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SOME NOTES ON FIRST STRONGLY GRADED RINGS

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Abstract. Let G be a group with identity e and R be an associative ring with a nonzero unity 1. Assume that R is first strongly G-graded and $H = supp(R,G)$. For $g \in H$, define $\alpha_g(x) = \sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)}$ where $x \in C_R(R_e) = \{r \in R : rx = xr \text{ for all } x \in R_e\}$, $r_g^{(i)} \in R_g$ and $t_{g^{-1}}^{(i)} \in R_{g^{-1}}$ for all $i = 1, \ldots\ldots, n_g$ for some positive integer n_g. In this article, we study $\alpha_g(x)$ and it's properties.

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Keywords: Graded rings, first strongly graded rings

1. Introduction

Throughout this article, R is an associative ring with nonzero unity 1. For a ring R and a subset T of R, $C_R(T) = \{r \in R : rt = tr \text{ for all } t \in T\}$. For a group G, $Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\}$. Let G be a group with identity e. Then R is said to be G-graded if there exist additive subgroups R_g of R such that $\mathbf{R} = \bigoplus_{g \in G} \mathbf{R}_g$ where $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. For $x \in R$, $x = \sum_{g \in G} x_g$ where x_g is the component of x in R_g. Also, $supp(R,G) = \{g \in G : R_g \neq 0\}$. Moreover, R_e is a subring of R and $1 \in R_e$. For more details, see [4]. Throughout this article, $H = supp(R,G)$.

First strongly graded rings have been introduced by Refai in [5]. A G-graded ring R is said to be first strongly graded if $1 \in R_g R_{g^{-1}}$ for all $g \in H$. R is first strongly G-graded if and only if H is a subgroup of G and $\mathbf{R}_g \mathbf{R}_h = \mathbf{R}_{gh}$ for all $g, h \in H$. For more details, see [5].

Definition 1 ([4])**.** Let R be a ring. Suppose that $\alpha : G \to Aut(R)$ and $\beta : G \times G \to U(R)$ where $Aut(R)$ is the group of automorphisms of R and $U(R)$ is the group of units of R. In [4], (R, G, α, β) is said to be crossed system if the following conditions hold for all $g, h, s \in G$ and $a \in R$.

(1) $\alpha_g(\alpha_h(a))\beta(g,h) = \beta(g,h)\alpha_{gh}(a)$.

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(2) $\beta(g,h)\beta(gh,s) = \alpha_g(\beta(h,s))\beta(g,hs)$.
(3) $\beta(g,e) = \beta(e,g) = 1$.

In [2], a G-graded ring R is said to be crossed product over the support if $R_g \bigcap U(R) \neq \phi$ for all $g \in H$. In [1], it was shown that if R is crossed product over the support, then R is first strongly graded and then by [5], H is a subgroup of G with $\mathbf{R}_g\mathbf{R}_h = \mathbf{R}_{gh}$ for all $g,h \in H$.

Suppose that R is crossed product over the support. We may choose the family $\{u_g\}_{g\in H}$ in R such that $u_g \in R_g \bigcap U(R)$ for all $g \in H$ and assume that $u_e = 1$. So, $R_g = R_e u_g = u_g R_e$ and $\{u_g\}_{g\in H}$ is a basis for the left (right) R_e-module R. Define the map $\alpha : H \to Aut(R_e)$ by $\alpha(g) = \alpha_g$ where $\alpha_g(a) = u_g a u_g^{-1}$ for all $g \in H$ and $a \in R_e$. Also, define $\beta : H \times H \to U(R_e)$ by $\beta(g,h) = u_g u_h u_{gh}^{-1}$ for all $g,h \in H$. Then α and β satisfy the conditions (1), (2) and (3) above (see [1]). Hence, (R_e, H, α, β) is a crossed system.

Assume that R is first strongly G-graded. For $g \in H$, define $\alpha_g(x) = \sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)}$ where $x \in C_R(R_e)$, $r_g^{(i)} \in R_g$ and $t_{g^{-1}}^{(i)} \in R_{g^{-1}}$ for all $i = 1, \ldots\ldots, n_g$ for some positive integer n_g. In this article, we study $\alpha_g(x)$ and it's properties.

2. Results

In this section, we introduce our results.

Let R be a G-graded ring and X be a non-empty subset of G. Then $R_X = \bigoplus_{g\in X} R_g$. If X is a subgroup of G, then R_X is a subring of R. For more details, see [3]. We begin our results by the following.

Theorem 1. *Consider the above crossed system (R_e, H, α, β). Suppose that X is a subgroup of H such that $X \subseteq Z(H) \bigcap Ker(\alpha)$ and $\beta(x,y) = \beta(y,x)$ for all $(x,y) \in X \times X$. If R_e is commutative, then R_X is commutative.*

Proof. Consider the family $\{u_g\}_{g\in H}$ above. Let $g,h \in X$ and $a_g, b_h \in R_e$. Then

$$(a_g u_g)(b_h u_h) = a_g \alpha_g(b_h)\beta(g,h)u_{gh} = a_g b_h \beta(g,h) u_{gh} = b_h \alpha_h(a_g)\beta(h,g)u_{hg} = (b_h u_h)(a_g u_g).$$

Hence, R_X is commutative. □

Let R be a first strongly G-graded ring (not necessary to be crossed product over the support). Then $R_g R_{g^{-1}} = R_e$ for all $g \in H$. So, for every $g \in H$, there exists $n_g \in \mathbf{Z}^+$, $r_g^{(i)} \in R_g$ and $t_{g^{-1}}^{(i)} \in R_{g^{-1}}$ such that $1 = \sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)}$ since $1 \in R_e$.

Define $\alpha_g(x) = \sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)}$ for all $x \in C_R(R_e)$.

Theorem 2. *Let R be a first strongly graded ring. Then α_g is independent of the choice of $r_g^{(i)}$'s and $t_{g^{-1}}^{(i)}$'s.*

Proof. Let $n_g, n'_g \in \mathbf{Z}^+$ and $r_g^{(i)}, s_g^{(i)} \in R_g$, $t_{g^{-1}}^{(i)}, w_{g^{-1}}^{(i)} \in R_{g^{-1}}$ such that

$$1 = \sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)} = \sum_{j=1}^{n'_g} s_g^{(j)} w_{g^{-1}}^{(j)}.$$

Let $x \in C_R(R_e)$. Then since $w_{g^{-1}}^{(j)} r_g^{(i)} \in R_e$,

$$\sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)} - \sum_{j=1}^{n'_g} s_g^{(j)} x w_{g^{-1}}^{(j)} = 1.\left(\sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)}\right) - \left(\sum_{j=1}^{n'_g} s_g^{(j)} x w_{g^{-1}}^{(j)}\right).1 =$$
$$\sum_{j=1}^{n'_g}\sum_{i=1}^{n_g} s_g^{(j)} w_{g^{-1}}^{(j)} r_g^{(i)} x t_{g^{-1}}^{(i)} - \sum_{j=1}^{n'_g}\sum_{i=1}^{n_g} s_g^{(j)} x w_{g^{-1}}^{(j)} r_g^{(i)} t_{g^{-1}}^{(i)} = 0.$$

□

The next lemma is fundamental for our next results.

Lemma 1. *Let R be a first strongly graded ring. If $r \in R$ such that $rR_h = \{0\}$ for some $h \in H$, then $r = 0$.*

Proof. Suppose that $r \in R$ and $h \in H$ such that $rR_h = \{0\}$. Then $r = r.1 \in rR_e = rR_h R_{h^{-1}} = (rR_h) R_{h^{-1}} = \{0\}$, i.e., $r = 0$. □

Theorem 3. *Let R be a first strongly graded ring and $g \in H$. Then $\alpha_g(x)$ is the only element of R satisfies $\alpha_g(x)a_g = a_g x$ for all $a_g \in R_g$. Moreover, $\alpha_g(x) \in C_R(R_e)$ and if $x \in Z(R_e)$, then $\alpha_g(x) \in Z(R_e)$.*

Proof. Let $a_g \in R_g$. Then $t_{g^{-1}}^{(i)} a_g \in R_{g^{-1}} R_g = R_e$ and $t_{g^{-1}}^{(i)} a_g$ commutes with $x \in C_R(R_e)$ for all $i = 1, \ldots\ldots, n_g$. So,

$$\alpha_g(x)a_g = \sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)} a_g = \sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)} a_g x = a_g \left(\sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)}\right) x$$

$$= a_g.1.x = a_g x.$$

Let $x \in C_R(R_e)$ and $y \in R$ such that $r_g^{(i)} x = y r_g^{(i)}$ for all $i = 1, \ldots\ldots, n_g$. Then

$$\alpha_g(x) = \sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)} = \sum_{i=1}^{n_g} y r_g^{(i)} t_{g^{-1}}^{(i)} = y\left(\sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)}\right) = y.1 = y,$$

i.e., $\alpha_g(x)$ is the only element satisfies $\alpha_g(x)a_g = a_g x$ for all $a_g \in R_g$. Since R is first strongly graded, if $x \in R_e$, then $\alpha_g(x) \in R_e$. In particular, if $x \in Z(R_e)$, then $\alpha_g(x) \in Z(R_e)$. So, for $x \in Z(R_e)$ and $s \in R_e$,

$$s\alpha_g(x) = 1.s\alpha_g(x) = \sum_{i=1}^{n_g}\sum_{j=1}^{n'_g} r_g'^{(j)} t_{g^{-1}}'^{(j)} s r_g^{(i)} x t_{g^{-1}}^{(i)}.$$

Since $t_{g^{-1}}'^{(j)} s r_g^{(i)} \in R_e$,

$$s\alpha_g(x) = \sum_{i=1}^{n_g}\sum_{j=1}^{n'_g} r_g'^{(j)} x t_{g^{-1}}'^{(j)} s r_g^{(i)} t_{g^{-1}}^{(i)} = \left(\sum_{j=1}^{n'_g} r_g'^{(j)} x t_{g^{-1}}'^{(j)}\right).s.\left(\sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)}\right) = \alpha_g(x).s.1 = \alpha_g(x)s.$$

If $a_g \in R_g$ and $w \in R_e$, then $wa_g \in R_e R_g = R_g$ and then

$$(\alpha_g(x)w)a_g = \alpha_g(x)(wa_g) = (wa_g)x = w(a_g x) = w(\alpha_g(x)a_g)$$

$$= (w\alpha_g(x))a_g$$

Which implies that $(\alpha_g(x)w - w\alpha_g(x))R_g = \{0\}$. By Lemma 1, $\alpha_g(x)w = w\alpha_g(x)$ and hence $\alpha_g(x) \in C_R(R_e)$. □

Theorem 4. *Let R be a first strongly graded ring. Then the following hold:*

(1) $\alpha_g \circ \alpha_h = \alpha_{gh}$ *for all* $g, h \in H$.
(2) $\alpha_g^{-1} = \alpha_{g^{-1}}$ *for all* $g \in H$.
(3) $\alpha_g(xb) = \alpha_g(x)\alpha_g(b)$ *for all* $g \in H$ *and* $x, b \in C_R(R_e)$.

Proof. Since $1 \in R_e$, $x = 1.x = \alpha_e(x).1 = \alpha_e(x)$ for all $x \in C_R(R_e)$. Let $g, h \in H$, $a_g \in R_g$ and $a_h \in R_h$. Then $a_g a_h \in R_g R_h = R_{gh}$ and then

$$\alpha_{gh}(x)(a_g a_h) = (a_g a_h)x = a_g(a_h x) = a_g(\alpha_h(x)a_h) = (a_g\alpha_h(x))a_h$$

$$= (\alpha_g(x)(\alpha_h(x))a_g)a_h = \alpha_g(x)(\alpha_h(x))(a_g a_h)$$

for all $x \in C_R(R_e)$ which implies that $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$ by Lemma 1 as $a_g a_h$ generates the R_e-submodule R_{gh}.

Let $g \in H$, $s_{g^{-1}}^{(j)} \in R_{g^{-1}}$ and $w_g^{(j)} \in R$, $j = 1, \ldots, n_{g^{-1}}$ such that $1 = \sum_{j=1}^{n_{g^{-1}}} s_{g^{-1}}^{(j)} w_g^{(j)}$.

Then for every $x \in C_R(R_e)$,

$$\alpha_{g^{-1}}(\alpha_g(x)) = \sum_{j=1}^{n_{g^{-1}}} s_{g^{-1}}^{(j)} \left(\sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)} \right) w_g^{(j)} =$$
$$\sum_{j=1}^{n_{g^{-1}}} s_{g^{-1}}^{(j)} \left(\sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)} w_g^{(j)} \right) = \sum_{j=1}^{n_{g^{-1}}} s_{g^{-1}}^{(j)} \left(\sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)} w_g^{(j)} x \right) =$$
$$\sum_{j=1}^{n_{g^{-1}}} s_{g^{-1}}^{(j)} \left(\sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)} \right) w_g^{(j)} x = \sum_{j=1}^{n_{g^{-1}}} s_{g^{-1}}^{(j)}.1.w_g^{(j)} x = \left(\sum_{j=1}^{n_{g^{-1}}} s_{g^{-1}}^{(j)} w_g^{(j)} \right) x$$
$$= 1.x = x$$

which implies that $(\alpha_g)^{-1} = \alpha_{g^{-1}}$. For $x, b \in C_R(R_e)$ and $a_g \in R_g$,

$$\alpha_g(xb)a_g = a_g(xb) = (a_g x)b = (\alpha_g(x)a_g)b = \alpha_g(x)(a_g b) = \alpha_g(x)(\alpha_g(b)a_g) = (\alpha_g(x)\alpha_g(b))a_g.$$

By Lemma 1, $\alpha_g(xb) = \alpha_g(x)\alpha_g(b)$.

□

Theorem 5. *Let R be a first strongly graded ring. If $x \in C_R(R_e)$ and $g \in H$, then $xR_g = R_g x$ (x centralizes R_g) if and only if $\alpha_g(x) = x$.*

Proof. Suppose that R_g is centralized by x. Then for every $a_g \in R_g$,

$$\alpha_g(x)a_g = \left(\sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)} \right) a_g = \sum_{i=1}^{n_g} r_g^{(i)} x t_{g^{-1}}^{(i)} a_g.$$

Since $t_{g^{-1}}^{(i)} r_g^{(i)} \in R_e$,

$$\alpha_g(x)a_g = \sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)} a_g x = \left(\sum_{i=1}^{n_g} r_g^{(i)} t_{g^{-1}}^{(i)} \right) a_g x = 1.a_g x = a_g x = x a_g$$

as x centralizes R_g. By Lemma 1, $\alpha_g(x) = x$. Conversely, for every $a_g \in R_g$, $a_g x = \alpha_g(x)a_g = x a_g$, i.e., x centralizes R_g □

Corollary 1. *Let R be a first strongly graded ring. Then $Z(R) = \{x \in C_R(R_e) : \alpha_g(x) = x$ for all $g \in H\}$.*

Proof. Since R is first strongly graded, $Z(R) = \bigcap_{g \in H} C_R(R_g) = \{x \in C_R(R_e) : x \in C_R(R_g)$ for all $g \in H\}$. Note that if $g \notin H$, then $R_g = \{0\}$ and then $C_R(R_g) = R$. By Theorem 5, $x \in C_R(R_g)$ if and only if $\alpha_g(x) = x$ and hence $Z(R) = \{x \in C_R(R_e) : \alpha_g(x) = x$ for all $g \in H\}$. □

Remark 1. Also, it is nice to see that If R is a first strongly graded ring, then $C_R(R_e) =$

$$\left\{x = \sum_{g\in H} x_g \in R : x_g \in R_g \text{ with } cx_g = x_g c \text{ for all } g \in H \text{ and for all } c \in R_e\right\}$$
$$= \left\{x = \sum_{g\in H} x_g \in R : x_g \in \left(R_g \bigcap C_R(R_e)\right) \text{ for all } g \in H\right\}$$
$$= \bigoplus_{g\in H} \left(R_g \bigcap C_R(R_e)\right).$$

Note that for $g \notin H$, $R_g = \{0\}$ and then $x_g = 0$.

The next result is a generalization of Corollary 1.

Theorem 6. *Let R be a first strongly graded ring and X be a subgroup of H. Then $C_R(R_X) =$*

$$\left\{t = \sum_{g\in H} t_g \in R : t_g \in C_R(R_e)\bigcap R_g, \alpha_x(t_g) = t_{xgx^{-1}} \textit{ for all } g \in H \textit{ and for all } x \in X\right\}$$
$$= \{t \in C_R(R_e) : \alpha_x(t) = t \textit{ for all } x \in X\}.$$

Proof. Let $t = \sum_{g\in H} t_g \in C_R(R_X)$ where $t_g \in R_g$. Since $R_e \subseteq R_X$, $t \in C_R(R_e)$ and then by Remark 1, $t_g \in C_R(R_e)$ for all $g \in H$. Let $x \in X$. Then for every $s_x \in R_x$,

$$s_x \sum_{g\in H} t_g = \sum_{g\in H} t_g s_x$$

since $t \in C_R(R_X)$. As $t_g \in C_R(R_e)$, by Theorem 3,

$$\sum_{g\in H} \alpha_x(t_g)s_x = \sum_{g\in H} t_g s_x.$$

Since $R_x R_g R_{x^{-1}} R_x = R_{xg}$, for all $g \in H$ and for all $x \in X$,

$$\alpha_x(t_g)s_x = t_{xgx^{-1}} s_x \text{ for all } s_x \in R_x.$$

Choose $a_x^{(i)} \in R_x$ and $b_{x^{-1}}^{(i)} \in R_{x^{-1}}$ where $i = 1, \ldots\ldots, n_x$ for some positive integer n_x such that

$$1 = \sum_{i=1}^{n_x} a_x^{(i)} b_{x^{-1}}^{(i)}.$$

Then

$$\alpha_x(t_g) = \alpha_x(t_g).1 = \alpha_x(t_g)\sum_{i=1}^{n_x} a_x^{(i)} b_{x^{-1}}^{(i)} = \sum_{i=1}^{n_x} \alpha_x(t_g) a_x^{(i)} b_{x^{-1}}^{(i)} =$$
$$\sum_{i=1}^{n_x} t_{xgx^{-1}} a_x^{(i)} b_{x^{-1}}^{(i)} = t_{xgx^{-1}} \sum_{i=1}^{n_x} a_x^{(i)} b_{x^{-1}}^{(i)} = t_{xgx^{-1}}.$$

For the converse, Suppose that $t = \sum_{g\in H} t_g \in R$ where $t_g \in C_R(R_e) \bigcap R_g$ and $\alpha_x(t_g) = t_{xgx^{-1}}$ for all $g \in H$ and for all $x \in X$. Then for every $s_x \in R_x$,

$$s_x t = \sum_{g\in H} s_x t_g = \sum_{g\in H} \alpha_x(t_g) s_x = \sum_{g\in H} t_{xgx^{-1}} s_x = \sum_{r\in H} t_r s_x = t s_x$$

which implies that $t \in C_R(R_X)$. □

Theorem 7. *Let R be a first strongly graded ring such that R_e is commutative. If X is a subgroup of H such that $X \subseteq Z(G)$ and $\alpha_g(a) = a$ for all $g \in H$ and $a \in R_e$, then $J \bigcap C_R(R_X) \neq \{0\}$ for every non-zero two sided ideal J of R.*

Proof. Let J be a nonzero two sided ideal of R. Let $x \in X$ and $s_x \in R_x$. Define $K_{s_x} : R \to R$ by

$$K_{s_x}(t) = K_{s_x}\left(\sum_{g\in H} t_g\right) = s_x \sum_{g\in H} t_g - \sum_{g\in H} t_g s_x = \sum_{j\in H} k_j.$$

Note that $k_x = s_x t_e - t_e s_x = \alpha_x(t_e) s_x - t_e s_x = t_e s_x - t_e s_x = 0$. On the other hand, $k_{xg} = s_x t_g - t_g s_x \in R_{gx} = R_{xg}$ might be zero or nonzero. Thus the number of elements in $supp(K_{s_x}(t)$ is less than $supp(t)$. Moreover,

$$C_R(R_X) = \bigcap_{x\in X, s_x\in R_x} Ker(K_{s_x}).$$

Let $t = \sum_{g\in H} t_g \in J$ be a nonzero element. We may assume that $t_e \neq 0$. Otherwise, there exists a nonzero $t' = \sum_{g\in H} t'_g \in J$ such that $t'_e \neq 0$. So, there exists $y \in G$ such that $t_y \neq 0$. Also, there exists $b_{y^{-1}}^{(i)} \in R_{y^{-1}}$ such that $b_{y^{-1}}^{(i)} t_y \neq 0$ where $i = 1, \ldots\ldots, n_y$ for some positive integer n_y, this is because if $b_{y^{-1}}^{(i)} t_y = 0$ for all $i = 1, \ldots\ldots, n_y$, then we choose $a_y^{(i)} \in R_y$ such that

$$\sum_{i=1}^{n_y} a_y^{(i)} b_{y^{-1}}^{(i)} = 1$$

and then

$$t_y = 1.t_y = \sum_{i=1}^{n_y} a_y^{(i)} b_{y^{-1}}^{(i)} t_y = 0.$$

Hence, for every $t \in J$ there exists $b_{y^{-1}}^{(j)} t = t' = \sum_{g \in H} t'_g$ in J such that $t'_e = b_{y^{-1}}^{(j)} t_y \neq 0$ and $|supp(t)| \geq |supp(t')| \geq 1$. Now, we assumed that $t = \sum_{g \in H} t_g \in J$ such that $t_e \neq 0$. If $t \in C_R(R_X)$, then it is done. Suppose that $t \notin C_R(R_X)$. Then there exists $x \in X$ and $s_x \in R_x$ such that $K_{s_x}(t) \neq 0$. Since $K_{s_x}(t) \in J$, we find an element in J with smaller support. Keep on this procedure, we will stop since $supp(t)$ is finite. Thus, we will find an element $\xi = \sum_{g \in H} \xi_g \in J \bigcap C_R(R_X)$ such that $\xi_e \neq 0$. □

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FABER POLYNOMIAL COEFFICIENT ESTIMATES FOR BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER BASED ON SUBORDINATE CONDITIONS INVOLVING OF THE JACKSON (p,q)-DERIVATIVE

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Abstract. In the present paper, new subclasses of bi-univalent functions of complex order associated with the (p,q)-derivative are introduced. Furthermore, using the Faber polynomial expansions, we get upper bounds for the coefficients of functions belonging to these classes.

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1. Introduction, definitions and notations

Denote by E the unit disc of the complex plane, $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, A the class of functions analytic in E, satisfying the conditions

$$f(0) = 0 \text{ and } \quad f'(0) = 1.$$

Then each function f in A has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

We denote by S the subclass of A consisting of functions the form (1.1) which are also univalent in E.

For f and F analytic in E, we say that f is subordinate to F, written $f \prec F$, if there exists a Schwarz function

$$t(z) = \sum_{n=1}^{\infty} c_n z^n$$

with $|t(z)| < 1$ in E, such that $f(z) = F(t(z))$. For the Schwarz function $t(z)$ we note that $|c_n| < 1$. (e.g. see Duren [11]).

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It is well known that every function $f \in S$ has an inverse f^{-1}, satisfying $f^{-1}(f(z)) = z\,(z \in E)$ and $f\left(f^{-1}(w)\right) = w\ \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right)$, where

$$f^{-1}(w) = w\ - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

A function $f \in A$ is said to be bi-univalent in E if both f and f^{-1} are univalent in E. Let Σ denote the class of bi-univalent functions in E given by (1.1). For a brief history and interesting examples in the class Σ, see [24] (see also [7], [8], [19], [20]). Furthermore, judging by the remarkable flood of papers on the subject (see, for example, [9], [5], [17], [23], [24], [25]).

The Faber polynomials introduced by Faber [12] play an important role in various areas of mathematical sciences, especially in Geometric Function Theory. Grunsky [14] succeeded in establishing a set of conditions for a given function which are necessary and in their totality sufficient for the univalency of this function, and in these conditions the coefficients of the Faber polynomials play an important role. Schiffer [22] gave a differential equations for univalent functions solving certain extremum problems with respect to coefficients of such functions; in this differential equation appears again a polynomial which is just the derivative of a Faber polynomial (Schaeffer-Spencer [21]).

Not much is known about the bounds on the general coefficient $|a_n|$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([6], [15], [16]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N}\backslash\{1,2\}$; $\mathbb{N} = \{1,2,3,...\}$) is still an open problem.

1.1. *The Jackson (p,q)-derivative*

For the convenience, we provide some basic definitions and concept details of q-calculus which are used in this paper. We suppose throughout the paper that $0 < q < p \leq 1$. We recall the definitions of fractional q-calculus operators of complex valued function f. We shall follow the notation and terminology in [13].

Definition 1 (see [10])**.** The (p,q)-derivative of the function f is defined as

$$(D_{p,q} f)(z) = \begin{cases} \frac{f(pz)-f(qz)}{(p-q)z} & \text{for } z \neq 0 \\ f'(0) & \text{for } z = 0 \end{cases} . \tag{1.3}$$

From (1.3), we deduce that

$$(D_{p,q} f)(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q}\, a_n z^{n-1},$$

where the symbol $[n]_{p,q}$ denotes the so-called (p,q) *-bracket* or *twin-basic number*

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}.$$

It happens clearly that $D_{p,q}z^n = [n]_{p,q}\, z^{n-1}$. Note also that for $p = 1$, the Jackson (p,q)-derivative reduces to the Jackson q-derivative given by (see [18])

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad z \neq 0.$$

The twin-basic number is a natural generalization of the q-number, that is

$$\lim_{p\to 1} [n]_{p,q} = [n]_q = \frac{1-q^n}{1-q}, \quad q \neq 1.$$

As with ordinary derivative, the action of the (p,q)-derivative of a function is a linear operator. More precisely, for any constants a and b,

$$D_{p,q}(af(z) + bh(z)) = aD_{p,q}f(z) + bD_{p,q}h(z).$$

The (p,q)-derivative fulfils the following product rules

$$D_{p,q}(f(z)h(z)) = f(pz)D_{p,q}h(z) + h(qz)D_{p,q}f(z),$$
$$D_{p,q}(f(z)h(z)) = h(pz)D_{p,q}f(z) + f(qz)D_{p,q}h(z).$$

Further, the (p,q)-derivative fulfils the following product rules

$$D_{p,q}\left(\frac{f(z)}{h(z)}\right) = \frac{h(qz)D_{p,q}f(z) - f(qz)D_{p,q}h(z)}{h(pz)h(qz)},$$
$$D_{p,q}\left(\frac{f(z)}{h(z)}\right) = \frac{h(pz)D_{p,q}f(z) - f(pz)D_{p,q}h(z)}{h(pz)h(qz)}.$$

From (1.2) and (1.3), we also deduce that

$$\begin{aligned}(D_{p,q}g)(w) &= \tfrac{g(pw)-g(qw)}{(p-q)w} \\ &= 1 - [2]_{p,q}\, a_2 w + [3]_{p,q}\left(2a_2^2 - a_3\right) w^2 \\ &\quad - [4]_{p,q}\left(5a_2^3 - 5a_2a_3 + a_4\right) w^3 + \cdots\end{aligned}$$

where the function g is given by (1.2).

2. Preliminary results

By using the Faber polynomial expansion of functions $f \in A$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as follows (see [3]):

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots)\, w^n,$$

where

$$K_{n-1}^{-n} = \frac{(-n)!}{(-2n+1)!\,(n-1)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!\,(n-3)!} a_2^{n-3} a_3$$

$$
\begin{aligned}
&+\frac{(-n)!}{(-2n+3)!\,(n-4)!}a_2^{n-4}a_4 \\
&+\frac{(-n)!}{[2(-n+2)]!\,(n-5)!}a_2^{n-5}\left[a_5+(-n+2)\,a_3^2\right] \\
&+\frac{(-n)!}{(-2n+5)!\,(n-6)!}a_2^{n-6}\left[a_6+(-2n+5)\,a_3a_4\right] \\
&+\sum_{j\geq 7}a_2^{n-j}V_j,
\end{aligned}
\tag{2.1}
$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $a_2, a_3, ..., a_n$ [4]. In particular, the first three terms of K_{n-1}^{-n} are given below:

$$
\begin{aligned}
\frac{1}{2}K_1^{-2} &= -a_2, \\
\frac{1}{3}K_2^{-3} &= 2a_2^2-a_3, \\
\frac{1}{4}K_3^{-4} &= -\left(5a_2^3-5a_2a_3+a_4\right).
\end{aligned}
\tag{2.2}
$$

In general, for any $p \in \mathbb{N}$ and $n \geq 2$, an expansion of K_{n-1}^{p} is as, [3],

$$
K_{n-1}^{p} = pa_n+\frac{p\,(p-1)}{2}E_{n-1}^{2}+\frac{p!}{(p-3)!3!}E_{n-1}^{3}+...+\frac{p!}{(p-n+1)!\,(n-1)!}E_{n-1}^{n-1},
\tag{2.3}
$$

where $E_{n-1}^{p} = E_{n-1}^{p}(a_2, a_3, ...)$ and by [1],

$$
E_{n-1}^{m}(a_2, ..., a_n) = \sum_{n=2}^{\infty}\frac{m!\,(a_2)^{\mu_1}...(a_n)^{\mu_{n-1}}}{\mu_1!...\mu_{n-1}!}, \quad \text{for } m \leq n
$$

while $a_1 = 1$, and the sum is taken over all nonnegative integers $\mu_1, ..., \mu_n$ satisfying

$$
\begin{aligned}
\mu_1+\mu_2+\ ...\ +\mu_{n-1} &= m, \\
\mu_1+2\mu_2+\ ...\ +(n-1)\,\mu_{n-1} &= n-1.
\end{aligned}
$$

Evidently, $E_{n-1}^{n-1}(a_2, ..., a_n) = a_2^{n-1}$,(see [2]); while $a_1 = 1$, and the sum is taken over all nonnegative integers $\mu_1, ..., \mu_n$ satisfying

$$
\begin{aligned}
\mu_1+\mu_2+\ ...\ +\mu_n &= m, \\
\mu_1+2\mu_2+\ ...\ +n\mu_n &= n.
\end{aligned}
$$

It is clear that $E_n^n(a_1, a_2, ..., a_n) = a_1^n$. The first and the last polynomials are:

$$
E_n^1 = a_n, \qquad E_n^n = a_1^n.
$$

In the following, let ϕ be an analytic function with positive real part in E, with $\phi(0)=1$ and $\phi'(0)>0$. Also, let $\phi(E)$ be starlike with respect to 1 and symmetric with respect to the real axis. Thus, ϕ has the Taylor series expansion

$$\phi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots \quad (B_1>0). \tag{2.4}$$

Suppose that $u(z)$ and $v(w)$ are analytic in the unit disk E with $u(0)=v(0)=0$, $|u(z)|<1$, $|v(w)|<1$, and suppose that

$$u(z)=p_1z+\sum_{n=2}^{\infty}p_nz^n,\quad v(w)=q_1w+\sum_{n=2}^{\infty}q_nw^n \quad (|z|<1). \tag{2.5}$$

It is well known that

$$|p_1|\le 1,\quad |p_2|\le 1-|p_1|^2,\quad |q_1|\le 1,\quad |q_2|\le 1-|q_1|^2. \tag{2.6}$$

Next, the equations (2.4) and (2.5) lead to

$$\begin{aligned}\phi(u(z))&=1+B_1u(z)+B_2(u(z))^2z^2+\cdots\\&=1+B_1p_1z+\left(B_1p_2+B_2p_1^2\right)z^2+\cdots\\&=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}B_kE_n^k(p_1,p_2,\ldots,p_n)z^n,\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\phi(v(w))&=1+B_1v(w)+B_2(v(w))^2z^2+\cdots\\&=1+B_1q_1w+\left(B_1q_2+B_2q_1^2\right)w^2+\cdots\\&=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}B_kE_n^k(q_1,q_2,\ldots,q_n)w^n.\end{aligned} \tag{2.8}$$

Definition 2. A function $f\in\Sigma$ is said to be in the class $R_{\Sigma,b}^{p,q}(\phi)$ if the following subordination relationships hold true:

$$\left[1+\frac{1}{b}\left((D_{p,q}f)(z)-1\right)\right]\prec\phi(z) \tag{2.9}$$

and

$$\left[1+\frac{1}{b}\left((D_{p,q}g)(w)-1\right)\right]\prec\phi(w) \tag{2.10}$$

where $b \in \mathbb{C}\backslash\{0\}$, $0 < q < p \leq 1$; $z, w \in E$ and the function g is given by (1.2).

We note from Definition 2 that

$$\lim_{p\to 1} R^{p,q}_{\Sigma,b}(\phi) = \left\{ f \in \Sigma : \left\{ \begin{array}{l} \lim\limits_{p\to 1} \left[1 + \frac{1}{b}\left((D_{p,q} f)(z) - 1\right)\right] \prec \phi(z) \\ \\ \lim\limits_{p\to 1} \left[1 + \frac{1}{b}\left((D_{p,q} g)(w) - 1\right)\right] \prec \phi(w) \end{array} \right. \right\} = R^{q}_{\Sigma,b}(\phi).$$

By suitably specializing the parameters b and p, we state the new subclasses of bi-univalent functions of complex order $R^{p,q}_{\Sigma,b}(\phi)$ as illustrated in the following Definitions.

Definition 3. For $b = 1$, a function $f \in \Sigma$ is said to be in the class $R^{p,q}_{\Sigma}(\phi)$ if it satisfies the following conditions respectively:

$$(D_{p,q} f)(z) \prec \phi(z)$$

and

$$(D_{p,q} g)(w) \prec \phi(w)$$

where $0 < q < p \leq 1; z, w \in E$ and the function g is given by (1.2).

Definition 4. For $b = 1$ and $p \to 1$, a function $f \in \Sigma$ is said to be in the class $R^{q}_{\Sigma}(\phi)$ if it satisfies the following conditions respectively:

$$(D_{q} f)(z) \prec \phi(z)$$

and

$$(D_{q} g)(w) \prec \phi(w)$$

where $z, w \in E$ and the function g is given by (1.2).

In this paper, we study the class $R^{p,q}_{\Sigma,b}(\phi)$ of analytic bi-univalent functions defined above by using the (p,q)-derivative operator. Moreover, we use the Faber polynomial expansions to obtain bounds for the general coefficients $|a_n|$ of bi-univalent functions in $R^{p,q}_{\Sigma,b}(\phi)$ as well as we provide estimates for the initial coefficients of these functions. Several new consequences of the results are also pointed out.

3. Main results

Our first main result is given by Theorem 1 below.

Theorem 1. *For $b \in \mathbb{C}\backslash\{0\}$, let $f \in \mathrm{R}^{p,q}_{\Sigma,b}(\phi)$. If $a_m = 0$; $2 \leq m \leq n-1$, then*

$$|a_n| \leq \frac{B_1 |b|}{[n]_{p,q}}, \quad n \geq 3.$$

Proof. Let f be given by (1.1). We have

$$(D_{p,q}f)(z)-1=\sum_{n=2}^{\infty}[n]_{p,q}\,a_n z^{n-1}, \tag{3.1}$$

and, for its inverse map $g=f^{-1}$, it is seen that

$$(D_{p,q}g)(w)-1=\sum_{n=2}^{\infty}K_{n-1}^{-n}(a_2,a_3,\ldots)\,w^{n-1}=\sum_{n=2}^{\infty}[n]_{p,q}\,b_n w^{n-1}. \tag{3.2}$$

From (2.9) and (2.10) yields

$$1+\frac{1}{b}\big((D_{p,q}f)(z)-1\big)=\varphi(u(z)) \tag{3.3}$$

and

$$1+\frac{1}{b}\big((D_{p,q}g)(w)-1\big)=\varphi(v(w)) \tag{3.4}$$

Comparing the corresponding coefficients of (3.3) and (3.4) yields

$$\frac{1}{b}[n]_{p,q}\,a_n=B_1\,p_{n-1}, \tag{3.5}$$

and

$$\frac{1}{b}[n]_{p,q}\,b_n=B_1 q_{n-1}. \tag{3.6}$$

Note that for $a_m=0$; $2\le m\le n-1$ we have $b_n=-a_n$ and so

$$\frac{1}{b}[n]_{p,q}\,a_n=B_1\,p_{n-1},$$
$$-\frac{1}{b}[n]_{p,q}\,a_n=B_1 q_{n-1}.$$

Now taking the absolute values of either of the above two equations and from (2.6), we obtain

$$|a_n|=\frac{B_1\,|p_{n-1}|\,|b|}{[n]_{p,q}}=\frac{B_1\,|q_{n-1}|\,|b|}{[n]_{p,q}}\le\frac{B_1\,|b|}{[n]_{p,q}}.$$

□

Theorem 2. *For $b\in\mathbb{C}\backslash\{0\}$, let $f\in \mathrm{R}^{q}_{\Sigma,b}(\phi)$. If $a_m=0$; $2\le m\le n-1$, then*

$$|a_n|\le\frac{B_1\,|b|\,(1-q)}{1-q^n};\quad n\ge 3.$$

Theorem 3. *Let $f\in \mathrm{R}^{p,q}_{\Sigma}(\phi)$. If $a_m=0$; $2\le m\le n-1$, then*

$$|a_n|\le\frac{B_1}{[n]_{p,q}};\quad n\ge 3.$$

Theorem 4. *Let* $f \in \mathrm{R}_{\Sigma}^{q}(\phi)$. *If* $a_m = 0$; $2 \le m \le n-1$, *then*

$$|a_n| \le \frac{B_1(1-q)}{1-q^n}; \quad n \ge 3.$$

4. Coefficient estimates

In this section we obtain coefficient estimates for functions belonging to the classes $R_{\Sigma,b}^{p,q}(\phi)$, $R_{\Sigma,b}^{q}(\phi)$, $R_{\Sigma}^{p,q}(\phi)$ and $R_{\Sigma}^{q}(\phi)$.

Theorem 5. *Let* $f \in \mathrm{R}_{\Sigma,b}^{p,q}(\phi)$ $(b \in \mathbb{C}\backslash\{0\})$. *Then*

$$|a_2| \le \min\left\{K(p,q), \frac{|b|\,B_1\sqrt{B_1}}{\sqrt{\left|(B_1^2 b - B_2)(p^2+q^2) + (B_1^2 b - 2B_2)pq\right| + B_1(p^2+2pq+q^2)}}\right\}$$

and

$$|a_3| \le \min\{L(p,q), M(p,q)\}$$

where

$$K(p,q) = \begin{cases} \sqrt{\frac{B_1|b|}{p^2+pq+q^2}}; & |B_2| \le B_1 \\ \sqrt{\frac{|B_2||b|}{p^2+pq+q^2}}; & |B_2| > B_1 \end{cases},$$

$$L(p,q) = \begin{cases} \frac{B_1|b|}{p^2+pq+q^2}; & |B_2| \le B_1 \\ \frac{|B_2||b|}{p^2+pq+q^2}; & |B_2| > B_1 \end{cases},$$

and

$$M(p,q) = \begin{cases} \frac{B_1|b|}{p^2+pq+q^2}; & B_1 \le \frac{p^2+2pq+q^2}{(p^2+pq+q^2)|b|} \\ \frac{|b|B_1\left[\left|(B_1^2b-B_2)(p^2+q^2)+(B_1^2b-2B_2)pq\right|+B_1^2|b|(p^2+pq+q^2)\right]}{\left[\left|(B_1^2b-B_2)(p^2+q^2)+(B_1^2b-2B_2)pq\right|+B_1(p^2+2pq+q^2)\right](p^2+pq+q^2)}; & \\ & B_1 > \frac{p^2+2pq+q^2}{(p^2+pq+q^2)|b|}. \end{cases}$$

Proof. Replacing n by 2 and 3 in (3.5) and (3.6), respectively, we find that

$$\frac{1}{b}[2]_{p,q}\,a_2 = B_1 p_1, \tag{4.1}$$

$$\frac{1}{b}[3]_{p,q}\,a_3 = B_1 p_2 + B_2 p_1^2, \tag{4.2}$$

$$-\frac{1}{b}[2]_{p,q}\,a_2 = B_1 q_1, \tag{4.3}$$

$$\frac{1}{b}[3]_{p,q}\,(2a_2^2 - a_3) = B_1 q_2 + B_2 q_1^2. \tag{4.4}$$

From (4.1) and (4.3), we obtain

$$p_1 = -q_1. \tag{4.5}$$

By adding (4.4) to (4.2), further computations using (4.5) lead to

$$\frac{2}{b}[3]_{p,q}\, a_2^2 = B_1\,(p_2+q_2) + 2B_2\, p_1^2. \tag{4.6}$$

Making use of (4.1) in the above equality (4.6), we get

$$\left[2bB_1^2\,[3]_{p,q} - 2B_2\,[2]_{p,q}^2\right] a_2^2 = b^2 B_1^3\,(p_2+q_2)\,. \tag{4.7}$$

Combining (4.7) and (2.6), we obtain

$$\begin{aligned} 2\left|bB_1^2\,[3]_{p,q} - B_2\,[2]_{p,q}^2\right| |a_2|^2 \leq & |b|^2\, B_1^3\,(|p_2|+|q_2|) \\ \leq & 2\,|b|^2\, B_1^3\left(1-|p_1|^2\right) \\ = & \, 2\,|b|^2\, B_1^3 - 2\,|b|^2\, B_1^3\,|p_1|^2\,. \end{aligned} \tag{4.8}$$

It follows from (4.1) that

$$|a_2| \leq \frac{|b|\, B_1\sqrt{B_1}}{\sqrt{\left|B_1^2 b\,[3]_{p,q} - B_2\,[2]_{p,q}^2\right| + B_1\,[2]_{p,q}^2}}. \tag{4.9}$$

Moreover, by (2.6) and (4.6)

$$\begin{aligned} \frac{2}{|b|}[3]_{p,q}\,|a_2|^2 \leq & B_1\,(|p_2|+|q_2|) + 2\,|B_2|\,|p_1|^2 \\ \leq & \, 2B_1\left(1-|p_1|^2\right) + 2\,|B_2|\,|p_1|^2 \\ = & \, 2B_1 + 2\,|p_1|^2\,(|B_2|-B_1) \end{aligned}$$

$$\frac{1}{|b|}[3]_{p,q}\,|a_2|^2 \leq \begin{cases} B_1; |B_2| \leq B_1 \\ |B_2|; |B_2| > B_1 \end{cases}.$$

Clearly, we can see that

$$|a_2| \leq \begin{cases} \sqrt{\frac{B_1|b|}{[3]_{p,q}}}; & |B_2| \leq B_1 \\ \sqrt{\frac{|B_2||b|}{[3]_{p,q}}}; & |B_2| > B_1 \end{cases}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (4.4) from (4.2), we obtain

$$\frac{2}{b}[3]_{p,q}\, a_3 = \frac{2}{b}[3]_{p,q}\, a_2^2 + B_1(p_2 - q_2). \tag{4.10}$$

Clearly, from (4.6), we have that

$$a_3 = \frac{b\left[B_1(p_2+q_2) + 2B_2 p_1^2\right]}{2[3]_{p,q}} + \frac{bB_1(p_2-q_2)}{2[3]_{p,q}}$$

$$= \frac{bB_1 p_2 + bB_2 p_1^2}{[3]_{p,q}}$$

and consequently

$$|a_3| \leq \frac{|b|\, B_1 |p_2| + |b||B_2||p_1|^2}{[3]_{p,q}}$$

$$\leq \frac{|b|\, B_1\left(1-|p_1|^2\right) + |b||B_2||p_1|^2}{[3]_{p,q}}$$

$$= \frac{|b|\, B_1 + |b||p_1|^2(|B_2| - B_1)}{[3]_{p,q}}.$$

Hence, we write

$$|a_3| \leq \begin{cases} \frac{B_1|b|}{[3]_{p,q}}; & |B_2| \leq B_1 \\ \frac{|B_2||b|}{[3]_{p,q}}; & |B_2| > B_1. \end{cases}$$

On the other hand, by using (2.6) and (4.10), we have

$$\frac{2}{|b|}[3]_{p,q}\,|a_3| \leq \frac{2}{|b|}[3]_{p,q}\,|a_2|^2 + B_1(|p_2| + |q_2|)$$

$$\leq \frac{2}{|b|}[3]_{p,q}\,|a_2|^2 + 2B_1\left(1-|p_1|^2\right).$$

Then, with the help of (4.1), we have

$$B_1 |b| [3]_{p,q} |a_3| \leq \left[B_1 |b| [3]_{p,q} - [2]_{p,q}^2 \right] |a_2|^2 + B_1^2 |b|^2 .$$

Now, from (4.9), we obtain

$$|a_3| \leq \frac{B_1 |b|}{[3]_{p,q}} \left\{ 1 + \frac{B_1 \left[B_1 |b| [3]_{p,q} - [2]_{p,q}^2 \right]}{\left| B_1^2 b [3]_{p,q} - B_2 [2]_{p,q}^2 \right| + B_1 [2]_{p,q}^2} \right\} .$$

□

Theorem 6. *Let* $f \in \mathrm{R}_{\Sigma,b}^{q}(\phi)$ $(b \in \mathbb{C} \backslash \{0\})$. *Then*

$$|a_2| \leq \min \left\{ K(q), \frac{|b| B_1 \sqrt{B_1}}{\sqrt{\left| (B_1^2 b - B_2)(1+q^2) + (B_1^2 b - 2B_2) q \right| + B_1 (1 + 2q + q^2)}} \right\}$$

and

$$|a_3| \leq \min \{ L(q), M(q) \}$$

where

$$K(q) = \begin{cases} \sqrt{\frac{B_1 |b|}{1+q+q^2}}; & |B_2| \leq B_1 \\ \sqrt{\frac{|B_2||b|}{1+q+q^2}}; & |B_2| > B_1 \end{cases},$$

$$L(q) = \begin{cases} \frac{B_1 |b|}{1+q+q^2}; & |B_2| \leq B_1 \\ \frac{|B_2||b|}{1+q+q^2}; & |B_2| > B_1 \end{cases},$$

and

$$M(q) = \begin{cases} \frac{B_1 |b|}{1+q+q^2}; & B_1 \leq \frac{1+2q+q^2}{(1+q+q^2)|b|} \\ \frac{|b| B_1 \left[\left| (B_1^2 b - B_2)(1+q^2) + (B_1^2 b - 2B_2) q \right| + B_1^2 |b| (1+q+q^2) \right]}{\left[\left| (B_1^2 b - B_2)(1+q^2) + (B_1^2 b - 2B_2) q \right| + B_1 (1+2q+q^2) \right] (1+q+q^2)}; & B_1 > \frac{1+2q+q^2}{(1+q+q^2)|b|} \end{cases}.$$

Theorem 7. *Let* $f \in \mathrm{R}_{\Sigma}^{p,q}(\phi)$. *Then*

$$|a_2| \leq \min \left\{ K(p,q), \frac{B_1 \sqrt{B_1}}{\sqrt{\left| (B_1^2 - B_2)(p^2 + q^2) + (B_1^2 - 2B_2) pq \right| + B_1 (p^2 + 2pq + q^2)}} \right\}$$

and

$$|a_3| \leq \min \{ L(p,q), M(p,q) \}$$

where

$$K(p,q)=\begin{cases}\sqrt{\frac{B_1}{p^2+pq+q^2}}; & |B_2|\leq B_1\\ \sqrt{\frac{|B_2|}{p^2+pq+q^2}}; & |B_2|>B_1\end{cases},$$

$$L(p,q)=\begin{cases}\frac{B_1}{p^2+pq+q^2}; & |B_2|\leq B_1\\ \frac{|B_2|}{p^2+pq+q^2}; & |B_2|>B_1\end{cases},$$

and

$$M(p,q)=\begin{cases}\frac{B_1}{p^2+pq+q^2}; & B_1\leq\frac{p^2+2pq+q^2}{p^2+pq+q^2}\\ \frac{B_1\left[\left|(B_1^2-B_2)(p^2+q^2)+(B_1^2-2B_2)pq\right|+B_1^2(p^2+pq+q^2)\right]}{\left[\left|(B_1^2-B_2)(p^2+q^2)+(B_1^2-2B_2)pq\right|+B_1(p^2+2pq+q^2)\right](p^2+pq+q^2)}; & B_1>\frac{p^2+2pq+q^2}{p^2+pq+q^2}\end{cases}.$$

Theorem 8. *Let* $f\in \mathrm{R}_{\Sigma}^{q}(\phi)$. *Then*

$$|a_2|\leq\min\left\{K(q),\frac{B_1\sqrt{B_1}}{\sqrt{\left|(B_1^2-B_2)(1+q^2)+(B_1^2-2B_2)q\right|+B_1(1+2q+q^2)}}\right\}$$

and

$$|a_3|\leq\min\{L(q),M(q)\}$$

where

$$K(q)=\begin{cases}\sqrt{\frac{B_1}{1+q+q^2}}; & |B_2|\leq B_1\\ \sqrt{\frac{|B_2|}{1+q+q^2}}; & |B_2|>B_1\end{cases},$$

$$L(q)=\begin{cases}\frac{B_1}{1+q+q^2}; & |B_2|\leq B_1\\ \frac{|B_2|}{1+q+q^2}; & |B_2|>B_1\end{cases},$$

and

$$M(q)=\begin{cases}\frac{B_1}{1+q+q^2}; & B_1\leq\frac{1+2q+q^2}{1+q+q^2}\\ \frac{B_1\left[\left|(B_1^2-B_2)(1+q^2)+(B_1^2-2B_2)q\right|+B_1^2(1+q+q^2)\right]}{\left[\left|(B_1^2-B_2)(1+q^2)+(B_1^2-2B_2)q\right|+B_1(1+2q+q^2)\right](1+q+q^2)}; & B_1>\frac{1+2q+q^2}{1+q+q^2}\end{cases}.$$

5. Corollaries and concluding Remarks

Corollary 1. *If we take*

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \ldots \quad (0 < \alpha \leq 1),$$

in Theorem 1 and Theorem 2, respectively, we have

$$|a_n| \leq \frac{2\alpha\,|b|}{[n]_{p,q}}; \quad n \geq 3$$

and

$$|a_n| \leq \frac{2\alpha\,|b|\,(1-q)}{1-q^n}; \quad n \geq 3.$$

Remark 1. Let $f \in R_{\Sigma}^{p,q}\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$. Then

$$|a_n| \leq \frac{2\alpha}{[n]_{p,q}}; \quad n \geq 3.$$

Remark 2. Let $f \in R_{\Sigma}^{q}\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$. Then

$$|a_n| \leq \frac{2\alpha(1-q)}{1-q^n}; \quad n \geq 3.$$

Corollary 2. *If we take*

$$\phi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \quad (0 \leq \beta < 1),$$

in Theorem 1 and Theorem 2, respectively, we have

$$|a_n| \leq \frac{2(1-\beta)\,|b|}{[n]_{p,q}}; \quad n \geq 3$$

and

$$|a_n| \leq \frac{2(1-\beta)\,|b|\,(1-q)}{1-q^n}; \quad n \geq 3.$$

Remark 3. Let $f \in R_{\Sigma}^{p,q}\left(\frac{1+(1-2\beta)z}{1-z}\right)$. Then

$$|a_n| \leq \frac{2(1-\beta)}{[n]_{p,q}}; \quad n \geq 3.$$

Remark 4. Let $f \in R_{\Sigma}^{p,q}\left(\frac{1+(1-2\beta)z}{1-z}\right)$. Then

$$|a_n| \leq \frac{2(1-\beta)(1-q)}{1-q^n}; \quad n \geq 3.$$

Taking $\phi(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}$ in Theorem 5 and Theorem 6, then we have the following results.

Corollary 3. *Let* $f\in \mathrm{R}^{p,q}_{\Sigma,b}(\phi)$ $(b\in\mathbb{C}\backslash\{0\})$. *Then*

$$|a_2|\leq \min\left\{\sqrt{\frac{2\alpha\,|b|}{p^2+pq+q^2}},\frac{2\alpha\,|b|}{\sqrt{\alpha\,|(2b-1)(p^2+q^2)+2(b-1)pq|+p^2+2pq+q^2}}\right\}$$

and

$$|a_3|\leq \min\left\{\frac{2\alpha\,|b|}{p^2+pq+q^2},H(p,q)\right\}$$

where

$$H(p,q)=\begin{cases}\frac{2\alpha|b|}{p^2+pq+q^2}; & 0<\alpha\leq\frac{p^2+2pq+q^2}{2(p^2+pq+q^2)|b|}\\ \frac{2\alpha^2|b|\left[\left|(2b-1)(p^2+q^2)+2(b-1)pq\right|+2|b|(p^2+pq+q^2)\right]}{\left[\alpha\left|(2b-1)(p^2+q^2)+2(b-1)pq\right|+p^2+2pq+q^2\right](p^2+pq+q^2)}; & \frac{p^2+2pq+q^2}{2(p^2+pq+q^2)|b|}<\alpha\leq 1\end{cases}.$$

Corollary 4. *Let* $f\in \mathrm{R}^{q}_{\Sigma,b}(\phi)$ $(b\in\mathbb{C}\backslash\{0\})$. *Then*

$$|a_2|\leq \min\left\{\sqrt{\frac{2\alpha\,|b|}{1+q+q^2}},\frac{2\alpha\,|b|}{\sqrt{\alpha\left|(2b-1)(1+q^2)+2(b-1)q\right|+1+2q+q^2}}\right\}$$

and

$$|a_3|\leq \min\left\{\frac{2\alpha\,|b|}{1+q+q^2},H(q)\right\}$$

where

$$H(q)=\begin{cases}\frac{2\alpha|b|}{1+q+q^2}; & 0<\alpha\leq\frac{1+2q+q^2}{2(1+q+q^2)|b|}\\ \frac{2\alpha^2|b|\left[\left|(2b-1)(1+q^2)+2(b-1)q\right|+2|b|(1+q+q^2)\right]}{\left[\alpha\left|(2b-1)(1+q^2)+2(b-1)q\right|+(1+2q+q^2)\right](1+q+q^2)}; & \frac{1+2q+q^2}{2(1+q+q^2)|b|}<\alpha\leq 1\end{cases}.$$

Corollary 5. *Let* $f\in \mathrm{R}^{p,q}_{\Sigma}\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$. *Then*

$$|a_2|\leq\frac{2\alpha}{\sqrt{(\alpha+1)(p^2+q^2)+2pq}}$$

and

$$|a_3|\leq \min\left\{\frac{2\alpha}{p^2+pq+q^2},H(p,q)\right\}$$

where

$$H(p,q)=\begin{cases}\frac{2\alpha}{p^2+pq+q^2}; & 0<\alpha\leq\frac{p^2+2pq+q^2}{2(p^2+pq+q^2)}\\ \frac{2\alpha^2\left[3(p^2+q^2)+2pq\right]}{\left[(\alpha+1)(p^2+q^2)+2pq\right](p^2+pq+q^2)}; & \frac{p^2+2pq+q^2}{2(p^2+pq+q^2)}<\alpha\leq 1\end{cases}.$$

Corollary 6. *Let* $f \in \mathrm{R}_{\Sigma}^{q}\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$. *Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\alpha+1)(1+q^2)+2q}}$$

and

$$|a_3| \leq \min\left\{\frac{2\alpha}{1+q+q^2}, H(q)\right\}$$

where

$$H(q) = \begin{cases} \frac{2\alpha}{1+q+q^2}; & 0 < \alpha \leq \frac{1+2q+q^2}{2(1+q+q^2)} \\ \frac{2\alpha^2\left[3(1+q^2)+2q\right]}{\left[(\alpha+1)(1+q^2)+2q\right](1+q+q^2)}; & \frac{1+2q+q^2}{2(1+q+q^2)} < \alpha \leq 1 \end{cases}.$$

By choosing $\phi(z) = \frac{1+(1-2\beta)z}{1-z}$ in Theorem 5 and Theorem 6, then we have the following results.

Corollary 7. *Let* $f \in \mathrm{R}_{\Sigma,b}^{p,q}(\phi)$ $(b \in \mathbb{C}\backslash\{0\})$. *Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{2(1-\beta)|b|}{p^2+pq+q^2}}, \frac{2(1-\beta)|b|}{\sqrt{\left|2(1-\beta)b[3]_{p,q}-[2]_{p,q}^2\right|+[2]_{p,q}^2}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{2(1-\beta)|b|}{p^2+pq+q^2}, R(p,q)\right\}$$

where

$$R(p,q) = \begin{cases} \frac{2(1-\beta)|b|}{p^2+pq+q^2}; & \frac{2(p^2+pq+q^2)|b|-(p^2+2pq+q^2)}{2(p^2+pq+q^2)|b|} \leq \beta < 1 \\ \frac{2(1-\beta)|b|\left[\left|2(1-\beta)b(p^2+pq+q^2)-(p^2+2pq+q^2)\right|+2(1-\beta)|b|(p^2+pq+q^2)\right]}{\left[\left|2(1-\beta)b(p^2+pq+q^2)-(p^2+2pq+q^2)\right|+(p^2+2pq+q^2)\right](p^2+pq+q^2)}; & \\ & 0 \leq \beta < \frac{2(p^2+pq+q^2)|b|-(p^2+2pq+q^2)}{2(p^2+pq+q^2)|b|}. \end{cases}$$

Corollary 8. *Let* $f \in \mathrm{R}_{\Sigma,b}^{q}(\phi)$ $(b \in \mathbb{C}\backslash\{0\})$. *Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{2(1-\beta)|b|}{1+q+q^2}}, \frac{2(1-\beta)|b|}{\sqrt{|2(1-\beta)b(1+q+q^2)-(1+2q+q^2)|+1+q+q^2}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{2(1-\beta)|b|}{1+q+q^2}, R(q)\right\}$$

where

$$R(q)=\begin{cases}\frac{2(1-\beta)|b|}{1+q+q^2}; & \frac{2(1+q+q^2)|b|-(1+2q+q^2)}{2(1+q+q^2)|b|}\leq\beta<1\\ \frac{2(1-\beta)|b|\left[\left|2(1-\beta)b(1+q+q^2)-(1+2q+q^2)\right|+2(1-\beta)|b|(1+q+q^2)\right]}{\left[\left|2(1-\beta)b(1+q+q^2)-(1+2q+q^2)\right|+1+2q+q^2\right](1+q+q^2)}; & \\ & 0\leq\beta<\frac{2(1+q+q^2)|b|-(1+2q+q^2)}{2(1+q+q^2)|b|}.\end{cases}$$

Corollary 9. *Let* $f\in \mathrm{R}_{\Sigma}^{p,q}\left(\frac{1+(1-2\beta)z}{1-z}\right)$. *Then*

$$|a_2|\leq\min\left\{\sqrt{\frac{2(1-\beta)}{p^2+pq+q^2}},\frac{2(1-\beta)}{\sqrt{|(1-2\beta)(p^2+q^2)-2\beta pq|+p^2+2pq+q^2}}\right\}$$

and

$$|a_3|\leq\min\left\{\frac{2(1-\beta)}{p^2+pq+q^2},R(p,q)\right\}$$

where

$$R(p,q)=\begin{cases}\frac{2(1-\beta)}{p^2+pq+q^2}; & \frac{p^2+q^2}{2(p^2+pq+q^2)}\leq\beta<1\\ \frac{(3-4\beta)(p^2+q^2)+2(1-2\beta)pq}{(p^2+pq+q^2)^2}; & 0\leq\beta<\frac{p^2+q^2}{2(p^2+pq+q^2)}\end{cases}.$$

Corollary 10. *Let* $f\in \mathrm{R}_{\Sigma}^{q}\left(\frac{1+(1-2\beta)z}{1-z}\right)$. *Then*

$$|a_2|\leq\min\left\{\sqrt{\frac{2(1-\beta)}{1+q+q^2}},\frac{2(1-\beta)}{\sqrt{\left|(1-2\beta)(1+q^2)-2\beta q)\right|+1+q+q^2}}\right\}$$

and

$$|a_3|\leq\min\left\{\frac{2(1-\beta)}{1+q+q^2},R(q)\right\}$$

where

$$R(q)=\begin{cases}\frac{2(1-\beta)}{1+q+q^2}; & \frac{1+q^2}{2(1+q+q^2)}\leq\beta<1\\ \frac{(3-4\beta)(1+q^2)+2(1-2\beta)q}{(1+q+q^2)^2}; & 0\leq\beta<\frac{1+q^2}{2(1+q+q^2)}\end{cases}.$$

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SOME REMARKS ON REGULARIZED MULTIVALUED NONCONVEX EQUILIBRIUM PROBLEMS

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Abstract. This work focuses on the class of multivalued regularized equilibrium problems in the context of uniformly prox-regular sets introduced and studied in [Noor, M.A.: Multivalued regularized equilibrium problems. J. Global Optim. 35, 483–492 (2006)]. The algorithms and results presented in the paper cited above are investigated and analyzed and some fatal errors in them are detected. Meanwhile, the correct version of the corresponding algorithm and results is given.

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Keywords: regularized multivalued nonconvex equilibrium problems, auxiliary principle technique, prox-regularity, predictor-corrector methods, jointly pseudomonotone mappings, convergence analysis

1. Introduction

It is well-known that the equilibrium problem (EP) is an unified model of several problems including variational inequality problems, optimization problems, problems of Nash equilibria, saddle point problems, fixed point problems and complementarity problems, etc, see, for example, [1, 10, 12, 19] and the references therein. Because of the wide applications to optimization, economics, finance, physics, image reconstruction, network, ecology, transportation and engineering sciences, EP have been extended and generalized in different directions, see, for example, [1, 2, 4, 6, 11, 23] and the references therein. An important and useful generalization of EP is the generalized multivalued equilibrium problem [23] involving a nonlinear bifunction. It has been shown that a wide class of unrelated odd order and nonsymmetric free, moving, obstacle and equilibrium problems can be studied via the multivalued equilibrium problems.

It is worth to mention that most of the results regarding the existence and iterative approximation of solutions to variational inequality problems and equilibrium problems have been investigated and considered so far to the case where the underlying set is a convex set. This is because all the techniques are based on the properties of the projection operator over convex sets, which may not hold in general, when the

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sets are nonconvex. In recent years, the concept of convex set has been generalized in many directions, which has potential and important applications in various fields. It is well known that the uniformly prox-regular sets are nonconvex and include the convex sets as special cases, for more details, see, for example, [8, 9, 17, 18, 25].

In the recent past, several authors considered and studied different classes of variational inequalities and equilibrium problems in the setting of uniformly prox-regular sets, see, for example, [3, 5, 7, 14, 21, 24] and the references therein.

Recently, Noor [23] considered and studied a class of equilibrium problems known as the multivalued regularized equilibrium problems in the context of uniformly prox-regular sets. He used the auxiliary principle technique [20] to suggest some iterative methods for solving the multivalued regularized equilibrium problems. He also studied the convergence analysis of the proposed methods under some certain conditions.

The main objective of this paper is to investigate and analyze the algorithms and results presented in [23]. Some fatal errors in the algorithms and main results of [23] are detected. Meanwile, the correct version of the algorithms and convergence results corresponding to the algorithms and results given in [23] is presented.

2. Notations and Preliminaries

Throughout the paper, unless otherwise specified, we use the following notations, terminology and assumptions. Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let K be a nonempty closed subset of $\mathcal{H}$. We denote by $d_K(.)$ or $d(.,K)$ the usual distance function from a point to a set K, that is, $d_K(u) = \inf_{v \in K} \|u - v\|$.

Definition 1. Let $u \in \mathcal{H}$ be a point not lying in K. A point $v \in K$ is called *a closest point or a projection of u onto K* if $d_K(u) = \|u - v\|$. The set of all such closest points is denoted by $P_K(u)$, that is,

$$P_K(u) := \{v \in K : d_K(u) = \|u - v\|\}.$$

Definition 2. The *proximal normal cone* of K at a point $u \in K$ is given by

$$N_K^P(u) := \{\xi \in \mathcal{H} : \exists \alpha > 0 \text{ such that } u \in P_K(u + \alpha\xi)\}.$$

The following lemma gives a characterization of the proximal normal cone.

Lemma 1 ([17, Proposition 1.1.5])**.** *Let K be a nonempty closed subset of $\mathcal{H}$. Then $\xi \in N_K^P(u)$ if and only if there exists a constant $\alpha = \alpha(\xi, u) > 0$ such that $\langle \xi, v - u \rangle \leq \alpha \|v - u\|^2$ for all $v \in K$.*

Definition 3 ([16])**.** Let $f : \mathcal{H} \to \mathbb{R}$ be locally Lipschitz near a point x. The *Clarke's directional derivative* of f at x in the direction v, denoted by $f^\circ(x; v)$, is defined by

$$f^\circ(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t},$$

where y is a vector in $\mathcal{H}$ and t is a positive scalar.

The *tangent cone* to K at a point $x \in K$, denoted by $T_K(x)$, is defined by

$$T_K(x) := \left\{v \in \mathcal{H} : d_K^\circ(x;v) = 0\right\}.$$

The *normal cone* to K at $x \in K$, denoted by $N_K(x)$, is defined by

$$N_K(x) := \{\xi \in \mathcal{H} : \langle \xi, v\rangle \leq 0 \text{ for all } v \in T_K(x)\}.$$

The *Clarke normal cone*, denoted by $N_K^C(x)$, is defined by

$$N_K^C(x) = \overline{co}[N_K^P(x)],$$

where $\overline{co}[S]$ denotes the closure of the convex hull of S.

Clearly, $N_K^P(x) \subseteq N_K^C(x)$. Note that $N_K^C(x)$ is a closed and convex cone, whereas $N_K^P(x)$ is convex, but may not be closed. For further details on this topic, we refer to [16, 17, 25] and the references therein.

In 1995, Clarke et al. [18] introduced a nonconvex set, called *proximally smooth set*. Subsequently, it has been investigated by Poliquin et al. [25] but under the name of uniformly prox-regular set. Such kind of sets are used in many nonconvex applications in optimization, economic models, dynamical systems, differential inclusions, etc. For further details and applications, we refer to [13] and the references therein. This class of nonconvex sets seems particularly well suited to overcome the difficulties which arise due to the nonconvexity assumption.

Definition 4 ([18])**.** For a given $r \in (0, +\infty]$, a subset K of $\mathcal{H}$ is said to be *normalized uniformly prox-regular* (or *uniformly r-prox-regular*) if for all $\bar{x} \in K$ and all $\mathbf{0} \neq \xi \in N_K^P(\bar{x})$,

$$\left\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \right\rangle \leq \frac{1}{2r}\|x - \bar{x}\|^2, \quad \text{for all } x \in K.$$

The class of normalized uniformly prox-regular sets includes the class of convex sets, p-convex sets [15], $C^{1,1}$ submanifolds (possibly with boundary) of $\mathcal{H}$, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets [18].

Lemma 2 ([18])**.** *A closed set $K \subseteq \mathcal{H}$ is convex if and only if it is uniformly r-prox-regular for every $r > 0$.*

If $r = +\infty$, then in view of Definition 4 and Lemma 2, the uniform r-prox-regularity of K is equivalent to the convexity of K.

The union of two disjoint intervals $[a,b]$ and $[c,d]$ is uniformly r-prox-regular with $r = \frac{c-b}{2}$ [14, 17, 25]. The finite union of disjoint intervals is also uniformly r-prox-regular and r depends on the distances between the intervals.

3. Regularized Multivalued Nonconvex Equilibrium Problems

Let $C(\mathcal{H})$ denote the family of all the nonempty compact subsets of $\mathcal{H}$. Suppose that $g : \mathcal{H} \to \mathcal{H}$ is a single-valued operator and $T : \mathcal{H} \to C(\mathcal{H})$ is a multivalued operator. For a given bifunction $F(.,.) : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, Noor [23] considered the problem of finding $u \in \mathcal{H} : g(u) \in K$ and $v \in T(u)$ such that

$$F(v, g(v)) + \gamma \|g(v) - g(u)\|^2 \geq 0, \quad \forall v \in \mathcal{H} : g(v) \in K, \tag{3.1}$$

where $\gamma = \frac{1}{2r}$. He called it *multivalued regularized equilibrium problem*.

It should be pointed out that there is a small mistake in the context of problem (2.1) in [23]. In fact, the bifunction $F : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ in problem (2.1) in [23] should be replaced by $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$.

Lemma 3 ([22])**.** *Let X be a complete metric space, and $T : X \to C(X)$ be a multivalued mapping. Then for any $x, y \in X$, $u \in T(x)$, there exists $v \in T(y)$ such that*

$$d(u, v) \leq M(T(x), T(y)), \tag{3.2}$$

where $M(.,.)$ is the Hausdorff metric on $C(X)$ defined by

$$M(A, B) = \max\left\{\sup_{x\in A}\inf_{y\in B}\|x - y\|, \sup_{y\in B}\inf_{x\in A}\|x - y\|\right\}, \quad \forall A, B \in C(X).$$

Noor [23] considered the following auxiliary multivalued regularized equilibrium problem: For $u \in \mathcal{H}$ with $g(u) \in K$ and $v \in T(u)$, find $w \in \mathcal{H}$ with $g(w) \in K$ and $\xi \in T(w)$ such that, for all $g(v) \in K$, the following relation holds

$$F(\xi, g(v)) + \langle g(w) - g(u), g(v) - g(w)\rangle + \gamma \|g(v) - g(w)\|^2 \geq 0. \tag{3.3}$$

He claimed that if $w = u$, then w is a solution of the multivalued regularized equilibrium problem (3.1). Based on this fact and by utilizing Lemma 3, he suggested the following predictor-corrector algorithm for solving problem (3.3).

Algorithm 1 ([23, Algorithm 3.1]). For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned} \rho F(\eta_{n+1}, g(v)) + \langle g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle \\ + \gamma \|g(u_{n+1}) - g(u_n)\|^2 \geq 0, \quad \forall g(v) \in K, \end{aligned} \tag{3.4}$$

$$\eta_n \in T(u_n) : \|\eta_{n+1} - \eta_n\| \leq M(T(u_{n+1}), T(u_n)), \tag{3.5}$$

where $\rho > 0$ is a constant and $n = 0, 1, 2, \ldots$.

In order to study the convergence analysis of Algorithm 1, Noor [23] used the following definition.

Definition 5 ([23, Definition 2.4])**.** The bifunction $F(.,.)$ is said to be:

(a) *partially relaxed strongly g-monotone* if there exists a constant $\alpha > 0$ such that, for all $u_1, u_2, z \in \mathcal{H},\ w_1 \in T(u_1), w_2 \in T(u_2)$ the following relation holds

$$F(w_1, g(u_2)) + F(w_2, g(z)) \leq \alpha \|g(z) - g(u_1)\|^2,$$

(b) *g-monotone* if

$$F(w_1, g(u_2)) + F(w_2, g(u_1)) \leq 0,$$

$$\forall u_1, u_2, z \in \mathcal{H},\ w_1 \in T(u_1), w_2 \in T(u_2);$$

(c) *g-pseudomonotone* if for all $u_1, u_2, z \in \mathcal{H}$, $w_1 \in T(u_1)$, $w_2 \in T(u_2)$,

$$F(w_1, g(u_2)) + \gamma \|g(u_2) - g(u_1)\|^2 \geq 0$$

$$\Rightarrow\ -F(w_2, g(u_1)) + \gamma \|g(u_2) - g(u_1)\|^2 \geq 0.$$

The following theorem played an important role in establishing the strong convergence of the iterative sequences generated by Algorithm 1.

Theorem 1 ([23, Theorem 3.1])**.** *Let $u \in \mathcal{H}$ be a solution of (3.1) and u_{n+1} be the approximate solution obtained from Algorithm 1. If $F(.,.)$ is g-pseudomonotone, then*

$$(1-\gamma)\|g(u_{n+1}) - g(u)\|^2 \leq \|g(u_n) - g(u)\|^2 - (1-\gamma)\|g(u_{n+1}) - g(u_n)\|^2. \tag{3.6}$$

By a careful reading of the proof of Theorem 1 (that is, [23, Theorem 3.1]), we found that there are fatal errors in it. In fact, by assuming $u \in \mathcal{H} : g(u) \in K$ and $v \in T(u)$ as a solution of the problem (3.1) and taking $v = u_{n+1}$ in (3.1), Noor [23] deduced the following inequality:

$$F(v, g(u_{n+1})) + \gamma \|g(u_{n+1}) - g(u)\|^2 \geq 0. \tag{3.7}$$

By (3.7) and with the help of the concept of g-pseudomonotonicity of the bifunction F presented in part (c) of Definition 5, he obtained the inequality (3.6) in [23] as follows:

$$-F(\eta_{n+1}, g(u)) + \gamma \|g(u_{n+1}) - g(u)\|^2 \geq 0. \tag{3.8}$$

By setting $v = u$ in (3.4), Noor [23] derived the following inequality:

$$\begin{aligned} \rho F(\eta_{n+1}, g(u)) + \langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle \\ + \gamma \|g(u) - g(u_{n+1})\|^2 \geq 0, \end{aligned} \tag{3.9}$$

and then relying on (3.8), he deduced the inequality (3.7) in [23] as follows:

$$\begin{aligned} &\langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle \\ &\geq -\rho F(\eta_{n+1}, g(u)) - \gamma \|g(u) - g(u_{n+1})\|^2 \\ &\geq -\gamma \|g(u) - g(u_{n+1})\|^2 - \gamma \|g(u_{n+1}) - g(u)\|^2. \end{aligned} \tag{3.10}$$

Letting $u = g(u) - g(u_{n+1})$ and $v = g(u_{n+1}) - g(u_n)$ and by utilizing the well-known property of the inner product, he obtained the inequality (3.8) in [23] as follows:

$$2\langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle = \|g(u) - g(u_n)\|^2 - \|g(u_{n+1}) - g(u_n)\|^2 - \|g(u) - g(u_{n+1})\|^2. \tag{3.11}$$

Combining (3.10) and (3.11), Noor [23] deduced the required inequality (3.6). We now show that under the conditions mentioned in Theorem 1, what yields is not the inequality (3.6). In fact, taking $v = u$ in (3.4), what we obtain is the inequality

$$\begin{aligned} \rho F(\eta_{n+1}, g(u)) + \langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle \\ + \gamma \|g(u_{n+1}) - g(u_n)\|^2 \geq 0, \end{aligned} \tag{3.12}$$

not the inequality (3.9). Furthermore, by virtue of (3.8) and (3.12), we obtain the inequality

$$\begin{aligned} &\langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle \\ &\geq -\rho F(\eta_{n+1}, g(u)) - \gamma \|g(u_{n+1}) - g(u_n)\|^2 \\ &\geq -\rho\gamma \|g(u) - g(u_{n+1})\|^2 - \gamma \|g(u_{n+1}) - g(u_n)\|^2, \end{aligned} \tag{3.13}$$

not the inequality (3.10). At long last, by (3.11) and (3.13), what we get is the inequality

$$(1 - 2\rho\gamma)\|g(u_{n+1}) - g(u)\|^2 \leq \|g(u) - g(u_n)\|^2 - (1 - 2\gamma)\|g(u_{n+1}) - g(u_n)\|^2,$$

not the inequality (3.6).

Noor [23] claimed that the sequence generated by Algorithm 1 is strongly convergent to a solution of the problem (3.1).

Theorem 2 ([23, Theorem 3.2])**.** *Let $\mathcal{H}$ be a finite dimensional space and let $g : \mathcal{H} \to \mathcal{H}$ be injective. Let $T : \mathcal{H} \to C(\mathcal{H})$ be M-Lipschitz continuous operator. If $\gamma \leq 1$, then the sequence $\{u_n\}$ given by Algorithm 1 converges to a solution u of (3.1).*

Now we analyze the proof of Theorem 2 (that is, [23, Theorem 3.2]).

Theorem 1 plays a crucial role in the proof of Theorem 2. But, as we have pointed out the assertion of Theorem 1 is not true necessarily. Beside this fact, we also found that, by a careful reading, there are some fatal errors in the proof of Theorem 2.

In the first place, Noor [23] asserted that the inequality (3.6) implies the boundedness of the sequence $\{g(u_n)\}$ and then he deduced the boundedness of the sequence $\{u_n\}$ under the injectivity assumption of the operator g.

In fact, relying on (3.6), Noor [23] claimed that

$$\|g(u_{n+1}) - g(u)\| \leq \|g(u_n) - g(u)\|, \tag{3.14}$$

that is, the sequence $\{\|g(u_n)-g(u)\|\}$ is nonincreasing. Taking into account of the facts that the sequence $\{\|g(u_n)-g(u)\}$ is nonincreasing and the operator g is injective, he deduced the boundedness of the sequences $\{g(u_n)\}$ and $\{u_n\}$. However, using the inequality (3.6), we obtain the inequality

$$\|g(u_{n+1})-g(u)\| \leq \frac{1}{\sqrt{1-\gamma}}\|g(u_n)-g(u)\|, \tag{3.15}$$

not the inequality (3.14). It goes without saying that the inequality (3.15) does not guarantee that the sequence $\{\|g(u_n)-g(u)\}$ is nonincreasing. Therefore, in light of the conditions mentioned in Theorem 2, the sequence $\{g(u_n)\}$ is not necessarily bounded and so the sequence $\{u_n\}$ is not also necessarily bounded.

In the second place, the author claimed that by applying the inequality (3.6), one can deduce the following inequality:

$$\sum_{n=0}^{\infty}(1-\gamma)\|g(u_{n+1})-g(u_n)\|^2 \leq \|g(u_0)-g(u)\|^2, \tag{3.16}$$

which implies that

$$\lim_{n\to\infty}\|g(u_{n+1})-g(u_n)\| = 0. \tag{3.17}$$

However, by using the inequality (3.6), what one can get is the following inequality:

$$\begin{aligned}&\sum_{n=0}^{\infty}(1-\gamma)\|g(u_{n+1})-g(u)\|^2\\ &\leq \|g(u_0)-g(u)\|^2+\sum_{n=0}^{\infty}\gamma\|g(u_{n+1})-g(u)\|^2,\end{aligned} \tag{3.18}$$

not the inequality (3.16). Obviously, the inequality (3.18) does not imply the relation (3.17). It is worth mentioning that the boundedness of the sequence $\{u_n\}$ and the relation (3.17) are main tools to establish the statement of Theorem 2.

Thirdly, on page 489, at the end of the proof of Theorem 2 (that is, [23, Theorem 3.2]), the author derived the following inequality by using M-Lipschitz continuity with constant δ of T:

$$\|v_n-v\| \leq M(T(u_n),T(u)) \leq \delta\|u_n-u\|.$$

Unfortunately, there is a fatal error in the above inequality. In fact, in view of the preceding inequality, the author used the fact that $v \in T(u)$ before proving it. Even, without considering this fact, the following example illustrates that for any given $x, y \in \mathcal{H}$, $u \in T(x)$ and $v \in T(y)$, the inequality (3.2) need not be hold.

Example 1. Let X be the space of all bounded real sequences, that is,

$$X = l^{\infty} = \Big\{x = (x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R}, \sup_{n\in\mathbb{N}}|x_n| < \infty\Big\}.$$

Then X equipped with the ∞-norm (or sup-norm) defined by

$$\|x\|_\infty = \|(x_n)_{n\in\mathbb{N}}\|_\infty = \sup_{n\in\mathbb{N}} |x_n|, \qquad \forall x = (x_n)_{n\in\mathbb{N}} \in l^\infty,$$

is a Banach space, and the set $\{e_1, e_2, \dots, e_n, \dots\}$, where for each $n \in \mathbb{N}$, $e_n = (0, 0, \dots, 1, 0, \dots)$, 1 at the nth coordinate and all other coordinates are zero, is a Schauder basis of l^∞. Let the multivalued mapping $T : X \to CB(X)$ be defined by

$$T(x) = \begin{cases} \{(\frac{\gamma}{n})_{n\in\mathbb{N}}, e_n : n = 1, 2, \dots\}, & x \neq e_l, \\ \{e_n\}_{n\in\mathbb{N}}, & x = e_l, \end{cases}$$

where $\gamma \in [-1, 0)$ is an arbitrary real constant and $l \in \mathbb{N}$ is arbitrary but fixed. Take $x \neq e_l$, an arbitrary element belonging to X, $y = e_l$ and $u = (\frac{\gamma}{n})_{n\in\mathbb{N}}$. If $a = (\frac{\gamma}{n})_{n\in\mathbb{N}}$, then considering the fact that $\gamma \in [-1, 0)$, for any $k \in \mathbb{N}$, we have

$$\begin{aligned} d(a, e_k) &= \left\| (\frac{\gamma}{n})_{n\in\mathbb{N}} - e_k \right\|_\infty = \sup_{i\in\mathbb{N}} \left| \frac{\gamma}{i} - e_{k,i} \right| \\ &= \sup \left\{ \left| \frac{\gamma}{i} \right|, \left| \frac{\gamma}{k} - 1 \right| : i \in \mathbb{N}, i \neq k \right\} \\ &= \left| \frac{\gamma}{k} - 1 \right| = 1 - \frac{\gamma}{k}, \end{aligned}$$

where

$$e_{k,i} = \delta_{k,i} = \begin{cases} 1 & i = k, \\ 0 & otherwise, \end{cases}$$

and so the fact that $\gamma \in [-1, 0)$ implies that

$$d(a, T(y)) = \inf_{b\in T(y)} d(a, b) = \inf \left\{ 1 - \frac{\gamma}{k} : k \in \mathbb{N} \right\} = 1.$$

If $a = e_n$, for some $n \in \mathbb{N}$, then for each $k \in \mathbb{N}$, $k \neq n$, we have

$$d(a, e_k) = \|e_n - e_k\|_\infty = \sup_{i\in\mathbb{N}} \left| e_{n,i} - e_{k,i} \right| = 1$$

and

$$d(a, e_n) = \|e_n - e_n\|_\infty = 0.$$

Therefore,

$$d(a, T(y)) = \inf_{b\in T(y)} d(a, b) = 0,$$

consequently,

$$\sup_{a\in T(x)} d(a, T(y)) = 1.$$

If $b = e_k$, for some $k \in \mathbb{N}$, then for $a = (\frac{\gamma}{n})_{n\in\mathbb{N}}$, we obtain

$$d(e_k, a) = \left\| e_k - (\frac{\gamma}{n})_{n\in\mathbb{N}} \right\|_\infty = 1 - \frac{\gamma}{k}$$

and

$$d(e_k, e_n) = \|e_k - e_n\|_\infty = \sup_{i \in \mathbb{N}} \left| e_{k,i} - e_{n,i} \right| = 1$$

for each $n \neq k$. For the case when $n = k$, we have

$$d(e_k, e_k) = \|e_k - e_k\|_\infty = 0.$$

Accordingly,

$$d(T(x), b) = \inf_{a \in T(x)} d(a, b) = 0,$$

and so

$$\sup_{b \in T(y)} d(T(x), b) = 0.$$

Hence,

$$M(T(x), T(y)) = \max\left\{ \sup_{a \in T(x)} d(a, T(y)), \sup_{b \in T(y)} d(T(x), b) \right\} = 1.$$

Taking into account of the fact that for each $k \in \mathbb{N}$,

$$\left\| (\frac{\gamma}{n})_{n \in \mathbb{N}} - e_k \right\|_\infty = 1 - \frac{\gamma}{k} > 1,$$

because $\gamma \in [-1, 0)$, it follows that for any $v \in T(y)$,

$$d(u, v) = \|u - v\|_\infty > M(T(x), T(y)).$$

In view of the above mentioned arguments, the statements of Theorems 1 and 2 are not valid in general.

In order to overcome these difficulties, we present the correct versions of the problems (3.1) and (3.3) and Algorithm 1.

Let $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a nonlinear bifunction and let $T : K \to C(\mathcal{H})$ be a multivalued operator. Replacing the operator $g : \mathcal{H} \to \mathcal{H}$ in the problem (3.1) by a surjective operator $g : K \to K$, we consider the problem of finding $u \in K$ and $v \in T(u)$ such that

$$F(v, g(v)) + \gamma \|g(v) - g(u)\|^2 \geq 0, \qquad \forall v \in K, \tag{3.19}$$

where $\gamma = \frac{1}{2r}$, and is called the *regularized multivalued nonconvex equilibrium problem* (in short, RMNEP). In the sequel, we denote by RMNEP(T, F, g, K) the set of solutions of RMNEP (3.19).

For given $u \in K$ and $v \in T(u)$, we consider the following auxiliary regularized multivalued nonconvex equilibrium problem of finding $w \in K$ and $\xi \in T(w)$ such that

$$\rho F(\xi, g(v)) + \langle w - u, v - w \rangle + \rho\gamma \|g(v) - g(w)\|^2 \geq 0, \qquad \forall v \in K, \tag{3.20}$$

where $\rho > 0$ is a constant. We observe that if $w = u$, then clearly w is a solution of RMNEP (3.19). This observation allows us to suggest the following iterative method for solving RMNEP (3.19).

Algorithm 2. For given $u_0 \in K$ and $\xi_0 \in T(u_0)$, compute the iterative sequences $\{u_n\}$ and $\{\xi_n\}$ by the iterative schemes

$$\rho F(\xi_n, g(v)) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle + \rho\gamma\|g(v) - g(u_{n+1})\|^2 \geq 0, \ \forall v \in K, \tag{3.21}$$

$$\xi_n \in T(u_n) : \|\xi_{n+1} - \xi_n\| \leq M(T(u_{n+1}), T(u_n)), \tag{3.22}$$

where $\rho > 0$ is a constant and $n = 0, 1, 2, \ldots$.

To establish the strong convergence of the sequence generated by Algorithm 2, we need the following definitions.

Definition 6. A multivalued operator $T : \mathcal{H} \to C(\mathcal{H})$ is said to be *M-Lipschitz continuous* with constant δ if there exists a constant $\delta > 0$ such that

$$M(T(u), T(v)) \leq \delta\|u - v\|, \qquad \forall u, v \in \mathcal{H},$$

where $M(.,.)$ is the Hausdorff metric on $C(\mathcal{H})$.

Definition 7. Let $T : K \to C(\mathcal{H})$ be a multivalued operator and $g : K \to K$ be a nonlinear operator. For a given positive real constant γ, the bifunction $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is said to be g-pseudomonotone with respect to T with constant γ, iff

$$F(\xi_1, g(u_2)) + \gamma\|g(u_2) - g(u_1)\|^2 \geq 0$$

implies that

$$F(\xi_2, g(u_1)) + \gamma\|g(u_2) - g(u_1)\|^2 \leq 0, \qquad \forall u_1, u_2 \in K, \xi_1 \in T(u_1), \xi_2 \in T(u_2).$$

The next proposition plays a key role in establishing the strong convergence of the iterative sequences generated by Algorithm 2.

Proposition 1. *Let T, F, g and γ be the same as in* RMNEP (3.19) *and let $u \in K$, $v \in T(u)$ be the solution of* RMNEP (3.19). *Suppose further that $\{u_n\}$ and $\{\xi_n\}$ are the sequences generated by Algorithm 2. If F is g-pseudomonotone with respect to T with constant γ, then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2, \qquad \forall n \geq 0. \tag{3.23}$$

Proof. Since $u \in K$ and $v \in T(u)$ are the solution of RMNEP (3.19), we have

$$\rho F(v, g(v)) + \rho\gamma\|g(v) - g(u)\|^2 \geq 0, \quad \forall v \in K, \tag{3.24}$$

where the real constant $\rho > 0$ is the same as in Algorithm 2.

Taking $v = u_n$ in (3.24), we obtain

$$\rho F(v, g(u_n)) + \rho\gamma\|g(u_n) - g(u)\|^2 \geq 0. \tag{3.25}$$

Thanks to the fact that F is g-pseudomonotone with respect to T with constant γ, from the inequality (3.25), it follows that

$$\rho F(\xi_n, g(u)) + \rho\gamma\|g(u_n) - g(u)\|^2 \leq 0. \tag{3.26}$$

Letting $v = u$ in (3.21), we get

$$\rho F(\xi_n, g(u)) + \langle u_{n+1} - u_n, u - u_{n+1}\rangle + \rho\gamma\|g(u) - g(u_n)\|^2 \geq 0. \tag{3.27}$$

By combining (3.26) and (3.27), we conclude that

$$\langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq -\rho F(\xi_n, g(u)) - \rho\gamma\|g(u) - g(u_n)\|^2 \geq 0. \tag{3.28}$$

On the other hand, by assuming $x = u_{n+1} - u_n$ and $y = u - u_{n+1}$, and by utilizing the well known property of the inner product, we obtain

$$2\langle u_{n+1} - u_n, u - u_{n+1}\rangle = \|u - u_n\|^2 - \|u - u_{n+1}\|^2 - \|u_{n+1} - u_n\|^2. \tag{3.29}$$

Making use of (3.28) and (3.29), we derive the required inequality (3.23). □

We now prove the strong convergence of the sequences generated by Algorithm 2 to a solution of RMNEP (3.19).

Theorem 3. *Let $\mathcal{H}$ be a finite dimensional real Hilbert space and let $g : K \to K$ be a continuous surjective operator. Suppose that the operator $T : K \to C(\mathcal{H})$ is M-Lipschitz continuous with constant δ and the bifunction $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is continuous in the first argument. Furthermore, let all the conditions of Proposition 1 hold and* RMNEP$(T, F, g, K) \neq \varnothing$. *Then, the iterative sequences $\{u_n\}$ and $\{\xi_n\}$ generated by Algorithm 2 converge strongly to $\hat{u} \in K$ and $\hat{v} \in T(\hat{u})$, respectively, and $(\hat{u}, \hat{v})$ is a solution of* RMNEP (3.19).

Proof. Let $u \in K$ and $v \in T(u)$ be the solution of RMNEP (3.19). From the inequality (3.23) it follows that the sequence $\{\|u - u_n\|\}$ is nonincreasing and so the sequence $\{u_n\}$ is bounded. Moreover, in virtue of the inequality (3.23), we yield

$$\sum_{n=0}^{\infty} \|u_n - u_{n+1}\|^2 \leq \|u - u_0\|^2,$$

which guarantees $\|u_n - u_{n+1}\| \to 0$, as $n \to \infty$. Let $\hat{u}$ be a cluster point of the sequence $\{u_n\}$. With the help of the boundedness of the sequence $\{u_n\}$, we deduce that there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \to \hat{u}$, as $i \to \infty$. Utilizing the inequality (3.22) and considering the fact that the operator T is M-Lipschitz continuous with constant δ, we obtain

$$\|\xi_{n_i+1} - \xi_{n_i}\| \leq M(T(u_{n_i+1}), T(u_{n_i})) \leq \delta\|u_{n_i+1} - u_{n_i}\|. \tag{3.30}$$

The inequality (3.30) implies that $\|\xi_{n_i+1} - \xi_{n_i}\| \to 0$, as $i \to \infty$. Therefore, $\{\xi_{n_i}\}$ is a Cauchy sequence in $\mathcal{H}$. Hence, $\xi_{n_i} \to \hat{v}$, as $i \to \infty$, for some $\hat{v} \in \mathcal{H}$. By (3.21), we get, for all $v \in K$, that

$$\rho F(\xi_{n_i}, g(v)) + \langle u_{n_i+1} - u_{n_i}, v - u_{n_i+1}\rangle + \rho\gamma\|g(v) - g(u_{n_i})\|^2 \geq 0. \tag{3.31}$$

Taking into consideration the facts that F is continuous in the first argument and g is continuous, by taking the limit in the relation (3.31) as $i \to \infty$, it follows that

$$F(\hat{v}, g(v)) + \gamma \|g(v) - g(\hat{u})\|^2 \geq 0, \quad \forall v \in K. \tag{3.32}$$

In the meanwhile, from M-Lipschitz continuity with constant δ of T, we deduce that

$$\begin{aligned} d(\hat{v}, T(\hat{u})) &= \inf\left\{\|\hat{v} - q\| : q \in T(\hat{u})\right\} \\ &\leq \|\hat{v} - \xi_{n_i}\| + d(\xi_{n_i}, T(\hat{u})) \\ &\leq \|\hat{v} - \xi_{n_i}\| + M(T(u_{n_i}), T(\hat{u})) \\ &\leq \|\hat{v} - \xi_{n_i}\| + \delta \|u_{n_i} - \hat{u}\|. \end{aligned}$$

We observe that the right-hand side of the above inequality tends to zero as $i \to \infty$. Thanks to the fact that $T(\hat{u}) \in CB(\mathcal{H})$, we conclude that $\hat{v} \in T(\hat{u})$. Then, the inequality (3.32) implies that $\hat{u} \in K$ and $\hat{v} \in T(\hat{u})$ are the solution of RMNEP (3.19). Thus, relying on Proposition 1, we have

$$\|u_{n+1} - \hat{u}\| \leq \|u_n - \hat{u}\|, \quad \forall n \geq 0.$$

The preceding inequality guarantees that $u_n \to \hat{u}$, as $n \to \infty$. Consequently, the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$. By (3.22) and M-Lipschitz continuity with constant δ of T, we conclude that for all $n \geq 0$,

$$\|\xi_{n+1} - \xi_n\| \leq M(T(u_{n+1}), T(u_n)) \leq \delta \|u_{n+1} - u_n\|. \tag{3.33}$$

The inequality (3.33) implies that $\{\xi_n\}$ is also a Cauchy sequence in $\mathcal{H}$. Considering the fact that $\hat{v}$ is a cluster point of the sequence $\{\xi_n\}$, we conclude that $\xi_n \to \hat{v}$, as $n \to \infty$, that is, the sequence $\{\xi_n\}$ has also exactly one cluster point $\hat{v}$. This gives us the desired result. □

It is well known that to implement the proximal point methods, one has to calculate the approximate solution implicity, which is in itself a difficult problem. To overcome this drawback, Noor [23] considered the following auxiliary uniformly regularized equilibrium problem: For a given $u \in \mathcal{H} : g(u) \in K$, $v \in T(u)$, find $w \in \mathcal{H} : g(w) \in K$ such that

$$\begin{aligned} \rho F(v, g(v)) + \langle g(w) - g(u), g(v) - g(w) \rangle \\ + \gamma \|g(v) - g(w)\|^2 \geq 0, \quad \forall g(v) \in K, \end{aligned} \tag{3.34}$$

where $\rho > 0$ is a constant.

Noor [23] claimed that if $w = u$, then clearly w is a solution of the problem (3.1). Based on this fact, he proposed the following iterative method for solving the problem (3.1).

Algorithm 3 ([23, Algorithm 3.2]). For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned} \rho F(v_n, g(v)) + \langle g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle \\ + \gamma \|g(v) - g(u_{n+1})\|^2 \geq 0, \ \forall g(v) \in K, \end{aligned} \tag{3.35}$$

$$v_n \in T(u_n) : \|v_{n+1} - v_n\| \leq M(T(u_{n+1}), T(u_n)), \quad n = 0, 1, 2, \ldots. \tag{3.36}$$

By an easy checking, we found that unlike the claim of the author in [23], if $w = u$ then w is not necessarily a solution of the problem (3.1). In fact, if $w = u$, then the auxiliary problem (3.34) reduces to the following regularized multivalued nonconvex equilibrium problem:

Find $u \in \mathcal{H} : g(u) \in K$ such that

$$\rho F(v, g(v)) + \gamma \|g(v) - g(u)\|^2 \geq 0, \quad \forall v \in \mathcal{H} : g(v) \in K. \tag{3.37}$$

However, the following example shows that every solution of the problem (3.37) is not necessary a solution of the problem (3.1).

Example 2. Let $\mathcal{H} = \mathbb{R}$ and $K = [\alpha, \beta] \cup [\delta, \sigma]$ be the union of two disjoint intervals $[\alpha, \beta]$ and $[\delta, \sigma]$ where $0 < \alpha < \beta < \delta < \sigma$. Then, K is a uniformly r-prox-regular set with $r = \frac{\delta - \beta}{2}$ and so we have $\gamma = \frac{1}{2r} = \frac{1}{\delta - \beta}$. Let us define the operators $T : \mathcal{H} \to C(\mathcal{H})$ and $g : \mathcal{H} \to \mathcal{H}$ by

$$T(x) = \left\{\lambda_i (a_i^{m_i x} + x^{n_i}) : i = 1, 2, \ldots, l\right\}, \quad g(x) = \mu \sqrt[p]{x^q}, \quad \forall x \in \mathcal{H}$$

and the bifunction $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ by

$$F(x, y) = \theta x (y - \beta) \qquad \forall x, y \in \mathcal{H},$$

where $p, q \in \mathbb{N} \backslash \{1\}$ and $n_i, l \in \mathbb{N}$ are arbitrary but fixed natural numbers, $\mu, \theta > 0$, $m_i \in \mathbb{R}$, $a_i > 1$ and

$$\lambda_i < \frac{\beta - \sigma}{\theta(\delta - \beta)\left(a_i^{m_i \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{p n_i}}\right)}, (\text{where } i = 1, 2, \ldots, l)$$

are arbitrary real constants. Take

$$u = \sqrt[q]{(\frac{\beta}{\mu})^p} \text{ and } v = \lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{p n_j}}\right),$$

where $j \in \{1, 2, \ldots, l\}$ is arbitrary but fixed. Meanwhile, let

$$\rho = \min\left\{-\frac{1}{\theta \lambda_i \left(a_i^{m_i \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{p n_i}}\right)} : i = 1, 2, \ldots, l\right\}.$$

Then, for all $v \in \mathcal{H}$, we have

$$\begin{aligned}
&\rho F(v, g(v)) + \gamma \|g(v) - g(u)\|^2 \\
&= \rho\theta\lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}} \right) (\mu \sqrt[p]{v^q} - \beta) + \frac{1}{\delta - \beta} (\mu \sqrt[p]{v^q} - \beta)^2 \\
&= (\mu \sqrt[p]{v^q} - \beta) \left[\rho\theta\lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}} \right) + \frac{1}{\delta - \beta} (\mu \sqrt[p]{v^q} - \beta) \right].
\end{aligned} \tag{3.38}$$

If $v \in \left[\sqrt[q]{(\frac{\alpha}{\mu})^p}, \sqrt[q]{(\frac{\beta}{\mu})^p} \right]$, then $\alpha - \beta \le \mu \sqrt[p]{v^q} - \beta \le 0$ and

$$\begin{aligned}
&\rho\theta\lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}} \right) + \frac{\alpha - \beta}{\delta - \beta} \\
&\le \rho\theta\lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}} \right) + \frac{1}{\delta - \beta} (\mu \sqrt[p]{v^q} - \beta) \\
&\le \rho\theta\lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}} \right).
\end{aligned} \tag{3.39}$$

For the case when $v \in \left[\sqrt[q]{(\frac{\delta}{\mu})^p}, \sqrt[q]{(\frac{\sigma}{\mu})^p} \right]$, we have $\mu \sqrt[p]{v^q} - \beta \in [\delta - \beta, \sigma - \beta]$ and

$$\begin{aligned}
&\rho\theta\lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}} \right) + 1 \\
&\le \rho\theta\lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}} \right) + \frac{1}{\delta - \beta} (\mu \sqrt[p]{v^q} - \beta) \\
&\le \rho\theta\lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}} \right) + \frac{\sigma - \beta}{\delta - \beta}.
\end{aligned} \tag{3.40}$$

By (3.39) and (3.40) and taking into consideration the fact that

$$\lambda_i < 0 < \rho \le -\frac{1}{\theta\lambda_i \left(a_i^{m_i \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_i}} \right)}, \text{ for each } i \in \{1, 2, \ldots, l\},$$

it follows that

$$(\mu \sqrt[p]{v^q} - \beta) \left[\rho\theta\lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}} \right) + \tfrac{1}{\delta - \beta} (\mu \sqrt[p]{v^q} - \beta) \right] \ge 0,$$
$$\forall v \in \left[\sqrt[q]{(\tfrac{\alpha}{\mu})^p}, \sqrt[q]{(\tfrac{\beta}{\mu})^p} \right] \cup \left[\sqrt[q]{(\tfrac{\delta}{\mu})^p}, \sqrt[q]{(\tfrac{\sigma}{\mu})^p} \right],$$

which leads to

$$(\mu \sqrt[p]{v^q}-\beta)\left[\rho\theta\lambda_j\left(a_j^{m_j\sqrt[q]{(\frac{\beta}{\mu})^p}}+\sqrt[q]{(\frac{\beta}{\mu})^{pn_j}}\right)+\tfrac{1}{\delta-\beta}(\mu\sqrt[p]{v^q}-\beta)\right]\geq 0,$$
$$\forall v\in\mathcal{H}:\mu\sqrt[p]{v^q}\in[\alpha,\beta]\cup[\delta,\sigma]. \tag{3.41}$$

Now, in virtue of (3.38) and (3.41), we deduce that

$$\rho F(v,g(v))+\gamma\|g(v)-g(u)\|^2\geq 0,\qquad \forall v\in\mathcal{H}:g(v)\in K.$$

On the other hand, for all $v\in\mathcal{H}$, we have

$$\begin{aligned}
&F(v,g(v))+\gamma\|g(v)-g(u)\|^2\\
&=\theta\lambda_j\left(a_j^{m_j\sqrt[q]{(\frac{\beta}{\mu})^p}}+\sqrt[q]{(\frac{\beta}{\mu})^{pn_j}}\right)(\mu\sqrt[p]{v^q}-\beta)+\frac{1}{\delta-\beta}(\mu\sqrt[p]{v^q}-\beta)^2\\
&=(\mu\sqrt[p]{v^q}-\beta)\left[\theta\lambda_j\left(a_j^{m_j\sqrt[q]{(\frac{\beta}{\mu})^p}}+\sqrt[q]{(\frac{\beta}{\mu})^{pn_j}}\right)+\frac{1}{\delta-\beta}(\mu\sqrt[p]{v^q}-\beta)\right].
\end{aligned}$$

The fact that $\lambda_i<\dfrac{\beta-\sigma}{\theta(\delta-\beta)\left(a_i^{m_i\sqrt[q]{(\frac{\beta}{\mu})^p}}+\sqrt[q]{(\frac{\beta}{\mu})^{pn_i}}\right)}$, for each $i\in\{1,2,\dots,l\}$, implies that

$$\theta\lambda_j\left(a_j^{m_j\sqrt[q]{(\frac{\beta}{\mu})^p}}+\sqrt[q]{(\frac{\beta}{\mu})^{pn_j}}\right)+\frac{1}{\delta-\beta}(\mu\sqrt[p]{v^q}-\beta)<0,$$
$$\forall v\in\left[\sqrt[q]{(\frac{\delta}{\mu})^p},\sqrt[q]{(\frac{\sigma}{\mu})^p}\right].$$

Since $\mu\sqrt[p]{v^q}-\beta\in[\delta-\beta,\sigma-\beta]$, for all $v\in\left[\sqrt[q]{(\frac{\delta}{\mu})^p},\sqrt[q]{(\frac{\sigma}{\mu})^p}\right]$, it follows that

$$(\mu\sqrt[p]{v^q}-\beta)\left[\theta\lambda_j\left(a_j^{m_j\sqrt[q]{(\frac{\beta}{\mu})^p}}+\sqrt[q]{(\frac{\beta}{\mu})^{pn_j}}\right)+\frac{1}{\delta-\beta}(\mu\sqrt[p]{v^q}-\beta)\right]<0,$$
$$\forall v\in\left[\sqrt[q]{(\frac{\delta}{\mu})^p},\sqrt[q]{(\frac{\sigma}{\mu})^p}\right],$$

that is,

$$(\mu\sqrt[p]{v^q}-\beta)\left[\theta\lambda_j\left(a_j^{m_j\sqrt[q]{(\frac{\beta}{\mu})^p}}+\sqrt[q]{(\frac{\beta}{\mu})^{pn_j}}\right)+\frac{1}{\delta-\beta}(\mu\sqrt[p]{v^q}-\beta)\right]<0,$$
$$\forall v\in\mathcal{H}:g(v)\in[\delta,\sigma].$$

Hence, the inequality

$$F(v, g(v)) + \gamma \|g(v) - g(u)\|^2 \geq 0$$

cannot hold for all $v \in \mathcal{H}$ with $g(v) \in K$. Relying on this fact, we conclude that every solution of the problem (3.37) is not a solution of the problem (3.1) necessarily. Accordingly, for given $u \in \mathcal{H} : g(u) \in K$ and $v \in T(u)$, if $w = u$ is a solution of the auxiliary regularized equilibrium problem (3.34), then w need not be a solution of the problem (3.1).

Noor [23] claimed that using essentially the technique of Theorem 2, one can study the convergence analysis of Algorithm 3. For this end, he first asserted that the following statement holds.

Theorem 4 ([23, Theorem 3.3])**.** *Let the bifunction $F(.,.)$ be partially relaxed strongly g-monotone with constant $\alpha > 0$. If u_{n+1} is the approximate solution obtained from Algorithm 3 and $u \in \mathcal{H}$ is a solution of (3.1), then*

$$\begin{aligned}&(1-\gamma)\|g(u) - g(u_{n+1})\|^2 \\ &\leq \|g(u) - g(u_n)\|^2 - (1 - 2\rho\alpha - \gamma)\|g(u_n) - g(u_{n+1})\|^2.\end{aligned} \tag{3.42}$$

Now we analyze Algorithm 3 and the proof of Theorem 4 (that is, [23, Theorem 3.3]).

Algorithm 3 is constructed based on the fact that if $w = u$, where $u \in \mathcal{H}$ with $g(u) \in K$ and $v \in T(u)$ are given elements satisfying the problem (3.1), then w is a solution of the problem (3.1). Whereas, it was shown that for given $u \in \mathcal{H}$: $g(u) \in K$ and $v \in T(u)$, if $w = u$ then the problem (3.20) does not reduce to the problem (3.1), that is, w is not a solution of the problem (3.1). Hence, by considering the problem (3.34) for a given $u \in \mathcal{H}$ with $g(u) \in K$ and $v \in T(u)$, and in light of the above mentioned arguments, it should be pointed out that unlike the claim in [23], Algorithm 3 cannot be used for solving the problem (3.1). In order to overcome these difficulties, we need to replace $\gamma\|g(v) - g(w)\|^2$ and $\gamma\|g(v) - g(u_{n+1})\|^2$ by $\rho\gamma\|g(v) - g(w)\|^2$ and $\rho\gamma\|g(v) - g(u_{n+1})\|^2$ in (3.34) and (3.35), respectively. In the meanwhile, in view of the proof of Theorem 4 (that is, Theorem 3.3 in [23]), η_n in (3.36) must be replaced by ν_n.

By a careful reading the proof of Theorem 4 (that is, [23, Theorem 3.3]), we discovered that under the assumptions mentioned in Theorem 4, the relation (3.42) does not hold necessarily. By assuming $u \in \mathcal{H}$ with $g(u) \in K$ and $v \in T(u)$ as the solution of the problem (3.1) and taking $v = u_{n+1}$ in (3.1), Noor [23] deduced the inequality (3.7) (which is the inequality (3.15) in [23]). Taking $v = u$ in (3.35), he obtained the following inequality:

$$\begin{aligned}\rho F(v_n, g(u)) + \langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle \\ + \gamma\|g(u) - g(u_{n+1})\|^2 \geq 0.\end{aligned} \tag{3.43}$$

Applying (3.7) and (3.43) and considering the fact that $F(.,.)$ is partially relaxed strongly g-monotone with constant α, the author derived the inequality (3.16) in [23] as follows:

$$\begin{aligned}
&\langle g(u_{n+1})-g(u_n), g(u)-g(u_{n+1})\rangle \\
&\geq -\rho F(v_n, g(u)) - \gamma\|g(u)-g(u_{n+1})\|^2 \\
&\geq \rho\Big(F(v_n, g(u)) + F(v, g(u_{n+1}))\Big) \\
&\quad -\gamma\|g(u)-g(u_{n+1})\|^2 - \gamma\|g(u_n)-g(u_{n+1})\|^2 \\
&\geq -\alpha\rho\|g(u_n)-g(u_{n+1})\|^2 - \gamma\|g(u)-g(u_{n+1})\|^2 \\
&\quad -\gamma\|g(u_n)-g(u_{n+1})\|^2.
\end{aligned} \tag{3.44}$$

In the end, by combining (3.11) and (3.44), he concluded the required result (3.42). However, unlike the claim in [23], by (3.7) and (3.43) and invoking the definition of partially relaxed strong g-monotonicity of the bifunction F given in part (a) of Definition 5, what obtain is the inequality

$$\begin{aligned}
&\langle g(u_{n+1})-g(u_n), g(u)-g(u_{n+1})\rangle \\
&\geq -\rho F(v_n, g(u)) - \gamma\|g(u)-g(u_{n+1})\|^2 \\
&\geq -\rho\Big(F(v_n, g(u)) + F(v, g(u_{n+1}))\Big) \\
&\quad -\rho\gamma\|g(u)-g(u_{n+1})\|^2 - \gamma\|g(u)-g(u_{n+1})\|^2 \\
&\geq -\alpha\rho\|g(u_n)-g(u_{n+1})\|^2 - (\rho+1)\gamma\|g(u)-g(u_{n+1})\|^2,
\end{aligned} \tag{3.45}$$

not the inequality (3.44). In the meantime, by combining (3.11) and (3.45), we get the inequality

$$\begin{aligned}
&(1-2(\rho+1)\gamma)\|g(u)-g(u_{n+1})\|^2 \\
&\leq \|g(u)-g(u_n)\|^2 - (1-2\alpha\rho)\|g(u_{n+1})-g(u_n)\|^2,
\end{aligned}$$

not the inequality (3.42).

Relying on the above mentioned arguments and considering the fact that Theorem 4 plays an important and key role in the study of the convergence analysis of Algorithm 3, by an argument analogous to the previous one, mentioned for the proof of Theorem 2, we can prove that unlike the claim of the author in [23], using essentially the technique of Theorem 2 (that is, [23, Theorem 3.2], one cannot study the convergence analysis of Algorithm 3.

In order to overcome these difficulties, we now present the correct versions of the problem (3.3), Algorithm 3 and Theorem 4.

Let T, F, g and γ be the same as in RMNEP (3.19). For given $u \in K$ and $v \in T(u)$, we consider the problem of finding $w \in K$ such that

$$\rho F(v, g(v)) + \langle w-u, v-w\rangle + \rho\gamma\|g(v)-g(w)\|^2 \geq 0, \qquad \forall v \in K, \tag{3.46}$$

where $\rho > 0$ is a constant. If $w = u$, then it goes without saying that w is a solution of RMNEP (3.19). This fact enables us to suggest a predictor-corrector method for solving RMNEP (3.19) as follows.

Algorithm 4. Let T, F, g and γ be the same as in RMNEP (3.19). For given $u_0 \in K$ and $v_0 \in T(u_0)$, define the iterative sequences $\{u_n\}$ and $\{v_n\}$ by the iterative schemes

$$\rho F(v_n, g(v)) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle + \rho\gamma \|g(v) - g(u_{n+1})\|^2 \geq 0, \quad \forall v \in K, \tag{3.47}$$

$$v_n \in T(u_n) : \|v_{n+1} - v_n\| \leq M(T(u_{n+1}), T(u_n)), \tag{3.48}$$

where $\rho > 0$ is a constant and $n = 0, 1, 2, \ldots$.

In order to establish the strong convergence of the sequences generated by Algorithm 4, we need the following definition.

Definition 8. Let $T : K \to CB(\mathcal{H})$ be a multivalued operator and let $g : K \to K$ be a nonlinear operator. The bifunction $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is said to be

(a) *g-monotone* with respect to T if

$$F(w_1, g(u_2)) + F(w_2, g(u_1)) \leq 0, \quad \forall u_1, u_2 \in K, w_1 \in T(u_1), w_2 \in T(u_2);$$

(b) *r-strongly g-monotone* with respect to T if there exists a constant $r > 0$ such that, for all $u_1, u_2 \in K, w_1 \in T(u_1), w_2 \in T(u_2)$, we have

$$F(w_1, g(u_2)) + F(w_2, g(u_1)) \leq -r\|g(u_1) - g(u_2)\|^2;$$

(c) *partially ς-strongly g-monotone* with respect to T if there exists a constant $\varsigma > 0$ such that, for all $u_1, u_2, z \in K, w_1 \in T(u_1), w_2 \in T(u_2)$, we have

$$F(w_1, g(u_2)) + F(w_2, g(z)) \leq -\varsigma\|g(z) - g(u_2)\|^2;$$

(d) *partially ζ-relaxed g-monotone* with respect to T of type (I) if there exists a constant $\zeta > 0$ such that, for all $u_1, u_2, z \in K, w_1 \in T(u_1), w_2 \in T(u_2)$, we have

$$F(w_1, g(u_2)) + F(w_2, g(z)) \leq \zeta\|z - u_1\|^2;$$

(e) *partially (ϱ, ϖ)-mixed relaxed and strongly g-monotone with respect to T of type* (I) if there exist two constants $\varrho, \varpi > 0$ such that, for all $u_1, u_2, z \in K, w_1 \in T(u_1), w_2 \in T(u_2)$, we have

$$F(w_1, g(u_2)) + F(w_2, g(z)) \leq \varrho\|z - u_1\|^2 - \varpi\|g(z) - g(u_2)\|^2.$$

It should be remarked that if $z = u_1$, then partially strong g-montonicity with respect to T and partially mixed relaxed and strong g-monotonicity with respect to T of type (I) of the bifunction F reduce to strong g-monotonicity with respect to T, and partially relaxed g-monotonicity with respect to T of type (I) reduces to g-monotonicity.

The following result plays a crucial role in the study of the convergence analysis of Algorithm 4.

Proposition 2. *Let T, F, g and γ be the same as in* RMNEP (3.19) *and let $u \in K$, $v \in T(u)$ be the solution of* RMNEP (3.19). *Moreover, assume that $\{u_n\}$ and $\{v_n\}$ are the sequences generated by Algorithm 4. If the bifunction F is partially $(\alpha, 2\rho\gamma)$-mixed relaxed and strongly g-monotone with respect to T of type* (I), *then*

$$\|u-u_{n+1}\|^2 \le \|u-u_n\|^2 - (1-2\alpha\rho)\|u_{n+1}-u_n\|^2, \qquad \forall n \ge 0. \tag{3.49}$$

Proof. Since $u \in K$ and $v \in T(u)$ are the solution of RMNEP (3.19), it follows that (u, v) satisfies (3.24). Taking $v = u_{n+1}$ in (3.24), we obtain

$$\rho F(v, g(u_{n+1})) + \rho\gamma\|g(u_{n+1}) - g(u)\|^2 \ge 0. \tag{3.50}$$

Setting $v = u$ in (3.47), we get

$$\rho F(v_n, g(u)) + \langle u_{n+1}-u_n, u-u_{n+1}\rangle + \rho\gamma\|g(u)-g(u_{n+1})\|^2 \ge 0. \tag{3.51}$$

By combining (3.50) and (3.51) and considering the fact that the bifunction F is partially $(\alpha, 2\gamma)$-mixed relaxed and strongly g-monotone with respect to T of type (I), we deduce that

$$\begin{aligned}
&\langle u_{n+1}-u_n, u-u_{n+1}\rangle \\
&\ge -\rho F(v_n, g(u)) - \rho\gamma\|g(u)-g(u_{n+1})\|^2 \\
&\ge -\rho\Big(F(v_n, g(u)) + F(v, g(u_{n+1}))\Big) - 2\rho\gamma\|g(u)-g(u_{n+1})\|^2 \\
&\ge -\alpha\rho\|u_{n+1}-u_n\|^2.
\end{aligned} \tag{3.52}$$

Applying (3.29) and (3.52), we conclude that for all $n \ge 0$,

$$\|u-u_n\|^2 - \|u-u_{n+1}\|^2 - \|u_{n+1}-u_n\|^2 \ge -2\alpha\rho\|u_{n+1}-u_n\|^2,$$

whence we obtain

$$\|u-u_{n+1}\|^2 \le \|u-u_n\|^2 - (1-2\alpha\rho)\|u_{n+1}-u_n\|^2,$$

the required result (3.49). □

The paper is closed by the next assertion that provides us the required conditions under which the iterative sequences generated by Algorithm 4 converges strongly to a solution of RMNEP (3.19).

Theorem 5. *Let $\mathcal{H}$ be a finite dimensional real Hilbert space and let $g : K \to K$ be a continuous surjective mapping. Assume that the bifunction $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is continuous in the first argument and the operator $T : K \to C(\mathcal{H})$ is M-Lipschitz continuous with constant δ. Suppose further that all the conditions of Proposition 2 hold and* RMNEP$(T, F, g, K) \neq \varnothing$. *If $\rho \in (0, \frac{1}{2\alpha})$, then the iterative sequences $\{u_n\}$ and $\{v_n\}$ generated by Algorithm 4 converge strongly to $\hat{u} \in K$ and $\hat{v} \in T(\hat{u})$, respectively, and $(\hat{u}, \hat{v})$ is a solution of* RMNEP (3.19).

Proof. Since RMNEP$(T, F, g, K) \neq \varnothing$, we can take $u \in K$ and $v \in T(u)$ as a solution of RMNEP (3.19). In light of the inequality (3.49), we deduce that the sequence $\{\|u_n - u\|\}$ is nonincreasing and so the sequence $\{u_n\}$ is bounded. In the meanwhile, by (3.49), it follows that

$$\sum_{n=0}^{\infty}(1-2\alpha\rho)\|u_{n+1}-u_n\|^2 \leq \|u-u_0\|^2,$$

which guarantees $\|u_{n+1} - u_n\| \to 0$, as $n \to \infty$. Let $\hat{u}$ be a cluster point of the sequence $\{u_n\}$. Considering the fact that the sequence $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \to \hat{u}$, as $i \to \infty$. By a similar proof as in Theorem 1, we conclude that $\{v_{n_i}\}$ is a Cauchy sequence in $\mathcal{H}$ and $v_{n_i} \to \hat{v}$, for some $\hat{v} \in \mathcal{H}$, as $i \to \infty$. Moreover, $\hat{u} \in K$ and $\hat{v} \in T(\hat{u})$ are the solution of RMNEP (3.19), and the sequences $\{u_n\}$ and $\{v_n\}$ have exactly one cluster point $\hat{u}$ and $\hat{v}$, respectively, and the proof is now complete. □

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SOME GENERALIZATIONS OF DARBO FIXED POINT THEOREM AND ITS APPLICATION

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Abstract. Recently, B. Samet et al. (Fixed Point Theory and Appl. 2013:5, 2013, doi:10.1186/1687-1812-2013-5) has introduced the concept of new contraction mappings and obtained results on the fixed point for nonlinear contractive mappings in a metric space. In the present paper, we introduce the concept of a new contraction via the measure of non-compactness on a Banach space, we obtain a few generalizations of Darbo fixed point theorem and extend some recent results of Aghajani et al. We also show the applicability of obtained results to the theory of integral equations.

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1. Introduction

Integral equation create a very important and significant part of the mathematical analysis and has various applications into real world problems. On the other hand, measures of noncompactness are very useful tools in the wide area of functional analysis such as the metric fixed point theory and the theory of operator equations in Banach spaces. They are also used in the study of functional equations, ordinary and partial differential equations, fractional partial differential equations, integral and integro-differential equations, optimal control theory, etc., see [1–6, 12–16]. In our investigations, we apply the method associated with the technique of measures of noncompactness to generalize the Darbo fixed point theorem [9] and to extend some recent results of Aghajani et al.[5]. Moreover, we are going to study the solvability of the following nonlinear quadratic integral equation of the fractional order

$$\begin{aligned} x(t) = a(t) + h(t,x(t)) \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} m s^{m-1} k_1(f(t,s)) x(s)\, ds \\ \times \int_0^t \frac{(t^n - s^n)^{\beta-1}}{\Gamma(\beta)} n s^{n-1} k_2(g(t,s)) x(s)\, ds, \\ t \in I = [0,1],\ 0 < \alpha, \beta \le 1,\ m,n > 0, \quad (1.1) \end{aligned}$$

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where, $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$. Let us recall that the function $h(t,x)$ involved in Eq. (1.1) generates the superposition operator H defined by the formula $(Hx)(t) = h(t,x(t))$ where $x = x(t)$ is an arbitrary function defined on I (cf.[7],[11]). We show that Eq. (1.1) has solutions in $C(I)$ which are nondecreasing on the interval I.
Recently, Samet et al. [17] obtained the following useful generalization of the Banach contraction principle.

Theorem 1. *Let (X,d) be a complete metric space, $\varphi : X \to \mathbb{R}_+$ be a lower semi continuous function and $T : X \to X$ be a given mapping. Suppose that for each $0 < a < b < \infty$, there exists $0 < \gamma(a,b) < 1$ such that*

$$\begin{aligned} &a \le d(x,y) + \varphi(x) + \varphi(y) \le b \\ &\qquad \Longrightarrow d(Tx,Ty) + \varphi(Tx) + \varphi(Ty) \le \gamma(a,b)[d(x,y) + \varphi(x) + \varphi(y)] \end{aligned} \tag{1.2}$$

for all $x, y \in X$. Then T has a unique fixed point $x \in X$.

2. Auxiliary facts

In this section, we recall some notations, definitions and theorems to obtain all results of this work. Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0,+\infty)$. Let $(E, \|\cdot\|)$ be a real Banach space. Let $\overline{B}(x,r)$ denote the closed ball centered at x with radius r. The symbol $\overline{B}_r$ stands for the ball $\overline{B}(0,r)$. For X, a nonempty subset of E, we denote by $\overline{X}$ and $ConvX$ the closure and the convex closure of X, respectively. Moreover, let us denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets. We use the following definition of the measure of noncompactness given in [9].

Definition 1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of non-compactness in E if it satisfies the following conditions:

- **(1^0):** The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{N}_E$,
- **(2^0):** $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$,
- **(3^0):** $\mu(\bar{X}) = \mu(X)$,
- **(4^0):** $\mu(ConvX) = \mu(X)$,
- **(5^0):** $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$,
- **(6^0):** If (X_n) is a sequence of closed sets from m_E such that $X_{n+1} \subset X_n (n = 1,2,...)$ and if $\lim_{n\longrightarrow\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^\infty X_n$ is nonempty.

The family $ker\mu$ defined in axiom (1^0) is called the kernel of the measure of noncompactness μ. One of the properties of the measure of noncompactness is $X_\infty \in ker\mu$. Indeed, from the inequality $\mu(X_\infty) \le \mu(X_n)$ for $n = 1,2,3,...$, we infer that

$\mu(X_\infty) = 0$. Further facts concerning measures of noncompactness and their properties may be found in [8,9]. Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem, and includes the existence part of Banach's fixed point theorem.

Theorem 2 (Schauder [3])**.** *Let C be a nonempty, bounded, closed, convex subset of a Banach space E. Then every compact, continuous map $T : C \to C$ has at least one fixed point.*

In the following we state a fixed-point theorem of Darbo type proved by Banaś and Goebel [9].

Theorem 3. *Let C be a nonempty, closed, bounded, and convex subset of the Banach space E and $F : C \to C$ be a continuous mapping. Assume that there exist a constant $k \in [0,1)$ such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset of C. Then F has a fixed-point in C.*

Remark 1. [9] Under the assumptions of the above theorem it can be shown that the set $FixF$ of fixed points of F belonging to Ω is an element of $ker\mu$.

3. Fixed point theorem

The main result of the present paper is the following fixed point theorem which is a generalization of Darbo fixed point theorem (cf. Theorem3).

Theorem 4. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and $T : C \to C$ be two continuous functions such that for any $0 < a < b < \infty$, there exists $0 < \gamma(a,b) < 1$ such that for all $X \subseteq C$,*

$$a \leq \mu(X) + \varphi(\mu(X)) \leq b \Longrightarrow \mu(TX) + \varphi(\mu(TX)) \leq \gamma(a,b)[\mu(X) + \varphi(\mu(X))], \tag{3.1}$$

where μ is an arbitrary measure of noncompactness. Then T has at least one fixed point in C.

Proof. Let$C_0 = C$, we construct a sequence $\{C_n\}$ such that $C_{n+1} = Conv(TC_n)$, for $n \geq 0.TC_0 = TC \subseteq C = C_0, C_1 = Conv(TC_0) \subseteq C = C_0$, therefore by continuing this process, we have

$$C_0 \supseteq C_1 \supseteq C_2 \supseteq ... \supseteq C_n \supseteq C_{n+1} \supseteq ...\cdot$$

If there exists a positive integer $N \in \mathbb{N}$ such that $\mu(C_N) + \varphi(\mu(C_N)) = 0$,i.e, $\mu(C_N) = 0$, then C_N is relatively compact. On the other hand, we have $T(C_N) \subseteq Conv(TC_N) = C_{N+1} \subseteq C_N$. Then Theorem 2 implies that T has a fixed point. So we assume that

$$0 < \mu(C_n) + \varphi(\mu(C_n)), \quad \forall n \geq 1. \tag{3.2}$$

Suppose that

$$\mu(C_{n_0}) + \varphi(\mu(C_{n_0})) < \mu(C_{n_0+1}) + \varphi(\mu(C_{n_0+1})) \tag{3.3}$$

for some $n_0 \in \mathbb{N}$. In addition, by (3.2) and (3.3), we have

$$0 < a := \mu(C_{n_0}) + \varphi(\mu(C_{n_0})) \leq \mu(C_{n_0}) + \varphi(\mu(C_{n_0})) \\ < \mu(C_{n_0+1}) + \varphi(\mu(C_{n_0}+1)) := b.$$

By using (3.1), there exists $0 < \gamma(a,b) < 1$ such that

$$\mu(C_{n_0+1}) + \varphi(\mu(C_{n_0}+1)) = \mu(convTC_{n_0}) + \varphi(\mu(convTC_{n_0})) \\ = \mu(TC_{n_0}) + \varphi(\mu(TC_{n_0})) \leq \gamma(a,b)[\mu(C_{n_0}) + \varphi(\mu(C_{n_0}))] \\ < \gamma(a,b)[\mu(C_{n_0+1}) + \varphi(\mu(C_{n_0}+1))],$$

which implies that $\gamma(a,b) > 1$, a contradiction. This implies that

$$\mu(C_{n+1}) + \varphi(\mu(Cn+1)) \leq \mu(C_n) + \varphi(\mu(C_n)),$$

for all $n \in \mathbb{N}$, that is, the sequence $\{\mu(C_n) + \varphi(\mu(C_n))\}$ is non-increasing and non-negative, we infer that

$$\lim_{n\to\infty} \mu(C_n) + \varphi(\mu(C_n)) = \delta. \tag{3.4}$$

Now, we show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Then

$$0 < a := \delta \leq \mu(C_n) + \varphi(\mu(C_n)) \leq \mu(C_0) + \varphi(\mu(C_0)) =: b, \quad for\ all\ n \geq 0.$$

By (3.1), there exists $0 < \gamma(a,b) < 1$ such that

$$\mu(C_{n+1}) + \varphi(\mu(C_n+1)) = \mu(convTC_n) + \varphi(\mu(convTC_n)) \\ = \mu(TC_n) + \varphi(\mu(TC_n)) \leq \gamma(a,b)[\mu(C_n) + \varphi(\mu(C_n))].$$

Letting $n \longrightarrow \infty$ in the above inequality and using (3.4), we get $\delta \leq \gamma(a,b)\delta$, which implies that $\gamma(a,b) \geq 1$, a contradiction. Thus, $\delta = 0$ and

$$\lim_{n\to\infty} \mu(C_n) + \varphi(\mu(C_n)) = 0.$$

It follows that

$$\lim_{n\to\infty} \mu(C_n) = 0.$$

Since $C_n \supseteq C_{n+1}$ and $TC_n \subseteq C_n$ for all $n = 1,2,...$, it follows from (6^0) that

$$C_\infty = \bigcap_{n=1}^{\infty} C_n$$

is nonempty convex closed set, invariant under T and belongs to $Ker\mu$. Therefore Theorem 2 completes the proof. □

An immediate consequence of Theorem 4 is the following.

Theorem 5. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E. Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and $T : C \to C$ be continuous functions. Suppose that there exists a constant $0 < \lambda < 1$, such that*

$$\mu(TX) + \varphi(\mu(TX)) \leq \lambda[\mu(X) + \varphi(\mu(X))],$$

for all $X \subseteq C$, where μ is an arbitrary measure of noncompactness. Then T has at least one fixed point in C.

Remark 2. Taking $\varphi \equiv 0$ in Theorem 5, we obtain the Darbo fixed point theorem.

Theorem 6. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : C \to C$ be a continuous operator satisfying*

$$\psi[\mu(TX) + \varphi(\mu(TX))] \leq \psi[\mu(X) + \varphi(\mu(X))] - \phi[\mu(X) + \varphi(\mu(X))], \tag{3.5}$$

for any nonempty $X \subseteq C$, where μ is an arbitrary measure of noncompactness, $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is continuous function and $\psi, \phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are given functions such that ψ is continuous and nondecreasing function and ϕ is lower semicontinuous, $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. Then T has at least one fixed point in C.

Proof. Similarly as in the proof of the Theorem 4, by using induction we define the sequence $\{C_n\}$, where $C_0 = C$ and $C_{n+1} = Conv(TC_n)$, for $n \geq 0$. If there exists a positive integer $N_0 \in \mathbb{N}$ such that $\mu(C_{N_0}) + \varphi(\mu(C_{N_0})) = 0$, i.e, $\mu(C_{N_0}) = 0$, then C_{N_0} is relatively compact. On the other hand, $T(C_{N_0}) \subseteq Conv(TC_{N_0}) = C_{N_0+1} \subseteq C_{N_0}$, then theorem 2 implies that T has a fixed point. So we assume that

$$0 < \mu(C_n) + \varphi(\mu(C_n)), \quad \forall n \geq 1.$$

From (3.1), we get

$$\begin{aligned}
\psi[\mu(C_{n+1}) + \varphi(\mu(C_n + 1))] &= \psi[\mu(convTC_n) + \varphi(\mu(convTC_n))] \\
&= \psi[\mu(TC_n) + \varphi(\mu(TC_n))] \\
&\leq \psi[\mu(C_n) + \varphi(\mu(C_n))] - \phi[\mu(C_n) + \varphi(\mu(C_n))] \\
&\leq \psi[\mu(C_n) + \varphi(\mu(C_n))].
\end{aligned} \tag{3.6}$$

On the other hand, ψ is nondecreasing, then

$$\mu(C_{n+1}) + \varphi(\mu(C_n + 1)) \leq \mu(C_n) + \varphi(\mu(C_n))$$

that is, the sequence $\{\mu(C_n) + \varphi(\mu(C_n))\}$ is nonincreasing and nonnegative. We infer that

$$\lim_{n\to\infty} \mu(C_n) + \varphi(\mu(C_n)) = \delta. \tag{3.7}$$

Now, we show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Passing to the (upper) limit in (3.6), when $n \longrightarrow \infty$, it follows that

$$\psi(\delta) \leq \psi(\delta) - \liminf_{n\longrightarrow\infty} \phi[\mu(C_n) + \varphi(\mu(C_n))] \leq \psi(\delta) - \phi(\delta),$$

i.e. $\phi(\delta) \leq 0$. Using the properties of ϕ function, we get that $\delta = 0$ which is a contradiction. We conclude that

$$\lim_{n\to\infty} \mu(C_n) + \varphi(\mu(C_n)) = 0,$$

therefore,

$$\lim_{n\to\infty} \mu(C_n) = 0.$$

Since $C_n \supseteq C_{n+1}$ and $TC_n \subseteq C_n$ for all $n = 1, 2, \ldots$. It follows from (6^0) that

$$C_\infty = \bigcap_{n=1}^{\infty} C_n$$

is nonempty convex closed set, invariant under T and belongs to $Ker\mu$. Therefore Theorem 2 completes the proof. □

Now, the following fixed point theorem follows immediately from Theorem 6 is a generalization of [5].

Theorem 7. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : C \to C$ be continuous operator satisfying*

$$\psi(\mu(TX)) \leq \psi(\mu(X)) - \phi(\mu(X)),$$

for any nonempety $X \subseteq C$, where μ is an arbitrary measure of noncompactness and $\psi, \phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are given functions such that ψ is continuous and nondecreasing function and ϕ is lower semicontinuous, $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. Then T has at least one fixed point in C.

4. An application to existence of solution for a functional integral equation.

Let $C(I) = C[0,1]$ be the Banach space of all continuous functions on $I = [0,1]$ equipped with the standard norm

$$\|x\| = max\{|x(t)| : t \geq 0\}.$$

Next, we recall the definition of a measure of noncompactness in $C(I)$ which will be used in this Section. This measure was introduced and studied in [10]. Let X be a fixed nonempty and bounded subset of $C(I)$. For $x \in X$ and $\epsilon \geq 0$, denote by $\omega(x,\epsilon)$ the modulus of continuity of the function x on the interval $[0,1]$, i.e.

$$\omega(x,\epsilon) := sup\{|x(t) - x(s)| : t, s \in [0,1], |t - s| \leq \epsilon\}.$$

Further, let us put

$$\omega(X,\epsilon) := sup\{\omega(x,\epsilon) : x \in X\},\ \omega_0(X) := \lim_{\epsilon\to 0} \omega(X,\epsilon).$$

Define

$$i(x) := sup\{|x(s) - x(t)| - [x(s) - x(t)] : t, s \in I, t \leq s\},$$

and

$$i(X) := sup\{i(x) : x \in X\}.$$

Observe that all functions belonging to X are nondecreasing on I if and only if $i(X) = 0$.

Now, let us define the function μ on the family $\mathfrak{M}_C(I)$ by the formula

$$\mu(X) := \omega_0(X) + i(X).$$

It can be shown [10] that the function μ is a measure of noncompactness in the space $C(I)$.

Now, equation (1.1) will be investigated under the assumptions:

(i): $a : I \to \mathbb{R}_+$ is a continuous, nondecreasing and nonnegative function on I;

(ii): $h : I \times \mathbb{R} \to \mathbb{R}$ is continuous function in t, x such that $h(I \times \mathbb{R}_+) \subseteq \mathbb{R}_+$ and there exists a nonnegative constant c such that

$$|h(t,x) - h(t,y)| \leq c|x - y| \tag{4.1}$$

for all $t \in I$ and all $x, y \in \mathbb{R}$;

(iii): The superposition operator H generated by the function $h(t,x)$ satisfies for any nonnegative function x the condition $i(Hx) \leq c\ i(x)$ where c is the same constant as in (ii);

(iv): $f, g : I \times I \to \mathbb{R}$ are continuous and the functions $f(t,s)$ and $g(t,s)$ are nondecreasing for each variable t and s, separately;

(v): $k_1 : Imf \to \mathbb{R}_+$ is a continuous and nondecreasing function on the compact set Imf;

(vi): $k_2 : Img \to \mathbb{R}_+$ is a continuous and nondecreasing function on the compact set Img;

(vii): The inequality

$$M_1\Gamma(\alpha+1)\Gamma(\beta+1) + (cr + M_2)\|k_1\|\|k_2\|r^2 \leq \Gamma(\alpha+1)\Gamma(\beta+1)r \tag{4.2}$$

has a positive solution r_0 such that $c\ \|k_1\|\|k_2\|r_0^2 \leq \Gamma(\alpha+1)\Gamma(\beta+1)$, where $M_1 = max\{|a(t)| : t \in I\}$ and $M_2 = max\{|h(t,0)| : t \in I\}$.

Theorem 8. *Under assumptions $(i) - (vii)$, the equation (1.1) has at least one solution $x = x(t)$ which belongs to the space $C(I)$ and is nondecreasing on the interval I.*

Proof. Consider the operators F, G and T defined on the space $C(I)$ by the formulas

$$(Fx)(t) = \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} ms^{m-1}k_1(f(t,s))x(s)\,ds,$$

$$(Gx)(t) = \int_0^t \frac{(t^n - s^n)^{\beta-1}}{\Gamma(\beta)} ns^{n-1}k_2(g(t,s))x(s)\,ds,$$

$$(Tx)(t) = a(t) + h(t,x(t))(Fx)(t)(Gx)(t).$$

Firstly, we prove that F and G are selfmaps on $C(I)$. To this end, it suffices to show that if $x \in C(I)$ then $Fx,\ Gx \in C(I)$. Fix $\epsilon > 0$, let $x \in C(I)$ and let $t_1, t_2 \in I$ (without loss of generality assume that $t_2 \geq t_1$) and $|t_2 - t_1| \leq \epsilon$. Then we get

$$\begin{aligned}
|(Fx)(t_2) - (Fx)(t_1)| \leq |&\int_0^{t_2} \frac{(t_2^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} m s^{m-1} k_1(f(t_2,s))x(s)ds \\
&- \int_0^{t_2} \frac{(t_2^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} m s^{m-1} k_1(f(t_1,s))x(s)ds| \\
&+ |\int_0^{t_2} \frac{(t_2^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} m s^{m-1} k_1(f(t_1,s))x(s)ds \\
&- \int_0^{t_1} \frac{(t_2^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} m s^{m-1} k_1(f(t_1,s))x(s)ds| \\
&+ |\int_0^{t_1} \frac{(t_2^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} m s^{m-1} k_1(f(t_1,s))x(s)ds \\
&- \int_0^{t_1} \frac{(t_1^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} m s^{m-1} k_1(f(t_1,s))x(s)ds| \\
\leq &\int_0^{t_2} \frac{(t_2^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} m s^{m-1} |k_1(f(t_2,s)) - k_1(f(t_1,s))||x(s)|ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} m s^{m-1} |k_1(f(t_1,s))||x(s)|ds \\
&+ \int_0^{t_1} \frac{|(t_2^m - s^m)^{\alpha-1} - (t_1^m - s^m)^{\alpha-1}|}{\Gamma(\alpha)} m s^{m-1} |k_1(f(t_1,s))||x(s)|ds.
\end{aligned}$$

Therefore, if we denote

$$\omega_{k_1 of}(\epsilon,.) = sup\{|k_1(f(t,s)) - k_1(f(t',s))| : t,t',s \in I \ and \ |t-t'| \leq \epsilon\}$$

then

$$\begin{aligned}
|(Fx)(t_2) - (Fx)(t_1)| \leq &\frac{\|x\|\omega_{k_1 of}(\epsilon,.)}{\Gamma(\alpha)} \frac{t_2^{m\alpha}}{\alpha} + \frac{\|x\|\|k_1\|}{\Gamma(\alpha)} \frac{(t_2^m - t_1^m)^\alpha}{\alpha} \\
&+ \frac{\|x\|\|k_1\|}{\Gamma(\alpha)} [\frac{(t_2^m - t_1^m)^\alpha}{\alpha} + \frac{t_1^{m\alpha}}{\alpha} - \frac{t_2^{m\alpha}}{\alpha}] \\
&\leq \frac{\|x\|\omega_{k_1 of}(\epsilon,.)}{\Gamma(\alpha+1)} + \frac{2\|x\|\|k_1\|}{\Gamma(\alpha+1)} (t_2^m - t_1^m)^\alpha.
\end{aligned}$$

Similarly, one can show that

$$|(Gx)(t_2) - (Gx)(t_1)| \leq \frac{\|x\|\omega_{k_2 og}(\epsilon,.)}{\Gamma(\beta+1)} + \frac{2\|x\|\|k_2\|}{\Gamma(\beta+1)} (t_2^n - t_1^n)^\beta,$$

where

$$\omega_{k_2og}(\epsilon,.) = sup\{|k_2(g(t,s)) - k_2(g(t',s))| : t,t',s \in I \ and \ |t-t'| \leq \epsilon\}.$$

Obviously, $\omega_{k_1of}(\epsilon,.) \longrightarrow 0$ as $\epsilon \longrightarrow 0$ which is a simple consequence of the uniform continuity of the function k_1of on the set $I \times I$. Thus $Fx \in C(I)$, and similar to the previous case $Gx \in C(I)$, and consequently, $Tx \in C(I)$. Also, we have

$$|(Fx)(t)| \leq \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} ms^{m-1}|k_1(f(t,s))||x(s)|ds \leq \frac{||k_1||||x||}{\Gamma(\alpha+1)} \tag{4.3}$$

for all $t \in I$. On the other hand,

$$|(Gx)(t)| \leq \int_0^t \frac{(t^n - s^n)^{\beta-1}}{\Gamma(\beta)} ns^{n-1}|k_2(g(t,s))||x(s)|ds \leq \frac{||k_2||||x||}{\Gamma(\beta+1)}. \tag{4.4}$$

Linking (4.3) and (4.4) we obtain

$$\begin{aligned}|(Tx)(t)| &\leq |a(t)| + |h(t,x)||Fx(t)||Gx(t)| \\ &\leq M_1 + [|h(t,x) - h(t,0)| + |h(t,0)|]\frac{\|k_1\|\|k_2\|\|x\|^2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \\ &\leq M_1 + (c\|x\| + M_2)\frac{\|k_1\|\|k_2\|\|x\|^2}{\Gamma(\alpha+1)\Gamma(\beta+1)}.\end{aligned}$$

Hence,

$$\|Tx\| \leq M_1 + (c\|x\| + M_2)\frac{\|k_1\|\|k_2\|\|x\|^2}{\Gamma(\alpha+1)\Gamma(\beta+1)}.$$

Thus, if $||x|| \leq r_0$ we obtain from assumption (vii) the estimate

$$\|Tx\| \leq M_1 + (cr_0 + M_2)\frac{\|k_1\|\|k_2\|r_0^2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \leq r_0.$$

Consequently, the operator T maps the ball $B_{r_0} \subset C(I)$ into itself. Next, we prove that the operator T is continuous on B_{r_0}. To do this, let $\{x_n\}$ be a sequence in B_{r_0} such that $x_n \to x$. We have to show that $Tx_n \to Tx$. In fact, for each $t \in I$, we have

$$\begin{aligned}|(Fx_n)(t) - (Fx)(t)| = |&\int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} ms^{m-1}k_1(f(t,s))x_n(s)ds \\ &- \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} ms^{m-1}k_1(f(t,s))x(s)ds| \\ &\leq \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} ms^{m-1}|k_1(f(t,s))||x_n(s) - x(s)|ds,\end{aligned}$$

thus

$$||Fx_n - Fx|| \leq \frac{||k_1||}{\Gamma(\alpha+1)}||x_n - x||.$$

Similarly, we have

$$||Gx_n - Gx|| \leq \frac{||k_2||}{\Gamma(\beta+1)}||x_n - x||.$$

As,

$$\begin{aligned}
&|(Tx_n)(t) - (Tx)(t)| = \\
&= |h(t,x_n(t))(Fx_n)(t)(Gx_n)(t) - h(t,x(t))(Fx)(t)(Gx)(t)| \\
&\leq |h(t,x_n(t))(Fx_n)(t)(Gx_n)(t) - h(t,x(t))(Fx_n)(t)(Gx_n)(t)| \\
&\quad + |h(t,x(t))(Fx_n)(t)(Gx_n)(t) - h(t,x(t))(Fx)(t)(Gx_n)(t)| \\
&\quad + |h(t,x(t))(Fx)(t)(Gx_n)(t) - h(t,x(t))(Fx)(t)(Gx)(t)| \\
&\leq |h(t,x_n(t)) - h(t,x(t))||(Fx_n)(t)||(Gx_n)(t)| \\
&\quad + |h(t,x(t))||(Fx_n)(t) - (Fx)(t)||(Gx_n)(t)| \\
&\quad + |h(t,x(t))||(Fx)(t)||(Gx_n)(t) - (Gx)(t)| \\
&\leq c|x_n(t) - x(t)| \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} ms^{m-1}|k_1(f(t,s))||x_n(s)|ds \\
&\quad \times \int_0^t \frac{(t^n - s^n)^{\beta-1}}{\Gamma(\beta)} ns^{n-1}|k_2(g(t,s))||x_n(s)|ds \\
&\quad + (c|x(t)| + M_2) \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} ms^{m-1}|k_1(f(t,s))||x_n(s) - x(s)|ds \\
&\quad \times \int_0^t \frac{(t^n - s^n)^{\beta-1}}{\Gamma(\beta)} ns^{n-1}|k_2(g(t,s))||x_n(s)|ds \\
&\quad + (c|x(t)| + M_2) \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} ms^{m-1}|k_1(f(t,s))||x(s)|ds \\
&\quad \times \int_0^t \frac{(t^n - s^n)^{\beta-1}}{\Gamma(\beta)} ns^{n-1}|k_2(g(t,s))||x_n(s) - x(s)|ds.
\end{aligned}$$

It follows that

$$\begin{aligned}
||Tx_n - Tx| \leq c||x_n - x|| &\frac{||k_1||}{\Gamma(\alpha+1)} \frac{||k_2||}{\Gamma(\beta+1)}||x_n||^2 \\
&+ (c||x|| + M_2)\frac{||k_1||}{\Gamma(\alpha+1)}||x_n - x||\frac{||k_2||}{\Gamma(\beta+1)}||x_n|| \\
&+ (c||x|| + M_2)\frac{||k_1||}{\Gamma(\alpha+1)}||x||\frac{||k_2||}{\Gamma(\beta+1)}||x_n - x||.
\end{aligned}$$

This proves that T is continuous on B_{r_0}. Consider the operator T on the subset $B_{r_0}^+$ of the ball B_{r_0} defined in the following way:

$$B_{r_0}^+ = \{x \in B_{r_0} : x(t) \geq 0, \ for \ t \in I\}.$$

Obviously, the set $B_{r_0}^+$ is nonempty, bounded, closed and convex. In view of our assumptions $(i),(iii),(v)$ and (vi) if $x(t) \geq 0$ then $(Tx)(t) \geq 0$ for all $t \in I$. Thus T transforms the set $B_{r_0}^+$ into itself. Moreover, T is continuous on $B_{r_0}^+$. Let X be a nonempty subset of $B_{r_0}^+$. Fix $\epsilon > 0$ and $t_1, t_2 \in I$ with $|t_2 - t_1| \leq \epsilon$. Without loss of generality assume that $t_2 \geq t_1$. Then we get

$$\begin{aligned}
|(Tx)(t_2) - (Tx)(t_1)| &= |a(t_2) + h(t_2, x(t_2))(Fx)(t_2)(Gx)(t_2) \\
&\quad - a(t_1) - h(t_1, x(t_1))(Fx)(t_1)(Gx)(t_1)| \\
&\leq |a(t_2) - a(t_1)| + |h(t_2, x(t_2))(Fx)(t_2)(Gx)(t_2) \\
&\quad - h(t_1, x(t_2))(Fx)(t_2)(Gx)(t_2)| \\
&\quad + |h(t_1, x(t_2))(Fx)(t_2)(Gx)(t_2) - h(t_1, x(t_1))(Fx)(t_2)(Gx)(t_2)| \\
&\quad + |h(t_1, x(t_1))(Fx)(t_2)(Gx)(t_2) - h(t_1, x(t_1))(Fx)(t_1)(Gx)(t_2)| \\
&\quad + |h(t_1, x(t_1))(Fx)(t_1)(Gx)(t_2) - h(t_1, x(t_1))(Fx)(t_1)(Gx)(t_1)| \\
&\leq |a(t_2) - a(t_1)| + |h(t_2, x(t_2)) - h(t_1, x(t_2))||(Fx)(t_2)||(Gx)(t_2)| \\
&\quad + |h(t_1, x(t_2)) - h(t_1, x(t_1))||(Fx)(t_2)||(Gx)(t_2)| \\
&\quad + |h(t_1, x(t_1))||(Fx)(t_2) - (Fx)(t_1)||(Gx)(t_2)| \\
&\quad + |h(t_1, x(t_1))||(Fx)(t_1)||(Gx)(t_2) - (Gx)(t_1)| \\
&\leq \omega(a, \epsilon) + \gamma_{r_0}(h, \epsilon) \frac{||x||||k_1||}{\Gamma(\alpha+1)} \frac{||x||||k_2||}{\Gamma(\beta+1)} + c|x(t_2) - x(t_1)| \frac{||x||||k_1||}{\Gamma(\alpha+1)} \frac{||x||||k_2||}{\Gamma(\beta+1)} \\
&\quad + (c||x|| + M_2)[\frac{||x||\omega_{k_1 of}(\epsilon, .)}{\Gamma(\alpha+1)} + \frac{2||x||||k_1||}{\Gamma(\alpha+1)}(t_2^m - t_1^m)^\alpha] \frac{||x||||k_2||}{\Gamma(\beta+1)} \\
&\quad + (c||x|| + M_2) \frac{||x||||k_1||}{\Gamma(\alpha+1)} [\frac{||x||\omega_{k_2 og}(\epsilon, .)}{\Gamma(\beta+1)} + \frac{2||x||||k_2||}{\Gamma(\beta+1)}(t_2^n - t_1^n)^\beta],
\end{aligned}$$

where we denoted

$$\gamma_{r_0}(h, \epsilon) = sup\{|h(t, x) - h(t', x)| : t, t' \in I, x \in [0, r_0], |t - t'| \leq \epsilon\}.$$

By applying the mean value theorem on $[t_1, t_2]$, we get

$$|t_2^m - t_1^m|^\alpha \leq m^\alpha |t_2 - t_1|^\alpha,$$

and

$$|t_2^n - t_1^n|^\beta \leq n^\beta |t_2 - t_1|^\beta.$$

Thus from the last inequality, we get

$$\begin{aligned}|(Tx)(t_2)-(Tx)(t_1)| &\leq \omega(a,\epsilon)+\gamma_{r_0}(h,\epsilon)\frac{||x||||k_1||}{\Gamma(\alpha+1)}\frac{||x||||k_2||}{\Gamma(\beta+1)}\\ &+c|x(t_2)-x(t_1)|\frac{||x||||k_1||}{\Gamma(\alpha+1)}\frac{||x||||k_2||}{\Gamma(\beta+1)}\\ &+(c||x||+M_2)[\frac{||x||\omega_{k_1of}(\epsilon,.)}{\Gamma(\alpha+1)}+\frac{2||x||||k_1||}{\Gamma(\alpha+1)}(m\epsilon)^{\alpha}]\frac{||x||||k_2||}{\Gamma(\beta+1)}\\ &+(c||x||+M_2)\frac{||x||||k_1||}{\Gamma(\alpha+1)}[\frac{||x||\omega_{k_2og}(\epsilon,.)}{\Gamma(\beta+1)}+\frac{2||x||||k_2||}{\Gamma(\beta+1)}(n\epsilon)^{\beta}],\end{aligned}$$

hence,

$$\begin{aligned}\omega(Tx,\epsilon) &\leq \omega(a,\epsilon)+\gamma_{r_0}(h,\epsilon)\frac{r_0^2||k_1||||k_2||}{\Gamma(\alpha+1)\Gamma(\beta+1)}+c\,\omega(x,\epsilon)|\frac{r_0^2||k_1||||k_2||}{\Gamma(\alpha+1)\Gamma(\beta+1)}\\ &+(cr_0+M_2)[\frac{r_0\omega_{k_1of}(\epsilon,.)}{\Gamma(\alpha+1)}+\frac{2r_0||k_1||}{\Gamma(\alpha+1)}(m\epsilon)^{\alpha}]\frac{r_0||k_2||}{\Gamma(\beta+1)}\\ &+(cr_0+M_2)\frac{r_0||k_1||}{\Gamma(\alpha+1)}[\frac{r_0\omega_{k_2og}(\epsilon,.)}{\Gamma(\beta+1)}+\frac{2r_0||k_2||}{\Gamma(\beta+1)}(n\epsilon)^{\beta}].\end{aligned}$$

Thus, taking the supremum on X, we obtain

$$\begin{aligned}\omega(TX,\epsilon) &\leq \omega(a,\epsilon)+\gamma_{r_0}(h,\epsilon)\frac{r_0^2||k_1||||k_2||}{\Gamma(\alpha+1)\Gamma(\beta+1)}+c\,\omega(X,\epsilon)|\frac{r_0^2||k_1||||k_2||}{\Gamma(\alpha+1)\Gamma(\beta+1)}\\ &+(cr_0+M_2)[\frac{r_0\omega_{k_1of}(\epsilon,.)}{\Gamma(\alpha+1)}+\frac{2r_0||k_1||}{\Gamma(\alpha+1)}(m\epsilon)^{\alpha}]\frac{r_0||k_2||}{\Gamma(\beta+1)}\\ &+(cr_0+M_2)\frac{r_0||k_1||}{\Gamma(\alpha+1)}[\frac{r_0\omega_{k_2og}(\epsilon,.)}{\Gamma(\beta+1)}+\frac{2r_0||k_2||}{\Gamma(\beta+1)}(n\epsilon)^{\beta}].\end{aligned}$$

From the uniform continuity of the functions k_1of and k_2og on the set $I\times I$ and h on the set $I\times[0,r_0]$ and the continuity of the function a on I, we have that $\omega_{k_1of}(\epsilon,.)\to 0$, $\omega_{k_2og}(\epsilon,.)\to 0$, $\gamma_{r_0}(h,\epsilon)\to 0$ and $\omega(a,\epsilon)\to 0$ as $\epsilon\to 0$. So, let $\epsilon\to 0$ to obtain

$$\omega_0(TX)\leq c\ \frac{r_0^2||k_1||||k_2||}{\Gamma(\alpha+1)\Gamma(\beta+1)}\ \omega_0(X). \tag{4.5}$$

Let $x \in X$ and $t_1, t_2 \in I$ with $t_1 < t_2$. Then

$$\begin{aligned}
&|(Tx)(t_2)-(Tx)(t_1)|-[(Tx)(t_2)-(Tx)(t_1)] = \\
&= |a(t_2)+h(t_2,x(t_2))(Fx)(t_2)(Gx)(t_2) \\
&\quad -a(t_1)-h(t_1,x(t_1))(Fx)(t_1)(Gx)(t_1)| \\
&\quad -[a(t_2)+h(t_2,x(t_2))(Fx)(t_2)(Gx)(t_2) \\
&\quad -a(t_1)-h(t_1,x(t_1))(Fx)(t_1)(Gx)(t_1)] \\
&\le \{|a(t_2)-a(t_1)|-[a(t_2)-a(t_1)]\}+|h(t_2,x(t_2))(Fx)(t_2)(Gx)(t_2) \\
&\quad -h(t_1,x(t_1))(Fx)(t_2)(Gx)(t_2)| \\
&\quad +|h(t_1,x(t_1))(Fx)(t_2)(Gx)(t_2)-h(t_1,x(t_1))(Fx)(t_1)(Gx)(t_2)| \\
&\quad +|h(t_1,x(t_1))(Fx)(t_1)(Gx)(t_2)-h(t_1,x(t_1))(Fx)(t_1)(Gx)(t_1)| \\
&\quad -\{[h(t_2,x(t_2))(Fx)(t_2)(Gx)(t_2)-h(t_1,x(t_1))(Fx)(t_2)(Gx)(t_2)] \\
&\quad +[h(t_1,x(t_1))(Fx)(t_2)(Gx)(t_2)-h(t_1,x(t_1))(Fx)(t_1)(Gx)(t_2)] \\
&\quad +[h(t_1,x(t_1))(Fx)(t_1)(Gx)(t_2)-h(t_1,x(t_1))(Fx)(t_1)(Gx)(t_1)]\} \\
&\le \{|h(t_2,x(t_2))-h(t_1,x(t_1))|-[h(t_2,x(t_2))-h(t_1,x(t_1))]\}(Fx)(t_2)(Gx)(t_2) \\
&\quad +h(t_1,x(t_1))\{|(Fx)(t_2)-(Fx)(t_1)|-[(Fx)(t_2)-(Fx)(t_1)]\}(Gx)(t_2) \\
&\quad +h(t_1,x(t_1))(Fx)(t_1)\{|(Gx)(t_2)-(Gx)(t_1)|-[(Gx)(t_2)-(Gx)(t_1)]\} \\
&\le i(Hx)\frac{r_0||k_1||}{\Gamma(\alpha+1)}\frac{r_0||k_2||}{\Gamma(\beta+1)}.
\end{aligned}$$

The above estimate implies that

$$i(Tx) \le c\, i(x)\frac{r_0^2||k_1||||k_2||}{\Gamma(\alpha+1)\Gamma(\beta+1)},$$

and consequently,

$$i(TX) \le \frac{c\, r_0^2||k_1||||k_2||}{\Gamma(\alpha+1)\Gamma(\beta+1)} i(X). \tag{4.6}$$

From (4.5) and (4.6) and the definition of the measure of noncompactness μ, we obtain

$$\begin{aligned}
\mu(TX) &= \omega_0(TX)+i(TX) \\
&\le c\,\frac{r_0^2||k_1||||k_2||}{\Gamma(\alpha+1)\Gamma(\beta+1)}\,\omega_0(X)+\frac{c\, r_0^2||k_1||||k_2||}{\Gamma(\alpha+1)\Gamma(\beta+1)} i(X) \\
&\le c\,\frac{r_0^2||k_1||||k_2||}{\Gamma(\alpha+1)\Gamma(\beta+1)}(\omega_0(X)+i(X)) \le c\,\frac{r_0^2||k_1||||k_2||}{\Gamma(\alpha+1)\Gamma(\beta+1)}\mu(X).
\end{aligned}$$

Now, taking into account the above inequality and the fact that $c\,\frac{r_0^2||k_1||||k_2||}{\Gamma(\alpha+1)\Gamma(\beta+1)} < 1$ and applying Remark 2, we complete the proof. Also, such

a solution is nondecreasing in view of Remark 1 and the definition of the measure of noncompactness μ given in Section 2. $\square$

In what follows we illustrate the above obtained result by the following example.

Example 1. Consider the following functional integral equation of fractional order

$$x(t) = \frac{1}{5}t^3 + \frac{2tx(t)}{5(1+t)} \int_0^t \frac{2s}{\Gamma(\frac{1}{2})\sqrt{t^2-s^2}} [\frac{1}{8}(t+s) + \frac{1}{4}]x(s)\,ds \times \int_0^t \frac{3s^2}{\Gamma(\frac{1}{3})\sqrt[3]{(t^3-s^3)^2}} \frac{(t+\sqrt{s})^2}{12} x(s)\,ds, \quad (4.7)$$

where $t \in I$. In this example, we have $a(t) = \frac{1}{5}t^3$ and this function satisfies assumption (i) and $M_1 = \frac{1}{5}$. Here $f(t,s) = \frac{1}{4}\sqrt{t+s}$ and $g(t,s) = t + \sqrt{s}$ and these functions satisfies assumption (iv). Let $k_1 : [0, \frac{\sqrt{2}}{4}] \to \mathbb{R}_+$ and $k_2 : [0,2] \to \mathbb{R}_+$ be given by $k_1(y) = 2y^2 + \frac{1}{4}$ and $k_2(y) = \frac{1}{12}y^2$, then k_1 and k_2 satisfying assumptions (v) and (vi) with $||k_1|| = \frac{1}{2}$ and $||k_2|| = \frac{1}{3}$. Moreover, the function $h(t,x) = \frac{2tx}{5(1+t)}$ satisfies hypothesis (ii) and

$$|h(t,x) - h(t,y)| \le \frac{1}{5}|x-y|$$

for all $x, y \in \mathbb{R}$ and $t \in I$. Moreover, the function h satisfies assumption (iii). Indeed, taking an arbitrary nonnegative function $x \in C(I)$ and $t_1, t_2 \in I$ such that $t_1 \le t_2$, we obtain

$$\begin{aligned}
&|(Hx)(t_2) - (Hx)(t_1)| - [(Hx)(t_2) - (Hx)(t_1)] \\
&\quad = |h(t_2, x(t_2)) - h(t_1, x(t_1))| - [h(t_2, x(t_2)) - h(t_1, x(t_1))] \\
&\quad = |\frac{2t_2x(t_2)}{5(1+t_2)} - \frac{2t_1x(t_1)}{5(1+t_1)}| - [\frac{2t_2x(t_2)}{5(1+t_2)} - \frac{2t_1x(t_1)}{5(1+t_1)}] \\
&\quad \le |\frac{2t_2x(t_2)}{5(1+t_2)} - \frac{2t_2x(t_1)}{5(1+t_2)}| + |\frac{2t_2x(t_1)}{5(1+t_2)} - \frac{2t_1x(t_1)}{5(1+t_1)}| \\
&\qquad - [\frac{2t_2x(t_2)}{5(1+t_2)} - \frac{2t_2x(t_1)}{5(1+t_2)} + \frac{2t_2x(t_1)}{5(1+t_2)} - \frac{2t_1x(t_1)}{5(1+t_1)}] \\
&\quad \le \frac{2t_2}{5(1+t_2)}|x(t_2) - x(t_1)| + |\frac{2t_2}{5(1+t_2)} - \frac{2t_1}{5(1+t_1)}|x(t_1) \\
&\qquad - \frac{2t_2}{5(1+t_2)}[x(t_2) - x(t_1)] - [\frac{2t_2}{5(1+t_2)} - \frac{2t_1}{5(1+t_1)}]x(t_1) \\
&\quad \le \frac{2t_2}{5(1+t_2)}\{|x(t_2) - x(t_1)| - [x(t_2) - x(t_1)]\} \le \frac{2t_2}{5(1+t_2)} i(x) \le \frac{1}{5} i(x).
\end{aligned}$$

In this case the first inequality (4.2) of assumption (vii) has the form

$$\frac{1}{5}\Gamma(\frac{3}{2})\Gamma(\frac{4}{3})+\frac{1}{30}r^3 \leq \Gamma(\frac{3}{2})\Gamma(\frac{4}{3})r,$$

and this admits $r_0 = 1$ as a positive solution. Moreover,

$$c\ ||k_1||\ ||k_2||\ r_0^2 = \frac{1}{5}\ \frac{1}{2}\ \frac{1}{3}.1 < \Gamma(\frac{3}{2})\ \Gamma(\frac{4}{3}) = \frac{1}{2}\Gamma(\frac{1}{2})\ \frac{1}{3}\Gamma(\frac{1}{3}).$$

On the other hand, we have $\Gamma(\frac{1}{2}) = 1.77245$ and $\Gamma(\frac{1}{3}) = 2.67893$. Then Theorem 8 guarantees that equation (4.7) has a nondecreasing solution.

Corollary 1. *Let the assumptions of Theorem 8 be satisfied (with $m = n = 1$), then the fractional-order quadratic integral equation*

$$x(t) = a(t) + h(t,x(t))\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} k_1(f(t,s))x(s)\,ds \times \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} k_2(g(t,s))x(s)\,ds,$$

has at least one solution $x \in C(I)$.

Corollary 2. *Let the assumptions of Theorem 8 be satisfied (with $h(t,x(t)) = 1$), then the fractional-order quadratic integral equation*

$$x(t) = a(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} k_1(f(t,s))x(s)\,ds\,.\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} k_2(g(t,s))x(s)\,ds,$$

has at least one solution $x \in C(I)$.

Corollary 3. *Let the assumptions of Theorem 8 be satisfied (with $k_1 = k_2 = I, \alpha = \beta, f = g, a(t) = 0$), then the fractional-order quadratic integral equation*

$$x(t) = [\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(t,s)x(s)\,ds]^2,$$

has at least one solution $x \in C(I)$.

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FIXED POINT RESULTS FOR GENERALIZED RATIONAL α-GERAGHTY CONTRACTION

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Abstract. In this paper, an effort has been made to improve the notion of α-Geraghty contraction type mappings and establish some common fixed point theorems for a pair of α-admissible mappings under the improved approach of generalized rational α-Geraghty contractive type condition in a complete metric space. An example has been constructed to demonstrate the novelty of these results.

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1. Preliminaries and scope

In 1973, Geraghty [3] studied a generalization of Banach contraction principle. He generalized the Banach contraction principle in a different way than it was done by different investigators. In 2012, Samet et al. [16], introduced a concept of $\alpha-\psi$- contractive type mappings and established various fixed point theorems for mappings in complete metric spaces. Afterwards, Karapinar [11], refined the notion and obtained various fixed point results. See more results in [9]. Hussain et al. [7], generalized the concept of α-admissible mappings and proved fixed point theorems. Subsequently, Abdeljawad [1] introduced a pair of $\alpha-$admissible mappings satisfying new sufficient contractive conditions different from those in [7], [16] and obtained fixed point and common fixed point theorems. Salimi et al. [15], modified the concept of $\alpha-\psi-$ contractive mappings and established fixed point results. Recently, Hussain et al. [8] proved some fixed point results for single and set-valued $\alpha-\eta-\psi$-contractive mappings in the setting of complete metric space. Mohammadi et al. [13], introduced a new notion of $\alpha-\phi-$contractive mappings and showed that it was a real generalization for some old results. Thereafter, many papers have published on geraghty contractions. For more detail see [4–6, 8, 11, 14] and references therein.

Definition 1 ([16])**.** Let $S : X \to X$ and $\alpha : X \times X \to \mathbb{R}$. We say that S is α-admissible if $x, y \in X$, $\alpha(x, y) \geq 1 \Rightarrow \alpha(Sx, Sy) \geq 1$.

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Example 1 ([12]). Consider $X = [0,\infty)$, and define $S : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by $Sx = 2x$, for all $x, y \in X$ and

$$\alpha(x,y) = \begin{cases} e^{\frac{y}{x}}, & \text{if } x \geq y, x \neq 0 \\ 0, & \text{if } x < y. \end{cases}$$

Then S is $\alpha-$admissible.

Definition 2 ([1]). Let $S, T : X \to X$ and $\alpha : X \times X \to [0,+\infty)$. We say that the pair (S,T) is α-admissible if $x, y \in X$ such that $\alpha(x,y) \geq 1$, then we have $\alpha(Sx,Ty) \geq 1$ and $\alpha(Tx,Sy) \geq 1$.

Definition 3 ([10]). Let $S : X \to X$ and $\alpha : X \times X \to [0,+\infty)$. We say that S is triangular α-admissible if $x, y \in X$, $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1 \Rightarrow \alpha(x,y) \geq 1$.

Definition 4 ([10]). Let $S : X \to X$ and $\alpha : X \times X \to (-\infty,+\infty)$. We say that S is a triangular α-admissible mapping if

(T1) $\alpha(x,y) \geq 1$ implies $\alpha(Sx,Sy) \geq 1$, $x, y \in X$,
(T2) $\alpha(x,z) \geq 1$, $\alpha(z,y) \geq 1$, implies $\alpha(x,y) \geq 1$, $x, y, z \in X$.

Definition 5 ([1]). Let $S, T : X \to X$ and $\alpha : X \times X \to (-\infty,+\infty)$. We say that a pair (S,T) is triangular α-admissible if

(T1) $\alpha(x,y) \geq 1$, implies $\alpha(Sx,Ty) \geq 1$ and $\alpha(Tx,Sy) \geq 1$, $x, y \in X$.
(T2) $\alpha(x,z) \geq 1$, $\alpha(z,y) \geq 1$, implies $\alpha(x,y) \geq 1$, $x, y, z \in X$.

Definition 6 ([15]). Let $S : X \to X$ and let $\alpha, \eta : X \times X \to [0,+\infty)$ be two functions. We say that T is α-admissible mapping with respect to η if $x, y \in X$, $\alpha(x,y) \geq \eta(x,y) \Rightarrow \alpha(Sx,Sy) \geq \eta(Sx,Sy)$. Note that if we take $\eta(x,y) = 1$, then this definition reduces to definition in [16]. Also if we take $\alpha(x,y) = 1$, then we says that S is an η-subadmissible mapping.

Lemma 1 ([2]). *Let $S : X \to X$ be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Sx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.*

Lemma 2. *Let $S, T : X \to X$ be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define sequence $x_{2i+1} = Sx_{2i}$, and $x_{2i+2} = Tx_{2i+1}$, where $i = 0, 1, 2, \ldots$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.*

We denote by Ω the family of all functions $\beta : [0,+\infty) \to [0,1)$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$.

Theorem 1 ([3]). *Let (X,d) be a metric space. Let $S : X \to X$ be a self mapping. Suppose that there exists $\beta \in \Omega$ such that for all $x, y \in X$,*

$$d(Sx,Sy) \leq \beta(d(x,y))\, d(x,y).$$

then S has a fixed unique point $p \in X$ and $\{S^n x\}$ converges to p for each $x \in X$.

2. Results

In this section, we prove some fixed point theorems satisfying generalized rational α-Geraghty contraction type mappings in complete metric space. Let (X,d) be a metric space, and let $\alpha : X \times X \to \mathbb{R}$ be a function. Let $S,T : X \to X$ is called a pair of generalized rational α-Geraghty contraction type mappings if there exists $\beta \in \Omega$ such that for all $x, y \in X$,

$$\alpha(x,y)d(Sx,Ty) \leq \beta\left(M(x,y)\right)M(x,y) \tag{2.1}$$

where

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,Sx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Sx)d(y,Ty)}{1+d(Sx,Ty)}\right\}.$$

If $S = T$ then T is called generalized rational α-Geraghty contraction type mappings if there exists $\beta \in \Omega$ such that for all $x, y \in X$,

$$\alpha(x,y)d(Tx,Ty) \leq \beta\left(N(x,y)\right)N(x,y)$$

where

$$N(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\right\}.$$

Theorem 2. *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function. Let $S,T : X \to X$ be two mappings then suppose that the following holds:*

(i) (S,T) is pair of generalized rational α-Geraghty contraction type mapping;

(ii) (S,T) is triangular α-admissible;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;

(iv) S and T are continuous;

Then (S,T) have common fixed point.

Proof. Let x_1 in X be such that $x_1 = Sx_0$ and $x_2 = Tx_1$. Continuing this process, we construct a sequence x_n of points in X such that,

$$x_{2i+1} = Sx_{2i}, \text{ and } x_{2i+2} = Tx_{2i+1}, \text{ where } i = 0,1,2,\ldots. \tag{2.2}$$

By assumption $\alpha(x_0,x_1) \geq 1$ and pair (S,T) is α -admissible, by Lemma 2, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.3}$$

Then

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) = d(Sx_{2i}, Tx_{2i+1}) &\leq \alpha(x_{2i}, x_{2i+1})d(Sx_{2i}, Tx_{2i+1}) \\ &\leq \beta\left(M(x_{2i}, x_{2i+1})\right)M(x_{2i}, x_{2i+1}), \end{aligned}$$

for all $i \in \mathbb{N} \cup \{0\}$. Now

$$\begin{aligned} M(x_{2i}, x_{2i+1}) &= \max\left\{d(x_{2i}, x_{2i+1}), \frac{d(x_{2i}, Sx_{2i})d(x_{2i+1}, Tx_{2i+1})}{1+d(x_{2i}, x_{2i+1})}, \frac{d(x_{2i}, Sx_{2i})d(x_{2i+1}, Tx_{2i+1})}{1+d(Sx_{2i}, Tx_{2i+1})}\right\} \\ &= \max\left\{d(x_{2i}, x_{2i+1}), \frac{d(x_{2i}, x_{2i+1})d(x_{2i+1}, x_{2i+2})}{1+d(x_{2i}, x_{2i+1})}, \frac{d(x_{2i}, x_{2i+1})d(x_{2i+1}, x_{2i+2})}{1+d(x_{2i+1}, x_{2i+2})}\right\} \end{aligned}$$

$$\leq \max\{d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2})\}.$$

Thus

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) &\leq \beta\left(M(x_{2i}, x_{2i+1})\right) M(x_{2i}, x_{2i+1}) \\ &\leq \beta\left(d(x_{2i}, x_{2i+1})\right) d((x_{2i}, x_{2i+1}) < d(x_{2i}, x_{2i+1}). \end{aligned}$$

so that,

$$d((x_{2i+1}, x_{2i+2}) < d(x_{2i}, x_{2i+1}). \tag{2.4}$$

This implies that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}), \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.5}$$

So, sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nonincreasing. Now, we prove that $d(x_n, x_{n+1}) \to 0$. It is clear that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence. Therefore, there exists some positive number r such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. From (2.4), we have

$$\frac{d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1})} \leq \beta(d(x_n, x_{n+1})) \leq 1.$$

Now by taking limit $n \to \infty$, we have

$$1 \leq \beta(d(x_n, x_{n+1})) \leq 1,$$

that is

$$\lim_{n\to\infty} \beta(d(x_n, x_{n+1})) = 1.$$

By the property of β, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.6}$$

Now, we show that sequence $\{x_n\}$ is Cauchy sequence. Suppose on contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ and sequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ such that, for all positive integers k, we have $m_k > n_k > k$,

$$d(x_{m_k}, x_{n_k}) \geq \epsilon \tag{2.7}$$

and

$$d(x_{m_k}, x_{n_{k-1}}) < \epsilon. \tag{2.8}$$

By the triangle inequality, we have

$$\begin{aligned} \epsilon &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) \\ &< \epsilon + d(x_{n_{k-1}}, x_{n_k}). \end{aligned}$$

That is,

$$\epsilon < \epsilon + d(x_{n_{k-1}}, x_{n_k}) \tag{2.9}$$

for all $k \in \mathbb{N}$. In the view of (2.9), (2.6), we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon. \tag{2.10}$$

Again using triangle inequality, we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k})$$

and

$$d(x_{m_{k+1}}, x_{n_{k+1}}) \leq d(x_{m_{k+1}}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}}).$$

Taking limit as $k \to +\infty$ and using (2.6) and (2.10), we obtain

$$\lim_{k\to+\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon. \tag{2.11}$$

By Lemma 2, $\alpha(x_{n_k}, x_{m_{k+1}}) \geq 1$, we have

$$\begin{aligned} d(x_{n_{k+1}}, x_{m_{k+2}}) &= d(Sx_{n_k}, Tx_{m_{k+1}}) \leq \alpha(x_{n_k}, x_{m_{k+1}}) d(Sx_{n_k}, Tx_{m_{k+1}}) \\ &\leq \beta(M(x_{n_k}, x_{m_{k+1}}))M(x_{n_k}, x_{m_{k+1}}). \end{aligned}$$

Finally, we conclude that

$$\frac{d(x_{n_{k+1}}, x_{m_{k+2}})}{M(x_{n_k}, x_{m_{k+1}})} \leq \beta(M(x_{n_k}, x_{m_{k+1}})).$$

By using (2.6), taking limit as $k \to +\infty$ in the above inequality, we obtain

$$\lim_{k\to\infty} \beta(d(x_{n_k}, x_{m_{k+1}})) = 1. \tag{2.12}$$

So, $\lim_{k\to\infty} d(x_{n_k}, x_{m_{k+1}}) = 0 < \epsilon$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete so there exists $p \in X$ such that $x_n \to p$ implies that $x_{2i+1} \to p$ and $x_{2i+2} \to p$. As S and T are continuous, so we get $Tx_{2i+1} \to Tp$ and $Sx_{2i+2} \to Sp$. Thus $p = Sp$ similarly, $p = Tp$, we have $Sp = Tp = p$. Then (S, T) have common fixed point. □

In the following Theorem, we dropped continuity.

Theorem 3. *Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function. Let $S, T : X \to X$ be two mappings then suppose that the following holds:*

(i) (S, T) is a pair of generalized rational α -Geraghty contraction type mapping;

(ii) (S, T) is triangular α-admissible;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq 1$ for all k.

Then (S, T) have common fixed point.

Proof. Follows the similar lines of the Theorem 2. Define a sequence $x_{2i+1} = Sx_{2i}$, and $x_{2i+2} = Tx_{2i+1}$, where $i = 0, 1, 2, \ldots$. converges to $p \in X$. By the hypotheses of (iv) there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{2n_k}, p) \geq 1$ for all k. Now by using (2.1) for all k, we have

$$d(x_{2n_k+1}, Tp) = d(Sx_{2n_k}, Tp) \leq \alpha(x_{2n_k}, p) d(Sx_{2n_k}, Tp)$$

$$\leq \beta\left(M(x_{2n_k}, p)\right) M(x_{2n_k}, p).$$

so that,

$$d(x_{2n_k+1}, Tp) \leq \beta\left(M(x_{2n_k}, p)\right) M(x_{2n_k}, p). \tag{2.13}$$

On the other hand, we obtain

$$M(x_{2n_k}, p) = \max\left\{d(x_{2n_k}, p), \frac{d(x_{2n_k}, Sx_{2n_k}), d(p, Tp)}{1 + d(x_{2n_k}, p)}, \frac{d(x_{2n_k}, Sx_{2n_k}), d(p, Tp)}{1 + d(Sx_{2n_k}, Tp)}\right\}.$$

Letting $k \to \infty$ then we have

$$\lim_{k\to\infty} M(x_{2n_k}, p) = \max\{d(p, Sp), d(p, Tp)\}. \tag{2.14}$$

Case I.
$\lim_{k\to\infty} M(x_{2n_k}, p) = d(p, Tp)$. Suppose that $d(p, Tp) > 0$. From (2.14), for a large k , we have $M(x_{2n_k}, p) > 0$, which implies that

$$\beta(M(x_{2n_k}, p)) < M(x_{2n_k}, p).$$

Then, we have

$$d(x_{2n_k}, Tp) < M(x_{2n_k}, p) \tag{2.15}$$

Letting $k \to \infty$ in (2.15), we obtain that $d(p, Tp) < d(p, Tp)$, which is a contradiction. Thus, we find that $d(p, Tp) = 0$, implies $p = Tp$.
Case II.
$\lim_{k\to\infty} M(x_{2n_k}, p) = d(p, Sp)$. Similarly $p = Sp$. Thus $p = Tp = Sp$. □

If $M(x, y) = \max\left\{d(x, y), \frac{d(x, Sx)d(y, Sy)}{1 + d(x, y)}, \frac{d(x, Sx)d(y, Sy)}{1 + d(Sx, Sy)}\right\}$ and $S = T$ in Theorem 2 and Theorem 3 , we have the following corollaries.

Corollary 1. *Let (X, d) be a complete metric space and let S is $\alpha-$ admissible mappings such that the following holds:*
(i) S is a generalized rational α-Geraghty contraction type mapping;
(ii) S is triangular α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, T(x_0)) \geq 1$;
(iv) S is continuous;
Then S has a fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

Corollary 2. *Let (X, d) be a complete metric space and let S is $\alpha-$ admissible mappings such that the following holds:*
(i) S is a generalized rational α-Geraghty contraction type mapping;
(ii) S is triangular α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;
(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq 1$ for all k.

Then S has a fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

If $M(x,y) = \max\{d(x,y), d(x,Sx), d(y,Sy)\}$ in Theorem 1, Theorem 2, we obtain the following corollaries.

Corollary 3 ([2]). *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function. Let $S : X \to X$ be a mapping then suppose that the following holds:*
(i) S is a generalized α-Geraghty contraction type mapping;
(ii) S is triangular α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;
(iv) S is continuous;
Then S has a fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

Corollary 4 ([2]). *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function. Let $S : X \to X$ be a mapping then suppose that the following holds:*
(i) S is a generalized α-Geraghty contraction type mapping;
(ii) S is triangular α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;
(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq 1$ for all k.
Then S has a fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

Let (X,d) be a metric space, and let $\alpha, \eta : X \times X \to \mathbb{R}$ be a function. A map $S, T : X \to X$ is called a pair of generalized rational α-Geraghty contraction type mappings if there exists $\beta \in \Omega$ such that for all $x, y \in X$,

$$\alpha(x,y) \geq \eta(x,y) \Rightarrow d(Sx,Ty) \leq \beta\left(M(x,y)\right) M(x,y) \tag{2.16}$$

where

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,Sx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Sx)d(y,Ty)}{1+d(Sx,Ty)}\right\}.$$

Theorem 4. *Let (X,d) be a complete metric space. Let S is $\alpha-$admissible mappings with respect to η such that the following holds:*
(i) (S,T) is a generalized rational α-Geraghty contraction type mapping;
(ii) (S,T) is triangular α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$;
(iv) S and T are continuous;
Then (S,T) have common fixed point.

Proof. Let x_1 in X be such that $x_1 = Sx_0$ and $x_2 = Tx_1$. Continuing this process, we construct a sequence x_n of points in X such that,

$$x_{2i+1} = Sx_{2i}, \text{ and } x_{2i+2} = Tx_{2i+1}, \text{ where} i = 0, 1, 2, \ldots. \tag{2.17}$$

By assumption $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$ and the pair (S, T) is α-admissible with respect to η, we have, $\alpha(Sx_0, Tx_1) \geq \eta(Sx_0, Tx_1)$ from which we deduce that $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ which also implies that $\alpha(Tx_1, Sx_2) \geq \eta(Tx_1, Sx_2)$. Continuing in this way we obtain $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$.

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) &= d(Sx_{2i}, Tx_{2i+1}) \leq \alpha(x_{2i}, x_{2i+1}) d(Sx_{2i}, Tx_{2i+1}) \\ &\leq \beta\left(M(x_{2i}, x_{2i+1})\right) M(x_{2i}, x_{2i+1}), \end{aligned}$$

Therefore,

$$d(x_{2i+1}, x_{2i+2}) \leq \alpha(x_{2i}, x_{2i+1}) d(Sx_{2i}, Tx_{2i+1}) \tag{2.18}$$

for all $i \in \mathbb{N} \cup \{0\}$. Now

$$\begin{aligned} M(x_{2i}, x_{2i+1}) &= \max\left\{d(x_{2i}, x_{2i+1}), \frac{d(x_{2i}, Sx_{2i}) d(x_{2i+1}, Tx_{2i+1})}{1 + d(x_{2i}, x_{2i+1})}, \frac{d(x_{2i}, Sx_{2i}) d(x_{2i+1}, Tx_{2i+1})}{1 + d(Sx_{2i}, Tx_{2i+1})}\right\} \\ &= \max\left\{d(x_{2i}, x_{2i+1}), \frac{d(x_{2i}, x_{2i+1}) d(x_{2i+1}, x_{2i+2})}{1 + d(x_{2i}, x_{2i+1})}, \frac{d(x_{2i}, Sx_{2i}) d(x_{2i+1}, Tx_{2i+1})}{1 + d(x_{2i+1}, x_{2i+2})}\right\} \\ &\leq \max\left\{d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2})\right\}. \end{aligned}$$

From the definition of β, the case $M(x_{2i}, x_{2i+1}) = d(x_{2i+1}, x_{2i+2})$ is impossible.

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) &\leq \beta\left(M(x_{2i}, x_{2i+1})\right) M(x_{2i}, x_{2i+1}) \\ &\leq \beta\left(d(x_{2i+1}, x_{2i+2})\right) d(x_{2i+1}, x_{2i+2}) < d(x_{2i+1}, x_{2i+2}). \end{aligned}$$

Which is a contradiction. Otherwise, in other case

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) &\leq \beta\left(M(x_{2i}, x_{2i+1})\right) M(x_{2i}, x_{2i+1}) \\ &\leq \beta\left(d(x_{2i}, x_{2i+1})\right) d((x_{2i}, x_{2i+1}) < d(x_{2i}, x_{2i+1}). \end{aligned}$$

This, implies that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}), \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.19}$$

Follows the similar lines of the Theorem 2. Hence p is common fixed point of S and T. □

Theorem 5. *Let (X, d) be a complete metric space and let (S, T) are $\alpha-$admissible mappings with respect to η such that the following holds:*

(i) (S, T) is a generalized rational α-Geraghty contraction type mapping;

(ii) (S, T) is triangular α-admissible;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$;

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq \eta(x_{n_k}, p)$ for all k.

Then S and T has common fixed point.

Proof. Follows the similar line of the Theorem 3. □

If $M(x,y) = \max\left\{d(x,y), \dfrac{d(x,Sx)d(y,Sy)}{1+d(x,y)}, \dfrac{d(x,Sx)d(y,Sy)}{1+d(Sx,Sy)}\right\}$ and $S = T$ in the Theorem 4, Theorem 5 , we get the following corollaries.

Corollary 5. *Let (X,d) be a complete metric space and let S is $\alpha-$admissible mappings with respect to η such that the following holds:*
(i) S is a generalized rational α-Geraghty contraction type mapping;
(ii) S is triangular α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$;
(iv) S is continuous;
Then S has a fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

Corollary 6. *Let (X,d) be a complete metric space and let S is $\alpha-$admissible mappings with respect to η such that the following holds:*
(i) S is a generalized rational α-Geraghty contraction type mapping;
(ii) S is triangular α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$;
(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq \eta(x_{n_k}, p)$ for all k.
Then S has a fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

Example 2. Let $X = \{1,2,3\}$ with metric

$$d(1,3) = d(3,1) = \frac{5}{7}\ d(1,1) = d(2,2) = d(3,3) = 0$$

$$d(1,2) = d(2,1) = 1,\ d(2,3) = d(3,2) = \frac{4}{7}$$

$$\alpha(x,y) = \left\{\begin{array}{ll} 1, & \text{if } x,y \in X, \\ 0, & \text{otherwise} \end{array}\right\}.$$

Define the mappings $S, T : X \to X$ as follows:

$$Sx = 1 \text{ for each } x \in X.$$

$$T(1) = T(3) = 1,\ T(2) = 3.$$

and $\beta : [0,+\infty) \to [0,1]$, then

$$\alpha(x,y)d(Tx,Ty) \leq \beta(M(x,y))M(x,y).$$

Let $x = 2$ and $y = 3$ then condition (2.1) is not satisfied.

$$d(T(2),T(3)) = d(3,1) = \frac{5}{7}$$

$$M(x,y) = \max\{d(2,3), d(2,T(2)), d(3,T(3))\}$$
$$= \max\left\{\frac{4}{7}, \frac{4}{7}, \frac{5}{7}\right\} = \frac{5}{7}$$

and

$$\alpha(2,3)d\left(T(2),T(3)\right) \not\leq \beta(M(x,y))M(x,y).$$

If

$$M(x,y) = \max\left\{d(2,3), \frac{d(2,T(2))d(3,T(3))}{1+d(2,3)}, \frac{d(2,T(2))d(3,T(3))}{1+d(T2,T3)}\right\}$$
$$= \max\left\{\frac{4}{7}, \frac{20}{77}, \frac{20}{84}\right\} = \frac{4}{7}$$

Then the contractions does not holds.

$$\alpha(2,3)d\left(T(2),T(3)\right) \not\leq \beta(M(x,y))M(x,y).$$

We prove that Theorem 1 can be applied to S and T. Let $x, y \in X$, clearly (S,T) is $\alpha-$admissible mapping such that $\alpha(x,y) \geq 1$. Let $x, y \in X$ and so that $Sx, Ty \in X$ and $\alpha(Sx,Ty) = 1$. Hence (S,T) is α-admissible. We show that condition (2.1) of Theorem 1 is satisfied. If $x, y \in X$ then $\alpha(x,y) = 1$, we have

$$\alpha(x,y)d(Sx,Ty) \leq \beta(M(x,y))\left(M(x,y)\right).$$

where

$$M(x,y) = \max\left\{d(2,3), \frac{d(2,S(2))d(3,T(3))}{1+d(2,3)}, \frac{d(2,S(2))d(3,T(3))}{1+d(S2,T3)}\right\}$$
$$= \max\left\{\frac{4}{7}, \frac{20}{77}, \frac{20}{49}\right\} = \frac{4}{7}$$

and

$$d(S2,T3) = d(1,1) = 0.$$
$$\alpha(x,y)d(Sx,Ty) \leq \beta(M(x,y))\left(M(x,y)\right).$$

Hence all the hypothesis of the Theorem 1 is satisfied, So S, T have a common fixed point.

Remark 1. More detail, applications and examples see in [2] and references there in. Our results are more general than those in [2], [15] and improve several results existing in the literature.

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FC-RINGS

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Abstract. We investigate properties of FC-rings (i.e. rings R in which the centralizer $C_R(a)$ of any element $a \in R$ is of finite index in R) and, in particular, characterize left Artinian rings with a finite set of all derivations $\operatorname{Der} R$ (respectively inner derivations $\operatorname{IDer} R$). We show that if R is a Jacobson radical ring in which its adjoint group R° has a finite number of conjugacy classes, then

$$R = R_{p_1} \bigoplus \cdots \bigoplus R_{p_t} \bigoplus D$$

is a ring direct sum of Jacobson radical rings R_{p_i} and D, where the additive group D^+ is a torsion-free divisible group, the adjoint group D° is a group with a finite number of conjugacy classes, $R_{p_i}^+$ is a finite p_i-group ($i = 1, \dots, t$) and $p_1, \dots, p_t$ are pairwise distinct primes.

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1. Introduction

Let R be an associative ring, not necessarily with identity.

It is well-known that a group G is called *an FC-group* if the centralizer $C_G(g)$ of any element $g \in G$ is of finite index in G [19]. Naturally, if the centralizer

$$C_R(a) = \{r \in R \mid ar = ra\}$$

of any element $a \in R$ is of finite index in the additive group R^+ of R, then R is called *an FC-ring*. Every finite ring and every commutative ring are FC-rings. Since

$$\partial_a : R \ni r \mapsto ar - ra = [a, r] \in R$$

(so-called *an inner* derivation of R induced by a) is an endomorphism of the group R^+, the kernel

$$\ker \partial_a = \{r \in R \mid ra = ar\} = C_R(a)$$

is the centralizer of $a \in R$ and the quotient group $R^+/\ker \partial_a$ is isomorphic to the image $\operatorname{Im} \partial_a$, we deduce that R is an FC-ring if and only if $\operatorname{Im} \partial_a$ is finite for any $a \in R$. A map $\delta : R \to R$ is said to be *a derivation* of R if

$$\delta(a + b) = \delta(a) + \delta(b) \text{ and } \delta(ab) = \delta(a)b + a\delta(b)$$

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for all $a, b \in R$. Clearly, the zero map $0_R : R \ni r \mapsto 0 \in R$ is a derivation of R.

An algebraic operation "$\circ$" determined by the rule

$$a \circ b = a + b + ab$$

for any $a, b \in R$ is associative with the neutral element $0 \in R$. The set of all invertible elements in R with respect to "$\circ$" is a group (which is called *the adjoint group* of R and denoted by R°).

Notations. For a group G, g^{-1} is the inverse of $g \in G$, $a^G := \{g^{-1}ag \mid g \in G\}$ is the conjugacy class of $a \in G$, G' is the commutator subgroup of G (i.e. a subgroup generated by all multiplicative commutators $g^{-1}h^{-1}gh$, where $g, h \in G$), $Z_0(G) := 1$ is a trivial subgroup of G, $Z(G) = Z_1(G) := \{z \in G \mid zg = gz \text{ for all } g \in G\}$ is the center of G, $Z_{\alpha+1}(G)/Z_\alpha(G) = Z(G/Z_\alpha(G))$ and $Z_\lambda(G) = \cup_{\beta<\lambda} Z_\beta(G)$, where α is an ordinal and λ a limit ordinal. Recall that a group G is called:

- *hypercentral* (respectively *nilpotent*) if $Z_\theta(G) = G$ for some ordinal (respectively non-negative integer) θ,
- *locally nilpotent* if every its finitely generated subgroup is nilpotent,
- *simple* in case $G \neq 1$ and 1, G are the only normal subgroups of G,
- *solvable* if $G^{(n)} = 1$ for some integer $n \geq 0$, where $G^{(0)} = G$ and $G^{(m+1)} = (G^{(m)})'$ for any integer $m \geq 1$,
- *locally solvable* if every its finitely generated subgroup is solvable.

Every nilpotent group is hypercentral. It is known that a group is hypercentral if and only if every its non-trivial homomorphic image has the non-trivial center. A group G with a finite number of conjugacy classes is called *a υ-group*. A subgroup A of an (additive) abelian group G is called *pure* if $A \cap nG = nA$ for any integer n. An (additive) abelian group G is *divisible* if, for every positive integer n and every $a \in G$, there exists $x \in G$ such that $nx = a$.

For any ring R, $[x, r] = xr - rx$ is an additive commutator of $x, r \in R$, $C(R)$ is the commutator ideal of R that is the ideal generated by the commutator set $[R, R] = \{[x, r] \mid x, r \in R\}$, $J(R)$ is the Jacobson radical of R, R^+ is the additive group of R, $N(R)$ is the set of all nilpotent elements of R, $U(R)$ is the unit group of R with identity, $Z(R)$ is the center of R, $F(R) = \{a \in R \mid a \text{ is of finite order in } R^+\}$ is the torsion part of R, $\exp F(R)$ is the exponent of the group $F(R)^+$, $g^{(-1)}$ is the inverse of g in the adjoint group R°, $x^{(n)}$ is the nth power of x in the adjoint group R°, $\operatorname{ann} X = \{a \in R \mid aX = 0 = Xa\}$ is the annihilator of $X \subseteq R$. By $\operatorname{Der} R$ we denote the set of all derivations of R. A subring S is *of finite index* in R (i.e. $|R : S| < \infty$) if the additive subgroup S^+ has a finite index in R^+. Recall that a ring R is called:

- *Jacobson radical* if $R^\circ = R$,
- *nil* if every its element x is nilpotent, i.e. there exists an integer $n = n(x) > 0$ such that $x^n = 0$; if there exists an integer $n > 0$ such that $x^n = 0$ for any $x \in R$, then R is *nil of bounded index*,

- *nilpotent* in case there is an integer $m > 0$ such that $x_1 x_2 \cdots x_m = 0$ for any $x_1, x_2, \ldots, x_m \in R$,
- *locally nilpotent* if every its finitely generated subring is nilpotent,
- *reduced* if it is without nonzero nilpotent elements,
- *simple* in case $R^2 \neq 0$ and 0, R are the only ideals of R,
- *local* if it has identity and the quotient ring $R/J(R)$ is simple,
- *semiprime* if it has no nonzero nilpotent ideals,
- *left Artinian* in case for every descending chain
$$I_1 \geq I_2 \geq \cdots \geq I_n \geq \cdots$$
of left ideals I_j of R, there is an integer $n \geq 1$ with $I_{n+1} = I_n$ $(j = 1, 2, \ldots)$.

Any unexplained terminology is standard as in [3, 10] and [19].

The purpose of this paper is to study associative FC-rings R and some related topics. Obviously, commutative rings and, in particular, differentially trivial rings (i.e. $\operatorname{Der} R = \{0_R\}$) are FC-rings. Associative rings R with finite sets $\operatorname{Der} R$, $\operatorname{IDer} R$ are very related to FC-rings. Rings R with the center $Z(R)$ of finite index are FC-rings. In [21, Problem 84] F. Szász asked: "In which rings R the additive group $Z(R)^+$ of the center $Z(R)$ has a finite group-theoretic index with respect to R^+?" From Corollary 2 it follows that this problem is equivalent to study of rings with a finite set of all inner derivations $\operatorname{IDer} R$. Y. Hirano [12, Proposition 1] has proved that the condition $|R : Z(R)| < \infty$ implies that the commutator ideal $C(R)$ is finite. F. Szász [21, Problem 83] asked: "In which rings the commutator ideal is finite or can be finitely generated?" H. Bell [5] has proved that if I is a nonzero right ideal of finite index in a prime ring R and $[I, I]$ is finite, then R is either finite or commutative (see also [17, Corollary 1.2]). C. Lanski [16] has showed that if T is a finite higher commutator of R containing no nonzero nilpotent element, then T generates a finite ideal of R.

We study left Artinian rings with the finite set of all derivations $\operatorname{Der} R$ (respectively inner derivations $\operatorname{IDer} R$) and prove the following with similar flavour.

Theorem 1. *Let R be a left Artinian ring. Then* $\operatorname{IDer} R$ *(respectively* $\operatorname{Der} R$*) is finite if and only if $R = A \oplus F$ is a direct sum of a finite ideal F and a commutative (respectively differentially trivial) reduced ideal A.*

An FC-ring R has the adjoint FC-group. It is easy to see that a Jacobson radical ring R is an FC-ring if and only if its adjoint group R° is an FC-group. In [21, Problem 88] F. Szász asked:" Let
$$\hat{a} = \{(1-x)a(1-x)^{-1} \mid x \in R\}$$
in a Jacobson radical ring R. When is every class $\hat{a}$ finite, and when is a number of the classes $\hat{a}$ finite?" About Jacobson radical rings we prove the following

Proposition 1. *If R is a Jacobson radical ring with the adjoint FC-group R°, then:*

(1) *every nonzero homomorphic image B of R is commutative or* ann B *is nonzero,*
(2) *R° is a hypercentral group,*
(3) *if R^+ is torsion-free, then R is commutative,*
(4) *if R is nonzero, then the commutator ideal $C(R)$ is proper in R; if, moreover, R is non-commutative, then R^2 is proper in R.*

We give a partial answer on the second part of [21, Problem 88] in the following

Proposition 2. *Let R be a Jacobson radical ring. Then the adjoint group R° is a υ-group if and only if*

$$R = R_{p_1} \bigoplus \cdots \bigoplus R_{p_t} \bigoplus D$$

is a ring direct sum of Jacobson radical rings R_{p_i} and D, where D^+ is a torsion-free divisible group, D° is a υ-group, $R_{p_i}^+$ is a finite p_i-group $(i = 1, \dots, t)$ and $p_1, \dots, p_t$ are pairwise distinct primes.

F. Szász [20] has investigated properties of infinite Jacobson radical rings R whose adjoint groups R° have only two conjugacy classes. If, moreover, $R^2 = R$, then it is a simple domain by Propositions 1 and 2 from [20]. We make this result more precise in the following

Corollary 1. *If R is a simple Jacobson radical ring, then the following hold:*
(1) *if R° is a υ-group, then either R is a domain or contains a nonzero nilpotent element,*
(2) *if the adjoint group R° has only two conjugacy classes, then R is a domain with the torsion-free divisible additive group R^+ and the simple adjoint group R°.*

2. Preliminaries

For a convenience of the reader and in order to have the paper more self-contained in this section we collect some results needed in the next.

Lemma 1 ([1], Lemma 2.4(1))**.** *If R is a nil-ring and p is prime, then the additive group R^+ is a p-group if and only if the adjoint group R° is a p-group.*

Lemma 2 (see [2], Corollary 1)**.** *Let G be a subgroup of the adjoint group of a radical ring. If G has finite exponent, then it is locally nilpotent.*

If $(R, +, \cdot)$ is an associative ring, then R is a Lie ring with respect to the addition "+" and the Lie multiplication "$[-,-]$" (denoted by R^L) defined by the rule $[a,b] = a \cdot b - b \cdot a$ for any $a, b \in R$. Then the center $Z(R)$ of R is an ideal of the Lie ring R^L.

Lemma 3. *If R is an associative ring, then there is a Lie ring isomorphism*

$$\mathrm{IDer}\, R \ni \partial_a \mapsto a + Z(R) \in R^L / Z(R).$$

Proof. Immediate. □

From this we have the following

Corollary 2. *Let R be a ring. Then the set* $\mathrm{IDer}\, R$ *is finite if and only if* $|R : Z(R)| < \infty$.

Lemma 4. *Let R be a ring, I an ideal and S a subring of R. If the set of all inner derivations* $\mathrm{IDer}\, R$ *is finite, then sets* $\mathrm{IDer}\, S$ *and* $\mathrm{IDer}(R/I)$ *are finite.*

Proof. Straightforward. □

Lemma 5 ([13], Theorem 1)**.** *Let S be a subring of a ring R. If S has a finite index in R, then there exists an ideal I of R contained in S such that R/I is a finite ring.*

If R has identity 1, then

$$R^\circ \ni a \mapsto 1 + a \in U(R)$$

is a group isomorphism of the adjoint group R° and the unit group $U(R)$ of R. As proved in [8], a division ring D with multiplicative FC-group $U(D)$ is commutative.

Lemma 6. *A local ring R is an FC-ring if and only if its adjoint group R° is an FC-group.*

Proof. The adjoint group of any FC-ring is an FC-group. A commutative ring is an FC-ring. Therefore we assume that a local ring R is not commutative and R° is an FC-group. Since R has a proper ideal of finite index in view of Lemma 5, we conclude that the quotient ring $R/J(R)$ is finite. Moreover, $J(R)^\circ \cong 1 + J(R)$ is a subgroup of the unit group $U(R)$,

$$|J(R) : C_{J(R)}(a)| = |1 + J(R) : C_{1+J(R)}(a)| < \infty$$

for any $a \in U(R)$ and therefore $|R : C_R(a)| < \infty$. Inasmuch $R = J(R) \cup U(R)$ and $J(R)$ is a Jacobson radical FC-ring, we deduce that R is an FC-ring. □

Lemma 7 ([19], Theorem 14.5.9)**.** *If G is an FC-group, then the commutator subgroup G' is torsion.*

Lemma 8 ([15], Theorem 2)**.** *Let R be a semiprime ring and $S = \{x \in R \mid x^2 = 0\}$. If the cardinality* $\mathrm{card}\, S$ *is finite, then $R = A \oplus F$ is a direct sum of ideals A and F, A is reduced and F is finite. In particular, R has only finitely many nilpotent elements.*

Lemma 9 (see [11, 18])**.** *Let A be an algebra over a field of characteristic zero. Suppose that there is a positive integer n such that $a^n = 0$ for all $a \in A$. Then there is an integer N such that $a_1 a_2 \cdots a_N = 0$ for all $a_1, a_2, \ldots, a_N \in A$.*

Theorem 1 of [23] implies the next

Lemma 10. *A finite Jacobson radical ring is nilpotent.*

Lemma 11 ([15], Lemma 3)**.** *If R is a finite ring which is not nilpotent, then R contains a nonzero idempotent.*

Lemma 12 ([19], 4.3.8)**.** *A pure subgroup A of finite index of an abelian group G is a direct summand.*

A ring R is *a subdirect product* of some rings S_i $(i \in I)$ if, for any $i \in I$, $S_i \cong R/K_i$, where K_i is an ideal of R and

$$\bigcap_{i \in I} K_i = 0.$$

Lemma 13 (see [4])**.** *A ring R is reduced if and only if it is a subdirect product of domains.*

Lemma 14 ([10], §1.4, Corollary 2)**.** *If R is a left Artinian ring and its Jacobson radical $J(R) = 0$ is zero, then R contains identity.*

If I is an ideal of a ring R, then we say that an idempotent $g + I$ of the quotient ring R/I *can be lifted (to e) modulo I* in case there is an idempotent $e \in R$ such that $g + I = e + I$.

Lemma 15 ([3], Proposition 27.1)**.** *If I is a nil-ideal of a ring R, then idempotents lift modulo I.*

Lemma 16 ([9], Theorem 18.13)**.** *A left Artinian ring is left Noetherian (i.e. every its ideal is finitely generated).*

A commutative ring V is called *a v-ring* if it is complete (in the $J(V)$-adic topology), discrete, unramified valuation ring of characteristic 0 with the quotient ring $V/J(V)$ of prime characteristic p [6, p.79]. Then $J(V) = pV$, V/pV is a field, $p^k V/p^{k+1}$ and V/pV are isomorphic as (V/pV)-linear spaces.

Lemma 17. *Let R be a complete (in the $J(R)$-adic topology) local Noetherian commutative ring of prime power characteristic p^n. Then the following hold:*

(1) (see [6, Theorem 9]) *if $n = 1$, then there exists a subfield C of R such that $R = J(R) + C$ is a group direct sum,*
(2) (see [6, Theorem 11]) *if $n \geq 2$, then there exists a subring C of R such that $R = J(R) + C$ is a group sum, where $C \cong V/p^n V$ for some v-ring V and $J(R) \cap V = pV$.*

Any local Artinian ring R is a complete local Noetherian ring.

3. Properties of FC-rings

Lemma 18. *If R is an infinite ring with a finite set of all inner derivations* IDer R, *then the following hold:*

(1) *the centralizer $C_R(a)$ is a subring of finite index in R for any $a \in R$ (i.e. R is an FC-ring),*
(2) *the adjoint group R° is an FC-group,*
(3) *if R is a simple ring, then R is a field,*
(4) *R contains a central ideal I of finite index such that $I \cdot C(R) = 0$,*
(5) *the commutator ideal $C(R)$ is finite.*

Proof. (1) Since the set $\{\partial_r(a) \mid r \in R\}$ is finite, the index $|R : C_R(a)|$ is finite.
(2) It follows from the part (1).
(3) It holds from Corollary 2 in view of Lemma 5.
(4) Corollary 2 and Lemma 5 imply that R contains an ideal I of finite index such that $I \leq Z(R)$. Moreover, for any $r, t \in R$ and $i \in I$ we obtain that

$$(rt)i = r(ti) = (ti)r = t(ir) = t(ri) = (tr)i,$$

and so $(rt - tr)i = 0$. As a consequence, $I \cdot C(R) = 0$.
(5) By Corollary 2, the center $Z(R)$ is of finite index in R and so, by the part (4), the annihilator $\operatorname{ann} \partial_x(y)$ has a finite index in R. Then the commutator ideal

$$C(R) = \sum_{x,y \in R \setminus Z(R)} R\partial_x(y)R$$

is finite.

□

Proof of Proposition 1. (1) Assume that B is a non-commutative homorphic image of R and $a \in B \setminus Z(B)$. Then $C_B(a)$ is of finite index in B and, by Lemma 5, the centralizer $C_B(a)$ contains a proper ideal I of B such that $|B : I| < \infty$. If $i \in I$ and $r \in B$, then

$$ari = ria = rai, \; iar = air = ira$$

and so

$$[a, r]i = 0 = i[a, r].$$

This gives that $[B, B] \subseteq \operatorname{ann} I$. Since B/I is a finite radical ring, it is nilpotent by Lemma 10. We have that $B^n \leq I$ for some positive integer n and $\operatorname{ann} I \subseteq \operatorname{ann}(B^n)$. From this it holds that $\operatorname{ann} B \neq 0$. Then every non-trivial homomorphic image of R is commutative or has a non-trivial annihilator.
(2) Since $\operatorname{ann} R \subseteq Z(R)$, we deduce that every proper quotient group of R° has the non-trivial center. This means that the adjoint group R° is hypercentral [19, Exercises 12.2.2].
(3) By Lemma 7, every torsion-free FC-group is abelian.
(4) The commutator ideal of a commutative ring R is zero (and so it is proper in $R \neq 0$). Assume that R is non-commutative. As in the part (1), we can prove that R contains a proper ideal of finite index and therefore R^2 is proper in R in view of Lemma 10. Obviously that $C(R) \subseteq R^2$. Hence $C(R)$ is proper in a non-commutative ring R.

□

4. Rings with a finite set of derivations

Lemma 19. *If e is an idempotent of a commutative ring R, then $d(e) = 0$ for any $d \in \operatorname{Der} R$.*

Proof. In fact, $d(e) = d(e^2) = d(e)e + ed(e)$ implies that $ed(e) = ed(e)e + ed(e)$ and so $d(e)e = ed(e) = e^2 d(e) = ed(e)e = 0$. Hence $d(e) = 0$. □

Lemma 20. *Let R be a ring. If the set $\operatorname{Der} R$ (respectively $\operatorname{IDer} R$) is finite, then $\operatorname{Der} R = \{0_R\}$ (respectively R is commutative) or $\delta(R) \subseteq F(R)$ for any $\delta \in \operatorname{Der} R$ (respectively $\delta \in \operatorname{IDer} R$).*

Proof. Assume that $\delta(a) \neq 0$ for some $\delta \in \operatorname{Der} R$ and $a \in R$. Since the set

$$\{n\delta(a) \mid n \text{ is an integer}\}$$

is finite, the torsion part $F(R) \neq 0$ is nonzero and $\delta(a) \in F(R)$. □

Lemma 21. *Any finite semiprime commutative ring R is differentially trivial.*

Proof. By Lemma 14, R contains identity 1 and, by the Artin-Wedderburn structure theorem, $R = R_1 \oplus \cdots \oplus R_m$ is a ring direct sum of finite fields R_i $(i = 1, \ldots, m)$. Since $1 = f_1 + \cdots + f_m$, where f_i is identity of R_i and $R_i = f_i R$, we obtain that

$$d(R_i) = d(f_i)R + f_i d(R) = f_i d(R) \subseteq f_i R = R_i$$

for any $d \in \operatorname{Der} R$ by Lemma 19. Every finite field is differentially trivial and we conclude that $d(R) = 0$. Hence R is differentially trivial. □

Proposition 3. *Let R be a reduced ring (respectively a ring with the torsion-free additive group R^+). Then the following hold:*

(1) *the set of all inner derivations $\operatorname{IDer} R$ is finite if and only if R is commutative,*
(2) *the set of all derivations $\operatorname{Der} R$ is finite if and only if R is differentially trivial.*

Proof. If the additive group R^+ is torsion-free and $\operatorname{IDer} R$ (respectively $\operatorname{Der} R$) is finite, the assertion follows in view of Lemma 20. Therefore we suppose that R is reduced.

(1) Assume that the set $\operatorname{IDer} R$ is finite. By Lemmas 13 and 20, R is a subdirect product of domains D with finite sets $\operatorname{IDer} D$ of inner derivations. If D is finite, then it is a field. If D is infinite, then it does not contain a proper ideal of finite index and therefore it is commutative in view of Lemma 18(4). This implies that R is commutative.

The converse is clear.

(2) If $\operatorname{Der} R$ is finite, then R is commutative in view of the part (1). Assume that $d(a) \neq 0$ for some $d \in \operatorname{Der} R$ and $a \in R$. The rule $rd : R \ni x \mapsto rd(x) \in R$ determines a derivation rd of R and so $Rd \subseteq \operatorname{Der} R$. Then the set $Rd(a)$ is a finite

ring and Lemma 11 implies that there exists a nonzero idempotent $e \in Rd(a)$ such that

$$Re \leq Rd(a)$$

and $e = td(a)$ for some $t \in R$. By Lemma 21, Re is differentially trivial. This gives that

$$0 = d(ae) = d(a)e = d(a)td(a).$$

Then $d(a)t$ is a nilpotent element and therefore $e = 0$, a contradiction. Hence $\operatorname{Der} R = \{0_R\}$.

The converse is clear. □

Corollary 3. *Let R be a semiprime ring. Then* $\operatorname{IDer} R$ *(respectively* $\operatorname{Der} R$*) is finite if and only if $R = A \oplus F$ is a direct sum of a finite ideal F and a commutative (respectively differentially trivial) reduced ideal A (in particular, $A = 0$).*

Proof. Assume that R is infinite and $\operatorname{IDer} R$ (respectively $\operatorname{Der} R$) is finite. If $a \in Z(R) \cap N(R)$, then aR is a nilpotent ideal of R and therefore $|N(R)| \leq |R : Z(R)|$. Since the index $|R : Z(R)|$ is finite by Corollary 2, we deduce that

$$R = A \bigoplus F$$

is a direct sum of a finite ideal F and a reduced ideal A by Lemma 8 (in particular, $A = 0$). If $A \neq 0$, then A is a commutative (respectively differentially trivial) ring by Proposition 3.

The converse is clear. □

Corollary 4. *A semiprime Jacobson radical FC-ring is commutative.*

Proof. In view of Lemma 10, R does not contain a nonzero finite ideal and so $C(R) = 0$ by Lemma 18(5). Hence R is commutative. □

Lemma 22. *If R is a commutative Artinian ring such that $|R : J(R)| < \infty$, then it is finite.*

Proof. Since R is a ring direct sum of local Artinian rings of prime power characteristics by the Artin-Wedderburn structure theorem, we may assume that R is local Artinian of characteristic p^n for some prime p and an integer $n \geq 1$. By Lemma 17, $R = J(R) + C$ is a group sum, where either C is a field (and consequently it is finite) or $C \cong V/p^n V$ for some v-ring V and $n \geq 2$. Since $R/J(R) \cong V/pV$ is a field, $p^k V/p^{k+1} V$ and V/pV are isomorphic as (V/pV)-linear spaces $(k = 1, \ldots, n-1)$, we deduce that C is finite. In view of Lemma 16,

$$J(R) = \sum_{s=1}^{t} j_s R$$

for some integer $t \geq 1$ and elements $j_1, \dots, j_s \in R$. Then

$$\begin{aligned} J(R) &= \sum_{s=1}^{t} (j_s J(R) + j_s C) \\ &= \sum_{s=1}^{t} (j_s \sum_{l=1}^{t} (j_l J(R) + j_l C) + j_s C) \\ \dots &= \sum_{u=1}^{w} g_w C \end{aligned}$$

for some integer $w \geq 1$ and $g_1, \dots, g_w \in J(R)$. Thus R is finite. □

As usual, if e is an idempotent od a ring R (not necessary with 1), then we write

$$eR(1-e) := \{er - ere \mid r \in R\},\ (1-e)Re := \{re - ere \mid r \in R\}$$

and

$$(1-e)R(1-e) := \{r - er - re + ere \mid r \in R\}.$$

Proof of Theorem 1. a) Suppose that the set $\mathrm{IDer}\, R$ is finite and R is infinite. By Lemma 18, the commutator ideal $C(R)$ is finite. The quotient ring

$$R/J(R) = F_1 \bigoplus A_1$$

is semisimple and so, by the Artin-Wedderburn structure theorem, it is a direct sum of ideals F_1 and A_1, where F_1 is finite. If $A_1 = 0$, then $R/C(R)$ (and consequently R) is finite by Lemma 22, a contradiction with the assumption. Thus $A_1 \neq 0$ and we may assume that A_1 does not contain a nonzero finite ideal and so, by the Artin-Wedderburn structure theorem, A_1 is a ring direct sum of finitely many infinite semisimple Artinian rings (which are fields). An idempotent that is identity of A_1 can be lifted to an idempotent $e \in R$ by Lemma 15. We denote eRe by A and $A \cap \mathrm{ann}\, C(R)$ by C_1. Then $|A : C_1| < \infty$ in view of Lemma 18(4) and $J(A) = A \cap J(R)$. Since A_1 does not contain a proper ideal of finite index and the quotient ring

$$A/J(A) \cong (A + J(R))/J(R) \cong A_1$$

is a ring direct sum of finitely many infinite fields, we deduce that

$$A/J(A) = (C_1 + J(A))/J(A) \text{ and } A = J(A) + C_1.$$

The Jacobson radical $J(A)$ does not contain e and, as a consequence, $A = C_1 \leq \mathrm{ann}\, C(R)$. Since $e \in A$, we have that $eC(R) = C(R)e = 0$ and hence

$$rea - erea = (re - er)ea = 0 \text{ and } aer - aere = ae(er - re) = 0$$

for any $a \in A$ and $r \in R$. If

$$I = (1-e)R(1-e) + eR(1-e) + (1-e)Re + C(R),$$

then I is an ideal of R and $R = I + A$. Inasmuch $IA = 0 = AI$, we conclude that A is an ideal of R. Moreover, if $z \in I \cap A$, then

$$exe = z = (u - eu - ue + eue) + (ev - eve) + (we - ewe) + c$$

for some $x, u, v, w \in R$, $c \in C(R)$ and $exe = e^2xe^2 = eze = ece = 0$ what forces that $I \cap A = 0$. Hence

$$R = I \bigoplus A \tag{4.1}$$

is a ring direct sum, where A is commutative.

b) Now assume that the set $\operatorname{Der} R$ is finite. Then the set $\operatorname{IDer} R$ is also finite and (4.1) it follows. Consequently, $\operatorname{Der} A$ is finite. If $\delta \in \operatorname{Der} A$, then the rule

$$b\delta : A \ni r \mapsto b\delta(r) \in A$$

determines a derivation $b\delta$ of A for any $b \in A$. Since sets $A\delta$ and $A\delta(a)$ are finite, the index $|A : \operatorname{ann}\delta(a)| < \infty$ for any $a \in A$. But $A/J(A)$ is a ring direct sum of finitely many infinite fields and so A does not contain a proper ideal of finite index. This yields that $\delta = 0_A$ is zero and consequently A is a differentially trivial ring.

The converse is clear.

□

5. On Jacobson radical rings with two conjugacy classes

It is well known that any linear group (over a field) and any SI-group (i.e. group which possess a normal series with abelian factors) with a finite number of conjugacy classes are finite (see e.g. [19] or [14]). A result of Cohn [7] says that there exists a radical ring whose adjoint group has only two conjugacy classes and it is a torsion-free simple group.

Lemma 23. *If R is a Jacobson radical ring with the adjoint υ-group R°, then:*

(1) *R has only the finite number of two-sided ideals,*
(2) *R has a simple homomorphic image,*
(3) *the center $Z(R)$ is finite.*

Proof. (1) If I is an ideal of R, then I° is a normal subgroup of R°. Since every normal subgroup is a set-theoretic sum of some conjugacy classes, the assertion holds.

(2) It follows from the part (1).

(3) If $a \in Z(R)$, then the conjugacy class a^G contains only one a. □

Lemma 24. *Let R be a Jacobson radical ring and $R^2 \neq R$. If the adjoint group R° has only two conjugacy classes, then R is a zero-ring that contains only two elements.*

Proof. Assume $G = R^\circ$ has only exactly two conjugacy classes. Now, $\{0\}$ is a conjugacy class by itself, and the other class is a^G for some $0 \neq a \in G$. Then $(R^2)^\circ$

is a normal subgroup of G, because R^2 is an ideal of R, thus $(R^2)^\circ$ is the union of conjugacy classes. As $0 \in R^2$, we have that $(R^2)^\circ$ is either $\{0\}$ or G. Since the ring R is Jacobson radical, we obtain that R^2 is either $\{0\}$ or R. From $R^2 \neq R$ it follows that $R^2 = \{0\}$, i.e. R is a zero-ring. Hence "$\circ$" is simply the addition, and then a number of conjugacy classes is card R from the commutativity of the addition "$+$". □

Lemma 25 ([22], Corollary, p. 332)**.** *Let G be a locally solvable group. If G has a finite number of conjugacy classes, then it is finite.*

Corollary 5. *Any Jacobson radical ring R with the torsion adjoint υ-group R° is finite.*

Proof. Since elements of the same conjugacy class have the same order, R° is of finite exponent. Then, by Lemma 2, the adjoint group R° is locally nilpotent. By Lemma 25, R is finite. □

Proof of Proposition 2. *a*) By Corollary 5, the torsion part $F(R)$ is finite and so D° is a υ-group.

b) Since $F(R)$ is an ideal of R and R° is a υ-group, we deduce that the exponent $\exp F(R)$ is finite. Then $F(R)$ is a ring direct sum of finitely many p-components for pairwise distinct primes p.

c) Suppose that the additive group R^+ is torsion-free. Assume that qR is proper in R for some prime q. Since the set

$$\{nR \mid n \text{ is a positive integer}\}$$

is finite by Lemma 23(1), we conclude that

$$q^l R = q^k R$$

for some positive integers l, k, where $l > k$. If $a \in R \setminus qR$, then $q^k a = q^l b$ for some $b \in R$ and

$$q^k(a - q^{l-k} b) = 0.$$

This yields $a = q^{l-k} b \in qR$, a contradiction. Hence R^+ is divisible.

d) Let R be any Jacobson radical ring with the adjoint υ-group R°. Then $R^+/F(R)$ is a divisible group and, by Lemma 12,

$$R^+ = F(R) \bigoplus D$$

is a group direct sum, where D is a divisible group. If $c, d \in D$, then $cd = f + h$ for some $f \in F(R)$ and $h \in D$. Since $c = nc_1$ and $h = nh_1$ for some $c_1, h_1 \in D$, where n is a positive integer, we conclude that

$$f \in \bigcap_{n=1}^{\infty} nR.$$

Therefore $f = 0$ and D is a ring. Moreover $F(R)D = 0 = DF(R)$. Then $R = F(R) \oplus D$ is a ring direct sum and so $F(R)$ is a ring with a finite number of conjugacy classes. □

6. Some corollaries

If B is the two-element zero ring, then B is Jacobson radical and its adjoint group B° has only two conjugacy classes. But B is not simple because $B^2 = 0$.

Proof of Corollary 1. (1) Assume that R is not a domain and does not contain a nonzero nilpotent element. If there exists a nonzero element $a \in R$ with a nonzero right annihilator $A = \operatorname{ann}_r a$, then $A \neq R$, $(Aa)^2 = 0$ and, as a consequence, $Aa = 0$. This means that A is contained in the two-sided annihilator $\operatorname{ann} a$. Since R does not contain a nonzero nilpotent element, $\operatorname{ann} a$ is a nonzero proper ideal of R, a contradiction.

(2) In view of the part (1), assume that R contains a nonzero nilpotent element $x \in R$ such that $x^2 = 0$. Then any $0 \neq y \in R$ is contained in the class x^{R° and so there exists $a \in R$ such that

$$y = a^{(-1)} \circ x \circ a.$$

Since

$$\begin{aligned}
y^2 &= (x + xa + a^{(-1)}x + a^{(-1)}xa)(x + xa + a^{(-1)}x + a^{(-1)}xa) \\
&= (xa^{(-1)}xa + xaxa + xaa^{(-1)}xa) + (xax + xa^{(-1)}x + xaa^{(-1)}x) \\
&\quad + (a^{(-1)}xa^{(-1)}x + a^{(-1)}xax + a^{(-1)}xaa^{(-1)}x) \\
&\quad + (a^{(-1)}xaxa + a^{(-1)}xa^{(-1)}xa + a^{(-1)}xaa^{(-1)}xa) \\
&= x(a^{(-1)} \circ a)xa + x(a^{(-1)} \circ a)x + a^{(-1)}x(a^{(-1)} \circ a)x \\
&\quad + a^{(-1)}x(a^{(-1)} \circ a)xa \\
&= 0,
\end{aligned}$$

we obtain that R is a nil-ring of bounded index. If $F(R) \neq 0$, then the p-part

$$F_p(R) = \{a \in R \mid a \text{ is of finite order } p^n \text{ for some non-negative integer } n\}$$

is a nonzero ideal of R for some prime p and therefore $R = F_p(R)$. Hence R^+ is a p-group. If $F(R) = 0$, then R^+ is torsion-free.

a) If R^+ is a p-group, then, by Lemma 1, R° is a p-group and, by Corollary 5, R is finite. Then R is a nilpotent ring, a contradiction with the simplicity of R.

b) Let R^+ be a torsion-free group. By Proposition 2, R is a $\mathbb{Q}$-algebra and, by Lemma 9, it is nilpotent, a contradiction.

Hence R is reduced and, by the part (1), R is a domain. As a consequence, R° is a simple group. By Proposition 2, R^+ is torsion-free divisible. □

Corollary 6. *If R is an infinite nil-ring, then $R^2 \neq R$ or R° is not a υ-group.*

Proof. By contrary. If R° is a υ-group and $R^2 = R$, then, in view of Lemma 23(2), there exists an ideal I of R with the simple homomorphic image R/I. If $(R/I)^+$ is a torsion group, then $(R/I)^\circ$ is torsion by Lemma 1 and so R/I is a finite nilpotent ring, which leads to a contradiction. If $(R/I)^+$ is a torsion-free group, then, as in the proof of Corollary 1, R/I is a nil $\mathbb{Q}$-algebra of bounded index and it is nilpotent by Lemma 9, a contradiction with $R^2 = R$. □

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FINITE DIFFERENCES OF EULER'S ZETA FUNCTION

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Abstract. We find accurate approximations for certain finite differences of the Euler zeta function, $\zeta(x)$.

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Keywords: Riemann zeta function, finite differences

1. Introduction

The Euler zeta function is the precursor of the Riemann zeta function [1, 5]. It is defined for real values greater than 1 by

$$\zeta(x) = 1 + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \cdots, \qquad x > 1. \tag{1.1}$$

As is well known, this infinite series converges for any real $x > 1$, and we have

$$\zeta(x) > \zeta(x+1) \qquad \text{and} \qquad \lim_{x\to\infty} \zeta(x) = 1.$$

A finite difference is an expression of the form

$$f(x+b) - f(x+a). \tag{1.2}$$

The use of finite differences can be traced back to the end of the 16th century. Newton's interpolation formula [4] is an early occurrence of finite differences. For more on the subject we refer the reader to [3].

For many applications, it is useful to consider three particular forms of (1.2): forward, backward, and central differences. A forward difference is an expression of the form

$$\Delta_h f(x) = f(x+h) - f(x).$$

A backward difference is given by

$$\nabla_h f(x) = f(x) - f(x-h).$$

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Finally, a central difference takes the form

$$\delta_h f(x) = f(x+h/2) - f(x-h/2).$$

Iterating, we obtain the n-th order forward, backward, and central differences:

$$\begin{aligned}
\Delta_h^n f(x) &= \Delta_h[\Delta_h^{n-1} f(x)] \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x+(n-i)h), \\
\nabla_h^n f(x) &= \nabla_h[\nabla_h^{n-1} f(x)] \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x-ih), \\
\delta_h^n f(x) &= \delta_h[\delta_h^{n-1} f(x)] \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} f\left(x+\left(\frac{n}{2}-i\right)h\right).
\end{aligned}$$

We note that

$$\nabla_h^n f(x) = \Delta_h^n f(x-nh) = \delta_h^n f(x-nh/2). \tag{1.3}$$

Dividing the finite difference (1.2) by $b-a$, leads to a difference quotient. Thus, finite differences can be used to approximate derivatives or higher order derivatives. Finite difference methods are fundamental in finding numerical solution to differential equations, especially boundary value problems. Higher order finite differences can also be used to express certain recurrences.

In this note, motivated by these facts, we provide accurate approximations for the n-th order forward, backward, and central finite differences of the classical Euler zeta function. We note that Flajolet and Vepstas [2] also explored finite differences of values of the zeta function at integers. Our main result, which is valid for all real x sufficiently large and is of a different flavor than the results of [2], is given below.

Theorem 1. *Let n be a positive integer. For any real numbers $h > 0$ and $x > 1+nh$,*

$$\frac{(2^h-1)^n}{2^x} < (-1)^n \nabla_h^n \zeta(x) < \frac{(2^h-1)^n}{2^x} + R_{n,h}(x),$$

where

$$R_{n,h}(x) = \frac{1}{2^x} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \frac{2^{ih+1}}{x-ih-1}.$$

Using (1.3), similar inequalities can be derived for the finite differences $\Delta_h^k \zeta(x)$ and $\delta_h^k \zeta(x)$. Clearly,

$$\lim_{x\to\infty} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \frac{2^{ih+1}}{x-ih-1} = 0.$$

Thus, we obtain the following asymptotic formulas for the n-th order forward, backward and central differences of Euler's zeta function.

Corollary 1. *Let n be a positive integer. For any real number $h > 0$,*

a) $2^x \Delta_h^n \zeta(x) \sim \left(\dfrac{1-2^h}{2^h}\right)^n$,

b) $2^x \nabla_h^n \zeta(x) \sim (1-2^h)^n$,

c) $2^x \delta_h^n \zeta(x) \sim \left(\dfrac{1-2^h}{\sqrt{2^h}}\right)^n$,

as $x \to \infty$.

The case $n = h = 1$ of this corollary can be used to derive the following (equivalent) limits:

$$\lim_{x\to\infty} 2^x (\zeta(x) - \zeta(x+1)) = \frac{1}{2},$$
$$\lim_{x\to\infty} 2^x (\zeta(x-1) - \zeta(x)) = 1,$$

and

$$\lim_{x\to\infty} 2^x \left(\zeta\left(x-\frac{1}{2}\right) - \zeta\left(x+\frac{1}{2}\right)\right) = \frac{1}{\sqrt{2}}.$$

2. Proof of Theorem 1

We have

$$\begin{aligned}
(-1)^n 2^x \nabla_h^n \zeta(x) &= 2^x \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \zeta(x-ih) \\
&= 2^x \sum_{k=2}^{\infty} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \frac{1}{k^{x-ih}} \\
&= 2^x \sum_{k=2}^{\infty} \left(\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} k^{ih} \right) \frac{1}{k^x} \\
&= 2^x \sum_{k=2}^{\infty} \frac{(k^h-1)^n}{k^x}
\end{aligned}$$

$$= (2^h - 1)^n + 2^x \sum_{k=3}^{\infty} \frac{(k^h - 1)^n}{k^x}.$$

The sum $\sum_{k=3}^{\infty} \frac{(k^h - 1)^n}{k^x}$ is a right Riemann sum for the function $g(t) = \frac{(t^h - 1)^n}{t^x}$ on the interval $[2, \infty)$. Since $x > 1 + nh$, the function $g(t)$ is positive and decreasing. Thus,

$$\sum_{k=3}^{\infty} \frac{(k^h - 1)^n}{k^x} < \int_2^{\infty} \frac{(t^h - 1)^n}{t^x} dt,$$

On the other hand,

$$\int_2^{\infty} \frac{(t^h - 1)^n}{t^x} dt = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \int_2^{\infty} t^{ih-x} dt$$
$$= \frac{1}{2^x} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \frac{2^{ih+1}}{x - ih - 1}.$$

Then, for $x > 1 + nh$, we deduce the double inequality

$$(2^h - 1)^n < (-1)^n 2^x \nabla_h^n \zeta(x)$$
$$< (2^h - 1)^n + \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \frac{2^{ih+1}}{x - ih - 1}.$$

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THE BERNSTEIN CUBATURE FORMULA REVISED

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Abstract. Let $a,b,c,d \in \mathbb{R}$ be given such that $a<b, c<d$. Starting with the Bernstein bivariate approximation formula on $[a,b]\times[c,d]$ a corresponding composite Bernstein type cubature formula is constructed. Its coefficients and an upper bound estimation for the remainder term are established. Numerical examples and comparisons with other known cubature formulas are also provided.

2010 *Mathematics Subject Classification:* 65D32; 41A10

Keywords: Bernstein operator, Bernstein bivariate approximation formula, remainder term

1. Preliminaries

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. It is well known (see for example [1,2,5,6,8,9]) that the Bernstein bivariate operators $B_{m,n} : C[0,1]\times[0,1] \to C[0,1]\times[0,1]$ are defined for any $f \in C[0,1]\times[0,1]$, any $(x,y)\in[0,1]\times[0,1]$ and any $m,n\in\mathbb{N}$ by:

$$B_{m,n}f(x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x)p_{n,j}(y)f\left(\frac{k}{m},\frac{j}{n}\right) \tag{1.1}$$

where

$$p_{m,k}(x) = \binom{m}{k}x^k(1-x)^{m-k}, \qquad p_{n,j}(x) = \binom{n}{j}y^j(1-y)^{n-j} \tag{1.2}$$

are the fundamental univariate Bernstein's polynomials.

Note that the bivariate polynomials (1.1) are known as the Bernstein bivariate polynomials of degree (m,n).

For any $f\in C[0,1]\times[0,1]$, $(x,y)\in[0,1]\times[0,1]$, $m,n\in\mathbb{N}$, the equality

$$f(x,y) = B_{m,n}f(x,y) + R_{m,n}f(x,y) \tag{1.3}$$

is known as the Bernstein bivariate approximation formula, $R_{m,n}f$ being its remainder term.

Regarding the remainder term of (1.3) were established the following results.

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Theorem 1 ([5]). *For any $f \in C[0,1]\times[0,1]$ the remainder term of (1.3) can be represented under the form:*

$$R_{m,n}f(x,y) = -\frac{x(1-x)}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{n} p_{m-1,k}(x)p_{n,j}(y)\left[\begin{array}{c} x,\frac{k}{m},\frac{k+1}{m} \\ \frac{j}{n} \end{array};f\right] \tag{1.4}$$
$$-\frac{y(1-y)}{n}\sum_{k=0}^{m}\sum_{j=0}^{n-1} p_{m,k}(x)p_{n-1,j}(y)\left[\begin{array}{c} \frac{k}{m} \\ y,\frac{j}{k},\frac{j+1}{n} \end{array};f\right]$$
$$+\frac{xy(1-x)(1-y)}{mn}\sum_{k=0}^{m-1}\sum_{j=0}^{n-1} p_{m-1,k}(x)p_{n-1,j}(y)\left[\begin{array}{c} x,\frac{k}{m},\frac{k+1}{m} \\ y,\frac{j}{n},\frac{j+1}{n} \end{array};f\right],$$

for any $(x,y)\in[0,1]\times[0,1]$, where the brackets denote bivariate divided differences.

Theorem 2 ([5]). *Let $p,q\in\mathbb{N}_0$, $a\le x_0<x_1<\cdots<x_p\le b, c\le y_0<y_1<\cdots<y_q\le d$ and $f:[a,b]\times[c,d]$ be given. Suppose $f\in C^{(p-1,q-1)}[a,b]\times[c,d]$ and there exists $\dfrac{\partial^{p+q}f}{\partial x^p\partial y^q}$ on $]a,b[\times]c,d[$. Then, there exists $(\xi,\eta)\in]a,b[\times]c,d[$ such that*

$$\left[\begin{array}{c} x_0,x_1,\ldots,x_p \\ y_0,y_1,\ldots,y_q \end{array};f\right] = \frac{1}{p!q!}\frac{\partial^{p+q}f}{\partial x^p\partial y^q}(\xi,\eta). \tag{1.5}$$

Theorem 3 ([5]). *Suppose $f\in C^{2,2}[0,1]\times[0,1]$ and there exists $\dfrac{\partial^4 f}{\partial x^2\partial y^2}$, bounded on $]0,1[\times]0,1[$. The following*

$$|(R_{m,n}f)(x,y)| \le \frac{x(1-x)}{2m}M_1[f]+\frac{y(1-y)}{2n}M_2[f]+\frac{xy(1-x)(1-y)}{4mn}M_3[f] \tag{1.6}$$

holds, for any $(x,y)\in[0,1]\times[0,1]$ and any $m,n\in\mathbb{N}$, where

$$M_1(f) = \sup_{(x,y)\in]0,1[\times]0,1[}\left|\frac{\partial^2 f(x,y)}{\partial x^2}\right|,$$
$$M_2(f) = \sup_{(x,y)\in]0,1[\times]0,1[}\left|\frac{\partial^2 f(x,y)}{\partial y^2}\right|, \tag{1.7}$$
$$M_3(f) = \sup_{(x,y)\in]0,1[\times]0,1[}\left|\frac{\partial^4 f(x,y)}{\partial x^2\partial y^2}\right|.$$

Applying the above results, in [4] was obtained the following Bernstein type cubature formula.

Theorem 4 ([3]). *Let $f \in C^{2,2}[0,1]\times[0,1]$ be given such that $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ are bounded on $]0,1[\times]0,1[$. Then, the following Bernstein type cubature formula*

$$\int_0^1\int_0^1 f(x,y)dx\,dy = \sum_{i=0}^{m}\sum_{j=0}^{n} A_{i,j} f\left(\frac{i}{m},\frac{j}{n}\right) + R_{m,n}[f] \tag{1.8}$$

holds, where the coefficients are expressed by

$$A_{ij} = \frac{1}{(m+1)(n+1)}, \quad (\forall)\, i = \overline{0,m},\ (\forall)\, j = \overline{0,n} \tag{1.9}$$

and the remainder term satisfies

$$|R_{m,n}[f]| \le \frac{1}{12m} M_1[f] + \frac{1}{12n} M_2[f] + \frac{1}{144mn} M_3[f]. \tag{1.10}$$

Note that the cubature formula (1.8) has the degree of exactness $(1,1)$ i.e. it is exact for the bivariate polynomials $\epsilon_{i,j}(x,y) = x^i y^j, 0 \le i \le j \le 1, i,j \in \mathbb{N}_0, i+j \le 1$.

Theorem 1.4 was generalized as follows.

Theorem 5 ([3]). *Let $f \in C^{2,2}[0,1]\times[0,1]$ be given such that $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ are bounded on $]0,1[\times]0,1[$. The following composite Bernstein cubature formula*

$$\int_0^1\int_0^1 f(x,y)dx\,dy = \frac{1}{mn(p+1)(q+1)} \sum_{k=1}^{m}\sum_{j=1}^{n}\sum_{h=0}^{p}\sum_{l=0}^{q} f\left(\frac{kp-p+h}{mp},\frac{jq-q+l}{nq}\right) + R_{m,n}[f]. \tag{1.11}$$

holds, were $R_{m,n}[f]$ satisfies (1.10).

Using (1.11) it is possible to approximate $\int_0^1\int_0^1 f(x,y)dx\,dy$ which the desired precision ε, imposing the condition $|R_{m,n}[f]| < \varepsilon$.

The focus of the present paper is to extend the results from [4] in order to obtain a composite Bernstein type cubature formula on any bidimensional interval $[a,b]\times[c,d]$. Finally, numerical examples and comparisons with other known cubature formulas will be provided.

2. Main results

Let $a,b,c,d \in \mathbb{R}$ be given such that $a<b, c<d$ and $\mathbb{N}$ be the set of positive integers.

Lemma 1. *The Bernstein bivariate polynomial associated to $f \in C[a,b]\times[c,d]$ is expressed for any $(x,y)\in[a,b]\times[c,d]$ and any $m,n\in\mathbb{N}$ by*

$$\overline{B}_{m,n}f(x,y)=\sum_{k=0}^{m}\sum_{j=0}^{n}\overline{p}_{m,k}(x)\overline{p}_{n,j}(y)f\left(a+k\cdot\frac{b-a}{m},c+j\cdot\frac{d-c}{n}\right) \tag{2.1}$$

where

$$\overline{p}_{m,k}(x)=\frac{1}{(b-a)^m}\binom{m}{k}(x-a)^k(b-x)^{m-k}, \tag{2.2}$$

$$\overline{p}_{n,j}(y)=\frac{1}{(d-c)^n}\binom{n}{j}(y-c)^j(d-y)^{n-j}$$

are the Bernstein fundamental polynomials.

Proof. The vectorial function $(t,s)\to\left(\frac{x-a}{b-a},\frac{y-c}{d-c}\right)$ transform the bidimensional interval $[a,b]\times[c,d]$ into $[0,1]\times[0,1]$. Taking the above and definitions (1.1),(1.2) into account one arrives to (2.1), (2.2). □

For any $f\in C[a,b]\times[c,d], m,n\in\mathbb{N}$ the equality

$$f(x,y)=\overline{B}_{m,n}f(x,y)+\overline{R}_{m,n}f(x,y) \tag{2.3}$$

is the Bernstein approximation formula on $[a,b]\times[c,d]$. Applying Theorem 1.3, for the remainder term of (2.3) follows

Lemma 2. *If $f\in C^{2,2}[a,b]\times[c,d]$ such that $\frac{\partial^2 f}{\partial x^2},\frac{\partial^2 f}{\partial y^2},\frac{\partial^4 f}{\partial x^2\partial y^2}$ are bounded on $]a,b[\times]c,d[$, the following*

$$\begin{aligned}|R_{m,n}f(x,y)|\le{}&\frac{(x-a)(b-x)}{2m(b-a)^2}M_1[f]\\&+\frac{(y-c)(d-y)}{2n(d-c)^2}M_2[f]\\&+\frac{(x-a)(b-x)(y-c)(d-y)}{4mn(b-a)^2(d-c)^2}M_3[f]\end{aligned} \tag{2.4}$$

holds, for any $(x,y)\in[a,b]\times[c,d]$, where

$$M_1[f] = \sup_{(x,y)\in I\times J} \left| \frac{\partial^2 f}{\partial x^2}(x,y) \right|, \tag{2.5}$$

$$M_2[f] = \sup_{(x,y)\in I\times J} \left| \frac{\partial^2 f}{\partial y^2}(x,y) \right|,$$

$$M_3[f] = \sup_{(x,y)\in I\times J} \left| \frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \right|$$

and $I \times J = [a,b]\times[c,d]$.

Consider now the bidimensional interval $[a,b]\times[c,d]$ divided in $m\cdot n$ equally spaced subintervals

$$I_k \times J_j = \left[a+(k-1)\cdot\frac{b-a}{m}, a+k\cdot\frac{b-a}{m}\right]\times\left[c+(j-1)\cdot\frac{d-c}{n}, c+j\cdot\frac{d-c}{n}\right],$$
$$k=\overline{1,m},\ j=\overline{1,n}.$$

In each such type of interval one considers the distinct knots (x_h, y_l), $h=\overline{0,p}, l=\overline{0,q}$, where

$$x_h = a+(kp-p+h)\cdot\frac{b-a}{mp},\ y_l = c+(jq-q+l)\cdot\frac{d-c}{nq},\ h=\overline{0,p},\ l=\overline{0,q}. \tag{2.6}$$

Applying Lemma 2.1 on $I_k\times J_j$, follows the following bivariate Bernstein type polynomial

$$\overline{B}_{p,k,q,j}; f(x,y) = \frac{m^p\cdot n^q}{(b-a)^p(d-c)^q}\sum_{k=0}^{p}\sum_{l=0}^{q}\binom{p}{k}\binom{q}{l} \tag{2.7}$$
$$\left(x-a-(k-1)\cdot\frac{b-a}{m}\right)^k\left(a+k\cdot\frac{b-a}{m}-x\right)^{p-k}\left(y-c-(j-1)\cdot\frac{d-c}{n}\right)^l\left(a+j\cdot\frac{d-c}{n}-y\right)^{q-l}$$
$$f\left(a+(kp-p+h)\cdot\frac{b-a}{mp}, c+(jq-q+l)\cdot\frac{d-c}{nq}\right).$$

On any bidimensional interval

$$I_k\times J_j = \left[a+(k-1)\cdot\frac{b-a}{m}, a+k\cdot\frac{b-a}{m}\right]\times\left[c+(j-1)\cdot\frac{d-c}{n}, c+j\cdot\frac{d-c}{n}\right]$$

holds the bivariate Bernstein approximation formula

$$f(x,y) = \overline{B}_{p,k,q,j} f(x,y) + \overline{R}_{p,k,q,j} f(x,y). \tag{2.8}$$

Applying Lemma 2.2 we get the following upper bound estimation for the remainder term of (2.8)

$$|\overline{R}_{p,k,q,j} f(x,y)| \tag{2.9}$$

$$\leq \frac{m\left(x-a-(k-1)\cdot\frac{b-a}{m}\right)\left(a+k\cdot\frac{b-a}{m}-x\right)}{2(b-a)^2}M_1''[f]+$$
$$+\frac{n\left(y-c-(j-1)\cdot\frac{d-c}{n}\right)\left(c+j\cdot\frac{d-c}{n}-y\right)}{2(d-c)^2}M_2''[f]+$$
$$+\frac{mn\left(x-a-(k-1)\cdot\frac{b-a}{m}\right)\left(a+k\cdot\frac{b-a}{m}-x\right)\left(y-c-(j-1)\cdot\frac{d-c}{n}\right)\left(c+j\cdot\frac{d-c}{n}-y\right)}{4(b-a)^2(d-c)^2}$$
$$M_3''[f]$$

where

$$M_1''[f]=\sup_{(x,y)\in I_k\times J_j}\left|\frac{\partial^2 f}{\partial x^2}(x,y)\right|, \tag{2.10}$$
$$M_2''[f]=\sup_{(x,y)\in I_k\times J_j}\left|\frac{\partial^2 f}{\partial y^2}(x,y)\right|,$$
$$M_3''[f]=\sup_{(x,y)\in I_k\times J_j}\left|\frac{\partial^4 f}{\partial x^2\partial y^2}(x,y)\right|.$$

Theorem 6. *Let* $I_k\times J_j=\left[a+(k-1)\cdot\frac{b-a}{m},a+k\cdot\frac{b-a}{m}\right]\times\left[c+(j-1)\cdot\frac{d-c}{n},c+j\cdot\frac{d-c}{n}\right]$ *and* x_k,y_l *given at* (2.6). *The coefficients of the Bernstein type cubature formula*

$$\iint\limits_{I_k\times J_j} f(x,y)dxdy=\sum_{h=0}^{p}\sum_{l=0}^{q}\overline{A}_{h,k,l,j}f(x_k,y_l)+\overline{R}_{kj}[f] \tag{2.11}$$

are expressed by

$$\overline{A}_{h,k,l,j}=\frac{(b-a)(d-c)}{mn(p+1)(q+1)}. \tag{2.12}$$

Proof. Integrating (2.8) on $I_k\times J_j$ and taking (2.7) into account, yields

$$\overline{A}_{h,k,l,j}=\frac{m^p\cdot n^q}{(b-a)^p(d-c)^q}\binom{p}{k}\binom{q}{l}\iint\limits_{I_k\times J_j}\left(x-a-(k-1)\cdot\frac{b-a}{m}\right)^k\left(a+k\cdot\frac{b-a}{m}-x\right)^{p-k}$$
$$\cdot\left(y-c-(j-1)\cdot\frac{d-c}{n}\right)^l\left(c+j+\frac{d-c}{n}-y\right)^{q-l}dxdy.$$

Denoting

$$A_p=\frac{m^p}{(b-a)^p}\binom{p}{h}\int\limits_{I_k}\left(x-a-(k-1)\cdot\frac{b-a}{m}\right)^k\left(a+k\cdot\frac{b-a}{m}-x\right)^{p-k}dx$$

respectively

$$A_q = \frac{n^q}{(d-c)^q}\binom{q}{l}\int\limits_{J_j}\left(y-c-(j-1)\cdot\frac{d-c}{n}\right)^l\left(c+j+\frac{d-c}{n}-y\right)^{q-l}dy$$

we have

$$\overline{A}_{h,k,l,j} = A_p \cdot A_q.$$

To computing A_p one makes the change of variable $x = a+(k-1)\frac{b-a}{m}+t\frac{b-a}{m}$, which leads to

$$\begin{aligned} A_p &= \frac{m^p}{(b-a)^p}\binom{p}{h}\frac{(b-a)^p}{m^p}\cdot\frac{b-a}{m}\int\limits_0^1 t^h(1-t)^{p-h}dt \\ &= \binom{p}{h}\frac{b-a}{m}B(h+1,p-h+1) \end{aligned}$$

were $B(h+1,p-h+1)$ is the Euler function of first kind. Taking its well known properties into account, one arrives to

$$A_p = \binom{p}{h}\frac{b-a}{m}\cdot\frac{h!(p-h)!}{(p+1)!} = \frac{b-a}{m(p+1)}.$$

In a similar way one obtains $A_q = \frac{d-c}{n(q+1)}$ and then, taking the equality $\overline{A}_{h,k,l,j} = A_p \cdot A_q$ into account follows (2.12). □

Theorem 7. *Let $f \in C^{2,2}[a,b]\times[c,d]$ such that exist $\dfrac{\partial^4 f}{\partial x^2\partial y^2}$ on $I_k \times J_j$ and $\dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}, \dfrac{\partial^4 f}{\partial x^2\partial y^2}$ are bounded on $I_k\times J_j$, $k=\overline{1,m}$, $j=\overline{1,n}$. Then, the following upper bound estimation for the remainder term of* (2.11)

$$\begin{aligned} |\overline{R}_{k.j}[f]| \le \frac{(b-a)(d-c)}{12m^2n}M_1''[f] + \frac{(b-a)(d-c)}{12mn^2}M_2''[f] + \\ + \frac{(b-a)(d-c)}{144m^2n^2}M_3''[f] \end{aligned} \tag{2.13}$$

holds, where $M_1''[f], M_2''[f], M_3''[f]$ are defined at (2.10).

Proof. The inequality (2.13) follows integrating the approximation formula (2.8) and taking the inequalities (2.9) into account. □

The main result of the paper is the following

Theorem 8. *Let $f \in C^{2,2}[a,b]\times[c,d[$ be given such that exist $\dfrac{\partial^4 f}{\partial x^2 \partial y^2}$ on $]a,b[\times]c,d[$ and $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$, $\dfrac{\partial^4 f}{\partial x^2 \partial y^2}$ are bounded on $]a,b[\times]c,d[$. Then, the following composite Bernstein type cubature formula*

$$\int_a^b\int_c^d f(x,y)dxdy = \frac{(b-a)(d-c)}{mn(p+1)(q+1)}\sum_{k=i}^{m}\sum_{j=i}^{n}\sum_{h=0}^{p}\sum_{l=0}^{q} f(x_h,y_l)+R_{m,n}[f] \tag{2.14}$$

holds, where x_h, y_l are defined at (2.6) while the remainder term verifies the inequality (1.10).

Proof. Adding the Bernstein type cubature formulas (2.11) for $k=\overline{1,m}$, $j=\overline{1,n}$ one arrives to (2.14). □

Remark 1.

(i) For $a=0, b=1$ one refinds the results from [7].

(ii) It is immediate that $\lim\limits_{m,n\to\infty}|R_{m,n}[f]|=0$, which proves that

$$\lim_{\substack{m\to\infty\\ n\to\infty}}\frac{(b-a)(d-c)}{mn(p+1)(q+1)}$$

$$\sum_{k=i}^{m}\sum_{j=i}^{n}\sum_{h=0}^{p}\sum_{l=0}^{q} f\left(a+(kp-p+h)\cdot\frac{b-a}{mp}, c+(jq-q+l)\cdot\frac{d-c}{nq}\right)=$$

$$=\int_a^b\int_c^d f(x,y)dxdy.$$

3. Numerical examples

In [9] the authors introduce cubature formulas for two-variables function with boundary-layer components to evaluate the integral

$$I_f=\int_a^b\int_c^d f(x,y)dxdy \tag{3.1}$$

For a uniform grid on the domain $[a,b]\times[c,d]$ with nodes $(x_i,y_j), i=0,1,\dots,m$, $j=0,1,\dots,n$, with steps h_1 and h_2 in x and y, respectively, let $u_{i,j}=f(x_i,y_j)$.

Let us denote:

Trapezoidal Rule (TR)

$$TR_f=\frac{h_1h_2}{4}\sum_{i,j}(u_{i+1,j+1}+u_{i,j+1}+u_{i,j}+u_{i+1,j}),\ \ 0\le i<m,\ \ 0\le j<n$$

Simpson's Rule (SR)

$$SR_f = \frac{h_1 h_2}{9} \sum_{i,j} (16u_{i,j} + 4u_{i+1,j} + 4u_{i-1,j} + 4u_{i,j+1} + 4u_{i,j-1} + 4u_{i+1,j+1}$$
$$+u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1}), \;\; i = 1,3,\cdots,m-1, \;\; j = 1,3,\cdots,n-1$$

Bernstein Rule (BR)

$$BR_f = \frac{(b-a)(d-c)}{mn(p+1)(q+1)}$$
$$\sum_{k=i}^{m} \sum_{j=i}^{n} \sum_{h=0}^{p} \sum_{l=0}^{q} f\left(a+(kp-p+h)\cdot \frac{b-a}{mp}, c+(jq-q+l)\cdot \frac{d-c}{nq}\right).$$

To demonstrate the accuracy of our new numerical cubature formula, we compare the *Bernstein Rule (BR)* with *Trapezoidal Rule (TR)* and *Simpson's Rule (SR)* by using the following test functions:

$$f_1 : [0,2]\times[0,2] \to \mathbb{R}, \; f_1(x,y) = e^{-(x^2+y^2)}$$

$$f_2 : [0,5]\times[0,3] \to \mathbb{R}, \; f_2(x,y) = e^{-2(x+y)} \sin(4x+4y)$$

$$f_\varepsilon [0,1]\times[0,1] \to \mathbb{R}, \; f_\varepsilon(x,y) = (1-e^{\frac{-x}{\varepsilon}})(1-e^{\frac{-2y}{\varepsilon}})(1-x)(1-y)+$$
$$+cos(\frac{\pi x}{2})e^{-y}, \;\; \varepsilon \in (,1].$$

The last function f_ε is studied in [9] for the values of $\varepsilon = 1$, $\varepsilon = 10^{-1}$, $\varepsilon = 10^{-2}$, $\varepsilon = 10^{-3}$, $\varepsilon = 10^{-4}$ and $\varepsilon = 10^{-5}$, by using Trapezoidal Rule (TR), Simpson's Rule (SR) and the new methods introduced by the author.

Let us denote the values of the integrals by

$$I_{f_1} = \int_0^2 \int_0^2 f_1(x,y)dxdy, \quad I_{f_2} = \int_0^5 \int_0^3 f_2(x,y)dxdy,$$
$$I_{f_\varepsilon} = \int_0^1 \int_0^1 f_\varepsilon(x,y)dxdy$$

and the errors by

$$eTR_f = |I_f - TR_f|, \;\; eSR_f = |I_f - SR_f|, \;\; eBR_f = |I_f - BR_f|.$$

The numerical results obtained for $m = 64$, $n = 64$, $p = 5$ and $q = 5$ are presented in Table 1 and Table 2. In these tables, $e-m$ means 10^{-m}.

The Mathcad 14.0 package was used to generate these numerical results.

Table 1. Errors for f_1 and f_2

	eTR$_f$	eSR$_f$	eBR$_f$
f_1	1.05e-5	6.84e-9	2.10e-6
f_2	1.40e-5	1.89e-6	1.47e-6

Table 2. Errors for f_ε

ε	eTR_{f_ε}	eSR_{f_ε}	eBR_{f_ε}
1	2.68e-5	6.65e-11	5.37e-6
10^{-1}	2.87e-4	1.42e-6	5.76e-5
10^{-2}	2.71e-3	6.95e-4	5.54e-4
10^{-3}	7.01e-3	4.43e-3	1.91e-3
10^{-4}	7.76e-3	5.11e-3	2.59e-3
10^{-5}	7.76e-3	5.18e-3	2.59e-3

4. Conclusions

In Table 1 and Table 2 one can see that Bernstein cubature formula revised is better than trapezoidal cubature formula for all functions studied. Even if the Simpson's cubature formula is of order (2,2) and Bernstein cubature formula is of order (1,1), the results obtained by using our revised formula are better for the cases of functions f_2 and f_ε, with $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$.

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EXISTENCE RESULTS FOR SYSTEMS OF ITERATIVE DIFFERENTIAL EQUATIONS

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Abstract. In this article we give conditions for the existence of solutions of the initial value problem for systems of differential equations depending not only on the value of solutions but also on iterations of their coordinates.

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1. Introduction

Several works deal with first iterative initial value problems (see M.Fečkan [5], V.Berinde [2], S.Staněk [11–17]), while M.Medved' [8], S.Staněk [10] and P.Zhang and X.Gong [18] studied a functional differential equation of the more general form

$$x'(t) = F(t, x^{[1]}(t), x^{[2]}(t), \dots, x^{[k]}(t)), \qquad x(t) \in \mathbb{R}.$$

Starting from results in [18] we prove an existence result for the following system of differential equations

$$x'(t) = f(t, x^{[1]}(t), x^{[2]}(t), \dots, x^{[k]}(t)), \tag{1.1}$$

with initial conditions

$$x(t_0) = x_0, \tag{1.2}$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t)) \in \mathbb{R}^n$, $x^{[i]}(t) = (x_1^{[i]}(t), x_2^{[i]}(t), \dots, x_n^{[i]}(t))$, $x_j^{[i]}(t)$ is the i-th iteration of self-mapping $x_j(t)$, $j = 1, 2, \dots, n$, $f : \mathbb{R}^{nk+1} \to \mathbb{R}^n$ is continuous and, in addition, we show how one can approximate the non-unique solution of such kind of iterative differential equations by means of iterative techniques. The main idea is to use the powerful and more reliable technique of non-expansive

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operators and to adapt and use several convergence theorems from the theory of iterative approximation of fixed points of non-expansive mappings (see the monograph [1]).

2. Main results

In the proof of the main result we need the following lemma.

Lemma 1. *Let*

$$\Phi_M = \{x \in C^0(I_h, \mathbb{R}^n) : \|x(t) - x(s)\| = \max_{1\le j\le n} |x_j(t) - x_j(s)| \le M|t-s|;\ t,s \in I_h\},$$

where $0 < M < 1$. *If* $g = (g_1, g_2, \dots, g_n) \in \Phi_M$ *and* $g^{[i]} = (g_1^{[i]}, g_2^{[i]}, \dots, g_n^{[i]})$, *where* $g_j^{[i]}(t)$ *the* i*-th iteration of* $g_j(t)$ *is for all* $i \in \{1,2,\dots,k\}$, $j \in \{1,2,\dots,n\}$ *well defined and the same for* $h = (h_1, h_2, \dots, h_n) \in \Phi_M$, *then*

$$\left\|g^{[i]} - h^{[i]}\right\|_{I_h} \le \frac{1-M^i}{1-M}\|g-h\|_{I_h}, \qquad i = 1,2,\dots,k.$$

Proof. Let $g,h \in \Phi_M$, $g = (g_1, g_2, \dots, g_n)$, $h = (h_1, h_2, \dots, h_n)$. Since $|g_j(t) - g_j(s)| \le \|g(t) - g(s)\| \le M|t-s|$ and similarly $|h_j(t) - h_j(s)| \le \|h(t) - h(s)\| \le M|t-s|$ for all $j \in \{1,2,\dots,n\}$ and $t,s \in I_h$, by [18, Lemma 2.1], we have the inequality

$$\left\|g_j^{[i]} - h_j^{[i]}\right\|_{I_h} \le \frac{1-M^i}{1-M}\|g_j - h_j\|_{I_h}.$$

This yields

$$\begin{aligned}\left\|g^{[i]} - h^{[i]}\right\|_{I_h} &= \max_{1\le j\le n}\left\|g_j^{[i]} - h_j^{[i]}\right\|_{I_h}\\ &\le \frac{1-M^i}{1-M}\max_{1\le j\le n}\|g_j - h_j\|_{I_h} = \frac{1-M^i}{1-M}\|g-h\|_{I_h}\end{aligned}$$

for all $i \in \{1,2,\dots,k\}$. □

Theorem 1. *Suppose that* $f : \mathbb{R}^{nk+1} \to \mathbb{R}^n$ *is a continuous mapping. Let there exists a positive constant* r *such that*

$$(1-N)r > L_0 = \max\{l_1^0, l_2^0, \dots, l_n^0\}, \tag{2.1}$$

where $l_j^0 = |x_j^0 - t_0|$, $j = 1,2,\dots,n$, $x_0 = (x_1^0, x_2^0, \dots, x_n^0)$,

$$N = \|f\|_{B(y_0,r)} < 1, \tag{2.2}$$

where $B(y_0,r)$ *denotes the closed ball centered at* $y_0 = (t_0, x_0, \dots, x_0)$ *with radius* r. *Then the initial value problem* (1.1), (1.2) *has a solution defined on* $[t_0 - l, t_0 + l]$ *for any* $l \in [\frac{L_0}{1-N}, r]$.

Proof. The initial value problem (1.1), (1.2) is equivalent to the system of integral equations

$$x_j(t) = x_j^0 + \int_{t_0}^{t} f_j(s, x_1^{[1]}(s), \dots, x_n^{[1]}(s), \dots, x_1^{[k]}(s), \dots, x_n^{[k]}(s))\,\mathrm{d}s, \tag{2.3}$$

where $j \in \{1,2,\dots,n\}$, $x(t_0) = x_0 = (x_1^0, x_2^0, \dots, x_n^0)$, $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$, $f = (f_1, f_2, \dots, f_n)$. Let $l \in [\frac{L_0}{1-N}, r]$ and $t \in [t_0 - l, t_0 + l]$. Since $\|f_j\|_{B(y_0,r)} \le N = \|f\|_{B(y_0,r)}$, we obtain from (2.1) the following estimate for $x_j(t)$, $j \in \{1,2,\dots,n\}$:

$$\begin{aligned} |x_j(t) - t_0| &\le |x_j(t) - x_j(t_0)| + |x_j(t_0) - t_0| \le N|t - t_0| + |x_j^0 - t_0| \\ &\le Nl + l_j^0 \le Nl + L_0 \le Nl + l(1-N) = l. \end{aligned}$$

We have obtained the inequalities

$$|x_j(t) - t_0| \le l, \qquad j = 1,2,\dots,n,\ t \in [t_0 - l, t_0 + l].$$

We will prove by induction the validity of the inequalities

$$|x_j^{[i]}(t) - t_0| \le l, \quad j = 1,2,\dots,n,\ i \in \{1,2,\dots,k\},\ t \in [t_0 - l, t_0 + l]. \tag{2.4}$$

We assume that the inequalities (2.4) hold for all integer $i \in \{1,2,\dots,k-1\}$. Then

$$\begin{aligned} |x_j^{[i+1]}(t) - t_0| &\le |x_j^{[i+1]}(t) - x_j(t_0)| + |x_j(t_0) - t_0| \\ &= |x_j(x_j^{[i]}(t)) - x_j(t_0)| + |x_j^0 - t_0| \le N|x_j^{[i]}(t) - t_0| + l_j^0 \\ &\le Nl + L_0 \le Nl + l(1-N) = l. \end{aligned}$$

This means that the iterates $x_j^{[i]}$ are well defined on the interval $[t_0 - l, t_0 + l]$ for any $x = (x_1, x_2, \dots, x_n) \in \Phi_N := \{x \in C^0(I_h, \mathbb{R}^n) : x_j(t_0) = x_j^0,\ j = 1,2,\dots,n;\ \|x(t) - x(s)\| = \max_{1 \le j \le n} |x_j(t) - x_j(s)| \le N|t - s|;\ t, s \in I_h\}$, and thus the operator

$$\mathcal{G} : \Phi_N \to C^0([t_0 - l, t_0 + l], \mathbb{R}^n), \qquad \mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n), \tag{2.5}$$

where

$$\mathcal{G}_j x(t) = x_j^0 + \int_{t_0}^{t} f_j(s, x_1^{[1]}(s), \dots, x_n^{[1]}(s), \dots, x_1^{[k]}(s), \dots, x_n^{[k]}(s))\,\mathrm{d}s,$$

and $j \in \{1,2,\dots,n\}$ is well defined and obviously

$$\mathcal{G}_j x(t_0) = x_j^0, \tag{2.6}$$

for any $x \in \Phi_N$ and $j \in \{1,2,\dots,n\}$. Now we will prove that $\mathcal{G}$ is a self-mapping operator. We need to estimate

$$\begin{aligned}
&\left\| \left(t, x^{[1]}(t), x^{[2]}(t), \dots, x^{[k]}(t)\right) - \left(t_0, x_0, \dots, x_0\right) \right\| \\
&= \max\left\{|t-t_0|, \|x^{[1]}(t)-x(t_0)\|, \dots, \|x^{[k]}(t)-x(t_0)\|\right\} \\
&= \max\Big\{|t-t_0|, \max_{1\le j\le n}|x_j(t)-x_j(t_0)|, \max_{1\le j\le n}|x_j(x_j^{[1]}(t))-x_j(t_0)|, \\
&\qquad \dots, \max_{1\le j\le n}|x_j(x_j^{[k-1]}(t))-x_j(t_0)|\Big\} \\
&\le \max\Big\{|t-t_0|, N|t-t_0|, N\max_{1\le j\le n}|x_j^{[1]}(t)-t_0|, \\
&\qquad \dots, N\max_{1\le j\le n}|x_j^{[k-1]}(t)-t_0|\Big\} \\
&\le \max\left\{l, Nl, \dots, Nl\right\} \le l \le r, \qquad t \in I_l,
\end{aligned}$$

i.e.

$$\left(t, x^{[1]}(t), x^{[2]}(t), \dots, x^{[k]}(t)\right) \in B(y_0, r).$$

This yields

$$\begin{aligned}
&\|\mathcal{G}x(t_2)-\mathcal{G}x(t_1)\| = \max_{1\le j\le n}|\mathcal{G}_j x(t_2)-\mathcal{G}_j x(t_1)| \\
&\le \max_{1\le j\le n}\left|\int_{t_1}^{t_2}|f_j(s, x_1^{[1]}(s), \dots, x_n^{[1]}(s), \dots, x_1^{[k]}(s), \dots, x_n^{[k]}(s))|\,\mathrm{d}s\right| \\
&\le \|f\|_{B(y_0,r)}\left|\int_{t_1}^{t_2}\mathrm{d}s\right| \le N|t_2-t_1|
\end{aligned} \tag{2.7}$$

for any $t_1, t_2 \in I_l = [t_0-l, t_0+l]$. Thus (2.4) and (2.5) yield $\mathcal{G} \in \Phi_N$, i.e., $\mathcal{G}$ is self-mapping operator. For any $x, y \in \Phi_N$, ($x(t_0) = x_0 = y(t_0)$) and any $t \in I_l$ we have

$$\begin{aligned}
\|\mathcal{G}x(t)-\mathcal{G}y(t)\| &= \max_{1\le j\le n}\left|\mathcal{G}_j x(t)-\mathcal{G}_j y(t)\right| \\
&\le \max_{1\le j\le n}\Big|\int_{t_0}^{t}|f_j(s, x_1^{[1]}(s), \dots, x_n^{[1]}(s), \dots, x_1^{[k]}(s), \dots, x_n^{[k]}(s)) \\
&\qquad - f_j(s, y_1^{[1]}(s), \dots, y_n^{[1]}(s), \dots, y_1^{[k]}(s), \dots, y_n^{[k]}(s))|\,\mathrm{d}s\Big|.
\end{aligned}$$

By the Lemma 1,

$$\begin{aligned}
&\Big\| \Big(s, x_1^{[1]}(s), \dots, x_n^{[1]}(s), \dots, x_1^{[k]}(s), \dots, x_n^{[k]}(s)\Big) \\
&\qquad - \Big(s, y_1^{[1]}(s), \dots, y_n^{[1]}(s), \dots, y_1^{[k]}(s), \dots, y_n^{[k]}(s)\Big)\Big\| \\
&= \max\Big\{|x_1^{[1]}(s) - y_1^{[1]}(s)|, \dots, |x_n^{[1]}(s) - y_n^{[1]}(s)|, \\
&\qquad \dots, |x_1^{[k]}(s) - y_1^{[k]}(s)|, \dots, |x_n^{[k]}(s) - y_n^{[k]}(s)|\Big\} \\
&= \max_{1\le i\le k}\Big\{\max_{1\le j\le n} |x_j^{[i]}(s) - y_j^{[i]}(s)|\Big\} = \max_{1\le i\le k}\Big\|x^{[i]}(s) - y^{[i]}(s)\Big\| \\
&\le \max_{1\le i\le k} \|x^{[i]} - y^{[i]}\|_{I_l} \le \max_{1\le i\le k}\Big\{\frac{1-N^i}{1-N}\|x-y\|_{I_l}\Big\} \\
&= \frac{1-N^k}{1-N}\|x-y\|_{I_l} < \frac{1}{1-N}\|x-y\|_{I_l}.
\end{aligned}$$

It follows from the uniform continuity of f on the closed ball $B(y_0, r)$ that there for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that the inequality

$$\|\mathcal{G}x - \mathcal{G}y\| = \max_{1\le j\le n} \|\mathcal{G}_j x - \mathcal{G}_j y\| \le \varepsilon l$$

holds for $\|x - y\|_{I_l} < \delta$ and therefore the operator $\mathcal{G}$ is continuous.

Since Φ_N is a convex compact subset of Banach space $C^0(I_l, \mathbb{R}^n)$ and $\mathcal{G}$ is continuous operator, by the *Schauder fixed point theorem* we obtain that $\mathcal{G}$ has a fixed point $g_0 \in \Phi_N$ and hence g_0 is a solution of the initial value problem for system of differential equations (1.1) associated with (1.2) on the interval $I_l = [t_0 - l, t_0 + l]$ and this completes the proof. □

2.1. *Existence theorem and approximation of solutions*

The following result is a fixed point theorem for nonexpansive mappings (see e.g. in Berinde [1, Theorem 2.1]):

Lemma 2. *Let C be a nonempty closed convex and bounded subset of a uniformly convex Banach space B. Then any nonexpansive mapping $F: C \to C$ has a fixed point.*

Definition 1. Let C be a convex subset of a normed linear space B and let $F: C \to C$ be a self-mapping. Given any $x_0 \in C$ and a real number $\lambda \in (0, 1]$, the sequence $\{x_n\}$ defined by the formula

$$x_{n+1} = (1-\lambda)x_n + \lambda F x_n, \qquad n = 0, 1, 2, \dots \tag{2.8}$$

is usually called the Krasnoselskij iteration, or Krasnoselskij-Mann iteration.

Definition 2. Let C be a convex subset of a normed linear space B and let $F:C \to C$ be a self-mapping. Given an $x_0 \in C$ and a real numbers $\lambda_n \in [0,1]$, the sequence $\{x_n\}$ defined by the formula

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n F x_n, \qquad n = 0,1,2,\dots \tag{2.9}$$

is usually called the Mann iteration.

It was shown by M.Krasnoselskij [7], in the case for $\lambda = 1/2$, and later by H. Schaefer [9] for an arbitrary $\lambda \in (0,1)$, that if B is a uniformly convex Banach space and C is a convex and compact subset of B, then the M.Krasnoselskij iteration converges to a fixed point of F. M.Edelstein [4] proved that strict convexity of B suffices for the same conclusion. The question if the assumption of the strict convexity of B can be removed was answered affirmatively by the following S.Ishikawa [6] result:

Lemma 3. *Let C be a subset of a Banach space B and let $F:C \to C$ be a nonexpansive mapping. For arbitrary $x_0 \in C$, consider the Mann iteration process $\{x_n\}$ given by (2.9) under the following assumptions:*

(a) *$x_n \in C$ for all positive integers n;*
(b) *$0 \le \lambda_n \le b < 1$ for all positive integers n;*
(c) *$\sum_{n=0}^{\infty} \lambda_n = \infty$.*

If $\{x_n\}$ is bounded, then $x_n - F x_n \to 0$ as $n \to \infty$.

The following corollary of Lemma 3 will be particularly important for the second main result of our paper.

Corollary 1 (see in C.Chidume [3, Theorem 6.17])**.** *Let C be a convex and compact subset of a Banach space B and let $F:C \to C$ be a nonexpansive mapping. If the Mann iteration process $\{x_n\}$ given by (2.9) satisfies the assumptions* (a)–(c) *of Lemma 3, then $\{x_n\}$ converges strongly to a fixed point of F.*

We consider the initial value problem

(1) $x'(t) = f(t, x^{[1]}(t), x^{[2]}(t), \dots, x^{[k]}(t)), \qquad t \in [a,b];$
(2) $x(t_0) = x_0$.

where $t_0 \in [a,b]$, $x_0 = (x_1^0, x_2^0, \dots, x_n^0) \in (a,b)^{nk}$, $f \in C^0([a,b]^{nk+1}, \mathbb{R}^n)$ are given and $[a,b]^{nk+1}$ denotes the product $nk+1$ intervals $[a,b]$. Let us put

$$A_t = \max\{t-a, b-t\}, \qquad B_x^j = \min\{x_j - a, b - x_j\}. \tag{2.10}$$

We are interested here in solution from the space $C^1([a,b],[a,b]^n)$ belonging to the class

$$\begin{aligned} \Phi_K := \{x \in C^0([a,b],[a,b]^n) :\ & x_j(t_0) = x_j^0,\ j = 1,2,\dots,n; \\ & \|x(t)-x(s)\| = \max_{1\le j\le n} |x_j(t)-x_j(s)| \le K|t-s|;\ t,s \in [a,b]\}, \end{aligned} \tag{2.11}$$

where $K = \|f\|_{[a,b]^{nk+1}}$.

Theorem 2. *Assume that the following conditions are satisfied for the initial value problem* (1.1), (1.2)*:*

(i) $f:\mathbb{R}^{nk+1}\to\mathbb{R}^n$ *is continuous and* $[a,b]$ *is a compact interval containing* t_0, x_j^0, $j=1,2,\dots,n$*;*

(ii) *there exist* $L>0$ *such that* $\|f(t,u)-f(t,v)\|\le L\|u-v\|$ *for all* $t\in[a,b]$ *and* $u,v\in[a,b]^{nk}$*;*

(iii) *let* $K=\|f\|_{[a,b]^{nk+1}}>0$ *and let one of the following conditions hold:*

(a) $KA_{t_0}\le B_{x_0}^j$, *for all* $j=1,2,\dots,n$, *where* A_{t_0}, $B_{x_0}^j$ *are defined by* (2.10)*;*

(b) $t_0=a, K(b-a)\le b-x_j^0$, $j=1,2,\dots,n$, $f(t,u)\ge 0$ *for all* $(t,u)\in[a,b]^{nk+1}$*;*

(c) $t_0=b, K(b-a)\le x_j^0-a$, $j=1,2,\dots,n$, $f(t,u)\ge 0$ *for all* $(t,u)\in[a,b]^{nk+1}$.

In addition

(iv) (A) $LA_{t_0}\dfrac{1-K^k}{1-K}\le 1$, *if* $K<1$,

(B) $LA_{t_0}k\le 1$, *if* $K=1$.

Then there exists at least one solution x^* *of problem* (1.1), (1.2) *in the class* Φ_K, *which can be approximated by the Mann iteration*

$$x_{m+1}=(1-\lambda_m)x_m+\lambda_m\mathcal{G}x_m,\qquad m=0,1,2,\dots,\tag{2.12}$$

where $\{\lambda_m\}$ *is a given sequence of real numbers satisfying the assumptions* (b)–(c) *formulated in Lemma 3 and the operator*

$$\mathcal{G}:\Phi_K\to C^0([a,b],\mathbb{R}^n),\qquad \mathcal{G}=(\mathcal{G}_1,\mathcal{G}_2,\dots,\mathcal{G}_n)$$

is defined by

$$\mathcal{G}_j x(t)=x_j^0+\int_{t_0}^t f_j(s,x_1^{[1]}(s),\dots,x_n^{[1]}(s),\dots,x_1^{[k]}(s),\dots,x_n^{[k]}(s))\,\mathrm{d}s,$$

where $j\in\{1,2,\dots,n\}$.

Proof. Φ_K is nonempty convex compact subset of Banach space $(C^0([a,b],[a,b]^n))$ with usual supremum norm. It is clear that $x^*\in\Phi_K$ is a solution of the initial value problem (1.1), (1.2) if and only if x^* is a fixed point of $\mathcal{G}$, i.e., $x^*=\mathcal{G}x^*$. Our aim is to shaw that $\mathcal{G}$ has a fixed point in Φ_K. We will apply Schauder fixed point theorem or Banach fixed point theorem.

First we prove that Φ_K is a invariant set with respect to $\mathcal{G}$, i.e., $\mathcal{G}(\Phi_K)\subseteq\Phi_K$.

If the condition (iii)(a) holds, then for any $x \in \Phi_K$, $t \in [a,b]$ and for all $j \in \{1,2,\dots,n\}$ we have

$$\begin{aligned}\mathcal{G}_j x(t) &= x_j^0 + \int_{t_0}^{t} f_j(s, x_1^{[1]}(s), \dots, x_n^{[1]}(s), \dots, x_1^{[k]}(s), \dots, x_n^{[k]}(s))\,\mathrm{d}s \\ &\le x_j^0 + \Big|\int_{t_0}^{t} f_j(s, x_1^{[1]}(s), \dots, x_n^{[1]}(s), \dots, x_1^{[k]}(s), \dots, x_n^{[k]}(s))\,\mathrm{d}s\Big| \\ &\le x_j^0 + K\,|t-t_0| \le x_j^0 + K\,A_{t_0} \le x_j^0 + B_{x_0}^j \le x_j^0 + (b - x_j^0) = b\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}_j x(t) &\ge x_j^0 - \Big|\int_{t_0}^{t} f_j(s, x_1^{[1]}(s), \dots, x_n^{[1]}(s), \dots, x_1^{[k]}(s), \dots, x_n^{[k]}(s))\,\mathrm{d}s\Big| \\ &\ge x_j^0 - K\,|t-t_0| \ge x_j^0 - K\,A_{t_0} \ge x_j^0 - B_{x_0}^j \ge x_j^0 - (x_j^0 - a) = a.\end{aligned}$$

The cases (iii)(b) and (iii)(c) can be treated in a similar way and so we have that for any $x \in \Phi_K$ and $t \in [a,b]$, one has $\mathcal{G}_j x(t) \in [a,b]$ for all $j \in \{1,2,\dots,n\}$.

Now, for any $t_1, t_2 \in [a,b]$ we have

$$\begin{aligned}&\|\mathcal{G}x(t_2) - \mathcal{G}x(t_1)\| = \max_{1\le j\le n} |\mathcal{G}_j x(t_2) - \mathcal{G}_j x(t_1)| \\ &\le \max_{1\le j\le n} \Big|\int_{t_1}^{t_2} |f_j(s, x_1^{[1]}(s), \dots, x_n^{[1]}(s), \dots, x_1^{[k]}(s), \dots, x_n^{[k]}(s))|\,\mathrm{d}s\Big| \\ &\le \|f\|_{[a,b]^{nk+1}} \Big|\int_{t_1}^{t_2} \mathrm{d}s\Big| \le K|t_2 - t_1|.\end{aligned}$$

By the formula (2.6) we have $\mathcal{G}_j x(t_0) = x_j^0$ for any $x \in \Phi_K$ and $j \in \{1,2,\dots,n\}$. Therefore $\mathcal{G}\colon \Phi_K \to \Phi_K$ (i.e., $\mathcal{G}$ is a selfmapping of Φ_K).

From the assumptions (iii) and the definitions of A_{t_0} and $B_{x_0}^j$, it follows that $K \le 1$. Therefore according to Lemma 1 for any $x, y \in \Phi_K$, $s \in [a,b]$, $i \in \{2,3,\dots,k\}$, $j \in \{1,2,\dots,n\}$ and $K < 1$ the following holds:

$$\begin{aligned}&\Big|x_j^{[i]}(s) - y_j^{[i]}(s)\Big| \\ &\le \Big|x_j(x_j^{[i-1]}(s)) - x_j(y_j^{[i-1]}(s))\Big| + \Big|x_j(y_j^{[i-1]}(s)) - y_j(y_j^{[i-1]}(s))\Big| \\ &\le K\Big|x_j^{[i-1]}(s) - y_j^{[i-1]}(s))\Big| + \max_{\xi\in[a,b]} |x_j(\xi) - y_j(\xi)| \\ &\le K\Big(K\Big|x_j^{[i-2]}(s) - y_j^{[i-2]}(s))\Big| + \max_{\xi\in[a,b]} |x_j(\xi) - y_j(\xi)|\Big) + \max_{\xi\in[a,b]} |x_j(\xi) - y_j(\xi)| \\ &\le \cdots \le \big(K^{i-1} + K^{i-2} + \cdots + 1\big) \max_{\xi\in[a,b]} |x_j(\xi) - y_j(\xi)| \le \frac{1-K^k}{1-K}\|x_j - y_j\|,\end{aligned}$$

and if $K = 1$ then

$$\left|x_j^{[i]}(s) - y_j^{[i]}(s)\right| \le k\|x_j - y_j\|.$$

Now if we take any $x, y \in \Phi_K$ and any $t \in [a,b]$, for $K < 1$ we have

$$\begin{aligned}
\|\mathcal{G}x(t) - \mathcal{G}y(t)\| &= \max_{1\le j\le n}\left|\mathcal{G}_j x(t) - \mathcal{G}_j y(t)\right| \\
&\le \max_{1\le j\le n}\Big|\int_{t_0}^{t} |f_j(s, x_1^{[1]}(s), \dots, x_n^{[1]}(s), \dots, x_1^{[k]}(s), \dots, x_n^{[k]}(s)) \\
&\qquad - f_j(s, y_1^{[1]}(s), \dots, y_n^{[1]}(s), \dots, y_1^{[k]}(s), \dots, y_n^{[k]}(s))|\,\mathrm{d}s\Big| \\
&\le \Big|\int_{t_0}^{t} L\Big\|\Big(x_1^{[1]}(s), \dots, x_n^{[1]}(s), \dots, x_1^{[k]}(s), \dots, x_n^{[k]}(s)\Big) \\
&\qquad - \Big(y_1^{[1]}(s), \dots, y_n^{[1]}(s), \dots, y_1^{[k]}(s), \dots, y_n^{[k]}(s)\Big)\Big\|\,\mathrm{d}s\Big| \\
&\le L\Big|\int_{t_0}^{t} \max_{1\le i\le k}\Big\{\max_{1\le j\le n}|x_j^{[i]}(s) - y_j^{[i]}(s)|\Big\}\mathrm{d}s\Big| \\
&\le L\Big|\int_{t_0}^{t} \max_{1\le j\le n}\Big\{\frac{1-K^k}{1-K}\|x_j - y_j\|\Big\}\mathrm{d}s\Big| \\
&\le L\,\frac{1-K^k}{1-K}\|x-y\||t-t_0| \le LA_{t_0}\,\frac{1-K^k}{1-K}\,\|x-y\|.
\end{aligned}$$

(In the case $K = 1$ we have $\|\mathcal{G}x(t) - \mathcal{G}y(t)\| \le LA_{t_0}\,k\|x-y\|$).

Now, by taking the supremum in the last inequalities, we get

$$\|\mathcal{G}x - \mathcal{G}y\| \le LA_{t_0}\,\frac{1-K^k}{1-K}\,\|x-y\|,\ (\ \|\mathcal{G}x - \mathcal{G}y\| \le LA_{t_0}\,k\|x-y\|\ )$$

and so from the conditions (iv) it follows that operator $\mathcal{G}$ is nonexpansive and hence continuous. Now it remains to apply the *Schauder fixed point theorem* to obtain the first part theorem (if $LA_{t_0}\frac{1-K^k}{1-K} < 1$ for $K < 1$, or $LA_{t_0}k < 1$ for $K = 1$, then $\mathcal{G}$ is a contraction mapping and hence in view *Banach fixed point theorem*, has a unique solution) and Corollary 1 to get the second one. This completes the proof. □

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ON GENERALIZED VECTOR EQUILIBRIUM-LIKE PROBLEM IN TOPOLOGICAL VECTOR SPACES

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Abstract. In this paper, some existence results for generalized vector equilibrium-like problems (for short, GVELPs) are established by virtue of a generalized KKM-Fan theorem in topological vector spaces. In addition, some applications to vector optimization problems are also given. The results presented in this paper are the extension and improvement of the corresponding results of other authors.

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1. Introduction and preliminaries

The vector variational inequality (VVI) was first introduced in [1] in the setting of finite dimensional Euclidean space $\mathbf{R}^{\mathbf{n}}$. Subsequently, the VVI was generalized to infinite dimensional spaces in [9, 11, 22, 30]. The vector variational-like inequality (VVLI), a generalization of VVI, was studied in [26, 27] with applications in the vector optimization problem (VOP). The VVI and VVLI with set-valued mappings have been considered in [16, 18, 31]. In recent years some generalizations of VVI and VVLI have been further studied by many authors. For details, see e.g., [2–8, 13–15, 17, 19–21, 23–25, 28, 32, 34] and references therein. In 2008, Jabarootian and Zafarani [21] considered an extension of the VVLI with set-valued mapping, called generalized vector variational-like inequality (GVVLI). By using a generalized KKM-Fan theorem in [12], they established some results on the existence of solutions for this class of GVVLIs.

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Next, we recall some concepts and notations. A nonempty subset C of a vector space Y is a convex cone if $\lambda C \subseteq C$ for all $\lambda \geq 0$ and $C + C \subseteq C$. A convex cone C is pointed if $C \cap (-C) = \{0\}$. A cone C is proper if it is properly contained in Y. Note that C is a proper cone if and only if $0 \notin \mathrm{int}C$, where $\mathrm{int}C$ denotes the interior of C. A pointed convex cone C induces a partial order $\leq_C$ on Y defined by $x \leq_C y$ whenever $y - x \in C$. In this case, $(Y, \leq_C)$ is an ordered vector space $(Y, \leq_C)$ with an order relation $\leq_C$. The weak order $\not\leq_{\mathrm{int}C}$ on an ordered vector space $(Y, \leq_C)$ with $\mathrm{int}C \neq \varnothing$ is defined by $x \not\leq_{\mathrm{int}C} y$ whenever $y - x \notin \mathrm{int}C$.

Let X and Y be two topological spaces. We say that a multifunction $\varphi : X \to 2^Y$ is closed, or has closed graph if its graph was given by

$$\mathcal{G}(\varphi) = \{(x, y) \in X \times Y : y \in \varphi(x)\}$$

which is a closed subset of $X \times Y$.

For any two Hausdorff topological vector spaces (t.v.s.) X and Y, let $\mathcal{L}(X, Y)$ denote the family of all continuous linear operators from X into Y. When Y is the set $\mathbf{R}$ of real numbers, $\mathcal{L}(X, Y)$ is the usual dual space X^* of X. For any $x \in X$ and any $u \in \mathcal{L}(X, Y)$, we shall write the value $u(x)$ as $\langle u, x\rangle$. We suppose throughout this paper that K is a nonempty closed convex subset of X, $T : K \to 2^{\mathcal{L}(X,Y)}$ is a set-valued mapping, $\Phi : \mathcal{L}(X, Y) \times K \times K \to Y$, $A : K \times \mathcal{L}(X, Y) \to \mathcal{L}(X, Y)$, $\alpha : X \to Y$ and $f : K \times K \to Y$ are four functions such that $f(x, x) = 0$ for all $x \in X$, and $\{C(x) : x \in K\}$ is a family of closed, convex and pointed cones of Y (i.e., $C : K \to 2^Y$ is a cone mapping) such that $\mathrm{int}C(x) \neq \varnothing$. for all $x \in K$.

Motivated and inspired by generalized vector equilibrium problems considered in Zeng and Yao [33, 35], this work is to further extend the results of Jabarootian and Zafarani [21] to the setting of the generalized vector equilibrium-like problems (GVELPs). The main purpose of this article is the solvability of the following GVELP and its related problems in topological vector spaces.

Here, $\Phi(A(a, T(z)), x, y) = \{\Phi(A(a, u), x, y) : u \in T(z)\}$, $\forall a, x, y, z \in K$.

Problem (I). Find $x_0 \in K$ and $w_0 \in A(x_0, T(x_0))$ such that

$$\Phi(w_0, x_0, y) + f(x_0, y) \not\leq_{\mathrm{int}C(x_0)} 0, \quad \text{for all } y \in K.$$

Problem (II). Find $x_0 \in K$ such that for each $y \in K$

$$\Phi(w, x_0, y) + f(x_0, y) \not\leq_{\mathrm{int}C(x_0)} \alpha(y - x_0), \quad \text{for some } w \in A(x_0, T(y)).$$

Problem (III). Find $x_0 \in K$ and $w_0 \in A(x_0, T(x_0))$ such that

$$\Phi(w_0, x_0, y) \not\leq_{C(x_0)\setminus\{0\}} 0, \quad \text{for all } y \in K.$$

Problem (IV). Find $x_0 \in K$ such that

$$0 \leq_{C(x_0)} \Phi(w, x_0, y), \quad \text{for all } y \in K \text{ and } w \in A(x_0, T(y)).$$

Problem (V). Find $x_0 \in K$ such that

$$0 \leq_{C(x_0)} \Phi(w, x_0, y), \quad \text{for all } y \in K \text{ and } w \in A(x_0, T(x_0)).$$

In particular, if $A(x,u) = u$ for each $(x,u) \in K \times \mathcal{L}(X,Y)$ and $\Phi(w,x,y) = \langle w, \eta(y,x)\rangle$ for each $(w,x,y) \in \mathcal{L}(X,Y) \times K \times K$, where $\eta : K \times K \to X$ is a function, then Problems (I)-(V) reduce to Problems (I)-(V) in [21], respectively. Problems (I) and (II) with $A(x,u) = u$ and $\Phi(w,x,y) = \langle w, \eta(y,x)\rangle$ become the simplified versions of the generalized vector variational-like inequality problems (GVVTIPs) in [24]. In this case, it is enough to replace $M(\cdot, T(\cdot)) : K \to 2^{\mathcal{L}(X,Y)}$ in [24] with our mapping T. For Problem (II), by putting $\alpha \equiv 0$, we deduce in a similar way GVVTIP (II) in [24]. Problem (II) with $A(x,u) = u$ and $\Phi(w,x,y) = \langle w, \eta(y,x)\rangle$ is also a vector version of Problem (II) in [19].

Lemma 1 (see [9]). *Let $(Y, \leq_C)$ be an ordered topological vector space with a closed, convex and pointed cone C with* $\mathrm{int}C \neq \varnothing$. *Then for each $x, y \in Y$, one has*

(1) $y - x \in \mathrm{int}C$ *and* $y \notin \mathrm{int}C \;\Rightarrow\; x \notin \mathrm{int}C$.
(2) $y - x \in C$ *and* $y \notin \mathrm{int}C \;\Rightarrow\; x \notin \mathrm{int}C$.
(3) $y - x \in -\mathrm{int}C$ *and* $y \notin -\mathrm{int}C \;\Rightarrow\; x \notin -\mathrm{int}C$.
(4) $y - x \in -C$ *and* $y \notin -\mathrm{int}C \;\Rightarrow\; x \notin -\mathrm{int}C$.

We denote by $\mathcal{F}(X)$ the family of all nonempty finite subsets of X. Let $F : Y \to 2^X$ be a set-valued mapping. Then F is said to be transfer closed-valued iff for each $(y,x) \in Y \times X$ with $x \notin F(y)$, there exists $y' \in Y$ such that $x \notin \mathrm{cl}F(y')$. If $B \subseteq Y$ and $A \subseteq X$, then we call $F : B \to 2^A$ transfer closed-valued iff the multi-valued mapping $y \mapsto F(y) \cap A$ is transfer closed-valued. When $X = Y$ and $A = B$, we call F transfer closed-valued on A. Let K be a convex subset of a vector space X. Then a mapping $F : K \to 2^X$ is called a KKM mapping iff for each nonempty finite subset A of K, $\mathrm{conv}A \subset F(A)$, where $\mathrm{conv}A$ denotes the convex hull of A, and $F(A) = \bigcup\{F(x) : x \in A\}$.

Theorem 1 (see [12]). *Let K be a nonempty and convex subset of a Hausdorff t.v.s. X. Suppose that $\Gamma, \hat{\Gamma} : K \to 2^K$ are two set-valued mappings such that the following conditions are satisfied:*

(A1) $\hat{\Gamma}(x) \subseteq \Gamma(x)$, $\forall x \in K$;

(A2) $\hat{\Gamma}$ is a KKM map;

(A3) for each $A \in \mathcal{F}(K)$, Γ is transfer closed-valued on $\mathrm{conv}A$*;*

(A4) for each $A \in \mathcal{F}(K)$, $\mathrm{cl}_K(\bigcap_{x \in \mathrm{conv}A} \Gamma(x)) \cap \mathrm{conv}A = (\bigcap_{x \in \mathrm{conv}A} \Gamma(x)) \cap \mathrm{conv}A$*;*

(A5) there is a nonempty compact convex set $B \subseteq K$ such that $\mathrm{cl}_K(\bigcap_{x \in B} \Gamma(x))$ *is compact.*

Then, $\bigcap_{x \in K} \Gamma(x) \neq \varnothing$.

Definition 1. Let $T : X \to 2^Y$ be a set-valued mapping. Then,

(i) T is said to be upper semicontinuous (u.s.c.) iff, for each closed set $B \subset Y$,

$$T^-(B) = \{x \in X : T(x) \cap B \neq \varnothing\} \text{ is clised in } X;$$

(ii) T is said to be lower semicontinuous (l.s.c.) iff, for each open set $B \subset Y$,

$$T^{-}(B) = \{x \in X : T(x) \cap B \neq \varnothing\} \text{ is open in } X;$$

(iii) T is said to be closed (resp. open) iff, the set $G_r(T) = \{(x,y) \in X \times Y : y \in T(x)\}$ is closed (resp. open) in $X \times Y$.

2. GVELPs WITH MONOTONICITY

Let $\mho$ be the family of all bounded subsets of X whose union is total in X, i.e., the linear hull of $\bigcup\{S : S \in \mho\}$ is dense in X. Let $\mathcal{B}$ be a neighborhood base of 0 in Y. When S runs through $\mho$ and V through $\mathcal{B}$, the family

$$M(S,V) = \{u \in \mathcal{L}(X,Y) : \langle u, x\rangle \in V, \forall x \in S\}$$

is a neighborhood base of 0 in $\mathcal{L}(X,Y)$ for a unique translation-invariant topology, called the topology of uniform convergence on the sets $S \in \mho$, or briefly the τ-topology; see [29].

Definition 2. Let $\alpha : X \to Y$ with $\alpha(tz) = t^p\alpha(z)$ $\forall t > 0$ and $\forall z \in X$, where $p > 1$. Then a function $\Phi : \mathcal{L}(X,Y) \times K \times K \to Y$ is said to be

(i) relaxed α-f-pseudomonotone with respect to T and A if for each x_1, x_2, $y \in K$,

$$\begin{aligned}&\Phi(w_1,x_1,x_2) + f(x_1,x_2) \not\leq_{\operatorname{int}C(x_1)} 0 \text{ for all } w_1 \in A(y,Tx_1)\\ &\Rightarrow \Phi(w_2,x_1,x_2) + f(x_1,x_2) \not\leq_{\operatorname{int}C(x_1)} \alpha(x_2 - x_1) \text{ for all } w_2 \in A(y,Tx_2).\end{aligned}$$

(ii) relaxed α-f-pseudomonotone-type with respect to T and A if for each x_1, x_2, $y \in K$,

$$\begin{aligned}&\Phi(w_1,x_1,x_2) + f(x_1,x_2) \not\leq_{\operatorname{int}C(x_1)} 0 \text{ for some } w_1 \in A(y,Tx_1)\\ &\Rightarrow \Phi(w_2,x_1,x_2) + f(x_1,x_2) \not\leq_{\operatorname{int}C(x_1)} \alpha(x_2 - x_1) \text{ for some } w_2 \in A(y,Tx_2).\end{aligned}$$

(iii) strongly pseudomonotone with respect to T and A if for each $x_1, x_2, y \in K$,

$$\begin{aligned}&\Phi(w_1,x_1,x_2) \not\leq_{C(x_1)\setminus\{0\}} 0 \text{ for some } w_1 \in A(y,Tx_1)\\ &\Rightarrow 0 \leq_{C(x_1)} \Phi(w_2,x_1,x_2) \text{ for all } w_2 \in A(y,Tx_2).\end{aligned}$$

In particular, if $A(x,u) = u$ for each $(x,u) \in K \times \mathcal{L}(X,Y)$ and $\Phi(w,x,y) = \langle w, \eta(y,x)\rangle$ for each $(w,x,y) \in \mathcal{L}(X,Y) \times K \times K$, where $\eta : K \times K \to X$ is a function, then the concepts in Definition 2 (i)-(iii) reduce to the ones in Definition 2 (i)-(iii) of [21], respectively.

Definition 3. A function $\Phi : \mathcal{L}(X,Y) \times K \times K \to Y$ is said to be upper hemicontinuous (u.h.c.) (resp. lower hemicontinuous (l.h.c.)) with respect to T and A if for each $x, y, z \in K$,

$$t \in [0,1] \mapsto \Phi(A(z, T(x + t(y - x))), x, y)$$

is u.s.c. (resp. l.s.c.) at 0^+.

Definition 4. A set-valued mapping $F : K \to 2^Y$ is said to be C-convex where C is a convex cone in Y if for each $x, y \in K$ and $t \in [0,1]$ one has

$$(1-t)F(x) + tF(y) \subseteq F((1-t)x + ty) + C.$$

A single valued function $F : K \to Y$ is said to be C-convex if for each $x, y \in K$ and $t \in [0,1]$,

$$F((1-t)x + ty) \leq_C (1-t)F(x) + tF(y).$$

Here, we consider a generalized vector version of Minty-type lemma.

Lemma 2. *Let $C : K \to 2^Y$ be a cone mapping such that* $\mathrm{int}C_- \neq \emptyset$, *where $C_- = \bigcap_{x\in K} C(x)$. Assume that*

(H1) for each $(w,x) \in \mathcal{L}(X,Y) \times K$, $\Phi(w,x,x) = 0$ and f is C_--convex in the second variable;

(H2) Φ is continuous in the first variable and C_--convex in the third variable, and A is continuous in the second variable;

(H3) Φ is relaxed α-f-pseudomonotone type and u.h.c. with respect to T and A, where $T : K \to 2^{\mathcal{L}(X,Y)}$ takes compact values.

Then, Problems (I) and (II) are equivalent.

Proof. Problem (I) $\Rightarrow$ Problem (II) follows from the definition of relaxed α-f-pseudomonotonicity type of Φ with respect to T and A.

Conversely, suppose that we can find $x_0 \in K$ such that for each $y \in K$

$$\Phi(w, x_0, y) + f(x_0, y) \not\leq_{\mathrm{int}C(x_0)} \alpha(y - x_0), \quad \text{for some } w \in A(x_0, T(y)).$$

This means that

$$\Phi(A(x_0, T(y)), x_0, y) + f(x_0, y) - \alpha(y - x_0) \not\subseteq \mathrm{int}C(x_0). \tag{2.1}$$

Assume to the contrary that, for any element $x_0 \in K$, there exists $y_0 \in K$ such that

$$\Phi(w_0, x_0, y_0) + f(x_0, y_0) \leq_{\mathrm{int}C(x_0)} 0, \quad \text{for all } w_0 \in A(x_0, T(x_0)).$$

That is,

$$\Phi(A(x_0, T(x_0)), x_0, y_0) + f(x_0, y_0) \subseteq -\mathrm{int}C(x_0).$$

Since A is continuous in the second variable, by the compactness of $T(x_0)$ we know that $A(x_0, T(x_0)$ is a compact set in $\mathcal{L}(X,Y)$. So, by the continuity of Φ in the first variable, we know that $\Phi(A(x_0, T(x_0)), x_0, y_0) + f(x_0, y_0)$ is a compact set in Y. Hence we can choose an open neighborhood V of zero in Y such that

$$\Phi(A(x_0, T(x_0)), x_0, y_0) + f(x_0, y_0) + V \subseteq -\mathrm{int}C(x_0).$$

We consider $x_t = x_0 + t(y_0 - x_0)$ for $t \in [0,1]$. By the upper hemicontinuity of Φ with respect to T and A, there exists $0 < \beta < 1$ such that, for $0 < t < \beta$,

$$\Phi(A(x_0, T(x_t)), x_0, y_0) + f(x_0, y_0) + V \subseteq -\mathrm{int}C(x_0). \tag{2.2}$$

Since $-C(x_0)$ is a cone, from condition (H1), we obtain

$$\frac{1}{t}f(x_0,x_t)-f(x_0,y_0)\in -C(x_0). \tag{2.3}$$

Then, by (2.2) and (2.3), we deduce that

$$\Phi(A(x_0,T(x_t)),x_0,y_0)+\frac{1}{t}f(x_0,x_t)+V\subseteq -\mathrm{int}C(x_0). \tag{2.4}$$

On the other hand, since Φ is C_--convex in the third variable, we deduce that for all $v_t\in T(x_t)$,

$$\frac{1}{t}[\Phi(A(x_0,v_t),x_0,x_t)-(1-t)\Phi(A(x_0,v_t),x_0,x_0)-t\Phi(A(x_0,v_t),x_0,y_0)]\in -C(x_0). \tag{2.5}$$

From condition (H1) and adding (2.4) and (2.5), for all $v_t\in T(x_t)$,

$$\Phi(A(x_0,v_t),x_0,x_t)+f(x_0,x_t)+tV\subseteq -\mathrm{int}C(x_0).$$

As $\lim_{t\to 0^+}\frac{1}{t}\alpha(t(y_0-x_0))=0$, for $t>0$ small enough, we have

$$\Phi(A(x_0,v_t),x_0,x_t)+f(x_0,x_t)-\alpha(x_t-x_0)\subseteq -\mathrm{int}C(x_0).$$

But this contradicts (2.1). □

Remark 1. Lemma 2 generalizes Lemma 2.1 of [25] to the GVELPs. It also improves and extends Theorem 3.1 of [24] and Lemma 2.5 of [19] because the VVLIs and GVVIs considered in [19, 24] are replaced by the more general problems, i.e., GVELPs.

Theorem 2. *Let $C:K\to 2^Y$ be a cone mapping such that* $\mathrm{int}C_-\neq\emptyset$, *where $C_-=\bigcap_{x\in K}C(x)$. Let hypotheses (H1)-(H3) of Lemma 2 hold and suppose that*

(i) for each finite subset D of K, the set-valued mapping $\Gamma:K\to 2^K$ defined by

$$\Gamma(y)=\{x\in K:\Phi(w,x,y)+f(x,y)\not\leq_{\mathrm{int}C(x)}\alpha(y-x),\ \text{for some } w\in A(x,T(y))\}$$

is transfer closed valued on $\mathrm{conv}(D)$*;*

(ii) for each finite subset D of K, $x,y\in\mathrm{conv}(D)$ and for each net $\{x_\lambda\}$ in K converging to x, if $x_\lambda\in\Gamma(z)$ for all $z\in\mathrm{conv}(D)$ and for all λ, then $x\in\Gamma(y)$;

(iii) there exist a nonempty compact set $M\subset K$ and a nonempty compact convex set $B\subset K$ such that for each $x\in K\setminus M$, there exists $y\in B$ such that $x\notin\Gamma(y)$.
Then, Problem (I) holds.

Proof. We define the set-valued mapping $\hat{\Gamma}:K\to 2^K$ by

$$\hat{\Gamma}(y):=\{x\in K:\Phi(w,x,y)+f(x,y)\not\leq_{\mathrm{int}C(x)}0,\ \text{for some } w\in A(x,T(x))\}.$$

We show that $\hat{\Gamma}$ is a KKM mapping. Indeed, assume $\hat{\Gamma}$ is not a KKM mapping. Then there exists $\{x_1,x_2,...,x_n\}\subset K$, $t_i\geq 0$, $i=1,2,...,n$ with $\sum_{i=1}^n t_i=1$ such that $x=\sum_{i=1}^n t_ix_i\notin\bigcup_{i=1}^n\hat{\Gamma}(x_i)$. Thus for any $w\in A(x,T(x))$, we have

$$\Phi(w,x,x_i)+f(x,x_i)\leq_{\mathrm{int}C(x)}0,\quad i=1,2,...,n;$$

therefore, we deduce that

$$\sum_{i=1}^{n} t_i \Phi(w,x,x_i) + \sum_{i=1}^{n} t_i f(x,x_i) \leq_{\mathrm{int}C(x)} 0. \tag{2.6}$$

On the other hand, since f is C_--convex in the second variable and Φ is C_--convex in the third variable, we know from Eq. (2.6) that for each $w \in A(x,T(x))$

$$\begin{aligned} 0 &= \Phi(w,x,x) + f(x,x) \\ &\leq_{C(x)} \sum_{i=1}^{n} t_i \Phi(w,x,x_i) + \sum_{i=1}^{n} t_i f(x,x_i) \\ &\leq_{\mathrm{int}C(x)} 0, \end{aligned}$$

which leads to a contradiction that $C(x) \neq Y$. Hence, $\hat{\Gamma}$ is a KKM mapping. Since Φ is relaxed α-f-pseudomonotone-type with respect to T and A, we have $\hat{\Gamma}(y) \subseteq \Gamma(y)$ for each $y \in K$. Hence, Γ is also a KKM mapping. Thus all the conditions of Theorem 1 are fulfilled by the mappings $\hat{\Gamma}$ and Γ. Therefore, $\bigcap_{y\in K} \Gamma(y) \neq \varnothing$. Hence, Problem (II) holds and from Lemma 2, Problem (I) is deduced. □

Remark 2. In the above theorem, if $\Gamma(y)$ is closed for all $y \in K$, then conditions (i) and (ii) hold. In the following results, we will establish the closedness of $\Gamma(y)$ for each $y \in K$.

Corollary 1. *Let X be a normed space and $(Y,\mathcal{T})$ be a t.v.s. Let $C : K \to 2^Y$ be a cone mapping such that* $\mathrm{int}C_- \neq \varnothing$, *where $C_- = \bigcap_{x\in K} C(x)$. Let hypotheses (H1)-(H3) of Lemma 2 hold and suppose that*

(i) the set-valued mapping $W : K \to 2^Y$ defined by $W(x) = Y \setminus (-\mathrm{int}C(x))$ has weakly closed graph;

(ii) f is weakly continuous in the first variable and α is weakly continuous;

(iii) A is continuous from $\sigma_X \times \tau$ to τ, where σ_X is the weak topology of X and τ is the uniform convergence topology of $\mathcal{L}(X,Y)$;

(iv) for each $z \in K$, $\Phi(\cdot,\cdot,z) : \mathcal{L}(X,Y) \times K \to Y$ is continuous from $\tau \times \sigma_X$ to σ_Y, where σ_Y is the weak topology of Y;

(v) there exist a nonempty weakly compact set $M \subset K$ and a nonempty weakly compact convex set $B \subset K$ such that for each $x \in K \setminus M$, there exists $y \in B$ such that $x \notin \Gamma(y)$.

Then, Problem (I) holds.

Proof. By Theorem 2, it is enough to show that for each $y \in K$, $\Gamma(y)$ is weakly closed. Let $\{x_\beta\}$ be a net in $\Gamma(y)$ weakly convergent to $x_0 \in K$. Since $x_\beta \in \Gamma(y)$, there exists $v_\beta \in \Gamma(y)$ satisfying

$$\Phi(A(x_\beta,v_\beta),x_\beta,y) + f(x_\beta,y) \not\leq_{\mathrm{int}C(x_\beta)} \alpha(y - x_\beta),$$

that is,

$$z_\beta = \Phi(A(x_\beta, v_\beta), x_\beta, y) + f(x_\beta, y) - \alpha(y - x_\beta) \notin -\mathrm{int}C(x_\beta). \tag{2.7}$$

Thus, $z_\beta \in W(x_\beta)$ and hence $(x_\beta, z_\beta) \in G_r(W)$. Since $T(y)$ is compact, $\{v_\beta\}$ has a convergent subnet in $T(y)$. Let $\{v_\lambda\}$ be a subnet of $\{v_\beta\}$ that converges to $v_0 \in T(y)$. Since A is continuous from $\sigma_X \times \tau$ to τ, we know that $\{A(x_\lambda, v_\lambda)\}$ converges to $A(x_0, v_0) \in \mathcal{L}(X, Y)$. Also, since $\Phi(\cdot, \cdot, y) : \mathcal{L}(X, Y) \times K \to Y$ is continuous from $\tau \times \sigma_X$ to σ_Y, we know that $\Phi(A(x_\lambda, v_\lambda), x_\lambda, y)$ converges weakly to $\Phi(A(x_0, v_0), x_0, y)$. Consequently, by the weak continuity of f in the first variable and the weak continuity of α we deduce that $\{\Phi(A(x_\lambda, v_\lambda), x_\lambda, y) + f(x_\lambda, y) - \alpha(y - x_\lambda)\}$ converges weakly to $z_0 = \Phi(A(x_0, v_0), x_0, y) + f(x_0, y) - \alpha(y - x_0)$. Since W has weakly closed graph and $(x_\lambda, z_\lambda) \in G_r(W)$ (due to (2.7)), we get $(x_0, z_0) \in G_r(W)$. Therefore,

$$\Phi(A(x_0, v_0), x_0, y) + f(x_0, y) - \alpha(y - x_0) \notin -\mathrm{int}C(x_0).$$

That is, there exists $\tilde{w}(= A(x_0, v_0)) \in A(x_0, T(y))$ such that

$$\Phi(\tilde{w}, x_0, y) + f(x_0, y) \not\leq_{\mathrm{int}C(x_0)} \alpha(y - x_0).$$

Thus, $x_0 \in \Gamma(y)$, which completes the proof. □

Remark 3. Corollary 1 generalizes Corollary 2.1 of [21] to the GVELPs. It also improves and extends Theorem 3.4 of [24] and Theorem 3.2 of [23] because the problems considered in our Corollary 1 are the more general problems, i.e., GVELPs.

Corollary 2. *Let X be metrizable. Let $C : K \to 2^Y$ be a cone mapping such that* $\mathrm{int}C_- \neq \emptyset$, *where $C_- = \bigcap_{x \in K} C(x)$. Let hypotheses (H1)-(H3) of Lemma 2 and condition (iii) of Theorem 2 hold, and suppose that*

(i) the set-valued mapping $W : K \to 2^Y$ defined by $W(x) = Y \setminus (-\mathrm{int}C(x))$ is closed;

(ii) f is continuous in the first variable and α is continuous;

(iii) for each $z \in K$, $\Phi(\cdot, \cdot, z) : \mathcal{L}(X, Y) \times K \to Y$ is continuous and A is continuous.

Then, Problem (I) holds.

Proof. By the similar proof to that of Corollary 1, we obtain the desired result. □

Remark 4. It is easy to see that in the case where K is a compact subset of X, we can omit the metrizability of X in the above corollary. Moreover, Corollary 2 generalizes Corollary 2.2 of [21] to the GVELPs. It also improves and extends Theorems 2.1, 2.2 and 3.1 of [19], Theorem 3.1 of [32] and Theorem 3.4 of [23] because the problems considered in our Corollary 2 are the more general problems, i.e., GVELPs.

As another consequence of Theorem 2, when Φ is relaxed α-f-pseudomonotone with respect to T and A, we also obtain a generalized version of Theorems 3.1-3.4 in [25] and hence Theorem 3.2 in [32].

Corollary 3. *Assume T has a continuous selection g and let $C : K \to 2^Y$ be a cone mapping such that* $\mathrm{int}C_- \neq \varnothing$, *where $C_- = \bigcap_{x\in K} C(x)$. Let hypotheses (H1)-(H3) of Lemma 2 and conditions (i)-(iii) of Corollary 2 hold. If there exist a nonempty compact set $M \subset K$ and a nonempty compact convex set $B \subset K$ such that, for each $x \in K \setminus M$, there exists $y \in B$ such that*

$$\Phi(A(x,g(y)),x,y) + f(x,y) \leq_{\mathrm{int}C(x)} \alpha(y-x).$$

Then Problem (I) holds.

Proof. Since Φ is relaxed α-f-pseudomonotone with respect to T and A, hence Φ is α-f-pseudomonotone-type with respect to g (selection) and A. Thus, by Theorem 2, it is enough to show that, for each $y \in K$, the set

$$\Gamma(y) = \{x \in K : \Phi(w,x,y) + f(x,y) \not\leq_{\mathrm{int}C(x)} \alpha(y-x) \quad \text{with } w = A(x,g(y))\}$$

is closed. Indeed, let $\{x_\beta\}$ be a net in $\Gamma(y)$ convergent to $x_0 \in K$. Since $x_\beta \in \Gamma(y)$,

$$\Phi(w_\beta,x_\beta,y) + f(x_\beta,y) \not\leq_{\mathrm{int}C(x_\beta)} \alpha(y-x_\beta) \quad \text{with } w_\beta = A(x_\beta,g(y));$$

that is,

$$z_\beta = \Phi(A(x_\beta,g(y)),x_\beta,y) + f(x_\beta,y) - \alpha(y-x_\beta) \notin -\mathrm{int}C(x_\beta).$$

Thus, $z_\beta \in W(x_\beta)$ and hence $(x_\beta,z_\beta) \in G_r(W)$. By the continuity of f in the first variable, continuity of α and A, and continuity of $\Phi(\cdot,\cdot,y)$, we have

$$z_0 = \lim_\beta z_\beta = \Phi(A(x_0,g(y)),x_0,y) + f(x_0,y) - \alpha(y-x_0).$$

Since $G_r(W)$ is closed, $(x_0,z_0) \in G_r(W)$ and hence,

$$\Phi(A(x_0,g(y)),x_0,y) + f(x_0,y) - \alpha(y-x_0) \notin -\mathrm{int}C(x_0);$$

that is,

$$\Phi(\hat{w},x_0,y) + f(x_0,y) \not\leq_{\mathrm{int}C(x_0)} \alpha(y-x_0) \quad \text{with } \hat{w} = A(x_0,g(y)).$$

Therefore, $x_0 \in \Gamma(y)$. This means that $\Gamma(y)$ is closed. □

Corollary 3 generalizes Corollary 2.3 of [21] to the GVELPs. In order to obtain a solution of strong vector equilibrium-like problem, we need the following result. This result is a generalization of Lemma 2.2 in [21], and also a generalized vector version of Lemma 2.2 in [15].

Lemma 3. *Let $C : K \to 2^Y$ be a cone mapping such that* $\mathrm{int}C_- \neq \varnothing$, *where $C_- = \bigcap_{x\in K} C(x)$. Let Φ be strongly pseudomonotone and l.h.c. with respect to T and A. Assume that, for each $(w,x) \in \mathcal{L}(X,Y) \times K$, $\Phi(w,x,x) = 0$ and Φ is affine in the third variable. Then Problems (III) and (IV) are equivalent.*

Proof. Problem (III) $\Rightarrow$ Problems (IV) follows from the definition of strong pseudomonotonicity of Φ with respect to T and A.

Conversely, suppose that we can find $x_0 \in K$ such that

$$0 \leq_{C(x_0)} \Phi(w, x_0, y), \quad \text{for all } y \in K \text{ and } w \in A(x_0, T(y)). \tag{2.8}$$

We consider $x_t = x_0 + t(y - x_0)$ for $t \in [0,1]$. Replacing y by x_t in Eq. (2.8), we have

$$0 \leq_{C(x_0)} \Phi(w, x_0, y_t), \quad \text{for all } w \in A(x_0, T(x_t)).$$

As Φ is affine in the third variable, $\Phi(w, x_0, x_0) = 0, \forall w \in A(x_0, T(x_t))$, and $C(x_0)$ is a cone, we deduce that for any $w \in A(x_0, T(x_t))$

$$\begin{aligned} 0 &\leq_{C(x_0)} \Phi(w, x_0, y_t) \\ &= (1-t)\Phi(w, x_0, x_0) + t\Phi(w, x_0, y) = t\Phi(w, x_0, y); \end{aligned}$$

that is,

$$\Phi(A(x_0, T(x_t)), x_0, y) \subseteq C(x_0). \tag{2.9}$$

Suppose on the contrary that

$$\Phi(A(x_0, T(x_0)), x_0, y) \cap (Y \setminus C(x_0)) \neq \varnothing.$$

As Φ is l.h.c. with respect to T and A, we have

$$\Phi(A(x_0, T(x_t)), x_0, y) \cap (Y \setminus C(x_0)) \neq \varnothing,$$

for sufficiently small t, which contradicts (2.9). Hence,

$$\Phi(A(x_0, T(x_0)), x_0, y) \subseteq C(x_0), \quad \forall y \in K,$$

which immediately yields

$$\Phi(A(x_0, T(x_0)), x_0, y) \nsubseteq -C(x_0) \setminus \{0\}, \quad \forall y \in K.$$

That is, there exists $w_0 \in A(x_0, T(x_0))$ such that

$$\Phi(w_0, x_0, y) \nleq_{C(x_0)\setminus\{0\}} 0, \quad \text{for all } y \in K.$$

This completes the proof. □

Theorem 3. *Let $C : K \to 2^Y$ be a cone mapping such that* $\operatorname{int} C_- \neq \varnothing$*, where $C_- = \bigcap_{x \in K} C(x)$. Let Φ be strongly pseudomonotone and l.h.c. with respect to T and A. Suppose that*

(i) for each finite subset D of K, the set-valued mapping $\Gamma : K \to 2^K$ defined by

$$\Gamma(y) = \{x \in K : 0 \leq_{C(x)} \Phi(w, x, y), \text{ for all } w \in A(x, T(y))\}$$

is transfer closed valued on $\operatorname{conv}(D)$*;*

(ii) for each finite subset D of K, $x, y \in \operatorname{conv}(D)$ and for each net $\{x_\lambda\}$ in K converging to x, if $x_\lambda \in \Gamma(z)$ for all $z \in \operatorname{conv}(D)$ and for all λ, then $x \in \Gamma(y)$;

(iii) for each $(w, x) \in \mathcal{L}(X, Y) \times K$, $\Phi(w, x, x) = 0$ and Φ is affine in the third variable;

(iv) there exist a nonempty compact set $M \subset K$ and a nonempty compact convex set $B \subset K$ such that for each $x \in K \setminus M$, there exists $y \in B$ such that $x \notin \Gamma(y)$.
Then, Problem (III) holds.

Proof. For each $y \in K$, we define the set-valued mapping

$$\hat{\Gamma}(y) := \{x \in K : \Phi(w,x,y) \not\leq_{C(x)\setminus\{0\}} 0, \quad \text{for some } w \in A(x,T(x))\}.$$

We claim that $\hat{\Gamma}$ is a KKM mapping. Indeed, assume $\hat{\Gamma}$ is not a KKM mapping. Then there exist $\{x_1, x_2, ..., x_n\} \subset K$, $t_i \geq 0$, $i = 1,2,...,n$ with $\sum_{i=1}^n t_i = 1$ such that $x = \sum_{i=1}^n t_i x_i \notin \bigcup_{i=1}^n \hat{\Gamma}(x_i)$. Thus for any $w \in A(x,T(x))$, we have

$$\Phi(w,x,x_i) \leq_{C(x)\setminus\{0\}} 0, \quad i = 1,2,...,n,$$

which yields $\sum_{i=1}^n t_i \Phi(w,x,x_i) \leq_{C(x)\setminus\{0\}} 0$. Since for each $w \in A(x,T(x))$, $\Phi(w,x,x) = 0$ and Φ is affine in the third variable, we obtain

$$0 = \Phi(w,x,x) = \sum_{i=1}^n t_i \Phi(w,x,x_i) \leq_{C(x)\setminus\{0\}} 0,$$

which is a contradiction. Hence, $\hat{\Gamma}$ is a KKM mapping. It is clear that strong pseudo-monotonicity of Φ with respect to T and A, guarantees that $\hat{\Gamma}(y) \subseteq \Gamma(y)$ for each $y \in K$. Thus all the conditions of Theorem 1 are fulfilled by the mappings $\hat{\Gamma}$ and Γ. Therefore, $\bigcap_{y\in K} \Gamma \neq \varnothing$. Consequently, Problem (IV) holds and from Lemma 3, Problem (III) is deduced. □

Remark 5. Theorem 3 generalizes Theorem 2.2 of [21] to the GVELPs. In Theorem 3, if the set $\Gamma(y)$ is closed for each $y \in K$, then conditions (i) and (ii) hold. In the following result, we will establish the closedness of $\Gamma(y)$ for each $y \in K$. This result generalizes Corollary 2.4 of [21] to the GVELPs. Moreover, it also extends and improves Theorem 2.3 in [15].

Corollary 4. *Assume in Theorem 3, we replace conditions (i)-(ii) by the following conditions:*
(i) the set-valued mapping $C : K \to 2^Y$ is closed;
(ii) for each $z \in K$, $\Phi(\cdot,\cdot,z) : \mathcal{L}(X,Y) \times K \to Y$ is continuous and A is continuous in the first variable.
Then, Problem (III) holds.

Proof. By Theorem 3, it is enough to show that for each $y \in K$, $\Gamma(y)$ is closed. Let $\{x_\beta\}$ be a net in $\Gamma(y)$ convergent to $x_0 \in K$. Since $x_\beta \in \Gamma(y)$, for each $v \in T(y)$, we have $z_\beta = \Phi(A(x_\beta, v), x_\beta, y) \in C(x_\beta)$, which hence yields $(x_\beta, z_\beta) \in G_r(C)$. Also, since $\Phi(\cdot,\cdot,y) : \mathcal{L}(X,Y) \times K \to Y$ is continuous and A is continuous in the first variable, we know that the net $\{\Phi(A(x_\beta, v), x_\beta, y)\}$ converges to $(A(x_0, v), x_0, y)$;

that is, $z_0 = \lim_\beta z_\beta = \Phi(A(x_0, v), x_0, y)$. Since $G_r(C)$ is closed, we get $(x_0, z_0) \in G_r(C)$ and hence, $\Phi(A(x_0, v), x_0, y) \in C(x_0)$ for all $v \in T(y)$; that is,

$$0 \leq_{C(x_0)} \Phi(w, x_0, y), \text{ for all } w \in A(x_0, T(y)).$$

This means that $x_0 \in \Gamma(y)$. Therefore, $\Gamma(y)$ is closed. □

3. GVELPs WITHOUT MONOTONICITY

In this section, some existence results for GVELPs without any monotonicity are obtained.

Theorem 4. *Let $C : K \to 2^Y$ be a cone mapping such that* $\mathrm{int}C_- \neq \varnothing$, *where $C_- = \bigcap_{x \in K} C(x)$. Suppose that*

(i) for each finite subset D of K, the set-valued mapping $\Gamma : K \to 2^K$ defined by

$$\Gamma(y) = \{x \in K : 0 \leq_{C(x)} \Phi(w, x, y), \text{ for all } w \in A(x, T(x))\}$$

is transfer closed valued on $\mathrm{conv}(D)$*;*

(ii) for each finite subset D of K, $x, y \in \mathrm{conv}(D)$ and for each net $\{x_\lambda\}$ in K converging to x, if $x_\lambda \in \Gamma(z)$ for all $z \in \mathrm{conv}(D)$ and for all λ, then $x \in \Gamma(y)$;

(iii) for any fixed $x \in K$, the set $\{y \in K : 0 \not\leq_{C(x)} \Phi(w, x, y),$ for some $w \in A(x, T(x))\}$ is convex;

(iv) For each $x \in K$, one has $0 \leq_{C(x)} \Phi(w, x, x), \forall w \in A(x, T(x))$;

(v) there exist a nonempty compact set $M \subset K$ and a nonempty compact convex set $B \subset K$ such that for each $x \in K \setminus M$, there exists $y \in B$ such that $x \notin \Gamma(y)$.
Then, Problem (V) holds.

Proof. We claim that Γ is a KKM mapping. Indeed, assume Γ is not a KKM mapping. Then there exist $\{x_1, x_2, ..., x_n\} \subset K$, $t_i \geq 0$, $i = 1, 2, ..., n$ with $\sum_{i=1}^n t_i = 1$ such that $x = \sum_{i=1}^n t_i x_i \notin \bigcup_{i=1}^n \Gamma(x_i)$. Thus, for any $i = 1, 2, ..., n$, we have

$$\Phi(A(x, T(x)), x, x_i) \not\subseteq C(x).$$

Note that $\{y \in K : \Phi(A(x, T(x)), x, y) \not\subseteq C(x)\} = \{y \in K : 0 \not\leq_{C(x)} \Phi(w, x, y),$ for some $w \in A(x, T(x))\}$. So, from condition (iii), it follows that $\Phi(A(x, T(x)), x, x) \not\subseteq C(x)$, that is, there exists $w \in A(x, T(x))$ such that $0 \not\leq_{C(x)} \Phi(w, x, x)$, which contradicts condition (iv). Hence, Γ is a KKM mapping. Thus all the conditions of Theorem 1 are fulfilled by the mappings $\hat{\Gamma} = \Gamma$. Therefore, $\bigcap_{y \in K} \Gamma(y) \neq \varnothing$. Consequently, Problem (V) holds. □

Remark 6. In Theorem 4, we can replace conditions (iii)-(iv) by the following condition: (iii)′ Φ is properly quasimonotone with respect to T and A, i.e., for all $n \in \mathbf{N}$ and for all vectors $x_1, x_2, ..., x_n \in K$, and scalars $t_i > 0$, $i = 1, 2, ..., n$ with $\sum_{i=1}^n t_i = 1$ and $x = \sum_{i=1}^n t_i x_i$, $\Phi(A(x, T(x)), x, x_i) \subseteq C(x)$ holds for some i.

In fact, by a similar proof, we can show that Γ is a KKM mapping. If $\Phi(w, x, y) = \langle w, \eta(y, x)\rangle$ and $A(x, u) = u$ then Theorem 4 reduces to Theorem 3.1 of [21]. In this

case, if we let $\eta(y,x) = y - g(x)$, then we conclude Theorem 2.1 in [14]. When $\Gamma(y)$ is closed for each $y \in Y$, then Γ satisfies conditions (i) and (ii).

As a consequence of Theorem 4, in the following result we establish the closedness of Γ.

Corollary 5. *Let X be metrizable and suppose that*
(i) for each $z \in K$, $\Phi(\cdot,\cdot,z) : \mathcal{L}(X,Y) \times K \to Y$ is continuous and A is continuous;
(ii) $T : K \to 2^{\mathcal{L}(X,Y)}$ is l.s.c.;
(iii) the set-valued mapping $W : K \to 2^Y$ defined by $W(x) = Y \setminus C(x)$ is open.
Then, Problem (V) holds.

Proof. By Theorem 4, it is enough to show that for each $y \in K$, $\Gamma(y)$ is closed. Indeed, let $\{x_n\}$ be a sequence in $\Gamma(y)$ convergent to $\bar{x} \in K$. By the lower semi-continuity of T, for each $v \in T(\bar{x})$ there exists a sequence $v_n \in T(x_n)$ converging to v. Set $z_n = \Phi(A(x_n,v_n),x_n,y)$. Then from $x_n \in \Gamma(y)$, we have $z_n \in C(x_n)$. Thus, $(x_n,z_n) \notin G_r(W)$. By the continuity of A, we know that $\{A(x_n,v_n)\}$ converges to $A(\bar{x},v)$. Since $\Phi(\cdot,\cdot,y) : \mathcal{L}(X,Y) \times K \to Y$ is continuous, it is clear that $\{\Phi(A(x_n,v_n),x_n,y)\}$ converges to $\{\Phi(A(\bar{x},v),\bar{x},y)$; that is, $z = \lim_{n\to\infty} z_n = \Phi(A(\bar{x},v),\bar{x},y)$. Since $G_r(W)$ is open, we have $(\bar{x},z) \notin G_r(W)$, and hence $\bar{x} \in \Gamma(y)$. This completes the proof. □

Remark 7. If $\Phi(w,x,y) = \langle w, \eta(y,x)\rangle$ and $A(x,u) = u$ then Corollary 5 reduces to Corollary 3.1 of [21]. In this case, if we let $\eta(y,x) = y - g(x)$, then Corollary 5 establishes an existence theorem for strong vector implicit variational inequality problem in [20]. Thus we extend and improve Theorem 3.2 in [20] in many aspects. Corollary 5 also extends and improves Theorem 3.3 in [20] for the existence theorem of weak vector implicit variational inequality problem by using Theorem 3.1 (i) in [20]. As already mentioned, when K is a compact subset of X, then we can omit the condition of metrizability of X.

We now establish an existence result for the GVELP without any monotonicity, which is a generalization of Theorem 3.2 in [21] for the strong vector variational-like inequality problem without any monotonicity.

Theorem 5. *Let $C : K \to 2^Y$ be a cone mapping such that* $\mathrm{int}C_- \neq \emptyset$, *where $C_- = \bigcap_{x\in K} C(x)$. Suppose that*
(i) for each finite subset D of K, the set-valued mapping $\Gamma : K \to 2^K$ defined by

$$\Gamma(y) = \{x \in K : \Phi(w,x,y) \not\leq_{C(x)\setminus\{0\}} 0, \text{ for some } w \in A(x,T(x))\}$$

is transfer closed valued on $\mathrm{conv}(D)$*;*
(ii) for each finite subset D of K, $x,y \in \mathrm{conv}(D)$ and for each net $\{x_\lambda\}$ in K converging to x, if $x_\lambda \in \Gamma(z)$ for all $z \in \mathrm{conv}(D)$ and for all λ, then $x \in \Gamma(y)$;
(iii) for each $(w,x) \in \mathcal{L}(X,Y) \times K$, $\Phi(w,x,x) = 0$ and Φ is affine in the third variable. ;

(iv) there exist a nonempty compact set $M \subset K$ and a nonempty compact convex set $B \subset K$ such that for each $x \in K \setminus M$, there exists $y \in B$ such that $x \notin \Gamma(y)$. Then, Problem (III) holds.

Proof. By the similar proof to that of Theorem 3, on can show that Γ is a KKM mapping. Thus all the conditions of Theorem 1 are fulfilled by the mapping $\hat{\Gamma} = \Gamma$. Therefore, $\bigcap_{y\in K} \Gamma(y) \neq \varnothing$. Hence, there exists $x_0 \in \bigcap_{y\in K} \Gamma(y)$ which is a solution for Problem (III). □

As a consequence of Theorem 5, we obtain the following result which generalizes Theorem 3.2 of [21], and hence Theorem 2.1 of [15].

Corollary 6. *Suppose that, in addition to conditions (iii) and (iv) of Theorem 5, for each $y \in K$, $\Gamma(y)$ is closed. Then, Problem (III) holds.*

4. Applications

In this section, we consider solutions for the generalized weak optimization problem (VOP). Let K be a convex subset of a Banach space X, $g : K \to \mathbf{R^n}$, $g(x) = (g_1(x), g_2(x), ..., g_n(x))$ for $x \in K$, and $\{C(x) : x \in K\}$ be a family of closed, convex and pointed cones of $\mathbf{R^n}$ (i.e., $C : K \to 2^{\mathbf{R^n}}$ is a cone mapping) such that $\mathbf{R^n_+} \subseteq C(x), \forall x \in K$. A point $y \in K$ is called a solution for the generalized weak VOP of g if

$$g(x) \not\leq_{\mathrm{int}C(y)} g(y), \quad \text{for all } x \in K.$$

Definition 5. Let $f : X \to \mathbf{R}$ be locally Lipschitz at a given point $x \in X$ and v be any vector in X. The Clarke generalized directional derivative of f at x in the direction v, denoted by $f^\circ(x; v)$, is defined by

$$f^\circ(x; v) = \limsup_{y\to x, t\downarrow 0} \frac{f(y + tv) - f(y)}{t}.$$

Then, the Clarke generalized subdifferential of f at x, denoted by $\partial^c f(x)$, is defined as

$$\partial^c f(x) = \{\xi \in X^* : f^\circ(x; v) \geq \langle \xi, v\rangle, \ \forall v \in X\}.$$

Lemma 4 (see [10])**.** *If $f : X \to \mathbf{R}$ is locally Lipschitz, then $\partial^c f(x)$ is a nonempty convex and weak*-compact valued. Moreover, $\partial^c f : X \to 2^{X^*}$ is norm-to-weak* upper semicontinuous.*

These definitions and properties can be extended to a locally Lipschitz vector-valued function $g : X \to \mathbf{R^n}$. Denote by g_i, $i = 1, 2, ..., n$ the components of g. The Clarke generalized subdifferential of g at $x \in X$ is the set

$$\partial^c g(x) = \partial^c g_1(x) \times \partial^c g_2(x) \times \cdots \times \partial^c g_n(x).$$

Throughout the remainder of this paper, let $\Phi : \mathcal{L}(X, \mathbf{R^n}) \times K \times K \to \mathbf{R^n}$ be a function and $g : K \to \mathbf{R^n}$ be a function which has the Clarke generalized subdifferential on K.

Definition 6. The function g is said to be invex with respect to Φ on K if for each $x, y \in K$

$$\Phi(u, x, y) \leq_{\mathbf{R^n_+}} g(y) - g(x), \quad \forall\, u \in \partial^c g(x).$$

Remark 8. Let $\eta : X \times X \to X$ be a function. Let the function Φ be defined as $\Phi(u, x, y) = \langle u, \eta(y, x)\rangle, \forall (u, x, y) \in \mathcal{L}(X, \mathbf{R^n}) \times K \times K$. Assume that g_i is locally Lipschitz on K for $i = 1, 2, ..., n$. If

$$\langle \zeta_i, \eta(y, x)\rangle \leq g_i(y) - g_i(x), \quad x, y \in K,\ \zeta_i \in \partial^c g_i(x), i = 1, 2, ..., n,$$

then the concept of invex function of g with respect to Φ on K, reduces to the one of invex function of g with respect to η on K (see Definition 6 in [21]).

Definition 7. Φ is said to be $\partial^c g$-monotone if for each $x_1, x_2 \in K$

$$\Phi(u_1, x_1, x_2) + \Phi(u_2, x_2, x_1) \leq_{C(x_1)} 0, \quad \forall u_i \in \partial^c g(x_i), i = 1, 2.$$

Definition 8. The function $\Phi : \mathcal{L}(X, \mathbf{R^n}) \times K \times K \to \mathbf{R^n}$ is called skew if $\Phi(u, x, y) + \Phi(u, y, x) = 0$ for each $(u, x, y) \in \mathcal{L}(X, \mathbf{R^n}) \times K \times K$.

Lemma 5. *Let Φ be skew and let g be invex with respect to Φ on K. Then Φ is $\partial^c g$-monotone on K.*

Proof. Suppose that g is invex with respect to Φ on K. Then, for any $x, y \in K$, by the invexity of g, we obtain that

$$\Phi(u, x, y) \leq_{\mathbf{R^n_+}} g(y) - g(x), \forall u \in \partial^c g(x) \quad \text{and}$$

$$\Phi(v, y, x) \leq_{\mathbf{R^n_+}} g(x) - g(y), \forall v \in \partial^c g(y).$$

By adding these last two relations, we have

$$\Phi(u, x, y) + \Phi(v, y, x) \leq_{\mathbf{R^n_+}} 0, \quad \text{for all } u \in \partial^c g(x),\ v \in \partial^c g(y).$$

Since $\mathbf{R^n_+} \subseteq C(x)$, we deduce that for each $x, y \in K$

$$\Phi(u, x, y) + \Phi(v, y, x) \leq_{C(x)} 0, \quad \text{for all } u \in \partial^c g(x),\ v \in \partial^c g(y).$$

This means that Φ is $\partial^c g$-monotone on K. □

Theorem 6. *Let $g : K \to \mathbf{R^n}$ be an invex function with respect to Φ on K. Given $x_0 \in K$. We consider the following problems:*

(P1) For each $y \in K$, there exists $u \in \partial^c g(x_0)$ such that $\Phi(u, x_0, y) \not\leq_{\mathrm{int}C(x_0)} 0$.

(P2) x_0 is a solution for the generalized weak VOP.

(P3) For each $y \in K$, there exists $v \in \partial^c g(y)$ such that $\Phi(v, x_0, y) \not\leq_{\mathrm{int}C(x_0)} 0$.

Then, (P1) $\Rightarrow$ (P2). If Φ is skew, (P2) $\Rightarrow$ (P3). If Φ is affine in the third variable and $\Phi(\partial^c g(y), x, x) = \{0\}$ for each $x, y \in K$, (P3) $\Rightarrow$ (P1).

Proof. (P1) $\Rightarrow$ (P2). Suppose that x_0 is not a solution for the generalized weak VOP. Then there exists $y \in K$ such that $g(y) \leq_{\mathrm{int}C(x_0)} g(x_0)$; that is,

$$g(x_0) - g(y) \in \mathrm{int}C(x_0).$$

On the other hand, by the invexity of g, we have

$$\Phi(u, x_0, y) \leq_{\mathbf{R}^{\mathbf{n}}_+} g(y) - g(x_0), \quad \forall u \in \partial^c g(x_0);$$

that is, for all $u \in \partial^c g(x_0)$,

$$\Phi(u, x_0, y) \in g(y) - g(x_0) - \mathbf{R}^{\mathbf{n}}_+ \subseteq -\mathrm{int}C(x_0) - C(x_0) \subseteq -\mathrm{int}C(x_0).$$

Hence,

$$\Phi(u, x_0, y) \leq_{\mathrm{int}C(x_0)} 0, \quad \forall u \in \partial^c g(x_0),$$

which contradicts (P1).

(P2) $\Rightarrow$ (P3). Suppose that (P3) is not true. Then, for each $x_0 \in K$, there exists $y_0 \in K$ such that

$$\Phi(v, x_0, y_0) \leq_{\mathrm{int}C(x_0)} 0, \quad \forall v \in \partial^c g(y_0);$$

that is,

$$\Phi(\partial^c g(y_0), x_0, y_0) \subseteq -\mathrm{int}C(x_0).$$

On the other hand, by the invexity of g, we have

$$\Phi(u, y_0, x_0) \leq_{\mathbf{R}^{\mathbf{n}}_+} g(x_0) - g(y_0), \quad \forall u \in \partial^c g(y_0);$$

hence, since Φ is skew, we have

$$-\Phi(u, x_0, y_0) \leq_{\mathbf{R}^{\mathbf{n}}_+} g(x_0) - g(y_0), \quad \forall u \in \partial^c g(y_0);$$

therefore,

$$g(x_0) - g(y_0) \in -\Phi(u, x_0, y_0) + \mathbf{R}^{\mathbf{n}}_+ \subseteq \mathrm{int}C(x_0) + C(x_0) \subseteq \mathrm{int}C(x_0).$$

That is, $g(y_0) \leq_{\mathrm{int}C(x_0)} g(x_0)$.

(P3) $\Rightarrow$ (P1). Since g is invex with respect to Φ on K, by Lemma 5 we know that Φ is $\partial^c g$-monotone on K. Now if in Lemma 2, put $A(u, x) = u$, $f = 0$, $\alpha = 0$ and $T = \partial^c g$, then we obtain the desired result. □

Theorem 6 generalizes Theorem 4.1 of [21] to the GVELPs. It also improves and extends the main result in [28]. As an application of Corollary 2, we establish an existence result for (P2). This result is the extension and refinement of Theorem 3.1 in [21].

Theorem 7. *Let $\Phi : \mathcal{L}(X, Y) \times K \times K \to \mathbf{R}^{\mathbf{n}}$ be a function and $g : K \to \mathbf{R}^{\mathbf{n}}$ be an invexity function with respect to Φ on K. Assume that the mapping $W : K \to 2^{\mathbf{R}^{\mathbf{n}}}$ defined by $W(x) = \mathbf{R}^{\mathbf{n}} \setminus (-\mathrm{int}C(x))$ is closed, Φ is skew and affine in the third variable, and for each $z \in K$, $\Phi(\cdot, \cdot, z) : \mathcal{L}(X, \mathbf{R}^{\mathbf{n}}) \times K \to \mathbf{R}^{\mathbf{n}}$ is continuous. If there exist a nonempty compact set $M \subset K$ and a nonempty compact convex set $B \subset$*

K such that for each $x \in K \setminus M$, there exists $y \in B$ such that $\Phi(\partial^c g(x), x, y) \subseteq -\mathrm{int}C(x)$. Then the generalized weak VOP has a solution.

Proof. Lemmas 4 and 5 imply that all the conditions of Corollary 2 are fulfilled. Hence, we deduce (P1) and then the proof follows from Theorem 6. □

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ORTHOMODULAR LATTICES CAN BE CONVERTED INTO LEFT RESIDUATED L-GROUPOIDS

IVAN CHAJDA AND HELMUT LÄNGER

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Abstract. We show that every orthomodular lattice can be considered as a left residuated l-groupoid satisfying divisibility, antitony, the double negation law and three more additional conditions expressed in the language of residuated structures. Also conversely, every left residuated l-groupoid satisfying the mentioned conditions can be organized into an orthomodular lattice.

2010 *Mathematics Subject Classification:* 06C15; 06A11

Keywords: orthomodular lattice, left residuated l-groupoid, divisibility, antitony, double negation law

It is well-known that residuated structures form an algebraic axiomatization of fuzzy logics, see e. g. [1] for an overview. The reader can find necessary concepts and definitions concerning residuated structures in [5], however this paper is self-contained. Orthomodular lattices were introduced by G. Birkhoff and J. von Neumann as an algebraic axiomatization of the logic of quantum mechanics, see e. g. [4], [6] or [2] for details. Hence it is a natural question if these two concepts have a common base, i. e. if orthomodular lattices can be considered as residuated structures and hence as an axiomatization of certain fuzzy logic and, conversely, if certain residuated structures can be converted into orthomodular lattices, i. e. if the logic of quantum mechanics can be considered as a kind of fuzzy logic. For the theory of orthomodular lattices cf. the monographs [6] and [2] as well as the paper [3].

We start with the definition of an orthomodular lattice.

Definition 1. An *orthomodular lattice* is an algebra $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ satisfying (i) – (v) for all $x, y \in L$:

(i) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice.
(ii) $x \vee x' = 1$
(iii) $x \leq y$ implies $y' \leq x'$.

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(iv) $(x')' = x$
(v) $x \leq y$ implies $y = x \vee (y \wedge x')$.

Remark 1. In every lattice $(L, \vee, \wedge)$ with a unary operation $'$ satisfying (iii) and (iv) the so-called de Morgan laws

$$(x \vee y)' = x' \wedge y' \text{ and } (x \wedge y)' = x' \vee y'$$

hold.

Remark 2. According to the de Morgan laws condition (v) can be replaced by

(vi) $x \leq y$ implies $x = y \wedge (x \vee y')$.

Now we introduce left residuated l-groupoids.

Definition 2. A *left residuated l-groupoid* is an algebra $\mathcal{A} = (A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2,2,2,2,0,0)$ satisfying (i) – (iii) for all $x, y, z \in A$:

(i) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice.
(ii) $x \odot 1 = 1 \odot x = x$.
(iii) $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$.

Condition (iii) is called *left adjointness*. $\mathcal{A}$ is said to satisfy *divisibility* if

$$(x \rightarrow y) \odot x = x \wedge y$$

for all $x, y \in A$. We define a unary operation $'$ on A by

$$x' := x \rightarrow 0$$

for all $x \in A$. $\mathcal{A}$ is said to satisfy *antitony* if

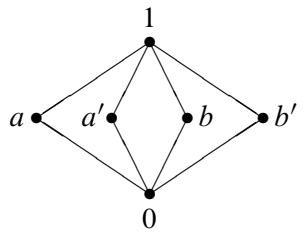
$$x \leq y \text{ implies } y' \leq x'$$

for all $x, y \in A$ and $\mathcal{A}$ is said to satisfy the *double negation law* if

$$(x')' = x$$

for all $x \in A$.

Example 1. If $A := \{0, a, a', b, b', 1\}$, $(A, \vee, \wedge, 0, 1)$ denotes the bounded lattice with the Hasse diagram



and the binary operations $\odot$ and $\to$ are defined by the tables

$\odot$	0	a	a'	b	b'	1
0	0	0	0	0	0	0
a	0	a	0	b	b'	a
a'	0	0	a'	b	b'	a'
b	0	a	a'	b	0	b
b'	0	a	a'	0	b'	b'
1	0	a	a'	b	b'	1

$\to$	0	a	a'	b	b'	1
0	1	1	1	1	1	1
a	a'	1	a'	a'	a'	1
a'	a	a	1	a	a	1
b	b'	b'	b'	1	b'	1
b'	b	b	b	b	1	1
1	0	a	a'	b	b'	1

then $(A, \vee, \wedge, \odot, \to, 0, 1)$ is a left residuated l-groupoid satisfying divisibility, antitony and the double negation law. The mentioned lattice is the smallest orthomodular lattice which is not a Boolean algebra and it is usually denoted by MO2.

The following theorem says that to every orthomodular lattice there can be assigned a left residuated l-groupoid in a natural way.

Theorem 1. *Let $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$ be an orthomodular lattice and define binary operations $\odot$ and $\to$ on L by the following formulas:*

$$x \odot y = (x \vee y') \wedge y, \tag{0.1}$$

$$x \to y = (y \wedge x) \vee x'. \tag{0.2}$$

Then $\mathbf{A}(\mathcal{L}) = (L, \vee, \wedge, \odot, \to, 0, 1)$ is a left residuated l-groupoid satisfying divisibility, antitony, the double negation law as well as the following identity:

$$x \odot (x \vee y) = x. \tag{0.3}$$

Moreover, $x' = x \to 0$ for all $x \in L$.

Proof. Let $a, b \in L$. We have

$$a \to 0 = (0 \wedge a) \vee a' = 0 \vee a' = a'.$$

Of course, $(L, \vee, \wedge, 0, 1)$ is a bounded lattice. Moreover,

$$a \odot 1 = (a \vee 1') \wedge 1 = (a \vee 0) \wedge 1 = a \wedge 1 = a \text{ and}$$
$$1 \odot a = (1 \vee a') \wedge a = 1 \wedge a = a.$$

If $a \odot b \leq c$ then $(a \vee b') \wedge b \leq c$ and hence

$$a \leq a \vee b' = ((a \vee b') \wedge b) \vee b' = (((a \vee b') \wedge b) \wedge b) \vee b' \leq (c \wedge b) \vee b' = b \to c.$$

If, conversely, $a \leq b \to c$ then $a \leq (c \wedge b) \vee b'$ and hence

$$a \odot b = (a \vee b') \wedge b \leq (((c \wedge b) \vee b') \vee b') \wedge b = ((c \wedge b) \vee b') \wedge b = c \wedge b \leq c.$$

Now, using orthomodularity (i. e. (v) of Definition 1), we have

$$(a \to b) \odot a = (((b \wedge a) \vee a') \vee a') \wedge a = ((b \wedge a) \vee a') \wedge a = a \wedge b.$$

In view of Definition 1, $a \leq b$ implies $b' \leq a'$ and we have $(a')' = a$. Finally, by applying (0.1) and (vi) of Remark 2 we obtain

$$a \odot (a \vee b) = (a \vee (a \vee b)') \wedge (a \vee b) = a.$$

□

Remark 3. The operation $x \odot y := (x \vee y') \wedge y$ is called the *Sasaki projection* of x onto y (cf. [6] and [2]).

Conversely, certain left residuated l-groupoids give rise to an orthomodular lattice.

Theorem 2. *Let $\mathcal{A} = (A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be a left residuated l-groupoid satisfying antitony, the double negation law as well as identities* (0.1) *and* (0.3) *of Theorem 1. Moreover, define $x' := x \rightarrow 0$ for all $x \in A$. Then $\mathbf{L}(\mathcal{A}) = (A, \vee, \wedge, ', 0, 1)$ is an orthomodular lattice.*

Proof. Let $a, b \in A$. Clearly, $(A, \vee, \wedge, 0, 1)$ is a bounded lattice and $a' = a \rightarrow 0$. Using antitony we see that $a \leq b$ implies $b' \leq a'$. Moreover, we have $(a')' = a$ according to the double negation law. Finally, if $a \leq b$ then, using (0.3) and (0.1), we have

$$b = (b')' = (b' \odot (b' \vee a'))' = (b' \odot a')' = ((b' \vee a) \wedge a')' = a \vee (b \wedge a')$$

and hence $a \vee a' = a \vee (1 \wedge a') = 1$. □

Finally, we prove that the correspondence described in the last two theorems is one-to-one.

Theorem 3. *We have $\mathbf{L}(\mathbf{A}(\mathcal{L})) = \mathcal{L}$ for every orthomodular lattice $\mathcal{L}$ and $\mathbf{A}(\mathbf{L}(\mathcal{A})) = \mathcal{A}$ for every left residuated l-groupoid satisfying antitony, the double negation law as well as identities* (0.1) – (0.3) *of Theorem 1.*

Proof. If $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$ is an orthomodular lattice, $\mathbf{A}(\mathcal{L}) = (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ and $\mathbf{L}(\mathbf{A}(\mathcal{L})) = (L, \vee, \wedge, ^*, 0, 1)$ then

$$x^* = x \rightarrow 0 = (0 \wedge x) \vee x' = 0 \vee x' = x'$$

for all $x \in L$, therefore we obtain $\mathbf{L}(\mathbf{A}(\mathcal{L})) = \mathcal{L}$. Conversely, if $\mathcal{A} = (A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a left residuated l-groupoid satisfying divisibility, antitony, the double negation law as well as identities (0.1) – (0.3) of Theorem 1, $\mathbf{L}(\mathcal{A}) = (A, \vee, \wedge, ', 0, 1)$ and $\mathbf{A}(\mathbf{L}(\mathcal{A})) = (A, \vee, \wedge, \circ, \Rightarrow, 0, 1)$ then

$$x \circ y = (x \vee y') \wedge y = x \odot y \text{ and}$$
$$x \Rightarrow y = (y \wedge x) \vee x' = x \rightarrow y$$

for all $x, y \in A$, therefore we obtain $\mathbf{A}(\mathbf{L}(\mathcal{A})) = \mathcal{A}$. □

Remark 4. We have shown that orthomodular lattices can be considered as special residuated lattices and hence the logic of quantum mechanics axiomatized by them has a common base with a certain fuzzy logic axiomatized just by means of residuated lattices as pointed out in [1]. This sheds a new light on the logic of quantum mechanics and yields new tools for its investigation.

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ON THE STRUCTURE OF BALANCED NEAR SEMIRINGS

IVAN CHAJDA AND HELMUT LÄNGER

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Abstract. The concept of a balanced near semiring was introduced by the authors in [5] in order to describe ring-like structures derived by means of lattices with involution, in particular with complementation. Now we show that every such balanced near semiring has an underlying structure which is a Boolean algebra. Based on this result, we describe ideals of these near semirings. A natural one-to-one correspondence between ideals and congruences is established.

2010 *Mathematics Subject Classification:* 16Y60; 16Y30; 06E25

Keywords: near semiring, antitone involution, balanced near semiring, Boolean algebra, ideal, congruence

As definition of a semiring, we take that of [7] resp. [9]. However, also other non-equivalent definitions can be found in the literature, but the present one is the most appropriate for our reasons because it corresponds in certain sense with various so-called quantum structures, see also [4]. The problem is that in these structures multiplication need not be associative and left distributivity need not be satisfied. This motivated us to introduce the concept of a near semiring, see [4] and [5], which bears some advantages of semirings but the mentioned properties are neglected.

In [5] we investigated mainly near semirings which are, moreover, equipped with an involution and can be derived from lattices having an antitone involution. In some particular cases for lattices with sectional involutions or for ortholattices (see e.g. [1]), we obtained interesting results. Hence, the natural question concerns algebraic constructions of these near semirings, in particular their factorization by congruences. This motivated us to study when a congruence on such an enriched near semiring is determined by its 0-class which is called the congruence kernel. Our first task is to describe the properties of these kernels and to introduce ideals which should correspond to congruences. For this sake, it was useful to reveal the structure of the underlying lattice which turns out to be a Boolean algebra. With this observation in hand, we can use the knowledge on congruences and ideals on Boolean algebras in order to obtain similar results for balanced near semirings.

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Definition 1. A *near semiring* is an algebra $(R,+,\cdot,0,1)$ of type $(2,2,0,0)$ such that $(R,+,0)$ is a commutative monoid and the following identities are satisfied:

- $(x+y)z \approx xz+yz$,
- $x0 \approx 0x \approx 0$,
- $x1 \approx 1x \approx x$.

A near semiring $(R,+,\cdot,0,1)$ is called *idempotent* if it satisfies the identity $x+x \approx x$.

Let $(R,+,\cdot,0,1)$ be an idempotent near semiring. Then $(R,+)$ is a semilattice which will be considered as a join-semilattice. The corresponding partial order relation $\leq$ is then defined by $x \leq y$ if $x+y=y$ $(x,y \in R)$. It will be called the *induced order* of $(R,+,\cdot,0,1)$. Clearly, 0 is the least element with respect to this order. A unary operation $'$ on R is called an *antitone involution* on $(R,\leq)$ if both $x \leq y$ implies $y' \leq x'$ (for $x,y \in R$) and the identity $(x')' \approx x$ is satisfied. Let $'$ be an antitone involution on $(R,\leq)$ and put $\mathbf{R} := (R,+,\cdot,',0,1)$. If one defines $x \wedge y := (x'+y')'$ for all $x,y \in R$ then $(R,+,\wedge)$ is a lattice with smallest element 0. The algebra $\mathbf{R}$ is called an *ortho near semiring* if it satisfies the identity $x+x' \approx 0'$. Let $\mathbf{R}$ be an ortho near semiring. Then $(R,+,\wedge,',0,0')$ is an ortholattice since also the identity $x \wedge x' \approx 0$ is satisfied due to the De Morgan laws. We denote this ortholattice by $\mathbb{L}(\mathbf{R})$ and call it the *ortholattice corresponding to* $\mathbf{R}$. The algebra $\mathbf{R}$ is called *a balanced near semiring* if it satisfies the identities

$$0' \approx 1 \text{ and } ((x+y)'+y)'+x \approx x.$$

This is the case if and only if the corresponding ortholattice $\mathbb{L}(\mathbf{R})$ satisfies the identities $0' \approx 1$ and

$$((x \vee y) \wedge y') \vee x \approx x. \tag{0.1}$$

In what follows we are going to specify the ortholattice corresponding to a balanced near semiring $\mathbf{R}$. For this sake we use the method developed by the first author and R. Padmanabhan in [6]. In fact, this result is implicitely contained in [2], but in a rather modified version and with different assumptions. For the reader's convenience we provide the full proof.

Theorem 1. *An ortholattice* $\mathbf{L} = (L,\vee,\wedge,',0,1)$ *is a Boolean algebra if and only if it satisfies* (0.1).

Proof. In accordance with (0.1) we have

$$x \wedge (x' \vee y) \approx ((y \vee x') \wedge x) \wedge (((y \vee x') \wedge x) \vee y) \approx ((y \vee x') \wedge x) \wedge y \approx x \wedge y,$$

i.e.

$$x \wedge (x' \vee y) \approx x \wedge y. \tag{0.2}$$

Due to the De Morgan laws also the following holds:

$$x \vee (x' \wedge y) \approx x \vee y. \tag{0.3}$$

Now (0.3) and (0.2) imply

$$\begin{aligned}(x\wedge y)\vee z &\approx z\vee(x\wedge y)\approx z\vee(z'\wedge(x\wedge y))\approx z\vee((z'\wedge x)\wedge y)\approx\\ &\approx z\vee((z'\wedge(z\vee x))\wedge y)\approx z\vee(z'\wedge((x\vee z)\wedge y))\approx z\vee((x\vee z)\wedge y)\approx\\ &\approx((x\vee z)\wedge y)\vee z,\end{aligned}$$

i.e.

$$(x\wedge y)\vee z\approx((x\vee z)\wedge y)\vee z. \tag{0.4}$$

Due to the De Morgan laws,

$$(x\vee y)\wedge z\approx((x\wedge z)\vee y)\wedge z. \tag{0.5}$$

Now (0.5) and (0.4) imply

$$\begin{aligned}(x\vee y)\wedge z &\approx((x\wedge z)\vee y)\wedge z\approx(x\wedge z)\vee(((x\wedge z)\vee y)\wedge z)\approx\\ &\approx((y\vee(x\wedge z))\wedge z)\vee(x\wedge z)\approx(y\wedge z)\vee(x\wedge z)\approx(x\wedge z)\vee(y\wedge z).\end{aligned}$$

Hence $\mathbf{L}$ is a distributive ortholattice, i.e. a Boolean algebra. The converse statement is evident. □

Corollary 1. *An ortho near semiring* $\mathbf{R}=(R,+,\cdot,',0,1)$ *is a balanced near semiring if and only if it satisfies the identity* $0'\approx 1$ *and its corresponding ortholattice* $\mathbb{L}(\mathbf{R})$ *is a Boolean algebra.*

Example 1. Every Boolean algebra $(B,\vee,\wedge,',0,1)$ is a balanced near semiring whose multiplication coincides with the operation of lattice meet.

There exist balanced near semirings whose multiplication does not coincide with the meet operation of the corresponding Boolean algebra as can be seen from the following example:

Example 2. If $\mathbf{B}=(B,\vee,\wedge,',0,1)$ is a Boolean algebra and

$$xy:=\begin{cases} x\wedge y & \text{if } \{x,y\}\cap\{0,1\}\neq\varnothing \text{ and}\\ y & \text{otherwise}\end{cases}$$

then $(B,\vee,\cdot,',0,1)$ is a balanced near semiring different from $\mathbf{B}$ in case $|B|>2$. This can be seen as follows: Let $a,b,c\in B$. It is easy to check that $(a+b)c=ac+bc$ in case $\{a,b,c\}\cap\{0,1\}\neq\varnothing$. If $\{a,b,c\}\cap\{0,1\}=\varnothing$ then we have $a+b\neq 0$ and hence $(a+b)c=c=c+c=ac+bc$. Moreover, in case $|B|>2$ we have $aa'=a'\neq 0=a\wedge a'$ if $a\neq 0,1$.

Theorem 2. *The balanced near semiring* $\mathbf{R}=(R,+,\cdot,',0,1)$ *of Example 2 is simple.*

Proof. Without loss of generality, assume $|R|>2$. Let $\mathbb{L}(\mathbf{R})=(R,\vee,\wedge,',0,1)$, $\Theta\in\operatorname{Con}\mathbf{R}$ and $(a,b)\in\Theta$ with $a\neq b$.
Put $c:=a\wedge b$ and $d:=a\vee b$. Then $(c,d)\in\Theta$ and $c<d$. Put $e:=c\vee d'$. Then

$(e,1) \in \Theta$. Now $e = 1$ would imply $c = (c \vee d) \wedge (c \vee d') = c \vee d \geq d$, a contradiction. Therefore $e \neq 1$. If $e = 0$ then $(x,1) = (x \vee 0, x \vee 1) \in \Theta$ for all $x \in R$ which implies $\Theta = R^2$. Hence assume $e \neq 0$. Then $(x,e) = (x1, xe) \in \Theta$ for all $x \in R \setminus \{0\}$. Moreover, $e' \in R \setminus \{0\}$ and $(e',0) = (1 \wedge e', e \wedge e') \in \Theta$. This shows $\Theta = R^2$ also in this case completing the proof of the theorem. □

It was shown by the authors in [5] that balanced near semirings $\mathbf{R}$ are congruence regular, i.e. every congruence Θ on $\mathbf{R}$ is fully determined by every single class $[a]\Theta$, in particular by its 0-class $[0]\Theta$ which is called the *kernel* of Θ. Hence, by describing congruence kernels we in fact obtain a complete description of the congruences on $\mathbf{R}$.

Remark 1. A non-empty subset I of the base set B of a Boolean algebra $\mathbf{B} = (B, \vee, \wedge, ', 0, 1)$ is called a *Boolean ideal* of $\mathbf{B}$ if it satisfies the following conditions for all $a, b \in B$:

- $a \in I$ implies $a \wedge b \in I$,
- $a, b \in I$ implies $a \vee b \in I$.

Let $\mathrm{Id}\,\mathbf{B}$ denote the set of all Boolean ideals of $\mathbf{B}$, $\mathbf{Id}\,\mathbf{B}$ the lattice $(\mathrm{Id}\,\mathbf{B}, \subseteq)$ of all Boolean ideals of $\mathbf{B}$, $\mathrm{Con}\,\mathbf{B}$ the set of all congruences on $\mathbf{B}$ and $\mathbf{Con}\,\mathbf{B}$ the congruence lattice $(\mathrm{Con}\,\mathbf{B}, \subseteq)$ of $\mathbf{B}$. It is well-known (see e.g. [8]) that $\Theta \mapsto [0]\Theta$ and $I \mapsto \Theta(I) := \{(x,y) \in B^2 \mid (x \wedge y') \vee (x' \wedge y) \in I\}$ are mutually inverse isomorphisms between $\mathbf{Con}\,\mathbf{B}$ and $\mathbf{Id}\,\mathbf{B}$.

Now we define ideals in balanced near semirings.

Definition 2. Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a balanced near semiring and I a non-empty subset of R. Then I is called an *ideal* of $\mathbf{R}$ if it satisfies the following conditions for all $a, b, c \in R$:

(i) $a \in I$ implies $(a' + b)' \in I$,
(ii) $a, b \in I$ implies $a + b \in I$,
(iii) $(a' + b)', (a + b')' \in I$ implies $((ac)' + (bc))', ((ca)' + (cb))' \in I$.

Let $\mathrm{Id}\,\mathbf{R}$ denote the set of all ideals of $\mathbf{R}$, $\mathbf{Id}\,\mathbf{R}$ the ideal lattice $(\mathrm{Id}\,\mathbf{R}, \subseteq)$ of $\mathbf{R}$, $\mathrm{Con}\,\mathbf{R}$ the set of all congruences on $\mathbf{R}$ and $\mathbf{Con}\,\mathbf{R}$ the congruence lattice $(\mathrm{Con}\,\mathbf{R}, \subseteq)$ of $\mathbf{R}$.

This means that the ideals of a balanced near semiring $\mathbf{R}$ are in fact those Boolean ideals of the corresponding Boolean algebra $\mathbb{L}(\mathbf{R})$ which satisfy condition (iii). Putting $b = 0$ in (iii) one obtains $IR \cup RI \subseteq I$.

Remark 2. It can be easily seen that if a balanced near semiring is a Boolean algebra then the concepts of ideals coincide since then multiplication coincides with meet.

Recall that a *variety* $\mathcal{V}$ is called *congruence permutable* if $\Theta \circ \Phi = \Phi \circ \Theta$ for all $\mathbf{A} \in \mathcal{V}$ and all $\Theta, \Phi \in \mathrm{Con}\,\mathbf{A}$, *congruence distributive* if $(\Theta \vee \Phi) \wedge \Psi = (\Theta \wedge$

$\Psi) \vee (\Phi \wedge \Psi)$ for all $\mathbf{A} \in \mathcal{V}$ and all $\Theta, \Phi, \Psi \in \operatorname{Con}\mathbf{A}$, *arithmetical* if it is both congruence permutable and congruence distributive and *congruence regular* if for all $\mathbf{A} = (A, F) \in \mathcal{V}$ and for all $\Theta, \Phi \in \operatorname{Con}\mathbf{A}$, $[a]\Theta = [a]\Phi$ implies $\Theta = \Phi$.

Remark 3. If $\mathbf{R}$ is a balanced near semiring then $\mathbf{Con}\,\mathbf{R}$ is a sublattice of $\mathbf{Con}\,\mathbb{L}(\mathbf{R})$ and $\mathbf{Id}\,\mathbf{R}$ is a sublattice of $\mathbf{Id}\,\mathbb{L}(\mathbf{R})$. Since the variety of Boolean algebras is arithmetical and congruence regular, the same is true for the variety of balanced near semirings (cf. [5]).

Due to the congruence regularity of $\mathbf{R}$ we have a one-to-one correspondence between ideals of $\mathbf{R}$ and congruences on $\mathbf{R}$.

Theorem 3. *Let* $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ *be a balanced near semiring. Then* $\Theta \mapsto [0]\Theta$ *and* $I \mapsto \Theta(I) := \{(x, y) \in R^2 \mid (x' + y)' + (x + y')' \in I\}$ *are mutually inverse isomorphisms between* $\mathbf{Con}\,\mathbf{R}$ *and* $\mathbf{Id}\,\mathbf{R}$.

Proof. Let $a, b, c \in R$, $\Theta \in \operatorname{Con}\mathbf{R}$ and $I \in \operatorname{Id}\mathbf{R}$.
Since $\Theta \in \operatorname{Con}\mathbb{L}(\mathbf{R})$ we have $[0]\Theta \in \operatorname{Id}\mathbb{L}(\mathbf{R})$. Thus $[0]\Theta$ satisfies (i) and (ii). Hence we need only prove (iii). If $(a' + b)', (a + b')' \in [0]\Theta$ then $(a \wedge b') \vee (a' \wedge b) \in [0]\Theta$ and hence $(a, b) \in \Theta([0]\Theta) = \Theta$. This shows $(ac, bc), (ca, cb) \in \Theta = \Theta([0]\Theta)$ and hence $((ac)' + (bc))' = (ac) \wedge (bc)' \in [0]\Theta$ and $((ca)' + (cb))' = (ca) \wedge (cb)' \in [0]\Theta$ proving (iii). This shows $[0]\Theta \in \operatorname{Id}\mathbf{R}$.
Since $I \in \operatorname{Id}\mathbb{L}(\mathbf{R})$ we have $\Theta(I) \in \operatorname{Con}\mathbb{L}(\mathbf{R})$. We need only prove the compatibility of $\Theta(I)$ with multiplication. If $(a, b) \in \Theta(I)$ then $(a' + b)', (a + b')' \in I$ and hence $((ac)' + (bc))', ((ca)' + (cb))' \in I$ and, due to symmetry, also $((ac) + (bc)')', ((ca) + (cb)')' \in I$. This shows

$$((ac) \wedge (bc)') \vee ((ac)' \wedge (bc)), ((ca) \wedge (cb)') \vee ((ca)' \wedge (cb)) \in I$$

and hence $(ac, bc), (ca, cb) \in \Theta(I)$. Therefore $\Theta(I) \in \operatorname{Con}\mathbf{R}$.
The rest follows from Remarks 1 and 3. □

Remark 4. Theorem 3 shows that ideals of a balanced near semiring $\mathbf{R}$ coincide with congruence kernels, not only within the underlying Boolean algebra $\mathbb{L}(\mathbf{R})$ but also within $\mathbf{R}$ itself.

In the following theorem we describe principal ideals in balanced near semirings.

If $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ is a balanced near semiring and $a, b \in R$ then let $\Theta(a, b)$ denote the congruence on $\mathbf{R}$ generated by (a, b) and $I(a)$ denote the principal ideal of $\mathbf{R}$ generated by a. Moreover, let $P_1(\mathbf{R})$ denote the set of all unary polynomial functions on $\mathbf{R}$.

Theorem 4. *Let* $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ *be a balanced near semiring and* $a \in R$. *Then*

$$I(a) = \{p(a) \mid p \in P_1(\mathbf{R}), p(0) = 0\}.$$

Proof. We first prove that $I(a)$ corresponds to $\Theta(a, 0)$. Since $[0]\Theta(a, 0)$ is an ideal of $\mathbf{R}$ containing a we have $I(a) \subseteq [0]\Theta(a, 0)$. Since $\Theta(I(a))$ is a congruence

on $\mathbf{R}$ containing $(a,0)$ we have $\Theta(a,0) \subseteq \Theta(I(a))$. Both assertions together show that $I(a)$ corresponds to $\Theta(a,0)$. Now let $b \in R$. Then $b \in I(a)$ if and only if $(b,0) \in \Theta(a,0)$. Since the variety of balanced near semirings is congruence permutable, $\Theta(a,0)$ coincides with the least reflexive compatible relation $R(a,0)$ containing $(a,0)$ (cf. [10]). According to [3], $(b,0) \in R(a,0)$ if and only if there exists a $p \in P_1(\mathbf{R})$ with $(p(a), p(0)) = (b,0)$. Hence $b \in I(a)$ if and only if there exists a $p \in P_1(\mathbf{R})$ with $p(0) = 0$ and $p(a) = b$ completing the proof of the theorem. □

It is evident that $\mathbf{Id}\,\mathbf{R}$ is a complete lattice with greatest element R and, according to Theorem 3 with smallest element $\{0\}$. Moreover, meet coincides with set-theoretical intersection and for $I, J \in \mathrm{Id}\,\mathbf{R}$, $I \vee J$ is the ideal generated by $I \cup J$. Due to Theorem 3, $\mathbf{Id}\,\mathbf{R}$ is isomorphic to $\mathbf{Con}\,\mathbf{R}$ and hence algebraic.

We recall the concept of a pseudocomplement in a lattice $\mathbf{L} = (L, \vee, \wedge)$ with smallest element 0. The element b of L is called the *pseudocomplement* of $a \in L$ if b is the greatest element x of L with $a \wedge x = 0$. The *lattice* $\mathbf{L}$ is called *pseudocomplemented* if every of its elements has a pseudocomplement.

Now we can prove the following:

Theorem 5. *The ideal lattice of a balanced near semiring is pseudocomplemented.*

Proof. Let $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ be a balanced near semiring and $I \in \mathrm{Id}\,\mathbf{R}$. Since $\mathbf{R}$ is congruence distributive according to Remark 3, $\mathbf{Id}\,\mathbf{R}$ is distributive. Since $\mathbf{Id}\,\mathbf{R}$ is also algebraic, meet distributes over arbitrary joins (see e.g. [8]). If $S := \{J \in \mathrm{Id}\,\mathbf{R} \mid I \wedge J = \{0\}\}$ and $K := \bigvee S$ then

$$I \wedge K = I \wedge \bigvee S = \bigvee \{I \wedge J \mid J \in S\} = \bigvee \{0\} = \{0\}$$

which shows that K is the greatest element of $(S, \subseteq)$ and hence the pseodocomplement of I. □

In the following let I^* denote the pseudocomplement of the ideal I of $\mathbf{R}$ in $\mathbf{Id}\,\mathbf{R}$.

Recall the concept of factor congruences of an algebra. Let $\mathbf{A} = (A, F)$ be an algebra and $\Theta, \Phi \in \mathrm{Con}\,\mathbf{A}$. Then (Θ, Φ) is called a *pair of factor congruences* of $\mathbf{A}$ if $x \mapsto ([x]\Theta, [x]\Phi)$ is an isomorphism from $\mathbf{A}$ to $(\mathbf{A}/\Theta) \times (\mathbf{A}/\Phi)$. It is well-known that this is equivalent to the fact that $\Theta \circ \Phi = \Phi \circ \Theta$, $\Theta \vee \Phi = \nabla$ and $\Theta \wedge \Phi = \Delta$.

Definition 3. Let $\mathbf{R}$ be a balanced near semiring and $I, J \in \mathrm{Id}\,\mathbf{R}$. Then we call (I, J) a *pair of factor ideals* of $\mathbf{R}$ if $(\Theta(I), \Theta(J))$ is a pair of factor congruences of $\mathbf{R}$.

Theorem 6. *Let* $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ *be a balanced near semiring and* $a \in R$. *Then* $(I(a), I(a'))$ *is a pair of factor ideals of* $\mathbf{R}$ *if and only if* $I(a) \cap I(a') = \{0\}$.

Proof. From Theorem 3 we know that $I \mapsto \Theta(I)$ is an isomorphism from $\mathbf{Id}\,\mathbf{R}$ to $\mathbf{Con}\,\mathbf{R}$ and from Remark 3 we have that any two congruences of $\mathbf{R}$ commute.

Moreover, $I(a) \vee I(a') = R$ since $1 = a + a' \in I(a) \vee I(a')$. Hence $(I(a), I(a'))$ is a pair of factor ideals of $\mathbf{R}$ if and only if $\Theta(I(a)) \cap \Theta(I(a')) = \Delta$, i.e. $I(a) \cap I(a') = \{0\}$. □

Corollary 2. *Let* $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ *be a balanced near semiring. Then for each* $x \in R$, $(I(x), I(x'))$ *is a pair of factor ideals of* $\mathbf{R}$ *if and only if the following condition is satisfied:*

$$\text{If } x \in R,\ p, q \in P_1(\mathbf{R}),\ p(0) = q(0) = 0 \text{ and } p(x) = q(x') \text{ then } p(x) = 0. \quad (0.6)$$

Proof. Let $a \in R$. According to Theorem 6, $(I(a), I(a'))$ is a pair of factor ideals of $\mathbf{R}$ if and only if $I(a) \cap I(a') = \{0\}$. If $I(a) \cap I(a') = \{0\}$, $p, q \in P_1(\mathbf{R})$, $p(0) = q(0) = 0$ and $p(a) = q(a')$ then, according to Theorem 4, $p(a) \in I(a)$ and $q(a') \in I(a')$ and hence $p(a) = 0$, i.e. (0.6) holds. If, conversely (0.6) holds and $b \in I(a) \cap I(a')$ then, again according to Theorem 4, there exist $p, q \in P_1(\mathbf{R})$ with $p(a) = b$ and $q(a') = b$ whence $b = p(a) = 0$ according to (0.6) showing $I(a) \cap I(a') = \{0\}$. □

Remark 5. In case $|B| > 2$ the balanced near semiring from Example 2 does not satisfy (0.6) since for $a \in B \setminus \{0, 1\}$ and $p(x) = q(x) := xa$ for all $x \in B$ we have $p, q \in P_1(\mathbf{B})$, $p(0) = q(0) = 0a = 0$ and $p(a) = aa = a = a'a = q(a')$, but $p(a) = a \neq 0$. On the contrary, consider the balanced near semiring from Example 1 which is in fact a Boolean algebra. Since Boolean algebras are in one-to-one correspondence with unitary Boolean rings and every unary polynomial function in a Boolean ring is of the form $p(x) = bx + c$ for some $b, c \in B$, where $+$ denotes the symmetric difference, we can rewrite it in the language of Boolean algebras as $p(x) = (x \wedge d) \vee (x' \wedge e) \vee f$ for some $d, e, f \in B$. Since we ask $p(0) = 0$, we obtain $e \vee f = 0$ whence $e = f = 0$, i.e. $p(x) = x \wedge d$. If now $a \in B$, $p(x) = x \wedge d$ and $q(x) = x \wedge g$ with some $g \in B$ and $p(a) = q(a')$ then $p(a) = p(a) \wedge q(a') \leq a \wedge a' = 0$ proving that (0.6) holds in this case.

Definition 4. We define the *annihilator* $Ann(A)$ of a subset A of a balanced near semiring $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ as the greatest ideal I of $\mathbf{R}$ with $I \cap A \subseteq \{0\}$ if this ideal exists. We will write $Ann(a)$ instead of $Ann(\{a\})$.

$Ann(A)$ need not exist as the following example shows:

Example 3. In the four-element Boolean algebra considered as a balanced near semiring $\mathbf{R} = (\{0, a, a', 1\}, +, \cdot, 0, 1)$ with join as addition and meet as multiplication $Ann(1)$ does not exist since the ideals I of $\mathbf{R}$ with $I \cap \{1\} \subseteq \{0\}$ are $\{0\}$, $\{0, a\}$ and $\{0, a'\}$ and none of them is the greatest one.

We have $Ann(\varnothing) = Ann(0) = R$, $Ann(R) = \{0\}$ and if $A \subseteq B \subseteq R$ and $Ann(A)$ and $Ann(B)$ exist then $Ann(B) \subseteq Ann(A)$.

As mentioned above, $\mathbf{Id}\, R$ is infinitely join-distributive. Hence for an ideal I of $\mathbf{R}$ the annihilator $Ann(I)$ of I always exists and coincides with I^*. For annihilators of ideals we have the following easy results:

Proposition 1. *Let* $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ *be a balanced near semiring,* $I \in \mathrm{Id}\,\mathbf{R}$ *and* $a \in R$*. Then the following hold:*

(i) $I \subseteq Ann(Ann(I))$
(ii) $Ann(Ann(Ann(I))) = Ann(I)$
(iii) $(I, Ann(I))$ *is a pair of factor ideals of* $\mathbf{R}$ *if and only if* $I \vee Ann(I) = R$.
(iv) $(I(a), I(a'))$ *is a pair of factor ideals of* $\mathbf{R}$ *if and only if* $a' \in Ann(I(a))$.

Proof. (i) and (iii) follow from $I \cap Ann(I) = \{0\}$. (ii) follows from (i) and from the fact that Ann is antitone. (iv) follows from $I(a) \vee I(a') = R$ and from the fact that the following are equivalent: $I(a) \cap I(a') = \{0\}$, $I(a') \subseteq Ann(I(a))$, $a' \in Ann(I(a))$. □

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COMPUTER ALGORITHM FOR SOLVING OF THE CHAZY EQUATION OF THE THIRD ORDER WITH SIX SINGULAR POINTS

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Abstract. The algorithm for calculating the coefficients of Chazy differential equation of the third order with six constant poles with respect to the unknown function is given. For such values of the poles a corresponding differential equation can be integrated in a symbolic form. When solving this problem the computational-algebraic algorithm to the construction of five non-linear differential equations of the third order, which are reduced to a linear inhomogeneous equations of the second order with six singular points is built. The algorithm is demonstrated on an example; five differential equations are obtained and their general solutions are found in elliptic functions. The calculations are implemented using *Mathematica* system.

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1. Introduction

The problem of finding third-order differential equations which solutions define new transcendental functions other than the Painleve transcendents [16], urged Chazy [4] to study the differential equations of the form

$$y''' = R(y'', y', y, x), \tag{1.1}$$

where R a rational function of y'', y', y with the analytic coefficients of x. The main problem, which is considered in the Chazy's fundamental paper [4], is the construction and study of new non-linear differential equations of the third order of P-type and in the description of new classes of transcendental functions defined by these equations. The study of the differential equations of the form (1.1), according to the Painleve method [16], reduces to the study of so-called "simplified" equations (a maximum number of poles in these equations is equal to six). Among the most interesting "simplified" equations Chazy emphasized a differential equation of the

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form

$$y''' = \frac{PQ'' - P''Q}{PQ' - P'Q}\, y'y'' - \frac{P'Q'' - P''Q'}{PQ' - P'Q}\, y'^3, \tag{1.2}$$

where P, Q are fourth degree polynomials of y with constant coefficients. For equation (1.2) the corresponding complete equation, according to the Painleve method, has the form [4]

$$\begin{aligned} y''' = \sum_{k=1}^{6} ((y' - a_k')(y'' - a_k'') + A_k (y' - a_k')^3 + \\ B_k (y' - a_k')^2 + \mathsf{C}_k (y' - a_k')) \times (y - a_k)^{-1} + \\ D y'' + E y' + \prod_{i=1}^{6} (y - a_i) \sum_{k=1}^{6} \frac{F_k}{y - a_k}, \end{aligned} \tag{1.3}$$

32 coefficients a_k, A_k, B_k, C_k, F_k $(k = \overline{1,6})$, D, E satisfy the system, consisting of 31 algebraic and differential equations [4] (Application, systems (5.1)-(5.6)).

Chazy has not completed integration of the system (5.1)-(5.6), and has not obtained the equations of the form (1.3), that was primarily due to the necessity of carrying out cumbersome symbolic computations and transformations. He only showed that some cases of degeneration of this equation are the canonical Painleve equations [11].

Research in the system (5.1) was conducted in the works [13, 14], and its solution has the form [8, 14]

$$\begin{aligned} A_k &= \frac{3\beta_3 - 6a_k^4 + 4\sigma_1 a_k^3 + 3(\alpha_2 - \sigma_2) a_k^2 - 3\beta_2 a_k - \sigma_4}{6a_k^5 - 5\sigma_1 a_k^4 + 4\sigma_2 a_k^3 - 3\sigma_3 a_k^2 + 2\sigma_4 a_k - \sigma_5}, \\ \beta_2 &= \frac{(3\alpha_2 - \sigma_2)(2\alpha_2\sigma_1 - 3\sigma_3) - 2\sigma_1\sigma_4 + 6\sigma_5}{\sigma_1^2 + 18\alpha_2 - 6\sigma_2}, \\ \beta_3 &= \frac{(\sigma_2 - 3\alpha_2)\left(12\alpha_2^2 - 4\sigma_2\alpha_2 + \sigma_1\sigma_3 - 4\sigma_4\right) + 2\sigma_1\sigma_5}{2\left(\sigma_1^2 + 18\alpha_2 - 6\sigma_2\right)}, \end{aligned} \tag{1.4}$$

where σ_k are basic symmetric polynomials, consisting of elements a_k $(k=\overline{1,6})$, function α_2 satisfies the equation of the fifth degree

$$\begin{aligned}
&1296\alpha_2^5-1296\sigma_2\alpha_2^4+(432\sigma_2^2+216\sigma_1\sigma_3-\\
&432\sigma_4)\alpha_2^3-(48\sigma_2^3+144\sigma_1\sigma_3\sigma_2-288\sigma_4\sigma_2-\\
&24\sigma_1^2\sigma_4+216\sigma_1\sigma_5-1296\sigma_6)\alpha_2^2+\\
&(9\sigma_3^2\sigma_1^2-4\sigma_5\sigma_1^3-8\sigma_2\sigma_4\sigma_1^2+144\sigma_6\sigma_1^2+\\
&(24\sigma_2^2\sigma_3-36\sigma_3\sigma_4+72\sigma_2\sigma_5)\sigma_1-48\sigma_2^2\sigma_4+\\
&108\sigma_3\sigma_5-864\sigma_2\sigma_6)\alpha_2-3\sigma_1^2\sigma_2\sigma_3^2-8\sigma_1^2\sigma_4^2-\\
&72\sigma_5^2+2\sigma_1^3\sigma_3\sigma_4+12\sigma_1\sigma_2\sigma_3\sigma_4-6\sigma_1^2\sigma_3\sigma_5-\\
&36\sigma_2\sigma_3\sigma_5+48\sigma_1\sigma_4\sigma_5+4\sigma_1^4\sigma_6+\\
&144\sigma_2^2\sigma_6-48\sigma_1^2\sigma_2\sigma_6=0
\end{aligned} \tag{1.5}$$

and the denominator in the relations (1.4) $\sigma_1^2+18\alpha_2-6\sigma_2$ is not equal to zero. If the condition

$$\sigma_1^2+18\alpha_2-6\sigma_2=0 \tag{1.6}$$

holds, then the values A_k have the form

$$\begin{gathered}
A_k=-\frac{6a_k-\sigma_1}{108}\times\\
\frac{108a_k^3-54\sigma_1a_k^2-6\left(\sigma_1^2-6\sigma_2\right)a_k-\sigma_1^3-108\alpha_3+6\sigma_1\sigma_2-27\sigma_3}{6a_k^5-5\sigma_1a_k^4+4\sigma_2a_k^3-3\sigma_3a_k^2+2\sigma_4a_k-\sigma_5}
\end{gathered} \tag{1.7}$$

and the equality still preserves

$$\begin{gathered}
\sigma_6=\frac{1}{2}\alpha_3\left(4\alpha_3+\sigma_3\right)+\frac{\sigma_1^2-6\sigma_2}{5832}\times\\
\left(\sigma_1\left(\sigma_1^3-6\sigma_2\sigma_1+108\alpha_3+27\sigma_3\right)-108\sigma_4\right).
\end{gathered} \tag{1.8}$$

If the coefficients a_k are constants and $D=0$, then equation (1.3) is simplified

$$y'''=\sum_{k=1}^{6}\frac{y'y''+A_ky'^3+\mathsf{C}_ky'}{y-a_k}+Ey'. \tag{1.9}$$

In the papers [5, 8, 9] by using solutions (1.4), (1.7) the new solutions of the equation (1.9) were found and properties of such solutions were also studied.

Let us note that the coefficients a_k, A_k, C_k, E of the equation (1.9) connected by the relations [4, 8, 13]

$$\sum_{\substack{k=1\\ j\neq k}}^{6} \frac{\mathsf{C}_j-\mathsf{C}_k}{a_k-a_j} = 2A_k + E \quad (k,j=\overline{1,6})\,. \tag{1.10}$$

The number of equations of the kind (1.9) is equal five, since the equation (1.5) has five, in the general case, complex roots. The essential fact is that for some sets of values of coefficients a_k the roots of the equation (1.5) in radicals can be found (this case is implemented in the example).

In those cases where the roots of the equation (1.5) can't be found in the radicals, the coefficients A_k, C_k are calculated only approximately. It's worthy to note that computer algebra systems have built-in functions that allow saving the symbolic form of the coefficients A_k, C_k and calculate them with a given accuracy (for example, the function Root-object [1]). For the differential equations built with such approximated coefficients the integration procedure used in this paper can be applied as well.

Note that the differential equation (1.3) can be studied by constructing an equivalent system of two differential equations. In paper [4] Chazy built such a system. Other equivalent systems were built by N.A. Lukashevich [12]. For equation (1.9) the equivalent systems were built in the papers [10, 18]. Here the coefficients of the obtained systems have been found with the help of an analytical method, which was implemented as a software module.

Study of the equation (1.3) can be carried out also by using the approximate method proposed by V.N. Orlov [15]. This method is based on the separation of the domain of construction solutions into a regular area and into neighborhoods of movable critical points, and on the construction of a sequence of analytic continuations.

The method of constructing one- and two-parameter families of solutions of the equation (1.3) for certain values of coefficients is considered in [6, 7].

An important circumstance facilitating the study of the equation (1.9) is that by substituting

$$\frac{dy}{dx} = \sqrt{z(y)}, \tag{1.11}$$

proposed by N.A.Lukashevich [13], this equation can be linearized. Indeed, as shown in [8] equation (1.9) reduced to the linear equation of the second order with six different poles with respect to the independent variable

$$\frac{d^2z}{dy^2} = \sum_{i=1}^{6} \frac{1}{y-a_i}\frac{dz}{dy} + 2\sum_{i=1}^{6} \frac{A_i}{y-a_i} z + 2\sum_{i=1}^{6} \frac{\mathsf{C}_i}{y-a_i} + E. \tag{1.12}$$

In the paper [8] proved that the solution of equation (1.9) with coefficients that satisfy the relations (1.4) - (1.8) and the system (1.10), is constructed by means of the

quadrature with using the inverse value of the general solution of equation (1.12). For the certain values of the coefficients a_k the general solution is written by means of elliptic functions, and in the simplest cases - even by means of elementary functions.

In this paper we propose a computational algorithm for constructing the Chazy equation of the form (1.9) with coefficients satisfying the relations (1.4) - (1.8), (1.10). This equation can be integrated in the symbolic form and depends on the set values of the six coefficients a_k ([8, 9, 17]). The integration of the obtained equations is demonstrated by the example. All analytical transformations and numerical calculations are performed using computer algebra system $Mathematica$.

2. An algorithm for constructing equations and their integration

The proposed algorithm consists of several steps.

a) We determine values of the coefficients a_k and calculate the values of the basic symmetric polynomials σ_k, composed of them. After this we find five roots α_{2i} $(i = \overline{1,5})$ of the equation (1.5).

b) Using formulas (1.4) or (1.7)-(1.8) with the condition (1.6) for the given values a_k and α_{2i} we find the corresponding values of the coefficients A_k $(k = \overline{1,6})$.

c) For each from the five found sets of values a_k, A_k $(k = \overline{1,6})$, we solve system (1.10) with respect to unknowns C_k; system (1.10) is degenerate and therefore we will solve it for the unknown C_j $(j = \overline{3,6})$, which will depend on the values of C_1, C_2, E).

d) We substitute the obtained values a_k, A_k $(k = \overline{1,6})$ and values C_j $(j = \overline{3,6})$ for each of the roots α_{2i} $(i = \overline{1,5})$ into equation (1.9). As a result, we obtain five differential equations.

e) Using substitution (1.11) we linearize each of the five equations and write them as equation (1.12). If the coefficient of E is constant, then the resulting linear equation can be integrated. Then, using the inverse of (1.11) substitution, we find a general solution for each of the five equations in the form (1.9). As a result, we obtain families of functions defined in the implicit form; they contain three parameters C_1, C_2, E and three arbitrary constants c_i $(i = 1,2,3)$. If the coefficient of E is not constant, it is required in each case to check the integrability of the resulting linear equation. If the linear equation can be integrated, then we find a general solution of equation (1.9) and use the inverse substitution for (1.11).

3. Example

Algorithm for constructing equations presented above and their integration we demonstrate by the example. We choose the values of a_k $(k = \overline{1,6})$ such that the roots of the equation (1.5) can be found by using radicals.

Consider the equation (1.9) where $E = const$ with coefficients of the form

$$a_1 = 1,\ a_2 = -1,\ a_3 = 2, a_4 = -2,\ a_5 = \frac{1}{2},\ a_6 = -\frac{1}{2}. \tag{3.1}$$

Follow the step a) of algorithm, we obtain a polynomial

$$1296\alpha_2^5+27216\alpha_2^4+154224\alpha_2^3-146448\alpha_2^2-2939328\alpha_2-4064256. \tag{3.2}$$

The roots of the polynomial (3.2) α_2 have the form

$$1,\ -\frac{\sqrt{57}+11}{8},\ \frac{\sqrt{57}-11}{8},-\frac{7}{4}, \tag{3.3}$$

and the root $-\frac{7}{4}$ is a double and satisfies the equation (1.6). Note that the first three roots do not satisfy the equation (1.6), therefore, to calculate coefficients A_k $(k=\overline{1,6})$ we use formulas (1.4).

For root $\alpha_{21}=1$ we find the following values of the coefficients:

$$\begin{aligned}
&A_1=-1,\ A_2=1,\ A_3=-\tfrac{13}{20},\\
&A_4=\tfrac{13}{20},\ A_5=-\tfrac{7}{5},\ A_6=\tfrac{7}{5},\\
&\mathsf{C}_3=\tfrac{45\mathsf{C}_1+19\mathsf{C}_2+24E}{40},\ \mathsf{C}_4=\tfrac{19\mathsf{C}_1+45\mathsf{C}_2-24E}{40},\\
&\mathsf{C}_5=\tfrac{18\mathsf{C}_1-10\mathsf{C}_2-3E}{20},\ \mathsf{C}_6=\tfrac{18\mathsf{C}_2-10\mathsf{C}_1+3E}{20},
\end{aligned} \tag{3.4}$$

where $\mathsf{C}_1,\ \mathsf{C}_2,\ E$ are arbitrary constants. Equation (1.9) with coefficients (3.1), (3.4) reduced by the substitution (1.11) to the equation (1.12) of the form

$$\begin{aligned}
&3\mathsf{C}_1(y+1)^2(8y^3-8y^2-14y+11)+\\
&3\mathsf{C}_2(y-1)^2(8y^3+8y^2-14y-11)+\\
&(8y^6-24y^4+24y^2-8)E-\\
&(48y^4-150y^2+66)z+\\
&(24y^5-84y^3+42y)z'-\\
&(4y^6-21y^4+21y^2-4)z''=0.
\end{aligned} \tag{3.5}$$

Equation (3.5) is linear nonhomogeneous differential equation of the second order with variable coefficients. Integration of this equation can be performed by the classical method. For this we integrate the first corresponding homogeneous equation, then by using quadratures we find its solution. These calculations are implemented by using computer algebra systems (for example, using the function *DSolve* [2, 17] in *Mathematica* system). One can also use the search algorithm of rational solutions of linear ordinary differential equations with polynomial coefficients presented in the work [3]. The general solution of equation (3.5) has the form

$$\begin{aligned}
&z=\tfrac{1}{2}(\mathsf{C}_1(2y+1)+\mathsf{C}_2(2y-1)+\\
&2(c_1(2y^2-5y+2))^2+y(c_2y^2+c_2+Ey))),
\end{aligned} \tag{3.6}$$

where $c_1,\ c_2$ are arbitrary constants. Substituting the expression (3.6) into formula (1.11), and integrating the resulting equation, we find the general solution of equation

(1.9), (3.1), (3.4) implicitly

$$\frac{2F\left(\arcsin\left(\sqrt{\frac{(\psi_2-\psi_4)(\psi_1-y(x))}{(\psi_1-\psi_4)(\psi_2-y(x))}}\right)|\frac{(\psi_2-\psi_3)(\psi_1-\psi_4)}{(\psi_1-\psi_3)(\psi_2-\psi_4)}\right)}{(\psi_1-\psi_4)\sqrt{z}}\times$$
$$\frac{(\psi_1-y(x))\sqrt{\frac{(\psi_1-\psi_2)(\psi_3-y(x))}{(\psi_1-\psi_3)(\psi_2-y(x))}}(\psi_4-y(x))}{\sqrt{\frac{(\psi_1-\psi_2)(\psi_2-\psi_4)(\psi_1-y(x))(\psi_4-y(x))}{(\psi_1-\psi_4)^2(\psi_2-y(x))^2}}}=\frac{x}{\sqrt{2}}+c_3, \tag{3.7}$$

where $\psi_i \equiv Root[8\sharp 1^4 c_1+\sharp 1^3(2c_2-40c_1)+\sharp 1^2(66c_1+2E)+\sharp 1(2\mathsf{C}_1+2\mathsf{C}_2-40c_1+2c_2)+\mathsf{C}_1-\mathsf{C}_2+8c_1\&, i]$ $(i=\overline{1,4})$ [1]; z denotes the right part of equation (3.6), c_3 is an arbitrary constant, function F is an elliptic integral of the first kind.

Although the function (3.7) has a sufficiently complicated form, its visualization is easily represented under a given set of parameters. For example, the plot of the solution

$$x=\frac{F\left(\sin^{-1}\left(\frac{2y}{5}\right)|\frac{25}{36}\right)\sqrt{25-4y^2}\sqrt{9-y^2}}{6\sqrt{4y^4-61y^2+225}},$$

which was obtained from the general solution of (3.7) for the following parameters:

$$c_1=1,\ c_2=20,\ c_3=0, \mathsf{C}_1=221, \mathsf{C}_2=-221, E=-94,$$

has the simple form (Fig. 1, continuous curve). These values of the parameters correspond to the initial condition $y(0)=0$ for the equation (1.11), where $z(y)$ is determined by the equation (3.6). We apply numerical integration to the Cauchy problem formulated in such a way and plot the corresponding graph (Fig. 1, dashed line). Figure 1 shows that the two curves (continuous and dashed) coincide.

Remark. Using the computer algebra systems for the analytical solutions (3.7) gives us the possibility to easily visualize the dependence of the six parameters c_1, c_2, c_3, C_1, C_2, E. We choose the certain values of these parameters and then can draw graph of the solution at the selected interval of variable x. For the root

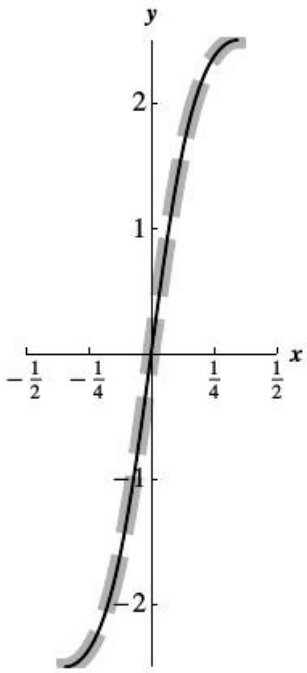

FIGURE 1. The graphs of the analytical and numerical solutions on the interval [-1/2, 1/2].

$\alpha_{22} = \frac{1}{8}(\sqrt{57} - 11)$ coefficients A_k, C_k have the form

$$
\begin{aligned}
&A_1 = -\tfrac{1}{6}\left(6+\sqrt{57}\right),\ A_2 = \tfrac{1}{6}\left(6+\sqrt{57}\right),\\
&A_3 = \tfrac{1}{24}\left(\sqrt{57}-27\right),\ A_4 = \tfrac{1}{24}\left(27-\sqrt{57}\right),\\
&A_5 = \tfrac{1}{6}\left(3+\sqrt{57}\right),\ A_6 = \tfrac{1}{6}\left(-3-\sqrt{57}\right),\\
&\mathsf{C}_3 = \frac{9(7\sqrt{57}-27)\mathsf{C}_1+(133-9\sqrt{57})\mathsf{C}_2+12(5\sqrt{57}-7)E}{4(79+5\sqrt{57})},\\
&\mathsf{C}_4 = \frac{(133-9\sqrt{57})\mathsf{C}_1+9(7\sqrt{57}-27)\mathsf{C}_2+12(7-5\sqrt{57})E}{4(79+5\sqrt{57})},\\
&\mathsf{C}_5 = \frac{-9(33+\sqrt{57})\mathsf{C}_1+(127-17\sqrt{57})\mathsf{C}_2+(45\sqrt{57}-321)E}{4(79+5\sqrt{57})},\\
&\mathsf{C}_6 = \frac{(127-17\sqrt{57})\mathsf{C}_1-9(33+\sqrt{57})\mathsf{C}_2+(321-45\sqrt{57})E}{4(79+5\sqrt{57})},
\end{aligned}
\tag{3.8}
$$

where C_1, C_2, E are arbitrary parameters. The general solution of the corresponding linear equation (1.12) has the form

$$\begin{aligned} z = c_1 y \left(y^2 + \tfrac{\sqrt{57}-11}{8}\right) + c_2 (4\left(509 + 31\sqrt{57}\right) y^4 + \\ \left(4941 + 903\sqrt{57}\right) y^2 - 425\sqrt{57} - 3683) + \\ \frac{(3+4\sqrt{57})(\mathsf{C}_1+\mathsf{C}_2)y + (7\sqrt{57}-27)(\mathsf{C}_1-\mathsf{C}_2)}{79+5\sqrt{57}} + E\left(y^2 + \tfrac{2\sqrt{57}-19}{7}\right), \end{aligned} \tag{3.9}$$

where c_1, c_2 are arbitrary constants. Substitute the relation (3.9) into formula (1.11). After integration we find the general solution of equation (1.9), (3.1), (3.8) using elliptic functions, similar to (3.7).

Under certain parameter sets the general solution can be expressed in the terms of elementary functions. For example, if

$$\begin{aligned} E = -\tfrac{3}{8}\left(51 + \sqrt{57}\right),\ \mathsf{C}_1 = 80 + 32\sqrt{\tfrac{19}{3}}, \\ \mathsf{C}_2 = -80 - 32\sqrt{\tfrac{19}{3}},\ c_1 = 0,\ c_2 = \tfrac{1}{2036+124\sqrt{57}}, \end{aligned}$$

then we obtain the one-parameter family of solutions of the form

$$y = \frac{3 - 3e^{6(c_3+x)}}{e^{6(c_3+x)} + 1},$$

where c_3 is arbitrary constant.

For root $\alpha_{23} = -\frac{1}{8}(\sqrt{57}+11)$ coefficients A_k, C_k have the form

$$
\begin{aligned}
&A_1 = -1 + \frac{\sqrt{57}}{6}, \quad A_2 = 1 - \frac{\sqrt{57}}{6},\\
&A_3 = -\frac{\sqrt{57}+27}{24}, \quad A_4 = \frac{27+\sqrt{57}}{24},\\
&A_5 = \frac{1}{6}\left(3-\sqrt{57}\right), \quad A_6 = \frac{1}{6}\left(\sqrt{57}-3\right),\\
&\mathsf{C}_3 = \frac{9\left(7\sqrt{57}+27\right)\mathsf{C}_1 - \left(133+9\sqrt{57}\right)\mathsf{C}_2 + 12\left(5\sqrt{57}+7\right)E}{4\left(5\sqrt{57}-79\right)},\\
&\mathsf{C}_4 = \frac{\left(133+9\sqrt{57}\right)\mathsf{C}_1 - 9\left(7\sqrt{57}+27\right)\mathsf{C}_2 + 12\left(7+5\sqrt{57}\right)E}{4\left(79-5\sqrt{57}\right)},\\
&\mathsf{C}_5 = \frac{9\left(\sqrt{57}-33\right)\mathsf{C}_1 + \left(127+17\sqrt{57}\right)\mathsf{C}_2 - 3\left(107+15\sqrt{57}\right)E}{4\left(79-5\sqrt{57}\right)},\\
&\mathsf{C}_6 = \frac{\left(127+17\sqrt{57}\right)\mathsf{C}_1 - 9\left(33-\sqrt{57}\right)\mathsf{C}_2 + (321+45\sqrt{57})E}{4\left(79-5\sqrt{57}\right)},
\end{aligned}
\tag{3.10}
$$

where C_1, C_2, E are arbitrary parameters. The general solution of the corresponding linear equation (1.12) has the form

$$
\begin{aligned}
z = c_1 y(y^2 - \frac{\sqrt{57}+11}{8}) + c_2(4(509-31\sqrt{57})y^4 +\\
(4941-903\sqrt{57})y^2 + 425\sqrt{57} - 3683) -\\
\frac{3+\sqrt{57}}{16}(\mathsf{C}_1+\mathsf{C}_2)y - \frac{6+\sqrt{57}}{7}(\mathsf{C}_1-\mathsf{C}_2) + E\left(y^2 - \frac{2\sqrt{57}+19}{7}\right),
\end{aligned}
\tag{3.11}
$$

where c_1, c_2 are arbitrary constants. We substitute the relation (3.11) into formula (1.11). After integration we find the general solution of equation (1.9), (3.1), (3.10) using elliptic functions, similar to (3.7).

Let's now turn to the study of the multiple root $\alpha_{24} = -\frac{7}{4}$. Since the condition (1.6) for the root is performed, then to calculate coefficients A_k, C_k we use formulas (1.7), (1.8). In this case, due to the multiplicity (is equal two) of the root, we will build two of the differential equations of the form (1.9) for which the corresponding

coefficients A_k have the form

$$\begin{gathered} A_1 = -1+\sqrt{\frac{11}{6}},\ A_2 = 1+\sqrt{\frac{11}{6}}, \\ A_3 = -\frac{36+\sqrt{66}}{30},\ A_4 = \frac{36-\sqrt{66}}{30}, \\ A_5 = \frac{12-2\sqrt{66}}{15},\ A_6 = -\frac{2}{15}\left(6+\sqrt{66}\right),\ \alpha_3 = -\frac{1}{4}\sqrt{\frac{33}{2}} \end{gathered} \tag{3.12}$$

or

$$\begin{gathered} A_1 = -1-\sqrt{\frac{11}{6}},\ A_2 = 1-\sqrt{\frac{11}{6}}, \\ A_3 = \frac{\sqrt{66}-36}{30},\ A_4 = \frac{36+\sqrt{66}}{30}, \\ A_5 = \frac{12+2\sqrt{66}}{15},\ A_6 = \frac{2}{15}\left(\sqrt{66}-6\right),\ \alpha_3 = \frac{1}{4}\sqrt{\frac{33}{2}}. \end{gathered} \tag{3.13}$$

Sets of coefficients (3.12) and (3.13) define the two different types of equations (1.9).

Substituting coefficients (3.1), (3.12) into system (1.10) and solving it, we calculate

$$\begin{gathered} \mathsf{C}_3 = \frac{-9(6+\sqrt{66})\mathsf{C}_1+(\sqrt{66}-14)\mathsf{C}_2-6(\sqrt{66}-4)E}{40}, \\ \mathsf{C}_4 = \frac{-(14+\sqrt{66})\mathsf{C}_1+9(\sqrt{66}-6)\mathsf{C}_2-6(4+\sqrt{66})E}{40}, \\ \mathsf{C}_5 = \frac{9(\sqrt{66}-6)\mathsf{C}_1-(14+\sqrt{66})\mathsf{C}_2-6(4+\sqrt{66})E}{160}, \\ \mathsf{C}_6 = \frac{(\sqrt{66}-14)\mathsf{C}_1-9(6+\sqrt{66})\mathsf{C}_2-6(\sqrt{66}-4)E}{160}, \end{gathered} \tag{3.14}$$

where C_1, C_2, E are arbitrary parameters. The general solution of the corresponding linear equation (1.12) has the form

$$\begin{gathered} z = c_1(y^2+1)(\sqrt{11}y^2-\sqrt{6}y-\sqrt{11})+ \\ c_2(11-\sqrt{57}-8y^2)(y((\sqrt{66}+\sqrt{418})y-44)+ \\ \sqrt{418}-\sqrt{66})+\mathsf{C}_1(-\frac{1}{8}\sqrt{\frac{33}{2}}(y+1)-\frac{3}{16}(2y+1))+ \\ \frac{1}{16}\mathsf{C}_2((\sqrt{66}-6)y-\sqrt{66}+3)+E(y^2-\frac{1}{8}(\sqrt{66}y+11)), \end{gathered} \tag{3.15}$$

where c_1, c_2 are arbitrary constants. Substituting relation (3.15) into formula (1.11) and integrating the obtained equation, we find the general solution of equation (1.9), (3.1), (3.12), (3.14) using the elliptic functions in the implicit form, similar to (3.7).

Set of coefficients (3.13) corresponds to the following values of the coefficients C_k

$$\mathsf{C}_3 = \frac{9(\sqrt{66}-6)\mathsf{C}_1 - (\sqrt{66}+14)\mathsf{C}_2 + 6(\sqrt{66}+4)E}{40},$$
$$\mathsf{C}_4 = \frac{(\sqrt{66}-14)\mathsf{C}_1 - 9(\sqrt{66}+6)\mathsf{C}_2 + 6(\sqrt{66}-4)E}{40},$$
$$\mathsf{C}_5 = \frac{-9(\sqrt{66}+6)\mathsf{C}_1 + (\sqrt{66}-14)\mathsf{C}_2 + 6(\sqrt{66}-4)E}{160},$$
$$\mathsf{C}_6 = \frac{-(\sqrt{66}+14)\mathsf{C}_1 + 9(\sqrt{66}-6)\mathsf{C}_2 + 6(\sqrt{66}+4)E}{160}$$

and the corresponding differential equation is integrated in a similar way; its solution is built in the implicit form, too.

4. Conclusions

The algorithm to determine the coefficients of the nonlinear third-order differential equation with six constant poles with respect to the unknown function - that satisfy the system of algebraic equations - is described. Five differential equations are integrated for the case when the roots of the corresponding algebraic equation of the fifth degree are expressed in radicals. Their general solutions are obtained using the elliptic functions and depend on three parameters and three arbitrary constants of integration.

It should be noted that the computational algorithm for the construction of differential equations and their integration procedure are connected with carrying out of cumbersome calculations, which have been performed using a computer algebra system *Mathematica*.

The results can be used in the study of the equation (1.3) in the case when the coefficients a_k are not constants.

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5. Application

The system of algebraic and differential equations determines the coefficients of the equation (1.3):

$$\begin{gathered}\sum_{k=1}^{6} A_k = 0, \sum_{k=1}^{6} a_k A_k = -6,\\ \sum_{k=1}^{6} {a_k}^2 A_k = -2\sum_{k=1}^{6} a_k,\\ 2A_k^2 + \sum_j \frac{A_k - A_j}{a_k - a_j} = 0 \ (k, j = \overline{1,6};\ k \neq j),\end{gathered} \tag{5.1}$$

$$\begin{gathered}\left(\frac{5}{2}A_k - \sum_j \frac{1}{a_k - a_j}\right) B_k + \sum_j \left(\frac{1}{2}A_k + \frac{1}{a_k - a_j}\right) B_j =\\ -A'_k + A_k \sum_j \frac{a'_k - a'_j}{a_k - a_j} - 3\sum_j A_j \frac{a'_k - a'_j}{a_k - a_j} + \frac{3}{2}A_k \sum_{i=1}^{6} a'_i A_i,\end{gathered} \tag{5.2}$$

$$2D + \sum_{k=1}^{6} (B_k - 3a'_k A_k) = 0, \tag{5.3}$$

$$\begin{gathered}-\left(2A_k + \sum_j \frac{1}{a_k - a_j}\right) \mathsf{C}_k + \sum_j \frac{\mathsf{C}_j}{a_k - a_j} = B_k^2 - B'_k -\\ B_k \sum_j \frac{a'_k - a'_j}{a_k - a_j} - \sum_j \frac{3A_j (a'_k - a'_j)^2 + 2B_j (a'_k - a'_j)}{a_k - a_j} +\\ B_k D - E - \sum_j \frac{a''_k - a''_j}{a_k - a_j},\end{gathered} \tag{5.4}$$

$$\begin{gathered}-a'''_k - B_k \mathsf{C}_k + \mathsf{C}'_k + \sum_j \frac{(a'_k - a'_j)(a''_k - a''_j - \mathsf{C}_k)}{a_k - a_j} +\\ \sum_j \frac{A_j (a'_k - a'_j)^3 + B_j (a'_k - a'_j)^2 + \mathsf{C}_k)(a'_k - a'_j)}{a_k - a_j} +\\ E a'_k + D(a''_k - \mathsf{C}_k) + F_k \prod_j (a_k - a_j) = 0,\end{gathered} \tag{5.5}$$

$$\sum_{k=1}^{6} F_k = \sum_{k=1}^{6} a_k F_k = \sum_{k=1}^{6} {a_k}^2 F_k = 0, \tag{5.6}$$

where $k, j = \overline{1,6}$; $k \neq j$.

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THE METRIC APPROACH TO FIXED POINT THEOREMS IN METRIC-LIKE SPACES

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Abstract. By constructing two metrics from a given metric-like space we show that some fixed point theorems in metric-like spaces may be obtained from the corresponding results in metric spaces.

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1. Introduction and preliminaries

In 1992 Matthews [18] introduced the notion of a partial metric space as a part of the study of denotational semantics of data flow networks. The main difference comparing to the standard metric is that the self-distance of an arbitrary point need not be equal to zero. After that, partial metric spaces were investigated by many authors, see [14, 19, 28], [16, Chapter 14] and the references given there.

In 2012 Harandi [9] introduced the notion of a metric-like space as a generalization of a partial metric space and initiated the fixed point theory in this space. As an application, the author derived some new fixed point theorems in partial metric spaces. Note that a metric-like was mentioned before as a dislocated metric in [10, 29].

After that metric-like spaces and its modifications were investigated by many others. Isik and Turkoglu [11] established some fixed point theorems for weakly contractive maps defined in ordered metric-like spaces. Malhotra *et al.* [17] obtained the fixed point results for F-type contractions which satisfy a weaker condition than the monotonicity of self-map of a partially ordered metric-like space. A fixed point result for F-expansive maps was also proved. Shukla *et al.* [25] introduced the notion of a 0-σ-complete metric-like space and proved some common fixed point theorems in such space. Shobkolaei *et al.* [25] demonstrated a fundamental lemma for the convergence of sequences in metric-like spaces, and by using it authors proved some Suzuki-type fixed point results. Karapinar and Salimi [15] discussed the existence and uniqueness of a fixed point of a cyclic map in the context of metric-like spaces. Recently, Salimi *et al.* [23] introduced the notions of α-$\psi\phi$-contractive and cyclic

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α-$\psi\phi$-contractive maps and established the existence and uniqueness of fixed points for such maps in complete metric-like spaces. The results generalized and extended several comparable results in the existing literature.

The above results unified and generalized some well-known fixed point results in the literature. To prove these results, authors used similar techniques as that used to prove fixed point theorems in metric spaces.

In this paper we present another approach to fixed point theorems in metric-like spaces. By introducing two metrics induced from a metric-like space we show that some fixed point theorems in metric-like spaces may be deduced immediately from fixed point theorems in metric spaces. In particular, we show that

(1) [9, Theorem 2.7] is a consequence of [22, Theorem 1].
(2) [9, Theorem 2.11] is a consequence of [21, Corollary on page 463].
(3) [11, Theorem 1], [11, Theorem 2] and [11, Theorem 3] are consequences of [20, Corollary 3.3] and [20, Theorem 4.3].
(4) [25, Theorem 9] is a consequence of [27, Theorem 2].
(5) Various fixed point theorems in metric-like spaces in [1, 2, 4, 5, 12, 13, 15, 17, 23, 26] may be deduced from certain fixed point theorems in metric spaces by using similar arguments.

This work may be seen as a continuation of [3, 8] and some others in obtaining fixed point theorems on generalized metric spaces from fixed point theorems on certain metric spaces.

First, we recall some notions and lemmas which will be used latter.

Definition 1 ([9, Definition 2.1])**.** Let X be a non-empty set and $\sigma : X \times X \longrightarrow \mathbb{R}^+$ be a function such that for all $x, y, z \in X$,

(1) If $\sigma(x, y) = 0$ then $x = y$;
(2) $\sigma(x, y) = \sigma(y, x)$;
(3) $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$.

Then σ is called a *metric-like* on X and the pair (X, σ) is called a *metric-like space*. Each metric-like σ on X generates a topology τ_σ on X whose base is the family of open σ-balls

$$B_\sigma(x, \varepsilon) = \{y \in X : |\sigma(x, y) - \sigma(x, x)| < \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$. A sequence $\{x_n\}$ in (X, σ) is called *convergent* to a point $x \in X$ if $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x)$, written $\lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in (X, σ) is called *σ-Cauchy* if $\lim_{n,m\to\infty} \sigma(x_n, x_m)$ exists and is finite. The metric-like space (X, σ) is called *complete* if for each σ-Cauchy sequence $\{x_n\}$, there exists some $x \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x) = \lim_{n,m\to\infty} \sigma(x_m, x_n).$$

Remark 1 ([9, page 2]). (1) If $f : X \longrightarrow Y$ is a continuous map from a metric-like space X to a metric-like space Y and $\lim_{n\to\infty} x_n = x$ in X then $\lim_{n\to\infty} f(x_n) = f(x)$ in Y.
(2) Every partial metric space [6, Definition 2] is a metric-like space.

Some interesting examples of metric-like spaces are as follows.

Example 1 ([9, Example 2.2]). Let $X = \{0,1\}$ and

$$\sigma(x,y) = \begin{cases} 2 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Then σ is a metric-like. Since $\sigma(0,0) > \sigma(0,1)$, σ is not a partial metric.

Example 2 ([17, Example 3]). Let $X = \mathbb{R}^+$ and

$$\sigma(x,y) = \begin{cases} 2x & \text{if } x = y \\ \max\{x,y\} & \text{if } x \neq y. \end{cases}$$

Then σ is a metric-like. Since $\sigma(1,1) > \sigma(0,1)$, σ is not a partial metric.

2. Metrics induced by a metric-like

First, we present some new notions and lemmas in metric-like spaces.

Definition 2. Let (X,σ) be a metric-like space.
(1) A sequence $\{x_n\}$ is called *0-σ-Cauchy* if $\lim_{n,m\to\infty} \sigma(x_n,x_m) = 0$.
(2) (X,σ) is called *0-complete* if for each 0-σ-Cauchy sequence $\{x_n\}$, there exists some $x \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n,x) = \sigma(x,x) = \lim_{n,m\to\infty} \sigma(x_m,x_n) = 0.$$

Lemma 1. *Let (X,σ) be a metric-like space.*
(1) *If (X,σ) is complete then it is 0-complete.*
(2) *If $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ and $\sigma(x,x) = \sigma(y,y) = 0$ then $\lim_{n\to\infty} \sigma(x_n,y_n) = \sigma(x,y)$.*

Proof. (1). It is straightforward from the definition.
(2). The inequality

$$\sigma(x_n,y_n) \leq \sigma(x_n,x) + \sigma(x,y) + \sigma(y,y_n)$$

implies

$$\sigma(x_n,y_n) - \sigma(x,y) \leq \sigma(x_n,x) + \sigma(y,y_n).$$

Similarly

$$\sigma(x,y) \leq \sigma(x,x_n) + \sigma(x_n,y_n) + \sigma(y_n,y)$$

implies

$$\sigma(x,y) - \sigma(x_n,y_n) \leq \sigma(x_n,x) + \sigma(y,y_n).$$

So that

$$|\sigma(x_n, y_n) - \sigma(x, y)| \leq \sigma(x_n, x) + \sigma(y_n, y). \tag{2.1}$$

Taking the limit as $n \to \infty$ in (2.1) and note that $\sigma(x,x) = \sigma(y,y) = 0$, we have $\lim_{n\to\infty} \sigma(x_n, y_n) = \sigma(x, y)$. □

The following example shows that the implication (1) in Lemma 1 can not be reversed.

Example 3. Let $X = (0,1)$ and $\sigma(x,y) = |x-y| + 1$ for all $x, y \in X$. Then we have

(1) σ is a metric-like.
(2) (X,σ) is 0-complete.
(3) (X,σ) is not complete.

Proof. (1). It is easy to check that σ is a metric-like on X.

(2). Since $\sigma(x,y) \geq 1$ for all $x, y \in X$, (X,σ) has no any 0-σ-Cauchy sequence. This shows that (X,σ) is 0-complete.

(3). We have that

$$\lim_{n,m\to\infty} \sigma\Big(\frac{1}{2n}, \frac{1}{2m}\Big) = \lim_{n,m\to\infty} \Big(\Big|\frac{1}{2n} - \frac{1}{2m}\Big| + 1\Big) = 1.$$

Then $\{\frac{1}{2n}\}$ is a σ-Cauchy sequence in (X,σ). On the contrary, suppose that $\lim_{n\to\infty} \frac{1}{2n} = x$ in (X,σ). Then

$$\lim_{n\to\infty} \sigma\Big(\frac{1}{2n}, x\Big) = \lim_{n\to\infty} \Big(\Big|\frac{1}{2n} - x\Big| + 1\Big) = \sigma(x,x) = 1.$$

This implies that $x = 0 \in X = (0,1)$. It is a contradiction. Then $\{\frac{1}{2n}\}$ is not convergent in (X,σ). Therefore, (X,σ) is not complete. □

From a given metric-like we construct two metrics as follows.

Theorem 1. *Let (X,σ) be a metric-like space and*

$$d_\sigma(x,y) = \begin{cases} 0 & \text{if } x = y \\ \sigma(x,y) & \text{if } x \neq y \end{cases}$$

$$m_\sigma(x,y) = \begin{cases} 0 & \text{if } x = y \\ \max\{\sigma(x,y), \sigma(x,x), \sigma(y,y)\} & \text{if } x \neq y. \end{cases}$$

Then we have

(1) *d_σ and m_σ are metrics on X.*
(2) *The metric-like space (X,σ) is 0-complete if and only if the metric space (X, d_σ) is complete.*
(3) *The metric-like space (X,σ) is 0-complete if and only if the metric space (X, m_σ) is complete.*

Proof. (1). It is clear that d_σ is a metric on X. For all $x, y \in X$, we have $m_\sigma(x,y) = 0$ if and only if $x = y$ and $m_\sigma(x,y) = m_\sigma(y,x)$. Now, for all $x, y, z \in X$, we may assume $x \neq y \neq z$. Then

$$\begin{aligned} m_\sigma(x,z) &= \max\{\sigma(x,z), \sigma(x,x), \sigma(z,z)\} \\ &\leq \max\{\sigma(x,y) + \sigma(y,z), \sigma(x,x), \sigma(z,z)\} \\ &\leq \max\{\sigma(x,y), \sigma(x,x)\} + \max\{\sigma(y,z), \sigma(z,z)\} \\ &\leq \max\{\sigma(x,y), \sigma(x,x), \sigma(y,y)\} + \max\{\sigma(y,z), \sigma(y,y), \sigma(z,z)\} \\ &= m_\sigma(x,y) + m_\sigma(y,z). \end{aligned}$$

Therefore, m_σ is also a metric on X.

(2). Suppose that (X,σ) is 0-complete and let $\{x_n\}$ be a Cauchy sequence in (X, d_σ).

If there exists $x \in X$ such that the set $\{n \in \mathbb{N} : x_n = x\}$ is infinite then we get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d_\sigma(x_{n_k}, x) = 0$ for all $k \in \mathbb{N}$. So $\lim_{k\to\infty} x_{n_k} = x$ in (X, d_σ). Since $\{x_n\}$ is a Cauchy sequence in (X, d_σ), we get $\lim_{n\to\infty} x_n = x$ in (X, d_σ).

If this situation does not happen then there exists $n_0 \in \mathbb{N}$ such that $x_n \neq x_m$ for all $n \neq m \geq n_0$. So $\sigma(x_n, x_m) = d_\sigma(x_n, x_m)$ for all $n \neq m \geq n_0$. In this case $\lim_{n,m\to\infty} \sigma(x_n, x_m) = \lim_{n,m\to\infty} d_\sigma(x_n, x_m) = 0$. Then $\{x_n\}$ is a 0-σ-Cauchy sequence. By hypothesis there exists $x \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x,x) = \lim_{n,m\to\infty} \sigma(x_n, x_m) = 0.$$

Then $\lim_{n\to\infty} d_\sigma(x_n, x) = \lim_{n\to\infty} \sigma(x_n, x) = 0$. So $\lim_{n\to\infty} x_n = x$ in (X, d_σ).

By the above, the metric space (X, d_σ) is complete.

Conversely, suppose that (X, d_σ) is complete and let $\{x_n\}$ be a 0-σ-Cauchy sequence in the metric-like space (X,σ). Then

$$d_\sigma(x_n, x_m) = \begin{cases} \sigma(x_n, x_m) & \text{if } x_n \neq x_m \\ 0 & \text{if } x_n = x_m \end{cases}$$

for all $n, m \in \mathbb{N}$. This implies that $\{x_n\}$ is a Cauchy sequence in (X, d_σ). So there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$ in (X, d_σ).

If the set $M_x = \{n \in \mathbb{N} : x_n = x\}$ is infinite then we get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\sigma(x_{n_k}, x_{m_k}) = \sigma(x,x)$ for all k. Since $\{x_n\}$ is 0-σ-Cauchy, we get $\lim_{k\to\infty} \sigma(x_{n_k}, x_{m_k}) = \sigma(x,x) = 0$. So $\lim_{k\to\infty} \sigma(x_{n_k}, x) = \sigma(x,x) = 0$. Note that $\sigma(x_k, x) \leq \sigma(x_k, x_{n_k}) + \sigma(x_{n_k}, x)$ for all $k \in \mathbb{N}$ and $\{x_n\}$ is a 0-σ-Cauchy in (X,σ). So $\lim_{k\to\infty} \sigma(x_k, x) = 0$. This shows that $\lim_{n\to\infty} x_n = x$ in (X,σ). For $n \in \mathbb{N} \setminus M_x$ we get $x_n \neq x$ and so $\sigma(x_n, x) = d_\sigma(x_n, x)$. This implies that $\lim_{n\to\infty} \sigma(x_n, x) =$

$\lim_{n\to\infty} d_\sigma(x_n,x)=0$. Note that

$$0\le \sigma(x,x)\le \lim_{n\to\infty}\big[\sigma(x,x_n)+\sigma(x_n,x)\big]=0.$$

Then we have

$$\lim_{n\to\infty}\sigma(x_n,x)=\sigma(x,x)=\lim_{n,m\to\infty}\sigma(x_n,x_m)=0.$$

By the above, the metric-like space (X,σ) is 0-complete.

(3). Suppose that (X,σ) is 0-complete and let $\{x_n\}$ be a Cauchy sequence in the metric space (X,m_σ). If there exists $x\in X$ such that $\{n\in\mathbb{N}: x_n=x\}$ is infinite then, as in the proof of (2), it follows $\lim_{n\to\infty} x_n=x$ in (X,m_σ).

Suppose now that $x_n\neq x_m$ for all $n\neq m\in\mathbb{N}$. Then we have

$$\begin{aligned}0\le \lim_{n,m\to\infty}\sigma(x_n,x_m)&\le \lim_{n,m\to\infty}\max\big\{\sigma(x_n,x_m),\sigma(x_n,x_n),\sigma(x_m,x_m)\big\}\\&=\lim_{n,m\to\infty} m_\sigma(x_n,x_m)=0.\end{aligned}$$

This shows that $\{x_n\}$ is a 0-σ-Cauchy sequence in the 0-complete metric-like space (X,σ). Then there exists $x\in X$ such that

$$\lim_{n\to\infty}\sigma(x_n,x)=\sigma(x,x)=\lim_{n,m\to\infty}\sigma(x_n,x_m)=0.$$

It follows that $\lim_{n\to\infty}\sigma(x_n,x_n)=0$ and so

$$\lim_{n\to\infty} m_\sigma(x_n,x)=\lim_{n\to\infty}\max\big\{\sigma(x_n,x),\sigma(x_n,x_n),\sigma(x,x)\big\}=0.$$

This shows that $\lim_{n\to\infty} x_n=x$ in (X,m_σ).

By the above, the metric space (X,m_σ) is complete.

Conversely, suppose that (X,m_σ) is complete and let $\{x_n\}$ be a 0-σ-Cauchy sequence in the metric-like space (X,σ). So $\lim_{n,m\to\infty}\sigma(x_n,x_m)=0$. Then

$$\lim_{n\to\infty}\sigma(x_n,x_n)=\lim_{m\to\infty}\sigma(x_m,x_m)=0.$$

So that

$$\lim_{n,m\to\infty} m_\sigma(x_n,x_m)=\lim_{n,m\to\infty}\max\big\{\sigma(x_n,x_m),\sigma(x_n,x_n),\sigma(x_m,x_m)\big\}=0.$$

Thus $\{x_n\}$ is a Cauchy sequence in the complete metric space (X,m_σ). Then there exists $x\in X$ such that

$$\lim_{n\to\infty} m_\sigma(x_n,x)=\lim_{n\to\infty}\max\big\{\sigma(x_n,x),\sigma(x_n,x_n),\sigma(x,x)\big\}=0.$$

It implies that $\lim_{n\to\infty}\sigma(x_n,x)=\sigma(x,x)=0$. Therefore,

$$\lim_{n\to\infty}\sigma(x_n,x)=\sigma(x,x)=\lim_{n,m\to\infty}\sigma(x_n,x_m)=0.$$

So $\lim_{n\to\infty} x_n = x$ in (X,σ). This shows that the metric-like space (X,σ) is 0-complete. □

Next, we prove a technical lemma on the relation between certain values relating the metric-like space (X,σ) and the metric space (X,m_σ) mentioned in Theorem 1.

Lemma 2. *Let (X,σ) be a metric-like space, $T : X \longrightarrow X$ be a map and for all $x,y \in X$,*

$$M_\sigma(x,y) = \max\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty),\sigma(x,Ty),\sigma(y,Tx),\sigma(x,x),\\ \sigma(y,y),\sigma(Tx,Tx),\sigma(Ty,Ty)\},$$

$$M_{m_\sigma}(x,y) = \max\{m_\sigma(x,y),m_\sigma(x,Tx),m_\sigma(y,Ty),m_\sigma(x,Ty),m_\sigma(y,Tx)\}.$$

Then $M_\sigma(x,y) = M_{m_\sigma}(x,y)$ for all $x \neq y \in X$.

Proof. We first show that $M_\sigma(x,y) \le M_{m_\sigma}(x,y)$ for all $x \neq y \in X$. We only consider the following four cases.

Case 1. $M_\sigma(x,y) = \max\{\sigma(x,y),\sigma(x,x),\sigma(y,y)\}$. Then

$$M_\sigma(x,y) = m_\sigma(x,y) \le M_{m_\sigma}(x,y).$$

Case 2. $M_\sigma(x,y) = \sigma(x,Tx)$. If $Tx = x$ then

$$M_\sigma(x,y) = \sigma(x,x) \le m_\sigma(x,y) \le M_{m_\sigma}(x,y).$$

If $Tx \neq x$ then

$$M_\sigma(x,y) = \sigma(x,Tx) \le \max\{\sigma(x,Tx),\sigma(x,x),\sigma(Tx,Tx)\} = m_\sigma(x,Tx)\\ \le M_{m_\sigma}(x,y).$$

Case 3. $M_\sigma(x,y) = \sigma(x,Ty)$. If $Ty = x$ then

$$M_\sigma(x,y) = \sigma(x,x) \le m_\sigma(x,y) \le M_{m_\sigma}(x,y).$$

If $Ty \neq x$ then

$$M_\sigma(x,y) = \sigma(x,Ty) \le \max\{\sigma(x,Ty),\sigma(x,x),\sigma(Ty,Ty)\} = m_\sigma(x,Ty)\\ \le M_{m_\sigma}(x,y).$$

Case 4. $M_\sigma(x,y) = \sigma(Tx,Tx)$. If $Tx = x$ then

$$M_\sigma(x,y) = \sigma(x,x) \le m_\sigma(x,y) \le M_{m_\sigma}(x,y).$$

If $Tx \neq x$ then

$$M_\sigma(x,y) = \sigma(Tx,Tx) \le \max\{\sigma(x,Tx),\sigma(x,x),\sigma(Tx,Tx)\} = m_\sigma(x,Tx)\\ \le M_{m_\sigma}(x,y).$$

We next show that $M_\sigma(x,y) \ge M_{m_\sigma}(x,y)$ for all $x \neq y \in X$. We only consider the following three cases.

Case 1. $M_{m_\sigma}(x,y) = m_\sigma(x,y)$. Then

$$M_{m_\sigma}(x,y) = \max\{\sigma(x,y), \sigma(x,x), \sigma(y,y)\} \leq M_\sigma(x,y).$$

Case 2. $M_{m_\sigma}(x,y) = m_\sigma(x,Tx)$. If $Tx = x$ then

$$M_{m_\sigma}(x,y) = 0 \leq M_\sigma(x,y).$$

If $Tx \neq x$ then

$$M_{m_\sigma}(x,y) = m_\sigma(x,Tx) = \max\{\sigma(x,Tx), \sigma(x,x), \sigma(Tx,Tx)\} \leq M_\sigma(x,y).$$

Case 3. $M_{m_\sigma}(x,y) = m_\sigma(x,Ty)$. If $Ty = x$ then

$$M_{m_\sigma}(x,y) = 0 \leq M_\sigma(x,y).$$

If $Ty \neq x$ then

$$M_{m_\sigma}(x,y) = m_\sigma(x,Ty) = \max\{\sigma(x,Ty), \sigma(x,x), \sigma(Ty,Ty)\} \leq M_\sigma(x,y).$$

By the above, we have $M_\sigma(x,y) = M_{m_\sigma}(x,y)$. □

3. Applications

In this section we show that some fixed point theorems in metric-like spaces may be obtained from the corresponding results in metric spaces. The following results are consequences of [7, Theorem 2.1].

Theorem 2. *Let (X,σ) be a complete metric-like space and $T : X \longrightarrow X$ be a map such that*

$$\sigma(Tx,Ty) \leq \psi\big(M_\sigma(x,y)\big) \tag{3.1}$$

for all $x, y \in X$, where $M_\sigma(x,y)$ is defined as in Lemma 2, and $\psi : [0,\infty) \longrightarrow [0,\infty)$ is a non-decreasing function satisfying $\psi(t) < t$, $\lim_{s\to t^+} \psi(s) < t$ for all $t > 0$ and $\lim_{t\to\infty} \big[t - \psi(t)\big] = \infty$. Then T has a fixed point.

Proof. Since ψ is non-decreasing, it follows from (3.1) that

$$\sigma(Tx,Tx) \leq \psi\big(\max\{\sigma(x,x), \sigma(x,Tx), \sigma(Tx,Tx)\}\big) \leq \psi\big(M_\sigma(x,y)\big) \tag{3.2}$$

for all $x, y \in X$. By using (3.1), (3.2) and Lemma 2, we have, for $Tx \neq Ty$,

$$\begin{aligned} m_\sigma(Tx,Ty) &= \max\{\sigma(Tx,Ty), \sigma(Tx,Tx), \sigma(Ty,Ty)\} \leq \psi\big(M_\sigma(x,y)\big) \\ &= \psi\big(M_{m_\sigma}(x,y)\big). \end{aligned}$$

Note that if $Tx = Ty$ then the inequality $m_\sigma(Tx,Ty) \leq \psi\big(M_{m_\sigma}(x,y)\big)$ is obviously true. Then, from Lemma 1.(1) and Theorem 1.(3), we obtain that (X, m_σ) is a complete metric space. Then, by using [7, Theorem 2.1], we get the conclusion. □

Theorem 3 ([9, Theorem 2.4])**.** *Let (X,σ) be a complete metric-like space and $T : X \longrightarrow X$ be a map such that*

$$\sigma(Tx,Ty) \leq \psi\big(M(x,y)\big) \tag{3.3}$$

for all $x, y \in X$, where

$$M(x,y) = \max\big\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty),\sigma(x,Ty),\sigma(y,Tx),\sigma(x,x),\sigma(y,y)\big\}$$

and $\psi : [0,\infty) \longrightarrow [0,\infty)$ is a non-decreasing function satisfying $\psi(t) < t$, $\lim_{s\to t^+} \psi(s) < t$ for all $t > 0$ and $\lim_{t\to\infty} \big[t - \psi(t)\big] = \infty$. Then T has a fixed point.

Proof. Since $M(x,y) \leq M_\sigma(x,y)$ for all $x, y \in X$ and ψ is non-decreasing, we see that (3.3) implies (3.1). Then the conclusion is a direct consequence of Theorem 2. □

The following is a consequence of [22, Theorem 1].

Theorem 4 ([9, Theorem 2.7])**.** *Let (X,σ) be a complete metric-like space and $T : X \longrightarrow X$ be a map such that*

$$\sigma(Tx,Ty) \leq \sigma(x,y) - \varphi\big(\sigma(x,y)\big) \tag{3.4}$$

for all $x, y \in X$, where $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is a non-decreasing continuous function satisfying $\varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Proof. We will prove that

$$d_\sigma(Tx,Ty) \leq d_\sigma(x,y) - \varphi\big(d_\sigma(x,y)\big). \tag{3.5}$$

by considering the following three cases.

Case 1. $Tx \neq Ty$. Then (3.5) is a direct consequence of (3.4).

Case 2. $Tx = Ty$ and $x = y$. Then (3.5) becomes $0 \leq 0 - \varphi(0)$ and that obviously holds.

Case 3. $Tx = Ty$ and $x \neq y$. Then (3.5) becomes $0 \leq \sigma(x,y) - \varphi\big(\sigma(x,y)\big)$ and that holds by (3.4).

By the above cases, we have that (3.5) holds for all $x, y \in X$. From Lemma 1.(1) and Theorem 1.(2), we obtain that (X, d_σ) is a complete metric space. Then, by using [22, Theorem 1], we get the conclusion. □

The following result is a consequence of [21, Corollary on page 463].

Theorem 5 ([9, Theorem 2.11])**.** *Let (X,σ) be a complete metric-like space and $T : X \longrightarrow X$ be a map such that*

$$\sigma(Tx,Ty) \leq \alpha\big(\sigma(x,y)\big)\sigma(x,y) \tag{3.6}$$

for all $x, y \in X$, where $\alpha : [0,\infty) \longrightarrow [0,\infty)$ is non-increasing. Then T has a unique fixed point.

Proof. Using [21, Corollary on page 463] and the argument as in the proof of Theorem 4. □

Remark 2. There exist a metric-like space (X,σ) and a map $T : X \longrightarrow X$ such that

$$\begin{aligned}
&\max\{\sigma(x_n,x_{n-1}),\sigma(x_{n+1},x_n)\}\\
&< \max\big\{\sigma(x_n,x_{n-1}),\sigma(x_n,Tx_n),\sigma(x_{n-1},Tx_{n-1}),\sigma(x_n,x_n),\sigma(x_{n-1},x_{n-1}),\\
&\qquad \frac{1}{2}\big[\sigma(x_n,Tx_{n-1})+\sigma(Tx_n,x_{n-1})\big]\big\}
\end{aligned}$$

where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ and all $x_0 \in X$.

Proof. Let $X = \{1,2\}$ and $\sigma(1,1) = \sigma(2,2) = 1$, $\sigma(1,2) = \sigma(2,1) = \frac{1}{2}$. Then (X,σ) is a metric-like space. Put $T1 = 2$, $T2 = 1$. Then we have

$$\begin{aligned}
\frac{1}{2} &= \max\{\sigma(x_n,x_{n-1}),\sigma(x_{n+1},x_n)\}\\
&< 1\\
&= \max\big\{\sigma(x_n,x_{n-1}),\sigma(x_n,Tx_n),\sigma(x_{n-1},Tx_{n-1}),\sigma(x_n,x_n),\sigma(x_{n-1},x_{n-1}),\\
&\qquad \frac{1}{2}\big[\sigma(x_n,Tx_{n-1})+\sigma(Tx_n,x_{n-1})\big]\big\}.
\end{aligned}$$

□

Remark 2 implies the argument on [11, page 4] that

$$\begin{aligned}
&\max\big\{\sigma(x_n,x_{n-1}),\sigma(x_n,Tx_n),\sigma(x_{n-1},Tx_{n-1}),\sigma(x_n,x_n),\sigma(x_{n-1},x_{n-1}),\\
&\qquad \frac{1}{2}\big[\sigma(x_n,Tx_{n-1})+\sigma(Tx_n,x_{n-1})\big]\big\}\\
&\le \max\{\sigma(x_n,x_{n-1}),\sigma(x_{n+1},x_n)\}
\end{aligned}$$

is not correct. Then [11, Theorem 1], [11, Theorem 2] and [11, Theorem 3] must to be restated as in Theorem 6, Theorem 7 and Theorem 8 respectively, where

$$\max\big\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty),\sigma(x,x),\sigma(y,y),\frac{1}{2}\big[\sigma(x,Ty)+\sigma(y,Tx)\big]\big\}$$

in mentioned theorems is replaced by

$$\max\big\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty),\frac{1}{2}\big[\sigma(x,Ty)+\sigma(y,Tx)\big]\big\}.$$

Note that they are consequences of [20, Corollary 3.3] and [20, Theorem 4.3].

Theorem 6. *Let $(X,\preceq,\sigma)$ be a partially ordered complete metric-like space and $T : X \longrightarrow X$ be a continuous and non-decreasing map such that for all comparable $x,y \in X$,*

$$\psi\big(\sigma(Tx,Ty)\big) \le \psi\big(N(x,y)\big) - \phi\big(N(x,y)\big) \tag{3.7}$$

where

$$N(x,y) = \max\left\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{1}{2}\left[\sigma(x,Ty)+\sigma(y,Tx)\right]\right\}$$

and

(1) $\psi : [0,\infty) \longrightarrow [0,\infty)$ *is continuous non-decreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$*;*
(2) $\phi : [0,\infty) \longrightarrow [0,\infty)$ *is a lower semi-continuous function with* $\phi(t) = 0$ *if and only if* $t = 0$.

If there exists $x_0 \in X$ *with* $x_0 \preceq Tx_0$ *then* T *has a fixed point.*

Proof. Using [20, Corollary 3.3] and the argument as in the proof of Theorem 4. □

Theorem 7. *Under the same hypotheses of Theorem 6 and without assuming the continuity of* T*, assume that whenever* $\{x_n\}$ *is a non-decreasing sequence in* X *such that* $\lim_{n\to\infty} x_n = x$ *implies* $x_n \preceq x$ *for all* $n \in \mathbb{N}$*, then* T *has a fixed point in* X.

Proof. By using [20, Corollary 3.3] and the argument as in the proof of Theorem 4. □

Theorem 8. *Let all conditions of Theorem 6 (resp. Theorem 7) be fulfilled and let the following condition be satisfied: for arbitrary two points* $x, y \in X$*, there exists* $z \in X$ *which is comparable with both* x *and* y*. Then the fixed point of* T *is unique.*

Proof. Using [20, Theorem 4.3] and the argument as in the proof of Theorem 4. □

Theorem 9 ([25, Theorem 9])**.** *Let* (X,σ) *be a complete metric-like space,* $\theta : [0,1) \longrightarrow (\frac{1}{2},1]$ *be a function defined by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le \frac{\sqrt{5}-1}{2} \\ \frac{1-r}{r^2} & \text{if } \frac{\sqrt{5}-1}{2} \le r \le \frac{1}{\sqrt{2}} \\ \frac{1}{1+r} & \text{if } \frac{}{\sqrt{2}} \le r \le 1 \end{cases}$$

and $T : X \longrightarrow X$ *be a map such that for all* $x, y \in X$*,*

$$\theta(r)\sigma(x,Tx) \le \sigma(x,y) \text{ implies } \sigma(Tx,Ty) \le r\sigma(x,y). \tag{3.8}$$

Then T *has a unique fixed point* $z \in X$ *and for each* $x \in X$*,* $\lim_{n\to\infty} T^n x = z$.

Proof. We will prove that for all $x, y \in X$,

$$\theta(r)d_\sigma(x,Tx) \le d_\sigma(x,y) \text{ implies } d_\sigma(Tx,Ty) \le rd_\sigma(x,y). \tag{3.9}$$

If $Tx = Ty$ then $d_\sigma(Tx,Ty) = 0$ and thus (3.9) obviously holds. If $Tx \neq Ty$ then $x \neq y$ and thus $d_\sigma(Tx,Ty) = \sigma(Tx,Ty)$, $d_\sigma(x,y) = \sigma(x,y)$. Note that if

$x = Tx$ then T obviously has fixed point. So we may assume that $x \neq Tx$ and then $d_\sigma(x, Tx) = \sigma(x, Tx)$. In this case we find that (3.8) becomes (3.9).

So (3.9) holds for all $x, y \in X$. By Theorem 1 we see that (X, d_σ) is complete since (X, σ) is complete. By [27, Theorem 2] we get the conclusion. □

Remark 3. By using similar arguments as in the proofs of above corollaries, we may obtain various fixed point theorems in metric-like spaces in [1, 2, 4, 5, 12, 13, 15, 17, 23, 26] from certain fixed point theorems in metric spaces. Then authors should be careful on stating fixed point theorems in metric-like spaces. Note that every partial metric space is a metric-like space and for more details on obtaining fixed point theorems on partial metric spaces from fixed point theorems in metric spaces, readers may refer to [8] and [24].

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ON COFINITELY WEAK $\delta-$SUPPLEMENTED MODULES

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Abstract. Let R be a ring and M be a left $R-$module. M is called *cofinitely weak $\delta-$supplemented* (or briefly $\delta-$ *CWS*-module) if every cofinite submodule of M has a weak $\delta-$supplement in M. In this paper, we give various properties of this kind of modules. It is shown that a module M is $\delta-$*CWS*-module if and only if every maximal submodule has a weak $\delta-$supplement in M. The class of cofinitely weak $\delta-$supplemented modules are closed under taking homomorphic images, arbitrary sums and short exact sequences. Also we give some conditions equivalent to being a $\delta-$*CWS*-module for a $\delta-$coatomic module.

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1. Introduction & preliminaries

Throughout this paper, R will be an associative ring with identity and M will be an unitary left $R-$module and the symbol "$\leq$" will denote submodule property. Let M be an $R-$module. A submodule N of M is called *small* in M and denoted by $N \ll M$, if for every submodule K of M the equality $M = N + K$ implies $K = M$. A submodule N of M is said to be *essential* in M and denoted by $N \trianglelefteq M$, if $N \cap K \neq 0$ for every nonzero submodule K of M. A module M is said to be *singular* if $M \cong \frac{N}{K}$ for some module N and a submodule $K \leq N$ with $K \trianglelefteq N$. Let N, L be submodules of M. We call L as a *supplement* of N in M, if $M = N + L$ and $N \cap L$ is small in L [12]. Also L is called a *weak supplement* of N in M, if $M = N + L$ and $N \cap L \ll M$ [9,15]. Clearly in this situation N is a weak supplement of K, too. A module M is called *(weakly) supplemented*, if every submodule of M has a *(weak) supplement*. By using this definition, Büyükaşık and Lomp showed that a ring R is left perfect if and only if every left $R-$module is weakly supplemented, if and only if R is semilocal and the radical of the countably infinite free left $R-$module has a weak supplement in [6]. Furthermore Alizade and Büyükaşık showed that a ring R is semilocal if and only if every direct product of simple modules is weakly supplemented in [4].

Following [14], recall that a submodule N of a module M is said to be $\delta-$*small* in M and written $N \ll_\delta M$, provided $M \neq N + X$ for any proper submodule X of

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M with $\frac{M}{X}$ singular. The sum of $\delta-$small submodules of a module M is denoted by $\delta(M)$. Let M be an $R-$module. M is called $\delta-$*coatomic module* whenever $N \leq M$ and $\delta\left(\frac{M}{N}\right) = \frac{M}{N}$ implies $\frac{M}{N} = 0$. For more detailed discussion on $\delta-$ coatomic modules we refer to [7]. Let L be a submodule of a module M. A submodule K of M is called a $\delta-$*supplement* of L in M, if $M = L + K$ and $L \cap K \ll_\delta K$. The module M is called $\delta-$*supplemented* if every submodule of M has a $\delta-$supplement in M. On the other hand, the submodule L is said to be a *weak supplement* of N in M, if $M = L + N$ and $L \cap N \ll_\delta M$. Also, the module M is called *weakly* $\delta-$*supplemented* if every submodule of M has a weak $\delta-$supplement in M. For more discussion on $\delta-$small submodules, $\delta-$supplemented and weakly $\delta-$supplemented modules, we refer to [8, 13, 14].

Alizade et al. studied certain modules whose maximal submodules have supplements, and introduced cofinitely supplemented modules in [3]. A submodule N of a module M is said to be *cofinite* if the factor module $\frac{M}{N}$ is finitely generated. M is called a *cofinitely (weak) supplemented* module if every cofinite submodule of M has a (weak) supplement in M (see [3, 5], respectively). Nevertheless, it is known by [3], Theorem 2.8 and [5], Theorem 2.11, an $R-$module M is cofinitely (weak) supplemented if and only if every maximal submodule of M has a (weak) supplement in M. Clearly, supplemented modules are cofinitely supplemented and weakly supplemented modules are cofinitely weak supplemented.

A module M is called *cofinitely* $\delta-$*supplemented*, if every submodule of M has a $\delta-$supplement in M. In [10], cofinitely $\delta-$supplemented modules are introduced as a generalization of cofinitely supplemented modules. On the other hand, some properties of these modules are given in [1, 2].

In this paper, we will call a module M is *cofinitely weak* $\delta-$*supplemented* (or briefly $\delta-$*CWS*-module) if every cofinite submodule of M has a weak $\delta-$supplement. We will introduce *cofinitely weak* $\delta-$*supplemented* modules and obtain some properties of these modules.

2. Cofinitely weak $\delta-$supplemented modules

Lemma 1. *Let M be a module and U be a cofinite (maximal) submodule of M. If V is a weak $\delta-$supplement of U in M, then U has a finitely generated (cyclic) weak $\delta-$supplement in M contained in V.*

Proof. If U is cofinite, then $\frac{M}{U} \cong \frac{V}{(V \cap U)}$ is finitely generated. Let $\frac{V}{(V \cap U)}$ be generated by elements $x_1 + V \cap U$, $x_2 + V \cap U$, ...$, x_n + V \cap U$ (for every $i = 1, 2, ...n$ $x_i \in V$). Then for the finitely generated submodule $W = Rx_1 + Rx_2 + ... + Rx_n$ of V, we have $W + U = W + V \cap U + U = V + U = M$ and $W \cap U \leq V \cap U \ll_\delta M$. Therefore W is a finitely generated weak $\delta-$supplement of U in M contained in V. If U is maximal, then $\frac{V}{(V \cap U)}$ is a cyclic module generated by some element $x + (V \cap U)$ and $W = Rx$ is a weak $\delta-$supplement of U. $\square$

Lemma 2. *Let M be a module. If, for every cofinite submodule U of M, there exists a submodule V of M such that $M = U + V$ and $U \cap V$ has a weak $\delta-$supplement in V, then M is a $\delta-$CWS-module.*

Proof. Let U be a cofinite submodule of M. By assumption, there is a submodule V in M such that $M = U + V$ and $U \cap V$ has a weak $\delta-$supplement X in V. Then $U \cap V + X = V$ and $(U \cap V) \cap X = U \cap X \ll_\delta V$. Note that $M = U + V = U + U \cap V + X = U + X$ and $U \cap X \ll_\delta M$. Hence X is a weak $\delta-$supplement of U in M. It follows that M is a $\delta-$CWS-module. □

Lemma 3. *Let M be a module and U be a cofinite submodule of M. If U has a weak $\delta-$supplement V in M and $\delta(K) = K \cap \delta(M)$ for every finitely generated submodule K of V, then U has a finitely generated $\delta-$supplement in M.*

Proof. V is a weak $\delta-$supplement of U in M, i.e. $U + V = M$ and $U \cap V \ll_\delta M$. Since $\frac{M}{U}$ is finitely generated, by Lemma 1 U has a finitely generated weak $\delta-$supplement $K \leq V$ in M, i.e. $M = U + K$ and $U \cap K \ll_\delta M$. Then $U \cap K \leq \delta(M)$. Therefore $U \cap K \leq K \cap \delta(M) = \delta(K)$ and so K is a $\delta-$supplement of U in M. □

Theorem 1. *Let M be a module such that for every finitely generated submodule K of M, $\delta(K) = K \cap \delta(M)$. Then M is cofinitely weak $\delta-$supplemented if and only if M is cofinitely $\delta-$supplemented.*

Proof. Let U be a cofinite submodule of M. Since M is a $\delta-\ CWS$-module, U has a weak $\delta-$supplement V in M and by Lemma 3, U has a $\delta-$supplement. Hence M is cofinitely $\delta-$supplemented.

The converse is obvious. □

Corollary 1. *Let M be a finitely generated module such that for every (finitely generated) submodule K of M, $\delta(K) = K \cap \delta(M)$. Then M is weakly $\delta-$supplemented if and only if M is $\delta-$supplemented.*

Proof. Follows from Theorem 1 as in a finitely generated module, every submodule is cofinite. □

Proposition 1. *A homomorphic image of a $\delta-$CWS-module is a $\delta-$CWS-module.*

Proof. Let $f : M \to N$ be a homomorphism and M be a $\delta - CWS$-module. Suppose that X is a cofinite submodule of $f(M)$. Then, we can easily get $\frac{M}{f^{-1}(X)} \cong \frac{(\frac{M}{Ker(f)})}{(\frac{f^{-1}(X)}{Ker(f)})} \cong \frac{f(M)}{X}$ which implies that $\frac{M}{f^{-1}(X)}$ is finitely generated. Since M is a $\delta - CWS$-module, $f^{-1}(X)$ has a weak $\delta-$supplement U in M, i.e. $f^{-1}(X) + U = M$ and $f^{-1}(X) \cap U \ll_\delta M$. So $f(f^{-1}(X) + f(U) = f(M)$ and since X is a submodule of $f(M)$, $f(f^{-1}(X)) = X$ and so $X + f(U) = f(M)$. Furthermore, $f(f^{-1}(X)) \cap f(U) \ll_\delta f(M)$ by Lemma 1.3(2) in [14]. Therefore $X \cap f(U) \ll_\delta f(M)$. □

Corollary 2. *Any factor module of a $\delta-$CWS-module is a $\delta-$CWS-module.*

To prove that an arbitrary sum of $\delta-CWS$-modules is a $\delta-CWS$-module, we use the following standard lemma.

Lemma 4. *Let M be a module, N and U be submodules of M with cofinitely weak $\delta-$suplemented N and cofinite U. If $N+U$ has a weak $\delta-$supplement in M, then U also has a weak $\delta-$supplement in M.*

Proof. Let X be a weak $\delta-$supplement of $N+U$ in module M. Then we have $\frac{N}{[N\cap(X+U)]} \cong \frac{N+(X+U)}{X+U} = \frac{M}{X+U} \cong \frac{\left(\frac{M}{U}\right)}{\left(\frac{(X+U)}{U}\right)}$. The last module is a finitely generated module. Hence $N\cap(X+U)$ has a weak $\delta-$supplement Y in N, i.e. $Y+[N\cap(X+U)]=N$ and $Y\cap[N\cap(X+U)]=Y\cap(X+U)\ll_\delta N\leq M$. Since $M=U+X+N=U+X+Y+[N\cap(X+U)]=X+U+Y$, Y is a weak $\delta-$supplement of $X+U$ in M. Therefore $U\cap(X+Y)\leq[X\cap(Y+U)]+[Y\cap(X+U)]\ll_\delta M$ by Lemma 1.3(1) of [14]. This means that $X+Y$ is a weak $\delta-$supplement of U in M. □

Proposition 2. *Any arbitrary sum of $\delta-$CWS-modules is a $\delta-$CWS-module.*

Proof. Let $M=\sum_{i\in I} M_i$ where each module M_i is a cofinitely weak $\delta-$supplemented and N be a cofinite submodule of M. Then $\frac{M}{N}$ is generated by some finite set $\{x_1+N, x_2+N, ..., x_n+N\}$ and therefore $M=Rx_1+Rx_2+...+Rx_n+N$. Since each x_i is contained in the sum $\sum_{j\in J} M_j$ for some finite subset $J=\{1_1,...,1_{s(1)},..,n_{s(n)}\}$ of I, $M=M_{1_1}+\sum_{j\in J-\{1_1\}} M_j+N$ has a trivial weak $\delta-$supplement 0 in M and since $M_{1_1}$is a $\delta-CWS$-module, $N+\sum_{j\in J} M_j$ has a weak $\delta-$supplement by Lemma 4. Continuing in this way, we will obtain (after we have used Lemma 4 $\sum_{i=1}^{n} s(i)$ times) N has a weak $\delta-$supplement in M. □

Let M and N be $R-$modules. If there is an epimorphism $f: M^{(\Lambda)}\longrightarrow N$ for some set Λ, then N is called an $M-$generated module. The following corollary follows from Corollary 2 and Proposition 2.

Corollary 3. *If M is a $\delta-$CWS-module, then any $M-$generated module is a $\delta-$CWS-module.*

Now we are going to prove that a module is cofinitely weak $\delta-$supplemented if and only if every maximal submodule has a weak $\delta-$supplement in M. Firstly we need the following lemma.

Lemma 5. *Let U, K be submodules of an $R-$module M. If K is a weak $\delta-$supplement of a maximal submodule N of M. If $K+U$ has a weak $\delta-$supplement in M, then U has a weak $\delta-$supplement in M.*

Proof. Let K be a weak $\delta-$supplement of a maximal submodule $N \leq M$, and X be a weak $\delta-$supplement of $K+U$ in M, i.e. $X+K+U=M$ and $X \cap (K+U) \ll_\delta M$. If $K \cap (X+U) \leq N$, then $(K+X) \cap U \leq [K \cap (X+U)] + [X \cap (K+U)] \ll_\delta M$. So, in this case $K+X$ is a weak $\delta-$supplement of U in M.

Now, suppose that $K \cap (X+U) \nleq N$, i.e. $K \cap (X+U) \nleq K \cap N$. Since $\frac{K}{(K \cap N)} \cong \left(\frac{(K+N)}{N}\right) = \frac{M}{N}$ and N is a maximal submodule of M, $K \cap N$ is a maximal submodule of K. Therefore $(K \cap N) + [K \cap (X+U)] = K$. Also, we get $M = U+K+X = U+(K \cap N)+[K \cap (X+U)]+X = U+(K \cap N)+X$ and $(U \cap [(K \cap N)+X]) \leq [(K \cap N) \cap (U+X)] + [((K \cap N)+U) \cap X] \leq (K \cap N) + [(K+U) \cap X] \ll_\delta M$ by Lemma 1.3(2) of [14]. So $((K \cap N)+X)$ is a weak $\delta-$supplement of U in M. Thus in both cases there is a weak $\delta-$supplement of U in M. □

For a module M, let E be the set of all submodules K such that K is a weak $\delta-$supplement for some maximal submodule of M and $CWS_\delta(M)$ denote the sum of all submodules from E.

Theorem 2. *Let M be a module. Then the following statements are equivalent:*

(i) *M is a $\delta-$CWS-module,*
(ii) *Every maximal submodule of M has a weak $\delta-$supplement,*
(iii) *$\frac{M}{CWS_\delta(M)}$ has no maximal submodules.*

Proof.
(i) $\Rightarrow$ (ii): Since every maximal submodule is cofinite, the proof is obvious.
(ii) $\Rightarrow$ (iii): Suppose that there is a maximal submodule of $\frac{N}{CWS_\delta(M)}$ of $\frac{M}{CWS_\delta(M)}$ and $CWS_\delta(M) \leq N$. Then N is a maximal submodule of M. By hypothesis, there is a weak $\delta-$supplement K of N. Then $K \in E$ and so $K \leq CWS_\delta(M) \leq N \leq M$. Hence $N = M$. This contradiction shows that $\frac{M}{CWS_\delta(M)}$ has no maximal submodules.
(iii) $\Rightarrow$ (i): Let U be a cofinite submodule of M. Since $\frac{\left(\frac{M}{U}\right)}{\left(\frac{(U+CWS_\delta(M))}{U}\right)} \cong \frac{M}{(U+CWS_\delta(M))}$, $U+CWS_\delta(M)$ is a cofinite submodule of M. If $\frac{M}{[U+CWS_\delta(M)]} \neq 0$ i.e. $U+CWS_\delta(M) \neq M$, then there is a maximal submodule $\frac{N}{[U+CWS_\delta(M)]}$ of the finitely generated $\frac{M}{[U+CWS_\delta(M)]}$. It follows that N is a maximal submodule of M and $\frac{N}{CWS_\delta(M)}$ is a maximal submodule of $\frac{M}{CWS_\delta(M)}$. This contradicts hypothesis. So $M = U+CWS_\delta(M)$. Now $\frac{M}{U}$ is finitely generated, say by elements $x_1+U, x_2+U, ..., x_m+U$, we have $M = Rx_1+Rx_2+...+Rx_m+U$.

Each element x_i $(i = 1, 2, .., m)$ can be written as $x_i = u_i + c_i$, where $u_i \in U, c_i \in CWS_\delta(M)$. Since each c_i is contained in the sum of finite number of submodules from E, $M = U + K_1 + K_2 + ... + K_n$ for some submodules $K_1, K_2, ..., K_n$ of M from E. Now $M = (U + K_1 + ... + K_{n-1}) + K_n$ has a weak $\delta-$supplement, namely 0. By Lemma 5, $U + K_1 + K_2 + ... + K_{n-1}$ has a weak $\delta-$supplement. Continuing in this way we obtain that U has a weak $\delta-$supplement in M. Hence M is a $\delta - CWS$-module. □

Proposition 3. *Let M be a module and $\frac{M}{\delta(M)}$ be a cofinitely weak $\delta-$supplemented. Then every cofinite submodule of $\frac{M}{\delta(M)}$ is a direct summand.*

Proof. Let $\frac{K}{\delta(M)}$ be a cofinite submodule of $\frac{M}{\delta(M)}$. By hypothesis, $\frac{K}{\delta(M)}$ has a weak $\delta-$supplement $\frac{L}{\delta(M)}$, i.e. $\left(\frac{K}{\delta(M)}\right) + \left(\frac{L}{\delta(M)}\right) = \frac{M}{\delta(M)}$ and $\left(\frac{K}{\delta(M)}\right) \cap \left(\frac{L}{\delta(M)}\right) \ll_\delta \frac{M}{\delta(M)}$. Since $\delta\left(\frac{M}{\delta(M)}\right) = 0$, $\left(\frac{K}{\delta(M)}\right) \cap \left(\frac{L}{\delta(M)}\right) = 0_{\frac{M}{\delta(M)}}$. Hence $\frac{K}{\delta(M)}$ is a direct summand. □

Theorem 3. *Let M be $\delta-$coatomic module. Then the following statements are equivalent:*

(i) *M is a $\delta-$CWS-module,*
(ii) *$\frac{M}{\delta(M)}$ is a $\delta-$CWS-module,*
(iii) *Every cofinite submodule of $\frac{M}{\delta(M)}$ is a direct summand,*
(iv) *Every maximal submodule of $\frac{M}{\delta(M)}$ is a direct summand,*
(v) *Every maximal submodule of M has a weak $\delta-$supplement.*

Proof.
(i)⇒(ii) By Corollary 2.
(ii)⇒(iii) By Proposition 3.
(iii)⇒(iv) Maximal submodules are cofinite so by the assumption they are direct summand.
(iv)⇒(v) If N is a maximal submodule of M, then $\frac{N}{\delta(M)}$ is a maximal submodule of $\frac{M}{\delta(M)}$. So there is a submodule $\frac{K}{\delta(M)}$ of $\frac{M}{\delta(M)}$ such that $\frac{M}{\delta(M)} = \left(\frac{K}{\delta(M)}\right) \oplus \left(\frac{N}{\delta(M)}\right)$. Therefore $K \cap N \leq \delta(M) \ll_\delta M$. Hence K is a weak $\delta-$supplement in M.

Let N be a maximal submodule of M which does not contain $\delta(M)$. In this case, we have $\delta(M) + N = M$. So $\delta(M)$ is a $\delta-$supplement of N in M.
(v)⇒(i) By Theorem 2 this proof holds for every module M. □

Theorem 4. *Let M be an $R-$module with $\delta(M) \ll_\delta M$ and $\frac{M}{\delta(M)}$ be a $\delta-$CWS-module. Then M is a $\delta-$CWS-module.*

Proof. Let U be a cofinite submodule of M. Then $\frac{M}{(U+\delta(M))} \cong \frac{\left(\frac{M}{U}\right)}{\left(\frac{(U+\delta(M))}{U}\right)}$ is finitely generated, i.e. $U+\delta(M)$ is cofinite. On the other hand

$$\frac{\left(\frac{M}{\delta(M)}\right)}{\left[\frac{(U+\delta(M))}{\delta(M)}\right]} \cong \frac{M}{(U+\delta(M))}$$

is finitely generated and so $\frac{(U+\delta(M))}{\delta(M)}$ is a cofinite submodule of $\frac{M}{\delta(M)}$. By assumption, there exists a submodule $\frac{V}{\delta(M)}$ of $\frac{M}{\delta(M)}$ such that $\left[\frac{(U+\delta(M))}{\delta(M)}\right]+\left(\frac{V}{\delta(M)}\right)=\frac{M}{\delta(M)}$ and $\left[\frac{(U+\delta(M))}{\delta(M)}\right]\cap\left(\frac{V}{\delta(M)}\right)=\frac{[(U\cap V)+\delta(M)]}{\delta(M)} \ll_\delta \frac{M}{\delta(M)}$. Now we get $M=U+\delta(M)+V=U+V$. Since $\delta\left(\frac{M}{\delta(M)}\right)=0_{\frac{M}{\delta(M)}}$, we obtain that $(U\cap V)+\delta(M)=\delta(M)$, that is $U\cap V\leq\delta(M)$ and since $\delta(M)\ll_\delta M$, $U\cap V$ is also $\delta-$small in M. Therefore M is a $\delta-CWS$-module. □

Let M and N be $R-$modules. We call an epimorphism $f:M\to N$ is a $\delta-$cover in case $Kerf \ll_\delta M$ [11].

Corollary 4. *A $\delta-$cover of a $\delta-$CWS-module is a $\delta-$CWS-module.*

Theorem 5. *Let $0\to L\to M\to N\to 0$ be a short exact sequence. If L and N are $\delta-$CWS-modules and L has a weak $\delta-$supplement in M, then M is a $\delta-$CWS-module.*

Proof. Without restriction of generality, we will assume that $L\leq M$. Let S be weak $\delta-$supplement of L in M, i.e. $L+S=M$ and $L\cap S\ll_\delta M$. Then we have, $\frac{M}{L\cap S}\cong\frac{L}{L\cap S}\oplus\frac{S}{L\cap S}$. $\frac{L}{L\cap S}$ is cofinitely weak $\delta-$supplemented as a factor module of L which is cofinitely weak $\delta-$supplemented. On the other hand, $\frac{S}{L\cap S}\cong\frac{M}{L}\cong N$ is cofinitely weak $\delta-$supplemented. Then $\frac{M}{L\cap S}$ is cofinitely weak $\delta-$supplemented as a sum of cofinitely weak $\delta-$ supplemented. If we take Theorem 4 into consideration, then M became a $\delta-CWS$-module since $f:M\to\frac{M}{L\cap S}$ is a $\delta-$cover. □

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PROXIMAL MONOTONICITY IN PARTIALLY ORDERED METRIC SPACES AND F-CYCLIC CONTRACTIONS

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1. Introduction and Preliminaries

Recently, D. Wardowski ([7]) provided a new class of contractive mappings as below.

Definition 1. Let $\mathcal{F}$ be the collection of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ which satisfy the following conditions:
$(F1)$ F is strictly increasing.
$(F2)$ For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\alpha_n \to 0 \Leftrightarrow F(\alpha_n) \to -\infty$.
$(F3)$ There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.
For a given metric space (X,d), a mapping $T : X \to X$ is said to be an F- contraction if there exists $\tau > 0$ such that

$$d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)), \quad \forall x,y \in X.$$

It is remarkable to note that every F- contraction self-mapping is continuous. The following result is an extension of *Banach contraction principe*.

Theorem 1 ([7])**.** *Let (X,d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.*

Another generalization of Banach contraction principe was established in [4] as follows.

Theorem 2. *Let A and B be two nonempty closed subsets of a complete metric space (X,d). Suppose that $T : A\cup B \to A\cup B$ is a cyclic mapping, that is $T(A) \subseteq B$ and $T(B) \subseteq A$, such that $d(Tx,Ty) \leq \alpha\, d(x,y)$ for some $\alpha \in (0,1)$and for all $x \in A, y \in B$. Then T has a unique fixed point in $A\cap B$.*

An interesting feature about the above observation is that the mapping T in Theorem 2 may not be continuous.

It might be interesting to ask if in Theorem 1.1 $A\cap B = \varnothing$, then what happens. It is clear that the fixed point equation $Tx = x$ has no fixed point. In this case, we have the following notion.

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Definition 2. Let A, B be two nonempty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic mapping. A point $p \in A \cup B$ is called a best proximity point of T if $d(p, Tp) = \operatorname{dist}(A, B)$, where $\operatorname{dist}(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$.

Indeed best proximity point theorems have been studied to find necessary conditions such that the minimization problem

$$\min_{x \in A \cup B} d(x, Tx),$$

has at least one solution, where T is a cyclic mapping defined on $A \cup B$.

Let A and B be two nonempty subsets of a metric space (X, d). We will use the following notations in our coming discussion.

$$d^*(x, y) = d(x, y) - \operatorname{dist}(A, B), \quad \forall (x, y) \in A \times B,$$

$$d(x, A) = \operatorname{dist}(\{x\}, A), \quad \forall x \in X,$$

$$\mathcal{P}_A(z) = \{x \in A : d(x, z) = d(z, A)\}, \quad \forall z \in X,$$

$$A_0 := \{x \in A : d(x, y) = \operatorname{dist}(A, B) \quad \text{for some} \quad y \in B\},$$

$$B_0 := \{y \in B : d(x, y) = \operatorname{dist}(A, B) \quad \text{for some} \quad x \in A\}.$$

Note that if (A, B) is a nonempty, weakly compact and convex pair of subsets of a Banach space X, then A_0 and B_0 are also nonempty, closed and convex subsets of X.

Definition 3. Let A and B be nonempty subsets of a metric space (X, d). We say that A is Chebyshev set w.r.t. B provided that $\mathcal{P}_A(x)$ is singleton for any $x \in B$.

For instance, if A and B are two nonempty, weakly compact and convex sets in a strictly convex Banach space X, then A is Chebyshev set w.r.t. B and B is also Chebyshev set w.r.t. A.

In [6] Suzuki et al. introduced a notion of property UC on metric spaces as follows.

Definition 4. ([6]) Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) is said to satisfy the property UC provided if $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_n d(x_n, y_n) = \operatorname{dist}(A, B)$ and $\lim_n d(z_n, y_n) = \operatorname{dist}(A, B)$, then $\lim_n d(x_n, z_n) = 0$.

Example 1.1.([2]) Let A and B be nonempty subsets of a uniformly convex Banach space X. Assume that A is convex. Then (A, B) satisfies the property UC.

After that a weaker notion of property UC was introduced in [3] as follows.

Definition 5. Let A and B be nonempty subsets of a metric space (X, d). The pair (A, B) is said to satisfies the property WUC if for any sequence $\{x_n\}$ in A such that for every $\varepsilon > 0$ there exists $y \in B$ satisfying that $d(x_n, y) \leq \operatorname{dist}(A, B) + \varepsilon$ for $n \geq N$, then it is the case that $\{x_n\}$ is convergent.

It was proved in [3] that if A and B are two nonempty subsets of a *complete pointwise uniformly convex geodesic metric space* (X,d) *with monotone modulus of convexity* such that A is convex, then (A,B) satisfies the property WUC (see Proposition 3.15 of [3]).

Here, we recall some notions in partially ordered metric spaces from [5]

Definition 6. Let $(X,\preceq)$ be a partially ordered set. A self mapping $T : X \to X$ is said to be monotone nondecreasing iff $T(x) \preceq T(y)$ whenever $x, y \in X, x \preceq y$.

Definition 7. Let $(X,\preceq)$ be a partially ordered set and d be a metric on X. A subset A of X is said to be a regular set provided that for any nondecreasing sequence $\{x_n\}$ in A so that $x_n \to x \in A$, then $x_n \preceq x$ for any $n \in \mathbb{N}$.

The following best proximity point theorem is the main result of [1].

Theorem 3. *(Theorem 3.5 of [1]) Let $(X,\preceq)$ be a partially ordered set and d be a metric on X. Let A, B be two nonempty subsets of X such that (A,B) satisfies the property* UC. *Suppose A is complete and is a regular set and let $T : A\cup B \to A\cup B$ be a cyclic mapping such that T and T^2 are nondecreasing on A. Moreover,*

$$d(T\acute{x}, T^2x) \leq \alpha d(\acute{x}, Tx) + (1-\alpha)\mathrm{dist}(A,B),$$

and

$$d(T\acute{y}, T^2y) \leq \alpha d(\acute{y}, Ty) + (1-\alpha)\mathrm{dist}(A,B),$$

for some $\alpha \in [0,1[$ and for all $(x,\acute{x}) \in A\times A, (y,\acute{y}) \in B\times B$ with $x \preceq \acute{x}, y \preceq \acute{y}$. If there exists $x_0 \in A$ such that $x_0 \preceq T^2x_0$ and $x_{n+1} = Tx_n$, then T has a best proximity point $p \in A$ and $x_{2n} \to p$.

In the current paper, we prove some existence and convergence results of best proximity points for a new class of mappings, called F- cyclic contractions in the setting on ordered metric spaces. Our conclusions extend and improve some recent results in the literature. We also provide some examples to illustrate our main results.

2. Main results

2.1. *Fixed point results*

In this section, we provide some new fixed point results in partially ordered metric spaces for cyclic mappings which satisfy the Wardowski type contraction.

We begin our main results with the following auxiliary lemma.

Lemma 1. *Let (A,B) be nonempty pair of subsets of a metric space (X,d) and "$\preceq$" be a partially ordered relation on A such that A is complete. Let $T : A\cup B \to A\cup B$ be a cyclic mapping such that T^2 is nondecreasing on A and there exists $\tau > 0$ such that*

$$d^*(T\acute{x}, T^2x) > 0 \Rightarrow \tau + F(d^*(T\acute{x}, T^2x) \leq F(d^*(x', Tx)),$$

for all $x, \acute{x} \in A$ *with* $x \preceq \acute{x}$. *If there exists* $x_0 \in A$ *with* $x_0 \preceq T^2 x_0$ *and if* $x_{n+1} = Tx_n$, *then* $d(x_n, x_{n+1}) \to \operatorname{dist}(A, B)$.

Proof. Since T^2 is nondecreasing on A and $x_0 \preceq T^2 x_0$,

$$x_0 \preceq T^2 x_0 \preceq \cdots \preceq T^{2n} x_0 \preceq \cdots.$$

Put $r_n = d^*(x_{n+1}, x_n)$. By the fact that $x_0 \preceq T^2 x_0$,

$$\begin{aligned} F(d^*(x_3, x_2)) &= F(d^*(T(T^2 x_0), T^2 x_0)) \\ &\le F(d^*(T^2 x_0, Tx_0)) - \tau = F(d^*(x_1, x_2)) - \tau. \end{aligned}$$

Also,

$$\begin{aligned} F(d^*(x_5, x_4)) &= F(d^*(T(T^4 x_0), T^2(T^2 x_0))) \\ &\le F(d^*(T^4 x_0, T^3 x_0)) - \tau = F(d^*(T(T^2 x_0), T^2(T^2 x_0))) - \tau \\ &\le \{F(d^*(T^2 x_0, T^3 x_0)) - \tau\} - \tau = F(d^*(x_3, x_2)) - 2\tau \\ &\le F(d^*(x_2, x_1)) - 3\tau. \end{aligned}$$

Continuing this process and by induction, we obtain

$$F(d^*(x_{2n+1}, x_{2n})) \le F(d^*(x_2, x_1)) - (2n-1)\tau, \quad \forall n \in \mathbb{N},$$

which deduces that $\lim_{n\to\infty} F(d^*(x_{2n+1}, x_{2n})) = -\infty$. From the condition (F2), $d^*(x_{2n+1}, x_{2n}) \to 0$ or $d(x_{2n+1}, x_{2n}) \to \operatorname{dist}(A, B)$. Similarly, we can see that

$$F(d^*(x_{2n+2}, x_{2n+1})) \le F(d^*(x_2, x_1)) - 2n\tau, \quad \forall n \in \mathbb{N},$$

which implies that $d(x_{2n+1}, x_{2n+2}) \to \operatorname{dist}(A, B)$. Hence, $d(x_n, x_{n+1}) \to \operatorname{dist}(A, B)$ and the result follows.

□

The following fixed point result is motivated by Theorem 2.

Theorem 4. *Let* (A, B) *be nonempty pair of subsets of a metric space* (X, d) *and "*$\preceq$*" be a partially ordered relation on* A. *Let* $T : A \cup B \to A \cup B$ *be a cyclic mapping such* $T|_A$ *is continuous and that* T^2 *is nondecreasing on* A *and there exists* $\tau > 0$ *such that*

$$d(T\acute{x}, T^2 x) > 0 \Rightarrow \tau + F(d(T\acute{x}, T^2 x) \le F(d(x', Tx)),$$

for all $x, \acute{x} \in A$ *with* $x \preceq \acute{x}$. *If there exists* $x_0 \in A$ *with* $x_0 \preceq T^2 x_0$ *and if* $x_{n+1} = Tx_n$, *then* $A \cap B$ *is nonempty and* T *has a fixed point in* $A \cap B$. *Moreover,* $\{x_{2n}\}$ *converges to the fixed point of* T.

Proof. We consider the following two cases:

Case 1. $T^2 x_0 = x_0$. Now if $d(x_0, Tx_0) > 0$, then

$$\begin{aligned} F(d(x_0, Tx_0)) &= F(d(T^2 x_0, T(T^2 x_0))) = F(d(T(T^2 x_0), T^2 x_0)) \\ &\le F(d(T^2 x_0, Tx_0)) - \tau = F(d(x_0, Tx_0)) - \tau < F(d(x_0, Tx_0)), \end{aligned}$$

which is impossible. That is, $x_0 = Tx_0$ and we are finished.

Case 2. $T^2x_0 \neq x_0$. Put $r_n = d(x_{n+1}, x_n)$. It now follows from a similar argument of Lemma 1 that

$$F(r_{2n}) \leq F(r_0) - 2n\tau, \quad F(r_{2n-1}) \leq F(r_1) - 2(n-1)\tau.$$

Hence, $\max\{r_{2n-1}, r_{2n}\} \to 0$. From (F3) there exist $k_1, k_2 \in (0,1)$ so that

$$\lim_{n\to\infty} r_{2n}^{k_1} F(r_{2n}) = 0, \quad \lim_{n\to\infty} r_{2n-1}^{k_2} F(r_{2n-1}) = 0.$$

We have

$$\begin{aligned} r_{2n}^{k_1} F(r_{2n}) - r_{2n}^{k_1} F(r_0) &\leq r_{2n}^{k_1}(F(r_0 - 2n\tau)) - r_{2n}^{k_1} F(r_0) \\ &= -r_{2n}^{k_1} 2n\tau \leq 0, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Thus $\lim_{n\to\infty} n r_{2n}^{k_1} = 0$. Equivalently, we can see that $\lim_{n\to\infty} n r_{2n-1}^{k_2} = 0$. Let $N \in \mathbb{N}$ be such that $\max\{n r_{2n}^{k_1}, n r_{2n-1}^{k_2}\} \leq 1$ for all $n \geq N$. Now for all $m > n \geq N$ we have

$$\begin{aligned} d(x_{2m}, x_{2n}) &\leq \sum_{j=n}^{m+1} [d(x_{2j}, x_{2j+1}) + d(x_{2j+1}, x_{2j+2})] \\ &= \sum_{j=n}^{m} (r_{2j} + r_{2j+1}) \leq \sum_{j=n}^{\infty} (r_{2j} + r_{2j+1}) \\ &\sum_{j=n}^{\infty} \left(\frac{1}{j^{\frac{1}{k_1}}} + \frac{1}{j^{\frac{1}{k_2}}}\right) < \sum_{j=1}^{\infty} \left(\frac{1}{j^{\frac{1}{k_1}}} + \frac{1}{j^{\frac{1}{k_2}}}\right) < \infty. \end{aligned}$$

Therefore, the sequence $\{x_{2n}\}$ is a Cauchy sequence and hence is convergent to a point $p \in A$. In view of the fact that T is continuous on A, $x_{2n+1} = Tx_{2n} \to Tp$ and so

$$d(p, Tp) = \lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = \lim_{n\to\infty} r_{2n} = 0,$$

that is, p is a fixed point of T in $A \cap B$.

□

Remark 1. The condition of continuity of the mapping T on A in Theorem 4 can be replaced with the condition of regularity of the set A.

Proof. As in the proof of Theorem 4, the nondecreasing sequence $\{x_{2n}\}$ converges to a point $p \in A$. Since A is regular, $x_{2n} \preceq p$ for all $n \in \mathbb{N}$. We now have

$$\begin{aligned} F(d(Tp, x_{2n})) &= F(d(Tp, T^2x_{2n-2})) \leq F(d(p, x_{2n-1})) - \tau \\ &\leq F(d(p, x_{2n}), d(x_{2n}, x_{2n-1})) - \tau \to -\infty \quad (n \to \infty), \end{aligned}$$

and this concludes that $\lim_{n\to\infty} d(Tp, x_{2n}) = 0$ by (F2) and so, $p = Tp$.

□

Next corollaries are the straightforward consequences of Theorem 4.

Corollary 1 (Theorems 2.2, 2.3 of [1]). *Let (A,B) be a nonempty pair of subsets of a metric space (X,d), such that A is complete, and let "$\preceq$" be a partially ordered relation on A. Let $T : A\cup B \to A\cup B$ be a cyclic mapping and T^2 be nondecreasing on A and $d(T\acute{x}, T^2x) \leq \alpha d(\acute{x}, Tx)$ for some $\alpha \in [0,1[$ and for all $(x,\acute{x}) \in A\times A$ with $x \preceq \acute{x}$. Assume that $T|_A$ is continuous or A is a regular set. If there exists $x_0 \in A$ with $x_0 \preceq T^2x_0$, then $A\cap B \neq \varnothing$, hence T has a fixed point $p \in A\cap B$. Moreover, if $x_{n+1} := Tx_n$, then $x_{2n} \to p$.*

Corollary 2 ([5]). *Let $(X,\preceq)$ be a partially ordered set and let there exists a metric d in X which makes (X,d) into a complete metric space. Suppose $T : X \to X$ is a nondecreasing mapping on X such that either T is continuous or X is a regular set. Assume there exists $\alpha \in [0,1[$ such that $d(Tx,Ty) \leq \alpha d(x,y)$ for each $y \preceq x$. If there exists $x_0 \in X$ with $x_0 \preceq T(x_0)$, then T has a fixed point.*

The following example illustrates Theorem 4.

Example 1. Let $(X,\preceq)$ be a partially ordered, where $X = [0,3]$ is complete metric space with metric $d : X\times X \to [0,\infty)$ defined as $d(x,y) = |x-y|$, $x,y \in X$, and $x \preceq y \Leftrightarrow x \leqslant y$, for each $x,y \in X$. Let $A = [0,1]\cup[2,3] \subset X$, and $B = (1,3] \subset X$. Of course $\preceq$ is a partially ordered relation on A. Let $T : A\cup B \to A\cup B$ be defined as follows

$$T(x) = \begin{cases} \frac{15}{8} & \text{if } x \in [0,1] \\ 2 & \text{if } x \in (1,3]. \end{cases} \tag{2.2}$$

Of course $T|_A$ is continuous, and iteration T^2 is nondecreasing on A.

Let $F : \mathbb{R}_+ \to \mathbb{R}$ be given by the formula $F(t) = \ln(t^2+t)$, where $t > 0$. It is easy to see, that the map F satisfies the conditions (F1)-(F3).

We show that T satisfies the condition (2.1). Let $\tau = \ln(2) > 0$. Let $x,\acute{x} \in A$ with $x \preceq \acute{x}$ (i.e. $x \leqslant \acute{x}$). We consider the following three cases:

Case 1. If $x,\acute{x} \in [0,1]$, then, by (2.2), we obtain $T(x) = T(\acute{x}) = \frac{15}{8}$, $T^2(x) = T^2(\acute{x}) = 2$. Hence $d(T(\acute{x}), T^2(x)) = d(\frac{15}{8}, 2) = \frac{1}{8} > 0$. Moreover, since $\acute{x} \leqslant 1$, by simply calculations, we have $\frac{15}{8} - \acute{x} \geqslant \frac{7}{8}$

$$[d(T(\acute{x}), T^2(x))]^2 + d(T(\acute{x}), T^2(x)) = \frac{1}{64} + \frac{1}{8} = \frac{9}{64}$$
$$\leqslant \frac{1}{2}[\frac{49}{64} + \frac{56}{64}] \leqslant \frac{1}{2}[(\frac{15}{8} - \acute{x})^2 + (\frac{15}{8} - \acute{x})] = \frac{1}{2}([d(\acute{x}, T(x))]^2 + d(\acute{x}, T(x)))$$
$$\leqslant ([d(\acute{x}, T(x))]^2 + d(\acute{x}, T(x)))e^{-\ln(2)}.$$

In consequence,

$$[d(T(\acute{x}), T^2(x))]^2 + d(T(\acute{x}), T^2(x)) \leqslant ([d(\acute{x}, T(x))]^2 + d(\acute{x}, T(x)))e^{-\ln(2)}.$$

Hence,

$$e^{\ln([d(T(\acute{x}),T^2(x))]^2+d(T(\acute{x}),T^2(x)))} \leqslant e^{\ln([d(\acute{x},T(x))]^2+d(\acute{x},T(x)))}e^{-\ln(2)}$$

and next

$$e^{F(d(T(\acute{x}),T^2(x)))} \leqslant e^{F(d(\acute{x},T(x)))}e^{-\ln(2)} = e^{F(d(\acute{x},T(x)))-\ln(2)}$$

Finally, we obtain

$$F(d(T(\acute{x}),T^2(x))) \leqslant F(d(\acute{x},T(x))) - \ln(2)$$

thus

$$\tau + F(d(T(\acute{x}),T^2(x))) \leqslant F(d(\acute{x},T(x)))$$

Case 2. If $x \in [0,1]$ and $\acute{x} \in [2,3]$, then, by (2.2), we obtain $T(x) = \frac{15}{8}$, $T(\acute{x}) = T^2(x) = T^2(\acute{x}) = 2$. Hence $d(T(\acute{x}),T^2(x)) = d(2,2) = 0$.

Case 3. If $x,\acute{x} \in [2,3]$, then, by (2.2), we obtain $T(x) = T(\acute{x}) = T^2(x) = T^2(\acute{x}) = 2$. Hence $d(T(\acute{x}),T^2(x)) = d(2,2) = 0$.

In consequence of considerations of each cases, we obtain that T satisfies the condition (2.1).

Of course, there exists $x_0 \in A$ (for example $x_0 = 1$) such that $x_0 \leqslant T^2(x_0)$. All assumption of Theorem 2.2 are satisfied. There exists a fixed point $w = 2$ of the map T in A. Moreover if we define the sequence $(x_n : n \in \mathbb{N})$ as $x_{n+1} = T(x_n)$, $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_{2n} = w$.

2.2. *Best proximity point results*

Motivated by Theorem 3, we establish an existence result of best proximity points for F- cyclic contractions under weaker assumptions.

We begin our main conclusions of this section with the following theorem.

Theorem 5. *Let (A,B) be nonempty pair of subsets of a metric space (X,d) and "$\preceq$" be a partially ordered relation on A such that A is complete. Let $T : A \cup B \to A \cup B$ be a cyclic mapping such that T^2 is nondecreasing on A and there exists $\tau > 0$ such that*

$$d^*(T\acute{x},T^2x) > 0 \Rightarrow \tau + F(d^*(T\acute{x},T^2x) \leq F(d^*(x',Tx)),$$

for all $x,\acute{x} \in A$ with $x \preceq \acute{x}$. Suppose there exists $x_0 \in A$ with $x_0 \preceq T^2x_0$ and define $x_{n+1} = Tx_n$. If A is a regular set and boundedly compact, then T has a best proximity point.

Proof. Since T^2 is nondecreasing, the sequence $\{x_{2n}\}$ is increasing and by the fact that A is boundedly compact, there exists a subsequence $\{x_{2n_k}\}$ of $\{x_{2n}\}$ converging to some $p \in A$. Thus

$$\operatorname{dist}(A,B) \leq d(p,x_{2n_k-1}) \leq d(p,x_{2n_k}) + d(x_{2n_k},x_{2n_k-1}).$$

Now if $k \to \infty$, then by Lemma 1 we have $d(p, x_{2n_k-1}) \to \operatorname{dist}(A, B)$. In view of the fact that A is a regular set, $x_{2n_k} \preceq p$ for all $k \in \mathbb{N}$. We have

$$F(d^*(x_{2n_k}, Tp)) = F(d^*(Tp, T^2 x_{2n_k-2}))$$

$$\leq F(d^*(p, Tx_{2n_k-2})) - \tau = F(d^*(p, x_{2n_k-1})) - \tau \to -\infty \quad (k \to \infty).$$

Therefore, $d^*(x_{2n_k}, Tp) \to 0$ and so $d(p, Tp) = \operatorname{dist}(A, B)$. □

Remark 2. It is worth noticing that if in Theorem 5 $\mathcal{P}_A(y)$ is singleton for any $y \in B_0$, then $T^2 p = p$, that is, p is a fixed point of T^2.

Proof. Since $d(p, Tp) = \operatorname{dist}(A, B)$, we have $\mathcal{P}_A(Tp) = \{p\}$. If $d^*(Tp, T^2 p) > 0$, then

$$F(d^*(Tp, T^2 p)) \leq F(d^*(p, Tp)) - \tau,$$

which is impossible by the fact that $F \in \mathcal{F}$ and so the domain of F is $(0, \infty)$. Thereby, $d(Tp, T^2 p) = \operatorname{dist}(A, B)$ and so $T^2 p \in \mathcal{P}_A(Tp)$ and this turns that $T^2 p = p$.

□

Definition 8. Let A and B be nonempty subsets of a partially metric space (X, d). The pair (A, B) is said to have monotone proximally property if for any increasing sequence $\{x_n\}$ in A such that for every $\varepsilon > 0$ there exist $y \in B$ and $N \in \mathbb{N}$ satisfying that $d(x_n, y) \leq \operatorname{dist}(A, B) + \varepsilon$ for $n \geq N$, then it is the case that $\{x_n\}$ has a convergent subsequence.

Next example shows that the monotone proximally property does not imply property WUC in general.

Example 2.2. Consider $X = \mathbb{R}$ with the usual metric and with the natural partially ordered relation $\leq$. Suppose $A = (-\infty, 0] \cup \{2\}$ and $B = \{1\}$. Then $\operatorname{dist}(A, B) = 1$ and $A_0 = \{0, 2\}$, $B_0 = \{1\}$. Let $\varepsilon > 0$ be given. We have the following two cases:

Case 1. Consider the nondecreasing sequence $\{x_n\}$ defined with $x_n = -\frac{1}{n}$. Then for $y = 1$ we have $\lim_{n\to\infty} d(x_n, y) = 1$ and so, there exists $N \in \mathbb{N}$ such that $d(x_n, y) \leq \operatorname{dist}(A, B) + \varepsilon$ for all $n \geq N$. We note that $x_n \to 0$.

Case 2. Let $\{x_n\}$ be a sequence in A for which $x_n = 2$ for all $n \in \mathbb{N}$ except perhaps finite numbers. In this case we have $x_n \to 2$ and that $\lim_{n\to\infty} d(x_n, y) = 1$.
Therefore, (A, B) has the monotone proximally property.
We now claim that (A, B) does not satisfy the property WUC. To this end, define the sequence $\{z_n\}$ in A as

$$z_n = \begin{cases} -\frac{1}{n} & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

The we have $\lim_{n\to\infty} d(z_n, y) = \text{dist}(A, B)$ but the sequence $\{z_n\}$ is not convergent.

Example 2.3. Suppose $X = \mathbb{R}^2$ and define the metric d on X by

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|, \quad \forall\ (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2.$$

Consider the partially ordered relation on X with $(x_1, y_1) \preceq (x_2, y_2) \Leftrightarrow x_1 \le x_2,\ y_1 \le y_2$. We define

$$A = \{(0, 1 - \frac{1}{2n}) : n \in \mathbb{N}\} \cup \{(0, n) : n \in \mathbb{N}\}, \quad B = \{(2, y) : 0 \le y \le 1\}.$$

By simply calculations, we obtain $\text{dist}(A, B) = 2$. By definition of A, we see that there exists only one increasing sequence $\{x_n\}$ in A such that for every $\varepsilon > 0$ there exist $y \in B$ and $N \in \mathbb{N}$ satisfying that $d(x_n, y) \le \text{dist}(A, B) + \varepsilon$ for $n \ge N$. The above assumption satisfies the sequence $\{x_n = (0, 1 - \frac{1}{2n}) : n \in \mathbb{N}\}$ for $y = (2, 1)$. Then, of course $\{x_n\}$ has a convergent subsequence (for example $\{x_k = (0, 1 - \frac{1}{4k}) : k \in \mathbb{N}\}$).

The following theorem is the main result of this section.

Theorem 6. *Let (A, B) be nonempty pair of subsets of a metric space (X, d) and "$\preceq$" be a partially ordered relation on A such that A is complete. Let $T : A \cup B \to A \cup B$ be a cyclic mapping such that T^2 is nondecreasing on A and there exists $\tau > 0$ such that*

$$d^*(T\acute{x}, T^2x) > 0 \Rightarrow \tau + F(d^*(T\acute{x}, T^2x) \le F(d^*(x', Tx)),$$

for all $x, \acute{x} \in A$ with $x \preceq \acute{x}$. Suppose there exists $x_0 \in A$ with $x_0 \preceq T^2x_0$ and define $x_{n+1} = Tx_n$. If A is bounded and a regular set and (A, B) has the monotone proximally property, then T has a best proximity point.

Proof. From Lemma 1 $d(x_n, x_{n+1}) \to \text{dist}(A, B)$. Since A is bounded, the sequence $\{x_{2n}\}$ and so the sequence $\{x_{2n-1}\}$ are also bounded. Consider $n \in \mathbb{N}$. Then for all $k \in \mathbb{N}$ by the fact that the sequences $\{x_{2n-1}\}$ and $\{x_{2n}\}$ are nondecreasing and bounded, we have

$$\begin{aligned}
&F(d^*(x_{2n+2k}, x_{2n+1})) \\
&= F(d^*(T^{2n+2k}x_0, T^{2n+1}x_0)) = F(d^*(T(x_{2n+2k-1}), T^2(x_{2n-1}))) \\
&\le F(d^*(x_{2n+2k-1}, x_{2n})) - \tau \quad (\text{since } x_{2n-1} \preceq x_{2n+2k-1}) \\
&= F(d^*(Tx_{2n+2k-2}, T^2x_{2n-2})) - \tau \\
&\le F(d^*(x_{2n+2k-2}, x_{2n-1})) - 2\tau \quad (\text{since } x_{2n-2} \preceq x_{2n+2k-2}) \\
&\le \ldots \le F(d^*(x_{2k}, x_1)) - 2n\tau \le F(M) - 2n\tau,
\end{aligned}$$

where $M := \sup\{d^*(x_{2k}, x_1) : k \in \mathbb{N}\}$. Thus for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d^*(x_{2n_0+2k}, x_{2n_0+1}) \le \varepsilon$ which implies that

$$x_{2n_0+2k} \in \mathcal{B}(x_{2n_0+1}, \varepsilon + \text{dist}(A, B)), \quad \forall k \in \mathbb{N}.$$

By the fact that (A,B) has the monotone proximally property, we conclude that the sequence $\{x_{2n}\}$ has a convergent subsequence to a point $p \in A$. It now follows from an equivalent argument of Theorem 5 that p is a best proximity point of T and the result follows.

□

Let us illustrate Theorem 6 with the following example.

Example 2.4. Suppose $X = \mathbb{R}^2$ and define the metric d on X by

$$d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|, \quad \forall\ (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2.$$

We know that X is not strictly convex. Consider the partially ordered relation on X with

$$(x_1,y_1) \preceq (x_2,y_2) \Leftrightarrow x_1 \le x_2,\ y_1 \le y_2.$$

Let

$$A = \{(0, 1 - \frac{1}{2n}) : n \in \mathbb{N}\} \cup \{(0,n) : n \in \mathbb{N}\}, \quad B = \{(2,y) : 0 \le y \le 1\}.$$

Then $\operatorname{dist}(A,B) = 2$ and $A_0 = \{(0, 1-\frac{1}{2n}) : n \in \mathbb{N}\} \cup \{(0,n) : n \in \{1,2,3\}\}$, $B_0 = B$. Also, it is easy to see that (A,B) has the monotone proximally property. Define the cyclic mapping $T : A \cup B \to A \cup B$ by formula

$$T(x) = \begin{cases} (2,1) & \text{for } x \in W := \{(x_1,y_1) \in X : x_1 = 0 \wedge y_1 \in \{1,2\} \cup \{1 - \frac{1}{2n} : n \in \mathbb{N}\}\} \\ (2,0) & \text{for } x \in V := \{(x_1,y_1) \in X : x_1 = 0 \wedge y_1 \in \{n \in \mathbb{N} : n > 2\}\} \\ (0,1) & \text{for } x = (x_1,y_1) = (2,k),\ k \in [0,1] \end{cases}$$

where $x \in A \cup B$. Then, we calculate

$$T^2(x) = \begin{cases} (0,1) & \text{for } x \in W \cup V \\ (2,1) & \text{for } x = (x_1,y_1) = (2,k),\ k \in [0,1] \end{cases}, x \in A \cup B.$$

It is easy to observe that the map T is continuous on A, and T^2 is nondecreasing on A.

Next, we show that, for $F(t) = \ln(t)$, $t > 0$, there exists $\tau > 0$ such that

$$d^*(T\acute{x}, T^2x) > 0 \Rightarrow \tau + F(d^*(T\acute{x}, T^2x) \le F(d^*(x', Tx)),$$

for all $x, \acute{x} \in A$ with $x \preceq \acute{x}$.

Indeed, let $x, \acute{x} \in A$ with $x \preceq \acute{x}$ and $\tau = \ln(2) > 0$. Since $x \preceq \acute{x}$, we may consider three cases:

Case 1. If $x, \acute{x} \in W$, then we have the following: $T(x) = (2,1)$, $T^2(x) = (0,1)$, $T(\acute{x}) = (2,1)$, $T^2(\acute{x}) = (0,1)$, thus $d(T(\acute{x}), T^2(x)) = d((2,1),(0,1)) = 2$, and $d^*(T(\acute{x}), T^2(x)) = d(T(\acute{x}), T^2(x)) - dist(A,B) = 0$. In consequence the contraction conditions is trivial satisfies.

Case 2. If $x, \acute{x} \in V$, then we have the following: $T(x) = (2,0)$, $T^2(x) = (0,1)$, $T(\acute{x}) = (2,0)$, $T^2(\acute{x}) = (0,1)$, thus $d(T(\acute{x}), T^2(x)) = d((2,0),(0,1)) = 3$, and $d^*(T(\acute{x}), T^2(x))$
$= d(T(\acute{x}), T^2(x)) - dist(A,B) = 1 > 0$. Moreover, $d(\acute{x}, T(x)) = d((0,m),(2,0)) = m+2$, and $d^*(\acute{x}, T(x)) = d(\acute{x}, T(x)) - dist(A,B) = m$, for some $m > 2$, such that $\acute{x} = (0,m) \in V$. On the other hand, we have $m \geq 3$, so $\frac{1}{m} \leq \frac{1}{3} < \frac{1}{2}$. In consequence, since $\frac{1}{2} = e^{-\ln(2)} = e^{-\tau}$, we obtain

$$d^*(T(\acute{x}), T^2(x)) = 1 \leq me^{-\tau} = d^*(\acute{x}, T(x))e^{-\tau}.$$

Therefore

$$e^{\tau} d^*(T(\acute{x}), T^2(x)) \leq d^*(\acute{x}, T(x)),$$

and next

$$e^{\tau} e^{\ln(d^*(T(\acute{x}), T^2(x)))} \leq e^{\ln(d^*(\acute{x}, T(x)))}.$$

Finally, we obtain

$$\tau + F(d^*(T(\acute{x}), T^2(x))) \leq F(d^*(\acute{x}, T(x))).$$

Hence, the contraction conditions is satisfies in this case.

Case 3. If $x \in W$ and $\acute{x} \in V$, then we have the following: $T(x) = (2,1)$, $T^2(x) = (0,1)$, $T(\acute{x}) = (2,0)$, $T^2(\acute{x}) = (0,1)$, thus $d(T(\acute{x}), T^2(x)) = d((2,0),(0,1)) = 3$, and $d^*(T(\acute{x}), T^2(x)) = d(T(\acute{x}), T^2(x)) - dist(A,B) = 1 > 0$. Moreover, $d(\acute{x}, T(x)) = d((0,m),(2,1)) = m+1$, and $d^*(\acute{x}, T(x)) = d(\acute{x}, T(x)) - dist(A,B)$ $= m-1$, for some $m > 2$, such that $\acute{x} = (0,m) \in V$. On the other hand, we have $m \geq 3$, so $m-1 \geq 2$ and next $1 \leq \frac{1}{2}(m-1)$. In consequence, since $\frac{1}{2} = e^{-\tau}$, we obtain

$$d^*(T(\acute{x}), T^2(x)) = 1 \leq (m-1)e^{-\tau} = d^*(\acute{x}, T(x))e^{-\tau}.$$

Therefore

$$e^{\tau} d^*(T(\acute{x}), T^2(x)) \leq d^*(\acute{x}, T(x)),$$

and next

$$e^{\tau} e^{\ln(d^*(T(\acute{x}), T^2(x)))} \leq e^{\ln(d^*(\acute{x}, T(x)))}.$$

Finally, we obtain

$$\tau + F(d^*(T(\acute{x}), T^2(x))) \leq F(d^*(\acute{x}, T(x))).$$

Hence, the contraction conditions is satisfies in this case.

In consequence, for the map T there exists $\tau > 0$ such that

$$d^*(T\acute{x}, T^2x) > 0 \Rightarrow \tau + F(d^*(T\acute{x}, T^2x) \leq F(d^*(x', Tx)),$$

for all $x, \acute{x} \in A$ with $x \preceq \acute{x}$. We observe that exists $x_0 \in A$ with $x_0 \preceq T^2x_0$ (for example $x_0 = (0, \frac{1}{2})$ and then $T^2x_0 = (0,1)$). Ofcourse, the set A is bounded and a regular set and (A,B) has the monotone proximally property (see Example 3.2). Finally, the point $(0,1)$ is the best proximity point of T (we recall $T(0,1) = (2,1)$,

so $d((0,1), T(0,1)) = 2 = \operatorname{dist}(A, B))$.

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DEGREE SUM CONDITION FOR FRACTIONAL ID-k-FACTOR-CRITICAL GRAPHS

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Abstract. A graph G is called a fractional ID-k-factor-critical graph if after deleting any independent set of G the resulting graph admits a fractional k-factor. In this paper, we prove that for $k \geq 2$, G is a fractional ID-k-factor-critical graph if $\delta(G) \geq \frac{n}{3} + k$, $\sigma_2(G) \geq \frac{4n}{3}$, $n \geq 6k - 8$. The result is best possible in some sense.

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Keywords: graph, fractional factor, fractional ID-k-factor-critical graph, degree sum

1. Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, the degree and the neighborhood of x in G are denoted by $d_G(x)$ and $N_G(x)$, respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. We use $N_G[x]$ to denote $N_G(x) \cup \{x\}$. We denote the minimum degree and the maximum degree of G by $\delta(G)$ and $\Delta(G)$, respectively.

Let $k \geq 1$ be an integer. A spanning subgraph F of G is called a *k-factor* if $d_F(x) = k$ for each $x \in V(G)$. Let $h : E(G) \to [0, 1]$ be a function. If $\sum_{x \in e} h(e) = k$ for any $x \in V(G)$, then we call $G[F_h]$ a *fractional k-factor* of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. The following result on degree condition for fractional k-factor is known.

Theorem 1 (Yu et al. [10]). *Let k be an integer with $k \geq 2$, and let G be a graph of order n with $n \geq 4k - 3$, $\delta(G) \geq k$. If*

$$\max\{d_G(u), d_G(v)\} \geq \frac{n}{2}$$

for each pair of non-adjacent vertices u and v of G, then G has a fractional k-factor.

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In what follows, we always assume that n is order of G, i.e., $n = |V(G)|$, and G is not complete. Chang et al. [1] introduced the concept of fractional independent-set-deletable k-factor critical (shortly, ID-k-factor critical) graph, that is, if removing any independent I from G, the resulting graph has a fractional k-factor. Also, Chang et al. [1] proved that if $n \geq 6k-8$ and $\delta(G) \geq \frac{2n}{3}$, then G is fractional ID-k-factor-critical. More results on fractional ID-k-factor-critical graphs can be found in Gao and Wang [2–6] and Jin [8].

In this paper, we focus on the degree sum condition for fractional ID-k-factor-critical graph. Let $\sigma_2(G) = \min\{d_G(u) + d_G(v)\}$ for each pair of non-adjacent vertices u and v of G. Niessen [9] researched the degree sum condition for a graph which exists regular factor. Iida and Nishimura [7] studied the existence of factor by virtue of $\sigma_2(G)$, and proved that if $n \geq 4k-5$, kn is even, $\delta(G) \geq k$, and $\sigma_2(G) \geq n$, then G has a k-factor. The main result in our paper study the degree sum condition for fractional ID-k-factor-critical graphs and give as follows:

Theorem 2. *Let $k \geq 2$ be an integer, and let G be a graph of order n with $n \geq 6k-8$. If $\delta(G) \geq \frac{n}{3} + k$ and $\sigma_2(G) \geq \frac{4n}{3}$, then G is a fractional ID-k-factor-critical graph.*

Also, we will show that Theorem 2 is sharp in some sense.

In order to prove our main result, we need the following lemma which is the necessary and sufficient condition for the existence of a fractional k-factor in a graph.

Lemma 1 (L. Zhang and G. Liu [11])**.** *Let $k \geq 1$ be an integer, and let G be a graph. Then G has a fractional k-factor if and only if for every subset S of $V(G)$,*

$$\delta_G(S,T) = k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| \geq 0$$

where $T = \{x : x \in V(G) - S, d_{G-S}(x) \leq k-1\}$.

2. Proof of Theorem 2

Suppose that G satisfies the conditions of Theorem 2, but is not a fractional ID-k-factor-critical graph. Then there exist an independent set I such that $G' = G - I$ has no fractional k-factor. By the argument of Lemma 1, there exists a subset S of $V(G')$ such that

$$\delta_{G'}(S,T) = k|S| + \sum_{x \in T} d_{G'-S}(x) - k|T| \leq -1. \tag{2.1}$$

Here, $T = \{x : x \in V(G') - S, d_{G'-S}(x) \leq k-1\}$.

If G' is a completed graph, then G' has fractional k-factor from the degree sum condition, the bound of n and the definition of fractional k-factor. This is a contradiction.

If $|I| = 1$, then $n' \geq 6k-9$. It is easy to verify that $\delta(G') \geq k$ and $\max\{d_{G'}(u), d_{G'}(v)\} \geq \frac{n'}{2} = \frac{n-1}{2}$ for each pair of non-adjacent vertices u and v of G'. Thus, the results holds from Theorem 1.

We now consider $|I| \geq 2$ and G' is not complete. Obviously, $T \neq \varnothing$ and $S \neq \varnothing$ by $|I| \geq 2$ and $\delta(G) \geq \frac{n}{3}+k$. Let $d_1 = \min\{d_{G'-S}(x) : x \in T\}$ and choose $x_1 \in T$ such that $d_{G'-S}(x_1) = d_1$. If $T - N_T[x_1] \neq \varnothing$, let $d_2 = \min\{d_{G'-S}(x) : x \in T - N_T[x_1]\}$ and choose $x_2 \in T - N_T[x_1]$ such that $d_{G'-S}(x_2) = d_2$. So, $d_1 \leq d_2$. Let $|S| = s$, $|T| = t$, $|N_T[x_1]| = p$. Then we have $p \leq d_1 + 1$, $d_{G'-S}(T) \geq d_1 p + d_2(t-p)$ and $ks - kt + d_1 p + d_2(t-p) \leq k|S| - k|T| + d_{G'-S}(T) < 0$. Thus,

$$|S| \leq \frac{k|T| - d_{G'-S}(T) - 1}{k} \leq \frac{k|T|-1}{k},$$

i.e., $1 \leq s \leq t - \frac{1}{k}$.

Let $|V(G')| = n'$. We obtain $2n' \geq \sigma_2(G) \geq \frac{4n}{3} \geq \frac{4}{3}(6k-8)$. Since n' is an integer, we get $n' \geq 4k-5$. If $\sigma_2(G') < n'$, then $\frac{4(n'+|I|)}{3} \leq \sigma_2(G) < n' + 2|I|$, i.e., $n' < 2|I| \leq \frac{2n}{3}$. This contradicts to $\sigma_2(G) \geq \frac{4n}{3}$ and $|I| \geq 2$. Therefore, $\sigma_2(G') \geq n'$. Furthermore, we obtain $\delta(G') \geq k$ by $|I| \geq 2$ and $\delta(G) \geq \frac{n}{3}+k$.

We consider following two cases:

Case 1. $T = N_T[x_1]$**.** In this case, $t = p \leq d_1 + 1$ and $d_2 = 0$. If $d_1 = k-1$, then $t \leq k$, $k|S| - k|T| + d_{G'-S}(T) \geq ks - kt + d_1 p = ks - kt + (k-1)t \geq ks - t \geq 0$, which contradicts (2.1). If $0 \leq d_1 \leq k-2$, then $t \leq d_1 + 1 \leq k-1$. By $\delta(G') \geq k$ and $d_{G'}(x_1) \leq s + d_1$, we have $s \geq k - d_1$. Thus, $k|S| - k|T| + d_{G'-S}(T) \geq ks - kt + d_1 p \geq k(k-d_1) + (d_1 - k)t = (k-d_1)(k-t) > 0$, which contradicts (2.1).

Case 2. $T - N_T[x_1] \neq \varnothing$**.** We consider following three subcases.

Case 2.1. $d_1 = d_2 = k-1$**.** In this subcase, $k|S| - k|T| + d_{G'-S}(T) \geq ks - kt + d_1 p + d_2(t-p) = ks - kt + (k-1)p + (k-1)(t-p) = ks - t \geq 0$, which contradicts (2.1). In fact, if $ks \leq t-1$, then $s + ks + 1 \leq s + t \leq n'$. Note that x_1, x_2 are not adjacent in G'. Thus, $2(s+k-1) \geq \sigma_2(G') \geq n' \geq s + sk + 1$. We get $s = 1$. Thus, $2(1+k-1) \geq 2(s+k-1) \geq \sigma_2(G') \geq n' \geq 4k-5$, i.e., $k = 2$. In this case, $d_1 = d_2 = 1$, $s = 1$, $t \geq 3$, $n' \geq 4$. We have $4 \leq n' \leq \sigma_2(G') \leq 2s + 2 = 4$, i.e., $t = 3$, $n' = 4$. Thus, the vertex $T - \{x_1, x_2\}$ has degree 2 in T, and we can check that $k|S| - k|T| + d_{G'-S}(T) = 0$. This is a contradiction.

Case 2.2. $0 \leq d_1 \leq k-2$ and $d_2 = k-1$. In this subcase, $p \leq d_1 + 1 \leq k-1$. Since x_1 and x_2 are not adjacent in G', we have $(s+k-1) + (s+d_1) \geq \sigma_2(G') \geq n' \geq 4k-5$, i.e., $n' \leq 2s + k - 1 + d_1$ and $s \geq \frac{3k-d_1-4}{2}$. Thus,

$$\begin{aligned}
&k|S| - k|T| + d_{G'-S}(T) \\
&\geq ks - kt + d_1 p + d_2(t-p) \\
&\geq ks - k(n'-s) + (d_1 - k + 1)(d_1 + 1) + (k-1)(n'-s) \\
&= (k+1)s - n' - k + 1 + d_1^2 + (2-k)d_1
\end{aligned}$$

$$\begin{aligned}
&\geq (k+1)s-(2s+k-1+d_1)-k+1+d_1^2+(2-k)d_1\\
&=(k-1)s-2k+2+d_1^2+(1-k)d_1\\
&\geq (k-1)\frac{3k-d_1-4}{2}-2k+2+d_1^2+(1-k)d_1\\
&=d_1^2+\frac{3}{2}(1-k)d_1+(k-1)\frac{3k-4}{2}-2k+2.
\end{aligned}$$

If $k\geq 5$, then $\frac{3}{4}(k-1)\leq k-2$ and d_1 can reach to $\frac{3}{4}(k-1)$. We get

$$\begin{aligned}
&d_1^2+\frac{3}{2}(1-k)d_1+(k-1)\frac{3k-4}{2}-2k+2\\
&\geq \frac{9}{16}(k-1)^2-\frac{9}{8}(k-1)^2+(k-1)\frac{3k-4}{2}-2k+2\\
&=\frac{15}{16}k^2-\frac{35}{8}k+\frac{55}{16}\\
&\geq \frac{15}{16}k^2-\frac{35}{8}k+\frac{55}{16}>0,
\end{aligned}$$

which contradicts (2.1).

If $k=2,3,4$, then

$$\begin{aligned}
&d_1^2+\frac{3}{2}(1-k)d_1+(k-1)\frac{3k-4}{2}-2k+2\\
&\geq (k-2)^2+\frac{3}{2}(1-k)(k-2)+(k-1)\frac{3k-4}{2}-2k+2\\
&=k^2-5k+5.
\end{aligned}$$

If $k=4$, then $k^2-5k+5\geq 0$, which contradicts (2.1).

If $k=3$, then $d_2=2$, $d_1=0$ or 1. If $d_1=0$, then $s\geq \frac{n'}{2}-1$ and $t\leq \frac{n'}{2}+1$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(\frac{n'}{2}-1)-k(\frac{n'}{2}+1)+2(\frac{n'}{2}+1-1)\geq 2k-5>0$, which contradicts (2.1). Assume $d_1=1$. If $n'\geq 8=4k-4$, then we get contradiction similarly as what we discuss above. If $n'=7$, then $n\leq 10$ since $n'\geq \frac{2n}{3}$. And, if $s\geq 3$, we obtain $k|S|-k|T|+d_{G'-S}(T)\geq 0$. The last situation is $k=3$, $n'=7$, $s=2$. Thus, $\sigma_2(G)\leq 13$ which contradicts $\sigma_2(G)\geq \frac{4n}{3}$.

Assume $k=2$. Then $d_1=0$ and $d_2=1$. If $G'-S-T\neq\varnothing$, then $t\leq n'-s-1$ and $k|S|-k|T|+d_{G'-S}(T)\geq 2s-2(n'-s-1)+(n'-s-p-1)\geq 3s-n'-p+1\geq 3s-n\geq 3s-(2s+1)=s-1\geq 0$, which contradicts (2.1). Suppose $G'-S-T=\varnothing$. If $n'\geq 4k-3=5$, then $s\geq \frac{n'-1}{2}$ and $t\leq \frac{n'+1}{2}$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k\frac{n'-1}{2}-k\frac{n'+1}{2}+(\frac{n'+1}{2}-1)\geq 0$, which contradicts (2.1). If $n'=4k-4=4$, then $s\geq 2$ and $t\leq 2$ by s is an integer. Thus, $k|S|-k|T|+d_{G'-S}(T)>0$, which contradicts (2.1). If $n=3=4k-5$, then $s\geq 1$ and $t\leq 2$. If $t\leq 1$, then $s\geq 2$ and we have $k|S|-k|T|+d_{G'-S}(T)\geq 0$, which contradicts (2.1). The last case is $s=1$ and $t=2$. Then at least one vertex in T is of degree at least 2 in $G'-S$. Thus, $k|S|-$

$k|T|+d_{G'-S}(T)\geq ks-kt+d_1p+d_2(t-p)\geq k-2k+(2-1)+1=0$, which contradicts (2.1).

Case 2.3. $0\leq d_1\leq d_2\leq k-2$**.** In this subcase, $k-1-d_2\geq 1$ and $n'-s-t\geq 0$. So, $(k-1-d_2)(n'-s-t)>ks-kt+d_1p+d_2(t-p)$. Thus, $(k-d_2)(n'-s)-ks>(d_1-d_2)p+(n'-s-t)\geq(d_1-d_2)(d_1+1)+(n'-s-t)$, i.e.,

$$(k-d_2)(n'-s)-ks\geq(d_1-d_2)(d_1+1)+1. \tag{2.2}$$

In terms of $n'\geq 4k-5$, we obtain

$$d_2\frac{n'}{2}\geq d_2(2k-\frac{5}{2}). \tag{2.3}$$

In view of $s\geq\frac{n'-d_1-d_2}{2}$, we have

$$(s-\frac{n'}{2})(2k-d_2)\geq-\frac{d_1+d_2}{2}(2k-d_2). \tag{2.4}$$

Adding (2.2), (2.3) and (2.4), we get

$$\begin{aligned}0&\geq d_1^2+\frac{d_2^2}{2}-\frac{d_1d_2}{2}+d_1-\frac{7}{2}d_2+1+(d_2-d_1)k\\&\geq d_1^2+\frac{d_2^2}{2}-\frac{d_1d_2}{2}+d_1-\frac{7}{2}d_2+1+(d_2-d_1)(d_2+2)\\&=d_1^2+\frac{3}{2}d_2^2-\frac{3}{2}d_1d_2-\frac{3}{2}d_2-d_1+1.\end{aligned}$$

Equivalent to

$$(d_1-(\frac{3}{4}d_2+\frac{1}{2}))^2+(\frac{\sqrt{15}}{4}d_2-\frac{9}{2\sqrt{15}})^2-\frac{3}{5}\leq 0.$$

We have

$$0\leq d_1\leq d_2\leq 2$$

by $(\frac{\sqrt{15}}{4}d_2-\frac{9}{2\sqrt{15}})^2-\frac{3}{5}\leq 0$.

- If $d_1=d_2=2$. In this case, if $n'\geq 4k-4$, then $s\geq\frac{n'}{2}-2$ and $t\leq n'-s\leq\frac{n'}{2}+2$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(\frac{n'}{2}-2)-k(\frac{n'}{2}+2)+2(\frac{n'}{2}+2)\geq 0$, which contradicts (2.1). If $n'=4k-5$, then $s\geq 2k-4$ and $t\leq n'-s\leq 2k-1$ since s is an integer. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(2k-4)-k(2k-1)+2(2k-1)=k-2\geq 0$, which contradicts (2.1).
- If $d_1=1$ and $d_2=2$. In this case, if $n'\geq 4k-3$, then $s\geq\frac{n'-3}{2}$ and $t\leq n'-s\leq\frac{n'+3}{2}$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k\frac{n'-3}{2}-k\frac{n'+3}{2}+2+2\frac{n'-1}{2}\geq k-2\geq 0$, which contradicts (2.1). If $n'=4k-4$, then $s\geq 2k-3$ and $t\leq n'-s\leq 2k-1$ since s is an integer. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(2k-3)-k(2k-1)+2+2(2k-1-2)=2k-4\geq 0$, which contradicts (2.1). If $n'=4k-5$, then $s\geq 2k-4$ and $t\leq n'-s\leq 2k-1$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(2k-4)-k(2k-1)+2+$

$2(2k-3)=k-4\geq 0$ if $k\geq 4$, which contradicts (2.1). If $k=2$, then $n'=4k-5=3$. In terms of $n'\geq \frac{2}{3}n$, we get n=4, which contradicts $|I|\geq 2$. In particular, for $k=3$. If $n'\geq 8=4k-4$, then we get $k|S|-k|T|+d_{G'-S}(T)\geq 0$. If $n'=7$, then $n\leq 10$ since $n'\geq \frac{2n}{3}$. And, if $s\geq 3$, we get $k|S|-k|T|+d_{G'-S}(T)\geq 0$. The last situation is $k=3$, $n'=7$, $s=2$. Thus, $\sigma_2(G)\leq 13$ which which contradicts $\sigma_2(G)\geq \frac{4n}{3}$.

- If $d_1=0$ and $d_2=2$. In this case, if $n'\geq 4k-4$, then $s\geq \frac{n'}{2}-1$ and $t\leq n-s\leq \frac{n'}{2}+1$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(\frac{n'}{2}-1)-k(\frac{n'}{2}+1)+2(\frac{n'}{2}+1-1)=n'-2k\geq 2k-4\geq 0$, which contradicts (2.1). If $n'=4k-5$, then $s\geq 2k-3$ and $t\leq n'-s\leq 2k-2$ since s is an integer. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(2k-3)-k(2k-2)+2(2k-2-1)=3k-6\geq 0$, which contradicts (2.1).
- If $d_1=d_2=1$. In this case, $s\geq \frac{n'}{2}-1$ and $t\leq n'-s\leq \frac{n'}{2}+1$. If $n'\geq 4k-2$, then $k|S|-k|T|+d_{G'-S}(T)\geq k(\frac{n'}{2}-1)-k(\frac{n'}{2}+1)+(\frac{n'}{2}+1)\geq 0$, which contradicts (2.1). If $n'=4k-3$, then $s\geq 2k-2$ and $t\leq n'-s\leq 2k-1$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(2k-2)-k(2k-1)+(2k-1)=k-1>0$, which contradicts (2.1). If $n'=4k-4$, then $s\geq 2k-3$ and $t\leq 2k-1$. If $s\geq 2k-2$ or $t\leq 2k-2$, then we have $k|S|-k|T|+d_{G'-S}(T)\geq 0$. If $s=2k-3$ and $t=2k-1$, then at least one vertex in T is of degree at least 2 in T since t is odd. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(2k-3)-k(2k-1)+(2k-1)+1=0$, which contradicts (2.1). If $n'=4k-5$, then $s\geq 2k-3$ and $t\leq 2k-2$ since s is an integer. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(2k-3)-k(2k-2)+(2k-2)=k-2\geq 0$, which contradicts (2.1).
- If $d_1=0$ and $d_2=1$. In this case, $s\geq \frac{n'-1}{2}$, $t\leq n'-s=\frac{n'+1}{2}$ and $p\leq d_1+1=1$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq k(\frac{n'-1}{2})-k(\frac{n'+1}{2})+(\frac{n'+1}{2}-1)\geq k-3\geq 0$ if $k\geq 3$, which contradicts (2.1). If $k=2$ and $n'\geq 5=4k-3$, then $k|S|-k|T|+d_{G'-S}(T)\geq k\frac{n'-1}{2}-k\frac{n'+1}{2}+(\frac{n'+1}{2}-1)\geq k-2=0$, which contradicts (2.1). If $n'=4=4k-4$, then $s\geq 2$ and $t\leq 2$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq 2k-2k+(2-1)>0$, which contradicts (2.1). The last situation is $k=2$ and $n'=3=4k-5$. Then $s\geq 1$ and $t\leq 2$. If $s\geq 2$ or $t\leq 1$, then we get $k|S|-k|T|+d_{G'-S}(T)\geq 0$, which contradicts (2.1). Otherwise, $s=1$ and $t=2$. Then at least one vertex in T has degree at least 2 in T since t is even and $d_1=0$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq 2-4+1+1=0$, which contradicts (2.1).
- If $d_1=d_2=0$. In this case, $s\geq \frac{n'}{2}$ and $t\leq \frac{n'}{2}$. Thus, $k|S|-k|T|+d_{G'-S}(T)\geq 0$, which contradicts (2.1).

Thus, we complete the proof of Theorem 2. □

Remark 1. We construct some graphs to show that the bounds in the Theorem 2 are best possible.

For $k\geq 3$, let $G=(2k-3)K_1\vee(K_{2k-4}\vee(k-1)K_2)$. Then $n=6k-9$, $\delta(G)=4k-6\geq \frac{n}{3}+k$ and $\sigma_2(G)=8k-12=\frac{4n}{3}$. Let $I=(2k-3)K_1$, $S=K_{2k-4}$. Then $T=(k-1)K_2$ and $k|S|-k|T|+d_{G'-S}(T)=-2<0$. So, G is not a fractional

ID-k-factor-critical graph. For $k = 2$ and $|G| = 3 = 6k - 9$, then G is not a fractional ID-k-factor-critical graph. Thus, the bound of n is best possible.

If $k \geq 3$. Let $G = (2k-2)K_1 \vee (K_{2k-3} \vee (2k-2)K_1)$. Then $n = 6k-7$, $\delta(G) = 4k-5 \geq \frac{n}{3} + k$, but $\sigma_2(G) = 8k-10 < \frac{4n}{3}$. Let $I = (2k-2)K_1$, $S = K_{2k-3}$. Then $T = (2k-2)K_1$, $d_{G'-S}(T) = 0$ and $k|S| + \sum_{x \in T} d_{G'-S}(x) - k|T| = -k < 0$. So, G is not a fractional ID-k-factor-critical graph. The condition $\sigma_2(G) \geq \frac{4n}{3}$ is best possible for $k \geq 3$.

At last, the condition that $\delta(G) \geq \frac{n}{3} + k$ cannot be replaced by $\frac{n}{3} + k - 1$. We consider a such graph G: n is divided by 3 and $G = \frac{n}{3}K_1 \vee G'$. Let $I = \frac{n}{3}K_1$. Deleting I form G, we have $\delta(G') = k-1$ if $\delta(G) = \frac{n}{3} + k - 1$. Therefore, $G - I$ has no fractional k-factor by the definition.

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OSCILLATORY BEHAVIOR OF SECOND ORDER DAMPED NEUTRAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS

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Abstract. The authors establish some new criteria for the oscillation of second order damped nonlinear neutral differential equations with distributed deviating arguments. Two examples are also provided to illustrate the results.

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1. Introduction

This paper deals with oscillatory behavior of all solutions of the second order damped nonlinear neutral differential equation with distributed deviating arguments

$$\left(r(t)\left(y'(t)\right)^{\alpha}\right)' + p(t)\left(y'(t)\right)^{\alpha} + \int_a^b q(t,\tau) f\left(t, x(g(t,\tau))\right) d\tau = 0,$$
$$t \geq t_0 \geq 0, \tag{1.1}$$

where $y(t) = x(t) + b(t)\phi\left(x(\sigma(t))\right)$, $0 < a < b$, and α is the quotient of odd positive integers.

In the remainder of the paper we assume that:

(i) $p, r : [t_0, \infty) \to \mathbb{R}^+$ are continuous functions such that $r(t)$ is nondecreasing, and

$$\int_{t_0}^{\infty} \left(\frac{1}{r(s)} A(s,t_0)\right)^{1/\alpha} ds < \infty, \tag{1.2}$$

where

$$A(t,t_0) = \exp\left(\int_{t_0}^{t} \frac{-p(s)}{r(s)} ds\right);$$

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(ii) $b : [t_0, \infty) \to \mathbb{R}$ is a continuous function such that $0 \le b(t) < 1$;
(iii) $q : [t_0, \infty) \times [a, b] \to \mathbb{R}^+$ is a continuous function;
(iv) $g : [t_0, \infty) \times [a, b] \to \mathbb{R}$ is a continuous function such that $g(t, \tau)$ is decreasing in τ, $g(t, \tau) \le t$, and $g(t, \tau) \to \infty$ as $t \to \infty$, $\tau \in [a, b]$;
(v) $f(t, u) : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $uf(t, u) > 0$ for all $u \ne 0$ and there exists a positive constant μ such that

$$f(t, u)/u^\alpha \ge \mu \quad \text{for } u \ne 0;$$

(vi) $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $u\phi(u) > 0$ for all $u \ne 0$ and there exists a real number β with $0 < \beta < 1$ such that $\phi(u)/u \le \beta$ for $u \ne 0$;
(vii) $\sigma : [t_0, \infty) \to \mathbb{R}$ is a continuous function such that $\sigma(t) \le t$, and $\sigma(t) \to \infty$ as $t \to \infty$.

By a *solution* of (1) we mean a function $x : [t_x, \infty) \to \mathbb{R}$ such that $y \in C^1([t_x, \infty), \mathbb{R})$ and $r(t)(y'(t))^\alpha \in C^1([t_x, \infty), \mathbb{R})$, and which satisfies equation (1) on $[t_x, \infty)$. Without further mention, we will assume throughout that every solution $x(t)$ of (1) under consideration here is continuable to the right and nontrivial, i.e., $x(t)$ is defined on some ray $[t_x, \infty)$, for some $t_x \ge t_0$, and $\sup\{|x(t)| : t \ge T\} > 0$ for every $T \ge t_x$. Moreover, we tacitly assume that (1) possesses such solutions. Such a solution is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x, \infty)$; otherwise it is called *nonoscillatory*. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various differential equations, and we refer the reader to the papers [1, 2, 4–7, 9, 10, 12, 13] and the references therein as examples of recent results on this topic. However, oscillation results for second order differential equations with distributed deviating arguments are relatively scarce in the literature; some results can be found, for example, in [3, 8, 11, 14] and the references contained therein. Our purpose here is to establish some new oscillation criteria for equation (1) different from those in [3, 8, 11, 14] and to contribute to the growing body of research on second order neutral differential equations in general and those with distributed delays and a damping term in particular.

2. Main results

In this section, we establish some new criteria for the oscillation of equation (1). It will be convenient to employ the following notations.

$$Q(t) = \int_a^b q(t, \tau) B^\alpha(g(t, \tau))\, d\tau, \quad g(t, b) = g(t), \quad B(t) = 1 - \beta b(t),$$

$$Q^*(t) = \int_a^b q(t,\tau) Q_1^\alpha(g(t,\tau))\, d\tau, \quad R(t) = \int_t^\infty \left(\frac{A(s,t)}{r(s)}\right)^{1/\alpha} ds,$$

and

$$Q_1(t) = \left[1 - \beta b(t)\left(\frac{R(\sigma(t))}{R(t)}\right)\right].$$

Throughout the paper we assume that

$$Q_1(t) \geq 0 \quad \text{for} \quad t \geq t_0. \tag{2.1}$$

We begin with the following lemma that will be used to prove our main results.

Lemma 1 ([4]). *Let $\tau(t) \in C([t_0,\infty),\mathbb{R})$, $\tau(t) \leq t$, and $\tau(t) \to \infty$ as $t \to \infty$. Assume $x \in C^2[T,\infty)$ satisfies*

$$x(t) > 0, \quad x'(t) > 0, \quad \text{and} \quad x''(t) \leq 0 \quad \text{for } t \geq T \geq t_0.$$

Then for each $k \in (0,1)$, there exists $T_k \geq T$ such that

$$\frac{x(\tau(t))}{\tau(t)} \geq k\frac{x(t)}{t} \quad \text{for } t \geq T_k. \tag{2.2}$$

Theorem 1. *Let conditions (i)-(vii), (1.2), and (2.1) hold. If there exists a positive function $\rho(t) \in C^1([t_0,\infty),\mathbb{R})$ such that*

$$p(t) \geq \frac{r(t)\rho'(t)}{\rho(t)} \quad \text{for } t \geq t_0, \tag{2.3}$$

$$\int_{t_0}^\infty \rho(s) Q(s) \left(\frac{g(s)}{s}\right)^\alpha ds = \infty, \tag{2.4}$$

and

$$\int_{t_0}^\infty \left[\frac{A(t,t_0)}{r(t)} \int_{t_0}^t Q^*(s) R^\alpha(s) ds\right]^{1/\alpha} dt = \infty, \tag{2.5}$$

then equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1), say $x(t) > 0$ and $x(\sigma(t)) > 0$ for $t \in [t_1,\infty)$, and $x(g(t,\tau)) > 0$ for $(t,\tau) \in [t_1,\infty) \times [a,b]$ for some $t_1 \in [t_0,\infty)$. From (1) and (v), we have

$$\left(r(t)\left(y'(t)\right)^\alpha\right)' + p(t)\left(y'(t)\right)^\alpha + \mu \int_a^b q(t,\tau) x^\alpha(g(t,\tau)) d\tau \leq 0 \quad \text{for } t \geq t_1,$$

so

$$\left(r(t)\left(y'(t)\right)^\alpha\right)' + p(t)\left(y'(t)\right)^\alpha \leq 0. \tag{2.6}$$

Setting $u(t) = r(t)\,(y'(t))^{\alpha}$, we see that

$$u'(t) + \frac{p(t)}{r(t)}u(t) \le 0,$$

which implies

$$\left(u(t)\exp\left(\int_{t_1}^{t} \frac{p(s)}{r(s)}ds\right)\right)' \le 0.$$

Thus, $u(t)\exp\left(\int_{t_1}^{t} \frac{p(s)}{r(s)}ds\right)$ is decreasing and so is eventually of one sign. Therefore, $y'(t)$ eventually has a fixed sign say for $t \ge t_2 \ge t_1$.

We shall then distinguish the following two cases:

Case (I): $y'(t) > 0$ for $t \ge t_2$;

Case (II): $y'(t) < 0$ for $t \ge t_2$.

Consider Case (I). From (1) and (v), we obtain

$$\left(r(t)\left(y'(t)\right)^{\alpha}\right)' + p(t)\left(y'(t)\right)^{\alpha} + \mu\int_a^b q(t,\tau)x^{\alpha}(g(t,\tau))d\tau \le 0 \quad \text{for } t \ge t_1. \tag{2.7}$$

Since $x(t) \le y(t)$, in view of (vi) and (vii), we have

$$y(t) \le x(t) + \beta b(t)x(\sigma(t)) \le x(t) + \beta b(t)y(\sigma(t)) \le x(t) + \beta b(t)y(t),$$

from which we see that

$$x(t) \ge (1-\beta b(t))\,y(t) = B(t)y(t). \tag{2.8}$$

Using (2.8) in (2.7) gives

$$\left(r(t)\left(y'(t)\right)^{\alpha}\right)' + p(t)\left(y'(t)\right)^{\alpha} + \mu\int_a^b q(t,\tau)B^{\alpha}(g(t,\tau))y^{\alpha}(g(t,\tau))d\tau \le 0.$$

In view of (iv) and the fact that $y(t)$ is increasing, the last inequality takes the form

$$\left(r(t)\left(y'(t)\right)^{\alpha}\right)' + p(t)\left(y'(t)\right)^{\alpha} + \mu Q(t)y^{\alpha}(g(t)) \le 0. \tag{2.9}$$

From (2.6) we see that

$$r'(t)\left(y'(t)\right)^{\alpha} + \alpha r(t)\left(y'(t)\right)^{\alpha-1}y''(t) + p(t)\left(y'(t)\right)^{\alpha} \le 0.$$

Since $r(t)$ is positive and nondecreasing, this implies $y''(t) < 0$. Therefore, by Lemma 1, for $k \in (0,1)$ fixed, there exits $t_3 \ge t_2$ such that

$$\frac{y(g(t))}{g(t)} \ge k\frac{y(t)}{t} \quad \text{for all } t \ge t_3. \tag{2.10}$$

Substituting (2.10) into (2.9) gives

$$\left(r(t)\left(y'(t)\right)^{\alpha}\right)' + p(t)\left(y'(t)\right)^{\alpha} + \mu k^{\alpha} Q(t)\left(\frac{g(t)}{t}\right)^{\alpha} y^{\alpha}(t) \le 0 \quad \text{for } t \ge t_3. \tag{2.11}$$

Now define the function $w(t)$ by

$$w(t) = \rho(t)\frac{r(t)\left(y'(t)\right)^{\alpha}}{y^{\alpha}(t)} \quad \text{for } t \ge t_3. \tag{2.12}$$

Clearly, $w(t) > 0$, and

$$w'(t) = \frac{\rho'(t)}{\rho(t)} w(t) + \rho(t)\frac{\left(r(t)\left(y'(t)\right)^{\alpha}\right)'}{y^{\alpha}(t)} - \rho(t)\frac{r(t)\left(y'(t)\right)^{\alpha}\left(y^{\alpha}(t)\right)'}{y^{2\alpha}(t)}. \tag{2.13}$$

Using (2.11) and (2.12) in (2.13), we obtain

$$w'(t) \le -\mu k^{\alpha}\rho(t)Q(t)\left(\frac{g(t)}{t}\right)^{\alpha} + \left(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\right) w(t) - \frac{\alpha\left(w(t)\right)^{1+1/\alpha}}{\left(r(t)\rho(t)\right)^{1/\alpha}}.$$

In view of (2.3) and $w(t) > 0$, the last inequality implies

$$w'(t) \le -\mu k^{\alpha}\rho(t)Q(t)\left(\frac{g(t)}{t}\right)^{\alpha} \quad \text{for } t \ge t_3.$$

Integrating this inequality from t_3 to t yields

$$0 \le w(t) \le w(t_3) - \mu k^{\alpha}\int_{t_3}^{t} \rho(s)Q(s)\left(\frac{g(s)}{s}\right)^{\alpha} ds \to -\infty$$

as $t \to \infty$, which is a contradiction.

Next, we consider Case (II). From (1.1), we obtain

$$\begin{aligned}\left(\frac{r(t)\left(y'(t)\right)^{\alpha}}{A(t,t_0)}\right)' &= \frac{\left(r(t)\left(y'(t)\right)^{\alpha}\right)'}{A(t,t_0)} - \frac{r(t)\left(y'(t)\right)^{\alpha} A'(t,t_0)}{A^2(t,t_0)} \\ &= \frac{1}{A(t,t_0)}\left[\left(r(t)\left(y'(t)\right)^{\alpha}\right)' + p(t)\left(y'(t)\right)^{\alpha}\right] \\ &= -\frac{1}{A(t,t_0)}\int_a^b q(t,\tau) f\left(t, x(g(t,\tau))\right) d\tau \le 0,\end{aligned}$$

i.e, $r(t)\left(y'(t)\right)^{\alpha}/A(t,t_0)$ is decreasing. Hence, we have

$$\frac{r(s)\left(y'(s)\right)^{\alpha}}{A(s,t_0)} \le \frac{r(t)\left(y'(t)\right)^{\alpha}}{A(t,t_0)} \quad \text{for } s \ge t \ge t_3, \tag{2.14}$$

from which we obtain

$$y'(s) \le r^{1/\alpha}(t)y'(t)\left(\frac{A(s,t)}{r(s)}\right)^{1/\alpha} \quad \text{for } s \ge t.$$

It follows that

$$y(u) - y(t) \le r^{1/\alpha}(t)y'(t)\int_t^u \left(\frac{A(s,t)}{r(s)}\right)^{1/\alpha} ds.$$

Letting $u \to \infty$ in the last inequality, we see that

$$y(t) \ge -r^{1/\alpha}(t)y'(t)\int_t^\infty \left(\frac{A(s,t)}{r(s)}\right)^{1/\alpha} ds = R(t)\left(-r^{1/\alpha}(t)y'(t)\right), \tag{2.15}$$

which implies

$$\left(\frac{y(t)}{R(t)}\right)' \ge 0,$$

and hence $y(t)/R(t)$ is nondecreasing. From this, the definition of $y(t)$, (vi), and (vii), we have

$$\begin{aligned} x(t) &\ge y(t) - \beta b(t)x(\sigma(t)) \ge y(t) - \beta b(t)y(\sigma(t) \\ &\ge \left[1 - \beta b(t)\left(\frac{R(\sigma(t))}{R(t)}\right)\right]y(t) = Q_1(t)y(t). \end{aligned} \tag{2.16}$$

Again using the fact that $r(t)\left(y'(t)\right)^\alpha / A(t,t_0)$ is decreasing, we have

$$r(t)\left(-y'(t)\right)^\alpha \ge \frac{A(t,t_0)}{A(t_3,t_0)} r(t_2)\left(-y'(t_2)\right)^\alpha = \gamma A(t,t_0) > 0 \quad \text{for } t \ge t_3, \tag{2.17}$$

for some positive constant γ.

Combining (2.17) and (2.15) gives

$$y(t) \ge \gamma^{1/\alpha} R(t) A^{1/\alpha}(t,t_0) \quad \text{for } t \ge t_3. \tag{2.18}$$

From (2.7), (2.16), (iv), and the fact that $y(t)$ is decreasing, we obtain

$$\begin{aligned} -\left(r(t)\left(y'(t)\right)^\alpha\right)' &\ge p(t)\left(y'(t)\right)^\alpha + \mu\int_a^b q(t,\tau)x^\alpha(g(t,\tau))d\tau \\ &\ge p(t)\left(y'(t)\right)^\alpha + \mu\int_a^b q(t,\tau)Q_1^\alpha(g(t,\tau))y^\alpha(g(t,\tau))d\tau \\ &\ge p(t)\left(y'(t)\right)^\alpha + \mu\int_a^b q(t,\tau)Q_1^\alpha(g(t,\tau))y^\alpha(t)d\tau \end{aligned}$$

$$= p(t)\left(y'(t)\right)^{\alpha} + \mu Q^*(t) y^{\alpha}(t). \tag{2.19}$$

Using (2.18) in (2.19), we arrive at

$$-\left(r(t)\left(y'(t)\right)^{\alpha}\right)' \geq p(t)\left(y'(t)\right)^{\alpha} + \mu\gamma Q^*(t) A(t,t_0) R^{\alpha}(t). \tag{2.20}$$

With $U(t) = r(t)\left(y'(t)\right)^{\alpha}$, (2.20) becomes

$$U'(t) \leq -\frac{p(t)}{r(t)} U(t) - \mu\gamma Q^*(t) A(t,t_0) R^{\alpha}(t),$$

which can be written as

$$\left(\frac{U(t)}{A(t,t_0)}\right)' \leq -\mu\gamma Q^*(t) R^{\alpha}(t), \quad \text{for } t \geq t_3.$$

Integrating this inequality from t_3 to t gives

$$\begin{aligned} U(t) &\leq U(t_3) A(t,t_3) - \mu\gamma A(t,t_0) \int_{t_3}^{t} Q^*(s) R^{\alpha}(s) ds \\ &\leq -\mu\gamma A(t,t_0) \int_{t_3}^{t} Q^*(s) R^{\alpha}(s) ds, \end{aligned}$$

which leads to

$$y'(t) \leq -(\mu\gamma)^{1/\alpha} \left(\frac{A(t,t_0)}{r(t)} \int_{t_3}^{t} Q^*(s) R^{\alpha}(s) ds\right)^{1/\alpha}.$$

An integration of the last inequality from t_3 to t implies

$$0 < y(t) \leq y(t_3) - (\mu\gamma)^{1/\alpha} \int_{t_3}^{t} \left(\frac{A(u,t_0)}{r(u)} \int_{t_3}^{u} Q^*(s) R^{\alpha}(s) ds\right)^{1/\alpha} du \to -\infty$$

as $t \to \infty$, by (2.5). This is contradicts the fact that $y(t) > 0$ and completes the proof of the theorem. □

Our next theorem gives conditions under which a solution will either oscillate or converge to zero as $t \to \infty$.

Theorem 2. *In Theorem 2.2, if condition (2.5) is replaced by*

$$\int_{t_0}^{\infty} \left(\frac{A(t,t_0)}{r(t)}\right)^{1/\alpha} \left(\int_{t_0}^{t} Q^*(s) ds\right)^{1/\alpha} dt = \infty, \tag{2.21}$$

then any solution x of equation (1) either oscillates or converges to zero as $t \to \infty$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1), say $x(t) > 0$, $x(\sigma(t)) > 0$ for $t \in [t_1, \infty)$, and $x(g(t,\tau)) > 0$ for $(t,\tau) \in [t_1,\infty) \times [a,b]$ for some $t_1 \in [t_0,\infty)$. We again distinguish the two cases:

$$\text{(I)} \quad y'(t) > 0 \qquad \text{or} \qquad \text{(II)} \quad y'(t) < 0$$

for $t \geq t_2$ for some $t_2 \geq t_1$.

The proof if Case (I) holds is similar to that of Case (I) in the proof of Theorem 1, and hence is omitted.

Next, we consider Case (II). Since $y(t) > 0$ and $y'(t) < 0$, there exists a constant c such that

$$\lim_{t\to\infty} y(t) = c < \infty,$$

where $c \geq 0$.

If $c > 0$, there exists $t_3 \geq t_2$ such that

$$y(t) \geq c \qquad \text{for } t \geq t_3. \tag{2.22}$$

From (2.7), (2.16), (2.22), and (iv), we see that

$$\begin{aligned}
-\left(r(t)\left(y'(t)\right)^{\alpha}\right)' &\geq p(t)\left(y'(t)\right)^{\alpha} + \mu \int_a^b q(t,\tau)x^{\alpha}(g(t,\tau))d\tau \\
&\geq p(t)\left(y'(t)\right)^{\alpha} + \mu \int_a^b q(t,\tau)Q_1^{\alpha}(g(t,\tau))y^{\alpha}(g(t,\tau))d\tau \\
&\geq p(t)\left(y'(t)\right)^{\alpha} + \mu \int_a^b q(t,\tau)Q_1^{\alpha}(g(t,\tau))y^{\alpha}(t))d\tau \\
&\geq p(t)\left(y'(t)\right)^{\alpha} + \mu c^{\alpha} Q^*(t),
\end{aligned}$$

which can be written as

$$\begin{aligned}
\left(\frac{\left(r(t)\left(y'(t)\right)^{\alpha}\right)}{A(t,t_0)}\right)' &\leq -\mu c^{\alpha}\frac{1}{A(t,t_0)}Q^*(t) \\
&\leq -\mu c^{\alpha} Q^*(t), \qquad \text{for } t \geq t_3.
\end{aligned} \tag{2.23}$$

Integrating (2.23) from t_3 to t, we obtain

$$\frac{r(t)\left(y'(t)\right)^{\alpha}}{A(t,t_0)} \leq \frac{r(t_3)\left(y'(t_3)\right)^{\alpha}}{A(t_3,t_0)} - \mu c^{\alpha}\int_{t_3}^{t} Q^*(s)ds \leq -\mu c^{\alpha}\int_{t_3}^{t} Q^*(s)ds,$$

which leads to

$$y'(t) \le -c\mu^{1/\alpha}\left(\frac{A(t,t_0)}{r(t)}\right)^{1/\alpha}\left(\int_{t_3}^{t} Q^*(s)ds\right)^{1/\alpha}.$$

Integrating from t_3 to $u \ge t_3$ and applying (2.21), we see that

$$0 \le y(u) \le y(t_3) - c\mu^{1/\alpha}\int_{t_3}^{u}\left(\frac{A(t,t_0)}{r(t)}\right)^{1/\alpha}\left(\int_{t_3}^{t} Q^*(s)ds\right)^{1/\alpha} dt \to -\infty$$

as $u \to \infty$ which is a contradiction. Hence, we have $c = 0$, that is, $\lim_{t\to\infty} y(t) = 0$. Since $0 < x(t) \le y(t)$ on $[t_1, \infty)$, we have $\lim_{t\to\infty} x(t) = 0$, and this completes the proof of the theorem. □

3. Discussion and examples

The results here appear to be one of the first attempts to look at equations in the form of (1) containing a damping term. We hope that this will encourage other researchers to explore similar problems. We conclude this paper with two examples to illustrate the applicability of our results.

Example 1. Consider the neutral differential equation

$$\left(t^2\left(y'(t)\right)^{1/3}\right)' + t\left(y'(t)\right)^{1/3} + \int_1^2 \left(t^6 + 2\tau\right) f\left(t, x(t/2 + 1/3\tau)\right) d\tau = 0, \quad t \ge 1. \tag{3.1}$$

Here we have $\alpha = 1/3$, $r(t) = t^2$, $p(t) = t$, $q(t,\tau) = t^6 + 2\tau$, $g(t,\tau) = t/2 + 1/3\tau$, $a = 1$, $b = 2$, and we take $y(t) = x(t) + \frac{1}{32}\phi(x(t/2))$ with $\phi(u) = \frac{1}{2}u$, and $f(t,u) = u^\alpha$. Then, $\beta = 1/2$, $\mu = 1$, $b(t) = 1/32$, $B(t) = 63/64$, $Q(t) = (63/64)^{1/3}\left(t^6 + 3\right)$, $Q^*(t) = (1/2)^{1/3}[t^6 + 3]$, $Q_1(t) = 1/2$, $A(t,t_0) = 1/t$, $A(s,t) = t/s$, and $R(t) = 1/8t^5$, $R(\sigma(t)) = 4/t^5$. It is clear that conditions (i)-(vii) and (1.2) hold. With $\rho(t) = t$, we see that condition (2.3) holds, and conditions (2.4) and (2.5) become

$$(63/64)^{1/3}\int_1^\infty s\left(s^6 + 3\right)\left(\frac{s/2 + 1/6}{s}\right)^{1/3} \ge (63/128)^{1/3}\int_1^\infty s^7 ds = \infty,$$

and

$$\int_1^\infty \left[\frac{1}{t^3}\int_1^t \left(\frac{1}{2}\right)^{1/3}\left(s^6 + 3\right)\left(\frac{1}{8s^5}\right)^{1/3} ds\right]^3 dt$$

$$\geq (3^3/2^{16}) \int_1^\infty \left[\frac{\left(t^{16/3}-1\right)}{t^3} \right]^3 dt = \infty,$$

respectively, so equation (1) is oscillatory by Theorem 1.

Example 2. In Example 1, if we take $q(t,\tau) = t^{5/2} + 2\tau$ instead of $q(t,\tau) = t^6 + 2\tau$, then we see that all conditions of Theorem 2 hold. Since condition (2.5) is not satisfied, Theorem 1 can not be applied to Example 2. In this case, any solution $x(t)$ of the equation either oscillates or converges to zero as $t \to \infty$.

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ALGORITHM DEFINING THE HYDRAULIC RESISTANCE COEFFICIENT BY LINES METHOD IN GAS-LIFT PROCESS

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Abstract. A mathematical model of the process of gas-lift in the oil production is considered. In this process the motions of gas and gas-liquid mixture (GLM) in annular space and in the lift are described by the system of partial differential equations of hyperbolic type. Applying the straight line method the initial problem is reduced to the system of ordinary differential equations relatively to the volume of gas, gas-liquid mixture and their pressures. To determine the coefficient of hydraulic resistance λ_c statistical data for the fixed well is used (volume of injected gas at the wellhead of the annular space and GLM at the end of lift) and on the basis of these results, constituting the corresponding functional, the functional gradient is derived that allows one to calculate λ_c. On the concrete example the statistical value of the coefficient of hydraulic resistance $\lambda_{\tilde{c}}$ and the new value of the coefficient of hydraulic resistance λ_c are compared, it is shown that λ_c differs from the real $\lambda_{\tilde{c}}$ on the order of 10^{-3}.

2010 *Mathematics Subject Classification:* 49J20; 65K05; 65N40

Keywords: gas-lift, gas-liquid mixture, hydraulic resistance coefficient, gradient of the functional

1. Introduction

As it is known [7, 8, 12, 18, 20], the gas-lift method for oil production is used when the fountain method impossible due to the lower reservoir pressure. The essence of the gas-lift method is that due to the energy of the injected gas, the liquid can be lifted to the ground surface. The motions [2, 18, 19] of the injected gas and GLM are described by the partial differential equations of hyperbolic type. To solve the corresponding optimal control problems (feeding on the estuary such minimum gas volume, that the debit would be the highest) the method of lines are used, which leads the initial problem to the linear - quadratic optimal control problem for a system of ordinary differential equations [1, 3, 6].

However, there are some significant parameters such as the coefficient of hydraulic resistance λ_c in the lift, included in the equation of fluid motion in the lift (the identifications or the inversions problems [16, 17]), which determines the generation rate of GLM. In this paper we focus on the choice the parameter λ_c, where choosing the corresponding quadratic functional, the posed problem is solved that allows one to

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calculate λ_c [8]. According to the obtained formulas is suggested the algorithm and the program is made using Matlab. Thus, the offered algorithm allows to find the coefficient of hydraulic resistance λ_c and the results coincide with accuracy 10^{-3} with data from mines.

2. Problem statement

As it is known [1, 3, 5, 7, 9], the motions of the gas and GLM in the tubes are described by the system of partial differential equations of hyperbolic type:

$$\begin{cases} \frac{\partial P}{\partial t} = -\frac{c_i^2}{F_i}\frac{\partial Q}{\partial x}, \\ \frac{\partial Q}{\partial t} = -F_i\frac{\partial P}{\partial x} - 2a_i Q, \end{cases} \quad i = 1,2, \tag{2.1}$$

where $P = P(x,t)$– is pressure, Q– is volume of gas and GLM, correspondingly, c_i- is the sound velocity in the gas and GLM, correspondingly, $2a_i = \frac{g}{\omega_c^i} + \frac{\lambda_c^i \omega_c^i}{2D_i}$ $(i = 1,2)$; g, λ_c^i, $(i = 1,2)$– acceleration of gravity and hydraulic resistance in the gas and GLM, correspondingly, ω_c^i, $(i = 1,2)$ - averaged over the cross section of the velocity of the gas and mixture in the annular zone and lift, correspondingly. D_i, $i = 1,2$- are effective internal diameters of lift and annular space, F_i, $i = 1,2$– cross-sectional area of pump-compressor pipe and is also a constant on axes. Applying the straight line method and denoting $l = 1/2n$, we obtain [14] from (2.1)

$$\begin{cases} \frac{dP_k}{dt} = -\frac{c_i^2}{F_i l}(Q_k - Q_{k-1}), \\ \frac{dQ_k}{dt} = -\frac{F_i}{l}(P_k - P_{k-1}) - 2a_i Q_k, \end{cases} \tag{2.2}$$

where

$$F_i = \begin{cases} F_1, \ 0 < k \le n, \\ F_2, \ n < k \le 2n, \end{cases} \quad c_i = \begin{cases} c_1, \ 0 < k \le n, \\ c_2, \ n < k \le 2n. \end{cases}$$

Note that for $k = n+1$ an equation (2.2) has the following form [10]

$$\begin{cases} \frac{dP_{n+1}}{dt} = -\frac{c_2^2}{F_2 l}Q_{n+1} + \frac{c_2^2}{F_2 l}Q_n + \frac{c_2^2}{F_2 l}Q_{pl}, \\ \frac{dQ_{n+1}}{dt} = -\frac{F_2}{l}P_{n+1} + \frac{F_2}{l}P_n - 2a_2 Q_{n+1} + \frac{F_2}{l}P_{pl}, \end{cases}$$

where Q_{pl}, P_{pl} - are volume flow and pressure of the reservoir at the bottom of the well.

After some transformations the system (2.2) can be reduced to

$$\dot{x} = A(\lambda_c)x + Bu + V, \tag{2.3}$$

with initial condition

$$X_0 = \left[P_1^0, Q_1^0, ..., P_{2n}^0, \ Q_{2n}^0\right]', \tag{2.4}$$

where X_0 is the initial position of the well, u is controlling influence and $A = \left[a_{ij}\right]$,

$B = [b_{ij}]$, $V = [v_{ij}]$,

$$a_{ij} = \begin{cases} a_{11}, \text{ if } i = j \quad \text{and} \quad 1 \le i \le n, \\ a_{12}, \text{ if } i = j+1 \text{ and } 1 \le i < n-1, \\ a_{22}, \text{ if } i = j \quad \text{and} \quad n < i < 2n, \\ a_{21}, \text{ if } i = j+1 \text{ and } n-1 < j \le 2n-1, \\ 0, \text{ in other cases,} \end{cases}$$

$$a_{11} = \begin{bmatrix} 0 & -c_1^2/F_1 l \\ -F_1/l & -2a_2 \end{bmatrix}, \; a_{12} = \begin{bmatrix} 0 & c_1^2/F_1 l \\ F_1/l & 0 \end{bmatrix},$$

$$a_{21} = \begin{bmatrix} 0 & c_2^2/F_2 l \\ F_2/l & 0 \end{bmatrix}, \; a_{22} = \begin{bmatrix} 0 & -c_2^2/F_2 l \\ -F_2/l & -2a_2 \end{bmatrix},$$

$$b_{ij} = \begin{cases} c_1^2/F_1 l, & \text{if } i = 1, j = 2, \\ F_1/l, & \text{if } i = 2, j = 1, \\ 0, & \text{in other cases,} \end{cases}$$

$$v_{ij} = \begin{cases} c_2^2/F_2 l, & \text{if } i = n+1, \; j = 2, \\ F_2/l, & \text{if } i = n+2, \; j = 1, \\ 0, & \text{in other cases.} \end{cases}$$

$$x = [P_1, Q_1, P_2, Q_2, \ldots, P_n, Q_n, P_{n+1}, Q_{n+1}, \ldots, P_{2n}, Q_{2n}]'$$

Let's have some statistics, which at the given initial volumes of gas the debit $\tilde{Q}^i_{2n}$ is measured at the output, i.e. $\tilde{Q}^i_0$, and $\tilde{Q}^i_{2n}$ are known ($i = \overline{1, N}$), where N is the number of measurements [4, 8].

It is required to find such values of the coefficient of hydraulic resistance λ_c [4, 8, 11, 13], which system (2.3) will describe the motion of GLM in the lift, closer to practice (adequate mathematical model). To solve this problem, the objective function based on the lowest square deviation from the real initial data $\tilde{Q}^i_{2n}$ is constructed, i.e. it is required to minimize the functional

$$f(a_2(\lambda_c)) = \sum_{i=1}^{N} \left[\tilde{Q}^i_{2n} - Q^i_{2n} \right]^2. \tag{2.5}$$

Thus, the problem of defining the coefficient of hydraulic resistance λ_c is reduced to the finding the gradient of functional (2.5).

3. Calculating the gradient of the functional (2.5)

The general solution of equation (2.2) can be represented as follows

$$\begin{aligned} x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\sigma)}Bud\sigma + \int_0^t e^{A(t-\sigma)}Vd\sigma = \\ &= e^{At}x_0 + A^{-1}(e^{At}-E)Bu + A^{-1}(e^{At}-E)V, \end{aligned} \tag{3.1}$$

where from the equation (3.1) Q_{2n}^i has the form

$$Q_{2n}^i = J \cdot x(T), \tag{3.2}$$

where $J = [0,\ 0,\ 0,\ 0,\ ...,\ 1]$, $E-$ is the identity matrix.

In fact, u is a function of the variable t. To get any value of the volume of gas or the corresponding value of the pressure, we can take them in the mouth of the well in the form of $\sin(bt)u_0$, that formula (3.1) turns into the following

$$\begin{aligned} x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\sigma)}Bud\sigma + \int_0^t e^{A(t-\sigma)}Vd\sigma = e^{At}x_0 + \\ &+ \int_0^t e^{A(t-\sigma)}B\sin(b\sigma)u_0 d\sigma + \int_0^t e^{A(t-\sigma)}Vd\sigma = \\ &= (e^{At}x_0 + E(e^{At} - u_0\cos t - A\sin t)Bu_0)(E-A^2)^{-1} + A^{-1}(e^{At}-E)V. \end{aligned}$$

Then, substituting (3.1) and (3.2) in (2.5) we have

$$\begin{aligned} f(a_2(\lambda_c)) &= \sum_{i=1}^{N}\left[\tilde{Q}_{2n}^i - J\cdot x(T)\right]^2 = \\ &= \sum_{i=1}^{N}[\tilde{Q}_{2n}^i - Je^A x_0^i - JA^{-1}(e^{AT}-E)Bu - JA^{-1}(e^{AT}-E)V]^2. \end{aligned} \tag{3.3}$$

For the solution of the optimization problem (2.3) - (2.5) we find the gradient of the functional $f(a_2(\lambda_c))$ and equating it to zero, we have

$$
\begin{aligned}
&\tfrac{\partial f(a_2(\lambda_c))}{\partial a_2(\lambda_c)} = 2\sum_{i=1}^{N}(\tilde{Q}_{2n}^{i} - Je^{AT}x_0 - \\
&-A^{-1}(e^{AT}-E)Bu - JA^{-1}(e^{AT}-E)V)\times \\
&\times\sum_{i=1}^{N}(-J(\sum_{k=1}^{\infty}\sum_{j=1}^{k}\tfrac{T^k}{k!}(A(a_2(\lambda_c)))^{j-1}\times \\
&\times\tfrac{\partial A(a_2(\lambda_c))}{\partial a_2(\lambda_c)}(A(a_2(\lambda_c)))^{j-1})x_0^i - J^{-1}\tfrac{\partial A(a_2(\lambda_c))}{\partial a_2(\lambda_c)}^{-1}Bu - \\
&-(\sum_{k=1}^{\infty}\sum_{j=1}^{k}J\tfrac{T^k}{k!}(A(a_2(\lambda_c)))^{j-1}\tfrac{\partial A(a_2(\lambda_c))}{\partial a_2(\lambda_c)}\times \\
&\times(A(a_2(\lambda_c)))^{j-1})Bu + J^{-1}\tfrac{\partial A(a_2(\lambda_c))}{\partial(a_2(\lambda_c))}^{-1}V + \\
&+J(-^{-1}\tfrac{\partial A(a_2(\lambda_c))}{\partial a_2(\lambda_c)}^{-1} + (\sum_{k=1}^{\infty}\sum_{j=1}^{k}J\tfrac{T^{k+1}}{(k+1)!}\times \\
&\times(A(a_2(\lambda_c)))^{j-1}\tfrac{\partial A(a_2(\lambda_c))}{\partial a_2(\lambda_c)}(A(a_2(\lambda_c)))^{k-j})V) = 0.
\end{aligned}
\tag{3.4}
$$

Using the method [4] for the solution of the nonlinear equation we define λ_c in the form

$$\lambda_c = 4a_2D/\omega_c + 2Dg/\omega_c^2.$$

Thus, we formulate the following algorithm for finding λ_c.

Algorithm:

1. The initial data and parameters A, B, V, u are introduced from (2.3);
2. The statistical (observations) data

$$x_0 = [P_1^{0i}, Q_1^{0i}, P_2^{0i}, Q_2^{0i}, \cdots, P_n^{0i}, Q_n^{0i}, P_{n+1}^{0i}, Q_{n+1}^{0i}, \cdots, P_{2n}^{0i}, Q_{2n}^{0i}]'$$

and the debit $\tilde{Q}_{2n}^{i}$ from the practice for the same well ($i = \overline{1, N}$, N-is the number of measurements) are introduced;

3. The matrix A, B are founded from (3.1)

$$x(t) = [P_1, Q_1, \ldots, P_{n+1}, Q_{n+1}, P_{2n}, Q_{2n}];$$

4. The functional $f(a_2(\lambda_c))$ is formulated from (3.3);
5. The solution of equation (3.4) is founded using the steepest descent method and univariate method (golden section search method) [15].
6. The condition $\left|\frac{\partial f(a_2(\lambda_c))}{\partial a_2(\lambda_c)}\right| < \varepsilon$ is checked out for sufficiently small number ε. If this condition is not satisfied we go to the step 2, else calculating procedure ends.

Example. Let's consider the realization of the proposed algorithm on the example [10]. After applying of the proposed algorithm we obtain that $\lambda_c = 0.2357157$. Here $\tilde{\lambda}_c = 0.23$, where $\tilde{\lambda}_c$ is the hydraulic resistance value from practice [4, 8]. Note that

λ_c differs from $\tilde{\lambda}_c$ to the order 10^{-3}, and it shows the efficiency of the proposed algorithm.

4. Conclusion

The quadratic functional is formed for the solution of the indicated problem and it allows to find the coefficient of hydraulic resistance. The offered algorithm confirm the adequacy of the constructed mathematical model with practice.

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ON A DIOPHANTINE EQUATION ON TRIANGULAR NUMBERS

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Abstract. A number N is a triangular number if it can be written as $N = 0+1+\cdots+n$ for some natural number n. We study the problem of finding all nonnegative integer solutions of the Diophantine equation $(0+1+\cdots+X)+(0+1+\cdots+Y) = (0+1+\cdots+Z)$. Using this equation, some new and curious increasing integer sequences are built.

2010 *Mathematics Subject Classification:* 11D25; 11D41

Keywords: Diophantine equation, triangular numbers

1. Introduction

A triangular number is a number of the form $T_n = \sum_{k=0}^{n} k = n(n+1)/2$, where n is a natural number.

So the first few triangular numbers are $0,1,3,6,10,15,21,28,\ldots$ (sequence A000217 in [6]). A well known fact about the triangular numbers is that X is a triangular number if and only if $8X+1$ is a perfect square. Triangular numbers can be thought of as the numbers of dots needed to make a triangle.

We are concerned by the Diophantine equation of the form

$$T_X + T_Y = T_Z \tag{1.1}$$

Papers [1–5] gave many interesting results concerning the problem of solvability of Diophantine equations related to triangular numbers. The aim of this paper is two-fold: on the one hand, it gives all solutions of the Diophantine equation (1.1), and on the other hand, it gives us a method of finding explicitly (and quickly) infinite families of solutions of (1.1) in where many new integer sequences are derived.

2. General solution

Let us start with some technical lemmas.

Lemma 1. *Let $a,b,c,d \in \mathbb{N}$. Then $ab = cd$ if and only if there exist $p,q,m,n \in \mathbb{N}$ such that $a = mn$, $b = pq$, $c = mp$ and $d = nq$.*

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Proof. Let $a = \prod_{i\in A\subset\mathbb{N}} p_i^{\alpha_i}$ and $b = \prod_{i\in B\subset\mathbb{N}} q_i^{\beta_i}$ where p_i, q_i are primes, $\alpha_i, \beta_i \in \mathbb{N}$ and A, B are finite sets. The fact $cd = \prod_{i\in A\subset\mathbb{N}} p_i^{\alpha_i} \prod_{i\in B\subset\mathbb{N}} q_i^{\beta_i}$ implies that $c = \prod_{i\in A'\subset A} p_i^{\alpha'_i} \prod_{i\in B'\subset B} q_i^{\beta'_i}$ and $d = \prod_{i\in A''\subset A} p_i^{\alpha''_i} \prod_{i\in B''\subset B} q_i^{\beta''_i}$, where $0 \le \alpha'_i, \alpha''_i \le \alpha_i$, $0 \le \beta'_i, \beta''_i \le \beta_i$ with $\alpha_i = \alpha'_i + \alpha''_i$ and $\beta_i = \beta'_i + \beta''_i$. If we put $m = \prod_{i\in A'\subset A} p_i^{\alpha'_i}$, $p = \prod_{i\in B'\subset B} q_i^{\beta'_i}$, $n = \prod_{i\in A''\subset A} p_i^{\alpha''_i}$ and $q = \prod_{i\in B''\subset B} q_i^{\beta''_i}$ then we obtain, $a = mn$, $b = pq$, $c = mp$ and $d = nq$. □

Lemma 2. *Equation (1.1) is equivalent to* $X(X+1) = (Z-Y)(Z+Y+1)$.

Proof. Replacing T_X by $\frac{X(X+1)}{2}$, T_Y by $\frac{Y(Y+1)}{2}$ and T_Z by $\frac{Z(Z+1)}{2}$ equation (1.1) becomes

$$\frac{X(X+1)}{2} + \frac{Y(Y+1)}{2} = \frac{Z(Z+1)}{2},$$

i.e.,

$$X^2 + X + Y^2 + Y = Z^2 + Z. \tag{2.1}$$

Thus

$$X(X+1) = (Z-Y)(Z+Y+1).$$

□

Now, we are able to establish the following theorem.

Theorem 1. *All nonnegative integer solutions of the equation* $T_X + T_Y = T_Z$ *are given by* $\begin{cases} X = mn \\ Y = \frac{1}{2}(nq - mp - 1) \\ Z = \frac{1}{2}(nq + mp - 1) \end{cases} : m, n, p, q \in \mathbb{N}$ *where* $pq - mn = 1$ *and* $nq - mp - 1 \in 2\mathbb{N}$.

Proof. According to Lemma 2, (1.1) is equivalent to $X(X+1) = (Z-Y)(Z+Y+1)$, thus thanks to Lemma 1, one has

$$\begin{aligned} X &= mn \\ X+1 &= pq \\ Z-Y &= mp \\ Z+Y+1 &= nq \quad \text{where } m, n, p, q \in \mathbb{N}. \end{aligned}$$

Therefore we get

$$\begin{aligned} X &= mn \\ Y &= \tfrac{1}{2}(nq - mp - 1) \end{aligned}$$

$$Z = \tfrac{1}{2}(nq + mp - 1) \text{ where } m, n, p, q \in \mathbb{N}$$

such that $pq - mn = 1$ and $nq - mp - 1 \in 2\mathbb{N}$. □

Example 1. $p = 4$, $q = 7$, $m = 3$ and $n = 9$. Clearly, $X = 27$, $Y = 25$ and $Z = 37$.

3. Solutions families

First of all, note that it is not easy to find integers m, n, p, q such that $pq - mn = 1$ and $nq - mp - 1 \in 2\mathbb{N}$.

The usage of matrices enables us to obtain a large number of solutions of (1.1). In this section, we will construct several infinite many families solutions of our triangular equation.

By completing squares in (2.1), we obtain

$$(X + \frac{1}{2})^2 + (Y + \frac{1}{2})^2 = (Z + \frac{1}{2})^2 + \frac{1}{4}.$$

Multiplying both sides by 4, we get

$$(2X + 1)^2 + (2Y + 1)^2 = (2Z + 1)^2 + 1.$$

By putting, $x = 2X + 1$, $y = 2Y + 1$ and $z = 2Z + 1$, we obtain

$$x^2 + y^2 = 1 + z^2. \tag{3.1}$$

Since the proof of the following proposition can be seen easily we omit it.

Proposition 1. *The two families* $\begin{pmatrix} 1 \\ a \\ a \end{pmatrix}, \begin{pmatrix} a \\ 1 \\ a \end{pmatrix}$ *are solutions of (3.1) for all integer* a.

Theorem 2. *There exist some 3×3 matrices* $\mathfrak{M}$, *with elements in* $\{1, 2, 3\}$, *such that if* $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ *is a solution of (3.1) then* $\mathfrak{M}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ *is also a solution.*

Proof. Let $\mathfrak{M} = \begin{pmatrix} A & B & C \\ E & F & G \\ L & M & N \end{pmatrix}$ and let $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ a solution of (3.1).

It is clear that $\begin{pmatrix} A & B & C \\ E & F & G \\ L & M & N \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} Ax_0 + By_0 + Cz_0 \\ Fy_0 + Gz_0 + Ex_0 \\ Lx_0 + My_0 + Nz_0 \end{pmatrix}$.

Now, if the following system holds

$$\begin{aligned} A^2 - L^2 + E^2 &= 1 \\ B^2 + F^2 - M^2 &= 1 \\ C^2 + G^2 - N^2 &= -1 \end{aligned} \tag{3.2}$$

$$\begin{aligned} AB - LM + EF &= 0 \\ AC - LN + EG &= 0 \\ BC - MN + FG &= 0 \end{aligned}$$

then

$$(Ax_0 + By_0 + Cz_0)^2 + (Fy_0 + Gz_0 + Ex_0)^2 - (Lx_0 + My_0 + Nz_0)^2 = 1.$$

It follows from this, $\begin{pmatrix} Ax_0 + By_0 + Cz_0 \\ Fy_0 + Gz_0 + Ex_0 \\ Lx_0 + My_0 + Nz_0 \end{pmatrix}$ is a solution of (3.1).

We will explore, in five steps, all solutions of the system (S) are in the space $\{1,2,3\}^9$.

Case 1: $A = L = E = 1$.

$$B^2 + F^2 - M^2 = 1 \tag{3.3}$$

$$\begin{aligned} C^2 + G^2 - N^2 &= -1 \\ B + F &= M \qquad (3.4) \\ C + G &= N \\ BC - MN + FG &= 0. \end{aligned}$$

From (3.4), we obtain $F = 1$ or $B = 1$, then (3.3) implies $B = M$ or $F = M$ then $F = 0$ or $B = 0$ which is impossible.

Case 2 : $A = 1,\ L = E = 2$.

$$\begin{aligned} B^2 + F^2 - M^2 &= 1 \\ C^2 + G^2 - N^2 &= -1 \\ B - 2M + 2F &= 0 \qquad (3.5) \\ C - 2N + 2G &= 0 \qquad (3.6) \\ BC - MN + FG &= 0. \end{aligned}$$

Equations (3.5) and (3.6) imply that B and C are even.

Then from

$$M^2 - F^2 = 3 \tag{3.7}$$

$$N^2 - G^2 = 5 \tag{3.8}$$

$$\begin{aligned} M - F &= 1 \\ N - G &= 1 \\ MN - FG &= 4. \end{aligned}$$

Equation (3.7) gives $M = 2, F = 1$ and (3.8) gives $N = 3, G = 2$,

then $\mathfrak{M}_1 = \begin{pmatrix} A & B & C \\ E & F & G \\ L & M & N \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$.

Case 3 : $A = 1, L = E = 3$.

$$\begin{aligned} B^2 + F^2 - M^2 &= 1 \\ C^2 + G^2 - N^2 &= -1 \\ B - 3M + 3F &= 0 \qquad (3.9) \\ C - 3N + 3G &= 0 \qquad (3.10) \\ BC - MN + FG &= 0. \end{aligned}$$

Equations (3.9) and (3.10) give us $B = C = 3$, then

$$\begin{aligned} F^2 - M^2 &= -8 \\ G^2 - N^2 &= -10 \qquad (3.11) \\ -M + F &= -1 \\ -N + G &= -1 \\ -MN + FG &= -9. \end{aligned}$$

It is easy to see that equation (3.11) is impossible in $\{1,2,3\}^2$

Case 4 : $A = 2$.
Equation 3.2 implies that $L = 2, E = 1$.
So, we get

$$\begin{aligned} B^2 + F^2 - M^2 &= 1 \\ C^2 + G^2 - N^2 &= -1 \\ 2B - 2M + F &= 0 \qquad (3.12) \\ 2C - 2N + G &= 0 \qquad (3.13) \\ BC - MN + FG &= 0. \end{aligned}$$

Equations (3.12) and (3.13) give us $F = G = 2$, thus

$$\begin{aligned} M^2 - B^2 &= 3 \qquad (3.14) \\ N^2 - C^2 &= 5 \qquad (3.15) \\ M - B &= 1 \\ N - C &= 1 \\ BC - MN + 4 &= 0. \end{aligned}$$

Equation (3.14) gives us $M = 2, B = 1$ and (3.15) gives $N = 3, C = 2$.

We obtain $\mathfrak{M}_2 = \begin{pmatrix} A & B & C \\ E & F & G \\ L & M & N \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix}$.

Case 5 : $A = 3$.

Equation (1_0) in (S) implies that : $L = 3, E = 1$. Now

$$\begin{aligned} B^2 + F^2 - M^2 &= 1 \\ C^2 + G^2 - N^2 &= -1 \\ 3B - 3M + F &= 0 \qquad (3.16) \\ 3C - 3N + G &= 0 \qquad (3.17) \\ BC - MN + FG &= 0. \end{aligned}$$

Equations (3.16) and (3.17) give us $F = G = 3$, thus

$$\begin{aligned} M^2 - B^2 &= 8 \\ N^2 - C^2 &= 10 \qquad (3.18) \\ B - M + 1 &= 0 \\ C - N + 1 &= 0 \\ BC - MN + 9 &= 0. \end{aligned}$$

It is easy to see that equation (3.18) is impossible in $\{1, 2, 3\}^2$. □

Corollary 1. *The equation (1.1) admits infinite many solutions.*

Proof. Let $\mathfrak{M} = \mathfrak{M}_1$ and $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ a solution of (3.1) then $\mathfrak{M}^n \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a solution of (3.1) for all integer $n \geq 0$. □

4. New integer sequences

Using the on-line Encyclopedia of integer sequences (see [6]), we can easily check that the following two integer sequences formed by solutions X of triangular equation (1.1) : $1, 5, 35, 203, 1179, 6929, 40391, 235415, 1372105, 7997213$... and $0, 8, 54, 322, 1884, 10988, 64050, 373318, 2175864, 12681872, \ldots$ are new.

4.1. *Constructions*

1: We start with $\begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$, a solution of (3.1), thanks to theorem 2, by multiplying recursively by $\mathfrak{M}_1$, we obtain

$$\begin{pmatrix} 11 \\ 13 \\ 17 \end{pmatrix}, \begin{pmatrix} 71 \\ 69 \\ 99 \end{pmatrix}, \begin{pmatrix} 407 \\ 409 \\ 577 \end{pmatrix}, \begin{pmatrix} 2379 \\ 2377 \\ 3363 \end{pmatrix}, \begin{pmatrix} 13\,859 \\ 13\,861 \\ 19\,601 \end{pmatrix}, \begin{pmatrix} 80\,783 \\ 80\,781 \\ 114\,243 \end{pmatrix},$$
$$\begin{pmatrix} 470\,831 \\ 470\,833 \\ 665\,857 \end{pmatrix}, \begin{pmatrix} 2744\,211 \\ 2744\,209 \\ 3880\,899 \end{pmatrix}, \begin{pmatrix} 15\,994\,427 \\ 15\,994\,429 \\ 22\,619\,537 \end{pmatrix}, \ldots \text{ are also solutions of (3.1).}$$

We get, $x = 3, 11, 71, 407, 2379, 13859, 80783, 470831, 2744211, 15\,994\,427, \ldots$
But $x = 2X + 1$, so

$$X = 1, 5, 35, 203, 1179, 6929, 40391, 235415, 1372105, 7997213, \ldots$$

is a new sequence of odd integers formed by solutions X of our triangular equation (1.1).

2: We start with $\begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$ a solution of (3.1), thanks to theorem 2, by multiplying recursively by $\mathfrak{M}_2$, we obtain

$$\begin{pmatrix} 17 \\ 21 \\ 27 \end{pmatrix}, \begin{pmatrix} 109 \\ 113 \\ 157 \end{pmatrix}, \begin{pmatrix} 645 \\ 649 \\ 915 \end{pmatrix}, \begin{pmatrix} 3769 \\ 3773 \\ 5333 \end{pmatrix}, \begin{pmatrix} 21\,977 \\ 21\,981 \\ 31\,083 \end{pmatrix}, \begin{pmatrix} 128\,101 \\ 128\,105 \\ 181\,165 \end{pmatrix},$$
$$\begin{pmatrix} 746\,637 \\ 746\,641 \\ 1055\,907 \end{pmatrix}, \begin{pmatrix} 4351\,729 \\ 4351\,733 \\ 6154\,277 \end{pmatrix}, \begin{pmatrix} 25\,363\,745 \\ 25\,363\,749 \\ 35\,869\,755 \end{pmatrix}, \ldots \text{ are also solutions of (3.1).}$$

We get, $x = 1, 17, 109, 645, 3769, 21977, 128101, 746637, 4351729, 25\,363\,745, \ldots$
But $x = 2X + 1$, then

$$X = 0, 8, 54, 322, 1884, 10988, 64050, 373318, 2175864, 12681872, \ldots$$

is a new sequence of even integers formed by solutions X of our triangular equation (1.1).

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A NUMERICAL EXAMPLE FOR THE INTERSECTION OF THE COMPATIBLE QUASILINEAR EXTENSIONS OF A PARTIAL ORDER

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Abstract. We present a numerical example for the intersection of the maximal compatible extensions of a partial order.

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1. Introduction

One of the typical problems in the theory of ordered algebras is to describe the intersection of the compatible linear extensions of $\leq_r$ in a partially ordered algebraic structure $(A, F, \leq_r)$. If $F = \emptyset$, then the answer can be found in [10]. Szpilrajn proved that any partial order $\leq_r$ (or r) on a set A can be extended to a linear order $\leq_R$ (or R). He also proved that any partial order is the intersection of its linear extensions. For $F = \{f\}$, where $f : A \to A$ is a unary operation with the natural compatibility property ($x \leq_r y$ implies $f(x) \leq_r f(y)$ for all $x, y \in A$) the above problem has been investigated. Szigeti and Nagy proved that the partially ordered mono-unary algebra $(A, f, \leq_r)$ has a compatible linear extension if and only if f is acyclic (see [8]). For an acyclic partially ordered unary algebra $(A, f, \leq_r)$ the intersection of the compatible linear extensions of r is determined in [7]. Foldes and Szigeti in [6] proved that any compatible partial order on the unary algebra (A, f) can be extended to a so called compatible f-quasilinear partial order. A consequence of this fact is that the maximal compatible partial orders are exactly the compatible f-quasilinear partial orders on (A, f). If the function f has no proper cycle (acyclic), then a compatible partial order is f-quasilinear if and only if it is linear. For an acyclic partially ordered unary algebra $(A, f, \leq_r)$ the intersection of the compatible linear extensions of r is determined in [7]. A new characterization of the maximal compatible extensions of r and the description of the intersection of these extensions in an arbitrary (possibly non-acyclic) triple $(A, f, \leq_r)$ can be seen in [9].

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In the first part of our paper we investigate a finite variant of the example for the mono-unary partially ordered algebra in [9]. The second part of our paper we use computer calculations to provide the explicit description of the intersection in the above mentioned finite example.

In order to present a self contained treatment, in section 2 we provide the necessary prerequisites.

2. Preliminary definitions and results

Let (A, f) be a unary algebra. A partial order $\leq_r$ on A is called *compatible* on (A, f) if the function (unary operation) $f : A \longrightarrow A$ is an order endomorphism with respect to $\leq_r$ ($x \leq_r y$ implies $f(x) \leq_r f(y)$ for all $x, y \in A$). The equivalence relation $\sim_f$ is defined in [9] as follows: for $x, y \in A$ let $x \sim_f y$ if $f^k(x) = f^l(y)$ for some integers $k \geq 0$ and $l \geq 0$. The equivalence class $[x]_f$ of an element $x \in A$ is called the *f-component* of x. Clearly, $[x]_f$ contains the *f-orbit* of x:

$$\{x, f(x), \ldots, f^k(x), \ldots\} \subseteq [x]_f \ .$$

An element $c \in A$ is called *cyclic* with respect to f, if $f^m(c) = c$ for some integer $m \geq 1$. The *period* of a cyclic element c is

$$n = n(c) = \min\{m \mid m \geq 1 \text{ and } f^m(c) = c\}$$

and $f^k(c) = f^l(c)$ holds if and only if $k - l$ is divisible by n. The *cycle* $C = \{c, f(c), \ldots, f^{n-1}(c)\}$ has exactly n elements and $f(C) = C$. The f-orbit of x is finite if and only if $[x]_f$ contains a cyclic element. The function f has a *proper cycle* if there exists a cyclic element $c \in A$ with respect to f such that $n(c) \geq 2$. When $n(c) = 1$ then $f(c) = c$. In this case $c \in A$ is a *fixed point* of f. The concept of the f-prohibited pair was introduced by Foldes and Szigeti in [6]. A pair $(x, y) \in A \times A$ is called *f-prohibited*, if we can find integers $k \geq 0$, $l \geq 0$ and $n \geq 2$ such that n is not a divisor of $k - l$, the elements $f^k(x), f^{k+1}(x), \ldots, f^{k+n-1}(x)$ are distinct and $f^{k+n}(x) = f^k(x) = f^l(y)$. For an f-prohibited pair (x, y) and for the integers $k \geq 0$, $n \geq 2$ as above, $f^k(x)$ is a cyclic element in $[x]_f = [y]_f$ of period n.

Proposition 1 (see [6]). *If $(A, f, \leq_r)$ is a partially ordered unary algebra, then we have the following.*

(1) *If (x, y) is f-prohibited, then (y, x) is also f-prohibited.*
(2) *If (x, y) is not f-prohibited, then $(f(x), f(y))$ is also not f-prohibited.*
(3) *If $[x]_f \neq [y]_f$, then (x, y) is not f-prohibited.*
(4) *If the f-orbit of x is infinite, then there is no cyclic element in $[x]_f$ and hence (x, y) is not f-prohibited (even in the case of $[x]_f = [y]_f$).*
(5) *If $(x, y) \in A \times A$ is an f-prohibited pair, then x and y are incomparable with respect to $\leq_r$ (in particular, any two distinct elements in a full cycle are incomparable with respect to $\leq_r$).*

A compatible partial order R on a unary algebra (A, f) is called *f-quasilinear*, if $(x, y) \in R$ or $(y, x) \in R$ for all non f-prohibited pairs $(x, y) \in A \times A$. We use of the following notations of [9]:

$$\mathcal{QL}(A, f, \leq_r) = \{R \mid r \subseteq R \subseteq A \times A \text{ is a compatible } f\text{-quasilinear partial order}\},$$

$$\mathcal{L}(A, f, \leq_r) = \{R \mid r \subseteq R \subseteq A \times A \text{ is a compatible linear order}\}.$$

The intersection

$$\mathbf{cl}(A, f, \leq_r) = \bigcap_{R \in \mathcal{QL}(A, f, \leq_r)} R$$

is called the *closure* of r with respect to (A, f). Since $\mathcal{QL}(A, f, \leq_r)$ is never empty by the Main Theorem of [6], the above definition gives a closure operator (with the monotone, idempotent and extensive properties) on the set of the compatible partial orders of (A, f).

An element $a \in A$ of the partially ordered unary algebra $(A, f, \leq_r)$ is

(1) *$\uparrow$-definite* if $f^p(a) <_r f^q(a)$ for some integers $0 \leq p < q$,
(2) *$\downarrow$-definite* if $f^q(a) <_r f^p(a)$ for some integers $0 \leq p < q$,
(3) *indefinite* if $f^p(a)$ and $f^q(a)$ are r-incomparable for all integers $0 \leq p, q$ with $f^p(a) \neq f^q(a)$.

It can be easily checked if $(A, f, \leq_r)$ is a partially ordered unary algebra and $a \in A$, then a satisfies exactly one of the above $\uparrow$-definite, $\downarrow$-definite and indefinite properties. Define the sets $M_\uparrow$, $M_\downarrow$ and M as in [7].

$$\begin{aligned} M_\uparrow = \{(x, y) \in A \times A \mid x \text{ is } \uparrow\text{-definite}, (\exists m)(\exists t) 0 \leq m \leq t, \\ f^t(x) \leq_r f^m(y) \neq f^m(x)\}, \\ M_\downarrow = \{(x, y) \in A \times A \mid x \text{ is } \downarrow\text{-definite}, (\exists m)(\exists t) 0 \leq t \leq m, \\ f^t(x) \leq_r f^m(y) \neq f^m(x)\}, \\ M = \{(x, y) \in A \times A \mid x \text{ is indefinite}, (\exists m)(\exists t)(\exists m')(\exists t') 0 \leq m \leq t, 0 \leq t' \leq m', \\ f^t(x) \leq_r f^m(y) \neq f^m(x), f^{t'}(x) \leq_r f^{m'}(y) \neq f^{m'}(x)\}. \end{aligned}$$

We shall make use of the set

$$P = \{(x, y) \in A \times A \mid (x, y) \text{ is not } f\text{-prohibited}\}.$$

One of the main theorems of [9] gives a complete description of the intersection of the maximal compatible extensions of r:

Theorem 1. *If $(A, f, \leq_r)$ is a partially ordered unary algebra, then*

$$\boldsymbol{cl}(A, f, \leq_r) = (M_\uparrow \cup M_\downarrow \cup M \cup \{(x, x) \mid x \in A\}) \cap P.$$

3. A NUMERICAL EXAMPLE

The following example can be found in [9]. Let

$$2 \le p_1 < p_2 < p_3 < q_1 < q_2 < q_3 < q_4 < r$$

be a sequence of prime numbers and consider the infinite set

$$A = \Big\{ p_1^{m_1} p_2^{m_2} p_3^{m_3} q_1^{n_1} q_2^{n_2} q_3^{n_3} q_4^{n_4} r^k \mid \\ m_i \ge 0, n_j \ge 0, i \in \{1,2,3\}, j \in \{1,2,3,4\}, k \ge 0 \Big\}$$

of integers. A natural partial order on A is the divisibility relation: for $x, y \in A$ we use the notation $x \le_d y$ if $x \mid y$.

Define the function $f : A \to A$ on $x = p_1^{m_1} p_2^{m_2} p_3^{m_3} q_1^{n_1} q_2^{n_2} q_3^{n_3} q_4^{n_4} r^k$ by

$$f(p_1^{m_1} p_2^{m_2} p_3^{m_3} q_1^{n_1} q_2^{n_2} q_3^{n_3} q_4^{n_4} r^k) = p_1^{m_3} p_2^{m_1} p_3^{m_2} q_1^{n_1+n_2} q_2^{n_3} q_3^{n_4} r^{\langle k-1 \rangle},$$

where $\langle l \rangle = l$ if $l \ge 0$ and $\langle l \rangle = 0$ if $l < 0$. Note that $\langle \langle l \rangle - 1 \rangle = \langle l - 1 \rangle$. Clearly, $x \le_d y$ implies that $f(x) \le_d f(y)$. We shall make use of the following calculations:

$$f^2(p_1^{m_1} p_2^{m_2} p_3^{m_3} q_1^{n_1} q_2^{n_2} q_3^{n_3} q_4^{n_4} r^k) = p_1^{m_2} p_2^{m_3} p_3^{m_1} q_1^{n_1+n_2+n_3} q_2^{n_4} r^{\langle k-2 \rangle},$$

$$f^3(p_1^{m_1} p_2^{m_2} p_3^{m_3} q_1^{n_1} q_2^{n_2} q_3^{n_3} q_4^{n_4} r^k) = p_1^{m_1} p_2^{m_2} p_3^{m_3} q_1^{n_1+n_2+n_3+n_4} r^{\langle k-3 \rangle},$$

$$f^4(p_1^{m_1} p_2^{m_2} p_3^{m_3} q_1^{n_1} q_2^{n_2} q_3^{n_3} q_4^{n_4} r^k) = p_1^{m_3} p_2^{m_1} p_3^{m_2} q_1^{n_1+n_2+n_3+n_4} r^{\langle k-4 \rangle},$$

$$f^5(p_1^{m_1} p_2^{m_2} p_3^{m_3} q_1^{n_1} q_2^{n_2} q_3^{n_3} q_4^{n_4} r^k) = p_1^{m_2} p_2^{m_3} p_3^{m_1} q_1^{n_1+n_2+n_3+n_4} r^{\langle k-5 \rangle},$$

$$f^6(p_1^{m_1} p_2^{m_2} p_3^{m_3} q_1^{n_1} q_2^{n_2} q_3^{n_3} q_4^{n_4} r^k) = p_1^{m_1} p_2^{m_2} p_3^{m_3} q_1^{n_1+n_2+n_3+n_4} r^{\langle k-6 \rangle}.$$

It follows that $f^6(x) \le_d f^3(x)$ and $f^{\langle k-3 \rangle + 6}(x) = f^{\langle k-3 \rangle + 3}(x)$ for all $x \in A$. Thus $f^6(x) \ne f^3(x)$ implies that x is $\downarrow$-definite. Obviously, $f^6(x) = f^3(x)$ holds if and only if $0 \le k \le 3$. For all $x \in A$ and $i \in \{1,2,4,5\}$ the following equivalences are straightforward consequences of the above calculations:

(P1) $x \le_d f^i(x) \iff m_1 = m_2 = m_3, n_2 = n_3 = n_4 = k = 0$ and $x = f(x)$,
(P2) $x \le_d f^3(x) \iff n_2 = n_3 = n_4 = k = 0$ and $x = f^3(x)$,
(P3) $f^i(x) \le_d x \iff m_1 = m_2 = m_3, n_2 = n_3 = n_4 = 0$ and $x = f(x) r^{k - \langle k-1 \rangle}$,
(P4) $f^3(x) \le_d x \iff n_2 = n_3 = n_4 = 0$ and $x = f^3(x) r^{k - \langle k-3 \rangle}$.

There is no $\uparrow$-definite element in the set A, so the set $M_\uparrow$ is empty. The element x is indefinite if and only if one of the following conditions holds:

(Q1) $k = 0$,
(Q2) $k = 1$ and $\{n_2, n_3, n_4\} \ne \{0\}$,
(Q3) $k = 2$ and $\{n_3, n_4\} \ne \{0\}$,
(Q4) $k = 3$ and $n_4 \ne 0$.

In all other cases x is $\downarrow$-definite.

Take

$x_1 = p_1^1 p_2^2 p_3^3 q_1 q_2 q_3 q_4 r^5, x_2 = f^5(x_1) = p_1^3 p_2^1 p_3^2 q_1^4$ and $y = p_1^2 p_2^3 p_3^4 q_1 q_2 q_3 q_4 r^4$.

Now $[x_1]_f = [x_2]_f \neq [y]_f$ ensures that (x_1, y) and (x_2, y) are non f-prohibited pairs. Since $f^6(x_1) <_d f^3(x_1)$ and $\{x_2, f(x_2), f^2(x_2)\}$ is an antichain, we obtain that x_1 is $\downarrow$-definite and x_2 is indefinite. The relations

$$f^5(x_1) \leq_d f^7(y) \neq f^7(x_1) \text{ and } f^0(x_2) \leq_d f^7(y) \neq f^7(x_2)\,,$$
$$f^6(x_2) \leq_d f^4(y) \neq f^4(x_2)$$

imply that $(x_1, y) \in M_\downarrow \cap P$ and $(x_2, y) \in M \cap P$. Observe that (x_1, y) and (x_2, y) are incomparable pairs with respect to $\leq_d$ and

$$(x_1, y), (x_2, y) \in (M_\downarrow \cup M \cup \{(x, x) \mid x \in A\}) \cap P = \mathbf{cl}(A, f, \leq_d)$$

by Theorem 2.2. The full description of $M_\downarrow$ and M would require a lengthy examination.

Now we consider the following finite subset of A:

$$B = \Big\{2^{m_1} 3^{m_2} 5^{m_3} 7^{n_1} 11^{n_2} 13^{n_3} 17^{n_4} 19^k \mid$$
$$0 \leq m_i, n_j, k \leq K, i \in \{1,2,3\}, j \in \{1,2,3,4\}\Big\}.$$

The pair $(B, \leq_d)$ is a partially ordered set, where $\leq_d$ is the divisibility relation as above. The function $g : B \to B$ on $x \in B$ as follows:

$$\begin{aligned} g(x) &= g\left(2^{m_1} 3^{m_2} 5^{m_3} 7^{n_1} 11^{n_2} 13^{n_3} 17^{n_4} 19^k\right) \\ &= 2^{m_3} 3^{m_1} 5^{m_2} 7^{\rangle n_1 + n_2 \langle} 11^{n_3} 13^{n_4} 19^{\langle k-1 \rangle}, \end{aligned}$$

where

$$\rangle t \langle = \begin{cases} t, & \text{if } t \leq K, \\ K, & \text{if } t > K. \end{cases}$$

and

$$\langle l \rangle = \begin{cases} l, & \text{if } l \geq 0, \\ 0, & \text{if } l < 0. \end{cases}$$

Note that $\rangle\rangle t \langle + s \langle = \rangle t + s \langle$ and $\langle \langle l \rangle - 1 \rangle = \langle l - 1 \rangle$ for all positive integers.

The function g has the natural compatibility property: $x \leq_d y$ implies $g(x) \leq_d g(y)$ for all $x, y \in B$. For the iterates of g we have:

$$\begin{aligned} g^2(x) &= 2^{m_2} 3^{m_3} 5^{m_1} 7^{\rangle n_1 + n_2 + n_3 \langle} 11^{n_4} 19^{\langle k-2 \rangle}, \\ g^3(x) &= 2^{m_1} 3^{m_2} 5^{m_3} 7^{\rangle n_1 + n_2 + n_3 + n_4 \langle} 19^{\langle k-3 \rangle}, \\ g^4(x) &= 2^{m_3} 3^{m_1} 5^{m_2} 7^{\rangle n_1 + n_2 + n_3 + n_4 \langle} 19^{\langle k-4 \rangle}, \\ g^5(x) &= 2^{m_2} 3^{m_3} 5^{m_1} 7^{\rangle n_1 + n_2 + n_3 + n_4 \langle} 19^{\langle k-5 \rangle}, \\ g^6(x) &= 2^{m_1} 3^{m_2} 5^{m_3} 7^{\rangle n_1 + n_2 + n_3 + n_4 \langle} 19^{\langle k-6 \rangle}. \end{aligned}$$

$x \in B$ is a fixed point of g if

$$\begin{aligned} x &= 2^{m_1}3^{m_2}5^{m_3}7^{n_1}11^{n_2}13^{n_3}17^{n_4}19^{k} \\ &= 2^{m_3}3^{m_1}5^{m_2}7^{\rangle n_1+n_2\langle}11^{n_3}13^{n_4}19^{\langle k-1\rangle} = g(x). \end{aligned}$$

The above equality holds if and only if $m_1 = m_2 = m_3$, $n_2 = n_3 = n_4 = k = 0$ and $m_1, n_1 \in \{0, 1, \ldots, K\}$, therefore the number of fixed points is $(K+1)^2$. If the exponent of every prime number is at most 1, then $|B| = 2^8 = 256$ and the function g has four fixed points:

(a) $m_1 = m_2 = m_3 = 0$ and $n_1 = 0$, then x_1=1;
(b) $m_1 = m_2 = m_3 = 0$ and $n_1 = 1$, then x_2=7;
(c) $m_1 = m_2 = m_3 = 1$ and $n_1 = 0$, then x_3=30;
(d) $m_1 = m_2 = m_3 = 1$ and $n_1 = 1$, then x_4=210.

Obviously, the fixed points of g are in different g-components. If $K = 1$ there it is easy to see that:

$$[1]_g = \{1, 19\} \qquad \text{and} \qquad [30]_g = \{30, 570\}.$$

The g-components of the other two fixed points contain more elements. If $x \in B$ is a fixed point, then the equivalence class $[x]_g$ has no proper cyclic elements. Clearly, any fixed point of g is an indefinite element in the partially ordered unary algebra $(B, g, \leq_d)$.

The proper cyclic elements can be obtained from the properties (P2) and (P4). If $K = 1$, it is deemed to be sufficient for the property (P2). $c \in A$ is a cyclic element of g, according to the property (P2), when $c = g^3(c)$, i. e. $n_2 = n_3 = n_4 = k = 0$ and $m_i, n_1 \in \{0, 1\}$, $i \in \{1, 2, 3\}$, furthermore $m_1 \neq m_3$. We have four proper cycle:

$$C_1 = \{2, 3, 5\}; \quad C_2 = \{6, 10, 15\}; \quad C_3 = \{14, 21, 35\}; \quad C_4 = \{42, 70, 105\}.$$

We note that $[2]_g$, $[6]_g$, $[14]_g$ and $[42]_g$ are disjoint g-components of $(B, g, \leq_d)$. It is clear, that there is no $\uparrow$-definite element in the set B and the set $M_\uparrow$ is empty as in the above infinite example.

As we can see it above, the calculation of the elements of the intersection is time consuming manually. These particular calculations do not provide us to see the underlying structures.

4. Calculating intersections by Maple, MAGMA and a parallel program

For calculating the pairs of intersection we had three programs. At first we calculated them for $K = 1$ by Maple, but it was very slow and a lot of resources were used.

Secondary we made it by MAGMA [3] which was stable and produced the correct results. E.g. MAGMA used 157 seconds only in CPU time. The version of MAGMA was 2.18-4.

The third one was a parallel program which was made in JCluster [2]. It is an environment using JAVA to make and run parallel algorithms (programs) on more then one workstations. This program used very simple technics e.g. exchange a lot of "for" cycles by parellel processes. To reduce the CPU time we used 1, 2, 4 , 8 and 16 workstations like a super PC with 1GB network. Beacuse of it we have to mention that in the case when large number of workstations or cores were used the CPU time has increased by the network communication delay and the data broadcasting (see the table below). Certainly this program used synchronization tools also to get correct results.

The running times (in CPU time) collected in the following table:

Environment	CPU time (sec)	Number of cores
Maple	425	4
MAGMA	157	4
Parallel program (JCluster)	67	1
Parallel program (JCluster)	55	2
Parallel program (JCluster)	42	4
Parallel program (JCluster)	35	8

5. Calculation of the finite example

The actual implementation of the calculation is primarily based on the previous *Maple* and *Magma* scripts. All implementations provide the same results but the new C++ version run more efficiently. We can see the dataflow on Figure 1.

The `prims` is the array of prime numbers. In this case it contains only eight elements.

$$\text{prims} = [2,3,5,7,11,13,17,19]$$

In the definition of the set B the exponents of the factors must be non-negative integers. The K value is the upper bound of them:

$$0 \le m_i, n_j, k \le K, \quad \forall i \in \{1,2,3\}, \quad j \in \{1,2,3,4\}.$$

We can consider the exponents of the factorization of the elements of B as the digits of a number in $K+1$ based numerical system. From this reason $|B| = (K+1)^8$. In most cases in the implementation we use lists instead of sets. The lists are indexed by non-negative integers. The first index is the 0. The index of any B element is the value of the exponents of prime factorization in the mentioned $K+1$ based numerical system. For example when $K = 1$ then $30 = 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^0 \cdot 11^0 \cdot 13^0 \cdot 17^0 \cdot 19^0$, the index is 7. Let denote $indices$ a function which calculate this index from elements of B.

We need to calculate often with powers of g function. The `CP0` contains the indices of following elements in `B`:

$$[x, g(x), g^2(x), g^3(x), g^4(x), g^5(x), g^6(x)],$$

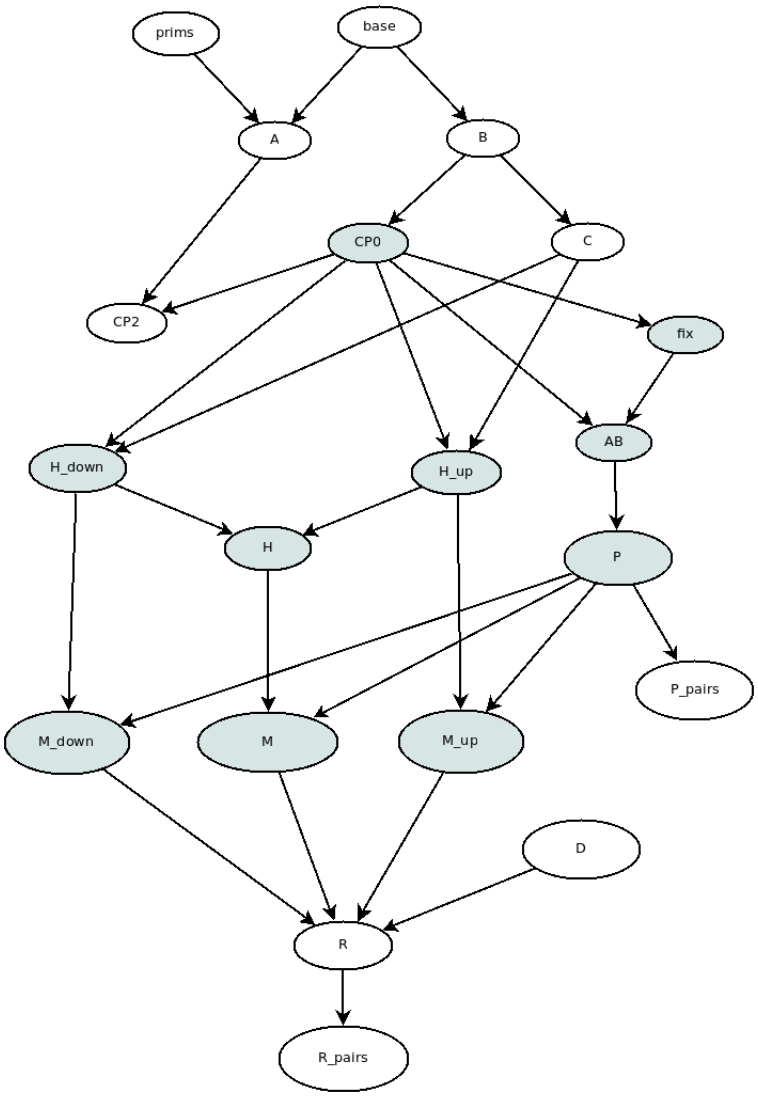

FIGURE 1. The dataflow of the calculation.

where $x \in B$. This list of lists does not stored instead it is a function of the index of the elements of B. The fix set contains only the indices of B elements where the values are the same in `CP0` vectors. We cache the indices of B where the power of $g(x)$ is not in set fix.

The P denotes the set of prohibited pairs. The implementation also considers the indices. We can define P as the following set.

$$P = \{(x,y)|x,y \in \mathrm{AB}, x_k = x_{k+3}, x_k = y_l, (k-l) \mod 3 \neq 0, \\ k \in [1;4], l \in [1;7]\}$$

The C matrix is only a boolean function in the implementation,

$$C(x,y) = \begin{cases} true, & x_i \leq y_i, i \in [1;8], \\ false, & else, \end{cases}$$

where $x, y \in \mathrm{B}$. The `H_down`, `H` and `H_up` sets contain the elements of the set B,

$$\mathrm{H_up} = \left\{x \in \mathrm{CP0}, x_j \neq x_k, C(x_j, x_k) = true, \forall j,k, j \neq k\right\},$$

$$\mathrm{H_down} = \left\{x \in \mathrm{CP0}, x_j \neq x_k, C(x_k, x_j) = true, \forall j,k, j \neq k\right\},$$

$$\mathrm{H} = \mathrm{CP0} \setminus (\mathrm{H_up} \cup \mathrm{H_down}).$$

The `M_down`, `M` and `M_up` are the sets of pairs from A and defined as the following:

$$\mathrm{M_down} = \{(x,y)|x \in \mathrm{H_down}, y \in \mathrm{CP0}, \exists m,t \in [1;7], t \leq m, \\ C(x_t, y_m) = true, x_m \neq y_m, (x,y) \notin \mathrm{P}\},$$

$$\mathrm{M} = \{(x,y)|x \in \mathrm{H}, y \in \mathrm{CP0}, \exists m,t,m',t' \in [1;7], m \leq t, t' \leq m', C(x_t, y_m) = true, \\ C(x_{t'}, y_{m'}) = true, x_m \neq y_m, x_{m'} \neq y_{m'}, (x,y) \notin \mathrm{P}\},$$

$$\mathrm{M_up} = \{(x,y)|x \in \mathrm{H_up}, y \in \mathrm{CP0}, \exists m,t \in [1;7], m \leq t, \\ C(x_t, y_m) = true, x_m \neq y_m, (x,y) \notin \mathrm{P}\}.$$

6. Visualization of the map of unary algebra

For visualizing the map of unary algebra for $K = 1$ we used GraphViz [1]. At the first step, we collected the elements of the map, and save them as a text file with `gv` extension.

Unfortunately, the default radial layout did not make possible to use those graph image as a figure. Instead, we have implemented an own graph layout calculation by using the `turtle` module of the Python library [4].

At the last step, we have converted the resulted layout to *TikZ* format [5]. We can see the results of these visualization process at the Figure 2, 3, 4, 5 and 6.

Thanks to the resulted map we have obtained eight disjunct sub-graphs. Moreover, the symmetry among the sub-graphs has revealed. The following sub-graphs are pairwise isomorph:

- $[1]_g$ and $[30]_g$,
- $[2]_g$ and $[6]_g$,
- $[7]_g$ and $[210]_g$,
- $[14]_g$ and $[42]_g$.

pair-wise isomorphism parts. There are four components which contain proper cycle with 3 elements.

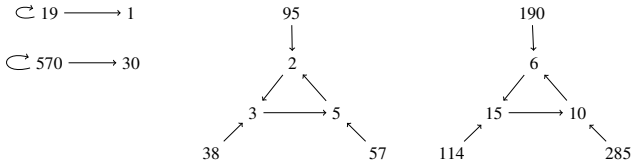

FIGURE 2. The graphs of $[1]_g$, $[30]_g$, $[2]_g$ and $[6]_g$ g-components

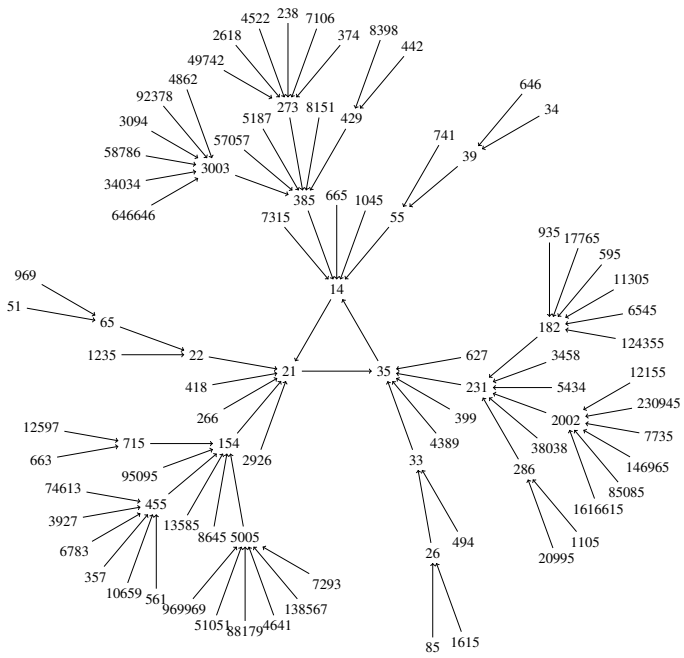

FIGURE 3. The graph of $[14]_g$ g-component

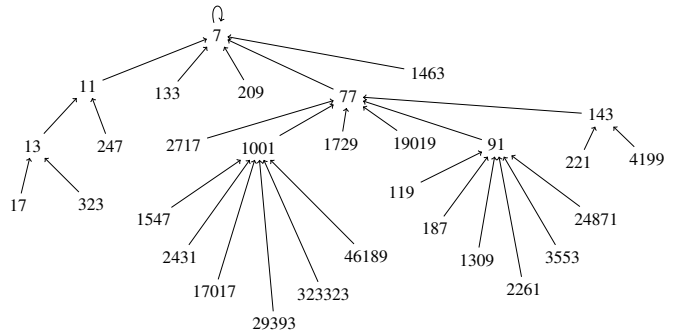

FIGURE 4. The graph of $[7]_g$ g-component

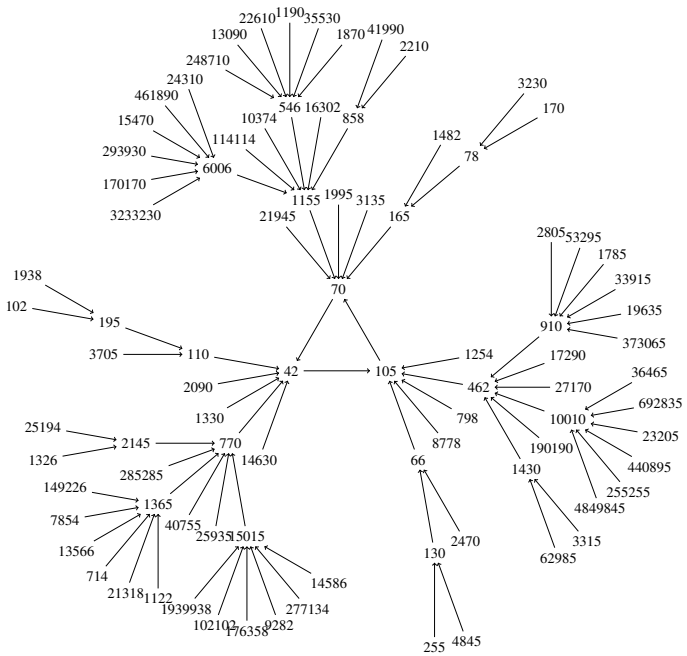

FIGURE 5. The graph of $[42]_g$ g-component

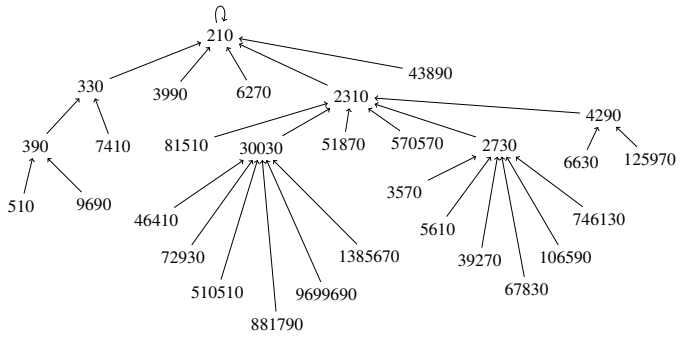

FIGURE 6. The graph of $[210]_g$ g-component

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SHARP POWER MEAN BOUNDS FOR THE SECOND NEUMAN MEAN

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Abstract. In this paper, we prove that the double inequality $M_\alpha(a,b) < N_{GQ}(a,b) < M_\beta(a,b)$ holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq 2\log 2/(5\log 2 - 2\log \pi) = 1.1785\cdots$ and $\beta \geq 4/3$,where $N_{GQ}(a,b) = [G(a,b) + Q^2(a,b)/U(a,b)]/2$ is the second Neuman mean, $G(a,b) = \sqrt{ab}$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$ and $U(a,b) = (a-b)/[\sqrt{2}\tan^{-1}((a-b)/\sqrt{2ab})]$ are the geometric, quadratic and Yang mean of a and b, respectively.

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Keywords: Schwab-Borchardt mean, second Neuman mean, power mean, geometric mean, Yang mean

1. Introduction

For $p \in \mathbb{R}$ and $a,b > 0$ with $a \neq b$, the pth power mean $M_p(a,b)$[14] of a and b is defined by

$$M_p(a,b) = \begin{cases} (\frac{a^p+b^p}{2})^{1/p} & \text{if } p \neq 0 \\ \sqrt{ab} & \text{if } p = 0. \end{cases}$$

It is well known that the power mean $M_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b > 0$ with $a \neq b$. Many bivariate means are the special cases of the power mean, for example, $M_0(a,b) = G(a,b) = \sqrt{ab}$, $M_1(a,b) = A(a,b) = (a+b)/2$ and $M_2(a,b) = Q(a,b) = \sqrt{(a^2+b^2)/2}$ are respectively the arithmetic, geometric and quadratic means. Many properties for the power mean can be found in the literature[2–5, 11, 22, 24, 26, 31, 36].

The Schwab-Borchardt mean $SB(a,b)$[16, 17] defined by

$$SB(a,b) = \begin{cases} \frac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)} & if\ a < b \\ \frac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)} & if\ a > b, \end{cases}$$

where $\cos^{-1}(x)$ and $\cosh^{-1}(x) = \log(x + \sqrt{x^2-1})$ are the inverse cosine and inverse hyperbolic cosine functions, respectively.It is well-known that $SB(a,b)$ is strictly

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increasing in both a and b, nonsymmetric and homogeneous of degree 1 with respect to a and b. Many symmetric bivariate means are special cases of the Schwab-Borchardt mean, for example, the first Seiffert mean $P(a,b)$, second Seiffert mean $T(a,b)$, Neuman-Sándor mean $M(a,b)$, logarithmic mean $L(a,b)$ and Yang mean $U(a,b)$[29] are respectively defined by

$$P(a,b)=\frac{a-b}{2\sin^{-1}\left[(a-b)/(a+b)\right]}=SB\big[G(a,b),A(a,b)\big],$$

$$T(a,b)=\frac{a-b}{2\tan^{-1}\left[(a-b)/(a+b)\right]}=SB\big[A(a,b),Q(a,b)\big],$$

$$M(a,b)=\frac{a-b}{2\sinh^{-1}\left[(a-b)/(a+b)\right]}=SB\big[Q(a,b),A(a,b)\big],$$

$$L(a,b)=\frac{a-b}{2\tanh^{-1}\left[(a-b)/(a+b)\right]}=SB\big[A(a,b),G(a,b)\big],$$

and

$$U(a,b)=\frac{a-b}{\sqrt{2}\tan^{-1}\left[(a-b)/\sqrt{2ab}\right]}=SB\big[G(a,b),Q(a,b)\big].$$

In 2014, Neuman [15] found a new bivariate means derived from the Schwab-Borchardt mean

$$N(a,b)=\frac{1}{2}\Big[a+\frac{b^2}{SB(a,b)}\Big].$$

We call $N(a,b)$ is the second Neuman mean[19]. Let $a>b$, $v=(a-b)/(a+b)\in(0,1)$, then Neuman [15] gave explicit formulas

$$N_{AG}(a,b)=\frac{1}{2}A(a,b)\Big[1+(1-v^2)\frac{\tanh^{-1}(v)}{v}\Big],N_{GA}(a,b)=\frac{1}{2}A(a,b)\Big[\sqrt{1-v^2}+\frac{\sin^{-1}(v)}{v}\Big],$$

$$N_{QA}(a,b)=\frac{1}{2}A(a,b)\Big[\sqrt{1+v^2}+\frac{\sinh^{-1}(v)}{v}\Big],N_{AQ}(a,b)=\frac{1}{2}A(a,b)\Big[1+(1+v^2)\frac{\tan^{-1}(v)}{v}\Big].$$

and proved that the inequalities

$$G(a,b)<N_{AG}(a,b)<N_{GA}(a,b)<A(a,b)<N_{QA}(a,b)<N_{AQ}(a,b)<Q(a,b)$$

for $a,b>0$ with $a\neq b$.

Very recently, Shen et. al. [21] found a new mean $N_{GQ}(a,b)$ derived from the Schwab- Borchardt mean. Let $a>b$, $u=(a-b)/\sqrt{2ab}\in(0,+\infty)$, then explicit formulas for $N_{GQ}(a,b)$ be in the following:

$$N_{GQ}(a,b)=\frac{1}{2}G(a,b)\Big[1+(1+u^2)\frac{\tan^{-1}(u)}{u}\Big].$$

Recently, the bounds involving the power and the Schwab-Borchardt means has been the subject of intensive research. In particular, many remarkable inequalities for the power mean, Schwab-Borchardt mean and their related means can be found in the literature [1, 6–10, 12, 13, 18–21, 23, 25, 27–30, 32–35].

Radó[20] (see also [13, 18, 23]) proved that the double inequalities

$$M_p(a,b) < L(a,b) < M_q(a,b), M_\lambda(a,b) < I(a,b) < M_\mu(a,b)$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $p \leq 0$, $q \geq 1/3$, $\lambda \leq 2/3$ and $\mu \geq \log 2$, where $I(a,b) = (a^a/b^b)^{1/(a-b)}/e$ is the indentric mean of a and b.

In [7–10, 12, 28], the authors proved that $p_1 = \log 2/\log \pi$, $q_1 = 2/3$, $p_2 = \log 2/(\log \pi - \log 2)$, $q_2 = 5/3$, $p_3 = \log 2/\log[2\log(1+\sqrt{2})]$ and $q_3 = 4/3$ are the best possible parameters such that the double inequalities

$$\begin{aligned} M_{p_1}(a,b) &< P(a,b) < M_{q_1}(a,b), \\ M_{p_2}(a,b) &< T(a,b) < M_{q_2}(a,b), \\ M_{p_3}(a,b) &< M(a,b) < M_{q_3}(a,b) \end{aligned}$$

hold for all $a,b > 0$ with $a \neq b$.

Chu [6] and Yang [30] proved that the double inequalities

$$M_{\lambda_1}(a,b) < X(a,b) < M_{\mu_1}(a,b), M_{\lambda_2}(a,b) < U(a,b) < M_{\mu_2}(a,b)$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\lambda_1 \leq 1/3$, $\mu_1 \geq \log 2/(1+\log 2)$, $\lambda_2 \leq 2\log 2/(2\log \pi - \log 2)$ and $\mu_2 \geq 4/3$, where $X(a,b) = Ae^{G/P-1}$is the Sándor mean of a and b.

In [21], the authors proved the double inequalities

$$\begin{gathered} \alpha_1 Q(a,b) + (1-\alpha_1)G(a,b) < N_{GQ}(a,b) < \beta_1 Q(a,b) + (1-\beta_1)G(a,b), \\ \frac{\alpha_2}{G(a,b)} + \frac{1-\alpha_2}{Q(a,b)} < \frac{1}{N_{GQ}(a,b)} < \frac{\beta_2}{G(a,b)} + \frac{1-\beta_2}{Q(a,b)}, \\ \alpha_3 Q(a,b) + (1-\alpha_3)U(a,b) < N_{GQ}(a,b) < \beta_3 Q(a,b) + (1-\beta_3)U(a,b) \end{gathered}$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 2/3$, $\beta_1 \geq \pi/4$, $\alpha_2 \leq 0$, $\beta_2 \geq 1/3$, $\alpha_3 \leq 0$ and $\beta_3 \geq (\pi^2-8)/[4(\pi-2)] = 0.4094\cdots$

The main purpose of this paper is to present the best possible parameter α and β such that the double inequalities $M_\alpha(a,b) < N_{GQ}(a,b) < M_\beta(a,b)$ hold for all $a,b > 0$ with $a \neq b$.

2. Main result

In order to prove our main result we need a lemma, which we present in this section.

Lemma 1. *Let $p \in \mathbb{R}$, and*

$$\begin{aligned} f(x) = x^{2p+2} + x^{2p+1} + 5x^{2p} + x^{2p-1} + (2p-3)x^{p+3} - 4x^{p+2} + 4x^p \\ - (2p-3)x^{p-1} - x^3 - 5x^2 - x - 1 \end{aligned} \tag{2.1}$$

Then the following statements are true:

(1) *If $p = 4/3$, then $f(x) > 0$ for all $x \in (1,+\infty)$;*

(2) *If* $p = 2\log 2/(5\log 2 - 2\log \pi) = 1.1785\cdots$, *then there exists* $\lambda \in (1,+\infty)$ *such that* $f(x) < 0$ *for* $x \in (1,\lambda)$ *and* $f(x) > 0$ *for* $x \in (\lambda,+\infty)$.

Proof. For part (1), if $p = 4/3$, then (2.1) becomes

$$\begin{aligned} f(x) &= \frac{1}{3}(x^{2/3}-1)^3(3x^{8/3}-x^{7/3}+9x^2+6x^{4/3}+9x^{2/3}-x^{1/3}+3) \\ &= \frac{1}{3}(x^{2/3}-1)^3\big[2x^{8/3}+x^{7/3}(x^{1/3}-1)+9x^2+6x^{4/3} \\ &\qquad +8x^{2/3}+x^{1/3}(x^{1/3}-1)+3\big] \\ &> \frac{1}{3}(x^{2/3}-1)^3(2x^{8/3}+9x^2+6x^{4/3}+8x^{2/3}+3) \end{aligned} \tag{2.2}$$

for $x \in (1,+\infty)$.

Therefore, part (1) follows from (2.2).

For part (2), let $p = 2\log 2/(5\log 2 - 2\log \pi) = 1.1785\cdots$, $f_1(x) = f'(x)$, $f_2(x) = f_1'(x)$, $f_3(x) = f_2'(x)$, $f_4(x) = x^{5-p} f_3'(x)$. Then elaborated computations lead to

$$\lim_{x\to 1} f(x) = 0, \lim_{x\to+\infty} f(x) = +\infty, \tag{2.3}$$

$$\begin{aligned} f_1(x) &= 2(p+1)x^{2p+1}+(2p+1)x^{2p}+10px^{2p-1}+(2p-1)x^{2p-2} \\ &\quad +(p+3)(2p-3)x^{p+2}-4(p+2)x^{p+1}+4px^{p-1} \\ &\quad -(p-1)(2p-3)x^{p-2}-3x^2-10x-1 \end{aligned}$$

$$\lim_{x\to 1} f_1(x) = 24\big(p-\frac{4}{3}\big) < 0, \lim_{x\to+\infty} f_1(x) = +\infty, \tag{2.4}$$

$$\begin{aligned} f_2(x) &= 2(p+1)(2p+1)x^{2p}+2p(2p+1)x^{2p-1}+10p(2p-1)x^{2p-2} \\ &\quad +2(p-1)(2p-1)x^{2p-3}+(p+2)(p+3)(2p-3)x^{p+1} \\ &\quad -4(p+1)(p+2)x^p+4p(p-1)x^{p-2} \\ &\quad -(p-1)(p-2)(2p-3)x^{p-3}-6x-10 \end{aligned}$$

$$\lim_{x\to 1} f_2(x) = 24(2p+1)\big(p-\frac{4}{3}\big) < 0, \lim_{x\to+\infty} f_2(x) = +\infty, \tag{2.5}$$

$$\begin{aligned} f_3(x) &= 4p(p+1)(2p+1)x^{2p-1}+2p(4p^2-1)x^{2p-2}+20p(p-1)(2p-1)x^{2p-3} \\ &\quad +2(p-1)(2p-1)(2p-3)x^{2p-4}+(p+1)(p+2)(p+3)(2p-3)x^p \\ &\quad -4p(p+1)(p+2)x^{p-1}+4p(p-1)(p-2)x^{p-3} \\ &\quad -(p-1)(p-2)(p-3)(2p-3)x^{p-4}-6 \end{aligned}$$

$$\lim_{x\to 1} f_3(x) = 4(22p^3-33p^2+17p-12) < 0, \lim_{x\to+\infty} f_3(x) = +\infty, \tag{2.6}$$

$$\begin{aligned}
f_4(x) &= 4p(p+1)(4p^2-1)x^{p+3} + 4p(p-1)(4p^2-1)x^{p+2} \\
&\quad + 20p(p-1)(2p-1)(2p-3)x^{p+1} + 4(p-1)(p-2)(2p-1)(2p-3)x^p \\
&\quad + p(p+1)(p+2)(p+3)(2p-3)x^4 - 4p(p^2-1)(p+2)x^3 \\
&\quad + 4p(p-1)(p-2)(p-3)x - (p-1)(p-2)(p-3)(p-4)(2p-3) \\
&= a_0x^{p+3} + a_2x^{p+2} + a_4x^{p+1} + a_5x^p + a_1x^4 + a_3x^3 + a_6x + a_7. \qquad (2.7)
\end{aligned}$$

Note that

$$p+3 > 4 > p+2 > 3 > p+1 > p > 1 > 0, \qquad (2.8)$$

$$a_0 > 0, a_1 < 0, a_2 > 0, a_3 < 0, a_4 < 0, a_5 > 0, a_6 > 0, a_7 < 0, \qquad (2.9)$$

$$23p^2 - 43p + 12 = -6.7311\cdots < 0, 2p^3 - 37p^2 + 89p - 48 = 8.7726\cdots > 0, \qquad (2.10)$$

$$2p^3 + 119p^2 - 125p + 70 = 91.2430\cdots > 0, \qquad (2.11)$$

$$a_0 + a_1 = p(p^2-1)(2p^2+25p+22) > 0, \qquad (2.12)$$

$$a_2 + a_3 + a_4 = 4p(p-1)(23p^2-43p+12) \qquad (2.13)$$

$$a_5 + a_6 + a_7 = (p-1)(2-p)(2p^3-37p^2+89p-48), \qquad (2.14)$$

$$\sum_{i=0}^{4} a_i = p(p-1)(2p^3+119p^2-125p+70), \qquad (2.15)$$

It follows from (2.7)-(2.15) that

$$\begin{aligned}
f_4(x) &> (a_0+a_1)x^4 + (a_2+a_3+a_4)x^3 + (a_5+a_6+a_7)x \\
&> \sum_{i=0}^{4} a_i x^4 + (a_5+a_6+a_7)x > 0 \qquad (2.16)
\end{aligned}$$

for $x \in (1, +\infty)$.

From (2.16) we clearly see that $f_3(x)$ is strictly increasing on $(1, +\infty)$. Then (2.6) leads to the conclusion that there exists $\lambda_1 > 1$ such that $f_2(x)$ is strictly decreasing on $(1, \lambda_1]$ and strictly increasing on $[\lambda_1, +\infty)$.

It follows from (2.5) and the piecewise monotonicity of $f_2(x)$, we conclude that there exists $\lambda_2 \in (1, +\infty)$ such that $f_1(x)$ is strictly decreasing on $(1, \lambda_2]$ and strictly increasing on $[\lambda_2, +\infty)$.

From (2.4) and the piecewise monotonicity of $f_1(x)$ that there exists $\lambda_3 \in (1, +\infty)$ such that $f(x)$ is strictly decreasing on $(1, \lambda_3]$ and strictly increasing on $[\lambda_3, +\infty)$.

Therefore, part (2) follows from (2.3) and the piecewise monotonicity of $f(x)$. □

Theorem 1. *The double inequality*

$$M_\alpha(a,b) < N_{GQ}(a,b) < M_\beta(a,b),$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha \leq 2\log 2/(5\log 2 - 2\log \pi) = 1.1785\cdots$ *and* $\beta \geq 4/3$.

Proof. Since $N_{GQ}(a,b)$ and $M_p(a,b)$ are symmetric and homogenous of degree 1, we assume that $a > b > 0$. Let $x = a/b \in (1,+\infty)$, $p \in \mathbb{R}_+$. Then we have

$$\begin{aligned}&\log\big[N_{GQ}(a,b)\big]-\log\big[M_p(a,b)\big]\\&=\log\big[2\sqrt{x}(x-1)+\sqrt{2}(x^2+1)\tan^{-1}\big(\frac{x-1}{\sqrt{2x}}\big)\big]-\log\big[4(x-1)\big]-\frac{1}{p}\log\big(\frac{x^p+1}{2}\big).\end{aligned} \tag{2.17}$$

Let

$$\begin{aligned}F(x)=&\log\big[2\sqrt{x}(x-1)+\sqrt{2}(x^2+1)\tan^{-1}\big(\frac{x-1}{\sqrt{2x}}\big)\big]\\&-\log\big[4(x-1)\big]-\frac{1}{p}\log\big(\frac{x^p+1}{2}\big)\end{aligned} \tag{2.18}$$

Then simple computations lead to

$$\lim_{x\to 1^+} F(x)=0, \tag{2.19}$$

$$\lim_{x\to+\infty} F(x)=\frac{1}{p}\log 2+\log\pi-5\log\sqrt{2}, \tag{2.20}$$

$$F'(x)=\frac{x^{p+1}+2x^p-x^{p-1}-x^2+2x+1}{(x-1)(x^p+1)\big[2\sqrt{x}(x-1)+\sqrt{2}(x^2+1)\tan^{-1}\big(\frac{x-1}{\sqrt{2x}}\big)\big]}F_1(x), \tag{2.21}$$

where

$$F_1(x)=\frac{2\sqrt{x}(x-1)(x^{p-1}+1)}{x^{p+1}+2x^p-x^{p-1}-x^2+2x+1}-\sqrt{2}\tan^{-1}\big(\frac{x-1}{\sqrt{2x}}\big),$$

$$\lim_{x\to 1} F_1(x)=0, \tag{2.22}$$

$$\lim_{x\to+\infty} F_1(x)=-\frac{\sqrt{2}}{2}\pi<0, \tag{2.23}$$

$$F_1'(x)=-\frac{2(x-1)}{\sqrt{x}(x^2+1)(x^{p+1}+2x^p-x^{p-1}-x^2+2x+1)^2}f(x), \tag{2.24}$$

where $f(x)$ is defined by (2.1).

We divide the proof into four cases.

Case 1. $p=2\log 2/(5\log 2-2\log\pi)$ Then it follows from Lemma 1(2) and (2.24) that there exists $\lambda\in(1,+\infty)$ such that $F_1(x)$ is strictly increasing on $(1,\lambda]$ and strictly decreasing on $[\lambda,+\infty)$.

Equations (2.21) and (2.22)-(2.23) together with the piecewise monotonicity of $F_1(x)$ lead to the conclusion that there exists $\lambda_0\in(1,+\infty)$ such that $F(x)$ is strictly increasing on $(1,\lambda_0]$ and strictly decreasing on $[\lambda_0,+\infty)$.

Note that (2.20) becomes

$$\lim_{x\to+\infty} F(x)=0, \tag{2.25}$$

Therefore,

$$N_{GQ}(a,b) > M_{2\log 2/(5\log 2-2\log \pi)}(a,b)$$

for all $a,b>0$ with $a\neq b$ follows from (2.17)-(2.19) and (2.25) together with the piecewise monotonicity of $F(x)$.

Case 2 . $p=4/3$ Then it follows from Lemma 1(1) and (2.24) that $F_1(x)$ is strictly decreasing on $(1,+\infty)$.

Therefore,

$$N_{GQ}(a,b) < M_{4/3}(a,b)$$

for all $a,b>0$ with $a\neq b$ follows from (2.17)-(2.19) and (2.21)-(2.22) together with the monotonicity of $F_1(x)$.

Case 3 . $p>2\log 2/(5\log 2-2\log \pi)$ Then (2.20) leads to

$$\lim_{x\to+\infty} F(x) < 0, \tag{2.26}$$

Equations (2.17)-(2.18) together with inequality (2.26) imply that there exists large enough $M_0>1$ such that

$$N_{GQ}(a,b) < M_p(a,b)$$

for all $a,b>0$ with $x\in(M_0,+\infty)$.

Case 4 . $p<4/3$ Let $x>0$, $x\to 0$, then making use the Taylor expansion we get

$$\begin{aligned} &N_{GQ}(1,1+x)-M_p(1,1+x)\\ &\quad=\frac{2x\sqrt{x+1}+\sqrt{2}\big[(x+1)^2+1\big]\tan^{-1}\left(\frac{x}{\sqrt{2(x+1)}}\right)}{4x}-\left[\frac{1+(1+x)^p}{2}\right]^{1/p}\\ &\qquad\qquad=\frac{4-3p}{24}x^2+o(x^2). \end{aligned} \tag{2.27}$$

Equation (2.27) implies that there exists small enough $\delta_0>0$ such that

$$N_{GQ}(1,1+x) > M_p(1,1+x)$$

for all $a,b>0$ with $x\in(0,\delta_0)$.

Therefore, Theorem 1 follows easily from Cases 1-4 and the monotonicity of the function $p\to M_p(a,b)$. □

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EXISTENCE OF SOLUTIONS FOR FRACTIONAL INTERVAL–VALUED DIFFERENTIAL EQUATIONS BY THE METHOD OF UPPER AND LOWER SOLUTIONS

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Abstract. In this work we firstly study some important properties of fractional calculus for interval-valued functions and introduce the concepts of upper and lower solutions for interval-valued Caputo fractional differential equations. Then, we prove an existence result for interval-valued Caputo fractional differential equations by use of the method of upper and lower solutions. Finally several examples will be presented to illustrate our abstract results.

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Keywords: interval-valued functions, upper and lower solutions, existence of solutions, generalized Hukuhara derivative, fractional calculus for interval-valued functions

1. Introduction

Let $R(I)$ be the set of nonempty bounded, closed interval of $\mathbb{R}$. The main purpose of this paper is to introduce and investigate the following interval-valued differential equation with fractional derivative operator of Caputo type ((IVCFDE), for short):

$$(IVCFDE)\begin{cases} {}^{C}\mathfrak{D}_t^{\alpha}x(t) = f(t,x(t)),\ t \in J := [0,T], \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $0 < \alpha \leq 1$, ${}^{C}\mathfrak{D}_t^{\alpha}$ stands the Caputo fractional derivative for interval-valued functions (see Definition 14 below), $x : J \to R(I)$, $f \in C(J \times R(I); R(I))$ and $x_0 \in R(I)$, which will be be specified in Section 4. Interval analysis was introduced as an attempt to handle interval (nonstatistical, non-probabilistic) uncertainty which appears in many mathematical or computer models of some deterministic real-world phenomena. The development has experienced a long historical time. The first celebrated monograph which mainly dealt with interval analysis was written by Moore [34]. With this monograph, it began a new era of applications to error analysis for digital computers and engineering problems. For more details, please, see [3, 12, 15, 25, 33, 35–37] and the references therein.

Based on interval analysis, the concepts of integral and Hukuhara derivative (H-derivative, for short) for set functions were introduced by Aumann [5] and Hukuhara

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[10], respectively. Recently, the interval valued differential equations and fuzzy differential equations have began to be intensively studied (see, e.g., [6, 13, 16, 26–31, 38]). Plotnikova in [38] has studied a class of linear differential equations with π-derivative and linear differential inclusion; Abbas and Lupulescu [1] have investigated and obtained some results about set functional differential equations. According to the concept of H-derivative, some new notions of derivatives for interval-valued functions have been introduced, such as G-derivative, gH-derivative and π-derivative. It is known that, if the set of switching points is finite, then the notions of G-differentiability, gH-differentiability and π-differentiability will be coincided with interval-valued functions (for more details, ones can refer to [6, 42]). In accordance with this statement, we just consider the concept of gH-differentiability in this paper.

On the other hand, fractional differential equations (FDEs, for short) have become more and more popular and importance due to their numerous applications in many fields of science and engineering. Which involved in physics, population dynamics, chemical technology, biotechnology, aerodynamics, electrodynamics of complex mediums, polymer rheology, control of dynamical systems, etc. (see, e.g., [2, 8, 9, 14, 18–24, 47, 48]).

Although fractional calculus and interval-valued differential equations both are playing important roles in our real life. Up to now, fractional calculus for interval valued functions have just been studied by Lupulescu in [28] and the interval-valued Caputo fractional differential equations have not yet been addressed. Based on this motivation, the purpose of this paper is to develop properties of interval-valued Caputo fractional derivative operator and theory of existence of solutions for interval-valued Caputo fractional differential equation (1.1). Also, we believe that the presented results will be useful for development about the theory of interval-valued fractional differential equations and the relevant applications, such as, digital computers.

The rest of this paper is organized as follows. In Section 2, some preliminaries are provided. In Section 3, we shall introduce the concepts of fractional calculus for interval-valued functions and present some important properties of partial order relationship '$\preceq_{LU}$'. Furthermore, in Section 4, we prove the existence and uniqueness results for interval-valued fractional integro-differential equations (IVFIE, for short) by employing the method of upper and lower solutions and then obtain the existence of solutions for (1.1). Finally, several examples will be illustrated.

2. Preliminaries

Let $R(I)$ be the space of nonempty bounded, closed interval of $\mathbb{R}$. For any element A of $R(I)$, we denote $A = [a^L, a^U]$, here a^L and a^U stand for the lower and upper bounds of A, respectively. Let $A = [a^L, a^U]$, $B = [b^L, b^U] \in R(I)$ and $\lambda \in \mathbb{R}$. The

usual interval operations, i.e., well-known as Minkowski addition and scalar multiplication, are defined as follows

$$A+B=[a^L,a^U]+[b^L,b^U]=[a^L+b^L,a^U+b^U],$$

and

$$\lambda A=\lambda[a^L,a^U]=\begin{cases}[\lambda a^L,\lambda a^U] & \text{if } \lambda>0,\\ \{0\} & \text{if } \lambda=0,\\ [\lambda a^U,\lambda a^L] & \text{if } \lambda<0,\end{cases}$$

respectively. Particularly, when $\lambda=-1$, which has

$$-A:=(-1)\cdot A=(-1)\cdot[a^L,a^U]=[-a^U,-a^L].$$

So, the Minkowski difference can be written as

$$A-B=A+(-1)\cdot B=[a^L-b^U,a^U-b^L].$$

In fact, based on Minkowski addition and scalar multiplication, $R(I)$ merely constructs a quasi-linear space (more detail ones can refer to Markov [33] and Schneider [40]). For $A=[a^L,a^U]\in R(I)$, we define

$$\|A\|:=\max\{|a^L|,|a^U|\}.$$

Then, it is easily to get that $\|\cdot\|$ is a norm of quasi-linear space $R(I)$, and therefore $(R(I),+,\cdot,\|\cdot\|)$ is a normed quasilinear space. At the same time, the Hausdorff–Pompeiu distance $\mathcal{H}:R(I)\times R(I)\to\mathbb{R}_+:=[0,\infty)$ will be defined by

$$\mathcal{H}(A,B)=\max\{|a^L-b^L|,|a^U-b^U|\},\ \text{for any } A=[a^L,a^U],B=[b^L,b^U]\in R(I).$$

By the concept of $\mathcal{H}$, we readily obtain its properties as follows

$$\begin{aligned}&\mathcal{H}(\lambda A,\lambda B)=|\lambda|\mathcal{H}(A,B)\text{ for all }\lambda\in\mathbb{R},\\&\mathcal{H}(A+C,B+C)=\mathcal{H}(A,B),\\&\mathcal{H}(A+B,C+D)\le\mathcal{H}(A,C)+\mathcal{H}(B,D).\end{aligned}$$

Indeed, it is obviously that the Hausdorff metric $\mathcal{H}$ is associated with the norm $\|\cdot\|$ by $\|A\|=\mathcal{H}(A,\{0\})$. Moreover, it follows from Li, Ogura and Kreinovich in [17], which knows that $(R(I),\mathcal{H})$ is a complete, locally compact and separable metric space.

For any $A\in R(I)$, it can be seen $A+(-A)\neq 0$ in general, thus the opposite of A is not the inverse of A with respect to the Minkowski addition unless $A=\{a\}$ is a singleton. To partially overcome this situation, the Hukuhara difference (or H-difference see c.f. [10]) has been introduced as a set C, for which is $A\ominus B=C$ if and only if $A=B+C$. The most important property of $\ominus$ is that $A\ominus A=\{0\}$, $\forall A\in R(I)$ and $(A+B)\ominus B=A$, $\forall A,B\in R(I)$. Despite, the H-difference is unique, but it does not always exist, for example $A=[1,2]$, $B=[0,4]$, it is impossible

$C = [1, -2]$. In order to solve this problem, the generalized Hukuhara difference is proposed in recently.

Definition 1 ([33, 41])**.** The generalized Hukuhara difference (or gH-difference) of two intervals $A = [a^L, a^U]$, $B = [b^L, b^U] \in R(I)$ is defined as follows

$$\begin{aligned} A \ominus_{gH} B &= [a^L, a^U] \ominus_{gH} [b^L, b^U] \\ &= [\min\{a^L - b^L, a^U - b^U\}, \max\{a^L - b^L, a^U - b^U\}]. \end{aligned} \tag{2.1}$$

Remark 1. For any $A = [a^L, a^U] \in R(I)$, the width of A can be denoted by $w(A) = a^U - a^L$ and the midpoint of A by $m(A) = \frac{a^U + a^L}{2}$. Then, (2.1) is equivalent to

$$A \ominus_{gH} B = \begin{cases} [a^L - b^L, a^U - b^U], & \text{if } w(A) \geq w(B) \\ [a^U - b^U, a^L - b^L], & \text{if } w(A) < w(B). \end{cases}$$

Hence, for any $A, B, C \in R(I)$, it has

$$\begin{aligned} A \ominus_{gH} B = C &\Leftrightarrow \begin{cases} A \ominus B = C, & \text{if } w(A) \geq w(B) \\ B \ominus A = (-C) & \text{if } w(A) < w(B). \end{cases} \\ &\Leftrightarrow \begin{cases} A = B + C, & \text{if } w(A) \geq w(B) \\ B = A + (-C) & \text{if } w(A) < w(B). \end{cases} \end{aligned}$$

and $\mathcal{H}(A, B) = \|A \ominus_{gH} B\|$.

Now, we shall recall some properties of gH-difference.

Proposition 1 ([32, 41])**.** *Let $A, B, C, D \in R(I)$, $\alpha, \beta \in \mathbb{R}$ and denote*

$$W_1 = \Big(w(A) - w(C)\Big)\Big(w(B) - w(D)\Big), \quad W_2 = \Big(w(A) - w(B)\Big)\Big(w(C) - w(D)\Big).$$

Then the following results hold

(i) $(-\mathrm{A}) \ominus_{\mathrm{gH}} \mathrm{B} = (-\mathrm{B}) \ominus_{\mathrm{gH}} \mathrm{A}$

(ii) $(\mathrm{A} + \mathrm{B}) \ominus_{\mathrm{gH}} (\mathrm{C} + \mathrm{D}) = \begin{cases} (A \ominus_{gH} C) + (B \ominus_{gH} D), & \text{if } W_1 \geq 0, \\ (A \ominus_{gH} C) \ominus_{gH} (-(B \ominus_{gH} D)), & \text{if } W_1 < 0, \end{cases}$

(iii) $(\mathrm{A} \ominus_{\mathrm{gH}} \mathrm{B}) + (\mathrm{C} \ominus_{\mathrm{gH}} \mathrm{D})$

$$= \begin{cases} (A \ominus_{gH} (-C)) \ominus_{gH} (B \ominus_{gH} (-D)), & \text{if } W_1 \geq 0,\ W_2 < 0, \\ (A \ominus_{gH} (-C)) + ((-B) \ominus_{gH} D), & \text{if } W_1 < 0,\ W_2 < 0, \\ (A + C) \ominus_{gH} (B + D), & \text{if } W_2 \geq 0, \end{cases}$$

(iv) $(\mathrm{A} \ominus_{\mathrm{gH}} \mathrm{B}) \ominus_{\mathrm{gH}} (\mathrm{C} \ominus_{\mathrm{gH}} \mathrm{D})$

$$= \begin{cases} (A \ominus_{gH} C) \ominus_{gH} (B \ominus_{gH} D), & \text{if } W_2 \geq 0,\ W_1 \geq 0, \\ (A \ominus_{gH} C) + (-(B \ominus_{gH} D)), & \text{if } W_2 \geq 0,\ W_1 < 0, \\ (A + (-C)) \ominus_{gH} (B + (-D)), & \text{if } W_2 < 0, \end{cases}$$

The function $f:[a,b]\to R(I)$, which defined on $[a,b]$, is named to be an interval-valued function, i.e., $f(t)$ is a closed interval in $\mathbb{R}$ for each $t\in[a,b]$ and can also be written as $f(t)=[f^L(t),f^U(t)]$, where f^L and f^U are two real valued functions defined on $[a,b]$ which satisfied $f^L(t)\le f^U(t)$ for every $t\in[a,b]$. In addition, an interval-valued function f is called to be w-monotone, if real valued function $w(f(t))=f^U(t)-f^L(t)$ is monotone.

Definition 2 ([32])**.** Let $f:[a,b]\to R(I)$ be an interval-valued function. Then $\lim_{t\to t_0} f(t)$ exists, if and only if $\lim_{t\to t_0} f^L(t)$ and $\lim_{t\to t_0} f^U(t)$ exist as finite numbers, i.e.,

$$\lim_{t\to t_0} f(t)=\left[\lim_{t\to t_0} f^L(t),\lim_{t\to t_0} f^U(t)\right].$$

Thus f is continuous if and only if f^L and f^U are both continuous.

Definition 3 ([41])**.** Let $f,g:[a,b]\to R(I)$ be two interval-valued functions, then we define the interval-valued function $f\ominus_{gH} g:[a,b]\to R(I)$ by

$$(f\ominus_{gH} g)(t):=f(t)\ominus_{gH} g(t),\ \text{for all } t\in[a,b].$$

Remark 2. By above definitions, we know that if there exist $\lim_{t\to t_0} f(t)=A$, $\lim_{t\to t_0} g(t)=B$, then $\lim_{t\to t_0}(f\ominus_{gH} g)(t)$ exists, and

$$\lim_{t\to t_0}(f\ominus_{gH} g)(t)=A\ominus_{gH} B.$$

Besides, it is well-known that if f and g are two continuous interval-valued functions then $f+g$, $\lambda f,\forall\lambda\in\mathbb{R}$ and the single-valued function $w(f(t))$ are continuous also (more details we can refer to [32]).

Let $C([a,b];R(I))$ be the set of continuous interval-valued functions from $[a,b]$ into $R(I)$. Then $C([a,b];R(I))$ is a complete metric space with respect to the metric

$$\mathcal{H}_{\mathcal{C}}(f,g):=\|f\ominus_{gH} g\|_{\mathcal{C}}=\sup_{a\le t\le b}\mathcal{H}(f(t),g(t)),$$

where $f,g\in C([a,b];R(I))$.

Definition 4 ([32,41])**.** Let $f:[a,b]\to R(I)$ be an interval-valued function and $t_0\in[a,b]$. We define $f'(t_0)\in R(I)$ (provided it exists) as

$$f'(t_0)=\lim_{h\to 0}\frac{f(t_0+h)\ominus_{gH} f(t_0)}{h}.$$

The $f'(t_0)$ is called the generalized Hukuhara derivative (gH-derivative, for short) of f at $t_0\in[a,b]$. Also, we define the left gH-derivative $f'_-(t_0)\in R(I)$ (provided it exists) as

$$f'_-(t_0)=\lim_{h\to 0^-}\frac{f(t_0+h)\ominus_{gH} f(t_0)}{h},$$

and the right gH-derivative $f'_+(t_0) \in R(I)$ (provided it exists) as

$$f'_+(t_0) = \lim_{h\to 0^+} \frac{f(t_0+h) \ominus_{gH} f(t_0)}{h}.$$

Saying that f is generalized Hukuhara differentiable (gH-differentiable, for short) on $[a,b]$ if $f'(t) \in R(I)$ exists at each point $t \in [a,b]$. At the end points of $[a,b]$ only one sided gH-derivatives would be considered. The interval-valued function $f' : [a,b] \to R(I)$ is then called the gH-derivative of f on $[a,b]$.

Next, let us recall some important properties about gH-differential of interval-valued functions.

Theorem 1 ([32,41])**.** *Let $f : [a,b] \to R(I)$ be such that $f(t) = [f^L(t), f^U(t)]$, $t \in [a,b]$. If the real valued functions f^L and f^U are both differential at $t \in [a,b]$, then f is gH-differential at $t \in [a,b]$ and*

$$f'(t) = \left[\min\left\{\frac{d}{dt}f^L(t), \frac{d}{dt}f^U(t)\right\}, \max\left\{\frac{d}{dt}f^L(t), \frac{d}{dt}f^U(t)\right\}\right], \ \forall t \in [a,b].$$

Theorem 2 ([32,41])**.** *Let $f : [a,b] \to R(I)$ be gH-differentiable on $[a,b]$. Then for all $C \in R(I)$ and for all $\lambda \in \mathbb{R}$, the interval-valued functions $f + C$, $f \ominus_{gH} C$ and λf are gH-differentiable on $[a,b]$, where $(f + C)' = f'$, $(f \ominus_{gH} C)' = f'$, $(\lambda f)' = \lambda f'$.*

Definition 5 ([28])**.** Let $f : [a,b] \to R(I)$ be an interval-valued function. f is called a simple interval-valued function, if it is constant on each sets of $I_k \subset [a,b]$, $k = 1,2,\cdots,m$, which produce a finite system of pair wise disjoint, Lebesgue measurable sets covering $[a,b]$.

Definition 6 ([28])**.** Let $f : [a,b] \to R(I)$ be an interval-valued function. f is said to be measurable if there exists a sequence $f_n : [a,b] \to R(I)$ of simple interval-valued functions which satisfies

$$\lim_{n\to\infty} \mathcal{H}(f_n(t), f(t)) = 0, \text{ for a.e. } t \in [a,b].$$

Definition 7 ([28])**.** Let $f : [a,b] \to R(I)$ be an interval-valued function such that $f(t) = [f^L(t), f^U(t)]$. Also, both f^L and f^U are measurable and Lebesgue integrable on $[a,b]$. Then, we defined the Lebesgue integral of f by $\int_a^b f(s)ds$, i.e.

$$\int_a^b f(s)ds := \left[\int_a^b f^L(s)ds, \int_a^b f^U(s)ds\right].$$

Remark 3. $\lim_{n\to\infty} \mathcal{H}(f_n(t), f(t)) = 0$ if and only if $\lim_{n\to\infty} |f_n^L(t) - f^L(t)| = 0$ and $\lim_{n\to\infty} |f_n^U(t) - f^U(t)| = 0$. Then it can be easily get that an interval-valued function f is measurable on $[a,b]$ if and only if f^L and f^U are measurable on $[a,b]$. Moreover, f is Lebesgue integrable on $[a,b]$ if and only if f is measurable and real

function $\|f(t)\|$ is Lebesgue integrable on $[a,b]$ (those results have been obtained by [5, 28]).

Definition 8 ([28])**.** Let $f:[a,b]\to R(I)$ be an interval-valued function. f is said to be absolutely continuous, if for each $\epsilon>0$, there exists $\delta>0$ such that for each family $\{(s_k,t_k)\subset[a,b];\ k=1,2,\dots,n\}$ of disjoint open intervals with $\sum_{k=1}^{n}(t_k-s_k)<\delta$, we have

$$\sum_{k=1}^{n}\mathcal{H}(f(s_k),f(t_k))<\epsilon.$$

Let $AC([a,b];R(I))$ denote the set of all absolutely continuous interval-valued functions from $[a,b]$ into $R(I)$. The following theorem gives us a necessary and sufficient condition for absolutely continuous interval-valued functions.

Proposition 2 ([4, 28, 32, 41])**.** *An interval-valued function $f:[a,b]\to R(I)$ is absolutely continuous if and only if f^L and f^U are both absolutely continuous.*

It follows well-known result (see Corollary 7.23 of [43]) that if $\phi:[a,b]\to\mathbb{R}$ is an absolutely continuous function, then $\frac{d}{dt}\phi(t)$ exists a.e. $t\in[a,b]$. Therefore, we have the following theorem.

Theorem 3 ([5, 28])**.** *If $f\in AC([a,b];R(I))$, then f is gH-difference a.e. $t\in[a,b]$ and $f'\in L^1([a,b];R(I))$.*

3. Fractional calculus for interval-valued functions and partial order relationship

In this section, we will discuss the properties of fractional calculus for interval-valued functions. J will be denoted as $J=:[0,T]$.

Definition 9 ([14, 39])**.** Let $\phi\in L^1(J;\mathbb{R})$. The Riemann-Liouville fractional integral of order $\alpha>0$ for ϕ is defined by (provided it exists)

$$I_t^{\alpha}\phi(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi(s)ds,\ \ t\in J,$$

where Γ is the well-known Gamma function.

In this paper, we denote $I_t^0\phi(t)=\phi(t)$ apparently.

Definition 10 ([14, 39])**.** Let $\phi\in L^1(J,\mathbb{R})$. The Riemann-Liouville derivative of order $\alpha\in(0,1]$ for ϕ is defined (provided exists)

$${}^{L}D_t^{\alpha}\phi(t)=\frac{d}{dt}I_t^{1-\alpha}\phi(t),\ \text{a.e. } t\in J,$$

thus is,

$${}^{L}D_t^{\alpha}\phi(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-s)^{-\alpha}\phi(s)ds,\ \text{a.e. } t\in J.$$

Definition 11 ([14, 39])**.** Let $\phi \in L^1(J,\mathbb{R})$ be such that ${}^L D_t^\alpha \phi(t)$ exists for a.e. $t \in J$. The Caputo fractional derivative of order $\alpha \in (0,1]$ for ϕ is defined as (provided exists)

$$ {}^C D_t^\alpha \phi(t) = {}^L D_t^\alpha [\phi(t) - \phi(0)], \text{ a.e. } t \in J, $$

thus is,

$$ {}^C D_t^\alpha \phi(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} [\phi(s) - \phi(0)] ds, \text{ a.e. } t \in J. $$

Remark 4. In particular, when $\alpha = 1$ and $\phi \in AC(J;\mathbb{R})$, then it has

$$ {}^L D_t^1 \phi(t) = \frac{d}{dt}\phi(t), \text{ a.e. } t \in J. $$

In the sequel, we denote $L^p(J;R(I))$ the set of all interval-valued functions $f : J \to R(I)$ such that the real function $t \mapsto \|f(t)\|$ belongs to $L^P(J;\mathbb{R})$. In fact, we readily get from [17] that $L^p(J;R(I))$ is a complete metric space with respect to the metric $\mathcal{H}_p$ defined by $\mathcal{H}_p(f,g) := \|f \ominus_{gH} g\|_p$ for any $f,g \in L^p(J;R(I))$, where

$$ \|f\|_p := \begin{cases} \left(\int_0^T \|f(t)\|^p dt \right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty, \\ \operatorname{ess\,sup}_{t \in J} \|f(t)\|, & \text{if } p = \infty. \end{cases} $$

Now, the definition of Riemann-Liouville fractional integral for the interval-valued functions will be given.

Definition 12 ([28])**.** Let $f \in L^1(J;R(I))$ be an interval-valued function such that $f(t) = [f^L(t), f^U(t)]$ and $\alpha > 0$. The interval-valued Riemann-Liouville fractional integral of order α for f is defined by (provided it exists)

$$ \mathfrak{I}_t^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \ t \in J, $$

that is,

$$ \mathfrak{I}_t^\alpha f(t) := [I_t^\alpha f^L(t), I_t^\alpha f^U(t)], \ t \in J, $$

Remark 5. In fact, operator $\mathfrak{I}_t^\alpha$ has good properties, such as (more details ones can refer to c.f. [28]) if $f,g \in L^p(J;R(I))$ and $\alpha,\beta > 0$

(i) $w(\mathfrak{I}_t^\alpha f(t)) = I_t^\alpha w(f(t))$ (see [28, Reamrk 5]);
(ii) $\mathfrak{I}_t^\alpha(\mathfrak{I}_t^\beta f(t)) = \mathfrak{I}_t^{\alpha+\beta} f(t)$ (see [28, Reamrk 5]);
(iii) $\mathfrak{I}_t^\alpha c f(t)) = c\mathfrak{I}_t^\alpha f(t)$ for every $c \in [0,\infty)$ (see [28, Reamrk 5]);
(iv) $\mathfrak{I}_t^\alpha(f+g)(t)) = \mathfrak{I}_t^\alpha f(t) + \mathfrak{I}_t^\alpha g(t)$ (see [28, Reamrk 5]);
(v) $\mathfrak{I}_t^\alpha(f \ominus_{gH} g)(t)) \supseteq \mathfrak{I}_t^\alpha f(t) \ominus_{gH} \mathfrak{I}_t^\alpha g(t)$ (see [28, Theorem 1]);
(vi) $\mathfrak{I}_t^\alpha(f \ominus_{gH} g)(t)) = \mathfrak{I}_t^\alpha f(t) \ominus_{gH} \mathfrak{I}_t^\alpha g(t)$, if $w(f(t)) - w(g(t))$ has a constant sign on J (see [28, Theorem 1]).

Definition 13 ([28])**.** Let $f \in L^1(J;R(I))$ be an interval-valued function such that $f(t) = [f^L(t), f^U(t)]$ and $0 < \alpha \leq 1$. The interval-valued Riemann-Liouville fractional derivative (or Riemann-Liouville gH-fractional derivative) of order α is defined by (provided it exists)

$$^L\mathfrak{D}_t^\alpha f(t)(t) := (\mathfrak{I}_t^{1-\alpha} f(t))', \text{ a.e. } t \in J,$$

thus is,

$$^L\mathfrak{D}_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\left(\int_0^t (t-s)^{-\alpha} f(s)ds\right)', \text{ a.e. } t \in J.$$

Remark 6. In this paper, we denote $f_{1-\alpha}(t) := \mathfrak{I}_t^{1-\alpha} f(t)$

$$f_{1-\alpha}(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} f(s)ds, \text{ for all } t \in J.$$

Hence, it knows from [28] that if $f \in C(J;R(I))$, then $f_{1-\alpha}$ is absolutely continuous and

(i) if either f is w-increasing on J or f is w-decreasing and $f_{1-\alpha}$ is w-increasing on J then

$$^L\mathfrak{D}_t^\alpha f(t) = [^L D_t^\alpha f^L(t), {}^L D_t^\alpha f^U(t)], \text{ for a.e. } t \in J; \tag{3.1}$$

(ii) if $f_{1-\alpha}$ is w-decreasing on J then

$$^L\mathfrak{D}_t^\alpha f(t) = [^L D_t^\alpha f^U(t), {}^L D_t^\alpha f^L(t)], \text{ for a.e. } t \in J. \tag{3.2}$$

Definition 14 ([28])**.** Let $f \in L^1(J;R(I))$ be an interval-valued function such that $f(t) = [f^L(t), f^U(t)]$ and the Riemann-Liouville gH-fractional derivative $^L\mathfrak{D}_t^\alpha f(t)$, $\alpha \in (0,1]$, exists a.e. $t \in J$. The interval-valued Caputo fractional derivative (or Caputo gH-fractional derivative) of order $\alpha \in (0,1]$ is defined by (provided it exists)

$$^C\mathfrak{D}_t^\alpha f(t) := {}^L\mathfrak{D}_t^\alpha [f(t) \ominus_{gH} f(0)], \text{ a.e. } t \in J.$$

Next, the following partial ordering $\preceq_{LU}$ will be introduced (see, Inuiguchi and Kume in [11]; Wu in [44,45]).

Definition 15 ([11,44,45])**.** Let $A = [a^L, a^U]$, $B = [b^L, b^U]$ be two closed intervals and $f, g \in C(J;R(I))$ be two interval-valued functions.

(i) We say $A \preceq_{LU} B$ if and only if

$$a^L \leq b^L \text{ and } a^U \leq b^U.$$

Also It can be written as $A \prec_{LU} B$ if and only if

$$A \preceq_{LU} B \text{ and } A \neq B,$$

thus is,

$$a^L < b^L,\ a^U < b^U;$$

$$\text{or }, a^L \leq b^L,\ a^U < b^U;$$
$$\text{or }, a^L < b^L,\ a^U \leq b^U.$$

(ii) We say that $f \preceq_{LU} g$ if and only if

$$f(t) \preceq_{LU} g(t),\ \forall t \in J.$$

In the sequel, $A \preceq_{LU} B$ also means $B \succeq_{LU} A$. In fact, by the definition of $\preceq_{LU}$, the following lemmas can be obtain

Lemma 1. *Let $A, B, C, D \in R(I)$ and $t \in \mathbb{R}$. Then, it has*

(i) *$A = B$ if and only if $A \preceq_{LU} B$ and $B \preceq_{LU} A$;*
(ii) *if $A \preceq_{LU} B$, then $A + C \preceq_{LU} B + C$;*
(iii) *if $A \preceq_{LU} B$ and $C \preceq_{LU} D$, then $A + C \preceq_{LU} B + D$;*
(iv) *if $A \preceq_{LU} B$ and $t \geq 0$, then $tA \preceq_{LU} tB$.*

Lemma 2. *Let $f, g \in C(J, R(I))$ be two interval-valued functions with $f \preceq_{LU} g$, i.e., $f(t) \preceq_{LU} g(t)$ for each $t \in J$. Then, for each $\alpha \in (0, 1]$ we get*

$$\mathfrak{I}_t^\alpha f(t) \preceq_{LU} \mathfrak{I}_t^\alpha g(t), \text{ for any } t \in J. \tag{3.3}$$

Proof. By the definition of $\mathfrak{I}_t^\alpha$, it has

$$\mathfrak{I}_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds = \frac{1}{\Gamma(\alpha)} \left[\int_0^t (t-s)^{\alpha-1} f^L(s)ds, \int_0^t (t-s)^{\alpha-1} f^U(s)ds \right],$$
$$\mathfrak{I}_t^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds = \frac{1}{\Gamma(\alpha)} \left[\int_0^t (t-s)^{\alpha-1} g^L(s)ds, \int_0^t (t-s)^{\alpha-1} g^U(s)ds \right].$$

Because of $f^L(t) \leq g^L(t)$, $f^U(t) \leq g^U(t)$ for all $t \in J$ and $(t-s)^{\alpha-1} > 0$, which can easily has $(t-s)^{\alpha-1} f^L(t) \leq (t-s)^{\alpha-1} g^L(t)$ and $(t-s)^{\alpha-1} f^U(t) \leq (t-s)^{\alpha-1} g^U(t)$, hence

$$\int_0^t (t-s)^{\alpha-1} f^L(s)ds \leq \int_0^t (t-s)^{\alpha-1} g^L(s)ds,$$
$$\int_0^t (t-s)^{\alpha-1} f^U(s)ds \leq \int_0^t (t-s)^{\alpha-1} g^U(s)ds.$$

thus is $\int_0^t (t-s)^{\alpha-1} f(s)ds \preceq_{LU} \int_0^t (t-s)^{\alpha-1} g(s)ds$. Besides, it is immediate from item (iv) of Lemma 1 that for each $\alpha \in (0, 1]$ it concludes $\mathfrak{I}_t^\alpha f(t) \preceq_{LU} \mathfrak{I}_t^\alpha g(t),\ \forall t \in J$. □

Lemma 3. *Let $f, g \in C(J; R(I))$ be two interval-valued functions with $f \preceq_{LU} g$ and $A \in R(I)$. Then, for every $\alpha \in (0, 1]$ the following results hold:*

$$A \ominus (-1) \cdot \mathfrak{I}_t^\alpha f(t) \preceq_{LU} A \ominus (-1) \cdot \mathfrak{I}_t^\alpha g(t),\ \forall t \in J,$$

providing $A \ominus (-1) \cdot \mathfrak{I}_t^\alpha f(t)$ and $A \ominus (-1) \cdot \mathfrak{I}_t^\alpha g(t)$ are well-defined for each $t \in J$.

Proof. For each $\alpha \in (0,1]$, which has

$$\mathfrak{I}_t^\alpha f(t) = [I_t^\alpha f^L(t), I_t^\alpha f^U(t)],$$
$$\mathfrak{I}_t^\alpha g(t) = [I_t^\alpha g^L(t), I_t^\alpha g^U(t)].$$

By the assumptions and Lemma 2, which deduce

$$\mathfrak{I}_t^\alpha f(t) \preceq_{LU} \mathfrak{I}_t^\alpha g(t),$$

i.e., for every $\alpha \in (0,1]$,

$$I_t^\alpha f^L(t) \le I_t^\alpha g^L(t) \text{ and } I_t^\alpha f^U(t) \le I_t^\alpha g^U(t),\ \forall t \in J. \tag{3.4}$$

By the definition of Hukuhara difference, we obtain

$$A \ominus (-1)\cdot \mathfrak{I}_t^\alpha f(t) = [a^L + I_t^\alpha f^U(t), a^U + I_t^\alpha f^L(t)],$$
$$A \ominus (-1)\cdot \mathfrak{I}_t^\alpha g(t) = [a^L + I_t^\alpha g^U(t), a^U + I_t^\alpha g^L(t)],$$

It follows from (3.4) that

$$a^L + I_t^\alpha f^U(t) \le a^L + I_t^\alpha g^U(t) \text{ and}$$
$$a^U + I_t^\alpha f^L(t) \le a^U + I_t^\alpha g^L(t).$$

Which means that for each $\alpha \in (0,1]$ it has $A \ominus (-1)\cdot \mathfrak{I}_t^\alpha f(t) \preceq_{LU} A \ominus (-1)\cdot \mathfrak{I}_t^\alpha g(t)$, $\forall t \in J$. □

Lemma 4. *Let $\{f_n\} \subset C(J;R(I))$ and $g \in C(J;R(I))$. Assuming that $\{f_n\}$ satisfies*

$$f_n \preceq_{LU} g,\ \forall n \in \mathbb{N},$$

and $f_n(t)$ converges to $f(t)$ in $R(I)$ for each $t \in J$, then which has $f \preceq_{LU} g$.

Proof. For each $n \in \mathbb{N}$, because of $f_n \preceq_{LU} g$ then for all $t \in J$ we have

$$f_n(t) \preceq_{LU} g(t),$$

thus is

$$f_n^L(t) \le g^L(t) \text{ and } f_n^U(t) \le g^U(t).$$

Due to for all $t \in J$, $f_n(t) \to f(t)$ in $R(I)$ as $n \to \infty$, this implies

$$\lim_{n\to\infty} \mathcal{H}(f_n(t), f(t)) = \lim_{n\to\infty} \max\{|f_n^L(t) - f^L(t)|, |f_n^U(t) - f^U(t)|\} = 0.$$

Thereby, it can easily get for each $t \in J$, $f_n^L(t) \to f^L(t)$ and $f_n^U(t) \to f^U(t)$ in $\mathbb{R}$ as $n \to \infty$. Hence, for any $t \in J$ we obtain

$$f^L(t) = \lim_{n\to\infty} f_n^L(t) \le g^L(t),$$
$$f^U(t) = \lim_{n\to\infty} f_n^U(t) \le g^U(t).$$

That means $f \preceq_{LU} g$. □

Now the concept of monotone sequences will be introduced.

Definition 16. Let $\{f_n\}$ be an interval-valued functions sequence. The sequence $\{f_n\}$ is said to be

(i) nondecreasing sequence iff,

$$f_n \preceq_{LU} f_m, \text{ for any } m \geq n;$$

(ii) nonincreasing sequence iff,

$$f_m \preceq_{LU} f_n, \text{ for any } m \geq n;$$

(iii) monotone sequence it is nondecreasing sequence or nonincreasing sequence.

Lemma 5. *Let $\{f_n\} \subset C(J; R(I))$ be a monotone sequence. If it has a subsequence which convergent to $f \in C(J; R(I))$, then $\{f_n\}$ converges to f in $C(J; R(I))$ and*

$$f_n \preceq_{LU} f \text{ if } \{f_n\} \text{ is nondecreasing},$$
$$f \preceq_{LU} f_n \text{ if } \{f_n\} \text{ is nonincreasing}.$$

Proof. According to the hypothesises, assuming that there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ convergent to f. Then, for any $\epsilon > 0$ there exists a $k_0 \in \mathbb{N}$ such that

$$\|f_n - f\|_{\mathcal{C}} = \sup_{t \in J} \mathcal{H}(f_{n_k}(t), f(t)) < \epsilon, \ \forall k \geq k_0. \tag{3.5}$$

Because $\{f_n\}$ is monotone. Hence, real valued function sequences $\{f_n^L(t)\}$ and $\{f_n^U(t)\}$ have the same monotonicity in $\mathbb{R}$. However, using the condition of (3.5), we know that for each $t \in J$

$$f_{n_k}^L(t) \to f^L(t) \text{ in } \mathbb{R},$$
$$f_{n_k}^U(t) \to f^U(t) \text{ in } \mathbb{R}.$$

Which means that $f^L(t)$ and $f^U(t)$ are supremum of $\{f_n^L(t)\}$ and $\{f_n^U(t)\}$, respectively, provided $\{f_n\}$ is nondecreasing; however if $\{f_n\}$ is nonincreasing then $f^L(t)$ and $f^U(t)$ are both infremum of $\{f_n^L(t)\}$ and $\{f_n^U(t)\}$ (depending on the nondecreasing or nonincreasing character of the sequence, see [43]). Besides, for any $n \in \mathbb{N}$ we can find $k^* \in \mathbb{N}$ such that $n_{k^*} > n$, $f_{n_{k^*}}^L(t) \in \{f_{n_k}^L(t)\}$ and $f_{n_{k^*}}^U(t) \in \{f_{n_k}^U(t)\}$ satisfied

$$f_n^L(t) \leq f_{n_{k^*}}^L \leq f^L(t) \text{ and } f_n^U(t) \leq f_{n_{k^*}}^U \leq f^U(t) \text{ if } \{f_n\} \text{ is nondecreasing},$$
$$f_n^L(t) \geq f_{n_{k^*}}^L \geq f^L(t) \text{ and } f_n^U(t) \geq f_{n_{k^*}}^U \geq f^U(t) \text{ if } \{f_n\} \text{ is nonincreasing}.$$

That implies for each $n \in \mathbb{N}$

$$f_n \preceq_{LU} f \text{ if } \{f_n\} \text{ is nondecreasing},$$
$$f \preceq_{LU} f_n \text{ if } \{f_n\} \text{ is nonincreasing}.$$

On the other hand, for each $m \in \mathbb{N}$ such that $m > n_{k_0}$ (see (3.5)) we have

$$f_{n_{k_0}} \preceq_{LU} f_m \preceq_{LU} f \text{ if } \{f_n\} \text{ is nondecreasing,}$$
$$f \preceq_{LU} f_m \preceq_{LU} f_{n_{k_0}} \text{ if } \{f_n\} \text{ is nonincreasing.}$$

So, it follows from (3.5) that

$$\| f_m - f \|_{\mathcal{C}} = \sup_{t \in J} \mathcal{H}(f_m(t), f(t)) \leq \sup_{t \in J} \mathcal{H}(f_{n_{k_0}}(t), f(t)) < \epsilon.$$

This concludes that $\{f_n\}$ converges to f in $C(J; R(I))$. □

4. Main results

In this section, we shall investigate the existence of solutions for (IVCFDE) (1.1). To do so, the existence of solutions for interval-valued fractional integral equation (IVFIE, for short) will be considered firstly

$$x(t) \ominus_{gH} x_0 = \mathfrak{I}_t^{\alpha} f(t, x(t)),\ t \in J, \tag{4.1}$$

by method of upper and lower solutions.

Remark 7. Indeed, if $x \in C(J; R(I))$ is such that $w(x(t)) \geq w(x(0))$ for all $t \in J$, then (4.1) becomes the type I as

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds,\ t \in J. \tag{4.2}$$

However, when $x \in C(J; R(I))$ satisfies $w(x(t)) \leq w(x(0))$ for all $t \in J$, then (4.1) can be written type II by

$$x(t) = x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds,\ t \in J. \tag{4.3}$$

In particularly, if $x \in C(J; R(I))$ is w-increasing then (4.1) can be written as (4.2). On the other hand, (4.1) also can be written as (4.3), when $x \in C(J; R(I))$ is w-decreasing.

Now the definition of upper and lower solutions for problem (4.1) will be given, which plays an essential role in this paper.

Definition 17. Let $(\underline{x}_1, \overline{x}_1) \in C(J; R(I)) \times C(J; R(I))$ and $(\underline{x}_2, \overline{x}_2) \in C(J; R(I)) \times C(J; R(I))$ be two pair given functions. It says that

(i) $\underline{x}_1$ and $\overline{x}_1$ are upper and lower solutions of the type I for problem (4.1), respectively, if

$$\overline{x}_1(t) \succeq_{LU} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{x}_1(s)) ds,\ t \in J,$$

and

$$\underline{x}_1(t) \preceq_{LU} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \underline{x}_1(s)) ds, \ t \in J,$$

(ii) $\underline{x}_2$ and $\overline{x}_2$ are upper and lower solutions of the type II for problem (4.1), respectively, if

$$\overline{x}_2(t) \succeq_{LU} x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{x}_2(s)) ds, \ t \in J,$$

and

$$\underline{x}_2(t) \preceq_{LU} x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \underline{x}_2(s)) ds, \ t \in J,$$

providing $x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{x}_2(s)) ds$ and $x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \underline{x}_2(s)) ds$ are well-defined for each $t \in J$.

Definition 18. Let $(\underline{x}_1, \overline{x}_1)$ and $(\underline{x}_2, \overline{x}_2)$ be a pair of upper and lower solutions of the type I and type II for problem (4.1), respectively. The admissible sets U_1 and U_2 of solutions for problem (4.1) governed by $(\underline{x}_1, \overline{x}_1)$ and $(\underline{x}_2, \overline{x}_2)$ respectively, which are defined as

$$U_1 := \left\{ x \in C(J; R(I)) : \ \underline{x}_1 \preceq_{LU} x \preceq_{LU} \overline{x}_1 \text{ and } x \text{ is a type I solution of (4.1)} \right\},$$

$$U_2 := \left\{ x \in C(J; R(I)) : \ \underline{x}_2 \preceq_{LU} x \preceq_{LU} \overline{x}_2 \text{ and } x \text{ is a type II solution of (4.1)} \right\}.$$

Based on the partial ordering $\preceq_{LU}$, we now consider the concepts of maximal and minimal solutions in admissible sets U_1 and U_2.

Definition 19. Let admissible sets U_1 and U_2 be governed by $(\underline{x}_1, \overline{x}_1)$ and $(\underline{x}_2, \overline{x}_2)$, respectively. It says that

(i) $x : J \to R(I)$ is a solution of (4.1) depended on $(\underline{x}_1, \overline{x}_1)$ or $(\underline{x}_2, \overline{x}_2)$, if $x \in U_1 \cup U_2$;

(ii) $x_M \in U_1$ (resp. U_2) is a maximal solution of problem (4.1) in U_1 (resp. U_2), if

$$x \preceq_{LU} x_M, \ \text{ for each } x \in U_1 \text{ (resp. } U_2);$$

(iii) $x_L \in U_1$ (resp. U_2) is a minimal solution of problem (4.1) in U_1 (resp. U_2), if

$$x_L \preceq_{LU} x, \ \text{ for each } x \in U_1 \text{ (resp. } U_2).$$

Then, we illustrate the existence of solutions for (4.1) delimited by a pair of upper and lower solutions.

Theorem 4. *Let $f \in C(J \times R(I); R(I))$ be given. Assuming that one of the following conditions holds:*

(A_1) *There exists a pair of upper and lower solutions $(\underline{x}_1, \overline{x}_1)$ of type I for problem (4.1) such that*

$$\underline{x}_1 \preceq_{LU} \overline{x}_1,$$

and $f(t,\cdot) : [\min_{t\in J} \underline{x}_1^L(t), \max_{t\in J} \overline{x}_1^U(t)] \to R(I)$ is nondecreasing for all $t \in J$, i.e. for any $A, B \in R(I)$ with $A \preceq_{LU} B$ then $f(t,A) \preceq_{LU} f(t,B)$ for all $t \in J$.

(A_2) *There exists a pair of upper and lower solutions $(\underline{x}_2, \overline{x}_2)$ of type II for problem (4.1) such that*

$$\underline{x}_2 \preceq_{LU} \overline{x}_2,$$

which is also the sequences satisfied

$$\begin{cases} y_0 = \underline{x}_2, \\ y_{n+1} = x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_n(s)) ds, \ t \in J, \ \ n = 0,1,2,\cdots, \end{cases}$$

and

$$\begin{cases} z_0 = \overline{x}_2, \\ z_{n+1} = x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z_n(s)) ds, \ t \in J, \ n = 0,1,2,\cdots, \end{cases}$$

They are well-defined, for all $t \in J$, and $f(t,\cdot) : [\min_{t\in J} \underline{x}_2^L(t), \max_{t\in J} \overline{x}_2^U(t)] \to R(I)$ is nondecreasing for all $t \in J$.

Then there exist maximal and minimal solutions $x_M, x_L \in U_1$ in the case (A_1) (or $x_M, x_L \in U_2$ in the case (A_2)) satisfied for (4.1).

Proof. For one thing, we assume that (A_1) satisfied, then construct the sequence $\{y_n\}_{n=0}^{\infty}$ as follows

$$y_0 = \underline{x}_1,$$
$$y_{n+1}(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_n(s)) ds, \ \forall t \in J \text{ and } n = 0,1,2,\cdots.$$

For another, constructing another sequence $\{z_n\}_{n=0}^{\infty}$ also as

$$z_0 = \overline{x}_1,$$
$$z_{n+1}(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z_n(s)) ds, \ \forall t \in J \text{ and } n = 1,2,\cdots.$$

Then, we have the next claims.

Claim 1. *The sequences* $\{y_n\}_{n=0}^{\infty}$ *and* $\{z_n\}_{n=0}^{\infty}$ *are both satisfied the following relationship*

$$\underline{x}_1 = y_0 \preceq_{LU} y_1 \preceq_{LU} \cdots \preceq_{LU} y_n \preceq_{LU} \cdots \preceq_{LU} z_n \preceq_{LU} \cdots \preceq_{LU} z_1 \preceq_{LU} z_0 = \overline{x}_1. \tag{4.4}$$

To do so, we start with proving that $\{y_n\}_{n=0}^{\infty}$ is nondecreasing sequence and satisfies

$$y_n \preceq_{LU} z_0,\ n = 0, 1, \cdots.$$

By (A_1), it has $\underline{x}_1 = y_0 \preceq_{LU} z_0 = \overline{x}_1$. However, for $n = 0$, since $y_0 = \underline{x}_1$ is a lower solution of type I for problem (4.1) then

$$y_1(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_0(s))ds \succeq_{LU} y_0,\ \forall t \in J.$$

On the other hand, because $f(t,\cdot) : [\min_{t\in J} \underline{x}_1^L(t), \max_{t\in J} \overline{x}_1^U(t)] \to R(I)$ is nondecreasing for each $t \in J$, then we readily get

$$f(s, y_0(s)) \preceq_{LU} f(s, z_0(s)),\ \forall s \in [0,t],\ t \in J.$$

It is of course immediate from Lemma 2 and (ii) of Lemma 1 that

$$\begin{aligned} &y_1(t) \\ &= x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_0(s))ds \\ &\preceq_{LU} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z_0(s))ds \\ &\preceq_{LU} z_0(t),\ \forall t \in J. \end{aligned}$$

That implies $y_0 \preceq_{LU} y_1 \preceq_{LU} z_0$. Now, applying the inductive method

$$y_{n-1} \preceq_{LU} y_n \preceq_{LU} z_0. \tag{4.5}$$

Hence, obtain that for each $t \in J$

$$\begin{aligned} y_n(t) &= x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_{n-1}(s))ds, \\ y_{n+1}(t) &= x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_n(s))ds. \end{aligned}$$

Due to the nondecreasing property of f and $y_{n-1} \preceq_{LU} y_n$, then we apply Lemma 1 and Lemma 2 again to obtain by (4.5)

$$y_n \preceq_{LU} y_{n+1} \preceq_{LU} z_0.$$

By the same arguments, one implies that $y_0 \preceq_{LU} z_{n+1} \preceq_{LU} z_n \preceq_{LU} z_0$, for each $n = 0, 1, 2, \cdots$.

To finish the proof of this claim, we need to show $y_n \preceq_{LU} z_n$, for all $n \in \mathbb{N}$. The induction can also be used to prove this conclusion. When $n = 0$, it is obvious $\underline{x}_1 = y_0 \preceq_{LU} z_0 = \overline{x}_1$ thanks to (A_1). So, assuming that

$$y_n \preceq_{LU} z_n. \tag{4.6}$$

Then, by the definitions of y_{n+1} and z_{n+1}, It has

$$y_{n+1}(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_n(s)) ds,$$
$$z_{n+1}(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z_n(s)) ds.$$

Similarly, we obtain

$$y_{n+1} \preceq_{LU} z_{n+1},$$

because f is nondecreasing with respect to the second variable and (4.6). Consequently, the proof of claim finished.

Claim 2. *$\{y_n\}$ and $\{z_n\}$ are relatively compact in $C(J; R(I))$.*

By Claim 1 and the continuity of f, we know that $\{y_n(t)\}$ and $\{z_n(t)\}$ are both bounded in $R(I)$ for each $t \in J$. Then, it deduces that there exists a constant $M_f > 0$ such that for each $n \in \mathbb{N}$ and $t \in J$

$$\mathcal{H}(f(t, y_n(t)), \{0\}) \le M_f, \quad \mathcal{H}(f(t, z_n(t)), \{0\}) \le M_f.$$

For each $n \in \mathbb{N}$ and $\forall t \in J$, we have the following estimate

$$\begin{aligned}
\mathcal{H}(y_{n+1}(t), x_0) &= \mathcal{H}\left(x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_n(s)) ds, x_0\right) \\
&= \mathcal{H}\left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_n(s)) ds, \{0\}\right) \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{H}(f(s, y_n(s)), \{0\}) ds \\
&\le \frac{M_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \le \frac{M_f T^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

which means that $\{y_n\}$ is uniformly bounded. In addition, for any $t_1, t_2 \in J$, without loss of generality we assume $t_1 \le t_2$, then calculate

$$\begin{aligned}
&\mathcal{H}(y_{n+1}(t_1), y_{n+1}(t_2)) \\
&= \mathcal{H}\left(x_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f(s, y_n(s)) ds, x_0 \right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} f(s, y_n(s)) ds\right)
\end{aligned} \tag{4.7}$$

$$
\begin{aligned}
&= \mathcal{H}\left(\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,y_n(s))ds, \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,y_n(s))ds\right)\\
&= \frac{1}{\Gamma(\alpha)}\mathcal{H}\left(\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,y_n(s))ds, \int_0^{t_2}(t_2-s)^{\alpha-1}f(s,y_n(s))ds\right)\\
&= \frac{1}{\Gamma(\alpha)}\max\left\{\left|\int_0^{t_2}(t_2-s)^{\alpha-1}f^L(s,y_n(s))ds-\int_0^{t_1}(t_1-s)^{\alpha-1}f^L(s,y_n(s))ds\right|,\right.\\
&\quad\left.\left|\int_0^{t_2}(t_2-s)^{\alpha-1}f^U(s,y_n(s))ds-\int_0^{t_1}(t_1-s)^{\alpha-1}f^U(s,y_n(s))ds\right|\right\}\\
&\le \frac{M_f}{\Gamma(\alpha)}\left(\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right)\\
&= \frac{M_f}{\Gamma(\alpha+1)}\left(t_1^\alpha-t_2^\alpha+2(t_2-t_1)^\alpha\right)\to 0, \text{ as } |t_1-t_2|\to 0.
\end{aligned}
$$

This means that $\{y_n\}$ is equicontinuous in $C(J;R(I))$. Therefore, utilizing Arzela-Ascoli Theorem (see [46]) it concludes that $\{y_n\}$ is relatively compact in $C(J;R(I))$. Similarly, we can also get that $\{z_n\}$ is relatively compact in $C(J;R(I))$.

By virtue of Claim 1 and Claim 2, it knows that $\{y_n\}$ and $\{z_n\}$ are monotone and relatively compact. Thereby, applying Lemma 5 that $\{y_n\}$ and $\{z_n\}$ are two Cauchy sequences, i.e., there exists continuous functions x_L and x_M such that $\{y_n\}$ and $\{z_n\}$ converge uniformly to x_L and x_M in $C(J;R(I))$, respectively. Hence, we readily obtain for each $t\in J$

$$
\begin{aligned}
x_L(t) &= x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x_L(s))ds,\\
x_M(t) &= x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x_M(s))ds.
\end{aligned}
$$

Also, according to Lemma 4 and Lemma 5, it has $\underline{x}_1 \preceq_{LU} x_L \preceq_{LU} x_M \preceq_{LU} \overline{x}_1$, i.e., $x_L, x_M \in U_1$. Furthermore, we get that x_L and x_M are minimal and maximal solutions in U_1, respectively. Assuming that $x\in U_1$ then, it has

$$\underline{x}_1 \preceq_{LU} x \preceq_{LU} \overline{x}_1.$$

Since f is nondecreasing, then using the similar methods, we obtain $y_1 \preceq_{LU} x \preceq_{LU} z_1$. So, that inducts

$$\underline{x}_1 \preceq_{LU} y_n \preceq_{LU} x \preceq_{LU} z_n \preceq_{LU} \overline{x}_1, \quad \text{for each } n\in\mathbb{N}. \tag{4.8}$$

Taking limits as $n\to\infty$ into (4.8), one yields

$$\underline{x}_1 \preceq_{LU} x_L \preceq_{LU} x \preceq_{LU} x_M \preceq_{LU} \overline{x}_1.$$

That finishes the proof, when (A_1) is satisfied.

However, if (A_2) holds. Then, it can also be written as $y_0 = \underline{x}_2$, $z_0 = \overline{x}_2$ and then construct sequences $\{y_n\}$ and $\{z_n\}$ as follows

$$y_{n+1}(t) = x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_n(s)) ds,$$
$$z_{n+1}(t) = x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z_n(s)) ds.$$

By the same arguments of the previous part, we assert that

$$\underline{x}_2 = y_0 \preceq_{LU} y_1 \preceq_{LU} \cdots \preceq_{LU} y_n \preceq_{LU} \cdots \preceq_{LU} z_n \preceq_{LU} \cdots \preceq_{LU} z_1 \preceq_{LU} z_0 = \overline{x}_2.$$

Following the same trend above in addition to Lemma 3 to prove the theorem. □

Theorem 5. *Let $f : J \times R(I) \to R(I)$ be a continuous interval-valued function. Then a w-monotone interval-valued function $x \in C(J; R(I))$ is a solution of (IVFIE) (4.1) then it also satisfies (IVCFDE) (1.1).*

Proof. Let $x \in C(J; R(I))$ be a w-monotone solution of (IVFIE) (4.1), then it can be written as

$$x(t) \ominus_{gH} x_0 = \mathfrak{I}_t^\alpha f(t, x(t)), \ \forall t \in J.$$

By the continuity of f, we easily get that $f^L(t, x(t))$ and $f^U(t, x(t))$ are both continuous. Then, $t \mapsto I_t^\alpha f^L(t, x(t))$ and $t \mapsto I_t^\alpha f^U(t, x(t))$ are absolutely continuous on J, i.e., $\mathfrak{I}_t^\alpha f(t)$ is absolutely continuous by Proposition 2. Hence, it has $x(0) \ominus_{gH} x_0 = 0$, thus is $x(0) = x_0$. Since x is w-monotone on J then it follows from [7, Lemma 1] that $x(t) \ominus_{gH} x_0$ is w-increasing on J. Therefore $t \mapsto \mathfrak{I}_t^\alpha f(t, x(t))$ is also w-increasing and

$$\mathfrak{I}_t^\alpha f(t, x(t)) = [I_t^\alpha f^L(t, x(t)), I_t^\alpha f^U(t, x(t))], \ \forall t \in J.$$

Hence,

$$\begin{aligned}
{}^C\mathfrak{D}_t^\alpha \mathfrak{I}_t^\alpha f(t, x(t)) &= {}^L\mathfrak{D}_t^\alpha [\mathfrak{I}_t^\alpha f(t, x(t)) \ominus_{gH} (\mathfrak{I}_t^\alpha f(t, x(t))|_{t=0})] \\
&= {}^L\mathfrak{D}_t^\alpha [\mathfrak{I}_t^\alpha f(t, x(t)) \ominus_{gH} \{0\}] \\
&= {}^L\mathfrak{D}_t^\alpha \mathfrak{I}_t^\alpha f(t, x(t)) \\
&= {}^L\mathfrak{D}_t^\alpha [I_t^\alpha f^L(t, x(t)), I_t^\alpha f^U(t, x(t))] \\
&= \frac{d}{dt} \mathfrak{I}_t^{1-\alpha} [I_t^\alpha f^L(t, x(t)), I_t^\alpha f^U(t, x(t))] \\
&= \frac{d}{dt} [I_t^{1-\alpha} I_t^\alpha f^L(t, x(t)), I_t^{1-\alpha} I_t^\alpha f^U(t, x(t))] \\
&= \frac{d}{dt} \left[\int_0^t f^L(s, x(s)) ds, \int_0^t f^U(s, x(s)) ds \right] \\
&= [f^L(t, x(t)), f^U(t, x(t))]
\end{aligned}$$

$$= f(t, x(t)) \text{ for a.e. } t \in J. \tag{4.9}$$

On the other hand, applying Theorem 2 it directly obtains that

$$\begin{aligned}
&{}^{C}\mathfrak{D}_t^{\alpha}[x(t) \ominus_{gH} x_0] \\
&= {}^{L}\mathfrak{D}_t^{\alpha}[(x(t) \ominus_{gH} x_0) \ominus_{gH} (x(t) \ominus_{gH} x_0|_{t=0})] \\
&= {}^{L}\mathfrak{D}_t^{\alpha}[(x(t) \ominus_{gH} x_0) \ominus_{gH} \{0\}] \\
&= {}^{L}\mathfrak{D}_t^{\alpha}[x(t) \ominus_{gH} x_0] \\
&= {}^{L}\mathfrak{D}_t^{\alpha}[x(t) \ominus_{gH} x(0)] \\
&= {}^{C}\mathfrak{D}_t^{\alpha} x(t), \text{ for a.e. } t \in J.
\end{aligned} \tag{4.10}$$

Combining (4.9) with (4.10) it concludes

$$\begin{cases} {}^{C}\mathfrak{D}_t^{\alpha} x(t) = f(t, x(t)), \text{ for a.e. } t \in J, \\ x(0) = x_0. \end{cases}$$

This means that $x \in C(J; R(I))$ is a solution of (IVCFDE) (1.1). □

Now, we denote U as the solution set of (IVCFDE) (1.1).

Theorem 6. *Assuming that the same hypothesises as in Theorem 4 hold. If, in addition, $t \mapsto \mathfrak{I}_t^{\alpha} f(t, x(t))$ is w-increasing, for each $\underline{x}_1 \preceq_{LU} x \preceq_{LU} \overline{x}_1$ (or $\underline{x}_2 \preceq_{LU} x \preceq_{LU} \overline{x}_2$). Then, it has $U_1 \subseteq U$ (or $U_2 \subseteq U$), thus is (IVCFDE) (1.1) has at least one w-monotone solution.*

Proof. By the assumptions, if (A_1) is satisfied, then we know that $U_1 \neq \varnothing$. For any $x \in U_1$, thus is

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds, \ \forall t \in J.$$

Hence, it calculate

$$w(x(t)) = w(x_0) + w(\mathfrak{I}_t^{\alpha} f(t, x(t))).$$

Therefore, x is w-increasing, because $t \mapsto \mathfrak{I}_t^{\alpha} f(t, x(t))$ is w-increasing, for each $\underline{x}_1 \preceq_{LU} x \preceq_{LU} \overline{x}_1$. Consequently, applying Theorem 5 we conclude that x is also a solution of (IVCFDE) (1.1). This means $U_1 \subseteq U$.

Similarly, when (A_2) holds, we also get that for each $x \in U_2$, x is w-decreasing. So, applying Theorem 5 again, it implies $U_2 \subseteq U$. □

5. Numerical examples

In this section, we mainly use foregoing analysis to present some simple examples to solve the interval-valued Caputo fractional differential equation.

Example 1. Consider the following interval-valued Caputo fractional differential equation

$$\begin{cases} {}^{C}\mathfrak{D}_t^{\frac{1}{2}} x(t) = [\frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}} - t^2 + x^L(t), \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}} - t - 1 + x^U(t)],\ t \in [0, \frac{1}{2}], \\ x(0) = [0, 1]. \end{cases} \tag{5.1}$$

Proof. Denote $f(t, x(t)) = [\frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}} - t^2 + x^L(t), \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}} - t - 1 + x^U(t)],\ t \in [0, \frac{1}{2}]$. In this probelm, we just discuss

$$x(t) = x_0 + \mathfrak{I}_t^{\frac{1}{2}} [\frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}} - t^2 + x^L(t), \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}} - t - 1 + x^U(t)],\ t \in [0, \frac{1}{2}]. \tag{5.2}$$

Let $\underline{x}_1 = [0, 1]$, $\overline{x}_1 = [t^2 + t^3, 1 + t + t^2]$ and

$$B_1 = \left[\min_{t \in [0,0.5]} \underline{x}_1^L(t), \max_{t \in [0,0.5]} \underline{x}_1^U(t) \right] = \left[0, \frac{7}{4} \right].$$

By the form of f, we readily get f is continuous and nondecreasing in B_1 with respect to the second variable. Also, $\underline{x}_1 \preceq_{LU} \overline{x}_1$, this means that all conditions of (A_1) in Theorem 4 are satisfied. So, it constructs the sequences $\{y_n\}$ and $\{z_n\}$ as follows

$$\begin{aligned} y_0 &= \underline{x}_1, \\ y_{n+1} &= x_0 + \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-s)^{-\frac{1}{2}} f(s, y_n(s)) ds, \end{aligned}$$

and

$$\begin{aligned} z_0 &= \overline{x}_1, \\ z_{n+1} &= x_0 + \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-s)^{-\frac{1}{2}} f(s, z_n(s)) ds. \end{aligned}$$

By Theorem 4, it knows that $y_n \to x_L$ in $C([0, 0.5]; R(I))$ and $z_n \to x_M$ in $C([0, 0.5]; R(I))$. In fact, may calculate that

$$x_L(t) = x_M(t) = x(t) = [t^2, 1 + t],\ t \in [0, 0.5].$$

However, $w(x(t)) = 1 + t - t^2$ is w-increasing on $[0, 0.5]$ then by Theorem 5, we conclude that x is a solution of (5.1). Moreover, by applying the numerical scheme described in the previous processes, we obtain Fig. 1 and Table 1. □

Example 2. Consider the following interval-valued Caputo fractional differential equation

$$\begin{cases} {}^{C}\mathfrak{D}_t^{\frac{1}{2}} x(t) = [\frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}} - t + x^L(t), \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}} - t^2 - 1 + x^U(t)], \\ x(0) = [0, 1]. \end{cases} \tag{5.3}$$

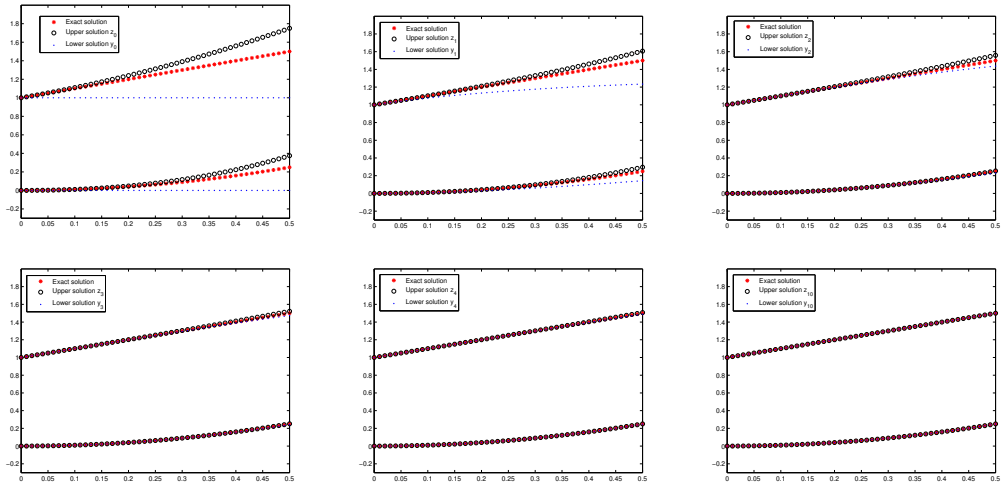

FIGURE 1. A plot of y_k and z_k, $k = 0,1,2,3,4,10$, for Example 1

TABLE 1. Error Analysis

Absolute Error \ Iterations	$n=5$	$n=10$	$n=15$	$n=20$
$\sup_{t\in[0,0.5]} \lvert y_n^L(t)-x^L(t)\rvert$	2.60×10^{-3}	5.90×10^{-6}	5.38×10^{-9}	2.52×10^{-12}
$\sup_{t\in[0,0.5]} \lvert y_n^U(t)-x^U(t)\rvert$	1.04×10^{-2}	3.84×10^{-5}	4.84×10^{-8}	2.90×10^{-11}
$\sup_{t\in[0,0.5]} \lvert z_n^L(t)-x^L(t)\rvert$	3.91×10^{-4}	5.90×10^{-7}	4.04×10^{-10}	1.51×10^{-13}
$\sup_{t\in[0,0.5]} \lvert z_n^U(t)-x^U(t)\rvert$	7.36×10^{-3}	1.67×10^{-5}	1.52×10^{-8}	7.14×10^{-12}

Proof. Let $f(t,x(t)) := [\frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}} - t + x^L(t), \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}} - t^2 - 1 + x^U(t)]$, $t \in [0,\frac{1}{2}]$, $t \in [0,0.5]$. For problem (5.3), we just discuss

$$x(t) = x_0 \ominus \frac{(-1)}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}} f(s,x(s))ds,\ t \in [0,0.5]. \tag{5.4}$$

Consider the upper and lower solutions of (5.4) as

$$\underline{x}_2(t) = [0,1],$$
$$\overline{x}_2(t) = [t+t^2, 1+t^2+t^3].$$

We may verify that all condition of (A_2) in Theorem 4 also hold. Consequently, it constructs the sequences $\{y_n\}$ and $\{z_n\}$ as follows

$$y_{n+1}(t) = x_0 \ominus (-1)\cdot\frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}} f(s,y_n(s))ds,$$

$$z_{n+1}(t) = x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-s)^{-\frac{1}{2}} f(s, z_n(s)) ds.$$

It directly calculate that $y_n \to [t, t^2+1]$ and $z_n \to [t, t^2+1]$. So, $x(t) = [t, t^2+1]$ is a solution of (5.4). Besides, $w(x(t)) = 1 + t^2 - t$ is w-decreasing, then applying Theorem 5 again to obtain that x is also a solution of (5.3). In addition, we also obtain Fig. 2 and Table 2. □

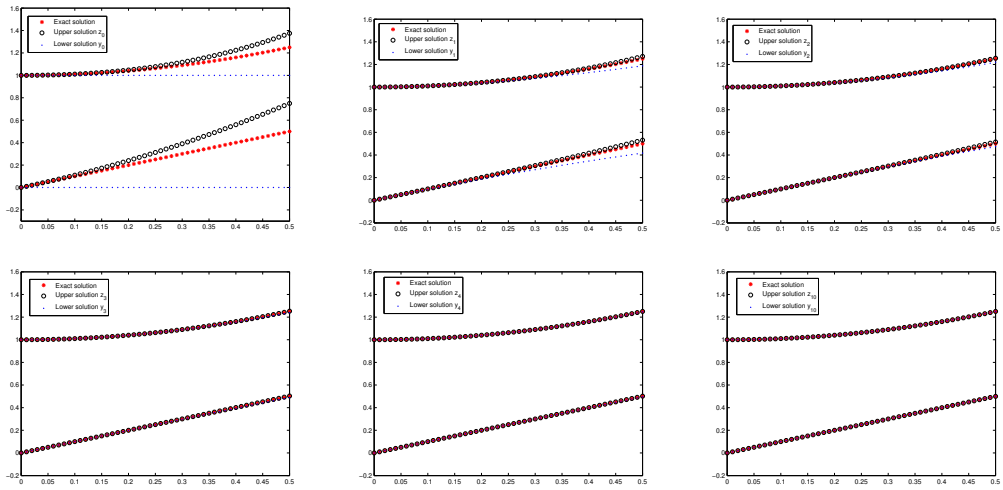

FIGURE 2. A plot of y_k and z_k, $k = 0, 1, 2, 3, 4, 10$, for Example 2

TABLE 2. Error Analysis

Absolute Error \ Iterations	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$\sup_{t\in[0,0.5]} \lvert y_n^L(t) - x^L(t)\rvert$	1.04×10^{-2}	3.84×10^{-5}	2.15×10^{-8}	2.90×10^{-11}
$\sup_{t\in[0,0.5]} \lvert y_n^U(t) - x^U(t)\rvert$	1.04×10^{-2}	2.36×10^{-5}	4.82×10^{-8}	1.01×10^{-11}
$\sup_{t\in[0,0.5]} \lvert z_n^L(t) - x^L(t)\rvert$	3.13×10^{-3}	1.18×10^{-5}	3.23×10^{-9}	5.05×10^{-12}
$\sup_{t\in[0,0.5]} \lvert z_n^U(t) - x^U(t)\rvert$	2.60×10^{-3}	2.36×10^{-6}	5.38×10^{-9}	6.06×10^{-13}

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SOME NEW HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR FUNCTIONS WHOSE nth DERIVATIVES ARE LOGARITHMICALLY RELATIVE $h-$PREINVEX

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Abstract. The authors have introduced the concept of logarithmically relative $h-$preinvex function which is a generalized form of previously known concepts [9, 11], and try to establish some new Hermite-Hadamard type integral inequalities for functions whose absolute values of nth derivatives are logarithmically relative $h-$preinvex.

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1. Introduction and preliminaries

In recent years, researchers have paid a lot of attention to study and investigate the theory of convex functions due to its extensive applications in different fields of pure and applied sciences. Weir et al. [15] had given a significant generalization of convex functions by introducing preinvex functions. Varosanec[12], introduced the $h-$convex functions. Noor et al. [9] have introduced the concept of logarithmically $h-$preinvex functions, generalizing the concept of logarithmically $s-$preinvex functions, logarithmically $P-$preinvex functions and logarithmically $Q-$preinvex functions.

There are many inequalities for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications. Hermit-Hadamard inequlaities for the convex functions and their variant forms are available in the literature [3–11, 13, 14].

Motivated by the works of Noor et al. [9], Fulga et al. [1], and using the concept of logarithmically relative $h-$preinvex functions, we have derived some new Hermite-Hadamard type inequalites.

The paper is arranged in such a way that after this Introduction,in Section 1, the authors have defined the logarithmically relative $h-$preinvex functions and its relating consequent concepts, in section 2, the authors have discussed their main results

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regarding Hermite-Hadamard type integral inequalities, using the concept of Young's inquality, Hölder's inequality, and power mean's inequality.

Let M be a nonempty closed set in $\mathbf{R}^n$. Let $f : M \to (0,\infty)$ be a continuous function and let $\theta : \mathbf{R^n} \times \mathbf{R^n} \to \mathbf{R}^n$ be a continuous bifunction.

Definition 1 ([1])**.** A set $M \subseteq \mathbf{R}^n$ is said to be a relative invex (or $g-$invex) with respect to the map θ, if there exists a function $g : \mathbf{R}^n \to \mathbf{R}^n$ such that,

$$g(a) + t\theta(g(b), g(a)) \in M;\ \forall\ a, b \in \mathbf{R}^n\ : g(a), g(b) \in M,\ t \in [0,1].$$

Definition 2. Let $M \subseteq \mathbf{R}^n$ be a relative invex set with respect to θ. Then $\theta-$relative path, $P_{g(a);g(c)}$, joining the points $g(a)$ and $g(c) = g(a) + \theta(g(b), g(a))$ is defined as:

$$P_{g(a);g(c)} = \{g(d) | g(d) = g(a) + t\theta(g(b), g(a));\ a, b \in M,\ t \in [0,1]\}.$$

Remark 1. For $g = I$, I an identity function, definition 2 coincides with definition of $\theta-$path [14].

Definition 3. Let $h : J \to \mathbf{R}$, where $(0,1) \subseteq J$ be an interval in $\mathbf{R}$; let $h \neq 0$ for all values of J; let $M \subseteq \mathbf{R}^n$ be a relative invex set with respect to θ. A function $f : M \to (0,\infty)$ is said to be logarithmically relative $h-$preinvex (or $\log-(g,h)$ preinvex) with respect to θ, if

$$f(g(a) + t\theta(g(b), g(a))) \leq [f(g(a))]^{h(1-t)}[f(g(b))]^{h(t)}, \tag{1.1}$$

for $a, b \in \mathbf{R}^n$ such that $g(a), g(b) \in M$, $t \in (0,1)$.

f is logarithmically relative $h-$preconcave (or $\log-(g,h)$ preconcave) with respect to θ whenever the inequality sign in (1.1) is reversed.

Remark 2. For $g(t) = I$, I an identity function, definition 3 coincides with definition of logarithmically $h-$preinvex function[9].

Example 1. Let $M = [-2,-1] \bigcup [1,2]$, then obviously M is relative invex for $g : \mathbf{R} \to \mathbf{R}^+ \bigcup \{0\}$ and $\theta : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ respectively defined by:

$$g(x) = \begin{cases} x^2 & \text{for } |x| \leq \sqrt{2} \\ 2 & \text{for } |x| > \sqrt{2}. \end{cases} \qquad \theta(x,y) = \begin{cases} x - y & \text{for } x.y > 0 \\ 2 - y & \text{for } x.y < 0. \end{cases}$$

Consider, $f : M \to (0,\infty)$ defined by

$$f(x) = \arctan(x + 2 + \sqrt{3})\ and\ h(t) = t^s,$$

where $s \leq 0$. Then f is logarithmically relative $h-$preinvex function with respect to θ.

Remark 3. For $x = \sqrt{2}$, $y = 1$, $t = \frac{1}{2}$, $s = 1$, the above function f is not logarithmically relative $h-$preinvex function.

Definition 4. A function $f : M \to (0,\infty)$ is said to be logarithmically relative $s-$preinvex (or $\log-(g,s)$ preinvex) with respect to θ, if

$$f\left(g(a)+t\theta\left(g(b),g(a)\right)\right) \leq [f(g(a))]^{(1-t)^s}[f(g(b))]^{t^s},$$

for $a,b \in \mathbf{R}^n$ such that $g(a),g(b) \in M$, $t \in [0,1]$, $s \in (0,1]$.

Remark 4. (1) For $g(t) = I$, I an identity function, definition 4 coincides with definition of logarithmically $s-$preinvex function [9].
(2) For $h(t) = t^s$, definition 3 coincides with definition 4.

Definition 5. A function $f : M \to (0,\infty)$ is said to be logarithmically relative $P-$preinvex (or $\log-(g,P)$ preinvex) with respect to θ, if

$$f\left(g(a)+t\theta\left(g(b),g(a)\right)\right) \leq [f(g(a))][f(g(b))],$$

for $a,b \in \mathbf{R}^n$ such that $g(a),g(b) \in M$, $t \in [0,1]$.

Remark 5. (1) For $g(t) = I$, I an identity function, definition 5 coincides with definition of logarithmically $P-$preinvex function [9].
(2) For $h(t) = 1$, definition 3 coincides with definition 5.

Definition 6. A function $f : M \to (0,\infty)$ is said to be logarithmically relative $Q-$preinvex (or $\log-(g,Q)$ preinvex) with respect to θ, if

$$f\left(g(a)+t\theta\left(g(b),g(a)\right)\right) \leq [f(g(a))]^{\frac{1}{1-t}}[f(g(b))]^{\frac{1}{t}},$$

for $a,b \in \mathbf{R}^n$ such that $g(a),g(b) \in M$, $t \in (0,1)$.

Remark 6. (1) For $g(t) = I$, I an identity function, definition 6 coincides with definition of logarithmically $Q-$preinvex function[9].
(2) For $h(t) = \frac{1}{t}$, definition 3 coincides with definition 6.

Example 2. Let $M = [-2,-1]\bigcup[1,2]$, then obviously M is relative invex for $g : \mathbf{R} \to \mathbf{R}^+\bigcup\{0\}$ and $\theta : \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ respectively defined by

$$g(x) = \begin{cases} x^2 & \text{for } |x| \leq \sqrt{2} \\ 1 & \text{for } |x| > \sqrt{2}. \end{cases} \qquad \theta(x,y) = x-y.$$

Consider, $f : M \to (0,\infty)$ defined by $f(x) = \arctan(x+2+\epsilon)$ for $\epsilon > 0$ and $h(t) = \frac{1}{t}$. Then f is logarithmically relative $Q-$preinvex function with respect to θ.

Definition 7. The beta function is defined as:

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x,y > 0.$$

Definition 8. The incomplete beta function is defined as:

$$\beta_z(x,y) = \int_0^z t^{x-1}(1-t)^{y-1}dt, \quad 0 \leq z \leq 1; \quad x,y > 0.$$

2. Results

For establishing some new Hermite-Hadamard type integral inequalities for $n-$times differentiable functions such that $|f^{(n)}|$ are logarithmically relative $h-$preinvex functions, we need the following result, which is a generalization of a result proved by Wang et al. [14, Lemma1].

Lemma 1 ([2]). *Let $A \subseteq \mathbf{R}$ be an open relative invex subset with respect to $\theta : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, with $a, b \in A$ and $\theta(g(a), g(b)) \neq 0$. If $f : \mathbf{R} \to \mathbf{R}$ is $n-$times differentiable function and $f^{(n)}$ is integrable on the $\theta-$relative path $P_{g(b);g(c)}$, then*

$$\begin{aligned}
&\frac{[\theta(g(a),g(b))]^n}{2^{n+2}\times n!}\int_0^1 (n-2t)t^{n-1}\left[f^{(n)}\left(g(b)+\frac{1-t}{2}\theta(g(a),g(b))\right)\right.\\
&\qquad\left.+f^{(n)}\left(g(b)+\frac{2-t}{2}\theta(g(a),g(b))\right)\right]dt\\
&=\frac{1}{2}\left[\frac{f(g(b))+f(g(b)+\theta(g(a),g(b)))}{2}+f\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\\
&\qquad-\frac{1}{\theta(g(a),g(b))}\int_{g(b)}^{g(b)+\theta(g(a),g(b))} f(g(x))\,dg(x)\\
&-\sum_{k=2}^{n-1}\frac{[\theta(g(a),g(b))]^k(k-1)}{2^{k+2}\times(k+1)!}\left[f^{(k)}(g(b))+f^{(k)}\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]
\end{aligned}\tag{2.1}$$

where the above summation is zero for $n = 1, 2$

Before going to our main results, we make the following assumptions:

(1) $\widetilde{A_n} := \frac{|\theta(g(a),g(b))|^n}{n!\times 2^{n+2}}$

(2) $\widetilde{B}_{n,k}(t) := \left|f^{(n)}(g(b))\right| \times \left|\frac{f^{(n)}(g(a))}{f^{(n)}(g(b))}\right|^{h(\frac{k-t}{2})}$

(3) $h(\frac{1+t}{2}) + h(\frac{1-t}{2}) = 1 = h(\frac{t}{2}) + h(\frac{2-t}{2})$.

Theorem 1. *Let $A \subseteq \mathbf{R}$ be a relative invex subset with respect to $\theta : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, with $a, b \in A$ and $\theta(g(a), g(b)) \neq 0$. Suppose that $f : A \to (0, \infty)$ is $n-$times differentiable function and $f^{(n)}$ is integrable on the $\theta-$relative path $P_{g(b);g(c)}$, such that $|f^{(n)}|$ is logarithmically relative $h-$preinvex on A; let $q_k > 1$, $k = 1, 2$, then*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(g(b))+f(g(b)+\theta(g(a),g(b)))}{2}+f\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\right.\\
&\qquad-\frac{1}{\theta(g(a),g(b))}\int_{g(b)}^{g(b)+\theta(g(a),g(b))} f(g(u))\,dg(u)
\end{aligned}$$

$$-\sum_{k=2}^{n-1}\frac{[\theta(g(a),g(b))]^k(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}(g(b))+f^{(k)}\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\Bigg|$$
$$\leq\begin{cases}\widetilde{A_1}\sum_{k=1}^{2}\frac{1}{q_k}\left[\frac{(q_k-1)^2}{2q_k-1}+\int_0^1\left(\widetilde{B}_{1,k}(t)\right)^{q_k}dt\right] & \text{for } n=1\\ \widetilde{A_n}\sum_{k=1}^{2}\frac{1}{q_k}\left[\frac{n^{\frac{nq_k}{q_k-1}+1}\times(q_k-1)}{2^{\frac{q_k(n-1)}{q_k-1}+1}}\beta_{\frac{2}{n}}\left(\frac{nq_k-1}{q_k-1},\frac{2q_k-1}{q_k-1}\right)+\int_0^1\left(\widetilde{B}_{n,k}(t)\right)^{q_k}dt\right] & \text{for } n\geq 2.\end{cases} \tag{2.2}$$

Proof. Since $g(b)+t\theta(g(a),g(b))\in A$ for every $t\in(0,1)$, by Lemma 1 and Young's inequality, we have

$$\Bigg|\frac{1}{2}\left[\frac{f(g(b))+f(g(b)+\theta(g(a),g(b)))}{2}+f\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]$$
$$-\frac{1}{\theta(g(a),g(b))}\int_{g(b)}^{g(b)+\theta(g(a),g(b))}f(g(u))\,dg(u)$$
$$-\sum_{k=2}^{n-1}\frac{[\theta(g(a),g(b))]^k(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}(g(b))+f^{(k)}\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\Bigg|$$
$$\leq\frac{|\theta(g(a),g(b))|^n}{n!\times 2^{n+2}}\int_0^1\left|(n-2t)t^{n-1}\right|\left[\left|f^{(n)}\left(g(b)+\frac{1-t}{2}\theta(g(a),g(b))\right)\right|\right.$$
$$\left.+\left|f^{(n)}\left(g(b)+\frac{2-t}{2}\theta(g(a),g(b))\right)\right|\right]dt$$
$$\leq\frac{|\theta(g(a),g(b))|^n}{n!\times 2^{n+2}}\times\int_0^1\left[\frac{q_1-1}{q_1}\left|(n-2t)t^{n-1}\right|^{\frac{q_1}{q_1-1}}\right.$$
$$+\frac{1}{q_1}\left|f^{(n)}\left(g(b)+\frac{1-t}{2}\theta(g(a),g(b))\right)\right|^{q_1}+\frac{q_2-1}{q_2}\left|(n-2t)t^{n-1}\right|^{\frac{q_2}{q_2-1}}$$
$$\left.+\frac{1}{q_2}\left|f^{(n)}\left(g(b)+\frac{2-t}{2}\theta(g(a),g(b))\right)\right|^{q_2}\right]dt \tag{2.3}$$

Using logarithmically relative $h-$preinvexity of $|f^{(n)}|$, we have

$$\int_0^1\left|f^{(n)}\left(g(b)+\frac{1-t}{2}\theta(g(a),g(b))\right)\right|^{q_1}dt$$
$$\leq\int_0^1\left(\left|f^{(n)}(g(b))\right|^{h(\frac{1+t}{2})}\left|f^{(n)}(g(a))\right|^{h(\frac{1-t}{2})}\right)^{q_1}dt$$
$$=\int_0^1\left(\left|f^{(n)}(g(b))\right|\left|\frac{f^{(n)}(g(a))}{f^{(n)}(g(b))}\right|^{h(\frac{1-t}{2})}\right)^{q_1}dt \tag{2.4}$$

Similarly

$$\int_0^1 \left| f^{(n)}\left(g(b)+\frac{2-t}{2}\theta\left(g(a),g(b)\right)\right)\right|^{q_2} dt$$
$$\leq \int_0^1 \left(\left|f^{(n)}\left(g(b)\right)\right| \left|\frac{f^{(n)}\left(g(a)\right)}{f^{(n)}\left(g(b)\right)}\right|^{h\left(\frac{2-t}{2}\right)}\right)^{q_2} dt \quad (2.5)$$

Also,

$$\int_0^1 \left|(n-2t)t^{n-1}\right|^{\frac{q_k}{q_k-1}} dt = \begin{cases} \frac{q_k-1}{2q_k-1} & \text{for } n=1 \\ \frac{n^{\frac{nq_k}{q_k-1}+1}}{2^{\frac{q_k(n-1)}{q_k-1}+1}}\beta_{\frac{2}{n}}\left(\frac{nq_k-1}{q_k-1},\frac{2q_k-1}{q_k-1}\right) & \text{for } n\geq 2. \end{cases} \quad (2.6)$$

A combination of (2.3)-(2.6) yields the desired result. □

Corollary 1. *Under the conditions of theorem 1 for $q_1=q_2$, we have*

$$\left|\frac{1}{2}\left[\frac{f\left(g(b)\right)+f\left(g(b)+\theta\left(g(a),g(b)\right)\right)}{2}+f\left(\frac{2g(b)+\theta\left(g(a),g(b)\right)}{2}\right)\right]\right.$$
$$-\frac{1}{\theta\left(g(a),g(b)\right)}\int_{g(b)}^{g(b)+\theta(g(a),g(b))} f\left(g(u)\right)dg(u)$$
$$\left.-\sum_{k=2}^{n-1}\frac{\left[\theta\left(g(a),g(b)\right)\right]^k(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}\left(g(b)\right)+f^{(k)}\left(\frac{2g(b)+\theta\left(g(a),g(b)\right)}{2}\right)\right]\right|$$
$$\leq \begin{cases} \frac{\widetilde{A}_1}{q_2}\left[\frac{2(q_2-1)^2}{2q_2-1}+\int_0^1\sum_{k=1}^2\left(\widetilde{B}_{1,k}(t)\right)^{q_2}dt\right] & \textit{for } n=1 \\ \frac{\widetilde{A}_n}{q_2}\left[\frac{n^{\frac{nq_2}{q_2-1}+1}\times(q_2-1)}{2^{\frac{q_2(n-1)}{q_2-1}}}\beta_{\frac{2}{n}}\left(\frac{nq_2-1}{q_2-1},\frac{2q_2-1}{q_2-1}\right)+\int_0^1\sum_{k=1}^2\left(\widetilde{B}_{n,k}(t)\right)^{q_2}dt\right] & \textit{for } n\geq 2. \end{cases}$$

Theorem 2. *Let $A\subseteq \mathbf{R}$ be a relative invex subset with respect to $\theta:\mathbf{R}\times\mathbf{R}\to\mathbf{R}$, with $a,b\in A$ and $\theta(g(a),g(b))\neq 0$. Suppose that $f:A\to(0,\infty)$ is $n-$times differentiable function and $f^{(n)}$ is integrable on the $\theta-$relative path $P_{g(b);g(c)}$, such that $|f^{(n)}|$ is logarithmically relative $h-$preinvex on A; let $q_3>1$, then*

$$\left|\frac{1}{2}\left[\frac{f\left(g(b)\right)+f\left(g(b)+\theta\left(g(a),g(b)\right)\right)}{2}+f\left(\frac{2g(b)+\theta\left(g(a),g(b)\right)}{2}\right)\right]\right.$$
$$-\frac{1}{\theta\left(g(a),g(b)\right)}\int_{g(b)}^{g(b)+\theta(g(a),g(b))} f\left(g(u)\right)dg(u)$$
$$\left.-\sum_{k=2}^{n-1}\frac{\left[\theta\left(g(a),g(b)\right)\right]^k(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}\left(g(b)\right)+f^{(k)}\left(\frac{2g(b)+\theta\left(g(a),g(b)\right)}{2}\right)\right]\right|$$

$$\leq \begin{cases} \frac{\widetilde{A_1}}{q_3}\left[\frac{(q_3-1)^2}{2q_3-1}+\int_0^1\left(\sum_{k=1}^{2}\widetilde{B}_{1,k}(t)\right)^{q_3}dt\right] & \text{for } n=1 \\ \frac{\widetilde{A_n}}{q_3}\left[\frac{n^{\frac{nq_3}{q_3-1}+1}\times(q_3-1)}{2^{\frac{q_3(n-1)}{q_3-1}+1}}\beta_{\frac{2}{n}}\left(\frac{nq_3-1}{q_3-1},\frac{2q_3-1}{q_3-1}\right)+\int_0^1\left(\sum_{k=1}^{2}\widetilde{B}_{n,k}(t)\right)^{q_3}dt\right] & \text{for } n\geq 2. \end{cases} \tag{2.7}$$

Proof. Since $g(b)+t\theta(g(a),g(b))\in A$ for every $t\in(0,1)$, by Lemma 1, Young's inequality and the logarithmically relative $h-$preinvexity of $|f^{(n)}|$, we have

$$\begin{aligned}
&\Bigg|\frac{1}{2}\left[\frac{f(g(b))+f(g(b)+\theta(g(a),g(b)))}{2}+f\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right] \\
&\qquad -\frac{1}{\theta(g(a),g(b))}\int_{g(b)}^{g(b)+\theta(g(a),g(b))}f(g(u))\,dg(u) \\
&-\sum_{k=2}^{n-1}\frac{[\theta(g(a),g(b))]^k(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}(g(b))+f^{(k)}\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\Bigg| \\
&\leq \frac{|\theta(g(a),g(b))|^n}{n!\times 2^{n+2}}\int_0^1\left|(n-2t)t^{n-1}\right|\left[\left|f^{(n)}\left(g(b)+\frac{1-t}{2}\theta(g(a),g(b))\right)\right|\right. \\
&\qquad \left.+\left|f^{(n)}\left(g(b)+\frac{2-t}{2}\theta(g(a),g(b))\right)\right|\right]dt \\
&\leq \frac{|\theta(g(a),g(b))|^n}{n!\times 2^{n+2}}\times\int_0^1\left[\frac{q_3-1}{q_3}\left|(n-2t)t^{n-1}\right|^{\frac{q_3}{q_3-1}}\right. \\
&\qquad +\frac{1}{q_3}\left\{\left|f^{(n)}\left(g(b)+\frac{1-t}{2}\theta(g(a),g(b))\right)\right|\right. \\
&\qquad \left.\left.+\left|f^{(n)}\left(g(b)+\frac{2-t}{2}\theta(g(a),g(b))\right)\right|\right\}^{q_3}\right]dt \\
&\leq \frac{|\theta(g(a),g(b))|^n}{n!\times 2^{n+2}\times q_3}\times\int_0^1\left[(q_3-1)\left|(n-2t)t^{n-1}\right|^{\frac{q_3}{q_3-1}}\right. \\
&\qquad \left.+\left(\sum_{k=1}^{2}\left|f^{(n)}(g(b))\right|\left|\frac{f^{(n)}(g(a))}{f^{(n)}(g(b))}\right|^{h(\frac{k-t}{2})}\right)^{q_3}\right]dt
\end{aligned}$$

□

Corollary 2. *Under the conditions of theorem 2 for $n=2$, we have*

$$\begin{aligned}
&\Bigg|\frac{1}{2}\left[\frac{f(g(b))+f(g(b)+\theta(g(a),g(b)))}{2}+f\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right] \\
&\qquad -\frac{1}{\theta(g(a),g(b))}\int_{g(b)}^{g(b)+\theta(g(a),g(b))}f(g(u))\,dg(u)\Bigg|
\end{aligned}$$

$$\leq \frac{\widetilde{A_2}}{q_3}\left[2^{\frac{q_3}{q_3-1}}\times(q_3-1)\times\beta\left(\frac{2q_3-1}{q_3-1},\frac{2q_3-1}{q_3-1}\right)+\int_0^1\left(\sum_{k=1}^{2}\widetilde{B}_{2,k}(t)\right)^{q_3}dt\right]$$

Theorem 3. *Let $A\subseteq\mathbf{R}$ be a relative invex subset with respect to $\theta:\mathbf{R}\times\mathbf{R}\to\mathbf{R}$, with $a,b\in A$ and $\theta(g(a),g(b))\neq 0$. Suppose that $f:A\to(0,\infty)$ is $n-$times differentiable function and $f^{(n)}$ is integrable on the $\theta-$relative path $P_{g(b);g(c)}$, such that $|f^{(n)}|$ is logarithmically relative $h-$preinvex on A; let $q_3>1$, and $q_3\geq r>0$, then*

$$\begin{aligned}
&\Bigg|\frac{1}{2}\left[\frac{f(g(b))+f(g(b)+\theta(g(a),g(b)))}{2}+f\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\\
&\qquad-\frac{1}{\theta(g(a),g(b))}\int_{g(b)}^{g(b)+\theta(g(a),g(b))}f(g(u))\,dg(u)\\
&-\sum_{k=2}^{n-1}\frac{[\theta(g(a),g(b))]^k(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}(g(b))+f^{(k)}\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\Bigg|\\
&\leq\begin{cases}\frac{\widetilde{A_1}}{q_3}\left[\frac{(q_3-1)^2}{2q_3-r-1}+\int_0^1|(n-2t)t^{n-1}|^r\left(\sum_{k=1}^{2}\widetilde{B}_{1,k}(t)\right)^{q_3}dt\right] & \text{for } n=1\\
\frac{\widetilde{A_n}}{q_3}\left[\frac{n^{\frac{n(q_3-r)}{q_3-1}+1}\times(q_3-1)}{2^{\frac{(q_3-r)(n-1)}{q_3-1}+1}}\beta_{\frac{2}{n}}\left(\frac{nq_3-nr+r-1}{q_3-1},\frac{2q_3-r-1}{q_3-1}\right)\right.\\
\left.+\int_0^1|(n-2t)t^{n-1}|^r\left(\sum_{k=1}^{2}\widetilde{B}_{n,k}(t)\right)^{q_3}dt\right] & \text{for } n\geq 2.\end{cases}
\end{aligned}\tag{2.8}$$

Proof. Since $g(b)+t\theta(g(a),g(b))\in A$ for every $t\in(0,1)$, by Lemma 1, Young's inequality and the logarithmically relative $h-$preinvexity of $|f^{(n)}|$, we have

$$\begin{aligned}
&\Bigg|\frac{1}{2}\left[\frac{f(g(b))+f(g(b)+\theta(g(a),g(b)))}{2}+f\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\\
&\qquad-\frac{1}{\theta(g(a),g(b))}\int_{g(b)}^{g(b)+\theta(g(a),g(b))}f(g(u))\,dg(u)\\
&-\sum_{k=2}^{n-1}\frac{[\theta(g(a),g(b))]^k(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}(g(b))+f^{(k)}\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\Bigg|\\
&\leq\frac{|\theta(g(a),g(b))|^n}{n!\times 2^{n+2}}\int_0^1\left|(n-2t)t^{n-1}\right|\left[\left|f^{(n)}\left(g(b)+\frac{1-t}{2}\theta(g(a),g(b))\right)\right|\right.\\
&\qquad\left.+\left|f^{(n)}\left(g(b)+\frac{2-t}{2}\theta(g(a),g(b))\right)\right|\right]dt\\
&\leq\frac{|\theta(g(a),g(b))|^n}{n!\times 2^{n+2}}\int_0^1\left[\frac{q_3-1}{q_3}\left|(n-2t)t^{n-1}\right|^{\frac{q_3-r}{q_3-1}}+\frac{1}{q_3}\left|(n-2t)t^{n-1}\right|^r\right.
\end{aligned}$$

$$\times\left\{\left|f^{(n)}\left(g(b)+\frac{1-t}{2}\theta\left(g(a),g(b)\right)\right)\right|\right.$$
$$\left.+\left|f^{(n)}\left(g(b)+\frac{2-t}{2}\theta\left(g(a),g(b)\right)\right)\right|\right\}^{q_3}\Bigg]dt$$
$$\leq\frac{\left|\theta\left(g(a),g(b)\right)\right|^{n}}{n!\times 2^{n+2}\times q_3}\times\int_0^1\left[(q_3-1)\left|(n-2t)t^{n-1}\right|^{\frac{q_3-r}{q_3-1}}\right.$$
$$\left.+\left|(n-2t)t^{n-1}\right|^{r}\times\left(\sum_{k=1}^{2}\left|f^{(n)}\left(g(b)\right)\right|\left|\frac{f^{(n)}\left(g(a)\right)}{f^{(n)}\left(g(b)\right)}\right|^{h\left(\frac{k-t}{2}\right)}\right)^{q_3}\right]dt$$

□

Corollary 3. *Under the conditions of theorem 3 for $r=q_3$, we have*

$$\left|\frac{1}{2}\left[\frac{f\left(g(b)\right)+f\left(g(b)+\theta\left(g(a),g(b)\right)\right)}{2}+f\left(\frac{2g(b)+\theta\left(g(a),g(b)\right)}{2}\right)\right]\right.$$
$$-\frac{1}{\theta\left(g(a),g(b)\right)}\int_{g(b)}^{g(b)+\theta(g(a),g(b))}f\left(g(u)\right)dg(u)$$
$$\left.-\sum_{k=2}^{n-1}\frac{\left[\theta\left(g(a),g(b)\right)\right]^{k}(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}\left(g(b)\right)+f^{(k)}\left(\frac{2g(b)+\theta\left(g(a),g(b)\right)}{2}\right)\right]\right|$$
$$\leq\begin{cases}\frac{\widetilde{A}_1}{q_3}\left[(q_3-1)+\int_0^1\left(\left|(n-2t)t^{n-1}\right|\sum_{k=1}^{2}\widetilde{B}_{1,k}(t)\right)^{q_3}dt\right] & \text{for } n=1\\ \frac{\widetilde{A}_n}{q_3}\left[\frac{n(q_3-1)}{2}\beta_{\frac{2}{n}}(1,1)+\int_0^1\left(\left|(n-2t)t^{n-1}\right|\sum_{k=1}^{2}\widetilde{B}_{n,k}(t)\right)^{q_3}dt\right] & \text{for } n\geq 2.\end{cases}$$

Theorem 4. *Let $A\subseteq\mathbf{R}$ be a relative invex subset with respect to $\theta:\mathbf{R}\times\mathbf{R}\to\mathbf{R}$, with $a,b\in A$ and $\theta(g(a),g(b))\neq 0$. Suppose that $f:A\to(0,\infty)$ is $n-$times differentiable function and $f^{(n)}$ is integrable on the $\theta-$relative path $P_{g(b);g(c)}$, such that $|f^{(n)}|$ is logarithmically relative $h-$preinvex on A; let $q_3>1$, and then*

$$\left|\frac{1}{2}\left[\frac{f\left(g(b)\right)+f\left(g(b)+\theta\left(g(a),g(b)\right)\right)}{2}+f\left(\frac{2g(b)+\theta\left(g(a),g(b)\right)}{2}\right)\right]\right.$$
$$-\frac{1}{\theta\left(g(a),g(b)\right)}\int_{g(b)}^{g(b)+\theta(g(a),g(b))}f\left(g(u)\right)dg(u)$$
$$\left.-\sum_{k=2}^{n-1}\frac{\left[\theta\left(g(a),g(b)\right)\right]^{k}(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}\left(g(b)\right)+f^{(k)}\left(\frac{2g(b)+\theta\left(g(a),g(b)\right)}{2}\right)\right]\right|$$

$$\leq \begin{cases} \widetilde{A}_1 \sum_{k=1}^{2} \left(\int_0^1 \left(\widetilde{B}_{1,k}(t) \right)^{q_3} dt \right)^{\frac{1}{q_3}} \left(\frac{q_3-1}{2q_3-1} \right)^{1-\frac{1}{q_3}} & \text{for } n=1 \\ \widetilde{A}_n \sum_{k=1}^{2} \left(\int_0^1 \left(\widetilde{B}_{n,k}(t) \right)^{q_3} dt \right)^{\frac{1}{q_3}} \left[\frac{n^{\frac{nq_3}{q_3-1}+1}}{2^{\frac{q_3(n-1)}{q_3-1}+1}} \beta_{\frac{2}{n}} \left(\frac{nq_3-1}{q_3-1}, \frac{2q_3-1}{q_3-1} \right) \right]^{1-\frac{1}{q_3}} & \text{for } n \geq 2. \end{cases} \tag{2.9}$$

Proof. Since $g(b)+t\theta(g(a),g(b)) \in A$ for every $t \in (0,1)$, by Lemma 1, Hölder's inequality and the logarithmically relative $h-$preinvexity of $|f^{(n)}|$, we have

$$\begin{aligned} &\Bigg| \frac{1}{2} \left[\frac{f(g(b)) + f(g(b)+\theta(g(a),g(b)))}{2} + f\left(\frac{2g(b)+\theta(g(a),g(b))}{2} \right) \right] \\ &\qquad - \frac{1}{\theta(g(a),g(b))} \int_{g(b)}^{g(b)+\theta(g(a),g(b))} f(g(u))\, dg(u) \\ &- \sum_{k=2}^{n-1} \frac{[\theta(g(a),g(b))]^k (k-1)}{(k+1)! \times 2^{k+2}} \left[f^{(k)}(g(b)) + f^{(k)}\left(\frac{2g(b)+\theta(g(a),g(b))}{2} \right) \right] \Bigg| \\ &\leq \frac{|\theta(g(a),g(b))|^n}{n! \times 2^{n+2}} \int_0^1 \left| (n-2t)t^{n-1} \right| \Bigg[\left| f^{(n)}\left(g(b) + \frac{1-t}{2}\theta(g(a),g(b)) \right) \right| \\ &\qquad + \left| f^{(n)}\left(g(b) + \frac{2-t}{2}\theta(g(a),g(b)) \right) \right| \Bigg] dt \\ &\leq \frac{|\theta(g(a),g(b))|^n}{n! \times 2^{n+2}} \left(\int_0^1 \left| (n-2t)t^{n-1} \right|^{\frac{q_3}{q_3-1}} dt \right)^{1-\frac{1}{q_3}} \\ &\qquad \times \left\{ \left[\int_0^1 \left| f^{(n)}\left(g(b) + \frac{1-t}{2}\theta(g(a),g(b)) \right) \right|^{q_3} dt \right]^{\frac{1}{q_3}} \right. \\ &\qquad \left. + \left[\int_0^1 \left| f^{(n)}\left(g(b) + \frac{2-t}{2}\theta(g(a),g(b)) \right) \right|^{q_3} dt \right]^{\frac{1}{q_3}} \right\} \\ &\qquad \leq \frac{|\theta(g(a),g(b))|^n}{n! \times 2^{n+2}} \left(\int_0^1 \left| (n-2t)t^{n-1} \right|^{\frac{q_3}{q_3-1}} dt \right)^{1-\frac{1}{q_3}} \\ &\qquad \times \sum_{k=1}^{2} \left(\int_0^1 \left(\left| f^{(n)}(g(b)) \right| \left| \frac{f^{(n)}(g(a))}{f^{(n)}(g(b))} \right|^{h\left(\frac{k-t}{2}\right)} \right)^{q_3} dt \right)^{\frac{1}{q_3}} \end{aligned}$$

□

Corollary 4. *Under the conditions of theorem 4 for $n=2$, we have*

$$\left|\frac{1}{2}\left[\frac{f\left(g(b)\right)+f\left(g(b)+\theta\left(g(a),g(b)\right)\right)}{2}+f\left(\frac{2g(b)+\theta\left(g(a),g(b)\right)}{2}\right)\right]\right.$$
$$\left.-\frac{1}{\theta\left(g(a),g(b)\right)}\int_{g(b)}^{g(b)+\theta(g(a),g(b))}f\left(g(u)\right)dg(u)\right|$$
$$\leq 2\widetilde{A}_2\sum_{k=1}^{2}\left(\int_0^1\left(\widetilde{B}_{2,k}(t)\right)^{q_3}dt\right)^{\frac{1}{q_3}}\left[\beta\left(\frac{2q_3-1}{q_3-1},\frac{2q_3-1}{q_3-1}\right)\right]^{1-\frac{1}{q_3}}$$

Theorem 5. *Let $A\subseteq\mathbf{R}$ be a relative invex subset with respect to $\theta:\mathbf{R}\times\mathbf{R}\to\mathbf{R}$, with $a,b\in A$ and $\theta(g(a),g(b))\neq 0$. Suppose that $f:A\to(0,\infty)$ is $n-$times differentiable function and $f^{(n)}$ is integrable on the $\theta-$relative path $P_{g(b);g(c)}$, such that $|f^{(n)}|$ is logarithmically relative $h-$preinvex on A; let $q_3\geq 1$, then*

$$\left|\frac{1}{2}\left[\frac{f\left(g(b)\right)+f\left(g(b)+\theta\left(g(a),g(b)\right)\right)}{2}+f\left(\frac{2g(b)+\theta\left(g(a),g(b)\right)}{2}\right)\right]\right.$$
$$-\frac{1}{\theta\left(g(a),g(b)\right)}\int_{g(b)}^{g(b)+\theta(g(a),g(b))}f\left(g(u)\right)dg(u)$$
$$\left.-\sum_{k=2}^{n-1}\frac{\left[\theta\left(g(a),g(b)\right)\right]^k(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}\left(g(b)\right)+f^{(k)}\left(\frac{2g(b)+\theta\left(g(a),g(b)\right)}{2}\right)\right]\right|$$
$$\leq\begin{cases}\dfrac{\widetilde{A}_1\sum_{k=1}^{2}\left(\int_0^1|(n-2t)t^{n-1}|\left(\widetilde{B}_{1,k}(t)\right)^{q_3}dt\right)^{\frac{1}{q_3}}}{2^{1-\frac{1}{q_3}}} & \text{for } n=1\\ \widetilde{A}_n\sum_{k=1}^{2}\left(\int_0^1|(n-2t)t^{n-1}|\left(\widetilde{B}_{n,k}(t)\right)^{q_3}dt\right)^{\frac{1}{q_3}}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q_3}} & \text{for } n\geq 2.\end{cases}$$

Proof. By using the weighted power mean inequality and the similar techniques used in theorem 4, we can prove this theorem. □

Corollary 5. *Under the conditions of theorem 5 for $q_3=1$, we have*

$$\left|\frac{1}{2}\left[\frac{f\left(g(b)\right)+f\left(g(b)+\theta\left(g(a),g(b)\right)\right)}{2}+f\left(\frac{2g(b)+\theta\left(g(a),g(b)\right)}{2}\right)\right]\right.$$
$$-\frac{1}{\theta\left(g(a),g(b)\right)}\int_{g(b)}^{g(b)+\theta(g(a),g(b))}f\left(g(u)\right)dg(u)$$
$$\left.-\sum_{k=2}^{n-1}\frac{\left[\theta\left(g(a),g(b)\right)\right]^k(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}\left(g(b)\right)+f^{(k)}\left(\frac{2g(b)+\theta\left(g(a),g(b)\right)}{2}\right)\right]\right|$$
$$\leq\begin{cases}\widetilde{A}_1\ \sum_{k=1}^{2}\int_0^1|(n-2t)t^{n-1}|\,\widetilde{B}_{1,k}(t)dt & \text{for } n=1\\ \widetilde{A}_n\ \sum_{k=1}^{2}\int_0^1|(n-2t)t^{n-1}|\,\widetilde{B}_{n,k}(t)dt & \text{for } n\geq 2.\end{cases}$$

Remark 7. If $g(t) = I$, where I is an identity function, then our results coincide the results for logarithmically $h-$preinvex functions.

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BALANCING WITH POWERS OF THE LUCAS SEQUENCE OF RECURRENCE $u_n = Au_{n-1} - u_{n-2}$

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Abstract. In this paper, we show that there is no solution of the diophantine equation

$$u_1^k + u_2^k + \cdots + u_{n-1}^k = u_{n+1}^l + u_{n+2}^l + \cdots + u_{n+r}^l$$

for special cases of k and l where the elements of sequence $\{u_n\}$ satisfy the relation $u_n = Au_{n-1} - u_{n-2}$ with $u_0 = 0$, $u_1 = 1$ and $A \geq 3$ is a positive integer.

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1. Introduction

Let the sequence $\{u_n\}$ is defined by the recurrence relation

$$u_n = Au_{n-1} - u_{n-2} \tag{1.1}$$

where A is a positive integer with $u_0 = 0$ and $u_1 = 1$. It's Binet form of the sequence $\{u_n\}$ is known as

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1.2}$$

where $\alpha = \frac{A+\sqrt{A^2-4}}{2}$ and $\beta = \frac{A-\sqrt{A^2-4}}{2}$. Assume the $\{v_n\}$ be the associate sequence of the sequence $\{u_n\}$. Namely, the elements of the sequence $\{v_n\}$ satisfies the following recurrence,

$$v_n = Av_{n-1} - v_{n-2}$$

with initial conditon $v_0 = 2$ and $v_1 = A$. The Binet formula of the sequnce $\{v_n\}$ is

$$v_n = \alpha^n + \beta^n. \tag{1.3}$$

The case $A = 6$ coincides with the sequence of balancing number (see [2, 4, 7, 8]) whose elements satisfy the equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r).$$

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Recently, several authors handled some diophantine equations including balancing-type rules. For example, Behera et al. [3] showed that the diophantine equation

$$F_1^k + F_2^k + \cdots + F_{n-1}^k = F_{n+1}^l + F_{n+2}^l + \cdots + F_{n+r}^l \tag{1.4}$$

has no solution in the positive integers n, r, k, l with $n \geq 2$ in the case

$$k \leq l \quad \text{and} \quad (k,l) = (2,1),\ (3,1),\ (3,2). \tag{1.5}$$

Here F_n denotes the nth Fibonacci number. They also conjectured in [3] that only the quadruple $(n,r,k,l) = (4,3,8,2)$ satisfies the equation (1.4). Their conjecture was proved by Alvarado et al. [1]. In the sequel, Irmak [5] replaced the Fibonacci numbers with balancing numbers in (1.4), and conjectured that there is no solution of the equation

$$B_1^k + B_2^k + \cdots + B_{n-1}^k = B_{n+1}^l + B_{n+2}^l + \cdots + B_{n+r}^l.$$

In this paper, we generalize the conjecture of Irmak [5]. Our conjecture is following,

Conjecture 1. *Assume that $A \geq 3$. For the sequence $\{u_n\}_{n\geq 0}$, there is no solution of the equation*

$$u_1^k + u_2^k + \cdots + u_{n-1}^k = u_{n+1}^l + u_{n+2}^l + \cdots u_{n+r}^l \tag{1.6}$$

for positive integers $n \geq 2$, r, k and l.

2. Preliminaries

In this section, we present several lemmas to confirm the conjecture. First lemma presents some formulas including the sums of the elements of the sequence $\{u_n\}$.

Lemma 1. *For any positive integer k, the following identities hold.*
(a) $\sum_{k=1}^n u_k = \frac{1}{A-2}(u_{n+1} - u_n - 1)$
(b) $\sum_{k=1}^n u_k^2 = \frac{1}{A^2-4}(u_{2n+1} - (2n+1))$
(c) $\sum_{k=1}^n u_k^3 = \frac{1}{A^2-4}\left\{\frac{u_{3n+3}-u_{3n}-u_3}{A^3-3A-2} - \frac{3}{A-2}(u_{n+1} - u_n - 1)\right\}$
(d) $\sum_{k=1}^n u_{2k-1} = u_n^2$.

Proof. We prove the third one. We follow the Binet formula (1.2). Other identities can be proven by similar way.

$$\begin{aligned}\sum_{k=1}^n u_k^3 &= \frac{1}{(\alpha-\beta)^3}\sum_{k=0}^{n-1}\left(\alpha^{k+1} - \beta^{k+1}\right)^3 \\ &= \frac{1}{(\alpha-\beta)^3}\sum_{k=0}^{n-1}\alpha^{3k+3} - \beta^{3k+3} - 3\left(\alpha^{k+1} - \beta^{k+1}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(\alpha-\beta)^3}\left\{\alpha^3\frac{1-\alpha^{3n}}{1-\alpha^3}-\beta^3\frac{1-\beta^{3n}}{1-\beta^3}-3\left(\alpha\frac{1-\alpha^n}{1-\alpha}-\beta\frac{1-\beta^n}{1-\beta}\right)\right\}\\
&= \frac{1}{(\alpha-\beta)^2}\left\{\frac{\alpha^{3n+3}-\beta^{3n+3}-\left(\alpha^{3n}-\beta^{3n}\right)-\left(\alpha^3-\beta^3\right)}{(\alpha-\beta)\left(\alpha^3+\beta^3-2\right)}\right.\\
&\left.-\frac{3}{A-2}\left(u_{n+1}-u_n-1\right)\right\}\\
&= \frac{1}{A^2-4}\left\{\frac{u_{3n+3}-u_{3n}-u_3}{A^3-3A-2}-\frac{3}{A-2}\left(u_{n+1}-u_n-1\right)\right\}
\end{aligned}
$$

as claimed. □

Lemma 2. *For positive integer n,*

$$u_n^2-u_{n-1}^2=u_{2n-1}$$

follows.

Proof. It can proven by the Binet formula of the sequence $\{u_n\}$ or Lemma 1 d. □

Lemma 3. *For the positive integer $A\geq 3$ and $n\geq 1$, then the inequalities*

$$(A-2)\,u_n<u_{n+1}-u_n<(A-1)\,u_n$$

and

$$\left(A^3-3A-2\right)u_{3n}<u_{3n+3}-u_{3n}-u_3<\left(A^3-3A-1\right)u_{3n}$$

hold.

Proof. By the Binet formulas of the sequences $\{u_n\}$ and $\{v_n\}$, we obtain the formula

$$u_{rn}=v_r u_{r(n-1)}+(-1)^r u_{r(n-2)}.$$

Assume that $r=3$. By the recurrence relation (1.1),

$$
\begin{aligned}
u_{3(n+1)}-u_{3n} &= \left(A^3-3A\right)u_{3n}-u_{3(n-1)}-u_{3n}\\
&= \left(A^3-3A-1\right)u_{3n}-u_{3(n-1)}\\
&< \left(A^3-3A-1\right)u_{3n}
\end{aligned}
$$

follows. Since the inequality $-u_{3n}<-u_{3(n-1)}$ holds, then

$$u_{3(n+1)}-u_{3n}=\left(A^3-3A-1\right)u_{3n}-u_{3(n-1)}>\left(A^3-3A-2\right)u_n$$

follows. Second one can be proven by similar way. □

Lemma 4. *Suppose that $A\geq 3$. Then for all integers $n\geq 3$, the inequality*

$$\alpha^{n-1}<u_n<\alpha^{n-0.83} \tag{2.1}$$

hold.

Proof. See Lemma 2.2 in [6]. □

Lemma 5. *Suppose that $a > 0$ and $b \geq 0$ are real numbers, and that t_0 is a positive integer. Then for all integers $t_0 \leq t \leq t_1$, the inequality*

$$\alpha^{t+\kappa_1} \leq a\alpha^t + b \leq \alpha^{t+\kappa_0}$$

where $\kappa_i = \log_\alpha \left(1 + \frac{b}{\alpha^{t_i}}\right)$ for $i = 0, 1$.

Proof. It is obvious. □

3. Results

In this section, we prove the several theorems which the special cases of the diophantine equation (1.6). These theorems confirm the conjecture.

Theorem 1. *If $l \geq k$, then there is no solution of the diophantine equation (1.6) for positive integers r and $n \geq 2$.*

Proof. By Lemma 1 (a),

$$\begin{aligned} u_1^k + u_2^k + \cdots + u_{n-1}^k &< (u_1 + u_2 + \cdots + u_{n-1})^k \\ &< \left(\frac{u_n - u_{n-1} - 1}{A-2}\right)^l < u_n^l \end{aligned}$$

which completes the proof. □

Theorem 2. *If $k = 2$ and $l = 1$ in (1.6), then there is no solution of the diophantine equation (1.6) for positive integers r and $n \geq 2$.*

Proof. The Lemma 1 (a) and (b) yield that

$$\frac{u_{2n-1} - (2n-1)}{A^2 - 4} = \frac{1}{A-2}\left(u_{n+r+1} - u_{n+r} - u_{n+1} + u_n\right)$$

which gives

$$u_{2n-1} - (2n-1) + (A+2)(u_{n+1} - u_n) = (A+2)(u_{n+r+1} - u_{n+r}).$$

Let

$$LS := u_{2n-1} - (2n-1) + (A+2)(u_{n+1} - u_n)$$

and

$$RS := (A+2)(u_{n+r+1} - u_{n+r}).$$

Together with Lemma 3, we have

$$u_{2n-1} - (2n-1) + \left(A^2 - 4\right)u_n < LS < u_{2n-1} + (A+2)(A-1)u_n.$$

So, $LS > \alpha^{2n-2}$ holds. The Lemma 4 gives

$$\begin{aligned} LS &< u_{2n-1} + (A+2)(A-1)u_n \\ &< \alpha^{2n-1.83} + \alpha^{n+1.57} \\ &= \alpha^{n+1.57}\left(\alpha^{n-3.4} + 1\right) < 2\alpha^{n+1.57}\alpha^{n-2.8} < \alpha^{2n-0.5}. \end{aligned} \tag{3.1}$$

Similarly, we have the followings with Lemma 3

$$(A+2)(A-2)u_{n+r} < RS < (A+2)(A-1)u_{n+r}.$$

So, the inequalities

$$\alpha^{n+r+0.67} < RS < \alpha^{n+r+1.57} \tag{3.2}$$

follow by Lemma 4. Combining the inequalities (3.1) and (3.2), we deduce that

$$\max\{2n-2, n+r+0.67\} < \min\{n+r+1.57, 2n-0.5\}$$

which yields that

$$1.17 < n-r < 3.57.$$

So, there are two possibilities which are $n = r+2$ and $n = r+3$.

If $n = r+2$, then the equation (1.6) turns to the equation

$$u_1^2 + u_2^2 + \cdots + u_{r+1}^2 = u_{r+1} + u_{r+2} + \cdots + u_{2r+2}.$$

Above equation yields that

$$\frac{1}{A^2-4}(u_{2r+3} - (2r+3)) = u_{r+1} + u_{r+2} + \cdots + u_{2r+2}.$$

Obviously, this is not possible. For the case $n = r+3$, we arrive at a contradiction similarly since we obtain the following equation.

$$\frac{1}{A^2-4}(u_{2r+5} - (2r+5)) = u_{r+1} + u_{r+2} + \cdots + u_{2r+3}.$$

Therefore, we complete the proof of Theorem 2. □

Theorem 3. *If $k = 3$ and $l = 1$ in (1.6), then there is no solution of the diophantine equation (1.6) for positive integers r and $n \geq 2$.*

Proof. By Lemma 1 (a) and (c), the equation turns to

$$\frac{1}{A-2}\left\{\frac{u_{3n} - u_{3n-3} - u_3}{A^3 - 3A - 2} - \frac{3}{A-2}(u_n - u_{n-1} - 1)\right\} + (u_{n+1} - u_n)$$
$$= u_{n+r+1} - u_{n+r}.$$

Let

$$LS := \frac{1}{(A-2)\left(A^3-3A-2\right)}(u_{3n} - u_{3n-3} - u_3)$$
$$-\left(\frac{3}{(A-2)^2} + 1\right)(u_n - u_{n-1} - 1) + \frac{3}{A-2}$$

and

$$RS := u_{n+r+1} - u_{n+r}$$

We have the followings by Lemma 3,

$$\frac{1}{(A-2)}u_{3n-3} - \left(\frac{3}{(A-2)^2} + 1\right)(A-1)u_{n-1} < LS$$

and

$$LS < \frac{\left(A^3-3A-1\right)}{(A-2)\left(A^3-3A-2\right)} u_{3n-3} - \left(\frac{3}{(A-2)^2}+1\right)(A-2)\,u_{n-1}.$$

Since $\alpha^{-1} < \frac{(A^3-3A-1)}{(A-2)(A^3-3A-2)} < \alpha^{0.63}$ and $1 < \left(\frac{3}{(A-2)^2}+1\right)(A-2) < \alpha^{1.45}$ follow, then the inequalities

$$\alpha^{3n-5} - \alpha^{n-0.38} < LS < \alpha^{3n-0.2} - \alpha^{n-1}$$

hold which gives that

$$\alpha^{3n-6.5} < LS < \alpha^{3n-0.2}. \tag{3.3}$$

By Lemma 3, we obtain

$$(A-2)\,u_{n+r} < RS < (A-1)\,u_{n+r}$$

which yields that

$$\alpha^{n+r-1} < RS < \alpha^{n+r-0.17}. \tag{3.4}$$

Together with the inequalities (3.3) and (3.4), we get that

$$\max\{3n-6.5, n+r-1\} < \min\{3n-0.2, n+r-0.17\}.$$

So there are seven possibilities which are $2n-r=j$ where $j \in \{0,1,2,3,4,5,6\}$.

If $r=2n$, the the diophantine equation turns the equation

$$u_1^3+u_2^3+\cdots+u_{n-1}^3 = u_{n+1}+u_{n+2}+\cdots+u_{3n}.$$

By Lemma 1 (c), we get

$$\begin{aligned}&\frac{1}{A^2-4}\left\{\frac{u_{3n}-u_{3n-3}-u_3}{A^3-3A-2} - \frac{3}{A-2}(u_n-u_{n-1}-1)\right\}\\ &= u_{n+1}+u_{n+2}+\cdots+u_{3n}.\end{aligned}$$

Since $\alpha^{-5} < \frac{1}{(A^2-4)(A^3-3A-2)} < \alpha^{-3.26}$ holds, then $\alpha^{3n-6} < \frac{u_{3n}}{(A^2-4)(A^3-3A-2)} < \alpha^{3n-4.09}$ follows. But this is not possible since the inequality $\alpha^{3n-1} < u_{3n} < \alpha^{3n-0.83}$ holds. Since we have similar calculations for the cases $j \in \{1,2,3,4,5,6\}$, we omit the proofs of these cases to cut unnecessary calculations. So, we complete the proof of Theorem 3. □

Theorem 4. *If $k=3$ and $l=2$ in (1.6), then there is no solution of the diophantine equation (1.6) for positive integers r and $n \geq 2$.*

Proof. By Lemma 1 (b) and (c), we have

$$\begin{aligned}&\frac{u_{3n}-u_{3n-3}-u_3}{A^3-3A-2} - \frac{3}{A-2}(u_n-u_{n-1}-1) + u_{2n+1}\\ &= u_{2(n+r)+1} - 2r.\end{aligned}$$

Let

$$RS := u_{2(n+r)+1} - 2r$$

and

$$LS := \frac{u_{3n} - u_{3n-3} - u_3}{A^3 - 3A - 2} - \frac{3}{A-2}(u_n - u_{n-1} - 1) + u_{2n+1}.$$

Together with Lemma 4, we have

$$\alpha^{2(n+r)} - 2r < RS < \alpha^{2(n+r)+1-0.83}. \tag{3.5}$$

The inequalities

$$\alpha^{3n-4} < u_{3n-3} < \frac{u_{3n} - u_{3n-3} - u_3}{A^3 - 3A - 2} < \frac{A^2 - 3A - 1}{A^2 - 3A - 2} u_{3n-3} < \alpha^{3n-3.89},$$

$$\alpha^{n-2} < 3u_{n-1} < \frac{3}{A-2}(u_n - u_{n-1} - 1) < \frac{3(A-1)}{A-2} u_{n-1} < \alpha^{n-1.83+1.15}$$

and

$$\alpha^{2n} < u_{2n+1} < \alpha^{2n+0.17}$$

yield that

$$\begin{aligned}
&\alpha^{n-0.68}\left(\alpha^{2n-3.32} - 1 + \alpha^{n+0.68}\right) \\
&= \alpha^{3n-4} - \alpha^{n-0.68} + \alpha^{2n} < LS \\
&< \alpha^{3n-3.89} - \alpha^{n-2} + \alpha^{2n+0.17} \\
&= \alpha^{n-2}\left(\alpha^{2n-2.89} - 1 + \alpha^{n+2.17}\right).
\end{aligned}$$

Since $\alpha^{2n-4.1} < \alpha^{2n-3.32} - 1$ and $\alpha^{2n-2.89} - 1 < \alpha^{2n-2.88}$ hold, then we obtain

$$\alpha^{n-0.68}\left(\alpha^{2n-4.1} + \alpha^{n+0.68}\right) < LS < \alpha^{n-2}\left(\alpha^{2n-2.88} + \alpha^{n+2.17}\right).$$

By the inequalities $\alpha^{2n}\alpha^{n-4.78} < \alpha^{n-0.68}\left(\alpha^{n+0.68}\left(\alpha^{n-4.78} + 1\right)\right) = \alpha^{n-0.68}\left(\alpha^{2n-4.1} + \alpha^{n+0.68}\right)$ and $\alpha^{n-2}\left(\alpha^{2n-2.89} + \alpha^{n+2.17}\right) = \alpha^{n-2}\left(\alpha^{n+2.17}\left(\alpha^{n-5.06} + 1\right)\right) < \alpha^{n-2+n+2.17+n-1.9}$, then

$$\alpha^{3n-4.78} < LS < \alpha^{3n-1.73} \tag{3.6}$$

follows. Together with inequalities (3.5) and (3.6), the condition

$$\max\{3n - 4.78, 2n + 2r\} < \min\{3n - 1.73, 2n + 2r + 0.17\}$$

gives that $1.73 < n - 2r < 4.95$. So, the possible cases are $n - 2r = i$ where $i \in \{2, 3, 4\}$. When we replace $n = 2r + i$ in the equation (1.6), we get that

$$u_1^3 + u_2^3 + \cdots u_{2r+i-1}^3 = u_{2n+2}^2 + u_{2n+3}^2 + \cdots + u_{3r+i}^2.$$

Obviously, the left hand side is less than the right hand side. Therefore, we arrive at a contradiction. Hence, the proof is completed. □

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APPROXIMATION BY q-DURRMEYER - STANCU POLYNOMIALS IN COMPACT DISKS IN THE CASE OF $q > 1$

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Abstract. In this paper, the order of simultaneous approximation and Voronovskaja-type results with quantitative estimate for complex q-Kantorovich polynomials ($q > 0$) attached to analytic functions on compact disks are obtained. In particular, it is proved that for functions analytic in $\{z \in \mathbb{C} : |z| < R\}$, $R > q$, the rate of approximation by the q-Durrmeyer - Stancu operators ($q > 1$) is of order q^{-n} versus $1/n$ for the classical q-Durrmeyer - Stancu operators.

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1. Introduction

For each integer $k \geq 0$, the q-integer $[k]_q$ and the q-factorial $[k]_q!$ are defined by

$$[k]_q := \begin{cases} \dfrac{1-q^k}{1-q} & \text{if} \quad q \in R^+ \backslash \{1\}, \\ k & \text{if} \quad q = 1 \end{cases} \quad \text{for } k \in N \quad \text{and} \quad [0]_q = 0,$$

$$[k]_q! := [1]_q [2]_q \ldots [k]_q \quad \text{for } k \in \mathbb{N} \quad \text{and} \quad [0]_q! = 1.$$

For integers $0 \leq k \leq n$, the q-binomial coefficient is defined by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

For fixed $1 \neq q > 0$, we denote the q-derivative $D_q f(z)$ of f by

$$D_q f(z) = \begin{cases} \frac{f(qz)-f(z)}{(q-1)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

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The q-analogue of integration in the interval $[0,A]$ is defined by

$$\int_0^A f(t)d_qt := A(1-q)\sum_{n=0}^{\infty} f(Aq^n)q^n, \quad 0<q<1.$$

For $0<q^{-1}<1$ we have

$$B_{q^{-1}}(m,n) = \frac{[m-1]_{q^{-1}}![n-1]_{q^{-1}}!}{[m+n-1]_{q^{-1}}!}$$

In [2], Durrmeyer introduced the following Bernstein-Durrmeyer operator and [8] obtained simultaneous and ordinary approximation for these operator

$$D_n(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(q;x)\int_0^1 p_{n,k}(t)f(t)dt, \quad x\in[0,1] \tag{1.1}$$

where

$$p_{n,k}(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k}.$$

For $0\le\alpha\le\beta$, Stancu [13] introduced the Bernstein-Stancu operator as follows:

$$B_{n,\alpha,\beta}(f;x) = \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right)p_{n,k}(x). \tag{1.2}$$

In [4] Gupta defined q-Durrmeyer type operators:

$$D_{n,q}(f;x) = [n+1]_q\sum_{k=0}^{n} q^{-k}p_{n,k}(q;x)\int_0^1 f(t)p_{n,k}(t)d_qt, \quad x\in[0,1] \tag{1.3}$$

where

$$p_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k\prod_{j=0}^{n-k-1}(1-q^jx).$$

In [4] and [3] (Gupta and Finta) some local and global direct results for different q-analogues of Durrmeyer operator is studied. In [6], Gupta and Wang defined the following $q-$Durrmeyer operator and estimate the rate of convergence for these operator;

$$M_{n,q}(f;x) \tag{1.4}$$

$$= [n+1]_q\sum_{k=1}^{n} q^{1-k}p_{n,k}(q;x)\int_0^1 f(t)p_{n,k-1}(q;qt)d_qt + f(0)p_{n,0}(q;x), \quad x\in[0,1]$$

Also Gupta and Finta [5] studied some direct results for the certain q-Durrmeyer operator(1). In [1], Agarwal and Gupta introduced and studied complex case of $q-$Durrmeyer operators (1). In [12], Gal, Gupta and Mahmudov obtained some approximation properties for the following q-Durrmeyer operators;

$$M_{n,q}(f;z)$$
$$= [n]_q \sum_{k=1}^{n} q^{1-k} p_{n,k}(q;x) \int_0^1 f(t) p_{n-1,k-1}(q;qt)\, d_q t + f(0) p_{n,0}(q;z), \quad z \in \mathbb{C}$$

In [9] Mahmudov and Gupta introduced the complex genuine Durrmeyer -Stancu operators as follows:

$$U_n^{\alpha,\beta}(f;z) = p_{n,0}(z) f\left(\frac{\alpha}{n+\beta}\right) + p_{n,n}(z) f\left(\frac{n+\alpha}{n+\beta}\right)$$
$$+(n-1)\sum_{k=1}^{n-1} p_{n,k}(z) \int_0^1 p_{n-2,k-1}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt.$$

where α, β are given two real parameters satisfying the condition $0 \le \alpha \le \beta$.In [11] , M. Ren, X. Zeng and L. Zeng introduced the following complex Stancu-type Durrmeyer operators and obtained the approximation properties of these operators.

$$M_n^{(\alpha,\beta)}(f;z) = p_{n,0}(z) f\left(\frac{\alpha}{n+\beta}\right) + n\sum_{k=1}^{n} p_{n,k}(z) \int_0^1 p_{n-1,k-1}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt.$$

In[7], Gupta and Karslı considered Stancu type generalization as follows

$$M_n^{(\alpha,\beta)}(f;z) = p_{n,0}(z) f\left(\frac{\alpha}{n+\beta}\right) + (n+1)\sum_{k=1}^{n} p_{n,k}(z) \int_0^1 p_{n,k-1}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt.$$

Recently, Mahmudov (2014) [10] introduced and obtained the following approximation results of the complex q-Durrmeyer type operator in the case of $q > 1$.

$$\mathfrak{D}_n(f;z) = p_{n,0}(q;q^2 z) f(0)$$
$$+ [n+1]_{q^{-1}} \sum_{k=1}^{n} q^{k-1} p_{n,k}(q;q^2 z) \int_0^1 p_{n,k-1}(q^{-1};q^{-1}t) f\left(q^{k-n}t\right) d_{1/q} t.$$

In the present paper, for $q > 1$,we introduce the complex q-Durrmeyer-Stancu operators as follows

$$\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z) = f\left(\frac{\alpha}{[n]_q+\beta}\right) p_{n,0}\left(q;q^2 z\right) \tag{1.5}$$

$$+[n+1]_{q^{-1}}\sum_{k=1}^{n}q^{k-1}p_{n,k}\left(q;q^2z\right)\int_0^1 p_{n,k-1}(q^{-1};q^{-1}t)f\left(\frac{q^{k-n-2}[n]_q t+\alpha}{[n]_q+\beta}\right)d_{\frac{1}{q}}t.$$

Notice that The case $\alpha=\beta=0$ and $q>1$, studied in [10].

Let $\mathbb{D}_R$ be a disc $\mathbb{D}_R:=\{z\in\mathbb{C}:|z|<R\}$ in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D}_R)$ the space of all analytic functions on $\mathbb{D}_R$. For $f\in H(\mathbb{D}_R)$ we assume that $f(z)=\sum_{m=0}^{\infty}a_m z^m$ for all $z\in\mathbb{D}_R$. The norm $\|f\|_r:=\max\{|f(z):|z|\leq r|\}$. We define $e_m(z)=z^m$ for all $m\in\mathbb{N}\cup\{0\}$.

The paper is organized as follows. In Section 2 we give some auxilary results, in particular we prove a which includes some properties of the complex $q-$Durrmeyer-Stancu operators in the case $q>1$. In Section 3 we study approximation properties (Theorem 1), Voronovskaja type quantitative estimation (Theorem 2) and exact order of approximation (Theorem 3).

2. Auxiliary results

In this section we find relation between the moments of $q-$Durrmeyer operators $\mathfrak{D}_{n,q}(e_j;z)$ and $q-$Durrmeyer-Stancu operators $\mathfrak{D}_{n,q}^{(\alpha,\beta)}(e_m;z)$.

Lemma 1 ([10]). *For $m=0,1,2$,we have*

$$\mathfrak{D}_{n,q}(e_0;z)=1,\quad \mathfrak{D}_{n,q}(e_1;z)=\frac{q^2[n]_q}{[n+2]_q}z,$$

$$\mathfrak{D}_{n,q}(e_2;z)=\frac{q^2[n]_q(1+q)}{[n+3]_q[n+2]_q}z+\frac{q^3[n]_q([n]_q-1)}{[n+3]_q[n+2]_q}z^2,$$

where

$$\mathfrak{D}_{n,q}(e_m;z)=\frac{[n+1]_q!}{[n+m+1]_q!}\sum_{k=0}^{n}p_{n,k}(q;q^2z)\frac{[k+m-1]_q!}{[k-1]_q!}. \tag{2.1}$$

In Lemma 2, we define the moments for complex q-Durrmeyer-Stancu operators $(q>1)$.

Lemma 2. *For all $m,n\in\mathbb{N}_0,z\in\mathbb{C}$ and $0\leq\alpha\leq\beta$, we have the following recurrence relation*

$$\mathfrak{D}_{n,q}^{(\alpha,\beta)}(e_m;z)=\sum_{j=0}^{m}\binom{m}{j}\frac{[n]_q^j\alpha^{m-j}}{\left([n]_q+\beta\right)^m}\mathfrak{D}_{n,q}(e_j;z),$$

where $e_m(z)=z^m$.

Proof. By using 2.1 and simple computation,

For $m=0$,

$$\mathfrak{D}_{n,q}^{(\alpha,\beta)}(e_0;z)$$

$$= [n+1]_{q^{-1}} \sum_{k=1}^{n} q^{k-1} p_{n,k}(q;q^2z) \int_0^1 p_{n,k-1}(q^{-1};q^{-1}t) d_{1/q}t + p_{n,0}(q;q^2z)$$
$$= \mathfrak{D}_{n,q}(e_0;z) = 1.$$

Secondly, we calculate the first order monomial

$$\mathfrak{D}_{n,q}^{(\alpha,\beta)}(e_1;z) = \frac{\alpha}{\left([n]_q+\beta\right)} p_{n,0}(q;q^2z)$$
$$+ [n+1]_{q^{-1}} \sum_{k=1}^{n} q^{k-1} p_{n,k}(q;q^2z) \int_0^1 p_{n,k-1}(q^{-1};q^{-1}t) \left(\frac{q^{k-n-2}t\,[n]_q+\alpha}{[n]_q+\beta} \right) d_{1/q}t$$
$$= \frac{[n]_q}{[n]_q+\beta} \left\{ [n+1]_{q^{-1}} \sum_{k=1}^{n} q^{k-1} p_{n,k}(q;q^2z) \int_0^1 q^{k-n-2} t p_{n,k-1}(q^{-1};q^{-1}t) d_{1/q}t \right\}$$
$$+ \frac{\alpha}{\left([n]_q+\beta\right)}$$
$$\left\{ [n+1]_{q^{-1}} \sum_{k=1}^{n} q^{k-1} p_{n,k}(q;q^2z) \int_0^1 p_{n,k-1}(q^{-1};q^{-1}t) d_{1/q}t + p_{n,0}(q;q^2z) \right\}$$
$$= \frac{[n]_q}{[n]_q+\beta} \mathfrak{D}_{n,q}(e_1;z) + \frac{\alpha}{\left([n]_q+\beta\right)} \mathfrak{D}_{n,q}(e_0;z)$$
$$= \frac{[n]_q}{\left([n]_q+\beta\right)} \frac{q^2\,[n]_q}{[n+2]_q} z + \frac{\alpha}{[n]_q+\beta}.$$

Finally, For $m = 2$,

$$\mathfrak{D}_{n,q}^{(\alpha,\beta)}(e_2;z) = \frac{\alpha^2}{\left([n]_q+\beta\right)^2} p_{n,0}(q;q^2z)$$
$$+ \left\{ [n+1]_{q^{-1}} \sum_{k=1}^{n} q^{k-1} p_{n,k}(q;q^2z) \int_0^1 p_{n,k-1}(q^{-1};q^{-1}t) \frac{\left(q^{k-n-2}t\,[n]_q+\alpha\right)^2}{\left([n]_q+\beta\right)^2} d_{1/q}t \right\}$$
$$= \frac{[n]_q^2}{\left([n]_q+\beta\right)^2} \mathfrak{D}_{n,q}(e_2;z) + \frac{2\,[n]_q\,\alpha}{\left([n]_q+\beta\right)^2} \mathfrak{D}_{n,q}(e_1;z) + \frac{\alpha^2}{\left([n]_q+\beta\right)^2} \mathfrak{D}_{n,q}(e_0;z)$$
$$= \frac{[n]_q^2}{\left([n]_q+\beta\right)^2} \left\{ \frac{q^2\,[n]_q\,(1+q)}{[n+3]_q\,[n+2]_q} z + \frac{q^3\,[n]_q\left([n]_q-1\right)}{[n+3]_q\,[n+2]_q} z^2 \right\}$$

$$+\frac{2[n]_q\alpha}{\left([n]_q+\beta\right)^2}\frac{q^2[n]_q}{[n+2]_q}z+\frac{\alpha^2}{\left([n]_q+\beta\right)^2}.$$

□

Next lemma estimation of the moments $\mathfrak{D}_{n,q}^{(\alpha,\beta)}(e_m;z)$ in the compact disks $\overline{\mathbb{D}_R}$.

Lemma 3. *For all $m,n\in\mathbb{N}_0$, $z\in\overline{\mathbb{D}_R}$, $0\le\alpha\le\beta$ and $r\ge 1$, we have*

$$\mathfrak{D}_{n,q}^{(\alpha,\beta)}(e_m;z)\le(q^2r)^m.$$

Proof. Using the inequality $\left|\mathfrak{D}_{n,q}(e_j;z)\right|\le(q^2r)^j$ ([10])

$$\begin{aligned}\left|\mathfrak{D}_{n,q}^{(\alpha,\beta)}(e_m;z)\right|&\le\sum_{j=0}^{m}\binom{m}{j}\frac{[n]_q^j\alpha^{m-j}}{\left([n]_q+\beta\right)^m}\left|\mathfrak{D}_{n,q}(e_j;z)\right|\\&\le(q^2r)^m\sum_{j=0}^{m}\binom{m}{j}\frac{[n]_q^j\alpha^{m-j}}{\left([n]_q+\beta\right)^m}\\&=(q^2r)^m\left(\frac{[n]_q+\alpha}{[n]_q+\beta}\right)^m\le(q^2r)^m.\end{aligned}$$

□

Remark 1. By simple computation, we have

$$\mathfrak{D}_{n,q}^{(\alpha,\beta)}((e_1-e_0);z)=\frac{\alpha-\left([2]_q+\beta\right)}{[n]_q+\beta}z.$$

3. Approximation by complex $q-$Durrmeyer-Stancu polynomials

We start with the following quantitative estimates of the convergence for complex q-Durrmeyer-Stancu operators attached to an analytic function in a disk of radius $R>1$ and center 0.

Theorem 1. *Let $0\le\alpha\le\beta, 1<q<R<\infty$ and $1\le r<\frac{R}{q^3}$. For all $z\in\mathbb{D}_r$ and $n\in\mathbb{N}$, we have the following inequality*

$$\left|\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z)-f(z)\right|\le\left(\frac{1+q^2r+\alpha+\beta}{[n]_q}\right)\sum_{m=1}^{\infty}|a_m|\,m(m+1)\left(q^3r\right)^m.$$

where $f\in H(\mathbb{D}_R)$.

Proof.

$$\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z)-f(z)=\sum_{j=0}^{m}\binom{m}{j}\frac{[n]_q^j\alpha^{m-j}}{\left([n]_q+\beta\right)^m}\left|\mathfrak{D}_{n,q}(e_j;z)-(e_j;z)\right|$$

$$+\sum_{j=0}^{m}\binom{m}{j}\frac{[n]_q^j\,\alpha^{m-j}}{\left([n]_q+\beta\right)^m}z^j-z^m.$$

Using the estimation

$$\left|\mathfrak{D}_{n,q}(e_j;z)-(e_j;z)\right|\leq\frac{1+q^2r}{[n+2]_q}j(j+1)(q^3r)^{j-1}.$$

(See [10]). Then we can easily obtain

$$\begin{aligned}\left|\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z)-f(z)\right|&\leq\frac{1+q^2r}{[n+2]_q}m(m+1)(q^3r)^{m-1}+\left(\frac{[n]^m}{([n]+\beta)^m}-1\right)z^m\\&+\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{m}{m-j}\frac{[n]_q^j\,\alpha^{m-j}}{\left([n]_q+\beta\right)^m}z^j\\&\leq\frac{1+q^2r}{[n+2]_q}m(m+1)(q^3r)^{m-1}+\frac{m\beta}{[n]_q+\beta}r^m+\frac{m\alpha}{[n]_q}r^{m-1}\\&\leq\frac{\left(1+q^2r+\alpha+\beta\right)}{[n]_q}m(m+1)(q^3r)^{m-1}.\end{aligned}$$

□

The next theorem gives Voronovskaja type result in compact disks, for complex q-Kantorovich operators attached to an analytic function in $\mathbb{D}_R$, $R>1$ and center 0. In orders to formulate the results we introduce

$$L_q(f;z)=\frac{(1-q^2z)D_qf(z)-(1-z)D_{q^{-1}}f(z)}{q-1}.$$

Theorem 2. *Let $0\leq\alpha\leq\beta$ and $1\leq r<\frac{R}{q4}$. Then for all $z\in\mathbb{D}_r$ and $n\in\mathbb{N}$, we have the following inequality*

$$\begin{aligned}&\left|\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z)-f(z)-\frac{\alpha-\left([2]_q+\beta\right)z}{[n]+\beta}-\frac{1}{[n+2]_q}L_q(f;z)\right|\\&\leq\frac{11\left(1+q^2r\right)^2}{[n+2]_q^2}\sum_{m=1}^{\infty}(m+1)^4(q^4r)^m\\&+\frac{\left(\alpha^2+\beta^2+\beta\left([2]_q+\beta\right)\right)}{([n]+\beta)^2}\sum_{m=2}^{\infty}|a_m|\,m(m-1)\left(q^4r\right)^m\\&+\frac{\left(1+q^2r\right)\left(\alpha+[2]_q+\beta\right)}{([n]+\beta)\,[n+2]_q}\sum_{m=1}^{\infty}|a_m|\,m^2(m+1)\left(q^4r\right)^m,\end{aligned}$$

where $f\in H(\mathbb{D}_R)$.

Proof. We immediately obtain

$$\begin{aligned}
&\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z)-f(z)-\frac{\alpha-\left([2]_q+\beta\right)z}{[n]+\beta}f'(z)-\frac{1}{[n+2]_q}L_q(f;z)\\
&=\mathfrak{D}_{n,q}(f;z)-f(z)-\frac{1}{[n+2]_q}L_q(f;z)+\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z)-\mathfrak{D}_{n,q}(f;z)\\
&\quad-\frac{\alpha-\left([2]_q+\beta\right)z}{[n]+\beta}f'(z)\\
&=\sum_{m=1}^{\infty}a_m\left(\mathfrak{D}_{n,q}(e_m;z)-e_m(z)-\frac{1}{[n+2]_q}L_q(e_m;z)\right)\\
&+\sum_{m=1}^{\infty}a_m\left(\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z)-\mathfrak{D}_{n,q}(f;z)-\frac{\alpha-\left([2]_q+\beta\right)z}{[n]+\beta}mz^{m-1}\right)
\end{aligned}$$

For the first estimate, using the Theorem 3 in [10], namely

$$\left|\mathfrak{D}_{n,q}(f;z)-f(z)-\frac{1}{[n+2]_q}L_q(f;z)\right|\leq\frac{11\left(1+q^2r\right)^2}{[n+2]_q^2}\sum_{m=1}^{\infty}(m+1)^4(q^4r)^m.$$

For the second series, we rewrite it as follows.

$$\begin{aligned}
&\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z)-\mathfrak{D}_{n,q}(f;z)-\frac{\alpha-\left([2]_q+\beta\right)z}{[n]+\beta}f'(z)\\
&=\sum_{j=0}^{m-1}\binom{m}{j}\frac{[n]_q^j\,\alpha^{m-j}}{([n]+\beta)^m}D_{n,q}(e_j;z)+\frac{[n]_q^m}{([n]+\beta)^m}D_{n,q}(e_m;z)-D_{n,q}(e_m;z)\\
&\qquad-\left[\frac{\alpha-\left([2]_q+\beta\right)z}{[n]+\beta}\right]mz^{m-1}\\
&=\sum_{j=0}^{m-1}\binom{m}{j}\frac{[n]_q^j\,\alpha^{m-j}}{([n]+\beta)^m}D_{n,q}(e_j;z)+\left(\frac{[n]_q^m}{([n]+\beta)^m}-1\right)D_{n,q}(e_m;z)\\
&\qquad-\left[\frac{\alpha-\left([2]_q+\beta\right)z}{[n]+\beta}\right]mz^{m-1}\\
&=\sum_{j=0}^{m-2}\binom{m}{j}\frac{[n]_q^j\,\alpha^{m-j}}{([n]+\beta)^m}D_{n,q}(e_j;z)+\frac{m\,[n]_q^{m-1}\,\alpha}{([n]+\beta)^m}D_{n,q}(e_{m-1};z)\\
&-\sum_{j=0}^{m-1}\binom{m}{j}\frac{[n]_q^j\,\beta^{m-j}}{([n]+\beta)^m}D_{n,q}(e_m;z)-\left[\frac{\alpha-\left([2]_q+\beta\right)z}{[n]+\beta}\right]mz^{m-1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{m-2} \binom{m}{j} \frac{[n]_q^j \alpha^{m-j}}{([n]+\beta)^m} D_{n,q}(e_j;z) + \frac{m\,[n]_q^{m-1}\,\alpha}{([n]+\beta)^m} \left(D_{n,q}(e_{m-1};z) - z^{m-1}\right) \\
&- \sum_{j=0}^{m-2} \binom{m}{j} \frac{[n]_q^j \beta^{m-j}}{([n]+\beta)^m} D_{n,q}(e_m;z) + \left(\frac{[n]_q^{m-1}}{([n]+\beta)^{m-1}} - 1\right) \frac{\alpha}{[n]+\beta} m z^{m-1} \\
&\qquad + \frac{m\,[n]_q^{m-1}\left([2]_q+\beta\right)}{([n]+\beta)^m} \left(z^m - D_{n,q}(e_m;z)\right) \\
&\qquad + \left(1 - \frac{[n]_q^{m-1}}{([n]+\beta)^{m-1}}\right) \left[\frac{\left([2]_q+\beta\right) z}{[n]+\beta}\right] m z^{m-1} \\
&\qquad := \sum_{k=1}^{6} I_k .
\end{aligned}
$$

We use the inequality

$$
1 - \frac{[n]_q^k}{([n]+\beta)^k} \le \sum_{j=1}^{k} \left(1 - \frac{[n]_q}{[n]_q+\beta}\right) = \frac{k\beta}{[n]_q+\beta}
$$

to estimate I_1, I_2, I_3, I_4, I_5 and I_6.

$$
\begin{aligned}
|I_1| &\le \left| \sum_{j=0}^{m-2} \binom{m}{j} \frac{[n]_q^j \alpha^{m-j}}{([n]+\beta)^m} D_{n,q}(e_j;z) \right| \\
&\le \sum_{j=0}^{m-2} \binom{m}{j} \frac{[n]_q^j \alpha^{m-j}}{([n]+\beta)^m} \left| D_{n,q}(e_j;z) \right| \\
&= \sum_{j=0}^{m-2} \binom{m-2}{j} \frac{m(m-1)}{(m-j-1)(m-j)} \frac{[n]_q^j \alpha^{m-j}}{([n]+\beta)^m} \left| D_{n,q}(e_j;z) \right| \\
&\le \frac{m(m-1)}{2} \frac{\alpha^2}{([n]+\beta)^2} \left(q^2 r\right)^{m-2},
\end{aligned}
$$

$$
\begin{aligned}
|I_2| &\le \frac{m\,[n]_q^{m-1}\,\alpha}{([n]+\beta)^m} \left| D_{n,q}(e_{m-1};z) - z^{m-1} \right| \\
&\le \frac{m\,[n]_q^{m-1}\,\alpha}{([n]+\beta)^m} \frac{\left(1+q^2 r\right)}{[n+2]_q} (m-1) m \left(q^3 r\right)^{m-2} \\
&\le \frac{m^2 (m-1)\,\alpha \left(1+q^2 r\right)}{([n]+\beta)\,[n+2]_q} \left(q^3 r\right)^{m-2},
\end{aligned}
$$

$$|I_3| \leq \sum_{j=0}^{m-2} \binom{m}{j} \frac{[n]_q^j \beta^{m-j}}{([n]+\beta)^m} \left|D_{n,q}(e_m;z)\right| \leq \frac{m(m-1)\beta^2}{([n]+\beta)^2} \left(q^2 r\right)^m,$$

$$|I_4| \leq \left(\frac{[n]_q^{m-1}}{([n]+\beta)^{m-1}} - 1\right) \frac{\alpha}{[n]+\beta} m z^{m-1} \leq \frac{m(m-1)\alpha\beta}{\left([n]_q+\beta\right)^2} r^{m-1},$$

$$\begin{aligned}
|I_5| &\leq \frac{m\,[n]_q^{m-1}\left([2]_q+\beta\right)}{([n]+\beta)^m} \left|z^m - D_{n,q}(e_m;z)\right| \\
&\leq \frac{m^2(m+1)\left([2]_q+\beta\right)\left(1+q^2 r\right)}{([n]+\beta)\,[n+2]_q} (q^3 r)^{m-1},
\end{aligned}$$

$$|I_6| \leq \frac{m\,(m-1)\,\beta\left([2]_q+\beta\right)}{([n]+\beta)^2} r^m.$$

Then we get

$$\begin{aligned}
&\left|\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z) - \mathfrak{D}_{n,q}(f;z) - \frac{\alpha - \left([2]_q+\beta\right) z}{[n]+\beta} f'(z)\right| \\
&\leq \frac{m(m-1)}{2} \frac{\alpha^2}{([n]+\beta)^2} \left(q^2 r\right)^{m-2} + \frac{m^2\,(m-1)\,\alpha\left(1+q^2 r\right)}{([n]+\beta)\,[n+2]_q} \left(q^3 r\right)^{m-2} \\
&+ \frac{m(m-1)\beta^2}{([n]+\beta)^2} \left(q^2 r\right)^m + \frac{m(m-1)\alpha\beta}{\left([n]_q+\beta\right)^2} r^{m-1} \\
&+ \frac{m^2(m+1)\left([2]_q+\beta\right)\left(1+q^2 r\right)}{([n]+\beta)\,[n+2]_q} (q^3 r)^m + \frac{m\,(m-1)\,\beta\left([2]_q+\beta\right)}{([n]+\beta)^2} r^m \\
&\leq \frac{m(m-1)\left(\alpha^2+\beta^2+\beta\left([2]_q+\beta\right)\right)}{([n]+\beta)^2} \left(q^4 r\right)^m \\
&+ \frac{m^2\,(m+1)\left(1+q^2 r\right)\left(\alpha+[2]_q+\beta\right)}{([n]+\beta)\,[n+2]_q} \left(q^4 r\right)^m.
\end{aligned}$$

□

In the following theorem, for q-Durrmeyer-Stancu polynomials, we obtain the exact order of approximation.

Theorem 3. *Let $1 < q < R$, $1 \leq r < \frac{R}{q^2}$ (or $0 < q \leq 1$, $1 \leq r < R$) and $f \in H(\mathbb{D}_R)$. If f is not a constant function then the estimate*

$$\left\|\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f) - f\right\|_r \geq \frac{1}{[n+1]_q} C_{r,q}(f), \quad n \in \mathbb{N},$$

holds, where the constant $C_{r,q}(f)$ depends on f, q and r but is independent of n.

Proof of Theorem 3. For all $z \in \mathbb{D}_R$ and $n \in \mathbb{N}$ we get

$$\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z) - f(z) = \frac{1}{[n+2]_q}$$

$$\left\{ \begin{array}{c} \frac{[n+2]_q}{[n]+\beta}\left(\alpha - \left([2]_q + \beta\right)\right) f'(z) + L_q(f;z) \\ + [n+2]_q \left(\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z) - f(z) - \frac{\alpha - ([2]_q+\beta)z}{[n]+\beta} f'(z) - \frac{1}{[n+2]_q} L_q(f;z) \right) \end{array} \right\}.$$

We apply

$$\|F+G\|_r \geq |\|F\|_r - \|G\|_r| \geq \|F\|_r - \|G\|_r$$

to get

$$\left\| \mathfrak{D}_{n,q}^{(\alpha,\beta)}(f) - f \right\|_r \geq \frac{1}{[n+2]_q}$$

$$\left\{ \begin{array}{c} \left\| \frac{[n+2]_q}{[n]+\beta}\left(\alpha e_0 - \left([2]_q + \beta\right) e_1\right) f' \right) + L_q(f) \right\|_r \\ - [n+2]_q \left\| \mathfrak{D}_{n,q}^{(\alpha,\beta)}(f) - f - \frac{\alpha e_0 - ([2]_q+\beta) e_1}{[n]+\beta} f' - \frac{1}{[n+2]_q} L_q(f;z) \right\|_r \end{array} \right\}.$$

Because by hypothesis f is not a constant in $\mathbb{D}_R$, it follows

$$\left\| \frac{n+2}{[n]+\beta}\left(\alpha e_0 - \left([2]_q + \beta\right) e_1\right) f' \right) + L_q(f) \right\|_r > 0.$$

Indeed, assuming the contrary it follows that

$$\frac{n+2}{[n]+\beta}\left(\alpha e_0 - \left([2]_q + \beta\right) e_1\right) f' \right) + L_q(f) = 0$$

for all $z \in \overline{\mathbb{D}}_R$ that is

$$\frac{n+2}{[n]+\beta} \sum_{m=1}^{\infty} a_m \left(\alpha - ([2]+\beta) z\right) m z^{m-1}$$

$$+ \sum_{m=1}^{\infty} a_m \left(\frac{[m]_q - m}{q-1} + q^{-1} \frac{[m]_{q^{-1}} - m}{q^{-1}-1} \right) z^{m-1} \left(1 - q^2 z\right)$$

$$- \sum_{m=1}^{\infty} a_m \frac{[m]_{q^{-1}}}{q-1} z^{m-1} (q^2-1) z = 0$$

$$\frac{[n+2]_q}{[n]+\beta} \sum_{m=1}^{\infty} a_m \alpha m z^{m-1} - \frac{[n+2]_q}{[n]+\beta} \sum_{m=1}^{\infty} a_m ([2]+\beta) m z^m$$

$$+ \sum_{m=1}^{\infty} a_m \left(\frac{[m]_q - m}{q-1} + q^{-1} \frac{[m]_{q^{-1}} - m}{q^{-1}-1} \right) z^{m-1}$$

$$+\sum_{m=1}^{\infty} a_m \left(\frac{[m]_q - m}{q-1} + q^{-1}\frac{[m]_q - m}{q^{-1}-1}\right) z^m q^2$$

$$-\sum_{m=1}^{\infty} a_m \frac{[m]_{q^{-1}}}{q-1} z^m (q^2-1) = 0$$

$$\left(\frac{[n+2]_q \alpha}{[n]+\beta}\right) a_1 + \frac{[n+2]_q}{[n]+\beta}\sum_{m=1}^{\infty} a_{m+1}\alpha(m+1) z^m - \frac{[n+2]_q}{[n]+\beta}\sum_{m=1}^{\infty} a_m ([2]+\beta) m z^m$$

$$+\sum_{m=1}^{\infty} a_{m+1}\left(\frac{[m+1]_q - (m+1)}{q-1} + q^{-1}\frac{[m+1]_{q^{-1}} - (m+1)}{q^{-1}-1}\right) z^m$$

$$+\sum_{m=1}^{\infty} a_m \left(\frac{[m]_q - m}{q-1} + q^{-1}\frac{[m]_q - m}{q^{-1}-1}\right) z^m q^2 - \sum_{m=1}^{\infty} a_m \frac{[m]_{q^{-1}}}{q-1} z^m (q^2-1) = 0$$

So,

$$a_1 = 0$$

$$a_{m+1}\left(\frac{n+2}{[n]+\beta}\alpha(m+1) + \left(\frac{[m+1]_q - (m+1)}{q-1} + q^{-1}\frac{[m+1]_{q^{-1}} - (m+1)}{q^{-1}-1}\right)\right)$$

$$= a_m \left(\frac{n+2}{[n]+\beta}([2]+\beta) m - \left(\frac{[m]_q - m}{q-1} + q^{-1}\frac{[m]_q - m}{q^{-1}-1}\right) q^2 + \frac{[m]_{q^{-1}}}{q-1}(q^2-1)\right)$$

$$a_{m+1} = \frac{\left(\frac{n+2}{[n]+\beta}\alpha(m+1) + \left(\frac{[m+1]_q - m+1}{q-1} + q^{-1}\frac{[m+1]_{q^{-1}} - (m+1)}{q^{-1}-1}\right)\right)}{\left(\frac{n+2}{[n]+\beta}([2]+\beta) m - \left(\frac{[m]_q - m}{q-1} + q^{-1}\frac{[m]_q - m}{q^{-1}-1}\right) q^2 + \frac{[m]_{q^{-1}}}{q-1}(q^2-1)\right)}$$

for all $z \in \overline{\mathbb{D}}_R \setminus \{0\}$. Thus $a_m = 0$, $m = 1, 2, 3, \ldots$Thus, f is constant, which is contradiction with the hypothesis.

Now, by Theorem 2 we have

$$[n+2]_q \left| \mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z) - f(z) - \frac{\alpha e_0 - ([2]_q + \beta) e_1}{[n]+\beta} m z^{m-1'} - \frac{1}{[n+2]_q} L_q(f;z) \right|$$

$$\leq \frac{11\left(1+q^2 r\right)^2}{[n+2]_q^2}\sum_{m=1}^{\infty}(m+1)^4 (q^4 r)^m$$

$$+\frac{\left(\alpha^2+\beta^2+\beta\left([2]_q+\beta\right)\right)}{([n]+\beta)^2}\sum_{m=2}^{\infty}|a_m| m(m-1)\left(q^4 r\right)^m$$

$$+\frac{\left(1+q^2 r\right)\left(\alpha+[2]_q+\beta\right)}{([n]+\beta)[n+2]_q}\sum_{m=1}^{\infty}|a_m| m^2 (m+1)\left(q^4 r\right)^m \to 0 \text{ as } n \to \infty.$$

Moreover,

$$\left\| \frac{[n+2]_q}{[n]+\beta}\left(\alpha e_0-\left([2]_q+\beta\right)e_1\right)f'\right)+L_q(f)\right\|_r$$
$$\to \left\| q^2\left(\alpha e_0-\left([2]_q+\beta\right)e_1\right)f'\right)+L_q(f)\right\|_r$$

Consequently, there exists n_1 (depending only on f and r) such that for all $n \geq n_1$ we have

$$\left\| \frac{[n+2]_q}{[n]+\beta}\left(\alpha e_0-\left([2]_q+\beta\right)e_1\right)f'\right)+L_q(f)\right\|_r$$
$$-[n+2]_q\left\| \mathfrak{D}_{n,q}^{(\alpha,\beta)}(f)-f-\frac{\alpha e_0-\left([2]_q+\beta\right)e_1}{[n]+\beta}f'-\frac{1}{[n+2]_q}L_q(f;z)\right\|_r$$
$$\geqslant \frac{1}{2}\left\| \frac{[n+2]_q}{[n]+\beta}\left(\alpha e_0-\left([2]_q+\beta\right)e_1\right)f'\right)+L_q(f)\right\|_r$$

which implies

$$\left\| \mathfrak{D}_{n,q}^{(\alpha,\beta)}(f)-f\right\|_r$$
$$\geq \frac{1}{[n+2]_q}\frac{1}{2}\left\| q^2\left(\alpha e_0-\left([2]_q+\beta\right)e_1\right)f'\right)+L_q(f)\right\|_r, \quad \text{for all } n \geq n_1.$$

For $1 \leq n \leq n_1-1$ we have

$$\left\| \mathfrak{D}_{n,q}^{(\alpha,\beta)}(f)-f\right\|_r \geq \frac{1}{[n+2]_q}\left([n+2]_q\left\| \mathfrak{D}_{n,q}^{(\alpha,\beta)}(f)-f\right\|_r\right)=\frac{1}{[n+1]_q}M_{r,n}(f)>0,$$

which finally implies that

$$\left\| \mathfrak{D}_{n,q}^{(\alpha,\beta)}(f)-f\right\|_r \geq \frac{1}{[n+1]_q}C_{r,q}(f),$$

for all n, with

$$C_{r,q}(f)$$
$$=\min\left\{M_{r,1}(f),\ldots,M_{r,n_1-1}(f),\frac{1}{2}\left\| q^2\left(\alpha e_0-\left([2]_q+\beta\right)e_1\right)f'\right)+L_q(f)\right\|_r\right\}.$$

□

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AN EXTENSION OF RATIONAL CONTRACTIVE CONDITION IN PARTIALLY ORDERED RECTANGULAR METRIC SPACES

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Abstract. In this paper, we discuss the concept of partially ordered rectangular metric spaces using the notation $(X, \preceq, d)$. Also, we give a generalized contractive condition in the context of partially ordered rectangular metric space for all comparable pair of X. Our results extend and generalize the well-known results of [13], [1], [14] and [11].

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1. Introduction

Banach contraction principle proved by mathematician S. Banach was published in 1922. This principle is one of the most useful tools in the study of nonlinear equations. This contraction mapping principle has been generalized in many directions ([3], [7], [10], [17], [19], [5]).

In 1973, Geraghty [7] introduced the following generalization of Banach's contraction principle that known Geraghty type contraction.

Let (X, d) be a complete metric space and let $f : X \to X$ be map. Suppose there exists $\beta \in \vartheta$ such that for each $x, y \in X$

$$d(fx, fy) \leq \beta(d(x, y))\, d(x, y)$$

where $\vartheta = \{\beta : \beta(t_n) \to 1 \text{ implies } t_n \to 0\}$. Then f has a unique fixed point $z \in X$ and $\{f^n x\}$ converges to z, for each $x \in X$.

In 1977, Jaggi [10] proved the following theorem satisfying a rational type contractive inequality.

Let (X, d) be a complete metric space and let $f : X \to X$ be a continuous mapping such that

$$d(fx, fy) \leq \alpha \frac{d(x, fx)\, d(y, fy)}{d(x, y)} + \beta d(x, y)$$

for all distinct points $x, y \in X$ where $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. *Then f has a unique fixed point.*

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In 2000, Branciari introduced the notation of a generalized metric spaces. A generalized metric is a semi-metric which does not satisfy the triangle inequality but satisfies a weaker condition called quadrilateral inequality.

A rectangular metric space needs not to

(1) have a continuous rectangular metric
(2) have a convergent sequence as a Cauchy sequence.

Branciari also established an analogue of Banach's fixed point theorem on rectangular metric spaces. Extension of this analogue has been made in many directions ([4], [18], [2], [12]).

In 2004, Ran and Reurings [16] established an analogue of Banach's fixed point theorem in partially ordered set.

Theorem 1. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Furthermore, X is such that*

$$\text{every pair } x, y \in X \text{ has a lower bound and an upper bound.} \tag{1.1}$$

If $f : X \to X$ is a continuous, monotone (i.e., either order-preserving or order-reversing) map from X into X such that

- $$\exists\, 0 < c < 1 : d(fx, fy) \leq c\, d(x, y), x \succeq y, \tag{1.2}$$
- $$\exists\, x_0 \in X : x_0 \preceq fx_0 \text{ or } x_0 \succeq fx_0,$$

then f has a unique fixed point $\overline{x}$. Moreover, for every $x \in X$ $\lim_{n\to\infty} f^n x = \overline{x}$.

One see that the difference of this theorem than the Banach contraction principle is that contractive inequality (1.2) satisfies only for comparable pair of the $(X, \preceq)$.

In 2005, Nieto and López [15] extended some results to study a problem of ordinary differential equations. Also, authors introduced the following concept to give some alternative conditions for the existence and uniqueness of the fixed point.

$$\text{"if any nondecreasing sequence } \{x_n\} \text{ in } X \text{ converges to } z, \text{ then } x_n \preceq z \text{ for all } n \geq 0\text{"} \tag{1.3}$$

$$\text{"every pair of elements has a lower bound or an upper bound"} \tag{1.4}$$

$$\text{" for every } x, y \in X; \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y\text{"}. \tag{1.5}$$

According to Nieto and López [15]:

(1) The mapping defined as $f : X \to X$ needs not continuity if $(X, \preceq)$ has the property (1.3).

(2) The partially ordered set $(X, \preceq)$ needs not the property (1.1). It is sufficient to consider that X has the property (1.4).
(3) The property (1.4) is equivalent to the property (1.5).

Remark 1 ([15]). considered the map $f : X \to X$ as nondecreasing, but Ran and Reurings [16] studied as a monotone. In the other words, according to Ran and Reurings [16] it is sufficient to consider f as monotone, that is, f may be nondecreasing or nonincreasing.

Papers [8], [9], [13], [1], [14] and [11] were particularly focus on the following three points:

(1) The mapping defined by $f : X \to X$ is nondecreasing and continuous or f is nondecreasing and the set $(X, \preceq)$ has the property (1.3).
(2) It is taken the assumption "if there exists $x_0 \in X$ such that $x_0 \preceq f x_0$".
(3) In order to guarantee the uniqueness of the fixed point, they needs the property (1.5).

In this paper we aim to

(1) introduce the concept of partially ordered rectangular metric space $(X, \preceq, d)$,
(2) give a generalized rational contractive condition that involve all results of the papers [1]-[5] and [7]-[19] without need some restrictions given in the above.
(3) Furthermore, we take the mapping defined by $f : X \to X$ is monotone and continuous or f is monotone and the set $(X, \preceq)$ is regular.

We shall also discuss some results from [1]-[5] and [7]-[19] after giving the proof of our main results.

2. Mathematical preliminaries

Now, we recall some notations and remarks which are needed in the sequel.

Definition 1 ([4])**.** Let X be a nonempty set and let $d : X \times X \to [0, \infty)$ satisfy the following conditions for all $x, y \in X$ and all distinct $u, v \in X$ each of them different from x and y.
RM1) $d(x, y) = 0$ if and only if $x = y$,
RM2) $d(x, y) = d(y, x)$,
RM3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (quadrilateral inequality).
Then the function d is called a rectangular metric and the pair (X, d) is called a rectangular metric spaces (for short RMS).

In literature, generalized metric spaces (or in short g.m.s) instead of for rectangular metric spaces (in short RMS) is used by some researchers.

Definition 2 ([4])**.** Let (X, d) a rectangular metric spaces and $\{x_n\}$ be sequence in X.

(1) $\{x_n\}$ is called g.m.s convergent to a limit x if and only if $d(x_n, x) \to 0$ as $n \to \infty$,
(2) $\{x_n\}$ is called g.m.s Cauchy sequence if and only if for every $\epsilon > 0$ there exists positive integer $N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for all $n > m > N(\epsilon)$,
(3) A rectangular metric spaces (X, d) is called complete if every g.m.s Cauchy sequence is g.m.s convergent.

Definition 3. Let $(X, \preceq, d)$ be a partially ordered rectangular metric space. X is called regular if it has the following properties:

•

$$\text{if a nondecreasing sequence } \{x_n\} \to x \text{ in } (X, d), \text{ then } x_n \preceq x \text{ for all } n \geq 0; \quad (2.1)$$

•

$$\text{if a nonincreasing sequence } \{x_n\} \to x \text{ in } (X, d), \text{ then } x_n \succeq x \text{ for all } n \geq 0. \quad (2.2)$$

Regularity the mapping with define by $f : X \to X$ need not be nondecreasing. Alias, it is sufficient to consider monotone, that is, f may be nondecreasing or nonincreasing.

Some notations are defined as follows.

(1) Let $\mathcal{F}$ be the set of functions $\xi : [0, \infty) \to [0, \infty)$ satisfying the condition $\xi(t) = 0$ if and only if $t = 0$.
(2) We denote by Ψ the set of functions $\psi \in \mathcal{F}$ such that ψ is continuous and nondecreasing.
(3) We denote by Φ the set of functions $\phi \in \mathcal{F}$ such that ϕ is lower semi-continuous.
(4) Let $(X, \preceq, d)$ be a metric ordered metric space and $f : X \to X$ be mappings for all distinct comparable points $x, y \in X$,

$$M(x, y) = \max\left\{d(x, y),\ d(x, fx),\ d(fx, fy),\ \frac{d(x, fx)\, d(y, fy)}{d(x, y)}\right\},$$

$$N_L(x, y) = L \min\{d(x, fy),\ d(x, fx),\ d(y, fy),\ d(y, fx)\} \text{ such that } L \geq 0.$$

3. Fixed point Theorem

Definition 4 ([6]). $F : [0, \infty)^2 \to \mathbb{R}$ is called C*-class* function if it is continuous and satisfies following axioms:
(1) $F(s, t) \leq s$;
(2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$;
for all $s, t \in [0, \infty)$.

We denote the set of C-class functions by $\mathcal{C}$. It is clear that if $F \in \mathcal{C}$ then $F(0, 0) = 0$.

Theorem 2. *Let $(X, \preceq, d)$ be a partially ordered Hausdorff and complete RMS. Let $f : X \to X$ be a monotone mapping satisfying for all distinct points $x, y \in X$ with $x \preceq y$,*

$$\psi(d(fx, fy)) \leq F(\psi(M(x, y)), \phi(M(x, y))) + N_L(x, y) \tag{3.1}$$

such that $\psi \in \Psi$, $\phi \in \Phi$, $F \in \mathcal{C}$. Also, suppose that either

(1) *C1) f is continuous or*
(2) *C2) $(X, \preceq, d)$ be regular.*

If there exists $x_0 \in X$ with $x_0 \preceq fx_0$ or $x_0 \succeq fx_0$, then f has a fixed point in X. Moreover, if $(X, \preceq)$ has the property (1.4) (or (1.5)), then the fixed point is unique.

Proof. Let $x_0 \in X$ such that $x_0 \preceq fx_0$ or $x_0 \succeq fx_0$. Define the sequence $\{x_n\}$ by $x_n = fx_{n-1}$ for $n \geq 1$. If there there exists n_0 such that $x_{n_0} = x_{n_0+1}$. Then f has a fixed point. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. In this case, we give the proof step by step.

Step 1: The sequence $\{x_n\}$ defined by $x_n = fx_{n-1}$ for $n \geq 1$ is monotone. In fact, since f is monotone, we have

$$x_0 \preceq x_1 = fx_0 \preceq x_2 = f^2x_0 \preceq \cdots \preceq x_n = f^nx_0 \preceq \cdots \tag{3.2}$$

or

$$x_0 \succeq x_1 = fx_0 \succeq x_2 = f^2x_0 \succeq \cdots \succeq x_n = f^nx_0 \succeq \cdots \tag{3.3}$$

Step 2: We claim that $d(x_n, x_{n+1}) \to 0$. Indeed, since $\{x_n\}$ is a monotone sequence, from contractive inequality, we have

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &= \psi(d(fx_{n-1}, fx_n)) && (3.4)\\ &\leq F(\psi(M(x_{n-1}, x_n)), \phi(M((x_{n-1}, x_n)))) + N_L((x_{n-1}, x_n)) \\ &\leq \psi(M(x_{n-1}, x_n)) + N_L((x_{n-1}, x_n)) && (3.5) \end{aligned}$$

where

$$M(x_{n-1}, x_n) =$$

$$\max\left\{d(x_{n-1}, x_n), d(x_{n-1}, fx_{n-1}), d(x_n, x_{n+1}), \frac{d(x_{n-1}, fx_{n-1})\, d(x_n, fx_n)}{d(x_{n-1}, x_n)}\right\}$$
$$= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$$

and

$$\begin{aligned} N_L(x_{n-1}, x_n) &= L\min\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_n, x_n)\} \\ &= 0. \end{aligned}$$

Thus, from (3.5), it is clear that $\{d(x_n, x_{n+1})\}$ is a monotone nonincreasing sequence of non-negative real numbers. Thus, there exists $r \geq 0$ such that $d(x_n, x_{n+1}) \to r$ as $n \to \infty$.

Taking limit as $n \to \infty$ in (3.4), we have

$$\begin{aligned}\lim_{n\to\infty} \psi\left(d\left(x_n, x_{n+1}\right)\right) &\leq \lim_{n\to\infty} F\left(\psi\left(M\left(x_{n-1}, x_n\right)\right), \phi\left(M\left(\left(x_{n-1}, x_n\right)\right)\right)\right) \\ &\leq \lim_{n\to\infty} \psi\left(M\left(x_{n-1}, x_n\right)\right)\end{aligned}$$

and

$$\psi(r) \leq F(\psi(r), \phi(r)) \leq \psi(r) \tag{3.6}$$

this implies that $F(\psi(r), \phi(r)) = \psi(r)$. From the definiton of F function, $\psi(r) = 0$ or $\phi(r) = 0$. So, we hold that

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0. \tag{3.7}$$

Step 3: We claim that $d(x_n, x_{n+2}) \to 0$. Indeed, from contractive inequality we have

$$\begin{aligned}\psi\left(d\left(x_n, x_{n+2}\right)\right) &= \psi\left(d\left(fx_{n-1}, \ fx_{n+1}\right)\right) \\ &\leq F\left(\psi\left(M\left(x_{n-1}, x_{n+1}\right)\right), \phi\left(M\left(\left(x_{n-1}, x_{n+1}\right)\right)\right)\right) \\ &\quad + N_L\left(\left(x_{n-1}, x_{n+1}\right)\right) \\ &\leq \psi\left(M\left(x_{n-1}, x_{n+1}\right)\right) + N_L\left(\left(x_{n-1}, x_{n+1}\right)\right)\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}M\left(x_{n-1}, x_{n+1}\right) &= \max\left\{\begin{array}{c} d\left(x_{n-1}, x_{n+1}\right), \ d\left(x_{n-1}, fx_{n-1}\right), \ d\left(x_n, x_{n+2}\right), \\ \frac{d\left(x_{n-1}, fx_{n-1}\right) d\left(x_{n+1}, fx_{n+1}\right)}{d\left(x_{n-1}, x_{n+1}\right)} \end{array}\right\} \\ &= \max\left\{d\left(x_{n-1}, x_{n+1}\right), \ d\left(x_{n-1}, x_n\right), \ d\left(x_n, x_{n+2}\right), \frac{d\left(x_{n-1}, x_n\right) d\left(x_{n+1}, x_{n+2}\right)}{d\left(x_{n-1}, x_{n+1}\right)}\right\}.\end{aligned}$$

As a convenience, let

$$a_n = d\left(x_{n-1}, x_{n+1}\right), b_n = d\left(x_{n-1}, x_{n+1}\right), \text{for all } n \geq 1.$$

Therefore, we obtain

$$M\left(x_{n-1}, x_{n+1}\right) = \max\left\{a_n, b_n, a_{n+1}, \frac{b_n b_{n+2}}{a_n}\right\}. \tag{3.9}$$

If

$$M\left(x_{n-1}, x_{n+1}\right) = b_n \text{ or } M\left(x_{n-1}, x_{n+1}\right) = \frac{b_n b_{n+2}}{a_n}$$

then thanks to properties of ψ function, we have $d(x_n, x_{n+2}) \to 0$ as $n \to \infty$. If

$$M\left(x_{n-1}, x_{n+1}\right) = a_n \text{ or } M\left(x_{n-1}, x_{n+1}\right) = a_{n+1}$$

then, we see that $\{d(x_n, x_{n+2})\}$ is a monotone nonincreasing sequence of non-negative real numbers. As shown above, we get that $d(x_n, x_{n+2}) \to 0$ as $n \to \infty$. As a result, for all cases in (3.9), we have

$$\lim_{n\to\infty} d\left(x_n, x_{n+2}\right) = 0. \tag{3.10}$$

Step 4: We claim that f has a periodic point. Assume that f has no periodic point; then $\{x_n\}$ is a sequence of distinct points, that is, $x_n \neq x_m$ for all $m \neq n$. In this case we will get that $\{x_n\}$ is a g.m.s Cauchy sequence. If not, then there exists some $\epsilon > 0$ for which we can find two sequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}, m(k) > n(k) > k$ for each $k \geq 0$ such that

$$d\left(x_{m(k)}, x_{n(k)}\right) \geq \epsilon \text{ and } d\left(x_{m(k)-1}, x_{n(k)}\right) < \epsilon.$$

Note that $\{x_n\}$ is a sequence of distinct points, then thanks to rectangular inequality, we have

$$\begin{aligned}\epsilon &\leq d\left(x_{m(k)}, x_{n(k)}\right)\\ &\leq d\left(x_{m(k)}, x_{m(k)-2}\right) + d\left(x_{m(k)-2}, x_{m(k)-1}\right) + d\left(x_{m(k)-1}, x_{n(k)}\right)\\ &\leq d\left(x_{m(k)}, x_{m(k)-2}\right) + d\left(x_{m(k)-2}, x_{m(k)-1}\right) + \epsilon.\end{aligned}$$

Thus,

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \epsilon.$$

Also, we get

$$\begin{aligned}\psi\left(d\left(x_{m(k)}, x_{n(k)}\right)\right) \leq F\left(\psi\left(M\left(x_{m(k)-1}, x_{n(k)-1}\right)\right), \phi\left(M\left(x_{m(k)-1}, x_{n(k)-1}\right)\right)\right)\\ + N_L\left(x_{m(k)-1}, x_{n(k)-1}\right)\end{aligned}$$

where

$$M\left(x_{m(k)-1}, x_{n(k)-1}\right) = \max\left\{\begin{array}{c} d\left(x_{m(k)-1}, x_{n(k)-1}\right), d\left(x_{m(k)-1}, x_{m(k)}\right), d\left(x_{m(k)}, x_{n(k)}\right)\\ , \frac{d\left(x_{m(k)-1}, x_{m(k)}\right) d\left(x_{n(k)-1}, x_{n(k)}\right)}{d\left(x_{m(k)-1}, x_{n(k)-1}\right)}\end{array}\right\}$$

and

$$N_L\left(x_{m(k)-1}, x_{n(k)-1}\right) = L\min\left\{d\left(x_{m(k)-1}, x_{n(k)}\right), d\left(x_{m(k)-1}, x_{m(k)}\right), d\left(x_{n(k)-1}, x_{n(k)}\right), d\left(x_{n(k)-1}, x_{m(k)}\right)\right\}.$$

Let $n \to \infty$ in (3), we have

$$\psi(\epsilon) \leq F(\psi(\epsilon), \phi(\epsilon)) \leq \psi(\epsilon).$$

Thus we have $F(\psi(\epsilon), \phi(\epsilon)) = \psi(\epsilon)$. From definition of F class function and ψ function we get that $\epsilon = 0$. This is a contradiction. Hence, $\{x_n\}$ is a g.m.s Cauchy sequence. Thus there exists $u \in X$ such that the sequence $\{x_n\}$ converges to u. In here, we have two cases:

- Let C1) holds, that is, f be continuous, then we have

$$u = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f x_{n-1} = f \lim_{n\to\infty} x_{n-1} = f u. \tag{3.11}$$

- Let C2) holds, that is, $(X, \preceq, d)$ be regular. From (3.2) and (3.3), it is clear that the sequence $\{x_n\}$ is a monotone sequence. Thus, by the regularity of X, we have $x_n \succeq u$ or $x_n \preceq u$ for all $n \geq 1$. This means that for each $n \geq 1$ x_n and u are comparable elements in the sense of partially order "$\preceq$". Note that f is a monotone mapping, then for each $n \geq 1$ fx_n and fu are comparable elements. This is lets to use contractive inequality (3.1);

$$\begin{aligned}\psi(d(fx_n, fu)) &= \psi(d(fx_{n-1}, fu)) \\ &\leq F(\psi(M(x_{n-1}, u)), \phi(M((x_{n-1}, u)))) + N_L((x_{n-1}, u)) \\ &\leq \psi(M(x_{n-1}, u)) + N_L((x_{n-1}, u)) \end{aligned} \tag{3.12}$$

where

$$M(x_{n-1}, u) = \max\left\{d(x_{n-1}, u), d(x_{n-1}, x_n), d(fx_{n-1}, fu), \frac{d(x_{n-1}, x_n)\, d(u, fu)}{d(x_{n-1}, u)}\right\} \tag{3.13}$$

and

$$N_L(x_{n-1}, u) = L \min\{d(x_{n-1}, fu), d(x_{n-1}, x_n), d(u, fu), d(u, x_n)\}. \tag{3.14}$$

Let $n \to \infty$ in (3.13) and (3.14), we obtain

$$\lim_{n\to\infty} M(x_{n-1}, u) = d(u, fu) \text{ and } \lim_{n\to\infty} N_L(x_{n-1}, u) = 0. \tag{3.15}$$

Also, let $n \to \infty$ in (3.12) and use (3.15), we have

$$\psi(d(u, fu)) \leq F(\psi(d(u, fu)), \phi(d(u, fu))) \leq \psi(d(u, fu)) \tag{3.16}$$

this implies $F(\psi(d(u, fu)), \phi(d(u, fu))) = \psi(d(u, fu))$. From the definition of F function $\psi(d(u, fu)) = 0$ or $\phi(d(u, fu)) = 0$. We arrive at

$$d(u, fu) = 0 \text{ and } fu = u. \tag{3.17}$$

Note that (3.11) and (3.17) are contradiction with assumption that f has no periodic point. Consequently, f has a periodic point, that is, there exists $a \in X$ such that $a = f^p a$.

Step 5: We claim that f has a fixed point. Note that there exists $a \in X$ such that $a = f^p a$. It is clear that $a \in X$ is a fixed point f for $p = 1$. We will prove that $\vartheta = f^{p-1}a$ is a fixed point of f in case of $p > 1$. If possible, assume the contrary, that is, let $f^{p-1}a \neq f^p a$. Thus, from the contractive inequality, we get

$$\begin{aligned}\psi(d(a, fa)) &= \psi\left(d\left(f^p a, f^{p+1}a\right)\right) \\ &\leq F\left(\psi\left(M\left(f^{p-1}a, f^p a\right)\right), \phi\left(M\left(f^{p-1}a, f^p a\right)\right)\right) + N_L\left(f^{p-1}a, f^p a\right)\end{aligned} \tag{3.18}$$

where

$$M\left(f^{p-1}a, f^{p}a\right) = \max\left\{\begin{array}{c} d\left(f^{p-1}a, f^{p}a\right), d\left(f^{p-1}a, f^{p}a\right), d\left(f^{p}a, f^{p+1}a\right), \\ \frac{d\left(f^{p-1}a, f^{p}a\right) d\left(f^{p}a, f^{p+1}a\right)}{d\left(f^{p-1}a, f^{p}a\right)} \end{array}\right\}$$
$$= \max\left\{d\left(f^{p-1}a, f^{p}a\right), d\left(f^{p}a, f^{p+1}a\right)\right\}.$$

Note that $\{d(x_n, x_{n+1})\}$ is a monotone nonincreasing sequence of non-negative real numbers, then we get that

$$M\left(f^{p-1}a, f^{p}a\right) = d\left(f^{p-1}a, f^{p}a\right) \text{ and } N_L\left(f^{p-1}a, f^{p}a\right) = 0.$$

Consider (3.18) and definition of F class function, then we have

$$\begin{aligned} \psi(d(a, fa)) &= \psi\left(d\left(f^{p}a, f^{p+1}a\right)\right) \\ &\leq F\left(\psi\left(d\left(f^{p-1}a, f^{p}a\right)\right), \phi\left(d\left(f^{p-1}a, f^{p}a\right)\right)\right) \\ &\leq \psi\left(d\left(f^{p-1}a, f^{p}a\right)\right). \end{aligned}$$

From the definition of ψ function, we have $d(a, fa) < d\left(f^{p-1}a, f^{p}a\right)$. Continuing the similar process, we get the following contradiction

$$d(a, fa) < d\left(f^{p-1}a, f^{p}a\right) \leq d\left(f^{p-2}a, f^{p-1}a\right) < \cdots \leq d(a, fa).$$

Hence, the assumption that $\vartheta = f^{p-1}a$ is not a fixed point of f is not true. Consequently, f has a fixed point.

Step 6: The fixed point is unique. Now, we show that this fixed point is unique. If possible, let $\vartheta, \vartheta' \in X$ be fixed point of f . We distinguish two cases:

- Case 1: If ϑ is comparable to ϑ' then $f\vartheta = \vartheta$ is comparable to $f\vartheta' = \vartheta'$. Note that

$$\begin{aligned} \psi\left(d\left(\vartheta, \vartheta'\right)\right) &= \psi\left(d\left(f\vartheta,\ f\vartheta'\right)\right) \\ &\leq F\left(\psi\left(M\left(\vartheta, \vartheta'\right)\right), \phi\left(M\left(\vartheta, \vartheta'\right)\right)\right) + N_L\left(\vartheta, \vartheta'\right) \\ &\leq \psi\left(M\left(\vartheta, \vartheta'\right)\right) + N_L\left(\vartheta, \vartheta'\right) \end{aligned} \tag{3.19}$$

  where

$$M\left(\vartheta, \vartheta'\right) = d\left(\vartheta, \vartheta'\right) \text{ and } N_L\left(\vartheta, \vartheta'\right) = 0. \tag{3.20}$$

  Thus, from (3.19), we have

$$\psi\left(d\left(\vartheta, \vartheta'\right)\right) \leq F\left(\psi\left(d\left(\vartheta, \vartheta'\right)\right), \phi\left(d\left(\vartheta, \vartheta'\right)\right)\right) \leq \psi\left(d\left(\vartheta, \vartheta'\right)\right). \tag{3.21}$$

  From definition of F function and property of ψ, ϕ functions, we obtain that $\vartheta = \vartheta'$.

- Case 2: If ϑ is not comparable to ϑ', then there exists an element $z \in X$ such that z comparable to ϑ and ϑ'. Without loss of generalty, we choose $z \preceq \vartheta$. Let define a sequence $\{z_n\}$ by

$$z_0 = z \text{ and } z_n = fz_{n-1}, \text{ for all } n \geq 1.$$

Note that f is monotone, then we have

$$f^n z \preceq f^n \vartheta = \vartheta \text{ or } f^n z \succeq f^n \vartheta = \vartheta$$

that is, we have

$$z_n \preceq \vartheta \text{ or } z_n \succeq \vartheta.$$

If there exists n_0 such that $\vartheta = z_{n_0}$, then $z_n = f z_{n-1} = f\vartheta = \vartheta$ for all $n \geq n_0 - 1$. So $\lim_{n\to\infty} z_n = \vartheta$. Analogously, we get $\lim_{n\to\infty} z_n = \vartheta'$. From the uniqueness of the limit $\vartheta = \vartheta'$.

If $\vartheta \neq z_n$ for all $n = 0, 1, 2, \cdots$, then from (3.1), we have

$$\begin{aligned}\psi(d(\vartheta, z_n)) &= \psi(d(f\vartheta, f z_{n-1})) \\ &\leq F(\psi(M(\vartheta, z_{n-1})), \phi(M(\vartheta, z_{n-1}))) + N_L(\vartheta, z_{n-1})\end{aligned} \tag{3.22}$$

where

$$M(\vartheta, z_{n-1}) = \max\{d(\vartheta, z_{n-1}), d(\vartheta, z_n)\} \text{ and } N_L(\vartheta, z_{n-1}) = 0. \tag{3.23}$$

Considering (3.23) with (3.1), we obtain

$$\psi(d(\vartheta, z_n)) \leq F(\psi(M(\vartheta, z_{n-1})), \phi(M(\vartheta, z_{n-1}))) \leq \psi(M(\vartheta, z_{n-1})). \tag{3.24}$$

This is implies that $\{d(\vartheta, z_n)\}$ is a monotone decreasing sequences of non-negative real numbers. Thus, there exists $\lambda \geq 0$ such that $\{d(\vartheta, z_n)\}$ converges to λ as $n \to \infty$. Letting $n \to \infty$ in (3.24), we have

$$\psi(\lambda) \leq F(\psi(\lambda), \phi(\lambda)) \leq \psi(\lambda) \tag{3.25}$$

and this implies that $\lambda = 0$, that is,

$$\lim_{n\to\infty} d(\vartheta, z_n) = 0.$$

Analogously, we have

$$\lim_{n\to\infty} d(\vartheta', z_n) = 0.$$

From the quadrilateral inequality of rectangular metric function we have

$$d(\vartheta, \vartheta') \leq d(\vartheta, z_n) + d(z_n, \vartheta') + d(\vartheta', \vartheta') \tag{3.26}$$

take $n \to \infty$ in (3.26), we hold that $\vartheta = \vartheta'$, that is fixed point is unique.

□

The following theorem obtained by picking out $F(s,t) = s - t$.

Theorem 3. *Let $(X, \preceq, d)$ be a partially ordered Hausdorff and complete RMS. Let $f : X \to X$ be a monotone mapping satisfying for all distinct points $x, y \in X$ with $x \preceq y$,*

$$\psi(d(fx, fy)) \leq \psi(M(x,y)) - \phi(M(x,y)) + N_L(x,y) \tag{3.27}$$

such that $\psi \in \Psi$, $\phi \in \Phi$. Also, suppose either

(1) *C1) f is continuous or*
(2) *C2) $(X, \preceq, d)$ be regular.*

If there exists $x_0 \in X$ with $x_0 \preceq f x_0$ or $x_0 \succeq f x_0$, then f has a fixed point in X. Moreover, if $(X, \preceq)$ has the property (1.4) (or (1.5)), then the fixed point is unique.

Example 1. Take the $X = \{0, 1, 2, 3\}$ and define usual order "$\leq$" on X. Define a function $d : X \times X \to [0, \infty)$ as following;

$$d(1,3) = d(3,1) = 4$$

$$d(1,2) = d(2,1) = 2$$

$$d(2,3) = d(3,2) = 1$$

$$d(x,y) = |x - y|, \text{ for other cases.}$$

Note that $d(1,3) > d(1,2) + d(2,3)$, then d is not a (usual) metric on X. On the other hand d is a generalized (rectangular) metric on X. Define $f : X \to X$ by

$$f = \begin{pmatrix} 0123 \\ 0231 \end{pmatrix}.$$

Case 1: Let $(x,y) = (1,3) \Longrightarrow d(f1, f3) = d(2,1) = 2$ and $M(1,3) = 4$, $N_L(1,3) = 0$.
Case 2: Let $(x,y) = (1,2) \Longrightarrow d(f1, f2) = d(2,3) = 1$ and $M(1,2) = 4$, $N_L(1,2) = 0$.
Case 3: Let $(x,y) = (2,3) \Longrightarrow d(f2, f3) = d(3,1) = 4$ and $M(2,3) = 4$, $N_L(2,3) = 0$.
Case 4: Let $(x,y) = (1,0) \Longrightarrow d(f1, f0) = d(2,0) = 2$ and $M(1,2) = 2$, $N_L(1,2) = 0$.
Case 5: Let $(x,y) = (2,0) \Longrightarrow d(f2, f0) = d(3,0) = 3$ and $M(2,0) = 3$, $N_L(2,0) = 0$.
Case 6: Let $(x,y) = (3,0) \Longrightarrow d(f3, f0) = d(1,0) = 1$ and $M(3,0) = 4$, $N_L(3,0) = 0$.
For all cases there exists $\psi \in \Psi$, $\phi \in \Phi$, $F \in \mathcal{C}$. One of the choices:

$$F : [0,\infty)^2 \to \mathbb{R}, \text{by } F(s,t) = s - t$$

$$\psi : [0,\infty) \to [0,\infty) \text{ is any continuous and nondecreasing,}$$

$$\phi : [0,\infty) \to [0,\infty) \text{ by } \phi(\delta) = 0.$$

Therefore, all conditions of Theorem 2 are satisfied. Hence, f has a unique fixed point. In fact $u = 0$ is the unique fixed point of f.

4. Some auxiliary results

In this section we aim to give some results that obtained from our main theorem.

Example 2. The following functions $F : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$.
(1)$F(s,t) = ks$, $0 < k < 1$,
(2) $F(s,t) = s - t$,
(3) $F(s,t) = \frac{s}{(1+t)^r}$, $r \in (0,\infty)$.

Theorem 4 (Lakzian and Samet [12])**.** *Let (X,d) be a Hausdorff and complete g.m.s and let $T : X \to X$ be a self-mapping satisfying*

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \varphi(d(x,y))$$

for all $x,y \in X$, where $\psi \in \Psi$ and $\varphi : [0,\infty) \to [0,\infty)$ is continuous and $\varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

In 2011, Luong & Thuan [13] introduced a generalized rational contractive inequality to obtain a fixed point result in partially ordered metric space. The following theorem extends Theorem 2.1 of [13] from partially ordered metric space to partially ordered rectangular metric space.

Theorem 5. *Let $(X,\preceq)$ be a partially ordered Hausdorff and complete RMS. Let $T : X \to X$ be an nondecreasing mapping satisfying the following inequality*

$$d(Tx,Ty) \leq M(x,y) - \varphi(M(x,y)), \text{ for all } x,y \in X \text{ with } x \succeq y, x \neq y$$

where

(1) $\phi : [0,\infty) \to [0,\infty)$ *is a lower semi-continuous function with* $\phi(t) = 0 \Leftrightarrow t = 0$
(2) $M(x,y) = \max\left\{\frac{d(x,Tx)d(y,Ty)}{d(x,y)}, d(x,y)\right\}$.

Also, assume either T is continuous or X has the property (1.3)*. If there exists $x_0 \preceq Tx_0$, then T has a fixed point. Moreover, if X has the property* (1.5)*, then the fixed point is unique.*

Arshad et.al [1], by introducing the concept of almost Jaggi contraction, they gave some fixed point theorems for rational contractions in partially ordered metric spaces. The following theorem extends Theorem 2 of [1] from partially ordered metric space to partially ordered rectangular metric space.

Definition 5. Let $(X,\preceq,d)$ be a partially ordered rectangular metric space. A self mapping f on X is called an almost Jaggi contraction if it satisfies the following condition:

$$d(fx,fy) \leq \alpha \frac{d(x,fx)\,d(y,fy)}{d(x,y)} + \beta d(x,y) + L \min\{d(x,fy), d(y,fx)\},$$

for any distinct $x,y \in X$ with $x \preceq y$, where $L \geq 0$ and $\alpha,\beta \in [0,1)$ with $\alpha + \beta < 1$.

Theorem 6. *Let $(X, \preceq, d)$ be a partially ordered Hausdorff and complete RMS. Suppose that a self-mapping f is an almost Jaggi contraction, continuous and non-decreasing. Suppose there exists $x_0 \in X$ with $x_0 \preceq f x_0$. Then f has a unique fixed point.*

Mustafa et. al.[14] used the definition of almost contraction to make a discussion on generalized almost contractions via rational expressions in partially ordered metric space. The following theorem extends Theorem 3 of [14] from partially ordered metric space to partially ordered rectangular metric space.

Theorem 7. *Let $(X, \preceq)$ be a partially ordered Hausdorff and complete RMS. Let $f : X \to X$ be a non-decreasing mapping which satisfies the inequality*

$$\phi(d(Tx, Ty)) \leq$$
$$\phi(M(x,y)) - \psi(M(x,y)) + L \min\{d(x, fy), d(x, fx), d(y, fy), d(y, fx)\},$$

for all distinct points $x, y \in X$ with $y \leq x$ where $\phi \in \Phi, \psi \in \Psi, L \geq 0$ and

$$M(x,y) = \max\left\{\frac{d(x, fx)\, d(y, fy)}{d(x,y)}, d(x,y)\right\}.$$

Also, assume either f is continuous or X has the feature (1.3). If there exists $x_0 \in X$ such that $x_0 \preceq f x_0$, then f has a fixed point. Moreover, if X has the feature (1.5), then the fixed point is unique.

5. Applications

Let Λ be the set of functions $\xi : [0, +\infty) \to [0, +\infty)$ such that

(1) ξ is Lebesgue integrable mapping on each compact subset of $[0, +\infty)$;
(2) $\int_0^{\varepsilon} \xi(s)\, ds > 0$ for every $\varepsilon > 0$.

For this class of functions, we can express the following results.

Theorem 8. *Let (X, d) be a Hausdorff and complete RMS, and let f be a self-map satisfying*

$$\int_0^{d(fx,fy)} \xi_1(s)\, ds \leq F\left(\int_0^{M(x,y)} \xi_1(s)\, ds, \int_0^{M(x,y)} \xi_2(s)\, ds\right) + N_L(x,y)$$

for all $x, y \in X$ and $\xi_1, \xi_2 \in \Lambda$, $F \in \mathcal{C}$ and

$$M(x,y) = \max\left\{d(x,y),\ d(x,fx),\ d(fx,fy),\ \frac{d(x,fx)\, d(y,fy)}{d(x,y)}\right\},$$
$$N_L(x,y) = L \min\{d(x,fy),\ d(x,fx),\ d(y,fy),\ d(y,fx)\} \text{ such that } L \geq 0.$$

Then f has a unique fixed point $\vartheta \in X$.

Proof. Let $\psi(s) = \int_0^s \xi_1(v)\, dv$ and $\phi(s) = \int_0^s \xi_2(v)\, dv$. Then $\psi \in \Psi$, $\phi \in \Phi$, and beside, the function ψ is nondecreasing. By Theorem 2, T has a unique fixed point. □

Taking $\xi_2(s) = (1-k)\xi_1(s)$ and $F(s,t) = s-t$ in Theorem 8, we obtain the following statement.

Corollary 1. *Let (X,d) be a Hausdorff and complete RMS, and let f be a self-map satisfying*

$$\int_0^{d(fx,fy)} \xi_1(s)\,ds \leq k\int_0^{\max\left\{d(x,y),\, d(x,fx),\, d(fx,fy),\, \frac{d(x,fx)d(y,fy)}{d(x,y)}\right\}} \xi_1(s)\,ds + N_L(x,y),$$

for all x, $y \in X$ and ξ_1, $\xi_2 \in \Lambda$, $F \in \mathcal{C}$.

Then f has a unique fixed point $\vartheta \in X$.

Taking $F(s,t) = s-t$ in Theorem 8, we have the following statement.

Corollary 2. *Let (X,d) be a Hausdorff and complete RMS, and let f be a self-map satisfying*

$$\int_0^{d(fx,fy)} \xi_1(s)\,ds \leq \int_0^{M(x,y)} \xi_1(s)\,ds - \int_0^{M(x,y)} \xi_2(s)\,ds + N_L(x,y),$$

for all x, $y \in X$ and ξ_1, $\xi_2 \in \Lambda$, $F \in \mathcal{C}$ and

$$M(x,y) = \max\left\{d(x,y),\, d(x,fx),\, d(fx,fy),\, \frac{d(x,fx)\,d(y,fy)}{d(x,y)}\right\},$$

$$N_L(x,y) = L\min\{d(x,fy),\, d(x,fx),\, d(y,fy),\, d(y,fx)\} \text{ such that } L \geq 0.$$

Then f has a unique fixed point $\vartheta \in X$.

Taking $F(s,t) = ks$, $0 < k < 1$ and $L = 0$ in Theorem 8, we get the following statement.

Corollary 3. *Let (X,d) be a Hausdorff and complete RMS, and let f be a self-map satisfying*

$$\int_0^{d(fx,fy)} \xi_1(s)\,ds \leq k\int_0^{\max\left\{d(x,y),\, d(x,fx),\, d(fx,fy),\, \frac{d(x,fx)d(y,fy)}{d(x,y)}\right\}} \xi_1(s)\,ds,$$

for all x, $y \in X$ and ξ_1, $\xi_2 \in \Lambda$, $F \in \mathcal{C}$.

Then f has a unique fixed point $\vartheta \in X$.

6. Conclusion

Our theorems and corolaries which include the corresponding results studied in Luong & Thuan (2011), Arshad et al. (2013), Mustafa et al. (2014), Kir & Kıziltunc (2015) as special cases fundamentally improve and generalize the results of ([3], [7], [10], [17], [19], [5], [4], [18], [2], [12], [16], [15], [8], [9]) in the following sense.

(i) Rectangular metric space has substituted the sum at the righthand side of the triangle inequality by a three-term expression.

(ii) Generalized from metric space to rectangular metric space.

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THETA FUNCTION IDENTITIES AND REPRESENTATION NUMBERS OF CERTAIN QUADRATIC FORMS IN TWELVE VARIABLES

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Abstract. In this paper using the (p,k)-parametrization of theta functions and Eisenstein Series, developed by Alaca, Alaca and Williams, we obtain some new theta function identities and then use them to derive explicit formulae for the number of representations of a positive integer n by certain quadratic forms

$$\sum_{k=1}^{6} a_k(x_{2k-1}^2 + x_{2k-1}x_{2k} + x_{2k}^2)$$

in twelve variables where $a_k \in \{1,2,4\}$.

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1. Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ denote the set of natural numbers, non-negative integers, integers, rational numbers and complex numbers respectively so that $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $a_1,...,a_6 \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we let $R(a_1,...,a_6;n)$ denote the representation numbers of n by the form $a_1(x_1^2 + x_1x_2 + x_2^2) + a_2(x_3^2 + x_3x_4 + x_4^2) + a_3(x_5^2 + x_5x_6 + x_6^2) + a_4(x_7^2 + x_7x_8 + x_8^2) + a_5(x_9^2 + x_9x_{10} + x_{10}^2) + a_6(x_{11}^2 + x_{11}x_{12} + x_{12}^2)$, that is

$$R(a_1,...,a_6;n) := \\ = \operatorname{card}\left\{ (x_1,...,x_{12}) \in \mathbb{Z}^{12} : n = \sum_{k=1}^{6} a_k(x_{2k-1}^2 + x_{2k-1}x_{2k} + x_{2k}^2) \right\}. \tag{1.1}$$

If l of $a_1,...,a_6$ are equal, say

$$a_i = a_{i+1} = \cdots = a_{i+l-1} = a \tag{1.2}$$

for convenience we indicate this in $R(a_1,...,a_6;n)$ by writing a^l for $a_i, a_{i+1},...,a_{i+l-1}$.

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For $k \in \mathbb{N}$ and $n \in \mathbb{Q}$ the sum of divisor function is defined by

$$\sigma_k(n) = \begin{cases} \sum\limits_{\substack{d \in \mathbb{N} \\ d|n}} d^k, & \text{if } n \in \mathbb{N}, \\ 0, & \text{if } n \in \mathbb{Q}, n \notin \mathbb{N}. \end{cases} \tag{1.3}$$

The Dedekind eta function $\eta(z)$ is the holomorphic function defined on the upper half plane $H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ by the product formula

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}). \tag{1.4}$$

Through the remainder of the paper we take $q = q(z) := e^{2\pi i z}$ with $z \in H$ and so by (1.4) we have

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{1.5}$$

Let N be a positive integer. An eta quotient is defined to be a finite product of the form

$$f(z) = \prod_{0 \le \delta \backslash N} \eta^{r_\delta}(\delta z). \tag{1.6}$$

When all of the exponents r_δ are nonnegative, $f(z)$ is said to be an eta product. We now define the following eta products.

$$B_1(q) := \sum_{n=1}^{\infty} b_1(n) q^n = \eta^{12}(2z), \tag{1.7}$$

$$B_2(q) := \sum_{n=1}^{\infty} b_2(n) q^n = \eta^5(z)\eta^5(2z)\eta(3z)\eta(6z), \tag{1.8}$$

$$B_3(q) := \sum_{n=1}^{\infty} b_3(n) q^n = \eta^6(z)\eta^6(3z), \tag{1.9}$$

$$B_4(q) := \sum_{n=1}^{\infty} b_4(n) q^n = \eta^5(2z)\eta^5(4z)\eta(6z)\eta(12z), \tag{1.10}$$

$$B_5(q) := \sum_{n=1}^{\infty} b_5(n) q^n = \eta(z)\eta^5(3z)\eta^5(4z)\eta(12z), \tag{1.11}$$

$$B_6(q) := \sum_{n=1}^{\infty} b_6(n) q^n = \eta(z)\eta(2z)\eta^5(3z)\eta^5(6z), \tag{1.12}$$

$$B_7(q) := \sum_{n=1}^{\infty} b_7(n)q^n = \eta^{12}(6z). \tag{1.13}$$

Determination of representation number formulae for quadratic forms which are sums of binary quadratic form $x_1^2 + x_1x_2 + x_2^2$ was considered before by many mathematicians. See for example [5, 9, 12]. Among the studies some contains formulae for the forms in twelve variables. In a recent publication Yao and Xia [14] obtained formula for $R(1^6;n)$. The authors proved that

$$R(1^6;n) = \frac{252}{13}\sigma_5(n) - \frac{6804}{13}\sigma_5(n/3) + \frac{216}{13}a(n), \tag{1.14}$$

where

$$\sum_{n=1}^{\infty} a(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6. \tag{1.15}$$

From (1.15) and (1.9) it is clear that $a(n) = b_3(n)$.

A similar formulae for $R(1^6;n)$ was given before by Lomadze [9]. Recently, Köklüce [7] has derived formulae for the convolution sums $W_{1,6}^{1,3}(n)$, $W_{2,3}^{1,3}(n)$, $W_{3,2}^{1,3}(n)$ and $W_{6,1}^{1,3}(n)$. As an application he used these evaluations to derive formulae for the representation numbers $R(1^5,2^1;n)$, $R(1^4,2^2;n)$ and $R(1^3,2^3;n)$. In another work he and his coauthor Eser [8] have found formulae for the convolutions sums $W_{1,12}^{1,3}(n)$, $W_{3,4}^{1,3}(n)$, $W_{4,3}^{1,3}(n)$ and $W_{12,1}^{1,3}(n)$ and then used them for deriving representation number formulae for some quadratic forms in twelve variables which are sums of squares. Köklüce has used another method developed by Lomadze to obtain explicit formulae for quadratic forms in twelve and sixteen variables which are direct sums of binary quadratic forms with discriminant -23 in [6].

In the present paper, we obtain formulae for 21 quadratic forms in twelve variables by using theta function identities. These formulae are given in terms of $\sigma_5(n)$ and the integers $b_i(n)$, ($i = 1,2,3,4,5,6,7$ and $n \in \mathbb{N}$) given by the equation (1.7)-(1.13). We compare our formula for $R(1^6;n)$ with results of Yao and Xia [14] and formulae for $R(1^5,2^1;n)$, $R(1^4,2^2;n)$ and $R(1^3,2^3;n)$ with previous work of Köklüce [7] and see that they are consistent. Similar methods have been used before for deriving representation numbers formulae for sextenary and octonary quadratic forms. See for example [4] and [13], respectively.

The rest of this paper is organized as follows. In Sec. 2, we state our main theorem. In Sec. 3, we give parametrization of theta functions and Eisenstein series in terms of p and k. In Sec. 4, we give a lemma which contains required theta function identities for the proof of the main theorem. In Sec. 5, we prove the main theorem. Finally, in Sec. 6, we give a concluding remark.

2. Statement of the Theorem

Theorem 1. *Let $n \in \mathbb{N}$ then,*

(i) $$R(1^6;n) = \frac{252}{13}\sigma_5(n) - \frac{6804}{13}\sigma_5(n/3) + \frac{216}{13}b_3(n), \tag{2.1}$$

(ii) $$R(1^5,2^1;n) = \frac{66}{7}\sigma_5(n) - \frac{192}{7}\sigma_5(n/2) + \frac{1782}{7}\sigma_5(n/3) - \frac{5184}{7}\sigma_5(n/6) + \frac{144}{7}b_2(n) + \frac{1296}{7}b_6(n), \tag{2.2}$$

(iii) $$R(1^4,2^2;n) = \frac{60}{13}\sigma_5(n) + \frac{192}{13}\sigma_5(n/2) - \frac{1620}{13}\sigma_5(n/3) - \frac{5184}{13}\sigma_5(n/6) - \frac{288}{13}b_2(n) + \frac{540}{13}b_3(n) + \frac{2592}{13}b_6(n), \tag{2.3}$$

(iv) $$R(1^3,2^3;n) = \frac{18}{7}\sigma_5(n) - \frac{144}{7}\sigma_5(n/2) + \frac{486}{7}\sigma_5(n/3) - \frac{3888}{7}\sigma_5(n/6) + \frac{108}{7}b_2(n) + \frac{972}{7}b_6(n), \tag{2.4}$$

(v) $$R(1^2,2^4;n) = \frac{12}{13}\sigma_5(n) + \frac{240}{13}\sigma_5(n/2) - \frac{324}{13}\sigma_5(n/3) - \frac{6480}{13}\sigma_5(n/6) - \frac{126}{13}b_2(n) + \frac{270}{13}b_3(n) + \frac{1134}{13}b_6(n), \tag{2.5}$$

(vi) $$R(1^1,2^5;n) = \frac{6}{7}\sigma_5(n) - \frac{132}{7}\sigma_5(n/2) + \frac{162}{7}\sigma_5(n/3) - \frac{3564}{7}\sigma_5(n/6) + \frac{36}{7}b_2(n) + \frac{324}{7}b_6(n), \tag{2.6}$$

(vii) $$R(1^5,4^1;n) = \frac{66}{13}\sigma_5(n) - \frac{198}{13}\sigma_5(n/2) - \frac{1782}{13}\sigma_5(n/3) + \frac{384}{13}\sigma_5(n/4) + \frac{5346}{13}\sigma_5(n/6) - \frac{10368}{13}\sigma_5(n/12) + \frac{774}{13}b_1(n) - \frac{1260}{13}b_2(n) + \frac{810}{13}b_3(n) - \frac{16128}{13}b_4(n) + \frac{24192}{13}b_5(n) - \frac{6804}{13}b_6(n) + \frac{33534}{13}b_7(n), \tag{2.7}$$

(viii) $$R(1^4,4^2;n) = \frac{15}{13}\sigma_5(n) + \frac{45}{13}\sigma_5(n/2) - \frac{405}{13}\sigma_5(n/3) + \frac{192}{13}\sigma_5(n/4) - \frac{1215}{13}\sigma_5(n/6) - \frac{5184}{13}\sigma_5(n/12) + \frac{387}{13}b_1(n) - \frac{846}{13}b_2(n) + \frac{756}{13}b_3(n) - \frac{8064}{13}b_4(n) + \frac{12096}{13}b_5(n) - \frac{1458}{13}b_6(n) + \frac{16767}{13}b_7(n), \tag{2.8}$$

(ix) $$R(1^3,4^3;n)=\frac{9}{26}\sigma_5(n)-\frac{81}{26}\sigma_5(n/2)-\frac{243}{26}\sigma_5(n/3)+\frac{288}{13}\sigma_5(n/4)$$
$$+\frac{2187}{26}\sigma_5(n/6)-\frac{7776}{13}\sigma_5(n/12)+\frac{459}{26}b_1(n)-\frac{3591}{104}b_2(n)+\frac{3591}{104}b_3(n)$$
$$-\frac{3672}{13}b_4(n)+\frac{5508}{13}b_5(n)-\frac{729}{104}b_6(n)+\frac{12393}{26}b_7(n), \tag{2.9}$$

(x) $$R(1^2,4^4;n)=\frac{3}{52}\sigma_5(n)+\frac{45}{52}\sigma_5(n/2)-\frac{81}{52}\sigma_5(n/3)+\frac{240}{13}\sigma_5(n/4)$$
$$-\frac{1215}{52}\sigma_5(n/6)-\frac{6480}{13}\sigma_5(n/12)+\frac{531}{52}b_1(n)-\frac{801}{52}b_2(n)+\frac{891}{52}b_3(n)$$
$$-\frac{1656}{13}b_4(n)+\frac{2484}{13}b_5(n)-\frac{243}{52}b_6(n)+\frac{8019}{52}b_7(n), \tag{2.10}$$

(xi) $$R(1^1,4^5;n)=\frac{3}{104}\sigma_5(n)-\frac{99}{104}\sigma_5(n/2)-\frac{81}{104}\sigma_5(n/3)+\frac{264}{13}\sigma_5(n/4)$$
$$+\frac{2673}{104}\sigma_5(n/6)-\frac{7128}{13}\sigma_5(n/12)+\frac{99}{26}b_1(n)-\frac{1035}{208}b_2(n)+\frac{1485}{208}b_3(n)$$
$$-\frac{558}{13}b_4(n)+\frac{837}{13}b_5(n)-\frac{729}{208}b_6(n)+\frac{2187}{52}b_7(n) \tag{2.11}$$

(xii) $$R(1^4,2^1,4^1;n)=\frac{15}{7}\sigma_5(n)+\frac{51}{7}\sigma_5(n/2)+\frac{405}{7}\sigma_5(n/3)-\frac{192}{7}\sigma_5(n/4)$$
$$+\frac{1377}{7}\sigma_5(n/6)-\frac{5184}{7}\sigma_5(n/12)+\frac{333}{7}b_1(n)+\frac{36}{7}b_2(n)-\frac{216}{7}b_3(n)$$
$$+\frac{3456}{7}b_5(n)-\frac{3564}{7}b_6(n)+\frac{8991}{7}b_7(n), \tag{2.12}$$

(xiii) $$R(1^3,2^2,4^1;n)=\frac{18}{13}\sigma_5(n)-\frac{150}{13}\sigma_5(n/2)-\frac{486}{13}\sigma_5(n/3)+\frac{384}{13}\sigma_5(n/4)$$
$$+\frac{4050}{13}\sigma_5(n/6)-\frac{10368}{13}\sigma_5(n/12)+\frac{306}{13}b_1(n)-\frac{909}{26}b_2(n)+\frac{729}{26}b_3(n)$$
$$-\frac{4896}{13}b_4(n)+\frac{7344}{13}b_5(n)-\frac{2835}{26}b_6(n)+\frac{8262}{13}b_7(n), \tag{2.13}$$

(xiv) $$R(1^3,2^1,4^2;n)=\frac{9}{14}\sigma_5(n)-\frac{69}{14}\sigma_5(n/2)+\frac{243}{14}\sigma_5(n/3)-\frac{96}{7}\sigma_5(n/4)$$
$$-\frac{1863}{14}\sigma_5(n/6)-\frac{2592}{7}\sigma_5(n/12)+\frac{333}{14}b_1(n)+9b_2(n)-\frac{108}{7}b_3(n)$$
$$+\frac{1728}{7}b_5(n)-\frac{1377}{7}b_6(n)+\frac{8991}{14}b_7(n), \tag{2.14}$$

(xv) $$R(1^2,2^3,4^1;n) = \frac{3}{7}\sigma_5(n) + 9\sigma_5(n/2) + \frac{81}{7}\sigma_5(n/3) - \frac{192}{7}\sigma_5(n/4) + 243\sigma_5(n/6) - \frac{5184}{7}\sigma_5(n/12) + \frac{81}{7}b_1(n) + \frac{27}{7}b_2(n) - \frac{27}{7}b_3(n) + \frac{432}{7}b_5(n) - \frac{243}{7}b_6(n) + \frac{2187}{7}b_7(n), \tag{2.15}$$

(xvi) $$R(1^2,2^2,4^2;n) = \frac{3}{13}\sigma_5(n) + \frac{57}{13}\sigma_5(n/2) - \frac{81}{13}\sigma_5(n/3) + \frac{192}{13}\sigma_5(n/4) - \frac{1539}{13}\sigma_5(n/6) - \frac{5184}{13}\sigma_5(n/12) + \frac{153}{13}b_1(n) - \frac{999}{52}b_2(n) + \frac{999}{52}b_3(n) - \frac{2448}{13}b_4(n) + \frac{3672}{13}b_5(n) - \frac{2025}{52}b_6(n) + \frac{4131}{13}b_7(n), \tag{2.16}$$

(xvii) $$R(1^2,2^1,4^3;n) = \frac{3}{28}\sigma_5(n) + \frac{69}{28}\sigma_5(n/2) + \frac{81}{28}\sigma_5(n/3) - \frac{144}{7}\sigma_5(n/4) + \frac{1863}{28}\sigma_5(n/6) - \frac{3888}{7}\sigma_5(n/12) + \frac{243}{28}b_1(n) + \frac{171}{28}b_2(n) - \frac{81}{28}b_3(n) + \frac{324}{7}b_5(n) + \frac{81}{28}b_6(n) + \frac{6561}{28}b_7(n), \tag{2.17}$$

(xviii) $$R(1^1,2^4,4^1;n) = \frac{6}{13}\sigma_5(n) - \frac{138}{13}\sigma_5(n/2) - \frac{162}{13}\sigma_5(n/3) + \frac{384}{13}\sigma_5(n/4) + \frac{3726}{13}\sigma_5(n/6) - \frac{10368}{13}\sigma_5(n/12) + \frac{72}{13}b_1(n) - \frac{243}{26}b_2(n) + \frac{243}{26}b_3(n) - \frac{1152}{13}b_4(n) + \frac{1728}{13}b_5(n) - \frac{405}{26}b_6(n) + \frac{1944}{13}b_7(n), \tag{2.18}$$

(xix) $$R(1^1,2^3,4^2;n) = \frac{3}{14}\sigma_5(n) - \frac{9}{2}\sigma_5(n/2) + \frac{81}{14}\sigma_5(n/3) - \frac{96}{7}\sigma_5(n/4) - \frac{243}{2}\sigma_5(n/6) - \frac{2592}{7}\sigma_5(n/12) + \frac{81}{14}b_1(n) + \frac{27}{14}b_2(n) - \frac{27}{14}b_3(n) + \frac{216}{7}b_5(n) - \frac{243}{14}b_6(n) + \frac{2187}{14}b_7(n), \tag{2.19}$$

(xx) $$R(1^1,2^2,4^3;n) = \frac{3}{26}\sigma_5(n) - \frac{75}{26}\sigma_5(n/2) - \frac{81}{26}\sigma_5(n/3) + \frac{288}{13}\sigma_5(n/4) + \frac{2025}{26}\sigma_5(n/6) - \frac{7776}{13}\sigma_5(n/12) + \frac{54}{13}b_1(n) - \frac{63}{8}b_2(n) + \frac{999}{104}b_3(n) - \frac{864}{13}b_4(n) + \frac{1296}{13}b_5(n) - \frac{405}{104}b_6(n) + \frac{1458}{13}b_7(n), \tag{2.20}$$

(xxi) $$R(1^1,2^1,4^4;n) = \frac{3}{56}\sigma_5(n) - \frac{51}{56}\sigma_5(n/2) + \frac{81}{56}\sigma_5(n/3) - \frac{120}{7}\sigma_5(n/4) - \frac{1377}{56}\sigma_5(n/6) - \frac{3240}{7}\sigma_5(n/12) + \frac{153}{56}b_1(n) + \frac{9}{4}b_2(n) + \frac{27}{28}b_3(n) - \frac{108}{7}b_5(n) + \frac{1053}{28}b_6(n) + \frac{4131}{56}b_7(n). \tag{2.21}$$

3. The (p,k)-parametrization of theta functions and Eisenstein series

In his second notebook [11] Ramanujan gives the definitions of Eisenstein series $L(q)$, $M(q)$ and $N(q)$ by

$$L(q) := 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \tag{3.1}$$

$$M(q) := 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} \tag{3.2}$$

and

$$N(q) := 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}. \tag{3.3}$$

It can be easily seen that

$$L(q) := 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n, \tag{3.4}$$

$$M(q) := 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n \tag{3.5}$$

and

$$N(q) := 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n. \tag{3.6}$$

For $q \in \mathbb{C}$, $|q| < 1$ the Jacobi one-dimensional theta function $\varphi(q)$ is defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{3.7}$$

The Borweins' two-dimensional theta function $a(q)$ is defined by

$$a(q) := \sum_{(m,n)\in\mathbb{Z}^2} q^{m^2+mn+n^2}, \tag{3.8}$$

so that for $a_1,\ldots,a_6 \in \mathbb{N}$ we have

$$\sum_{n=0}^{\infty} N(a_1,...,a_6;n)q^n = a(q^{a_1})...a(q^{a_6}). \tag{3.9}$$

Alaca, Alaca and Williams [2] defined p and k respectively by

$$p = p(q) := \frac{\varphi^2(q)-\varphi^2(q^3)}{2\varphi^2(q^3)} \tag{3.10}$$

and

$$k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{3.11}$$

Alaca, Alaca and Williams [2] derived formulae for $a(q), a(q^2)$ and $a(q^4)$ in terms of p and k. They have proved that

$$a(q) = (1+4p+p^2)k, \tag{3.12}$$

$$a(q^2) = (1+p+p^2)k \tag{3.13}$$

and

$$a(q^4) = (1+p-\frac{1}{2}p^2)k. \tag{3.14}$$

Formulae for the series $N(q), N(q^2), N(q^3), N(q^4), N(q^6)$ and $N(q^{12})$ in terms of p and k are determined by the same authors [1]. Equations (3.22)-(3.27) are respectively as follows

$$\begin{aligned} N(q) = (1-246p-5532p^2-38614p^3-135369p^4-276084p^5-348024p^6 \\ -276084p^7-135369p^8-38614p^9-5532p^{10}-246p^{11}+p^{12})k^6, \end{aligned} \tag{3.15}$$

$$\begin{aligned} N(q^2) = (1+6p-114p^2-625p^3-\frac{4059}{2}p^4-4302p^5-5556p^6-4302p^7 \\ -\frac{4059}{2}p^8-625p^9-114p^{10}+6p^{11}+p^{12})k^6, \end{aligned} \tag{3.16}$$

$$\begin{aligned} N(q^3) =&(1+6p+12p^2-58p^3-297p^4-396p^5-264p^6-396p^7-297p^8 \\ &-58p^9+12p^{10}+6p^{11}+p^{12})k^6, \end{aligned} \tag{3.17}$$

$$N(q^4) = (1+6p+12p^2+5p^3-45p^4-144p^5-\frac{1167}{8}p^6+\frac{171}{8}p^7+\frac{2151}{32}p^8 - \frac{739}{16}p^9-\frac{345}{8}p^{10}+\frac{129}{32}p^{11}+\frac{1}{64}p^{12})k^6, \tag{3.18}$$

$$N(q^6) = (1+6p+12p^2+5p^3-\frac{27}{2}p^4-18p^5-12p^6-18p^7-\frac{27}{2}p^8+5p^9 + 12p^{10}+6p^{11}+p^{12})k^6, \tag{3.19}$$

and

$$N(q^{12}) = (1+6p+12p^2+5p^3-\frac{27}{2}p^4-18p^5-\frac{33}{8}p^6+\frac{45}{8}p^7+\frac{135}{32}p^8 + \frac{17}{16}p^9+\frac{3}{16}p^{10}+\frac{3}{32}p^{11}+\frac{1}{64}p^{12})k^6. \tag{3.20}$$

The Ramanujan discriminant function $\triangle(q)$ [10] is defined by

$$\triangle(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

Alaca, Alaca and Williams [1] expressed $\triangle(q^i)$ ($i = 1,2,3,4,6$ and 12) in terms of p and k. They solved these equations for $\prod_{n=1}^{\infty}(1-q^{in})$, ($i = 1,2,3,4,6$ and 12) and obtained the following equations in [3].

$$\prod_{n=1}^{\infty}(1-q^n) = q^{-\frac{1}{24}}2^{-\frac{1}{6}}p^{\frac{1}{24}}(1-p)^{\frac{1}{2}}(1+p)^{\frac{1}{6}}(1+2p)^{\frac{1}{8}}(2+p)^{\frac{1}{8}}k^{\frac{1}{2}}, \tag{3.21}$$

$$\prod_{n=1}^{\infty}(1-q^{2n}) = q^{-\frac{1}{12}}2^{-\frac{1}{3}}p^{\frac{1}{12}}(1-p)^{\frac{1}{4}}(1+p)^{\frac{1}{12}}(1+2p)^{\frac{1}{4}}(2+p)^{\frac{1}{4}}k^{\frac{1}{2}}, \tag{3.22}$$

$$\prod_{n=1}^{\infty}(1-q^{3n}) = q^{-\frac{1}{8}}2^{-\frac{1}{6}}p^{\frac{1}{8}}(1-p)^{\frac{1}{6}}(1+p)^{\frac{1}{2}}(1+2p)^{\frac{1}{24}}(2+p)^{\frac{1}{24}}k^{\frac{1}{2}}, \tag{3.23}$$

$$\prod_{n=1}^{\infty}(1-q^{4n}) = q^{-\frac{1}{6}}2^{-\frac{2}{3}}p^{\frac{1}{6}}(1-p)^{\frac{1}{8}}(1+p)^{\frac{1}{24}}(1+2p)^{\frac{1}{8}}(2+p)^{\frac{1}{2}}k^{\frac{1}{2}}, \tag{3.24}$$

$$\prod_{n=1}^{\infty}(1-q^{6n}) = q^{-\frac{1}{4}}2^{-\frac{1}{3}}p^{\frac{1}{4}}(1-p)^{\frac{1}{12}}(1+p)^{\frac{1}{4}}(1+2p)^{\frac{1}{12}}(2+p)^{\frac{1}{12}}k^{\frac{1}{2}} \tag{3.25}$$

and

$$\prod_{n=1}^{\infty}(1-q^{12n}) = q^{-\frac{1}{2}}2^{-\frac{2}{3}}p^{\frac{1}{2}}(1-p)^{\frac{1}{24}}(1+p)^{\frac{1}{8}}(1+2p)^{\frac{1}{24}}(2+p)^{\frac{1}{6}}k^{\frac{1}{2}}. \quad (3.26)$$

Using (3.21)-(3.26) we obtained the following identities.

$$\eta^{12}(2z) = \frac{1}{16}p(1-p)^3(1+p)(1+2p)^3(2+p)^3k^6, \quad (3.27)$$

$$\eta^5(z)\eta^5(2z)\eta(3z)\eta(6z) = \frac{1}{8}p(1-p)^4(1+p)^2(1+2p)^2(2+p)^2k^6, \quad (3.28)$$

$$\eta^6(z)\eta^6(3z) = \frac{1}{4}p(1-p)^4(1+p)^4(1+2p)(2+p)k^6, \quad (3.29)$$

$$\eta^5(2z)\eta^5(4z)\eta(6z)\eta(12z) = \frac{1}{64}p^2(1-p)^2(1+p)(1+2p)^2(2+p)^4k^6, \quad (3.30)$$

$$\eta(z)\eta^5(3z)\eta^5(4z)\eta(12z) = \frac{1}{32}p^2(1-p)^2(1+p)^3(1+2p)(2+p)^3k^6, \quad (3.31)$$

$$\eta(z)\eta(2z)\eta^5(3z)\eta^5(6z) = \frac{1}{8}p^2(1-p)^2(1+p)^4(1+2p)(2+p)k^6, \quad (3.32)$$

$$\eta^{12}(6z) = \frac{1}{16}p^3(1-p)(1+p)^3(1+2p)(2+p)k^6. \quad (3.33)$$

4. Theta function identities

Theorem 2.

(i) $$a^6(q) = -\frac{1}{26}N(q) + \frac{27}{26}N(q^3) + \frac{216}{13}B_3(q), \quad (4.1)$$

(ii) $$a^5(q)a(q^2) = \frac{-11}{588}N(q) + \frac{8}{147}N(q^2) - \frac{99}{196}N(q^3) + \frac{72}{49}N(q^6) + \frac{144}{7}B_2(q) + \frac{1296}{7}B_6(q), \quad (4.2)$$

(iii) $$a^4(q)a^2(q^2) = -\frac{5}{546}N(q) - \frac{8}{273}N(q^2) + \frac{45}{182}N(q^3) + \frac{72}{91}N(q^6) - \frac{288}{13}B_2(q) + \frac{540}{13}B_3(q) + \frac{2592}{13}B_6(q), \quad (4.3)$$

(iv) $$a^3(q)a^3(q^2) = -\frac{1}{196}N(q) + \frac{2}{49}N(q^2) - \frac{27}{196}N(q^3) + \frac{54}{49}N(q^6) + \frac{108}{7}B_2(q) + \frac{972}{7}B_6(q), \tag{4.4}$$

(v) $$a^2(q)a^4(q^2) = -\frac{1}{546}N(q) - \frac{10}{273}N(q^2) + \frac{9}{182}N(q^3) + \frac{90}{91}N(q^6) - \frac{126}{13}B_2(q) + \frac{270}{13}B_3(q) + \frac{1134}{13}B_6(q), \tag{4.5}$$

(vi) $$a(q)a^5(q^2) = -\frac{1}{588}N(q) + \frac{11}{294}N(q^2) - \frac{9}{196}N(q^3) + \frac{99}{98}N(q^6) + \frac{36}{7}B_2(q) + \frac{324}{7}B_6(q), \tag{4.6}$$

(vii) $$a^5(q)a(q^4) = -\frac{11}{1092}N(q) + \frac{11}{364}N(q^2) + \frac{99}{364}N(q^3) - \frac{16}{273}N(q^4) - \frac{297}{364}N(q^6) + \frac{144}{91}N(q^{12}) + \frac{774}{13}B_1(q) - \frac{1260}{13}B_2(q) + \frac{810}{13}B_3(q) - \frac{16128}{13}B_4(q) + \frac{24192}{13}B_5(q) - \frac{6804}{13}B_6(q) + \frac{33534}{13}B_7(q), \tag{4.7}$$

(viii) $$a^4(q)a^2(q^4) = -\frac{5}{2184}N(q) - \frac{5}{728}N(q^2) + \frac{45}{728}N(q^3) - \frac{8}{273}N(q^4) + \frac{135}{728}N(q^6) + \frac{72}{91}N(q^{12}) + \frac{387}{13}B_1(q) - \frac{846}{13}B_2(q) + \frac{756}{13}B_3(q) - \frac{8064}{13}B_4(q) + \frac{12096}{13}B_5(q) - \frac{1458}{13}B_6(q) + \frac{16767}{13}B_7(q), \tag{4.8}$$

(ix) $$a^3(q)a^3(q^4) = -\frac{1}{1456}N(q) + \frac{9}{1456}N(q^2) + \frac{27}{1456}N(q^3) - \frac{4}{91}N(q^4) - \frac{243}{1456}N(q^6) + \frac{108}{91}N(q^{12}) + \frac{459}{26}B_1(q) - \frac{3591}{104}B_2(q) + \frac{3591}{104}B_3(q) - \frac{3672}{13}B_4(q) + \frac{5508}{13}B_5(q) - \frac{729}{104}B_6(q) + \frac{12393}{26}B_7(q), \tag{4.9}$$

(x) $$a^2(q)a^4(q^4) = -\frac{1}{8736}N(q) - \frac{5}{2912}N(q^2) + \frac{9}{2912}N(q^3) - \frac{10}{273}N(q^4) + \frac{135}{2912}N(q^6) + \frac{90}{91}N(q^{12}) + \frac{531}{52}B_1(q) - \frac{801}{52}B_2(q) + \frac{891}{52}B_3(q) - \frac{1656}{13}B_4(q) + \frac{2484}{13}B_5(q) - \frac{243}{52}B_6(q) + \frac{8019}{52}B_7(q), \tag{4.10}$$

$$\text{(xi)}\quad a(q)a^5(q^4) = -\frac{1}{17472}N(q) + \frac{11}{5824}N(q^2) + \frac{9}{5824}N(q^3) - \frac{11}{273}N(q^4)$$
$$-\frac{297}{5824}N(q^6) + \frac{99}{91}N(q^{12}) + \frac{99}{26}B_1(q) - \frac{1035}{208}B_2(q) + \frac{1485}{208}B_3(q)$$
$$-\frac{558}{13}B_4(q) + \frac{837}{13}B_5(q) - \frac{729}{208}B_6(q) + \frac{2187}{52}B_7(q), \tag{4.11}$$

$$\text{(xii)}\quad a^4(q)a(q^2)a(q^4) = -\frac{5}{1176}N(q) - \frac{17}{1176}N(q^2) - \frac{45}{392}N(q^3)$$
$$+\frac{8}{147}N(q^4) - \frac{153}{392}N(q^6) + \frac{72}{49}N(q^{12}) + B_1(q) + \frac{36}{7}B_2(q) - \frac{216}{7}B_3(q)$$
$$+\frac{3456}{7}B_5(q) - \frac{3564}{7}B_6(q) + \frac{8991}{7}B_7(q), \tag{4.12}$$

$$\text{(xiii)}\quad a^3(q)a^2(q^2)a(q^4) = -\frac{1}{364}N(q) + \frac{25}{1092}N(q^2) + \frac{27}{364}N(q^3) - \frac{16}{273}N(q^4)$$
$$-\frac{225}{364}N(q^6) + \frac{144}{91}N(q^{12}) + \frac{306}{13}B_1(q) - \frac{909}{26}B_2(q) + \frac{729}{26}B_3(q)$$
$$-\frac{4896}{13}B_4(q) + \frac{7344}{13}B_5(q) - \frac{2835}{26}B_6(q) + \frac{8262}{13}B_7(q), \tag{4.13}$$

$$\text{(xiv)}\quad a^3(q)a(q^2)a^2(q^4) = -\frac{1}{784}N(q) + \frac{23}{2352}N(q^2) - \frac{27}{784}N(q^3) + \frac{4}{147}N(q^4)$$
$$+\frac{207}{784}N(q^6) + \frac{36}{49}N(q^{12}) + \frac{333}{14}B_1(q) + 9B_2(q) - \frac{108}{7}B_3(q)$$
$$+\frac{1728}{7}B_5(q) - \frac{1377}{7}B_6(q) + \frac{8991}{14}B_7(q), \tag{4.14}$$

$$\text{(xv)}\quad a^2(q)a^3(q^2)a(q^4) = -\frac{1}{1176}N(q) - \frac{1}{56}N(q^2) - \frac{9}{392}N(q^3) + \frac{8}{147}N(q^4)$$
$$-\frac{27}{56}N(q^6) + \frac{72}{49}N(q^{12}) + B_1(q) + \frac{27}{7}B_2(q) - \frac{27}{7}B_3(q)$$
$$+\frac{432}{7}B_5(q) - \frac{243}{7}B_6(q) + \frac{2187}{7}B_7(q), \tag{4.15}$$

$$\text{(xvi)}\quad a^2(q)a^2(q^2)a^2(q^4) = -\frac{1}{2184}N(q) - \frac{19}{2184}N(q^2) + \frac{9}{728}N(q^3) - \frac{8}{273}N(q^4)$$
$$+\frac{171}{728}N(q^6) + \frac{72}{91}N(q^{12}) + \frac{153}{13}B_1(q) - \frac{999}{52}B_2(q) + \frac{999}{52}B_3(q)$$
$$-\frac{2448}{13}B_4(q) + \frac{3672}{13}B_5(q) - \frac{2025}{52}B_6(q) + \frac{4131}{13}B_7(q), \tag{4.16}$$

(xvii) $$a^2(q)a(q^2)a^3(q^4) = -\frac{1}{4704}N(q) - \frac{23}{4704}N(q^2) - \frac{9}{1568}N(q^3) + \frac{2}{49}N(q^4) - \frac{207}{1568}N(q^6) + \frac{54}{49}N(q^{12}) + \frac{243}{28}B_1(q) + \frac{171}{28}B_2(q) - \frac{81}{28}B_3(q) + \frac{324}{7}B_5(q) + \frac{81}{28}B_6(q) + \frac{6561}{28}B_7(q), \tag{4.17}$$

(xviii) $$a(q)a^4(q^2)a(q^4) = -\frac{1}{1092}N(q) + \frac{23}{1092}N(q^2) + \frac{9}{364}N(q^3) - \frac{16}{273}N(q^4) - \frac{207}{364}N(q^6) + \frac{144}{91}N(q^{12}) + \frac{72}{13}B_1(q) - \frac{243}{26}B_2(q) + \frac{243}{26}B_3(q) - \frac{1152}{13}B_4(q) + \frac{1728}{13}B_5(q) - \frac{405}{26}B_6(q) + \frac{1944}{13}B_7(q), \tag{4.18}$$

(xix) $$a(q)a^3(q^2)a^2(q^4) = -\frac{1}{2352}N(q) + \frac{1}{112}N(q^2) - \frac{9}{784}N(q^3) + \frac{4}{147}N(q^4) + \frac{27}{112}N(q^6) + \frac{36}{49}N(q^{12}) + \frac{81}{14}B_1(q) + \frac{27}{14}B_2(q) - \frac{27}{14}B_3(q) + \frac{216}{7}B_5(q) - \frac{243}{14}B_6(q) + \frac{2187}{14}B_7(q), \tag{4.19}$$

(xx) $$a(q)a^2(q^2)a^3(q^4) = -\frac{1}{4368}N(q) + \frac{25}{4368}N(q^2) + \frac{9}{1456}N(q^3) - \frac{4}{91}N(q^4) - \frac{225}{1456}N(q^6) + \frac{108}{91}N(q^{12}) + \frac{54}{13}B_1(q) - \frac{63}{8}B_2(q) + \frac{999}{104}B_3(q) - \frac{864}{13}B_4(q) + \frac{1296}{13}B_5(q) - \frac{405}{104}B_6(q) + \frac{1458}{13}B_7(q), \tag{4.20}$$

(xxi) $$a(q)a(q^2)a^4(q^4) = -\frac{1}{9408}N(q) + \frac{17}{9408}N(q^2) - \frac{9}{3136}N(q^3) + \frac{5}{147}N(q^4) + \frac{153}{3136}N(q^6) + \frac{45}{49}N(q^{12}) + \frac{153}{56}B_1(q) + \frac{9}{4}B_2(q) + \frac{27}{28}B_3(q) - \frac{108}{7}B_5(q) + \frac{1053}{28}B_6(q) + \frac{4131}{56}B_7(q). \tag{4.21}$$

Proof. We just prove part (ii). The rest can be proved similarly. From (3.12), (3.13), (3.15), (3.16), (3.17), (3.19), (1.8) and (1.12) we see that

$$\begin{aligned}
&\frac{-11}{588}N(q) + \frac{8}{147}N(q^2) - \frac{99}{196}N(q^3) + \frac{72}{49}N(q^6) \\
&+ \frac{144}{7}\eta^5(z)\eta^5(2z)\eta(3z)\eta(6z) + \frac{1296}{7}\eta(z)\eta(2z)\eta^5(3z)\eta^5(6z) \\
&= (1 + 21p + 186p^2 + 905p^3 + 2655p^4 + 4914p^5 + 5964p^6 + 4914p^7 \\
&+ 2655p^8 + 905p^9 + 186p^{10} + 21p^{11} + p^{12})k^6
\end{aligned}$$

$$= (1+4p+p^2)^5(1+p+p^2)k^6$$
$$= a^5(q)a(q^2).$$

□

5. Proof of Theorem 1

We just prove part (ii) as the remaining parts can be proved similarly.

Proof. From (3.9), (3.12), (3.13) and Theorem 2 we have

$$\sum_{n=0}^{\infty} R(1^5,2^1;n)q^n = a^5(q)a(q^2) = (1+4p+p^2)^5(1+p+p^2)k^6$$
$$= \frac{-11}{588}N(q) + \frac{8}{147}N(q^2) - \frac{99}{196}N(q^3) + \frac{72}{49}N(q^6)$$
$$+\frac{144}{7}\eta^5(z)\eta^5(2z)\eta(3z)\eta(6z) + \frac{1296}{7}\eta(z)\eta(2z)\eta^5(3z)\eta^5(6z)$$
$$= \frac{-11}{588}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right) + \frac{8}{147}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}\right)$$
$$-\frac{99}{196}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^{3n}\right) + \frac{72}{49}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^{6n}\right)$$
$$+\frac{144}{7}\sum_{n=1}^{\infty}b_2(n)q^n + \frac{1296}{7}\sum_{n=1}^{\infty}b_6(n)q^n$$
$$= 1+\sum_{n=1}^{\infty}\left(\frac{66}{7}\sigma_5(n) - \frac{192}{7}\sigma_5\left(\frac{n}{2}\right) + \frac{1782}{7}\sigma_5\left(\frac{n}{3}\right) - \frac{5184}{7}\sigma_5\left(\frac{n}{6}\right)\right.$$
$$\left.+\frac{144}{7}b_2(n) + \frac{1296}{7}b_6(n)\right)q^n.$$

For $n \in \mathbb{N}$, equating the coefficients of q^n on both sides in the above equations, we obtain (2.2). □

Denoting the right hand side of (2.2) by $S(1^5,2^1;n)$ we give the first ten values of $R(1^5,2^1;n)$ and $S(1^5,2^1;n)$ in Table 1 to illustrate the equation.

6. Concluding remark

We note that there is no non-trivial linear relationships between the series $N(q^i)$($i = 1,2,3,4,6$ and 12) and $B_j(q)$, ($j = 1,2,...7$). However, it is clear from (1.7), (1.8), (1.10) and (1.13) that $B_7(q) = B_1(q^3)$ and $B_4(q) = B_2(q^2)$. Thus

$$b_7(n) = b_1(\frac{n}{3}), b_4(n) = b_2(\frac{n}{2}) \tag{6.1}$$

and hence the number of cusp forms used in formulae given in Theorem 1 could be reduced to 5.

TABLE 1. The first ten values of $R(1^5,2^1;n)$ and $S(1^5,2^1;n)$.

n	$R(1^5,2^1;n)$	$\sigma_5(n)$	$\sigma_5(\frac{n}{2})$	$\sigma_5(\frac{n}{3})$	$\sigma_5(\frac{n}{6})$	$b_2(n)$	$b_6(n)$	$S(1^5,2^1;n)$
1	30	1	0	0	0	1	0	30
2	366	33	1	0	0	-5	1	366
3	2370	244	0	1	0	0	-1	2370
4	9390	1057	33	0	0	34	-2	9390
5	28116	3126	0	0	0	-30	-4	28116
6	76146	8052	244	33	1	-81	5	76146
7	162096	16808	0	0	0	68	12	162096
8	291246	33825	1057	0	0	100	-4	291246
9	622830	59293	0	244	0	81	0	622830
10	881460	103158	3126	0	0	-174	-10	881460

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IDENTITIES RELATED TO THE STIRLING NUMBERS AND MODIFIED APOSTOL-TYPE NUMBERS ON UMBRAL CALCULUS

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Abstract. In this paper, by using umbral calculus and umbral algebra methods, we derive several interesting identities and relations related to the modified and unification of the Bernoulli, Euler and Genocchi polynomials and numbers and the generalized (β-) Stirling numbers of the second kind. We also give some applications and remarks related to these numbers and polynomials.

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1. Introduction, definitions and preliminaries

As usual, let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{C}$ denote the sets of positive integers, integers, rational numbers, real numbers, positive real numbers and complex numbers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Assume that $\log z$ denotes the principal branch of the multi-valued function $\log z$ with the imaginary part $\Im(\log z)$ constrained by $-\pi < \Im(\log z) < \pi$. Denote the falling factorial by $(n)_k = n(n-1)\cdots(n-k+1)$ $(k \geq 1)$ with $(n)_0 = 1$.

The unification of the Bernoulli, Euler and Genocchi polynomials is defined by Ozden [17]:

$$g_\beta(x,t;k,a,l) := \frac{2^{1-k}t^k e^{tx}}{\beta^l e^t - a^l} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}(x;k,a,l)\frac{t^n}{n!}. \tag{1.1}$$

If $\beta = a$ then $|t| < 2\pi$, and if $\beta \neq a$, $k \in \mathbb{N}_0$ and $a, l \in \mathbb{C}\backslash\{0\}$ then $|t| < l \log\left(\frac{\beta}{a}\right)$.

Remark 1. Note that Equation (1.1) with $x = 1$ is reduced to the generating functions for the unification of the Bernoulli, Euler and Genocchi numbers.

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Remark 2. Using the special values of a,l,k and β in (1.1), the polynomials $\mathcal{Y}_{n,\beta}(x;k,a,l)$ provide us a generalization and unification of the Apostol-Bernoulli polynomials, Apostol-Euler polynomials and Apostol-Genocchi polynomials, given by

$$\mathcal{B}_n(x,\beta) = \mathcal{Y}_{n,\beta}(x;1,1,1),$$
$$\mathcal{E}_n(x,\beta) = \mathcal{Y}_{n,\beta}(x;0,-1,1)$$

and

$$\mathcal{G}_n(x,\beta) = \mathcal{Y}_{n,\beta}(x;1,-1,1),$$

respectively. Moreover, for the classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$, given by

$$B_n(x) = \mathcal{B}_n(x,1),$$
$$E_n(x) = \mathcal{E}_n(x,1)$$

and

$$G_n(x) = \mathcal{G}_n(x,1),$$

respectively. Substituting $x = 0$ into these polynomials, one also has the classical Bernoulli numbers B_n, the classical Euler numbers E_n and the classical Genocchi numbers G_n, given by

$$B_n = B_n(0),$$
$$E_n = E_n(0),$$

and

$$G_n = G_n(0),$$

respectively. In [9], Bernoulli polynomial can be expressed as an affine combination of the polynomials $(x-1)^n$, $(x-2)^n$, $\ldots$, $(x-k-1)^n$. In [25], generating functions for some families of Genocchi type polynomials were constructed. In [8], a further investigation for the Apostol-Bernoulli polynomials and the Apostol-Genocchi polynomials was presented. In [7], some new formulas for such a kind of sums of the products of an arbitrary number of the Apostol-Bernoulli, Euler, and Genocchi polynomials were established by making use of the generating function methods and summation transform techniques. See [1]-[26] and the references therein.

In [15], Ozden and Simsek modified the polynomials $\mathcal{Y}_{n,\beta}(x;k,a,l)$ as follows:

$$f(t;k,a,b,\beta) = \left(\frac{t^k 2^{1-k}}{\beta b^t - a^t}\right)^v b^{xt} = \sum_{n=0}^{\infty} Y_{n,\beta}^{(v)}(x,k,a,b)\frac{t^n}{n!}, \tag{1.2}$$

where the polynomials $Y_{n,\beta}^{(v)}(x,k,a,b)$ are called *modification and unification of the Apostol-type polynomials of order* v. One easily sees that

$$Y_{n,\beta}^{(v)}(0,k,a,b) = Y_{n,\beta}^{(v)}(k,a,b),$$

which denotes *modification and unification of the Apostol-type numbers* of order v, and

$$\mathcal{Y}_{n,\beta}^{(v)}(x;k,1,1) = Y_{n,\beta}^{(v)}(x,k,1,e)$$

which denotes Apostol-type polynomials (See [16, 17]). In [11], several symmetry identities for generalized Apostol type polynomials were proved by using their generating functions.

Ozden and Simsek [15] gave an explicit formula for the polynomials $Y_{n,\beta}^{(v)}(x,k,a,b)$ as

$$Y_{n,\beta}^{(v)}(x,k,a,b) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} \, (x \ln b)^{n-j} \, Y_{j,\beta}^{(v)}(k,a,b).$$

and also gave the recurrence relation for the numbers $Y_\beta(k,a,b)$ as

$$\beta \left(Y_\beta(k,a,b) + \ln b\right)^n - \left(Y_\beta(k,a,b) + \ln a\right)^n = \begin{cases} 2^{1-k} k! & n = k, \\ 0 & n \neq k, \end{cases}$$

where $\left(Y_\beta(k,a,b)\right)^m$ is replaced by $Y_{m,\beta}(k,a,b)$.

Remark 3. $Y_{m,\beta}(k,a,b)$ is a generalization of the classical Bernoulli, Euler and Genocchi numbers. If we substitute $k = a = \beta = v = 1$ and $b = e$ into (1.2), we have

$$Y_{n,1}^{(1)}(1,1,e) = B_n.$$

If we substitute $k = 0$, $a = v = 1$, $\beta = -1$ and $b = e$ into (1.2), we have

$$Y_{n,-1}^{(1)}(0,1,e) = -E_n.$$

If we substitute $k = a = v = 1$, $\beta = -1$ and $b = e$ into (1.2), we have

$$Y_{n,-1}^{(v)}(1,1,e) = -\frac{1}{2} G_n.$$

The generalized β-Stirling type numbers of the second kind are given by the following definition:

Definition 1 (see [21])**.** Let a, $b \in \mathbb{R}^+$ with $a \neq b$, $\beta \in \mathbb{C}$ and $v \in \mathbb{N}_0$. The generalized Stirling numbers of the second kind $\mathcal{S}(n,v;a,b;\beta)$ are defined by the generating function

$$f_{S,v}(t;a,b;\beta) = \frac{\left(\beta b^t - a^t\right)^v}{v!} = \sum_{n=0}^{\infty} \mathcal{S}(n,v;a,b;\beta) \frac{t^n}{n!}. \tag{1.3}$$

Substituting $a = 1$ and $b = e$ into (1.3), we have the β-Stirling numbers of the second kind

$$\mathcal{S}(n,v;1,e;\beta) = S(n,v;\beta)$$

(See [13, 23, 24]). If $\beta = 1$, then we get the classical Stirling numbers of the second kind by

$$S(n,v;1) = S(n,v)$$

(See [1]-[26]).

We have the following theorem in [22].

Theorem 1. *We have*

$$\mathcal{S}(n,v;a,b;\beta) = \frac{1}{v!}\sum_{j=0}^{v}(-1)^j \binom{v}{j} \beta^{v-j}\left(j\ln a + (v-j)\ln b\right)^n \tag{1.4}$$

and

$$\mathcal{S}(n,v;a,b;\beta) = \frac{1}{v!}\sum_{j=0}^{v}(-1)^{v-j} \binom{v}{j} \beta^{j}\left(j\ln b + (v-j)\ln a\right)^n . \tag{1.5}$$

Remark 4. Note that by setting $a = 1$ and $b = e$ in the identity (1.4), we have

$$S(n,v;\beta) = \frac{1}{v!}\sum_{j=0}^{v}\binom{v}{j}\beta^{v-j}(-1)^j (v-j)^n .$$

This relation has been studied by Srivastava [23] and Luo [13]. If $\beta = 1$, then this relation is further reduced to

$$S(n,v) = \frac{1}{v!}\sum_{j=0}^{v}\binom{v}{j}(-1)^j (v-j)^n$$

(see [1]-[26]).

Definition 2 ([22])**.** Let $a, b \in \mathbb{R}^+$ with $a \neq b$, $x \in \mathbb{R}$, $\beta \in \mathbb{C}$ and $v \in \mathbb{N}_0$. The generalized array type polynomials $\mathcal{S}_v^n(x;a,b;\beta)$ are defined by means of the following generating function:

$$g_v(x,t;a,b;\beta) = \frac{1}{v!}\left(\beta b^t - a^t\right)^v b^{xt} = \sum_{n=0}^{\infty}\mathcal{S}_v^n(x;a,b;\beta)\frac{t^n}{n!}. \tag{1.6}$$

By using (1.6), we have

$$\mathcal{S}_v^n(x;a,b;\beta) = \frac{1}{v!}\sum_{j=0}^{v}(-1)^{v-j}\binom{v}{j}\beta^j\left(\ln\left(a^{v-j}b^{x+j}\right)\right)^n \tag{1.7}$$

(see [22]).

Note that the polynomials $\mathcal{S}_v^n(x;a,b;\beta)$ are called the *generalized λ-array type polynomials*.

Substituting $x = 0$ into (1.7), we get (1.5) by

$$\mathcal{S}_v^n(0;a,b;\beta) = \mathcal{S}(n,v;a,b;\beta).$$

Setting $a = \beta = 1$ and $b = e$ in (1.7), we have

$$S_v^n(x) = \frac{1}{v!}\sum_{j=0}^{v}(-1)^{v-j}\binom{v}{j}(x+j)^n,$$

which is a result in Chang and Ha [2, Eq.(3.1)] and Simsek [21]. It is easy to see that $S_0^0(x) = S_n^n(x) = 1$, $S_0^n(x) = x^n$, and for $v > n$, $S_v^n(x) = 0$ (see [2, Eq.(3.1)]).

Theorem 2 ([22])**.** *We have*

$$\mathcal{S}_v^n(x;a,b;\lambda) = \sum_{j=0}^{n}\binom{n}{j}\mathcal{S}(j,v;a,b;\lambda)\left(\ln b^x\right)^{n-j}. \tag{1.8}$$

The theory of the umbral calculus and the umbral algebra has been applied and expanded by many authors. By using the action of a linear functional on a polynomial Sheffer sequences and Appell sequences, one can obtain many properties of various polynomials. In [4] by using methods of umbral calculus and algebraic nature, some progress in the theory of generating functions involving harmonic numbers were made. In [18] by investigating some umbral-calculus generalizations of the addition formulas, several addition formulas for a general class of Appell sequences are given, involving the generalized Bernoulli, Euler, Genocchi and Srivastava polynomials. In [26] umbral calculus presentations of the Chan-Chyan-Srivastava polynomials and of their substantially more general form, the Erkuş-Srivastava polynomials were investigated.

In this paper, by using umbral calculus and umbral algebra methods, we derive several interesting identities and relations related to the modified and unification of the Bernoulli, Euler and Genocchi polynomials and numbers and the generalized (β-) Stirling numbers of the second kind. We also give some applications and remarks related to these numbers and polynomials.

Some results in this paper are extracted from the preprint [20], and the conference proceeding and survey can be seen in [10].

2. Identities on umbral calculus and umbral algebra

In this section, we give several relations between the modification and unification of the Apostol-type polynomials of order v and the Stirling numbers of the second kind.

We need some identities of the umbral algebra and calculus. The following formulas and notations are given in work of Roman [19].

Let P be the algebra of polynomials in the single variable x over the complex number field. Let P^* be the vector space of all linear functionals on P. Let $\langle L \mid p(x)\rangle$ be the action of a linear functional L on a polynomial $p(x)$. Let F denote the algebra

of formal power series

$$f(t)=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k. \tag{2.1}$$

Let $f\in F$ define a linear functional on P and for all $k\in\mathbb{N}_0$,

$$a_k=\left\langle f(t)\mid x^k\right\rangle. \tag{2.2}$$

The order $o(f(t))$ of a power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. A series $f(t)$ for which $o(f(t))=1$ is called a delta series. And a series $f(t)$ for which $o(f(t))=0$ is called an invertible series.

Let $f(t)$, $g(t)$ be in F. Then we have

$$\langle f(t)g(t)\mid p(x)\rangle=\langle f(t)\mid g(t)p(x)\rangle. \tag{2.3}$$

For all $p(x)$ in P, we have

$$\left\langle e^{yt}\mid p(x)\right\rangle=p(y) \tag{2.4}$$

and

$$e^{yt}p(x)=p(x+y). \tag{2.5}$$

The Sheffer polynomials are defined by the generating function

$$\sum_{k=0}^{\infty}\frac{s_k(x)}{k!}t^k=\frac{1}{g(t)}e^{xt}. \tag{2.6}$$

(see [19, Theorem 2.3.4]).

Theorem 3 ([19, Theorem 2.3.6])**.** *Let $f(t)$ be a delta series and let $g(t)$ be an invertible series. Then there exist a unique sequence $s_n(x)$ of polynomials satisfying the orthogonality condition*

$$\left\langle g(t)f(t)^k\mid s_n(x)\right\rangle=n!\delta_{n,k} \tag{2.7}$$

for all $n,k\in\mathbb{N}_0$.

Let

$$s_n(x)=g(t)^{-1}x^n \tag{2.8}$$

with the derivative formula

$$ts_n(x)=ns_{n-1}(x). \tag{2.9}$$

We can obtain the following lemma.

Lemma 1. *For $n\in\mathbb{N}_0$, we have*

$$Y_{n,\beta}^{(v)}(x;k,a,b)=\left(\frac{2^{1-k}t^k}{\beta b^t-a^t}\right)^v(\ln b)^n x^n.$$

Proof. By (1.2)

$$\sum_{n=0}^{\infty} Y_{n,\beta}^{(v)}(x,k,a,b)\frac{t^n}{n!} = \left(\frac{t^k 2^{1-k}}{\beta b^t - a^t}\right)^v e^{xt\ln b}$$
$$= \left(\frac{t^k 2^{1-k}}{\beta b^t - a^t}\right)^v \sum_{n=0}^{\infty}(x\ln b)^n \frac{t^n}{n!}.$$

Applying (2.2) with (2.1) to both sides of this equation, we get the result. □

Lemma 2. *For $n \in \mathbb{N}_0$, we have*

$$\left\langle (\beta b^t - a^t)^j \mid Y_{n,\beta}(x;k,a,b)\right\rangle$$
$$= \sum_{m=0}^{j}\binom{j}{m}(-1)^{j-m}\beta^m Y_{n,\beta}(m\ln b + (j-m)\ln a, k, a, b).$$

Proof.

$$\left\langle (\beta b^t - a^t)^j \mid Y_{n,\beta}(x;k,a,b)\right\rangle$$
$$= \left\langle \sum_{m=0}^{j}(-1)^{j-m}\binom{j}{m}\beta^m e^{t(m\ln b+(j-m)\ln a)} \mid Y_{n,\beta}(x;k,a,b)\right\rangle$$
$$= \sum_{m=0}^{j}(-1)^{j-m}\binom{j}{m}\beta^m \left\langle e^{t(m\ln b+(j-m)\ln a)} \mid Y_{n,\beta}(x;k,a,b)\right\rangle.$$

Substituting Equation (2.4) into the above equation, we get the desired result. □

Remark 5. We see that

$$\left\langle (\beta b^t - 1)^j \mid Y_{n,\beta}(x;k,1,b)\right\rangle = \left\langle (\beta e^t - 1)^j \mid \mathcal{Y}_{n,\beta}(x,k,1,1)\right\rangle$$

([6, Lemma 2. Eq.(3.1)]).

Lemma 3. *We have*

$$v^n S(n,l) = \frac{1}{l!}\left\langle \left(e^{vt}-1\right)^l \mid x^n\right\rangle,$$

where $S(n,l)$ are the Stirling numbers of the second kind.

Proof. Since

$$\frac{1}{l!}\left(e^{vt}-1\right)^l = \sum_{j=0}^{\infty} v^n S(n,l)\frac{t^n}{n!},$$

by using (2.1) and (2.2), we get the desired result. □

Remark 6. Substituting $v=1$ into Lemma 3, we have

$$S(n,l)=\frac{1}{l!}\left\langle \left(e^t-1\right)^l \mid x^n\right\rangle$$

([19, p.59]).

Theorem 4. *We have*

$$\begin{aligned}
&\sum_{m=0}^{j}\binom{j}{m}(-1)^{j-m}\beta^m Y_{n,\beta}(m\ln b+(j-m)\ln a,k,a,b)\\
&=2^{1-k}(\ln b)^n\beta^{j-1}k!\binom{n}{k}\sum_{l=0}^{j-1}l!\binom{j-1}{l}\left(1-\frac{1}{\beta}\right)^{j-l-1}\\
&\quad\times\sum_{v=0}^{n-k}\binom{n-k}{v}((j-1)\ln a)^{n-k-v}\left(\ln\frac{b}{a}\right)^v S(v,l).
\end{aligned}$$

Proof. By Lemma 1, we get

$$\left\langle (\beta b^t-a^t)^j \mid Y_{n,\beta}(x;k,a,b)\right\rangle=\left\langle (\beta b^t-a^t)^j \mid \frac{2^{1-k}t^k}{\beta b^t-a^t}(\ln b)^n x^n\right\rangle.$$

Substituting (2.3) and (2.9) into this equation and applying Lemma 2, we get

$$\left\langle (\beta b^t-a^t)^j \mid Y_{n,\beta}(x;k,a,b)\right\rangle=2^{1-k}(\ln b)^n\left\langle (\beta b^t-a^t)^{j-1} \mid k!\binom{n}{k}x^{n-k}\right\rangle.$$

After some elementary calculations in this equation, we obtain

$$\begin{aligned}
&\left\langle (\beta b^t-a^t)^j \mid Y_{n,\beta}(x;k,a,b)\right\rangle\\
&=2^{1-k}(\ln b)^n k!\binom{n}{k}\beta^{j-1}\sum_{l=0}^{j-1}\binom{j-1}{l}\left(1-\frac{1}{\beta}\right)^{j-l-1}e^{t((j-1)\ln a)}\\
&\quad\times\left\langle \left(e^{t\left(\ln\frac{b}{a}\right)}-1\right)^l \mid x^{n-k}\right\rangle.
\end{aligned}$$

By applying Lemma 3 with (2.3) in this equation, we obtain the desired result. □

Remark 7. Substituting $a=1$ and $b=e$ into the identity in Theorem 4, we get the work of Dere et al. [6, Theorem 3, Corollary 2]

$$\sum_{m=0}^{j}\binom{j}{m}(-1)^{j-m}\beta^m y_{n,\beta}(m;k,1,1)$$

$$= \frac{\beta^{j-1}}{2^{k-1}} k! \binom{n}{k} \sum_{l=0}^{j-1} \frac{(j-1)!}{(j-l-1)!} \left(1-\frac{1}{\beta}\right)^{j-l-1} S(n-k,l).$$

Remark 8. By setting $\beta = k = a = 1$ and $b = e$ in Theorem 4, we get the following well-known result in [19, P. 94]:

$$\left\langle (e^t-1)^j \mid B_n(x) \right\rangle = n(j-1)!S(n-1,j-1)$$

or

$$\sum_{m=0}^{j} \binom{j}{m} (-1)^{j-m} B_n(m) = n(j-1)!S(n-1,j-1).$$

Substituting $a = 1$, $k = 0$, $\beta = -1$ and $b = e$ into the identity in Theorem 4, we get the following:

Corollary 1.

$$\sum_{m=0}^{j} \binom{j}{m} E_n(m) = \sum_{l=0}^{j-1} \binom{j-l}{l} 2^{j-l} l! S(n,l).$$

Substituting $a = 1$, $k = 1$, $\beta = -1$ and $b = e$ into Theorem 4, we get the following results:

Corollary 2.

$$\sum_{m=0}^{j} \binom{j}{m} G_n(m) = n \sum_{l=0}^{j-1} \binom{j-l}{l} 2^{j-l-2} l! S(n-1,l).$$

Corollary 3.

$$\left\langle (e^t+1)^j \mid G_n(x) \right\rangle = \sum_{m=0}^{j} \binom{j}{m} G_n(m).$$

Proof. In the work of Dere et al. [6, p. 3255], it is proved that

$$\begin{aligned}\left\langle (e^t-1)^j \mid y_{n,-1}(x,1,-1,e) \right\rangle &= \left\langle (e^t+1)^j \mid G_n(x) \right\rangle \\ &= n \sum_{l=0}^{j-1} \binom{j-l}{l} 2^{j-l} l! S(n-1,l).\end{aligned}$$

Combining this equation with Corollary 2, we have the desired result. □

Theorem 5. (Recurrence relation) *For $v \geq 2$, we have*

$$\begin{aligned}\beta Y_{n,\beta}^{(v)}(x+\ln b;k,a,b) &= Y_{n,\beta}^{(v)}(x+\ln a;k,a,b) \\ &\quad + 2^{1-k}(n)_k (\ln b)^k Y_{n-k,\beta}^{(v-1)}(x;k,a,b).\end{aligned}$$

Proof. By using Lemma 1, we get

$$(\beta b^t - a^t) Y_{n,\beta}^{(v)}(x;k,a,b) = 2^{1-k}(n)_k (\ln b)^k Y_{n-k,\beta}^{(v-1)}(x;k,a,b). \tag{2.10}$$

By using (2.5), we obtain

$$(\beta b^t - a^t) Y_{n,\beta}^{(v)}(x;k,a,b) = \beta Y_{n,\beta}^{(v)}(x+\ln b;k,a,b) - Y_{n,\beta}^{(v)}(x+\ln a;k,a,b). \tag{2.11}$$

By combining (2.10) and (2.11), we get the desired result. □

Remark 9. If we set $a = 1$ and $b = e$ in Theorem 5, we obtain

$$\beta \mathcal{Y}_{n,\beta}^{(v)}(x+1;k,1,1) = \mathcal{Y}_{n,\beta}^{(v)}(x;k,1,1) + 2^{1-k}(n)_k \mathcal{Y}_{n-k,\beta}^{(v-1)}(x;k,1,1)$$

([6, Theorem 6]).

Remark 10. By substituting $a = 1$, $k = 0$, $b = e$ and $\beta = -1$ into the identity in Theorem 5, we get the recurrence relations for the Euler polynomials of higher-order:

$$E_n^{(v)}(x+1) = -E_n^{(v)}(x) + 2E_n^{(v-1)}(x) \quad (v \geq 2,\ n \in \mathbb{N})$$

([6], [19, p.103]).

Remark 11. By substituting $a = k = 1$, $b = e$ and $\beta = -1$ into the identity in Theorem 5, we have the recurrence relation for the Genocchi polynomials of higher-order:

$$(e^t + 1) G_n^{(v)}(x+1) = 2n G_{n-1}^{(v-1)}(x) \quad (v \geq 2,\ n \in \mathbb{N})$$

([5, Theorem7]).

Remark 12. By substituting $a = k = 1$, $b = e$ and $\beta = 1$ into the identity in Theorem 5, we have the recurrence relation for the Bernoulli polynomials of higher-order:

$$B_n^{(v)}(x+1) = B_n^{(v)}(x) + n B_{n-1}^{(v-1)}(x) \quad (v \geq 2,\ n \in \mathbb{N})$$

([6] and [19, Eq.(4.2.6)]).

By using Lemma 1 with (2.10), for $v = 1$, we get the following theorem, which is very useful in the theory of the Diophantine equation:

Theorem 6. *For $n, k \in \mathbb{N}_0$ with $n \geq k$, we have*

$$\left(\beta b^t - a^t\right) Y_{n,\beta}(x;k,a,b) = 2^{1-k}(n)_k (\ln b)^k x^{n-k}. \tag{2.12}$$

Remark 13. Substituting $a = 1$ and $b = e$ into (2.12), we have

$$\beta \mathcal{Y}_{n,\beta}(x+1;k,1,e) - \mathcal{Y}_{n,\beta}(x;k,1,e) = 2^{1-k}(n)_k x^{n-k}$$

([14]). By substituting $a = k = 1$, $b = e$ and $\beta = 1$ into (2.12), we have

$$B_n(x+1) - B_n(x) = n x^{n-1}$$

([14], [19, p.95], [24]). By substituting $a = 1$, $k = 0$, $b = e$ and $\beta = -1$ into (2.12), we have

$$E_n(x+1) + E_n(x) = 2x^n$$

([14], [19, p. 95], [24]). By substituting $a = k = 1$, $b = e$ and $\beta = -1$ into (2.12), we have

$$G_n(x+1) + G_n(x) = 2nx^n$$

([5, Corollary 1], [14]).

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ON THE DEGREE OF APPROXIMATION BY MODIFIED GAMMA OPERATORS

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Abstract. We consider certain linear operators associated with gamma function in polynomial weighted spaces of functions of one variable and study approximation properties of these operators, including theorems on the degree of approximation.

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Keywords: gamma type operator, linear operators, approximation order

1. Introduction

In [11] Lupas and Müller introduced the sequence of linear positive operators $\{G_n\}$ defined by

$$G_n(f;x) = \int_0^\infty g_n(x,u) f\left(\frac{n}{u}\right) du,$$

which is called gamma operator, where $g_n(x,u) = \frac{x^{n+1}}{n!} e^{-xu} u^n$, $x \in \mathbb{R}_+ := (0,\infty)$. Approximation problems for G_n in some function spaces were examined in many papers, for example in [1, 11, 13, 15].

The above operators were modified by several authors (e.g. [3, 4, 12]) which showed that new operators have similar approximation properties to G_n (see [2, 5, 8–10, 14]).

The approximation of functions by gamma type operators

$$L_n(f;x) = \frac{(2n+3)!x^{n+3}}{n!(n+2)!} \int_0^\infty \frac{t^n}{(x+t)^{2n+4}} f(t)\, dt, \quad x \in \mathbb{R}_+$$

in polynomial weighted spaces C_p were studied by Karsli [6]. The space C_p, $p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, is associated with the weight function

$$w_0(x) = 1 \quad \text{and} \quad w_p(x) = \frac{1}{1+x^p}, \quad p \in \mathbb{N},$$

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and consists of all real-valued functions f for which fw_p is uniformly continuous and bounded on $\mathbb{R}_0 = [0,\infty)$. The norm on C_p is defined by

$$||f||_p = \sup_{x\in\mathbb{R}_0} w_p(x)|f(x)|. \tag{1.1}$$

Moreover, if f is right-side continuous at $x = 0$, we define $L_n(f;0) = f(0)$, $n \in \mathbb{N}$.

We shall use the modulus of continuity of $f \in C_p$,

$$\omega_p(f,\delta) = \sup_{h\in[0,\delta]} ||\Delta_h f||_p, \quad \delta \geq 0,$$

and the modulus of smoothness of $f \in C_p$,

$$\omega_p^2(f,\delta) = \sup_{h\in[0,\delta]} ||\Delta_h^2 f||_p, \quad \delta \geq 0,$$

where

$$\Delta_h f(x) = f(x+h) - f(x), \quad \Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$$

for $x, h \in \mathbb{R}_0$.

In [6], it was showed that L_n defines a positive linear operator $C_p \to C_p$. For $f \in C_p$, $p \in \mathbb{N}_0$ and $x \in (0,\infty)$, it was proved that

$$w_p(x)\,|L_n(f;x) - f(x)| \leq M_p\,\omega_p^2\left(f, \frac{x}{\sqrt{n+2}}\right) + \omega_p\left(f, \frac{x}{n+2}\right), \tag{1.2}$$

where M_p is a positive constant.

From (1.2) we conclude that if $f \in C_p$, $p \in \mathbb{N}_0$, then

$$\|L_n(f;\cdot) - f\|_p = O\left(n^{-1/2}\right).$$

Thus the question arises, whether the rate of approximation given in the paper [6] cannot be improved. In connection with this question we propose a new family of linear operators. The method was inspired by Kirov [7].

Let D_p, $p \in \mathbb{N}$, be the set of all real-valued continuous functions f on $\mathbb{R}_+$, such that

$$w_p(x)x^k f^{(k)}(x), \quad w_{p-k}(x) f^{(k)}(x), \quad k = 0,1,2,\ldots,p$$

are continuous and bounded on $\mathbb{R}_+$, and $f^{(p)}$ is uniformly continuous on $\mathbb{R}_+$. The norm on D_p is given by (1.1).

We introduce the following class of operators in D_p, $p \in \mathbb{N}$.

Definition 1. Fix $p \in \mathbb{N}$. For functions $f \in D_p$ we define the operators

$$A_n(f;p;x) = \int_0^\infty \frac{(2n+3)!x^{n+3}t^n}{n!(n+2)!(x+t)^{2n+4}} \sum_{j=0}^{p} \frac{f^{(j)}(t)(x-t)^j}{j!}\,dt, \tag{1.3}$$

$x \in \mathbb{R}_+$, $n \in \mathbb{N}$.

Throughout this paper we shall denote by $M_{\alpha,\beta}$ positive constants depending only on indicated parameters α,β, and point out that they are not the same at each occurrence.

2. Auxiliary results

In this section we give some preliminary results which will be used in the rest part of this paper.

In the sequel the following functions will be meaningful:

$$e_m(t) = t^m, \quad \phi_{x,m}(t) = (t-x)^m, \quad m \in \mathbb{N}_0,\ x,t \in \mathbb{R}_0.$$

Using the definition (1.3) and the equality

$$\left(x^n\right)^{(k)} = \frac{n!}{(n-k)!}x^{n-k},$$

we can prove the following lemma.

Lemma 1. *Fix $p \in \mathbb{N}_0$. Then we have*

$$A_n(e_0;p;x) = 1, \tag{2.1}$$

$$A_n(e_m;p;x) = e_m(x) \quad \textit{for} \quad p \le m.$$

In [4] the author obtained the following result.

Lemma 2 ([4])**.** *For any $m \in \mathbb{N}_0$, $m \le n+2$, we have*

$$L_n(e_m;x) = \frac{(n+m)!(n+2-m)!}{n!(n+2)!}x^m \tag{2.2}$$

and there exists a positive constant M_m, $m \in \mathbb{N}_0$, such that

$$L_n\left(\phi_{x,m};x\right) \le M_m \frac{x^m}{n^{[(m+1)/2]}}, \tag{2.3}$$

where $[a]$ denotes the integral part of a.

Lemma 3. *Fix $p \in \mathbb{N}_0$. Then there exists a positive constant M_p such that*

$$\left\|L_n\left(1/w_p;x\right)\right\|_p \le M_p, \quad n \in \mathbb{N}. \tag{2.4}$$

Proof. For $p = 0$ the inequality (2.4) is obvious.

Let $p \in \mathbb{N}$. Using (2.2) we obtain

$$\omega_p(x)|L_n(1/\omega_p;x)| = \frac{1}{1+x^p} + \frac{1}{1+x^p}\cdot\frac{(n+p)!(n+2-p)!}{n!(n+2)!}x^p \le M_p,$$

which gives the assertion. □

Similarly we can prove

Lemma 4. *Fix $p \in \mathbb{N}_0$. Then there exists a positive constant M_p such that*

$$\sup_{x\in\mathbb{R}_+} w_p(x)x^k L_n\left(1/w_{p-k};x\right) \le M_p, \quad n \in \mathbb{N}, k = 0,1,2,\dots,p. \tag{2.5}$$

Next we shall prove the following result.

Lemma 5. *Fix $p \in \mathbb{N}$. Then there exists a positive constant $M_{f,p}$ such that*

$$\|A_n(f;p;\cdot)\|_p \le M_{f,p} \tag{2.6}$$

for all $f \in D_p$ and $n \in \mathbb{N}$.

Formulas (1.3) and (2.6) show that $A_n(f;p)$ is well-defined on the space D_p, $p \in \mathbb{N}$.

Proof. Let $f \in D_p$, $p \in \mathbb{N}$. From this, using the elementary inequality

$$(a+b)^k \le 2^{k-1}(a^k+b^k), \quad a,b \ge 0, k \in \mathbb{N}_0,$$

we obtain

$$|x-t|^k \left|f^{(k)}(t)\right| \le 2^{k-1}\left|f^{(k)}(t)\right|\left(x^k+t^k\right) \le M_{f,p,k}\left(\frac{1}{w_p(t)}+\frac{x^k}{w_{p-k}(t)}\right),$$

$x,t \in \mathbb{R}_+$, $k = 0,1,2,\dots,p$. By linearity of A_n we have

$$\begin{aligned} &w_p(x)|A_n(f;p;x)| \\ &\quad \le M_{f,p}w_p(x)\int_0^\infty \frac{(2n+3)!x^{n+3}t^n}{n!(n+2)!(x+t)^{2n+4}}\left\{\frac{1}{w_p(t)}+\sum_{j=0}^{p}\frac{x^j}{w_{p-j}(t)}\right\}dt \\ &\quad = M_{f,p}w_p(x)\left\{L_n\left(1/w_p;x\right)+\sum_{j=0}^{p}x^jL_n\left(1/w_{p-j};x\right)\right\}. \end{aligned}$$

Thus, using (2.4) and (2.5) we conclude that

$$w_p(x)|A_n(f;p;x)| \le M_{f,p},$$

which gives the result. □

3. Main results

In this section we give theorems on the degree of approximation of the function $f \in D_p$, $p \in \mathbb{N}$, by the operators $A_n(f;p)$.

Theorem 1. *Fix $p \in \mathbb{N}_0$. Then there exists a positive constant M_p such that, for every $f \in D_{2p+1}$, we have*

$$\|A_n(f;2p+1;\cdot)-f\|_{2p+1} \le \frac{M_p}{n^{p+\frac{1}{2}}}\|f^{(2p+1)}\|_0, \quad n \in \mathbb{N}.$$

Proof. Let $f \in D_{2p+1}$, $p \in \mathbb{N}_0$. This implies that $f^{(2p+1)} \in C_0$. Observe that we can write

$$f(x) = \sum_{j=0}^{2p+1} \frac{f^{(j)}(t)(x-t)^j}{j!} + \frac{(x-t)^{2p+1}}{(2p)!} \int_0^1 (1-s)^{2p} \left\{ f^{(2p+1)}(t+s(x-t)) - f^{(2p+1)}(t) \right\} ds.$$

Using (2.1) and (1.3), we obtain

$$\begin{aligned} & w_{2p+1}(x)|A_n(f;2p+1;x) - f(x)| \\ & \le w_{2p+1}(x) \int_0^\infty \frac{(2n+3)!x^{n+3}t^n}{n!(n+2)!(x+t)^{2n+4}} \left| \sum_{j=0}^{2p+1} \frac{f^{(j)}(t)(x-t)^j}{j!} - f(x) \right| dt \\ & \le w_{2p+1}(x) \int_0^\infty \frac{(2n+3)!x^{n+3}t^n}{n!(n+2)!(x+t)^{2n+4}} \cdot \frac{|x-t|^{2p+1}}{(2p)!} \\ & \quad \times \int_0^1 (1-s)^{2p} \left\{ \left| f^{(2p+1)}(t+s(x-t)) \right| + \left| f^{(2p+1)}(t) \right| \right\} ds\, dt. \end{aligned}$$

By our assumption we have

$$\begin{aligned} & w_{2p+1}(x)|A_n(f;2p+1;x) - f(x)| \\ & \le M_p w_{2p+1}(x) \| f^{(2p+1)} \|_0 \int_0^\infty \frac{(2n+3)!x^{n+3}t^n}{n!(n+2)!(x+t)^{2n+4}} \cdot \frac{|x-t|^{2p+1}}{(2p+1)!} dt \\ & \le M_p w_{2p+1}(x) \| f^{(2p+1)} \|_0 L_n \left(|\phi_{x,2p+1}|; x \right). \end{aligned} \tag{3.1}$$

Moreover, by the Cauchy-Schwarz inequality and using (2.3), we get

$$\begin{aligned} w_{2p+1}(x) L_n \left(|\phi_{x,2p+1}|; x \right) & \le w_{2p+1}(x) \left(L_n \left(\phi_{x,4p+2}; x \right) \right)^{1/2} \\ & \le \left(M_p \frac{x^{4p+2}}{(1+x^{2p+1})^2} \cdot \frac{1}{n^{[(4p+3)/2]}} \right)^{1/2} \\ & \le \frac{M_p}{n^{p+\frac{1}{2}}}. \end{aligned} \tag{3.2}$$

Combining (3.2) with (3.1) we immediately conclude

$$\| A_n(f;2p+1;\cdot) - f \|_{2p+1} \le \frac{M_p}{n^{p+\frac{1}{2}}} \| f^{(2p+1)} \|_0$$

for $n \in \mathbb{N}$ and $p \in \mathbb{N}_0$. The theorem is proved. □

Theorem 2. *Fix $p \in \mathbb{N}_0$. Then there exists a positive constant M_p such that, for every $f \in D_{2p+2}$, we have*

$$\|A_n(f;2p+2;\cdot)-f\|_{2p+2} \le \frac{M_p}{n^{p+1}}\|f^{(2p+2)}\|_0, \quad n \in \mathbb{N}.$$

Proof. Let $f \in D_{2p+2}$, $p \in \mathbb{N}_0$. Then $f^{(2p+2)} \in C_0$. Similarly as in the proof of Theorem 1, we can write

$$\begin{aligned} &w_{2p+2}(x)|A_n(f;2p+2;x)-f(x)| \\ &\quad \le w_{2p+2}(x)\int_0^\infty \frac{(2n+3)!x^{n+3}t^n}{n!(n+2)!(x+t)^{2n+4}}\cdot\frac{(x-t)^{2p+2}}{(2p+1)!} \\ &\qquad \times \int_0^1 (1-s)^{2p+1}\left\{\left|f^{(2p+2)}(t+s(x-t))\right|+\left|f^{(2p+2)}(t)\right|\right\}ds\,dt. \end{aligned}$$

From this, by our assumption and using (2.3), we obtain

$$\begin{aligned} &w_{2p+2}(x)|A_n(f;2p+2;x)-f(x)| \\ &\quad \le M_p w_{2p+2}(x)\|f^{(2p+2)}\|_0\int_0^\infty \frac{(2n+3)!x^{n+3}t^n}{n!(n+2)!(x+t)^{2n+4}}(x-t)^{2p+2}dt \\ &\quad = M_p w_{2p+2}(x)\|f^{(2p+2)}\|_0 L_n\left(\phi_{x,2p+2};x\right) \\ &\quad \le M_p\|f^{(2p+2)}\|_0\frac{x^{2p+2}}{1+x^{2p+2}}\cdot\frac{1}{n^{[(2p+3)/2]}} \le \frac{M_p}{n^{p+1}}\|f^{(2p+2)}\|_0 \end{aligned}$$

for $x \in \mathbb{R}_+$, $n \in \mathbb{N}$ and $p \in \mathbb{N}_0$. This completes the proof of Theorem 2. □

Corollary 1. *For every fixed $f \in D_p$, $p \in \mathbb{N}$ we have*

$$\|A_n(f;p;\cdot)-f\|_p = O\left(n^{-p/2}\right).$$

Remark 1. Corollary 1 shows that the operators A_n, $n \in \mathbb{N}$, give a better order of approximation of functions $f \in D_p$, $p \in \mathbb{N}$, than L_n.

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A COMMUTATIVITY CRITERION FOR δ-PRIME RINGS

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Abstract. We prove that under some natural assumptions, a δ-prime ring is commutative whenever so is the range of the derivation δ. We also provide a few useful observations on δ-prime rings, related to the main theorem.

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1. Preliminaries and introduction

Throughout the present note, R is an associative ring, possibly without identity, and $\delta : R \longrightarrow R$ is a derivation. The latter means that the map δ is additive and satisfies the Leibniz rule

$$\forall\, x, y \in R : \delta(xy) = \delta(x)y + x\delta(y).$$

We will denote by $\mathbb{N}$ the set of non-negative integers (and by $\mathbb{Z}$ the set of all integers). We define

$$\delta^n = \begin{cases} \mathrm{id}_R, & \text{if } n = 0, \\ \underbrace{\delta \circ \ldots \circ \delta}_{n}, & \text{if } n \in \mathbb{N} \setminus \{0\}. \end{cases}$$

The commutator of elements $x, y \in R$ is defined by $[x, y] = xy - yx$. Let us recall that a set $E \subseteq R$ is said to be δ-*stable*, if $\delta(E) \subseteq E$. A left, right, or two-sided ideal I of the ring R is called a δ-*ideal*, if it is a δ-stable set. Observe that the intersection of an arbitrary family of two-sided δ-ideals of the ring R is again a two-sided δ-ideal of R. This fact enables us to consider two-sided δ-ideals generated by subsets of R. In the latter two sentences, the word "two-sided" can obviously be replaced by "left" or "right". The ring R is said to be 2-*torsion free*, if

$$\forall\, x \in R : 2x = 0 \Longrightarrow x = 0.$$

The centralizer, the left annihilator and the right annihilator of a set $E \subseteq R$ are defined by

$$\mathrm{C}_R(E) = \{a \in R : ax = xa \text{ for all } x \in E\},$$
$$\mathrm{ann}_\ell(E) = \{a \in R : ax = 0 \text{ for all } x \in E\},$$

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$$\mathrm{ann}_r(E) = \{a \in R : xa = 0 \text{ for all } x \in E\},$$

respectively. It is easy to see that $\mathrm{ann}_\ell(E)$ is a left ideal of the ring R and $\mathrm{ann}_r(E)$ is a right ideal of R. Moreover, if E is δ-stable, then both annihilators are δ-ideals. We denote by $\mathrm{Z}(R)$ the center of R, i.e.,

$$\mathrm{Z}(R) = \mathrm{C}_R(R) = \{a \in R : ax = xa \text{ for all } x \in R\}.$$

The present note is concerned with δ-prime rings. Let us recall that the ring R is said to be *δ-prime*, if it is nonzero and the product of any two its nonzero two-sided δ-ideals is again nonzero. Notice that R is prime if and only if it is 0-prime (i.e., δ-prime with δ being the zero derivation). A typical example of a δ-prime ring which is not even semiprime can be found, for instance, in [8]. The theorem below provides among other things a useful characterization of δ-prime rings in terms of annihilators. For the proof of the theorem we refer to [9].

Theorem 1. *Suppose that R is a nonzero ring. The following conditions are equivalent:*

(1) *R is δ-prime,*
(2) *for any $a, b \in R$, if $\forall\, n \in \mathbb{N} : aR\delta^n(b) = \{0\}$, then $a = 0$ or $b = 0$,*
(3) *for any $a, b \in R$, if $\forall\, n \in \mathbb{N} : \delta^n(a)Rb = \{0\}$, then $a = 0$ or $b = 0$,*
(4) *for any $a, b \in R$, if the product of the two-sided δ-ideal of R generated by a and the two-sided δ-ideal of R generated by b is equal to $\{0\}$, then $a = 0$ or $b = 0$,*
(5) *the product of any two nonzero left δ-ideals of R is again nonzero,*
(6) *the product of any two nonzero right δ-ideals of R is again nonzero,*
(7) *for an arbitrary nonzero left δ-ideal I of the ring R we have $\mathrm{ann}_\ell(I) = \{0\}$,*
(8) *for an arbitrary nonzero right δ-ideal J of the ring R we have $\mathrm{ann}_r(J) = \{0\}$.*

Some further information about δ-prime rings can be found in [2]. For an exhaustive historical account on derivations in abstract algebra we refer to [3].

Derivations on rings such that the elements in the range commute have been studied extensively over the years. The study of such derivations on prime rings of characteristic different from 2 was initiated by Herstein in [6]. The results of Herstein were then generalized in Daif's paper [4] to semiprime rings. At the present time, a considerable number of various modifications and extensions of Herstein and Daif's original results are known (see, for instance, [1, 5, 11]; we also refer to [10] for a general survey of commutativity conditions for rings). Most of these modifications and extensions require the underlying ring to be prime or at least semiprime. The present note generalizes some of the results in [6, 7] to δ-prime rings. Our main purpose is to prove that if the ring R is δ-prime, then under certain natural assumptions the commutativity of $\delta(R)$ implies the commutativity of R.

2. The range of a derivation

Let us begin with a slightly enhanced version of [7, Theorem 1].

Proposition 1. *Suppose that $\delta^3 \neq 0$. Then the subring S of the ring R generated by $\delta(R)$ contains a nonzero two-sided δ-ideal of R.*

Proof. Let I be the two-sided δ-ideal of R generated by the set $\delta^3(R)$. Obviously, $I \neq \{0\}$. We will show that $I \subseteq S$. Observe first that I coincides with the totality of finite sums of elements of the set $A \cup B \cup C \cup \delta^3(R)$, where $A = \{a\delta^3(x) : a, x \in R\}$, $B = \{\delta^3(x)b : b, x \in R\}$ and $C = \{a\delta^3(x)b : a, b, x \in R\}$. Consequently, it is enough to prove that $A \cup B \cup C \cup \delta^3(R) \subseteq S$. Notice that $\delta^3(R) \subseteq \delta(R)$. Now, pick any $r, y \in R$. Then

$$\delta(r\delta(y)) = \delta(r)\delta(y) + r\delta^2(y).$$

Since $\delta(r\delta(y))$ and $\delta(r)\delta(y)$ belong to the subring S, we therefore get $r\delta^2(y) \in S$. In the same way one can prove that $\delta^2(y)r \in S$. Hence

$$A = \{a\delta^2(\delta(x)) : a, x \in R\} \subseteq S,$$

and analogously $B \subseteq S$. It remains to show that $C \subseteq S$. For arbitrary $a, b, x \in R$ we have

$$\delta(a\delta^2(x)b) = \delta(a)\delta^2(x)b + a\delta^3(x)b + a\delta^2(x)\delta(b).$$

Since $a\delta^2(x)$ and $\delta^2(x)b$ are elements of S, the above equality implies that $a\delta^3(x)b \in S$. The desired inclusion follows. □

The slight enhancement thus consists in replacing a "usual" ideal of the ring R with a δ-ideal of R. However, the proof which we have presented is in fact Herstein's original proof, with a few technical modifications. (We include this proof for the sake of completness). This is because the "usual" two-sided ideal of R generated by $\delta^3(R)$ is δ-stable.

3. Auxiliary results

We now start with a simple lemma concerning two derivations.

Lemma 1. *Let I be a nonzero left or right δ-ideal of the ring R. Moreover, let $d : R \longrightarrow R$ be a derivation. Suppose that R is δ-prime and d vanishes on I. Then $d = 0$.*

Proof. Assume that I is a left δ-ideal. Pick any $a \in I$ and any $x \in R$. Since $RI \subseteq I$, the Leibniz rule yields $0 = d(xa) = d(x)a$. Consequently, $d(R) \subseteq \mathrm{ann}_\ell(I)$. It follows therefore from the δ-primeness that $d(R) = \{0\}$. If I is a right δ-ideal, we can proceed analogously. □

Recall that for any fixed $a \in R$, the map $\partial_a : R \ni x \longmapsto [a, x] \in R$ is a derivation. This derivation is referred to as the *inner derivation* of the ring R induced by a.

Lemma 2. *Let I be as in the previous lemma. Suppose, again, that the ring R is δ-prime. Then R is commutative whenever so is I.*

Proof. Assume that the δ-ideal I is commutative. Pick an arbitrary $a \in I$. By the commutativity of I, the inner derivation $\partial_a : R \longrightarrow R$ vanishes on I. The previous lemma therefore implies that $\partial_a(x) = [a,x] = 0$ for any $x \in R$. Thus we have $I \subseteq \mathrm{Z}(R)$. Now, pick an arbitrary $b \in R$. Since $I \subseteq \mathrm{Z}(R)$, the inner derivation ∂_b vanishes on I. Consequently, $[b,x] = 0$ for all $x \in R$. The commutativity of R follows. □

The next lemma is a generalized version of [6, Lemma 1.1.7].

Lemma 3. *Let $u \in R$ and $k \in \mathbb{N}$. Then the set*

$$A_\delta^k(u) = \mathrm{ann}_\ell(\{[\delta^m(u), x] : x \in R, m \in \mathbb{N}, m \geqslant k\})$$

is a two-sided δ-ideal of the ring R. Moreover, if R is δ-prime and $A_\delta^0(u) \neq \{0\}$, then $u \in \mathrm{Z}(R)$.

Proof. Obviously, $A_\delta^k(u)$ is a left ideal of R. Pick arbitrary elements $a \in A_\delta^k(u)$ and $x, y \in R$. Let $m \in \mathbb{N}$ be such that $m \geqslant k$. Then we have

$$0 = a[\delta^m(u), yx] = a[\delta^m(u), y]x + ay[\delta^m(u), x] = ay[\delta^m(u), x].$$

This yields that $ay \in A_\delta^k(u)$ for any $a \in A_\delta^k(u)$ and any $y \in R$. Hence $A_\delta^k(u)$ is a right ideal of R. We will show that $A_\delta^k(u)$ is δ-stable. If $a \in A_\delta^k(u)$, $x \in R$, $m \in \mathbb{N}$ and $m \geqslant k$, then

$$\begin{aligned} 0 &= \delta(a[\delta^m(u), x]) = \\ &= \delta(a)[\delta^m(u), x] + a[\delta^{m+1}(u), x] + a[\delta^m(u), \delta(x)] = \\ &= \delta(a)[\delta^m(u), x]. \end{aligned}$$

Consequently, $\delta(a) \in A_\delta^k(u)$ for an arbitrary $a \in A_\delta^k(u)$.

Let us turn to the "moreover" part. Suppose that R is a δ-prime ring and $A_\delta^0(u) \neq \{0\}$. Since $A_\delta^0(u)$ is a two-sided δ-ideal of R, it follows that $\mathrm{ann}_r(A_\delta^0(u)) = \{0\}$. But

$$\{[u, x] : x \in R\} = \{[\delta^0(u), x] : x \in R\} \subseteq \mathrm{ann}_r(A_\delta^0(u)).$$

Therefore $[u, x] = 0$ for all $x \in R$. This means that $u \in \mathrm{Z}(R)$. □

In [6] (proof of Lemma 1.1.9), Herstein showed that if R is a semiprime ring of characteristic different from 2 and $\delta^2 = 0$, then $\delta = 0$. The first part of the proposition below is a modification of the result: we impose a weaker condition on δ and assume that R is δ-prime. Before starting the proposition, notice that R^2 is a two-sided δ-ideal of the ring R.

Proposition 2. *Let R be a 2-torsion free δ-prime ring.*

(*i*) *If $\delta^2(R^2) = \{0\}$, then $\delta = 0$.*

(*ii*) *If $\delta \neq 0$, then $\mathrm{C}_R(\delta(R)) = \mathrm{Z}(R)$.*

Proof. Suppose that δ^2 vanishes on R^2. Define $K \subseteq R$ to be the totality of finite sums of elements of the form $a\delta(x)b$, where $a, b \in R$ and $x \in R^2$. Then K is a two-sided δ-ideal of the ring R. Now, pick arbitrary $r, s \in R^2$. Since R is 2-torsion free and

$$0 = \delta^2(rs) = \delta^2(r)s + 2\delta(r)\delta(s) + r\delta^2(s) = 2\delta(r)\delta(s),$$

we get $\delta(r)\delta(s) = 0$. Consequently, if $x, y, z \in R^2$, then

$$0 = \delta(x)\delta(zy) = \delta(x)\delta(z)y + \delta(x)z\delta(y) = \delta(x)z\delta(y).$$

It follows that $a\delta(x)bf\delta(y)g = 0$ for any $a, b, f, g \in R$ and any $x, y \in R^2$. This yields $K^2 = \{0\}$. In view of the δ-primeness of R, we have therefore $K = \{0\}$. Again let $x \in R^2$. Since

$$\forall\, a, b \in R : a\delta(x)b = 0,$$

we obtain $\delta(x)R \subseteq \mathrm{ann}_r(R)$. The δ-primeness implies that $\mathrm{ann}_r(R) = \{0\}$. Thus $\delta(x) \in \mathrm{ann}_\ell(R)$. But $\mathrm{ann}_\ell(R) = \{0\}$ too. We have therefore proved that $\delta(R^2) = \{0\}$. Finally, pick an arbitrary $a \in R$ and an arbitrary $x \in R^2$. Then

$$0 = \delta(ax) = \delta(a)x + a\delta(x) = \delta(a)x.$$

It follows that $\delta(R) \subseteq \mathrm{ann}_\ell(R^2)$. Using the δ-primeness again, we get $R^2 \neq \{0\}$ and hence $\mathrm{ann}_\ell(R^2) = \{0\}$. We have therefore $\delta = 0$.

Let us turn to property (ii). We will proceed by contraposition. Suppose that $\mathrm{C}_R(\delta(R)) \neq \mathrm{Z}(R)$. In other words, there is an element $u \in R \setminus \mathrm{Z}(R)$ such that $[u, \delta(a)] = 0$ for any $a \in R$. Define $C = \mathrm{C}_R(\{\delta^m(u) : m \in \mathbb{N}\})$. An easy induction shows that

$$\forall\, m \in \mathbb{N}\, \forall\, a \in R : [\delta^m(u), \delta(a)] = 0.$$

Hence $\delta(R) \subseteq C$. Moreover, if $x \in R$, $c \in C$ and $m \in \mathbb{N}$, then

$$\begin{aligned} 0 &= [\delta^m(u), \delta(cx)] = [\delta^m(u), \delta(c)x + c\delta(x)] = [\delta^m(u), \delta(c)x] + [\delta^m(u), c\delta(x)] = \\ &= [\delta^m(u), \delta(c)]x + \delta(c)[\delta^m(u), x] + [\delta^m(u), c]\delta(x) + c[\delta^m(u), \delta(x)] = \\ &= \delta(c)[\delta^m(u), x]. \end{aligned}$$

This yields that $\delta(C) \subseteq A^0_\delta(u)$. Since $u \notin \mathrm{Z}(R)$, it follows from Lemma 3 that $A^0_\delta(u) = \{0\}$. Consequently, $\delta(C) = \{0\}$. We therefore obtain

$$\delta^2(R^2) \subseteq \delta^2(R) = \delta(\delta(R)) \subseteq \delta(C) = \{0\}$$

(because $\delta(R) \subseteq C$). Thus by property (i), we have $\delta = 0$. □

4. The main theorem

Our main result is a generalization of Herstein's [7, Theorem 2].

Theorem 2. *Let R be a δ-prime ring. Suppose that*

$$\forall\, x, y \in R : \delta(x)\delta(y) = \delta(y)\delta(x).$$

Then R is commutative whenever one of the following conditions is satisfied:
(1) $\delta^3 \neq 0$,
(2) $\delta \neq 0$ *and R is 2-torsion free.*

Proof. Assume that condition (1) is satisfied. Let S be the subring of the ring R generated by $\delta(R)$. Proposition 1 guarantees that there exists a nonzero two-sided δ-ideal I of the ring R contained in S. Since the elements of $\delta(R)$ commute pairwise, the subring S is commutative. Therefore so is I. The commutativity of the ring R now follows from Lemma 2.

Assume that condition (2) is satisfied. Since the elements of $\delta(R)$ commute pairwise, we have $\delta(R) \subseteq \mathrm{C}_R(\delta(R))$. Consequently, Proposition 2 (ii) yields $\delta(R) \subseteq \mathrm{Z}(R)$. Notice that $E = \{x \in R : \delta(x) \neq 0\}$ is a nonempty set and pick any $x \in E$. Let $y \in R$. Since $\delta(y) \in \mathrm{Z}(R)$ and $\delta(xy) \in \mathrm{Z}(R)$, we get

$$\begin{aligned}\delta(x)yx + x^2\delta(y) = \delta(x)yx + x\delta(y)x = \delta(xy)x = x\delta(xy) = \\ = x\delta(x)y + x^2\delta(y).\end{aligned}$$

It follows that $\delta(x)yx = x\delta(x)y$. Hence

$$\delta(x)[x, y] = \delta(x)xy - \delta(x)yx = x\delta(x)y - x\delta(x)y = 0.$$

An easy inductive argument now shows that

$$(*) \quad \forall\, n \in \mathbb{N} \setminus \{0\}\, \forall\, y \in R : \delta^n(x)[x, y] = 0.$$

Consider the two-sided δ-ideal J of the ring R generated by $\delta(x)$. Notice that $J \neq \{0\}$. Since $\delta^n(x) \in \mathrm{Z}(R)$ for all $n \in \mathbb{N} \setminus \{0\}$, the ideal J coincides with the totality of finite sums of elements of the set $A \cup B$, where

$$A = \{a\delta^n(x) : a \in R, n \in \mathbb{N} \setminus \{0\}\}, \quad B = \{k\delta^n(x) : k \in \mathbb{Z}, n \in \mathbb{N} \setminus \{0\}\}$$

(cf. [9, Proposition 2.6]). Property $(*)$ therefore yields $\{[x, y] : y \in R\} \subseteq \mathrm{ann}_r(J)$. It follows now from the δ-primeness of R that $[x, y] = 0$ for all $y \in R$. Hence we have proved that

$$(**) \quad \forall\, x \in R : \delta(x) \neq 0 \Longrightarrow x \in \mathrm{Z}(R).$$

Finally, let $z \in R$ be such that $\delta(z) = 0$. Pick some $x_0 \in E$. Then $\delta(z + x_0) \neq 0$. By property $(**)$, we therefore get $z + x_0 \in \mathrm{Z}(R)$. Since x_0 also belongs to $\mathrm{Z}(R)$, we obtain that $z \in \mathrm{Z}(R)$. The commutativity of the ring R follows. $\square$

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SOME ESTIMATES ON THE HERMITE-HADAMARD INEQUALITY THROUGH GEOMETRICALLY QUASI-CONVEX FUNCTIONS

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Abstract. In present paper, by proving a new integral identity, using Hölder's inequality and mathematical analysis, we prove some Hermite-Hadamard type integral inequalities for geometrically quasi-convex functions which give better estimates to those already proven for the right-side of a Hermite-Hadamard type inequality established for geometrically convex functions in earlier works. A numerical example is also provided to support our claim. Applications of the results to special means of positive real numbers are given.

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1. Introduction

A function $g : J \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on J if

$$g(\sigma l + (1-\sigma) m) \leq \sigma g(l) + (1-\sigma) g(m)$$

holds for every $l, m \in J$ and $\sigma \in [0,1]$. There are several accomplishments of the role of convexity towards the field of inequalities and in the other branches of pure and applied mathematics but one of them is the celebrated Hermite Hadamard inequality, which is expressed as follows

$$g\left(\frac{r+w}{2}\right) \leq \frac{1}{w-r}\int_r^w g(l)\,dl \leq \frac{g(r)+g(w)}{2}, \tag{1.1}$$

where $g : \varphi \neq J \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function and $r, w \in J$ with $r < w$. Undeniably, the inequality (1.1) specifies a necessary and sufficient conditions for a function g to be convex on the interval $[r, w]$.

The approach of quasi-convex functions speculates the concept of convex functions, that is a function $g : J \subseteq \mathbb{R} \to \mathbb{R}$ is quasi-convex on J if the following relaxation of Jensen's inequality holds

$$g(\sigma l + (1-\sigma) m) \leq \max\{g(l), g(m)\}$$

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for $0 \leq \sigma \leq 1$.

A large number of papers have been written to provide refinements and generalizations of the inequalities (1.1) during the past few years by using convexity, quasi-convexity and their generalizations, see for instance [1–9], [11–17] and the references therein.

Many researchers have tried to generalize the notions of usual convexity and quasi-convexity in a number of approaches, one of the generalizations of the usual convexity is the geometric convexity which is avowed in the definition below.

Definition 1 ([10])**.** A function $g : J \subseteq (0,\infty) \to (0,\infty)$ is said to be geometrically convex on J if

$$g\left(l^{\sigma} m^{1-\sigma}\right) \leq [g(l)]^{\sigma} [g(m)]^{1-\sigma}$$

holds for $x, y \in J$ and $\sigma \in [0,1]$.

The definition was further generalized by Qi and Xi in [12] as follows.

Definition 2 ([12])**.** A function $g : J \subseteq [0,\infty) \to [0,\infty)$ is called geometrically quasi-convex function on J if

$$g\left(l^{\sigma} m^{1-\sigma}\right) \leq \sup\{g(l), g(m)\}$$

holds for $l, m \in J$ and $\sigma \in [0,1]$.

Remark 1. [12] If $g : J \subseteq [0,\infty) \to [0,\infty)$ is decreasing and geometrically quasi-convex on J, then it is quasi-convex on J. If $g : J \subseteq [0,\infty) \to [0,\infty)$ is increasing and quasi-convex on J, then it is geometrically quasi-convex on J.

Example 1. The function $g(l) = l^{s}$, $l \in \mathbb{R}_{+}$ for $s \in \mathbb{R}$ geometrically quasi-convex on $\mathbb{R}_{+}$ because for all $\sigma \in [0,1]$, the following inequality holds

$$\left|g\left(l^{\sigma} m^{1-\sigma}\right)\right|^{s} = l^{s\sigma} m^{s(1-\sigma)} \leq \begin{cases} m^{s}, & s \geq 0 \\ l^{s}, & s < 0. \end{cases}$$

Some of the main results from [12] for geometrically quasi-convex functions are given in following theorems.

Theorem 1 ([12])**.** *Let $g : J \subseteq \mathbb{R}_{+} \to \mathbb{R}$ be a differentiable on J° and $g' \in L([r,w])$ for $r, w \in J^{\circ}$ with $r < w$. If $\left|g'\right|^{q}$ is a convex function on $[r,w]$, then*

$$\left|\frac{(\ln w) g(w) - (\ln r) g(r)}{\ln w - \ln r} - \frac{1}{\ln w - \ln r} \int_{r}^{w} \frac{g(l)}{l} dl\right| \leq N(r,w) \sup\left\{\left|g'(r)\right|, \left|g'(w)\right|\right\}, \quad (1.2)$$

where

$$N(r,w)=\int_0^1 r^{1-\sigma}w^{\sigma}\left|\ln\left(r^{1-\sigma}w^{\sigma}\right)\right|d\sigma$$
$$=\begin{cases}\frac{w\ln w-r\ln r-(w-r)}{2(\ln w-\ln r)}, & r\geq 1,\\ \frac{w\ln w+r\ln r+2-w-r}{\ln w-\ln r}, & r<1<w,\\ \frac{w-r-(w\ln w-r\ln r)}{2(\ln w-\ln r)}, & w\leq 1.\end{cases}$$

Theorem 2 ([12]). *Let $g:J\subseteq\mathbb{R}_+\to\mathbb{R}$ be a differentiable mapping on J° and $g'\in L[r,w]$ for $r,w\in J^\circ$ with $r<w$. If $\left|g'\right|^q$ is a geometrically quasi-convex on $[r,w]$ for $q>1$, then*

$$\left|\frac{(\ln w)g(w)-(\ln r)g(r)}{\ln w-\ln r}-\frac{1}{\ln w-\ln r}\int_r^w\frac{g(l)}{l}dl\right|$$
$$\leq\left\{\frac{q-1}{q}N\left(r^{\frac{q}{q-1}},w^{\frac{q}{q-1}}\right)\right\}^{1-\frac{1}{q}}[M(r,w)]^{\frac{1}{q}}\sup\left\{\left|g'(r)\right|,\left|g'(w)\right|\right\},\quad(1.3)$$

where $N(r,w)$ is defined in Theorem 1 and

$$M(r,w)=\int_0^1\left|\ln\left(r^{1-\sigma}w^{\sigma}\right)\right|d\sigma$$
$$=\begin{cases}\frac{w\ln w+r\ln r}{2}, & r\geq 1,\\ \frac{(\ln w)^2+(\ln r)^2}{\ln w-\ln r}, & r<1<w,\\ -\frac{w\ln w+r\ln r}{2}, & w\leq 1.\end{cases}$$

Theorem 3 ([12]). *Let $g:J\subseteq\mathbb{R}_+\to\mathbb{R}$ be a differentiable on J° and $g'\in L[r,w]$ for $r,w\in J^\circ$ with $r<w$. If $\left|g'\right|^q$ is a geometrically convex on $[r,w]$ for $q>1$ and ℓ is a real number such that $q>\ell>0$. Then*

$$\left|\frac{(\ln w)g(w)-(\ln r)g(r)}{\ln w-\ln r}-\frac{1}{\ln w-\ln r}\int_r^w\frac{g(l)}{l}dl\right|\leq\left(\frac{q-1}{q-\ell}\right)^{1-\frac{1}{q}}\left(\frac{1}{\ell}\right)^{\frac{1}{q}}$$
$$\times\left[N\left(r^{\frac{q-\ell}{q-1}},w^{\frac{q-\ell}{q-1}}\right)\right]^{1-\frac{1}{q}}\left[N\left(r^{\ell},w^{\ell}\right)\right]^{\frac{1}{q}}\sup\left\{\left|g'(r)\right|,\left|g'(w)\right|\right\},\quad(1.4)$$

where $M(r,w)$ and $N(r,w)$ are defined in Theorem 1 and Theorem 2.

The main objective of the present paper is to provide new Hermite-Hadamard type inequalities for geometrically quasi-convex functions which provide refinements of

the results given in [12]. A numerical example is provided to justify our claim. Applications of our results to special means are presented as well in Section 3.

2. Inequalities for geometrically quasi-convex functions

The following two Lemmas are requisite to authorize our major determinations in this section:

Lemma 1. *Suppose $J \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable on J° and $g' \in L([r,w])$, for r, $w \in J^\circ$ and $r < w$, then*

$$\begin{aligned}\frac{(\ln w)g(w)-(\ln r)g(r)}{\ln w-\ln r}-\frac{1}{\ln w-\ln r}\int_r^w \frac{g(l)}{l}\,dl \\ =\frac{1}{2}\int_0^1 r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)g'\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)d\sigma \\ +\frac{1}{2}\int_0^1 r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)g'\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)d\sigma. \quad (2.1)\end{aligned}$$

Proof. Let

$$J_1=\frac{1}{2}\int_0^1 r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)g'\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)d\sigma$$

and

$$J_2=\frac{1}{2}\int_0^1 r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)g'\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)d\sigma.$$

Let $l=r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}$ in J_1 and integrating by parts, we get

$$\begin{aligned}J_1&=\frac{1}{\ln w-\ln r}\int_{\sqrt{rw}}^w(\ln l)\,g'(l)\,dl\\ &=\frac{1}{\ln w-\ln r}\left[(\ln w)\,g(w)-\ln\left(\sqrt{rw}\right)g\left(\sqrt{rw}\right)-\int_{\sqrt{rw}}^w\frac{g(l)}{l}\,dl\right]. \quad (2.2)\end{aligned}$$

Now let $l=r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}$ in J_2 and integrating by parts, we have

$$\begin{aligned}J_2&=\frac{1}{\ln w-\ln r}\int_r^{\sqrt{rw}}(\ln l)\,g'(l)\,dl\\ &=\frac{1}{\ln w-\ln r}\left[\ln\left(\sqrt{rw}\right)g\left(\sqrt{rw}\right)-(\ln r)\,g(r)-\int_r^{\sqrt{rw}}\frac{g(l)}{l}\,dl\right]. \quad (2.3)\end{aligned}$$

By adding (2.3) and (2.2), we get required result. □

Lemma 2. *For $w > r > 0$, we have*

$$\xi(r,w) := \int_0^1 \left|\ln\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right| d\sigma$$
$$= \begin{cases} -\frac{\ln r + 3\ln w}{4}, & r < w < 1, \\ \frac{\ln r + 3\ln w}{4}, & 1 < r < w, \\ \frac{5(\ln w)^2 + (\ln r)^2 + 2(\ln r)(\ln w)}{4(\ln w - \ln r)}, & r < 1 < w, rw < 1, \\ \frac{(\ln w)^2}{\ln w - \ln r}, & r < 1 < w, rw = 1 \end{cases}$$

and

$$\eta(r,w) := \int_0^1 r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}} \left|\ln\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right| d\sigma$$
$$= \begin{cases} \frac{2\left[w - \sqrt{rw} + \sqrt{rw}\ln\left(\sqrt{rw}\right) - w\ln w\right]}{\ln w - \ln r}, & r < w < 1, \\ \frac{2\left[-w + \sqrt{rw} - \sqrt{rw}\ln\left(\sqrt{rw}\right) + w\ln w\right]}{\ln w - \ln r}, & 1 < r < w, \\ \frac{2\left[2 + \sqrt{rw}\ln\left(\sqrt{rw}\right) - \sqrt{rw} + w\ln w - w\right]}{\ln w - \ln r}, & r < 1 < w, rw < 1, \\ \frac{2[w\ln w - w + 1]}{\ln w - \ln r}, & r < 1 < w, rw = 1. \end{cases}$$

Proof. The proof can be done by simple computations. □

Theorem 4. *Let $g : J \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on J° and $g' \in L[r,w]$ for $r, w \in J^\circ$ with $r < w$. If $\left|g'\right|$ is a geometrically quasi-convex on $[r,w]$, then*

$$\left|\frac{(\ln w)g(w) - (\ln r)g(r)}{\ln w - \ln r} - \frac{1}{\ln w - \ln r}\int_r^w \frac{g(l)}{l} dl\right|$$
$$\leq \frac{\eta(r,w)}{2}\left(\sup\left\{\left|g'\left(\sqrt{rw}\right)\right|, \left|g'(w)\right|\right\}\right)$$
$$+ \frac{\eta(w,r)}{2}\left(\sup\left\{\left|g'(r)\right|, \left|g'\left(\sqrt{rw}\right)\right|\right\}\right), \quad (2.4)$$

where $\eta(r,w)$ is defined in Lemma 2.

Proof. From Lemma 1 and using Hölder inequality, It is easy to observe that the following inequality holds

$$\left|\frac{(\ln w)g(w)-(\ln r)g(r)}{\ln w-\ln r}-\frac{1}{\ln w-\ln r}\int_r^w \frac{g(l)}{l}dl\right|$$
$$\leq \frac{1}{2}\int_0^1 r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\left|\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|\left|g'\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|d\sigma$$
$$+\frac{1}{2}\int_0^1 r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\left|\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|\left|g'\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|d\sigma. \quad (2.5)$$

By using the geometric quasi-convexity of $\left|g'\right|$ on $[r,w]$, we have

$$\left|g'\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|\leq \sup\left\{|g'(\sqrt{rw})|,|g'(w)|\right\}$$

and

$$\left|g'\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|\leq \sup\left\{\left|g'(r)\right|,\left|g'(\sqrt{rw})\right|\right\}.$$

By using Lemma 2 and the above results, we obtain

$$\int_0^1 r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\left|\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|\left|g'\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|d\sigma$$
$$\leq \sup\left\{\left|g'(\sqrt{rw})\right|,\left|g'(w)\right|\right\}\int_0^1 r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\left|\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|d\sigma$$
$$=\eta(r,w)\left(\sup\left\{\left|g'(\sqrt{rw})\right|,\left|g'(w)\right|\right\}\right) \quad (2.6)$$

and

$$\int_0^1 r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\left|\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|\left|g'\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|d\sigma$$
$$\leq \sup\left\{\left|g'(r)\right|,\left|g'(\sqrt{rw})\right|\right\}\int_0^1 r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\left|\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|d\sigma$$
$$=\eta(w,r)\left(\sup\left\{\left|g'(r)\right|,\left|g'(\sqrt{rw})\right|\right\}\right). \quad (2.7)$$

The inequality (2.4) can be obtained by using (2.6) and (2.7) in (2.5). □

Theorem 5. *Let $g: J\subseteq \mathbb{R}_+\to\mathbb{R}$ be a differentiable mapping on J° and $g'\in L[r,w]$ for $r,w\in J^\circ$ with $r<w$. If $\left|g'\right|^q$ is a geometrically quasi-convex on $[r,w]$ for $q>1$, then*

$$\left|\frac{(\ln w)g(w)-(\ln r)g(r)}{\ln w-\ln r}-\frac{1}{\ln w-\ln r}\int_r^w \frac{g(l)}{l}dl\right|$$
$$\leq \frac{1}{2}\left\{\frac{q-1}{q}\eta\left(r^{\frac{q}{q-1}},w^{\frac{q}{q-1}}\right)\right\}^{1-\frac{1}{q}}\left[\xi(r,w)\left(\sup\left\{\left|g'(\sqrt{rw})\right|^q,\left|g'(w)\right|^q\right\}\right)\right]^{\frac{1}{q}}$$

$$+\frac{1}{2}\left\{\frac{q-1}{q}\eta\left(w^{\frac{q}{q-1}},r^{\frac{q}{q-1}}\right)\right\}^{1-\frac{1}{q}}\left[\xi(w,r)\left(\sup\left\{\left|g'(w)\right|^{q},\left|g'\left(\sqrt{rw}\right)\right|^{q}\right\}\right)\right]^{\frac{1}{q}}, \tag{2.8}$$

where $\xi(r,w)$ and $\eta(r,w)$ are defined in Lemma 2.

Proof. From Lemma 1 and using Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{(\ln w)g(w)-(\ln r)g(r)}{\ln w-\ln r}-\frac{1}{\ln w-\ln r}\int_r^w\frac{g(l)}{l}dl\right| \\
&\quad\le\frac{1}{2}\int_0^1 r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\left|\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|\left|g'\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|d\sigma \\
&\quad+\frac{1}{2}\int_0^1 r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\left|\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|\left|g'\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|d\sigma \\
&\quad\le\frac{1}{2}\left(\int_0^1 r^{\frac{q(1-\sigma)}{2(q-1)}}w^{\frac{q(1+\sigma)}{2(q-1)}}\left|\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|d\sigma\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1\left|\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|\left|g'\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|^{q}d\sigma\right)^{\frac{1}{q}} \\
&\quad+\frac{1}{2}\left(\int_0^1 r^{\frac{q(1+\sigma)}{2(q-1)}}w^{\frac{q(1-\sigma)}{2(q-1)}}\left|\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|d\sigma\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1\left|\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|\left|g'\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|^{q}d\sigma\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.9}$$

By using the geometric quasi-convexity of $\left|g'\right|^{q}$ on $[r,w]$, we obtain

$$\begin{aligned}
&\int_0^1\left|\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|\left|g'\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|^{q}d\sigma \\
&\quad\le\left(\sup\left\{\left|g'\left(\sqrt{rw}\right)\right|^{q},\left|g'(w)\right|^{q}\right\}\right)\int_0^1\left|\ln\left(r^{\frac{1-\sigma}{2}}w^{\frac{1+\sigma}{2}}\right)\right|d\sigma \\
&\quad=\xi(r,w)\left(\sup\left\{\left|g'\left(\sqrt{rw}\right)\right|^{q},\left|g'(w)\right|^{q}\right\}\right)
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&\int_0^1\left|\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|\left|g'\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|^{q}d\sigma \\
&\quad\le\left(\sup\left\{\left|g'(r)\right|^{q},\left|g'\left(\sqrt{rw}\right)\right|^{q}\right\}\right)\int_0^1\left|\ln\left(r^{\frac{1+\sigma}{2}}w^{\frac{1-\sigma}{2}}\right)\right|d\sigma \\
&\quad=\xi(w,r)\left(\sup\left\{\left|g'(r)\right|^{q},\left|g'\left(\sqrt{rw}\right)\right|^{q}\right\}\right).
\end{aligned} \tag{2.11}$$

Now an application of Lemma 2, gives us the following results

$$\int_0^1 r^{\frac{q(1-\sigma)}{2(q-1)}} w^{\frac{q(1+\sigma)}{2(q-1)}} \left|\ln\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right| d\sigma = \frac{q-1}{q}\eta\left(r^{\frac{q}{q-1}}, w^{\frac{q}{q-1}}\right) \tag{2.12}$$

$$\int_0^1 r^{\frac{q(1+\sigma)}{2(q-1)}} w^{\frac{q(1-\sigma)}{2(q-1)}} \left|\ln\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right| d\sigma = \frac{q-1}{q}\eta\left(w^{\frac{q}{q-1}}, r^{\frac{q}{q-1}}\right), \tag{2.13}$$

$$\int_0^1 \left|\ln\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right| d\sigma = \xi(r,w) \tag{2.14}$$

and

$$\int_0^1 \left|\ln\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right| d\sigma = \xi(w,r). \tag{2.15}$$

A combination of the results (2.9)-(2.15) gives us the required inequality (2.8). Hence the proof of the theorem is complete. □

Theorem 6. *Let $g : J \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable on J° and $g' \in L[r,w]$ for $r, w \in J^\circ$ with $r < w$. If $\left|g'\right|^q$ is a geometrically convex on $[r,w]$ for $q > 1$ and ℓ is a real number such that $q > \ell > 0$. Then*

$$\begin{aligned}
&\left|\frac{(\ln w)\, g(w) - (\ln r)\, g(r)}{\ln w - \ln r} - \frac{1}{\ln w - \ln r}\int_r^w \frac{g(l)}{l} dl\right| \le \frac{1}{2}\left(\frac{q-1}{q-\ell}\right)^{1-\frac{1}{q}}\left(\frac{1}{\ell}\right)^{\frac{1}{q}} \\
&\times\left\{\left[\eta\left(r^{\frac{q-\ell}{q-1}}, w^{\frac{q-\ell}{q-1}}\right)\right]^{1-\frac{1}{q}}\left[\eta\left(r^{\ell}, w^{\ell}\right)\left(\sup\left\{\left|g'\left(\sqrt{rw}\right)\right|^q, \left|g'(w)\right|^q\right\}\right)\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\eta\left(w^{\frac{q-\ell}{q-1}}, r^{\frac{q-\ell}{q-1}}\right)\right]^{1-\frac{1}{q}}\left[\eta\left(w^{\ell}, r^{\ell}\right)\left(\sup\left\{\left|g'(r)\right|^q, \left|g'\left(\sqrt{rw}\right)\right|^q\right\}\right)\right]^{\frac{1}{q}}\right\},
\end{aligned} \tag{2.16}$$

where $\eta(r,w)$ is defined in Lemma 2.

Proof.

$$\begin{aligned}
&\left|\frac{(\ln w)\, g(w) - (\ln r)\, g(r)}{\ln w - \ln r} - \frac{1}{\ln w - \ln r}\int_r^w \frac{g(l)}{l} dl\right| \\
&\le \frac{1}{2}\int_0^1 r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}} \left|\ln\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right| \left|g'\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right| d\sigma \\
&+ \frac{1}{2}\int_0^1 r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}} \left|\ln\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right| \left|g'\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right| d\sigma \\
&\le \frac{1}{2}\left(\int_0^1 r^{\frac{(q-\ell)(1-\sigma)}{2(q-1)}} w^{\frac{(q-\ell)(1+\sigma)}{2(q-1)}} \left|\ln\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right| d\sigma\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 r^{\frac{\ell(1-\sigma)}{2}} w^{\frac{\ell(1+\sigma)}{2}} \left|\ln\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right| \left|g'\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right|^q d\sigma\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\frac{1}{2}\left(\int_0^1 r^{\frac{(q-\ell)(1+\sigma)}{2(q-1)}} w^{\frac{(q-\ell)(1-\sigma)}{2(q-1)}} \left|\ln\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right| d\sigma\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 r^{\frac{\ell(1+\sigma)}{2}} w^{\frac{\ell(1-\sigma)}{2}} \left|\ln\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right| \left|g'\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right|^q d\sigma\right)^{\frac{1}{q}}. \quad (2.17)$$

Now by using the quasi-geometric convexity of $\left|g'\right|^q$ on $[r,w]$ and Lemma 2, we have

$$\int_0^1 r^{\frac{\ell(1-\sigma)}{2}} w^{\frac{\ell(1+\sigma)}{2}} \left|\ln\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right| \left|g'\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right|^q d\sigma$$
$$\leq \frac{1}{\ell}\eta\left(r^\ell, w^\ell\right)\left(\sup\left\{\left|g'\left(\sqrt{rw}\right)\right|^q, \left|g'(w)\right|^q\right\}\right), \quad (2.18)$$

$$\int_0^1 r^{\frac{\ell(1+\sigma)}{2}} w^{\frac{\ell(1-\sigma)}{2}} \left|\ln\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right| \left|g'\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right|^q d\sigma$$
$$\leq \frac{1}{\ell}\eta\left(w^\ell, r^\ell\right)\left(\sup\left\{\left|g'(r)\right|^q, \left|g'\left(\sqrt{rw}\right)\right|^q\right\}\right), \quad (2.19)$$

$$\int_0^1 r^{\frac{(q-\ell)(1-\sigma)}{2(q-1)}} w^{\frac{(q-\ell)(1+\sigma)}{2(q-1)}} \left|\ln\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)\right| d\sigma$$
$$= \left(\frac{q-1}{q-\ell}\right)\eta\left(r^{\frac{q-\ell}{q-1}}, w^{\frac{q-\ell}{q-1}}\right) \quad (2.20)$$

and

$$\int_0^1 r^{\frac{(q-\ell)(1+\sigma)}{2(q-1)}} w^{\frac{(q-\ell)(1-\sigma)}{2(q-1)}} \left|\ln\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right| d\sigma$$
$$= \left(\frac{q-1}{q-\ell}\right)\eta\left(w^{\frac{q-\ell}{q-1}}, r^{\frac{q-\ell}{q-1}}\right). \quad (2.21)$$

Applying (2.18)-(2.21) in (2.17), we get (2.16). This completes the proof of the theorem. □

Theorem 7. *Let $g:\mathbb{R}_+\to\mathbb{R}_0$ be a geometrically quasi-convex function on $[r,w]$ and $g\in L([r,w])$. Then*

$$g\left((rw)^{\frac{1}{2}}\right) \leq \frac{1}{\ln w-\ln r}\int_r^w \frac{g(l)}{l}\,dl$$
$$\leq \frac{1}{2}\left[\sup\left\{g(r), g\left(\sqrt{rw}\right)\right\}+\sup\left\{g(\sqrt{rw}), g(w)\right\}\right]. \quad (2.22)$$

Proof. It is easy to observe that

$$(rw)^{\frac{1}{2}} = r^{\frac{1}{2}\left(\frac{1-\sigma}{2}\right)} w^{\frac{1}{2}\left(\frac{1+\sigma}{2}\right)} r^{\frac{1}{2}\left(\frac{1+\sigma}{2}\right)} w^{\frac{1}{2}\left(\frac{1-\sigma}{2}\right)}$$

for $0 \leq \sigma \leq 1$. Since $g(l)$ is geometrically quasi- convex on $[r, w]$, we have

$$g\left((rw)^{\frac{1}{2}}\right) \leq \sup\left\{g\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right), g\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right\}. \tag{2.23}$$

Now we consider the two cases:
Case I
If

$$\sup\left\{g\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right), g\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right\} = g\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)$$

then

$$g\left((rw)^{\frac{1}{2}}\right) \leq g\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right). \tag{2.24}$$

Integrating (2.24) with respect to σ over $[0, 1]$, we obtain

$$g\left((rw)^{\frac{1}{2}}\right) \leq \frac{2}{\ln w - \ln r} \int_{\sqrt{rw}}^{w} \frac{g(l)}{l} dl = \int_0^1 g\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right) d\sigma$$
$$\leq \sup\left\{g\left(\sqrt{rw}\right), g(w)\right\}. \tag{2.25}$$

$$\int_0^1 g(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}})\, d\sigma = \int_0^1 g(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}})\, d\sigma = \frac{1}{\ln w - \ln r} \int_r^w \frac{g(l)}{l} dl \tag{2.26}$$

Case II
If

$$\sup\left\{g\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right), g\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right)\right\} = g\left(r^{\frac{1+\sigma}{2}} w^{\frac{1+\sigma}{2}}\right)$$

then

$$g\left((rw)^{\frac{1}{2}}\right) \leq g\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right). \tag{2.27}$$

Integrating (2.27) with respect to σ over $[0, 1]$, we get

$$g\left((rw)^{\frac{1}{2}}\right) \leq \frac{2}{\ln w - \ln r} \int_r^{\sqrt{rw}} \frac{g(l)}{l} dl = \int_0^1 g\left(r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}\right) d\sigma$$
$$\leq \sup\left\{g(r), g\left(\sqrt{rw}\right)\right\}. \tag{2.28}$$

Adding (2.25) and (2.28) and dividing the resulting inequality by 2, we get the inequality (2.22). Thus the proof of the theorem is completed. □

Theorem 8. *Let $g, h : [r, w] \subseteq \mathbb{R} \to \mathbb{R}$ be a geometrically convex function on $[r, w]$ and $gh \in L([r, w])$, then the following inequality holds*

$$\frac{1}{\ln w - \ln r} \int_r^w \frac{g(l)}{l} h(l)\, dl$$
$$\leq \frac{1}{2}\left[\sup\left\{g(\sqrt{rw})h(\sqrt{rw}), g(\sqrt{rw})h(w), g(w)h(\sqrt{rw}), g(w)h(w)\right\}\right.$$
$$\left. + \sup\left\{g\left(\sqrt{rw}\right)h\left(\sqrt{rw}\right), g(r)h\left(\sqrt{rw}\right), g\left(\sqrt{rw}\right)h(r), g(r)h(r)\right\}\right]. \tag{2.29}$$

Proof. Let $l = r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}$, $0 \le \sigma \le 1$ and using the geometric quasi-convex of g and h on $[r, w]$, we have

$$\frac{2}{\ln w - \ln r} \int_{\sqrt{rw}}^{w} \frac{g(l)h(l)}{l} dl = \int_0^1 g\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right) h\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right) d\sigma$$
$$\le \sup\left\{g\left(\sqrt{rw}\right)h\left(\sqrt{rw}\right), g(w)h\left(\sqrt{rw}\right), g\left(\sqrt{rw}\right)h(w), g(w)h(w)\right\}. \quad (2.30)$$

Similarly, the substitution $l = r^{\frac{1+\sigma}{2}} w^{\frac{1-\sigma}{2}}$, $0 \le \sigma \le 1$, gives us the following result

$$\frac{2}{\ln w - \ln r} \int_{r}^{\sqrt{rw}} \frac{g(l)h(l)}{l} dl = \int_0^1 g\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right) h\left(r^{\frac{1-\sigma}{2}} w^{\frac{1+\sigma}{2}}\right) d\sigma$$
$$\le \sup\left\{g\left(\sqrt{rw}\right)h\left(\sqrt{rw}\right), g(r)h\left(\sqrt{rw}\right), g\left(\sqrt{rw}\right)h(r), g(r)h(r)\right\}. \quad (2.31)$$

Adding (2.30) and (2.31) and dividing the resulting inequality by 2, we get (2.29). □

3. Applications to special means

Let $w > r > 0$ and $s \in \mathbb{R}$. Consider the following means

$$G(r, w) = \sqrt{rw},$$

$$I(r, w) = \begin{cases} \frac{1}{e}\left(\frac{w^w}{r^r}\right)^{\frac{1}{w-r}}, & r \ne w, \\ a, & r = w, \end{cases}$$

$$L(a, b) = \begin{cases} \frac{w-r}{\ln w - \ln r}, & r \ne w, \\ r, & r = w, \end{cases}$$

and

$$L_s(r, w) = \begin{cases} \left[\frac{w^{s+1} + r^{s+1}}{(s+1)(w-r)}\right]^{\frac{1}{s}}, & s \ne -1, 0, \\ L(r, w), & s = -1, \\ I(r, w), & s = 0, \end{cases}$$

The above means are known as respectively the geometric, the exponential, the logarithmic and the generalized logarithmic means of two positive real numbers a and b.

We now give applications of our results to the above means.

Theorem 9. *Let $w > r > 0$ and $s \in \mathbb{R}$.*

(1) *If $r < w < 1$ and $s > 0$, then*

$$\ln I\left(r^{s+1}, w^{s+1}\right) \leq \frac{|s+1| \ln I\left(G(a,b), w\right)}{(\ln w - \ln r)\left[L_s\left(r^{s+1}, w^{s+1}\right)\right]^s} \times \left\{w^s\left(G(a,b) - w\right) + [G(a,b)]^s\left(G(a,b) - r\right)\right\}. \quad (3.1)$$

(2) *If $1 < r < w$ and $s < -1$, then*

$$\ln I\left(r^{s+1}, w^{s+1}\right) \leq -\frac{|s+1| \ln I\left(G(a,b), w\right)}{(\ln w - \ln r)\left[L_s\left(r^{s+1}, w^{s+1}\right)\right]^s} \times \left\{[G(a,b)]^s\left(G(a,b) - w\right) + r^s\left(G(a,b) - r\right)\right\}. \quad (3.2)$$

Proof. Let $g(l) = l^{s+1}$, $l \in \mathbb{R}_+$ and $s \in \mathbb{R}$, $s \neq -1$. If $m > l > 0$, then

$$\left|g'\left(l^{\frac{\sigma-1}{2}} m^{\frac{\sigma+1}{2}}\right)\right|^s \leq \begin{cases} |s+1| m^s, & , s \geq 0 \\ |s+1| (lm)^{\frac{s}{2}}, & s < 0. \end{cases}$$

Also

$$\left|g'\left(l^{\frac{\sigma+1}{2}} m^{\frac{\sigma-1}{2}}\right)\right|^s \leq \begin{cases} |s+1| (lm)^{\frac{s}{2}}, & , s \geq 0 \\ |s+1| l^s, & s < 0. \end{cases}$$

This shows that the function $\left|g'(l)\right| = |s+1| l^s$ for $s \in \mathbb{R}$, $s \neq -1$, is geometrically quasi-convex on $\mathbb{R}_+$.

Now

$$\frac{(\ln w) g(w) - (\ln r) g(r)}{\ln w - \ln r} - \frac{1}{\ln w - \ln r} \int_r^w \frac{g(l)}{l} dl = \left[L_s\left(r^{s+1}, w^{s+1}\right)\right]^s \ln I\left(r^{s+1}, w^{s+1}\right), \quad (3.3)$$

$$\begin{aligned} &\frac{1}{2}\eta(r,w)\left(\sup\left\{\left|g'\left(\sqrt{rw}\right)\right|, \left|g'(w)\right|\right\}\right) \\ &= \frac{|s+1| w^s\left[w - \sqrt{rw} + \sqrt{rw}\ln\left(\sqrt{rw}\right) - w\ln w\right]}{\ln w - \ln r} \\ &= \frac{|s+1| w^s\left(\sqrt{rw} - w\right)\ln I\left(\sqrt{rw}, w\right)}{\ln w - \ln r} \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} &\frac{1}{2}\eta(w,r)\left(\sup\left\{\left|g'(r)\right|, \left|g'\left(\sqrt{rw}\right)\right|\right\}\right) \\ &= \frac{|s+1| (rw)^{\frac{s}{2}}\left[r - \sqrt{rw} + \sqrt{rw}\ln\left(\sqrt{rw}\right) - r\ln r\right]}{\ln w - \ln r} \end{aligned} \quad (3.5)$$

$$= \frac{|s+1|(rw)^{\frac{s}{2}}\left(\sqrt{rw}-r\right)\ln I\left(\sqrt{rw},r\right)}{\ln w-\ln r}.$$

Substituting (3.3), (3.4) and (3.5) in Theorem 4, we get (3.1). The inequality (3.2) can be obtained in similar way. □

The following numerical example illustrates that our results provide refinements of the results given in [12].

For the geometrically quasi-convex function $g(x) = x^3$, $x \in \mathbb{R}_+$, the following table is prepared using mathematica.

Function	Error Bound of Theorem 4	Error Bound of Theorem 1
$g(x) = x^3, a = \frac{1}{2}, b = \frac{2}{3}$	0.363473	0.418996
$g(x) = x^3, a = \frac{3}{2}, b = 2$	10.4408	11.6002
$g(x) = x^3, a = \frac{1}{2}, b = \frac{3}{2}$	0.389078	1.60745

Similarly we can make tables for comparison of the results given in Theorem 2 and Theorem 3 with those given in Theorem 5 and Theorem 6 respectively.

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LIMIT CYCLES AND INVARIANT CENTERS FOR AN EXTENDED KUKLES SYSTEM

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Abstract. A class of polynomial systems of odd degree with limit cycles, invariant centers and invariant straight lines, is examined. The limit cycles can be obtain as a bifurcation of a non hyperbolic focus at the origin as Hopf bifurcations. We will also obtain the necessary and sufficient conditions for the critical point at the interior of bounded region to be a center.

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1. Introduction

Let us consider a real autonomous system of ordinary differential equations on the plane with polynomial nonlinearities.

$$\begin{cases} \dot{x} = P(x,y) = \sum_{i+j=0}^{n} a_{ij} x^i y^j \\ \dot{y} = Q(x,y) = \sum_{i+j=0}^{n} b_{ij} x^i y^j \end{cases}, \quad \text{with } a_{ij},\ b_{ij} \in \mathbb{R} \tag{1.1}$$

Suppose that the origin of (1.1) is a critical point of center-focus type. We are concerned with two closely related questions, both of which are significant elements in work on Hilbert's 16th problem. The first is the number of limit cycles (that is, isolated periodic solutions) which bifurcate from a critical point and the second is the derivation of necessary and sufficient conditions for a critical point to be a center (that is, all orbits in the neighborhood of the critical point are closed). The limit cycles problem and the center problem is concentrated on specific classes of systems, those may be systems of a given degree (quadratic systems or a cubic systems, for example), or they may be of a particular form; for instance, a Kukles systems - that is, a systems of the form

$$\dot{x} = -y\ , \qquad \dot{y} = x + \lambda y + g(x,y)\ .$$

Recall that systems of this type were studied, for the first time, by Isaak Solomonovich Kukles [4], who studied a linear center with cubic nonhomogeneous nonlinearities.
A closely related problem is the derivation of conditions under which the system is

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integrable and other questions of interest relate to the coexistence of limit cycles and algebraic invariant curves.

Let $n = \max(\partial P, \partial Q)$, where the symbol ∂ denotes 'degree of'. A function F is said to be *invariant* with respect to (1.1) if there is a polynomial G called cofactor, with $\partial G < m$, such that $\dot{F} = FG$. Here $\dot{F} = F_x P + F_y Q$ is the rate of change of F along orbits. It is well known that the existence of invariant curves has significant repercussions on the possible phase-portraits of the system.

Recently in [3], the authors considered a Kukles extended system of degree three, namely, a system of the form

$$\begin{cases} \dot{x} = P(x,y) = \lambda x + y + kxy \\ \dot{y} = Q(x,y) = -x + \lambda y + \displaystyle\sum_{i+j=2}^{3} a_{ij} x^i y^j \end{cases}, \quad \text{with } k,\ a_{ij} \in \mathbb{R}$$

and the authors obtained some conditions for the existence of centre and limit cycles.

With the same idea, we consider an extended Kukles system of degree $2n+5$, $n \geq 1$, with limit cycles, invariant parabolas and invariant straight lines, given by

$$X_\mu : \begin{cases} \dot{x} = -y(1-k^2x^2) \\ \dot{y} = \frac{1}{k^2}\big[2x(1-k^2x^2)^2 + (-1+k^2x^2-k^2y)(-1+k^2x^2+k^2y) \\ \qquad \big(k^2x - 2x + k^2\lambda y + \sum_{i=1}^{n} a_{2i+1} y^{2i+1}\big)\big] \end{cases} \tag{1.2}$$

where $\mu = (k, \lambda, a_3, \ldots, a_{2n+1}) \in \mathbb{R}^{n+2}$, $k \neq 0$. In this paper, for $n \geq 1$, we consider another class of extended Kukles system of degree $2n+3$, with an invariant non–degenerate centers and two invariant straight line. For these kind of second-order differential systems, we show that for certain values of the parameters, the invariant centers and the invariant straight lines, can coexist with at least n small amplitude limit cycles which are constructed by Hopf bifurcation. We conclude with some numerical simulations of the results.

2. Main results

Let us consider the extended Kukles of degree $2n+3$

$$X_\mu : \begin{cases} \dot{x} = -y + k^2x^2y \\ \dot{y} = x - k^2x^3 + (k^2x^2 + k^2y^2 - 1) \\ \qquad \left(-\dfrac{1}{2}\sum_{i=1}^{n} a_{2i+1} y^{2i+1} + \lambda y(1-k^2x^2)\right) \end{cases} \tag{2.1}$$

where $\mu = (k, \lambda, a_3, \ldots, a_{2n+1}) \in \mathbb{R}^{n+2}$, $k \neq 0$

Lemma 1. *For all $n \geq 1$ and for all $\mu = (k, \lambda, a_3, \ldots, a_{2n+1}) \in \mathbb{R}^{n+2}$ the straight lines $L_1(x) = 1 - kx$, $L_2(x) = 1 + kx$ and the center $F(x,y) = k^2x^2 + k^2y^2 - 1$ are invariant algebraic curve of (2.1).*

Proof. It is easy to verify that

$$\dot{L_1} = L_{1_x}\dot{x} + L_{1_y}\dot{y} = kyL_1R(x,y),$$

where the cofactor is $R(x,y) = ky(1+kx)$,

$$\dot{L_2} = L_{2_x}\dot{x} + L_{2_y}\dot{y} = -kyL_2L(x,y),$$

where the cofactor is $R(x,y) = -ky(1-kx)$, and for $F(x,y) = k^2x^2 + k^2y^2 - 1$

$$\dot{F} = F_x\dot{x} + F_y\dot{y} = F(x,y)R(x,y),$$

where the cofactor is $R(x,y) = 2k^2y(-\frac{1}{2}\sum_{i=1}^{n} a_{2i+1}y^{2i+1} + \lambda yL_1(x)L_2(x))$. Then the straight lines $L_1(x)$, $L_2(x)$ and the center $F(x,y)$ are invariant algebraic curve of (2.1). □

Theorem 1. *If $a_{2j+1} = 0$, $\forall j = 1,2,...,n-1$ and $a_{2n+1} \neq 0$, system (2.1) at the origin has a weak focus of order n.*

Proof. The linear part of (2.1) at the singularity $(0,0)$ is

$$DX_\mu(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & -\lambda \end{pmatrix}.$$

If $\lambda = 0$, we have $\operatorname{div} X_\mu(0,0) = 0$ and $\det DX_\mu(0,0) = 1$, then the critical point $(0,0)$ is a weak focus. As $\operatorname{div} X_\mu(0,0) = -\lambda$, we have the Liapunov quantities $l_0 = -\lambda$, and using Mathematica Software [1], we are able to compute the Liapunov quantities l_j, for $j \geq 1$. If $\lambda = 0$, $l_0 = 0$ and $l_1 = \frac{3a_3}{16}$, if $a_3 = 0$, $l_1 = 0$ and $l_2 = \frac{5a_5}{32}$, if $a_5 = 0$, $l_2 = 0$ and $l_3 = \frac{35a_7}{256}$ and so on, then, If $\lambda = a_3 = a_5 = ... = a_{2n-1} = 0$, and $a_{2n+1} \neq 0$, we have $l_0 = l_1 = l_3 = ... = l_{n-1} = 0$, and for $n \geq 1$, $l_n = a_{2n+1}\frac{(2n+1)!!}{2(2n+2)!!}$. Finally for all $j \geq n$, $l_j = a_{2j+1}R_j(k)$ where $R_j(k)$ is a polynomial on k, then system (2.1) at the origin has a weak focus of order n. □

Theorem 2. *System (2.1) has a local center at the origin if and only if*

$$\lambda = a_3 = a_5 = ... = a_{2n+1} = 0.$$

Proof. From the linear part at the origin, it is clear that the condition $\lambda = 0$ is necessary for a center. By Theorem 1, the Liapunov quantities are given by $l_0 = -\lambda$ and for $n \geq 1$, $l_n = a_{2n+1}\frac{(2n+1)!!}{2(2n+2)!!}$. As all calculated Liapunov quantities are zero, this shows that the conditions are necessary.

If $a_{2j+1} = 0$, $j = 1,2,3,...,n$, system (2.1) is given by

$$\begin{cases} \dot{x} = P(x,y) = -y + k^2x^2y \\ \dot{y} = Q(x,y) = x - k^2x^3 \end{cases} \tag{2.2}$$

As the symmetries $P(-x,y) = P(x,y)$ and $Q(-x,y) = -Q(x,y)$ are satisfied, this proves the sufficiency that system (2.2) has a local center at the origin. □

The solutions are on the circles centered at the origin and on the invariant straight lines $x = \frac{1}{k}$ and $x = -\frac{1}{k}$.

The small circles (local center), and large circles are solutions, and the large ones contain four nontrivial solutions plus four singularities. In between there is a circle containing two nontrivial solutions and two singularities.

In order to illustrate the result stated in Theorem 1, Figure 1 shows a numerical simulation of the system (2.1) with $k = 1$, $a_3 = 0$ and $\lambda = 0$, which corresponds to the case of center. The simulation was obtained using the Pplane7 Software with MATLAB [2].

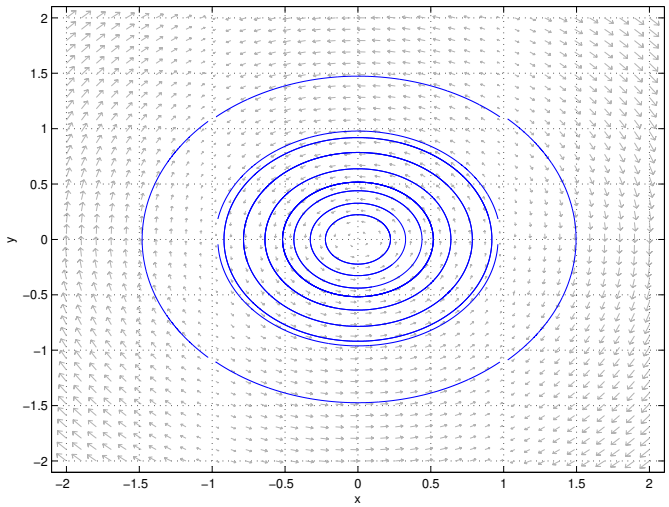

FIGURE 1. Local Center ($n = 1$).

Theorem 3. *In the parameters space $\mathbb{R}^{n+2}$, there exists an open set $\mathcal{N} \neq \phi$, such that for all $\mu = (k, \lambda, a_3, \ldots, a_{2n+1}) \in \mathcal{N}$, $k \neq 0$, system (2.1) has at least n small-amplitude limit cycle bounded by the invariant conic.*

Proof. If $\lambda = 0$ and $a_{2j-1} = 0$, $\forall j = 2, 3, \ldots, n$ and $a_{2n+1} \neq 0$, by Theorem 1, system (2.1) has at the singularity $(0,0)$ a repelling or attracting weak focus of order n.

Let us consider the case $a_{2n+1} > 0$ and n odd number, that is an attracting weak focus of order n. Taking $a_{2n-1} < 0$, so that $l_n > 0$ and $l_{n-1} < 0$, the origin of (2.1) is reversed and a hyperbolic repelling small amplitude limit cycle is created (Hopf Bifurcation), the limit cycle created persist under new small perturbation. Furthermore the singularity at the origin is an attracting weak focus of order $n-1$.

If we adjust the parameter a_{2n-3}, so that $a_{2n-3} > 0$ small enough and $l_n > 0$, $l_{n-1} < 0$ and $l_{n-2} > 0$, the stability at the origin is reversed and new hyperbolic attracting small amplitude limit cycle is created, that persists under new small perturbation. Furthermore the singularity at the origin is an repelling weak focus of order $n-2$. By following the same process until all Liapunov quantities are non-zero, such that in each step the stability of the singularity is inverted, n hyperbolic small amplitude limit cycles are created and we obtain $n+1$ inequalities which define an a open set $\mathcal{N}$ in the parameter space, given by $\mathcal{N} = \{(k, \lambda, a_3, ..., a_{2n+1}) \in \mathbb{R}^{n+2} \mid k \neq 0, l_n > 0, l_{n-1} < 0, l_{n-2} > 0, ..., l_2 < 0, l_1 > 0, l_0 < 0\}$. □

In order to illustrate the result stated in Theorem 3, Figure 2 shows a numerical simulation of the system (2.1) with $k = 1$, $a_3 = 2$ and $\lambda = 0.01$, which corresponds to the case of attracting limit cycle. The simulation was obtained using the Pplane7 Software with MATLAB [2].

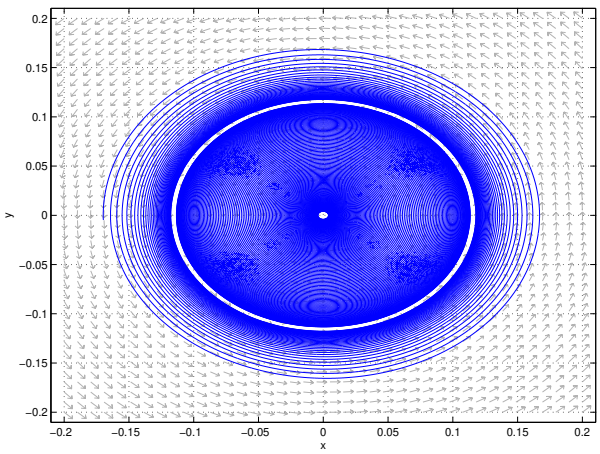

FIGURE 2. Repelling limit cycle ($n = 1$).

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SOME ORTHOGONALITY EQUATION WITH TWO FUNCTIONS

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Abstract. The aim of this paper is to describe the solution (f,g) of the equation

$$\langle f(x)|f(y)\rangle = \langle g(x)|y\rangle\,,\ x,y\in X,$$

where $f\colon X\to Y$, $g\colon X\to X$, X,Y are inner product spaces over the same field $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$.

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Keywords: orthogonality equation, Hilbert space, bounded linear operator, self-adjoint operator

1. Introduction

Throughout this paper X,Y are Hilbert spaces over the same field $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, $\langle\cdot|\cdot\rangle$ denotes the inner product and $\|\cdot\|$ the norm associated with it. We shall not distinguish between the symbols used for X and Y, $D(T)$ denotes the domain of the operator T.

It is known that $h\colon X\to Y$ is a solution of the orthogonality equation:

$$\langle h(x)|h(y)\rangle = \langle x|y\rangle\,,\ x,y\in X \tag{1.1}$$

if and only if h is a linear isometry (see, e.g. [6, Lemma 2.1.1 and the following Remark]).

J. Chmieliński [3] studied the generalized orthogonality equation

$$\langle f(x)|g(y)\rangle = \langle x|y\rangle\,,\ x,y\in X,$$

with two unknown functions $f,g\colon X\to Y$. The form of solutions of the above equation was presented by R. Łukasik and P. Wójcik [5], R. Łukasik [4].

Also in the paper of M. M. Sadr [8] we can find the Banach-orthogonality equation

$$\langle f(x)|g(\alpha)\rangle = \langle x|\alpha\rangle\,,\ x\in E,\ \alpha\in E^*,$$

where $f\colon E\to F$, $g\colon E^*\to F^*$, E,F are Banach spaces, F is reflexive.

In this paper we take a different approach. Instead take two different functions on the left side of (1.1), we change only the right side of (1.1).

Now we recall some facts from the theory of adjoint operators.

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Definition 1. Let $D(f)$ be a linear subspace of X, $f: D(f) \to Y$ be a linear operator. Let further

$$D(f^*) = \{y \in Y : \bigvee_{z \in X} \bigwedge_{x \in D(f)} \langle f(x)|y\rangle = \langle x|z\rangle\}.$$

A function $f^*: D(f^*) \to X$ is called an adjoint operator (to f) if and only if

$$\langle f(x)|y\rangle = \left\langle x \middle| f^*(y)\right\rangle,\ x \in D(f),\ y \in D(f^*).$$

Remark 1 (see [1, §39]). Let $D(f)$ be a dense linear subspace of X, $f: D(f) \to Y$ be a linear operator. Then the adjoint operator exists and it is unique. Moreover $D(f^*)$ is a linear subspace of Y, f^* is a linear operator and

$$\ker f^* = (\operatorname{im} f)^{\perp}, \quad \operatorname{cl} \operatorname{im} f = (\ker f^*)^{\perp}.$$

Definition 2. Let $D(f)$ be a linear subspace of X. A linear operator $f: D(f) \to Y$ is called bounded if and only if there exists $M > 0$ such that

$$\|f(x)\| \leq M \|x\|,\ x \in D(f).$$

Remark 2 (see [1, §19]). Let $D(f)$ be a dense linear subspace of X, $f: D(f) \to Y$ be a linear and bounded operator. Then f can be uniquely extended to a linear and bounded operator on X.

Definition 3. Let $D(g)$ be a dense linear subspace of X. A function $g: D(g) \to X$ is called a symmetric operator if and only if

$$\langle g(x)|y\rangle = \langle x|g(y)\rangle,\ x, y \in D(g).$$

Definition 4. Let $D(g)$ be a dense linear subspace of X. A function $g: D(g) \to X$ is called a self-adjoint operator if and only if $D(g^*) = D(g)$ and $g^* = g$.

Remark 3. For bounded operators definitions of the symmetric operator and the self-adjoint operator are equivalent.

Definition 5. Let $D(g)$ be a dense linear subspace of X. A linear operator $g: D(g) \to X$ is called positive if and only if

$$\langle g(x)|x\rangle \geq 0,\ x \in D(g).$$

Theorem 1 (see [7, §104] and [2])**.** *Let $g: X \to X$ be a self-adjoint and positive operator. Then there exists a unique self-adjoint and positive operator $T: X \to X$ such that $g = T \circ T$.*

Because every symmetric and positive operator can be extended to a self-adjoint operator we have the following result.

Theorem 2. *Let $D(g)$ be a dense linear subspace of X, $g: D(g) \to X$ be a symmetric and positive operator. Then there exists a symmetric and positive operator $h: D(h) \to X$ such that $D(g) \subset D(h)$ and $g = h \circ h$.*

Definition 6. Let $D(T)$ be a dense linear subspace of X. A linear operator $T\colon D(T) \to Y$ is called a linear isometry if and only if

$$\|T(x)\| = \|x\|, \; x \in D(T).$$

Remark 4 (see Remark 2). Let $D(T)$ be a dense linear subspace of X. Let $T\colon D(T) \to Y$ be a linear isometry. Then T can be uniquely extended to a linear isometry from X to Y.

Remark 5. Let $D(T)$ be a dense linear subspace of X. Let $T\colon D(T) \to Y$ be a linear isometry. Then

$$\langle T(x)|T(y)\rangle = \langle x|y\rangle, \; x, y \in D(T).$$

We would like to divide our consideration into two cases: f, g are bounded or unbounded. In view of Lemma 1 we do not have other cases.

2. Main result for bounded operators

Because the domain of a bounded operator can be extended to the whole space, we assume in this section that operators are defined on the whole X.
First we describe properties of solutions of equation (2.1).

Theorem 3. *Let $f\colon X \to Y$, $g\colon X \to X$ satisfy the equation*

$$\langle f(x)|f(y)\rangle = \langle g(x)|y\rangle, \; x, y \in X. \tag{2.1}$$

Then f is linear and bounded, g is a self-adjoint and positive operator, $\ker f = \ker g$.

Proof. We observe that

$$\langle g(x)|y\rangle = \langle f(x)|f(y)\rangle = \overline{\langle f(y)|f(x)\rangle} = \overline{\langle g(y)|x\rangle} = \langle x|g(y)\rangle, \; x, y \in X.$$

We have also

$$\langle g(x)|x\rangle = \langle f(x)|f(x)\rangle \geq 0, \; x \in X.$$

Hence g is a self-adjoint and positive operator. In particular g is continuous. Since

$$\begin{aligned}\langle f(z)|f(\alpha x + y) - \alpha f(x) - f(y)\rangle &= \langle g(z)|\alpha x + y\rangle - \alpha \langle g(z)|x\rangle - \langle g(z)|y\rangle \\ &= 0, \; x, y, z \in X, \; \alpha \in \mathbb{K},\end{aligned}$$

then $f(\alpha x + y) - \alpha f(x) - f(y) \in \operatorname{im} f^{\perp}$, $x, y \in X$, $\alpha \in \mathbb{K}$, so f is linear. We notice that

$$\|f(x)\|^2 = \langle f(x)|f(x)\rangle = \langle g(x)|x\rangle \leq \|g(x)\| \cdot \|x\| \leq \|g\| \cdot \|x\|^2, \; x \in X,$$

which means that f is bounded.
Finally, for $x \in X$ we have

$$x \in \ker g \iff \bigwedge_{y \in X} \langle g(x)|y\rangle = 0 \iff \bigwedge_{y \in X} \langle f(x)|f(y)\rangle = 0 \iff x \in \ker f.$$

□

Theorem 4. *Let $f: X \to Y$ be linear and bounded. Then there exists a unique function $g: X \to X$ such that f and g satisfy equation (2.1). Moreover $g = f^* \circ f$.*

Proof. Let $g = f^* \circ f$. Hence

$$\langle f(x)|f(y)\rangle = \left\langle f^*(f(x))\middle|y\right\rangle = \langle g(x)|y\rangle\,,\ x, y \in X.$$

Assume that there exists $h: X \to X$ such that

$$\langle f(x)|f(y)\rangle = \langle h(x)|y\rangle\,,\ x, y \in X.$$

Hence

$$\langle g(x) - h(x)|y\rangle = \langle f(x)|f(y)\rangle - \langle f(x)|f(y)\rangle = 0,\ x, y \in X,$$

so $h = g$. □

Theorem 5. *Let $g: X \to X$ be a self-adjoint and positive operator,* $\operatorname{im} g$ *be isometric with some subspace of Y. Then there exists a function $f: X \to Y$ such that f and g satisfy equation (2.1).*

Proof. Let $X_0 = \operatorname{cl}\operatorname{im} g$, $T: X_0 \to Y$ be a linear isometry (as an extension of a linear isometry from $\operatorname{im} g$ to Y). In view of Theorem 1 there exists a self-adjoint operator $h: X \to X$ such that $g = h^2$. It is easy to see that $\ker h = \ker g = X_0^{\perp}$ and hence $\operatorname{cl}\operatorname{im} h = \operatorname{cl}\operatorname{im} g = X_0$. We define function $f: X \to Y$ by the formula

$$f(x) = T(h(x)),\ x \in X.$$

Then

$$\begin{aligned}\langle f(x)|f(y)\rangle &= \langle T(h(x))|T(h(y))\rangle = \langle h(x)|h(y)\rangle \\ &= \left\langle h^2(x)\middle|y\right\rangle = \langle g(x)|y\rangle\,,\ x, y \in X.\end{aligned}$$

□

In the above theorem the assumption that $\operatorname{im} g$ is isometric with some subspace of Y is necessary. If the space Y is too small, then this theorem does not hold, which shows the below example.

Example 1. Let $\mathbb{K} = \mathbb{R} = Y$, $X = \mathbb{R}^2$, $g = Id_X$. Suppose that there exists $f: X \to Y$ such that (2.1) holds. Then

$$f(x_1, x_2) \cdot f(y_1, y_2) = x_1 y_1 + x_2 y_2,\ x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

Hence

$$\begin{aligned} f(1,0)^2 &= 1, \\ f(0,1)^2 &= 1, \\ f(1,0) \cdot f(0,1) &= 0, \end{aligned}$$

which is impossible.

3. Main result for unbounded operators

Theorem 6. *Let D be a dense linear subspace of X, $f: D \to Y$, $g: D \to X$ satisfy the equation*

$$\langle f(x)|f(y)\rangle = \langle g(x)|y\rangle, \ x, y \in D. \tag{3.1}$$

Then f is linear, $\operatorname{im} f \subset D(f^*)$*, g is a symmetric and positive operator,* $\ker f = \ker g$.

Proof. Since

$$\begin{aligned}\langle f(z)|f(\alpha x+y)-\alpha f(x)-f(y)\rangle &= \langle g(z)|\alpha x+y\rangle - \alpha\langle g(z)|x\rangle - \langle g(z)|y\rangle \\ &= 0, \ x, y, z \in D, \ \alpha \in \mathbb{K},\end{aligned}$$

then $f(\alpha x+y)-\alpha f(x)-f(y) \in \operatorname{im} f^{\perp}$, $x, y \in D$, $\alpha \in \mathbb{K}$, so f is linear. Hence

$$\begin{aligned}\langle g(\alpha x+y)-\alpha g(x)-g(y)|z\rangle &= \langle f(\alpha x+y)|f(z)\rangle - \alpha\langle f(x)|f(z)\rangle - \langle f(y)|f(z)\rangle \\ &= 0, \ x, y, z \in D, \ \alpha \in \mathbb{K},\end{aligned}$$

so g is also linear.
We observe that

$$\langle g(x)|y\rangle = \langle f(x)|f(y)\rangle = \overline{\langle f(y)|f(x)\rangle} = \overline{\langle g(y)|x\rangle} = \langle x|g(y)\rangle, \ x, y \in D.$$

We have also

$$\langle g(x)|x\rangle = \langle f(x)|f(x)\rangle \geq 0, \ x \in X.$$

Hence g is a symmetric and positive operator.
Let $z \in \operatorname{im} f$. Then $z = f(y)$ for some $y \in D$. We have

$$\langle f(x)|z\rangle = \langle f(x)|f(y)\rangle = \langle g(x)|y\rangle = \langle x|g(y)\rangle, \ x \in D,$$

so $z \in D(f^*)$.
Finally, for $x \in D$ we have

$$x \in \ker g \iff \bigwedge_{y\in X} \langle g(x)|y\rangle = 0 \iff \bigwedge_{y\in X} \langle f(x)|f(y)\rangle = 0 \iff x \in \ker f.$$

□

Lemma 1. *Let $f: D \to Y$, $g: D \to X$ satisfy the equation*

$$\langle f(x)|f(y)\rangle = \langle g(x)|y\rangle, \ x, y \in D.$$

Then f, g are linear and f is bounded if and only if g is bounded.

Proof. In view of the previous theorem f is linear, g is symmetric, so it is also linear.
($\Rightarrow$) Let f be bounded. We have

$$\begin{aligned}|\langle g(x)|y\rangle| = |\langle f(x)|f(y)\rangle| &\leq \|f(x)\|\cdot\|f(y)\| \\ &\leq \|f\|^2\cdot\|x\|\cdot\|y\|, \ x, y \in D.\end{aligned}$$

Fix $x \in D$. Let $(y_n)_{n\in\mathbb{N}} \subset D$ be a sequence convergent to $g(x)$. Hence

$$\begin{aligned}\|g(x)\|^2 = |\langle g(x)|g(x)\rangle| &= \lim_{n\to\infty} |\langle g(x)|y_n\rangle| \le \lim_{n\to\infty} \|f\|^2 \cdot \|x\| \cdot \|y_n\| \\ &= \|f\|^2 \cdot \|x\| \cdot \|g(x)\|,\end{aligned}$$

which means that g is bounded.
($\Leftarrow$) Let g be bounded. We have

$$\begin{aligned}\|f(x)\|^2 = \langle f(x)|f(x)\rangle = \langle g(x)|x\rangle &\le \|g(x)\| \cdot \|x\| \\ &\le \|g\| \cdot \|x\|^2,\ x \in D,\end{aligned}$$

so f is bounded. □

Theorem 7. *Let D be a dense linear subspace of X, $f: D \to Y$ be linear and* $\operatorname{im} f \subset D(f^*)$. *Then there exists a unique function $g: D \to X$ such that f and g satisfy equation* (3.1). *Moreover $g = f^* \circ f$.*

Proof. Because $\operatorname{im} f \subset D(f^*)$, then $g = f^* \circ f$ is well-defined. Hence

$$\langle f(x)|f(y)\rangle = \left\langle f^*(f(x))\middle|y\right\rangle = \langle g(x)|y\rangle,\ x, y \in D.$$

Assume that there exists $h: D \to X$ such that

$$\langle f(x)|f(y)\rangle = \langle h(x)|y\rangle,\ x, y \in D.$$

Hence

$$\langle g(x) - h(x)|y\rangle = \langle f(x)|f(y)\rangle - \langle f(x)|f(y)\rangle = 0,\ x, y \in D,$$

so $h = g$. □

Theorem 8. *Let D be a dense linear subspace of X, $g: D \to X$ be a symmetric and positive operator,* $\operatorname{im} g$ *be isometric with some subspace of Y. Then there exists a function $f: D \to Y$ such that f and g satisfy equation* (3.1).

Proof. Let $X_0 = \operatorname{cl} \operatorname{im} g$, $T: X_0 \to Y$ be a linear isometry (as an extension of a linear isometry from $\operatorname{im} g$ to Y). In view of Theorem 2 there exists a symmetric operator $h: D(h) \to X$ such that $D \subset D(h)$ and $g = h^2$. It is easy to see that $\operatorname{cl} X_0^\perp = \operatorname{cl} \ker g \subset \operatorname{cl} \ker h$ and hence $\operatorname{cl} \operatorname{im} h = (\ker h)^\perp \subset (\ker g)^\perp = \operatorname{cl} \operatorname{im} g = X_0$. We define a function $f: D \to Y$ by the formula

$$f(x) = T(h(x)),\ x \in D.$$

Then

$$\begin{aligned}\langle f(x)|f(y)\rangle &= \langle T(h(x))|T(h(y))\rangle = \langle h(x)|h(y)\rangle \\ &= \left\langle h^2(x)\middle|y\right\rangle = \langle g(x)|y\rangle,\ x, y \in D.\end{aligned}$$

□

The following example shows that we do not have uniqueness of f in Theorems 5 and 8.

Example 2. Let $f,g,h\colon \mathbb{K}^3 \to \mathbb{K}^3$ be maps given by formulas

$$\begin{aligned} f(x_1,x_2,x_3) &= (x_2,x_1,x_3),\ x_1,x_2,x_3 \in \mathbb{K},\\ h(x_1,x_2,x_3) &= (x_1,x_3,x_2),\ x_1,x_2,x_3 \in \mathbb{K},\\ g &= Id_{\mathbb{K}^3}. \end{aligned}$$

It is easy to see that $f^2 = h^2 = Id_{\mathbb{K}^3}$ and $f^* = f$, $h^* = h$. Hence

$$\langle f(x)|f(y)\rangle = \langle h(x)|h(y)\rangle = \langle x|y\rangle = \langle g(x)|y\rangle\,,\ x,y \in \mathbb{K}^3.$$

But we can also say something about the family of functions f which satisfies equation (3.1) with the same g.

Theorem 9. *Let D be a dense linear subspace of X, $f\colon D \to Y$, $g\colon D \to X$ satisfy equation (3.1). Then $h\colon D \to Y$ satisfies the equation*

$$\langle h(x)|h(y)\rangle = \langle g(x)|y\rangle\,,\ x,y \in D, \tag{3.2}$$

if and only if there exists a linear isometry $I\colon \operatorname{im} f \to Y$ such that $h = I \circ f$.

Proof. ($\Longrightarrow$) Let $h\colon D \to Y$ satisfy equation (3.2). We know that f,h are linear. We define $I\colon \operatorname{im} f \to Y$ by the formula

$$I(f(x)) = h(x),\ x \in D.$$

We observe that

$$\begin{aligned} f(x) = f(y) &\Longleftrightarrow x-y \in \ker f \Longleftrightarrow x-y \in \ker g\\ &\Longleftrightarrow x-y \in \ker h \Longleftrightarrow h(x) = h(y),\ x,y \in D, \end{aligned}$$

so I is well-defined. We have also

$$\begin{aligned} I(\alpha f(x) + f(y)) &= I(f(\alpha x + y)) = h(\alpha x + y) = \alpha h(x) + h(y)\\ &= \alpha I(f(x)) + I(f(y)),\ x,y \in D,\ \alpha \in \mathbb{K} \end{aligned}$$

and

$$\|I(f(x))\|^2 = \|h(x)\|^2 = \langle g(x)|x\rangle = \|f(x)\|^2\,,\ x \in D.$$

Hence I is a linear isometry and $h = I \circ f$.
($\Longleftarrow$) Let $I\colon \operatorname{im} f \to Y$ be a linear isometry, $h = I \circ f$. Then

$$\langle h(x)|h(y)\rangle = \langle I(f(x))|I(f(y))\rangle = \langle f(x)|f(y)\rangle = \langle g(x)|y\rangle\,,\ x,y \in D.$$

□

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PRIMARY-LIKE SUBMODULES AND A SCHEME OVER THE PRIMARY-LIKE SPECTRUM OF MODULES

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Abstract. Let R be a commutative ring with identity and M be a unitary R-module. In this paper, we obtain a scheme $(\mathcal{X}(M), \mathbb{O}_{\mathcal{X}(M)})$ over the primary-like spectrum $\mathcal{X}(M)$ of M and prove that $(\mathcal{X}(M), \mathbb{O}_{\mathcal{X}(M)})$ is a Noetherian scheme when R is a Noetherian ring.

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1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unital. For a submodule N of an R-module M, $(N : M)$ denotes the ideal $\{r \in R \mid rM \subseteq N\}$ and the annihilator of M, denoted by $Ann(M)$, is the ideal $(0 : M)$. An R-module M is called faithful if $Ann(M) = (0)$. A submodule P of an R-module M is said to be p-prime if $P \neq M$ and for $p = (P : M)$, whenever $rm \in P$ (where $r \in R$ and $m \in M$) then $m \in P$ or $r \in p$ [7, 11]. The collection of all prime submodules of M is denoted by $Spec(M)$. If N is a submodule of M, then the radical of N, denoted by $\operatorname{rad} N$, is the intersection of all prime submodules of M containing N, unless no such primes exist, in which case $\operatorname{rad} N = M$ [8].

A submodule Q of M is said to be primary-like if $Q \neq M$ and whenever $rm \in Q$ (where $r \in R$ and $m \in M$) implies $r \in (Q : M)$ or $m \in \operatorname{rad} Q$ [4]. An R-module M is said to be primeful or ψ-module if either $M = (0)$ or $M \neq (0)$ and the map $: Spec(M) \to Spec(R/Ann(M))$, defined by $\psi(P) = (P : M)/Ann(M)$ is surjective [10]. If M/N is a ψ-module over R, then $\sqrt{(N : M)} = (\operatorname{rad} N : M)$ [10, Proposition 5.3]. It is easily seen that, if Q is a primary-like submodule of M such that M/Q is a ψ-module over R, then $(Q : M)$ is a primary ideal of R and so $p = \sqrt{(Q : M)}$ is a prime ideal of R [4, Lemma 2.1], and in this case Q is called a p-primary-like submodule of M. The primary-like spectrum of M denoted by $\mathcal{X}(M)$ is defined to be the set of all primary-like submodules Q of M, where M/Q is a ψ-module.

An R-module M is said to be a ϕ-module if either $M = (0)$ or $M \neq (0)$ and the

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map $\phi : \mathcal{X}(M) \to Spec(R/Ann(M))$ defined by $\phi(Q) = \sqrt{(Q : M)}/Ann(M)$ is surjective. If M is a ϕ-module and p is a prime ideal of R containing $Ann(M)$, then there exists $Q \in \mathcal{X}(M)$ such that $\psi(S_p(Q + pM)) = \phi(Q) = p/Ann(M)$, where $S_p(Q + pM) = \{m \in M \mid cm \in Q + pM$ for some $c \in R \setminus p\}$ is the saturation of $Q + pM$ in M with respect to p. Thus every ϕ-module is a ψ-module; but the following example shows that the converse is not true.

Example 1 (cf. [10, Example 1]). Let Ω be the set of all prime integers, $M = \prod_{p\in\Omega} \frac{\mathbb{Z}}{p\mathbb{Z}}$ and $M' = \bigoplus_{p\in\Omega} \frac{\mathbb{Z}}{p\mathbb{Z}}$, where p runs through Ω. Hence M is a faithful ψ-module over $\mathbb{Z}$ and $Spec(M) = \{M' = S_0(0)\} \cup \{pM : p \in \Omega\}$. Now if ϕ is surjective, then there exists $N \in \mathcal{X}(M)$ such that $\phi(N) = \sqrt{(N : M)} = 0$. It follows that $(N : M) = 0$. Since M/N is a ψ-module, we have $N \subseteq \bigcap_{p\in\Omega} pM = 0$. But 0 is not prime and so is not primary-like because $\mathrm{rad}\,0 = 0$. Hence $N \notin \mathcal{X}(M)$, a contradiction. Thus M is not a ϕ-module.

The Zariski topology on the spectrum of prime ideals of a commutative ring is one of the main tools in algebraic geometry. Recall that the spectrum $Spec(R)$ of a ring R consists of all prime ideals of R and is non-empty. For each ideal I of R, we set $V(I)$ (or $V^R(I)$) $= \{p \in Spec(R) \mid p \supseteq I\}$). Then the sets $V(I)$, where I is an ideal of R, satisfy the axioms for the closed sets of a topology on $Spec(R)$, called the Zariski topology (for example see [3]). It is well-known that for any ring R, there is a sheaf of rings on $Spec(R)$, called the structure sheaf, denoted by $\mathcal{O}_{Spec(R)}$, defined as follows: for each prime ideal p of R, let R_p be the localization of R at p. For an open set $U \subseteq Spec(R)$ with respect to the Zariski topology, we define $\mathcal{O}_{Spec(R)}(U)$ to be the set of functions $r : U \to \coprod_{p\in U} R_p$, such that $r(p) \in R_p$, for each $p \in U$, and such that r is a quotient of elements of R locally: to be precise, we require that for each $p \in U$, there is a neighborhood V of p, contained in U, and there are elements $a, s \in R$, such that for each $p' \in V$, $s \notin p'$ and $r(p') = \frac{a}{s}$ in $R_{p'}$ (see for example [5], for definition and basic properties of the sheaf $\mathcal{O}_{Spec(R)}$).

In the literature, there are many different generalizations of the Zariski topology for modules over commutative rings. For example, Lu has introduced a Zariski topology on $Spec(M)$ whose closed sets are $V(N) = \{P \in Spec(M) \mid (N : M) \subseteq (P : M)\}$ for any submodule N of M [9]. This topological space has been investigated from several point of views (see for example [1, 2, 6, 12]).

As a new generalization of the Zariski topology, the Zariski topology $\mathcal{T}$ on $\mathcal{X}(M)$ is a topology in which closed sets are of the form $\nu(N) = \{Q \in \mathcal{X}(M) \mid \sqrt{(N : M)} \subseteq \sqrt{(Q : M)}\}$ (Lemma 1). There are various generalizations of sheaves from rings to modules in which the sheaf on $Spec(M)$ is the set of all functions $r : Spec(M) \to \coprod_{p\in U} M_p$ with the property similar to that for $Spec(R)$ (some of these types of sheaves have been given and studied in [6, 12]). In parallel, we introduce a sheaf $\mathbb{O}_{\mathcal{X}(M)}$ over $\mathcal{X}(M)$.

We show that the set $\mathcal{B} = \{\mathcal{X}_r : r \in R\}$, where $\mathcal{X}_r = \mathcal{X}(M) - \nu(rM)$ is a basis

for the Zariski topology over $\mathcal{X}(M)$ (Lemma 5). In particular, if M is a ϕ-module, then the elements $\mathcal{X}_r$ of $\mathcal{B}$ are quasi-compact (Corollary 3). This basis is used to show that $\mathcal{O}_{\mathcal{X}(M)}(\mathcal{X}_s) \cong R_s$ for each $s \in R$, where M is a faithful ϕ-module and $R_s = \{\frac{a}{s^n} : a \in R, n \in \mathbb{N}\}$ (Theorem 4). Finally we show that if M is a ϕ-module over a Noetherian ring R and $\mathcal{X}(M)$ is a T_0-space, then $(\mathcal{X}(M), \mathcal{O}_{\mathcal{X}(M)})$ is a Noetherian scheme (Theorem 5).

2. The Zariski topology on $\mathcal{X}(M)$

We begin with a lemma to see that the sets $\nu(N) = \{Q \in \mathcal{X}(M) \mid \sqrt{(N:M)} \subseteq \sqrt{(Q:M)}\}$ satisfy the axioms of closed sets for a topology.

Lemma 1. *Let M be an R-module. Then*

(1) $\nu(0) = \mathcal{X}(M)$ *and* $\nu(M) = \varnothing$.
(2) $\bigcap_{i\in I} \nu(N_i) = \nu(\sum_{i\in I}(N_i : M)M)$, *for each family* $\{N_i \mid i \in I\}$ *of submodules of* M.
(3) $\nu(N) \cup \nu(N') = \nu(N \cap N')$, *for each pair* N, N' *of submodules of* M.

Proof. (1) and (3) are trivial.
(2) Since M/Q is a ψ-module, $(\operatorname{rad} Q : M) = \sqrt{(Q:M)}$ [10, Proposition 5.3]. Also it is easily verified that $((\operatorname{rad} Q : M)M : M) = (\operatorname{rad} Q : M))$. Using these facts we have the following implications.

$$\begin{aligned}
Q \in \bigcap_{i\in I} \nu(N_i) &\Rightarrow \sqrt{(Q:M)} \supseteq \sum_{i\in I}(N_i : M) \\
&\Rightarrow \sqrt{(Q:M)}M \supseteq (\sum_{i\in I}(N_i : M))M \\
&\Rightarrow (\sqrt{(Q:M)}M : M) \supseteq ((\sum_{i\in I}(N_i : M))M : M) \\
&\Rightarrow ((\operatorname{rad} Q : M)M : M) \supseteq ((\sum_{i\in I}(N_i : M))M : M) \\
&\Rightarrow (\operatorname{rad} Q : M) \supseteq ((\sum_{i\in I}(N_i : M))M : M) \\
&\Rightarrow \sqrt{(Q:M)} \supseteq \sqrt{((\sum_{i\in I}(N_i : M))M : M)} \\
&\Rightarrow Q \in \nu((\sum_{i\in I}(N_i : M))M).
\end{aligned}$$

For the reverse inclusion we have

$$\begin{aligned} Q \in \nu(\sum_{i\in I}(N_i : M)M) &\Rightarrow \sqrt{(Q : M)} \supseteq ((\sum_{i\in I}(N_i : M))M : M) \\ &\Rightarrow \sqrt{(Q : M)} \supseteq ((N_i : M)M : M) \ \ \forall i \in I \\ &\Rightarrow \sqrt{(Q : M)} \supseteq (N_i : M) \ \ \forall i \in I \\ &\Rightarrow \sqrt{(Q : M)} \supseteq \sqrt{(N_i : M)} \ \ \forall i \in I \\ &\Rightarrow Q \in \bigcap_{i\in I} \nu(N_i) \end{aligned}$$

□

We will use $\overline{R}$ and $X^{\overline{R}}$ to represent $R/Ann(M)$ and $Spec(R/Ann(M))$ respectively.

Proposition 1. *Let M be an R-module. Then $\phi^{-1}(V^{\overline{R}}(\overline{I})) = \nu(IM)$, for every ideal $I \in V(Ann(M))$. Therefore the map ϕ is continuous with respect to the Zariski topology on $\mathcal{X}(M)$.*

Proof. Suppose $Q \in \phi^{-1}(V^{\overline{R}}(\overline{I}))$. Then $\phi(Q) \in V^{\overline{R}}(\overline{I})$ and so $\sqrt{(Q : M)} \supseteq I$. Hence $\sqrt{(Q : M)} \supseteq \sqrt{(IM : M)}$. Thus $Q \in \nu(IM)$. The argument is reversible and so ϕ is continuous. □

Theorem 1. *Let M be a ϕ-module over a ring R. Then $\phi(\nu(N)) = V^{\overline{R}}(\overline{(N : M)})$ and $\phi(\mathcal{X}(M) - \nu(N)) = X^{\overline{R}} - V^{\overline{R}}(\overline{(N : M)})$ for every submodule N of M, i.e., ϕ is both closed and open.*

Proof. As we have seen in Proposition 1, $\phi^{-1}(V^{\overline{R}}(\overline{I})) = \nu(IM)$, for every ideal $I \in V(Ann(M))$. Hence for every submodule N of M, $\phi^{-1}(V^{\overline{R}}(\overline{(N : M)})) = \nu((N : M)M) = \nu(N)$. So $\phi(\nu(N)) = \phi \circ \phi^{-1}(V^{\overline{R}}(\overline{(N : M)})) = V^{\overline{R}}(\overline{(N : M)})$ as ϕ is surjective. Thus

$$\phi(\mathcal{X}(M) - \nu(N)) = \phi(\phi^{-1}(X^{\overline{R}}) - \phi^{-1}(V^{\overline{R}}(\overline{(N : M)}))) = X^{\overline{R}} - V^{\overline{R}}(\overline{(N : M)})$$

□

Corollary 1. *Let M be an R-module. Then ϕ is a bijection if and only if ϕ is a homeomorphism.*

Proposition 2. *Let M be an R-module and Q, $Q' \in \mathcal{X}(M)$. Then the following statements are equivalent.*

(1) *If $\nu(Q) = \nu(Q')$, then $Q = Q'$;*
(2) *For each $p \in Spec(R)$, the set $\{Q \in \mathcal{X}(M) : \sqrt{(Q : M)} = p\}$ is empty or a singleton set;*
(3) *ϕ is injective.*

Proof. (1) $\Rightarrow$ (2) Let Q, $Q' \in \mathcal{X}(M)$ and $\sqrt{(Q:M)} = \sqrt{(Q':M)} = p$. Then $\nu(Q) = \nu(Q')$. Thus $Q = Q'$ by (1).
(2) $\Rightarrow$ (3) Suppose Q, $Q' \in \mathcal{X}(M)$ and $\phi(Q) = \phi(Q')$. Hence $\sqrt{(Q:M)} = \sqrt{(Q':M)} = p$. Thus $Q = Q'$ by (2).
(3) $\Rightarrow$ (1) is clear. □

Let $\mathcal{Y}$ be a subset of $\mathcal{X}(M)$ for a module M. We will denote the closure of $\mathcal{Y}$ in $\mathcal{X}(M)$ by $\overline{\mathcal{Y}}$.

Proposition 3. *Let M be an R-module and let $\mathcal{Y} = \{Q_1, Q_2, ..., Q_n\}$ be a finite subset of $\mathcal{X}(M)$. Then $\overline{\mathcal{Y}} = \nu(Q_1) \cup ... \cup \nu(Q_n)$.*

Proof. Clearly, $\mathcal{Y} \subseteq \nu(Q_1) \cup ... \cup \nu(Q_n)$. Assume that $\mathcal{F}$ is any closed subset of $\mathcal{X}(M)$ such that $\mathcal{Y} \subseteq \mathcal{F}$. Hence $\mathcal{F} = \nu(N)$ for the submodule N of M. Let $Q \in \nu(Q_1) \cup ... \cup \nu(Q_n)$. Then there exists i $(1 \leq i \leq n)$ such that $Q \in \nu(Q_i)$ and so $\sqrt{(Q_i:M)} \subseteq \sqrt{(Q:M)}$. Since $Q_i \in \mathcal{F}$, $\sqrt{(N:M)} \subseteq \sqrt{(Q_i:M)} \subseteq \sqrt{(Q:M)}$ and hence $Q \in \mathcal{F}$. Hence $\nu(Q_1) \cup ... \cup \nu(Q_n) \subseteq \mathcal{F}$. Thus $\overline{\mathcal{Y}} = \nu(Q_1) \cup ... \cup \nu(Q_n)$. □

The above proposition immediately yields that the following result.

Corollary 2. *Let M be an R-module. Then*

(1) $\overline{Q} = \nu(Q)$ *for every* $Q \in \mathcal{X}(M)$.
(2) $Q' \in \overline{Q}$ *if and only if* $\sqrt{(Q':M)} \supseteq \sqrt{(Q:M)}$ *if and only if* $\nu(Q') \subseteq \nu(Q)$.

Proof. By Proposition 3 is clear. □

A topological space $\mathbb{X}$ is a T_0-space if and only if for any two distinct points in $\mathbb{X}$ there exists an open subset of $\mathbb{X}$ which contains one of the points but not the other. We know that, for any ring R, $Spec(R)$ is always a T_0-space for the usual Zariski topology. In [9, P. 429], it has been shown that if M is a vector space, then $(Spec(M),)$ is not a T_0-space. This example can be used again to show that $(\mathcal{X}(M), \mathcal{T})$ is not also a T_0-space. In fact $\nu(N) = \mathcal{X}(M)$ for every proper vector subspace N of M so that the Zariski topology on $\mathcal{X}(M)$ is the trivial topology even when $|\mathcal{X}(M)| > 1$.

Theorem 2. *Let M be an R-module. Then $\mathcal{X}(M)$ is a T_0-space if and only if one of the statements* (1) − (3) *in Proposition 2 holds.*

Proof. First suppose $\mathcal{X}(M)$ is a T_0-space. We prove the item(1) of proposition 2. For this assume $\nu(Q) = \nu(Q')$ and $Q \neq Q'$. Since $\mathcal{X}(M)$ is a T_0-space, $\overline{Q} \neq \overline{Q'}$. Thus by Corollary 2 we have $\nu(Q) \neq \nu(Q')$, a contradiction. Conversely, suppose that $Q \neq Q' \in \mathcal{X}(M)$ and $\nu(Q) \neq \nu(Q')$. Therefore by Corollary 2, $\overline{Q} \neq \overline{Q'}$. Thus $\mathcal{X}(M)$ is a T_0-space. □

For each $r \in R$, we set $\mathcal{X}_r = \mathcal{X}(M) - \nu(rM)$ and $D_{\overline{r}} = X^{\overline{R}} - V(\overline{R}\overline{r})$. It is easily seen that $\mathcal{X}_{0_R} = \varnothing$, $\mathcal{X}_{1_R} = \mathcal{X}(M)$.

Lemma 2. *Let M be an R-module. Then $\phi(\mathcal{X}_r) \subseteq D_{\overline{r}}$; the equality holds if M is a ϕ-module.*

Proof. By Proposition 1, $\phi^{-1}(D_{\overline{r}}) = \phi^{-1}(X^{\overline{R}} - V(\overline{R}\overline{r})) = \mathcal{X}(M) - \nu(rM) = \mathcal{X}_r$. The equality follows form Theorem 1. □

Lemma 3. *Let $r, s \in R$. Then the following hold.*

(1) $\mathcal{X}_{rs} = \mathcal{X}_r \cap \mathcal{X}_s$.
(2) $\mathcal{X}_{r^n} = \mathcal{X}_r$ *for all* $n \in \mathbb{N}$.
(3) *If r is nilpotent, then* $\mathcal{X}_r = \varnothing$.

Proof. (1) By Proposition 1, $\mathcal{X}_{rs} = \phi^{-1}(D_{\overline{rs}})$. Hence $\mathcal{X}_{rs} = \phi^{-1}(D_{\overline{r}}) \cap \phi^{-1}(D_{\overline{s}}) = \mathcal{X}_r \cap \mathcal{X}_s$.
(2) follows from (1).
(3) Since r is nilpotent, $r^n = 0$ for some $n \in \mathbb{N}$. Hence by (2), $\mathcal{X}_r = \mathcal{X}_{r^n} = \mathcal{X}_0 = \varnothing$. □

Lemma 4. *Let $r, s \in R$ and M be a faithful ϕ-module over R. If $\mathcal{X}_s \subseteq \mathcal{X}_r$, then $s \in \sqrt{Rr}$.*

Proof. Suppose $\mathcal{X}_s \subseteq \mathcal{X}_r$. Hence $\phi(\mathcal{X}_s) \subseteq \phi(\mathcal{X}_r)$. Since M is a ϕ-module, $D_{\overline{s}} \subseteq D_{\overline{r}}$ by Lemma 2. Now since M is faithful, $D_s \subseteq D_r$. Thus we have $s \in \sqrt{Rr}$. □

Lemma 5. *Let M be an R-module. Then the set $\mathcal{B} = \{\mathcal{X}_r : r \in R\}$ forms a basis for the Zariski topology on $\mathcal{X}(M)$.*

Proof. If $\mathcal{X}(M) = \varnothing$, then $\mathcal{B} = \varnothing$ and the proposition is trivially true. Hence we assume that $\mathcal{X}(M) \neq \varnothing$ and let $\mathcal{U}$ be any open set in $\mathcal{X}(M)$. Hence $\mathcal{U} = \mathcal{X}(M) - \nu(IM)$ for some ideal I of R. Note that $\nu(IM) = \nu(\sum_{a_i \in I} a_i M) = \nu(\sum_{a_i \in I} (a_i M : M)M) = \cap_{a_i \in I} \nu(a_i M)$. Hence $\mathcal{U} = \mathcal{X}(M) - \cap_{a_i \in I} \nu(a_i M) = \cup_{a_i \in I} \mathcal{X}_{a_i}$. This proves that $\mathcal{B}$ is a basis for the Zariski topology on $\mathcal{X}(M)$. □

Theorem 3. *Let M be a ϕ-module over a ring R. Then $\mathcal{X}_r$ and $\mathcal{X}_{r_1} \cap \ldots \cap \mathcal{X}_{r_n}$ are quasi-compact subsets of $\mathcal{X}(M)$.*

Proof. For any open covering of $\mathcal{X}_r$, there is a family $\{r_\lambda \in R : \lambda \in \Lambda\}$ of elements of R such that $\mathcal{X}_r \subseteq \cup_{\lambda \in \Lambda} \mathcal{X}_{r_\lambda}$ by Lemma 5. $D_{\overline{r}} = \phi(\mathcal{X}_r) \subseteq \bigcup_{\lambda \in \Lambda} \phi(\mathcal{X}_{r_\lambda}) = \cup_{\lambda \in \Lambda} D_{\overline{r}_\lambda}$ by Proposition 2. It follows that there exists a finite subset Λ' of Λ such that $D_{\overline{r}} \subseteq \cup_{\lambda \in \Lambda} D_{\overline{r}_\lambda}$ as $D_{\overline{r}}$ is quasi-compact, whence by Proposition 2, $\mathcal{X}_r = \phi^{-1}(D_{\overline{r}}) \subseteq \cup_{\lambda \in \Lambda'} \mathcal{X}_{r_\lambda}$. Thus $\mathcal{X}_r$ is quasi-compact. For the other part, it suffices to show that the intersection $\mathcal{X}_{r_1} \cap \mathcal{X}_{r_2}$ is a quasi-compact set. Let Ω be any open covering of $\mathcal{X}_{r_1} \cap \mathcal{X}_{r_2}$. Then Ω also covers each $\mathcal{X}_{r_i}$ $(i = 1, 2)$ which is quasi-compact. Hence each $\mathcal{X}_{r_i}$ has a finite subcover and so $\mathcal{X}_{r_1} \cap \mathcal{X}_{r_2}$ has a finite subcover. □

Corollary 3. *Let M be a ϕ-module over a ring R. Then $\mathcal{X}(M)$ is quasi-compact and has a basis of quasi-compact open subsets.*

3. SHEAF, LOCALLY RINGED SPACE AND SCHEME

Let M be an R-module. For every open subset $\mathcal{U}$ of $\mathcal{X}(M)$ we define $\mathbb{O}_{\mathcal{X}(M)}(\mathcal{U})$ to be a subring of $\prod\{R_p \mid p = \sqrt{(Q:M)}, Q \in \mathcal{U}\}$, the ring of functions $r : \mathcal{U} \to \coprod\{R_p \mid p = \sqrt{(Q:M)}, Q \in \mathcal{U}\}$, where $r(Q) \in R_p$, for each $Q \in \mathcal{U}$ and $p = \sqrt{(Q:M)}$ such that for each $Q \in \mathcal{U}$, there is a neighborhood $\mathcal{V}$ of Q , contained in $\mathcal{U}$, and elements $s, t \in R$, such that for each $Q' \in \mathcal{V}$, $t \notin p' = \sqrt{(Q':M)}$, and $r(Q') = \frac{s}{t}$ in $R_{p'}$. It is easy to check that $\mathbb{O}_{\mathcal{X}(M)}(\mathcal{U})$ is a commutative ring with identity. Furthermore, for open sets $\mathcal{V} \subseteq \mathcal{U}$ we define $\vartheta_{\mathcal{U},\mathcal{V}} : \mathbb{O}_{\mathcal{X}(M)}(\mathcal{U}) \to \mathbb{O}_{\mathcal{X}(M)}(\mathcal{V})$ given by $\{r_p\}_{Q\in\mathcal{U}} \mapsto \{r'_{p'}\}_{Q'\in\mathcal{V}}$, where $p = \sqrt{(Q:M)}$ and $p' = \sqrt{(Q':M)}$. It is easy to check that $\mathbb{O}_{\mathcal{X}(M)}$ is a sheaf of rings.
For any sheaf $\mathbb{O}$ on a topological space $\mathbb{X}$ and for any $x \in \mathbb{X}$, the stalk of $\mathbb{O}$ at x, denoted by $\mathbb{O}_x$, is $\mathbb{O}_x = \{m \mid$ there exists a neighborhood $\mathbb{U}$ of x and $r \in \mathbb{O}_{\mathbb{X}}(\mathbb{U})$ such that m is the germ of r at the point $x\}$. We say that m is the germ of r at the point x if there exists a neighborhood $\mathbb{V}$ containing x such that $\mathbb{V} \subseteq \mathbb{U}$ and $\vartheta_{\mathcal{U},\mathcal{V}}(r) = m$. Two such pairs $< \mathbb{U}, r >$ and $< \mathbb{V}, s >$ define the same element for m of $\mathbb{O}_x$ if and only if there is an open neighborhood $\mathbb{W}$ of x with $\mathbb{W} \subseteq \mathbb{U} \cap \mathbb{V}$ such that $x \in \mathbb{W}$ $r|_{\mathbb{W}} = s|_{\mathbb{W}}$.

Lemma 6. *Let M be an R-module. Then for each $Q \in \mathcal{X}(M)$, the stalk $\mathbb{O}_Q$ of the sheaf $\mathbb{O}_{\mathcal{X}(M)}$ is isomorphic to R_p, where $p = \sqrt{(Q:M)}$.*

Proof. Assume $Q \in \mathcal{X}(M)$ and $m \in \mathbb{O}_Q$. Therefore there exists a neighborhood $\mathcal{U}$ of Q and $r \in \mathbb{O}_{\mathcal{X}(M)}(\mathcal{U})$ such that m is the germ of r at the point Q. For $p = \sqrt{(Q:M)}$ we define $\mu : \mathbb{O}_Q \to R_p$ given by $m \mapsto r(Q)$. It is easy to check that μ is a well-defined homomorphism. Suppose $\mathcal{V}$ is another neighborhood of Q and $s \in \mathbb{O}_{\mathcal{X}(M)}(\mathcal{V})$ such that m is the germ of s at the point Q. Hence there is an open neighborhood $\mathcal{W}$ of Q with $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$ such that $r|_{\mathcal{W}} = s|_{\mathcal{W}}$. Since $Q \in \mathcal{W}$, then $r(Q) = s(Q)$. The map μ is surjective, because any element of R_p can be represented as a quotient $\frac{a}{s}$ with $a \in R$ and $s \in R\backslash p$. Now we define $r(Q') = \frac{a}{s}$ in $R_{p'}$, where $p' = \sqrt{(Q':M)}$ for all $Q' \in \mathcal{X}_s$. Then $r \in \mathbb{O}(\mathcal{X}_s)$. If m is the equivalent class of r in $\mathbb{O}_Q$, then $\mu(m) = \frac{a}{s}$. To show that μ is injective, let $\mathcal{U}$ be a neighborhood of Q, and let $r, r' \in \mathbb{O}_{\mathcal{X}(M)}(\mathcal{U})$ be elements having the same value $r(Q) = r'(Q)$ at Q. By the definition of our sheaf, we may assume that $r = \frac{a}{s}$ and $r' = \frac{a'}{s'}$, where $a, a' \in R$ and $s, s' \in R\backslash p$. Since $\frac{a}{s}$ and $\frac{a'}{s'}$ have the same image in R_p, it follows from the definition of localization that there is an $s'' \in R\backslash p$ such that $s''(s'a - sa') = 0$ in R. Therefore $\frac{a}{s} = \frac{a'}{s'}$ in every local ring $R_{p'}$ such that $s, s', s'' \in R\backslash p'$. But the set of such Q', where $p' = \sqrt{(Q':M)}$ is the open set $\mathcal{X}_s \cap \mathcal{X}_{s'} \cap \mathcal{X}_{s''}$, which contains Q. Hence $r = r'$ in a whole neighborhood of Q, so they have the same stalk at Q. $\square$

A locally ringed space $(\mathbb{X}, \mathbb{O}_{\mathbb{X}})$ is a pair consisting of a topological space $\mathbb{X}$ and a sheaf of rings $\mathbb{O}_{\mathbb{X}}$ all of whose stalks are local rings.

Corollary 4. *Let M be an R-module. Then $(\mathcal{X}(M), \mathbb{O}_{\mathcal{X}(M)})$ is a locally ringed space.*

Proof. Use Lemma 6. □

Let $(\mathbb{X}, \mathbb{O}_{\mathbb{X}})$ be a locally ringed space. The stalk $\mathbb{O}_{\mathbb{X},x}$ of $\mathbb{X}$ at x is said to be the local ring of $\mathbb{X}$ at x. A morphism of ringed spaces $(f, f^\sharp) : (\mathbb{X}, \mathbb{O}_{\mathbb{X}}) \to (\mathbb{Y}, \mathbb{O}_{\mathbb{Y}})$ is given by a continuous map $f : \mathbb{X} \to \mathbb{Y}$ and an f-map of sheaves of rings $f^\sharp : \mathbb{O}_{\mathbb{Y}} \to \mathbb{O}_{\mathbb{X}}$. You can think of $f^\sharp$ as a map $\mathbb{O}_{\mathbb{Y}} \to f_*\mathbb{O}_{\mathbb{X}}$, where $f_*\mathbb{O}_{\mathbb{X}}$ is a sheaf over $\mathbb{X}$ defined by $f_*\mathbb{O}_{\mathbb{X}}(\mathbb{V}) = \mathbb{O}_{\mathbb{X}}(f^{-1}(\mathbb{V}))$ for any open subset $\mathbb{V} \subseteq \mathbb{Y}$. Moreover the restriction map on an inclusion of open sets of $\mathbb{Y}$ is defined naturally. A morphism of locally ringed spaces $(f, f^\sharp) : (\mathbb{X}, \mathbb{O}_{\mathbb{X}}) \to (\mathbb{Y}, \mathbb{O}_{\mathbb{Y}})$ is a morphism of ringed spaces such that for all $x \in \mathbb{X}$ the induced ring map $\mathbb{O}_{\mathbb{Y},f(x)} \to \mathbb{O}_{\mathbb{X},x}$ is a local ring map.

Proposition 4. *Let M and M' be R-modules and $\pi : M \to M'$ be an isomorphism of modules. Then π induces a morphism of locally ringed spaces $(f, f^\sharp) : (\mathcal{X}(M'), \mathbb{O}_{\mathcal{X}(M')}) \to (\mathcal{X}(M), \mathbb{O}_{\mathcal{X}(M)})$.*

Proof. We define $f(Q') = \pi^{-1}(Q')$ For any $Q' \in \mathcal{X}(M')$. It is easily seen that f is well-defined. In the following it is shown that $f^{-1}(\nu(N)) = \nu((N : M)M')$ for any closed set $\nu(N)$ of $\mathcal{X}(M)$ and so f is continuous.

$$\begin{aligned}
Q' \in f^{-1}(\nu(N)) &\Leftrightarrow f(Q') \in \nu(N) \\
&\Leftrightarrow \sqrt{(f(Q') : M)} \supseteq \sqrt{(N : M)} \\
&\Leftrightarrow \sqrt{(f(Q') : M)} \supseteq \sqrt{((N : M)M : M)} \\
&\Leftrightarrow \sqrt{(\pi^{-1}(Q') : M)} \supseteq \sqrt{((N : M)M : M)} \\
&\Leftrightarrow (rad(\pi^{-1}(Q')) : M)M \supseteq (N : M)M \\
&\Leftrightarrow \pi^{-1}(radQ') \supseteq (N : M)M \\
&\Leftrightarrow radQ' \supseteq (N : M)M' \\
&\Leftrightarrow Q' \in \nu((N : M)M').
\end{aligned}$$

Assume $\mathcal{U}$ is an open subset of $\mathcal{X}(M)$ and $r \in \mathbb{O}_{\mathcal{X}(M)}(\mathcal{U})$. Let $Q \in f^{-1}(\mathcal{U})$. Then $f(Q) = \pi^{-1}(Q) \in \mathcal{U}$. Assume that $\mathcal{W}$ is an open neighborhood of $\pi^{-1}(Q)$ with $\mathcal{W} \subseteq \mathcal{U}$ and $a, s \in R$ such that for each $Q' \in \mathcal{W}$, $s \notin p' = \sqrt{(Q' : M)}$ and $r(Q') = \frac{a}{s}$ in $R_{p'}$. Since $\pi^{-1}(Q) \in \mathcal{W}$, then $Q \in f^{-1}(\mathcal{W})$. Since f is continuous, $f^{-1}(\mathcal{W})$ is an open subset of $\mathcal{X}(M')$. We show that for each $Q'' \in f^{-1}(\mathcal{W})$ we have $s \notin \sqrt{(Q'' : M')}$. Suppose, on the contrary, $s \in \sqrt{(Q'' : M')}$ for some $Q'' \in f^{-1}(\mathcal{W})$. So $\pi^{-1}(Q'') = f(Q'') \in \mathcal{W}$. Since π is an epimorphism, $\sqrt{(Q'' : M')} = \sqrt{(\pi^{-1}(Q'') : M)}$. Hence $s \in \sqrt{(\pi^{-1}(Q'') : M)}$, a contradiction. Therefore, we can

define $f^\sharp(\mathcal{U}) : \mathbb{O}_{\mathcal{X}(M)}(\mathcal{U}) \to \mathbb{O}_{\mathcal{X}(M')}(f^{-1}(\mathcal{U})$ given by $f^\sharp(\mathcal{U})(r) = r \circ f$. Suppose $\mathcal{V} \subseteq \mathcal{U}$ and $Q \in f^{-1}(\mathcal{V})$. According to the commutativity of the following diagram:

$$\begin{tikzcd}
f^{-1}(\mathcal{V}) \arrow[r, "f"] \arrow[d, "i"'] & \mathcal{V} \arrow[r, "r'|_{\mathcal{V}}"] \arrow[d, "i"] & R_{\sqrt{Q:M}} \\
f^{-1}(\mathcal{U}) \arrow[r, "f"'] & \mathcal{U} \arrow[ur, "r'"'] &
\end{tikzcd}$$

We have $(r' \circ f)|_{f^{-1}(\mathcal{V})}(Q) = r'|_{\mathcal{V}} \circ f(Q)$. Now, we show that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{O}_{\mathcal{X}(M)}(\mathcal{U}) \arrow[r, "f^\sharp(\mathcal{U})"] \arrow[d, "\rho_{\mathcal{U},\mathcal{V}}"'] & \mathbb{O}_{\mathcal{X}(M')}(f^{-1}(\mathcal{U})) \arrow[d, "\rho'_{f^{-1}(\mathcal{U}),f^{-1}(\mathcal{V})}"] \\
\mathbb{O}_{\mathcal{X}(M)}(\mathcal{V}) \arrow[r, "f^\sharp(\mathcal{V})"'] & \mathbb{O}_{\mathcal{X}(M')}(f^{-1}(\mathcal{V})
\end{tikzcd}$$

Suppose that $r' \in \mathbb{O}_{\mathcal{X}(M)}(\mathcal{U})$. For each $Q \in \mathcal{U}$, we have

$$\rho'_{f^{-1}(\mathcal{U}),f^{-1}(\mathcal{V})} f^\sharp(\mathcal{U})(r')(Q) = \rho'_{f^{-1}(\mathcal{U}),f^{-1}(\mathcal{V})}(r' \circ f)(Q) =$$
$$(r' \circ f)|_{f^{-1}(\mathcal{V})}(Q) = r'|_{\mathcal{V}} \circ f(Q) = \rho_{\mathcal{U},\mathcal{V}}(r') \circ f(Q) = f^\sharp(\mathcal{V})\rho_{\mathcal{U},\mathcal{V}}(r')(Q).$$

It follows that $f^\sharp : \mathbb{O}_{\mathcal{X}(M)} \to f_*\mathbb{O}_{\mathcal{X}(M')}$ is a morphism of sheaves. By Lemma 6, the map $f^\sharp_Q : \mathbb{O}_{\mathcal{X}(M),f(Q)} \to \mathbb{O}_{\mathcal{X}(M'),Q}$ on stalks is clearly the map of local rings $R_{\sqrt{(f(Q):M)}} \to R_{\sqrt{(Q:M')}}$. Thus the proof is completed. □

Proposition 5. *Let $g : R \to R'$ be a ring homomorphism, M' be an R'-module and M be a ϕ-module over R such that $\mathcal{X}(M)$ is a T_0-space and $Ann_R(M) \subseteq Ann_R(M')$. Then g induces a morphism of locally ringed spaces $(f, f^\sharp) : (\mathcal{X}(M'), \mathbb{O}_{\mathcal{X}(M')}) \to (\mathcal{X}(M), \mathbb{O}_{\mathcal{X}(M)})$.*

Proof. Since $Ann_R(M) \subseteq Ann_R(M')$, then $\overline{g} : \overline{R} \to \overline{R'}$ is induced by g. It is well known $h : Spec(R') \to Spec(R)$ given by $p \mapsto g^{-1}(p)$ and $\overline{h} : X^{\overline{R'}} \to X^{\overline{R}}$ given by $\overline{p} \mapsto \overline{g}^{-1}(\overline{p})$ are continuous maps. Also by Proposition 1, $\phi_{M'} : \mathcal{X}(M') \to X^{\overline{R'}}$ is a continuous map and by Corollary 1 and Theorem 2, $\phi_M : \mathcal{X}(M) \to X^{\overline{R}}$ is a homeomorphism. Therefore the map $f : \mathcal{X}(M') \to \mathcal{X}(M)$ given by $Q \mapsto \phi_M^{-1} \circ \overline{h} \circ \phi_{M'}(Q)$ is continuous. For each $Q \in \mathcal{X}(M')$, we get a local homomorphism

$$g_{\sqrt{(Q:_{R'}M')}} : R_{h(\sqrt{(Q:_{R'}M')})} \to R'_{\sqrt{(Q:_{R'}M')}}$$

given by $\frac{r}{s} \mapsto \frac{g(r)}{g(s)}$. This map is well-defined, because if $s \notin h(\sqrt{(Q :_{R'} M')}) = g^{-1}(\sqrt{(Q :_{R'} M')})$, then $g(s) \notin \sqrt{(Q :_{R'} M')}$. Let $\mathcal{U} \subseteq \mathcal{X}(M)$ be an open subset and $r \in \mathcal{O}_{\mathcal{X}(M)}(\mathcal{U})$. Suppose $Q \in f^{-1}(\mathcal{U})$. Then $f(Q) \in \mathcal{U}$ and there exists a neighborhood $\mathcal{W}$ of $f(Q)$ with $\mathcal{W} \subseteq \mathcal{U}$ and elements $a, s \in R$ such that for each $Q' \in \mathcal{W}$, we have $s \notin \sqrt{(Q' :_R M)}$ and $r(Q') = \frac{a}{s} \in R_{\sqrt{(Q':_R M)}}$. Hence $s \notin \sqrt{(f(Q) :_R M)}$. By definition of f, we have

$$\begin{aligned} f(Q) &= (\phi_M^{-1} \circ \bar{h} \circ \phi_{M'}^{-1})(Q) = (\phi_M^{-1} \circ \bar{h})(\overline{\sqrt{(Q :_{R'} M')}}) \\ &= \phi_M^{-1}(\bar{g}^{-1}(\overline{(\sqrt{(Q :_{R'} M')})}) = \phi_M^{-1}(\overline{g^{-1}(\sqrt{(Q :_{R'} M')})}) \\ &= K, \end{aligned}$$

for some $K \in \mathcal{X}(M)$. Now since M is a ϕ-module, $\overline{\sqrt{(K :_R M)}} = \phi_M(K) = \overline{g^{-1}(\sqrt{(Q :_{R'} M')})}$ and hence $\sqrt{(f(Q) :_R M)} = \sqrt{(K :_R M)} = g^{-1}(\sqrt{(Q :_{R'} M')})$. Therefore $s \notin \sqrt{(f(Q) :_R M)}$ follows that $g(s) \notin \sqrt{(Q :_{R'} M')}$. Thus $g_{\sqrt{(Q:_{R'}M')}}(\frac{a}{s})$ define a section on $\mathcal{O}_{\mathcal{X}(M')}(f^{-1}(\mathcal{W}))$. Since

$$\begin{tikzcd} R_s \arrow[r] \arrow[d] & R'_{g(s)} \arrow[d] \\ R_{g^{-1}(\sqrt{(Q:_{R'}M')})} \arrow[r] & R'_{\sqrt{(Q:_{R'}M')}} \end{tikzcd}$$

is a commutative diagram of natural maps, we define

$$f^\sharp(\mathcal{U}) : \mathcal{O}_{\mathcal{X}(M)}(\mathcal{U}) \to f_*\mathcal{O}_{\mathcal{X}(M')}(\mathcal{U}) = \mathcal{O}_{\mathcal{X}(M')}(f^{-1}(\mathcal{U}))$$

which is given by $f^\sharp(\mathcal{U})(r)(Q) = g_{\sqrt{(Q:_{R'}M')}}(r(f(Q)))$ for each $r \in \mathcal{O}_{\mathcal{X}(M)}(\mathcal{U})$ and $Q \in f^{-1}(\mathcal{U})$. Suppose $\mathcal{V} \subseteq \mathcal{U}$ and $Q \in f^{-1}(\mathcal{V})$. According to the following commutative diagram
We have $g_{\sqrt{(Q:_{R'}M')}} \circ r|_{\mathcal{V}} \circ f(Q) = (g_{\sqrt{(Q:_{R'}M')}} \circ r \circ f)|_{f^{-1}(\mathcal{V})}(Q)$. Considering the diagram
It is easy to check that

$$\rho'_{f^{-1}(\mathcal{U}),f^{-1}(\mathcal{V})} f^\sharp(\mathcal{U})(r)(Q) = \rho'_{f^{-1}(\mathcal{U}),f^{-1}(\mathcal{V})} g_{\sqrt{(Q:_{R'}M')}} r \circ f(Q) = (g_{\sqrt{(Q:_{R'}M')}} r \circ f)|_{f^{-1}(\mathcal{V})}(Q) = g_{\sqrt{(Q:_{R'}M')}} r|_{\mathcal{V}} \circ f(Q) = f^\sharp(\mathcal{V})(r|_{\mathcal{V}})(Q) = f^\sharp(\mathcal{V})\rho_{\mathcal{U},\mathcal{V}}(r)(Q).$$

$$\begin{tikzcd}
f^{-1}(\mathcal{U}) \arrow[r, "f"] & \mathcal{U} \arrow[dr, "r"] & & \\
f^{-1}(\mathcal{V}) \arrow[u, "i"] \arrow[r, "f"] \arrow[drr, "g_{\sqrt{(Q:_{R'}M')}} \circ r\,|_{\mathcal{V}} \circ f"'] & \mathcal{V} \arrow[u, "i"] \arrow[r, "r\,|_{\mathcal{V}}"] & R_{g^{-1}(\sqrt{(Q:_{R'}M')})} \arrow[d, "g_{\sqrt{(Q:_{R'}M')}}"] & \\
& & R'_{\sqrt{(Q:_{R'}M')}} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{O}_{\mathcal{X}(M)}(\mathcal{U}) \arrow[r, "f^\sharp(\mathcal{U})"] \arrow[d, "\rho_{\mathcal{U},\mathcal{V}}"'] & \mathbb{O}_{\mathcal{X}(M')}(f^{-1}(\mathcal{U})) \arrow[d, "\rho'_{f^{-1}(\mathcal{U}),f^{-1}(\mathcal{V})}"] \\
\mathbb{O}_{\mathcal{X}(M)}(\mathcal{V}) \arrow[r, "f^\sharp(\mathcal{V})"'] & \mathbb{O}_{\mathcal{X}(M')}(f^{-1}(\mathcal{V})
\end{tikzcd}$$

Thus the diagram is commutative and it follows that $f^\sharp : \mathbb{O}_{\mathcal{X}(M)} \to f_*\mathbb{O}_{\mathcal{X}(M')}$ is a morphism of sheaves. By Lemma 6, the map $f^\sharp_Q : \mathbb{O}_{\mathcal{X}(M),f(Q)} \to \mathbb{O}_{\mathcal{X}(M'),Q}$ on stalks is clearly $R_{h(\sqrt{(Q:_{R'}M')})} \to R'_{\sqrt{(Q:_{R'}M')}}$. Thus the proof is completed. $\square$

Theorem 4. *Let $s \in R$ and M be a faithful ϕ-module over a ring R. Then $\mathbb{O}_{\mathcal{X}(M)}(\mathcal{X}_s) \cong R_s$.*

Proof. Suppose $\mu : R_s \to \mathbb{O}_{\mathcal{X}(M)}(\mathcal{X}_s)$ given by $\frac{a}{s^n} \mapsto (r : Q \mapsto \frac{a}{s^n} \in R_{\sqrt{(Q:M)}})$. Indeed μ sends that $\frac{a}{s^n}$ to the section $r \in \mathbb{O}_{\mathcal{X}(M)}(\mathcal{X}_s)$ which assigns to each Q the image of $\frac{a}{s^n} \in R_{\sqrt{(Q:M)}}$. It is clear that $\mu(\frac{a}{s^n})$ is unique, since the range of r is $\frac{a}{s^n}$. Therefore to show that μ is well-defined, it suffices to verify that $s^n \notin \sqrt{(Q:M)}$. Since $Q \in \mathcal{X}_s = \mathcal{X}(M) - \nu(sM)$, we have $\sqrt{(sM:M)} \nsubseteq \sqrt{(Q:M)}$. Now if $s^n \in \sqrt{(Q:M)}$ (or equivalently $s \in \sqrt{(Q:M)}$), we have

$$\begin{aligned}
r \in \sqrt{(sM:M)} &\Rightarrow r^n M \subseteq sM \subseteq \sqrt{(Q:M)}M \quad \text{for some } n > 0 \\
&\Rightarrow r^n \in (\sqrt{(Q:M)}M : M) = ((\mathrm{rad}(Q):M)M : M) \\
&\Rightarrow r^n \in (\mathrm{rad}(Q):M) = \sqrt{(Q:M)} \\
&\Rightarrow r \in \sqrt{(Q:M)}
\end{aligned}$$

which gives the contradiction $\sqrt{(sM:M)} \subseteq \sqrt{(Q:M)}$. Moreover μ is a homomorphism, since $\mathbb{O}_{\mathcal{X}(M)}(\mathcal{X}_s)$ is a ring with the operations $(r_1 + r_2)(Q) = r_1(Q) + r_2(Q)$ and $(r_1 r_2)(Q) = r_1(Q) r_2(Q)$. Now we are going to show that μ is injective. Let $\mu(\frac{a}{s^n}) = \mu(\frac{a'}{s^m})$, then for every $Q \in \mathcal{X}_s$, $\frac{a}{s^n}$ and $\frac{a'}{s^m}$ have the same image in

R_p, where $p = \sqrt{(Q:M)}$. Thus there exists $t \in R\backslash p$ such that $t(s^m a - s^n a') = 0$. Let $I = Ann(s^m a - s^n a')$. Then $t \in I$ and $t \notin p$, so $I \not\subseteq p$. This happens for any $Q \in \mathcal{X}_s$. Hence we conclude that $V(I) \cap \{\sqrt{(Q:M)} \mid Q \in \mathcal{X}_s\} = \varnothing$ and so $\{\sqrt{(Q:M)} \mid Q \in \mathcal{X}_f\} \subseteq Spec(R) - V(I)$. Since M is a ϕ-module, by Lemma 2 we have

$$D_s = \{\sqrt{(Q:M)} \mid Q \in \mathcal{X}_s\} \subseteq D(I).$$

Therefore $s \in \sqrt{I}$ and so $s^l \in I$ for some positive integer l. Now we have $s^l(s^m a - s^n a') = 0$ which shows that $\frac{a}{s^n} = \frac{a'}{s^m}$ in R_p. Thus μ is injective. Now we show that μ surjective. Assume $r \in \mathcal{O}_{\mathcal{X}(M)}(\mathcal{X}_s)$. Then we can cover $\mathcal{X}_s$ with open subset $\mathcal{V}_i$, on which s is represented by $\frac{a_i}{b_i}$, with $b_i \notin \sqrt{(Q:M)}$ for all $Q \in \mathcal{V}_i$ and so $\mathcal{V}_i \subseteq \mathcal{X}_{b_i}$. By Lemma 5, the open sets of the form $\mathcal{X}_k$ form a basis for the Zariski topology. So, we may assume that $\mathcal{V}_i = \mathcal{X}_{k_i}$ for some $k_i \in R$. Since $\mathcal{X}_{k_i} \subseteq \mathcal{X}_{b_i}$, by Lemma 4, $k_i \in \sqrt{Rb_i}$. Thus $k_i^n \in Rb_i$ for some $n \in \mathbb{N}$. So $k_i^n = cb_i$ and $\frac{a_i}{b_i} = \frac{ca_i}{cb_i} = \frac{ca_i}{k_i^n}$. We see that r is represented by $\frac{a_i'}{h_i}$, $(a_i' = ca_i, h_i = k_i^n)$ on $\mathcal{X}_{h_i}$ and (since $\mathcal{X}_{k_i} = \mathcal{X}_{k_i^n}$) the $\mathcal{X}_{h_i}$ cover $\mathcal{X}_s$. The open cover $\mathcal{X}_s = \cup \mathcal{X}_{h_i}$ has a finite subcover by Theorem 3. Assume $\mathcal{X}_s \subseteq \mathcal{X}_{h_1} \cup \cdots \cup \mathcal{X}_{h_n}$. For $1 \leq i, j \leq n$, $\frac{a_i'}{h_i}$ and $\frac{a_j'}{h_j}$ both represent r on $\mathcal{X}_{h_i} \cap \mathcal{X}_{h_j}$. By Lemma 3 $\mathcal{X}_{h_i} \cap \mathcal{X}_{h_j} = \mathcal{X}_{h_i h_j}$ and by injectivity of μ, we get $\frac{a_i'}{h_i} = \frac{a_j'}{h_j}$ in $R_{h_i h_j}$. Hence for some n_{ij}, we have $(h_i h_j)^{n_{ij}}(h_j a_i' - h_i a_j') = 0$. Let $m = max\{n_{ij} \mid 1 \leq i, j \leq n\}$. Then

$$h_j^{m+1}(h_i a_i') - h_i^{m+1}(h_j a_j') = 0.$$

By replacing each h_i by h_i^{m+1}, and a_i' by $h_i a_i'$, we still see that r is represented on $\mathcal{X}_{h_i}$ by $\frac{a_i'}{h_i}$, and furthermore, we have $h_j a_i' = h_i a_j'$ for all i, j. Since $\mathcal{X}_s \subseteq \mathcal{X}_{h_1} \cup \cdots \cup \mathcal{X}_{h_n}$, by Lemma 2 we have

$$D_s = \phi(\mathcal{X}_s) \subseteq \cup_{i=1}^n \phi(\mathcal{X}_{h_i}) = \cup_{i=1}^n D_{h_i}.$$

Hence there are $c_1, \cdots, c_n \in R$ and $n' \in \mathbb{N}$, such that $s^{n'} = \sum_i c_i h_i$. Let $a = \sum_i c_i a_i'$. Then for each j we have

$$h_j a = \sum_i c_i a_i' h_j = \sum_i c_i h_i a_j' = a_j' s^{n'}.$$

It follows that $\frac{a}{s^{n'}} = \frac{a_j'}{h_j}$ on $\mathcal{X}_{h_j}$. So $\mu(\frac{a}{s^{n'}}) = r$ everywhere, which shows that μ is surjective. □

Corollary 5. *Let M be a faithful ϕ-module over a ring R. Then $\mathcal{O}_{\mathcal{X}(M)}(\mathcal{X}(M)) \cong R$.*

Proof. Use Theorem 4. □

An affine scheme is a locally ringed space isomorphic as a locally ringed space to $Spec(R)$ for some ring R. A scheme is a locally ringed space with the property that every point has an open neighborhood which is an affine scheme. A scheme is locally Noetherian if it can be covered by open affine subsets $Spec(R_i)$, where each R_i is a Noetherian ring. A scheme is Noetherian if it is locally Noetherian and quasi-compact [5].

Theorem 5. *Let M be a ϕ-module over a ring R such that $\mathcal{X}(M)$ is a T_0-space. Then $(\mathcal{X}(M), \mathcal{O}_{\mathcal{X}(M)})$ is a scheme. Moreover, if R is Noetherian, then $(\mathcal{X}(M), \mathcal{O}_{\mathcal{X}(M)})$ is a Noetherian scheme.*

Proof. Suppose $r \in R$. Therefore by Proposition 1, $\phi|_{\mathcal{X}_r}$ is continuous. Also by Theorem 2, $\phi|_{\mathcal{X}_r}$ is a bijection. Let $\mathcal{F}$ be a closed subset of $\mathcal{X}_r$. Then $\mathcal{F} = \mathcal{X}_r \cap \nu(N)$ for some submodule N of M. Hence $\phi(\mathcal{F}) = \phi(\mathcal{X}_r) \cap V(\sqrt{(N : M)})$ is a closed subset of $\phi(\mathcal{X}_r)$. Thus $\phi|_{\mathcal{X}_r}$ is a homeomorphism. Assume that $\mathcal{X}(M) = \cup_{i \in I} \mathcal{X}_{r_i}$. Since ϕ is a bijection, then for $i \in I$ we have $\mathcal{X}_{r_i} \cong \phi(\mathcal{X}_{r_i}) = \{\sqrt{(Q : M)} \mid Q \in \mathcal{X}_{r_i}\} = D_{r_i} \cong Spec(R_{r_i})$. Thus by Theorem 4, $\mathcal{X}_{r_i}$ is an affine scheme. So it implies that $(\mathcal{X}(M), \mathcal{O}_{\mathcal{X}(M)})$ is a scheme. For the last statement, since R is Noetherian, so is R_{r_i} for each $i \in I$. Hence $(\mathcal{X}(M), \mathcal{O}_{\mathcal{X}(M)})$ is a locally Noetherian scheme. By Corollary 3, $\mathcal{X}(M)$ is quasi-compact. Thus $(\mathcal{X}(M), \mathcal{O}_{\mathcal{X}(M)})$ is a Noetherian scheme. □

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THE MIXED BVP FOR THE SECOND ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATION AT RESONANCE

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Abstract. Efficient sufficient conditions are established for the solvability of the mixed problem

$$u''(t) = p(t)u(t) + f(t,u(t)) + h(t), \qquad u(a) = 0, \quad u'(b) = 0,$$

where $h, p \in L([a,b];R)$ and $f \in K([a,b] \times R; R)$, in the case where the homogeneous linear problem $w''(t) = p(t)w(t)$, $w(a) = 0$, $w'(b) = 0$ has nontrivial solutions.

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1. Introduction

Consider on the set $I = [a,b]$ the second order nonlinear ordinary differential equation

$$u''(t) = p(t)u(t) + f(t,u(t)) + h(t) \quad \text{for} \quad t \in I \tag{1.1}$$

with the boundary conditions

$$u(a) = 0, \quad u'(b) = 0, \tag{1.2}$$

where $h, p \in L(I;R)$ and $f \in K(I \times R; R)$. By a solution of problem (1.1), (1.2) we understand a function $u \in \widetilde{C}'(I,R)$, which satisfies equation (1.1) almost everywhere on I and satisfies conditions (1.2).

Along with (1.1), (1.2) we consider the homogeneous problem

$$w''(t) = p(t)w(t) \quad \text{for} \quad t \in I, \tag{1.3}$$

$$w(a) = 0, \quad w'(b) = 0. \tag{1.4}$$

At present, the foundations of the general theory of two-point boundary value problems are already laid and problems of this type are studied by many authors and investigated in detail (see, for instance, [3, 4, 10, 11, 13, 14] and references therein). On the other hand, in all of these works, only the non-resonance case is considered. An analysis of the available literature shows that, in contrast to the Dirichlet problem, the case where the problem (1.3), (1.4) has nontrivial solutions is practically unstudied. It should be noted that, in the majority of works on this subject, the Dirichlet

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boundary value problem for the second order ordinary differential equation with the corresponding homogeneous problem possessing a nontrivial solution studied in the case where the first coefficient of the homogeneous linear problem is a constant and, more precisely, only in the simplest case where this constant is the first eigenvalue of the homogeneous linear problem (see, for instance, [1, 2, 4–8, 10, 15] and references therein). In [16], we developed a technique which allowed us to establish efficient sufficient conditions (Landesman–Lazer's type conditions) for the solvability of Dirichlet BVP for second order ODE in the case where the first coefficient of the homogeneous linear equation is a Lebesgue integrable function (not necessarily constant) and no information is assumed on the number of zeros of the solution. (In particular, if the first coefficient in homogeneous linear equation is constant, we are able to study the cases where this constant not necessarily coincides with the first eigenvalue of the corresponding homogeneous linear problem). The theorems proved there significantly generalize and improve a number of previous results of other authors (see [1, 2, 4, 6, 15]).

In the present paper we generalize the method developed in article [16] for the Dirichlet boundary value problem, and prove Landesman–Lazer's type efficient sufficient conditions for solvability of problem (1.1), (1.2) in the case when the function $p \in L(I;R)$ is not necessarily constant, under the assumption that the homogeneous problem (1.3), (1.4) has a nontrivial solution which may have arbitrarily many zeros in the interval $]a,b]$.

The results presented here are new and generalize Fredholm's third theorem for nonlinear ODE in the sense that the known Fredholm theorem is obtained in the special case where $f(t,x) \equiv 0$.

Throughout the paper we use the following notations:

N is the set of all natural numbers. R is the set of all real numbers, $R_+ = [0,+\infty[$.

$C(I;R)$ is the Banach space of continuous functions $u : I \to R$ with the norm $\|u\|_C = \max\{|u(t)| : t \in I\}$.

$\widetilde{C}'(I;R)$ is the set of functions $u : I \to R$ which are absolutely continuous together with their first derivatives.

$L(I;R)$ is the Banach space of the Lebesgue integrable functions $p : I \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|ds$.

$K(I \times R;R)$ is the set of the functions $f : I \times R \to R$ satisfying the Carathéodory conditions, i.e., $f(\cdot,x) : I \to R$ is a measurable function for all $x \in R$, $f(t,\cdot) : R \to R$ is a continuous function for almost all $t \in I$, and for every $r > 0$ there exists $q_r \in L(I;R_+)$ such that $|f(t,x)| \le q_r(t)$ for almost all $t \in I$, $|x| \le r$.

Having $w : I \to R$, we put:

$$N_w \overset{\text{def}}{=} \{t \in]a,b] : w(t) = 0\},$$

$$\Omega_w^+ \overset{\text{def}}{=} \{t \in I : w(t) > 0\}, \quad \Omega_w^- \overset{\text{def}}{=} \{t \in I : w(t) < 0\},$$

and $[w(t)]_+ = (|w(t)| + w(t))/2$, $[w(t)]_- = (|w(t)| - w(t))/2$ for $t \in I$.

Definition 1. Let A be a finite (eventually empty) subset of I. We say that $f \in E(A)$, if $f \in K(I \times R; R)$ and, for any measurable set $G \subseteq I$ and an arbitrary constant $r > 0$, we can choose $\varepsilon > 0$ such that if

$$\int_G |f(s,x)|ds \neq 0 \ \text{ for } \ x \geq r \ (x \leq -r)$$

then

$$\int_{G \setminus U_\varepsilon} |f(s,x)|ds - \int_{U_\varepsilon} |f(s,x)|ds \geq 0 \quad \text{for} \quad x \geq r \ (x \leq -r),$$

where $U_\varepsilon = I \cap \left(\cup_{k=1}^n]t_k - \varepsilon/2n, t_k + \varepsilon/2n[\right)$ if $A = \{t_1, t_2, \ldots, t_n\}$, and $U_\varepsilon = \varnothing$ if $A = \varnothing$.

Remark 1. If $f \in K(I \times R; R)$ then $f \in E(\varnothing)$.

Remark 2. It is clear that if $f(t,x) \overset{\text{def}}{\equiv} f_0(t)g_0(x)$, where $f_0 \in L(I;R)$ and $g_0 \in C(I;R)$, then $f \in E(A)$ for every finite set $A \subset I$.

The example below shows that there exists a function $f \in K(I \times R; R)$ such that $f \notin E(\{t_1, \ldots, t_k\})$ for some points $t_1, \ldots, t_k \in I$.

Example 1. Let $f(t,x) = |t|^{-1/2} g(t,x)$ for $t \in [-1, 0[\cup]0, 1]$, $x \in R$, and $f(0,.) \equiv 0$, where $g(-t,x) = g(t,x)$ for $t \in]-1,1]$, $x \in R$, and

$$g(t,x) = \begin{cases} x & \text{for} \quad x \leq 1/t, \ t > 0 \\ 1/t & \text{for} \quad x > 1/t, \ t > 0 \end{cases}.$$

Then $f \in K([0,1] \times R; R)$ and it is clear that $f \notin E(\{0\})$ because, for every $\varepsilon > 0$, if $x \geq 1/\varepsilon$ then $\int_\varepsilon^1 f(s,x)ds - \int_0^\varepsilon f(s,x)ds = 4(\varepsilon^{-1/2} - x^{1/2}) - 2 < 0$.

2. Main results

Theorem 1. *Let $i \in \{0,1\}$, w be a nonzero solution of the problem* (1.3), (1.4), *$f \in E(N_w)$, there exists a constant $r > 0$ such that the function $(-1)^i f$ is nondecreasing in the second argument for $|x| \geq r$,*

$$(-1)^i f(t,x) \operatorname{sgn} x \geq 0 \quad \textit{for} \quad t \in I, \ |x| \geq r, \tag{2.1}$$

$$\int_{\Omega_w^+} |f(s,r)|ds + \int_{\Omega_w^-} |f(s,-r)|ds \neq 0, \tag{2.2}$$

and

$$\lim_{|x| \to +\infty} \frac{1}{|x|} \int_a^b |f(s,x)|ds = 0. \tag{2.3}$$

Then there exists $\delta > 0$ such that the problem (1.1), (1.2) has at least one solution for every h satisfying the condition

$$\Big|\int_a^b h(s)w(s)ds\Big| < \delta\|w\|_C. \tag{2.4}$$

Corollary 1. *Let the assumptions of Theorem 1 be satisfied and*

$$\int_a^b h(s)w(s)ds = 0. \tag{2.5}$$

Then the problem (1.1), (1.2) has at least one solution.

Example 2. From Theorem 1 it follows that the problem

$$u''(t) = -\lambda^2 u(t) + \sigma|u(t)|^\alpha \operatorname{sgn} u(t) + h(t) \quad \text{for} \quad 0 \le t \le \pi/2 \tag{2.6}$$

$$u(0) = 0, \quad u'(\pi/2) = 0, \tag{2.7}$$

with $\lambda = 2k-1\,(k \in N)$, $\sigma \in \{-1,1\}$, and $\alpha \in]0,1[$ has at least one solution if $h \in L([0,\pi/2],R)$ is such that $\int_0^{\pi/2} h(s)\sin\lambda s ds = 0$.

Theorem 2. *Let $i \in \{0,1\}$, w be a nonzero solution of the problem (1.3), (1.4), $f(t,x) \overset{\text{def}}{\equiv} f_0(t)g_0(x)$ with $f_0 \in L(I;R_+)$, $g_0 \in C(R;R)$, there exists a constant $r > 0$ such that $(-1)^i g_0$ is non-decreasing for $|x| \ge r$ and*

$$(-1)^i g_0(x)\operatorname{sgn} x \ge 0 \qquad \textit{for} \quad |x| \ge r. \tag{2.8}$$

Let, moreover,

$$|g_0(r)|\int_{\Omega_w^+} f_0(s)ds + |g_0(-r)|\int_{\Omega_w^-} f_0(s)ds \ne 0 \tag{2.9}$$

and

$$\lim_{|x|\to+\infty} |g_0(x)| = +\infty, \qquad \lim_{|x|\to+\infty} \frac{g_0(x)}{x} = 0. \tag{2.10}$$

Then, for every $h \in L(I;R)$, the problem (1.1), (1.2) has at least one solution.

Example 3. From Theorem 2 it follows that the equation

$$u''(t) = p_0(t)u(t) + p_1(t)|u(t)|^\alpha \operatorname{sgn} u(t) + h(t) \quad \text{for} \quad t \in I, \tag{2.11}$$

with the conditions (1.2) has at least one solution for arbitrary $\alpha \in]0,1[$ $p_0, h \in L(I;R)$, and such $p_1 \in L(I;R)$ that the condition $\sigma p_1(t) > 0$ for $t \in I$ holds, where $\sigma \in \{-1,1\}$.

Theorem 3. *Let $i \in \{0,1\}$ and w be a nonzero solution of the problem (1.3), (1.4). Let, moreover, there exist constants $r > 0$, $\varepsilon > 0$, and functions $\alpha, f^+, f^- \in L(I;R_+)$ such that the conditions*

$$\begin{aligned} (-1)^i f(t,x) &\le -f^-(t) \quad \text{for} \quad x \le -r, \\ f^+(t) &\le (-1)^i f(t,x) \quad \text{for} \quad x \ge r, \end{aligned} \tag{2.12$_i$}$$

$$\sup\{|f(t,x)| : x \in R\} \le \alpha(t) \tag{2.13}$$

hold on I, and let

$$\begin{aligned} &-\int_a^b (f^+(s)[w(s)]_- + f^-(s)[w(s)]_+)ds + \varepsilon||\alpha||_L \\ &\le (-1)^{i+1}\int_a^b h(s)w(s)ds \\ &\le \int_a^b (f^-(s)[w(s)]_- + f^+(s)[w(s)]_+)ds - \varepsilon||\alpha||_L. \end{aligned} \tag{2.14}$$

Then the problem (1.1), (1.2) has at least one solution.

Remark 3. If $f \not\equiv 0$ then the condition (2.12$_i$) ($i = 1,2$) of Theorem 3 can be replaced by

$$\begin{aligned} &-\int_a^b (f^+(s)[w(s)]_- + f^-(s)[w(s)]_+)ds \\ &< (-1)^{i+1}\int_a^b h(s)w(s)ds \\ &< \int_a^b (f^-(s)[w(s)]_- + f^+(s)[w(s)]_+)ds. \end{aligned} \tag{2.15}$$

because from (2.15) there follows the existence of a constant $\varepsilon > 0$ such that the condition (2.12$_i$) is satisfied.

Remark 4. If $\widetilde{f}(t) = \min\{f^+(t), f^-(t)\}$ then the condition (2.14) of Theorem 3 can be replaced by

$$\left|\int_a^b h(s)w(s)ds\right| \le \int_a^b \widetilde{f}(s)|w(s)|ds - \varepsilon||\alpha||_L.$$

Example 4. From Theorem 3 it follows that the equation

$$u''(t) = -\lambda^2 u(t) + \frac{|u(t)|^\alpha}{1+|u(t)|^\alpha}\operatorname{sgn} u(t) + h(t) \quad \text{for} \quad 0 \le t \le \pi/2, \tag{2.16}$$

where $\lambda = 2k-1$ ($k \in N$) and $\alpha \in]0,+\infty[$, with the conditions (2.7) has at least one solution if $h \in L([0,\pi/2],R)$ is such that $|h(t)| < 1$ for $0 \le t \le \pi/2$.

3. Auxiliary propositions

Let $u_n \in \widetilde{C}'(I;R)$, $\|u_n\|_C \neq 0$ $(n \in N)$, w be an arbitrary solution of the problem (1.3), (1.4), and $r > 0$. Then, for every $n \in N$, we define:

$$A_{n,1} \overset{\text{def}}{=} \{t \in I : |u_n(t)| \le r\}, \qquad A_{n,2} \overset{\text{def}}{=} \{t \in I : |u_n(t)| > r\},$$

$$B_{n,i} \overset{\text{def}}{=} \{t \in A_{n,2} : \operatorname{sgn} u_n(t) = (-1)^{i-1} \operatorname{sgn} w(t)\} \ \ (i = 1,2),$$

$$C_{n,1} \overset{\text{def}}{=} \{t \in A_{n,2} : |w(t)| \ge 1/n\}, \qquad C_{n,2} \overset{\text{def}}{=} \{t \in A_{n,2} : |w(t)| < 1/n\},$$

$$D_n \overset{\text{def}}{=} \{t \in I : |w(t)| > r||u_n||_C^{-1} + 1/2n\},$$

$$A_{n,2}^{\pm} \overset{\text{def}}{=} \{t \in A_{n,2} : \pm u_n(t) > r\}, \quad B_{n,i}^{\pm} \overset{\text{def}}{=} A_{n,2}^{\pm} \cap B_{n,i},$$

$$C_{n,i}^{\pm} \overset{\text{def}}{=} A_{n,2}^{\pm} \cap C_{n,i} \ \ (i = 1,2), \ D_n^{\pm} \overset{\text{def}}{=} \{t \in I : \pm w(t) > r||u_n||_C^{-1} + 1/2n\},$$

From these definitions it is clear that, for any $n \in N$, we have

$$A_{n,1} \cap A_{n,2} = \varnothing, A_{n,2}^{+} \cap A_{n,2}^{-} = \varnothing, \ \ B_{n,1} \cap B_{n,2} = \varnothing, \ C_{n,1} \cap C_{n,2} = \varnothing,$$
$$D_n^{+} \cap D_n^{-} = \varnothing, \ B_{n,2}^{+} \cap B_{n,2}^{-} = \varnothing, \ C_{n,i}^{+} \cap C_{n,i}^{-} = \varnothing \ (i = 1,2), \tag{3.1}$$

and

$$\begin{gathered} A_{n,1} \cup A_{n,2} = I, \ A_{n,2}^{+} \cup A_{n,2}^{-} = A_{n,2}, \ B_{n,1} \cup B_{n,2} = A_{n,2} \setminus N_w, \\ C_{n,1} \cup C_{n,2} = A_{n,2}, \ B_{n,2}^{+} \cup B_{n,2}^{-} = B_{n,2}, \ C_{n,1}^{\pm} \cup C_{n,2}^{\pm} = A_{n,2}^{\pm}, \\ C_{n,i}^{+} \cup C_{n,i}^{-} = C_{n,i} \ (i = 1,2), \ D_n^{+} \cup D_n^{-} = D_n. \end{gathered} \tag{3.2}$$

The proofs of the following two lemmas are given in [16].

Lemma 1. *Let $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$, $r > 0$, w be an arbitrary nonzero solution of the problem (1.3), (1.4), and*

$$||u_n||_C \ge 2rn \quad \textit{for} \quad n \in N, \tag{3.3}$$

$$||v_n - w||_C \le 1/2n \quad \textit{for} \quad n \in N, \tag{3.4}$$

where $v_n(t) = u_n(t)||u_n||_C^{-1}$. Then there exists $n_0 \in N$ such that

$$D_{n_0}^{+} \subset A_{n,2}^{+}, \quad D_{n_0}^{-} \subset A_{n,2}^{-} \quad \textit{for} \quad n \ge n_0, \tag{3.5}$$

$$C_{n_0,1}^{+} \subset D_n^{+} \quad C_{n_0,1}^{-} \subset D_n^{-} \quad \textit{for} \quad n \ge n_0. \tag{3.6}$$

Moreover

$$\lim_{n\to+\infty} \operatorname{mes} A_{n,1} = 0, \qquad \lim_{n\to+\infty} \operatorname{mes} A_{n,2} = \operatorname{mes} I, \tag{3.7}$$

$$C_{n,1} \subset B_{n,1}, \qquad B_{n,2} \subset C_{n,2}, \tag{3.8}$$

$$B_{n,2}^{+} \subset C_{n,2}^{+}, \qquad B_{n,2}^{-} \subset C_{n,2}^{-}, \tag{3.9}$$

$$C_{n,1}^{+} \subset B_{n,1}^{+}, \qquad C_{n,1}^{-} \subset B_{n,1}^{-}, \tag{3.10}$$

$$\begin{aligned}\lim_{n\to+\infty}\operatorname{mes} C_{n,1} &= \lim_{n\to+\infty}\operatorname{mes} B_{n,1} = \operatorname{mes} I,\\ \lim_{n\to+\infty}\operatorname{mes} C_{n,2} &= \lim_{n\to+\infty}\operatorname{mes} B_{n,2} = 0,\end{aligned} \tag{3.11}$$

$$r < |u_n(t)| \le ||u_n||_C/2n \qquad \textit{for} \quad t \in B_{n,2}, \tag{3.12}$$

$$|u_n(t)| \ge ||u_n||_C/2n > r \qquad \textit{for} \quad t \in C_{n,1}, \tag{3.13}$$

$$C^{\pm}_{n,2} = \{t \in A_{n,2} : 0 \le \pm w(t) < 1/n\}, \tag{3.14}$$

$$C^{\pm}_{n,1} \subset \Omega^{\pm}_w, \quad \lim_{n\to+\infty}\operatorname{mes} C^{\pm}_{n,1} = \operatorname{mes}\Omega^{\pm}_w. \tag{3.15}$$

Lemma 2. *Let $i \in \{1,2\}$, $r > 0$, $k \in N$, w_0 be a nonzero solution of the problem (1.3), (1.4), $N_{w_0} = \{t_1,\dots,t_k\}$, the function $f_1 \in E(N_{w_0})$ be non-decreasing in the second argument for $|x| \ge r$, and*

$$f_1(t,x)\operatorname{sgn} x \ge 0 \qquad \textit{for} \quad t \in I,\ |x| \ge r. \tag{3.16}$$

Then:

(a) *If $G \subset I$ and*

$$\int_G |f_1(s,(-1)^i r)w_0(s)|ds \ne 0, \tag{3.17}$$

then there exist $\delta_0 > 0$ and $\varepsilon_1 > 0$ such that

$$\mathbb{I}(G,U_\varepsilon,x) \overset{\text{def}}{\equiv} \int_{G\setminus U_\varepsilon} |f_1(s,x)w_0(s)|ds - \int_{U_\varepsilon} |f_1(s,x)w_0(s)|ds \ge \delta_0 \tag{3.18}$$

for $(-1)^i x \ge r$ and $0 < \varepsilon \le \varepsilon_1$, where $U_\varepsilon = I \cap \big(\cup_{j=1}^k [t_j - \varepsilon/2k, t_j + \varepsilon/2k]\big)$.

(b) *If $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$, $r > 0$, w is an arbitrary nonzero solution of the problem (1.3), (1.4), and the condition (3.3) holds, then there exist $\varepsilon_2 \in]0,\varepsilon_1]$ and $n_0 \in N$ such that*

$$\mathbb{I}(D^+_n, U^+_\varepsilon, x) \ge -\frac{\delta_0}{2} \quad \textit{for} \quad x \ge r, \tag{3.19$_1$}$$

$$\mathbb{I}(D^-_n, U^-_\varepsilon, x) \ge -\frac{\delta_0}{2} \quad \textit{for} \quad x \le -r \tag{3.19$_2$}$$

for $n \ge n_0$ and $0 < \varepsilon \le \varepsilon_2$, where $U^{\pm}_\varepsilon = \{t \in U_\varepsilon : \pm w(t) \ge 0\}$.

Lemma 3. *Let all the conditions of Lemma 1 be fulfilled and there exist $r > 0$ such that the condition (3.16) holds, where $f_1 \in K(I \times R; R)$. Then*

$$\lim_{n\to+\infty}\inf \int_s^t f_1(\xi,u_n(\xi))\operatorname{sgn} u_n(\xi)d\xi \ge 0 \quad \textit{for} \quad a \le s < t \le b. \tag{3.20}$$

Proof. Let

$$\gamma_r^*(t) \stackrel{\text{def}}{=} \sup\{|f_1(t,x)| : |x| \le r\} \quad \text{for} \quad t \in I. \tag{3.21}$$

Then, according to (3.1), (3.2), and (3.16), we obtain the estimate

$$\int_s^t f_1(\xi, u_n(\xi)) \operatorname{sgn} u_n(\xi) d\xi$$
$$\ge -\int_{[s,t]\cap A_{n,1}} \gamma_r^*(\xi) d\xi + \int_{[s,t]\cap A_{n,2}} |f_1(\xi, u_n(\xi))| d\xi$$

for $a \le s < t \le b$, $n \in N$. This estimate and (3.7) imply (3.20). □

Lemma 4. *Let w_0 be a nonzero solution of the problem (1.3), (1.4), $r > 0$, the function $f_1 \in E(N_{w_0})$ be non-decreasing in the second argument for $|x| \ge r$, condition (3.16) hold, and*

$$\int_{\Omega_{w_0}^+} |f_1(s,r)| ds + \int_{\Omega_{w_0}^-} |f_1(s,-r)| ds \ne 0. \tag{3.22}$$

Then there exist $\delta > 0$ and $n_1 \in N$ such that if

$$\Big| \int_a^b h_1(s) w_0(s) ds \Big| < \delta ||w_0||_C \tag{3.23}$$

then, for every nonzero solution w of the problem (1.3), (1.4), and functions $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$ such that the conditions (3.3),

$$|v_n^{(i)}(t) - w^{(i)}(t)| \le 1/2n \quad \text{for} \quad t \in I,\ n \in N,\ (i = 0,1) \tag{3.24}$$

where $v_n(t) = u_n(t)||u_n||_C^{-1}$ for $t \in I$ and

$$u_n(a) = 0, \ \ u_n'(b) = 0 \tag{3.25}$$

are fulfilled, there exists $n_1 \in N$ such that

$$\mathbb{M}_n(w) \stackrel{\text{def}}{=} \int_a^b (h_1(s) + f_1(s, u_n(s))) w(s) ds \ge 0 \quad \text{for} \quad n \ge n_1. \tag{3.26}$$

Proof. Without loss of generality we can assume that $||w_0||_C = 1$. Also it is not difficult to verify that all the assumption of Lemma 1 are satisfied. Then, by the definition of the sets $B_{n,1}$, $B_{n,2}$, the conditions (3.1), (3.2), and (3.16), we obtain the estimate

$$\int_a^b f_1(s, u_n(s)) w(s) ds \ge -\int_{A_{n,1}} \gamma_r^*(s) |w(s)| ds + \widehat{\mathbb{M}}_n(w), \tag{3.27}$$

where

$$\widehat{\mathbb{M}}_n(w) \stackrel{\text{def}}{=} -\int_{B_{n,2}} |f_1(s, u_n(s)) w(s)| ds + \int_{B_{n,1}} |f_1(s, u_n(s)) w(s)| ds.$$

On the other hand, from the unique solvability of the Cauchy problem for the equation (1.3) it is clear that

$$w'(a) \neq 0, \quad w'(b) \neq 0, \quad w'(t_i) \neq 0 \qquad \text{for } i = 1, \dots, k \tag{3.28}$$

if $N_{w_0} = \{t_1, \dots, t_k\}$. Now note that, for any nonzero solution w of the problem (1.3), (1.4), there exists $\beta \neq 0$ such that $w(t) = \beta w_0(t)$. Consequently

$$\Omega_w^{\pm} = \Omega_{w_0}^{\pm} \quad \text{if} \quad \beta > 0 \quad \text{and} \quad \Omega_w^{\mp} = \Omega_{w_0}^{\pm} \quad \text{if} \quad \beta < 0. \tag{3.29}$$

Then in view of (3.15) and (3.22), there exists $n_2 \geq n_0$ such that

$$\int_{C_{n_2,1}^+} |f_1(s,r)w_0(s)|ds \neq 0 \text{ and/or } \int_{C_{n_2,1}^-} |f_1(s,-r)w_0(s)|ds \neq 0. \tag{3.30}$$

From (3.30), in view of (3.6), it follows that

$$\int_{D_n^+} |f_1(s,r)w_0(s)|ds \neq 0 \quad \text{for} \quad n \geq n_2 \tag{3.31$_1$}$$

and/or

$$\int_{D_n^-} |f_1(s,-r)w_0(s)|ds \neq 0 \quad \text{for} \quad n \geq n_2. \tag{3.31$_2$}$$

Consequently, all the assumptions of Lemma 3.2 are satisfied with $G = D_n^+$ and/or $G = D_n^-$. Therefore, there exist $\varepsilon_0 \in]0, \varepsilon_2[$, $n_3 \geq n_2$, and $\delta_0 > 0$ such that

$$\begin{aligned} &\mathbb{I}(D_n^+, U_{\varepsilon_0}^+, x) \geq \delta_0 \text{ for } x \geq r, \quad n \geq n_3, \\ &\mathbb{I}(D_n^-, U_{\varepsilon_0}^-, x) \geq -\delta_0/2 \text{ for } x \leq -r, \quad n \geq n_3 \end{aligned} \tag{3.32}$$

if (3.31$_1$) holds, and

$$\begin{aligned} &\mathbb{I}(D_n^-, U_{\varepsilon_0}^-, x) \geq \delta_0 \quad \text{for} \quad x \leq -r, \ n \geq n_3, \\ &\mathbb{I}(D_n^+, U_{\varepsilon_0}^+, x) \geq -\delta_0/2 \quad \text{for} \quad x \geq r, \quad n \geq n_3 \end{aligned} \tag{3.33}$$

if (3.31$_2$) holds.

On the other hand, the definition of the set U_ε and (3.14), imply that there exists $n_4 > n_3$, such that

$$C_{n,2}^+ \subset U_{\varepsilon_0}^+, \quad C_{n,2}^- \subset U_{\varepsilon_0}^- \quad \text{for} \quad n \geq n_4. \tag{3.34}$$

By these inclusions, (3.2), and (3.5) we obtain

$$C_{n,1}^+ = A_{n,2}^+ \setminus C_{n,2}^+ \supset D_{n_0}^+ \setminus U_{\varepsilon_0}^+, \ C_{n,1}^- = A_{n,2}^- \setminus C_{n,2}^- \supset D_{n_0}^- \setminus U_{\varepsilon_0}^+ \tag{3.35}$$

for $n \geq n_4$. First suppose that $N_{w_0} \neq \varnothing$ and there exists $n \geq n_4$ such that

$$B_{n,2} \neq \varnothing. \tag{3.36}$$

Then, by taking into account that f_1 is non-decreasing in the second argument for $|x| \geq r$, (3.3), (3.12), (3.16) and the definitions of the sets $B^+_{n,2}, B^-_{n,2}$, we get

$$\begin{aligned}
&|f_1(t,u_n(t))| = f_1(t,u_n(t)) \\
&\leq f_1\Big(t,\frac{||u_n||_C}{2n}\Big) = \Big|f_1\Big(t,\frac{||u_n||_C}{2n}\Big)\Big| \quad \text{for } t \in B^+_{n,2}, \\
&|f_1(t,u_n(t))| = -f_1(t,-u_n(t)) \\
&\leq -f_1\Big(t,-\frac{||u_n||_C}{2n}\Big) = \Big|f_1\Big(t,-\frac{||u_n||_C}{2n}\Big)\Big| \quad \text{for } t \in B^-_{n,2}.
\end{aligned} \tag{3.37}$$

Analogously, from (3.3), (3.13), (3.16), and the definitions of the sets $C^+_{n,1}, C^-_{n,1}$, we obtain the estimates

$$\begin{aligned}
&|f_1(t,u_n(t))| \geq \Big|f_1\Big(t,\frac{||u_n||_C}{2n}\Big)\Big| \quad \text{for} \quad t \in C^+_{n,1}, \\
&|f_1(t,u_n(t))| \geq \Big|f_1\Big(t,-\frac{||u_n||_C}{2n}\Big)\Big| \quad \text{for} \quad t \in C^-_{n,1}.
\end{aligned} \tag{3.38}$$

Then from (3.1), (3.2), (3.9), (3.37) and respectively from (3.1), (3.2), (3.8), and (3.38) we have

$$\begin{aligned}
&\int_{B_{n,2}} |f_1(s,u_n(s))w(s)|ds \\
&\leq \int_{B^+_{n,2}} |f_1(s,\frac{||u_n||_C}{2n})w(s)|ds + \int_{B^-_{n,2}} |f_1(s,-\frac{||u_n||_C}{2n})w(s)|ds \\
&\leq \int_{C^+_{n,2}} |f_1(s,\frac{||u_n||_C}{2n})w(s)|ds + \int_{C^-_{n,2}} |f_1(s,-\frac{||u_n||_C}{2n})w(s)|ds
\end{aligned} \tag{3.39}$$

and respectively

$$\begin{aligned}
&\int_{B_{n,1}} |f_1(s,u_n(s))w(s)|ds \geq \int_{C_{n,1}} |f_1(s,u_n(s))w(s)|ds \\
&\geq \int_{C^+_{n,1}} |f_1(s,\frac{||u_n||_C}{2n})w(s)|ds + \int_{C^-_{n,1}} |f_1(s,-\frac{||u_n||_C}{2n})w(s)|ds.
\end{aligned} \tag{3.40}$$

If the condition (3.36) holds, from (3.39) and (3.40) we obtain

$$\begin{aligned}
\frac{\widehat{\mathbb{M}}_n(w)}{|\beta|} \geq &\left(\int_{C^+_{n,1}} \Big|f_1\Big(s,\frac{||u_n||_C}{2n}\Big)w_0(s)\Big|ds - \int_{C^+_{n,2}} \Big|f_1\Big(s,\frac{||u_n||_C}{2n}\Big)w_0(s)\Big|ds\right) \\
&+\left(\int_{C^-_{n,1}} \Big|f_1\Big(s,-\frac{||u_n||_C}{2n}\Big)w_0(s)\Big|ds - \int_{C^-_{n,2}} \Big|f_1\Big(s,-\frac{||u_n||_C}{2n}\Big)w_0(s)\Big|ds\right),
\end{aligned}$$

whence, by (3.34) and (3.35) we get

$$\frac{\widehat{\mathbb{M}}_n(w)}{|\beta|} \geq \mathbb{I}\Big(D_n^+, U_{\varepsilon_0}^+, \frac{||u_n||_C}{2n}\Big) + \mathbb{I}\Big(D_n^-, U_{\varepsilon_0}^-, -\frac{||u_n||_C}{2n}\Big) \tag{3.41}$$

for $n \geq n_4$. From (3.41) by (3.32) and (3.33) we obtain

$$\widehat{\mathbb{M}}_n(w) \geq \frac{\delta_0|\beta|}{2} \qquad \text{for} \quad n \geq n_4. \tag{3.42}$$

On the other hand, in view of (3.10), (3.16), the definition of the sets $A_{n,2}, B_{n,1}$, and the fact that f_1 is non-decreasing in the second argument, we obtain the estimate

$$\begin{aligned}\int_{B_{n,1}} |f_1(s,u_n(s))w(s)|ds &\geq \int_{B_{n,1}^+} |f_1(s,r)w(s)|ds + \int_{B_{n,1}^-} |f_1(s,-r)w(s)|ds \\ &\geq \int_{C_{n,1}^+} |f_1(s,r)w(s)|ds + \int_{C_{n,1}^-} |f_1(s,-r)w(s)|ds.\end{aligned} \tag{3.43}$$

Now suppose that there exists $n \geq n_4$ such that

$$B_{n,2} = \varnothing. \tag{3.44}$$

Then from (3.30) and (3.43), (3.44) there follows the existence of $\delta^* > 0$ such that $\widehat{\mathbb{M}}_n(w) \geq |\beta|\delta^*$. From this inequality and (3.42) it follows that, in both cases when (3.36) or (3.44) are fulfilled, the inequality

$$\widehat{\mathbb{M}}_n(w) \geq |\beta|\delta \quad \text{for} \quad n \geq n_4 \tag{3.45}$$

holds with $\delta = \min\{\delta_0/2, \delta^*\}$. From (3.27) by (3.7) and (3.45), we see that for any $\varepsilon \in]0, \delta[$ there exists $n_1 > n_4$ such that

$$\int_a^b f_1(s,u_n(s))w(s)ds \geq |\beta|(\delta - \varepsilon) \quad \text{for} \quad n \geq n_1,$$

and thus

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq \delta - \varepsilon - \left|\int_a^b h_1(s)w_0(s)ds\right| \quad \text{for} \quad n \geq n_1. \tag{3.46}$$

If $N_{w_0} = \varnothing$ then $|w(t)| > 0$ for $a < t < b$ and in view of (3.3), (3.24), (3.25) and (3.28), the condition (3.44) holds, i.e., the inequality (3.46) also holds.

Consequently since $\varepsilon > 0$ is arbitrary, the inequality (3.26) from (3.46) and (3.23) follows. □

Lemma 5. *Let w_0 be a nonzero solution of the problem (1.3), (1.4), $r > 0$, and the conditions (3.16), (3.23) hold with $f_1(t,x) \overset{\text{def}}{\equiv} f_0(t)g_1(x)$, where $f_0 \in L(I;R_+)$ and a non-decreasing function $g_1 \in C(R;R)$ be such that*

$$\lim_{|x|\to+\infty} |g_1(x)| = +\infty. \tag{3.47}$$

Then, for every nonzero solution w of the problem (1.3), (1.4) *and functions* $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$ *fulfilling the conditions* (3.3), (3.24), (3.25), *the inequality* (3.26) *holds.*

Proof. From the assumptions of our lemma it is clear that the relations (3.27)–(3.35), (3.37)-(3.40) and (3.43) with $f_1(t,x) = f_0(t)g_1(x)$ and $w(t) = \beta w_0(t)$ $(\beta \neq 0)$ are fulfilled.

Assuming $\int_{C^+_{n_2,1}} |f_1(s,r)w_0(s)|ds \neq 0$, the condition (3.31_1) is satisfied i.e., (3.32) holds.

Now notice that from (3.15) and the equality $C^+_{n,1} = \Omega^+_w \setminus (\Omega^+_w \setminus C^+_{n,1})$ it follows that there exist $\varepsilon > 0$ and $n_0 \in N$ such that

$$\int_{C^+_{n,1}} |f_0(s)w_0(s)|ds \geq \int_{\Omega^+_w} |f_0(s)w_0(s)|ds - \varepsilon > 0 \tag{3.48}$$

for $n \geq n_0$.

First consider the case when there exists $n \geq n_4$ such that the condition (3.44) holds. Without loss of generality we can assume that $n_4 > n_0$. Then by (3.29), (3.43), (3.44) and (3.48), we obtain

$$\widehat{\mathbb{M}}_n(w) \geq |\beta||g_1(r)|\left(\int_{\Theta_\beta} |f_0(s)w_0(s)|ds - \varepsilon\right) > 0, \tag{3.49}$$

where $\Theta_\beta = \Omega^+_{w_0}$ if $\beta > 0$ and $\Theta_\beta = \Omega^-_{w_0}$ if $\beta < 0$.

Consider now the case when there exists $n \geq n_4$ such that (3.36) holds. From (3.3) and the definition of the set D^+_n it follows that $D^+_n \subset D^+_{n+1}$, and since g_1 is non-decreasing, from (3.32) we obtain

$$\begin{aligned}\mathbb{I}(D^+_n, U^+_{\varepsilon_0}, x) &= |g_1(x)|\left(\int_{D^+_n \setminus U^+_{\varepsilon_0}} |f_0(t)w_0(s)|ds - \int_{U^+_{\varepsilon_0}} |f_0(t)w_0(s)|ds\right)\\ &\geq |g_1(r)|\mu = \mathbb{I}(D^+_{n_4}, U^+_{\varepsilon_0}, r) \geq \delta_0\end{aligned}$$

for $x \geq r$, with $\mu = \int_{D^+_{n_4} \setminus U^+_{\varepsilon_0}} |f_0(s)w_0(s)|ds - \int_{U^+_{\varepsilon_0}} |f_0(s)w_0(s)|ds > 0$. By the last inequality, (3.3), (3.32), and (3.41) we get

$$\widehat{\mathbb{M}}_n(w) \geq |\beta|(|g_1(r)|\mu - \delta_0/2). \tag{3.50}$$

Applying (3.49), (3.50) in (3.27) and taking (3.7) into account, we conclude that there exist $\varepsilon_1 > 0$ and $n_1 \geq n_4$ such that

$$|\beta|\left(|g_1(r)|\mu_1 - \frac{\delta_0}{2} - \varepsilon_1\right) \leq \int_a^b f_1(s,u_n(s))w(s)ds \quad \text{for} \quad n \geq n_1$$

with $\mu_1 = \min(\mu, \int_{\Omega^+_{w_0}} |f_0(s)w_0(s)|ds - \varepsilon)$. From (3.47) and the last inequality it is clear that, for any function h_1, we can choose $r > 0$ such that the inequality (3.26) will be true. In a similar manner one can prove (3.26) in the case when $\int_{C^-_{n_2,1}} |f_1(s,r)w_0(s)|ds \neq 0$.

□

Lemma 6. *Let $r > 0$, there exist functions $\alpha, f^-, f^+ \in L(I, R_+)$ such that the conditions*

$$\begin{aligned} f_1(t,x) &\le -f^-(t) \quad \text{for} \quad x \le -r, \\ f^+(t) &\le f_1(t,x) \quad \text{for} \quad x \ge r \end{aligned} \tag{3.51}$$

are satisfied,

$$\sup\{|f_1(t,x)| : x \in R\} = \alpha(t) \quad \text{for} \quad t \in I, \tag{3.52}$$

and there exist a nonzero solution w_0 of the problem (1.3), (1.4) and $\varepsilon > 0$ such that

$$\begin{aligned} &-\int_a^b (f^+(s)[w_0(s)]_- + f^-(s)[w_0(s)]_+)ds + \varepsilon||\alpha||_L \\ &\le -\int_a^b h_1(s)w_0(s)ds \\ &\le \int_a^b (f^-(s)[w_0(s)]_- + f^+(s)[w_0(s)]_+)ds - \varepsilon||\alpha||_L. \end{aligned} \tag{3.53}$$

Then, for every nonzero solution w of the problem (1.3), (1.4) and functions $u_n \in \tilde{C}'(I;R)$ $(n \in N)$ fulfilling the conditions (3.3), (3.24), and (3.25), there exists $n_1 \in N$ such that the inequality (3.26) holds.

Proof. First note that, for any nonzero solution w of the problem (1.3), (1.4), there exists $\beta \neq 0$ such that $w(t) = \beta w_0(t)$. Moreover, it is not difficult to verify that all the assumptions of Lemma1 are satisfied for the function $w(t) = \beta w_0(t)$. From (3.1), (3.2), and (3.52) we get

$$\begin{aligned} \mathbb{M}_n(w) \ge &-\int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w(s)|ds + \int_{B_{n,1}} f_1(s,u_n)w(s)ds \\ &+ \int_a^b h_1(s)w(s)ds. \end{aligned} \tag{3.54}$$

On the other hand, by the definition of the set $B_{n,1}$ we have

$$\operatorname{sgn} u_n(t) = \operatorname{sgn} w(t) \quad \text{for} \quad t \in B^+_{n,1} \cup B^-_{n,1}. \tag{3.55}$$

Hence, by (3.1), (3.2), (3.10), (3.51), and (3.55), from (3.54) we obtain the estimate

$$\mathbb{M}_n(w) - \int_a^b h_1(s)w(s)ds$$

$$\geq -\int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w(s)|ds + + \int_{B^+_{n,1}} f^+(s)|w(s)|ds + \int_{B^-_{n,1}} f^-(s)|w(s)|ds \tag{3.56}$$

$$\geq -\int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w(s)|ds + \int_{C^+_{n,1}} f^+(s)|w(s)|ds + \int_{C^-_{n,1}} f^-(s)|w(s)|ds.$$

Now, note that $f^- \equiv 0$ and $f^+ \equiv 0$ if $f_1(t,x) \equiv 0$. Therefore by (3.7), (3.11), (3.15), and the inclusions $C^+_{n,1} \subset \Omega^+_w$, $C^-_{n,1} \subset \Omega^-_w$, we see that there exist $\varepsilon > 0$ and $n_1 \in N$ such that

$$\begin{gathered} \frac{1}{3}\varepsilon||\alpha||_L \geq \int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w_0(s)|ds \\ \int_{\Omega^\pm_w} f^\pm(s)|w_0(s)|ds - \frac{1}{3}\varepsilon||\alpha||_L \leq \int_{C^\pm_{n,1}} f^\pm(s)|w_0(s)|ds \end{gathered} \tag{3.57}$$

for $n \geq n_1$. By virtue of (3.56) and (3.57), we obtain

$$\begin{aligned} \frac{\mathbb{M}_n(w)}{|\beta|} \geq -\varepsilon||\alpha||_L &+ \int_{\Omega^+_w} f^+(s)|w_0(s)|ds \\ &+ \int_{\Omega^-_w} f^-(s)|w_0(s)|ds + \sigma\int_a^b h_1(s)w_0(s)ds \end{aligned}$$

for $n \geq n_1$, where $\sigma = \operatorname{sgn}\beta$. Now, by taking into account that

$$\int_{\Omega^\pm_w} l(s)|w_0(s)|ds = \int_{\Omega^\pm_{w_0}} l(s)|w_0(s)|ds = \int_a^b l(s)[w_0(s)]_\pm ds$$

if $\beta > 0$ and

$$\int_{\Omega^\pm_w} l(s)|w_0(s)|ds = \int_{\Omega^\mp_{w_0}} l(s)|w_0(s)|ds = \int_a^b l(s)[w_0(s)]_\mp ds$$

if $\beta < 0$ for an arbitrary $l \in L(I,R)$, from the last inequalities we get

$$\begin{aligned} \frac{\mathbb{M}_n(w)}{|\beta|} \geq -\varepsilon||\alpha||_L &+ \int_a^b (f^+(s)[w_0(s)]_+ + f^-(s)[w_0(s)]_-)ds \\ &+ \int_a^b h_1(s)w_0(s)ds \quad \text{for} \quad n \geq n_1 \end{aligned}$$

if $\sigma = 1$, and

$$\begin{aligned} \frac{\mathbb{M}_n(w)}{|\beta|} \geq -\varepsilon||\alpha||_L &+ \int_a^b (f^+(s)[w_0(s)]_- + f^-(s)[w_0(s)]_+)ds \\ &- \int_a^b h_1(s)w_0(s)ds \quad \text{for} \quad n \geq n_1 \end{aligned}$$

if $\sigma = -1$. From the last inequalities and (3.53) we immediately obtain (3.26). □

Lemma 7. *Let problem* (1.3), (1.4) *has the nontrivial solution. Than there exists* $\varepsilon > 0$ *suth that the equation*

$$w''(t) = \lambda p(t) w \quad \textit{for} \quad t \in I, \tag{3.58}$$

under boundari conditions (1.4) *has only the trivial solution if* $\lambda \in]1, 1+\varepsilon]$.

Proof. Let G be the Green's function of the boundary value problem $u''(t) = 0$, $u(a) = 0$, $u'(b) = 0$, then problem (3.58), (1.4) is equivalent to the equation $w(t) = \lambda \Gamma(w)(t)$, where the operator $\Gamma : C(I;R) \to C(I;R)$ is defined by the equality $\Gamma(x)(t) = \int_a^b G(t,s)p(s)x(s)ds$. As it is well-known $\Gamma : C(I;R) \to C(I;R)$ is a compact operator, and then for every $r > 0$ the disc $|\lambda| \le r$, contains at most finite number of characteristic values [see [9], Capitol XIII, §3, Theorem 1]. From this fact the existence of $\varepsilon > 0$ such that the set $]1, 1+\varepsilon]$ does not contain the characteristic values of the equation $w(t) = \lambda \Gamma(w)(t)$, it follows. Consequently this equation, i.e., problem (3.58), (1.4) has only the trivial solution if $\lambda \in]1, 1+\varepsilon]$. □

4. Proof of the main results

Proof of Theorem 1. Let $p_n(t) = (1 + (-1)^i/n)p(t)$ and for any $n \in N$, consider the problems

$$u_n''(t) = p_n(t)u_n(t) + f(t, u_n(t)) + h(t) \quad \text{for} \quad t \in I, \tag{4.1}$$

$$u_n(a) = 0, \quad u_n'(b) = 0. \tag{4.2}$$

and (3.58). In view of the condition (2.3) and the fact that $(-1)^i f(t,x)$ is non-decreasing in the second argument for $|x| \ge r$, we obtain

$$\lim_{n\to+\infty} \frac{1}{||z_n||_C} \int_a^b |f(s, z_n(s))| ds = 0 \tag{4.3}$$

for an arbitrary sequence $z_n \in C(I;R)$ with $\lim_{n\to+\infty} ||z_n||_C = +\infty$. Moreover, in view of Lemma 7, the problem (3.58) has only the zero solution for every $n \ge n_0$. Therefore, as it is well-known (see [12], Corollary 2.1, p. 2271), from the inequality (4.3) it follows that the problems (4.1), (4.2) has at least one solution, suppose u_n.

Assume that

$$\lim_{n\to+\infty} ||u_n||_C = +\infty \tag{4.4}$$

and $v_n(t) = u_n(t)||u_n||_C^{-1}$, then the conditions

$$v_n(a) = 0 \qquad v_n'(b) = 0, \tag{4.5}$$

$$||v_n||_C = 1 \tag{4.6}$$

are fulfilled, and

$$v_n''(t) = p_n(t)v_n(t) + \frac{1}{||u_n||_C}(f(t, u_n(t))) + h(t)). \tag{4.7}$$

Hence, by the conditions (4.3) and (4.6), from (4.7) we get the existence of $r_0 > 0$ such that $||v_n'||_C \le r_0$. Consequently, in view of (4.6) by the Arzela-Ascoli lemma, without loss of generality we can assume that there exists a function $w \in \widetilde{C}'(I,R)$ such that $\lim_{n\to+\infty} v_n^{(i)}(t) = w^{(i)}(t)$ $(i = 0,1)$ uniformly on I. From the last equality and (4.4) there follows the existence of an increasing sequence $\{\alpha_k\}_{k=1}^{+\infty}$ of a natural numbers, such that $||u_{\alpha_k}||_C \ge 2rk$ and $||v_{\alpha_k}^{(i)} - w^{(i)}||_C \le 1/2k$ for $k \in N$. Without loss of generality we can suppose that $u_n \equiv u_{\alpha_n}$ and $v_n \equiv v_{\alpha_n}$. In this case we see that u_n and v_n are the solutions of the problems (4.1), (4.2) and (4.7), (4.5) respectively, and the inequalities

$$||u_n||_C \ge 2rn, \quad ||v_n^{(i)} - w^{(i)}||_C \le 1/2n \quad \text{for} \quad n \in N \tag{4.8}$$

are fulfilled.

From (4.7), by virtue of (4.5), (4.8) and (2.3), we obtain that w is a solution of the problem (1.3), (1.4). Multiplying the equations (4.1) and (1.3) respectively by w and $-u_n$, and therefore integrating their sum from a to b, in view of conditions (4.2) and (1.4), we obtain

$$(-1)^{i+1}\frac{||u_n||_C}{\alpha_n}\int_a^b p(s)w(s)v_n(s)ds = \int_a^b (h(s) + f(s,u_n(s)))w(s)ds \tag{4.9}$$

for $n \ge n_0$, where in view of conditions (4.8) the equality

$$\lim_{n\to+\infty}\int_a^b p(s)w(s)v_n(s)ds = \int_a^b p(s)w^2(s)ds$$

holds. On the other hand multiplying equation (1.3) by w, and therefore integrating from a to b, in view of condition (1.4), we obtain

$$\int_a^b p(s)w^2(s)ds = \int_a^b w''(s)w(s)ds = -\int_a^b w'^2(s)ds < 0,$$

and from (4.9) by the last two relations we get

$$(-1)^i\int_a^b (h(s) + f(s,u_n(s)))w(s)ds > 0. \tag{4.10}$$

for $n \in N \ge n_0$. Now note that, in view the conditions (2.1), (2.2), (2.4), (4.2), and (4.8), all the assumptions of Lemma 4 with $f_1(t,x) = (-1)^i f(t,x)$, $h_1(t) = (-1)^i h(t)$ are satisfied. Therefore, the inequality (3.26) is true, which contradicts (4.10). This contradiction proves that (4.4) does not hold and thus there exists $r_1 > 0$ such that $||u_n||_C \le r_1$ for $n \in N$. Consequently, from (4.1) and (4.2) it is clear that there exists $r_1' > 0$ such that $||u_n'||_C \le r_1'$ and $|u_n''(t)| \le \sigma(t)$ for $t \in I$, $n \in N$, where $\sigma(t) = 2|p(t)|r_1 + |h(t)| + \gamma_{r_1}(t)$. Hence, by Arzela-Ascoli lemma, without loss of generality we can assume that there exists a function $u_0 \in \widetilde{C}'(I;R)$ such that $\lim_{n\to+\infty} u_n^{(i)}(t) = u_0^{(i)}(t)$ $(i = 0,1)$ uniformly on I. Therefore, it follows from (4.1) and (4.2) that u_0 is a solution of the problem (1.1), (1.2). □

Proof of Theorem 2. The proof is the same as the proof of Theorem 1. The only difference is that we use Lemma 5 instead of Lemma 4. □

Proof of Theorem 3. From (2.13) it is clear that, for an arbitrary sequence $z_n \in C(I;R)$ such that $\lim_{n\to+\infty} ||z_n||_C = +\infty$, the equality (4.3) holds. From (4.3) and Lemma 6, analogously as in the proof of Theorem 1, we show that the problem (1.1), (1.2) has at least one solution. □

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β_* RELATION ON LATTICES

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Abstract. In this paper, we generalize β^* relation on submodules of a module (see [1]) to elements of a complete modular lattice. Let L be a complete modular lattice. We say $a, b \in L$ are β_* *equivalent*, $a\beta_* b$, if and only if for each $t \in L$ such that $a \vee t = 1$ then $b \vee t = 1$ and for each $k \in L$ such that $b \vee k = 1$ then $a \vee k = 1$, this is equivalent to $a \vee b \ll 1/a$ and $a \vee b \ll 1/b$. We show that the β_* relation is an equivalence relation. Then, we examine β_* relation on weakly supplemented lattices. Finally, we show that L is weakly supplemented if and only if for every $x \in L$, x is equivalent to a weak supplement in L.

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1. Introduction

Throughout this paper, L denotes a complete modular lattice with the smallest element 0 and the greatest element 1. A lattice we will mean a complete modular lattice. In a lattice L, an element $1 \neq m \in L$ is called *maximal* in L if there is no element between m and 1. An element a of L called *small* in L, if $a \vee b \neq 1$ holds for every $b \neq 1$. This is denoted by $a \ll L$. An element c of L is called a *supplement* of b in L if it is minimal relative to the property $b \vee c = 1$. Equivalently, an element c is a supplement of b in L if and only if $b \vee c = 1$ and $b \wedge c \ll c/0$. An element c of L is called a *weak supplement* of b in L if $b \vee c = 1$ and $b \wedge c \ll L$. A lattice L is called *supplemented* (respectively, *weakly supplemented*) if each element of L has a supplement (respectively, weak supplement) in L. For $a \in L$, we said that $b \in L$ is a *complement* of a in L if $a \wedge b = 0$ and $a \vee b = 1$ (see [3]). It is denoted by $a \oplus b = 1$ (see [2]). A lattice L is called *complemented* if each element in L has at least one complement in L (see [2]). A lattice L is called *hollow* if every element with distinct from 1 small in L. An element a of L has *ample supplements* in L if for every $t \in L$ with $a \vee t = 1$, there is a supplement t' of a with $t' \leq t$. L is called *amply supplemented* if all elements of L have ample supplements in L. In a lattice L, the meet of all maximal elements in L is called *radical* of L, denoted by $rad(L)$. If $a \in L$ such that $a \ll L$ then $a \leq rad(L)$ (see [4], Proposition 6). For

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$a,b \in L$ such that $a \leq b$, we said that b *lies above* a if $b \ll 1/a$. A lattice L is called *distributive* if for any elements a,b,c of L, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ holds.

2. β_* Relation

Definition 1. Let a,b be elements of L. We define a relation β_* on the elements of L by $a\beta_* b$ if and only if for each $t \in L$ such that $a \vee t = 1$ then $b \vee t = 1$ and for each $k \in L$ such that $b \vee k = 1$ then $a \vee k = 1$.

Lemma 1. *β_* is an equivalence relation.*

Proof. The reflexive and symmetric properties are clear. For transitivity, assume $a\beta_* b$ and $b\beta_* c$. Let $t \in L$ such that $a \vee t = 1$. Since $a\beta_* b$, $b \vee t = 1$. So, by $b\beta_* c$, $c \vee t = 1$. Similarly, for each $k \in L$ such that $c \vee k = 1$ then $a \vee k = 1$. Finally $a\beta_* c$. □

Theorem 1. *Let a,b be elements of L. Then,*

(1) *$a\beta_* b$ if and only if $a \vee c = 1$ and $b \vee c = 1$ for each $c \in L$ such that $a \vee b \vee c = 1$.*
(2) *$a\beta_* b$ if and only if $a \vee b \ll 1/a$ and $a \vee b \ll 1/b$.*

Proof. (1) ($\Rightarrow$) Let $a\beta_* b$ and $c \in L$ such that $a \vee b \vee c = 1$. Since $a \vee (b \vee c) = 1$ and $a\beta_* b$, $b \vee (b \vee c) = 1$. Hence $b \vee c = 1$. Similarly, $a \vee c = 1$.
($\Leftarrow$) Let $t \in L$ such that $a \vee t = 1$. Then $a \vee b \vee t = 1$. By hypothesis, $b \vee t = 1$. Similarly, if $k \in L$ with $b \vee k = 1$ then $a \vee k = 1$. So $a\beta_* b$.

(2) ($\Rightarrow$) Let $a\beta_* b$ and $t \in 1/a$ such that $a \vee b \vee t = 1$. Since $a\beta_* b$, $a \vee t = 1$. Then $t = 1$. Therefore $a \vee b \ll 1/a$. Similarly, $a \vee b \ll 1/b$.
($\Leftarrow$) Let $a \vee b \ll 1/a$, $a \vee b \ll 1/b$ and $t \in L$ such that $a \vee t = 1$. So $(a \vee b) \vee (b \vee t) = a \vee b \vee t = 1$. Since $a \vee b \ll 1/b$ and $b \vee t \in 1/b$, $b \vee t = 1$. Similarly, for each $k \in L$ such that $b \vee k = 1$ then $a \vee k = 1$. □

Theorem 2. *Let a,b be elements of L. Then,*

(1) *If $a \ll L$ and $a\beta_* b$ then $b \ll L$.*
(2) *All small elements in L is equivalent with β_* equivalence relation.*

Proof. (1) Let $a\beta_* b$, $a \ll L$ and $t \in L$ such that $b \vee t = 1$. Hence $a \vee t = 1$. Since $a \ll L$, $t = 1$. Thus $b \ll L$.

(2) Let $a \ll L$ and $b \ll L$ for $a,b \in L$. Since $a \ll L$, if $a \vee t = 1$ then $t = 1$. Therefore $b \vee t = 1$. Similarly, $a \vee k = 1$ for each $k \in L$ such that $b \vee k = 1$. Thus $a\beta_* b$. □

Corollary 1. *L is hollow if and only if all elements with distinct from 1 in L are equivalent with β_* relation.*

Proof. ($\Rightarrow$) Let L be hollow. Then all elements of L with distinct from 1 are small in L. Then by Theorem 2 (2), all elements with distinct from 1 in L are equivalent with β_* relation.
($\Leftarrow$) Let all elements of L with distinct from 1 be equivalent to each other. Let $a, t \in L$, $a \neq 1$ and $a \vee t = 1$. If $t \neq 1$, then by hyphothesis $a\beta_* t$ and $t = t \vee t = 1$. This is a contradiction. Hence $t = 1$ and $a \ll L$. Therefore L is hollow. □

Theorem 3. *Let a, b be elements of L such that $a \leq b$. If b lies above a, then $a\beta_* b$.*

Proof. Assume b lies above a. Then, $b \ll 1/a$. Since $a \leq b$ for any $t \in L$ such that $a \vee t = 1$, $b \vee t = 1$. Conversely, let $k \in L$ with $b \vee k = 1$. Then $b \vee a \vee k = 1$. Since $b \ll 1/a$ and $a \vee k \in 1/a$, $a \vee k = 1$. Hence $a\beta_* b$.

□

Lemma 2. *Let a, b, c be elements of L. If $a \vee b = 1$ and $(a \wedge b) \vee c = 1$, then $a \vee (b \wedge c) = b \vee (a \wedge c) = 1$.*

Proof. Assume $a \vee b = 1$ and $(a \wedge b) \vee c = 1$. Since $(a \wedge b) \vee c = 1$, $a = a \wedge 1 = a \wedge [(a \wedge b) \vee c] = (a \wedge b) \vee (a \wedge c)$. Then $1 = a \vee b = (a \wedge b) \vee (a \wedge c) \vee b = b \vee (a \wedge c)$. Similarly $a \vee (b \wedge c) = 1$. □

Theorem 4. *Let $a, b \in L$. If $a\beta_* b$ then the following conditions hold.*

(1) *If there exist supplements of a and b then these are the same.*
(2) *If there exist weak supplements of a and b then these are the same.*

Proof. (1) Let c be a supplement of a. Then $a \vee c = 1$. Since $a\beta_* b$, $b \vee c = 1$. Let $d \in L$ such that $d \leq c$ and $b \vee d = 1$. Therefore $a \vee d = 1$. Since c is a supplement of a and $d \leq c$, $d = c$. Then c is a supplement of b. Similarly, interchanging the roles of a and b we can show that each supplement of b is also a supplement of a.

(2) Let $a\beta_* b$ and c be a weak supplement of a in L. Therefore $a \vee c = 1$ and $a \wedge c \ll L$. Since $a\beta_* b$ and $a \vee c = 1$, $b \vee c = 1$. Let t be an element of L such that $(b \wedge c) \vee t = 1$. By Lemma 2, $b \vee (c \wedge t) = 1$ and since $a\beta_* b$, $a \vee (c \wedge t) = 1$. Then by also Lemma 2, $(a \wedge c) \vee t = 1$ and since $a \wedge c \ll L$, $t = 1$. Therefore $b \wedge c \ll L$ and so c is also a weak supplement of b. Similarly, interchanging the roles of a and b we can show that each weak supplement of b is also a weak supplement of a.

□

Theorem 5. *Let L be an amply supplemented lattice and $a, b \in L$. If supplements of a and b in L are the same then $a\beta_* b$.*

Proof. Let $t \in L$ such that $a \vee t = 1$. Since L is amply supplemented, there exists a supplement r of a in L such that $r \leq t$. By the hypothesis, r is also a supplement of b. Then $b \vee r = 1$. Since $r \leq t$, $b \vee t = 1$. Similarly, interchanging the roles of a and

b we can show that $a \vee k = 1$ for any element k of L such that $b \vee k = 1$. Therefore $a\beta_* b$. □

Corollary 2. *Let $x, y, c \in L$ such that $x \leq y$ and c is a weak supplement of x in L. Then $x\beta_* y$ if and only if $y \wedge c \ll L$.*

Proof. ($\Rightarrow$): Clear from Theorem 4 (2).
($\Leftarrow$): Since $x \leq y$, for any element t of L such that $x \vee t = 1$, $y \vee t = 1$. Let $k \in L$ such that $y \vee k = 1$. Since c is weak supplement of x in L, $x \vee c = 1$ and $x \wedge c \ll L$. Therefore $y \wedge (x \vee c) = 1 \wedge y$, and so $y = x \vee (y \wedge c)$. Hence $x \vee (y \wedge c) \vee k = 1$. Since $y \wedge c \ll L$, $x \vee k = 1$. Thus $x\beta_* y$. □

Theorem 6. *Let $x, y, z, a, b \in L$ such that $a \oplus b = 1$ and y is a supplement of x in L. Then*

(1) *If $z\beta_* y$ then $z/(z \wedge x) \cong y/(y \wedge x)$.*
(2) *If $z\beta_* b$ then $z/(z \wedge a) \cong b/0$.*
(3) *Let $z \leq b$. Then $z\beta_* b$ if and only if $z = b$.*
(4) *Let $b \leq z$. Then $z\beta_* b$ if and only if $z \wedge a \ll L$.*

Proof. (1) Since y is a supplement of x in L and $z\beta_* y$, $x \vee y = x \vee z = 1$. Since $z/(z \wedge x) \cong (z \vee x)/x$ and $(y \vee x)/x \cong y/(y \wedge x)$, $z/(z \wedge x) \cong y/(y \wedge x)$. Thus $z/(z \wedge x) \cong y/(y \wedge x)$.
(2) By 6 (1), $z/(z \wedge a) \cong b/(a \wedge b)$. Since $a \oplus b = 1$, $z/(z \wedge a) \cong b/0$.
(3) ($\Rightarrow$): Since $a \oplus b = 1$ and $z\beta_* b$, $a \vee z = 1$. Also, since b is a supplement of a in L and $z \leq b$, $z = b$.
($\Leftarrow$) Clear from reflexive property of β_*.
(4) ($\Rightarrow$): Since a is a weak supplement of b and $z\beta_* b$, it follows from Theorem 4 (2) that a is also a weak supplement of z in L. Hence $z \wedge a \ll L$.
($\Leftarrow$) It is clear from Corollary 2.

□

Theorem 7. *Let L be a distributive lattice and $a, b \in L$. If $a \oplus b = 1$ and $a\beta_* x$, then $a \leq x$ and $b \wedge x \ll L$.*

Proof. Since $a \oplus b = 1$ and $a\beta_* x$, $x \vee b = 1$. Hence $a \wedge (x \vee b) = a \wedge 1 = a$. By the distributive property, $(a \wedge x) \vee (a \wedge b) = a$. Since $a \wedge b = 0$, $a \wedge x = a$, and so $a \leq x$. Also since $x\beta_* a$ and $a \leq x$, $b \wedge x \ll L$ by Theorem 6 (4). □

Theorem 8. *Let L be a distributive lattice and $x \in L$. If $x\beta_* y$ and there exists a decomposition $a \oplus b = 1$ such that $a \leq x$ and $b \wedge x \ll L$, then $a \leq y$ and $b \wedge y \ll L$.*

Proof. Since $a \leq x$ and $b \wedge x \ll L$, $a\beta_* x$ by Theorem 6 (4). Since $x\beta_* y$ and $a\beta_* x$, $a\beta_* y$. By Theorem 7, $a \leq y$ and $b \wedge y \ll L$. □

Theorem 9. *Let $x \in L$ and k be a maximal element of L.*

(1) *If $a, b \in L$ such that $a \vee b = 1$, $b \neq 1$ and $x\beta_* a$ then $x \not\leq b$.*

(2) *If* $x\beta_* y$ *and* $x \leq k$ *then* $y \leq k$.
(3) *If* $x\beta_* k$ *then* $x \leq k$.
(4) *If* $x\beta_* k$ *and* w *is a weak supplement of* x *in* L *then* $k = x \vee (k \wedge x)$ *and* $k \wedge w \ll L$.

Proof. (1) Assume that $x \leq b$. Since $a \vee b = 1$ and $x\beta_* a$, $x \vee b = 1$ so $b = 1$. This is a contradiction with b is distinct from 1. Therefore $x \not\leq b$.
(2) Let $y \not\leq k$. Then $k \vee y = 1$. Since $x\beta_* y$, $k = k \vee x = 1$. This is a contradiction. Therefore $y \leq k$.
(3) Let $x\beta_* k$ and $x \not\leq k$. Since k is a maximal element, $x \vee k = 1$. Moreover $k = 1$, because $x\beta_* k$. This is a contradiction. Therefore $x \leq k$.
(4) Let $x\beta_* k$ and w be a weak supplement of x in L. From Theorem 4 (2), w is a weak supplement of k in L. Hence $k \vee w = 1$ and $k \wedge w \ll L$. Since $x\beta_* k$, $x \leq k$ by 9 (3). Since $x \vee w = 1$ and $x \leq k$, the modular law yields $k = x \vee (k \wedge w)$.

□

Theorem 10. *Let* $a, b \in L$ *and* $a \oplus b = 1$. *For* $x, s \in a/0$, *if* $x\beta_* s$ *in* L, *then* $x\beta_* s$ *in* $a/0$.

Proof. Let $k \in a/0$ such that $x \vee k = a$. Then $(x \vee k) \oplus b = 1$, and so $x \vee (k \vee b) = 1$. Hence $s \vee (k \vee b) = 1$ since $x\beta_* s$ in L. Then $[(s \vee k) \vee b] \wedge a = a$, it follows that $(s \vee k) \vee (b \wedge a) = a$. We obtain that $s \vee k = a$. Similarly, interchanging roles of x and s, we can show that $x \vee t = a$ for any element t of $a/0$ such that $s \vee t = a$. Therefore $x\beta_* s$ in $a/0$. □

Theorem 11. *Let* $x, y, k \in L$ *such that* $x \vee k = y \vee k = 1$, $k \wedge y \leq k \wedge x$ *and* $x \vee y \ll 1/y$. *Then* $x \vee y \ll 1/x$.

Proof. Let $t \in 1/x$ such that $(x \vee y) \vee t = 1$. Since $x \vee k = 1$, $t \wedge (x \vee k) = t \wedge 1$, and so $x \vee (t \wedge k) = t$ by the modular law. Hence $x \vee y \vee (t \wedge k) = 1$. It follows that $x \vee y \vee [y \vee (t \wedge k)] = 1$. Since $x \vee y \ll 1/y$, $y \vee (t \wedge k) = 1$. Then by Lemma 2, $t \vee (k \wedge y) = 1$. Since $k \wedge y \leq k \wedge x$, $1 = t \vee (k \wedge y) = t \vee (k \wedge x) = t$. Therefore $x \vee y \ll 1/x$. □

Theorem 12. *Let* $x, y, a, b \in L$. *If* $a, b \ll L$, $x \leq y \vee b$ *and* $y \leq x \vee a$, *then* $x\beta_* y$.

Proof. Let $k \in L$ such that $x \vee y \vee k = 1$. Since $x \leq y \vee b$, $y \vee b \vee k = 1$. From $b \ll L$, $y \vee k = 1$. Similarly, $x \vee k = 1$. Hence, by Theorem 1 (1), $x\beta_* y$. □

Theorem 13. *Let* $x_1, x_2, y_1, y_2 \in L$. *If* $x_1\beta_* y_1$ *and* $x_2\beta_* y_2$ *then* $(x_1 \vee x_2)\beta_*(y_1 \vee y_2)$.

Proof. Let $k \in L$ such that $(x_1 \vee x_2) \vee (y_1 \vee y_2) \vee k = 1$. Since $x_1\beta_* y_1$, $y_1 \vee x_2 \vee y_2 \vee k = 1$ and $x_1 \vee x_2 \vee y_2 \vee k = 1$. Also since $x_2\beta_* y_2$, $y_1 \vee y_2 \vee k = 1$ and $x_1 \vee x_2 \vee k = 1$. By Theorem 1 (1), $(x_1 \vee x_2)\beta_*(y_1 \vee y_2)$. □

Corollary 3. *Let $x, y_1, y_2, ..., y_n \in L$. If $x\beta_* y_i$ for $i = 1, 2, ..., n$, then $x\beta_* \bigvee_{i=1}^{n} y_i$.*

Proof. Clear from Theorem 13. □

Theorem 14. *Let $x, y \in L$ and $j \ll L$. Then $x\beta_* y$ if and only if $x\beta_*(y \vee j)$.*

Proof. ($\Rightarrow$) Let $k \in L$ with $x \vee k = 1$. Since $x\beta_* y$, $y \vee k = 1$. Then $y \vee j \vee k = 1$. Let $t \in L$ such that $(y \vee j) \vee t = 1$. Since $j \ll L$, $y \vee t = 1$. $x \vee t = 1$ from $x\beta_* y$. Hence $x\beta_*(y \vee j)$.
($\Leftarrow$) Let $k \in L$ with $x \vee k = 1$. Since $x\beta_*(y \vee j)$, $y \vee j \vee k = 1$. Also, since $j \ll L$, $y \vee k = 1$. Let $t \in L$ such that $y \vee t = 1$. Then $y \vee j \vee t = 1$. Since $x\beta_*(y \vee j)$, $x \vee t = 1$. Hence $x\beta_* y$.

□

Theorem 15. *Let $rad(L) = 0$ and $a \oplus b = 1$. If $x\beta_* a$ for some $x \in L$ then $x \oplus b = 1$.*

Proof. Since $a \oplus b = 1$, b is a supplement of a in L. Since $x\beta_* a$, b is also a supplement of x in L by Theorem 4 (1). Therefore $b \vee x = 1$ and $b \wedge x \ll x/0$. Since $rad(L) = 0$, $x \wedge b \leq rad(L) = 0$. Hence $x \oplus b = 1$. □

Theorem 16. *L is weakly supplemented if and only if for every $x \in L$, x is β_* equivalent to a weak supplement in L.*

Proof. ($\Rightarrow$) Let $x \in L$. Since L is weakly supplemented, there exists $z \in L$ such that $x \vee z = 1$ and $x \wedge z \ll L$. Also x is a weak supplement of z in L. Since β_* relation is reflexive, $x\beta_* x$. So, every element of L is β_* equivalent to a weak supplement element in L.
($\Leftarrow$) Let $x \in L$. By the hypothesis, there exists a weak supplement $z \in L$ such that $x\beta_* z$. Let z be a weak supplement of a in L. Thus $a \vee z = 1$ and $a \wedge z \ll L$. Also, a is a weak supplement element of z in L. Since $x\beta_* z$, a is also a weak supplement of x in L by Theorem 4 (2). □

Remark 1. The converse of Theorem 3 is not always true. We can give an example about that. Let K be a hollow module which is not simple. L be the lattice of the set of all submodules of $K \times K$ with respect to the ordering relation of inclusion. Since K is not simple, K has a submodule T with $T \neq 0$ and $T \ll K$. Clearly we see that $T \times 0 \ll L$ and $0 \times T \ll L$. But $T \times 0$ and $0 \times T$ don't lie above each other.

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PARTIALLY SOLVED DIFFERENTIAL SYSTEMS WITH TWO-POINT NON-LINEAR BOUNDARY CONDITIONS

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Abstract. We suggest a new constructive approach for the solvability analysis and approximate solution of certain types of partially solved Lipschitzian differential systems with two-point non-linear boundary conditions. The practical application of the suggested technique is shown on a numerical example.

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1. Introduction and subsidiary statements

The solvability analysis and approximate construction of solutions of various types of regular and singular boundary value problems were successfully done mainly in case of an explicit form of differential systems

$$\frac{dx(t)}{dt} = f(t, x(t)).$$

There is a large gap in the study of solutions of boundary value problems given for systems of differential equations of implicit form, in particular partially resolved with respect to the derivative. This work in a certain form fills this shortcoming.

We study the following boundary value problem on a compact interval

$$\frac{dx(t)}{dt} = f\left(t, x(t), \frac{dx(t)}{dt}\right), t \in [a,b], \tag{1.1}$$

$$g(x(a), x(b)) = d. \tag{1.2}$$

Here we suppose that $f : [a,b] \times D \times D_1 \to \mathbb{R}^n$ and $g : D \times D \to \mathbb{R}^n$ are continuous functions defined on a bounded sets $D \subset \mathbb{R}^n$ and $D^1 \subset \mathbb{R}^n$ (domain D will be concretized later, see (1.8), D_1 is given), and the function f is Lipschitzian with respect to the second and third variables in the following form:

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$$\left|\frac{du}{dt}-\frac{dv}{dt}\right| = \left|f\left(t,u,\frac{du}{dt}\right)-f\left(t,v,\frac{dv}{dt}\right)\right| \leq K_1\left|u-v\right|+K_2\left|\frac{du}{dt}-\frac{dv}{dt}\right| \quad (1.3)$$

for any $t \in [a,b]$ fixed and all $\{u,v\} \subset D$, $\left\{\frac{du}{dt},\frac{dv}{dt}\right\} \subset D_1$, where K_1, K_2 are a non-negative constant matrix of dimension $n \times n$.

Here and below, the absolute value sign and inequalities between vectors are understood componentwise. A similar convention is adopted for the operations "max", "min". The symbol 1_n stands for the unit matrix of dimension n, $r(K)$ denotes a spectral radius of a square matrix K.

If the maximal in modulus eigenvalue of matrix K_2 is less then one

$$r(K_2) < 1,$$

then from (1.3), if $u \neq v$, we obtain

$$[1_n - K_2]\left|\frac{du}{dt}-\frac{dv}{dt}\right| \leq K_1\left|u-v\right|,$$

or

$$\left|f\left(t,u,\frac{du}{dt}\right)-f\left(t,v,\frac{dv}{dt}\right)\right| \leq K\left|u-v\right|, \quad (1.4)$$

where

$$K = [1_n - K_2]^{-1} K_1 = K_1 + K_2 [1_n - K_2]^{-1} K_1.$$

Moreover, we suppose that for the maximal in modulus eigenvalue of matrix

$$Q = \frac{3(b-a)}{10} K \quad (1.5)$$

holds

$$r(Q) < 1. \quad (1.6)$$

If $z \in \mathbb{R}^n$ and ρ is a vector with non-negative components, $B(z,\rho)$ stands for the componentwise ρ -neighbourhood of z :

$$B(z,\rho) := \{\xi \in \mathbb{R}^n : |\xi - z| \leq \rho\}.$$

Similarly, for the given bounded connected set $\Omega \subset \mathbb{R}^n$, we define its componentwise $\rho-$neighbourhood by putting

$$B(\Omega,\rho) := \underset{\xi\in\Omega}{\cup} B(\xi,\rho).$$

Let us fix certain closed bounded sets $D_a \subset \mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$ and focus on the continuously differentiable solutions $x : [a,b] \to D$, $x' : [a,b] \to D_1$ of problem (1.1)-(1.2) with values $x(a) \in D_a$ and $x(b) \in D_b$. For given two bounded connected sets $D_a \subset \mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$, introduce the set

$$D_{a,b} := (1-\theta)z + \theta\eta,\ z \in D_a, \eta \in D_b, \theta \in [0,1] \quad (1.7)$$

and its componentwise $\rho-$neighbourhood

$$D := B(D_{a,b}, \rho)\,. \tag{1.8}$$

It is important to emphasize that D and D_1 are supposed to be bounded and, thus, the Lipschitz condition for f is not assumed globally. The boundary conditions (1.2), generally speaking, non-separated and non-linear.

With the function f involved in equation (1.1), we associate the vector

$$\delta_{[a,b],D,D_1}(f) := \frac{\max\limits_{(t,x,\frac{dx}{dt})\in[a,b]\times D\times D_1} f\left(t,x,\frac{dx}{dt}\right) - \min\limits_{(t,x,\frac{dx}{dt})\in[a,b]\times D\times D_1} f\left(t,x,\frac{dx}{dt}\right)}{2}. \tag{1.9}$$

We recall some subsidiary statements which are needed below.

Lemma 1 ([1], Lemma.3.13)**.** *Let $f : [\tau, \tau + I] \to \mathbb{R}^n$ be a continuous function. Then, for an arbitrary $t \in [\tau, \tau + I]$, the inequality*

$$\left|\int_\tau^t \left[f(\tau) - \frac{1}{T}\int_\tau^{\tau+I} f(s)ds \right] d\tau \right| \le \frac{1}{2}\alpha_1(t,\tau,I)\left[\max_{t\in[\tau,\tau+I]} f(t) - \min_{t\in[\tau,\tau+I]} f(t)\right] \tag{1.10}$$

holds, where

$$\alpha_1(t,\tau,I) = 2(t-\tau)\left(1-\frac{t-\tau}{I}\right),\ |\alpha_1(t,\tau,I)| \le \frac{I}{2},\ t\in[\tau,\tau+I]. \tag{1.11}$$

Lemma 2 ([1], Lemma 3.16)**.** *Let the sequence of continuous functions $\{\alpha_m(t,\tau,I)\}_{m=0}^{\infty}$, for $t \in [\tau, \tau + I]$, $m = 0, 1, 2, \ldots$ be defined by the recurrence relation*

$$\alpha_{m+1}(t,\tau,I) = \left(1-\frac{t-\tau}{I}\right)\int_\tau^t \alpha_m(s,\tau,I)ds + \frac{t-\tau}{I}\int_t^{\tau+I} \alpha_m(s,\tau,I)ds, \tag{1.12}$$

$\alpha_0(t,\tau,I) = 1$.

Then the following estimates hold for $t \in [\tau, \tau + I]$:

$$\alpha_{m+1}(t,\tau,I) \le \frac{10}{9}\left(\frac{3I}{10}\right)^m \alpha_1(t,\tau,I),\ m \geqslant 0, \tag{1.13}$$

$$\alpha_{m+1}(t,\tau,I) \le \frac{3I}{10}\alpha_m(t,\tau,I), m \geqslant 2,$$

2. Parametrization and convergence of successive approximations

The idea that we are going to employ is based on the reduction to a family of simple auxiliary boundary value problems [5]. This approach was used also in [2–4, 6, 7]. Namely, we introduce the vectors of parameters

$$z = col(z_1, z_2, ..., z_n),\ \eta = col(\eta_1, \eta_2, ..., \eta_n) \tag{2.1}$$

by formally putting

$$z = x(a),\ \eta = x(b). \tag{2.2}$$

Instead of boundary value problem (1.1)-(1.2) we will consider the following "model" problem with two-point linear separated parametrized conditions at a and b:

$$\frac{dx}{dt} = f\left(t, x, \frac{dx}{dt}\right),\ t \in [a,b], \tag{2.3}$$

$$x(a) = z,\ x(b) = \eta. \tag{2.4}$$

As will be seen from statements below, one can then go back to the original problem by choosing the values of the introduced parameters appropriately.

Let us connect with the two-point parametrized boundary value problem (2.3)-(2.4) the sequence of functions

$$x_{m+1}(t,z,\eta) = z + \int_a^t f\left(s, x_m(s,z,,\eta), \frac{dx_m(s,z,\eta)}{ds}\right) ds - \tag{2.5}$$

$$-\frac{t-a}{b-a}\int_a^b f\left(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{ds}\right) ds + \frac{t-a}{b-a}[\eta - z],\ t \in [a,b],$$

$m = 1, 2, ...$, satisfying (2.4) for arbitrary $z, \eta \in \mathbb{R}^n$, where

$$x_0(t,z,\eta) = z + \frac{t-a}{b-a}[\eta - z] = \left(1 - \frac{t-a}{b-a}\right) z + \frac{t-a}{b-a}\eta,\ t \in [a,b]. \tag{2.6}$$

It is easy to see from (2.6) that $x_0(t,z,\eta)$ is a linear combination of vectors z and η, when $z \in D_a,\ \eta \in D_b$.

Theorem 1. *Assume that*

$$\exists \text{ non negative vector } \rho: \ \rho \geq \frac{b-a}{2}\delta_{[a,b],D,D_1}(f), \tag{2.7}$$

where D is the $\rho-$neighhourhood of the set $D_{a,b}$ defined according to (1.7), (1.8) and $\delta_{[a,b],D,D_1}(f)$ is given as in (1.9):

$$\delta_{[a,b],D,D_1}(f) := \frac{\max\limits_{(t,x,\frac{dx}{dt})\in[a,b]\times D\times D_1} f(t,x,\frac{dx}{dt}) - \min\limits_{(t,x,\frac{dx}{dt})\in[a,b]\times D\times D_1} f(t,x,\frac{dx}{dt})}{2}, \tag{2.8}$$

the function $f \in C([a,b]\times D\times D_1,\mathbb{R}^n)$ is Lipschitzian with respect to the second and third variables according to condition (1.3) and for the matrix Q of form (1.5) holds an inequality (1.6).

Then, for all fixed $z \in D_a$, and $\eta \in D_b$:

1. The functions of the sequence (2.5) belonging to the domain D are continuously differentiable on the interval $[a,b]$, and satisfy the two-point separated boundary conditions (2.4).

2. The sequence of functions (2.5) for $t \in [a,b]$ converges as $m \to \infty$ to the limit function uniformly

$$x_\infty(t,z,\eta) = \lim_{m\to\infty} x_m(t,z,\eta). \tag{2.9}$$

3. The limit function satisfies the two-point separated boundary conditions (2.4).

4. The limit function $x_\infty(t,z,\eta)$ for all $t \in [a,b]$ is a unique continuously differentiable solution of the integral equation

$$x(t) = z + \int_a^t f(s,x(s),\frac{dx(s)}{ds})ds - \frac{t-a}{b-a}\int_a^b f(s,x(s),\frac{dx(s)}{ds})ds + \frac{t-a}{b-a}[\eta - z], \tag{2.10}$$

i.e. it is the solution of the Cauchy problem for the modified system of integro-differential equations:

$$\frac{dx}{dt} = f\left(t,x,\frac{dx(t)}{dt}\right) + \frac{1}{b-a}\Delta(z,\eta), \ x(a) = z \tag{2.11}$$

where $\Delta(z,\eta)) : D_a \times D_b \to \mathbb{R}^n$ is a mapping given by formula

$$\Delta(z,\eta) := [\eta - z] - \int_a^b f\left(s,x_\infty(s,z,\eta),\frac{dx_\infty(s,z,\eta)}{ds}\right)ds. \tag{2.12}$$

5. The following error estimation holds:

$$|x_\infty(\cdot,z,\eta)-x_m(\cdot,z,\eta)|\leqslant$$

$$\leqslant\frac{10}{9}\alpha_1(t,a,b-a)Q^m(1_n-Q)^{-1}\delta_{[a,b],D,D_1}(f),\ t\in[a,b],m\geq 0. \tag{2.13}$$

Proof. We will prove that for $z\in D_a$, $\eta\in D_b$ and $t\in[a,b]$ the values of the functions (2.5) belong to the domain D and it is a Cauchy sequence in the Banach space $C([a,b],\mathbb{R}^n)$. Indeed, using the estimation (1.10) of Lemma 1, relation (2.5) for $m=0,t\in[a,b]$ implies that

$$|x_1(t,z,\eta)-x_0(t,z,\eta)|\leq$$

$$\leq\alpha_1(t,a,b-a)\left[\frac{\max\limits_{(t,x,\frac{dx}{dt})\in[a,b]\times D\times D_1}f(t,x,\frac{dx}{dt})-\min\limits_{(t,x,\frac{dx}{dt})\in[a,b]\times D\times D_1}f(t,x,\frac{dx}{dt})}{2})\right] \tag{2.14}$$

$$\leq\alpha_1(t,a,b-a)\,\delta_{[a,b],D,D_1}(f)\leq\frac{b-a}{2}\,\delta_{[a,b],D,D_1}(f),$$

which means according to (2.7), that $x_1(t,z,\eta)\in D$, whenever $(t,z,\eta)\in[a,b]\times D_a\times D_b$.

Using this and arguing by induction according to Lemma 1 we can easily establish that

$$|x_m(t,z,\eta)-x_0(t,z,\eta)|\leq\alpha_1(t,a,b-a)\,\delta_{[a,b],D,D_1}(f)\leq\frac{b-a}{2}\,\delta_{[a,b],D,D_1}(f),$$

$m=2,3,...$, which means that all functions (2.5) are also contained in the domain D, for all $m=1,2,3,..$ and $(t,z,\eta)\in[a,b]\times D_a\times D_b$.

Now, consider the difference of functions

$$x_{m+1}(t,z,\eta)-x_m(t,z,\eta)=$$

$$=\int_a^t\left[f\left(s,x_m(s,z,\eta),\frac{dx_m(s,z,\eta)}{ds}\right)-f\left(s,x_{m-1}(s,z,\eta),\frac{dx_{m-1}(s,z,\eta)}{ds}\right)\right]ds \tag{2.15}$$

$$-\frac{t-a}{b-a}\int_a^b\left[f\left(s,x_m(s,z,\eta),\frac{dx_m(s,z,\eta)}{ds}\right)-f\left(s,x_{m-1}(s,z,\eta),\frac{dx_{m-1}(s,z,\eta)}{ds}\right)\right]ds,$$

$m=1,2,...$ and introduce the notation

$$r_m(t,z,\eta)=|x_m(t,z,\eta)-x_{m-1}(t,z,\eta)|,m=1,2,.... \tag{2.16}$$

By virtue of the Lipschitz condition (1.4), for $m = 1$ from (2.15), we have

$$r_2(t,z,\eta) \le K\left[\left(1-\frac{t-a}{b-a}\right)\int_a^t r_1(s,z,\eta)ds + \frac{t-a}{b-a}\int_t^b r_1(s,z,\eta)ds\right]. \tag{2.17}$$

According to the recurrence relation (1.12) and estimation (1.13) from (2.14) and (2.17) follows that

$$\begin{aligned} r_2(t,z,\eta) &\le K\left(1-\frac{t-a}{b-a}\right)\int_a^t \alpha_1(s,a,b-a)\ \delta_{[a,b],D,D_1}(f)ds + \\ &+\frac{t-a}{b-a}\int_t^b \alpha_1(s,a,b-a)\ \delta_{[a,b],D,D_1}(f)ds \le \\ &\le K\alpha_2(t,a,b-a)\ \delta_{[a,b],D,D_1}(f). \end{aligned}$$

By induction using estimation (1.13), we can easily establish that

$$r_{m+1}(t,z,\eta) \le K^m\alpha_{m+1}(t,a,b-a)\ \delta_{[a,b],D,D_1}(f) \le \frac{10}{9}Q^m\alpha_1(t,a,b-a)\ \delta_{[a,b],D,D_1}(f) \tag{2.18}$$

Therefore, in view of (2.18)

$$\begin{aligned} &\left|x_{m+j}(t,z,\eta)-x_m(t,z,\eta)\right| \le \\ &\le \left|x_{m+j}(t,z,\eta)-x_{m+j-1}(t,z,\eta)\right| + \left|x_{m+j-1}(t,z,\eta)-x_{m+j-2}(t,z,\eta)\right| + \dots \\ &+\left|x_{m+1}(t,z,\eta)-x_m(t,z,\eta)\right| = \sum_{i=1}^{j} r_{m+i}(t,z,\eta) \le \\ &\le \frac{10}{9}\alpha_1(t,a,b-a)\sum_{i=1}^{j} Q^{m+i-1}\delta_{[a,b],D,D_1}(f) = \\ &= \frac{10}{9}\alpha_1(t,a,b-a)Q^m\sum_{i=0}^{j-1} Q^i\delta_{[a,b],D,D_1}(f), \end{aligned} \tag{2.19}$$

where $\delta_{[a,b],D,D_1}(f)$ is given by (2.8). Since, due to (1.6), the maximum eigenvalue of the matrix (1.5) does not exceed the unity, we have

$$\sum_{i=0}^{j-1} Q^i \le (1_n - Q)^{-1}, \quad \lim_{m\to\infty} Q^m = 0_n. \tag{2.20}$$

Therefore, we conclude from (2.19) that, according to Cauchy criterium, the sequence $\{x_m(t,z,\eta)\}_{m=0}^{\infty}$ of the form (2.5) uniformly converges in the domain $(t,z,\eta) \in [a,b]\times D_a \times D_b$ to the limit function $x_\infty(t,z,\eta)$. Since all functions of the sequence

(2.5) satisfy the linear separated parametrized conditions (2.4) for all values of the introduced parameter $z \in D_a$, $\eta \in D_b$ the limit function $x_\infty(t,z,\eta)$ also satisfies these conditions. Passing to the limit as $m \to \infty$ in equality (2.5) we show that the limit function satisfies both the integral equation (2.10) and the Cauchy problem (2.11), where $\Delta(z,\eta)$ is given by (2.12). Passing to the limit as $j \to \infty$ in (2.19) we get the estimation (2.13). □

3. Connection of the limit function $x_\infty(t,z,\eta)$ to the solution of the original problem

Theorem 2. *Under the assumptions of Theorem 1, the limit function*

$$x_\infty(t,z^*,\eta^*) = \lim_{m\to\infty} x_m(t,z^*,\eta^*) \tag{3.1}$$

of the sequence (2.5) is a solution of the non-linear boundary value problem (1.1)-(1.2) if and only if the pair of parameters (z^*,η^*) from (2.2) satisfies the system of $2n$ algebraic or transcendental equations

$$\Delta(z,\eta) := [\eta - z] - \int_a^b f(s, x_\infty(s,z,\eta), \frac{dx_\infty(s,z,\eta)}{ds})ds = 0,$$
$$\Lambda(z,\eta) := g(x_\infty(a,z,\eta), x_\infty(b,z,\eta)) - d = 0. \tag{3.2}$$

Proof. The proof can be carried out similarly as in Theorems 2 and 3 from [8]. □

Remark 1. The system of equations (3.2) is usually referred to as a *determining equations.* In such a manner, the original infinite-dimensional problem (1.1)- (1.2) is reduced to a system of $2n$ equations numerical equations.

The method thus consists of two parts, namely, the analytic part, when the integral equation (2.10) is dealt with by using the method of successive approximations (2.5), and the numerical one, which consists in finding values of the $2n$ unknown parameters from equations (3.2).

The next statement proves that the system of determining equations (3.2) defines all possible solutions of the original non-linear boundary value problem (1.1)-(1.2).

Theorem 3. *Let the assumptions of Theorem 1 hold. Furthermore, assume there exist some pair of vectors $(z^0,\eta^0) \in D_a \times D_b$ that satisfy the system of determining equations (3.2).*

Then:

1. The non-linear boundary value problem (1.1)-(1.2) has a solution $x^0(\cdot)$ such that

$$x^0(a) = z^0,\ x^0(b) = \eta^0. \tag{3.3}$$

Moreover, this solution is given by the limit function of the sequence (2.5):

$$x^0(\cdot) = x_\infty(\cdot, z^0, \eta^0) = \lim_{m\to\infty} x_m(\cdot, z^0, \eta^0),\ t \in [a,b]. \tag{3.4}$$

2. *If the non-linear boundary value problem (1.1)-(1.2) has a solution $x^0(\cdot)$, then the system of determining equations (3.2) is satisfied with*

$$z = x^0(a), \eta = x^0(b). \tag{3.5}$$

Proof. The proof can be carried out similarly as in Theorem 4 from [8]. □

4. Solvability analysis based on the approximate determining system

Although Theorem 2 provides a theoretical answer to the question on the construction of a solution of the original non-linear boundary value problem (1.1)-(1.2), its application faces certain difficulties due to the fact that the explicit form of the limit function $x_\infty(\cdot, z, \eta)$ and consequently the explicit form of the functions

$$\Delta : D_a \times D_b \to \mathbb{R}^n, \ \Lambda : D_a \times D_b \to \mathbb{R}^n$$

in (3.2) is usually unknown. This complication can be overcome by using the so-called *approximate determining equations*

$$\Delta_m(z,\eta) := [\eta - z] - \int_a^b f(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{ds})ds = 0,$$

$$\Lambda_m(z,\eta) := g(x_m(a,z,\eta), x_m(b,z,\eta))) - d = 0 \tag{4.1}$$

for a fixed m.

Lemma 3. *Under the assumptions of Theorem 1 and if the function $g : D_a \times D_b \to \mathbb{R}^n$ in the boundary restrictions (1.2) satisfy the Lipschitz condition*

$$|g(u_1,u_2) - g(v_1,v_2)| \le K_{g1}|u_1 - u_2| + K_{g2}|v_1 - v_2|, \tag{4.2}$$

for all $\{u_1,u_2\} \subset D_a \times D_b$ and $\{v_1,v_2\} \subset D_a \times D_b$, where K_{g1}, K_{g2} are a non-negative constant matrix of dimension $n \times n$, then for the exact and approximate determining functions defined by (3.2) and (4.1) for any $(z,\eta) \in D_a \times D_b$ and $m \ge 1$ hold the following estimates:

$$|\Delta(z,\eta) - \Delta_m(z,\eta)| \le \frac{10}{27} K Q^m (1_n - Q)^{-1} \delta_{[a,b],D,D_1}(f),$$

$$|\Lambda(z,\eta) - \Lambda_m(z,\eta)| \le \left[K_{g1} + K_{g2}\right] \frac{5}{9}(b-a) Q^m (1_n - Q)^{-1} \delta_{[a,b],D,D_1}(f), \tag{4.3}$$

where the matrix Q and the vector $\delta_{[a,b],D,D_1}(f)$ are given respectively in (1.5) and (2.8).

Proof. Let us fix an arbitrary $(z,\eta) \in D_a \times D_b$. Using the Lipschitz condition (1.4), estimate (2.13) and the equality

$$\int_a^b \alpha_1(t,a,b-a)dt = \frac{(b-a)^2}{3}, \tag{4.4}$$

we have

$$\begin{aligned}
&|\Delta(z,\eta) - \Delta_m(z,\eta)| = \\
&= \left| \int_a^b f(s,x_\infty(s,z,\eta))ds - \int_a^b f(s,x_m(s,z,\eta))ds \right| \\
&\leq K \int_a^b \frac{10}{9}\alpha_1(s,a,b) Q^m (1_n - Q)^{-1} \delta_{[a,b],D,D_1}(f)ds = \\
&= \frac{10}{27} K Q^m (1_n - Q)^{-1} \delta_{[a,b],D,D_1}(f),
\end{aligned}$$

which proves the first inequality in (4.3).

From (3.2) and (4.1) using the Lipschitz condition (4.2) and (2.13), we obtain

$$\begin{aligned}
|\Lambda(z,\eta) - \Lambda_m(z,\eta)| &= |g(x_\infty(a,z,\eta), x_\infty(b,z,\eta)) - g(x_m(a,z,\eta), x_m(b,z,\eta))| \\
&\leq K_{g1} |x_\infty(a,z,\eta) - x_m(a,z,\eta)| + K_{g2} |x_\infty(a,z,\eta) - x_m(a,z,\eta)| \\
&\leq \left[K_{g1} + K_{g2}\right] \frac{5}{9}(b-a) Q^m (1_n - Q)^{-1} \delta_{[a,b],D,D_1}(f),
\end{aligned}$$

i.e. the second estimate in (4.3) holds also. □

Based on both exact an approximate determining systems (3.2) and (4.1) let us introduce the mappings $H : D_a \times D_b \to \mathbb{R}^{2n}$ and $H_m : D_a \times D_b \to \mathbb{R}^{2n}$ by setting

$$H(z,\eta) := \begin{bmatrix} [\eta - z] - \int_a^b f(s, x_\infty(s,z,\eta), \frac{dx_\infty(s,z,\eta)}{ds})ds, & z \in D_a \times D_b \\ g(x_\infty(a,z,\eta), x_\infty(b,z,\eta)) - d \end{bmatrix}, \tag{4.5}$$

$$H_m(z,\eta) := \begin{bmatrix} [\eta - z] - \int_a^b f(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{ds})ds, & z \in D_a \times D_b \\ g(x_m(a,z,\eta), x_m(b,z,\eta)) - d \end{bmatrix}, \tag{4.6}$$

We see from Theorem 2 that the critical points of the vector field H of the form (4.5) determine solutions of the non-linear boundary value problem (1.1)-(1.2). The

next statement establishes a similar result based upon properties of vector field H_m explicitly known from (4.6).

Theorem 4. *Assume that the conditions of Lemma 3 hold. Moreover, one can specify an $m \geq 1$ and a set*

$$\Omega := D_1 \times D_2 \subset \mathbb{R}^{2n},$$

where $D_1 \subset D_a, D_2 \subset D_b$ are certain bounded open sets such that the mapping H_m satisfies the relation

$$|H_m(z,\eta)| \rhd_{\partial\Gamma} \begin{bmatrix} \frac{10}{27} K Q^m (1_n - Q)^{-1} \delta_{[a,b],D,D_1}(f) \\ \left[K_{g1} + K_{g2}\right] \frac{5}{9}(b-a) Q^m (1_n - Q)^{-1} \delta_{[a,b],D,D_1}(f) \end{bmatrix} \tag{4.7}$$

on the boundary $\partial\Omega$ of the set Ω. If, in addition

$$\deg(H_m, \Omega, 0) \neq 0, \tag{4.8}$$

then there exists a pair $(z^*, \eta^*) \in D_1 \times D_2$ for which the function

$$x^*(\cdot) := x_\infty\left(\cdot, z^*, \eta^*\right)$$

is a solution of the non-linear boundary value problem (1.1)-(1.2).

In (4.7) the binary relation $\rhd_{\partial\Gamma}$ is defined in [1] as a kind of strict inequality for vector functions and it means that at every point on the boundary $\partial\Omega$ at least one of the components of the vector $|H_m(z,\eta)|$ is greater than the corresponding component of the vector in the right-hand side. The degree in (4.8) is the Brouwer degree because all the vectors fields are finite-dimensional. Likewise, all the terms in the right-hand side of (4.7) are computed explicitly e.g. by using computer algebra system.

Proof. The proof can be carried out similarly as in Theorem 4 from [6]. □

5. Example

Let us apply the approach described above to the system of differential equations

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1(t)\, x_2(t) - \frac{dx_2(t)}{dt} + x_2^2(t) = f_1\left(t, x_1(t)\,, x_2(t), \frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt}\right) \\ \frac{dx_2(t)}{dt} = \frac{dx_1(t)}{dt}\frac{dx_2(t)}{dt} + \frac{1}{2}\, x_2(t) + \frac{t}{4} = f_2\left(t, x_1(t)\,, x_2(t), \frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt}\right) \end{cases}, \tag{5.1}$$

$t \in [a,b] = \left[0, \frac{1}{2}\right]$, considered under the two- point non-linear boundary conditions

$$\begin{cases} x_1^2(a) - x_2(b) = -\frac{1}{32} \\ x_2^2(a) - x_1(b) = \frac{1}{32} \end{cases} \tag{5.2}$$

Following (2.1), (2.2), introduce the parameters $z = col(z_1, z_2)$, $\eta = col(\eta_1, \eta_2)$.

Let us consider the following choice of subsets D_a, D_b and D_1, where one looks for the values $x(a)$, $x(b)$ and the values of the derivatives $\frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt}$:

$$D_a = D_b = \{(x_1, x_2) : \ -0.2 \le x_1 \le 0.2, \ -0.2 \le x_1 \le 0.2\}, \tag{5.3}$$

$$D_1 = \left\{(x_1, x_2) : \ -0.2 \le \frac{dx_1}{dt} \le 0.2 \ , \ -0,2 \le \frac{dx_2}{dt} \le 0.2\right\}. \tag{5.4}$$

This choice of the sets D_a and D_b is motivated by the fact that the zero-th approximate determining system (i. e., (4.1) with m = 0) has roots lying in these sets (5.3), see the second line in Table 1. Recall that, in order to obtain it, only function (2.6) are used, and no iteration is yet carried out. We see that this piecewise linear function provides quite reasonable approximate values of the parameters. In this case, according to (1.7), we have

$$D_{a,b} = D_a = D_b. \tag{5.5}$$

For ρ involved in (2.7), we choose the vector

$$\rho := col(0.3, 0.3). \tag{5.6}$$

Then, in view of (5.3), (5.5), (5.6), the set (1.8) takes the form

$$D = \{(x_1, x_2) : \ -0.5 \le x_1 \le 0.5, \ -0.5 \le x_1 \le 0.5\}. \tag{5.7}$$

A direct computation shows that the Lipschitz condition (1.4) for f given by (5.1) on D and D_1 of forms (5.7) and (5.4) holds with matrices

$$K_1 = \begin{pmatrix} 0.5 & 1 \\ 0 & 0.5 \end{pmatrix}, \ K_2 = \begin{pmatrix} 0 & 1 \\ 0.2 & 0.2 \end{pmatrix},$$

$$[1_n - K_2]^{-1} = \begin{pmatrix} 1.333333333 & 1.666666667 \\ 0.333333333 & 1.666666667 \end{pmatrix},$$

$$K = [1_n - K_2]^{-1} K_1 = \begin{pmatrix} 0.6666666665 & 2.166666666 \\ 0.1666666666 & 1.166666667 \end{pmatrix}.$$

Therefore, by (1.5)

$$Q = \begin{pmatrix} 0.09999999998 & 0.3249999999 \\ 0.02499999999 & 0.1750000000 \end{pmatrix}$$

and $r(Q) = 0.235128120913226 < 1$.

Furthermore, in view of (1.9)

$$\delta_{[a,b],D,D_1}(f) := \frac{\max\limits_{(t,x,\frac{dx}{dt}) \in [a,b] \times D \times D_1} f(t, x, \frac{dx}{dt}) - \min\limits_{(t,x,\frac{dx}{dt}) \in [a,b] \times D \times D_1} f(t, x, \frac{dx}{dt})}{2} =$$

$$= \begin{pmatrix} 0.4812500000 \\ 0.3525000000 \end{pmatrix}$$

and by (5.6) we have

$$\frac{b-a}{2} \delta_{[a,b],D,D_1}(f) = \begin{pmatrix} 0.1203125000 \\ 0.0881250000 \end{pmatrix} \le \rho.$$

TABLE 1.

m	z_1	z_2	η_1	η_2
Exact	0	0	$-\frac{1}{32} = -0.03125$	$\frac{1}{32} = 0.03125$
0	0.001561487459	-0.001556662026	-0.0312475768	0.03125243824
1	0.0004122967488	-0.0003873893367	-0.03124984993	0.03125016999
2	-0.000157576675	0.0001562942819	-0.03124997556	0.03125002482
3	$-0.8640511032 \cdot 10^{-5}$	$0.6489136242 \cdot 10^{-5}$	-0.03124999997	0.03125000006
4	$0.1264993624 \cdot 10^{-4}$	$-0.1236038310 \cdot 10^{-4}$	-0.03124999985	0.03125000015
5	$-4.892901202 \cdot 10^{-7}$	$6.586000019 \cdot 10^{-7}$	-0.03125000000	0.03125000001
6	$-0.1073874769 \cdot 10^{-5}$	$0.1030957108 \cdot 10^{-5}$	-0.03125000001	0.03125000000
7	$1.587193848 \cdot 10^{-7}$	$-1.712004563 \cdot 10^{-7}$	-0.03124999999	0.03124999999
8	$8.595697086 \cdot 10^{-8}$	$-8.040876912 \cdot 10 - 8$	-0.03125000000	0.03125000000
9	$-2.502073667 \cdot 10^{-8}$	$2.574184204 \cdot 10^{-8}$	-0.03124999999	0.03125000001

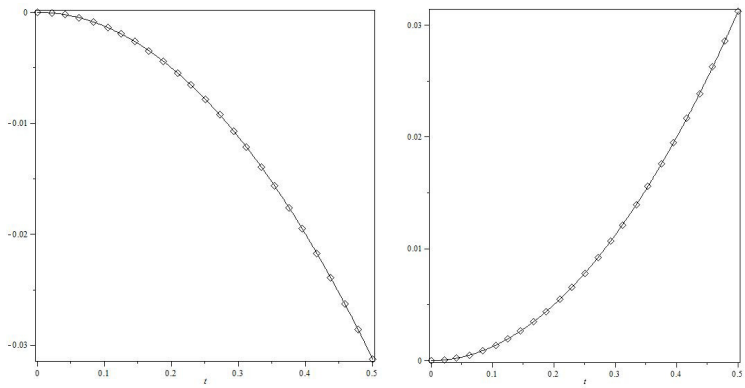
We thus see that all the conditions of Theorem 1 are fulfilled, and the sequence of functions (2.5) for this example is convergent.

It is easy to verify that the pair of functions

$$x_1^*(t) = -\frac{t^2}{8},\ x_2^*(t) = \frac{t^2}{8}$$

is a solution of the given boundary value problem (5.1)-(5.2).

Using (2.5) and applying Maple 13 for different values of m to implement the approximations $x_m(t,z,\eta) = col(x_{m1}(t,z,\eta), x_{m2}(t,z,\eta))$ and solving the approximate determining system (4.1), we find the following values of introduced parameters, which are presented in Table 1. The graphs of the exact and approximate solution for $m = 9$ for the first and second components are shown on the Fig. 1.



(a) First component (b) Second component

FIGURE 1. The exact solution $(x_1^*(t), x_2^*(t))$ (solid line) and its nineth approximation (dots)

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HYBRID DIFFERENCE-DIFFERENTIAL BOUNDARY-VALUE PROBLEM

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Abstract. We study the problems of finding the conditions for the existence of solutions of a linear Noetherian boundary-value problem for a hybrid system of ordinary differential equations and difference equations and the construction of these solutions. We also propose an algorithm for the construction of solutions of linear Noetherian boundary-value problems for hybrid systems of difference-differential equations in the critical and noncritical cases.

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1. Introduction

The mathematical description of continuous processes with short-term perturbations whose duration can be neglected leads to the investigation of boundary-value problems for impulsive ordinary differential equations [7, 13, 14, 19, 22, 24, 25]. The analysis of discrete processes requires the study of boundary-value problems for difference equations [3,5,20,21,23]. The proposed hybrid difference-differential system contains an unknown function of piecewise continuous argument and an unknown function of discrete argument. To find these unknowns, we propose to use a system of ordinary differential equations containing the unknown function of discrete argument and a system of difference equations containing the unknown function of piecewise continuous argument. The proposed approach to the mathematical description of both continuous and discrete processes enables one to combine the efficient methods of the theory of boundary-value problems for ordinary differential and difference equations.

2. Statement of the problem

We consider the problem of finding conditions for the existence of solutions

$$x(t) \in \mathbb{R}^n, \quad t \in \Omega \cup \Theta, \quad \Omega := [a, \tau_1[\cup [\tau_{p_1}, \tau_{p_1+1}[\cup \ldots \cup [\tau_{p_q}, \tau_{p_q+1}] \subset [a, b],$$

$$y(k) \in \mathbb{R}^m, \quad k \in \Theta := \{a, \tau_1, \tau_2, \ldots, \tau_{p_1}, \tau_{p_1+1}, \ldots, \tau_{p_2}, \tau_{p_2+1}, \ldots, b\}$$

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and the problem of construction of these solutions for the hybrid Noetherian ($m+n \neq \lambda+\mu$) difference-differential boundary-value problem

$$x'(t) = A(t,k)x(t) + B(t,k)y(k) + \varphi(t,k), \quad t \in \Omega, \tag{2.1}$$

$$y(k+1) = C(k)x(k) + D(k)y(k) + \psi(k), \quad k \in \Theta, \tag{2.2}$$

$$\Delta x(\tau_{p_i+1}, \tau_{p_{i+1}}) := S_{\tau_{p_i+1}} x(\tau_{p_i+1}-0) + b_{\tau_{p_i+1}}, \quad \ell x(\cdot) = \alpha, \quad \wp\, y(\cdot) = \beta. \tag{2.3}$$

We seek the first component of the required solution $x(t) \in \mathbb{C}^1[\Omega_I]$ of the boundary-value problem (2.1)–(2.3) in a class of functions continuously differentiable on the set $\Omega \subset [a,b]$, except the points $\tau_1,\ \tau_{p_1+1},\ \tau_{p_2+1},\ \dots,\ \tau_{p_{q-1}}$ at which the solution $x(t)$ may have finite discontinuities

$$\Delta x(\tau_{p_i+1}, \tau_{p_{i+1}}) := x(\tau_{p_{i+1}}) - x(\tau_{p_i+1}-0) \leq \rho, \quad i = 0,\ 1,\ 2,\ \dots,\ q-1,$$

where $a := \tau_0 := \tau_{p_0},\ \tau_{p_q+1} := b,\ \alpha \in \mathbb{R}^\mu,\ \beta \in \mathbb{R}^\nu$. The second component of the required solution $y(k) \in \mathbb{R}^m,\ k \in \Theta$ of the Noetherian boundary-value problem (2.1)–(2.3) is sought in the set of bounded sequences. The matrices $A(t,k) \in \mathbb{R}^{n\times n}$, $B(t,k) \in \mathbb{R}^{n\times m}$ and the vector function $\varphi(t,k) \in \mathbb{R}^n$ are assumed to be continuous in the first and second arguments on the segment $[a,b]$. The matrices $C(k) \in \mathbb{R}^{m\times n},\ D(k) \in \mathbb{R}^{m\times m},\ S_{\tau_i} \in \mathbb{R}^{m\times m}$ and the vector function $\psi(k) \in \mathbb{R}^m$ are assumed to be bounded on the set Θ by functions, $\ell x(\cdot)$ is the linear bounded vector functional:

$$\ell x(\cdot) :\ \mathbb{C}^1[\Omega_I] \to \mathbb{R}^\mu,$$

$\wp\, y(\cdot)$ is the linear bounded vector functional given in the space of vector functions bounded on the set Θ as follows:

$$\wp\, y(\cdot) :\ \{y(k) :\ \Theta \to \mathbb{R}^m\} \to \mathbb{R}^\nu, \quad \mu \neq \nu.$$

Various hybrid boundary-value problems for systems of ordinary differential and functional-differential equations are extensively investigated (see, e.g., [3]). The posed hybrid boundary-value problem (2.1)–(2.3) is a generalization of boundary-value problems for ordinary impulsive differential equations [7, 13, 19, 22, 24, 25], boundary-value problems for difference equations [5,20,21,23], and different boundary-value problems for systems of ordinary differential equations [7].

3. Generalized Green operator of the Cauchy problem

For the sake of definiteness, we set

$$x(a) = \zeta \in \mathbb{R}^n, \quad y(a) = \xi \in \mathbb{R}^m, \quad m \neq n. \tag{3.1}$$

Let $X_a(t)$ be the normal ($X_a(a) = I_n$) fundamental matrix of the homogeneous part $x'(t) = A(t,a)x(t)$ of system (2.1) on the segment $[a;\tau_1]$. The solution of system (2.1) in the interval $[a;\tau_1[$ is represented in the form

$$x(t,\zeta,\xi) = X_a(t)\zeta + V_a(t)\xi + K_a\Big[\varphi(s,a)\Big](t),$$

$$y(a,\zeta,\xi) = Z(a)\zeta + W(a)\xi + K\Big[\psi(\tau_0)\Big](a),\ Z(\tau_0) := 0,\ W(\tau_0) := I_m,$$

where

$$V_a(t) := X_a(t)\int_a^t X_a^{-1}(s)B(s,a)W(a)\,ds,\ K\Big[\psi(\tau_0)\Big](a) = 0$$

and

$$K_a\Big[\varphi(s,a)\Big](t) := X_a(t)\int_a^t X_a^{-1}(s)\varphi(s,a)\,ds$$

is the generalized Green operator of the Cauchy problem (3.1) for system (2.1) in the interval $[a;\tau_1[$. We represent the solution of system (2.2) for $k = \tau_1$ in the form

$$y(\tau_1,\zeta,\xi) = Z(\tau_1)\zeta + W(\tau_1)\xi + K\Big[\psi(a)\Big](\tau_1),\ Z(\tau_1) := C(\tau_0),\quad W(\tau_1) := D(\tau_0),$$

where

$$K\Big[\psi(a)\Big](\tau_1) = \psi(\tau_0) = \psi(a)$$

is the generalized Green operator of the Cauchy problem $y(a) = \xi$ for system (2.2) with $k = \tau_1$. Similarly, we represent the solution of system (2.2) for $k = \tau_2$ in the form

$$y(\tau_2,\zeta,\xi) = Z(\tau_2)\zeta + W(\tau_2)\xi + K\Big[\psi(\tau_1)\Big](\tau_2),$$

where

$$Z(\tau_2) := C(\tau_1)X_a(\tau_1) + D(\tau_1)Z(\tau_1),\quad W(\tau_2) := C(\tau_1)V_a(\tau_1) + D(\tau_1)W(\tau_1),$$

$$K\Big[\psi(\tau_1)\Big](\tau_2) := C(\tau_1)K_a\Big[\varphi(s,a)\Big](\tau_1) + D(\tau_1)K\Big[\psi(a)\Big](\tau_1) + \psi(\tau_1).$$

If we continue our reasoning, then we get a solution of system (2.2) with $k = \tau_{p_1}$:

$$y(\tau_{p_1},\zeta,\xi) = Z(\tau_{p_1})\zeta + W(\tau_{p_1})\xi + K\Big[\psi(\tau_{p_1}-1)\Big](\tau_{p_1}),$$

where

$$Z(\tau_{p_1}) := C(\tau_{p_1-1})X_a(\tau_{p_1-1}) + D(\tau_{p_1-1})Z(\tau_{p_1-1}),$$
$$W(\tau_{p_1}) := C(\tau_{p_1-1})V_a(\tau_{p_1-1}) + D(\tau_{p_1-1})W(\tau_{p_1-1})$$

and

$$K\Big[\psi(\tau_{p_1-1})\Big](\tau_{p_1}) := C(\tau_{p_1-1})K_a\Big[\varphi(s,\tau_0)\Big](\tau_{p_1-1})$$
$$+ D(\tau_{p_1-1})K\Big[\psi(\tau_{p_1-2})\Big](\tau_{p_1-1}) + \psi(\tau_{p_1-1})$$

is the generalized Green operator of the Cauchy problem $y(a) = \xi$ for system (2.2) with $k = \tau_{p_1}$. Let $X_1(t)$ be the normal ($X_1(\tau_{p_1}) = I_n$) fundamental matrix of the homogeneous part

$$x'(t) = A(t, \tau_{p_1})x(t), \quad t \in [\tau_{p_1}; \tau_{p_1+1}[$$

of system (2.1). We represent the solution of system (2.1) satisfying the first condition in (2.3) on the interval $[\tau_{p_1}; \tau_{p_1+1}[$ in the form

$$x(t, \zeta, \xi) = X_{\tau_{p_1}}(t)\zeta + V_{\tau_{p_1}}(t)\xi + K_{\tau_{p_1}}\Big[\varphi(s, \tau_{p_1}); b_1\Big](t),$$

where

$$K_{\tau_{p_1}}\Big[\varphi(s, \tau_{p_1})\Big](t) := X_1(t)\int_{\tau_{p_1}}^{t} X_1^{-1}(s)\varphi(s, \tau_{p_1})ds, \quad t \in [\tau_{p_1}; \tau_{p_1+1}[$$

is the Green operator of the Cauchy problem (3.1) for system (2.1),

$$X_{\tau_{p_1}}(t) := X_1(t)(I_n + S_1)X_a(\tau_1) + K_{\tau_{p_1}}\Big[B(s, \tau_{p_1})Z(\tau_{p_1})\Big](t)$$

is the normal ($X_{\tau_{p_1}}(\tau_{p_1}) = (I_n + S_1)X_a(\tau_1)$) fundamental matrix of the homogeneous part of system (2.1) for $t \in [\tau_{p_1}; \tau_{p_1+1}[$ satisfying the first condition in (2.3)

$$V_{\tau_{p_1}}(t) := X_1(t)(I_n + S_1)V_a(\tau_1) + K_{\tau_{p_1}}\Big[B(s, \tau_{p_1})W(\tau_{p_1})\Big](t),$$

$$\begin{aligned}\mathcal{K}_{\tau_{p_1}}\Big[\varphi(s, \tau_{p_1}); b_1\Big](t) &:= X_1(t)b_1 + X_1(t)(I_n + S_1)K_a\Big[\varphi(s, a)\Big](\tau_1)\\ &\quad + K_{\tau_{p_1}}\Big[\varphi(s, \tau_{p_1})\Big](t) + K_{\tau_{p_1}}\Big\{B(s, \tau_{p_1})K\Big[\psi(\tau_{p_1-1})\Big](\tau_{p_1})\Big\}(t).\end{aligned}$$

Note that the function $x(t, \zeta, \xi)$ is defined both on the segment $[\tau_{p_1}; \tau_{p_1+1}]$ and at the points

$$t = k \in \{\tau_{p_1}, \tau_{p_1+1}, \dots, \tau_{p_2-1}\} \in \Theta.$$

We represent the solution of system (2.2) for $k = \tau_{p_1+1}$ in the form

$$y(\tau_{p_1+1}, \zeta, \xi) = Z(\tau_{p_1+1})\zeta + W(\tau_{p_1+1})\xi + \mathcal{K}\Big[\psi(\tau_{p_1}); b_1\Big](\tau_{p_1+1}),$$

where

$$\begin{aligned}Z(\tau_{p_1+1}) &:= C(\tau_{p_1})X_{\tau_{p_1}}(\tau_{p_1}) + D(\tau_{p_1})Z(\tau_{p_1}),\\ W(\tau_{p_1+1}) &:= C(\tau_{p_1})V_{\tau_{p_1}}(\tau_{p_1}) + D(\tau_{p_1})W(\tau_{p_1})\end{aligned}$$

and

$$\mathcal{K}\Big[\psi(\tau_{p_1}); b_1\Big](\tau_{p_1+1}) := C(\tau_{p_1})\mathcal{K}_{\tau_{p_1}}\Big[\varphi(s, \tau_{p_1}); b_1\Big](\tau_{p_1})$$

$$+ D(\tau_{p_1})K\Big[\psi(\tau_{p_1-1})\Big](\tau_{p_1}) + \psi(\tau_{p_1})$$

is the generalized Green operator of the Cauchy problem $y(a) = \xi$ for system (2.2). Continuing our reasoning, we determine a solution of system (2.2) with $k = \tau_{p_2}$:

$$y(\tau_{p_2}, \zeta, \xi) = Z(\tau_{p_2})\zeta + W(\tau_{p_2})\xi + \mathcal{K}\Big[\psi(\tau_{p_2} - 1); b_1\Big](\tau_{p_2}),$$

where

$$Z(\tau_{p_2}) := C(\tau_{p_2-1})X_{\tau_{p_1}}(\tau_{p_2-1}) + D(\tau_{p_2-1})Z(\tau_{p_2-1}),$$
$$W(\tau_{p_2}) := C(\tau_{p_2-1})V_{\tau_{p_1}}(\tau_{p_2-1}) + D(\tau_{p_2-1})W(\tau_{p_2-1})$$

and

$$\mathcal{K}\Big[\psi(\tau_{p_2} - 1); b_1\Big](\tau_{p_2}) := C(\tau_{p_2-1})\mathcal{K}_{\tau_{p_1}}\Big[\varphi(s, \tau_{p_2-2}); b_1\Big](\tau_{p_2-1})$$
$$+ D(\tau_{p_2-1})K\Big[\psi(\tau_{p_2-2})\Big](\tau_{p_2-1}) + \psi(\tau_{p_2-1})$$

is the generalized Green operator of the Cauchy problem $y(a) = \xi$ for system (2.2). Further, we continue our reasoning and assume that $X_q(t)$ is the normal ($X_q(\tau_{p_q}) = I_n$) fundamental matrix of the homogeneous part

$$x'(t) = A(t, \tau_{p_q})x(t), \quad t \in [\tau_{p_q}; b],$$

of system (2.1). We represent the solution of system (2.1) satisfying the first condition in (2.3) on the segment $[\tau_{p_q}; b]$ in the form

$$x(t, \zeta, \xi) = X_{\tau_{p_q}}(t)\zeta + V_{\tau_{p_q}}(t)\xi + \mathcal{K}_{\tau_{p_q}}\Big[\varphi(s, \tau_{p_1}); b_q\Big](t),$$

where

$$K_{\tau_{p_q}}\Big[\varphi(s, \tau_{p_q})\Big](t) := X_q(t)\int_{\tau_{p_q}}^{t} X_q^{-1}(s)\varphi(s, \tau_{p_1})ds, \quad t \in [\tau_{p_q}; b],$$

is the Green operator of the Cauchy problem (3.1) for system (2.1),

$$X_{\tau_{p_q}}(t) := X_q(t)(I_n + S_q)X_{p_{q-1}}(\tau_q) + K_{\tau_{p_q}}\Big[B(s, \tau_{p_q})Z(\tau_{p_q})\Big](t)$$

is the normal ($X_{\tau_{p_q}}(\tau_{p_q}) = (I_n + S_q)X_{p_{q-1}}(\tau_q)$) fundamental matrix of the homogeneous part of system (2.1) for $t \in [\tau_{p_q}; b]$ satisfying the first condition in (2.3),

$$V_{\tau_{p_q}}(t) := X_q(t)(I_n + S_q)V_{\tau_{p_{q-1}}}(\tau_{p_q}) + \mathcal{K}_{\tau_{p_q}}\Big[B(s, \tau_{p_q})W(\tau_{p_q})\Big](t),$$

$$\mathcal{K}_{\tau_{p_q}}\Big[\varphi(s, \tau_{p_q}); b_q\Big](t) := X_q(t)b_q + X_q(t)(I_n + S_q)\mathcal{K}_{\tau_{p_q}}\Big[\varphi(s, \tau_{p_{q-1}}); b_{q-1}\Big](\tau_{p_q})$$

$$+K_{\tau_{p_q}}\Big[\varphi(s,\tau_{p_q})\Big](t)+K_{\tau_{p_q}}\Big\{B(s,\tau_{p_q})\mathcal{K}\Big[\psi(\tau_{p_1-1});b_{q-1}\Big](\tau_{p_q})\Big\}(t).$$

Lemma 1. *The general solution of the Cauchy problem (3.1) for system (2.1), (2.2) satisfying the first condition in (2.3) can be represented in the form*

$$x(t,\zeta,\xi)=X_{\tau_{p_i}}(t)\zeta+V_{\tau_{p_i}}(t)\xi+\mathcal{K}_{\tau_{p_i}}\Big[\varphi(s,\tau_i)\Big](t),$$

$$y(k,\zeta,\xi)=Z(k)\zeta+W(k)\xi+\mathcal{K}\Big[\psi(k-1,\tau_i)\Big](k),$$

where the matrices

$$V_a(t):=X_a(t)\int_a^t X_a^{-1}(s)B(s,a)W(a)\,ds,\quad t\in[a;\tau_1[,$$

$$Z(\tau_{p_1}):=C(\tau_{p_1-1})X_a(\tau_{p_1-1})+D(\tau_{p_1-1})Z(\tau_{p_1-1}),\quad Z(a):=0,$$

$$W(\tau_{p_1}):=C(\tau_{p_1-1})V_a(\tau_{p_1-1})+D(\tau_{p_1-1})W(\tau_{p_1-1}),\quad W(a):=I_m,\ \dots,$$

$$X_{\tau_{p_q}}(t)=X_q(t)(I_n+S_q)X_{p_{q-1}}(\tau_q)+\mathcal{K}_{\tau_{p_q}}\Big[B(s,\tau_{p_q})Z(\tau_{p_q})\Big](t),\quad t\in[\tau_{p_q};b],$$

$$V_{\tau_{p_q}}(t)=X_q(t)(I_n+S_q)V_{\tau_{p_{q-1}}}(\tau_{p_q})+\mathcal{K}_{\tau_{p_q}}\Big[B(s,\tau_{p_q})W(\tau_{p_q})\Big](t),\quad t\in[\tau_{p_q};b],$$

specify the general solution

$$x(t,\zeta,\xi)=X_{\tau_{p_i}}(t)\zeta+V_{\tau_{p_i}}(t)\xi,\quad y(k,\zeta,\xi)=Z(k)\zeta+W(k)\xi$$

of the homogeneous part of system (2.1), (2.2) satisfying the first condition in (2.3) and

$$\mathcal{K}_a\Big[\varphi(s,a)\Big](t):=X_a(t)\int_a^t X_a^{-1}(s)\varphi(s,a)\,ds,t\in[a;\tau_1],$$

$$K\Big[\psi(a)\Big](\tau_1):=\psi(\tau_1),$$

$$K\Big[\psi(\tau_1)\Big](\tau_2):=C(\tau_1)K_a\Big[\varphi(s,a)\Big](\tau_1)+D(\tau_1)K\Big[\psi(a)\Big](\tau_1)+\psi(\tau_1),\ \dots,$$

$$K\Big[\psi(\tau_{p_1-1})\Big](\tau_{p_1}):=C(\tau_{p_1-1})K_a\Big[\varphi(s,\tau_0)\Big](\tau_{p_1-1})$$

$$+D(\tau_{p_1-1})K\Big[\psi(\tau_{p_1-2})\Big](\tau_{p_1-1})+\psi(\tau_{p_1-1}),\quad \mathcal{K}\Big[\psi(\tau_{p_1});b_1\Big](\tau_{p_1+1})$$

$$:=C(\tau_{p_1})\mathcal{K}_{\tau_{p_1}}\Big[\varphi(s,\tau_{p_1});b_1\Big](\tau_{p_1})+D(\tau_{p_1})K\Big[\psi(\tau_{p_1-1})\Big](\tau_{p_1})+\psi(\tau_{p_1}),\ \dots,$$

$$\mathcal{K}_{\tau_{p_q}}\Big[\varphi(s,\tau_{p_q});b_q\Big](t)=X_q(t)b_q+X_q(t)(I_n+S_q)\mathcal{K}_{\tau_{p_q}}\Big[\varphi(s,\tau_{p_{q-1}});b_{q-1}\Big](\tau_{p_q})$$

$$+ K_{\tau_{p_q}}\Big[\varphi(s,\tau_{p_q})\Big](t) + K_{\tau_{p_q}}\Big\{B(s,\tau_{p_q})\mathcal{K}\Big[\psi(\tau_{p_1-1});b_{q-1}\Big](\tau_{p_q})\Big\}(t), \quad t\in[\tau_{p_q};b],$$

$$\mathcal{K}\Big[\psi(\tau_{p_q}-1);b_q\Big](\tau_{p_q}) := C(\tau_{p_q-1})\mathcal{K}_{\tau_{p_q}}\Big[\varphi(s,\tau_{p_q-2});b_q\Big](\tau_{p_q-1})$$

$$+ D(\tau_{p_q-1})K\Big[\psi(\tau_{p_q-2})\Big](\tau_{p_q-1}) + \psi(\tau_{p_q-1})$$

is the generalized Green operator of the Cauchy problem (3.1) for system (2.1), (2.2) satisfying the first condition in (2.3).

4. Generalized Green operator of the hybrid difference-differential boundary-value problem

In what follows, we study the problem of determination of conditions for the existence of solutions of the linear boundary-value problem (2.1)–(2.3) and construct these solutions. Assume that the conditions of the lemma are satisfied. We represent the linear bounded vector functional $\ell x(\cdot)$ in the form

$$\ell x(\cdot) := \sum_{i=0}^{q} \ell_i x(\cdot),$$

where

$$\ell_i x(\cdot) : \mathbb{C}^1[\tau_{p_i},\tau_{p_i+1}] \to \mathbb{R}^m, \quad i = 0,\ 1,\ 2,\ \dots,\ q$$

are linear bounded vector functionals. Further, we enumerate elements of the set Θ :

$$k_i \in \Theta : k_0 := a,\ \ k_1 := \tau_1,\ \dots,\ \ k_\sigma := b, \quad \sigma := p_1 + \dots p_q + 2.$$

Similarly, we represent the linear bounded vector functional $\wp y(\cdot)$ in the form

$$\wp y(\cdot) := \sum_{k\in\Theta} \wp_k y(\cdot),$$

where $\wp_k y(\cdot) :\ \{y(k) :\ \Theta \to \mathbb{R}^m\} \to \mathbb{R}^\nu$ are linear bounded vector functionals. Denote the vectors

$$\lambda_\ell := \sum_{i=0}^{q} \ell_i \mathcal{K}_{\tau_{p_i}}\Big[\varphi(s,\tau_i)\Big](\cdot) \in \mathbb{R}^\mu$$

and

$$\lambda_\wp := \sum_{i=0}^{\sigma} \wp_i \mathcal{K}\Big[\psi(k_i);b_j\Big](k_i+1), \quad j = 0,\ 1,\ 2,\ \dots,\ q.$$

Substituting the solution of the Cauchy problem (3.1) for system (2.1), (2.2) satisfying the first condition in (2.3) in the second and third conditions in (2.3), we arrive

at the problem of determination of solutions $\zeta \in \mathbb{R}^n$, $\xi \in \mathbb{R}^m$ of the linear matrix equation

$$\mathcal{Q}\begin{pmatrix} \zeta \\ \xi \end{pmatrix} = \lambda, \quad \lambda := \begin{pmatrix} \alpha - \lambda_\ell \\ \beta - \lambda_\wp \end{pmatrix}.$$

In the critical case ($P_{\mathcal{Q}^*} \neq 0$), under the condition

$$P_{\mathcal{Q}^*_d}\lambda = 0, \tag{4.1}$$

the solution of the last equation is given by the vector

$$\begin{pmatrix} \zeta \\ \xi \end{pmatrix} = \mathcal{Q}^+\lambda + P_{\mathcal{Q}_r}c_r, \quad c_r \in \mathbb{R}^r,$$

where

$$\mathcal{Q} := \begin{pmatrix} \ell X(\cdot) & \ell V(\cdot) \\ \wp Z(\cdot) & \wp W(\cdot) \end{pmatrix}$$

is an $((\mu+\nu)\times(m+n))$ constant matrix. Here, $P_{\mathcal{Q}^*}$ is the $((\mu+\nu)\times(\mu+\nu))$ matrix orthoprojector $P_{\mathcal{Q}^*} : \mathbb{R}^{\mu+\nu} \to \mathbb{N}(\mathcal{Q}^*)$; the matrix $P_{\mathcal{Q}_r}$ is formed by r linearly independent columns of the $((m+n)\times(m+n))$ matrix orthoprojector $P_{\mathcal{Q}}$, and the matrix $P_{\mathcal{Q}^*_d}$ is formed by d linearly independent rows of the matrix orthoprojector $P_{\mathcal{Q}^*}$. In addition,

$$\ell X(\cdot) := \sum_{i=0}^{q} \ell_i X_{\tau_{p_i}}(\cdot), \quad \ell V(\cdot) := \sum_{i=0}^{q} \ell_i V_{\tau_{p_i}}(\cdot),$$

$$\wp Z(\cdot) := \sum_{i=0}^{\sigma} \wp_i Z(k_i), \quad \wp W(\cdot) := \sum_{i=0}^{\sigma} \wp_i W(k_i).$$

In the critical case, under condition (4.1), the solution of the boundary-value problem (2.1)–(2.3) specifies an r-parameter family of vectors $\zeta \in \mathbb{R}^n$, $\xi \in \mathbb{R}^m$. We denote the blocks of the matrix orthoprojector $P_{\mathcal{Q}}$ as follows:

$$P_{\mathcal{Q}_{r_{(11)}}} := \begin{pmatrix} I_n & O \end{pmatrix} P_{\mathcal{Q}_r}, \quad P_{\mathcal{Q}_{r_{(12)}}} := \begin{pmatrix} O & I_n \end{pmatrix} P_{\mathcal{Q}_r},$$
$$P_{\mathcal{Q}_{r_{(21)}}} := \begin{pmatrix} I_m & O \end{pmatrix} P_{\mathcal{Q}_r}, \quad P_{\mathcal{Q}_{r_{(22)}}} := \begin{pmatrix} O & I_m \end{pmatrix} P_{\mathcal{Q}_r}.$$

For the matrices, we write

$$X_r(t) := \begin{cases} X_a(t)P_{\mathcal{Q}_{r_{(11)}}}, & t \in [a;\tau_1[, \\ X_{\tau_{p_1}}(t)P_{\mathcal{Q}_{r_{(11)}}}, & t \in [\tau_{p_1};\tau_{p_1+1}[, \\ X_{\tau_{p_2}}(t)P_{\mathcal{Q}_{r_{(11)}}}, & t \in [\tau_{p_2};\tau_{p_2+1}[, \\ \dots\dots\dots\dots & \dots\dots\dots\dots \\ X_{\tau_{p_q}}(t)P_{\mathcal{Q}_{r_{(11)}}}, & t \in [\tau_{p_q};b], \end{cases}$$

$$V_r(t) := \begin{cases} V_a(t)P_{\mathcal{Q}_{r_{(12)}}}, & t \in [a;\tau_1[, \\ V_{\tau_{p_1}}(t)P_{\mathcal{Q}_{r_{(12)}}}, & t \in [\tau_{p_1};\tau_{p_1+1}[, \\ X_{\tau_{p_2}}(t)P_{\mathcal{Q}_{r_{(12)}}}, & t \in [\tau_{p_2};\tau_{p_2+1}[, \\ \dots\dots\dots\dots & \dots\dots\dots\dots \\ V_{\tau_{p_q}}(t)P_{\mathcal{Q}_{r_{(12)}}}, & t \in [\tau_{p_q};b] \end{cases}$$

$$Z_r(k) := Z(k)P_{\mathcal{Q}_{r_{(21)}}}, \quad W_r(k) := W(k)P_{\mathcal{Q}_{r_{(22)}}}, \quad k \in \Theta.$$

We also denote the blocks of the matrix $\mathcal{Q}^+$ as

$$\mathcal{Q}^+_{(11)} := \begin{pmatrix} I_n & O \end{pmatrix}\mathcal{Q}^+, \quad \mathcal{Q}^+_{(12)} := \begin{pmatrix} O & I_n \end{pmatrix}\mathcal{Q}^+,$$
$$\mathcal{Q}^+_{(21)} :=:= \begin{pmatrix} I_m & O \end{pmatrix}\mathcal{Q}^+, \quad \mathcal{Q}^+_{(22)} :=:= \begin{pmatrix} O & I_m \end{pmatrix}\mathcal{Q}^+.$$

The theorem presented below gives necessary and sufficient conditions for the solvability of the hybrid difference-differential boundary-value problem (2.1)–(2.3).

Theorem 1. *In the critical case* $(P_{\mathcal{Q}^*} \neq 0)$, *under condition (4.1), the solution of the hybrid difference-differential boundary-value problem (2.1)–(2.3) can be represented in the form*

$$x(t,c_r) = X_r(t)c_r + V_r(t)c_r + G\Big[\varphi(s,\tau_i);\alpha\Big](t), \quad c_r \in \mathbb{R}^r,$$
$$y(k,c_r) = Z_r(k)c_r + W_r(k)c_r + G\Big[\psi(k-1,\tau_i);\beta\Big](k)$$

with the help of the generalized Green operator of the linear boundary-value problem (2.1)–(2.3)

$$G\Big[\varphi(s,\tau_i);\alpha\Big](t) :=$$
$$= \begin{cases} X_a(t)\mathcal{Q}^+_{(11)}\lambda + V_a(t)\mathcal{Q}^+_{(12)}\mu + \mathcal{K}_a\Big[\varphi(s,a)\Big](t), & t \in [a;\tau_1[, \\ X_{\tau_{p_1}}(t)\mathcal{Q}^+_{(11)}\lambda + V_{\tau_{p_1}}(t)\mathcal{Q}^+_{(12)}\mu + \mathcal{K}_{\tau_{p_1}}\Big[\varphi(s,\tau_{p_1};b_1)\Big](t), & t \in [\tau_{p_1};\tau_{p_1+1}[, \\ \dots\dots\dots\dots & \dots\dots\dots\dots \\ X_{\tau_{pq}}(t)\mathcal{Q}^+_{(11)}\lambda + V_{\tau_{pq}}(t)\mathcal{Q}^+_{(12)}\mu + \mathcal{K}_{\tau_{pq}}\Big[\varphi(s,\tau_{pq};b_q)\Big](t), & t \in [\tau_{pq};b], \end{cases}$$

$$G\Big[\psi(k-1,\tau_i);\beta\Big](k) := Z(k)\mathcal{Q}^+_{(21)}\lambda + W(k)\mathcal{Q}^+_{(22)}\mu + K[\psi(k-1)](k), \quad k \in \Theta,$$

where the matrices $X_r(t)$, $V_r(t)$, $Z_r(k)$, *and* $W_r(k)$ *specify the general solution of the homogeneous part of the boundary-value problem (2.1)–(2.3); here,* $K[\varphi(s,k)](t)$, $K[\psi(k-1)](k)$ *is the generalized Green operator of the Cauchy problem (3.1) for system (2.1), (2.2) satisfying the first condition in (2.3).*

The condition of solvability (4.1) and the structure of the generalized Green operator of the hybrid difference-differential boundary-value problem (2.1)–(2.3) generalize the conventional results of the theory of Noetherian boundary-value problems for impulsive ordinary differential equations [7, 13, 22, 24] and the corresponding results of the theory of boundary-value problems for difference equations [5, 21, 23].

Example 1. The problem of construction of solutions of the hybrid difference-differential system

$$x'(t) = Ax(t) + By(k) + \varphi(t), \quad y(k+1) = Cx(k) + Dy(k) + \psi(k), \tag{4.2}$$

satisfying the antiperiodic boundary condition with pulsed action

$$\Delta x(\tau_1, \tau_3) := S_1 x(\tau_1 - 0) + b_{\tau_1}, \quad \ell x(\cdot) = \alpha \in \mathbb{R}^2, \quad \wp\, y(\cdot) = \beta \in \mathbb{R}^2, \tag{4.3}$$

where $\tau_1 := \pi$, $\tau_{p_1} := 2\pi$,

$$A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad D := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\varphi(t) := \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}, \quad \psi(k) := \begin{pmatrix} \sin k \\ \cos k \end{pmatrix}, \quad \Omega := [0, \pi[\cup [2\pi, 3\pi], \quad \tau_2 := \frac{3\pi}{2},$$

$$S_1 := -2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b_{\tau_1} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \alpha := \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \beta := \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

$$\Theta := \left\{0, \pi, \frac{3\pi}{2}, 2\pi, 3\pi\right\}, \quad \ell x(\cdot) := x(0) + x(3\pi), \quad \wp\, y(\cdot) := y(0) + y(3\pi)$$

satisfies the conditions of the proved Theorem 1.

The matrices

$$X_0(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad V_0(t) = \begin{pmatrix} \sin t & 0 \\ \cos t - 1 & 0 \end{pmatrix}, \quad Z(0) = W(0) = I_2$$

specify the general solution

$$x(t, \zeta, \xi) = X_0(t)\zeta + V_0(t)\xi, \quad y(0, \zeta, \xi) = Z(0)\zeta + W(0)\xi, \quad \zeta \in \mathbb{R}^2, \quad \xi \in \mathbb{R}^2$$

of the homogeneous part of system (4.2) on the interval $[0; \pi[$. Similarly, the matrices

$$Z(\pi) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad W(\pi) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$Z\left(\frac{3\pi}{2}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad W\left(\frac{3\pi}{2}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

specify the general solution

$$y(\pi, \zeta, \xi) = Z(\pi)\zeta + W(\pi)\xi, \quad y\left(\frac{3\pi}{2}, \zeta, \xi\right) = Z\left(\frac{3\pi}{2}\right)\zeta + W\left(\frac{3\pi}{2}\right)\xi$$

of the homogeneous part of system (4.2) with $k \in \{\pi, \frac{3\pi}{2}\}$. The matrices

$$X_{2\pi}(t) := X_0(t),\ V_{2\pi}(t) = \begin{pmatrix} 2\sin t & 0 \\ 2\cos t & 0 \end{pmatrix},$$

$$Z(2\pi) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},\ W(2\pi) = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$$

specify the general solution

$$x(t,\zeta,\xi) = X_{2\pi}(t)\zeta + V_{2\pi}(t)\xi, \quad y(2\pi,\zeta,\xi) = Z(2\pi)\zeta + W(2\pi)\xi, \quad \zeta \in \mathbb{R}^2, \quad \xi \in \mathbb{R}^2$$

of the homogeneous part of system (4.2) on the segment $[2\pi; 3\pi]$ satisfying the first boundary condition in (4.3). Similarly, the matrices

$$Z(3\pi) = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \quad W(3\pi) = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$$

specify the general solution

$$y(3\pi,\zeta,\xi) = Z(3\pi)\zeta + W(3\pi)\xi, \quad \zeta \in \mathbb{R}^2, \quad \xi \in \mathbb{R}^2$$

of the homogeneous part of system (4.2) with $k = 3\pi$. Since

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 2 \end{pmatrix},$$

we have $P_{\mathcal{Q}^*} \neq 0$. Therefore, the problem of construction of solutions of the hybrid difference-differential boundary-value problem (4.2), (4.3) corresponds to the critical case; moreover, the condition of its solvability (4.1) is satisfied. A solution of the hybrid difference-differential boundary-value problem (4.2), (4.3) can be represented in the form

$$x(t,c_r) = X_r(t)c_r + V_r(t)c_r + G\Big[\varphi(s,\tau_i);\alpha\Big](t), \quad c_r \in \mathbb{R}^1,$$

$$y(k,c_r) = Z_r(k)c_r + W_r(k)c_r + G\Big[\psi(k-1,\tau_i);\beta\Big](k)$$

with the help of the generalized Green operator of the linear boundary-value problem (4.2), (4.3)

$$G\Big[\varphi(s,\tau_i);\alpha\Big](t) = K_{\tau_{p_i}}\Big[\varphi(s,\tau_i);b_i\Big](t),$$

$$G\Big[\psi(k-1,\tau_i);\beta\Big](k) = K\Big[\psi(k-1,\tau_i);b_i\Big](k).$$

Here,

$$K_{\tau_{p_i}}[\varphi(s,k);b_i](t)=\begin{cases}\frac{1}{2}\begin{pmatrix}2\sin t\sin 2t\\ -\sin t+\sin 3t\end{pmatrix}, & t\in[0,\pi[,\\ \frac{1}{2}\begin{pmatrix}3\cos t-\cos 3t-2\sin t\\ 2-2\cos t-3\sin t+\sin 3t\end{pmatrix}, & t\in[2\pi,3\pi],\end{cases}$$

$$K[\psi](0)=\begin{pmatrix}0\\0\end{pmatrix},\quad K[\psi(0)](\pi)=\begin{pmatrix}0\\1\end{pmatrix},\quad K\Big[\psi(\pi)\Big]\Big(\frac{3\pi}{2}\Big)=\begin{pmatrix}0\\0\end{pmatrix},$$

$$K\Big[\psi\Big(\frac{3\pi}{2}\Big)\Big](2\pi)=\begin{pmatrix}-1\\0\end{pmatrix},\quad K[\psi(2\pi)](3\pi)=\begin{pmatrix}0\\0\end{pmatrix}$$

is the generalized Green operator of the Cauchy problem

$$x(a)=\zeta\in\mathbb{R}^2,\ y(a)=\xi\in\mathbb{R}^2$$

for system (4.2), (4.3). Since rank $P_{\mathcal{Q}}=1$, we get

$$P_{\mathcal{Q}_r}=\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}^*$$

and, hence,

$$X_r(t)=\begin{pmatrix}\sin t\\ \cos t\end{pmatrix},\quad V_r(t)\equiv 0,\quad t\in\Omega,$$

$$Z_r(0)=-Z_r(3\pi)=\begin{pmatrix}0\\1\end{pmatrix},\quad W_r(0)=W_r(3\pi)=\begin{pmatrix}0\\0\end{pmatrix}.$$

In the noncritical case ($P_{\mathcal{Q}^*}=0$), condition (4.1) is satisfied for any inhomogeneities $\varphi(t,k)\in\mathbb{R}^n$, $t\in\Omega$, and $\psi(k)\in\mathbb{R}^m$, $k\in\Theta$. Moreover, the solution of the hybrid difference-differential boundary-value problem (2.1)–(2.3) is determined by the following assertion:

Corollary 1. *In the noncritical case ($P_{\mathcal{Q}^*}=0$), the solution of the hybrid difference-differential boundary-value problem (2.1)–(2.3) can be represented in the form*

$$x(t,c_r)=U_r(t)c_r+V_r(t)c_r+G\Big[\varphi(s,\tau_i);\alpha\Big](t),\quad c_r\in\mathbb{R}^r,$$

$$y(k,c_r)=Z_r(k)c_r+W_r(k)c_r+G\Big[\psi(k-1,\tau_i);\beta\Big](k)$$

with the help of the generalized Green operator of the linear boundary-value problem (2.1)–(2.3)

$$G\Big[\varphi(s,\tau_i);\alpha\Big](t),\quad t\in\Omega,\quad G\Big[\psi(k-1,\tau_i);\beta\Big](k),\quad k\in\Theta,$$

where the matrices $X_r(t)$, $V_r(t)$, $Z_r(k)$, and $W_r(k)$ specify the general solution of the homogeneous part of the boundary-value problem (2.1)–(2.3); here, $K[\varphi(s,k)](t)$, $K[\psi(k-1)](k)$ is the generalized Green operator of the Cauchy problem (3.1) for system (2.1), (2.2) satisfying the first condition in (2.3).

Example 2. The problem of construction of the solutions of the hybrid difference-differential system

$$x'(t) = Ax(t) + By(k) + \varphi(t), \quad y(k+1) = Cx(k) + Dy(k) + \psi(k) \tag{4.4}$$

satisfying the boundary condition

$$\Delta x(\tau_1, \tau_{p_1}) := S_1 x(\tau_1 - 0) + b_1, \quad \ell x(\cdot) = \alpha \in \mathbb{R}^2, \quad \wp y(\cdot) = \beta \in \mathbb{R}^1, \tag{4.5}$$

where

$$A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C := \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D := \begin{pmatrix} 1 \end{pmatrix}, \quad \tau_1 := \pi,$$

$$\varphi(t) := \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}, \quad \psi(k) := \begin{pmatrix} \sin k \end{pmatrix}, \quad \tau_2 := \frac{3\pi}{2}, \quad \Omega := [0, \pi[\cup [2\pi, 3\pi],$$

$$\Theta := \left\{0, \pi, \frac{3\pi}{2}, 2\pi, 3\pi\right\}, \quad S_{\tau_1} := -2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b_{\tau_1} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \alpha := \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

$$\ell x(\cdot) := x(0) - x(3\pi), \quad \wp y(\cdot) := y(0) + y(3\pi), \quad \beta := \begin{pmatrix} 3 \end{pmatrix}, \quad \tau_{p_1} := 2\pi$$

satisfies the conditions of the proved Corollary 1.

The matrices

$$X_0(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \; V_0(t) = \begin{pmatrix} \sin t \\ -1 + \cos t \end{pmatrix},$$

$$Z(0) = \begin{pmatrix} 0 & 0 \end{pmatrix}, \; W(0) = \begin{pmatrix} 1 \end{pmatrix}$$

specify the general solution

$$x(t, \zeta, \xi) = X_0(t)\zeta + V_0(t)\xi, \quad y(0, \zeta, \xi) = Z(0)\zeta + W(0)\xi, \quad \zeta \in \mathbb{R}^2, \quad \xi \in \mathbb{R}^1$$

of the homogeneous part of the difference system (4.4) in the interval $[0; \pi[$. Similarly, the matrices

$$Z(\pi) = \begin{pmatrix} 1 & 0 \end{pmatrix}, \; W(\pi) = \begin{pmatrix} 1 \end{pmatrix}, \; Z\left(\frac{3\pi}{2}\right) = \begin{pmatrix} 0 & 0 \end{pmatrix}, \; W\left(\frac{3\pi}{2}\right) = \begin{pmatrix} 3 \end{pmatrix}$$

specify the general solution

$$y(\pi, \zeta, \xi) = Z(\pi)\zeta + W(\pi)\xi, \quad y\left(\frac{3\pi}{2}, \zeta, \xi\right) = Z\left(\frac{3\pi}{2}\right)\zeta + W\left(\frac{3\pi}{2}\right)\xi$$

of the homogeneous part of the difference system (4.4) for $k \in \left\{\pi, \frac{3\pi}{2}\right\}$. The matrices

$$X_{2\pi}(t) := \begin{pmatrix} \cos t & 0 \\ -\sin t & 1 \end{pmatrix}, \; V_{2\pi}(t) = \begin{pmatrix} 2\sin t \\ 2\cos t \end{pmatrix},$$

$$W\left(\frac{3\pi}{2}\right) = 1, \; Z\left(\frac{3\pi}{2}\right) = \begin{pmatrix} 0 & 0 \end{pmatrix}, \; Z(2\pi) = \begin{pmatrix} 0 & -1 \end{pmatrix},$$

$$W(2\pi) = \begin{pmatrix} 0 \end{pmatrix}, \; Z(3\pi) = \begin{pmatrix} 1 & -1 \end{pmatrix}, \; W(3\pi) = \begin{pmatrix} 0 \end{pmatrix}$$

specify the general solution

$$x(t,\zeta,\xi) = X_{2\pi}(t)\zeta + V_{2\pi}(t)\xi, \quad y(k,\zeta,\xi) = Z(k)\zeta + W(k)\xi, \quad k \in \{\pi, 3\pi\}$$

of the homogeneous part of system (4.2) on the segment $[2\pi; 3\pi]$ satisfying the first boundary condition in (4.3). Since

$$\mathcal{Q} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix},$$

we have $P_{\mathcal{Q}} = P_{\mathcal{Q}^*} = 0$. Therefore, the problem of construction of solutions of the hybrid difference-differential boundary-value problem (4.4), (4.5) corresponds to the noncritical case. Moreover, the unique ($r = 0$) solution

$$x(t) = G\Big[\varphi(s,\tau_i);\alpha\Big](t), \quad t \in \Omega, \quad y(k) = G\Big[\psi(k-1,\tau_i);\beta\Big](k), \quad k \in \Theta$$

of the hybrid difference-differential boundary-value problem (4.4), (4.5) can be represented by using the generalized Green operator of the linear boundary-value problem (4.4), (4.5)

$$G\Big[\varphi(s,\tau_i);\alpha\Big](t) = \begin{cases} \frac{1}{2}\begin{pmatrix} 4\sin t + \cos t - \cos 3t \\ -2 + 4\cos t - \sin t + \sin 3t \end{pmatrix}, & t \in [0,\pi[, \\ \frac{1}{2}\begin{pmatrix} 3\cos t + 2\sin t - \cos 3t \\ 4 + 2\cos t - 3\sin t + \sin 3t \end{pmatrix}, & t \in [2\pi, 3\pi], \end{cases}$$

$$G\Big[\psi(0);\beta\Big](0) = 0, \ \dots, \ G\Big[\psi(2\pi);\beta\Big](3\pi) = 0.$$

Note that scheme of construction and the structure of the generalized Green operator for the hybrid difference-differential boundary-value problem (2.1), (2.2) is independent of degeneracy [18] or nondegeneracy [7, 22, 24] of the matrices $X_{\tau_{p_i}}(t)$, $t \in \Omega$, and $W(k)$, $k \in \Theta$, and the matrix

$$\Gamma_{\tau_{p_i}}(t,k) := \begin{pmatrix} X_{\tau_{p_i}}(t) & V_{\tau_{p_i}}(t) \\ Z(k) & W(k) \end{pmatrix}, \quad t \in \Omega, \quad k \in \Theta, \quad i = 0, 1, 2, \dots, q,$$

specifying the general solution of the homogeneous part of the boundary-value problem (2.1), (2.2). In particular, the matrices $X(t)$, $W(k)$, and $\Gamma_{\tau_{p_i}}(t,k)$ in Examples 1 and 2 are nondegenerate.

Example 3. The matrices $X_{\tau_{p_1}}(t)$, $W_{\tau_{p_1}}(k)$, and $\Gamma_{\tau_{p_1}}(t,k)$ degenerate in the problem of construction of solutions of the hybrid difference-differential problem (4.2), (4.3) as a result of the degenerate $\det(I_2 + S_1) = 0$ pulsed action for $S_1 := -I_2$.

The matrices $X_0(t)$, $V_0(t)$, $Z(0)$, and $W(0)$ specifying the general solution

$$x(t,\zeta,\xi) = X_0(t)\zeta + V_0(t)\xi, \quad y(0,\zeta,\xi) = Z(0)\zeta + W(0)\xi, \quad \zeta \in \mathbb{R}^2, \quad \xi \in \mathbb{R}^2$$

of the homogeneous part of system (4.2) in the interval $[0;\pi[$ and the matrices

$$Z(\pi),\ W(\pi),\ Z\left(\frac{3\pi}{2}\right),\ W\left(\frac{3\pi}{2}\right)$$

specifying the general solution

$$y(\pi,\zeta,\xi)=Z(\pi)\zeta+W(\pi)\xi,\quad y\left(\frac{3\pi}{2},\zeta,\xi\right)=Z\left(\frac{3\pi}{2}\right)\zeta+W\left(\frac{3\pi}{2}\right)\xi$$

of the homogeneous part of system (4.2) for $k\in\left\{\pi,\frac{3\pi}{2}\right\}$ are given in Example 1. Moreover,

$$\det\Gamma_0(0,0)=\det\Gamma_0(\pi,\pi)=1\neq 0.$$

Since

$$\mathcal{Q}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\0&0&-1&2\end{pmatrix},$$

we have $P_{\mathcal{Q}}=P_{\mathcal{Q}^*}=0$. Therefore, the problem of construction of solutions of the hybrid difference-differential boundary-value problem (4.2), (4.3) corresponds to the noncritical case. Furthermore,

$$G\Big[\varphi(s,\tau_i);\alpha\Big](t)=\begin{cases}\frac{1}{2}\begin{pmatrix}\cos t-\cos 3t\\-\sin t+\sin 3t\end{pmatrix}, & t\in[0,\pi[,\\ \frac{1}{2}\begin{pmatrix}3\cos t-\cos 3t-2\sin t\\2-2\cos t-3\sin t+\sin 3t\end{pmatrix}, & t\in[2\pi,3\pi],\end{cases}$$

$$G\Big[\psi(0);\beta\Big](0)=\begin{pmatrix}0\\0\end{pmatrix},\ G\Big[\psi\left(\frac{3\pi}{2}\right);\beta\Big](2\pi)=\begin{pmatrix}-1\\0\end{pmatrix},$$

$$G\Big[\psi(2\pi);\beta\Big](3\pi)=\begin{pmatrix}0\\2\end{pmatrix}.$$

Note that, as a result of the degenerate $\det(I_2+S_1)=0$ pulsed action, we get the equality $\det\Gamma_0(2\pi,2\pi)=\det\Gamma_{2\pi}(3\pi,3\pi)=0$.

Since the set $\Omega\cup\Theta$ is a special case of the set of the "time scale"-type, the accumulated results can be deduced in a similar way [1, 2, 4, 11]. The scheme of investigation of hybrid boundary-value problems for systems of difference-differential equations can be applied to the analysis of the problems of bifurcation of solutions of boundary-value problems for difference-differential systems with pulsed action [7], hybrid difference-differential boundary-value problems for systems with pulsed action of a more general form [13, 14], matrix difference-differential boundary-value problems for systems with pulsed action [6, 12, 15, 16], hybrid difference-integro-differential boundary-value problems for systems with pulsed action [8, 9], and hybrid difference-differential boundary-value problems for systems with pulsed action in abstract spaces [10, 17].

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INEQUALITIES FOR log−CONVEX FUNCTIONS AND P−FUNCTIONS

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Abstract. In this paper we obtain some new integral inequalities like Hermite-Hadamard type for log−convex functions and P−functions.

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1. Introduction

We shall recall the definitions of log−convex functions and P−functions.

A function is called log−convex or multiplicatively convex on a real interval $I = [a,b]$, if $\log f$ is convex , or, equivalently if for all $x, y \in I$ and all $\alpha \in [0,1]$, one has the inequality

$$f(\alpha x + (1-\alpha)y) \leq [f(x)]^{\alpha}[f(y)]^{(1-\alpha)}. \tag{1.1}$$

It is said to be log−concave if the inequality in (1.1) is reversed.

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is P−function or f belongs to the class of $P(I)$, if it is nonnegative and for all $x, y \in I$ and $\alpha \in [0,1]$, satisfies the following inequality

$$f(\alpha x + (1-\alpha)y) \leq f(x) + f(y).$$

For further information about log−convex functions and P−functions see [1,2,4–7, 9–11].

The following inequality is called Hermite-Hadamard inequality:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then the double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

holds.

The aim of this paper is to obtain some new integral inequalities like Hermite-Hadamard type for twice differentiable log−convex functions and P−functions.

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In order to prove our main results we need the following Lemma from [8]:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $a,b \in I$ with $a < b$. If $f'' \in L([a,b])$, then*

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \\
&= \frac{(b-a)^2}{54}\left[\int_0^1 \left(2-t-t^2\right) f''\left(ta+(1-t)\frac{2a+b}{3}\right)dt\right. \\
&\quad + \int_0^1 \left(2+t-t^2\right) f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)dt \\
&\quad \left. + \int_0^1 \left(3t-t^2\right) f''\left(t\frac{a+2b}{3}+(1-t)b\right)dt\right].
\end{aligned}$$

2. INEQUALITIES VIA log−CONVEXITY AND $P-$ FUNCTIONS

We shall start with the following result:

Theorem 1. *Let $f : I \to [0,\infty)$ be a twice differentiable mapping on I° such that $f'' \in L[a,b]$ where $a,b \in I^\circ$ with $a < b$. If $|f''|$ is $\log-$convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{54}\left\{\left|f''\left(\frac{2a+b}{3}\right)\right|\mu_K + \left|f''\left(\frac{a+2b}{3}\right)\right|\mu_M + \left|f''(b)\right|\mu_N\right\}
\end{aligned}$$

where

$$\begin{aligned}
\mu_K &= -\frac{2}{\ln K} + \frac{3K-1}{(\ln K)^2} + \frac{2-2K}{(\ln K)^3}, \\
\mu_M &= \frac{2M-2}{\ln M} + \frac{M+1}{(\ln M)^2} + \frac{2-2M}{(\ln M)^3} \\
\mu_N &= \frac{2N}{\ln N} + \frac{3-N}{(\ln N)^2} + \frac{2-2N}{(\ln N)^3}
\end{aligned}$$

and

$$K = \frac{|f''(a)|}{\left|f''\left(\frac{2a+b}{3}\right)\right|}, \quad M = \frac{\left|f''\left(\frac{2a+b}{3}\right)\right|}{\left|f''\left(\frac{a+2b}{3}\right)\right|}, N = \frac{\left|f''\left(\frac{a+2b}{3}\right)\right|}{|f''(b)|}$$

In the sequel of the paper, we set $K, M, N \neq 1$

Proof. From Lemma 1, property of the modulus and log−convexity of $|f''|$ we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{54}\left\{\int_0^1 (2-t-t^2)\left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right|dt\right. \\
&\quad+\int_0^1 (2+t-t^2)\left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right|dt \\
&\quad\left.+\int_0^1 (3t-t^2)\left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right|dt\right\} \\
&\leq \frac{(b-a)^2}{54}\left\{\left|f''\left(\frac{2a+b}{3}\right)\right|\int_0^1 (2-t-t^2)\left[\frac{|f''(a)|}{\left|f''\left(\frac{2a+b}{3}\right)\right|}\right]^t dt\right. \\
&\quad+\left|f''\left(\frac{a+2b}{3}\right)\right|\int_0^1 (2+t-t^2)\left[\frac{\left|f''\left(\frac{2a+b}{3}\right)\right|}{\left|f''\left(\frac{a+2b}{3}\right)\right|}\right]^t dt \\
&\quad\left.+|f''(b)|\int_0^1 (3t-t^2)\left[\frac{\left|f''\left(\frac{a+2b}{3}\right)\right|}{|f''(b)|}\right]^t dt\right\}.
\end{aligned}$$

The proof is completed by making use of the necessary computation. □

Theorem 2. *Let $f : I \to [0,\infty)$ be a twice differentiable mapping on I° such that $f'' \in L[a,b]$ where $a,b \in I^\circ$ with $a < b$. If $|f''|^q$ is* log−*convex on $[a,b]$, then the following inequality holds for some fixed $q > 1$*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{54}\left\{\left(\frac{3^{p+2}-2^{p+1}(p+4)}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left|f''\left(\frac{2a+b}{3}\right)\right|\left(\frac{K^q-1-q\ln K}{(q\ln K)^2}\right)^{\frac{1}{q}}\right. \\
&\quad+\left(\frac{2^{p+1}(p+4)-2p-5}{(p+1)(p+2)}\right)\left|f''\left(\frac{a+2b}{3}\right)\right|\left(\frac{(2M^q-1)q\ln M+1-M^q}{(q\ln M)^2}\right)^{\frac{1}{q}} \\
&\quad\left.+\left(\frac{3^{p+2}-2^{p+1}(p+4)}{(p+1)(p+2)}\right)^{\frac{1}{p}}|f''(b)|\left(\frac{N^q q\ln N-N^q+1}{(q\ln N)^2}\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* K,M,N *are as in the aforementioned Theorem.*

Proof. From Lemma 1 and using the weighted version of Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{54} \\
&\quad\times\left\{\left(\int_0^1 (1-t)(2+t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad\left(\int_0^1 (1+t)(2-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1+t)\left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad\left.+\left(\int_0^1 t(3-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

If we use the log-convexity of $|f''|^q$ above and calculate the integrals we get the desired result. □

Theorem 3. *Let* $f:I\to[0,\infty)$ *be a twice differentiable mapping on* I° *such that* $f''\in L[a,b]$ *where* $a,b\in I^\circ$ *with* $a<b$. *If* $|f''|$ *is* $P-$*function on* $[a,b]$ *, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{54}\left[\frac{7|f''(a)|+20\left[\left|f''\left(\frac{2a+b}{3}\right)\right|+\left|f''\left(\frac{a+2b}{3}\right)\right|\right]+7|f''(b)|}{6}\right].
\end{aligned}$$

Proof. Since $|f''|^q$ is $P-$function on $[a,b]$, from Lemma 1 and properties of the modulus we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{54}\left\{\int_0^1 \left(2-t-t^2\right)\left[|f''(a)|+\left|f''\left(\frac{2a+b}{3}\right)\right|\right]\right. \\
&\quad+\int_0^1 \left(2+t-t^2\right)\left[\left|f''\left(\frac{2a+b}{3}\right)\right|+\left|f''\left(\frac{a+2b}{3}\right)\right|\right]
\end{aligned}$$

$$+\int_0^1 (3t-t^2)\left[\left|f''\left(\frac{a+2b}{3}\right)\right|+|f''(b)|\right]\Bigg\}.$$

The proof is completed by making use of the necessary computation. □

Theorem 4. *Let $f : I \to [0,\infty)$ be a twice differentiable mapping on I° such that $f'' \in L[a,b]$ where $a,b \in I^\circ$ with $a<b$. If $|f''|^q$ is P−function on $[a,b]$, then the following inequality holds for some fixed $q>1$*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{54}\left\{\left(\frac{3^{p+2}-2^{p+1}(p+4)}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left[\left(\frac{|f''(a)|^q+\left|f''\left(\frac{2a+b}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.\right. \\
&\quad\left.+\left(\frac{\left|f''\left(\frac{a+2b}{3}\right)\right|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}\right] \\
&\quad+\left(\frac{2^{p+1}(p+4)-2p-5}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{3\left[\left|f''\left(\frac{a+2b}{3}\right)\right|^q+\left|f''\left(\frac{2a+b}{3}\right)\right|^q\right]}{2}\right)^{\frac{1}{q}}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since $|f''|^q$ is P−function on $[a,b]$, from Lemma 1 and using the weighted version of Hölder's integral inequality, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{54}\left\{\left(\int_0^1 (1-t)(2+t)^p\,dt\right)^{\frac{1}{p}}\right. \\
&\quad\times\left(\int_0^1 (1-t)\left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad+\left(\int_0^1 (1+t)(2-t)^p\,dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1 (1+t)\left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_0^1 t\,(3-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg\}\\
&\leq \frac{(b-a)^2}{54}\Bigg\{\left(\int_0^1 (1-t)\,(2+t)^p\,dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1 (1-t)\left[\left|f''(a)\right|^q+\left|f''\left(\frac{2a+b}{3}\right)\right|^q\right]dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 (1+t)\,(2-t)^p\,dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1 (1+t)\left[\left|f''\left(\frac{a+2b}{3}\right)\right|^q+\left|f''\left(\frac{2a+b}{3}\right)\right|^q\right]dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 t\,(3-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left[\left|f''\left(\frac{a+2b}{3}\right)\right|^q+\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}\Bigg\}.
\end{aligned}
$$

If we calculate the above integrals, we get the desired result. □

Theorem 5. *Under the assumptions of above Theorem, we can write*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\leq \frac{(b-a)^2}{54}\Bigg\{\left(\frac{3^{p+1}-2^{p+1}}{p+1}\right)^{\frac{1}{p}}\Bigg[\left(\frac{|f''(a)|^q+\left|f''\left(\frac{2a+b}{3}\right)\right|^q}{q+1}\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{\left|f''\left(\frac{a+2b}{3}\right)\right|^q+|f''(b)|^q}{q+1}\right)^{\frac{1}{q}}\Bigg]\\
&\quad+\left(\frac{2^{p+1}-1}{p+1}\right)^{\frac{1}{p}}\left[(2^{q+1}-1)\left(\frac{\left|f''\left(\frac{a+2b}{3}\right)\right|^q+\left|f''\left(\frac{2a+b}{3}\right)\right|^q}{q+1}\right)\right]^{\frac{1}{q}}\Bigg\}.
\end{aligned}
$$

Proof. Since $|f''|^q$ is $P-$function on $[a,b]$, from Lemma 1 and using the Hölder's integral inequality, we obtain

$$
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|
$$

$$\leq \frac{(b-a)^2}{54}\left\{\left(\int_0^1 (2+t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^q\left[|f''(a)|^q+\left|f''\left(\frac{2a+b}{3}\right)\right|^q\right]dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^1 (2-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1+t)^q\left[\left|f''\left(\frac{a+2b}{3}\right)\right|^q+\left|f''\left(\frac{2a+b}{3}\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_0^1 (3-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t^q\left[\left|f''\left(\frac{a+2b}{3}\right)\right|^q+|f''(b)|^q\right]dt\right)^{\frac{1}{q}}\right\}.$$

The proof is completed by making use of the necessary computation. □

Theorem 6. *Let $f : I \to [0,\infty)$, be a twice differentiable mapping on I° such that $f'' \in L[a,b]$ where $a,b \in I^\circ$ with $a<b$. If $|f''|^q$ is P−function on $[a,b]$, then the following inequality holds for some fixed $q>1$,*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{54}\left\{\left[|f''(a)|+\left|f''\left(\frac{2a+b}{3}\right)\right|\right]\left(\frac{1}{(p+1)^{\frac{1}{p}}}\right)\left(\frac{3^{q+1}-2^{q+1}}{(q+1)}\right)^{\frac{1}{q}}\right.$$
$$+\left[\left|f''\left(\frac{a+2b}{3}\right)\right|+\left|f''\left(\frac{2a+b}{3}\right)\right|\right]\left(\frac{2^{p+1}-1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2^{q+1}-1}{q+1}\right)^{\frac{1}{q}}$$
$$\left.\left[\left|f''\left(\frac{a+2b}{3}\right)\right|+|f''(b)|\right]\left(\frac{3^{q+1}-2^{q+1}}{q+1}\right)^{\frac{1}{q}}\left(\frac{1}{(p+1)^{\frac{1}{p}}}\right)\right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since $|f''|^q$ is P−function on $[a,b]$, from Lemma 1 and the weighted version of Hölder's inequality (see [3, pp117]), we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{54}\left\{\int_0^1 (1-t)(2+t)\left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right|dt\right.$$
$$+\int_0^1 (1+t)(2-t)\,dt\left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right|dt$$
$$\left.+\int_0^1 t(3-t)\left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right|dt\right\}$$

$$
\begin{aligned}
&\leq \frac{(b-a)^2}{54}\left\{\left(\int_0^1 (1-t)^p \left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right| dt\right)^{\frac{1}{p}}\right. \\
&\times\left(\int_0^1 (2+t)^q \left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right| dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 (1+t)^p \left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right| dt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1 (2-t)^q \left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right| dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 t^p \left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right| dt\right)^{\frac{1}{p}} \\
&\times\left.\left(\int_0^1 (3-t)^q \left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right| dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

The proof is completed by making use of the necessary computation. □

Remark 1. One can obtain inequalities for $quasi-$convex functions. The details are omitted to the interested reader.

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ON GENERALIZATIONS RELATED TO THE LEFT SIDE OF FEJER'S INEQUALITY VIA FRACTIONAL INTEGRAL OPERATOR

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Abstract. In the paper, firstly, a new fractional integral identity is obtained. Then, some new results related to the left side of Fejér's inequality for differentiable mappings whose derivatives in absolute value are convex via fractional integral operator, using this identity with fundamental inequalities such as Hölder's integral inequality, power-mean inequality and triangle inequality for integral, are presented. The results presented here would provide extensions of those proved in [11].

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1. Introduction and preliminaries

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality (*see e.g.* [5]). The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities. In [6], Fejér established the following inequality:

Let $f : [a, b] \to \mathbb{R}$ be convex function. Then the inequality

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{a+b}{2}\int_a^b g(x)dx \tag{1.2}$$

holds, where $g : [a, b] \to \mathbb{R}$ is nonnegative,integrable and symmetric to $\frac{a+b}{2}$.

For some results which generalize, improve, and extend the inequalities (1.1) and (1.2), (*see e.g.* [7, 10–13]).

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In the following, we will give some necessary definitions and preliminary results which are used and referred to throughout this paper.

Definition 1. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}u^{\alpha-1}du$. Here is $J^{0}_{a+}f(x) = J^{0}_{b-}f(x) = f(x)$.

Some recent result and properties concerning the this integral operators can be found in [2–4, 8].

In [9], Raina introduced a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = \mathcal{F}^{\sigma(0),\sigma(1),\dots}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k+\lambda)} x^k \quad (\rho,\lambda > 0; |x| < \mathbf{R}), \tag{1.3}$$

where the coefficients $\sigma(k)\,(k \in \mathbb{N} = \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers and $\mathbf{R}$ is the set of real numbers. With the help of (1.3), Raina [9] and Agarwal *et al.* [1] defined the following left-sided and right-sided fractional integral operators respectively, as follows:

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right)(x) = \int_a^x (x-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(x-t)^{\rho}]\varphi(t)dt \quad (x > a), \tag{1.4}$$

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right)(x) = \int_x^b (t-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\varphi(t)dt \quad (x < b), \tag{1.5}$$

where $\lambda, \rho > 0$, $w \in \mathbb{R}$ and $\varphi(t)$ is such that the integral on the right side exits.

It is easy to verify that $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi(x)$ and $\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi(x)$ are bounded integral operators on $L(a,b)$, if

$$\mathfrak{M} := \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}] < \infty. \tag{1.6}$$

In fact, for $\varphi \in L(a,b)$, we have

$$||\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi(x)||_1 \leq \mathfrak{M}(b-a)^{\lambda}||\varphi||_1 \tag{1.7}$$

and

$$||\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi(x)||_1 \leq \mathfrak{M}(b-a)^{\lambda}||\varphi||_1 \tag{1.8}$$

where

$$||\varphi||_p := \left(\int_a^b |\varphi(t)|^p dt\right)^{\frac{1}{p}}.$$

Here, many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. For instance the classical Riemann-Liouville fractional integrals J^{α}_{a+} and J^{α}_{b-} of order α follow easily by setting $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in (1.4) and (1.5).

Set *et al.* established a new identity and some Hermite-Hadamard-Fejér type inequalities for differentiable mappings whose derivatives in absolute value are convex via Riemann-Liouville fractional integrals as the following:

Lemma 1 ([11]). *Let $f : [a,b] \to \mathbb{R}$ be a differentiable on mapping (a,b) with $a < b$ and let $g : [a,b] \to \mathbb{R}$. If $f', g \in L[a,b]$, then the following identity for fractional integrals holds:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right] \\ & -\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right] \\ & = \frac{1}{\Gamma(\alpha)}\int_a^b k(t)f'(t)dt, \end{aligned} \tag{1.9}$$

where

$$k(t) = \begin{cases} \int_a^t (s-a)^{\alpha-1}g(s)ds, & t \in \left[a, \frac{a+b}{2}\right], \\ \int_b^t (b-s)^{\alpha-1}g(s)ds, & t \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

Theorem 1 ([11]). *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$ and $g : [a,b] \to \mathbb{R}$ is continuous. If $|f'|$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} & \Bigg| f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right] \\ & -\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\Bigg| \\ & \leq \frac{(b-a)^{\alpha+1}||g||_{[a,b],\infty}}{2^{\alpha+1}(\alpha+1)\Gamma(\alpha+1)}\left(|f'(a)| + |f'(b)|\right) \end{aligned} \tag{1.10}$$

with $\alpha > 0$.

Theorem 2. ([11]) *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$ and let $g : [a,b] \to \mathbb{R}$ is continuous. If $|f'|^q$ is convex on $[a,b]$,*

$q \geq 1$, then the following inequality for fractional integrals holds:

$$\left| f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right] - \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right|$$
$$\leq \frac{(b-a)^{\alpha+1}||g||_{[a,b],\infty}}{2^{\alpha+1+\frac{1}{q}}(\alpha+1)(\alpha+2)^{\frac{1}{q}}\Gamma(\alpha+1)}$$
$$\times\left\{\left((\alpha+3)|f'(a)|^q+(\alpha+1)|f'(b)|^q\right)^{\frac{1}{q}}+\left((\alpha+1)|f'(a)|^q+(\alpha+3)|f'(b)|^q\right)^{\frac{1}{q}}\right\} \tag{1.11}$$

with $\alpha > 0$.

Theorem 3 ([11]). *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$ and let $g : [a,b] \to \mathbb{R}$ is continuous. If $|f'|^q$ is convex on $[a,b]$, $q > 1$, then the following inequality for fractional integrals holds:*

$$\left| f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right] - \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right|$$
$$\leq \frac{(b-a)^{\alpha+1}||g||_{[a,b],\infty}}{2^{\alpha+1+\frac{2}{q}}(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}$$
$$\times\left[\left(3|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}+\left(|f'(a)|^q+3|f'(b)|^q\right)^{\frac{1}{q}}\right], \tag{1.12}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Motivated by results works done in [1,9,11], in this paper we show that Fejér type inequalities containing the Reimann-Liouville fractional integral operator and given in [11] can be extended to fractional integral operator introduced in [1,9].

2. Main results

Lemma 2. *Let $f : [a,b] \to \mathbb{R}$ be differentiable mapping on (a,b) with $a < b$ and let $g : [a,b] \to \mathbb{R}$. If $f', g \in L[a,b]$, then the following identity for fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right)\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}g(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}g(a)\right] \tag{2.1}$$

$$
\begin{aligned}
&-\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}(fg)(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}(fg)(a)\right]\\
&=\int_a^{\frac{a+b}{2}}\left(\int_a^t (s-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}]g(s)ds\right)f'(t)dt\\
&+\int_{\frac{a+b}{2}}^b\left(\int_b^t (b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)ds\right)f'(t)dt.
\end{aligned}
$$

Proof. It suffices to note that

$$
\begin{aligned}
I &= \int_a^{\frac{a+b}{2}}\left(\int_a^t (s-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}]g(s)ds\right)f'(t)dt\\
&+\int_{\frac{a+b}{2}}^b\left(\int_b^t (b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)ds\right)f'(t)dt\\
&= I_1+I_2.
\end{aligned}
\tag{2.2}
$$

By integration by parts, we get

$$
\begin{aligned}
I_1 &= \left(\int_a^t (s-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}]g(s)ds\right)f(t)dt\Bigg|_a^{\frac{a+b}{2}}\\
&-\int_a^{\frac{a+b}{2}}(t-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(t-a)^{\rho}]g(t)f(t)dt\\
&=\left(\int_a^{\frac{a+b}{2}}(s-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}]g(s)ds\right)f\left(\frac{a+b}{2}\right)\\
&-\int_a^{\frac{a+b}{2}}(t-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(t-a)^{\rho}](fg)(t)dt\\
&=f\left(\frac{a+b}{2}\right)\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}g(a)-\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}(fg)(a)
\end{aligned}
\tag{2.3}
$$

and similarly

$$
\begin{aligned}
I_2 &= \left(\int_b^t (b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)ds\right)f(t)dt\Bigg|_{\frac{a+b}{2}}^{b}\\
&-\int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-t)^{\rho}]g(t)f(t)dt\\
&=\left(\int_{\frac{a+b}{2}}^{b}(b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)ds\right)f\left(\frac{a+b}{2}\right)
\end{aligned}
\tag{2.4}
$$

$$-\int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-t)^{\rho}](fg)(t)dt$$
$$=f\left(\frac{a+b}{2}\right)\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}g(b)-\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}(fg)(b).$$

Putting (2.3) and (2.4) in (2.2), we obtain (2.1) which completes the proof. □

Theorem 4. *Let $f:I\to\mathbb{R}$ be a differentiable mapping on I° and $f'\in L[a,b]$ with $a<b$ and $g:[a,b]\to\mathbb{R}$ is continuous. If $|f'|$ is convex on $[a,b]$, then the following for fractional integrals holds:*

$$\left|f\left(\frac{a+b}{2}\right)\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}g(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}g(a)\right]\right. \tag{2.5}$$
$$\left.-\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}(fg)(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}(fg)(a)\right]\right|$$
$$\leq(b-a)^{\alpha+1}||g||_{[a,b],\infty}\mathcal{F}^{\sigma_1}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]\left(|f'(a)|+|f'(b)|\right)$$

where

$$\sigma_1(k):=\sigma(k)\frac{1}{2^{\alpha+\rho k+1}(\alpha+\rho k+1)}$$

with $\alpha>0$.

Proof. Since $|f'|$ is convex on $[a,b]$, we know that for $t\in[a,b]$

$$|f'(t)|=\left|f'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|\leq\frac{b-t}{b-a}|f'(a)|+\frac{t-a}{b-a}|f'(b)|.$$

From Lemma 2 we have

$$\left|f\left(\frac{a+b}{2}\right)\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}g(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}g(a)\right]\right.$$
$$\left.-\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}(fg)(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}(fg)(a)\right]\right|$$
$$\leq\int_{a}^{\frac{a+b}{2}}\left|\int_{a}^{t}(s-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}]g(s)ds\right||f'(t)|dt$$
$$+\int_{\frac{a+b}{2}}^{b}\left|\int_{b}^{t}(b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)ds\right||f'(t)|dt$$
$$\leq\frac{||g||_{[a,\frac{a+b}{2}],\infty}}{b-a}\int_{a}^{\frac{a+b}{2}}\left(\int_{a}^{t}(s-a)^{\alpha-1}\left(\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}}{\Gamma(\alpha+\rho k)}(s-a)^{\rho k}\right)ds\right)$$
$$\times\left[(b-t)|f'(a)|+(t-a)|f'(b)|\right]dt$$

$$+\frac{||g||_{[\frac{a+b}{2},b],\infty}}{b-a}\int_{\frac{a+b}{2}}^{b}\left(\int_{b}^{t}(b-s)^{\alpha-1}\left(\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}}{\Gamma(\alpha+\rho k)}(b-s)^{\rho k}\right)ds\right)$$
$$\times\left[(b-t)|f'(a)|+(t-a)|f'(b)|\right]dt$$
$$=\frac{||g||_{[a,\frac{a+b}{2}],\infty}}{b-a}\int_{a}^{\frac{a+b}{2}}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}}{\Gamma(\alpha+\rho k)}\left(\int_{a}^{t}(s-a)^{\alpha+\rho k-1}ds\right)$$
$$\times\left[(b-t)|f'(a)|+(t-a)|f'(b)|\right]dt$$
$$+\frac{||g||_{[\frac{a+b}{2},b],\infty}}{b-a}\int_{\frac{a+b}{2}}^{b}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}}{\Gamma(\alpha+\rho k)}\left(\int_{b}^{t}(b-s)^{\alpha+\rho k-1}ds\right)$$
$$\times\left[(b-t)|f'(a)|+(t-a)|f'(b)|\right]dt$$
$$=\frac{||g||_{[a,\frac{a+b}{2}],\infty}}{b-a}$$
$$\times\sum_{k=0}^{\infty}\left(\int_{a}^{\frac{a+b}{2}}(t-a)^{\alpha+\rho k}\left[(b-t)|f'(a)|+(t-a)|f'(b)|\right]dt\right)\frac{\sigma(k)|w|^{k}}{\Gamma(\alpha+\rho k+1)}$$
$$+\frac{||g||_{[\frac{a+b}{2},b],\infty}}{b-a}$$
$$\times\sum_{k=0}^{\infty}\left(\int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha+\rho k}\left[(b-t)|f'(a)|+(t-a)|f'(b)|\right]dt\right)\frac{\sigma(k)|w|^{k}}{\Gamma(\alpha+\rho k+1)}$$

$$=\frac{||g||_{[a,\frac{a+b}{2}],\infty}}{b-a}$$
$$\times\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}}{\Gamma(\alpha+\rho k+1)}\left[\left(\frac{(\alpha+\rho k+3)(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+1)(\alpha+\rho k+2)}\right)|f'(a)|\right.\right.$$
$$\left.\left.+\left(\frac{(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+2)}\right)|f'(b)|\right]\right\}$$
$$+\frac{||g||_{[\frac{a+b}{2},b],\infty}}{b-a}$$
$$\times\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}}{\Gamma(\alpha+\rho k+1)}\left[\left(\frac{(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+2)}\right)|f'(a)|\right.\right.$$
$$\left.\left.+\left(\frac{(\alpha+\rho k+3)(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+1)(\alpha+\rho k+2)}\right)|f'(b)|\right]\right\}$$

$$\leq \frac{||g||_{[a,b],\infty}}{b-a}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)}\frac{(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+1}(\alpha+\rho k+1)}\left(|f'(a)|+|f'(b)|\right)\right\}$$
$$= ||g||_{[a,b],\infty}(b-a)^{\alpha+1}\mathcal{F}^{\sigma_1}_{\rho,\alpha+1}[w(b-a)^{\rho}](|f'(a)|+|f'(b)|),$$

where

$$\int_a^{\frac{a+b}{2}}(t-a)^{\alpha+\rho k+1}dt = \int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha+\rho k+1}dt = \frac{(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+2)},$$

$$\int_a^{\frac{a+b}{2}}(t-a)^{\alpha+\rho k}(b-t)dt = \int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha+\rho k}(t-a)dt$$
$$= \frac{(\alpha+\rho k+3)(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+1)(\alpha+\rho k+2)}.$$

This completes the proof. □

Remark 1. If we choose $\sigma(0)=1$ and $w=0$ in Theorem 4, then the inequality (2.5) reduces to (1.10).

Theorem 5. *Let $f: I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a<b$ and let $g:[a,b]\to\mathbb{R}$ is continuous. If $|f'|^q$ is convex on $[a,b]$, $q\geq 1$, then the following inequality for fractional integrals holds:*

$$\left|f\left(\frac{a+b}{2}\right)\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}g(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}g(a)\right]\right. \tag{2.6}$$
$$\left.-\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}(fg)(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}(fg)(a)\right]\right|$$
$$\leq ||g||_{[a,b],\infty}((b-a)^{\alpha+1})^{\frac{1}{q}}\left((b-a)^{\alpha+1}\mathcal{F}^{\sigma_1}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left[\mathcal{F}^{\sigma_2}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]|f'(a)|^q+\mathcal{F}^{\sigma_3}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]|f'(b)|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\mathcal{F}^{\sigma_3}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]|f'(a)|^q+\mathcal{F}^{\sigma_2}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]|f'(b)|^q\right]^{\frac{1}{q}}\right\}$$

where

$$\sigma_1(k) := \sigma(k)\frac{1}{(\alpha+\rho k+1)2^{\alpha+\rho k+1}},$$
$$\sigma_2(k) := \sigma(k)\frac{\alpha+\rho k+3}{(\alpha+\rho k+1)(\alpha+\rho k+2)2^{\alpha+\rho k+2}}$$

and

$$\sigma_3(k) := \sigma(k)\frac{1}{(\alpha+\rho k+2)2^{\alpha+\rho k+2}}$$

with $\alpha > 0$.

Proof. Since $|f'|^q$ is convex on $[a,b]$, we know that for $t \in [a,b]$

$$|f'(t)|^q = \left| f'\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b\right)\right|^q \leq \frac{b-t}{b-a}|f'(a)|^q + \frac{t-a}{b-a}|f'(b)|^q.$$

Using Lemma 2, power-mean inequality and the convexity of $|f'|^q$, it follows that

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right)\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}g(b) + \mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}g(a)\right]\right.\\
&\left. -\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}(fg)(b) + \mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}(fg)(a)\right]\right|\\
&\leq \left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}]g(s)ds\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-a)^{\rho}]g(s)ds\right||f'(t)|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{a+b}{2}}^b\left|\int_b^t (b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)ds\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_{\frac{a+b}{2}}^b\left|\int_b^t (b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)ds\right||f'(t)|^q dt\right)^{\frac{1}{q}}\\
&\leq ||g||_{[a,\frac{a+b}{2}],\infty}\left(\int_a^{\frac{a+b}{2}}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\left|\int_a^t (s-a)^{\alpha+\rho k-1}ds\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_a^{\frac{a+b}{2}}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\left|\int_a^t (s-a)^{\alpha+\rho k-1}ds\right||f'(t)|^q dt\right)^{\frac{1}{q}}\\
&+||g||_{[\frac{a+b}{2},b],\infty}\left(\int_{\frac{a+b}{2}}^b\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\left|\int_b^t (b-s)^{\alpha+\rho k-1}ds\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_{\frac{a+b}{2}}^b\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\left|\int_b^t (b-s)^{\alpha+\rho k-1}ds\right||f'(t)|^q dt\right)^{\frac{1}{q}}\\
&\leq\left((b-a)^{\alpha+1}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\frac{(b-a)^{\rho k}}{(\alpha+\rho k)(\alpha+\rho k+1)2^{\alpha+\rho k+1}}\right)^{1-\frac{1}{q}}
\end{aligned}$$

$$\times\left\{\frac{||g||_{[a,\frac{a+b}{2}],\infty}}{(b-a)^{\frac{1}{q}}}\left(\int_a^{\frac{a+b}{2}}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)(\alpha+\rho k)}\right.\right.$$

$$\times\left[(t-a)^{\alpha+\rho k}(b-t)|f'(a)|^q+(t-a)^{\alpha+\rho k+1}|f'(b)|^q\right]dt\Bigg)^{\frac{1}{q}}$$

$$+\frac{||g||_{[\frac{a+b}{2},b],\infty}}{(b-a)^{\frac{1}{q}}}\left(\int_{\frac{a+b}{2}}^{b}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)(\alpha+\rho k)}\right.$$

$$\left.\left.\times\left[(b-t)^{\alpha+\rho k+1}|f'(a)|^q+(b-t)^{\alpha+\rho k}(t-a)|f'(b)|^q\right]dt\right)^{\frac{1}{q}}\right\}$$

$$=\left((b-a)^{\alpha+1}\mathcal{F}^{\sigma_1}_{\rho,\alpha+1}[w(b-a)^{\rho}]\right)^{1-\frac{1}{q}}$$

$$\times\left\{\frac{||g||_{[a,\frac{a+b}{2}],\infty}}{(b-a)^{\frac{1}{q}}}\left(\sum_{k=0}^{\infty}\left(\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)(\alpha+\rho k)}\right.\right.\right.$$

$$\times\left[\frac{(\alpha+\rho k+3)(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+1)(\alpha+\rho k+2)}|f'(a)|^q\right.$$

$$\left.\left.\left.+\frac{(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+2)}|f'(b)|^q\right]\right)\right)^{\frac{1}{q}}$$

$$+\frac{||g||_{[\frac{a+b}{2},b],\infty}}{(b-a)^{\frac{1}{q}}}\left(\sum_{k=0}^{\infty}\left(\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)(\alpha+\rho k)}\right.\right.$$

$$\times\left[\frac{(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+2)}|f'(a)|^q\right.$$

$$\left.\left.\left.+\frac{(\alpha+\rho k+3)(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+1)(\alpha+\rho k+2)}|f'(b)|^q\right]\right)\right)^{\frac{1}{q}}\right\}$$

$$\leq||g||_{[a,b],\infty}\left((b-a)^{\alpha+1}\right)^{\frac{1}{q}}\left((b-a)^{\alpha+1}\mathcal{F}^{\sigma_1}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left[\mathcal{F}^{\sigma_2}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]|f'(a)|^q+\mathcal{F}^{\sigma_3}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]|f'(b)|^q\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\mathcal{F}^{\sigma_3}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]|f'(a)|^q+\mathcal{F}^{\sigma_2}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]|f'(b)|^q\right]^{\frac{1}{q}}\right\},$$

where it is easily seen that

$$\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha+\rho k-1} ds \right| dt = \int_{\frac{a+b}{2}}^b \left| \int_b^t (b-s)^{\alpha+\rho k-1} ds \right| dt$$
$$= \frac{(b-a)^{\alpha+\rho k+1}}{2^{\alpha+\rho k+1}(\alpha+\rho k)(\alpha+\rho k+1)},$$

$$\int_a^{\frac{a+b}{2}} (t-a)^{\alpha+\rho k}(b-t)dt = \int_{\frac{a+b}{2}}^b (b-t)^{\alpha+\rho k}(t-a)dt$$
$$= \frac{(\alpha+\rho k+3)(b-a)^{\alpha+\rho k+2}}{2^{\alpha+\rho k+2}(\alpha+\rho k+1)(\alpha+\rho k+2)}.$$

Hence the proof is completed. □

Remark 2. If we choose $\sigma(0) = 1$ and $w = 0$ in Theorem 5, then the inequality (2.6) reduces to (1.11).

Theorem 6. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$ and let $g : [a,b] \to \mathbb{R}$ is continuous. If $|f'|^q$ is convex on $[a,b]$, $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right)\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w} g(b) + \mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w} g(a)\right] \right. \\ &\left. -\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}(fg)(b) + \mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}(fg)(a)\right]\right| \\ &\leq \frac{(b-a)^{\alpha+1}||g||_{[a,b]_\infty}}{2^{\frac{3}{q}}}(\mathcal{F}^{\sigma_1}_{\rho,\alpha+1}[|w|(b-a)^\rho])^{\frac{1}{p}} \\ &\times \left\{[3|f'(a)|^q + |f'(b)|^q]^{\frac{1}{q}} + [|f'(a)|^q + 3|f'(b)|^q]^{\frac{1}{q}}\right\} \end{aligned} \tag{2.7}$$

where

$$\sigma_1(k) := \sigma(k)\frac{1}{2^{\alpha+\rho k+\frac{1}{p}}(\alpha p+\rho k p+1)^{\frac{1}{p}}}$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Lemma 2, Hölder's inequality and the convexity of $|f'|^q$, it follows that

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right)\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w} g(b) + \mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w} g(a)\right] \right. \\ &\left. -\left[\mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}+;w}(fg)(b) + \mathcal{J}^{\sigma}_{\rho,\alpha,\frac{a+b}{2}-;w}(fg)(a)\right]\right| \end{aligned}$$

$$\leq \left| \int_a^{\frac{a+b}{2}} \left(\int_a^t (s-a)^{\alpha-1} \mathcal{F}_{\rho,\alpha}^{\sigma}[w(s-a)^{\rho}] g(s) ds \right) f'(t) dt \right.$$

$$\left. + \int_{\frac{a+b}{2}}^b \left(\int_b^t (b-s)^{\alpha-1} \mathcal{F}_{\rho,\alpha}^{\sigma}[w(b-s)^{\rho}] g(s) ds \right) f'(t) dt \right|$$

$$\leq \int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} \mathcal{F}_{\rho,\alpha}^{\sigma}[w(s-a)^{\rho}] g(s) ds \right| |f'(t)| dt$$

$$+ \int_{\frac{a+b}{2}}^b \left| \int_b^t (b-s)^{\alpha-1} \mathcal{F}_{\rho,\alpha}^{\sigma}[w(b-s)^{\rho}] g(s) ds \right| |f'(t)| dt$$

$$= \int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} \sum_{k=0}^{\infty} \frac{\sigma(k) w^k (s-a)^{\rho k}}{\Gamma(\alpha+\rho k)} g(s) ds \right| |f'(t)| dt$$

$$+ \int_{\frac{a+b}{2}}^b \left| \int_b^t (b-s)^{\alpha-1} \sum_{k=0}^{\infty} \frac{\sigma(k) w^k (b-s)^{\rho k}}{\Gamma(\alpha+\rho k)} g(s) ds \right| |f'(t)| dt$$

$$= \int_a^{\frac{a+b}{2}} \left| \sum_{k=0}^{\infty} \frac{\sigma(k) w^k}{\Gamma(\alpha+\rho k)} \int_a^t (s-a)^{\alpha+\rho k-1} g(s) ds \right| |f'(t)| dt$$

$$+ \int_{\frac{a+b}{2}}^b \left| \sum_{k=0}^{\infty} \frac{\sigma(k) w^k}{\Gamma(\alpha+\rho k)} \int_b^t (b-s)^{\alpha+\rho k-1} g(s) ds \right| |f'(t)| dt$$

$$\leq \int_a^{\frac{a+b}{2}} \left(\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)} \left| \int_a^t (s-a)^{\alpha+\rho k-1} g(s) ds \right| \right) |f'(t)| dt$$

$$+ \int_{\frac{a+b}{2}}^b \left(\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)} \left| \int_b^t (b-s)^{\alpha+\rho k-1} g(s) ds \right| \right) |f'(t)| dt$$

$$\leq ||g||_{[a,\frac{a+b}{2}],\infty} \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)} \left(\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha+\rho k-1} ds \right| |f'(t)| dt \right)$$

$$+ ||g||_{[\frac{a+b}{2},b],\infty} \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)} \left(\int_{\frac{a+b}{2}}^b \left| \int_b^t (b-s)^{\alpha+\rho k-1} ds \right| |f'(t)| dt \right)$$

$$\leq ||g||_{[a,b],\infty} \left\{ \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)} \left(\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha+\rho k-1} ds \right|^p dt \right)^{\frac{1}{p}} \left(\int_a^{\frac{a+b}{2}} |f'(t)|^q dt \right)^{\frac{1}{q}} \right.$$

$$\left. + \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)} \left(\int_{\frac{a+b}{2}}^b \left| \int_b^t (b-s)^{\alpha+\rho k-1} ds \right|^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{a+b}{2}}^b |f'(t)|^q dt \right)^{\frac{1}{q}} \right\}$$

$$= ||g||_{[a,b],\infty}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\left(\int_a^{\frac{a+b}{2}}\left|\frac{(s-a)^{\alpha+\rho k}}{(\alpha+\rho k)}\Big|_a^t\right|^p dt\right)^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}|f'(t)|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\left(\int_{\frac{a+b}{2}}^{b}\left|\frac{(b-s)^{\alpha+\rho k}}{(\alpha+\rho k)}\Big|_b^t\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^{b}|f'(t)|^q dt\right)^{\frac{1}{q}}\right\}$$

$$= ||g||_{[a,b],\infty}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\left(\int_a^{\frac{a+b}{2}}\frac{(t-a)^{(\alpha+\rho k)p}}{(\alpha+\rho k)^p}dt\right)^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}|f'(t)|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)}\left(\int_{\frac{a+b}{2}}^{b}\frac{(b-t)^{(\alpha+\rho k)p}}{(\alpha+\rho k)^p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^{b}|f'(t)|^q dt\right)^{\frac{1}{q}}\right\}$$

$$= ||g||_{[a,b],\infty}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)}\left(\frac{(t-a)^{\alpha p+\rho kp+1}}{\alpha p+\rho kp+1}\Big|_a^{\frac{a+b}{2}}\right)^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}|f'(t)|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)}\left(-\frac{(b-t)^{\alpha p+\rho kp+1}}{\alpha p+\rho kp+1}\Big|_{\frac{a+b}{2}}^{b}\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^{b}|f'(t)|^q dt\right)^{\frac{1}{q}}\right\}$$

$$= ||g||_{[a,b],\infty}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)}\left(\frac{(b-a)^{\alpha p+\rho kp+1}}{2^{\alpha p+\rho kp+1}(\alpha p+\rho kp+1)}\right)^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}|f'(t)|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)}\left(\frac{(b-a)^{\alpha p+\rho kp+1}}{2^{\alpha p+\rho kp+1}(\alpha p+\rho kp+1)}\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^{b}|f'(t)|^q dt\right)^{\frac{1}{q}}\right\}$$

$$= ||g||_{[a,b],\infty}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)}\left(\frac{(b-a)^{\alpha+\rho k+\frac{1}{p}}}{2^{\alpha+\rho k+\frac{1}{p}}(\alpha p+\rho kp+1)^{\frac{1}{p}}}\right)\left(\int_a^{\frac{a+b}{2}}|f'(t)|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)}\left(\frac{(b-a)^{\alpha+\rho k+\frac{1}{p}}}{2^{\alpha+\rho k+\frac{1}{p}}(\alpha p+\rho kp+1)^{\frac{1}{p}}}\right)\left(\int_{\frac{a+b}{2}}^{b}|f'(t)|^q dt\right)^{\frac{1}{q}}\right\}$$

$$\leq ||g||_{[a,b],\infty}(b-a)^{\alpha+1}\mathcal{F}^{\sigma_1}_{\rho,\alpha+1}[|w|(b-a)^{\rho}]$$

$$\times\left\{\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{\frac{1}{q}}\right\}.$$

Here we use

$$\int_a^{\frac{a+b}{2}}\left|\int_a^t(s-a)^{\alpha+\rho k-1}ds\right|^p dt = \int_{\frac{a+b}{2}}^{b}\left|\int_b^t(b-s)^{\alpha+\rho k-1}ds\right|^p dt$$

$$= \frac{(b-a)^{\alpha p+\rho kp+1}}{2^{\alpha p+\rho kp+1}(\alpha+\rho k)^p(\alpha p+\rho kp+1)},$$

$$\int_a^{\frac{a+b}{2}} |f'(t)|^q dt \leq \frac{1}{b-a} \int_a^{\frac{a+b}{2}} \left[(b-t)|f'(a)|^q + (t-a)|f'(b)|^q\right] dt$$
$$= (b-a)\frac{3|f'(a)|^q + |f'(b)|^q}{8}$$

and

$$\int_{\frac{a+b}{2}}^b |f'(t)|^q dt \leq \frac{1}{b-a} \int_{\frac{a+b}{2}}^b \left[(b-t)|f'(a)|^q + (t-a)|f'(b)|^q\right] dt$$
$$= (b-a)\frac{|f'(a)|^q + 3|f'(b)|^q}{8}.$$

Hence the inequality (2.7) is proved. □

Remark 3. If we choose $\sigma(0) = 1$ and $w = 0$ in Theorem 6, then the inequality (2.7) reduces to (1.12).

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ON CONVERSION OF RESONANT PROBLEM TO NON-RESONANT ONE

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Abstract. We consider a nonlinear resonant boundary value problem. To prove the existence of a solution to a given boundary value problem we replace the linear part of a given equation by non-resonant linear part. First, to modify a resonant problem to a regular one we use the Taylor expansion for $f(t,x,x')$ with respect to x. The second way of conversion a given problem to non-resonant one is based on an appropriate choice of "good" approximation to expected solution. We provide the existence results illustrating both ways.

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1. Introduction

When studying two-point boundary value problems for a nonlinear second order equation

$$x'' = f(t,x,x') \tag{1.1}$$

we first look at the right side. If f is continuous and bounded (in modulus) function then the problem is solvable if the respective homogeneous problem

$$x'' = 0, \quad \text{+boundary conditions} \tag{1.2}$$

has only the trivial solution. Otherwise the problem may be unsolvable and this depends on the right hand side of an equation.

Similarly, if we consider the problem with the linear part

$$x'' + p(t)x' + q(t)x = F(t,x,x'), \tag{1.3}$$

$$x(a) = A, \quad x(b) = B, \tag{1.4}$$

we are sure that the problem is solvable (for F bounded) if the respective homogeneous problem

$$x'' + p(t)x' + q(t)x = 0, \quad x(a) = 0,\ x(b) = 0 \tag{1.5}$$

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has only the trivial solution. If the above problem has a nontrivial solution also, then the following question arises: which right hand sides $F(t,x,x')$ are "good" and which are "bad" for the existence of a solution.

The answer is given by the Fredholm theory for $F = F(t)$: if the right hand side is "orthogonal" to an eigenfunction of a conjugate linear part, then this right side function $F(t)$ is "good".

The problem (1.3) with the property that the respective homogeneous problem has a nontrivial solution, is called *resonant.* There is intensive literature on solvability of resonant problems. Usually the appropriate conditions on the linear part and on the nonlinearity are imposed in order the problem to be solvable. One may consult recent articles [1, 5, 12] and references therein for the respective bibliography.

In the paper [9] a resonant problem for the second order nonlinear equation is treated from "inside". First the *type* of a solution is introduced. It is shown afterwards that a resonant problem that has a solution of definite type in fact can be reduced to a non-resonant problem with the similar type linear part.

In this note we would like to analyze this process of conversion of a resonant problem to non-resonant one in details. It is to be mentioned that the natural idea of "a shift" from a resonant linear part to a non-resonant one was employed in the papers [4, 5, 13]. "Shift" arguments in a broad sense (replacing the linear part of a given equation multiply by different non-resonant linear parts and proving the existence of multiple solutions to a given boundary value problem) were applied for the study of resonant problems in [12] in context of the quasi-linearization approach [3]. In Section 4 of this paper we consider two ways of relaxing the quasilinearization arguments used in [12].

In order to make exposition self-contained first we introduce notions of the type of a linear part of an equation and that of the type of a solution.

2. Types of linear parts and types of solutions

Consider quasi-linear problem (1.3), (1.4). We say that the linear part $(l_2x)(t) = x'' + p(t)x' + q(t)$ has type i, if a solution $y(t)$ of the Cauchy problem

$$y'' + p(t)y' + q(t)y = 0, \quad y(a) = 0,\ y'(a) = 1 \tag{2.1}$$

has exactly i zeros in the interval (a,b) and $y(b) \neq 0$.

If $y(b) = 0$ then the linear part $(l_2x)(t)$ is *resonant.*

For instance, the linear part $x'' + k^2x$, where $3\pi < k < 4\pi$, has type 3.

Let us pass to definition of the type of a solution.

Let $\xi(t)$ be a solution of the BVP

$$x'' = f(t,x,x'), \quad x(a) = A,\ x(b) = B. \tag{2.2}$$

The right hand side function f may contain a linear part.

Consider the respective equation of variations

$$y'' = f_x(t,\xi(t),\xi'(t))y + f_{x'}(t,\xi(t),\xi'(t))y'. \tag{2.3}$$

Definition 1. Let $\xi(t)$ be a solution of BVP. We say that the type of $\xi(t)$ is i, if equation of variations (2.3) with respect to $\xi(t)$ is such that a solution $y(t)$ with the initial conditions

$$y(a) = 0, \quad y'(a) = 1 \tag{2.4}$$

has exactly i zeros in the interval (a,b) and $y(b) \neq 0$. Denote this:

$$type(\xi) = i.$$

If moreover $y(b) = 0$, denote the intermediate type

$$type(\xi) = (i, i+1). \tag{2.5}$$

Remark 1. Therefore, a solution of type $(i,i+1)$ is a solution $\xi(t)$ such that the respective $y(t)$ has exactly i zeros in (a,b) and $y(b) = 0$.

Remark 2. The study of solutions of BVP in terms of solutions of equations of variations was initiated in early papers [6–8]. The authors have observed the existence of solutions of zero type (in our terminology) in problems satisfying some standard requirements.

Example 1. The trivial solution $\xi(t) \equiv 0$ of the problem $x'' = -x + x^3$, $x(0) = 0$, $x(\pi) = 0$ is of type $(0,1)$ since a solution $y(t)$ of the Cauchy problem $y'' = -y$ (equation of variations with respect to ξ), $y(0) = 0$, $y'(0) = 1$ has the first zero at $t = \pi$.

3. Reduction to quasi-linear problem

Return to the problem (1.1), (1.4). The right side f may contain linear part. The case of the problem (1.1), (1.4) being resonant is not excluded.

Theorem 1. *Suppose the problem (1.1), (1.4) has a solution $\xi(t)$ of type i.*

Then the problem can be reformulated in the quasi-linear form (1.3), (1.4), where the linear part $x'' + p(t)x' + q(t)x$ in (1.3) has the type i.

Proof. One has that

$$\begin{aligned} x'' - \xi''(t) &= f(t,x,x') - f(t,\xi(t),\xi'(t)) \\ &= f_x(t,\xi(t),\xi'(t))(x-\xi(t)) + f_{x'}(t,\xi(t),\xi'(t))(x'-\xi'(t)) + \varphi(t,x,x') \end{aligned} \tag{3.1}$$

or

$$\begin{aligned} x'' &- f_{x'}(t,\xi(t),\xi'(t))x' - f_x(t,\xi(t),\xi'(t))x \\ &= \xi''(t) - f_{x'}(t,\xi(t),\xi'(t))\xi'(t) - f_x(t,\xi(t),\xi'(t))\xi(t) + \varphi(t,x,x') \end{aligned} \tag{3.2}$$

and finally

$$(l_2x)(t) = h(t) + \widetilde{\varphi}(t,x,x'), \tag{3.3}$$

where

$$(l_2x)(t) = x'' - f_{x'}(t,\xi(t),\xi'(t))x' - f_x(t,\xi(t),\xi'(t))x,$$
$$h(t) = \xi''(t) - f_{x'}(t,\xi(t),\xi'(t))\xi'(t) - f_x(t,\xi(t),\xi'(t))\xi(t),$$

$\widetilde{\varphi}(t,x,x')$ is a smooth (C^1) bounded function which coincides with $\varphi(t,x,x')$ in some vicinity of $(t,\xi(t),\xi'(t))$, $t \in [a,b]$.

The linear part $(l_2x)(t)$ has type i since a solution $\xi(t)$ has type i. Function ξ, by construction, is a solution of BVP (3.3), (1.4). □

Remark 3. The above reduction is not possible if a solution $\xi(t)$ is of indefinite type $(i,i+1)$.

4. Resonant problems

Consider, for simplicity, the problem

$$x'' + k^2x = f(t,x), \quad x(0) = 0, \quad x(1) = 0. \tag{4.1}$$

If f is continuous function (together with f_x) and bounded, then the problem above is solvable provided that k is not multiple of π. If $k = i\pi$, where i is an integer, then the homogeneous problem

$$x'' + k^2x = 0, \quad x(0) = 0, \quad x(1) = 0 \tag{4.2}$$

has a nontrivial solution and solvability of the problem (4.1) is not guaranteed.

The Fredholm alternative gives an answer in case $f = f(t)$. If not, the following approach was proposed. Change the left side in equation so that it is not resonant yet, add the same term to the right, and finally truncate modified right side appropriately obtaining nonresonant quasi-linear problem. This problem has a solution and if some estimates can be proved for a solution, then this solution may solve also the original problem.

Look how this works. Consider instead of (4.1) the equivalent problem

$$x'' + k^2x + \varepsilon^2x = f(t,x) + \varepsilon^2x, \quad x(0) = 0, \quad x(1) = 0. \tag{4.3}$$

Truncate the right side in equation (4.3) so that the truncated right side function $F(t,x)$ coincides with $f(t,x) + \varepsilon^2x$ for $x \in [-N,N]$ and $t \in [0,1]$.

The modified problem

$$x'' + k^2x + \varepsilon^2x = F(t,x), \quad x(0) = 0, \quad x(1) = 0 \tag{4.4}$$

has a solution $x(t)$ and the representation

$$x(t) = \int_0^1 G(t,s)F(s,x(s))\,ds$$

is valid, where G is the Green's function associated with new (nonresonant) left side. If the key inequality

$$\Gamma \cdot M \leq N \tag{4.5}$$

holds, where Γ and M are respectively bounds (the estimate constants) for the Green's function $|G|$ and $|F(t,x)|$, then $|x(t)| \leq N$ and $x(t)$ is a solution of (4.1). Due to the results in [10] and [11], there is a solution $x(t)$ that has definite type induced by the linear part in (4.4). Therefore multiple application of this scheme using multiple different linear parts can prove the existence of multiple solutions of the (resonant) problem (4.1). This scheme was tested on equations of the Emden - Fowler type in [12] (see also [3]).

In the sequel we would like to discuss two ways of relaxing inequality (4.5).

4.1. *Relaxing (4.5) - first way*

It is known (see [12]) that the above approach is applicable to resonant problems of the type

$$x'' + \pi^2 x = -|x|^p \operatorname{sign} x, \quad x(0) = x(1) = 0, \quad t \in [0,1]. \tag{4.6}$$

The reason is, probably, that after modifying the equation in (4.6) looks

$$x'' + \pi^2 x + \varepsilon^2 x = -|x|^p \operatorname{sign} x + \varepsilon^2 x \tag{4.7}$$

and the right hand side function

$$-|x|^p \operatorname{sign} x + \varepsilon^2 x \tag{4.8}$$

for appropriate choice of ε is uniformly small on an interval $[-N, N]$. In other words, the right side of (4.6) is well approximated by a linear term $\varepsilon^2 x$ in the interval $[-N, N]$. Then a constant M in (4.5) is relatively small also and the inequality (4.5) is likely to be satisfied.

Let us extend this technique to more general cases. Consider the problem

$$x'' + \pi^2 x = g(x), \quad x(0) = x(1) = 0, \quad t \in [0,1]. \tag{4.9}$$

It follows from the Taylor expansion that

$$g(x) - g(x_0) \simeq g'(x_0)(x - x_0). \tag{4.10}$$

Adding $-g(x_0) - g'(x_0)(x - x_0)$ to both sides, one obtains

$$x'' + \pi^2 x - g(x_0) - g'(x_0)(x - x_0) = g(x) - g(x_0) - g'(x_0)(x - x_0). \tag{4.11}$$

The right hand side is small if an expected solution $x(t)$ is close to a constant x_0. Suppose $x_0 = 0$ and assume we are looking for solutions with small amplitude of oscillations. Equation (4.11) becomes

$$x'' + \pi^2 x - g(0) - g'(0)x = g(x) - g(0) - g'(0)x \tag{4.12}$$

or, provided that $g(0) = 0$,

$$x'' + \pi^2 x - g'(0)x = g(x) - g'(0)x. \tag{4.13}$$

Example 2. $g(x) = -k \arctan x$

Consider the boundary value problem

$$x'' + \pi^2 x = -k \arctan x, \; x(0) = x(1) = 0. \tag{4.14}$$

Reduce it to the form (4.13)

$$x'' + \pi^2 x - g'(0)x = g(x) - g'(0)x,$$

$$x'' + \pi^2 x + \left.\frac{k}{1+x^2}\right|_{x=0} x = -k \arctan x + \left.\frac{k}{1+x^2}\right|_{x=0} x,$$

$$x'' + \pi^2 x + kx = -k \arctan x + kx.$$

We obtain the differential equation

$$x'' + \left(\pi^2 + k\right) x = -k \arctan x + kx,$$

where the linear part is no more resonant with respect to the given boundary conditions.

The appropriate estimate of the Green's function of the corresponding homogeneous boundary value problem is a constant

$$\Gamma = \frac{1}{\sqrt{\pi^2 + k}\,|\sin\sqrt{\pi^2 + k}|}. \tag{4.15}$$

We define $N > 0$ and $M(N) = |-k \arctan N + kN|$.

Therefore if the inequality

$$\frac{|-k \arctan N + kN|}{\sqrt{\pi^2 + k}\,|\sin\sqrt{\pi^2 + k}|} \le N \tag{4.16}$$

holds, then the boundary value problem (4.14) has a solution $|x(t)| \le N$.

For example, if $k = 1$, then the inequality (4.16) holds for $N \le 2.4$.

Example 3. $g(x) = -k \sin x$. Consider the boundary value problem

$$x'' + \pi^2 x = -k \sin x, \; x(0) = x(1) = 0. \tag{4.17}$$

Analogously, as in the previous example we reduce the equation (4.17) to the form (4.13). We obtain the differential equation

$$x'' + \left(\pi^2 + k\right) x = -k \sin x + kx,$$

where the linear part is no more resonant with respect to the given boundary conditions.

The estimate of the Green's function for the corresponding homogeneous boundary value problem is given by (4.15).

Analogously, we define N positive and $M(N) = |-k \sin N + kN|$.

Therefore if the inequality

$$\frac{|-k\sin N + kN|}{\sqrt{\pi^2+k}\,|\sin\sqrt{\pi^2+k}|} \le N. \tag{4.18}$$

holds, then the boundary value problem (4.17) has a solution $|x(t)| \le N$.

And, for example, if $k = 1$, then the inequality (4.18) holds for $N \le 1.92$.

Example 4. $g(x) = -k(e^x - 1)$.

Consider the boundary value problem

$$x'' + \pi^2 x = -k(e^x - 1),\ x(0) = x(1) = 0. \tag{4.19}$$

Analogously, as in the previous examples we reduce the equation (4.19) to the form (4.13). We obtain the differential equation

$$x'' + \left(\pi^2 + k\right)x = -k(e^x - 1) + kx,$$

where the linear part is no more resonant with respect to the given boundary conditions.

The estimate of the Green's function for the corresponding homogeneous boundary value problem is given by (4.15).

Analogously, we define N positive and $M(N) = |-k(e^N - 1) + kN|$.

Therefore if the inequality

$$\frac{|-k(e^N - 1) + kN|}{\sqrt{\pi^2+k}\,|\sin\sqrt{\pi^2+k}|} \le N \tag{4.20}$$

holds, then the boundary value problem (4.19) has a solution $|x(t)| \le N$.

And, for example, if $k = 1$, then the inequality (4.20) holds for $N \le 0.77$.

4.2. *Relaxing (4.5) - second way*

The second way of relaxing the inequality (4.5) is based on an appropriate choice of "good" approximation to an expected solution.

Consider the (resonant) problem

$$x'' = f(t,x), \quad x(0) = x(1) = 0, \quad t \in [0,1], \tag{4.21}$$

where $f(t,x) = -\pi^2 x + g(x)$ and $g(x)$ is a continuous generally nonlinear function.

Assume that there exists $u \in C^2([0,1])$ and $N \ge 0$ such that the estimate

$$\Gamma \cdot M(N) \le N \tag{4.22}$$

holds, where Γ is an estimate of the Green's function $G(t,s)$ (G is the Green's function for $x'' = 0$, $x(0) = x(1) = 0$ and $M(N) = \max\{|f(t,x) - u''(t)| :\ 0 \le t \le 1,\ |x - u(t)| \le N\}$.

We will show that then the problem (4.21) has a solution.

Function $u(t)$ plays the role of an approximation to a solution $x(t)$ of the problem (4.21). Introduce new variable $y = x - u(t)$. Define

$$\begin{aligned} f_1(t,x) &= f(t,x) - u''(t), \\ F_1(t,y) &= f_1(t,u(t)+y), \\ F(t,y) &= F_1(t,\delta(-N,y,N)), \end{aligned}$$

where $\delta(-N,y,N)$ is the "cut off" function. Notice that

$$\max\{|F(t,y)| : \ 0 \le t \le 1, \ |y| \le N\} = M(N).$$

Consider the problem

$$y'' = F(t,y) \quad y(0) = 0, \ y(1) = 0, \tag{4.23}$$

which is solvable (by R. Conti's theorem, [2]). By virtue of (4.22) any solution $y(t)$ of (4.23) satisfies the estimate $|y(t)| \le N \ \ \forall t \in I$. Then $y''(t) = F_1(t,y(t)) = f_1(t,u(t)+y(t)) = f(t,x(t)) - u''(t)$ and the function $y(t)+u(t)$ solves the problem (4.21).

This technique will work out if the approximation $u(t)$ is close to a solution $x(t)$. Then $M(N)$ in (4.22) may be small for $|x-u(t)| \le N$, where N is small also.

This approach is illustrated by the examples below. First a solution $x(t)$ should be found numerically. Second, a solution $x(t)$ may be approximated by an appropriate function (the function $\alpha \sin \beta t$ will be tested) and all the estimations (4.22) are to be proved.

Example 5. $g(x) = -\arctan x$.

Consider the boundary value problem

$$x'' = -\pi^2 x - \arctan x, \ x(0) = x(1) = 0. \tag{4.24}$$

Denote $u(t) = 32 \sin \pi t$. Introduce new variable $y = x - u(t)$. Therefore $x = y + u(t)$.

Define

$$\begin{aligned} f_1(t,x) &= g(t,x) - u''(t) = -\pi^2 x - \arctan x + 32\pi^2 \sin \pi t, \\ F_1(t,y) &= f_1(t,u(t)+y) \\ &= -\pi^2(y + 32\sin \pi t) - \arctan(y + 32 \sin \pi t) + 32\pi^2 \sin \pi t \\ &= -\pi^2 y - \arctan(y + 32 \sin \pi t), \\ F(t,y) &= F_1(t,\delta(-N,y,N)) \\ &= -\pi^2 \delta(-N,y,N) - \arctan(\delta(-N,y,N) + 32 \sin \pi t), \end{aligned}$$

where $\delta(-N,y,N)$ is the "cut off" function.

Notice that

$$\max\{|F(t,y)| : 0 \le t \le 1, |y| \le N\} = M(N).$$

Consider the problem

$$y'' = F(t,y), \quad y(0) = 0, \quad y(1) = 0. \tag{4.25}$$

The Green's function of the corresponding homogeneous boundary value problem is

$$G(t,y) = \begin{cases} s(t-1), & 0 \le s \le t \le 1, \\ t(s-1), & 0 < t < s \le 1. \end{cases} \tag{4.26}$$

The estimate of Green's function is $\Gamma = \dfrac{1}{4}$.

Suppose that $N = 1$, then

$$M = |-\pi^2 - \arctan(1 + 32\sin\pi t)| \approx 11.41.$$

The inequality (4.22) does not hold since

$$\Gamma \cdot M \approx 0.25 * 11.41 \approx 2.85 > 1 = N.$$

The inequality

$$\frac{1}{4}\left|-\pi^2 N - \arctan(N + 160)\right| \le N$$

is not satisfied. Using this method we cannot prove that the boundary value problem (4.24) is solvable.

Example 6. $g(x) = -\sin x$. Consider the boundary value problem

$$x'' = -\pi^2 x - \sin x, \quad x(0) = x(1) = 0. \tag{4.27}$$

Denote $u(t) = 4\sin\pi t$. Introduce the new variable $y = x - u(t)$. Therefore $x = y + u(t)$.

Analogously as in the previous example we define

$$\begin{aligned} f_1(t,x) &= g(t,x) - u''(t) = -\pi^2 x - \sin x + 4\pi^2 \sin\pi t, \\ F_1(t,y) &= f_1(t, u(t) + y) \\ &= -\pi^2(y + 4\sin\pi t) - \sin(y + 4\sin\pi t) + 4\pi^2 \sin\pi t \\ &= -\pi^2 y - \sin(y + 4\sin\pi t), \\ F(t,y) &= F_1(t, \delta(-N, y, N)) \\ &= -\pi^2 \delta(-N, y, N) - \sin(\delta(-N, y, N) + 4\sin\pi t), \end{aligned}$$

where $\delta(-N, y, N)$ is the "cut off" function.

Notice that

$$\max\{|F(t,y)| : 0 \le t \le 1, |y| \le N\} = M(N).$$

Consider the homogeneous boundary value problem (4.25) with the Green's function (4.26).

The estimate of Green's function is $\Gamma = \dfrac{1}{4}$.

Suppose that $N = 0.1$, then

$$M = |-0.1\pi^2 - \sin(0.1 + 4\sin\pi t)| \approx 0.17.$$

The inequality

$$\Gamma \cdot M \approx 0.25 * 0.17 \approx 0.04 < 0.1 = N$$

holds, therefore the resonant boundary value problem (4.27) has a solution.

The inequality

$$\frac{1}{4}|-\pi^2 N - \sin(N+4)| \le N$$

is valid, if $N \le 0.14$, then the boundary value problem (4.27) has a solution $|x(t) - u(t)| \le N$.

Example 7. $g(x) = -(e^x - 1)$.

Consider the boundary value problem

$$x'' = -\pi^2 x - (e^x - 1),\ x(0) = x(1) = 0. \tag{4.28}$$

Denote $u(t) = 32000\sin\pi t$. Introduce the new variable $y = x - u(t)$. Then $x = y + u(t)$.

Analogously we define

$$\begin{aligned}
f_1(t,x) &= g(t,x) - u''(t) = -\pi^2 x - (e^x - 1) - 32000\pi^2 \sin\pi t,\\
F_1(t,y) &= f_1(t, u(t) + y)\\
&= -\pi^2(y - 32000\sin\pi t) - (e^{y - 32000\sin\pi t} - 1) - 32000\pi^2 \sin\pi t\\
&= -\pi^2 y - (e^{y - 32000\sin\pi t} - 1),\\
F(t,y) &= F_1(t, \delta(-N, y, N))\\
&= -\pi^2 \delta(-N, y, N) - (e^{\delta(-N,y,N) - 32000\sin\pi t} - 1),
\end{aligned}$$

where $\delta(-N, y, N)$ is the "cut off" function.

Notice that

$$\max\{|F(t,y)| : 0 \le t \le 1, |y| \le N\} = M(N).$$

Consider the homogeneous boundary value problem (4.25) with the Green's function (4.26).

The estimate of the Green's function is $\Gamma = \dfrac{1}{4}$.

Suppose that $N = 0.15$, then

$$M = |-0.15\pi^2 - (e^{0.15 - 32000\sin\pi t} - 1)| \approx 0.48.$$

The inequality

$$\Gamma \cdot M \approx 0.25 * 0.48 \approx 0.12 < 0.15 = N$$

holds, therefore the resonant boundary value problem is solvable.

The inequality

$$\frac{1}{4}|-\pi^2 N-(e^{N-32000}-1)|\le N$$

is valid, if $N\le 0.17$, then the boundary value problem (4.28) has a solution $|x(t)-u(t)|\le N$.

Finally, consider example concerning equations with the linear part depending on the derivative x'. Namely, look at the problem

$$x''+2x'+(\pi^2+1)x=\varphi(x,x'),\ \ x(0)=0,\ \ x(1)=0, \tag{4.29}$$

where $\varphi(x,x')$ will be defined later.

This problem is resonant since the homogeneous boundary value problem

$$x''+2x'+(\pi^2+1)x=0,\ \ x(0)=0,\ \ x(1)=0 \tag{4.30}$$

has the nontrivial solution $x_0=e^{-t}\sin\pi t$.

Consider the modified problem

$$x''+2x'+(1+\pi^2)x+\pi x-2\pi x'=\varphi(x,x')+\pi x-2\pi x',$$

where the linear part is no more resonant with respect to the boundary conditions $x(0)=x(1)=0$. We will need the Green's function

$$G(t,s)=\begin{cases}\dfrac{e^{(1-\pi)(s-t)}\sin(\sqrt{3\pi}s)\sin\big(\sqrt{3\pi}(t-1)\big)}{\sqrt{3\pi}\sin\sqrt{3\pi}}, & 0\le s\le t\le 1,\\[2ex] \dfrac{e^{(1-\pi)(s-t)}\sin\big(\sqrt{3\pi}(s-1)\big)\sin(\sqrt{3\pi}t)}{\sqrt{3\pi}\sin\sqrt{3\pi}}, & 0<t<s\le 1.\end{cases}$$

and

$$\frac{\partial G(t,s)}{\partial t}=\begin{cases}\dfrac{e^{(1-\pi)(s-t)}\sin(\sqrt{3\pi}s)\Big((\pi-1)\sin\big(\sqrt{3\pi}(t-1)\big)+\sqrt{3\pi}\cos\big(\sqrt{3\pi}(t-1)\big)\Big)}{\sqrt{3\pi}\sin\sqrt{3\pi}},\\[2ex] \dfrac{e^{(1-\pi)(s-t)}\sin\big(\sqrt{3\pi}(s-1)\big)\Big((\pi-1)\sin(\sqrt{3\pi}t)+\sqrt{3\pi}\cos\sqrt{3\pi}t\Big)}{\sqrt{3\pi}\sin\sqrt{3\pi}}.\end{cases}$$

Computations accomplished by Wolfram Mathematica shows that

$$\big|G(t,s)\big|<7\ \ \text{and}\ \ \left|\frac{\partial G}{\partial t}(t,s)\right|<50. \tag{4.31}$$

Let $\Gamma=7$ and $\Gamma_1=50$. We wish to define $N\le 0$ and $N_1\le 0$ that

$$\begin{aligned}&\Gamma\cdot M\le N,\\ &\Gamma_1\cdot M\le N_1,\end{aligned} \tag{4.32}$$

where

$$M=\max_{|x|\le N,|x'|\le N_1}\Big\{\big|\varphi(x,x')+\pi x-2\pi x'\big|\Big\}.$$

Let $N = 7$, $N_1 = 50$. Let $\varphi(x, x')$ be a function such that

$$M = \max_{|x|\le 7, |x'|\le 50} \left\{ \left| \varphi(x, x') + \pi x - 2\pi x' \right| \right\}$$

satisfies (4.32). Then the problem

$$x'' + (2-2\pi)x' + (\pi^2 + \pi + 1)x = \varphi\big(\delta(-7, x, 7), \delta(-50, x', 50)\big) + \pi\delta(-7, x, 7) \\ -2\pi\delta(-50, x', 50), \quad x(0) = x(1) = 0 \quad (4.33)$$

has a solution $x(t)$ such that

$$\left| x(t) \right| = \left| \int_o^1 G(t,s) F\big(s, x(s), x'(s)\big) ds \right| \le \Gamma \cdot M \le N,$$
$$\left| x'(t) \right| = \left| \int_o^1 \frac{\partial G(t,s)}{\partial t} F\big(s, x(s), x'(s)\big) ds \right| \le \Gamma_1 \cdot M \le N_1,$$

where F is the right side of the equation in (4.33), and $x(t)$ is therefore a solution of (4.29). Any continuous function $\varphi(x, x')$ such that

$$M = \max_{|x|\le N, |x'|\le N_1} \left\{ \left| \varphi(x, x') + \pi x - 2\pi x' \right| \right\}$$

satisfy

$$M \le \frac{N}{\Gamma} = 1 \quad \text{and} \quad M \le \frac{N_1}{\Gamma_1} = 1,$$

is suitable for (4.29).

5. Conclusion

If the second order BVP is known to have a solution of type i, then the problem can be reduced to a quasi-linear problem "around" a solution, irrespectively of either the original problem is resonant or not.

If the second order formally resonant BVP has not a solution of indefinite type then either it has not a solution at all or it can be reduced to quasi-linear problem.

It is reasonable to combine the "shift" type arguments with multiple different (with different linear parts) quasi-linearizations of a given problem since multiple solutions can be obtained.

Using the "shift" arguments when making quasi-linearization of a resonant problem it is advisable to approximate the nonlinearity by its Taylor approximation.

If an approximation of the expected solution is known then the process of quasi-linearization can be made easier and more effective.

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NULL CONTROLLABILITY OF NONLOCAL HILFER FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study exact null controllability of Hilfer fractional semilinear stochastic differential equations in Hilbert spaces. By using fractional calculus and fixed point approach, sufficient conditions of exact null controllability for such fractional systems are established. An example is given to show the application of our results.

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1. Introduction

The stochastic differential equations arise in many mathematical models [5, 14, 18, 20]. The problem of controllability of nonlinear stochastic or deterministic system has been discussed in [3, 4, 6, 8, 9, 16, 19].

Recently, basic theory of differential equations involving Caputo and Riemann-Liouville fractional derivatives can be found in [1, 2, 13, 21–24, 26–30] and the references cited therein. Beside Caputo and Riemann-Liouville fractional derivatives, there exists a new definition of fractional derivative introduced by Hilfer, which generalized the concept of Riemann-Liouville derivative and has many application in physics, for more details, see [10–12, 25].

In this paper, we investigate the exact null controllability of Hilfer fractional semilinear stochastic differential equation of the form

$$\begin{cases} D_{0+}^{\nu,\mu}x(t) = Ax(t) + Bu(t) \\ +F(t,x(t)) + G(t,x(t))\frac{d\omega(t)}{dt}, \quad t \in J = [0,b], \\ I_{0+}^{(1-\nu)(1-\mu)}x(0) + h(x) = x_0, \end{cases} \tag{1.1}$$

where $D_{0+}^{\nu,\mu}$ is the Hilfer fractional derivative, $0 \le \nu \le 1$, $\frac{1}{2} < \mu < 1$, A is the infinitesimal generator of strongly continuous semigroup of bounded linear operators

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$S(t)$, $t \geq 0$, on a separable Hilbert space H with inner product $\langle .,. \rangle$ and norm $\| \, . \, \|$. There exists a $M \geq 1$ such that $\sup_{t \geq 0} \|S(t)\| \leq M$. The control function $u(\cdot)$ is given in $L_2(J,U)$, the Hilbert space of admissible control functions with U as a separable Hilbert space. The symbol B stands for a bounded linear operator from U into H. Here ω is an H-valued Wiener process associated with a positive, nuclear covariance operator Q, F is an H-valued map and G is a $L(K,H)$-valued map both defined on $J \times H$ (where K is a real separable Hilbert space with norn $\|\cdot\|_K$ and $L(K,H)$ is the space of all bounded, linear operators from K to H, we write simply $L(H)$ if $H = K$) and $h : C(J,H) \to H$.

2. Preliminaries

In this section, some definitions and results are given which will be used throughout this paper.

Definition 1 (see [15, 17])**.** The fractional integral operator of order $\mu > 0$ for a function f can be defined as

$$I^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(s)}{(t-s)^{1-\mu}} ds, \;\; t > 0$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2 (see [11])**.** The Hilfer fractional derivative of order $0 \leq \nu \leq 1$ and $0 < \mu < 1$ for a function f is defined by

$$D_{0+}^{\nu,\mu} f(t) = I_{0+}^{\nu(1-\mu)} \frac{d}{dt} I_{0+}^{(1-\nu)(1-\mu)} f(t).$$

Let (Ω, Υ, P) be a complete probability space furnished with complete family of right continuous increasing sub σ-algebras $\{\Upsilon_t : t \in J\}$ satisfying $\Upsilon_t \subset \Upsilon$. An H-valued random variable is an Υ- measurable function $x(t) : \Omega \to H$ and a collection of random variables $\Psi = \{x(t,\omega) : \Omega \to H | t \in J\}$ is called a stochastic process. Usually we suppress the dependence on $\omega \in \Omega$ and write $x(t)$ instead of $x(t,\omega)$ and $x(t) : J \to H$ in the place of Ψ. Let $\beta_n(t)$ $(n = 1,2,...)$ be a sequence of real valued one-dimensional standard Brownian motions mutually independent over (Ω, Υ, P). Set

$$\omega(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n(t) e_n, \; t \geq 0,$$

where λ_n, $(n = 1,2,...)$ are nonnegative real numbers and $\{e_n\}$ $(n = 1,2,...)$ is a complete orthonormal basis in K. Let $Q \in L(K,K)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite $Tr(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$, (Tr denotes the trace of the operator). Then the above K-valued stochastic process $\omega(t)$ is called Q-Wiener process.

We assume that $\Upsilon_t = \sigma\{\omega(s) : 0 \leq s \leq t\}$ is the σ-algebra generated by ω.

For $\phi \in L(K,H)$ we define

$$\| \phi \|_Q^2 = Tr(\phi Q \phi^*) = \sum_{n=1}^{\infty} \| \sqrt{\lambda_n} \phi e_n \|^2 .$$

If $\| \phi \|_Q^2 < \infty$, then ϕ is called a Q-Hilbert-Schmidt operator. Let $L_Q(K,H)$ denote the space of all Q-Hilbert-Schmidt operators $\phi : K \to H$. The completion $L_Q(K,H)$ of $L(K,H)$ with respect to the topology induced by the norm $\| \, . \, \|_Q$ where $\| \phi \|_Q^2 = \|\phi, \phi\|$ is a Hilbert space with the above norm topology. The collection of all strongly-measurable, square-integrable, H-valued random variables, denoted by $L_2(\Omega, H)$, is a Banach space equipped with norm $\| x(\cdot) \|_{L_2(\Omega,H)} = (E \| x(.,\omega) \|^2)^{\frac{1}{2}}$, where the expectation, E is defined by $E(x) = \int_\Omega x(\omega) dP$. An important subspace of $L_2(\Omega, H)$ is given by $L_2^0(\Omega, H) = \{x \in L_2(\Omega, H), x$ is Υ_0-measurable $\}$.

Let $C(J, L_2(\Omega, H))$ be the Banach space of all continuous maps from J into $L_2(\Omega, H)$ satisfying the condition $\sup_{t \in J} E\|x(t)\|^2 < \infty$.

Define $Y = \{x : t^{(1-\nu)(1-\mu)} x(t) \in C(J, L_2(\Omega, H))\}$, with norm $\|\cdot\|_Y$ defined by $\|\cdot\|_Y = (\sup_{t \in J} E\|t^{(1-\nu)(1-\mu)} x(t)\|^2)^{\frac{1}{2}}$. Obviously, Y is a Banach space.

For $x \in H$, we define two families of operators $\{S_{\nu,\mu}(t) : t \geq 0\}$ and $\{P_\mu(t) : t \geq 0\}$ by

$$S_{\nu,\mu}(t) = I_{0+}^{\nu(1-\mu)} P_\mu(t), \; P_\mu(t) = t^{\mu-1} T_\mu(t), \; T_\mu(t) = \int_0^\infty \mu\theta \Psi_\mu(\theta) S(t^\mu \theta) d\theta,$$

where

$$\Psi_\mu(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)! \Gamma(1-n\mu)}, \; 0 < \mu < 1, \; \theta \in (0, \infty)$$

is a function of Wright-type which satisfies

$$\int_0^\infty \theta^\Psi \Psi_\mu(\theta) d\theta = \frac{\Gamma(1+\Psi)}{\Gamma(1+\mu\Psi)}$$

for $\theta \geq 0$.

Lemma 1 (see [10]). *The operator $S_{\nu,\mu}$ and P_μ have the following properties.*
(i) $\{P_\mu(t) : t > 0\}$ is continuous in the uniform operator topology.
(ii) For any fixed $t > 0$, $S_{\nu,\mu}(t)$ and $P_\mu(t)$ are linear and bounded operators, and

$$\|P_\mu(t)x\| \leq \frac{M t^{\mu-1}}{\Gamma(\mu)} \|x\|, \;\; \|S_{\nu,\mu}(t)x\| \leq \frac{M t^{(\nu-1)(\mu-1)}}{\Gamma(\nu(1-\mu)+\mu)} \|x\|.$$

(iii) $\{P_\mu(t) : t > 0\}$ and $\{S_{\nu,\mu}(t) : t > 0\}$ are strongly continuous.

To study the exact null controllability of (1.1) we consider the fractional linear system

$$\begin{cases} D_{0+}^{\nu,\mu} y(t) = Ay(t) + Bu(t) + F(t) + G(t)\frac{d\omega(t)}{dt}, \quad t \in J = [0,b], \\ I_{0+}^{(1-\nu)(1-\mu)} y(0) = y_0, \end{cases} \tag{2.1}$$

associated with the system (1.1).

Define the operator

$$L_0^b u = \int_0^b P_\mu(b-s)Bu(s)ds \ : L_2(J,U) \to H,$$

where $L_0^b u$ has a bounded inverse operator $(L_0)^{-1}$ with values in $L_2(J,U)/ker(L_0^b)$, and

$$\begin{aligned} N_0^b(y,F,G) = S_{\nu,\mu}(b)y + \int_0^b P_\mu(b-s)F(s)ds \\ + \int_0^b P_\mu(b-s)G(s)d\omega(s) : H \times L_2(J,U) \to H. \end{aligned}$$

Definition 3. The system (2.1) is said to be exactly null controllable on J if

$$\text{Im } L_0^b \supset \text{Im } N_0^b.$$

By [7], the system (2.1) is exactly null controllable if there exists $\gamma > 0$ such that

$$\|(L_0^b)^* y\|^2 \geq \gamma \|(N_0^b)^* y\|^2$$

for all $y \in H$.

Lemma 2 (see [16])**.** *Suppose that the linear system* (2.1) *is exactly null controllable on J. Then the linear operator*

$$W = (L_0)^{-1} N_0^b : H \times L_2(J,H) \to L_2(J,U)$$

is bounded and the control

$$\begin{aligned} u(t) &= -(L_0)^{-1}\left[S_{\nu,\mu}(b)y_0 + \int_0^b P_\mu(b-s)F(s)ds + \int_0^b P_\mu(b-s)G(s)d\omega(s)\right] \\ &= -W(y_0,F,G) \end{aligned}$$

transfers the system (2.1) *from y_0 to* 0, *where L_0 is the restriction of L_0^b to* $[\ker L_0^b]^\perp$, *$F \in L_2(J,H)$ and $G \in L_2(J,L(K,H))$.*

3. Exact null controllability

In this section, we formulate sufficient conditions for exact null controllability for the system (1.1). First, we give the definitions of mild solution and exact null controllability for it.

Definition 4. We say $x \in C(J, L_2(\Omega, H))$ is a mild solution to (1.1) if it satisfies that

$$x(t) = S_{\nu,\mu}(t)[x_0 - h(0)] + \int_0^t P_\mu(t-s)[F(s,x(s)) + Bu(s)]ds + \int_0^t P_\mu(t-s)G(s,x(s))d\omega(s),\ t \in J.$$

Definition 5. The system (1.1) is said to be exact null controllable on the interval J if there exists a stochastic control $u \in L_2(J,U)$ such that the solution x of the system (1.1) satisfies $x(b) = 0$.

To prove the main result, we need the following hypotheses:

$(H1)$ The fractional linear system (2.1) is exactly null controllable on J.

$(H2)$ The function $F : J \times H \to H$ is locally Lipschitz continuous, for all $t \in J,\ x,\ x_1,\ x_2 \in H$, there exist constant $c_1 > 0$, such that

$$\|F(t,x_2) - F(t,x_1)\|^2 \le c_1\|x_2 - x_1\|^2, \quad \|F(t,x)\|^2 \le c_1(1 + \|x\|^2).$$

$(H3)$ The function $G : J \times H \to L(K,H)$ is locally Lipschitz continuous, for all $t \in J,\ x,\ x_1,\ x_2 \in H$, there exist constant $c_2 > 0$, such that

$$\|G(t,x_2) - G(t,x_1)\|_Q^2 \le c_2\|x_2 - x_1\|^2, \quad \|G(t,x)\|_Q^2 \le c_2(1 + \|x\|^2).$$

$(H4)$ The function $h : C(J,H) \to H$ is continuous, for any $x,\ x_1,\ x_2 \in C(J,H)$, there exist constant $c_3 > 0$, such that

$$\|h(x_2) - h(x_1)\|^2 \le c_3\|x_2 - x_1\|^2, \quad \|h(x)\|^2 \le c_3(1 + \|x\|^2).$$

Set $\varrho_1 := \frac{4M^2c_3}{\Gamma^2(\nu(1-\mu)+\mu)} + \frac{M^2b^{1+2\nu(\mu-1)}}{(2\mu-1)\Gamma^2(\mu)}(c_1 + c_2Tr(Q))$ and $\varrho_2 := 1 + \frac{4M^2b^{2\mu-1}\|W\|^2\|B\|^2}{(2\mu-1)\Gamma^2(\mu)}$.

Theorem 1. *If the hypotheses $(H1)$-$(H4)$ are satisfied, then the system (1.1) is exactly null controllable on J provided that*

$$\varrho := \varrho_1\varrho_2 < 1. \tag{3.1}$$

Proof. For an arbitrary x define the operator Φ on Y as follows

$$(\Phi x)(t) = S_{\nu,\mu}(t)[x_0 - h(x)] + \int_0^t P_\mu(t-s)[F(s,x(s)) - BW(x_0 - h(x), F, G)]ds \tag{3.2}$$

$$+\int_0^t P_\mu(t-s)G(s,x(s))d\omega(s), \quad t\in J,$$

where

$$\begin{aligned}
u(t) &= W(x_0-h(x),F,G)(t)\\
&= -(L_0)^{-1}\{S_{\nu,\mu}(b)[x_0-h(x)]\\
&+\int_0^b P_\mu(b-s)F(s,x(s))ds+\int_0^b P_\mu(b-s)G(s,x(s))d\omega(s)\}.
\end{aligned}$$

It will be shown that the operator Φ from Y into itself has a fixed point.
Step 1. The control $u(\cdot)=-W(x_0-h(x),F,G)$ is bounded on Y.
Indeed,

$$\begin{aligned}
\|u\|_Y^2 &= \sup_{t\in J} t^{2(1-\nu)(1-\mu)}E\|u\|^2 \qquad (3.3)\\
&\le \sup_{t\in J} t^{2(1-\nu)(1-\mu)}E\|W(x_0-h(x),F,G)(s)\|^2\\
&\le \|W\|^2\Bigg\{\frac{M^2}{\Gamma^2(\nu(1-\mu)+\mu)}[E\|x_0\|^2+c_3(1+E\|x\|^2)]\\
&+\frac{M^2b^{1+2\nu(\mu-1)}}{(2\mu-1)\Gamma^2(\mu)}(1+E\|x\|^2)(c_1+c_2Tr(Q))\Bigg\}.
\end{aligned}$$

Step 2. We show that Φ maps Y into itself.
From (3.2) and (3.3) for $t\in J$, we have

$$\begin{aligned}
\|(\Phi x)(t)\|_Y^2 &= \sup_{t\in J} t^{2(1-\nu)(1-\mu)}E\|(\Phi x)(t)\|^2\\
&\le 4\sup_{t\in J} t^{2(1-\nu)(1-\mu)}\Bigg\{E\|S_{\nu,\mu}(t)[x_0-h(x)]\|^2\\
&+E\left\|\int_0^t P_\mu(t-s)F(s,x(s))ds\right\|^2\Bigg\}\\
&+4\sup_{t\in J} t^{2(1-\nu)(1-\mu)}\\
&\times\Bigg\{E\left\|\int_0^t P_\mu(t-s)[BW(x_0-h(x),F,G)(s)]ds\right\|^2\\
&+E\left\|\int_0^t P_\mu(t-s)G(s,x(s))d\omega(s)\right\|^2\Bigg\}\\
&\le\Bigg[\frac{4M^2}{\Gamma^2(\nu(1-\mu)+\mu)}[E\|x_0\|^2+c_3(1+E\|x\|^2)]
\end{aligned}$$

$$+\frac{4M^2b^{1+2\nu(\mu-1)}}{(2\mu-1)\Gamma^2(\mu)}(1+E\|x\|^2)(c_1+c_2Tr(Q))\Bigg]$$
$$\times\left[1+\frac{M^2\|B\|^2\|W\|^2\|b^{2\mu-1}}{(2\mu-1)\Gamma^2(\mu)}\right]<\infty.$$

Therefore Φ maps Y into itself.

Step 3. We prove $(\Phi x)(t)$ is continuous on J for any $x\in Y$.

Let $0<t\le b$ and $\epsilon>0$ be sufficiently small, then,

$$\begin{aligned}
&\|(\Phi x)(t+\epsilon)-(\Phi x)(t)\|_Y^2 \qquad (3.4)\\
&=\sup_{t\in J} t^{2(1-\nu)(1-\mu)}E\|(\Phi x)(t+\epsilon)-(\Phi x)(t)\|^2\\
&\le 4\sup_{t\in J} t^{2(1-\nu)(1-\mu)}\\
&\times E\|(S_{\nu,\mu}(t+\epsilon)-S_{\nu,\mu}(t))[x_0-h(x)]\|^2\\
&+4\sup_{t\in J} t^{2(1-\nu)(1-\mu)}\\
&\times E\left\|\int_0^{t+\epsilon}P_\mu(t+\epsilon-s)[BW(x_0-h(x),F,G)(s)]ds\right.\\
&\left.-\int_0^t P_\mu(t-s)[BW(x_0-h(x),F,G)(s)]ds\right\|^2\\
&+4\sup_{t\in J} t^{2(1-\nu)(1-\mu)}\\
&\times E\left\|\int_0^{t+\epsilon}P_\mu(t+\epsilon-s)F(s,x(s))ds-\int_0^t P_\mu(t-s)F(s,x(s))ds\right\|^2\\
&+4\sup_{t\in J} t^{2(1-\nu)(1-\mu)}\\
&\times E\left\|\int_0^{t+\epsilon}P_\mu(t+\epsilon-s)G(s,x(s))d\omega(s)-\int_0^t P_\mu(t-s)G(s,x(s))d\omega(s)\right\|^2.
\end{aligned}$$

Clearly, from Lemma 1, $(H2)$ and $(H3)$, the right hand side of (3.4) tends to zero as $\epsilon\to 0$. Hence, $(\Phi x)(t)$ is continuous on J.

Step 4. We show that $(\Phi x)(t)$ is a contraction on Y.

Let $x_1,\ x_2\in Y$, for any $t\in(0,b]$ be fixed, then

$$\begin{aligned}
&\|(\Phi x_2)(t)-(\Phi x_1)(t)\|_Y^2\\
&=\sup_{t\in J} t^{2(1-\nu)(1-\mu)}E\|(\Phi x_2)(t)-(\Phi x_1)(t)\|^2\\
&\le 4\sup_{t\in J} t^{2(1-\nu)(1-\mu)}E\|(S_{\nu,\mu}(t)[h(x_2)-h(x_1)]\|^2
\end{aligned}$$

$$
\begin{aligned}
&+4\sup_{t\in J} t^{2(1-\nu)(1-\mu)} E\left\| \int_0^t P_\mu(t-s)[BW(x_0-h(x_2),F,G)(s) \right. \\
&\left. -BW(x_0-h(x_1),F,G)(s)]ds \right\|^2 \\
&+4\sup_{t\in J} t^{2(1-\nu)(1-\mu)} E\left\| \int_0^t P_\mu(t-s)[F(s,x_2(s))-F(s,x_1(s))]ds \right\|^2 \\
&+4\sup_{t\in J} t^{2(1-\nu)(1-\mu)} E\left\| \int_0^t P_\mu(t-s)[G(s,x_2(s))-G(s,x_1(s))]d\omega(s) \right\|^2 \\
&\le \varrho E\|x_2-x_1\|^2.
\end{aligned}
$$

Hence, Φ is a contraction in Y via (3.1). From the Banach fixed point theorem, Φ has a unique fixed point. Therefore the system (1.1) is exact null controllable on J. □

4. An example

Consider the following Hilfer fractional stochastic partial differential system

$$
\begin{cases}
D_{0+}^{\nu,\frac{2}{3}} x(t,z) = \frac{\partial^2}{\partial z^2} x(t,z) + u(t,z) \\
+ f(t,x(t,z)) + g(t,x(t,z)) \frac{d\omega(t)}{dt}, \ \ t\in J,\ 0<z<1, \\
x(t,0) = x(t,1) = 0,\ t\in J, \\
I_{0+}^{\frac{1}{3}(1-\nu)}(x(0,z)) + \sum_{i=1}^{p} k_i x(t_i,z) = x_0(z), \ \ 0\le z\le 1,
\end{cases}
\tag{4.1}
$$

where p is a positive integer, $0<t_0<t_1<\ldots<t_p<b$ and $\omega(t)$ is Wiener process, $u\in L_2(0,b)$, and $H=L_2([0,1])$. Let $f: R\times R\to R$ and $g: R\times R\to R$ are continuous and global Lipschitz continuous in the second variable. Also, let $A: H\to H$ be defined by $Ay=\frac{\partial^2}{\partial z^2}y$ with domain $D(A)=\{y\in H: y,\ \frac{\partial y}{\partial z}$ are absolutely continuous, and $\frac{\partial^2 y}{\partial z^2}\in H, y(0)=y(1)=0\}$.

It is known that A is self-adjoin and has the eigenvalues $\lambda_n=-n^2\pi^2$, $n\in N$, with the corresponding normalized eigenvectors $e_n(z)=\sqrt{2}\ \sin(n\pi z)$. Furthermore, A generates an analytic compact semigroup of bounded linear operator $S(t),\ t\ge 0$, on a separable Hilbert space H which is given by

$$
S(t)y=\sum_{n=1}^{\infty}(y_n,e_n)e_n=\sum_{n=1}^{\infty}2e^{-n^2\pi^2 t}\ \sin(n\pi z)\int_0^1 \sin(n\pi\xi)y(\xi)d\xi,\ y\in H.
$$

If $u\in L_2(J,H)$, then $B=I,\ B^*=I$.

Now we consider

$$\begin{cases} D_{0+}^{\nu,\frac{2}{3}} y(t,z) = \frac{\partial^2}{\partial z^2} y(t,z) + u(t,z) \\ + f(t,z) + g(t,z) d\omega(t), \ \ t \in J, \ 0 < z < 1, \\ y(t,0) = y(t,1) = 0, \ t \in J, \\ I_{0+}^{\frac{1}{3}(1-\nu)}(y(0,z)) = y_0(z), \ \ 0 \le z \le 1, \end{cases} \tag{4.2}$$

The system (4.2) is exact null controllability if there is a $\gamma > 0$, such that

$$\int_0^b \|B^* P_\mu^*(b-s)y\|^2 ds \ge \gamma \left[\|S_{\mu,\nu}^*(b)y\|^2 + \int_0^b \|P_\mu^*(b-s)y\|^2 ds \right],$$

or equivalently

$$\int_0^b \|P_\mu(b-s)y\|^2 ds \ge \gamma \left[\|S_{\mu,\nu}(b)y\|^2 + \int_0^b \|P_\mu(b-s)y\|^2 ds \right].$$

If $f = 0$ and $g = 0$ in (4.2), then the fractional linear system is exactly null controllable if

$$\int_0^b \|P_\mu(b-s)y\|^2 ds \ge b\|S_{\mu,\nu}(b)y\|^2.$$

Therefore,

$$\int_0^b \|P_\mu(b-s)y\|^2 ds \ge \frac{b}{1+b} \left[\|S_{\mu,\nu}(b)y\|^2 + \int_0^b \|P_\mu(b-s)y\|^2 ds \right].$$

Hence, the linear fractional system (4.2) is exactly null controllable on $[0,b]$. So the hypothesis (H_1) is satisfied.

We define $F : J \times H \to H$, $G : J \times H \to L(K,H)$ and $h : C(J,H) \to H$ as follows: $F(t,x) = f(t,x(t,z))$, $G(t,x) = g(t,x(t,z))$ and $h(x) = \sum_{i=1}^{p} k_i x(t_i,z)$ Then $h(\cdot)$ satisfies $(H4)$.

By choosing the constants k_i, $i = 1,2,...,p, M, c_1$, c_2 and c_3 such that $\varrho < 1$. Hence, all the hypotheses of Theorem 1 are satisfied, so the Hilfer fractional stochastic partial differential system (4.1) is exact null controllable on $[0,b]$.

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