(X, Y, Z) + ZT(X, Y) + YT(X, Z) \\ &\quad - T(T(X, Y), Z) + T(T(X, Z), Y) - 2T(T(Y, Z), X). \end{aligned} \quad (2.13)$$

Example 1. Let a generalized Riemannian manifold $\mathbb{G}\mathbb{R}_3$ be determined by the non-symmetric basic matrix

$$[G(X, Y)] = \begin{bmatrix} 1 & e^{x^1} & -e^{-x^2} \\ -e^{x^1} & 1 & -e^{x^3} \\ e^{-x^2} & e^{x^3} & 1 \end{bmatrix}$$

and let $f : \mathbb{G}\mathbb{R}_3 \rightarrow \mathbb{G}\bar{\mathbb{R}}_3$ be an equitortion geodesic mapping of the manifold $\mathbb{G}\mathbb{R}_3$.

The symmetric and anti-symmetric part of the basic matrix $(G(X, Y))$ are, respectively:

$$[g(X, Y)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [F(X, Y)] = \begin{bmatrix} 0 & e^{x^1} & -e^{-x^2} \\ -e^{x^1} & 0 & -e^{x^3} \\ e^{-x^2} & e^{x^3} & 0 \end{bmatrix}.$$

The symmetric part $(g(X, Y))$ of the basic tensor $(G(X, Y))$ is constant. For this reason, the coordinates of the curvature tensor $R(X; Y, Z)$ of the associated manifold $\mathbb{R}_3$ are $R_{jmn}^i = 0$. Furthermore, the coordinates of the Weyl projective tensor $W(X, Y, Z)$ of the manifold $\mathbb{R}_3$ are $W_{jmn}^i = 0$. The coordinates of the torsion tensor $T(X, Y)$ of the manifold $\mathbb{G}\mathbb{R}_3$ are

$$\begin{aligned} T_{23}^1 &= \frac{1}{2}e^{-x^2}, T_{32}^1 = -\frac{1}{2}e^{-x^2}, T_{13}^2 = \frac{1}{2}e^{-x^2}, \\ T_{31}^2 &= -\frac{1}{2}e^{-x^2}, T_{12}^3 = \frac{1}{2}e^{-x^2}, T_{21}^3 = -\frac{1}{2}e^{-x^2} \end{aligned}$$

and $T_{jk}^i = 0, i, j, k = 1, 2, 3$, in all other cases.

The coordinates $\mathcal{W}_1^i, \mathcal{W}_2^i, \mathcal{W}_3^i, i, j, m, n = 1, 2, 3$, of the invariants $\mathcal{W}_1(X, Y, Z), \mathcal{W}_2(X, Y, Z), \mathcal{W}_3(X, Y, Z)$ of the mapping f are:

$$\begin{aligned} \mathcal{W}_1^i &= W_{jmn}^i + T_{jm,n}^i - T_{jn,m}^i + T_{jm}^\alpha T_{\alpha n}^i - T_{jn}^\alpha T_{\alpha m}^i, \\ \mathcal{W}_2^i &= W_{jmn}^i - T_{jm,n}^i + T_{jn,m}^i + T_{jm}^\alpha T_{\alpha n}^i - T_{jn}^\alpha T_{\alpha m}^i, \\ \mathcal{W}_3^i &= W_{jmn}^i + T_{jm,n}^i + T_{jn,m}^i - T_{jm}^\alpha T_{\alpha n}^i + T_{jn}^\alpha T_{\alpha m}^i - 2T_{mn}^\alpha T_{\alpha j}^i \end{aligned}$$

for partial derivations denoted by commas and the above obtained coordinates T_{jk}^i of the torsion tensor $T(X, Y)$ of the manifold $\mathbb{G}\mathbb{R}_3$. After a bit of computation, we obtain coordinates $\mathcal{W}_{1jmn}^i, \mathcal{W}_{2jmn}^i, \mathcal{W}_{3jmn}^i$ of the invariants $\mathcal{W}_1(X, Y, Z), \mathcal{W}_2(X, Y, Z), \mathcal{W}_3(X, Y, Z)$ are:

$$\begin{aligned} \mathcal{W}_1 : & \begin{cases} \mathcal{W}_{1223}^1 = \mathcal{W}_{1123}^2 = \mathcal{W}_{1312}^2 = \mathcal{W}_{1221}^3 = \frac{1}{2}e^{-x^2} \\ \mathcal{W}_{1221}^1 = \mathcal{W}_{1331}^1 = \mathcal{W}_{1112}^2 = \mathcal{W}_{1332}^2 = \mathcal{W}_{1131}^3 = \mathcal{W}_{1232}^3 = \frac{1}{4}e^{-2x^2} \\ \mathcal{W}_{1232}^1 = \mathcal{W}_{1132}^2 = \mathcal{W}_{1321}^2 = \mathcal{W}_{1212}^3 = -\frac{1}{2}e^{-x^2} \\ \mathcal{W}_{1212}^1 = \mathcal{W}_{1313}^1 = \mathcal{W}_{1121}^2 = \mathcal{W}_{1323}^2 = \mathcal{W}_{1113}^3 = \mathcal{W}_{1223}^3 = -\frac{1}{4}e^{-2x^2} \end{cases} \\ \mathcal{W}_2 : & \begin{cases} \mathcal{W}_{2232}^1 = \mathcal{W}_{2132}^2 = \mathcal{W}_{2321}^2 = \mathcal{W}_{2212}^3 = \frac{1}{2}e^{-x^2} \\ \mathcal{W}_{2221}^1 = \mathcal{W}_{2331}^1 = \mathcal{W}_{2112}^2 = \mathcal{W}_{2332}^2 = \mathcal{W}_{2131}^3 = \mathcal{W}_{2232}^3 = \frac{1}{4}e^{-2x^2} \\ \mathcal{W}_{2223}^1 = \mathcal{W}_{2123}^2 = \mathcal{W}_{2312}^2 = \mathcal{W}_{2221}^3 = -\frac{1}{2}e^{-x^2} \\ \mathcal{W}_{2212}^1 = \mathcal{W}_{2313}^1 = \mathcal{W}_{2121}^2 = \mathcal{W}_{2323}^2 = \mathcal{W}_{2113}^3 = \mathcal{W}_{2223}^3 = -\frac{1}{4}e^{-2x^2} \end{cases} \\ \mathcal{W}_3 : & \begin{cases} \mathcal{W}_{3322}^1 = e^{-x^2}, \mathcal{W}_{3122}^3 = -e^{-x^2}, \mathcal{W}_{3212}^1 = \mathcal{W}_{3121}^2 = \mathcal{W}_{3131}^3 = \mathcal{W}_{3232}^3 = \frac{3}{4}e^{-2x^2} \\ \mathcal{W}_{3312}^2 = \mathcal{W}_{3321}^2 = \frac{1}{2}e^{-x^2}, \\ \mathcal{W}_{3221}^1 = \mathcal{W}_{3331}^1 = \mathcal{W}_{3112}^2 = \mathcal{W}_{3332}^2 = \frac{1}{4}e^{-2x^2} \\ \mathcal{W}_{3223}^1 = \mathcal{W}_{3232}^2 = \mathcal{W}_{3123}^2 = \mathcal{W}_{3132}^2 = \mathcal{W}_{3212}^3 = \mathcal{W}_{3221}^3 = \mathcal{W}_{3311}^3 = \mathcal{W}_{3322}^3 = -\frac{1}{2}e^{-x^2} \\ \mathcal{W}_{3313}^1 = \mathcal{W}_{3323}^2 = \mathcal{W}_{3113}^3 = \mathcal{W}_{3223}^3 = -\frac{1}{4}e^{-2x^2} \end{cases} \end{aligned}$$

and $\mathcal{W}_{\theta jmn}^i = 0, \theta, i, j, m, n = 1, 2, 3$, in all other cases.

3. CONCLUSION

We expanded the concept of geodesic mappings defined on spaces with torsion in this paper. Invariants of equitortion geodesic mappings are obtained above. We found another generalization of Weyl projective tensor in this way.

The results, which are obtained in this paper, are presented in the corresponding operator forms. Moreover, some of these results are presented coordinately for different applications in physics.

In further research, we will stay focused on NGT, generalized Riemannian spaces in the sense of Eisenhart, as well as semi-symmetric and quarter-symmetric spaces. We will also study conformal invariants of a generalized Riemannian manifold GR_N .

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Author's address

Nenad O. Vesić

University of Niš, Faculty of Science and Mathematics, Višegradska 33, 18000 Niš, Serbia.

E-mail address: vesko1985@pmf.ni.ac.rs



ON COMPUTABLE CLASSES OF EQUIDISTANT SETS: EQUIDISTANT FUNCTIONS

CS. VINCZE, A. VARGA, M. OLÁH, AND L. FÓRIÁN

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Abstract. In their paper [6] the authors posed the problem of the characterization of closed subsets in the Euclidean plane that can be realized as the equidistant set of two connected disjoint closed sets. We are going to solve the problem for a special class of equidistant sets. As a main result we give necessary and sufficient conditions for a function to be a so-called equidistant function. This means that its graph is the equidistant set of a line (the first coordinate axis) and the (convex) epigraph of a positive valued convex function satisfying some regularity conditions. In the first step we present a parametric expression for equidistant functions. The parametric expression allows us to use the basic differential geometric tools. We also give a sufficient and necessary condition for a function to be equidistant. In the concluding remarks section some perspectives and extensions of such a special approach will be presented together with some open problems.

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1. INTRODUCTION: NOTATIONS AND PRELIMINARIES

Let $K \subset \mathbb{R}^2$ be a subset in the Euclidean coordinate plane. The distance between a point (x, y) and K is measured by the usual infimum formula:

$$d((x, y), K) := \inf\{d((x, y), (a, b)) \mid (a, b) \in K\}.$$

Let us define the equidistant set of K and $L \subset \mathbb{R}^2$ as the set all of whose points have the same distance from K and L :

$$\{K = L\} := \{(x, y) \in \mathbb{R}^2 \mid d((x, y), K) = d((x, y), L)\}.$$

The equidistant sets can be considered as a kind of the generalization of conics [6]: K and L are called the focal sets. Equidistant sets are often called midsets too. Their investigations have been started by Wilker's and Loveland's fundamental works [12] and [4]. For another generalization of the classical conics and their applications see

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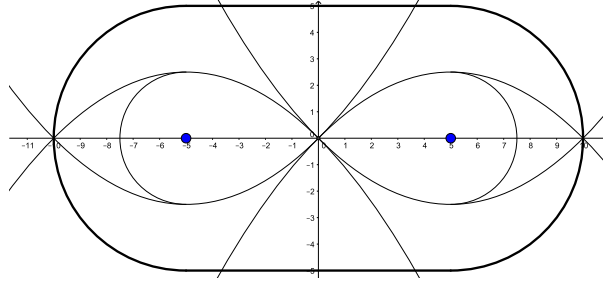


FIGURE 1. The 2D-version of Loveland's example

e.g. [1, 5] (polyellipses and their applications), [2, 8, 9] and [10]. "We find equidistant sets as conventionally defined frontiers in territorial domain controversies: for instance, the United Nations Convention on the Law of the Sea (Article 15) establishes that, in absence of any previous agreement, the delimitation of the territorial sea between countries occurs exactly on the median line every point of which is equidistant of the nearest points to each country"; for the citation see [6]. The points of an equidistant set are difficult to determine in general because there are no simple formulas to compute the distance between a point and a set. On the other hand the equidistant set can have strange and exotic properties even if it has relatively simple focal sets (dimension, connectedness etc.). For some amusing examples see [12] and [4]. Figure 1 shows the 2D-version of Loveland's example in [4]. The equidistant set forms a curve with self-intersection in the middle part of the figure. It consists of two semicircles (left and right hand parts) and four congruent parabolic arcs (the equidistant set of a point and a line). The focal set K contains two different points and L is the boundary of a parallel body¹ of the segment joining the points in K . Therefore the investigation of special classes of equidistant sets seems to be as important as the investigation of the general properties; see e.g. [11], where the authors characterize the equidistant points of finite focal sets in terms of computable constants and parametrization. The process is also implemented in MAPLE. Its motivation is a kind of continuity property of equidistant sets; see Theorem 11 in [6]. Since any compact subset can be approximated by finite subsets with respect to the Hausdorff metric we can approximate the equidistant points of K and L with the equidistant points of finite subsets K_n and L_n ; for the details see [11]. In what follows we consider the case of the focal sets

$$K := \{(t, 0) \in \mathbb{R}^2 \mid t \in \mathbb{R}\} \text{ and } L := \{(t, f(t)) \in \mathbb{R}^2 \mid t \in \mathbb{R}\},$$

where $f: \mathbb{R} \rightarrow \mathbb{R}$ is a positive valued twice continuously differentiable convex function. For the sake of simplicity the equidistant set $\{K = L\}$ will be denoted by E_f .

¹The parallel body of L with radius R means the union of the closed disks with radius R centered at the points of L ; see e.g. [3] and [7].

Definition 1. A function $G: \mathbb{R} \rightarrow \mathbb{R}$ is an equidistant function if its graph is the equidistant set of K and L for some positive valued twice continuously differentiable convex function f .

In the next section the equidistant functions are given in terms of a special parametric expression (2.4). Theorem 1 contains the characterization of a parametric expression to be the parametrization of the graph of an equidistant function. Some applications of the basic differential geometric tools (arclength, inclination angle, curvature) are also presented (Subsections 3.1 and 3.2). Theorem 2 contains a simple relationship between the inclination angles of f and its equidistant function at the corresponding positions by the parametric expression. We also give necessary and sufficient conditions for a function $G: \mathbb{R} \rightarrow \mathbb{R}$ to be an equidistant function (Theorem 3). The results give the solution of the problem posed in [6] for a special class of equidistant sets (equidistant functions): *characterize all closed sets of the plane that can be realized as the equidistant set of two connected disjoint closed sets*. In the concluding remarks section some perspectives and extensions of such a special approach will be presented together with some open problems.

2. THE PARAMETRIC EXPRESSION OF AN EQUIDISTANT FUNCTION

Since f is (twice) differentiable, the supporting line of its (convex) epigraph coincides with the tangent line of the function at any $t \in \mathbb{R}$. Using the outer unit normal of the epigraph the equidistant points can be characterized by the formula

$$(x(t), y(t)) = (t, f(t)) + y(t) \frac{1}{\sqrt{1 + f'^2(t)}} (f'(t), -1), \quad (2.1)$$

where $y(t) > 0$ denotes the (common) distance of the point $(x(t), y(t))$ to the focal sets. Therefore

$$x(t) = t + \frac{y(t)f'(t)}{\sqrt{1 + f'^2(t)}} \quad \text{and} \quad y(t) = f(t) - \frac{y(t)}{\sqrt{1 + f'^2(t)}}, \quad (2.2)$$

$$f(t) = y(t) \cdot \left(1 + \frac{1}{\sqrt{1 + f'^2(t)}} \right) = y(t) \left(\frac{1 + \sqrt{1 + f'^2(t)}}{\sqrt{1 + f'^2(t)}} \right), \quad (2.3)$$

i.e.

$$x(t) = t + \frac{f(t)f'(t)}{1 + \sqrt{1 + f'^2(t)}} \quad \text{and} \quad y(t) = \frac{f(t)\sqrt{1 + f'^2(t)}}{1 + \sqrt{1 + f'^2(t)}}. \quad (2.4)$$

2.1. The geometric interpretation of equidistancy

Figure 2 shows the case of $f(t) = \frac{1+t^2}{3}$ ($t \in \mathbb{R}$) at $t = 3$. The equidistant point at $t = 3$ (i.e. $x(t) = 5$) is given as the intersection of the normal line at the tangency point and the bisector of the angle enclosed by the tangent line and the first coordinate axis.

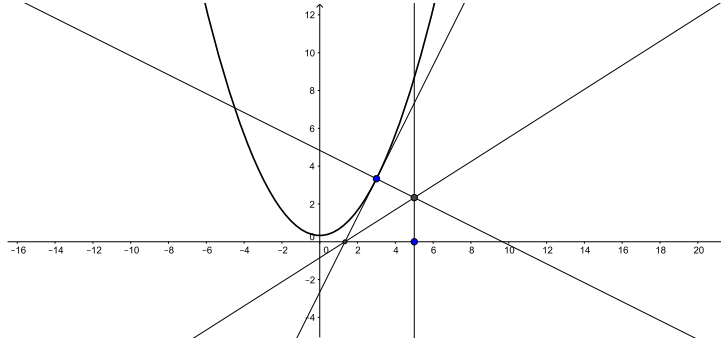


FIGURE 2. Formula (2.1) and Theorem 2

3. THE CHARACTERIZATION OF THE PARAMETRIC EXPRESSION OF THE EQUIDISTANT FUNCTIONS

Let

$$\theta_f(t) := \arctan f'(t)$$

be the inclination angle of the function f ; note that $-\pi/2 < \theta_f(t) < \pi/2$. In terms of the parametrization (2.4)

$$\begin{aligned} \sin \theta_f(t) &= \frac{f'(t)}{\sqrt{1 + f'^2(t)}} = \frac{x(t) - t}{y(t)} \\ \cos \theta_f(t) &= \frac{1}{\sqrt{1 + f'^2(t)}} = \frac{1}{\sqrt{1 - \left(\frac{x(t) - t}{y(t)}\right)^2}} = \frac{\sqrt{y^2(t) - (x(t) - t)^2}}{y(t)} \\ \tan \theta_f(t) &= f'(t) = \frac{x(t) - t}{\sqrt{y^2(t) - (x(t) - t)^2}}. \end{aligned} \quad (3.1)$$

Using the auxiliary function²

$$\varphi(t) := \tan \frac{\theta_f(t)}{2} \quad (3.2)$$

it follows that

$$\begin{aligned} \sin \theta_f(t) &= \frac{2\varphi(t)}{1 + \varphi^2(t)}, \quad \cos \theta_f(t) = \frac{1 - \varphi^2(t)}{1 + \varphi^2(t)}, \quad \tan \theta_f(t) = \frac{2\varphi(t)}{1 - \varphi^2(t)}, \\ \varphi(t) &= \frac{\sin \theta_f(t)}{1 + \cos \theta_f(t)}, \quad \frac{1 + \varphi^2(t)}{2} = \frac{1}{1 + \cos \theta_f(t)}; \end{aligned} \quad (3.3)$$

²It is the usual substitution to transform trigonometric expressions into algebraic ones.

note that $-1 < \varphi(t) < 1$. By (3.1) and (3.3), the parametrization (2.4) can be written as follows:

$$\begin{aligned} x(t) &= t + f(t) \frac{\frac{f'(t)}{\sqrt{1+f'^2(t)}}}{1 + \frac{1}{\sqrt{1+f'^2(t)}}} = t + f(t) \frac{\sin \theta_f(t)}{1 + \cos \theta_f(t)} = t + f(t)\varphi(t), \\ y(t) &= f(t) \frac{1}{1 + \frac{1}{\sqrt{1+f'^2(t)}}} = f(t) \frac{1}{1 + \cos \theta_f(t)} = \frac{1}{2} f(t)(1 + \varphi^2(t)). \end{aligned} \quad (3.4)$$

Exercise 1. Find a geometric argument to conclude (3.4). Hint: see Subsection 2.1 and Figure 2.

Lemma 1. For any $t \in \mathbb{R}$: $y'(t) = x'(t)\varphi(t)$.

Proof. Using (3.1) and (3.4)

$$\begin{aligned} x'(t) &= 1 + \tan \theta_f(t)\varphi(t) + f(t)\varphi'(t) \stackrel{(3.3)}{=} 1 + \frac{2\varphi^2(t)}{1 - \varphi^2(t)} + f(t)\varphi'(t) \\ &= \frac{1 + \varphi^2(t)}{1 - \varphi^2(t)} + f(t)\varphi'(t) \\ y'(t) &= \frac{1}{2} (\tan \theta_f(t)(1 + \varphi^2(t)) + 2f(t)\varphi(t)\varphi'(t)) \\ &\stackrel{(3.3)}{=} \frac{\varphi(t)}{1 - \varphi^2(t)} (1 + \varphi^2(t)) + f(t)\varphi(t)\varphi'(t) = \varphi(t) \left(\frac{1 + \varphi^2(t)}{1 - \varphi^2(t)} + f(t)\varphi'(t) \right) \end{aligned} \quad (3.5)$$

as was to be proved. $\square$

Theorem 1. Let $x: \mathbb{R} \rightarrow \mathbb{R}$ and $y: \mathbb{R} \rightarrow \mathbb{R}$ be continuously differentiable functions. They give the parametric expression of an equidistant function if and only if

(EF1) for any $t \in \mathbb{R}$: $y(t) > 0$,

(EF2) for any $t \in \mathbb{R}$: $-1 < \frac{x(t)-t}{y(t)} < 1$ and the function $t \mapsto \frac{x(t)-t}{y(t)}$ is monotone increasing,

(EF3)

$$y'(t) \left(1 + \frac{y(t)}{\sqrt{y^2(t) - (x(t)-t)^2}} \right) - x'(t) \frac{x(t)-t}{\sqrt{y^2(t) - (x(t)-t)^2}} = 0.$$

Proof. The necessity of (EF1) is clear. (EF2) follows from (3.1) and the convexity of the function f . (EF3) is an equivalent formulation of Lemma 1:

$$y'(t) - x'(t)\varphi(t) = 0 \Rightarrow y'(t) - x'(t) \frac{\sin \theta_f(t)}{1 + \cos \theta_f(t)} = 0 \quad (3.6)$$

because of (3.3). Therefore

$$y'(t) \left(1 + \frac{1}{\cos \theta_f(t)} \right) - x'(t) \tan \theta_f(t) = 0 \quad (3.7)$$

and we have the necessity of (EF3) by (3.1). To see the converse statement suppose that (EF1) - (EF3) are satisfied and let us define the function

$$f(t) = \int_0^t \frac{x(s) - s}{\sqrt{y^2(s) - (x(s) - s)^2}} ds + y(0) + \sqrt{y^2(0) - x^2(0)}. \quad (3.8)$$

Using (EF3), a simple calculation shows that the derivative of the function

$$y(t) + \sqrt{y^2(t) - (x(t) - t)^2} - f(t) \quad (3.9)$$

is zero. By definition (3.8)

$$y(t) + \sqrt{y^2(t) - (x(t) - t)^2} - f(t) = y(0) + \sqrt{y^2(0) - x^2(0)} - f(0) = 0$$

for any $t \in \mathbb{R}$ and, consequently,

$$y(t) + \sqrt{y^2(t) - (x(t) - t)^2} = f(t). \quad (3.10)$$

Therefore f is a positively valued function with monotone increasing derivative:

$$f'(t) = \frac{x(t) - t}{\sqrt{y^2(t) - (x(t) - t)^2}} = \tan \left(\arcsin \frac{x(t) - t}{y(t)} \right) \quad (3.11)$$

because of (3.1) and (EF2). Formula (3.11) implies that

$$\frac{1}{\sqrt{1 + f'^2(t)}} = \frac{\sqrt{y^2(t) - (x(t) - t)^2}}{y(t)} \quad (3.12)$$

and, by substituting in (3.10) and (3.11), it follows that

$$y(t) = \frac{f(t) \sqrt{1 + f'^2(t)}}{1 + \sqrt{1 + f'^2(t)}} \quad \text{and} \quad x(t) = t + \frac{f(t) f'(t)}{1 + \sqrt{1 + f'^2(t)}}, \quad (3.13)$$

i.e. they are just the parametrization of E_f in the sense of (2.4). $\square$

3.1. Inclination angle and curvature

Let $\theta_e(t)$ be the inclination angle of the equidistant function at the parameter $t \in \mathbb{R}$. This means that $\theta_e(t)$ is the inclination angle belonging to the point $(x(t), y(t))$; it is given by the polar coordinate expression of the tangent vector:

$$x'(t) = v_e(t) \cos \theta_e(t) \quad \text{and} \quad y'(t) = v_e(t) \sin \theta_e(t),$$

where $v_e(t) := \sqrt{x'^2(t) + y'^2(t)}$.

Theorem 2. For any $t \in \mathbb{R}$: $\theta_e(t) = \frac{\theta_f(t)}{2}$

Proof. By Lemma 1

$$\begin{aligned} v_e(t) \sin \theta_e(t) &= y'(t) = x'(t) \varphi(t) = v_e(t) \cos \theta_e(t) \varphi(t) \\ &= v_e(t) \cos \theta_e(t) \tan \frac{\theta_f(t)}{2}. \end{aligned}$$

Therefore

$$\varphi(t) = \tan \frac{\theta_f(t)}{2} = \tan \theta_e(t) \Rightarrow \theta_e(t) = \frac{\theta_f(t)}{2}$$

as was to be proved. $\square$

Lemma 2. *The parametrization (2.4) is regular because there exists a positive number ε such that $x'(t) \geq \varepsilon > 0$ for any $t \in \mathbb{R}$, i.e. x is strictly monotone increasing and its range is the entire real line.*

Proof. According to Theorem 2,

$$-\pi/4 < \theta_e(t) < \pi/4.$$

Since $x'(t) = v_e(t) \cos(\theta_e(t))$,

$$\frac{v_e(t)}{\sqrt{2}} \leq x'(t) \leq v_e(t),$$

i.e. x' is non-negative. If $x'(t) = 0$ then (EF3) implies that $y'(t) = 0$. Differentiating

$$\frac{f'(t)}{\sqrt{1+f'^2(t)}} = \frac{x(t)-t}{y(t)} \quad (3.14)$$

we have

$$\frac{f''(t) \left(\sqrt{1+f'^2(t)} - \frac{f'^2(t)}{\sqrt{1+f'^2(t)}} \right)}{1+f'^2(t)} = \frac{(x'(t)-1)y(t) - (x(t)-t)y'(t)}{y^2(t)} \quad (3.15)$$

and, consequently,

$$\frac{f''(t)}{\sqrt{(1+f'^2(t))^3}} = -\frac{1}{y(t)} < 0 \quad (3.16)$$

provided that $x'(t) = y'(t) = 0$. This contradicts to the convexity of f . Since $-1 < \varphi(t) < 1$, the absolute value of $y'(t)$ is dominated by $x'(t)$ in the sense of Lemma 1. On the other hand formula (3.15) gives a contradiction like (3.16) by a simple continuity argument provided that $x'(t)$ is close enough to the zero for some $t \in \mathbb{R}$. Therefore $x'(t)$ is uniformly bounded by 0, i.e. there exists a positive ε such that $x'(t) \geq \varepsilon > 0$ for any $t \in \mathbb{R}$ and the range of the function is the entire real line because of the uniform lower bound for the local rate of increasing of the function. $\square$

For the computation of the curvature we use the following formulas:

$$\kappa_f(t) = \frac{\theta'_f(t)}{v_f(t)} \text{ and } \kappa_e(t) = \frac{\theta'_e(t)}{v_e(t)}. \quad (3.17)$$

By the previous theorem,

$$\kappa_e(t) = \frac{\theta'_e(t)}{v_e(t)} = \frac{\theta'_f(t)}{2v_e(t)} = \frac{1}{2} \frac{v_f(t)}{v_e(t)} \kappa_f(t), \quad (3.18)$$

where

$$v_f(t) = \sqrt{1 + f'^2(t)} = \frac{1}{\cos \theta_f(t)} = \frac{1 + \varphi^2(t)}{1 - \varphi^2(t)}, \quad (3.19)$$

$$\begin{aligned} \kappa_f(t) &= \frac{\theta'_f(t)}{v_f(t)} = 2 \frac{(\arctan \varphi)'(t)}{v_f(t)} = 2 \frac{\varphi'(t)}{1 + \varphi^2(t)} \frac{1}{v_f(t)} \\ &\stackrel{(3.19)}{=} 2\varphi'(t) \frac{1 - \varphi^2(t)}{(1 + \varphi^2(t))^2}, \end{aligned} \quad (3.20)$$

$$v_e(t) = \sqrt{x'^2(t) + y'^2(t)} = \sqrt{x'^2(t)(1 + \varphi^2(t))} = x'(t) \sqrt{1 + \varphi^2(t)} \quad (3.21)$$

because of Lemma 1 and Lemma 2. As formula (3.5) shows

$$v_e(t) = \left(\frac{1 + \varphi^2(t)}{1 - \varphi^2(t)} + f(t)\varphi'(t) \right) \sqrt{1 + \varphi^2(t)}. \quad (3.22)$$

Therefore

$$\kappa_e(t) = \frac{1}{2} \frac{\frac{1 + \varphi^2(t)}{1 - \varphi^2(t)}}{\left(\frac{1 + \varphi^2(t)}{1 - \varphi^2(t)} + f(t)\varphi'(t) \right) \sqrt{1 + \varphi^2(t)}} 2\varphi'(t) \frac{1 - \varphi^2(t)}{(1 + \varphi^2(t))^2}, \quad (3.23)$$

i.e.

$$\kappa_e(t) = \frac{\varphi'(t)}{(1 + \varphi^2(t))^{3/2} \left(\frac{1 + \varphi^2(t)}{1 - \varphi^2(t)} + f(t)\varphi'(t) \right)} \quad (3.24)$$

Table 1 contains the numerical values of the inclination angle, the magnitude of the velocity and the curvature in case of the function

$$\begin{aligned} f(t) &= t^2 + 1 \Rightarrow \tan \theta_f(t) = f'(t) = 2t, \\ \varphi(t) &= \tan \frac{\arctan(2t)}{2} \quad \text{and} \quad \varphi'(t) = \frac{1}{\cos^2 \frac{\arctan(2t)}{2}} \frac{1}{1 + 4t^2}. \end{aligned}$$

3.2. An estimation for the arclength between equidistant points

The arclength between the parameters $a < b$ can be computed as

$$\mathcal{L}(E_f)_a^b = \int_a^b v_e(t) dt;$$

note that it is the arclength between the equidistant points $(x(a), y(a))$ and $(x(b), y(b))$. Since $-1 < \varphi(t) < 1$

$$\int_a^b \frac{1 + \varphi^2(t)}{1 - \varphi^2(t)} + f(t)\varphi'(t) dt \leq \mathcal{L}(E_f)_a^b \leq \sqrt{2} \int_a^b \frac{1 + \varphi^2(t)}{1 - \varphi^2(t)} + f(t)\varphi'(t) dt$$

TABLE 1. Some numerical values

	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$
$x(t)$	0	2.2361	5.9039	11.4713	19.0073
$y(t)$	$\frac{1}{2}$	1.3820	4.0240	8.5881	15.1241
$\theta_e(t)$	0	0.5536	0.6629	0.7028	0.7232
$v_e(t)$	2	3.2785	5.8316	8.5804	11.3750
$\kappa_e(t)$	$\frac{1}{2}$	0.0610	0.0101	0.0031	0.0014

because of (3.22). Here

$$\int_a^b f(t)\varphi'(t) dt = f(b)\varphi(b) - f(a)\varphi(a) - \int_a^b f'(t)\varphi(t) dt,$$

$$\int_a^b f'(t)\varphi(t) dt = \int_a^b \varphi(t) \tan \theta_f(t) dt \stackrel{(3.3)}{=} \int_a^b \frac{2\varphi^2(t)}{1-\varphi^2(t)} dt$$

and, consequently,

$$b - a + f(b)\varphi(b) - f(a)\varphi(a) \leq \mathcal{L}(E_f)_a^b \leq \sqrt{2}(b - a + f(b)\varphi(b) - f(a)\varphi(a)).$$

Example 1. If $f(t) = 1 + t^2$ then, by using the data of Table 1, we have that

$$4 + 17 \tan \frac{0.7232}{2} \approx 10.4299 \leq \mathcal{L}(E_f)_0^4 \leq 14.7501 \approx \sqrt{2} \left(4 + 17 \tan \frac{0.7232}{2} \right).$$

4. THE CHARACTERIZATION OF THE EQUIDISTANT FUNCTIONS

In what follows we are looking for the equidistant function in the form

$$y(t) = G(x(t)), \text{ where } G : \mathbb{R} \mapsto \mathbb{R}$$

Lemma 3. *G is a positive valued, twice continuously differentiable convex function.*

Proof. Using Lemma 2, it follows that $G(t) = y \circ x^{-1}(t)$. Therefore

$$G'(t) = y'(x^{-1}(t)) \cdot \frac{1}{x'(x^{-1}(t))} = \varphi \circ x^{-1}(t)$$

because of Lemma 1. Since f is twice continuously differentiable and

$$\varphi = \tan \frac{\arctan f'}{2},$$

we have that $\phi \circ x^{-1}$ is continuously differentiable. The convexity of G follows from the curvature formula (3.18) with $\kappa_f \geq 0$ because of the convexity of the function f . $\square$

Exercise 2. By Lemma 3 we can take the sequence G_n , where $G_0 := f$ and $G_{n+1}: \mathbb{R} \rightarrow \mathbb{R}$ is the equidistant function with respect to G_n . Prove that $G_n(t)$ tends to zero at each $t \in \mathbb{R}$. Hint: see Theorem 2.

Using that

$$y'(t) = G'(x(t)) \cdot x'(t)$$

and $x'(t) > 0$, condition (EF3) says that

$$G'(x(t)) \cdot \left(G(x(t)) + \sqrt{G^2(x(t)) - (x(t) - t)^2} \right) = x(t) - t, \quad (4.1)$$

$$G'(x(t)) \cdot \sqrt{G^2(x(t)) - (x(t) - t)^2} = x(t) - t - G'(x(t)) \cdot G(x(t)). \quad (4.2)$$

Taking the square of both sides

$$-G'^2(x(t)) \cdot (x(t) - t)^2 = (x(t) - t)^2 - 2(x(t) - t) \cdot G'(x(t)) \cdot G(x(t)) \quad (4.3)$$

We have two possible cases: if $x(t) \neq t$ then

$$-G'^2(x(t)) \cdot (x(t) - t) = (x(t) - t) - 2G'(x(t)) \cdot G(x(t)). \quad (4.4)$$

If $x(t) = t$ then, by condition (EF3) in Theorem 1, $y'(t) = 0$. Since

$$y'(t) = G'(x(t))x'(t)$$

we have that $G'(x(t)) = 0$ because of Lemma 2. This means that equation (4.4) holds in this exceptional case too. Therefore

$$x(t) - t = \frac{2G'(x(t)) \cdot G(x(t))}{1 + G'^2(x(t))} \Rightarrow x(t) - \frac{2G'(x(t)) \cdot G(x(t))}{1 + G'^2(x(t))} = t. \quad (4.5)$$

Using the auxiliary function

$$H(x) := x - \frac{2G'(x) \cdot G(x)}{1 + G'^2(x)} \quad (4.6)$$

it follows that $H(x(t)) = t$, i.e. $H^{-1}(t) = x(t)$; note that $x'(t) > 0$ implies the existence of the inverse function. In the sense of Lemma 2 the inverse of H must be of the form $H^{-1}: \mathbb{R} \rightarrow \mathbb{R}$.

Theorem 3. *A twice continuously differentiable function $G : \mathbb{R} \rightarrow \mathbb{R}$ is an equidistant function if and only if*

$$H(x) := x - \frac{2G'(x) \cdot G(x)}{1 + G'^2(x)} \quad (4.7)$$

has a strictly monotone increasing inverse function of the form $H^{-1} : \mathbb{R} \rightarrow \mathbb{R}$ and the parametric expression

$$x(t) := H^{-1}(t) \text{ and } y(t) := G \circ H^{-1}(t)$$

of G satisfies conditions (EF1), (EF2) and (EF3) of Theorem 1.

Remark 1. The substitution $t = H(x)$ in (EF1) implies that G must be a positive valued function. We also have that G is convex and $-1 < G'(x) < 1$ for any $x \in \mathbb{R}$ because of the monotonicity property (EF2); see the behavior of the function $\frac{2x}{1+x^2}$ on the real line. Since the parametrization in Theorem 3 is based on condition (EF3), it is automatically satisfied: a twice continuously differentiable, positive valued convex function $G : \mathbb{R} \rightarrow \mathbb{R}$ is an equidistant function if and only if

$$H(x) := x - \frac{2G'(x) \cdot G(x)}{1 + G'^2(x)} \quad (4.8)$$

has a strictly monotone increasing inverse function of the form $H^{-1} : \mathbb{R} \rightarrow \mathbb{R}$ and $-1 < G'(x) < 1$ for any $x \in \mathbb{R}$. Theorem 3 gives the solution of the problem posed in [6] for a special class of equidistant sets (equidistant functions): *characterize all closed sets of the plane that can be realized as the equidistant set of two connected disjoint closed sets.*

5. CONCLUDING REMARKS

5.1. An open problem

It can be easily seen that conditions (EF1) - (EF3) are invariant under orientation preserving transformations of the parameter. Can we generalize the characterization of equidistant functions as plane curves in terms of conditions that are invariant under (orientation preserving) Euclidean motions?

5.2. Generalizations and extensions of the process

As a simple conclusion of the geometric interpretation of equidistancy (see Subsection 2.1) we have that the equidistant function is the envelope of the family of parabolas such that the first coordinate axis is the common directrix and the focal point is running through the graph of the function f . Since the equidistant function takes exactly the form of a parabolic arc in the outer normal cone belonging to any singularity on the graph, the singularities (if they exist at all) do not affect the equidistant function. Moreover, any convex function can be approximated by convex functions of class $\mathcal{C}^2$, i.e. the regularity conditions for the function f can be

used without loss of generality; see also Theorem 11 in [6] (continuity property of equidistant sets). Another technical simplification is to take the entire real line as the domain of the function f . It can be easily seen that if the domain is bounded from above (for example), i.e. $f: (-\infty, b) \rightarrow \mathbb{R}$ then the equidistant function takes the form of an infinite parabolic arc for any $t > \lim_{t \rightarrow b-} x(t)$. We can also take most of unbounded closed convex set in the plane as the epigraph of a convex function by choosing the first coordinate axis parallel to the orthogonal complement of a ray contained in the set. Finally the process presented in the paper gives a local approximation of the equidistant sets in some cases as follows. Consider two disjoint closed convex sets K and L in the plane. For the sake of simplicity we suppose that their boundaries are of class $\mathcal{C}^2$. If P is an equidistant point of K and L then there exist uniquely determined closest points on K and L to P : $A \in K$ and $B \in L$. The tangent line k of K at the point A provides a linear approximation around A of K containing the closest points of K to the equidistant points around P because of the continuity of the orthogonal projection of the outer points onto the boundary of a closed convex set. If the perpendicular line from B to k intersects the interior of L then we can use the local version of the process to approximate the equidistant set of K and L by an equidistant function around P . The previous argument motivates us to use the $\mathcal{C}^2$ boundary in a more effective way by substituting the coordinate axis k with the osculating circle to the boundary of K at the point A . This circle provides a higher order contact than the tangent line. Therefore we propose the solution of the problem of the characterization of equidistant sets belonging to the epigraph of a positive valued convex function and the circle given by $x^2 + (y + R)^2 = R^2$.

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Authors' addresses

Cs. Vincze

Csaba Vincze, Univ. of Debrecen, Inst. of Math., H-4002 Debrecen, P.O.Box 400, Debrecen, Hungary

E-mail address: `csvincze@science.unideb.hu`

A. Varga

Adrienn Varga, Univ. of Debrecen, Fac. of Eng., H-4002 Debrecen, P.O.Box 400, Debrecen, Hungary

E-mail address: `vargaa@eng.unideb.hu`

M. Oláh

Márk Oláh, Univ. of Debrecen, BSC mathematics, H-4002 Debrecen, P.O.Box 400, Debrecen, Hungary

E-mail address: `olma4000@euromail.hu`

L. Fórián

L. Fórián, Univ. of Debrecen, BSC mathematics, H-4002 Debrecen, P.O.Box 400, Debrecen, Hungary

E-mail address: `laci.forian@gmail.com`



A SMALLER COVER FOR CLOSED UNIT CURVES

WACHARIN WICHIRAMALA

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Abstract. Forty years ago Schaer and Wetzel showed that a $\frac{1}{\pi} \times \frac{1}{2\pi} \sqrt{\pi^2 - 4}$ rectangle, whose area is about 0.12274, is the smallest rectangle that is a cover for the family of all closed unit arcs. More recently Füredi and Wetzel showed that one corner of this rectangle can be clipped to form a pentagonal cover having area 0.11224 for this family of curves. Here we show that then the opposite corner can be clipped to form a hexagonal cover of area less than 0.11023 for this same family. This irregular hexagon is the smallest cover currently known for this family of arcs.

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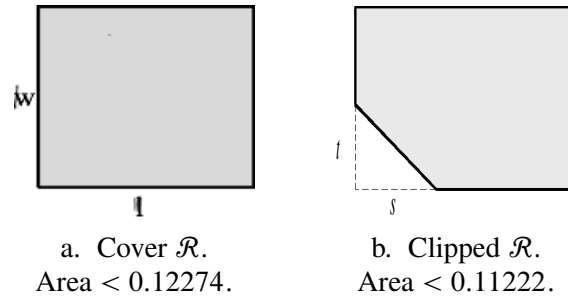
Keywords: covering of unit arcs, covering by convex sets, worm problem

1. INTRODUCTION

Forty years ago Schaer and Wetzel [7] (and independently Chakarian and Klamkin [2]) showed that the smallest rectangular region $\mathcal{R}$ that is a cover for the family $\mathcal{F}_0$ of all closed unit arcs in the Euclidean plane, that is to say, contains a congruent (i.e., an isometric) copy of each closed unit arc, is $l = \frac{1}{\pi}$ by $w = \sqrt{\frac{1}{4} - \frac{1}{\pi^2}}$ (Figure 1a.) This rectangle has area about 0.12274.

The underlying problem here is a variant for closed unit arcs of a well-known unsolved problem posed in 1966 by Leo Moser (see [5, pp. 211, 218-219]): find the area of the smallest (convex) region in the plane that contains a congruent (i.e., isometric) copy of every arc of unit length. The existence of a convex cover of least area α_2 for the family $\mathcal{F}_0$ of all closed unit arcs follows from standard compactness arguments, but its uniqueness is not known. Problems of these kinds are closely related to the well-known Lebesgue Universal Cover problem (see [3, §D15]), and they seem equally intractable.

Both [7] and [2] made the elementary observation that the least area α_2 must be greater than the smallest convex hull of a circle of unit circumference and a unit line segment, a set having area 0.09632. Using geometric methods Füredi and Wetzel [4] recently raised the lower bound to 0.09670, and they showed that the pentagonal region formed by suitably clipping one corner of $\mathcal{R}$ (Figure 1b) is a cover for $\mathcal{F}_0$ with area less than 0.11222. They achieved a further slight reduction in the upper bound

FIGURE 1. Two covers for $\mathcal{F}_0$.

by replacing a portion of the clipping line segment by a short elliptic arc, forming a curvilinear convex hexagonal cover having area less than 0.11213. These are the best published bounds for α_2 at present:

$$0.09670 < \alpha_2 < 0.11213.$$

2. A SMALLER COVER

In recent years computational methods have increasingly been employed to attack geometric questions. For example, Brass and his student Sharifi [1] used a grid search numerical method to improve the known lower bound for the Lebesgue Universal Cover problem. In this article we employ numerical convex optimization to reduce the known upper bound for α_2 by nearly 1.7 per cent. More precisely, we establish the following theorem.

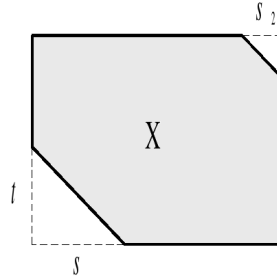
Theorem 1. *Let $s = 0.1420171$, $t = 0.1481552$, and $s_2 = 0.0617141$, and using these data, let $\mathfrak{X}$ be the rectangular region $\mathcal{R}$ clipped by two parallel line segments as pictured in Figure 2. Then $\mathfrak{X}$ is a cover for the family $\mathcal{F}_0$ of all closed curves of unit length, and its area is about 0.1102299.*

The area of $\mathfrak{X}$ is $lw - \frac{1}{2}st(1 + (s_2/s)^2) \approx 0.1102299$. Showing that $\mathfrak{X}$ is a cover for $\mathcal{F}_0$ is the objective of this article. The value of s_2 is chosen so that every closed arc that cannot be covered by $\mathfrak{X}$ must be longer than 1.00001 numerically.

3. PLACING A CLOSED ARC IN THE RECTANGLE

From [8], it suffices for $\mathfrak{X}$ to cover all simple closed unit arcs. From [6], every closed unit arc is contained in the convex hull of another convex closed unit arc. Since $\mathfrak{X}$ is convex, we may assume that we are dealing with convex closed unit arcs. Since every convex closed unit arc has diameter at least $1/\pi$, we may place such arc in the rectangle $\mathcal{R}$ so that it touches the upper and lower boundaries [7]

For convenient, let $\theta = \arctan m$ where $m = t/s$. Let $p_1, p_2, \dots, p_8$ be the (chosen) lower left, bottom, lower right, right, upper right, top, upper left, left points touched



The cover $\mathfrak{X}$.
Area < 0.11023 .

FIGURE 2. A smaller cover for $\mathcal{F}_0$.

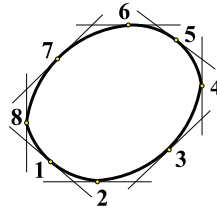


FIGURE 3. All touched points $p_1, p_2, \dots, p_8$.

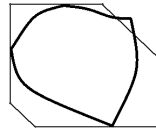


FIGURE 4. The situation OUTL5.

by support lines (coming) from angles $-90^\circ - \theta, -90^\circ, \dots, 180^\circ$ (with slopes $-m, 0, m, \infty, -m, 0, m, \infty$). The points p_i appear in counter clockwise order since the closed arc is simple. Note that 2 consecutive points, including p_1 and p_8 , may coincide (see Figure 3). For $i = 1, \dots, 8$, let (x_i, y_i) be the coordinate of p_i . Let x_L and x_R be the left and right x-coordinate of a concerned rectangle, called **box**. As we move the box to the right, we end up with $x_L = x_8$. For the opposite direction, we have $x_R = x_4$.

When the box is moving to the right (the arc is relatively moving to the left), the situation that p_5 is in the big corner is called OUTL5 (see Figure 4), and the situation that p_7 is in the small corner is called outL7 (see Figure 5). When the box is moving to the left (the arc is relatively moving to the right), the situation that p_7 is in the big corner is called OUTR7 (see Figure 7), and the situation that p_5 is in the small corner

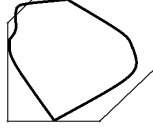


FIGURE 5. The situation outL7.

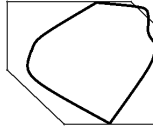


FIGURE 6. The situation outR5.

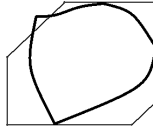
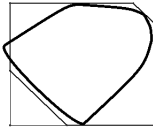


FIGURE 7. The situation OUTR7.

FIGURE 8. The situation that p_5 is on the border of the small corner and the arc leaves the box.

is called outR5 (see Figure 6). When the box is moving to the situation that p_5 is on the border of the small corner, the arc may leave the box (see Figure 8).

4. THE MAIN THEOREM

To prove Theorem 1, we start by suppose that a convex closed unit arc γ cannot be covered. We will finally find a contradiction, mostly by that γ is longer than 1.00001, using numerical optimization.

When we translate the box horizontally, consider a big corner and the opposite small corner, the arc is either passing through the big one or the small one. When the arc is placed to the left, the points p_3 and p_5 may visit the big corners. Similarly for the right justification, the point p_1 and p_7 may visit the big corners. We then divide into cases as follows (see Figure 9).

Case 1: OUTL3 and OUTL5.

Case 2: OUTL5 and (not OUTL3, then) outL7.

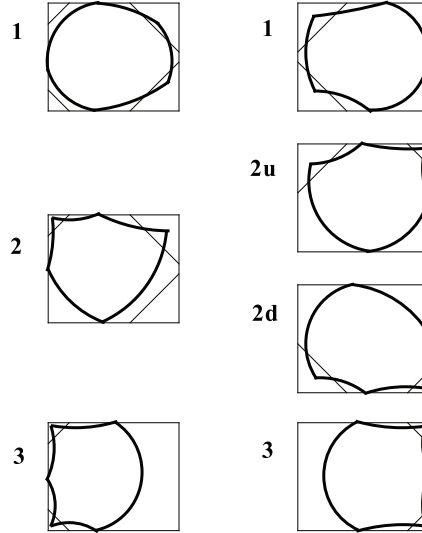


FIGURE 9. All cases.

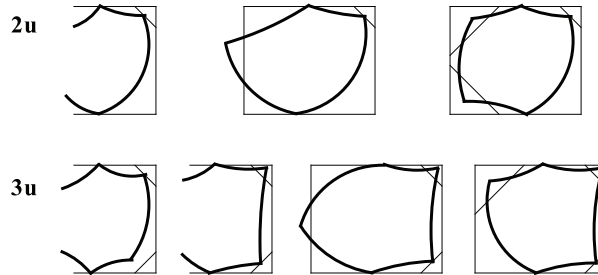


FIGURE 10. Translations in subcases ending with .2u and .3u.

Case 3: (not OUTL3 nor OUTL5, then) outL1 and outL7.

Now we are going into each big case where we will encounter many subcases.

Case 1: OUTL3 and OUTL5. Now consider the right justification. Due to the symmetry of case 1, we have the following subcases.

Subcase 1.1: OUTR7 and OUTR1. From [4], $l(\gamma) \geq l(p_1 p_2 \dots p_8) > 1$.

Subcase 1.2u: (2up) OUTR7 and (not OUTR1, then) outR5. Now we translate the box so that p_5 is on the border of the small corner (see Figure 10). Consequently $x_R = \frac{w-t_2-y_5}{-m} + x_5$, called $\mathbf{x}_{R5}$. Then the arc leaves these covers on the left side. Suppose the arc leave the box on the left side, i.e. $x_8 < x_L$, from numerical work 1.2uL, its length is at least 1.001. Hence the arc does not leave the box and then it leaves the cover from corners. Thus it is OUT7 and OUT1. Note that we may regard OUTL5 as OUT5 with $x_L = x_8$ and regard OUTR7 as OUT7 with $x_R = x_4$. Here

OUT7 and OUT1 are with specific x_R above. From numerical work 1.2uC, the length is at least 1.02231.

Subcase 1.3: (not OUTR7 nor OUTR1, then) outR3 and outR5. Now we translate the box so that p_5 is on the border of the small corner and (without of loss of generality) p_3 does not visit the small corner. Suppose the arc does not leave the box, it leaves the cover through corners. Now it is OUTL7 and OUTL1 with this x_R . Now translate the box so that p_3 is on the border of the small corner where $x_R = \frac{t_2 - y_3}{m} + x_3$, called $\mathbf{x_{R3}}$. Here p_5 visits the small corner and the arc does not leave the left of the box. Hence it is OUT7 with this x_R . From numerical work 1.3C, the length is at least 1.00852. Now suppose the arc leaves the box when $x_R = x_{R5}$. We translate the box so that p_3 is on the border of the small corner. If the arc does not leave the box, from numerical work 1.3LC, the length is at least 1.00994. If the arc leaves the box, from numerical work 1.3LL, the length is at least 1.04008.

Case 2: OUTL5 and (not OUTL3, then) outL7. We start by translating so that p_7 is on the border of the left corner. Here $x_L = \frac{w - t_2 - y_7}{m} + x_7$, called $\mathbf{x_{L7}}$. Subcases are according to right justification.

Subcase 2.2u: (2up) OUTR7 and (not OUTR1, then) outR5. Now we translate so that $x_R = x_{R5}$. If both translations keep the arc in side the boxes, we have OUT3 and OUT5 when $x_L = x_{L7}$ and have OUT1 and OUT7 when $x_R = x_{R5}$. From numerical work 2C.2uC, the length is at least 1.0093. If the arc leaves the box when $x_R = x_{R5}$, we have OUT1 and OUT7. (2C.2uL) The length is at least 1.01069. If the arc leaves the box when $x_L = x_{L7}$ and when $x_R = x_{R5}$, (2R.2uL) the length is at least 1.03344.

Subcase 2.2d: (2down) OUTR1 and (not OUTR7, then) outR3. From 2.2d, the length is at least 1.00584.

Subcase 2.3: (not OUTR7 nor OUTR1, then) outR3 and outR5. When we translate so that $x_R = x_{R5}$. Now we consider whether p_3 is in the small corner.

Subsubcase 2.3u: p_3 is not in the small corner (see Figure 10). We will translate so that $x_L = x_{L7}$ and so that $x_R = x_{R5}$. If the arc are in both boxes, we have OUT3 and OUT5 when $x_L = x_{L7}$. (2C.3uC) The length is at least 1.00596. If the arc leaves the box when $x_R = x_{R5}$, (2.3uL) the length is at least 1.00318. If the arc leaves the box when $x_L = x_{L7}$, (2R.3u) the length is at least 1.00392.

Subsubcase 2.3d: p_3 is in the small corner. So we may think that when we translate so that $x_R = x_{R3}$, we have p_5 not in the small corner. Similarly to the previous subsubcase, from 2C.3dC, 2.3dL and 2R.3d, we have the length is at least 1.00854, 1.00504 and 1.00392 respectively.

Case 3: (not OUTL3 nor OUTL5, then) outL1 and outL7. We now have to consider only when the right justification causes (not OUTR7 nor OUTR1, then) outR3 and outR5. We will divide into subcase similar to the subcase 2.3 but with fewer subcases due to symmetry.

Subcase 3.3u: From 3C.3uC (OUT3 and OUT5 when $x_L = x_{L7}$, OUT1 and OUT7 when $x_R = x_{R5}$) and 3.3uL ($x_8 < x_4 - l$ when $x_R = x_{R5}$), the length is at least 1.00001 and 1.05382 respectively.

Subcase 3.3d: From 3C.3dC and 3.3dL, the length is at least 1.00001 and 1.05367 respectively.

In every case, the length is greater than 1 (or numerically greater than 1.00001). This is a contradiction. Therefore $\mathfrak{X}$ is a cover for closed unit arcs. The theorem is proved completely.

5. FURTHER IMPROVEMENTS

Repeating all the numerical process with threshold smaller than 1.00001 would yield a smaller cover.

We tried to make a significant improvement by subdividing the subsubcases according to numerical works 3C.3uC and 3C.3dC using the same method in subcase 1.3. The result is negative.

As the cover has to accommodate the straight back-and-forth segment of length $\frac{1}{2}$, it is clearly that there is no way to improve to another smaller cover by cutting the third corner (clipping 3 corners).

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APPENDIX: NUMERICAL WORK

Each numerical minimization used here is called a *convex programming* where we minimize a convex function over a convex domain. It is theoretically confirmed that the minimum may be obtained numerically with high accuracy.

We numerically compute those minimums using Wolfram Mathematica. The Mathematica notebook file can be found at

www.math.sc.chula.ac.th/wacharin/optimization/closed_arcs and at <https://www.researchgate.net/publication/322266021>

A smaller cover for closed unit curves.

The explanation can be found at <http://arxiv.org/abs/1801.08405> [9] and at https://www.researchgate.net/publication/322266021_A_smaller_cover_for_closed_unit_curves with DOI 10.13140/RG.2.2.24191.51364.

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Author’s address

Wacharin Wichiramala

Chulalongkorn University, Faculty of Science, Department of Mathematics and Computer Science,
254 Phayathai Road, Pathumwan, Bangkok 10330, Thailand

E-mail address: wacharin.w@chula.ac.th



SOME INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE FOR s -GEOMETRICALLY CONVEX FUNCTIONS

HONG-PING YIN, JING-YU WANG, AND FENG QI

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Abstract. In the paper, the authors present some integral inequalities of the Hermite–Hadamard type for s -geometrically convex functions and for the product of two s -geometrically convex functions.

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Keywords: integral inequality, Hermite–Hadamard type, s -geometrically convex function, product

1. INTRODUCTION

We recall the definitions of classical convex functions and geometrically convex functions.

Definition 1. Let $f : I \subseteq \mathbb{R} = (-\infty, \infty) \rightarrow \mathbb{R}$. If the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad (1.1)$$

is valid for all $x, y \in I$ and $\lambda \in [0, 1]$, then f is called the convex function on I ; if the inequality (1.1) reverses, then f is called the concave function on I .

Definition 2. Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \rightarrow \mathbb{R}_+$. If the inequality

$$f(x^\lambda y^{1-\lambda}) \leq f^\lambda(x) f^{1-\lambda}(y) \quad (1.2)$$

is sound for any $x, y \in I$ and $\lambda \in [0, 1]$, then f is called the geometrically convex function; if the inequality of (1.2) reverses, then f is called the geometrically concave function.

The concept of classical convex functions has been generalized or extended widely in recent decades. Some of them can be reformulated as follows.

Definition 3 ([3, 5]). Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}_0 = [0, \infty)$ and $s \in (0, 1]$. If the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$, then f is called the s -convex function on I .

Definition 4 ([14, 15]). Let $s \in (0, 1]$ and $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}_+$. If the inequality

$$f(x^\lambda y^{1-\lambda}) \leq f^{\lambda^s}(x) f^{(1-\lambda)^s}(y)$$

validates for any $x, y \in I$ and $\lambda \in [0, 1]$, then f is called the s -geometrically convex function on I .

For classical convex functions, we have the famous Hermite–Hadamard integral inequality below.

Theorem 1. Let $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on $[a, b]$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \quad (1.3)$$

If f is a concave function on I , then the inequality (1.3) reverses.

In the literature, there have existed some integral inequalities of the Hermite–Hadamard type on classical convex functions and s -convex functions. Some of them can be reformulated as follows.

Theorem 2 ([4, Theorem 2.2]). Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping and $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{8} (|f'(a)| + |f'(b)|).$$

Theorem 3 ([7, Theorems 2.3 and 2.4]). Let $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'|^p$ is s -convex on $[a, b]$ for some $s \in (0, 1]$ and $p > 1$, then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{16} \left(\frac{4}{p+1}\right)^{1/p} (|f'(a)| + |f'(b)|)$$

and

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{b-a}{4} \left(\frac{4}{p+1}\right)^{1/p} \left\{ [|f'(a)|^{p/(p-1)} \right. \\ &\quad \left. + 3|f'(b)|^{p/(p-1)}]^{1-1/p} + [3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}]^{1-1/p} \right\}. \end{aligned}$$

Theorem 4 ([6, Theorem 3]). Let $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'|^q$ is s -convex on $[a, b]$ for some $s \in (0, 1]$ and $q > 1$, then

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{b-a}{2} \left[\frac{q-1}{2(2q-1)} \right]^{1/p} \left(\frac{1}{s+1} \right)^{1/q} \\ &\quad \times \left\{ \left[|f'(a)|^q + \left| f'\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} + \left[|f'(b)|^q + \left| f'\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} \right\}. \end{aligned}$$

In recent decades, many new integral inequalities of the Hermite–Hadamard type for diverse new kinds of convex functions have been created and established. For detailed information, please refer to [1–4, 6–15] and closely related references therein.

In this paper, we will establish some new integral inequalities of the Hermite–Hadamard type for s -geometrically convex functions.

2. SOME NEW INTEGRAL INEQUALITIES

We now start out to establish some integral inequalities of the Hermite–Hadamard type for s -geometrically convex functions.

Theorem 5. *Let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an integrable function, $a, b \in I$ with $a < b$, and $s \in (0, 1]$. If f is an s -geometrically convex function, then*

$$f^{(1/2)^{1-s}}(\sqrt{ab}) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \leq [f(a)f(b)]^{1-s} L(f^s(a), f^s(b)),$$

where the logarithmic mean $L(u, v)$ is defined by

$$L(u, v) = \begin{cases} \frac{v-u}{\ln v - \ln u}, & u \neq v; \\ u, & u = v. \end{cases} \quad (2.1)$$

Proof. By changing the variable $x = a^t b^{1-t}$ for $t \in [0, 1]$ and by the s -geometric convexity, we have

$$\frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx = \int_0^1 f(a^t b^{1-t}) dt \leq \int_0^1 f^{ts}(a) f^{(1-t)s}(b) dt.$$

In [2], it was obtained that the inequality

$$\eta^{ts} \leq \eta^{st+1-s} \quad (2.2)$$

is valid for $\eta \geq 1$, $0 \leq t \leq 1$, and $0 < s \leq 1$. When $s \in (0, 1)$, the s -geometrically convex function satisfies $f(x) \geq 1$. Consequently, since $f(a), f(b) \geq 1$, we arrive at

$$\begin{aligned} \int_0^1 f^{ts}(a) f^{(1-t)s}(b) dt &\leq \int_0^1 f^{st+1-s}(a) f^{s(1-t)+1-s}(b) dt \\ &= [f(a)f(b)]^{1-s} L(f^s(a), f^s(b)). \end{aligned}$$

Due to $\sqrt{ab} = \sqrt{a^t b^{1-t} b^t a^{1-t}}$ for all $t \in [0, 1]$, by the s -geometric convexity, we find

$$f^{(1/2)^{1-s}}(\sqrt{ab}) \leq [f(a^t b^{1-t}) f(b^t a^{1-t})]^{1/2} \leq \frac{f(a^t b^{1-t}) + f(b^t a^{1-t})}{2}.$$

Integrating with respect to $t \in [0, 1]$ on both sides of the above inequality gains

$$f^{(1/2)^{1-s}}(\sqrt{ab}) \leq \frac{1}{2} \int_0^1 [f(a^t b^{1-t}) + f(b^t a^{1-t})] dt = \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx.$$

The proof of Theorem 5 is complete. $\square$

Theorem 6. Let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an integrable function, $a, b \in I$ with $a < b$, and $s \in (0, 1]$. If f is an s -geometrically convex function on $[a, b]$, then

$$\sqrt{ab} f^{(1/2)^{1-s}}(\sqrt{ab}) \leq \frac{1}{\ln b - \ln a} \int_a^b f(x) dx \leq [f(a)f(b)]^{1-s} L(af^s(a), bf^s(b)).$$

where $L(u, v)$ is the logarithmic mean defined by (2.1).

Proof. Taking $x = a^t b^{1-t}$ for $t \in [0, 1]$, using Definition 4, and employing the inequality (2.2) lead to

$$\begin{aligned} \frac{1}{\ln b - \ln a} \int_a^b f(x) dx &= \int_0^1 a^t b^{1-t} f(a^t b^{1-t}) dt \leq \int_0^1 a^t b^{1-t} f^{ts}(a) f^{(1-t)s}(b) dt \\ &\leq \int_0^1 a^t b^{1-t} f^{st+1-s}(a) f^{s(1-t)+1-s}(b) dt \\ &= [f(a)f(b)]^{1-s} L(af^s(a), bf^s(b)) \end{aligned}$$

and

$$\begin{aligned} \sqrt{ab} f^{(1/2)^{1-s}}(\sqrt{ab}) &= \int_0^1 \sqrt{a^t b^{1-t} b^t a^{1-t}} f^{(1/2)^{1-s}}(\sqrt{a^t b^{1-t} b^t a^{1-t}}) dt \\ &\leq \int_0^1 \sqrt{a^t b^{1-t} b^t a^{1-t}} [f(a^t b^{1-t}) f(b^t a^{1-t})]^{1/2} dt \\ &\leq \frac{1}{2} \int_0^1 [a^t b^t f(a^t b^{1-t}) + b^t a^{1-t} f(b^t a^{1-t})] dt \\ &= \frac{1}{\ln b - \ln a} \int_a^b f(x) dx. \end{aligned}$$

The proof of Theorem 6 is complete. $\square$

Theorem 7. Let $f, g : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be integrable functions, $a, b \in I$ with $a < b$, and $s_1, s_2 \in (0, 1]$. If f is an s_1 -geometrically convex function and g is an s_2 -geometrically convex function on $[a, b]$, then

$$\begin{aligned} f^{(1/2)^{1-s_1}}(\sqrt{ab}) g^{(1/2)^{1-s_2}}(\sqrt{ab}) &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)g(x)}{x} dx \\ &\leq [f(a)f(b)]^{1-s_1} [g(a)g(b)]^{1-s_2} L(f^{s_1}(a)g^{s_2}(a), f^{s_1}(b)g^{s_2}(b)), \end{aligned}$$

where $L(u, v)$ is the logarithmic mean defined by (2.1).

Proof. Letting $x = a^t b^{1-t}$ for $t \in [0, 1]$, using the s -geometric convexity, and utilizing the inequality (2.2) result in

$$\frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)g(x)}{x} dx = \int_0^1 f(a^t b^{1-t}) g(a^t b^{1-t}) dt$$

$$\begin{aligned}
 &\leq \int_0^1 f^{t^{s_1}}(a) f^{(1-t)^{s_1}}(b) g^{t^{s_1}}(a) g^{(1-t)^{s_2}}(b) dt \\
 &\leq \int_0^1 f^{s_1 t + 1 - s_1}(a) f^{s_2(1-t) + 1 - s_2}(b) g^{s_1 t + 1 - s_1}(a) g^{s_2(1-t) + 1 - s_2}(b) dt \\
 &= [f(a)f(b)]^{1-s_1} [g(a)g(b)]^{1-s_2} L(f^{s_1}(a)g^{s_2}(a), f^{s_1}(b)g^{s_2}(b)).
 \end{aligned}$$

For $t \in [0, 1]$, we have

$$\begin{aligned}
 &f^{(1/2)^{1-s_1}}(\sqrt{ab}) g^{(1/2)^{1-s_2}}(\sqrt{ab}) \\
 &= f^{(1/2)^{1-s_1}}(\sqrt{a^t b^{1-t} b^t a^{1-t}}) g^{(1/2)^{1-s_2}}(\sqrt{a^t b^{1-t} b^t a^{1-t}}) \\
 &\leq [f(a^t b^{1-t}) f(b^t a^{1-t}) g(a^t b^{1-t}) g(b^t a^{1-t})]^{1/2} \\
 &\leq \frac{f(a^t b^{1-t}) g(a^t b^{1-t}) + f(b^t a^{1-t}) g(b^t a^{1-t})}{2}.
 \end{aligned}$$

Integrating on both sides with respect to $t \in [0, 1]$ leads to

$$\begin{aligned}
 &f^{(1/2)^{1-s_1}}(\sqrt{ab}) g^{(1/2)^{1-s_2}}(\sqrt{ab}) \\
 &\leq \frac{1}{2} \int_0^1 [f(a^t b^{1-t}) g(a^t b^{1-t}) + f(b^t a^{1-t}) g(b^t a^{1-t})] dt \\
 &= \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)g(x)}{x} dx.
 \end{aligned}$$

The proof of Theorem 7 is complete. $\square$

Theorem 8. Let $f, g : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be integrable functions, $a, b \in I$ with $a < b$, and $s_1, s_2 \in (0, 1]$. If f is an s_1 -geometrically convex function and g is an s_2 -geometrically convex function on $[a, b]$, then

$$\begin{aligned}
 &\sqrt{ab} f^{(1/2)^{1-s_1}}(\sqrt{ab}) g^{(1/2)^{1-s_2}}(\sqrt{ab}) \leq \frac{1}{\ln b - \ln a} \int_a^b f(x)g(x) dx \\
 &\leq [f(a)f(b)]^{1-s_1} [g(a)g(b)]^{1-s_2} L(f^{s_1}(a)g^{s_2}(a), f^{s_1}(b)g^{s_2}(b)),
 \end{aligned}$$

where $L(u, v)$ is the logarithmic mean defined by (2.1).

Proof. Setting $x = a^t b^{1-t}$ for $t \in [0, 1]$ and making use of Definition 4 arrive at

$$\begin{aligned}
 &\frac{1}{\ln b - \ln a} \int_a^b f(x)g(x) dx = \int_0^1 a^t b^{1-t} f(a^t b^{1-t}) g(a^t b^{1-t}) dt \\
 &\leq \int_0^1 a^t b^{1-t} f^{t^{s_1}}(a) f^{(1-t)^{s_1}}(b) g^{t^{s_1}}(a) g^{(1-t)^{s_2}}(b) dt \\
 &\leq [f(a)f(b)]^{1-s_1} [g(a)g(b)]^{1-s_2} L(f^{s_1}(a)g^{s_2}(a), f^{s_1}(b)g^{s_2}(b))
 \end{aligned}$$

and

$$\begin{aligned}
 & \sqrt{ab} f^{(1/2)^{1-s_1}}(\sqrt{ab}) g^{(1/2)^{1-s_2}}(\sqrt{ab}) \\
 & \leq \int_0^1 (a^t b^{1-t} b^t a^{1-t})^{1/2} [f(a^t b^{1-t}) f(b^t a^{1-t}) g(a^t b^{1-t}) g(b^t a^{1-t})]^{1/2} dt \\
 & \leq \frac{1}{2} \int_0^1 [a^t b^{1-t} f(a^t b^{1-t}) g(a^t b^{1-t}) + b^t a^{1-t} f(b^t a^{1-t}) g(b^t a^{1-t})] dt \\
 & = \frac{1}{\ln b - \ln a} \int_a^b f(x) g(x) dx.
 \end{aligned}$$

The proof of Theorem 8 is thus complete. $\square$

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Authors' addresses

Hong-Ping Yin

College of Mathematics, Inner Mongolia University for Nationalities, Tongliao, Inner Mongolia, 028043, China

E-mail address: hongpingyin@qq.com

Jing-Yu Wang

College of Mathematics, Inner Mongolia University for Nationalities, Tongliao, Inner Mongolia, 028043, China

E-mail address: jing-yu.wang@qq.com

Feng Qi

Corresponding author, Institute of Mathematics, Henan Polytechnic University, Jiaozuo, Henan, 454010, China, Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin, 300387, China and College of Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan, 610054, China

E-mail address: qifeng618@gmail.com; qifeng618@hotmail.com

EDITOR-IN-CHIEF

Miklós Rontó, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matronto@uni-miskolc.hu

EDITORS

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Michal Fečkan, Comenius University, Department of Mathematical Analysis; Mlyn-ská dolina, 842 15 Bratislava, Slovakia; Michal.Feckan@fmph.uniba.sk

Sándor Fegyverneki, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matfs@uni-miskolc.hu

Stephan Foldes, University of Tampere; 33100 Tampere, Finland; mmn.foldes@gmail.com

Aurél Galántai, Óbuda University, John von Neumann Faculty of Informatics; H-1034 Budapest, Bcsi u. 96/b, Hungary; galantai.aurel@nik.uni-obuda.hu

István Györi, Department of Mathematics and Computing, University of Veszprém; P. O. Box 158, H-8201 Veszprém, Hungary; gyori@almos.vein.hu

László Hatvani, Bolyai Institute, University of Szeged; 1 Aradi vértanúk tere, H-6720 Szeged, Hungary; hatvani@math.u-szeged.hu

Attila Házy, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matha@uni-miskolc.hu

Imre Juhász, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; agtji@uni-miskolc.hu

Judit Makó, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matjudit@uni-miskolc.hu

TECHNICAL EDITORS

Krisztián Hriczó, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; mathk@uni-miskolc.hu

Attila Kőrei, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matka@uni-miskolc.hu

Nino Partsvania, Andrea Razmadze Mathematical Institute; 2 University St., Tbilisi 0186 Georgia; ninopa@rmi.ge

Nikolai Perestyuk, National Shevchenko University of Kiev, Faculty of Mechanics and Mathematics; 64 Vladimirska St., 01006 Kiev, Ukraine; pmo@univ.kiev.ua

Adrian Petrusel, Department of Mathematics, Babes-Bolyai University of Cluj-Napoca; No. 1, 400084, Cluj-Napoca, Romania; petrusel@math.ubbcluj.ro

Sándor Radeleczki, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matradi@uni-miskolc.hu

Csaba Rakaczki, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matrcs@uni-miskolc.hu

András Rontó, Institute of Mathematics, Academy of Sciences of Czech Republic; Žižkova 22, 616 62 Brno, Czech Republic; ronto@math.cas.cz

Nikolai Rozov, Moscow State University, Faculty of Mechanics and Mathematics; The Main Building, Vorobiovy Gory, 119899 Moscow, Russia; rozov@rozov.mccme.ru

Anatoly Samoilenko, Institute of Mathematics, National Academy of Sciences of Ukraine; 3 Tereshchenkivska St., 01601 Kiev, Ukraine; sam@imath.kiev.ua

Jenő Szigeti, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matjeno@uni-miskolc.hu

Jiří Šremr, Institute of Mathematics, Academy of Sciences of Czech Republic; Žižkova 22, 616 62 Brno, Czech Republic; sremr@ipm.cz

Hari Mohan Srivastava, Department of Mathematics and Statistics, University of Victoria; Victoria, British Columbia V8W 3R4, Canada; harimsri@math.uvic.ca

Sergei Trofimchuk, Instituto de Matemática y Física, Universidad de Talca; 2 Norte 685, Talca, Chile; trofimch@inst-mat.utalca.cl

Péter Olajos, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matolaj@uni-miskolc.hu

Péter Varga, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matvarga@uni-miskolc.hu

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