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CONTENTS

<i>M. Abbas, Y.I. Suleiman, and C. Vetro</i> A simulation function approach for best proximity point and variational inequality problems	3
<i>A. Abbasi and A. Ramin</i> An extension of total graph over a module	17
<i>Ghasem A. Afrouzi, John R. Graef, and Saeid Shokooh</i> Multiple solutions for Neumann systems in an Orlicz-Sobolev space setting	31
<i>Gholamreza Aghamollaei and Mohammad Ali Nourollahi</i> Some results on the joint higher numerical ranges and radii of matrices . .	47
<i>Muhammad Usman Ali, Fahimuddin, Tayyab Kamran, and Erdal Karapinar</i> Fixed point theorems in uniform space endowed with graph	57
<i>Margareta-Eliza Balazs</i> A Maia type fixed point theorem for Prešić-Kannan operators	71
<i>Sh. Banaei, M. B. Ghaemi, and R. Saadati</i> An extension of Darbo's theorem and its application to system of neutral differential equations with deviating argument	83
<i>Rovshan A. Bandaliyev</i> Compactness criteria in weighted variable Lebesgue spaces	95
<i>S. Bazine, A. Aliouche, and F. Ellaggoune</i> Common fixed point theorems on spaces with vector-valued b -metrics . .	103
<i>V. E. Berezovski, S. Bácsó, and J. Mikeš</i> Diffeomorphism of affine connected spaces which preserved Riemannian and Ricci curvature tensors	117



CONTENTS

<i>Barkat Ali Bhayo, Riku Klén, and József Sándor</i>	
New trigonometric and hyperbolic inequalities	125
<i>M. Bombardelli, L. Nikolova, and S. Varošanec</i>	
Unified treatment of fractional integral inequalities via linear functionals	139
<i>T. P. Calci, S. Halicioglu, and A. Harmanci</i>	
A Generalization of J -Quasipolar Rings	155
<i>Miguel Couceiro and Gerasimos C. Meletiou</i>	
On a special class of median algebras	167
<i>Ayhan Dil and Mirac Cetin Firengiz</i>	
Shifted Euler-Seidel matrices	173
<i>Melánia-Iulia Dobrican</i>	
Coupled and tripled fixed point theorems on a metric space endowed with a binary relation	189
<i>Radim Hošek</i>	
Walks in path graph on four vertices and Fibonacci sequence	199
<i>Yun Hua</i>	
Bounds for the arithmetic mean in terms of the toader mean and other bi- variate means	203
<i>Gürhan İçöz and Bayram Çekim</i>	
q -analogue of Mittag-Leffler operators	211
<i>Levent Kargin and Veli Kurt</i>	
On generalized two-index Hermite matrix polynomials	223
<i>Refik Keskin and Merve Güney Duman</i>	
A note on some Diophantine equations	235



CONTENTS

<i>E. Kılıç and H. Prodinger</i>	
Closed form evaluation of Melham's reciprocal sums	251
<i>Emine Koç and Öznur Gölbaşı</i>	
Multiplicative generalized derivations on Lie ideals in semiprime rings II .	265
<i>Bülent Köklüce and Müberra Gürel</i>	
Representation numbers by sums of quadratic forms $x_1^2 + x_1x_2 + x_2^2$ in sixteen variables	277
<i>Dorđe Krstinić and Marija Mikić</i>	
Note on asymptotical behavior of solutions of Emden-Fowler equation and the existence and uniqueness of solution of some Cauchy problem .	285
<i>A. R. Fiuj Laali and H. Haj Seyyed Javadi</i>	
Spectra of some special bipartite graphs	295
<i>Diffalah Laissaoui, Fouad Bounebirat, and Mourad Rahmani</i>	
On the hyper-sums of powers of integers	307
<i>G. M. Mahmoud and M. E. Ahmed</i>	
Analysis of chaotic and hyperchaotic conservative complex nonlinear systems	315
<i>I. Martinjak and I. Vrsaljko</i>	
On polynomial identities for recursive sequences	327
<i>Mohammad Vahdani Mehrabadi, Shervin Sahebi, and Hamid H. S. Javadi</i>	
On a generalization of NC-McCoy rings	337
<i>Dijana Mosić</i>	
On the Moore–Penrose inverse in rings with involution	347
<i>M. Nunokawa, H. M. Srivastava, N. Tuneski, and B. Jolevska-Tuneska</i>	
Some Marx-Strohhäcker type results for a class of multivalent functions .	353



CONTENTS

Süleyman Öğrekçi, Adil Mısıır, and Aydın Tiryaki

On the oscillation of second-order nonlinear differential equations with
damping 365

M.A. Özarıslan, R. Srivastava, and C. Kaanođlu

Certain families of multivariable Chan-Chyan-Srivastava polynomials . . . 379

H. N. Özgen

On equivalence of two integrability methods 391

Páll-Szabó Ágnes Orsolya

Modified Hadamard product properties of certain class of analytic func-
tions with varying arguments defined by Ruscheweyh derivative . . . 397

Ákos Pintér and Csaba Rakaczki

On the decomposability of linear combinations of Euler polynomials . . . 407

Ravinder Krishna Raina and Janusz Sokół

Some coefficient properties relating to a certain class of starlike functions 417

Shafiq Ur Rehman

A note on a characterization theorem for a certain class of domains 427

A. Rontó, M. Rontó, and N. Shchobak

On boundary value problems with prescribed number of zeroes of solutions 431

B. Şahin

Generic Riemannian maps 453

Elif Tan and A. Bulent Ekin

Some identities on conditional sequences by using matrix method 469

Jitka Vacková



CONTENTS

On the solvability of boundary value problems for some nonlinear systems of ordinary differential equations on $\mathbb{R}_+$	479
<i>Hasret Yazarli</i> Permuting triderivations of prime and semiprime rings	489
<i>Monika Zakany</i> Fixed point theorems for local almost contractions	499
<i>Chang-Jian Zhao and Mihály Bencze</i> Brunn-Minkowski inequality for L_p -mixed intersection bodies	507
<i>Hou Yu Zhao and Michal Fečkan</i> Pseudo almost periodic solutions of an iterative equation with variable co- efficients	515
<i>Milan Lj. Zlatanović, Svetislav M. Minčič, and Vladislava M. Stanković</i> New integrability conditions of derivation equations in a subspace of asym- metric affine connection space	525



A SIMULATION FUNCTION APPROACH FOR BEST PROXIMITY POINT AND VARIATIONAL INEQUALITY PROBLEMS

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Abstract. We study sufficient conditions for existence of solutions to the global optimization problem $\min_{x \in A} d(x, fx)$, where A, B are nonempty subsets of a metric space (X, d) and $f : A \rightarrow B$ belongs to the class of proximal simulative contraction mappings. Our results unify, improve and generalize various comparable results in the existing literature on this topic. As an application of the obtained theorems, we give some solvability theorems of a variational inequality problem.

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1. INTRODUCTION

Let A and B be nonempty subsets of a metric space (X, d) and $f : A \rightarrow B$ an arbitrary mapping on A . A fixed point problem defined by a pair (A, B) of sets and a mapping f is to find a point x^* in A such that $d(x^*, fx^*) = 0$. It is clear that if $A \cap B = \emptyset$, the fixed point problem defined by (A, B) and f has no solution. In this case, a point x^* in A satisfying

$$d(x^*, fx^*) \leq d(a, fx^*)$$

for every a in A is the nearest point to $fx^* \in B$ in A . Such a point is called an approximate fixed point of f .

It is a reasonable demand to settle down with such a point when an operator equation $fx = x$ does not admit a solution.

The study of conditions that assure existence and uniqueness of approximate fixed point of a mapping f is an important area of research.

Suppose that $\Delta_{AB} = d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$ is the measure of a distance between two sets A and B . A point x^* in A is called a best proximity point of f if

$$d(x^*, fx^*) = \Delta_{AB}.$$

Best proximity point results deal with sufficient conditions under which the nonlinear minimization problem

$$\min_{x \in A} d(x, fx)$$

has at least one solution. Thus a best proximity point problem defined by a mapping f and a pair (A, B) of sets is to find a point x^* in A such that

$$d(x^*, fx^*) = \Delta_{AB}.$$

If $A = B$, the best proximity point problem reduces to a fixed point problem.

In light of this, a best proximity point problem can be viewed as a natural generalization of the fixed point problem. Also, results dealing with existence and uniqueness of best proximity point of certain mappings are more general than the ones dealing with approximate fixed point problem of those mappings.

The theory of best proximity point has proved to be simple and applicable in solving real world problems in nonlinear analysis, optimization, economics, game theory, and so forth. See, for example, [10] and [13].

The purpose of this paper is to study best proximity point results of proximal simulative contraction mappings. These results extend, unify and strengthen various known results in [6, 7, 12, 17] among others. As an application of the obtained theorems, we give some solvability theorems of a variational inequality problem, see [18].

2. PRELIMINARIES

Throughout this paper, we assume that (X, d) is a metric space and (A, B) a pair of nonempty subsets of X .

Consistent with [6], the following definitions and results will be needed in the sequel. Let

$$\begin{aligned} A_0 &= \{x \in A : d(x, y) = \Delta_{AB} \text{ for some } y \in B\}, \\ B_0 &= \{y \in B : d(x, y) = \Delta_{AB} \text{ for some } x \in A\}. \end{aligned}$$

If $A \cap B \neq \emptyset$, then A_0 and B_0 are nonempty. Moreover, A_0 and B_0 are contained in the boundaries of A and B , respectively whenever A and B are closed subsets of a normed linear space such that $\Delta_{AB} > 0$.

Definition 1. A set B is said to be *approximatively compact* with respect to A if every sequence $\{y_n\}$ in B , satisfying $d(x, y_n) \rightarrow d(x, B)$ for some $x \in A$, has a convergent subsequence.

Sankar Raj [16] introduced the concept of *P-property* and obtained a best proximity point result for a class of weakly contractive mappings as an interesting generalization of Banach contraction principle.

Definition 2. Let A_0 be nonempty. The pair (A, B) is said to have the *P-property* if and only if

$$d(x_1, y_1) = d(x_2, y_2) = \Delta_{AB} \text{ implies that } d(x_1, x_2) = d(y_1, y_2)$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Example 1 ([1]). Let $X = \mathbb{R}^2$. Define the metric d on X by

$$d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

If $A := \{(x, 0) : -1 \leq x \leq 1\}$ and $B := \{(0, y) : -1 \leq y \leq 1\}$, then the pair (A, B) has the *P-property*.

Several authors studied best proximity point results for different contractive mappings employing the notion of P-property, see [8] and the references mentioned therein.

Abkar and Gabeleh [2] proved that most of best proximity point results based on P-property can be deduced from existing comparable fixed point results in the literature.

Sadiq Basha [6] modified the concept of P-property and introduced the concept of proximal contractions of first and second kinds as follows:

Definition 3. A mapping $f : A \rightarrow B$ is said to be a *proximal Banach contraction of first kind* if there exists a non-negative number $\alpha < 1$ such that for all u_1, u_2, x_1, x_2 in A ,

$$d(u_1, f x_1) = d(u_2, f x_2) = \Delta_{AB} \text{ implies that } d(u_1, u_2) \leq \alpha d(x_1, x_2).$$

Definition 4. A mapping $f : A \rightarrow B$ is said to be a *proximal Banach contraction of second kind* if there exists a non-negative number $\alpha < 1$ such that for all u_1, u_2, x_1, x_2 in A ,

$$d(u_1, f x_1) = d(u_2, f x_2) = \Delta_{AB} \text{ implies that } d(f u_1, f u_2) \leq \alpha d(f x_1, f x_2).$$

Then, Sanhan et al. [17] introduced the concept of proximal φ -contraction of the first and second kinds.

Definition 5. A mapping $f : A \rightarrow B$ is said to be a *proximal φ -contraction of first kind* if there exists an upper semi-continuous function from the right, with $\varphi(t) < t$ for all $t > 0$, such that for all u_1, u_2, x_1, x_2 in A ,

$$d(u_1, f x_1) = d(u_2, f x_2) = \Delta_{AB} \text{ implies that } d(u_1, u_2) \leq \varphi(d(x_1, x_2)).$$

Definition 6. A mapping $f : A \rightarrow B$ is said to be a *proximal φ -contraction of second kind* if there exists an upper semi-continuous function from the right, with $\varphi(t) < t$ for all $t > 0$, such that for all u_1, u_2, x_1, x_2 in A ,

$$d(u_1, f x_1) = d(u_2, f x_2) = \Delta_{AB} \text{ implies that } d(f u_1, f u_2) \leq \varphi(d(f x_1, f x_2)).$$

Note that if $\varphi(t) = \alpha t$, $\alpha \in [0, 1)$ then proximal φ -contraction of first and second kinds reduce to proximal Banach contraction of first and second kinds, respectively.

Another interesting generalization of proximal Banach contractions was obtained by Sadiq Basha and Shahzad [7].

Definition 7. A mapping $f : A \rightarrow B$ is said to be a *generalized proximal contraction of first kind* if there exist non-negative numbers $\alpha, \beta, \gamma, \delta$ with $\alpha + \beta + \gamma + 2\delta < 1$ such that for all $u_1, u_2, x_1, x_2 \in A$,

$$d(u_1, fx_1) = d(u_2, fx_2) = \Delta_{AB}$$

implies that

$$d(u_1, u_2) \leq \alpha d(x_1, x_2) + \beta d(x_1, u_1) + \gamma d(x_2, u_2) + \delta [d(x_1, u_2) + d(x_2, u_1)].$$

Definition 8. A mapping $f : A \rightarrow B$ is said to be a *generalized proximal contraction of second kind* if there exist non-negative numbers $\alpha, \beta, \gamma, \delta$ with $\alpha + \beta + \gamma + 2\delta < 1$ such that for all $u_1, u_2, x_1, x_2 \in A$,

$$d(u_1, fx_1) = d(u_2, fx_2) = \Delta_{AB}$$

implies that

$$\begin{aligned} d(fu_1, fu_2) &\leq \alpha d(fx_1, fx_2) + \beta d(fx_1, fu_1) + \gamma d(fx_2, fu_2) \\ &\quad + \delta [d(fx_1, fu_2) + d(fx_2, fu_1)]. \end{aligned}$$

For more results in this direction, we refer to [3, 11, 14] and the references mentioned therein.

Very recently, Khojasteh et al. [12] introduced the notion of *simulation function* and Argoubi et al. [5] (see also [15]) modified the definition as follows:

Definition 9. A mapping $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ is called a *simulation function* if it satisfies the following conditions: (ζ_1) $\zeta(t, s) < s - t$ for all $t, s > 0$; (ζ_2) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n \in (0, \infty)$, then

$$\limsup_{n \rightarrow \infty} \zeta(t_n, s_n) < 0.$$

Example 2 ([12]). If $\varphi : [0, \infty) \rightarrow [0, \infty)$ is an upper semi-continuous function such that $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$, then $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ defined by $\zeta(t, s) = \varphi(s) - t$ for all $s, t \in [0, \infty)$ is a simulation function.

Let $\mathfrak{S}$ be the family of all functions $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ satisfying the conditions (ζ_1) and (ζ_2) .

Definition 10 ([12]). Let X be a metric space. A mapping $f : X \rightarrow X$ is said to be $\mathfrak{S}$ -contraction (or *simulative contraction*) if there exists $\zeta \in \mathfrak{S}$ such that

$$\zeta(d(fx, fy), d(x, y)) \geq 0, \quad \text{for all } x, y \in X.$$

Note that Banach contractions are $\mathfrak{S}$ -contractions but the converse does not hold in general; see [4, 12].

We introduce the following notions of *proximal simulative contractions*.

Definition 11. A mapping $f : A \rightarrow B$ is said to be *proximal simulative contraction of first kind* if there exists a mapping $\zeta \in \mathfrak{S}$ such that for all u_1, u_2, x_1, x_2 in A , $d(u_1, fx_1) = d(u_2, fx_2) = \Delta_{AB}$ implies that

$$\zeta(d(u_1, u_2), d(x_1, x_2)) \geq 0.$$

Definition 12. A mapping $f : A \rightarrow B$ is said to be *proximal simulative contraction of second kind* if there exists a mapping $\zeta \in \mathfrak{S}$ such that for all u_1, u_2, x_1, x_2 in A , $d(u_1, fx_1) = d(u_2, fx_2) = \Delta_{AB}$ implies that

$$\zeta(d(fu_1, fu_2), d(fx_1, fx_2)) \geq 0.$$

We note that

- (i) if $A = B$, then proximal simulative contractions of the first kind are $\mathfrak{S}$ -contractions.
- (ii) If $\zeta(t, s) = \varphi(s) - t$ for all $s, t \in [0, \infty)$, where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is an upper semi-continuous function such that $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$. Then proximal simulative contractions of first and second kinds reduce to proximal φ -contractions of first and second kinds, respectively.

3. MAIN RESULTS

We start with the following result dealing with a continuous proximal simulative contraction of first kind.

Theorem 1. Let A and B be nonempty closed subsets of a complete metric space (X, d) . Suppose that A_0 and B_0 are nonempty, $f : A \rightarrow B$ is a continuous proximal simulative contraction of first kind and $f(A_0) \subseteq B_0$. Then there exists a unique element x in A such that $d(x, fx) = \Delta_{AB}$. Moreover, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ satisfying $d(x_{n+1}, fx_n) = \Delta_{AB}$ for all $n \in \mathbb{N} \cup \{0\}$ converges to the best proximity point x of f .

Proof. Let x_0 be a given point in A_0 . Since $f(A_0) \subseteq B_0$, we can choose $x_1 \in A_0$ such that $d(x_1, fx_0) = \Delta_{AB}$. As $fx_1 \in B_0$, there exists a point $x_2 \in A_0$ such that $d(x_2, fx_1) = \Delta_{AB}$. Continuing in this manner, we can obtain a sequence $\{x_n\}$ in A_0 such that

$$d(x_n, fx_{n-1}) = \Delta_{AB} \quad \text{and} \quad d(x_{n+1}, fx_n) = \Delta_{AB} \quad \text{for all } n \in \mathbb{N}. \quad (3.1)$$

Since f is proximal simulative contraction of first kind, we have

$$\zeta(d(x_n, x_{n+1}), d(x_{n-1}, x_n)) \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

If for some $m \in \mathbb{N}$, we have $x_{m-1} = x_m$, then

$$d(x_m, fx_{m-1}) = d(x_m, fx_m) = \Delta_{AB},$$

that is, x_m is a best proximity point of f . Thus, we can assume that $d(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$. Now, the property (ζ_1) of a simulation function assures that

$$0 \leq \zeta(d(x_n, x_{n+1}), d(x_{n-1}, x_n)) < d(x_{n-1}, x_n) - d(x_n, x_{n+1})$$

and hence

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \quad \text{for all } n \in \mathbb{N}.$$

It follows that $\{d(x_n, x_{n+1})\}$ is a monotonically decreasing sequence of non-negative real numbers which is bounded below, then there exists $r \geq 0$ such that $\{d(x_n, x_{n+1})\}$ converges to r . We claim that $r = 0$. Assume on the contrary that $r > 0$. Obviously,

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = \lim_{n \rightarrow \infty} d(x_{n-1}, x_n) = r. \quad (3.2)$$

From (3.2) and the property (ζ_2) of a simulation function, we get

$$0 \leq \limsup_{n \rightarrow \infty} \zeta(d(x_n, x_{n+1}), d(x_{n-1}, x_n)) < 0$$

and hence $r = 0$, that is

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (3.3)$$

Next, to prove that $\{x_n\}$ is a Cauchy sequence in X , it suffices to show that $\{x_{2n}\}$ is a Cauchy sequence in X . If not, there exists $\epsilon > 0$ and two subsequences $\{x_{2m_k}\}$ and $\{x_{2n_k}\}$ of $\{x_{2n}\}$ with $n_k > m_k \geq k$ such that

$$d(x_{2m_k}, x_{2n_k}) \geq \epsilon \quad \text{for all } k \in \mathbb{N}. \quad (3.4)$$

Without any loss of generality, we assume that for all $k \in \mathbb{N}$, n_k is the smallest positive integer greater than m_k for which this inequality holds, then

$$d(x_{2m_k}, x_{2n_k-2}) < \epsilon \quad \text{for all } k \in \mathbb{N}. \quad (3.5)$$

By (3.4) and (3.5), we have

$$\begin{aligned} \epsilon &\leq d(x_{2m_k}, x_{2n_k}) \\ &\leq d(x_{2m_k}, x_{2n_k-2}) + d(x_{2n_k-2}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k}) \\ &< \epsilon + d(x_{2n_k-2}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k}). \end{aligned} \quad (3.6)$$

On taking the limit as $k \rightarrow \infty$ in the previous inequality (3.6) and using (3.3), we obtain that

$$\lim_{k \rightarrow \infty} d(x_{2m_k}, x_{2n_k}) = \epsilon. \quad (3.7)$$

Similarly, we have

$$\begin{aligned} \epsilon &\leq d(x_{2m_k}, x_{2n_k}) \\ &\leq d(x_{2m_k}, x_{2m_k+1}) + d(x_{2m_k+1}, x_{2n_k+1}) + d(x_{2n_k+1}, x_{2n_k}). \end{aligned} \quad (3.8)$$

Also,

$$d(x_{2m_k+1}, x_{2n_k+1}) \leq d(x_{2m_k+1}, x_{2m_k}) + d(x_{2m_k}, x_{2n_k}) + d(x_{2n_k}, x_{2n_k+1}). \quad (3.9)$$

On taking the limit as $k \rightarrow \infty$ on both sides of (3.8) and (3.9), using (3.3) and (3.7), we have

$$\lim_{k \rightarrow \infty} d(x_{2m_k+1}, x_{2n_k+1}) = \epsilon. \quad (3.10)$$

Since $\{x_{2m_k}\}$ and $\{x_{2n_k}\}$ are subsequences of $\{x_{2n}\}$ with $n_k > m_k \geq k$, then by (3.1) we get

$$d(x_{2m_k+1}, f x_{2m_k}) = \Delta_{AB} = d(x_{2n_k+1}, f x_{2n_k}) \quad \text{for all } k \in \mathbb{N}. \quad (3.11)$$

By using (3.7) and (3.10), it follows from (3.11) and the property (ζ_2) of a simulation function that

$$0 \leq \limsup_{k \rightarrow \infty} \zeta(d(x_{2m_k+1}, x_{2n_k+1}), d(x_{2m_k}, x_{2n_k})) < 0,$$

a contradiction and hence $\{x_n\}$ is a Cauchy sequence in X . Since A is a closed subset of a complete metric space (X, d) , then there exists $x \in A$ such that

$$\lim_{n \rightarrow \infty} x_n = x. \quad (3.12)$$

On taking the limit as $n \rightarrow \infty$ on one of equalities in (3.1), by (3.12) and the continuity of f , we obtain that

$$d(x, f x) = \Delta_{AB}.$$

Therefore, x is a best proximity point of f in A . To prove uniqueness, suppose there exists another best proximity point x^* of f in A , other than x . That is

$$d(x^*, x) > 0, \quad d(x^*, f x^*) = \Delta_{AB} \quad \text{and} \quad d(x, f x) = \Delta_{AB}.$$

By the property (ζ_1) of a simulation function, since f is proximal simulative contraction of first kind, we get $d(x^*, x) < d(x^*, x)$, a contradiction, and hence the best proximity point of f in A is unique. This completes the proof. $\square$

Example 3. Let $X = \mathbb{R}^2$ be endowed with the Euclidean metric

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2| \quad \text{for all } x_1, y_1, x_2, y_2 \in X.$$

Let $A = \{(0, y) : 0 \leq y \leq 1\}$ and $B = \{(1, y) : 0 \leq y \leq 1\}$. Then, define $f : A \rightarrow B$ and $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ by

$$f((0, y)) = (1, y - \frac{y^2}{2}) \quad \text{and} \quad \zeta(t, s) = \begin{cases} \frac{1}{2}s - t & \text{if } t < s, \\ 0 & \text{if } t \geq s. \end{cases}$$

Indeed the pair (X, d) is a complete metric space, $A = A_0$, $B = B_0$, $\Delta_{AB} = 1$, and f is a continuous proximal simulative contraction of first kind which is not a proximal contraction of first kind. Moreover f has no fixed point, but the point $z = (0, 0) \in A$ is a unique best proximity point of f .

In the next result, we consider a continuous proximal simulative contraction mapping of second kind.

Theorem 2. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A is approximatively compact with respect to B . Suppose that A_0 and B_0 are nonempty, $f : A \rightarrow B$ is a continuous proximal simulative contraction of second kind and $f(A_0) \subseteq B_0$. Then there exists an element x in A such that $d(x, fx) = \Delta_{AB}$. Moreover, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ satisfying $d(x_{n+1}, fx_n) = \Delta_{AB}$ for all $n \in \mathbb{N} \cup \{0\}$ converges to a best proximity point x of f . Further, if x, x^* are best proximity points of f , then $fx = fx^*$ and f has a unique best proximity point if it is an injective function on A_0 .*

Proof. Let x_0 be a given point in A_0 . Following arguments similar to those in the proof of Theorem 1, we obtain that $\{x_n\}$ is a sequence in A_0 satisfying

$$d(x_n, fx_{n-1}) = \Delta_{AB} \quad \text{and} \quad d(x_{n+1}, fx_n) = \Delta_{AB} \quad \text{for all } n \in \mathbb{N}.$$

We assume that $fx_{n-1} \neq fx_n$ for all $n \in \mathbb{N}$; in fact, if $fx_{m-1} = fx_m$ for some $m \in \mathbb{N}$, from

$$d(x_m, fx_m) = d(x_m, fx_{m-1}) = \Delta_{AB},$$

we deduce that x_m is a proximity point of f . Since f is proximal simulative contraction of second kind, we have

$$\zeta(d(fx_n, fx_{n+1}), d(fx_{n-1}, fx_n)) \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

It follows from the property (ζ_1) of a simulation function that

$$0 \leq \zeta(d(fx_n, fx_{n+1}), d(fx_{n-1}, fx_n)) < d(fx_{n-1}, fx_n) - d(fx_n, fx_{n+1}),$$

which implies that

$$d(fx_n, fx_{n+1}) < d(fx_{n-1}, fx_n) \quad \text{for all } n \in \mathbb{N}.$$

Therefore $\{d(fx_n, fx_{n+1})\}$ is a monotonically decreasing sequence of non-negative real numbers which is bounded below, then there exists $r \geq 0$ such that $\{d(fx_n, fx_{n+1})\}$ converges to r . Following arguments similar to those in the proof of Theorem 1 we obtain that $r = 0$. Thus

$$\lim_{n \rightarrow \infty} d(fx_{n-1}, fx_n) = 0 \quad \text{for all } n \in \mathbb{N}.$$

Also, $\{fx_n\}$ is a Cauchy sequence in X . Since B is a closed subset of a complete metric space (X, d) , then there exists $y \in B$ such that

$$\lim_{n \rightarrow \infty} fx_n = y. \tag{3.13}$$

Now, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(y, A) &\leq d(y, x_{n+1}) \leq d(y, fx_n) + d(fx_n, x_{n+1}) \\ &= d(y, fx_n) + \Delta_{AB} \\ &\leq d(y, fx_n) + d(y, A), \end{aligned}$$

which implies that

$$d(y, A) \leq d(y, x_{n+1}) \leq d(y, f x_n) + d(y, A). \quad (3.14)$$

On taking the limit as $n \rightarrow \infty$ in (3.14) and using (3.13), we have

$$\lim_{n \rightarrow \infty} d(y, x_{n+1}) = d(y, A).$$

Since A is approximatively compact with respect to B , the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ convergent to some element $x \in A$. It follows, from the continuity of f , that

$$d(x, f x) = \lim_{k \rightarrow \infty} d(x_{n_k}, f x_{n_k-1}) = \Delta_{AB}.$$

Thus $d(x, f x) = \Delta_{AB}$ and hence x is a best proximity point of f in A . If there is another best proximity point x^* of f in A other than x with $f x^* \neq f x$, then we have

$$d(f x^*, f x) > 0 \quad \text{and} \quad d(x^*, f x^*) = d(x, f x) = \Delta_{AB}.$$

By the property (ζ_1) of a simulation function, since f is proximal simulative contraction of the second kind, we have $d(f x^*, f x) < d(f x^*, f x)$, a contradiction, and hence $f x = f x^*$. Thus, f has a unique best proximity point if it is an injective function on A_0 . $\square$

Finally, we relax the requirement of continuity for the mapping f in the following result.

Theorem 3. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that B is approximatively compact with respect to A . Suppose that A_0 and B_0 are nonempty, $f : A \rightarrow B$ is a proximal simulative contraction of first kind and $f(A_0) \subseteq B_0$. Then there exists a unique element x in A such that $d(x, f x) = \Delta_{AB}$. Moreover, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ satisfying $d(x_{n+1}, f x_n) = \Delta_{AB}$ for all $n \in \mathbb{N} \cup \{0\}$ converges to the best proximity point x of f .*

Proof. Let x_0 be a given point in A_0 . Following arguments similar to those in the proof of Theorem 1 we build a Cauchy sequence $\{x_n\}$ of points of A_0 satisfying

$$d(x_{n+1}, x_n) > 0 \quad \text{and} \quad d(x_{n+1}, f x_n) = \Delta_{AB} \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \quad (3.15)$$

Since A is a closed subset of a complete metric space (X, d) , then there exists $x \in A$ such that

$$\lim_{n \rightarrow \infty} x_n = x. \quad (3.16)$$

Now, for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} d(x, B) &\leq d(x, f x_n) \leq d(x, x_{n+1}) + d(x_{n+1}, f x_n) \\ &= d(x, x_{n+1}) + \Delta_{AB} \\ &\leq d(x, x_{n+1}) + d(x, B). \end{aligned}$$

Thus

$$d(x, B) \leq d(x, fx_n) \leq d(x, x_{n+1}) + d(x, B). \quad (3.17)$$

On taking the limit as $n \rightarrow \infty$ in (3.17) and using (3.16), we obtain that

$$\lim_{n \rightarrow \infty} d(x, fx_n) = d(x, B).$$

Since B is approximatively compact with respect to A , $\{fx_n\}$ has a subsequence $\{fx_{n_k}\}$ convergent to some element $y \in B$. Thus,

$$d(x, y) = \lim_{k \rightarrow \infty} d(x_{n_k+1}, fx_{n_k}) = \Delta_{AB}.$$

This implies that $x \in A_0$. Since $f(A_0) \subseteq B_0$, we can choose $u \in A$ such that

$$d(u, fx) = \Delta_{AB}. \quad (3.18)$$

We shall show that $u = x$. If $\{x_n\}$ has a subsequence convergent to u , then $u = x$, so we can assume that $x_n \neq u$ for all $n \in \mathbb{N} \cup \{0\}$. By (3.15), we deduce that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $x_{n_k} \neq x$ for all $k \in \mathbb{N}$. Then by (3.15), (3.18) and the property (ζ_1) of a simulation function, we obtain

$$0 \leq \zeta(d(x_{n_k+1}, u), d(x_{n_k}, x)) < d(x_{n_k}, x) - d(x_{n_k+1}, u),$$

that is,

$$d(x_{n_k+1}, u) < d(x_{n_k}, x) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \quad (3.19)$$

On taking the limit as $n \rightarrow \infty$ on both sides of (3.19) and using (3.16), we get that

$$0 \leq d(x, u) \leq 0.$$

Therefore $u = x$ and hence $d(x, fx) = \Delta_{AB}$. Suppose that there exists another best proximity point x^* in A of f , other than x . Then we have

$$d(x^*, x) > 0 \quad \text{and} \quad d(x^*, fx^*) = d(x, fx) = \Delta_{AB}.$$

By the property (ζ_1) of a simulation function, since f is proximal simulative contraction of first kind, we get

$$d(x^*, x) < d(x^*, x),$$

a contradiction. Therefore the best proximity point of f in A is unique. $\square$

Example 4. Let $X = \mathbb{R}^2$ be endowed with the Euclidean metric

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2| \quad \text{for all } x_1, y_1, x_2, y_2 \in X.$$

Let $A = \{(0, y) : 0 \leq y < \infty\}$ and $B = \{(1, y) : 0 \leq y < \infty\}$. Then, define $f : A \rightarrow B$ and $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ by

$$f((0, y)) = \begin{cases} (1, \frac{\ln(1+y)}{y}) & \text{if } y > 0, \\ (1, 0) & \text{otherwise,} \end{cases} \quad \text{and} \quad \zeta(t, s) = \begin{cases} \frac{1}{2}s - t & \text{if } t < s, \\ 0 & \text{if } t \geq s. \end{cases}$$

Indeed $A = A_0$, $B = B_0$, $\Delta_{AB} = 1$, and f is a proximal simulative contraction of first kind which is not continuous. Further, B is approximatively compact with

respect to A and $z = (0, 0)$ is a unique point in A satisfying the conclusion of Theorem 3.

We conclude with the following remarks:

- (i) If we put $A = B$ in Theorem 3, we obtain the main result in [12].
- (ii) If we put $A = B$ and $\zeta(s, t) = \alpha s - t$ for all $s, t \in [0, \infty)$ where $\alpha \in [0, 1)$ in Theorem 3, we obtain a well known Banach fixed point theorem.
- (iii) If we choose $\zeta(t, s) = \varphi(s) - t$ for all $s, t \in [0, \infty)$ where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is an upper semi-continuous function such that $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$, we obtain the result in [17].

4. VARIATIONAL INEQUALITIES

Let K be a nonempty closed and convex subset of a real Hilbert space H , with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$. A variational inequality problem can be stated as follows:

Find $u \in K$ such that $\langle gu, v - u \rangle \geq 0$ for all $v \in K$, where $g : H \rightarrow H$ is a given operator.

The interest for such a kind of mathematical tool is due to the fact that a wide class of equilibrium and economic problems, arising in the applied sciences, can be described by variational inequalities. Here, we consider the metric projection operator, say $P_K : H \rightarrow K$, for solving the variational inequality problem. Indeed, it is well-known that, for each $u \in H$, there exists a unique nearest point $P_K u \in K$ such that $\|u - P_K u\| \leq \|u - v\|$ for all $v \in K$, see [9].

The following Lemmas hold.

Lemma 1. *Let $z \in H$. Then $u \in K$ satisfies the inequality $\langle u - z, y - u \rangle \geq 0$, for all $y \in K$ if and only if $u = P_K z$.*

Lemma 2. *Let $g : H \rightarrow H$. Then $u \in K$ is a solution of $\langle gu, v - u \rangle \geq 0$, for all $v \in K$, if and only if $u = P_K(u - \lambda gu)$, with $\lambda > 0$.*

It is obvious that any set is approximatively compact with respect to itself. Therefore, we prove our first result.

Theorem 4. *Let K be a nonempty closed and convex subset of a real Hilbert space H . Suppose that $g : H \rightarrow H$ is such that $P_K(I - \lambda g) : K \rightarrow K$ is a proximal simulative contraction of first kind. Then there exists a unique element $u \in K$ such that $\langle gu, v - u \rangle \geq 0$ for all $v \in K$. Moreover, for any fixed element $u_0 \in K$, the sequence $\{u_n\}$, with $u_{n+1} = P_K(u_n - \lambda gu_n)$ where $\lambda > 0$ and $n \in \mathbb{N} \cup \{0\}$, converges to u .*

Proof. Define $f : K \rightarrow K$ by $fx = P_K(x - \lambda gx)$ so that, by Lemma 2, $u \in K$ is a solution of $\langle gu, v - u \rangle \geq 0$ for all $v \in K$, if and only if $u = fu$. Now, f satisfies all the hypotheses of Theorem 1 with $A = B = K$, then the conclusions of Theorem 4 hold true as a particular case of Theorem 3. $\square$

Similarly, the following result is related to Theorem 2.

Theorem 5. *Let K be a nonempty closed and convex subset of a real Hilbert space H . Suppose that $g : H \rightarrow H$ is such that $P_K(I - \lambda g) : K \rightarrow K$ is a continuous proximal simulative contraction of second kind. Then there exists a unique element $u \in K$ such that $\langle gu, v - u \rangle \geq 0$ for all $v \in K$. Moreover, for any fixed element $u_0 \in K$, the sequence $\{u_n\}$, with $u_{n+1} = P_K(u_n - \lambda gu_n)$ where $\lambda > 0$ and $n \in \mathbb{N} \cup \{0\}$, converges to u .*

Of course, these type-theorems can be extended and particularized for different variational inequality problems. For instance, we think to the case of a variational inequality of the second kind, that is, find $u \in K$ such that $\langle gu, v - u \rangle + \psi_K(v) - \psi_K(u) \geq 0$ for all $v \in H$, where $g : H \rightarrow H$ is a given operator and $\psi_K : H \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper, convex and continuous function. Clearly, if we assume

$$\psi_K(v) = \begin{cases} +\infty & \text{if } v \notin K, \\ 0 & \text{if } v \in K, \end{cases}$$

that is the indicator function of nonempty closed and convex set K , then we retrieve the classical variational inequality problem at the beginning of this section.

5. CONCLUSIONS

In this paper, we present an abstract approach to the solution of best proximity point and variational inequality problems, via classical arguments of fixed point theory. The motivation in the use of known fixed point techniques is their usefulness in covering a wide range of situations, without artificial expedients. On the other hand, we give sufficient flexibility to our theory, by using classes of simulation functions. As shown in dealing with variational inequality problems, the approach give us the possibility to design a convergent algorithm for the solution of the problem under investigation. Precisely, by using the metric projection, we are able to construct a sequence of approximations, say $\{u_n\}$, which is convergent to the solution of the problem.

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AN EXTENSION OF TOTAL GRAPH OVER A MODULE

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Abstract. Let R be a commutative ring with nonzero identity and $U(R)$ its multiplicative group of units. Let M be an R -module where the collection of prime submodules is non-empty and let N_A be an arbitrary union of prime submodules. Also, suppose that $c \in U(R)$ such that $c^{-1} = c$. We define the extended total graph of M as a simple graph $T\Gamma_c(M, N_A)$ with vertex set M , and two distinct elements $x, y \in M$ are adjacent if and only if $x + cy \in N_A$. In this paper, we will study some graph theoretic results of $T\Gamma_c(M, N_A)$.

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1. INTRODUCTION

Let R be commutative ring with $1 \neq 0$, $U(R)$ its multiplicative group of units and $Z(R)$ its set of zero-divisors. A proper submodule N of M is said to be a *prime submodule* if whenever $rm \in N$ for some $r \in R$ and $m \in M$, then either $m \in N$ or $r \in (N :_R M)$. Clearly, if N is a prime submodule of M , then $P = (N :_R M)$ is a prime ideal of R . Let M be an R -module, $T(M)$ its set of torsion elements and $\{N_\lambda\}_{\lambda \in \Omega}$ its set of all prime submodules. The R -module M is said to be *primeless* if $\Omega = \emptyset$. For a submodule L of an R -module M , the ideal $\{r \in R | rM \subseteq L\}$ and submodule $\{m \in M | rm \subseteq L\}$ will be denoted by $(L :_R M)$ and $(L :_M r)$, respectively. Let $N_A = \bigcup_{\lambda \in A} N_\lambda$ be a proper subset of M , and let $H_A = (N_A :_R M)$ for $\emptyset \neq A \subseteq \Omega$. It can be shown that $H_A = \bigcup_{\lambda \in A} P_\lambda$.

The total graph of R was introduced by Anderson and Badawi in [4], as the graph with all elements of R as vertices, and two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in Z(R)$. Also they introduced in [5] the generalized total graph of R in which $Z(R)$ is extended to H , a *multiplicative – prime* subset of R , in such away that $ab \in H$ for every $a \in H$ and $b \in R$, and whenever $ab \in H$ for all $a, b \in R$, then either $a \in H$ or $b \in H$. In fact, it is easily seen that H is a multiplicative-prime subset of R if and only if $R \setminus H$ is a saturated multiplicatively closed subset of R . Thus H is a multiplicative-prime subset of R if and only if H is a union of prime ideals of R .

In what follows, we extend the generalized total graph such that the extended total graph of M is a simple graph with vertex set M , and two distinct elements $x, y \in M$ are adjacent if and only if $x + cy \in N_A$ where $c \in U(R)$ and $c^{-1} = c$. This graph is denoted by $T\Gamma_c(M, N_A)$. In general, for $A, B \subseteq M$, $T\Gamma_c(A, B)$ is a simple graph with vertices all element of A , and two distinct vertices x and y are adjacent if and only if $x + cy \in B$.

The authors in [7] and [10] generalized the notion of a total graph to an R -module M . They considered the vertex set of a graph $T(\Gamma(M))$ as the elements of M such that two vertices are adjacent if and only if $x + y \in T(M)$. In [3], D. D. Anderson and Sangmin Chun proved that if $M \neq T(M)$, then $T(M)$ is a union of prime submodules of M . Consequently, $T\Gamma_c(M, N_A)$ is a generalization of $T(\Gamma(M))$ too. Let $c = -1$, then $\overline{T\Gamma}_c(M, N_A)$, the complement graph of $T\Gamma_c(M, N_A)$ (i.e., $T\Gamma_c(M, M \setminus N_A)$), is a Cayley graph, also let $M = R$ and N_A be the union of all the maximal ideals of R (i.e., $N_A = R \setminus U(R)$), then observe that $\overline{T\Gamma}_1(M, N_A)$ (i.e., $T\Gamma_1(M, M \setminus N_A)$) is the unit graph of R in the sense of [6] and $\overline{T\Gamma}_{-1}(M, N_A)$ (i.e., $T\Gamma_{-1}(M, M \setminus N_A)$) is the unitary Cayley graph in the sense of [2] and [9].

For a proper submodule L of M , $M(L) = \{m \in M \mid rm \in L \text{ for some } r \in R \setminus (L :_R M)\}$. In section 2, we will show that if $M \neq M(L)$, then there is $\Lambda \subseteq \Omega$ such that $M(L) = N_\Lambda$. In [1], the authors introduce a generalization of total graph of a module with respect to the set $M(L)$. For $\Lambda \subseteq \Omega$, N_Λ is more general than $M(L)$ since there are R -modules M and $\Lambda \subseteq \Omega$ such that N_Λ is not of the form $M(L)$ for all submodules L of M .

In section 3, we determine some basic properties of extended total graph specifically identifying its regularity and vertex transitivity. Since N_Λ is a union of prime submodules of M , the study of $T\Gamma_c(M, N_\Lambda)$ breaks naturally into two cases depending on whether or not N_Λ is a (prime) submodule of M . In section 4, we study the case when N is a (prime) submodule of M . In the final section, we do the case when N_Λ is not a submodule of M , and we improve Theorem 3.3 in [7] and Theorem 4.3 in [1] by Theorem 14, and Theorem 3.12 in [5] by Theorem 15.

Let Γ be a simple graph. We say that Γ is *totally disconnected* if none of two vertices of Γ are adjacent. A subgraph Γ_1 of Γ is an *induced subgraph* if vertex set of Γ_1 is contained in vertex set of Γ and two vertices of Γ_1 are adjacent if and only if they are adjacent in Γ . Throughout this paper, all subgraphs are induced. We say that two subgraphs Γ_1 and Γ_2 of Γ are *disjoint* if Γ_1 and Γ_2 have no common vertices and no vertex of Γ_1 (resp., Γ_2) is adjacent (in Γ) to any vertex not in Γ_1 (resp., Γ_2). If vertex x is an end point of edge e , then x and e are called incident. The degree of a vertex x in a graph Γ , written $\deg_\Gamma(x)$ is the number of edges incident to x . Graph Γ is called k -*regular* if degree of each vertex of Γ is k . For vertices x and y of Γ , we define $d(x, y)$ to be the length of the shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The *diameter* of Γ is $\text{diam}(\Gamma) = \sup\{d(x, y) \mid x \text{ and } y \text{ are vertices of } \Gamma\}$. The *girth* of Γ , denoted by

$gr(\Gamma)$, is the length of a shortest cycle in Γ ($gr(\Gamma) = \infty$ if Γ contains no cycles). We denote the complete graph on n vertices by K^n and the complete bipartite graph on m and n vertices by $K^{m,n}$. We will call a $K^{1,n}$, a *stargraph*. A graph Γ is called *vertex transitive* if for every two vertices x and y there exists $\tau \in Aut(\Gamma)$ such that $\tau(x) = y$.

Throughout, all rings R are commutative with $1 \neq 0$, and M is an R -module with at least one prime submodule. For $A \subseteq M$, let $A^* = A \setminus \{0\}$. As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the integers and integers modulo n , respectively.

2. $M(L)$ AS A UNION OF PRIME SUBMODULES

In this section, we consider the question of when for an R -module M , the set $M(L)$ is a union of prime submodules and determine a family of non-primeless R -modules. We refer the reader to [1] for some properties concerning $M(L)$. Throughout this section, L is a proper submodule of M over the commutative ring R . Let $M(L) = \{m \in M \mid rm \in L \text{ for some } r \in R \setminus (L :_R M)\}$. It is easy to see that $M(L)$ is closed under the multiplication of scalars. However $M(L)$ may not be an additive subgroup of M . Let $M = M(L)$, then $M(L)$ may or may not be a union of prime submodules. Also let $M(L)$ be a proper submodule of M , then it is a prime submodule of M , by [1, Theorem 2.1]. For our main result in this section, we need the below theorem.

Theorem 1. *Let M be an R -module with $M \neq M(L)$, and let $\mathcal{A} = \{L_\Delta \mid L_\Delta \text{ is a submodule of } M \text{ with } L_\Delta \subseteq M(L), \text{ and } L_\Delta = \bigcup_{h \in \Delta} (L :_M h) \text{ for some } \Delta \subset R\}$. Then a maximal element of $\mathcal{A}$ is a prime submodule.*

Proof. Let $L_\Delta = \bigcup_{h \in \Delta} (L :_M h)$ be a maximal element of $\mathcal{A}$. Suppose that $rm \in L_\Delta$ for some $r \in R$ and $m \in M$ such that $m \notin L_\Delta$. First, assume that $rh \in R \setminus (L :_R M)$ for every $h \in \Delta$. So each $(L :_M h) \subseteq (L :_M rh)$ and hence $L_\Delta \subseteq L'_\Delta = \bigcup_{h \in \Delta} (L :_M rh)$. Also, suppose $l_1, l_2 \in L'_\Delta$. Then $l_i \in (L :_M rh_i)$ for $i = 1, 2$. So $rl_i \in (L :_M h_i) \subseteq L_\Delta$ and hence $rl_1 + rl_2 \in L_\Delta$. Thus $rl_1 + rl_2 \in (L :_M h_j)$ for some $h_j \in \Delta$; so $l_1 + l_2 \in (L :_M rh_j) \subseteq L'_\Delta$. Since L'_Δ is clearly closed under scalar product, L'_Δ is submodule of M with $L'_\Delta \subseteq M(L)$. Thus by the maximality of L_Δ , $L_\Delta = L'_\Delta$. Now $rm \in L_\Delta$ implies $rm \in (L :_M h)$ for some $h \in \Delta$. Hence $m \in (L :_M rh) \subseteq L'_\Delta = L_\Delta$; a contradiction. Thus $rh \in (L :_R M)$ for some $h \in \Delta$ and hence $rM \subseteq (L :_M h) \subseteq L_\Delta$. Therefore L_Δ is a prime submodule of M . $\square$

Theorem 2. *Let M be an R -module with $M \neq M(L)$, then $M(L)$ is a union of prime submodules.*

Proof. Let $l \in M(L)$, $\mathcal{A}_l = \{L_\Delta \mid L_\Delta \text{ is a submodule of } M, l \in L_\Delta \subseteq M(L), \text{ and } L_\Delta = \bigcup_{h \in \Delta} (L :_M h) \text{ for some } \Delta \subset R\}$, and $rl \in L$ where $r \in R \setminus (L :_R M)$. Then $l \in (L :_M r)$; thus $\mathcal{A}_l \neq \emptyset$. By Zorn's Lemma, $\mathcal{A}_l$ has a maximal element L_l . By above theorem, L_l is a prime submodule of $M(L)$. Therefore $M(L) = \bigcup_{l \in M(L)} L_l$ is a union of prime submodules of M . $\square$

In the next corollary, we determine a family of non-primeless R -modules.

Corollary 1. *Let M be an R -module such that $M \neq M(L)$ for some proper submodule L of M , then M is not primeless.*

Proof. This is clear from above theorem. $\square$

3. BASIC PROPERTIES OF THE EXTENDED TOTAL GRAPH

The basic properties of the extended total graph are given below, independent of whether or not N_Δ is a submodule of M . Since $(c+1)(c-1) = 0 \in H_\Delta = \bigcup_{\lambda \in \Lambda} P_\lambda$, either $c+1 \in H_\Delta$ or $c-1 \in H_\Delta$. First, we determine properties of $T\Gamma_c(M, N_\Delta)$ for some $c \in U(R)$.

Theorem 3. *Let there is $c \in U(R)$ where $c \neq 1, -1$ and $c^2 = 1$.*

- (1) *If $x + y \in N_\Delta$, then either $x + cy \in N_\Delta$ or $x - cy \in N_\Delta$.*
- (2) *If $c + 1 \notin H_\Delta$, then $T\Gamma_c(M, N_\Delta) \cong T\Gamma_1(M, N_\Delta)$.*
- (3) *If $c - 1 \notin H_\Delta$, then $T\Gamma_c(M, N_\Delta) \cong T\Gamma_{-1}(M, N_\Delta)$.*

Proof. (1) Let $x + y \in N_\lambda$ for some $\lambda \in \Lambda$, then $(c-1)(x - cy) \in N_\lambda$. Let $x - cy \notin N_\lambda$, then $c-1 \in P_\lambda$ hence $x + y + (c-1)y = x + cy \in N_\lambda$.
 (2) Let $c + 1 \notin H_\Delta$ and $x + y \in N_\Delta$, then $x + cy \in N_\Delta$ since $(c+1)(x + cy) \in N_\Delta$. Also if $x + cy \in N_\Delta$, then $x + y \in N_\Delta$, since $(c+1)(x + y) \in N_\Delta$ and $c + 1 \notin H_\Delta$.
 (3) Let $c - 1 \notin H_\Delta$ and $x - y \in N_\Delta$, then $x + cy \in N_\Delta$ since $(c-1)(x + cy) \in N_\Delta$. Also if $x + cy \in N_\Delta$, then $x - y \in N_\Delta$, since $(c-1)(x - y) \in N_\Delta$ and $c - 1 \notin H_\Delta$. $\square$

Remark 1. Suppose that $x, y \in M$ are adjacent in $T\Gamma_1(M, N_\Delta)$, then x and cy are adjacent in $T\Gamma_c(M, N_\Delta)$, if $x \neq cy$.

Theorem 4. *Let M be an R -module with $|N_\Delta| = \alpha$ and $m \in M$. If $(c+1)m \notin N_\Delta$, then $\deg_{T\Gamma_c(M, N_\Delta)}(m) = \alpha$; otherwise, $\deg_{T\Gamma_c(M, N_\Delta)}(m) = \alpha - 1$. In particular, $c + 1 \in H_\Delta$ if and only if $T\Gamma_c(M, N_\Delta)$ is a $(\alpha - 1)$ -regular graph.*

Proof. There is a unique $x = n - cm \in M$ for every $n \in N_\Delta$. Hence m is adjacent to x unless $x = m$. If $(c+1)m \notin N_\Delta$, then $n - cm \neq m$ for every $n \in N_\Delta$. Therefore $\deg_{T\Gamma_c(M, N_\Delta)}(m) = \alpha$. Otherwise, let $(c+1)m \in N_\Delta$, then $\deg_{T\Gamma_c(M, N_\Delta)}(m) = \alpha - 1$ since m cannot be adjacent to itself. In particular, if $c + 1 \in H_\Delta$, then $(c+1)m \in N_\Delta$ and $\deg_{T\Gamma_c(M, N_\Delta)}(m) = \alpha - 1$ for every $m \in M$. Now suppose that $T\Gamma_c(M, N_\Delta)$ is a $(\alpha - 1)$ -regular graph, $c + 1 \notin H_\Delta$ and $m' \in M \setminus N_\Delta$, then $\deg_{T\Gamma_c(M, N_\Delta)}(m') = \alpha$ since $(c+1)m' \notin N_\Delta$ for every $m' \in M \setminus N_\Delta$, a contradiction. so $c + 1 \in H_\Delta$. $\square$

Theorem 5. *If $c - 1 \notin H_\Delta$, then $T\Gamma_c(M, N_\Delta)$ is vertex transitive.*

Proof. For each $m \in M$ the mapping $\tau_m : x \mapsto x + m$ is a permutation of the elements of M . Let $c - 1 \notin H_A$, then $x + cy \in N_A$ if and only if $x + m + c(y + m) \in N_A$ since $(c - 1)(x + m + c(y + m)) = (c - 1)(x + cy) \in N_A$. Hence τ_m is an automorphism of $T\Gamma_c(M, N_A)$. The permutations τ_m form a subgroup of the automorphism group of $T\Gamma_c(M, N_A)$. This subgroup acts transitively on the vertices of $T\Gamma_c(M, N_A)$ because for any two vertices m and m' , the automorphism $\tau_{m-m'}$ maps m to m' . $\square$

The previous theorem gave an important property of $T\Gamma_c(M, N_A)$. The vertex transitivity identifies some properties of graph such as the edge connectivity. For further investigation on vertex transitivity and edge connectivity, see [8]. Hence if $c - 1 \notin H_A$ and $T\Gamma_c(M, N_A)$ is connected (i.e., $M = \langle N_A \rangle$ by Theorem 12), then the edge connectivity of $T\Gamma_c(M, N_A)$ is equal to $\alpha - 1$.

Example 1. Let $M = \mathbb{Z}_3 \times \mathbb{Z}_3$, $R = \mathbb{Z}_{12}$, $c = 5$, $N_1 = \mathbb{Z}_3 \times 0$, $N_2 = 0 \times \mathbb{Z}_3$, and $N_A = N_1 \cup N_2$, then by Theorem 3(3), $T\Gamma_5(M, N_A) \cong T\Gamma_{11}(M, N_A)$. $T\Gamma_5(M, N_A)$ is a 4-regular graph by Theorem 4. Also, by Theorem 5, $T\Gamma_5(M, N_A)$ is vertex transitive.

4. THE CASE WHEN N_A IS A SUBMODULE OF M

We know that $N_A = \bigcup_{\lambda \in \Lambda} N_\lambda$ is a proper submodule of M if and only if N_A is a prime submodule of M . Moreover $P_\lambda = (N_\lambda :_R M)$ and $H_A = (N_A :_R M)$ are prime ideals of R .

Theorem 6. *Let M be an R -module such that N_A is a proper submodule of M . Then $T\Gamma_c(N_A, N_A)$ is a complete subgraph of $T\Gamma_c(M, N_A)$ and $T\Gamma_c(N_A, N_A)$ is disjoint from $T\Gamma_c(M \setminus N_A, N_A)$.*

Proof. It is clear that $T\Gamma_c(N_A, N_A)$ is a complete subgraph of $T\Gamma_c(M, N_A)$. Suppose that $y \in M \setminus N_A$ and $x \in N_A$ are adjacent. Then $x + cy \in N_A$ and $cy \in N_A$ hence $c \in H_A$ which is a contradiction since $c \in U(R)$. Therefore, $T\Gamma_c(N_A, N_A)$ is disjoint from $T\Gamma_c(M \setminus N_A, N_A)$. $\square$

Definition 1. By a slice of $T\Gamma_c(M, N_A)$ we mean a subgraph of $T\Gamma_c(M, N_A)$ with a vertex set as the form $x + N_A$, denoted by S_x , for some $x \in M$.

It is easy to see that a slice of $T\Gamma_c(M, N_A)$ is an induced subgraph. So, two slices S_x and S_y are the same is and only if $x - y \in N_A$.

The next theorem gives a complete description of $T\Gamma_c(M, N_A)$. It also shows that non-isomorphic modules may have isomorphic total graphs. We allow α and β to be infinite cardinals.

Theorem 7. *Let M be an R -module such that N_A is a proper submodule of M , and let $H_A = (N_A :_R M)$, $|N_A| = \alpha$ and $|M/N_A| = \beta$.*

- (1) *If $c + 1 \in H_A$, then $T\Gamma_c(M \setminus N_A, N_A)$ is the union of $\beta - 1$ disjoint K^α 's.*

- (2) If $c + 1 \notin H_\Delta$, then $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ is the union of $(\beta - 1)/2$ disjoint $K^{\alpha, \alpha}_s$.

Proof. (1) Assume that $c + 1 \in H_\Delta$, and let $x \in M \setminus N_\Delta$. Then the slice S_x is a complete subgraph of $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ since $(x + n_1) + c(x + n_2) = (c + 1)x + n_1 + n_2 \in N_\Delta$ for all $n_1, n_2 \in N_\Delta$ since $c + 1 \in H_\Delta$ and N_Δ is a submodule of M . Note that distinct slices form disjoint subgraphs of $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ since if $x + n_1$ and $y + n_2$ are adjacent for some $y \in M \setminus N_\Delta$ and $n_1, n_2 \in N_\Delta$, then $x + cy = (x + n_1) + c(y + n_2) - (n_1 + cn_2) \in N_\Delta$ if and only if $x - y = (x + cy) - (c + 1)y \in N_\Delta$ since $c + 1 \in H_\Delta$ and N_Δ is a submodule of M . Then $S_x = S_y$, a contradiction. Thus $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ is the union of $\beta - 1$ disjoint subgraphs S_x , each of which is a K^α , where $\alpha = |N_\Delta| = |x + N_\Delta|$.

(2) Assume that $c + 1 \notin H_\Delta$, and let $x \in M \setminus N_\Delta$. Then no two distinct elements in S_x are adjacent, suppose not. So, $(x + n_1) + c(x + n_2) \in N_\Delta$ for $n_1, n_2 \in N_\Delta$. This implies that $(c + 1)x \in N_\Delta$ hence $c + 1 \in H_\Delta$ since N_Δ is a prime submodule of M , a contradiction. On the other hand, two slices S_x and S_{-cx} are disjoint and each vertex of S_{-cx} is adjacent to all vertex of S_x . Thus $S_x \cup S_{-cx}$ is a complete bipartite subgraph of $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$. Furthermore, if $x + n_1$ is adjacent to $y + n_2$ for some $y \in M \setminus N_\Delta$ and $n_1, n_2 \in N_\Delta$, then $y + cx \in N_\Delta$ as in part(1) above, and hence $y + N_\Delta = -cx + N_\Delta$. Thus $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ is the union of $(\beta - 1)/2$ disjoint subgraphs $S_x \cup S_{-cx}$, each of which is $K^{\alpha, \alpha}$, where $\alpha = |N_\Delta| = |x + N_\Delta|$

□

From the above theorem, one can easily deduce when $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ is a complete or connected graph. The next theorem determines when $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ is either complete or connected.

Theorem 8. Let M be an R -module such that N_Δ is a proper submodule of M , and let $H_\Delta = (N_\Delta :_R M)$.

- (1) Let $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ be a complete graph, then either $|M/N_\Delta| = |M| = 3$ or $|M/N_\Delta| = 2$. Its converse is true when either $|M/N_\Delta| = |M| = 3$ and $c + 1 \notin (0 :_R M)$ or $|M/N_\Delta| = 2$.
- (2) Let $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ be a connected graph, then either $|M/N_\Delta| = 3$ or $|M/N_\Delta| = 2$. Its converse is true when either $|M/N_\Delta| = 3$ and $c - 1 \in H_\Delta$ or $|M/N_\Delta| = 2$.
- (3) $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ (and hence $T\Gamma_c(N_\Delta, N_\Delta)$ and $T\Gamma_c(M, N_\Delta)$) is a totally disconnected graph if and only if $N_\Delta = \{0\}$ and $c + 1 \in (0 :_R M)$.

Proof. Let $|M/N_\Delta| = \beta$ and $\alpha = |N_\Delta|$

- (1) Let $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ be a complete graph. Then by Theorem 7, $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ is a single K^α or $K^{1,1}$. If $c + 1 \in H_\Delta$, then $\beta - 1 = 1$ thus $\beta = 2$,

and hence $|M/N_\Delta| = 2$. If $c + 1 \notin H_\Delta$, then $\alpha = 1$ and $(\beta - 1)/2 = 1$. Thus $N_\Delta = \{0\}$ and $\beta = 3$; hence $|M/N_\Delta| = |M| = 3$. Conversely, suppose first that $M/N_\Delta = \{N_\Delta, x + N_\Delta\}$, where $x \notin N_\Delta$. Since $cx \notin N_\Delta$, so, $x + N_\Delta = -cx + N_\Delta$ which implies that $(c + 1)x \in N_\Delta$. Let $m, m' \in M \setminus N_\Delta$. Then $m + x, m' + x \in N_\Delta$ since none of $m + x + N_\Delta$ and $m' + x + N_\Delta$ are equal to $x + N_\Delta$ so $m + cm' = (m + x) + c(m' + x) - (c + 1)x \in N_\Delta$ since N_Δ is a submodule of M . Thus $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ is complete. Next, suppose that $|M/N_\Delta| = |M| = 3$; hence $H_\Delta = (0 :_R M)$ since $N_\Delta = \{0\}$. By Theorem 7, $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ is a complete graph since $c + 1 \notin (0 :_R M)$.

- (2) Let $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ be connected. Then by Theorem 7, $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ is a single K^α or $K^{\alpha, \alpha}$. If $c + 1 \in H_\Delta$, then $\beta - 1 = 1$, and hence $|M/N_\Delta| = 2$. If $c + 1 \notin H_\Delta$, then $(\beta - 1)/2 = 1$, and hence $|M/N_\Delta| = 3$. Conversely, by part (1) above if $|M/N_\Delta| = 2$ then $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ is complete and so it is connected. Suppose that $|M/N_\Delta| = 3$ and $c - 1 \in H_\Delta$. First, we show that $c + 1 \notin H_\Delta$. Suppose not. Let $c + 1 \in H_\Delta$ and $M/N_\Delta = \{N_\Delta, x + N_\Delta, y + N_\Delta\}$, where $x, y \notin N_\Delta$; it is easy to see that $x + cy \in N_\Delta$. This yields that x and y are adjacent, a contradiction, by the proof of Theorem 7(1). Thus $c + 1 \notin H_\Delta$. Therefore, $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ is the complete bipartite graph $K^{\alpha, \alpha}$, by Theorem 7(2).
- (3) $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ is totally disconnected if and only if it be a disjoint union of K^1 's. so by Theorem 7, $|N_\Delta| = 1$. Further, since m and $-cm$ are adjacent for all $m \in M$, it follows that $m = -cm$, hence $c + 1 \in (0 :_R M)$.

□

The next theorem gives a more explicit description of the diameter of $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ when N_Δ is a proper submodule of M .

Theorem 9. *Let M be an R -module such that N_Δ is a proper submodule of M , and let $H_\Delta = (N_\Delta :_R M)$.*

- (1) *$\text{diam}(T\Gamma_c(M \setminus N_\Delta, N_\Delta)) = 0$ if and only if $|M| = 2$.*
- (2) *$\text{diam}(T\Gamma_c(M \setminus N_\Delta, N_\Delta)) = 1$ if and only if either $N_\Delta \neq \{0\}$ and $|M/N_\Delta| = 2$ or $|M| = 3$ and $c + 1 \notin (0 :_R M)$.*
- (3) *$\text{diam} T\Gamma_c(M \setminus N_\Delta, N_\Delta) = 2$ if and only if $N_\Delta \neq \{0\}$, $|M/N_\Delta| = 3$ and $c - 1 \in H_\Delta$.*
- (4) *Otherwise, $\text{diam}(T\Gamma_c(M \setminus N_\Delta, N_\Delta)) = \infty$.*

Proof. It is clear by the proof of the Theorem 8. □

The next theorem describes the girth of $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ and $T\Gamma_c(M, N_\Delta)$ when N_Δ is a proper submodule of M .

Theorem 10. *Let M be an R -module such that N_Δ is a proper submodule of M , and let $H_\Delta = (N_\Delta :_R M)$.*

- (1) (a) *$\text{gr}(T\Gamma_c(M \setminus N_\Delta, N_\Delta)) = 3$ if and only if $c + 1 \in H_\Delta$ and $|N_\Delta| \geq 3$.*

- (b) $gr(T\Gamma_c(M \setminus N_\Delta, N_\Delta)) = 4$ if and only if $c + 1 \notin H_\Delta$ and $|N_\Delta| \geq 2$.
- (c) Otherwise, $gr(T\Gamma_c(M \setminus N_\Delta, N_\Delta)) = \infty$.
- (2) (a) $gr(T\Gamma_c(M, N_\Delta)) = 3$ if and only if $|N_\Delta| \geq 3$.
- (b) $gr(T\Gamma_c(M, N_\Delta)) = 4$ if and only if $c + 1 \notin H_\Delta$ and $|N_\Delta| = 2$.
- (c) Otherwise, $gr(T\Gamma_c(M, N_\Delta)) = \infty$.

Proof. Apply Theorem 7 and Theorem 6. □

Example 2. Let $M = \mathbb{Z}_2 \times \mathbb{Z}_3$, $R = \mathbb{Z}_{12}$ and $N_\Delta = \mathbb{Z}_2 \times 0$, then $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ is the union of 2 disjoint K^2 's, if $c = 5$ or 11, by Theorem 7(1) and $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ is a $K^{2,2}$, if $c = 1$ or 7, by Theorem 7(2). Now, if $c = 1$ or 7, then $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ is connected with $diam(T\Gamma_c(M \setminus N_\Delta, N_\Delta)) = 2$ and $gr(T\Gamma_c(M \setminus N_\Delta, N_\Delta)) = 4$ by Theorem 8(2), 9(3) and 10(1)(b), respectively, and $gr(T\Gamma_c(M, N_\Delta)) = 4$ by Theorem 10(2)(b).

5. THE CASE WHEN N_Δ IS NOT A SUBMODULE OF M

In this section, we consider the remaining case when N_Δ is not a submodule of M , this implies that $|N_\Delta| \geq 3$. So there are distinct $x, y \in N_\Delta^*$ such that $x + cy \in M \setminus N_\Delta$. In this case, we show that $T\Gamma_c(N_\Delta, N_\Delta)$ is always connected but never complete. Moreover $T\Gamma_c(N_\Delta, N_\Delta)$ and $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ are never disjoint subgraphs of $T\Gamma_c(M, N_\Delta)$. We first show that $T\Gamma_c(M, N_\Delta)$ is connected when $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ is connected. However, we give an example to show that the converse fails.

Theorem 11. *Let M be an R -module and N_Δ a union of prime submodules of M that is not a submodule of M .*

- (1) $T\Gamma_c(N_\Delta, N_\Delta)$ is connected with $diam(T\Gamma_c(N_\Delta, N_\Delta)) = 2$.
- (2) Some vertices of $T\Gamma_c(N_\Delta, N_\Delta)$ are adjacent to a vertex of $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$.
In particular, the subgraphs $T\Gamma_c(N_\Delta, N_\Delta)$ and $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ of $T\Gamma_c(M, N_\Delta)$ are not disjoint.
- (3) If $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ is connected, then $T\Gamma_c(M, N_\Delta)$ is connected.

Proof. (1) Every $x \in N_\Delta^*$ is adjacent to 0. Thus $x - 0 - y$ is a path in $T\Gamma_c(N_\Delta, N_\Delta)$ of length two between any two distinct $x, y \in N_\Delta^*$. Moreover, there are nonadjacent $x, y \in N_\Delta^*$ since N_Δ is not a submodule of M ; so $diam(T\Gamma_c(N_\Delta, N_\Delta)) = 2$.

(2) Since N_Δ is not a submodule of M , there are distinct $x, y \in N_\Delta^*$ such that $x + cy \in M \setminus N_\Delta$. Then $-y \in N_\Delta^*$ and $x + cy \in M \setminus N_\Delta$ are adjacent vertices in $T\Gamma_c(M, N_\Delta)$ since $x + cy - cy = x \in N_\Delta$. The “in particular” statement is clear.

(3) By part (1) above, it suffices to show that there is a path from x to y in $T\Gamma_c(M, N_\Delta)$ for every $x \in N_\Delta$ and $y \in M \setminus N_\Delta$. By part (2) above, there are adjacent vertices u and v in $T\Gamma_c(N_\Delta, N_\Delta)$ and $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$, respectively. Since $T\Gamma_c(N_\Delta, N_\Delta)$ is connected, there is a path from x to

u in $T\Gamma_c(N_\Delta, N_\Delta)$; and since $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ is connected, there is a path from v to y in $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$, then there is a path from x to y in $T\Gamma_c(M, N_\Delta)$ since u and v are adjacent in $T\Gamma_c(M, N_\Delta)$. Thus $T\Gamma_c(M, N_\Delta)$ is connected. $\square$

Next, we determine an equivalent condition for connectedness of $T\Gamma_c(M, N_\Delta)$ and compute $\text{diam}(T\Gamma_c(M, N_\Delta))$. As usual, if $A \subseteq M$, then $\langle A \rangle$ denotes the submodule of M generated by A .

Theorem 12. *Let M be an R -module and N_Δ a union of prime submodule of M . Then $T\Gamma_c(M, N_\Delta)$ is connected if and only if $M = \langle N_\Delta \rangle$ (i.e., $m = n_1 + n_2 + \dots + n_k$ for every $m \in M$ and for some $n_1, \dots, n_k \in N_\Delta$ and $k \in \mathbb{N}$).*

Proof. Suppose that $T\Gamma_c(M, N_\Delta)$ is connected, let $m \in M$. Then there is a path $0 - m_1 - \dots - m_t - m$ from 0 to m in $T\Gamma_c(M, N_\Delta)$. Thus $m_1, m_2 + cm_1, \dots, m + cm_t \in N_\Delta$. Hence $m \in \langle m_1, m_2 + cm_1, \dots, m + cm_t \rangle \subseteq \langle N_\Delta \rangle$; thus $M = \langle N_\Delta \rangle$. Conversely, suppose that $M = \langle N_\Delta \rangle$. We show that there is a path from 0 to m in $T\Gamma_c(M, N_\Delta)$ for every $0 \neq m \in M$. By hypothesis, $m = n_1 + n_2 + \dots + n_k$ for some $n_1, \dots, n_k \in N_\Delta$ and $k \in \mathbb{N}$. Also we define a function, denoted by $f(t)$, which is equal to 0 where t is even and equal to 1 where t is odd. Let $a_0 = 0$ and $a_j = (-1)^{f(k-j)} c^{f(k-j)} (n_1 + \dots + n_j)$ for every integer j with $1 \leq j \leq k$. Then $a_j + ca_{j+1} \in N_\Delta$ for every integer j with $0 \leq j \leq k-1$, and thus $0 = a_0 - a_1 - \dots - a_{k-1} - a_k = m$ is a path from 0 to m in $T\Gamma_c(M, N_\Delta)$ of length at most k . Now, let $0 \neq u, w \in M$. Then by the preceding argument, there are paths from u to 0 and 0 to w in $T\Gamma_c(M, N_\Delta)$. Hence there is a path from u to w in $T\Gamma_c(M, N_\Delta)$; so $T\Gamma_c(M, N_\Delta)$ is connected. $\square$

In the next theorem and its corollary, we determine a relation between $T\Gamma_c(R, H_\Delta)$ and $T\Gamma_c(M, N_\Delta)$, where $H_\Delta = (N_\Delta :_R M)$.

Theorem 13. *Let M be an R -module, and N_Δ a union of prime submodules of M where $H_\Delta = (N_\Delta :_R M)$. If $T\Gamma_c(R, H_\Delta)$ is connected, then $T\Gamma_c(M, N_\Delta)$ is connected as well.*

Proof. Suppose that $T\Gamma_c(R, H_\Delta)$ is connected and let $m \in M$. Then there is a path $0 - r_1 - \dots - r_k - 1$ from 0 to 1 in $T\Gamma_c(R, H_\Delta)$. Then $r_1, r_2 + cr_1, \dots, r_k + cr_{k-1}, 1 + cr_k \in H_\Delta$. Hence $0 - r_1 m - \dots - r_k m - m$ is a path from 0 to m in $T\Gamma_c(M, N_\Delta)$. Since all vertices may be connected via 0, $T\Gamma_c(M, N_\Delta)$ is a connected graph. $\square$

Corollary 2. *Let M be an R -module, and N_Δ a union of prime submodules of M and $H_\Delta = (N_\Delta :_R M)$. If $R = \langle H_\Delta \rangle$, then $T\Gamma_c(M, N_\Delta)$ is connected.*

Proof. This follows directly from Theorem 12 and 13. $\square$

Remark 2. According to the proof of the Theorem 13, if $d(0, 1) = k$ in $T\Gamma_c(R, H_\Delta)$, then $d(0, m) \leq k$ in $T\Gamma_c(M, N_\Delta)$ for every $m \in M$.

In next theorem, we improve Theorem 3.3 of [7] and Theorem 4.3 of [1] without considering M as a finitely generated R -module.

Theorem 14. *Let M be an R -module and N_Δ a union of prime submodule of M that is not a submodule of M such that $M = \langle N_\Delta \rangle$ (i.e., $T\Gamma_c(M, N_\Delta)$ is connected). If there is $k \geq 2$ which is a greatest integer i such that $m = n_1 + n_2 + \dots + n_i$ for every $m \in M$ and for some $n_1, \dots, n_i \in N_\Delta$ where $n_1 + n_2 + \dots + n_i$ is a shortest representation of the element m , then $\text{diam}(T\Gamma_c(M, N_\Delta)) = k$. Otherwise, $\text{diam}(T\Gamma_c(M, N_\Delta)) = \infty$.*

Proof. Let m and m' be distinct elements in M such that they are not adjacent. We show that there is a path from m to m' in $T\Gamma_c(M, N_\Delta)$ with length at most k . We define a function, denoted by $f(t)$, which is equal to 0 where t is even and equal to 1 where t is odd. By the proof of Theorem 12, we can consider two path $(m - m') - x_1 - \dots - x_{k-1} - 0$ and $(m + cm') - y_1 - \dots - y_{k-1} - 0$ of lengths at most k . Let $a_0 = m$, $a_k = m'$, $a_j = (-1)^{f(k-j)} c^{f(k-j)} m' + x_j f(k-1) + y_j f(k)$ for every integers j with $1 \leq j \leq k-1$. Then $a_j + ca_{j+1} \in N_\Delta$ for every integers j with $0 \leq j \leq k-1$, and thus $m = a_0 - a_1 - \dots - a_{k-1} - a_k = m'$ is a path from m to m' in $T\Gamma_c(M, N_\Delta)$ of length at most k . Now we show that any path from 0 to m in $T\Gamma_c(M, N_\Delta)$ has length at least k . Suppose that $0 - b_1 - \dots - b_{l-1} - m$ is a path from 0 to m in $T\Gamma_c(M, N_\Delta)$ of length l and let $m = n_1 + \dots + n_k$ be a shortest representation of the element m . Thus $b_1, b_2 + cb_1, \dots, b_{l-1} + cb_{l-2}, m + cb_{l-1} \in N_\Delta$, and hence $m \in \langle b_1, b_2 + cb_1, \dots, b_{l-1} + cb_{l-2}, m + cb_{l-1} \rangle$. Thus $l \geq k$ and so the shortest path between 0 and m in $T\Gamma_c(M, N_\Delta)$ has length k . Therefore $\text{diam}(T\Gamma_c(M, N_\Delta)) = k$. If there is no such k , then we show that $\text{diam}(T\Gamma_c(M, N_\Delta)) = \infty$. Suppose not. Let $\text{diam}(T\Gamma_c(M, N_\Delta)) = t$ where $t < \infty$. Since $k = \infty$, there is $m \in M$ such that $m = n_1 + \dots + n_{t+1}$ is a shortest representation of the element m . According what is proved, it is contradiction since there is a path from 0 to m in $T\Gamma_c(M, N_\Delta)$ of length at most t . $\square$

Remark 3. Let $N_\Delta = \bigcup_{\lambda \in \Delta} N_\lambda$ and $M = \langle N_\Delta \rangle$, then it is clear that $1 \leq k \leq |\Delta|$ where k is as mentioned in Theorem 14.

Corollary 3. *Let M be an R -module and N_Δ a union of prime submodule of M such that $T\Gamma_c(M, N_\Delta)$ is connected. Let $m = n_1 + \dots + n_k$ be a shortest representation of the element m and k is as mentioned in Theorem 14.*

- (1) $\text{diam}(T\Gamma_c(M, N_\Delta)) = d(0, m)$
- (2) If $\text{diam}(T\Gamma_c(M, N_\Delta)) = k$ then $\text{diam}(T\Gamma_c(M \setminus N_\Delta, N_\Delta)) \geq k - 2$.

Proof. (1) This is clear from the proof of Theorem 14.

- (2) Since $k = \text{diam}(T\Gamma_c(M, N_\Delta)) = d(0, m)$ by part (1) above, let $0 - b_1 - \dots - b_{k-1} - m$ be a shortest path from 0 to m in $T\Gamma_c(M, N_\Delta)$. Clearly $b_1 \in N_\Delta$. If $b_i \in N_\Delta$ for some integer i with $2 \leq i \leq k-1$, then the path $0 - b_i -$

$\cdots - b_{k-1} - m$ from 0 to m has length less than k , a contradiction. Thus $b_i \in M \setminus N_\Delta$ for every integer i with $2 \leq i \leq k-1$. Hence $b_2 - \cdots - b_{k-1} - m$ is a shortest path from b_2 to m in $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ of length $k-2$. Thus $\text{diam}(T\Gamma_c(M \setminus N_\Delta, N_\Delta)) \geq k-2$. $\square$

We next investigate the girth of $T\Gamma_c(N_\Delta, N_\Delta)$, $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$, and $T\Gamma_c(M, N_\Delta)$ when N_Δ is not a submodule of M . Recall that $H_\Delta = (N_\Delta :_R M) = \bigcup_{\lambda \in \Delta} P_\lambda$ where N_Δ is proper subset of M with $|N_\Delta| \geq 3$ and $P_\lambda = (N_\lambda :_R M)$.

Theorem 15. *Let M be an R -module, and let $N_\Delta = \bigcup_{\lambda} N_\lambda$ for prime submodules N_λ of M , that is not a submodule of M and $H_\Delta = (N_\Delta :_R M)$. Suppose that $m_1 - m_2 - m_3$ is a path of length two in $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ for distinct elements $m_1, m_2, m_3 \in M \setminus N_\Delta$.*

- (1) *If $c+1 \in H_\Delta$ and $\bigcap_{\lambda} N_\lambda \neq \{0\}$, then $\text{gr}(T\Gamma_c(M \setminus N_\Delta, N_\Delta)) = 3$.*
- (2) *If $(c+1)m_i \neq 0$ for all integers i with $1 \leq i \leq 3$, then $\text{gr}(T\Gamma_c(M \setminus N_\Delta, N_\Delta)) \leq 4$.*

Proof. (1) Suppose that there is a $0 \neq h \in \bigcap_{\lambda} N_\lambda$. If $m_2 \neq m_1 + h$, then $m_1 - m_2 - (m_1 + h) - m_1$ is a cycle of length three in $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ since $(c+1)m_1 \in N_\Delta$. Hence, assume that $m_2 = m_1 + h$. Since $(m_1 + h) + cm_3 = m_2 + cm_3 \in N_\Delta$ and $h \in \bigcap_{\lambda} N_\lambda$, we have $m_1 + cm_3 \in N_\Delta$. Thus $m_1 - m_2 - m_3 - m_1$ is a cycle of length three in $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$. Thus $\text{gr}(T\Gamma_c(M \setminus N_\Delta, N_\Delta)) = 3$.

- (2) Suppose that $(c+1)m_i \neq 0$ for all integers i with $1 \leq i \leq 3$. Then $m_i \neq -cm_i$ for every i with $1 \leq i \leq 3$. There are distinct integer j, k with $1 \leq j, k \leq 3$ such that they are adjacent and $m_j \neq -cm_k$ since if $m_1 + cm_2 = m_3 + cm_2 = 0$, then $m_1 = m_3$, a contradiction. Thus $m_j - m_k - (-cm_k) - (-cm_j) - m_j$ is a 4-cycle in $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$; so $\text{gr}(T\Gamma_c(M \setminus N_\Delta, N_\Delta)) \leq 4$. $\square$

In the above theorem, we improve the proof of Theorem 3.12(1) in [5]. Also in the part (2) of the above theorem, $0 \neq (c+1)m_i$ can belongs to N_Δ unlike Theorem 3.12(3) in [5].

Corollary 4. *Let M be an R -module, and let $N_\Delta = \bigcup_{\lambda} N_\lambda$ for prime submodules N_λ of M , that is not a submodule of M and $H_\Delta = (N_\Delta :_R M)$. Suppose that $m_1 - m_2 - m_3$ is a path of length two in $T\Gamma_c(M \setminus N_\Delta, N_\Delta)$ for distinct elements $m_1, m_2, m_3 \in M \setminus N_\Delta$. If $(c+1)m_i = 0$ for some integer i with $1 \leq i \leq 3$ and $c+1 \notin (0 :_R M)$, then $\text{gr}(T\Gamma_c(M \setminus N_\Delta, N_\Delta)) = 3$.*

Proof. Suppose that $c+1 \notin (0 :_R M)$. Thus $c+1 \neq 0$. Since $m_i \in M \setminus N_\Delta$ for all integers i with $1 \leq i \leq 3$ and $(c+1)m_i = 0$ for some i with $1 \leq i \leq 3$, we have $c+1 \in P_\lambda$ for every P_λ where $P_\lambda = (N_\lambda :_R M)$. Hence $0 \neq c+1 \in \bigcap_{\lambda} P_\lambda$.

Since $c + 1 \notin (0 :_R M)$, there is a $m \in M$ such that $0 \neq (c + 1)m \in \bigcap_{\lambda} N_{\lambda}$. Thus $gr(T\Gamma_c(M \setminus N_{\Delta}, N_{\Delta})) = 3$ by Theorem 15(1). $\square$

Corollary 5. *Let M be an R -module, and let $N_{\Delta} = \bigcup_{\lambda} N_{\lambda}$ for prime submodules N_{λ} of M and $H_{\Delta} = (N_{\Delta} :_R M)$. Suppose that $m_1 - m_2 - m_3$ is a path of length two in $T\Gamma_c(M \setminus N_{\Delta}, N_{\Delta})$ for distinct elements $m_1, m_2, m_3 \in M \setminus N_{\Delta}$, $c + 1 \in H_{\Delta}$, $\bigcap_{\lambda} N_{\lambda} \neq \{0\}$ and $|N_{\Delta}| \geq 3$, then $gr(T\Gamma_c(M \setminus N_{\Delta}, N_{\Delta})) = 3$.*

Proof. This follows directly from Theorem 10(1)(a) and Theorem 15(1). $\square$

Theorem 16. *Let M be an R -module and N_{Δ} a union of prime submodule of M that is not a submodule of M .*

- (1) *Either $gr(T\Gamma_c(N_{\Delta}, N_{\Delta})) = 3$ or $gr(T\Gamma_c(N_{\Delta}, N_{\Delta})) = \infty$. Moreover, if one has $gr(T\Gamma_c(N_{\Delta}, N_{\Delta})) = \infty$, then $|N_{\lambda}| = 2$ for any $\lambda \in \Lambda$ where N_{λ} is a non-zero submodule of M . Also, $T\Gamma_c(N_{\Delta}, N_{\Delta})$ is a star graph.*
- (2) *$gr(T\Gamma_c(M, N_{\Delta})) = 3$ if and only if $gr(T\Gamma_c(N_{\Delta}, N_{\Delta})) = 3$.*
- (3) *$gr(T\Gamma_c(M, N_{\Delta})) \leq 4$.*
- (4) *$gr(T\Gamma_c(M, N_{\Delta})) = 4$ if and only if $gr(T\Gamma_c(N_{\Delta}, N_{\Delta})) = \infty$.*

Proof. (1) If $n + cn' \in N_{\Delta}$ for some distinct $n, n' \in N_{\Delta}^*$, then $0 - n - n' - 0$ is a 3-cycle in $T\Gamma_c(N_{\Delta}, N_{\Delta})$; so $gr(T\Gamma_c(N_{\Delta}, N_{\Delta})) = 3$. Otherwise, $n + cn' \in M \setminus N_{\Delta}$ for all distinct $n, n' \in N_{\Delta}^*$. So in this case, every $n \in N_{\Delta}$ is adjacent to 0, and no two distinct $n, n' \in N_{\Delta}^*$ are adjacent. Thus $T\Gamma_c(N_{\Delta}, N_{\Delta})$ is a star graph with center 0; so $gr(T\Gamma_c(N_{\Delta}, N_{\Delta})) = \infty$. Moreover, let $N_{\Delta} = \bigcup_{\lambda \in \Lambda} N_{\lambda}$ is not a submodule of M where N_{λ} is a prime submodule of M so $|\Lambda| \geq 2$. Assume that $gr(T\Gamma_c(N_{\Delta}, N_{\Delta})) = \infty$. Then $x + cy \in M \setminus N_{\Delta}$ for all distinct $x, y \in N_{\Delta}^*$, and thus if $N_{\lambda} \neq \{0\}$, then each $|N_{\lambda}| = 2$.

- (2) It suffices to show that $gr(T\Gamma_c(N_{\Delta}, N_{\Delta})) = 3$ when $gr(T\Gamma_c(M, N_{\Delta})) = 3$. Let $(c + 1)n \neq 0$ for some $n \in N_{\Delta}^*$, then $0 - n - (-cn) - 0$ is a 3-cycle in $T\Gamma_c(N_{\Delta}, N_{\Delta})$. Otherwise, $(c + 1)n = 0$ for all $n \in N_{\Delta}$. Since N_{Δ} is not a submodule of M , there are distinct elements $n, n' \in N_{\Delta}$ such that $n + n' \in M \setminus N_{\Delta}$. Then $(c + 1)(n + n') = 0$, thus $c + 1 \in H_{\Delta}$. Let $m - m_1 - m_2 - m$ be a 3-cycle in $T\Gamma_c(M, N_{\Delta})$. Then $n_1 = cm + m_1, n_2 = cm + m_2, m_1 + cm_2 \in N_{\Delta}$. Thus $0 - n_1 - n_2 - 0$ is a 3-cycle in $T\Gamma_c(N_{\Delta}, N_{\Delta})$; therefore $gr(T\Gamma_c(N_{\Delta}, N_{\Delta})) = 3$.
- (3) Since N_{Δ} is not a submodule of M , there are distinct elements $n, n' \in N_{\Delta}$ such that $n + n' \in M \setminus N_{\Delta}$. Then $0 - (-cn) - n + n' - (-cn') - 0$ is a 4-cycle in $T\Gamma_c(M, N_{\Delta})$.
- (4) This follows by parts (1), (2) and (3) above. $\square$

Example 3. (a) Let $M = \mathbb{Z}[X]$ be an $\mathbb{Z}[X]$ -module. Then $N_{\Delta} = \mathbb{Z}[X] \setminus \mathbb{Z}^*$ is a union of prime submodule of M , that is not a submodule of M . Thus $T\Gamma_c(N_{\Delta}, N_{\Delta})$ is connected with $diam(T\Gamma_c(N_{\Delta}, N_{\Delta})) = 2$ by Theorem

- 11(1). Moreover, by Theorem 12 and 14, $T\Gamma_c(M, N_A)$ is connected with $\text{diam}(T\Gamma_c(M, N_A)) = 2$ since $z = X + z - X$ for $X, z - X \in N_A$ for every $z \in \mathbb{Z}^*$. However, $T\Gamma_c(M \setminus N_A, N_A)$ is not connected since there is no path from 1 to 2 in $T\Gamma_c(M \setminus N_A, N_A)$. Thus the converse of Theorem 11(3) need not hold.
- (b) Let $k \in \mathbb{N}$, $M = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $R = \mathbb{Z}_{2k}$, $N_1 = \mathbb{Z}_2 \times 0 \times 0$, $N_2 = 0 \times \mathbb{Z}_2 \times 0$, $N_3 = 0 \times 0 \times \mathbb{Z}_2$ and $N_A = \bigcup N_j$ for $1 \leq j \leq 3$, then by Theorem 12 and 14, $T\Gamma_c(M, N_A)$ is connected with $\text{diam}(T\Gamma_c(M, N_A)) = 3$, note that $(1, 1, 0)$ is the sum of two elements of N_A but $(1, 1, 1)$ is the sum of three element of N_A , and $T\Gamma_c(R, H_A)$ is disconnected so the converse of Theorem 13 is fails, also $\text{gr}(T\Gamma_c(N_A, N_A)) = \infty$ and $\text{gr}(T\Gamma_c(M, N_A)) = 4$ by Theorem 16.

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MULTIPLE SOLUTIONS FOR NEUMANN SYSTEMS IN AN ORLICZ-SOBOLEV SPACE SETTING

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Abstract. In this paper, the authors improve some results on the existence of at least three weak solutions for non-homogeneous systems. The proof of the main result relies on a recent variational principle due to Ricceri.

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1. INTRODUCTION

This paper is motivated by phenomena that are described by non-homogeneous Neumann double eigenvalue systems of the type

$$\begin{cases} -\operatorname{div}(\alpha_i(|\nabla u_i|)\nabla u_i) + \alpha_i(|u_i|)u_i = \lambda F_{u_i}(x, u_1, \dots, u_n) \\ \quad + \mu G_{u_i}(x, u_1, \dots, u_n), & \text{in } \Omega, \\ \frac{\partial u_i}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

for $1 \leq i \leq n$, where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 3$) with a smooth boundary $\partial\Omega$, ν is the outward unit normal to $\partial\Omega$, λ and μ are positive parameters, $F, G : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ are measurable functions with respect to $x \in \Omega$ for every $(t_1, \dots, t_n) \in \mathbb{R}^n$ and are C^1 with respect to $(t_1, \dots, t_n) \in \mathbb{R}^n$ for a.e. $x \in \Omega$. Here, F_{u_i} and G_{u_i} denote the partial derivative of F and G with respect to u_i , respectively, and the functions $\alpha_i(t) : \mathbb{R} \rightarrow \mathbb{R}$ for $1 \leq i \leq n$ will be specified later.

For any function $H : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$, we denote by (H0) the following assumption on H :

(H0) H is measurable with respect to $x \in \Omega$ for every $(t_1, \dots, t_n) \in \mathbb{R}^n$, and for every $M > 0$ and for every $1 \leq i \leq n$,

$$\sup_{|(t_1, \dots, t_n)| \leq M} |H_{u_i}(x, t_1, \dots, t_n)| \in L^1(\Omega).$$

Our approach in this paper relies on variational methods in Orlicz-Sobolev spaces. Such spaces originated with Nakano [26] and were further developed by Musielak

and Orlicz [25]. Many properties of Sobolev spaces have been extended to Orlicz-Sobolev spaces, mainly by Dankert [11], Donaldson and Trudinger [14], and O'Neill [27] (see also [1] for an excellent account of those works). Orlicz-Sobolev spaces have been used in the last two decades to model various phenomena. These kind of spaces play a significant role in many fields of mathematics, such as approximation theory, partial differential equations, calculus of variations, nonlinear potential theory, quasiconformal mappings, differential geometry, geometric function theory, and probability theory. Due to this, several authors have widely studied the existence of solutions for eigenvalue problems involving non-homogeneous operators (see, for example, [2, 3, 5–10, 15, 17, 18, 20–22, 33]).

Notice that if $i = 1$ and $\mu = 0$, then problem (1.1) becomes

$$\begin{cases} -\operatorname{div}(\alpha(|\nabla u|)\nabla u) + \alpha(|u|)u = \lambda f(x, u), & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

Bonanno, Molica Bisci, and Rădulescu [5] used a recent critical points result in order to prove the existence of an open interval of positive eigenvalues for which the problem (1.2) admits at least three weak solutions. The same authors in [6] established an open interval of positive parameters for which the problem (1.2) admits infinitely many weak solutions that strongly converges to zero.

In [8] Cammaroto and Vilasi proved the existence of three weak solutions to the nonhomogeneous boundary value problem:

$$\begin{cases} -\operatorname{div}(\alpha(|\nabla u|)\nabla u) = \lambda f(x, u) + \mu g(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

Note that if we take $\alpha(t) = |t|^{p-2}$ ($p > 1$) in problems (1.2) and (1.3), the operator on the left hand side reduces to the classical p -Laplacian; these types of problems have been widely investigated and a large variety of existence results can be found in the literature.

The classical three critical point theorem of Pucci and Serrin [28, 29] asserts that if $h : Y \rightarrow \mathbb{R}$ is of class C^1 (Y is a Banach space), satisfies the Palais-Smale condition, and has two local minima, then h has a third critical point. The general variational principle of Ricceri [31, 32] extends the Pucci-Serrin theorem and provides alternatives for the multiplicity of critical points of certain functionals depending on a parameter. We refer to [19] and the monograph of Graef and Kong [16] for several applications of Ricceri's variational principles.

In this paper, motivated by the above facts and the recent paper of Molica Bisci and Rădulescu [23], under appropriate conditions on F and G , we present sufficient conditions for the system (1.1) to possesses three weak solutions in an Orlicz-Sobolev space.

Our paper is organized as follows. In Section 2, some preliminaries and the abstract Orlicz-Sobolev space settings are presented. In Section 3, we discuss the existence of three weak solutions for the system (1.1). We also point out special cases of the results and we illustrate the results by presenting an example.

2. FUNCTIONAL SETTING

We now recall the following three critical points theorem that follows from a combination of [4, Theorem 3.6] and [32, Theorem 1].

Theorem 1. *Let X be a reflexive real Banach space, $J : X \rightarrow \mathbb{R}$ be a sequentially weakly lower semicontinuous and continuously Gâteaux differentiable functional that is bounded on bounded subsets of X and whose Gâteaux derivative admits a continuous inverse on the dual space X^* , and let $I : X \rightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact and satisfies*

$$J(0) = I(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$ with $r < J(\bar{x})$ such that

- (j) $\sup_{J(x) \leq r} I(x) < rI(\bar{x})/J(\bar{x})$,
- (jj) for each λ in

$$\Lambda_r := \left(\frac{J(\bar{x})}{I(\bar{x})}, \frac{r}{\sup_{J(x) \leq r} I(x)} \right),$$

the functional $J - \lambda I$ is coercive.

Then, for each compact interval $[a, b] \in \Lambda_r$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a, b]$ and every C^1 functional $\Gamma : X \rightarrow \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation $J'(x) - \lambda I'(x) - \mu \Gamma'(x) = 0$ has at least three solutions in X whose norms are less than ρ .

In order to study the problem (1.1), let us introduce the function spaces settings to be used. We will give just a brief review of some basic concepts and facts of the theory of Orlicz-Sobolev spaces that will be useful in what follows; for more details we refer the reader to Adams [1], Diening [12], Musielak [24], and Rao and Ren [30].

We begin by recalling some facts from the theory of Orlicz-Sobolev spaces that will be used in the present paper. Suppose that the function $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is such that the mapping $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\varphi(t) = \begin{cases} \alpha(|t|)t, & \text{for } t \neq 0, \\ 0, & \text{for } t = 0, \end{cases}$$

is an odd and strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. Consider the functions

$$\Phi(t) = \int_0^t \varphi(s) ds \quad \text{and} \quad \Phi^*(t) = \int_0^t \varphi^{-1}(s) ds \quad \text{for } t \in \mathbb{R}.$$

We observe that Φ is a Young's function, that is, $\Phi(0) = 0$, Φ is convex, and

$$\lim_{t \rightarrow \infty} \Phi(t) = +\infty.$$

Furthermore, since $\Phi(t) = 0$ if and only if $t = 0$,

$$\lim_{t \rightarrow 0} \frac{\Phi(t)}{t} = 0, \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{\Phi(t)}{t} = +\infty,$$

Φ is called an N -function. The function Φ^* is called the complementary function of Φ and satisfies

$$\Phi^*(t) = \sup\{st - \Phi(s) : s \geq 0\} \quad \text{for all } t \geq 0.$$

We note that Φ^* is also an N -function and the following Young's inequality holds:

$$st \leq \Phi(s) + \Phi^*(t), \quad \text{for all } s, t \geq 0.$$

Assume that Φ satisfies the structural hypotheses

$$(\Phi 0) \quad 1 < \liminf_{t \rightarrow \infty} \frac{t\varphi(t)}{\Phi(t)} \leq \varphi^+ := \sup_{t > 0} \frac{t\varphi(t)}{\Phi(t)} < \infty$$

and

$$(\Phi 1) \quad N < \varphi^- := \inf_{t > 0} \frac{t\varphi(t)}{\Phi(t)} < \liminf_{t \rightarrow \infty} \frac{\log(\Phi(t))}{\log(t)}.$$

We also assume that the function

$$(\Phi 2) \quad t \rightarrow \Phi(\sqrt{t}) \quad \text{for } t \in [0, \infty)$$

is convex. To illustrate these notions, we have the following example.

Example 1. Let $p > N + 1$. Define

$$\varphi(t) = \frac{|t|^{p-2}}{\log(1+|t|)} t \quad \text{for } t \neq 0, \quad \varphi(0) = 0,$$

and

$$\Phi(t) = \int_0^t \varphi(s) ds.$$

By [9, Example 3, p. 243], we have

$$\varphi^- = p - 1 < \varphi^+ = p = \liminf_{t \rightarrow \infty} \frac{\log(\Phi(t))}{\log(t)}.$$

Thus, conditions $(\Phi 0)$ and $(\Phi 1)$ are satisfied. Furthermore, by a direct computation, we see that the function $t \mapsto \Phi(\sqrt{t})$ for $t \in [0, \infty)$ is convex, so $(\Phi 2)$ is also satisfied.

The Orlicz space $L_\Phi(\Omega)$ defined by the N -function Φ (see for instance [1] and [13]) is the space of measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that

$$\|u\|_{L_\Phi} := \sup \left\{ \left| \int_\Omega u(x)v(x) dx \right| : \int_\Omega \Phi^*(|v(x)|) dx \leq 1 \right\} < \infty.$$

The space $(L_\Phi(\Omega), \|\cdot\|_{L_\Phi})$ is a Banach space whose norm is equivalent to the Luxemburg norm

$$\|u\|_{1,\Phi} := \inf \left\{ k > 0 : \int_\Omega \Phi \left(\frac{u(x)}{k} \right) dx \leq 1 \right\}.$$

We denote by $W^1 L_\Phi(\Omega)$ the corresponding Orlicz-Sobolev space for problem (1.1) defined by

$$W^1 L_\Phi(\Omega) = \left\{ u \in L_\Phi(\Omega) : \frac{\partial u}{\partial x_j} \in L_\Phi(\Omega), j = 1, \dots, N \right\}.$$

This is a Banach space with respect to the norm

$$\|u\|_{2,\Phi} = \|\nabla u\|_{1,\Phi} + \|u\|_{1,\Phi};$$

see [1, 10, 15].

The following two lemmas can be found in [18].

Lemma 1. ([18, Lemma 2.2]) *In $W^1 L_\Phi(\Omega)$, the norms*

$$\|u\|_{2,\Phi} = \|\nabla u\|_{1,\Phi} + \|u\|_{1,\Phi},$$

$$\|u\|_{3,\Phi} = \max\{\|\nabla u\|_{1,\Phi}, \|u\|_{1,\Phi}\},$$

and

$$\|u\|_\Phi = \inf \left\{ \mu > 0 : \int_\Omega \left[\Phi \left(\frac{|u(x)|}{\mu} \right) + \Phi \left(\frac{|\nabla u(x)|}{\mu} \right) \right] dx \leq 1 \right\}$$

are equivalent. More precisely, for every $u \in W^1 L_\Phi(\Omega)$ we have

$$\|u\|_\Phi \leq 2\|u\|_{3,\Phi} \leq 2\|u\|_{2,\Phi} \leq 4\|u\|_\Phi.$$

Lemma 2. ([18, Lemma 2.3]) *Let $u \in W^1 L_\Phi(\Omega)$. Then:*

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)] dx \geq \|u\|_\Phi^{\varphi^-}, \quad \text{if } \|u\|_\Phi > 1;$$

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)] dx \geq \|u\|_\Phi^{\varphi^+}, \quad \text{if } \|u\|_\Phi < 1.$$

These spaces generalize the usual spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$, in which the role played by the convex mapping $t \mapsto |t|^p/p$ is assumed by a more general convex function $\Phi(t)$.

Proposition 1. (See [1, p. 241 and p. 247]) *The spaces $L_\Phi(\Omega)$ and $W^1 L_\Phi(\Omega)$ are separable and reflexive Banach spaces.*

Remark 1. In view of condition $(\Phi 1)$, from [10, Lemma D.2] it follows that $W^1 L_{\Phi}(\Omega)$ is continuously embedded in $W^{1, \varphi^-}(\Omega)$. On the other hand, since we assumed that $\varphi^- > N$, we can conclude that $W^{1, \varphi^-}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Thus, we have that $W^1 L_{\Phi}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$.

Throughout this paper, the functions $\alpha_i : \mathbb{R} \rightarrow \mathbb{R}$ are such that the mappings $\varphi_i : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\varphi_i(t) = \begin{cases} \alpha_i(|t|)t, & \text{for } t \neq 0, \\ 0, & \text{for } t = 0, \end{cases}$$

for $1 \leq i \leq n$ are odd and strictly increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$. Also, assume that the functions

$$\Phi_i(t) = \int_0^t \varphi_i(s) ds \quad \text{for } t \in \mathbb{R}$$

for $1 \leq i \leq n$ satisfying the structural hypotheses $(\Phi 0)$, $(\Phi 1)$, and $(\Phi 2)$.

We let E be the Cartesian product of n Orlicz-Sobolev spaces $W^1 L_{\Phi_i}(\Omega)$ for $1 \leq i \leq n$, i.e., $E = \prod_{i=1}^n W^1 L_{\Phi_i}(\Omega)$ endowed with the norm

$$\|u\| := \sum_{i=1}^n \|u\|_{\Phi_i}, \quad u = (u_1, \dots, u_n) \in E.$$

In the space $C^0(\overline{\Omega})$, we consider the norm $\|u\|_{\infty} := \sup_{x \in \overline{\Omega}} |u(x)|$. Set

$$m := \max \left\{ \sup_{u \in W^1 L_{\Phi_i}(\Omega) \setminus \{0\}} \frac{\|u\|_{\infty}}{\|u\|_{\Phi_i}} : \text{for } 1 \leq i \leq n \right\}. \quad (2.1)$$

Since $N < \varphi_i^-$ for $1 \leq i \leq n$, from Remark 1, the embedding $E \hookrightarrow C^0(\overline{\Omega}) \times \dots \times C^0(\overline{\Omega})$ is compact, so that $m < +\infty$.

We say that $u = (u_1, \dots, u_n) \in E$ is a *weak solution* of the problem (1.1) if

$$\begin{aligned} & \int_{\Omega} \sum_{i=1}^n (\alpha_i(|\nabla u_i(x)|) \nabla u_i(x) \nabla v_i(x) + \alpha_i(|u_i(x)|) u_i(x) v_i(x)) dx \\ & - \lambda \int_{\Omega} \sum_{i=1}^n F_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) dx \\ & - \mu \int_{\Omega} \sum_{i=1}^n G_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) dx = 0 \end{aligned}$$

for every $v = (v_1, \dots, v_n) \in E$.

3. MAIN RESULTS

We begin this section with our main existence result.

Theorem 2. *Assume that $F : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies (H0) and the following conditions:*

(F1) *there exist $d \in L^1(\Omega)$ and n positive constants β_i , with $\beta_i < \varphi_i^-$ for $1 \leq i \leq n$, such that*

$$0 \leq F(x, t_1, \dots, t_n) \leq d(x) \left(1 + \sum_{i=1}^n |t_i|^{\beta_i} \right)$$

for a.e. $x \in \Omega$ and all $(t_1, \dots, t_n) \in \mathbb{R}^n$;

(F2) *$F(x, 0, \dots, 0) = 0$ for a.e. $x \in \Omega$;*

(F3) *there exist $0 < c_i < m$ and $s_i \in \mathbb{R}$ for $1 \leq i \leq n$ with*

$$|\Omega| \sum_{i=1}^n \Phi_i(|s_i|) > \min \left\{ \left(\frac{c_i}{m} \right)^{\varphi_i^+} : 1 \leq i \leq n \right\},$$

such that

$$\begin{aligned} & \int_{\Omega} \sup_{|t_1| \leq c_1, \dots, |t_n| \leq c_n} F(x, t_1, \dots, t_n) dx \\ & < \frac{\min \left\{ \left(\frac{c_i}{m} \right)^{\varphi_i^+} : 1 \leq i \leq n \right\}}{|\Omega| \sum_{i=1}^n \Phi_i(|s_i|)} \int_{\Omega} F(x, s_1, \dots, s_n) dx, \end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of the set Ω .

Then, setting

$$\Lambda := \left(\frac{|\Omega| \sum_{i=1}^n \Phi_i(|s_i|)}{\int_{\Omega} F(x, s_1, \dots, s_n) dx}, \frac{\min \left\{ \left(\frac{c_i}{m} \right)^{\varphi_i^+} : 1 \leq i \leq n \right\}}{\int_{\Omega} \sup_{|t_1| \leq c_1, \dots, |t_n| \leq c_n} F(x, t_1, \dots, t_n) dx} \right),$$

for each compact interval $[a, b] \subset \Lambda$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a, b]$ and for every function $G : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying condition (H0), there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the system (1.1) admits at least three weak solutions in E whose norms are less than ρ .

Proof. In order to prove the theorem, we will apply Theorem 1. For each $(u_1, \dots, u_n) \in E$, let the functionals $J, I : E \rightarrow \mathbb{R}$ be defined by

$$J(u) := \int_{\Omega} \sum_{i=1}^n (\Phi_i(|\nabla u_i(x)|) + \Phi_i(|u_i(x)|)) dx$$

and

$$I(u) := \int_{\Omega} F(x, u_1(x), \dots, u_n(x)) \, dx.$$

Clearly, J is bounded on each bounded subset of E and similar arguments as those used in [21, Lemma 4.2] imply that J and I are continuously Gâteaux differentiable functionals whose derivatives at the point $u = (u_1, \dots, u_n) \in E$ are the functional $J'(u)$ and $I'(u)$ given by

$$J'(u)(v) = \int_{\Omega} \sum_{i=1}^n (\alpha_i(|\nabla u_i(x)|) \nabla u_i(x) \nabla v_i(x) + \alpha_i(|u_i(x)|) u_i(x) v_i(x)) \, dx$$

and

$$I'(u)(v) = \int_{\Omega} \sum_{i=1}^n F_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) \, dx$$

for every $v = (v_1, \dots, v_n) \in E$. Moreover, since Φ_i for $1 \leq i \leq n$ are convex, it follows that J is a convex functional, and hence it is sequentially weakly lower semi-continuous. By Lemma 2, we see that J is coercive, and arguing as in the proof of [18, Lemma 3.2], we have that $J' : E \rightarrow E^*$ is a uniformly monotone operator in E . By applying the Minty-Browder theorem (Theorem 26.A of [34]), J' admits a continuous inverse on E^* .

We claim that $I' : E \rightarrow E^*$ is a compact operator. To this end, it suffices to show that I' is strongly continuous on E , so for fixed $(u_1, \dots, u_n) \in E$, let $(u_{1m}, \dots, u_{nm}) \rightarrow (u_1, \dots, u_n)$ weakly in E as $m \rightarrow +\infty$. Then we have $(u_{1m}, \dots, u_{nm})$ converges uniformly to $(u_1, \dots, u_n)$ on Ω as $m \rightarrow +\infty$ (see [34]). Since $F(x, \cdot, \dots, \cdot)$ is C^1 in $\mathbb{R}^n$ for every $x \in \Omega$, the partial derivatives of F are continuous in $\mathbb{R}^n$ for every $x \in \Omega$, so for $1 \leq i \leq n$, $F_{u_i}(x, u_{1m}, \dots, u_{nm}) \rightarrow F_{u_i}(x, u_1, \dots, u_n)$ strongly as $m \rightarrow +\infty$. By the Lebesgue dominated convergence theorem, $I'(u_{1m}, \dots, u_{nm}) \rightarrow I'(u_1, \dots, u_n)$ strongly as $m \rightarrow +\infty$. Thus, I' is strongly continuous on E , which implies that I' is a compact operator by [34, Proposition 26.2]. Hence, our claim is true.

From the definitions of J and I and from (F2), we have

$$J(0) = I(0) = 0.$$

Next, set $w(x) := (s_1, \dots, s_n)$ for any $x \in \Omega$, and

$$r := \min \left\{ \left(\frac{c_i}{m} \right)^{\varphi_i^+} : 1 \leq i \leq n \right\}.$$

Clearly, $w \in E$, and we have

$$J(w) = |\Omega| \sum_{i=1}^n \Phi_i(|s_i|) > r > 0.$$

Let $J(u) \leq r$ for $u = (u_1, \dots, u_n) \in E$. By Lemma 2, for $1 \leq i \leq n$, we have

$$\|u\|_{\Phi_i} \leq \max\{r^{1/\varphi_i^-}, r^{1/\varphi_i^+}\}.$$

Then, for $1 \leq i \leq n$,

$$\|u\|_{\Phi_i} \leq \frac{c_i}{m}.$$

Taking into account that for each $u \in W^1 L_{\Phi_i}(\Omega)$,

$$\|u\|_{\infty} \leq m\|u\|_{\Phi_i}$$

for $1 \leq i \leq n$ (see (2.1)), we obtain

$$\|u_i\|_{\infty} \leq c_i \quad \text{for } 1 \leq i \leq n.$$

Therefore, for every $u = (u_1, \dots, u_n) \in E$,

$$\begin{aligned} \sup_{u \in J^{-1}((-\infty, r))} I(u) &= \int_{\Omega} F(x, u_1(x), \dots, u_n(x)) dx \\ &\leq \int_{\Omega} \sup_{|t_1| \leq c_1, \dots, |t_n| \leq c_n} F(x, t_1, \dots, t_n) dx. \end{aligned}$$

Condition (F3) implies

$$\begin{aligned} \frac{\sup_{u \in J^{-1}((-\infty, r))} I(u)}{r} &\leq \frac{\int_{\Omega} \sup_{|t_1| \leq c_1, \dots, |t_n| \leq c_n} F(x, t_1, \dots, t_n) dx}{\min \left\{ \left(\frac{c_i}{m} \right)^{\varphi_i^+} : 1 \leq i \leq n \right\}} \\ &< \frac{\int_{\Omega} F(x, s_1, \dots, s_n) dx}{|\Omega| \sum_{i=1}^n \Phi_i(|s_i|)} \\ &= \frac{I(w)}{J(w)}. \end{aligned}$$

Thus, assumption (j) of Theorem 1 is satisfied.

In view of (F1), the functional $J - \lambda I$ is coercive for every positive parameter λ , in particular, for every

$$\lambda \in \Lambda \subseteq \left(\frac{J(w)}{I(w)}, \frac{r}{\sup_{J(u) \leq r} I(u)} \right),$$

so condition (jj) of Theorem 1 holds. In addition, for every function $G : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying condition (H0), the functional

$$\Gamma(u) := \int_{\Omega} G(x, u_1(x), \dots, u_n(x)) dx$$

is well defined and continuously Gâteaux differentiable on E with a compact derivative given by

$$\Gamma'(u)(v) = \int_{\Omega} \sum_{i=1}^n G_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) dx$$

for every $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n) \in E$. Thus, all the conditions of Theorem 1 are satisfied. Also, note that the solutions of the equation

$$J'(u) - \lambda I'(u) - \mu \Gamma'(u) = 0$$

are exactly the weak solutions of (1.1). So, the conclusion follows from Theorem 1. $\square$

Next, we consider the special case of Theorem 2 where F does not depend on $x \in \Omega$.

Theorem 3. Assume that $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is a C^1 -function satisfying

(F4) there exist $n+1$ positive constants η and β_i with $\beta_i < \varphi_i^-$ for $1 \leq i \leq n$, such that

$$0 \leq F(t_1, \dots, t_n) \leq \eta \left(1 + \sum_{i=1}^n |t_i|^{\beta_i} \right)$$

for all $(t_1, \dots, t_n) \in \mathbb{R}^n$;

(F5) $F(0, \dots, 0) = 0$;

(F6) there exist $0 < c_i < m$ and $s_i \in \mathbb{R}$ for $1 \leq i \leq n$ with

$$|\Omega| \sum_{i=1}^n \Phi_i(|s_i|) > \min \left\{ \left(\frac{c_i}{m} \right)^{\varphi_i^+} : 1 \leq i \leq n \right\},$$

such that

$$\begin{aligned} & \sup_{|t_1| \leq c_1, \dots, |t_n| \leq c_n} F(t_1, \dots, t_n) \\ & < \frac{\min \left\{ \left(\frac{c_i}{m} \right)^{\varphi_i^+} : 1 \leq i \leq n \right\}}{|\Omega| \sum_{i=1}^n \Phi_i(|s_i|)} F(s_1, \dots, s_n). \end{aligned}$$

Then, setting

$$\Lambda := \left(\frac{\sum_{i=1}^n \Phi_i(|s_i|)}{F(s_1, \dots, s_n)}, \frac{\min \left\{ \left(\frac{c_i}{m} \right)^{\varphi_i^+} : 1 \leq i \leq n \right\}}{|\Omega| \sup_{|t_1| \leq c_1, \dots, |t_n| \leq c_n} F(t_1, \dots, t_n)} \right),$$

for each compact interval $[a, b] \subset \Lambda$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a, b]$ and for every function $G : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying condition (H0), there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the system

$$\begin{cases} -\operatorname{div}(\alpha_i(|\nabla u_i|)\nabla u_i) + \alpha_i(|u_i|)u_i = \lambda F_{u_i}(u_1, \dots, u_n) \\ \quad + \mu G_{u_i}(x, u_1, \dots, u_n), & \text{in } \Omega, \\ \frac{\partial u_i}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

for $1 \leq i \leq n$, admits at least three weak solutions in E whose norms are less than ρ .

Now, consider the problem

$$\begin{cases} -\operatorname{div}(\alpha(|\nabla u|)\nabla u) + \alpha(|u|)u = \lambda f(u) + \mu g(x, u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.2)$$

where λ and μ are positive parameters, $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is an L^1 -Carathéodory function, and α is as in Section 2.

Let

$$k := \sup_{u \in W^1 L_\Phi(\Omega) \setminus \{0\}} \frac{\|u\|_\infty}{\|u\|_\Phi}$$

and

$$F(t) := \int_0^t f(\xi) d\xi \quad \text{for } t \in \mathbb{R}. \quad (3.3)$$

The following result is a special case of Theorem 3.

Theorem 4. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative continuous function, and assume that the function F in (3.3) satisfies:*

(F7) *there exist two positive constants η and $\beta < \varphi^-$ such that*

$$F(t) \leq \eta(1 + |t|^\beta) \quad \text{for } t \in \mathbb{R};$$

(F8) *there exist $0 < c < k$ and $s \in \mathbb{R}$ with*

$$|\Omega| \Phi(|s|) > \left(\frac{c}{k}\right)^{\varphi^+}$$

such that

$$\sup_{|t| \leq c} F(t) < \frac{\left(\frac{c}{k}\right)^{\varphi^+}}{|\Omega| \Phi(|s|)} F(s).$$

Then, setting

$$\Lambda := \left(\frac{\Phi(|s|)}{F(s)}, \frac{\left(\frac{c}{k}\right)^{\varphi^+}}{|\Omega| \sup_{|t| \leq c} F(t)} \right),$$

for each compact interval $[a, b] \subset \Lambda$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a, b]$ and for every L^1 -Carathéodory function $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, there

exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the system (3.2) admits at least three weak solutions in $W^1 L_\Phi(\Omega)$ whose norms are less than ρ .

As an example of our main result, we state a special case of Theorem 4.

Corollary 1. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative (not identically zero) continuous function such that*

$$(\ell_0) \quad \lim_{t \rightarrow 0^+} \frac{f(t)}{t^{\varphi^+ - 1}} = 0.$$

Assume that

$$F(t) \leq \eta \left(1 + |t|^\beta\right) \quad \text{for } t \in \mathbb{R},$$

and set

$$\Lambda := \left(|\Omega| \inf_{s \in S} \frac{\Phi(s)}{F(s)}, +\infty \right),$$

where

$$S := \{s > 0 : F(s) > 0\}.$$

Then, for each compact interval $[a, b] \subset \Lambda$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a, b]$ and for every L^1 -Carathéodory function $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the system (3.2) admits at least three weak solutions in $W^1 L_\Phi(\Omega)$ whose norms are less than ρ .

Proof. Fix $\lambda > |\Omega| \inf_{s \in S} \frac{\Phi(s)}{F(s)}$. Then, there exists $\bar{s}$ such that $F(\bar{s}) > 0$ and

$$\lambda > |\Omega| \frac{\Phi(\bar{s})}{F(\bar{s})}.$$

From condition (ℓ_0) , we have

$$\lim_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^{\varphi^+}} = 0.$$

Therefore, we can find a positive constant $\bar{c}$ such that

$$\bar{c} < k \min \left\{ 1, (|\Omega| \Phi(|s|))^{1/\varphi^+} \right\},$$

and

$$\frac{F(\bar{c})}{\bar{c}^{\varphi^+}} < \frac{1}{k^{\varphi^+}} \min \left\{ \frac{F(\bar{s})}{|\Omega| \Phi(|s|)}, \frac{1}{\lambda |\Omega|} \right\}.$$

Hence,

$$\lambda \in \left(\frac{\Phi(|s|)}{F(s)}, \frac{\left(\frac{\bar{c}}{k}\right)^{\varphi^+}}{|\Omega| F(\bar{c})} \right).$$

The hypotheses of Theorem 4 are satisfied, so the conclusion follows. $\square$

Next, we give a concrete application of Corollary 1.

Example 2. Let Ω be a non-empty bounded open subset of the Euclidean Space $\mathbb{R}^N$ ($N \geq 3$) with smooth boundary $\partial\Omega$, and let Φ be a Young's function that satisfies hypotheses $(\Phi 0)$ – $(\Phi 2)$. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(t) := \begin{cases} 0, & \text{if } t < 0, \\ t^{\varphi^+}, & \text{if } 0 \leq t \leq 1, \\ t^s, & \text{if } t > 1, \end{cases}$$

where $s \in (0, \varphi^- - 1)$. Setting

$$\Lambda := \left(\frac{\Phi(|s|)}{F(s)}, \frac{(\frac{c}{k})^{\varphi^+}}{|\Omega| \sup_{|t| \leq c} F(t)} \right),$$

from Corollary 1 for each compact interval $[a, b] \subset \Lambda$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a, b]$ and for every L^1 -Carathéodory function $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the system (3.2) admits at least three weak solutions in $W^1 L_\Phi(\Omega)$ whose norms are less than ρ .

In particular, let $\Omega \subset \mathbb{R}^3$ with $|\Omega| = 1$. Consider $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(t) := \begin{cases} 0, & \text{if } t < 0, \\ t^5, & \text{if } 0 \leq t \leq 1, \\ t^2, & \text{if } t > 1. \end{cases}$$

Set $\Phi(s) := \int_0^s \frac{t|t|^3}{\log(1+|t|)} dt$. Then, for every $s > 0$,

$$\frac{\Phi(s)}{F(s)} := \begin{cases} 6 \frac{\int_0^s \frac{t|t|^3}{\log(1+|t|)} dt}{s^6}, & \text{if } 0 \leq s \leq 1, \\ 6 \frac{\int_0^s \frac{t|t|^3}{\log(1+|t|)} dt}{2s^3 - 1}, & \text{if } s > 1. \end{cases}$$

Moreover, by direct computations, since the function $\frac{\Phi(s)}{F(s)}$ attains its minimum at $s_0 \approx 1.189089126$, it follows that

$$\inf_{s>0} \frac{\Phi(s)}{F(s)} \approx 1.804670144.$$

So, for every L^1 -Carathéodory function $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the non-homogeneous Neumann problem

$$\begin{cases} -\operatorname{div}\left(\frac{|\nabla u|^3}{\log(1+|\nabla u|)}\nabla u\right) + \frac{|u|^3}{\log(1+|u|)}u = 2f(u) + \mu g(x, u), & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

admits at least three weak solutions in $W^1 L_\Phi(\Omega)$.

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SOME RESULTS ON THE JOINT HIGHER NUMERICAL RANGES AND RADII OF MATRICES

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Abstract. In this paper, some algebraic properties of the joint k -numerical radius, joint k -radius and the joint k -norm of matrices are investigated. Moreover, using the joint higher numerical ranges of diagonal matrices which are convex polyhedrons, a description for the shape of the higher numerical ranges of matrix polynomials is given.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{M}_{n \times m}$ be the vector space of all $n \times m$ complex matrices. For the case $n = m$, $\mathbb{M}_{n \times n}$ is denoted by $\mathbb{M}_n$; namely, the algebra of all $n \times n$ complex matrices. Throughout the paper, k, m and n are considered as positive integers, and $k \leq n$. Moreover, I_k denotes the $k \times k$ identity matrix. The set of all $n \times k$ isometry matrices is denoted by $\mathcal{X}_{n \times k}$; i.e., $\mathcal{X}_{n \times k} = \{X \in \mathbb{M}_{n \times k} : X^*X = I_k\}$. Also, the group of $n \times n$ unitary matrices is denoted by $\mathcal{U}_n$; namely,

$$\mathcal{U}_n = \{U \in \mathbb{M}_n : U^*U = I_n\} = \mathcal{X}_{n \times n}.$$

The notion of k -numerical range of $A \in \mathbb{M}_n$, which was first introduced by P. R. Halmos in [8], is defined and denoted by

$$W_k(A) = \left\{ \frac{1}{k} \operatorname{tr}(X^*AX) : X \in \mathcal{X}_{n \times k} \right\},$$

where $\operatorname{tr}(\cdot)$ denotes the trace. The sets $W_k(A)$, where $k \in \{1, 2, \dots, n\}$, are generally called the *higher numerical ranges* of A . When $k = 1$, we have the classical numerical range $W_1(A) = W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}$, which has been studied extensively; see for example [7] and [9, Chapter 1]. Motivation of our study comes from finite-dimensional quantum systems. In quantum physics, e.g., see [6], the quantum states are represented by density matrices, i.e., positive semidefinite matrices with trace one. If a quantum state $D \in \mathbb{M}_n$ has rank one, i.e., $D = xx^*$

for some $x \in \mathbb{C}^n$ with $x^*x = 1$, then D is called a pure quantum state; otherwise, D is said to be a mixed quantum state which can be written as a convex combination of pure quantum states. So, for $A \in \mathbb{M}_n$, we have $W(A) = \{tr(AD) : D \in \mathbb{M}_n \text{ is a pure quantum state}\}$. It is known that A is Hermitian if and only if $W(A) \subseteq \mathbb{R}$, and also A is a positive semidefinite matrix if and only if $W(A) \subseteq [0, \infty]$. Moreover, by the fact that the convex hull of the set

$$\left\{ \frac{1}{k}P : P \in \mathbb{M}_n \text{ is Hermitian, } P^2 = P, \text{ and } tr(P) = k \right\}$$

equals to the set $\mathcal{S}_k$ of density matrices $D \in \mathbb{M}_n$ satisfying $\frac{1}{k}I_n - D$ is positive semidefinite, we have

$$\begin{aligned} W_k(A) &= \left\{ \frac{1}{k}tr(AP) : P \in \mathbb{M}_n, P^2 = P = P^*, tr(P) = k \right\} \\ &= \{tr(AD) : D \in \mathcal{S}_k\}. \end{aligned}$$

Let $A \in \mathbb{M}_n$ have eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, counting multiplicities. The set of all k -averages of eigenvalues of A is denoted by $\sigma^{(k)}(A)$; namely,

$$\sigma^{(k)}(A) = \left\{ \frac{1}{k}(\lambda_{i_1} + \lambda_{i_2} + \dots + \lambda_{i_k}) : 1 \leq i_1 < i_2 < \dots < i_k \leq n \right\}.$$

Notice that if $k = 1$, then $\sigma^{(1)}(A) = \sigma(A)$, i.e., the spectrum of A . Next, we list some properties of the k -numerical range of matrices which will be useful in our discussion. For more details, see [8, 13] and their references.

Proposition 1. *Let $A \in \mathbb{M}_n$. Then the following assertions are true:*

- (i) $W_k(A)$ is a compact and convex set in $\mathbb{C}$;
- (ii) $conv(\sigma^{(k)}(A)) \subseteq W_k(A)$, where $conv(S)$ denotes the convex hull of a set $S \subseteq \mathbb{C}$. The equality holds if A is normal;
- (iii) $\{\frac{1}{n}tr(A)\} = W_n(A) \subseteq W_{n-1}(A) \subseteq \dots \subseteq W_2(A) \subseteq W_1(A) = W(A)$;
- (iv) If $V \in \mathcal{X}_{n \times s}$, where $k \leq s \leq n$, then $W_k(V^*AV) \subseteq W_k(A)$. The equality holds if $s = n$, i.e., $W_k(U^*AU) = W_k(A)$, where $U \in \mathcal{U}_n$;
- (v) For any $\alpha, \beta \in \mathbb{C}$, $W_k(\alpha A + \beta I_n) = \alpha W_k(A) + \beta$, and for the case $k < n$, $W_k(A) = \{\alpha\}$ if and only if $A = \alpha I_n$.

Let $A \in \mathbb{M}_n$. The k -numerical radius, k -spectral radius and the k -spectral norm of A are defined and denoted, respectively, by

$$r_k(A) := \max\{|z| : z \in W_k(A)\}, \quad (1.1)$$

$$\rho_k(A) := \max\{|z| : z \in \sigma^{(k)}(A)\}, \quad (1.2)$$

and

$$\|A\|_{(k)} := \max\left\{ \left| \frac{1}{k}tr(X^*AY) \right| : X, Y \in \mathcal{X}_{n \times k} \right\}. \quad (1.3)$$

It is clear that $r_1(A) = r(A)$, $\rho_1(A) = \rho(A)$, and $\|A\|_{(1)} = \|A\|$, which are the numerical radius, the spectral radius, and the spectral matrix norm (i.e., the matrix norm

subordinate to the Euclidean vector norm) of A , respectively. Also, $r_k(\cdot)$ for the case $k < n$, and $\|\cdot\|_{(k)}$ are vector norms on $\mathbb{M}_n$. Now, in the following proposition, we state some other properties of $\rho_k(\cdot)$, $r_k(\cdot)$ and $\|\cdot\|_{(k)}$, and their relations. For more details, see [10, 12].

Proposition 2. *Let $A \in \mathbb{M}_n$. Then the following assertions are true:*

- (i) $r_k(V^*AV) \leq r_k(A)$ and $\|V^*AV\|_{(k)} \leq \|A\|_{(k)}$ for all $V \in \mathcal{X}_{n \times s}$, where $k \leq s \leq n$. The equality holds if $s = n$;
- (ii) $\|A\|_{(k)} = \frac{1}{k} \sum_{i=1}^k s_i(A)$, where $s_1(A) \geq s_2(A) \geq \dots \geq s_n(A)$ are the singular values of A ;
- (iii) $\rho_k(A) \leq r_k(A) \leq \|A\|_{(k)}$;
- (iv) $r_k(A) = \|A\|_{(k)}$ if and only if there exists a number $\theta \in \mathbb{R}$ such that $\alpha_j = a_j e^{i\theta}$ for all $j = 1, 2, \dots, k$, where $a_1 \geq a_2 \geq \dots \geq a_n$ are the singular values and $\alpha_1, \alpha_2, \dots, \alpha_n$, where $|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_n|$, are the eigenvalues of A ;
- (v) If $k < n$, then $\frac{1}{2(2k-1)} \|A\| \leq r_k(A) \leq \|A\|$;
- (vi) $\|A\|_{(n)} \leq \|A\|_{(n-1)} \leq \dots \leq \|A\|_{(1)} = \|A\|$.

In Section 2, we study some algebraic properties of the joint k -numerical radius, joint k -spectral radius and the joint k -norm of matrices. In Section 3, using the joint higher numerical ranges of diagonal matrices which are convex polyhedrons, we give a description for the shape of the higher numerical ranges of matrix polynomials.

2. MAIN RESULTS

We begin this section by introducing the notion of the joint k -numerical radius of matrices.

Definition 1. Let $(A_1, A_2, \dots, A_m) \in \mathbb{M}_n^m$. The joint k -numerical radius of $(A_1, A_2, \dots, A_m)$ is defined and denoted by

$$r_k(A_1, A_2, \dots, A_m) := \sup \{ l_2(a_1, a_2, \dots, a_m) : (a_1, a_2, \dots, a_m) \in W_k(A_1, A_2, \dots, A_m) \},$$

where

$$l_2(a_1, a_2, \dots, a_m) = \left(\sum_{i=1}^m |a_i|^2 \right)^{\frac{1}{2}}, \text{ and}$$

$$W_k(A_1, A_2, \dots, A_m) = \left\{ \left(\frac{1}{k} \text{tr}(X^* A_1 X), \dots, \frac{1}{k} \text{tr}(X^* A_m X) \right) : X \in \mathcal{X}_{n \times k} \right\}$$

is the joint k -numerical range of $(A_1, A_2, \dots, A_m)$.

Next, we are going to state a new description of $r_k(A_1, A_2, \dots, A_m)$ which is one of our main results in this section. For this, we need the following lemma.

Lemma 1. *Let $(a_1, a_2, \dots, a_m) \in \mathbb{C}^m$. Then*

$$l_2(a_1, a_2, \dots, a_m) = \sup_{(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1} \left| \sum_{j=1}^m \lambda_j a_j \right|,$$

where $S^1 = \{(z_1, z_2, \dots, z_m) \in \mathbb{C}^m : \sum_{i=1}^m |z_i|^2 = 1\}$.

Proof. To avoid of trivial case, we assume that $(a_1, a_2, \dots, a_m) \neq (0, 0, \dots, 0)$. Let $(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1$ be arbitrary. By the Cauchy-Schwartz inequality, we have

$$\left| \sum_{i=1}^m \lambda_i a_i \right| \leq \left(\sum_{i=1}^m |\lambda_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^m |a_i|^2 \right)^{\frac{1}{2}} = l_2(a_1, a_2, \dots, a_m).$$

So,

$$\sup_{(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1} \left| \sum_{j=1}^m \lambda_j a_j \right| \leq l_2(a_1, a_2, \dots, a_m).$$

By setting $\lambda_i = \frac{\overline{a_i}}{l_2(a_1, \dots, a_m)}$, where $i = 1, 2, \dots, m$, we have $(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1$. Moreover,

$$\left| \sum_{j=1}^m \lambda_j a_j \right| = \frac{\sum_{j=1}^m |a_j|^2}{l_2(a_1, a_2, \dots, a_m)} = l_2(a_1, a_2, \dots, a_m).$$

Therefore, the result holds. $\square$

Theorem 1. *If $(A_1, A_2, \dots, A_m) \in \mathbb{M}_n^m$, then*

$$r_k(A_1, A_2, \dots, A_m) = \sup_{(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1} r_k(\lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_m A_m).$$

Proof. For every $(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1$, we have

$$\begin{aligned} W_k(\lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_m A_m) &= \left\{ \frac{1}{k} \text{tr} \left(X^* \left(\sum_{j=1}^m \lambda_j A_j \right) X \right) : X \in \mathcal{X}_{n \times k} \right\} \\ &= \left\{ \sum_{j=1}^m \lambda_j a_j : (a_1, a_2, \dots, a_m) \in W_k(A_1, A_2, \dots, A_m) \right\}. \end{aligned}$$

Thus,

$$\begin{aligned} r_k(\lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_m A_m) &= \sup \left\{ \left| \sum_{j=1}^m \lambda_j a_j \right| : (a_1, a_2, \dots, a_m) \in W_k(A_1, A_2, \dots, A_m) \right\}. \end{aligned}$$

Now, using Definition 1 and Lemma 1, we have:

$$\begin{aligned}
 r_k(A_1, A_2, \dots, A_m) &= \sup_{(a_1, a_2, \dots, a_m) \in W_k(A_1, A_2, \dots, A_m)} l_2(a_1, a_2, \dots, a_m) \\
 &= \sup_{(a_1, a_2, \dots, a_m) \in W_k(A_1, A_2, \dots, A_m)} \sup_{(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1} \left| \sum_{j=1}^m \lambda_j a_j \right| \\
 &= \sup_{(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1} \sup_{(a_1, a_2, \dots, a_m) \in W_k(A_1, A_2, \dots, A_m)} \left| \sum_{j=1}^m \lambda_j a_j \right| \\
 &= \sup_{(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1} r_k(\lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_m A_m).
 \end{aligned}$$

So, the proof is complete. $\square$

In the following proposition which follows from Theorem 1 and Propositions 1 and 2, we state some basic properties of the joint k -numerical radius of matrices.

Proposition 3. *Let $(A_1, A_2, \dots, A_m) \in \mathbb{M}_n^m$. Then the following assertions are true:*

- (i) *If $k < n$, then $r_k(A_1, A_2, \dots, A_m) = 0$ if and only if $A_1 = A_2 = \dots = A_m = 0$;*
- (ii) *$r_k(\lambda A_1, \lambda A_2, \dots, \lambda A_m) = |\lambda| r_k(A_1, A_2, \dots, A_m)$ for all $\lambda \in \mathbb{C}$;*
- (iii) *$r_k(A_1 + A'_1, \dots, A_m + A'_m) \leq r_k(A_1, \dots, A_m) + r_k(A'_1, \dots, A'_m)$, where $A'_1, A'_2, \dots, A'_m \in \mathbb{M}_n$;*
- (iv) *$r_k(S^* A_1 S, S^* A_2 S, \dots, S^* A_m S) \leq r_k(A_1, A_2, \dots, A_m)$ for all $S \in \mathcal{X}_{n \times t}$, where $k \leq t \leq n$. The equality holds if $t = n$. Consequently, for every $U \in \mathcal{U}_n$,*

$$r_k(U^* A_1 U, U^* A_2 U, \dots, U^* A_m U) = r_k(A_1, A_2, \dots, A_m).$$

In view of Proposition 3, $r_k(\cdot, \dots, \cdot)$, where $k < n$, is a vector norm on $\mathbb{M}_n^m$. Moreover, Theorem 1 leads us to introduce the notion of the joint k -norm of matrices.

Definition 2. Let $(A_1, \dots, A_m) \in \mathbb{M}_n^m$. The joint k -norm of $(A_1, \dots, A_m)$ is defined and denoted by

$$\|(A_1, A_2, \dots, A_m)\|_{(k)} := \sup_{(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1} \|\lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_m A_m\|_{(k)}.$$

Using Definition 2 and Proposition 2, we can show that the joint k -norm of matrices satisfies in the following basic properties.

Proposition 4. *Let $(A_1, A_2, \dots, A_m) \in \mathbb{M}_n^m$. Then the following assertions are true:*

- (i) *If $k < n$, then $\|(A_1, A_2, \dots, A_m)\|_{(k)} = 0$ if and only if $A_1 = A_2 = \dots = A_m = 0$;*

- (ii) $\|(\lambda A_1, \lambda A_2, \dots, \lambda A_m)\|_{(k)} = |\lambda| \|(A_1, A_2, \dots, A_m)\|_{(k)}$ for all $\lambda \in \mathbb{C}$;
- (iii) $\|(A_1 + A'_1, \dots, A_m + A'_m)\|_{(k)} \leq \|(A_1, \dots, A_m)\|_{(k)} + \|(A'_1, \dots, A'_m)\|_{(k)}$,
where $(A'_1, A'_2, \dots, A'_m) \in \mathbb{M}_n^m$;
- (iv) $\|(S^* A_1 S, S^* A_2 S, \dots, S^* A_m S)\|_{(k)} \leq \|(A_1, A_2, \dots, A_m)\|_{(k)}$ for all $S \in \mathcal{X}_{n \times t}$, where $k \leq t \leq n$. The equality holds if $t = n$. Consequently, for every $U \in \mathcal{U}_n$,

$$\|(U^* A_1 U, U^* A_2 U, \dots, U^* A_m U)\|_{(k)} = \|(A_1, A_2, \dots, A_m)\|_{(k)}.$$

In view of Proposition 4, for the case $k < n$, the joint k -norm is a vector norm on $\mathbb{M}_n^m$. In the final result of this section, we are going to study the relations between the joint k -numerical radius and the joint k -norm of matrices. For this, we need the following lemma.

Lemma 2. *Let $A \in \mathbb{M}_n$. Then*

$$\frac{1}{2(2k-1)} \|A\|_{(k)} \leq r_k(A) \leq \|A\|_{(k)}.$$

Proof. The first inequality follows from Proposition 2((v) and (vi)). Also, by (1.1) and (1.3), it is clear that $r_k(A) \leq \|A\|_{(k)}$, and so, the proof is complete. $\square$

Theorem 2. *Let $(A_1, A_2, \dots, A_m) \in \mathbb{M}_n^m$. Then*

$$\begin{aligned} \frac{1}{2(2k-1)} \|(A_1, \dots, A_m)\|_{(k)} &\leq r_k(A_1, \dots, A_m) \\ &\leq \min\{\|(A_1, \dots, A_m)\|_{(k)}, l_2(r_k(A_1), \dots, r_k(A_m))\}. \end{aligned}$$

Moreover, the right inequality is sharp.

Proof. Applying Lemma 2 to the matrix $A := \lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_m A_m$, where $(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1$, and using Definition 2 and Theorem 1, we have

$$\frac{1}{2(2k-1)} \|(A_1, \dots, A_m)\|_{(k)} \leq r_k(A_1, \dots, A_m) \leq \|(A_1, \dots, A_m)\|_{(k)}.$$

Moreover, it is clear that $W_k(A_1, A_2, \dots, A_m) \subseteq W_k(A_1) \times W_k(A_2) \times \dots \times W_k(A_m)$. Then,

$$\begin{aligned} r_k(A_1, \dots, A_m) &= \sup\{l_2(a_1, \dots, a_m) : (a_1, \dots, a_m) \in W_k(A_1, \dots, A_m)\} \\ &\leq l_2(r_k(A_1), \dots, r_k(A_m)). \end{aligned}$$

Therefore, the right inequality also holds.

To prove that the right inequality is sharp, we consider the matrix $A \in \mathbb{M}_n$ with singular values $a_1 \geq a_2 \geq \dots \geq a_n$ and eigenvalues $\alpha_1, \alpha_2, \dots, \alpha_n$, where $|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_n|$. Moreover, we assume that there exist a $\theta \in \mathbb{R}$ such that $\alpha_j = a_j e^{i\theta}$ for all $j = 1, 2, \dots, n$. Then by Proposition 2(v), we have $r_k(A) = \|A\|_{(k)}$. Now, suppose

that $A_j = t_j A$, where $t_j \in \mathbb{C}$ for all $j = 1, 2, \dots, m$. Then by Theorem 1, Proposition 4 and Definition 2, we have:

$$\begin{aligned}
 r_k(A_1, A_2, \dots, A_m) &= \sup_{(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1} r_k(\lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_m A_m) \\
 &= \sup_{(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1} \left| \sum_{j=1}^m t_j \lambda_j \right| r_k(A) \\
 &= \sup_{(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1} \left| \sum_{j=1}^m t_j \lambda_j \right| \|A\|_{(k)} \\
 &= \sup_{(\lambda_1, \lambda_2, \dots, \lambda_m) \in S^1} \|\lambda_1 A_1 + \lambda_2 A_2 + \dots + \lambda_m A_m\|_{(k)} \\
 &= \|(A_1, A_2, \dots, A_m)\|_{(k)}.
 \end{aligned}$$

Now, if we choose $t_j \geq 0$, then by Lemma 1 and the above inequalities, we have:

$$\|(A_1, A_1, \dots, A_m)\|_{(k)} = l_2(r_k(A_1), r_k(A_2), \dots, r_k(A_m)).$$

Hence, the right inequality changes to equality. This completes the proof. $\square$

3. ADDITIONAL RESULTS

In this section, by using the joint higher numerical ranges of diagonal matrices, we find an approximation for the higher numerical ranges of matrix polynomials. For this, suppose that

$$P(\lambda) = A_m \lambda^m + A_{m-1} \lambda^{m-1} + \dots + A_1 \lambda + A_0 \quad (3.1)$$

is a matrix polynomial, where $A_i \in \mathbb{M}_n$ ($i = 0, 1, \dots, m$), $A_m \neq 0$ and λ is a complex variable. The numbers m and n are referred as the *degree* and the *order* of $P(\lambda)$, respectively. The matrix polynomial $P(\lambda)$ is said to be a *diagonal matrix polynomial* if all the coefficients A_i are diagonal matrices. A scalar $\lambda_0 \in \mathbb{C}$ is called an *eigenvalue* of $P(\lambda)$ if the system $P(\lambda_0)x = 0$ has a nonzero solution $x_0 \in \mathbb{C}^n$. This solution x_0 is known as an *eigenvector* of $P(\lambda)$ corresponding to λ_0 , and the set of all eigenvalues of $P(\lambda)$ is said to be the *spectrum* of $P(\lambda)$; namely,

$$\sigma[P(\lambda)] = \{\mu \in \mathbb{C} : \det(P(\mu)) = 0\}.$$

The (classical) numerical range of $P(\lambda)$ is defined and denoted by

$$W[P(\lambda)] := \{\mu \in \mathbb{C} : x^* P(\mu) x = 0 \text{ for some nonzero } x \in \mathbb{C}^n\},$$

which is closed and contains $\sigma[P(\lambda)]$. The numerical range of matrix polynomials plays an important role in the study of overdamped vibration systems with finite number of degrees of freedom, and it is also related to the stability theory; e.g., see [5, 11] and its references. Notice that the notion of $W[P(\lambda)]$ is generalization of the *classical numerical range* of a matrix $A \in \mathbb{M}_n$; namely, $W[\lambda I - A] = W(A)$. In the last

few years, the generalization of the numerical range of matrices and matrix polynomials has attracted much attention and many interesting results have been obtained; e.g., see [1–4, 15]. One of these generalizations is the notion of higher numerical ranges. The k -numerical range of $P(\lambda)$ is defined and denoted, e.g., see [2], by

$$W_k[P(\lambda)] = \{\mu \in \mathbb{C} : \text{tr}(X^* P(\mu) X) = 0 \text{ for some } X \in \mathcal{X}_{n \times k}\}. \quad (3.2)$$

Also, the k -spectrum of $P(\lambda)$ is defined as

$$\sigma^{(k)}[P(\lambda)] = \{\mu \in \mathbb{C} : 0 \in \sigma^{(k)}(P(\mu))\}. \quad (3.3)$$

It is clear that $\sigma^{(k)}[P(\lambda)] \subseteq W_k[P(\lambda)] = \{\mu \in \mathbb{C} : 0 \in W_k(P(\mu))\}$, $\sigma^{(1)}[P(\lambda)] = \sigma[P(\lambda)]$, and $W_1[P(\lambda)] = W[P(\lambda)]$. Moreover, if $P(\lambda) = \lambda I - A$, where $A \in \mathbb{M}_n$, then $W_k[P(\lambda)] = W_k(A)$ and $\sigma^{(k)}[P(\lambda)] = \sigma^{(k)}(A)$. The sets $W_k[P(\lambda)]$, where $k \in \{1, 2, \dots, n\}$, are generally called the *higher numerical ranges* of $P(\lambda)$. The *joint k -numerical range* of $P(\lambda)$ is defined as the joint k -numerical range of its coefficients; namely, $JW_k[P(\lambda)] := W_k(A_0, A_1, \dots, A_m)$. It is known, e.g., see [2, Theorem 2.2(iii)], that:

$$\begin{aligned} JW_k[P(\lambda)] &= \{\mu \in \mathbb{C} : c_m \mu^m + \dots + c_0 = 0, (c_0, \dots, c_m) \in JW_k[P(\lambda)]\} \\ &= \{\mu \in \mathbb{C} : c_m \mu^m + \dots + c_0 = 0, (c_0, \dots, c_m) \in \text{conv}(JW_k[P(\lambda)])\}. \end{aligned} \quad (3.4)$$

So, if $Q(\lambda)$ is a matrix polynomial of degree m and arbitrary order such that $JW_k[P(\lambda)] \subseteq JW_k[Q(\lambda)]$ or if $\text{conv}(JW_k[P(\lambda)]) \subseteq \text{conv}(JW_k[Q(\lambda)])$, then we have $W_k[P(\lambda)] \subseteq W_k[Q(\lambda)]$. In the following proposition, we characterize the joint k -numerical range of a diagonal matrix polynomial.

Proposition 5. *Let $P(\lambda)$, as in (3.1), be a diagonal matrix polynomial. Then $JW_k[P(\lambda)]$ is a convex polyhedron. Conversely, every convex polyhedron $H \subseteq \mathbb{C}^{m+1}$ is the joint k -numerical range of a diagonal matrix polynomial of degree m .*

Proof. Suppose that $A_i = \text{diag}(a_1^{(i)}, a_2^{(i)}, \dots, a_n^{(i)})$ for $i = 0, 1, \dots, m$ and $S_1, S_2, \dots, S_{c(n,k)}$ are all the subsets of $\{1, 2, \dots, n\}$ with k elements. By considering $q_{S_i}^{(j)} = \frac{\sum_{t \in S_i} a_t^{(j)}}{k}$ for $j = 0, 1, \dots, m$, we have

$$JW_k[P(\lambda)] = \text{conv}(\{q_{S_i}^{(1)}, q_{S_i}^{(2)}, \dots, q_{S_i}^{(m)} : i = 1, 2, \dots, c(n,k)\}).$$

Conversely, suppose that $H \subseteq \mathbb{C}^{m+1}$ is a convex polyhedron with n vertices $v_1, v_2, \dots, v_n$. Now, we can find the points $(c_{i,0}, c_{i,1}, \dots, c_{i,m}) \in \mathbb{C}^{m+1}$, $i = 1, 2, \dots, q$ with $q \geq n$ such that their k -averages are $v_1, v_2, \dots, v_n$. Then $H = JW_k[D(\lambda)]$, where

$$D(\lambda) = \text{diag}(c_{1,m}, \dots, c_{q,m})\lambda^m + \dots + \text{diag}(c_{1,1}, \dots, c_{q,1})\lambda + \text{diag}(c_{1,0}, \dots, c_{q,0}).$$

So, the proof is complete. $\square$

The following theorem is a generalization of [14, Theorem 2.4].

Theorem 3. *Let $P(\lambda)$ be a matrix polynomial as in (3.1). Then*

$$\bigcup W_k[D_1(\lambda)] = W_k[P(\lambda)] = \bigcap W_k[D_2(\lambda)],$$

where the union [intersection] is taken over all diagonal matrix polynomials $D_1(\lambda)$ [$D_2(\lambda)$] of degree m for which $JW_k[D_1(\lambda)] \subseteq JW_k[P(\lambda)] \subseteq JW_k[D_2(\lambda)]$.

Proof. By [1, Theorem 3.1(ii)], the left equality holds.

For the right equality, note that for every diagonal matrix polynomial $D_2(\lambda)$ of degree m for which $JW_k[D_2(\lambda)] \supseteq JW_k[P(\lambda)]$, we have $W_k[D_2(\lambda)] \supseteq W_k[P(\lambda)]$. Thus

$$W_k[P(\lambda)] \subseteq \bigcap_{JW_k[D_2(\lambda)] \supseteq JW_k[P(\lambda)]} W_k[D_2(\lambda)].$$

Conversely, let

$$\lambda_0 \in \bigcap_{JW_k[D_2(\lambda)] \supseteq JW_k[P(\lambda)]} W_k[D_2(\lambda)].$$

By considering $T_1(\lambda_0) = \{(c_0, c_1, \dots, c_m) \in \mathbb{C}^{m+1} : c_m \lambda_0^m + \dots + c_1 \lambda_0 + c_0 = 0\}$, we have $T_1(\lambda_0) \cap JW_k[D_2(\lambda)] \neq \emptyset$ for every diagonal matrix polynomial $D_2(\lambda)$ satisfying $JW_k[D_2(\lambda)] \supseteq JW_k[P(\lambda)]$. Thus, by [14, lemma 4.1] and Proposition 5, we have

$$T_1(\lambda_0) \cap \text{conv}(JW_k[P(\lambda)]) \neq \emptyset, .$$

So, by (3.4), $\lambda_0 \in W_k[P(\lambda)]$. Therefore,

$$W_k[P(\lambda)] \supseteq \bigcap_{JW_k[D_2(\lambda)] \supseteq JW_k[P(\lambda)]} W_k[D_2(\lambda)],$$

and hence, the proof is complete. $\square$

Remark 1. In view of Proposition 5 and Theorem 3, one can estimate the shape of the k -numerical range of matrix polynomials by convex polyhedrons.

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FIXED POINT THEOREMS IN UNIFORM SPACE ENDOWED WITH GRAPH

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Abstract. In this paper, we shall introduce the concepts of F_G -contraction and ψ_G -contraction in uniform space endowed with graph to investigate the existence of a fixed point of mappings satisfying these notions. We shall also introduce a common fixed point theorem for pair of mappings satisfying the notion of ψ_G -contraction in uniform space endowed with graph.

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1. INTRODUCTION

Very recently, Wardowski [19] categorized the family of mappings into a new family, denoted by F or $\mathfrak{F}$, to introduce a new contraction condition, the F -contraction. As it is expected, it extend the famous Banach contraction mapping principle. Inspired the result of Wardowski, [19], a number of different approaches has been reported, see [8, 13, 15–17] and related references therein.

On the other hand, Czerwik [9] extended the notion of metric by introducing a new notion, b -metric. Indeed, Czerwik [9] modified the triangle inequality that causes to several differences in the related topology, and hence in convergence of a sequence, Hausdorffness of the topology etc. We also mention that Jachymski [10] successively set-up the Banach contraction mapping principle in the frame of a complete metric space endowed with a directed graph, Jachymski [10]. Later, this interesting contribution has been appreciated by several authors [5, 6, 11, 12, 14, 18].

The purpose of this paper is to introduce and discuss some new fixed point theorems and common fixed point theorems for mappings in uniform space endowed with graph satisfying the F_G -contraction, and ψ_G -contraction. Further, we introduce and use the notion of E_S -distance in this paper.

Some basic definitions and fundamental results are recollected for the sake of completeness. Let X be a nonempty set. A nonempty family, ϑ of subsets of $X \times X$ is called the uniform structure of X , if it satisfies the following properties:

- (i) if G is in ϑ , then G contains the diagonal $\{(x, x) | x \in X\}$;
- (ii) if G is in ϑ and H is a subset of $X \times X$ which contains G , then H is in ϑ ;
- (iii) if G and H are in ϑ , then $G \cap H$ is in ϑ ;
- (iv) if G is in ϑ , then there exists H in ϑ , such that, whenever (x, y) and (y, z) are in H , then (x, z) is in G ;
- (v) if G is in ϑ , then $\{(y, x) | (x, y) \in G\}$ is also in ϑ .

The pair (X, ϑ) is called a uniform space and the element of ϑ is called entourage or neighborhood or surrounding. The pair (X, ϑ) is called a quasiuniform space (see e.g. [20]) if property (v) is omitted

Let $\Delta = \{(x, x) | x \in X\}$ be the diagonal of a non-empty set X . For $V, W \in X \times X$, we shall use the following setting in the sequel

$$V \circ W = \{(x, y) | \text{there exists } z \in X : (x, z) \in W \text{ and } (z, y) \in V\}$$

and

$$V^{-1} = \{(x, y) | (y, x) \in V\}.$$

For a subset $V \in \vartheta$, a pair of points x and y are said to be V -close if $(x, y) \in V$ and $(y, x) \in V$. Moreover, a sequence $\{x_n\}$ in X is called a Cauchy sequence for ϑ , if for any $V \in \vartheta$, there exists $N \geq 1$ such that x_n and x_m are V -close for $n, m \geq N$. For (X, ϑ) , there is a unique topology $\tau(\vartheta)$ on X generated by $V(x) = \{y \in X | (x, y) \in V\}$ where $V \in \vartheta$.

A sequence $\{x_n\}$ in X is convergent to x for ϑ , denoted by $\lim_{n \rightarrow \infty} x_n = x$, if for any $V \in \vartheta$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in V(x)$ for every $n \geq n_0$. A uniform space (X, ϑ) is called Hausdorff if the intersection of all the $V \in \vartheta$ is equal to Δ of X , that is, if $(x, y) \in V$ for all $V \in \vartheta$ implies $x = y$. If $V = V^{-1}$ then we shall say that a subset $V \in \vartheta$ is symmetrical. Throughout the paper, we shall assume that each $V \in \vartheta$ is symmetrical. For more details, see e.g. [1–4]

Now, we shall recall the notions of A -distance and E -distance.

Definition 1. [2,3] Let (X, ϑ) be a uniform space. A function $p : X \times X \longrightarrow [0, \infty)$ is said to be an A -distance if for any $V \in \vartheta$ there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in V$.

Definition 2. [2,3] Let (X, ϑ) be a uniform space. A function $p : X \times X \longrightarrow [0, \infty)$ is said to be an E -distance if

- (i) p is an A -distance,
- (ii) $p(x, y) \leq p(x, z) + p(z, y)$, $\forall x, y, z \in X$.

Example 1. [2,3] Let (X, ϑ) be a uniform space and let d be a metric on X . It is evident that (X, ϑ_d) is a uniform space where ϑ_d is a set of all subsets of $X \times X$ containing a "band" $U_\epsilon = \{(x, y) \in X^2 | d(x, y) < \epsilon\}$ for some $\epsilon > 0$. Moreover, if $\vartheta \subseteq \vartheta_d$, then d is an E -distance on (X, ϑ) .

Lemma 1. [2, 3] Let (X, ϑ) be a Hausdorff uniform space and p be an A -distance on X . Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[0, \infty)$ converging to 0. Then, for $x, y, z \in X$, the following results hold:

- (a) If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.
- (b) If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to z .
- (c) If $p(x_n, x_m) \leq \alpha_n$ for all $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence in (X, ϑ) .

Let p be an A -distance. A sequence in a uniform space (X, ϑ) with an A -distance is said to be a p -Cauchy if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $p(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.

Definition 3. [2, 3] Let (X, ϑ) be a uniform space and p be an A -distance on X .

- (i) X is S -complete if for every p -Cauchy sequence $\{x_n\}$, there exists x in X with $\lim_{n \rightarrow \infty} p(x_n, x) = 0$.
- (ii) X is p -Cauchy complete if for every p -Cauchy sequence $\{x_n\}$, there exists x in X with $\lim_{n \rightarrow \infty} x_n = x$ with respect to $\tau(\vartheta)$.
- (iii) $T : X \rightarrow X$ is p -continuous if $\lim_{n \rightarrow \infty} p(x_n, x) = 0$ implies $\lim_{n \rightarrow \infty} p(T(x_n), T(x)) = 0$.

Remark 1. Let (X, ϑ) be a Hausdorff uniform space which is S -complete. If a sequence $\{x_n\}$ be a p -Cauchy sequence, then we have $\lim_{n \rightarrow \infty} p(x_n, x) = 0$. Regarding Lemma 1(b), We derive that $\lim_{n \rightarrow \infty} x_n = x$ with respect to the topology $\tau(\vartheta)$ and hence S -completeness implies p -Cauchy completeness.

Cosentino *et al.* [7] modify the $\mathfrak{F}$ -family introduced by Wardowski [19], in the setting of b -metric spaces as follows:

Definition 4. [7] Let $s \geq 1$ be a real number. Denote by $\mathfrak{F}_s$ the family of all functions $F : (0, \infty) \rightarrow \mathbb{R}$ satisfying the following conditions:

- (F₁) F is strictly increasing, that is, for each $a_1, a_2 \in (0, \infty)$ with $a_1 < a_2$, we have $F(a_1) < F(a_2)$;
- (F₂) for each sequence $\{\vartheta_n\}$ of positive real numbers we have $\lim_{n \rightarrow \infty} \vartheta_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\vartheta_n) = -\infty$;
- (F₃) for each sequence $\{\vartheta_n\}$ of positive real numbers with $\lim_{n \rightarrow \infty} \vartheta_n = 0$, there exists $k \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \vartheta_n^k F(\vartheta_n) = 0$.
- (F₄) for each sequence $\{\vartheta_n\}$ of positive real numbers such that $\tau + F(s\vartheta_n) \leq F(\vartheta_{n-1})$ for each $n \in \mathbb{N}$ and some $\tau > 0$, then $\tau + F(s^n \vartheta_n) \leq F(s^{n-1} \vartheta_{n-1})$ for each $n \in \mathbb{N}$.

Cosentino *et al.* [7] also showed that the following functions belong to $\mathfrak{F}_s$.

- $F(x) = x + \ln x$, for each $x > 0$.
- $F(x) = \ln x$, for each $x > 0$.

2. MAIN RESULTS

We will start this section with the following definition:

Definition 5. Let (X, ϑ) be a uniform space. A function $p : X \times X \longrightarrow [0, \infty)$ is said to be an E_s -distance if

- (i) p is an A -distance,
- (ii) $p(x, y) \leq s[p(x, z) + p(z, y)]$, $\forall x, y, z \in X$ and for some $s \geq 1$.

Example 2. Let (X, ϑ) be a uniform space and let d be a b -metric on X . Then clearly, (X, ϑ_d) is a uniform space where ϑ_d is a set of all subsets of $X \times X$ containing a "band" $U_\epsilon = \{(x, y) \in X^2 \mid d(x, y) < \epsilon\}$ for some $\epsilon > 0$. Moreover, if $\vartheta \subseteq \vartheta_d$, then d is an E_s -distance on (X, ϑ) .

Throughout this section, we assume that $G = (V, E)$ is a directed graph such that the set of its vertices V coincides with X (i.e., $V = X$) and the set of its edges E is such that $E \supseteq \Delta$, where $\Delta = \{(x, x) : x \in X\}$. Further, assume that G has no parallel edges. A mapping $T : X \rightarrow X$ is p_G -continuous if for each sequence $\{x_n\} \subseteq X$ with $(x_n, x_{n+1}) \in E$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} p(x_n, x) = 0$, then we have $\lim_{n \rightarrow \infty} p(Tx_n, Tx) = 0$.

Definition 6. Let (X, ϑ) be a uniform space endowed with the graph G and p is an E_s -distance on X . A mapping $T : X \rightarrow X$ is a F_G -contraction, if there exist $F \in \mathfrak{F}_s$ and $\tau > 0$, such that, for each $(x, y) \in E$, we have

$$\tau + F(sp(Tx, Ty)) \leq F(p(x, y)), \quad (2.1)$$

whenever $\min\{p(Tx, Ty), p(x, y)\} > 0$.

Theorem 1. Let (X, ϑ) be a S -complete Hausdorff uniform space endowed with the graph G and p is an E_s -distance on X . Let $T : X \rightarrow X$ is an F_G -contraction satisfying the following conditions:

- (i) T is edge preserving, that is, for $(x, y) \in E$, we have $(Tx, Ty) \in E$;
- (ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E$ and $(Tx_0, x_0) \in E$;
- (iii) T is p_G -continuous, or, for any sequence $\{x_n\} \subseteq X$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$ and $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$, we have $(x_n, x) \in E$ for each $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. By hypothesis (ii), there exists $x_0 \in X$ such that $(x_0, x_1) = (x_0, Tx_0) \in E$. From (2.1), we have

$$\tau + F(sp(x_1, x_2)) = \tau + F(sp(Tx_0, Tx_1)) \leq F(p(x_0, x_1)) \quad (2.2)$$

As T is edge preserving, for $(x_0, x_1) \in E$, we have $(x_1, x_2) \in E$, From (2.1), we have

$$\tau + F(sp(x_2, x_3)) = \tau + F(sp(Tx_1, Tx_2)) \leq F(p(x_1, x_2)) \quad (2.3)$$

Continuing in the same way, we get a sequence $\{x_n\} \subset X$ such that

$$x_n = Tx_{n-1}, x_{n-1} \neq x_n \text{ and } (x_{n-1}, x_n) \in E \text{ for each } n \in \mathbb{N}.$$

Furthermore,

$$\tau + F(sp(x_n, x_{n+1})) \leq F(p(x_{n-1}, x_n)) \text{ for each } n \in \mathbb{N}. \quad (2.4)$$

By using the property F_4 , we have

$$\tau + F(s^n p(x_n, x_{n+1})) \leq F(s^{n-1} p(x_{n-1}, x_n)).$$

Letting $p_n = p(x_n, x_{n+1})$, for each $n \in \mathbb{N}$ and after some simplification, we have

$$F(s^n p_n) \leq F(p_0) - n\tau \text{ for each } n \in \mathbb{N}. \quad (2.5)$$

Letting $n \rightarrow \infty$ in (2.5), we get $\lim_{n \rightarrow \infty} F(s^n p_n) = -\infty$. Thus, by property (F_2) , we have $\lim_{n \rightarrow \infty} s^n p_n = 0$. From (F_3) there exists $k \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} (s^n p_n)^k F(s^n p_n) = 0.$$

From (2.5) we have

$$(s^n p_n)^k F(s^n p_n) - (s^n p_n)^k F(p_0) \leq -(s^n p_n)^k n\tau \leq 0 \text{ for each } n \in \mathbb{N}. \quad (2.6)$$

Letting $n \rightarrow \infty$ in (2.6), we get

$$\lim_{n \rightarrow \infty} n(s^n p_n)^k = 0. \quad (2.7)$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $n(s^n p_n)^k \leq 1$ for each $n \geq n_1$. Thus, we have

$$s^n p_n \leq \frac{1}{n^{1/k}}, \text{ for each } n \geq n_1. \quad (2.8)$$

To show that $\{x_n\}$ is a p -Cauchy sequence, consider

$$S_n = \sum_{i=n_1}^n \frac{1}{i^{1/k}}.$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series. Thus, there exists $S \in [0, \infty)$ such that $\lim_{n \rightarrow \infty} S_n = S$. Consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By using the triangular inequality and (2.8), we have

$$\begin{aligned} p(x_n, x_m) &\leq s^n p(x_n, x_{n+1}) + s^{n+1} p(x_{n+1}, x_{n+2}) + \cdots + s^{m-1} p(x_{m-1}, x_m) \\ &= \sum_{i=n_1}^{m-1} s^i p_i - \sum_{i=n_1}^{n-1} s^i p_i \\ &\leq S_{m-1} - S_{n-1}. \end{aligned}$$

Thus, $\lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0$. In a similar way, we show that $\lim_{n, m \rightarrow \infty} p(x_m, x_n) = 0$. Thus, $\{x_n\}$ is a p -Cauchy sequence. As (X, ϑ) is S -complete, there exists $x^* \in X$ such that $\lim_{n \rightarrow \infty} p(x_n, x^*) = 0$. By condition (iii), when T is p_G -continuous,

we have $\lim_{n \rightarrow \infty} p(x_{n+1}, Tx^*) = 0$. As $\lim_{n \rightarrow \infty} p(x_n, x^*) = 0$ and $\lim_{n \rightarrow \infty} p(x_n, Tx^*) = 0$. Thus by Lemma 1-(a) we have $x^* = Tx^*$. By condition (iii), when we have $(x_n, x^*) \in E$ for each $n \in \mathbb{N}$. From (2.1), we have

$$\tau + F(sp(Tx_n, Tx^*)) \leq F(p(x_n, x^*)).$$

This implies that $sp(x_{n+1}, Tx^*) < p(x_n, x^*)$. Letting $n \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} p(x_{n+1}, Tx^*) = 0$. Again, by Lemma 1-(a) we have $x^* = Tx^*$. $\square$

Let Ψ be the family of functions $\psi : [0, \infty) \rightarrow [0, \infty)$ satisfying the following conditions:

(Ψ_1) ψ is nondecreasing;

(Ψ_2) $\sum_{n=1}^{+\infty} s^n \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n^{th} iterate of ψ .

Definition 7. Let (X, ϑ) be a uniform space endowed with the graph G and p is an E_s -distance on X . A mapping $T : X \rightarrow X$ is a ψ_G -contraction mapping if for each $(x, y) \in E$, we have

$$p(Tx, Ty) \leq \psi(p(x, y)) \quad (2.9)$$

where $\psi \in \Psi$.

Theorem 2. Let (X, ϑ) be a S -complete Hausdorff uniform endowed with the graph G and p is an E_s -distance on X . Let $T : X \rightarrow X$ is a ψ_G -contraction mapping satisfying the following conditions:

- (i) T is edge preserving, that is, for $(x, y) \in E$, we have $(Tx, Ty) \in E$;
- (ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E$ and $(Tx_0, x_0) \in E$;
- (iii) T is p_G -continuous, or, for any sequence $\{x_n\} \subseteq X$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$ and $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$, we have $(x_n, x) \in E$ for each $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. By hypothesis (ii) of theorem we have $x_0 \in X$ such that $(x_0, Tx_0) \in E$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_{n_0} = x_{n_0+1}$ for some n_0 , then x_{n_0} is a fixed point of T . So, we can assume that $x_n \neq x_{n+1}$ for all n . Since T is edge preserving, we have

$$(x_0, x_1) = (x_0, Tx_0) \in E \Rightarrow (Tx_0, Tx_1) = (x_1, x_2) \in E.$$

Inductively, we have

$$(x_n, x_{n+1}) \in E, \text{ for all } n \in \mathbb{N} \cup \{0\}. \quad (2.10)$$

From (2.9) and (2.10), it follows that for all $n \in \mathbb{N} \cup \{0\}$, we have

$$p(x_{n+1}, x_{n+2}) = p(Tx_n, Tx_{n+1}) \leq \psi(p(x_n, x_{n+1})). \quad (2.11)$$

Iteratively, we get

$$p(x_n, x_{n+1}) \leq \psi^n(p(x_0, x_1)), \text{ for all } n \in \mathbb{N}.$$

Since p is an E_S -distance then for $m > n$, we have

$$\begin{aligned} p(x_n, x_m) &\leq s^n p(x_n, x_{n+1}) + s^{n+1} p(x_{n+1}, x_{n+2}) + \cdots + s^{m-1} p(x_{m-1}, x_m) \\ &\leq s^n \psi^n(p(x_0, x_1)) + s^{n+1} \psi^{n+1}(p(x_0, x_1)) + \cdots + s^{m-1} \psi^{m-1}(p(x_0, x_1)). \end{aligned}$$

To show that $\{x_n\}$ is a p -Cauchy sequence, consider

$$S_n = \sum_{k=0}^n s^k \psi^k(p(x_0, x_1)).$$

Thus from (2) we have

$$p(x_n, x_m) \leq S_{m-1} - S_{n-1}. \quad (2.12)$$

Since $\psi \in \Psi$, there exists $S \in [0, \infty)$ such that $\lim_{n \rightarrow \infty} S_n = S$. Thus by (2.12) we have

$$\lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0. \quad (2.13)$$

Since p is not symmetric then by repeating the same argument we have

$\lim_{n, m \rightarrow \infty} p(x_m, x_n) = 0$. Hence the sequence $\{x_n\}$ is a p -Cauchy in the S -complete space X . Thus, there exists $x^* \in X$ such that $\lim_{n \rightarrow \infty} p(x_n, x^*) = 0$. By condition (iii), when we have T is p_G -continuous, we get $\lim_{n \rightarrow \infty} p(Tx_n, Tx^*) = 0$ which implies that $\lim_{n \rightarrow \infty} p(x_{n+1}, Tx^*) = 0$. Hence we have $\lim_{n \rightarrow \infty} p(x_n, x^*) = 0$ and $\lim_{n \rightarrow \infty} p(x_n, Tx^*) = 0$. Thus by Lemma 1-(a) we have $x^* = Tx^*$. By condition (iii), when we have $(x_n, x^*) \in E$ for each $n \in \mathbb{N}$, then from (2.9)

$$p(x_{n+1}, Tx^*) = p(Tx_n, Tx^*) \leq \psi(p(x_n, x^*)) < p(x_n, x^*). \quad (2.14)$$

Letting $n \rightarrow \infty$ in the above inequality, we have $\lim_{n \rightarrow \infty} p(x_{n+1}, Tx^*) = 0$. Thus by repeating the same arguments as above we have $x^* = Tx^*$. $\square$

Example 3. Let $X = [0, 1]$ be endowed with a graph $G = (V, E)$ with $V = X$ and $E = \{(x, y) : x, y \in \{\frac{1}{n+1} : n \in \mathbb{N}\} \cup \{0\}\} \cup \{(x, x) : x \in X\}$, and b -metric $d(x, y) = (x - y)^2$ with $s = 2$. Define $\vartheta = \{U_\epsilon | \epsilon > 0\}$. It is easy to see that (X, ϑ) is a uniform space. Define $T : X \rightarrow X$ by

$$Tx = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{3n+1} & \text{if } x = \frac{1}{n} : n > 1 \\ \sqrt{x} & \text{otherwise.} \end{cases} \quad (2.15)$$

Take $\psi(t) = \frac{t}{3}$ for all $t \geq 0$. It is easy to see that T is edge preserving and ψ_G -contraction. Also for $x_0 = \frac{1}{2}$ we have $(x_0, Tx_0) \in E$ and $(Tx_0, x_0) \in E$. Moreover for any sequence $\{x_n\}$ in X with $x_n \rightarrow x$ as $n \rightarrow \infty$ and $(x_{n-1}, x_n) \in E$ for each $n \in \mathbb{N}$ we have $(x_n, x) \in E$ for each $n \in \mathbb{N}$. Therefore by Theorem 2, T has a fixed point.

To investigate the uniqueness of a fixed point, we consider the following condition:

(H): For all $x, y \in \text{Fix}(T)$, there exists $z \in X$ such that $(z, x) \in E$ and $(z, y) \in E$.

Here, $\text{Fix}(T)$ denotes the set of all fixed points of T .

The following theorem guarantees the uniqueness of a fixed point.

Theorem 3. *Adding the condition (H) in the hypothesis of Theorem 2, we obtain the uniqueness of fixed point of T .*

Proof. Suppose, on the contrary, that $u, v \in X$ are two distinct fixed points of T . From (H), there exists $z \in X$ such that

$$(z, u) \in E \text{ and } (z, v) \in E. \quad (2.16)$$

By using the fact that T is edge preserving, from (2.16), we have

$$(T^n z, u) \in E \text{ and } (T^n z, v) \in E, \text{ for all } n \in \mathbb{N} \cup \{0\}. \quad (2.17)$$

We define the sequence $\{z_n\}$ in X by $z_{n+1} = Tz_n = T^n z_0$ for all $n \in \mathbb{N} \cup \{0\}$ and $z_0 = z$. From (2.17) and (2.9), we have

$$p(z_{n+1}, u) = p(Tz_n, Tu) \leq \psi(p(z_n, u)), \quad (2.18)$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that

$$p(z_n, u) \leq \psi^n(p(z_0, u)), \text{ for all } n \in \mathbb{N}.$$

Letting $n \rightarrow \infty$ in the above inequality, we obtain

$$\lim_{n \rightarrow \infty} p(z_n, u) = 0. \quad (2.19)$$

Similarly, we have

$$\lim_{n \rightarrow \infty} p(z_n, v) = 0. \quad (2.20)$$

From (2.19) and (2.20) together with Lemma 1-(a), it follows that $u = v$. Thus, fixed point of T is unique. $\square$

Definition 8. Let (X, ϑ) be a uniform space endowed with the graph G . A pair of two self mappings $T, S : X \rightarrow X$ is said to be a ψ_G -contraction pair if for each $(x, y) \in E$, we have

$$\max\{p(Tx, Sy), p(Sx, Ty)\} \leq \psi(p(x, y)), \quad (2.21)$$

where $\psi \in \Psi$.

Theorem 4. *Let (X, ϑ) be a S -complete Hausdorff uniform space endowed with the graph G and p is an E_S -distance on X . Suppose that the pair of $T, S : X \rightarrow X$ is ψ_G -contraction pair satisfying the following conditions.*

- (i) (T, S) is edge preserving pair, that is, for each $(x, y) \in E$, we have $(Tx, Sy) \in E$ and $(Sx, Ty) \in E$;
- (ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E$ and $(Tx_0, x_0) \in E$;

- (iii) for any sequence $\{x_n\}$ in X with $x_n \rightarrow x$ as $n \rightarrow \infty$ and $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N} \cup \{0\}$, then $(x_n, x) \in E$ for each $n \in \mathbb{N} \cup \{0\}$.

Then T and S have a common fixed point.

Proof. By hypothesis (ii) of theorem, we have $x_0 \in X$ such that $(x_0, Tx_0) \in E$ and $(Tx_0, x_0) \in E$. Since (T, S) is an edge preserving pair, then we can construct a sequence such that

$Tx_{2n} = x_{2n+1}$, $Sx_{2n+1} = x_{2n+2}$ and $(x_n, x_{n+1}) \in E$, $(x_{n+1}, x_n) \in E$, $\mathbb{N} \cup \{0\}$, for all $n \in \mathbb{N} \cup \{0\}$. From (2.21) for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} p(x_{2n+1}, x_{2n+2}) &= p(Tx_{2n}, Sx_{2n+1}) \\ &\leq \max\{p(Tx_{2n}, Sx_{2n+1}), p(Sx_{2n}, Tx_{2n+1})\} \\ &\leq \psi(p(x_{2n}, x_{2n+1})). \end{aligned}$$

Hence, we conclude that

$$p(x_{2n+1}, x_{2n+2}) \leq \psi(p(x_{2n}, x_{2n+1})). \quad (2.22)$$

Similarly, we get

$$\begin{aligned} p(x_{2n+2}, x_{2n+3}) &= p(Sx_{2n+1}, Tx_{2n+2}) \\ &\leq \max\{p(Tx_{2n+1}, Sx_{2n+2}), p(Sx_{2n+1}, Tx_{2n+2})\} \\ &\leq \psi(p(x_{2n+1}, x_{2n+2})). \end{aligned}$$

Hence, we have

$$p(x_{2n+2}, x_{2n+3}) \leq \psi(p(x_{2n+1}, x_{2n+2})). \quad (2.23)$$

Thus from (2.22) and (2.23), and by induction, we get

$$p(x_n, x_{n+1}) \leq \psi^n(p(x_0, x_1)), \text{ for all } n \in \mathbb{N}. \quad (2.24)$$

Now we show that $\{x_n\}$ is a p -Cauchy sequence. Since p is an E_s -distance then for $m > n$, we have

$$\begin{aligned} p(x_n, x_m) &\leq s^n p(x_n, x_{n+1}) + s^{n+1} p(x_{n+1}, x_{n+2}) + \cdots + s^{m-1} p(x_{m-1}, x_m) \\ &\leq s^n \psi^n(p(x_0, x_1)) + s^{n+1} \psi^{n+1}(p(x_0, x_1)) + \cdots + s^{m-1} \psi^{m-1}(p(x_0, x_1)). \end{aligned}$$

Now, we shall consider

$$S_n = \sum_{k=0}^n s^k \psi^k(p(x_0, x_1)).$$

Thus, from (2) we have

$$p(x_n, x_m) \leq S_{m-1} - S_{n-1}. \quad (2.25)$$

Since $\psi \in \Psi$, there exists $S \in [0, \infty)$ such that $\lim_{n \rightarrow \infty} S_n = S$. Thus, by (2.25) we have

$$\lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0. \quad (2.26)$$

Since p is not symmetric then by repeating the same argument we have

$$\lim_{n,m \rightarrow \infty} p(x_m, x_n) = 0. \quad (2.27)$$

Hence the sequence $\{x_n\}$ is a p -Cauchy in the S -complete space X . Thus, there exists $x^* \in X$ such that $\lim_{n \rightarrow \infty} p(x_n, x^*) = 0$ which implies $\lim_{n \rightarrow \infty} Tx_{2n} = \lim_{n \rightarrow \infty} Sx_{2n+1} = x^*$. By assumption (iii), we have $(x_n, x^*) \in E$. Thus, by using the triangular inequality and (2.21), we have

$$\begin{aligned} p(x_n, Tx^*) &\leq sp(x_n, x_{2n+2}) + sp(x_{2n+2}, Tx^*) \\ &= sp(x_n, x_{2n+2}) + sp(Sx_{2n+1}, Tx^*) \\ &\leq sp(x_n, x_{2n+2}) + s \max\{p(Tx_{2n+1}, Sx^*), p(Sx_{2n+1}, Tx^*)\} \\ &\leq sp(x_n, x_{2n+2}) + s\psi(p(x_{2n+1}, x^*)) \end{aligned} \quad (2.28)$$

Letting $n \rightarrow \infty$ in (2.28), we have $p(x_n, Tx^*) = 0$. Hence we have $\lim_{n \rightarrow \infty} p(x_n, x^*) = 0$ and $\lim_{n \rightarrow \infty} p(x_n, Tx^*) = 0$. Thus by Lemma 1-(a) we have $x^* = Tx^*$. Analogously, we can drive $x^* = Sx^*$. Therefore $x^* = Tx^* = Sx^*$. $\square$

Remark 2. Note that Theorem 4 is valid if one replace condition (ii) with

(ii)': there exists $x_0 \in X$ such that $(x_0, Sx_0) \in E$ and $(Sx_0, x_0) \in E$.

Example 4. Let $X = [0, 1]$ be endowed with a graph $G = (V, E)$ with $V = X$ and $E = \{(x, y) : x, y \in \{\frac{1}{n+1} : n \in \mathbb{N}\} \cup \{0\}\} \cup \{(x, x) : x \in X\}$, and dislocated metric space $d(x, y) = \max\{x, y\}$. Define $\vartheta = \{U_\epsilon | \epsilon > 0\}$, where $U_\epsilon = \{(x, y) \in X^2 : d(x, y) < d(x, x) + \epsilon\}$. It is easy to see that (X, ϑ) is a uniform space. Define $T : X \rightarrow X$ by

$$Tx = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2n+1} & \text{if } x = \frac{1}{n} : n > 1 \\ x^2 & \text{otherwise} \end{cases} \quad (2.29)$$

and $S : X \rightarrow X$ by

$$Sx = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2n} & \text{if } x = \frac{1}{n} : n > 1 \\ \sqrt{x} & \text{otherwise} \end{cases} \quad (2.30)$$

Take $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. Further, it is easy to see that (T, S) is edge preserving and ψ_G -contraction pair. Also for $x_0 = \frac{1}{2}$ we have $(x_0, Tx_0) \in E$ and $(Tx_0, x_0) \in E$. Moreover for any sequence $\{x_n\}$ in X with $x_n \rightarrow x$ as $n \rightarrow \infty$ and $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N} \cup \{0\}$ we have $(x_n, x) \in E$ for each $n \in \mathbb{N}$. Therefore by Theorem 4, T and S have a common fixed point.

To investigate the uniqueness of a common fixed point, we use the following condition.

(I) For each $x, y \in CFix(T, S)$, we have $(x, y) \in E$, where $CFix(T, S)$ is the set of all common fixed points of T and S .

Theorem 5. *Adding the condition (I) in the hypothesis of Theorem 4, we obtain the uniqueness of common fixed point of T and S .*

Proof. On the contrary suppose that $u, v \in X$ are two distinct common fixed points of T and S . From (I) and (2.21) we have

$$p(u, v) = \max\{p(Tu, Sv), p(Su, Tv)\} \leq \psi(p(u, v)) < p(u, v),$$

which is impossible for $p(u, v) > 0$. Consequently, we have $p(u, v) = 0$. Analogously, one can show that $p(u, u) = 0$. Thus we have $u = v$, which is a contradiction to our assumption. Hence T and S have a unique common fixed point. $\square$

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A MAIA TYPE FIXED POINT THEOREM FOR PREŠIĆ-KANNAN OPERATORS

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Abstract. Starting from Prešić's fixed point theorem, established in [Prešić, Slaviša B. Sur une classe d'inéquations aux différences finies et sur la convergence de certaines suites. (French) Publ. Inst. Math. (Beograd) (N.S.) 5 (19) 1965 75–78] on the one hand, and Maia's fixed point theorem obtained in [Maia, Maria Grazia. Un'osservazione sulle contrazioni metriche. (Italian) Rend. Sem. Mat. Univ. Padova 40 1968 139–143] on the other hand, two different extensions of Banach fixed point theorem are extended, generalised and unified.

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1. INTRODUCTION AND PRELIMINARIES

In 1965, S. Prešić [4] extended the famous Banach contraction principle [1] to the case of product spaces and obtained convergence results for some particular sequences. Prešić type operators have applications in solving nonlinear difference equations, cyclic systems and in the study of convergence of sequences. Prešić's result may be stated as follows:

Theorem 1 ([5]). *Let (X, d) be a complete metric space, k a positive integer and $f : X^k \rightarrow X$ a Prešić operator, it is, a mapping for which there exists $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}_+$, $\sum_{i=1}^k \alpha_i = \alpha < 1$ such that:*

$$d(f(x_0, \dots, x_{k-1}), f(x_1, \dots, x_k)) \leq \sum_{i=1}^k \alpha_i d(x_{i-1}, x_i), \quad (1.1)$$

for all $x_0, \dots, x_k \in X$.

Then:

- 1) *f has a unique fixed point x^* ;*
- 2) *the sequence $\{y_n\}_{n \geq 0}$,*

$$y_{n+1} = f(y_n, y_n, \dots, y_n), n \geq 0,$$

converges to x^* ;

3) the sequence $\{x_n\}_{n \geq 0}$ with $x_0, \dots, x_{k-1} \in X$ and

$$x_n = f(x_{n-k}, x_{n-k+1}, \dots, x_{n-1}), n \geq k,$$

also converges to x^* , with a rate estimated by

$$d(x_{n+1}, x^*) \leq \alpha d(x_n, x^*) + M \cdot \theta^n, n \geq 0,$$

where $M > 0$ is constant.

The following result was given by M. G. Maia in 1968 and is also a generalization of Banach contraction mapping principle for sets endowed with two comparable metrics.

Theorem 2 ([8], [3]). *Let X be a nonempty set, d and ρ two metrics on X and $f : X \rightarrow X$ an operator. We suppose that:*

- (i) $d(x, y) \leq \rho(x, y)$ for all $x, y \in X$;
- (ii) (X, d) is a complete metric space;
- (iii) $f : (X, d) \rightarrow (X, d)$ is continuous;
- (iv) $f : (X, \rho) \rightarrow (X, \rho)$ is an α -contraction.

Then:

- (a) $F_f = \{x^*\}$;
- (b) $f^n(x) \xrightarrow{d} x^*$ as $n \rightarrow \infty$, for all $x \in X$;
- (c) $f^n(x) \xrightarrow{\rho} x^*$ as $n \rightarrow \infty$, for all $x \in X$;
- (d) $\rho(x, x^*) \leq \frac{1}{1-\alpha} \rho(x, f(x))$, for each $x \in X$.

Remark 1. Particular cases:

1. From Maia's fixed point theorem when $d \equiv \rho$, we get Banach's fixed point theorem.
2. From Prešić's fixed point theorem when $k = 1$, we get Banach's fixed point theorem.
3. A Prešić-Maia type fixed point theorem has been obtained by Shukla, S. and Radenović, S. in [9].

Prešić's fixed point theorem has been extended by Păcurar to the case when the Banach type contraction condition (1.1) is replaced by a Prešić-Kannan type contraction.

Theorem 3 ([5], [2]). *Let (X, d) be a complete metric space, k a positive integer, $a \in \mathbb{R}$ a constant such that $0 \leq a \cdot k \cdot (k + 1) < 1$ and $f : X^k \rightarrow X$ an operator satisfying the following condition:*

$$\begin{aligned}
& d(f(x_0, x_1, \dots, x_{k-1}), f(x_1, x_2, \dots, x_k)) \leq \\
& \leq a \cdot d(x_0, f(x_0, \dots, x_0)) + a \cdot d(x_1, f(x_1, \dots, x_1)) + \dots + \\
& \quad + a \cdot d(x_k, f(x_k, \dots, x_k)),
\end{aligned} \tag{1.2}$$

for any $x_0, x_1, \dots, x_k \in X$.

Then:

- 1) f has a unique fixed point x^* ;
- 2) the sequence $\{y_n\}_{n \geq 0}$,

$$y_{n+1} = f(y_n, y_n, \dots, y_n), n \geq 0,$$

converges to x^* ;

- 3) the sequence $\{x_n\}_{n \geq 0}$ with $x_0, \dots, x_{k-1} \in X$ and

$$x_n = f(x_{n-k}, x_{n-k+1}, \dots, x_{n-1}), n \geq k,$$

also converges to x^* , with a rate estimated by

$$d(x_{n+1}, x^*) \leq C \cdot \theta^n, n \geq 0,$$

where C is a positive constant.

Starting from these results, the aim of this paper is to extend Theorem 3 to the case of a set endowed with two comparable metrics.

We shall need in our proof the following lemma given by S. Prešić in [4].

Lemma 1. Let $k \in \mathbb{N}, k \neq 0$ and $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}_+$ such that $\sum_{i=1}^k \alpha_i = \alpha < 1$. If $\{\Delta_n\}_{n \geq 1}$ is a sequence of positive numbers satisfying

$$\Delta_{n+k} \leq \alpha_1 \Delta_n + \alpha_2 \Delta_{n+1} + \dots + \alpha_k \Delta_{n+k-1}, n \geq 1, \tag{1.3}$$

then there exist $L > 0$ and $\theta \in (0, 1)$ such that

$$\Delta_n \leq L \cdot \theta^n, \text{ for all } n \geq 1. \tag{1.4}$$

Remark 2. The following consideration shall simplify our approach in this proof [7].

For any operator $f : X^k \rightarrow X$, k a positive integer, we can define its *associate operator* $F : X \rightarrow X$ by

$$F(x) = f(x, \dots, x), x \in X.$$

Obviously, $x \in X$ is a fixed point of $f : X^k \rightarrow X$ if and only if it is a fixed point of its associate operator F . This enables the study of f by means of the operator F .

2. THE MAIN RESULT

Theorem 4 (Maia fixed point theorem for Prešić-Kannan operators). *Let X be a nonempty set, d and ρ two metrics on X and k a positive integer, $a \in \mathbb{R}$ a constant such that $0 \leq a \cdot k \cdot (k+1) < 1$ and $f : X^k \rightarrow X$ an operator satisfying the following condition:*

$$\begin{aligned} \rho(f(x_0, x_1, \dots, x_{k-1}), f(x_1, x_2, \dots, x_k)) \leq \\ \leq a \cdot \rho(x_0, f(x_0, \dots, x_0)) + a \cdot \rho(x_1, f(x_1, \dots, x_1)) + \\ + \dots + a \cdot \rho(x_k, f(x_k, \dots, x_k)), \end{aligned} \quad (2.1)$$

for any $x_0, x_1, \dots, x_k \in X$.

We suppose that:

- (i) $d(x, y) \leq \rho(x, y)$ for all $x, y \in X$;
- (ii) (X, d) is a complete metric space;
- (iii) $f : (X^k, d) \rightarrow (X, d)$ is continuous;

Then:

- (a) f has a unique fixed point x^* , $Ff = x^*$, $f(x^*, x^*, \dots, x^*) = x^*$;
- (b) the sequence $\{x_n\}_{n \geq 0}$ with $x_0, \dots, x_{k-1} \in X$ and

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k+1}), n \geq k-1,$$

converges to x^* with respect to d ;

- (c) the sequence $\{y_n\}_{n \geq 0}$,

$$y_{n+1} = f(y_n, y_n, \dots, y_n), n \geq 0,$$

converges to x^* with respect to ρ ;

- (d) the following estimate holds

$$\rho(x_{n+1}, x^*) \leq C \cdot \theta^n, n \geq 0, \quad (2.2)$$

where C is a positive constant.

Proof. (c): From $\{y_n\}_{n \geq 0}$, $y_{n+1} = f(y_n, y_n, \dots, y_n)$, $n \geq 0$ and considering the associate operator $F : X \rightarrow X$, $F(x) = f(x, x, \dots, x)$, for any $x, y \in X$ we have:

$$\begin{aligned} \rho(F(x), F(y)) &= \rho(f(x, x, \dots, x), f(y, y, \dots, y)) \leq \\ &\leq \rho(f(x, x, \dots, x), f(x, x, \dots, y)) + \\ &+ \rho(f(x, x, \dots, y), f(x, \dots, x, y)) + \\ &+ \dots + \\ &+ \rho(f(x, y, \dots, y), f(y, y, \dots, y)). \end{aligned}$$

By condition (1.2) and inequality (2.5) it follows

$$\rho(F(x), F(y)) \leq a \cdot \frac{k \cdot (k+1)}{2} \cdot [\rho(x, F(x)) + \rho(y, F(y))]$$

for any $x, y \in X$. As a was assumed to satisfy $0 \leq a \cdot k \cdot (k+1) < 1$, it follows that

$$0 \leq a \cdot \frac{k \cdot (k+1)}{2} < \frac{1}{2},$$

so F is a Kannan operator.

According to Kannan's theorem [2], there exists a unique $x^* \in X$ such that $F(x^*) = x^*$, namely that $x^* = f(x^*, x^*, \dots, x^*)$ and this can be obtained as limit of the sequence of successive approximations of F , which is exactly the sequence $\{y_n\}_{n \geq 0}$, defined by $y_{n+1} = f(y_n, y_n, \dots, y_n)$, $n \geq 0$, so

$$\lim_{n \rightarrow \infty} F(y_n) = x^*, \quad F(y_n) \text{ converges to } x^* = f(x^*, x^*, \dots, x^*) \text{ in } (X, \rho)$$

and

$$y_{n+1} = f(y_n, y_n, \dots, y_n) \text{ converges to } x^* = f(x^*, x^*, \dots, x^*).$$

(a), (b) : Let $\{x_n\}_{n \geq 0}$, $x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k+1})$, $n \geq k-1$.

For $n \geq 0$ we have:

$$\begin{aligned} \rho(x_{n+1}, x^*) &= \rho(f(x_n, x_{n-1}, \dots, x_{n-k+1}), f(x^*, x^*, \dots, x^*)) \leq \\ &\leq \rho(f(x_n, x_{n-1}, \dots, x_{n-k+1}), f(x_{n-1}, \dots, x_{n-k+1}, x^*)) + \\ &+ \rho(f(x_{n-1}, \dots, x_{n-k+1}, x^*), f(x_{n-2}, \dots, x_{n-k+1}, x^*, x^*)) + \\ &\quad + \dots + \\ &+ \rho(f(x_{n-k+1}, x^*, \dots, x^*), f(x^*, x^*, \dots, x^*)). \end{aligned}$$

Let $F : X \rightarrow X$, $F(x) = f(x, x, \dots, x)$, $x \in X$ and using the condition (1.2) in the inequality above, we get:

$$\begin{aligned} \rho(x_{n+1}, x^*) &\leq a \cdot [\rho(x_n, F(x_n)) + \dots + \\ &+ \rho(x_{n-k+1}, F(x_{n-k+1})) + \rho(x^*, F(x^*))] + \\ &+ a \cdot [\rho(x_{n-1}, F(x_{n-1})) + \dots + \rho(x_{n-k+1}, F(x_{n-k+1})) + \\ &+ \rho(x^*, F(x^*)) + \rho(x^*, F(x^*))] + \\ &\quad + \dots + \\ &+ a \cdot [\rho(x_{n-k+1}, F(x_{n-k+1})) + \rho(x^*, F(x^*)) + \dots + \rho(x^*, F(x^*))]. \end{aligned}$$

Since $\rho(x^*, F(x^*)) = 0$, this implies:

$$\begin{aligned} \rho(x_{n+1}, x^*) &\leq a \cdot [1 \cdot \rho(x_n, F(x_n)) + 2 \cdot \rho(x_{n-1}, F(x_{n-1})) + \\ &\quad \dots + k \cdot \rho(x_{n-k+1}, F(x_{n-k+1}))] \end{aligned} \tag{2.3}$$

For each $j \in \mathbb{N}$ the following holds:

$$\rho(x_j, F(x_j)) \leq \rho(x_j, x^*) + \rho(x^*, F(x_j)) \tag{2.4}$$

and $\rho(x^*, F(x_j)) = \rho(F(x^*), F(x_j))$

$$\rho(F(x^*), F(x_j)) \leq$$

$$\begin{aligned}
& a \cdot \underbrace{[\rho(x^*, f(x^*, x^*, \dots, x^*)) + \dots + \rho(x^*, f(x^*, x^*, \dots, x^*))]}_{k \text{ times}} + \\
& \quad + \rho(x_j, f(x_j, x_j, \dots, x_j)) + \\
& + a \cdot \underbrace{[\rho(x^*, f(x^*, x^*, \dots, x^*)) + \dots + \rho(x^*, f(x^*, x^*, \dots, x^*))]}_{k-1 \text{ times}} + \\
& \quad + \underbrace{\rho(x_j, f(x_j, x_j, \dots, x_j)) + \rho(x_j, f(x_j, x_j, \dots, x_j))}_{2 \text{ times}} + \\
& \quad + \dots + \\
& \quad + a \cdot [\rho(x^*, f(x^*, x^*, \dots, x^*)) + \\
& \quad + \underbrace{\rho(x_j, f(x_j, x_j, \dots, x_j)) + \dots + \rho(x_j, f(x_j, x_j, \dots, x_j))}_{k \text{ times}}]
\end{aligned}$$

so

$$\begin{aligned}
\rho(F(x^*), F(x_j)) & \leq a \cdot \rho(x^*, f(x^*, x^*, \dots, x^*)) \cdot [k + (k-1) + \dots + 1] + \\
& \quad + a \cdot \rho(x_j, f(x_j, x_j, \dots, x_j)) \cdot [1 + 2 + \dots + k]
\end{aligned}$$

and finally

$$\begin{aligned}
\rho(F(x^*), F(x_j)) & \leq a \cdot \frac{k \cdot (k+1)}{2} \cdot [\rho(x^*, f(x^*, x^*, \dots, x^*)) + \\
& \quad \rho(x_j, f(x_j, x_j, \dots, x_j))].
\end{aligned}$$

Thus

$$\rho(F(x^*), F(x_j)) \leq a \cdot \frac{k \cdot (k+1)}{2} \cdot [\rho(x^*, F(x^*)) + \rho(x_j, F(x_j))] \quad (2.5)$$

and

$$\begin{aligned}
\rho(x^*, F(x_j)) & = \rho(F(x^*), F(x_j)) \leq \\
& \leq a \cdot \frac{k \cdot (k+1)}{2} \cdot [\rho(x^*, F(x^*)) + \rho(x_j, F(x_j))] = \\
& = a \cdot \frac{k \cdot (k+1)}{2} \cdot \rho(x_j, F(x_j)).
\end{aligned}$$

Thus (2.5) becomes:

$$\rho(x_j, F(x_j)) \leq \rho(x_j, x^*) + a \cdot \frac{k \cdot (k+1)}{2} \cdot \rho(x_j, F(x_j)).$$

By denoting $A = a \cdot \frac{k \cdot (k+1)}{2}$ we have

$$\rho(x_j, F(x_j)) \leq \frac{1}{1-A} \cdot \rho(x_j, x^*) \quad (2.6)$$

Using (2.6) in inequality (2.3) we obtain

$$\rho(x_{n+1}, x^*) \leq$$

$$\leq \frac{a}{1-A} \cdot \rho(x_n, x^*) + \frac{2 \cdot a}{1-A} \cdot \rho(x_{n-1}, x^*) + \cdots + \frac{k \cdot a}{1-A} \cdot \rho(x_{n-k+1}, x^*) \quad (2.7)$$

Now by denoting

$$\begin{aligned} \Delta_n &= \rho(x_n, x^*), n \geq 0 \\ \alpha_i &= \frac{i \cdot a}{1-A}, i = \overline{1, k}, \end{aligned}$$

the inequality (2) becomes

$$\Delta_{n+1} \leq \alpha_1 \cdot \Delta_n + \alpha_2 \cdot \Delta_{n-1} + \cdots + \alpha_k \cdot \Delta_{n-k+1}, n \geq k. \quad (2.8)$$

The coefficients $\alpha_1, \alpha_2, \dots, \alpha_k$ are all positive, as $0 < a < \frac{1}{k \cdot (k+1)}$.

Besides

$$\sum_{i=1}^k \alpha_i = \sum_{i=1}^k \frac{i \cdot a}{1-A} = \frac{a}{1-A} \cdot \sum_{i=1}^k i = \frac{a}{1-A} \cdot \frac{k \cdot (k+1)}{2} = \frac{A}{1-A}.$$

So from

$$\begin{aligned} 0 &< a \cdot k \cdot (k+1) < 1 \\ 0 &< \frac{a \cdot k \cdot (k+1)}{2} < \frac{1}{2} \\ 0 &< A < \frac{1}{2} < 1 \end{aligned}$$

we have

$$\sum_{i=1}^k \alpha_i = \frac{A}{1-A} < 1.$$

Now the conditions required in Lemma (1.1) are fulfilled. Consequently there exist $L \geq 0$ and $\theta \in (0, 1)$ such that $\Delta_n \leq L \cdot \theta^n$, $n \geq 1$, namely such that

$$\rho(x_n, x^*) \leq L \cdot \theta^n, n \geq 1. \quad (2.9)$$

Using (2.9) in the inequality (2) we have:

$$\begin{aligned} \rho(x_{n+1}, x^*) &\leq \\ &\leq \frac{a}{1-A} \cdot L \cdot \theta^n + \frac{2 \cdot a}{1-A} \cdot L \cdot \theta^{n-1} + \cdots + \frac{k \cdot a}{1-A} \cdot L \cdot \theta^{n-k+1} \leq \\ &\leq \theta^n \cdot \frac{a \cdot L}{1-A} \cdot (1 + 2 \cdot \theta^{-1} + \cdots + k \cdot \theta^{1-k}) = \\ &= \theta^n \cdot \frac{a \cdot L}{1-A} \cdot \sum_{i=1}^k i \cdot \theta^{1-i}, L \geq 0, \theta > 0. \end{aligned}$$

This inequality shall lead to estimate (d), where:

$$C = \frac{a \cdot L}{1-A} \cdot \sum_{i=1}^k i \cdot \theta^{1-i} > 0$$

$$\rho(x_{n+1}, x^*) \leq C \cdot \theta^n, \quad n \geq 0. \quad (2.10)$$

It follows immediately that $\rho(x_{n+1}, x^*) \rightarrow 0$ as $n \rightarrow \infty$, so the sequence $\{x_n\}_{n \geq 0}$ converges to x^* in (X, ρ) , and x^* is the unique fixed point of the operator f .

For $n \geq 0$, from (i) and (2.10) we obtain

$$d(x_{n+1}, x^*) \leq \rho(x_{n+1}, x^*) < \varepsilon.$$

Since $\theta \in (0, 1)$, it follows that $\{x_n\}_{n \geq 0}$ is a Cauchy sequence in (X, d) , which in the complete metric space (X, d) is also convergent.

From the continuity of f in (X, d) ,

$$\lim_{n \rightarrow \infty} d(f(x_n, x_{n-1}, \dots, x_{n-k+1}), f(x^*, x^*, \dots, x^*)) = 0$$

we get

$$\lim_{n \rightarrow \infty} f(x_n, x_{n-1}, \dots, x_{n-k+1}) = f(x^*, x^*, \dots, x^*) \text{ in } (X, d)$$

and $\{x_n\}_{n \geq 0}$ converges to x^* in (X, d) and $F_f = x^*, x^* = f(x^*, x^*, \dots, x^*)$. □

Remark 3. We have the following important particular cases of Theorem 4:

1. If $k = 1$, by Theorem 4 we get a Maia fixed point theorem corresponding to Kannan contraction condition. Kannan fixed point theorem is obtained in the particular case when $d = \rho$. Note that the original Maia's fixed point theorem is established for the case of Banach's contraction condition.
2. If $d = \rho$, by Theorem 4 we get Păcurar's fixed point theorem [6].

Example 1. Let $X = [0, 1]$ endowed with two metrics d and ρ . For $k = 2$ define $f : X^2 \rightarrow X$ by

$$f(x, y) = \begin{cases} \frac{x+y}{6}, & (x, y) \in [0, 1], x \geq y, \\ 0, & \text{otherwise.} \end{cases} \quad (2.11)$$

Then f is a Prešić-Kannan contraction.

Proof. In the first part of the proof we will show that f is a Prešić-Kannan operator. In this particular case condition (2.1) becomes:

$$\begin{aligned} \rho(f(x_0, x_1), f(x_1, x_2)) &\leq a \cdot \rho(x_0, f(x_0, x_0)) + \\ &+ a \cdot \rho(x_1, f(x_1, x_1)) + a \cdot \rho(x_2, f(x_2, x_2)), \end{aligned}$$

with $x_0 > x_1 > x_2$.

If $k = 2$ then $0 \leq a < \frac{1}{6}$.

For $x_0 = 1, x_1 = x_2 = 0$

$$\rho(f(x_0, x_1), f(x_1, x_2)) = \rho(f(1, 0), f(0, 0)) = \rho\left(\frac{1}{6}, 0\right) = \frac{1}{36} \leq$$

$$\begin{aligned} &\leq a \cdot \rho(1, f(1, 1)) + a \cdot \rho(0, f(0, 0)) + a \cdot \rho(0, f(0, 0)) = \\ &= a \cdot [\rho(1, \frac{1}{3}) + 0 + 0] = a \cdot \frac{2}{9} \end{aligned}$$

so

$$\frac{1}{36} \leq a \cdot \frac{2}{9}$$

it follows that

$$a \geq \frac{1}{8}.$$

The conclusion is that f given by (2.11) is a Prešić-Kannan operator, that is, it satisfies (2.1) for any $x_0, x_1, x_2 \in [0, 1]$, $x_0 > x_1 > x_2$, with constant $a \in [\frac{1}{8}, \frac{1}{6})$.

Let $\{x_n\}_{n \geq 0}$, $x_{n+2} = \{x_{n+1}, x_n\}$, $n \geq 0$. For any $x, y \in X$ we have:

$$\rho(x_{n+2}, x^*) \leq C \cdot \theta^n, n \geq 0, \quad (2.12)$$

where

$$C = \frac{a \cdot L}{1 - 3 \cdot a} \cdot (1 + 2 \cdot \frac{1}{\theta}) > 0.$$

It follows immediately that $\rho(x_{n+2}, x^*) \rightarrow 0$ when $n \rightarrow \infty$, so the sequence $\{x_n\}_{n \geq 0}$ converges to x^* in (X, ρ) , and $x^* = f(x^*, x^*)$ is the unique fixed point of the operator f .

$$d(x_{n+2}, x^*) \leq \rho(x_{n+2}, x^*) < \epsilon$$

Since $\theta \in (0, 1)$, it follows that $\{x_n\}_{n \geq 0}$ is a Cauchy sequence in (X, d) , which in the complete metric space (X, d) is also convergent.

From the continuity of f in (X, d) ,

$$\lim_{n \rightarrow \infty} d(f(x_{n+1}, x_n), f(x^*, x^*)) = 0$$

we get

$$\lim_{n \rightarrow \infty} f(x_{n+1}, x_n) = f(x^*, x^*) \text{ in } (X, d)$$

and $\{x_n\}_{n \geq 0}$ converges to x^* in (X, d) and $F_f = x^*$, $x^* = f(x^*, x^*)$.

From $\{y_n\}_{n \geq 0}$, $y_{n+2} = f(y_n, y_n)$, $n \geq 0$ and considering the associate operator $F : X \rightarrow X$, $F(x) = f(x, x)$, for any $x, y \in X$ we have:

$$\rho(F(x), F(y)) \leq 3 \cdot a \cdot [\rho(x, F(x)) + \rho(y, F(y))]$$

for any $x, y \in X$. As a was assumed to satisfy $0 \leq a \cdot k \cdot (k + 1) < 1$, it follows that

$$0 \leq 3 \cdot a < \frac{1}{2},$$

so F is a Kannan operator and due to Kannan's theorem, there exist a unique fixed point $x^* \in X$ such that $F(x^*) = x^*$, so $\{y_n\}_{n \geq 0}$ converges to $x^* = f(x^*, x^*)$. $\square$

The following theorem is an extension of Theorem 4 to the case of metric spaces endowed with a partial order.

Theorem 5. *Let X be a nonempty set equipped with partial order $\leq$, d and ρ two metrics on X and k a positive integer, $a \in \mathbb{R}$ a constant such that $0 \leq a \cdot k \cdot (k+1) < 1$ and $f : X^k \rightarrow X$ an operator satisfying the following condition:*

$$\begin{aligned} & \rho(f(x_0, x_1, \dots, x_{k-1}), f(x_1, x_2, \dots, x_k)) \leq \\ & \leq a \cdot \rho(x_0, f(x_0, \dots, x_0)) + a \cdot \rho(x_1, f(x_1, \dots, x_1)) + \\ & + \dots + a \cdot \rho(x_k, f(x_k, \dots, x_k)), \end{aligned} \quad (2.13)$$

for any $x_0, x_1, \dots, x_k \in X$ with $x_0 \leq x_1 \leq \dots \leq x_k$.

We suppose that:

- (i) $d(x, y) \leq \rho(x, y)$ for all $x, y \in X$ with $x \leq y$;
- (ii) (X, d) is a complete metric space;
- (iii) $f : (X^k, d) \rightarrow (X, d)$ is continuous and nondecreasing with respect to " $\leq$ ";

Then:

- (a) f has a unique fixed point x^* , $F_f = x^*$, $f(x^*, x^*, \dots, x^*) = x^*$;
- (b) the sequence $\{x_n\}_{n \geq 0}$ with $x_0, \dots, x_{k-1} \in X$ and

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k+1}), n \geq k-1,$$

converges to x^* ;

- (c) the sequence $\{y_n\}_{n \geq 0}$,

$$y_{n+1} = f(y_n, y_n, \dots, y_n) = F(y_n),$$

for all $n \geq 0$, $y_0 \in X$ such that $y_0 \leq x^*$, converges to x^* as well;

- (d) the following estimate holds

$$\rho(x_{n+1}, x^*) \leq C \cdot \theta^n, n \geq 0, \quad (2.14)$$

where C is a positive constant and $\theta \in (0, 1)$.

Proof of this theorem follows from a similar process as used in the proof of theorem 4.

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AN EXTENSION OF DARBO'S THEOREM AND ITS APPLICATION TO SYSTEM OF NEUTRAL DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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Abstract. In this paper we extend Darbo's fixed point theorem in Banach spaces and obtain a tripled fixed point theorem. The technique of measure of noncompactness is the main tool in carrying out our proof. Finally, as an application of our results, we analyze the existence of solutions for a system of neutral differential equations.

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1. INTRODUCTION AND PRELIMINARIES

The notion of a measure of non compactness (MNC) was introduced by Kuratowski [9] in 1930. Darbo's fixed point theorem [7] which ensures the existence of fixed point is an important application of this measure, since it generalizes both Schauder fixed point and Banach contraction principle. Thereafter, many investigations and papers have been appeared in the MNC and it's application, see for example [1, 2, 5, 8, 10]. The aim of this paper is to generalize the Darbo's fixed point theorem via the concept of the class of operators $O(f; \cdot)$ and extend the results of the theorem presented by Samadi and Ghaemi in [10]. Further, we apply our extension to obtain a tripled fixed point and study the existence of solutions for the system

$$\begin{cases} x'(t) = f\left(t, x(\zeta(t)), y(\zeta(t)), z(\zeta(t)), x'(\beta(t)), y'(\beta(t)), z'(\beta(t))\right) \\ y'(t) = f\left(t, y(\zeta(t)), x(\zeta(t)), z(\zeta(t)), y'(\beta(t)), x'(\beta(t)), z'(\beta(t))\right) \\ z'(t) = f\left(t, z(\zeta(t)), y(\zeta(t)), x(\zeta(t)), z'(\beta(t)), y'(\beta(t)), x'(\beta(t))\right) \end{cases} \quad (1.1)$$

where $t \in [0, T]$ with the initial condition

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0.$$

First, we introduce some notations and definitions which are used throughout this paper. Let R be the set of real numbers, $R_+ = [0, \infty)$ and $(E, \|\cdot\|)$ be a real Banach space with the zero element 0. We write $\overline{B}(x, r)$ to denote the closed ball centered at x with radius r . If X be a nonempty subset of E then the symbols $\overline{X}$ and $ConvX$ stand for the closure and closed convex hull of X , respectively. Moreover, $\mathfrak{M}_E$ is the family of nonempty bounded subset of E and $\mathfrak{N}_E$ denote its subfamily consisting of all relatively compact sets.

Definition 1 ([6]). A mapping $\mu : \mathfrak{M}_E \longrightarrow R_+$ is said to be a measure of non-compactness in E if it satisfies the following conditions:

- 1° The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{N}_E$;
- 2° $X \subset Y \implies \mu(X) \leq \mu(Y)$;
- 3° $\mu(\overline{X}) = \mu(X)$;
- 4° $\mu(ConvX) = \mu(X)$;
- 5° $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$;
- 6° If $\{X_n\}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \dots$, and if $\lim_{n \rightarrow \infty} \mu(X_n) = 0$, then $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

The following concept of $O(f; \cdot)$ was given by Altun and Turkoglu [4].

Let $F([0, \infty))$ be the class of all functions $f : [0, \infty) \rightarrow [0, \infty)$ and let Θ be the class of all operators

$$O(\bullet; \cdot) : F([0, \infty)) \rightarrow F([0, \infty)), \quad f \rightarrow O(f; \cdot)$$

satisfying the following conditions:

- (1) $O(f; t) > 0$ for $t > 0$ and $O(f; 0) = 0$.
- (2) $O(f; t) \leq O(f; s)$ for $t \leq s$.
- (3) $\lim_{n \rightarrow \infty} O(f; t_n) = O(f; \lim_{n \rightarrow \infty} t_n)$.
- (4) $O(f; \max\{t, s\}) = \max\{O(f; t), O(f; s)\}$ for some $f \in F([0, \infty))$.

Example 1. ([4]) If $f : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing, continuous function such that $f(0) = 0$ and $f(t) > 0$ for $t > 0$, then the operator defined by

$$O(f; t) = \frac{f(t)}{1 + f(t)}$$

satisfies the above conditions.

Definition 2 ([8]). A triple (x, y, z) is called a triple fixed point of a mapping $T : X \times X \times X \rightarrow X$ if

$$T(x, y, z) = x, \quad T(y, x, z) = y, \quad T(z, y, x) = z.$$

The following theorems are basic for our main results.

Theorem 1 (Schauder [3]). *Let C be a nonempty, bounded, closed and convex subset of a Banach space E . Then every compact and continuous map $T : C \rightarrow C$ has at least one fixed point.*

Theorem 2 (Darbo [7]). *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and $T : C \rightarrow C$ be a continuous mapping. Assume that there exists a constant $K \in [0, 1)$ such that $\mu(TX) \leq K\mu(X)$ for any nonempty subset X of C , where μ is a MNC defined in E . Then T has at least a fixed point in C .*

Theorem 3 ([10]). *Suppose $\mu_1, \mu_2, \dots, \mu_n$ are measures of noncompactness in Banach spaces $E_1, E_2, \dots, E_n$ respectively. Moreover assume that the function $F : R_+^n \rightarrow R_+$ is convex and $F(x_1, \dots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, \dots, n$. Then*

$$\tilde{\mu}(X) = F(\mu_1(X_1), \mu_2(X_2), \dots, \mu_n(X_n)),$$

defines a measure of noncompactness in $E_1 \times E_2 \times \dots \times E_n$ where X_i denotes the natural projection of X into E_i , for $i = 1, 2, \dots, n$.

Remark 1. Aghajani et al. [1] illustrated the Theorem 3 by the following example. Let F be as follows:

$$F(x, y, z) = \max\{x, y, z\}, \text{ or } F(x, y, z) = x + y + z, \text{ for any } (x, y, z) \in [0, \infty)^3.$$

They showed that

$$\tilde{\mu}(X) = \max(\mu_1(X_1), \mu_2(X_2), \mu_3(X_3)), \text{ or } \tilde{\mu}(X) = \mu_1(X_1) + \mu_2(X_2) + \mu_3(X_3),$$

defines an MNC in the space $E_1 \times E_2 \times E_3$ where X_i , ($i = 1, 2, 3$) are the natural projection of X into E_i .

Samadi and Ghaemi [10] generalized Darbo's fixed point theorem as follow.

Theorem 4. *Let U be a nonempty, bounded, closed and convex subset of a Banach space E . Assume $F : U \rightarrow U$ be a continuous operator such that satisfying*

$$\psi(\mu(F(X))) \leq \phi(\psi(\mu(X)))\psi(\mu(X))$$

for all nonempty subset X of U , where μ is an arbitrary MNC in E , $\psi : R_+ \rightarrow R_+$ is nondecreasing function such that $\psi(t) = 0$ if and only if $t=0$ and $\phi : [0, \infty) \rightarrow [0, 1)$ is a continuous function such that $\limsup_{t \rightarrow r} \phi(t) < 1$ for all $r \geq 0$. Then F has a fixed point in U .

The Darbo's fixed point theorem is followed if $\psi = I$ (identity map) and $\phi = K$ in the Theorem 4.

2. MAIN RESULTS

Now we state one of the main results in this paper which extends and generalizes Darbo's fixed point theorem by using concept of $O(f; \cdot)$.

Theorem 5. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and $T : C \rightarrow C$ be a continuous operator such that*

$$\psi[O(f; \mu(TX))] \leq \Phi[\psi(O(f; \mu(X)))]\psi(\mu(X)),$$

for X of C , $O(\bullet; \cdot) \in \Theta$ and let $\psi: R_+ \rightarrow R_+$ be a nondecreasing function such that $\psi(t) = 0$ if and only if $t=0$. Let $\Phi: [0, \infty) \rightarrow [0, 1]$ is a continuous function such that $\lim_{n \rightarrow \infty} \Phi^n(t) = 0$ for each $t \geq 0$, where μ is an arbitrary MNC. Then T has at least one fixed point in C .

Proof. By induction, we construct a sequence $\{C_n\}$ such that $C_0 = C$ and $C_{n+1} = \text{Conv}(TC_n)$ for $n \geq 1$. We have $TC_0 = TC \subseteq C = C_0$, $C_1 = \text{Conv}(TC_0) \subseteq C = C_0$, therefore by continuing this process deduce that

$$C_0 \supseteq C_1 \supseteq \dots \supseteq C_n \supseteq C_{n+1} \supseteq \dots$$

If there exists an integer $N \geq 0$ such that $\mu(C_N) = 0$, then C_N is relatively compact. In this case Theorem 1 implies that T has a fixed point. Now we suppose that $\mu(C_n) \neq 0$ for $n \geq 0$. We have

$$\begin{aligned} \psi[O(f; \mu(C_{n+1}))] &= \psi[O(f; \mu(\text{Conv}(TC_n)))] \\ &= \psi[O(f; \mu(TC_n))] \\ &\leq \Phi[\psi(O(f; \mu(C_n)))]\psi(\mu(C_n)) \\ &\leq \Phi^2[\psi(O(f; \mu(C_{n-1})))]\psi(\mu(C_{n-1})) \\ &\vdots \\ &\leq \Phi^n[\psi(O(f; \mu(C_0)))]\psi(\mu(C_0)) \\ &= \Phi^n[\psi(O(f; \mu(C)))]\psi(\mu(C)) \end{aligned} \quad (2.1)$$

By letting $n \rightarrow \infty$ in (1.2) we infer that $\lim_{n \rightarrow \infty} \psi[O(f; \mu(C_{n+1}))] = 0$. Therefore,

$$\lim_{n \rightarrow \infty} \psi[O(f; \mu(C_{n+1}))] = \psi[O(f; \lim_{n \rightarrow \infty} \mu(C_{n+1}))] = 0,$$

we know that $O(f; 0) = 0$. Thus,

$\lim_{n \rightarrow \infty} \mu(C_{n+1}) = 0$. Since $C_n \supseteq C_{n+1}$ and $TC_n \subseteq C_n$ for all $n = 1, 2, 3, \dots$ then from 6° of definition MNC, $X_\infty = \bigcap_{n=1}^\infty X_n$ is a nonempty, closed and convex set, invariant under T and belongs to $\text{Ker } \mu$. Consequently, from Theorem 1 we deduce that T has at least a fixed point. $\square$

Remark 2. The Theorem 4 is followed if $O(f; t) = t$ and $f = I$ in Theorem 5.

The following Corollary is immediate of Theorem 5.

Corollary 1. Let C be a nonempty, bounded, closed and convex subset of a Banach space E , $T: C \rightarrow C$ and $\psi: R_+ \rightarrow R_+$ are continuous functions. Suppose that there exists a constant $0 < \lambda < 1$ such that for all $X \subseteq C$,

$$\psi[O(f; \mu(TX))] \leq \lambda[\psi(O(f; \mu(X)))]\psi(\mu(X)),$$

where μ is an arbitrary measure of noncompactness and $O(\bullet; \cdot) \in \Theta$. Then T has at least one fixed point in C .

Remark 3. The Darbo's fixed point theorem is followed if $O(f; t) = t$, $f = I$ and $\psi = I$ in Corollary 1.

Theorem 6. Let C be a nonempty, bounded, closed and convex subset of a Banach space E and $T : C \times C \times C \rightarrow C$ be a continuous function such that

$$\begin{aligned} \psi[O(f; \mu(T(x_1 \times x_2 \times x_3)))] &\leq \frac{1}{3} \Phi[\psi(O(f; \mu(x_1) + \mu(x_2) + \mu(x_3)))] \\ &\quad \times \psi[\mu(x_1) + \mu(x_2) + \mu(x_3)] \end{aligned} \quad (2.2)$$

for any subset x_1, x_2, x_3 of C , where μ is an arbitrary MNC, $\Phi : [0, \infty) \rightarrow [0, 1)$ is nondecreasing function such that $\lim_{n \rightarrow \infty} \Phi^n(t) = 0$ for $t \geq 0$ and $\psi : R_+ \rightarrow R_+$ is a continuous function such that $\psi(t) = 0$ if and only if $t=0$. Also, $O(\bullet; \cdot) \in \Theta$ and $O(f; t + r + s) \leq O(f; t) + O(f; s) + O(f; r)$ for all $t, s, r \geq 0$. Then T has at least a tripled fixed point.

Proof. From Remark 1, we have $\tilde{\mu}(x) = \mu(x_1) + \mu(x_2) + \mu(x_3)$. Now we define a mapping $\tilde{T} : C \times C \times C \rightarrow C \times C \times C$ by

$$\tilde{T}(x, y, z) = (T(x, y, z), T(y, x, z), T(z, y, x)).$$

It is clear that $\tilde{T}$ is continuous. We prove that $\tilde{T}$ satisfies all the conditions of Theorem 5. Let $X \subseteq C \times C \times C$ be any nonempty subset by in (2.2) and 2° we obtain

$$\begin{aligned} \psi[O(f; \tilde{\mu}(\tilde{T}(x)))] &\leq \psi[O(f; \tilde{\mu}(T(x_1 \times x_2 \times x_3)) \times T(x_2 \times x_1 \times x_3) \\ &\quad \times T(x_3 \times x_2 \times x_1))] \\ &= \psi[O(f; \mu(T(x_1 \times x_2 \times x_3)) + \mu(T(x_2 \times x_1 \times x_3)) \\ &\quad + \mu(T(x_3 \times x_2 \times x_1)))] \\ &= \psi[O(f; \mu(T(x_1 \times x_2 \times x_3)))] + \psi[O(f; \mu(T(x_2 \times x_1 \times x_3)))] \\ &\quad + \psi[O(f; \mu(T(x_3 \times x_2 \times x_1)))] \\ &\leq \frac{1}{3} \Phi[\psi(O(f; \mu(x_1) + \mu(x_2) + \mu(x_3)))] \psi[\mu(x_1) + \mu(x_2) + \mu(x_3)] \\ &\quad + \frac{1}{3} \Phi[\psi(O(f; \mu(x_2) + \mu(x_1) + \mu(x_3)))] \psi[\mu(x_2) + \mu(x_1) + \mu(x_3)] \\ &\quad + \frac{1}{3} \Phi[\psi(O(f; \mu(x_3) + \mu(x_2) + \mu(x_1)))] \psi[\mu(x_3) + \mu(x_2) + \mu(x_1)] \\ &= \Phi[\psi(O(f; \mu(x_1) + \mu(x_2) + \mu(x_3)))] \psi[\mu(x_1) + \mu(x_2) + \mu(x_3)]. \end{aligned}$$

From Theorem 5 deduce that $\tilde{T}$ has at least a fixed point in $C \times C \times C$ and T has at least a tripled fixed point. $\square$

As application of Theorem 6 we can get the following corollary.

Corollary 2. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and $T : C \times C \times C \rightarrow C$ be a continuous function such that*

$$\mu(T(x_1 \times x_2 \times x_3)) \leq \frac{1}{3} \Phi(\mu(x_1) + \mu(x_2) + \mu(x_3)),$$

for any subset x_1, x_2, x_3 of C , where μ is an arbitrary MNC. Also, $\Phi : [0, \infty) \rightarrow [0, 1]$ is a nondecreasing function such that $\lim_{n \rightarrow \infty} \Phi^n(t) = 0$ for $t \geq 0$. Then T has at least a tripled fixed point.

3. APPLICATION

In this section as an application of Theorem 6 and Corollary 2 we study the existence of solutions for the system of equations (1.1). Let the space $C[0, T]$ consisting of all real functions defined, bounded and continuous on the interval $[0, T]$ equipped with the standard norm

$$\|x\| = \sup\{|x(t)| : t \in [0, T]\}.$$

Recall that the modulus of continuity of a function $x \in C[0, T]$ is defined by

$$\omega(x, \epsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, T], |t - s| \leq \epsilon\}.$$

Since x is uniformly continuous on $[0, T]$ then $\omega(x, \epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ and the Hausdorff measure of noncompactness for all bounded sets Ω of $C[0, T]$ is equivalent to

$$\chi(\Omega) = \lim_{\epsilon \rightarrow 0} \left\{ \sup_{x \in X} \omega(x, \epsilon) \right\}.$$

See more detail [3].

Now we consider the following assumptions:

- (A₁) $\zeta, \beta : [0, T] \rightarrow [0, T]$ are continuous functions.
- (A₂) The function $f : [0, T] \times \mathbb{R}^6 \rightarrow \mathbb{R}$ is continuous and there exist continuous and nondecreasing functions $\varphi, \theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\lim_{n \rightarrow \infty} \Phi^n(t) = 0$ for $t \geq 0$ and

$$|f(t, x_1, x_2, \dots, x_6) - f(t, y_1, y_2, \dots, y_6)| \leq \theta\left(\max_{1 \leq i \leq 3} |x_i - y_i|\right) + \quad (3.1)$$

$$\frac{1}{3} \varphi(|x_4 - y_4| + |x_5 - y_5| + |x_6 - y_6|)$$

- (A₃) There exists a positive solution r_0 of the inequality

$$M + \theta(Tr) + \frac{1}{3} \varphi(3r) \leq r \quad (3.2)$$

where

$$M = \sup\{|f(t, x_0, y_0, z_0, 0, 0, 0)| : t \in [0, T]\}.$$

Now, let us put $x'(t) = X(t)$, $y'(t) = Y(t)$ and $z'(t) = Z(t)$. Then the problem (1.1) can be replaced equivalently by the following functional-integral equation

$$\begin{cases} X(t) = f\left(t, x_0 + \int_0^{\xi(t)} X(s)ds, y_0 + \int_0^{\xi(t)} Y(s)ds, z_0 + \int_0^{\xi(t)} Z(s)ds, X(\beta(t)), Y(\beta(t)), Z(\beta(t))\right) \\ Y(t) = f\left(t, y_0 + \int_0^{\xi(t)} Y(s)ds, x_0 + \int_0^{\xi(t)} X(s)ds, z_0 + \int_0^{\xi(t)} Z(s)ds, Y(\beta(t)), X(\beta(t)), Z(\beta(t))\right) \\ Z(t) = f\left(t, z_0 + \int_0^{\xi(t)} Z(s)ds, y_0 + \int_0^{\xi(t)} Y(s)ds, x_0 + \int_0^{\xi(t)} X(s)ds, Z(\beta(t)), Y(\beta(t)), X(\beta(t))\right) \end{cases} \quad (3.3)$$

where $t \in [0, T]$. In the sequel we will examine the equation (3.3).

Theorem 7. *Under the assumptions $(A_1) - (A_3)$ the equation (3.3) has at least one solution $x \in \{C[0, T]\}^3$*

Proof. Let us consider the operator T defined on the space $C[0, T]$ by the formula

$$\begin{aligned} T(x, y, z)(t) = & f\left(t, x_0 + \int_0^{\xi(t)} X(s)ds, y_0 + \int_0^{\xi(t)} Y(s)ds, z_0 + \int_0^{\xi(t)} Z(s)ds, X(\beta(t)), Y(\beta(t)), Z(\beta(t))\right) \end{aligned} \quad (3.4)$$

for all $t \in [0, T]$. Observe that in view of assumptions (A_1) and (A_2) , the function Tx is continuous on $[0, T]$ for any $x \in C[0, T]$, i.e. T transforms the space $C[0, T]$ into itself. For arbitrarily fixed $t \in [0, T]$, we have

$$\begin{aligned} |T(x, y, z)(t)| & \leq |f\left(t, x_0 + \int_0^{\xi(t)} x(s)ds, y_0 + \int_0^{\xi(t)} y(s)ds, z_0 + \int_0^{\xi(t)} z(s)ds, x(\beta(t)), y(\beta(t)), z(\beta(t))\right) \\ & \quad - f(t, x_0, y_0, z_0, 0, 0, 0)| + |f(t, x_0, y_0, z_0, 0, 0, 0)| \\ & \leq \theta \left(\max \left\{ \left| \int_0^{\xi(t)} x(s)ds \right|, \left| \int_0^{\xi(t)} y(s)ds \right|, \left| \int_0^{\xi(t)} z(s)ds \right| \right\} \right) \\ & \quad + \frac{1}{3} \varphi(|x(\beta(t))| + |y(\beta(t))| + |z(\beta(t))|) \\ & \quad + |f(t, x_0, y_0, z_0, 0, 0, 0)| \\ & \leq \theta \left(T \max \{ \|x\|, \|y\|, \|z\| \} \right) + \frac{1}{3} \varphi(\|x\| + \|y\| + \|z\|) + M \end{aligned}$$

Thus,

$$\|T(x, y, z)\| \leq \theta \left(T \max \{\|x\|, \|y\|, \|z\|\} \right) + \frac{1}{3} \varphi(\|x\| + \|y\| + \|z\|) + M \quad (3.5)$$

and $T(x, y, z) \in C[0, T]$ for any $(x, y, z) \in (C[0, T])^3$. Due to Inequality (3.2) the function T maps $(\bar{B}_{r_0})^3$ into $\bar{B}_{r_0}$. Now we show that T is a continuous function on $(\bar{B}_{r_0})^3$. To do this, let us fix $\varepsilon > 0$ and take arbitrary $(x, y, z), (u, v, w) \in (\bar{B}_{r_0})^3$ such that $\max \{\|x - u\|, \|y - v\|, \|z - w\|\} < \varepsilon$. Then, for $t \in [0, T]$, we have

$$\begin{aligned} \left| T(x, y, z)(t) - T(u, v, w)(t) \right| &\leq \left| f \left(t, x_0 + \int_0^{\xi(t)} x(s) ds, y_0 + \int_0^{\xi(t)} y(s) ds, z_0 \right. \right. \\ &\quad \left. \left. + \int_0^{\xi(t)} z(s) ds, x(\beta(t)), y(\beta(t)), z(\beta(t)) \right) \right. \\ &\quad \left. - f \left(t, u_0 + \int_0^{\xi(t)} u(s) ds, v_0 + \int_0^{\xi(t)} v(s) ds, w_0 \right. \right. \\ &\quad \left. \left. + \int_0^{\xi(t)} w(s) ds, u(\beta(t)), v(\beta(t)), w(\beta(t)) \right) \right| \\ &\leq \theta \left(\max \left\{ \int_0^{\xi(t)} |x(s) - u(s)| ds, \int_0^{\xi(t)} |y(s) - v(s)| ds, \right. \right. \\ &\quad \left. \left. \int_0^{\xi(t)} |z(s) - w(s)| ds \right\} \right) \\ &\quad + \frac{1}{3} \varphi \left(|x(\beta(t)) - u(\beta(t))| + |y(\beta(t)) - v(\beta(t))| \right. \\ &\quad \left. + |z(\beta(t)) - w(\beta(t))| \right) \\ &\leq \theta(T\varepsilon) + \frac{1}{3} \varphi(3\varepsilon) \end{aligned}$$

So we have $\theta(T\varepsilon) + \frac{1}{3} \varphi(3\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Thus T is a continuous function on $(\bar{B}_{r_0})^3$. Now we show that T satisfies all the conditions of Corollary 2. Let X_1, X_2 and X_3 be nonempty and bounded subsets of $\bar{B}_{r_0}$. Assume that $\varepsilon > 0$ is arbitrary constant. Also we take $t_1, t_2 \in [0, T]$, with $|t_2 - t_1| \leq \varepsilon$ and $x \in X_1, y \in X_2$ and $z \in X_3$. Then we have

$$\begin{aligned}
& \left| T(x, y, z)(t_1) - T(x, y, z)(t_2) \right| \leq \left| f \left(t_1, x_0 + \int_0^{\xi(t_1)} x(s) ds, y_0 + \int_0^{\xi(t_1)} y(s) ds, z_0 \right. \right. \\
& \quad \left. \left. + \int_0^{\xi(t_1)} z(s) ds, x(\beta(t_1)), y(\beta(t_1)), z(\beta(t_1)) \right) \right. \\
& \quad \left. - f \left(t_1, x_0 + \int_0^{\xi(t_1)} x(s) ds, y_0 + \int_0^{\xi(t_1)} y(s) ds, z_0 \right. \right. \\
& \quad \left. \left. + \int_0^{\xi(t_1)} z(s) ds, x(\beta(t_2)), y(\beta(t_2)), z(\beta(t_2)) \right) \right| \\
& \quad + \left| f \left(t_1, x_0 + \int_0^{\xi(t_1)} x(s) ds, y_0 + \int_0^{\xi(t_1)} y(s) ds, z_0 \right. \right. \\
& \quad \left. \left. + \int_0^{\xi(t_1)} z(s) ds, x(\beta(t_2)), y(\beta(t_2)), z(\beta(t_2)) \right) \right. \\
& \quad \left. - f \left(t_2, x_0 + \int_0^{\xi(t_1)} x(s) ds, y_0 + \int_0^{\xi(t_1)} y(s) ds, z_0 \right. \right. \\
& \quad \left. \left. + \int_0^{\xi(t_1)} z(s) ds, x(\beta(t_2)), y(\beta(t_2)), z(\beta(t_2)) \right) \right| \\
& \quad + \left| f \left(t_2, x_0 + \int_0^{\xi(t_1)} x(s) ds, y_0 + \int_0^{\xi(t_1)} y(s) ds, z_0 \right. \right. \\
& \quad \left. \left. + \int_0^{\xi(t_1)} z(s) ds, x(\beta(t_2)), y(\beta(t_2)), z(\beta(t_2)) \right) \right. \\
& \quad \left. - f \left(t_2, x_0 + \int_0^{\xi(t_1)} x(s) ds, y_0 + \int_0^{\xi(t_1)} y(s) ds, z_0 \right. \right. \\
& \quad \left. \left. + \int_0^{\xi(t_1)} z(s) ds, x(\beta(t_2)), y(\beta(t_2)), z(\beta(t_2)) \right) \right| \\
& \quad + \left| f \left(t_2, x_0 + \int_0^{\xi(t_1)} x(s) ds, y_0 + \int_0^{\xi(t_1)} y(s) ds, z_0 \right. \right. \\
& \quad \left. \left. + \int_0^{\xi(t_1)} z(s) ds, x(\beta(t_2)), y(\beta(t_2)), z(\beta(t_2)) \right) \right. \\
& \quad \left. - f \left(t_2, x_0 + \int_0^{\xi(t_1)} x(s) ds, y_0 + \int_0^{\xi(t_1)} y(s) ds, z_0 \right. \right. \\
& \quad \left. \left. + \int_0^{\xi(t_1)} z(s) ds, x(\beta(t_2)), y(\beta(t_2)), z(\beta(t_2)) \right) \right| \\
& \leq \frac{1}{3} \varphi \left(|x(\beta(t_1)) - x(\beta(t_2))| + |y(\beta(t_1)) - y(\beta(t_2))| \right. \\
& \quad \left. + |z(\beta(t_1)) - z(\beta(t_2))| \right) + \omega(f, \varepsilon) \\
& \quad + \theta \left(\max \left\{ \int_{\xi(t_2)}^{\xi(t_1)} |x(s)| ds, \int_{\xi(t_2)}^{\xi(t_1)} |y(s)| ds, \right. \right. \\
& \quad \left. \left. \int_{\xi(t_2)}^{\xi(t_1)} |z(s)| ds \right\} \right) \\
& \leq \frac{1}{3} \varphi \left(\omega(x, \omega(\beta, \varepsilon)) + \omega(y, \omega(\beta, \varepsilon)) + \omega(z, \omega(\beta, \varepsilon)) \right) \\
& \quad + \omega(f, \varepsilon) + \theta \left(\max \{ r_0 \omega(\xi, \varepsilon) \} \right),
\end{aligned}$$

where

$$\begin{aligned}\omega(\beta, \varepsilon) &= \sup \left\{ |\beta(t_2) - \beta(t_1)| : |t_1 - t_2| \leq \varepsilon, t_1, t_2 \in [0, T] \right\} \\ \omega(\zeta, \varepsilon) &= \sup \left\{ |\zeta(t_2) - \zeta(t_1)| : t_1, t_2 \in [0, T], |t_1 - t_2| \leq \varepsilon \right\} \\ \omega(x, \omega(\beta, \varepsilon)) &= \sup \left\{ |x(t_2) - x(t_1)| : |t_1 - t_2| \leq \omega(\beta, \varepsilon), t_1, t_2 \in [0, \beta(T)] \right\}, \\ \omega(f, \varepsilon) &= \sup \left\{ |f(t_1, x_1, x_2, \dots, x_6) - f(t_2, x_1, x_2, \dots, x_6)| : |t_1 - t_2| \leq \varepsilon, \right. \\ &\quad \left. t_1, t_2 \in [0, T], x_4, x_5, x_6 \in [-r_0, r_0], x_1, x_2, x_3 \in [-\beta(T)r_0, \beta(T)r_0] \right\}.\end{aligned}$$

We obtained

$$\begin{aligned}\left| T(x, y, z)(t_1) - T(x, y, z)(t_2) \right| &\leq \frac{1}{3} \varphi \left(\omega(x, \omega(\beta, \varepsilon)) + \omega(y, \omega(\beta, \varepsilon)) + \omega(z, \omega(\beta, \varepsilon)) \right) \\ &\quad + \omega(f, \varepsilon) + \theta \left(\max \{ r_0 \omega(\zeta, \varepsilon) \} \right)\end{aligned}$$

Therefore, we have

$$\begin{aligned}\omega \left(T(X_1 \times X_2 \times X_3), \varepsilon \right) &\leq \frac{1}{3} \varphi \left(\omega(X_1, \omega(\beta, \varepsilon)) + \omega(X_2, \omega(\beta, \varepsilon)) + \omega(X_3, \omega(\beta, \varepsilon)) \right) \\ &\quad + \omega(f, \varepsilon) + \theta \left(\max \{ r_0 \omega(\zeta, \varepsilon) \} \right).\end{aligned}$$

On the other hand by the uniform continuity of f , β and ζ on $[0, T] \times [-\beta(T)r_0, \beta(T)r_0]^3 \times [-r_0, r_0]^3$, $[0, T]$ and $[0, T]$ respectively, we have $\omega(f, \varepsilon) \rightarrow 0$, $\omega(\beta, \varepsilon)$ and $\omega(\zeta, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Therefore we obtain

$$\mu \left(T(X_1 \times X_2 \times X_3) \right) \leq \frac{1}{3} \varphi \left(\mu(X_1) + \mu(X_2) + \mu(X_3) \right).$$

By applying Corollary 2, T has a fixed point. □

Example 2. Consider the following system of functional differential equations

$$\begin{cases} x'(t) = t^2 + \sqrt[3]{x(t)} + \sqrt[5]{y(t)} + \sqrt[7]{z(t)} + \frac{1}{3} \ln(1 + |x'(t) + y'(t) + z'(t)|) \\ y'(t) = t^2 + \sqrt[3]{y(t)} + \sqrt[5]{x(t)} + \sqrt[7]{z(t)} + \frac{1}{3} \ln(1 + |x'(t) + y'(t) + z'(t)|) \\ z'(t) = t^2 + \sqrt[3]{z(t)} + \sqrt[5]{y(t)} + \sqrt[7]{x(t)} + \frac{1}{3} \ln(1 + |x'(t) + y'(t) + z'(t)|) \end{cases} \quad (3.6)$$

where $t \in [0, 3]$ with the initial condition

$$x(0) = 3, \quad y(0) = 1, \quad z(0) = 2.$$

Eq. (3.6) is a special case of Eq. (1.1) where

$$\beta(t) = \zeta(t) = t,$$

$$f(t, x_1, \dots, x_6) = t^2 + \sqrt[3]{x_1} + \sqrt[5]{x_2} + \sqrt[7]{x_3} + \frac{1}{3} \ln(1 + |x_4 + x_5 + x_6|).$$

From the definitions of β and ζ , hypothesis (A₁) are obviously satisfied. Suppose that $t \in [0, 3]$. Now, by taking $\varphi(t) = \ln(1 + t)$, $\theta(t) = \max_{i=3,5,7} \{\sqrt[i]{t}\}$ we have

$$\begin{aligned} |f(t, x_1, \dots, x_6) - f(t, y_1, \dots, y_6)| &\leq |\sqrt[3]{x_1} - \sqrt[3]{y_1}| + |\sqrt[5]{x_2} - \sqrt[5]{y_2}| + |\sqrt[7]{x_3} - \sqrt[7]{y_3}| \\ &\quad + \frac{1}{3} |\ln(1 + |x_4 + x_5 + x_6|) - \ln(1 + |y_4 + y_5 + y_6|)| \\ &\leq |\sqrt[3]{x_1} - \sqrt[3]{y_1}| + |\sqrt[5]{x_2} - \sqrt[5]{y_2}| + |\sqrt[7]{x_3} - \sqrt[7]{y_3}| \\ &\quad + \frac{1}{3} \ln(1 + \frac{|x_4 + x_5 + x_6| - |y_4 + y_5 + y_6|}{1 + |y_4 + y_5 + y_6|}) \\ &\leq \sqrt[3]{|x_1 - y_1|} + \sqrt[5]{|x_2 - y_2|} + \sqrt[7]{|x_3 - y_3|} \\ &\quad + \frac{1}{3} \ln(1 + |x_4 + x_5 + x_6 - (y_4 + y_5 + y_6)|) \\ &\leq \sqrt[3]{|x_1 - y_1|} + \sqrt[5]{|x_2 - y_2|} + \sqrt[7]{|x_3 - y_3|} \\ &\quad + \frac{1}{3} \ln(1 + |x_4 - y_4| + |x_5 - y_5| + |x_6 - y_6|) \\ &= \frac{1}{3} \varphi(|x_4 - y_4| + |x_5 - y_5| + |x_6 - y_6|) \\ &\quad + \theta(\max_{i=1,2,3} \{|x_i - y_i|\}). \end{aligned} \tag{3.7}$$

Thus, from (3.7) we infer that condition (A₂) holds. Furthermore,

$$\begin{aligned} M &= \sup\{|f(t, x_0, y_0, z_0, 0, 0, 0)| : t \in [0, T]\} \\ &= \sup\{t^2 + \sqrt[3]{3} + \sqrt[5]{1} + \sqrt[7]{2} : t \in [0, 3]\} \\ &\leq 12. \end{aligned}$$

It is easy to see that each number $r \geq 20$ satisfies the inequality in condition (A₃), i.e.,

$$12 + \theta(3r) + \frac{1}{3} \varphi(3r) \leq r.$$

Thus, as the number r_0 we can take $r_0 = 20$. Consequently, all the conditions of Theorem 7 are satisfied. Hence the system equations 3.6 has at least one solution which belong to the space $\{C[0, T]\}^3$.

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COMPACTNESS CRITERIA IN WEIGHTED VARIABLE LEBESGUE SPACES

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Abstract. In this paper a compactness criterion in weighted variable Lebesgue spaces is proved. In particular, are proved a compactness criterion in variable exponent sequence Lebesgue spaces.

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Keywords: weighted variable Lebesgue spaces, variable exponent sequence Lebesgue spaces, compactness, convolution type operator, maximal function

1. INTRODUCTION

Compactness results in the usually Lebesgue spaces are often vital in existence proofs for nonlinear partial differential equations. A necessary and sufficient condition for a subset of usually Lebesgue spaces to be compact is given in what is often called the Kolmogorov compactness theorem, or Frechet-Kolmogorov theorem. Furthermore, we trace out the historical roots of Kolmogorov compactness theorem, which originated in [12] (see also [19]).

In this paper we extend Kolmogorov compactness criterion to the case of weighted variable Lebesgue spaces. The aim of this paper is to give a characterization of precompact sets in weighted variable Lebesgue space. Moreover, we study the precompactness of weighted sequence spaces. The theory of variable Lebesgue spaces was intensively developed during the last two decades, inspired both by difficult open problems in this theory and possibly applications shown in [16]. We refer to detail in [1–4, 13, 17, 18] and e.t.c.

Let us mention some generalizations of the Riesz-Kolmogorov theorem. Recently in [7] and [11] was shown the Riesz-Kolmogorov compactness theorem on metric spaces. In [8] full characterization of relatively compact sets is given in the case of variable Lebesgue spaces on metric measure spaces. In [20] the compactness theorem

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in $L^p(G)$ on locally compact group G is shown. In [14] Kolmogorov theorem for $p = 2$ in terms of the Fourier transform is given (see also [5, 6]).

2. PRELIMINARIES

An ε -cover of a metric space is a cover of the space consisting of sets of diameter at most ε . A metric space is called totally bounded if it admits a finite ε -cover for all ε . It is well known that a metric space is compact if and only if it is complete and totally bounded (see [21]). Since we are interested in compactness results for subsets of Banach spaces, we concentrate our attention on total boundedness.

Let R^n be the n -dimensional Euclidean space of points $x = (x_1, \dots, x_n)$. Let p be a Lebesgue measurable function on R^n such that $1 \leq \underline{p} \leq p(x) \leq \bar{p} \leq \infty$, $\underline{p} = \text{ess inf}_{x \in R^n} p(x)$, $\bar{p} = \text{ess sup}_{x \in R^n} p(x)$, and ω is a weight function on R^n , i.e. ω is non-negative, almost everywhere (a.e.) positive function on R^n . The Lebesgue measure of a set Ω will be denoted by $|\Omega|$.

Definition 1. By $L_{p(x), \omega}(R^n)$ we denote the set of all measurable functions f on R^n such that for some $\lambda_0 > 0$

$$I_{p, \omega}(f) = \int_{R^n} \left(\frac{|f(x)|}{\lambda_0} \right)^{p(x)} \omega(x) dx < \infty.$$

Note that the expression

$$\|f\|_{L_{p(\cdot), \omega}(R^n)} = \|f\|_{p(\cdot), \omega} = \inf \left\{ \lambda > 0 : \int_{R^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} \omega(x) dx \leq 1 \right\}$$

defines the norm in the space $L_{p(x), \omega}(R^n)$. The spaces $L_{p(x), \omega}(R^n)$ is a Banach function space with respect to the norm (see [3]).

By $P^{log}(R^n)$ we denote the class of variable exponents satisfying following condition:

$$|p(x) - p(y)| \leq \frac{C_1}{-\ln|x-y|}, \quad 0 < |x-y| \leq \frac{1}{2},$$

$$|p(x) - p_\infty| \leq \frac{C_2}{\ln(e+|x|)}, \quad |x| > 2,$$

where C_1 and C_2 are positive constants independents on x, y and $\lim_{|y| \rightarrow \infty} p(y) = p_\infty$.

Let us define the class $A_{p(\cdot)}$ consisting of those weights $\omega \in L_1^{loc}(R^n)$ such that

$$\sup_B |B|^{-1} \left\| \omega^{1/p(\cdot)} \right\|_{L_{p(\cdot)}} \left\| \omega^{-1/p(\cdot)} \right\|_{L_{p'(\cdot)}} < \infty,$$

where the supremum is taken over all balls $B \subset R^n$ and $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Here is the key lemma for many compactness results.

Lemma 1. [9] *Let X be a metric space. Assume that, for every $\varepsilon > 0$, there exists some $\delta > 0$, a metric space W , and a mapping $\Phi : X \mapsto W$ so that $\Phi[X]$ is totally bounded, and whenever $x, y \in X$ are such that $d(\Phi(x), \Phi(y)) < \delta$, then $d(x, y) < \varepsilon$. Then X is totally bounded.*

Let $f \in L_1^{loc}(R^n)$ and we define the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y)| dy$$

and

$$(f * g)(x) = \int_{R^n} f(y)g(x-y) dy$$

denoted the usual convolution operator.

Theorem 1. [10] *Let $p \in P^{log}(R^n)$ with $1 < \underline{p} \leq \bar{p} < \infty$ and ω is a weight function.*

Then $M : L_{p(x), \omega}(R^n) \mapsto L_{p(x), \omega}(R^n)$ is bounded if and only if $\omega \in A_{p(\cdot)}$. The embedding constant depends on p and ω .

Theorem 2. *Let $p \in P^{log}(R^n)$ and let $\psi : R^n \mapsto R$ be an integrable function. Assume that $\psi_\varepsilon(x) := \varepsilon^{-n} \psi\left(\frac{x}{\varepsilon}\right)$ for all $\varepsilon > 0$ and $\Psi(x) := \sup_{|y| \geq |x|} \psi(y) \in L_1(R^n)$.*

Then

a) *Then exists $A > 0$ such that $\sup_{\varepsilon > 0} |(f * \psi_\varepsilon)(x)| \leq 2A Mf(x)$, where A depends on Ψ , $f \in L_{p(x), \omega}(R^n)$ and M is Hardy-Littlewood maximal operator;*

b) *If $\int_{R^n} \psi(x) dx = 1$, then for $f \in L_{p(x), \omega}(R^n)$ we have $f * \psi_\varepsilon \rightarrow f$ for $\varepsilon \rightarrow 0$*

a. e. $x \in R^n$. If additionally $\bar{p} < \infty$, then $\lim_{\varepsilon \rightarrow 0+} \|f * \psi_\varepsilon - f\|_{p(\cdot), \omega} = 0$.

Proof of Theorem 2 analogously to the proof of non-weighted case (see [4]).

Thus, from Theorem 1 and Theorem 2 (a) we obtain the following

Theorem 3. *Let $p \in P^{log}(R^n)$ with $1 < \underline{p} \leq \bar{p} < \infty$ and $\omega \in A_{p(\cdot)}$. Then*

$$\|f * \psi_\varepsilon\|_{p(\cdot), \omega} \leq 2A \|f\|_{p(\cdot), \omega},$$

where A is independent of ε and f .

3. MAIN RESULTS.

Let $w = \{w_n\}$ be a sequence of positive numbers. Let

$$l_{p_n}(w) := \left\{ x : x = (x_1, \dots, x_n, \dots), \sum_{k=1}^{\infty} \left(\frac{|x_k|}{\lambda_0} \right)^{p_k} w_k < \infty \right\}$$

denote weighted variable sequence Lebesgue spaces with the norm

$$\|x\|_{l_{p_n}(w)} := \left\| x w^{\frac{1}{p_n}} \right\|_{l_{p_n}} = \inf \left\{ \lambda > 0 : \sum_{k=1}^{\infty} \left(\frac{|x_k|}{\lambda} \right)^{p_k} w_k \leq 1 \right\}.$$

Theorem 4. Let $1 \leq p_n \leq \sup_n p_n < \infty$, $w = \{w_n\}$ be a sequence of positive numbers and $S \subset l_{p_n}(w)$. A subset S is totally bounded if and only if:

- (i) it is pointwise bounded;
- (ii) for every $\varepsilon > 0$ there is some n such that, for every $x \in S$

$$\left\| x w^{\frac{1}{p_k}} \right\|_{l_{p_k}(k>n)} < \varepsilon.$$

Proof of Theorem 4. Let $S \subset l_{p_n}(w)$ satisfies the two condition. Given $\varepsilon > 0$, fix n as in the second condition, and define a mapping $\Phi : S \mapsto R^n$ by

$$\Phi(x) = (x_1, \dots, x_n).$$

By the pointwise boundedness of S , the image $\Phi(S)$ is totally bounded. Let $x, y \in S$ with

$$|\Phi(x) - \Phi(y)|_{l_{p_n}(w)} = \inf \left\{ \lambda > 0 : \sum_{k \leq n} \left(\frac{|x_k - y_k|}{\lambda} \right)^{p_k} w_k \leq 1 \right\} < \varepsilon.$$

We have

$$\begin{aligned} & \|x - y\|_{l_{p_n}(w)} \\ & \leq 2 \left(|\Phi(x) - \Phi(y)|_{l_{p_n}(w)} + \inf \left\{ \lambda > 0 : \sum_{k=n+1}^{\infty} \left(\frac{|x_k - y_k|}{\lambda} \right)^{p_k} w_k \leq 1 \right\} \right) \\ & < 2\varepsilon + 2\varepsilon = 4\varepsilon. \end{aligned}$$

By Lemma 1, S is totally bounded.

Note that conditions (i) and (ii) are also necessary. $\square$

Remark 1. Note that in the case $w = 1$ Theorem 4 was proved in [8]. Also, for $p(x) = p = \text{const}$, $\omega = 1$ Theorem 4 was proved in [9].

Theorem 5. Let $p \in P^{log}(R^n)$ and $1 \leq \underline{p} \leq p(x) \leq \bar{p} < \infty$. Suppose that ω is a weight function on R^n and $\omega \in A_{p(\cdot)}$. Then a subset S of $L_{p(x),\omega}(R^n)$ is totally bounded if and only if:

- 1) S is bounded in $L_{p(x),\omega}(R^n)$, i.e. $\sup_{f \in S} \|f\|_{p(\cdot),\omega} < \infty$;
- 2) for every $\varepsilon > 0$ there is some $\eta > 0$ such that, for every $f \in S$ $\|f\|_{L_{p(\cdot),\omega}(|x|>\eta)} < \varepsilon$;
- 3) $\lim_{\varepsilon \rightarrow 0+} \|f * \psi_\varepsilon - f\|_{p(\cdot),\omega} = 0$, uniformly for $f \in S$.

Proof of Theorem 5. Necessity. Assume that S is totally bounded. The existence of a finite ε -cover for S , for any ε , clearly implies the boundedness of S , thus establishing condition 1).

Now we prove the necessity of 2). To establish condition 2), let $\varepsilon > 0$ be given and let $\{U_1, \dots, U_m\}$ be an ε -cover of S and choose g_j for $j = 1, \dots, m$. Select $\eta > 0$ such that

$$\|g_j\|_{L_{p(\cdot),\omega}(|x|>\eta)} < \varepsilon, \quad j = 1, \dots, m.$$

If $f \in U_j$, then $\|f - g_j\|_{p(\cdot),\omega} \leq \varepsilon$, and so

$$\|f\|_{L_{p(\cdot),\omega}(|x|>\eta)} \leq \|f - g_j\|_{L_{p(\cdot),\omega}(|x|>\eta)} + \|g_j\|_{L_{p(\cdot),\omega}(|x|>\eta)} \leq 2\varepsilon,$$

thus establishing condition 2).

To prove 3) we first note that, by Theorem 2 b), given $\varepsilon > 0$, there exists an h_k indexed to each g_k such that

$$\|\psi_h * g_k - g_k\|_{p(\cdot),\omega} < \varepsilon$$

where $h < h_k$. Put $h_0 = \min_{1 \leq k \leq l} h_k$, we have $\|\psi_h * g_k - g_k\|_{p(\cdot),\omega} < \varepsilon$ for all $k = 1, \dots, l$ where $h < h_0$. Then for $h < h_0$ and all $f \in S$ by Theorem 3 we have a suitable g_r such that

$$\begin{aligned} \|\psi_h * f - f\|_{p(\cdot),\omega} &\leq \|\psi_h * (f - g_r)\|_{p(\cdot),\omega} + \|f - g_r\|_{p(\cdot),\omega} + \|\psi_h * g_r - g_r\|_{p(\cdot),\omega} \\ &\leq (2A + 1) \|f - g_r\|_{p(\cdot),\omega} + \varepsilon < (2A + 2)\varepsilon, \end{aligned}$$

which proves the necessity of 3).

Sufficiency part of Theorem 5 is proved analogously to the non-weighted case, i.e. when $\omega = 1$ (see [15]).

This completes the proof of Theorem 5. □

Remark 2. Note that in the case $\omega = 1$ and when $\Omega \subset R^n$ is a bounded open set Theorem 5 was proved in [15]. Also, for $p(x) = p = \text{const}$, $\omega = 1$ and other condition than 3) Theorem 5 was proved in [9].

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COMMON FIXED POINT THEOREMS ON SPACES WITH VECTOR-VALUED B -METRICS

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Abstract. The purpose of this paper is to prove common fixed point theorems for two operators without the assumption of continuity on a set endowed with vector-valued b -metric and we also study the case of two vector-valued b -metrics. The well-posedness of the fixed point problem is also discussed and an application is presented for systems of operator equations in complete b -normed vector space. Our results improve and generalize theorems 2 and 5 of M. Boriceanu [5].

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1. INTRODUCTION

The classical Banach contraction principle was extended for contraction mappings on spaces endowed with vector-valued metrics by Perov [9] in 1964. In [2] Bakhtin introduced the notion of b -metric spaces as a generalization of metric spaces and this notion was extensively used by Czerwik in [6]. Bakhtin proved the contraction mapping principle in b -metric spaces that generalized the famous Banach contraction principle in metric spaces. Since then, several papers have dealt with fixed point theory or the variational principle for single-valued operators in b -metric spaces see V. Berinde [3]. In the first section, we prove some results for two operators on a generalized b -metric space without the assumption of continuity that has been used in theorem 2 of M. Boriceanu [5] and the well-posedness of the fixed point problem is also discussed. As an application we present a theorem for systems of operator equations in complete b -normed vector space and in the second section we generalize theorem 5 given by M. Boriceanu [5] using two generalized b -metrics.

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2. BASIC DEFINITIONS AND AUXILIARY RESULTS

The aim of this section is to present some definitions and results that will be needed in the sequel.

Definition 1. [6] Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \rightarrow \mathbb{R}_+^n$ is said to be a vector-valued b -metric on X if and only if for all $x, y, z \in X$ the following conditions are satisfied:

- (1) $d(x, y) = 0$ if and only if $x = y$,
- (2) $d(x, y) = d(y, x)$,
- (3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

A pair (X, d) is called a generalized b -metric space.

Remark 1. If $\alpha, \beta \in \mathbb{R}^n$ with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ and $c \in \mathbb{R}$, then by $\alpha \leq \beta$ (respectively $\alpha < \beta$), we mean that $\alpha_i \leq \beta_i$ (respectively $\alpha_i < \beta_i$), for all $i = \overline{1, n}$ and by $\alpha \leq c$ we mean that $\alpha_i \leq c$, for all $i = \overline{1, n}$.

The advantage for using a b -metric is that it allows us to obtain concrete results, not only regarding the existence of the fixed point, but also regarding the convergence of sequences of successive approximations, data dependence of the fixed point set and the study of well-posedness of the fixed point problem.

Notice that in a generalized b -metric space (X, d) the notions of convergent sequence, Cauchy sequence, completeness, compactness, open and closed subset are similar to those for usual metric spaces, but in general a generalized b -metric function d is not continuous, for examples, see [8].

Throughout this paper we denote by $\mathcal{M}_{n \times n}(\mathbb{R}_+)$ the set of all $n \times n$ matrices with positive elements, by Θ the zero $n \times n$ matrix and by I the identity $n \times n$ matrix. Notice also that, for the sake of simplicity, we will make an identification between row and column vectors in $\mathbb{R}^n$.

Definition 2. [14] A matrix $C \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ is said to be convergent to zero if and only if

$$C^n \rightarrow \Theta, \quad \text{as } n \rightarrow \infty.$$

For other examples and considerations on matrices which converge to zero, see Turinici [13].

Notice that, for the proof of the main results, we need the following theorem, part of which being a classical result in matrix analysis, see [10, 11].

Theorem 1. [10] Let $C \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$. The following statements are equivalent:

- i) C is a matrix convergent to zero,
- ii) the matrix $(I - C)$ is nonsingular and

$$(I - C)^{-1} = I + C + C^2 + \dots + C^n + \dots,$$

- iii) the matrix $(I - C)$ is nonsingular and $(I - C)^{-1}$ has nonnegative elements.

Lemma 1. [6] Let (X, d) be a generalized b -metric space and let $\{x_k\}_{k=0}^n \subset X$. Then:

$$d(x_n, x_0) \leq sd(x_0, x_1) + \cdots + s^{n-1}d(x_{n-2}, x_{n-1}) + s^{n-1}d(x_{n-1}, x_n).$$

The notion of well-posedness was treated in many articles, for example the article of M. Akkouchi, V. Popa [1] and M. Boriceanu [4].

Definition 3. Let (X, d) be a generalized b -metric space and $f, g : (X, d) \rightarrow (X, d)$ be two mappings. The fixed point problem of f and g is said to be well posed if:

- i) f and g have a unique common fixed point z in X ,
- ii) for any sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} d(f(x_n), x_n) = 0$
or $\lim_{n \rightarrow \infty} d(g(x_n), x_n) = 0$, we have $\lim_{n \rightarrow \infty} d(x_n, z) = 0$.

Now, we give this lemma which is a generalization of lemma 3.1 of [12].

Lemma 2. Let (X, d) be a generalized b -metric space and $\{x_n\}$ a sequence in X such that

$$d(x_n, x_{n+1}) \leq Cd(x_{n-1}, x_n), \quad n = 1, 2, \dots,$$

where $C \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$. Then the sequence $\{x_n\}$ is Cauchy sequence in X provided that sC is convergent towards zero.

Proof. For any n , we have:

$$\begin{aligned} d(x_n, x_{n+1}) &\leq Cd(x_{n-1}, x_n) \\ &\leq C^2d(x_{n-2}, x_{n-1}) \leq \dots \leq C^nd(x_0, x_1). \end{aligned}$$

To prove that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, we estimate $d(x_n, x_{n+p})$ using the triangle inequality:

$$\begin{aligned} d(x_n, x_{n+p}) &\leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + \cdots + s^{p-2}d(x_{n+p-3}, x_{n+p-2}) \\ &\quad + s^{p-1}d(x_{n+p-2}, x_{n+p-1}) + s^{p-1}d(x_{n+p-1}, x_{n+p}) \\ &\leq sC^nd(x_0, x_1) + s^2C^{n+1}d(x_0, x_1) + \cdots + s^{p-2}C^{n+p-3}d(x_0, x_1) \\ &\quad + s^{p-1}C^{n+p-2}d(x_0, x_1) + s^{p-1}C^{n+p-1}d(x_0, x_1) \\ &= sC^nd(x_0, x_1)[I + sC + \cdots + s^{p-2}C^{p-2} + s^{p-2}C^{p-1}] \\ &\leq sC^nd(x_0, x_1)[I + sC + \cdots + s^{p-2}C^{p-2} + s^{p-1}C^{p-1}] \\ &\leq sC^nd(x_0, x_1)(I - sC)^{-1} \\ &\leq (sC)^nd(x_0, x_1)(I - sC)^{-1}. \end{aligned}$$

Note that $(I - sC)$ is nonsingular since sC is convergent to zero. This implies that the sequence $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. $\square$

In the case of b -metric space, the previous lemma will be as follows:

Lemma 3. [12] Let (X, d) be a b -metric space, $\{x_n\}$ a sequence in X such that:

$$d(x_n, x_{n+1}) \leq C d(x_{n-1}, x_n),$$

where $0 \leq C < 1$. Then the sequence $\{x_n\}$ is a Cauchy sequence in X provided that $sC < 1$.

For the case of generalized metric space or metric space it is enough to replace s by 1 in lemma 2 and lemma 3 respectively.

In the next section, we generalize and improve the following theorem of M. Boriceanu [5]:

Theorem 2. Let (X, d) be a complete generalized b -metric space. Assume that the operator $f : X \rightarrow X$ satisfies the following conditions:

- (a) f is continuous;
- (b) there exists matrices $M, N, P \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ with:
 - (i) $(I - N - Ps)$ is nonsingular and $(I - N - Ps)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$;
 - (ii) sC is convergent towards zero, where $C = (I - N - Ps)^{-1}(M + N + Ps)$;
 - (iii) $d(f(x), f(y)) \leq M d(x, y) + N[d(x, f(x)) + d(y, f(y))] + P[d(x, f(y)) + d(y, f(x))]$, for all $x, y \in X$.

Then:

- (1) f has a fixed point x^* in X .
- (2) If, in addition, $(I - M - 2P)$ is nonsingular and $(I - M - 2P)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$, then x^* is unique.

3. COMMON FIXED POINT THEOREMS FOR TWO OPERATORS ON GENERALIZED b -METRIC SPACE WITHOUT THE ASSUMPTION OF CONTINUITY

The first main result of this paper is the following:

Theorem 3. Let (X, d) be a complete generalized b -metric space. Assume that the operators $f, g : X \rightarrow X$ satisfy the following conditions:

there exists matrices $M, N, P \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ with:

- (i) $(I - N - Ps)$ is nonsingular and $(I - N - Ps)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ and $(I - sN - s^2P)$ is nonsingular and $(I - sN - s^2P)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$;
- (ii) sC is convergent towards zero, where $C = (I - N - Ps)^{-1}(M + N + Ps)$;
- (iii) $d(f(x), g(y)) \leq M d(x, y) + N[d(x, f(x)) + d(y, g(y))] + P[d(x, g(y)) + d(y, f(x))]$, for all $x, y \in X$.

Then:

- (1) f and g have a common fixed point z in X .
- (2) If, in addition, $(I - M - 2P)$ is nonsingular and $(I - M - 2P)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$, then z is unique.

(3) If $(I - s(M + P) - s^2 P)$ is nonsingular and $(I - s(M + P) - s^2 P)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$, then the fixed point problem (of f and g) is well-posed.

Proof. 1. Let x_0 be some point in X , we consider $(x_n)_{n \in \mathbb{N}}$ the sequence of successive approximations for f and g , defined by:

$$\begin{aligned} x_{2n+1} &= f(x_{2n}), & n &= 0, 1, \dots \\ x_{2n+2} &= g(x_{2n+1}), & n &= 0, 1, \dots \end{aligned}$$

We have:

$$\begin{aligned} & d(x_{2n}, x_{2n+1}) \\ &= d(g(x_{2n-1}), f(x_{2n})) \\ &\leq Md(x_{2n-1}, x_{2n}) + N[d(x_{2n}, f(x_{2n})) + d(x_{2n-1}, g(x_{2n-1}))] \\ &+ P[d(x_{2n}, g(x_{2n-1})) + d(x_{2n-1}, f(x_{2n}))] \\ &= Md(x_{2n-1}, x_{2n}) + N[d(x_{2n}, x_{2n+1}) + d(x_{2n-1}, x_{2n})] + Pd(x_{2n-1}, x_{2n+1}) \\ &\leq Md(x_{2n-1}, x_{2n}) + N[d(x_{2n}, x_{2n+1}) + d(x_{2n-1}, x_{2n})] \\ &+ Ps[d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})] \end{aligned}$$

this implies that:

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &\leq (I - N - Ps)^{-1}(M + N + Ps)d(x_{2n-1}, x_{2n}) \\ &= Cd(x_{2n-1}, x_{2n}). \end{aligned}$$

Similarly, we have:

$$\begin{aligned} & d(x_{2n+1}, x_{2n+2}) \\ &= d(f(x_{2n}), g(x_{2n+1})) \\ &\leq Md(x_{2n}, x_{2n+1}) + N[d(x_{2n}, f(x_{2n})) + d(x_{2n+1}, g(x_{2n+1}))] \\ &+ P[d(x_{2n}, g(x_{2n+1})) + d(x_{2n+1}, f(x_{2n}))] \\ &= Md(x_{2n}, x_{2n+1}) + N[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + Pd(x_{2n}, x_{2n+2}) \\ &\leq Md(x_{2n}, x_{2n+1}) + N[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \\ &+ Ps[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]. \end{aligned}$$

Thus

$$\begin{aligned} & d(x_{2n+1}, x_{2n+2}) \\ &\leq (I - N - Ps)^{-1}(M + N + Ps)d(x_{2n}, x_{2n+1}) = Cd(x_{2n}, x_{2n+1}). \end{aligned}$$

We obtain that:

$$d(x_n, x_{n+1}) \leq C^n d(x_0, x_1), \quad \text{for each } n \in \mathbb{N}.$$

Following the same steps of lemma 2, we get that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. Using the fact that (X, d) is complete we get that $(x_n)_{n \in \mathbb{N}}$ is convergent in X . Thus, there exists $z \in X$ such that $d(x_n, z) \rightarrow 0$, as $n \rightarrow \infty$.

Now, we show that z is a fixed point for f by esteeming $d(f(z), z)$ we obtain:

$$\begin{aligned} d(f(z), z) &\leq sd(f(z), g(x_{2n+1})) + sd(x_{2n+2}, z) \\ d(f(z), z) - sd(x_{2n+2}, z) &\leq sd(f(z), g(x_{2n+1})) \end{aligned}$$

and

$$\begin{aligned} d(f(z), g(x_{2n+1})) &\leq Md(z, x_{2n+1}) + N[d(z, f(z)) + d(x_{2n+1}, g(x_{2n+1}))] \\ &\quad + P[d(z, g(x_{2n+1})) + d(x_{2n+1}, f(z))] \\ &= Md(z, x_{2n+1}) + N[d(z, f(z)) + d(x_{2n+1}, x_{2n+2})] \\ &\quad + P[d(z, x_{2n+2}) + d(x_{2n+1}, f(z))]. \end{aligned}$$

We obtain:

$$\begin{aligned} d(f(z), z) - sd(x_{2n+2}, z) &\leq sMd(z, x_{2n+1}) + sN[d(z, f(z)) + d(x_{2n+1}, x_{2n+2})] \\ &\quad + sP[d(z, x_{2n+2}) + s(d(x_{2n+1}, z) + d(z, f(z)))] \end{aligned}$$

Passing to the limit and taking into account that $(I - sN - s^2P)$ is nonsingular and $(I - sN - s^2P)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ we get that z is a fixed point for f .

Let us show that $g(z) = z$ using the condition (iii), we have:

$$\begin{aligned} d(z, g(z)) &= d(f(z), g(z)) \\ &\leq Md(z, z) + N[d(z, f(z)) + d(z, g(z))] + P[d(z, g(z)) + d(z, f(z))]. \end{aligned}$$

We obtain:

$$(I - N - P)d(z, g(z)) \leq 0.$$

Note that $(I - N - P)$ is nonsingular and $(I - N - P)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ we conclude that $z = g(z)$ so we obtain that z is a common fixed point for f and g .

2. Now, we show that f and g have a unique common fixed point. For this, we assume that there exists another point w fixed by f . Using the condition (iii), we have:

$$\begin{aligned} d(w, z) &= d(f(w), g(z)) \\ &\leq Md(w, z) + N[d(w, f(w)) + d(z, g(z))] + P[d(w, g(z)) + d(z, f(w))] \\ &\leq (M + 2P)d(w, z). \end{aligned}$$

We obtain:

$$(I - M - 2P)d(w, z) \leq 0.$$

Taking into account that $(I - M - 2P)$ is nonsingular and $(I - M - 2P)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ this implies that z is a unique common fixed point for f and g .

3. From the previous results, we know that f and g have a unique common fixed point z in X . Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of points in X such that $\lim_{n \rightarrow \infty} d(f(x_n), x_n) = 0$ or $\lim_{n \rightarrow \infty} d(g(x_n), x_n) = 0$, we have:

$$d(x_n, z) \leq s(d(x_n, g(x_n)) + d(g(x_n), f(z)))$$

and

$$\begin{aligned} & d(g(x_n), f(z)) \\ & \leq Md(x_n, z) + N[d(x_n, g(x_n)) + d(z, f(z))] + P[d(x_n, f(z)) + d(z, g(x_n))] \\ & \leq Md(x_n, z) + Nd(x_n, g(x_n)) + Pd(x_n, z) + Ps[d(z, x_n) + d(x_n, g(x_n))] \end{aligned}$$

we obtain:

$$\begin{aligned} d(x_n, z) & \leq sd(x_n, g(x_n)) + s[Md(x_n, z) + Nd(x_n, g(x_n)) + Pd(x_n, z) \\ & \quad + Ps(d(z, x_n) + d(x_n, g(x_n)))]. \end{aligned}$$

This implies that:

$$(I - s(M + P) - s^2P)d(z, x_n) \leq (sI + sN + s^2P)d(x_n, g(x_n)).$$

Passing to the limit and taking into account that $(I - s(M + P) - s^2P)$ is nonsingular and $(I - s(M + P) - s^2P)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$, we get that $d(z, x_n) \rightarrow 0$, as $n \rightarrow \infty$. $\square$

We get the concept of generalized metric space in the previous theorem if $s = 1$, in this case we have:

Corollary 1. *Let (X, d) be a complete generalized metric space. Assume that the operators $f, g : X \rightarrow X$ satisfy the following conditions:*

there exists matrices $M, N, P \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ with:

- (i) *$(I - N - P)$ is nonsingular and $(I - N - P)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$;*
- (ii) *C is convergent towards zero, where $C = (I - N - P)^{-1}(M + N + P)$;*
- (iii) *$d(f(x), g(y)) \leq Md(x, y) + N[d(x, f(x)) + d(y, g(y))] + P[d(x, g(y)) + d(y, f(x))]$, for all $x, y \in X$.*

Then:

- (1) *f and g have a common fixed point z in X .*
- (2) *If, in addition, $(I - M - 2P)$ is nonsingular and $(I - M - 2P)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$, then*
 - *z is unique.*
 - *The fixed point problem (of f and g) is well-posed.*

Proof. As in the lemma 2, we have:

$$d(x_n, x_{n+p}) \leq C^n d(x_0, x_1)(I - C)^{-1}.$$

Note that $(I - C)$ is nonsingular since C is convergent to zero. This implies that the sequence $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. Using the fact that (X, d) is complete we

get that $(x_n)_{n \in \mathbb{N}}$ is convergent in X . Thus, there exists $z \in X$ such that $d(x_n, z) \rightarrow 0$, as $n \rightarrow \infty$.

Now, we show that z is a fixed point for f for all positive integers n , using the condition (iii) we have:

$$\begin{aligned} d(f(z), x_{2n+2}) &= d(f(z), g(x_{2n+1})) \\ &\leq Md(z, x_{2n+1}) + N[d(z, f(z)) + d(x_{2n+1}, g(x_{2n+1}))] \\ &\quad + P[d(z, g(x_{2n+1})) + d(x_{2n+1}, f(z))] \\ &= Md(z, x_{2n+1}) + N[d(z, f(z)) + d(x_{2n+1}, x_{2n+2})] \\ &\quad + P[d(z, x_{2n+2}) + d(x_{2n+1}, f(z))]. \end{aligned}$$

Passing to the limit (the generalized metric function d is continuous) and taking into account that $(I - N - P)$ is nonsingular and $(I - N - P)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ this implies that $z = f(z)$, so z is a fixed point for f .

The rest of the proof follows as in theorem 3. $\square$

If $n = 1$ in the previous theorem then we get the concept of b -metric introduced by Bakhtin, in this case we have:

Corollary 2. *Let (X, d) be a complete b -metric space. Assume that the operators $f, g : X \rightarrow X$ satisfy the following conditions:*

there exists constants $M, N, P \in \mathbb{R}_+$ with:

- (i) $d(f(x), g(y)) \leq Md(x, y) + N[d(x, f(x)) + d(y, g(y))] + P[d(x, g(y)) + d(y, f(x))]$, for all $x, y \in X$;
- (ii) $sC < 1$ with $C = \frac{M + N + Ps}{1 - N - Ps}$, $sN + s^2P < 1$ and $N + Ps < 1$.

Then:

- (1) f and g have a common fixed point z in X .
- (2) If, in addition, $M + 2P < 1$ then z is unique.
- (3) If $(s(M + P) + s^2P) < 1$, then the fixed point problem (of f and g) is well-posed.

Proof. Applying lemma 3 (we can also see [7, 12] and following the same steps as in theorem 3, we get our results. $\square$

Next, We give the definition of a b -normed vector space.

Definition 4. Let X be a vector space. A b -norm on X is a function:

$$\|\cdot\| : X \rightarrow \mathbb{R}_+$$

that satisfies the following three conditions:

- (1) $\|x\| = 0 \Leftrightarrow x = 0$, $\forall x \in X$;
- (2) $\|\alpha x\| = |\alpha|^s \|x\|$, $\forall x \in X$, $\forall \alpha \in \mathbb{R}$ and $s \geq 1$;
- (3) $\|x + y\| \leq s(\|x\| + \|y\|)$, $\forall x, y \in X$, $s \geq 1$.

A vector space equipped with a b -norm is called a b -normed vector space.

Remark 2. Let us notice here that some advantages of a vector-valued norm with respect to the usual scalar norms were very nice pointed out, by several examples, see [10]. More precisely, one can show that, in general, the condition that C is a matrix convergent to zero is weaker than the contraction conditions for operators given in terms of the scalar norms on X of the following type:

$$\begin{aligned}\|x\|_M &= \|x_1\| + \|x_2\| \\ \|x\|_C &= \max\{\|x_1\|, \|x_2\|\} \\ \|x\|_E &= (\|x_1\|^2 + \|x_2\|^2)^{1/2}\end{aligned}$$

As an application of theorem 3, we present this theorem for a system of operator equations.

Theorem 4. *Let $(X, |\cdot|)$ be a b -normed complete vector space and let $f, g : X \times X \rightarrow X$ be two operators. Suppose that there exist $m_{ij}, n_{ij}, p_{ij} \in \mathbb{R}_+, i, j \in \{1, 2\}$ such that,*

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, N = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}, P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

for each $x = (x_1, x_2), y = (y_1, y_2) \in X \times X$, one has:

$$\begin{aligned}(1) \quad & |f_1(x_1, x_2) - g_1(y_1, y_2)| \leq m_{11}|x_1 - y_1| + m_{12}|x_2 - y_2| \\ & + n_{11}(|x_1 - f_1(x_1, x_2)| + |y_1 - g_1(y_1, y_2)|) \\ & + n_{12}(|x_2 - f_2(x_1, x_2)| + |y_2 - g_2(y_1, y_2)|) \\ & + p_{11}(|x_1 - g_1(y_1, y_2)| + |y_1 - f_1(x_1, x_2)|) \\ & + p_{12}(|x_2 - g_2(y_1, y_2)| + |y_2 - f_2(x_1, x_2)|) \\ (2) \quad & |f_2(x_1, x_2) - g_2(y_1, y_2)| \leq m_{21}|x_1 - y_1| + m_{22}|x_2 - y_2| \\ & + n_{21}(|x_1 - f_1(x_1, x_2)| + |y_1 - g_1(y_1, y_2)|) \\ & + n_{22}(|x_2 - f_2(x_1, x_2)| + |y_2 - g_2(y_1, y_2)|) \\ & + p_{21}(|x_1 - g_1(y_1, y_2)| + |y_1 - f_1(x_1, x_2)|) \\ & + p_{22}(|x_2 - g_2(y_1, y_2)| + |y_2 - f_2(x_1, x_2)|).\end{aligned}$$

In addition, we assume that the matrix $(I - N - Ps)$ is nonsingular and $(I - N - Ps)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$, $(I - sN - s^2P)$ is nonsingular and $(I - sN - s^2P)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ and sC is convergent to zero.

Then, the system

$$\begin{aligned}u_1 &= f_1(u_1, u_2) = g_1(u_1, u_2), \\ u_2 &= f_2(u_1, u_2) = g_2(u_1, u_2),\end{aligned}$$

has at least one solution $z \in X \times X$. Moreover, if, in addition, the matrix $(I - M - 2P)$ is nonsingular and $(I - M - 2P)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$, then the above solution is unique.

Proof. Consider $E = X \times X$ and the operators f, g given by the expression:

$$\begin{aligned} f(x_1, x_2) &= (f_1(x_1, x_2), f_2(x_1, x_2)) \\ g(y_1, y_2) &= (g_1(y_1, y_2), g_2(y_1, y_2)). \end{aligned}$$

Then our system is now represented as a fixed point equation of the following form: $w = f(w) = g(w), w \in E$. Notice also that the conditions (1)+(2) can be jointly represented as follows:

$$\begin{aligned} \|f(x) - g(y)\| &\leq M\|x - y\| + N(\|x - f(x)\| + \|y - g(y)\|) \\ &\quad + P(\|x - g(y)\| + \|y - f(x)\|) \text{ for all } x, y \in X \times X. \end{aligned}$$

Hence, theorem 3 applies in (E, d) with $d(u, v) = \|u - v\| = \begin{pmatrix} |u_1 - v_1| \\ |u_2 - v_2| \end{pmatrix}$. $\square$

4. FIXED POINT THEOREMS FOR TWO OPERATORS ON GENERALIZED b -METRIC SPACE WITH TWO b -METRICS

The next result improves this theorem of M. Boriceanu[5]:

Theorem 5. *Let (X, δ) be a complete generalized b -metric space and d another vector-valued b -metric on X . Assume that the operator $f : X \rightarrow X$ satisfies the following conditions:*

- (a) f is (d, δ) -uniformly continuous or there exists a matrix $U \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ such that $\delta(x, y) \leq U \cdot d(x, y)$, for all $x, y \in X$;
- (b) f is (δ, δ) -continuous;
- (c) there exists matrices $M, N, P \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ with:
 - (i) $(I - N - Ps)$ is nonsingular and $(I - N - Ps)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$;
 - (ii) sC is convergent towards zero, where $C = (I - N - Ps)^{-1}(M + N + Ps)$;
 - (iii) $d(f(x), f(y)) \leq Md(x, y) + N[d(x, f(x)) + d(y, f(y))] + P[d(x, f(y)) + d(y, f(x))]$, for all $x, y \in X$.

Then:

- (1) For any $x_0 \in X$ we have $\delta(f^k(x_0), x^*) \rightarrow 0$, as $k \rightarrow \infty$, where x^* is a fixed point for f .
- (2) If, in addition, $(I - M - 2P)$ is nonsingular and $(I - M - 2P)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$, then x^* is unique.

In this section, we present another result in the case of a generalized b -metric space but endowed with two b -metrics.

Theorem 6. Let (X, δ) be a complete generalized b -metric space and d another vector-valued b -metric on X . Assume that the operators $f, g : X \rightarrow X$ satisfy the following conditions:

- (a) There exists a matrix $U \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ such that $\delta(x, y) \leq U \cdot d(x, y)$, for all $x, y \in X$;
- (b) f is (δ, δ) -continuous;
- (c) There exists matrices $M, N, P \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ with:
 - (i) $(I - N - Ps)$ is nonsingular and $(I - N - Ps)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$;
 - (ii) sC is convergent towards zero, where $C = (I - N - Ps)^{-1}(M + N + Ps)$;
 - (iii) $d(f(x), g(y)) \leq Md(x, y) + N[d(x, f(x)) + d(y, g(y))] + P[d(x, g(y)) + d(y, f(x))]$, for all $x, y \in X$.

Then:

- (1) f and g have a common fixed point z in X .
- (2) If, in addition, $(I - M - 2P)$ is nonsingular and $(I - M - 2P)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$, then z is a unique common fixed point of f and g .
- (3) If $(I - s(M + P) - s^2P)$ is nonsingular and $(I - s(M + P) - s^2P)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$, then the fixed point problem (of f and g) is well-posed.

Proof. As in the proof of theorem 3, we obtain that $(x_n)_{n \in \mathbb{N}}$ is d -Cauchy. It follows from (a) that $(x_n)_{n \in \mathbb{N}}$ is δ -Cauchy sequence. Since (X, δ) is a complete generalized b -metric space, there exists $z \in X$ such that $\delta(x_{2n+1}, z) \rightarrow 0$, as $n \rightarrow \infty$.

By (b), we have that $\delta(f(x_{2n}), f(z)) \rightarrow 0$, as $n \rightarrow \infty$. But $\delta(f(x_{2n}), f(z)) = \delta(x_{2n+1}, f(z))$. Hence we have $z = f(z)$. Thus z is a fixed point for f and

$$\delta(f^n(x_0), z) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

The rest of the proof follows as in theorem 3. □

Now, we present the previous theorem in the case of generalized metric space.

Corollary 3. Let (X, δ) be a complete generalized metric space and d another vector-valued metric on X . Assume that the operators $f, g : X \rightarrow X$ satisfy the following conditions:

- (a) There exists a matrix $U \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ such that $\delta(x, y) \leq U \cdot d(x, y)$, for all $x, y \in X$;
- (b) f is (δ, δ) -continuous;
- (c) there exists matrices $M, N, P \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ with:
 - (i) $(I - N - P)$ is nonsingular and $(I - N - P)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$;
 - (ii) C is convergent toward zero, where $C = (I - N - P)^{-1}(M + N + P)$;
 - (iii) $d(f(x), g(y)) \leq Md(x, y) + N[d(x, f(x)) + d(y, g(y))] + P[d(x, g(y)) + d(y, f(x))]$, for all $x, y \in X$.

Then :

- (1) z is a common fixed point for f and g .
- (2) If, in addition, $(I - M - 2P)$ is nonsingular and $(I - M - 2P)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$, then
 - z is a unique common fixed point of f and g .
 - The fixed point problem (of f and g) is well-posed.

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DIFFEOMORPHISM OF AFFINE CONNECTED SPACES WHICH PRESERVED RIEMANNIAN AND RICCI CURVATURE TENSORS

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Abstract. In this paper we study the preserving of Riemannian and Ricci tensors with respect to a diffeomorphism of spaces with affine connection. We consider geodesic and almost geodesic mappings of the first type. The basic equations of these maps form a closed system of Cauchy type in covariant derivatives. We determine the quantity of essential (substantial) parameters on which the general solution of this problem depends.

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1. INTRODUCTION

The theory of geodesic mappings was developed in the works of T. Thomas, H. Weyl, L.P. Eisenhart, P.A. Shirokov, K. Yano, A.Z. Petrov, A.S. Solodovnikov, N.S. Sinyukov, A.V. Aminova, J. Mikeš, I.G. Shandra, S.E. Stepanov and others, see [10–12, 14–16, 18–20, 22, 24].

The problems raised in the study of geodesic maps were developed by V.F. Kagan, V. Vranceanu, P.K. Rashevsky, L.Ya. Shapiro, V.D. Vedenyapin, A.Z. Petrov and others. In particular, the concept of quasi-geodesic mapping was introduced by A.Z. Petrov. Close to it there are holomorphically projective mappings of Kähler spaces, considered originally by T. Otsuki and Y. Tashiro. See [15, 17, 19, 20, 23].

It natural generalization of these classes there are almost geodesic mappings introduced by N.S. Sinyukov [20]. Recently investigated almost geodesic mappings V.S. Shadny, V.S. Sobchuk, N.V. Yablonska, J. Mikeš, V.E. Berezovski, M.S. Stankovic, L.S. Velimirovic, M.L. Zlatanovic [1–9, 19, 21].

2. BASIC CONCEPTS OF THE THEORY OF DIFFEOMORPHISM OF AFFINE CONNECTED SPACES AND THEIR TYPES

We consider an n -dimensional torsion-free affine connected space in the coordinate system (x) , and assume that $n \geq 2$, further on all functions are sufficiently smooth.

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We can find the basic concepts of the theory of geodesic and of almost geodesic mappings in [15, 19, 20].

Suppose, that an affine connected space A_n admits a diffeomorphism f onto an affine connected space $\bar{A}_n$ in a common coordinate system $x = (x^1, \dots, x^n)$.

Let us

$$P_{ij}^h(x) = \bar{\Gamma}_{ij}^h(x) - \Gamma_{ij}^h(x) \quad (2.1)$$

given, where $\bar{\Gamma}_{ij}^h(x)$ and $\Gamma_{ij}^h(x)$ are components of objects of connections of affine connected spaces A_n and $\bar{A}_n$ respectively in common coordinate system (x) , P_{ij}^h is a the *deformation tensor*.

Definition 1. A curve, defined in an n -dimensional affine connected space A_n , is called *geodesic*, if its tangent vector is parallel along it.

Definition 2. A curve, defined in an affine connected space A_n ($n > 2$), is called *almost geodesic*, if exists along it a two-dimensional parallel plane containing its tangent vector.

Definition 3. Diffeomorphism f between two manifolds A_n and $\bar{A}_n$ with affine connections is called *geodesic mapping* if any geodesic curve of A_n is mapped onto a geodesic curve of $\bar{A}_n$ and vice versa.

Diffeomorphism f is geodesic mapping if and only if the deformation tensor P_{ij}^h in a common coordinate system has the following forms

$$P_{ij}^h(x) = \psi_i(x)\delta_j^h + \psi_j(x)\delta_i^h, \quad (2.2)$$

where δ_i^h is Kronecker symbol and ψ_i is a covariant vector field.

If $\psi_i \equiv 0$, then f is an *affine mapping*.

Definition 4. Diffeomorphism $f: A_n \rightarrow \bar{A}_n$ ($n > 2$) is called *almost geodesic mapping*, if any geodesic curve of A_n is mapped onto an almost geodesic curve of $\bar{A}_n$.

Diffeomorphism $f: A_n \rightarrow \bar{A}_n$ is almost geodesic if and only if the deformation tensor P_{ij}^h in a common coordinate system has the following forms

$$A_{\alpha\beta\gamma}^h \lambda^\alpha \lambda^\beta \lambda^\gamma = a P_{\alpha\beta}^h \lambda^\alpha \lambda^\beta + b \lambda^h, \quad (2.3)$$

where

$$A_{ijk}^h = P_{ij,k}^h + P_{ij}^\alpha P_{\alpha k}^h, \quad (2.4)$$

$\bar{\Gamma}_{ij}^h(x)$ and $\Gamma_{ij}^h(x)$ are affine connections of spaces A_n and $\bar{A}_n$, λ^h is an arbitrary vector, a and b are some functions dependent on x^h and λ^h .

Here and after the symbol “,” means covariant derivation in A_n .

N.S. Sinyukov [20] defined three types of almost geodesic mappings π_1, π_2, π_3 from the basic equations. It is well known [3, 6, 8], if $n > 5$ there exist only these three types.

Almost geodesic mapping of type π_1 is characterized by the conditions

$$A_{(ijk)}^h = a_{(ij}\delta_{k)}^h + b_{(i}P_{jk)}^h, \quad (2.5)$$

where a_{ij} is some symmetric tensor, b_i is some covector, the bracket (ijk) denotes the symmetrization of the indices i, j, k (without division).

If in (2.5) the covector b_i is equal to zero, then the mapping is called *canonical*. It is well known, that arbitrary almost geodesic mapping of type π_1 can be represented in the form of a composition of a canonical almost geodesic mapping of type π_1 and a geodesic mapping [20].

3. CONDITION OF REPRESENTATION OF RIEMANNIAN AND RICCI TENSORS WITH RESPECT TO THE DIFFEOMORPHISM OF THE AFFINE CONNECTED SPACE

From equations (2.1) we obtain a relation between Riemannian tensors R_{ijk}^h and $\bar{R}_{ijk}^h$ of spaces A_n and $\bar{A}_n$ respectively:

$$\bar{R}_{ijk}^h = R_{ijk}^h + P_{i[k,j]}^h + P_{\alpha[j}^h P_{k]i}^\alpha, \quad (3.1)$$

where $[jk]$ denotes alternation of the indices j, k (without division).

Then the relation (3.1) can be written as

$$\bar{R}_{ijk}^h = R_{ijk}^h - A_{i[jk]}^h. \quad (3.2)$$

Therefore, we have

Theorem 1. *The Riemannian tensor is preserved under diffeomorphism $f: A_n \rightarrow \bar{A}_n$ if and only if*

$$A_{ijk}^h = A_{ikj}^h. \quad (3.3)$$

Contracting by indices h and k we obtain

$$\bar{R}_{ij} = R_{ij} - A_{i[jk]}^h. \quad (3.4)$$

where R_{ij} and $\bar{R}_{ij}$ are Ricci tensors of spaces A_n and $\bar{A}_n$ respectively.

So we get

Theorem 2. *The Ricci tensor is preserved under diffeomorphism $f: A_n \rightarrow \bar{A}_n$ if and only if tensor A_{ijk}^h satisfies the following conditions*

$$A_{ij\alpha}^\alpha = A_{i\alpha j}^\alpha. \quad (3.5)$$

4. PRESERVATION OF THE RIEMANN AND RICCI TENSORS WITH RESPECT TO THE GEODESIC MAPPINGS OF THE AFFINE CONNECTED SPACE

Consider the geodesic mappings which are characterized by the equations (2.2). Form such mappings tensors of the form

$$A_{ijk}^h = \delta_i^h \psi_{j,k} + \delta_j^h \psi_{i,k} + 2\delta_k^h \psi_i \psi_j + \delta_i^h \psi_j \psi_k + \delta_j^h \psi_i \psi_k. \quad (4.1)$$

On the basis of Theorem 1 Riemannian tensor R_{ijk}^h is invariant with respect to geodesic mapping, if condition (3.3) satisfies.

Substituting (4.1) into (3.3) we have

$$\delta_j^h (\psi_{i,k} - \psi_i \psi_k) + \delta_i^h (\psi_{j,k} - \psi_j \psi_k) - \delta_i^h (\psi_{k,j} - \psi_k \psi_j) - \delta_k^h (\psi_{i,j} - \psi_i \psi_j) = 0. \quad (4.2)$$

After contraction (4.2) by indices h and i we obtain

$$\psi_{j,k} - \psi_{k,j} = 0. \quad (4.3)$$

Transvecting (4.2) by indices h and j , we get

$$n(\psi_{i,k} - \psi_i \psi_k) - (\psi_{i,k} - \psi_k \psi_i) = 0. \quad (4.4)$$

From (4.3) and (4.4) follows

$$\psi_{i,k} = \psi_i \psi_k \quad (4.5)$$

Therefore

Theorem 3. *An affine connected space A_n admits geodesic mapping into an another affine connected space $\bar{A}_n$ preserving Riemmanian tensor if and only if the Cauchy type system (4.5) has a solution with respect to functions $\psi_i(x)$.*

On the basis of Theorem 2, the Ricci tensor R_{ij} of an affine connected space A_n is invariant with respect to geodesic mappings if satisfies condition (3.5).

Substituting (4.1) into (3.5) we get

$$\psi_{i,j} - \psi_j \psi_i = n(\psi_{i,j} - \psi_i \psi_j). \quad (4.6)$$

From (4.5) and (4.6) we obtain (4.4).

The above leads to

Theorem 4. *An affine connected space A_n admits geodesic mapping into an another affine connected space $\bar{A}_n$ preserving the Ricci tensor if and only if Cauchy type system (4.4) has a solution with respect to functions $\psi_i(x)$.*

Integrability conditions of equations (4.4) are the following

$$\psi_a R_{ijk}^h = 0. \quad (4.7)$$

The general solution of Cauchy type system (4.4) depends on no more than n substantial parameters.

From (2.3) and (2.4) follows

Theorem 5. *If under a geodesic mapping of an affine connected space A_n into an affine connected space $\bar{A}_n$ Ricci tensor is an invariant geometric object, then Riemannian tensor also is an invariant geometric object.*

This result can be achieved, if we consider the Weyl tensor.

$$W_{ijk}^h = R_{ijk}^h + \frac{1}{n+1} \delta_i^h R_{[jk]} - \frac{1}{n^2-1} [(nR_{ij} + R_{ji})\delta_k^h - (nR_{ik} + R_{ki})\delta_j^h], \quad (4.8)$$

which is an invariant object under geodesic mappings.

Space A_n which admits geodesic mapping preserving Riemannian and Ricci tensors, is characterized by the following interesting property.

Theorem 6. *In an affine connected space which admits geodesic mappings preserving Riemannian and Ricci tensors, there exists a covariant constant vector field.*

Indeed, from (4.3) we have that covector ψ_i is locally gradient, that is, there exists function $\Psi(x)$ such that $\psi_i(x) = \partial_i \Psi(x)$ where $\partial_i = \partial/\partial x^i$.

We introduce function $\Phi(x) = e^{-\Psi(x)}$ and we can see on the basis of condition (4.5) satisfies $\Phi_{i,j} = 0$. Thus covector field $\varphi_i = \partial_i \Phi(x)$ is a covariant constant in space A_n that is

$$\varphi_{i,j} = 0. \quad (4.9)$$

On the other hand, if in A_n there exists a covariant constant vector field φ_i , then using of the substitution $\Psi(x) = -\ln(\Phi(x))$, $\partial_i \Phi(x) = \varphi_i(x)$ we can construct vector field $\psi_i(x)$, which satisfies (4.5) Thus Theorem 6 is proved.

In addition, we note that equation (4.9) forms linear system of Cauchy type in covariant derivatives with respect to unknown functions $\varphi_i(x)$. Integrability conditions of (4.9) has the form $\varphi_\alpha R_{ijk}^\alpha = 0$. There are linear algebraic equations with respect to $\varphi_i(x)$. Their differential prolongations are also linear.

The global existence of these fields was discussed in [13].

5. PRESERVATION OF RIEMANN AND RICCI TENSORS WITH RESPECT TO THE SPECIAL CASE OF CANONICAL ALMOST GEODESIC MAPPINGS OF THE FIRST TYPE OF AFFINE CONNECTED SPACE

Consider the special case of canonical almost geodesic mappings, which is characterized by the following equation

$$P_{ij,k}^h + P_{ij}^\alpha P_{\alpha k}^h = \delta_{(i}^h a_{jk)}, \quad (5.1)$$

where a_{ij} is some symmetric tensor.

Indeed, mappings which are characterized by equation (5.1), are canonical almost geodesic mappings of the first type, namely after symmetrization of equations (5.1) we obtain (2.5).

For mappings, which are characterized by equations (5.1), tensor A has the following structure

$$A_{ijk}^h = \delta_{(i}^h a_{jk)}, \quad (5.2)$$

Then, obviously, conditions (3.3) and (3.5) are fulfilled for the tensor A . Therefore, there is a

Theorem 7. *Riemannian and Ricci tensors are invariant geometrical objects under canonical almost geodesic mappings of the first type, which are characterized by (5.1).*

Because in a projective-euclidean space Riemannian tensor vanishes, and in a Ricci-flat space Ricci tensor vanishes, then by Theorem (3.2) we have

Theorem 8. *If projective euclidean or Ricci-flat space admits canonical almost geodesic mapping of the first type, which is characterized by equations (5.1), into space $\bar{A}_n$, then $\bar{A}_n$ is projective-euclidean or Ricci-flat space respectively.*

Considering (5.1) as a Cauchy-type system with respect to deformation tensor P_{ij}^h we can determine their condition of integrability.

For this we covariantly differentiate (5.1) on x^m , then we alternate it by indices k and m . Using of the Ricci identity we have

$$\begin{aligned} \delta_i^h a_{j[k,m]} + \delta_j^h a_{i[k,m]} + \delta_{[k|i j| m]}^h = \\ - P_{ij}^\alpha R_{\alpha km}^h + P_{\alpha(j}^h R_{i)km}^\alpha + a_{j[m} P_{k]i}^h + a_{j[m} P_{k]j}^h. \end{aligned} \quad (5.3)$$

Contracting condition of integrability (5.3) by indices h and m , we get

$$\begin{aligned} a_{jk,i} + a_{ik,j} - (n+1)a_{ij,k} = \\ - P_{ij}^\alpha R_{\alpha km} + P_{\alpha j}^\beta R_{jk\beta}^\alpha + a_{j\alpha} P_{ki}^\alpha - a_{jk} P_{\alpha i}^\alpha + a_{i\alpha} P_{jk}^\alpha - a_{ik} P_{j\alpha}^\alpha. \end{aligned} \quad (5.4)$$

We alternate equations (5.4) by indices j and k , then we obtain

$$\begin{aligned} a_{ij,k} = a_{ik,j} + \frac{1}{(n+2)} (P_{ij}^\alpha R_{\alpha k} + P_{ik}^\alpha R_{\alpha j} - P_{\alpha j}^\beta R_{ik\beta}^\alpha + P_{\alpha k}^\beta R_{ij\beta}^\alpha \\ - P_{\alpha i}^\beta R_{jk\beta}^\alpha + P_{\alpha i}^\beta R_{kj\beta}^\alpha - a_{j\alpha} P_{ki}^\alpha + a_{k\alpha} P_{ij}^\alpha + a_{ik} P_{j\alpha}^\alpha - a_{ij} P_{k\alpha}^\alpha). \end{aligned} \quad (5.5)$$

Observing (5.5), we can write equations in the following form

$$\begin{aligned} a_{ik,j} = \frac{1}{(n-1)(n+2)} (n(P_{ik}^\alpha R_{\alpha j} - P_{\alpha(k}^\beta R_{i)j\beta}^\alpha) + R_{\alpha(k} P_{i)j}^\alpha R_{ij\beta}^\alpha - P_{\alpha j}^\beta R_{(ik)\beta}^\alpha \\ - P_{\alpha(i}^\beta R_{|j|k)\beta}^\alpha + (n+1)(a_{j(i} P_{k)\alpha}^\alpha - a_{\alpha(i} P_{k)j}^\alpha) + 2(a_{ik} P_{j\alpha}^\alpha - a_{j\alpha} P_{ik}^\alpha)). \end{aligned} \quad (5.6)$$

Obviously, equations (5.1) and (5.6) in a given space represent a Cauchy type system with respect to functions $P_{ij}^h(x)$ and $a_{ij}(x)$ which, of course, are satisfied even algebraic conditions.

$$P_{ij}^h(x) = P_{ji}^h(x), \quad a_{ij}(x) = a_{ji}(x). \quad (5.7)$$

So we proved

Theorem 9. *An affine connected space A_n admits almost geodesic mapping, defined by equation (5.1) onto affine connected space $\bar{A}_n$ if and only if there exist solution of mixed equation system of Cauchy type (5.1), (5.6) and (5.7) with respect to function $P_{ij}^h(x)$ and $a_{ij}(x)$.*

The general solution of Cauchy type mixed system depends on (no more than) $\frac{1}{2}n(n+1)^2$ real parameters.

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NEW TRIGONOMETRIC AND HYPERBOLIC INEQUALITIES

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Abstract. The aim of this paper is to prove new trigonometric and hyperbolic inequalities, which constitute refinements or analogs of famous Cusa-Huygens, Wu-Srivastava, and related inequalities. Most of the results are sharp.

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1. INTRODUCTION

Since the last decade many authors have been interested in finding upper and lower bounds for expression $f(x)/x$, where $f(x)$ is a trigonometric or a hyperbolic function. We continue this line of research.

The inequalities

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{\cos x + 2}{3} \quad (1.1)$$

hold for $0 < |x| < \pi/2$. The left hand side inequality is due to by D.D. Adamović and D.S. Mitrinović [9, p. 238], and the right hand side one was obtained by N. Cusa and C. Huygens in 2005 [16]. The hyperbolic version of (1.1) is

$$(\cosh x)^{1/3} < \frac{\sinh x}{x} < \frac{\cosh x + 2}{3}, \quad (1.2)$$

for $x \neq 0$. The left hand side inequality in (1.2) was obtained by Lazarević [9, p. 270], and the right hand side inequality is called the hyperbolic Cusa-Huygens inequality [11].

In 1989 J. Wilker [10] discovered the following inequality

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2, \quad 0 < |x| < \frac{\pi}{2}. \quad (1.3)$$

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Numerous authors have studied this inequality by giving simpler proofs and generalization, e.g, see [2, 3, 11, 12, 17–19, 21, 22]. The following inequality is due to Huygens [5]

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 3, \quad 0 < |x| < \frac{\pi}{2} \quad (1.4)$$

and is called Huygens inequality or the first Wilker inequality. In [18], Wu and Srivastava introduced the following inequality

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2, \quad 0 < |x| < \frac{\pi}{2}, \quad (1.5)$$

which is called the second Wilker inequality.

We study inequalities (1.1)–(1.5) and our main results are the following five theorems. Our first main result is a generalization of Cusa-Huygens inequality (1.1).

Theorem 1. *For $x \in [-\pi/2, \pi/2]$, we have*

$$\frac{\cos x + \alpha - 1}{\alpha} \leq \frac{\sin x}{x} \leq \frac{\cos x + \beta - 1}{\beta}, \quad (1.6)$$

with the best possible constants $\alpha = \pi/(\pi - 2) \approx 2.75194$ and $\beta = 3$. The lower bound is sharp for $x \in \{-\pi/2, 0, \pi/2\}$ and the upper bound is sharp for $x = 0$.

Remark 1. The upper bound of Theorem 1 is sharp at point $x = 0$ and the lower bound at points $x = -\pi/2$, $x = 0$ and $x = \pi/2$. For values $x \in [-\pi/2, \pi/2]$ the difference between the function and the lower bound is less than 0.01 and the difference between the function and the upper bound is less than 0.031. The right hand side inequality holds true for all real numbers x and as a sharp inequality for all $x \neq 0$.

In the following two theorems we introduce Wu-Srivastava type inequalities for the trigonometric functions.

Theorem 2. *For $x \in [-\pi/2, \pi/2]$, we have*

$$\left(\frac{x}{\sin x}\right)^2 + \left(\frac{\pi^2}{4} - 1\right) \frac{x}{\tan x} \leq \frac{\pi^2}{4} \quad (1.7)$$

and the equality is attained for $x = -\pi/2$, $x = 0$ and $x = \pi/2$.

Remark 2. The upper bound of Theorem 2 is sharp at points $x = -\pi/2$, $x = 0$ and $x = \pi/2$. For values $x \in [-\pi/2, \pi/2]$ the difference between the function and the upper bound is less than 0.13.

Theorem 3. *For $x \in [-\pi/2, \pi/2]$, we have*

$$(\alpha - 1) \frac{x}{\sin x} + \frac{x}{\tan x} \leq \alpha, \quad (1.8)$$

$$\left(\frac{x}{\sin x}\right)^\alpha + \frac{x}{\tan x} < \left(\frac{\pi}{2}\right)^\alpha, \quad (1.9)$$

with the best possible constant $\alpha = \pi/(\pi - 2)$. The inequality (1.8) holds as equality for $x \in \{-\pi/2, 0, \pi/2\}$ and the inequality (1.9) holds as equality for $x \in \{-\pi/2, \pi/2\}$.

Remark 3. For values $x \in [-\pi/2, \pi/2]$ the difference between the function α and the lower bound is less than 0.031. In inequality (1.9) the difference between the function and the lower bound is between 1.45 and 1.9.

The next theorem is a generalization of the Huygens inequality.

Theorem 4. For $x \in (-\pi/2, \pi/2)$, we have

$$3 \cos x \leq \frac{x}{\sin x} + 2 \frac{x}{\tan x} \leq 2 + \cos x. \quad (1.10)$$

The inequalities hold as equalities for $x = 0$.

The following result is a Wilker and Wu-Srivastava type result for the hyperbolic functions.

Theorem 5. For $x > 0$, we have

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} < \left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} < \frac{1 + \cosh(2x/3)}{2} \left(\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x}\right).$$

Our last result is a counterpart of the inequality

$$\exp(-x^2/6) < \frac{2 + \cos x}{3}, \quad (1.11)$$

$x \in (0, \infty)$, which recently appeared in [20, Thm 2].

Theorem 6. For $x \in (0, \pi/2)$, the following inequalities hold

$$\exp\left(\alpha - \frac{(\pi-2)x^2}{2\pi}\right) < \frac{(\pi-2)\cos(x) + 2}{\pi} < \exp\left(\beta - \frac{(\pi-2)x^2}{2\pi}\right),$$

with the best possible constants $\alpha = (\pi^2 + 8\log(2/\pi) - 2\pi)/8 \approx -0.00328$ and $\beta = 0$.

2. PRELIMINARIES AND LEMMAS

The following lemma was originally proved in [13], but can also be found in [4].

Lemma 1. For $0 < R \leq \infty$. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $C(x) = \sum_{n=0}^{\infty} c_n x^n$ be two real power series converging on the interval $(-R, R)$. If the sequence $\{a_n/c_n\}$ is increasing (decreasing) and $c_n > 0$ for all n , then the function $A(x)/C(x)$ is also increasing (decreasing) on $(0, R)$.

For $|x| < \pi$, the following power series expansions can be found in [7, 1.3.1.4 (2)–(3)],

$$x \cot x = 1 - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n}, \quad (2.1)$$

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1}, \quad (2.2)$$

and

$$\coth x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1}, \quad (2.3)$$

where B_{2n} are the even-indexed Bernoulli numbers, see [6, p. 231]. We can get the following expansions directly from (2.2) and (2.3),

$$\frac{1}{(\sin x)^2} = -(\cot x)' = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| (2n-1) x^{2n-2}, \quad (2.4)$$

$$\frac{1}{(\sinh x)^2} = -(\coth x)' = \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} (2n-1) |B_{2n}| x^{2n-2}. \quad (2.5)$$

For the following expansion formula

$$\frac{x}{\sin x} = 1 + \sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!} |B_{2n}| x^{2n} \quad (2.6)$$

see [8].

Lemma 2 (Theorem 2 in [1]). *For $-\infty < a < b < \infty$, let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$, and differentiable on (a, b) . Let $g'(x) \neq 0$ on (a, b) . If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b) , then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 3. *The following function*

$$f_1(x) = \frac{(x/\sin x)^2 - x \cot x}{1 - x \cot x}$$

is strictly increasing from $(0, \pi/2)$ onto $(\pi^2/4)$.

Proof. Let $f_1(x) = A_1(x)/C_1(x)$, where

$$A_1(x) = (x/\sin x)^2 - x \cot x \quad \text{and} \quad C_1(x) = 1 - x \cot x.$$

By using expansion formulas (2.2) and (2.4) we get

$$A_1(x) = 1 + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} (2n-1) |B_{2n}| x^{2n} - 1 + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n}$$

$$= \sum_{n=1}^{\infty} \frac{2^{2n} 2n}{(2n)!} |B_{2n}| x^{2n} = \sum_{n=1}^{\infty} a_n x^{2n},$$

and

$$C_1(x) = \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n} = \sum_{n=1}^{\infty} c_n x^{2n}.$$

Let $d_n = a_n/c_n = 2n$, which is increasing in $n \in \mathbb{N}$. Thus, by Lemma 1 $f_1(x)$ is strictly increasing in $x \in (0, \pi/2)$. Applying l'Hôpital rule, we get $\lim_{x \rightarrow 0} f_1(x) = 2$ and $\lim_{x \rightarrow \pi/2} f_1(x) = \pi^2/4$. This completes the proof. $\square$

Lemma 4. *The following inequalities hold*

$$\frac{\tanh x}{x} \leq \frac{2}{\sqrt{9 + 4x^2} - 1}, \quad x \in \mathbb{R}, \quad (2.7)$$

$$\frac{\sinh x}{x} < \frac{\cosh x + 2}{3} < (\cosh x)^{1/3} \frac{\cosh(2x/3) + 1}{2}, \quad x > 0. \quad (2.8)$$

Proof. Clearly both sides of the inequality (2.7) get value 1 at $x = 0$. By symmetry of the function we need to consider only the positive values of x . Let

$$f_2(x) = \left(\frac{2x}{\tanh x} + 1 \right)^2 - 4x^2 - 9.$$

By differentiation we get

$$\begin{aligned} f_2'(x) &= 4 \left(\coth x + \frac{x(1 - 2x \coth x)}{(\sinh x)^2} \right) \\ &= 4 \frac{x^2 \cosh x}{(\sinh x)^3} \left(\left(\frac{\sinh x}{x} \right)^2 + \frac{\tanh x}{x} - 2 \right) \\ &= 4 \frac{x^2 \cosh x}{(\sinh x)^3} f_3(x) > 0, \end{aligned}$$

where the last inequality follows since $f_3(x) > 0$ is equivalent to

$$\frac{\sinh x}{x} + \frac{1}{\cosh x} > \frac{x}{\sinh x},$$

which is clearly true because $(\sinh x)/x > 1$. Now f_2 is strictly increasing, and $\lim_{x \rightarrow 0} f_2(x) = 0 < f_2(x)$. This implies the proof of (2.7).

The first inequality in (2.8) is well known, for the second inequality we define

$$f_4(x) = \frac{\cosh x + 2}{3} - (\cosh x)^{1/3} \frac{\cosh(2x/3) + 1}{2}.$$

Simple computation gives

$$f_4'(x) = \frac{\sinh \frac{x}{3} - 2 \sinh x + 4 \cosh^{2/3} x \sinh x - 3 \sinh \frac{5x}{3}}{12(\cosh x)^{2/3}}$$

and clearly $\sinh \frac{x}{3} - 2 \sinh x < 0$. We prove that $4 \cosh^{2/3} x \sinh x - 3 \sinh \frac{5x}{3} < 0$, which is equivalent to

$$f_5(x) = 4 \sinh x + 2 \sinh(3x) + \frac{11}{8} \sinh(5x) - \frac{81}{8} \sinh \frac{5x}{3} > 0.$$

Since

$$\begin{aligned} f_5''(x) &= 4 \sinh x + 18 \sinh(3x) + \frac{275}{8} \sinh(5x) - \frac{225}{8} \sinh \frac{5x}{3} \\ &\geq \frac{275}{8} \sinh(5x) - \frac{225}{8} \sinh \frac{5x}{3} > 0 \end{aligned}$$

for all $x > 0$, it is clear that $f_5'(x) > f_5'(0) = 0$ and $f_5(x)$ is increasing. Thus $f_5(x) > f_5(0) = 0$. $\square$

Remark 4. In Lemma (2.7) the difference between the function and the upper bound is less than 0.02. The upper bound is asymptotically sharp $\frac{2}{\sqrt{9+4x^2}-1} - \frac{\tanh x}{x} \rightarrow 0$ as $x \rightarrow \pm\infty$.

3. PROOFS OF THE MAIN RESULTS

In this section we prove our main theorems.

Proof of Theorem 1. Sharpness of the bounds is obvious. Since the bounds and the function $\frac{\sin x}{x}$ are even we need to prove sharp inequality for $x \in (0, \pi/2)$. Let

$$f_6(x) = \frac{\cos x - 1}{(\sin x)/x - 1} = \frac{(\sin x)/x - x \cot x}{x/\sin x - 1} = \frac{A_2(x)}{C_2(x)}.$$

Using series expansion formulas (2.1) and (2.6), we get

$$\begin{aligned} A_2(x) &= \sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!} |B_{2n}| x^{2n} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n} \\ &= \sum_{n=1}^{\infty} \frac{2(2^{2n}-1)}{(2n)!} |B_{2n}| x^{2n} = \sum_{n=1}^{\infty} \tilde{a}_n x^{2n}, \end{aligned}$$

and

$$C_2(x) = \sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!} |B_{2n}| x^{2n} = \sum_{n=1}^{\infty} \tilde{c}_n x^{2n}.$$

We denote $\tilde{d} = \tilde{a}/\tilde{c} = 2(2^{2n}-1)/(2^{2n}-2)$. Clearly $\tilde{d}$ is a decreasing function of $n \in \mathbb{N}$. Hence f_6 is decreasing by Lemma 1. Applying l'Hôspital rule, we get $\lim_{x \rightarrow 0} f_6(x) = 3$ and $\lim_{x \rightarrow \pi/2} f_6(x) = \pi/(\pi-2)$, this finishes the proof. $\square$

Proof of Theorem 2. Equality in the claim is clearly attained at points $x = -\pi/2$, $x = 0$ and $x = \pi/2$. Since the left hand side of the inequality is an even function we need to show that the inequality is sharp for $x \in (0, \pi/2)$. Let

$$\frac{x/\sin x - \cos x}{(\sin x)/x - x \cot x} = \frac{(x/\sin x)^2 - x \cot x}{1 - x \cot x} = f_1(x).$$

By Lemma 3, we get

$$2 < \frac{x/\sin x - \cos x}{(\sin x)/x - x \cot x} < \frac{\pi^2}{4},$$

which implies the following inequalities

$$\frac{4}{\pi^2} \left(\frac{x}{\sin x} + \left(\frac{\pi^2}{4} - 1 \right) \cos x \right) < \frac{\sin x}{x} < \frac{1}{2} \left(\frac{x}{\sin x} + \cos x \right). \quad (3.1)$$

The first inequality of (3.1) can be written as (1.7). This completes the proof. $\square$

The second inequality of (3.1) is also proved by Neuman and Sándor, see [11, Theorem 2.3]. They pointed out that it can be written as the Wu-Srivastava inequality.

Lemma 5. Let $\alpha = \pi/(\pi - 2)$ as in Theorem 1. The function

$$f_\alpha(b) = \left(\frac{\alpha}{\alpha + b - 1} \right)^\alpha + \frac{\alpha b}{\alpha + b - 1}$$

is decreasing from $(0, 1)$ onto $(2, k)$, where $k = (\pi/2)^\alpha \approx 3.46505$.

Proof. By differentiation we obtain

$$\begin{aligned} f'_\alpha(b) &= \frac{a(a-1-(a+b-1)(a/(a+b-1))^a)}{(a+b-1)^2} \\ &= -\frac{\pi(\pi^{\pi/(\pi-2)}(2+\pi(\pi-2))^{(2-\pi)/2}-2)}{(2+b(\pi-2))^2}, \end{aligned}$$

which is negative, and $f_\alpha(b) \rightarrow k$ as $b \rightarrow 0$ and $f_\alpha(b) \rightarrow 2$ as $b \rightarrow 1$. $\square$

Proof of Theorem 3. The inequality (1.8) follows from the first inequality of (1.6). Again, utilizing the same inequality

$$\frac{x}{\sin x} < \frac{\alpha}{\alpha - 1 + \cos x},$$

we get

$$\left(\frac{x}{\sin x} \right)^\alpha + \frac{x}{\tan x} < \left(\frac{\alpha}{\alpha - 1 + \cos x} \right)^\alpha + \frac{\alpha \cos x}{\alpha - 1 + \cos x} = f_\alpha(\cos x) < \left(\frac{\pi}{2} \right)^\alpha$$

by Lemma 5, because $f_\alpha(\cos x)$ is strictly increasing and $\lim_{x \rightarrow 1} f_\alpha(\cos x) = (\pi/2)^\alpha$. $\square$

An other proof for (1.8). Let

$$f(x) = (\alpha - 1) \frac{x}{\sin x} + \frac{x}{\tan x}$$

for $x \in (0, \pi/2)$. An easy computation gives

$$(\sin x)^2 \cdot f'(x) = (\alpha - 1) \sin x - (\alpha - 1)x \cos x + \sin x \cos x - x = g(x).$$

Now $g'(x) = (2(\sin x)/x) \cdot h(x)$, where $h(x) = (\alpha - 1)/2 - (\sin x)/x$. The function $(\sin x)/x$ is strictly decreasing as the function of x , and thus the equation $(\alpha - 1)/2 = (\sin x)/x$ has at most a single root in $(0, \pi/2)$.

Suppose that $(\alpha - 1)/2 < 1$, then the equation has exactly one root $x_0 \approx 0.8795$. As $h(x) < 0$ for $x < x_0$ and $h(x) > 0$ for $x > x_0$, the function $g(x)$ will be strictly decreasing for x in $(0, x_0)$, and strictly increasing in $(x_0, \pi/2)$.

Suppose that $\alpha > \pi/2 + 1$, which is equivalent to $(\alpha - 1)/2 > \pi/4$. Then we get $g(\pi/2) = \alpha - 1 - \pi/2 > 0$. As $g(0) = 0$ we have $g(x_0) < 0$, and g will have a single root x_1 in interval $(x_0, \pi/2)$. Then we get $g(x) < 0$ for x in $(0, x_1)$, and $g(x) > 0$ for x in $(x_1, \pi/2)$, where $x_1 \approx 1.1559$. This means that at the points 0 and $\pi/2$ the function $f(x)$ will obtain its maximum values. If $f(\pi/2) \leq f(0)$, the inequality $f(x) \leq \alpha$ is true. The inequality $f(\pi/2) \leq f(0)$ is equivalent to $\alpha \leq \pi/(\pi - 2)$. Remark that for the best possible $\alpha = \pi/(\pi - 2)$ one has also $\alpha > \pi/2 + 1$ and $(\alpha - 1)/2 < 1$, so the assumed properties in the proof are valid. This finishes the proof of the inequality $f(x) \leq \alpha$, with best possible $\alpha = \pi/(\pi - 2)$. $\square$

Remark 5. Indeed, the point x_1 above is the minimum point of $f(x)$ on $(0, \pi/2)$. In fact the following converse of inequality (1.8)

$$(\alpha - 1) \frac{x}{\sin x} + \frac{x}{\tan x} \geq f(x_1) \approx 2.7219 \quad (3.2)$$

holds true for $\alpha = \pi/(\pi - 2)$, $(\alpha - 1)/2 < 1$ and $\alpha > \pi/2 + 1$ (as $\alpha < 3$, it is sufficient to suppose $\pi/2 + 1 < \alpha \leq \pi/(\pi - 2)$).

Proof of Theorem 4. Clearly the function, its lower and upper bound get value 3 at origin. By symmetry of the function we consider only values $x \in (0, \pi/2)$. The second inequality in (1.10) is equivalent to write

$$f_7(x) = \left(\frac{2 + \cos x}{3 \cos x} \right) - \left(\frac{2x}{\sin x} + \frac{x}{\sin x \cos x} \right) > 0.$$

It is sufficient to prove that $f_7 > 0$. Using the the following inequalities

$$(\cos(x/2))^{4/3} < \frac{\sin x}{x} < \frac{2 + \cos x}{3}$$

we get

$$f_7(x) > \frac{3}{\cos x} \frac{\sin x}{x} - \frac{x}{\sin x} \left(2 + \frac{1}{\cos x} \right)$$

$$\begin{aligned}
&= \frac{3x}{\sin x \cos x} \left(\left(\frac{\sin x}{x} \right)^2 - \frac{2 \cos x + 1}{3} \right) \\
&> \frac{3x}{\sin x \cos x} \left(\left(\frac{1 + \cos x}{2} \right)^{4/3} - \frac{2 \cos x + 1}{3} \right) \\
&= \frac{6x}{\sin 2x} f_8(x).
\end{aligned}$$

For showing that f_8 is positive, we define the function

$$f_9(y) = \frac{(1+y)^4}{(2y+1)^3}, \quad y \in (0, 1),$$

and get

$$f_9'(y) = \frac{(1-y)(1+y)^3}{(1+2y)^4} < 0,$$

with $f_9(1) = 16/27$. This implies that f_8 is positive, and this completes the proof of the second inequality. $\square$

Remark 6. The upper bound of Theorem 4 holds true for values $x \in (-\pi, \pi)$. The difference between the function and the lower bound is less than 1.6 and between the function and the upper bound is less than 0.55. In both cases the difference is less than x^2 .

Corollary 1. For $x \in (0, \pi/2)$, we have

$$\frac{\pi}{2} + \cos x < \frac{x}{\sin x} + 2 \frac{x}{\tan x} < 2 + \cos x. \quad (3.3)$$

Proof. Let $f(x) = x/\sin x + 2x/\tan x - \cos x$ for $x \in (0, \pi/2)$. After elementary computations, we get

$$f'(x) \cdot (\sin x)^2 = \sin x - x \cos x + 2 \sin x \cos x - 2x + (\sin x)^3 = h(x).$$

Now $h'(x) = \sin x \cdot k(x)$, where

$$k(x) = x - 4 \sin x + 3 \sin x \cos x.$$

As $k'(x) = 4(\cos x)^2 - 4 \cos x - 2(\sin x)^2$, and $0 < \cos x < 1$ we obtain $k'(x) < 0$. Thus $k(x) < k(0) = 0$, implying $h(x) < h(0) = 0$. Finally, we get $f'(x) < 0$. Thus $f(x)$ is strictly decreasing, this implies $f(\pi/2) < f(x) < f(0)$, so the result follows. $\square$

We remark here that the first inequality of (3.3) cannot be compared with the first inequality of (1.10), and obviously the right hand sides of both inequalities are equal.

Corollary 2. The inequality (2.7) implies the hyperbolic version of the Wu-Srivastava inequality (1.5).

Proof. It is easy to see that the inequality (2.7) is equivalent to

$$\left(\frac{x}{\tanh x}\right)^2 + \frac{x}{\tanh x} - x^2 - 2 > 0.$$

The above inequality can be written as

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} > 0,$$

by observing that

$$\left(\frac{x}{\sinh x}\right)^2 - x^2 = \frac{x}{\sinh x} > 0. \quad \square$$

Proof of Theorem 5. The first inequality is well known and follows from (1.2). Similarly the second inequality follows from (2.8). $\square$

In 2010 J. Sándor (see [15]) proved that the best positive constants a and b such that

$$\left(\frac{\cos x + 2}{3}\right)^a < \frac{\sin x}{x} < \left(\frac{\cos x + 2}{3}\right)^b,$$

are $a = \log(\pi/2)/\log(3/2) \approx 1.113$ and $b = 1$.

Now, we state two similar results:

Theorem 7. For $x \in (-\pi/2, \pi/2)$, we have

$$(1) \quad \frac{\cos x + 2}{3^{\alpha_1}} < \frac{\sin x}{x} < \frac{\cos x + 2}{3^{\beta_1}},$$

$$(2) \quad \frac{\cos x + 2^{\alpha_2}}{3} < \frac{\sin x}{x} < \frac{\cos x + 2^{\beta_2}}{3},$$

with best possible constants $\alpha_1 = \log(\pi)/\log(3) \approx 1.04198$, $\alpha_2 = \log(\pi/6)/\log(2) \approx 0.93345$, $\beta_1 = 1$ and $\beta_2 = 1$.

Proof. For (1), let $f_{10}(x) = \frac{x(2+\cos x)}{\sin x}$, with $x \in (0, \pi/2)$. One has

$$(\sin x)^2 \cdot f'_{10}(x) = 2 \sin x - 2x \cos x + \sin x \cos x - x = f_{11}(x).$$

As $f'_{11}(x) = 2(\sin x)(x - \sin x) > 0$, we get $f_{11}(x) > f_{11}(0) = 0$, so $f'_{10}(x) > 0$, proving that $f_{10}(x)$ is strictly increasing. This implies that the function

$$f_{12}(x) = \frac{1}{\log 3} \log \left(\frac{x(\cos x + 2)}{\sin x} \right)$$

is strictly increasing in $x \in (0, \pi/2)$, and we get

$$\beta_1 = f_{12}(0+) < f_{12}(x) < f_{12}(\pi/2) = \alpha_1,$$

thus (1) follows.

To prove (2) we write $f_{13} = 3(\sin x)/x - \cos x$, for $x \in (0, \pi/2)$. Now

$$x^2 f'_{13}(x) = 3x \cos x + x^2 \sin x - 3 \sin x = f_{14}(x),$$

and $f'_{14}(x) = x(x \cos x - \sin x) < 0$, since $x \cos x < \sin x$. This shows that $f_{14}(x) < f_{14}(0) = 0$, implying $f'_{13}(x) < 0$ and proving that $f_{13}(x)$ is strictly decreasing. Now $f_{13}(x) < f_{13}(0+) = 2$ and $f_{13}(x) > f_{13}(\pi/2) = 6/\pi$. As $2 = 2^{\beta_2}$ for $\beta_2 = 1$, and $6/\pi = 2^{\alpha_2}$ for $\alpha_2 = \log(6/\pi)/\log(2)$, the result follows. $\square$

Proof of Theorem 6. Let

$$f(x) = \log\left(\frac{(\pi-2)\cos x + 2}{\pi}\right) + \frac{(\pi-2)x^2}{2\pi}.$$

Simple calculation yields

$$f'(x) = (\pi-2)\left(\frac{x}{\pi} - \frac{\sin x}{2 + (\pi-2)\cos x}\right),$$

which is negative by Theorem 1. Thus, the f function is strictly decreasing, and

$$\alpha = \lim_{x \rightarrow \pi/2} f(x) < f(x) < 0 = \beta = \lim_{x \rightarrow 0} f(x).$$

This implies the proof. $\square$

We finish the paper by giving a new type of Kober's inequality [9, 14], which follows easily from (1.11) and Theorem 6.

Corollary 3. For $x \in (0, \pi/2)$, the following inequalities hold

$$3 \exp\left(-\frac{x^2}{6}\right) - 2 < \cos x < \frac{\pi \exp(-(\pi-2)x^2/(2\pi)) - 2}{\pi-2}.$$

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UNIFIED TREATMENT OF FRACTIONAL INTEGRAL INEQUALITIES VIA LINEAR FUNCTIONALS

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Abstract. In the paper we prove several inequalities involving two isotonic linear functionals. We consider inequalities for functions with variable bounds, for Lipschitz and Hölder type functions etc. These results give us an elegant method for obtaining a number of inequalities for various kinds of fractional integral operators such as for the Riemann-Liouville fractional integral operator, the Hadamard fractional integral operator, fractional hypergeometric integral and corresponding q -integrals.

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1. INTRODUCTION

Recently several papers involving inequalities for fractional integral operators have been published, see [3, 4, 7, 8, 19, 21, 22] and references therein. Certain similarity of those inequalities shows that those results have a common origin. In this paper we give a unified treatment of several known inequalities for fractional integral operators via theory of isotonic linear functionals. In fact, we prove general inequalities involving isotonic linear functionals from which some interesting results are followed.

The paper is organized in the following way. The rest of this section contains definitions and some examples of isotonic linear functionals connected with fractional integration and integration on time scales. The Chebyshev inequality for one and two isotonic functionals are given. Inequalities for Lipschitz functions are given in the second section. The third section is devoted to new inequalities involving two isotonic functionals and functions with variable upper and lower bounds. The fourth section is devoted to results involving more than two functions. Applications or references where we can find applications in the theory of fractional operators and calculus on time scales are also given.

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Isotonic linear functionals

Definition 1 (Isotonic linear functional). Let E be a non-empty set and L be a class of real-valued functions on E having the properties:

L1. If $f, g \in L$, then $(af + bg) \in L$ for all $a, b \in \mathbf{R}$;

L2. The function **1** belongs to L . ($\mathbf{1}(t) = 1$ for $t \in E$).

A functional $A : L \rightarrow \mathbf{R}$ is called an isotonic linear functional if

A1. $A(af + bg) = aA(f) + bA(g)$ for $f, g \in L, a, b \in \mathbf{R}$;

A2. $f \in L, f(t) \geq 0$ on E implies $A(f) \geq 0$.

There exist a lot of interesting examples of linear functionals which play some role in different parts of mathematics. In the following example we describe the most mentioned functionals - discrete and integral -, and we give several functionals which appear in the theory of fractional calculus and calculus on time scales.

Example 1. (i) **Discrete functional.** If $E = \{1, 2, \dots, n\}$ and $f : E \rightarrow \mathbf{R}$, then $A(f) = \sum_{i=1}^n f(i)$ is an isotonic linear functional.

(ii) **Integral functional.** If $E = [a, b] \subset \mathbf{R}$ and $L = L(a, b)$, then $A(f) = \int_a^b f(t)dt$ is an isotonic linear functional. If $A_1(f) = \frac{1}{b-a}A(f)$, then A_1 is a normalized isotonic linear functional.

(iii) **Fractional hypergeometric operator.** If $t > 0$, $\alpha > \max\{0, -\beta - \mu\}$, $\mu > -1, \beta - 1 < \eta < 0$, then

$$A(f) = I_t^{\alpha, \beta, \eta, \mu} \{f(t)\}$$

is an isotonic linear functional, ([3]), where $I_t^{\alpha, \beta, \eta, \mu} \{f(t)\}$ is a fractional hypergeometric operator defined as

$$\begin{aligned} I_t^{\alpha, \beta, \eta, \mu} \{f(t)\} \\ = \frac{t^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} \int_0^t \sigma^\mu (t-\sigma)^{\alpha-1} {}_2F_1(\alpha + \beta + \mu, -\eta, \alpha; 1 - \frac{\sigma}{t}) f(\sigma) d\sigma \end{aligned}$$

where the function ${}_2F_1(a, b, c, t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}$ is the Gaussian hypergeometric function and $(a)_n$ is the Pochhammer symbol:
 $(a)_n = a(a+1) \dots (a+n-1)$, $(a)_0 = 1$.

- Putting $\mu = 0$, then the fractional hypergeometric operator reduces to the Saigo fractional integral operator $I^{\alpha, \beta, \eta} \{f(t)\}$.
- The Erdélyi-Kober fractional integral operator $I^{\alpha, \eta} \{f(t)\}$ is a particular case of $I_t^{\alpha, \beta, \eta, \mu} \{f(t)\}$ when $\beta = \mu = 0$.

- One of the earliest defined and the most investigated fractional integral operator is the so-called Riemann-Liouville operator defined as

$$J^\alpha f(t) = I_t^{\alpha, -\alpha, 0, 0} \{f(t)\} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\sigma)^{\alpha-1} f(\sigma) d\sigma, \quad \alpha > 0, \quad (1.1)$$

and it is a particular case of a fractional hypergeometric operator for $\beta = -\alpha$, $\eta = \mu = 0$.

(iv) **q-analogues.** The above-mentioned operators have the so-called q -analogues. We describe a q -analogue of Saigo's fractional integral, [2]. Let $\Re(\alpha) > 0$, $\beta, \eta \in \mathbb{C}$, $0 < q < 1$. A q -analogue of Saigo's fractional integral $I_q^{\alpha, \beta, \eta}$ is given for $|\tau/t| < 1$ by equation

$$\begin{aligned} I_q^{\alpha, \beta, \eta} \{f(t)\} &= \frac{t^{-\beta-1}}{\Gamma_q(\alpha)} \int_0^t \left(q \frac{\tau}{t}; q\right)_{\alpha-1} \\ &\times \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta}; q)_m (q^{-\eta}; q)_m}{(q^\alpha; q)_m (q; q)_m} \cdot q^{(\eta-\beta)m} (-1)^m q^{-m(m-1)/2} \left(\frac{\tau}{t} - 1\right)_q^m f(\tau) d_q \tau, \end{aligned}$$

where

$$(a; q)_\alpha = \frac{\prod_{k=0}^{\infty} (1 - aq^k)}{\prod_{k=0}^{\infty} (1 - aq^{\alpha+k})} \quad \text{and} \quad (t-a)_q^n = t^n \left(\frac{a}{t}; q\right)_n.$$

If $\alpha > 0$, $\beta, \eta \in \mathbb{R}$ with $\alpha + \beta > 0$ and $\eta < 0$, then $I_q^{\alpha, \beta, \eta}$ is isotonic, [9].

(v) **The Hadamard fractional integral.** The Hadamard fractional integral of order $\alpha > 0$ of function f is defined as

$${}_H J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_1^x \left(\log \frac{x}{y}\right)^{\alpha-1} \frac{f(y) dy}{y}, \quad 1 < x.$$

For further reading about fractional calculus we recommend, for example, [14].

(vi) In 1988 S. Hilger introduced the calculus on time scales, a strong tool for unified treatment of differential and difference equations. Among different kinds of integrals the most investigated is Δ -integral, see for example, [1, 15]. A Δ -integral was followed by ∇ -integral, $\diamond_\alpha$ integral, α, β -symmetric integral etc. All of them are isotonic linear functionals.

Chebyshev-type inequalities for isotonic linear functionals

After that short text about various kinds of isotonic linear functionals, let us say few words about some inequalities of Chebyshev type involving isotonic linear functionals.

We say that functions f and g on E are similarly ordered (or synchronous) if for each $x, y \in E$

$$(f(x) - f(y))(g(x) - g(y)) \geq 0.$$

If the reversed inequality holds, then we say that f and g are oppositely ordered or asynchronous. The most famous inequality which involve similarly or oppositely ordered functions is the Chebyshev inequality for integrals. It states that if p, f and g are integrable real functions on $[a, b] \subset \mathbf{R}$ and if f and g are similarly ordered, then

$$\int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx \geq \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx. \quad (1.2)$$

If f and g are oppositely ordered then the reverse of the inequality in (1.2) is valid.

During the last century a lot of results about the Chebyshev inequality appear. Here, we only give the most recent results involving two isotonic linear functionals, [16].

Theorem 1 (The Chebyshev inequality for two functionals). *Let A and B be two isotonic linear functionals on L and let $p, q \in L$ be non-negative functions. Let f, g be two functions on E such that $pf, pg, qf, qg, pfg, qfg \in L$.*

If f and g are similarly ordered functions, then

$$A(pfg)B(q) + A(p)B(qfg) \geq A(pf)B(qg) + A(pg)B(qf). \quad (1.3)$$

If f and g are oppositely ordered functions, then the reverse inequality in (1.3) holds.

Putting $A = B$, $p = q$ in (1.3) and divided by 2 we get that for similarly ordered functions f and g such that $pf, pg, pfg \in L$, the following holds

$$A(p)A(pfg) \geq A(pf)A(pg).$$

If f and g are oppositely ordered functions, then the reverse inequality holds.

It is, in fact, the Chebyshev inequality for one isotonic positive functional.

One of the most investigated question related to the Chebyshev integral inequality is the question of finding bounds for the so-called Chebyshev difference which is defined as a difference between two sides from inequality (1.2). Results related to that question are called Grüss type inequalities. In [16] Grüss type inequalities for the Chebyshev difference $T(A, B, p, q, f, g)$ which arise from inequality (1.3) are given where

$$T(A, B, p, q, f, g) = B(q)A(pfg) + A(p)B(qfg) - A(pf)B(qg) - A(pg)B(qf).$$

2. INEQUALITIES FOR $M - g$ -LIPSCHITZ AND HÖLDER-TYPE FUNCTIONS

In this section $M - g$ -Lipschitz functions are considered. We say that f is an $M - g$ -Lipschitz function if

$$|f(x) - f(y)| \leq M|g(x) - g(y)|$$

for all $x, y \in E$. If $g = id$, then f is simple called an M -Lipschitz function.

In the following theorem we consider two functions f and g which are h_1 - and h_2 -Lipschitz functions with constants M_1 and M_2 respectively.

Theorem 2. Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let M_1, M_2 be real numbers and let f, g, h_1, h_2 be functions such that f is $M_1 - h_1$ -Lipschitz and g is $M_2 - h_2$ -Lipschitz, i.e. for all $x, y \in E$

$$|f(x) - f(y)| \leq M_1 |h_1(x) - h_1(y)|, \quad (2.1)$$

$$|g(x) - g(y)| \leq M_2 |h_2(x) - h_2(y)|. \quad (2.2)$$

If all the terms in the below inequality exist and h_1 and h_2 are or similarly ordered, or oppositely ordered, then

$$|T(A, B, p, q, f, g)| \leq M_1 M_2 T(A, B, p, q, h_1, h_2). \quad (2.3)$$

Proof. Let h_1 and h_2 be similarly ordered. Multiplying the inequalities (2.1) and (2.2) we get

$$|(f(x) - f(y))(g(x) - g(y))| \leq M_1 M_2 (h_1(x) - h_1(y))(h_2(x) - h_2(y)).$$

It means that

$$(f(x) - f(y))(g(x) - g(y)) \leq M_1 M_2 (h_1(x) - h_1(y))(h_2(x) - h_2(y)) \quad \text{and}$$

$$(f(x) - f(y))(g(x) - g(y)) \geq -M_1 M_2 (h_1(x) - h_1(y))(h_2(x) - h_2(y)).$$

Since

$$(f(x) - f(y))(g(x) - g(y)) = f(x)g(x) + f(y)g(y) - f(x)g(y) - f(y)g(x)$$

multiplying with $p(x)q(y)$ and acting on the first inequality by functional A with respect to x and then by functional B with respect to y we get

$$\begin{aligned} & A(pfg)B(q) + A(p)B(qfh) - A(pf)B(qg) - A(pg)B(qf) \\ & \leq M_1 M_2 (A(ph_1 h_2)B(q) + A(p)B(qh_1 h_2) - A(ph_1)B(qh_2) - A(ph_2)B(qh_1)), \\ & T(A, B, p, q, f, g) \leq M_1 M_2 T(A, B, p, q, h_1, h_2). \end{aligned}$$

Similarly, from the second inequality we obtain

$$T(A, B, p, q, f, g) \geq -M_1 M_2 T(A, B, p, q, h_1, h_2)$$

and we get the claimed result. The case when h_1 and h_2 are oppositely ordered is proven similiary. $\square$

Theorem 3. Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let M be real number and let f, g be functions such that

$$|f(x) - f(y)| \leq M |g(x) - g(y)|, \quad \forall x, y.$$

If all the terms in the below inequality exist, then

$$\begin{aligned} & |A(pfg)B(q) + A(p)B(qfh) - A(pf)B(qg) - A(pg)B(qf)| \\ & \leq M (A(pg^2)B(q) - 2A(pg)B(qg) + A(p)B(qg^2)). \end{aligned}$$

Proof. Since f is an $M - g$ -Lipschitz function and g is $1 - g$ -Lipschitz the desired inequality is a simple consequence of Theorem 2. $\square$

In the following table we give a list of papers where particular cases of Theorems from this section can be found.

TABLE 1. Known applications in particular cases of fractional integral operators

Particular cases	Theorem 2	Theorem 3
Riem.-Liouv. oper.	[12], $A = B, p \neq q, h_1 = h_2 = id$	[11, Thm 3.5, Thm 3.7], $p \neq q, A = B$ and $A \neq B$,
Saigo oper.	[23, Thm 2.20] $A \neq B, p \neq q, h_1 = h_2 = id$	[23, Thm 2.19] $A \neq B, p \neq q$
q -Saigo oper.	[23, Thm 3.17] $A \neq B, p \neq q, h_1 = h_2 = id$	[23, Thm 3.16] $A \neq B, p \neq q$,
q -R-L. int. operator	[6, Thm 3.4] $A \neq B, p \neq q, h_1 = h_2 = id$, [6, Thm 3.5] $A = B, p \neq q, h_1 = h_2 = id$	[6, Thm 3.3] $A \neq B, p \neq q$,
Riemann int.	[13, Thm 2.1], $h_1 = h_2 = id$ $A = B, p = q = 1$	[13, Thm 4.1], $A = B, p = q = 1$
time scale $\diamond_\alpha$ integral	[5, Thm 4.1], $A = B, p = q, h_1 = h_2 = id$	[5, Thm 5.1], $A = B, p = q$

Let us say few words how to read the above table. In the first column we write a list of isotonic linear functionals. In the corresponding row we give a reference where applications of our Theorem 2 and Theorem 3 occur. For example, Theorem 2 for two Saigo operators, but with functions h_1 and h_2 equal to identity id can be find in paper [23] as Theorem 2.20 etc.

As we can see, our results enable us to give analogue results for other cases of linear functionals such as for fractional hypergeometric operators, for the Hadamard operators, for other kinds of integrals on time scales etc. Also, we improve existing results by using two different, more general functions h_1 and h_2 instead of identity function id . For example, here we give a result for the Hadamard integral operators. Let $p = q = 1$, $h_1 = h_2 = id$, $\alpha, \beta > 0$. If functions f, g are M_1- , M_2 -Lipschitz respectively, then the following inequality holds

$$\begin{aligned} & \frac{\log^\beta t}{\Gamma(\beta+1)} {}_H J^\alpha(fg(t)) + \frac{\log^\alpha t}{\Gamma(\alpha+1)} {}_H J^\beta(fg(t)) \\ & - {}_H J^\alpha(f(t)) {}_H J^\beta(g(t)) - {}_H J^\alpha(g(t)) {}_H J^\beta(f(t)) \end{aligned}$$

$$\leq M_1 M_2 \frac{t^2}{\Gamma(\alpha)\Gamma(\beta)} \times \left[\frac{\log^\alpha t \cdot \gamma(\beta, 2\log t)}{2^\beta \alpha} + \frac{\log^\beta t \cdot \gamma(\alpha, 2\log t)}{2^\alpha \beta} - 2\gamma(\alpha, \log t)\gamma(\beta, \log t) \right]$$

where $\gamma(s, x)$ is incomplete Gamma function, i.e. $\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt$.

The procedure when we firstly apply on some function $F(x, y)$ functional A with respect to variable x and then apply functional B with respect to variable y occurs very often in this paper. So, we use the following notation: if $F(x, y)$ is a function, then the number which appears after the above-described procedure is written as

$$B_y A_x(F(x, y)) \text{ or } B_y A_x(F).$$

It is worthless to say that if A and B are isotonic linear functionals, then a functional

$$F \mapsto B_y A_x(F)$$

is also an isotonic linear functional.

Theorem 4. *Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . If f is of r -Hölder-type and g is of s -Hölder-type, i.e.*

$$|f(x) - f(y)| \leq H_1 |x - y|^r, \quad |g(x) - g(y)| \leq H_2 |x - y|^s$$

for all $x, y \in E$, where $H_1, H_2 > 0$, $r, s \in (0, 1]$ fixed, then

$$|T(A, B, p, q, f, g)| \leq H_1 H_2 \cdot B_y A_x(p(x)q(y)|x - y|^{r+s}).$$

Proof. It is proved in similar manner as Theorem 2. □

In particular case, when $p = q = 1$, $A(f) = B(f) = \int_a^b f(x)dx$, a factor $B_y A_x(p(x)q(y)|x - y|^{r+s})$ was calculated in [13] and it is equal to

$$\frac{2(b-a)^{r+s+2}}{(r+s+1)(r+s+2)}.$$

3. INEQUALITIES FOR FUNCTIONS WITH VARIABLE BOUNDS

In this section we collect different results for functions with variable and constant bounds. For example, if we look at the paper [19], their Theorem 4 can be seen as a particular case of the following theorem.

Theorem 5. *Let A and B be isotonic linear functionals on L , p, q be non-negative functions from L . Let f, ϕ_1, ϕ_2 , be functions such that*

$$\phi_1(t) \leq f(t) \leq \phi_2(t)$$

and all terms in the below inequality exist. Then

$$A(p\phi_2)B(qf) + A(pf)B(q\phi_1) \geq A(p\phi_2)B(q\phi_1) + A(pf)B(qf).$$

Proof. We consider the inequality

$$(\phi_2(x) - f(x))(f(y) - \phi_1(y)) \geq 0.$$

It is equivalent to the following

$$\phi_2(x)f(y) + f(x)\phi_1(y) \geq \phi_1(y)\phi_2(x) + f(x)f(y).$$

After multiplying with $p(x)q(y)$ and acting on this inequality first by functional A with respect to x and then by B with respect to y we get the wanted inequality. $\square$

If functions ϕ_1, ϕ_2 become constant functions, then we get the following corollary.

Corollary 1. *Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let m, M be real numbers and let f be function such that*

$$m \leq f(t) \leq M$$

and all terms in the below inequality exist. Then

$$MA(p)B(qf) + mA(pf)B(q) \geq MmA(p)B(q) + A(pf)B(qf).$$

Proof. Follows from Theorem 5 for $\phi_1(t) = m, \phi_2(t) = M$. $\square$

Corollary 2. *Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let $M > 0$ and let φ, f be function such that*

$$|f(t) - \varphi(t)| < M$$

and all terms in the below inequality exist. Then

$$\begin{aligned} A(p\varphi)B(qf) + A(pf)B(q\varphi) + MA(p)B(qf) + MA(p\varphi)B(q) + M^2A(p)B(q) \\ \geq A(p\varphi)B(q\varphi) + MA(p)B(q\varphi) + MA(pf)B(q) + A(pf)B(qf). \end{aligned}$$

Proof. Follows from Theorem 5 for $\phi_1(t) = \varphi(t) - M, \phi_2(t) = \varphi(t) + M$. $\square$

Theorem 6. *Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let $\phi_1, \phi_2, \psi_1, \psi_2, f$ and g be functions such that all terms in the below inequality exist and conditions*

$$\phi_1(t) \leq f(t) \leq \phi_2(t) \quad \text{and} \quad \psi_1(t) \leq g(t) \leq \psi_2(t)$$

hold. Then

$$\begin{aligned} A(p\phi_1)B(q\psi_1) + A(pf)B(qg) &\geq A(p\phi_1)B(qg) + A(pf)B(q\psi_1), \\ A(p\phi_1)B(q\psi_2) + A(pf)B(qg) &\leq A(p\phi_1)B(qg) + A(pf)B(q\psi_2), \\ A(p\phi_2)B(q\psi_1) + A(pf)B(qg) &\leq A(p\phi_2)B(qg) + A(pf)B(q\psi_1), \\ A(p\phi_2)B(q\psi_2) + A(pf)B(qg) &\geq A(p\phi_2)B(qg) + A(pf)B(q\psi_2). \end{aligned}$$

Proof. The inequality $(f(x) - \phi_1(x))(g(y) - \psi_1(y)) \geq 0$ which can be written as

$$\phi_1(x)\psi_1(y) + f(x)g(y) \geq \phi_1(x)g(y) + f(x)\psi_1(y)$$

is obviously true.

After multiplying it with $p(x)q(y)$ and acting on this inequality first by functional A with respect to x and then by B with respect to y , we get

$$\begin{aligned} B_y A_x(p(x)q(y)(\phi_1(x)\psi_1(y) + f(x)g(y))) \\ \geq B_y A_x(p(x)q(y)(\phi_1(x)g(y) + f(x)\psi_1(y))) \end{aligned}$$

and applying properties of isotonic linear functionals A and B we obtain the first inequality. The other three inequalities are obtained in a similar way starting from

$$\begin{aligned} (f(x) - \phi_1(x))(\psi_2(y) - g(y)) &\geq 0, \\ (\phi_2(x) - f(x))(g(y) - \psi_1(y)) &\geq 0, \\ (\phi_2(x) - f(x))(\psi_2(y) - g(y)) &\geq 0 \end{aligned}$$

respectively. $\square$

Corollary 3. Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let m, M, n, N be real numbers and let f and g be functions such that

$$m \leq f(t) \leq M \quad \text{and} \quad n \leq g(t) \leq N$$

and all terms in the below inequalities exist. Then

$$\begin{aligned} mnA(p)B(q) + A(pf)B(qg) &\geq mA(p)B(qg) + nA(pf)B(q), \\ mNA(p)B(q) + A(pf)B(qg) &\leq mA(p)B(qg) + NA(pf)B(q), \\ MnA(p)B(q) + A(pf)B(qg) &\leq MA(p)B(qg) + nA(pf)B(q), \\ MNA(p)B(q) + A(pf)B(qg) &\geq MA(p)B(qg) + NA(pf)B(q). \end{aligned}$$

Proof. Follows from Theorem 6 for $\phi_1 = m, \phi_2 = M, \psi_1 = n, \psi_2 = N$. $\square$

Theorem 7. Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let θ_1 and θ_2 be a positive real numbers satisfying $\frac{1}{\theta_1} + \frac{1}{\theta_2} = 1$. Let f, ϕ_1, ϕ_2 , be functions such that

$$\phi_1(t) \leq f(t) \leq \phi_2(t).$$

and all terms in the below inequality exist. Then

$$\begin{aligned} \frac{1}{\theta_1} B(q) A(p(\phi_2 - f)^{\theta_1}) + \frac{1}{\theta_2} A(p) B(q(f - \phi_1)^{\theta_2}) + A(p\phi_2) B(q\phi_1) \\ + A(pf) B(qf) \geq A(p\phi_2) B(qf) + A(pf) B(q\phi_1). \end{aligned} \quad (3.1)$$

Proof. Let us mention the Young inequality which holds for non-negative a, b and for positive θ_1 and θ_2 with property $\frac{1}{\theta_1} + \frac{1}{\theta_2} = 1$:

$$\frac{1}{\theta_1}a^{\theta_1} + \frac{1}{\theta_2}b^{\theta_2} \geq ab.$$

Setting in the previous inequality

$$a = \phi_2(x) - f(x), \quad b = f(y) - \phi_1(y)$$

we have

$$\frac{1}{\theta_1}(\phi_2(x) - f(x))^{\theta_1} + \frac{1}{\theta_2}(f(y) - \phi_1(y))^{\theta_2} \geq (\phi_2(x) - f(x))(f(y) - \phi_1(y)).$$

Applying usual procedure we get

$$\begin{aligned} B_y A_x \left(p(x)q(y) \left(\frac{1}{\theta_1} (\phi_2(x) - f(x))^{\theta_1} + \frac{1}{\theta_2} (f(y) - \phi_1(y))^{\theta_2} \right) \right. \\ \left. + B_y A_x \left(p(x)q(y)(\phi_2(x) - f(x))(f(y) - \phi_1(y)) \right) \right) \end{aligned}$$

and after applying properties of A and B we obtain

$$\frac{1}{\theta_1} B(q) A(p(\phi_2 - f)^{\theta_1}) + \frac{1}{\theta_2} A(p) B(q(f - \phi_1)^{\theta_2}) \geq A(p(\phi_2 - f)) B(q(f - \phi_1)).$$

Using result from Theorem 5 we get inequality (3.1). $\square$

Corollary 4. Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let $m, n \in \mathbf{R}$ and let f be function such that $m \leq f(t) \leq M$ and all terms in the below inequality exist. Then

$$\begin{aligned} (M + m)^2 A(p) B(q) + A(pf^2) B(q) + 2A(pf) B(qf) + A(p) B(qf^2) \\ \geq 2(M + m)[A(p) B(qf) + A(pf) B(q)]. \end{aligned}$$

Proof. Follows from Theorem 7 for $\phi_1 = m, \phi_2 = M$ and $\theta_1 = \theta_2 = 2$. $\square$

Theorem 8. Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let θ_1 and θ_2 be a positive real numbers satisfying $\frac{1}{\theta_1} + \frac{1}{\theta_2} = 1$. Let $\phi_1, \phi_2, \psi_1, \psi_2, f$ and g be functions such that all terms in the below inequality exist and conditions

$$\phi_1(t) \leq f(t) \leq \phi_2(t) \quad \text{and} \quad \psi_1(t) \leq g(t) \leq \psi_2(t)$$

hold. Then

$$\begin{aligned} \frac{1}{\theta_1} A(p(\phi_2 - f)^{\theta_1}) B(q) + \frac{1}{\theta_2} A(p) B(q(\psi_2 - g)^{\theta_2}) &\geq A(p(\phi_2 - f)) B(q(\psi_2 - g)), \\ \frac{1}{\theta_1} A(p(\phi_2 - f)^{\theta_1}) B(q) + \frac{1}{\theta_2} A(p) B(q(g - \psi_1)^{\theta_2}) &\geq A(p(\phi_2 - f)) B(q(g - \psi_1)), \end{aligned}$$

$$\begin{aligned} \frac{1}{\theta_1} A(p(f - \phi_1)^{\theta_1}) B(q) + \frac{1}{\theta_2} A(p) B(q(\psi_2 - g)^{\theta_2}) &\geq A(p(f - \phi_1)) B(q(\psi_2 - g)), \\ \frac{1}{\theta_1} A(p(f - \phi_1)^{\theta_1}) B(q) + \frac{1}{\theta_2} A(p) B(q(g - \psi_1)^{\theta_2}) &\geq A(p(f - \phi_1)) B(q(g - \psi_1)). \end{aligned}$$

Proof. Using the Young inequality for $a = \phi_2(x) - f(x)$, $b = \psi_2(y) - g(y)$ we get

$$\frac{1}{\theta_1} (\phi_2(x) - f(x))^{\theta_1} + \frac{1}{\theta_2} (\psi_2(y) - g(y))^{\theta_2} \geq (\phi_2(x) - f(x)) (\psi_2(y) - g(y)).$$

Multiplying both sides with $p(x)q(y)$ and acting on the inequality by functional A with respect to x and then by B with respect to y , we get

$$\begin{aligned} \frac{1}{\theta_1} A(p(\phi_2 - f)^{\theta_1}) B(q) + \frac{1}{\theta_2} A(p) B(q(\psi_2 - g)^{\theta_2}) \\ \geq A(p(\phi_2 - f)) B(q(\psi_2 - g)). \end{aligned}$$

The other three inequalities can be proved in the similar way. $\square$

In the following table we give a list of papers where particular cases of some Theorems from this section can be found.

TABLE 2. Known applications for particular cases of fractional integral operators

Particular cases	Theorem 5	Corollary 1	Theorem 6
R-L oper.	[21, Thm 2], $A \neq B, p = q = 1$	[21, Cor 3], $A \neq B, p = q = 1$	[21, Thm 5], $A \neq B, p = q = 1$
E-K oper.	[22, Cor 2] $A \neq B, p = q = 1$		[22, Cor 3] $A \neq B, p = q = 1$
Saigo oper.	[8, Thm 8] $A \neq B, p = q = 1$ [22, Thm1] $A \neq B, p = q = 1$	[8, remark 9] $A \neq B, p = q = 1$ [22, Cor 4] $A \neq B, p = q = 1$	[8, Thm 10] $A \neq B, p = q = 1$ [22, Thm2] $A \neq B, p = q = 1$
Had. oper.	[19, Thm 4], $A \neq B, p = q = 1$	[19, Cor 5], $A \neq B, p = q = 1$	[19, Thm 18], $A \neq B, p = q = 1$

Let us mention that the non-weighted versions of Theorems 7, 8 and Corollary 4 for two Hadamard operators $A = {}_H J^\alpha$ and $B = {}_H J^\beta$ can be found in paper [19].

As we can see, we did not find similar results for hypergeometric operators in literature. But, it is obvious that our results can be applied on two fractional hypergeometric operator or on q -analogues of those integral operators.

4. INEQUALITIES FOR THREE FUNCTIONS

This section is devoted to the results involving three or more functions and in some sense it generalize the previous section.

Theorem 9. *Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let f, g be similarly ordered functions and let h be function with positive values. If all terms in the below inequality exist, then*

$$\begin{aligned} & A(p f g h) B(q) + A(p f g) B(q h) + A(p h) B(q f g) + A(p) B(q f g h) \\ & \geq A(p f h) B(q g) + A(p f) B(q g h) + A(p g h) B(q f) + A(p g) B(q f h). \end{aligned} \quad (4.1)$$

If f and g are oppositely ordered, then the reversed inequality holds.

Proof. For similarly ordered functions f, g we have $(f(x) - f(y))(g(x) - g(y)) \geq 0$, but then also

$$(f(x) - f(y))(g(x) - g(y))(h(x) + h(y)) \geq 0.$$

This can be written as

$$\begin{aligned} & f(x)g(x)h(x) + f(x)g(x)h(y) + f(y)g(y)h(x) + f(y)g(y)h(y) \\ & \geq f(x)g(y)h(x) + f(x)g(y)h(y) + f(y)g(x)h(x) + f(y)g(x)h(y). \end{aligned}$$

Multiplying both sides by $p(x)q(y)$ and acting on this inequality first by functional A with respect to x and then by B with respect to y we get the desired inequality. $\square$

Remark 1. For $p = q$ from previous theorem we get

$$\begin{aligned} & A(p f g h) B(p) + A(p f g) B(p h) + A(p h) B(p f g) + A(p) B(p f g h) \\ & \geq A(p f h) B(p g) + A(p f) B(p g h) + A(p g h) B(p f) + A(p g) B(p f h). \end{aligned}$$

If also $A = B$ then

$$A(p f g h) A(p) + A(p f g) A(p h) \geq A(p f h) A(p g) + A(p f) A(p g h)$$

and for $h = \text{const}$ it reduces to the Chebyshev inequality.

Remark 2. Particular cases of inequality (4.1) are appeared in several papers for different kinds of linear functionals. For example, if A and B are Riemann-Liouville's operators, then a non-weighted inequality is given in Theorem 2.1 in paper [20]. If A and B are different fractional q -integral of the Riemann-Liouville-type, then (4.1) is given in [18, Thm 2.1] for $p = q$. Inequalities involving two q -analogues of Saigo's fractional integral operators which are particular cases of (4.1) are given in [2] as Theorems 5 and 6, while similar results for generalized q -Erdélyi-Kober fractional integral operators are given in [17, Thm 1 and 2].

Lemma 1. Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let

$$\begin{aligned} H_{f,g,h}(x, y) &= (f(x) - f(y))(g(x) - g(y))(h(x) - h(y)) \\ &= f(x)g(x)h(x) + f(x)g(y)h(y) + f(y)g(x)h(y) + f(y)g(y)h(x) \\ &\quad - f(y)g(x)h(x) - f(x)g(y)h(x) - f(x)g(x)h(y) - f(y)g(y)h(y). \end{aligned}$$

Then

$$\begin{aligned} A_x B_y(p(x)q(y)H_{f,g,h}(x, y)) \\ &= A(pfgh)B(q) + A(pf)B(qgh) + A(pg)B(qfh) + A(ph)B(qfg) \\ &\quad - A(pgh)B(qf) - A(pfh)B(qg) - A(pfg)B(qh) - A(p)B(qfgh). \end{aligned}$$

Theorem 10. Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let m, M, n, N, k, K be real numbers and let f, g, h be functions such that

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N, \quad k \leq h(x) \leq K, \quad \forall x.$$

If all the terms in the below inequality exist, then

$$\begin{aligned} &|A(pfgh)B(q) + A(pf)B(qgh) + A(pg)B(qfh) + A(ph)B(qfg) \\ &\quad - A(pfg)B(qh) - A(pfh)B(qg) - A(pgh)B(qf) - A(p)B(fgh)| \\ &\leq (M - m)(N - n)(K - k)A(p)B(q). \end{aligned}$$

Proof. From $m \leq f(x) \leq M$ it follows $|f(x) - f(y)| \leq M - m$. Therefore

$$|(f(x) - f(y))(g(x) - g(y))(h(x) - h(y))| \leq (M - m)(N - n)(K - k), \text{ i.e.}$$

$$|H_{f,g,h}(x, y)| \leq (M - m)(N - n)(K - k).$$

Multiplying both sides by $p(x)q(y)$ and using Lemma 1 we get

$$\begin{aligned} &|A(pfgh)B(q) - A(pfg)B(qh) - A(pfh)B(qg) + A(pf)B(qgh) \\ &\quad - A(pgh)B(qf) + A(pg)B(qfh) + A(ph)B(qfg) - A(p)B(fgh)| \\ &\leq (M - m)(N - n)(K - k)A(p)B(q), \end{aligned}$$

which proves the theorem. $\square$

Remark 3. Particular cases of Theorem 10 are appeared in [2, Thm 8 and 9] for two q -analogues of Saigo's fractional integral operators and in [17, Thm 3 and 4] for generalized q -Erdélyi-Kober fractional integral operators.

Theorem 11. Let A and B be isotonic linear functionals on L and let p, q be non-negative functions from L . Let M_1, M_2, M_3 be real numbers and let f_i , ($i = 1, 2, 3$) be $M_i - g$ -Lipschitz functions. If all the terms in the below inequality exist, then

$$|A(pf_1 f_2 f_3)B(q) + A(pf_1)B(qf_2 f_3) + A(pf_2)B(qf_1 f_3) + A(pf_3)B(qf_1 f_2)$$

$$-A(pf_1 f_2)B(qf_3) - A(pf_1 f_3)B(qf_2) - A(pf_2 f_3)B(qf_1) - A(p)B(qf_1 f_2 f_3)| \\ \leq M_1 M_2 M_3 \cdot B_y A_x(p(x)q(y)|g(x) - g(y)|^3).$$

Proof. If f_i , ($i = 1, 2, 3$) are $M_i - g$ -Lipschitz functions, then

$$|f_1(x) - f_1(y)| \leq M_1 |g(x) - g(y)|, \quad |f_2(x) - f_2(y)| \leq M_2 |g(x) - g(y)|, \\ |f_3(x) - f_3(y)| \leq M_3 |g(x) - g(y)|$$

for all x, y . Multiplying those inequalities we get

$$|H_{f_1, f_2, f_3}(x, y)| \leq M_1 M_2 M_3 |g(x) - g(y)|^3.$$

This is equivalent to

$$H_{f_1, f_2, f_3}(x, y) \leq M_1 M_2 M_3 |g(x) - g(y)|^3 \quad \text{and} \\ -H_{f_1, f_2, f_3}(x, y) \leq M_1 M_2 M_3 |g(x) - g(y)|^3.$$

Multiplying both inequalities with $p(x)q(y)$ and acting on the resulting inequalities by A with respect to x and then by B with respect to y , we get the desired result. $\square$

Remark 4. In [2, Thm 11 and 12] and [17, Thm 5 and 6] the authors attempted to give corresponding results for q -analogues of Saigo's fractional integral operators and for generalized q -Erdélyi-Kober fractional integral operators, respectively. But they used assumptions $|f_i(x) - f_i(y)| \leq M_i(x - y)$, $i = 1, 2, 3$, $x, y > 0$ which leads to conclusion that $f_i \equiv 0$.

Remark 5. Considering results from this and from the previous section it is clear how Theorems 2 and 11 can be generalized for n $M_i - g$ -Lipschitz functions f_i , $i = 2, \dots, n$. We leave it to a reader.

Results with two functions and three weights

The following result is based on the successive using of the Chebyshev inequality for pairs of weights.

Theorem 12. *Let A and B be isotonic linear functionals on L and let p, q, r be non-negative functions from L . If f and g are similarly ordered functions, then*

$$A(p)[2A(q)B(rfg) + A(r)B(qfg) + B(r)A(qfg)] \\ + A(pfg)[A(q)B(r) + A(r)B(q)] \\ \geq A(p)[A(qf)B(rg) + A(qg)B(rf)] + A(q)[A(pf)B(rg) + A(pg)B(rf)] \\ + A(r)[A(pf)B(qg) + A(pg)B(qf)],$$

under assumptions that all terms are well-defined.

If f and g are oppositely ordered functions, then the reversed inequality holds.

Proof. Replacing in (1.3) p by q and q by r and multiplying by $A(p)$ we get

$$A(p)[A(q)B(rfg) + B(r)A(qfg)] \geq A(p)[A(qf)B(rg) + A(qg)B(rf)].$$

Replacing in (1.3) q by r and multiplying by $A(q)$ we get

$$A(q)[A(p)B(rfg) + B(r)A(pfg)] \geq A(q)[A(pf)B(rg) + A(pg)B(rf)].$$

Multiplying (1.3) by $A(r)$ we get

$$A(r)[A(p)B(qfg) + B(q)A(pfg)] \geq A(r)[A(pf)B(qg) + A(pg)B(qf)].$$

Adding the above inequalities we get the statement of the theorem. $\square$

Remark 6. Theorem 12 are proved in several papers for different kinds of linear operators. For example, if A and B are the Riemann-Liouville operators, then it is given in [10]. Result involving Hadamard operators is given in [7], while an analogue result for the Saigo operators and q -analogue of Saigo's operators are given in [9] and [23].

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A GENERALIZATION OF J -QUASIPOLAR RINGS

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Abstract. In this paper we introduce a class of quasipolar rings which is a generalization of J -quasipolar rings. Let R be a ring with identity. An element $a \in R$ is called δ -quasipolar if there exists $p^2 = p \in \text{comm}^2(a)$ such that $a + p$ is contained in $\delta(R)$, and the ring R is called δ -quasipolar if every element of R is δ -quasipolar. We use δ -quasipolar rings to extend some results of J -quasipolar rings. Then some of the main results of J -quasipolar rings are special cases of our results for this general setting. We give many characterizations and investigate general properties of δ -quasipolar rings.

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1. INTRODUCTION

Throughout this paper all rings are associative with identity unless otherwise stated. Let R be a ring. According to Koliha and Patricio [10], the *commutant* and *double commutant* of an element $a \in R$ are defined by $\text{comm}(a) = \{x \in R \mid xa = ax\}$, $\text{comm}^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in \text{comm}(a)\}$, respectively. If $R^{qnil} = \{a \in R \mid 1 + ax \in U(R) \text{ for every } x \in \text{comm}(a)\}$ and $a \in R^{qnil}$, then a is said to be *quasinilpotent* (see [9]). The element a is called *quasipolar* if there exists $p^2 = p \in R$ such that $p \in \text{comm}^2(a)$, $a + p$ is invertible in R and $ap \in R^{qnil}$. Any idempotent p satisfying the above conditions is called a *spectral idempotent* of a , and this term is borrowed from spectral theory in Banach algebra and it is unique for a .

Quasipolar rings have been studied by many ring theorists (see [1, 2, 5–7, 9, 10] and [15]). In [7], the element $a \in R$ is called *nil-quasipolar* if there exists $p^2 = p \in \text{comm}^2(a)$ such that $a + p$ is nilpotent, the idempotent p is called a *nil-spectral idempotent* of a . The ring R is said to be *nil-quasipolar* if every element of R is nil-quasipolar. Recently, J -quasipolar rings are studied in [4]. The element a is called *J -quasipolar* if there exists $p^2 = p \in R$ such that $p \in \text{comm}^2(a)$ and $a + p \in J(R)$, p is called a *J -spectral idempotent* of a . The ring R is said to be *J -quasipolar* if

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every element of R is J -quasipolar. Motivated by these, we introduce a new class of quasipolar rings which is a generalization of J -quasipolar rings. By using δ -quasipolar rings, we extend some results of J -quasipolar rings.

An outline of the paper is as follows: Section 2 deals with δ -quasipolar rings. We prove various basic characterizations and properties of δ -quasipolar rings. It is proven that every J -quasipolar ring is δ -quasipolar. We supply an example to show that all δ -quasipolar rings need not be J -quasipolar. Among others the δ -quasipolarity of Dorroh extensions and some classes of matrix rings are investigated. In Section 3, we introduce an upper class of δ -quasipolar rings, namely, weakly δ -quasipolar rings. We show that every direct summand of a weakly δ -quasipolar ring is weakly δ -quasipolar and every direct product of weakly δ -quasipolar rings is weakly δ -quasipolar, and we give some properties of such rings.

In what follows, $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and the ring of rational numbers and for a positive integer n , $\mathbb{Z}_n$ is the ring of integers modulo n . For a positive integer n , let $Mat_n(R)$ denote the ring of all $n \times n$ matrices and $T_n(R)$ the ring of all $n \times n$ upper triangular matrices with entries in R . We write $J(R)$ and $nil(R)$ for the Jacobson radical of R and the set of nilpotent elements of R , respectively.

2. δ -QUASIPOLAR RINGS

In this section we introduce the concept of δ -quasipolar rings and investigate some properties of such rings. We show that every quasipolar ring need not be δ -quasipolar (Example 2). It is proven that every J -quasipolar ring is δ -quasipolar and the converse does not hold in general (see Example 3). Among others we extend some results of J -quasipolar rings for this general setting.

A right ideal I of the ring R is said to be δ -small in R if whenever $R = I + K$ with R/K singular right R -module for any right ideal K then $R = K$. In [16], the ideal $\delta(R)$ is introduced as a sum of δ -small right ideals of R . We begin with the equivalent conditions for $\delta(R)$ which is proved in [16, Theorem 1.6] for an easy reference for the reader.

Lemma 1. *Given a ring R , each of the following sets is equal to $\delta(R)$.*

- (1) $R_1 =$ the intersection of all essential maximal right ideals of R .
- (2) $R_2 =$ the unique largest δ -small right ideal of R .
- (3) $R_3 = \{x \in R \mid xR + K_R = R \text{ implies } K_R \text{ is a direct summand of } R_R\}$.
- (4) $R_4 = \bigcap \{\text{ideals } P \text{ of } R \mid R/P \text{ has a faithful singular simple module}\}$.
- (5) $R_5 = \{x \in R \mid \text{for all } y \in R \text{ there exists a semisimple right ideal } Y \text{ of } R \text{ such that } (1 + xy)R \oplus Y = R_R\}$.

Now we give our main definition.

Definition 1. Let R be a ring. An element $a \in R$ is called δ -quasipolar if there exists $p^2 = p \in \text{comm}^2(a)$ such that $a + p \in \delta(R)$ and p is called a δ -spectral idempotent. The ring R is called δ -quasipolar if every element of R is δ -quasipolar.

The following are examples for δ -quasipolar rings.

Example 1. (1) Every semisimple ring and every Boolean ring is δ -quasipolar.
(2) Since $\delta(\mathbb{Q}) = \mathbb{Q}$, $\mathbb{Q}$ is δ -quasipolar. On the other hand, $\mathbb{Z}$ is not δ -quasipolar since $\delta(\mathbb{Z}) = 0$.

One may suspects that every quasipolar ring is δ -quasipolar. But the following example erases the possibility.

Example 2. Let p be a prime integer with $p \geq 3$ and $R = \mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ at the ideal (p) . By [4, Example 2.8], R is a quasipolar ring. Since $J(R) = \delta(R)$, it is not δ -quasipolar.

Let S_r denote the right socle of the ring R , that is, S_r is the sum of minimal right ideals of R . We now prove that the class of J -quasipolar rings is a subclass of δ -quasipolar rings.

Lemma 2. *If R is a J -quasipolar ring, then R is δ -quasipolar. The converse holds if $S_r \subseteq J(R)$.*

Proof. The first assertion is clear since $J(R) \subseteq \delta(R)$. Assume that R is δ -quasipolar. If $S_r \subseteq J(R)$, then $J(R)/S_r = J(R/S_r) = \delta(R)/S_r$ by [16, Corollary 1.7] and we have $J(R) = \delta(R)$. Hence, R is J -quasipolar. $\square$

The converse of Lemma 2 is not true in general as the following example shows.

Example 3. Let F be a field and consider the ring $R = \begin{bmatrix} F & F \\ F & F \end{bmatrix}$. Then R is a semisimple ring and $R = \delta(R)$ and $J(R) = 0$. Hence R is δ -quasipolar and it is not J -quasipolar.

Lemma 3. *Let R be a ring. Then we have the following.*

- (1) *If $a, u \in R$ and u is invertible, then a is δ -quasipolar if and only if $u^{-1}au$ is δ -quasipolar.*
- (2) *The element $a \in R$ is δ -quasipolar if and only $-1 - a$ is δ -quasipolar.*
- (3) *If R is a δ -quasipolar ring with $\delta(R) = J(R)$, then the spectral idempotent for any invertible element in R is the identity of R .*

Proof. (1) Assume that a is δ -quasipolar. Let $p^2 = p \in \text{comm}^2(a)$ such that $a + p \in \delta(R)$. Let $x \in \text{comm}(u^{-1}au)$. Then $(uxu^{-1})a = a(uxu^{-1})$. Since $p \in \text{comm}^2(a)$, $(uxu^{-1})p = p(uxu^{-1})$. Hence $(u^{-1}pu)^2 = u^{-1}pu \in \text{comm}^2(u^{-1}au)$. Since $\delta(R)$ is an ideal of R , $u^{-1}(a + p)u = u^{-1}au + u^{-1}pu \in \delta(R)$. Thus $u^{-1}au$ is δ -quasipolar. Conversely, if $u^{-1}au$ is δ -quasipolar, then by the preceding proof

$u(u^{-1}au)u^{-1} = a$ is δ -quasipolar.

(2) Assume that a is δ -quasipolar. Let $p^2 = p \in \text{comm}^2(a)$ such that $a + p = r \in \delta(R)$. Then $-1 - a + (1 - p) = -r \in \delta(R)$. Then $1 - p \in \text{comm}^2(-1 - a)$ and $1 - p$ is the spectral idempotent of $-1 - a$. Conversely, if $-1 - a$ is δ -quasipolar, then from what we have proved that $-1 - (-1 - a) = a$ is quasipolar.

(3) Assume that $\delta(R) = J(R)$. Then δ -quasipolarity of R implies J -quasipolarity of R . So its proof can be directly obtained from [4, Example 2.2]. $\square$

In [4, Corollary 2.3], it is proved that if R is a J -quasipolar ring, then $2 \in J(R)$. In this direction we prove the following.

Lemma 4. *If R is a δ -quasipolar ring, then $2 \in \delta(R)$.*

Proof. For the identity 1, there exists $p^2 = p \in R$ such that $1 + p \in \delta(R)$. Multiplying the latter by p , we have $2p \in \delta(R)$. So $2 = 2(1 + p) - 2p \in \delta(R)$. $\square$

Lemma 4 can be used to determine whether given rings are δ -quasipolar.

Example 4. (1) The ring $\mathbb{Z}_3$ is a semisimple ring and δ -quasipolar but the ring $R = \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{bmatrix}$ is not δ -quasipolar since $\delta(R) = \begin{bmatrix} 0 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{bmatrix}$ and 2 does not contained in $\delta(R)$.

(2) Let $R = \{(a_{ij}) \in T_n(\mathbb{Z}_3) \mid a_{11} = a_{22} = \dots = a_{nn}\}$. $\mathbb{Z}_3$ is δ -quasipolar but R is not since $\delta(R) = \{(a_{ij}) \in T_n(\mathbb{Z}_3) \mid a_{11} = a_{22} = \dots = a_{nn} = 0\}$ and 2 does not contained in $\delta(R)$.

Recall that a ring R is called *local* if it has only one maximal left ideal, equivalently, maximal right ideal.

Proposition 1. *Let R be a local ring. If $R/J(R) \cong \mathbb{Z}_2$, then R is δ -quasipolar.*

Proof. Let $a \in R$. If $a \in J(R)$, it is clear. Assume that $a \notin J(R)$. Since R is local, a is invertible. Hence $a + 1 \in \delta(R)$ by $\delta(R) = J(R)$. $\square$

A ring R is said to be *clean* [12] if for each $a \in R$ there exists $e^2 = e \in R$ such that $a - e$ is invertible, and R is called *strongly clean* [13] provided that every element of R can be written as the sum of an idempotent and an invertible element that commute.

Example 5. Let $R = \{(q_1, q_2, q_3, \dots, q_n, a, a, a, \dots) \mid n \geq 1; q_i \in \mathbb{Q}; a \in \mathbb{Z}_{(2)}\}$. Then R is strongly clean but not quasipolar (see [15, Example 3.4(3)]). Therefore R is not J -quasipolar since every J -quasipolar ring is quasipolar. On the other hand, since $S_r = 0$ and $\delta(R)/S_r = J(R)/S_r$, $\delta(R) = J(R)$. Thus R is not δ -quasipolar.

In [4, Theorem 2.9], it is shown that if the ring R is J -quasipolar, then $R/J(R)$ is Boolean and idempotents in $R/J(R)$ lift R . We have the following result for δ -quasipolar rings.

Theorem 1. *If R is a δ -quasipolar ring, then $R/\delta(R)$ is a Boolean ring and idempotents in $R/\delta(R)$ lift R .*

Proof. Let $\bar{a} \in R/\delta(R)$. There exists $p^2 = p \in \text{comm}^2(-1 + a)$ such that $-1 + a + p \in \delta(R)$. Hence $\bar{a} = \bar{1} - \bar{p}$ is an idempotent in $R/\delta(R)$ and $R/\delta(R)$ is a Boolean ring. Let $\bar{a}^2 = \bar{a} \in R/\delta(R)$. Then there exists $p^2 = p \in \text{comm}^2(-a)$ such that $-a + p \in \delta(R)$. This yields $\bar{a} = \bar{p}$, as asserted. $\square$

The concept of δ_r -clean rings are defined in [8]. A ring R is called δ_r -clean if for every element $a \in R$ there exists an idempotent $e \in R$ such that $a - e \in \delta(R)$. A ring is *abelian* if all idempotents are central.

Lemma 5. *If R is a δ -quasipolar ring, then it is δ_r -clean. The converse holds if R is abelian.*

Proof. Let R be a δ -quasipolar ring and $a \in R$. There exists $p^2 = p \in \text{comm}^2(-1 + a)$ such that $-1 + a + p \in \delta(R)$. Then $a - (1 - p) \in \delta(R)$. For the converse, assume that R is abelian. Let $a \in R$. There exists an idempotent e such that $1 + a - e \in \delta(R)$. By assumption, $1 - e$ is a central idempotent and so $1 - e \in \text{comm}^2(a)$. $\square$

Recall that a ring R is *exchange* if for every $a \in R$, there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$. Namely, von Neumann regular rings and clean rings are exchange.

Corollary 1. *Let R be a δ -quasipolar ring. Then*

- (1) R is an exchange ring.
- (2) $R/\delta(R)$ is a clean ring.

Proof. (1) Let R be a δ -quasipolar ring. By Lemma 5, R is a δ_r -clean ring. By [8, Theorem 2.2(2)], every δ_r -clean ring is an exchange ring.
 (2) By Theorem 1, $R/\delta(R)$ is Boolean, therefore, it is clean. $\square$

Corollary 2. *Consider following conditions for a ring R .*

- (1) R is δ -quasipolar and $\delta(R) = 0$.
- (2) R is Boolean.
- (3) R is von Neumann regular and δ -quasipolar.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) Assume that R is δ -quasipolar and $\delta(R) = 0$. By Theorem 1, R is Boolean.

(2) $\Rightarrow$ (3) Assume that R is Boolean. Then it is commutative with characteristic 2 and $a^2 + a = 0 \in \delta(R)$ and $a^2 = a = a^3$ for all $a \in R$. Hence R is von Neumann regular and δ -quasipolar. $\square$

Strongly J -clean rings were introduced by Chen in [3]. For a ring R the element $a \in R$ is called *J -clean* if a is the sum of an idempotent and a radical element in

its Jacobson radical. The ring R is called J -clean if every element is a sum of an idempotent and a radical element.

Theorem 2. *If R is an abelian J -clean ring, then it is δ -quasipolar.*

Proof. Let $a \in R$. Then we have $-a \in R$. Since R is J -clean, there exist $e^2 = e \in R$ and $j \in J(R)$ such that $-a = e + j$. Hence $a + e \in J(R)$. Since R is abelian, $e^2 = e \in \text{comm}^2(a)$ and $J(R) \subseteq \delta(R)$, R is δ -quasipolar as asserted. $\square$

All δ -quasipolar rings need not be Boolean and the converse statement of Theorem 2 is not true in general.

Example 6. The ring $\mathbb{Z}_3$ is semisimple and so $\mathbb{Z}_3 = \delta(\mathbb{Z}_3)$. Therefore $\mathbb{Z}_3$ is δ -quasipolar, but it is neither Boolean nor J -clean.

In [4, Proposition 2.11], it is shown that a ring R is local and J -quasipolar if and only if R is J -quasipolar with only trivial idempotents if and only if $R/J(R) \cong \mathbb{Z}_2$. We have the following for δ -quasipolar rings.

Proposition 2. *Let R be a ring with only trivial idempotents. Then R is δ -quasipolar if and only if $R/\delta(R) \cong \mathbb{Z}_2$.*

Proof. Assume that R is δ -quasipolar. Let $a \in R$. There exists an idempotent $p \in \text{comm}^2(a)$ such that $-a + p \in \delta(R)$. By hypothesis $p = 1$ or $p = 0$. If $\delta(R) = 0$, then $R/\delta(R) \cong \mathbb{Z}_2$. Suppose that $\delta(R) \neq 0$. For any $a \in R \setminus \delta(R)$, $\bar{a} = \bar{1} \in R/\delta(R)$. Hence $R/\delta(R) \cong \mathbb{Z}_2$. Conversely, suppose that $R/\delta(R)$ is isomorphic to $\mathbb{Z}_2$ by isomorphism f . Let $a \in R \setminus \delta(R)$. Then $f(-\bar{a}) = \bar{1} \in \mathbb{Z}_2$. Then $f(-\bar{a}) = f(\bar{1})$ implies $-\bar{a} - \bar{1} \in \text{Ker } f = 0$. Hence $-\bar{a} = \bar{1}$. That is, $a + 1 \in \delta(R)$. Thus R is δ -quasipolar. $\square$

Recall that a ring R is called *strongly π -regular* if for every element a of R there exist a positive integer n (depending on a) and an element x of R such that $a^n = a^{n+1}x$, equivalently, an element y of R such that $a^n = ya^{n+1}$. In spite of the fact that $J(R)$ is contained in both $\delta(R)$ and R^{qnil} , no comparings between $\delta(R)$ and R^{qnil} exist. Strongly π -regular rings play crucial role in this direction.

Proposition 3. *Let R be a δ -quasipolar ring and $\delta(R) = J(R)$. Then R is strongly π -regular if and only if $J(R) = R^{qnil} = \text{nil}(R) = \delta(R)$.*

Proof. Necessity. Let $a \in R^{qnil}$. Then for any $x \in \text{comm}(a)$, $1 - ax$ is invertible. By hypothesis, there exist a positive integer m and $b \in R$ such that $a^m = a^{m+1}b$. Since $b \in \text{comm}(a)$ by [11, Page 347, Exercise 23.6(1)], $a^m = 0$. Hence $a \in \text{nil}(R)$ and so $R^{qnil} \subseteq \text{nil}(R)$. To prove $\text{nil}(R) \subseteq \delta(R)$, let $a \in \text{nil}(R)$. By hypothesis there exists $p^2 = p \in \text{comm}^2(1 - a)$ such that $1 - a + p \in \delta(R)$. Since $1 - a$ is invertible, $p = 1$ by Lemma 3 (3). Hence $2 - a \in \delta(R)$. Also $2 \in \delta(R)$ by Lemma 4, we then have $a \in \delta(R)$.

Sufficiency. Let $a \in R$. There exists $p^2 = p \in \text{comm}^2(-1 + a)$ such that $-1 + a + p \in \delta(R)$. Set $u = -1 + a + p \in \text{nil}(R)$. Then $a + p$ is invertible and $ap = up$ is nilpotent so that $a^n p = 0$ for some positive integer n . So $a^n = a^n(1 - p) = (u + (1 - p))^n(1 - p) = (u + 1)^n(1 - p) = (a + p)^n(1 - p) = (1 - p)(a + p)^n$. By [13, Proposition 1], a is strongly π -regular. This completes the proof. $\square$

Let R and V be rings and V be an (R, R) -bimodule that is also a ring with $(vw)r = v(wr)$, $(vr)w = v(rw)$, and $(rv)w = r(vw)$ for all $v, w \in V$ and $r \in R$. The Dorroh extension $D(R, V)$ of R by V defined as the ring consisting of the additive abelian group $R \oplus V$ with multiplication $(r, v)(s, w) = (rs, rw + vs + vw)$ where $r, s \in R$ and $v, w \in V$.

Uniquely clean rings were introduced by Nicholson and Zhou in [14]. A ring R is *uniquely clean* in case for any $a \in R$ there exists a unique idempotent $e \in R$ such that $a - e \in R$ is invertible. In [8], among others, uniquely δ_r -clean rings are studied. A ring R is called *uniquely δ_r -clean* if for every element $a \in R$ there exists a unique idempotent $e \in R$ such that $a - e \in \delta(R)$. Uniquely clean Dorroh extensions in [14, Proposition 7] and uniquely δ_r -clean Dorroh extensions in [8, Proposition 3.11] are considered. Now we consider δ -quasipolar Dorroh extensions.

Proposition 4. *Let R be a ring. Then we have the following.*

- (1) *If $D(R, V)$ is δ -quasipolar, then R is δ -quasipolar.*
- (2) *If the following conditions are satisfied, then $D(R, V)$ is δ -quasipolar.*
 - (i) *R is δ -quasipolar;*
 - (ii) *$e^2 = e \in R$, then $ev = ve$ for all $v \in V$;*
 - (iii) *$V = \delta(V)$.*

Proof. (1) Let $r \in R$. There exists $e^2 = e \in D(R, V)$ such that $e \in \text{comm}^2(r, 0)$ and $(r, 0) + e \in \delta(D(R, V))$. Since $e \in D(R, V)$, e has the form such that $(p, v)^2 = (p, v)$ and $p^2 = p$. Then $e = (p, v) \in \text{comm}^2(r, 0)$ implies that $p \in \text{comm}^2(r)$ and $r + p \in \delta(R)$ since $(r + p, v) \in \delta(D(R, V))$ and by [8, Proposition 3.11]. Hence R is δ -quasipolar.

(2) Assume that (i), (ii) and (iii) hold. Let $(r, v) \in D(R, V)$. There exists $p^2 = p \in \text{comm}^2(r)$ such that $r + p \in \delta(R)$. By (iii), $(0, V) \subseteq \delta(D(R, V))$. Then $(r, v) + (p, 0) = (r + p, v) \in \delta(D(R, V))$. To see that $(p, 0) \in \text{comm}^2((r, v))$, let $(a, b) \in D(R, V)$ and $(a, b)(r, v) = (r, v)(a, b)$. Then $ar = ra$ and so $ap = pa$ since $p \in \text{comm}^2(r)$. Also $pb = bp$ by (ii). Therefore we have $(p, 0)(a, b) = (a, b)(p, 0)$ that is $(p, 0) \in \text{comm}^2((r, v))$. $\square$

As an application of Dorroh extensions we consider the following example. This example also shows that in Proposition 4 (2), the conditions (i), (ii) and (iii) are not superfluous.

Example 7. Consider the ring $D(\mathbb{Z}, \mathbb{Q})$. Then $D(\mathbb{Z}, \mathbb{Q}) \cong \mathbb{Z} \times \mathbb{Q}$. Then $\delta(\mathbb{Z} \times \mathbb{Q}) = (0) \times \mathbb{Q}$. Since $\mathbb{Z}$ is not δ -quasipolar, $D(\mathbb{Z}, \mathbb{Q})$ is not δ -quasipolar.

Let R and S be any ring and M an (R, S) -bimodule. Consider the ring of the formal upper triangular matrix ring $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$. It is well known that $\delta(T) \subseteq \begin{bmatrix} \delta(R) & M \\ 0 & \delta(S) \end{bmatrix}$. However, if $M = R = S = F$ is a field, then $\delta(T) = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$.

The following example illustrates the δ -quasipolarity of full matrix rings and upper triangular matrix rings depend on the coefficient ring.

- Example 8.* (1) Consider the ring $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. Then $J(R) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ and $\delta(R) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. R is δ -quasipolar.
- (2) As noted in Example 4, the ring $\mathbb{Z}_3$ is semisimple and therefore δ -quasipolar. However, the ring $\begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{bmatrix}$ is not δ -quasipolar.
- (3) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \text{Mat}_2(\mathbb{Z})$. For any $P^2 = P \in \text{comm}^2(A)$, the matrix P has the form $P = \begin{bmatrix} x & y \\ 0 & x \end{bmatrix}$ with $x^2 = x$ and $2xy = y$ where $x, y \in \mathbb{Z}$. This would imply that P is the zero matrix or the identity matrix. Since $\delta(\mathbb{Z}) = 0$, $\delta(\text{Mat}_2(\mathbb{Z})) = 0$. In consequence, $A + P$ can not be in $\delta(\text{Mat}_2(\mathbb{Z}))$. Therefore $\text{Mat}_2(\mathbb{Z})$ is not δ -quasipolar.
- (4) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in T_2(\mathbb{Z})$. The idempotents of $T_2(\mathbb{Z})$ are zero, identity, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & y \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix}$ where y is an arbitrary integer. Since A commutes with only zero, identity, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, among these idempotents there is no idempotent P such that $A + P \in \delta(T_2(\mathbb{Z}))$ since $\delta(T_2(\mathbb{Z})) = \begin{bmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$. Hence $T_2(\mathbb{Z})$ is not δ -quasipolar.

3. WEAKLY δ -QUASIPOLAR RINGS

In this section, we introduce an upper class of δ -quasipolar rings, namely, weakly δ -quasipolar rings, and we give some properties of such rings.

Definition 2. Let R be a ring and $a \in R$. The element a is called *weakly δ -quasipolar* if there exists $p^2 = p \in \text{comm}(a)$ such that $a + p \in \delta(R)$, and p is called a *weakly δ -spectral idempotent*. A ring R is called *weakly δ -quasipolar* if every element of R is weakly δ -quasipolar.

An element of a ring is called *strongly J -clean* [3] provided that it can be written as the sum of an idempotent and an element in its Jacobson radical that commute. A ring is *strongly J -clean* in case each of its elements is strongly J -clean.

Example 9. (1) Every semisimple ring and every Boolean ring is weakly δ -quasipolar, since δ -quasipolar rings are weakly δ -quasipolar.
(2) Every strongly J -clean ring is weakly δ -quasipolar.

Proposition 5. *Let $f : R \rightarrow S$ be a surjective ring homomorphism. If R is weakly δ -quasipolar, then S is weakly δ -quasipolar.*

Proof. Let $s \in S$ with $s = f(r)$ where $r \in R$. There exists an idempotent $p \in \text{comm}(r)$ such that $r + p \in \delta(R)$. Let $q = f(p)$. Then $q^2 = q \in \text{comm}(f(r)) = \text{comm}(s)$. By [16], $f(\delta(R)) \subseteq \delta(S)$. Then $s + q = f(r) + f(p) = f(r + p) \in f(\delta(R)) \subseteq \delta(S)$. Hence S is weakly δ -quasipolar. $\square$

Corollary 3. *Every direct summand of a weakly δ -quasipolar ring is weakly δ -quasipolar.*

Proposition 6. *Let $R = \prod_{i=1}^n R_i$ be a finite direct product of rings. R is weakly δ -quasipolar if and only if each R_i is weakly δ -quasipolar for $(i = 1, 2, \dots, n)$.*

Proof. One way is clear from Corollary 3. We may assume that $n = 2$ and R_1 and R_2 are weakly δ -quasipolar. Let $a = (x_1, x_2) \in R$. There exist idempotents $p_i \in \text{comm}(x_i)$ such that $x_i + p_i \in \delta(R_i)$ for $(i = 1, 2)$. Then $p = (p_1, p_2)$ is an idempotent in R and $p \in \text{comm}(a)$ and $a + p \in \delta(R)$. Hence R is weakly δ -quasipolar. $\square$

In [8], Gurgun and Ozcan introduce and investigate properties of δ_r -clean rings. Motivated by this work strongly δ_r -clean rings can be defined as follows.

Definition 3. An element $x \in R$ is called *strongly δ_r -clean* provided that there exist an idempotent $e \in R$ and an element $w \in \delta_r$ such that $x = e + w$ and $ew = we$. A ring R is called *strongly δ_r -clean* in case every element in R is strongly δ_r -clean.

Any strongly J -clean ring is strongly δ_r -clean. But the converse need not be true, for example any commutative semisimple ring which is not a Boolean ring is such a ring.

Note that in the following theorem it is proved that the notions of strongly δ_r -clean rings and weakly δ -quasipolar rings coincide.

Theorem 3. *Let R be a ring. Then R is a weakly δ -quasipolar ring if and only if it is strongly δ_r -clean.*

Proof. Let R be a weakly δ -quasipolar ring and $a \in R$. There exists $p^2 = p \in \text{comm}(-1 + a)$ such that $-1 + a + p \in \delta(R)$. Then $a - (1 - p) \in \delta(R)$ and $a(1 - p) = (1 - p)a$. Hence R is a strongly δ_r -clean ring. Conversely, assume that R is a strongly

δ_r -clean ring. Let $a \in R$. Since $-a \in R$, by assumption there exists an idempotent $p \in R$ such that $-a - p \in \delta(R)$ and $(-a)p = p(-a)$. So R is a weakly δ -quasipolar ring. $\square$

Theorem 3 states that the weakly δ -quasipolarity of a ring is equivalent to the strongly δ_r -cleanness of this ring. The following example reveals that a weakly δ -quasipolar element is different from a strongly δ_r -clean element.

Example 10. Let $R = \mathbb{Z}$ and $a = 1 \in R$. There exists no idempotent p such that $a + p \in \delta(R)$. Then a is not weakly δ -quasipolar. Let $p = 1 \in R$. Since $a - p \in \delta(R)$, a is strongly δ_r -clean. On the other hand, if $a = -1 \in R$, then there exists no idempotent p such that $a - p \in \delta(R)$. Then a is not strongly δ_r -clean. Let $p = 1 \in R$. Since $a + p \in \delta(R)$, a is weakly δ -quasipolar.

Theorem 4. *Let R be a local ring with non-zero maximal ideal. Then the following are equivalent.*

- (1) R is weakly δ -quasipolar;
- (2) R is strongly J -clean;
- (3) R is uniquely clean;
- (4) $R/J(R) \cong \mathbb{Z}_2$;
- (5) $R/\delta(R) \cong \mathbb{Z}_2$.

Proof. Let R be a local ring with non-zero maximal ideal.

(1) $\Leftrightarrow$ (2) Assume that R is weakly δ -quasipolar. Let $a \in R$. There exists $p^2 = p \in \text{comm}(-1 + a)$ such that $-1 + a + p \in \delta(R)$. Then $a - (1 - p) \in \delta(R)$. Since $p \in \text{comm}(-1 + a)$, $pa = ap$. Hence R is strongly J -clean by $J(R) = \delta(R)$. Similarly, the rest is clear.

(2) $\Leftrightarrow$ (3) follows from [3, Lemma 4.2].

(3) $\Leftrightarrow$ (4) follows from [14, Theorem 15].

(1) $\Rightarrow$ (5) Let R be weakly δ -quasipolar and $\bar{0} \neq \bar{a} = a + \delta(R) \in R/\delta(R)$, we show that $\bar{a} = \bar{1}$. Then there exists an idempotent $p \in R$ such that $-a + p \in \delta(R)$ and $p^2 = p \in \text{comm}(-a)$. Since R is a local, $p = 0$ or $p = 1$. If $p = 0$, this contradicts $\bar{0} \neq \bar{a}$. Therefore $p = 1$. It follows that $\bar{a} = \bar{1}$.

(5) $\Rightarrow$ (1) It follows from Proposition 2. $\square$

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ON A SPECIAL CLASS OF MEDIAN ALGEBRAS

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Abstract. In this short note we consider a class of median algebras, called $(1, 2 : 3)$ -semilattices, that is pertaining to cluster analysis. Such median algebras arise from a natural generalization of conservativeness, and their description is given in terms of forbidden substructures.

2010 *Mathematics Subject Classification:* 03C05; 06D75

Keywords: median algebra, median semilattice, generalized conservativeness

1. PRELIMINARIES: MEDIAN ALGEBRAS

Recall that a *median algebra* is a structure $\mathbf{A} = \langle A, \mathbf{m} \rangle$ for a nonempty set A and a ternary symmetric operation $\mathbf{m}: A^3 \rightarrow A$, called *median*, that is a near-unanimity (i.e., $\mathbf{m}(x, x, y) = x$) and such that for every $x, y, z, t, u \in A$,

$$\mathbf{m}(\mathbf{m}(x, y, z), t, u) = \mathbf{m}(x, \mathbf{m}(y, t, u), \mathbf{m}(z, t, u)).$$

Alternative axiomatizations were also proposed [2, 7, 8]. In particular, it can be shown that the following equation

$$\mathbf{m}(\mathbf{m}(x, y, z), y, z) = \mathbf{m}(x, y, z)$$

holds in every median algebra.

Such algebras have been studied by several authors and they were shown to be tightly related to several ordered structures such as semilattices and distributive lattices. Indeed, Sholander [9] showed that each element a of a median algebra $\mathbf{A}$ gives rise to a *median semilattice* $\langle A, \leq_a \rangle$ where $\leq_a$ defined by

$$x \leq_a y \iff \mathbf{m}(a, x, y) = x.$$

In such a semilattice, a is the bottom element $\mathbf{A}_a := \langle A, \leq_a \rangle$ and the induced semilattice operation $\wedge_a$ is then given by $x \wedge_a y = \mathbf{m}(a, x, y)$. Furthermore, its principal ideals

$$\downarrow x := \{y \in A : y \leq_a x\}$$

are distributive lattices. Conversely, if a $\wedge$ -semilattice has the property that for any $a, b, c \in A$, the three elements $a \wedge b, b \wedge c, c \wedge a$ have a supremum whenever each pair

of them is bounded above, then a median operation can be defined by

$$\mathbf{m}_{\leq}(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (z \wedge y), \quad \text{for every } x, y, z \in A. \quad (1.1)$$

Here $\vee$ denotes the supremum so that $\mathbf{m}_{\leq}(x, y, z)$ is defined as the supremum of $x \wedge y$, $x \wedge z$ and $z \wedge y$. As shown in [1], for every median algebra $\mathbf{A} = \langle A, \mathbf{m} \rangle$ and every $a \in A$, we have that $\mathbf{m} = \mathbf{m}_{\leq_a}$.

Similarly, every distributive lattice gives rise to a median algebra using (1.1), and the converse also holds whenever there are $a, b \in A$ such that $\mathbf{m}(a, x, b) = x$ for every $x \in A$.

In the case of discrete structures, median algebras can also be thought of as certain special graphs. A *median graph* is a connected graph (i.e., for every pair of vertices there is a path connecting them) that has the property that for any three vertices a, b, c , there is exactly one vertex x that minimizes the sum of the distances to a, b and c . It was shown in [1] that every median semilattice whose intervals are finite has a *covering graph* (i.e., undirected HASSE diagram) that is median, and that every median graph is the covering graph of a median semilattice. For further background see, e.g., [2].

In this short note we are interested in some special classes of median algebras that are pertaining to cluster analysis, e.g., in the case of overlapping clusters. We say that a median semilattice $\mathbf{A} := \langle A, \leq \rangle$ (thought of as a median algebra $\mathbf{A} = \langle A, \mathbf{m} \rangle$) is

(i) a $(1 : 3)$ -median semilattice if for every $x, y, z \in A$,

$$\mathbf{m}(x, y, z) \in \{x, y, z\};$$

(ii) a $(2 : 3)$ -median semilattice if for every $x, y, z \in A$,

$$\mathbf{m}(x, y, z) \in \{x \wedge y, y \wedge z, z \wedge x\};$$

(iii) a $(1, 2 : 3)$ -median semilattice if for every $x, y, z \in A$,

$$\mathbf{m}(x, y, z) \in \{x, y, z, x \wedge y, y \wedge z, z \wedge x\}.$$

Note that, apart from the 4-element Boolean lattice, every $(1 : 3)$ -median semilattice is a $(2 : 3)$ -median semilattice. Also, every $(1 : 3)$ -median semilattice and every $(2 : 3)$ -median semilattice is a $(1, 2 : 3)$ -median semilattice.

However, the converse statements are not true as it is illustrated by Figure 1.

Remark 1. Note that $(1 : 3)$ -median semilattices are exactly those median algebras whose subsets are themselves median (sub)algebras. In the terminology of [3], they are median algebras with *msd* (median stabilization degree) equal to 0.

The median semilattices in (i) and (ii) were described in terms of forbidden substructures in [5] and [4], respectively. In this paper we provide a similar description for $(1, 2 : 3)$ -median semilattices.

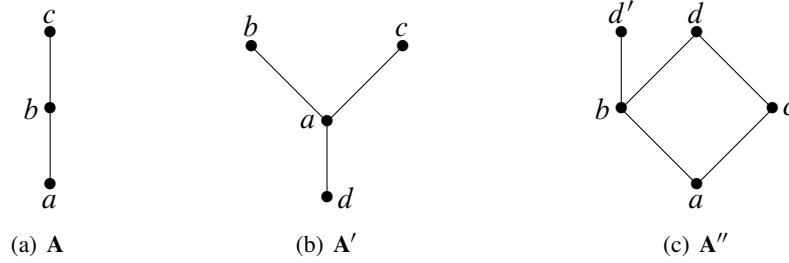


FIGURE 1. Examples of median semilattices: $\mathbf{A}$ is a $(1 : 3)$ -median semilattice, $\mathbf{A}'$ is a $(2 : 3)$ -median semilattice that is not $(1 : 3)$, and $\mathbf{A}''$ is a $(1, 2 : 3)$ -median semilattice that is not $(2 : 3)$.

2. MAIN RESULT

In [5], $(1 : 3)$ -median semilattices were referred to as *conservative* median algebras, and it was shown that, apart from the 4-element Boolean algebra, they can be thought of as chains. (Curiously, the 4-element Boolean algebra also appears as an exceptional in another area of the theory of ordered sets; see e.g. [6].) Equivalently, $(1 : 3)$ -median semilattices were shown to coincide exactly with those ordered structures that do not contain the 4-element star $\mathbf{A}'$ (see Figure 1(b)).

Recently, $(2 : 3)$ -median semilattices were explicitly described in [4]. It was shown that $(2 : 3)$ -median semilattices coincide with those median semilattices that are trees: a $\wedge$ -semilattice (resp. $\vee$ -semilattice) is said to be a *tree* if no pair of incomparable elements have an upper (resp. lower) bound.

In particular, it follows that if a median algebra is a $(1 : 3)$ - or a $(2 : 3)$ -semilattice, then the same holds for any associated semilattice order $\leq_a$. As illustrated by the ordered structures in Figure 2, this invariance no longer holds for $(1, 2 : 3)$ -median semilattices.

As mentioned above, $(1 : 3)$ -median semilattices and $(2 : 3)$ -median semilattices can be thought of as ordered structures that do not contain the 4-element star and the 4-element Boolean algebra, respectively. The following proposition provides a similar description for $(1, 2 : 3)$ -median semilattices.

Proposition 1. *Let $\mathbf{A} := \langle A, \leq \rangle$ be a median semilattice. Then $\mathbf{A}$ is a $(1, 2 : 3)$ -median semilattice if and only if it does not contain a copy of the median semilattice in Figure 2(b) as a semilattice (in fact, only need to consider b, c, d and d').*

Proof. Note that if $\mathbf{A} = \langle A, \leq \rangle$ contains a copy of the median semilattice in Figure 2(b), then taking $\{x, y, z\} = \{d', b, c\}$ have that

$$\mathbf{m}(x, y, z) \notin \{x, y, z, x \wedge y, y \wedge z, z \wedge x\}.$$

Hence the condition is necessary.

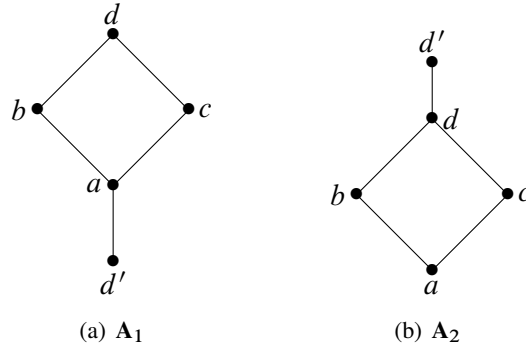


FIGURE 2. $\mathbf{A}_1$ is a $(1, 2 : 3)$ -median semilattice, whereas the semilattice $\mathbf{A}_2$ is not.

To show that the condition is sufficient, suppose that $\mathbf{A}$ is not a $(1, 2 : 3)$ -median semilattice. Hence, there are $x, y, z, d \in A$ such that

$$d = \mathbf{m}(x, y, z) \notin \{x, y, z, x \wedge y, y \wedge z, z \wedge x\}.$$

Using the fact that

$$d = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x),$$

we have that $b \wedge c < d$, for every $b, c \in \{x, y, z\}$ with $b \neq c$. Consider

$$d' = (x \wedge y) \vee (y \wedge z).$$

Clearly, $d' \leq d$.

Now if $d' < d$, then $\{x \wedge y, y \wedge z, x \wedge y \wedge z, d, d'\}$ is a copy of Figure 2(b). So we may assume that $d = d' = (x \wedge y) \vee (y \wedge z) = y \wedge (x \vee z)$. But in this case, $\{y, x \wedge y, y \wedge z, x \wedge y \wedge z, d\}$ is a copy of Figure 2(b). As these were the only two possible cases, the proof of Proposition 1 is now complete. $\square$

It is noteworthy that new examples of $(1, 2 : 3)$ -median semilattices can be obtained from old ones by taking any non trivial bottom-rooted tree and replacing some of its leaves (maximal elements) by any $(1, 2 : 3)$ -semilattices, e.g., as in Figure 2(a). A further example can be obtained by taking, e.g., the $(1, 2 : 3)$ -median semilattice in Figure 1(c) and replacing d' by the 4-element Boolean lattice.

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SHIFTED EULER-SEIDEL MATRICES

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Abstract. In this study defining Shifted Euler-Seidel matrices we generalize the Euler-Seidel matrices method. Owing to this generalization one can investigate any sequences (s_n) which have two term linear recurrences as $s_{m+n} = \alpha s_{m+n-1} + \beta s_{n-1}$ (α and β are real parameters and $n, m \in \mathbb{Z}^+$). By way of illustration, we give some examples related to the Fibonacci p -numbers.

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Keywords: Fibonacci p -numbers, (incomplete) bivariate Fibonacci p -polynomials, generating functions

1. INTRODUCTION

Let (a_n) be a sequence. The Euler-Seidel matrix associated with this sequence is determined recursively by the formula (see [6]):

$$\begin{aligned} a_n^0 &= a_n \quad (n \geq 0) \\ a_n^k &= a_n^{k-1} + a_{n+1}^{k-1} \quad (n \geq 0, k \geq 1). \end{aligned} \quad (1.1)$$

From relation (1.1), it can be seen that the first row and the first column can be transformed into each other via the well known binomial inverse pair as,

$$a_0^n = \sum_{k=0}^n \binom{n}{k} a_k^0, \quad (1.2)$$

$$a_n^0 = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} a_0^k. \quad (1.3)$$

Also any entry a_n^k can be written in terms of the initial sequence as:

$$a_n^k = \sum_{i=0}^k \binom{k}{i} a_{n+i}^0. \quad (1.4)$$

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Proposition 1 (Euler). *Let*

$$a(t) = \sum_{n=0}^{\infty} a_n^0 t^n$$

be the generating function of the initial sequence (a_n^0) . Then the generating function of the sequence (a_n) is

$$\bar{a}(t) = \sum_{n=0}^{\infty} a_n^0 t^n = \frac{1}{1-t} a\left(\frac{t}{1-t}\right). \quad (1.5)$$

Proposition 2 (Seidel). *Let*

$$A(t) = \sum_{n=0}^{\infty} a_n^0 \frac{t^n}{n!}$$

be the exponential generating function of the initial sequence (a_n^0) . Then the exponential generating function of the sequence (a_n) is

$$\bar{A}(t) = \sum_{n=0}^{\infty} a_n^0 \frac{t^n}{n!} = e^t A(t). \quad (1.6)$$

The Euler-Seidel matrices are useful and rather elementary technique to investigate properties of some special numbers and polynomials. In [4] one can see applications related to the Bernoulli and Euler polynomials. Using the Euler-Seidel matrix authors obtained some properties of the geometric and exponential polynomials and numbers in [3]. Mező and Dil in [12] use the Euler-Seidel method for deriving new identities for the hyperharmonic and r-Stirling numbers. In [7] present authors obtained identities for the generalized second order recurrence relation by using the generalized Euler-Seidel matrix. Barry and Hennessy [1] studied the Euler-Seidel matrix of certain integer sequences, using the binomial transform and the Hankel matrices. For moment sequences, they gave an integral representation of the Euler-Seidel matrix. Chen [2] investigated the summation form of Bernoulli numbers which can form an Euler-Seidel matrix. The upper diagonal elements of this Euler-Seidel matrix are called “the median Bernoulli numbers”. Chen determined the prime divisors of their numerators and denominators also obtained their ordinary generating function. Tutaş [15] defined the period of a Euler-Seidel matrix over a field $\mathbb{F}_p$ with p elements, where p is a prime number and gave applications on the generalized Franel numbers.

There are also similar matrices related to the Euler-Seidel matrices. One of them is the symmetric infinite matrix which is defined by Dil and Mező in [5]. They established this matrix especially to investigate properties of the hyperharmonic numbers. In [8] authors gave some identities for the Fibonacci and incomplete Fibonacci p -numbers via the symmetric matrix method.

The Fibonacci p -numbers had been discovered by Stakhov while investigating “diagonal sums” of the Pascal triangle (see [13]). In [14] the Fibonacci p -numbers $F_p(n)$ are defined by the following recurrence relation for $n > p$

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \quad (1.7)$$

with initial conditions

$$F_p(0) = 0, F_p(n) = 1 \quad (n = 1, 2, \dots, p),$$

and the Lucas p -numbers $L_p(n)$ are defined by the following recurrence relation for $n > p$

$$L_p(n) = L_p(n-1) + L_p(n-p-1) \quad (1.8)$$

with initial conditions

$$L_p(0) = p+1, L_p(n) = 1 \quad (n = 1, 2, \dots, p).$$

Note that for $p = 1$ the Fibonacci and Lucas p -numbers are reduced to the well-known Fibonacci and Lucas sequences $\{F_n\}$, $\{L_n\}$, respectively.

In [10] the Pell p -numbers $P_p(n)$ are defined by the following recurrence relations for $n > p$

$$P_p(n) = 2P_p(n-1) + P_p(n-p-1) \quad (1.9)$$

with initial conditions

$$P_p(0) = 0, P_p(n) = 2^{n-1} \quad (n = 1, 2, \dots, p)$$

and the Pell-Lucas p -numbers $Q_p(n)$ are defined by the following recurrence relations for $n > p$

$$Q_p(n) = 2Q_p(n-1) + Q_p(n-p-1) \quad (1.10)$$

with initial conditions

$$Q_p(0) = p+1, Q_p(n) = 2^n \quad (n = 1, 2, 3, \dots, p).$$

Note that for $p = 1$ the Pell and Pell-Lucas p -numbers are reduced to the well-known Pell and Pell-Lucas sequences $\{P_n\}$, $\{Q_n\}$, respectively.

The generalized bivariate Fibonacci p -polynomials $F_{p,n}(a, b)$ and the generalized bivariate Lucas p -polynomials $L_{p,n}(a, b)$ are defined (see [16]) the recursion for $p \geq 1$

$$F_{p,n}(a, b) = aF_{p,n-1}(a, b) + bF_{p,n-p-1}(a, b); \quad n > p \quad (1.11)$$

with

$$F_{p,0}(a, b) = 0, F_{p,n}(a, b) = a^{n-1} \text{ for } n = 1, 2, \dots, p$$

and

$$L_{p,n}(a, b) = aL_{p,n-1}(a, b) + bL_{p,n-p-1}(a, b); \quad n > p \quad (1.12)$$

with

$$L_{p,0}(a, b) = p+1, L_{p,n}(a, b) = a^n \text{ for } n = 1, 2, \dots, p$$

When $a = b = 1$, $F_{p,n}(a, b) = F_p(n)$ and $L_{p,n}(a, b) = L_p(n)$.

For $p \geq 1, n \geq 1$, the incomplete bivariate Fibonacci and Lucas p -polynomials are defined as

$$F_{p,n}^k(a,b) = \sum_{j=0}^k \binom{n-jp-1}{j} a^{n-j(p+1)-1} b^j; \quad 0 \leq k \leq \left\lfloor \frac{n-1}{p+1} \right\rfloor. \quad (1.13)$$

and

$$L_{p,n}^k(a,b) = \sum_{j=0}^k \frac{n}{n-jp} \binom{n-jp}{j} a^{n-j(p+1)} b^j; \quad 0 \leq k \leq \left\lfloor \frac{n}{p+1} \right\rfloor. \quad (1.14)$$

Moreover the following properties of the incomplete bivariate Fibonacci and Lucas p -polynomials are given in [8] as

$$\sum_{j=0}^h \binom{h}{j} b^{h-j} a^j F_{p,n+p(j-1)}^{k+j}(a,b) = F_{p,n+(p+1)h-p}^{k+h}(a,b); \quad 0 \leq k \leq \frac{n-h-p-1}{p+1} \quad (1.15)$$

and

$$\sum_{j=0}^h \binom{h}{j} b^{h-j} a^j L_{p,n+p(j-1)}^{k+j}(a,b) = L_{p,n+(p+1)h-p}^{k+h}(a,b); \quad 0 \leq k \leq \frac{n-p-h}{p+1}. \quad (1.16)$$

2. SHIFTED EULER-SEIDEL MATRICES WITH TWO PARAMETERS

So far we have mentioned about the Euler-Seidel matrix and its applications. But this method is useful for only linear sequences which have a recurrence related to two consecutive terms. Here we generalize this method to the sequence having linear recurrence with an arbitrary “gap”. Owing to this generalization one can investigate for example the Fibonacci p -numbers or get informations about such subsequences indexed by multiples of some natural numbers

$$\sum_{r=0}^n \binom{n}{r} x^{n-r} y^r F_{rm}.$$

Let us consider a given sequence $(a_n)_{n \geq 0}$. We define “Shifted Euler-Seidel matrix with two parameters” corresponding to this sequence recursively by the formulae

$$\begin{aligned} a_n^0 &= a_n \quad (n \geq 0), \\ m a_n^k &= x_m a_n^{k-1} + y_m a_{n+m}^{k-1} \quad (n \geq 0, k \geq 1 \text{ and } m \text{ is a fixed positive integer}) \end{aligned} \quad (2.1)$$

where $m a_n^k$ represents the k th row and n th column entry (here the left below index m shows that all entries depend on m) and x, y are real parameters; i.e;

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & x_m a_n^{k-1} & \xrightarrow{\cdot} & y_m a_{n+m}^{k-1} & \cdot & \cdot \\ \cdot & \cdot & \cdot & m a_n^k & \swarrow & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

The following proposition gives the relation between an arbitrary entry of the matrix and the initial sequence.

Proposition 3. *We have*

$$m a_n^k = \sum_{i=0}^k \binom{k}{i} x^{k-i} y^i a_{n+im}^0 \quad (2.2)$$

where m is a fixed positive integer.

Proof. By induction on $n + k$. □

Corollary 1.

$$m a_0^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i a_{im}^0 \quad (2.3)$$

and

$$a_{nm}^0 = \frac{1}{y^n} \sum_{i=0}^n \binom{n}{i} (-x)^{n-i} m a_0^i. \quad (2.4)$$

2.1. Generating Functions

In this subsection we give connections between the generating functions of the initial sequences and the first column entries of the Shifted Euler-Seidel matrix with two parameters.

2.1.1. Ordinary generating functions

Proposition 4. *For the real parameters x and y the following relation holds:*

$$\overline{m a_{x,y}}(t) = \frac{1}{1-xt} \pi m a_{x,y} \left(\frac{yt}{1-xt} \right) \quad (2.5)$$

where

$$\overline{m a_{x,y}}(t) = \sum_{n=0}^{\infty} m a_0^n t^n \quad \text{and} \quad \pi m a_{x,y}(t) = \sum_{n=0}^{\infty} a_{nm}^0 t^n.$$

Proof. Considering (2.3) we write

$$\overline{ma_{x,y}}(t) = \sum_{n=0}^{\infty} \left(\sum_{r=0}^n \binom{n}{r} x^{n-r} y^r a_{rm}^0 \right) t^n.$$

By changing the order of the above sums we get

$$\begin{aligned} \overline{ma_{x,y}}(t) &= \sum_{r=0}^{\infty} (yt)^r a_{rm}^0 \sum_{n=0}^{\infty} \binom{n+r}{r} (xt)^n \\ &= \frac{1}{1-xt} \sum_{r=0}^{\infty} a_{rm}^0 \left(\frac{yt}{1-xt} \right)^r. \end{aligned}$$

In the last step we used the Newton binomial formula. □

Applications

If we take $a_n^0 = F_n$ in (2.5), then we have

$$\pi m a_{x,y}(t) = \sum_{n=0}^{\infty} F_n m t^n.$$

By using the following well known identity (see [11, p.230])

$$\sum_{n=0}^{\infty} F_n m t^n = \frac{F_m t}{1 - L_m t + (-1)^m t^2},$$

we have

$$\sum_{n=0}^{\infty} \left(\sum_{r=0}^n \binom{n}{r} x^{n-r} y^r F_{rm} \right) t^n = \frac{y F_m t}{(1-xt)^2 - yt(1-xt) L_m + (-1)^m (yt)^2}$$

which is the ordinary generating function of the sequence

$$\sum_{r=0}^n \binom{n}{r} x^{n-r} y^r F_{rm}.$$

Remark 1. Special case of this relation for $x = y = 1$ obtained by Hoggatt in [9].

Similarly, using Binet's formula we obtain generating function for every m -th terms of the Lucas numbers as:

$$\sum_{n=0}^{\infty} L_{nm} t^n = \frac{L_m t}{1 - L_m t + (-1)^m t^2}. \quad (2.6)$$

Considering this generating function and setting $a_n^0 = L_n$ in (2.5) we get

$$\sum_{n=0}^{\infty} \left(\sum_{i=0}^n \binom{n}{i} x^{n-i} y^i L_{im} \right) t^n = \frac{y L_m t}{(1-xt)^2 - yt(1-xt) L_m + (-1)^m (yt)^2}.$$

2.1.2. Exponential generating functions

The following proposition also provides the connection between the exponential generating functions of the initial sequence and the first column entries of the Shifted Euler-Seidel matrix with two parameters.

Proposition 5. *The following relation holds:*

$$\overline{m}A_{x,y}(t) = e^{xt} {}_{m\pi}A(yt) \quad (2.7)$$

where

$$\overline{m}A_{x,y}(t) = \sum_{n=0}^{\infty} m a_0^n \frac{t^n}{n!} \quad \text{and} \quad {}_{m\pi}A_{x,y}(t) = \sum_{n=0}^{\infty} a_{nm}^0 \frac{t^n}{n!}.$$

Proof. Using (2.3) we get

$$\begin{aligned} \overline{m}A_{x,y}(t) &= \sum_{n=0}^{\infty} \left(\sum_{r=0}^n \binom{n}{r} x^{n-r} y^r a_{rm}^0 \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^n \frac{a_{rm}^0 x^{n-r} y^r}{(n-r)! r!} t^n. \end{aligned}$$

Hence we have

$$\overline{m}A_{x,y}(t) = \left(\sum_{n=0}^{\infty} a_{rn}^0 \frac{(yt)^n}{n!} \right) \times \left(\sum_{n=0}^{\infty} \frac{(xt)^n}{n!} \right)$$

and this completes the proof. $\square$

The above results on generating functions enable us to transfer informations of the sequence $(a_n)_n$ to the subsequence $(a_{nm})_n$, and vice versa.

Applications

Now we apply this method to the Fibonacci numbers by setting $a_n^0 = F_n$. Then considering (see [11])

$$\sum_{n=0}^{\infty} F_{nm} \frac{t^n}{n!} = \frac{e^{\alpha^m t} - e^{\beta^m t}}{\alpha - \beta}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, we get immediately that

$${}_{m\pi}A_{x,y}(t) = \frac{e^{\alpha^m t} - e^{\beta^m t}}{\alpha - \beta}.$$

This equation together with (2.7) gives

$$\sum_{n=0}^{\infty} \left(\sum_{r=0}^n \binom{n}{r} x^{n-r} y^r F_{rm} \right) \frac{t^n}{n!} = \frac{e^{(\alpha^m y + x)t} - e^{(\beta^m y + x)t}}{\alpha - \beta}.$$

Hence we obtain the exponential generating function of the sequence

$$\sum_{r=0}^n \binom{n}{r} x^{n-r} y^r F_{rm}.$$

With a similar approach we get

$$\sum_{n=0}^{\infty} \left(\sum_{r=0}^n \binom{n}{r} x^{n-r} y^r L_{rm} \right) \frac{t^n}{n!} = e^{(\alpha^m y + x)t} + e^{(\beta^m y + x)t}.$$

Remark 2. Considering other special sequences as Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers and applying the Shifted Euler-Seidel matrices one can obtain similar results.

2.2. Applications of the Generalized Bivariate Fibonacci and Lucas p -polynomials

In this subsection, we obtain some results on the generalized bivariate Fibonacci and Lucas p -polynomials using the Shifted Euler-Seidel matrices with two parameters.

Proposition 6. For the integers $n \geq 0$, $k \geq 0$ and $p \geq 1$ we have

$$F_{p,n+k(p+1)}(a,b) = \sum_{i=0}^k \binom{k}{i} b^{k-i} a^i F_{p,n+ip}(a,b). \quad (2.8)$$

Proof. By setting $x = b$ and $y = a$ in (2.1), we get

$${}_m a_n^k = b \left({}_m a_n^{k-1} \right) + a \left({}_m a_{n+m}^{k-1} \right). \quad (2.9)$$

Let us consider the initial sequences $a_n^0 = F_{p,n}(a,b)$, $n \geq 0$. In view of equations (1.11) and (2.9) by induction on k we obtain:

$$a_n^k = F_{p,n+k(p+1)}(a,b).$$

Using (2.2), we have

$$a_n^k = \sum_{i=0}^k \binom{k}{i} b^{k-i} a^i F_{p,n+ip}(a,b).$$

Combining these results we get

$$F_{p,n+k(p+1)}(a,b) = \sum_{i=0}^k \binom{k}{i} b^{k-i} a^i F_{p,n+ip}(a,b).$$

□

We may use (2.3) and (2.4) to conclude that:

Corollary 2.

$$F_{p,n(p+1)}(a, b) = \sum_{i=0}^n \binom{n}{i} b^{n-i} a^i F_{p,ip}(a, b) \quad (2.10)$$

and

$$F_{p,np}(a, b) = \frac{1}{a^n} \sum_{i=0}^n \binom{n}{i} (-b)^{n-i} F_{p,i(p+1)}(a, b). \quad (2.11)$$

Similar results for the generalized bivariate Lucas p -polynomials can be given as follows.

Corollary 3.

$$L_{p,n+k(p+1)}(a, b) = \sum_{i=0}^k \binom{k}{i} b^{k-i} a^i L_{p,n+ip}(a, b), \quad (2.12)$$

$$L_{p,n(p+1)}(a, b) = \sum_{i=0}^n \binom{n}{i} b^{n-i} a^i L_{p,ip}(a, b) \quad (2.13)$$

and

$$L_{p,np}(a, b) = \frac{1}{a^n} \sum_{i=0}^n \binom{n}{i} (-b)^{n-i} L_{p,i(p+1)}(a, b). \quad (2.14)$$

2.3. Applications for the incomplete bivariate Fibonacci and Lucas p -polynomials

Proposition 7. For $0 \leq k \leq \frac{t-p-n-1}{p+1}$ we have

$$F_{p,t+p(n-1)}^{k+n}(a, b) = \frac{1}{a^n} \sum_{i=0}^n \binom{n}{i} (-b)^{n-i} F_{p,t+(p+1)i-p}^{k+i}(a, b).$$

Proof. Let choose $a_{np}^0 = F_{p,t+p(n-1)}^{k+n}(a, b)$ in (2.9). By (2.3) we have

$$a_0^n = \sum_{i=0}^n \binom{n}{i} b^{n-i} a^i F_{p,t+p(i-1)}^{k+i}(a, b).$$

From (1.15) we write

$$a_0^n = F_{p,t+(p+1)n-p}^{k+n}(a, b).$$

Therefore we obtain the dual formula of (1.15) using the equation (2.4) as:

$$F_{p,t+p(n-1)}^{k+n}(a, b) = \frac{1}{a^n} \sum_{i=0}^n \binom{n}{i} (-b)^{n-i} F_{p,t+(p+1)i-p}^{k+i}(a, b). \quad (2.15)$$

□

Remark 3. We can write similar result for the Lucas p -polynomials as:

$$L_{p,t+p(n-1)}^{k+n}(a,b) = \frac{1}{a^n} \sum_{i=0}^n \binom{n}{i} (-b)^{n-i} L_{p,t+(p+1)i-p}^{k+i}(a,b). \quad (2.16)$$

Example 1. In (2.15) and (2.16) by taking $a = 2$ and $b = 1$ we obtain the following properties of the incomplete Pell and Pell-Lucas p -numbers

$$P_p^{k+n}(t+p(n-1)) = \frac{1}{2^n} \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} P_p^{k+j}(t+(p+1)j-p)$$

and

$$Q_p^{k+n}(t+p(n-1)) = \frac{1}{2^n} \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} Q_p^{k+j}(t+(p+1)j-p),$$

respectively.

3. TABLES

In this section we summarize similar results that we obtained in the previous sections.

a	b	p	$F_{p,n}(a,b)$
a	b	1	Bivariate Fibonacci polynomials $F_n(a,b)$
a	1	p	Fibonacci p -polynomials $F_{p,n}(a)$
a	1	1	Fibonacci polynomials $f_n(a)$
1	1	p	Fibonacci p -numbers $F_p(n)$
1	1	1	Fibonacci numbers F_n
$2a$	b	p	Bivariate Pell p -polynomials $F_{p,n}(2a,b)$
$2a$	b	1	Bivariate Pell polynomials $F_n(2a,b)$
$2a$	1	p	Pell p -polynomials $P_{p,n}(a)$
$2a$	1	1	Pell polynomials $P_n(a)$
2	1	1	Pell numbers P_n
$2a$	-1	1	Second kind Chebyshev polynomials $U_{n-1}(a)$
a	$2b$	p	Bivariate Jacobsthal p -polynomials $F_{p,n}(a,2b)$
a	$2b$	1	Bivariate Jacobsthal polynomials $F_n(a,2b)$
1	$2b$	1	Jacobsthal polynomials $J_n(b)$
1	2	1	Jacobsthal numbers J_n

Sequence	For $F_{p,n+k(p+1)}(a,b) = \sum_{i=0}^k \binom{k}{i} b^{k-i} a^i F_{p,n+ip}(a,b)$, particular cases are
$F_n(a,b)$	$F_{n+2k}(a,b) = \sum_{i=0}^k \binom{k}{i} b^{k-i} a^i F_{n+i}(a,b)$
$F_{p,n}(a)$	$F_{p,n+k(p+1)}(a) = \sum_{i=0}^k \binom{k}{i} a^i F_{p,n+ip}(a)$
$f_n(a)$	$f_{n+2k}(a) = \sum_{i=0}^k \binom{k}{i} a^i f_{n+i}(a); F_{n+2k} = \sum_{i=0}^k \binom{k}{i} F_{n+i}$
$F_p(n)$	$F_p(n+k(p+1)) = \sum_{i=0}^k \binom{k}{i} F_p(n+ip)$
$F_{p,n}(2a,b)$	$F_{p,n+k(p+1)}(2a,b) = \sum_{i=0}^k \binom{k}{i} b^{k-i} (2a)^i F_{p,n+ip}(2a,b)$
$F_n(2a,b)$	$F_{n+2k}(2a,b) = \sum_{i=0}^k \binom{k}{i} b^{k-i} (2a)^i F_{n+i}(2a,b)$
$P_{p,n}(a)$	$P_{p,n+k(p+1)}(a) = \sum_{i=0}^k \binom{k}{i} (2a)^i P_{p,n+ip}(a)$
$P_n(a)$	$P_{n+2k}(a) = \sum_{i=0}^k \binom{k}{i} (2a)^i P_{n+i}(a); P_{n+2k} = \sum_{i=0}^k \binom{k}{i} 2^i P_{n+i}$
$U_{n-1}(a)$	$U_{n+2k-1}(a) = \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} (2a)^i U_{n+i-1}(a)$
$F_{p,n}(a,2b)$	$F_{p,n+k(p+1)}(a,2b) = \sum_{i=0}^k \binom{k}{i} (2b)^{k-i} a^i F_{p,n+ip}(a,2b)$
$F_n(a,2b)$	$F_{n+2k}(a,2b) = \sum_{i=0}^k \binom{k}{i} (2b)^{k-i} a^i F_{n+i}(a,2b)$
$J_n(b)$	$J_{n+2k}(b) = \sum_{i=0}^k \binom{k}{i} (2b)^{k-i} J_{n+i}(b); J_{n+2k} = \sum_{i=0}^k \binom{k}{i} 2^{k-i} J_{n+i}$

Sequence	For $F_{p,n(p+1)}(a,b) = \sum_{i=0}^n \binom{n}{i} b^{n-i} a^i F_{p,ip}(a,b)$, particular cases are
$F_n(a,b)$	$F_{2n}(a,b) = \sum_{i=0}^n \binom{n}{i} b^{n-i} a^i F_i(a,b)$
$F_{p,n}(a)$	$F_{p,n(p+1)}(a) = \sum_{i=0}^n \binom{n}{i} a^i F_{p,ip}(a,b)$
$f_n(a)$	$f_{2n}(a) = \sum_{i=0}^n \binom{n}{i} a^i f_i(a); F_{2n} = \sum_{i=0}^n \binom{n}{i} F_i$
$F_p(n)$	$F_p(n(p+1)) = \sum_{i=0}^n \binom{n}{i} F_p(ip)$
$F_{p,n}(2a,b)$	$F_{p,n(p+1)}(2a,b) = \sum_{i=0}^n \binom{n}{i} b^{n-i} (2a)^i F_{p,ip}(2a,b)$
$F_n(2a,b)$	$F_{2n}(a,b) = \sum_{i=0}^n \binom{n}{i} b^{n-i} (2a)^i F_i(2a,b)$
$P_{p,n}(a)$	$P_{p,n(p+1)}(a) = \sum_{i=0}^n \binom{n}{i} (2a)^i P_{p,ip}(a)$
$P_n(a)$	$P_{2n}(a) = \sum_{i=0}^n \binom{n}{i} (2a)^i P_i(a); P_{2n} = \sum_{i=0}^n \binom{n}{i} 2^i P_i$
$U_{n-1}(a)$	$U_{2n-1}(a) = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} (2a)^i U_{i-1}(a)$
$F_{p,n}(a,2b)$	$F_{p,n(p+1)}(a,2b) = \sum_{i=0}^n \binom{n}{i} (2b)^{n-i} a^i F_{p,ip}(a,2b)$
$F_n(a,2b)$	$F_{2n}(a,2b) = \sum_{i=0}^n \binom{n}{i} (2b)^{n-i} a^i F_i(a,2b)$
$J_n(b)$	$J_{2n}(b) = \sum_{i=0}^n \binom{n}{i} (2b)^{n-i} J_i(b); J_{2n} = \sum_{i=0}^n \binom{n}{i} 2^{n-i} J_i$

Sequence	For $F_{p,np}(a,b) = \frac{1}{a^n} \sum_{i=0}^n \binom{n}{i} (-b)^{n-i} F_{p,i(p+1)}(a,b)$, particular cases are
$F_n(a,b)$	$F_n(a,b) = \frac{1}{a^n} \sum_{i=0}^n \binom{n}{i} (-b)^{n-i} F_{2i}(a,b)$
$F_{p,n}(a)$	$F_{p,np}(a) = \frac{1}{a^n} \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} F_{p,i(p+1)}(a)$
$f_n(a)$	$f_n(a) = \frac{1}{a^n} \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} f_{2i}(a); F_n = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} F_{2i}$
$F_p(n)$	$F_p(np) = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} F_p(i(p+1))$
$F_{p,n}(2a,b)$	$F_{p,np}(2a,b) = \frac{1}{(2a)^n} \sum_{i=0}^n \binom{n}{i} (-b)^{n-i} F_{p,i(p+1)}(2a,b)$
$F_n(2a,b)$	$F_n(2a,b) = \frac{1}{(2a)^n} \sum_{i=0}^n \binom{n}{i} (-b)^{n-i} F_{2i}(2a,b)$
$P_{p,n}(a)$	$P_{p,np}(a) = \frac{1}{(2a)^n} \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} P_{p,i(p+1)}(a)$
$P_n(a)$	$P_n(a) = \frac{1}{(2a)^n} \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} P_{2i}(a); P_n = \frac{1}{2^n} \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} P_{2i}$
$U_{n-1}(a)$	$U_{n-1}(a) = \frac{1}{(2a)^n} \sum_{i=0}^n \binom{n}{i} U_{2i}(a)$
$F_{p,n}(a,2b)$	$F_{p,np}(a,2b) = \frac{1}{a^n} \sum_{i=0}^n \binom{n}{i} (-2b)^{n-i} F_{p,i(p+1)}(a,2b)$
$F_n(a,2b)$	$F_n(a,2b) = \frac{1}{a^n} \sum_{i=0}^n \binom{n}{i} (-2b)^{n-i} F_{2i}(a,2b)$
$J_n(b)$	$J_n(b) = \sum_{i=0}^n \binom{n}{i} (-2b)^{n-i} J_{2i}(b); J_n = \sum_{i=0}^n \binom{n}{i} (-2)^{n-i} J_{2i}$

a	b	p	$L_{p,n}(a,b)$
a	b	1	Bivariate Lucas polynomials $L_n(a,b)$
a	1	p	Lucas p -polynomials $L_{p,n}(a)$
a	1	1	Lucas polynomials $\ell_n(a)$
1	1	p	Lucas p -numbers $L_p(n)$
1	1	1	Lucas numbers L_n
$2a$	b	p	Bivariate Pell-Lucas p -polynomials $L_{p,n}(2a,b)$
$2a$	b	1	Bivariate Pell-Lucas polynomials $L_n(2a,b)$
$2a$	1	p	Pell-Lucas p -polynomials $Q_{p,n}(a)$
$2a$	1	1	Pell-Lucas polynomials $Q_n(a)$
2	1	1	Pell-Lucas numbers Q_n
$2a$	-1	1	First kind Chebyshev polynomials $T_n(a)$
a	$2b$	p	Bivariate Jacobsthal-Lucas p -polynomials $L_{p,n}(a,2b)$
a	$2b$	1	Bivariate Jacobsthal-Lucas polynomials $L_n(a,2b)$
1	$2b$	1	Jacobsthal polynomials $j_n(b)$
1	2	1	Jacobsthal numbers j_n

Sequence	For $L_{p,n+k(p+1)}(a,b) = \sum_{i=0}^k \binom{k}{i} b^{k-i} a^i L_{p,n+ip}(a,b)$, particular cases are
$L_n(a,b)$	$L_{n+2k}(a,b) = \sum_{i=0}^k \binom{k}{i} b^{k-i} a^i L_{n+i}(a,b)$
$L_{p,n}(a)$	$L_{p,n+k(p+1)}(a) = \sum_{i=0}^k \binom{k}{i} a^i L_{p,n+ip}(a)$
$\ell_n(a)$	$\ell_{n+2k}(a) = \sum_{i=0}^k \binom{k}{i} a^i \ell_{n+i}(a); \quad L_{n+2k} = \sum_{i=0}^k \binom{k}{i} L_{n+i}$
$L_p(n)$	$L_p(n+k(p+1)) = \sum_{i=0}^k \binom{k}{i} L_p(n+ip)$
$L_{p,n}(2a,b)$	$L_{p,n+k(p+1)}(2a,b) = \sum_{i=0}^k \binom{k}{i} b^{k-i} (2a)^i L_{p,n+ip}(2a,b)$
$L_n(2a,b)$	$L_{n+2k}(2a,b) = \sum_{i=0}^k \binom{k}{i} b^{k-i} (2a)^i L_{n+i}(2a,b)$
$\mathcal{Q}_{p,n}(a)$	$\mathcal{Q}_{p,n+k(p+1)}(a) = \sum_{i=0}^k \binom{k}{i} (2a)^i \mathcal{Q}_{p,n+ip}(a)$
$\mathcal{Q}_n(a)$	$\mathcal{Q}_{n+2k}(a) = \sum_{i=0}^k \binom{k}{i} (2a)^i \mathcal{Q}_{n+i}(a); \quad \mathcal{Q}_{n+2k} = \sum_{i=0}^k \binom{k}{i} 2^i \mathcal{Q}_{n+i}$
$T_n(a)$	$T_{n+2k}(a) = \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} (2a)^i T_{n+i}(a)$
$L_{p,n}(a,2b)$	$L_{p,n+k(p+1)}(a,2b) = \sum_{i=0}^k \binom{k}{i} (2b)^{k-i} a^i L_{p,n+ip}(a,2b)$
$L_n(a,2b)$	$L_{n+2k}(a,2b) = \sum_{i=0}^k \binom{k}{i} (2b)^{k-i} a^i L_{n+i}(a,2b)$
$j_n(b)$	$j_{n+2k}(b) = \sum_{i=0}^k \binom{k}{i} (2b)^{k-i} j_{n+i}(b); \quad j_{n+2k} = \sum_{i=0}^k \binom{k}{i} 2^{k-i} j_{n+i}$

Sequence	For $L_{p,n(p+1)}(a,b) = \sum_{i=0}^n \binom{n}{i} b^{n-i} a^i L_{p,ip}(a,b)$, particular cases are
$L_n(a,b)$	$L_{2n}(a,b) = \sum_{i=0}^n \binom{n}{i} b^{n-i} a^i L_i(a,b)$
$L_{p,n}(a)$	$L_{p,n(p+1)}(a) = \sum_{i=0}^n \binom{n}{i} a^i L_{p,ip}(a)$
$\ell_n(a)$	$\ell_{2n}(a) = \sum_{i=0}^k \binom{k}{i} a^i \ell_i(a); \quad L_{2n} = \sum_{i=0}^n \binom{n}{i} L_i$
$L_p(n)$	$L_p(n(p+1)) = \sum_{i=0}^k \binom{k}{i} L_p(ip)$
$L_{p,n}(2a,b)$	$L_{p,n(p+1)}(2a,b) = \sum_{i=0}^n \binom{n}{i} b^{n-i} (2a)^i L_{p,ip}(2a,b)$
$L_n(2a,b)$	$L_{2n}(2a,b) = \sum_{i=0}^n \binom{n}{i} b^{n-i} (2a)^i L_i(2a,b)$
$Q_{p,n}(a)$	$Q_{p,n(p+1)}(a) = \sum_{i=0}^n \binom{n}{i} (2a)^i Q_{p,ip}(a)$
$Q_n(a)$	$Q_{2n}(a) = \sum_{i=0}^n \binom{n}{i} (2a)^i Q_i(a); \quad Q_{2n} = \sum_{i=0}^n \binom{n}{i} 2^i Q_i$
$T_n(a)$	$T_{2n}(a) = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} (2a)^i T_i(a)$
$L_{p,n}(a,2b)$	$L_{p,n(p+1)}(a,2b) = \sum_{i=0}^n \binom{n}{i} (2b)^{n-i} a^i L_{p,ip}(a,2b)$
$L_n(a,2b)$	$L_{2n}(a,2b) = \sum_{i=0}^n \binom{n}{i} (2b)^{n-i} a^i L_i(a,2b)$
$j_n(b)$	$j_{2n}(b) = \sum_{i=0}^n \binom{n}{i} (2b)^{n-i} j_i(b); \quad j_{2n} = \sum_{i=0}^n \binom{n}{i} 2^{n-i} j_i$

Sequence	For $L_{p,np}(a,b) = \frac{1}{a^n} \sum_{i=0}^n \binom{n}{i} (-b)^{n-i} L_{p,i(p+1)}(a,b)$, particular cases are
$L_n(a,b)$	$L_n(a,b) = \frac{1}{a^n} \sum_{i=0}^n \binom{n}{i} (-b)^{n-i} L_{2i}(a,b)$
$L_{p,n}(a)$	$L_{p,np}(a) = \frac{1}{a^n} \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} L_{p,i(p+1)}(a)$
$\ell_n(a)$	$\ell_n(a) = \frac{1}{a^n} \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} \ell_{2i}(a)$; $L_n = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} L_{2i}$
$L_p(n)$	$L_p(np) = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} L_p(i(p+1))$
$L_{p,n}(2a,b)$	$L_{p,np}(2a,b) = \frac{1}{(2a)^n} \sum_{i=0}^n \binom{n}{i} (-b)^{n-i} L_{p,i(p+1)}(2a,b)$
$L_n(2a,b)$	$L_n(2a,b) = \frac{1}{(2a)^n} \sum_{i=0}^n \binom{n}{i} (-b)^{n-i} L_{2i}(2a,b)$
$Q_{p,n}(a)$	$Q_{p,np}(a) = \frac{1}{(2a)^n} \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} Q_{p,i(p+1)}(a)$
$Q_n(a)$	$Q_n(a) = \frac{1}{(2a)^n} \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} Q_{2i}(a)$; $Q_n = \frac{1}{2^n} \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} Q_{2i}$
$T_n(a)$	$T_n(a) = \frac{1}{(2a)^n} \sum_{i=0}^n \binom{n}{i} T_{2i}(a)$
$L_{p,n}(a,2b)$	$L_{p,np}(a,2b) = \frac{1}{a^n} \sum_{i=0}^n \binom{n}{i} (-2b)^{n-i} L_{p,i(p+1)}(a,2b)$
$L_n(a,2b)$	$L_n(a,2b) = \frac{1}{a^n} \sum_{i=0}^n \binom{n}{i} (-2b)^{n-i} L_{2i}(a,2b)$
$j_n(b)$	$j_n(b) = \sum_{i=0}^n \binom{n}{i} (-2b)^{n-i} j_{2i}(b)$; $j_n = \sum_{i=0}^n \binom{n}{i} (-2)^{n-i} j_{2i}$

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COUPLED AND TRIPLED FIXED POINT THEOREMS ON A METRIC SPACE ENDOWED WITH A BINARY RELATION

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Abstract. The purpose of this paper is to transpose some of the results regarding the coupled fixed point theory, developed by Bhaskar and Lakshmikantham [7], using the approaches of Ben-El-Mechaiekh [2], Asgari, Mousavi [1] and Berinde [3]. The first approach replaces the partial order relation from the Ran-Reurings fixed point theorem [10] with a transitive one, the second one uses a reflexive-only relation and whereas the third one introduces the tripled fixed points. Our aim is to obtain some generalized fixed point theorems, containing a more simple symmetrical condition, also extending them for the case of tripled fixed point.

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Keywords: coupled fixed point, tripled fixed point, symmetrical contractivity condition, binary relation, reflexivity

1. INTRODUCTION

Ran and Reurings [10], in 2003, extended the famous fixed point theorem of Banach, endowing the metric space X with a partial order relation. Thus, the contractivity condition wasn't satisfied unless the elements x and y chosen from the metric space were comparable (i.e. $x \geq y$). Their work inspired many others who started studying the existence and uniqueness of fixed points in partially ordered metric spaces.

In 2006 Bhaskar and Lakshmikantham [7] have studied the coupled fixed points introduced by Guo and Lakshmikantham [8], in the context of a weak contractive condition, providing some fixed point theorems with respect to it.

These results, at their turn, produced enthusiasm among the mathematicians who continued studying the simple and coupled fixed point, obtaining new important results, with applications in different types of equations. In 2011 Berinde [3] extends the coupled fixed point theorems for mixed monotone operators obtained by Bhaskar and Lakshmikantham by significantly weakening the contractivity condition involved. He also introduces a new concept: the tripled fixed point, in collaboration with Borcut [4].

In 2012, Samet and Turinici [11] establish fixed point theorems for contractive mappings on a metric space endowed with an amorphous binary relation. They also show that some fixed point theorems for cyclical contractive mappings can be deduced from their main results.

Ben-el-Mechaiekh [2] observes that, in the proof of Ran and Reurings fixed point theorem, neither the antisymmetry, nor the reflexivity of the partial order relation are being used. In 2014 he replaces the relation used by the two mathematicians with a transitive binary relation. His work is essentially based on the extension that Edelstein brings to Banach's contraction principle in 1961 [6].

One of the latest results in this field is provided by Asgari and Mousavi [1] who replaced the partial order relation used by Ran and Reurings, with a *reflexive relation* that is not necessarily neither transitive, nor antisymmetric. Their paper was published in the spring of 2015 and the key feature of their results is the fact that the contractivity condition holds only on comparable elements with respect to the binary relation.

2. RESULTS

2.1. Preliminary considerations

Now, we start by recalling the definition of (R-)coupled and tripled fixed point; see for instance [7], [4], [1].

Definition 1 ([7]). We call an element $(x, y) \in X \times X$ a **coupled fixed point** of the mapping F , if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2 ([1]). Let X be a nonempty set and let R be a reflexive relation on X , $f : X \times X \rightarrow X$. An element $(x, y) \in X \times X$ is called **R -coupled fixed point** of f , if $f \times f(x, y) \in X_R(x, y)$.

Definition 3 ([4]). We call an element $(x, y, z) \in X \times X$ a **tripled fixed point** of the mapping F , if $F(x, y, z) = x$, $F(y, x, z) = y$ and $F(z, y, x) = z$.

Asgari and Mousavi [1] also redefine the mixed monotone property of a mapping, introduced by Bhaskar and Lakshmikantham [7], endowing the metric space with a reflexive relation. This is given as follows:

Definition 4 ([1]). Let X be a nonempty set and let R be a reflexive relation on X , $f : X \times X \rightarrow X$. The mapping f has the mixed **R -monotone property** on X if $f \times f(X_R(x, y)) \subseteq X_R(f \times f(x, y))$, for all $(x, y) \in X \times X$, where $X_R(x, y) = \{(z, t) \in X \times X : zRx \wedge yRt\}$, $\forall (x, y) \in X \times X$.

In [1] Asgari and Mousavi also prove a coupled fixed point theorem with respect to a reflexive relation.

Theorem 1 ([1]). Let (X, d) be a metric space and R a reflexive relation on X . If $f : X \times X \rightarrow X$ is a mapping such that:

- f has the mixed R -monotone property on X .
- (X, d) is a complete metric space.
- f has an R -coupled fixed point, i.e. there exists $(x_0, y_0) \in X \times X$ such that $f \times f(x_0, y_0) \in X_R(x_0, y_0)$.
- there exists a constant $k \in [0, 1)$ such that:

$$d(f(x, y), f(z, t)) \leq \frac{k}{2}[d(x, z) + d(y, t)], \forall (x, y) \in X_R(z, t).$$

- f is orbitally continuous.

Then:

- There exists $x^*, y^* \in X$ such that $f(x^*, y^*) = x^*$ and $f(y^*, x^*) = y^*$.
- The sequences $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ defined by $x_{n+1} = f(x_n, y_n)$ and $y_{n+1} = f(y_n, x_n)$ converge respectively to x^* and y^* .
- The error estimation is given by :

$$\max_{n \in \mathbb{N}} \{d(x_n, x^*), d(y_n, y^*)\} \leq \frac{k^n}{2(1-k)} [d(f(x_0, y_0), x_0) + d(f(y_0, x_0), y_0)].$$

In [9], Rus and Petruşel emphasize the fact that, when working on a set that is endowed with a metric and a binary relation (an order structure in [9]), we should add an additional assumption, namely the compatibility between the two structures:

Remark 1 ([9]). If $\{x_n\}_{n \in \mathbb{N}} \rightarrow x, \{y_n\}_{n \in \mathbb{N}} \rightarrow y$ and $x_n \leq y_n$ then $x \leq y, \forall n \in \mathbb{N}$.

Thus, the compatibility condition between the metric and the reflexive relation R used in the theorem above [1] should be :

Remark 2. If $x_n R y_n$ then $\lim_{n \rightarrow \infty} x_n R \lim_{n \rightarrow \infty} y_n, \forall n \in \mathbb{N}$.

2.2. A coupled fixed point theorem

The following result generalizes the main result of Asgari and Mousavi [1], using a more general symmetrical condition, like the one used by Berinde [3].

Theorem 2. Let (X, d) be a metric space, R be a binary reflexive relation on X such that R and d are compatible. If $F : X^2 \rightarrow X$ is a mapping having an R -coupled fixed point, the mixed R -monotone property on X , it is orbitally continuous and $\exists k \in [0, 1)$ such that

$$d(F(x, y), F(z, t)) + d(F(y, x), F(t, z)) \leq k[d(x, z) + d(y, t)], \quad \forall (x, y) \in X_R(z, t). \quad (2.1)$$

Then:

- (1) $\exists (\bar{x}, \bar{y}) \in X^2$ such that $F(\bar{x}, \bar{y}) = \bar{x}$ and $F(\bar{y}, \bar{x}) = \bar{y}$.
- (2) The sequences $\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}}$, defined by $x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n)$, converge to $\bar{x}$ and $\bar{y}$, respectively.

(3) The error estimation that holds is:

$$\max_{n \in \mathbb{N}} \{d(x_n, \bar{x}), d(y_n, \bar{y})\} \leq \frac{k^n}{1-k} [d(f(x_0, y_0), x_0) + d(f(y_0, x_0), y_0)].$$

Proof. Since the mapping F admits an R -coupled fixed point, let $(x_0, y_0) \in X \times X$ be it, we have $F \times F(x_0, y_0) \in X_R(x_0, y_0)$. Further, using the mixed R -monotone property of F , we have $F \times F(x_0, y_0) \in X_R(F(x_0, y_0), F(y_0, x_0))$. Using the induction, we can easily prove that:

$$(F^n(x_0, y_0), F^n(y_0, x_0)) \in X_R(F^{n-1}(x_0, y_0), F^{n-1}(y_0, x_0)). \quad (2.2)$$

We define $d_2 : X^2 \times X^2 \rightarrow \mathbb{R}_+$ $d_2(Y, Z) = \frac{1}{2}[d(x, z) + d(y, t)]$, $\forall Y = (x, y), Z = (z, t) \in X^2$.

d_2 is a metric on X^2 because:

- $d_2(Y, Z) = 0 \Leftrightarrow Y = Z$ is a simple task to check, using the definition of d_2 and the fact that d is a metric.
- $d_2(Y, Z) = d_2(Z, Y)$, $\forall Y, Z \in X^2$ holds, because d is a metric, and the sum in d_2 's definition is commutative.
- $d_2(Y, Z) \leq d_2(Y, T) + d_2(T, Z)$, $\forall Y, T, Z \in X^2$ can also be easily checked.

Therefore (X^2, d_2) is a complete metric space.

We consider the operator:

$T : X^2 \rightarrow X^2$ defined by $T(Y) = (F(x, y), F(y, x))$, $\forall Y = (x, y) \in X^2$.

For $Y = (x, y), Z = (z, t) \in X^2$, considering the definition for d_2 , we have:

$$d_2(T(Y), T(Z)) = \frac{d(F(x, y), F(z, t)) + d(F(y, x), F(t, z))}{2}$$

and

$$d_2(Y, Z) = \frac{d(x, z) + d(y, t)}{2}.$$

By the contractivity condition (2.1) we have

$$d_2(T(Y), T(Z)) \leq k \cdot d_2(Y, Z), \forall Y, Z \in X^2, Y \geq Z. \quad (2.3)$$

From (2.2), we have the monotony of T and that $\{Z_n\}_{n \geq 0}$ is nondecreasing, we denote $Z_n = (F^{n-1}(x_0, y_0), F^{n-1}(y_0, x_0))$.

We denote $Y = Z_n \geq Z_{n-1} = V$.

We replace this in (2.3), obtaining:

$$\begin{aligned} d_2(T(Z_n), T(Z_{n-1})) &\leq k \cdot d_2(Z_n, Z_{n-1}), n \geq 1 \Leftrightarrow \\ &\Leftrightarrow d_2(Z_{n+1}, Z_n) \leq k \cdot d_2(Z_n, Z_{n-1}), n \geq 1. \end{aligned}$$

Using the induction, we have:

$$d_2(Z_{n+1}, Z_n) \leq k^n \cdot d_2(Z_1, Z_0), n \geq 1.$$

Let $i < j$. We get:

$$\begin{aligned} d_2(Z_i, Z_j) &\leq \sum_{l=i+1}^j d_2(Z_l, Z_{l-1}) \leq (k^i + k^{i+1} + \dots + k^{j-i-1}) \cdot d_2(Z_1, Z_0) \leq \\ &\leq k^i \frac{1 - k^{j-i-1}}{1 - k} \cdot d_2(Z_1, Z_0) \end{aligned} \quad (2.4)$$

$\Rightarrow \{Z_n\}_{n \geq 0}$ is a Cauchy sequence in the complete metric space $(X^2, d_2) \Rightarrow$

$$\Rightarrow \lim_{n \rightarrow \infty} Z^n = \bar{Z}.$$

We now use (2.2): $T(\bar{Z}) = \bar{Z} \Leftrightarrow (F(\bar{x}, \bar{y}), F(\bar{y}, \bar{x})) = (\bar{x}, \bar{y})$

$\Leftrightarrow F(\bar{x}, \bar{y}) = \bar{x}, F(\bar{y}, \bar{x}) = \bar{y} \Leftrightarrow (\bar{x}, \bar{y})$ is the coupled fixed point for F .

Since (X, d) is a complete metric space, $\exists \bar{x}, \bar{y} \in X$ such that

$F^n(x_0, y_0) \rightarrow \bar{x}, F^n(y_0, x_0) \rightarrow \bar{y}, n \rightarrow \infty$. Using the fact that F is orbitally continuous, we have:

$$\begin{aligned} \{x_n\}_{n \in \mathbb{N}} &\rightarrow \bar{x}, x_{n+1} = F(x_n, y_n) \\ \{y_n\}_{n \in \mathbb{N}} &\rightarrow \bar{y}, y_{n+1} = F(y_n, x_n) \end{aligned}$$

So, by (2.4) we have:

$$d_2((x_n, y_n), (\bar{x}, \bar{y})) \leq \frac{k^n}{1 - k} \cdot d_2((x_1, y_1), (x_0, y_0)), n \geq 0.$$

We return to the original metric d :

$$\begin{aligned} \frac{d(x_n, \bar{x}) + d(y_n, \bar{y})}{2} &\leq \frac{k^n}{1 - k} \cdot \frac{d(x_1, x_0) + d(y_1, y_0)}{2} \Leftrightarrow \\ \Leftrightarrow d(x_n, \bar{x}) + d(y_n, \bar{y}) &\leq \max_{n \in \mathbb{N}} \{d(x_n, \bar{x}), d(y_n, \bar{y})\} \leq \frac{k^n}{1 - k} \cdot [d(x_1, x_0) + d(y_1, y_0)]. \end{aligned}$$

But $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$. We get:

$$\max_{n \in \mathbb{N}} \{d(x_n, \bar{x}), d(y_n, \bar{y})\} \leq \frac{k^n}{1 - k} \cdot [d(F(x_0, y_0), x_0) + d(F(y_0, x_0), y_0)].$$

□

2.3. A tripled fixed point theorem

We now extend the results of Asgari and Mousavi [1], using the tripled fixed points concept introduced by Berinde and Borcut [4]:

Definition 5. Let X be a topological space and $f, g : X \times X \times X \rightarrow X$. Then:

- An element $(x, y, z) \in X \times X \times X$ is called a **tripled attractor basin element** of f with respect to $(x^*, y^*, z^*) \in X \times X \times X$, if $f^n(x, y, z) \rightarrow x^*$, $f^n(y, x, z) \rightarrow y^*$ and $f^n(z, x, y) \rightarrow z^*$, when $n \rightarrow \infty$. The set of these points (x^*, y^*, z^*) will be denoted with $A_f^c(x^*, y^*, z^*)$, and for x^* , with $A_f(x^*)$.

- The mapping f is called **orbitally continuous** if $(x, y, z), (a, b, c) \in X \times X \times X$ and $f^{n_k}(x, y, z) \rightarrow a, f^{n_k}(y, x, z) \rightarrow b, f^{n_k}(z, x, y) \rightarrow c$, when $k \rightarrow \infty$, implies $f^{n_k+1}(x, y, z) \rightarrow f(a, b, c), f^{n_k+1}(y, x, z) \rightarrow f(b, a, c)$ and $f^{n_k+1}(z, x, y) \rightarrow f(c, a, b)$, when $k \rightarrow \infty$.
- Let X be a nonempty set and let R be a reflexive relation on X . For every $(t, u, v) \in X \times X \times X$ we define $X_R(t, u, v) = \{(x, y, z) \in X \times X \times X : xRt \wedge uRy \wedge zRv\}$.

Definition 6. Let X be a nonempty set and let R be a reflexive relation on X , $f : X \times X \times X \rightarrow X$.

- The mapping f has the **mixed R -monotone property** on X if $f \times f(X_R(x, y, z)) \subseteq X_R(f \times f(x, y, z))$, for all $(x, y, z) \in X \times X \times X$.
- An element $(x, y, z) \in X \times X \times X$ is called **R -tripled fixed point** of f , if $f \times f(x, y, z) \in X_R(x, y, z)$.

Theorem 3. Let (X, d) be a metric space and R be a binary reflexive relation on X such that R and d are compatible. If $F : X^3 \rightarrow X$ is a mapping having an R -tripled fixed point, the mixed R -monotone property on X , it is orbitally continuous and $\exists k \in [0, 1)$ such that

$$\begin{aligned} d(F(x, y, z), F(t, u, v)) &\leq k \cdot [d(x, t) + d(y, u) + d(z, v)], \\ \forall (x, y, z) \in X_R(t, u, v), k &< 1. \end{aligned} \quad (2.5)$$

Then:

- (1) $\exists (\bar{x}, \bar{y}, \bar{z}) \in X^3$ such that $F(\bar{x}, \bar{y}, \bar{z}) = \bar{x}, F(\bar{y}, \bar{x}, \bar{z}) = \bar{y}, F(\bar{z}, \bar{y}, \bar{x}) = \bar{z}$.
- (2) The sequences $\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}}, \{z_n\}_{n \in \mathbb{N}}$, defined by $x_{n+1} = F(x_n, y_n, z_n), y_{n+1} = F(y_n, x_n, z_n), z_{n+1} = F(z_n, y_n, x_n)$, converge to $\bar{x}, \bar{y}$ and $\bar{z}$, respectively.
- (3) The error estimation that holds is: $\max_{n \in \mathbb{N}} \{d(x_n, \bar{x}), d(y_n, \bar{y}), d(z_n, \bar{z})\} \leq \frac{k^n}{1-k} [d(F(x_0, y_0, z_0), x_0) + d(F(y_0, x_0, z_0), y_0) + d(F(z_0, y_0, x_0), z_0)]$.

Proof. Since the mapping F admits an R -tripled fixed point, let $(x_0, y_0, z_0) \in X \times X \times X$ be it, we have $F \times F(x_0, y_0, z_0) \in X_R(x_0, y_0, z_0)$. Further, using the mixed R -monotone property of F , we have

$$F \times F(x_0, y_0, z_0) \in X_R(F(x_0, y_0, z_0), F(y_0, x_0, z_0), F(z_0, y_0, x_0)).$$

Using the induction, we can easily prove that:

$$\begin{aligned} (F^n(x_0, y_0, z_0), F^n(y_0, x_0, z_0), F^n(z_0, y_0, x_0)) &\in \\ \in X_R(F^{n-1}(x_0, y_0, z_0), F^{n-1}(y_0, x_0, z_0), F^{n-1}(z_0, y_0, x_0)). \end{aligned} \quad (2.6)$$

We claim that, for $n \in \mathbb{N}$

$$\begin{aligned} d(F^{n+1}(x_0, y_0, z_0), F^n(x_0, y_0, z_0)) &\leq \\ k^n \cdot [d(F(x_0, y_0, z_0), y_0) + d(F(y_0, z_0, x_0), y_0) + d(F(z_0, x_0, y_0), z_0)]. \end{aligned} \quad (2.7)$$

For $n = 1$, we get:

$$\begin{aligned} d(F^2(x_0, y_0, z_0), F(x_0, y_0, z_0)) &= d(F(F(x_0, y_0, z_0), F(y_0, z_0, x_0), F(z_0, x_0, y_0)), \\ &\quad F(x_0, y_0, z_0)) \\ &\leq k \cdot [d(F(x_0, y_0, z_0), x_0) + d(F(y_0, z_0, x_0), y_0) + d(F(z_0, x_0, y_0), z_0)]. \end{aligned}$$

Now, we assume that (2.7) holds. Using the forth assumption from the hypothesis we get:

$$\begin{aligned} d(F^{n+2}(x_0, y_0, z_0), F^{n+1}(x_0, y_0, z_0)) &= \\ d(F(F^{n+1}(x_0, y_0, z_0), F^{n+1}(y_0, z_0, x_0), F^{n+1}(z_0, x_0, y_0)), \\ &\quad F(F^n(x_0, y_0, z_0), F^n(y_0, z_0, x_0), F^n(z_0, x_0, y_0))) \leq \\ &\leq k \cdot [d(F^{n+1}(x_0, y_0, z_0), F^n(x_0, y_0, z_0)) + \\ &\quad d(F^{n+1}(y_0, z_0, x_0), F^n(y_0, z_0, x_0)) + d(F^{n+1}(z_0, x_0, y_0), F^n(z_0, x_0, y_0))] \leq \\ &\leq k^{n+1} \cdot [d(F(x_0, y_0, z_0), x_0) + d(F(y_0, z_0, x_0), y_0) + d(F(z_0, x_0, y_0), z_0)]. \end{aligned}$$

This implies that $\{F^n(x_0, y_0, z_0)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X .

Similarly, following the same steps, we can prove that $\{F^n(y_0, z_0, x_0)\}_{n \in \mathbb{N}}$ and $\{F^n(z_0, x_0, y_0)\}_{n \in \mathbb{N}}$ are also Cauchy sequences in X .

Since X is a complete metric space, there exist $\bar{x}, \bar{y}, \bar{z} \in X$ such that $F^n(x_0, y_0, z_0) \rightarrow \bar{x}$, $F^n(y_0, z_0, x_0) \rightarrow \bar{y}$, $F^n(z_0, x_0, y_0) \rightarrow \bar{z}$, $n \rightarrow \infty$.

Let $m > n$. Then:

$$\begin{aligned} d(F^m(x_0, y_0, z_0), F^n(x_0, y_0, z_0)) &\leq \sum_{j=n}^{m-1} d(F^{j+1}(x_0, y_0, z_0), F^j(x_0, y_0, z_0)) \leq \\ &\leq (k^{n-1} + k^n + \dots + k^{m-n}) \\ &\quad \cdot [d(F(x_0, y_0, z_0), x_0) + d(F(y_0, z_0, x_0), y_0) + d(F(z_0, x_0, y_0), z_0)] \\ &= \frac{k^n - k^m}{1 - k} \cdot [d(F(x_0, y_0, z_0), x_0) + d(F(y_0, z_0, x_0), y_0) + d(F(z_0, x_0, y_0), z_0)] < \\ &< \frac{k^n}{1 - k} \cdot [d(F(x_0, y_0, z_0), x_0) + d(F(y_0, z_0, x_0), y_0) + d(F(z_0, x_0, y_0), z_0)]. \end{aligned}$$

But $x_{n+1} = F(x_n, y_n, z_n)$, $y_{n+1} = F(y_n, x_n, z_n)$ and $z_{n+1} = F(z_n, y_n, x_n)$, $F^n(x_0, y_0, z_0) \rightarrow \bar{x}$, $F^n(y_0, z_0, x_0) \rightarrow \bar{y}$, $F^n(z_0, x_0, y_0) \rightarrow \bar{z}$, $n \rightarrow \infty$ and F is orbitally continuous. Thus, applying maximum to the last relation, we get:

$$\begin{aligned} \max_{n \in \mathbb{N}} \{d(x_n, \bar{x}), d(y_n, \bar{y}), d(z_n, \bar{z})\} &\leq \frac{k^n}{1 - k} \cdot [d(F(x_0, y_0, z_0), x_0) + \\ &\quad + d(F(y_0, x_0, z_0), y_0) + d(F(z_0, y_0, x_0), z_0)]. \end{aligned}$$

□

3. EXAMPLES AND CONCLUSIONS

Example 1. The Theorem 2 is more general than the fixed point theorem of Asgari and Mousavi [1] because of the new contractivity condition, fact illustrated by the following example:

Let $X = \mathbb{R}$, the metric $d(x, y) = |x - y|$, the relation R on X given by

$$xRy \Leftrightarrow \frac{x-2y}{3} = \frac{y-2x}{3}.$$

Let $F : X \times X \rightarrow X$ be defined by $F(x, y) = \frac{x-2y}{3}$. So, $\forall (x, y) \in X^2$, F has the mixed R -monotone property, satisfies (2.1), but does not satisfy the contractivity condition in Asgari and Mousavi's fixed point theorem [1]. Indeed, assume that there exists $k \in [0, 1)$ such that (1) holds. This means

$$\left| \frac{x-2y}{3} - \frac{z-2t}{3} \right| \leq \frac{k}{2} [|x-z| + |y-t|], \forall (x, y) \in X_R(z, t)$$

by which, for $x = z$, we get

$$\frac{2}{3} |y-t| \leq \frac{k}{2} |y-t|$$

which, for $y \neq t$, would imply $\frac{4}{3} \leq k$, where $k < 1$, a contradiction. Now, we prove that (2.1) holds. We have

$$\left| \frac{x-2y}{3} - \frac{z-2t}{3} \right| \leq \frac{1}{2} |x-z| + \frac{1}{3} |y-t|, \forall (x, y) \in X_R(z, t)$$

and

$$\left| \frac{y-2x}{3} - \frac{t-2z}{3} \right| \leq \frac{1}{2} |y-t| + \frac{1}{3} |x-z|, \forall (x, y) \in X_R(z, t).$$

By summing up these two relations, we get (2.1) with $k = \frac{5}{6} < 1$.

Example 2. Let $X = \mathbb{R}$, the metric $d(x, y) = |x - y|$, the relation R on X given by

$$(x, y, z)R(t, u, v) \Leftrightarrow xRt \wedge yRu \wedge zRv,$$

where $xRt \Leftrightarrow x^2 + 2x = t^2 + 2t$. Let $F : X \times X \times X \rightarrow X$ be defined by $F(x, y, z) = x^2 - 2x - 4$. So, $\forall (x, y, z) \in X^3$, we have :

$$X_R(x, y, z) = \{(x, y, z), (x, y+2, z), (x+2, y, z), (x+2, y+2, z), (x, y, z+2), (x+2, y+2, z+2), (x+2, y, z+2), (x, y+2, z+2)\}.$$

$$F \times F(X_R(x, y, z)) = \{F \times F(x, y, z)\} \subseteq X_R(F \times F(x, y, z))$$

So, F has the mixed R -monotone property. Furthermore, F is continuous and $\exists (-3, 5, 2) \in X \times X \times X$ such that $F(-3, 5, 2) \in X_R(-3, 5, 2)$. Therefore, the hypotheses of the Theorem 3 is satisfied. The tripled fixed points are: $(4, 4, 4)$, $(4, -1, 4)$, $(-1, 4, 4)$, $(-1, -1, 4)$, $(4, 4, -1)$, $(-1, 4, -1)$, $(4, -1, -1)$, $(-1, -1, -1)$.

In their research, Asgari and Mousavi point and use the fact that the binary relation used in the previous results in the field doesn't have to be necessarily antisymmetric, reflexive or transitive. Thus, they use a reflexive binary relation, obtaining new, original results. In this paper, we took advantage of the concept of R-coupled fixed point and mixed R-monotone property of a mapping [1] to introduce new coupled and tripled fixed point results with respect to a reflexive relation, using symmetrical contractivity conditions, in the sense of Berinde et. al. [3], [5].

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WALKS IN PATH GRAPH ON FOUR VERTICES AND FIBONACCI SEQUENCE

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Abstract. Using elementary knowledge of graph theory, we show that a path graph on four vertices exhibits *Fibonacci structure*. For arbitrary start and end vertices, the number of walks of any length is given by a Fibonacci number.

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Keywords: path graph, walk in graph, Fibonacci sequence, Fibonacci structure

The Fibonacci sequence is a well known mathematical object, having interesting connections to different features in nature. It is given by a simple recurrence formula.

Definition 1. A sequence $\Phi : \mathbb{N} \rightarrow \mathbb{N}$ defined with

$$\Phi(n+1) = \Phi(n) + \Phi(n-1), \quad \forall n \in \mathbb{N}, \quad (0.1)$$

with initial values $\Phi(0) = 0$, $\Phi(1) = 1$ is called the *Fibonacci sequence*.

In this short comment we reveal the link between Fibonacci numbers and number of walks in a path graph on four vertices, i.e. of length three, see graph P_3 in Figure 1. Note that the Fibonacci structure is linked with P_3 and cannot be generalized to P_N .



FIGURE 1. Path-graph of length three.

To avoid any confusion, we recall the definition of a walk in a graph, see [2, p. 9]. A *walk* of length k in graph G is a non-empty alternating sequence $v_0 e_0 v_1 e_1 \dots e_{k-1} v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$.

We denote with

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (0.2)$$

the *adjacency* matrix of P_3 . An elementary assertion of graph theory (see [1, Lemma 2.5]) gives the *number of walks of length k between vertices i and j* as the appropriate element of k -th power of the adjacency matrix B of the graph, i.e.

$$w_{k,i,j} := \{B^k\}_{i,j}. \quad (0.3)$$

The following lemma enables us to compute powers of adjacency matrix of a path graph (regardless the number of vertices).

Lemma 1. *Let $B \in \mathbb{R}^{N \times N}$ be an adjacency matrix of a path graph of length $N - 1$. Then for any $k \in \mathbb{N}$ it holds that*

$$\{B^{k+1}\}_{i,j} = \{B^k\}_{i-1,j} + \{B^k\}_{i+1,j}, \quad (0.4)$$

and

$$\{B^{k+1}\}_{i,j} = \{B^k\}_{i,j-1} + \{B^k\}_{i,j+1}, \quad (0.5)$$

where we set $\{B^p\}_{l,m} = 0$ for any $p \in \mathbb{N}$ and $(l,m) \notin \{1, \dots, N\}^2$.

Proof. The claim immediately follows from graph theory interpretation of (0.3): Any walk of length $k + 1$ between vertices i and j contains a unique subwalk of length k from either $i - 1$ or $i + 1$ (if such vertex exists) to j . Hence we can conclude (0.4), similar argument proves also (0.5). $\square$

Note that any path-graph is a bipartite graph, which implies non-existence of walk of odd length between vertices of the same parity (and walk of even length between vertices of different parity), i.e. $\{B^k\}_{i,j} = 0$ if the parity of k and $i + j$ differs. Using this property we define the following split of k -th power of the adjacency matrix of P_3 , to matrices of corner, edge and middle elements, as they represent three qualitatively distinct cases.

We set matrices C_k, E_k, M_k for $k \in \mathbb{N}$ with

$$\{C_k\}_{i,j} := \begin{cases} \{A^k\}_{i,j} & (i,j) \in \{1,4\} \times \{1,4\} \text{ and } (i+j+k) \text{ even,} \\ 0 & \text{otherwise,} \end{cases} \quad (0.6)$$

$$\{E_k\}_{i,j} := \begin{cases} \{A^k\}_{i,j} & (i,j) \in (\{2,3\} \times \{1,4\}) \cup (\{1,4\} \times \{2,3\}) \\ & \text{and } (i+j+k) \text{ even,} \\ 0 & \text{otherwise,} \end{cases} \quad (0.7)$$

$$\{M_k\}_{i,j} := \begin{cases} \{A^k\}_{i,j} & (i,j) \in \{2,3\} \times \{2,3\} \text{ and } (i+j+k) \text{ even,} \\ 0 & \text{otherwise.} \end{cases} \quad (0.8)$$

They create a *complete split* of A^k , i.e. $A^k = C_k + E_k + M_k$ and maximally one of them has a non-zero contribution for each element $\{A^k\}_{i,j}$, as it is illustrated in the following scheme.

$$A^k = \begin{pmatrix} 0 & E_k & 0 & C_k \\ E_k & 0 & M_k & 0 \\ 0 & M_k & 0 & E_k \\ C_k & 0 & E_k & 0 \end{pmatrix}, \quad A^l = \begin{pmatrix} C_l & 0 & E_l & 0 \\ 0 & M_l & 0 & E_l \\ E_l & 0 & M_l & 0 \\ 0 & E_l & 0 & C_l \end{pmatrix},$$

for $k = 2s - 1, l = 2s, s \in \mathbb{N}$.

We call elements defined by first lines in (0.6), (0.7) or (0.8) *structural non-zeros*. Note that C_1 is a zero matrix, the only one whose structural non-zeros attain zero value.

Matrix C_k of corner elements represents walks between vertices of the degree 1 (not necessarily distinct), E_k represents the walks connecting vertices with different degrees and finally M_k represents the walks between vertices with degree equal to 2.

We use the previously established split in the next Theorem.

Theorem 1. *For given $k \in \mathbb{N}$, the structural non-zeros of matrix $C_k(E_k, M_k)$ are equal to a constant, which we denote by $c_k(e_k, m_k)$. Moreover, it holds that*

$$c_k = \Phi(k - 1), \quad e_k = \Phi(k), \quad m_k = \Phi(k + 1). \quad (0.9)$$

Proof. Applying (0.5) to all structural non-zeros of C_{k+1} we get

$$c_{k+1} = e_k. \quad (0.10)$$

Applying (0.4) and (0.5) to structural non-zeros of E_{k+1} gives two relations

$$e_{k+1} = m_k, \quad (0.11)$$

$$e_{k+1} = c_k + e_k. \quad (0.12)$$

Combining (0.10) and (0.12) one gets

$$c_{k+2} = e_{k+1} = c_k + e_k = c_{k+1} + c_k.$$

From (0.2) we deduce initial values $c_1 = 0$ and, with the help of (0.10), $c_2 = e_1 = 1$. Hence we conclude $c_k = \Phi(k - 1)$ for any $k \in \mathbb{N}$. Then (0.10) and (0.11) imply $e_k = \Phi(k)$ and $m_k = \Phi(k + 1)$, for $k \in \mathbb{N}$. $\square$

We can conclude from Theorem 1, that any non-zero entry in A^k is a Fibonacci number. In other words, number of k -walks in P_3 from i to j is represented by a Fibonacci number $\Phi(n) = \Phi(n(i, j, k))$, where $n(i, j, k) \in \{k - 1, k, k + 1\}$ according to (0.6)–(0.8) and Theorem 1.

Remark 1. The link between Fibonacci sequence and walks in path graph was revealed during the research on a generalization of bistable equation with non-smooth minima, see [3] or [4]. When substituting the double-well potential with a *quadruple-well* one, then the nontrivial and qualitatively distinct stationary solutions can be

represented by walks in graph P_3 , provided additional geometrical constraints are met by the potential to ensure the transitions between minima have the equal length.

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BOUNDS FOR THE ARITHMETIC MEAN IN TERMS OF THE TOADER MEAN AND OTHER BIVARIATE MEANS

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Abstract. We find the greatest values α_1 and α_2 , and the least values β_1 and β_2 , such that the double inequalities $\alpha_1 T(a, b) + (1 - \alpha_1)H(a, b) < A(a, b) < \beta_1 T(a, b) + (1 - \beta_1)H(a, b)$ and $\alpha_2 T(a, b) + (1 - \alpha_2)\overline{H}(a, b) < A(a, b) < \beta_2 T(a, b) + (1 - \beta_2)\overline{H}(a, b)$ hold for all $a, b > 0$ with $a \neq b$. Here, $\overline{H}(a, b) = \sqrt{2ab}/\sqrt{a^2 + b^2}$, $H(a, b) = 2ab/(a + b)$, $A(a, b) = (a + b)/2$, and $T(a, b) = \frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta$ denote the harmonic root-square, harmonic, arithmetic and Toader means of a and b , respectively.

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Keywords: harmonic root-square mean, harmonic, arithmetic, Toader mean, complete elliptic integrals

1. INTRODUCTION

In [16], Toader introduced the Toader mean $T(a, b)$ of two positive numbers a and b as follows:

$$\begin{aligned} T(a, b) &= \frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta \\ &= \begin{cases} 2a\mathcal{E}(\sqrt{1 - (b/a)^2})/\pi, & a > b, \\ 2b\mathcal{E}(\sqrt{1 - (a/b)^2})/\pi, & a < b, \\ a, & a = b. \end{cases} \end{aligned} \quad (1.1)$$

where $\mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{1/2} d\theta$ ($r \in [0, 1]$) is the complete elliptic integral of the second kind.

Recently, the Toader mean has been the object of intensive research. In particular, many remarkable inequalities for the Toader mean $T(a, b)$ can be found in the literatures [9–12, 14–16]

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Let $\overline{H}(a, b) = \sqrt{2ab}/\sqrt{a^2+b^2}$, $H(a, b) = 2ab/(a+b)$, $G(a, b) = \sqrt{ab}$, $A(a, b) = (a+b)/2$, $S(a, b) = \sqrt{(a^2+b^2)/2}$, $C(a, b) = (a^2+b^2)/(a+b)$ be the harmonic root-square, harmonic, geometric, arithmetic, quadratic, and contraharmonic means of a and b , respectively. Then it is known that the inequalities

$$\overline{H}(a, b) < H(a, b) < G(a, b) < A(a, b) < S(a, b) < C(a, b) \quad (1.2)$$

hold for all $a, b > 0$ with $a \neq b$.

For $p \in \mathbb{R}$, $a, b > 0$ with $a \neq b$ the power mean $M_p(a, b)$ is defined by

$$M_p(a, b) = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \quad (1.3)$$

It is well known that the power mean $M_p(a, b)$ is continuous and strictly increasing with respect to p . Many means are special case of $M_p(a, b)$, for example, $M_{-1}(a, b) = H(a, b) = 2ab/(a+b)$, $M_0(a, b) = G(a, b) = \sqrt{ab}$, $M_1(a, b) = A(a, b) = (a+b)/2$, $M_2(a, b) = S(a, b) = \sqrt{(a^2+b^2)/2}$.

Vuorinen [17] conjectured that

$$M_{3/2}(a, b) < T(a, b) \quad (1.4)$$

for all $a, b > 0$ with $a \neq b$. This conjecture was solved by Qiu and Shen[14], and Barnard, Pearce and Richards in [6], respectively.

In [1], Alzer and Qiu presented that

$$T(a, b) < M_{\log 2 / \log(\pi/2)}(a, b) \quad (1.5)$$

for all $a, b > 0$ with $a \neq b$, which gives a best possible upper bound for Toader mean in terms of the power mean.

From (1.4) and (1.5) one concludes that

$$A(a, b) < T(a, b) < S(a, b) \quad (1.6)$$

for all $a, b > 0$ with $a \neq b$.

In [11], the authors demonstrated that the double inequality

$$\alpha S(a, b) + (1-\alpha)A(a, b) < T(a, b) < \beta S(a, b) + (1-\beta)A(a, b) \quad (1.7)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq \frac{1}{2}$ and $\beta \geq \frac{4-\pi}{(\sqrt{2}-1)\pi}$.

In [15], the authors proved that the double inequalities

$$\begin{aligned} \alpha_1 C(a, b) + (1-\alpha_1)A(a, b) < T(a, b) < \beta_1 C(a, b) + (1-\beta_1)A(a, b), \\ \frac{\alpha_2}{A(a, b)} + \frac{1-\alpha_2}{C(a, b)} < \frac{1}{T(a, b)} < \frac{\beta_2}{A(a, b)} + \frac{1-\beta_2}{C(a, b)} \end{aligned} \quad (1.8)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1/4$, $\beta_1 \geq 4/\pi - 1$ and $\alpha_2 \leq \pi/2 - 1$, $\beta_2 \geq 3/4$.

The main purpose of this paper is to find the greatest values α_1, α_2 and the least values β_1, β_2 in $(0, 1)$ such that the double inequalities

$$\begin{aligned}\alpha_1 T(a, b) + (1 - \alpha_1) H(a, b) &< A(a, b) < \beta_1 T(a, b) + (1 - \beta_1) H(a, b), \\ \alpha_2 T(a, b) + (1 - \alpha_2) \overline{H}(a, b) &< A(a, b) < \beta_2 T(a, b) + (1 - \beta_2) \overline{H}(a, b),\end{aligned}$$

hold for all $a, b > 0$ with $a \neq b$.

As an application, we get a new lower bound for the complete elliptic integral of the second kind in terms of elementary functions, which improves some well-known results.

2. PRELIMINARIES AND LEMMAS

In order to establish our main result we need several lemmas, which we present in this section.

For $0 < r < 1$ and $r' = \sqrt{1 - r^2}$ Legendre's complete elliptic integrals of the first and second kinds [7, 8] are defined by

$$\begin{cases} \mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{-1/2} d\theta, \\ \mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r'), \\ \mathcal{K}(0) = \pi/2, \quad \mathcal{K}(1) = \infty \end{cases}$$

and

$$\begin{cases} \mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{1/2} d\theta, \\ \mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r'), \\ \mathcal{E}(0) = \pi/2, \quad \mathcal{E}(1) = 1, \end{cases}$$

respectively.

For $0 < r < 1$, the following formulas were presented in [4, Appendix E, pp. 474-475]:

$$\begin{aligned}\frac{d\mathcal{K}}{dr} &= \frac{\mathcal{E} - r'^2 \mathcal{K}}{r r'^2}, & \frac{d\mathcal{E}}{dr} &= \frac{\mathcal{E} - \mathcal{K}}{r}, \\ \frac{d(\mathcal{E} - r'^2 \mathcal{K})}{dr} &= r \mathcal{K}, & \frac{d(\mathcal{K} - \mathcal{E})}{dr} &= \frac{r \mathcal{E}}{r'^2}, \\ \mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) &= \frac{2\mathcal{E} - r'^2 \mathcal{K}}{1+r}.\end{aligned}\tag{2.1}$$

The following lemma can be found in [4, Theorem 3.21 (1)].

Lemma 1. $(\mathcal{E} - r'^2 \mathcal{K})/r^2$ is strictly increasing from $(0, 1)$ onto $(\pi/4, 1)$.

Lemma 2 (Theorem 1.25 in [4]). For $-\infty < a < b < \infty$, let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$, and be differentiable on (a, b) , let $g'(x) \neq 0$ on (a, b) . If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b) , then so are

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

3. MAIN RESULTS

Now we are in a position to state and prove our main results.

Theorem 1. *The double inequality*

$$\alpha_1 T(a, b) + (1 - \alpha_1) H(a, b) < A(a, b) < \beta_1 T(a, b) + (1 - \beta_1) H(a, b) \quad (3.1)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq \frac{\pi}{4}$ and $\beta_1 \geq \frac{4}{5}$.

Proof. Without loss of generality, we assume that $a > b$. Let $t = b/a \in (0, 1)$ and $r = \frac{1-t}{1+t}$, then

$$\begin{aligned} \frac{A(a, b) - H(a, b)}{T(a, b) - H(a, b)} &= \frac{\frac{1+t}{2} - \frac{2t}{1+t}}{\frac{2}{\pi} \mathcal{E}'(t) - \frac{2t}{1+t}} \\ &= \frac{\frac{1}{1+r} - (1-r)}{\frac{2}{\pi} \mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) - (1-r)} \\ &= \frac{r^2}{\frac{2}{\pi} (2\mathcal{E} - r'^2 \mathcal{K}) - 1 + r^2}. \end{aligned} \quad (3.2)$$

Let $f_1(r) = r^2$, $f_2(r) = \frac{2}{\pi} (2\mathcal{E} - r'^2 \mathcal{K}) - 1 + r^2$, and

$$f(r) = \frac{f_1(r)}{f_2(r)} = \frac{r^2}{\frac{2}{\pi} (2\mathcal{E} - r'^2 \mathcal{K}) - 1 + r^2}. \quad (3.3)$$

Then simple computations lead to

$$f_1(0) = f_2(0) = 0, \quad (3.4)$$

$$f_1'(r) = 2r, \quad (3.5)$$

$$f_2'(r) = \frac{2}{\pi} \frac{\mathcal{E} - r'^2 \mathcal{K}}{r} + 2r, \quad (3.6)$$

$$\frac{f_1'(r)}{f_2'(r)} = \frac{1}{\frac{1}{\pi} \frac{\mathcal{E} - r'^2 \mathcal{K}}{r^2} + 1}. \quad (3.7)$$

It follows from (3.7) with Lemma 1 and Lemma 2 that $f(r)$ is strictly decreasing in $(0, 1)$. Moreover, using L'Hôpital's rule we get

$$\lim_{x \rightarrow 0^+} f(x) = \frac{4}{5}, \quad (3.8)$$

and

$$\lim_{x \rightarrow 1^-} f(x) = \frac{\pi}{4}. \quad (3.9)$$

Therefore, Theorem 1 follows from (3.8) and (3.9) together with the monotonicity of $f(r)$. $\square$

From Theorem 1 we get a new bound for the complete elliptic integral $\mathcal{E}(r)$ of the second kind in terms of elementary functions as follows.

Corollary 1. For $r \in (0, 1)$ and $r' = \sqrt{1-r^2}$, we have

$$\frac{\pi}{2} \left[\frac{5}{8}(1+r') - \frac{r'}{2(1+r')} \right] < \mathcal{E}(r) < \frac{\pi}{2} \left[\frac{2(1+r')}{\pi} - \left(1 - \frac{4}{\pi}\right) \frac{2r'}{1+r'} \right]. \quad (3.10)$$

Proof. Without loss of generality, assume that $a > b$. Substituting $r' = \frac{b}{a}, \alpha_1 = \frac{\pi}{4}, \beta_1 = \frac{4}{5}$ into Theorem 1 produces Corollary 1. $\square$

Theorem 2. The double inequality

$$\alpha_2 T(a, b) + (1 - \alpha_2) \overline{H}(a, b) < A(a, b) < \beta_2 T(a, b) + (1 - \beta_2) \overline{H}(a, b), \quad (3.11)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq \frac{\pi}{4}$ and $\beta_2 \geq \frac{6}{7}$.

Proof. Without loss of generality, we assume that $a > b$. Let $t = b/a \in (0, 1)$ and $r = \frac{1-t}{1+t}$, then

$$\begin{aligned} \frac{A(a, b) - \overline{H}(a, b)}{T(a, b) - \overline{H}(a, b)} &= \frac{\frac{1+t}{2} - \frac{\sqrt{2}t}{\sqrt{1+t^2}}}{\frac{\pi}{2} \mathcal{E}'(t) - \frac{\sqrt{2}t}{\sqrt{1+t^2}}} \\ &= \frac{\frac{1}{1+r} - \frac{1-r}{\sqrt{1+r^2}}}{\frac{2}{\pi} \mathcal{E} \left(\frac{2\sqrt{r}}{1+r} \right) - \frac{1-r}{\sqrt{1+r^2}}} \\ &= \frac{1 - \frac{1-r^2}{\sqrt{1+r^2}}}{\frac{2}{\pi} (2\mathcal{E} - r'^2 \mathcal{K}) - \frac{1-r^2}{\sqrt{1+r^2}}} \end{aligned} \quad (3.12)$$

Let $g_1(r) = 1 - \frac{1-r^2}{\sqrt{1+r^2}}$, $g_2(r) = \frac{2}{\pi} (2\mathcal{E} - r'^2 \mathcal{K}) - \frac{1-r^2}{\sqrt{1+r^2}}$ and

$$g(r) = \frac{g_1(r)}{g_2(r)} = \frac{1 - \frac{1-r^2}{\sqrt{1+r^2}}}{\frac{2}{\pi} (2\mathcal{E} - r'^2 \mathcal{K}) - \frac{1-r^2}{\sqrt{1+r^2}}}. \quad (3.13)$$

Simple calculations show that

$$g_1(0) = g_2(0) = 0, \quad (3.14)$$

$$g'_1(r) = \frac{r(r^2 + 3)}{(1 + r^2)^{3/2}}, \quad (3.15)$$

$$g'_2(r) = \frac{2}{\pi} \frac{\mathcal{E} - r'^2 \mathcal{K}}{r} + \frac{r(r^2 + 3)}{(1 + r^2)^{3/2}}, \quad (3.16)$$

$$\frac{g'_1(r)}{g'_2(r)} = \frac{\frac{(r^2+3)}{(1+r^2)^{3/2}}}{\frac{2}{\pi} \frac{\mathcal{E}-r'^2 \mathcal{K}}{r^2} + \frac{(r^2+3)}{(1+r^2)^{3/2}}} = \frac{1}{\frac{\frac{2}{\pi} \frac{\mathcal{E}-r'^2 \mathcal{K}}{r^2}}{\frac{(r^2+3)}{(1+r^2)^{3/2}}} + 1}. \quad (3.17)$$

It is easy to verify that the function $\frac{(r^2+3)}{(1+r^2)^{3/2}}$ is positive and strictly decreasing in $(0, 1)$, then (3.17) and Lemma 1 lead to the conclusion that $\frac{g'_1(r)}{g'_2(r)}$ is strictly decreasing in $(0, 1)$. Hence, $g(r)$ is strictly decreasing directly from Lemma 2. Moreover, the usage of L'Hôpital's rule and standard argument show that

$$\lim_{x \rightarrow 0^+} g(x) = \frac{6}{7}, \quad (3.18)$$

and

$$\lim_{x \rightarrow 1^-} g(x) = \frac{\pi}{4}. \quad (3.19)$$

Thus, Theorem 2 follows from (3.18) and (3.19) together with the monotonicity of $g(r)$. □

4. COMPARISON WITH SOME WELL-KNOWN RESULTS

Recently, the complete elliptic integrals have attracted the attention of numerous mathematicians. In particular, many remarkable properties and inequalities for the complete elliptic integrals can be found in the literature [1–3, 5, 11, 13, 18–21].

In [11], the authors obtained that

$$\frac{\pi}{2} \left[\frac{1}{2} \sqrt{\frac{1+r'^2}{2}} + \frac{1+r'}{4} \right] < \mathcal{E}(r) < \frac{\pi}{2} \left[\left(\frac{4-\pi}{(\sqrt{2}-1)\pi} \sqrt{\frac{1+r'^2}{2}} + \frac{(\sqrt{2}\pi-4)(1+r')}{2(\sqrt{2}-1)\pi} \right) \right], \quad (4.1)$$

for all $r \in (0, 1)$ and $r' = \sqrt{1-r^2}$.

Guo and Qi [13] proved that

$$\frac{\pi}{2} - \frac{1}{2} \log \frac{(1+r)^{1-r}}{(1-r)^{1+r}} < \mathcal{E}(r) < \frac{\pi-1}{2} + \frac{1-r^2}{4r} \log \frac{1+r}{1-r}, \quad (4.2)$$

for all $r \in (0, 1)$.

It was pointed out in [11] that the bounds in (4.1) for $\mathcal{E}(r)$ are better than that in (4.2) for some $r \in (0, 1)$.

Let $g(x) = \frac{5}{8}(1+x) - \frac{x}{2(1+x)} - \left[\frac{1}{2} \sqrt{\frac{1+x^2}{2}} + \frac{1+x}{4} \right]$, $x \in (0, 1)$. Then simple computations lead to

$$g(x) = \frac{3x^2 + 2x + 3 - 2\sqrt{2(1+x^2)}(1+x)}{8(1+x)}. \quad (4.3)$$

Since

$$(3x^2 + 2x + 3)^2 - \left(2\sqrt{2(1+x^2)}(1+x) \right)^2 = (1-x)^4 > 0,$$

we clearly see that $g(x) > 0$, which shows that the lower bounds in (3.10) for $\mathcal{E}(r)$ is better than the one in (4.1).

Very recently, Yin and Qi obtained in [20] that

$$\frac{\pi}{2} \frac{\sqrt{6+2\sqrt{1-r^2}-3r^2}}{2\sqrt{2}} \leq \mathcal{E}(r) \leq \frac{\pi}{2} \frac{\sqrt{10-2\sqrt{1-r^2}-5r^2}}{2\sqrt{2}}. \quad (4.4)$$

Since

$$\frac{5}{8}(1+x) - \frac{x}{2(1+x)} > \frac{\sqrt{6+2x-3(1-x^2)}}{2\sqrt{2}}$$

is equivalent to

$$(5x^2 + 6x + 5)^2 > 8(x+1)^2(3x^2 + 2x + 3)$$

and $(x-1)^4 > 0$, the lower bound in (3.10) for $\mathcal{E}(r)$ is better than the one in (4.4).

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q -ANALOGUE OF MITTAG-LEFFLER OPERATORS

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Abstract. In this paper, we study q -extension of Mittag-Leffler operators. We establish moments of these operators and estimate convergence results with the help of classical modulus of continuity. Also, we give weighted approximation property and A -statistically convergence of the operators $L_{n,q}^{(\beta)}$.

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1. INTRODUCTION

We first mention some notations of q -calculus. Throughout the present paper, assume that q is a real number satisfying the inequality $0 < q \leq 1$. For $n \in \mathbb{N}$, we have

$$[n]_q = [n] := \begin{cases} (1 - q^n) / (1 - q) & , \quad q \neq 1 \\ n & , \quad q = 1 \end{cases} ,$$
$$[n]_q! = [n]! := \begin{cases} [n][n-1] \dots [1] & , \quad n \geq 1 \\ 1 & , \quad n = 0 \end{cases} ,$$

and

$$(1+x)_q^n := \begin{cases} \prod_{j=0}^{n-1} (1 + q^j x) & , \quad n = 1, 2, \dots \\ 1 & , \quad n = 0. \end{cases}$$

For integers $0 \leq k \leq n$, the q -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} .$$

The q -analogue of integration, introduced by Jackson [7] in the interval $[0, a]$, is defined by

$$\int_0^a f(x) d_q x := a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n, \quad 0 < q < 1 \text{ and } a > 0.$$

The q -improper integral used in the present paper is defined as

$$\int_0^{\infty/A} f(x) d_q x := (1-q) \sum_{n=0}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}, \quad A > 0,$$

provided the sum converges absolutely. The two q -Gamma functions are defined as

$$\Gamma_q(x) = \int_0^{1/1-q} t^{x-1} E_q(-qt) d_q t \quad \text{and} \quad \gamma_q^A(x) = \int_0^{\infty/A(1-q)} t^{x-1} e_q(-t) d_q t. \quad (1.1)$$

There are two q -analogues of the exponential function e^x , see [8],

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]!} = \frac{1}{(1-(1-q)x)_q^{\infty}}, \quad |x| < \frac{1}{1-q}, \quad |q| < 1,$$

and

$$E_q(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{[k]!} = (1 + (1-q)x)_q^{\infty}, \quad |q| < 1.$$

By Jackson [7], it was shown that the q -Beta function defined in the usual formula

$$B_q(t, s) = \frac{\Gamma_q(s) \Gamma_q(t)}{\Gamma_q(s+t)}.$$

The q -integral representation of q -Beta function, which is a q -analogue of Euler's formula, is

$$B_q(t, s) = \int_0^1 x^{t-1} (1-qx)_q^{s-1} d_q x, \quad t, s > 0. \quad (1.2)$$

In 1903, G.M. Mittag-Leffler [11] defined the Mittag-Leffler function by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}; \quad (z \in \mathbb{C}, \quad R(\alpha) > 0).$$

In 1905, A. Wiman [15] gave the two-index Mittag-Leffler function definition by

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}; \quad (z, \beta \in \mathbb{C}, \quad R(\alpha) > 0).$$

Note that, for $\beta = 1$ then we get $E_{\alpha, 1}(z) = E_{\alpha}(z)$.

M.A. Özarslan [12] investigated properties of the following Mittag-Leffler operators

$$L_n^{(\beta)}(f; x) = \frac{1}{E_{1, \beta}\left(\frac{nx}{b_n}\right)} \sum_{k=0}^{\infty} f\left(\frac{kb_n}{n}\right) \frac{(nx)^k}{b_n^k \Gamma(k + \beta)}, \quad (1.3)$$

M.H. Annaby and Z.S. Mansour [1] defined q -analogues of the Mittag-Leffler functions by

$$e_{\alpha,\beta}(z;q) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(\alpha k + \beta)}, \quad |z(1-q)^\alpha| < 1,$$

and

$$E_{\alpha,\beta}(z;q) = \sum_{k=0}^{\infty} \frac{q^{\alpha \frac{k(k-1)}{2}} z^k}{\Gamma_q(\alpha k + \beta)}, \quad z \in \mathbb{C}$$

where $\beta \in \mathbb{C}$ and $R(\alpha) > 0$. Here we will use q -Mittag-Leffler function for $\alpha = 1$ as

$$E_{1,\beta}\left(\frac{nz}{b_n}; q\right) = \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k + \beta)} \left(\frac{nz}{b_n}\right)^k.$$

Now we introduce the following operators

$$L_{n,q}^{(\beta)}(f;x) = \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} f\left(\frac{[k]b_n}{n}\right) \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^k \quad (1.4)$$

where (b_n) is a sequence of positive real numbers, $\beta > 1$ is fixed, $n \in \mathbb{N}$, and

$$f \in E := \left\{ f \in C[0, \infty) : \lim_{x \rightarrow \infty} \frac{f(x)}{1+x^2} \text{ is finite} \right\},$$

where $C[0, \infty)$ is the space of continuous functions defined on $[0, \infty)$. The norm of Banach lattice E is given by

$$\|f\|_* := \sup_{x \in [0, \infty)} \frac{|f(x)|}{1+x^2}.$$

It is obvious that the operators $L_{n,q}^{(\beta)}(f;x)$ are linear and positive.

When $q = 1$, we have the Mittag-Leffler operators given by Özarslan [12]. When $q = 1$ and $\beta = 1$, we get the modified Szász-Mirakjan operators.

Lemma 1. For each $0 < q < 1$, $x \geq 0$ and $n \in \mathbb{N}$, we have

$$L_{n,q}^{(\beta)}(1;x) = 1, \quad (1.5)$$

$$\left| L_{n,q}^{(\beta)}(t;x) - x \right| \leq \frac{[\beta-1]b_n}{n}, \quad (1.6)$$

$$\left| L_{n,q}^{(\beta)}(t^2;x) - x^2 \right| \leq \frac{b_n}{n} \left(x + \frac{[\beta-1]b_n}{n} \right) + 2x \frac{b_n[\beta-1]}{n}, \quad (1.7)$$

and

$$L_{n,q}^{(\beta)}((t-x)^2;x) \leq \frac{b_n}{n} (1 + 4[\beta-1])x + \left(\frac{b_n}{n}\right)^2 [\beta-1]. \quad (1.8)$$

Proof. From definition of the operators $L_{n,q}^{(\beta)}$ given by (1.4), we get

$$L_{n,q}^{(\beta)}(1; x) = 1.$$

Using $[k] = [k + \beta - 1] - q^k[\beta - 1]$ and from (1.4), we have

$$\begin{aligned} L_{n,q}^{(\beta)}(t; x) &= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \frac{[k] b_n}{n} \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^k \\ &= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=1}^{\infty} \frac{b_n}{n} \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k + \beta - 1)} \left(\frac{nx}{b_n}\right)^k \\ &\quad - \frac{[\beta - 1]}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \frac{b_n}{n} \frac{q^{\frac{k(k-1)}{2}} q^k}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^k \\ &\leq \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \frac{b_n}{n} \frac{q^{\frac{k(k+1)}{2}}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^{k+1} \\ &\quad + \frac{[\beta - 1]}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \frac{b_n}{n} \frac{q^{\frac{k(k-1)}{2}} q^k}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^k. \end{aligned}$$

For $q^k \leq 1$, we get

$$\begin{aligned} L_{n,q}^{(\beta)}(t; x) &\leq \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \frac{b_n}{n} \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^{k+1} \\ &\quad + \frac{[\beta - 1]}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \frac{b_n}{n} \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^k \\ &\leq x + \frac{[\beta - 1]b_n}{n}. \end{aligned}$$

So we can write

$$\left| L_{n,q}^{(\beta)}(t; x) - x \right| \leq \frac{[\beta - 1]b_n}{n}.$$

Using definition of the operators $L_{n,q}^{(\beta)}$, we have

$$L_{n,q}^{(\beta)}(t^2; x) = \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \left(\frac{[k] b_n}{n}\right)^2 \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k + \beta)} \left(\frac{nx}{b_n}\right)^k.$$

Using the fact that $[k] = q[k-1] + 1$ and $[k] = [k + \beta - 1] - q^k[\beta - 1]$, we obtain

$$\begin{aligned}
 L_{n,q}^{(\beta)}(t^2; x) &= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \left(\frac{b_n}{n}\right)^2 (q[k-1][k] + [k]) \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k+\beta)} \left(\frac{nx}{b_n}\right)^k \\
 &= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=2}^{\infty} \left(\frac{b_n}{n}\right)^2 [k-1][k] \frac{q^{\frac{k(k-1)}{2}+1}}{\Gamma_q(k+\beta)} \left(\frac{nx}{b_n}\right)^k \\
 &\quad + \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \left(\frac{b_n}{n}\right)^2 [k] \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k+\beta)} \left(\frac{nx}{b_n}\right)^k \\
 &= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=2}^{\infty} \left(\frac{b_n}{n}\right)^2 [k-1] ([k + \beta - 1] - q^k[\beta - 1]) \frac{q^{\frac{k(k-1)}{2}+1}}{\Gamma_q(k+\beta)} \left(\frac{nx}{b_n}\right)^k \\
 &\quad + \frac{b_n}{n} L_{n,q}^{(\beta)}(t; x) \\
 &= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=2}^{\infty} \left(\frac{b_n}{n}\right)^2 [k-1] \frac{q^{\frac{k(k-1)}{2}+1}}{\Gamma_q(k+\beta-1)} \left(\frac{nx}{b_n}\right)^k + \frac{b_n}{n} L_{n,q}^{(\beta)}(t; x) \\
 &\quad - \frac{[\beta-1]}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=1}^{\infty} \left(\frac{b_n}{n}\right)^2 [k-1] \frac{q^{\frac{k(k-1)}{2}+1+k}}{\Gamma_q(k+\beta)} \left(\frac{nx}{b_n}\right)^k \\
 &= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=2}^{\infty} \left(\frac{b_n}{n}\right)^2 ([k + \beta - 2] - q^{k-1}[\beta - 1]) \frac{q^{\frac{k(k-1)}{2}+1}}{\Gamma_q(k+\beta-1)} \left(\frac{nx}{b_n}\right)^k \\
 &\quad - \frac{[\beta-1]}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=2}^{\infty} \left(\frac{b_n}{n}\right)^2 [k-1] \frac{q^{\frac{k(k-1)}{2}+1+k}}{\Gamma_q(k+\beta)} \left(\frac{nx}{b_n}\right)^k + \frac{b_n}{n} L_{n,q}^{(\beta)}(t; x) \\
 &= \frac{b_n}{n} L_{n,q}^{(\beta)}(t; x) + \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=2}^{\infty} \left(\frac{b_n}{n}\right)^2 \frac{q^{\frac{k(k-1)}{2}+1}}{\Gamma_q(k+\beta-2)} \left(\frac{nx}{b_n}\right)^k \\
 &\quad - \frac{[\beta-1]}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=2}^{\infty} \left(\frac{b_n}{n}\right)^2 \frac{q^{\frac{k(k-1)}{2}+k}}{\Gamma_q(k+\beta-1)} \left(\frac{nx}{b_n}\right)^k \\
 &\quad - \frac{[\beta-1]}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=1}^{\infty} \left(\frac{b_n}{n}\right)^2 [k-1] \frac{q^{\frac{k(k-1)}{2}+1+k}}{\Gamma_q(k+\beta)} \left(\frac{nx}{b_n}\right)^k
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{b_n}{n} L_{n,q}^{(\beta)}(t; x) + \frac{x^2}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} q^{3k+3}}{\Gamma_q(k+\beta)} \left(\frac{nx}{b_n}\right)^k \\ &\quad + \frac{[\beta-1]}{E_{1,\beta}\left(\frac{nx}{b_n}; q\right)} \sum_{k=2}^{\infty} \left(\frac{b_n}{n}\right)^2 \frac{q^{\frac{k(k-1)}{2}+k}}{\Gamma_q(k+\beta-1)} \left(1 + \frac{q[k-1]}{[k+\beta-1]}\right) \left(\frac{nx}{b_n}\right)^k. \end{aligned}$$

For $q^k \leq 1$ and $\frac{[k-1]}{[k+\beta-1]} \leq 1$, we have

$$L_{n,q}^{(\beta)}(t^2; x) \leq x^2 + \frac{b_n}{n} \left(x + \frac{[\beta-1]b_n}{n}\right) + 2x \frac{b_n[\beta-1]}{n}. \quad (1.9)$$

So we can write

$$\left|L_{n,q}^{(\beta)}(t^2; x) - x^2\right| \leq \frac{b_n}{n} \left(x + \frac{[\beta-1]b_n}{n}\right) + 2x \frac{b_n[\beta-1]}{n}.$$

Finally, using the (1.6) and (1.7), we acquire that

$$\begin{aligned} L_{n,q}^{(\beta)}((t-x)^2; x) &= L_{n,q}^{(\beta)}(t^2; x) - 2x L_{n,q}^{(\beta)}(t; x) + x^2 L_{n,q}^{(\beta)}(1; x) \\ &\leq \left|L_{n,q}^{(\beta)}(t^2; x) - x^2\right| + 2x \left|L_{n,q}^{(\beta)}(t; x) - x\right| \\ &\leq \frac{b_n}{n} \left(x + \frac{[\beta-1]b_n}{n}\right) + 2x \frac{b_n[\beta-1]}{n} + \frac{2[\beta-1]b_n}{n} x \\ &= \frac{b_n}{n} (1 + 4[\beta-1])x + \left(\frac{b_n}{n}\right)^2 [\beta-1]. \end{aligned}$$

□

2. WEIGHTED APPROXIMATION

Here, we give weighted approximation theorem for the operator $L_{n,q}^{(\beta)}$.

Theorem 1. Let $q = q_n \in (0, 1)$ such that $q_n \rightarrow 1$ as $n \rightarrow \infty$, then for each $f \in E$ and $\lim_{n \rightarrow \infty} \frac{b_n}{n} = 0$, we get

$$\lim_{n \rightarrow \infty} \left\|L_{n,q_n}^{(\beta)}(f; x) - f(x)\right\|_* = 0$$

where $\beta > 1$.

Proof. From (1.5), we know that $L_{n,q}^{(\beta)}(1; x) = 1$, so

$$\left\|L_{n,q_n}^{(\beta)}(1; x) - 1\right\|_* = 0.$$

From (1.6), for $n > 1$, we get

$$\left\|L_{n,q_n}^{(\beta)}(t; x) - x\right\|_* = \sup_{x \in [0, \infty)} \frac{\left|L_{n,q_n}^{(\beta)}(t; x) - x\right|}{1 + x^2}$$

$$\begin{aligned} &\leq \frac{[\beta-1]b_n}{n} \sup_{x \in [0, \infty)} \frac{1}{1+x^2} \\ &\leq \frac{[\beta-1]b_n}{n}. \end{aligned}$$

So we can write

$$\lim_{n \rightarrow \infty} \|L_{n,q_n}^{(\beta)}(t; x) - x\|_* = 0.$$

Finally, for (1.7), we get

$$\begin{aligned} \|L_{n,q_n}^{(\beta)}(t^2; x) - x^2\|_* &= \sup_{x \in [0, \infty)} \frac{|L_{n,q_n}^{(\beta)}(t^2; x) - x^2|}{1+x^2} \\ &\leq (1+2[\beta-1]) \frac{b_n}{n} \sup_{x \in [0, \infty)} \frac{x}{1+x^2} \\ &\quad + \left(\frac{b_n}{n}\right)^2 [\beta-1] \sup_{x \in [0, \infty)} \frac{1}{1+x^2} \\ &\leq (1+2[\beta-1]) \frac{b_n}{2n} + \left(\frac{b_n}{n}\right)^2 [\beta-1]. \end{aligned}$$

Then, we acquire that

$$\|L_{n,q_n}^{(\beta)}(t^2; x) - x^2\|_* \rightarrow 0 \text{ as } n \rightarrow \infty.$$

□

3. RATE OF CONVERGENCE

Lemma 2. For all $x \in [0, \infty)$ and $n \in \mathbb{N}$, we get

$$\omega(x) L_{n,q}^{(\beta)}\left(\frac{1}{\omega}; x\right) \leq \mu(\beta)$$

where $\left(\frac{b_n}{n}\right)$ is a bounded sequence of positive numbers and $\beta > 1$ is fixed. Also $\mu(\beta)$ is a constant and $\omega(x) = \frac{1}{1+x^2}$. Additionally for all $f \in E$ and $q \in (0, 1)$, we can obtain

$$\|L_{n,q}^{(\beta)}(f)\|_* \leq \mu(\beta) \|f\|_*.$$

Proof. Using (1.5) and (1.9), we get

$$\begin{aligned} \omega(x) L_{n,q}^{(\beta)}\left(\frac{1}{\omega}; x\right) &= \frac{1}{1+x^2} \left[L_{n,q}^{(\beta)}(1; x) + L_{n,q}^{(\beta)}(t^2; x) \right] \\ &\leq \frac{1}{1+x^2} \left(1+x^2 + \frac{b_n}{n} (1+2[\beta-1])x + \left(\frac{b_n}{n}\right)^2 [\beta-1] \right) \end{aligned}$$

$$\begin{aligned} &\leq 1 + \frac{b_n}{n} (1 + 2[\beta - 1]) + \left(\frac{b_n}{n}\right)^2 [\beta - 1] \\ &= \mu(\beta). \end{aligned}$$

Also we have

$$\begin{aligned} \omega(x) \left| L_{n,q}^{(\beta)}(f; x) \right| &= \omega(x) \left| L_{n,q}^{(\beta)}\left(\omega \frac{f}{\omega}; x\right) \right| \\ &\leq \|f\|_* \omega(x) L_{n,q}^{(\beta)}\left(\frac{1}{\omega}; x\right) \\ &\leq \mu(\beta) \|f\|_*. \end{aligned}$$

If we take supremum both sides of above inequality over $x \in [0, \infty)$, then the proof is done. $\square$

Now, we acquire that the rate of convergence of the operators $L_{n,q}^{(\beta)} f$ to f with the help of usual modulus of convergence.

Firstly, we remember that the modulus of continuity of f on the closed interval $[0, D]$ is given by

$$\omega_D(f, \delta) = \sup_{\substack{x, t \in [0, D] \\ |t-x| \leq \delta}} |f(t) - f(x)|.$$

We always know that, for a function f belongs to E , there is a limit equation as $\lim_{\delta \rightarrow \infty} \omega_D(f, \delta) = 0$.

Theorem 2. Assume that $\left(\frac{b_n}{n}\right)$ is a bounded sequence of positive numbers with $\lim_{n \rightarrow \infty} \frac{b_n}{n} = 0$ and $\beta > 1$. For each $f \in E$, $q \in (0, 1)$ and $x \in [0, D] \subset [0, \infty)$; ($D > 0$),

$$\left\| L_{n,q}^{(\beta)}(f; x) - f(x) \right\|_{C[0, D]} \leq M_f(\beta, D) \epsilon_n^2(\beta, D) + 2\omega_{D+1}(f, \epsilon_n(\beta, D))$$

where

$$\epsilon_n(\beta, D) = \frac{b_n}{n} (1 + 4[\beta - 1]) D + \left(\frac{b_n}{n}\right)^2 [\beta - 1]$$

and $M_f(\beta, D)$ is an absolute constant.

Proof. From the hypothesis, $x \in [0, D]$ and $t \leq D + 1$, we get

$$|f(t) - f(x)| \leq \omega_{D+1}(f, |t-x|) \leq \left(1 + \frac{|t-x|}{\delta}\right) \omega_{D+1}(f, \delta)$$

where $\delta > 0$. Also we can write

$$\begin{aligned} |f(t) - f(x)| &\leq K_f(1 + x^2 + t^2) \leq K_f(2 + 3x^2 + 2(t-x)^2) \\ &\leq 6K_f(1 + D^2)(t-x)^2 \end{aligned}$$

for $x \in [0, D]$ and $t > D + 1$, getting $t - x > 1$ and K_f is a constant depending on f . From the above, we have

$$|f(t) - f(x)| \leq 6K_f(1 + D^2)(t - x)^2 + \left(1 + \frac{|t - x|}{\delta}\right) \omega_{D+1}(f, \delta)$$

for $x \in [0, D]$ and $t \geq 0$. By the well-known Cauchy-Schwarz inequality and from (1.8), we have

$$\begin{aligned} |L_{n,q}^{(\beta)}(f; x) - f(x)| &\leq 6K_f(1 + D^2)L_{n,q}^{(\beta)}((t - x)^2; x) \\ &\quad + \omega_{D+1}(f, \delta) \left(1 + \frac{1}{\delta} \left(L_{n,q}^{(\beta)}((t - x)^2; x)\right)^{1/2}\right) \\ &\leq 6K_f(1 + D^2) \left[\frac{b_n}{n} (1 + 4[\beta - 1])x + \left(\frac{b_n}{n}\right)^2 [\beta - 1] \right] \\ &\quad + \omega_{D+1}(f, \delta) \left[1 + \frac{1}{\delta} \left(\frac{b_n}{n} (1 + 4[\beta - 1])x + \left(\frac{b_n}{n}\right)^2 [\beta - 1] \right)^{\frac{1}{2}} \right] \\ &\leq M_f(\beta, D) \epsilon_n^2(\beta, D) + 2\omega_{D+1}(f, \epsilon_n(\beta, D)), \end{aligned}$$

where

$$\epsilon_n(\beta, D) = \left(\frac{b_n}{n} (1 + 4[\beta - 1]) D + \left(\frac{b_n}{n}\right)^2 [\beta - 1] \right)^{1/2}$$

and $M_f(\beta, D) = 6(1 + D^2)K_f$. So, the proof is done. $\square$

4. A-STATISTICAL CONVERGENCE

Recently, some authors use A -statistical convergence by [2, 3, 12, 13]. Here for a non-negative regular summability matrix $A = (a_{jk})$, the definition of A -density of a subset K of $\mathbb{N}$ is as

$$\delta_A(K) = \lim_j \sum_{k \in K} a_{j,k},$$

where limit exists (see [5]). If a sequence $x = (x_n)$ convergent to l meaning A -statistically converges symbolized by $st_A - \lim x = l$ where every $\varepsilon > 0$, $\delta_A \{n \in \mathbb{N} : |x - l| \geq \varepsilon\} = 0$ (see [4, 14]).

Substituting $A = C_1$ which C_1 is the Cesaro matrix of order one, then we obtain the statistical convergence instead of A -statistically convergence [6, 10]. Choosing $A = I$, which is the identity matrix, then we get the ordinary convergence instead of A -statistically convergence. In the case of $\lim_j \max_n |a_{j,n}| = 0$, Kolk [9] showed that A -statistical convergence is stronger than ordinary convergence.

Assuming that

$$st_A - \lim_n \frac{b_n}{n} = 0,$$

then one can easily see that

$$st_A - \lim_n \frac{b_n^2}{n^2} = 0.$$

Taking $A = C_1$ and defining

$$b_n := \begin{cases} n; & n = m^2 \ (m \in \mathbb{N}) \\ 1; & \text{otherwise} \end{cases},$$

so we can easily obtain that $st_A - \lim_n \frac{b_n}{n} = st_A - \lim_n \frac{b_n^2}{n^2} = 0$.

Theorem 3. Assume that $A = (a_{jk})$ is a non-negative regular summability matrix, $q \in (0, 1)$ and $\beta > 1$. We have

$$st_A - \left\| L_{n,q}^{(\beta)}(f; x) - f(x) \right\|_{C[0,D]} = 0,$$

if $st_A - \lim_n \frac{b_n}{n} = 0$ and each $f \in E$.

Proof. We choose $\varepsilon > 0$ satisfying $\varepsilon < r$ where $r > 0$. Defining the following sets as

$$\begin{aligned} T &:= \{n : \epsilon_n(\beta, D) \geq r\}, \\ T_1 &:= \left\{ n : \frac{b_n}{n} (1 + 4[\beta - 1]) D \geq \frac{r - \varepsilon}{2} \right\}, \\ T_2 &:= \left\{ n : \left(\frac{b_n}{n} \right)^2 [\beta - 1] \geq \frac{r - \varepsilon}{2} \right\}. \end{aligned}$$

Then one can see $T \subset T_1 \cup T_2$. Thus we can write

$$\sum_{k \in T} a_{jk} \leq \sum_{k \in T_1} a_{jk} + \sum_{k \in T_2} a_{jk}.$$

We get $\lim_j \sum_{k \in T} a_{jk} = 0$ for $j \rightarrow \infty$ in the above and $st_A - \lim_n \frac{b_n}{n} = 0$. Hence we have that $st_A - \lim_n \epsilon_n(\beta, D) = 0$ and this showed that

$$st_A - \lim_n \omega_{D+1}(f, \epsilon_n(\beta, D)) = 0.$$

From the Theorem 2, the proof is done. □

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ON GENERALIZED TWO-INDEX HERMITE MATRIX POLYNOMIALS

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Abstract. In this study, we give a new generalization for Hermite matrix polynomials. We obtain some formulas related to an explicit representation, matrix recurrence relations and the Rodrigues formula for two-index Hermite matrix polynomials. Also we generalize Runge's addition formula and Nielsen's identity for these new generalizations. Moreover we give Burchnell's operational formula and Nielsen's identity for Hermite matrix polynomials by means of two-index Hermite matrix polynomials.

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1. INTRODUCTION AND NOTATION

Throughout this paper, if A is a matrix in $\mathbb{C}^{r \times r}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A . Its 2-norm is denoted by $\|A\|$ and defined by ([10])

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

where for a vector y in $\mathbb{C}^r$, $\|y\|_2 = (y^T, y)^{\frac{1}{2}}$ is the Euclidean norm of y . Moreover the zero matrix and identity matrix will be denoted by 0 and I , respectively. For a matrix $A(k, n)$ in $\mathbb{C}^{r \times r}$ for $n \geq 0$ and $k \geq 0$, then in an analogous way to the proof of Lemma 11 of [26], it follows that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^n A(k, n-k). \quad (1.1)$$

If A is a matrix in $\mathbb{C}^{r \times r}$ such that $\operatorname{Re}(z) > 0$ for every eigenvalue $z \in \sigma(A)$, we say a positive stable matrix. Also throughout this paper, we will use a positive stable matrix A in $\mathbb{C}^{r \times r}$. Then Hermite matrix polynomials are defined by means of the

following generating function [13]

$$\sum_{n=0}^{\infty} H_n(x, A) \frac{t^n}{n!} = \exp\left(xt\sqrt{2A} - t^2 I\right). \quad (1.2)$$

From the above equation we have

$$H_n(x, A) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n! \left(x\sqrt{2A}\right)^{n-2k}}{k!(n-2k)!}, \quad n \geq 0$$

and

$$\frac{d^k}{dx^k} H_n(x, A) = \left(\sqrt{2A}\right)^k \frac{n!}{(n-k)!} H_{n-k}(x, A), \quad 0 \leq k \leq n. \quad (1.3)$$

Also, Hermite matrix polynomials satisfy Runge's addition formula ([1])

$$2^{\frac{n}{2}} H_n(x+y, A) = \sum_{k=0}^n \binom{n}{k} H_k\left(\sqrt{2}x, A\right) H_{n-k}\left(\sqrt{2}y, A\right),$$

a generating matrix function ([14])

$$\sum_{n=0}^{\infty} H_n(x, A) H_n(y, A) \frac{t^n}{n!} = (1-4t^2)^{-\frac{1}{2}} \exp\left(\frac{2A(xy t - (x^2 + y^2)t^2)}{1-4t^2}\right)$$

and the summation formula ([17])

$$H_n(x, A) H_m(x, A) = m!n! \sum_{k=0}^{\min(m,n)} \frac{2^k H_{m+n-2k}(x, A)}{(m-k)!(n-k)!k!}. \quad (1.4)$$

Some other properties of Hermite matrix polynomials can be found in [1, 3, 7, 15, 22, 29, 33].

In the recent two decades, special matrix functions are an emergent field whose development is reaching important results both the theoretical and practical points of view. Some important results for classical scalar special functions are generalized to special matrix functions, see for example [2, 5, 8, 9, 16, 21, 30, 31]. Also, some generalized form of these matrix polynomials are introduced and studied in [4, 18–20, 23–25, 27, 28, 32]. Our purpose here is to introduce and study a new generalization which is called two-index Hermite matrix polynomials. The organization of the paper is as follows. In Section 2, the class of two-index Hermite matrix polynomials is introduced. Starting from an appropriate matrix generating function, matrix recurrence relations and Runge's additional theorem and multiplication theorem for these matrix polynomials are established. Section 3 deals with the operational identity which simplifies the studying properties of two-index Hermite matrix polynomials rather than generating matrix function. In the sense of that we construct Rodrigues formula. We give Nielsen's identity for these matrix polynomials as an

application of this formula. Finally, we focus on Hermite matrix polynomials. We establish Burchnell's operational formula and Nielsen's identity.

2. TWO-INDEX HERMITE MATRIX POLYNOMIALS: DEFINITION AND PROPERTIES

One of the most direct ways of exploring generalized classes of Hermite matrix polynomials is to start from modifying the form of the ordinary generating function. So, we define two-index Hermite polynomials $H_{n,m}(x, A)$ by

$$\begin{aligned} F(x, u, v, A) &= \exp\left(\sqrt{\frac{A}{2}}x(u+v) - uvI\right) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}(x, A) \frac{u^n}{n!} \frac{v^m}{m!}, \quad |u| < \infty, |v| < \infty. \end{aligned} \quad (2.1)$$

We can write

$$\exp\left(\sqrt{\frac{A}{2}}x(u+v) - uvI\right) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \left(x\sqrt{\frac{A}{2}}\right)^{n+m} \frac{u^{n+k} v^{m+k}}{k!n!m!}.$$

Using the relation (1.1) yields

$$\begin{aligned} \exp\left(\sqrt{\frac{A}{2}}x(u+v) - uvI\right) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\min(m,n)} (-1)^k \binom{m}{k} \binom{n}{k} k! \left(x\sqrt{\frac{A}{2}}\right)^{m+n-2k} \frac{u^n v^m}{n!m!}. \end{aligned} \quad (2.2)$$

From (2.1) and (2.2), we obtain an explicit representation for the two-index Hermite matrix polynomials in the form:

$$H_{n,m}(x, A) = \sum_{k=0}^{\min(m,n)} (-1)^k \binom{m}{k} \binom{n}{k} k! \left(x\sqrt{\frac{A}{2}}\right)^{m+n-2k}. \quad (2.3)$$

It follows that $H_{n,m}(x, A)$ is a polynomial of degree $m+n$ and $H_{n,m}(x, A) = H_{m,n}(x, A)$. For the unity of the formulations, we define trivially

$$H_{n,m}(x, A) = \mathbf{0},$$

whenever $m < 0$ or $n < 0$. Note that $H_{n,0}(x, A) = \left(x\sqrt{\frac{A}{2}}\right)^n$. From (2.3), it is clear that

$$H_{n,m}(0, A) = \begin{cases} 0 & ; m \neq n \\ (-1)^n n! I & ; m = n \end{cases} \quad (2.4)$$

and

$$H_{1,m}(x, A) = \left(x\sqrt{\frac{A}{2}}\right)^{m+1} - m \left(x\sqrt{\frac{A}{2}}\right)^{m-1}, \quad (2.5)$$

for every integer $m \geq 1$.

The first few values of $H_{n,m}(x, A)$ are given by

$H_{n,m}(x, A)$	$m = 1$	$m = 2$
$n = 1$	$\left(x\sqrt{\frac{A}{2}}\right)^2 - I$	$\left(x\sqrt{\frac{A}{2}}\right)^3 - 2\left(x\sqrt{\frac{A}{2}}\right)$
$n = 2$	$\left(x\sqrt{\frac{A}{2}}\right)^3 - 2\left(x\sqrt{\frac{A}{2}}\right)$	$\left(x\sqrt{\frac{A}{2}}\right)^4 - 4\left(x\sqrt{\frac{A}{2}}\right)^2 + 2I$
$n = 3$	$\left(x\sqrt{\frac{A}{2}}\right)^4 - 3\left(x\sqrt{\frac{A}{2}}\right)$	$\left(x\sqrt{\frac{A}{2}}\right)^5 - 6\left(x\sqrt{\frac{A}{2}}\right)^3 + 6\left(x\sqrt{\frac{A}{2}}\right)$

From (2.3), one can deduce easily the symmetry formula

$$H_{n,m}(-x, A) = (-1)^{n+m} H_{n,m}(x, A).$$

So that the $H_{n,m}(x, A)$ is odd or even if and only if $n + m$ is odd or even respectively. By taking $u = v$ in (2.1), we obtain a relation between Hermite matrix polynomials and our generalization as

$$H_n(x, A) = \sum_{m=0}^n \binom{n}{m} H_{n-m,m}(x, A). \quad (2.6)$$

Now we study the uniform convergence of the matrix series (2.1) in the complex domain $|x| < a$ for a fixed value of u and v . By taking 2-norm in (2.3), we get

$$\|H_{n,m}(x, A)\| \leq H_{n,m}\left(a \left\|\sqrt{\frac{A}{2}}\right\|\right), \quad (2.7)$$

where $H_{n,m}(x)$ is the classical scalar two-index Hermite polynomials in the form ([12])

$$H_{n,m}(x) = \sum_{q=0}^{\min(m,n)} (-1)^q q! \binom{m}{q} \binom{n}{q} x^{m+n-2q}.$$

Moreover

$$\exp\left(\left\|\sqrt{\frac{A}{2}}\right\| a |u + v| + |u| |v|\right) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}\left(a \left\|\sqrt{\frac{A}{2}}\right\|\right) \frac{|u|^n}{n!} \frac{|v|^m}{m!}. \quad (2.8)$$

In view of (2.7) and (2.8), the matrix series (2.1) is termwise differentiable with respect to x .

Proposition 1. *The following recurrence formula for $H_{n,m}(x, A)$ holds*

$$\frac{d^s}{dx^s} H_{n,m}(x, A) = s! \left(\sqrt{\frac{A}{2}} \right)^s \sum_{r=0}^s \binom{n}{s-r} \binom{m}{r} H_{n-s+r, m-r}(x, A). \quad (2.9)$$

Proof. Differentiating the both sides of (2.1) with respect to x and making appropriate changes of indices, we get

$$\frac{d}{dx} H_{n,m}(x, A) = \sqrt{\frac{A}{2}} [n H_{n-1,m}(x, A) + m H_{n,m-1}(x, A)]. \quad (2.10)$$

To end the proof we apply mathematical induction to (2.10). $\square$

Proposition 2. *Two-index Hermite matrix polynomials satisfy the following recurrences formulas*

$$H_{n+1,m}(x, A) = x \sqrt{\frac{A}{2}} H_{n,m}(x, A) - m H_{n,m-1}(x, A), \quad (2.11)$$

$$H_{n,m+1}(x, A) = x \sqrt{\frac{A}{2}} H_{n,m}(x, A) - n H_{n-1,m}(x, A), \quad (2.12)$$

$$(n-m) H_{n,m}(x, A) = x \sqrt{\frac{A}{2}} (H_{n+1,m}(x, A) - H_{n,m+1}(x, A)). \quad (2.13)$$

Proof. Taking termwise differentiation with respect to u to (2.1), we have

$$\frac{\partial}{\partial u} F(x, u, v, A) = \left(x \sqrt{\frac{A}{2}} - v I \right) F(x, u, v, A)$$

and, so that

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} n H_{n,m}(x, A) \frac{u^{n-1}}{n!} \frac{v^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x \sqrt{\frac{A}{2}} H_{n,m}(x, A) \frac{u^n}{n!} \frac{v^m}{m!} - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}(x, A) \frac{u^n}{n!} \frac{v^{m+1}}{m!}. \end{aligned}$$

Making appropriate changes of indices yields (2.11). For the proof of (2.12), we apply the method after differentiating (2.1) with respect to v . Finally, subtracting (2.11) from (2.12) gives (2.13). $\square$

Proposition 3. *Two-index Hermite matrix polynomials satisfy the multiplication and addition formulas as follow*

$$H_{n,m}(\mu x, A) = \mu^{m+n} \sum_{k=0}^{\min(n,m)} \binom{n}{k} \binom{m}{k} k! \left(1 - \frac{1}{\mu^2} \right)^k H_{n-k, m-k}(x, A), \quad (2.14)$$

$$\begin{aligned}
& (\lambda^2 + \mu^2)^{\frac{n+m}{2}} H_{n,m} \left(\frac{\lambda z_1 + \mu z_2}{(\lambda^2 + \mu^2)^{\frac{1}{2}}}, A \right) \\
&= \lambda^{m+n} \sum_{k=0}^n \sum_{l=0}^m \binom{n}{k} \binom{m}{l} \left(\frac{\mu}{\lambda} \right)^{k+l} H_{n-k,m-l}(z_1, A) H_{k,l}(z_2, A),
\end{aligned} \tag{2.15}$$

where λ and μ are any complex numbers.

Proof. Taking μx for x , $\frac{u}{\mu}$ for u and $\frac{v}{\mu}$ for v in (2.1), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{H_{n,m}(\mu x, A)}{\mu^{m+n}} \frac{u^n}{n!} \frac{v^m}{m!} &= \exp \left(\sqrt{\frac{A}{2}} x (u+v) - \frac{uv}{\mu^2} I \right) \\
&= \exp \left(\sqrt{\frac{A}{2}} x (u+v) - \left(uv + uv - \frac{uv}{\mu^2} \right) I \right).
\end{aligned}$$

Then using (1.1) and comparing the coefficients of $\frac{u^n}{n!} \frac{v^m}{m!}$ in both sides of the identity, we get (2.15). Eq. (2.14) can be proved similarly. $\square$

Corollary 1. *Two-index Hermite matrix polynomials satisfy the following equations:*

$$2^{\frac{n+m}{2}} H_{n,m}(\sqrt{2}x, A) = \sum_{k=0}^n \sum_{l=0}^m \binom{n}{k} \binom{m}{l} H_{n-k,m-l}(x, A) H_{k,l}(x, A), \tag{2.16}$$

$$2^{\frac{n+m}{2}} H_{n,m}(x+y, A) = \sum_{k=0}^n \sum_{l=0}^m \binom{n}{k} \binom{m}{l} H_{n-k,m-l}(\sqrt{2}x, A) H_{k,l}(\sqrt{2}y, A), \tag{2.17}$$

$$2^{\frac{n+m}{2}} H_{n,m}(x, A) = \sum_{k=0}^n \sum_{l=k}^m (-1)^k k! \binom{n}{k} \binom{m}{l} H_{n-k,m-l}(\sqrt{2}x, A). \tag{2.18}$$

Equation (2.17) is the Runge's addition formula for two-index Hermite matrix polynomials.

3. SOME RESULTS FOR TWO-INDEX HERMITE MATRIX POLYNOMIALS

In this section, we obtain an operational formula and give its applications such as Rodrigues formula and Nielsen's identity for two-index Hermite matrix polynomials. Moreover we establish Burchnell's operational formula and Nielsen's identity for Hermite matrix polynomials. Firstly, we have the following main theorem.

Theorem 1. For $n, m \in \mathbb{N} \cup \{0\}$, we have the following operational formula

$$H_{n,m}(x, A) = \left(x \sqrt{\frac{A}{2}} - \left(\sqrt{\frac{A}{2}} \right)^{-1} \frac{d}{dx} \right)^n \left(x \sqrt{\frac{A}{2}} \right)^m. \quad (3.1)$$

Proof. It is clear that

$$\begin{aligned} \exp \left(-u \left(\sqrt{\frac{A}{2}} \right)^{-1} \frac{d}{dx} \right) e^{x \sqrt{\frac{A}{2}} v} &= \sum_{n=0}^{\infty} \frac{(-u)^n}{n!} \left(\sqrt{\frac{A}{2}} \right)^{-n} \frac{d^n}{dx^n} e^{x \sqrt{\frac{A}{2}} v} \\ &= e^{x \sqrt{\frac{A}{2}} v - uvI}. \end{aligned}$$

Multiplying both side of the above equation with $e^{x \sqrt{\frac{A}{2}} u}$, we get

$$\exp \left[\left(x \sqrt{\frac{A}{2}} - \left(\sqrt{\frac{A}{2}} \right)^{-1} \frac{d}{dx} \right) u \right] e^{x \sqrt{\frac{A}{2}} v} = \exp \left(\sqrt{\frac{A}{2}} x (u + v) - uvI \right).$$

So we have

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(x \sqrt{\frac{A}{2}} - \left(\sqrt{\frac{A}{2}} \right)^{-1} \frac{d}{dx} \right)^n \left(x \sqrt{\frac{A}{2}} \right)^m \frac{u^n}{n!} \frac{v^m}{m!} \\ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}(x, A) \frac{u^n}{n!} \frac{v^m}{m!}. \end{aligned}$$

Comparing the coefficients of $\frac{u^n}{n!} \frac{v^m}{m!}$ in both sides of the above equation, we get (3.1). $\square$

In the following corollary, we obtain another recurrence formula for the two-index Hermite matrix polynomials.

Corollary 2. For $n, m, r \in \mathbb{N} \cup \{0\}$, we have

$$H_{n+r,m}(x, A) = \left(x \sqrt{\frac{A}{2}} - \left(\sqrt{\frac{A}{2}} \right)^{-1} \frac{d}{dx} \right)^n H_{r,m}(x, A). \quad (3.2)$$

For the proof of the Rodrigues formula and the Nielsen's identity for $H_{n,m}(x, A)$ we need the following lemma.

Lemma 1. For any n -times differentiable functions $f(x)$ and $g(x)$ we have

$$\left(x \sqrt{\frac{A}{2}} - \left(\sqrt{\frac{A}{2}} \right)^{-1} \frac{d}{dx} \right)^n [f(x) g(x)] \quad (3.3)$$

$$= e^{\frac{A}{4}x^2} (-1)^n \left(\sqrt{\frac{A}{2}} \right)^{-n} \frac{d^n}{dx^n} \left[e^{-\frac{A}{4}x^2} f(x) g(x) \right].$$

Proof. One can prove (3.3) by mathematical induction. $\square$

Setting $f(x) = \left(x \sqrt{\frac{A}{2}} \right)^m$ and $g(x) = 1$ in (3.3) we have the following corollary.

Corollary 3. *The Rodrigues formula for $H_{n,m}(x, A)$ is given as follows:*

$$H_{n,m}(x, A) = e^{\frac{A}{4}x^2} (-1)^n \left(\sqrt{\frac{A}{2}} \right)^{m-n} \frac{d^n}{dx^n} \left[e^{-\frac{A}{4}x^2} x^m \right]. \quad (3.4)$$

As a conclusion of the above corollary, it is good to note that two-index Hermite matrix polynomials constitute a subclass of matrix analogues of Gould-Hopper polynomials defined by ([11])

$$H_n^\gamma(x, \alpha, p) = (-1)^n x^{-\alpha} e^{px^\gamma} \frac{d^n}{dx^n} \left(x^\alpha e^{-px^\gamma} \right).$$

In fact, we have

$$H_{n,m}(x, A) = x^m \left(\sqrt{\frac{A}{2}} \right)^{m-n} H_n^2 \left(x, m, \frac{A}{4} \right).$$

Now, we give a formula that calculates the weighted sum of the product of the same polynomials in the following proposition.

Proposition 4. *For $n, m, r \in \mathbb{N} \cup \{0\}$, the Nielsen's identity for $H_{n,m}(x, A)$ holds:*

$$\begin{aligned} H_{n+r,m}(x, A) &= (-1)^n \sum_{k,s,l=0}^{n,k,s} \left(\sqrt{\frac{A}{2}} \right)^{s-m-k} s! \binom{n}{k} \binom{s}{k} \binom{r}{s-l} \binom{m}{l} (m)_{n-k} \\ &\quad \times \frac{x^k}{x^{m+n}} H_{n-k,m}(x, A) H_{r-s+k,m-l}(x, A), \end{aligned}$$

where $(z)_n$ is Pochhammer symbol.

Proof. Taking $g(x) = \left(x \sqrt{\frac{A}{2}} \right)^m$ in (3.3), applying the Leibniz formula and using (3.4), we get

$$\begin{aligned} &\left(x \sqrt{\frac{A}{2}} - \left(\sqrt{\frac{A}{2}} \right)^{-1} \frac{d}{dx} \right)^n \left[f(x) \left(x \sqrt{\frac{A}{2}} \right)^m \right] \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\sqrt{\frac{A}{2}} \right)^{-k} H_{n-k,m}(x, A) \frac{d^k}{dx^k} [f(x)]. \end{aligned} \quad (3.5)$$

Then setting $f(x) = \left(x\sqrt{\frac{A}{2}}\right)^{-m} H_{r,m}(x, A)$ in (3.5) and using (3.2) we get

$$\begin{aligned} H_{n+r,m}(x, A) \\ = \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\sqrt{\frac{A}{2}}\right)^{-k} H_{n-k,m}(x, A) \frac{d^k}{dx^k} \left[\left(x\sqrt{\frac{A}{2}}\right)^{-m} H_{r,m}(x, A) \right]. \end{aligned}$$

Applying the Leibniz formula and using (2.9), we get the proof. $\square$

Now, we focus on Burchnell's operational formula and Nielsen's identity for Hermite matrix polynomials. Setting (3.1) in (2.6) we have

$$\begin{aligned} H_n(x, A) &= \sum_{m=0}^n \binom{n}{m} H_{n-m,m}(x, A) \\ &= \sum_{m=0}^n \binom{n}{m} \left(x\sqrt{\frac{A}{2}} - \left(\sqrt{\frac{A}{2}}\right)^{-1} \frac{d}{dx} \right)^{n-m} \left(x\sqrt{\frac{A}{2}} \right)^m. \end{aligned}$$

Thus we have the following corollary.

Corollary 4. *Burchnell's operational formula for $H_n(x, A)$ holds*

$$H_n(x, A) = \left(x\sqrt{2A} - \left(\sqrt{\frac{A}{2}}\right)^{-1} \frac{d}{dx} \right)^n (I). \quad (3.6)$$

In the following corollary, we obtain another recurrence formula for Hermite matrix polynomials.

Corollary 5. *For $n, r \in \mathbb{N} \cup \{0\}$,*

$$H_{n+r}(x, A) = \left(x\sqrt{2A} - \left(\sqrt{\frac{A}{2}}\right)^{-1} \frac{d}{dx} \right)^n H_r(x, A). \quad (3.7)$$

As an application of the above corollary, we state the Nielsen's identity for Hermite matrix polynomials in the following theorem.

Theorem 2. *For $n, r \in \mathbb{N} \cup \{0\}$, we have the following equation*

$$H_{n+r}(x, A) = \sum_{k=0}^{\min(n,r)} (-2)^k \binom{n}{k} \binom{r}{k} k! H_{n-k}(x, A) H_{r-k}(x, A). \quad (3.8)$$

Proof. Using mathematical induction one can show that

$$\left(x\sqrt{2A} - \left(\sqrt{\frac{A}{2}}\right)^{-1} \frac{d}{dx} \right)^n [f(x)] \quad (3.9)$$

$$= \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\sqrt{\frac{A}{2}} \right)^{-k} H_{n-k}(x, A) \frac{d^k}{dx^k} [f(x)].$$

For $f(x) = H_r(x, A)$ in (3.9), we obtain

$$H_{n+r}(x, A) = \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\sqrt{\frac{A}{2}} \right)^{-k} H_{n-k}(x, A) \frac{d^k}{dx^k} [H_r(x, A)].$$

Therefore, using (1.3) and $H_{-1}(x, A) = 0$ we get

$$H_{n+r}(x, A) = \sum_{k=0}^{\min(n,r)} (-2)^k \binom{n}{k} \binom{r}{k} k! H_{n-k}(x, A) H_{r-k}(x, A).$$

□

From the above theorem we can immediately derive the following identity as a particular case

$$H_{2n}(x, A) = (-2)^n (n!)^2 \sum_{k=0}^n \frac{(-1)^k [H_k(x, A)]^2}{2^k (k!)^2 (n-k)!}. \quad (3.10)$$

Moreover, the use of the binomial transform see for details ([6]) to the identity (3.10) gives the inverse of the relation (3.10). So, $[H_n(x, A)]^2$ can be written as

$$[H_n(x, A)]^2 = H_n(x, A) H_n(x, A) = (2)^n (n!)^2 \sum_{k=0}^n \frac{H_{2k}(x, A)}{2^k (k!)^2 (n-k)!}.$$

It is good to note that the above equation is the special case of (1.4).

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A NOTE ON SOME DIOPHANTINE EQUATIONS

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Abstract. Let $k \geq 3$ be an odd integer. In this paper we investigate all positive integer solutions of the equations $x^4 - kx^2y + y^2 = \mp A$, $x^4 - kx^2y + y^2 = \mp A(k^2 - 4)$, $x^4 - (k^2 - 4)y^2 = \mp 4A$, and $x^2 - (k^2 - 4)y^4 = \mp 4A$ with $A = \mp(k \mp 2)$. We show that if $k \equiv 1 \pmod{8}$ and $k^2 - 4$ be a square-free integer, then the equation $x^4 - kx^2y + y^2 = (k - 2)(k^2 - 4)$ has no positive integer solutions. Moreover, if $k^2 - 4$ be a square-free integer, then the equation $x^4 - kx^2y + y^2 = -(k + 2)(k^2 - 4)$ has no positive integer solutions.

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1. INTRODUCTION

Let k and s be two nonzero integers with $k^2 + 4s > 0$. The generalized Fibonacci sequence is defined by $U_0(k, s) = 0$, $U_1(k, s) = 1$, and $U_{n+1}(k, s) = kU_n(k, s) + sU_{n-1}(k, s)$ for $n \geq 1$ and generalized Lucas sequence is defined by $V_0(k, s) = 2$, $V_1(k, s) = k$, and $V_{n+1}(k, s) = kV_n(k, s) + sV_{n-1}(k, s)$ for $n \geq 1$, respectively. Moreover, generalized Fibonacci and Lucas numbers for negative subscripts are defined as $U_{-n} = -(-s)^{-n}U_n$ and $V_{-n} = (-s)^{-n}V_n$ for $n \in \mathbb{N}$. Especially $U_{-n}(k, -1) = -U_n(k, -1)$ and $V_{-n}(k, -1) = V_n(k, -1)$ for every natural number n .

In the literature, there are many papers dealing with the number of the solutions of the equation $ax^4 + bx^2y + cy^2 - d = 0$. In [7], the author investigated all non-negative integer solutions of the equations $x^4 - kx^2y + y^2 = 1, 4$, under the assumption that k is even. If $k \neq 318$ and k is not a perfect square, then all non-negative integer solutions of the equation $x^2 - kxy^2 + y^4 = 1$ are $(x, y) = (k, 1)$, $(1, 0)$, $(0, 1)$. If $k = 338$, then all positive integer solutions of this equation are $(x, y) = (13051348805, 6214)$, $(114243, 6214)$. If k is a perfect square, then all positive integer solutions of this equation are $(x, y) = (1, \sqrt{k})$, $(k^2 - 1, \sqrt{k})$. Moreover, the author showed that if $k = 2v^2$ for some integer v , then all positive integer solutions of the equation $x^2 - kxy^2 + y^4 = 4$ are $(x, y) = (2, \sqrt{2k})$, $(2k^2 - 2, \sqrt{2k})$.

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Otherwise, the only nonnegative integer solution of this equation is $(x, y) = (2, 0)$. In [2], wherein the author investigated all positive integer solutions of the equation $x^2 - kxy^2 + y^4 = c$ for $c \in \{\mp 1, \mp 2, \mp 4\}$, with the assumption that k is odd for the particular cases $c = 1$ or 4 . He showed that the equation $x^2 - kxy^2 + y^4 = 4$ has no positive integer solutions. Moreover, he showed that if $y = k^2$ for some odd integer k , then the only integer solution of the equation $x^2 - kxy^2 + y^4 = 1$ is $y = 1$. Otherwise, all positive integer solutions of this equation are $y = 1$ or k . Moreover, he showed that the equation

$$x^2 - kxy^2 + y^4 = -1 \quad (1.1)$$

has no integer solutions when $k = 3$. Otherwise, the only integer solution of the equation (1.1) is $y = 1$. In addition, he showed that the equation

$$x^2 - kxy^2 + y^4 = -4 \quad (1.2)$$

has a solution only when $k = 3$ or 6 . If $k = 3$, then all positive integer solutions of equation (1.2) are given by $(x, y) = (2, 2)$ or $(10, 2)$. If $k = 6$, then all positive integer solutions of equation (1.2) are given by $(x, y) = (1, 1), (5, 1), (29, 13)$ or $(985, 13)$.

Let k be an odd integer. In this paper, firstly, we solve the equations $x^2 = V_n(k, -1) \mp V_{n-1}(k, -1)$ and $x^2 = U_n(k, -1) \mp U_{n-1}(k, -1)$, respectively. We show that if $k \equiv 3 \pmod{8}$, then the equation $x^2 = V_n(k, -1) + V_{n-1}(k, -1)$ has no solutions and if $k \equiv 1 \pmod{8}$, then the equation $x^2 = V_n(k, -1) - V_{n-1}(k, -1)$ has no solutions. Lastly, we find all positive integer solutions of the following equations

$$\begin{aligned} x^4 - (k^2 - 4)y^2 &= 4(k + 2), \\ x^4 - (k^2 - 4)y^2 &= -4(k - 2), \\ x^2 - (k^2 - 4)y^4 &= 4(k + 2), \\ x^2 - (k^2 - 4)y^4 &= -4(k - 2), \\ x^4 - kx^2y + y^2 &= -(k - 2), \end{aligned}$$

and

$$x^4 - kx^2y + y^2 = k + 2.$$

We show that if $k \equiv 1 \pmod{8}$, then the equation $x^4 - (k^2 - 4)y^2 = -4(k - 2)$ has no integer solutions. Moreover, if $k^2 - 4$ is square-free, then we investigate all positive integer solutions of the equations

$$x^4 - kx^2y + y^2 = (k^2 - 4)(k - 2)$$

and

$$x^4 - kx^2y + y^2 = -(k^2 - 4)(k + 2).$$

We show that if $k \equiv 1 \pmod{8}$ and $k^2 - 4$ is square-free, then the equation $x^4 - kx^2y + y^2 = (k - 2)(k^2 - 4)$ has no positive integer solutions. Moreover, if $k^2 - 4$

is square-free, then we show that the equation $x^4 - kx^2y + y^2 = -(k+2)(k^2-4)$ has no positive integer solutions. Let $\left(\frac{a}{b}\right)$ represent Jacobi symbol. Then

$$\left(\frac{a^2}{k}\right) = 1$$

for every $a \in \mathbb{Z}$. Moreover, we have

$$\left(\frac{-1}{k}\right) = 1 \text{ if and only if } k \equiv 1, 5 \pmod{8}, \quad (1.3)$$

$$\left(\frac{2}{k}\right) = 1 \text{ if and only if } k \equiv 1, 7 \pmod{8}, \quad (1.4)$$

and

$$\left(\frac{-2}{k}\right) = 1 \text{ if and only if } k \equiv 1, 3 \pmod{8}. \quad (1.5)$$

2. PRELIMINARY RESULTS

From now on, $\square$ represents a perfect square. Now we give the following lemma from [5].

Lemma 1. *Let k be odd. If $V_n(k, -1) = \square$ for some natural number n , then $n = 1$ and k is a perfect square.*

The following two theorems are given in [3].

Theorem 1. *Let $k \geq 3$ be an odd integer. If $V_n(k, -1) = k\square$ for some natural number n , then $n = 1$.*

Theorem 2. *Let $k \geq 3$ be an odd integer. If $V_n(k, -1) = q\square$ for some $q \mid k$ with $q > 1$ and $n \geq 1$, then $n = 1$.*

The following theorem is given in [6].

Theorem 3. *Let $n \in \mathbb{N} \cup \{0\}$, $m, r \in \mathbb{Z}$, and m be a nonzero integer. Then*

$$U_{2mn+r} \equiv U_r \pmod{U_m}, \quad (2.1)$$

$$U_{2mn+r} \equiv (-1)^n U_r \pmod{V_m}, \quad (2.2)$$

$$V_{2mn+r} \equiv V_r \pmod{U_m}, \quad (2.3)$$

and

$$V_{2mn+r} \equiv (-1)^n V_r \pmod{V_m}, \quad (2.4)$$

where $U_n = U_n(k, -1)$ and $V_n = V_n(k, -1)$.

When k is odd, it is seen that $V_{2^r} \equiv 7(\text{mod } 8)$ and thus

$$\left(\frac{-1}{V_{2^r}}\right) = -1 \quad (2.5)$$

for $r \geq 1$. It can be easily seen that

$$\left(\frac{k+1}{V_{2^r}}\right) = \left(\frac{k-1}{V_{2^r}}\right) = 1 \quad (2.6)$$

for $r \geq 1$. Moreover, it can be shown that if $k \equiv 1, 7(\text{mod } 8)$, then

$$\left(\frac{k+2}{V_{2^r}}\right) = 1 \quad (2.7)$$

and

$$\left(\frac{k-2}{V_{2^r}}\right) = \begin{cases} -1 & \text{if } k \equiv 1, 7(\text{mod } 8), \\ 1 & \text{if } k \equiv 3(\text{mod } 8). \end{cases} \quad (2.8)$$

From now on, instead of $U_n(k, -1)$ and $V_n(k, -1)$, we will write U_n and V_n , respectively. The following seven theorems are given in [4].

Theorem 4. *Let $k \geq 3$ be an integer and $k+2$ be not a perfect square. Then all positive integer solutions of the equation $x^2 - kxy + y^2 = k+2$ are given by $(x, y) = (U_{n+1} + U_n, U_n + U_{n-1})$ with $n \geq 1$.*

Theorem 5. *Let $k \geq 3$. Then all positive integer solutions of the equation $x^2 - kxy + y^2 = -(k-2)$ are given by $(x, y) = (U_{n+1} - U_n, U_n - U_{n-1})$ with $n \geq 0$.*

Theorem 6. *Let $k \geq 3$ an integer and $k+2$ be not a perfect square. Then all positive integer solutions of the equation $x^2 - (k^2 - 4)y^2 = 4(k+2)$ are given by $(x, y) = (V_n + V_{n-1}, U_n + U_{n-1})$ with $n \geq 1$.*

Theorem 7. *Let $k \geq 3$ an integer and $k+2$ be a perfect square. Then all positive integer solutions of the equation $x^2 - (k^2 - 4)y^2 = 4(k+2)$ are given by $(x, y) = (\sqrt{k+2}V_{n-2}, U_{n-2})$ with $n \geq 1$.*

Theorem 8. *Let $k \geq 3$. Then all positive integer solutions of the equation $x^2 - (k^2 - 4)y^2 = -4(k-2)$ are given by $(x, y) = (V_n - V_{n-1}, U_n - U_{n-1})$ with $n \geq 1$.*

Theorem 9. *Let $k \geq 3$ an integer and $k^2 - 4$ be square-free. Then all positive integer solutions of the equation $x^2 - kxy + y^2 = -(k+2)(k^2 - 4)$ are given by $(x, y) = (V_{n+1} + V_n, V_n + V_{n-1})$ with $n \geq 0$.*

Theorem 10. *Let $k \geq 3$ an integer and $k^2 - 4$ be square-free. Then all positive integer solutions of the equation $x^2 - kxy + y^2 = (k-2)(k^2 - 4)$ are given by $(x, y) = (V_{n+1} - V_n, V_n - V_{n-1})$ with $n \geq 1$.*

3. SOME THEOREMS AND LEMMAS

From now on, we will assume that $n \geq 0$ and $k \geq 3$ is odd.

Lemma 2. *If $k \equiv 3 \pmod{8}$, then $\left(\frac{k^2-k-1}{V_{2r}}\right) = -1$.*

Proof. An induction method shows that

$$V_{2r} \equiv \begin{cases} k-1 \pmod{k^2-k-1} & \text{if } r \text{ is even,} \\ -k \pmod{k^2-k-1} & \text{if } r \text{ is odd.} \end{cases}$$

Let r be even. Then we obtain

$$\begin{aligned} \left(\frac{k^2-k-1}{V_{2r}}\right) &= (-1)^{\left(\frac{k^2-k-2}{2}\right)\left(\frac{V_{2r}-1}{2}\right)} \left(\frac{V_{2r}}{k^2-k-1}\right) \\ &= \left(\frac{k-1}{k^2-k-1}\right) = \left(\frac{2}{k^2-k-1}\right) \left(\frac{(k-1)/2}{k^2-k-1}\right) \\ &= (-1)^{\frac{(k^2-k-1)^2-1}{8}} (-1)^{\left(\frac{k^2-k-2}{2}\right)\left(\frac{k-3}{4}\right)} \left(\frac{k^2-k-1}{(k-1)/2}\right) \\ &= (-1) \left(\frac{-1}{(k-1)/2}\right) = (-1)(-1)^{\frac{k-3}{4}} = -1. \end{aligned}$$

Let r be odd. Then we have

$$\begin{aligned} \left(\frac{k^2-k-1}{V_{2r}}\right) &= (-1)^{\left(\frac{k^2-k-2}{2}\right)\left(\frac{V_{2r}-1}{2}\right)} \left(\frac{V_{2r}}{k^2-k-1}\right) \\ &= \left(\frac{-k}{k^2-k-1}\right) = \left(\frac{-1}{k^2-k-1}\right) \left(\frac{k}{k^2-k-1}\right) \\ &= (-1)^{\frac{k^2-k-2}{2}} (-1)^{\left(\frac{k^2-k-2}{2}\right)\left(\frac{k-1}{2}\right)} \left(\frac{k^2-k-1}{k}\right) \\ &= \left(\frac{-1}{k}\right) = (-1)^{\left(\frac{k-1}{2}\right)} = -1. \end{aligned}$$

□

Since the proof of the following lemma is easy, we omit it.

Lemma 3. $V_{2r} \equiv \begin{cases} k \pmod{k^2+k-1} & \text{if } r \text{ is even,} \\ -k-1 \pmod{k^2+k-1} & \text{if } r \text{ is odd.} \end{cases}$

By using the above lemma, we can give the following lemma.

Lemma 4. *If $k \equiv 1, 5 \pmod{8}$, then $\left(\frac{k^2+k-1}{V_{2r}}\right) = 1$.*

Theorem 11. *Let $k \equiv 1, 5, 7 \pmod{8}$. If the equation $x^2 = V_n + V_{n-1}$ has a solution, then $n = 0$ or 1 . If $k \equiv 3 \pmod{8}$, then we have two solutions $(n, k) = (3, 3), (3, 43)$.*

Proof. Assume that $x^2 = V_n + V_{n-1}$ for some positive integer x . Firstly, let $k \equiv 1, 5, 7 \pmod{8}$ and $n > 2$. We divide the remainder of the proof into four cases.

Case 1: Assume that $n = 4q - 1$ with $q > 0$. Then $n = 2 \cdot 2^r a - 1$ with a odd and $r \geq 1$. By (2.4) and (2.3), it follows that

$$x^2 = V_n + V_{n-1} \equiv -(V_{-1} + V_{-2}) \equiv -(k^2 + k - 2) \pmod{V_{2r}}$$

and

$$x^2 = V_n + V_{n-1} \equiv V_{-1} + V_{-2} \equiv k^2 + k - 2 \equiv -2 \pmod{U_2}.$$

The above congruences give $\left(\frac{-(k^2+k-2)}{V_{2r}}\right) = 1$ and $\left(\frac{-2}{k}\right) = 1$. Since $\left(\frac{-2}{k}\right) = 1$, we obtain $k \equiv 1, 3 \pmod{8}$ by (1.5). Since $k \equiv 1, 5, 7 \pmod{8}$, it follows that $k \equiv 1 \pmod{8}$. By (2.6), (2.5), and (2.7), we get

$$1 = \left(\frac{-(k^2+k-2)}{V_{2r}}\right) = \left(\frac{-1}{V_{2r}}\right) \left(\frac{k+2}{V_{2r}}\right) \left(\frac{k-1}{V_{2r}}\right) = (-1) \cdot 1 \cdot 1 = -1,$$

a contradiction.

Case 2: Assume that $n = 4q$ with $q \geq 1$. Then $n = 2 \cdot 2^r a$ with a odd and $r \geq 1$. By (2.4), we get

$$x^2 = V_n + V_{n-1} \equiv -(V_0 + V_{-1}) \equiv -(k+2) \pmod{V_{2r}}.$$

This shows that $\left(\frac{-(k+2)}{V_{2r}}\right) = 1$. Then by (2.5), it follows that $\left(\frac{-(k+2)}{V_{2r}}\right) = \left(\frac{-1}{V_{2r}}\right) \left(\frac{k+2}{V_{2r}}\right) = (-1) \left(\frac{k+2}{V_{2r}}\right) = 1$, i.e., $\left(\frac{k+2}{V_{2r}}\right) = -1$. Moreover, since

$$x^2 = V_{4q} + V_{4q-1} \equiv V_0 + V_{-1} \equiv k+2 \equiv 2 \pmod{U_2},$$

we obtain $\left(\frac{2}{U_2}\right) = \left(\frac{2}{k}\right) = 1$. By (1.4), we get $k \equiv 1, 7 \pmod{8}$. Then by (2.7), it follows that $\left(\frac{k+2}{V_{2r}}\right) = 1$, which contradicts the fact that $\left(\frac{k+2}{V_{2r}}\right) = -1$.

Case 3: Assume that $n = 4q + 1$ with $q > 0$. Then $n = 2 \cdot 2^r a + 1$ with a odd and $r \geq 1$. By (2.4) and (2.3), it follows that

$$x^2 = V_n + V_{n-1} \equiv -(V_1 + V_0) \equiv -(k+2) \pmod{V_{2r}}$$

and

$$x^2 = V_{4q+1} + V_{4q} \equiv V_1 + V_0 \equiv k+2 \equiv 2 \pmod{U_2}.$$

Both the congruences above give $\left(\frac{-(k+2)}{V_{2r}}\right) = 1$ and $\left(\frac{2}{U_2}\right) = \left(\frac{2}{k}\right) = 1$. Then by (1.4) and (2.7), we have a contradiction.

Case 4: Assume that $n = 4q + 2$ for some $q > 0$. Then $n = 2 \cdot 2^r a + 2$ with a odd and $r \geq 1$. By (2.4) and (2.3), it follows that

$$x^2 = V_n + V_{n-1} \equiv -(V_2 + V_1) \equiv -(k^2 + k - 2) \pmod{V_{2r}}$$

and

$$x^2 = V_n + V_{n-1} \equiv V_2 + V_1 \equiv -2 \pmod{U_2}.$$

These show that $\left(\frac{-(k^2+k-2)}{V_{2r}}\right) = 1$ and $\left(\frac{-2}{k}\right) = 1$. Since $\left(\frac{-2}{k}\right) = 1$, we obtain $k \equiv 1, 3 \pmod{8}$ by (1.5). It follows that $k \equiv 1 \pmod{8}$ since $k \equiv 1, 5, 7 \pmod{8}$. By (2.6), (2.5), and (2.7), we get

$$1 = \left(\frac{-(k^2+k-2)}{V_{2r}}\right) = \left(\frac{-1}{V_{2r}}\right) \left(\frac{k+2}{V_{2r}}\right) \left(\frac{k-1}{V_{2r}}\right) = (-1) \cdot 1 \cdot 1 = -1,$$

a contradiction. Thus we get $n \leq 2$. If $n = 2$, then we obtain $x^2 = V_2 + V_1 = k^2 + k - 2$. The case that $k = 2$ is impossible since $k \geq 3$. Thus we get $n = 0$ or 1 .

Now, assume that $k \equiv 3 \pmod{8}$. Let $n = 6q + r$ with $0 \leq r \leq 5$. Then we get

$$V_n = V_{6q+r} \equiv V_r \pmod{U_3}$$

by using (2.3). Since $8 \mid U_3$, it follows that $V_n \equiv V_r \pmod{8}$ with $0 \leq r \leq 5$ and since $k \equiv 3 \pmod{8}$, it can be easily seen that $V_n \equiv V_0, V_1, V_2, V_3, V_4, V_5 \equiv 2, 3, 7 \pmod{8}$. Then we obtain $x^2 = V_n + V_{n-1} \equiv 5, 2, 1 \pmod{8}$. Since $x^2 \equiv 0, 1, 4 \pmod{8}$, we get $r = 3$ or 4 .

Assume that $r = 3$. Then $n = 6q + 3$ with $q \geq 0$. Let q be odd. Then by (2.4), it follows that

$$x^2 = V_n + V_{n-1} = V_{6q+3} + V_{6q+2} \equiv -(V_3 + V_2) \equiv -V_2 \equiv -(k^2 - 2) \pmod{V_3}.$$

Since $V_3 = k(k^2 - 3)$, we obtain

$$x^2 \equiv -(k^2 - 2) \equiv -k^2 + 2 \equiv -1 \pmod{k^2 - 3}.$$

This is impossible since $\frac{k^2-3}{2} \equiv -1 \pmod{4}$.

Let q be even and $q > 0$. Then $n = 6q + 3 = 2 \cdot 2^r a + 3$ with a odd and $r \geq 1$. By (2.4), it follows that

$$\begin{aligned} x^2 = V_n + V_{n-1} &= V_{2 \cdot 2^r a + 3} + V_{2 \cdot 2^r a + 2} \\ &\equiv -(V_3 + V_2) \equiv -(k + 2)(k^2 - k - 1) \pmod{V_{2r}}. \end{aligned}$$

This shows that $\left(\frac{-(k+2)(k^2-k-1)}{V_{2r}}\right) = 1$. However, this is impossible since

$$\left(\frac{-(k+2)(k^2-k-1)}{V_{2r}}\right) = \left(\frac{-1}{V_{2r}}\right) \left(\frac{k+2}{V_{2r}}\right) \left(\frac{k^2-k-1}{V_{2r}}\right) = (-1)(-1)(-1) = -1$$

by (2.5), (2.7), and Lemma 2. If $q = 0$, then we get $x^2 = V_3 + V_2 = k^3 + k^2 - 3k - 2$. But the integer points on $Y^2 = X^3 + X^2 - 3X - 2$ are easily determined by using MAGMA [1] to be $(X, Y) = (-2, 0), (-1, \pm 1), (3, \pm 5), (2, \pm 2), (43, \pm 285)$. Then it follows that $k = 3$ or 43 .

Now, assume that $r = 4$. Then $n = 6q + 4$ with $q \geq 0$. Thus we can write $n = 6(q + 1) - 2 = 6t - 2$ and therefore,

$$V_n + V_{n-1} = V_{6t-2} + V_{6t-3} \equiv -(V_3 + V_2) \equiv -V_2 \equiv -(k^2 - 2) \pmod{V_3}.$$

Since $V_3 = k(k^2 - 3)$, we obtain

$$x^2 \equiv -(k^2 - 2) \equiv -k^2 + 2 \equiv -1 \pmod{k^2 - 3}.$$

This is impossible since $\frac{k^2-3}{2} \equiv -1 \pmod{4}$. $\square$

Theorem 12. *Let $k \equiv 3, 5, 7 \pmod{8}$. If the equation $x^2 = V_n - V_{n-1}$ has a solution, then $n = 1$ or 2 . Moreover, if $k \equiv 1 \pmod{8}$, then the equation $x^2 = V_n - V_{n-1}$ has no solutions.*

Proof. Let $k \equiv 3, 5, 7 \pmod{8}$. Assume that $x^2 = V_n - V_{n-1}$ for some positive integer x . Let $n > 2$. We divide the remainder of the proof into four cases.

Case 1: Assume that $n = 4q - 1$ with $q > 0$. Then $n = 2 \cdot 2^r a - 1$ with a odd and $r \geq 1$. By (2.4) and (2.3), it follows that

$$x^2 = V_{4q-1} - V_{4q-2} \equiv -(V_{-1} - V_{-2}) \equiv k^2 - k - 2 \pmod{V_{2^r}}$$

and

$$x^2 = V_{4q-1} - V_{4q-2} \equiv V_{-1} - V_{-2} \equiv 2 \pmod{U_2}.$$

These show that $\left(\frac{k^2-k-2}{V_{2^r}}\right) = 1$ and $\left(\frac{2}{k}\right) = 1$. Since $\left(\frac{2}{k}\right) = 1$, it follows that $k \equiv 1, 7 \pmod{8}$ by (1.4). Since $k \equiv 3, 5, 7 \pmod{8}$, we get $k \equiv 7 \pmod{8}$. Then by using (1.4) and (2.8), we obtain

$$1 = \left(\frac{k^2-k-2}{V_{2^r}}\right) = \left(\frac{k-2}{V_{2^r}}\right) \left(\frac{k+1}{V_{2^r}}\right) = (-1) \cdot 1 = -1,$$

a contradiction.

Case 2: Assume that $n = 4q$ with $q \geq 1$. Then $n = 2 \cdot 2^r a$ with a odd and $r \geq 1$. By (2.4), we get

$$x^2 = V_{4q} - V_{4q-1} \equiv -(V_0 - V_{-1}) \equiv k - 2 \pmod{V_{2^r}}.$$

This shows that $\left(\frac{k-2}{V_{2^r}}\right) = 1$. On the other hand, by using (2.3), we obtain

$$x^2 = V_{4q} - V_{4q-1} \equiv V_0 - V_{-1} \equiv -k + 2 \pmod{U_2},$$

which implies that $\left(\frac{2}{k}\right) = 1$. Then it follows that $k \equiv 1, 7 \pmod{8}$ by (1.4). Since $k \equiv 3, 5, 7 \pmod{8}$, we have $k \equiv 7 \pmod{8}$. Then by (2.8), we get $\left(\frac{k-2}{V_{2^r}}\right) = -1$, which contradicts the fact that $\left(\frac{k-2}{V_{2^r}}\right) = 1$.

Case 3: Assume that $n = 4q + 1$ with $q > 0$. Then $n = 2 \cdot 2^r a + 1$ with a odd and $r \geq 1$. By (2.4) and (2.3), it follows that

$$x^2 = V_{4q+1} - V_{4q} \equiv -(V_1 - V_0) \equiv -(k - 2) \pmod{V_{2^r}}$$

and

$$x^2 = V_{4q+1} - V_{4q} \equiv V_1 - V_0 \equiv -2 \pmod{U_2}.$$

These show that $\left(\frac{-(k-2)}{V_{2r}}\right) = 1$ and $\left(\frac{-2}{k}\right) = 1$. Since $\left(\frac{-2}{k}\right) = 1$, it follows that $k \equiv 1, 3 \pmod{8}$ by (1.5). Since $k \equiv 3, 5, 7 \pmod{8}$, we get $k \equiv 3 \pmod{8}$. Then by (2.5) and (2.8), we obtain

$$\left(\frac{-(k-2)}{V_{2r}}\right) = \left(\frac{-1}{V_{2r}}\right) \left(\frac{k-2}{V_{2r}}\right) = -1,$$

which contradicts the fact that $\left(\frac{-(k-2)}{V_{2r}}\right) = 1$.

Case 4: Assume that $n = 4q + 2$ with $q > 0$. Then $n = 2 \cdot 2^r a + 2$ with a odd and $r \geq 1$. By (2.4) and (2.3), it follows that

$$x^2 = V_{4q+2} - V_{4q+1} \equiv -(V_2 - V_1) \equiv -(k^2 - k - 2) \pmod{V_{2r}}$$

and

$$x^2 = V_{4q+2} - V_{4q+1} \equiv V_2 - V_1 \equiv -2 \pmod{U_2}.$$

The above congruences give $\left(\frac{-(k^2-k-2)}{V_{2r}}\right) = 1$ and $\left(\frac{-2}{k}\right) = 1$. Since $\left(\frac{-2}{k}\right) = 1$, it follows that $k \equiv 1, 3 \pmod{8}$ by (1.5). Since $k \equiv 3, 5, 7 \pmod{8}$, we get $k \equiv 3 \pmod{8}$. Then by using (1.4), (2.5), and (2.8), we obtain $1 = \left(\frac{-(k^2-k-2)}{V_{2r}}\right) = \left(\frac{-1}{V_{2r}}\right) \left(\frac{k-2}{V_{2r}}\right) \left(\frac{k+1}{V_{2r}}\right) = -1$, a contradiction. Therefore, $n \leq 2$. If $n = 0$, then we have $x^2 = V_0 - V_{-1} = 2 - k$, which is impossible. So, we get $n = 1$ or 2 .

Now, assume that $k \equiv 1 \pmod{8}$. Let $n = 6q + r$ with $0 \leq r \leq 5$. Then by using (2.3), we get

$$V_n = V_{6q+r} \equiv V_r \pmod{U_3}.$$

Since $8 \mid U_3$, it follows that $V_n \equiv V_r \pmod{8}$ with $0 \leq r \leq 5$. Moreover, since $k \equiv 1 \pmod{8}$, it can be easily seen that $V_n \equiv V_0, V_1, V_2, V_3, V_4, V_5 \equiv 2, 1, 7, 6 \pmod{8}$. Then we get $x^2 = V_n - V_{n-1} \equiv 1, 6, 7 \pmod{8}$. It can be seen that $r = 0$ or 4 .

Assume that $r = 0$. Then $n = 6q$ with $q \geq 0$. Let q be odd. Then it follows that

$$x^2 = V_n - V_{n-1} = V_{6q} - V_{6q-1} \equiv -(V_0 - V_{-1}) \equiv k - 2 \pmod{V_3}$$

by (2.4). Since $V_3 = k(k^2 - 3)$, we obtain

$$x^2 \equiv k - 2 \pmod{k^2 - 3},$$

which implies that $x^2 \equiv k - 2 \pmod{\frac{k^2-3}{2}}$. Therefore, $\left(\frac{k-2}{(k^2-3)/2}\right) = 1$. Since $k \equiv 1 \pmod{8}$, we get $\frac{k^2-3}{2} \equiv 3 \pmod{4}$. These show that

$$\begin{aligned} 1 &= \left(\frac{k-2}{(k^2-3)/2}\right) = (-1)^{\left(\frac{k-3}{2}\right)\left(\frac{k^2-5}{4}\right)} \left(\frac{(k^2-3)/2}{k-2}\right) \\ &= (-1)^{\left(\frac{2}{k-2}\right)\left(\frac{k^2-3}{k-2}\right)} \end{aligned}$$

$$= (-1) \cdot 1 \cdot \left(\frac{1}{k-2} \right) = -1,$$

a contradiction. Let q be even. Then $n = 6q = 2 \cdot 2^r a$ with a odd and $r \geq 1$. By (2.4), it follows that

$$x^2 = V_n - V_{n-1} = V_{2 \cdot 2^r a} - V_{2 \cdot 2^r a - 1} \equiv -(V_0 - V_{-1}) \equiv k - 2 \pmod{V_{2^r}}.$$

This shows that $\left(\frac{k-2}{V_{2^r}} \right) = 1$, which is impossible by (2.8).

Now, assume that $r = 4$. Then $n = 6q + 4$ with $q \geq 0$. We can write $n = 6(q + 1) - 2 = 6t - 2$. Let t be odd. Then it follows that

$$x^2 = V_n - V_{n-1} = V_{6t-2} - V_{6t-3} \equiv -(V_{-2} - V_{-3}) \equiv -V_2 \equiv -(k^2 - 2) \pmod{V_3}$$

by (2.4). Since $V_3 = k(k^2 - 3)$, we obtain

$$x^2 \equiv -(k^2 - 2) \equiv -1 \pmod{k^2 - 3},$$

which shows that $x^2 \equiv -1 \pmod{\frac{k^2-3}{2}}$. But this is impossible since $\frac{k^2-3}{2} \equiv 3 \pmod{4}$.

Let t be even. Then $n = 6t - 2 = 2 \cdot 2^r a - 2$ with a odd and $r \geq 1$. It follows that

$$\begin{aligned} x^2 &= V_n - V_{n-1} = V_{2 \cdot 2^r a - 2} - V_{2 \cdot 2^r a - 3} \\ &\equiv -(V_{-2} - V_{-3}) \equiv (k - 2)(k^2 + k - 1) \pmod{V_{2^r}} \end{aligned}$$

by (2.4). This shows that $\left(\frac{(k-2)(k^2+k-1)}{V_{2^r}} \right) = 1$. Then by (2.8) and Lemma 4, we obtain

$$\left(\frac{k-2}{V_{2^r}} \right) \left(\frac{k^2+k-1}{V_{2^r}} \right) = -1,$$

which contradicts the fact that $\left(\frac{(k-2)(k^2+k-1)}{V_{2^r}} \right) = 1$. This completes the proof. $\square$

Theorem 13. *If the equation $x^2 = U_{n+1} + U_n$ has a solution, then $n = 0$ or 1 .*

Proof. Assume that $x^2 = U_{n+1} + U_n$ for some positive integer x . Now, assume that $n > 2$. We distinguish four cases.

Case 1: Assume that $n = 4q - 1$ with $q > 0$. Then by (2.1), we get

$$x^2 = U_{4q} + U_{4q-1} \equiv U_0 + U_{-1} \equiv -1 \pmod{U_2}.$$

This shows that $\left(\frac{-1}{k} \right) = 1$. Then it follows that $k \equiv 1, 5 \pmod{8}$ by (1.3).

Let $q = 3u - 1$ with $u > 0$. Then $n = 12u - 5$. By using (2.1), we obtain

$$x^2 = U_{12u-4} + U_{12u-5} \equiv U_{-4} + U_{-5} \equiv k - 3 \pmod{U_3}.$$

Since $8 \mid U_3$, we get $x^2 \equiv k - 3 \pmod{8}$. Moreover, since $k \equiv 1, 5 \pmod{8}$, it follows that $x^2 \equiv -2, 2 \pmod{8}$, which is impossible.

Let $q = 3u$ with $u > 0$. Then $n = 12u - 1$. Here, we get

$$x^2 = U_{12u-1} + U_{12u} \equiv U_{-1} + U_0 \equiv -1 \pmod{U_3}$$

by (2.1). Since $8 \mid U_3$, it follows that $x^2 \equiv -1 \pmod{8}$, which is impossible.

Let $q = 3u + 1$ with $u \geq 0$. Then $n = 12u + 3$. Here, we obtain

$$x^2 = U_{12u+4} + U_{12u+3} \equiv U_4 + U_3 \equiv U_4 \equiv -k \pmod{U_3}$$

by (2.1). Since $8 \mid U_3$, it follows that $x^2 \equiv -k \pmod{8}$. Moreover, since $k \equiv 1, 5 \pmod{8}$, we obtain $x^2 \equiv -1, -5 \pmod{8}$. However, this is impossible.

Case 2: Assume that $n = 4q$ with $q \geq 1$. Then $n = 2 \cdot 2^r a$ with a odd and $r \geq 1$. By (2.2), we get

$$x^2 = U_{4q+1} + U_{4q} \equiv -(U_1 + U_0) \equiv -1 \pmod{V_{2r}},$$

which is impossible by (2.5).

Case 3: Assume that $n = 4q + 1$ with $q > 0$. Then $n = 2 \cdot 2^r a + 1$ with a odd and $r \geq 1$. By (2.2), we get

$$x^2 = U_{4q+2} + U_{4q+1} \equiv -(U_2 + U_1) \equiv -(k + 1) \pmod{V_{2r}}.$$

This shows that $\left(\frac{-(k+1)}{V_{2r}}\right) = 1$. Since $\left(\frac{-1}{V_{2r}}\right) = -1$ and $\left(\frac{-(k+1)}{V_{2r}}\right) = \left(\frac{-1}{V_{2r}}\right) \left(\frac{k+1}{V_{2r}}\right)$, it follows that $\left(\frac{k+1}{V_{2r}}\right) = -1$. This is impossible by (1.5).

Case 4: Assume that $n = 4q + 2$. Then $n = 2 \cdot 2^r a + 2$ with a odd and $r \geq 1$. By (2.1), we obtain

$$x^2 = U_{4q+3} + U_{4q+2} \equiv U_3 + U_2 \equiv -1 \pmod{U_2},$$

i.e., $\left(\frac{-1}{k}\right) = 1$. It follows that $k \equiv 1, 5 \pmod{8}$ by (1.3). On the other hand, by (2.2), we get

$$x^2 = U_{4q+3} + U_{4q+2} \equiv -(U_3 + U_2) \equiv -(k^2 + k - 1) \pmod{V_{2r}}.$$

This shows that $\left(\frac{-(k^2+k-1)}{V_{2r}}\right) = 1$. Since $\left(\frac{-(k^2+k-1)}{V_{2r}}\right) = \left(\frac{-1}{V_{2r}}\right) \left(\frac{k^2+k-1}{V_{2r}}\right) = 1$ and $\left(\frac{-1}{V_{2r}}\right) = -1$, we have $\left(\frac{k^2+k-1}{V_{2r}}\right) = -1$. This is impossible by Lemma 4. Therefore, $n \leq 2$. If $n = 2$, then $x^2 = U_3 + U_2 = k^2 + k - 1$, which shows that $k = 1$. This is impossible since $k \geq 3$. Thus we get $n = 0$ or 1 . $\square$

Since the proof of the following theorem is similar to that of above theorem, we omit its proof.

Theorem 14. *If the equation $x^2 = U_{n+1} - U_n$ has a solution, then $n = 0$ or 1 .*

4. MAIN THEOREMS

Now, we consider positive integer solutions of some fourth-order Diophantine equations.

Theorem 15. *Let $k + 2$ be not a perfect square. If $k + 1$ is a perfect square, then all positive integer solutions of the equation $x^4 - kx^2y + y^2 = k + 2$ are given by $(x, y) = (\sqrt{k+1}, 1), (\sqrt{k+1}, k^2 + k - 1), (1, k + 1)$. If $k + 1$ is not a perfect square, then the only positive integer solution of this equation is $(x, y) = (1, k + 1)$.*

Proof. Assume that $x^4 - kx^2y + y^2 = k + 2$ for some positive integers x and y . Then by Theorem 4, we obtain $x^2 = U_{n+1} + U_n$, $y = U_n + U_{n-1}$ or $x^2 = U_n + U_{n-1}$, $y = U_{n+1} + U_n$ with $n > 0$. Let $x^2 = U_{n+1} + U_n$. Then it follows that $n = 0$ or 1 by Theorem 13. Since $n > 0$, we have $n = 1$. Then we have $x^2 = U_{n+1} + U_n = U_2 + U_1 = k + 1$ and $y = U_1 + U_0 = 1$. If $k + 1$ is a perfect square, then it follows that $x = \sqrt{k+1}$ and $y = 1$. Now, let $x^2 = U_n + U_{n-1}$. Then we obtain $n - 1 = 0$ or 1 by Theorem 13. Assume that $n = 1$. Then $x^2 = U_1 + U_0 = 1$ and $y = U_2 + U_1 = k + 1$. Thus we have $x = 1$ and $y = k + 1$. Now, assume that $n = 2$. Then we get $x^2 = U_2 + U_1 = k + 1$ and $y = U_3 + U_2 = k^2 + k - 1$. If $k + 1$ is a perfect square, then it follows that $x = \sqrt{k+1}$ and $y = k^2 + k - 1$. Thus if $k + 1$ is not a perfect square, then this equation has no positive integer solutions. Then the proof follows. $\square$

Theorem 16. *If $k - 1$ is a perfect square, then all positive integer solutions of the equation $x^4 - kx^2y + y^2 = -(k - 2)$ are given by $(x, y) = (1, 1), (\sqrt{k-1}, 1), (\sqrt{k-1}, k^2 - k - 1), (1, k - 1)$. Moreover, if $k - 1$ is not a perfect square, then all positive integer solutions of this equation are given by $(x, y) = (1, 1)$ or $(1, k - 1)$.*

Proof. Assume that $x^4 - kx^2y + y^2 = -(k - 2)$ for some positive integers x and y . Then by Theorem 5, we obtain $x^2 = U_{n+1} - U_n$, $y = U_n - U_{n-1}$ or $x^2 = U_n - U_{n-1}$, $y = U_{n+1} - U_n$ with $n \geq 0$. Let $x^2 = U_{n+1} - U_n$. Then it follows that $n = 0$ or 1 by Theorem 14. Assume that $n = 0$. Then we have $x^2 = U_{n+1} - U_n = U_1 - U_0 = 1$ and $y = U_0 - U_{-1} = 1$. Thus we have $x = 1$ and $y = 1$. Now, assume that $n = 1$. Then we get $x^2 = U_2 - U_1 = k - 1$ and $y = U_1 - U_0 = 1$. If $k - 1$ is a perfect square, then it follows that $x = \sqrt{k-1}$ and $y = 1$. Now, let $x^2 = U_n - U_{n-1}$. Then we obtain $n = 1$ or 2 by Theorem 14. Assume that $n = 1$. Then $x^2 = U_1 - U_0 = 1$ and $y = U_2 - U_1 = k - 1$. Thus we have $x = 1$ and $y = k - 1$. Now, assume that $n = 2$. Then we get $x^2 = U_2 - U_1 = k - 1$ and $y = U_3 - U_2 = k^2 - k - 1$. If $k - 1$ is a perfect square, then it follows that $x = \sqrt{k-1}$ and $y = k^2 - k - 1$. Thus if $k - 1$ is not a perfect square, then this equation has no positive integer solutions other than $(1, 1)$ and $(1, k - 1)$. $\square$

Theorem 17. *If $k \equiv 1, 5, 7 \pmod{8}$, then the equation $x^4 - (k^2 - 4)y^2 = 4(k + 2)$ has a solution if and only if $k + 2$ is a perfect square. Moreover, if $k \equiv 1, 5, 7 \pmod{8}$ and $k + 2$ is a perfect square, then the only positive integer solution of this equation is $(x, y) = (\sqrt{k+2}, 1)$. If $k \equiv 3 \pmod{8}$, then the equation $x^4 - (k^2 - 4)y^2 = 4(k + 2)$ has positive integer solutions only when $k = 3$ or 43.*

Proof. Assume that $x^4 - (k^2 - 4)y^2 = 4(k + 2)$ for some positive integers x and y . Firstly, let $k \equiv 1, 5, 7 \pmod{8}$ and $k + 2$ be not a perfect square. Then by Theorem 6, it follows that $x^2 = V_n + V_{n-1}$ with $n \geq 1$. Therefore, by Theorem 11, it is seen that $n = 0$ or 1 . Since $n \geq 1$, we get $n = 1$. Then it follows that $x^2 = k + 2$. But this is impossible since $k + 2$ is not a perfect square. Let $k \equiv 1, 5, 7 \pmod{8}$ and $k + 2$ be a perfect square. Then by Theorem 7, it follows that $x^2 = \sqrt{k + 2}V_{n-2}$. Therefore, we obtain $V_{n-2}(\sqrt{k + 2}, -1) = \square, \sqrt{k + 2}\square$ or $a\square$ with $a \mid \sqrt{k + 2}$. If $V_{n-2}(\sqrt{k + 2}, -1) = \sqrt{k + 2}\square$ or $a\square$ with $a \mid \sqrt{k + 2}$, then it follows that $n = 3$ by Theorem 2 and Theorem 1. Then it can be seen that $x = \sqrt{k + 2}$ and $y = 1$. If $V_{n-2}(\sqrt{k + 2}, -1) = \square$, we have $n = 3$ and $k + 2$ is a perfect square by Lemma 1. Thus if $k + 2$ is a perfect square, then we obtain $x = \sqrt{k + 2}$ and $y = 1$. Now, let $k \equiv 3 \pmod{8}$. Then $k + 2$ is not a perfect square and it follows that $k = 3$ or 43 by Theorem 6 and Theorem 11. When $k = 3$, we get the solution $(x, y) = (5, 11)$. When $k = 43$, we get the solution $(x, y) = (285, 1891)$. $\square$

Theorem 18. *If $k + 1$ is not a perfect square, then the only positive integer solution of the equation $x^2 - (k^2 - 4)y^4 = 4(k + 2)$ is $(x, y) = (k + 2, 1)$. Moreover, if $k + 1$ is a perfect square, then all positive integer solutions of this equation are given by $(x, y) = (k^2 + k - 2, \sqrt{k + 1})$ or $(k + 2, 1)$.*

Proof. Assume that $x^2 - (k^2 - 4)y^4 = 4(k + 2)$ for some positive integers x and y . Then by Theorem 6, it follows that $x = V_n + V_{n-1}$, $y^2 = U_n + U_{n-1}$ with $n > 0$. Therefore, by Theorem 13, we have $n = 1$ or 2 . Let $n = 1$. Then it follows that $x = V_1 + V_0 = k + 2$ and $y^2 = U_1 + U_0 = 1$. Therefore, we obtain $(x, y) = (k + 2, 1)$. Let $n = 2$. Then it is seen that $x = V_2 + V_1 = k^2 - 2 + k$ and $y^2 = U_2 + U_1 = k + 1$. If $k + 1$ is a perfect square, then we obtain $x = k^2 + k - 2$ and $y = \sqrt{k + 1}$. $\square$

Theorem 19. *Let $k \equiv 3, 5, 7 \pmod{8}$. If $k = 3$, then all positive integer solutions of the equation $x^4 - (k^2 - 4)y^2 = -4(k - 2)$ are given by $(x, y) = (2, 2), (1, 1)$. If $k \neq 3$ and $k - 2$ is a perfect square, then the only positive integer solution of the equation $x^4 - (k^2 - 4)y^2 = -4(k - 2)$ is $(x, y) = (\sqrt{k - 2}, 1)$. If $k \neq 3$ and $k - 2$ is not a perfect square, then this equation has no positive integer solutions.*

Proof. Assume that $x^4 - (k^2 - 4)y^2 = -4(k - 2)$ for some positive integers x and y . Then by Theorem 8, we obtain $x^2 = V_n - V_{n-1}$ and $y = U_n - U_{n-1}$ with $n > 0$. Since $k \equiv 3, 5, 7 \pmod{8}$ and $x^2 = V_n - V_{n-1}$, it follows that $n = 1$ or 2 by Theorem 12. Let $n = 1$. Then we have $x^2 = V_1 - V_0 = k - 2$ and $y = U_1 - U_0 = 1$. If $k - 2$ is a perfect square, then we obtain $(x, y) = (\sqrt{k - 2}, 1)$. Otherwise this equation has no solutions. Let $n = 2$. Then it follows that $x^2 = V_2 - V_1 = k^2 - 2 - k$ and $y = U_2 - U_1 = k - 1$. Since $x^2 = k^2 - k - 2$, we get $4x^2 = 4k^2 - 4k - 8$, i.e., $(2k - 1)^2 - (2x)^2 = 9$. This shows that $k = 3$. Hence, we get $x = 2$ and $y = 2$. $\square$

Theorem 20. *Let $k \equiv 1 \pmod{8}$. Then the equation $x^4 - (k^2 - 4)y^2 = -4(k - 2)$ has no integer solutions.*

Proof. Assume that $x^4 - (k^2 - 4)y^2 = -4(k - 2)$ for some positive integers x and y . Then by Theorem 8, we obtain $x^2 = V_n - V_{n-1}$ and $y = U_n - U_{n-1}$. This is impossible by Theorem 12. $\square$

Theorem 21. *If $k - 1$ is not a perfect square, then the only positive integer solution of the equation $x^2 - (k^2 - 4)y^4 = -4(k - 2)$ is $(x, y) = (k - 2, 1)$. Moreover, if $k - 1$ is a perfect square, then all positive integer solutions of the equation $x^2 - (k^2 - 4)y^4 = -4(k - 2)$ are given by $(x, y) = (k^2 - 2 - k, \sqrt{k - 1})$, $(k - 2, 1)$.*

Proof. Assume that $x^2 - (k^2 - 4)y^4 = -4(k - 2)$ for some positive integers x and y . Then by Theorem 8, we obtain $x = V_n - V_{n-1}$ and $y^2 = U_n - U_{n-1}$ with $n > 0$. Therefore, by Theorem 14, it follows that $n = 1$ or 2 . Let $n = 1$. Then we have $x = V_1 - V_0 = k - 2$ and $y^2 = U_1 - U_0 = 1$. Therefore, $(x, y) = (k - 2, 1)$. Now, let $n = 2$. Then it follows that $x = V_2 - V_1 = k^2 - 2 - k$ and $y^2 = U_2 - U_1 = k - 1$. Therefore, if $k - 1$ is a perfect square, then it follows that $(x, y) = (k^2 - k - 2, \sqrt{k - 1})$ is a solution. $\square$

Theorem 22. *Let $k \equiv 1, 5, 7 \pmod{8}$ and $k^2 - 4$ be square-free. Then the equation $x^4 - kx^2y + y^2 = -(k + 2)(k^2 - 4)$ has no positive integer solutions.*

Proof. Assume that $x^4 - kx^2y + y^2 = -(k + 2)(k^2 - 4)$ for some positive integers x and y . Then by Theorem 9, we obtain $x^2 = V_{n+1} + V_n$, $y = V_n + V_{n-1}$ or $x^2 = V_n + V_{n-1}$, $y = V_{n+1} + V_n$ with $n \geq 0$. Let $x^2 = V_{n+1} + V_n$. Since $k \equiv 1, 5, 7 \pmod{8}$, we obtain $n + 1 = 0$ or 1 by Theorem 11. Since $n \geq 0$, we get $n = 0$. Then we have $x^2 = V_1 + V_0 = k + 2$ and $y = V_0 + V_{-1} = k + 2$. This is impossible since $k^2 - 4$ is square-free. Let $x^2 = V_n + V_{n-1}$. Then by Theorem 11, we obtain $n = 0$ or 1 . Thus it follows that $x^2 = V_0 + V_{-1} = V_1 + V_0 = k + 2$. This is impossible since $k^2 - 4$ is square-free. $\square$

From Theorem 9 and Theorem 11, we can give the following theorem.

Theorem 23. *Let $k \equiv 3 \pmod{8}$ and $k^2 - 4$ be square-free. Then the equation $x^4 - kx^2y + y^2 = -(k + 2)(k^2 - 4)$ has no positive integer solutions.*

Theorem 24. *Let $k \equiv 3, 5, 7 \pmod{8}$ and $k^2 - 4$ be square-free. Then the equation $x^4 - kx^2y + y^2 = (k - 2)(k^2 - 4)$ has positive integer solutions only when $k = 3$ and in which case all positive integer solutions are given by $(x, y) = (2, 1)$, $(1, 4)$ or $(2, 11)$.*

Proof. Assume that $x^4 - kx^2y + y^2 = (k - 2)(k^2 - 4)$ for some positive integers x and y . Then by Theorem 10, we obtain $x^2 = V_{n+1} - V_n$, $y = V_n - V_{n-1}$ or $x^2 = V_n - V_{n-1}$, $y = V_{n+1} - V_n$ with $n \geq 1$. Assume that $x^2 = V_{n+1} - V_n$. Then by Theorem 12, we obtain $n + 1 = 1$ or 2 . Since $n \geq 1$, we get $n = 1$. Then it follows that $x^2 = V_2 - V_1 = k^2 - k - 2$ and $y = V_1 - V_0 = k - 2$. Since $x^2 = k^2 - k - 2$, we get $4x^2 = 4k^2 - 4k - 8$, i.e., $(2k - 1)^2 - (2x)^2 = 9$. Thus it follows that $k = 3$. Then it can be seen that $(x, y) = (2, 1)$. Now, assume that $x^2 = V_n - V_{n-1}$. Then by Theorem

12, it follows that $n = 1$ or 2 . Let $n = 1$. Then we get $x^2 = V_1 - V_0 = k - 2$ and $y = V_2 - V_1 = k^2 - 2 - k$. Thus $k - 2$ must be a perfect square. But this is impossible in the case that $k > 3$. Let $k = 3$. Then we obtain $x = 1$ and $y = 4$. Let $n = 2$. Then we obtain $x^2 = V_2 - V_1 = k^2 - k - 2$. It can be shown that $k = 3$. Then we have $(x, y) = (2, 11)$. This completes the proof. $\square$

From Theorem 10 and Theorem 12, we can give the following theorem.

Theorem 25. *Let $k \equiv 1 \pmod{8}$ and $k^2 - 4$ be square-free. Then the equation $x^4 - kx^2y + y^2 = (k - 2)(k^2 - 4)$ has no positive integer solutions.*

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CLOSED FORM EVALUATION OF MELHAM'S RECIPROCAL SUMS

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Abstract. Recently Melham [1] gives closed formulæ for certain finite reciprocal sums. In this paper, we present a different approach to compute these sums in closed form. Our approach is straight-forward and simple. First we convert the sums in q -notation, then use partial fraction decomposition and telescoping to derive closed formulæ.

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1. INTRODUCTION

Let $a \geq 0$ and $b \geq 0$ be integers with $(a, b) \neq (0, 0)$. For p a positive integer, define the sequences $\{W_n\}$ and $\{\overline{W}_n\}$ by

$$W_n = pW_{n-1} + W_{n-2} \quad \text{and} \quad \overline{W}_n = W_{n-1} + W_{n+1},$$

where $W_0 = a$, $W_1 = b$.

When $(a, b, p) = (0, 1, 1)$, we have $\{W_n\} = \{F_n\}$, and $\{\overline{W}_n\} = \{L_n\}$, which are the Fibonacci and Lucas numbers, respectively. When $(a, b, p) = (0, 1, p)$, we denote $\{W_n\} = \{U_n\}$, and $\{\overline{W}_n\} = \{V_n\}$, which are the generalized Fibonacci and Lucas numbers, respectively.

For $k \geq 1$, $m \geq 0$, and $n \geq 2$, the author [1] gave closed formulæ for the following finite reciprocal sums:

- (1) For nonnegative integers $m_1 < m_2$ and $m_3 < m_4$ with $m_1 + m_2 = m_3 + m_4$,

$$\sum_{t=1}^n \frac{1}{W_{k(t+m_1)+m} W_{k(t+m_2)+m} \overline{W}_{k(t+m_3)+m} \overline{W}_{k(t+m_4)+m}},$$

- (2) For nonnegative integers m_i with $m_i < m_{i+1}$ for $1 < i < 4$ and positive integer c ,

$$\text{i) } \sum_{t=1}^n \frac{U_{2kt+2m}}{P_6(W, \overline{W})} \quad \text{ii) } \sum_{t=1}^n \frac{U_{2k(t+m_1)+2m}}{P_6(W, \overline{W})} \quad \text{iii) } \sum_{t=1}^n \frac{U_{2k(t+m_2)+2m}}{P_6(W, \overline{W})},$$

(3)

$$\text{i) } \sum_{t=1}^n \frac{1}{P_8(W, \overline{W})} \quad \text{ii) } \sum_{t=1}^n \frac{U_{2k(t+c)+2m} U_{2k(t+2c)+2m}}{P_8(W, \overline{W}, c, 2c, 3c)},$$

(4)

$$\text{i) } \sum_{t=1}^n \frac{U_{2k(t+m_1)+2m} U_{2k(t+m_2)+2m} U_{2k(t+m_3)+2m}}{P_{10}(W, \overline{W})} \quad \text{ii) } \sum_{t=1}^n \frac{1}{P_{12}(W, \overline{W})},$$

where $P_{2r}(W, \overline{W}, m_1, m_2, \dots, m_{r-1})$, or briefly $P_{2r}(W, \overline{W})$, is given by

$$P_{2r}(W, \overline{W}) = W_{kt+m} \overline{W}_{kt+m} \prod_{i=1}^{r-1} W_{k(t+m_i)+m} \overline{W}_{k(t+m_i)+m}.$$

In this paper, we present a different approach to compute these sums. Our approach is straight-forward and simple and works in all instances. First we convert these sums in q -notation and use partial fraction decomposition. Using telescoping, we derive closed formulæ for them.

The Binet forms are

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} = A\alpha^{n-1} \frac{(1 - q^n B/A)}{(1 - q)},$$

$$\overline{W}_n = A\alpha^n + B\beta^n = A\alpha^n (1 + q^n B/A)$$

and

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} \frac{(1 - q^n)}{(1 - q)}, \quad V_n = \alpha^n + \beta^n = \alpha^n (1 + q^n),$$

where $\alpha, \beta = (p \pm \sqrt{p^2 + 4})/2$, $q = \beta/\alpha$, $\alpha = iq^{-\frac{1}{2}}$, $A = b - a\beta$ and $B = b - a\alpha$.

We frequently denote the sequences $\{W_n\}$ and $\{\overline{W}_n\}$ by $\{W_n(A, B)\}$ and $\{\overline{W}_n(A, B)\}$, respectively.

2. SIMPLE EVALUATION OF MELHAM'S SUMS

1. We start with the first kind of sums by converting them into q -notation:

$$\begin{aligned} & \sum_{t=1}^n \frac{1}{W_{k(t+m_1)+m} W_{k(t+m_2)+m} \overline{W}_{k(t+m_3)+m} \overline{W}_{k(t+m_4)+m}} \\ &= \frac{(1-q)^2}{A^4 \alpha^{k(m_1+m_2+m_3+m_4)+4m-2}} \\ & \times \sum_{t=1}^n \left(\frac{q^{2kt}}{(1 - q^{k(t+m_1)+m} B/A)(1 - q^{k(t+m_2)+m} B/A)} \right. \\ & \left. \times \frac{1}{(1 + q^{k(t+m_3)+m} B/A)(1 + q^{k(t+m_4)+m} B/A)} \right), \end{aligned}$$

where m_1, m_2, m_3 and m_4 are defined as before.

Without constant factor and writing $z = q^{tk}$ and $q^{km_i+m} = q^{c_i}$ and $B/A = a$ for $1 \leq i \leq 4$, we define

$$S_n := \sum_{t=1}^n \frac{z^2}{(1-azq^{c_1})(1-azq^{c_2})(1+azq^{c_3})(1+azq^{c_4})}.$$

Let

$$T(z) := \frac{z^2}{(1-azq^{c_1})(1-azq^{c_2})(1+azq^{c_3})(1+azq^{c_4})}.$$

The partial fraction decomposition of $T(z)$ is

$$\begin{aligned} & \frac{z^2}{(1-azq^{c_1})(1-azq^{c_2})(1+azq^{c_3})(1+azq^{c_4})} \\ &= -\frac{q^{c_4}}{a^2(q^{c_2}+q^{c_4})(q^{c_1}+q^{c_4})(q^{c_3}-q^{c_4})(1+aq^{c_4}z)} \\ & \quad -\frac{q^{c_2}}{a^2(q^{c_2}+q^{c_3})(q^{c_2}+q^{c_4})(q^{c_1}-q^{c_2})(1-aq^{c_2}z)} \\ & \quad +\frac{q^{c_1}}{a^2(q^{c_1}+q^{c_4})(q^{c_1}+q^{c_3})(q^{c_1}-q^{c_2})(1-aq^{c_1}z)} \\ & \quad +\frac{q^{c_3}}{a^2(q^{c_3}-q^{c_4})(q^{c_2}+q^{c_3})(q^{c_1}+q^{c_3})(1+aq^{c_3}z)}. \end{aligned}$$

The assumption $m_1 < m_2$ and $m_3 < m_4$ with $m_1 + m_2 = m_3 + m_4$ means $c_1 < c_2$ and $c_3 < c_4$ with $c_1 + c_2 = c_3 + c_4$. We write

$$\begin{aligned} & \frac{z^2}{(1-azq^{c_1})(1-azq^{c_2})(1+azq^{c_3})(1+azq^{c_4})} \\ & \quad \times a^2(q^{c_2}+q^{c_4})(q^{c_1}+q^{c_4})(q^{c_3}-q^{c_4})(q^{c_2}+q^{c_3})(q^{c_1}-q^{c_2})(q^{c_1}+q^{c_3}) \\ &= -q^{c_3}(q^{c_2}+q^{c_4})(q^{c_1}+q^{c_4})(q^{c_1}-q^{c_2}) \left[\frac{1}{1+aq^{c_4}z} - \frac{1}{1+aq^{c_3}z} \right] \\ & \quad -q^{c_2}(q^{c_1}+q^{c_4})(q^{c_3}-q^{c_4})(q^{c_1}+q^{c_3}) \left[\frac{1}{1-aq^{c_2}z} - \frac{1}{1-aq^{c_1}z} \right] \end{aligned}$$

and so

$$\begin{aligned} & a^2q^{c_1+c_2}(1+q^{c_4-c_2})(1-q^{c_4-c_3})(1+q^{c_3-c_2})(1-q^{c_2-c_1})(1+q^{c_3-c_1})S_n \\ &= -(1+q^{c_4-c_2})(1-q^{c_2-c_1}) \left[\sum_{t=1}^n \frac{1}{1+aq^{c_3+d}z} - \sum_{t=1}^n \frac{1}{1+aq^{c_3}z} \right] \\ & \quad - (1-q^{c_4-c_3})(1+q^{c_3-c_1}) \left[\sum_{t=1}^n \frac{1}{1-aq^{c_1+c}z} - \sum_{t=1}^n \frac{1}{1-aq^{c_1}z} \right], \end{aligned}$$

which, by the telescoping sum identity

$$\sum_{t=1}^n \left(\frac{1}{1+azq^{b+c}} - \frac{1}{1+azq^c} \right) = \sum_{t=1}^b \left(\frac{1}{1+azq^{n+c}} - \frac{1}{1+azq^c} \right)$$

gives us

$$\begin{aligned} & a^2 q^{c_1+c_2} (1+q^{c_4-c_2})(1-q^{c_4-c_3})(1+q^{c_3-c_2})(1-q^{c_2-c_1})(1+q^{c_3-c_1}) S_n \\ &= -(1+q^{c_4-c_2})(1-q^{c_2-c_1}) \sum_{t=1}^{c_4-c_3} \left(\frac{1}{1+azq^{n+c_3}} - \frac{1}{1+azq^{c_3}} \right) \\ & \quad - (1-q^{c_4-c_3})(1+q^{c_3-c_1}) \sum_{t=1}^{c_2-c_1} \left(\frac{1}{1-azq^{n+c_1}} - \frac{1}{1-azq^{c_1}} \right). \end{aligned}$$

We write it in original form

$$\begin{aligned} & (B/A)^2 q^{k(m_1+m_2)+2m} (1+q^{k(m_4-m_2)})(1-q^{k(m_4-m_3)}) \\ & \times (1+q^{k(m_3-m_2)})(1-q^{k(m_2-m_1)})(1+q^{k(m_3-m_1)}) S_n \\ &= -(1+q^{k(m_4-m_2)})(1-q^{k(m_2-m_1)}) \\ & \times \sum_{t=1}^{k(m_4-m_3)} \left(\frac{1}{1+q^{tk+n+km_3+m} B/A} - \frac{1}{1+q^{tk+km_3+m} B/A} \right) \\ & \quad - (1-q^{k(m_4-m_3)})(1+q^{k(m_3-m_1)}) \\ & \times \sum_{t=1}^{k(m_2-m_1)} \left(\frac{1}{1-q^{tk+n+km_1+m} B/A} - \frac{1}{1-q^{tk+km_1+m} B/A} \right) \end{aligned}$$

or

$$\begin{aligned} S_n &= \frac{A^3 B^{-2} \alpha^{2(2m+km_1+km_2)}}{\Delta^2 V_{k(m_4-m_2)} V_{k(m_3-m_1)} V_{k(m_3-m_2)} U_{k(m_2-m_1)} U_{k(m_4-m_3)}} \\ & \times \left[\Delta V_{k(m_4-m_2)} U_{k(m_2-m_1)} \sum_{t=1}^{k(m_4-m_3)} \left(\frac{\alpha^{tk+n+km_3+m}}{\overline{W}_{t+n+km_3+m}} - \frac{\alpha^{tk+km_3+m}}{\overline{W}_{tk+km_3+m}} \right) \right. \\ & \quad \left. + U_{k(m_4-m_3)} V_{k(m_3-m_1)} \sum_{t=1}^{k(m_2-m_1)} \left(\frac{\alpha^{tk+n+km_1+m}}{W_{tk+n+km_1+m}} - \frac{\alpha^{tk+km_1+m}}{W_{tk+km_1+m}} \right) \right] \end{aligned}$$

and so

$$\sum_{t=1}^n \frac{1}{W_{k(t+m_1)+m} W_{k(t+m_2)+m} \overline{W}_{k(t+m_3)+m} \overline{W}_{k(t+m_4)+m}}$$

$$\begin{aligned}
&= \frac{1}{AB^2 V_{k(m_4-m_2)} V_{k(m_3-m_1)} V_{k(m_3-m_2)} U_{k(m_2-m_1)} U_{k(m_4-m_3)}} \\
&\times \left[\Delta V_{k(m_4-m_2)} U_{k(m_2-m_1)} \sum_{t=1}^{k(m_4-m_3)} \left(\frac{\alpha^{tk+n+km_3+m}}{\overline{W}_{t+n+km_3+m}} - \frac{\alpha^{tk+km_3+m}}{\overline{W}_{tk+km_3+m}} \right) \right. \\
&\quad \left. + U_{k(m_4-m_3)} V_{k(m_3-m_1)} \sum_{t=1}^{k(m_2-m_1)} \left(\frac{\alpha^{tk+n+km_1+m}}{\overline{W}_{tk+n+km_1+m}} - \frac{\alpha^{tk+km_1+m}}{\overline{W}_{tk+km_1+m}} \right) \right],
\end{aligned}$$

which is the desired evaluation of the first type of sums.

For example, we have

$$\sum_{t=1}^n \frac{1}{F_{t+1} F_{t+2} L_{t+1} L_{t+2}} = -\frac{1}{2} \left[\sqrt{5} \left(\frac{\alpha^{n+2}}{L_{n+2}} - \frac{\alpha^2}{L_2} \right) + \left(\frac{\alpha^{n+2}}{F_{n+2}} - \alpha^2 \right) \right]$$

and

$$\sum_{t=1}^n \frac{1}{F_t F_{t+3} L_{t+1} L_{t+2}} = -\frac{1}{6} \left[2\sqrt{5} \left(\frac{\alpha^{n+2}}{L_{n+2}} - \frac{\alpha^2}{3} \right) - \sum_{t=1}^3 \left(\frac{\alpha^{t+n}}{F_{t+n}} - \frac{\alpha^t}{F_t} \right) \right].$$

2. i) The sums take in q -notation the form

$$\begin{aligned}
\sum_{t=1}^n \frac{U_{2kt+2m}}{P_6(W, \overline{W})} &= A^{-6} \alpha^{5-2km_1-2km_2-4m} (1-q)^2 \\
&\times \sum_{t=1}^n \left(\frac{\alpha^{-4kt} (1-q^{2k(t+m_2)+2m})}{(1-q^{2kt+2m}(B/A)^2)} \right. \\
&\quad \left. \frac{1}{(1-q^{2k(t+m_1)+2m}(B/A)^2)(1-q^{2k(t+m_2)+2m}(B/A)^2)} \right).
\end{aligned}$$

Without constant factor and writing $z = q^{2tk}$, $q^{2km_i} = q^{c_i}$, $q^{2m} = c$ and $(B/A)^2 = a$ for $1 \leq i \leq 2$, we consider

$$S_n := \sum_{t=1}^n \frac{z(1-cz)}{(1-acz)(1-aczq^{c_1})(1-aczq^{c_2})}.$$

Let

$$T(z) := \frac{z(1-cz)}{(1-acz)(1-aczq^{c_1})(1-aczq^{c_2})}.$$

The partial fraction decomposition of $T(z)$ is

$$\begin{aligned}
T(z) &= \frac{z(1-cz)}{(1-acz)(1-aczq^{c_1})(1-aczq^{c_2})} \\
&= -\frac{(1-a)}{a^2 c (1-acz)(1-q^{c_1})(1-q^{c_2})}
\end{aligned}$$

$$\begin{aligned}
& - \frac{(1 - aq^{c_2})}{a^2cq^{c_1}(1 - acq^{c_2}z)(1 - q^{c_2})(1 - q^{c_2-c_1})} \\
& + \frac{(1 - aq^{c_1})}{a^2cq^{c_1}(1 - q^{c_1})(1 - acq^{c_1}z)(1 - q^{c_2-c_1})}.
\end{aligned}$$

The assumption $0 < m_1 < m_2$ means $c_1 < c_2$. By telescoping, we write

$$\begin{aligned}
S_n &= \frac{(1-a)}{a^2c(1-q^{c_1})(1-q^{c_2})} \sum_{t=1}^{c_1} \left(\frac{1}{1-acq^n z} - \frac{1}{1-acz} \right) \\
& - \frac{(1-aq^{c_2})}{a^2cq^{c_1}(1-q^{c_2})(1-q^{c_2-c_1})} \sum_{t=1}^{c_2-c_1} \left(\frac{1}{1-aczq^{n+c_1}} - \frac{1}{1-aczq^{c_1}} \right)
\end{aligned}$$

and so we obtain

$$\begin{aligned}
& \sum_{t=1}^n \frac{U_{2kt+2m}}{P_6(W, \bar{W})} = \frac{(A^2 - B^2)}{A^2 B^4 \Delta U_{2km_1} U_{2km_2}} \\
& \times \sum_{t=1}^{2km_1} \left(\frac{\alpha^{2tk+n+2m+3}}{W_{2tk+n+2m}(A^2, B^2)} - \frac{\alpha^{2tk+2m+3}}{W_{2tk+2m}(A^2, B^2)} \right) \\
& - \frac{W_{2km_2}(A^2, B^2)}{A^2 B^4 U_{2km_2} U_{2k(m_2-m_1)}} \\
& \times \sum_{t=1}^{2k(m_2-m_1)} \left(\frac{\alpha^{2tk+n+2km_1+2m+3}}{W_{2tk+n+2km_1+2m}(A^2, B^2)} - \frac{\alpha^{2tk+2km_1+2m+3}}{W_{2tk+2km_1+2m}(A^2, B^2)} \right),
\end{aligned}$$

where Δ is defined as before.

ii) Now we write the second class of sums in q -notation as

$$\begin{aligned}
& \sum_{t=1}^n \frac{U_{2k(t+m_1)+2m}}{P_6(W, \bar{W})} = A^{-6} \alpha^{5-2km_2-4m} (1-q)^2 \\
& \times \sum_{t=1}^n \left(\frac{q^{2kt}(1-q^{2k(t+m_1)+2m})}{(1-q^{2kt+2m}(B/A)^2)(1-q^{2k(t+m_1)+2m}(B/A)^2)} \right) \\
& \times \frac{1}{(1-q^{2k(t+m_2)+2m}(B/A)^2)}.
\end{aligned}$$

Without constant factor and writing $z = q^{2tk}$, $q^{2km_i} = q^{c_i}$ and $(B/A)^2 = a$ for $1 \leq i \leq 2$, we define the sum

$$S_n := \sum_{t=1}^n \frac{z(1-czq^{c_1})}{(1-acz)(1-aczq^{c_1})(1-aczq^{c_2})}.$$

Let

$$T(z) := \frac{z(1 - czq^{c_1})}{(1 - acz)(1 - aczq^{c_1})(1 - aczq^{c_2})}.$$

Similarly as before, by using the partial fraction decomposition of $T(z)$ and telescoping, since $c_2 > c_1$, we obtain

$$S_n = \frac{1}{a^2c(1 - q^{c_2 - c_1})} \left[\frac{1 - a}{1 - q^{c_1}} \sum_{t=1}^{c_1} \left(\frac{1}{1 - aczq^n} - \frac{1}{1 - acz} \right) - \frac{1 - aq^{c_2 - c_1}}{1 - q^{c_2}} \sum_{t=1}^{c_2} \left(\frac{1}{1 - aczq^n} - \frac{1}{1 - acz} \right) \right].$$

Thus we get the result

$$\begin{aligned} \sum_{t=1}^n \frac{U_{2k(t+m_1)+2m}}{P_6(W, \overline{W})} &= \frac{\alpha^{2m+3}}{B^4 A^2 U_{2k(m_2-m_1)}} \\ &\times \left[\frac{(A^2 - B^2)}{\Delta U_{2km_1}} \sum_{t=1}^{2km_1} \left(\frac{\alpha^{2kt+n}}{W_{2tk+2m+n}(A^2, B^2)} - \frac{\alpha^{2kt}}{W_{2tk+2m}(A^2, B^2)} \right) \right. \\ &\left. - \frac{W_{2k(m_2-m_1)}(A^2, B^2)}{U_{2km_2}} \sum_{t=1}^{2km_2} \left(\frac{\alpha^{2kt+n}}{W_{2tk+2m+n}(A^2, B^2)} - \frac{\alpha^{2kt}}{W_{2tk+2m}(A^2, B^2)} \right) \right], \end{aligned}$$

where Δ is defined as before.

For the next sums, we only present the key steps, as the procedure is always the same.

iii) Consider the sums and its q -form

$$\begin{aligned} \sum_{t=1}^n \frac{U_{2k(t+m_2)+2m}}{P_6(W, \overline{W})} &= A^{-6} \alpha^{5-2km_1-4m} (1 - q)^2 \\ &\times \sum_{t=1}^n \left(\frac{q^{2kt} (1 - q^{2k(t+m_2)+2m})}{(1 - q^{2kt+2m}(B/A)^2)(1 - q^{2k(t+m_1)+2m}(B/A)^2)} \right. \\ &\left. \times \frac{1}{(1 - q^{2k(t+m_2)+2m}(B/A)^2)} \right). \end{aligned}$$

Without constant factor and rewriting the parameters, we consider the sums

$$S_n := \sum_{t=1}^n \frac{z(1 - czq^{c_2})}{(1 - acz)(1 - aczq^{c_1})(1 - aczq^{c_2})}.$$

Here by the partial fraction decomposition

$$\frac{z(1-czq^{c_2})}{(1-acz)(1-aczq^{c_1})(1-aczq^{c_2})} = \frac{1}{a^2c} \left(\frac{(a-q^{c_2})}{(1-acz)(1-q^{c_1})(1-q^{c_2})} - \frac{q^{c_2-c_1}(1-a)}{(1-acq^{c_2}z)(1-q^{c_2})(1-q^{c_2-c_1})} - \frac{(a-q^{c_2-c_1})}{(1-q^{c_1})(1-acq^{c_1}z)(1-q^{c_2-c_1})} \right),$$

and using telescoping, since $c_2 > c_1$, we write

$$S_n = -\frac{1}{a^2c(1-q^{c_2-c_1})} \left[\frac{a(1-q^{c_2-c_1}a^{-1})}{1-q^{c_1}} \sum_{t=1}^{c_1} \left(\frac{1}{1-aczq^n} - \frac{1}{1-acz} \right) + \frac{q^{c_2-c_1}(1-a)}{1-q^{c_2}} \sum_{t=1}^{c_2} \left(\frac{1}{1-aczq^n} - \frac{1}{1-acz} \right) \right]$$

and obtain the result

$$\sum_{t=1}^n \frac{U_{2k(t+m_2)+2m}}{P_6(W, \overline{W})} = -\frac{1}{B^4 A^2 U_{2k(m_2-m_1)}} \times \left[\frac{W_{2k(m_2-m_1)}(B^2, A^2)}{U_{2km_1}} \sum_{t=1}^{2km_1} \left(\frac{\alpha^{2kt+2m+n+5}}{W_{2kt+2m+n}(A^2, B^2)} - \frac{\alpha^{2kt+2m+5}}{W_{2kt+2m}(A^2, B^2)} \right) + \frac{(A^2 - B^2)}{(\alpha - \beta) U_{2km_2}} \sum_{t=1}^{2km_2} \left(\frac{\alpha^{2kt+2m+n+3}}{W_{2kt+2m+n}(A^2, B^2)} - \frac{\alpha^{2kt+2m+3}}{W_{2kt+2m}(A^2, B^2)} \right) \right].$$

3. i) For $0 < m_1 < m_2 < m_3$, we consider the following sums in its q -form

$$\sum_{t=1}^n \frac{1}{P_8(W, \overline{W})} = A^{-8} \alpha^{8-2km_1-2km_2-2km_3-8m} (1-q)^4 \times \sum_{t=1}^n \frac{z^2}{(1-acz)(1-aczq^{c_1})(1-aczq^{c_2})(1-aczq^{c_3})}.$$

Without constant factor, we only consider the sum

$$S_n := \sum_{t=1}^n \frac{z^2}{(1-acz)(1-aczq^{c_1})(1-aczq^{c_2})(1-aczq^{c_3})}.$$

By partial fraction decomposition of the summand and using telescoping, we write

$$\begin{aligned} & a^2 c^2 q^{c_1+c_2} (1-q^{c_1})(1-q^{c_2})(1-q^{c_3})(1-q^{c_3-c_2})(1-q^{c_2-c_1})(1-q^{c_3-c_1}) S_n \\ &= q^{c_3} (1-q^{c_1+c_2-c_3})(1-q^{c_3})(1-q^{c_2-c_1}) \sum_{t=1}^{c_2} \left(\frac{1}{1-acq^n z} - \frac{1}{1-acz} \right) \\ & - q^{c_3} (1-q^{c_2-c_1})(1-q^{c_1})(1-q^{c_2}) \sum_{t=1}^{c_3} \left(\frac{1}{1-acq^n z} - \frac{1}{1-acz} \right) \end{aligned}$$

$$+ q^{c_2}(1-q^{c_3-c_2})(1-q^{c_2})(1-q^{c_3}) \sum_{t=1}^{c_2-c_1} \left(\frac{1}{1-acq^{n+c_1}z} - \frac{1}{1-acq^{c_1}z} \right).$$

Consequently after converting it back into the original form,

$$\begin{aligned} & \sum_{t=1}^n \frac{1}{P_8(W, \overline{W})} \\ &= \frac{\alpha^{2m+4}}{A^2 B^4 U_{2km_1} U_{2km_2} U_{2km_3} U_{2k(m_3-m_2)} U_{2k(m_2-m_1)} U_{2k(m_3-m_1)}} \\ & \times \left[U_{2k(m_1+m_2-m_3)} U_{2km_3} U_{2k(m_2-m_1)} \right. \\ & \times \sum_{t=1}^{2km_2} \left(\frac{\alpha^{n+2tk}}{W_{n+2tk+2m}(A^2, B^2)} - \frac{\alpha^{2tk}}{W_{2tk+2m}(A^2, B^2)} \right) \\ & - U_{2k(m_2-m_1)} U_{2km_1} U_{2km_2} \sum_{t=1}^{2km_3} \left(\frac{\alpha^{n+2tk}}{W_{n+2tk+2m}(A^2, B^2)} - \frac{\alpha^{2tk}}{W_{2tk+2m}(A^2, B^2)} \right) \\ & + U_{2k(m_3-m_2)} U_{2km_2} U_{2km_3} \\ & \times \left. \sum_{t=1}^{2k(m_2-m_1)} \left(\frac{\alpha^{n+2tk+2km_1}}{W_{n+2tk+2km_1+2m}(A^2, B^2)} - \frac{\alpha^{2tk+2km_1}}{W_{2tk+2km_1+2m}(A^2, B^2)} \right) \right]. \end{aligned}$$

3. ii) We now consider the sums

$$\sum_{t=1}^n \frac{U_{2k(t+g)+2m} U_{2k(t+2g)+2m}}{P_8(W, \overline{W}, g, 2g, 3g)}.$$

Converting it into q -form and rewriting the parameters, that is, $z = q^{2tk}$, $q^{2m} = c$, $q^{2kg} = q^d$ and $(B/A)^2 = a$, we get

$$A^{-8} \alpha^{6-6gk-4m} (1-q)^2 \sum_{t=1}^n \frac{z(1-czq^d)(1-czq^{2d})}{(1-acz)(1-aczq^d)(1-aczq^{2d})(1-aczq^{3d})}.$$

Without constant factor, we consider

$$S_n := \sum_{t=1}^n \frac{z(1-czq^d)(1-czq^{2d})}{(1-acz)(1-aczq^d)(1-aczq^{2d})(1-aczq^{3d})}.$$

By partial fraction decomposition of the summand of S_n and using telescoping, we obtain

$$S_n = \frac{1}{a^3 c (1-q^d)^2 (1-q^{2d})}$$

$$\begin{aligned}
& \times \left[(a - q^d)(1 - a)(1 - q^{3d}) \sum_{t=1}^d \left(\frac{1}{1 - aczq^n} - \frac{1}{1 - acz} \right) \right. \\
& - (1 - aq^d)(1 - aq^{2d})(1 - q^d) \sum_{t=1}^{3d} \left(\frac{1}{1 - aczq^n} - \frac{1}{1 - acz} \right) \\
& \left. + (1 - aq^d)(1 - a)(1 - q^{3d}) \sum_{t=1}^{2d} \left(\frac{1}{1 - aczq^n} - \frac{1}{1 - acz} \right) \right].
\end{aligned}$$

Thus we get

$$\begin{aligned}
& \sum_{t=1}^n \frac{U_{2k(t+g)+2m} U_{2k(t+2g)+2m}}{P_8(W, \overline{W}, g, 2g, 3g)} \\
& = \frac{1}{B^6 A^4 U_{2kg}^2 U_{4kg}} \left[W_{2kg}(B^2, A^2)(A^2 - B^2) U_{6kg} \right. \\
& \times \sum_{t=1}^d \left(\frac{\alpha^{2m+n-6gk+2kt+4}}{W_{n+2kt+2m}(A^2, B^2)} - \frac{\alpha^{2m-6gk+2kt+4}}{W_{2kt+2m}(A^2, B^2)} \right) \\
& - \Delta W_{2kg}(A^2, B^2) W_{4kg}(A^2, B^2) U_{2kg} \\
& \times \sum_{t=1}^{3d} \left(\frac{\alpha^{2m+n-6gk+2kt+4}}{W_{n+2kt+2m}(A^2, B^2)} - \frac{\alpha^{2m-6gk+2kt+4}}{W_{2kt+2m}(A^2, B^2)} \right) \\
& + W_{2kg}(A^2, B^2)(A^2 - B^2) U_{6kg} \\
& \left. \times \sum_{t=1}^{2d} \left(\frac{\alpha^{2m+n-6gk+2kt+4}}{W_{n+2kt+2m}(A^2, B^2)} - \frac{\alpha^{2m-6gk+2kt+4}}{W_{2kt+2m}(A^2, B^2)} \right) \right],
\end{aligned}$$

where Δ is defined as before.

4. i) For $0 < m_1 < m_2 < m_3 < m_4$, we consider the sums

$$\sum_{t=1}^n \frac{U_{2k(t+m_1)+2m} U_{2k(t+m_2)+2m} U_{2k(t+m_3)+2m}}{P_{10}(W, \overline{W})}.$$

If we convert it into q -notation and write $z = q^{2tk}$, $q^{2m} = c$, $q^{2km_i} = q^{c_i}$ and $(B/A)^2 = a$ for $1 \leq i \leq 4$, then the sums above equals

$$\begin{aligned}
& A^{-10} \alpha^{7-2km_4-4m} (1 - q)^2 \\
& \times \sum_{t=1}^n \frac{z(1 - czq^{c_1})(1 - czq^{c_2})(1 - czq^{c_3})}{(1 - acz)(1 - aczq^{c_1})(1 - aczq^{c_2})(1 - aczq^{c_3})(1 - aczq^{c_4})}.
\end{aligned}$$

Without constant factor, we consider the sums

$$S_n := \sum_{t=1}^n \frac{z(1-czq^{c_1})(1-czq^{c_2})(1-czq^{c_3})}{(1-acz)(1-aczq^{c_1})(1-aczq^{c_2})(1-aczq^{c_3})(1-aczq^{c_4})}.$$

The partial fraction decomposition of the summand is

$$\begin{aligned} & \frac{z(1-czq^{c_1})(1-czq^{c_2})(1-czq^{c_3})a^4c}{(1-acz)(1-aczq^{c_1})(1-aczq^{c_2})(1-aczq^{c_3})(1-aczq^{c_4})} \\ &= B\left(\frac{1}{1-acq^{c_1}z} - \frac{1}{1-acz}\right) - C\left(\frac{1}{1-acq^{c_2}z} - \frac{1}{1-acz}\right) \\ &+ D\left(\frac{1}{1-acq^{c_3}z} - \frac{1}{1-acz}\right) - E\left(\frac{1}{1-acq^{c_4}z} - \frac{1}{1-acz}\right), \end{aligned}$$

where

$$\begin{aligned} A &= \frac{a^3(1-a^{-1}q^{c_1})(1-a^{-1}q^{c_2})(1-a^{-1}q^{c_3})}{(1-q^{c_1})(1-q^{c_2})(1-q^{c_3})(1-q^{c_4})}, \\ B &= \frac{a^2(1-a^{-1}q^{c_2-c_1})(1-a^{-1}q^{c_3-c_1})(1-a)}{(1-q^{c_1})(1-q^{c_2-c_1})(1-q^{c_3-c_1})(1-q^{c_4-c_1})}, \\ C &= -\frac{a(1-aq^{c_2-c_1})(1-a^{-1}q^{c_3-c_2})(1-a)}{(1-q^{c_2})(1-q^{c_2-c_1})(1-q^{c_3-c_2})(1-q^{c_4-c_2})}, \\ D &= \frac{(1-aq^{c_3-c_1})(1-aq^{c_3-c_2})(1-a)}{(1-q^{c_3})(1-q^{c_3-c_1})(1-q^{c_3-c_2})(1-q^{c_4-c_3})}, \\ E &= \frac{(1-aq^{c_4-c_1})(1-aq^{c_4-c_2})(1-aq^{c_4-c_3})}{(1-q^{c_4})(1-q^{c_4-c_1})(1-q^{c_4-c_2})(1-q^{c_4-c_3})}; \end{aligned}$$

note that $A = C + E - B - D$.

Then by telescoping and converting the sums into the original notion and taking care again about the omitted factor, we obtain

$$\begin{aligned} & \sum_{t=1}^n \frac{U_{2k(t+m_1)+2m} U_{2k(t+m_2)+2m} U_{2k(t+m_3)+2m}}{P_{10}(W, \bar{W})} \\ &= \frac{(A^2 - B^2)\alpha^{2m+5}}{\Delta A^6 B^8} \left(\frac{W_{2k(m_2-m_1)}(B^2, A^2) W_{2k(m_3-m_1)}(B^2, A^2)}{U_{2km_1} U_{2k(m_2-m_1)} U_{2k(m_3-m_1)} U_{2k(m_4-m_1)}} T_1 \right. \\ &+ \frac{W_{2k(m_2-m_1)}(A^2, B^2) W_{2k(m_3-m_2)}(B^2, A^2)}{U_{2km_2} U_{2k(m_2-m_1)} U_{2k(m_3-m_2)} U_{2k(m_4-m_2)}} T_2 \\ &+ \frac{W_{2k(m_3-m_1)}(A^2, B^2) W_{2k(m_3-m_2)}(A^2, B^2)}{U_{2km_3} U_{2k(m_3-m_1)} U_{2k(m_3-m_2)} U_{2k(m_4-m_3)}} T_3 \\ &\left. - \frac{\Delta W_{2k(m_4-m_2)}(A^2, B^2) W_{2k(m_4-m_1)}(A^2, B^2) W_{2k(m_4-m_3)}(A^2, B^2)}{(A^2 - B^2) U_{2km_4} U_{2k(m_4-m_1)} U_{2k(m_4-m_2)} U_{2k(m_4-m_3)}} T_4 \right), \end{aligned}$$

with $\alpha - \beta = \Delta$ and the sums

$$T_i = \sum_{t=1}^{2km_i} \left(\frac{\alpha^{n+2tk}}{W_{n+2tk+2m}(A^2, B^2)} - \frac{\alpha^{2tk}}{W_{2tk+2m}(A^2, B^2)} \right).$$

4. ii) For $0 < m_1 < m_2 < m_3 < m_4 < m_5$, we consider the sums

$$\sum_{t=1}^n \frac{1}{P_{12}(W, \overline{W})}.$$

If we convert it into q -notation and write $z = q^{2tk}$, $q^{2m} = c$, $q^{2km_i} = q^{c_i}$ and $(B/A)^2 = a$ for $1 \leq i \leq 5$, then it equals

$$\frac{(1-q)^6}{A^{12} \alpha^{2(6m+km_1+km_2+km_3+km_4+km_5-6)}} \times \sum_{t=1}^n \frac{z^3}{(1-acz)(1-aczq^{c_1})(1-aczq^{c_2})(1-aczq^{c_3})(1-aczq^{c_4})(1-aczq^{c_5})}.$$

Without constant factor, we consider the sums

$$S_n := \sum_{t=1}^n \left(\frac{z^3}{(1-acz)(1-aczq^{c_1})(1-aczq^{c_2})} \times \frac{1}{(1-aczq^{c_3})(1-aczq^{c_4})(1-aczq^{c_5})} \right).$$

By partial fraction decomposition of the summand and using telescoping, we write

$$\begin{aligned} a^3 c^3 S_n = & -F \sum_{t=1}^n \left(\frac{1}{1-acq^{c_5}z} - \frac{1}{1-acz} \right) \\ & + E \sum_{t=1}^n \left(\frac{1}{1-acq^{c_4}z} - \frac{1}{1-acz} \right) - D \sum_{t=1}^n \left(\frac{1}{1-acq^{c_3}z} - \frac{1}{1-acz} \right) \\ & + C \sum_{t=1}^n \left(\frac{1}{1-acq^{c_2}z} - \frac{1}{1-acz} \right) - B \sum_{t=1}^n \left(\frac{1}{1-acq^{c_1}z} - \frac{1}{1-acz} \right), \end{aligned}$$

where

$$\begin{aligned} A &= \frac{1}{(1-q^{c_1})(1-q^{c_2})(1-q^{c_3})(1-q^{c_4})(1-q^{c_5})}, \\ B &= \frac{q^{2c_1}}{(1-q^{c_1})(q^{c_1}-q^{c_2})(q^{c_1}-q^{c_3})(q^{c_1}-q^{c_4})(q^{c_1}-q^{c_5})}, \end{aligned}$$

$$\begin{aligned}
C &= \frac{q^{2c_2}}{(1-q^{c_2})(q^{c_1}-q^{c_2})(q^{c_2}-q^{c_3})(q^{c_2}-q^{c_4})(q^{c_2}-q^{c_5})}, \\
D &= \frac{q^{2c_3}}{(1-q^{c_3})(q^{c_1}-q^{c_3})(q^{c_2}-q^{c_3})(q^{c_3}-q^{c_4})(q^{c_3}-q^{c_5})}, \\
E &= \frac{q^{2c_4}}{(1-q^{c_4})(q^{c_1}-q^{c_4})(q^{c_2}-q^{c_4})(q^{c_3}-q^{c_4})(q^{c_4}-q^{c_5})}, \\
F &= \frac{q^{2c_5}}{(1-q^{c_5})(q^{c_1}-q^{c_5})(q^{c_2}-q^{c_5})(q^{c_3}-q^{c_5})(q^{c_4}-q^{c_5})};
\end{aligned}$$

note that $A = F - E + D - C + B$.

Consequently, we obtain

$$\begin{aligned}
&\sum_{t=1}^n \frac{1}{P_{12}(W, \bar{W})} = \frac{\alpha^6 \Delta}{A^6 B^6} \\
&\times \left(-\frac{1}{U_{2km_5} U_{2k(m_5-m_1)} U_{2k(m_5-m_2)} U_{2k(m_5-m_3)} U_{2k(m_5-m_4)}} T_5 \right. \\
&+ \frac{1}{U_{2km_3} U_{2k(m_4-m_1)} U_{2k(m_4-m_2)} U_{2k(m_4-m_3)} U_{2k(m_5-m_4)}} T_4 \\
&- \frac{1}{U_{2km_3} U_{2k(m_3-m_1)} U_{2k(m_3-m_2)} U_{2k(m_4-m_3)} U_{2k(m_5-m_3)}} T_3 \\
&+ \frac{1}{U_{2km_2} U_{2k(m_2-m_1)} U_{2k(m_3-m_2)} U_{2k(m_4-m_2)} U_{2k(m_5-m_2)}} T_2 \\
&\left. - \frac{1}{U_{2km_1} U_{2k(m_2-m_1)} U_{2k(m_3-m_1)} U_{2k(m_4-m_1)} U_{2k(m_5-m_1)}} T_1 \right),
\end{aligned}$$

where the sum T_i is defined as before.

Of course, we could invent many more examples, but we think that the message is clear now.

It should be noted that our elementary method can always be used to simplify sums of the type considered here; even if they do not telescope, they lead to simpler answers.

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MULTIPLICATIVE GENERALIZED DERIVATIONS ON LIE IDEALS IN SEMIPRIME RINGS II

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Abstract. Let R be a semiprime ring and L is a Lie ideal of R such that $L \not\subseteq Z(R)$. A map $F : R \rightarrow R$ is called a multiplicative generalized derivation if there exists a map $d : R \rightarrow R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. In the present paper, we shall prove that d is a commuting map on L if any one of the following holds: i) $F(uv) = \pm uv$, ii) $F(uv) = \pm vu$, iii) $F(u)F(v) = \pm uv$, iv) $F(u)F(v) = \pm vu$, v) $F(u)F(v) \pm uv \in Z$, vi) $F(u)F(v) \pm vu \in Z$, vii) $[F(u), v] \pm [u, G(v)] = 0$, for all $u, v \in L$.

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1. INTRODUCTION

Let R will be an associative ring with center Z . For any $x, y \in R$, as usual $[x, y] = xy - yx$ and $xoy = xy + yx$ will denote the well-known Lie and Jordan products respectively. Recall that a ring R is prime if for $x, y \in R$, $xRy = 0$ implies either $x = 0$ or $y = 0$ and R is semiprime if for $x \in R$, $xRx = 0$ implies $x = 0$. An additive subgroup L of R is said to be a Lie ideal of R if $[u, r] \in L$, for all $u \in L, r \in R$. An additive mapping $d : R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F : R \rightarrow R$ is called a generalized derivation if there exists a derivation $d : R \rightarrow R$ such that

$$F(xy) = F(x)y + xd(y), \text{ for all } x, y \in R.$$

This definition was given by M. Brešar in [4].

Following [5], a multiplicative derivation is a map $d : R \rightarrow R$ which satisfies $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. The concept of multiplicative derivation was introduced by Daif. Of course these maps are not additive. Motivated by this work, the definition of a multiplicative generalized derivation was extended by Daif and Tamman El-Saiyad in [6] as follows:

$F : R \rightarrow R$ is called a multiplicative generalized derivation if there exists a derivation $d : R \rightarrow R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. Dhara and

Ali gave a slight generalization of this definition taking d is any map (not necessarily an additive map or a derivation) in [7]. Hence, one may observe that the concept of multiplicative generalized derivations includes the concept of derivations, multiplicative derivation and the left multipliers (i.e., $F(xy) = F(x)y$ for all $x, y \in R$). So, it should be interesting to extend some results concerning these notions to multiplicative generalized derivations. Every generalized derivation is a multiplicative generalized derivation. But the converse is not true in general.

During the past few years several authors have proved commutativity theorems for prime rings or semiprime rings admitting automorphisms or derivations on appropriate subsets of R . In [1], Ashraf and Rehman showed that R is prime ring with a nonzero ideal L of R and d is a derivation of R such that $d(xy) \pm xy \in Z$, for all $x, y \in L$, then R is commutative. This theorem considered for generalized derivations in [10]. Being inspired by these results, recently Dhara and Ali discuss the commutativity theorems for prime rings or semiprime rings involving multiplicative generalized derivations in [7].

In the present paper, we shall extend the above result for Lie ideals of semiprime rings with multiplicative generalized derivation of R . Our obtained results are more useful than [7]. Throughout this paper, R will be a 2-torsion free semiprime ring admitting two multiplicative generalized derivations F and G , L a square-closed Lie ideal of R such that $F(2x) = 2F(x)$, $G(2x) = 2G(x)$ for all $x \in R$.

2. RESULTS

We will make some extensive use of the basic commutator identities:

$$\begin{aligned} [x, yz] &= y[x, z] + [x, y]z \\ [xy, z] &= [x, z]y + x[y, z] \\ xo(yz) &= (xoy)z - y[x, z] = y(xoz) + [x, y]z \\ (xy)oz &= x(yoz) - [x, z]y = (xoz)y + x[y, z]. \end{aligned}$$

Moreover, we shall require the following lemmas.

Lemma 1 ([3, Lemma 4]). *If $L \not\subseteq Z(R)$ is a Lie ideal of a 2-torsion free prime ring R and $a, b \in R$ such that $aLb = \{0\}$, then $a = 0$ or $b = 0$.*

Lemma 2 ([3, Lemma 2]). *Let R be a prime ring with characteristic not two. If L a noncentral Lie ideal of R , then $C_R(L) = Z$.*

Lemma 3 ([3, Lemma 5]). *Let R be a prime ring with characteristic not two and L a nonzero Lie ideal of R . If d is a nonzero derivation of R such that $d(L) = 0$, then $L \subseteq Z$.*

Lemma 4 ([2, Theorem 7]). *Let R be a prime ring with characteristic not two and L a nonzero Lie ideal of R . If d is a nonzero derivation of R such that $[d(u), u] \in Z$ for all $u \in L$, then $L \subseteq Z$.*

Lemma 5 ([9, Lemma 2]). *Let R be a 2-torsion free semiprime ring, L is a Lie ideal of R such that $L \not\subseteq Z(R)$ and $a \in L$. If $aLa = 0$, then $a^2 = 0$ and there exists a nonzero ideal $K = R[L, L]R$ of R generated by $[L, L]$ such that $[K, R] \subseteq L$ and $Ka = aK = 0$.*

Corollary 1 ([8, Corollary 2.1]). *Let R be a 2-torsion free semiprime ring, L a Lie ideal of R such that $L \not\subseteq Z(R)$ and $a, b \in L$.*

- (i) *If $aLa = 0$, then $a = 0$.*
- (ii) *If $aL = 0$ (or $La = 0$), then $a = 0$*
- (ii) *If L is square-closed and $aLb = 0$, then $ab = 0$ and $ba = 0$.*

Theorem 1. *Let R be a 2-torsion free semiprime ring and L a square-closed Lie ideal of R such that $L \not\subseteq Z(R)$. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero map d such that $d(x) \in L$, for all $x \in L$. If $F(uv) = \pm uv$, for all $u, v \in L$, then $d(L) = (0)$ and $F(uv) = F(u)v$, for all $u, v \in L$.*

Proof. By the hypothesis, we have

$$F(uv) = \pm uv \text{ for all } u, v \in L. \quad (2.1)$$

Replacing v by vw , $w \in L$ in (2.1), we get

$$F(uvw) = \pm uvw \text{ for all } u, v, w \in L.$$

This implies that

$$F(uv)w + uvd(w) = \pm uvw.$$

Using equation (2.1), we obtain

$$uvd(w) = 0, \text{ for all } u, v, w \in L.$$

By Corollary 1, we have

$$ud(w) = 0, \text{ for all } u, w \in L.$$

Again using Corollary 1, the last expression forces that $d(L) = (0)$. Hence we obtain that $F(uv) = F(u)v + ud(v) = F(u)v$, for all $u, v \in L$. $\square$

Corollary 2. *Let R be a 2-torsion free prime ring and L a square-closed Lie ideal of R . Suppose that R admits a multiplicative generalized derivation F associated with a nonzero derivation d . If $F(uv) = \pm uv$, for all $u, v \in L$, then $L \subseteq Z$.*

Proof. By the same techniques in the proof of Theorem 1, we obtain that

$$uvd(w) = 0, \text{ for all } u, v, w \in L.$$

By Lemma 1, we have $u = 0$ or $d(w) = 0$ for all $u, w \in L$. In the first case, we have $u = 0$. Hence, $d(u) = 0$. As a result of both two situation, we get $d(u) = 0$, for all $u \in L$. By Lemma 3, we conclude that $L \subseteq Z$. $\square$

Theorem 2. *Let R be a 2-torsion free semiprime ring and L a square-closed Lie ideal of R such that $L \not\subseteq Z(R)$. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero map d such that $d(x) \in L$, for all $x \in L$. If $F(uv) = \pm vu$, for all $u, v \in L$, then $d(L) = (0)$ and $F(uv) = F(u)v$, for all $u, v \in L$.*

Proof. Suppose that

$$F(uv) - vu = 0, \text{ for all } u, v \in L. \quad (2.2)$$

Replacing v by vw , $w \in L$ in the above equation, we get

$$0 = F(uvw) - vwu = F(uv)w + uvd(w) - vwu = (F(uv) - vu)w + v[u, w] + uvd(w).$$

Using (2.2), we have

$$v[u, w] + uvd(w) = 0 \quad (2.3)$$

Writing w by u in the last equation, we get

$$uvd(u) = 0, \text{ for all } u, v \in L. \quad (2.4)$$

By Corollary 1, we get

$$ud(u) = 0, \text{ for all } u \in L.$$

Replacing w by v in (2.3) and using the last equation, we obtain

$$v[u, v] = 0 \text{ for all } u, v \in L. \quad (2.5)$$

Writing u by wu , we have

$$0 = v[wu, v] = vw[u, v] + v[w, v]u = vw[u, v], \text{ for all } u, v, w \in L.$$

This implies that

$$vw[u, v] = 0 \text{ for all } u, v, w \in L. \quad (2.6)$$

Replacing w by uw in the above equation, we get

$$vuw[u, v] = 0 \text{ for all } u, v, w \in L.$$

Left multiplying by u this (2.6), we obtain that

$$uvw[u, v] = 0 \text{ for all } u, v, w \in L.$$

Commuting the two last equation, we have

$$[u, v]w[u, v] = 0 \text{ for all } u, v, w \in L.$$

By Corollary 1, we have

$$[u, v] = 0, \text{ for all } u, v \in L.$$

Returning the equation (2.3) and using the last equation, we obtain

$$uvd(w) = 0, \text{ for all } u, v, w \in L.$$

Again using Corollary 1, we conclude that $d(L) = (0)$, and so $F(uv) = F(u)v$.

In a similar manner, we can prove that the same conclusion holds for $F(uv) + vu = 0$ for all $u, v \in L$. $\square$

Corollary 3. *Let R be a 2-torsion free prime ring and L a square-closed Lie ideal of R . Suppose that R admits a multiplicative generalized derivation F associated with a nonzero derivation d . If $F(uv) = \pm vu$, for all $u, v \in L$, then $L \subseteq Z$.*

Proof. Applying the same techniques in the proof of Theorem 2, we get

$$uvd(u) = 0, \text{ for all } u, v \in L.$$

By Lemma 1, we have $u = 0$ or $d(u) = 0$, and so, $d(u) = 0$, for all $u \in L$. By Lemma 3, we conclude that $L \subseteq Z$. $\square$

Theorem 3. *Let R be a 2-torsion free semiprime ring and L a square-closed Lie ideal of R such that $L \not\subseteq Z(R)$. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero map d such that $d(x) \in L$, for all $x \in L$. If $F(u)F(v) = \pm uv$, for all $u, v \in L$, then $d(L) = (0)$ and $F(uv) = F(u)v$, for all $u, v \in L$.*

Proof. By the hypothesis

$$F(u)F(v) = \pm uv \text{ for all } u, v \in L. \quad (2.7)$$

Replacing v by vw in (2.7) and using the hypothesis, we find that

$$0 = F(u)F(vw) \mp uvw = F(u)(F(v)w + vd(w)) \mp uvw = (F(u)F(v) \mp uv)w + F(u)vd(w).$$

Using (2.7), we have

$$F(u)vd(w) = 0, \text{ for all } u, v, w \in L. \quad (2.8)$$

Replacing u by tu , $t \in L$, we have

$$0 = F(tu)vd(w) = (F(t)u + td(u))vd(w), \text{ for all } u, v, w, t \in L.$$

Application (2.8), we have

$$td(u)vd(w) = 0, \text{ for all } u, v, w, t \in L.$$

In particular, we get $vd(u)wvd(u) = 0$, for all $u, v \in L$. This implies that $vd(u) = 0$, for all $u, v \in L$ by Corollary 1, and so $d(u) = 0$, for all $u \in L$. Hence we obtain $F(uv) = F(u)v$. $\square$

Corollary 4. *Let R be a 2-torsion free prime ring and L a square-closed Lie ideal of R . Suppose that R admits a multiplicative generalized derivation F associated with a nonzero derivation d . If $F(u)F(v) = \pm uv$, for all $u, v \in L$, then $L \subseteq Z$.*

Proof. Using the same methods in the proof of Theorem 3, we have

$$vd(u) = 0, \text{ for all } u, v \in L.$$

By Lemma 1, we have $d(u) = 0$, for all $u \in L$, and so, $L \subseteq Z$, by Lemma 3. $\square$

Theorem 4. *Let R be a 2-torsion free semiprime ring and L a square-closed Lie ideal of R such that $L \not\subseteq Z(R)$. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero map d such that $d(x) \in L$, for all $x \in L$. If $F(u)F(v) = \pm vu$, for all $u, v \in L$, then $d(L) = (0)$ and $F(uv) = F(u)v$, for all $u, v \in L$.*

Proof. Assume that

$$F(u)F(v) - vu = 0 \text{ for all } u, v \in L. \quad (2.9)$$

Replacing v with vu in (2.9), we obtain

$$0 = F(u)F(vu) - vu^2 = F(u)(F(v)u + vd(u)) - vu^2 = (F(u)F(v) - vu)u + F(u)vd(u).$$

Using (2.9), we get

$$F(u)vd(u) = 0, \text{ for all } u, v \in L.$$

Left multiplying by $F(w)$ for $w \in L$ this equation and using this, we obtain

$$F(w)F(u)vd(u) = 0, \text{ for all } u, v, w \in L.$$

By the hypothesis, we can write the last equation

$$u w v d(u) = 0, \text{ for all } u, v, w \in L.$$

Replacing w by $d(u)$, we have

$$ud(u)vd(u) = 0, \text{ for all } u, v, w \in L.$$

Taking v by vu , we get

$$ud(u)vud(u) = 0, \text{ for all } u, v \in L.$$

Hence we arrive at $ud(u) = 0$, for all $u \in L$, by Corollary 1.

Now, by the hypothesis, we get

$$\begin{aligned} 0 &= F(u)F(vw) - (vw)u = F(u)(F(v)w + vd(w)) - vw u \\ &= (F(u)F(v) - vu)w + v[u, w] + F(u)vd(w), \end{aligned}$$

and so

$$v[u, w] + F(u)vd(w) = 0, \text{ for all } u, v, w \in L. \quad (2.10)$$

Replacing w by v in this equation and using $vd(v) = 0$ for all $v \in L$, we get

$$v[u, v] = 0, \text{ for all } u, v \in L.$$

This equation is same as equation (2.5) in the proof of Theorem 2. Hence, using the same arguments in there, we get

$$[u, v] = 0, \text{ for all } u, v \in L.$$

Using the last equation in (2.10), we have

$$F(u)vd(w) = 0, \text{ for all } u, v, w \in L.$$

Putting u by uw , we get

$$0 = F(uw)vd(w) = (F(u)w + ud(w))vd(w) = ud(w)vd(w)$$

and so

$$ud(w)vd(w) = 0, \text{ for all } u, v, w \in L.$$

Writing vu instead of v , we get

$$ud(w)vud(w) = 0, \text{ for all } u, v, w \in L.$$

By Corollary 1, we have $ud(w) = 0$, for all $u, w \in L$, and so $d(L) = 0$. Hence, we conclude that $F(uv) = F(u)v$, for all $u, v \in L$.

In a similar manner, we can prove that the same conclusion holds for $F(u)F(v) + vu = 0$ for all $u, v \in L$. $\square$

Corollary 5. *Let R be a 2-torsion free prime ring and L a square-closed Lie ideal of R . Supposed that R admits a multiplicative generalized derivation F associated with the derivation d . If $F(u)F(v) = \pm vu$, for all $u, v \in L$, then $L \subseteq Z$.*

Proof. By the same techniques in the proof of Theorem 4, we obtain that

$$u w v d(u) = 0, \text{ for all } u, v, w \in L.$$

By Lemma 1, we have $u = 0$ or $d(u) = 0$ for each $u \in L$, and so $d(u) = 0$, for all $u \in L$. By Lemma 3, we conclude that $L \subseteq Z$. $\square$

Theorem 5. *Let R be a 2-torsion free semiprime ring and L a square-closed Lie ideal of R such that $L \not\subseteq Z(R)$. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero map d such that $d(x) \in L$, for all $x \in L$. If $F(u)F(v) \pm uv \in Z$, for all $u, v \in L$, then d is a commuting map on L .*

Proof. We have

$$F(u)F(v) \pm uv \in Z, \text{ for all } u, v \in L. \quad (2.11)$$

Replacing v with vw , $w \in L$, we have

$$F(u)F(vw) \pm u(vw) \in Z, \text{ for all } u, v, w \in L. \quad (2.12)$$

That is

$$F(u)(F(v)w + vd(w)) \pm uvw \in Z, \text{ for all } u, v, w \in L.$$

Commuting the last equation with w and using (2.12), we get

$$[F(u)vd(w), w] = 0, \text{ for all } u, v, w \in L. \quad (2.13)$$

Putting v by wv in the above relation, we obtain

$$[F(u)wvd(w), w] = 0, \text{ for all } u, v, w \in L. \quad (2.14)$$

Replacing u by uw in (2.13), we get

$$[(F(u)w + ud(w))vd(w), w] = 0.$$

Using (2.14), we have

$$[ud(w)vd(w), w] = 0, \text{ for all } u, v, w \in L. \quad (2.15)$$

That is

$$ud(w)v[d(w), w] + [ud(w)v, w]d(w) = 0, \text{ for all } u, v, w \in L.$$

Replacing u with ut in the last equation and using the last equation, we have

$$[u, w]td(w)vd(w) = 0, \text{ for all } u, v, w, t \in L.$$

Taking v by $v[u, w]t$ in this equation, we obtain

$$[u, w]td(w)v[u, w]td(w) = 0, \text{ for all } u, v, w, t \in L.$$

By Corollary 1, we have

$$[u, w]td(w) = 0, \text{ for all } u, w, t \in L.$$

Writing u by $d(w)$ in the above equation, we see that

$$[d(w), w]td(w) = 0, \text{ for all } w, t \in L. \quad (2.16)$$

Multiplying (2.16) on the right by w , we get

$$[d(w), w]td(w)w = 0, \text{ for all } w, t \in L.$$

Replacing t by tw in equation (2.16), we obtain that

$$[d(w), w]tw d(w) = 0, \text{ for all } w, t \in L.$$

Commuting two last equation, we have $[d(w), w]L[d(w), w] = 0$, for all $w \in L$. By Corollary 1, we see that $[d(w), w] = 0$, for all $w \in L$. Hence, d is a commuting map on L . $\square$

Corollary 6. *Let R be a 2-torsion free prime ring and L a square-closed Lie ideal of R . Supposed that R admits a multiplicative generalized derivation F associated with the derivation d . If $F(u)F(v) \pm uv \in Z$, for all $u, v \in L$, then $L \subseteq Z$.*

Proof. Using the same methods in the proof of Theorem 5, we have

$$[u, w]td(w) = 0, \text{ for all } u, w, t \in L.$$

By Lemma 1, we get either $[u, w] = 0$ or $d(w) = 0$, for each $w \in L$. We set $K = \{w \in L \mid [u, w] = 0, \text{ for all } u \in L\}$ and $T = \{w \in L \mid d(w) = 0\}$. Clearly each of K and T is additive subgroup of L . Moreover, L is the set-theoretic union of K and T . But a group can not be the set-theoretic union of two proper subgroups, hence $K = L$ or $T = L$. In the first case, we have $L \subseteq Z$ by Lemma 2. In the latter case, we have $L \subseteq Z$ by Lemma 3. This completes the proof. $\square$

Theorem 6. *Let R be a semiprime ring with characteristic not two and L a square-closed Lie ideal of R such that $L \not\subseteq Z(R)$. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero map d such that $d(x) \in L$, for all $x \in L$. If $F(u)F(v) \pm vu \in Z$, for all $u, v \in L$, then d is commuting map on L .*

Proof. By the hypothesis, we have

$$F(u)F(v) - vu \in Z, \text{ for all } u, v \in L.$$

Replacing v with vw , we get

$$F(u)(F(v)w + vd(w)) - vwu \in Z$$

That is

$$(F(u)F(v) - vu)w + v[u, w] + F(u)vd(w) \in Z$$

Commuting this equation with w and using the hypothesis, we obtain

$$[v[u, w], w] + [F(u)vd(w), w] = 0 \quad (2.17)$$

Replacing u by uw in the above relation, we get

$$[v[u, w], w]w + [(F(u)w + ud(w))vd(w), w] = 0 \quad (2.18)$$

Taking v by wv in equation (2.17), we get

$$w[v[u, w], w] + [F(u)wvd(w), w] = 0 \quad (2.19)$$

Subtracting (2.18) from (2.19), we have

$$[[v[u, w], w], w] + [ud(w)vd(w), w] = 0 \quad (2.20)$$

Writing u by uw in the above relation, we have

$$[[v[u, w], w], w] + [uwd(w)vd(w), w] = 0 \quad (2.21)$$

Right multiplying (2.20) by w and subtracting it from (2.21), we get

$$[u[d(w)vd(w), w], w] = 0, \text{ for all } u, v, w \in L. \quad (2.22)$$

Replacing u by $d(w)vd(w)u$ in the last equation, we have

$$\begin{aligned} 0 &= [d(w)vd(w)u[d(w)vd(w), w], w] \\ &= d(w)vd(w)[u[d(w)vd(w), w], w] + [d(w)vd(w), w]u[d(w)vd(w), w]. \end{aligned}$$

Using (2.22), it reduces to

$$[d(w)vd(w), w]u[d(w)vd(w), w] = 0, \text{ for all } u, v, w \in L.$$

By Corollary 1, we have

$$[d(w)vd(w), w] = 0, \text{ for all } v, w \in L.$$

That is,

$$d(w)vd(w)w - wd(w)vd(w) = 0, \text{ for all } v, w \in L. \quad (2.23)$$

Taking v by $vd(w)u$, $u \in L$ in this equation, we have

$$d(w)vd(w)ud(w)w - wd(w)vd(w)ud(w) = 0.$$

Using (2.23), we obtain

$$d(w)vwd(w)ud(w) - d(w)vd(w)wud(w) = 0.$$

That is,

$$d(w)v[d(w), w]ud(w) = 0, \text{ for all } u, v, w \in L.$$

This implies that

$$[d(w), w]v[d(w), w]u[d(w), w] = 0, \text{ for all } u, v, w \in L.$$

Right multiplying this equation by $v[d(w), w]$, we get

$$[d(w), w]v[d(w), w]u[d(w), w]v[d(w), w] = 0, \text{ for all } u, v, w \in L.$$

By Corollary 1, we obtain

$$[d(w), w]v[d(w), w] = 0, \text{ for all } v, w \in L.$$

That is, $[d(w), w] = 0$, for all $w \in L$ by Corollary 1. Hence, d is commuting on L . In a similar manner, we can prove that the same conclusion holds for $F(u)F(v) + vu \in Z$ for all $u, v \in L$. $\square$

Corollary 7. *Let R be a prime ring with characteristic not two and L a square-closed Lie ideal of R . Supposed that R admits a multiplicative generalized derivation F associated with the derivation d . If $F(u)F(v) \pm vu \in Z$, for all $u, v \in L$, then $L \subseteq Z$.*

Proof. By the same methods in the proof of Theorem 6, we have

$$[u[d(w)vd(w), w], w] = 0, \text{ for all } u, v, w \in L. \quad (2.24)$$

Replacing u by $d(w)vd(w)u$, $w \in [L, L]$ in the last equation, we have

$$[d(w)vd(w), w] = 0, \text{ for all } v \in L, w \in [L, L].$$

Again using the same techniques after the equation (2.23) in the proof of Theorem 6, we have $[d(w), w] = 0$, for all $w \in [L, L]$. By Lemma 5 and Lemma 2, we conclude that $L \subseteq Z$. $\square$

Theorem 7. *Let R be a 2-torsion free semiprime ring and L a square-closed Lie ideal of R such that $L \not\subseteq Z(R)$. Suppose that R admit two multiplicative generalized derivations F, G associated with nonzero maps d and g such that $g(x) \in L$, for all $x \in L$. If $[F(u), v] \pm [u, G(v)] = 0$ for all $u, v \in L$, then g is commuting map on L .*

Proof. By the hypothesis, we have

$$[F(u), v] \pm [u, G(v)] = 0, \text{ for all } u, v \in L.$$

Replacing v by vu in the hypothesis, we have

$$v[F(u), u] + [F(u), v]u \pm [u, G(v)]u \pm [u, v]g(v) \pm v[u, g(u)] = 0, \text{ for all } u, v \in L.$$

Application the hypothesis, we get

$$v[F(u), u] \pm [u, v]g(u) \pm v[u, g(u)] = 0, \text{ for all } u, v \in L. \quad (2.25)$$

Writing v by wv in (2.25) and using this equation, we get

$$[u, w]vg(u) = 0, \text{ for all } u, v \in L.$$

Taking w by $g(u)$ in the last equation, we obtain

$$[u, g(u)]vg(u) = 0, \text{ for all } u, v \in L. \quad (2.26)$$

Replacing v by vu in (2.26), we find that

$$[u, g(u)]vug(u) = 0, \text{ for all } u, v \in L. \quad (2.27)$$

Multiplying (2.26) on the right by u , we get

$$[u, g(u)]vg(u)u = 0, \text{ for all } u, v \in L. \quad (2.28)$$

Subtracting (2.27) and (2.28), we have

$$[u, g(u)]v[u, g(u)] = 0, \text{ for all } u, v \in L.$$

By Corollary 1, we obtain that $[u, g(u)] = 0$, for all $u, v \in L$. $\square$

Corollary 8. *Let R be a 2-torsion free prime ring, L be a square-closed Lie ideal of R and $F, G : R \rightarrow R$ two generalized derivations associated with derivations d and g . If $[F(u), v] \pm [u, G(v)] = 0$ for all $u, v \in L$, then $L \subseteq Z$.*

Proof. Using the same methods in the proof of Theorem 7, we obtain that

$$[u, w]vg(u) = 0, \text{ for all } u, v \in L.$$

By Lemma 1, we get $[u, w] = 0$ or $g(u) = 0$, for all $u, v, w \in L$. We set $K = \{u \in L \mid [u, w] = 0, \text{ for all } w \in L\}$ and $T = \{u \in L \mid g(u) = 0\}$. We had done Corollary 6. Hence we obtain that $L \subseteq Z$. $\square$

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REPRESENTATION NUMBERS BY SUMS OF QUADRATIC FORMS $X_1^2 + X_1X_2 + X_2^2$ IN SIXTEEN VARIABLES

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Abstract. Let $R(a_1, \dots, a_8; n)$ denote the number of representations of an integer n by the form $a_1(x_1^2 + x_1x_2 + x_2^2) + a_2(x_3^2 + x_3x_4 + x_4^2) + a_3(x_5^2 + x_5x_6 + x_6^2) + a_4(x_7^2 + x_7x_8 + x_8^2) + a_5(x_9^2 + x_9x_{10} + x_{10}^2) + a_6(x_{11}^2 + x_{11}x_{12} + x_{12}^2) + a_7(x_{13}^2 + x_{13}x_{14} + x_{14}^2) + a_8(x_{15}^2 + x_{15}x_{16} + x_{16}^2)$. In this article, we derive formulae for $R(1, 2, 2, 2, 2, 2, 2, 2; n)$, $R(1, 1, 1, 2, 2, 2, 2, 2; n)$, $R(1, 1, 1, 1, 1, 2, 2, 2; n)$ and $R(1, 1, 1, 1, 1, 1, 1, 2; n)$. These formulae are given in terms of the function $\sigma_7(n)$ and the numbers $\tau_{8,2}(n)$ and $\tau_{8,6}(n)$.

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1. INTRODUCTION

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ denote the set of positive integers, integers and rational numbers respectively and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $a_1, \dots, a_8 \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we let $R(a_1, \dots, a_8; n)$ denote the representation number of n by the form $a_1(x_1^2 + x_1x_2 + x_2^2) + a_2(x_3^2 + x_3x_4 + x_4^2) + a_3(x_5^2 + x_5x_6 + x_6^2) + a_4(x_7^2 + x_7x_8 + x_8^2) + a_5(x_9^2 + x_9x_{10} + x_{10}^2) + a_6(x_{11}^2 + x_{11}x_{12} + x_{12}^2) + a_7(x_{13}^2 + x_{13}x_{14} + x_{14}^2) + a_8(x_{15}^2 + x_{15}x_{16} + x_{16}^2)$, that is

$$R(a_1, \dots, a_8; n) \quad (1.1)$$

$$:= \text{card} \left\{ (x_1, \dots, x_{16}) \in \mathbb{Z}^{16} : n = \sum_{k=1}^8 a_k (x_{2k-1}^2 + x_{2k-1}x_{2k} + x_{2k}^2) \right\}.$$

If l of $a_1, \dots, a_8$ are equal, say

$$a_i = a_{i+1} = \dots = a_{i+l-1} = a \quad (1.2)$$

for convenience we indicate this in $R(a_1, \dots, a_8; n)$ by writing a^l for $a_i, a_{i+1}, \dots, a_{i+l-1}$. For $k \in \mathbb{N}$ and $n \in \mathbb{Q}$ the sum of divisor function is defined by

$$\sigma_k(n) = \begin{cases} \sum_{\substack{d \in \mathbb{N} \\ d|n}} d^k, & \text{if } n \in \mathbb{N}, \\ 0, & \text{if } n \in \mathbb{Q}, n \notin \mathbb{N}. \end{cases} \quad (1.3)$$

Ramakrishnan and Sahu [8] have recently proved formulae for $R(1^8; n)$ and $R(1^4, 2^4; n)$ for all $n \in \mathbb{N}$. In the present paper, motivated from the work of Ramakrishnan and Sahu we derive formulae for $R(1^1, 2^7; n)$, $R(1^3, 2^5; n)$, $R(1^5, 2^3; n)$ and $R(1^7, 2^1; n)$ by using the method of Alaca, Alaca and Williams (see for example [1]). These formulae are given in terms of the function $\sigma_7(n)$ and the numbers $\tau_{8,2}(n)$ and $\tau_{8,6}(n)$ which are introduced in [8]. Köklüce [4] has derived formulae for the quadratic forms in twelve and sixteen variables which are sums of quadratic forms with discriminant -23 by using the method developed by Lomadze [7].

2. PRELIMINARIES AND STATEMENT OF THE THEOREM

For $a, b, r, s, n \in \mathbb{N}$ with $a \leq b$ we define the convolution sum $W_{a,b}^{r,s}(n)$ by

$$W_{a,b}^{r,s}(n) := \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ al+bm=n}} \sigma_r(l) \sigma_s(m). \quad (2.1)$$

In recent years, some of the convolution sums have been evaluated explicitly. In this study, we require the evaluations for $W_{1,1}^{3,3}(n)$, $W_{1,2}^{3,3}(n)$, $W_{1,3}^{3,3}(n)$, $W_{2,3}^{3,3}(n)$, $W_{1,6}^{3,3}(n)$. The convolution sum

$$W_{1,1}^{3,3}(n) = \frac{1}{120} \sigma_7(n) - \frac{1}{120} \sigma_3(n), \quad (2.2)$$

was evaluated explicitly by Ramanujan [9]. Formulae for the sum $W_{1,2}^{3,3}(n)$ was firstly given by Cheng and Williams [2]. Formulae for many other convolution sums of this type have also been given by the authors in that study. Ramakrishnan and Sahu [8] have recently evaluated the following four formulae containing formulae for $W_{1,2}^{3,3}(n)$ as well.

$$W_{1,2}^{3,3}(n) = -\frac{1}{240} \sigma_3(n) - \frac{1}{240} \sigma_3\left(\frac{n}{2}\right) + \frac{1}{2040} \sigma_7(n) + \frac{2}{255} \sigma_7\left(\frac{n}{2}\right) + \frac{1}{272} \tau_{8,2}(n), \quad (2.3)$$

$$W_{1,3}^{3,3}(n) = -\frac{1}{240} \sigma_3(n) - \frac{1}{240} \sigma_3\left(\frac{n}{3}\right) + \frac{1}{9840} \sigma_7(n) + \frac{81}{9840} \sigma_7\left(\frac{n}{3}\right) + \frac{1}{246} \tau_{8,3}(n), \quad (2.4)$$

$$\begin{aligned}
W_{2,3}^{3,3}(n) &= -\frac{1}{240}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{167280}\sigma_7(n) + \frac{1}{10455}\sigma_7\left(\frac{n}{2}\right) + \frac{27}{55760}\sigma_7\left(\frac{n}{3}\right) \\
&+ \frac{27}{3485}\sigma_7\left(\frac{n}{6}\right) + \frac{7}{8160}\tau_{8,2}(n) + \frac{189}{2720}\tau_{8,2}\left(\frac{n}{3}\right) + \frac{1}{820}\tau_{8,3}(n) + \frac{4}{205}\tau_{8,3}\left(\frac{n}{2}\right) \\
&- \frac{1}{480}\tau_{8,6}(n),
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
W_{1,6}^{3,3}(n) &= -\frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3\left(\frac{n}{6}\right) + \frac{1}{167280}\sigma_7(n) + \frac{1}{10455}\sigma_7\left(\frac{n}{2}\right) + \frac{27}{55760}\sigma_7\left(\frac{n}{3}\right) \\
&+ \frac{27}{3485}\sigma_7\left(\frac{n}{6}\right) + \frac{7}{8160}\tau_{8,2}(n) + \frac{189}{2720}\tau_{8,2}\left(\frac{n}{3}\right) + \frac{1}{820}\tau_{8,3}(n) + \frac{4}{205}\tau_{8,3}\left(\frac{n}{2}\right) \\
&+ \frac{1}{480}\tau_{8,6}(n),
\end{aligned} \tag{2.6}$$

where $\tau_{8,2}(n)$, $\tau_{8,3}(n)$, $\tau_{8,6}(n)$ are n -th Fourier coefficients of the newforms $\Delta_{8,2}(n)$, $\Delta_{8,3}(n)$, $\Delta_{8,6}(n)$ given in [8]. In a recent publication Köklüce and Eser [5] have completed the evaluations of the convolution sums $W_{1,12}^{1,3}(n)$, $W_{3,4}^{1,3}(n)$, $W_{4,3}^{1,3}(n)$, and $W_{12,1}^{1,3}(n)$.

We now state our main result.

Theorem 1. *Let $n \in \mathbb{N}$ then,*

(i)

$$\begin{aligned}
R(1^1, 2^7; n) &= \frac{6}{85}\sigma_7(n) - \frac{516}{85}\sigma_7\left(\frac{n}{2}\right) - \frac{486}{85}\sigma_7\left(\frac{n}{3}\right) + \frac{41796}{85}\sigma_7\left(\frac{n}{6}\right) + \frac{9}{17}\tau_{8,2}(n) \\
&- \frac{729}{17}\tau_{8,2}\left(\frac{n}{3}\right) + \frac{27}{5}\tau_{8,6}(n),
\end{aligned} \tag{2.7}$$

(ii)

$$\begin{aligned}
R(1^3, 2^5; n) &= \frac{18}{85}\sigma_7(n) - \frac{528}{85}\sigma_7\left(\frac{n}{2}\right) - \frac{1458}{85}\sigma_7\left(\frac{n}{3}\right) + \frac{42768}{85}\sigma_7\left(\frac{n}{6}\right) + \frac{27}{17}\tau_{8,2}(n) \\
&- \frac{2187}{17}\tau_{8,2}\left(\frac{n}{3}\right) + \frac{81}{5}\tau_{8,6}(n),
\end{aligned} \tag{2.8}$$

(iii)

$$R(1^5, 2^3; n) = \frac{66}{85}\sigma_7(n) - \frac{576}{85}\sigma_7\left(\frac{n}{2}\right) - \frac{5346}{85}\sigma_7\left(\frac{n}{3}\right) + \frac{46656}{85}\sigma_7\left(\frac{n}{6}\right) - \frac{54}{17}\tau_{8,2}(n)$$

$$+ \frac{4374}{17} \tau_{8,2}\left(\frac{n}{3}\right) + \frac{162}{5} \tau_{8,6}(n), \quad (2.9)$$

(iv)

$$\begin{aligned} R(1^7, 2^1; n) = & \frac{258}{85} \sigma_7(n) - \frac{768}{85} \sigma_7\left(\frac{n}{2}\right) - \frac{20898}{85} \sigma_7\left(\frac{n}{3}\right) + \frac{62208}{85} \sigma_7\left(\frac{n}{6}\right) - \frac{72}{17} \tau_{8,2}(n) \\ & + \frac{5832}{17} \tau_{8,2}\left(\frac{n}{3}\right) + \frac{216}{5} \tau_{8,6}(n). \end{aligned} \quad (2.10)$$

3. PROOF OF THEOREM 1

To prove the theorem we need the representation number formulae for the following octonary quadratic forms,

$$f_1 := x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2, \quad (3.1)$$

$$f_2 := x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + x_5^2 + x_5x_6 + x_6^2 + 2(x_7^2 + x_7x_8 + x_8^2) \quad (3.2)$$

and

$$f_3 := x_1^2 + x_1x_2 + x_2^2 + 2(x_3^2 + x_3x_4 + x_4^2) + 2(x_5^2 + x_5x_6 + x_6^2) + 2(x_7^2 + x_7x_8 + x_8^2). \quad (3.3)$$

For $l \in \mathbb{N}_0$ we set $r_i(l) = \text{card}\{(x_1, \dots, x_8) \in \mathbb{Z}^8 : l = f_i(x_1, \dots, x_8)\}$. Obviously $r_i(0) = 1$ for $i \in \{1, 2, 3\}$. The representation number formula

$$r_1(l) = 24\sigma_3(l) + 216\sigma_3\left(\frac{l}{3}\right), \quad l \in \mathbb{N}, \quad (3.4)$$

is given by Lomadze [6]. Köklüce [3] has obtained formulae for the number of representation of a positive integer l by the forms f_2 and f_3 . He has proved that,

$$r_2(l) = 18\sigma_3(l) - 48\sigma_3\left(\frac{l}{2}\right) - 162\sigma_3\left(\frac{l}{3}\right) + 432\sigma_3\left(\frac{l}{6}\right), \quad l \in \mathbb{N}, \quad (3.5)$$

$$r_3(l) = 6\sigma_3(l) - 36\sigma_3\left(\frac{l}{2}\right) - 54\sigma_3\left(\frac{l}{3}\right) + 324\sigma_3\left(\frac{l}{6}\right), \quad l \in \mathbb{N}. \quad (3.6)$$

Proof. (i) We just prove part (i) in detail as the rest can be proved similarly. It is clear that

$$R(1^1, 2^7; n) = \sum_{\substack{l, m \in \mathbb{N}_0 \\ 2l + m = n}} r_1(l) r_3(m)$$

$$= r_1(0)r_3(n) + r_1\left(\frac{n}{2}\right)r_3(0) + \sum_{\substack{l,m \in \mathbb{N} \\ 2l+m=n}} r_1(l)r_3(m).$$

Thus using (3.4) and (3.6) we have

$$\begin{aligned} & R(1^1, 2^7; n) - (6\sigma_3(n) - 36\sigma_3\left(\frac{n}{2}\right) - 54\sigma_3\left(\frac{n}{3}\right) + 324\sigma_3\left(\frac{n}{6}\right) + 24\sigma_3\left(\frac{n}{2}\right) + 216\sigma_3\left(\frac{n}{6}\right)) \\ &= \sum_{\substack{l,m \in \mathbb{N} \\ 2l+m=n}} (24\sigma_3(l) + 216\sigma_3\left(\frac{l}{3}\right))(6\sigma_3(m) - 36\sigma_3\left(\frac{m}{2}\right) - 54\sigma_3\left(\frac{m}{3}\right) + 324\sigma_3\left(\frac{m}{6}\right)) \\ &= 144 \sum_{\substack{l,m \in \mathbb{N} \\ 2l+m=n}} \sigma_3(l)\sigma_3(m) - 864 \sum_{\substack{l,m \in \mathbb{N} \\ 2l+m=n}} \sigma_3(l)\sigma_3\left(\frac{m}{2}\right) \\ &\quad - 1296 \sum_{\substack{l,m \in \mathbb{N} \\ 2l+m=n}} \sigma_3(l)\sigma_3\left(\frac{m}{3}\right) + 7776 \sum_{\substack{l,m \in \mathbb{N} \\ 2l+m=n}} \sigma_3(l)\sigma_3\left(\frac{m}{6}\right) \\ &\quad + 1296 \sum_{\substack{l,m \in \mathbb{N} \\ 2l+m=n}} \sigma_3\left(\frac{l}{3}\right)\sigma_3(m) - 7776 \sum_{\substack{l,m \in \mathbb{N} \\ 2l+m=n}} \sigma_3\left(\frac{l}{3}\right)\sigma_3\left(\frac{m}{2}\right) \\ &\quad - 11664 \sum_{\substack{l,m \in \mathbb{N} \\ 2l+m=n}} \sigma_3\left(\frac{l}{3}\right)\sigma_3\left(\frac{m}{3}\right) + 69984 \sum_{\substack{l,m \in \mathbb{N} \\ 2l+m=n}} \sigma_3\left(\frac{l}{3}\right)\sigma_3\left(\frac{m}{6}\right) \\ &= 144W_{1,2}^{3,3}(n) - 864W_{1,1}^{3,3}\left(\frac{n}{2}\right) - 1296W_{2,3}^{3,3}(n) + 7776W_{1,3}^{3,3}\left(\frac{n}{2}\right) + 1296W_{1,6}^{3,3}(n) \\ &\quad - 7776W_{1,3}^{3,3}\left(\frac{n}{2}\right) - 11664W_{1,2}^{3,3}\left(\frac{n}{3}\right) + 69984W_{1,1}^{3,3}\left(\frac{n}{6}\right). \end{aligned}$$

Appealing to (2.2)-(2.6) and adding $6\sigma_3(n) - 36\sigma_3\left(\frac{n}{2}\right) - 54\sigma_3\left(\frac{n}{3}\right) + 324\sigma_3\left(\frac{n}{6}\right) + 24\sigma_3\left(\frac{n}{2}\right) + 216\sigma_3\left(\frac{n}{6}\right)$ to both sides of the equation we obtain the asserted formula.

(ii) It is clear that

$$R(1^3, 2^5; n) = \sum_{\substack{l,m \in \mathbb{N}_0 \\ 2l+m=n}} r_1(l)r_2(m).$$

Using equations (3.4), (3.5) and (2.1) we have

$$\begin{aligned} & R(1^3, 2^5; n) - (18\sigma_3(n) - 48\sigma_3\left(\frac{n}{2}\right) - 162\sigma_3\left(\frac{n}{3}\right) + 432\sigma_3\left(\frac{n}{6}\right) + 24\sigma_3\left(\frac{n}{2}\right) \\ &\quad + 216\sigma_3\left(\frac{n}{6}\right)) \\ &= 432W_{1,2}^{3,3}(n) - 1152W_{1,1}^{3,3}\left(\frac{n}{2}\right) - 3888W_{2,3}^{3,3}(n) + 10368W_{1,3}^{3,3}\left(\frac{n}{2}\right) + 3888W_{1,6}^{3,3}(n) \end{aligned}$$

$$-10368W_{1,3}^{3,3}\left(\frac{n}{2}\right) - 34992W_{1,2}^{3,3}\left(\frac{n}{3}\right) + 93312W_{1,1}^{3,3}\left(\frac{n}{6}\right).$$

Appealing to (2.2)-(2.6) we obtain the desired result .

(iii) It is clear that

$$R(1^5, 2^3; n) = \sum_{\substack{l, m \in \mathbb{N}_0 \\ l+m=n}} r_1(l)r_3(m).$$

Using equations (3.4), (3.6) and (2.1) we have

$$\begin{aligned} & R(1^5, 2^3; n) - (6\sigma_3(n) - 36\sigma_3\left(\frac{n}{2}\right) - 54\sigma_3\left(\frac{n}{3}\right) + 324\sigma_3\left(\frac{n}{6}\right) + 24\sigma_3(n) \\ & + 216\sigma_3\left(\frac{n}{3}\right)) \\ & = 144W_{1,1}^{3,3}(n) - 864W_{1,2}^{3,3}(n) - 1296W_{1,3}^{3,3}(n) + 7776W_{1,6}^{3,3}(n) + 1296W_{1,3}^{3,3}(n) \\ & - 7776W_{2,3}^{3,3}(n) - 11664W_{1,1}^{3,3}\left(\frac{n}{3}\right) + 69984W_{1,2}^{3,3}\left(\frac{n}{3}\right). \end{aligned}$$

Appealing to (2.2)-(2.6) we obtain the desired result.

(iv) Clearly

$$R(1^7, 2^1; n) = \sum_{\substack{l, m \in \mathbb{N}_0 \\ l+m=n}} r_1(l)r_2(m).$$

Using equations (3.4), (3.5) and (2.1) we have

$$\begin{aligned} & R(1^7, 2^1; n) - (18\sigma_3(n) - 48\sigma_3\left(\frac{n}{2}\right) - 162\sigma_3\left(\frac{n}{3}\right) + 432\sigma_3\left(\frac{n}{6}\right) + 24\sigma_3(n) \\ & + 216\sigma_3\left(\frac{n}{3}\right)) \\ & = 432W_{1,1}^{3,3}(n) - 1152W_{1,2}^{3,3}(n) - 3888W_{1,3}^{3,3}(n) + 10368W_{1,6}^{3,3}(n) + 3888W_{1,3}^{3,3}(n) \\ & - 10368W_{2,3}^{3,3}(n) - 34992W_{1,1}^{3,3}\left(\frac{n}{3}\right) + 93312W_{1,2}^{3,3}\left(\frac{n}{3}\right). \end{aligned}$$

Appealing to (2.2)-(2.6) we obtain the desired result. $\square$

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NOTE ON ASYMPTOTICAL BEHAVIOR OF SOLUTIONS OF EMDEN-FOWLER EQUATION AND THE EXISTENCE AND UNIQUENESS OF SOLUTION OF SOME CAUCHY PROBLEM

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Abstract. We study differential equation of Emden-Fowler type

$$y'' - x^a y^\sigma = 0, \quad \text{for } a \in \mathbb{R} \text{ and } \sigma < 0.$$

We describe the conditions on parameters a and σ which assure that this equation has infinitely many solutions defined in some neighborhood of zero and the conditions which guarantee existence of infinitely solutions with certain asymptotic behavior.

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Keywords: Emden-Fowler differential equation, asymptotic behavior of solutions

1. INTRODUCTION

In this paper we study the very important non-linear second-order differential equation

$$y'' - x^a y^\sigma = 0.$$

This equation came first into prominence in connection with the astrophysical researcher Emden. A number of results obtained by Emden in the usual half-intuitive, wholly ingenious fashion of the physicist were made by Fowler, who was then stimulated to continue and give a complete discussion of solutions of this equation for all values of the parameters. The equation has several very interesting physical applications, occurring in astrophysics in the form of the Emden equation and in atomic physics in the form of Fermi-Thomas equation.

Mathematically, the equation has great potential. It is a nontrivial, nonlinear, differential equation with a large class of solutions whose behavior can be ascertained with astonishing accuracy, despite the fact that the solutions, in general, can't be obtained explicitly. The Emden-Fowler type of equation has significant applications in

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many fields of scientific and technical world and this equation has been investigated by many researchers. Such equations were considered, e.g. in [1–5]. In [1,2] asymptotic properties of solutions of the Emden-Fowler equations at infinity were obtained. In papers [3–5] some asymptotic properties of solutions of the Emden-Fowler equations near zero were obtained. These papers motivate us to fully examine the behavior of solutions Emden-Fowler equation near zero in the case when $\sigma < 0$ and $a \in \mathbb{R}$.

Our paper is divided into two parts. In Section 2, we discuss conditions on parameters which ensure that the solution of Emden-Fowler equation is defined in some neighborhood of zero and it is shown that this result can not be proven under weaker assumptions.

In Section 3, we examine the asymptotic behavior of the solutions of this equation. We proved the existence of infinitely many solutions of this equation for some value of parameters. Also, we proved uniqueness of solution which satisfies appropriate conditions.

With some changes of the variables, we can reduce equation

$$(t^\rho \cdot u'(t))' - t^\alpha \cdot u^\sigma(t) = 0, \quad \text{for } \rho \neq 1, \quad (1.1)$$

to the equation

$$y'' - x^a y^\sigma = 0. \quad (1.2)$$

Our results give appropriate results for equation (1.1). Namely, if $\rho > 1$, set $x = \frac{t^{\rho-1}}{\rho-1}$, $u = \frac{y}{x} \cdot (\rho-1)^{\frac{\rho-\alpha-2}{(\rho-1)(\sigma-1)}}$, the equation for y is $y''(x) - x^a y^\sigma = 0$, where $a = \frac{\alpha+\rho}{\rho-1} - (\sigma+3)$. If $\rho < 1$, set $x = \frac{t^{1-\rho}}{1-\rho}$, $u = y \cdot (1-\rho)^{-\frac{\rho+\alpha}{(1-\rho)(\sigma-1)}}$, the equation for y is $y''(x) - x^a y^\sigma = 0$, where $a = \frac{\alpha+\rho}{1-\rho}$.

2. ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF EMDEN-FOWLER EQUATION

Consider the equation

$$y'' - x^a y^\sigma = 0, \quad (2.1)$$

where $\sigma < 0$ and $a \in \mathbb{R}$.

Using Picard theorem we can conclude existence of the local defined solution, so the natural question is to find conditions which assure that the solution is defined globally. First we will prove that for some values of parameters a and σ every monotone solution of equation (2.1) is defined on the interval containing zero environment, i.e. that there are no monotone solutions such that the graph of the solution "tends to the point on x -axis".

Lemma 1. *If $\sigma \leq -1$ and $x_1 > 0$, then every positive monotone solution of differential equation (2.1) is defined on some interval I such that $(0, x_1] \subset I$, $x_1 > 0$.*

Proof. It is enough to observe only the nondecreasing solutions. Let observe the solution such that $y(x_1) > 0$, $y'(x_1) > 0$. Suppose that $x_0 > 0$ and $y(x)$ is the solution defined on the interval $(x_0, x_1]$, but $y(x)$ is not defined on the interval $(x', x_0]$, for arbitrarily $x' < x_0$. In this situation we have $\lim_{x \rightarrow x_0+} y(x) = 0$. It follows from equation (2.1) that $y''(x) > 0$, and from that we can conclude that $y'(x)$ is an increasing function, so there exists finite $\lim_{x \rightarrow x_0+} y'(x)$, so we have $y(x) \leq y(x_1)$ and $y'(x) \leq y'(x_1)$ for every $x \in (x_0, x_1]$. The function $y(x)$ can be continuously extended in point x_0 such that $y(x_0) = 0$, then for every $x \in (x_0, x_1)$ and some $\xi \in (x_0, x)$ the inequality $\frac{y(x)-0}{x-x_0} = y'(\xi) \leq y'(x_1)$ holds. Since $\sigma < 0$, it follows that

$$y''(x) = x^a y^\sigma(x) \geq (y'(x_1))^\sigma x^a (x - x_0)^\sigma,$$

and for every $x \in (x_0, x_1)$ we have

$$y'(x_1) \geq y'(x_1) - y'(x) = \int_x^{x_1} y''(t) dt \geq (y'(x_1))^\sigma \int_x^{x_1} t^a (t - x_0)^\sigma dt.$$

Since $\sigma \leq -1$, the right side of the inequality tends to ∞ when $x \rightarrow x_0$, but the left side of the inequality is constant, which is a contradiction, so it follows that such solution is defined on the interval $(0, x_1]$. $\square$

Example 1. Let's

$$x(u) = x_0 \cdot e^{\frac{2}{\sigma+1} \int_0^u \frac{dv}{\sqrt{v^2+8 \cdot \frac{v^{\sigma+1}}{\sigma+1}}}}, \quad y(u) = \sqrt{x_0} \cdot u \cdot e^{\frac{1}{\sigma+1} \int_0^u \frac{dv}{\sqrt{v^2+8 \cdot \frac{v^{\sigma+1}}{\sigma+1}}}}, \quad x_0 > 0, u \in (0, \infty),$$

where $a = -\frac{\sigma+3}{2}$ and $-1 < \sigma < 0$. We notice that $x(u)$ and $y(u)$ are well defined on the interval $(0, \infty)$. Also, $x(u)$ is an injective function from $(0, \infty)$ to (x_0, ∞) , so we can observe $x(u)$ and $y(u)$ as parametric representation of function $y(x)$, defined on (x_0, ∞) . Straightforward calculation shows that

$$\begin{aligned} x'(u) &= \frac{2x_0 \cdot e^{\frac{2}{\sigma+1} \int_0^u \frac{dv}{\sqrt{v^2+8 \cdot \frac{v^{\sigma+1}}{\sigma+1}}}}}{\sqrt{u^2+8 \cdot \frac{u^{\sigma+1}}{\sigma+1}}}, \\ y'(u) &= \left(\sqrt{x_0} + \frac{u \cdot \sqrt{x_0}}{\sqrt{u^2+8 \cdot \frac{u^{\sigma+1}}{\sigma+1}}} \right) \cdot e^{\frac{1}{\sigma+1} \int_0^u \frac{dv}{\sqrt{v^2+8 \cdot \frac{v^{\sigma+1}}{\sigma+1}}}}, \\ x''(u) &= \frac{2x_0}{u^2+8 \cdot \frac{u^{\sigma+1}}{\sigma+1}} \cdot e^{\frac{2}{\sigma+1} \int_0^u \frac{dv}{\sqrt{v^2+8 \cdot \frac{v^{\sigma+1}}{\sigma+1}}}} \cdot \left(2 - \frac{u+4u^\sigma}{\sqrt{u^2+8 \cdot \frac{u^{\sigma+1}}{\sigma+1}}} \right), \end{aligned}$$

$$y''(u) = \frac{\sqrt{x_0} \cdot e^{\int_0^u \frac{dv}{\sqrt{v^2+8 \cdot \frac{v^{\sigma+1}}{\sigma+1}}}}}{\sqrt{u^2+8 \cdot \frac{u^{\sigma+1}}{\sigma+1}}} \cdot \left(1 + \frac{u}{\sqrt{u^2+8 \cdot \frac{u^{\sigma+1}}{\sigma+1}}} + \frac{\frac{4(1-\sigma)}{1+\sigma} \cdot u^{\sigma+1}}{u^2+8 \cdot \frac{u^{\sigma+1}}{\sigma+1}} \right).$$

It follows that

$$\begin{aligned} \frac{d^2 y}{dx^2}(u) &= \frac{1}{(x'(u))^3} (y''(u) \cdot x'(u) - x''(u) \cdot y'(u)) \\ &= x_0^{-\frac{3}{2}} \cdot u^{\sigma} \cdot e^{-3 \int_0^u \frac{dv}{\sqrt{v^2+8 \cdot \frac{v^{\sigma+1}}{\sigma+1}}}} \cdot \text{and} \\ (x(u))^a \cdot (y(u))^{\sigma} &= (x(u))^{-\frac{\sigma+3}{2}} \cdot (y(u))^{\sigma} = x_0^{-\frac{3}{2}} \cdot u^{\sigma} \cdot e^{-3 \int_0^u \frac{dv}{\sqrt{v^2+8 \cdot \frac{v^{\sigma+1}}{\sigma+1}}}}, \end{aligned}$$

so with $x(u)$ and $y(u)$ a parametric representation of function $y(x)$ is given, which is a solution of equation (2.1). Notice that

$$x(u) \rightarrow x_0, \quad y(u) \rightarrow 0,$$

when $u \rightarrow 0+$. It follows that the result of Lemma 1 can not be extended for every $-1 < \sigma < 0$ and $a + \sigma + 1 \leq 0$.

Lemma 1 provides existence of the solution in some neighborhood of zero, so the natural question is to investigate asymptotic behavior of such solutions. In the next theorem we show that for some values of parameters a and σ there are no solutions $y(x)$ of differential equation (2.1), such that $\lim_{x \rightarrow 0+} y(x) = 0$, i.e. that there are no solutions such that the graph of the solution "tends to the point $(0, 0)$ ".

Theorem 1. *If $\sigma < 0$ and $a + \sigma + 1 \leq 0$, then there are no positive solutions $y(x)$ of differential equation (2.1) defined in the some neighborhood of zero such that $\lim_{x \rightarrow 0+} y(x) = 0$.*

Proof. If there was a positive solution with the above characteristics, then it must be nondecreasing. Extending the function $y(x)$ such that $y(0) = 0$, we derive that the function $y(x)$ is continuous on the segment $[0, x_1]$ and differentiable on the interval $(0, x_1)$ for some $x_1 > 0$. It follows that for every $x \in (0, x_1)$ the inequality $\frac{y(x)-0}{x-0} = y'(\xi) \leq y'(x)$ holds for some $\xi \in (0, x)$. Since $y'(x)$ is an increasing function, follows that $y(x) \leq xy'(x) \leq xy'(x_1)$. From differential equation (2.1) and from above line, it follows that

$$y''(x) = x^a (y(x))^{\sigma} \geq x^{a+\sigma} (y'(x_1))^{\sigma},$$

for each $x \in (0, x_1)$. From the integration we obtain

$$y'(x_1) \geq y'(x_1) - y'(x) = \int_x^{x_1} y''(t) dt \geq (y'(x_1))^{\sigma} \int_x^{x_1} t^{a+\sigma} dt,$$

i.e. $(y'(x_1))^{1-\sigma} \geq \int_x^{x_1} t^{a+\sigma} dt$, which is impossible, since $a + \sigma \leq -1$, so that $\int_x^{x_1} t^{a+\sigma} dt \rightarrow \infty$ when $x \rightarrow 0+$. $\square$

If $\sigma < 0$ and $a + \sigma + 1 > 0$, then there are infinitely many solutions $y(x)$ of differential equation (2.1), such that $\lim_{x \rightarrow 0+} y(x) = 0$, i.e. there are infinitely many solutions such that the graph of the solution "tends to the point $(0, 0)$ ". This result is from [4].

Example 2. Results of Lemma 1 and Theorem 1 can not be extended for every $a > -2$ and $a + \sigma \leq -1$. For example, for $a = -1$, $\sigma = 0$ solutions $y(x) = x \ln x + C(x - x_0)$ of the equation $y'' = x^a y^\sigma$ are nondecreasing on the interval (x_0, ∞) for every $x_0, C \geq 0$. Specially, for $x_0 = 0$ and every $C \geq 0$ we have $\lim_{x \rightarrow 0+} y(x) = 0$.

Finally, at the end of this section we give conditions on parameters a and σ which assure that the positive solutions of (2.1) have no vertical asymptote at $x = 0$.

Theorem 2. *If $\sigma < 0$, $a > -2$ and $y(x)$ is positive solution of differential equation (2.1) defined on the interval $(0, x_0]$, such that $y'(x_0) \leq 0$, then*

$$\lim_{x \rightarrow +0} y(x) < +\infty.$$

Proof. It follows from equation (2.1) that $y''(x) > 0$ for $x \in [x_1, x_2]$, $x_2 > x_1 > 0$, i.e. $y(x)$ is a convex function. From this and from the condition $y'(x_0) \leq 0$, we can conclude that $y'(x) < 0$ for $x \in (0, x_0)$ and $y(x) \geq y(x_0)$ for $x \in (0, x_0]$. This, together with (2.1) implies the inequality

$$y''(x) = x^a y^\sigma(x) \leq x^a y^\sigma(x_0). \quad (2.2)$$

First consider the case $a \neq -1$. From the inequality (2.2), after integration, we have

$$y'(x_0) - y'(x) = \int_x^{x_0} y''(t) dt \leq \int_x^{x_0} t^a y^\sigma(x_0) dt = \frac{y^\sigma(x_0)}{a+1} \cdot (x_0^{a+1} - x^{a+1}),$$

for $x \in (0, x_0]$, i.e. after another integration we have

$$-y(x_0) + y(x) \leq \frac{y^\sigma(x_0)}{a+1} \left(x_0^{a+1}(x_0 - x) + \frac{x_0^{a+2} - x^{a+2}}{a+2} \right) - y'(x_0)(x_0 - x).$$

Therefore

$$y(x) \leq \frac{y^\sigma(x_0)}{a+1} \left(x_0^{a+1}(x_0 - x) + \frac{x_0^{a+2} - x^{a+2}}{a+2} \right) - y'(x_0)(x_0 - x) + y(x_0).$$

Since $a > -2$ we conclude that $\lim_{x \rightarrow +0} y(x) < +\infty$.

Now, consider the case $a = -1$. Similarly, from the inequality (2.2) we have

$$y'(x_0) - y'(x) \leq y^\sigma(x_0) \cdot (\ln x_0 - \ln x), \quad x \in (0, x_0],$$

$$\text{i.e. } y(x) \leq y^\sigma(x_0) \cdot (x_0 - x) \cdot (\ln x_0 - \ln \bar{x}) - y'(x_0)(x_0 - x) + y(x_0),$$

for some $\bar{x} \in (x, x_0)$. Therefore, we conclude that $\lim_{x \rightarrow +0} y(x) < +\infty$. $\square$

In [3] it was shown that if $a \leq -2$, the result of the previous theorem can not be applied. Also in that paper asymptotic behavior near zero is obtained of each positive solution of equation (2.1) for $a \leq -2$.

3. CAUCHY PROBLEM FOR AN EMDEN-FOWLER EQUATION

Consider the Cauchy problem

$$y'' - x^a y^\sigma = 0, \quad y(0) = c, \quad y'(0) = \lambda, \quad (3.1)$$

where $\sigma < 0$, $a \in \mathbb{R}$ and λ, c are arbitrary constants such that $c > 0$.

First we must precise what is the meaning of Cauchy problem in this situation. Emden-Fowler equation is defined for $x > 0$, but, for example, under assumptions of Lemma 1 and Theorem 2 it follows that $\lim_{x \rightarrow 0+} y(x) = c$ exists in $[0, \infty)$, so we will write this shortly as $y(0) = c$. Similarly, since $y''(x) > 0$ for $x > 0$, we have that $\lim_{x \rightarrow 0+} y'(x) = \lambda$ exists in $\mathbb{R} \cup \{-\infty\}$ and we will consider situation when $\lambda \in \mathbb{R}$.

The next theorem provides us the existence and uniqueness of solution of the Cauchy problem (3.1).

Theorem 3. *If $\sigma < 0$ and $a > -1$, then Cauchy problem (3.1) has a unique solution defined on some interval $(0, h]$, $h \leq 1$ (solution is defined on $(0, h]$ and solution can continuously differentiable extend to 0, such that $y(0) = c > 0$, $y'(0) = \lambda$), for h small enough.*

Proof. First consider the case $\lambda > 0$. Instead of Cauchy problem (3.1) we consider the integral equation

$$y(x) = \lambda x + c + \int_0^x (x-t)t^a y^\sigma(t) dt. \quad (3.2)$$

The sequence of functions is defined as follows:

$$y_0(x) = \lambda x + c, \quad y_{n+1}(x) = \lambda x + c + \int_0^x (x-t)t^a y_n^\sigma(t) dt. \quad (3.3)$$

Let us show that all functions $y_n(x)$ satisfy the inequalities

$$c + \lambda x \leq y_n(x) \leq c + 2\lambda x, \quad x \in [0, h], \quad n \in \mathbb{N}_0, \quad (3.4)$$

for some h independent of n . We prove this assertion by induction. If $n = 0$ then $c + \lambda x = y_0(x) \leq 2\lambda x + c$, for $x \in [0, h]$. Let inequalities (3.4) hold, then

$$(c + \lambda x)^\sigma \geq y_n^\sigma(x) \geq (2\lambda x + c)^\sigma,$$

for $x \in [0, h]$. Then

$$y_{n+1}(x) \leq \lambda x + c + \int_0^x (x-t)t^a (c + \lambda t)^\sigma dt$$

$$\leq \lambda x + c + x \int_0^h t^a (c + \lambda t)^\sigma dt \leq 2\lambda x + c,$$

where h is sufficiently small. Since

$$\lambda x + c \leq \lambda x + c + \int_0^x (x-t) t^a y_n^\sigma(t) dt = y_{n+1}(x),$$

so (3.4) holds.

We will show uniform convergence of the sequence $y_n(x)$ on $[0, h]$. Consider the function series $y_0(x) + \sum_{n=1}^{\infty} (y_n(x) - y_{n-1}(x))$. The sequence of partial sums of that series is $S_n(x) = y_0(x) + (y_1(x) - y_0(x)) + \dots + (y_n(x) - y_{n-1}(x)) = y_n(x)$, i.e. uniform convergence of functional sequence $y_n(x)$ is equivalent to uniform convergence of that function series.

Let us show that

$$|y_n(x) - y_{n-1}(x)| \leq x^n \lambda^n, \quad x \in [0, h], \quad n \in \mathbb{N}, \quad (3.5)$$

for some h independent of n . We prove this assertion by induction. For $n = 1$, we have

$$|y_1(x) - y_0(x)| \leq x \int_0^x t^a (\lambda t + c)^\sigma dt \leq x \int_0^h t^a (\lambda t + c)^\sigma dt \leq \lambda x,$$

where h is sufficiently small. Let inequality (3.5) hold, then

$$\begin{aligned} |y_{n+1}(x) - y_n(x)| &\leq x \int_0^x t^a |y_n^\sigma(t) - y_{n-1}^\sigma(t)| dt \\ &= x |\sigma| \int_0^x t^a |y_n(t) - y_{n-1}(t)| \xi^{\sigma-1}(t) dt, \end{aligned}$$

where $\min_{x \in [0, h]} (y_n(x), y_{n-1}(x)) \leq \xi(x) \leq \max_{x \in [0, h]} (y_n(x), y_{n-1}(x))$, but

$$(c + \lambda x)^{\sigma-1} \geq \xi^{\sigma-1}(x) \geq (2\lambda x + c)^{\sigma-1},$$

so we have

$$\begin{aligned} |y_{n+1}(x) - y_n(x)| &\leq x |\sigma| \int_0^x t^a (c + \lambda t)^{\sigma-1} x^n \lambda^n dt \\ &\leq x^{n+1} \lambda^n |\sigma| \int_0^h t^a (c + \lambda t)^{\sigma-1} dt, \end{aligned}$$

$$\text{i.e.} \quad |y_{n+1}(x) - y_n(x)| \leq x^{n+1} \lambda^{n+1},$$

where h is sufficiently small. We conclude that

$$|y_n(x) - y_{n-1}(x)| \leq x^n \lambda^n \leq (\lambda h)^n, \quad x \in [0, h], \quad n \in \mathbb{N}.$$

From lines above we can conclude that sequence $y_n(x)$ uniformly converges to some function $y(x)$.

Let $y(x) = \lim_{n \rightarrow +\infty} y_n(x)$, $x \in [0, h]$. Then $y(0) = c$ and $y'(0) = \lambda$. Since we have

$$|y_n^\sigma(x) - y^\sigma(x)| \leq \xi^{\sigma-1}(x) \cdot |\sigma| \cdot |y_n(x) - y(x)|,$$

where $\min_{x \in [0, h]} (y_n(x), y(x)) \leq \xi(x) \leq \max_{x \in [0, h]} (y_n(x), y(x))$, sequence $y_n^\sigma(x)$ uniformly converges to function $y^\sigma(x)$ on $[0, h]$. Because of that we have

$$\lim_{n \rightarrow +\infty} \int_0^x (x-t)t^a y_n^\sigma(t) dt = \int_0^x \lim_{n \rightarrow +\infty} ((x-t)t^a y_n^\sigma(t)) dt = \int_0^x (x-t)t^a y^\sigma(t) dt.$$

From this we conclude that sequence $y_n(x)$ uniformly converges to function given in (3.2).

Let us prove the uniqueness. Let $y_1(x)$ and $y_2(x)$ be solutions of Cauchy problem (3.1). Then

$$|y_1(x) - y_2(x)| \leq x|\sigma| \int_0^x t^a |y_1(t) - y_2(t)| \xi^{\sigma-1}(t) dt,$$

where $\min_{x \in [0, h]} (y_1(x), y_2(x)) \leq \xi(x) \leq \max_{x \in [0, h]} (y_1(x), y_2(x))$, so

$$y_1(x) \geq \lambda x + c, \quad y_2(x) \geq \lambda x + c, \quad x \in [0, h],$$

because $y_1''(x) > 0$ and $y_2''(x) > 0$, for $x \in [0, h]$, we can conclude that

$$y_1^{\sigma-1}(x) \leq (\lambda x + c)^{\sigma-1}, \quad y_2^{\sigma-1}(x) \leq (\lambda x + c)^{\sigma-1}, \quad x \in [0, h],$$

$$\text{i.e.} \quad \xi^{\sigma-1}(x) \leq (\lambda x + c)^{\sigma-1}, \quad x \in [0, h].$$

Finally, we have

$$|y_1(x) - y_2(x)| \leq x \cdot \max_{x \in [0, h]} |y_1(x) - y_2(x)| \cdot |\sigma| \int_0^h t^a (\lambda t + c)^{\sigma-1} dt,$$

$$\text{i.e.} \quad |y_1(x) - y_2(x)| \leq \frac{1}{2} \max_{x \in [0, h]} |y_1(x) - y_2(x)|, \quad x \in [0, h],$$

where h is sufficiently small. It follows that $y_1 \equiv y_2$.

Now, let us consider the case $\lambda \leq 0$. Instead of Cauchy problem (3.1) we consider the integral equation (3.2), where $x \in [0, h]$, for h sufficiently small ($h\lambda + c > 0$). The sequence of functions $y_n(x)$ is defined as (3.3). Let us show that all functions $y_n(x)$ satisfy the inequalities

$$c + \lambda x \leq y_n(x) \leq c + x, \quad x \in [0, h], \quad n \in \mathbb{N}_0, \quad (3.6)$$

for some h independent of n . We prove this assertion by induction. If $n = 0$ then $c + \lambda x = y_0(x) \leq x + c$, for $x \in [0, h]$. Let inequalities (3.6) hold, then

$$(c + \lambda x)^\sigma \geq y_n^\sigma(x) \geq (x + c)^\sigma,$$

for $x \in [0, h]$. Then

$$y_{n+1}(x) \leq c + \int_0^x (x-t)t^a(c+\lambda t)^\sigma dt \leq c + x \int_0^h t^a(c+\lambda t)^\sigma dt \leq x + c,$$

where h is sufficiently small. How it is

$$\lambda x + c \leq \lambda x + c + \int_0^x (x-t)t^a y_n^\sigma(t) dt = y_{n+1}(x),$$

then (3.6) holds.

We will show uniform convergence of the sequence $y_n(x)$ on $[0, h]$. For proof of uniform convergence of functional sequence $y_n(x)$ is enough to prove uniform convergence of function series $y_0(x) + \sum_{n=1}^{\infty} (y_n(x) - y_{n-1}(x))$.

Let us show that

$$|y_n(x) - y_{n-1}(x)| \leq x^n, \quad x \in [0, h], \quad n \in \mathbb{N}, \quad (3.7)$$

for some h independent of n . We prove this assertion by induction. For $n = 1$, we have

$$|y_1(x) - y_0(x)| \leq x \int_0^x t^a(\lambda t + c)^\sigma dt \leq x \int_0^h t^a(\lambda t + c)^\sigma dt \leq x,$$

where h is sufficiently small. Let inequality (3.7) hold, then

$$\begin{aligned} |y_{n+1}(x) - y_n(x)| &\leq x \int_0^x t^a |y_n^\sigma(t) - y_{n-1}^\sigma(t)| dt \\ &= x |\sigma| \int_0^x t^a |y_n(t) - y_{n-1}(t)| \xi^{\sigma-1}(t) dt, \end{aligned}$$

where $\min_{x \in [0, h]} (y_n(x), y_{n-1}(x)) \leq \xi(x) \leq \max_{x \in [0, h]} (y_n(x), y_{n-1}(x))$, so

$$(c + \lambda x)^{\sigma-1} \geq \xi^{\sigma-1}(x) \geq (x + c)^{\sigma-1},$$

we have

$$|y_{n+1}(x) - y_n(x)| \leq x |\sigma| \int_0^x t^a (c + \lambda t)^{\sigma-1} x^n dt \leq x^{n+1} |\sigma| \int_0^h t^a (c + \lambda t)^{\sigma-1} dt,$$

$$\text{i.e.} \quad |y_{n+1}(x) - y_n(x)| \leq x^{n+1},$$

where h is sufficiently small. We conclude that

$$|y_n(x) - y_{n-1}(x)| \leq x^n \leq h^n, \quad x \in [0, h], \quad n \in \mathbb{N}.$$

From lines above we can conclude that sequence $y_n(x)$ uniformly converges to some function $y(x)$.

In the same way as $\lambda > 0$, it can be shown that sequence $y_n^\sigma(x)$ uniformly converges to function $y^\sigma(x)$ on $[0, h]$. Because of that we have

$$\lim_{n \rightarrow +\infty} \int_0^x (x-t)t^a y_n^\sigma(t) dt = \int_0^x \lim_{n \rightarrow +\infty} ((x-t)t^a y_n^\sigma(t)) dt = \int_0^x (x-t)t^a y^\sigma(t) dt.$$

From this we conclude that sequence $y_n(x)$ uniformly converges to function

$$y(x) = \lambda x + c + \int_0^x (x-t)t^a y^\sigma(t) dt.$$

The proof of uniqueness in this case is the same as the proof when $\lambda > 0$. It follows that $y_1 \equiv y_2$. $\square$

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SPECTRA OF SOME SPECIAL BIPARTITE GRAPHS

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Abstract. Let $G = (P, Q)$ be a bipartite graph and G' be a graph obtained by joining each vertex of P and Q with m and s new vertices respectively. We obtain the characteristic, Laplacian and signless Laplacian polynomial of G' . As an application, we give a simple proof for Csikvari's lemma on eigenvalues of graphs.

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1. INTRODUCTION

Throughout the paper $0_{n \times n}$ and I_n are used respectively, for square zero and identity matrix of order n , 0_n and j_n , respectively, stand for length- n column vectors consisting entirely of 0's and 1's. The Kronecker product $A \otimes B$ of two matrices $A(a_{ij})$ and $B(b_{ij})$ of order $m \times n$ and $p \times q$, respectively, is the $mp \times nq$ matrix obtained from A by replacing each element a_{ij} by $a_{ij}B$. This operation has the properties $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ whenever the products AC and BD exist. The latter implies $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ for nonsingular matrices A and B . Moreover, if A and B are $n \times n$ and $p \times p$ matrices, then $\det(A \otimes B) = (\det A)^p \cdot (\det B)^n$. $\bar{K}_n$, K_n and $K_{n,m}$ stand for the graph with n isolated vertices, complete graph and complete bipartite graph, respectively. Also we use these two notations

$$T_w = \begin{bmatrix} I_w & 0 \\ 0 & 0_{(n-w) \times (n-w)} \end{bmatrix} \quad S_w = \begin{bmatrix} 0_{(n-w) \times (n-w)} & 0 \\ 0 & I_w \end{bmatrix}$$

where $T_n = S_n = I_n$.

Let G be a graph with the vertex set $\{v_1, \dots, v_n\}$. The adjacency matrix of G is an $(n \times n)$ matrix $A(G)$ whose (i, j) -entry is 1 if v_i is adjacent to v_j and 0, otherwise. The characteristic polynomial of G , denoted by $f_G(\lambda)$, is the characteristic polynomial of $A(G)$. We will write it simply f_G when there is no confusion. The roots of f_G are called the eigenvalues of G . The notation $(a)^{[m]}$, means that multiplicity of the root a is m . The Laplacian matrix of G and the signless Laplacian matrix of G are defined as $L(G) = \Delta(G) - A(G)$ and $Q(G) = \Delta(G) + A(G)$, respectively,

where $\Delta(G)$ is the diagonal matrix whose entries are the degrees of G . We denote the Laplacian polynomial of G by $f_{L(G)}(\mu)$ and the signless Laplacian polynomial of G by $f_{Q(G)}(\nu)$. Denote the eigenvalues of $A(G)$, $L(G)$ and $Q(G)$, respectively, by

$$\begin{aligned}\lambda_1(G) &\geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \\ \mu_1(G) &\leq \mu_2(G) \leq \cdots \leq \mu_n(G) \\ \nu_1(G) &\leq \nu_2(G) \leq \cdots \leq \nu_n(G).\end{aligned}$$

Spectral graph theory is an area of intense study, see for instance [1, 2, 4–6]. Graph transformations are natural techniques for producing new graphs from old ones, and their spectra have received considerable attention in recent years. In this paper, we find the characteristic, Laplacian and signless Laplacian polynomial of some special bipartite graphs obtained from other bipartite graphs by a specific graph transformation. We describe it as follows. Let $G = (P, Q)$ be a bipartite graph with n vertices which $|P| = p$, $|Q| = n - p$. We join each vertex of P and Q with m and s new vertices, respectively. We add $(n - p)(m - s)$ isolated vertices. This only results adding 0's to the various spectrums, but will simplify our computation. We denote the resulting graph by G_{ms} . By a proper labeling, the adjacency matrix of G_{ms} is

$$A(G_{ms}) = \begin{bmatrix} A(G) & D \\ D^T & 0 \end{bmatrix},$$

where $D = j_m^T \otimes I_n - (0_s^T, j_{m-s}^T) \otimes S_{n-p}$.

We obtain the characteristic, Laplacian and signless Laplacian polynomial of G_{ms} . Let $G = (P, Q)$ be a bipartite graph with n vertices which $|P| = p$, $|Q| = n - p$. We join each vertex of P and Q with m and s new vertices respectively. From now on, we denote this graph by G' . In Section 2 we give some preliminaries. In Section 3 we obtain the characteristic polynomial of G_{ms} . The results on G_{ms} enable us to obtain the characteristic polynomial of G' . As an application, we give a simple proof for Csikvari's lemma on eigenvalues of graphs ([3] Lemma 2.8). Finally, we construct some graphs whose all nonzero eigenvalues are of the form $\pm\sqrt{m \pm \sqrt{m}}$ for any natural number m . In Section 4 we obtain Laplacian and signless Laplacian polynomials of G_{ms} and G' .

2. PRELIMINARIES

Lemma 1 (Schur complement [1]). *Let A be an $n \times n$ matrix partitioned as*

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are square matrices. If A_{11} and A_{22} are invertible, then

$$\begin{aligned} \det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} &= \det(A_{22}) \det(A_{11} - A_{12} A_{22}^{-1} A_{21}) \\ &= \det(A_{11}) \det(A_{22} - A_{21} A_{11}^{-1} A_{12}). \end{aligned}$$

Notation 1. Let $G = (P, Q)$ be a bipartite graph with n vertices such that $|P| = p$, $|Q| = n - p$. Also C is a matrix of order $p \times (n - p)$ such that

$$A(G) = \begin{pmatrix} 0 & C \\ C^T & 0 \end{pmatrix}.$$

For

$$f_G(\lambda) = \det \begin{pmatrix} \lambda I_p & -C \\ -C^T & \lambda I_{n-p} \end{pmatrix},$$

with two methods of Schur complement we have:

$$\begin{aligned} f_G(\lambda) &= \lambda^{n-2p} \det(\lambda^2 I_p - C C^T) = \lambda^{n-2p} g_G(\lambda) \\ &= \lambda^{2p-n} \det(\lambda^2 I_{n-p} - C^T C) = \lambda^{2p-n} q_G(\lambda). \end{aligned}$$

where $g_G(\lambda) = \det(\lambda^2 I_p - C C^T)$ and $q_G(\lambda) = \det(\lambda^2 I_{n-p} - C^T C)$. It is clear that if $n = 2p$, then $f_G(\lambda) = g_G(\lambda) = q_G(\lambda)$.

3. CHARACTERISTIC POLYNOMIAL OF G_{ms}

Theorem 1. Let $G = (P, Q)$ be a bipartite graph with n vertices such that $|P| = p$, $|Q| = n - p$. Then

$$f_{G_{ms}}(\lambda) = \begin{cases} \lambda^{mn-n+2p} (\lambda^2 - s)^{n-2p} g_G(\sqrt{\frac{(\lambda^2 - m)(\lambda^2 - s)}{\lambda^2}}) & n \geq 2p \\ \lambda^{mn+n-2p} (\lambda^2 - m)^{2p-n} q_G(\sqrt{\frac{(\lambda^2 - m)(\lambda^2 - s)}{\lambda^2}}) & n \leq 2p. \end{cases}$$

Proof. Let $D = j_m^T \otimes I_n - (0_s^T, j_{m-s}^T) \otimes S_{n-p}$. By using of Schur complement we have

$$\begin{aligned} f_{G_{ms}}(\lambda) &= \det \begin{pmatrix} \lambda I_n - A(G) & -D \\ -D^T & \lambda I_{nm} \end{pmatrix} \\ &= \lambda^{mn} \det((\lambda I_n - A(G)) - D(\lambda I_{nm})^{-1} D^T) \\ &= \lambda^{mn} \det((\lambda I_n - A(G)) - \frac{1}{\lambda} D D^T) \\ &= \lambda^{mn-n} \det((\lambda^2 I_n - \lambda A(G)) - D D^T). \end{aligned}$$

It is easy to see that $D D^T = m I_n - (m-s) S_{n-p}$. Since G is bipartite there is a matrix C of order $p \times (n-p)$ such that

$$f_{G_{ms}}(\lambda) = \lambda^{mn-n} \det \begin{pmatrix} (\lambda^2 - m) I_p & -\lambda C \\ -\lambda C^T & (\lambda^2 - s) I_{n-p} \end{pmatrix}.$$

There are two cases:

CASE 1. Let $n \geq 2p$. In this case, let $(\lambda^2 - s) I_{n-p}$ be invertible. Then by Schur complement, we have

$$\begin{aligned} f_{G_{ms}}(\lambda) &= \lambda^{mn-n} (\lambda^2 - s)^{n-p} \det((\lambda^2 - m) I_p - (\lambda C ((\lambda^2 - s) I_{n-p})^{-1} \lambda C^T)) \\ &= \lambda^{mn-n} (\lambda^2 - s)^{n-p} \det((\lambda^2 - m) I_p - \frac{\lambda^2}{(\lambda^2 - s)} (C C^T)) \\ &= \lambda^{mn-n} (\lambda^2 - s)^{n-p} \left(\frac{\lambda^2}{(\lambda^2 - s)} \right)^p \det\left(\frac{(\lambda^2 - m)(\lambda^2 - s)}{\lambda^2} I_p - (C C^T) \right). \end{aligned}$$

By Notation 1 we obtain

$$f_{G_{ms}}(\lambda) = \lambda^{mn-n+2p} (\lambda^2 - s)^{n-2p} g_G \left(\sqrt{\frac{(\lambda^2 - m)(\lambda^2 - s)}{\lambda^2}} \right).$$

CASE 2. Let $n \leq 2p$. In this case, let $(\lambda^2 - m) I_p$ be invertible. Then

$$\begin{aligned}
f_{G_{ms}}(\lambda) &= \lambda^{mn-n}(\lambda^2 - m)^p \det((\lambda^2 - s)I_{n-p} - (\lambda C^T((\lambda^2 - m)I_p)^{-1}\lambda C)) \\
&= \lambda^{mn-n}(\lambda^2 - m)^p \det((\lambda^2 - s)I_{n-p} - \frac{\lambda^2}{(\lambda^2 - m)}(C^T C)) \\
&= \lambda^{mn-n}(\lambda^2 - m)^p (\frac{\lambda^2}{(\lambda^2 - m)})^{n-p} \det(\frac{(\lambda^2 - s)(\lambda^2 - m)}{\lambda^2} I_{n-p} - (C^T C)).
\end{aligned}$$

By Notation 1 we obtain

$$f_{G_{ms}}(\lambda) = \lambda^{mn+n-2p}(\lambda^2 - m)^{2p-n} q_G(\sqrt{\frac{(\lambda^2 - s)(\lambda^2 - m)}{\lambda^2}}).$$

□

Corollary 1. For graph G' we have

$$f_{G'}(\lambda) = \begin{cases} \lambda^{n(s-1)+p(m-s+2)}(\lambda^2 - s)^{n-2p} g_G(\sqrt{\frac{(\lambda^2 - m)(\lambda^2 - s)}{\lambda^2}}) & n \geq 2p \\ \lambda^{n(s+1)+p(m-s-2)}(\lambda^2 - m)^{2p-n} q_G(\sqrt{\frac{(\lambda^2 - m)(\lambda^2 - s)}{\lambda^2}}) & n \leq 2p. \end{cases}$$

Proof. The result follows by removing $(n-p)(m-s)$ number of λ 's from $f_{G_{ms}}(\lambda)$. □

For $s = 0$, we obtain a nice result which is due to Csikvari ([3] Lemma 2.8).

Corollary 2. For graph G' , let $s=0$. Then

$$f_{G'}(\lambda) = \begin{cases} \lambda^{n+p(m-2)} g_G(\sqrt{\lambda^2 - m}) & n \geq 2p \\ \lambda^{n+p(m-2)}(\lambda^2 - m)^{2p-n} q_G(\sqrt{\lambda^2 - m}) & n \leq 2p. \end{cases}$$

Corollary 3. All nonzero eigenvalues of the graph G' are of the form $\pm\sqrt{m}$, $\pm\sqrt{s}$ and $\pm\sqrt{\frac{m+s+\lambda_i^2 \pm \sqrt{(m+s+\lambda_i^2)^2 - 4ms}}{2}}$, where λ_i for $i = 1, \dots, t$ are all positive eigenvalues of G .

Proof. It is clear from $\lambda^2 - m = 0$, $\lambda^2 - s = 0$ and $\frac{(\lambda^2 - m)(\lambda^2 - s)}{\lambda^2} = \lambda_i^2$. □

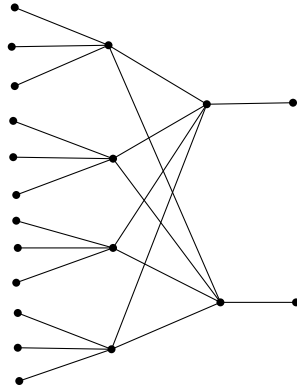
Example 1. Let $G = K_{4,2}$, $m = 3$ and $s = 1$. Then $f_G(\lambda) = \lambda^2(\underbrace{\lambda^2(\lambda^2 - 8)}_{q_G(\lambda)})$ and

$8 = 2p > n = 6$. We have

$$\begin{aligned} f_{G'}(\lambda) &= \lambda^{12}(\lambda^2 - 3)^2 \left(\frac{(\lambda^2 - 3)(\lambda^2 - 1)}{\lambda^2} \right) \left(\frac{(\lambda^2 - 3)(\lambda^2 - 1)}{\lambda^2} - 8 \right) \\ &= \lambda^8(\lambda^2 - 3)^3(\lambda^2 - 1)(\lambda^4 - 12\lambda^2 + 3). \end{aligned}$$

So the eigenvalues of the graph G' are

$$\begin{aligned} \{ &(0)^{[8]}, (\sqrt{3})^{[3]}, (-\sqrt{3})^{[3]}, (1)^{[1]}, (-1)^{[1]}, (\sqrt{6 + \sqrt{33}})^{[1]}, (-\sqrt{6 + \sqrt{33}})^{[1]} \\ &, (\sqrt{6 - \sqrt{33}})^{[1]}, (-\sqrt{6 - \sqrt{33}})^{[1]} \}. \end{aligned}$$



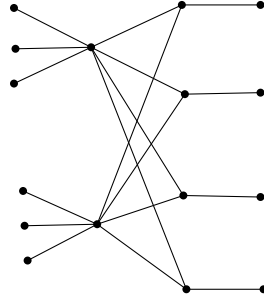
Example 2. Let $G = K_{2,4}$, $m = 3$ and $s = 1$. Then $f_G(\lambda) = \lambda^2(\underbrace{\lambda^2(\lambda^2 - 8)}_{g_G(\lambda)})$ and

$4 = 2p < n = 6$. We have

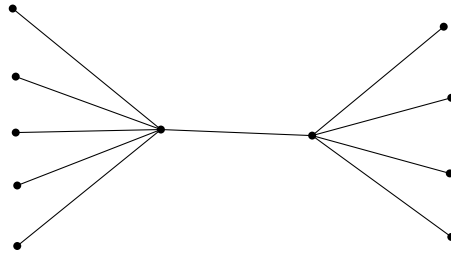
$$\begin{aligned} f_{G'}(\lambda) &= \lambda^8(\lambda^2 - 1)^2 \left(\frac{(\lambda^2 - 3)(\lambda^2 - 1)}{\lambda^2} \right) \left(\frac{(\lambda^2 - 3)(\lambda^2 - 1)}{\lambda^2} - 8 \right) \\ &= \lambda^4(\lambda^2 - 1)^3(\lambda^2 - 3)(\lambda^4 - 12\lambda^2 + 3). \end{aligned}$$

So the eigenvalues of the graph G' are

$$\begin{aligned} \{ &(0)^{[4]}, (1)^{[3]}, (-1)^{[3]}, (\sqrt{3})^{[1]}, (-\sqrt{3})^{[1]}, (\sqrt{6 + \sqrt{33}})^{[1]}, (-\sqrt{6 + \sqrt{33}})^{[1]} \\ &, (\sqrt{6 - \sqrt{33}})^{[1]}, (-\sqrt{6 - \sqrt{33}})^{[1]} \}. \end{aligned}$$



Example 3. (Graphs with eigenvalues $\pm\sqrt{m \pm \sqrt{m}}$.) For any positive integer m , let $G = K_2$ and $s = m - 1$. Then by using Corollary 3, G' has eigenvalue 0 with multiplicity $2m - 3$ and eigenvalues $\sqrt{m + \sqrt{m}}$, $\sqrt{m - \sqrt{m}}$, $-\sqrt{m + \sqrt{m}}$ and $-\sqrt{m - \sqrt{m}}$ all with multiplicity 1. The graph below has eigenvalue zero with multiplicity 7, and $\sqrt{5 + \sqrt{5}}$, $\sqrt{5 - \sqrt{5}}$, $-\sqrt{5 + \sqrt{5}}$ and $-\sqrt{5 - \sqrt{5}}$, all with multiplicity 1.



4. LAPLACIAN AND SIGNLESS LAPLACIAN POLYNOMIAL OF G_{ms}

In this section we assume that G is an r -regular bipartite graph with $|G| = n = 2k$ where k is the size of each part and G' is obtained from G by joining each vertex of one part to m and each vertex of another part to s new vertices. We denote the Laplacian polynomial of G' by $f_{L(G')}(\mu)$ and the signless Laplacian polynomial of G' by $f_{Q(G')}(v)$.

Theorem 2. *Let G be an r -regular bipartite graph with $|G| = n = 2k$ where k is the size of each part. Then*

$$f_{L(G_{ms})}(\mu) = \mu^{(m-s)k} (\mu - 1)^{ns + (m-s)k} f_G\left(\sqrt{(\mu - m - r - \frac{m}{\mu - 1})(\mu - s - r - \frac{s}{\mu - 1})}\right)$$

Proof. We have

$$\Delta(G_{ms}) = \begin{bmatrix} (m+r)I_n - (m-s)S_k & 0 \\ 0 & B \end{bmatrix}, \quad B = \begin{bmatrix} I_{sn} & 0 \\ 0 & I_{m-s} \otimes T_k \end{bmatrix},$$

where Δ is diagonal matrix whose entries are the degrees of G_{ms} . Since $L(G_{ms}) = \Delta(G_{ms}) - A(G_{ms})$ we have

$$\begin{aligned} f_{L(G_{ms})}(\mu) &= \det(\mu I - L(G_{ms})) = \det(\mu I - \Delta(G_{ms}) + A(G_{ms})) \\ &= \det \begin{bmatrix} \mu I_n - (m+r)I_n + (m-s)S_k + A(G) & D \\ D^T & \mu I_{nm} - B \end{bmatrix}. \end{aligned}$$

By Schur complement we have

$$\begin{aligned} f_{L(G_{ms})}(\mu) &= \det(\mu I_{mn} - B) \cdot \det(\mu I_n - (m+r)I_n + (m-s)S_k \\ &\quad + A(G) - D(\mu I_{mn} - B)^{-1} D^T) \end{aligned}$$

where $D = j_m^T \otimes I_n - (0_s^T, j_{m-s}^T) \otimes S_{n-p}$. It is easily seen

$$\det(\mu I_{mn} - B) = \mu^{(m-s)k} (\mu - 1)^{ns + (m-s)k}.$$

Now we need to compute $D(\mu I_{mn} - B)^{-1} D^T$. It is easy to see that

$$(\mu I_{mn} - B)^{-1} = \begin{bmatrix} \frac{1}{\mu-1} I_{sn} & 0 \\ 0 & I_{m-s} \otimes E \end{bmatrix},$$

where $E = \frac{1}{\mu-1} T_k + \frac{1}{\mu} S_k$. So

$$\begin{aligned} &D(\mu I_{mn} - B)^{-1} D^T \\ &= \begin{bmatrix} \underbrace{I_n, \dots, I_n}_{s \text{ times}}, \underbrace{T_k, \dots, T_k}_{m-s \text{ times}} \end{bmatrix} \begin{bmatrix} \frac{1}{\mu-1} I_{sn} & 0 \\ 0 & I_{m-s} \otimes E \end{bmatrix} \begin{bmatrix} \underbrace{I_n, \dots, I_n}_{s \text{ times}}, \underbrace{T_k, \dots, T_k}_{m-s \text{ times}} \end{bmatrix}^T \\ &= \begin{bmatrix} \underbrace{\frac{1}{\mu-1} I_n, \dots, \frac{1}{\mu-1} I_n}_{s \text{ times}}, \underbrace{\frac{1}{\mu-1} T_k, \dots, \frac{1}{\mu-1} T_k}_{m-s \text{ times}} \end{bmatrix} \begin{bmatrix} \underbrace{I_n, \dots, I_n}_{s \text{ times}}, \underbrace{T_k, \dots, T_k}_{m-s \text{ times}} \end{bmatrix}^T \\ &= \frac{s}{\mu-1} I_n + \frac{m-s}{\mu-1} T_k. \end{aligned}$$

Since G is bipartite, there is a matrix C of order $n \times (n-p)$ such that

$$\begin{aligned}
& \det(\mu I_n - (m+r)I_n + (m-s)S_k + A(G) - D(\mu I_{mn} - B)^{-1}D^T) \\
&= \det(\mu I_n - (m+r)I_n + (m-s)S_k + A(G) - (\frac{s}{\mu-1}I_n + \frac{m-s}{\mu-1}T_k)) \\
&= \det \begin{bmatrix} (\mu - m - r - \frac{m}{\mu-1})I_k & C \\ C^T & (\mu - s - r - \frac{s}{\mu-1})I_k \end{bmatrix}.
\end{aligned}$$

By Schur complement we have

$$\begin{aligned}
& \det \begin{bmatrix} (\mu - m - r - \frac{m}{\mu-1})I_k & C \\ C^T & (\mu - s - r - \frac{s}{\mu-1})I_k \end{bmatrix} \\
&= (\mu - s - r - \frac{s}{\mu-1})^k \det((\mu - m - r - \frac{m}{\mu-1})I_k - \frac{1}{\mu - s - r - \frac{s}{\mu-1}}CC^T) \\
&= \det((\mu - m - r - \frac{m}{\mu-1})(\mu - s - r - \frac{s}{\mu-1})I_k - CC^T) \\
&= f_G(\sqrt{(\mu - m - r - \frac{m}{\mu-1})(\mu - s - r - \frac{s}{\mu-1})}).
\end{aligned}$$

So we obtain

$$\begin{aligned}
& f_{L(G_{ms})}(\mu) \\
&= \mu^{(m-s)k} (\mu-1)^{ns+(m-s)k} f_G(\sqrt{(\mu - m - r - \frac{m}{\mu-1})(\mu - s - r - \frac{s}{\mu-1})})
\end{aligned}$$

□

By the same argument of Theorem 2 we have the following theorem.

Theorem 3. Let G be an r -regular bipartite graph with $|G| = n = 2k$ where k is the size of each part. Then

$$\begin{aligned}
& f_{Q(G_{ms})}(v) \\
&= v^{(m-s)k} (v-1)^{ns+(m-s)k} f_G(\sqrt{(v - m - r - \frac{m}{v-1})(v - s - r - \frac{s}{v-1})})
\end{aligned}$$

Corollary 4. For graph G' we have

$$\begin{aligned}
f_{L(G')}(\mu) &= (\mu-1)^{ns+(m-s)k} f_G(\sqrt{(\mu - m - r - \frac{m}{\mu-1})(\mu - s - r - \frac{s}{\mu-1})}), \\
f_{Q(G')}(v) &= (v-1)^{ns+(m-s)k} f_G(\sqrt{(v - m - r - \frac{m}{v-1})(v - s - r - \frac{s}{v-1})}).
\end{aligned}$$

Proof. The proof follows immediately from Theorems 2 and 3. $\square$

Corollary 5. For graph G' , let $s = 0$. Then

$$f_{L(G')}(\mu) = (\mu - 1)^{mk} f_G(\sqrt{(\mu - m - r - \frac{m}{\mu-1})(\mu - r)}),$$

$$f_{Q(G')}(v) = (v - 1)^{mk} f_G(\sqrt{(v - m - r - \frac{m}{v-1})(v - r)}).$$

Proof. Apply Corollary 4, for $s = 0$. $\square$

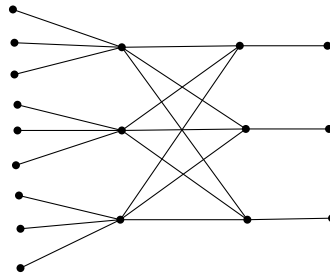
Example 4. Let $G = K_{3,3}$, $m = 3$ and $s = 1$. Since $f_G(\lambda) = \lambda^4(\lambda^2 - 9)$ we have

$$\begin{aligned} f_{L(G')}(\mu) &= \\ (\mu - 1)^{12} (\mu - 6 - \frac{3}{\mu-1})^2 (\mu - 4 - \frac{1}{\mu-1})^2 ((\mu - 6 - \frac{3}{\mu-1})(\mu - 4 - \frac{1}{\mu-1}) - 9) \\ &= (\mu - 1)^6 (\mu^2 - 7\mu + 3)^2 (\mu^2 - 5\mu + 3)^2 (\mu^4 - 12\mu^3 + 8\mu^2 - 18\mu) \\ &= \mu(\mu - 1)^6 (\mu^2 - 7\mu + 3)^2 (\mu^2 - 5\mu + 3)^2 (\mu^3 - 12\mu^2 + 8\mu - 18). \end{aligned}$$

So the Laplacian and signless Laplacian eigenvalues of the graph G' are

$$\begin{aligned} \{(0)^{[1]}, (1)^{[6]}, (\frac{7+\sqrt{37}}{2})^{[2]}, (\frac{7-\sqrt{37}}{2})^{[2]}, (\frac{5+\sqrt{13}}{2})^{[2]}, (\frac{5-\sqrt{13}}{2})^{[2]}, \\ (8.47467)^{[1]}, (2.75414)^{[1]}, (0.77119)^{[1]}\}. \end{aligned}$$

The last three roots are found by *newgraph*.



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ON THE HYPER-SUMS OF POWERS OF INTEGERS

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Abstract. The main object of this paper is to study an old problem concerning the hyper-sums of powers of integers. First, we establish some important properties of this problem (generating function, explicit formula, congruence). Finally, an explicit formula for the hyper-sums of powers of integers involving the generalized Bernoulli polynomials are also given.

2010 Mathematics Subject Classification: 11B73; 11B68; 33C05

Keywords: Bernoulli numbers, sums of powers of integers, Stirling numbers

1. INTRODUCTION

Following the usual notations (see [4]), the falling factorial $x^{\underline{n}}$ ($x \in \mathbb{C}$) is defined by $x^{\underline{0}} = 1$, $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$ for $n > 0$ and, the rising factorial denoted by $x^{\overline{n}}$, is defined by $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$ with $x^{\overline{0}} = 1$. The (signed) Stirling numbers of the first kind $s(n, k)$ are the coefficients in the expansion

$$x^{\underline{n}} = \sum_{k=0}^n s(n, k) x^k.$$

The Stirling numbers of the second kind, denoted $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ are the coefficients in the expansion

$$x^n = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} x^{\underline{k}}.$$

The Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ count the number of ways to partition a set of n elements into exactly k nonempty subsets.

The hyper-sums (sums of sums) of powers of integers $S_p^{(r)}(n)$ ($p \geq 0$) (or the r -fold summation of p th powers) are defined recursively as

$$\begin{cases} S_0^{(r)}(n) = \binom{n+r}{r+1}, \quad r \geq 0, \\ S_p^{(0)}(n) = \sum_{i=1}^n i^p ; \quad S_p^{(r)}(n) = \sum_{j=1}^n S_p^{(r-1)}(j), \quad n, r, p \geq 1. \end{cases}$$

As mentioned by Knuth [8], the hyper-sums of powers of integers was introduced for the first time by Faulhaber as a generalization of sums of powers of integers

$$S_p^{(0)}(n) = 1^p + 2^p + \cdots + n^p.$$

The sums $S_p^{(0)}(n)$ have been of interest for a long time and different methods have been proposed to obtain them, see [1–3, 5, 6, 9, 10].

In 2005, Inaba [7] extended the well-known formulas for $S_p^{(0)}(n)$ which is based on Stirling numbers of the second kind to obtain the explicit formulas of the hyper-sums of powers of integers

$$S_p^{(r)}(n) = \sum_{k=1}^p k! \binom{n+r+1}{r+k+1} \left\{ \begin{matrix} p \\ k \end{matrix} \right\}, \quad n, p \geq 1. \quad (1.1)$$

Note that when $p \geq 0$, we can write $S_p^{(r)}(n)$ as

$$S_p^{(r)}(n) = \sum_{k=0}^p \left\{ \begin{matrix} p \\ k \end{matrix} \right\} \left(k! \binom{n+r+1}{k+r+1} - \binom{n+r}{r} \delta_{0,k} \right), \quad (1.2)$$

where $\delta_{i,j}$ denotes the Kronecker symbol.

The purpose of the present paper is to give some results concerning the hyper-sums of powers of integers $S_p^{(r)}(n)$.

2. MAIN RESULTS

We begin by the exponential generating function of the hyper-sums of powers of integers.

Theorem 1. *The exponential generating function of the hyper-sums of powers of integers $S_p^{(r)}(n)$ is*

$$\sum_{p \geq 0} S_p^{(r)}(n) \frac{z^p}{p!} = \binom{n+r+1}{r+1} {}_2F_1 \left(\begin{matrix} 1, -n \\ r+2 \end{matrix}; 1-e^z \right) - \binom{n+r}{r},$$

where ${}_2F_1\left(\begin{smallmatrix} a, b \\ c \end{smallmatrix}; z\right)$ denotes the Gaussian hypergeometric function defined by

$$\sum_{n \geq 0} \frac{(a)^{\overline{n}} (b)^{\overline{n}} z^n}{(c)^{\overline{n}} n!}.$$

Proof. First, we invert the formula (1.2) to obtain

$$\sum_{k=0}^p s(p, k) S_k^{(r)}(n) = p! \binom{n+r+1}{p+r+1} - \binom{n+r}{r} \delta_{0,p}.$$

Further, we have

$$\begin{aligned} A_0(z) &= \sum_{p \geq 0} \left(p! \binom{n+r+1}{p+r+1} - \binom{n+r}{r} \delta_{0,p} \right) \frac{z^p}{p!} \\ &= \sum_{p \geq 0} \binom{n+r+1}{p+r+1} z^p - \binom{n+r}{r} \\ &= \binom{n+r+1}{r+1} \sum_{p \geq 0} \frac{(1)^{\overline{p}} (-n)^{\overline{p}} (-z)^p}{(r+2)^{\overline{p}} p!} - \binom{n+r}{r}. \end{aligned}$$

Finally, the result follows from [11, Theorem 3]

$$\sum_{p \geq 0} S_p^{(r)}(n) \frac{z^p}{p!} = A_0(e^z - 1).$$

This completes the proof. $\square$

We will now derive a few further consequences of Theorem 1.

Corollary 1. *The exponential generating function of the hyper-sums of powers of integers $S_p^{(r)}(n)$ is given by*

$$\sum_{p \geq 0} S_p^{(r)}(n) \frac{z^p}{p!} = \sum_{i=1}^n \binom{n+r-i}{r} e^{iz}. \quad (2.1)$$

Proof. It follows from the theory of hypergeometric functions that the Gaussian hypergeometric function ${}_2F_1\left(\begin{smallmatrix} 1, -n \\ r+2 \end{smallmatrix}; 1-e^z\right)$ has an integral representation given by

$${}_2F_1\left(\begin{smallmatrix} 1, -n \\ r+2 \end{smallmatrix}; 1-e^z\right) = (r+1) \int_0^1 (1-x)^r (1-x+xe^z)^n dx.$$

Hence, we obtain

$$\begin{aligned}
 \sum_{p \geq 0} S_p^{(r)}(n) \frac{z^p}{p!} &= \frac{(n+r+1)!}{n!r!} \sum_{k=0}^n \binom{n}{k} e^{(n-k)z} \int_0^1 (1-x)^{r+k} x^{n-k} dx - \binom{n+r}{r} \\
 &= \frac{(n+r+1)!}{n!r!} \sum_{k=0}^n \binom{n}{k} e^{(n-k)z} \frac{(r+k)!(n-k)!}{(n+r+1)!} - \binom{n+r}{r} \\
 &= \sum_{k=0}^n \binom{k+r}{r} e^{(n-k)z} - \binom{n+r}{r} \\
 &= \sum_{k=0}^{n-1} \binom{k+r}{r} e^{(n-k)z},
 \end{aligned}$$

as claimed. $\square$

An explicit formula for $S_p^{(r)}(n)$ is given in the following corollary.

Corollary 2. *The hyper-sums of powers of integers $S_p^{(r)}(n)$ ($p \geq 0$) is given by*

$$S_p^{(r)}(n) = \sum_{i=1}^n \binom{n+r-i}{r} i^p.$$

Proof. Since

$$\begin{aligned}
 \sum_{p \geq 0} S_p^{(r)}(n) \frac{z^p}{p!} &= \sum_{i=1}^n \binom{n+r-i}{r} \sum_{p \geq 0} \frac{(iz)^p}{p!} \\
 &= \sum_{p \geq 0} \left(\sum_{i=1}^n \binom{n+r-i}{r} i^p \right) \frac{z^p}{p!}.
 \end{aligned}$$

Equating the coefficient of $\frac{z^p}{p!}$, we get the result. $\square$

In the next theorem we will present some congruence relations for $S_p^{(r)}(n)$.

Theorem 2. *Let n be a prime number and for $p > 0$, $r \geq 0$, we have*

- (1) *If $n \mid r+1$, then $S_p^{(r)}(n) \equiv 0 \pmod{n}$,*
- (2) *If $n \nmid r+1$ and $n-1 \mid p$, then $S_p^{(r)}(n) \equiv -1 \pmod{n}$.*

Proof. If $n \mid r+1$ then by Lucas' congruence we get

$$\begin{aligned}
 S_p^{(r)}(n) &\equiv \sum_{i=1}^{n-1} \binom{r-i}{r} i^p \pmod{n} \\
 &\equiv 0 \pmod{n}.
 \end{aligned}$$

Since $n - 1 \mid p$, the Fermat's little theorem and Lucas' congruence gives

$$\begin{aligned} S_p^{(r)}(n) &\equiv \sum_{i=1}^{n-1} \binom{n+r-i}{r} \pmod{n} \\ &\equiv -1 + \binom{n+r}{r+1} \pmod{n} \\ &\equiv -1 \pmod{n}. \end{aligned}$$

This completes the proof. $\square$

Theorem 3. *The ordinary generating function of the hyper-sums of powers of integers $S_p^{(r)}(n)$ is given by*

$$\sum_{r \geq 0} S_p^{(r)}(n) z^r = \frac{1}{(1-z)^{n+1}} \sum_{i=1}^n (1-z)^i i^p. \quad (2.2)$$

Proof. Since

$$\sum_{r \geq 0} \binom{n+r-i}{r} z^r = (1-z)^{i-n-1},$$

which implies (2.2). $\square$

Further, from (2.1) and (2.2), we deduce the double generating function of the hyper-sums of powers of integers $S_p^{(r)}(n)$

$$\begin{aligned} \sum_{r \geq 0} \sum_{p \geq 0} S_p^{(r)}(n) \frac{z^p}{p!} t^r &= \sum_{i=1}^n \sum_{r \geq 0} \binom{n+r-i}{r} t^r e^{iz} \\ &= \frac{1}{(1-t)^{n+1}} \sum_{i=1}^n (e^z (1-t))^i \\ &= \frac{e^{(n+1)z} (1-t)^n - e^z}{(1-t)^n (e^z (1-t) - 1)}. \end{aligned}$$

Now, according to the well-known formula, for $n \in \mathbb{N}$ and $m \in \mathbb{N}^*$

$${}_2F_1 \left(\begin{matrix} -n, 1 \\ m \end{matrix}; z \right) = \frac{n! (z-1)^{m-2}}{(m)^{\overline{n}} z^{m-1}} \left[\sum_{k=0}^{m-2} \frac{(n+1)^{\overline{k}}}{k!} \left(\frac{z}{z-1} \right)^k - (1-z)^{n+1} \right],$$

we can rewrite the exponential generating function of the hyper-sums of powers of integers $S_p^{(r)}(n)$ as

$$\sum_{p \geq 0} S_p^{(r)}(n) \frac{z^p}{p!} = \frac{(-1)^r e^{rz}}{(1-e^z)^{r+1}} \left(\sum_{k=0}^r \binom{n+k}{k} (1-e^{-z})^k - e^{(n+1)z} \right)$$

$$- \binom{n+r}{r}. \quad (2.3)$$

The next result gives an explicit formula for $S_p^{(r)}(n)$ involving the generalized Bernoulli polynomials. Recall that the generalized Bernoulli polynomials $\mathbb{B}_n^{(\alpha)}(x)$ of degree n in x are defined by the exponential generating function [12, 13]

$$\left(\frac{z}{e^z - 1}\right)^\alpha e^{xz} = \sum_{n \geq 0} \mathbb{B}_n^{(\alpha)}(x) \frac{z^n}{n!} \quad (2.4)$$

for arbitrary parameter α .

In particular, $\mathbb{B}_n^{(1)}(x) := \mathbb{B}_n(x)$ denotes the classical Bernoulli polynomials with $\mathbb{B}_1(0) = -\frac{1}{2}$.

Theorem 4. *For all $n, p, r \geq 0$, we have*

$$\begin{aligned} S_p^{(r)}(n) &= \frac{p!}{(p+r+1)!} \mathbb{B}_{p+r+1}^{(r+1)}(n+r+1) \\ &\quad - p! \sum_{j=0}^r \binom{n+k}{k} \frac{\mathbb{B}_{p+j+1}^{(j+1)}(j)}{(p+j+1)!} - \binom{n+r}{r} \delta_{0,p}. \end{aligned}$$

Proof. By (2.3) and (2.4) we have

$$\begin{aligned} \sum_{p \geq 0} S_p^{(r)}(n) \frac{z^p}{p!} &= - \sum_{k=0}^r \binom{n+k}{k} \frac{e^{(r-k)z}}{(e^z - 1)^{r-k+1}} + \frac{e^{(r+n+1)z}}{(e^z - 1)^{r+1}} - \binom{n+r}{r} \\ &= - \sum_{k=0}^r \binom{n+k}{k} \sum_{p \geq 0} \mathbb{B}_p^{(r-k+1)}(r-k) \frac{z^{p-r-1}}{p!} \\ &\quad + \sum_{p \geq 0} \mathbb{B}_p^{(r+1)}(n+r+1) \frac{z^{p-r-1}}{p!} - \binom{n+r}{r}. \end{aligned}$$

After some rearrangement, we find

$$\begin{aligned} \sum_{p \geq 0} S_p^{(r)}(n) \frac{z^p}{p!} &= \sum_{p \geq 0} \frac{z^p}{p!} \left(p! \frac{\mathbb{B}_{p+r+1}^{(r+1)}(n+r+1)}{(p+r+1)!} \right. \\ &\quad \left. - \sum_{k=0}^r p! \frac{\mathbb{B}_{p+r-k+1}^{(r-k+1)}(r-k)}{(p+r-k+1)!} \binom{n+k}{k} - \binom{n+r}{r} \delta_{0,p} \right). \end{aligned}$$

Equating the coefficient of $\frac{z^p}{p!}$, we get the result. $\square$

When $r = 0$, Theorem 4 reduces to the known result

$$S_p^{(0)}(n) = \frac{1}{p+1} (\mathbb{B}_{p+1}(n+1) - \mathbb{B}_{p+1}(0)) - \delta_{0,p}.$$

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ANALYSIS OF CHAOTIC AND HYPERCHAOTIC CONSERVATIVE COMPLEX NONLINEAR SYSTEMS

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Abstract. The aim of this paper is to introduce and analyze chaotic and hyperchaotic conservative complex nonlinear systems. These systems appear in several branches of applied sciences. Lyapunov exponents are calculated to observe chaotic and hyperchaotic behaviors. The wide range of systems parameters at which chaotic and hyperchaotic solutions exist is calculated. The modules for complex variables are computed and plotted. The projections of chaotic and hyperchaotic solutions are shown in 3- spaces and 2-planes. The systems of this paper leave rooms for further studies in the near future, e.g. control and several types of synchronization of the solutions of these systems.

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1. INTRODUCTION

In recent years chaotic and hyperchaotic dissipative complex nonlinear systems have been proposed and studied in the literature [1–16, 18]. However, there also are many interesting cases of conservative complex nonlinear systems have not yet been as actively explored. As examples, of dissipative systems we state here the chaotic complex Lorenz, Chen and Lü systems, and hyperchaotic complex Lorenz, Chen and Lü systems. More recently, simple conservative, autonomous, second-order chaotic complex variable systems are studied [17]. Conservative nonlinear systems with real variables are introduced and studied in the literature and references therein [15, 17, 19–21]. Complex systems appear in several areas of applied sciences, for example, high-energy accelerators, rotor dynamics, elastic pendulums with two degrees of freedom, robots and beams and plates under static and dynamics loadings [6]. For more details we refer the reader to the review paper [6] and references therein. Complex Lorenz, Chen and Lü systems many be used to describe and simulate physics of detuned lasers and thermal convection of liquid flows [1, 2, 13]. A nonlinear system is defined as a chaotic system if it has one positive Lyapunov exponent. It is called hyperchaotic system or hyperchaotic of order m ($m \geq 2$) if m of its Lyapunov exponents are positive. A nonlinear dynamical system is called conservative system if the

sum of its Lyapunov exponents is zero. For complex nonlinear dynamical systems, the observable or measurable physical quantities usually are modules. For example, in alternating current machine, the current amplitude is represented by module [18] so the module is more important than that of real part and imaginary part.

In this paper, we introduce and study the basic properties of n -dimensional (nD) chaotic and hyperchaotic conservative complex nonlinear systems, and give examples on which we can illustrate our investigations. Consider the n -dimensional chaotic (or hyperchaotic) conservative complex nonlinear systems:

$$\dot{z} = Az + Bf(z), \quad (1.1)$$

where, $z = (z_1, z_2, \dots, z_n)^T$ is a state complex vector with $z_i = u_{2i-1} + ju_{2i}$, $i = 1, 2, \dots, n$, T denotes transpose, and $j = \sqrt{-1}$, $A, B \in C^{n \times n}$ are real (or complex) matrices of system parameters and $f = (f_1, f_2, \dots, f_n)^T$ is a vector of nonlinear complex functions, dots represent derivative with respect to t . The module of z_i can be defined as $M_{z_i} = \sqrt{u_{2i-1}^2 + u_{2i}^2}$, $i = 1, 2, \dots, n$.

For example, we propose a conservative chaotic complex system with $n = 2$ as:

$$\dot{z}_1 = \alpha z_2, \quad (1.2)$$

$$\dot{z}_2 = \beta \bar{z}_1^2 - \gamma z_1^2 \bar{z}_1, \quad (1.3)$$

where α , β and γ are positive real parameters. The function $\bar{z}$ means the complex conjugate of z . This system has three terms with two nonlinear terms in the right hand side. If system (1.3) contains only two terms, it becomes nonchaotic one as we checked numerically.

For hyperchaotic complex conservative systems, we introduce the following example ($n=3$):

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= -\mu z_1 + z_2 z_3, \\ \dot{z}_3 &= 1 - z_2^2, \end{aligned} \quad (1.4)$$

where μ is a positive real parameter. The counter real part of system (2.1) with $\mu = 1$ is a chaotic system as shown in Ref. [19].

The paper is organized as follows: In the next Section the complex behavior of the chaotic conservative complex nonlinear system (1.3) is studied. Based on the calculation of Lyapunov exponents, we calculated numerically the range of parameter values of the system (1.3) at which chaotic solution exist. This system is chaotic for a wide range of the system parameters. In Section 3, we constructed different examples of chaotic conservative complex nonlinear systems, and their Lyapunov exponents are calculated. The hyperchaotic conservative complex nonlinear system (2.1) has been studied in Section 4. Finally, concluding remarks were given in the last Section.

2. THE CHAOTIC CONSERVATIVE COMPLEX SYSTEM (1.3)

In this section, we study the basic properties of the proposed chaotic conservative complex nonlinear system (1.3).

The real version of (1.3) which is a 4D real chaotic autonomous conservative system is:

$$\begin{aligned}\dot{u}_1 &= \alpha u_3, \\ \dot{u}_2 &= \alpha u_4, \\ \dot{u}_3 &= \beta(u_1^2 - u_2^2) - \gamma(u_1^3 + u_1 u_2^2), \\ \dot{u}_4 &= -\beta(2u_1 u_2) - \gamma(u_2^3 + u_2 u_1^2).\end{aligned}\tag{2.1}$$

System (1.3) is conservative since

$$\frac{\partial \dot{u}_1}{\partial u_1} + \frac{\partial \dot{u}_2}{\partial u_2} + \frac{\partial \dot{u}_3}{\partial u_3} + \frac{\partial \dot{u}_4}{\partial u_4} = 0.\tag{2.2}$$

(2.1) Lyapunov exponents of system (1.3)

In this subsection we calculate Lyapunov exponents of system (1.3). Based on these exponents, we show that our system (1.3) is a chaotic one.

System (2.2) in vector notation can be written as:

$$\dot{U}(t) = h(U(t); \zeta),\tag{2.3}$$

where $U(t) = [u_1(t), u_2(t), u_3(t), u_4(t)]^T$ is the state space vector, $h = [h_1, h_2, h_3, h_4]^T$, ζ is a set of parameters and $[...]^T$ denoting transpose. The equations for small deviations δU from the trajectory $U(t)$ are:

$$\delta \dot{U}(t) = L_{ij}(U(t); \zeta) \delta U, \quad i, j = 1, 2, 3, 4\tag{2.4}$$

where $L_{i,j} = \frac{\partial h_i}{\partial u_j}$ is the Jacobian matrix of the form:

$$L_{i,j} = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ 2\beta u_1 - \gamma(3u_1^2 + u_2^2) & -2\beta u_2 - 2\gamma u_1 u_2 & 0 & 0 \\ -2\beta u_2 - 2\gamma u_1 u_2 & -2\beta u_1 - \gamma(3u_2^2 + u_1^2) & 0 & 0 \end{pmatrix}.\tag{2.5}$$

The Lyapunov exponents λ_i of the system is defined by:

$$\lambda_i = \lim_{t \rightarrow \infty} \frac{1}{t} \log \frac{\|\delta u_i(t)\|}{\|\delta u_i(0)\|}.\tag{2.6}$$

To find λ_i , Eqs. (2.3) and (2.4) must be numerically solved simultaneously. Runge-Kutta method of order 4 is used to calculate λ_i .

For the choice $\alpha = 10$, $\beta = 3$, and $\gamma = 2$ and the initial conditions $t_0 = 0$, $u_1(0) = 0.5$, $u_2(0) = 0.5$, $u_3(0) = 1$, and $u_4(0) = 0.0$, we calculate the Lyapunov exponents which are: $\lambda_1 = 2.1004$, $\lambda_2 = 0.0$, $\lambda_3 = 0.0$ and $\lambda_4 = -2.1004$.

This means that our system (1.3) for this choice of α , β , and γ has chaotic behavior since one of its Lyapunov exponents λ_1 is positive and conservative system since the sum of its Lyapunov exponents is zero ($\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$). It is clear that from the calculation of Lyapunov exponents that λ_1 equals λ_4 with negative sign.

(2.2) The range of system parameters of system (1.3)

In this subsection we calculate the range of the parameters α , β and γ at which chaotic behavior exist based on the calculations of Lyapunov exponents (2.6). The Lyapunov exponents versus the system parameters are plotted in Figure 1 using the initial conditions $t_0 = 0$, $u_1(0) = 0.5$, $u_2(0) = 0.5$, $u_3(0) = 1$, and $u_4(0) = 0$.

(i) Fix $\beta = 3$, $\gamma = 2$, and vary α :

In Figure 1a we plotted the corresponding Lyapunov exponents λ_i , $i = 1, 2, \dots, 4$ of system (1.3) versus α . It is clear that from this Figure when $\alpha \in (0, 50]$ system (1.3) has chaotic solutions.

(ii) Fix $\alpha = 10$, $\gamma = 2$, and vary β :

One can concludes from Figure 1b, that system (1.3) exhibits chaotic solutions for $\beta \in [3, 5.8]$, $[7.3, 7.45]$, and $[7.8, 11]$, where λ_1 is positive in these intervals .

(iii) Fix $\alpha = 10$, $\beta = 3$, and vary γ :

As we did for α , β , we plotted λ_i , versus γ in Figure 1c and we see that (1.3) has chaotic solutions for $\gamma \in [0.56, 0.98]$, $[1.54, 10]$.

To show the chaotic solutions of system (1.3), we plotted them in different projections in 3-spaces as shown in Figure 2 and 2-planes as in Figure 3. In Figure 3g, we plotted the module M_{z_1} - M_{z_2} plane where $M_{z_1} = \sqrt{u_1^2 + u_2^2}$, and $M_{z_2} = \sqrt{u_3^2 + u_4^2}$.

3. DIFFERENT EXAMPLES OF CHAOTIC COMPLEX SYSTEMS

In this section we can construct different examples of the chaotic conservative complex nonlinear systems. We present here, three examples and calculate their Lyapunov exponents with the same initial conditions of Figure 1 as:

$$\begin{aligned} \dot{z}_1 &= \alpha z_2, \\ \dot{z}_2 &= \bar{z}_1^2 - z_1^2 \bar{z}_1. \end{aligned} \quad (3.1)$$

For the case $\alpha = 10$, the Lyapunov exponents for system (3.1) are:
 $\lambda_1 = 0.5713$, $\lambda_2 = 0.0$, $\lambda_3 = 0.0$, $\lambda_4 = -0.5713$.

$$\begin{aligned} \dot{z}_1 &= \alpha z_2, \\ \dot{z}_2 &= \beta \bar{z}_1^2 - z_1^2 \bar{z}_1. \end{aligned} \quad (3.2)$$

For the case $\alpha = 10$, $\beta = 2$, the Lyapunov exponents for system (3.2) are:
 $\lambda_1 = 1.8522$, $\lambda_2 = 0.0$, $\lambda_3 = 0.0$, $\lambda_4 = -1.8522$.

$$\begin{aligned} \dot{z}_1 &= \alpha z_2, \\ \dot{z}_2 &= \bar{z}_1^2 - \beta z_1^2 \bar{z}_1. \end{aligned} \quad (3.3)$$

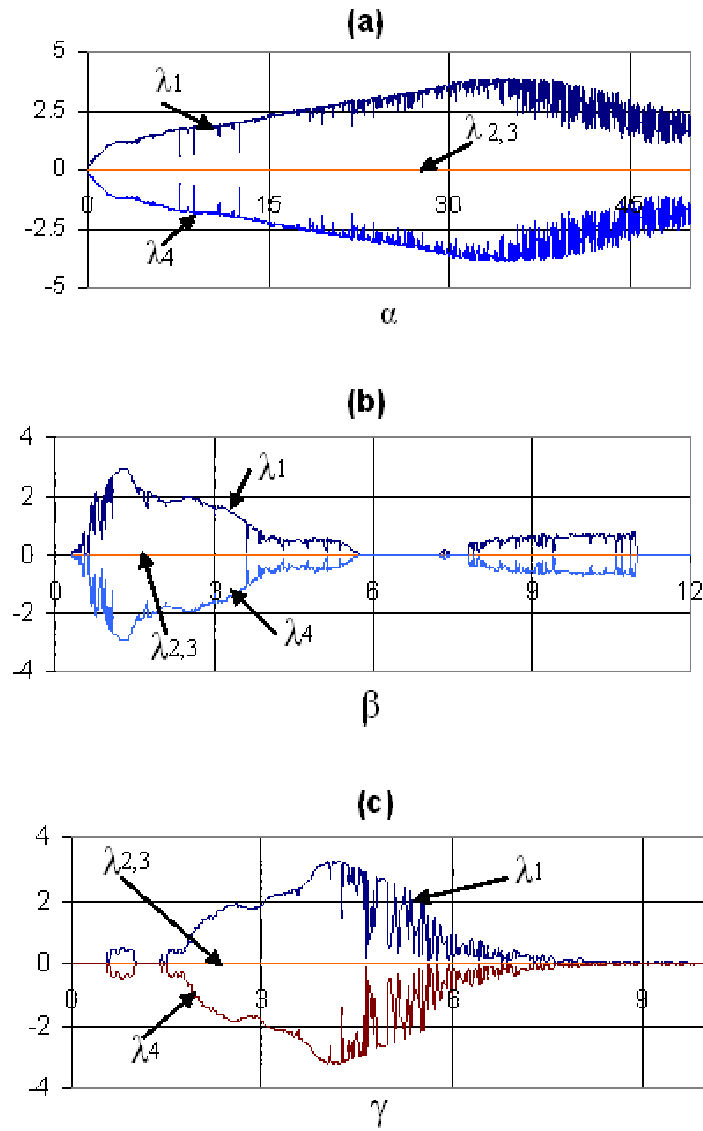


FIGURE 1. Lyapunov exponents of system (1.3) for the choice $\alpha = 10$, $\beta = 3$, and $\gamma = 2$ and the initial conditions $t_0 = 0$, $u_1(0) = 0.5$, $u_2(0) = 0.5$, $u_3(0) = 1$, and $u_4(0) = 0.0$, (a) λ_1 , λ_2 , λ_3 and λ_4 versus α , (b) λ_1 , λ_2 , λ_3 and λ_4 versus β , (c) λ_1 , λ_2 , λ_3 and λ_4 versus γ .

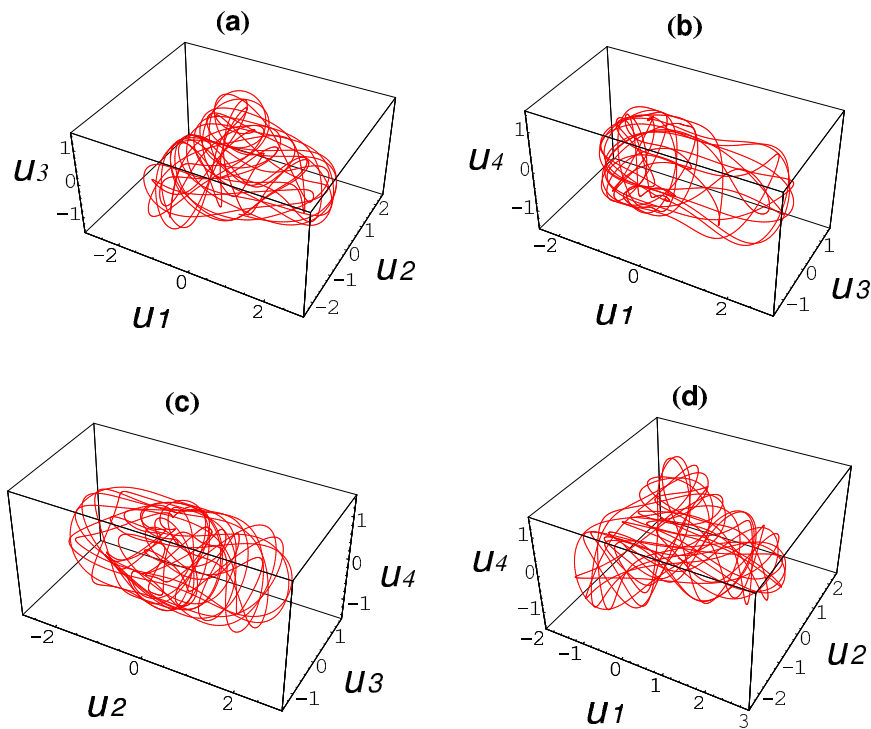


FIGURE 2. The chaotic solution of system (1.3) with the same initial conditions as in Figure 1 for the choice $\alpha = 10$, $\beta = 3$, and $\gamma = 2$. (a) (u_1, u_2, u_3) space, (b) (u_1, u_3, u_4) space, (c) (u_2, u_3, u_4) space, (d) (u_1, u_2, u_4) space.

For the case $\alpha = 6$, $\beta = 3$, the Lyapunov exponents for system (3.3) are:

$$\lambda_1 = 0.3650, \lambda_2 = 0.0, \lambda_3 = 0.0, \lambda_4 = -0.3650.$$

As we had for system (1.3), λ_1 and λ_4 are the same but with negative sign for our systems (3.1), (3.2) and (3.3). Therefore 2D complex systems may be not hyperchaotic ones. The systems of this section leave rooms for further investigations, for example,

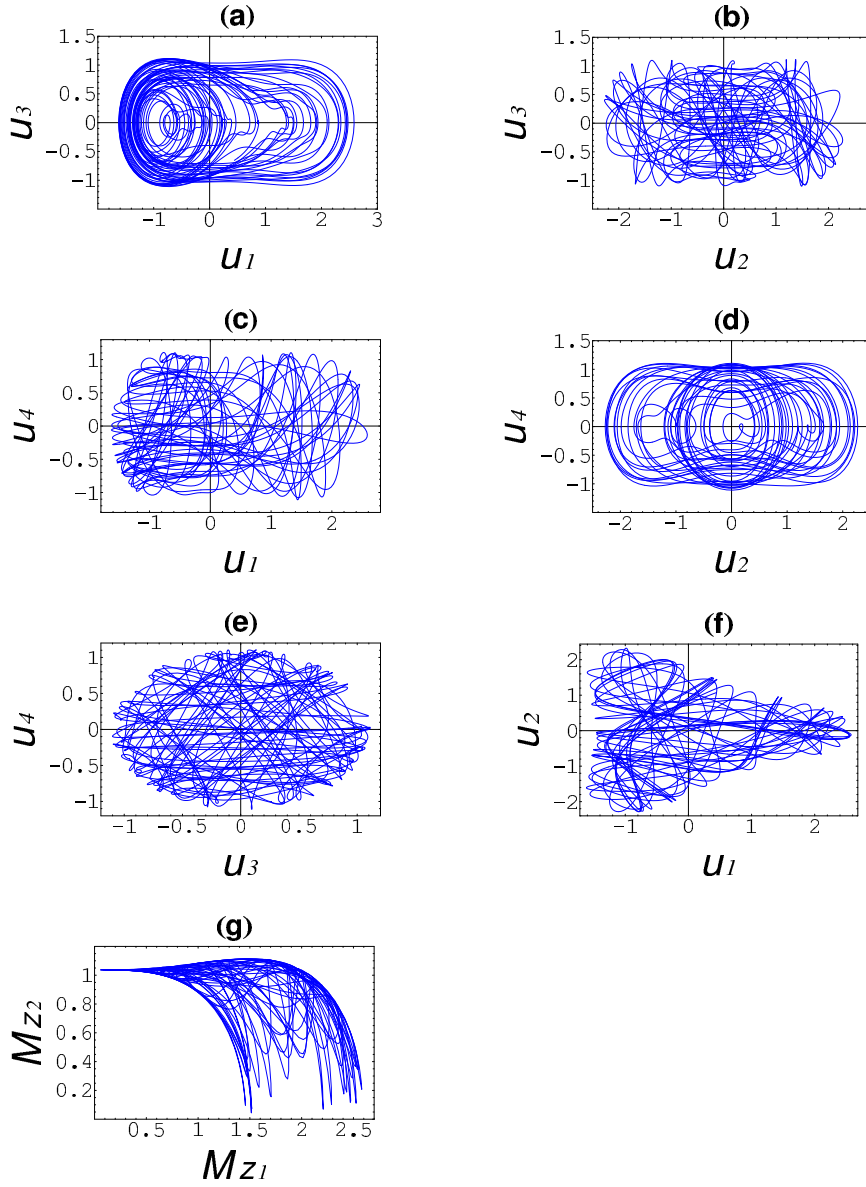


FIGURE 3. The chaotic solution of system (1.3) with the same initial conditions as in Figure 1 for $\alpha = 10$, $\beta = 3$, and $\gamma = 2$. (a) $u_1 - u_3$ plane, (b) $u_2 - u_3$ plane, (c) $u_1 - u_4$ plane, (d) $u_2 - u_4$ plane, (e) $u_3 - u_4$ plane, (f) $u_1 - u_2$ plane, (g) $M_{z1} - M_{z2}$ plane.

control and several types of synchronization (complete , modified projective and generalized synchronization) of solutions of these systems.

4. THE HYPERCHAOTIC CONSERVATIVE COMPLEX SYSTEM (2.1)

This section is devoted to analyze the hyperchaotic conservative complex nonlinear system (2.1).

The real version of (2.1) reads:

$$\begin{aligned}\dot{u}_1 &= u_3, \\ \dot{u}_2 &= u_4, \\ \dot{u}_3 &= -\mu u_1 + u_3 u_5 - u_4 u_6, \\ \dot{u}_4 &= -\mu u_2 + u_3 u_6 + u_4 u_5, \\ \dot{u}_5 &= 1 - (u_3^2 - u_4^2), \quad \dot{u}_6 = -2u_3 u_4.\end{aligned}\tag{4.1}$$

(4.1) Lyapunov exponents of system (2.1)

Based on Lyapunov exponents, we compute parameters values of our system (2.1) at which hyperchaotic solutions exist. The Jacobian matrix of our system (4.2) is of the form:

$$L_{i,j} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\mu & 0 & u_5 & -u_6 & u_3 & -u_4 \\ 0 & -\mu & u_6 & u_5 & u_4 & u_3 \\ 0 & 0 & -2u_3 & 2u_4 & 0 & 0 \\ 0 & 0 & -2u_4 & -2u_3 & 0 & 0 \end{pmatrix}.\tag{4.2}$$

For the choice of $\mu = 1.5$, and the initial conditions $t_0 = 0$, $u_1(0) = 0$, $u_2(0) = 0$, $u_3(0) = 5$, $u_4(0) = 0$, $u_5(0) = 0$, $u_6(0) = 0$, we calculate the Lyapunov exponents as $\lambda_1 = 0.055$, $\lambda_2 = 0.055$, $\lambda_3 = 0.0$, $\lambda_4 = 0.0$, $\lambda_5 = -0.055$, $\lambda_6 = -0.055$. This means that our system (2.1) is hyperchaotic (λ_1, λ_2 are positive) and conservative system since the sum of its Lyapunov exponents is zero ($\lambda_1 + \lambda_2 + \dots + \lambda_6 = 0$).

In Figure 4a we plotted the corresponding Lyapunov exponents λ_i , $i = 1, 2, \dots, 6$ of system (2.1) versus μ using the initial conditions $t_0 = 0$, $u_1(0) = 0$, $u_2(0) = 0$, $u_3(0) = 5$, $u_4(0) = 0$, and $u_5(0) = 0$. It is clear that from Figure 4a, when $\mu \in (0.4, 1.82]$, $(1.85, 1.92]$ and $(2.58, 2.81]$ the system (2.1) has hyperchaotic solutions since $\lambda_1 = \lambda_2 > 0$, $\lambda_3 = \lambda_4 = 0$ and $\lambda_5 = \lambda_6 < 0$. In Figures 4b, 4c, 4d, 4e we plotted hyperchaotic solutions in 3-spaces. Figure 5 contains hyperchaotic solutions in 3-spaces. The modules of the complex variables z_1 , z_2 and z_3 , are displayed in 2-planes in Figures 5d, 5e and 5f.

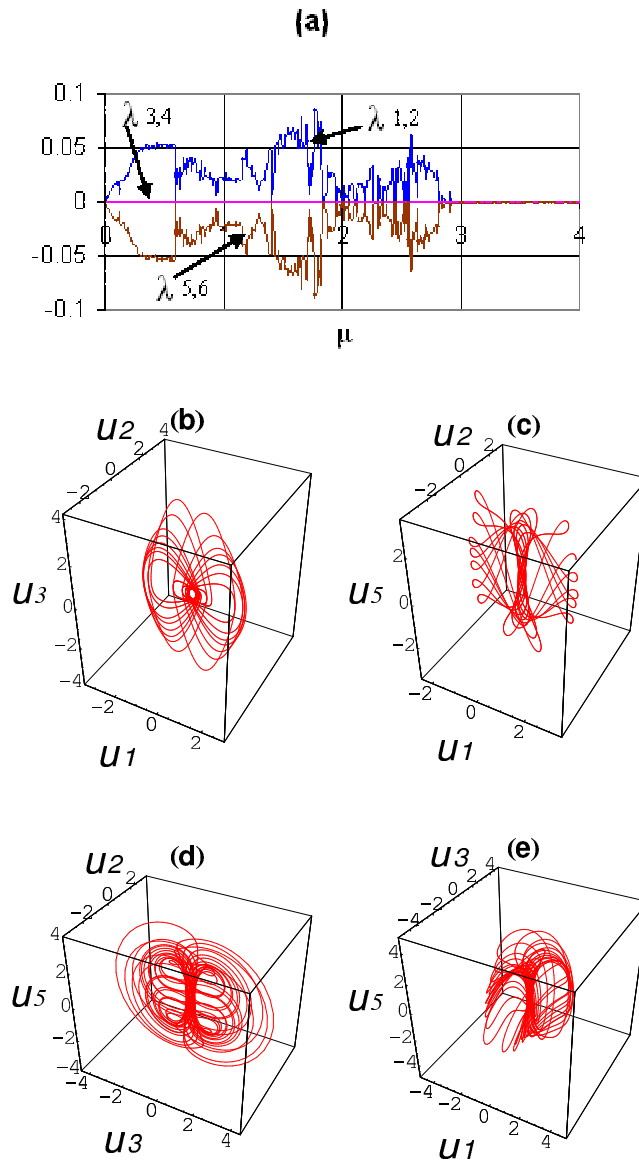


FIGURE 4. Lyapunov exponents and the hyperchaotic solution of (2.1) for $\mu = 1.5$ and the initial conditions $t_0 = 0, u_1(0) = 0, u_2(0) = 0, u_3(0) = 5, u_4(0) = 0, u_5(0) = 0, u_6(0) = 0$. (a) $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$, and λ_6 versus μ , (b) (u_1, u_2, u_3) space, (c) (u_1, u_2, u_5) space, (d) (u_3, u_2, u_5) space, (e) (u_1, u_3, u_5) space.

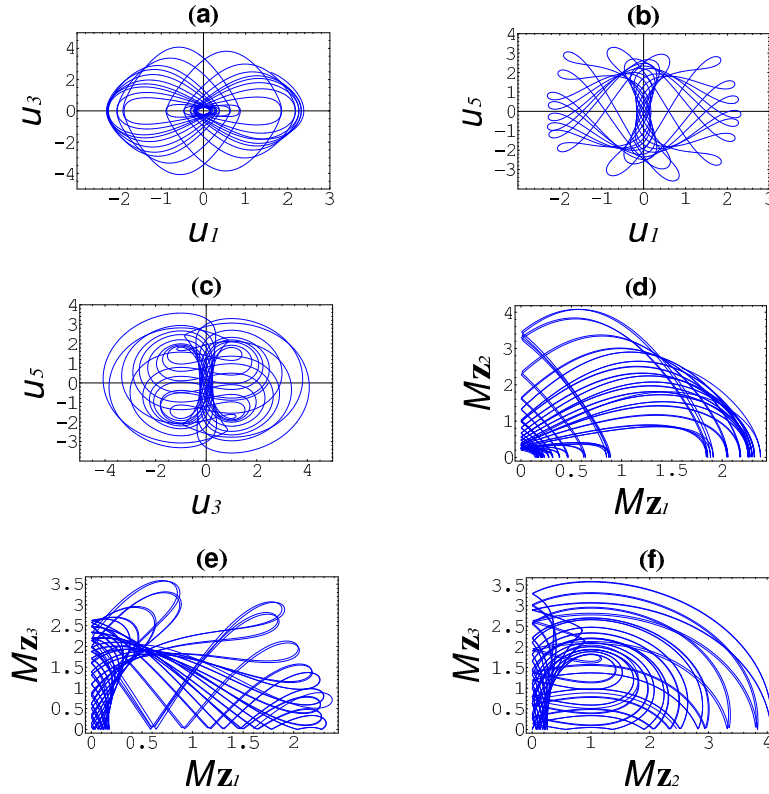


FIGURE 5. A hyperchaotic solution of system (2.1) with the same initial conditions as in Figure 4 and for $\mu = 1.5$: (a) $u_1 - u_3$ plane, (b) $u_1 - u_5$ plane, (c) $u_3 - u_5$ plane, (d) $M_{z_1} - M_{z_2}$ plane, (e) $M_{z_1} - M_{z_3}$ plane, (f) $M_{z_2} - M_{z_3}$ plane.

5. CONCLUSIONS

Our main goal in this paper was to introduce and analyze chaotic and hyperchaotic conservative complex nonlinear systems (1.3), (2.1), (3.1), (3.2) and (3.3). The sum of their Lyapunov exponents is zero. These systems have not yet been actively explored in the literature. We calculated numerically the values of the parameters at which chaotic and hyperchaotic solutions of system (1.3) and (2.1) exist in Figures 1 and 4a respectively. These solutions exist for large values of system parameters, for example $\alpha \in (0.0, 50]$ and $\gamma \in (1.54, 10]$. The modules of the complex variables z_1 , z_2 and z_3 of our systems (1.3) and (2.1) are displayed in Figure 3g, 5d, 5e, and 5f. Control and synchronization (e.g. complete synchronization, anti-synchronization, projective and modified projective synchronization and

lag synchronization) of chaotic and hyperchaotic solutions of our systems, need further investigations and will appear in the near future.

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ON POLYNOMIAL IDENTITIES FOR RECURSIVE SEQUENCES

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Abstract. In this paper we extend the notion of Melham sum to the Pell and Pell-Lucas sequences. While the proofs of general statements rely on the binomial theorem, we prove some spacial cases by the known Pell identities. We also give extensions of obtained expressions to the other recursive sequences.

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1. INTRODUCTION

The Pell sequence $(P_n)_{n \geq 0}$ and the Pell-Lucas sequence $(Q_n)_{n \geq 0}$ are defined as the second order recurrences,

$$P_{n+2} = 2P_{n+1} + P_n, \quad P_0 = 0, \quad P_1 = 1 \quad (1.1)$$

$$Q_{n+2} = 2Q_{n+1} + Q_n, \quad Q_0 = 2, \quad Q_1 = 2. \quad (1.2)$$

Equivalently, these sequences can be defined as the solutions of Diophantine equations

$$x^2 - dy^2 = \pm 1$$

for $d = 2$. More precisely, the pairs $(Q_n/2, P_n)$ are all solutions of these equations. The n -th term of the Pell sequence can also be expressed by the closed form equation. The Pell-Lucas sequence is sometimes called *companion Pell sequence* and there is also similar closed form for this sequence. We let γ denote the silver ratio, $\gamma := 1 + \sqrt{2}$ and we set $\delta := 1 - \sqrt{2}$. Then the closed formula for Pell sequence can be written as

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad (1.3)$$

while for the companion Pell numbers we have $Q_n = \gamma^n + \delta^n$.

Both the Pell sequence and Pell equation are the subject of numerous papers. Among the most remarkable theoretical number properties let us mention the facts that

- P_{2n+1} divides the sum $\sum_{k=0}^{2n} P_{2k+1}$,

- P_{2n} divides the sum $\sum_{k=1}^{2n} P_{2k-1}$,

and sum of the first $4n + 1$ Pell numbers $\sum_{k=1}^{4n+1} P_k$ is a perfect square [7]. Furthermore, in [6] Dujella found quadruples of the Pell and Pell-Lucas numbers that have the *property of Diophantus* of a certain order. In particular, the quadruples

$$\{P_{2n}, P_{2n+2}, 2P_{2n}, Q_{2n}P_{2n+1}Q_{2n+1}\},$$

$$\{P_{2n}, P_{2n+2}, 2P_{2n+2}, P_{2n+1}Q_{2n+1}Q_{2n+2}\}$$

have the property of Diophantus of order 1, meaning that $a_i a_j + 1$, is a perfect square where $a_i, a_j, i \neq j$ are the elements of a quadruple.

Some recent surveys on the Pell equation one can find in [1, 3].

There are also many known combinatorial properties and identities for Pell and Pell-Lucas sequences [4, 5, 12]. This includes several identities encountering both of the sequences,

$$Q_n = P_{n-1} + P_{n+1} \quad (1.4)$$

being the basic one. Recall that the Cassini identity [11] for Pell numbers has form

$$P_{n-1}P_{n+1} - P_n^2 = (-1)^n. \quad (1.5)$$

An elegant proof is based on the fact that

$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}^n = \begin{pmatrix} P_{n-1} & P_n \\ P_n & P_{n+1} \end{pmatrix},$$

which can be proved by induction. When applying the Cauchy-Binet theorem for determinants, the statement follows immediately. We will also use relation

$$P_{m+n} = P_{m-1}P_n + P_mP_{n+1}, \quad (1.6)$$

for the purpose to prove some polynomial identities for Pell numbers. Identity (1.6) can be proved by induction.

This paper aims at finding Pell identities and polynomial representation for the Pell numbers. In what follows, firstly we prove that $(2m + 1)n$ -th Pell number is represented as a polynomial in P_n . Then we extend the notion of Melham sum [10] to the Pell and Pell-Lucas sequences and find related expansions into the power series of P_n , where exponents are odd. Finally, we give extensions of the obtained identities for a certain, more general class of recursive sequences.

2. THE $(2m + 1)n$ -TH PELL NUMBER AS A POLYNOMIAL IN P_n

Proposition 1. *For the Pell sequence $(P_n)_{n \geq 0}$ we have*

$$i) \quad P_{3n} = 8P_n^3 + 3(-1)^n P_n \quad (2.1)$$

$$ii) \quad P_{5n} = 64P_n^5 + 40(-1)^n P_n^3 + 5P_n. \quad (2.2)$$

Proof. According to relations (1.5) and (1.6) we get

$$\begin{aligned} P_{3n} &= P_{2n+n} = P_{2n-1}P_n + P_{2n}P_{n+1} \\ &= P_{n-1}^2P_n + P_n^3 + (P_n - 1P_n + P_nP_{n+1})(2P_n + P_{n-1}) \\ &= P_n(P_{n-1}^2 + 2P_nP_{n-1} + P_n^2 + 2P_nP_{n+1} + P_{n-1}^2 + P_n^2 + (-1)^n) \end{aligned}$$

which finally gives

$$\begin{aligned} P_{3n} &= P_n(3P_n^2 + 2(-1)^2 + 2P_nP_{n+1} + P_{n-1}^2) \\ &= P_n(3P_n^2 + 2(-1)^2 + 4P_n^2 + P_{n-1}(2P_n + P_{n-1})) = P_n(8P_n^2 + 3(-1)^n). \end{aligned}$$

Application of the same relations also proves identities for P_{5n} . $\square$

Furthermore, for the next instance when n is odd we have

$$P_{7n} = 512P_n^7 - 448P_n^5 + 112P_n^3 - 7P_n \quad (2.3)$$

while all coefficients are positive when n is even.

Theorem 1. For the Pell sequence $(P_n)_{n \geq 0}$

$$P_{(2m+1)n} = \sum_{i=0}^m (-1)^{n(m+i)} 2^{3i} \frac{2m+1}{2i+1} \binom{m+i}{2i} P_n^{2i+1}. \quad (2.4)$$

Proof. We use equalities (2.5) and (2.6), which are results of D. Jennings available in [9] and which can be proved by induction.

$$\begin{aligned} \left(x^{2m} + \frac{1}{x^{2m}}\right) + \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + \left(x^2 + \frac{1}{x^2}\right) + 1 \\ = \sum_{i=0}^m \frac{2m+1}{m+i+1} \binom{m+i+1}{2i+1} \left(x - \frac{1}{x}\right)^{2i} \end{aligned} \quad (2.5)$$

$$\begin{aligned} \left(x^{2m} + \frac{1}{x^{2m}}\right) - \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + (-1)^{m+1} \left(x^2 + \frac{1}{x^2}\right) + (-1)^m \\ = \sum_{i=0}^m (-1)^{m+i} \frac{2m+1}{m+i+1} \binom{m+i+1}{2i+1} \left(x + \frac{1}{x}\right)^{2i} \end{aligned} \quad (2.6)$$

Having in mind Binet formula for the Pell numbers (1.3) and the fact that

$$\gamma \cdot \delta = -1 \quad (2.7)$$

$$\gamma - \delta = 2\sqrt{2} \quad (2.8)$$

we have

$$\frac{P_{pn}}{P_n} = \frac{\gamma^{pn} - \delta^{pn}}{\gamma^n - \delta^n} = x^{p-1} + x^{p-2}y + \cdots + xy^{p-2} + y^{p-1}, \quad (2.9)$$

where $x = \gamma^n$ and $y = \delta^n = \frac{(-1)^n}{x}$. When p is odd, the r.h.s. of (2.9) reduces to

$$\left(x^{p-1} + \frac{1}{x^{p-1}}\right) + (-1)^n \left(x^{p-1} + \frac{1}{x^{p-1}}\right) + \cdots + \left(x^{p-1} + \frac{1}{x^{p-1}}\right) + (-1)^n \quad (2.10)$$

when $p \equiv 3 \pmod{4}$ or to

$$\left(x^{p-1} + \frac{1}{x^{p-1}}\right) + (-1)^n \left(x^{p-1} + \frac{1}{x^{p-1}}\right) + \cdots + (-1)^n \left(x^{p-1} + \frac{1}{x^{p-1}}\right) + 1 \quad (2.11)$$

when $p \equiv 1 \pmod{4}$. Now, we have

$$x + \frac{1}{x} = \gamma^n + \frac{1}{\gamma^n} = \gamma^n + (-1)^n \delta^n$$

which gives

$$x + \frac{1}{x} = (\gamma - \delta)P_n, \quad n \equiv 1 \pmod{2} \quad (2.12)$$

$$x - \frac{1}{x} = (\gamma - \delta)P_n, \quad n \equiv 0 \pmod{2} \quad (2.13)$$

and furthermore

$$\left(x + \frac{1}{x}\right)^2 = 8P_n^2, \quad n \equiv 1 \pmod{2} \quad (2.14)$$

$$\left(x - \frac{1}{x}\right)^2 = 8P_n^2, \quad n \equiv 0 \pmod{2} \quad (2.15)$$

Since we get expression (2.10) assuming that p is odd we now substitute $p = 2m + 1$. Now, when n is even we obtain all positive terms in (2.10) and then r.h.s. of (2.9) is equal to the l.h.s. of equality (2.5),

$$P_{(2m+1)n} = \sum_{i=0}^m (-1)^{n(m+i)} 2^{3i} \frac{2m+1}{m+i+1} \binom{m+i+1}{2i+1} P_n^{2i+1}. \quad (2.16)$$

Analogue reasoning when n is odd gives the same relation, thus (2.16) holds true for any natural number n . Finally, a simple manipulation with (2.16) leads to the final form of the theorem. $\square$

One can easily see that relations (2.1), (2.2) and (2.3) appear from Theorem 1 for $m=1, 2$ and 3 , respectively. When $m=4$ Theorem 1 gives

$$P_{9n} = 2^{12} P_n^9 - 9 \cdot 2^9 P_n^7 + 1728 P_n^5 - 240 P_n^3 + 9 P_n \quad (2.17)$$

when n is odd while all coefficients are positive otherwise. Note that the leading coefficient in (2.4) is always a power of 2, 2^{3m} , while the absolute value of the coefficient in the term of the smallest degree is $2m+1$.

3. MELHAM SUM FOR THE PELL AND PELL-LUCAS SEQUENCE

Proposition 2. *Twice the sum of the Pell numbers having even indexes from 2 to n is equal to the $(2n+1)$ -st Pell number diminished by 1,*

$$1 + 2 \sum_{k=1}^n P_{2k} = P_{2n+1}. \quad (3.1)$$

Proof. The statement follows immediately from defining properties of Pell sequence,

$$\begin{aligned} P_{2n+1} &= 2P_{2n} + P_{2n-1} \\ &= 2P_{2n} + 2P_{2n-2} + P_{2n-3} \\ &= 2P_{2n} + 2P_{2n-2} + \cdots + 2P_2 + P_1. \end{aligned}$$

□

Note that relation (3.1) can be seen as the expansion of the expression $Q_1 \sum_{k=1}^n P_{2k}$ into polynomial in P_{2n+1} ,

$$Q_1 \sum_{k=1}^n P_{2k} = P_{2n+1} - 1.$$

In what follows we extend this idea to full generality. The expression

$$Q_1 Q_2 \cdots Q_{2m+1} \sum_{k=1}^n P_{2k}^{2m+1},$$

we shall call the *Melham sum for Pell and Pell-Lucas sequences*, because there is analogy with established term for Fibonacci and Lucas sequences. More on the Fibonacci sequence one can find in the classic book by S. Vajda. Introduction to Fibonacci polynomials one can find in [8], and some recent development in [2].

Lemma 1. *For the sequences $(P_n)_{n \geq 0}$, $(Q_n)_{n \geq 0}$ and $m \in \mathbb{N}$*

$$Q_m \sum_{k=1}^n P_{2mk} = P_{m(2n+1)} - P_m. \quad (3.2)$$

Proof. By relation (1.6) we have

$$\begin{aligned} P_{m+n} &= (P_{m-n} - P_n P_{n-1} (-1)^n) (-1)^{n+1} + P_m P_{n-1} \\ &= P_{m-1} (-1)^{n+1} + P_m P_{n-1} + P_m P_{n+1} \\ &= P_{m-1} (-1)^{n+1} + P_m (P_{n-1} + P_{n+1}) \\ &= P_m Q_n - (-1)^n P_{m-n}. \end{aligned}$$

Now we prove the statement of lemma by induction where this result is used in a step of induction. Thus, from the fact that the statement holds true for $n = 1$ we have to derive equality $Q_n \sum_{k=1}^{n+1} P_{2mk} = P_{m(2n+3)} - P_m$. We have

$$\begin{aligned} Q_n \sum_{k=1}^{n+1} P_{2mk} &= Q_n \left(\sum_{k=1}^n P_{2mk} + P_{2m(n+1)} \right) \\ &= P_{m(2n+1)} - P_m + Q_n P_{2m(n+1)} \\ &= P_{m(2n+1)} + Q P_{2m(n+1)} - P_m \\ &= P_{2m(n+1)+m} - P_m \\ &= P_{m(2n+3)} - P_m, \end{aligned}$$

which completes the statement of the lemma. $\square$

Lemma 2. For the sequences $(P_n)_{n \geq 0}$, $(Q_n)_{n \geq 0}$ and $m \in \mathbb{N}$

$$P_n^{2m+1} = \frac{1}{2^{3m}} \sum_{j=0}^m (-1)^{j(n+1)} \binom{2m+1}{j} P_{(2m+1-2j)n}. \quad (3.3)$$

Proof. By means of binomial theorem and using (2.7) as well as (2.8) we have

$$\begin{aligned} P_n^{2m+1} &= \left(\frac{\gamma^n - \delta^n}{\gamma - \delta} \right)^{2m+1} \\ &= \frac{1}{(\gamma - \delta)^{2m+1}} \sum_{j=0}^{2m+1} (-1)^{j+1} \binom{2m+1}{j} \gamma^{jn} \delta^{(2m+1-j)n} \\ &= \frac{1}{8^m (\gamma - \delta)} \sum_{j=0}^m (-1)^j \binom{2m+1}{j} (\gamma^{(2m+1-j)n} \delta^{jn} - \gamma^{jn} \delta^{(2m+1-j)n}) \\ &= \frac{1}{2^{3m}} \sum_{j=0}^m (-1)^j \binom{2m+1}{j} \gamma^{jn} \delta^{jn} \left(\frac{\gamma^{(2m+1-2j)n} - \delta^{(2m+1-2j)n}}{\gamma - \delta} \right) \\ &= \frac{1}{2^{3m}} \sum_{j=0}^m (-1)^{j(n+m)} P_{(2m+1-2j)n} \end{aligned}$$

which completes the statement of lemma. $\square$

Theorem 2. For $m \in \mathbb{N}$ and the sequences $(P_n)_{n \geq 0}$, $(Q_n)_{n \geq 0}$

$$\sum_{k=1}^n P_{2k}^{2m+1} = \frac{1}{2^{3m}} \sum_{j=0}^m \frac{(-1)^j}{Q_{2m+1-2j}} \binom{2m+1}{j} (P_{(2m+1-2j)(2n+1)} - P_{2m+1-2j}). \quad (3.4)$$

Proof. In Lemma 2 we substitute $n = 2k$ and then sum both sides of equality from $k = 1$ through n . It follows

$$P_{2k}^{2m+1} = \frac{1}{2^{3m}} \sum_{j=0}^m (-1)^j \binom{2m+1}{j} \sum_{k=1}^n P_{(2m+1-2j)2k}.$$

When we substitute $\sum_{k=1}^n P_{(2m+1-2j)2k}$ by the expression in Lemma 1, the proof is completed. $\square$

Theorem 3. For $m \in \mathbb{N}$ and sequences $(P_n)_{n \geq 0}$, $(Q_n)_{n \geq 0}$

$$\begin{aligned} \sum_{k=1}^n P_{2k}^{2m+1} &= \sum_{i=0}^m P_{2n+1}^{2i+1} \sum_{j=0}^{m-i} \frac{(-1)^{m+i} 2^{3(i-m)} (2m-2j+1) \binom{2m+1}{j} \binom{m-j+i}{2i}}{Q_{2m+1-2j} (2i+1)} \\ &\quad + \sum_{j=0}^m \frac{(-1)^{j+1} P_{2m+1-2j} \binom{2m+1}{j}}{2^{3m} Q_{2m+1-2j}}. \end{aligned} \quad (3.5)$$

Proof. When substitute m with $m-j$ and n with $2n+1$ in Theorem 1 one get

$$P_{(2m+1-2j)(2n+1)} = \sum_{i=0}^{m-j} (-1)^{(2n+1)(m-j+i)} 2^{3i} \frac{2m-2j+1}{2i+1} \binom{m-j+i}{2i} P_{2n+1}^{2i+1}.$$

We substitute this expression in Theorem 2 and the statement follows immediately. $\square$

Now we consider some particular cases of Theorem 3. When $m = 1$ we obtain

$$\sum_{k=1}^n P_{2k}^3 = \frac{1}{14} (P_{2n+1}^3 - 3P_{2n+1} + 2).$$

When multiply this relation with $Q_1 Q_3$ we get polynomial identity for the Melham sum in case $m = 1$

$$Q_1 Q_3 \sum_{k=1}^n P_{2k}^3 = 2P_{2n+1}^3 - 6P_{2n+1} + 4. \quad (3.6)$$

The next case, when $m = 2$ gives

$$Q_1 Q_3 Q_5 \sum_{k=1}^n P_{2k}^5 = 28P_{2n+1}^5 - 120P_{2n+1}^3 + 220P_{2n+1} - 128. \quad (3.7)$$

4. FURTHER EXTENSIONS

Given $s, t \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we define the second order recurrence with the relation

$$a_{n+2} = sa_{n-1} + ta_n \quad (4.1)$$

and initial values a_0 and a_1 . We say that a sequence $(a_n)_{n \geq 0}$ is a solution of (4.1) if its terms satisfies this recurrence. Here we consider a class of (4.1) defined by $t = 1$

and initial terms $a_0 = 0$, $a_1 = 1$. We let $(A_n)_{n \geq 0}$ denote the sequence defined by this class. It is worth mentioning that two notable representatives of this class are Fibonacci and Pell numbers.

Proposition 3. *For the sequence of numbers $(A_n)_{n \geq 0}$ we have*

$$i) \quad A_{3n} = (s^2 + 4)A_n^3 + 3(-1)^n A_n \quad (4.2)$$

$$ii) \quad A_{5n} = (s^2 + 4)^2 A_n^5 + 5(s^2 + 4)(-1)^n A_n^3 + 5A_n. \quad (4.3)$$

Proof. By induction we prove that

$$A_{n-1}A_{n+1} - A^2 = (-1)^2 \quad (4.4)$$

and also

$$A_{m+n} = A_{m-1}A_n + A_m A_{n+1}. \quad (4.5)$$

Now we employ (4.5) to get

$$\begin{aligned} A_{3n} &= A_{2n+n} = A_{2n-1}A_n + A_{2n}A_{n+1} \\ &= A_{n-1}^2 A_n + A_n^3 + (A_{n-1}A_n + A_n A_{n+1})(A_2 A_n + A_{n-1}) \\ &= A_n(A_{n-1}^2 + A_n^2 + sA_{n-1}A_n + sA_n A_{n+1} + A_{n-1}^2 + A_{n-1}A_{n+1}). \end{aligned}$$

Having in mind that

$$A_{n-1}^2 + sA_{n-1}A_n = A_n^2 + (-1)^n$$

by (4.4), we obtain

$$A_{3n} = A_n(2A_n^2 + 2(-1)^n + A_n^2 + sA_n A_{n+1} + A_{n-1}^2).$$

When applying again (4.4) to the terms $sA_n A_{n+1}$ and A_{n-1}^2 we finally have

$$\begin{aligned} A_{3n} &= A_n[4A_n^2 + s^2 A_n^2 + 3(-1)^n] \\ &= A_n[(s^2 + 4)A_n^2 + 3(-1)^n]. \end{aligned}$$

The second relation can be proved by analogue calculation. $\square$

Clearly, further identities can be proved in the same fashion as Proposition 3 was proved. Instead, we give a more elegant family of identities (4.6) that generalize Proposition 3. It follows as a corollary of Theorem 1.

Corollary 1. *For $m \in \mathbb{N}$ and the sequence of numbers $(A_n)_{n \geq 0}$ we have*

$$A_{(2m+1)n} = \sum_{i=0}^m (-1)^{n(m+i)} (s^2 + 4)^i \frac{2m+1}{2i+1} \binom{m+i}{2i} A_n^{2i+1}. \quad (4.6)$$

In order to prove Corollary 1 we use the fact that the closed form relation for the terms of sequence $(A_n)_{n \geq 0}$ is

$$A_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where

$$\alpha = \frac{1}{2}(s + \sqrt{s^2 + 4}), \quad \beta = \frac{1}{2}(s - \sqrt{s^2 + 4}).$$

Furthermore, it holds $\alpha \cdot \beta = -1$, $\alpha - \beta = \sqrt{s^2 + 4}$ which generalize relations (2.7) and (2.8) in the proof of Theorem 1. This completes the statement of Corollary 1.

Further generalizations and extensions of expressions presented in this work are also possible.

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ON A GENERALIZATION OF NC-MCCOY RINGS

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Abstract. In the present paper we concentrate on a natural generalization of NC-McCoy rings that is called J-McCoy and investigate their properties. We prove that local rings are J-McCoy. For a ring R , $R[[x]]$ is J-McCoy if and only if R is J-McCoy. Also, for an abelian ring R , we show that R is J-McCoy if and only if eR is J-McCoy, where e is an idempotent element of R . Moreover, we give an example to show that the J-McCoy property does not pass $M_n(R)$, but $S(R, n)$, $A(R, n)$, $B(R, n)$ and $T(R, n)$ are J-McCoy.

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1. INTRODUCTION

Throughout this paper, R denotes an associative ring with identity. For a ring R , $N(R)$, $M_n(R)$ and e_{ij} denote the set of all nilpotent elements in R , the $n \times n$ matrix ring over R , and the matrix with (i, j) -entry 1 and elsewhere 0, respectively. Rege-Chhawchharia [8] called a noncommutative ring R right McCoy if whenever polynomials $f(x) = \sum_{i=0}^n a_i x^i$, $g(x) = \sum_{j=0}^m b_j x^j \in R[x] \setminus \{0\}$ satisfy $f(x)g(x) = 0$, there exists a nonzero element $r \in R$ such that $a_i r = 0$. Left McCoy rings are defined similarly. A number of papers have been written on McCoy property of rings (see, e.g., [1, 4, 6, 7, 9]). The name "McCoy" was chosen because McCoy [6] had noted that every commutative ring satisfies this condition. Victor Camillo, Tai Keun Kwak, and Yang Lee [2] called a ring R right nilpotent coefficient McCoy (simply, right NC-McCoy) if whenever polynomials $f(x) = \sum_{i=0}^n a_i x^i$, $g(x) = \sum_{j=0}^m b_j x^j \in R[x] \setminus \{0\}$ satisfy $f(x)g(x) = 0$, there exists a nonzero element $r \in R$ such that $f(x)r \in N(R)[x]$. Left NC-McCoy rings are defined analogously, and a ring R is called NC-McCoy if it is both left and right NC-McCoy. They proved for a reduced

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ring R and $n \geq 2$, $M_n(R)$ is neither right nor left NC-McCoy, but $T_n(R)$ is a NC-McCoy ring for $n \geq 2$. Moreover, it is shown that R is right NC-McCoy if the polynomial ring $R[x]$ is right NC-McCoy and the converse holds if $N(R)[x] \subset N(R[x])$.

Motivated by the above results, we investigate a generalization of the right NC-McCoy rings. The Jacobson radical is an important tool for studying the structure of noncommutative rings, and denoted by $J(R)$. A ring R is said to be right J-McCoy (respectively left J-McCoy) if for each pair of nonzero polynomials $f(x) = \sum_{i=0}^n a_i x^i$ and $g(x) = \sum_{j=0}^m b_j x^j \in R[x] \setminus \{0\}$ with $f(x)g(x) = 0$, then there exists a nonzero element $r \in R$ such that $a_i r \in J(R)$ (respectively $rb_j \in J(R)$). A ring R is called J-McCoy if it is both left and right J-McCoy. It is clear that NC-McCoy rings are J-McCoy, but the converse is not always true. If R is J-semisimple (namely, $J(R) = 0$), then R is right J-McCoy if and only if R is right McCoy. Moreover, for Artinian rings, the concepts of NC-McCoy and J-McCoy rings are the same.

2. RESULTS

Definition 1. A ring R is said to be right J-McCoy (respectively left J-McCoy) if for each pair of nonzero polynomials $f(x) = \sum_{i=0}^n a_i x^i$ and $g(x) = \sum_{j=0}^m b_j x^j \in R[x] \setminus \{0\}$, $f(x)g(x) = 0$ implies that there exists a nonzero element $r \in R$ with $a_i r \in J(R)$ (respectively $rb_j \in J(R)$). A ring R is called J-McCoy if it is both left and right J-McCoy.

It is clear that NC-McCoy rings are J-McCoy, but the converse is not always true by the following example.

Example 1. Let A be the 3 by 3 full matrix ring over the power series ring $F[[t]]$ over a field F . Let

$B = \{M = (m_{ij}) \in A \mid m_{ij} \in tF[[t]] \text{ for } 1 \leq i, j \leq 2 \text{ and } m_{ij} = 0 \text{ for } i = 3 \text{ or } j = 3\}$
and

$$C = \{M = (m_{ij}) \in A \mid m_{ij} \in F \text{ and } m_{ij} = 0 \text{ for } i \neq j\}.$$

Let R be the subring of A generated by B and C . Let $F = \mathbb{Z}_2$. Note that every element of R is of the form $(a + f_1)e_{11} + f_2e_{12} + f_3e_{21} + (a + f_4)e_{22} + ae_{33}$ for some $a \in F$ and $f_i \in tF[[t]]$ ($i = 1, 2, 3, 4$). Consider two polynomials over R , $f(x) = te_{11} + te_{12}x + te_{21}x^2 + te_{22}x^3$ and $g(x) = -t(e_{21} + e_{22}) + t(e_{11} + e_{12})x \in R[x]$. Then $f(x)g(x) = 0$. If there exists $0 \neq r \in R$ such that $f(x)r \in N(R[x])$, then $r = 0$. Thus R is not right NC-McCoy.

Next we will show that R is right J-McCoy. Let $f(x) = \sum_{i=0}^n M_i x^i$ and $g(x) = \sum_{j=0}^m N_j x^j$ be nonzero polynomials in $R[x]$ such that $f(x)g(x) = 0$. Since $M_i = (a_i + f_{i1})e_{11} + f_{i2}e_{12} + f_{i3}e_{21} + (a_i + f_{i4})e_{22} + a_i e_{33}$ for some $a_i \in F$ and $f_{ij} \in tF[[t]]$ ($j = 0, 1, 2, 3, 4$), then for $C = te_{11}$ we have $M_i C = (a_i + f_{i1})te_{11} + f_{i3}te_{21} \in J(R)$. Thus R is right J-McCoy ring.

Proposition 1. *Let R be a ring and I an ideal of R such that R/I is a right (resp. left) J-McCoy ring. If $I \subseteq J(R)$, then R is a right (resp. left) J-McCoy ring.*

Proof. Suppose that $f(x) = \sum_{i=0}^m a_i x^i$ and $g(x) = \sum_{j=0}^n b_j x^j \in R[x] \setminus \{0\}$ such that $f(x)g(x) = 0$. Then $(\sum_{i=0}^m \bar{a}_i x^i)(\sum_{j=0}^n \bar{b}_j x^j) = \bar{0}$ in R/I . Thus there exists $\bar{c} \in R/I$ such that $\bar{a}_i \bar{c} \in J(R/I)$ and so $a_i c \in J(R)$. This means R is right J-McCoy ring. $\square$

Corollary 1. *Let R be any local ring. Then R is J-McCoy.*

The following example shows that, if R is a right J-McCoy ring, then $R/J(R)$ is not necessary right J-McCoy.

Example 2. Let R denote the localization of the ring $\mathbb{Z}$ of integers at the prime ideal (3). Consider the quaternions $\mathbb{Q}$ over R , that is, a free R -module with basis $1, i, j, k$ and multiplication satisfying $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$. Then $\mathbb{Q}$ is a noncommutative domain, and so $J(\mathbb{Q}) = 3\mathbb{Q}$ and $\mathbb{Q}/J(\mathbb{Q})$ is isomorphic to the 2-by-2 full matrix ring over $\mathbb{Z}/(3)$. Thus $\mathbb{Q}$ is a right J-McCoy ring, but $\mathbb{Q}/J(\mathbb{Q})$ is not right J-McCoy.

Proposition 2. *Let R_k be a ring, where $k \in I$. Then R_k is right (resp. left) J-McCoy for each $k \in I$ if and only if $R = \prod_{k \in I} R_k$ is right (resp. left) J-McCoy.*

Proof. Let each R_k be a right J-McCoy ring and $f(x) = \sum_{i=0}^m a_i x^i, g(x) = \sum_{j=0}^n b_j x^j \in R[x] \setminus \{0\}$ such that $f(x)g(x) = 0$, where $a_i = (a_i^{(k)})$, $b_j = (b_j^{(k)})$. If there exists $t \in I$ such that $a_i^{(t)} = 0$ for each $0 \leq i \leq m$, then we have $a_i c = 0 \in J(R)$ where $c = (0, 0, \dots, 1_{R_t}, 0, \dots, 0)$. Now suppose for each $k \in I$, there exists $0 \leq i_k \leq m$ such that $a_{i_k}^{(k)} \neq 0$. Since $g(x) \neq 0$, there exists $t \in I$ and $0 \leq j_t \leq n$ such that $b_{j_t}^{(t)} \neq 0$. Consider $f_t(x) = \sum_{i=0}^m a_i^{(t)} x^i$ and $g_t(x) = \sum_{j=0}^n b_j^{(t)} x^j \in R_t[x] \setminus \{0\}$. We have $f_t(x)g_t(x) = 0$. Thus there exists nonzero $c_t \in R_t$ such that $a_i^{(t)} c_t \in J(R_t)$, for each $0 \leq i \leq m$, since R_t is right J-McCoy ring. Therefore, $a_i(0, 0, \dots, c_t, 0, \dots, 0) \in \prod_{k \in I} J(R_k) = J(R)$, for each $0 \leq i \leq m$. Thus, R is right J-McCoy. Conversely, suppose R is right J-McCoy and $t \in I$. Let $f(x) = \sum_{i=0}^m a_i x^i, g(x) = \sum_{j=0}^n b_j x^j$ be nonzero polynomials in $R_t[x]$ such that $f(x)g(x) = 0$. Set

$$F(x) = \sum_{i=0}^m (0, 0, \dots, 0, a_i, 0, \dots, 0) x^i,$$

$$G(x) = \sum_{j=0}^n (0, 0, \dots, 0, b_j, 0, \dots, 0) x^j \in R[x] \setminus \{0\}.$$

Hence $F(x)G(x) = 0$ and so there exists $0 \neq c = (c_i)$ such that $(0, 0, \dots, 0, a_i, 0, \dots, 0)c \in J(R) = \prod_{k \in I} J(R_k)$. Therefore, $a_i c_t \in J(R_t)$ and so R_t is right J-McCoy $\square$

Corollary 2. *Let D be a ring and C a subring of D with $1_D \in C$. Let*

$$R(C, D) = \{(d_1, \dots, d_n, c, c, \dots) \mid d_i \in D, c \in C, n \geq 1\}$$

with addition and multiplication defined component-wise, $R(D, C)$ is a ring. Then D is right (resp. left) J-McCoy if and only if $R(D, C)$ is right (resp. left) J-McCoy.

Theorem 1. *The class of right (resp. left) J-McCoy rings is closed under direct limits with injective maps.*

Proof. Let $D = \{R_i, \alpha_{ij}\}$ be direct system of right J-McCoy rings R_i , for $i \in I$ and ring homomorphisms $\alpha_{ij} : R_i \rightarrow R_j$ for each $i \leq j$ satisfying $\alpha_{ij}(1) = 1$, where I is a directed partially ordered set. Set $R = \varinjlim R_i$ be a direct limit of D with $L_i : R_i \rightarrow R$ and $L_j \alpha_{ij} = L_i$ where every L_i is injective. We will show that R is an right J-McCoy ring. Take $a, b \in R$. Then $a = L_i(a_i), b = L_j(b_j)$ for some $i, j \in I$ and there is $k \in I$ such that $i \leq k, j \leq k$. Define

$$a + b = L_k(\alpha_{ik}(a_i) + \alpha_{jk}(b_j)) \text{ and } ab = L_k(\alpha_{ik}(a_i)\alpha_{jk}(b_j))$$

where $\alpha_{ik}(a_i)$ and $\alpha_{jk}(b_j)$ are in R_k . Then R forms a ring with $0 = L_i(0)$ and $1 = L_i(1)$. Now let $f(x) = \sum_{i=0}^n a_i x^i$ and $g(x) = \sum_{j=0}^m b_j x^j \in R[x]$ be nonzero polynomials such that $f(x)g(x) = 0$. There is $k \in I$ such that $f(x), g(x) \in R_k[x]$. Hence we get $f(x)g(x) = 0$ in $R_k[x]$. Since R_k is right J-McCoy, there exist $0 \neq c_k$ in R_k such that $a_i c_k \in J(R_k)$. Put $c = L_k(c_k)$. Then $a_i c \in \varinjlim J(R_k) = J(R)$ with a nonzero c in R . Thus R is right J-McCoy ring. $\square$

Theorem 2. *For a ring R , $R[[x]]$ is right (resp. left) J-McCoy if and only if R is right (resp. left) J-McCoy.*

Proof. Let R be a right J-McCoy ring. Since $R \cong \frac{R[[x]]}{\langle x \rangle}$ and $\langle x \rangle \subset J(R[[x]])$, then by Proposition 1, $R[[x]]$ is right J-McCoy. Conversely, assume that $R[[x]]$ is right J-McCoy. Let $f(y) = \sum_{i=0}^n a_i y^i$ and $g(y) = \sum_{j=0}^m b_j y^j$ be nonzero polynomials $\in R[y]$, such that $f(y)g(y) = 0$. Since $R[[x]]$ is right J-McCoy and $R \subseteq R[[x]]$, then there exists $0 \neq c(x) = c_0 + c_1 x + c_2 x^2 + \dots \in R[[x]]$ such that $a_i c(x) \in J(R[[x]])$ and so $a_i c_i \in J(R[[x]]) \cap R \subseteq J(R)$ for all $i = 0, 1, \dots, n$. Since $c(x)$ is nonzero, there exists $c_l \neq 0$ such that $a_i c_l \in J(R)$ for $i = 0, 1, \dots, n$ and so R is J-McCoy. $\square$

Theorem 3. *For a ring R , if $R[x]$ is right (resp. left) J-McCoy, then R is right (resp. left) J-McCoy. The converse holds if $J(R)[x] \subseteq J(R[x])$.*

Proof. Suppose that $R[x]$ is right J-McCoy. Let $f(y) = \sum_{i=0}^n a_i y^i$ and $g(y) = \sum_{j=0}^m b_j y^j$ be nonzero polynomials in $R[y]$, such that $f(y)g(y) = 0$. Since $R[x]$ is right J-McCoy and $R \subseteq R[x]$, then there exists $0 \neq c(x) = c_0 + c_1 x + \dots + c_k x^k \in R[x]$ such that $a_i c(x) \in J(R[x])$ and so $a_i c_i \in J(R[x]) \cap R \subseteq J(R)$ for all $i = 0, 1, \dots, n$. Since $c(x)$ is nonzero, there exists $c_l \neq 0$ such that $a_i c_l \in J(R)$ for

$i = 0, 1, \dots, n$ and so R is J-McCoy. Conversely, suppose that R is right J-McCoy and $f(y)g(y) = 0$ for nonzero polynomials $f(y) = f_0 + f_1y + \dots + f_my^m$ and $g(y) = g_0 + g_1y + \dots + g_ny^n$ in $(R[x])[y]$. Take the positive integer k with $k = \sum_{i=0}^m \deg f_i + \sum_{j=0}^n \deg g_j$ where the degree of the zero polynomial is taken to be zero. Then $f(x^k)$ and $g(x^k)$ are nonzero polynomials in $R[x]$ and $f(x^k)g(x^k) = 0$, since the set of coefficients of the f_i 's and g_j 's coincide with the set of coefficients of $f(x^k)$ and $g(x^k)$. Since R is right J-McCoy, there exists a nonzero element $c \in R$ such that $a_ic \in J(R)$, for any coefficient a_i of $f_i(x)$. So $f_ic \in J(R)[x] \subseteq J(R[x])$. Thus $R[x]$ is right J-McCoy. $\square$

Recall that a ring R is said to be abelian if every idempotent of it is central.

Proposition 3. *Let R be a right (resp. left) J-McCoy ring and e be an idempotent element of R . Then eRe is a right (resp. left) J-McCoy ring. The converse holds if R is an abelian ring.*

Proof. Consider $f(x) = \sum_{i=0}^n ea_iex^i, g(x) = \sum_{j=0}^m eb_jex^j \in (eRe)[x] \setminus \{0\}$ such that $f(x)g(x) = 0$. Since R is a right J-McCoy ring, there exists $s \in R$ such that $(ea_ie)s \in J(R)$. So $(ea_ie)ese \in eJ(R)e = J(eRe)$. Hence eRe is right J-McCoy. Now, assume that eRe is a right J-McCoy ring. Consider $f(x) = \sum_{i=0}^n a_ix^i, g(x) = \sum_{j=0}^m b_jx^j \in R[x] \setminus \{0\}$ such that $f(x)g(x) = 0$. Clearly, $ef(x)e, eg(x)e \in (eRe)[x]$ and $(ef(x)e)(eg(x)e) = 0$, since e is a central idempotent element of R . Then there exists $s \in eRe$ such that $(ea_ie)s = (a_i)s \in J(eRe) = eJ(R) \subset J(R)$. Hence, R is right J-McCoy. $\square$

The following example shows that, if R is a right J-McCoy ring, then $M_n(R)$ is not necessary right J-McCoy for $n \geq 2$, i.e. the J-McCoy property is not Morita invariant.

Example 3. Let $\mathbb{Z}$ be the set of integers. It's clear that $\mathbb{Z}$ is J-McCoy, but $M_3(\mathbb{Z})$ is not right J-McCoy. For

$$f(x) = \begin{pmatrix} 1 & x & x^2 \\ x^3 & x^4 & x^5 \\ x^6 & x^7 & x^8 \end{pmatrix} \text{ and } g(x) = \begin{pmatrix} x & x & x \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

in $M_3(\mathbb{Z})[x]$, we have $f(x)g(x) = 0$. Assume to the contrary that $M_3(\mathbb{Z})$ is right J-McCoy, then there exists $c = (c_{ij}) \in M_3(\mathbb{Z})$ such that $(E_{ij}c) \in J(M_3(\mathbb{Z})) = M_3(J(\mathbb{Z})) = 0$ for $i, j = 1, 2, 3$. This implies $c = 0$, which is a contradiction.

Let R be a ring and σ denote an endomorphism of R with $\sigma(1) = 1$. In [3] the authors introduced skew triangular matrix ring as a set of all triangular matrices with addition point-wise and a new multiplication subject to the condition $E_{ij}r = \sigma^{j-i}E_{ij}$. So $(a_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = a_{ij}b_{ij} + a_{i,i+1}\sigma(b_{i+1,j}) + \dots + a_{ij}\sigma^{j-i}(b_{jj})$, for each $i \leq j$ and denoted it by $T_n(R, \sigma)$. The subring of the skew triangular matrices with constant main diagonal is denoted by $S(R, n, \sigma)$; and the subring of the skew

triangular matrices with constant diagonals is denoted by $T(R, n, \sigma)$. We can denote $A = (a_{ij}) \in T(R, n, \sigma)$ by $(a_{11}, \dots, a_{1n})$. Then $T(R, n, \sigma)$ is a ring with addition point-wise and multiplication given by $(a_0, \dots, a_{n-1})(b_0, \dots, b_{n-1}) = (a_0b_0, a_0 * b_1 + a_1 * b_0, \dots, a_0 * b_{n-1} + \dots + a_{n-1} * b_0)$, with $a_i * b_j = a_i \sigma^i(b_j)$, for each i and j . Therefore, clearly one can see that $T(R, n, \sigma) \cong R[x; \sigma]/(x^n)$ is the ideal generated by x^n in $R[x; \sigma]$. we consider the following two subrings of $S(R, n, \sigma)$, as follows (see [3]):

$$A(R, n, \sigma) = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=1}^{n-j+1} a_j E_{i, i+j-1} + \sum_{j=\lfloor \frac{n}{2} \rfloor + 1}^n \sum_{i=1}^{n-j+1} a_{i, i+j-1} E_{i, i+j-1},$$

$$B(R, n, \sigma) = \{A + rE_{1k} \mid A \in A(R, n, \sigma) \text{ and } r \in R\} \quad n = 2k \geq 4$$

In the special case, when $\sigma = id_R$, we use $S(R, n)$, $A(R, n)$, $B(R, n)$ and $T(R, n)$ instead of $S(R, n, \sigma)$, $A(R, n, \sigma)$, $B(R, n, \sigma)$ and $T(R, n, \sigma)$, respectively.

Proposition 4. *Let R be a ring. Then S is right J-McCoy ring, for $n \geq 2$, where S is one of the rings $T_n(R, \sigma)$, $S(R, n, \sigma)$, $T(R, n, \sigma)$, $A(R, n, \sigma)$ or $B(R, n, \sigma)$.*

Proof. Let $f(x) = A_0 + A_1x + \dots + A_px^p$, $g(x) = B_0 + B_1x + \dots + B_qx^q$ be elements of $S[x]$ satisfying $f(x)g(x) = 0$ where the $(1, 1)$ -th entry of A_i is $a_{11}^{(i)}$. Then $A_i E_{1n} = a_{11}^{(i)} E_{1n} \in J(S)$ and the proof is complete. $\square$

Let R and S be two rings, and Let M be an (R, S) -bimodule. This means that M is a left R -module and a right S -module such that $(rm)s = r(ms)$ for all $r \in R$, $m \in M$, and $s \in S$. Given such a bimodule M we can form

$$T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} : r \in R, m \in M, s \in S \right\}$$

and define a multiplication on T by using formal matrix multiplication:

$$\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \begin{pmatrix} r' & m' \\ 0 & s' \end{pmatrix} = \begin{pmatrix} rr' & rm' + ms' \\ 0 & ss' \end{pmatrix}.$$

This ring construction is called triangular ring T .

Proposition 5. *Let R and S be two rings and T be the triangular ring $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ (where M is an (R, S) -bimodule). Then the rings R and S are right (resp. left) J-McCoy if and only if T is right (resp. left) J-McCoy.*

Proof. Assume that R and S are two right J-McCoy rings. Take $I = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$, then $T/I \simeq \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}$. Let

$$f(x) = \begin{pmatrix} r_0 & 0 \\ 0 & s_0 \end{pmatrix} + \begin{pmatrix} r_1 & 0 \\ 0 & s_1 \end{pmatrix}x + \dots + \begin{pmatrix} r_n & 0 \\ 0 & s_n \end{pmatrix}x^n,$$

$$g(x) = \begin{pmatrix} r'_0 & 0 \\ 0 & s'_0 \end{pmatrix} + \begin{pmatrix} r'_1 & 0 \\ 0 & s'_1 \end{pmatrix}x + \dots + \begin{pmatrix} r'_m & 0 \\ 0 & s'_m \end{pmatrix}x^m \in T[x]$$

satisfy $f(x)g(x) = 0$. Define

$$f_r(x) = r_0 + r_1x + \cdots + r_nx^n, g_r(x) = r'_0 + r'_1x + \cdots + r'_mx^m \in R[x]$$

and

$$f_s(x) = s_0 + s_1x + \cdots + s_nx^n, g_s(x) = s'_0 + s'_1x + \cdots + s'_mx^m \in S[x].$$

From $f(x)g(x) = 0$, we have $f_r(x)g_r(x) = f_s(x)g_s(x) = 0$. Since R and S are right J-McCoy rings, then there exists $c \in R$ and $d \in S$ such that $r_i c \in J(R)$ and $s_i d \in J(S)$ for any $1 \leq i \leq n$ and $1 \leq j \leq m$. Hence if we put $I = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ then T/I is right J-McCoy and so T is right J-McCoy by Proposition 1. Conversely, let T be a right J-McCoy ring, $f_r(x) = r_0 + r_1x + \cdots + r_nx^n$, $g_r(x) = r'_0 + r'_1x + \cdots + r'_mx^m \in R[x]$, such that $f_r(x)g_r(x) = 0$, and $f_s(x) = s_0 + s_1x + \cdots + s_nx^n$, $g_s(x) = s'_0 + s'_1x + \cdots + s'_mx^m \in S[x]$, such that $f_s(x)g_s(x) = 0$. Let

$$\begin{aligned} f(x) &= \begin{pmatrix} r_0 & 0 \\ 0 & s_0 \end{pmatrix} + \begin{pmatrix} r_1 & 0 \\ 0 & s_1 \end{pmatrix}x + \cdots + \begin{pmatrix} r_n & 0 \\ 0 & s_n \end{pmatrix}x^n \text{ and} \\ g(x) &= \begin{pmatrix} r'_0 & 0 \\ 0 & s'_0 \end{pmatrix} + \begin{pmatrix} r'_1 & 0 \\ 0 & s'_1 \end{pmatrix}x + \cdots + \begin{pmatrix} r'_m & 0 \\ 0 & s'_m \end{pmatrix}x^m \in T[x]. \end{aligned}$$

Then $f_r(x)g_r(x) = 0$ and $f_s(x)g_s(x) = 0$ implies that $f(x)g(x) = 0$. Since T is a right J-McCoy ring then there exists $\begin{pmatrix} c & m \\ 0 & d \end{pmatrix} \in T$ such that $\begin{pmatrix} r_i & 0 \\ 0 & s_i \end{pmatrix} \begin{pmatrix} c & m \\ 0 & d \end{pmatrix} \in J(T) = \begin{pmatrix} J(R) & M \\ 0 & J(S) \end{pmatrix}$. Thus $r_i c \in J(R)$ and $s_i d \in J(S)$ for any i, j . This shows that R and S are right J-McCoy. $\square$

Given a ring R and a bimodule ${}_R M_R$, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Corollary 3. *A ring R is right (resp. left) J-McCoy if and only if the trivial extension $T(R, R)$ is a right (resp. left) J-McCoy ring.*

Let S denote a multiplicatively closed subset of a ring R consisting of central regular elements. Let RS^{-1} be the localization of R at S . Then we have:

Theorem 4. *For a ring R , if R is right (resp. left) J-McCoy, then RS^{-1} is right (resp. left) J-McCoy.*

Proof. Suppose that R is right J-McCoy. Let $f(x) = \sum_{i=0}^n a_i c_i^{-1} x^i$, $g(x) = \sum_{j=0}^m b_j d_j^{-1} x^j$ be nonzero elements in $(RS^{-1})[x]$ such that $f(x)g(x) = 0$. Let $a_i c_i^{-1} = c^{-1} a'_i$ and $b_j d_j^{-1} = d^{-1} b'_j$ with c, d regular elements in R . So $f'(x)g'(x) = 0$ such that $f'(x) = \sum_{i=0}^n a'_i x^i$ and $g'(x) = \sum_{j=0}^m b'_j x^j \in R[x] \setminus \{0\}$. Since R is right J-McCoy, there exists $r \in R \setminus \{0\}$ such that $a'_i r \in J(R)$ for each i , equivalently we have $1 - t a'_i r$ is left invertible in R for each $t \in R$. So $c^{-1} w^{-1} (1 - t w^{-1} a_i c_i^{-1} r c w) = c^{-1} w^{-1} -$

$tw^{-1}a_ic_i^{-1}r$ is left invertible in RS^{-1} , for each $tw^{-1} \in RS^{-1}$ and so $a_ic_i^{-1}rcw \in J(RS^{-1})$. Thus RS^{-1} is right J-McCoy. $\square$

Corollary 4. *For a ring R , let $R[x]$ be a right (resp. left) J-McCoy ring. Then $R[x, x^{-1}]$ is a right (resp. left) J-McCoy ring.*

Proof. Let $\Delta = \{1, x, x^2, \dots\}$. Then clearly Δ is multiplicatively closed subset of $R[x]$. Since $R[x, x^{-1}] = \Delta^{-1}R$, it follows that $R[x, x^{-1}]$ is right J-McCoy. $\square$

A ring R is called right (left) quasi-duo if every maximal right (left) ideal of R is two-sided. It is clear that a ring R is right (left) quasi-duo if and only if $R/J(R)$ is right (left) quasi-duo. Also $R/J(R)$ is a reduced ring in case it is right (left) quasi-duo.

Proposition 6. *If R is a right (left) quasi-duo ring, then R is right (resp. left) J-McCoy, the converse does not hold in general.*

Proof. Since R is right quasi-duo ring, then $R/J(R)$ is reduced by [7] and so R is right J-McCoy by Proposition 1. But there exists a right J-McCoy ring R which is not right (left) quasi-duo. For instance, take any right primitive domain R that is not division ring (e.g. the free algebra $R = Q \langle x, y \rangle$). Then $R/J(R) = R$ is right J-McCoy, but R is not right quasi-duo by [5]. $\square$

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ON THE MOORE–PENROSE INVERSE IN RINGS WITH INVOLUTION

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Abstract. We study equivalent conditions to those given in Penrose equations for an element to be the Moore–Penrose inverse of a given element in a ring with involution, using the concept of EP, normal, bi-EP, bi-normal, l -(or r -)quasi-EP, l -(or r -)quasi-normal and $*$ -cancellable elements. The mentioned conditions are weaker than the one of being self-adjoint.

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1. INTRODUCTION

Let $\mathcal{R}$ be an associative ring with the unit 1. An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a + b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

An element $a \in \mathcal{R}$ is self-adjoint if $a^* = a$. An element $a \in \mathcal{R}$ satisfying $aa^* = a^*a$ is called normal.

The Moore–Penrose inverse of $a \in \mathcal{R}$ is the element $x \in \mathcal{R}$, if the following Penrose equations hold [7]:

$$(1) \, axa = a, \quad (2) \, xax = x, \quad (3) \, (ax)^* = ax, \quad (4) \, (xa)^* = xa.$$

There is at most one x such that above conditions hold, and such x is denoted by $a^\dagger$. The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$. In the last decades, the Moore–Penrose inverse has found a wide range of applications in many areas of science and became a useful tool for dealing with optimization problems, data analysis, the solution of linear integral equations, etc.

If $\delta \subseteq \{1, 2, 3, 4\}$ and x satisfies the equations (i) for all $i \in \delta$, then x is a δ -inverse of a . The set of all δ -inverse of a is denoted by $a\{\delta\}$. Notice that $a\{1, 2, 3, 4\} = \{a^\dagger\}$. Recall that $x \in a\{1, 4\}$ if and only if $x^* \in a^*\{1, 3\}$.

The following result is frequently used in the rest of the paper.

Theorem 1 ([1]). *For any $a \in \mathcal{R}^\dagger$, the following is satisfied:*

- (a) $(a^\dagger)^\dagger = a$;
- (b) $(a^*)^\dagger = (a^\dagger)^*$.

An element $a \in \mathcal{R}$ is said to be EP if $a \in \mathcal{R}^\dagger$ and $aa^\dagger = a^\dagger a$. Observe that a is EP if and only if a^* is EP.

We recall the reader that $a \in \mathcal{R}$ is called (i) bi-normal if $aa^*a^*a = a^*aaa^*$; (ii) bi-EP if $a \in \mathcal{R}^\dagger$ and $aa^\dagger a^\dagger a = a^\dagger aaa^\dagger$; (iii) l -quasi-normal if $aa^*a = a^*aa$; (iv) r -quasi-normal if $aaa^* = aa^*a$; (v) l -quasi-EP if $a \in \mathcal{R}^\dagger$ and $aa^\dagger a = a^\dagger aa$; (vi) r -quasi-EP if $a \in \mathcal{R}^\dagger$ and $aaa^\dagger = aa^\dagger a$ [2, 4, 6].

An element $a \in \mathcal{R}$ is: left $*$ -cancellable if $a^*ax = a^*ay$ implies $ax = ay$; it is right $*$ -cancellable if $xaa^* = yaa^*$ implies $xa = ya$; and it is $*$ -cancellable if it is both left and right $*$ -cancellable. Notice that a is left $*$ -cancellable if and only if a^* is right $*$ -cancellable. In C^* -algebras all elements are $*$ -cancellable. A ring $\mathcal{R}$ is called $*$ -reducing if every element of $\mathcal{R}$ is $*$ -cancellable. This is equivalent to the implication $a^*a = 0 \Rightarrow a = 0$ for all $a \in \mathcal{R}$.

Malik and Thome [6] investigated weaker conditions than those given by Penrose for an operator to be the Moore–Penrose inverse of a given bounded linear operator between two Hilbert spaces, using operator matrices.

In this paper, applying a purely algebraic technique, we characterize the Moore–Penrose inverse of an element in a ring with involution in terms of EP, normal, bi-EP, bi-normal, l -quasi-EP, r -quasi-EP, l -quasi-normal, r -quasi-normal and $*$ -cancellable elements, generalizing the results from [6]. Thus, we show that the properties of operator matrices are not necessary for the characterization of the Moore–Penrose inverse.

2. RESULTS

In the first theorem, if x is an 1-inverse of a , we present necessary and sufficient conditions for x to be a 3-inverse of a .

Theorem 2. *Let $a \in \mathcal{R}$ such that $a\{1\} \neq \emptyset$. If $x \in a\{1\}$, then the following statements are equivalent:*

- (i) ax is self-adjoint;
- (ii) ax is EP;
- (iii) ax is right $*$ -cancellable and normal;
- (iv) ax is bi-EP;
- (v) ax is $*$ -cancellable and bi-normal;
- (vi) ax is l -quasi-EP;
- (vii) ax is r -quasi-EP;
- (viii) ax is left $*$ -cancellable and l -quasi-normal;
- (ix) ax is right $*$ -cancellable and r -quasi-normal.

Proof. (i) $\Rightarrow$ (ii)–(ix): Because ax is self-adjoint idempotent, we deduce that $ax \in \mathcal{R}^\dagger$ and $(ax)^\dagger = ax$. So, the statements (ii)–(ix) hold.

(ii) $\Rightarrow$ (i): Since ax is EP, then $ax \in \mathcal{R}^\dagger$ and $(ax)^\dagger ax = ax(ax)^\dagger$. Thus,

$$ax = ax(ax)^\dagger ax = axax(ax)^\dagger = ax(ax)^\dagger$$

is self-adjoint.

(iii) $\Rightarrow$ (i): If ax is right $*$ -cancellable and normal, then $(ax)^*$ is left $*$ -cancellable and $ax(ax)^* = (ax)^*ax$. Multiplying this equality by ax from the left side, it follows

$$ax(ax)^* = ax(ax)^*ax.$$

Using $*$ -cancellation, we get $(ax)^* = (ax)^*ax$, i.e. $ax = (ax)^*ax$ is self-adjoint.

(iv) $\Rightarrow$ (i): Suppose that ax is bi-EP, that is, $ax \in \mathcal{R}^\dagger$ and

$$ax(ax)^\dagger(ax)^\dagger ax = (ax)^\dagger axax(ax)^\dagger.$$

Now, we get

$$\begin{aligned} [(ax)^\dagger]^* &= [(ax)^\dagger ax(ax)^\dagger]^* = [(ax)^\dagger axax(ax)^\dagger]^* \\ &= [ax(ax)^\dagger(ax)^\dagger ax]^* = (ax)^\dagger axax(ax)^\dagger \\ &= (ax)^\dagger ax(ax)^\dagger = (ax)^\dagger, \end{aligned}$$

which implies $(ax)^* = [(ax)^\dagger]^* = [(ax)^\dagger]^\dagger = ax$.

(v) $\Rightarrow$ (i): Let ax be $*$ -cancellable and bi-normal. Then

$$ax(ax)^*(ax)^*ax = (ax)^*axax(ax)^*$$

gives

$$ax(ax)^*ax = (ax)^*ax(ax)^*.$$

Multiplying the previous equality by ax from the left side, we obtain

$$ax(ax)^*ax = ax(ax)^*ax(ax)^*. \quad (2.1)$$

Since ax and $(ax)^*$ are $*$ -cancellable, by (2.1), $ax = ax(ax)^*$ is self-adjoint.

(vi) $\Rightarrow$ (i): Because ax is l -quasi-EP, we have that $ax \in \mathcal{R}^\dagger$ and

$$ax(ax)^\dagger ax = (ax)^\dagger axax$$

implying $ax = (ax)^\dagger ax$ is self-adjoint.

(vii) $\Rightarrow$ (i): Similarly as (vi) $\Rightarrow$ (i).

(viii) $\Rightarrow$ (i): Assume that ax is left $*$ -cancellable and l -quasi-normal. We deduce that $(ax)^*$ is right $*$ -cancellable and

$$ax(ax)^*ax = (ax)^*axax = (ax)^*ax,$$

which yield $ax(ax)^* = (ax)^*$. Hence, $ax(ax)^* = ax$ is self-adjoint.

(ix) $\Rightarrow$ (i): In the same way as in the part (viii) $\Rightarrow$ (i). □

The following result can be checked similarly as Theorem 2.

Corollary 1. *Let $a \in \mathcal{R}$ such that $a\{1\} \neq \emptyset$. If $x \in a\{1\}$, then the following statements are equivalent:*

- (i) ax is self-adjoint;
- (ii) ax is EP;
- (iii) $ax \in \mathcal{R}^\dagger$ and ax is normal;
- (iv) ax is bi-EP;
- (v) $ax \in \mathcal{R}^\dagger$ and ax is bi-normal;
- (vi) ax is l -quasi-EP;
- (vii) ax is r -quasi-EP;
- (viii) $ax \in \mathcal{R}^\dagger$ and ax is l -quasi-normal;
- (ix) $ax \in \mathcal{R}^\dagger$ and ax is r -quasi-normal.

Proof. Since $ax \in \mathcal{R}^\dagger$, by [5, Theorem 5.3], we deduce that ax is $*$ -cancellable. The rest of this proof follows as in the proof of Theorem 2. $\square$

For an 1-inverse x of a , several equivalent conditions which insure that x is 4-inverse of a are given now.

Theorem 3. *Let $a \in \mathcal{R}$ such that $a\{1\} \neq \emptyset$. If $x \in a\{1\}$, then the following statements are equivalent:*

- (i) xa is self-adjoint;
- (ii) xa is EP;
- (iii) xa is left $*$ -cancellable and normal;
- (iv) xa is bi-EP;
- (v) xa is $*$ -cancellable and bi-normal;
- (vi) xa is r -quasi-EP;
- (vii) xa is l -quasi-EP;
- (viii) xa is right $*$ -cancellable and r -quasi-normal;
- (ix) xa is left $*$ -cancellable and l -quasi-normal.

Proof. For $x \in a\{1\}$ such that xa satisfies any of the conditions (i)–(ix), we have that $x^* \in a^*\{1\}$ such that a^*x^* satisfies any of the conditions (i)–(ix) of Theorem 2. Applying Theorem 2, we verify this result. $\square$

In the main result of this article, weaker conditions than those given in Penrose equations for an element x to be the Moore–Penrose inverse of a are given.

Theorem 4. *Let $a \in \mathcal{R}$ such that $a\{1, 2\} \neq \emptyset$ and $x \in a\{1, 2\}$. If ax satisfies any of the conditions (ii)–(ix) of Theorem 2 and xa satisfies any of the conditions (ii)–(ix) of Theorem 3, then x is the Moore–Penrose inverse of a .*

Proof. This result is a consequence of Theorem 2 and Theorem 3. $\square$

The preceding theorems hold in rings with involution assuming in some conditions that ax and xa are left or (and) right $*$ -cancellable which is automatically satisfied in C^* -algebras and $*$ -reducing rings. Thus, we recover the results in [6] for Hilbert space operators.

In a unital C^* -algebra $\mathcal{A}$, since $a \in \mathcal{A}$ is Moore–Penrose invertible if and only if $a\{1\} \neq \emptyset$ (see [3]), we get the following result as a consequence of Theorem 4.

Corollary 2. *Let $a \in \mathcal{A}$ such that $a\{1\} \neq \emptyset$ and $x \in a\{1, 2\}$. If both ax and xa satisfy any of the statements:*

- (i) *EP*;
- (ii) *normal*;
- (iv) *bi-EP*;
- (v) *bi-normal*;
- (vi) *l-quasi-EP*;
- (vii) *r-quasi-EP*;
- (viii) *l-quasi-normal*;
- (ix) *r-quasi-normal*,

then x is the Moore–Penrose inverse of a .

By corresponding examples in [6], it is showed that conditions (i)–(ix) of Corollary 2 are weaker than one of being self-adjoint and can be adopted to define the Moore–Penrose inverse.

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SOME MARX-STROHHÄCKER TYPE RESULTS FOR A CLASS OF MULTIVALENT FUNCTIONS

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Abstract. In this paper, two Marx-Strohhäcker type results are proven for a certain class of analytic and multivalent functions in the open unit disk $\mathbb{D}$. The first result gives the order of multivalent starlikeness for multivalently convex functions of some specified order. The second result provides a lower bound over the unit disk $\mathbb{D}$ of $\Re\left(\frac{f(z)}{z^p}\right)$ for functions $f(z)$ that are multivalently starlike of a given order. Relevant connections of the results presented here with earlier results are also indicated.

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let $\mathcal{H}(\mathbb{D})$ denote the class of all functions which are *analytic* in the open unit disk

$$\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

For $n \in \mathbb{N}$ ($\mathbb{N} := \{1, 2, 3, \dots\}$) and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f : f \in \mathcal{H}(\mathbb{D}) \text{ and } f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}.$$

We also let $\mathcal{A}_p$ ($p \in \mathbb{N}$) be the subclass of $\mathcal{H}(\mathbb{D})$ consisting of functions of the following *normalized* form:

$$f(z) = z^p + a_{p+1} z^{p+1} + a_{p+2} z^{p+2} + \dots$$

and write $\mathcal{A} \equiv \mathcal{A}_1$. The functions (mappings) in $\mathcal{S} \subset \mathcal{A}$ are one-to-one and are called *normalized univalent functions in $\mathbb{D}$* . For more details about analytic, univalent and multivalent functions, see (for example) [1, 5, 9].

The classes of *p-valently starlike functions of order α in $\mathbb{D}$* and *p-valently convex functions of order α in $\mathbb{D}$* ($p \in \mathbb{N}$; $0 \leq \alpha < p$) are defined by

$$\mathcal{S}_p^*(\alpha) = \left\{ f \in \mathcal{A}_p : \Re\left(\frac{z f'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{D}) \right\}$$

and

$$\mathcal{K}_p(\alpha) = \left\{ f \in \mathcal{A}_p : \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathbb{D}) \right\},$$

respectively. In the case when $p = 1$, we respectively have the usual classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order α in $\mathbb{D}$. Moreover, in the case when $p \in \mathbb{N} \setminus \{1\}$, we are led to p -valently analytic function classes which correspond essentially to the classes $\mathcal{S}_p^*$ and $\mathcal{K}_p$ of p -valently starlike and p -valently convex functions of order α in $\mathbb{D}$. For simplicity, in the case when $\alpha = 0$, we will use the notations $\mathcal{S}_p^*$ and $\mathcal{K}_p$ and write

$$\mathcal{S}^* = \mathcal{S}_1^* = \mathcal{S}_1^*(0) \quad \text{and} \quad \mathcal{K} = \mathcal{K}_1 = \mathcal{K}_1(0)$$

in the case when $\alpha = 0$ and $p = 1$.

It is a well-known result due to Marx and Stroh  cker [4, 11] that convex functions have the order $\frac{1}{2}$ of starlikeness, that is, for $f \in \mathcal{A}$, we have

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0 \quad (z \in \mathbb{D}) \quad \implies \quad \Re \left(\frac{zf'(z)}{f(z)} \right) > \frac{1}{2} \quad (z \in \mathbb{D}). \quad (1.1)$$

The function $f(z) = \frac{z}{1-z}$ shows that this result is sharp, that is, the number $\frac{1}{2}$ cannot be replaced by a larger one. Let us call this the Marx-Stroh  cker result of Type I. It was proven in [6] that this implication is not true for p -valent functions ($p \geq 2$). Namely, it was proven that, if $p \in \mathbb{N} \setminus \{1\}$, then there exists $f \in \mathcal{K}_p$ such that $f \in \mathcal{S}_p^*$, but $f \notin \mathcal{S}_p^*(\alpha)$ for all $\alpha > 0$, that is, there exists $f \in \mathcal{A}_p$ such that

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0 \quad (z \in \mathbb{D}) \quad \text{and} \quad \Re \left(\frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{D}),$$

but the following inequality:

$$\Re \left[\frac{zf'(z)}{f(z)} \right] > \alpha \quad (z_0 \in \mathbb{D})$$

is not satisfied for any $\alpha > 0$.

In view of the above observation, it is natural to ask the following question: If $f(z)$ is a p -valently convex function of order α in $\mathbb{D}$ ($p \in \mathbb{N} \setminus \{1\}$; $0 \leq \alpha < p$), what is the order β of p -valent starlikeness of the function $f(z)$? For the case when $\frac{p-1}{2} \leq \alpha < p$, this problem is solved in a sharp way (that is, the best way possible) by Srivastava *et al.* [10], in which they showed that

$$\mathcal{K}_p(\alpha) \subset \mathcal{S}_p^*(\widehat{\beta}_1(\alpha, p)),$$

where

$$\widehat{\beta}_1(\alpha, p) = \frac{p}{{}_2F_1\left(1, 2(p-\alpha); p+1; \frac{1}{2}\right)} \quad (1.2)$$

is the largest number with such property. In this paper, we extend this result to the case when $0 \leq \alpha < \frac{p-1}{2}$. Unfortunately, so far we have not succeeded in proving

sharpness of results which we have obtained in this investigation. There are indications that they are not sharp. However, finding their sharp versions remains an *open problem*.

The analogue problem for the class of univalent functions was posed by Jack [2]: What is the largest number $\beta = \beta(\alpha)$ so that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\beta(\alpha))$? MacGregor [3] determined the exact value of $\beta(\alpha)$ for each α ($0 \leq \alpha < 1$) as the infimum over the disk $\mathbb{D}$ of the real part of a specific analytic function. It has been conjectured that this infimum is attained on the boundary of $\mathbb{D}$ at $z = -1$. Wilken and Feng in [12] asserted MacGregor's conjecture and obtained

$$\widehat{\beta}_2(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2(1-\alpha)}-2} & (\alpha \neq \frac{1}{2}) \\ \frac{1}{2\ln 2} & (\alpha = \frac{1}{2}). \end{cases} \quad (1.3)$$

We will also give a generalization for the class of multivalent functions of another classical Marx-Strohhäcker result ([4, 11]):

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \frac{1}{2} \quad (z \in \mathbb{D}) \implies \Re\left(\frac{f(z)}{z}\right) > \frac{1}{2} \quad (z \in \mathbb{D}). \quad (1.4)$$

Let us call this the Marx-Strohhäcker result of Type II. For several other investigations involving the Marx-Strohhäcker type results for other classes of multivalent functions, see (for example) the earlier works by Srivastava *et al.* [8, Section 3] and [7, Section 3]).

For proving the results in this paper, we will use the following Lemma from the theory of differential subordination between analytic functions.

Lemma 1. (see [5, p. 35, Theorem 2.3i(i)]). *Let $\Omega \subset \mathbb{C}$ and suppose that the function $\psi : \mathbb{C}^2 \times \mathbb{D} \rightarrow \mathbb{C}$ satisfies $\psi(ix, y; z) \notin \Omega$ for all $x \in \mathbb{R}$, $y \leq -\frac{n}{2}(1+x^2)$ and $z \in \mathbb{D}$. If $q \in \mathcal{H}[1, n]$ and $\psi(q(z), zq'(z); z) \in \Omega$ for all $z \in \mathbb{D}$, then*

$$\Re(q(z)) > 0 \quad (z \in \mathbb{D}).$$

2. A MARX-STROHHÄCKER RESULT OF TYPE I

Our Marx-Strohhäcker result of Type I is contained in Theorem 1 below.

Theorem 1. *Let $p \in \mathbb{N}$ and $0 \leq \beta < p$. Also let*

$$\alpha \equiv \alpha(\beta, p) = \begin{cases} \beta - \frac{1}{2} \left(\frac{\beta}{p-\beta} \right) & (0 \leq \beta < \frac{p}{2}) \\ \beta - \frac{1}{2} \left(\frac{p-\beta}{\beta} \right) & (\frac{p}{2} \leq \beta < p). \end{cases} \quad (2.1)$$

If $f \in \mathcal{A}_p$ and

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathbb{D}), \quad (2.2)$$

then

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > \beta \quad (z \in \mathbb{D}) \quad (2.3)$$

or, equivalently,

$$\mathcal{K}_p(\alpha(\beta, p)) \subset \mathcal{S}_p^*(\beta),$$

that is, p -valently convex functions of order $\alpha(\beta, p)$ in $\mathbb{D}$ have the β order of p -valent starlikeness in $\mathbb{D}$.

Proof. If we let the function $q(z)$ be given by

$$q(z) = \frac{1}{p-\beta} \left(\frac{zf'(z)}{f(z)} - \beta \right),$$

then $q(z) \in \mathcal{H}[1, 1]$. Further, for the function

$$\psi(r, s; z) = \frac{s(p-\beta)}{r(p-\beta) + \beta} + r(p-\beta) + \beta,$$

we find by using (2.2) that

$$\psi(q(z), zq'(z); z) = 1 + \frac{zf''(z)}{f'(z)} \in \Omega \equiv \{\omega : \Re \omega > \alpha\} \quad (z \in \mathbb{D}).$$

So, by the Lemma in Section 1, for proving (2.3) or, equivalently, that

$$\Re(q(z)) > 0 \quad (z \in \mathbb{D}),$$

it is enough to show that

$$\psi(ix, y; z) = \frac{y(p-\beta)}{ix(p-\beta) + \beta} + ix(p-\beta) + \beta \notin \Omega$$

for all real $x, y \leq -\frac{1+x^2}{2}$ ($n=1$ in the Lemma in Section 1) and for all $z \in \mathbb{D}$. Indeed, we have

$$\begin{aligned} \Re(\psi(ix, y; z)) &\leq \beta \left(1 + \frac{-\left(\frac{1+x^2}{2}\right)(p-\beta)}{\beta^2 + x^2(p-\beta)^2} \right) \\ &= \beta \left(1 + \frac{1}{p-\beta} \cdot \frac{p(\beta - \frac{p}{2})}{\beta^2 + x^2(p-\beta)^2} - \frac{\frac{1}{2}}{p-\beta} \right) \equiv \varphi(\beta, p, x), \end{aligned}$$

and from

$$\frac{\partial}{\partial x} \varphi(\beta, p, x) = \frac{\beta p(p-\beta)}{[\beta^2 + x^2(p-\beta)^2]^2} (p-2\beta)x$$

we observe that $\varphi(\beta, p, x)$ is a piecewise monotone function of x , which increases for $x \geq 0$ and decreases for $x < 0$ when $0 \leq \beta < \frac{p}{2}$ and which decreases for $x \geq 0$ and increases for $x < 0$ when $0 \leq \beta < \frac{p}{2}$. So, we get

$$\varphi(\beta, p, x) \leq \begin{cases} \lim_{x \rightarrow \infty} \varphi(\beta, p, x) = \beta - \frac{1}{2} \left(\frac{\beta}{p - \beta} \right) & (0 \leq \beta < \frac{p}{2}) \\ \varphi(\beta, p, 0) = \beta - \frac{1}{2} \left(\frac{p - \beta}{\beta} \right) & (\frac{p}{2} \leq \beta < p) \end{cases} \\ = \alpha(\beta, p). \quad (2.4)$$

This evidently completes the proof of Theorem 1. $\square$

Remark 1. Each of the following observations is worth recording here.

- (i) The value $\alpha(\beta, p)$ is well defined, that is, $0 \leq \alpha(\beta, p) < p$. In the case when $p = 1$, this can be directly verified. In the case when $p \in \mathbb{N} \setminus \{1\}$, it follows from the fact that $\alpha(\beta, p)$ is a piecewise strictly increasing function of β on the interval $[0, p)$ (which will be shown in the proof of Theorem 2 below). Thus, clearly, we have

$$0 = \alpha(0, p) \leq \alpha(\beta, p) < \lim_{\beta \rightarrow p} \alpha(\beta, p) = \beta < p.$$

- (ii) For $p = 1$ and $\beta = \frac{1}{2}$ in Theorem 1, we obtain $\alpha = 0$, that is, we obtain the Marx-Strohhäcker's result as in the implication (1.1).

Theorem 1 can be rewritten in the following equivalent form, which gives the Marx-Strohhäcker result of Type I for multivalent functions for any $\alpha \in [0, p)$.

Theorem 2. Let $p \in \mathbb{N}$, $0 \leq \alpha < p$ and

$$\beta \equiv \beta(\alpha, p) = \begin{cases} \frac{2(\alpha + p) - 1 - \sqrt{[2(\alpha + p) - 1]^2 - 16\alpha p}}{4} & (0 \leq \alpha < \frac{p-1}{2}) \\ \frac{2\alpha - 1 + \sqrt{(2\alpha - 1)^2 + 8p}}{4} & (\frac{p-1}{2} \leq \alpha < p) \end{cases}. \quad (2.5)$$

If $f \in \mathcal{A}_p$ and

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathbb{D}),$$

then

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > \beta \quad (z \in \mathbb{D})$$

or, equivalently,

$$\mathcal{K}_p(\alpha) \subset \mathcal{S}_p^*(\beta(\alpha, p)),$$

that is, p -valently convex functions of order α have the $\beta(\alpha, p)$ order of p -valent starlikeness.

Proof. For the function $\alpha(\beta, p)$ defined by (2.1), we have

$$\frac{\partial}{\partial \beta} \alpha(\beta, p) = \begin{cases} 1 - \frac{p}{2(p-\beta)^2} & (0 < \beta < \frac{p}{2}) \\ 1 + \frac{p}{2\beta^2} & (\frac{p}{2} < \beta < p) \end{cases}. \quad (2.6)$$

The partial derivative in (2.6) is obviously positive in the case when $\frac{p}{2} \leq \beta < p$. It is also positive when $p \in \mathbb{N} \setminus \{1, 2\}$ and $0 < \beta < \frac{p}{2}$, because

$$p \geq 2 \implies \beta < \frac{p}{2} \leq p - \sqrt{\frac{p}{2}} \implies 1 - \frac{p}{2(p-\beta)^2} > 0.$$

Thus, $\alpha(\beta, p)$ is a piecewise strictly increasing function of β on the interval $[0, p)$. This means that there exist its piecewise inverse functions. Moreover, it is not difficult to check that $\beta(\alpha, p)$, given by (2.5), is that inverse function.

For $p = 1$, it is easy to verify that $\alpha(\beta, 1)$ is continuous on $(0, 1)$, concave down on the interval $(0, \frac{1}{2})$ and, as stated before, increasing on $(\frac{1}{2}, 1)$ with $\alpha(\frac{1}{2}, 1) = 0$ and $\alpha(1, 1) = 1$. Therefore, for its inverse function, we can choose the one that corresponds to $\frac{1}{2} \leq \beta < 1$, that is, the inverse of $\beta - \frac{1}{2} \left(\frac{1-\beta}{\beta} \right)$, which is

$$\frac{2\alpha - 1 + \sqrt{(2\alpha - 1)^2 + 8p}}{4} = \beta(\alpha, 1)$$

with domain $[0, 1)$. □

Remark 2. Theorem 2 is an extension of a *sharp* result from the earlier work by Srivastava *et al.* [10] to the case when $0 \leq \alpha < \frac{p-1}{2}$. Unfortunately, we were not able to prove sharpness of this extension. Thus, the problem of obtaining its sharp versions remains an open problem.

The sharp Marx-Strohhäcker's result for the case when $\frac{p-1}{2} \leq \alpha < p$ is given by (1.2); for the case when $p = 1$ and $0 \leq \alpha < 1$, it is given by (1.3). Comparing the values of $\beta(\alpha, p)$ given by (2.5), $\hat{\beta}_1(\alpha, p)$ given by (1.2) and $\hat{\beta}_2(\alpha)$ given by (1.3), for the case when $\frac{p-1}{2} \leq \alpha < p$, we obtain the following consequences:

- (i) If $p = 1$ and $\alpha = 0$, then

$$\hat{\beta}_1(0, 1) = \hat{\beta}_2(0) = \beta(0, 2) = \frac{1}{2},$$

which is the Marx-Strohhäcker's result in (1.1);

- (ii) If $p = 1$ and $0 < \alpha < 1$, then

$$\hat{\beta}_1(\alpha, 1) = \hat{\beta}_2(\alpha) > \beta(\alpha, 1);$$

(iii) If $p \in \mathbb{N} \setminus \{1\}$ and $\alpha = \frac{p-1}{2}$, then

$$\beta(\alpha, p) = \widehat{\beta}_1(\alpha, p) = \frac{p}{2},$$

that is, the results are equivalent;

(iv) If $p \geq 2$ and $\frac{p-1}{2} < \alpha < p$, then

$$\beta(\alpha, p) \neq \widehat{\beta}_1(\alpha, p),$$

that is,

$$\beta(\alpha, p) < \widehat{\beta}_1(\alpha, p)$$

due to the sharpness of $\widehat{\beta}_1(\alpha, p)$.

3. A MARX-STROHHÄCKER RESULT OF TYPE II

In this section, we state and prove a generalization of the implication (1.4) for p -valent functions when p is any positive integer.

Theorem 3. Let $p \in \mathbb{N}$, $0 < \gamma < 1$ and

$$\beta \equiv \beta(\gamma, p) = \begin{cases} p - \frac{1}{2} \left(\frac{\gamma}{1-\gamma} \right) & (0 < \gamma < \frac{1}{2}) \\ p - \frac{1}{2} \left(\frac{1-\gamma}{\gamma} \right) & (\frac{1}{2} \leq \gamma < 1). \end{cases}$$

If $f \in \mathcal{A}_p$ and

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > \beta \quad (z \in \mathbb{D}), \quad (3.1)$$

then

$$\Re \left(\frac{f(z)}{z^p} \right) > \gamma \quad (z \in \mathbb{D}).$$

Proof. The proof of the implication asserted by Theorem 3 runs parallel to that given in the proof of Theorem 1. It makes use of the functions $q(z)$ and $\psi(r, s; z)$ given by

$$q(z) = \frac{1}{1-\gamma} \cdot \left(\frac{f(z)}{z^p} - \gamma \right) \in \mathcal{H}[1, 1]$$

and

$$\psi(r, s; z) = \frac{s(1-\gamma)}{r(1-\gamma) + \gamma} + p,$$

respectively. Thus, from (3.1), we find that

$$\psi(q(z), zq'(z); z) = \frac{zf'(z)}{f(z)} \in \Omega \equiv \{\omega : \Re \omega > \beta\} \quad (z \in \mathbb{D}).$$

Furthermore, by the Lemma in Section 1, for proving that

$$\Re\left(\frac{f(z)}{z^p}\right) > \gamma \quad (z \in \mathbb{D}),$$

that is, that

$$\Re(q(z)) > 0 \quad (z \in \mathbb{D}),$$

it is sufficient to show that

$$\psi(ix, y; z) = \frac{y(1-\gamma)}{ix(1-\gamma) + \gamma} + p \notin \Omega$$

for all real $x, y \leq -\frac{1+x^2}{2}$ ($n = 1$ in the Lemma in Section 1) and for all $z \in \mathbb{D}$. Indeed, we have

$$\begin{aligned} \Re(\psi(ix, y; z)) &\leq \frac{-\left(\frac{1+x^2}{2}\right)(1-\gamma)\gamma}{\gamma^2 + x^2(1-\gamma)^2} + p \\ &= -\frac{\gamma}{2(1-\gamma)} + \frac{\frac{\gamma(2\gamma-1)}{2(1-\gamma)}}{\gamma^2 + x^2(1-\gamma)^2} + p \equiv \varphi(\gamma, p, x). \end{aligned}$$

Also, from

$$\frac{\partial}{\partial x} \varphi(\gamma, p, x) = \frac{\gamma(1-\gamma)(1-2\gamma)x}{[\gamma^2 + x^2(1-\gamma)^2]^2},$$

we find that $\varphi(\gamma, p, x)$ is a piecewise monotone function of x (which increases for $x \geq 0$ and decreases for $x < 0$ when $0 < \gamma < \frac{1}{2}$ and which decreases for $x \geq 0$ and increases for $x < 0$ when $\frac{1}{2} < \gamma < 1$). So, we obtain

$$\varphi(\gamma, p, x) \leq \begin{cases} \lim_{x \rightarrow +\infty} \varphi(\gamma, p, x) = p - \frac{\gamma}{2(1-\gamma)} & (0 < \gamma < \frac{1}{2}) \\ \varphi(\gamma, p, 0) = p - \frac{1-\gamma}{2\gamma} & (\frac{1}{2} \leq \gamma < 1) \end{cases} = \beta(\gamma, p).$$

□

Remark 3. The value $\beta(\gamma, p)$ is well-defined, that is, $0 \leq \beta(\gamma, p) < p$. It can be verified by using the fact that, for $0 < \gamma < 1$, we have

$$\frac{1}{2(1-\gamma)} > \frac{1}{2} \quad \text{and} \quad \frac{1}{2\gamma} > \frac{1}{2}.$$

The function $\beta(\gamma, p)$, with respect to the variable γ , is continuous on $(0, 1)$ and has the same limit value at the end points of the interval:

$$\lim_{\gamma \rightarrow 0+} \beta(\gamma, p) = \lim_{\gamma \rightarrow 1-} \beta(\gamma, p) = p.$$

Also, from

$$\frac{\partial}{\partial \gamma} \beta(\gamma, p) = \begin{cases} -\frac{1}{2(1-\gamma)^2} & (0 < \gamma < \frac{1}{2}) \\ \frac{1}{2\gamma^2} & (\frac{1}{2} < \gamma < 1), \end{cases}$$

we realize that $\beta(\gamma, p)$ is a strictly decreasing function of the variable γ on the interval $(0, \frac{1}{2})$ and strictly increasing on the interval $(\frac{1}{2}, 1)$. Therefore, for the inverse function of $\beta(\gamma, p)$ (over the variable γ), there are two choices: $\frac{2(p-\beta)}{1+2(p-\beta)}$ (which corresponds to $0 < \gamma < \frac{1}{2}$) and $\frac{1}{1+2(p-\beta)}$ (which corresponds to $\frac{1}{2} < \gamma < 1$). Since we are interested in the one that gives larger values, for the inverse function, we choose the one that corresponds to $\frac{1}{2} < \gamma < 1$ and gives values in the interval $(\frac{1}{2}, 1)$, instead of the other that gives values in $(0, \frac{1}{2})$, that is,

$$\gamma(\beta, p) \equiv \beta^{-1}(\gamma, p) = \frac{1}{1+2(p-\beta)},$$

with domain $(p - \frac{1}{2}, p)$ for the variable β .

We now rewrite Theorem 3 in the following form.

Theorem 4. *Let $p \in \mathbb{N}$, $p - \frac{1}{2} \leq \beta < p$ and*

$$\gamma \equiv \gamma(\beta, p) = \frac{1}{1+2(p-\beta)}.$$

If $f \in \mathcal{A}_p$ and

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > \beta \quad (z \in \mathbb{D}),$$

then

$$\Re \left(\frac{f(z)}{z^p} \right) > \gamma \quad (z \in \mathbb{D}).$$

Upon setting $p = 1$ in Theorem 3 and Theorem 4, we get the following Corollary.

Corollary 1. *Let $f \in \mathcal{A}$. Then the following assertions hold true.*

(i) *If $0 < \gamma < 1$, then*

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > \begin{cases} \frac{3}{2} - \frac{1}{2(1-\gamma)} & (0 < \gamma < \frac{1}{2}) \\ \frac{3}{2} - \frac{1}{2\gamma} & (\frac{1}{2} \leq \gamma < 1) \end{cases} \implies \Re \left(\frac{f(z)}{z} \right) > \gamma.$$

(ii) *If $\frac{1}{2} \leq \beta < 1$, then*

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > \beta \implies \Re \left(\frac{f(z)}{z} \right) > \frac{1}{3-2\beta}.$$

All of the above inequalities correspond to the whole unit disk $\mathbb{D}$.

Remark 4. Each of the following observations is worthy of mention here.

- (i) For $\gamma = \frac{1}{2}$ in the above Corollary, we get $\beta = \frac{1}{2}$, which is the well-known Marx-Strohhäcker result implied in (1.4). The same conclusion follows for $\beta = \frac{1}{2}$ in the above Corollary.
- (ii) Srivastava *et al.* [10] studied the functions $f \in \mathcal{A}_p$ satisfying the following condition:

$$(1-\lambda) \left(\frac{zf^{(j)}(z)}{f^{(j-1)}(z)} \right) + \lambda \left(1 + \frac{zf^{(j+1)}(z)}{f^{(j)}(z)} \right) \prec (p-j+1) \left(\frac{1+Az}{1+Bz} \right)$$

$$(1 \leq j \leq p; -1 \leq B < A \leq 1),$$

where the symbol $\prec$ denotes the usual subordination between analytic functions. In fact, Srivastava *et al.* [10] obtained their result over

$$\left(\frac{f^{(j-1)}(z)}{z^{p-j+1}} \right)^v,$$

but only in the case when $\lambda > 0$ (see [10, p. 332, Theorem 5]). Therefore, for $j = A = -B = v = 1$, the results presented in this section provide an extension of those given in [10].

4. A UNIFIED PRESENTATION OF THE MARX-STROHHÄCKER TYPE RESULTS

Theorem 4, when combined with the aforementioned *sharp* result of Srivastava *et al.* [10], implies the following theorem.

Theorem 5. Let $p \in \mathbb{N}$ and $\frac{p-1}{2} \leq \alpha < p$. Suppose also that

$$\beta \equiv \widehat{\beta}_1(\alpha, p) = \frac{p}{{}_2F_1(1, 2(p-\alpha); p+1; \frac{1}{2})} \geq p - \frac{1}{2}$$

and

$$\gamma \equiv \gamma(\beta, p) = \frac{1}{1+2(p-\beta)}.$$

If $f \in \mathcal{A}_p$, then

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \implies \Re \left(\frac{zf'(z)}{f(z)} \right) > \beta \implies \Re \left(\frac{f(z)}{z^p} \right) > \gamma.$$

All of the above inequalities hold true in the whole unit disk $\mathbb{D}$.

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ON THE OSCILLATION OF SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DAMPING

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Abstract. In this paper, we are concerned with the oscillations in forced second order nonlinear differential equations with nonlinear damping terms. By using classical variational principle and averaging technique, new oscillation criteria are established, which revise, improve and extend some recent results. Furthermore our study answers the comment [16]. Examples are also given to illustrate the results.

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1. INTRODUCTION

Consider the second-order nonlinear forced differential equation for $t \geq t_0 > 0$

$$(r(t)k_1(x, x'))' + p(t)k_2(x, x')x' + q(t)f(x) = e(t), \quad (1.1)$$

where $p, q \in C([t_0, \infty), \mathbb{R})$, $r \in C^1([t_0, \infty), (0, \infty))$, $f, e \in C(\mathbb{R}, \mathbb{R})$, $k_1 \in C^1(\mathbb{R}^2, \mathbb{R})$ and $k_2 \in C(\mathbb{R}^2, \mathbb{R})$. We restrict our attention to solutions of Eq. (1.1) which exists on $[t_0, \infty)$. As usual, such a solution, $x(t)$, is said to be oscillatory if it has arbitrarily zeros for all $t_0 \geq 0$, otherwise, it is called nonoscillatory. Eq. (1.1) is called oscillatory if all solutions are oscillatory.

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions for different classes of second-order nonlinear forced (unforced) differential equations with damping see, for instance, [1, 3, 5–8, 12–15, 17, 18, 20, 21, 23].

Many results are established for the particular cases of Eq. (1.1); for example, Wong [20] studied the equation for $t \geq t_0$

$$x''(t) + p(t)x'(t) + q(t)f(x(t)) = 0. \quad (1.2)$$

In [15, 21], the authors obtained oscillation criteria for the equation for $t \geq t_0$

$$(r(t)x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0. \quad (1.3)$$

Tiryaki and Zafer [17], Mustafa, Rogovchenko and Rogovchenko [12] obtained several oscillation results for the nonlinear equation for $t \geq t_0$,

$$(r(t)\psi(x(t))x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0. \quad (1.4)$$

The general nonlinear differential equation with damping for $t \geq t_0$

$$(r(t)k(x(t), x'(t))x'(t))' + p(t)k(x(t), x'(t))x'(t) + q(t)f(x(t)) = 0, \quad (1.5)$$

has been considered recently by Ayanlar and Tiryaki [1] and S. P. Rogovchenko, Yu. V. Rogovchenko [13].

The more general nonlinear differential equation with damping for $t \geq t_0$,

$$(r(t)k_1(x(t), x'(t)))' + p(t)k_2(x(t), x'(t))x'(t) + q(t)f(x(t)) = 0 \quad (1.6)$$

has been first studied by Rogovchenko and Rogovchenko [14], which coincides our main equation with $e(t) = 0$. Later, Rogovchenko's study [14] has been extended by Tiryaki and Zafer [19]. They obtained several oscillation criteria for solution of Eq. (1.6) under some relationships between the functions k_1 and k_2 .

In 2006, Zhao and Meng [22] obtained some oscillation results for the nonlinear differential equation Eq. (1.6). They established new oscillation criteria which are extension and generalization of some known results by using the Riccati technique and the kernel functions of Philos' type under the following assumptions

(A1) $p(t) \geq 0$ for all $t \geq t_0$, $xf(x) > 0$ for all $x \neq 0$;

(A2) $vk_1(u, v) \geq \beta_1 |k_1(u, v)|^{\frac{\alpha+1}{\alpha}}$ for some $\beta_1 > 0$, and all $(u, v) \in \mathbb{R}^2$;

(A3) $vk_2(u, v)f^{\frac{1}{\alpha}}(u) \geq \beta_2 |k_1(u, v)|^{\frac{\alpha+1}{\alpha}}$ for some $\beta_2 > 0$, and all $(u, v) \in \mathbb{R}^2$;

(A4) $f'(x)$ exists and $\frac{f'(x)}{|f(x)|^{\frac{\alpha-1}{\alpha}}} \geq \beta_3 > 0$ for some positive constant β_3 and for all $x \in \mathbb{R} \setminus \{0\}$;

or

(A5) $q(t) \geq 0$ for all $t \geq t_0$, f satisfies $\frac{f(x)}{x} \geq L$ for some positive constant L and for all $x \neq 0$;

(A6) $vk_1(u, v) \geq \beta_4 |k_1(u, v)|^{\frac{\alpha+1}{\alpha}} u^{\frac{\alpha-1}{\alpha}}$ for some positive constant β_4 and for all $v \in \mathbb{R} \setminus \{0\}$ and all $u \in \mathbb{R}$;

(A7) $vu^{\frac{1}{\alpha}}k_2(u, v) \geq \beta_5 |k_1(u, v)|^{\frac{\alpha+1}{\alpha}}$ for some positive constant β_5 and for all $(u, v) \in \mathbb{R}^2$.

In 2007, Çakmak and Tiryaki [2] showed that the proof given by Zhao and Meng [22] are inaccurate when $x(t) < 0$ for $t \geq t_0$, because if we take $x(t) < 0$ then by assumptions (A1) and (A5) $f(x)$ becomes negative, therefore (A3), (A6) and (A7) are not satisfied. Thus, although, Zhao and Meng's study [22] is very interesting and well-organized, there are some important mistakes. Therefore, Çakmak and Tiryaki [2] suggested to change the conditions (A3), (A6) and (A7) by replacing with (A3a), (A6a) and (A7a) such that

(A3a) $vk_2(u, v)|f(u)|^{\frac{1}{\alpha}} \geq \beta_2 |k_1(u, v)|^{\frac{\alpha+1}{\alpha}}$ for some $\beta_2 > 0$, and all $(u, v) \in \mathbb{R}^2$;

(A6a) $vk_1(u, v) \geq \beta_4 |k_1(u, v)|^{\frac{\alpha+1}{\alpha}} |u|^{\frac{\alpha-1}{\alpha}}$ for some positive constant β_4 , and for all $v \in \mathbb{R} \setminus \{0\}$ and all $u \in \mathbb{R}$;

(A7a) $v|u|^{\frac{1}{\alpha}} k_2(u, v) \geq \beta_5 |k_1(u, v)|^{\frac{\alpha+1}{\alpha}}$ for some positive constant β_5 and for all $(u, v) \in \mathbb{R}^2$.

In 2008, Huang and Meng [9], take into considerations of Çakmak and Tiryaki's paper [2] and obtained some oscillation results for the nonlinear equation Eq. (1.6) under the same assumptions (A1)-(A7) in Zhao and Meng's paper [21] but replace the conditions (A3), (A6) and (A7) with (A3a), (A6a) and (A7a).

In 2011, Shang and Qin [16] showed that if Huang and Meng's conditions are taken into consideration there exists restriction on f and k_1 due to (A4) $f(u)$ and k_1 becomes $f(u) \geq \left(\frac{\beta_3}{\alpha}\right)^\alpha |u|^\alpha$ and $|k_1(u, v)| = \beta_1^{-\alpha} |v|^\alpha$ by the assumption (A2) and by the choice of $vk_1(u, v) = \beta_1 |k_1(u, v)|^{\frac{\alpha+1}{\alpha}}$. Then they showed that if (A3a) holds and letting $f(u) = \left(\frac{\beta_3}{\alpha}\right)^\alpha |u|^{\alpha-1} u$, $k_1(u, v) = |v|^{\alpha-1} v$, we obtain $vk_2(u, v) \geq \beta_2 \beta_3^{-1} |v|^{\frac{\alpha+1}{\alpha}} |u|^{-1}$. On the other hand k_2 become discontinuous at $u = 0$, which leads to contradiction by the assumption of k_2 . Additionally, in their Example 5.2 in [9], the authors considered the equation

$$(r(t)k_1(x, x'))' + p(t)k_2(x, x')x' + q(t)f(x) = 0,$$

with

$$k_1(u, v) = |v|^{p-1}v \quad \text{and} \quad k_2(u, v) = \frac{|v|^{p-1}v}{u(1+u^2)^{\frac{1}{p}}}.$$

But Shang and Qin [16] also showed that

$$vk_2(u, v) = \frac{|v|^{p-1}v^2}{u(1+u^2)^{\frac{1}{p}}}$$

is discontinuous at $u = 0$ and

$$k_1(u, v) = |v|^{p-1}v$$

is not continuously differentiable at $v = 0$. Therefore, oscillatory solutions of the given equation in Example 5.2 in [9] does not exist. Thus it seems that the conditions (A3a), (A6a) and (A7a) still need reconsideration.

Since the conditions of [9] used in the proofs, the recent papers [11] and [10] also need revision. Motivated by this fact, in this paper, first we will investigate the oscillatory behavior of second-order nonlinear forced differential equation Eq. (1.1) by revising the conditions (A3a), (A6a) and (A7a) to overcome the difficulties that we mentioned above. Secondly, we define a new form of the functional $A_{s_i}^{t_i}(h; t)$ as $A_{s_i}^{t_i}(h; \alpha + 1)$. By this way we overcome the problems of the singularity of the functional $A_{s_i}^{t_i}(h; t)$ at the points s_i, t_i ($i = 1, 2$) and the problems of the inapplicability of

the functional $A_{s_i}^{t_i}(h; t)$ for some α as mentioned in Theorem 2.1 in [20], again Theorem 2.1 in [2] and theorem 1.2 in [15] and so forth. Finally we give some illustrated examples.

2. MAIN RESULTS

Firstly we introduce the general mean and some well known properties that will be used in the proofs of our results.

Let

$$D(s_i, t_i) = \{u \in C^1[s_i, t_i] : u(t) \neq 0 \text{ for } t \in (s_i, t_i), u(s_i) = u(t_i) = 0\}, \quad (2.1)$$

for $i = 1, 2$. We define the functional $A_{s_i}^{t_i}(\cdot; n)$ for $H \in D(s_i, t_i)$ and $n \geq 0$ such as;

$$A_{s_i}^{t_i}(h; n) = \int_{s_i}^{t_i} |H(t)|^n h(t) dt, \quad s_i \leq t \leq t_i, i = 1, 2, \quad (2.2)$$

where $h \in C([t_0, \infty), [0, \infty))$. It is easily seen that the linear functional $A_{s_i}^{t_i}(\cdot; n)$ satisfies the conditions

- (1) $A_{s_i}^{t_i}(h; n) = A_{s_i}^{t_i}(|H|^k h; n - k)$, for $i = 1, 2$ and $k \in \mathbb{R}$;
- (2) $A_{s_i}^{t_i}(h'; n) \geq -A_{s_i}^{t_i}(n|H'h|; n - 1)$, for $i = 1, 2$.

In this section, we shall make use of the following conditions:

- (B₁) $f(x)$ is differentiable and $xf'(x) > 0$ for all $x \neq 0$,
- (B₂) $vk_1(u, v) \geq \beta_1 |k_1(u, v)|^{(\alpha+1)/\alpha}$, for some $\alpha > 0$, $\beta_1 > 0$ and for all $(u, v) \in \mathbb{R}^2$.
- (B₃) $uvk_2(u, v) \geq 0$ for all $(u, v) \in \mathbb{R}^2$,
- (B₄) $f'(x)$ exists and $\frac{f'(x)}{|f(x)|^{\frac{\alpha-1}{\alpha}}} \geq \beta_2 > 0$ for some positive constant β_2 and for all $x \in \mathbb{R} \setminus \{0\}$,
- (B₅) $\frac{f(x)}{x} \geq K|x|^{\gamma-1}$ for some $K > 0$, $\gamma \geq 1$ and for all $x \neq 0$,
- (B₆) $vk_1(u, v) \geq \beta_3 |k_1(u, v)|^{(\alpha+1)/\alpha} |u|^{(\alpha-1)/\alpha}$, for some $\alpha > 0$, $\beta_3 > 0$ and for all $u \in \mathbb{R}$, $v \neq 0$.

Theorem 1. Suppose the conditions (B₁) – (B₄) holds and for any $T \geq t_0$, there exists $T \leq s_1 < t_1 \leq s_2 < t_2$ such that

$$e(t) \leq 0 \text{ for } t \in [s_1, t_1], e(t) \geq 0 \text{ for } t \in [s_2, t_2] \text{ and } p(t) > 0 \text{ on } [s_1, t_1] \cup [s_2, t_2]. \quad (2.3)$$

Let $D(s_i, t_i)$ and $A_{s_i}^{t_i}(\cdot; n)$ are defined by (2.1) and (2.2) respectively. If there exists $H \in D(s_i, t_i)$ and a nonnegative constant n such that

$$A_{s_i}^{t_i}(q; n + \alpha + 1) > A_{s_i}^{t_i}(\delta r |H'|^{\alpha+1}; n), \quad (2.4)$$

for $i = 1, 2$, then Eq. (1.1) is oscillatory. Where $\delta = \left(\frac{\alpha}{\beta_1\beta_2}\right)^\alpha \left(\frac{n+\alpha+1}{\alpha+1}\right)^{\alpha+1}$.

Proof. On the contrary, suppose that Eq. (1.1) has a nonoscillatory solution $x(t)$. Then $x(t)$ eventually must have one sign, i.e. $x(t) \neq 0$ on $[T_0, \infty)$ for some large $T_0 \geq t_0$. By the assumptions, there exists s_1, t_1, s_2 and t_2 such that $T_0 \leq s_1 < t_1 \leq s_2 < t_2$ and (2.3) holds. Define

$$w(t) = \frac{r(t)k_1(x(t), x'(t))}{f(x(t))}, \quad t \in [s_1, t_1] \cup [s_2, t_2]. \quad (2.5)$$

Then differentiating (2.5) and using Eq. (1.1) we obtain

$$\begin{aligned} w'(t) = & -q(t) - \frac{p(t)k_2(x(t), x'(t))f(x(t))x'(t)}{f^2(x(t))} \\ & - \frac{r(t)k_1(x(t), x'(t))x'(t)f'(x(t))}{f^2(x(t))} + \frac{e(t)}{f(x(t))}. \end{aligned}$$

By using assumptions $(B_1) - (B_4)$ we obtain for $t \in [s_1, t_1] \cup [s_2, t_2]$

$$w'(t) \leq -q(t) - \frac{\beta_1\beta_2}{r^{1/\alpha}(t)} |w(t)|^{(\alpha+1)/\alpha} + \frac{e(t)}{f(x(t))}. \quad (2.6)$$

On the intervals $[s_1, t_1]$ (if $x(t) > 0$) or $[s_2, t_2]$ (if $x(t) < 0$), $w(t)$ satisfies

$$w'(t) \leq -q(t) - a(t) |w(t)|^{(\alpha+1)/\alpha}, \quad (2.7)$$

where $a(t) = \beta_1\beta_2 r^{-1/\alpha}(t)$.

Now first we assume that $x(t) > 0$ on $[T_0, \infty)$ for some large $T_0 \geq t_0$. Multiplying $|H(t)|^{n+\alpha+1}$ throughout Eq.(2.7) and integrating from s_i to t_i for $i = 1$, we obtain

$$A_{s_1}^{t_1}(q; n + \alpha + 1) \leq A_{s_1}^{t_1}((n + \alpha + 1) |H|^\alpha |H'| |w| - a |H|^{\alpha+1} |w|^{(\alpha+1)/\alpha}; n), \quad (2.8)$$

where $D(s_i, t_i)$ is given by hypotheses. Setting

$$F(v) := (n + \alpha + 1) |H|^\alpha |H'| |v - a |H|^{\alpha+1} v^{(\alpha+1)/\alpha}, \quad v > 0,$$

we have $F'(v^*) = 0$ and $F''(v^*) < 0$, where $v^* = \left(\frac{\alpha(n+\alpha+1)}{\alpha+1} \frac{1}{a} |H'| \frac{H'}{H}\right)^\alpha$, which implies that $F(v)$ obtains its maximum at v^* . So we have

$$F(v) \leq F(v^*) = \left(\frac{\alpha}{a}\right)^\alpha \left(\frac{n + \alpha + 1}{\alpha + 1}\right)^{\alpha+1} |H'|^{\alpha+1}. \quad (2.9)$$

Then we get, by using (2.9) in (2.8)

$$A_{s_i}^{t_i}(q; n + \alpha + 1) \leq A_{s_i}^{t_i}(\delta r |H'|^{\alpha+1}; n), \quad (2.10)$$

which contradicts to (2.4) for $i = 1$.

If $x(t) < 0$ on $[T_0, \infty)$ for some large $T_0 \geq t_0$, we get the inequality (2.6) again, which implies that (2.7) holds on the interval $[s_2, t_2]$. Applying operator $A_{s_i}^{t_i}(\cdot; n)$ for

$i = 2$ as mentioned above to (2.7) this time, we get the same contradiction to (2.4) for $i = 2$. Thus the proof is complete. $\square$

Lemma 1. [4] If A and B are non-negative constants and $m, n \in \mathbb{R}$ such that $\frac{1}{m} + \frac{1}{n} = 1$, then

$$\frac{1}{m}A + \frac{1}{n}B \geq A^{1/m} B^{1/n}.$$

Theorem 2. Suppose the conditions (B_1) , (B_3) , (B_5) , (B_6) holds and for any $T \geq t_0$, there exists $T \leq s_1 < t_1 \leq s_2 < t_2$ such that condition (2.3) holds and $q(t) \geq 0$ on $[s_1, t_1] \cup [s_2, t_2]$. If there exists a $H \in D(s_i, t_i)$ and a nonnegative constant n such that

$$A_{s_i}^{t_i}(Q; n + \alpha + 1) > A_{s_i}^{t_i}(\delta_1 r |H'|^{\alpha+1}; n), \quad (2.11)$$

for $i = 1, 2$, where

$$Q(t) = \gamma(\gamma - 1)^{(1-\gamma)/\gamma} [Kq(t)]^{1/\gamma} |e(t)|^{(\gamma-1)\gamma}, \quad \delta_1 = \left(\frac{\alpha}{\beta_3}\right)^\alpha \left(\frac{n + \alpha + 1}{\alpha + 1}\right)^{\alpha+1},$$

with the convention $0^0 = 1$. Then Eq. (1.1) is oscillatory.

Proof. On the contrary, suppose that Eq. (1.1) has a nonoscillatory solution $x(t)$. Then $x(t)$ eventually must have one sign. First we assume that $x(t) > 0$ on $[T_0, \infty)$ for some large $T_0 \geq t_0$. Define

$$\theta(t) = \frac{r(t)k_1(x(t), x'(t))}{x(t)}, \quad t \in [s_1, t_1] \cup [s_2, t_2]. \quad (2.12)$$

Then differentiating (2.12) and using Eq. (1.1) we obtain

$$\begin{aligned} \theta'(t) = & -\frac{q(t)f(x(t))}{x(t)} - \frac{p(t)k_2(x(t), x'(t))x(t)}{x^2(t)} \\ & - \frac{r(t)k_1(x(t), x'(t))x'(t)}{x^2(t)} + \frac{e(t)}{x(t)}. \end{aligned}$$

By using (B_1) , (B_2) , (B_3) , (B_6) and Eq. (2.3) we obtain for $t \in [s_1, t_1] \cup [s_2, t_2]$,

$$\theta'(t) \leq -q(t)K|x(t)|^{\gamma-1} - \frac{\beta_3}{r^{1/a}(t)}|\theta(t)|^{(\alpha+1)/\alpha} + \frac{e(t)}{x(t)}. \quad (2.13)$$

On the interval $[s_1, t_1]$, inequality (2.13) implies that $\theta(t)$ satisfies

$$q(t)K|x(t)|^{\gamma-1} + \left|\frac{e(t)}{x(t)}\right| \leq -\theta'(t) - \frac{\beta_3}{r^{1/a}(t)}|\theta(t)|^{(\alpha+1)/\alpha}. \quad (2.14)$$

For $\gamma > 1$, by setting $m = \gamma$, $n = \gamma/(\gamma - 1)$, $A = \gamma K q(t) |x(t)|^{\gamma-1}$,
 $B = \left(\frac{\gamma}{\gamma-1}\right) \left|\frac{e(t)}{x(t)}\right|$ and using Lemma 1, we obtain

$$q(t) K |x(t)|^{\gamma-1} + \left|\frac{e(t)}{x(t)}\right| \geq Q(t). \quad (2.15)$$

Hence, on the interval $[s_1, t_1]$, $\theta(t)$ satisfies

$$\theta'(t) \leq -Q(t) - \frac{\beta_3}{r^{1/a}(t)} |\theta(t)|^{(\alpha+1)/\alpha}. \quad (2.16)$$

Note that the inequality holds for $\gamma = 1$ also with the convention $0^0 = 1$.

Applying operator $A_{s_i}^{t_i}(\cdot; n)$ for $i = 1$, to (2.15) we obtain a contradiction to (2.11), this part of the proof is similar to Theorem 1 and hence omitted.

If $x(t) < 0$ on $[T_0, \infty)$ for some large $T_0 \geq t_0$, it is easy to see that (2.16) holds for $t \in [s_2, t_2]$. Then applying operator $A_{s_i}^{t_i}(\cdot; n)$ for $i = 2$, we still obtain contradiction. Thus the proof is complete. $\square$

Now we will give some other oscillation criteria for (1.1) through some new averaging functions $G(t, s) \in C(D_1, \mathbb{R})$, which satisfy:

- (i) $G(t, t) = 0$, $G(t, s) > 0$ for $t > s$,
- (ii) G has partial derivatives $\partial G/\partial t$ and $\partial G/\partial s$ on D_1 such that

$$\frac{\partial G}{\partial t} = g_1(t, s) \sqrt{G(t, s)}, \quad \frac{\partial G}{\partial s} = -g_2(t, s) \sqrt{G(t, s)}$$

where $D_1 = \{(t, s) : t_0 \leq s \leq t < \infty\}$ and $g_1, g_2 \in L_{loc}(D_1, \mathbb{R}^+)$.

Theorem 3. Suppose the conditions $(B_1) - (B_4)$ holds and for any $T \geq t_0$, there exists $T \leq s_1 < t_1 \leq s_2 < t_2$ such that condition (2.3) holds. If there exist some $\varepsilon_i \in (s_i, t_i)$, $i = 1, 2$, $G(t, s)$ satisfying (i)-(ii) and a positive function $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$ such that

$$\begin{aligned} & \frac{1}{G^{\alpha+1}(\varepsilon_i, s_i)} \int_{s_i}^{\varepsilon_i} [G^{\alpha+1}(\tau, s_i) q(\tau) \rho(\tau) - \delta_2 G_1^{\alpha+1}(\tau, s_i) r(\tau) \rho(\tau)] d\tau \\ & + \frac{1}{G^{\alpha+1}(t_i, \varepsilon_i)} \int_{\varepsilon_i}^{t_i} [G^{\alpha+1}(t_i, \tau) q(\tau) \rho(\tau) - \delta_2 G_2^{\alpha+1}(t_i, \tau) r(\tau) \rho(\tau)] d\tau \quad (2.17) \\ & > 0 \end{aligned}$$

for $i = 1, 2$ where

$$\delta_2 = \frac{\alpha^\alpha}{\beta_1^\alpha \beta_2^\alpha (\alpha + 1)^{\alpha+1}},$$

$$\begin{aligned} G_1(t, s) &= \left| (\alpha + 1) g_1(t, s) \sqrt{G(t, s)} + G(t, s) \frac{\rho'(s)}{\rho(s)} \right|, \\ G_2(t, s) &= \left| (\alpha + 1) g_2(t, s) \sqrt{G(t, s)} - G(t, s) \frac{\rho'(s)}{\rho(s)} \right|. \end{aligned}$$

Then Eq. (1.1) is oscillatory.

Proof. On the contrary, suppose that Eq. (1.1) has a nonoscillatory solution $x(t)$. Then $x(t) \neq 0$ on $[T, \infty)$ for some sufficiently large $T \geq t_0$. Define

$$w_1(t) = \rho(t) \frac{r(t) k_1(x(t), x'(t))}{f(x(t))}, t \in [s_1, t_1] \cup [s_2, t_2]. \quad (2.18)$$

Differentiating (2.18), using conditions $(B_1) - (B_4)$ and Eq. (1.1) we obtain for $t \in [s_1, t_1] \cup [s_2, t_2]$

$$\begin{aligned} w_1'(t) &\leq \\ -q(t) \rho(t) &+ \frac{\rho'(t)}{\rho(t)} w_1(t) - \beta_1 \beta_2 r^{-1/\alpha}(t) \rho^{-1/\alpha}(t) |w_1(t)|^{(\alpha+1)/\alpha} + \frac{e(t)}{f(x(t))} \rho(t). \end{aligned}$$

This implies that on the intervals $[s_1, t_1]$ (if $x(t) > 0$) or $[s_2, t_2]$ (if $x(t) < 0$), $w_1(t)$ satisfies the inequality

$$w_1'(t) \leq -q(t) \rho(t) + \frac{\rho'(t)}{\rho(t)} w_1(t) - \beta_1 \beta_2 r^{-1/\alpha}(t) \rho^{-1/\alpha}(t) |w_1(t)|^{(\alpha+1)/\alpha}. \quad (2.19)$$

On the one hand, multiplying (2.19) with $G^{\alpha+1}(t, s)$ and integrating (with t replaced by s) over $[\varepsilon_i, t]$ for $t \in [\varepsilon_i, t_i]$ and $i = 1, 2$ we have

$$\begin{aligned} \int_{\varepsilon_i}^t G^{\alpha+1}(t, s) q(s) \rho(s) ds &\leq G^{\alpha+1}(t, \varepsilon_i) w_1(\varepsilon_i) \\ &+ \int_{\varepsilon_i}^t [G^\alpha(t, s) G_2(t, s) |w_1(s)|] ds \\ &- \int_{\varepsilon_i}^t \left[\beta_1 \beta_2 r^{-1/\alpha}(t) \rho^{-1/\alpha}(t) G^{\alpha+1}(t, s) |w_1(s)|^{(\alpha+1)/\alpha} \right] ds. \end{aligned} \quad (2.20)$$

For a given t and s , set

$$F(v) = G^\alpha G_2 v - \beta_1 \beta_2 r^{-1/\alpha} \rho^{-1/\alpha} G^{\alpha+1} v^{(\alpha+1)/\alpha}, \quad v > 0.$$

F yields its maximum at the point $v^* = \left(\frac{\alpha}{\alpha+1} \frac{G_2}{\beta_1 \beta_2 G r^{-1/\alpha} \rho^{-1/\alpha}} \right)^\alpha$ and

$$F(v) \leq F_{\max} = F(v^*) = \delta_2 G_2 r \rho. \quad (2.21)$$

Then, by using (2.21) and letting $t \rightarrow t_i^-$ in (2.20), we get

$$\begin{aligned} & \int_{\varepsilon_i}^{t_i} G^{\alpha+1}(t_i, s) q(s) \rho(s) ds \\ & \leq G^{\alpha+1}(t_i, \varepsilon_i) w_1(\varepsilon_i) + \delta_2 \int_{\varepsilon_i}^{t_i} G_2^{\alpha+1}(t_i, s) r(s) \rho(s) ds. \end{aligned} \quad (2.22)$$

On the other hand, multiplying (2.19) with $G^{\alpha+1}(s, t)$, then integrating (with t replaced by s) over $[t, \varepsilon_i)$ for $t \in [t_i, \varepsilon_i)$, $i = 1, 2$ and using similar calculations with the proof of (2.21) we get

$$\begin{aligned} & \int_t^{\varepsilon_i} G^{\alpha+1}(s, s_i) q(s) \rho(s) ds \leq \\ & -G^{\alpha+1}(\varepsilon_i, s_i) w_1(\varepsilon_i) + \delta_2 \int_t^{\varepsilon_i} G_1^{\alpha+1}(s, s_i) r(s) \rho(s) ds. \end{aligned} \quad (2.23)$$

Letting $t \rightarrow s_i^+$ in (2.23), it follows that

$$\begin{aligned} & \int_{s_i}^{\varepsilon_i} G^{\alpha+1}(s, s_i) q(s) \rho(s) ds \\ & \leq -G^{\alpha+1}(\varepsilon_i, s_i) w_1(\varepsilon_i) + \delta_2 \int_{s_i}^{\varepsilon_i} G_1^{\alpha+1}(s, s_i) r(s) \rho(s) ds. \end{aligned} \quad (2.24)$$

Finally, dividing (2.22) and (2.24) by $G^{\alpha+1}(t_i, \varepsilon_i)$ and $G^{\alpha+1}(\varepsilon_i, s_i)$ respectively, and then adding them, we have the desired contradiction with (2.17). Thus the proof is complete. $\square$

Corollary 1. Suppose the conditions $(B_1) - (B_4)$ holds and for any $T \geq t_0$, there exists $T \leq s_1 < t_1 \leq s_2 < t_2$ such that condition (2.3) holds. If there exist some $\varepsilon_i \in (s_i, t_i)$, $i = 1, 2$, $G(t, s)$ satisfying (i)-(ii) and a positive function $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$ such that

$$\int_{s_i}^{\varepsilon_i} [G^{\alpha+1}(\tau, s_i) q(\tau) \rho(\tau) - \delta_2 G_1^{\alpha+1}(\tau, s_i) r(\tau) \rho(\tau)] d\tau > 0, \quad (2.25)$$

$$\int_{\varepsilon_i}^{t_i} [G^{\alpha+1}(t_i, \tau) q(\tau) \rho(\tau) - \delta_2 G_2^{\alpha+1}(t_i, \tau) r(\tau) \rho(\tau)] d\tau > 0, \quad (2.26)$$

for $i = 1, 2$. Then Eq. (1.1) is oscillatory.

Theorem 4. Suppose the conditions (B_1) , (B_3) , (B_5) , (B_6) holds and for any $T \geq t_0$, there exists $T \leq s_1 < t_1 \leq s_2 < t_2$ such that condition (2.3) holds and $q(t) \geq 0$

on $[s_1, t_1] \cup [s_2, t_2]$. If there exist some $\varepsilon_i \in (s_i, t_i)$, $i = 1, 2$, $G(t, s)$ satisfying (i)-(ii) and a positive function $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$ such that

$$\begin{aligned} & \frac{1}{G^{\alpha+1}(\varepsilon_i, s_i)} \int_{s_i}^{\varepsilon_i} [G^{\alpha+1}(\tau, s_i) Q(\tau) \rho(\tau) - \delta_3 G_1^{\alpha+1}(\tau, s_i) r(\tau) \rho(\tau)] d\tau \\ & + \frac{1}{G^{\alpha+1}(t_i, \varepsilon_i)} \int_{\varepsilon_i}^{t_i} [G^{\alpha+1}(t_i, \tau) Q(\tau) \rho(\tau) - \delta_3 G_2^{\alpha+1}(t_i, \tau) r(\tau) \rho(\tau)] d\tau \quad (2.27) \\ & > 0 \end{aligned}$$

for $i = 1, 2$ where

$$\delta_3 = \frac{\alpha^\alpha}{\beta_3^\alpha (\alpha + 1)^{\alpha+1}},$$

and H_1, H_2, Q are defined in previous theorems. Then Eq. (1.1) is oscillatory.

Proof. On the contrary, suppose that Eq. (1.1) has a nonoscillatory solution $x(t)$. Then $x(t) \neq 0$ on $[T, \infty)$ for some sufficiently large $T \geq t_0$. Define

$$\theta_1(t) = \rho(t) \frac{r(t) k_1(x(t), x'(t))}{x(t)}, t \in [s_1, t_1] \cup [s_2, t_2]. \quad (2.28)$$

Differentiating (2.28), using conditions (B_1) , (B_3) , (B_5) , (B_6) and Eq. (1.1) we obtain

$$\begin{aligned} & \rho(t) \left(Kq(t) |x(t)|^{\gamma-1} + \left| \frac{e(t)}{x(t)} \right| \right) \leq \\ & -\theta_1'(t) + \frac{\rho'(t)}{\rho(t)} \theta_1(t) - \beta_3 r^{-1/\alpha}(t) \rho^{-1/\alpha}(t) |\theta_1(t)|^{(\alpha+1)/\alpha} \end{aligned}$$

for $t \in [s_1, t_1]$ or $t \in [s_2, t_2]$. By choosing $m = \gamma$, $n = \gamma/(\gamma-1)$, $A = \gamma Kq(t) |x(t)|^{\gamma-1}$ and $B = \gamma/(\gamma-1) |e(t)/x(t)|$ in the Lemma 1 we get

$$\rho(t) Q(t) \leq -\theta_1'(t) + \frac{\rho'(t)}{\rho(t)} \theta_1(t) - \beta_3 r^{-1/\alpha}(t) \rho^{-1/\alpha}(t) |\theta_1(t)|^{(\alpha+1)/\alpha} \quad (2.29)$$

for $t \in [s_1, t_1]$ or $t \in [s_2, t_2]$ and for $\gamma > 1$. Note that the inequality (2.29) holds trivially for $\gamma = 1$.

The rest of the proof is similar with the previous theorem, hence omitted. $\square$

Corollary 2. Suppose the conditions (B_1) , (B_3) , (B_5) , (B_6) holds and for any $T \geq t_0$, there exists $T \leq s_1 < t_1 \leq s_2 < t_2$ such that condition (2.3) holds and $q(t) \geq 0$ on $[s_1, t_1] \cup [s_2, t_2]$. If there exist some $\varepsilon_i \in (s_i, t_i)$, $i = 1, 2$, $G(t, s)$ satisfying (i)-(ii)

and a positive function $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$ such that

$$\int_{s_i}^{\varepsilon_i} [G^{\alpha+1}(\tau, s_i) Q(\tau) \rho(\tau) - \delta_3 G_1^{\alpha+1}(\tau, s_i) r(\tau) \rho(\tau)] d\tau > 0, \quad (2.30)$$

$$\int_{\varepsilon_i}^{t_i} [G^{\alpha+1}(t_i, \tau) Q(\tau) \rho(\tau) - \delta_3 G_2^{\alpha+1}(t_i, \tau) r(\tau) \rho(\tau)] d\tau > 0, \quad (2.31)$$

for $i = 1, 2$. Then Eq. (1.1) is oscillatory.

Now we consider the more general equation

$$(r(t)k_1(x, x'))' + p(t)[Ak_2(x, x')x' + Bk_3(x, x')] + q(t)f(x) = e(t) \quad (2.32)$$

for $t \geq t_0 > 0$, where A, B are nonnegative constants and, $k_3 \in C(\mathbb{R}^2, \mathbb{R})$ with $uk_3(u, v) \geq 0$ for all $(u, v) \in \mathbb{R}^2$. By using same substitutions with the proofs above, it is easy to proof following results. Since all the possible proofs differ from the previous proofs with only the term $uk_3(u, v)$, we rely on nonnegativity of this term.

Theorem 5. Suppose the conditions $(B_1) - (B_4)$ holds and for any $T \geq t_0$, there exists $T \leq s_1 < t_1 \leq s_2 < t_2$ such that (2.3) holds. Let $D(s_i, t_i)$ and $A_{s_i}^{t_i}(\cdot; n)$ are defined by (2.1) and (2.2) respectively. If there exists $H \in D(s_i, t_i)$ and a nonnegative constant n such that the inequality (2.4) holds for $i = 1, 2$, then Eq. (2.32) is oscillatory.

Theorem 6. Suppose the conditions $(B_1), (B_3), (B_5), (B_6)$ holds and for any $T \geq t_0$, there exists $T \leq s_1 < t_1 \leq s_2 < t_2$ such that condition (2.3) holds and $q(t) \geq 0$ on $[s_1, t_1] \cup [s_2, t_2]$. Let $D(s_i, t_i)$ and $A_{s_i}^{t_i}(\cdot; n)$ are defined by (2.1) and (2.2) respectively. If there exists $H \in D(s_i, t_i)$ and a nonnegative constant n such that the inequality (2.11) holds for $i = 1, 2$ with the convention $0^0 = 1$. Then Eq. (2.32) is oscillatory.

Theorem 7. Suppose the conditions $(B_1) - (B_4)$ holds and for any $T \geq t_0$, there exists $T \leq s_1 < t_1 \leq s_2 < t_2$ such that condition (2.3) holds. If there exist some $\varepsilon_i \in (s_i, t_i)$, $i = 1, 2$, $G(t, s)$ satisfying (i)-(ii) and a positive function $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$ such that the inequality (2.17) holds for $i = 1, 2$. Then Eq. (2.32) is oscillatory.

Theorem 8. Suppose the conditions $(B_1), (B_3), (B_5), (B_6)$ holds and for any $T \geq t_0$, there exists $T \leq s_1 < t_1 \leq s_2 < t_2$ such that condition (2.3) holds and $q(t) \geq 0$ on $[s_1, t_1] \cup [s_2, t_2]$. If there exist some $\varepsilon_i \in (s_i, t_i)$, $i = 1, 2$, $G(t, s)$ satisfying (i)-(ii) and a positive function $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$ such that the inequality (2.27) holds for $i = 1, 2$. Then Eq. (2.32) is oscillatory.

Corollary 3. Suppose the conditions $(B_1) - (B_4)$ holds and for any $T \geq t_0$, there exists $T \leq s_1 < t_1 \leq s_2 < t_2$ such that condition (2.3) holds. If there exist some $\varepsilon_i \in (s_i, t_i)$, $i = 1, 2$, $G(t, s)$ satisfying (i)-(ii) and a positive function $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$

such that the inequalities (2.25) and (2.26) holds for $i = 1, 2$. Then Eq. (2.32) is oscillatory.

Corollary 4. Suppose the conditions (B_1) , (B_3) , (B_5) , (B_6) holds and for any $T \geq t_0$, there exists $T \leq s_1 < t_1 \leq s_2 < t_2$ such that condition (2.3) holds and $q(t) \geq 0$ on $[s_1, t_1] \cup [s_2, t_2]$. If there exist some $\varepsilon_i \in (s_i, t_i)$, $i = 1, 2$, $G(t, s)$ satisfying (i)-(ii) and a positive function $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$ such that the inequalities (2.30) and (2.31) holds for $i = 1, 2$. Then Eq. (2.32) is oscillatory.

Remark 1. If the hypotheses on the function $e(t)$ in condition (2.3) is replaced by the following condition

$$e(t) \geq 0 \text{ for } t \in [s_1, t_1] \text{ and } e(t) \leq 0 \text{ for } t \in [s_2, t_2]$$

we will find the condition of the all above theorems and corollaries are valid as well.

Example 1. Consider the equation

$$\left(t^{3\lambda+1}x'(t)\right)' + p(t)x(t)(x'(t))^2 + Mt^{3\lambda}x(t) = \sin t, \quad (2.33)$$

where $t \geq t_0 > 1$, $M > 0$, $\lambda > 0$ and $p(t) > 0$ is any function. It is easy to verify that the conditions $(B_1) - (B_3)$ hold for the functions

$$k_1(u, v) = v, \quad k_2(u, v) = uv, \quad f(u) = u$$

for $\alpha = \beta_1 = 1$.

Moreover let $H(t) = t^{-\lambda} \sin t$ and $s_1 = k\pi$, $t_1 = (k+1)\pi$, $s_2 = (k+1)\pi$, $t_2 = (k+2)\pi$. Then we have

$$A_{s_i}^{t_i}(q; 3) = \int_{s_i}^{t_i} |H(t)|^3 q(t) dt = M \int_{s_i}^{t_i} \sin^6 t dt = \frac{5M}{16} \pi, \text{ for } i = 1, 2.$$

On the other hand,

$$\begin{aligned} A_{s_1}^{t_1}(\delta r |H'|^{\alpha+1}; 1) &= A_{s_2}^{t_2}(\delta r |H'|^{\alpha+1}; 1) \leq \frac{9}{4} \int_{\pi}^{2\pi} (\lambda^2 t^{-1} + 4t - 4\lambda \sin^5 t \cos t) dt \\ &= \frac{27}{2} \pi^2 + \frac{9}{4} \lambda^2 \ln 2 \end{aligned}$$

So, the inequality (2.4) hold for $M > \frac{9}{10\pi} (6\pi^2 + \lambda^{2c} \ln 2)$. Thus Eq. (2.33) is oscillatory if $M > \frac{9}{10\pi} (6\pi^2 + \lambda^2 \ln 2)$ by Theorem 1.

Example 2. Consider the equation

$$\left(t^{3\lambda+1} \frac{x'(t)}{1+[x'(t)]^2}\right)' + \frac{p(t)x(t)(x'(t))^2}{1+[x'(t)]^2} + Nt^{3\lambda}x(t)(\cos x(t) + 2) = \sin t, \quad (2.34)$$

where $t \geq t_0 > 1$, $N > 0$, $\lambda > 0$ and $p(t) > 0$ is any function. Where

$$k_1(u, v) = \frac{v}{1+v^2}, k_2(u, v) = \frac{uv}{1+v^2} \text{ and } f(u) = u(\cos u + 2).$$

It is easy to verify that the conditions (B_1) , (B_3) , (B_5) and (B_6) hold for $\alpha = \beta_3 = K = \gamma = 1$. It is easy to check that $Q = q$ for $K = \gamma = 1$. Thus, by a similar calculating we can show that if $M > \frac{9}{10\pi} (6\pi^2 + \lambda^2 \ln 2)$ then Eq. (2.34) is oscillatory by Theorem 2.

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CERTAIN FAMILIES OF MULTIVARIABLE CHAN-CHYAN-SRIVASTAVA POLYNOMIALS

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Abstract. In this paper we introduce r -parameter Srivastava polynomials in r -variable by inserting new indices. These polynomials include the Lagrange polynomials in several variables, which are also known as Chan-Chyan-Srivastava polynomials (W.-C.C. Chan, C.-J. Chyan and H.M. Srivastava, The Lagrange polynomials in several variables, *Integral Transforms and Special Functions*, 12 (2001) 139–148). We prove several two sided linear generating relations between r -variable and $(r-1)$ -variable Chan-Chyan-Srivastava polynomials. Some special cases of the main results are also presented.

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1. INTRODUCTION

In 1972, Srivastava [21] introduced the following family of polynomials,

$$S_n^N(z) := \sum_{k=0}^{\left[\frac{n}{N}\right]} \frac{(-n)_{Nk}}{k!} A_{n,k} z^k \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; N \in \mathbb{N}), \quad (1.1)$$

where $\mathbb{N}$ is the set of positive integers, $\{A_{n,k}\}_{n,k=0}^{\infty}$ is a bounded double sequence of real or complex numbers, $[a]$ denotes the greatest integer in $a \in \mathbb{R}$, and $(\lambda)_\nu$, $(\lambda)_0 \equiv 1$, denotes the Pochhammer symbol defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)}$$

by means of familiar Gamma functions. After, González et al. [13] extended the Srivastava polynomials $S_n^N(z)$ as follows:

$$S_{n,m}^N(z) := \sum_{k=0}^{\left[\frac{n}{N}\right]} \frac{(-n)_{Nk}}{k!} A_{n+m,k} z^k \quad (m, n \in \mathbb{N}_0; N \in \mathbb{N}), \quad (1.2)$$

and investigated their properties extensively. In [3], the following family of bivariate polynomials has been introduced

$$S_n^{m,N}(x, y) = \sum_{k=0}^{\left[\frac{n}{N}\right]} A_{m+n,k} \frac{x^{n-Nk}}{(n-Nk)!} \frac{y^k}{k!} \quad (m, n \in \mathbb{N}_0, N \in \mathbb{N}),$$

and it has shown that the polynomials $S_n^{m,N}(x, y)$ includes many well known polynomials such as Lagrange-Hermite polynomials, Lagrange polynomials and Hermite-Kampé de Fériet polynomials (see also [20]). In [23], Srivastava et al. have introduced the three-variable polynomials

$$S_n^{m,M,N}(x, y, z) = \sum_{k=0}^{\left[\frac{n}{N}\right]} \sum_{l=0}^{\left[\frac{k}{M}\right]} A_{m+n,k,l} \frac{x^l}{l!} \frac{y^{k-Ml}}{(k-Ml)!} \frac{z^{n-Nk}}{(n-Nk)!} \quad (m, n \in \mathbb{N}_0; M, N \in \mathbb{N}), \quad (1.3)$$

where $\{A_{m,n,k}\}$ be a triple sequence of complex numbers. Suitable choices of $\{A_{m,n,k}\}$ in equation (1) gives three variable version of well-known polynomials (see also [14]). In [15], the multivariable extension of the Srivastava polynomials in r -variable was introduced:

$$\begin{aligned} & S_n^{m,N_1,N_2,\dots,N_{r-1}}(x_1, x_2, \dots, x_r) \\ &= \sum_{k_{r-1}=0}^{\left[\frac{n}{N_{r-1}}\right]} \sum_{k_{r-2}=0}^{\left[\frac{k_{r-1}}{N_{r-2}}\right]} \dots \sum_{k_1=0}^{\left[\frac{k_2}{N_1}\right]} A_{m+n,k_{r-1},k_1,\dots,k_{r-2}} \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2-N_1k_1}}{(k_2-N_1k_1)!} \dots \frac{x_r^{n-N_{r-1}k_{r-1}}}{(n-N_{r-1}k_{r-1})!} \\ & \quad (m, n \in \mathbb{N}_0; N_1, N_2, \dots, N_{r-1} \in \mathbb{N}), \end{aligned} \quad (1.4)$$

where $\{A_{m,k_{r-1},k_1,k_2,\dots,k_{r-2}}\}$ be a sequence of complex numbers. Recently, in [16], two-parameter one-variable Srivastava polynomials

$$S_n^{m_1,m_2}(x) := \sum_{k=0}^n \frac{(-n)_k}{k!} A_{m_1+m_2+n, m_2+k} x^k \quad (1.5)$$

$(m_1, m_2, n, k \in \mathbb{N}_0),$

two-parameter two-variable Srivastava polynomials

$$S_n^{m_1,m_2}(x, y) = \sum_{k=0}^n A_{m_1+m_2+n, m_2+k} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!} \quad (1.6)$$

$(m_1, m_2, n, k \in \mathbb{N}_0),$

and two-parameter three-variable Srivastava polynomials

$$S_n^{m_1, m_2, M}(x, y, z) = \sum_{k=0}^n \sum_{l=0}^{\lfloor \frac{k}{M} \rfloor} A_{m_1+m_2+n, m_2+k, l} \frac{x^l}{l!} \frac{y^{k-Ml}}{(k-Ml)!} \frac{z^{n-k}}{(n-k)!} \quad (1.7)$$

$$(m_1, m_2, n, k, l \in \mathbb{N}_0, M \in \mathbb{N})$$

were introduced. These polynomials include the family of polynomials which were introduced or investigated in [3, 13, 15, 17, 19, 21, 23]. In this paper we introduce r -parameter Srivastava polynomials in r -variables by inserting new indices. These polynomials include the Lagrange polynomials in several variables, which are also known as Chan-Chyan-Srivastava polynomials [4]. We prove several two sided linear generating relations and obtain various generating relations for the polynomials.

2. R-PARAMETER R-VARIABLE SRIVASTAVA POLYNOMIALS

In this section we define r -parameter r -variable Srivastava polynomials as follows:

$$S_n^{m_1, m_2, \dots, m_{r-1}, M}(x_1, x_2, \dots, x_r) \quad (2.1)$$

$$= \sum_{k_{r-1}=0}^n \sum_{k_{r-2}=0}^{k_{r-1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{\lfloor \frac{k_2}{M} \rfloor} A_{\Omega} \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2-Mk_1}}{(k_2-Mk_1)!} \dots \frac{x_r^{n-k_{r-1}}}{(n-k_{r-1})!},$$

$$(m_1, m_2, \dots, m_{r-1}, n \in \mathbb{N}_0; M \in \mathbb{N}),$$

where $\Omega = m_1 + m_2 + \dots + m_{r-1} + n$, $m_2 + \dots + m_{r-1} + k_{r-1}$, k_1 , $m_{r-1} + k_2$, $m_{r-2} + m_{r-1} + k_3, \dots, m_3 + \dots + m_{r-1} + k_{r-2}$. Note that appropriate choices of the sequence $\{A_{m_1+m_2, m_2, k_1, m_{r-1}, \dots, m_3}\}$ in (2.1) give the r -variable versions of the well known polynomials.

Remark 1. The Lagrange polynomials in several variables, which are known as Chan-Chyan-Srivastava polynomials [4] are defined by the generating function relation

$$\prod_{j=1}^r (1 - x_j t)^{-\alpha_j} = \sum_{n=0}^{\infty} g_n^{(\alpha_1, \alpha_2, \dots, \alpha_r)}(x_1, x_2, \dots, x_r) t^n$$

$$(\alpha_j \in \mathbb{C} (j = 1, \dots, r); |t| < \min \{|x_1|^{-1}, \dots, |x_r|^{-1}\}).$$

The polynomials are given explicitly by

$$g_n^{(\alpha_1, \alpha_2, \dots, \alpha_r)}(x_1, x_2, \dots, x_r)$$

$$= \sum_{k_{r-1}=0}^n \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} (\alpha_1)_{k_1} (\alpha_2)_{k_2-k_1} \dots (\alpha_{r-1})_{k_{r-1}-k_{r-2}} (\alpha_r)_{n-k_{r-1}}$$

$$\times \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2-k_1}}{(k_2-k_1)!} \cdots \frac{x_{r-1}^{k_{r-1}-k_{r-2}}}{(k_{r-1}-k_{r-2})!} \frac{x_r^{n-k_{r-1}}}{(n-k_{r-1})!}.$$

Many authors have studied the properties of these polynomials. For example, the bilateral generating functions for these polynomials and miscellaneous properties are given in Liu et al. [12, 18]. In [8], the orthogonality properties and various integral representations for these polynomials are given (see also [1, 2, 5–7]). Furthermore, these polynomials are used in approximation theory. In [11], Duman et al. investigated some approximation properties of positive linear operators constructed by these polynomials (see also [9, 10]). In [22] Srivastava et al. investigated umbral calculus presentations of these polynomials.

Remark 2. If we set $M = 1$ and

$A_{m,n,k_2,\dots,k_{r-2},k_{r-1}} = (\alpha_1)_{k_2}(\alpha_2)_{k_3-k_2}\cdots(\alpha_{r-2})_{k_{r-1}-k_{r-2}}(\alpha_{r-1})_{n-k_{r-1}}(\alpha_r)_{m-n}$
in equation (2.1), we find that

$$\begin{aligned} & A_{m_1+\dots+m_{r-1}+n, m_2+\dots+m_{r-1}+k_{r-1}, k_1, m_{r-1}+k_2, m_{r-2}+m_{r-1}+k_3, \dots, m_3+\dots+m_{r-1}+k_{r-2}} \\ &= (\alpha_1)_{k_1}(\alpha_2)_{m_{r-1}+k_2-k_1}(\alpha_3)_{m_{r-2}+k_3-k_2}\cdots(\alpha_{r-1})_{m_2+k_{r-1}-k_{r-2}}(\alpha_r)_{m_1+n-k_{r-1}} \\ &= (\alpha_2)_{m_{r-1}}(\alpha_3)_{m_{r-2}}\cdots(\alpha_{r-1})_{m_2}(\alpha_r)_{m_1}(\alpha_1)_{k_1}(\alpha_2+m_{r-1})_{k_2-k_1}(\alpha_3+m_{r-2})_{k_3-k_2} \\ &\quad \times \cdots (\alpha_{r-1}+m_2)_{k_{r-1}-k_{r-2}}(\alpha_r+m_1)_{n-k_{r-1}} \end{aligned}$$

and therefore

$$\begin{aligned} & S_n^{m_1, m_2, \dots, m_{r-1}, 1}(x_1, x_2, \dots, x_r) = \\ & (\alpha_2)_{m_{r-1}}(\alpha_3)_{m_{r-2}}\cdots(\alpha_{r-1})_{m_2}(\alpha_r)_{m_1} g_n^{(\alpha_1, \alpha_2+m_{r-1}, \alpha_3+m_{r-2}, \dots, \alpha_r+m_1)}(x_1, \dots, x_r). \end{aligned}$$

Now, let recall an infinite series identities which were obtained by Srivastava et al. [23].

Lemma 1 (see [23], Lemma 1). *Let $N_1, N_2, \dots, N_{r-1} \in \mathbb{N}$, $r = \{2, 3, \dots\}$. Then*

$$\begin{aligned} & \sum_{n_r=0}^{\infty} \sum_{n_{r-1}=0}^{\infty} \cdots \sum_{n_1=0}^{\infty} A(n_1, n_2, \dots, n_r) \\ &= \sum_{n_r=0}^{\infty} \sum_{n_{r-1}=0}^{\left[\frac{n_r}{N_{r-1}}\right]} \cdots \sum_{n_1=0}^{\left[\frac{n_2}{N_1}\right]} A(n_1, n_2 - N_1 n_1, \dots, n_r - N_{r-1} n_{r-1}) \end{aligned}$$

and

$$\sum_{n_r=0}^{\infty} \sum_{n_{r-1}=0}^{\left[\frac{n_r}{N_{r-1}}\right]} \cdots \sum_{n_1=0}^{\left[\frac{n_2}{N_1}\right]} A(n_1, n_2, \dots, n_r)$$

$$= \sum_{n_r=0}^{\infty} \sum_{n_{r-1}=0}^{\infty} \dots \sum_{n_1=0}^{\infty} A(n_1, n_2 + N_1 n_1, n_3 + N_2 N_1 n_1 + N_2 n_2, \dots, \\ n_r + \prod_{j=1}^{r-1} N_j n_1 + \prod_{j=2}^{r-1} N_j n_2 + \dots + N_{r-1} n_{r-1})$$

where $\{A(n_1, n_2, \dots, n_r)\}$ is a bounded r -tuple sequence of real or complex numbers.

The main result of this paper is given by the following theorem.

Theorem 1. Let $\{f(n)\}_{n=0}^{\infty}$ be a bounded sequence of complex numbers. Then

$$\sum_{m_1, \dots, m_{r-1}, n=0}^{\infty} f(m_1 + \dots + m_{r-1} + n) \quad (2.2) \\ \times S_n^{m_1, m_2, \dots, m_{r-1}, M}(x_1, x_2, \dots, x_r) \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} t^n \\ = \sum_{k_1, m_1, \dots, m_{r-1}=0}^{\infty} f(m_1 + \dots + m_{r-1} + M k_1) \\ \times A_{m_1+m_2+\dots+m_{r-1}+M k_1, m_2+\dots+m_{r-1}+M k_1, k_1, m_{r-1}+M k_1, \dots, m_3+\dots+m_{r-1}+M k_1} \\ \times \frac{(x_1 t^M)^{k_1}}{k_1!} \frac{(x_r t + w_1)^{m_1}}{m_1!} \frac{(x_{r-1} t + w_2)^{m_2}}{m_2!} \dots \frac{(x_2 t + w_{r-1})^{m_{r-1}}}{m_{r-1}!}$$

provided that each member of the series identity (2.2) exists.

Proof. Let the left hand side of (2.2) be denoted by $\Psi(x_1, \dots, x_r)$. Then using the definition of $S_n^{m_1, m_2, \dots, m_{r-1}, M}(x_1, x_2, \dots, x_r)$ on the left hand side of (2.2), we have

$$\Psi(x_1, \dots, x_r) \\ = \sum_{m_1, \dots, m_{r-1}, n=0}^{\infty} f(m_1 + \dots + m_{r-1} + n) \\ \times \sum_{k_{r-1}=0}^n \sum_{k_{r-2}=0}^{k_{r-1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{\left[\frac{k_2}{M}\right]} A_{\Omega} \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2-M k_1}}{(k_2-M k_1)!} \dots \frac{x_r^{n-k_{r-1}}}{(n-k_{r-1})!} \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} t^n.$$

Let define

$$F(r-1) := M k_1 + k_2 + k_3 + \dots + k_{r-1}.$$

By applying Lemma 1, we find

$$\Psi(x_1, \dots, x_r)$$

$$= \sum_{k_1, \dots, k_{r-1}, m_1, \dots, m_{r-1}, n=0}^{\infty} f(m_1 + \dots + m_{r-1} + n + F(r-1)) \\ \times A_{\Omega_1} \frac{x_1^{k_1}}{k_1!} \dots \frac{x_{r-1}^{k_{r-1}}}{k_{r-1}!} \frac{x_r^n}{n!} \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} t^{n+Mk_1+k_2+\dots+k_{r-1}}$$

where $\Omega_1 = m_1 + m_2 + \dots + m_{r-1} + n + F(r-1), m_2 + \dots + m_{r-1} + k_{r-1} + F(r-2)$,

$k_1, m_{r-1} + k_2 + Mk_1, \dots, m_3 + \dots + m_{r-1} + k_{r-2} + F(r-3)$.

Let $m_1 \rightarrow m_1 - n, m_2 \rightarrow m_2 - k_{r-1}, m_3 \rightarrow m_3 - k_{r-2},$

$\dots, m_{r-2} \rightarrow m_{r-2} - k_3, m_{r-1} \rightarrow m_{r-1} - k_2$ then

$$\Psi(x_1, \dots, x_r) \\ = \sum_{k_1, m_1, \dots, m_{r-1}=0}^{\infty} f(m_1 + \dots + m_{r-1} + Mk_1) \\ \times A_{m_1+\dots+m_{r-1}+Mk_1, m_2+\dots+m_{r-1}+Mk_1, k_1, m_{r-1}+Mk_1, \dots, m_3+\dots+m_{r-1}+Mk_1} \\ \times \frac{(x_1 t^M)^{k_1}}{k_1!} \frac{\sum_{n=0}^{m_1} \binom{m_1}{n} (x_r t)^n w_1^{m_1-n}}{m_1!} \frac{\sum_{k_{r-1}=0}^{m_2} \binom{m_2}{k_{r-1}} (x_{r-1} t)^{k_{r-1}} w_2^{m_2-k_{r-1}}}{m_2!} \\ \times \dots \frac{\sum_{k_2=0}^{m_{r-1}} \binom{m_{r-1}}{k_2} (x_2 t)^{k_2} w_{r-1}^{m_{r-1}-k_2}}{m_{r-1}!}$$

and

$$\Psi(x_1, \dots, x_r) \\ = \sum_{k_1, m_1, \dots, m_{r-1}=0}^{\infty} f(m_1 + \dots + m_{r-1} + Mk_1) \\ \times A_{m_1+m_2+\dots+m_{r-1}+Mk_1, m_2+\dots+m_{r-1}+Mk_1, k_1, m_{r-1}+Mk_1, \dots, m_3+\dots+m_{r-1}+Mk_1} \\ \times \frac{(x_1 t^M)^{k_1}}{k_1!} \frac{(x_r t + w_1)^{m_1}}{m_1!} \frac{(x_{r-1} t + w_2)^{m_2}}{m_2!} \dots \frac{(x_2 t + w_{r-1})^{m_{r-1}}}{m_{r-1}!}.$$

Whence the result. $\square$

Corollary 1. Let the polynomials $S_n^{m_1, m_2, \dots, m_{r-1}, M}(x_1, x_2, \dots, x_r)$ be defined by equation (2.1). Suppose also that $r-1$ -variable polynomials $P_{m_2}^M(x_1, \dots, x_{r-1})$ are defined by

$$P_{m_2}^M(x_1, \dots, x_{r-1}) \tag{2.3} \\ = \sum_{m_3=0}^{m_2} \dots \sum_{m_{r-1}=0}^{m_{r-2}} \sum_{k_1=0}^{\lfloor \frac{m_{r-1}}{M} \rfloor} A_{m_1+m_2, m_2, k_1, m_{r-1}, \dots, m_3}$$

$$\times \frac{(x_1)^{k_1}}{k_1!} \frac{(x_2)^{m_{r-1}-Mk_1}}{(m_{r-1}-Mk_1)!} \frac{(x_3)^{m_{r-2}-m_{r-1}}}{(m_{r-2}-m_{r-1})!} \cdots \frac{(x_{r-1})^{m_2-m_3}}{(m_2-m_3)!}.$$

Then, for a suitably bounded sequence $\{f(n)\}_{n=0}^\infty$, the following family of two-sided linear generating relations holds true between the r -variable polynomials $S_n^{m_1, m_2, \dots, m_{r-1}, M}(x_1, \dots, x_r)$ and $(r-1)$ -variable polynomials $P_{m_2}^M(x_1, \dots, x_{r-1})$:

$$\sum_{m_1, \dots, m_{r-1}, n=0}^{\infty} f(m_1 + \dots + m_{r-1} + n) S_n^{m_1, \dots, m_{r-1}, M}(x_1, \dots, x_r) \frac{w_1^{m_1}}{m_1!} \cdots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} t^n \quad (2.4)$$

$$= \sum_{m_1, m_2=0}^{\infty} f(m_1 + m_2) \frac{(x_r t + w_1)^{m_1}}{m_1!} P_{m_2}^M(x_1 t^M, x_2 t + w_{r-1}, \dots, x_{r-1} t + w_2)$$

provided that each member of the series identity (2.4) exists.

Proof. If we set $m_{r-1} \rightarrow m_{r-1} - Mk_1$, $m_{r-2} \rightarrow m_{r-2} - m_{r-1}$, $\dots$, $m_2 \rightarrow m_2 - m_3$ in the right side of the equation (2.2) then

$$\begin{aligned} & \sum_{m_1, \dots, m_{r-1}, n=0}^{\infty} f(m_1 + \dots + m_{r-1} + n) S_n^{m_1, \dots, m_{r-1}, M}(x_1, \dots, x_r) \frac{w_1^{m_1}}{m_1!} \cdots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} t^n \\ &= \sum_{m_1, m_2=0}^{\infty} f(m_1 + m_2) \frac{(x_r t + w_1)^{m_1}}{m_1!} \\ &\times \sum_{m_3=0}^{m_2} \cdots \sum_{m_{r-1}=0}^{m_{r-2}} \sum_{k_1=0}^{\lfloor \frac{m_{r-1}}{M} \rfloor} A_{m_1+m_2, m_2, k_1, m_{r-1}, \dots, m_3} \frac{(x_1 t^M)^{k_1}}{k_1!} \\ &\times \frac{(x_2 t + w_{r-1})^{m_{r-1}-Mk_1}}{(m_{r-1}-Mk_1)!} \frac{(x_3 t + w_{r-2})^{m_{r-2}-m_{r-1}}}{(m_{r-2}-m_{r-1})!} \cdots \frac{(x_{r-1} t + w_2)^{m_2-m_3}}{(m_2-m_3)!} \\ &= \sum_{m_1, m_2=0}^{\infty} f(m_1 + m_2) \frac{(x_r t + w_1)^{m_1}}{m_1!} P_{m_2}^M(x_1 t^M, x_2 t + w_{r-1}, \dots, x_{r-1} t + w_2). \end{aligned}$$

□

By setting $t = -\frac{w_1}{x_r}$ in (2.4), we get the following result.

Corollary 2. Under the hypotheses of Corollary 1, we get the following generating relation

$$\sum_{m_1, \dots, m_{r-1}, n=0}^{\infty} f(m_1 + \dots + m_{r-1} + n)$$

$$\begin{aligned}
& \times S_n^{m_1, \dots, m_{r-1}, M}(x_1, \dots, x_r) \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} \left(-\frac{w_1}{x_r}\right)^n \\
& = \sum_{m_2=0}^{\infty} f(m_2) P_{m_2}^M(x_1(-\frac{w_1}{x_r})^M, x_2(-\frac{w_1}{x_r}) + w_{r-1}, \dots, x_{r-1}(-\frac{w_1}{x_r}) + w_2).
\end{aligned}$$

Remark 3. Choosing $M = 1$ and

$$A_{m,n,k_2,k_3,\dots,k_{r-1}} = (\alpha_1)_{k_2}(\alpha_2)_{k_3-k_2} \dots (\alpha_{r-1})_{n-k_{r-1}}(\alpha_r)_{m-n} \quad (2.5)$$

in (2.3) we observe that

$$\begin{aligned}
& A_{m_1+m_2, m_2, k_1, m_{r-1}, m_{r-2}, \dots, m_4, m_3} \\
& = (\alpha_1)_{k_1}(\alpha_2)_{m_{r-1}-k_1}(\alpha_3)_{m_{r-2}-m_{r-1}} \dots (\alpha_{r-1})_{m_2-m_3}(\alpha_r)_{m_1}
\end{aligned}$$

then

$$P_n^1(x_1, \dots, x_{r-1}) = (\alpha_r)_{m_1} g_n^{(\alpha_1, \alpha_2, \dots, \alpha_{r-1})}(x_1, x_2, \dots, x_{r-1}) \quad (2.6)$$

where $g_n^{(\alpha_1, \alpha_2, \dots, \alpha_{r-1})}(x_1, x_2, \dots, x_{r-1})$ is the Lagrange polynomials of $(r-1)$ -variables.

Hence, upon setting $M = 1$ and considering (2.5) and (2.7) in Corollary 2, we get

$$\begin{aligned}
& \sum_{n=0}^{\infty} f(n) g_n^{(\alpha_1, \alpha_2, \dots, \alpha_{r-1})}(x_1(-\frac{w_1}{x_r}), x_2(-\frac{w_1}{x_r}) + w_{r-1}, \dots, x_{r-1}(-\frac{w_1}{x_r}) + w_2) \\
& = \sum_{m_1, \dots, m_{r-1}, n=0}^{\infty} f(m_1 + \dots + m_{r-1} + n) (\alpha_2)_{m_{r-1}} (\alpha_3)_{m_{r-2}} \dots (\alpha_{r-1})_{m_2} (\alpha_r)_{m_1} \\
& \times g_n^{(\alpha_1, \alpha_2+m_{r-1}, \alpha_3+m_{r-2}, \dots, \alpha_r+m_1)}(x_1, x_2, \dots, x_r) \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} \left(-\frac{w_1}{x_r}\right)^n.
\end{aligned} \quad (2.7)$$

Let choose $f(n) = 1$ in (2.7) and consider Remark 1 then we have

$$\begin{aligned}
& \sum_{n=0}^{\infty} g_n^{(\alpha_1, \alpha_2, \dots, \alpha_{r-1})}(x_1(-\frac{w_1}{x_r}), x_2(-\frac{w_1}{x_r}) + w_{r-1}, \dots, x_{r-1}(-\frac{w_1}{x_r}) + w_2) \quad (2.8) \\
& = \sum_{m_1, \dots, m_{r-1}, n=0}^{\infty} \sum_{k_{r-1}=0}^n \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} (\alpha_2)_{m_{r-1}} (\alpha_3)_{m_{r-2}} \dots (\alpha_{r-1})_{m_2} (\alpha_r)_{m_1} \\
& \times (\alpha_1)_{k_1} (\alpha_2 + m_{r-1})_{k_2-k_1} \dots (\alpha_{r-1} + m_2)_{k_{r-1}-k_{r-2}} (\alpha_r + m_1)_{n-k_{r-1}} \\
& \times \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2-k_1}}{(k_2-k_1)!} \dots \frac{x_{r-1}^{k_{r-1}-k_{r-2}}}{(k_{r-1}-k_{r-2})!} \frac{x_r^{n-k_{r-1}}}{(n-k_{r-1})!} \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} \left(-\frac{w_1}{x_r}\right)^n.
\end{aligned}$$

Let Φ denote the right side of (2.8). Applying Lemma 1 into the right side of the equation (2.8) we get

$$\begin{aligned} \Phi = & \sum_{k_1, \dots, k_{r-1}, m_1, \dots, m_{r-1}, n=0}^{\infty} (\alpha_1)_{k_1} (\alpha_2 + m_{r-1})_{k_2} \dots (\alpha_{r-1} + m_2)_{k_{r-1}} \\ & \times (\alpha_r)_{m_1} (\alpha_{r-1})_{m_2} \dots (\alpha_3)_{m_{r-2}} (\alpha_2)_{m_{r-1}} (\alpha_r + m_1)_n \\ & \times \frac{((- \frac{w_1}{x_r})x_1)^{k_1}}{k_1!} \frac{((- \frac{w_1}{x_r})x_2)^{k_2}}{k_2!} \dots \frac{((- \frac{w_1}{x_r})x_{r-1})^{k_{r-1}}}{k_{r-1}!} \frac{((- \frac{w_1}{x_r})x_r)^n}{n!} \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} \end{aligned}$$

and

$$\begin{aligned} \Phi &= \sum_{m_1, \dots, m_{r-1}=0}^{\infty} (\alpha_r)_{m_1} (\alpha_{r-1})_{m_2} \dots (\alpha_3)_{m_{r-2}} (\alpha_2)_{m_{r-1}} \\ &\quad \times (1 + (\frac{w_1}{x_r})x_1)^{-\alpha_1} (1 + (\frac{w_1}{x_r})x_2)^{-(\alpha_2 + m_{r-1})} \dots (1 + (\frac{w_1}{x_r})x_{r-1})^{-(\alpha_{r-1} + m_2)} \\ &\quad \times (1 + (\frac{w_1}{x_r})x_r)^{-(\alpha_r + m_1)} \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} \\ &= (1 + (\frac{w_1}{x_r})x_1)^{-\alpha_1} (1 + (\frac{w_1}{x_r})x_2)^{-\alpha_2} \dots (1 + (\frac{w_1}{x_r})x_{r-1})^{-\alpha_{r-1}} (1 + (\frac{w_1}{x_r})x_r)^{-\alpha_r} \\ &\quad \times \sum_{m_1, \dots, m_{r-1}=0}^{\infty} (\alpha_r)_{m_1} (\alpha_{r-1})_{m_2} \dots (\alpha_3)_{m_{r-2}} (\alpha_2)_{m_{r-1}} \\ &\quad \times (1 + (\frac{w_1}{x_r})x_2)^{-m_{r-1}} \dots (1 + (\frac{w_1}{x_r})x_{r-1})^{-m_2} (1 + (\frac{w_1}{x_r})x_r)^{-m_1} \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} \\ &= (1 + (\frac{w_1}{x_r})x_1)^{-\alpha_1} (1 + (\frac{w_1}{x_r})x_2)^{-\alpha_2} \dots (1 + (\frac{w_1}{x_r})x_{r-1})^{-\alpha_{r-1}} (1 + (\frac{w_1}{x_r})x_r)^{-\alpha_r} \\ &\quad \times (1 - \frac{w_1}{1 + (\frac{w_1}{x_r})x_r})^{-\alpha_r} (1 - \frac{w_2}{1 + (\frac{w_1}{x_r})x_{r-1}})^{-\alpha_{r-1}} \dots (1 - \frac{w_{r-1}}{1 + (\frac{w_1}{x_r})x_2})^{-\alpha_2} \\ &= (1 + (\frac{w_1}{x_r})x_1)^{-\alpha_1} (1 + (\frac{w_1}{x_r})x_2 - w_{r-1})^{-\alpha_2} \dots (1 + (\frac{w_1}{x_r})x_{r-1} - w_2)^{-\alpha_{r-1}} \\ &\quad \times (1 + (\frac{w_1}{x_r})x_r - w_1)^{-\alpha_r}. \end{aligned}$$

Corollary 3. *The generating relation*

$$\begin{aligned} & \sum_{n=0}^{\infty} g_n^{(\alpha_1, \alpha_2, \dots, \alpha_{r-1})} (x_1(-\frac{w_1}{x_r}), x_2(-\frac{w_1}{x_r}) + w_{r-1}, \dots, x_{r-1}(-\frac{w_1}{x_r}) + w_2) \\ &= (1 + (\frac{w_1}{x_r})x_1)^{-\alpha_1} (1 + (\frac{w_1}{x_r})x_2 - w_{r-1})^{-\alpha_2} \dots (1 + (\frac{w_1}{x_r})x_{r-1} - w_2)^{-\alpha_{r-1}} \end{aligned}$$

is given for $g_n^{(\alpha_1, \alpha_2, \dots, \alpha_{r-1})}(x_1, x_2, \dots, x_{r-1})$.

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ON EQUIVALENCE OF TWO INTEGRABILITY METHODS

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Abstract. In this paper, we introduce the concept of $|R, p|_k, k \geq 1$ integrability of improper integrals and by this definition we prove a theorem, that generalizes a theorem of Orhan [3].

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1. INTRODUCTION

Throughout this paper we assume that f is a real valued function which is continuous on $[0, \infty)$ and $s(x) = \int_0^x f(t)dt$. By $\sigma(x)$, we denote the Cesàro mean of $s(x)$. The integral $\int_0^\infty f(t)dt$ is said to be integrable $|C, 1|_k, k \geq 1$, in the sense of Flett [2], if

$$\int_0^\infty x^{k-1} |\sigma'(x)|^k dx = \int_0^\infty \frac{|v(x)|^k}{x} dx \quad (1.1)$$

is convergent. Here, $v(x) = \frac{1}{x} \int_0^x t f(t)dt$ is called a generator of the integral $\int_0^\infty f(t)dt$. Let p be a real valued, non-decreasing function on $[0, \infty)$ such that

$$P(x) = \int_0^x p(t)dt, p(x) \neq 0, p(0) = 0.$$

The Riesz mean of $s(x)$ is defined by

$$\sigma_p(x) = \frac{1}{P(x)} \int_0^x p(t)s(t)dt.$$

We say that the integral $\int_0^\infty f(t)dt$ is integrable $|R, p|_k, k \geq 1$, if

$$\int_0^\infty x^{k-1} |\sigma_p'(x)|^k dx \quad (1.2)$$

is convergent. In the special case if we take $p(x) = 1$ for all values of x , then $|R, p|_k$ integrability reduces to $|C, 1|_k$ integrability of improper integrals.

Given any functions f, g , it is customary to write $g(x) = O(f(x))$, if there exist η

and N , for every $x > N$, $|\frac{g(x)}{f(x)}| \leq \eta$.

The difference between $s(x)$ and its n th weighted mean $\sigma_p(x)$, which is called the weighted Kronecker identity, is given by the identity

$$s(x) - \sigma_p(x) = v_p(x), \quad (1.3)$$

where

$$v_p(x) = \frac{1}{P(x)} \int_0^x P(u) f(u) du.$$

We note that if we take $p(x) = 1$, for all values of x then we have the following identity (see [1])

$$s(x) - \sigma(x) = v(x).$$

Since

$$\sigma'_p(x) = \frac{p(x)}{P(x)} v_p(x),$$

condition (1.3) can be rewritten as

$$s(x) = v_p(x) + \int_0^x \frac{p(u)}{P(u)} v_p(u) du. \quad (1.4)$$

In view of the identity (1.4), the function $v_p(x)$ is called the generator function of $s(x)$.

Condition (1.1) can also be written as

$$\int_0^\infty x^{k-1} \left(\frac{P(x)}{P(x)} \right)^k |v_p(x)|^k dx \quad (1.5)$$

is convergent. We note that for infinite series, an analogous definition was introduced by Orhan [3]. Using this definition, Orhan [3] proved the following theorem dealing with $|R, p_n|_k$ and $|R, q_n|_k$ summability methods.

Theorem 1. *The $|R, p_n|_k, (k \geq 1)$ summability implies the $|R, q_n|_k, (k \geq 1)$ summability provided that*

$$n q_n = O(Q_n), \quad (1.6)$$

$$P_n = O(np_n), \quad (1.7)$$

$$Q_n = O(nq_n). \quad (1.8)$$

2. MAIN RESULT

The aim of this paper is to state Orhan's theorem for $|R, p|_k$ and $|R, q|_k$ integrability of improper integrals. Now we shall prove the following theorem.

Theorem 2. *Let p and q be real valued, non-decreasing functions on $[0, \infty)$ such that as $x \rightarrow \infty$*

$$xq(x) = O(Q(x)), \quad (2.1)$$

$$P(x) = O(xp(x)), \quad (2.2)$$

$$Q(x) = O(xq(x)). \quad (2.3)$$

If $\int_0^\infty f(t)dt$ is integrable $|R, p|_k$, then it is also integrable $|R, q|_k, k \geq 1$.

3. PROOF OF THE THEOREM

Let $\sigma_p(x)$ and $\sigma_q(x)$ be the functions of (R, p) and (R, q) means of the integral $\int_0^\infty f(t)dt$. Since $\int_0^\infty f(t)dt$ is integrable $|R, p|_k$, we can write

$$\int_0^\infty x^{k-1} \left(\frac{p(x)}{P(x)} \right)^k |v_p(x)|^k dx$$

is convergent. Differentiating the equation (1.4), we have

$$f(x) = v'_p(x) + \frac{p(x)}{P(x)} v_p(x).$$

By definition, we obtain

$$\sigma_q(x) = \frac{1}{Q(x)} \int_0^x q(t)s(t)dt = \frac{1}{Q(x)} \int_0^x (Q(x) - Q(t))f(t)dt$$

and

$$\begin{aligned} \sigma'_q(x) &= \frac{q(x)}{Q^2(x)} \int_0^x Q(t)f(t)dt \\ &= \frac{q(x)}{Q^2(x)} \int_0^x Q(t) \left[v'_p(t) + \frac{p(t)}{P(t)} v_p(t) \right] dt \\ &= \frac{q(x)}{Q^2(x)} \int_0^x Q(t)v'_p(t)dt + \frac{q(x)}{Q^2(x)} \int_0^x Q(t) \frac{p(t)}{P(t)} v_p(t)dt. \end{aligned}$$

Integrating by parts of the first statement, we have

$$\begin{aligned} \sigma'_q(x) &= \frac{q(x)}{Q^2(x)} \left[Q(x)v_p(x) - \int_0^x q(t)v_p(t)dt \right] + \frac{q(x)}{Q^2(x)} \int_0^x Q(t) \frac{p(t)}{P(t)} v_p(t)dt \\ &= \frac{q(x)}{Q(x)} v_p(x) + \frac{q(x)}{Q^2(x)} \int_0^x Q(t) \frac{p(t)}{P(t)} v_p(t)dt - \frac{q(x)}{Q^2(x)} \int_0^x q(t)v_p(t)dt \\ &= \sigma_{q,1}(x) + \sigma_{q,2}(x) + \sigma_{q,3}(x), \text{ say.} \end{aligned}$$

To complete the proof of the theorem, it is sufficient to show that

$$\int_0^m x^{k-1} |\sigma_{q,r}(x)|^k dx = O(1) \text{ as } m \rightarrow \infty, \text{ for } r = 1, 2, 3.$$

Using conditions (2.1) and (2.2), we have

$$\begin{aligned} \int_0^m x^{k-1} |\sigma_{q,1}(x)|^k dx &= \int_0^m x^{k-1} \left| \frac{q(x)}{Q(x)} v_p(x) \right|^k dx \\ &= \int_0^m x^{k-1} \left(\frac{q(x)}{Q(x)} \right)^k |v_p(x)|^k dx \\ &= O(1) \int_0^m x^{k-1} \left(\frac{p(x)}{P(x)} \right)^k |v_p(x)|^k dx \\ &= O(1) \int_0^m x^{k-1} |\sigma'_p(x)|^k dx \\ &= O(1) \text{ as } m \rightarrow \infty \end{aligned}$$

by virtue of the hypotheses of Theorem 2.

Applying Hölder's inequality with $k > 1$, we get

$$\begin{aligned} \int_0^m x^{k-1} |\sigma_{q,2}(x)|^k dx &= \\ &= O(1) \int_0^m x^{k-1} \left(\frac{q(x)}{Q^2(x)} \right)^k \left(\int_0^x \frac{Q(t)p(t)}{P(t)} |v_p(t)| dt \right)^k dx \\ &= O(1) \int_0^m \frac{q(x)}{Q^{k+1}(x)} \left(\int_0^x \frac{Q(t)p(t)}{P(t)} |v_p(t)| dt \right)^k dx \\ &= O(1) \int_0^m \frac{q(x)}{Q^2(x)} \left(\int_0^x \left(\frac{Q(t)}{q(t)} \right)^k q(t) \left(\frac{p(t)}{P(t)} \right)^k |v_p(t)|^k dt \right) \\ &\quad \times \left(\frac{1}{Q(x)} \int_0^x q(t) dt \right)^{k-1} dx \\ &= O(1) \int_0^m t^k q(t) \left(\frac{p(t)}{P(t)} \right)^k |v_p(t)|^k dt \int_t^m \frac{q(x)}{Q^2(x)} dx \\ &= O(1) \int_0^m t^k \frac{q(t)}{Q(t)} \left(\frac{p(t)}{P(t)} \right)^k |v_p(t)|^k dt \\ &= O(1) \int_0^m t^{k-1} \left(\frac{p(t)}{P(t)} \right)^k |v_p(t)|^k dt \\ &= O(1) \int_0^m t^{k-1} |\sigma'_p(t)|^k dt \end{aligned}$$

$$= O(1) \text{ as } m \rightarrow \infty$$

by virtue of the hypotheses of Theorem 2.

Finally, again by Hölder's inequality with $k > 1$, we have

$$\begin{aligned} \int_0^m x^{k-1} |\sigma_{q,3}(x)|^k dx &= O(1) \int_0^m x^{k-1} \left(\frac{q(x)}{Q^2(x)} \right)^k \left(\int_0^x q(t) |v_p(t)|^k dt \right)^k dx \\ &= O(1) \int_0^m \frac{q(x)}{Q^2(x)} \left(\int_0^x q(t) |v_p(t)|^k dt \right) \\ &\quad x \left(\frac{1}{Q(x)} \int_0^x q(t) dt \right)^{k-1} dx \\ &= O(1) \int_0^m q(t) |v_p(t)|^k dt \int_t^m \frac{q(x)}{Q^2(x)} dx \\ &= O(1) \int_0^m \frac{q(t)}{Q(t)} |v_p(t)|^k dt \\ &= O(1) \text{ as } m \rightarrow \infty \end{aligned}$$

by virtue of the hypotheses of Theorem 2.

This completes the proof of the theorem.

Theorem 3. Let p and q be real valued, non-decreasing functions on $[0, \infty)$ such that as $x \rightarrow \infty$

$$xp(x) = O(P(x)), \quad (3.1)$$

$$Q(x) = O(xq(x)), \quad (3.2)$$

$$P(x) = O(xp(x)). \quad (3.3)$$

If $\int_0^\infty f(t)dt$ is integrable $|R, q|_k$, then it is also integrable $|R, p|_k, k \geq 1$.

Proof. In Theorem 2 if we take $p(x) = q(x)$ and $q(x) = p(x)$, then we get Theorem 3. $\square$

Theorem 4. Let p and q be real valued, non-decreasing functions on $[0, \infty)$ such that as $x \rightarrow \infty$

$$xp(x) = O(P(x)), \quad (3.4)$$

$$P(x) = O(xp(x)), \quad (3.5)$$

$$xq(x) = O(Q(x)), \quad (3.6)$$

$$Q(x) = O(xq(x)). \quad (3.7)$$

Then the $|R, p|_k$ integrability of $\int_0^\infty f(t)dt$ is equivalent to the $|R, q|_k$ integrability of $\int_0^\infty f(t)dt$, where $k \geq 1$.

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MODIFIED HADAMARD PRODUCT PROPERTIES OF CERTAIN CLASS OF ANALYTIC FUNCTIONS WITH VARYING ARGUMENTS DEFINED BY RUSCHEWEYH DERIVATIVE

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Abstract. In this paper we study the modified Hadamard product properties of certain class of analytic functions with varying arguments defined by Ruscheweyh derivative. The obtained results are sharp and they improve known results.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1.1)$$

which are analytic and univalent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $g \in \mathcal{A}$ where

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \quad (1.2)$$

Ruscheweyh [5] defined that

$$D^\gamma f(z) = \frac{z}{(1-z)^{\gamma+1}} * f(z), (\gamma \geq -1) \quad (1.3)$$

which implies

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!} \quad (n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}). \quad (1.4)$$

The symbol $D^n f(z)$ ($n \in \mathbb{N}_0$) was called the n -th order Ruscheweyh derivative of $f(z)$ by Al-Amiri [1]. It is easy to see that

$$D^0 f(z) = f(z), D^1 f(z) = z f'(z)$$

$$D^n f(z) = z + \sum_{k=2}^{\infty} \delta(n, k) a_k z^k \quad (1.5)$$

where

$$\delta(n, k) = \binom{n+k-1}{n}. \quad (1.6)$$

Definition 1. Let f and g be analytic functions in U . We say that the function f is subordinate to the function g , if there exists a function w , which is analytic in U and $w(0) = 0; |w(z)| < 1; z \in U$, such that $f(z) = g(w(z)); \forall z \in U$. We denote by $\prec$ the subordination relation.

Definition 2. [2,4] For $\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}_0$ let $Q(n, \lambda, A, B)$ denote the subclass of $\mathcal{A}$ which contain functions $f(z)$ of the form (1.1) such that

$$(1 - \lambda)(D^n f(z))' + \lambda(D^{n+1} f(z))' \prec \frac{1 + Az}{1 + Bz}. \quad (1.7)$$

Definition 3 ([7]). A function $f(z)$ of the form (1.1) is said to be in the class $V(\theta_k)$ if $f \in A$ and $\arg(a_k) = \theta_k, \forall k \geq 2$. If $\exists \delta \in \mathbb{R}$ such that $\theta_k + (k-1)\delta \equiv \pi \pmod{2\pi}, \forall k \geq 2$ then $f(z)$ is said to be in the class $V(\theta_k, \delta)$. The union of $V(\theta_k, \delta)$ taken over all possible sequences $\{\theta_k\}$ and all possible real numbers δ is denoted by V . Let $VQ(n, \lambda, A, B)$ denote the subclass of V consisting of functions $f(z) \in Q(n, \lambda, A, B)$.

Theorem 1 ([4]). Let the function $f(z)$ defined by (1.1) be in V . Then $f(z) \in VQ(n, \lambda, A, B)$, if and only if

$$\sum_{k=2}^{\infty} k \delta(n, k) C_k |a_k| \leq (B - A)(n + 1) \quad (1.8)$$

where

$$C_k = (1 + B)[n + 1 + \lambda(k - 1)].$$

The extremal functions are

$$f_k(z) = z + \frac{(B - A)(n + 1)}{k C_k \delta(n, k)} e^{i\theta_k} z^k, (k \geq 2).$$

The modified Hadamard product of two functions f and g of the form (1.1) and (1.2), and which belong to $V(\theta_k, \delta)$ is defined by (see also [3, 6, 8])

$$(f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \quad (1.9)$$

2. MAIN RESULTS

Theorem 2. If $f \in VQ(n, \lambda, A_1, B)$, $g \in VQ(n, \lambda, A_2, B)$ then $f * g \in VQ(n, \lambda, A^*, B)$, where $A^* = B - \frac{(B-A_1)(B-A_2)(n+1)}{2C_2\delta(n,2)}$. The result is sharp.

Proof. Let $f \in VQ(n, \lambda, A_1, B)$, $g \in VQ(n, \lambda, A_2, B)$ and suppose they have the form (1.1). Since $f \in VQ(n, \lambda, A_1, B)$ we have

$$\frac{\sum_{k=2}^{\infty} k\delta(n, k)C_k |a_k|}{B - A_1} \leq n + 1 \quad (2.1)$$

and for $g \in VQ(n, \lambda, A_2, B)$ we have

$$\frac{\sum_{k=2}^{\infty} k\delta(n, k)C_k |b_k|}{B - A_2} \leq n + 1. \quad (2.2)$$

We know from Theorem 1 that $f * g \in VQ(n, \lambda, A^*, B)$ if and only if

$$\frac{\sum_{k=2}^{\infty} k\delta(n, k)C_k |a_k b_k|}{B - A^*} \leq n + 1. \quad (2.3)$$

By using the Cauchy-Schwarz inequality for (2.1) and (2.2) we have

$$\frac{\sum_{k=2}^{\infty} k\delta(n, k)C_k \sqrt{|a_k b_k|}}{\sqrt{(B - A_1)(B - A_2)}} \leq n + 1.$$

We note that

$$\frac{\sum_{k=2}^{\infty} k\delta(n, k)C_k |a_k b_k|}{B - A^*} \leq \frac{\sum_{k=2}^{\infty} k\delta(n, k)C_k \sqrt{|a_k b_k|}}{\sqrt{(B - A_1)(B - A_2)}}$$

implies (2.3). But this is equivalent to

$$\frac{|a_k b_k|}{B - A^*} \leq \frac{\sqrt{|a_k b_k|}}{\sqrt{(B - A_1)(B - A_2)}}$$

or

$$\sqrt{|a_k b_k|} \leq \frac{B - A^*}{\sqrt{(B - A_1)(B - A_2)}}. \quad (2.4)$$

From Theorem 1 we have:

$$|a_k| \leq \frac{(B - A_1)(n + 1)}{kC_k\delta(n, k)} \text{ and } |b_k| \leq \frac{(B - A_2)(n + 1)}{kC_k\delta(n, k)}, (k \geq 2)$$

this implies that

$$\sqrt{|a_k b_k|} \leq \frac{\sqrt{(B-A_1)(B-A_2)(n+1)}}{k C_k \delta(n, k)}, (k \geq 2). \quad (2.5)$$

From (2.5) we obtain that (2.4) holds if

$$\frac{\sqrt{(B-A_1)(B-A_2)(n+1)}}{k C_k \delta(n, k)} \leq \frac{B-A^*}{\sqrt{(B-A_1)(B-A_2)}}$$

or equivalently

$$A^* \leq B - \frac{(B-A_1)(B-A_2)(n+1)}{k C_k \delta(n, k)}.$$

But $k C_k \delta(n, k) < (k+1) C_{k+1} \delta(n, k+1)$, $(k \geq 2)$ so

$$\begin{aligned} B - \frac{(B-A_1)(B-A_2)(n+1)}{k C_k \delta(n, k)} &\geq B - \frac{(B-A_1)(B-A_2)(n+1)}{2 C_2 \delta(n, 2)}, (k \geq 2) \\ \Rightarrow A^* &= B - \frac{(B-A_1)(B-A_2)(n+1)}{2 C_2 \delta(n, 2)}. \end{aligned}$$

The result is sharp, because if

$$\begin{aligned} f(z) &= z + \frac{(B-A_1)(n+1)}{2 C_2 \delta(n, 2)} e^{i\theta_1} z^2 \in VQ(n, \lambda, A_1, B) \\ g(z) &= z + \frac{(B-A_2)(n+1)}{2 C_2 \delta(n, 2)} e^{i\theta_2} z^2 \in VQ(n, \lambda, A_2, B) \\ f * g &\in VQ(n, \lambda, A^*, B) \end{aligned}$$

and satisfy (8) with equality. Indeed,

$$2\delta(n, 2) C_2 \frac{(B-A_1)(B-A_2)(n+1)^2}{2^2 C_2^2 [\delta(n, 2)]^2} = (B-A^*)(n+1)$$

because

$$B-A^* = \frac{(B-A_1)(B-A_2)(n+1)}{2 C_2 \delta(n, 2)}.$$

□

Corollary 1. *If $f, g \in VQ(n, \lambda, A, B)$ then $f * g \in VQ(n, \lambda, A^*, B)$, where $A^* = B - \frac{(B-A)^2(n+1)}{2 C_2 \delta(n, 2)}$. The result is sharp.*

Theorem 3. *If $f \in VQ(n, \lambda, A, B_1)$, $g \in VQ(n, \lambda, A, B_2)$ then $f * g \in VQ(n, \lambda, A, B^*)$, where $B^* = A + \frac{(B_1-A)(B_2-A)(n+1)}{2 C_2 \delta(n, 2)}$. The result is sharp.*

Proof. Let $f \in VQ(n, \lambda, A, B_1), g \in VQ(n, \lambda, A, B_2)$ and suppose they have the form (1.1). Since $f \in VQ(n, \lambda, A, B_1)$ we have

$$\frac{\sum_{k=2}^{\infty} k\delta(n, k)C_k |a_k|}{B_1 - A} \leq n + 1 \quad (2.6)$$

and for $g \in VQ(n, \lambda, A, B_2)$ we have

$$\frac{\sum_{k=2}^{\infty} k\delta(n, k)C_k |b_k|}{B_2 - A} \leq n + 1. \quad (2.7)$$

We know from Theorem 1 that $f * g \in VQ(n, \lambda, A, B^*)$ if and only if

$$\frac{\sum_{k=2}^{\infty} k\delta(n, k)C_k |a_k b_k|}{B^* - A} \leq n + 1. \quad (2.8)$$

By using the Cauchy-Schwarz inequality for (2.6) and (2.7) we have

$$\frac{\sum_{k=2}^{\infty} k\delta(n, k)C_k \sqrt{|a_k b_k|}}{\sqrt{(B_1 - A)(B_2 - A)}} \leq n + 1.$$

We note that

$$\frac{\sum_{k=2}^{\infty} k\delta(n, k)C_k |a_k b_k|}{B^* - A} \leq \frac{\sum_{k=2}^{\infty} k\delta(n, k)C_k \sqrt{|a_k b_k|}}{\sqrt{(B_1 - A)(B_2 - A)}}$$

implies (2.8). But this is equivalent to

$$\frac{|a_k b_k|}{B^* - A} \leq \frac{\sqrt{|a_k b_k|}}{\sqrt{(B_1 - A)(B_2 - A)}}$$

or

$$\sqrt{|a_k b_k|} \leq \frac{B^* - A}{\sqrt{(B_1 - A)(B_2 - A)}}. \quad (2.9)$$

From Theorem 1 we have:

$$|a_k| \leq \frac{(B_1 - A)(n + 1)}{k C_k \delta(n, k)} \text{ and } |b_k| \leq \frac{(B_2 - A)(n + 1)}{k C_k \delta(n, k)}, (k \geq 2)$$

this implies that

$$\sqrt{|a_k b_k|} \leq \frac{\sqrt{(B_1 - A)(B_2 - A)}(n + 1)}{k C_k \delta(n, k)}, (k \geq 2). \quad (2.10)$$

from (2.10) we obtain that (2.9) holds if

$$\frac{\sqrt{(B_1 - A)(B_2 - A)}(n + 1)}{kC_k\delta(n, k)} \leq \frac{B^* - A}{\sqrt{(B_1 - A)(B_2 - A)}}$$

or equivalently

$$B^* \geq A + \frac{(B_1 - A)(B_2 - A)(n + 1)}{kC_k\delta(n, k)}.$$

But $kC_k\delta(n, k) < (k + 1)C_{k+1}\delta(n, k + 1)$, $(k \geq 2)$ so :

$$\begin{aligned} A + \frac{(B_1 - A)(B_2 - A)(n + 1)}{kC_k\delta(n, k)} &\leq A + \frac{(B_1 - A)(B_2 - A)(n + 1)}{2C_2\delta(n, 2)}, (k \geq 2) \\ \Rightarrow B^* &= A + \frac{(B_1 - A)(B_2 - A)(n + 1)}{2C_2\delta(n, 2)}. \end{aligned}$$

The result is sharp, because if

$$\begin{aligned} f(z) &= z + \frac{(B_1 - A)(n + 1)}{2C_2\delta(n, 2)} e^{i\theta_1} z^2 \in VQ(n, \lambda, A, B_1) \\ g(z) &= z + \frac{(B_2 - A)(n + 1)}{2C_2\delta(n, 2)} e^{i\theta_2} z^2 \in VQ(n, \lambda, A, B_2) \\ f * g &\in VQ(n, \lambda, A, B^*) \end{aligned}$$

and satisfy (8) with equality. Indeed,

$$2\delta(n, 2)C_2 \frac{(B_1 - A)(B_2 - A)(n + 1)^2}{2^2 C_2^2 [\delta(n, 2)]^2} = (B^* - A)(n + 1)$$

because

$$B^* - A = \frac{(B_1 - A)(B_2 - A)(n + 1)}{2C_2\delta(n, 2)}.$$

□

Corollary 2. If $f, g \in VQ(n, \lambda, A, B)$ then $f * g \in VQ(n, \lambda, A, B^*)$, where $B^* = A + \frac{(B-A)^2(n+1)}{2C_2\delta(n,2)}$. The result is sharp.

Theorem 4. If $f_j \in VQ(n, \lambda, A_j, B)$, $j = \overline{1, s}$, $s \in \{2, 3, 4, \dots\}$ then

$$f_1 * f_2 * \dots * f_s \in VQ(n, \lambda, A^{(s-1)*}, B), \text{ where } A^{(s-1)*} = B - \frac{(n+1)^{s-1} \prod_{j=1}^s (B-A_j)}{2^{s-1} C_2^{s-1} [\delta(n, 2)]^{s-1}}.$$

The result is sharp.

Proof. For the proof we use the mathematical induction method and suppose that $f_j, \forall j$ have the form (1.1).

Let $s = 2$. If $f_j \in VQ(n, \lambda, A_j, B)$, $j = \overline{1, 2}$ then $f_1 * f_2 \in VQ(n, \lambda, A^*, B)$ where $A^* = B - \frac{(B-A_1)(B-A_2)(n+1)}{kC_k\delta(n,k)}$, from Theorem 2 is true.

Assume, for $s = m$, that the formula displayed below holds.

If $f_j \in VQ(n, \lambda, A_j, B)$, $j = \overline{1, m}$, $m \in \{2, 3, 4, \dots\}$ then

$f_1 * f_2 * \dots * f_m \in VQ(n, \lambda, A^{(m-1)*}, B)$, where

$$A^{(m-1)*} = B - \frac{(n+1)^{m-1} \prod_{j=1}^m (B - A_j)}{k^{m-1} C_k^{m-1} [\delta(n, k)]^{m-1}}.$$

Let $s = m + 1$: if $f_1 * f_2 * \dots * f_m \in VQ(n, \lambda, A^{(m-1)*}, B)$, $m \in \{2, 3, 4, \dots\}$ and $f_{m+1} \in VQ(n, \lambda, A_{m+1}, B)$ then we have to prove

$$f_1 * f_2 * \dots * f_m * f_{m+1} \in VQ(n, \lambda, A^{m*}, B), \text{ where } A^{m*} = B - \frac{(n+1)^m \prod_{j=1}^{m+1} (B - A_j)}{k^m C_k^m [\delta(n, k)]^m}.$$

For the proof we use the result of Theorem 2:

$$\begin{aligned} A^{m*} &\leq B - \frac{(B - A^{(m-1)*})(B - A_{m+1})(n+1)}{k C_k \delta(n, k)} \\ A^{m*} &\leq B - \frac{(n+1)^{m-1} \prod_{j=1}^m (B - A_j)}{k^{m-1} C_k^{m-1} [\delta(n, k)]^{m-1}} (B - A_{m+1})(n+1) \\ A^{m*} &\leq B - \frac{(n+1)^m \prod_{j=1}^{m+1} (B - A_j)}{k^m C_k^m [\delta(n, k)]^m}. \end{aligned}$$

But $k C_k \delta(n, k) < (k+1) C_{k+1} \delta(n, k+1)$, $(k \geq 2)$ so:

$$\begin{aligned} B - \frac{(n+1)^m \prod_{j=1}^{m+1} (B - A_j)}{k^m C_k^m [\delta(n, k)]^m} &\geq B - \frac{(n+1)^m \prod_{j=1}^{m+1} (B - A_j)}{2^m C_2^m [\delta(n, 2)]^m}, (k \geq 2) \Rightarrow \\ A^{m*} &= B - \frac{(n+1)^m \prod_{j=1}^{m+1} (B - A_j)}{2^m C_2^m [\delta(n, 2)]^m}. \end{aligned}$$

The result is sharp, because if

$$f(z) = z + \frac{(B - A^{(s-1)*})(n+1)}{2 C_2 \delta(n, 2)} e^{i\theta_1} z^2 \in VQ(n, \lambda, A^{(s-1)*}, B)$$

$$g(z) = z + \frac{(B - A_s)(n+1)}{2 C_2 \delta(n, 2)} e^{i\theta_2} z^2 \in VQ(n, \lambda, A_s, B)$$

$$f * g \in VQ(n, \lambda, A^{s*}, B)$$

and satisfy (8) with equality. Indeed,

$$2\delta(n, 2)C_2 \frac{(B - A^{(s-1)*})(B - A_s)(n+1)^2}{2^2 C_2^2 [\delta(n, 2)]^2} = (B - A^{s*})(n+1)$$

because

$$\begin{aligned} B - A^{s*} &= \frac{(B - A^{(s-1)*})(B - A_s)(n+1)}{2C_2\delta(n, 2)} \Leftrightarrow \\ &= \frac{(n+1)^s \prod_{j=1}^{s+1} (B - A_j)}{2^s C_2^s [\delta(n, 2)]^s} \\ \Leftrightarrow B - A^{s*} &= \frac{(n+1)^s \prod_{j=1}^{s+1} (B - A_j)}{2^s C_2^s [\delta(n, 2)]^s}. \end{aligned}$$

□

Theorem 5. If $f_j \in VQ(n, \lambda, A, B_j)$, $j = \overline{1, s}$, $s \in \{2, 3, 4, \dots\}$ then

$$f_1 * f_2 * \dots * f_s \in VQ(n, \lambda, A, B^{(s-1)*}), \text{ where } B^{(s-1)*} = A + \frac{(n+1)^{s-1} \prod_{j=1}^s (B_j - A)}{2^{s-1} C_2^{s-1} [\delta(n, 2)]^{s-1}}.$$

The result is sharp.

Proof. For the proof we use the mathematical induction method and suppose that $f_j, \forall j$ have the form (1.1).

Let $s = 2$. If $f_j \in VQ(n, \lambda, A, B_j)$, $j = \overline{1, 2}$ then $f_1 * f_2 \in VQ(n, \lambda, A, B^*)$ where $B^* = A + \frac{(B_1 - A)(B_2 - A)(n+1)}{k C_k \delta(n, k)}$, from Theorem 3 is true.

Assume, for $s = m$, that the formula displayed below holds.

If $f_j \in VQ(n, \lambda, A, B_j)$, $j = \overline{1, m}$, $m \in \{2, 3, 4, \dots\}$ then

$$f_1 * f_2 * \dots * f_m \in VQ(n, \lambda, A, B^{(m-1)*}), \text{ where } B^{(m-1)*} = A + \frac{(n+1)^{m-1} \prod_{j=1}^m (B_j - A)}{k^{m-1} C_k^{m-1} [\delta(n, k)]^{m-1}}.$$

Let $s = m+1$: if $f_1 * f_2 * \dots * f_m \in VQ(n, \lambda, A, B^{(m-1)*})$, $m \in \{2, 3, 4, \dots\}$ and $f_{m+1} \in VQ(n, \lambda, A, B_{m+1})$ then we have to prove

$$f_1 * f_2 * \dots * f_m * f_{m+1} \in VQ(n, \lambda, A, B^{m*}), \text{ where } B^{m*} = A + \frac{(n+1)^m \prod_{j=1}^{m+1} (B_j - A)}{k^m C_k^m [\delta(n, k)]^m}.$$

For the proof we use the result of Theorem 3:

$$\begin{aligned} B^{m*} &\geq A + \frac{(B^{(m-1)*} - A)(B_{m+1} - A)(n+1)}{k C_k \delta(n, k)} \\ &\geq A + \frac{\frac{(n+1)^{m-1} \prod_{j=1}^m (B_j - A)}{k^{m-1} C_k^{m-1} [\delta(n, k)]^{m-1}} (B_{m+1} - A)(n+1)}{k C_k \delta(n, k)} \\ &\geq A + \frac{(n+1)^m \prod_{j=1}^{m+1} (B_j - A)}{k^m C_k^m [\delta(n, k)]^m}. \end{aligned}$$

But $kC_k\delta(n, k) < (k+1)C_{k+1}\delta(n, k+1)$, $(k \geq 2)$ so :

$$A + \frac{(n+1)^m \prod_{j=1}^{m+1} (B_j - A)}{k^m C_k^m [\delta(n, k)]^m} \leq A + \frac{(n+1)^m \prod_{j=1}^{m+1} (B_j - A)}{2^m C_2^m [\delta(n, 2)]^m}, (k \geq 2)$$

$$\Rightarrow B^{m*} = A + \frac{(n+1)^m \prod_{j=1}^{m+1} (B_j - A)}{2^m C_2^m [\delta(n, 2)]^m}.$$

The result is sharp, because if

$$f(z) = z + \frac{(B^{(s-1)*} - A)(n+1)}{2C_2\delta(n, 2)} e^{i\theta_1} z^2 \in VQ(n, \lambda, A, B^{(s-1)*})$$

$$g(z) = z + \frac{(B_s - A)(n+1)}{2C_2\delta(n, 2)} e^{i\theta_2} z^2 \in VQ(n, \lambda, A, B_s)$$

$$f * g \in VQ(n, \lambda, A, B^{s*})$$

and satisfy (8) with equality. Indeed,

$$2\delta(n, 2)C_2 \frac{(B^{(s-1)*} - A)(B_s - A)(n+1)^2}{2^2 C_2^2 [\delta(n, 2)]^2} = (B^{s*} - A)(n+1)$$

because

$$B^{s*} - A = \frac{(B^{(s-1)*} - A)(B_s - A)(n+1)}{2C_2\delta(n, 2)} \Leftrightarrow$$

$$\Leftrightarrow B^{s*} - A = \frac{(n+1)^s \prod_{j=1}^{s+1} (B_j - A)}{2^s C_2^s [\delta(n, 2)]^s}.$$

□

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ON THE DECOMPOSABILITY OF LINEAR COMBINATIONS OF EULER POLYNOMIALS

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Abstract. Considering the effect of addition to the polynomial decomposition, we present some results on the decomposability of linear combinations of Euler polynomials.

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1. INTRODUCTION

Let $\mathbb{K}$ be a field. If $f(x) \in \mathbb{K}[x]$ has degree at least 2, we say that $f(x)$ is *decomposable* over the field $\mathbb{K}$ if we can write $f(x) = f_1(f_2(x))$ for some nonlinear polynomial $f_1(x)$, $f_2(x) \in \mathbb{K}[x]$. Otherwise, we say that $f(x)$ is *indecomposable* over $\mathbb{K}$. Two decompositions $f(x) = f_1(f_2(x))$ and $f(x) = F_1(F_2(x))$ are said to be *equivalent* over the field $\mathbb{K}$, written $f_1 \circ f_2 \sim_{\mathbb{K}} F_1 \circ F_2$, if there exists a linear polynomial $l(x) \in \mathbb{K}[x]$ such that

$$f_1(x) = F_1(l(x)) \text{ and } F_2(x) = l(f_2(x)).$$

For a given $f(x) \in \mathbb{K}[x]$ with degree at least 2, a *complete decomposition* of $f(x)$ over $\mathbb{K}$ is a decomposition $f = f_1 \circ \dots \circ f_k$, where the polynomials $f_i \in \mathbb{K}[x]$ are indecomposable over $\mathbb{K}$ for $i = 1, \dots, k$. A polynomial of degree greater than 1 always has a complete decomposition. However, it is not unique even up to equivalence.

Euler polynomials are defined by the following generating function

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2}{e^t + 1} e^{tx}.$$

Euler polynomials appear in statistical physics as well as in semi-classical approximations to quantum probability distributions (see [2]), in various approximation and

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expansion formulas in discrete mathematics and in number theory (see for instance [1], [4]), in p -adic analysis (see [8], Chapter 2).

In their paper, Kreso and Rakaczki [9] proved that every Euler polynomial with odd degree is indecomposable, while if the degree is even, they characterized all the possible decompositions. For further results for the decomposability of an infinite family of polynomials we refer to [3], [6] and [5].

The purpose of this note is to investigate the decomposability of the family of polynomials $P_n(x)$ generated by the following function

$$\sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!} = g(t) e^{tx}, \quad (1.1)$$

where

$$g(t) = \frac{p(t)}{e^t + 1} = \frac{a_k t^k + \dots + a_1 t + a_0}{e^t + 1}, \quad a_k, \dots, a_0 \in \mathbb{Q}.$$

If $p(t) = 2$ we get back the Euler polynomials. From the above generating function (1.1) one can deduce that

$$P_n(x) = \begin{cases} \frac{a_0}{2}, & \text{if } n = 0; \\ \frac{a_0}{2} E_n(x) + \sum_{j=1}^n \frac{a_j}{2} n(n-1) \dots (n-j+1) E_{n-j}(x), & \text{if } 1 \leq n < k; \\ \frac{a_0}{2} E_n(x) + \sum_{j=1}^k \frac{a_j}{2} n(n-1) \dots (n-j+1) E_{n-j}(x), & \text{if } n \geq k. \end{cases} \quad (1.2)$$

Actually, the polynomials $P_n(x)$ are linear combinations of Euler polynomials with rational coefficients.

One can begin with the simple case when the polynomial $p(t) = a_2 t^2 + a_0$. We obtain that

$$P_0(x) = \frac{a_0}{2}, P_1(x) = \frac{a_0}{2} E_1(x) \text{ and } P_n(x) = \frac{a_0}{2} E_n(x) + \frac{a_2}{2} n(n-1) E_{n-2}(x)$$

for $n \geq 2$. In this paper we prove that the polynomial

$$P_n(x) = E_n(x) + c E_{n-2}(x), \quad n \in \mathbb{N}$$

is always indecomposable provided that n is odd. If n is even then we give a possible decomposition of $P_n(x)$ and formulate a conjecture about the complete decomposition of $P_n(x)$. Our results and the conjecture will be presented in the next Section. In Section 3 we collect the auxiliary results and lemmas. The proofs of our theorems are in Section 4.

2. NEW RESULTS

Theorem 1. *Let $P_n(x) = E_n(x) + c E_{n-2}(x)$, where c is an arbitrary rational number, $n \geq 2$ is an integer. Then the polynomials $P_n(x)$ are indecomposable for*

all odd n . In the case when the degree n is even then the polynomial $P_n(x)$ has the following decomposition:

$$P_n(x) = F(x^2 - x), \quad (2.1)$$

where $F(x) \in \mathbb{Q}[x]$.

We note that this theorem is a generalization of the result of Kreso and Rakaczki [9] when n is odd since we get back their result taking $c = 0$ in our theorem. The proof of the second part of Theorem 1 is trivial, because for even n , we have $E_n(x) = \tilde{E}_n(x^2 - x)$ with $\tilde{E}_n(x) \in \mathbb{Z}[x]$.

Let

$$S^+ = \{f(x) \in \mathbb{C}[x] \mid f(x) = f(1-x)\}$$

and

$$S^- = \{f(x) \in \mathbb{C}[x] \mid f(x) = -f(1-x)\}.$$

Theorem 2. *Let $P(x) \in \mathbb{Q}[x]$ be a monic polynomial. If $P(x) \in S^+$ then $P(x)$ is always decomposable and has the following decomposition*

$$P(x) = F(x^2 - x),$$

where $F(x) \in \mathbb{Q}[x]$.

The following conjecture is based on a long and straightforward computation.

Conjecture 1. *Let n be an even positive integer and c be an arbitrary rational number. Then every decomposition of the polynomial $P_n(x) = E_n(x) + cE_{n-2}(x)$ is equivalent to (2.1).*

In [10] we proved a similar result for the linear combinations of Bernoulli polynomials, however, in Bernoulli case there are some exceptional cases.

Another conjecture based on our computational experiments, Theorem 1 above and Theorem in [10] is the next

Conjecture 2. *Let m, n be odd positive integers and c be an arbitrary rational number. Then the polynomials $E_m(x) + cE_n(x)$ and $B_m(x) + cB_n(x)$ are indecomposable.*

3. AUXILIARY RESULTS

Lemma 1. *Let $f(x), g(x) \in \mathbb{R}[x]$ be monic polynomials with $\deg f(x), \deg g(x) \geq 2$ and $a \in \mathbb{R}$. Denote by $\text{mult}(\alpha, f(x))$ the multiplicity of the root α of the polynomial $f(x)$. Suppose that*

$$\text{mult}(a/2, g(a-x) - g(x)) < \text{mult}(a/2, f(g(a-x)) - f(g(x))).$$

Then $f'(g(a/2)) = 0$.

Proof. Let $f(x) = x^k + b_{k-1}x^{k-1} + \dots + b_1x + b_0$. By expanding $f(g(a-x)) - f(g(x))$ we get

$$f(g(a-x)) - f(g(x)) = (g(a-x) - g(x))H(x), \quad (3.1)$$

where

$$\begin{aligned} H(x) = & g(a-x)^{k-1} + g(a-x)^{k-2}g(x) + \dots + g(a-x)g(x)^{k-2} + g(x)^{k-1} + \\ & b_{k-1} \left(g(a-x)^{k-2} + g(a-x)^{k-3}g(x) + \dots + g(a-x)g(x)^{k-3} + g(x)^{k-2} \right) \\ & + \dots + b_3 (g(a-x)^2 + g(a-x)g(x) + g(x)^2) + b_2(g(a-x) + g(x)) + b_1. \end{aligned} \quad (3.2)$$

We know that $H(a/2) = 0$ by the condition of the Lemma. But from (3.2) we get that $H(a/2) = kg(a/2)^{k-1} + (k-1)b_{k-1}g(a/2)^{k-2} + \dots + 3b_3g(a/2)^2 + 2b_2g(a/2) + b_1 = f'(g(a/2))$. $\square$

Lemma 2. Let $f(x) = b_kx^k + \dots + b_1x + b_0 \in S^+ \cup S^-$. Then

$$b_{k-1} = -\frac{k}{2}b_k \text{ and } b_{k-3} = \frac{k(k-1)(k-2)}{24}b_k - \frac{k-2}{2}b_{k-2}. \quad (3.3)$$

Further, if $f(x) \in S^+$, then $\deg f(x)$ is even, while if $f(x) \in S^-$ then $\deg f(x)$ is odd and $f(1/2) = 0$.

Proof. Compare the coefficients of x^k , x^{k-1} and x^{k-3} in the equality $f(x) = \pm f(1-x)$. $\square$

Lemma 3. Let $f(x), g(x) \in S^+$ be polynomials with $g \neq 0$. Then there exist unique polynomials $q(x), r(x) \in S^+$ such that

$$f(x) = q(x)g(x) + r(x) \text{ and } \deg r(x) < \deg g(x). \quad (3.4)$$

Proof. From the long division algorithm for polynomials we know that there exist unique polynomials $q(x), r(x) \in \mathbb{C}[x]$ satisfying (3.4). Since $f(x) = f(1-x)$ and $g(x) = g(1-x)$, (3.4) gives that

$$f(x) = q(1-x)g(x) + r(1-x). \quad (3.5)$$

It follows from (3.4) and (3.5) that

$$0 = g(x)(q(1-x) - q(x)) + r(1-x) - r(x). \quad (3.6)$$

If $q(1-x) \neq q(x)$ then

$$\deg(g(x)(q(1-x) - q(x))) \geq \deg g(x) > \deg(r(1-x) - r(x))$$

which contradicts (3.6). This means that $q(x) \in S^+$ and so $r(x) \in S^+$. $\square$

The following two results are general theorems from the theory of decomposability.

Lemma 4 (Kreso and Rakaczki [9]). *Let $F(x) \in \mathbb{K}[x]$ be a monic polynomial such that $\deg F$ is not divisible by the characteristic of the field $\mathbb{K}$. Then for every nontrivial decomposition $F = F_1 \circ F_2$ over any field extension $\mathbb{L}$ of $\mathbb{K}$, there exists a decomposition $F = \tilde{F}_1 \circ \tilde{F}_2$ such that the following conditions are satisfied*

- $\tilde{F}_1 \circ \tilde{F}_2$ and $F_1 \circ F_2$ are equivalent over $\mathbb{L}$,
- $\tilde{F}_1(x)$ and $\tilde{F}_2(x)$ are monic polynomials with coefficients in $\mathbb{K}$,
- $\text{coeff}(x^{\deg \tilde{F}_1 - 1}, \tilde{F}_1(x)) = 0$.

Moreover, such decomposition $\tilde{F}_1 \circ \tilde{F}_2$ is unique.

Lemma 5 (I. Gusić [7]). *Let $\mathbb{K}$ be a field of characteristic zero. Suppose that the nonconstant polynomials $g, h, G, H \in \mathbb{K}[x]$ satisfy $g \circ h = G \circ H$ and $\deg h = \deg H$. Then there exists $a, b \in \mathbb{K}$ such that*

$$H(x) = ah(x) + b, \quad G(x) = g\left(\frac{1}{a}x - b\right).$$

Lemma 6. *Let $P(x) \in \mathbb{Q}[x]$ be a monic polynomial. Assume that $P(x) \in S^-$ and $P(x) = f(g(x))$, where $f(x), g(x) \in \mathbb{Q}[x]$ and $\deg(f(x)), \deg(g(x)) > 1$. Then we can assume that $f(x), g(x)$ are monic, $g(x) \in S^-$ and $f(x) = -f(-x)$.*

Proof. From Lemma 4 we can assume that $f(x), g(x) \in \mathbb{Q}[x]$ are monic and $\text{coeff}(x^{\deg f(x) - 1}, f(x)) = 0$. Let $f(x) = x^k + b_{k-2}x^{k-2} + \dots + b_1x + b_0$ and $g(x) = x^t + c_{t-1}x^{t-1} + \dots + c_1x + c_0$. Since

$$f(g(x)) = P(x) = -P(1-x) = -f(g(1-x)),$$

from Lemma 5 we obtain that there exist $a, b \in \mathbb{Q}$ such that

$$g(x) = ag(1-x) + b, \quad f(x) = -f\left(\frac{1}{a}x - b\right). \quad (3.7)$$

It follows from $\deg P(x) = kt$ and Lemma 2 that k and t are odd. Investigation of the coefficient of x^t in the equality $g(x) = ag(1-x) + b$ yields that $1 = (-1)^t a = -a$. Now, comparing the coefficient of x^{k-1} in the equality $f(x) = -f(-x - b)$ we obtain that $0 = kb$, that is $b = 0$. This proves our assertion. $\square$

Finally, we need the following lemma, in which we collect some well known properties of the Euler polynomials which will be used in the sequel, sometimes without particular reference.

Lemma 7.

- (a) $E_n(x) = (-1)^n E_n(1-x)$;
- (b) $E_n(x+1) + E_n(x) = 2x^n$;
- (c) $E'_n(x) = n E_{n-1}(x)$;
- (d) $E_{2n-1}(1/2) = E_{2n}(0) = E_{2n}(1) = 0$ for $n \in \mathbb{N}$;
- (e) $E_n(x) = \sum_{k=0}^n \binom{n}{k} E_k(0) x^{n-k}$;
- (f) $E_{2n-1}(0) E_{2n+1}(0) < 0$ for $n \in \mathbb{N}$.

Proof. See [4]. □

4. PROOFS OF THE THEOREMS

Proof of Theorem 2. We prove the assertion by using induction on n , the degree of the polynomial $P(x)$. If $n = 0$, that is $P(x)$ is a constant polynomial then the assertion is trivial. Now assume that the result is true for polynomial of degree less than n . From Lemma 3 we can deduce that $P(x) = q(x)(x^2 - x) + r(x)$, where $q(x), r(x) \in S^+$ and $\deg r(x) < 2$. But this means that $r(x)$ is a constant polynomial and $P(x) = q(x)(x^2 - x) + p_0$, where p_0 is a rational number. By the induction hypothesis, $q(x) = Q(x^2 - x)$ with $Q(x) \in \mathbb{Q}[x]$, and thus our statement is proved.

Proof of Theorem 1. Let n be odd. From the part (a) of Lemma 7 we have $P_n(x) \in S^-$. Suppose that $P_n(x) = f(g(x))$. From Lemma 6 we can assume that $f(x), g(x) \in \mathbb{Q}[x]$ are monic, $g(x) \in S^-$, $f(x) = -f(-x)$. Let

$$\begin{aligned} f(x) &= x^k + b_{k-2}x^{k-2} + b_{k-4}x^{k-4} + \dots + b_3x^3 + b_1x, \\ g(x) &= x^t + c_{t-1}x^{t-1} + \dots + c_1x + c_0. \end{aligned}$$

Using (b) of Lemma 7 one can deduce that

$$f(g(x+1)) + f(g(x)) = P_n(x+1) + P_n(x) = 2x^n + 2cx^{n-2}. \quad (4.1)$$

Since $P_n(x) \in S^-$ thus $P_n(x+1) = -P_n(-x)$. From (4.1) we infer that the polynomial $g(x) - g(-x)$ divides the polynomial $P_n(x) - P_n(-x) = 2x^n + 2cx^{n-2}$ and so one of the cases

$$g(x) - g(-x) = \begin{cases} (i) & dx^s; \\ (ii) & dx^s(x - e), \text{ where } e \in \{\pm\sqrt{c}\}; \\ (iii) & dx^s(x^2 + c). \end{cases} \quad (4.2)$$

holds. We know that

$$g(x) - g(-x) = 2x^t + 2c_{t-2}x^{t-2} + \dots + 2c_3x^3 + 2c_1x. \quad (4.3)$$

Consider the above three cases in (4.2). The case (ii) is simple. In this case $x = e$ is a root of the odd polynomial $g(x) - g(-x)$, but then $x = -e$ must be also a root which only possible if $c = 0$. If $c = 0$ we get back the case (i).

In the case (i) we have $d = 2$, $s = t$ and

$$c_{t-2} = c_{t-4} = \dots = c_3 = c_1 = 0 \quad (4.4)$$

by (4.3). Since the polynomial $f(x)$ is odd, it is easy to see that the coefficient of the monomial x^{n-4} depends only on the term $g(x)^k$ in $f(g(x)) = g(x)^k + b_{k-2}g(x)^{k-2} + \dots + b_1g(x)$. By expanding $g(x)^k$ and using (4.4) and Lemma 2 we get that this coefficient is

$$\frac{t^4}{16} \binom{k}{4} - \frac{t}{4} \binom{k}{2} \binom{t}{3}.$$

On the other hand the coefficient of x^{n-4} of $P_n(x) = E_n(x) + cE_{n-2}(x)$ is 0 by (d) and (e) of Lemma 7. Thus we get the following equation

$$0 = \frac{t^4}{16} \binom{k}{4} - \frac{t}{4} \binom{k}{2} \binom{t}{3} = \frac{t^2 k(k-1)}{384} (t^2(k-2)(k-3) - 8(t-1)(t-2)). \quad (4.5)$$

From this one can deduce that t divides $8(t-1)(t-2)$ which is impossible because $t \geq 3$ is an odd integer.

Investigate now the case (iii). It is easy to see that $d = 2$ and $s = t - 2$. If we assume that $t > 3$ then we get from (4.3) that $c_1 = g'(0) = 0$ and $c_{t-2} = c$. It is follows from

$$f(g(x)) - f(g(-x)) = 2x^n + 2cx^{n-2} \text{ and } n > t \geq 3$$

that

$$n - 2 = \text{mult}(0, f(g(x)) - f(g(-x))) > \text{mult}(0, g(x) - g(-x)) = t - 2.$$

By Lemma 1 one can obtain that $f'(g(0)) = 0$. From

$$P'_n(x) = nE_{n-1}(x) + c(n-2)E_{n-3}(x) = f'(g(x))g'(x) \quad (4.6)$$

we infer that $x = 0$ is a zero of $P''_n(x)$, that is

$$0 = n(n-1)E_{n-2}(0) + c(n-2)(n-3)E_{n-4}(0). \quad (4.7)$$

After a simple rearrange we obtain from (f) of Lemma 7 that

$$c = -\frac{n(n-1)E_{n-2}(0)}{(n-2)(n-3)E_{n-4}(0)} > 0. \quad (4.8)$$

On the other hand, if we compare the coefficient of x^{n-2} in the equality $P_n(x) = f(g(x))$ we have that

$$c = kc_{t-2} + \binom{k}{2} c_{t-1}^2 = ck + \binom{k}{2} \frac{t^2}{4}.$$

But this means that

$$c = \frac{1}{1-k} \binom{k}{2} \frac{t^2}{4} = -k \frac{t^2}{8} < 0, \quad (4.9)$$

which contradicts (4.8).

In the remaining case when $t = \deg g(x) = 3$ one can deduce from $g(x) \in S^-$, $c_{t-2} = c$ and Lemma 2 that

$$g(x) = x^3 - \frac{3}{2}x^2 + cx + \frac{1}{4} - \frac{1}{2}c. \quad (4.10)$$

Computing the coefficients of x^{n-4} in the equality $E_n(x) + cE_{n-2}(x) = f(g(x))$ we obtain that

$$0 = \frac{81}{16} \binom{k}{4} + c^2 \binom{k}{2} + \frac{27}{4} c \binom{k}{3} - 3 \binom{k}{2} \left(\frac{1}{4} - \frac{1}{2} c \right). \quad (4.11)$$

Combining (4.11) with (4.9) we have that

$$0 = -\frac{27}{64} k^4 + \frac{135}{128} k^3 + \frac{33}{128} k^2 - \frac{57}{64} k,$$

whence

$$k = 0, 1, \frac{3}{4} \pm \frac{\sqrt{385}}{12}.$$

In the case when n is even the assertion follows from (a) of Lemma 7 and Theorem 2.

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SOME COEFFICIENT PROPERTIES RELATING TO A CERTAIN CLASS OF STARLIKE FUNCTIONS

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Abstract. This paper considers the problem of determining coefficients in a class Δ^* of normalized starlike functions f analytic in the open unit disk $|z| < 1$ satisfying the inequality that

$$\left| \left\{ \frac{zf'(z)}{f(z)} \right\}^2 - 1 \right| < 2 \left| \frac{zf'(z)}{f(z)} \right|.$$

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1. INTRODUCTION

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Also, let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ comprising of functions f normalized by $f(0) = 0$, $f'(0) = 1$, and let $\mathcal{S} \subset \mathcal{A}$ denote the class of functions which are univalent in $\mathbb{U}$. Let a function f be analytic univalent in the unit disc $\mathbb{U} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$ with the normalization $f(0) = 0$, then f maps $\mathbb{U}$ onto a starlike domain with respect to $w_0 = 0$ if and only if

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in \mathbb{U}). \quad (1.1)$$

It is well known that if an analytic function f satisfies (1.1) and $f(0) = 0$, $f'(0) \neq 0$, then f is univalent and starlike in $\mathbb{U}$. The set of all functions $f \in \mathcal{A}$ that are starlike univalent in $\mathbb{U}$ will be denoted by $\mathcal{S}^*$.

For the purpose of this paper, we represent by Δ^* a class which is defined by

$$\Delta^* = \left\{ f \in \mathcal{S}^* : \left| \left\{ \frac{zf'(z)}{f(z)} \right\}^2 - 1 \right| < 2 \left| \frac{zf'(z)}{f(z)} \right|, z \in \mathbb{U} \right\} \quad (1.2)$$

and a related class studied by Rønning [8] was defined by

$$\mathcal{S}_p = \left\{ f \in \mathcal{S} : \left| \frac{zf'(z)}{f(z)} - 1 \right| < \Re \frac{zf'(z)}{f(z)}, z \in \mathbb{U} \right\}. \quad (1.3)$$

Interpreting geometrically the condition in (1.3), we note that $zf'(z)/f(z)$ lies inside the parabola

$$(\Im w)^2 < 2\Re w - 1,$$

and in this way the class $\mathcal{S}_p$ was observed to be connected with certain conic domains. In recent papers [1–4, 6, 10], certain function classes were considered and were defined under the condition that $zf'(z)/f(z)$ lies in a domain which possesses some geometric properties. If we interpret the condition in (1.2) geometrically, then we observe that the product of the distances of $zf'(z)/f(z)$ from the foci -1 and 1 is less than twice the distance of $zf'(z)/f(z)$ from the origin. The shape of the domain for $Q(z) = zf'(z)/f(z)$ is described in Theorem 1 below and the shape of $Q(\mathbb{U})$ is depicted in Figure 1.

Theorem 1 ([9]). *If $f(z) \in \Delta^*$, then*

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \quad z \in \mathbb{U} \quad (1.4)$$

and

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \sqrt{2} \quad \text{and} \quad \left| \frac{zf'(z)}{f(z)} + 1 \right| > \sqrt{2}, \quad z \in \mathbb{U}. \quad (1.5)$$

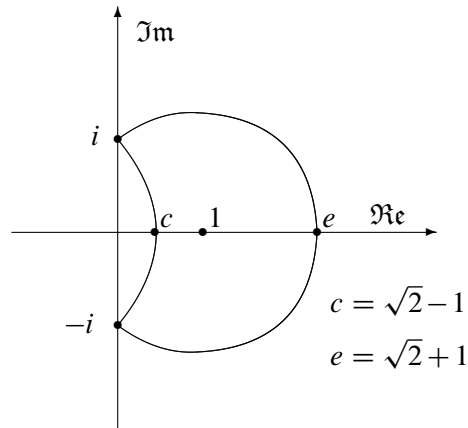


FIGURE 1. The domain for $zf'(z)/f(z)$, $f \in \Delta^*$.

2. COEFFICIENT ESTIMATES

Theorem 2. If $f(z) \in \Delta^*$ and

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{U}, \quad (2.1)$$

then

$$|a_2| \leq 1, \quad |a_3| \leq 3/4, \quad |a_4| \leq 1/2. \quad (2.2)$$

Proof. In view of (1.2), we have

$$\left\{ \frac{zf'(z)}{f(z)} \right\}^2 - 1 = 2w(z) \frac{zf'(z)}{f(z)},$$

where

$$|w(z)| < 1 \quad z \in \mathbb{U}, \quad w(z) = \sum_{k=1}^{\infty} c_k z^k. \quad (2.3)$$

Thus, we obtain

$$(zf'(z) - f(z))(zf'(z) + f(z)) = 2w(z)zf'(z)f(z). \quad (2.4)$$

If we assume that $a_1 = 1$, then from (2.1) and (2.3), we at once have

$$\begin{aligned} & \left(\sum_{k=1}^{\infty} (k-1)a_k z^k \right) \left(\sum_{k=1}^{\infty} (k+1)a_k z^k \right) \\ &= 2 \left(\sum_{k=1}^{\infty} c_k z^k \right) \left(\sum_{k=1}^{\infty} k a_k z^k \right) \left(\sum_{k=1}^{\infty} a_k z^k \right). \end{aligned} \quad (2.5)$$

Hence, we obtain

$$\begin{aligned} & \left(\sum_{k=1}^{\infty} (k+1)a_k z^k \right) \left(\sum_{k=1}^{\infty} (k-1)a_k z^k \right) \\ &= (2z + 3a_2 z^2 + 4a_3 z^3 + \cdots)(a_2 z^2 + 2a_3 z^3 + 3a_4 z^4 + \cdots) \\ &= 2a_2 z^3 + (4a_3 + 3a_2^2)z^4 + (6a_4 + 10a_2 a_3)z^5 + \cdots \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} & 2 \left(\sum_{k=1}^{\infty} c_k z^k \right) \left(\sum_{k=1}^{\infty} k a_k z^k \right) \left(\sum_{k=1}^{\infty} a_k z^k \right) \\ &= 2(c_1 z + c_2 z^2 + c_3 z^3 + \cdots)(z + 2a_2 z^2 + 3a_3 z^3 + \cdots)(z + a_2 z^2 + a_3 z^3 + \cdots) \\ &= 2c_1 z^3 + 2(3a_2 c_1 + c_2)z^4 + 2(4a_3 c_1 + 2a_2^2 c_1 + 3a_2 c_2 + c_3)z^5 + \cdots \end{aligned} \quad (2.7)$$

Equating now the coefficients of like powers of z in (2.6) and (2.7), we have

$$(i) \quad a_2 = c_1,$$

$$(ii) \quad 4a_3 + 3a_2^2 = 6a_2c_1 + 2c_2,$$

$$(iii) \quad 6a_4 + 10a_2a_3 = 8a_3c_1 + 4a_2^2c_1 + 6a_2c_2 + 2c_3.$$

It is well known that the coefficients of the bounded function $w(z)$ satisfies the inequality that $|c_k| \leq 1$, ($k = 1, 2, 3, \dots$), and hence from (i), we have the first inequality of (2.2) that $|a_2| \leq 1$. Now, from (i) and (ii), we have

$$|4a_3| = 2 \left| c_2 + \frac{3}{2}c_1^2 \right|. \quad (2.8)$$

Using the estimate (see [7]) that if $w(z)$ has the form (2.3), then

$$|c_2 - \mu c_1^2| \leq \max\{1, |\mu|\}, \quad \text{for all } \mu \in \mathbb{C}, \quad (2.9)$$

and we obtain from (2.8) and (2.9) that

$$|a_3| \leq \frac{3}{4},$$

which gives the second inequality of (2.2). From (iii), we find that

$$|6a_4| = |-10a_2a_3 + 8a_3c_1 + 4a_2^2c_1 + 6a_2c_2 + 2c_3|. \quad (2.10)$$

Because $a_2 = c_1$, (2.10) becomes

$$|6a_4| = |-2a_3c_1 + 4c_1^3 + 6c_1c_2 + 2c_3|. \quad (2.11)$$

Moreover, from (i) – (ii), we have

$$a_3 = \frac{1}{2}c_2 + \frac{3}{4}c_1^2, \quad (2.12)$$

and from (2.11) and (2.12), we obtain that

$$\begin{aligned} |6a_4| &= \left| -2 \left(\frac{1}{2}c_2 + \frac{3}{4}c_1^2 \right) c_1 + 4c_1^3 + 6c_1c_2 + 2c_3 \right| \\ &= \left| \frac{5}{2}c_1^3 + 5c_1c_2 + 2c_3 \right| \\ &= \left| \frac{5}{2}(c_1^3 + 2c_1c_2 + c_3) - \frac{1}{2}c_3 \right|. \end{aligned} \quad (2.13)$$

To find the bound for the coefficient a_4 , we next derive some properties of the coefficients c_k involved in (2.13). It is known that the function $p(z)$ given by

$$\frac{1+w(z)}{1-w(z)} = 1 + p_1z + p_2z^2 + \dots =: p(z) \quad (2.14)$$

defines a Caratheodory function with the property that $\Re\{p(z)\} > 0$ in $\mathbb{U}$ and that $|p_k| \leq 2$ ($k = 1, 2, 3, \dots$).

Using (2.3) and equating the coefficients of like powers of z in (2.14), we get

$$p_2 = 2(c_1^2 + c_2) \quad \text{and} \quad p_3 = 2(c_1^3 + 2c_1c_2 + c_3).$$

Hence $|c_1^2 + c_2| \leq 1$ and

$$|c_1^3 + 2c_1c_2 + c_3| \leq 1, \quad (2.15)$$

and upon using (2.13) and (2.15), we finally find that

$$\begin{aligned} |6a_4| &\leq \left| \frac{5}{2} (c_1^3 + 2c_1c_2 + c_3) \right| + \left| \frac{1}{2} c_3 \right| \\ &\leq \frac{5}{2} + \frac{1}{2} = 3, \end{aligned}$$

which gives the third inequality of (2.2) that $|a_4| \leq 1/2$. $\square$

Remark. We deem it worthwhile to point out here the sharpness of the estimates of the coefficients given by (2.2) of Theorem 2. Therefore, let us consider the function $f_1(z)$ by

$$\begin{aligned} f_1(z) &= z \exp \int_0^z \frac{\sqrt{1+t^2} + t - 1}{t} dt \\ &= \frac{2\sqrt{1+z^2} - 2}{z} \exp \left\{ z - 1 + \sqrt{1+z^2} \right\} \\ &= z + z^2 + \frac{3}{4}z^3 + \frac{5}{12}z^4 + \frac{1}{6}z^5 + \frac{1}{20}z^6 + \frac{49}{2880}z^7 \dots \quad (z \in \mathbb{U}). \end{aligned} \quad (2.16)$$

To show that $f_1(z) \in \Delta^*$, we need to show that

$$f_1(z) \in \mathcal{S}^* \quad \text{and} \quad \left| \left\{ \frac{zf_1'(z)}{f_1(z)} \right\}^2 - 1 \right| < 2 \left| \frac{zf_1'(z)}{f_1(z)} \right|. \quad (2.17)$$

We note that

$$0 \leq \Re \left\{ e^{it} + \sqrt{e^{2it} + 1} \right\} = \begin{cases} \cos t + \sqrt{|2 \cos t|} \cos t / 2 & \text{for } t \in [0, \pi/2], \\ \cos t + \sqrt{|2 \cos t|} \sin t / 2 & \text{for } t \in (\pi/2, 3\pi/2], \\ \cos t - \sqrt{|2 \cos t|} \cos t / 2 & \text{for } t \in (3\pi/2, 2\pi), \end{cases}$$

and therefore, for the above function f_1 , we have

$$\Re \left\{ \frac{zf_1'(z)}{f_1(z)} \right\} = \Re \left\{ z + \sqrt{1+z^2} \right\} > 0 \quad (z \in \mathbb{U}).$$

Hence, by (1.1), $f_1(z) \in \mathcal{S}^*$ and the left-hand side of the second condition in (2.17) becomes

$$\begin{aligned} \left| \left\{ \frac{zf_1'(z)}{f_1(z)} \right\}^2 - 1 \right| &= \left| \left\{ z + \sqrt{1+z^2} \right\}^2 - 1 \right| \\ &= 2|z| \left| z + \sqrt{1+z^2} \right| \\ &< 2 \left| z + \sqrt{1+z^2} \right| \end{aligned}$$

$$= 2 \left| \frac{zf_1'(z)}{f_1(z)} \right|,$$

which implies that $f_1(z) \in \Delta^*$. Thus, from (2.16), we see that the first and the second estimations in (2.2) are sharp. The question is whether the third estimation $|a_4| \leq 1/2$ is sharp in the class. The function (2.16) suggest the following conjecture.

Conjecture 1. *If $f(z)$ defined by (2.18) belongs to Δ^* , then*

$$|a_4| \leq \frac{5}{12} = 0.416\dots$$

In the sequel, we find somewhat weaker estimation for $|a_n|$, for the next coefficients too by applying another method.

Theorem 3. *If $f(z) \in \Delta^*$ and*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{U}, \quad (2.18)$$

then for $n = 2, 3, 4, \dots$, we have

$$(n-1)^2 |a_n|^2 \leq \sum_{k=1}^{n-1} |a_k|^2 (1 + 2k - k^2). \quad (2.19)$$

Proof. In view of (1.5), we have

$$\frac{zf'(z)}{f(z)} - 1 = \sqrt{2}w(z),$$

where

$$|w(z)| < 1 \quad z \in \mathbb{U}, \quad w(z) = \sum_{k=1}^{\infty} c_k z^k. \quad (2.20)$$

Thus, we obtain

$$\frac{1}{\sqrt{2}}(zf'(z) - f(z)) = w(z)f(z), \quad (2.21)$$

and from (2.18) and (2.20), we at once have

$$\frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} (k-1)a_k z^k = \sum_{k=1}^{\infty} c_k z^k \sum_{k=1}^{\infty} a_k z^k, \quad a_1 = 1.$$

Thus, we get

$$\frac{1}{\sqrt{2}} \sum_{k=1}^n (k-1)a_k z^k + \frac{1}{\sqrt{2}} \sum_{k=n+1}^{\infty} (k-1)a_k z^k = w(z) \left\{ \sum_{k=1}^{n-1} a_k z^k + \sum_{k=n}^{\infty} a_k z^k \right\},$$

which gives

$$\sum_{k=1}^n \frac{k-1}{\sqrt{2}} a_k z^k + \sum_{k=n+1}^{\infty} \frac{k-1}{\sqrt{2}} a_k z^k - \sum_{k=1}^{\infty} c_k z^k \sum_{k=n}^{\infty} a_k z^k = w(z) \left\{ \sum_{k=1}^{n-1} a_k z^k \right\}.$$

Therefore, we can write

$$\sum_{k=1}^n \frac{k-1}{\sqrt{2}} a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k = w(z) \sum_{k=1}^{n-1} a_k z^k,$$

for some b_k , $n+1 \leq k < \infty$, where b_k can be expressed in terms of the following relation involving the coefficients a_k and c_k :

$$b_k = \frac{k-1}{\sqrt{2}} a_k - \sum_{j=1}^{k-n} c_j a_{k-j}.$$

This gives

$$\begin{aligned} \left| \sum_{k=1}^n \frac{k-1}{\sqrt{2}} a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k \right|^2 &= \left| w(z) \sum_{k=1}^{n-1} a_k z^k \right|^2 \\ &\leq \left| \sum_{k=1}^{n-1} a_k z^k \right|^2, \end{aligned} \quad (2.22)$$

where

$$\sum_{k=1}^n \frac{k-1}{\sqrt{2}} a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k := \sum_{k=1}^{\infty} d_k z^k$$

is an analytic function in the unit disc. Making use of the known formula (see, for instance [5])

$$\int_0^{2\pi} \left| \sum_{k=1}^{\infty} d_k (r e^{i\theta})^k \right|^2 d\theta = 2\pi \sum_{n=1}^{\infty} |d_n|^2 r^{2n},$$

and integrating on $z = r e^{i\theta}$, $0 < r < 1$, $0 \leq \theta < 2\pi$, both the sides of (2.22), we obtain

$$\sum_{k=1}^n \frac{(k-1)^2}{2} |a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |b_k|^2 r^{2k} \leq \sum_{k=1}^{n-1} |a_k|^2 r^{2k}.$$

Therefore,

$$\frac{1}{2} \sum_{k=1}^n (k-1)^2 |a_k|^2 r^{2k} \leq \sum_{k=1}^{n-1} |a_k|^2 r^{2k},$$

which upon letting $r \rightarrow 1$ gives

$$\frac{1}{2} \sum_{k=1}^n (k-1)^2 |a_k|^2 \leq \sum_{k=1}^{n-1} |a_k|^2,$$

and this leads to the desired result (2.19). $\square$

Theorem 4. Let $f(z)$ defined by (2.18) belong to Δ^* . Then for $n \geq 3$, we have

$$|a_n| \leq \frac{\sqrt{3}}{n-1}. \quad (2.23)$$

Proof. From (2.19), we have (for $n \geq 3$)

$$\begin{aligned} (n-1)^2 |a_n|^2 &\leq \sum_{k=1}^{n-1} |a_k|^2 (1+2k-k^2) \\ &= 2|a_1|^2 + |a_2|^2 - 2|a_3|^2 - 7|a_4|^2 - \dots \\ &\leq 2|a_1|^2 + |a_2|^2 \\ &= 2 + |a_2|^2. \end{aligned}$$

Furthermore, $|a_2|^2 \leq 1$ by Theorem 2, and we have then

$$(n-1)^2 |a_n|^2 \leq 3$$

and finally we obtain (2.23). $\square$

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A NOTE ON A CHARACTERIZATION THEOREM FOR A CERTAIN CLASS OF DOMAINS

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Abstract. We have introduced and studied in [2] the class of *Globalized multiplicatively pinched-Dedekind domains (GMPD domains)*. This class of domains could be characterized by a certain factorization property of the non-invertible ideals, (see [2, Theorem 4]). In this note a simplification of the characterization theorem [2, Theorem 4] is provided in more general form.

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Keywords: invertible ideal, Prüfer domain, h-local domain

Let D be an integral domain. By an *MNI ideal* of D we mean an ideal of D which is maximal among the nonzero noninvertible ideals of D . By [6, Exercise 36, page 44], every MNI ideal is a prime ideal. Moreover, using standard Zorn's Lemma arguments, one can show that every nonzero non-invertible ideal is contained in some MNI ideal. D is said to be *h-local* provided every nonzero ideal of D is contained in at most finitely many maximal ideals of D and each nonzero prime ideal of D is contained in a unique maximal ideal of D . D is called a *pseudo-valuation domain (PVD)* if D is quasi-local with maximal ideal M and $M : M$ is a valuation domain with maximal ideal M , cf. [5] and [1, Proposition 2.5]. A *two-generated* domain is a domain whose ideals are two-generated. Let D be a quasi-local domain with maximal ideal M . By [5, Theorems 2.7 and 3.5], D is a two-generated PVD if and only if D is a field, a DVR, or a Noetherian domain such that its integral closure D' is a DVR with maximal ideal M and D'/M is a quadratic field extension of D/M .

In [2], we introduced and study the class of *Globalized multiplicatively pinched-Dedekind domains (GMPD domains)*. A domain D is called a *globalized multiplicatively pinched-Dedekind domain (GMPD domain)* if D is h-local and for each maximal ideal M , D_M is a two-generated PVD, or a valuation domain with value group $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{R}$, cf. [2, Definition 2]. A Dedekind domain is a GMPD domain and the integrally closed Noetherian GMPD domain are exactly the Dedekind domains. This class of domains could be characterized by a certain factorization property of the non-invertible ideals. An h-local domain D is a GMPD domain if and only if

every two MNI ideals are comaximal and every nonzero non-invertible ideal I of D can be written as $I = JP_1 \cdots P_k$ for some invertible ideal J and distinct MNI ideals $P_1, \dots, P_k$ uniquely determined by I ([2, Theorem 4, Remark 5]). In this note a more general simplification of this characterization theorem is provided (Theorem 2).

Throughout this note all rings are (commutative unitary) integral domains. For a domain D , D' (resp. $\bar{D}$) denotes the integral closure (resp. complete integral closure) of D . Any unexplained material is standard like in [4] or [6].

Lemma 1. *Every two distinct MNI ideals of a domain D are comaximal.*

Proof. Deny. Let $P_1 \neq P_2$ be the MNI ideals of D both contained in the maximal ideal M . Then M is invertible and so $P_i \subsetneq M$ implies that $P_i \subsetneq \bigcap_{n \geq 1} M^n = Q$. The ideal Q is invertible and prime. Indeed, if $ab \in Q$ with both $a, b \notin Q$, then there exist integers k, l and the ideals U, V such that $(a) = M^k U$ with $U \not\subseteq M$ and $(b) = M^l V$ with $V \not\subseteq M$. Since $ab \in M^{k+l+1}$, so $(ab) = M^{k+l+1} N$ for some ideal N . Combining, we get that $M^{k+l} UV = M^{k+l+1} N$. This implies that $UV = MN$ which is not possible because U, V are not contained in M . Hence Q is prime. As $P_i \subsetneq Q$, so Q is invertible. Since any two invertible prime ideals are not comparable, so $Q = M$. This implies that $M = M^2$ and hence $M = D$, a contradiction. $\square$

Recall [3, Section 5.1] that a domain D has pseudo-Dedekind factorization if for each nonzero non-invertible ideal I , there is an invertible ideal B (which might be D) and finitely many pairwise comaximal primes $P_1, P_2, \dots, P_n$ such that $I = BP_1 P_2 \cdots P_n$ (the requirement that $n > 0$ comes for free).

Theorem 1. *Let D be a domain such that every nonzero non-invertible ideal I of D can be written as $I = JP_1 \cdots P_k$ for some invertible ideal J and distinct MNI ideals $P_1, \dots, P_k$. Then D is h -local.*

Proof. By [6, Exercise 36, page 44], every MNI ideal is a prime ideal and by Lemma 1, $P_1, \dots, P_k$ are pairwise comaximal. Hence D has pseudo-Dedekind factorization, cf. [3, Section 5.1]. Now Apply [3, Corollary 5.2.14]. $\square$

Theorem 2. *A domain D is a GMPD domain if and only if every nonzero non-invertible ideal I of D can be written as $I = JP_1 \cdots P_k$ for some invertible ideal J and distinct MNI ideals $P_1, \dots, P_k$.*

Proof. Apply Theorem 1 and [2, Theorem 4]. $\square$

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ON BOUNDARY VALUE PROBLEMS WITH PRESCRIBED NUMBER OF ZEROES OF SOLUTIONS

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Abstract. We show how appropriate parametrisation technique and successive approximations can help to investigate solutions of Emden-Fowler type equations with a given number of zeroes. The technique can be efficiently applied for more general equations with non-linearities involving absolute value signs and various types of boundary conditions.

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Keywords: Emden-Fowler equation, model-type boundary conditions, successive approximations, prescribed number of zeroes

1. INTRODUCTION

The need to construct solutions with a prescribed number of zeroes arises quite often in the theory of ordinary differential equations (see, e. g., [2, 3, 9, 11, 12, 14] and references therein). Due to the overall complexity of the problem, it is natural to observe that the proof of the existence of such solutions is usually of qualitative character and is seldom constructive. One can cite considerably less works where the solution in question is found in a straightforward way.

The aim of this note is to demonstrate that the numerical-analytic techniques with interval divisions [17–19] can serve this purpose. Being far from claiming that, in this way, a complete description is possible (of course, every approach has its pros and cons), we should like to show that this idea provides one a rather general “machinery” for practical finding solutions with a given number of sign changes.

We demonstrate the approach on a particular case of the Emden-Fowler type equation

$$u''(t) = -q(t)|u(t)|^3 u(t) + r(t), \quad t \in [a, b], \quad (1.1)$$

where $q(t) > 0$ for all $t \in [a, b]$, for which the question mentioned has long been of interest. Note that, despite their seemingly simple form, these equations are known to exhibit very complicated behaviour of solutions [7]. For this class of equations, the existence of solutions with a given number of zeroes is studied, in particular, in [1, 5, 13].

The Nehari theory for Emden-Fowler type equations (1.1) provides a way to detect the existence of solutions with prescribed number of zeros [4–6, 10, 14]. For instance, in the case of the Dirichlet boundary value problem

$$\begin{aligned} u''(t) &= -q(t)(u(t))^3, \quad t \in [-1, 1], \\ u(-1) &= 0, \quad u(1) = 0, \end{aligned} \tag{1.2}$$

where q is the step function

$$q(t) = \begin{cases} 2 & \text{if } t \in [-1, -1 + \alpha] \\ 0 & \text{if } t \in (-1 + \alpha, 1 - \alpha) \\ 2 & \text{if } t \in [1 - \alpha, 1] \end{cases}$$

and α is chosen in a specific way, the Nehari theory gives three positive solutions of (1.2). In [4], the authors construct three positive solutions of problem (1.2) for $q(t) = 2/(1 + h\xi(t))^6$ with an $h > 0$ and the “ $\wedge$ -shaped” piecewise linear ξ ,

$$\xi(t) = \begin{cases} 1 + t & \text{for } -1 \leq t < 0, \\ 1 - t & \text{for } 0 \leq t \leq 1. \end{cases}$$

The methods used to construct the solution usually strongly depend on the specific form of the equation. In particular, the approach of [4] is based on the explicit use of the lemniscate functions $\operatorname{sl} t$ and $\operatorname{cl} t$, i. e., the respective solutions of the initial value problems $u(0) = 0, u'(0) = 1$ and $u(0) = 1, u'(0) = 0$ for the equation $u'' = -2u^3$.

Motivated, in particular, by the work [4] on the above-mentioned problem of form (1.2), we show here how parametrisation techniques can be used to construct solutions of this kind of equations. We describe the approach for the Dirichlet problem

$$u''(t) = g(t, u(t), u'(t)), \quad t \in [a, b], \tag{1.3}$$

$$u(a) = 0, \quad u(b) = 0, \tag{1.4}$$

focusing on several concrete Emden-Fowler type equations of form (1.1), which quite well illustrates the idea (the techniques are, however, not limited to this case). Depending on the choice of q and r in (1.1), we study several cases where sign-constant and sign-changing solutions are detected.

We use here the numerical-analytic approach from [19], which allows one to construct approximate the solutions of problem (4.4), (4.5) and, moreover, rigorously prove the existence of an exact solutions by using the results of computation. More details on the techniques discussed and used here can be found, e. g., in [15–18]. We describe how these techniques can be efficiently used to find either sign-constant or sign-changing solutions.

2. THE METHOD

Consider the two-point problem

$$u_1'(t) = f_1(t, u_1(t), u_2(t)), \quad (2.1)$$

$$u_2'(t) = f_2(t, u_1(t), u_2(t)), \quad t \in [a, b], \quad (2.2)$$

$$u_1(a) = 0, \quad u_1(b) = 0. \quad (2.3)$$

The Dirichlet problem (1.3), (1.4) obviously corresponds to the case where system (2.1), (2.2) has the form

$$u_1'(t) = u_2(t), \quad (2.4)$$

$$u_2'(t) = g(t, u_1(t), u_2(t)), \quad t \in [a, b]. \quad (2.5)$$

System (2.1), (2.2), (1.4) will be treated using parametrisation technique by “freezing” the values of the solution at suitable time instants.

2.1. Auxiliary problems

Let t_1 be a point from (a, b) . Consider the two-point problems

$$\begin{aligned} x_1'(t) &= f_1(t, x_1(t), x_2(t)) \\ x_2'(t) &= f_2(t, x_1(t), x_2(t)), \quad t \in [a, t_1], \end{aligned} \quad (2.6)$$

$$\begin{aligned} x_1(a) &= 0, \quad x_2(a) = z_2, \\ x_1(t_1) &= \eta_1, \quad x_2(t_1) = \eta_2, \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} y_1'(t) &= f_1(t, y_1(t), y_2(t)), \\ y_2'(t) &= f_2(t, y_1(t), y_2(t)), \quad t \in [t_1, b], \end{aligned} \quad (2.8)$$

$$\begin{aligned} y_1(t_1) &= \eta_1, \quad y_2(t_1) = \eta_2, \\ y_1(b) &= 0, \quad y_2(b) = \lambda_2, \end{aligned} \quad (2.9)$$

where z_2 , η_1 , η_2 and λ_2 are some real parameters. The form of problems (2.6), (2.7) and (2.8), (2.9) suggests that the functions

$$u_i(t) := \begin{cases} x_i(t) & \text{for } t \in [a, t_1], \\ y_i(t) & \text{for } t \in [t_1, b], \end{cases} \quad (2.10)$$

where $i = 1, 2$, can be used to construct a solution of the original problem (1.3). Obviously, (u_1, u_2) defined by (2.10) is a C^2 solution of (1.3) if and only if

$$x_1(t_1) = y_1(t_1), \quad x_2(t_1) = y_2(t_1). \quad (2.11)$$

It is thus natural to study continuously differentiable solutions of the boundary value problems (2.6), (2.7) and (2.8), (2.9) for which the continuous “gluing” property (2.11) holds.

2.2. Notation and symbols, some subsidiary statements

In the sequel, for any vector $x = \text{col}(x_1, x_2) \in \mathbb{R}^2$, the obvious notation $|x| = \text{col}(|x_1|, |x_2|)$ is used and the inequalities between vectors are understood componentwise. Maxima and minima of vector-valued functions are likewise understood componentwise.

The symbol $\mathbb{1}_2$ stands for the unit matrix of dimension 2, $r(K)$ denotes the maximal, in modulus, eigenvalue of a square matrix K .

For any non-negative vector $\rho \in \mathbb{R}^2$, by a componentwise ρ -neighbourhood of a point $z \in \mathbb{R}^2$, we understand the set

$$O_\rho(z) := \{\xi \in \mathbb{R}^2 : |\xi - z| \leq \rho\}. \quad (2.12)$$

Similarly, the componentwise ρ -neighbourhood of a set $\Omega \subset \mathbb{R}^2$ is defined as

$$O_\rho(\Omega) := \bigcup_{\xi \in \Omega} O_\rho(\xi). \quad (2.13)$$

For given two bounded sets $D_0 \subset \mathbb{R}^2$ and $D_1 \subset \mathbb{R}^2$, introduce the set

$$\mathcal{B}(D_0, D_1) := \{(1-\theta)z + \theta\eta : z \in D_0, \eta \in D_1, \theta \in [0, 1]\}. \quad (2.14)$$

For any a_0, b_0 , $a \leq a_0 < b_0 \leq b$, and a continuous function $f : [a, b] \times D \rightarrow \mathbb{R}^k$, $k \in \{1, 2\}$, $D \subset \mathbb{R}^2$, we put

$$\delta_{[a_0, b_0], D}(f) := \max_{(t, x) \in [a_0, b_0] \times D} f(t, x) - \min_{(t, x) \in [a_0, b_0] \times D} f(t, x) \quad (2.15)$$

and $\delta_{[a_0, b_0], \mathbb{R}^n}(f) := \delta_{[a_0, b_0]}(f)$.

For a non-negative 2×2 square matrix K , the notation $f \in \text{Lip}_K(\Omega, [a_0, b_0])$ means that the componentwise inequality

$$|f(t, u_1) - f(t, u_2)| \leq K |u_1 - u_2|$$

holds for all $t \in [a_0, b_0]$ and u_1, u_2 from Ω .

Let us put

$$\alpha_1(t, a_0, b_0) := 2(t - a_0) \left(1 - \frac{t - a_0}{b_0 - a_0} \right), \quad t \in [a_0, b_0], \quad (2.16)$$

for any a_0, b_0 , $a \leq a_0 < b_0 \leq b$.

2.3. Iterations for auxiliary problems

Let D_a , D_b , and D_{t_1} be certain fixed closed bounded sets in $\mathbb{R}^2$. The notation reflects the fact that we shall look for solutions u with $u(s) \in D_s$, $s \in \{a, t_1, b\}$. In view of the form of the auxiliary boundary conditions (2.7), (2.9), it is natural to suppose that

$$D_a \cup D_b \subset \{0\} \times \mathbb{R}. \quad (2.17)$$

For any non-negative vectors ρ_0 and ρ_1 , put

$$\Omega_0(\rho_0) := O_{\rho_0}(\mathcal{B}(D_a, D_{t_1})), \quad \Omega_1(\rho_1) := O_{\rho_1}(\mathcal{B}(D_{t_1}, D_b)). \quad (2.18)$$

According to (2.13) and (2.14), $\Omega_0(\rho_0)$ and $\Omega_1(\rho_1)$ are, in fact, neighbourhoods of the sets of straight line segments joining, respectively, D_a with D_{t_1} and D_{t_1} with D_b . Conditions on the equation will be imposed in the sequel on these wider sets with suitable ρ_0 and ρ_1 (see Assumptions 1 – 3 below).

The auxiliary two-point boundary value problems (2.6), (2.7) and (2.8), (2.9) will be treated by using iteration schemes suitable for the case of separated two-point conditions. More precisely, for any $m = 0, 1, \dots$, put

$$x_0(t, z, \eta) = \left(1 - \frac{t-a}{t_1-a}\right)z + \frac{t-a}{t_1-a}\eta, \quad (2.19)$$

$$x_{m+1}(t, z, \eta) = x_0(t, z, \eta) + \int_a^t f(s, x_m(s, z, \eta))ds - \frac{t-a}{t_1-a} \int_a^{t_1} f(\tau, x_m(s, z, \eta))ds \quad (2.20)$$

for $t \in [a, t_1]$ and

$$y_0(t, \eta, \lambda) = \left(1 - \frac{t-t_1}{b-t_1}\right)\eta + \frac{t-t_1}{b-t_1}\lambda, \quad (2.21)$$

$$y_{m+1}(t, \eta, \lambda) = y_0(t, \eta, \lambda) + \int_{t_1}^t f(s, y_m(s, \eta, \lambda))ds - \frac{t-t_1}{b-t_1} \int_{t_1}^b f(s, y_m(s, \eta, \lambda))ds \quad (2.22)$$

for $t \in [t_1, b]$. Here, $f = \text{col}(f_1, f_2)$. The values $z \in D_a$, $\eta \in D_{t_1}$, $\lambda \in D_b$ in (2.19)–(2.22) are considered as parameters.

It is clear that, for any fixed t_1 , the graphs of $x_0(\cdot, z, \eta)$ and $y_0(\cdot, \eta, \lambda)$ given by (2.19), (2.21) are the straight line segments joining the points (a, z) , (t_1, η) and (b, λ) . Moreover, the following proposition is obvious from (2.19)–(2.22).

Proposition 1. *All the functions $\{x_m(\cdot, z, \eta) : m \geq 0\}$ (resp., $\{y_m(\cdot, z, \eta) : m \geq 0\}$) are continuously differentiable on $[a, t_1]$ (resp., $[t_1, b]$) and satisfy conditions (2.7) (resp., (2.9)).*

Let us assume the following conditions.

Assumption 1. There exist non-negative vectors ρ_0 and ρ_1 such that

$$\rho_0 \geq \frac{t_1-a}{4} \delta_{[a, t_1], \Omega_0(\rho_0)}(f), \quad \rho_1 \geq \frac{b-t_1}{4} \delta_{[t_1, b], \Omega_1(\rho_1)}(f). \quad (2.23)$$

Recall that $\delta_{[a, t_1], \Omega_0(\rho_0)}(f)$ and $\delta_{[t_1, b], \Omega_1(\rho_1)}(f)$ are given by (2.15), whereas $\Omega_0(\rho_0)$ and $\Omega_1(\rho_1)$ are sets (2.18).

Assumption 2. There exist non-negative 2×2 matrices K_0 and K_1 such that $f \in \text{Lip}_{K_0}(\Omega_0(\rho_0), [a, t_1])$ and $f \in \text{Lip}_{K_1}(\Omega_1(\rho_1), [t_1, b])$.

The meaning of these two assumptions is that f is Lipschitzian on a sufficiently wide neighbourhood of the set which is expected to contain the values of potential solutions.

Assumption 3. The maximal in modulus eigenvalues of the matrices K_0, K_1 satisfy the inequalities

$$r(K_0) < \frac{10}{3(t_1 - a)}, \quad r(K_1) < \frac{10}{3(b - t_1)}. \quad (2.24)$$

Under these assumptions, sequences (2.20) and (2.22) are uniformly convergent.

Theorem 1. Let Assumptions 1 – 3 hold. Then, for all fixed $z \in D_a$, $\eta \in D_{t_1}$ and $\lambda \in D_b$:

1. For any $m = 0, 1, \dots$,

$$\{x_m(t, z, \eta) : t \in [a, t_1]\} \subset \Omega_0(\rho_0), \quad \{y_m(t, z, \eta) : t \in [t_1, b]\} \subset \Omega_1(\rho_1). \quad (2.25)$$

2. The limit functions $x_\infty(t, z, \eta) := \lim_{m \rightarrow \infty} x_m(t, z, \eta)$ and $y_\infty(t, \eta, \lambda) := \lim_{m \rightarrow \infty} y_m(t, \eta, \lambda)$ exist uniformly on $[a, t_1]$ and $[t_1, b]$ respectively.

3. $x_\infty(\cdot, z, \eta)$ and $y_\infty(\cdot, z, \eta)$ satisfy the boundary conditions (2.7), (2.9) and the initial conditions

$$x_\infty(a, z, \eta) = z, \quad y_\infty(t_1, \eta, \lambda) = \eta.$$

4. The functions $x_\infty(\cdot, z, \eta)$ and $y_\infty(\cdot, \eta, \lambda)$ are unique continuously differentiable solutions of the problems

$$x'(t) = f(t, x(t)) + \frac{1}{t_1 - a} \Delta^0(z, \eta), \quad t \in [a, t_1], \quad (2.26)$$

$$x(a) = z, \quad x(t_1) = \eta, \quad (2.27)$$

and

$$y'(t) = f(t, y(t)) + \frac{1}{b - t_1} \Delta^1(z, \eta), \quad t \in [t_1, b], \quad (2.28)$$

$$y(t_1) = \eta, \quad y(b) = \lambda, \quad (2.29)$$

respectively, where

$$\Delta^0(z, \eta) := \eta - z - \int_a^{t_1} f(s, x_\infty(s, z, \eta)) ds, \quad (2.30)$$

$$\Delta^1(\eta, \lambda) := \lambda - \eta - \int_{t_1}^b f(s, y_\infty(s, \eta, \lambda)) ds \quad (2.31)$$

for all $z \in D_a$, $\eta \in D_{t_1}$, $\lambda \in D_b$.

5. The estimates

$$|x_\infty(t, z, \eta) - x_m(t, z, \eta)| \leq \frac{5}{9} \alpha_1(t, a, t_1) Q_0^m (\mathbb{I}_n - Q_0)^{-1} \delta_{[a, t_1], \Omega_0(\rho_0)}(f)$$

for $t \in [a, t_1]$, $m \geq 0$ and

$$|y_\infty(t, \eta, \lambda) - y_m(t, \eta, \lambda)| \leq \frac{5}{9} \alpha_1(t, t_1, b) Q_1^m (\mathbb{I}_n - Q_1)^{-1} \delta_{[t_1, b], \Omega_1(\rho_1)}(f)$$

for $t \in [t_1, b]$, $m \geq 0$, hold, where $Q_0 := \frac{3}{10}(t_1 - a)K_0$, $Q_1 := \frac{3}{10}(b - t_1)K_1$, and the functions $\alpha_1(\cdot, a, t_1)$, $\alpha_1(\cdot, t_1, b)$ are defined according to (2.16).

According to this theorem, the limits of sequences (2.19), (2.20) and (2.21), (2.22) can be interpreted as solutions of the Cauchy problems (2.26), (2.27) and (2.28), (2.29) involving the free parameters z , η , and λ . The proof is carried out by analogy to Theorems 1 and 2 from [21].

2.4. Relation of $x_\infty(\cdot, z, \eta)$ and $y_\infty(\cdot, \eta, \lambda)$ to the original problem

It follows from Theorem 1 that the functions $D_a \times D_{t_1} \ni (z, \eta) \mapsto \Delta^0(z, \eta)$ and $D_{t_1} \times D_b \ni (\eta, \lambda) \mapsto \Delta^1(\eta, \lambda)$ are well defined.

Theorem 2. Under the assumptions of Theorem 1, the function

$$u(t) := \begin{cases} x_\infty(t, z, \eta) & \text{for } t \in [a, t_1], \\ y_\infty(t, \eta, \lambda) & \text{for } t \in [t_1, b], \end{cases} \quad (2.32)$$

is a C^2 solution of the original boundary value problem (2.1), (2.2), (2.3) if and only if the parameters (z, η, λ) satisfy the determining equations

$$\Delta^0(z, \eta) = 0, \quad \Delta^1(\eta, \lambda) = 0. \quad (2.33)$$

This statement is proved by analogy to [22, Theorem 3]. Note that, by virtue of the given Dirichlet boundary conditions, the first component of z and λ involved in (2.33) is equal to zero. Therefore, (2.33) is a system of four equations

$$\Delta^0\left(\begin{pmatrix} 0 \\ z_2 \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}\right) = 0, \quad \Delta^1\left(\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}\right) = 0 \quad (2.34)$$

with respect to four unknown scalar variables z_2 , η_1 , η_2 and λ_2 .

The next statement establishes that the system of determining equations (2.33) detects all solutions of the boundary value problem (4.4), (4.5) with graphs contained in the given domain.

Theorem 3. Assume that conditions of Theorem 1 are satisfied. If equations (2.33) hold for some $z = \text{col}(0, z_2^*) \in D_a$, $\eta = \text{col}(\eta_1^*, \eta_2^*) \in D_{t_1}$, $\lambda = \text{col}(0, \lambda_2^*) \in D_b$, then function (2.32) is a solution of the boundary value problem (2.1)–(2.3) satisfying the conditions

$$u(a) = \begin{pmatrix} 0 \\ z_2^* \end{pmatrix}, \quad u(t_1) = \begin{pmatrix} \eta_1^* \\ \eta_2^* \end{pmatrix}, \quad u(b) = \begin{pmatrix} 0 \\ \lambda_2^* \end{pmatrix}. \quad (2.35)$$

Conversely, if problem (2.1)–(2.3) has a solution $u(\cdot)$ such that $u(a) \in D_a$, $u(t_1) \in D_{t_1}$, $u(b) \in D_b$, and¹

$$\{u(t) : t \in [a, t_1]\} \subset \Omega_0(\rho_0), \quad \{u(t) : t \in [t_1, b]\} \subset \Omega_1(\rho_1),$$

then it necessarily has form (2.32) with

$$z = \begin{pmatrix} 0 \\ u_2(a) \end{pmatrix}, \quad \eta = u(t_1), \quad \lambda = \begin{pmatrix} 0 \\ u_2(b) \end{pmatrix} \quad (2.36)$$

and values (2.36) satisfy the system of determining equations (2.33).

Proof. The proof is similar to that of [22, Theorem 4]. $\square$

The solvability of the determining system (2.33) can be studied by analogy to [20] using topological degree methods [8] based on properties of approximate determining functions

$$\begin{aligned} \Delta_m^0(z, \eta) &:= \eta - z - \int_a^{t_1} f(s, x_m(s, z, \eta)) ds, \\ \Delta_m^1(\eta, \lambda) &:= \lambda - \eta - \int_{t_1}^b f(s, y_m(s, \eta, \lambda)) ds \end{aligned} \quad (2.37)$$

for fixed m . Formulae (2.37), unlike (2.30), (2.31), involve only known functions and the equations

$$\Delta_m^0(z, \eta) = 0, \quad \Delta_m^1(\eta, \lambda) = 0 \quad (2.38)$$

form an approximation to the determining system (2.33). For the Dirichlet problem, similarly to (2.34), system (2.38) is rewritten as

$$\Delta_m^0\left(\begin{pmatrix} 0 \\ z_2 \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}\right) = 0, \quad \Delta_m^1\left(\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}\right) = 0. \quad (2.39)$$

2.5. Approximation of a solution

If $(\tilde{z}, \tilde{\eta}, \tilde{\lambda}) \in D_a \times D_{t_1} \times D_b$ is a root of the approximate determining system (2.38) for a certain fixed m , then the function

$$U_m(t) = \begin{cases} x_m(t, \tilde{z}, \tilde{\eta}) & \text{for } t \in [a, t_1], \\ y_m(t, \tilde{\eta}, \tilde{\lambda}) & \text{for } t \in [t_1, b], \end{cases} \quad (2.40)$$

defined according to (2.20), (2.22) can be regarded as the m th approximation to a solution of problem (2.4), (2.5), (2.3). This is justified by the estimates

$$|x_\infty(t, \tilde{z}, \tilde{\eta}) - x_m(t, \tilde{z}, \tilde{\eta})| \leq \frac{5}{9} \alpha_1(t, a, t_1) Q_0^m (\mathbb{1}_n - Q_0)^{-1} \delta_{[a, t_1], \Omega_0(\rho_0)}(f)$$

¹Recall that ρ_0, ρ_1 are the values involved in Assumption 1 of Theorem 1 and $\Omega_0(\rho_0), \Omega_1(\rho_1)$ are the corresponding sets (2.18).

for $t \in [a, t_1]$, $m \geq 0$, and

$$|y_\infty(t, \tilde{\eta}, \tilde{\lambda}) - y_m(t, \tilde{\eta}, \tilde{\lambda})| \leq \frac{5}{9} \alpha_1(t, t_1, b) Q_1^m (\mathbb{1}_n - Q_1)^{-1} \delta_{[t_1, b], \Omega_1(\rho_1)}(f)$$

for $t \in [t_1, b]$, $m \geq 0$, which follow directly from Theorem 1.

2.6. More general boundary conditions

With appropriate modifications, the approach is also applicable for other types of boundary conditions. If, for example, instead of the Dirichlet conditions (1.4), one considers the condition

$$\phi(u(a), u'(a), u(b), u'(b)) = 0 \quad (2.41)$$

with $\phi : \mathbb{R}^4 \rightarrow \mathbb{R}^2$, then the approximate values of the parameters should be found from (2.38) complemented by the equations

$$\phi(z_1, z_2, \lambda_1, \lambda_2) = 0. \quad (2.42)$$

Relations (2.42) are deduced from the requirement that the iterations should satisfy the boundary conditions (2.41):

$$\phi(x_{m,1}(a, z, \eta), x_{m,2}(a, z, \eta), y_{m,1}(b, \eta, \lambda), y_{m,2}(b, \eta, \lambda)) = 0,$$

whence, in view of the recurrence relations (2.19)–(2.22) and transformation of the second order equation (1.3) into (2.4), (2.5), one arrives at (2.42). Then (2.38), (2.42) is a system of six equations with the same number of unknowns.

3. $(\sigma_0, \sigma_1; t_1)$ SOLUTIONS OF THE DIRICHLET PROBLEM

The argument from the preceding section can be adopted for finding solutions with prescribed number of zeroes. Let us explain the idea in application to the Dirichlet problem (1.3), (1.4).

Definition 1. Let $l \in \mathbb{N}$, $\{\sigma_0, \sigma_1, \dots, \sigma_l\} \subset \{-1, 1\}$, and $t_1, \dots, t_l$ are points such that $a < t_1 < t_2 < \dots < t_l < b$. We say that a continuous function $u : [a, b] \rightarrow \mathbb{R}$ is of type $(\sigma_0, \sigma_1, \dots, \sigma_l; t_1, t_2, \dots, t_l)$ if

$$\sigma_{k-1} u(t) > 0 \quad \text{for } t \in (t_{k-1}, t_k), \quad k = 1, 2, \dots, l+1,$$

where $t_0 := a$, $t_{l+1} := b$.

In other words, u possessing this property has zero value at the points $t_1, \dots, t_l$ and its sign before vanishing at t_k is σ_{k-1} , $k = 1, 2, \dots, l$. For $l = 0$, it is natural to understand type (σ_0) as the case where u has no zeroes on (a, b) and $\text{sign } u \equiv \sigma_0$.

In relation to problem (1.3), (1.4), the property $(\sigma_0, \sigma_1, \dots, \sigma_l; t_1, t_2, \dots, t_l)$, in fact, describes solutions with prescribed zeros and fixed signs around them. Here, we are interested in such solutions of equations similar to (1.2), (1.1) focusing on the case where $l = 0$ or $l = 1$, i.e., we will consider solutions of type (σ_0) and $(\sigma_0, \sigma_1; t_1)$. For

$l = 1$, a solution of problem (2.4)–(2.3) is composed from solutions of two auxiliary boundary value problems.

The techniques described above allow one to look for $(\sigma_0, \sigma_1; t_1)$ solutions of the Dirichlet problem (1.3), (1.4) by rewriting it as (2.4), (2.5), (2.3), i. e., (2.1)–(2.3) with $f = \text{col}(f_1, f_2)$ of the form

$$f_1(t, u_1, u_2) = u_2, \quad f_2(t, u_1, u_2) = g(t, u_1, u_2) \quad (3.1)$$

and studying the corresponding auxiliary problems on $[a, t_1]$ and $[t_1, b]$. Note that, for a $(\sigma_0, \sigma_1; t_1)$ solution, we have $\text{sign } u_1 = \sigma_0$ on $[a, t_1]$ and $\text{sign } u_1 = \sigma_1$ on $[t_1, b]$; this fact can be used to simplify possible terms with absolute value signs in (3.1).

Suppose that we are looking for $(\sigma_0, \sigma_1; t_1)$ solutions of problem (1.3), (1.4). The corresponding solution $u = \text{col}(u_1, u_2)$ of (2.4), (2.5), (2.3) then has the property $u_1(t_1) = 0$ and, therefore, Theorem 3 “detects” solutions u with

$$u(a) = \begin{pmatrix} 0 \\ z_2^* \end{pmatrix}, \quad u(t_1) = \begin{pmatrix} 0 \\ \eta_2^* \end{pmatrix}, \quad u(b) = \begin{pmatrix} 0 \\ \lambda_2^* \end{pmatrix}, \quad (3.2)$$

where z_2^* , η_2^* , λ_2^* , and t_1 are such that equalities (2.33) hold. Note that, with fixed $\eta_1 = 0$ and unknown t_1 , the number of variables and equations in (2.33) is four, i. e., is the same as in Theorem 2. Thus, formally the same equations (2.33) which have been deduced in Section 2.4 for the parameters $(z_2, \eta_1, \eta_2, \lambda_2)$ serve in this case to determine the unknown values $(z_2, \eta_2, \lambda_2, t_1)$. We should only take into account different roles of η_1 and t_1 due to which, instead of (2.34), system (2.33) is now rewritten in the form

$$\Delta^0\left(\begin{pmatrix} 0 \\ z_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix}\right) = 0, \quad \Delta^1\left(\begin{pmatrix} 0 \\ \eta_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}\right) = 0 \quad (3.3)$$

and involves the unknowns $(z_2, \eta_2, \lambda_2, t_1)$. The approximate values of the parameters $(z_2, \eta_2, \lambda_2, t_1)$ are looked for from the approximate determining equations

$$\Delta_m^0\left(\begin{pmatrix} 0 \\ z_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix}\right) = 0, \quad \Delta_m^1\left(\begin{pmatrix} 0 \\ \eta_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}\right) = 0, \quad (3.4)$$

which, in view of (3.1), have the form

$$\int_a^{t_1} x_{m,2}\left(s, \begin{pmatrix} 0 \\ z_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix}\right) ds = 0, \quad (3.5)$$

$$\int_a^{t_1} g\left(s, x_{m,1}\left(s, \begin{pmatrix} 0 \\ z_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix}\right), x_{m,2}\left(s, \begin{pmatrix} 0 \\ z_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix}\right)\right) ds = \eta_2 - z_2, \quad (3.6)$$

$$\int_{t_1}^b y_{m,2}\left(s, \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}\right) ds = 0, \quad (3.7)$$

$$\int_{t_1}^b g\left(s, y_{m,1}\left(s, \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}\right), y_{m,2}\left(s, \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}\right)\right) ds = \lambda_2 - \eta_2. \quad (3.8)$$

Here, (3.5), (3.6) (resp., (3.7), (3.8)) correspond to the auxiliary problem on $[a, t_1]$ (resp., $[t_1, b]$).

4. COMPUTATION FOR CONCRETE EXAMPLES

The technique described above can be efficiently used, in particular, for finding (σ_0) and $(\sigma_0, \sigma_1; t_1)$ solutions of Emden-Fowler type equations. In order to apply it to the Dirichlet problem (1.3), (1.4), we first rewrite (1.3) in the form (2.4), (2.5). The sets $\Omega_0(\rho_0)$, $\Omega_1(\rho_1)$ are defined according to (2.18) with some fixed sets D_a , D_b satisfying (2.17). It is not difficult to verify that the vectors ρ_0 , ρ_1 , according to (2.23), should satisfy the inequalities

$$\rho_0 \geq \frac{t_1 - a}{4} \left(\text{diam pr}_2 \Omega_0(\rho_0) \right), \quad \rho_1 \geq \frac{b - t_1}{4} \left(\text{diam pr}_2 \Omega_1(\rho_1) \right). \quad (4.1)$$

Here, for $\Omega \subset \mathbb{R}^2$ and $J \subset \mathbb{R}$, we use the notation $\text{diam } J := \sup J - \inf J$ and $\text{pr}_2 \Omega := \{z_2 \mid (z_1, z_2) \in \Omega \text{ for some } z_1\}$.

The Lipschitz condition from Theorem 1 in this case means that

$$|g(t, u_1, u_2) - g(t, v_1, v_2)| \leq k_{i1}|u_1 - v_1| + k_{i2}|u_2 - v_2| \quad (4.2)$$

for $t \in [(1-i)a + ib, (1-i)t_1 + ib]$, $\{u_j, v_j \mid j = 1, 2\} \subset \Omega_i(\rho_i)$, $i = 0, 1$, with some positive constants $\{k_{ij} \mid i = 0, 1, j = 1, 2\}$ satisfying the inequalities

$$k_{02} + \sqrt{k_{02}^2 + 4k_{01}} < \frac{20}{3(t_1 - a)}, \quad k_{12} + \sqrt{k_{12}^2 + 4k_{11}} < \frac{20}{3(b - t_1)}. \quad (4.3)$$

Thus, Assumptions 1–3 of Section 2 reformulated for problem (2.4), (2.5) consist in conditions (4.1)–(4.3). They mean that the function g from (1.3) is Lipschitzian on a set which is sufficiently wide (existence of ρ_0 , ρ_1 satisfying (4.1)) with constants which are sufficiently small (condition (4.3)). We omit the details of verification of these conditions in the examples below.

4.1. Two sign-constant solutions of (4.4)–(4.6), $p = 1$

Consider the Dirichlet problem

$$u''(t) = -\gamma q(t)u^3(t), \quad t \in [a, b], \quad (4.4)$$

$$u(a) = 0, \quad u(b) = 0, \quad (4.5)$$

where $a = -1$, $b = 1$, and q is a $\vee$ -shaped piecewise linear function,

$$q(t) = \begin{cases} \left(-\frac{2}{3}t + \frac{1}{3}\right)^p & \text{for } t \in [a, 0], \\ \left(\frac{2}{3}t + \frac{1}{3}\right)^p & \text{for } t \in [0, b], \end{cases} \quad (4.6)$$

with $p = 1$, $\gamma = 2$. We will show by computation that this problem has two solutions (positive and negative, which are almost symmetric to one another).

We take, e. g., $t_1 = 0^2$ and study the auxiliary boundary value problems (2.6)–(2.7) and (2.8)–(2.9) using the successive approximations (2.20), (2.22).

Constructing $x_1(\cdot, z, \eta)$ and $y_1(\cdot, \eta, \lambda)$ according to (2.20), (2.22) and solving the approximate algebraic determining system (2.38) for $m = 1$, we obtain the following numerical values for the unknown parameters:

$$\eta_1 \approx 1.953587407, \quad \eta_2 \approx 0, \quad \lambda_2 \approx -2.995901364, \quad z_2 \approx 2.995901364. \quad (4.7)$$

Inserting values (4.7) into the expressions for $x_1(\cdot, z, \eta)$ and $y_1(\cdot, \eta, \lambda)$, we obtain the first approximation (2.40) which has the form

$$U_1(t) = \begin{cases} X_1(t) = \text{col}(X_{1,1}(t), X_{1,2}(t)) & \text{for } t \in [a, t_1], \\ Y_1(t) = \text{col}(Y_{1,1}(t), Y_{1,2}(t)) & \text{for } t \in [a, b], \end{cases} \quad (4.8)$$

with

$$\begin{aligned} X_{1,1}(t) &\approx 1.953587407 + 0.455636725t - 1.497950682t^2, \\ X_{1,2}(t) &\approx 1.988232981t^5 + 6.213228065t^4 + 4.970582452t^3 \\ &\quad - 2.485291226t^2 - 6.226779958t \end{aligned}$$

for $t \in [a, t_1]$ and

$$\begin{aligned} Y_{1,1}(t) &\approx 1.953587407 - 0.455636725t - 1.497950682t^2, \\ Y_{1,2}(t) &\approx 1.988232981t^5 - 6.213228065t^4 + 4.970582452t^3 \\ &\quad + 2.485291226t^2 - 6.226779958t \end{aligned}$$

for $t \in [t_1, b]$.

Higher approximations ($m = 2, 3, \dots$) are obtained by analogy to (4.8). Note that the most difficult part is the construction of the functions $x_m(\cdot, z, \eta)$ and $y_m(\cdot, \eta, \lambda)$ in (2.20) and (2.22). If the explicit integration impossible or difficult, one can use suitable modifications of formulae (2.20) and (2.22) which, at the expense of a certain loss in accuracy, lead one to iterations better suited for practical computations.

In [19], we mention two natural modifications of this kind, namely the version of “frozen” parameters and the version with polynomial interpolation used in [16]. Let us apply here the “frozen” parameters version, which means that the approximations X_m and Y_m obtained on the m th step are used directly instead of $x_m(\cdot, z, \eta)$ and $y_m(\cdot, \eta, \lambda)$ (i. e., the roots of the m th determining system are substituted into the corresponding expressions before passing to the next step of iteration).

In our case, to obtain the second parametrized iterations (denote these functions by $\hat{x}_2(\cdot, z, \eta)$, $\hat{y}_2(\cdot, \eta, \lambda)$ to distinguish them from those given by (2.20), (2.22)), instead

²In this case, the value of t_1 can be chosen directly since sign-constant solutions are looked for and, thus, t_1 does not have the meaning of a zero of the solution. The choice $t_1 = 0$ is also natural because of the form of the forcing term (4.6).

of $x_1(\cdot, z, \eta)$ and $y_1(\cdot, \eta, \lambda)$, we use the respective components of (4.8):

$$\begin{aligned}\hat{x}_2(t, z, \eta) &= \int_a^t f(s, X_{1,1}(s), X_{1,2}(s))ds - \frac{t-a}{t_1-a} \int_a^{t_1} f(s, X_{1,1}(s), X_{1,2}(s))ds \\ &\quad + z + \frac{t-a}{t_1-a} (\eta - z), \quad t \in [a, t_1], \\ \hat{y}_2(t, \eta, \lambda) &= \int_{t_1}^t f(s, Y_{1,1}(s), Y_{1,2}(s))ds - \frac{t-t_1}{b-t_1} \int_{t_1}^b f(s, Y_{1,1}(s), Y_{1,2}(s))ds + \\ &\quad + \eta + \frac{t-t_1}{b-t_1} (\lambda - \eta), \quad t \in [t_1, b].\end{aligned}$$

Then we need to solve the second approximate determining system (2.38) which, for $m = 2$, has the form

$$\eta - z = \int_a^{t_1} f(s, X_{11}(s), X_{12}(s))ds, \quad \lambda - \eta = \int_{t_1}^b f(s, Y_{11}(s), Y_{12}(s))ds.$$

Note that the right-hand side terms of the latter system does not involve z , η , and λ , which is a major simplification. For $m > 2$, approximations X_m , Y_m are obtained by analogy using $\hat{x}_m(\cdot, z, \eta)$, $\hat{y}_m(\cdot, \eta, \lambda)$ constructed as above.

For our problem, carrying out computations by the “frozen” technique and solving the respective equations, we obtain the parameter values presented in Table 1.

m	η_1	η_2	λ_2	z_2
3	1.851388676	$3.483457908 \cdot 10^{-10}$	-2.622412592	2.622412592
4	1.8557702	$1.295121405 \cdot 10^{-9}$	-2.613402579	2.613402576
5	1.857259119	$-1.810452349 \cdot 10^{-9}$	-2.617838558	2.617838555
7	1.856920628	$-1.173443333 \cdot 10^{-9}$	-2.613691252	2.613691240

TABLE 1.

Computing the residual for the approximate (positive) solution $U_{7,1}$, we find that it does not exceed 0.0005. The graph of $U_{7,1}$ is shown on Figure 1.

According to Theorems 2 and 3, the number of solutions of the problem under consideration with the graphs lying in the given region is equal to the number of roots of the determining system (2.33). A computation shows that the approximate determining system (2.38) for $m = 1$, along with (4.7), has another root

$$\eta_1 \approx -1.953587407, \quad \eta_2 \approx 0, \quad \lambda_2 \approx 2.995901364, \quad z_2 \approx -2.995901364,$$

which indicates the presence of another (negative) solution of the problem. Refining these values on the next steps by using the “frozen” technique, we obtain the approximate solution corresponding to the parameter values shown in Table 2.

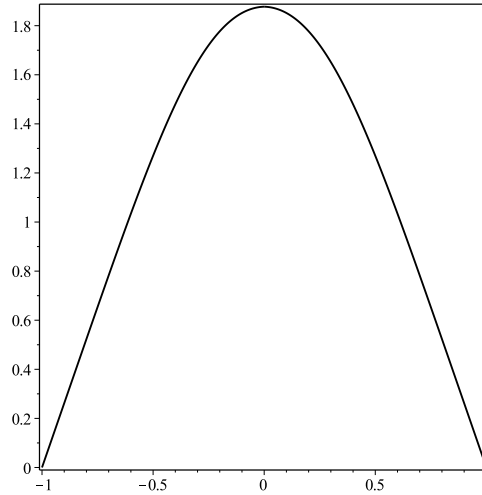


FIGURE 1. The seventh approximation to the positive solution.

m	η_1	η_2	λ_2	z_2
4	-1.862639362	$-3.695522622 \cdot 10^{-9}$	2.631889316	-2.631889323
5	-1.859693541	$1.253485922 \cdot 10^{-9}$	2.620422812	-2.620422815
7	-1.8571756	$-2.953304145 \cdot 10^{-9}$	2.613550388	-2.613550391
8	-1.856620650	$-1.408647961 \cdot 10^{-10}$	2.612670399	-2.612670398

TABLE 2.

The residual for the approximate (negative) solution $U_{8,1}$ does not exceed 0.001. The graph of $U_{8,1}$ is shown on Figure 3.

4.2. Two sign-constant solutions of (4.4)–(4.6), $p = 1/3$

Consider the Dirichlet problem (4.4)–(4.6) for $a = -1$, $b = 1$, $p = 1/3$, $\gamma = 2$. In this case we also obtain two solutions (positive and negative ones).

m	η_1	η_2	λ_2	z_2
1	1.543922367	0	-2.270535575	2.270535575
2	1.516958742	$-1.438012983 \cdot 10^{-12}$	-2.062927555	2.062927555
3	1.484912528	$1.783555583 \cdot 10^{-10}$	-2.017809966	2.017809966

TABLE 3.

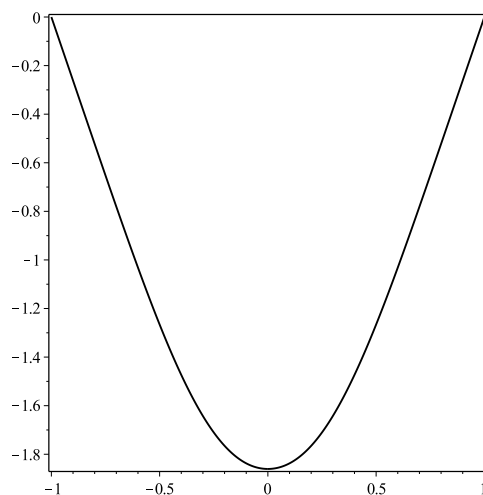


FIGURE 2. The eighth approximation to the negative solution.

The results of computation using formulae (2.20) and (2.22) (without “frozen” technique) for several steps of iteration are given in Table 3. The corresponding approximation positive solution $U_{3,1}$ ($m = 3$) is shown on Figure 3. The residual for $U_{3,1}$ does not exceed 0.02.

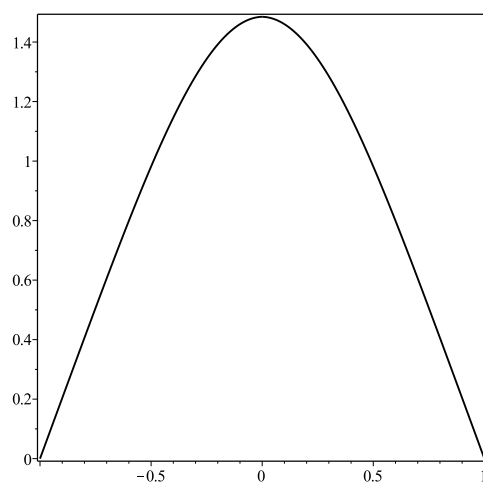


FIGURE 3. The third approximation to the positive solution.

The approximate negative solution (which appears to be symmetric to the positive one) is specified by the roots of the approximate determining equations (without frozen technique) shown in Table 4. The residual of this approximate solution for $m = 3$ does not exceed 0.025, its graph is shown on Figure 4.

m	η_1	η_2	λ_2	z_2
1	-1.543922367	0	2.270535575	-2.2705355756
3	-1.484912528	$2.569148168 \cdot 10^{-10}$	2.017809966	-2.017809967

TABLE 4.

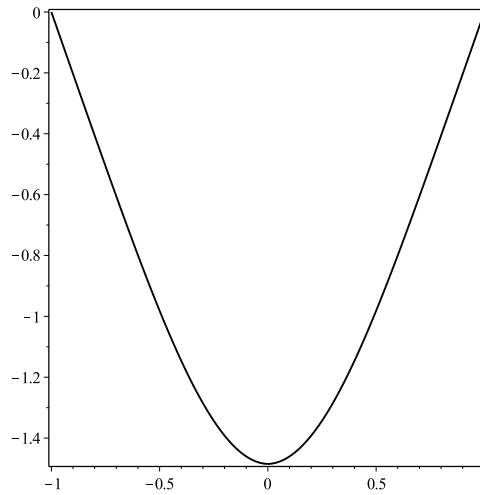


FIGURE 4. The third approximation to the negative solution.

4.3. $(-1, 1; t_1)$ solution of (4.9) with $a = -1/2$, $b = 1/2$, and the Dirichlet conditions (4.10)

Consider the Dirichlet problem

$$u''(t) = -\frac{1}{3}(8t^2 + 1)|u(t)|^3 u(t) + r(t), \quad t \in (a, b), \quad (4.9)$$

$$u(a) = 0, \quad u(b) = 0 \quad (4.10)$$

with $a = -1/2$, $b = 1/2$, and

$$r(t) = \begin{cases} 2 - \frac{1}{3}(8t^2 + 1)\left(t^2 + \frac{1}{4}t - \frac{1}{8}\right)^4 & \text{for } t \in \left[-\frac{1}{2}, t_1\right], \\ -6 + \frac{1}{3}(8t^2 + 1)\left(-3t^2 + \frac{9}{4}t - \frac{3}{8}\right)^4 & \text{for } t \in \left[t_1, \frac{1}{2}\right], \end{cases} \quad (4.11)$$

where t_1 is a point from $(-1/2, 1/2)$. Let us focus on $(-1, 1; t_1)$ solutions of problem (4.9), (4.10). According to the definition from Section 3, this means that the solution u should be negative on (a, t_1) , positive on (t_1, b) , and zero at t_1 . The value of $t_1 \in (a, b)$ is *a priori* unknown and has to be determined.

Let us proceed as described in Section 3. We pass to the corresponding problem (2.4), (2.5), (2.3) by rewriting (4.9) in the form of system (2.4), (2.5) with

$$g(t, u_1, u_2) := -\frac{1}{3}(8t^2 + 1)|u_1|^3 u_1 + r(t). \quad (4.12)$$

Since we focus on $(-1, 1; t_1)$ solutions, (4.12) can be replaced by the formula

$$g(t, u_1, u_2) = r(t) + \begin{cases} \frac{1}{3}(8t^2 + 1)u_1^4 & \text{if } t \in [-\frac{1}{2}, t_1], \\ -\frac{1}{3}(8t^2 + 1)u_1^4 & \text{if } t \in [t_1, \frac{1}{2}]. \end{cases} \quad (4.13)$$

A $(-1, 1; t_1)$ solution of (4.9), (4.10) is composed from solutions x, y of the auxiliary boundary value problems (2.6), (2.7) and (2.8), (2.9) for $f = \text{col}(f_1, f_2)$ of form (3.1) with g given by (4.13). For the auxiliary problems, the corresponding iteration sequences $\{x_m(\cdot, z, \eta) \mid m \geq 0\}$ and $\{y_m(\cdot, \eta, \lambda) \mid m \geq 0\}$ are constructed according to (2.19)–(2.22). Note that the functions depend on the unknown t_1 .

Computing the functions $x_1(\cdot, (\frac{0}{z_2}), (\frac{0}{\eta_2}))$, $y_1(\cdot, (\frac{0}{\eta_2}), (\frac{0}{\lambda_2}))$ according to (2.20), (2.22) for $m = 1$ and solving the first approximate determining system (3.5)–(3.8), we find the numerical values of the unknown parameters shown in Table 5.

m	η_2	λ_2	t_1	z_2
1	0.7500010172	-0.7500010060	0.2499996629	-0.7499983085
2	0.7499999991	-0.7499999996	0.2500000003	-0.7499999990
3	0.7500000001	-0.7500000006	0.2500000000	-0.7499999999

TABLE 5.

The values from Table 5 determine the corresponding approximate solutions (2.40). Note that, as is easy to verify, the function

$$u^*(t) = \begin{cases} t^2 + \frac{1}{4}t - \frac{1}{8} & \text{for } t \in [-\frac{1}{2}, \frac{1}{4}] \\ -3t^2 + \frac{9}{4}t - \frac{3}{8} & \text{for } t \in [\frac{1}{4}, \frac{1}{2}] \end{cases} \quad (4.14)$$

is an exact solution of the boundary value problem (4.9), (4.10). Clearly, solution (4.14) is of type $(-1, 1; 1/4)$.

The graph of the exact solution (4.14) and the approximate $(-1, 1; t_1)$ solution for $m = 2$ are shown on the Figure 5. The residual of the approximate solution is about $4 \cdot 10^{-9}$.

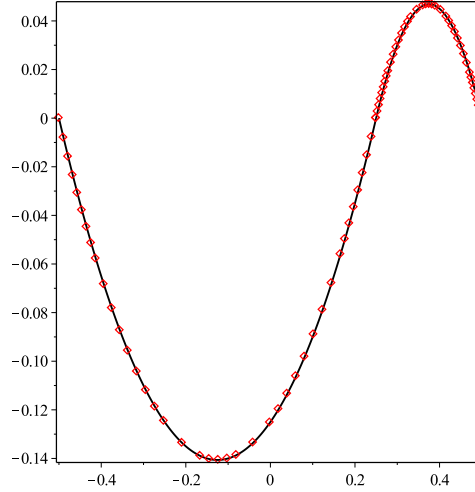


FIGURE 5. The exact solution (4.14) and its second approximation

4.4. $(1, -1; t_1)$ solution of (4.9) with the non-linear two-point conditions (4.16) and $b = 1/4$

Consider equation (4.9) with $a = -1/2$, $b = 1/4$,

$$r(t) := \begin{cases} 2 + \left(\frac{8}{3}t^2 + \frac{1}{3}\right) \left(t^2 - \frac{1}{12}t - \frac{1}{12}\right)^4 & \text{if } t \in \left[-\frac{1}{2}, t_1\right], \\ 2 - \left(\frac{8}{3}t^2 + \frac{1}{3}\right) \left(t^2 - \frac{1}{12}t - \frac{1}{12}\right)^4 & \text{if } t \in \left[t_1, \frac{1}{4}\right], \end{cases} \quad (4.15)$$

where t_1 is an unknown point from $(-1/2, 1/4)$. Let us look for $(1, -1; t_1)$ solutions of this equation satisfying the two-point boundary conditions

$$u\left(-\frac{1}{2}\right)u'\left(\frac{1}{4}\right) = \frac{25}{288}, \quad u'\left(-\frac{1}{2}\right) + u'\left(\frac{1}{4}\right) = -\frac{2}{3}. \quad (4.16)$$

Similarly to Section 4.3, we rewrite equation (4.9) as system (2.4), (2.5) with

$$g(t, u_1, u_2) = r(t) + \begin{cases} -\frac{1}{3}(8t^2 + 1)u_1^4 & \text{if } t \in \left[-\frac{1}{2}, t_1\right], \\ \frac{1}{3}(8t^2 + 1)u_1^4 & \text{if } t \in \left[t_1, \frac{1}{4}\right] \end{cases} \quad (4.17)$$

and consider (2.4), (2.5) under the boundary conditions

$$u_1\left(-\frac{1}{2}\right)u_2\left(\frac{1}{4}\right) = \frac{25}{288}, \quad u_2\left(-\frac{1}{2}\right) + u_2\left(\frac{1}{4}\right) = -\frac{2}{3}. \quad (4.18)$$

In (4.17), we have used the information on the sign of u_1 contained in the $(1, -1; t_1)$ property.

Having constructed the functions $x_1(\cdot, (\frac{z_1}{z_2}), (\frac{0}{\eta_2}))$ and $y_1(\cdot, (\frac{0}{\eta_2}), (\frac{\lambda_1}{\lambda_2}))$ according to (2.20) and (2.22), we write the corresponding equations (2.38):

$$\Delta_m^0((\frac{z_1}{z_2}), (\frac{0}{\eta_2})) = 0, \quad \Delta_m^1((\frac{0}{\eta_2}), (\frac{\lambda_1}{\lambda_2})) = 0 \quad (4.19)$$

with $m = 1$. Note that, in this case, we look for the values of $(z_1, z_2, \eta_2, \lambda_1, \lambda_2, t_1)$ since no *a priori* information the values of u_1 at a and b is available.

Recalling Section 2.6 and complementing (2.37) by the equations

$$z_1 \lambda_2 = \frac{25}{288}, \quad z_2 + \lambda_2 = -\frac{2}{3} \quad (4.20)$$

generated by the boundary conditions (4.18)³, we get a system of six equations with six unknowns. Solving (4.19), (4.20) with $m = 1$, we obtain the first approximation to the values of the unknown parameters. Repeating the same for $m = 2, 3$, we arrive at Table 6.

	$m = 1$	$m = 2$	$m = 3$
z_1	0.2083355451	0.2083355451	0.2083355451
z_2	-1.083328910	-1.083328910	-1.083328910
η_2	-0.5833204107	-0.58331754047	-0.5833175405
λ_1	-0.04166396646	-0.04166295247	-0.04166295248
λ_2	0.416662243	0.4166622431	0.4166622432
t_1	-0.2499957487	-0.2499943135	-0.2499943135

TABLE 6.

One can check that the function

$$u(t) = t^2 - \frac{t+1}{12}, \quad t \in [-\frac{1}{2}, \frac{1}{4}], \quad (4.21)$$

is an exact $(1, -1; -\frac{1}{4})$ solution of our problem (4.9), (4.16). The graphs of (4.21) and the approximate solution generated by the values of parameters from Table 6 ($m = 3$) are presented on Figure 6. The residual of the approximate solution is about $6 \cdot 10^{-10}$.

4.5. $(1, -1; t_1)$ solution of (4.9) with two-point conditions (4.22) and $b = 1/4$

Let us look for a $(1, -1; t_1)$ solution of equation (4.9) with $a = -1/2$, $b = 1/4$, and r given by (4.15) (see Section 4.4) under the boundary conditions

$$u\left(-\frac{1}{2}\right) u'\left(\frac{1}{4}\right) = \frac{25}{288}, \quad \left|u\left(\frac{1}{4}\right)\right| = \frac{1}{24}. \quad (4.22)$$

³We should recall the meaning of the parameters involved in (2.19)–(2.22); see (2.7), (2.9).

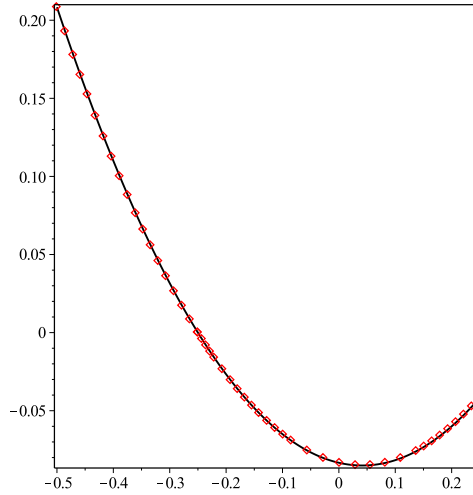


FIGURE 6. The exact solution (4.21) and its third approximation.

The approximate determining system consists in (2.38) and two additional equations (see Section 2.6)

$$z_1 z_2 = \frac{25}{288}, \quad |\lambda_1| = \frac{1}{24}. \quad (4.23)$$

Solving system (2.38), (4.23) for $m = 1, 2, 3$, we obtain the numerical values of parameters given in Table 7. Note that the exact solution of this problem has form (4.21);

	$m = 1$	$m = 2$	$m = 3$
z_1	0.2083409826	0.2083429898	0.2083429758
z_2	-1.083339776	-1.083343787	-1.083343759
η_2	-0.5833219625	-0.5833196642	-0.5833196604
λ_1	-0.04166668456	-0.04166667360	-0.04166666666
λ_2	0.4166513687	0.416647354	0.4166473826
t_1	-0.2499910875	-0.249987931	-0.2499879432

TABLE 7.

we see that the last row in the table gives approximate values of the exact value $t_1 = -1/4$.

The graphs of the exact and approximate $(1, -1; t_1)$ solutions of problem (4.9), (4.22) generated by the values of parameters from Table 7 for $m = 3$ practically coincide with those presented on Figure 6. The residual of the approximate solution in this case is about $3 \cdot 10^{-10}$.

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GENERIC RIEMANNIAN MAPS

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Abstract. As a generalization of semi-invariant Riemannian maps from almost Hermitian manifolds, we first introduce generic Riemannian maps from almost Hermitian manifolds to Riemannian manifolds, give examples, obtain decomposition theorems and investigate harmonicity and totally geodesicity of such maps. Then as a generalization of semi-invariant Riemannian maps to almost Hermitian manifolds, we introduce generic Riemannian maps from Riemannian manifolds to almost Hermitian manifolds, give examples and find necessary and sufficient conditions for such maps to be totally geodesic. The harmonicity of generic Riemannian maps to Kähler manifolds is also investigated.

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1. INTRODUCTION

As indicated in [9], a major flaw in Riemannian geometry (as compared to other subjects) is a shortage of suitable types of maps between Riemannian manifolds that will compare their geometric properties. In this direction, Fischer introduced Riemannian maps between Riemannian manifolds in [8] as a generalization of the notions of isometric immersions and Riemannian submersions. Isometric immersions and Riemannian submersions have been widely studied in differential geometry (see for examples [6] and [7]), however the theory of Riemannian maps is a new research field. Let $F : (M_1, g_1) \longrightarrow (M_2, g_2)$ be a smooth map between Riemannian manifolds. Then we denote the kernel space of F_* by $\ker F_*$ and consider the orthogonal complementary space $(\ker F_*)^\perp$ to $\ker F_*$. Then the tangent bundle of M_1 has the following decomposition

$$TM_1 = \ker F_* \oplus (\ker F_*)^\perp.$$

We denote the range of F_* by $\text{range} F_*$ and consider the orthogonal complementary space $(\text{range} F_*)^\perp$ to $\text{range} F_*$ in the tangent bundle TM_2 of M_2 . Thus the tangent bundle TM_2 of M_2 has the following decomposition

$$TM_2 = (\text{range} F_*) \oplus (\text{range} F_*)^\perp.$$

Now, a smooth map $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ is called Riemannian map at $p_1 \in M_1$ if the horizontal restriction $F_{*p_1}^h : (ker F_{*p_1})^\perp \longrightarrow (range F_{*p_1})$ is a linear isometry between the inner product spaces $((ker F_{*p_1})^\perp, g_1(p_1)|_{(ker F_{*p_1})^\perp})$ and $(range F_{*p_1}, g_2(p_2)|_{(range F_{*p_1})})$, $p_2 = F(p_1)$. It follows that isometric immersions are particular Riemannian maps with $ker F_* = \{0\}$. It also follows that Riemannian submersions are particular Riemannian maps with $(range F_*)^\perp = \{0\}$. If $rank F < \min\{m, n\}$, we always have $(range F_*)^\perp \neq \{0\}$, in this case, we say that F is a proper Riemannian map. It is known that a Riemannian map is a subimmersion which implies that the rank of the linear map $F_{*p} : T_p M_1 \longrightarrow T_{F(p)} M_2$ is constant for p in each connected component of M_1 , [1] and [8]. It is also important to note that Riemannian maps satisfy the eikonal equation which is a bridge between geometric optics and physical optics. For Riemannian maps and their applications in spacetime geometry, see: [9].

In [19], we introduced semi-invariant Riemannian maps from almost Hermitian manifolds onto Riemannian manifolds as a generalization of holomorphic Riemannian maps and anti-invariant Riemannian maps, then we studied the geometry of such maps. We also introduced and studied semi-invariant Riemannian maps to almost Hermitian manifolds in [18]. Recently, there are many research papers on the geometry of Riemannian maps between various Riemannian manifolds [10], [11], [14], [15], [16].

In this paper, we introduce and study both generic Riemannian maps from almost Hermitian manifolds and generic Riemannian maps to almost Hermitian manifolds.

The paper is organized as follows: In Section 2, we present the basic background needed for this paper. In section 3, we define generic Riemannian maps from almost Hermitian manifolds to Riemannian manifolds, give examples and obtain integrability conditions for distributions defined by generic Riemannian maps. We also find necessary and sufficient conditions for a semi-invariant map to be totally geodesic map and harmonic map. In the last section of this paper, we study generic Riemannian maps to almost Hermitian manifolds and obtain main properties of such maps.

2. PRELIMINARIES

In this section we recall some basic materials from [3] and [24]. A $2k$ -dimensional Riemannian manifold $(\bar{M}, \bar{g}, \bar{J})$ is called an almost Hermitian manifold if there exists a tensor field $\bar{J}$ of type $(1,1)$ on $\bar{M}$ such that $\bar{J}^2 = -I$ and

$$\bar{g}(X, Y) = \bar{g}(\bar{J}X, \bar{J}Y), \forall X, Y \in \Gamma(T\bar{M}), \quad (2.1)$$

where I denotes the identity transformation of $T_p \bar{M}$. Consider an almost Hermitian manifold $(\bar{M}, \bar{J}, \bar{g})$ and denote by $\bar{\nabla}$ the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then $\bar{M}$ is called a Kähler manifold [24] if $\bar{J}$ is parallel with respect to $\bar{\nabla}$, i.e.,

$$(\bar{\nabla}_X \bar{J})Y = 0 \quad (2.2)$$

for $X, Y \in \Gamma(T\tilde{M})$.

Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $F : M \longrightarrow N$ is a smooth map between them. Then the differential F_* of F can be viewed a section of the bundle $Hom(TM, F^{-1}TN) \longrightarrow M$, where $F^{-1}TN$ is the pullback bundle which has fibres $(F^{-1}TN)_p = T_{F(p)}N, p \in M$. $Hom(TM, F^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of F is given by

$$(\nabla F_*)(X, Y) = \nabla_X^F F_*(Y) - F_*(\nabla_X^M Y) \quad (2.3)$$

for $X, Y \in \Gamma(TM)$. It is known that the second fundamental form is symmetric. First note that in [17] we showed that the second fundamental form $(\nabla F_*)(X, Y), \forall X, Y \in \Gamma((ker F_*)^\perp)$, of a Riemannian map has no components in $range F_*$. More precisely we have the following.

Lemma 1 ([17]). *Let F be a Riemannian map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2) . Then*

$$g_2((\nabla F_*)(X, Y), F_*(Z)) = 0, \forall X, Y, Z \in \Gamma((ker F_*)^\perp).$$

Let F be a Riemannian map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2) . We will use the letters $\mathcal{H}$ and $\mathcal{V}$ to denote the orthogonal projections onto distributions $(ker F_*)^\perp$ and $ker F_*$, respectively. Then we define $\mathcal{T}$ and $\mathcal{A}$ as

$$\mathcal{A}_E F = \mathcal{H} \nabla_{\mathcal{H}E} \mathcal{V}F + \mathcal{V} \nabla_{\mathcal{H}E} \mathcal{H}F, \mathcal{T}_E F = \mathcal{H} \nabla_{\mathcal{V}E} \mathcal{V}F + \mathcal{V} \nabla_{\mathcal{V}E} \mathcal{H}F, \quad (2.4)$$

for vector fields E, F on M_1 , where ∇ is the Levi-Civita connection of g_1 . In fact, one can see that these tensor fields are O'Neill's tensor fields which were defined for Riemannian submersions. For any $E \in \Gamma(TM_1)$, $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on $(\Gamma(TM_1), g)$ reversing the horizontal and the vertical distributions. It is also easy to see that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}$ is horizontal, $\mathcal{A} = \mathcal{A}_{\mathcal{H}E}$. We note that the tensor field $\mathcal{T}$ satisfies

$$\mathcal{T}_U W = \mathcal{T}_W U, \forall U, W \in \Gamma(ker F_*). \quad (2.5)$$

On the other hand, from (2.4) we have

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W \quad (2.6)$$

$$\nabla_V X = \mathcal{H} \nabla_V X + \mathcal{T}_V X \quad (2.7)$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V} \nabla_X V \quad (2.8)$$

$$\nabla_X Y = \mathcal{H} \nabla_X Y + \mathcal{A}_X Y \quad (2.9)$$

for $X, Y \in \Gamma((ker F_*)^\perp)$ and $V, W \in \Gamma(ker F_*)$, where $\hat{\nabla}_V W = \mathcal{V} \nabla_V W$. In fact, these equations are O'Neill's formulas for Riemannian submersions [13].

From now on, for simplicity, we denote by ∇^2 both the Levi-Civita connection of (M_2, g_2) and its pullback along F . Then according to [12], for any vector field X on M_1 and any section V of $(\text{range} F_*)^\perp$, where $(\text{range} F_*)^\perp$ is the subbundle of $F^{-1}(TM_2)$ with fiber $(F_*(T_p M))^\perp$ -orthogonal complement of $F_*(T_p M)$ for g_2 over p , we have $\nabla_X^{F^\perp} V$ which is the orthogonal projection of $\nabla_X^2 V$ on $(F_*(TM))^\perp$. In [12], the author also showed that $\nabla^{F^\perp}$ is a linear connection on $(F_*(TM))^\perp$ such that $\nabla^{F^\perp} g_2 = 0$. We now define $\mathcal{S}_V$ as

$$\nabla_X^F V = -\mathcal{S}_V F_* X + \nabla_X^{F^\perp} V, \quad (2.10)$$

where $\mathcal{S}_V F_* X$ is the tangential component (a vector field along F) of $\nabla_{F_* X}^2 V$. It is easy to see that $\mathcal{S}_V F_* X$ is bilinear in V and $F_* X$ and $\mathcal{S}_V F_* X$ at p depends only on V_p and $F_{*p} X_p$. By direct computations, we obtain

$$g_2(\mathcal{S}_V F_* X, F_* Y) = g_2(V, (\nabla F_*)(X, Y)), \quad (2.11)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma((\text{range} F_*)^\perp)$. Since (∇F_*) is symmetric, it follows that $\mathcal{S}_V$ is a symmetric linear transformation of $\text{range} F_*$.

3. GENERIC RIEMANNIAN MAPS FROM ALMOST HERMITIAN MANIFOLDS

Let F be a Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Define

$$\mathcal{D}_p = (\ker F_{*p} \cap J(\ker F_{*p})), p \in M$$

the complex subspace of the vertical subspace $\mathcal{V}_p$

Definition 1. Let F be a Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . If the dimension D_p is constant along M and it defines a differentiable distribution on M then we say that F is a generic Riemannian map

A generic Riemannian map is purely real (respectively, complex) if $D_p = \{0\}$ (respectively, $D_p = \ker F_{*p}$). For a generic Riemannian map, the orthogonal complementary distribution $\mathcal{D}^\perp$, called purely real distribution, satisfies

$$\ker F_* = \mathcal{D} \oplus \mathcal{D}^\perp \quad (3.1)$$

and

$$\mathcal{D} \cap \mathcal{D}^\perp = \{0\}. \quad (3.2)$$

Let F be a generic Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then for $U \in \Gamma(\ker F_*)$, we write

$$JU = \phi U + \omega U, \quad (3.3)$$

where $\phi U \in \Gamma(\ker F_*)$ and $\omega U \in \Gamma((\ker F_*)^\perp)$. Now we consider the complementary orthogonal distribution μ to $\omega \mathcal{D}^\perp$ in $(\ker F_*)^\perp$. It is obvious that we have

$$\phi \mathcal{D}^\perp \subseteq \mathcal{D}^\perp, (\ker F_*)^\perp = \omega \mathcal{D}^\perp \oplus \mu.$$

Also for $X \in \Gamma((\ker F_*)^\perp)$, we write

$$JX = \mathcal{B}X + \mathcal{C}X, \quad (3.4)$$

where $\mathcal{B}X \in \Gamma(\mathcal{D}^\perp)$ and $\mathcal{C}X \in \Gamma(\mu)$. Then it is clear that we get

$$\mathcal{B}((\ker F_*)^\perp) = \mathcal{D}^\perp. \quad (3.5)$$

Considering (3.1), for $U \in \Gamma(\ker F_*)$, we can write

$$JU = P_1U + P_2U + \omega U, \quad (3.6)$$

where P_1 and P_2 are the projections from $\ker F_*$ to $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively.

We now give some examples of generic Riemannian maps from almost Hermitian manifolds to Riemannian manifolds.

Example 1. Every semi-invariant submersion [20] F is a generic Riemannian map with $(\text{range } F_*)^\perp = \{0\}$ and $\mathcal{D}^\perp$ is a totally real distribution.

Example 2. Every generic submersion [2] F is a generic Riemannian map with $(\text{range } F_*)^\perp = \{0\}$.

Example 3. Every semi-invariant Riemannian map [19] F from an almost Hermitian manifold to a Riemannian manifold is a generic Riemannian map such that $\mathcal{D}^\perp$ is a totally real distribution.

Example 4. Every slant- Riemannian map [21] F from an almost Hermitian manifold to a Riemannian manifold is a generic Riemannian map such that $\mathcal{D} = \{0\}$ and $\mathcal{D}^\perp$ is a slant distribution.

Example 5. Every semi-slant Riemannian map [15] F from an almost Hermitian manifold to a Riemannian manifold is a generic Riemannian map such that $\mathcal{D}^\perp$ is a slant distribution.

Since semi-invariant Riemannian maps include invariant Riemannian maps and anti-invariant Riemannian maps, such Riemannian maps are also examples of generic Riemannian maps. We say that a generic Riemannian map is proper if $\mathcal{D}^\perp$ is neither complex nor purely real. We now present an example of proper generic Riemannian maps. In the following $\mathbb{R}^{2m}$ denotes the Euclidean $2m$ -space with the standard metric. An almost complex structure J on $\mathbb{R}^{2m}$ is said to be compatible if $(\mathbb{R}^{2m}, J)$ is complex analytically isometric to the complex number space C^m with the standard flat Kählerian metric. We denote by J the compatible almost complex structure on $\mathbb{R}^{2m}$ defined by

$$J(a^1, \dots, a^{2m}) = (-a^2, a^1, \dots, -a^{2m}, a^{2m-1}).$$

Example 6. Consider the following map defined by

$$F : \begin{matrix} \mathbb{R}^8 \\ (x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8) \end{matrix} \longrightarrow \begin{matrix} \mathbb{R}^5 \\ (x^2, x^1, \frac{x^5+x^6+x^4}{\sqrt{3}}, 0, \frac{x^5-x^6}{\sqrt{2}}) \end{matrix}.$$

Then we have

$$\ker F_* = \text{span}\{U_1 = \frac{\partial}{\partial x^8}, U_2 = \frac{\partial}{\partial x^7}, U_3 = \frac{\partial}{\partial x^3}, U_4 = -2\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^5} + \frac{\partial}{\partial x^6}\}$$

and

$$\begin{aligned} (\ker F_*)^\perp &= \text{span}\{Z_1 = \frac{\partial}{\partial x^1}, Z_2 = \frac{\partial}{\partial x^2}, Z_3 = \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^5} + \frac{\partial}{\partial x^6}, \\ Z_4 &= \frac{\partial}{\partial x^5} - \frac{\partial}{\partial x^6}\}. \end{aligned}$$

Hence it is easy to see that

$$g_{\mathbb{R}^5}(F_*(Z_i), F_*(Z_i)) = g_{\mathbb{R}^8}(Z_i, Z_i), i = 1, 2, 3, 4$$

and

$$g_{\mathbb{R}^5}(F_*(Z_i), F_*(Z_j)) = g_{\mathbb{R}^8}(Z_i, Z_j) = 0,$$

$i \neq j$, Thus F is a Riemannian map. On the other hand, we have $JU_1 = U_2$ and $JU_3 = -\frac{1}{3}U_4 + \frac{1}{3}Z_3$ and $JU_4 = 2U_3 - Z_4$, where J is the complex structure of $\mathbb{R}^8$. Thus F is a generic Riemannian map with $\mathcal{D} = \text{span}\{U_1, U_2\}, \mathcal{D}^\perp = \text{span}\{U_3, U_4\}$ and $\mu = \text{span}\{Z_1, Z_2\}$.

We now investigate the effect of a proper generic Riemannian map on the geometry of the total manifold, the base manifold and the map itself.

Lemma 2. *Let F be a proper generic Riemannian map from a Kähler manifold (M, J, g_M) to a Riemannian manifold (N, g_N) . Then the distribution $\mathcal{D}$ is integrable if and only if the following expression is satisfied*

$$\mathcal{T}_X JY = \mathcal{T}_Y JX \quad (3.7)$$

for $X, Y \in \Gamma(\mathcal{D})$.

Proof. From (2.2), (2.6), (3.4) and (3.6) we have

$$\hat{\nabla}_X JY + \mathcal{T}_X JY = P_1 \hat{\nabla}_X Y + P_2 \hat{\nabla}_X Y + \omega \hat{\nabla}_X Y + \mathcal{B} \mathcal{T}_X Y + \mathcal{C} \mathcal{T}_X Y \quad (3.8)$$

for $X, Y \in \Gamma(\mathcal{D})$. Taking the vertical parts and the horizontal parts of (3.8) we get

$$\hat{\nabla}_X JY = P_1 \hat{\nabla}_X Y + P_2 \hat{\nabla}_X Y + \mathcal{B} \mathcal{T}_X Y \quad (3.9)$$

$$\mathcal{T}_X JY = \omega \hat{\nabla}_X Y + \mathcal{C} \mathcal{T}_X Y \quad (3.10)$$

Interchanging the role of X and Y in (3.10), and taking into account that $\mathcal{T}$ is symmetric on the vertical distribution, we derive

$$\mathcal{T}_X JY - \mathcal{T}_Y JX = \omega[X, Y]$$

which gives proof. $\square$

In a similar way, we have the following lemma.

Lemma 3. *Let F be a proper generic Riemannian map from a Kähler manifold (M, J, g_M) to a Riemannian manifold (N, g_N) . Then the distribution $\mathcal{D}^\perp$ is integrable if and only if the following expression is satisfied*

$$\hat{\nabla}_{Z_1} P_2 Z_2 - \hat{\nabla}_{Z_2} P_2 Z_1 = P_2[Z_1, Z_2] + \mathcal{T}_{Z_1} \omega Z_2 - \mathcal{T}_{Z_2} \omega Z_1 \quad (3.11)$$

for $Z_1, Z_2 \in \Gamma(\mathcal{D}^\perp)$.

We now investigate the geometry of leaves of $\mathcal{D}$ and $\mathcal{D}^\perp$.

Lemma 4. *Let F be a proper generic Riemannian map from a Kähler manifold (M, J, g_M) to a Riemannian manifold (N, g_N) . Then the distribution $\mathcal{D}$ defines a totally geodesic foliation in M if and only if*

- (1) $\hat{\nabla}_X P_2 Z + \mathcal{T}_X \omega Z$ has no components in $\mathcal{D}$ for $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$.
- (2) $\hat{\nabla}_X \mathcal{B}W + \mathcal{T}_X \mathcal{C}W$ has no components in $\mathcal{D}$ for $X \in \Gamma(\mathcal{D})$ and $W \in \Gamma((\ker F_*)^\perp)$.

Proof. From (2.2), (2.6), (3.4) and (3.6) we have

$$g_M(\nabla_X Y, Z) = -g_M(\hat{\nabla}_X P_2 Z + \mathcal{T}_X \omega Z, JY)$$

for $X, Y \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$. This gives (1). Also from (2.2), (3.4) and (3.6) we get

$$g_M(\nabla_X Y, W) = -g_M(\nabla_X \mathcal{B}W + \mathcal{C}W, JY)$$

for $X, Y \in \Gamma(\mathcal{D})$ and $W \in \Gamma((\ker F_*)^\perp)$. Now using (2.6) and (2.7)

$$g_M(\nabla_X Y, W) = -g_M(\hat{\nabla}_X \mathcal{B}W + \mathcal{T}_X \mathcal{C}W, JY)$$

which gives (2). $\square$

In a similar way, we have the following result.

Lemma 5. *Let F be a proper generic Riemannian map from a Kähler manifold (M, J, g_M) to a Riemannian manifold (N, g_N) . Then the distribution $\mathcal{D}^\perp$ defines a totally geodesic foliation in M if and only if*

- (1) $\hat{\nabla}_{Z_1} P_2 Z_2 + \mathcal{T}_{Z_1} \omega Z_2 = 0$ for $Z_1, Z_2 \in \Gamma(\mathcal{D}^\perp)$.
- (2) $\mathcal{C}\mathcal{H}\nabla_{Z_1} \omega Z_2 + \mathcal{C}\mathcal{T}_{Z_1} P_2 Z_2$ has no components in μ .

From Lemma 4 and Lemma 5 we obtain the following decomposition theorem.

Theorem 1. *Let F be a proper generic Riemannian map from a Kähler manifold (M, J, g_M) to a Riemannian manifold (N, g_N) . Then the fibres are locally product Riemannian manifold of the form $M_{\mathcal{D}} \times M_{\mathcal{D}^\perp}$ if*

- (1) $\hat{\nabla}_Y P_2 Z_2 + \mathcal{T}_Y \omega Z_2 = 0$ for $Y \in \Gamma(\ker F_*)$ and $Z_2 \in \Gamma(\mathcal{D}^\perp)$.
- (2) $\mathcal{C}\mathcal{H}\nabla_{Z_1} \omega Z_2 + \mathcal{C}\mathcal{T}_{Z_1} P_2 Z_2$ has no components in μ $Z_1 \in \Gamma(\mathcal{D}^\perp)$.
- (3) $\hat{\nabla}_X \mathcal{B}W + \mathcal{T}_X \mathcal{C}W$ has no components in $\mathcal{D}$ for $X \in \Gamma(\mathcal{D})$ and $W \in \Gamma((\ker F_*)^\perp)$.

Lemma 6. *Let F be a proper generic Riemannian map from a Kähler manifold (M, J, g_M) to a Riemannian manifold (N, g_N) . Then the distribution $\ker F_*$ defines a totally geodesic foliation in M if and only if*

- (1) $\hat{\nabla}_X P_2 Z + \mathcal{T}_X \omega Z$ has no components in $\mathcal{D}$ for $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$.
- (2) $\hat{\nabla}_X \mathcal{B}W + \mathcal{T}_X \mathcal{C}W$ has no components in $\mathcal{D}$ for $X \in \Gamma(\mathcal{D})$ and $W \in \Gamma((\ker F_*)^\perp)$.

Proof. From (2.2), (3.4) and (3.6) we have

$$g_M(\nabla_U V, \xi) = g_M(\hat{\nabla}_U \phi V, \mathcal{B}\xi) + g_M(\mathcal{T}_U \omega V, \mathcal{B}\xi) \\ + g_M(\mathcal{T}_U \omega V, \mathcal{C}\xi) + g_M(\hbar \nabla_U \phi V, \mathcal{C}\xi)$$

for $U, V \in \Gamma(\ker F_*)$ and $\xi \in \Gamma((\ker F_*)^\perp)$. Also from (2.2), (3.4) and (3.6) we get

$$g_M(\nabla_X Y, W) = -g_M(\nabla_X \mathcal{B}W + \mathcal{C}W, JY)$$

for $X, Y \in \Gamma(\mathcal{D})$ and $W \in \Gamma((\ker F_*)^\perp)$. Now using (2.6) and (2.7)

$$g_M(\nabla_X Y, W) = -g_M(\hat{\nabla}_X \mathcal{B}W + \mathcal{T}_X \mathcal{C}W, JY)$$

which gives (2). $\square$

In a similar way, we have the following result for distribution $(\ker F_*)^\perp$.

Lemma 7. *Let F be a proper generic Riemannian map from a Kähler manifold (M, J, g_M) to a Riemannian manifold (N, g_N) . Then the distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation in M if and only if*

$$\mathcal{B}\mathcal{A}_X \mathcal{B}Y + \mathcal{B}\mathcal{H}\nabla_X \mathcal{C}Y = -\phi \mathcal{V}\nabla_X \mathcal{B}Y - \phi \mathcal{A}_X \mathcal{C}Y$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$.

From Lemma 6 and Lemma 7, we have the following corollary.

Corollary 1. *Let F be a proper generic Riemannian map from a Kähler manifold (M, J, g_M) to a Riemannian manifold (N, g_N) . Then M is a locally product Riemannian manifold if and only if*

- (1) $\hat{\nabla}_U P_2 Z + \mathcal{T}_U \omega Z$ has no components in $\mathcal{D}$ for $U \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$.
- (2) $\hat{\nabla}_U \mathcal{B}W + \mathcal{T}_U \mathcal{C}W$ has no components in $\mathcal{D}$ for $U \in \Gamma(\mathcal{D})$ and $W \in \Gamma((\ker F_*)^\perp)$.

$$(3) \quad \mathcal{B}\mathcal{A}_X\mathcal{B}Y + \mathcal{B}\mathcal{H}\nabla_X\mathcal{C}Y = -\phi\mathcal{V}\nabla_X\mathcal{B}Y - \phi\mathcal{A}_X\mathcal{C}Y$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$.

In the rest of this section, we are going to find necessary and sufficient conditions for proper generic Riemannian maps from Kähler manifolds to Riemannian manifolds to be totally geodesic and harmonic, respectively. We recall that a map between Riemannian manifolds is totally geodesic if $(\nabla F_*) = 0$ on the total manifold. A geometric interpretation of a totally geodesic map is that it maps every geodesic in the total manifold into a geodesic in the base manifold in proportion to arc lengths. We also recall that a map is harmonic if $\text{trace} \nabla F_* = 0$. Geometrically, a harmonic map F is a critical point of the energy functional E .

Theorem 2. *Let F be a proper generic Riemannian map from a Kähler manifold (M, J, g_M) to a Riemannian manifold (N, g_N) . Then F is totally geodesic if and only if the following conditions are satisfied*

- (1) $g_M(\mathcal{A}_X\phi U + \mathcal{H}\nabla_X\omega U, \mathcal{C}Y) = -g_M(\mathcal{A}_X\omega U + \mathcal{V}\nabla_X\phi U, \mathcal{B}Y)$,
- (2) $g_M(\hat{\nabla}_U\mathcal{B}X + \mathcal{T}_U\mathcal{C}X, \phi V) = -g_M(\mathcal{T}_U\mathcal{B}X + \mathcal{H}\nabla_U\mathcal{C}X, \omega V)$,
- (3) $(\nabla F_*)(X, \omega P_2Y + \mathcal{C}\omega Y)$ has no components in $(F_*(TM))^\perp$,

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U, V \in \Gamma(\ker F_*)$.

Proof. First of all, from Lemma 1, we have $g_M((\nabla F_*)(X, Y), F_*(Z)) = 0$ for $X, Y, Z \in \Gamma((\ker F_*)^\perp)$. For $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$, using (2.3) and (2.2), we get

$$g_N((\nabla F_*)(U, X), F_*(Y)) = g_M(\nabla_X JU, JY).$$

Then from (3.4) and (3.3) we derive

$$g_N((\nabla F_*)(U, X), F_*(Y)) = g_M(\nabla_X\phi U + \omega U, \mathcal{B}Y + \mathcal{C}Y).$$

Using (2.9) and (2.8) we have

$$\begin{aligned} g_N((\nabla F_*)(U, X), F_*(Y)) &= g_M(\mathcal{A}_X\phi U + \mathcal{H}\nabla_X\omega U, \mathcal{C}Y) \\ &\quad + g_M(\mathcal{A}_X\omega U + \mathcal{V}\nabla_X\phi U, \mathcal{B}Y) \end{aligned}$$

which gives (1). In a similar way, we obtain (2). Now, for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $\xi \in \Gamma((F_*(TM))^\perp)$, we have

$$g_N((\nabla F_*)(X, Y), \xi) = -g_N(F_*(Y), \nabla_X^F \xi).$$

Using (2.10) and (2.1) we get

$$g_N((\nabla F_*)(X, Y), \xi) = -g_N(F_*(J^2Y), \mathcal{S}_\xi F_*(X)).$$

Thus from (3.4) and (3.3), we obtain

$$g_N((\nabla F_*)(X, Y), \xi) = -g_N(F_*(\omega P_2Y), \mathcal{S}_\xi F_*(X)) - g_N(F_*(\mathcal{C}\omega Y), \mathcal{S}_\xi F_*(X)).$$

Now using (2.11), we arrive at

$$g_N((\nabla F_*)(X, Y), \xi) = -g_N((\nabla F_*)(X, \omega P_2 Y) + (\nabla F_*)(X, \mathcal{C}\omega Y), \xi)$$

which gives (3). $\square$

Theorem 3. *Let F be a proper generic Riemannian map from a Kähler manifold (M, J, g_M) to a Riemannian manifold (N, g_N) . Then F is harmonic if and only if*

$$\begin{aligned} & \text{trace } |_{(ker F_*)} F_*(\mathcal{C}\mathcal{T}_{(\cdot)}\phi(\cdot) + \mathcal{C}\mathcal{H}\nabla_{(\cdot)}\omega(\cdot) + \omega\hat{\nabla}_{(\cdot)}\phi(\cdot) + \omega\mathcal{T}_{(\cdot)}\omega(\cdot)) \\ &= \text{trace } |_{(ker F_*)^\perp} (\nabla_{(\cdot)}^F F_*(\omega\mathcal{B}(\cdot) + \mathcal{C}^2(\cdot)) - F_*(\mathcal{C}\mathcal{A}_{(\cdot)}\mathcal{B}(\cdot) \\ &+ \mathcal{C}\mathcal{H}\nabla_{(\cdot)}\mathcal{C}(\cdot) + \omega\mathcal{V}\nabla_{(\cdot)}\mathcal{B}(\cdot) + \omega\mathcal{A}_{(\cdot)}\mathcal{C}(\cdot))) \end{aligned}$$

Proof. For $X \in \Gamma((ker F_*)^\perp)$ and $U \in \Gamma(ker F_*)$, from (2.2), (2.3), (3.3) and (3.4) we have

$$\begin{aligned} (\nabla F_*)(X, X) + (\nabla F_*)(U, U) &= -\nabla_X^F F_*(\omega\mathcal{B}X + \mathcal{C}^2 X) \\ &+ F_*(J(\nabla_X \mathcal{B}X + \mathcal{C}X)) + F_*(J(\nabla_U \phi U + \omega U)) \end{aligned}$$

Using (3.3), (3.4) and (2.6)-(2.9) we have

$$\begin{aligned} (\nabla F_*)(X, X) + (\nabla F_*)(U, U) &= -\nabla_X^F F_*(\omega\mathcal{B}X + \mathcal{C}^2 X) \\ &+ F_*(\mathcal{C}\mathcal{T}_U \phi U + \mathcal{C}\mathcal{H}\nabla_U \omega(U) + \omega\hat{\nabla}_U \phi U + \omega\mathcal{T}_U \omega U) \\ &+ F_*(\mathcal{C}\mathcal{A}_X \mathcal{B}X + \mathcal{C}\mathcal{H}\nabla_X \mathcal{C}X + \omega\mathcal{V}\nabla_X \mathcal{B}X + \omega\mathcal{A}_X \mathcal{C}X) \end{aligned}$$

which gives our assertion. $\square$

4. GENERIC RIEMANNIAN MAPS TO KÄHLER MANIFOLDS

In this section, we define generic Riemannian maps from Riemannian manifolds to almost hermitian manifolds, provide examples and investigate their geometric properties.

Let F be a Riemannian map from a Riemannian manifold (M_1, g_1) to an almost Hermitian manifold (M_2, g_2, J) . Define

$$\mathfrak{D}_p = (range F_{*p} \cap J(range F_{*p})), p \in M_1$$

the complex subspace of the image subspace $F_*(T_p M_1)$

Definition 2. Let F be a Riemannian map from a Riemannian manifold (M_1, g_1) to an almost Hermitian manifold (M_2, g_2, J) . If the dimension $\mathfrak{D}_p$ is constant along F and it defines a differentiable distribution on M_2 then we say that F is a generic Riemannian map.

A generic Riemannian map is purely real (respectively, complex) if $\mathfrak{D}_p = \{0\}$ (respectively, $\mathfrak{D}_p = range F_{*p}$). For a generic Riemannian map, the orthogonal complementary distribution $\mathfrak{D}^\perp$, called purely real distribution, satisfies

$$range F_* = \mathfrak{D} \oplus \mathfrak{D}^\perp \quad (4.1)$$

$$\mathfrak{D} \cap \mathfrak{D}^\perp = \{0\} \quad (4.2)$$

Let F be a generic Riemannian map from a Riemannian manifold (M_1, g_1) to an almost Hermitian manifold (M_2, g_2, J) . Then for $F_*(X) \in \Gamma(\text{range} F_*)$, $X \in \Gamma((\ker F_*)^\perp)$ we write

$$JF_*(X) = \varphi F_*(X) + \varpi F_*(X), \quad (4.3)$$

where $\varphi F_*(X) \in \Gamma(\text{range} F_*)$ and $\varpi F_*(X) \in \Gamma((\text{range} F_*)^\perp)$. Now we consider the complementary orthogonal distribution ν to $\varpi \mathfrak{D}^\perp$ in $(\text{range} F_*)^\perp$. Then it is obvious that

$$(\text{range} F_*)^\perp = \varpi \mathfrak{D}^\perp \oplus \nu, \varphi \mathfrak{D}^\perp \subseteq \mathfrak{D}^\perp.$$

Also for $V \in \Gamma((\text{range} F_*)^\perp)$, we write

$$JV = \mathfrak{B}V + \mathfrak{C}V, \quad (4.4)$$

where $\mathfrak{B}V \in \Gamma(\mathfrak{D}^\perp)$ and $\mathfrak{C}V \in \Gamma(\nu)$. We also have

$$\mathfrak{B}(\text{range} F_*)^\perp = \mathfrak{D}^\perp. \quad (4.5)$$

We now give some examples of generic Riemannian maps from Riemannian manifolds to almost Hermitian manifolds.

Example 7. Every CR-submanifold [4] F is a generic Riemannian map with $(\ker F_*) = \{0\}$ and $\mathfrak{D}^\perp$ is a totally real distribution.

Example 8. Every generic submanifold [5] F is a generic Riemannian map with $(\ker F_*) = \{0\}$.

Example 9. Every semi-invariant Riemannian map [18] F from a Riemannian manifold to an almost Hermitian manifold is a generic Riemannian map such that $\mathfrak{D}^\perp$ is a totally real distribution.

Since semi-invariant Riemannian maps include invariant Riemannian maps and anti-invariant Riemannian maps, such Riemannian maps are also examples of generic Riemannian maps.

Example 10. Every slant Riemannian map [22] F from a Riemannian manifold to an almost Hermitian manifold is a generic Riemannian map such that $\mathfrak{D} = \{0\}$ and $\mathfrak{D}^\perp$ is a slant distribution.

Example 11. Every semi-slant Riemannian map [16] F from a Riemannian manifold to an almost Hermitian manifold is a generic Riemannian map such that $\mathfrak{D}^\perp$ is a slant distribution.

Example 12. Every holomorphic Riemannian map [23] is a generic Riemannian map between almost Hermitian manifolds with $\mathfrak{D} = \text{range} F_*$.

We say that a generic Riemannian map is proper if $\mathfrak{D}^\perp$ is neither complex nor purely real. We now present an example of a proper generic Riemannian map from a Riemannian manifold to a Kähler manifold.

Example 13. Consider the following map defined by

$$F : \mathbb{R}^9 \longrightarrow \mathbb{R}^6 \\ (x^1, \dots, x^9) \longrightarrow (x^1, x^9, x^3, \frac{x^4+x^5}{\sqrt{2}}, \frac{x^4+x^5}{\sqrt{2}}, 0).$$

Then $\ker F_*$ is spanned by

$$U_1 = \frac{\partial}{\partial x^2}, U_2 = \frac{\partial}{\partial x^4} - \frac{\partial}{\partial x^5}, U_3 = \frac{\partial}{\partial x^6}, U_4 = \frac{\partial}{\partial x^7}, U_5 = \frac{\partial}{\partial x^8}.$$

$(\ker F_*)^\perp$ is spanned by

$$Z_1 = \frac{\partial}{\partial x^1}, Z_2 = \frac{\partial}{\partial x^9}, Z_3 = \frac{\partial}{\partial x^3}, Z_4 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^5}). \quad (4.6)$$

Hence it is easy to see that

$$g_{\mathbb{R}^6}(F_*(Z_i), F_*(Z_i)) = g_{\mathbb{R}^9}(Z_i, Z_i), i = 1, 2, 3, 4$$

and

$$g_{\mathbb{R}^6}(F_*(Z_i), F_*(Z_j)) = g_{\mathbb{R}^9}(Z_i, Z_j) = 0,$$

$i \neq j$, Thus F is a Riemannian map. Moreover, $(\text{range } F_*)^\perp$ is spanned by

$$V_1 = \frac{\partial}{\partial y^4} - \frac{\partial}{\partial y^5}, V_2 = \frac{\partial}{\partial y^6}.$$

On the other hand, we have $JF_*(Z_1) = F_*(Z_2)$ and $JF_*(Z_3) = \frac{1}{2}F_*(Z_4) + \frac{1}{2}V_1$ and $JF_*(Z_4) = -F_*(Z_3) + V_2$, where J is the complex structure of $\mathbb{R}^6$. Thus F is a generic Riemannian map with $\mathfrak{D} = \text{span}\{F_*(Z_1), F_*(Z_2)\}$, $\mathfrak{D}^\perp = \text{span}\{F_*(Z_3), F_*(Z_4)\}$ and $\mathfrak{v} = \text{span}\{V_1, V_2\}$.

We now find necessary and sufficient conditions for generic Riemannian maps from Riemannian manifolds to Kähler manifolds to be totally geodesic and harmonic. We first give the following result for totally geodesicity.

Theorem 4. *Let F be a proper generic Riemannian map from a Riemannian manifold (M_1, g_1) to a Kähler manifold (M_2, g_2, J) . Then F is totally geodesic if and only if*

$$(1) \text{ for } X, Y, \acute{Y} \in \Gamma((\ker F_*)^\perp),$$

$$g_2(\mathcal{S}_{\varpi F_*(Y)} F_*(X), \mathfrak{B}V) = g_2((\nabla F_*)(X, \acute{Y}), \mathfrak{C}V) + g_1(\mathcal{H} \nabla_X \acute{Y}, {}^* F_* \mathfrak{B}V) \\ + g_2(\nabla_X^{F^\perp} \varpi F_*(Y), \mathfrak{C}V)$$

is satisfied, where $F_(\acute{Y}) = \varphi F_*(Y)$,*

$$(2) \ker F_* \text{ is totally geodesic,}$$

$$(3) (\ker F_*)^\perp \text{ is totally geodesic.}$$

Proof. For $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma((\operatorname{range} F_*)^\perp)$, using (2.2), (2.1), (4.4) and (4.3) we have

$$g_2((\nabla F_*)(X, Y), V) = g_2(\nabla_X^F \varphi F_*(Y) + \varpi F_*(Y), \mathfrak{B} + V \mathfrak{C}V).$$

Then from (2.10) we get

$$\begin{aligned} g_2((\nabla F_*)(X, Y), V) &= g_2(\nabla_X^F F_*(Y), \mathfrak{B}V + \mathfrak{C}V) \\ &\quad - g_2(\mathcal{S}_{\varpi F_*(Y)} F_*(X), \mathfrak{B}V) + g_2(\nabla_X^{F^\perp} \varpi F_*(Y), \mathfrak{C}V). \end{aligned}$$

Then (2.3) and (2.9) imply

$$\begin{aligned} g_2((\nabla F_*)(X, Y), V) &= g_2((\nabla F_*)(X, Y), \mathfrak{C}V) + g_1(\mathcal{H} \nabla_X Y, {}^* F_* \mathfrak{B}V) \\ &\quad - g_2(\mathcal{S}_{\varpi F_*(Y)} F_*(X), \mathfrak{B}V) + g_2(\nabla_X^{F^\perp} \varpi F_*(Y), \mathfrak{C}V). \end{aligned}$$

which gives (1). (2) and (3) can be obtained in a similar way. $\square$

For the harmonicity, we have the following result.

Theorem 5. *Let F be a proper generic Riemannian map from a Riemannian manifold (M_1, g_1) to a Kähler manifold (M_2, g_2, J) . Then F is a harmonic map if and only if*

$$\begin{aligned} \operatorname{trace} |_{(\ker F_*)^\perp} \{ & -\mathfrak{C}(\nabla F_*)(\cdot, {}^* F \varphi F_*(\cdot)) - \varpi F_*(\mathcal{H} \nabla_{(\cdot)} {}^* F \varphi F_*(\cdot)) \\ & - \varpi \mathcal{S}_{\varpi F_*(\cdot)} F_*(\cdot) + \mathfrak{C} \nabla_{(\cdot)}^{F^\perp} \varpi F_*(\cdot) \} = 0 \end{aligned} \quad (4.7)$$

and the fibers are minimal.

Proof. From (2.3), (2.2) and (4.3) we have

$$(\nabla F_*)(X, X) = -J[\nabla_X^F \varphi F_*(X) + \varpi F_*(X)] - F_*(\nabla_X^1 X)$$

for $X \in \Gamma((\ker F_*)^\perp)$. For $F_*(\acute{X}) = \varphi F_*(X)$, from (2.3), (2.9), (2.10), (4.3) and (4.4) we get

$$\begin{aligned} (\nabla F_*)(X, X) &= -\mathfrak{B}(\nabla F_*)(X, \acute{X}) - \mathfrak{C}(\nabla F_*)(X, \acute{X}) \\ &\quad - \varphi F_*(\nabla_X \acute{X}) - \varpi F_*(\nabla_X \acute{X}) + \varphi \mathcal{S}_{\varpi F_*(X)} F_*(\acute{X}) \\ &\quad + \varpi \mathcal{S}_{\varpi F_*(X)} F_*(\acute{X}) + \mathfrak{B} \nabla_X^{F^\perp} \varpi F_*(X) + \mathfrak{C} \nabla_X^{F^\perp} \varpi F_*(X) \\ &\quad - F_*(\nabla_X \acute{X}), \end{aligned}$$

where ∇ is the Levi-Civita connection on M_1 . Now considering $(\operatorname{range} F_*)$ parts of this equation and taking trace on resulting equation we get (4.7). Second assertion comes from $(\nabla F_*)(U, U) = -F_*(\nabla_U U)$ for $U \in \Gamma(\ker F_*)$. $\square$

5. CONCLUSION REMARKS

Generic Riemannian maps we have studied in this paper include holomorphic submanifolds, totally real submanifolds, slant submanifolds, semi-slant submanifolds, holomorphic submersions, anti-invariant submersions, semi-invariant submersions, slant submersions, semi-slant submersions, invariant Riemannian maps, anti-invariant Riemannian maps, semi-invariant Riemannian maps, slant Riemannian maps and semi-slant Riemannian maps. Thus we have constructed two wide classes of Riemannian maps. Although we give some main properties of these Riemannian maps, there are still many research problems to investigate for interested readers.

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SOME IDENTITIES ON CONDITIONAL SEQUENCES BY USING MATRIX METHOD

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Abstract. In this paper, we consider the Fibonacci conditional sequence $\{f_n\}$ and the Lucas conditional sequence $\{l_n\}$. We derive some properties of Fibonacci and Lucas conditional sequences by using the matrix method. Our results are elegant as the results for ordinary Fibonacci and Lucas sequences.

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Keywords: Fibonacci sequence, conditional sequence, matrix method

1. INTRODUCTION

The Fibonacci numbers F_n are the terms of the sequence $0, 1, 1, 2, 3, 5, \dots$, where

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$. The sequence L_n of Lucas numbers, which follows the same recursive pattern as the Fibonacci numbers, but begins with $L_0 = 2$ and $L_1 = 1$. These numbers are famous for possessing wonderful properties, see also [5] and [8] for additional references and history.

This sequences has been generalized in various ways. Horadam [3] has considered a generalized sequence $\{W_n\}$ defined by

$$W_n = pW_{n-1} - qW_{n-2}, \quad n \geq 2$$

with the initial conditions $W_0 = a$ and $W_1 = b$, where a, b, p, q are arbitrary integers. If we take $a = 0, b = 1$ in $\{W_n\}$, we get the generalized Fibonacci sequence and if we take $a = 2, b = p$ in $\{W_n\}$, we get the generalized Lucas sequence. In [1], the author use matrix techniques to give proofs of well-known properties of the generalized Fibonacci and Lucas sequences.

This paper is in final form and no version of it will be submitted for publication elsewhere.

In [7], author introduced a new generalization of the Fibonacci sequence $\{f_n\}$, defined by for $n \geq 2$

$$f_n = \begin{cases} a_1 f_{n-1} + b_1 f_{n-2}, & \text{if } n \text{ is even} \\ a_2 f_{n-1} + b_2 f_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad (1.1)$$

with initial values $f_0 = 0$ and $f_1 = 1$, where a_1, a_2, b_1, b_2 are nonzero numbers. If we take $a_1 = a_2 = b_1 = b_2 = 1$ in $\{f_n\}$, we get the classical Fibonacci sequence. Here we call $\{f_n\}$, the Fibonacci conditional sequence. We can easily define the Lucas conditional sequence $\{l_n\}$ with the initial conditions $l_0 = 2$ and $l_1 = a_2$ similar to (1.1). In [9], in the case of $b_1 = b_2$, the author gave some well known identities for $\{f_n\}$ such as Cassini's, Catalan's identities, etc. Without the case of $b_1 = b_2$, Gelin-Cesaro identity and Catalan identity for the even indices of $\{f_n\}$ are given in [7]. Also, in [4], for the case of $b_1 = b_2 = 1$ author defined a generalization of Lucas sequence and derived several identities involving the generalized Fibonacci and Lucas sequences. All of these proofs are based on the generating function.

In this paper, we take the non-zero numbers a_1, a_2, b_1, b_2 completely arbitrary and we derive some properties of Fibonacci and Lucas conditional sequences by using the matrix method. For Lucas conditional sequences, we just need the case of $b_1 = b_2$. We collect our main results in Theorem 1, 2, 3 and 4 which are elegant as the results for ordinary Fibonacci and Lucas sequences. We also obtain some more new results (4.10)-(4.13) and (4.15).

We need the following results which can be obtained by taking $r = 2$ in [6, Theorem 5,6,9].

For $n \geq 4$,

$$f_n = A f_{n-2} - B f_{n-4} \quad (1.2)$$

where $A := a_1 a_2 + b_1 + b_2$ and $B := b_1 b_2$. The same result for $\{l_n\}$ also holds.

The generating function of the sequence $\{f_n\}$ is

$$F(x) = \frac{x + a_1 x^2 - b_1 x^3}{1 - A x^2 + B x^4} \quad (1.3)$$

and the generating function of the sequence $\{l_n\}$ is

$$L(x) = \frac{2 + a_2 x - (a_1 a_2 + 2b_2)x^2 + b_1 a_2 x^3}{1 - A x^2 + B x^4}. \quad (1.4)$$

By using (1.3) and (1.4), the Binet's formulas for the sequence $\{f_n\}$ and $\{l_n\}$ are given;

$$f_{2m} = a_1 \frac{\alpha^m - \beta^m}{\alpha - \beta} \quad (1.5)$$

$$f_{2m+1} = (a_1 a_2 + b_2) \frac{\alpha^m - \beta^m}{\alpha - \beta} - (b_1 b_2) \frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta} \quad (1.6)$$

$$l_{2m} = (a_1 a_2 + 2b_1) \frac{\alpha^m - \beta^m}{\alpha - \beta} - 2(b_1 b_2) \frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta} \quad (1.7)$$

$$l_{2m+1} = (a_1 a_2^2 + 2b_1 a_2 + a_2 b_2) \frac{\alpha^m - \beta^m}{\alpha - \beta} - (b_1 b_2 a_2) \frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta} \quad (1.8)$$

where $\alpha = \frac{A + \sqrt{A^2 - 4B}}{2}$ and $\beta = \frac{A - \sqrt{A^2 - 4B}}{2}$ that is, α and β are the roots of the polynomial $p(z) = z^2 - Az + B$.

2. FIBONACCI CASE

It is known that for the matrix $Q := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$. See [2] for a detailed history of the Q matrix.

Similar to this well-known result, setting $S := \begin{pmatrix} A & -B \\ 1 & 0 \end{pmatrix}$, we also have

$$S^n = \frac{1}{a_1} \begin{pmatrix} f_{2(n+1)} & -Bf_{2n} \\ f_{2n} & -Bf_{2(n-1)} \end{pmatrix} \quad (2.1)$$

which can be easily proven by induction and by the help of (1.2). It should be noted that S is invertible. We use the matrix identity (2.1) to get the following theorem easily.

Theorem 1. *The sequence $\{f_n\}$ satisfy*

$$-a_1^2 B^{m-1} = f_{2(m-1)} f_{2(m+1)} - f_{2m}^2 \quad (2.2)$$

$$a_1 B^m f_{2(n-m)} = f_{2n} f_{2(m+1)} - f_{2m} f_{2(n+1)} \quad (2.3)$$

$$a_1 f_{2n} = f_{2m} f_{2(n-m+1)} - B f_{2(n-m)} f_{2(m-1)} \quad (2.4)$$

$$a_1 f_{2(n+m+1)} = f_{2(n+1)} f_{2(m+1)} - B f_{2n} f_{2m} \quad (2.5)$$

$$a_1 f_{2(n+m)} = f_{2(n+1)} f_{2m} - B f_{2n} f_{2(m-1)} \quad (2.6)$$

$$a_1 f_{2(n+m-1)} = f_{2n} f_{2m} - B f_{2(n-1)} f_{2(m-1)} \quad (2.7)$$

for any positive integers m and n .

Proof. By taking the determinant of both sides of (2.1), we get the result (2.2) which is Cassini's identity for the even indices of the sequence $\{f_n\}$.

Since S is invertible,

$$S^{-n} = \frac{1}{a_1 B^n} \begin{pmatrix} -Bf_{2(n-1)} & Bf_{2n} \\ -f_{2n} & f_{2(n+1)} \end{pmatrix}.$$

Considering $S^{n-m} = S^n S^{-m}$ and by using the matrix identity (2.1), we get;

$$\frac{1}{a_1} \begin{pmatrix} f_{2(n-m+1)} & -Bf_{2(n-m)} \\ f_{2(n-m)} & -Bf_{2(n-m-1)} \end{pmatrix}$$

$$= \frac{-1}{a_1^2 B^{m-1}} \begin{pmatrix} f_{2(n+1)} f_{2(m-1)} - f_{2n} f_{2m} & f_{2n} f_{2(m+1)} - f_{2m} f_{2(n+1)} \\ f_{2(m-1)} f_{2n} - f_{2(n-1)} f_{2m} & f_{2(n-1)} f_{2(m+1)} - f_{2n} f_{2m} \end{pmatrix}$$

and by equating the $(1, 2)$ entries of the matrices on both sides, we get the result (2.3), which corresponds to the *d'Ocagne's identity* for the even indices of $\{f_n\}$.

Also, considering $S^{n-m} S^{m-1} = S^{n-1}$ and by using the matrix identity (2.1), we get;

$$\begin{pmatrix} f_{2(n-m+1)} f_{2m} - B f_{2(n-m)} f_{2(m-1)} & -B f_{2(m-1)} f_{2(n-m+1)} + B^2 f_{2(n-m)} f_{2(m-2)} \\ f_{2(n-m)} f_{2m} - B f_{2(n-m-1)} f_{2(m-1)} & -B f_{2(m-1)} f_{2(n-m)} + B^2 f_{2(n-m-1)} f_{2(m-2)} \end{pmatrix} \\ = a_1 \begin{pmatrix} f_{2n} & -B f_{2(n-1)} \\ f_{2(n-1)} & -B f_{2(n-2)} \end{pmatrix}.$$

By equating the $(1, 1)$ entries of the matrices on both sides, we get the result (2.4) which corresponds to the *Convolution property* for the even indices of $\{f_n\}$.

Similarly, since $S^{n+m} = S^n S^m$ and by using the matrix identity (2.1), we get the equalities (2.5), (2.6) and (2.7). $\square$

Special Cases:

(1) If we take $n = m$ in (2.5), we get

$$a_1 f_{2(2n+1)} = f_{2(n+1)}^2 - B f_{2n}^2. \quad (2.8)$$

(2) If we take $n = m$ in (2.6), we obtain the formula;

$$\begin{aligned} a_1 f_{4n} &= f_{2n} (f_{2(n+1)} - B f_{2(n-1)}) \\ &= f_{2n} ((b_2 - b_1) f_{2n} + a_1 l_{2n}). \end{aligned} \quad (2.9)$$

Note that for $2n = m$, if we take $a_1 = a_2 = b_1 = b_2 = 1$ in (2.9), it reduces to the well-known identity;

$$F_{2m} = F_m L_m.$$

We can get the Binet formula for the even indices of $\{f_n\}$ by using matrix method. For this, we assume $A^2 - 4B \neq 0$ to guarantee the existence of two distinct roots. Let α and β denote the roots of $p(z) = z^2 - Az + B$ that is the characteristic polynomial of S .

Theorem 2. *The Binet formula of $\{f_{2n}\}$ is*

$$f_{2n} = a_1 \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Proof. The eigenvalues and corresponding eigenvectors of the matrix S are $\lambda_1 = \alpha$, $\lambda_2 = \beta$ and $v_1 = \begin{pmatrix} \alpha \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} \beta \\ 1 \end{pmatrix}$. Therefore; $P^{-1} S P = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ where $P = \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix}$. It follows $S^n = P \begin{pmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{pmatrix} P^{-1}$. By equating corresponding

entries of the matrices on both sides and by using the matrix identity (2.1), we obtain the result. $\square$

3. LUCAS CASE

In this section, let $b_1 = b_2$. We also have a similar matrix identity for the conditional Lucas sequences $\{l_n\}$. Similar to the well-known result,

$$\begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$$

we also have

$$\begin{pmatrix} l_{2(n+1)} & -Bl_{2n} \\ l_{2n} & -Bl_{2(n-1)} \end{pmatrix} = \begin{pmatrix} A & -2B \\ 2 & -A \end{pmatrix} \begin{pmatrix} A & -B \\ 1 & 0 \end{pmatrix}^n \quad (3.1)$$

which can be easily proven by induction and by the help of (1.2).

Theorem 3. *The sequence $\{l_n\}$ satisfy*

$$l_{2(n-1)}l_{2(n+1)} - l_{2n}^2 = B^{n-1}(\alpha - \beta)^2$$

for any positive integer n .

Proof. By taking the determinant of both sides of (3.1), we get

$$(l_{2(n-1)}l_{2(n+1)} - l_{2n}^2) = B^{n-1}(A^2 - 4B)$$

and since $A^2 - 4B = (\alpha - \beta)^2$, we get the result which is Cassini's identity for even indices of $\{l_n\}$. $\square$

4. FIBONACCI-LUCAS CASE

Here, we also assume that $b_1 = b_2$ and $\Delta := A^2 - 4B \neq 0$. We can also use matrix techniques to prove some relations between the $\{f_n\}$ and $\{l_n\}$.

It is known that for the matrix $R := \frac{1}{2} \begin{pmatrix} 1 & 5 \\ 1 & 1 \end{pmatrix}$, $R^n = \frac{1}{2} \begin{pmatrix} L_n & 5F_n \\ F_n & L_n \end{pmatrix}$.

Similar to this well-known result, setting $T := \frac{1}{2} \begin{pmatrix} A & \Delta \\ 1 & A \end{pmatrix}$, we also have

$$T^n = \frac{1}{2} \begin{pmatrix} l_{2n} & \Delta \frac{f_{2n}}{a_1} \\ \frac{f_{2n}}{a_1} & l_{2n} \end{pmatrix} \quad (4.1)$$

which can be easily proven by induction and by the help of (1.5) and (1.7).

Theorem 4. *The sequences $\{f_n\}$ and $\{l_n\}$ satisfy*

$$4B^n = l_{2n}^2 - \frac{\Delta}{a_1^2} f_{2n}^2 \quad (4.2)$$

$$2f_{2(m+n)} = f_{2m}l_{2n} + f_{2n}l_{2m} \quad (4.3)$$

$$2l_{2(m+n)} = l_{2m}l_{2n} + \frac{\Delta}{a_1^2} f_{2n} f_{2m} \quad (4.4)$$

$$2B^m f_{2(n-m)} = f_{2n}l_{2m} - l_{2n} f_{2m} \quad (4.5)$$

$$2B^m l_{2(n-m)} = l_{2n}l_{2m} - \frac{\Delta}{a_1^2} f_{2n} f_{2m} \quad (4.6)$$

$$l_{2n}l_{2m} = l_{2(n+m)} + B^m l_{2(n-m)} \quad (4.7)$$

$$f_{2n}l_{2m} = f_{2(n+m)} + B^m f_{2(n-m)} \quad (4.8)$$

for any positive integers m and n .

Proof. By taking the determinant of both sides of (4.1), we get the result (4.2). Considering $T^{n+m} = T^n T^m$ and by using the matrix identity (4.1), we get;

$$\begin{aligned} & \frac{1}{2} \begin{pmatrix} l_{2(n+m)} & \Delta \frac{f_{2(n+m)}}{a_1} \\ \frac{f_{2(n+m)}}{a_1} & l_{2(n+m)} \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} l_{2n}l_{2m} + \frac{\Delta}{a_1^2} f_{2n} f_{2m} & \frac{\Delta}{a_1} (f_{2m}l_{2n} + f_{2n}l_{2m}) \\ \frac{1}{a_1} (f_{2n}l_{2m} + f_{2m}l_{2n}) & \frac{\Delta}{a_1^2} f_{2m} f_{2n} + l_{2n}l_{2m} \end{pmatrix}. \end{aligned}$$

By equating the corresponding entries of the matrices on both sides, we get the identities (4.3) and (4.4).

Similarly, considering $T^{n-m} = T^n T^{-m} = T^n (T^m)^{-1}$ and by using the matrix identity (4.1), we get;

$$\begin{aligned} & \frac{1}{2} \begin{pmatrix} l_{2(n-m)} & \frac{\Delta}{a_1} f_{2(n-m)} \\ \frac{1}{a_1} f_{2(n-m)} & l_{2(n-m)} \end{pmatrix} \\ &= \frac{1}{4B^m} \begin{pmatrix} l_{2n}l_{2m} - \frac{\Delta}{a_1^2} f_{2n} f_{2m} & \frac{-\Delta}{a_1} (l_{2n} f_{2m} - f_{2n}l_{2m}) \\ \frac{-1}{a_1} (l_{2n} f_{2m} - f_{2n}l_{2m}) & l_{2n}l_{2m} - \frac{\Delta}{a_1^2} f_{2n} f_{2m} \end{pmatrix} \end{aligned}$$

By equating the corresponding entries of the matrices on both sides, we get the results (4.5) and (4.6).

Finally, consider the equation

$$T^{n+m} + B^m T^{n-m} = T^n T^m + B^m T^n (T^m)^{-1}$$

then we get (4.7) and (4.8). □

Up to now, we only consider the even indices of the conditional sequences. Now, we develop another matrix identity for both even and odd indices of Fibonacci and Lucas conditional sequences.

We know that there is a relation between continued fractions and 2×2 matrices. By considering this relation and setting

$$C := \begin{pmatrix} a_2 & b_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_2 & a_2 b_1 \\ a_1 & b_1 \end{pmatrix}$$

then we have

$$C^n = \begin{pmatrix} f_{2n+1} & \frac{a_2 b_1}{a_1} f_{2n} \\ f_{2n} & \frac{b_1}{a_1} (f_{2n} - b_2 f_{2(n-1)}) \end{pmatrix} \quad (4.9)$$

by induction and also by the help of (1.1) and (1.2).

We can get the Binet formula for odd and even indices of $\{f_n\}$ by using this matrix identity. For this, again we assume $A^2 - 4B \neq 0$ to guarantee the existence of two distinct roots. Let α and β denote the roots of $p(z) = z^2 - Az + B$ that is the characteristic polynomial of C . The eigenvalues and corresponding eigenvectors of the matrix

C are $\lambda_1 = \alpha$, $\lambda_2 = \beta$ and $v_1 = \begin{pmatrix} \frac{\alpha - b_1}{a_1} \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} \frac{\beta - b_1}{a_1} \\ 1 \end{pmatrix}$. Therefore; $P^{-1}CP = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ where $P = \begin{pmatrix} \frac{\alpha - b_1}{a_1} & \frac{\beta - b_1}{a_1} \\ 1 & 1 \end{pmatrix}$. It follows $C^n = P \begin{pmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{pmatrix} P^{-1}$.

By equating (1, 1) and (2, 1) entries of the matrices on both sides, we obtain the Binet formula for odd and even indices of $\{f_n\}$.

By taking the determinat of both sides of (4.9), we get

$$\frac{b_1}{a_1} (f_{2n+1} [f_{2n} - b_2 f_{2(n-1)}] - a_2 f_{2n}^2) = B^n. \quad (4.10)$$

For $a_1 = a_2 = b_1 = b_2 = 1$ and $2n = m$; the above result reduces to the Cassini's identity for the ordinary Fibonacci sequence:

$$F_{m+1} F_{m-1} - F_m^2 = (-1)^m.$$

By using the matrix identity (4.9), if we consider $C^{n-m} = C^n C^{-m}$ and equating the corresponding entries of the matrices on both sides, we get the formula;

$$B^m f_{2(n-m)} = f_{2n} f_{2m+1} - f_{2m} f_{2n+1}. \quad (4.11)$$

For $a_1 = a_2 = b_1 = b_2 = 1$ and $2n = r$, $2m = s$; the above result reduces to the d'Ocagne's identity for the ordinary Fibonacci sequence:

$$(-1)^s F_{r-s} = F_r F_{s+1} - F_s F_{r+1}.$$

And if we consider $C^{n+m} = C^n C^m$ and equating the corresponding entries of the matrices on both sides, we get the formulas;

$$f_{2(n+m)} = f_{2n} f_{2m+1} + f_{2m} \left(\frac{b_1}{a_1} f_{2n} - \frac{B}{a_1} f_{2(n-1)} \right) \quad (4.12)$$

$$f_{2(n+m)+1} = f_{2n+1} f_{2m+1} + \frac{a_2 b_1}{a_1} f_{2n} f_{2m}. \quad (4.13)$$

For $a_1 = a_2 = b_1 = b_2 = 1$ and $2n = r$, $2m = s$; the above results reduce to the following well-known results

$$\begin{aligned}F_{r+s} &= F_r F_{s+1} + F_s F_{r-1} \\F_{r+s+1} &= F_{r+1} F_{s+1} + F_r F_s.\end{aligned}$$

We also have a similar matrix identity for both even and odd indices of $\{l_n\}$. For this, assume that $b_1 = b_2$. By induction and by the help of (1.1) and (1.2), we have

$$\begin{pmatrix} a_2 & 2\frac{a_2 b_1}{a_1} \\ 2 & -a_2 \end{pmatrix} \begin{pmatrix} a_1 a_2 + b_2 & a_2 b_1 \\ a_1 & b_1 \end{pmatrix}^n = \begin{pmatrix} l_{2n+1} & \frac{a_2 b_1}{a_1} l_{2n} \\ l_{2n} & \frac{b_1}{a_1} (l_{2n} - b_1 l_{2(n-1)}) \end{pmatrix} \quad (4.14)$$

and by taking the determinat of both sides of (4.14), we get

$$a_1 b_1 (l_{2n+1} [l_{2n} - b_1 l_{2(n-1)}] - a_2 l_{2n}^2) = -(\alpha - \beta)^2 B^n. \quad (4.15)$$

If we take $a_1 = a_2 = b_1 = b_2 = 1$ and $2n = m$ in (4.15), it reduces to the well-known identity for Lucas numbers:

$$L_{m+1} L_{m-1} - L_m^2 = (-1)^{m+1} 5.$$

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ON THE SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR SOME NONLINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS ON $\mathbb{R}_+$

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Abstract. Sufficient conditions guaranteeing the unique existence of a bounded solution of quasi-linear boundary value problems are established and a successive approximation method for constructing the solution is described, together with the proof of the stability of the used method.

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Keywords: boundary value problems, existence, uniqueness and construction of solution, Carathéodory conditions, stability

1. INTRODUCTION

On $\mathbb{R}_+$ we consider the boundary value problem for the quasi-linear differential system:

$$x'(t) = P(t)x(t) + f(t, x(t)), \quad (1.1)$$

$$\ell(x) = h(x), \quad (1.2)$$

where $P \in L_{loc}(\mathbb{R}_+, \mathbb{R}^{n \times n})$, $\ell : C_{loc}(\mathbb{R}_+, \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is linear bounded functional, $h : C_{loc}(\mathbb{R}_+, \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is continuous and generally nonlinear operator and $f : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a function satisfying the local Caratheodory conditions :

- (1) $f(\cdot, x) : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is a measurable function for an arbitrary $x \in \mathbb{R}^n$,
- (2) $f(t, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function for almost every $t \in \mathbb{R}_+$,
- (3) $\forall r > 0 \exists k_r \in L(\mathbb{R}_+; \mathbb{R}_+)$ such that $\|f(t, x)\| \leq k_r(t)$ for almost every $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$ such that $\|x\| \leq r$.

If we put $f(t, x(t)) \equiv q(t)$ and $h(x) \equiv c_0$ we get from the problem (1.1), (1.2) the linear boundary value problem for ordinary differential equations on $\mathbb{R}_+$ which was studied in [3].

The solvability of boundary value problems for ordinary differential systems on $\mathbb{R}_+$ is studied in much bibliography, for example in monograph [1] and publications [3, 5, 7] and in references mentioned there. The construction methods of the solution

of boundary value problems for systems of ordinary linear differential equations on closed interval is studied for example in §4.5. of monograph [4].

In Miskolc Mathematical Notes there are authors studying similar problems, the solutions of quasilinear differential equations on $\mathbb{R}_+$ in publication [2] and problems of integral-differential equations in publication [8].

In this work Theorems 1 and 2 provide the criteria of solvability and uniqueness solvability including the construction method of successive approximations for generally nonlinear boundary value problems on $\mathbb{R}_+$. Theorem 3 proves the stability of the method used for constructing the solution.

By a solution of system (1.1) on $\mathbb{R}_+$ it is understood a local absolutely continuous vector function $x \in C_{loc}(\mathbb{R}_+, \mathbb{R}^n)$ which is bounded on $\mathbb{R}_+$, i.e.

$$\sup\{\|x(t)\| : t \in \mathbb{R}_+\} < +\infty,$$

and satisfies the equation (1.1) for almost every $t \in \mathbb{R}_+$. A solution of system (1.1) satisfying condition (1.2) is called a solution of problem (1.1), (1.2) on $\mathbb{R}_+$.

Basic notation:

$\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, \infty)$,

$\mathbb{R}^n$ – the space of n -dimensional column vectors $x = (x_i)_{i=1}^n$ with the norm

$$\|x\| = \sum_{i=1}^n |x_i| \quad \text{and} \quad |x| = (|x_i|)_{i=1}^n,$$

$\mathbb{R}_+^n = \{x = (x_i)_{i=1}^n \in \mathbb{R}^n : x_i \in \mathbb{R}_+ (i = 1, \dots, n)\}$,

$\mathbb{R}^{n \times n}$ – the space of $n \times n$ matrices $X = (x_{ij})_{i,j=1}^n$ with the norm

$$\|X\| = \sum_{i,j=1}^n |x_{ij}| \quad \text{and} \quad |X| = (|x_{ij}|)_{i,j=1}^n,$$

$\mathbb{R}_+^{n \times n} = \{X = (x_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n} : x_{ij} \in \mathbb{R}_+ (i, j = 1, \dots, n)\}$.

if $x, y \in \mathbb{R}^n$ and $X, Y \in \mathbb{R}^{n \times n}$, then

$$x \leq y \iff y - x \in \mathbb{R}_+^n, \quad X \leq Y \iff Y - X \in \mathbb{R}_+^{n \times n}.$$

For the matrix $X = (x_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n}$ by the notation X_d and $X_{\bar{d}}$ we mean

$$X_d = \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & x_n \end{pmatrix}, \quad X_{\bar{d}} = \begin{pmatrix} 0 & x_{12} & \dots & x_{1n} \\ x_{21} & 0 & \dots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{n1} & x_{n2} & \dots & 0 \end{pmatrix},$$

$\det(X)$ – the determinant of the matrix X ,

$r(X)$ – the spectral radius of the matrix X ,

X^{-1} – the inverse matrix to X ,

E – the unit matrix,

$C(\mathbb{R}_+, \mathbb{R}^n)$ – the space of continuous and bounded vector functions¹ $x : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ with the norm

$$\|x\|_C = \sup\{\|x(t)\| : t \in \mathbb{R}_+\} \quad \text{and} \quad |x|_C = (\|x_i\|_C)_{i=1}^n,$$

$L(\mathbb{R}_+, \mathbb{R}^n)$ – the space of integrable vector functions $x : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ with the norm

$$\|x\|_L = \int_0^{+\infty} \|x(t)\| dt \quad \text{and} \quad |x|_L = (|x_i|_L)_{i=1}^n,$$

$L(\mathbb{R}_+, \mathbb{R}^{n \times n})$ – the space of integrable matrix functions $X : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$ with the norm

$$\|X\|_L = \int_0^{+\infty} \|X(t)\| dt \quad \text{and} \quad |X|_L = (|x_{ij}|_L)_{i,j=1}^n,$$

if $X \in C(\mathbb{R}_+, \mathbb{R}^{n \times n})$ is a matrix function with columns $x_1, \dots, x_n$ and $\ell : C(\mathbb{R}_+, \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is a linear functional, then $\ell(X)$ stands for the matrix function with columns $\ell(x_1), \dots, \ell(x_n)$.

2. RESULTS

To formulate the Theorem on solvability of the problem (1.1), (1.2) we need to formulate the following estimates $\forall x, y \in C_0(\mathbb{R}_+, \mathbb{R}^n)$:

$$|f(t, x) - f(t, y)| \leq \bar{L}(t)|x - y|_C$$

where $\bar{L}(t) \in L(\mathbb{R}_+, \mathbb{R}^{n \times n})$,

$$|h(x) - h(y)| \leq H|x - y|_C$$

where $H \in \mathbb{R}^{n \times n}$ is constant matrix and

$$|\ell(x)| \leq \bar{\ell}(|x|),$$

where $\bar{\ell} : C_0(\mathbb{R}_+, \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is linear positive bounded functional. Moreover we use the notations:

$$\begin{aligned} P_d^* &= \sup \left\{ \int_{t_0}^t P_d(s) ds : t \in \mathbb{R}_+ \right\}, \\ P^* &= \sup \left\{ \left| \int_{t_0}^t \exp \left(\int_s^t P_d(\xi) d\xi \right) |P_{\bar{d}}(s)| ds \right| : t \in \mathbb{R}_+ \right\}, \\ f^* &= \sup \left\{ \left| \int_{t_0}^t \exp \left(\int_s^t P_d(\xi) d\xi \right) |\bar{L}(s)| ds \right| : t \in \mathbb{R}_+ \right\}, \end{aligned}$$

where $t_0 \in \mathbb{R}_+$.

¹A vector or matrix function is said to be continuous, bounded, integrable etc., if such are its elements

Theorem 1. Assume that P_d^* , P^* and $\bar{\ell}(E)$, defined previously, satisfy the condition

$$\|P_d^*\| + \|P^*\| + \|\bar{\ell}(E)\| < +\infty, \quad (2.1)$$

the matrix

$$\Lambda = \ell(\exp \int_{t_0}^{\cdot} P_d(s) ds) \quad \text{is nonsingular} \quad (2.2)$$

and

$$r(S) < 1 \quad (2.3)$$

where

$$S = P^* + f^* + \Theta_d^* \Lambda^{-1} [H + \bar{\ell}(E)(P^* + f^*)], \quad (2.4)$$

and

$$\Theta_d^* = \sup \left\{ \exp \int_{t_0}^t P_d(\xi) d\xi : t \in \mathbb{R}_+ \right\}.$$

Then the problem (1.1), (1.2) has just one solution.

To prove Theorem 1 we shall need to formulate the following statement that corresponds with the methods used in publications of I.Kiguradze (see [4, 5, 7] and the bibliography mentioned there). To get the effective criteria of solvability of the considered problem we use one of the methods presented in §1.3. of [6].

Proposition 1. Every solution of the problem (1.1), (1.2) is also a solution of an operator equation

$$x = Tx \quad (2.5)$$

where

$$\begin{aligned} Tx = & \int_{t_0}^t \exp \left(\int_s^t P_d(\xi) d\xi \right) [P_{\bar{d}}(s)x(s) + f(s, x(s))] ds + \exp \left(\int_{t_0}^t P_d(\xi) d\xi \right) \\ & \times \Lambda^{-1} \left[h(x) - \ell \left(\int_{t_0}^{\cdot} \exp \left(\int_s^{\cdot} P_d(\xi) d\xi \right) [P_{\bar{d}}(s)x(s) + f(s, x(s))] ds \right) \right] \end{aligned} \quad (2.6)$$

where $t_0 \in \mathbb{R}_+$ is an arbitrary fixed point. And from the other side, every solution of the operator equation (2.5) is also a solution of the problem (1.1), (1.2).

Proof. Let $x(t)$ be a solution of the problem (1.1), (1.2). Then by integration of (1.1) we get

$$\begin{aligned} x(t) = & x(t_0) \exp \left(\int_{t_0}^t P_d(\xi) d\xi \right) \\ & + \int_{t_0}^t \exp \left(\int_s^t P_d(\xi) d\xi \right) [P_{\bar{d}}(s)x(s) + f(s, x(s))] ds. \end{aligned} \quad (2.7)$$

To calculate $x(t_0)$ we shall substitute $x(t)$ in the left side of the boundary condition (1.2) and with use of the linearity of the functional ℓ we get

$$h(x) = \ell \left(\exp \left(\int_{t_0}^{\cdot} P_d(\xi) d\xi \right) \right) x(t_0) + \ell \left(\int_{t_0}^{\cdot} \exp \left(\int_s^{\cdot} P_d(\xi) d\xi \right) \left[P_{\bar{d}}(s)x(s) + f(s, x(s)) \right] ds \right).$$

Then according to (2.2) we obtain

$$x(t_0) = \Lambda^{-1} \left(h(x) - \ell \left(\int_{t_0}^{\cdot} \exp \left(\int_s^{\cdot} P_d(\xi) d\xi \right) \left[P_{\bar{d}}(s)x(s) + f(s, x(s)) \right] ds \right) \right).$$

By substituting $x(t_0)$ back in (2.7) we get (2.6).

From the other side, if $x(t)$ is a solution of (2.5) then it satisfies the condition (1.2), because by substituting t_0 for t in (2.6) we get:

$$x(t_0) = \Lambda^{-1} \left[h(x) - \ell \left(x(\cdot) - x(t_0) \exp \left(\int_{t_0}^{\cdot} P_d(\xi) d\xi \right) \right) \right],$$

and then with use of linearity of ℓ we get (1.2). By derivating of (2.5) we get equation (1.1). It means that the solution $x(t)$ is also the solution of the problem (1.1), (1.2). $\square$

Proof of Theorem 1. To deduce the sufficient criterias of the solvability of the equation (2.5) we have to prove the contractivity of the operator $T : C_0(\mathbb{R}_+, \mathbb{R}^n) \rightarrow C_0(\mathbb{R}_+, \mathbb{R}^n)$, where $C_0(\mathbb{R}_+, \mathbb{R}^n)$ is a subspace of the space $C_{loc}(\mathbb{R}_+, \mathbb{R}^n)$ with bounded functions on $\mathbb{R}_+$. This space is a Banach space with a supreme norm and thus the operator T displays a Banach space into itself. Let's make the following estimation:

$$\begin{aligned} |Tx - Ty|_C &\leq \left| \int_{t_0}^t \exp \left(\int_s^t P_d(\xi) d\xi \right) P_{\bar{d}}(s) [x(s) - y(s)] \right|_C \\ &\quad + \left| \int_{t_0}^t \exp \left(\int_s^t P_d(\xi) d\xi \right) [f(s, x(s)) - f(s, y(s))] ds \right|_C \\ &\quad + \left| \exp \left(\int_{t_0}^t P_d(\xi) d\xi \right) \Lambda^{-1} [h(x) - h(y)] - \exp \left(\int_{t_0}^t P_d(\xi) d\xi \right) \Lambda^{-1} \right. \\ &\quad \times \ell \left(\int_{t_0}^t \exp \left(\int_s^t P_d(\xi) d\xi \right) P_{\bar{d}}(s) [x(s) - y(s)] ds \right. \\ &\quad \left. \left. + \int_{t_0}^t \exp \left(\int_s^t P_d(\xi) d\xi \right) [f(s, x(s)) - f(s, y(s))] ds \right) \right|_C \\ &\leq (P^* + f^*)|x - y|_C \\ &\quad + \Theta_d^* |\Lambda^{-1}| \left[H|x - y|_C + \bar{\ell}(E)(P^* + f^*)|x - y|_C \right] \end{aligned}$$

$$= S|x - y|_C$$

where

$$S = P^* + f^* + \Theta_d^* \Lambda^{-1} [H + \bar{\ell}(E)(P^* + f^*)].$$

Thus, for $x, y \in C_0(\mathbb{R}_+, \mathbb{R}^n)$, we get

$$\|Tx - Ty\|_C \leq \|S\| \|x - y\|_C \quad (2.8)$$

and since for S is valid $r(S) < 1$ we get $\|S\| < 1$. Thus we proved that T is a contraction. The existence of the unique solution of the operator equation (2.5) follows from Banach Theorem and by Lemma 1 this solution is also the solution of the problem (1.1), (1.2). $\square$

The construction of the solution of problem (1.1), (1.2) follows from Banach Theorem.

Theorem 2. Suppose that all assumptions of Theorem 1 are fulfilled. Then the problem (1.1), (1.2) has the solution x on $\mathbb{R}_+$ which is a limit of the sequence of the solutions of ancillary system:

$$x'_{m+1}(t) = P_d(t)x_{m+1}(t) + P_{\bar{d}}(t)x_m(t) + f(t, x_m(t)), \quad (2.9)$$

$$\ell(x_{m+1}) = h(x_m), \quad (2.10)$$

where $m \in \mathbb{N}$ and $x_1(t) \in C_0(\mathbb{R}_+, \mathbb{R}^n)$ is arbitrary.

Proof. It is obvious that the solution to the ancillary system (2.9), (2.10) is equivalent with the solution to the operator equation

$$x_{m+1} = Tx_m \quad (2.11)$$

where operator T is defined by (2.6), $m \in \mathbb{N}$ and $t^0 \in \mathbb{R}_+$ is an arbitrary given point. Let $x_1(t)$ be arbitrary. With respect to the contractivity of operator T and to Theorem 1 we shall prove that $x_2 = Tx_1, x_3 = Tx_2, \dots, x_m = Tx_{m-1}, x_{m+1} = Tx_m \dots$ is the Cauchy sequence:

$$\begin{aligned} \|x_{m+1} - x_m\|_C &= \|Tx_m - Tx_{m-1}\|_C \leq \|S\| \|x_m - x_{m-1}\|_C \\ &= \|S\| \|Tx_{m-1} - Tx_{m-2}\|_C \leq \|S\|^2 \|x_{m-1} - x_{m-2}\|_C \dots \\ &\leq \|S\|^{m-1} \|x_2 - x_1\|_C \end{aligned}$$

where $m \in \mathbb{N}$ is arbitrary. By virtue of this we get for $n > m, n \in \mathbb{N}$

$$\begin{aligned} \|x_n - x_m\|_C &\leq \|x_{m+1} - x_m\|_C + \dots + \|x_n - x_{n-1}\|_C \\ &\leq \|S\|^{m-1} \|x_2 - x_1\|_C + \|S\|^m \|x_2 - x_1\|_C + \dots + \|S\|^{n-2} \|x_2 - x_1\|_C \\ &= \|x_2 - x_1\|_C \|S\|^{m-1} (1 + \|S\| + \dots + \|S\|^{n-m-1}) \\ &\leq \|x_2 - x_1\|_C \|S\|^{m-1} (1 + \|S\| + \dots) = \frac{\|S\|^{m-1}}{1 - \|S\|} \|x_2 - x_1\|_C \end{aligned}$$

Let $\epsilon > 0$ be arbitrary. Since $r(S) < 1$ then $\|S\|^{m-1} \rightarrow 0$ for $m \rightarrow \infty$. Thus there exists $n_0 \in \mathbb{N}$ such that for $m \geq n_0$ the following holds:

$$\|S\|^{m-1} \|x_2 - x_1\|_C \frac{1}{1 - \|S\|} < \epsilon.$$

This implies $\|x_n - x_m\|_C < \epsilon$ for $m, n \geq n_0$. Thus we proved that $\{x_m\}$ is a Cauchy sequence. Since C_0 is a complete space $\{x_m\}$ is convergent. That means $x_m \rightarrow x_0$, $x_0 \in C_0$. With respect to the Banach Theorem x_0 is the only one fixed point of the operator T and also the solution of the operator equation (2.11). This solution is also the solution of the problem (2.9), (2.10). $\square$

The following Theorem proves that the process of constructing the solution of problem (2.9), (2.10) described above is stable in some sense, analogously to the assertions published by I.Kiguradze in [4].

Theorem 3. Consider a sequence of vectors $\gamma_m \in \mathbb{R}^n$ ($m = 1, 2, \dots$) and a sequence of vector-valued functions $\Delta_m \in L(\mathbb{R}_+, \mathbb{R}^n)$ ($m = 1, 2, \dots$). Let $\sum_{m=1}^{\infty} \epsilon_m$ be convergent where

$$\epsilon_m = \left(\|\gamma_m\| + \sup \left\{ \int_{t_0}^t \|\Delta_m(s)\| ds : t \in \mathbb{R} \right\} \right)$$

and assume that all the conditions of Theorem 1 are satisfied. Then the solution x on $\mathbb{R}_+$ of problem (1.1), (1.2) is the limit of the sequence of solutions of ancillary system:

$$x'_{m+1}(t) = P_d(t)x_{m+1}(t) + P_{\bar{d}}(t)x_m(t) + f(t, x_m(t)) + \Delta_m(t), \quad (2.12)$$

$$\ell(x_{m+1}) = h(x_m) + \gamma_m, \quad (2.13)$$

where $m \in \mathbb{N}$ and $x_1(t) \in C_0(\mathbb{R}_+, \mathbb{R}^n)$ is arbitrary.

Proof. To prove the theorem we have to show that the operator equation (2.11) enriched with error function doesn't breach the Cauchy property of the sequence occurred in Theorem 2. Therefore we consider the operator equation

$$x_{m+1} = Tx_m + \widehat{T}(\Delta_m, \gamma_m)$$

where operator T is defined by (2.6) and

$$\begin{aligned} \widehat{T}(\Delta_m, \gamma_m) = & \int_{t_0}^t \exp \left(\int_s^t P_d(\xi) d\xi \right) \Delta_m(s) ds \\ & + \exp \left(\int_{t_0}^t P_d(\xi) d\xi \right) \Lambda^{-1} \left[\gamma_m - \ell \left(\int_{t_0}^{\cdot} \exp \left(\int_s^{\cdot} P_d(\xi) d\xi \right) \Delta_m(s) ds \right) \right] \end{aligned}$$

where $m \in \mathbb{N}$ and $t^0 \in \mathbb{R}_+$ is an arbitrary given point. Now, to prove the stability of the solution we show that also the sequence $x_2 = Tx_1 + \widehat{T}(\Delta_1, \gamma_1), \dots, x_m = Tx_{m-1} + \widehat{T}(\Delta_{m-1}, \gamma_{m-1}), x_{m+1} = Tx_m + \widehat{T}(\Delta_m, \gamma_m) \dots$ is a Cauchy sequence,

with respect to Theorem 1, 2 and to the contractivity of operator $(T + \widehat{T})$, analogously to the proof of Theorem 2. In addition to (2.8) we need to make the following estimation:

$$\begin{aligned}
& \|\widehat{T}(\Delta_m, \gamma_m) - \widehat{T}(\Delta_{m-1}, \gamma_{m-1})\| \\
& \leq \left\| \int_{t_0}^t \exp\left(\int_s^t P_d(\xi) d\xi\right) [\Delta_m(s) ds - \Delta_{m-1}(s)] ds \right\| \\
& \quad + \left\| \exp\left(\int_{t_0}^t P_d(\xi) d\xi\right) \Lambda^{-1} [\gamma_m - \gamma_{m-1}] \right\| \\
& \quad + \left\| \exp\left(\int_{t_0}^t P_d(\xi) d\xi\right) \Lambda^{-1} \ell\left(\int_{t_0}^t \exp\left(\int_s^t P_d(\xi) d\xi\right) [\Delta_m(s) - \Delta_{m-1}(s)] ds\right) \right\| \\
& \leq \|\Theta_d^* |\Lambda^{-1}| [\gamma_m - \gamma_{m-1}]\| \\
& \quad + \|\Theta_d^* (1 + |\Lambda^{-1}| \bar{\ell}(E) \Theta_d^*) \int_{t_0}^t [\Delta_m(s) ds - \Delta_{m-1}(s)] ds \| \\
& \leq \|c^*\| \|\gamma_m - \gamma_{m-1}\| + \|S_\Delta\| \int_{t_0}^t \|\Delta_m(s) ds - \Delta_{m-1}(s)\| ds
\end{aligned}$$

where

$$c^* = \Theta_d^* |\Lambda^{-1}|, \quad S_\Delta = \Theta_d^* (1 + |\Lambda^{-1}| \bar{\ell}(E) \Theta_d^*).$$

Then

$$\begin{aligned}
\|x_{m+1} - x_m\|_C &= \|Tx_m + \widehat{T}(\Delta_m, \gamma_m) - (Tx_{m-1} + \widehat{T}(\Delta_{m-1}, \gamma_{m-1}))\|_C \\
&\leq \|Tx_m - Tx_{m-1}\|_C + \|\widehat{T}(\Delta_m, \gamma_m) - \widehat{T}(\Delta_{m-1}, \gamma_{m-1})\| \\
&\leq \|S\| \|x_m - x_{m-1}\| + \|c^*\| \|\gamma_m - \gamma_{m-1}\| \\
&\quad + \|S_\Delta\| \int_{t_0}^t \|\Delta_m(s) ds - \Delta_{m-1}(s)\| ds \\
&\leq \|S\| \|Tx_{m-1} - Tx_{m-2}\|_C + \|\widehat{T}(\Delta_{m-1}, \gamma_{m-1}) - \widehat{T}(\Delta_{m-2}, \gamma_{m-2})\| \\
&\quad + \|c^*\| \|\gamma_m - \gamma_{m-1}\| + \|S_\Delta\| \int_{t_0}^t \|\Delta_m(s) ds - \Delta_{m-1}(s)\| ds \leq \dots \\
&\leq \|S\|^{m-1} \|x_2 - x_1\| \\
&\quad + \|c^*\| \sum_{i=2}^m \|\gamma_i - \gamma_{i-1}\| + \|S_\Delta\| \sum_{i=2}^m \int_{t_0}^t \|\Delta_i(s) - \Delta_{i-1}(s)\| ds
\end{aligned}$$

where $m \in \mathbb{N}$ is arbitrary. For $n > m, n \in \mathbb{N}$ we get

$$\|x_n - x_m\|_C = \frac{\|S\|^{m-1}}{1 - \|S\|} \|x_2 - x_1\|_C + \|c^*\| \sum_{i=2}^m \|\gamma_i - \gamma_{i-1}\|$$

$$\begin{aligned}
& + \|S_\Delta\| \sum_{i=2}^m \int_{t_0}^t \|\Delta_i(s) - \Delta_{i-1}(s)\| ds \\
& + \|c^*\| \sum_{i=2}^{m+1} \|\gamma_i - \gamma_{i-1}\| + \|S_\Delta\| \sum_{i=2}^{m+1} \int_{t_0}^t \|\Delta_i(s) - \Delta_{i-1}(s)\| ds + \\
& \vdots \\
& + \|c^*\| \sum_{i=2}^{n-1} \|\gamma_i - \gamma_{i-1}\| + \|S_\Delta\| \sum_{i=2}^{n-1} \int_{t_0}^t \|\Delta_i(s) - \Delta_{i-1}(s)\| ds \\
& + \|c^*\| \sum_{i=2}^n \|\gamma_i - \gamma_{i-1}\| + \|S_\Delta\| \sum_{i=2}^n \int_{t_0}^t \|\Delta_i(s) - \Delta_{i-1}(s)\| ds \\
& \leq \frac{\|S\|^{m-1}}{1 - \|S\|} \|x_2 - x_1\|_C \\
& + \sum_{j=m}^n \left(\|c^*\| \|\gamma_j\| + \|S_\Delta\| \sup \left\{ \int_{t_0}^t \|\Delta_j(s)\| ds : t \in \mathbb{R} \right\} \right).
\end{aligned}$$

Convergency of $\sum_{j=1}^n \epsilon_j$ implies the convergency of the series of both addends of ϵ_m and their linear combinations as well. Then by virtue of Theorem 2 the sequence $\{x_m\} = \{Tx_m + \widehat{T}(\Delta_m, \gamma_m)\}$ is Cauchy and $x_m = Tx_m + \widehat{T}(\Delta_m, \gamma_m) \rightarrow x_0 \in C_0$ which is the solution of the problem (2.12), (2.13). $\square$

Remark 1. If $\gamma_m = 0$ and $\Delta_m \equiv 0$ then from Theorem 3 we get immediately Theorem 2. Thus Theorem 2 could be perceived as the corollary of Theorem 3. For clearer view on calculations in proofs of both theorems it is convenient to formulate first Theorem 2 separately and then Theorem 3 including the error function.

All consequences for linear boundary value problems published in [3] and analogous statements for solutions of linear boundary value problems for nonlinear systems of ordinary differential equations follow directly from Theorem 1 and Theorem 2.

The following example illustrates the case of nonlinear boundary value problem for nonlinear system of ordinary differential equations:

$$\begin{aligned}
x_i'(t) &= -x_i(t) + g_i(t) \sin x_i(t) + q_i(t), \quad (i = 1, \dots, n), \\
x_i(0) &= c_i \cos x_i(1) + c_{0i}, \quad (i = 1, \dots, n),
\end{aligned}$$

where $g_i, q_i \in L_{loc}(\mathbb{R}_+, \mathbb{R})$, $c_i, c_{0i} \in \mathbb{R}$ ($i = 1, \dots, n$). Then we have

$$\begin{aligned}
P(t) &\equiv -E \equiv P_d(t), \quad P_{\bar{d}}(t) \equiv 0, \\
f(t, x) &= (g_i(t) \sin x_i(t) + q_i(t))_{i=1}^n,
\end{aligned}$$

$$l(x) = (x_i(0))_{i=1}^n, \quad h(x) = (c_i \cos x_i(1) + c_{0i})_{i=1}^n$$

and then

$$|f(t, x)| \leq (|g_i(t)| + |q_i(t)|)_{i=1}^n, \quad |h(x)| \leq (|c_i| + |c_{0i}|)_{i=1}^n, \quad \bar{L}(t) \equiv 0, \quad \bar{h} = 0.$$

Because $P_d^* = 0, P^* = 0, \bar{\ell}(E) = E$ then it follows that condition (2.1) is fulfilled on assumption

$$\|g\|_L + \|q\|_L < +\infty.$$

Then $S = 0$ which means that the differential equation has a unique solution.

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PERMUTING TRIDERIVATIONS OF PRIME AND SEMIPRIME RINGS

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Abstract. The purpose of this paper is to prove some results concerning permuting triderivations and permuting generalized triderivations on prime and semiprime rings which partially extend some results contained in [9] and [10].

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1. INTRODUCTION AND PRELIMINARIES

Throughout this work, R will represent an associative ring and Z will denote the center of R . We shall write $[x, y]$ for $xy - yx$. Recall that a ring R is prime, if $xRy = \{0\}$ implies $x = 0$ or $y = 0$, and it is semiprime if $xRx = \{0\}$ implies $x = 0$.

It is very interesting and important that the similar properties of derivation which is the one of the basic theory in analysis and applied mathematics are also satisfied in the ring theory. The commutativity of prime rings with derivations was introduced by Posner in [12]. An additive map $d : R \rightarrow R$ is called derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Recently, a lot of work has been done on commutativity of prime rings with derivation (see [1], [2], [3], [14]).

In [4], Bresar defined concept of generalized derivation. An additive map $d : R \rightarrow R$ is called generalized derivation if there exists a derivation α of R such that $d(xy) = d(x)y + x\alpha(y)$ for all $x, y \in R$. Thus the concept of generalized derivation contains both the concepts of a derivation and of a left multiplier (i.e., additive maps satisfying $f(xy) = f(x)y$ for all $x, y \in R$). Basic examples are derivations and generalized inner derivations (i.e., maps of type $x \mapsto ax + xb$ for some $a, b \in R$). In [4], Bresar showed that if R has the property that $Rx = \{0\}$ implies $x = 0$ and $h : R \rightarrow R$ is any function, $d : R \rightarrow R$ is any additive map satisfying $d(xy) = d(x)y + xh(y)$ for all $x, y \in R$, then d is uniquely determined by h and moreover h must be derivation. In [5], Bresar defined concept of generalized bi-derivation and investigated some its properties.

In [7] and [8], Maksa defined biderivation in ring theory mutually to partial derivations and examined some properties of this derivation. A map $D(.,.) : R \times R \rightarrow R$ is said to be symmetric if $D(x, y) = D(y, x)$ for all $x, y \in R$. A map $d : R \rightarrow R$ defined by $d(x) = D(x, x)$ is called the trace of $D(.,.)$ where $D(.,.) : R \times R \rightarrow R$ is a symmetric map. It is clear that if $D(.,.)$ is biadditive (i.e., additive in all arguments), then the trace d of $D(.,.)$ satisfies the identity $d(x + y) = d(x) + d(y) + 2D(x, y)$ for all $x, y \in R$. A symmetric biadditive map $D(.,.) : R \times R \rightarrow R$ is called symmetric biderivation if $D(xz, y) = D(x, y)z + xD(z, y)$ for all $x, y, z \in R$. For any $y \in R$, the map $x \mapsto D(x, y)$ is a derivation. The trace of $D(.,.)$ is an even function. In [11] and [13], the authors investigated some properties of symmetric biderivation.

In [10], Öztürk defined concept of permuting triderivation. A map $D(.,.,.) : R \times R \times R \rightarrow R$ is called permuting if $D(x, y, z) = D(x, z, y) = D(z, x, y) = D(z, y, x) = D(y, z, x) = D(y, x, z)$ hold for all $x, y, z \in R$. A map $d : R \rightarrow R$ defined by $d(x) = D(x, x, x)$ is called trace of $D(.,.,.)$, where $D(.,.,.) : R \times R \times R \rightarrow R$ is a permuting map. It is obvious that, if $D(.,.,.) : R \times R \times R \rightarrow R$ is permuting triadditive (i.e., additive in both arguments), then the trace of $D(.,.,.)$ satisfies the relation $d(x + y) = d(x) + d(y) + 3D(x, x, y) + 3D(x, y, y)$ for all $x, y \in R$. A permuting triadditive map $D(.,.,.) : R \times R \times R \rightarrow R$ is called permuting triderivation if $D(xw, y, z) = D(x, y, z)w + xD(w, y, z)$ for all $x, y, z, w \in R$. The trace of $D(.,.,.)$ is an odd function. Let $D(.,.,.)$ be a permuting triderivation of R . In this case, for any fixed $a \in R$ and for all $x, y \in R$, a map $D_1(.,.) : R \times R \rightarrow R$ defined by $D_1(x, y) = D(a, x, y)$ and a map $d_2 : R \rightarrow R$ defined by $d_2(x) = D(a, a, x)$ are a symmetric biderivation (in this meaning, permuting 2-derivation is a symmetric biderivation) and a derivation, respectively. In [9] and [15], the authors investigated some properties of permuting triderivation.

Lemma 1 ([6, page 6, Corollary 2]). *If R is a semiprime ring and I is an ideal of R , then $I \cap \text{Ann} I = \{0\}$.*

Lemma 2 ([10, Lemma 5]). *Let R be a 2,3-torsion free ring, D a permuting triadditive map of R and d the trace of D . If $d(x) = 0$ for all $x \in R$, then $D = 0$.*

In this paper, our aim is to prove some results concerning permuting triderivations and permuting generalized triderivations on prime and semiprime rings which partially extend some results contained in [9] and [10].

2. RESULTS

Lemma 3. *Let R be a 2,3-torsion free prime ring and $\{0\} \neq I$ be an ideal of R . If D is a permuting triderivation such that $D(x, x, x) = 0$ for all $x \in I$, then either $D = 0$ or R is commutative.*

Proof. Let $D(x, x, x) = 0$ for all $x \in I$. Replacing x by $x + y$ for all $x, y \in I$, we get

$$D(x, x, y) + D(x, y, y) = 0 \quad (2.1)$$

Replacing y by $-y$ in 2.1 and comparing with 2.1, we have

$$D(x, x, y) = 0, \text{ for all } x, y \in I.$$

Replacing y by ry , $r \in R$, we get $D(x, x, r)y = 0$. Replacing x by $x + z$, $z \in I$, we get $D(x, z, r)y = 0$. Substituting sx for x , we obtain that $D(s, z, r)xy = 0$, $s \in R$. Substituting tz for z , we get $D(s, t, r)zxy = 0$. From here $D(s, t, r)I[x, y] = \{0\}$. Primeness of R yields $[x, y] = 0$ or $D(s, t, r) = 0$, for all $x, y \in I$ and $s, t, r \in R$. Thus we have $[I, I] = \{0\}$ and R is commutative or $D = 0$. $\square$

Theorem 1. *Let R be a 2, 3-torsion free prime ring and $\{0\} \neq I$ be an ideal of R . If D is a permuting triderivation with trace d such that $D(d(x), x, x) = 0$ for all $x \in I$, then either $D = 0$ or R is commutative ring.*

Proof. For any $x, y \in I$,

$$D(d(x+y), x+y, x+y) + D(d(-x+y), -x+y, -x+y) = 0$$

and since R is 2, 3-torsion free we get

$$\begin{aligned} 0 &= 2D(d(x), x, y) + D(d(y), x, x) + 6D(D(x, y, y), x, y) \\ &\quad + 3D(D(x, x, y), x, x) + 3D(D(x, x, y), y, y). \end{aligned} \quad (2.2)$$

Replacing y by $y + z$ in 2.2 and using 2.2, we obtain

$$\begin{aligned} 0 &= D(D(y, z, z), x, x) + D(D(y, y, z), x, x) + 4D(D(x, y, z), x, y) \\ &\quad + 2D(D(x, y, y), x, z) + 4D(D(x, y, z), x, z) + 2D(D(x, z, z), x, y) \\ &\quad + 2D(D(x, y, y), x, z) + D(D(x, x, y), z, z) + D(D(x, x, z), y, y) \\ &\quad + 2D(D(x, x, z), y, z) \end{aligned} \quad (2.3)$$

since R is 3-torsion free.

Replacing y by $-y$ in 2.3 and comparing with 2.3, we get

$$\begin{aligned} 0 &= D(D(y, y, z), x, x) + 4D(D(x, y, z), x, y) + 4D(D(x, y, y), x, z) \\ &\quad + D(D(x, x, z), y, y) \end{aligned} \quad (2.4)$$

since R is 2-torsion free.

Replacing z by yz in 2.4 and using 2.4, we get

$$\begin{aligned} 0 &= d(y)D(x, x, z) + D(x, x, y)D(y, y, z) + 4D(x, y, y)D(x, y, z) \\ &\quad + 4D(x, x, y)D(x, y, z) + D(x, x, y)D(y, y, z) + d(y)D(x, x, z). \end{aligned} \quad (2.5)$$

Replacing y by x in 2.5, we get

$$d(x)D(x, x, z) = 0, \text{ for all } x, z \in I \quad (2.6)$$

since R is 2, 3-torsion free.

Replacing z by yz , $y \in I$ in 2.6 and using 2.6, we have

$$d(x)yD(x, x, z) = 0, \text{ for all } x, y, z \in I.$$

Therefore $d(x)yd(x) = 0$ for all $x, y \in I$. And so, $yd(x)Ryd(x) = \{0\}$. Since R is prime ring, we get $yd(x) = 0$ for all $x, y \in I$. That is, $d(x) = 0$ for all $x \in I$. From Lemma 3, we obtain that $D = 0$ or R is commutative ring. $\square$

Theorem 2. *Let R be a 2,3,5-torsion free prime ring and I be an ideal of R . If D_1, D_2 are the permuting triderivations of R with traces d_1, d_2 respectively, such that $d_1(x)d_2(x) = 0$ for all $x \in I$, then either $D_1 = 0$ or $D_2 = 0$ unless R is commutative.*

Proof. Assume that R is not commutative. Then linearization of the relation

$$d_1(x)d_2(x) = 0 \text{ for all } x \in I \quad (2.7)$$

give as

$$\begin{aligned} 0 = & d_1(x)d_2(y) + 3d_1(x)D_2(x, x, y) + 3d_1(x)D_2(x, y, y) \\ & + d_1(y)d_2(x) + 3d_1(y)D_2(x, x, y) + 3d_1(y)D_2(x, y, y) \\ & + 3D_1(x, x, y)d_2(x) + 3D_1(x, x, y)d_2(y) + 9D_1(x, x, y)D_2(x, x, y) \\ & + 9D_1(x, x, y)D_2(x, y, y) + 3D_1(x, y, y)d_2(x) + 3D_1(x, y, y)d_2(y) \\ & + 9D_1(x, y, y)D_2(x, x, y) + 9D_1(x, y, y)D_2(x, y, y). \end{aligned} \quad (2.8)$$

Replacing y by $-y$ in 2.8, and comparing with 2.8, we have

$$\begin{aligned} 0 = & d_1(x)D_2(x, y, y) + d_1(y)D_2(x, x, y) + D_1(x, x, y)d_2(y) \\ & + 3D_1(x, x, y)D_2(x, x, y) + D_1(x, y, y)d_2(x) + 3D_1(x, y, y)D_2(x, y, y), \end{aligned} \quad (2.9)$$

since R is 2,3-torsion free.

Replacing y by $y + z$, $z \in I$ in 2.9 and using 2.9, we get

$$\begin{aligned} 0 = & 2d_1(x)D_2(x, y, z) + d_1(y)D_2(x, x, z) + d_1(z)D_2(x, x, y) \\ & + 3D_1(y, y, z)D_2(x, x, y) + 3D_1(y, y, z)D_2(x, x, z) + 3D_1(y, z, z)D_2(x, x, y) \\ & + 3D_1(y, z, z)D_2(x, x, z) + D_1(x, x, y)d_2(z) + 3D_1(x, x, y)D_2(y, y, z) \\ & + 3D_1(x, x, y)D_2(y, z, z) + D_1(x, x, z)d_2(y) + 3D_1(x, x, z)D_2(y, y, z) \\ & + 3D_1(x, x, z)D_2(y, z, z) + 3D_1(x, x, y)D_2(x, x, z) + 3D_1(x, x, z)D_2(x, x, y) \\ & + 2D_1(x, y, z)d_2(x) + 6D_1(x, y, y)D_2(x, y, z) + 3D_1(x, y, y)D_2(x, z, z) \\ & + 6D_1(x, y, z)D_2(x, y, y) + 12D_1(x, y, z)D_2(x, y, z) + 6D_1(x, y, z)D_2(x, z, z) \\ & + 3D_1(x, z, z)D_2(x, y, y) + 6D_1(x, z, z)D_2(x, y, z). \end{aligned} \quad (2.10)$$

Replacing y by $-y$ in 2.10 and comparing with 2.10, we get

$$\begin{aligned} 0 = & D_1(y, y, z)D_2(x, x, z) + D_1(y, z, z)D_2(x, x, y) + D_1(x, x, y)D_2(y, z, z) \\ & + D_1(x, x, z)D_2(y, y, z) + D_1(x, y, y)D_2(x, z, z) + 4D_1(x, y, z)D_2(x, y, z) \\ & + D_1(x, z, z)D_2(x, y, y), \end{aligned} \quad (2.11)$$

since R is 2,3-torsion free.

Replacing z by $z + w$, $w \in I$ in 2.11 and using 2.11, we get

$$\begin{aligned} 0 = & D_1(y, y, z) D_2(x, x, w) + D_1(y, y, w) D_2(x, x, z) + 2D_1(y, z, w) D_2(x, x, y) \\ & + 2D_1(x, x, y) D_2(y, z, w) + D_1(x, x, w) D_2(y, y, z) + D_1(x, x, z) D_2(y, y, w) \\ & + 2D_1(x, y, y) D_2(x, z, w) + 4D_1(x, y, z) D_2(x, y, w) + 4D_1(x, y, w) D_2(x, y, z) \\ & + 2D_1(x, z, w) D_2(x, y, y). \end{aligned} \quad (2.12)$$

Replacing y, z by x in 2.12 and using 2.12, we have

$$0 = d_1(x) D_2(x, x, w) + D_1(x, x, w) d_2(x), \quad (2.13)$$

since R is 2, 5-torsion free.

Replacing w by wt , $t \in I$ in 2.13, we have

$$\begin{aligned} 0 = & d_1(x) D_2(x, x, w)t + d_1(x) w D_2(x, x, t) \\ & + D_1(x, x, w) t d_2(x) + w D_1(x, x, t) d_2(x). \end{aligned} \quad (2.14)$$

Using 2.13 in 2.14, we get

$$0 = D_1(x, x, w) [t, d_2(x)] + [d_1(x), w] D_2(x, x, t). \quad (2.15)$$

Replacing w by $d_1(x)w$ in 2.15 and using 2.15, we get

$$D_1(x, x, d_1(x)) w [t, d_2(x)] = 0 \text{ for all } x, w, t \in I.$$

This implies that $D_1(x, x, d_1(x)) R w [t, d_2(x)] = \{0\}$ for all $x, w, t \in I$. Primeness of R yields that either $D_1(x, x, d_1(x)) = 0$ or $w [t, d_2(x)] = 0$ for all $x, w, t \in I$. If $D_1(x, x, d_1(x)) = 0$ for all $x \in I$, then conclusion follows from Theorem 1. Now consider the case when $w [t, d_2(x)] = 0$ for all $x, w, t \in I$. Primeness of R yields that $[t, d_2(x)] = 0$ for all $x, t \in I$. By linearizing we get $[t, D_2(x, x, y)] = 0$ for all $x, y, t \in I$. Replacing y by yz , we have $D_2(x, x, y) [t, z] + [t, y] D_2(x, x, z) = 0$ for all $x, y, z, t \in I$. In particular, $[t, y] D_2(x, x, y) = 0$ for all $x, y, t \in I$. This implies that $[t, y] w D_2(x, x, y) = 0$ for all $x, y, t, w \in I$. Since R is not commutative, we also have $[I, I] \neq \{0\}$. Hence primeness of R yields that $D_2(x, x, y) = 0$ for all $x, y \in I$. From Lemma 3, we get $D_2 = 0$. $\square$

Definition 1. Let R be a ring and $D : R \times R \times R \rightarrow R$ be a triadditive map. A triadditive map $\Delta : R \times R \times R \rightarrow R$ is called generalized derivation of R associated with D if for every $x, y \in R$, the map $z \mapsto \Delta(x, y, z)$ is a generalized derivation of R associated with D , for every $y, z \in R$, the map $x \mapsto \Delta(x, y, z)$ is a generalized derivation of R associated with D and for every $x, z \in R$, the map $y \mapsto \Delta(x, y, z)$ is a generalized derivation of R associated with D , i. e. for all $x, y, z, w \in R$,

$$\begin{aligned} \Delta(xw, y, z) &= \Delta(x, y, z)w + xD(w, y, z) \\ \Delta(x, yw, z) &= \Delta(x, y, z)w + yD(x, w, z) \\ \Delta(x, y, zw) &= \Delta(x, y, z)w + zD(x, y, w). \end{aligned}$$

Example 1. Let R be a ring. If D is any triderivation of R and $\alpha : R \times R \times R \rightarrow R$ is a triadditive map such that $\alpha(x, y, zw) = \alpha(x, y, z)w$, $\alpha(x, yw, z) = \alpha(x, y, z)w$ and $\alpha(xw, y, z) = \alpha(x, y, z)w$ for all $x, y, z, w \in R$, then $D + \alpha$ is a generalized D triderivation of R .

Lemma 4. Let R be a semiprime ring. Let $D : R \times R \times R \rightarrow R$ and $\Delta : R \times R \times R \rightarrow R$ be two triadditive maps. If Δ is a generalized triderivation associated with D , then D is a triderivation.

Proof. We compute $\Delta((xy)z, w, t)$ in two different ways. Then we have

$$\begin{aligned}\Delta((xy)z, w, t) &= \Delta(xy, w, t)z + xyD(z, w, t) \\ &= \Delta(x, w, t)yz + xD(y, w, t)z + xyD(z, w, t)\end{aligned}$$

and

$$\Delta(x(yz), w, t) = \Delta(x, w, t)yz + xD(yz, w, t)$$

for all $x, y, z, w, t \in R$. Comparing relations we obtain

$$x(D(y, w, t)z + yD(z, w, t) - D(yz, w, t)) = 0.$$

Since R is semiprime, we get $D(yz, w, t) = D(y, w, t)z + yD(z, w, t)$ for all $y, z, w, t \in R$. Similarly we obtain $D(x, zw, t) = D(x, z, t)w + zD(x, w, t)$ and $D(x, z, wt) = D(x, z, w)t + wD(x, w, t)$ for all $y, z, w, t \in R$. $\square$

Theorem 3. Let R be a 2, 3-torsion free prime ring and I be a nonzero ideal of R . If Δ is a permuting generalized triderivation associated with permuting triderivation D of R with trace d such that $\Delta(d(x), x, x) = 0$ for all $x \in I$, then the following hold:

- (i) $D = 0$,
- (ii) R is commutative,
- (iii) d is commuting on I .

Proof. Let

$$\Delta(d(x), x, x) = 0, \text{ for all } x \in I \quad (2.16)$$

Replacing x by $x + y$, $y \in I$ in 2.16 and using 2.16, we get

$$\begin{aligned}0 &= 2\Delta(d(x), x, y) + \Delta(d(x), y, y) + \Delta(d(y), x, x) + 2\Delta(d(y), x, y) \\ &\quad + 3\Delta(D(x, x, y), x, x) + 6\Delta(D(x, x, y), x, y) + 3\Delta(D(x, x, y), y, y) \\ &\quad + 3\Delta(D(x, y, y), x, x) + 6\Delta(D(x, y, y), x, y) + 3\Delta(D(x, y, y), y, y).\end{aligned} \quad (2.17)$$

Replacing y by $-y$ in 2.17 and comparing with 2.17, we have

$$\begin{aligned}0 &= \Delta(d(x), y, y) + 2\Delta(d(y), x, y) + 6\Delta(D(x, x, y), x, y) \\ &\quad + 3\Delta(D(x, y, y), x, x) + 3\Delta(D(x, y, y), y, y),\end{aligned} \quad (2.18)$$

since R is 2-torsion free.

Replacing y by $y + z$, $z \in I$ in 2.18 and using 2.18, we get

$$\begin{aligned} 0 = & 2\Delta(d(x), y, z) + 2\Delta(d(y), x, z) + 2\Delta(d(z), x, y) + 6\Delta(D(y, y, z), x, y) \\ & + 6\Delta(D(y, y, z), x, z) + 6\Delta(D(y, z, z), x, y) + 6\Delta(D(y, z, z), x, z) \\ & + 6\Delta(D(x, x, y), x, z) + 6\Delta(D(x, x, z), x, y) + 6\Delta(D(x, y, z), x, x) \\ & + 6\Delta(D(x, y, y), y, z) + 3\Delta(D(x, y, y), z, z) + 6\Delta(D(x, y, z), y, y) \\ & + 12\Delta(D(x, y, z), y, z) + 6\Delta(D(x, y, z), z, z) + 3\Delta(D(x, z, z), y, y) \\ & + 6\Delta(D(x, z, z), y, z). \end{aligned} \quad (2.19)$$

Replacing y by $-y$ in 2.19 and comparing with 2.19, we obtain

$$\begin{aligned} 0 = & 2\Delta(D(y, y, z), x, z) + 2\Delta(D(y, z, z), x, y) + \Delta(D(x, y, y), z, z) \\ & + 4\Delta(D(x, y, z), y, z) + \Delta(D(x, z, z), y, y), \end{aligned} \quad (2.20)$$

since R is 3-torsion free.

Replacing z by y in 2.20, we get

$$\begin{aligned} 0 = & 2\Delta(d(y), x, y) + 2\Delta(d(y), x, y) + \Delta(D(x, y, y), y, y) \\ & + 4\Delta(D(x, y, y), y, y) + \Delta(D(x, y, y), y, y) \end{aligned}$$

and so,

$$0 = 2\Delta(d(y), x, y) + 3\Delta(D(x, y, y), y, y), \quad (2.21)$$

since R is 2-torsion free.

Replacing x by xz , $z \in I$ in 2.21 and using 2.21, we obtain that

$$\begin{aligned} 0 = & 2xD(d(y), z, y) + 3D(x, y, y)D(z, y, y) \\ & + 3\Delta(x, y, y)D(z, y, y) + 3xD(D(z, y, y), y, y). \end{aligned} \quad (2.22)$$

Replacing x by ux , $u \in I$ in 2.22 and using 2.22, we get

$$\begin{aligned} 0 = & D(u, y, y)xD(z, y, y) + \Delta(u, y, y)xD(z, y, y) \\ & + uD(x, y, y)D(z, y, y) - u\Delta(x, y, y)D(z, y, y) \end{aligned} \quad (2.23)$$

since R is 3-torsion free.

Replacing u by x in 2.23, we have

$$0 = D(x^2, y, y)D(z, y, y) + [\Delta(x, y, y), x]D(z, y, y) \quad (2.24)$$

Replacing z by zu in 2.24 and using 2.24, we get

$$(D(x^2, y, y) + [\Delta(x, y, y), x])zD(u, y, y) = 0 \text{ for all } x, y, z, u \in I.$$

Since R is prime, we get either $D(x^2, y, y) + [\Delta(x, y, y), x] = 0$ or $D(u, y, y) = 0$. In the last case and Lemma 3, we have that either R is commutative or $D = 0$.

If $D(x^2, y, y) + [\Delta(x, y, y), x] = 0$ for all $x, y \in I$, then replacing y by $y + z$, $z \in I$, we obtain that

$$0 = D(x^2, y, y) + 2D(x^2, y, z) + D(x^2, z, z) + [\Delta(x, y, y), x]$$

$$+ 2[\Delta(x, y, z), x] + [\Delta(x, z, z), x].$$

This implies that $0 = D(x^2, y, z) + [\Delta(x, y, z), x]$, since R is 2-torsion free. Replacing z by zw , $w \in I$, we get

$$0 = \Delta(x, y, z)[w, x] + [z, x]D(x, y, w).$$

In the last relation, substituting x for w , y , we get

$$[z, x]d(x) = 0 \quad (2.25)$$

Replacing z by rz , $r \in R$ in 2.25 and using 2.25, we obtain that

$$[r, x]zd(x) = 0, \text{ for all } x, z \in I, r \in R \quad (2.26)$$

Replacing z by zx , we have

$$[r, x]zxd(x) = 0, \text{ for all } x, z \in I, r \in R \quad (2.27)$$

Multiplying by x on the right side in 2.26, we get

$$[r, x]zd(x)x = 0, \text{ for all } x, z \in I, r \in R \quad (2.28)$$

Comparing 2.27 and 2.28, we obtain that

$$[r, x]z[d(x), x] = 0, \text{ for all } x, z \in I, r \in R \quad (2.29)$$

Replacing r by $d(x)$ in 2.29, we get $[d(x), x]z[d(x), x] = 0$ for all $x, z \in I$. This implies that $z[d(x), x]Rz[d(x), x] = \{0\}$ for all $x, z \in I$. Since R is prime, we get $z[d(x), x] = 0$ for all $x, z \in I$. And so, $[d(x), x] \in \text{Ann}(I)$. This implies that $[d(x), x] \in I \cap \text{Ann}(I) = \{0\}$ for all $x \in I$. Hence d is commuting on I . $\square$

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FIXED POINT THEOREMS FOR LOCAL ALMOST CONTRACTIONS

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Abstract. This paper introduces a new class of contraction: the almost local contractions. Then, we prove the existence and uniqueness of a fixed-point for local almost contractions in two cases: with constant and variable coefficients of contraction.

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1. INTRODUCTION

The concept of local contraction was presented by Martins da Rocha and Filipe Vailakis in [4], meanwhile the almost contractive mappings was introduced by V. Berinde in [2]. The aim of this paper is to combine this two concepts and to study the fixed points of almost local contractions. First, we present the concept of almost contraction, following V. Berinde in [2] (2004).

Definition 1. Let (X, D) be a metric space and $T : X \rightarrow X$ is called almost contraction or (δ, L) - contraction if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + L \cdot d(y, Tx), \forall x, y \in X. \quad (1.1)$$

Remark 1. The term of almost contraction is equivalent to weak contraction, and it was first introduced by V. Berinde in [2].

Remark 2. Because of the symmetry of the distance, the almost contraction condition (1.1) includes the following dual one:

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + L \cdot d(x, Ty), \forall x, y \in X \quad (1.2)$$

obtained from (1.1) by replacing $d(Tx, Ty)$ by $d(Ty, Tx)$ and $d(x, y)$ by $d(y, x)$, and after that step, changing x with y , and viceversa.

Obviously, to prove the almost contractiveness of T , it is necessary to check both (1.1) and (1.2).

A strict contraction satisfies (1.1), with $\delta = a$ and $L = 0$, therefore is an almost contraction with a unique fixed point.

The concept of local contraction was first introduced by Martins da Rocha and Filipe Vailakis in [4] (2010)

Definition 2. Let F be a set and let $\mathcal{D} = (d_j)_{j \in J}$ a family of semidistances defined on F . We let σ be the weak topology on F defined by the family $\mathcal{D}$. Let r be a function from J to J . An operator $T : F \rightarrow F$ is a *local contraction* with respect $(\mathcal{D}, r)$ if, for every j , there exists $\beta_j \in [0, 1)$ such that

$$\forall f, g \in F, \quad d_j(Tf, Tg) \leq \beta_j d_{r(j)}(f, g)$$

We present an existence theorem (Theorem 1), then an existence and uniqueness theorem (Theorem 2), as they are presented in [2]. Their main merit is that they extend uniqueness and existence theorems for contractions to the larger class of almost contractions. They show us a method for approximating the fixed point, for which both a priori and a posteriori error estimates are available.

Theorem 1. Let (X, D) be a complete metric space and $T : X \rightarrow X$ a weak (almost) contraction. Then

- (1) $Fix(T) = \{x \in X : Tx = x\} \neq \emptyset$;
- (2) For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (3) converges to some $x^* \in Fix(T)$;

$$x_{n+1} = Tx_n, n = 0, 1, 2, \dots \quad (1.3)$$

- (3) The following estimates

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1), \quad n = 0, 1, 2, \dots \quad (1.4)$$

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n), \quad n = 1, 2, \dots \quad (1.5)$$

hold, where δ is the constant appearing in (1.1).

It is possible to force the uniqueness of the fixed point of an almost contraction, see [1] and [2], by imposing an additional contractive condition, quite similar to (1.1), as shown by the next theorem.

Theorem 2. Let (X, d) be a complete metric space and $T : X \rightarrow X$ be an almost contraction for which there exist $\theta \in (0, 1)$ and some $L_1 \geq 0$ such that

$$d(Tx, Ty) \leq \theta \cdot d(x, y) + L_1 \cdot d(x, Tx), \forall x, y \in X \quad (1.6)$$

Then

- (1) T has a unique fixed point, i.e., $Fix(T) = \{x^*\}$
- (2) For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (3) converges to x^* ;

(3) *The a priori and a posteriori error estimates*

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1), \quad n = 0, 1, 2, \dots$$

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n), \quad n = 1, 2, \dots$$

hold.

(4) *The rate of convergence of the Picard iteration is given by*

$$d(x_n, x^*) \leq \theta \cdot d(x_{n-1}, x^*), \quad n = 1, 2, \dots \quad (1.7)$$

The existence and uniqueness of fixed points for the local contractions was studied in [4].

Theorem 3. Assume that the space F is σ -Hausdorff, which means: for each pair $f, g \in F, f \neq g$, there exists $j \in J$ such that $d_j(f, g) > 0$.

Consider a function $r : J \rightarrow J$ and let $T : F \rightarrow F$ be a local contraction with respect to $(\mathcal{D}, r)$. Consider a nonempty, σ -bounded, sequentially σ -complete, and T -invariant subset $A \subset F$.

(1) (E) If the condition

$$\forall j \in J, \quad \lim_{n \rightarrow \infty} \beta_j \beta_{r(j)} \cdots \beta_{r^{n+1}(j)} \text{diam}_{r^{n+1}(j)}(A) = 0 \quad (1.8)$$

is satisfied, then the operator T admits a fixed point f^* in A .

(2) (S) Moreover, if $h \in F$ satisfies

$$\forall j \in J, \quad \lim_{n \rightarrow \infty} \beta_j \beta_{r(j)} \cdots \beta_{r^{n+1}(j)} d_{r^{n+1}(j)}(h, A) = 0 \quad (1.9)$$

then the sequence $(T^n h)_{n \in \mathbb{N}}$ is σ -convergent to f^* .

If A is a nonempty subset of F , then for each h in F , we let

$$d_j(h, A) \equiv \inf \{d_j(h, g) : g \in A\}.$$

Remark 3. There exists other interpretations of the local contractions, different from Martins da Rocha and Filipe Vailakis in [4], for example the definition given by Fang Jin-xuan in [3]:

Definition 3. A distribution function is a function $F : [-\infty, \infty] \rightarrow [0, 1]$ which is left continuous on $\mathbb{R}$, non-decreasing and $F(-\infty) = 0, F(\infty) = 1$.

We will denote by Δ the family of all distribution functions on $[-\infty, \infty]$. H is a special element of Δ defined by

$$H(t) = 0, \text{ if } t \leq 0 \quad \text{and} \quad 1, \text{ if } t > 0$$

If X is a nonempty set, $F : X \times X \rightarrow \Delta$ is called a probabilistic distance on X and $F(x, y)$ is usually denoted by F_{xy} .

Definition 4 (Schweizer and Sklar [5]). The ordered pair (X, F) is called a probabilistic metric space (shortly PM-space) if X is a nonempty set and F is a probabilistic distance satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

- (1) (FM-0) $F_{xy}(t) = 1 \Leftrightarrow x = y$
- (2) (FM-1) $F_{xy}(0) = 0$
- (3) (FM-2) $F_{xy} = F_{yx}$
- (4) (FM-3) $F_{xz}(t) = 1, F_{zy}(s) = 1 \Rightarrow F_{xy}(t + s) = 1$.

The ordered triple $(X, F, *)$ is called Menger space if (X, F) is a PM-space, $*$ is a t - norm and the following condition is also satisfied: for all $x, y, z, \in X$ and $t, s > 0$

- (5) (FM-4) $F_{xy}(t + s) \geq F_{xz}(t) * F_{zy}(s)$.

Definition 5. The PM- space (X, F) is said to be ε - chainable if for given $\varepsilon > 0$ and any $x, y \in X$ there is a finite set of points in X : $x = x_0, x_1, \dots, x_n = y$ such that $F_{x_{i-1}, x_i}(\varepsilon) = 1, \quad i = 1, 2, \dots, n$

We denote by $\mathfrak{T}$ the topology on X and it is called the (ε, λ) topology of (X, F, Δ) . Thus we can induce the concept of $\mathfrak{T}$ - Cauchy sequence and also $\mathfrak{T}$ - complete sequence.

Theorem 4 (The Fixed Point Theorem of One-valued Local Contraction Mappings, Fang [3]). *Let (X, F, Δ) be an ε - chainable and $\mathfrak{T}$ - complete Menger space. Let the mapping $T : X \rightarrow X$ satisfy the following condition: there exists $\delta \in (0, 1)$ so that for each $\alpha \in (0, \delta)$ there is a function $\Phi_\alpha(t)$ satisfying the condition Φ such that*

$$F_{Tx, Ty}(\Phi_\alpha)(t) \geq F_{x, y}(t) \quad (1.10)$$

for $F_{x, y}(\varepsilon) \neq 0$ and $F_{x, y}(t) > 1 - \alpha$.

Then T has a unique fixed point x_* in X and $T^n x_0 \rightarrow x_*$ for any $x_0 \in X$

Now, we shall try to combine these two different type of contractive mappings: the almost and local contractions, to study their fixed points.

2. ALMOST LOCAL CONTRACTIONS

Definition 6. The mapping $d(x, y) : X \times X \rightarrow \mathbb{R}_+$ is said to be a pseudometric if:

- (1) $d(x, y) = d(y, x)$
- (2) $d(x, y) \leq d(x, z) + d(z, y)$
- (3) $x = y$ implies $d(x, y) = 0$
(instead of $x = y \Leftrightarrow d(x, y) = 0$ in the metric case)

Definition 7. Let X be a set and let $\mathcal{D} = (d_j)_{j \in J}$ be a family of pseudometrics defined on X . We let σ be the weak topology on X defined by the family $\mathcal{D}$. A sequence $(x_n)_{n \in \mathbb{N}^*}$ is said to be σ -Cauchy if it is d_j -Cauchy, $\forall j \in J$. The subset A of X is said to be sequentially σ -complete if every σ -Cauchy sequence in X converges in X for the σ -topology. The subset $A \subset X$ is said to be σ -bounded if $\text{diam}_j(A) \equiv \sup\{d_j(x, y) : x, y \in A\}$ is finite for every $j \in J$.

Definition 8. Let r be a function from J to J . An operator $T : X \rightarrow X$ is called an *almost local contraction* with respect to $(\mathcal{D}, r)$ if, for every j , there exist the constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$d_j(Tx, Ty) \leq \theta \cdot d_j(x, y) + L \cdot d_{r(j)}(y, Tx), \forall x, y \in X \quad (2.1)$$

Remark 4. The almost contractions represent a particular case of almost local contractions, by taking (X, d) metric space instead of the pseudometrics d_j and $d_{r(j)}$ defined on X . Also, to obtain the almost contractions, we take in (1.1) for r the identity function, so we have $r(j)=j$.

Definition 9. The space X is σ -Hausdorff if the following condition is valid: for each pair $x, y \in X$, $x \neq y$, there exists $j \in J$ such that $d_j(x, y) > 0$. If A is a nonempty subset of X , then for each z in X , we let $d_j(z, A) \equiv \inf\{d_j(z, y) : y \in A\}$.

Theorem 5 is an existence fixed point theorem for almost local contractions.

Theorem 5. Consider a function $r : J \rightarrow J$ and let $T : X \rightarrow X$ be an almost local contraction with respect to $(\mathcal{D}, r)$. Consider a nonempty, σ -bounded, sequentially σ -complete, and T -invariant subset $A \subset X$. If the condition

$$\forall j \in J, \quad \lim_{n \rightarrow \infty} \theta^{n+1} \text{diam}_{r^{n+1}(j)}(A) = 0 \quad (2.2)$$

is satisfied, then the operator T admits a fixed point x^* in A .

Proof. Let $x_0 \in X$ be arbitrary and $\{x_n\}_{n=0}^\infty$ be the Picard iteration defined by

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}$$

Take $x := x_{n-1}, y := x_n$ in (2.1) to obtain

$$d_j(Tx_{n-1}, Tx_n) \leq \theta \cdot d_{r(j)}(x_{n-1}, x_n)$$

which yields

$$d_j(x_n, x_{n+1}) \leq \theta \cdot d_{r(j)}(x_{n-1}, x_n), \forall j \in J \quad (2.3)$$

Using (2.3), we obtain by induction with respect to n :

$$d_j(x_n, x_{n+1}) \leq \theta^n \cdot d_{r(j)}(x_0, x_1), \quad n = 0, 1, 2, \dots \quad (2.4)$$

According to the triangle rule, by (2.4) we get:

$$d_j(x_n, x_{n+p}) \leq \theta^n (1 + \theta + \cdots + \theta^{p-1}) d_{r(j)}(x_0, x_1) = \quad (2.5)$$

$$= \frac{\theta^n}{1 - \theta} (1 - \theta^p) \cdot d_{r(j)}(x_0, x_1), \quad n, p \in \mathbb{N}, p \neq 0 \quad (2.6)$$

Conditions (2.5) show us that the sequence $(x_n)_{n \in \mathbb{N}}$ is d_j -Cauchy for each j . The subset A is assumed to be sequentially σ -complete, there exists f^* in A such that $(T^n x)_{n \in \mathbb{N}}$ is σ -convergent to x^* . Besides, the sequence $(T^n x)_{n \in \mathbb{N}}$ converges for the topology σ to x^* , which implies

$$\forall j \in J, \quad d_j(Tx^*, x^*) = \lim_{n \rightarrow \infty} d_j(Tx^*, T^{n+1}x).$$

Recall that the operator T is an almost local contraction with respect to $(\mathcal{D}, r)$. From that, we have

$$\forall j \in J, \quad d_j(Tx^*, x^*) \leq \beta_j \lim_{n \rightarrow \infty} d_{r(j)}(x^*, T^n x).$$

The convergence for the σ -topology implies convergence for the pseudometric $d_{r(j)}$, we obtain $d_j(Tx^*, x^*) = 0$ for every $j \in J$.

This way, we prove that $Tf^* = f^*$, since σ is Hausdorff.

So, we prove the existence of the fixed point for almost local contractions. $\square$

Remark 5. For T verifies (2.1) with $L = 0$, we find Theorem Vailakis [4] by taking $\theta_j = \theta$.

Further, for the case $d_j = d, \forall j \in J$, with $d = \text{metric on } X$, we obtain the well known Banach contraction, with his unique fixed point.

Remark 6. In Theorem 5, the coefficient of contraction $\theta \in (0, 1)$ is constant, but local contractions have a coefficient of contraction $\theta_j \in [0, 1)$ which depends on $j \in J$. Our first goal is to extend the local almost contractions to the most general case of $\theta_j \in (0, 1)$.

The next Theorem represent an existence and uniqueness theorem for the almost local contractions with constant coefficient of contraction.

Theorem 6. *If to the conditions of Theorem 5, we add:*

(U) *for every fixed $j \in J$ there exists:*

$$\lim_{n \rightarrow \infty} (\theta + L)^n \text{diam}_{r^n(j)}(z, A) = 0, \forall x, y \in X \quad (2.7)$$

then the fixed point x^ of T is unique.*

Proof. Suppose, by contradiction, there are two different fixed points x^* and y^* of T . Then for every fixed $j \in J$ we have:

$$0 < d_j(x^*, y^*) = d_j(Tx^*, Ty^*) \leq \theta d_{r(j)}(x^*, y^*) + L d_{r(j)}(y^*, Tx^*) =$$

$$\begin{aligned}
&= (\theta + L) \cdot d_{r(j)}(x^*, y^*) \leq \cdots \leq (\theta + L)^n d_{r^n(j)}(x^*, y^*) \leq \\
&\leq (\theta + L)^n \text{diam}_{r^n(j)}(z, A)
\end{aligned}$$

Now, letting $n \rightarrow \infty$, we obtained a contradiction with condition (2.7), i.e. the fixed point is unique. $\square$

Remark. If the function r is the identity (i.e., $r(j) = j$), then the operator T is said to be a 0- local contraction and, in that case, conditions (2.1) and (2.7) are automatically satisfied. In particular, if a fixed point exists, it is unique on the whole space X .

We extend the Definition 8. to the case of almost local contractions with variable coefficient of contraction.

Definition 10. Let r be a function from J to J . An operator $T : X \rightarrow X$ is called *almost local contraction* with respect $(\mathcal{D}, r)$ or (θ_j, L_j) - contraction, if there exist a constant $\theta_j \in (0, 1)$ and some $L_j \geq 0$ such that

$$d_j(Tx, Ty) \leq \theta_j \cdot d_j(x, y) + L_j \cdot d_{r(j)}(y, Tx), \forall x, y \in X \quad (2.8)$$

Theorem 7. With the presumptions of Theorem 7, if we modify the condition (2.2) by the following one:

$$\forall j \in J, \quad \lim_{n \rightarrow \infty} \theta_j \theta_{r(j)} \cdots \theta_{r^n(j)} \text{diam}_{r^{n+1}(j)}(A) = 0, \quad (2.9)$$

then the operator T admits a fixed point x^* in A .

Proof. Let x an element in A . From the definition of almost local contraction T , for every pair of integers $q > n > 0$, we have

$$\begin{aligned}
&d_j(T^q x, T^n x) \\
&\leq \theta_j d_{r(j)}(T^{q-1} x, T^{n-1} x) + L_j d_{r(j)}(T^{n-1} x, T^q x) \leq \\
&\leq \theta_j [\theta_{r(j)} d_{r^2(j)}(T^{q-2} x, T^{n-2} x) + L_j d_{r^2(j)}(T^{n-2} x, T^{q-1} x)] + \\
&+ L_j [\theta_{r(j)} d_{r^2(j)}(T^{n-2} x, T^{q-1} x) + L_{r(j)} d_{r^2(j)}(T^{q-1} x, T^{n-1} x)] = \\
&= \theta_j [\theta_{r(j)} d_{r^2(j)}(T^{q-2} x, T^{n-2} x) + L_j (\theta_j + \theta_{r(j)}) d_{r^2(j)}(T^{n-2} x, T^{q-1} x)] + \\
&+ L_j L_{r(j)} d_{r^2(j)}(T^{q-1} x, T^{n-1} x) \leq \cdots \leq \\
&\leq \theta_j \theta_{r(j)} \cdots \theta_{r^{n-1}(j)} d_{r^n(j)}(T^{q-n} x, x) \\
&+ \cdots + L_j L_{r(j)} \cdots L_{r^{n-1}(j)} d_{r^n(j)}(T^{q-n} x, x)
\end{aligned}$$

Since A is T -invariant, $T^{q-n} x$ belongs to A , which yields

$$d_j(T^q x, T^n x) \leq \theta_j \theta_{r(j)} \cdots \theta_{r^{n-1}(j)} \text{diam}_{r^n(j)}(A).$$

That means: the sequence $(T^n x)_{n \in \mathbb{N}}$ is d_j -Cauchy for each j . The subset A is assumed to be sequentially σ -complete, there exists x^* in A such that $(T^n x)_{n \in \mathbb{N}}$ is σ -

convergent to x^* . Besides, the sequence $(T^n x)_{n \in \mathbb{N}}$ converges for the topology σ to x^* , which implies

$$\forall j \in J, \quad d_j(Tx^*, x^*) = \lim_{n \rightarrow \infty} d_j(Tx^*, T^{n+1}x).$$

Recall that the operator T is an almost local contraction with respect to $(\mathcal{D}, r)$. From that, we have

$$\forall j \in J, \quad d_j(Tx^*, x^*) \leq \theta_j \lim_{n \rightarrow \infty} d_{r(j)}(x^*, T^n x) + L_j d_{r(j)}(T^n x, Tx^*).$$

The convergence for the σ -topology implies convergence for the pseudodistance $d_{r(j)}$, we obtain $d_j(Tx^*, x^*) = 0$ for every $j \in J$. This way, we prove that $Tx^* = x^*$ since σ is Hausdorff. So, we prove the existence of the fixed point. $\square$

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BRUNN-MINKOWSKI INEQUALITY FOR L_p -MIXED INTERSECTION BODIES

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Abstract. In this paper, we establish L_p -Brunn-Minkowski inequality for dual Quermassintegral of L_p -mixed intersection bodies. As application, we give the well-known Brunn-Minkowski inequality for mixed intersection bodies.

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Keywords: the Brunn-Minkowski inequality, L_p -dual mixed volumes, L_p -mixed intersection bodies

1. INTRODUCTION

The intersection operator and the class of intersection bodies were defined by Lutwak [9]. The closure of the class of intersection bodies was studied by Goody, Lutwak, and Weil [5]. The intersection operator and the class of intersection bodies played a critical role in Zhang [12] and Gardner [2] on the solution of the famous Busemann-Petty problem (See also Gardner, Koldobsky, Schlumprecht [4]).

As Lutwak [9] shows (and as is further elaborated in Gardner's book [3]), there is a kind of duality between projection and intersection bodies. Consider the following illustrative example: It is well known that the projections (onto lower dimensional subspaces) of projection bodies are themselves projection bodies. Lutwak conjectured the “duality”: When intersection bodies are intersected with lower dimensional subspaces, the results are intersection bodies (within the lower dimensional subspaces). This was proven by Fallert, Goodey and Weil [1].

In [7] (see also [10] and [8]), Lutwak introduced mixed projection bodies and proved the following Brunn-Minkowski inequality for mixed projection bodies:

Theorem 1. *If $K, L \in \mathcal{K}^n$ and $0 \leq i < n$, then*

$$W_i(\mathbf{P}(K + L))^{1/(n-i)(n-1)} \geq W_i(\mathbf{P}K)^{1/(n-i)(n-1)} + W_i(\mathbf{P}L)^{1/(n-i)(n-1)}, \quad (1.1)$$

with equality if and only if K and L are homothetic.

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Where, $\mathcal{K}^n$ denotes the set of convex bodies in $\mathbb{R}^n$.

$$W_i(K) = V(\underbrace{K, \dots, K}_{n-i}, \underbrace{B, \dots, B}_i)$$

denotes the classical Quermassintegral of convex body K . $\mathbf{P}K$ denotes the projection body of convex body K .

In 2008, the Brunn-Minkowski inequality for mixed intersection bodies was established as follows [13].

Theorem 2. *If $K, L \in \varphi^n$, $0 \leq i < n$, then*

$$\tilde{W}_i(\mathbf{I}(K \tilde{+} L))^{1/(n-i)(n-1)} \leq \tilde{W}_i(\mathbf{I}K)^{1/(n-i)(n-1)} + \tilde{W}_i(\mathbf{I}L)^{1/(n-i)(n-1)}, \quad (1.2)$$

with equality if and only if K and L are dilates.

Where, φ^n denotes the set of star bodies in $\mathbb{R}^n$. Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K, \cdot) : S^{n-1} \rightarrow \mathbb{R}$, defined for $u \in S^{n-1}$, by

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda u \in K\}.$$

If $\rho(K, \cdot)$ is positive and continuous, K will be called a star body. Moreover, $\mathbf{I}K$ denotes the intersection body of star body K and the sum $\tilde{+}$ denotes the radial Minkowski sum and $\tilde{W}_i(K) = \tilde{V}(\underbrace{K, \dots, K}_{n-i}, \underbrace{B, \dots, B}_i)$ denotes the classical dual

Quermassintegral of star body K .

In 2006, Haberl and Ludwig [6] introduced L_p -intersection bodies ($p < 1$). For $K \in \mathcal{P}_0^n$, where $\mathcal{P}_0^n$ denotes the set of convex polytopes in $\mathbb{R}^n$ that contain the origin in their interiors. The star body $\mathbf{I}_p^+ K$ is defined for $u \in S^{n-1}$ by

$$\rho(\mathbf{I}_p^+ K, u)^p = \int_{K \cap u^+} |u \cdot x|^{-p} dx, \quad (1.3)$$

where $u^+ = \{x \in \mathbb{R}^n : u \cdot x \geq 0\}$, and define $\mathbf{I}_p^- K = \mathbf{I}_p^+(-K)$. For $p < 1$, the centrally symmetric star body $\mathbf{I}_p K = \mathbf{I}_p^+ K + \mathbf{I}_p^- K$ is called as the L_p intersection body of K . So for $u \in S^{n-1}$,

$$\rho^p(\mathbf{I}_p K, u) = \int_K |u \cdot x|^{-p} dx. \quad (1.4)$$

The purpose of this paper is to establish Brunn-Minkowski inequality for L_p -mixed intersection bodies as follows

Theorem 3. *If $K, L \in \varphi^n$, and $0 \leq i < n$, then for $p < 1$*

$$\tilde{W}_i(\mathbf{I}_p(K \tilde{+} L))^{1/(n-i)(n-1)} \leq \tilde{W}_i(\mathbf{I}_p K)^{1/(n-i)(n-1)} + \tilde{W}_i(\mathbf{I}_p L)^{1/(n-i)(n-1)}, \quad (1.5)$$

with equality if and only if K and L are dilates.

Where, $\mathbf{I}_p K$ denotes the above L_p -intersection body of star body K which was defined by Haberl and Ludwig [6].

Remark 1. Let $p \rightarrow 1^-$ in (1.5), (1.5) changes to (1.2).

To prove Theorem 3, the paper first introduce a new notion L_p -dual mixed volumes, then generalize Haberl and Ludwig's L_p -intersection bodies to L_p -mixed intersection bodies ($p < 1$). Moreover, we use a new way which is different from the way of [13].

2. PRELIMINARIES

The setting for this paper is n -dimensional Euclidean space $\mathbb{R}^n$ ($n > 2$). Let $\mathbb{C}^n$ denote the set of non-empty convex figures(compact, convex subsets) and $\mathcal{K}^n$ denote the subset of $\mathbb{C}^n$ consisting of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1} . For $u \in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u . We will use K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u . We use $V(K)$ for the n -dimensional volume of convex body K . The support function of $K \in \mathcal{K}^n$, $h(K, \cdot)$, defined on $\mathbb{R}^n$ by $h(K, \cdot) = \max\{x \cdot y : y \in K\}$. Let δ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$, $\delta(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$. Let $\tilde{\delta}$ denote the radial Hausdorff metric, as follows, if $K, L \in \varphi^n$, then $\tilde{\delta}(K, L) = |\rho_K - \rho_L|_\infty$.

2.1. L_p -dual mixed volumes

We define vector addition $\tilde{+}$ on $\mathbb{R}^n$, which we shall call the radial addition, as follows. For any $x_1, \dots, x_r \in \mathbb{R}^n$, $x_1 \tilde{+} \dots \tilde{+} x_r$ is defined to be the usual vector sum of $x_1, \dots, x_r$ if they all lie in a 1-dimensional subspace of $\mathbb{R}^n$, and as the zero vector otherwise.

If $K_1, \dots, K_r \in \varphi^n$ and $\lambda_1, \dots, \lambda_r \in \mathbb{R}$, then the radial Minkowski linear combination, $\lambda_1 K_1 \tilde{+} \dots \tilde{+} \lambda_r K_r$, is defined by

$$\lambda_1 K_1 \tilde{+} \dots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \dots \tilde{+} \lambda_r x_r : x_i \in K_i\}.$$

The following property will be used later. If $K, L \in \varphi^n$ and $\lambda, \mu \geq 0$

$$\rho(\lambda K \tilde{+} \mu L, \cdot) = \lambda \rho(K, \cdot) + \mu \rho(L, \cdot). \quad (2.1)$$

For $K_1, \dots, K_r \in \varphi^n$ and $\lambda_1, \dots, \lambda_r \geq 0$, the volume of the radial Minkowski linear combination $\lambda_1 K_1 \tilde{+} \dots \tilde{+} \lambda_r K_r$ is a homogeneous n th-degree polynomial in the λ_i [11],

$$V(\lambda_1 K_1 \tilde{+} \dots \tilde{+} \lambda_r K_r) = \sum \tilde{V}_{i_1, \dots, i_n} \lambda_{i_1} \dots \lambda_{i_n} \quad (2.2)$$

where the sum is taken over all n -tuples $(i_1, \dots, i_n)$ whose entries are positive integers not exceeding r . If we require the coefficients of the polynomial in (2.1.2) to be symmetric in their arguments, then they are uniquely determined. The coefficient $\tilde{V}_{i_1, \dots, i_n}$ is nonnegative and depends only on the bodies $K_{i_1}, \dots, K_{i_n}$. It is written as

$\tilde{V}(K_{i_1}, \dots, K_{i_n})$ and is called the *dual mixed volume* of $K_{i_1}, \dots, K_{i_n}$. If $K_1 = \dots = K_{n-i} = K$, $K_{n-i+1} = \dots = K_n = L$, the dual mixed volumes is written as $\tilde{V}_i(K, L)$. The dual mixed volumes $\tilde{V}_i(K, B)$ is written as $\tilde{W}_i(K)$.

If $K_i \in \varphi^n (i = 1, 2, \dots, n-1)$, then the dual mixed volume of $K_i \cap E_u (i = 1, 2, \dots, n-1)$ will be denoted by $\tilde{v}(K_1 \cap E_u, \dots, K_{n-1} \cap E_u)$. If $K_1 = \dots = K_{n-1-i} = K$ and $K_{n-i} = \dots = K_{n-1} = L$, then $\tilde{v}(K_1 \cap E_u, \dots, K_{n-1} \cap E_u)$ is written $\tilde{v}_i(K \cap E_u, L \cap E_u)$. If $L = B$, then $\tilde{v}_i(K \cap E_u, B \cap E_u)$ is written $\tilde{w}_i(K \cap E_u)$.

L_p -dual mixed volumes was defined as follows [14].

$$\tilde{V}_p(K_1, \dots, K_n) = \omega_n \left(\frac{1}{n\omega_n} \int_{S^{n-1}} \rho^p(K_1, u) \cdots \rho^p(K_n, u) dS(u) \right)^{1/p}, \quad p \neq 0, \quad (2.3)$$

where $K_1, \dots, K_n \in \varphi^n$.

If $K_1 = \dots = K_{n-1-i} = K$ and $K_{n-i} = \dots = K_{n-1} = L$, will write $\tilde{V}_p(\underbrace{K, \dots, K}_{n-1-i}, \underbrace{L, \dots, L}_i)$ as $\tilde{V}_{p,i}(K, L)$. If $K_1 = \dots = K_n = K$, will write $\tilde{V}_p(\underbrace{K, \dots, K}_n)$ as $\tilde{V}_p(K)$. If $L = B$, then write $\tilde{V}_p(\underbrace{K, \dots, K}_{n-i}, \underbrace{B, \dots, B}_i)$ as $\tilde{V}_{p,i}(K)$ and is called L_p -dual Quermassintegral as follows.

$$\tilde{V}_{p,i}(K) = \omega_n \left(\frac{1}{n\omega_n} \int_{S^{n-1}} \rho^{p(n-i)}(K, u) dS(u) \right)^{1/p}, \quad p \neq 0. \quad (2.4)$$

Remark 2. Apparently, let $p = 1$, then L_p -dual mixed volumes $\tilde{V}_p$ and L_p -dual Quermassintegral $\tilde{V}_{p,i}$ change to the classical dual mixed volumes $\tilde{V}$ and dual Quermassintegral $\tilde{W}_i$, respectively.

2.2. L_p -mixed intersection bodies

Since [6]

$$v(K \cap u^+) = \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{2} \int_K |u \cdot x|^{-1+\varepsilon} dx. \quad (2.5)$$

and

$$\rho(\mathbf{I}K, u) = \lim_{p \rightarrow 1^-} \frac{1-p}{2} \rho^p(\mathbf{I}_p K, u), \quad (2.6)$$

that is, the intersection body of K is obtained as a limit of L_p intersection bodies of K . Also note that a change to polar coordinates in (2.6) shows that up to a normalization factor $\rho^p(\mathbf{I}_p K, u)$ equals the Cosine transform of $\rho(K, u)^{n-p}$.

Here, we introduce the L_p -mixed intersection bodies of $K_1, \dots, K_{n-1}$. It is written as $\mathbf{I}_p(K_1, \dots, K_{n-1}) (p < 1)$, whose radial function is defined by

$$\rho^p(\mathbf{I}_p(K_1, \dots, K_{n-1}), u) = \frac{2}{1-p} \tilde{v}_p^*(K_1 \cap E_u, \dots, K_{n-1} \cap E_u), \quad (2.7)$$

where, $\tilde{v}_p^*(K_1 \cap E_u, \dots, K_{n-1} \cap E_u)$ denotes the p -dual mixed volumes of $K_1 \cap E_u, \dots, K_{n-1} \cap E_u$ in $(n-1)$ -dimensional space. If $K_1 = \dots = K_{n-i-1} = K, K_{n-i} = \dots = K_{n-1} = L$, then $\tilde{v}_p^*(K_1 \cap E_u, \dots, K_{n-1} \cap E_u)$ is written as $\tilde{v}_{p,i}^*(K \cap E_u, L \cap E_u)$. If $L = B$, then $\tilde{v}_{p,i}^*(K \cap E_u, L \cap E_u)$ is written as $\tilde{v}_{p,i}^*(K \cap E_u)$.

Remark 3. From the definition, which introduces a new star body, namely the L_p -mixed intersection body of $n-1$ given bodies.

From the definition, $V_p(K_1, \dots, K_n)$ is continuous function for any $K_i \in \varphi^n$, $i = 1, 2, \dots, n$, then

$$\begin{aligned} \lim_{p \rightarrow 1^-} \frac{1-p}{2} \rho^p(\mathbf{I}_p(K_1, \dots, K_{n-1}), u) \\ = \lim_{p \rightarrow 1^-} \omega_n \left(\frac{1}{n\omega_n} \int_{S^{n-1}} \rho^p(K_1, u) \cdots \rho^p(K_{n-1}, u) dS(u) \right)^{1/p} \\ = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_{n-1}, u) dS(u). \end{aligned}$$

On the other hand, by using definition of mixed intersection bodies (see [3] and [14]), we have

$$\begin{aligned} \rho(\mathbf{I}(K_1, \dots, K_{n-1}), u) &= \tilde{v}(K_1 \cap E_u, \dots, K_{n-1} \cap E_u) \\ &= \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_{n-1}, u) dS(u). \end{aligned}$$

Hence

$$\lim_{p \rightarrow 1^-} \frac{1-p}{2} \rho^p(\mathbf{I}_p(K_1, \dots, K_{n-1}), u) = \rho(\mathbf{I}(K_1, \dots, K_{n-1}), u).$$

For the L_p -mixed intersection bodies, $\mathbf{I}_p(K_1, \dots, K_{n-1})$, if $K_1 = \dots = K_{n-i-1} = K, K_{n-i} = \dots = K_{n-1} = L$, then $\mathbf{I}_p(K_1, \dots, K_{n-1})$ is written as $\mathbf{I}_p(K, L)_i$. If $L = B$, then $\mathbf{I}_p(K, L)_i$ is written as $\mathbf{I}_p K_i$ is called the i th L_p -intersection body of K . For $\mathbf{I}_p K_0$ simply write $\mathbf{I}_p K$, this is just the L_p -intersection bodies of star body K .

The following properties will be used later: If $K, L, M, K_1, \dots, K_{n-1} \in \varphi^n$, and $\lambda, \mu, \lambda_1, \dots, \lambda_{n-1} > 0$, then

$$\mathbf{I}_p(\lambda K \tilde{+} \mu L, M) = \lambda \mathbf{I}_p(K, M) \tilde{+} \mu \mathbf{I}_p(L, M), \quad (2.8)$$

where $M = (K_1, \dots, K_{n-2})$.

3. MAIN RESULTS

3.1. Some Lemmas

The following results will be required to prove our main Theorems.

Lemma 1. If $K, L \in \varphi^n$, $0 \leq i < n$, $0 \leq j < n-1$, $i, j \in \mathbb{N}$ and $p < 1$, then

$$\tilde{W}_i(\mathbf{I}_p(K, L)_j) = \frac{1}{n} \left(\frac{2}{1-p} \right)^{\frac{n-i}{p}} \int_{S^{n-1}} \tilde{v}_{p,j}^*(K \cap E_u, L \cap E_u)^{\frac{(n-i)}{p}} dS(u). \quad (3.1)$$

From (2.4) and (2.7), identity (3.1) in Lemma 1 easy follows.

Lemma 2. If $K_1, \dots, K_n \in \varphi^n$, $1 < r \leq n$, $0 \leq j < n-1$, $j \in \mathbb{N}$ and $p \neq 0$, then

$$\tilde{V}_p(K_1, \dots, K_n)^r \leq \prod_{j=1}^r \tilde{V}_p(\underbrace{K_j, \dots, K_j}_r, K_{r+1}, \dots, K_n), \quad (3.2)$$

with equality if and only if $K_1, \dots, K_n$ are all dilations [14].

From (3.1), (3.2) and in view of Hölder inequality for integral, we obtain

Lemma 3. If $K, L \in \varphi^n$, $0 \leq i < n$, $0 < j < n-1$, and $p < 1$, then

$$\tilde{W}_i(\mathbf{I}_p(K, L))^{n-1} \leq \tilde{W}_i(\mathbf{I}_p K)^{n-j-1} \cdot \tilde{W}_i(\mathbf{I}_p L)^j, \quad (3.3)$$

with equality if and only if K and L are dilates.

3.2. Brunn-Minkowski inequality for L_p -mixed intersection bodies

The Brunn-Minkowski inequality for L_p -intersection bodies, which will be established is: If $K, L \in \varphi^n$, $p < 1$ then

$$V(\mathbf{I}_p(K \tilde{+} L))^{1/n(n-1)} \leq V(\mathbf{I}_p K)^{1/n(n-1)} + V(\mathbf{I}_p L)^{1/n(n-1)}, \quad (3.4)$$

with equality if and only if K and L are dilates.

This is just the special case $i = 0$ of:

Theorem 4. If $K, L \in \varphi^n$, and $0 \leq i < n$, then

$$\tilde{W}_i(\mathbf{I}_p(K \tilde{+} L))^{1/(n-i)(n-1)} \leq \tilde{W}_i(\mathbf{I}_p K)^{1/(n-i)(n-1)} + \tilde{W}_i(\mathbf{I}_p L)^{1/(n-i)(n-1)}, \quad (3.5)$$

with equality if and only if K and L are dilates.

Proof. Let $M = (L_1, \dots, L_{n-2})$, from (2.1), (2.4), (2.8) and in view of Minkowski inequality for integral, we obtain that

$$\begin{aligned} \tilde{W}_i(\mathbf{I}_p(K \tilde{+} L, M))^{1/(n-i)} &= n^{-1/(n-i)} \|\rho(\mathbf{I}_p(K \tilde{+} L, M), u)\|_{n-i} \\ &= n^{-1/(n-i)} \|\rho(\mathbf{I}_p(K, M) \tilde{+} \mathbf{I}_p(L, M), u)\|_{n-i} \\ &= n^{-1/(n-i)} \|\rho(\mathbf{I}_p(K, M), u) + \rho(\mathbf{I}_p(L, M), u)\|_{n-i} \\ &\leq n^{-1/(n-i)} (\|\rho(\mathbf{I}_p(K, M), u)\|_{n-i} + \|\rho(\mathbf{I}_p(L, M), u)\|_{n-i}) \\ &= \tilde{W}_i(\mathbf{I}_p(K, M))^{1/(n-i)} + \tilde{W}_i(\mathbf{I}_p(L, M))^{1/(n-i)}. \end{aligned} \quad (3.6)$$

On the other hand, taking $L_1 = \dots = L_{n-2} = K \tilde{+} L$ to (3.6) and apply Lemma 2 and Lemma 3, and get

$$\begin{aligned}
\tilde{W}_i(\mathbf{I}_p(K \tilde{+} L))^{1/(n-i)} &\leq \\
&\tilde{W}_i(\mathbf{I}_p(K, K \tilde{+} L)_{n-2})^{1/(n-i)} + \tilde{W}_i(\mathbf{I}_p(L, K \tilde{+} L)_{n-2})^{1/(n-i)} \\
&\leq \tilde{W}_i(\mathbf{I}_p K)^{1/(n-1)(n-i)} \tilde{W}_i(\mathbf{I}_p(K \tilde{+} L))^{(n-2)/(n-1)(n-i)} \\
&\quad + \tilde{W}_i(\mathbf{I}_p L)^{1/(n-1)(n-i)} \tilde{W}_i(\mathbf{I}_p(K \tilde{+} L))^{(n-2)/(n-1)(n-i)}, \quad (3.7)
\end{aligned}$$

with equality if and only if K , L and $M = K \tilde{+} L$ are dilates, combine this with the equality condition of (3.6), it follows that the condition holds if and only if K and L are dilates.

Dividing both sides of (3.7) by $\tilde{W}_i(\mathbf{I}_p(K \tilde{+} L))^{(n-2)/(n-1)(n-i)}$, we get the inequality (3.5).

The proof is complete. $\square$

Remark 4. Let $i = 0$ and $p \rightarrow 1^-$ in (2.6), we get the well-known Brunn-Minkowski inequality for mixed intersection bodies as follows:

$$\tilde{V}(\mathbf{I}(K \tilde{+} L))^{1/n(n-1)} \leq \tilde{V}(\mathbf{I}K)^{1/n(n-1)} + \tilde{V}(\mathbf{I}L)^{1/n(n-1)}$$

with equality if and only if K and L are dilates.

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PSEUDO ALMOST PERIODIC SOLUTIONS OF AN ITERATIVE EQUATION WITH VARIABLE COEFFICIENTS

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Abstract. In this paper, the exponential dichotomy, and Tikhonov and Banach fixed point theorems are used to study the existence and uniqueness of pseudo almost periodic solutions of nonhomogeneous iterative functional differential equations of the form $x'(t) = \lambda_1(t)x(t) + \lambda_2(t)x^{[2]}(t) + \dots + \lambda_n(t)x^{[n]}(t) + f(t)$.

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Keywords: iterative functional differential equation, pseudo almost periodic solutions, fixed point theorem

1. INTRODUCTION

Recently, iterative functional differential equations of the form

$$x'(t) = H(x^{[0]}(t), x^{[1]}(t), x^{[2]}(t), \dots, x^{[n]}(t))$$

have appeared in several papers, where $x^{[0]}(t) = t$, $x^{[i]}(t) = x(x^{[i-1]}(t))$, $i \geq 1$. In [4], Cooke points out that it is highly desirable to establish the existence and stability properties of periodic solutions for equations of the form

$$x'(t) + ax(t - h(t, x(t))) = F(t),$$

in which the lag $h(t, x(t))$ implicitly involves $x(t)$. Stephan [14] studies the existence of periodic solutions of equation

$$x'(t) + ax(t - r + \mu h(t, x(t))) = F(t).$$

Eder [6] considers the iterative functional differential equation

$$x'(t) = x^{[2]}(t)$$

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and obtains that every solution either vanishes identically or is strictly monotonic. Fečkan [7] studies the equation

$$x'(t) = f(x^{[2]}(t))$$

by obtaining an existence theorem for solutions satisfying $x(0) = 0$. Later, Wang and Si [17] study

$$x'(x^{[r]}(t)) = c_0 z + c_1 x(t) + c_2 x^{[2]}(t) + \dots + x^{[n]}(t),$$

and show the existence theorem of analytic solutions. In particular, Si and Cheng [12] discusses the smooth solutions of equation of

$$x'(t) = \lambda_1 x(t) + \lambda_2 x^{[2]}(t) + \dots + \lambda_n x^{[n]}(t) + f(t).$$

For some various properties of solutions for several iterative functional differential equations, we refer the interested reader to [10, 11, 13, 15].

On the other hand, the existence of pseudo almost periodic solutions is among the most attractive topics in qualitative theory of differential equations due to their applications, especially in biology, economics and physics [1, 3, 5, 18, 20]. As pointed out by Ait Das and Ezzinbi [1], it is an interesting thing to study the pseudo almost periodic systems with delays. It is obviously that iterative functional differential equations are special type state-dependent delay differential equations. In [9], Liu pointed that the properties of the almost periodic functions do not always hold in the set of pseudo almost periodic functions and given an example: when $x(t)$ is a pseudo almost periodic function, $x(x(t))$ may not be a pseudo almost periodic function. To the best of our knowledge, there are few results about pseudo almost periodic solutions for iterative functional differential equations except [9] and [16].

In the present work, we propose an existence result for pseudo almost periodic solutions of iterative functional differential equations

$$x'(t) = \lambda_1(t)x(t) + \lambda_2(t)x^{[2]}(t) + \dots + \lambda_n(t)x^{[n]}(t) + f(t). \quad (1.1)$$

by using Tikhonov fixed theorem. Uniqueness of the solution is achieved by Banach fixed point theorem.

This paper is organized as follows. In Section 2 we give two theorems which are used in Section 3 to establish the main existence result. In Section 4, we show that (1.1) has a unique pseudo almost periodic solution under some suitable conditions and present an example to illustrate the theory. Related problems are also studied in [2].

2. PRELIMINARIES

Let $C(\mathbb{R}, \mathbb{R})$ denote the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ endowed with the usual metric

$$d(f, g) = \sum_{m=1}^{\infty} 2^{-m} \frac{\|f - g\|_m}{1 + \|f - g\|_m} \quad \text{for} \quad \|f - g\|_m = \max_{t \in [-m, m]} |f(t) - g(t)|,$$

so the topology on $C(\mathbb{R}, \mathbb{R})$ is the uniform convergence on each compact intervals of $\mathbb{R}$. We also consider the set $BC(\mathbb{R}, \mathbb{R})$ of all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with the norm $\|f\| = \sup_{t \in \mathbb{R}} |f(t)|$, so the topology on $BC(\mathbb{R}, \mathbb{R})$ is the uniform convergence on $\mathbb{R}$. We denote by $AP(\mathbb{R}, \mathbb{R})$ the set of all almost periodic functions from $\mathbb{R}$ to $\mathbb{R}$. Define the set

$$PAP_0(\mathbb{R}, \mathbb{R}) = \left\{ g \in BC(\mathbb{R}, \mathbb{R}) \left| \lim_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^T |g(t)| dt = 0 \right. \right\}.$$

A function $\varphi \in BC(\mathbb{R}, \mathbb{R})$ is called pseudo almost periodic if it can be expressed as $\varphi = h + g$, where $h \in AP(\mathbb{R}, \mathbb{R})$ and $g \in PAP_0(\mathbb{R}, \mathbb{R})$. The collection of such functions will be denoted by $PAP(\mathbb{R}, \mathbb{R})$. In particular, $(PAP(\mathbb{R}, \mathbb{R}), \|\cdot\|)$ is a Banach space [19]. For $M, L > 0$, define

$$B_C(M, L) = \left\{ \varphi \in C(\mathbb{R}, \mathbb{R}) \left| \begin{aligned} &|\varphi(t)| \leq M, |\varphi(t_2) - \varphi(t_1)| \leq L|t_2 - t_1|, \\ &\text{for all } t, t_1, t_2 \in \mathbb{R} \end{aligned} \right. \right\},$$

$$B_S(M, L) = S(\mathbb{R}, \mathbb{R}) \cap B_C(M, L), \quad S \in \{PAP, AP, PAP_0\}.$$

From [9], it is easy to see that $B_S(M, L)$, $S \in \{PAP, AP, PAP_0, C\}$ are closed convex and bounded subsets of $BC(\mathbb{R}, \mathbb{R})$ and $C(\mathbb{R}, \mathbb{R})$, respectively. Furthermore, by the Arzelà-Ascoli theorem, the subset $B_C(M, L)$ is compact in $C(\mathbb{R}, \mathbb{R})$.

On the other hand, the subset $B_C(M, L)$ is not precompact in $BC(\mathbb{R}, \mathbb{R})$, since a sequence $\left\{ M \operatorname{sech} \frac{L(t+n)}{M} \right\}_{n=1}^{\infty} \in B_C(M, L)$ has no a convergent subsequence in $BC(\mathbb{R}, \mathbb{R})$.

Furthermore, the subset $B_{AP}(M, L)$ is not compact in $C(\mathbb{R}, \mathbb{R})$, since we consider the function from [8]

$$g(t) = \sum_{k=1}^{\infty} \frac{1}{k} \sin^2 \frac{t}{2^k}.$$

From $\frac{1}{k} \sin^2 \frac{t}{2^k} \leq \frac{1}{k} \frac{t^2}{4^k}$ and that the series $\sum_{k=1}^{\infty} \frac{1}{k} \frac{t^2}{4^k}$ convergents, we see that

$$g_n(t) = \sum_{k=1}^n \frac{1}{k} \sin^2 \frac{t}{2^k} \rightarrow g(t)$$

in $C(\mathbb{R}, \mathbb{R})$. Similarly we derive

$$g'_n(t) = \sum_{k=1}^n \frac{1}{k2^{k-1}} \sin \frac{t}{2^k} \cos \frac{t}{2^k} \rightarrow g'(t)$$

in $BC(\mathbb{R}, \mathbb{R})$ and $\|g'_n\| \leq 2$, $\|g'\| \leq 2$. Take

$$\varphi_n(t) = \min\{1, g_n(t)\} = \frac{1 + g_n(t) - |1 - g_n(t)|}{2}.$$

Clearly $\varphi_n \in B_{AP}(1, 2)$ and $\varphi_n \rightarrow \varphi = \min\{1, g\}$ in $C(\mathbb{R}, \mathbb{R})$. Next we know from [8] that $\limsup_{t \rightarrow \infty} g(t) = \infty$ and $\liminf_{t \rightarrow \infty} g(t) = 0$. So applying [8, Lemma 2.1], we know that $\varphi \notin AP(\mathbb{R}, \mathbb{R})$. Hence the subset $B_{AP}(1, 2)$ is not complete in $C(\mathbb{R}, \mathbb{R})$. By taking the sequence $\left\{\varphi_n\left(\frac{L}{2M}t\right)\right\}_{n=1}^{\infty}$, we see that $B_{AP}(M, L)$ is not complete in $C(\mathbb{R}, \mathbb{R})$ for any $M > 0$ and $L > 0$. Since $B_{AP}(M, L) \subset B_C(M, L)$, $B_{AP}(M, L)$ is just precompact in $C(\mathbb{R}, \mathbb{R})$. We have also shown that $B_{AP}(M, L)$ are not precompact in $BC(\mathbb{R}, \mathbb{R})$, but they are complete.

Furthermore, for any $k \in \mathbb{N}$, we have $\frac{1}{2^k\pi} \int_0^{2^k\pi} \varphi(t)dt \geq \frac{1}{2^k\pi} \int_0^{2^k\pi} \sin^2 \frac{t}{2} = \frac{1}{2}$, so $\varphi \notin PAP_0(\mathbb{R}, \mathbb{R})$. But we do not know $\varphi \notin PAP(\mathbb{R}, \mathbb{R})$.

Finally, taking

$$\psi_n(t) = \begin{cases} 0 & \text{for } |t| \geq n+1, \\ 1 & \text{for } |t| \leq n, \\ t+n+1 & \text{for } t \in [-n-1, -n], \\ n+1-t & \text{for } t \in [n, n+1], \end{cases}$$

we see that $\psi_n \in B_{PAP_0}(1, 1)$ and $\psi_n \rightarrow 1$ in $C(\mathbb{R}, \mathbb{R})$, so $B_{PAP_0}(M, L)$ are not complete in $C(\mathbb{R}, \mathbb{R})$. We have also shown that $B_{PAP_0}(M, L)$ are not precompact in $BC(\mathbb{R}, \mathbb{R})$, but they are complete.

Summarizing, we arrive at the following result.

Theorem 1. *The subsets $B_S(M, L)$, $S \in \{PAP, AP, PAP_0, C\}$ are not precompact in $BC(\mathbb{R}, \mathbb{R})$. They are precompact in $C(\mathbb{R}, \mathbb{R})$. Furthermore, $B_{AP}(M, L)$ and $B_{PAP_0}(M, L)$ are just precompact in $C(\mathbb{R}, \mathbb{R})$, while $B_C(M, L)$ is compact in $C(\mathbb{R}, \mathbb{R})$.*

So (non-)compactness of $B_{PAP}(M, L)$ in $C(R, R)$ is still open.

Finally, we recall Tikhonov fixed point theorem.

Theorem 2. (Tikhonov) *Let Ω be a non-empty compact convex subset of a locally convex topological vector space X . Then any continuous function $A : \Omega \rightarrow \Omega$ has a fixed point.*

3. EXISTENCE RESULT

In this section, the existence of pseudo almost periodic solutions of equation (1.1) will be studied. Throughout this paper, it will be assumed that $\lambda_1 : \mathbb{R} \rightarrow (-\infty, 0)$ is an almost periodic function, $\lambda_i : \mathbb{R} \rightarrow \mathbb{R}$ are pseudo almost periodic functions, and

$$\lambda_1^+ = \sup_{t \in \mathbb{R}} \lambda_1(t) < 0, \lambda_i^+ = \sup_{t \in \mathbb{R}} |\lambda_i(t)|,$$

where $i = 2, \dots, n$.

Now we apply Theorem 2. Let X be either $PAP(\mathbb{R}, \mathbb{R})$ or $BC(\mathbb{R}, \mathbb{R})$ but fixed. If $\varphi \in X$, from Corollary 5.4 in [19], it follows $\varphi^{[2]} \in X$. Consider an auxiliary equation

$$x'(t) = \lambda_1(t)x(t) + \lambda_2(t)\varphi^{[2]}(t) + \dots + \lambda_n(t)\varphi^{[n]}(t) + f(t), \quad (3.1)$$

where $f \in PAP(\mathbb{R}, \mathbb{R})$ and $\lambda_1(t) < 0$. It is easy to see that the linear equation

$$x'(t) = \lambda_1(t)x(t)$$

admits an exponential dichotomy on $\mathbb{R}$, by Theorem 2.3 in [18], we know that (3.1) has exactly one solution

$$x_\varphi(t) = \int_{-\infty}^t e^{\int_s^t \lambda_1(u)du} \left(\sum_{i=2}^n \lambda_i(s)\varphi^{[i]}(s) + f(s) \right) ds$$

in X .

Now take $f \in B_{PAP}(M', L')$ for constants M' and L' . Let $A : B_C(M, L) \rightarrow BC(\mathbb{R}, \mathbb{R})$ be defined by

$$(A\varphi)(t) = \int_{-\infty}^t e^{\int_s^t \lambda_1(u)du} \left(\sum_{i=2}^n \lambda_i(s)\varphi^{[i]}(s) + f(s) \right) ds, \quad t \in \mathbb{R}. \quad (3.2)$$

Note $A : B_{PAP}(M, L) \rightarrow PAP(\mathbb{R}, \mathbb{R})$.

Lemma 1. For any $\varphi, \psi \in B_C(M, L)$, $t_1, t_2 \in \mathbb{R}$, the following inequality hold,

$$\|\varphi^{[n]} - \psi^{[n]}\| \leq \sum_{j=0}^{n-1} L^j \|\varphi - \psi\|, \quad n = 1, 2, \dots \quad (3.3)$$

Proof. It can be obtained by direct calculation by the definition of $B_C(M, L)$. $\square$

Lemma 2. Operator $A : B_C(M, L) \rightarrow C(\mathbb{R}, \mathbb{R})$ is continuous.

Proof. Let $\varphi_j \rightarrow \varphi_0$ as $j \rightarrow \infty$ for $\varphi_j \in B_C(M, L)$, $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ uniformly on any compact interval $[-m, m]$, $m \in \mathbb{N}$ of $\mathbb{R}$. Set

$$h_j(s) = \sum_{i=2}^n \lambda_i(s)\varphi_j^{[i]}(s) + f(s), \quad j \in \mathbb{N}_0.$$

Then $h_j \rightarrow h_0$ uniformly on $[-m, m]$. Next we have

$$\left| e^{\int_s^{-m} \lambda_1(u) du} h_j(s) \right| \leq \left(M \sum_{i=2}^n \lambda_i^+ + \|f\| \right) e^{-\lambda_1^+(s+m)}, \quad s \in [-\infty, -m].$$

Since

$$\int_{-\infty}^{-m} e^{-\lambda_1^+(s+m)} ds = -\frac{1}{\lambda_1^+} < \infty,$$

we can apply the Lebesgue dominated convergence theorem to obtain $(A\varphi_j)(-m) \rightarrow (A\varphi_0)(-m)$. From (3.1),

$$x_j'(t) - x_0'(t) = \lambda_1(t)(x_j(t) - x_0(t)) + h_j(t) - h_0(t) \quad (3.4)$$

for $x_j(t) = (A\varphi_j)(t)$, $k \in \mathbb{N}_0$. Integrating the both sides of (3.4) from $-m$ to t , we have

$$\begin{aligned} x_j(t) - x_0(t) &= x_j(-m) - x_0(-m) + \int_{-m}^t \lambda_1(s)(x_j(s) - x_0(s)) ds \\ &\quad + \int_{-m}^t (h_j(s) - h_0(s)) ds \end{aligned}$$

and

$$|x_j(t) - x_0(t)| \leq |x_j(-m) - x_0(-m)| + 2m\|h_j - h_0\|_m - \lambda_1^+ \int_{-m}^t |x_j(s) - x_0(s)| ds$$

for any $t \in [-m, m]$. Then Gronwall's inequality implies

$$\|x_j - x_0\|_m \leq (|x_j(-m) - x_0(-m)| + 2m\|h_j - h_0\|_m) e^{-2\lambda_1^+ m},$$

which means

$$\|A\varphi_j - A\varphi_0\|_m \leq (|A\varphi_j(-m) - A\varphi_0(-m)| + 2m\|h_j - h_0\|_m) e^{-2\lambda_1^+ m}.$$

Hence $A\varphi_j(t) \rightarrow A\varphi_0(t)$ uniformly on $t \in [-m, m]$. Since $m \in \mathbb{N}$ is arbitrarily, we get $A\varphi_j \rightarrow A\varphi_0$ in $C(\mathbb{R}, \mathbb{R})$, i.e., $A : B_C(M, L) \rightarrow C(\mathbb{R}, \mathbb{R})$ is continuous. This proves the continuity of A . $\square$

Of course, then $A : B_{PAP}(M, L) \rightarrow C(\mathbb{R}, \mathbb{R})$ is continuous as well.

Theorem 3. Suppose

$$-\lambda_1^+ - \sum_{i=2}^n \lambda_i^+ > 0. \quad (3.5)$$

Then for any $f \in B_{PAP}(M', L')$, Eq. (1.1) has a solution

$$\varphi \in \overline{B_{PAP}(M, L)} \subset C(\mathbb{R}, \mathbb{R})$$

for the constants M and L given by

$$M = -\frac{M'}{\lambda_1^+ + \sum_{i=2}^n \lambda_i^+}, \quad L = \frac{2M'\lambda_1^+}{\lambda_1^+ + \sum_{i=2}^n \lambda_i^+}. \quad (3.6)$$

Proof. For any $\varphi, \psi \in B_C(M, L), t, t_1, t_2 \in \mathbb{R}$, from (3.6) we have

$$\begin{aligned} |(A\varphi)(t)| &\leq \sum_{i=2}^n \lambda_i^+ \left| \int_{-\infty}^t e^{\int_s^t \lambda_1(u) du} \varphi^{[i]}(s) ds \right| + \left| \int_{-\infty}^t e^{\int_s^t \lambda_1(u) du} f(s) ds \right| \\ &\leq -\frac{M}{\lambda_1^+} \sum_{i=2}^n \lambda_i^+ - \frac{M'}{\lambda_1^+} = M. \end{aligned}$$

Without loss of generality, assume $t_2 \geq t_1$, then

$$\begin{aligned} &|(A\varphi)(t_2) - (A\psi)(t_1)| \\ &\leq \sum_{i=2}^n \lambda_i^+ \left| \int_{-\infty}^{t_2} e^{\int_s^{t_2} \lambda_1(u) du} \varphi^{[i]}(s) ds - \int_{-\infty}^{t_1} e^{\int_s^{t_1} \lambda_1(u) du} \varphi^{[i]}(s) ds \right| \\ &\quad + \left| \int_{-\infty}^{t_2} e^{\int_s^{t_2} \lambda_1(u) du} f(s) ds - \int_{-\infty}^{t_1} e^{\int_s^{t_1} \lambda_1(u) du} f(s) ds \right| \\ &\leq \sum_{i=2}^n \lambda_i^+ \left[\left| \int_{-\infty}^{t_1} e^{\int_s^{t_1} \lambda_1(u) du} (e^{\int_{t_1}^{t_2} \lambda_1(u) du} - 1) \varphi^{[i]}(s) ds \right| \right. \\ &\quad \left. + \left| \int_{t_1}^{t_2} e^{\int_s^{t_2} \lambda_1(u) du} \varphi^{[i]}(s) ds \right| \right] \\ &+ \left[\left| \int_{-\infty}^{t_1} e^{\int_s^{t_1} \lambda_1(u) du} (e^{\int_{t_1}^{t_2} \lambda_1(u) du} - 1) f(s) ds \right| + \left| \int_{t_1}^{t_2} e^{\int_s^{t_2} \lambda_1(u) du} f(s) ds \right| \right] \\ &\leq M \sum_{i=2}^n \lambda_i^+ \left[|\lambda_1^+| |t_2 - t_1| \left| \int_{-\infty}^{t_1} e^{\lambda_1^+(t_1-s)} ds \right| + \left| \int_{t_1}^{t_2} e^{\lambda_1^+(t_2-s)} ds \right| \right] \\ &\quad + M' \left[|\lambda_1^+| |t_2 - t_1| \left| \int_{-\infty}^{t_1} e^{\lambda_1^+(t_1-s)} ds \right| + \left| \int_{t_1}^{t_2} e^{\lambda_1^+(t_2-s)} ds \right| \right] \\ &\leq 2(M \sum_{i=2}^n \lambda_i^+ + M') |t_2 - t_1| = L |t_2 - t_1|. \end{aligned}$$

This shows that $A\varphi \in B_C(M, L)$. Hence $A : B_C(M, L) \rightarrow B_C(M, L)$ and also $A : B_{PAP}(M, L) \rightarrow B_{PAP}(M, L)$. By Lemma 2 we get $A : \overline{B_{PAP}(M, L)} \rightarrow \overline{B_{PAP}(M, L)}$. So all conditions of Tikhonov fixed theorem are satisfied for A with $\Omega = \overline{B_{PAP}(M, L)}$ and $X = C(\mathbb{R}, \mathbb{R})$. Thus there exists a fixed point φ in $\overline{B_{PAP}(M, L)}$ of A . This is

equivalent to say that φ is a solution of (1.1) in $\overline{B_{PAP}(M, L)}$. This completes the proof. $\square$

The solution φ of Theorem 3 is a limit in $C(\mathbb{R}, \mathbb{R})$ of a sequence of pseudo almost periodic functions in $B_{PAP}(M, L)$, i.e., φ is a uniform approximation on each compact intervals of $\mathbb{R}$ of a sequence of pseudo almost periodic functions. If $f \in B_{AP}(M', L')$ then we know by Theorem 1 only that in general, the solution φ of Theorem 3 belongs just to $f \in \overline{B_{AP}(M, L)}$.

4. UNIQUENESS RESULT

In this section, uniqueness of (1.1) will be proved, moreover, an example will be showed.

Theorem 4. Suppose (3.5). Then for any $f \in B_{PAP}(M', L')$, Eq. (1.1) has the unique solution $\varphi \in B_{PAP}(M, L)$ provided that it holds

$$\Gamma = -\frac{1}{\lambda_1^+} \sum_{i=2}^n \sum_{j=0}^{i-1} \lambda_i^+ L^j < 1 \quad (4.1)$$

for the constants M and L given by (3.6).

Proof. We already known that $A : B_{PAP}(M, L) \rightarrow B_{PAP}(M, L)$ where we consider $B_{PAP}(M, L)$ in $BC(\mathbb{R}, \mathbb{R})$. For $\varphi, \psi \in B_{PAP}(M, L)$, by (3.2) and (3.3)

$$\begin{aligned} |\varphi(t) - \psi(t)| &= |(A\varphi)(t) - (A\psi)(t)| \leq \sum_{i=2}^n \lambda_i^+ \int_{-\infty}^t e^{\int_s^t \lambda_1(u) du} |\varphi^{[i]}(s) - \psi^{[i]}(s)| ds \\ &\leq -\frac{1}{\lambda_1^+} \sum_{i=2}^n \sum_{j=0}^{i-1} \lambda_i^+ L^j \|\varphi - \psi\| = \Gamma \|\varphi - \psi\|, \end{aligned}$$

thus

$$\|\varphi - \psi\| \leq \Gamma \|\varphi - \psi\|.$$

From (4.1), we know $\Gamma < 1$ and $B_{PAP}(M, L)$ is closed in the Banach space $BC(\mathbb{R}, \mathbb{R})$. The Banach fixed point theorem gives a unique fixed point of A in $B_{PAP}(M, L)$. The proof is finished. $\square$

Example 1. Now, we apply Theorem 4. Consider the following equation:

$$\begin{aligned} x'(t) &= -(|\sin t + \sin \sqrt{3}t| + 6)x(t) + \frac{1}{5}(|\cos t + \cos \sqrt{5}t| + 8)x(x(t)) \\ &\quad + \frac{1}{7}(\sin t + \sin \sqrt{2}t + \operatorname{sech} t), \end{aligned} \quad (4.2)$$

where $\lambda_1(t) = -(|\sin t + \sin \sqrt{3}t| + 6)$, $\lambda_2(t) = \frac{1}{5}(|\cos t + \cos \sqrt{5}t| + 8)$ and $f(t) = \frac{1}{7}(\sin t + \sin \sqrt{2}t + \operatorname{sech} t)$. Take $M' = \frac{3}{7}$ and $L' = \frac{4}{7}$. A simple calculation yields

$\lambda_1^+ = -6$ and $\lambda_2^+ = 2$, so (3.5) holds: $-\lambda_1^+ - \lambda_2^+ = 4 > 0$. Then by (3.6) we derive $M = \frac{3}{28}$ and $L = \frac{9}{7}$. Thus (4.1) gives $\Gamma = \frac{16}{21} < 1$. Hence by Theorem 4, Eq. (4.2) has a unique solution in $B_{PAP}(\frac{3}{28}, \frac{9}{7})$.

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NEW INTEGRABILITY CONDITIONS OF DERIVATION EQUATIONS IN A SUBSPACE OF ASYMMETRIC AFFINE CONNECTION SPACE

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Abstract. In the work [10] we obtained derivational equations of a submanifold of a space L_N with asymmetric affine connection. Based on asymmetry of the connection we define four kinds of covariant derivative and obtain four kinds of derivational equations.

In [20] are examined integrability conditions of derivational equations, using the 1st and the 2nd kind of derivative, and in the present work we do it on the base of the 3rd and 4th kind.

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1. INTRODUCTION

In 1922 Cartan was put forward a modification of General Relativity Theory (GRT), by relaxing the assumption that the affine connection has vanishing the antisymmetric part (torsion tensor), and relating the torsion to the density of intrinsic angular momentum. Also, the torsion is implicit in the 1928 Einstein theory of gravitation with teleparallelism.

From 1923 to the end of his life Einstein worked on various variants of Unified Field Theory (Non-symmetric Gravitational Theory-NGT) [3]. This theory had the aim to unite gravitation theory and the theory of electromagnetism. Introducing different variants of his NGT, Einstein used a complex basic tensor, with a symmetric real part and a skew-symmetric imaginary part. Starting from 1950, Einstein used the real non-symmetric basic tensor g , sometimes called *generalized Riemannian metric/manifold*.

Notice that in NGT the symmetric part g_{ij} of the basic tensor g_{ij} ($g_{ij} = g_{ij} + g_{ij}$) is related to gravitation, and the skew-symmetric one g_{ij} to electromagnetism.

While on a Riemannian space the connection coefficients are expressed by virtue of the metric, g_{ij} , in Einstein's work on NGT the connection between these

magnitudes is determined by the so-called *Einstein metricity condition*, i.e. the non-symmetric metric tensor g and the connection components L_{ij}^k are connected with the equations

$$\frac{\partial g_{ij}}{\partial x^m} - L_{im}^p g_{pj} - L_{mj}^p g_{ip} = 0. \quad (1.1)$$

The choice of a connection in NGT is not uniquely determined. In particular, in NGT there exist two kinds of the covariant derivative. For example, for tensor a_j^i :

$$a_{j|m}^+ = \frac{\partial a_j^i}{\partial x^m} + L_{pm}^i a_j^p - L_{jm}^p a_p^i; \quad a_{j|m}^- = \frac{\partial a_j^i}{\partial x^m} + L_{mp}^i a_j^p - L_{mj}^p a_p^i,$$

where the lowering and the rising of indices one defines by equations

$$g_{pi} g^{pj} = g_{ip} g^{jp} = \delta_i^j, \quad (1.2)$$

Einstein considered only one curvature tensor:

$$R_{klm}^i = L_{kl,m}^i - L_{km,l}^i - L_{sl}^i L_{km}^s + L_{sm}^i L_{kl}^s \quad (1.3)$$

and proved Bianchi type identity for covariant curvature tensor (see [2])

$$R_{-+--+}^{iklm|n} + R_{-++-+}^{ikmn|l} + R_{-+--+}^{iknl|m} = 0,$$

where is $R_{iklm} = g_{si} R^s_{klm}$.

Afterwards, several mathematicians dealt with non-symmetric affine connection, for example, Eisenhart [4], Prvanović [13], Minčić [9–12], Stanković [16, 17] etc. Sinyukov [14] introduced the concept of almost geodesic mappings between affine connected spaces without torsion. Mikeš [1], [5–8, 13], [15], [18] gave some significant contributions to the study of geodesic and almost geodesic mappings of affine connected, Riemannian and Einstein spaces.

Let L_N be a space with asymmetric affine connection L_{jk}^i (in local coordinates), and torsion tensor T_{jk}^i .

In L_N one can define four kinds of covariant derivatives. For example, for a tensor a_j^i we have

$$\begin{aligned} a_{j|_1 m}^i &= a_{j,m}^i + L_{pm}^i a_j^p - L_{jm}^p a_p^i, & a_{j|_2 m}^i &= a_{j,m}^i + L_{mp}^i a_j^p - L_{mj}^p a_p^i, \\ a_{j|_3 m}^i &= a_{j,m}^i + L_{pm}^i a_j^p - L_{mj}^p a_p^i, & a_{j|_4 m}^i &= a_{j,m}^i + L_{mp}^i a_j^p - L_{jm}^p a_p^i. \end{aligned}$$

In [6] is proved very important theorem

Theorem 1. *Let $(L_N, g_{ij} = g_{ij} + g_{ij})$ be an asymmetric affine connection space and Γ_{jk}^i be the Levi-Civita connection of g_{ij} . Let L_{jk}^i be a linear connection with torsion T_{jk}^i . Then L_{jk}^i is uniquely determined by the following formula*

$$L_{i,jk} = \Gamma_{i,jk} + \frac{1}{2}(T_{i,jk} + T_{k,ij} - T_{j,ki}) - \frac{1}{2}(g_{jk|i} + g_{ki|j} - g_{ji|k}), \quad (1.4)$$

where $L_{i,jk} = g_{pi} L_{jk}^p$.

A submanifold $X_M \subset L_N$ is defined by equations

$$x^i = x^i(u^1, \dots, u^M) = x^i(u^\alpha), \quad i = \overline{1, N}.$$

Partial derivatives $B_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}$ ($\text{rank}(B_\alpha^i) = M$) define tangent vectors on X_M .

Consider $N - M$ contravariant vectors C_A^i ($A, B, C, \dots, \in \{M+1, \dots, N\}$) defined on X_M and linearly independent, and let the matrix $\begin{pmatrix} B_\alpha^i \\ C_A^i \end{pmatrix}$ be inverse for the matrix $\begin{pmatrix} B_\alpha^i \\ C_A^i \end{pmatrix}$ provided that the following conditions are satisfied [19, 20]:

$$\begin{aligned} a) B_\alpha^i B_i^\beta &= \delta_\alpha^\beta; & b) B_\alpha^i C_i^A &= 0; & c) B_i^\alpha C_A^i &= 0; \\ d) C_A^i C_i^B &= \delta_A^B; & e) B_\alpha^i B_j^\alpha + C_A^i C_j^A &= \delta_j^i; \end{aligned}$$

The magnitudes B_α^i, B_i^α are **projection factors (tangent vectors)**, and the magnitudes C_A^i, C_i^A are **affine pseudonormals** [10, 20].

The **induced connection** on X_M is [10, 19, 20]:

$$\tilde{L}_{\beta\gamma}^\alpha = B_i^\alpha (B_{\beta,\gamma}^i + L_{jk}^i B_\beta^j B_\gamma^k),$$

where $B_{\beta,\gamma}^i = \partial B_\beta^i / \partial u^\gamma = \partial^2 x^i / \partial u^\beta \partial u^\gamma$. Because L is asymmetric by virtue of j, k , $\tilde{L}$ is asymmetric in β, γ too. The submanifold X_M endowed with $\tilde{L}$ becomes L_M and we write $L_M \subset L_N$.

The set of pseudonormals of the submanifold $X_M \subset L_N$ makes a **pseudonormal bundle** of X_M , and we note it X_{N-M}^N . We have defined in [10] **induced connections of pseudonormal bundle** with coefficients

$$\bar{L}_{B\mu}^A = C_i^A (C_{B,\mu}^i + L_{jk}^i C_B^j B_\mu^k), \quad \bar{L}_{B\mu}^A = C_i^A (C_{B,\mu}^i + L_{kj}^i C_B^j B_\mu^k).$$

As the coefficients $L, \tilde{L}, \bar{L}$ are generally asymmetric, we can define four kinds of covariant derivative for a tensor, defined in the points of X_M . For example:

$$\begin{aligned} t_{j\beta B}^{i\alpha A}|_\mu &= t_{j\beta B,\mu}^{i\alpha A} + L_{pm}^i t_{j\beta B}^{p\alpha A} - L_{jm}^p t_{p\beta B}^{i\alpha A} + \tilde{L}_{\pi\mu}^\alpha t_{j\beta B}^{i\pi A} - \tilde{L}_{\beta\mu}^\pi t_{j\pi B}^{i\alpha A} + \bar{L}_{P\mu}^A t_{j\beta B}^{i\alpha P} - \bar{L}_{\beta\mu}^P t_{j\beta P}^{i\alpha A}, \\ t_{j\beta B}^{i\alpha A}|_\mu &= t_{j\beta B,\mu}^{i\alpha A} + L_{mp}^i t_{j\beta B}^{p\alpha A} - L_{mj}^p t_{p\beta B}^{i\alpha A} + \tilde{L}_{\mu\pi}^\alpha t_{j\beta B}^{i\pi A} - \tilde{L}_{\mu\beta}^\pi t_{j\pi B}^{i\alpha A} + \bar{L}_{P\mu}^A t_{j\beta B}^{i\alpha P} - \bar{L}_{\beta\mu}^P t_{j\beta P}^{i\alpha A} \\ t_{j\beta B}^{i\alpha A}|_\mu &= t_{j\beta B,\mu}^{i\alpha A} + L_{pm}^i t_{j\beta B}^{p\alpha A} - L_{mj}^p t_{p\beta B}^{i\alpha A} + \tilde{L}_{\pi\mu}^\alpha t_{j\beta B}^{i\pi A} - \tilde{L}_{\mu\beta}^\pi t_{j\pi B}^{i\alpha A} + \bar{L}_{P\mu}^A t_{j\beta B}^{i\alpha P} - \bar{L}_{\beta\mu}^P t_{j\beta P}^{i\alpha A} \\ t_{j\beta B}^{i\alpha A}|_\mu &= t_{j\beta B,\mu}^{i\alpha A} + L_{mp}^i t_{j\beta B}^{p\alpha A} - L_{jm}^p t_{p\beta B}^{i\alpha A} + \tilde{L}_{\mu\pi}^\alpha t_{j\beta B}^{i\pi A} - \tilde{L}_{\mu\beta}^\pi t_{j\pi B}^{i\alpha A} + \bar{L}_{P\mu}^A t_{j\beta B}^{i\alpha P} - \bar{L}_{\beta\mu}^P t_{j\beta P}^{i\alpha A} \end{aligned}$$

In this manner four connections ∇_{θ} on $X_M \subset L_N$ are defined. We shall note the obtained structures $(X_M \subset L_N, \nabla_{\theta}, \theta \in \{1, \dots, 4\})$.

2. NEW INTEGRABILITY CONDITIONS OF DERIVATIONAL EQUATIONS FOR TANGENTS

2.0. We have obtained in [20] integrability conditions of derivational equations and corresponding Gauss-Codazzi equations in the structure $(X_M \subset L_N, \nabla_{\theta}, \theta \in \{1, 2\})$. In the present work we are solving this problem for $\theta \in \{3, 4\}$. Based on the Theorem 2.2. in [10], the following derivational equations for tangents are in force

$$B_{\alpha|\mu}^i = \Omega_{\alpha\mu}^P C_P^i, \quad \theta \in \{3, 4\} \quad (2.1)$$

$$B_{i|\mu}^{\alpha} = \widehat{\Omega}_{P\mu}^{\alpha} C_i^P, \quad \theta \in \{3, 4\}, \quad (2.2)$$

and, (see [20]), for pseudonormals

$$C_{A|\mu}^i = -\widehat{\Omega}_{A\mu}^{\pi} B_{\pi}^i, \quad \theta \in \{3, 4\} \quad (2.3)$$

$$C_{i|\mu}^A = -\Omega_{\pi\mu}^A B_i^{\pi}, \quad \theta \in \{3, 4\}. \quad (2.4)$$

In this manner, one obtains

$$B_{\alpha|\mu|\omega}^i - B_{\alpha|\omega|\mu}^i = (\widehat{\Omega}_{P\mu}^{\pi} \Omega_{\omega}^P - \widehat{\Omega}_{P\omega}^{\pi} \Omega_{\mu}^P) B_{\pi}^i + (\Omega_{\alpha\mu|\omega}^P - \Omega_{\alpha\omega|\mu}^P) C_P^i, \quad \theta, \omega \in \{3, 4\}. \quad (2.5)$$

and analogously

$$B_{i|\mu|\omega}^{\alpha} - B_{i|\omega|\mu}^{\alpha} = (\Omega_{\pi\mu}^P \widehat{\Omega}_{P\omega}^{\alpha} - \Omega_{\pi\omega}^P \widehat{\Omega}_{P\mu}^{\alpha}) B_i^{\pi} + (\widehat{\Omega}_{P\mu|\omega}^{\alpha} - \widehat{\Omega}_{P\omega|\mu}^{\alpha}) C_i^P, \quad \theta \in \{3, 4\}.$$

where [10]

$$\begin{aligned} \Omega_{\alpha\mu}^P &= C_i^P (B_{\alpha,\mu}^i + L_{pm}^i B_{\alpha}^p B_{\mu}^m) = \Omega_{\alpha\mu}^P, \\ \Omega_{\alpha\mu}^P &= C_i^P (B_{\alpha,\mu}^i + L_{mp}^i B_{\alpha}^p B_{\mu}^m) = \Omega_{\alpha\mu}^P, \end{aligned} \quad (2.6)$$

$$\widehat{\Omega}_{P\mu}^{\alpha} = C_P^i (B_{i,\mu}^{\alpha} + L_{mi}^p B_P^{\alpha} B_{\mu}^m) = \widehat{\Omega}_{P\mu}^{\alpha}, \quad \widehat{\Omega}_{P\mu}^{\alpha} = C_P^i (B_{i,\mu}^{\alpha} + L_{im}^p B_P^{\alpha} B_{\mu}^m) = \widehat{\Omega}_{P\mu}^{\alpha}.$$

From the Theorem 2.2. in [10] the induced connection $\widetilde{L}$ in the structure $(X_M \subset L_N, \nabla_{\theta}, \theta \in \{3, 4\})$ is symmetric, i.e.

$$\widetilde{T}_{\beta\gamma}^{\alpha} = 0, \quad (2.7)$$

and based on the Theorem 3.2. in this case is

$$\overline{L}_1 = \overline{L}_2 = \overline{L}, \quad (2.8)$$

that is there exists an unique connection in the pseudonormal bundle.

2.1. Using the Ricci-type identities (12,13) in [9], by virtue of (2.7), we obtain

$$B_{\alpha|\mu|\nu}^i - B_{\alpha|\nu|\mu}^i = R_{\theta-2}^i{}_{pmn} B_{\alpha}^p B_{\mu}^m B_{\nu}^n - \widetilde{R}_{\alpha\mu\nu}^{\pi} B_{\pi}^i + (-1)^{\theta} \widetilde{T}_{\mu\nu}^{\pi} B_{\alpha|\pi}^i, \quad \theta \in \{3, 4\}. \quad (2.9)$$

where

$$\begin{aligned} a) \quad R_1^i{}_{jmn} &= L_{jm,n}^i - L_{jn,m}^i + L_{jm}^p L_{pn}^i - L_{jn}^p L_{pm}^i, \\ b) \quad R_2^i{}_{jmn} &= L_{mj,n}^i - L_{nj,m}^i + L_{mj}^p L_{np}^i - L_{nj}^p L_{mp}^i, \end{aligned} \quad (2.10)$$

are **curvature tensors of the 1st respectively the 2nd kind** of the L_N and $\widetilde{R}_{\beta\mu\nu}^{\alpha}$ is, with respect of (2.8), curvature tensor of $L_M^o \subset L_N$, where L_M^o is a subspace with symmetric affine connection.

Further, we examine integrability conditions for derivational equations of B_{α}^i and B_i^{α} , i.e. for $B_{\alpha|\mu}^i$, $B_{i|\mu}^{\alpha}$, $\theta \in \{3, 4\}$.

Substituting $\theta = \omega \in \{3, 4\}$ into (2.5) and comparing with (2.9), taking into consideration (2.7), we get **the 1st and the 2nd kind integrability condition of derivational equation (2.1)** in the structure $(X_M \subset L_N, \nabla, \theta \in \{3, 4\})$:

$$R_{\theta-2}^i{}_{pmn} B_{\alpha}^p B_{\mu}^m B_{\nu}^n - \widetilde{R}_{\alpha\mu\nu}^{\pi} B_{\pi}^i = (\widehat{\Omega}_{P\mu}^{\pi} \Omega_{\theta}^P{}_{\alpha\nu} - \widehat{\Omega}_{P\nu}^{\pi} \Omega_{\theta}^P{}_{\alpha\mu}) B_{\pi}^i + (\Omega_{\theta}^P{}_{\alpha\mu|\nu} - \Omega_{\theta}^P{}_{\alpha\nu|\mu}) C_{P}^i, \quad \theta \in \{3, 4\}. \quad (2.11)$$

Multiplying this equation equation with B_i^{λ} and taking into consideration (1.2), we obtain

$$R_{\theta-2}^i{}_{pmn} B_i^{\lambda} B_{\alpha}^p B_{\mu}^m B_{\nu}^n - \widetilde{R}_{\alpha\mu\nu}^{\lambda} = \widehat{\Omega}_{P\mu}^{\lambda} \Omega_{\theta}^P{}_{\alpha\nu} - \widehat{\Omega}_{P\nu}^{\lambda} \Omega_{\theta}^P{}_{\alpha\mu}, \quad \theta \in \{3, 4\}. \quad (2.12)$$

i.e the **Gauss equation of the 1st and the 2nd kind** in the structure $(X_M \subset L_N, \nabla, \theta \in \{3, 4\})$.

If one multiplies (2.11) with C_i^L , it follows that

$$R_{\theta-2}^i{}_{pmn} C_i^L B_{\alpha}^p B_{\mu}^m B_{\nu}^n = \Omega_{\theta}^L{}_{\alpha\mu|\nu} - \Omega_{\theta}^L{}_{\alpha\nu|\mu}, \quad \theta \in \{3, 4\}. \quad (2.13)$$

which are **the 1st Codazzi equation of the 1st and the 2nd kind** for the cited structure.

2.1' Further, if we use the Ricci type identities ([9], equation (12))

$$\begin{aligned} B_{i|\mu|_3}^\alpha - B_{i|_4}^\alpha|_{\mu|_4} &= -R_{imn}^P B_p^\alpha B_\mu^m B_v^n + \widetilde{R}_{\pi\mu\nu}^\alpha B_i^\pi, \\ B_{i|\mu|_4}^\alpha - B_{i|_4}^\alpha|_{\mu|_4} &= -R_{imn}^P B_p^\alpha B_\mu^m B_v^n + \widetilde{R}_{\pi\mu\nu}^\alpha B_i^\pi, \end{aligned} \quad (2.14)$$

and (2.3') for $\theta = \omega \in \{3, 4\}$, we have

$$\begin{aligned} -R_{imn}^P B_p^\alpha B_\mu^m B_v^n + \widetilde{R}_{\pi\mu\nu}^\alpha B_i^\pi &= (\Omega_{3\pi\mu}^P \widehat{\Omega}_{3P\nu}^\alpha - \Omega_{\theta\pi\nu}^P \widehat{\Omega}_{\theta P\mu}^\alpha) B_i^\pi + (\widehat{\Omega}_{3P\mu|_3}^\alpha - \widehat{\Omega}_{3P\nu|_3}^\alpha) C_i^P, \\ -R_{imn}^P B_p^\alpha B_\mu^m B_v^n + \widetilde{R}_{\pi\mu\nu}^\alpha B_i^\pi &= (\Omega_{4\pi\mu}^P \widehat{\Omega}_{4P\nu}^\alpha - \Omega_{\theta\pi\nu}^P \widehat{\Omega}_{\theta P\mu}^\alpha) B_i^\pi + (\widehat{\Omega}_{4P\mu|_4}^\alpha - \widehat{\Omega}_{4P\nu|_4}^\alpha) C_i^P \end{aligned} \quad (2.15)$$

which are **the 1st and the 2nd kind integrability conditions of derivational equation (2.2)** in the structure $(X_M \subset L_N, \nabla, \theta \in \{3, 4\})$.

a') By multiplying the previous equation with B_λ^i , one obtains

$$\begin{aligned} \widetilde{R}_{\lambda\mu\nu}^\alpha - R_{imn}^P B_p^\alpha B_\lambda^i B_\mu^m B_v^n &= \Omega_{3\lambda\mu}^P \widehat{\Omega}_{3P\nu}^\alpha - \Omega_{3\lambda\nu}^P \widehat{\Omega}_{3P\mu}^\alpha, \\ \widetilde{R}_{\lambda\mu\nu}^\alpha - R_{imn}^P B_p^\alpha B_\lambda^i B_\mu^m B_v^n &= \Omega_{4\lambda\mu}^P \widehat{\Omega}_{4P\nu}^\alpha - \Omega_{4\lambda\nu}^P \widehat{\Omega}_{4P\mu}^\alpha, \end{aligned} \quad (2.16)$$

which is **another form of (2.12)**.

If we exchange at (2.12) $i \leftrightarrow p$, $\alpha \leftrightarrow \lambda$, for $\theta = 3$ from that equation one gets

$$R_{imn}^P B_p^\alpha B_\mu^m B_v^n - \widetilde{R}_{\lambda\mu\nu}^\alpha = \widehat{\Omega}_{3P\mu}^\alpha \Omega_{3\lambda\nu}^P - \widehat{\Omega}_{3P\nu}^\alpha \Omega_{3\lambda\mu}^P.$$

Summing this equation with 1st case in (2.16), one concludes:

$$(R_{imn}^P - R_{imn}^P) B_p^\alpha B_\lambda^i B_\mu^m B_v^n = 0. \quad (2.17)$$

Putting $\theta = 4$ at (2.12), we get the 1st case from (2.16), and together with 2nd case it follows.

If we multiply (2.15) with C_L^i , it follows that

$$\begin{aligned} -R_{imn}^P B_p^\alpha C_L^i B_\mu^m B_v^n &= \widehat{\Omega}_{3L\mu|_3}^\alpha - \widehat{\Omega}_{3L\nu|_3}^\alpha, \\ -R_{imn}^P B_p^\alpha C_L^i B_\mu^m B_v^n &= \widehat{\Omega}_{4L\mu|_4}^\alpha - \widehat{\Omega}_{4L\nu|_4}^\alpha, \end{aligned} \quad (2.18)$$

and this is **another form for the 1st Codazzi equation of the 1st and the 2nd kind in the cited structure**.

2.2. Further we use the Ricci-type identity (equation (46) from [9])

$$B_{\alpha|\mu|_3}^i - B_{\alpha|_4}^i|_{\mu|_4} = R_{p\mu\nu}^i B_\alpha^p - \widetilde{R}_{\alpha\mu\nu}^\pi B_\pi^i, \quad (2.19)$$

where

$$R_{4j\mu\nu}^i = (L_{jm,n}^i - L_{jn,m}^i + L_{jm}^p L_{np}^i - L_{nj}^p L_{pm}^i) B_\mu^m B_\nu^n + T_{jm}^i (B_{\mu,\nu}^m - \tilde{L}_{\mu\nu}^\pi B_\pi^m), \quad (2.20)$$

is the 4th kind curvature tensor of L_N with respect to $X_M \subset L_N$. On the other hand, putting into (2.5) $\theta = 3, \omega = 4$ and comparing with (2.19), we get the 3rd kind integrability condition of derivational equation (2.1) in the structure $(X_M \subset L_N, \nabla, \theta \in \{3, 4\})$:

$$R_{4p\mu\nu}^i B_\alpha^p - \tilde{R}_{\alpha\mu\nu}^\pi B_\pi^i = (\hat{\Omega}_{3P\mu}^\pi \Omega_{4\alpha\nu}^P - \hat{\Omega}_{4P\nu}^\pi \Omega_{3\alpha\mu}^P) B_\pi^i + (\Omega_{3\alpha\mu|v}^P - \Omega_{4\alpha\nu|3}^P) C_P^i. \quad (2.21)$$

a) If one multiplies this equation with B_i^λ , we have

$$R_{4p\mu\nu}^i B_\alpha^p B_i^\lambda - \tilde{R}_{\alpha\mu\nu}^\lambda = \hat{\Omega}_{3P\mu}^\pi \Omega_{4\alpha\nu}^P - \hat{\Omega}_{4P\nu}^\pi \Omega_{3\alpha\mu}^P \quad (2.22)$$

and this is Gauss equation of the 3rd kind in the cited structure.

b) Multiplying (2.21) with C_i^L , we get

$$R_{4p\mu\nu}^i C_i^L B_\alpha^p = \Omega_{3\alpha\mu|v}^P - \Omega_{4\alpha\nu|3}^P, \quad (2.23)$$

i.e. the 1st Codazzi equation of the 3rd kind in the same structure.

2.2'. Based on equation (46) in [9], we have

$$B_{34|\mu|v}^\alpha - B_{43|\nu|\mu}^\alpha = R_{3i\mu\nu}^p B_p^\alpha + \tilde{R}_{\pi\mu\nu}^\alpha B_i^\pi,$$

where

$$R_{3j\mu\nu}^i = (L_{jm,n}^i - L_{nj,m}^i + L_{jm}^p L_{np}^i - L_{nj}^p L_{pm}^i) B_\mu^m B_\nu^n + T_{jm}^i (B_{\mu,\nu}^m - \tilde{L}_{\nu\mu}^\pi B_\pi^m), \quad (2.24)$$

is the 3rd kind curvature tensor of L_N with respect to $X_M \subset L_N$.

Simultaneously, putting into (2) $\theta = 3, \omega = 4$, and comparing with (2), we obtain the 3rd integrability condition of derivational equation (2.2) in the structure $(X_M \subset L_N, \nabla, \theta \in \{3, 4\})$:

$$R_{3i\nu\mu}^p B_p^\alpha + \tilde{R}_{\pi\mu\nu}^\alpha B_i^\pi = (\Omega_{3\pi\mu}^P \hat{\Omega}_{4P\nu}^\alpha - \Omega_{4\pi\nu}^P \hat{\Omega}_{3P\mu}^\alpha) B_i^\pi + (\hat{\Omega}_{3P\mu|v}^\alpha - \hat{\Omega}_{4P\nu|3}^\alpha) C_i^P. \quad (2.25)$$

a') From here, multiplying with B_λ^i

$$R_{3i\nu\mu}^p B_p^\alpha B_\lambda^i + \tilde{R}_{\pi\mu\nu}^\alpha = \Omega_{3\pi\mu}^P \hat{\Omega}_{4P\nu}^\alpha - \Omega_{4\pi\nu}^P \hat{\Omega}_{3P\mu}^\alpha, \quad (2.26)$$

which is another form of the 3rd kind Gauss equation in the cited structure.

b') Similarly as the equation (2.23), we get

$$R_{4\ p\nu\mu}^i C_i^L B_\alpha^p = \Omega_{3\ \alpha\mu|_\nu}^P - \Omega_{4\ \alpha\nu|_\mu}^P, \quad (2.27)$$

and that is another form of (2.23).

Now, we can state the following theorems

Theorem 2. *The 1st and 2nd kind integrability conditions derivational equations (2.1) and (2.1') for submanifold $X_M \subset L_N$ with the structure $(X_M \subset L_N, \nabla, \theta \in \{3, 4\})$, for the connections ∇_θ are given in (2.11) and (2.15) respectively. The 3rd kind of these conditions are given in (2.21).*

Theorem 3. *Gauss equations of the 1st and the 2nd kind are given in (2.12), and of the 3rd one in (2.22). The 1st Codazzi equations of the 1st and the 2nd kind are given in (2.18), and of the 3rd kind in (2.23). The equations (2.16), (2.18), (2.26), (2.27) are another forms of previous equations.*

3. INTEGRABILITY CONDITIONS OF DERIVATIONAL EQUATIONS FOR PSEUDONORMALS

3.0. Further, we use the similar procedure on derivational equation of pseudonormals.

Using (2.3), (2.1), we get

$$C_{\theta|\mu|_\nu}^i - C_{\omega|_\nu|_\mu}^i = -(\widehat{\Omega}_{\theta A\mu|_\nu}^\pi - \widehat{\Omega}_{\omega A\nu|_\mu}^\pi) B_\pi^i - (\widehat{\Omega}_{\theta A\mu}^\pi \widehat{\Omega}_{\omega}^P \pi_\nu - \widehat{\Omega}_{\omega A\nu}^\pi \widehat{\Omega}_{\theta}^P \pi_\mu) C_P^i \quad (3.1)$$

and from (2.4), (2.2):

$$C_{\theta|\mu|_\nu}^A - C_{\omega|_\nu|_\mu}^A = -(\Omega_{\theta \pi\mu|_\nu}^A - \Omega_{\omega \pi\nu|_\mu}^A) B_\pi^i - (\Omega_{\theta \pi\mu}^A \widehat{\Omega}_{\omega}^P \pi_\nu - \Omega_{\omega \pi\nu}^A \widehat{\Omega}_{\theta}^P \pi_\mu) C_i^P \\ \theta, \omega \in \{3, 4\}.$$

We need also the Ricci type identities ([11], Equation (2.19))

$$C_{\theta|\mu|_\nu}^i - C_{\omega|_\nu|_\mu}^i = R_{\theta-2\ pmn}^i C_A^p B_\mu^m B_\nu^n - \bar{R}_{A\mu\nu}^P C_P^i, \quad \theta \in \{3, 4\}, \quad (3.2)$$

and in the same way

$$C_{3\ 3}^A|_{\mu|_\nu} - C_{3\ 3}^A|_{\nu|_\mu} = \bar{R}_{P\mu\nu}^A C_i^P - R_{1\ imn}^p C_p^A B_\mu^m B_\nu^n, \\ C_{4\ 4}^A|_{\mu|_\nu} - C_{4\ 4}^A|_{\nu|_\mu} = \bar{R}_{P\mu\nu}^A C_i^P - R_{2\ imn}^p C_p^A B_\mu^m B_\nu^n, \quad (3.3)$$

where $R, \bar{R}$ are given at (2.10) and $\bar{R}$ at [11]

$$\widetilde{R}_{B\mu\nu}^A = \bar{L}_{B\mu,\nu}^A - \bar{L}_{B\nu,\mu}^A + \bar{L}_{B\mu}^P \bar{L}_{P\nu}^A - \bar{L}_{B\nu}^P \bar{L}_{P\mu}^A. \quad (3.4)$$

It follows that

$$C_{A|\mu|v}^i - C_{A|v|\mu}^i = R_{pmn}^i C_A^p - \bar{R}_{A\mu\nu}^P C_P^i, \quad (3.5)$$

and analogously

$$C_{i|\mu|v}^A - C_{i|v|\mu}^A = R_{imn}^P C_A^p + \bar{R}_{P\nu\mu}^A C_i^P,$$

The magnitude $\bar{R}_{B\mu\nu}^A$ is **curvature tensor of L_N with respect to the pseudonormal submanifold X_{N-M}^N** in the structure $(X_M \subset L_N, \nabla, \theta \in \{3, 4\})$.

3.1. Taking $\theta = \omega \in \{3, 4\}$ in (3.1) and comparing with (3.2), we obtain **the 1st and the 2nd kind integrability condition of derivational equation (2.3)** (for pseudonormals) in the structure $(X_M \subset L_N, \nabla, \theta \in \{3, 4\})$:

$$\begin{aligned} R_{\theta-2}^i{}_{pmn} C_A^p B_\mu^m B_\nu^n - \bar{R}_{A\mu\nu}^P C_P^i \\ = -(\hat{\Omega}_{A\mu|v}^\pi + \hat{\Omega}_{\theta A\nu|\mu}^\pi) B_\pi^i - (\hat{\Omega}_{\theta A\mu}^\pi \hat{\Omega}_{\theta \pi\nu}^P + \hat{\Omega}_{\theta A\nu}^\pi \hat{\Omega}_{\theta \pi\mu}^P) C_P^i, \quad \theta \in \{3, 4\}, \end{aligned} \quad (3.6)$$

a) Multiplying (3.6) with B_i^λ , one gets:

$$R_{\theta-2}^i{}_{pmn} B_i^\lambda C_A^p B_\mu^m B_\nu^n = -\hat{\Omega}_{A\mu|v}^\lambda + \hat{\Omega}_{\theta A\nu|\mu}^\lambda, \quad \theta \in \{3, 4\}. \quad (3.7)$$

which is **one more form of the 1st Codazzi equation (2.13)**.

b) If we multiply (3.6) with C_i^L , one obtains

$$R_{\theta-2}^i{}_{pmn} C_i^L C_A^p B_\mu^m B_\nu^n - \bar{R}_{A\mu\nu}^L = -\hat{\Omega}_{\theta A\mu}^\pi \hat{\Omega}_{\theta \pi\nu}^L + \hat{\Omega}_{\theta A\nu}^\pi \hat{\Omega}_{\theta \pi\mu}^L, \quad \theta \in \{3, 4\}. \quad (3.8)$$

and that is **the 2nd Codazzi equation of the 1st and the 2nd kind** in the cited structure.

3.1'. If one takes $\theta = \omega \in \{3, 4\}$ in (3) and compare with (3.3) we obtain **the 1st and the 2nd kind integrability condition of derivational equation (2.4)** in the structure $(X_M \subset L_N, \nabla, \theta \in \{3, 4\})$:

$$\begin{aligned} R_{2}^p{}_{imn} C_P^A B_\mu^m B_\nu^n - \bar{R}_{P\mu\nu}^A C_i^P &= (\hat{\Omega}_{3\pi\mu|v}^A - \hat{\Omega}_{3\pi\nu|\mu}^A) B_i^\pi + (\hat{\Omega}_{3\pi\mu}^A \hat{\Omega}_{3P\nu}^\pi - \hat{\Omega}_{3\pi\nu}^A \hat{\Omega}_{3P\mu}^\pi) C_i^P \\ R_{1}^p{}_{imn} C_P^A B_\mu^m B_\nu^n - \bar{R}_{P\mu\nu}^A C_i^P &= (\hat{\Omega}_{4\pi\mu|v}^A - \hat{\Omega}_{4\pi\nu|\mu}^A) B_i^\pi + (\hat{\Omega}_{4\pi\mu}^A \hat{\Omega}_{4P\nu}^\pi - \hat{\Omega}_{4\pi\nu}^A \hat{\Omega}_{4P\mu}^\pi) C_i^P. \end{aligned} \quad (3.9)$$

which is **the 1st and the 2nd kind integrability conditions ($\theta = 1, 2$) of derivational equation (2.4)**.

a') By multiplying of the previous equation with B_λ^i we get

$$\begin{aligned} R_{2imn}^p C_p^A B_i^\lambda B_\mu^m B_\nu^n &= \Omega_{\lambda\mu|_3}^A - \Omega_{\lambda\nu|_3}^A \\ R_{1imn}^p C_p^A B_i^\lambda B_\mu^m B_\nu^n &= \Omega_{\lambda\mu|_4}^A - \Omega_{\lambda\nu|_4}^A. \end{aligned} \quad (3.10)$$

that is **one more form of the 1st Codazzi equation (2.18)**.

b') Multiplying (3.9) with C_L^i , we get

$$\begin{aligned} R_{2imn}^p C_p^A C_L^i B_\mu^m B_\nu^n - \bar{R}_{L\mu\nu}^A &= \Omega_{\pi\mu}^A \hat{\Omega}_{L\nu}^\pi - \Omega_{\pi\nu}^A \hat{\Omega}_{L\mu}^\pi \\ R_{1imn}^p C_p^A C_L^i B_\mu^m B_\nu^n - \bar{R}_{L\mu\nu}^A &= \Omega_{\pi\mu}^A \hat{\Omega}_{L\nu}^\pi - \Omega_{\pi\nu}^A \hat{\Omega}_{L\mu}^\pi \end{aligned} \quad (3.11)$$

and this is **another form of the 2nd Codazzi equation of the 1st and the 2nd kind**.
3.2. If one takes $\theta = 3, \omega = 4$ in (3.1) and compares obtained equation with (3.5), we obtain **3rd integrability condition of derivational equation (2.3)** in the structure $(X_M \subset L_N, \nabla, \theta \in \{3, 4\})$:

$$R_{4p\mu\nu}^i C_A^p - \bar{R}_{A\mu\nu}^P C_P^i = -(\hat{\Omega}_{A\mu|_4}^\pi - \hat{\Omega}_{A\nu|_3}^\pi) B_\pi^i - (\hat{\Omega}_{A\mu}^P \hat{\Omega}_{4\pi\nu}^P - \hat{\Omega}_{A\nu}^P \hat{\Omega}_3^P \pi_\mu) C_P^i. \quad (3.12)$$

a) Multiplying (3.12) with B_i^λ , we get

$$R_{4p\mu\nu}^i B_i^\lambda C_A^p = -\hat{\Omega}_{A\mu|_4}^\lambda + \hat{\Omega}_{A\nu|_4}^\lambda, \quad (3.13)$$

which is **one more form of (2.17)**.

b) Multiplying (3.12) with C_i^L , we have

$$R_{4p\mu\nu}^i C_i^L C_A^p - \bar{R}_{A\mu\nu}^L = \Omega_{\pi\mu}^L \hat{\Omega}_{A\nu}^\pi - \Omega_{\pi\nu}^L \hat{\Omega}_{A\mu}^\pi. \quad (3.14)$$

which is **the 2nd Codazzi equation of the 3rd kind**.

3.2'. Endly, we put $\theta = 3, \omega = 4$ into (3) and compare obtained equation with (3.4'). In that manner, one obtains **the 3rd kind integrability condition of derivational equation (2.4)** in the structure $(X_M \subset L_N, \nabla, \theta \in \{3, 4\})$:

$$R_{3iv\mu}^p C_p^A + \bar{R}_{P\mu\nu}^A C_i^P = -(\Omega_{\pi\mu|_4}^A - \Omega_{\pi\nu|_3}^A) B_i^\pi - (\Omega_{\pi\mu}^A \hat{\Omega}_{P\nu}^\pi - \Omega_{\pi\nu}^A \hat{\Omega}_3^P \pi_\mu) C_i^P.$$

a') If one multiplies (3) with B_λ^i , it follows that

$$R_{3iv\mu}^p C_p^A B_\lambda^i = -\Omega_{\pi\mu|_4}^A + \Omega_{\pi\nu|_3}^A, \quad (3.15)$$

and this is **another form of (2.27) or (3.13)**.

b') Multiplying (3) with C_L^i , we have

$$R_{i\mu\nu}^p C_A^p C_L^i - \bar{R}_{L\mu\nu}^A = \hat{\Omega}_{3L\mu}^\pi \Omega_{\pi\nu}^A - \hat{\Omega}_{4L\nu}^\pi \Omega_{3\pi\mu}^A. \quad (3.16)$$

which is **another form of the 2nd Codazzi equation of the 3rd kind** i.e. of (3.14).

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